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MEHLER HEMIGROUPS AND EMBEDDING OF DISCRETE SKEW CONVOLUTION SEMIGROUPS ON SIMPLY CONNECTED NILPOTENT LIE GROUPS

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Abstract. It is shown how discrete skew convolution semigroups of probability measures on a simply connected nilpotent Lie group can be embedded into Lipschitz continuous semistable hemigroups by means of their generating functionals. These hemigroups are the distributions of increments of additive semi-selfsimilar processes. Considering these on an enlarged space-time group, we obtain Mehler hemigroups corresponding to periodically stationary processes of Ornstein-Uhlenbeck type, driven by certain additive processes with periodically stationary increments. The background driving processes are further represented by generalized Lie-Trotter formulas for convolutions, corresponding to a random integral approach known for finite-dimensional vector spaces.

1. Introduction

In the last decades there has been considerable interest in selfsimilar stochastic processes obeying certain space-time scaling properties. These processes are useful to model a wide variety of scaling phenomenas in diverse fields. Our focus is on additive processes, additionally assuming independent increments. In this case the family of distributions of the increments builds a stable hemigroup of probability measures. By Lamperti's [17] transformation the processes are closely connected with stationary Ornstein-Uhlenbeck type processes. On $\mathbb{R}^d$ a selfsimilar additive process can be represented by random integrals with respect to a background driving Lévy process and this representation extends to the Ornstein-Uhlenbeck process; see [14]. On groups such integral representations are not available, but there exist weak representations by Lie-Trotter formulas for convolutions on an enlarged space-time group; see [8, 9]. The resulting objects on groups are a convolution semigroup corresponding to the background driving Lévy process and a Mehler semigroup corresponding to the Ornstein-Uhlenbeck process. There has also been drawn attention to Mehler semigroups as Markovian transition operators on infinite dimensional vector spaces and its interplay to Ornstein-Uhlenbeck processes and skew convolution semigroups; see

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[3, 4, 5, 7, 18, 22]. A skew convolution semigroup arises as the family of cofactors for the selfdecomposable one-dimensional marginal distributions of a selfsimilar additive process. A random integral representation for operator-selfdecomposable measures has already been obtained by Jurek and Vervaat [15]. Conversely, given a skew convolution semigroup it is possible to reconstruct the stable hemigroup and thus all other distributional families mentioned above; see [8].

Our aim is to generalize these results for additive processes with the weaker scaling property of semi-selfsimilarity on a discrete scale. We focus on the question of reconstructing a semi-stable hemigroup (distributions of the increments) and other objects from a discrete skew convolution semigroup on a locally compact group $\mathbb{G}$. To motivate our studies, we first survey on results in this respect for operator-semi-selfsimilar processes on $\mathbb{R}^d$.

Let $\{X_t\}_{t\geq 0}$ be an **additive stochastic process** on $\mathbb{R}^d$, i.e. $X_0 = 0$, $t \mapsto P_{X_t}$ is weakly continuous and $\{X_t\}_{t\geq 0}$ has independent increments. Let $Q \in \mathrm{GL}(\mathbb{R}^d)$ be such that $e^{-tQ} \to 0$ as $t \to \infty$. The additive process $\{X_t\}_{t\geq 0}$ is called **operator-semi-selfsimilar** with exponent Q if $\{c^Q X_t\}_{t\geq 0} = \{X_{ct}\}_{t\geq 0}$ for some $c > 1$ in the sense of equality of all finite-dimensional distributions. Due to the construction of random integrals in [15] the processes $\{Y_t^{(+)}\}_{t\geq 0}$ and $\{Y_t^{(-)}\}_{t\geq 0}$ defined by

$$Y_t^{(+)} = \int_1^{e^t} s^{-Q}\, dX_s \quad \text{and} \quad Y_t^{(-)} = \int_{e^{-t}}^1 s^{Q}\, dX_s$$

are i.i.d. additive processes with $\log c$-stationary increments, i.e. $Y_{t+\log c}^{(\pm)} - Y_{s+\log c}^{(\pm)}$ is equal in distribution to $Y_t^{(\pm)} - Y_s^{(\pm)}$ for all $0 \leq s \leq t$, and with a certain finite logarithmic moment condition, from which the operator-semi-selfsimilar process can be almost surely pathwise recovered by

$$X_t = \begin{cases} \displaystyle\int_{-\log t}^{\infty} e^{-sQ}\, dY_s^{(-)} & \text{if } 0 \leq t \leq 1, \\ \displaystyle X_1 + \int_0^{\log t} e^{sQ}\, dY_s^{(+)} & \text{if } t > 1. \end{cases} \tag{1.1}$$

The process $\{Y_t^{(\pm)}\}_{t\geq 0}$ is called the **background driving additive periodic process**. For details see [2, 20]. Conversely, any additive process with $\log c$-stationary increments and with certain finite logarithmic moment defines an additive operator-semi-selfsimilar process in this way. The random integral representation (1.1) easily carries over to **Ornstein-Uhlenbeck type** processes $\{U_t^{(+)} = e^{-tQ} X_{e^t}\}_{t\geq 0}$ and $\{U_t^{(-)} = e^{tQ} X_{e^{-t}}\}_{t\geq 0}$ given by Lamperti's [17] transformation. These processes are periodically stationary Markov processes with period $\log c$, i.e. $\{U_{t+\log c}^{(\pm)}\}_{t\geq 0} = \{U_t^{(\pm)}\}_{t\geq 0}$ again in the sense of equality of all finite-dimensional distributions. Their Markov transition operators $P_{s,t}(f)(x) = \mathbb{E}(f(U_t^{(\pm)}) | U_s^{(\pm)} = x)$ for $0 \leq s \leq t$ and bounded measurable $f : \mathbb{R}^d \to \mathbb{R}$ can easily be shown to be $\log c$-periodic Feller hemigroups,

i.e. for $0 \leq s \leq r \leq t$ we have $P_{s,r}P_{r,t} = P_{s,t}$, $P_{s+\log c,t+\log c} = P_{s,t}$, and $P_{s,t}(f) \in \mathcal{C}_b(\mathbb{R}^d)$ for every $f \in \mathcal{C}_b(\mathbb{R}^d)$, which we call **Mehler hemigroups** in analogy to Mehler semigroups for stationary Ornstein-Uhlenbeck processes.
Turning back to the operator-semi-selfsimilar additive process $\{X_t\}_{t\geq 0}$, the family of distributions of the increments $(\mu_{s,t} = P_{X_t - X_s})_{0\leq s\leq t}$ builds a **continuous semistable convolution hemigroup**, i.e. for $0 \leq s \leq r \leq t$ we have $\mu_{s,r} * \mu_{r,t} = \mu_{s,t}$, $c^Q\mu_{s,t} = \mu_{cs,ct}$, and $(s,t) \mapsto \mu_{s,t}$ is weakly continuous. Especially, $\mu = \mu_{0,1}$ is **operator-semi-selfdecomposable**, i.e. for $n \in \mathbb{N}$ we have $\mu = c^{-nQ}\mu * \nu_n$ for some cofactors $(\nu_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}^1(\mathbb{R}^d)$, namely $\nu_n = \mu_{c^{-n},1}$. The cofactors build a **discrete skew convoluton semigroup**, i.e. $\nu_{n+m} = \nu_n * c^{-nQ}\nu_m$ for all $n, m \in \mathbb{N}$, and we further have $\nu_n \to \mu$ weakly. For the details we refer to [19]. Conversely, let $(\nu_n)_{n\in\mathbb{N}}$ be a discrete skew convolution semigroup with infinitely divisible $\nu = \nu_1 \in \mathcal{M}^1(\mathbb{R}^d)$ and assume that $\nu_n \to \mu$ weakly (equivalently, ν possesses a finite logarithmic moment). Then μ is operator-semi-selfdecomposable and there exists a continuous semistable hemigroup $(\mu_{s,t})_{0\leq s\leq t}$ with $\mu_{0,1} = \mu$ and $\nu_n = \mu_{c^{-n},1}$. The following construction is due to [1, 19]. For $s > 0$ let $n_s = \lfloor \log_c s \rfloor \in \mathbb{Z}$ and $r_s = s/c^{n_s} \in [1, c)$. With $\nu_0 = \varepsilon_0$ we define

$$\mu_{s,t} = \begin{cases} c^{(n_s+1)Q}\nu^{\log_c \frac{c}{r_s}} * c^{n_tQ}\nu_{n_t-n_s-1} * c^{(n_t+1)Q}\nu^{\log_c r_t} & \text{if } n_t > n_s \\ c^{(n_t+1)Q}\nu^{\log_c \frac{r_t}{r_s}} & \text{if } n_t = n_s \\ c^{n_tQ}\mu * c^{(n_t+1)Q}\nu^{\log_c r_t} & \text{if } s = 0 \end{cases} \tag{1.2}$$

It is a straightforward calculation that $(\mu_{s,t})_{0\leq s\leq t}$ is indeed a continuous semistable hemigroup with the desired properties. Note that for the above construction the assumption that ν is infinitely divisible is essential. In contrast to stable hemigroups, Theorem 1.1 in [21] shows the existence of an infinitely divisible semi-selfdecomposable μ with cofactor $\nu = \nu_1$ not being infinitely divisible. Hence infinite divisibility of ν is a sufficient but not necessary condition for embeddability into a continuous semistable hemigroup.
Instead of using the embedding hemigroup (1.2) known from [1], it is advantageous to use an additive rather than a multiplicative parametrization. We will use the (additive) semistable hemigroup $(\lambda_{s,t} = \mu_{c^{-t},c^{-s}})_{0\leq s\leq t}$ with $\nu_n = \mu_{c^{-n},1} = \lambda_{0,n} \to \mu$ weakly. One can easily show that (1.2) carries over to the simpler form

$$\lambda_{s,t} = \begin{cases} c^{-\lfloor s\rfloor Q}\nu^{t-s} & \text{if } \lfloor s\rfloor = \lfloor t\rfloor \\ c^{-\lfloor t\rfloor Q}\lambda_{0,t-\lfloor t\rfloor} * c^{-Q}\nu_{\lfloor t\rfloor-1} * \nu^{1-s} & \text{if } 0 \leq s \leq 1 \leq t \\ c^{-\lfloor s\rfloor Q}\lambda_{s-\lfloor s\rfloor,t-\lfloor s\rfloor} & \text{else} \end{cases} \tag{1.3}$$

In the following we extend and generalize these results to locally compact groups $\mathbb{G}$. In fact, for investigations of (semi-)selfdecomposability the assumption that the norming operators act contracting is essential. Furthermore the existence of continuous one-parameter groups of automorphisms implies connectedness. Therefore, without loss of generality, we assume $\mathbb{G}$ to be connected and contractible, hence a homogeneous group; cf. [9], 3.1.5 and Theorem 2.1.12. We first focus on the question of embeddability of a discrete skew convolution semigroup into a Lipschitz continuous semistable hemigroup in Section 2. As in the case of a stable hemigroup, the space-time enlargement enables us in Section 3 to obtain a Mehler hemigroup as a weak representation of the periodically stationary Ornstein-Uhlenbeck type process, and further a periodic hemigroup as a weak representation of the background driving additive periodic process. Finally, we show in Section 4 how to obtain weak analogues of random integral representations on $\mathbb{R}^d$ by generalized Lie-Trotter formulas for convolutions on $\mathbb{G}$.

2. Hemigroup embedding

A close look at the embedding semistable hemigroup (1.3) on $\mathbb{R}^d$ shows that due to infinite divisibility of $\nu = \nu_1$ we fill the gaps left by the discrete skew convolution semigroup $(\nu_n)_{n\in\mathbb{N}}$ with the help of the semigroup $(\nu^t)_{t\geq 0}$. On non-Abelian groups $\mathbb{G}$ the assumption that ν is embeddable into a convolution semigroup is too restrictive and we rather prefer a more general hemigroup embedding. Recall that now we use additive parametrization and thus on $\mathbb{G}$ the objects under use have slightly different definitions below than given in the Introduction. In the following let $\mathbb{G}$ denote a homogeneous group, i.e. a contractible simply connected nilpotent Lie group. Let throughout $\tau \in \operatorname{Aut}(\mathbb{G})$ and let $(T_t)_{t\in\mathbb{R}}$ be a continuous one-parameter group in $\operatorname{Aut}(\mathbb{G})$.

Definition 2.1. (a) A family $(\nu_{s,t})_{0\leq s\leq t} \subseteq \mathcal{M}^1(\mathbb{G})$ is called a **continuous hemigroup** if $(s,t) \mapsto \nu_{s,t}$ is weakly continuous, $\nu_{s,s} = \varepsilon_e$ for all $s \geq 0$ and we have $\nu_{s,r} * \nu_{r,t} = \nu_{s,t}$ for all $0 \leq s \leq r \leq t$. The hemigroup is called **τ-semistable** if $\tau(\nu_{s,t}) = \nu_{s+1,t+1}$ for all $0 \leq s \leq t$.
(b) A **discrete hemigroup** is a family $\{\nu(k,n)\}_{0\leq k\leq n} \subseteq \mathcal{M}^1(\mathbb{G})$ with $k, n \in \mathbb{Z}_+$ satisfying $\nu(k,k) = \varepsilon_e$ and $\nu(k,m) * \nu(m,n) = \nu(k,n)$ for all $0 \leq k \leq m \leq n$. Obviously, in this case $\nu(k,n) = \ast_{j=k+1}^{n} \nu(j-1,j)$, hence any sequence $\{\nu_j = \nu(j-1,j)\}_{j\in\mathbb{Z}_+}$ generates a discrete hemigroup. The discrete hemigroup is called **τ-semistable** if $\tau(\nu(k,n)) = \nu(k+1,n+1)$ for all $0 \leq k \leq n$.

Definition 2.2. (a) A weakly continuous family $(\nu_t)_{t\in\mathbb{R}_+} \subseteq \mathcal{M}^1(\mathbb{G})$ is called a **skew convolution semigroup** with respect to $(T_t)_{t\in\mathbb{R}}$ (or M-semigroup in Hazod [8]) if $\nu_{s+t} = \nu_s * T_s(\nu_t)$ for all $s, t \geq 0$.
(b) A sequence $\{\nu(k)\}_{k\in\mathbb{Z}_+} \subseteq \mathcal{M}^1(\mathbb{G})$ is called a **discrete skew convolution semigroup** with respect to $\tau \in \operatorname{Aut}(\mathbb{G})$ if $\nu(0) = \varepsilon_e$ and $\nu(k+n) = \nu(k) * \tau^k(\nu(n))$ for all $k, n \in \mathbb{Z}_+$.

As in the continuous case, discrete semistable hemigroups and discrete skew convolution semigroups are closely related. One immediately verifies the following relations.

Proposition 2.3. *(a) $\{\nu(k)\}_{k\in\mathbb{Z}_+}$ is a discrete skew convolution semigroup iff $\nu(0) = \varepsilon_e$ and $\nu(k) = \ast_{j=1}^{k} \tau^{j-1}(\nu)$ for all $k \in \mathbb{N}$ with $\nu = \nu(1)$.*

(b) $\{\nu(k,n)\}_{0\le k\le n}$ *is a discrete* τ*-semistable hemigroup iff* $\nu(k,k)=\varepsilon_e$ *and* $\nu(k,n)=\ast_{j=k+1}^{n}\tau^{j-1}(\nu)$ *for all* $0\le k<n$ *with* $\nu=\nu(0,1)$*.*
(c) $\{\nu(k)\}_{k\in\mathbb{Z}_+}$ *is a discrete skew convolution semigroup with respect to* τ *iff* $\{\nu(k,n)=\tau^k(\nu(n-k))\}_{0\le k\le n}$ *is a discrete* τ*-semistable hemigroup.*

Remark 2.4. According to Proposition 2.3 any $\nu\in\mathcal{M}^1(\mathbb{G})$ may be embedded into a discrete skew convolution semigroup, respectively a discrete τ-semistable hemigroup. Therefore we first concentrate on the problem under which conditions a discrete τ-semistable hemigroup may be embedded into a continuous τ-semistable hemigroup. We call $\nu\in\mathcal{M}^1(\mathbb{G})$ embeddable into a hemigroup (for short: **h-embeddable**) if there exists a continuous hemigroup $(\mu_{s,t})_{0\le s\le t\le 1}$ with $\mu_{0,1}=\nu$.
Note that h-embeddable laws $\nu=\mu_{0,1}$ with a commuting hemigroup, i.e. $\mu_{s,t}\ast\mu_{u,v}=\mu_{u,v}\ast\mu_{s,t}$ for $(s,t]\cap(u,v]=\emptyset$, are infinitely divisible as limits of commuting infinitesimal triangular arrays, and hence embeddable into a continuous convolution semigroup; cf. Shah [23], Theorem 1.1.
A hemigroup $(\nu_{s,t})_{0\le s\le t}$ is called **1-periodic** if $\nu_{s+1,t+1}=\nu_{s,t}$ for all $0\le s\le t$. Obviously, any h-embeddable law ν is embeddable into a 1-periodic hemigroup. If ν is embeddable into a continuous convolution semigroup $(\rho_t)_{t\ge 0}$ with $\rho_1=\nu$ then $(\nu_{s,t}=\rho_{t-s})_{0\le s\le t}$ is obviously 1-periodic.

Proposition 2.5. *(a) Embedding of a discrete hemigroup* $\{\nu(k,n)\}_{0\le k\le n}$ *into a continuous hemigroup is possible iff all* $\nu_j=\nu(j-1,j)$ *are h-embeddable.*
(b) Embedding of a discrete semistable hemigroup $\{\nu(k,n)\}_{0\le k\le n}$ *into a continuous semistable hemigroup is possible iff* $\nu=\nu(0,1)$ *is h-embeddable.*

Proof. The proof of (a) is obvious. To prove (b) first observe that if $(\lambda_{s,t})_{0\le s\le t}$ is a semistable hemigroup with $\lambda_{k,n}=\nu(k,n)$ for all $k,n\in\mathbb{Z}_+$, $k\le n$, then $(\mu_{s,t}=\lambda_{s,t})_{0\le s\le t\le 1}$ is a continuous hemigroup with $\mu_{0,1}=\lambda_{0,1}=\nu(0,1)=\nu$. Conversely, let ν be h-embeddable with $\nu=\mu_{0,1}$ and $\nu(k,n)=\ast_{j=k+1}^{n}\tau^{j-1}(\nu)$ for all $k,n\in\mathbb{Z}_+$, $k<n$. Define

$$\lambda_{s,t}=\begin{cases}\tau^{\lfloor s\rfloor}(\mu_{s-\lfloor s\rfloor,t-\lfloor s\rfloor}) & \text{, if } \lfloor s\rfloor=\lfloor t\rfloor\\ \mu_{s,1}\ast\nu(1,\lfloor t\rfloor)\ast\tau^{\lfloor t\rfloor}(\mu_{0,t-\lfloor t\rfloor}) & \text{, if } 0\le s\le 1\le t\\ \tau^{\lfloor s\rfloor}(\lambda_{s-\lfloor s\rfloor,t-\lfloor s\rfloor}) & \text{, else.}\end{cases} \tag{2.1}$$

As easily verified, $(\lambda_{s,t})_{0\le s\le t}$ is a continuous hemigroup with $\lambda_{k,n}=\nu(k,n)$ for all $k,n\in\mathbb{Z}_+$, $k\le n$, and we have $\tau(\lambda_{s,t})=\lambda_{s+1,t+1}$ by construction. □

Note that (2.1) coincides with the semistable hemigroup (1.3) on $\mathbb{G}=\mathbb{R}^d$ if we set $\tau=c^{-Q}$ and $\mu_{s,t}=\nu^{t-s}$ in case ν is infinitely divisible.

Now, according to Siebert [24], let $(X_i)_{i=1}^d$ be a basis of the Lie algebra $\mathfrak{G}$ of $\mathbb{G}$, and let $(\xi_i)_{i=1}^d$ be a local coordinate system, i.e. $\xi_i\in\mathcal{C}_c^\infty(\mathbb{G})$ with $|\xi_i|\le 1$, $\xi_i(e)=0$, $\xi_i(x^{-1})=-\xi_i(x)$ and $X_i(\xi_j)=\delta_{ij}$. Furthermore choose a Hunt function $\varphi=\varphi_{\mathbb{G}}$ with $\varphi=\sum_{i=1}^d\xi_i^2$ on a compact neighbourhood U_0 of e such that $0\le\varphi\le 1$ and

$1-\varphi \in \mathcal{C}_c^\infty(\mathbb{G})$. Let $\mathbb{A}(\mathbb{G})$ denote the cone of generating functionals of continuous convolution semigroups. For the background of probabilities on groups and Lévy-Khintchine formulas see e.g. [11, 9].

Definition 2.6. (a) For $\mu \in \mathcal{M}^1(\mathbb{G})$ we define $q(\mu) = \sum_{i=1}^d |\langle \mu, \xi_i \rangle| + \langle \mu, \varphi \rangle$ called the q**-functional**. Similarly, for a generating functional $A \in \mathbb{A}(\mathbb{G})$ we define $\|A\| = \sum_{i=1}^d |\langle A, \xi_i \rangle| + \langle A, \varphi \rangle$ and hence for the Poisson generator $A = \mu - \varepsilon_e$ we may write $q(\mu) = \|A\| = \|\mu - \varepsilon_e\|$.
(b) A hemigroup $(\mu_{s,t})_{0\le s\le t}$ is called **Lipschitz continuous** on $[0,R]$ if $q(\mu_{s,t}) \le C(t-s)$ for all $0 \le s \le t \le R$, where $C > 0$ depends on the hemigroup and on R. We simply call the hemigroup Lipschitz continuous if this condition is fulfilled for every $R > 0$.
(c) A hemigroup $(\mu_{s,t})_{0\le s\le t}$ is called of **bounded variation** on $[0,R]$ if for all decompositions $0 = r_1 < r_2 < \cdots < r_n < r_{n+1} = R$ we have $\sum_{i=1}^n q(\mu_{r_i,r_{i+1}}) \le \gamma$, where $\gamma > 0$ depends on the hemigroup and on R. We simply call the hemigroup of bounded variation if this condition is fulfilled for every $R > 0$.

Remark 2.7. (a) As mentioned in 2.5 of Siebert [24], $\|A\|$ is equivalent to $|A|_2$ and $|A|_2^\sim$, the norms of the functional A on the spaces of twice differentiable functions $\mathcal{C}_2(\mathbb{G})$, respectively $\mathcal{C}_2^\sim(\mathbb{G})$. Therefore it easily follows that for $T \in \mathrm{Aut}(\mathbb{G})$ there exists $C(T) > 0$ such that $q(T(\mu)) = \|T(\mu - \varepsilon_e)\| \le C(T)q(\mu)$ for $\mu \in \mathcal{M}^1(\mathbb{G})$ and $\|T(A)\| \le C(T)\|A\|$ for $A \in \mathbb{A}(\mathbb{G})$. Hence, according to Proposition 2.5, the semistable hemigroup $(\lambda_{s,t})_{0\le s\le t}$ is Lipschitz continuous on any interval $[0,R]$ iff $(\mu_{s,t} = \lambda_{s,t})_{0\le s\le t\le 1}$ is Lipschitz continuous.
(b) Lipschitz continuous hemigroups are almost surely differentiable by Siebert [24], Theorem 4.3 and Lemma 2.8. This fact will be important in Section 3 for the construction of the background driving process. For continuously differentiable hemigroups a different approach is given by Kunita [16].

Theorem 2.8. *Let $\nu \in \mathcal{M}^1(\mathbb{G})$ be h-embeddable with $\nu = \mu_{0,1}$ for some continuous hemigroup $(\mu_{s,t})_{0\le s\le t\le 1}$. Then there exists a Lipschitz continuous hemigroup $(\mu'_{s,t})_{0\le s\le t\le 1}$ with $\mu'_{0,1} = \nu = \mu_{0,1}$.*

Proof. Step 1: We first show that there exists a continuous hemigroup of bounded variation $(\bar\mu_{s,t})_{0\le s\le t\le 1}$ with $\bar\mu_{0,1} = \nu$.
The continuous hemigroup $(\mu_{s,t})_{0\le s\le t\le 1}$ may be represented as the family of distributions of the increments $X_s^{-1}X_t$ of a stochastically continuous additive process $(X_t)_{t\in[0,1]}$ with $X_0 = e$ almost surely. According to Feinsilver [6], Section 3e), there exists a decomposition $X_t = Z_t \cdot m_t$, where $t \mapsto m_t \in \mathbb{G}$ is continuous and $(Z_t)_{t\in[0,1]}$ is an additive process with $Z_0 = e$ almost surely such that the Lévy-Khintchine characteristics of the corresponding generating functionals are of bounded variation. In fact, $(Z_t)_{t\in[0,1]}$ is characterized by the property that $t \mapsto f(Z_t) - \int_0^t f(Z_s)\, L(ds)$ is a martingale for every $f \in \mathcal{C}_c^\infty(\mathbb{G})$, where L is given by the covariance function and the Lévy-measure function. For details cf. Feinsilver [6]. Let $(\nu_{s,t})_{0\le s\le t\le 1}$ denote

the family of distributions of the increments of $(Z_t)_{t\in[0,1]}$. Since the generating functionals of $(\nu_{s,t})_{0\le s\le t\le 1}$ are of bounded variation, the hemigroup itself is of bounded variation in the sense of Definiton 2.6(c), cf. Heyer and Pap[12], Theorems 6 and 7. For $0\le s\le t\le 1$ we have $X_s^{-1}X_t = m_s^{-1}Z_s^{-1}Z_t\cdot m_t$ and hence $\mu_{s,t}=\varepsilon_{m_s^{-1}}*\nu_{s,t}*\varepsilon_{m_t}$, especially $\mu_{0,1}=\nu_{0,1}*\varepsilon_{m_1}$, since $m_0=e$. Now choose a continuous one-parameter group $(u(t))_{t\in\mathbb{R}}\subseteq\mathbb{G}$ with $u(1)=m_1$ and define

$$\bar\mu_{s,t}=\begin{cases}\nu_{2s,2t} & \text{, if } 0\le s\le t\le \frac12\\ \varepsilon_{u(2(t-s))} & \text{, if } \frac12\le s\le t\le 1\\ \bar\mu_{s,\frac12}*\bar\mu_{\frac12,t} & \text{, if } 0\le s\le \frac12\le t\le 1.\end{cases}$$

Obviously, by construction $(\bar\mu_{s,t})_{0\le s\le t\le 1}$ is a continuous hemigroup of bounded variation with $\bar\mu_{0,1}=\nu_{0,1}*\varepsilon_{m_1}=\mu_{0,1}=\nu$.
Step 2: There exists a Lipschitz continuous hemigroup $(\mu'_{s,t})_{0\le s\le t\le 1}$ with $\mu'_{0,1}=\bar\mu_{0,1}=\nu$. This follows from Siebert [24], 7.4, with $\mu'_{s,t}=\bar\mu_{U^{-1}(s),U^{-1}(t)}$ for a suitable function $U:[0,1]\to[0,1]$. □

We now turn to the behaviour at infinity which is closely related to semi-selfdecomposability. Essentially, in the following we will need τ to be **contracting**, i.e. $\tau^n(x)\to e$ for all $x\in\mathbb{G}$.

Definition 2.9. A probability measure $\mu\in\mathcal{M}^1(\mathbb{G})$ is called **τ-semi-selfdecomposable** if there exists a τ-semistable hemigroup $(\lambda_{s,t})_{0\le s\le t}$ such that $\lambda_{0,t}\to\mu$ weakly as $t\to\infty$.

Obviously, for $n\in\mathbb{N}$ and $t>n$ we get

$$\lambda_{0,t}=\lambda_{0,n}*\lambda_{n,t}=\nu(n)*\tau^n(\lambda_{0,t-n})\to\nu(n)*\tau^n(\mu)=\mu$$

weakly as $t\to\infty$, coinciding with the usual definition of semi-selfdecomposability. Since $\lambda_{\lfloor t\rfloor,t}=\tau^{\lfloor t\rfloor}(\lambda_{0,t-\lfloor t\rfloor})\to\varepsilon_e$ by contractivity, due to the above hemigroup embedding we obtain that μ is τ-semi-selfdecomposable iff there exists an h-embeddable law ν such that $\nu(n)=*_{k=0}^{n-1}\tau^k(\nu)\to\mu$ weakly. According to [10], $\nu(n)$ converges weakly iff ν possesses a finite logarithmic moment, i.e. $\int_{\mathbb{G}}\log(1+\|x\|)\,d\nu(x)<\infty$, where $\|\cdot\|$ is a norm on the homogeneous group. On $\mathbb{G}=\mathbb{R}^d$ this coincides with the assumption of a finite logarithmic moment in [2, 20].

3. Space-time enlargement and Mehler hemigroups

According to Proposition 2.5(b) and Theorem 2.8, in the sequel we assume without loss of generality that for h-embeddable laws the underlying semistable hemigroup is Lipschitz continuous, in particular the semistable hemigroup constructed in (2.1). Moreover, for $\tau\in\mathrm{Aut}(\mathbb{G})$ there exists a continuous one-parameter group $(T_t)_{t\in\mathbb{R}}\subseteq\mathrm{Aut}(\mathbb{G})$ such that $\tau^k=T_1$ for some $k\in\mathbb{N}$; cf. [9], Proposition 2.8.14. If τ is contracting then also $(T_t)_{t\in\mathbb{R}}$ is contracting. Hence in the sequel we further assume without loss of generality τ to be embedded into a continuous one-parameter group $(T_t)_{t\in\mathbb{R}}$ with $\tau = T_1$.

Following an idea of Hofmann and Jurek [13], let $\mathbb{H} = \mathbb{G} \rtimes \mathbb{R}$ denote the **space-time group**, where the semidirect product is defined by the action $(T_t)_{t\in\mathbb{R}}$ as

$$(x, s) \cdot (y, t) = (x \cdot T_s(y), s + t) \quad \text{for all } x, y \in \mathbb{G} \text{ and } s, t \in \mathbb{R},$$

cf.,e.g., [9], §2.14 III. Let $\mathcal{M}^1_\star(\mathbb{H}) = \{\mu \otimes \varepsilon_u : \mu \in \mathcal{M}^1(\mathbb{G}),\, u \in \mathbb{R}\}$ be the closed, convex subsemigroup of $\mathcal{M}^1(\mathbb{H})$ with convolution given by

$$(\mu \otimes \varepsilon_s) \star (\nu \otimes \varepsilon_t) = (\mu * T_s(\nu), \varepsilon_{s+t}) \quad \text{for } \mu, \nu \in \mathcal{M}^1(\mathbb{G}) \text{ and } s, t \in \mathbb{R}.$$

Let $(\lambda_{s,t})_{0\le s\le t} \subseteq \mathcal{M}^1(\mathbb{G})$ be a continuous hemigroup and define

$$\begin{aligned} \lambda^\bullet_{s,t} &= T_{-s}(\lambda_{s,t}) \subseteq \mathcal{M}^1(\mathbb{G}) \\ \Lambda_{s,t} &= \lambda^\bullet_{s,t} \otimes \varepsilon_{t-s} \subseteq \mathcal{M}^1_\star(\mathbb{H}) \end{aligned}$$

then one can easily verify that $(\Lambda_{s,t})_{0\le s\le t}$ is a continuous hemigroup. Conversely, let $(\Lambda'_{s,t} = \lambda'_{s,t} \otimes \varepsilon_{\varphi(s,t)})_{0\le s\le t} \subseteq \mathcal{M}^1_\star(\mathbb{H})$ be a continuous hemigroup. Then for $0 \le s \le r \le t$ we have $\varphi(s, r) + \varphi(r, t) = \varphi(s, t)$ and hence, with $\psi(t) = \varphi(0, t)$ we get $\varphi(s, t) = \psi(t) - \psi(s)$. Furthermore $\lambda'_{s,r} * T_{\psi(r)-\psi(s)}(\lambda'_{r,t}) = \lambda'_{s,t}$ and thus $(\lambda''_{s,t} = T_{\psi(s)}(\lambda'_{s,t}))_{0\le s\le t}$ is a continuous hemigroup in $\mathcal{M}^1(\mathbb{G})$.

For bounded measurable functions $g : \mathbb{H} \to \mathbb{R}$ we obtain the convolutions

$$\begin{aligned} \Lambda_{s,t} \star g(x, r) &= \int_{\mathbb{H}} g\big((y, u) \cdot (x, r)\big)\, d\Lambda_{s,t}(y, u) \\ &= \int_{\mathbb{G}} g\big(y \cdot T_{t-s}(x), t - s + r\big)\, d\lambda^\bullet_{s,t}(y) \\ &= \int_{\mathbb{G}} g\big(T_{-s}(y \cdot T_t(x)), t - s + r\big)\, d\lambda_{s,t}(y). \end{aligned}$$

Thus for the space component we may define a family $(P_{s,t})_{0\le s\le t}$ of bounded linear operators as

$$P_{s,t}(f)(x) = \int_{\mathbb{G}} f\big(T_{-s}(y \cdot T_t(x))\big)\, d\lambda_{s,t}(y)$$

for bounded measurable functions $f : \mathbb{G} \to \mathbb{R}$. As easily verified we have the following properties.

Lemma 3.1. *For a τ-semistable hemigroup $(\lambda_{s,t})_{0\le s\le t}$ the above family $(P_{s,t})_{0\le s\le t}$ is a 1-periodic Feller hemigroup, i.e.*

$$\begin{aligned} &P_{r,t}P_{s,r} = P_{s,t} \quad \textit{for all } 0 \le s \le r \le t. \\ &P_{s+1,t+1} = P_{s,t} \quad \textit{for all } 0 \le s \le t. \\ &P_{s,t}(f) \in \mathcal{C}_b(\mathbb{G}) \quad \textit{for every } f \in \mathcal{C}_b(\mathbb{G}) \textit{ and } 0 \le s \le t. \end{aligned}$$

In analogy to Mehler semigroups for stable hemigroups, we call $(P_{s,t})_{0\le s\le t}$ a **Mehler hemigroup** of linear operators. These operators are closely related to the background driving process, which will be of our interest in the sequel.

Proposition 3.2. *(a) The hemigroup $(\lambda_{s,t})_{0\le s\le t}\subseteq\mathcal{M}^1(\mathbb{G})$ is Lipschitz continuous iff $(\Lambda_{s,t})_{0\le s\le t}\subseteq\mathcal{M}^1_\star(\mathbb{H})$ shares this property. This is the case iff the function $(s,t)\mapsto\lambda^\bullet_{s,t}\in\mathcal{M}^1(\mathbb{G})$ is Lipschitz continuous.*
(b) The hemigroup $(\lambda_{s,t})_{0\le s\le t}$ is τ-semistable iff $(\Lambda_{s,t})_{0\le s\le t}$ is 1-periodic. This is the case iff the function $(s,t)\mapsto\lambda^\bullet_{s,t}\in\mathcal{M}^1(\mathbb{G})$ is 1-periodic.

Proof. As in Definition 2.6, let $(\xi_i)_{i=1}^d$ be a local coordinate system on $\mathbb{G}$ and let $\varphi_{\mathbb{G}}$ be a Hunt function. Further let ξ_{d+1} be a local coordinate function on $\mathbb{R}$, e.g. $\xi_{d+1}(t)=\frac{t}{1+t^2}$. Then $(\bar\xi_i)_{i=1}^{d+1}$ with $\bar\xi_i(x,t)=\xi_i(x)$ for $1\le i\le d$ and $\bar\xi_{d+1}(x,t)=\xi_{d+1}(t)$ defines a local coordinate system on $\mathbb{H}$ and $\varphi_{\mathbb{H}}(x,t)=\varphi_{\mathbb{G}}(x)+\xi^2_{d+1}(t)$ is a Hunt function on $\mathbb{H}$. The q-functional on the space-time group is then given by $q_{\mathbb{H}}(\mu\otimes\varepsilon_t)=q_{\mathbb{G}}(\mu)+|\xi_{d+1}(t)|+\xi^2_{d+1}(t)$. Together with Remark 2.7 this proves (a), and (b) is easily verified by direct calculation. □

According to Siebert [24], Theorem 4.3 and Lemma 2.8, the Lipschitz continuous semistable hemigroup $(\lambda_{s,t})_{0\le s\le t}$ constructed in (2.1) is almost surely differentiable. For $0\le s\le 1$ let $C(s)=\frac{\partial^+}{\partial t}\mu_{s,t}\big|_{t=s}\in\mathbb{A}(\mathbb{G})$. Then, by the construction in (2.1), for $s>0$ we obtain

$$A(s)=\frac{\partial^+}{\partial t}\lambda_{s,t}\Big|_{t=s}=T_{\lfloor s\rfloor}\Big(\frac{\partial^+}{\partial t}\mu_{s-\lfloor s\rfloor,t-\lfloor s\rfloor}\Big|_{t=s}\Big)=T_{\lfloor s\rfloor}(C(s-\lfloor s\rfloor)).$$

The almost everywhere defined mapping $s\mapsto A(s)\in\mathbb{A}(\mathbb{G})$ is admissible in the sense of Siebert [24], 2.6, and we have

$$B(s,t)=B(t)-B(s)=\int_s^t A(u)\,du=\int_s^t T_{\lfloor u\rfloor}(C(u-\lfloor u\rfloor))\,du\in\mathbb{A}(\mathbb{G}),$$

where $t\mapsto B(t)=B(0,t)$ is increasing and Lipschitz continuous. On the other hand, again by Siebert [24], Theorem 4.3 and Lemma 2.8, and by Proposition 3.2(a), $(\Lambda_{s,t})_{0\le s\le t}$ is almost surely differentiable, hence in particular, $(\lambda^\bullet_{s,t})_{0\le s\le t}$ is almost surely differentiable. Put $\overline{A}(s)=\frac{\partial^+}{\partial t}\Lambda_{s,t}\big|_{t=s}\in\mathbb{A}(\mathbb{H})$ then for the space component we obtain

$$A^\bullet(s)=\frac{\partial^+}{\partial t}\lambda^\bullet_{s,t}\Big|_{t=s}=T_{-s}A(s)=T_{-(s-\lfloor s\rfloor)}(C(s-\lfloor s\rfloor)).$$

As above, we further define the generating functionals

$$\overline{B}(s,t)=\overline{B}(t)-\overline{B}(s)=\int_s^t\overline{A}(u)\,du\in\mathbb{A}(\mathbb{H})\quad\text{with }\overline{B}(t)=\overline{B}(0,t)$$

$$B^\bullet(s,t)=B^\bullet(t)-B^\bullet(s)=\int_s^t A^\bullet(u)\,du\in\mathbb{A}(\mathbb{G})\quad\text{with }B^\bullet(t)=B^\bullet(0,t)$$

and we easily obtain the following relations.

Proposition 3.3. *τ-semistability, respectively 1-periodicity of the hemigroups imply for all $0 \le s \le t$*

$$\begin{aligned} A(s+1) = \tau(A(s)) \quad &\textit{and} \quad B(s+1,t+1) = \tau(B(s,t)), \\ \overline{A}(s+1) = \overline{A}(s) \quad &\textit{and} \quad \overline{B}(s+1,t+1) = \overline{B}(s,t), \\ A^\bullet(s+1) = A^\bullet(s) \quad &\textit{and} \quad B^\bullet(s+1,t+1) = B^\bullet(s,t). \end{aligned}$$

Now we are ready to prove

Theorem 3.4. *There exists a bijection between Lipschitz continuous τ-semistable hemigroups $(\lambda_{s,t})_{0\le s\le t}$ and Lipschitz continuous 1-periodic hemigroups $(\bar\lambda_{s,t})_{0\le s\le t}$ on $\mathcal{M}^1(\mathbb{G})$ given by their families of generating functionals $(B(s,t))_{0\le s\le t}$, respectively $(B^\bullet(s,t))_{0\le s\le t}$.*

Remark 3.5. In analogy to background driving Lévy processes for stable hemigroups, we call $(\bar\lambda_{s,t})_{0\le s\le t}$ the (family of distributions of the increments of the) **background driving additive periodic process**.

Proof. According to Proposition 3.2 we have a 1-1-correspondence between Lipschitz continuous τ-semistable hemigroups $(\lambda_{s,t})_{0\le s\le t} \subseteq \mathcal{M}^1(\mathbb{G})$ and Lipschitz continuous 1-periodic hemigroups $(\Lambda_{s,t})_{0\le s\le t} \subseteq \mathcal{M}^1(\mathbb{H})$. According to Siebert [24], Section 4, these hemigroups are uniquely determined by the families of generating functionals $(B(s,t))_{0\le s\le t}$, repectively $(\overline{B}(s,t))_{0\le s\le t}$, or the corresponding admissible families $(A(u))_{u\ge 0} \subseteq \mathbb{A}(\mathbb{G})$, respectively $(\overline{A}(u))_{u\ge 0} \subseteq \mathbb{A}(\mathbb{H})$, satisfying the evolution equations (EE2) (Siebert [24], 4.3), respectively condition (E) (Siebert [24], 3.6). As easily seen, since $\Lambda_{s,t} \in \mathcal{M}^1_\star(\mathbb{H})$, $(\overline{B}(t))_{t\ge 0}$ satisfies (E), respectively $(\overline{A}(u))_{u\ge 0}$ satisfies (EE2) iff $(B^\bullet(t))_{t\ge 0}$, respectively $(A^\bullet(u))_{u\ge 0}$ satisfy these conditions. Therefore, by Siebert [24], 5.7 or 5.10, there exists a uniquely determined Lipschitz continuous hemigroup $(\bar\lambda_{s,t})_{0\le s\le t}$ with generating functionals $(B^\bullet(s,t) = B^\bullet(t) - B^\bullet(s))_{0\le s\le t}$, i.e. $A^\bullet(s) = \frac{\partial^+}{\partial t}\bar\lambda_{s,t}\big|_{t=s}$ almost everywhere. By Proposition 3.3 we conclude $A^\bullet(s+1) = A^\bullet(s)$ and $B^\bullet(s+1,t+1) = B^\bullet(s,t)$ for all $0 \le s \le t$. Furthermore (cf. Siebert [24], 6.1), for $R > 0$ and a sequence of decompositions $0 = c_0^{(n)} < c_1^{(n)} < \cdots < c_{n-1}^{(n)} < c_n^{(n)} = R$ with $\max_{1\le i\le n} |c_i^{(n)} - c_{i-1}^{(n)}| \to 0$, we have

$$\bar\lambda_{s,t} = \lim_{n\to\infty} \mathop{\huge *}_{i=r_n(s)+1}^{r_n(t)} \operatorname{Exp}\left(B^\bullet(c_i^{(n)}, c_{i+1}^{(n)})\right), \tag{3.1}$$

where $(\operatorname{Exp}(tU))_{t\ge 0}$ denotes the convolution semigroup generated by $U \in \mathbb{A}(\mathbb{G})$ and $r_n(u) = k$ iff $c_k^{(n)} \le u < c_{k+1}^{(n)}$. Therefore, 1-periodicity of $(B^\bullet(s,t))_{0\le s\le t}$ implies 1-periodicity of $(\bar\lambda_{s,t})_{0\le s\le t}$ as asserted.

The converse is proved analogously. Given $A^\bullet(s) = \frac{\partial^+}{\partial t}\bar\lambda_{s,t}\big|_{t=s}$ we define $A(s) =$

$T_s(A^\bullet(s))$ and $B(s,t) = \int_s^t A(u)\,du$. Then, observing (again by Siebert [24], 6.1)

$$\lambda_{s,t} = \lim_{n\to\infty} \mathop{\ast}_{i=r_n(s)+1}^{r_n(t)} \operatorname{Exp}\left(B(c_i^{(n)}, c_{i+1}^{(n)})\right)$$

and noting that periodicity $B^\bullet(s+1,t+1) = B^\bullet(s,t)$ yields $B(s+1,t+1) = \tau(B(s,t))$, we conclude semistability $\tau(\lambda_{s,t}) = \lambda_{s+1,t+1}$. □

4. Representations by generalized Lie-Trotter formulas

For vector spaces $\mathbb{G} = \mathbb{R}^d$ the additive periodic driving process can be represented by pathwise random integrals; cf. [2, 20]. For stable hemigroups on homogeneous groups the background driving process is a Lévy process and a weak version of random integrals is obtained by the Lie-Trotter formula for convolution semigroups, see [8, 9]. In order to obtain similar results for semistable hemigroups on homogeneous groups $\mathbb{G}$ we have to analyze Section 3 of Siebert [24]. There the hemigroups are represented as limits of row-products of infinitesimal arrays $\mu_{s,t} = \lim_{n\to\infty} \ast_{k=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \sigma_{n,k}$. Crucial are the following conditions (S') and (T) in Siebert [24]

$$\sum_{k=1}^{n} q(\sigma_{n,k}) \le \gamma \quad \text{for some } \gamma > 0 \text{ and all } n \in \mathbb{N}. \tag{4.1}$$

For every $\varepsilon > 0$ there exists a compact $K_\varepsilon \subseteq \mathbb{G}$ such that

$$\sum_{k=1}^{n} \sigma_{n,k}(\complement K_\varepsilon) < \varepsilon \quad \text{for all } n \in \mathbb{N}. \tag{4.2}$$

Here, in place of $\sigma_{n,k}$, for $k, n \in \mathbb{N}$ we consider the arrays given by

$$\mu(n,k) = \lambda_{\frac{k-1}{n},\frac{k}{n}} \quad \text{and} \quad \mu^\bullet(n,k) = \lambda^\bullet_{\frac{k-1}{n},\frac{k}{n}} = T_{\frac{k-1}{n}}(\mu(n,k)),$$

which obviously are infinitesimal.

Proposition 4.1. *The arrays* $\{\mu(n,k)\}_{k,n\in\mathbb{N}}$ *and* $\{\mu^\bullet(n,k)\}_{k,n\in\mathbb{N}}$ *fulfill conditions* (4.1) *and* (4.2) *(conditions (S') and (T) in Siebert* [24]*).*

Proof. Since $(\lambda_{s,t})_{0\le s\le t}$ is Lipschitz continuous, the array $\{\mu(n,k)\}_{k,n\in\mathbb{N}}$ satisfies condition (4.1); cf. Siebert [24], 5.3 and 5.4. Therefore, also $\{\mu^\bullet(n,k)\}_{k,n\in\mathbb{N}}$ satisfies condition (4.1) by Remark 2.7. To prove the tightness condition (4.2) we switch to the space-time hemigroup $(\Lambda_{s,t})_{0\le s\le t}$. Let $\varphi_{\mathbb{G}}$ and $\varphi_{\mathbb{H}}$ be Hunt functions as in the proof of Proposition 3.2, and let η denote the Lévy-measure of $\overline{B}(0,1) = \overline{B}(1)$. Put

$$\kappa(n,k) = \Lambda_{\frac{k-1}{n},\frac{k}{n}} - \varepsilon_{(e,0)} \quad \text{and} \quad \kappa_n(s,t) = \sum_{k=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \kappa(n,k).$$

Then for all $0 \le s \le t$ we have $\kappa_n(s,t) \to \overline{B}(s,t)$ by Siebert [24], Theorem 3.6, hence for nonnegative $f \in \mathcal{C}_2(\mathbb{H})$ with $f(e,0) = 0$ we get

$$\langle \overline{B}(1), f\rangle = \lim_{n\to\infty} \Big\langle \sum_{k=1}^{n} \kappa(n,k), f \Big\rangle = \lim_{n\to\infty} \Big\langle \sum_{k=1}^{n} \mu^{\bullet}(n,k) \otimes \varepsilon_{n^{-1}}, f \Big\rangle.$$

Especially, for $f = \varphi_{\mathbb{H}} = \varphi_{\mathbb{G}} + \xi_{d+1}^2$ we have

$$\langle \overline{B}(1), \varphi_{\mathbb{H}}\rangle = \lim_{n\to\infty} \Big\langle \sum_{k=1}^{n} \mu^{\bullet}(n,k), \varphi_{\mathbb{G}} \Big\rangle,$$

since $\xi_{d+1}^2(n^{-1}) \to 0$. Furthermore, since $\Lambda_{s,t} \in \mathcal{M}^1_\star(\mathbb{H})$, the Lévy-measure η of $\overline{B}(1)$ is concentrated on $\mathbb{G} \times \{0\} \subseteq \mathbb{H}$, and for nonnegative $g \in \mathcal{C}_2(\mathbb{H})$ with $g(e,0) = 0$ we have $\langle \overline{B}(1), g \cdot \varphi_{\mathbb{H}}\rangle = \langle \eta, g \cdot \varphi_{\mathbb{H}}\rangle$. Therefore, for $g = h \otimes 1_{\mathbb{R}}$ with $h \in \mathcal{C}_2(\mathbb{G})$, we obtain

$$\langle \eta, g \cdot \varphi_{\mathbb{H}}\rangle = \lim_{n\to\infty} \Big\langle \sum_{k=1}^{n} \mu^{\bullet}(n,k), h \cdot \varphi_{\mathbb{G}} \Big\rangle = \lim_{n\to\infty} \Big\langle \varphi_{\mathbb{G}} \cdot \sum_{k=1}^{n} \mu^{\bullet}(n,k), h \Big\rangle, \tag{4.3}$$

where $\varphi_{\mathbb{G}} \cdot \mu$ denotes the measure ν with Radon-Nikodym derivative $\frac{d\nu}{d\mu} = \varphi_{\mathbb{G}}$. Now for any neighbourhood V of e we have $\varphi_{\mathbb{G}}\big|_{\complement V} \ge \delta$ for some $\delta > 0$. Therefore (4.3) yields weak convergence of the bounded measures $\sum_{k=1}^{n} \mu^{\bullet}(n,k)\big|_{\complement V} \to \eta\big|_{\complement V}$ and by Prohorov's theorem the sequence $\{\sum_{k=1}^{n} \mu^{\bullet}(n,k)\big|_{\complement V}\}$ is uniformly tight. Whence, (4.2) follows. □

Now we are ready to prove the announced generalized Lie-Trotter formulas that can be seen as weak random integral representations.

Theorem 4.2. *With the above notations we have*

$$\bar{\lambda}_{s,t} = \lim_{n\to\infty} \mathop{\ast}_{k=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} T_{-\frac{k-1}{n}}(\lambda_{\frac{k-1}{n},\frac{k}{n}}) = \lim_{n\to\infty} \mathop{\ast}_{k=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \lambda^{\bullet}_{\frac{k-1}{n},\frac{k}{n}}$$

and conversely

$$\lambda_{s,t} = \lim_{n\to\infty} \mathop{\ast}_{k=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} T_{\frac{k-1}{n}}(\lambda^{\bullet}_{\frac{k-1}{n},\frac{k}{n}}).$$

Proof. According to Siebert [24], 3.6, the conditions (4.1) and (4.2) imply that $\big\{\lambda_n(s,t) = \ast_{k=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \lambda^{\bullet}_{\frac{k-1}{n},\frac{k}{n}} : n \in \mathbb{N}, 0 \le s \le t\big\}$ is uniformly tight, hence weakly relatively compact. Let (n') denote a universal net such that $\lambda_n(s,t) \to \lambda^*(s,t)$ along (n') for all $0 \le s \le t$. Then, by Siebert [24], 3.6, $(\lambda^*(s,t))_{0\le s\le t}$ is a Lipschitz continuous hemigroup with generating functionals $B^*(s,t) = \lim_{(n')} \sum_{k=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \big(\lambda^{\bullet}_{\frac{k-1}{n},\frac{k}{n}} - \varepsilon_e\big)$. Hence $B^*(s,t) = B^{\bullet}(s,t)$ for all $0 \le s \le t$ and, since the hemigroup is uniquely determined by the generating functionals (cf. Siebert [24], 5.7), we have $\lambda^*(s,t) = \bar{\lambda}_{s,t}$. The converse limit representation simply follows by $\lambda^{\bullet}_{s,t} = T_{-s}\lambda_{s,t}$. □

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