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AN ANALYSIS OF THE CIRCULARIZATION OF ELLIPTIC

SATELLITE ORBITS CAUSED BY ATMOSPHERIC DRAG

A THESIS

Presented to

The Faculty of the Graduate Division

by

John R. Cowley, Jr.

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AN ANALYSIS OF THE CIRCULARIZATION OF ELLIPTIC

SATELLITE ORBITS CAUSED BY ATMOSPHERIC DRAG

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LIST OF SYMBOLS

a	semimajor axis
A	satellite's average frontal area
A, B, C	constants in the altitude-density equation
C_D	drag coefficient
e	eccentricity
$\bar{e}_r, \bar{e}_\theta, \bar{e}_\varphi$	radial, transverse, and orthogonal (to radial-transverse plane) unit vectors
ΔE_D	change in total energy due to drag
E_T	total energy of an orbiting body
f	polar angle (true anomaly)
F_D	resisting force of air drag
F_g	gravitational force
G	universal constant of gravitation
h	altitude above spherical earth
H	angular momentum of satellite
ΔH	change in angular momentum due to drag
i	inclination angle of the orbit plane with respect to the plane of the equator
K	ballistic coefficient
m	satellite mass
M	mass of the earth
p	perigee point
P	period of orbit
r	radial distance measured from earth's center

LIST OF SYMBOLS (Continued)

r_p radial distance to perigee

R radius of a spherical satellite

R_e average earth radius

v velocity of satellite

v_p velocity of satellite at perigee

Greek Letters

μ planetary constant for the earth

ρ air density as a function of altitude

ω angular position of perigee, measured from line of intersection of orbital plane with equatorial plane; argument of perigee

Ω angular position of line of nodes measured from a reference direction (vernal equinox)

Subscripts

r radial component

θ transverse component

φ component normal to radial and transverse

0 values at beginning (at perigee) of any particular revolution

1 values at end (again at perigee) of that revolution

Bars over variables signify vector values.

Dots over variables indicate differentiation with respect to time.

CHAPTER I

INTRODUCTION

Background

Studies of the effect of atmospheric drag on the orbits of near-earth satellites began in earnest soon after the 1957 launch of Sputnik I. Many theoretical descriptions of the effects of drag have since been published.

Several disturbances are attributed to air drag. As suggested by the term "circularization," an elliptical orbit becomes more rounded: drag gradually shrinks the major axis, reduces the eccentricity, and decreases the orbital period. Detailed analyses have unveiled periodic variations superimposed on these steady changes. By taking into account the oblateness and rotation of the atmosphere, other effects are explained, particularly the changes in the angles (i, ω, Ω) that position an orbital plane with reference to the earth.

Purpose of the Thesis

This thesis presents an approach, on a basic level, to the problem of describing what atmospheric drag does to a satellite orbit. With the assumptions that will be made here, the orbit stays in a fixed plane. The elements which do change are the orbit's eccentricity, period, and the length of the major axis. Equations are derived to predict these changes, and they are compared with the generally accepted theory as applied to the decay of the orbit of the Explorer IX satellite.

The Assumptions

Six assumptions are needed to put the real problem in a form that the present method can handle. They are:

1) The earth is a perfect sphere.

2) The sun and the moon have no gravitational effect on the orbit.

3) The atmosphere does not rotate. It is fixed relative to an inertial coordinate system with origin at the earth's center.

4) Solar radiation pressure is negligible.

5) The atmosphere is spherically symmetric about the earth. At a given altitude above any point of the earth's surface the atmosphere density is the same.

6) The relation between density and altitude does not change with time.

The first four assumptions together restrict the motion to a plane fixed in space, because all forces with components perpendicular to the plane of motion have been ruled out. Except for the intrusion of an atmosphere, the problem would be one of a central-force attraction between two bodies. The fourth and sixth assumptions are discussed in Chapter II. The last two simplify the mathematics needed to formulate an altitude-density relation.

Preliminary Orbital Mechanics

There are many equations associated with the ideal Keplerian two-body orbit. The ones that will be needed are summarized here.

Polar coordinates (r,f) with origin at the earth's center, along with radial and transverse unit vectors, $\bar{e}_r$ and $\bar{e}_\theta$, are used to locate the satellite. The force of gravitational attraction between the earth and a satellite of mass m is given in vector form by

$$\bar{F}_G = -\frac{\mu m}{r^2} \bar{e}_r . \quad (1)$$

A satellite in orbit around the earth follows an elliptical path with the earth at one focus, and its equation is

$$r = \frac{a(1 - e^2)}{1 + e \cos f} . \quad (2)$$

The polar angle, f, is often called the "true anomaly." At "perigee," point p in Figure 1, the true anomaly is zero, and from equation (2),

$$r_p = a(1 - e) \quad (3)$$

where r_p is the distance from the focus of attraction to perigee. The velocity at perigee is given by

$$v_p^2 = \frac{\mu}{a} \left[\frac{1 + e}{1 - e}\right] . \quad (4)$$

In Chapter IV the "eccentric anomaly," E, is mentioned, and often equations for orbital problems are written using it instead of the polar angle f. The angle E is defined by Figure 2.

In the absence of air drag and other outside forces, it can be shown that the total energy of an orbiting body is a constant, given by the equation

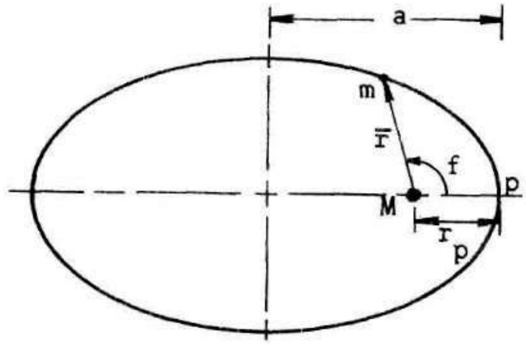


Figure 1. The Keplerian Ellipse.

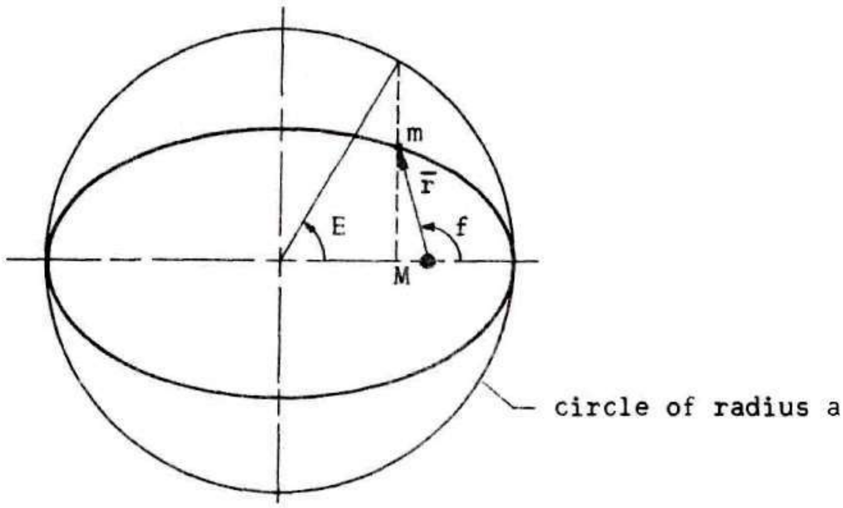


Figure 2. Location of the Angle E.

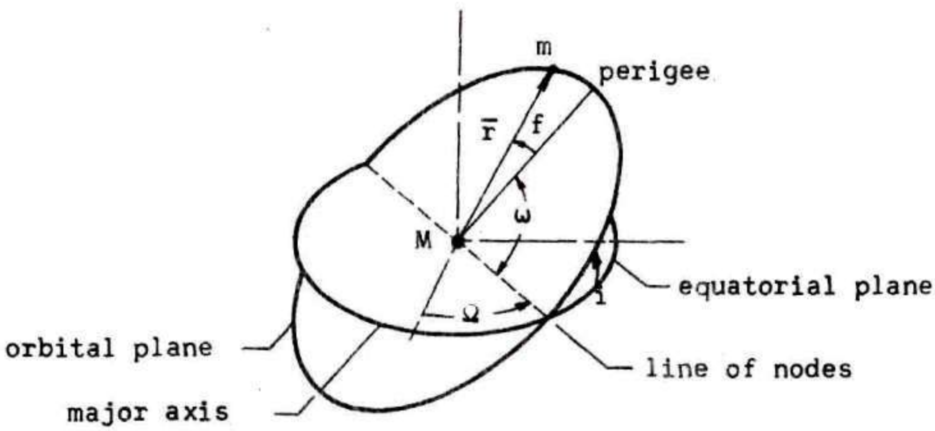


Figure 3. The Angles Positioning the Orbital Plane.

$$\frac{1}{2} mv^2 - \frac{\mu m}{r} = E_T \ , \qquad (5)$$

where v is the satellite's velocity measured with respect to an inertial reference frame located at the earth's center. The total energy, E_T, is the sum of the kinetic and potential energies of the satellite with respect to the earth. The value of the total energy depends on the planetary constant, the mass of the satellite, and the length of the major axis:

$$E_T = -\frac{\mu m}{2a} \ . \qquad (6)$$

The time required for one revolution is the period, P, of the orbit, and it is given by

$$P = 2\pi \left[\frac{a^3}{\mu}\right]^{1/2} \ . \qquad (7)$$

The angles i, ω, and Ω that position the orbital plane (when using geocentric equatorial coordinates) are shown in Figure 3.

Outline of the Approach

The problem can be divided into four parts:

1) Find an equation that gives air density as a function of altitude above the earth.

2) Derive an expression for energy lost to drag friction for each satellite revolution.

3) Establish an equation for the change in angular momentum of a satellite that occurs during a revolution in drag.

4) Employ the orbit equations and the energy and momentum relations to predict a, e, and P after one complete revolution. These should match values given by an accepted theory and actual satellite data.

The first two sub-problems are covered in Chapter II, and the last two comprise Chapter III.

Much of the practicability of this work depends on the use of a digital computer. Computer-based numerical methods are especially convenient for calculating the values of integrals that can not be integrated analytically.

CHAPTER II

ENERGY LOST TO AIR DRAG

Variation of Air Density with Altitude

Before the era of satellites, little was known about the density of the atmosphere above 100 km. Rocket probes sent up at irregular intervals ejected chemical vapors and small spheres whose behavior suggested properties of the upper atmosphere, often with large errors. With the appearance of satellites, ground observations of their deviations from predicted motion have made it possible to obtain more precise measurements of atmospheric densities.

One of the surprises found from tracking satellites has been the discovery of the wide variation of air density above 300 km: at 500 km the air is 5 times less dense on the summy side of the earth than on the shaded side. There is an atmospheric bulge associated with sunlight that lags the overhead sun position by about two hours. Also, sunspot activity can change the density at 500 km by a factor of 15 [1]. These and other effects, which increase with height, make it difficult to associate a specific density with an altitude. Generally, density has been observed to wander within the limits shown in Figure 4. Some advanced papers incorporate a "dynamic" density model, one with factors that imitate known periodic variations, but the use of such a model is a special study in itself. As the sixth assumption on page 2 indicates, fluctuations in atmospheric density due to differences in day-night sunlight heating, sunspot activity, magnetic storms, etc., will not be considered in this study. The important fact to note is that the "standard" and "average" density tabulations mentioned below, though not necessarily accurate for a particular time, are offered as points of departure. The actual density profile at the time under consideration should be used if it is available.

Model Atmospheres

Three atmosphere models are used as references:

1) The U. S. Standard Atmosphere, 1962, (USSA 1962) [2];

2) The COSPAR International Reference Atmosphere, 1961, (CIRA 1961) [3];

3) The Air Research and Development Command 1959 model, (ARDC 1959) [4].

The USSA 1962 average values are used here because they are the most recent of the three, and the listing is the most complete. Figure 4 is a plot of this model, and the values of some points are included in the Appendix, Table 1.

Solar Pressure

The fourth assumption, page 2, stated that solar pressure will not be considered. It is often not negligible, but it is ignored here because the purpose of the present analysis is to evaluate the effect of drag alone. To carefully describe actual satellite motion, solar radiation pressure should be accounted for, as recommended by L. G. Jacchia [5]:

> Thus, when ρ (density) is of the order of 10^{-16} g/cc, the effect of solar-radiation pressure may equal that of atmospheric drag. At times of sunspot maximum, this will

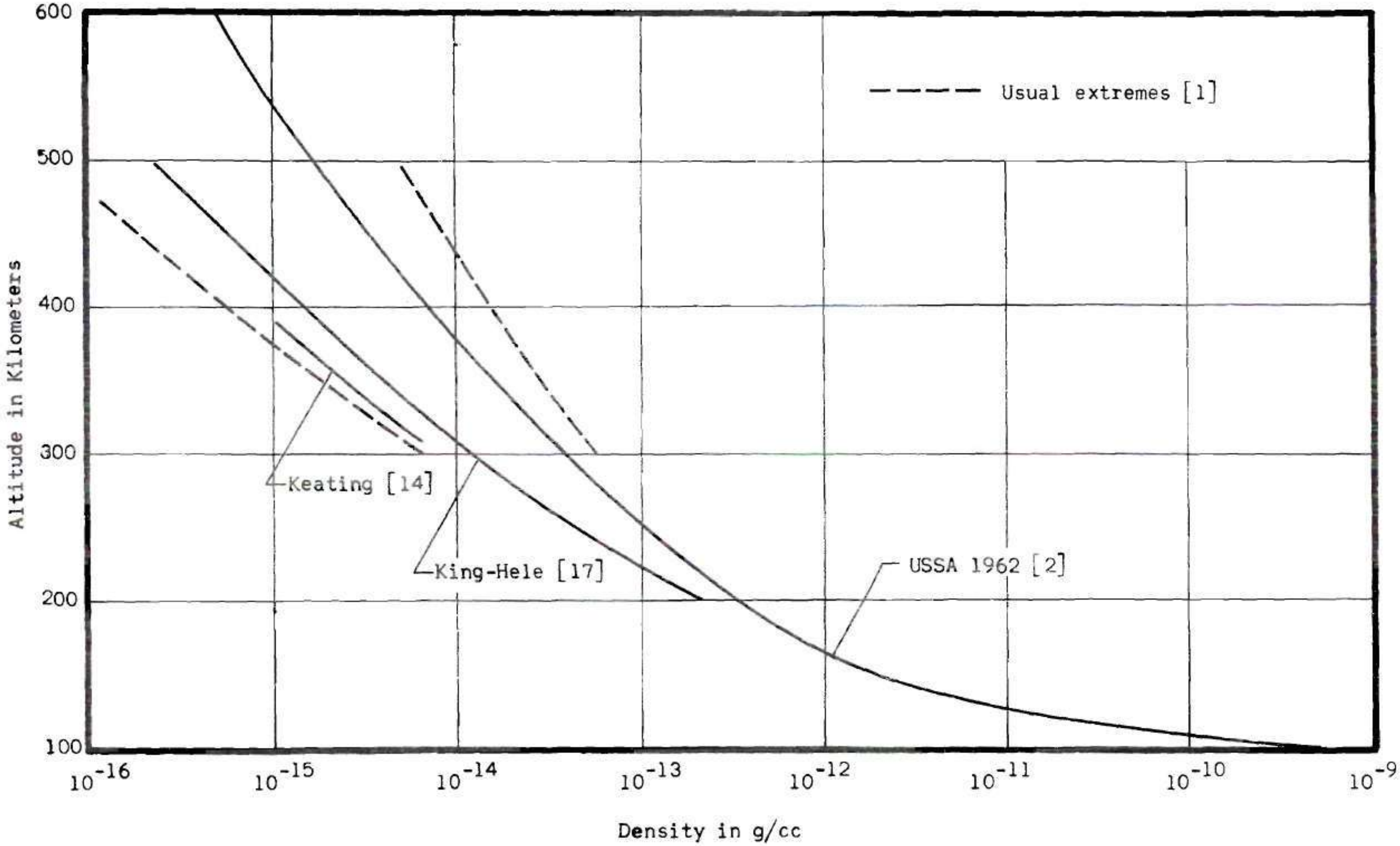


Figure 4. The USSA 1962 and Other Atmospheres.

occur at a height of 900 km; at times of low solar activity, however, when the atmosphere is appreciably contracted, it will occur as low as 500 km above the earth. If we want to determine atmospheric drag with a 10-percent accuracy or better, we must take into account solar-radiation pressure whenever the pergiee height of the satellite is greater than 400 km.

An Altitude-Density Equation

The present theory requires that the density curve be in equation form. Semi-log plots of the curve are often approximated simply with straight-line segments. Here the curve will be matched with part of a parabola; such a second degree function is more realistic than a straight line and is still convenient because its inverse is not hard to find.

Suppose coefficients A, B, and C can be found such that the USSA 1962 curve of Figure 4 is reasonably approximated by

$$h = Ax^2 + Bx + C \ , \tag{8}$$

where h is the altitude above the earth's spherical surface and $x = \ln \rho$. Then an equation for density in terms of altitude can be extracted from equation (8) by completing the square:

$$h - C = A(\ln \rho)^2 + B \ln \rho$$

$$\frac{h-C}{A} + \left(\frac{B}{2A}\right)^2 = (\ln \rho)^2 + \frac{B}{A} \ln \rho + \left(\frac{B}{2A}\right)^2$$

$$\pm\left[\frac{h-C}{A} + \left(\frac{B}{2A}\right)^2\right]^{1/2} = \left(\ln \rho + \frac{B}{2A}\right)$$

$$\rho = \exp\left\{-\frac{B}{2A} - \left[\frac{h-C}{A} + \left(\frac{B}{2A}\right)^2\right]^{1/2}\right\} .$$

The negative root is chosen because ρ must decrease as h increases. The altitude may be replaced by $r - R_e$, where R_e is the radius of the spherical earth:

$$\rho = \exp\left\{-\frac{B}{2A} - \left[\frac{r - R_e - C}{A} + \left(\frac{B}{2A}\right)^2\right]^{1/2}\right\} \tag{9}$$

Once values for A, B, and C are determined, equation (9) gives the density function $\rho(r)$ that will be needed later.

The values of the three coefficients are best found by the method of least squares, for which a computer procedure is convenient. To obtain a better equation for the densities in the region of interest, usually near the perigee of an orbit, it may be necessary to weight the least squares method by including additional points there. The coefficients should be reevaluated if the density changes drastically after they have been calculated.

An Equation for Energy Lost to Drag

The atmospheric drag force is commonly given by

$$F_D = -\frac{1}{2} A C_D \rho v^2 \tag{10}$$

where A is the average cross-sectional area of the satellite facing the direction of motion. For a sphere of radius R, A is simply the constant πR^2. The symbol ρ denotes the air density at the satellite's altitude, and v is the satellite's velocity relative to the surrounding air.

The dimensionless drag coefficient, C_D, is ideally 2.0 for a sphere [6], but King-Hele puts a practical value between 2.1 and 2.2 [7]. C_D is further discussed in Chapter IV. The area A and the coefficient C_D are usually combined into a single constant, K, called the "ballistic coefficient," where $K = AC_D$. Then

$$F_D = -\frac{1}{2} K \rho v^2$$

The drag force is directed opposite to the velocity. The velocity vector, $\bar{v}$, is now used to find the direction of $\bar{F}_D$ in terms of radial and transverse unit vectors. The velocity in polar coordinates is

$$\bar{v} = v_r \bar{e}_r + v_\theta \bar{e}_\theta$$

with components

$$v_r = \dot{r} \qquad v_\theta = r\dot{f}$$

From Figure 5 the drag force can be written:

$$\bar{F}_D = -\frac{1}{2} K \rho v^2 \left(\frac{v_r}{v}\right) \bar{e}_r - \frac{1}{2} K \rho v^2 \left(\frac{v_\theta}{v}\right) \bar{e}_\theta$$

$$\bar{F}_D = -\frac{1}{2} K \rho v (\dot{r}) \bar{e}_r - \frac{1}{2} K \rho v (r\dot{f}) \bar{e}_\theta \qquad (11)$$

The work done, W, by a force $\bar{F}$ that moves along a path C is given by the line integral

$$W = \int_C \bar{F} \cdot d\bar{S}$$

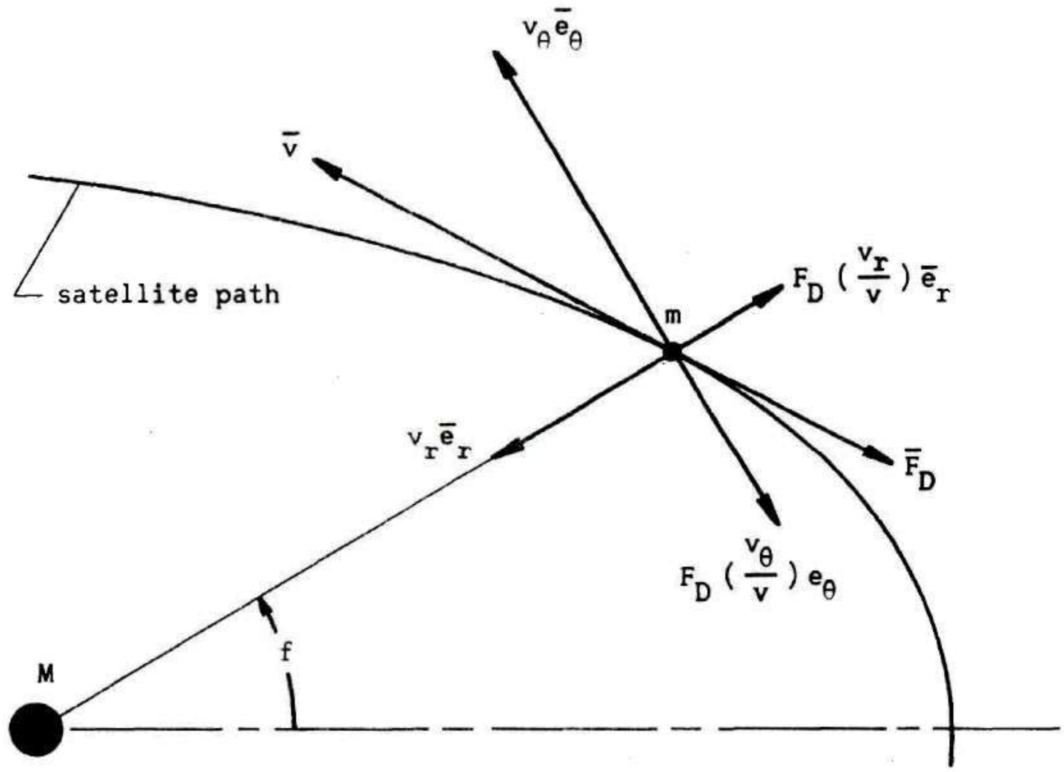


Figure 5. Relation Between $\bar{F}_D$ and $\bar{v}$.

where

$$d\bar{S} = dr\,\bar{e}_r + r\,df\,\bar{e}_\theta$$

The work done on a body equals its change in total energy. If ΔE_D is the change in energy of an orbiting body due to the drag force, then

$$\Delta E_D = \int_C \bar{F}_D \circ d\bar{S}$$

$$\Delta E_D = \int_C \left[-\frac{1}{2} K \rho v(\mathring{r})dr - \frac{1}{2} K \rho v(r\mathring{f})r\, df \right]$$

$$\Delta E_D = \int_C -\frac{1}{2} K \rho v[\mathring{r}\, dr + (r\mathring{f})r\, df] \qquad (12)$$

The actual path C that a satellite follows is a deteriorating ellipse, one that gradually spirals inward. An approximate path is chosen for C (for each revolution) in order to evaluate the above line integral. The one that is used is the perfect ellipse that would be followed by a satellite in the absence of air resistance. It has constant elements a and e, and with it r can be related to f by equation (2):

$$r = \frac{a(1 - e^2)}{1 + e \cos f} \qquad (2)$$

Except during the final few revolutions of a satellite's lifetime, the actual radial distance departs only a few hundred meters during a single revolution from that given by equation (2).

Expressions for the velocity components, $\dot{r}$ and $r\dot{f}$, in terms of v are now derivable by differentiating (2).

$$\dot{r} = \frac{a(1 - e^2)e \sin f}{(1 + e \cos f)^2} \dot{f} \tag{13}$$

$$v^2 = \bar{v} \cdot \bar{v} = \dot{r}^2 + (r\dot{f})^2$$

$$v^2 = \frac{a^2(1 - e^2)^2 e^2 \sin^2 f}{(1 + e \cos f)^4} (\dot{f})^2 + \frac{a^2(1 - e^2)^2}{(1 + e \cos f)^2} (\dot{f})^2$$

$$v^2 = \left[\frac{a^2(1 - e^2)^2 e^2 \sin^2 f + a^2(1 - e^2)^2(1 + e \cos f)^2}{(1 + e \cos f)^4}\right] (\dot{f})^2$$

$$\dot{f}^2 = \left[\frac{(1 + e \cos f)^4}{a^2(1 - e^2)^2 e^2 \sin^2 f + a^2(1 - e^2)^2(1 + e \cos f)^2}\right] v^2$$

$$\dot{f} = \frac{(1 + e \cos f)^2 v}{a(1 - e^2)(1 + 2e \cos f + e^2)^{1/2}} \tag{14}$$

Again use (2) and multiply both sides of it by $\dot{f}$; substitute (14):

$$r\dot{f} = \frac{a(1 - e^2)}{(1 + e \cos f)} \dot{f}$$

$$r\dot{f} = \frac{(1 + e \cos f)v}{(1 + 2e \cos f + e^2)^{1/2}} \tag{15}$$

Substitute (14) in (13):

$$\dot{r} = \frac{a(1 - e^2)e \sin f}{(1 + e \cos f)^2} \frac{(1 + e \cos f)^2 v}{a(1 - e^2)(1 + 2e \cos f + e^2)^{1/2}}$$

$$\dot{r} = \frac{e \sin f \quad v}{(1 + 2e \cos f + e^2)^{1/2}} . \tag{16}$$

Another statement that is needed follows from equation (2):

$$dr = \frac{a(1 - e^2)e \sin f}{(1 + e \cos f)^2} df . \tag{17}$$

Now that the path is specified, the line integral can be transformed into definite integrals of functions of f. Substitute equations (16), (17), (15), and (2) into equation (12):

$$\Delta E_D = -\frac{1}{2} K \left[\int_{f_o}^{f_1} \rho v^2 \frac{e \sin f}{(1 + 2e \cos f + e^2)^{1/2}} \frac{a(1 - e^2)e \sin f}{(1 + e \cos f)^2} df + \int_{f_o}^{f_1} \rho v^2 \frac{(1 + e \cos f)}{(1 + 2e \cos f + e^2)^{1/2}} \frac{a(1 - e^2)}{(1 + e \cos f)} df \right]$$

$$\Delta E_D = -\frac{1}{2} K \int_{f_o}^{f_1} \rho v^2 \frac{a(1 - e^2)}{(1 + 2e \cos f + e^2)^{1/2}} \left[\frac{e^2 \sin^2 f}{(1 + e \cos f)^2} + 1 \right] df$$

$$\Delta E_D = -\frac{1}{2} K \int_{f_o}^{f_1} \rho v^2 \frac{a(1 - e^2)}{(1 + 2e \cos f + e^2)^{1/2}} \left[\frac{e^2 \sin^2 f + 1 + 2e \cos f + e^2 \cos^2 f}{(1 + e \cos f)^2} \right] df \tag{18}$$

Combining equations (5) and (6), the "energy equation" is obtained:

$$\frac{1}{2} mv^2 - \frac{\mu m}{r} = -\frac{\mu m}{2a} ,$$

or

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \ . \tag{19}$$

Use this and equation (2) in equation (18):

$$\Delta E_D = -\frac{1}{2} K \int_{f_o}^{f_1} \rho\mu\left(\frac{2}{r} - \frac{1}{a}\right) r \left[\frac{(1 + 2e\cos f + e^2)^{1/2}}{(1 + e\cos f)}\right] df$$

$$\Delta E_D = -\frac{1}{2} K \int_{f_o}^{f_1} \rho\mu\left(2 - \frac{r}{a}\right) \frac{(1 + 2e\cos f + e^2)^{1/2}}{(1 + e\cos f)} \ df$$

Again use equation (2) for r:

$$\Delta E_D = -\frac{1}{2} K\mu \int_{f_o}^{f_1} \rho\left[2 - \frac{(1 - e^2)}{(1 + e\cos f)}\right] \frac{(1 + 2e\cos f + e^2)^{1/2}}{(1 + e\cos f)} \ df$$

$$\Delta E_D = -\frac{1}{2} K\mu \int_{f_o}^{f_1} \rho\left[\frac{(1 + 2e\cos f + e^2)}{(1 + e\cos f)}\right] \frac{(1 + 2e\cos f + e^2)^{1/2}}{(1 + e\cos f)} \ df$$

Air density, ρ, is a function of r, given in approximate form by equation (9). Since elliptical paths are being considered, r, in turn, becomes a function of f by using equation (2). The equation for ΔE_D is then an integral of a function of the variable f:

$$\Delta E_D = -\frac{1}{2} K\mu \int_{f_o}^{f_1} \rho \ \frac{(1 + 2e\cos f + e^2)^{3/2}}{(1 + e\cos f)^2} \ df \ .$$

This equation gives the energy lost to drag by a satellite that has completed the segment between f_o and f_1 of an elliptical orbit with elements a and e. The analysis in the next chapter deals with whole revolutions, beginning and ending at perigee, so that the energy lost to drag per revolution is given approximately by

$$\Delta E_D = -\frac{1}{2} K\mu \int_o^{2\pi} \frac{(1 + 2e \cos f + e^2)}{(1 + e \cos f)^2} \, df \; . \tag{20}$$

CHAPTER III

EQUATIONS FOR CHANGES IN SEMIMAJOR AXIS AND ECCENTRICITY

If, in addition to the assumptions made in Chapter I atmospheric drag were negligible, the orbital elements of a satellite $(a, e, i, \omega, \Omega, P)$ could be evaluated once at any point and would not change. With drag present, neither total energy nor angular momentum is conserved. On the basis of the six assumptions, a, e, and P then continuously decrease while i, ω, and Ω remain essentially fixed.

Equations are now derived to predict the new values of a, e, and P after a single revolution in drag. A continuous history is formed by letting the end conditions of one revolution be the initial conditions of the next.

Consider a satellite at the perigee of the ellipse specified by a_0, e_0, and T_0. After one revolution in drag the satellite is again at perigee, but the path it is following is then, only for an instant, a new ellipse given by a_1, e_1, and T_1. These elements are constantly (though slowly) changing. As an approximation they are assumed fixed for a single revolution until new values can be determined at the end of each complete revolution. This assumption made possible the evaluation of the drag integral in Chapter II and will help with the angular impulse integral to follow. The model of the path of a satellite in drag is then a series of diminishing ellipses with common major axis directions and

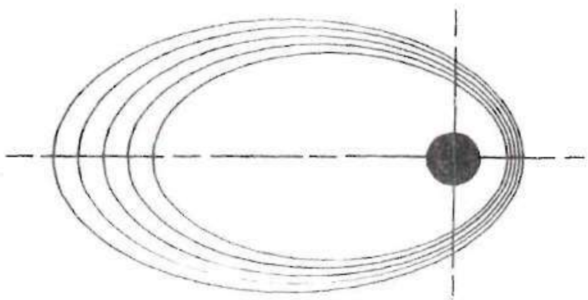


Figure 6. The Proposed Model of Orbital Decay
Due to Air Drag.

foci of attraction (see Figure 6).

The Semimajor Axis after a Revolution in Drag

Equation (6) indicates that the total energy of an orbit is a function of $\underline{a}$ alone, once the planetary constant and satellite mass are fixed. The energy lost per revolution, E_D, is known from equation (20) and can be used directly to obtain a_1 from a_o:

$$E_{T_o} = -\frac{\mu m}{2a_o}$$

$$E_{T_1} = -\frac{\mu m}{2a_1}$$

$$E_{T_1} = E_{T_o} + \Delta E_D$$

where ΔE_D is negative. From this,

$$-\frac{\mu m}{2a_1} = -\frac{\mu m}{2a_o} + \Delta E_D$$

$$a_1 = \left[\frac{1}{a_o} - \frac{2\Delta E_D}{\mu m}\right]^{-1} \qquad (21)$$

This equation gives the length of the semimajor axis after a satellite has followed the ellipse described by a_o and e_o through an atmosphere of density $\rho(r)$.

Eccentricity after a Revolution in Drag

Since two quantities, a and e, are needed to determine an ellipse uniquely (P is directly related to $\underline{a}$), two independent principles are required to find both elements. The work-energy approach was used to calculate a_1. The principle of angular impulse and angular momentum provides a second equation:

$$\bar{r} \times \bar{F}_D = \frac{d}{dt} (\bar{r} \times m\bar{v})$$

where $\bar{F}_D$ is the drag resistance vector. This leads to a relation between r_1 and v_1 (values at perigee after revolution). They must also conform to the energy equation,

$$v_1^2 = \mu \left[\frac{2}{r_1} - \frac{1}{a_1} \right]$$

and elimination of v_1 between the two gives an equation for r_1. With the assumption that at perigee v_1 is perpendicular to r_1, as v_o was to r_o, e_1 follows from r_1 using equation (3):

$$r_1 = a_1(1 - e_1)$$

$$e_1 = 1 - \frac{r_1}{a_1} \qquad (22)$$

Proceeding from the principle of angular impulse and angular momentum, and using the definition of angular momentum,

$$\bar{H} = \bar{r} \times m\bar{v} ,$$

write

$$(\bar{r} \times \bar{F}_D)dt = d\bar{H} .$$

Integrating the last equation,

$$\bar{H}_1 - \bar{H}_o = \int_o^1 (\bar{r} \times \bar{F}_D) dt \ .$$

From equation (11),

$$\bar{F}_D = - \frac{1}{2} K \rho v \mathring{r} \bar{e}_r - \frac{1}{2} K \rho v r \mathring{f} \bar{e}_\theta \ ,$$

and $\bar{r} = r \bar{e}_r$. Then

$$\bar{r} \times \bar{F}_D = - \frac{1}{2} K \rho v r^2 \mathring{f} \bar{e}_\varphi \ ,$$

where $\bar{e}_\varphi = \bar{e}_r \times \bar{e}_\theta$. Similarly,

$$\bar{H} = m r^2 \mathring{f} \bar{e}_\varphi$$

By equating vector components the previous equations may be written in scalar form:

$$H_1 - H_o = \int_o^1 (- \frac{1}{2} K \rho v r^2 \mathring{f}) dt \ .$$

Define

$$\Delta H = H_1 - H_o \ .$$

From the chain rule write $dt = \frac{dr}{\mathring{r}}$. Then

$$\Delta H = - \frac{1}{2} K \int_o^1 \rho v r^2 \frac{\dot{f}}{\mathring{r}} dr \ .$$

Conveniently,

$$\frac{\dot{f}}{\dot{r}}\, dr = \frac{df}{dt}\frac{dt}{dr}\, dr = df .$$

Then

$$\Delta H = -\frac{1}{2} K \int_0^1 \rho\, v\, r^2\, df .$$

Replace v with its value along the ellipse (a_o, e_o) that the energy equation requires for a Keplerian orbit,

$$v = \left[\mu\left(\frac{2}{r} - \frac{1}{a_o}\right)\right]^{1/2}$$

and eliminate r with equation (2),

$$r = \frac{a_o(1 - e_o^2)}{1 + e_o \cos f} .$$

Then the integral for the change in angular momentum per revolution becomes:

$$\Delta H = -\frac{1}{2} K \int_0^{2\pi} \rho\left[\mu\left(\frac{2}{r} - \frac{1}{a_o}\right)\right]^{1/2} r^2\, df$$

$$\Delta H = -\frac{1}{2} K \int_0^{2\pi} \rho\mu^{1/2}\, r^2\left[\frac{2(1 + e_o \cos f)}{a_o(1 - e_o^2)} - \frac{1}{a_o}\right]^{1/2} df$$

$$\Delta H = -\frac{1}{2} K\mu^{1/2} \int_o^{2\pi} \rho r^2 \left[\frac{1 + 2e_o \cos f + e_o^2}{a_o(1 - e_o^2)}\right]^{1/2} df$$

$$\Delta H = -\frac{1}{2} K\mu^{1/2} \int_o^{2\pi} \rho \frac{a_o^2 (1 - e_o^2)^2}{(1 + e_o \cos f)^2} \left[\frac{1 + 2e_o \cos f + e_o^2}{a_o(1 - e_o^2)}\right]^{1/2} df$$

$$\Delta H = -\frac{1}{2} K\mu^{1/2} \left[a_o(1 - e_o^2)\right]^{3/2} \int_o^{2\pi} \rho \frac{(1 + 2e_o \cos f + e_o^2)^{1/2}}{(1 + e_o \cos f)^2} df$$

This equation calculates the change in angular momentum of a satellite that has made a complete revolution along the ellipse defined by a_o and e_o through an atmosphere whose density is $\rho(r)$ as discussed in Chapter II.

If H_o and H_1 are evaluated at perigee assuming v_o and v_1 are perpendicular to r_o and r_1, respectively, then

$$\Delta H = mr_1^2 \dot{f}_1 - mr_o \dot{f}_o = mr_1 v_1 - mr_o v_o$$

$$v_1 = \frac{1}{r_1}\left[r_o v_o + \frac{\Delta H}{m}\right] \qquad (23)$$

The velocity at perigee is given by equation (4),

$$v_o^2 = \frac{\mu}{a_o}\left[\frac{1 + e_o}{1 - e_o}\right],$$

and the perigee distance, from equation (3), is

$$r_o = a_o(1 - e_o) .$$

From these the product $r_o v_o$ is

$$r_o v_o = [\mu a_o(1 - e_o^2)]^{1/2} .$$

Using this relationship, equation (23) becomes

$$v_1 = \frac{1}{r_1}\left\{[\mu a_o(1 - e_o^2)]^{1/2} + \frac{\Delta H}{m}\right\} .$$

The energy equation demands that v_1 and r_1 also be related according to

$$v_1^2 = \mu\left[\frac{2}{r_1} - \frac{1}{a_1}\right]$$

Eliminating v_1 between these last two gives:

$$\mu\left[\frac{2}{r_1} - \frac{1}{a_1}\right] = \frac{1}{r_1^2}\left\{[\mu a_o(1 - e_o^2)]^{1/2} + \frac{\Delta H}{m}\right\}^2$$

$$2r_1 - \frac{r_1^2}{a_1} = \left\{[a_o(1 - e_o^2)]^{1/2} + \frac{\Delta H}{m\mu^{1/2}}\right\}^2$$

Complete the square in order to solve for r_1:

$$r_1^2 - 2a_1 r_1 + a_1^2 = a_1^2 - a_1\left[a_o^{1/2}(1 - e_o^2)^{1/2} + \frac{\Delta H}{m\mu^{1/2}}\right]^2$$

$$r_1 = a_1 \pm \left\{a_1^2 - a_1\left[a_o^{1/2}(1 - e_o^2)^{1/2} + \frac{\Delta H}{m\mu^{1/2}}\right]^2\right\}^{1/2}$$

The negative root is chosen because with no drag ($\Delta H = 0$ and $a_1 = a_o$) the equation should reduce to $r_1 = r_o = a_o(1 - e_o)$.

$$r_1 = a_1 - a_1 \left\{ 1 - \frac{1}{a_1} \left[a_o^{1/2} (1 - e_o^2)^{1/2} + \frac{\Delta H}{m\mu^{1/2}} \right]^2 \right\}^{1/2}$$

Use equation (22) to relate r_1 to e_1,

$$e_1 = 1 - \frac{r_1}{a_1}$$

$$e_1 = \left\{ 1 - \frac{1}{a_1} \left[a_o^{1/2} (1 - e_o^2)^{1/2} + \frac{\Delta H}{m\mu^{1/2}} \right]^2 \right\}^{1/2} \qquad (24)$$

where

$$\frac{\Delta H}{m\mu^{1/2}} = -\frac{1}{2} \frac{K}{m} \left[a_o (1 - e_o^2) \right]^{3/2} \int_o^{2\pi} \rho \frac{(1 + 2e_o \cos f + e_o^2)^{1/2}}{(1 + e_o \cos f)^2} df \qquad (25)$$

These last two equations together give the eccentricity at the end of a revolution in air drag of an orbit initially described by a_o and e_o.

The Period after a Revolution in Drag

The new period, P_1, can be obtained immediately from a_1 by using the fundamental equation (7):

$$P_1 = 2\pi \left[\frac{a_1^3}{\mu} \right]^{1/2} \qquad (26)$$

CHAPTER IV

LITERATURE EQUATIONS, SATELLITE INFORMATION, AND CONSTANTS

Equations from the Literature

Many of the recent analyses of the atmospheric drag problem are summarized in references [8] and [9]. The most widely accepted theory results from the use of perturbation techniques, originally applied by Theodore Sterne [10, 11, 12, 13]. It is with this that the theory of Chapter III will be compared.

After transforming the variable from the generally used eccentric anomaly, E, to the true anomaly, f, the equations from perturbation theory for the changes per revolution in semimajor axis and eccentricity due to drag are:

$$\Delta a = - \frac{Ka^2}{m} \int_o^{2\pi} \rho \frac{(1 + 2e \cos f + e^2)^{3/2}}{(1 + e \cos f)^2} df$$

$$\Delta e = - \frac{Ka}{m} (1 - e^2) \int_o^{2\pi} \rho \frac{(1 + 2e \cos f + e^2)^{1/2}}{(1 + e \cos f)^2} (e + \cos f)\, df$$

where $K = AC_D$. After each revolution the new period is obtained by letting $a = (a + \Delta a)$ in equation (7):

$$P = 2\pi \left[\frac{(a + \Delta a)^3}{\mu}\right]^{1/2}$$

Explorer IX

Explorer IX (1961 δ1), the 12-foot diameter inflated satellite launched 16 February 1961, has been chosen for testing the theory because of its spherical shape and the amount of information available on it. It was put in orbit specifically for obtaining air density data, and its position was continuously monitored during its three-year lifetime.

Orbital observations of many satellites are published in a series of Special Reports by the Smithsonian Astrophysics Observatory. Special Report No. 84 in particular is devoted to the first seven months of Explorer IX's time in orbit and gives its weight and a value for C_D. A NASA paper, "Determination of Mean Atmospheric Densities from the Explorer IX Satellite" [14], gives the average semimajor axis, eccentricity, and density logarithm every few days up to the satellite's final decay. Tables 2 and 3 contain some of these values.

Values of Constants

The metric system of units has become common and will be used throughout. The average earth radius is

$$R_e = 6371.2 \text{ km} ,$$

and the planetary constant for the earth is

$$\mu = 3.986094 \times 10^5 \text{ km}^3/\text{sec}^2 .$$

The satellite's 12-foot diameter makes the cross-sectional area

$$A = 10.51 \text{ m}^2 .$$

Because of the spherical shape there is no question of orientation affecting the frontal area. The mass is given in reference [15] as

$$m = 6631.5 \text{ g} .$$

The drag coefficient for Explorer IX is not agreed upon. Reference [15] gives $C_D = 2.2$, and this sets the ratio

$$\frac{K}{m} = 3.49 \times 10^{-9} \text{ km}^2/\text{g} .$$

Reference [16] attributes the value $K/m = 500 \text{ ft}^2/\text{slug}$ to D. G. King-Hele, which upon conversion to metric units is

$$\frac{K}{m} = 3.19 \times 10^{-9} \text{ km}^2/\text{g} .$$

The effect of the choice of K/m will be considered later.

There are two sources for the atmosphere density profile during the period of interest (February-March, 1964), and both are plotted in Figure 4. One is the set of values for 1962-1964 determined from observations of many satellites by King-Hele [17], listed in Table 4 of the Appendix. The other source of density information is the satellite Explorer IX itself, from whose motion G. M. Keating [14] obtained logarithms of the density. Table 3 contains the last three months of his tabulation.

Keating's data leads, by the method of least squares (a combination of the entries marked "*"), to the density coefficients

$$A = 6.11496$$

$$B = 370.432$$

$$C = 5887.061$$

King-Hele's data give A, B, and C as

$$A = 2.32618$$

$$B = 108.551$$

$$C = 1388.40$$

The USSA 1962 Standard Atmosphere is fitted by the coefficients

$$A = 2.74573$$

$$B = 123.781$$

$$C = 1497.82$$

The results given by these three groups of constants are compared in the next chapter.

CHAPTER V

CONCLUSIONS

The Final Equations

The theory of Chapters II and III has produced the following: If a satellite at perigee of an elliptical orbit with elements a_o, e_o, and P_o encounters an atmosphere of density profile $\rho(r)$, then, when perigee is again reached after the upcoming revolution is completed, the elements of the satellite's orbit will have changed to a_1, e_1, and P_1, and their approximate values can be obtained from:

$$a_1 = \left[\frac{1}{a_o} - \frac{2\Delta E_D}{\mu m}\right]^{-1} \tag{21}$$

$$e_1 = \left\{1 - \frac{1}{a_1}\left[a_o^{1/2}(1 - e_o)^{1/2} + \frac{\Delta H}{m\mu^{1/2}}\right]^2\right\}^{1/2} \tag{24}$$

$$P_1 = 2\pi\left[\frac{a_1^3}{\mu}\right]^{1/2} \tag{26}$$

where

$$\frac{2\Delta E_D}{\mu m} = -\frac{K}{m}\int_o^{2\pi} \rho \frac{(1 + 2e_o \cos f + e_o^2)^{3/2}}{(1 + e_o \cos f)^2} df$$

$$\frac{\Delta H}{m\mu^{1/2}} = -\frac{1}{2}\frac{K}{m}\left[a_o(1 - e_o^2)\right]^{3/2}\int_o^{2\pi} \rho \frac{(1 + 2e_o \cos f + e_o^2)^{1/2}}{(1 + e_o \cos f)^2} df$$

The density is given by the function

$$\rho = \exp\left\{-\frac{B}{2A} - \left[\frac{r - R_e - C}{A} + \left(\frac{B}{2A}\right)^2\right]^{1/2}\right\} \quad (9)$$

where

$$r = \frac{a_o(1 - e_o^2)}{1 + e_o \cos f} \quad (2)$$

Comparison with Perturbation Theory

The two computer programs in the Appendix are for making calculations needed for the thesis and perturbation theories. They are arranged so that their outputs compare the two theories up to 300 revolutions, both using the data:

program notation		thesis notation		value	units
KM	=	K/m	=	3.19×10^{-9}	km^2/g
A[1]	=	a_o	=	7505.084	km
E[1]	=	e_o	=	0.104990	
AA	=	A	=	2.326179	
BB	=	B	=	108.5507	
CC	=	C	=	1388.399	

The last three constants are associated with the altitude-density equation (9) and are determined by the method of least squares from King-Hele's atmosphere data (Table 4). Incorporated in the perturbation theory program is the "least squares polynomial" procedure which calculates A, B, and C.

The thesis equations give results that agree well with the predictions from the accepted perturbation theory. They are compared in Figures 7, 8, 9, and 10 after 290 revolutions (about 20 days), using the above constants. These graphs cover a small interval of time because the scale required for plots of the entire period does not allow the differences in the two theories to be seen. The following evaluation is made at revolution 300:

	From Perturbation Theory	From Proposed Theory	Numerical Difference	Percent Difference
a	7323.082 km	7323.145	0.063	0.0009
e	0.083371	0.083376	0.000005	0.006
r_p	6712.549 km	6712.573	0.014	0.00021
P	103.944 $\frac{\text{min.}}{\text{rev.}}$	103.946 $\frac{\text{min.}}{\text{rev.}}$	0.002 $\frac{\text{min.}}{\text{rev.}}$	0.002

The theory developed in the thesis and the perturbation theory yield, then, essentially the same information. Although it is not as general in its applications, the proposed theory is derived with less mathematical work from the basic principles of mechanics.

<u>Comparison with the Orbit of Explorer IX</u>

Application of both the perturbation theory and the proposed theory to the actual satellite is complicated by four factors: the presence of solar pressure, the influence of the moon's force field, uncertainties in the value of C_D, and the lack of exact density data at the time of interest.

For most of the satellite's history, solar pressure and the

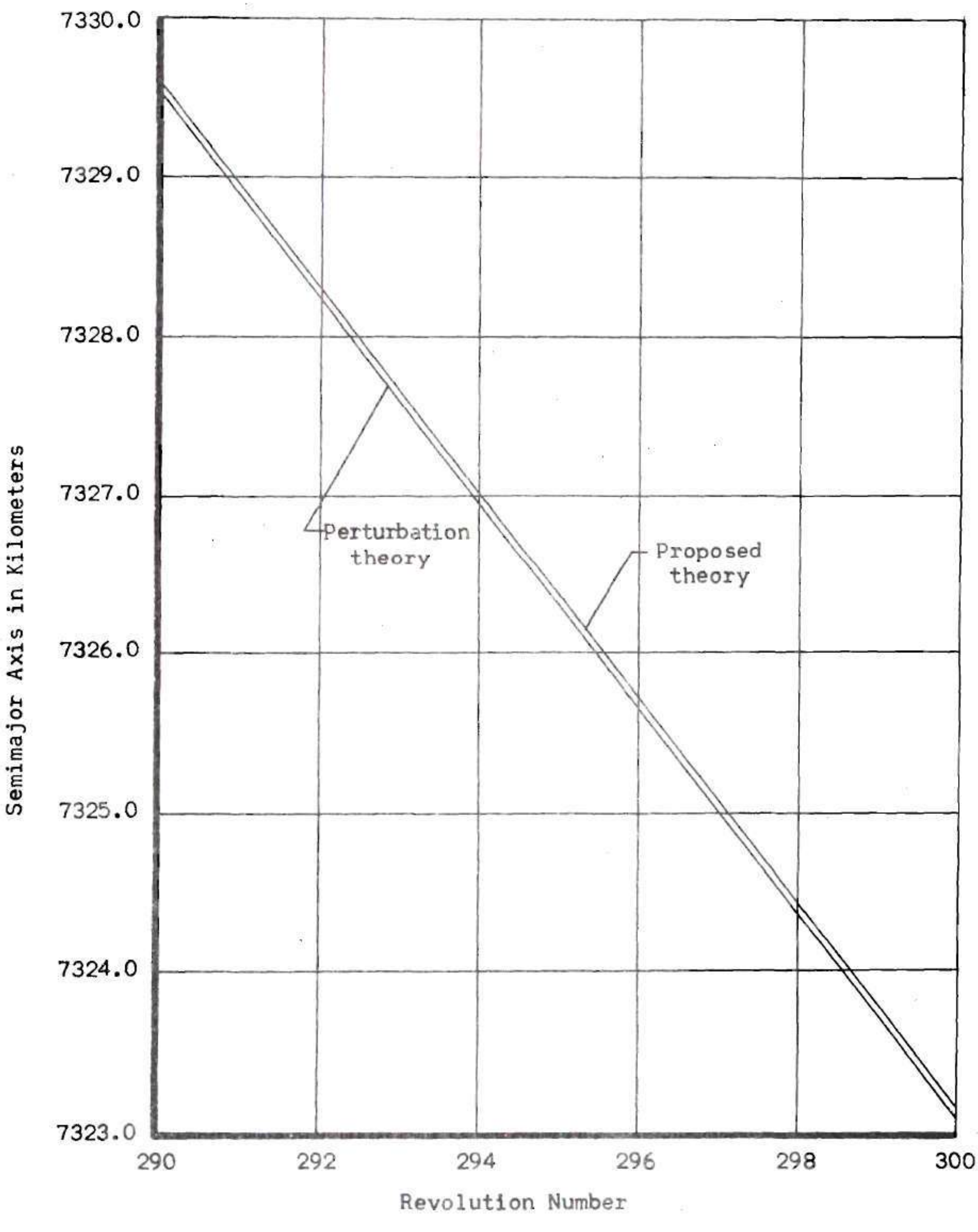


Figure 7. Semimajor Axis as Given by Perturbation Method and Proposed Theory after 290 Revolutions.

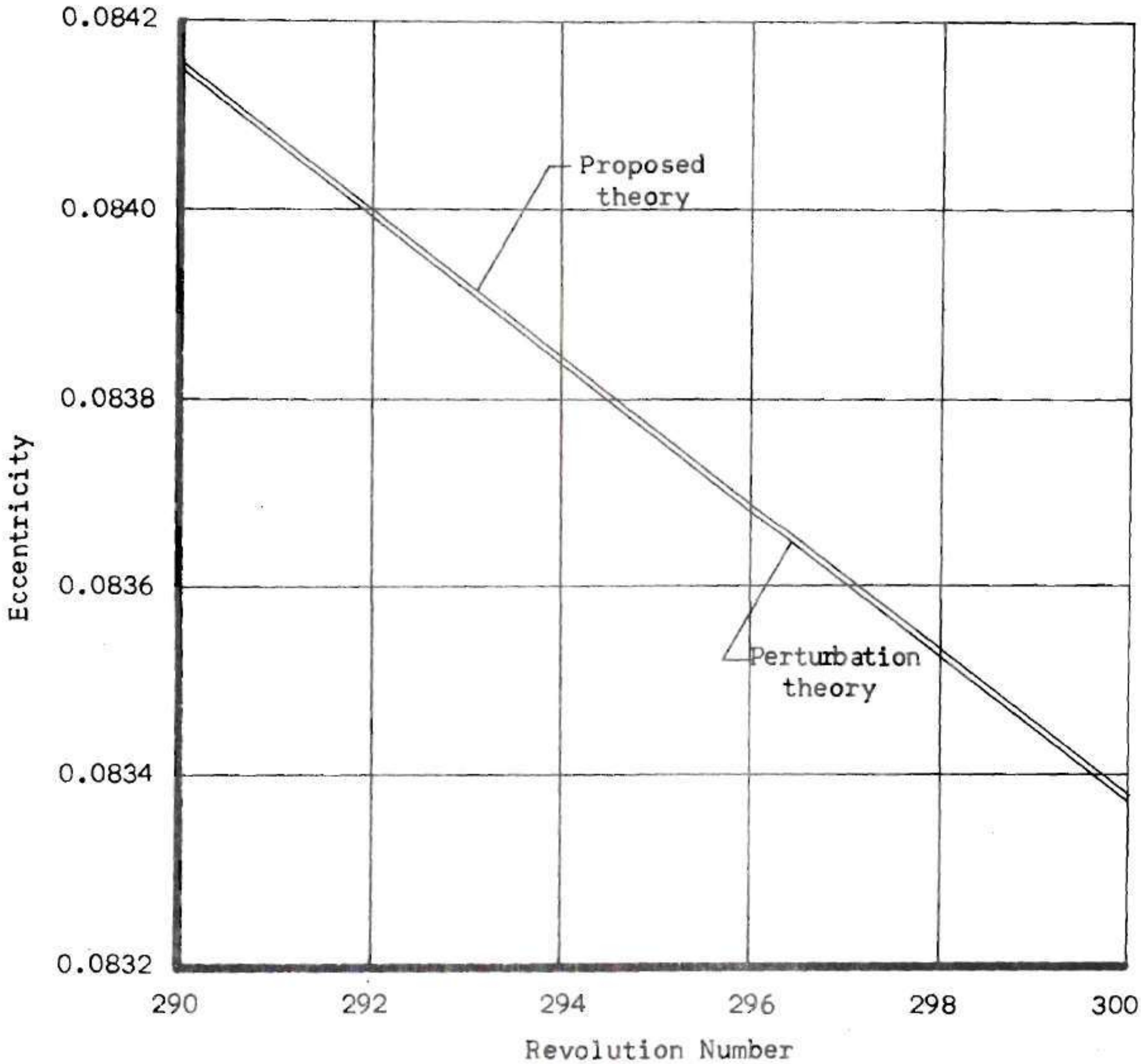


Figure 8. Eccentricity as Given by Perturbation Method and Proposed Theory after 290 Revolutions.

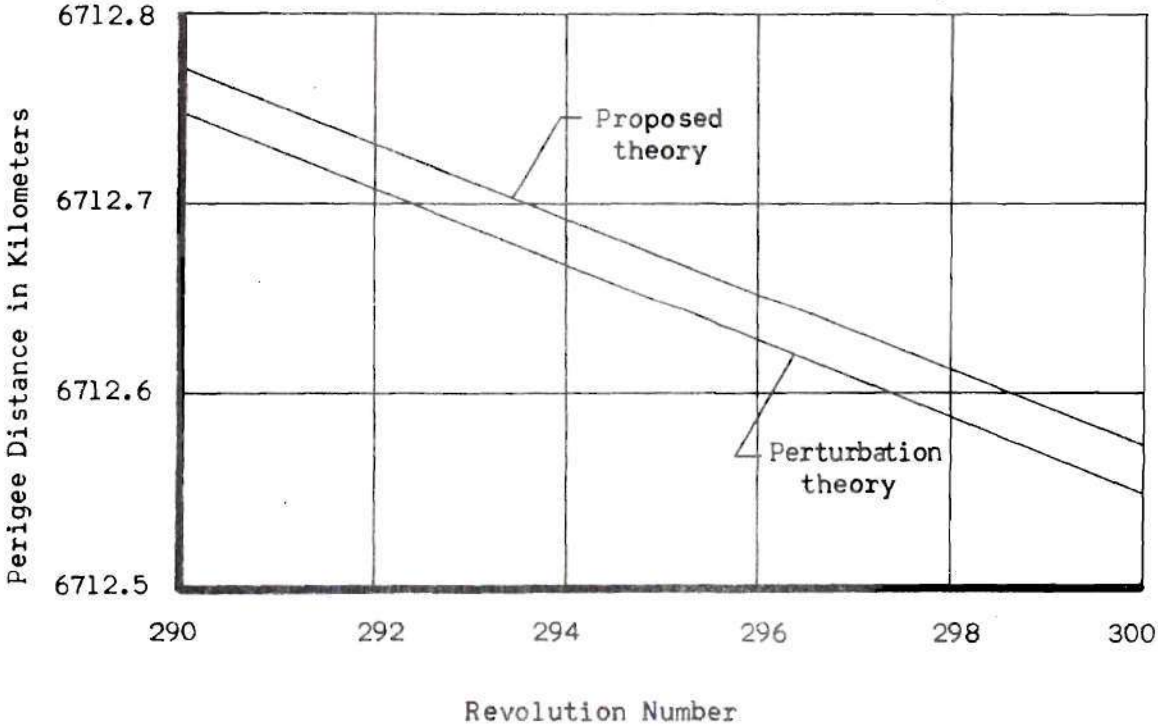


Figure 9. Perigee Distance as Given by Perturbation Method and Proposed Theory after 290 Revolutions.

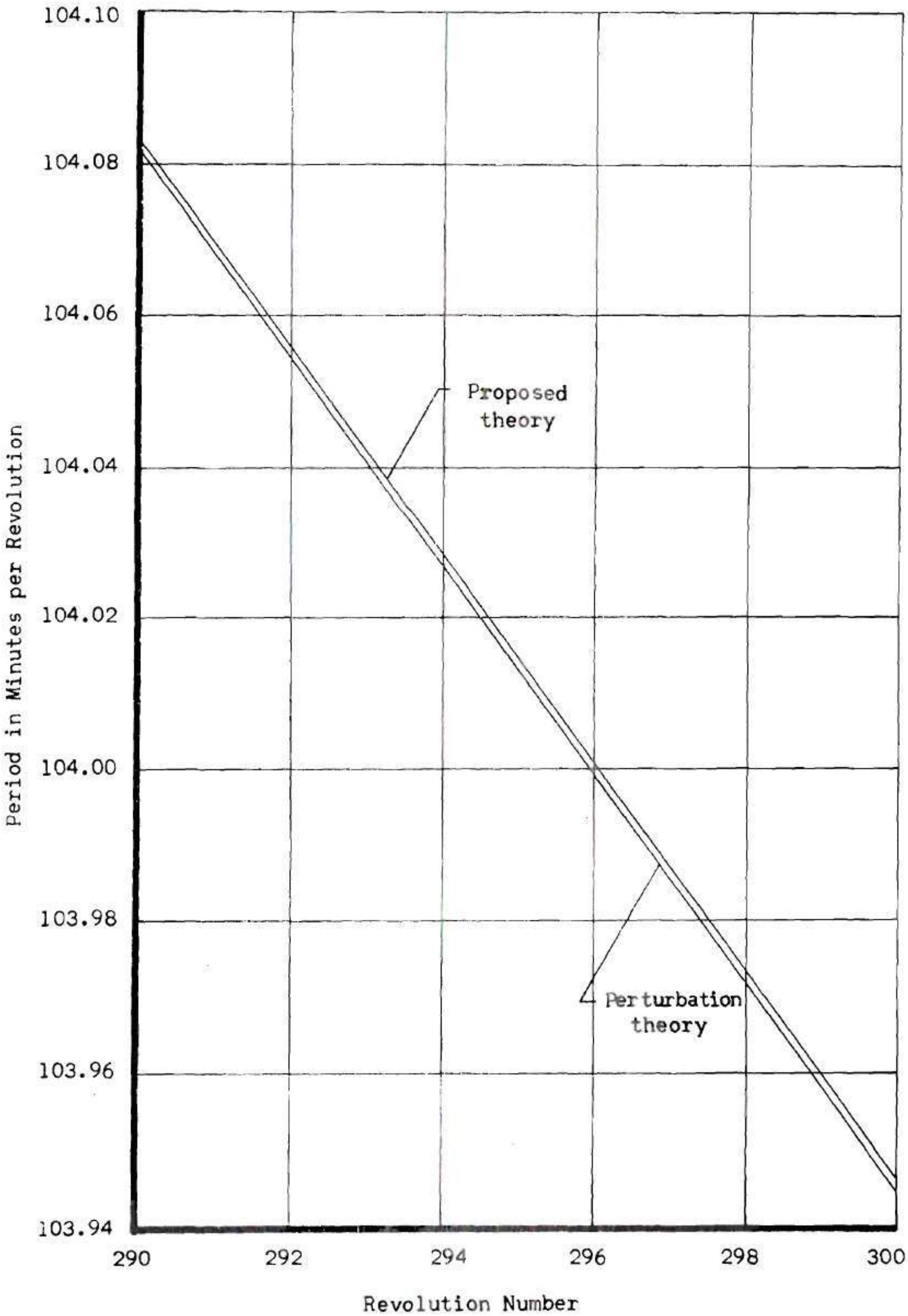


Figure 10. Period as Given by Perturbation Method and Proposed Theory after 290 Revolutions.

gravitational influence of the moon had large effects on the orbit, as the periodic change in eccentricity indicates (see Figure 11). Rather than calculating and subtracting the complex lunar and solar contributions to Δa and Δe, attention is confined to the last two months of orbit. Explorer IX's perigee was then in more dense atmosphere where drag effects predominated.

The two propositions for the value of K/m,

$$3.19 \times 10^{-9}\ km^2/g \text{ and}$$

$$3.49 \times 10^{-9}\ km^2/g\ ,$$

and the altitude-density profiles from the three sources,

USSA 1962 (Table 1),

G. M. Keating (Table 3), and

D. King-Hele (Table 4) ,

are all used in comparing the theory developed in the thesis with actual satellite data. The effect of the choice of K/m and density condition on the prediction of the semimajor axis and eccentricity are shown in Figure 12 and Figure 13, respectively.

The use of $K/m = 3.49 \times 10^{-9}\ km^2/g$ with Keating's density values is seen to be the best combination for application of the theory to Explorer IX's orbit. The results of applying the USSA 1962 model demonstrate the importance of a reasonable knowledge of the density situation at the time of orbit.

The success of the application of theory to satellite motion is affected by the method of numerical integration used to evaluate the integrals. The Lagrange process was chosen because of its dependability and availability as a prepared computer procedure, although other methods may be just as suitable. The 2π interval of integration was broken into 20 sub-intervals. More sub-intervals would lead to more accurate integration at the expense of computer time.

Considering the four complicating factors, particularly the constantly changing atmosphere, a and e as given by the proposed theory are of practical value in foretelling the behavior of a satellite in air drag. Accurate predictions for extended lengths of time are generally not necessary, except for estimations of lifetimes, because orbital positions are usually updated frequently from ground observations. For many cases the theory developed from the simplifying assumptions listed in Chapter I should prove useful as a first approach to the problem of drag-caused orbital decay.

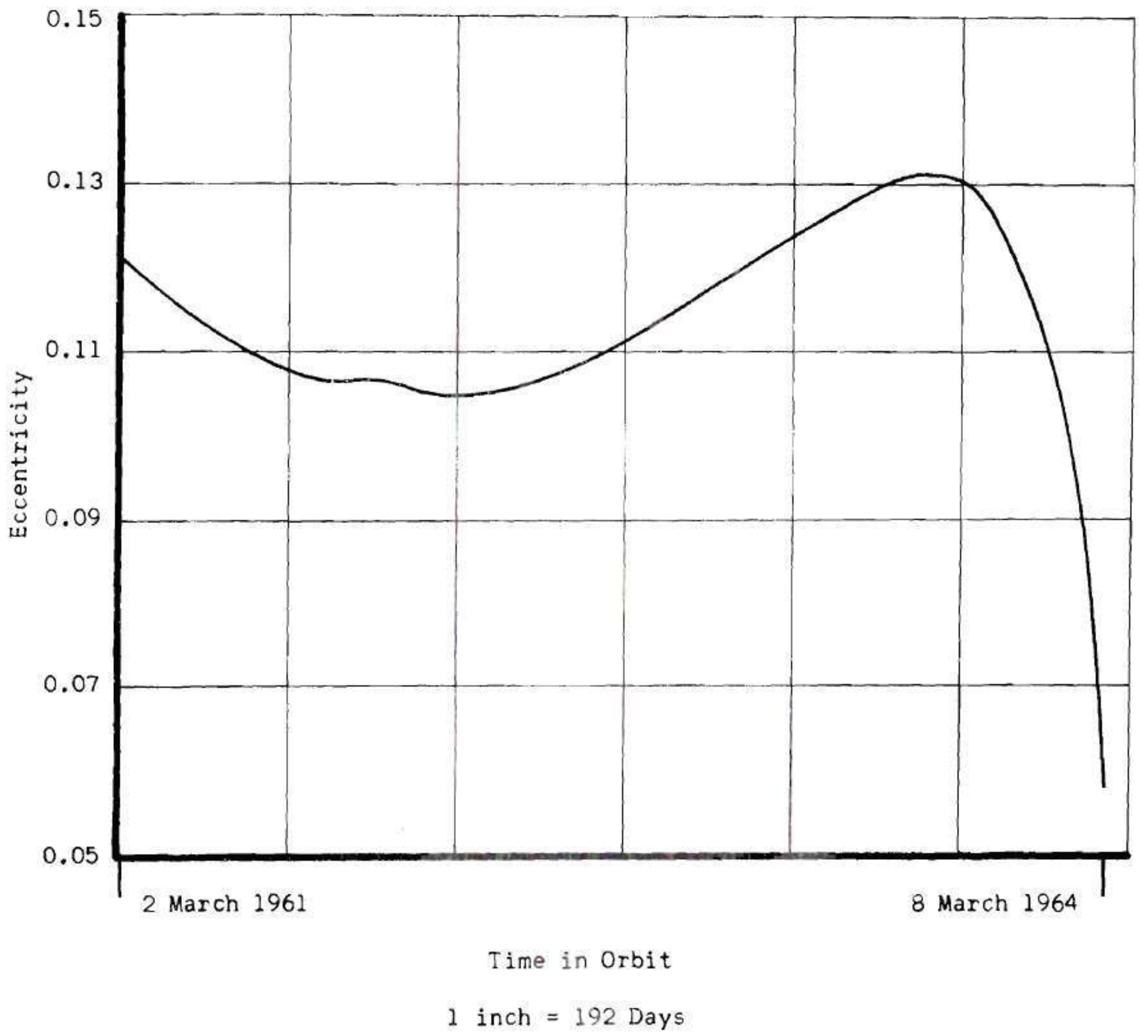


Figure 11. Eccentricity of the Orbit of Explorer IX Throughout Its Lifetime.

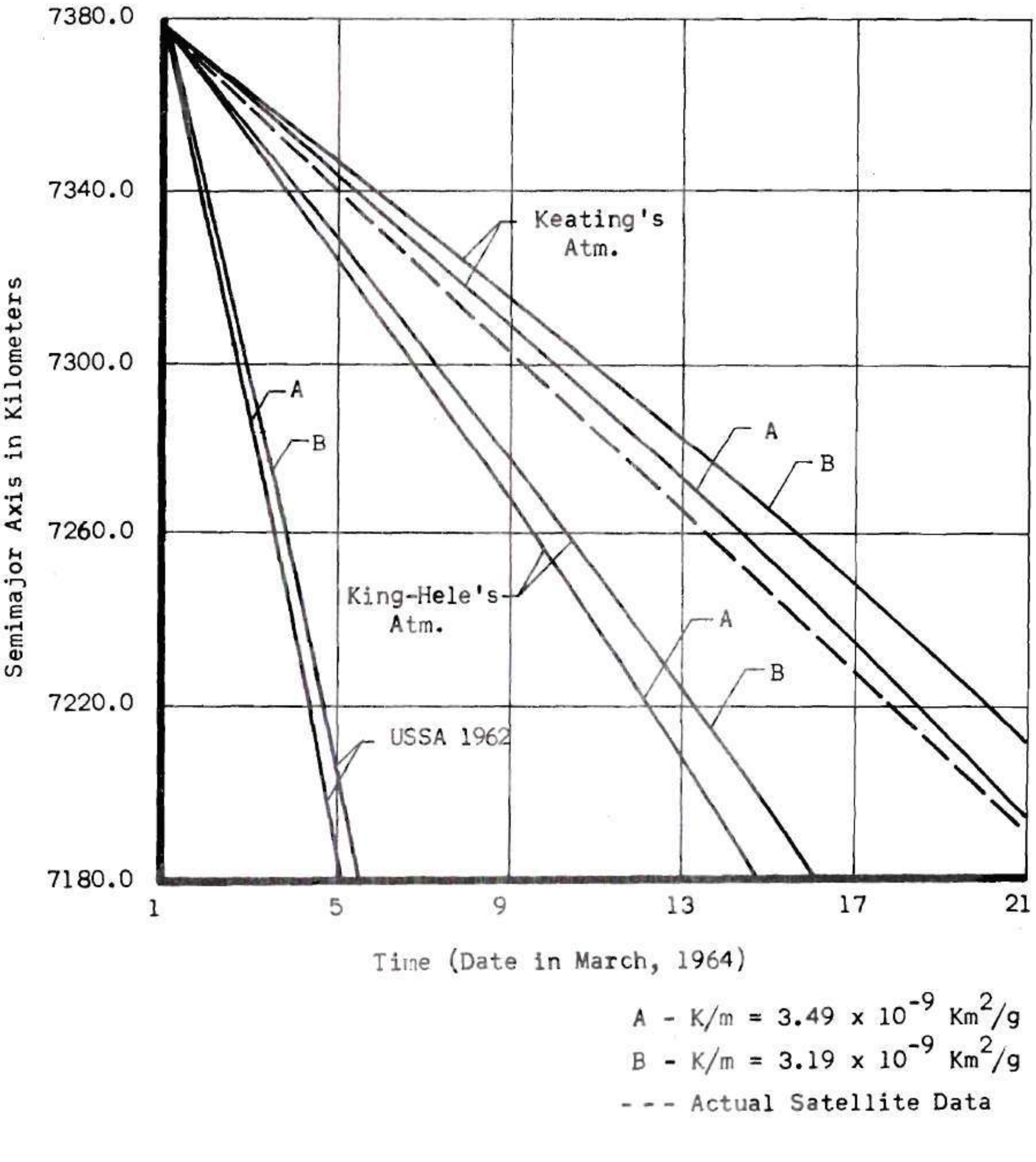
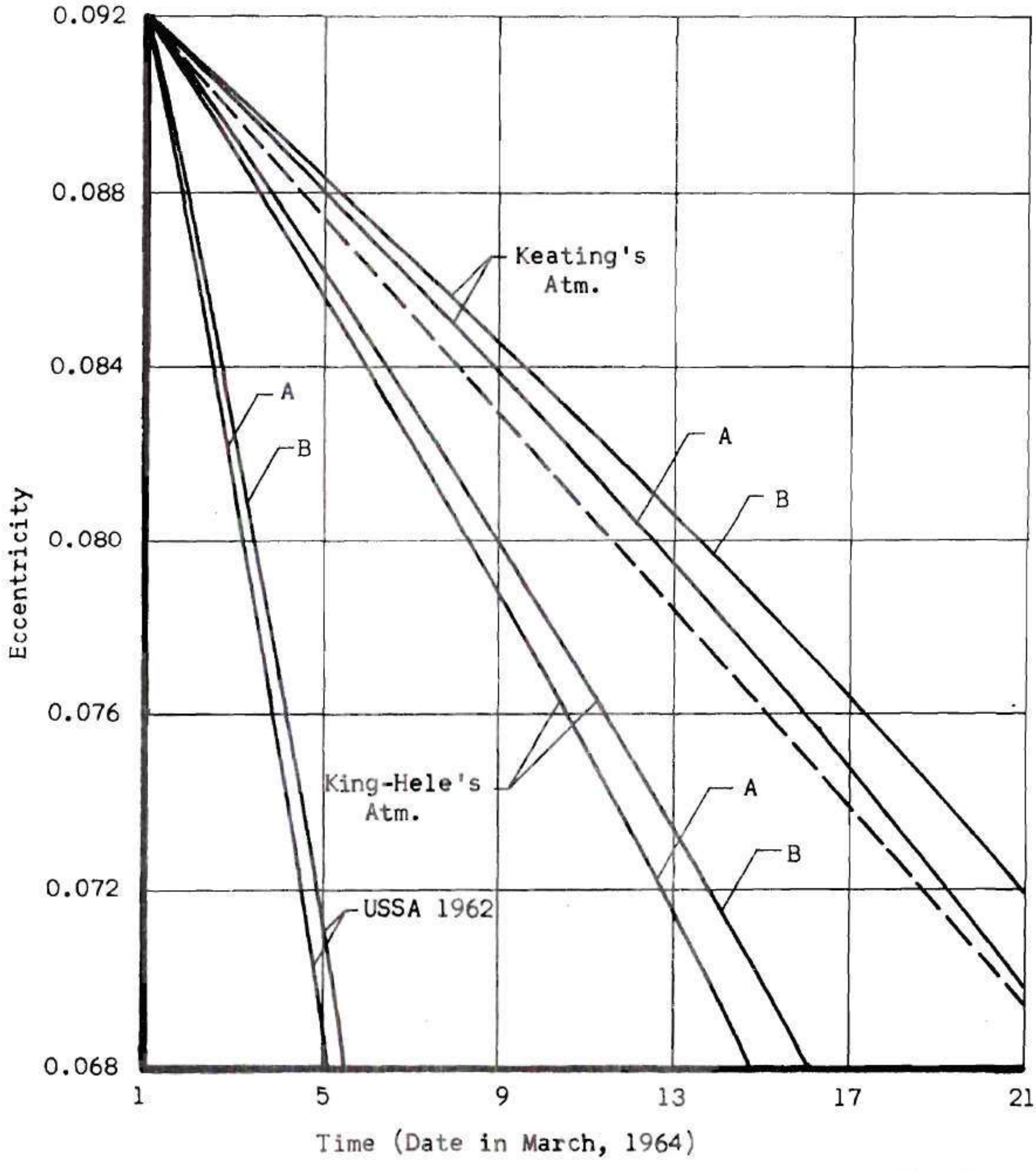


Figure 12. The Influence of K/m and the Density Profile on Theoretical Predictions of the Semimajor Axis.



A - $K/m = 3.49 \times 10^{-9}\ Km^2/g$
B - $K/m = 3.19 \times 10^{-9}\ Km^2/g$
- - - Actual Satellite Data

Figure 13. The Influence of $\frac{K}{m}$ and the Density Profile on Theoretical Predictions of Eccentricity.

APPENDIX

The Appendix contains tables of atmosphere densities from several sources and orbital elements of Explorer IX. Also there are computer programs and outputs comparing the perturbation theory and the theory presented in this thesis.

Table 1. Some Values of the USSA 1962 Atmosphere [2]

Altitude in km	Density in g/cc
100	4.97×10^{-10}
150	1.84×10^{-12}
200	3.32×10^{-13}
250	9.98×10^{-14}
300	3.58×10^{-14}
350	1.46×10^{-14}
400	6.50×10^{-15}
450	3.12×10^{-15}
500	1.58×10^{-15}
550	8.43×10^{-16}
600	4.64×10^{-16}
650	2.64×10^{-16}
700	1.54×10^{-16}

Table 2. Orbital Elements of Explorer IX [14]

Date (1964)	Semimajor Axis kilometers	Eccentricity
Jan. 3.0	7634.336	0.116690
11.0	7610.620	0.114210
17.0	7591.163	0.112310
25.0	7565.940	0.110030
Feb. 4.0	7532.842	0.107810
10.0	7505.084	0.104990
23.0	7433.590	0.098400
Mar. 1.0	7376.745	0.091926
8.0	7312.577	0.083690
14.0	7256.718	0.077040
22.0	7182.572	0.068716
29.0	7090.577	0.057570

Table 3. Densities Inferred from Explorer IX [14]

	Keating's Data		Conversion to Altitude and Density	
Date	r_p in km	$\log_{10}\rho$	h, in km	ρ in g/cc
Jan. 1.0	6743.694	-14.86	372.9	1.38×10^{-15} *
7.0	6742.463	-14.83	371.3	1.48×10^{-15}
14.0	6740.014	-14.80	368.8	1.58×10^{-15}
21.0	6736.044	-14.81	364.8	1.55×10^{-15} *
Feb. 1.0	6727.111	-14.79	355.9	1.62×10^{-15}
7.0	6718.945	-14.64	347.7	2.29×10^{-15} *
16.0	6709.743	-14.57	338.5	2.69×10^{-15} *
Mar. 2.5	6700.469	-14.40	329.3	3.98×10^{-15} *
4.5	6699.741	-14.35	328.5	4.47×10^{-15}
11.0	6699.217	-14.36	328.0	4.37×10^{-15} *
18.0	6693.492	-14.38	322.3	4.17×10^{-15} *
25.5	6685.950	-14.25	314.7	5.62×10^{-15} *

(The entries marked with an asterisk were used in the least squares calculation.)

Table 4. King-Hele's 1962-1964 Average Densities [17]

Altitude	Density in g/cc		
kilometers	1963-1964 day	1962-1964 night	numerical average
200	2.7×10^{-13}	1.8×10^{-13}	2.3×10^{-13}
300	1.5×10^{-14}	6.8×10^{-15}	1.1×10^{-14}
400	2.1×10^{-15}	7.6×10^{-16}	1.4×10^{-15}
500	3.4×10^{-16}	8.4×10^{-17}	2.1×10^{-16}

```
% PERTURBATION THEORY CALCULATIONS
BEGIN
$ CARD
$$ A A009                                                                00000000
                                                                         99999999
$$ A A066                                                                00000000
                                                                         99999999
$ CARD LIST
FILE IN  STAT (2,10) ; %
FILE OUT PRINT 6(3,15) ; %
REAL     AA,BB,CC,RE,X0,XP,XN,XQ,F,R,H,Z1,Z2,KM  ;
%
INTEGER  I,J,P,N ; %
LIST     LSTIN(P,N) ; %
WRITE   (PRINT[NO]) ;
READ    (STAT,/,LSTIN) ; %
CLOSE (STAT, RELEASE)  ;
BEGIN %
REAL ARRAY R1,A,E,T,DELA,DELE,COEFFA,COEFFE,DELALAG,DELELAG[0:P],
      ARGW,F4,RHO,F1,F2,F3,ARGA,ARGE[0:N] ; %
FORMAT   FMTOUT(//,X42,"NUMBER OF",X4 ,"SEMIMAJOR AXIS",X5 ,
              "ECCENTRICITY",X4,"PERIGEE DISTANCE",X6,"PERIOD") ;
FORMAT   FMTOUT1(X41,"REVOLUTIONS",X5 ,"KILOMETERS",X26,"KILOMETERS",
              X8,"MIN./REV.")  ;
FORMAT   FMTOUT2(/,X45,"RESULTS FROM PERTURBATION THEORY ",X1,
              " (K/M = 3.19@-9, DENSITY DATA FROM [17])",//)  ;
FORMAT   FMT(X45,I3,X10,F8.3,X10,F8.6,X10,F8.3,X10,F7.3)   ;
FORMAT   HEAD(X60,"CONSTANTS IN ALTITUDE-DENSITY EQUATION:              ");
FORMAT   FMTA(X70,"A =")  ;
FORMAT   FMTB(X70,"B =")  ;
FORMAT   FMTC(X70,"C =")  ;
FORMAT   FMAT(X75,F8.6) ;
FORMAT   FMBT(X75,F8.4) ;
FORMAT   FMCT(X75,F8.3) ;
LIST     LST(I-1,A[I],E[I],R1[I],T[I]) ;
WRITE    (PRINT[NO]) ;
BEGIN
```

```
% METHOD OF LEAST SQUARES
% ATMOSPHERIC DENSITY FUNCTION
%
INTEGER         I ; %
REAL ARRAY      X,H,A[0:15] ; %
X[1] ←  LN( 2.30@−13 ) ; %
X[2] ←  LN( 1.10@−14 ) ; %
X[3] ←  LN( 1.40@−15 ) ; %
X[4] ←  LN( 2.10@−16 ) ; %
H[1]  ←  200.0   ;
H[2]  ←  300.0   ;
H[3]  ←  400.0   ;
H[4]  ←  500.0   ;
          LSQPOLY(2, 4,X,H,A) ; %
FOR I ← 0 STEP 1 UNTIL 2 DO
A[I] ← A[I] ;
AA ←  A[2]  ;
BB  ←  A[1]  ;
CC  ←  A[0]  ;
END ;
A[1] ← 7505.084 ; %
E[1] ← 0.104990 ; %
KM  ←  3.19@−9 ;
R1[1] ← A[1] × ( 1 − E[1])  ;
RE ← 6371.200 ; %
XO ← XP ←  0 ; %
XN ← XQ ← 2 × 3.1415926536 ; %
H ← (XN − XO)/N ; %
%
   Z1       ← COEFFA[I] × LAGRANGE(N,3,1,XO,XN,XP,XQ,ARGA) ; %
   Z2       ← COEFFE[I] × LAGRANGE(N,3,1,XO,XN,XP,XQ,ARGE) ; %
FOR I ← 1 STEP 1 UNTIL P − 1 DO %
BEGIN %
%
FOR J ← 0 STEP 1 UNTIL N DO %
BEGIN %
F ←  J × 2 × 3.1415926536 /N ; %
```

```
R ← A[I] × (1 - E[I]*2)/(1 + E[I]× COS(F)) ; %
RHO[J] ← EXP( -(BB/(2×AA))- SQRT(((R-RE-CC)/AA) +((BB×BB)/(4×AA×AA))));
F1[J] ←  1 + 2 × E[I]×COS(F) + E[I]*2 ; %
F2[J] ← (1 + E[I]×COS(F))*2 ; %
F3[J] ← E[I] + COS(F) ; %
F4[J] ← SIN(F)  ;  %
ARGA[J] ← RHO[J] × ( SQRT(F1[J]))*3 / F2[J] ; %
ARGE[J] ← RHO[J] × SQRT(F1[J]) × F3[J]/F2[J] ; %
ARGW[J] ← RHO[J] × SQRT(F1[J]) × F4[J]/F2[J] ; %
END ;
COEFFA[I] ← -KM × 1@15 × A[I]*2  ;
COEFFE[I] ← -KM × 1@15 × A[I] × (1 - E[I]*2)  ;
%
DELALAG[I] ← COEFFA[I] × LAGRANGE(N,3,2,XO,XN,XP,XQ,ARGA) ; %
DELELAG[I] ← COEFFE[I] × LAGRANGE(N,3,2,XO,XN,XP,XQ,ARGE) ; %
%
DELA[I] ← DELALAG[I] ;
DELE[I] ← DELELAG[I] ;
%
A[I+1] ← A[I] + DELA[I] ; %
E[I+1] ← E[I] + DELE[I] ; %
R1[I+1]  ←  A[I+1]× ( 1 - E[I+1] )  ;
T[I] ← 2× 3.14159265 × SQRT( A[I]*3/3.986@5) /60  ;
T[P] ← 2× 3.14159265 × SQRT( A[P]*3/3.986@5) /60  ;
END ;
WRITE(PRINT,FMTOUT2) ;
WRITE(PRINT,HEAD) ;
WRITE(PRINT[NO],FMTA) ;   WRITE(PRINT,FMAT,AA)  ;
WRITE(PRINT[NO],FMTB) ;   WRITE(PRINT,FMBT,BB)  ;
WRITE(PRINT[NO],FMTC) ;   WRITE(PRINT,FMCT,CC)  ;
WRITE(PRINT,FMTOUT) ;
WRITE(PRINT,FMTOUT1) ;
FOR I ← 1 STEP 1 UNTIL P DO %
WRITE(PRINT,FMT,LST) ; %
```

RESULTS FROM PERTURBATION THEORY (K/M = 3.19@-9, DENSITY DATA FROM [17])

CONSTANTS IN ALTITUDE-DENSITY EQUATION:
A = 2.326179
B = 108.5507
C = 1388.400

NUMBER OF REVOLUTIONS	SEMIMAJOR AXIS KILOMETERS	ECCENTRICITY	PERIGEE DISTANCE KILOMETERS	PERIOD MIN./REV.
0	7505.084	0.104990	6717.125	107.843
1	7504.506	0.104923	6717.113	107.831
2	7503.928	0.104855	6717.102	107.818
3	7503.350	0.104788	6717.090	107.806
4	7502.771	0.104720	6717.078	107.793
5	7502.193	0.104653	6717.066	107.781
6	7501.614	0.104585	6717.054	107.769
7	7501.035	0.104518	6717.042	107.756
8	7500.456	0.104450	6717.030	107.744
9	7499.877	0.104383	6717.019	107.731
10	7499.298	0.104315	6717.007	107.719
11	7498.718	0.104248	6716.995	107.706
12	7498.139	0.104180	6716.983	107.694
13	7497.559	0.104112	6716.971	107.681
14	7496.979	0.104045	6716.959	107.669
15	7496.399	0.103977	6716.947	107.656
16	7495.819	0.103909	6716.935	107.644
17	7495.239	0.103841	6716.923	107.631
18	7494.658	0.103774	6716.910	107.619
19	7494.077	0.103706	6716.898	107.606
20	7493.497	0.103638	6716.886	107.594
21	7492.916	0.103570	6716.874	107.581
22	7492.335	0.103502	6716.862	107.569
23	7491.753	0.103434	6716.850	107.556
24	7491.172	0.103366	6716.838	107.544

25	7490.590	0.103298	6716.825	107.531
26	7490.009	0.103230	6716.813	107.519
27	7489.427	0.103162	6716.801	107.506
28	7488.845	0.103094	6716.789	107.494
29	7488.263	0.103026	6716.777	107.481
30	7487.680	0.102958	6716.764	107.468
31	7487.098	0.102890	6716.752	107.456
32	7486.515	0.102822	6716.740	107.443
33	7485.932	0.102753	6716.727	107.431
34	7485.349	0.102685	6716.715	107.418
35	7484.766	0.102617	6716.703	107.406
36	7484.183	0.102549	6716.690	107.393
37	7483.600	0.102480	6716.678	107.381
38	7483.016	0.102412	6716.665	107.368
39	7482.432	0.102344	6716.653	107.355
40	7481.848	0.102275	6716.641	107.343
41	7481.264	0.102207	6716.628	107.330
42	7480.680	0.102138	6716.616	107.318
43	7480.096	0.102070	6716.603	107.305
44	7479.511	0.102001	6716.591	107.293
45	7478.926	0.101933	6716.578	107.280
46	7478.341	0.101864	6716.565	107.267
47	7477.756	0.101796	6716.553	107.255
48	7477.171	0.101727	6716.540	107.242
49	7476.586	0.101658	6716.528	107.230
50	7476.000	0.101590	6716.515	107.217
51	7475.415	0.101521	6716.502	107.204
52	7474.829	0.101452	6716.490	107.192
53	7474.243	0.101384	6716.477	107.179
54	7473.657	0.101315	6716.464	107.167
55	7473.070	0.101246	6716.451	107.154
56	7472.484	0.101177	6716.439	107.141
57	7471.897	0.101108	6716.426	107.129
58	7471.310	0.101039	6716.413	107.116
59	7470.723	0.100971	6716.400	107.104
60	7470.136	0.100902	6716.388	107.091
61	7469.549	0.100833	6716.375	107.078

62	7468.961	0.100764	6716.362	107.066
63	7468.373	0.100695	6716.349	107.053
64	7467.786	0.100625	6716.336	107.040
65	7467.198	0.100556	6716.323	107.028
66	7466.609	0.100487	6716.310	107.015
67	7466.021	0.100418	6716.297	107.002
68	7465.432	0.100349	6716.284	106.990
69	7464.844	0.100280	6716.271	106.977
70	7464.255	0.100211	6716.258	106.964
71	7463.666	0.100141	6716.245	106.952
72	7463.077	0.100072	6716.232	106.939
73	7462.487	0.100003	6716.219	106.927
74	7461.898	0.099933	6716.206	106.914
75	7461.308	0.099864	6716.193	106.901
76	7460.718	0.099794	6716.180	106.888
77	7460.128	0.099725	6716.166	106.876
78	7459.538	0.099656	6716.153	106.863
79	7458.947	0.099586	6716.140	106.850
80	7458.357	0.099517	6716.127	106.838
81	7457.766	0.099447	6716.113	106.825
82	7457.175	0.099377	6716.100	106.812
83	7456.584	0.099308	6716.087	106.800
84	7455.992	0.099238	6716.074	106.787
85	7455.401	0.099168	6716.060	106.774
86	7454.809	0.099099	6716.047	106.762
87	7454.217	0.099029	6716.033	106.749
88	7453.625	0.098959	6716.020	106.736
89	7453.033	0.098889	6716.007	106.723
90	7452.441	0.098820	6715.993	106.711
91	7451.848	0.098750	6715.980	106.698
92	7451.255	0.098680	6715.966	106.685
93	7450.662	0.098610	6715.953	106.672
94	7450.069	0.098540	6715.939	106.660
95	7449.476	0.098470	6715.926	106.647
96	7448.882	0.098400	6715.912	106.634
97	7448.289	0.098330	6715.899	106.621
98	7447.695	0.098260	6715.885	106.609

99	7447.101	0.098190	6715.871	106.596
100	7446.506	0.098120	6715.858	106.583
101	7445.912	0.098050	6715.844	106.570
102	7445.317	0.097979	6715.830	106.558
103	7444.723	0.097909	6715.816	106.545
104	7444.128	0.097839	6715.803	106.532
105	7443.532	0.097769	6715.789	106.519
106	7442.937	0.097698	6715.775	106.507
107	7442.341	0.097628	6715.761	106.494
108	7441.746	0.097558	6715.747	106.481
109	7441.150	0.097487	6715.734	106.468
110	7440.554	0.097417	6715.720	106.455
111	7439.957	0.097346	6715.706	106.443
112	7439.361	0.097276	6715.692	106.430
113	7438.764	0.097205	6715.678	106.417
114	7438.167	0.097135	6715.664	106.404
115	7437.570	0.097064	6715.650	106.391
116	7436.973	0.096993	6715.636	106.379
117	7436.375	0.096923	6715.622	106.366
118	7435.778	0.096852	6715.608	106.353
119	7435.180	0.096781	6715.594	106.340
120	7434.582	0.096711	6715.580	106.327
121	7433.983	0.096640	6715.565	106.314
122	7433.385	0.096569	6715.551	106.302
123	7432.786	0.096498	6715.537	106.289
124	7432.187	0.096427	6715.523	106.276
125	7431.588	0.096356	6715.509	106.263
126	7430.989	0.096285	6715.494	106.250
127	7430.390	0.096214	6715.480	106.237
128	7429.790	0.096143	6715.466	106.225
129	7429.190	0.096072	6715.451	106.212
130	7428.590	0.096001	6715.437	106.199
131	7427.990	0.095930	6715.423	106.186
132	7427.389	0.095859	6715.408	106.173
133	7426.789	0.095788	6715.394	106.160
134	7426.188	0.095716	6715.379	106.147
135	7425.587	0.095645	6715.365	106.134

136	7424.986	0.095574	6715.350	106.122
137	7424.384	0.095503	6715.336	106.109
138	7423.782	0.095431	6715.321	106.096
139	7423.180	0.095360	6715.307	106.083
140	7422.578	0.095289	6715.292	106.070
141	7421.976	0.095217	6715.277	106.057
142	7421.374	0.095146	6715.263	106.044
143	7420.771	0.095074	6715.248	106.031
144	7420.168	0.095003	6715.233	106.018
145	7419.565	0.094931	6715.219	106.005
146	7418.961	0.094859	6715.204	105.992
147	7418.358	0.094788	6715.189	105.979
148	7417.754	0.094716	6715.174	105.967
149	7417.150	0.094644	6715.159	105.954
150	7416.546	0.094573	6715.144	105.941
151	7415.941	0.094501	6715.130	105.928
152	7415.337	0.094429	6715.115	105.915
153	7414.732	0.094357	6715.100	105.902
154	7414.127	0.094285	6715.085	105.889
155	7413.522	0.094213	6715.070	105.876
156	7412.916	0.094141	6715.055	105.863
157	7412.310	0.094069	6715.040	105.850
158	7411.704	0.093997	6715.024	105.837
159	7411.098	0.093925	6715.009	105.824
160	7410.492	0.093853	6714.994	105.811
161	7409.885	0.093781	6714.979	105.798
162	7409.279	0.093709	6714.964	105.785
163	7408.671	0.093637	6714.949	105.772
164	7408.064	0.093564	6714.933	105.759
165	7407.457	0.093492	6714.918	105.746
166	7406.849	0.093420	6714.903	105.733
167	7406.241	0.093347	6714.887	105.720
168	7405.633	0.093275	6714.872	105.707
169	7405.025	0.093203	6714.857	105.694
170	7404.416	0.093130	6714.841	105.681
171	7403.807	0.093058	6714.826	105.668
172	7403.198	0.092985	6714.810	105.655

173	7402.589	0.092913	6714.795	105.642
174	7401.979	0.092840	6714.779	105.629
175	7401.370	0.092767	6714.764	105.616
176	7400.760	0.092695	6714.748	105.603
177	7400.149	0.092622	6714.732	105.589
178	7399.539	0.092549	6714.717	105.576
179	7398.928	0.092477	6714.701	105.563
180	7398.317	0.092404	6714.685	105.550
181	7397.706	0.092331	6714.670	105.537
182	7397.095	0.092258	6714.654	105.524
183	7396.483	0.092185	6714.638	105.511
184	7395.871	0.092112	6714.622	105.498
185	7395.259	0.092039	6714.606	105.485
186	7394.647	0.091966	6714.590	105.472
187	7394.035	0.091893	6714.575	105.459
188	7393.422	0.091820	6714.559	105.446
189	7392.809	0.091747	6714.543	105.432
190	7392.195	0.091674	6714.527	105.419
191	7391.582	0.091600	6714.511	105.406
192	7390.968	0.091527	6714.494	105.393
193	7390.354	0.091454	6714.478	105.380
194	7389.740	0.091380	6714.462	105.367
195	7389.126	0.091307	6714.446	105.354
196	7388.511	0.091234	6714.430	105.340
197	7387.896	0.091160	6714.414	105.327
198	7387.281	0.091087	6714.397	105.314
199	7386.665	0.091013	6714.381	105.301
200	7386.049	0.090940	6714.365	105.288
201	7385.434	0.090866	6714.348	105.275
202	7384.817	0.090792	6714.332	105.262
203	7384.201	0.090719	6714.316	105.248
204	7383.584	0.090645	6714.299	105.235
205	7382.967	0.090571	6714.283	105.222
206	7382.350	0.090497	6714.266	105.209
207	7381.733	0.090424	6714.250	105.196
208	7381.115	0.090350	6714.233	105.182
209	7380.497	0.090276	6714.216	105.169

210	7379.879	0.090202	6714.200	105.156
211	7379.260	0.090128	6714.183	105.143
212	7378.641	0.090054	6714.166	105.129
213	7378.022	0.089980	6714.150	105.116
214	7377.403	0.089906	6714.133	105.103
215	7376.784	0.089832	6714.116	105.090
216	7376.164	0.089757	6714.099	105.077
217	7375.544	0.089683	6714.082	105.063
218	7374.924	0.089609	6714.065	105.050
219	7374.303	0.089535	6714.048	105.037
220	7373.682	0.089460	6714.031	105.024
221	7373.061	0.089386	6714.014	105.010
222	7372.440	0.089311	6713.997	104.997
223	7371.818	0.089237	6713.980	104.984
224	7371.196	0.089162	6713.963	104.970
225	7370.574	0.089088	6713.946	104.957
226	7369.952	0.089013	6713.929	104.944
227	7369.329	0.088939	6713.911	104.931
228	7368.706	0.088864	6713.894	104.917
229	7368.083	0.088789	6713.877	104.904
230	7367.459	0.088714	6713.860	104.891
231	7366.836	0.088640	6713.842	104.877
232	7366.212	0.088565	6713.825	104.864
233	7365.587	0.088490	6713.807	104.851
234	7364.963	0.088415	6713.790	104.837
235	7364.338	0.088340	6713.772	104.824
236	7363.713	0.088265	6713.755	104.811
237	7363.087	0.088190	6713.737	104.797
238	7362.462	0.088115	6713.720	104.784
239	7361.836	0.088040	6713.702	104.771
240	7361.209	0.087965	6713.684	104.757
241	7360.583	0.087889	6713.666	104.744
242	7359.956	0.087814	6713.649	104.730
243	7359.329	0.087739	6713.631	104.717
244	7358.702	0.087663	6713.613	104.704
245	7358.074	0.087588	6713.595	104.690
246	7357.446	0.087513	6713.577	104.677

247	7356.818	0.087437	6713.559	104.663
248	7356.189	0.087362	6713.541	104.650
249	7355.560	0.087286	6713.523	104.637
250	7354.931	0.087210	6713.505	104.623
251	7354.302	0.087135	6713.487	104.610
252	7353.672	0.087059	6713.469	104.596
253	7353.042	0.086983	6713.451	104.583
254	7352.412	0.086907	6713.432	104.569
255	7351.781	0.086832	6713.414	104.556
256	7351.150	0.086756	6713.396	104.543
257	7350.519	0.086680	6713.378	104.529
258	7349.888	0.086604	6713.359	104.516
259	7349.256	0.086528	6713.341	104.502
260	7348.624	0.086452	6713.322	104.489
261	7347.991	0.086376	6713.304	104.475
262	7347.359	0.086300	6713.285	104.462
263	7346.726	0.086223	6713.267	104.448
264	7346.092	0.086147	6713.248	104.435
265	7345.459	0.086071	6713.229	104.421
266	7344.825	0.085994	6713.211	104.408
267	7344.191	0.085918	6713.192	104.394
268	7343.556	0.085842	6713.173	104.381
269	7342.921	0.085765	6713.154	104.367
270	7342.286	0.085689	6713.135	104.353
271	7341.651	0.085612	6713.116	104.340
272	7341.015	0.085536	6713.097	104.326
273	7340.379	0.085459	6713.078	104.313
274	7339.743	0.085382	6713.059	104.299
275	7339.106	0.085305	6713.040	104.286
276	7338.469	0.085229	6713.021	104.272
277	7337.832	0.085152	6713.002	104.259
278	7337.194	0.085075	6712.983	104.245
279	7336.556	0.084998	6712.964	104.231
280	7335.918	0.084921	6712.944	104.218
281	7335.279	0.084844	6712.925	104.204
282	7334.640	0.084767	6712.906	104.191
283	7334.001	0.084690	6712.886	104.177

284	7333.361	0.084613	6712.867	104.163
285	7332.721	0.084535	6712.847	104.150
286	7332.081	0.084458	6712.828	104.136
287	7331.440	0.084381	6712.808	104.122
288	7330.800	0.084303	6712.788	104.109
289	7330.158	0.084226	6712.769	104.095
290	7329.517	0.084148	6712.749	104.081
291	7328.875	0.084071	6712.729	104.068
292	7328.233	0.083993	6712.709	104.054
293	7327.590	0.083916	6712.689	104.040
294	7326.947	0.083838	6712.669	104.027
295	7326.304	0.083760	6712.650	104.013
296	7325.660	0.083683	6712.629	103.999
297	7325.016	0.083605	6712.609	103.986
298	7324.372	0.083527	6712.589	103.972
299	7323.727	0.083449	6712.569	103.958
300	7323.082	0.083371	6712.549	103.944

```
%  PROPOSED THEORY CALCULATIONS
BEGIN
$ CARD
$$ A A009                                                                        00000000
                                                                                 99999999
$$ A A019                                                                        00000000
               Y1  ←  SQRT(ABS(XH))  ;                                           00004500
                                                                                 99999999
$ CARD LIST
FILE IN MECH (2,10) ; %
FILE OUT PRINT 6(3,15) ; %
REAL     AA,BB,CC,F,F1,F2,F3,B,H,R,RHO,RE, KM ,X0,XP,XN,XQ,Z1,Z2       ;
INTEGER  I,J,M,N ; %
LIST     LSTIN(M,N) ; %
WRITE(PRINT[NO]) ;
READ     (MECH,/,LSTIN) ;
CLOSE(MECH,RELEASE) ;
BEGIN
REAL ARRAY R1,A,E,DELELAG,DELE,DELHLAG,DELH,T[0:M+1],
         ARGH,ARG[0:N] ;
FORMAT   FMTOUT(//,X42,"NUMBER OF",X4 ,"SEMIMAJOR AXIS",X5 ,
              "ECCENTRICITY",X4,"PERIGEE DISTANCE",X6,"PERIOD") ;
FORMAT   FMTOUT1(X41,"REVOLUTIONS",X5 ,"KILOMETERS",X26,"KILOMETERS",
              X8,"MIN./REV.") ;
FORMAT   FMTOUT2(/,X45,"RESULTS FROM PROPOSED THEORY ",X1,
              " (K/M = 3.19@-9, DENSITY DATA FROM [17])",//) ;
FORMAT   FMT(X45,I3,X10,F8.3,X10,F8.6,X10,F8.3,X10,F7.3)  ;
FORMAT   HEAD(X60,"CONSTANTS IN ALTITUDE-DENSITY EQUATION:          ");
FORMAT   FMTA(X70,"A =")  ;
FORMAT   FMTB(X70,"B =")  ;
FORMAT   FMTC(X70,"C =")  ;
FORMAT   FMAT(X75,F8.6) ;
FORMAT   FMBT(X75,F8.4) ;
FORMAT   FMCT(X75,F8.3) ;
LIST     LST(I-1,A[I],E[I],R1[I],T[I]) ;
AA ← 2.326179   ;
BB ← 108.5507   ;
```

```
CC ← 1388.3999    ;
A[1] ← 7505.084 ; %
E[1] ← 0.104990 ; %
KM  ←  3.19@-9 ;
R1[1] ← A[1] × ( 1 - E[1])   ;
RE ← 6371.2 ; %
XO ← XP ← 0 ;
XN ← XQ ← 2 × 3.1415926536 ;
H ← (XN - XO)/N ;
   Z1        ← LAGRANGE(N,3,1,XO,XN,XP,XQ,ARG) ; %
   Z2        ← LAGRANGE(N,3,1,XO,XN,XP,XQ,ARGH)   ;
FOR I ← 1 STEP 1 UNTIL M DO
BEGIN
FOR J ← 0 STEP 1 UNTIL N DO
BEGIN
F ← J × 2 × 3.1415926536/N ;
R ← A[I] × (1 - E[I]*2)/(1 + E[I]×COS(F)) ;
RHO ← EXP( -BB/(2×AA) - SQRT((R - RE - CC)/AA + BB*2/(4×AA*2))) ;     28
F1 ←  (SQRT(1 + 2×E[I]×COS(F) + E[I]*2))*3 ; %
F2 ← (1 + E[I]×COS(F))*2 ;
F3 ←  SQRT(1 + 2×E[I]×COS(F) + E[I]*2)   ;
%
ARG[J] ← RHO×F1/F2 ; %
ARGH[J] ← RHO × F3/F2    ;
END ; %
DELELAG[I] ← LAGRANGE(N,3,2,XO,XN,XP,XQ,ARG) ; %
DELHLAG[I] ← LAGRANGE(N,3,2,XO,XN,XP,XQ,ARGH)   ;
B ← SQRT(A[I] × (1 - (E[I]*2)))    ;
DELE[I] ← -KM × (1@15) × DELELAG[I]   ;
DELH[I] ← -0.5×KM × B*3× DELHLAG[I]  ×@15 ;
A[I+1] ← 1/(1/A[I] - DELE[I])   ;
E[I+1]          ←          SQRT(1 - (1/A[I+1]) × ((B + DELH[I])*2))    ;
R1[I+1]  ←  A[I+1]× ( 1 - E[I+1] )   ;
T[I] ← 2× 3.14159265 × SQRT( A[I]*3/3.986@5) /60   ;
END ;
WRITE(PRINT,FMTOUT2) ;
WRITE(PRINT,HEAD) ;
```

```
WRITE(PRINT[NO],FMTA) ;    WRITE(PRINT,FMAT,AA)   ;
WRITE(PRINT[NO],FMTB) ;    WRITE(PRINT,FMBT,BB)   ;
WRITE(PRINT[NO],FMTC) ;    WRITE(PRINT,FMCT,CC)   ;
WRITE(PRINT,FMTOUT) ;
WRITE(PRINT,FMTOUT1) ;
FOR I ← 1 STEP 1 UNTIL M DO %
WRITE(PRINT,FMT,LST) ; %
END ;
END.
```

RESULTS FROM PROPOSED THEORY (K/M = 3.19e-9, DENSITY DATA FROM [17])

CONSTANTS IN ALTITUDE-DENSITY EQUATION:

A = 2.326179

B = 108.5507

C = 1388.400

NUMBER OF REVOLUTIONS	SEMIMAJOR AXIS KILOMETERS	ECCENTRICITY	PERIGEE DISTANCE KILOMETERS	PERIOD MIN./REV.
0	7505.084	0.104990	6717.125	107.843
1	7504.506	0.104923	6717.114	107.831
2	7503.928	0.104855	6717.102	107.818
3	7503.350	0.104788	6717.090	107.806
4	7502.771	0.104720	6717.078	107.793
5	7502.193	0.104653	6717.066	107.781
6	7501.614	0.104585	6717.055	107.769
7	7501.036	0.104518	6717.043	107.756
8	7500.457	0.104450	6717.031	107.744
9	7499.878	0.104383	6717.019	107.731
10	7499.298	0.104315	6717.007	107.719
11	7498.719	0.104248	6716.995	107.706
12	7498.139	0.104180	6716.983	107.694
13	7497.560	0.104112	6716.972	107.681
14	7496.980	0.104045	6716.960	107.669
15	7496.400	0.103977	6716.948	107.656
16	7495.820	0.103909	6716.936	107.644
17	7495.240	0.103841	6716.924	107.631
18	7494.659	0.103774	6716.912	107.619
19	7494.079	0.103706	6716.900	107.606
20	7493.498	0.103638	6716.888	107.594
21	7492.917	0.103570	6716.876	107.581
22	7492.336	0.103502	6716.863	107.569
23	7491.755	0.103434	6716.851	107.556
24	7491.173	0.103366	6716.839	107.544

25	7490.592	0.103298	6716.827	107.531
26	7490.010	0.103230	6716.815	107.519
27	7489.429	0.103162	6716.803	107.506
28	7488.847	0.103094	6716.791	107.494
29	7488.265	0.103026	6716.779	107.481
30	7487.682	0.102958	6716.766	107.468
31	7487.100	0.102890	6716.754	107.456
32	7486.517	0.102822	6716.742	107.443
33	7485.935	0.102753	6716.730	107.431
34	7485.352	0.102685	6716.717	107.418
35	7484.769	0.102617	6716.705	107.406
36	7484.186	0.102549	6716.693	107.393
37	7483.602	0.102480	6716.680	107.381
38	7483.019	0.102412	6716.668	107.368
39	7482.435	0.102344	6716.656	107.356
40	7481.851	0.102275	6716.643	107.343
41	7481.267	0.102207	6716.631	107.330
42	7480.683	0.102138	6716.618	107.318
43	7480.099	0.102070	6716.606	107.305
44	7479.514	0.102001	6716.594	107.293
45	7478.930	0.101933	6716.581	107.280
46	7478.345	0.101864	6716.569	107.268
47	7477.760	0.101796	6716.556	107.255
48	7477.175	0.101727	6716.543	107.242
49	7476.590	0.101658	6716.531	107.230
50	7476.004	0.101590	6716.518	107.217
51	7475.419	0.101521	6716.506	107.205
52	7474.833	0.101452	6716.493	107.192
53	7474.247	0.101384	6716.481	107.179
54	7473.661	0.101315	6716.468	107.167
55	7473.075	0.101246	6716.455	107.154
56	7472.488	0.101177	6716.443	107.142
57	7471.902	0.101108	6716.430	107.129
58	7471.315	0.101039	6716.417	107.116
59	7470.728	0.100971	6716.404	107.104
60	7470.141	0.100902	6716.392	107.091
61	7469.554	0.100833	6716.379	107.078

62	7468,966	0,100764	6716,366	107,066
63	7468,379	0,100695	6716,353	107,053
64	7467,791	0,100626	6716,340	107,041
65	7467,203	0,100556	6716,328	107,028
66	7466,615	0,100487	6716,315	107,015
67	7466,027	0,100418	6716,302	107,003
68	7465,438	0,100349	6716,289	106,990
69	7464,850	0,100280	6716,276	106,977
70	7464,261	0,100211	6716,263	106,965
71	7463,672	0,100141	6716,250	106,952
72	7463,083	0,100072	6716,237	106,939
73	7462,494	0,100003	6716,224	106,927
74	7461,904	0,099933	6716,211	106,914
75	7461,314	0,099864	6716,198	106,901
76	7460,725	0,099795	6716,185	106,889
77	7460,135	0,099725	6716,172	106,876
78	7459,545	0,099656	6716,159	106,863
79	7458,954	0,099586	6716,145	106,851
80	7458,364	0,099517	6716,132	106,838
81	7457,773	0,099447	6716,119	106,825
82	7457,182	0,099378	6716,106	106,813
83	7456,591	0,099308	6716,093	106,800
84	7456,000	0,099238	6716,079	106,787
85	7455,409	0,099169	6716,066	106,774
86	7454,817	0,099099	6716,053	106,762
87	7454,225	0,099029	6716,040	106,749
88	7453,633	0,098959	6716,026	106,736
89	7453,041	0,098890	6716,013	106,724
90	7452,449	0,098820	6716,000	106,711
91	7451,857	0,098750	6715,986	106,698
92	7451,264	0,098680	6715,973	106,685
93	7450,671	0,098610	6715,959	106,673
94	7450,078	0,098540	6715,946	106,660
95	7449,485	0,098470	6715,932	106,647
96	7448,892	0,098400	6715,919	106,634
97	7448,298	0,098330	6715,905	106,622
98	7447,704	0,098260	6715,892	106,609

99	7447.110	0.098190	6715.878	106.596
100	7446.516	0.098120	6715.865	106.583
101	7445.922	0.098050	6715.851	106.571
102	7445.328	0.097980	6715.837	106.558
103	7444.733	0.097909	6715.824	106.545
104	7444.138	0.097839	6715.810	106.532
105	7443.543	0.097769	6715.796	106.520
106	7442.948	0.097699	6715.783	106.507
107	7442.352	0.097628	6715.769	106.494
108	7441.757	0.097558	6715.755	106.481
109	7441.161	0.097487	6715.741	106.468
110	7440.565	0.097417	6715.728	106.456
111	7439.969	0.097347	6715.714	106.443
112	7439.372	0.097276	6715.700	106.430
113	7438.776	0.097206	6715.686	106.417
114	7438.179	0.097135	6715.672	106.404
115	7437.582	0.097064	6715.658	106.392
116	7436.985	0.096994	6715.644	106.379
117	7436.388	0.096923	6715.630	106.366
118	7435.790	0.096852	6715.616	106.353
119	7435.193	0.096782	6715.602	106.340
120	7434.595	0.096711	6715.588	106.328
121	7433.997	0.096640	6715.574	106.315
122	7433.398	0.096569	6715.560	106.302
123	7432.800	0.096498	6715.546	106.289
124	7432.201	0.096428	6715.532	106.276
125	7431.602	0.096357	6715.518	106.263
126	7431.003	0.096286	6715.503	106.251
127	7430.404	0.096215	6715.489	106.238
128	7429.804	0.096144	6715.475	106.225
129	7429.205	0.096073	6715.461	106.212
130	7428.605	0.096002	6715.446	106.199
131	7428.005	0.095931	6715.432	106.186
132	7427.405	0.095859	6715.418	106.173
133	7426.804	0.095788	6715.403	106.160
134	7426.203	0.095717	6715.389	106.148
135	7425.602	0.095646	6715.375	106.135

136	7425.001	0.095575	6715.360	106.122
137	7424.400	0.095503	6715.346	106.109
138	7423.799	0.095432	6715.331	106.096
139	7423.197	0.095361	6715.317	106.083
140	7422.595	0.095289	6715.302	106.070
141	7421.993	0.095218	6715.288	106.057
142	7421.390	0.095146	6715.273	106.044
143	7420.788	0.095075	6715.258	106.032
144	7420.185	0.095003	6715.244	106.019
145	7419.582	0.094932	6715.229	106.006
146	7418.979	0.094860	6715.214	105.993
147	7418.376	0.094788	6715.200	105.980
148	7417.772	0.094717	6715.185	105.967
149	7417.168	0.094645	6715.170	105.954
150	7416.564	0.094573	6715.155	105.941
151	7415.960	0.094502	6715.140	105.928
152	7415.356	0.094430	6715.126	105.915
153	7414.751	0.094358	6715.111	105.902
154	7414.146	0.094286	6715.096	105.889
155	7413.541	0.094214	6715.081	105.876
156	7412.936	0.094142	6715.066	105.863
157	7412.330	0.094070	6715.051	105.850
158	7411.725	0.093998	6715.036	105.837
159	7411.119	0.093926	6715.021	105.824
160	7410.512	0.093854	6715.006	105.811
161	7409.906	0.093782	6714.991	105.798
162	7409.299	0.093710	6714.976	105.785
163	7408.693	0.093638	6714.961	105.772
164	7408.086	0.093565	6714.945	105.759
165	7407.478	0.093493	6714.930	105.746
166	7406.871	0.093421	6714.915	105.733
167	7406.263	0.093348	6714.900	105.720
168	7405.655	0.093276	6714.884	105.707
169	7405.047	0.093204	6714.869	105.694
170	7404.439	0.093131	6714.854	105.681
171	7403.830	0.093059	6714.838	105.668
172	7403.221	0.092986	6714.823	105.655

173	7402.612	0.092914	6714.808	105.642
174	7402.003	0.092841	6714.792	105.629
175	7401.393	0.092769	6714.777	105.616
176	7400.784	0.092696	6714.761	105.603
177	7400.174	0.092623	6714.746	105.590
178	7399.563	0.092551	6714.730	105.577
179	7398.953	0.092478	6714.714	105.564
180	7398.342	0.092405	6714.699	105.551
181	7397.731	0.092332	6714.683	105.538
182	7397.120	0.092259	6714.667	105.525
183	7396.509	0.092186	6714.652	105.512
184	7395.897	0.092113	6714.636	105.498
185	7395.286	0.092040	6714.620	105.485
186	7394.673	0.091967	6714.604	105.472
187	7394.061	0.091894	6714.588	105.459
188	7393.449	0.091821	6714.573	105.446
189	7392.836	0.091748	6714.557	105.433
190	7392.223	0.091675	6714.541	105.420
191	7391.610	0.091602	6714.525	105.407
192	7390.996	0.091529	6714.509	105.394
193	7390.382	0.091455	6714.493	105.381
194	7389.768	0.091382	6714.477	105.367
195	7389.154	0.091309	6714.461	105.354
196	7388.540	0.091235	6714.445	105.341
197	7387.925	0.091162	6714.428	105.328
198	7387.310	0.091088	6714.412	105.315
199	7386.695	0.091015	6714.396	105.302
200	7386.079	0.090941	6714.380	105.288
201	7385.464	0.090868	6714.363	105.275
202	7384.848	0.090794	6714.347	105.262
203	7384.232	0.090720	6714.331	105.249
204	7383.615	0.090647	6714.314	105.236
205	7382.998	0.090573	6714.298	105.223
206	7382.381	0.090499	6714.282	105.209
207	7381.764	0.090425	6714.265	105.196
208	7381.147	0.090352	6714.249	105.183
209	7380.529	0.090278	6714.232	105.170

210	7379.911	0.090204	6714.216	105.157
211	7379.293	0.090130	6714.199	105.143
212	7378.675	0.090056	6714.182	105.130
213	7378.056	0.089982	6714.166	105.117
214	7377.437	0.089908	6714.149	105.104
215	7376.818	0.089834	6714.132	105.091
216	7376.198	0.089759	6714.115	105.077
217	7375.578	0.089685	6714.099	105.064
218	7374.958	0.089611	6714.082	105.051
219	7374.338	0.089537	6714.065	105.038
220	7373.718	0.089462	6714.048	105.024
221	7373.097	0.089388	6714.031	105.011
222	7372.476	0.089313	6714.014	104.998
223	7371.854	0.089239	6713.997	104.984
224	7371.233	0.089165	6713.980	104.971
225	7370.611	0.089090	6713.963	104.958
226	7369.989	0.089015	6713.946	104.945
227	7369.366	0.088941	6713.929	104.931
228	7368.744	0.088866	6713.912	104.918
229	7368.121	0.088791	6713.894	104.905
230	7367.498	0.088717	6713.877	104.891
231	7366.874	0.088642	6713.860	104.878
232	7366.250	0.088567	6713.843	104.865
233	7365.626	0.088492	6713.825	104.851
234	7365.002	0.088417	6713.808	104.838
235	7364.378	0.088342	6713.790	104.825
236	7363.753	0.088267	6713.773	104.811
237	7363.128	0.088192	6713.755	104.798
238	7362.502	0.088117	6713.738	104.785
239	7361.877	0.088042	6713.720	104.771
240	7361.251	0.087967	6713.703	104.758
241	7360.624	0.087892	6713.685	104.745
242	7359.998	0.087817	6713.667	104.731
243	7359.371	0.087741	6713.650	104.718
244	7358.744	0.087666	6713.632	104.705
245	7358.117	0.087591	6713.614	104.691
246	7357.489	0.087515	6713.596	104.678

247	7356,861	0,087440	6713,579	104,664
248	7356.233	0,087364	6713,561	104,651
249	7355,604	0,087289	6713,543	104,638
250	7354,976	0,087213	6713,525	104,624
251	7354,346	0,087138	6713,507	104,611
252	7353,717	0,087062	6713,489	104,597
253	7353,087	0,086986	6713,470	104,584
254	7352,457	0,086910	6713,452	104,570
255	7351,827	0,086835	6713,434	104,557
256	7351,197	0,086759	6713,416	104,543
257	7350,566	0,086683	6713,398	104,530
258	7349,935	0,086607	6713,379	104,517
259	7349,303	0,086531	6713,361	104,503
260	7348,671	0,086455	6713,343	104,490
261	7348,039	0,086379	6713,324	104,476
262	7347,407	0,086303	6713,306	104,463
263	7346,774	0,086227	6713,287	104,449
264	7346,142	0,086150	6713,269	104,436
265	7345,508	0,086074	6713,250	104,422
266	7344,875	0,085998	6713,232	104,409
267	7344,241	0,085921	6713,213	104,395
268	7343,607	0,085845	6713,194	104,382
269	7342,972	0,085769	6713,176	104,368
270	7342,337	0,085692	6713,157	104,355
271	7341,702	0,085616	6713,138	104,341
272	7341,067	0,085539	6713,119	104,327
273	7340,431	0,085462	6713,100	104,314
274	7339,795	0,085386	6713,081	104,300
275	7339,159	0,085309	6713,062	104,287
276	7338,522	0,085232	6713,043	104,273
277	7337,885	0,085155	6713,024	104,260
278	7337,248	0,085079	6713,005	104,246
279	7336,610	0,085002	6712,986	104,232
280	7335,972	0,084925	6712,967	104,219
281	7335,334	0,084848	6712,947	104,205
282	7334,696	0,084771	6712,928	104,192
283	7334,057	0,084694	6712,909	104,178

284	7333.418	0,084616	6712,889	104.164
285	7332.778	0,084539	6712,870	104.151
286	7332.138	0,084462	6712,851	104.137
287	7331.498	0,084385	6712,831	104.124
288	7330.857	0,084307	6712,812	104.110
289	7330.217	0,084230	6712,792	104.096
290	7329.575	0,084153	6712,772	104.083
291	7328.934	0,084075	6712,753	104.069
292	7328.292	0,083998	6712,733	104.055
293	7327.650	0,083920	6712,713	104.042
294	7327.007	0,083842	6712,693	104.028
295	7326.364	0,083765	6712,673	104.014
296	7325.721	0,083687	6712,654	104.001
297	7325.078	0,083609	6712,634	103.987
298	7324.434	0,083531	6712,614	103.973
299	7323.790	0,083454	6712,594	103.959
300	7323.145	0,083376	6712,573	103.946

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