SFB 823

A generalized functional delta method

Dominik Wied

Discussion Paper

Nr. 6/2010

SFB 823

A generalized functional delta method

Dominik Wied[*+]

[+]Fakultät Statistik, Technische Universität Dortmund, Germany

February 23, 2010

Abstract

We develop a generalized functional delta method, where the considered random function is not multiplied by a scalar, but by another function. It bases on a generalized Hadamard differentiability between special function spaces. For a certain class of functions, we calculate the Hadamard differential explicitely. We give an example, where the method allows for calculations that are not possible with previous methods.

Keywords. Hadamard differentiability, Asymptotics, Fluctuation tests.

JEL numbers:

1 Introduction

This paper presents a generalized functional delta method that bases on a generalized Hadamard differentiability between function spaces. The delta method is used in the proof of the asymptotic null distribution of the fluctuation test for constant correlation proposed by Wied and Arnold (2009) and can also be used for other CUSUM-type tests.

Wied and Arnold (2009) examine, somewhat simplified, the limit behavior of

$$s_n \cdot (f(M_n) - f(\theta)),$$

where s_n is a function sequence from $D[\epsilon, 1]$, $M_n : \Omega \to D[\epsilon, 1]^k$, $f : D[\epsilon, 1]^k \to D[\epsilon, 1]^l$ and $\theta \in D[\epsilon, 1]^k$, basing on the known limit behavior of

$$s_n \cdot (M_n - \theta),$$

[*]corresponding author: Dominik Wied, Institut für Wirtschafts- und Sozialstatistik, Fakultät Statistik, TU Dortmund, 44221 Dortmund, Germany, E-Mail: dominik.wied@tu-dortmund.de, Phone: +49/231/755 3869, Fax: +49/231/755 5284.

which is given by a functional central limit theorem. More concretely, the function f maps the properly standardized partial sums of the 5-dimensional random vector $(X_i^2, Y_i^2, X_i, Y_i, X_iY_i)'$ on the Bravais-Pearson correlation coefficient based on the first observations.
In an easier setting, the delta method is known for a long time, see e.g. Oehlert (1992): If X_n is a sequence of real random variables, $\theta \in \mathbb{R}$, $s_n = \sqrt{n}$ and it holds

$$s_n \cdot (X_n - \theta) \to_d N(0, \sigma^2),$$

then, for a continuous differentiable function g with $g'(\theta) \neq 0$,

$$s_n \cdot (g(X_n) - g(\theta)) \to_d N(0, \sigma^2(g'(\theta))^2).$$

This concept can be extended to other sequences s_n or to the multivariate case. The delta method is also known in the context of function spaces, see van der Vaart (1998) or van der Vaart (1991). It bases on the usual definition of Hadamard differentiability. Other forms of differentiability are Fréchet and Gateaux differentiability.
In all these definitions, however, the factor s_n in front of the considered random function is a scalar. We do not know any application in which the function is not multiplied by a scalar, but by another function. In this paper, we present a new form of delta method and Hadamard differentiability in which the latter holds.
We present the new definitions and main results in Section 2, while Section 3 shows, how the result can be used in applications, where previous methods are not applicable. Section 4 gives proofs to the results.

2 Definitions and main results

We consider different function spaces, either $D[\epsilon, 1]$ for $\epsilon \geq 0$, the space of càdlàg-functions, or a product space in which each component is either $D[\epsilon, 1]$ or $D^+[\epsilon, 1]$, the space of càdlàg-functions whose values are bounded away from zero. This is a more general setting than in the introduction. We show in the end of Section 3, why this is useful.
We always use the supremum norm together with the σ-field generated by the open balls, see Davidson (1994, p. 435), Gill (1989) or Pollard (1984, chapter 4).
We introduce the spaces

$$\mathbb{G}_1 := \mathbb{H}_1 \times \ldots \times \mathbb{H}_k \ (k \text{ times}, k \geq 1, \mathbb{H}_i \in \{D[\epsilon, 1], D^+[\epsilon, 1], \epsilon \geq 0\}), \tag{1}$$

$$\mathbb{G}_2 := \mathbb{H}_1 \times \ldots \times \mathbb{H}_l \ (l \text{ times}, l \geq 1, \mathbb{H}_i \in \{D[\epsilon, 1], D^+[\epsilon, 1], \epsilon \geq 0\}). \tag{2}$$

The space $\mathbb{G}_1^*$ is another function space with the same structure as $\mathbb{G}_1$. The difference is that some of the $D^+[\epsilon, 1]$ (if existing) are replaced by $D[\epsilon, 1]$ so that $\mathbb{G}_1^*$ is a subset of $\mathbb{G}_1$.
The special case is

$$\mathbb{G}_1 = D[\epsilon, 1]^k := D[\epsilon, 1] \times \ldots \times D[\epsilon, 1] \ (k \text{ times}).$$

Definition 2.1 (Generalized Hadamard differentiability). *Let $\mathbb{G}_1^* \subseteq \mathbb{G}_1$, $\mathbb{G}_2$ from* (1) *and* (2)*, $\theta \in \mathbb{G}_1^*$ and $\epsilon \geq 0$. A function $f : \mathbb{G}_1^* \to \mathbb{G}_2$ is generalized Hadamard differentiable in θ, if there is a continuous, linear map $f'_\theta : \mathbb{G}_1 \to \mathbb{G}_2$ (the generalized Hadamard differential), such that*

$$\lim_{n\to\infty} \left\| \tfrac{f(\theta+r_n h_n)-f(\theta)}{r_n} - f'_\theta(h) \right\|_{\mathbb{G}_2} = 0$$

for all $r_n \in D[\epsilon, 1]$ with $r_n(z) \neq 0 \ \forall z \in [\epsilon, 1] \, \forall n$, $h_n, h \in \mathbb{G}_1$ with $||r_n||_{D[\epsilon,1]} \to 0$ and $||h_n - h||_{\mathbb{G}_1} \to 0$, such that $\theta + r_n h_n \in \mathbb{G}_1^*$ for all n.

The main difference to the common Hadamard differentiability, as explained e.g. in van der Vaart (1998), is that here, r_n is an element of $D[\epsilon, 1]$ and not just a sequence of real numbers. Hence, we need the stronger assumption that r_n goes to 0 in the supremum norm on $D[\epsilon, 1]$. Another difference is that the spaces between which f operates are not arbitrary normed spaces, but special function spaces. Note that a function being generalized Hadamard differentiable is also normal Hadamard differentiable with respect to the function spaces.

Theorem 2.2 (Generalized delta method). *Let the assumptions of Definition 2.1 be fulfilled such that $f : \mathbb{G}_1^* \to \mathbb{G}_2$ is generalized Hadamard differentiable in θ. Let $M_n : \Omega \to \mathbb{G}_1^*$ be random functions such that*

$$s_n \cdot (M_n - \theta) \to_d M$$

as $n \to \infty$ for a sequence $s_n \in D[\epsilon, 1]$ with $||\frac{1}{s_n}||_{D[\epsilon,1]} \to 0, s_n(z) \neq 0 \ \forall z, \forall n$, and a random function M in $\mathbb{G}_1$. Then,

$$s_n \cdot (f(M_n) - f(\theta)) \to_d f'_\theta(M)$$

where f'_θ is the generalized Hadamard differential of f at θ.

The main difference to the delta method, as explained e.g. in van der Vaart (1998, p. 297), is that here, s_n is an element of $D[\epsilon, 1]$ and not just a sequence of real numbers. Hence, we need the stronger assumption that $\frac{1}{s_n}$ goes to 0 in the supremum norm on $D[\epsilon, 1]$. In Definition 2.1, ϵ may also take the value 0, while in the applications in Section 3, it must be larger than 0.
A corollary of Theorem 2.2 is furthermore needed in some situations.

Corollary 2.3. *If in Theorem 2.2 M_n even converges $\mathbb{P}$-almost surely against θ, the condition $M_n : \Omega \to \mathbb{G}_1$ is sufficient.*

We now consider a special type of functions, i.e.

$$f(\theta)(\cdot) = \psi(\theta(\cdot)) \tag{3}$$

for a function $\psi \in C^2$ from an open subset of $\mathbb{R}^k$ to $\mathbb{R}^l$. Thus, the function f maps $\theta(z)$ and θ is not integrated or mapped in other forms. In this case, the generalized Hadamard differential has a general form.

Theorem 2.4. *Let f from* (3) *such that $\psi = \begin{pmatrix} \psi^1 & \dots & \psi^l \end{pmatrix}'$ and θ from 2.1. For each $i \in \{1, \dots, l\}$ the Hessian matrix of ψ^i is continuous and bounded in an open neighborhood of the set $\{\theta(z), z \in [\epsilon, 1]\}$. Then f is generalized Hadamard differentiable in θ and the Hadamard differential has the same form as the usual derivative of ψ, i.e.*

$$f'_\theta(h)(\cdot) = D\psi_{\theta(\cdot)}(h(\cdot)).$$

3 Applications

In this section, we present two applications of the theory in the context of CUSUM-type tests. First, we give an example, where the generalized delta method allows for calculating a special limit function, while the normal delta method is not applicable. Second, we underline the importance of Corollary 2.3. Let for both examples $(X_i)_{i\in\mathbb{N}}$ be a sequence of random variables on $(\Omega, \mathfrak{A}, \mathsf{P})$.

First, suppose that $(X_i)_{i\in\mathbb{N}}$ is an independent sequence with the property

$$X_i = \begin{cases} \mu \ \mathsf{P} - \text{a.s.,} & 2^{2k} \leq i < 2^{2k+1}, k = 0, 1, 2, 3, \dots \\ Y_i, & \text{otherwise,} \end{cases}$$

where $Y_i \sim N(\mu, \sigma^2)$, see also Davidson (1994, p.489) in a slightly different form. Thus, $X_1 = \mu, X_4 = X_5 = X_6 = X_7 = \mu, X_{16} = X_{17} = \ldots = X_{31} = \mu$, and so forth.

Suppose that $g \in C^2$ is a real function with $g'(\mu) \neq 0$, e.g. $g(x) = \exp(x)$, and we want to calculate the limit of the random function

$$A_n(z) = \frac{s_n(z)}{\sqrt{n}} \left(g(\overline{X}_{s_n(z)}) - g(\mu) \right),$$

where

$$\overline{X}_{s_n(z)} = \frac{1}{s_n(z)} \sum_{i=1}^{s_n(z)} X_i.$$

The idea is to consider the random function

$$B_n(z) = \frac{s_n(z)}{\sqrt{n}} \left(\overline{X}_{s_n(z)} - \mu\right) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[zn]} (X_i - \mu)$$

at first and apply the delta method. But this function has no limit for $s_n(z) = [nz]$, the natural choice in the context of functional central limit theorems, because, for $z \in (\frac{1}{2}, 1]$,

$$B_n(z) = B_n\left(\frac{1}{2}\right)$$

with probability 1, when $n = 2^k - 1$ for k odd, and

$$\mathsf{Var}\left(B_n(z) - B_n\left(\frac{1}{2}\right)\right) \to \left(z - \frac{1}{2}\right)\sigma^2 > 0$$

when $n = 2^k - 1$ for k even and $k \to \infty$. However, the functional central limit theorem from Wooldridge and White (1988) is applicable on

$$B_n^*(z) = \frac{s_n^*(z)}{\sqrt{n}} \left(\overline{X}_{s_n(z)} - \mu\right),$$

where $s_n^*(z)$ is the minimal integer that satisfies

$$\sum_{i=2}^{s_n^*(z)} \mathbf{1}(2^{2k-1} \leq i < 2^{2k}, k \in \mathbb{N}) = [nz].$$

With this arrangement, n counts the actual number of increments in the sum, while $s_n^*(1)$ counts the nominal number, including the zeros. It holds $s_n^*(z) \geq [nz]$. We have

$$B_n^*(\cdot) \to_d W_1(\cdot),$$

where $W_1(z)$ is a one-dimensional Brownian motion on $D[\epsilon, 1]$ with covariance structure depending on $(X_i)_{i\in\mathbb{N}}$.

Introducing the function

$$f : D[\epsilon, 1] \to D[\epsilon, 1],$$
$$f(x_1)(\cdot) = g(x_1(\cdot))$$

and applying Theorem 2.2, we now get for

$$A_n^*(z) = \frac{s_n^*(z)}{\sqrt{n}} \left(g(\overline{X}_{s_n^*(z)}) - g(\mu)\right),$$

that

$$A_n^*(\cdot) \to_d f'_\mu(W_1)(\cdot),$$

making use of the fact that

$$\left|\left|\frac{\sqrt{n}}{s_n^*(z)}\right|\right|_{[\epsilon,1]} \to 0. \tag{4}$$

We use the constant function $\theta(z) = \mu$ in $D[\epsilon, 1]$. The Hadamard differential f_μ' can be easily calculated with Theorem 2.4, because f is of the form in (3) with

$$\psi : \mathbb{R} \to \mathbb{R},$$

$$\psi(x_1) = g(x_1).$$

It holds

$$f_\mu'(W_1)(\cdot) = g'(\mu)W_1(\cdot).$$

It is not possible to achieve the limit result for $A_n^*(\cdot)$ with the common functional delta method, because there is no appropriately explicite formula for $s_n^*(z)$ that would allow for separating n and z and applying the delta method. If it were, we could find a function $a(z)$ and a function $b(n)$ such that

$$\frac{s_n^*(z)}{\sqrt{n}} - a(z)b(n) \to 0 \tag{5}$$

uniformly in z. Let for the moment be $z = 1$ and consider the case $s_n^*(1) = 2^{2k+1}$ which corresponds to $n = 2^0 + 2^1 + 2^3 + \ldots + 2^{2k-1}$. Then, it holds, for $k \to \infty$,

$$\frac{n}{s_n^*(1)} \to \frac{1}{3},$$

so that $\frac{s_n^*(1)}{\sqrt{n}}$ is asymptotically equivalent to $3\sqrt{n}$. This means that $b(n)$ is asymptotically equivalent to $\sqrt{n}$, so that a necessary condition for (5) is that

$$\sqrt{n}\left(\frac{s_n^*(z)}{n} - a(z)\right) \to 0,$$

uniformly in z. Yet, we cannot find such a function a: For $n = 2^0 + 2^1 + 2^3 + \ldots + 2^{2k-1}$ and $s_n^*(1) = 2^{2k+1}$, it holds

$$\lim_{z\to 1^-} s_n^*(z) = 2^{2k}$$

and thus,

$$\lim_{n\to\infty}\left(\lim_{z\to 1^-}\frac{s_n^*(z)}{n} - \frac{s_n^*(1)}{n}\right) = -\frac{3}{2}.$$

But, for $s_n^*(1) = 2^{2k} - 1$ and $n = 2^1 + 2^3 + \ldots + 2^{2k-1}$, it holds

$$\lim_{z\to 1^-} s_n^*(z) = 2^{2k} - 2$$

and thus,

$$\lim_{n\to\infty}\left(\lim_{z\to 1^-}\frac{s_n^*(z)}{n}-\frac{s_n^*(1)}{n}\right)=0.$$

It is not possible to find a single function a, not depending on n, which fulfills these two characteristics.

Note that we cannot show convergence on $[0,1]$ with this method, because (4) does not hold then.

Second, suppose that $(X_i)_{i\in\mathbb{N}}$ is a sequence with $\mathsf{E}(X_i)=\mu$ for all $i\in\mathbb{N}$. The goal is to calculate the limit of the random function

$$C_n(z)=\frac{[zn]}{\sqrt{n}}\left(\left(\frac{1}{\sqrt{\overline{(X^2)}_{[zn]}-\left(\overline{X}_{[zn]}\right)^2}}\right)-\left(\frac{1}{\sqrt{m_2-\mu^2}}\right)\right) \tag{6}$$

on $D[\epsilon,1]$ for $\epsilon>0$. The term $\sqrt{\overline{(X^2)}_{[zn]}-\left(\overline{X}_{[zn]}\right)^2}$ is part of the successively estimated correlation coefficient. For the calculation, we become somewhat more general. Suppose that $h\in C^2$ is a real function with $\mathsf{E}(h(X_i))=:h_x<\infty$ for all $i\in\mathbb{N}$ and $h_x-h(\mu)>0$. Suppose that we want to calculate the limit of the random function

$$D_n(z)=\frac{[zn]}{\sqrt{n}}\left(\left(\overline{(h(X))}_{[zn]}-h(\overline{X}_{[zn]})\right)-(h_x-h(\mu))\right).$$

For $h(x)=x^2$, this is the standardized empirical variance, based on the first observations, on $D[\epsilon,1]$ for $\epsilon>0$. This is later needed for (6). Under suitable conditions on $(X_i)_{i\in\mathbb{N}}$, it holds with a functional central limit theorem for the random function

$$E_n(z)=\frac{[zn]}{\sqrt{n}}\begin{pmatrix}\overline{(h(X))}_{[zn]} & - & h_x\\ \overline{X}_{[zn]} & - & \mu\end{pmatrix}$$

that

$$E_n(\cdot)\to_d W_2(\cdot),$$

where $W_2(z)$ is a two-dimensional Brownian motion on $D[\epsilon,1]^2$ with covariance structure depending on $(X_i)_{i\in\mathbb{N}}$. Introducing the function

$$s: D[\epsilon,1]^2\to D[\epsilon,1],$$
$$s(x_1,x_2)(\cdot)=x_1(\cdot)-h(x_2(\cdot))$$

and applying Theorem 2.2, we now get analoguesly to the first example

$$D_n(\cdot)\to_d f'_{gx,\mu}(W)(\cdot)$$

with

$$f'_{m_2,\mu}(W_2)(\cdot) = (1, -g'(\mu))^T \; W_2(\cdot).$$

We suppose that this result can also be achieved with the common functional delta method because $[zn]$ has a rather simple form, but we believe that our method is more elegant and more intuitive.

Now, we want to apply the delta method on the function $D(z)$ with $g(x) = x^2$ from the first example. However, we cannot immediately apply Theorem 2.2 with the function

$$\begin{aligned} t : D[\epsilon, 1] &\to D[\epsilon, 1], \\ t(x_1)(\cdot) &= \frac{1}{\sqrt{x_1(\cdot)}}, \end{aligned}$$

because t is not properly defined (x_1 must not become 0). One possibility is to define the function on $D^+[\epsilon, 1]$, i.e.

$$t : D^+[\epsilon, 1] \to D[\epsilon, 1].$$

The problem is, that

$$F_n(z) := \left(\overline{(X^2)}_{[zn]} - \left(\overline{X}_{[zn]}\right)^2\right)$$

does not lie in $D^+[\epsilon, 1]$. If a strong of law numbers is applicable on F_n, however, we can apply Corollary 2.3, because, uniformly in z, F_n converges to a constant function which is bounded away from 0. Then, we can make comparable calculations as in the first example.

4 Proofs

Proof of Theorem 2.2

For each n, we define a function

$$g_n(h) = s_n \cdot \left(f\left(\theta + \frac{1}{s_n}h\right) - f(\theta)\right)$$

on $\mathbb{G}_n := \{h : \theta + \frac{1}{s_n}h \in \mathbb{G}_1^*\}$. Since f is generalized Hadamard differentiable, it holds

$$\lim_{n\to\infty} ||g_n(h_n) - f'_\theta(h)||_{\mathbb{G}_2} = 0$$

for each sequence h_n with $||h_n - h||_{\mathbb{G}_1} \to 0$ and $h \in \mathbb{G}_1$. With the CMT, it follows

$$s_n \cdot (f(M_n) - f(\theta)) = g_n(s_n \cdot (M_n - \theta)) \to_d f'_\theta(M).$$

Proof of Corollary 2.3

By Egoroff's Theorem, for each $\eta > 0$ there is a $n(\eta)$ and a set $\Omega_\eta \subset \Omega$ with $\mathbb{P}(\Omega_\eta) \geq 1 - \eta$, such that $M_n(\omega) \in \mathbb{G}_1^*$ for all $\omega \in \Omega_\eta$. With the characterization of convergence in distribution from Billingsley (1968, Theorem 2.1.iii) and Theorem 2.2 it holds for a closed set $A \in \mathbb{G}_2$ for $n \geq n(\eta)$

$$\begin{aligned}
&\mathbb{P}(\omega \in \Omega | (r_n \cdot (f(M_n) - f(\theta)))(\omega) \in A) \\
&= \mathbb{P}(\omega \in \Omega_\eta | (r_n \cdot (f(M_n) - f(\theta)))(\omega) \in A) \\
&+ \mathbb{P}(\omega \in \Omega / \Omega_\eta | (r_n \cdot (f(M_n) - f(\theta)))(\omega) \in A) \\
&\leq \mathbb{P}(\omega \in \Omega_\eta | (r_n \cdot (f(M_n) - f(\theta)))(\omega) \in A) + \eta
\end{aligned}$$

and hence

$$\begin{aligned}
\limsup_{n\to\infty} \mathbb{P}(\omega \in \Omega | (r_n \cdot (f(M_n) - f(\theta)))(\omega) \in A) &\leq \mathbb{P}(\omega \in \Omega_\eta | (f'_\theta(M))(\omega) \in A) + \eta \\
&\leq \mathbb{P}(\omega \in \Omega | (f'_\theta(M))(\omega) \in A) + \eta.
\end{aligned}$$

Since η is arbitrary, the corollary follows.

Proof of Theorem 2.4

First, we consider just one component from ψ, w.l.o.g. ψ^1 and keep $z \in [\epsilon, 1]$ fixed. With Taylor's Theorem, see Kaballo (1997, S. 128), and the expressions from Definition 2.1 it holds

$$\begin{aligned}
\psi^1(\theta(z) + r_n(z)h_n(z)) =& \psi^1(\theta(z)) + D\psi^1_{\theta(z)}(r_n(z)h_n(z)) + \frac{1}{2} \\
& < r_n(z)h_n(z), H\psi^1_{\theta(z)+\tau(r_n(z)h_n(z))r_n(z)h_n(z)}(r_n(z)h_n(z)) > \\
=& \psi^1(\theta(z)) + r_n(z) D\psi^1_{\theta(z)}(h_n(z)) + \frac{1}{2}(r_n(z))^2 \\
& < h_n(z), H\psi^1_{\theta(z)+\tau(r_n(z)h_n(z))r_n(z)h_n(z)}(h_n(z)) >
\end{aligned}$$

with suitable $\tau(r_n(z)h_n(z)) \in [0, 1]$ where $D\psi^1_{\theta(z)}$ and $H\psi^1_{\theta(z)+\tau(r_n(z)h_n(z))r_n(z)h_n(z)}(h_n(z))$ are continuous linear maps. For sufficiently large n,

$$\big|\big| < h_n(z), H\psi^1_{\theta(z)+\tau(r_n(z)h_n(z))r_n(z)h_n(z)}(h_n(z)) > \big|\big|_{D[\epsilon,1]} < \infty,$$

because $||r_n(z)||_{D[\epsilon,1]} \to 0$ and $||h_n - h||_{\mathbb{G}_1} \to 0$. After subtracting $\psi^1(\theta(z))$, dividing by $r_n(z)$ and taking the supremum ($z \in [\epsilon, 1]$) the claim follows for ψ^1 directly with the definition of the generalized Hadamard differentiability and the continuity of $D\psi^1_{\theta(z)}$.

Analogously, the same follows for the other components and the claim for the whole function then follows with the choice of

$$D\psi_\theta(h) = \Big(D\psi_\theta^1(h) \quad \dots \quad D\psi_\theta^l(h)\Big)'$$

and the definition of the multidimensional supremum norm.

Acknowledgement: We thank Herold Dehling for helpful comments. The financial support by the Deutsche Forschungsgemeinschaft (SFB 823, Statistik nichtlinearer dynamischer Prozesse) is gratefully acknowledged.

References

P. Billingsley. *Convergence of probability measures.* Wiley, New York, 1968.

J. Davidson. *Stochastic limit theory.* Oxford University Press, 1994.

R.D. Gill. Non- and semi-parametric maximum likelihood estimators and the von Mises method - Part 1. *Scandinavian Journal of Statistics*, 16:97–128, 1989.

W. Kaballo. *Einführung in die Analysis II.* Spektrum, Heidelberg, 1997.

G.W. Oehlert. A note on the delta method. *The American Statistician*, 46(1):27–29, 1992.

D. Pollard. *Convergence of stochastic processes.* Series in Statistics, Springer, New York, 1984.

A.W. van der Vaart. Efficiency and Hadamard differentiability. *Scandinavian Journal of Statistics*, 18(1):63–75, 1991.

A.W. van der Vaart. *Asymptotic statistics.* Cambridge University Press, Cambridge, 1998.

D. Wied and M. Arnold. A fluctuation test for constant correlation. *Technical Report, 03/09, SFB823, TU Dortmund University*, 2009.

J.M. Wooldridge and H. White. Some invariance principles and central limit theorems for dependent heterogeneous processes. *Econometric Theory*, 4:210–230, 1988.