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Bessel convolutions on matrix cones: Algebraic properties and random walks

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Abstract

Bessel-type convolution algebras of measures on the matrix cones of positive semidefinite $q \times q$-matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ were introduced recently by Rösler. These convolutions depend on a continuous parameter, generate commutative hypergroups and have Bessel functions of matrix argument as characters. In this paper, we study the algebraic structure of these hypergroups. In particular, the subhypergroups, quotients, and automorphisms are classified. The algebraic properties are partially related to properties of random walks on these matrix Bessel hypergroups. In particular, known properties of Wishart distributions, which form Gaussian convolution semigroups on these hypergroups, are put into a new light. Moreover, limit theorems for random walks are presented. In particular, we obtain strong laws of large numbers and a central limit theorem with Wishart distributions as limits.

1 Introduction

Recently, Rösler [20] introduced positivity-preserving convolution algebras on the matrix cones $\Pi_q(\mathbb{F})$ of positive semidefinite $q \times q$-matrices over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ which are related with Bessel functions of matrix argument and depend on some continuous parameter. With respect to this parameter, they interpolate the radial convolution algebras on non-squared matrix spaces $M_{p,q}(\mathbb{F})$ with $p \geq q$ which are, to some extent, studied in [8]. The convolutions of [20] generate commutative hypergroup structures on $\Pi_q(\mathbb{F})$ with Bessel functions of matrix argument as characters; see [8], [7] [12] and [5] for matrix Bessel functions and [1], [14] for hypergroups. The present paper is devoted to algebraic and probabilistic aspects of these hypergroups. In particular we shall show that these hypergroups admit many subhypergroups and hypergroup automorphisms.

Before going into detail, we recall the case $q = 1$: For any dimension $p \geq 1$, the Banach-$*$-algebra $M_b(\mathbb{R}^p)$ of bounded Borel measures on $\mathbb{R}^p$ with the usual convolution contains the space of all radial measures

$$M_b^{rad}(\mathbb{R}^p) := \{\mu \in M_b(\mathbb{R}^p) : \; u(\mu) = \mu \quad \text{for all} \quad u \in O(p)\}$$

as a Banach-$*$-subalgebra. If we identify the set of all orbits under the standard action of the orthogonal group $O(p)$ on $\mathbb{R}^p$ with $\Pi_1 := [0,\infty[$ via

$$p : x \mapsto |x| = (x_1^2 + \ldots + x_p^2)^{1/2},$$

then p induces an isomorphism between the Banach spaces $M_b^{rad}(\mathbb{R}^p)$ and $M_b(\Pi_1)$. We thus may transfer the Banach-$*$-algebra structure of $M_b^{rad}(\mathbb{R}^p)$ to $M_b(\Pi_1)$ which inherits a commutative, associative, probability preserving and weakly continuous convolution $*_p$. Calculation in polar coordinates shows that for $p \geq 1$,

$$\delta_r *_p \delta_s(f) = c_p \int_0^\pi f\big(\sqrt{r^2+s^2-2rs\cos\theta}\big)\sin^{p-2}\theta\, d\theta, \quad r,s \geq 0,\ f \in C(\Pi_1) \tag{1.1}$$

with a normalization constant $c_p > 0$ where for $p = 1$ (1.1) degenerates to

$$\delta_r * \delta_s = \frac{1}{2}(\delta_{|r-s|} + \delta_{r+s}) \qquad (r,s \in \mathbb{R},\ r,s \geq 0). \tag{1.2}$$

The convolution on $M_b(\Pi_1)$ is then obtained by linear, weakly continuous extension. It is well-known that for all real $p > 1$, Eq.(1.1) generates a commutative, probability preserving, weakly continuous convolution algebra on $M_b(\mathbb{R}_+)$ which interpolates the integer cases. These convolutions have no group interpretation and are related with the known product formulas

$$j_\alpha(r)j_\alpha(s) = \delta_r *_p \delta_s(j_\alpha).$$

for the normalized Bessel functions $j_\alpha(z) = {}_0F_1(\alpha+1; -z^2/4)$ with index $\alpha = p/2 - 1 \geq -1/2$ ([26]). The space $\mathbb{R}_+$ with the convolutions $*_p$ for $p \geq 1$ provides a prominent class of commutative hypergroups, called Bessel-Kingman hypergroups ([1]). For $p > 1$, these hypergroups have no nontrivial subhypergroups while in the degenerated case $p = 1$ the sets $c\mathbb{Z}_+$ for $c > 0$ form the nontrivial subhypergroups. Moreover, for $p \geq 1$, all hypergroup automorphisms are given by $x \mapsto cx$ for $c > 0$, see [27]. Random walks on Bessel-Kingman hypergroups, i.e., Markov chains on $\mathbb{R}_+$ with transition probabilities given in terms of $*_p$ were investigated first by Kingman [16]; for the later development we refer to [1] and references therein. Kingman in particular obtained laws of large numbers and a central limit theorem. For integers p, these limit theorems on $\mathbb{R}_+$ are just radial reformulations of classical limit theorems on $\mathbb{R}^p$.

We now turn to the higher rank case in [20]. For $p,q \in \mathbb{N}$ with $p \geq q$, consider the space $M_{p,q} = M_{p,q}(\mathbb{F})$ of $p \times q$ matrices over one of the division algebras $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or the quaternions $\mathbb{H}$ with real dimension $d = 1, 2$ or 4 respectively. $M_{p,q}$ is a real Euclidean vector space of dimension dpq with scalar product $(x|y) = \Re\mathrm{tr}(x^*y)$ where $x^* = \overline{x}^t$, $\Re t = \frac{1}{2}(t+\overline{t})$ is the real part of $t \in \mathbb{F}$, and tr the trace in $M_q(\mathbb{F}) := M_{q,q}(\mathbb{F})$. A measure on $M_{p,q}$ is called radial if it is invariant under the action of the unitary group $U_p = U_p(\mathbb{F})$ on $M_{p,q}$ by left multiplication,

$$U_p \times M_{p,q} \to M_{p,q}, \quad (u,x) \mapsto ux. \tag{1.3}$$

This action is orthogonal w.r.t. the scalar product above, and x,y are in the same U_p-orbit if and only if $x^*x = y^*y$. Thus the space of U_p-orbits is naturally parametrized by the cone $\Pi_q = \Pi_q(\mathbb{F})$ of positive semidefinite $q \times q$-matrices over $\mathbb{F}$. For $q = 1$ and $\mathbb{F} = \mathbb{R}$, we have $\Pi_1 = \mathbb{R}_+$ and end up with the one-dimensional case above. We now use the projection

$$p : M_{p,q} \to \Pi_q, \quad x \mapsto (x^*x)^{1/2},$$

with the usual unique square root on Π_q. Via this mapping the convolution algebra of radial measures on $M_{p,q}$ is transferred to a commutative, associative, probability preserving, weakly continuous convolution $*_p$ of measures on Π_q which forms a commutative hypergroup. By construction (and results of [8], [12]) this convolution corresponds to a product formula for Bessel functions $\mathcal{J}_\mu$ on Π_q with index $\mu = pd/2$. In [20], the convolution $*_p$ and the product formula for the corresponding $\mathcal{J}_\mu$ is written down in a way which allows for analytic continuation with respect to μ. This leads to "interpolating" commutative hypergroup structures $X_{q,\mu}$ on Π_q with a continuous real index $p \geq 2q$, i.e. $\mu \geq d(q-1/2)$ and with matrix Bessel functions of index μ as hypergroup characters. These hypergroups are self-dual with the identity as involution. The product formulas degenerate for $p = q, q+1\ldots, 2q$. For non-integer $p \in]q, 2q[$ there is unfortunately only a guess for explicit product formulas; see [20].

The present paper continues [20]. In the first place, we study algebraic properties of the matrix Bessel hypergroups of [20]. We first show that for each $a \in GL(q, \mathbb{F})$, the map $T_a(r) := (ar^2a^*)^{1/2}$ is a hypergroup automorphism of $X_{q,\mu}$ This reveals that matrix Bessel hypergroups in higher rank admit a rich structure of automorphisms similar to the Euclidean spaces $\mathbb{F}^d$. The deepest result will be the classification of all hypergroup automorphisms $Aut(X_{q,\mu})$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$. Indeed, we shall prove that for $\mathbb{F} = \mathbb{R}$, $Aut(X_{q,\mu})$ is the transformation group $\{T_a : a \in GL(q, \mathbb{F})\}$, and that for $\mathbb{F} = \mathbb{C}$ in addition the maps $\tau \circ T_a$ appear with $a \in GL(q, \mathbb{F})$ and τ the complex conjugation. We expect a similar result for $\mathbb{F} = \mathbb{H}$, but we are unable to prove it here. In addition, we will classify all subhypergroups in the general case, that is for all $\mathbb{F}$ and $\mu > d(q-1/2)$. More precisely, we prove that all subhypergroups are of the form

$$H_{k,u} := \left\{ u \begin{pmatrix} \tilde{r} & 0 \\ 0 & 0 \end{pmatrix} u^* : \tilde{r} \in \Pi_k \right\}$$

with $k \geq 0$ and a unitary matrix $u \in U_q$ (where $H_{0,u} = \{0\}$). We also show that for fixed μ, $H_{k,u}$ is canonically isomorphic with the hypergroup $X_{k,\mu}$, and that the quotient $X_{k,\mu}/H_{k,u}$ carries a quotient hypergroup structure and is isomorphic with $X_{q-k,\mu}$. The proofs of these algebraic properties will rely more on the properties of matrix Bessel functions (which form the hypergroup characters) rather than the explicit form of the convolution.

The second part of this paper is devoted to probability theory on matrix Bessel hypergroups. We introduce convolution semigroups of probability measures and random walks and show how Wishart distributions fit into this concept. In particular, some known facts about Wishart distributions and Wishart processes will appear under a new light. Moreover, Wishart distributions appear as limits in a central limit theorem. We also derive strong laws of large numbers for random walks on $X_{q,\mu}$. The proofs of both limit theorems relies on the concept of moment functions developed by Zeuner [27], [28] and the author; see also the monograph [1]. We point out that for the group case $\mu = pd/2$, all limit theorems are radial reformulations of classical limit theorems on $M_{p,q}$. We finally point out that the nice algebraic structure of these Bessel-type allows to develop an algebraic probability theory similar to vector spaces (for this topic see e.g. the monograph [11]).

This paper is organized as follows: In Section 2 we collect some basic facts about matrix Bessel functions and the corresponding hypergroups from [20]. Section 3 contains some general facts about hypergroups which will be useful lateron. In Section 4, the algebraic properties of matrix Bessel hypergroups are studied. The remaining sections are then devoted to probability theory on matrix Bessel hypergroups. Section 5 contains basic properties of convolution semigroups of probability measures, random walks, and Wishart distributions. In Section 6 we then derive a central limit theorem as well as strong laws of large numbers.

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2 Bessel convolutions on matrix cones

Here we collect some basic notions and facts about matrix Bessel functions and matrix Bessel hypergroups from [7], [1], [14], [20].

2.1 Bessel functions associated with matrix cones

Let $\mathbb{F}$ be one of the real division algebras $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ with real dimension $d = 1, 2$ or 4 respectively. Denote the usual conjugation in $\mathbb{F}$ by $t \mapsto \overline{t}$, the real part of $t \in \mathbb{F}$ by $\Re t = \frac{1}{2}(t + \overline{t})$, and $|t| = (t\overline{t})^{1/2}$ its norm.

For $p, q \in \mathbb{N}$ we denote by $M_{p,q} := M_{p,q}(\mathbb{F})$ the vector spaces of all $p \times q$-matrices over $\mathbb{F}$, and we put $M_q := M_{q,q}$. We consider the set

$$H_q = H_q(\mathbb{F}) = \{x \in M_q(\mathbb{F}) : x = x^*\}$$

of Hermitian $q \times q$-matrices over $\mathbb{F}$ as a Euclidean vector space with scalar product $(x|y) := \Re tr(x^*y)$ and the associated norm $\|x\| = (x|x)^{1/2}$. Here $x^* := \overline{x}^t$ and tr denote the trace. Its dimension is given by $dim_{\mathbb{R}} H_q = n := q + \frac{d}{2}q(q-1)$. Let further

$$\Pi_q := \{x^2 : \ x \in H_q\} = \{x^*x : x \in H_q\}$$

be the set of all positive semidefinite matrices in H_q, and Ω_q its topological interior which consists of all strictly positive definite matrices. Ω_q is a symmetric cone, i.e. an open convex self-dual cone whose linear automorphism group acts transitively, see [7] for details.

To define the Bessel functions associated with the symmetric cone Ω_q we first introduce their basic building blocks, the so-called spherical polynomials. These are just the polynomial spherical functions of Ω_q considered as a Riemannian symmetric space. They are indexed by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_q) \in \mathbb{N}_0^q$ (we write $\lambda \geq 0$ for short) and given by

$$\Phi_\lambda(x) = \int_{U_q} \Delta_\lambda(uxu^{-1})du, \quad x \in H_q$$

where du is the normalized Haar measure of U_q and Δ_λ is the power function on H_q with

$$\Delta_\lambda(x) := \Delta_1(x)^{\lambda_1-\lambda_2}\Delta_2(x)^{\lambda_2-\lambda_3} \cdot \ldots \cdot \Delta_q(x)^{\lambda_q};$$

the $\Delta_i(x)$ are the principal minors of the determinant $\Delta(x)$, see [7] for details. The Φ_λ are homogeneous of degree $|\lambda| = \lambda_1 + \ldots + \lambda_q$. There is a renormalization $Z_\lambda = c_\lambda \Phi_\lambda$ with constants $c_\lambda > 0$ depending on the underlying cone such that

$$(tr\, x)^k \ = \sum_{|\lambda|=k} Z_\lambda(x) \qquad \text{for } \ k \in \mathbb{N}_0, \tag{2.1}$$

see Section XI.5. of [7] (the Z_λ are called zonal polynomials there). By construction, they are invariant under conjugation by U_q and thus depend only on the eigenvalues of their argument. More precisely, for $x \in H_q$ with eigenvalues $\xi = (\xi_1, \ldots, \xi_q) \in \mathbb{R}^q$,

$$Z_\lambda(x) = C_\lambda^\alpha(\xi), \quad \alpha = \frac{2}{d}$$

where the C_λ^α are the Jack polynomials of index α in a suitable normalization (c.f. [7], [15], [20]). They are homogeneous of degree $|\lambda|$ and symmetric in their arguments.

The matrix Bessel functions associated with the cone Ω_q are now defined as ${}_0F_1$-hypergeometric series in terms of the Z_λ, as follows:

$$\mathcal{J}_\mu(x) = \sum_{\lambda\geq 0} \frac{(-1)^{|\lambda|}}{(\mu)_\lambda |\lambda|!} Z_\lambda(x), \quad x \in H_q$$

where for $\lambda = (\lambda_1, \dots \lambda_q) \in \mathbb{N}_0^q$, the generalized Pochhammer symbol $(\mu)_\lambda$ is

$$(\mu)_\lambda = (\mu)_\lambda^{2/d} = \prod_{j=1}^{q} \big(\mu - \frac{d}{2}(j-1)\big)_{\lambda_j}$$

and $\mu \in \mathbb{C}$ is an index with $(\mu)_\lambda \neq 0$ for $\lambda \geq 0$. This series converges absolutely for $x \in H_q$, see [7]. Later on, we need the linear terms in the expansion of $\mathcal{J}_\mu$. By (2.1) we have

$$\mathcal{J}_\mu(x) = 1 - \frac{1}{\mu}\, tr\, x + O(\|x\|^2). \tag{2.2}$$

To describe the main results of [20], we need the notion of a hypergroup, which will be recapitulated in the following section.

2.2 Hypergroups

A hypergroup $(X, *)$ consists of a locally compact Hausdorff space X and a multiplication $*$, called convolution, on the Banach space $M_b(X)$ of all bounded regular complex Borel measures with the total variation norm as norm, such that $(M_b(X), *)$ becomes a Banach algebra, and such that $*$ is weakly continuous and probability preserving and preserves compact supports of measures. Moreover, there exists an identity $e \in X$ with $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for $x \in X$, as well as a continuous involution $x \mapsto \bar{x}$ on X such that for $x, y \in X$, $e \in supp(\delta_x * \delta_y)$ is equivalent to $x = \bar{y}$, and $\delta_{\bar{x}} * \delta_{\bar{y}} = (\delta_y * \delta_x)^-$. Here for $\mu \in M_b(X)$, the measure μ^- is given by $\mu^-(A) = \mu(A^-)$ for Borel sets $A \subset X$.

A hypergroup $(X, *)$ is called commutative if and only if so is the convolution $*$, and hermitian, if the hypergroup involution is the identity. Hermitian hypergroups are commutative.

It is well-known that each commutative hypergroup admits a (up to normalization) unique Haar measure $\omega \in M^+(X)$ which is characterized by $\omega(f) = \int_X f(x * y)\, d\omega(y)$ for all $x \in X$ and all compactly supported, continuous functions $f \in C_c(X)$ where we use the notation

$$f(x * y) := \int_X f\, d(\delta_x * \delta_y).$$

Similar to the dual of a locally compact abelian group, we define the dual

$$\widehat{X} := \{\alpha \in C_b(X) : \alpha \neq 0,\ \alpha(\bar{x} * y) = \overline{\alpha(x)}\alpha(y) \text{ for all } x, y \in X\}.$$

$\widehat{X}$ is a locally compact Hausdorff space w.r.t. the topology of compact-uniform convergence, and its elements are called characters. The Fourier transform on $L^1(X, \omega)$ is defined by $\widehat{f}(\alpha) := \int_X f(x)\overline{\alpha(x)}\, d\omega(x)$, $\alpha \in \widehat{X}$, and the Fourier-Stieltjes transform of measures is defined in the same way. It is well-known that for a fixed Haar measure ω on X there is a unique Plancherel measure π on $\hat{X}$ such that the Fourier transform becomes an L^2-isometry between $L^2(X, \omega)$ and $L^2(\hat{X}, \pi)$.

Interesting examples are given as follows.

2.3 Bessel convolutions on matrix cones

For natural numbers p, q, consider the matrix space $M_{p,q} = M_{p,q}(\mathbb{F})$ of $p\times q$-matrices over $\mathbb{F}$. We regard $M_{p,q}$ as a real vector space with the Euclidean scalar product $(x|y) := \Re\text{tr}(x^*y)$ and norm $\|x\| = \sqrt{\text{tr}(x^*x)}$. Consider the action of the unitary group U_p on $M_{p,q}$ by left multiplication,

$$U_p \times M_{p,q} \to M_{p,q}\,, \quad (u,x) \mapsto ux.$$

This action is orthogonal w.r.t. the scalar product above, and x, y are in the same U_p-orbit if and only if $x^*x = y^*y$. The space $M_{p,q}^{U_p}$ of all orbits for this action can therefore be identified with the space $\Pi_q = \Pi_q(\mathbb{F})$ of positive semidefinite $q\times q$ matrices over $\mathbb{F}$ via

$$U_p.x \mapsto \sqrt{x^*x} =: |x|.$$

Here for $r \in \Pi_q$, $\sqrt{r}$ is the unique positive semidefinite square root of r. This bijection is a homeomorphism w.r.t. the quotient topology on $M_{p,q}^{U_p}$.

Now consider the Banach-$*$-algebra

$$M_b^{rad}(M_{p,q}) := \{\mu \in M_b(M_{p,q}):\ u(\mu) = \mu \quad \text{for all } \ u \in U_p\}$$

of all radial regular Borel measures on $M_{p,q}$, and the canonical projection

$$p: M_{p,q} \to \Pi_q, \quad x \mapsto (x^*x)^{1/2},$$

with the usual unique square root on Π_q. Via this mapping, the convolution on $M_b^{rad}(M_{p,q})$ is transferred to a commutative, associative, probability preserving and weakly continuous convolution $*_p$ of measures on Π_q which forms a commutative hypergroup. By construction (and results of [8], [12]) this convolution corresponds to a product formula for Bessel functions $\mathcal{J}_\mu$ on Π_q with index $\mu = pd/2$. In [20], the convolution $*_p$ and the product formula for the corresponding $\mathcal{J}_\mu$ is written down explicitly (see Eq. (2.3) below) in a way which allows for analytic continuation with respect to the index μ. This leads to "interpolating" commutative hypergroup structures $X_{q,\mu}$ on Π_q for all indices $\mu > \rho - 1$ with

$$\rho := d\big(q - \frac{1}{2}\big) + 1.$$

For $\mu \le \rho - 1$ having the form $\mu = pd/2$ with $p \in \mathbb{N}$, there exist also degenerated versions of the product formula (2.3) below; it is however not clear at the moment whether these discrete cases can be embedded into a continuous family of convolution and product formulas; see the discussion in [20].

In the following we use the abbreviations $D_q = \{v \in M_{q,q} : v^*v < I\}$ and

$$\kappa_\mu := \int_{D_q} \Delta(I - v^*v)^{\mu-\rho} dv$$

of [20]; for an explicit formula for κ_μ see [20]. The following result contains some of the main results of [20].

2.1 Theorem. *Let $\mu \in \mathbb{R}$ with $\mu > \rho - 1$. Then*

(a) *The assignment*

$$(\delta_r *_\mu \delta_s)(f) := \frac{1}{\kappa_\mu} \int_{D_q} f\big(\sqrt{r^2 + s^2 + svr + rv^*s}\,\big)\, \Delta(I - vv^*)^{\mu-\rho}\, dv \qquad (2.3)$$

for $f \in C_c(\Pi_q)$ *defines a hermitian hypergroup structure on* Π_q *with neutral element* $0 \in \Pi_q$. *The support of* $\delta_r *_\mu \delta_s$ *satisfies*

$$supp(\delta_r *_\mu \delta_s) \subseteq \{t \in \Pi_q : \|t\| \le \|r\| + \|s\|\}.$$

(b) *A Haar measure of this hypergroup* $X_{q,\mu} := (\Pi_q, *_\mu)$ *is given by*

$$\omega_\mu(f) = \frac{\pi^{q\mu}}{\Gamma_{\Omega_q}(\mu)} \int_{\Omega_q} f(\sqrt{r})\Delta(r)^\gamma dr$$

with $\gamma = \mu - \frac{d}{2}(q-1) - 1 \;=\; \mu - \frac{n}{q}$.

(c) *The dual space of* $X_{q,\mu}$ *is given by*

$$\widehat{X_{q,\mu}} = \{\varphi_s = \varphi_s^\mu : s \in \Pi_q\}$$

with

$$\varphi_s(r) := \mathcal{J}_\mu(\frac{1}{4} rs^2 r) = \varphi_r(s).$$

The hypergroup $X_{q,\mu}$ *is self-dual via the homeomorphism* $s \mapsto \varphi_s$. *Under this identification of* $\widehat{X_{q,\mu}}$ *with* $X_{q,\mu}$, *the Plancherel measure on* $X_{q,\mu}$ *is* $(2\pi)^{-2\mu q}\omega_\mu$.

Notice that in our normalization of the Haar measure for $\mu = pd/2$, ω_μ is just the image of the Lebesgue measure on $M_{p,q} \simeq \mathbb{R}^{dpq}$ under the canonical projection $p : M_{p,q} \simeq \mathbb{R}^{dpq} \to \Pi_q$. This normalization differs from [20], but is better suited for probability theory below, as here the constants correspond to the classical Euclidean setting.

The explicit product formula (2.3) may be used to describe the support of convolution products in more detail. Some special cases will be considered in Lemmas 4.4 and 4.17 below.

3 Some results about hypergroups

In this section we collect some further general notions and results about commutative hypergroups. We also prove some more or less straightforward results about hypergroup automorphisms which will be useful for the Bessel convolutions below.

We start with the following simple observation which will be useful several times.

3.1 Lemma. *If a locally compact space* X *carries two commutative hypergroup convolutions* $*_1$ *and* $*_2$ *such that the dual* $(X, *_1)^\wedge$ *is contained in* $(X, *_2)^\wedge$, *then* $*_1 = *_2$.

Proof. Let $\alpha \in (X, *_1)^\wedge \subset (X, *_2)^\wedge$. Then, for all $x, y \in X$,

$$(\delta_x *_1 \delta_y)^\wedge(\alpha) = \bar{\alpha}(x)\bar{\alpha}(y) = (\delta_x *_2 \delta_y)^\wedge(\alpha).$$

As the Fourier-Stieltjes transform of measures on $(X, *_1)$ is injective (see [14] or [1]), we obtain $\delta_x *_1 \delta_y = \delta_x *_2 \delta_y$ for $x, y \in X$. □

3.2 Definition. Let X, Y be commutative hypergroups. A closed set $H \subset X$ is called a subhypergroup, if for all $x, y \in H$, we have $\bar{x} \in H$ and $\{x\} * \{y\} := supp(\delta_x * \delta_y) \subset H$. Moreover, a continuous mapping $T : X \to Y$ is a hypergroup homomorphism, if $T(\delta_x * \delta_{\bar{y}}) = \delta_{T(x)} * \delta_{\overline{T(y)}}$ for all $x, y \in X$ where the mapping $T : X \to Y$ is extended to bounded Borel measures by taking images of measures. The notions of hypergroup isomorphisms and automorphisms are similar.

We need the following observations:

3.3 Lemma. *Let $(T_k)_{k\geq 1}$ be a sequence of hypergroup automorphisms on a hypergroup $(X, *)$ which converges pointwise to some continuous mapping $T : X \to X$ such that $T(X)$ is closed in X. Then T is a hypergroup homomorphism from X onto the subhypergroup $T(X)$.*

Proof. Let $x, y \in X$ and $f \in C_b(X)$. Then, by the weak continuity of the convolution,

$$\lim_k \int f\, dT_k(\delta_x * \delta_{\bar{y}}) = \lim_k \int f\, d(\delta_{T_k(x)} * \delta_{\overline{T_k(y)}}) = \int f\, d(\delta_{T(x)} * \delta_{\overline{T(y)}}).$$

On the other hand, by the transformation formula and dominated convergence,

$$\lim_k \int f\, dT_k(\delta_x * \delta_{\bar{y}}) = \lim_k \int f \circ T_k\, d(\delta_x * \delta_{\bar{y}}) = \int f \circ T\, d(\delta_x * \delta_{\bar{y}}) = \int f\, dT(\delta_x * \delta_{\bar{y}}).$$

This proves $T(\delta_x * \delta_{\bar{y}}) = \delta_{T(x)} * \delta_{\overline{T(y)}}$, and that $T(X)$ is a subhypergroup. □

The following results are well-known from the group case.

3.4 Proposition. *Let X be a commutative hypergroup and $T \in Aut(X)$.*

(1) *The Haar measure ω of X satisfies $T(\omega) = c_T\omega$ for some constant $c_T > 0$.*
(2) *For $f, g \in C_c(X)$ and $x \in X$, $((f \circ T) * (g \circ T))(x) = c_T \cdot (f * g)(T(x))$.*
(3) *There exists a dual homeomorphism $T^* : \hat{X} \to \hat{X}$ with $(T^*(\alpha))(x) = \alpha(T(x))$ for $\alpha \in \hat{X}, x \in X$. This T^* satisfies $\hat{f} \circ T^* = c_T \cdot (f \circ T^{-1})^\wedge$ for $f \in L^1(X, \omega)$ and $T^*(\pi) = c_T^{-1}\pi$ for the Plancherel measure on $\hat{X}$.*
(4) *If $\hat{X}$ carries the dual hypergroup structure $(\hat{X}, *)$ with*

$$\int_{\hat{X}} \gamma(x)\, d(\delta_\alpha * \delta_\beta)(\gamma) = \alpha(x)\beta(x) \quad \textit{for} \quad \alpha, \beta \in \hat{X},\ x \in X,$$

then T^* is a hypergroup automorphism on $\hat{X}$.

Proof. (1) For $f \in C_c(X)$ and $x \in X$,

$$\begin{aligned} T(\omega)(f_x) :&= \int_X f(x * y)\, dT(\omega)(y) = \int_X f(x * T(w))\, d\omega(w) \\ &= \int_X f \circ T(T^{-1}(x) * w)\, d\omega(w) = \int_X f \circ T(w)\, d\omega(w) = T(\omega)(f). \end{aligned}$$

Thus $T(\omega)$ is a Haar measure. As ω is unique ([14]), the result follows.

(2) For $x \in X$,

$$((f \circ T) * (g \circ T))(x) = \int_X (f \circ T)(x * \bar{y})(g \circ T)(y)\, d\omega(y)$$
$$= \int_X f(T(x) * \overline{T(y)})g(T(y))\, d\omega(y)$$
$$= \int_X f(T(x) * \bar{w})g(w)\, dT(\omega)(w) = c_T \cdot (f * g)(T(x)).$$

(3) As for $x, y \in X$ and $\alpha \in \hat{X}$

$$\int_X \alpha(T(z))\, d(\delta_x * \delta_{\bar{y}})(z) = \int_X \alpha(w)\, d(T(\delta_x * \delta_{\bar{y}}))(w)$$
$$= \int_X \alpha(w)\, d(\delta_{T(x)} * \delta_{\bar{T}(y)}))(w) = \alpha(T(x))\overline{\alpha(T(y))},$$

T^* defines a character $T^*(\alpha) \in \hat{X}$ for $\alpha \in \hat{X}$. Moreover, $T^* : \hat{X} \to \hat{X}$ is obviously a homeomorphism as $\hat{X}$ is equipped with the topology of locally uniform convergence. Furthermore, for $f \in L^1(X, \omega)$ and $\alpha \in \hat{X}$,

$$c_T \cdot (f \circ T^{-1})^\wedge(\alpha) = c_T \int_X f(T^{-1}(x))\overline{\alpha(x)}\, d\omega(x)$$
$$= \int_X f(x)\overline{T^*(\alpha)(x)}\, d\omega(x) = (\hat{f} \circ T^*)(\alpha).$$

Moreover, as π is characterized by the fact that the Fourier transform becomes an isometry between $L^2(X, \omega)$ and $L^2(\hat{X}, \pi)$, and as

$$\int_{\hat{X}} |\hat{f}|^2\, dT^*(\pi) = \int_{\hat{X}} |\hat{f} \circ T^*|^2\, d\pi = \int_X |f \circ T^{-1}|^2\, d\omega = c_T^{-1} \int_X |f|^2\, d\omega,$$

it follows that $T^*(\pi) = c_T^{-1}\pi$.

(4) For all $x \in X$ and $\alpha, \beta \in \hat{X}$ we have

$$T^*(\alpha * \bar{\beta})(x) = \int_{\hat{X}} \gamma(T(x))\, d(\delta_\alpha * \delta_{\bar{\beta}})(\gamma) = \alpha(T(x))\overline{\beta(T(x))}$$

and

$$(T^*(\alpha) * \overline{T^*(\beta)})(x) = \int_{\hat{X}} \gamma(x)\, d(\delta_{T^*(\alpha)} * \delta_{\overline{T^*(\beta)}})(\gamma) = T^*(\alpha)(x) \cdot \overline{T^*(\beta)(x)}$$
$$= \alpha(T(x))\overline{\beta(T(x))},$$

and thus $T^*(\alpha * \bar{\beta}) = T^*(\alpha) * \overline{T^*(\beta)}$.

□

4 Automorphisms, subhypergroups, and quotients

In this section we collect algebraic properties of the Bessel convolutions on matrix cones. We fix parameters d, q, μ, ρ as in Section in Section 2 and consider the associated hypergroup structure $X_{q,\mu}$ on Π_q. One major task will be the classification of all hypergroup automorphisms of $X_{q,\mu}$. For this, we first determine a group of hypergroup automorphisms. For this, we recall that $GL(q) := GL(q,\mathbb{F})$ acts on Π_q as a group of homeomorphisms via

$$T_a(r) := \sqrt{ar^2a^*} \qquad \text{for} \quad a \in GL(q),\ r \in \Pi_q.$$

To check that the T_a are in fact hypergroup automorphisms, we observe:

4.1 Lemma. *Let $s, r \in \Pi_q$ and $a \in M_q$. Then $\varphi_s(T_a(r)) = \varphi_{T_{a^*}(s)}(r)$.*

Proof. As the Bessel function $\mathcal{J}_\mu(r)$ depends on the spectrum of $r \in \Pi_q$ only, we have

$$\varphi_s(T_a(r)) = \mathcal{J}_\mu(\frac{1}{4}sar^2a^*s) = \mathcal{J}_\mu(\frac{1}{4}a^*s^2ar^2) = \varphi_{T_{a^*}(s)}(r).$$

□

4.2 Corollary. *Let $\mu \in M_b(\Pi_q)$, $a \in M_q$, and $s \in \Pi_q$. Then $\widehat{T_a(\mu)}(s) = \hat{\mu}(T_{a^*}(s))$.*

Proof. $\widehat{T_a(\mu)}(s) = \int \varphi_s(r)\, dT_a(\mu)(r) = \int \varphi_s(T_a(r))\, d\mu(r) = \int \varphi_{T_{a^*}(s)}(r)\, d\mu(r) = \hat{\mu}(T_{a^*}(s))$. □

4.3 Proposition. *$\{T_a :\ a \in GL(q)\}$ is a group of hypergroup automorphisms of $X_{q,\mu}$.*

Proof. Fix $a \in GL(q)$. Then the homeomorphism T_a induces a further commutative hypergroup structure $(\Pi_q, *_a)$ on Π_q by

$$\delta_x *_a \delta_y := T_a(\delta_{T_a^{-1}(x)} * \delta_{T_a^{-1}(y)}) \qquad (x, y \in \Pi_q).$$

It is easy to check that the dual space of this commutative hypergroup is $\{\varphi_s \circ T_{a^{-1}} :\ s \in \Pi_q\}$ where this space agrees with the dual of $X_{q,\mu}$ by Lemma 4.1. Lemma 3.1 now shows that the hypergroups $(\Pi_q, *_a)$ and $X_{q,\mu}$ agree, and hence T_a is a hypergroup automorphism. As the T_a obviously form a group, the proof is complete. □

The preceding proposition may be also checked directly via the explicit product formula (2.3), but in our eyes this approach is much more involved. Moreover, our approach works also for $\mu \le \rho - 1$ in which Eq. (2.3) degenerates (or is even unknown).

We prove below that for $\mathbb{F} = \mathbb{R}$ and $\mu > \rho - 1$, the group $\{T_a :\ a \in GL(q)\}$ is the group of all hypergroup automorphisms of $X_{q,\mu}$. This is however not correct for $\mathbb{F} = \mathbb{C}, \mathbb{H}$ and $q \ge 2$. For instance, for $\mathbb{F} = \mathbb{C}$ and $q \ge 2$, complex conjugation on Π_q is an automorphism which is not of the form above; for details see Theorems 4.11 and 4.12 below.

We next determine all subhypergroups. For this we need:

4.4 Lemma. *Let $\mu > \rho - 1$. Then for all $r \in \Pi_q$ and $c \in]0, 1]$,*

$$\{r\} * \{cr\} := supp(\delta_r * \delta_{cr}) = \{s \in \Pi_q :\ (1-c)r \le s \le (1+c)r\}.$$

Proof. We find a suitable automorphism T_a which maps r into the diagonal matrix $I_j := diag(1, \ldots, 1, 0, \ldots, 0)$ with $j := rank\, r$. We therefore may assume without loss of generality $r = I_j$.

By Eq. (2.3), we have

$$\{I_j\} * \{cI_j\} = \{\sqrt{(1+c^2)I_j + c(v + v^*)} \in \Pi_q :\ v \in M_q,\ vv^* \le I_j\}.$$

To simplify this set, we observe for $s \in \Pi_q$ and $h \in H_q$ that

$$s = \sqrt{(1+c^2)I_j + 2ch} \quad \Longleftrightarrow \quad h = (s^2 - ((1+c^2)I_j)/(2c),$$

where for s, h coupled in this way, $(1-c)I_j \le s \le (1+c)I_j$ is equivalent to $-I_j \le h \le I_j$. Therefore, for a given s with this property we may take $v = h \in H_q$ and obtain the inclusion $\supset$ in the statement of the lemma. Conversely, for $v \in M_q$ with $vv^* \le I_j$, the spectral norm of v is bounded by 1; hence the spectral norm of $h := (v+v^*)/2$ is also bounded by 1 which means for this hermitian matrix $-I_j \le h \le I_j$. This proves the converse inclusion and completes the proof. □

4.5 Remark. For $\mu = \rho - 1$, Lemma 4.4 is no longer correct. In fact, the degenerated explicit product formula in Proposition 3.16 of [20] and some matrix computation shows that here for instance, for the identity matrix I the set $\{I\} * \{I\}$ consists of those $s \in \Pi_q$ with eigenvalues $\lambda_1, \ldots, \lambda_q \in [0,2]$ with $\sum_{i=1}^q (\lambda_i^2/2 - 1)^2 \ge 1$, i.e., $\{I\} * \{I\}$ contains a hole.

In particular, for $q = 1$ and $\mathbb{F} = \mathbb{R}$ (i.e. $d = 1$) and $\mu = \rho - 1 = 1/2$ we have the degenerated Bessel convolution (1.2). This cosine hypergroup on $[0, \infty[$ has the discrete subhypergroups $c\mathbb{N}_0$ for $c > 0$. This example shows in particular that in the following proposition we partially must restrict our attention to the case $\mu > \rho - 1$.

4.6 Proposition. *Let $\mu \ge \rho - 1$, $k \in \{0, 1, \ldots, q\}$, and $u \in U_q$. Then*

$$H_{k,u} := \left\{ u \begin{pmatrix} \tilde{r} & 0 \\ 0 & 0 \end{pmatrix} u^* : \tilde{r} \in \Pi_k \right\}$$

(with $H_{0,u} = \{0\}$) is a subhypergroup of $X_{q,\mu}$, and the mapping $\tilde{r} \mapsto u \begin{pmatrix} \tilde{r} & 0 \\ 0 & 0 \end{pmatrix} u^*$ is a hypergroup isomorphism between $X_{k,\mu}$ and $H_{k,u}$. Moreover, for $\mu > \rho - 1$ all subhypergroups of $X_{q,\mu}$ are given in this way.

Proof. The $H_{k,I}$ are obviously subhypergroups by Eq. (2.3). Using suitable automorphisms, it is also clear that the $H_{k,u}$ are subhypergroups for arbitrary u.

In order to check that the $H_{k,u}$ are isomorphic with $X_{k,\mu}$, we may assume $u = I$ without loss of generality. In this case, we observe that the Jack polynomials C_λ^α in q and k variables respectively satisfy $C_\lambda^\alpha(0, \ldots, 0, \xi_1, \ldots, \xi_k) = C_\lambda^\alpha(\xi_1, \ldots, \xi_k)$ for $0 \le \xi_1 \le \ldots \le \xi_k$ by their very definition; see Stanley [23]. Hence, by the definition of the q- and k-dimensional Bessel functions respectively, we obtain the important relation

$$\mathcal{J}_\mu^q \left(\begin{pmatrix} \tilde{r} & 0 \\ 0 & 0 \end{pmatrix} \right) = \mathcal{J}_\mu^k(\tilde{r}) \qquad \text{for} \qquad \tilde{r} \in \Pi_k. \tag{4.1}$$

Therefore, all characters of $X_{k,\mu}$ appear as restrictions of characters on $X_{q,\mu}$ to $H_{k,u}$ where these restrictions are obviously characters on $H_{k,u}$. Lemma 3.1 now shows that the hypergroup structures on $H_{k,u}$ and $X_{k,\mu}$ are equal as claimed.

We still have to show that for $\mu > \rho - 1$, all subhypergroups of $X_{q,\mu}$ appear as some $H_{k,u}$. For this we show that each subhypergroup of $X_{q,\mu}$ which is not contained in some $H_{q-1,u}$ with $u \in U_q$ must be equal to $X_{q,\mu}$. As this says that each proper subhypergroup is contained in some $H_{q-1,u}$, we conclude from the first part of the proposition and induction that each subhypergroup appears as some $H_{k,u}$. In order to prove the claim above, consider some subhypergroup H which is not contained in some $H_{q-1,u}$. Let $a \in H$ be an arbitrary element with rank $k < q$ where we may assume without loss of generality $a \in H_{k,I}$ after using a suitable automorphism. We then find some $b = \begin{pmatrix} * & * \\ * & c \end{pmatrix} \in H$ with $c \in \Pi_{q-k}$ and $c \neq 0$. Then, by Eq. (2.3), $a + b \in \{a\} * \{b\} \subset H$, and $a + b$ has rank at least $k + 1$. Iterating this argument, we find some $r \in H$ with full rank. Lemma 4.4 now shows that $\{r\} * \{r\} \subset H$ contains a neighborhood U of 0 in Π_q. Applying 4.4 to elements of U several times, finally implies $H = \Pi_q$ as claimed. □

4.7 Remark. The identification of the convolution on subhypergroups above can be derived also directly from the Bessel convolution (1.1). For this one needs a relation between the measures

$$d\tau_{q,\mu}(v) := \frac{1}{\kappa_\mu} \Delta(I - vv^*)^{\mu-\rho}\, dv|_{D_q} \in M1(M_{q,q} \tag{4.2}$$

for $\mu > \rho - 1$ and different dimensions q where κ_μ and ρ depend on q. In fact, if we consider the projection

$$Q_k^q : M_{q,q} \to M_{k,k}, \quad \begin{pmatrix} r & * \\ * & * \end{pmatrix} \mapsto r$$

for dimensions $1 \leq k \leq q$, then we have for all $\mu > \rho - 1$,

$$Q_k^q(\tau_{q,\mu}) = \tau_{k,\mu} \tag{4.3}$$

which readily leads to this identification. A proof of Eq. (4.3) without almost no computation is based on Eq. (4.1) and the Fourier integral representation

$$J_\mu(x^*x) = \frac{1}{\kappa_\mu} \int_{D_q} e^{-2i\langle v|x\rangle} \Delta(I - v^*v)^{\mu-\rho} dv. \tag{4.4}$$

(see Eq. (3.12) of [20]) of $\mathcal{J}_\mu$. In fact, for $y \in M_{k,k}$ we have

$$\begin{aligned} (Q_k^q(\tau_{q,\mu}))^\wedge(2y) &= \int_{M_{k,k}} e^{-2i\langle y|x\rangle}\, dQ_k^q(\tau_{q,\mu})(x) = \int_{M_{q,q}} e^{-2i\langle y|Q_k^q(z)\rangle}\, d\tau_{q,\mu})(z) \\ &= \mathcal{J}_\mu\left(\begin{pmatrix} y^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} \right) = \mathcal{J}_\mu(y^*y) = \hat{\tau}_{k,\mu}(2y) \end{aligned} \tag{4.5}$$

which yields (4.3) by injectivity of the Fourier transformation.

Proposition 4.6 has several applications. As a first application we prove that the Bessel hypergroups $X_{q,\mu}$ are not isomorphic for different parameters μ. For this we start with the case $q = 1$ which was already handled in Zeuner [27]. We include the proof for sake of completeness.

4.8 Lemma. *Let $q = 1$, $\mathbb{F}$ arbitrary. and let $\mu_1 > \mu_2 \geq -1/2$ be different parameters. Then*

$$Aut(X_{1,\mu_1}) = \{T_a :\ a > 0\},$$

and there exists no hypergroup isomorphism from X_{1,μ_1} onto X_{1,μ_2}.

Proof. We first check the first statement. It suffices to prove $\subset$. For this we observe that the convolution on $X_{1,\mu_1} = [0, \infty[$ satisfies $\{|a - b|, a + b\} \subset \{a\} * \{b\} \subset [|a - b|, a + b]$ for all $a, b \geq 0$. Therefore, if $T \in Aut(X_{1,\mu_1})$ satisfies $T(1) = c$ for some $c > 0$, we obtain

$T(1/n) = c/n$ for all $n \in \mathbb{Z}_+$ and thus, $T(m/n) = cm/n$ for all $m, n \in \mathbb{Z}_+$. Continuity then yields $T(x) = cx$ for all $x \geq 0$.

Finally, if $\varphi : X_{1,\mu_1} \to X_{1,\mu_2}$ is an isomorphism, then we obtain with the arguments above also that $\varphi = T_c$ for some $c > 0$, i.e., the identity mapping $T_c^{-1} \circ \varphi$ is a hypergroup isomorphism. This is only possible if the both convolutions are equal which holds only for $\mu_1 = \mu_2$. □

4.9 Theorem. *For all fields* $\mathbb{F}$*, dimensions* $q \geq 1$*, and indices* $\mu_1 > \mu_2 \geq \rho - 1$*, the hypergroups* X_{q,μ_1} *and* X_{q,μ_2} *are not isomorphic.*

Proof. Assume that there exists an isomorphism $\varphi : X_{q,\mu_1} \to X_{q,\mu_2}$. Then, with the notion of Proposition 4.6, φ maps the one-dimensional subhypergroup $H_{1,I}$ of X_{q,μ_1} onto some subhypergroup $H_{1,u}$ of X_{q,μ_2} for a suitable $u \in U_q$. In fact, $\varphi(H_{1,u})$ is a connected subhypergroup that becomes disconnected after removing one point different from the identity 0, and this is possible by the classification in Proposition 4.6 for the subhypergroups $H_{1,u}$ of X_{q,μ_2} only. On the other hand, we know also from Proposition 4.6 that these both one-dimensional subhypergroups are isomorphic with X_{1,μ_1} and X_{1,μ_2} respectively. This leads to a contradiction with Lemma 4.8 and completes the proof. □

We next turn to a application of Proposition 4.6 to quotient hypergroups.

4.10 Remark. Let $a \in M_q$ be a matrix with rank $k \in \{0, \ldots, q\}$. We find $u, v \in U_q$ and a diagonal matrix $b = diag(b_1, \ldots, b_k, 0, \ldots, 0)$ with $b_1, \ldots, b_k \neq 0$ such that $a = ubv^*$ holds. The mapping T_a with $T_a(r) := \sqrt{ar^2a^*}$ then obviously is a continuous and open mapping from Π_q onto the subhypergroup $H_{k,u}$. Moreover, T_a is a hypergroup homomorphism. To check this, choose a sequence $(a_k)_k \subset GL(q)$ with $a_k \to a$. As then $T_{a_k} \to T_a$ pointwise on $X_{q,\mu}$, the assertion follows from Proposition 4.3 and Lemma 3.3. We next notice that the kernel of T_a is

$$kern\, T_a := \{r \in \Pi_q : \ T_a(r) = 0\} \ = \ \left\{ v \begin{pmatrix} 0 & 0 \\ 0 & \tilde{r} \end{pmatrix} v^* : \ \tilde{r} \in \Pi_{q-k} \right\} \ = \ H_{q-k,\tilde{v}}$$

for

$$\tilde{v} := \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ & & \cdots & & \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \cdot v \in U_q,$$

and that $kern\, T_a$ is a subhypergroup isomorphic with $X_{q-k,\mu}$ by Proposition 4.6.

Now let $H_{k,u}$ ($0 \leq k \leq q$, $u \in U_q$) be an arbitrary subhypergroup of $X_{q,\mu}$ As $H_{k,u}$ appears as kernel of some hypergroup homomorphism T from $X_{q,\mu}$ onto $X_{q-k,\mu}$ by the preceding considerations, we conclude from abstract results on hypergroup homomorphisms (see for instance [24]) that the quotient space

$$X_{q,\mu}/H_{k,u} := \{\{x\} * H_{k,u} : \ x \in X_{q,\mu}\}$$

(equipped with the quotient topology) carries a canonical quotient hypergroup structure with the convolution

$$\delta_{\{x\}*H_{k,u}} * \delta_{\{y\}*H_{k,u}} = \int_{X_{q,\mu}} \delta_{\{z\}*H_{k,u}} \, d(\delta_x * \delta_y)(z) \qquad (x, y \in X_{q,\mu})$$

(where this convolution is independent of the representatives x, y of the cosets; this may fail for arbitrary subhypergroups of arbitrary commutative hypergroups). Moreover, as in the group case, the hypergroup $X_{q,\mu}/H_{k,u}$ is isomorphic with $X_{q-k,\mu}$. This fact implies (see [25] and references there) that all subhypergroups of $X_{q,\mu}$ have a number of nice analytic properties which are obvious in the case of locally compact abelian groups, but which may fail for general commutative hypergroups. We therefore may say that our Bessel hypergroups on matrix cones are hypergroups which are in several respects quite close to lca groups.

In the end of this section, we classify all automorphisms for $\mathbb{F} = \mathbb{R}, \mathbb{C}$. For this we denote the group of all hypergroup automorphisms of $X_{q,\mu}$ by $Aut(X_{q,\mu})$. For $\mathbb{F} = \mathbb{R}$ we prove:

4.11 Theorem. *Let $\mathbb{F} = \mathbb{R}$ and $\mu > \rho - 1$. Then $Aut(X_{q,\mu}) = \{T_a : a \in GL(q)\}$.*

To describe all automorphisms for $\mathbb{F} = \mathbb{C}$ and $q \geq 2$, $\mu > \rho - 1$, we denote the transposition $x \mapsto x^t$ on the space of Hermitian matrices by τ. We know from Eq. (2.3) that its restriction to Π_q is contained in $Aut(X_{q,\mu})$. Moreover, for $q \geq 2$, $\tau \notin \{T_a : a \in GL(q, \mathbb{C})\}$. (In fact, for the proof of this fact we may restrict our attention to the case $q = 2$, in which case the statement can be checked by a direct computation.)

Moreover, as $\tau \circ T_a \circ \tau = T_{\tau(a)}$ for all $a \in GL(q, \mathbb{C})$, it follows that

$$\{\sigma \circ T_a : a \in GL(q, \mathbb{C}), \sigma \in \{Id, \tau\}\}$$

is a group of automorphisms of $X_{q,\mu}$.

4.12 Theorem. *Let $\mathbb{F} = \mathbb{C}$ and $\mu > \rho - 1$. Then*

$$Aut(X_{q,\mu}) = \{\sigma \circ T_a : a \in GL(q, \mathbb{C}), \sigma \in \{Id, \tau\}\}.$$

Our proof of this classification is quite complicated, covers the remaining part of Section 4 and may be skipped at a first reading. The proof is divided into several steps which partially work for all $\mathbb{F}$. In the first steps we deal with the multi-dimensional case for arbitrary fields $\mathbb{F}$. The first main result will be Proposition 4.16 below where we show that each $T \in Aut(X_{1,\mu})$, which preserves diagonal matrices, preserves the norm for all matrices in Π_q.

4.13 Lemma. *Let $\mu > \rho - 1$ and $T \in Aut(X_{q,\mu})$. Then for each $u \in U_q$ and $k = 0, \ldots, q$ there exists $\tilde{u} \in U_q$ with $T(H_{k,u}) = H_{k,\tilde{u}}$.*

Proof. Consider the maximal chain

$$\{0\} = H_{0,u} \subset H_{1,u} \subset \ldots \subset H_{q,u} = \Pi_q$$

of subhypergroups such that all inclusions are proper. The classification of all subhypergroups in Proposition 4.6 now leads to the claim. □

In the following, we denote the diagonal matrix $diag(0, \ldots, 0, 1, 0, \ldots, 0) \in \Pi_q$ with 1 at the i-th element by e_i $(i = 1, \ldots, q)$.

4.14 Lemma. *Let $\mu > \rho - 1$ and $T \in Aut(X_{q,\mu})$. Then there exists $a \in Gl(q)$ such that $T_a \circ T(c \cdot e_i) = c \cdot e_i$ for all $c \geq 0$ and $i = 1, \ldots, q$.*

Proof. For $i = 1, \dots, q$ consider $r_i := T(e_i) \in \Pi_q$. These matrices have rank 1 by Lemma 4.13. We thus find vectors $x_i \in \mathbb{F}^q$ with $r_i^2 = x_i x_i^*$. We claim that the x_i are linearly independent. In fact, if they would be dependent, we would find $x \in \mathbb{F}^q \setminus \{0\}$ with $r_i x = x_i x_i^* x = 0$ for all i. In other words, $r_1, \dots, r_q$ would be contained in a proper subhypergroup of Π_q. But this is impossible by Lemma 4.13, as Π_q is the only subhypergroup containing all e_i. We thus see that the x_i are linearly independent. Hence we find a unique $a \in Gl(q)$ such that for all i, ax_i is the ith unit vector. This implies $T_a \circ T(e_i) = e_i$ for all i. We thus conclude from Lemma 4.13 that $T_a \circ T$ is an automorphism on the one-dimensional hypergroups $\{c \cdot e_i : c \geq 0\}$ with $T_a \circ T(e_i) = e_i$. Therefore, by Lemma 4.8, $T_a \circ T$ is the identity on these subhypergroups. This proves the lemma. □

4.15 Lemma. *Let $T \in Aut(X_{q,\mu})$ with $T(c \cdot e_i) = c \cdot e_i$ for all $c \geq 0$ and $i = 1, \dots, q$. Then $T(r) = r$ for all diagonal matrices $r \in \Pi_q$.*

Proof. Let T^* be the dual automorphism according to Proposition 3.4. We first fix $i \in \{1, \dots, q\}$, $c \geq 0$ and $s \in \Pi_q$ and notice

$$\varphi_{T^*(s)}(c \cdot e_i) = \varphi_s(T(c \cdot e_i)) = \varphi_s(c \cdot e_i).$$

The Taylor expansion (2.2) of $\mathcal{J}_\mu$ now yields

$$1 - \frac{c^2}{4\mu} tr(e_i^2 T^*(s)^2) + O(c^4) = 1 - \frac{c^2}{4\mu} tr(e_i^2 s^2) + O(c^4)$$

for $c \to 0$. As this holds for all i, the matrices $T^*(s)^2$ and s^2 have the same diagonal parts for any s.

Now let $r = \sum_{i=1}^q c_i e_i \in \Pi_q$ be an arbitrary diagonal matrix. Using $\varphi_{T^*(cs)}(r) = \varphi_{cs}(T(r))$ and the Taylor expansion (2.2), we obtain

$$1 - \frac{c^2}{4\mu} tr(T(r)^2 s^2) + O(c^4) = J_\mu(\frac{1}{4} r^2 T^*(cs)^2)$$

where, by our considerations above, $r^2 T^*(cs)^2 = c^2 r^2 s^2 + h$ for some matrix $h = h(c, s, r)$ with zeros on the diagonal. Moreover, as $T^*(cs)$ as well as cs are both positive semidefinite, and as the absolute values of all entries of a positive semidefinite matrix are bounded by the maximum of the absolute values of the diagonal entries, we have $h(c, r, s) = O(c^2)$ for $c \to 0$. Hence, again by (2.2) ,

$$J_\mu(\frac{1}{4} r^2 T^*(cs)^2) = J_\mu(\frac{1}{4} c^2 r^2 s^2) + O(c^4).$$

Combining all results, we obtain

$$1 - \frac{c^2}{4\mu} tr(r^2 s^2) + O(c^4) = J_\mu(\frac{1}{4} c^2 r^2 s^2) + O(c^4) = 1 - \frac{c^2}{4\mu} tr(T(r)^2 s^2) + O(c^4)$$

for $c \to 0$. Hence, $tr(r^2 s^2) = tr(T(r)^2 s^2)$ for all $s \in \Pi_q$ and thus all $s \in H_q$. As the trace forms a scalar product, we obtain $T(r) = r$ as claimed. □

4.16 Proposition. *Let $\mu > \rho - 1$ and $T \in Aut(X_{q,\mu})$. Then there exist $a \in Gl(q)$ and a mapping $h : U_q \to U_q$ with $h(I) = I$ such that $T_a \circ T(uru^*) = T_{h(u)}(uru^*)$ for all diagonal matrices $r \in \Pi_q$ and all $u \in U_q$. In particular, for this a, we have $\|T_a \circ T(x)\| = \|x\|$ for all $x \in \Pi_q$.*

Proof. By Lemmas 4.14 and 4.15 we find $a \in Gl(q)$ such that $T_a \circ T(r) = r$ for all diagonal matrices $r \in \Pi_q$. The same argument together with change of basis show that for each $u \in U_q$ there exists $h(u) \in GL(q)$ such that $T_a \circ T(uru^*) = T_{h(u)}(uru^*)$ for all diagonal matrices $r \in \Pi_q$. Taking $r = I$, we see that for any $u \in U_q$, $I = T_a \circ T(I) = T_a \circ T(uIu^*) = T_{h(u)}(uIu^*) = T_{h(u)}(I)$, and hence $h(u) \in U_q$. The proposition is now obvious. □

We next restrict our attention to the case $q = 2$ where we derive the classification for $\mathbb{F} = \mathbb{R}, \mathbb{C}$. We also always assume $\mu > \rho - 1$. The key will be:

4.17 Lemma. *Let $q = 2$ and $\mu > \rho - 1$. Then, for all $a, c \in]0, \infty[$,*

$$\left(\left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right\} * \left\{\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}\right\}\right) \cap \{r \in \Pi_q : \ rank\, r = 1\} = \{s(\beta) : \ |\beta| = 1\},$$

where, for $\beta \in \mathbb{F}$ with $|\beta| = 1$,

$$s(\beta) := \begin{pmatrix} a^2 & a\beta c \\ a\bar{\beta}c & c^2 \end{pmatrix}^{1/2} = \frac{1}{a^2 + c^2} \begin{pmatrix} a & \beta c \\ -\bar{\beta}c & a \end{pmatrix} \begin{pmatrix} \sqrt{a^2 + c^2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & -\beta c \\ \bar{\beta}c & a \end{pmatrix}.$$

Proof. Let $v = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2$ with $vv^* \leq I$. Using the convolution formula Eq. (2.3) and

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}^2 + \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} v \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} v^* \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} a^2 & a\beta c \\ a\bar{\beta}c & c^2 \end{pmatrix},$$

the statement of the lemma follows easily. □

4.18 Corollary. *Let $T \in Aut(X_{2,\mu})$ with $T(r) = r$ for diagonal matrices $r \in \Pi_2$. Then, for all $a, c > 0$ and $\beta \in \mathbb{F}$ with $|\beta| = 1$ and with the notion of Lemma 4.17, $T(s(\beta)) \in \{s(\gamma) : \gamma \in \mathbb{F}, \ |\gamma| = 1\}$.*

Proof. T preserves the matrices $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$ for $a, c \geq 0$. As T also preserves the rank by 4.13, the statement follows from Lemma 4.17. □

Corollary 4.18 now leads to the classification for $q = 2$ and $\mathbb{F} = \mathbb{R}$.

4.19 Proposition. *For $\mathbb{F} = \mathbb{R}$, $Aut(X_{2,\mu}) = \{T_a : \ a \in GL(2)\}$.*

Proof. According to Lemmas 4.14 and 4.15, it suffices to prove that any $T \in Aut(X_{2,\mu})$ with $T(r) = r$ for diagonal matrices $r \in \Pi_2$ has the form $T = T_u$ for some $u \in O(2)$. For this take $a, c \geq 0$ and $\beta = \pm 1$ and consider the matrices $s(\beta)$ as above. Then, by Corollary 4.18, $T(s(\beta)) \in \{s(1), s(-1)\}$.

Assume now that $s_0 := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ satisfies $T(s_0) = s_0$. The continuity of T then implies that $T(s(\beta)) = s(\beta)$ for all a, c and $\beta = 1$. Clearly, this statement then must also hold for all a, c and $\beta = -1$. As by the diagonalization of $s(\beta)$ in Lemma 4.17 each rank one matrix in Π_2 appears as some $s(\beta)$, we conclude that T is the identity for all rank one matrices. Proposition 4.16 now implies that T is the identity for all matrices in Π_q.

Furthermore, if $T(s_0) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, then we get by the same arguments $T(s(\beta)) = s(-\beta)$ for all a, c and β and thus $T = T_u$ for $u = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on Π_q. This proves the claim. □

We next deal with $q = 2$ and $\mathbb{F} = \mathbb{C}, \mathbb{H}$.

4.20 Lemma. *Let $T \in Aut(X_{2,\mu})$ with $T(r) = r$ for all diagonal matrices $r \in \Pi_2$. Then there exists $v \in U_2$ such that $T_v \circ T(r) = r$ for all $r \in \Pi_2$ which are diagonal or which have the form $r = \begin{pmatrix} s+t & s-t \\ s-t & s+t \end{pmatrix}$ with $s, t \geq 0$.*

Proof. By Corollary 4.18 there exist numbers $\beta_1(s), \beta_2(t) \in \mathbb{F}$ with $|\beta_1(s)| = |\beta_2(t)| = 1$ such that

$$T\left(\begin{pmatrix} s & s \\ s & s \end{pmatrix}\right) = \begin{pmatrix} s & s\beta_1(s) \\ s\overline{\beta_1(s)} & s \end{pmatrix}, \tag{4.6}$$
$$T\left(\begin{pmatrix} t & -t \\ -t & t \end{pmatrix}\right) = \begin{pmatrix} t & -t\beta_2(t) \\ -t\overline{\beta_2(t)} & t \end{pmatrix}$$

for all $s, t \geq 0$. On the other hand,

$$< \begin{pmatrix} s & s \\ s & s \end{pmatrix}, \begin{pmatrix} t & -t \\ -t & t \end{pmatrix} > = \{u_0 r u_0^* : \ r \in \Pi_2 \text{ diagonal}\} =: H$$

for $u_0 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \in U_2$, and thus, by Proposition 4.16, $T(w) = v^* w v$ for all $w \in H$ and some $v \in U_2$. In particular, T is $\mathbb{R}$-linear on H which ensures that β_1, β_2 in Eq.(4.6) are constants independent of $s, t \geq 0$. Therefore,

$$T\left(\begin{pmatrix} s+t & s-t \\ s-t & s+t \end{pmatrix}\right) = \begin{pmatrix} s+t & \beta_1 s - \beta_2 t \\ \bar{\beta}_1 s - \bar{\beta}_2 t & s+t \end{pmatrix}$$

for $s, t \geq 0$. As T is norm-preserving, it follows that $|s - t| = |s - t\beta_1/\beta_2|$ for all $s, t \geq 0$. This yields $\beta_1 = \beta_2 =: \beta$, and thus $T(w) = v^* w v$ for $v = \begin{pmatrix} \beta & 0 \\ 0 & \bar{\beta} \end{pmatrix} \in U_2$, Therefore, v has the properties claimed in the lemma. □

4.21 Proposition. *For $\mathbb{F} = \mathbb{C}$,*

$$Aut(X_{2,\mu}) = \{\sigma \circ T_a : \ a \in GL(2, \mathbb{C}), \ \sigma \in \{Id, \tau\}\}.$$

Proof. According to Lemmas 4.14 and 4.15 and 4.20, it suffices to consider $T \in Aut(X_{2,\mu})$ with $T(r) = r$ for all $r \in \Pi_2$ which are diagonal or which have the form $r = \begin{pmatrix} s+t & s-t \\ s-t & s+t \end{pmatrix}$ with $s, t \geq 0$. Let $s, t \geq 0$ and let $u_0 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \in U_2$ with $u_0^{-1} = u_0$. As T_{u_0} a hypergroup automorphism, we conclude from Lemma 4.17 that

$$\left(\left\{T_{u_0}\left(\begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix}\right)\right\} * \left\{T_{u_0}\left(\begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix}\right)\right\}\right) \cap \{r \in \Pi_q : \ rank\, r = 1\}$$
$$= \{t(\beta) : \ \beta \in \mathbb{C}, \ |\beta| = 1\} \tag{4.7}$$

with

$$t(\beta) = T_{u_0}\left(\begin{pmatrix} s^2 & st\beta \\ st\bar{\beta} & t^2 \end{pmatrix}^{1/2}\right)$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} s^2+t^2+st(\beta+\bar{\beta}) & s^2-t^2+st(\bar{\beta}-\beta) \\ s^2-t^2+st(\bar{\beta}-\beta) & s^2+t^2-st(\bar{\beta}+\beta) \end{pmatrix}^{1/2}.$$

As

$$T\circ T_{u_0}\left(\begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix}\right) = T_{u_0}\left(\begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix}\right)$$

and

$$T\circ T_{u_0}\left(\begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix}\right) = T_{u_0}\left(\begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix}\right)$$

by our assumption, we see that for all $\beta \in \mathbb{C}$ with $|\beta| = 1$ there exists $\gamma = \gamma(\beta) \in \mathbb{C}$ with $|\gamma| = 1$ such that $T(t(\beta)) = t(\gamma)$. On the other hand, $t(\beta)$ is a rank one matrix and has thus the form $s(\delta)$ for some $\delta \in \mathbb{C}$ with $|\delta| = 1$ in the notion of Lemma 4.17. Therefore, by Corollary 4.18, the diagonal entries of $t(\beta)^2$ are preserved under T. Hence, β and $\gamma(\beta)$ have the same real parts, and thus $T(t(\beta)) \in \{t(\beta), t(\bar{\beta})\}$. A continuity argument shows that we have either $T(t(\beta)) = t(\beta)$ for all $s, t \geq 0$ and all $|\beta| = 1$, or that we always have the other case. As each rank one matrix $r \in \Pi_2$ appears as some $t(\beta)$ (for suitable s, t, β), we conclude from Proposition 4.16 that T is either the identity or the transposition τ on Π_2. □

We next restate Propositions 4.19 and 4.21. For this, we define for $i, j = 1, \ldots, q$ with $i \neq j$ the space $U^{i,j}(q)$ of all unitary $v \in U_q$ with $v_{k,k} = 1$ for all $k \neq i, j$, i.e., there are at most two possible non-trivial non-diagonal entries of v in the positions (i, j) and (j, i). The following statement now follows immediately from the proof of Propositions 4.19 and 4.21 by a suitable basis change.

4.22 Lemma. *Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$, $q \geq 2$, $u \in U_q$, and $T \in Aut(X_{q,\mu})$ with $T(uru^*) = uru^*$ for all diagonal matrices r. Let $i, j = 1, \ldots, q$ with $i \neq j$. Then there exists $u(i,j) \in U_q$ and $\varphi \in \{Id, \tau\}$ such that $T(vuru^*v^*) = \varphi \circ T_{u(i,j)}(vuru^*v^*)$ for all $v \in U^{i,j}(q)$ and all diagonal matrices $r \in \Pi_q$. For $\mathbb{F} = \mathbb{R}$ only the case $\varphi = Id$ appears.*

We are now ready to complete the classification.

Proof of Theorems 4.11 and 4.12. It suffices to consider the slightly more complicated case $\mathbb{F} = \mathbb{C}$. Moreover, as in the proof of Propositions 4.19 and 4.21, it suffices to prove that each $T \in Aut(X_{q,\mu})$ with $T(r) = r$ for diagonal matrices $r \in \Pi_q$ has the form $T = T_u$ or $T = \tau \circ T_u$ for some $u \in U(q)$. To prove this we recapitulate that there exist $N = N(q)$ and $i_1, \ldots, i_N, j_1, \ldots, j_N \in \{1, \ldots, q\}$ with $i_n \neq j_n$ for $n = 1, \ldots, N$ such that

$$U_q = U^{i_1,j_1}(q) \cdot U^{i_2,j_2}(q) \cdots U^{i_N,j_N}(q).$$

Moreover, Lemma 4.22 and induction show that for $n = 0, 1, \ldots, N$ there exist $u_n \in U_q$ and $\varphi_1, \ldots, \varphi_N \in \{Id, \tau\}$ such that

$$T(v_n v_{n-1} \ldots v_1 r v_1^* \ldots v_{n-1}^* v_n^*) = \varphi_n \circ T_{u_n}(v_n v_{n-1} \ldots v_1 r v_1^* \ldots v_{n-1}^* v_n^*)$$

for all diagonal matrices $r \in \Pi_q$ and all $v_1 \in O^{i_1,j_1}(q), \ldots, v_n \in O^{i_n,j_n}(q)$. The theorem now follows for $n = N$. □

We finally consider the case $\mathbb{F} = \mathbb{H}$. By the preceding proof, the classification only depends on the computation of $Aut(X_{2,\mu})$. We suggest that here the study of concrete additional matrices as in Lemma 4.20 and Proposition 4.21 leads to the following conjecture:

Consider the group $G_{\mathbb{H}}$ of automorphisms of the field $\mathbb{H} =< 1, i, j, k >_{\mathbb{R}}$ which fix the real line, which is generated by the 3 automorphisms which switch two of the i, j, k and change the sign of the third component. In this D_3-case we then we have $|G_{\mathbb{H}}| = 24$, and we may let act $G_{\mathbb{H}}$ on $X_{2,\mu}$ by using the same transformation in each component of a matrix. It can be easily checked by Eq. (2.3) that $G_{\mathbb{H}}$ then forms a group of hypergroup automorphisms on $X_{2,\mu}$. Moreover, for $\tau \in G_{\mathbb{H}}$ and $a \in GL(q)$ we have $\tau^{-1} \circ T_a \circ \tau = T_{\tau(a)}$. Therefore, $\{\tau \circ T_a : \ a \in GL(q), \ \tau \in G_{\mathbb{H}}\}$ forms a group of hypergroup automorphisms on $X_{2,\mu}$. We expect that

$$Aut(X_{q,\mu}) = \{\tau \circ T_a : \ a \in GL(q), \ \tau \in G_{\mathbb{H}}\}.$$

5 Convolution semigroups and Wishart distributions

In this section we first introduce convolution semigroups and associated random walks on $X_{q,\mu}$. This concept is well-known for commutative hypergroups; see [1], [19], and references there. We shall see that in particular general so-called squared Wishart distributions form such convolution semigroups. In this way several known results about Wishart distributions and Wishart processes may be partially seen under a new light, see [2],[3],[4],[5],[9], [7],[12], and in particular,[13] and [18]. We here notice that these Wishart distributions will appear later as limits in a central limit theorem in the next section. We first recapitulate the notions of convolution semigroups and associated random walks.

5.1 Definition. (1) A family $(\mu_t)_{t\geq 0} \subset M^1(X_{q,\mu})$ of probability measures on $X_{q,\mu}$ is called a (continuous) convolution semigroup on $X_{q,\mu}$, if $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \geq 0$ with $\mu_0 = \delta_e$, and if the mapping $[0, \infty[\to M^1(X_{q,\mu})$, $t \mapsto \mu_t$ is weakly continuous.

(2) A convolution semigroup $(\mu_t)_{t\geq 0}$ is called Gaussian if

$$\lim_{t\to 0} \frac{1}{t}\mu_t(X_{q,\mu} \setminus U) = 0 \qquad \text{for all open subsets } U \subset X_{q,\mu} \text{ with } 0 \in U.$$

(3) Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on $X_{q,\mu}$. A $X_{q,\mu}$-valued time-homogeneous Markov process $(X_t)_{t\geq 0}$ is called a Leṽy process on $X_{q,\mu}$ associated with $(\mu_t)_{t\geq 0}$, if its transition probabilities satisfy

$$P(X_t \in A|\ X_s = x) = (\mu_{t-s} * \delta_x)(A)$$

for all $0 \leq s \leq t$, $x \in X_{q,\mu}$, and Borel sets $A \subset X_{q,\mu}$. By well-known general principles for Feller processes, a Leṽy process on $X_{q,\mu}$ always admits a version with rcll paths, and a Lévy process is Gaussian, i.e., is associated with a Gaussian convolution semigroup, if and only if it admits a version with continuous paths; see [19].

(4) Similar to the continuous case, we say that a random walk $(S_n)_{n\geq 0}$ on $X_{q,\mu}$ associated with a sequence $(\mu_n)_{n\geq 1} \subset M^1(\Pi_q)$ is a Markov chain with initial distribution $P_{S_0} = \delta_0$ and the transition probabilities

$$P(S_n \in A|\ S_{n-1} = x) \ = \ (\delta_x * \mu_n)(A) \tag{5.1}$$

for $n \geq 1$, $x \in \Pi_q$ and Borel sets $A \subset \Pi_q$. If all μ_n are equal to some μ, then $(S_n)_{n\geq 0}$ is time-homogeneous, and we say that it is associated with the measure μ.

It is easy to check that for a random walk $(S_n)_{n\geq 0}$ on $X_{q,\mu}$ associated with $(\mu_n)_{n\geq 1}$ and $n \geq 0$, S_n has distribution $P_{S_n} = \mu_1 * \mu_2 * \cdots * \mu_n * P_{S_0}$.

We next turn to Wishart distributions which form examples of Gaussian convolution semigroups. Before defining them, we point out at the beginning that our notion of Wishart distributions is equivalent to, but slightly different from the classical one, as here in the group case $\mu = dp/2$, a positive semidefinite matrix $r \in \Pi_q$ corresponds to $\sqrt{x^*x}$ for $x \in M_{p,q}$ and not to x^*x as usual. In this way, images of U_p-invariant normal distributions on $M_{p,q}$ under the projection $M_{p,q} \to \Pi_q$, $x \mapsto \sqrt{x^*x}$, will be images of classical Wishart distributions on Π_q under $r \mapsto \sqrt{r}$ on Π_q. Also for general parameters μ, we use these images of classical Wishart distributions under this square root mapping, and call these distributions squared Wishart distributions. For instance, for $q = d = 1$, classical Wishart distributions are gamma distributions while squared ones are Rayleigh distributions. In this way, our notion is in agreement with Kingman [16] for $q = d = 1$ and close to the classical Euclidean setting. This notion has also the advantage that the limit theorems in Sections 6 and 7 will be in our notion very close to the classical Euclidean setting.

5.2 Definition. The standard squared Wishart distribution $W = W(d, q, \mu)$ on $\Pi_q = \Pi_q(\mathbb{F})$ with shape parameter $\mu \geq \rho - 1$ is the probability measure

$$(2\pi)^{-q\mu} e^{-tr(r^2)/2}\, d\omega_\mu(r) \qquad (r \in \Omega_q)$$

on Π_q. This is in fact a probability measure; this follows for instance from Lemma 5.4 below for $s = 0$:

We next turn to general squared Wishart distributions and observe first that ω_μ and hence W are invariant under the unitary transforms $r \mapsto uru^*$ on Π_q for $u \in U_q$. As any $a \in M_q$ may be written as $a = su$ with $s = \sqrt{aa^*} \in \Pi_q$ and $u \in U_q$, the image $T_a(W)$ under the mapping $T_a(r) = \sqrt{ar^2a^*}$ agrees with $T_s(W)$, i.e., $T_a(W)$ depends only on $s = \sqrt{aa^*}$.

5.3 Definition. The squared Wishart distribution $W(s^2) = W(d, q, \mu; s^2)$ on $\Pi_q = \Pi_q(\mathbb{F})$ with shape parameter $\mu \geq \rho - 1$ and covariance s^2 for $s \in \Pi_q$ is defined as the image of W under T_s (or, by the preceding discussion, under T_a for any $a \in M_q$ with $s^2 = aa^*$).

The transformation formula yields that for regular $s \in \Pi_q$, the distribution $W(s^2)$ has the ω_μ-density

$$f_{s^2}(r) := \frac{1}{\Delta(s)^\mu (2\pi)^{q\mu}} e^{-tr(s^{-1}r^2s^{-1})/2} \qquad (r \in \Pi_q). \tag{5.2}$$

Moreover, if $s \in \Pi_q$ is singular with rank $k < q$, then $W(s^2) = T_s(W)$ is supported by the proper subhypergroup $T_s(\Pi_q)$ which can be identified with $X_{k,\mu}$; cf. Proposition 4.6. We show below that if we regard $W(s^2) = T_s(W)$ as a measure on $X_{k,\mu}$, it again admits a density like Eq. (5.2) with respect to the Haar measure on $X_{k,\mu}$.

We next determine the Fourier transforms of squared Wishart distributions (in the hypergroup sense).

5.4 Lemma. *For $a \in M_q$, the Fourier transform of $W(aa^*)$ is given by*

$$\widehat{W(aa^*)}(s) = e^{-tr(a^*s^2a)/2} \qquad (s \in \Pi_q).$$

Proof. Proposition XV.2.1 of [7] yields

$$\int_{\Omega_q} e^{-tr(xy)} \mathcal{J}_\mu(x)\Delta(x)^{\mu-n/q}\,dx = \Gamma_{\Omega_q}(\mu)\Delta(y)^{-\mu}e^{-tr(y^{-1})}.$$

Change of variables $y^{-1} = s^2/2$ and $x = sr^2s/4$ readily leads to the claim for the standard case $a = I$. Finally Corollary 4.2 shows that for any $a \in M_q$,

$$\widehat{W(aa^*)}(r) = \widehat{T_a(W)}(r) = \hat{W}(T_{a^*}(s)) = e^{-tr(a^*s^2a)/2} = e^{-tr(aa^*s^2)/2}.$$

□

If we introduce the exponentials $e_z \in L^2(\Omega_\mu)$ with $e_z(r) := e^{-tr(r^2z^2)/2}$ for $a \in \Pi_q$, we may write the preceding lemma briefly as

$$\widehat{e_z} = (2\pi)^{q\mu}\Delta(z)^{-\mu}e_{z^{-1}}. \tag{5.3}$$

As announced above, we now briefly discuss the density of degenerated squared Wishart distributions. We restrict our attention to a special case without loss of generality (cf. Remark 4.10); the general case would need too much additional notation.

Let $a = diag(1,\dots,1,0,\dots,0)$ be a diagonal matrix with rank $k \in \{0,\dots,q\}$. Then the squared Wishart distribution $W(a)$ is supported by the subhypergroup $H_{k,I}$ which can be identified with $X_{k,\mu}$ via $\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} \simeq r$; see Proposition 4.6. Moreover, characters of $H_{k,I}$ can be written as

$$r \in H_{k,I} \mapsto \varphi_s(r) = \mathcal{J}_\mu(\frac{1}{4}s^2r^2) = \mathcal{J}_\mu\left(\frac{1}{4}\begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix}^2 \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}^2\right) \qquad (s \in H_{k,I});$$

see the proof of Proposition 4.6. Therefore, for $s \in H_{k,I}$,

$$\widehat{W(a)}(s) = \widehat{T_a(W)}(s) = \int_{H_{k,I}} \varphi_s\,dT_a(W) = \int_{\Pi_q} \varphi_s \circ T_a\,dW = \widehat{W}\begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, by Lemma 5.4, the Fourier transforms of $W(a)$ and the standard squared Wishart distribution on $H_{k,I}$ are equal. The injectivity of the hypergroup Fourier transform then yields that, under the identification above, $W(a)$ is just the standard squared Wishart distribution on $H_{k,I}$.

We note that this result may be also obtained by direct computation. The details here (e.g. regarding the constants) are however in our opinion more complicated than in our approach. Singular Wishart distributions are also considered (after the transformation $r \mapsto \sqrt{r}$) in [3], [17].

We next collect some trivial properties of squared Wishart distributions.

5.5 Lemma. *For all $a, b \in \Pi_q$:*

(1) $T_a(W(b^2)) = W(ab^2a)$;

(2) $W(b^2) * W(a^2) = W(a^2 + b^2)$;

(3) $(\mu_t := W(ta^2))_{t\geq 0}$ *is a Gaussian convolution semigroup.*

Proof. The first two statements follow from injectivity of the Fourier transform. Moreover, (2), the explicit formulas for the Fourier transforms, and Lévy's continuity theorem for the hypergroup Fourier transform (Ch. 4.2 of [1]) imply that $(\mu_t := W(ta^2))_{t\geq 0}$ is a convolution semigroup. For the proof of being Gaussian, we may assume a as identity matrix, in which case the definition may be checked easily by using the transformation formula. □

We expect that all Gaussian convolution semigroups on $X_{q,\mu}$ are given by squared Wishart distributions in this way. In fact, for the group cases $\mu = dp/2$ this can be easily deduced from the well-known corresponding result on the group $M_{q,p} \simeq \mathbb{R}^{dpq}$. In the other cases we shall investigate this point in a forthcoming paper. We also point out that the generator of the Wishart semigroups is known in princple; see Bru [2].

In the end of this section we determine translates $\delta_x * W(s^2)$ of squared Wishart distributions, as these shifted squared Wishart distributions appear in the transition kernels of Gaussian processes; cf. Definition 5.1(2). We here follow ideas of C. Herz [12] and use the following generalization of a result of Tricomi (see also [5]):

5.6 Lemma. *Let $f, g \in L^2(\Omega_\mu)$, and $\varphi(z) := \langle e_z, f\rangle$ and $\psi(z) := \langle e_z, g\rangle$ for $z \in \Omega_\mu$. Then $\hat{f} = g$ if and only if $\varphi(z) = (2\pi)^{-q\mu}\Delta(z)^{-\mu}\psi(z^{-1})$.*

Proof. Let $g = \hat{f}$. Eq. (5.3), and the Plancherel formula 2.1(c) imply

$$\begin{aligned}\varphi(z) := \langle e_z, f\rangle &= (2\pi)^{-2q\mu}\langle \widehat{e_z}, g\rangle \\ &= (2\pi)^{-q\mu}\Delta(z)^{-\mu}\langle e_{z^{-1}} g\rangle = (2\pi)^{-q\mu}\Delta(z)^{-\mu}\psi(z^{-1}).\end{aligned}$$

Conversely, a Stone-Weierstrass argument shows that the e_z span a dense subspace of $L^2(\Omega_\mu)$ which yields the converse statement. □

5.7 Lemma. *For any $s \in \Omega_q$ and $x \in \Pi_q$,*

$$d(\delta_x * W(s^2))(y) = \frac{1}{\Delta(s)^\mu (2\pi)^{q\mu}} e^{-tr(x^2+s^{-1}y^2s^{-1})/2}\mathcal{J}_\mu(-\frac{1}{4}x^2s^{-1}y^2s^{-1})\, d\omega_\mu(y)$$

Proof. Again, using a suitable automorphism, we may restrict our attention to the standard case. Fix $x \in \Pi_q$ and consider the functions

$$f(s) := e^{-tr(s^2+x^2)/2}\mathcal{J}_\mu(-\frac{1}{4}x^2s^2)$$

and $g(r) := e^{-tr(r^2)/2}\mathcal{J}_\mu(\frac{1}{4}x^2r^2)$ in $L^2(\omega_\mu)$. Then the associated functions φ, ψ according to the preceding lemma are given by

$$\varphi(z) := \int e^{-tr(s^2+x^2)/2}\mathcal{J}_\mu(-\frac{1}{4}x^2s^2)e^{-tr(s^2z^2)/2}\, d\omega(s) = e^{-tr(x^2)/2}\widehat{e_{\sqrt{z^2+1}}}(ix)$$

and

$$\psi(z) := \int e^{-tr(r^2)/2}\mathcal{J}_\mu(\frac{1}{4}x^2r^2)e^{-tr(r^2z^2)/2}\, d\omega(r) = \widehat{e_{\sqrt{z^2+1}}}(x).$$

Eq. (5.3) and analytic continuation yield $\varphi(z) = \Delta(z)^{-\mu}\psi(z^{-1})$, and hence, by Lemma 5.6, $(2\pi)^{-q\mu}\hat{f} = g = (W * \delta_x)^\wedge$ on Π_q. As the hypergroup Fourier transform is injective, the proof is complete. □

6 Limit theorems

In this section we derive a central limit theorem as well as strong laws of large numbers for random walks on matrix Bessel hypergroups which reduces in the group cases $\mu = pd/2$ just to radial parts of the classical central limit theorem on the vector space $M_{p,q}$ for sums of iid random variables and the corresponding classical strong laws of large numbers of Kolmogorov. The proof of the central limit theorem is standard and uses a Taylor expansion of the Fourier transforms as well as Lévy's continuity theorem for hypergroups. Before stating the CLT, we introduce so-called moment functions on matrix Bessel hypergroups. Such moment functions on hypergroups were introduced by Zeuner and used later for several limit theorems on hypergroups; see the monograph [1] for more details and references.

To introduce moment functions, we recapitulate that we regard M_q and H_q as real vector spaces with scalar product $(x|y) := \Re tr(xy^*)$ and norm $\|x\| = (x|x)^{1/2}$. For $k \in \mathbb{Z}_+$ and a function g in the variable $s \in H_q$, we denote the k-th differential of g by $d_s^k g(s)$ where this is a k-linear map on H_q. Following the literature on limit theorems on hypergroups (see Chapter 7 of [1], [28] and references cited there), we introduce moment functions and moments of probability measures on Π_m.

6.1 Definition. For $k \in \mathbb{Z}_+$ and $s_1, \ldots, s_k \in H_q$ define the moment function

$$m_k^{s_1,\ldots,s_k}(r) := i^k \cdot d_s^k \varphi_s(r)|_{s=0}(s_1, \ldots, s_k)$$

on Π_q. As $\varphi_s(r) = \mathcal{J}_\mu(\frac{1}{4}sr^2 s)$ and $\mathcal{J}_\mu(x) = 1 - \frac{tr(x)}{\mu} + o(\|x\|^2)$ for $x \to 0$, we have for $r \in \Pi_q$:

(1) $m_{2k+1}^{s_1,\ldots,s_{2k+1}} = 0$ for all k and $s_1, \ldots, s_{2k+1} \in H_q$, and
(2) $m_2^{s_1,s_2}(r) = \frac{1}{2\mu}\Re tr(s_1 r^2 s_2)$.
(3) In particular, $m_2^{I,I}(r) = \frac{1}{2\mu}\|r\|^2$.
(4) $m_0 \equiv 1$.

We say that a probability measure $\mu \in M^1(\Pi_q)$ has a k-th moment, if

$$\int_{\Pi_q} \|r\|^k \, d\mu(r) < \infty.$$

In order to get estimates for moment functions and derivatives of $\hat{\mu}$, we use the following Bochner-type integral representation for J_μ of Rösler; see Remark 3.7 of [20]:

$$\varphi_s(r) = J_\mu(\frac{1}{4}(sr)^*(sr)) = \frac{1}{\kappa_\mu} \int_D e^{-i(rv|s)} \Delta(I - vv^*)^{\mu-\rho} dv. \tag{6.1}$$

6.2 Lemma. *For $k \in \mathbb{Z}_+$ and $s_0, s_1, \ldots, s_k, r \in H_q$,*

$$\left| d_s^k \varphi_s(r)|_{s=s_0}(s_1, \ldots, s_k) \right| \le \|r\|^k \cdot \prod_{l=1}^{k} \|s_l\|,$$

and in particular, $|m_{2k}^{s_1,\ldots,s_{2k}}(r)| \le \|r\|^{2k} \cdot \prod_{l=1}^{2k} \|s_l\|$.

Proof. As D is compact, we may interchange derivatives and integration in Eq. (6.1). Thus,

$$d_s^k \varphi_s(r)|_{s=s_0}(s_1, \ldots, s_k) = \frac{(-i)^k}{\kappa_\mu} \int_D \prod_{l=1}^k (rv|s_l) \cdot e^{-i(rv|s)} \Delta(I - vv^*)^{\mu-\rho} dv.$$

As for $v \in D$, $0 \leq vv^* \leq I$ and hence $0 \leq rvv^*r \leq r^2$, we obtain

$$|(rv|s_l)| \leq \|rv\| \cdot \|s_l\| \leq \|r\| \cdot \|s_l\|,$$

and the lemma follows by taking absolute values. □

6.3 Proposition. *Let $\nu \in M^1(\Pi_q)$ and $k \geq 1$. If the $2k$-th moment of ν exists, then $\hat{\nu}$ is $2k$-times continuously differentiable on H_q with $d^{2k-1}\hat{\nu}(0) \equiv 0$. Moreover, for $l \leq 2k$,*

$$d^l \hat{\nu}(s) = \int_{\Pi_q} d_s^l \varphi_s(r)\, d\nu(r) \qquad (s \in \Pi_q),$$

and, in particular for $l \leq k$ and $s_1, \ldots, s_{2l} \in \Pi_q$,

$$d^{2l}\hat{\nu}(0)(s_1, \ldots, s_{2l}) = \int_{\Pi_q} m_{2l}^{s_1,\ldots,s_{2l}}(r)\, d\nu(r).$$

Proof. The estimation in Lemma 6.2 and standard results on derivatives of parameter integrals ensure that partial differentiation up to order $2k$ and integration may be interchanged in $\hat{\nu}(s) = \int_{\Pi_q} \varphi_s(r)\, d\nu(r)$. Therefore, under this condition, all statements are clear by the definition of moment functions. □

We are now in the position to prove the following central limit theorem. In the group case $\mu = dp/2$ it is equivalent to the classical central limit theorem on the Euclidean space $M_{p,q}$ for sums of i.i.d. random variables with a distribution which is invariant under the action of U_p on $M_{p,q}$.

6.4 Theorem. *Let $\nu \in M^1(\Pi_q)$ such that the second moment of ν exists. Then, the matrix*

$$\sigma^2 := \frac{1}{2\mu} \int_{\Pi_q} r^2\, d\nu(r) \in \Pi_q$$

exists, and the probability measures $T_{n^{-1/2}I}(\nu^{(n)}) = (T_{n^{-1/2}I}(\nu))^{(n)}$ tend weakly to the squared Wishart distribution $W(\sigma^2)$ for $n \to \infty$.

Proof. We first note that

$$\int_{\Pi_q} m_2^{s,s}(r)\, d\nu(r) = \frac{1}{2\mu} \int_{\Pi_q} tr(sr^2 s)\, d\nu(r) = \frac{1}{2\mu} tr\left(s \cdot \int_{\Pi_q} r^2 d\nu(r) \cdot s \right)$$
$$= tr(s\sigma^2 s).$$

Hence, the preceding proposition and the Taylor formula imply that $\hat{\nu}$ is twice continuously differentiable with

$$\hat{\nu}(s) = 1 - \frac{1}{2} d^2\hat{\nu}(0)(s,s) + o(\|s\|^2) = 1 - \frac{1}{2} tr(s\sigma^2 s) + o(\|s\|^2)$$

for $s \to 0$. Hence, by Corollary 4.2, for any $s \in \Pi_q$,

$$\lim_{n\to\infty} ((T_{n^{-1/2}I}(\nu))^{(n)})^\wedge(s) = \lim_{n\to\infty} \left(1 - \frac{1}{2n} tr(s\sigma^2 s) + o(n^{-1})\right)^n = e^{-tr(s\sigma^2 s)/2}.$$

Lemma 5.4 and Lévy's continuity theorem for the hypergroup Fourier transform (see Section 4.2 of [1]) now complete the proof. □

We next turn to strong laws of large numbers for random walks on $X_{q,\mu}$. For this we use the algebraic properties of the moment functions on $X_{q,\mu}$ which then will be used to construct martingales.

6.5 Lemma. *For all $k \geq 0$ and $s_1, \ldots, s_k, x, y \in \Pi_q$,*

$$\int m_k^{s_1,\ldots,s_k} \, d(\delta_x * \delta_y) = \sum_{l=0}^{k} \sum_{1 \leq i_1 < i_2 < \ldots < i_l \leq k} m_l^{s_{i_1},\ldots,s_{i_l}}(x) m_{k-l}^{s_1,\ldots,s_k \setminus s_{i_1},\ldots,s_{i_l}}(y)$$

where $s_1, \ldots, s_k \setminus s_{i_1}, \ldots, s_{i_l}$ stands for the s_j with $j \notin \{i_1, \ldots, i_l\}$. In particular,

$$\int m_2^{s_1,s_2} \, d(\delta_x * \delta_y) = m_2^{s_1,s_2}(x) + m_2^{s_1,s_2}(y)$$

and, in the language of matrix valued integrals,

$$\int r^2 \, d(\delta_x * \delta_y)(r) = x^2 + y^2.$$

Proof. As the φ_s are multiplicative, the definition of moment functions yields

$$\int m_k^{s_1,\ldots,s_k} \, d(\delta_x * \delta_y) = i^k \cdot d_s^k(\varphi_s(x)\varphi_s(y))|_{s=0}(s_1, \ldots, s_k).$$

The first statement now follows readily from a multivariate version of the Leibniz product rule for derivatives. The last equation follows from the preceding one by taking matrices $s_1, s_2 \in \Pi_q$ which have only zero entries except for precisely one 1 on the diagonal. □

We now construct martingales from random walks $(S_n)_{n\geq 0}$ on $X_{q,\mu}$ associated with a sequence $(\mu_n)_{n\geq 1} \subset M^1(\Pi_q)$. For this we realize $(S_n)_{n\geq 0}$ on some probability space which carries the canonical filtration associated with $(S_n)_{n\geq 0}$. Expectations will be denoted by $\mathbb{E}$.

Based on the preceding observations, it is standard to derive the following observations (cf. Section 7.3 of [1] or [28]):

6.6 Lemma. (1) *For each $s \in \Pi_q$, the $\mathbb{R}$-valued process*

$$\left(\varphi_s(S_n) \cdot \prod_{k=1}^{n} (\hat{\mu}_k(s))^{-1}\right)_{n\geq 0}$$

is a martingale.

(2) *Assume that all* μ_n *admit second moments. Then for all* $s_1, s_2 \in \Pi_q$,

$$\mathbb{E}(m_2^{s_1,s_2}(S_n)) = \sum_{k=1}^{n} \int m_2^{s_1,s_2}\, d\mu_k \qquad (n \geq 0),$$

and $(m_2^{s_1,s_2}(S_n) - \mathbb{E}(m_2^{s_1,s_2}(S_n)))_{n\geq 0}$ *is a martingale.*
In matrix language, $\mathbb{E}(S_n^2) = \sum_{k=1}^{n} \int r^2\, d\mu_k(r)$ *for* $n \geq 0$*, and the process* $(S_n^2 - \mathbb{E}(S_n^2))_{n\geq 0}$ *is a matrix-valued martingale.*

Also higher moment functions can be used to construct martingales under suitable moment conditions. This was worked out for instance in [21] for the closely related case of Markov chains on Weyl chambers which are associated with Dunkl operators. For a general discussion of moment functions and associated martingales see also [1], [19], [28].

Based on strong laws for martingales and the concept of moment functions, Zeuner [28] derived general strong laws of large numbers for random walks on general commutative hypergroups; see also Section 7.3 of [1]. In the present setting, Zeuner's results lead to the following strong laws which correspond in the group case $\mu = pd/2$ precisely to the classical strong laws of large numbers of Kolmogorov on the Euclidean spaces $M_{m,p}$.

6.7 Theorem. *Let* $(S_n)_{n\geq 0}$ *be a random walk on* $X_{q,\mu}$ *associated with the measures* $(\mu_n)_{n\geq 1} \subset M^1(X_{q,\mu})$.

(1) *If* $(a_n)_{n\geq 1} \subset]0,\infty[$ *satisfies* $a_n \to \infty$ *and*

$$\sum_{n=1}^{\infty} \frac{1}{a_n^2} \cdot \int \|r\|^2\, d\mu_n(r) \;<\; \infty,$$

then $\lim_{n\to\infty} S_n/a_n = 0$ *almost surely.*

(2) *Let* $\lambda \in]0,2[$ *and* $(S_n)_{n\geq 0}$ *time-homogeneous with* $\int \|r\|^\lambda d\mu(r) < \infty$*. Then* $n^{-1/\lambda} S_n \to 0$ *for* $n \to \infty$ *almost surely.*

Proof. Apply Theorem 7.8 and Corollary 7.11 of [28] respectively to the moment function $m_2^{I,I}(r) = \frac{1}{2\mu}\|r\|^2$. □

6.8 Remark. Let $\mu \geq \rho - 1$. The compact group $U_q \subset GL(q)$ acts as group of automorphisms on the hypergroup $X_{q,\mu}$ such that the space $X_{q,\mu}^{q_m}$ of orbits may be identified with the Weyl chamber

$$W_q := \{(\lambda_1, \ldots, \lambda_q) \in \mathbb{R}^q :\ 0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_q\}$$

of type B_q. It is shown in [20] that $X_{q,\mu}^{U_q} \simeq W_q$ carries a commutative orbit hypergroup structure whose characters are symmetric Dunkl kernels of type B_q; cf. [6]. With the canonical projection from $X_{q,\mu}$ onto $X_{q,\mu}^{U_q} \simeq W_q$, the preceding limit theorems can be immediately transferred into limit theorems for random walks on this B_q-Dunkl-type hypergroup structure on W_q. This leads to connections with limit results in [21].

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