The Dickey-Fuller-test for exponential random walks[1]

by

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Abstract

We derive the probability limit of the standard Dickey-Fuller-test in the context of an exponential random walk. This result might be useful in interpreting tests for unit roots when the test is inadvertantly applied to the levels of the data when the "true" random walk is in the logs.

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1 Introduction

Consider a stochastic process $y_t, t = 0, \ldots, T$ given by

$$y_t = \exp(z_t)$$

where

$$z_t = z_{t-1} + \sigma\epsilon_t$$

and the ϵ_t are i.i.d. random variables. The random variables z_t describe a random walk and fulfil the unit root hypothesis so that it would be appropriate to apply the logarithmic transformation to the original data y_t. We investigate what happens if this is not done and the Dickey-Fuller test is applied to the process y_t.

The issue appears to be empirically relevant, as it is often not a priori clear whether a unit root, if any, is present in the logs or in the levels of the data (Guerre and Jouneau 1995, Ermini and Hendry 1995, Franses and Knoop 1998, Kobayashi and McAlear 1999 and many others). The present paper provides some analytical underpinning to the suspicion first voiced by Granger and Hallmann (1991) that test for unit roots tend to overreject a correct null hypothesis of a unit root when the one forgets to take the logs.

2 A nonlinear transformation of an $I(1)$–process

The test statistic of the Dickey-Fuller-test is

$$DF(T) = T(\hat{\rho}(T) - 1)$$

where

$$\hat{\rho}(T) = \frac{\sum_1^T y_{t-1} y_t}{\sum_1^T y_{t-1}^2}.$$

The Dickey-Fuller-statistic is a nonlinear transformation of the partial sums of ϵ_t process. Park and Phillips (1998) have developed a general method of treating nonlinear functionals of integrated time series but unfortunately their results do not cover the asymptotic behaviour of the Dickey-Fuller-statistic. There are two reasons for this. Firstly if we write the statistic in the form

$$DF(T) = T\frac{\sum_1^T y_{t-1}^2(\exp(\sigma\epsilon_t) - 1)}{\sum_1^T y_{t-1}^2} = T\frac{S_1(T)}{S_0(T)} \tag{1}$$

it is seen that the functional does not fit into the Park and Phillips framework because of the explicit appearance of the increment ϵ_t. The second reason is not quite so obvious. Park and Phillips develop a theory which is applicable to the the sum $S_0(T)$. Suppose that the ϵ_t are i.i.d with mean zero and finite non-zero variance. On writing

$$M_T = \max_{0 \le t \le T} z_t. \tag{2}$$

Theorem 5.5 of Park and Phillips (1998) gives

$$\frac{S_0(T)}{\sqrt{T}\exp(2M_T)} \Rightarrow L(1, smax) \tag{3}$$

where $\Rightarrow$ denotes weak convergence, $L(x,t)$ is the local time of Brownian motion W on $[0,1]$ and

$$smax = \max_{0 \le t \le 1} W(t).$$

As $L(1, smax) = 0$ (3) reduces to

$$S_0(T) = o_{\mathbb{P}}(\sqrt{T}\exp(2M_T)) \tag{4}$$

so that (3) is of no help in analysing the Dickey-Fuller statistic. In order to analyse the asymptotic behaviour of the Dickey-Fuller-statistic we require the exact order of magnitude of $S_0(T)$. This we do in the next section. In Section 4 we apply the result to the Dickey-Fuller-statistic. In the particular case of the simple random walk we obtain the exact limiting behaviour.

3 The asymptotic behaviour of $\sum_0^T \exp(\sigma z_t))$

We prove the following theorem.

Theorem 1

Suppose the increments $(\epsilon_i)_1^\infty$ are i.i.d. and satisfy

$$\mathbb{E}(\epsilon_i) = 0\ , \quad \mathbb{V}(\epsilon_i) = 1\ , \quad \mathbb{E}(|\epsilon_i|^3) < \infty. \tag{5}$$

Then

$$\lim_{C \to \infty} \mathbb{P}\left(\exp(\sigma M_T) \le \sum_0^T \exp(\sigma z_t) \le C\exp(\sigma M_T)\right) = 1\ . \tag{6}$$

Proof: Clearly $\exp(M_T) \le \sum_0^T \exp(\sigma z_t)$ so it is sufficient to prove

$$\lim_{C \to \infty} \mathbb{P}\left(\sum_0^T \exp(\sigma z_t) \le C\exp(\sigma M_T)\right) = 1$$

which is equivalent to

$$\lim_{C\to\infty} \mathbb{P}\left(\sum_{0}^{T} \exp(-\sigma(M_T - z_t)) \le C\right) = 1\ .$$

This in turn follows from

$$\mathbb{E}\left(\sum_{0}^{T} \exp(-\sigma(M_T - z_t))\right) \le A \tag{7}$$

for some constant A. Consider the term

$$\begin{aligned} \mathbb{E}(\exp(-\sigma(M_T - z_t))) &= \int_0^\infty \exp(-\sigma x) dF_{t,T}(x) \\ &= F_{t,T}(\{0\}) + \sigma \int_0^\infty F_{t,T}(x) \exp(-\sigma x) dx \end{aligned} \tag{8}$$

where $F_{t,T}$ denotes the distribution function of $M_T - z_t$. We have

$$\begin{aligned} F_{t,T}(x) &= \mathbb{P}(M_T - z_t \le x) \\ &= \mathbb{P}(\{M_t - z_t \le x\} \cap \{\max_{j=1,\dots,n-t} \{z_{t+j} - z_t\} \le x\}) \\ &= \mathbb{P}(M_t - z_t \le x)\mathbb{P}(M_{n-t} - z_{n-t} \le x) \\ &= \mathbb{P}(m_t \ge -x)\mathbb{P}(m_{n-t} \ge -x) \\ &\le \mathbb{P}(z_t \ge -x)\mathbb{P}(z_{n-t} \ge -x) \end{aligned} \tag{9}$$

where $m_t = \min_{1\le s\le t} z_s$. To obtain upper bounds for $\mathbb{P}(z_t \ge -x)$ we note

$$\begin{aligned} \mathbb{P}(z_t \ge -x) &= \mathbb{P}\left(\frac{z_t}{\sqrt{t}} \ge -\frac{x}{\sqrt{t}}\right) \\ &\le \Phi\left(\frac{x}{\sqrt{t}}\right) + \frac{B}{\sqrt{t}} \end{aligned} \tag{10}$$

where we have used the central limit theorem and the Berry-Esséen bound. From (10), (9) and (8) we obtain

$$\mathbb{E}(\exp(-\sigma(M_T - z_t))) \le \frac{B}{\sqrt{t(T-t)}} \tag{11}$$

for some contstant B where we have used the same argument for $F_{t,T}(\{0\})$ as for $F_{t,T}(x)$. From (11) we conclude

$$\begin{aligned} \mathbb{E}\left(\sum_{0}^{T} \exp(-\sigma(M_T - z_t))\right) &\le B \sum_{1}^{T-1} \frac{1}{\sqrt{t(T-t)}} \\ &\le A. \end{aligned}$$

This proves (7) and with it the theorem. □

4 The asymptotic behaviour of the Dickey-Fuller-statistic

Theorem 2

Suppose the increments $(\epsilon_i)_1^\infty$ *are i.i.d. and satisfy*

$$\mathbb{E}(\epsilon_i) = 0\ , \quad \mathbb{V}(\epsilon_i) = 1\ , \quad \mathbb{E}(|\epsilon_i|^3) < \infty.$$

Then

$$\lim_{C \downarrow 0} \lim_{T \to \infty} \mathbb{P}(DF(T) \leq -CT) = 1\ . \tag{12}$$

Proof: We denote the path $(t, z_t)_0^T$ with $z_0 = 0$ by $B(T)$ and write $S_1(B(T)) = S_1(T)$. A local maximum of $B(T)$ is a point $(\tau, z_\tau, 1 \leq \tau \leq T-1$ with

$$\min\{z_{\tau-1}, z_{\tau+1}\} < z_\tau \geq \max\{z_{\tau-1}, z_{\tau+1}\}.$$

A new path of length is constructed by setting $\tilde{z}_t = z_t, 0 \leq s \leq \tau - 1$ and $\tilde{z}_\tau = \max\{z_{\tau-1}, z_{\tau+1}\}$ and $\tilde{z}_t = z_{t+1}, \tau+1 \leq t \leq T-1$. We denote this new path of length by $\tilde{B}(T-1)$. We describe the effect on $S_1(B(T))$ of removing the local maximum at τ. To ease the notation we set $\sigma = 1$ .Without loss of generality we assume that $z_{\tau-1} \leq z_{\tau+1} \leq z_\tau$ and write $z_\tau - z_{\tau+1} = \Delta$ and $z_\tau - z_{\tau-1} = \gamma + \Delta$ with γ and Δ both non-negative. We have

$$\begin{aligned} S_1(B(T)) - S_1(\tilde{B}(T-1)) &= \exp(2z_\tau)(\exp(-2(\gamma+\Delta))(\exp(\gamma+\Delta)-1) + \exp(-\Delta) - 1 \\ &\quad - \exp(-2\gamma - 2\Delta)(\exp(\gamma) - 1)) \\ &= -\exp(2z_\tau)(1-\exp(-\Delta))^2 \\ &\quad - \exp(2z_\tau)\exp(-\Delta)(1-\exp(-\Delta))(1-\exp(-\gamma)) \qquad (13) \\ &\leq -\exp(2z_\tau)(1-\exp(-\Delta))^2. \qquad (14) \end{aligned}$$

We note that the value of the final point of the path is not altered i.e. $\tilde{z}_{T-1} = z_T$. Under the conditions of the theorem $\lim_{T\to\infty} \mathbb{P}(M_T > \max\{0, S_T\} + a) = 1$ for all $a > 0$ so that there exists at least one local maximum with $z_\tau = M_T$. If we remove all such local maxima one by one then the final one is a strict local maximum and derives from local maxima of $B(T)$ satisfying $\tau_1 \leq \tau_2$ and

$$z_{\tau_1} - \epsilon_{\tau_1} < z_{\tau_1} = M_T = z_{\tau_2} > z_{\tau_2+1} + \epsilon_{\tau_2+1}.$$

It follows from (14) that

$$S_1(B(T)) \leq -\exp(2M_T)(1-\exp(-\Delta))^2 + S_1(\tilde{B}(T-k))$$

with

$$\Delta = \min\{\epsilon_{\tau_1}, -\epsilon_{\tau_2+1}\} > 0. \tag{15}$$

We now continue to remove local maxima until we reach a final path $B^*(T-S)$ which has no more local maxima. We have

$$S_1(B(T)) \le -\exp(2M_T)(1-\exp(-\Delta))^2 + S_1(B^*(T-S)).$$

As $B^*(T-S)$ has no local maxima it is either monontone or has at most one local minimum. It follows that the global maximum of this path is either located at 0 or the last point. As the removal of local maxima does not alter the value of the last point we have $S_1(B^*(T-S)) \le (T-S)\max\{0, z_T\}$ and hence

$$S_1(B(T)) \le -\exp(2M_T)(1-\exp(-\Delta))^2 + T\exp(\max\{0, z_T\}).$$

As $\lim_{T\to\infty} \mathbb{P}(M_T > z_T + a_T) = 1$ for any sequence a_T with $\lim_{T\to\infty} \frac{a_T}{\sqrt{T}} = 0$ it follows that $\tilde{B}_{T-k} \le o_{\mathbb{P}}(\exp(2\sigma M_T)$ and hence

$$S_1(B_T) \le -\exp(2M_T)((1-\exp(-\Delta))^2 + o_{\mathbb{P}}(1)). \tag{16}$$

with Δ given by (15). From Theorem 1 it follows that

$$DF(T) = T\frac{S_1(T)}{S_0(T)} \le -C((1-\exp(-\Delta))^2 + o_{\mathbb{P}}(1))$$

with high probability as $C \downarrow 0$. To complete the proof of the theroem it suffices to show that

$$\lim_{x\downarrow 0}\ \lim\ \inf_{T\to\infty} \mathbb{P}(\Delta \ge x) = 1\ . \tag{17}$$

To do this we first consider the distribution of ϵ_{τ_1} conditioned on $\tau_1 = t_1$. In this case $\epsilon_{\tau_1} = \epsilon_{t_1}$ and is defined by the inequalities

$$\epsilon_{t_1} > 0, \epsilon_{t_1} \ge -\sum_{t}^{t_1-1} \epsilon_s, t = t_1-1, \dots, 0$$

and

$$\sum_{t_1+1}^{t} \epsilon_s \le 0, t = t_1+1, \dots, T.$$

As the ϵ_t are assumed to be independently distributed these latter inequalities have no effect on the conditional distribution of ϵ_{t_1}. The other may be written in the form

$$\epsilon_{t_1} > \max\{0, f(\epsilon_1, \dots, \epsilon_{t_1-1})\}.$$

On conditioning on $\epsilon_1, \dots, \epsilon_{t_1-1}$ and using the fact that

$$\mathbb{P}(X \ge x | X \ge a) \ge \mathbb{P}(X \ge x)$$

for any random variable X and for any x and a we deduce

$$\begin{aligned}
\mathbb{P}(\epsilon_{\tau_1} \geq x | \tau_1 = t_1) &= \mathbb{P}(\epsilon_{t_1} \geq x | \epsilon_{t_1} > \max\{0, f(\epsilon_1, \ldots, \epsilon_{t_1 - 1})) \\
&= \mathbb{P}(\epsilon_{t_1} \geq x, \epsilon_{t_1} > 0 | \epsilon_{t_1} > f(\epsilon_1, \ldots, \epsilon_{t_1 - 1})) / \mathbb{P}(\epsilon_{t_1} > 0) \\
&\geq \mathbb{P}(\epsilon_{t_1} \geq x, \epsilon_{t_1} > 0) / \mathbb{P}(\epsilon_{t_1} > 0) \\
&= \mathbb{P}(\epsilon_{t_1} \geq x | \epsilon_{t_1} > 0).
\end{aligned}$$

On summing over t_1 we obtain

$$\mathbb{P}(\epsilon_{\tau_1} \geq x) \geq \mathbb{P}(\epsilon_1 \geq x | \epsilon_1 > 0)$$

and hence $\lim_x \downarrow 0 \mathbb{P}(\epsilon_{\tau_1} \geq x) = 1$ .This together with the corresponding result for ϵ_{τ_2} implies that Δ satifies (17) and completes the proof of the theorem. □

In the special case of the simple random walk where the ϵ_i are either 1 or -1 a more precise result is available.

Theorem 3

If the z_t describe a simple random walk then

$$plim_{T \to \infty} \frac{DF(T)}{T} = -\frac{(1 - \exp(-\sigma))^2}{\exp(2\sigma) + 1}. \tag{18}$$

Proof:

We note that for the simple random walk the γ in (13) is always zero. Because of this we can modify the procedure of removing local maxima as follows. If a local maximum ccurrs at the point t then this point and the point $t+1$ are removed and the reming part of the path is translated to the left. The effect of this is on S_1 described by

$$S_1(B(T)) = -\exp(2z_t)(1 - \exp(-\sigma))^2 + S_1(\tilde{B}(T-2)$$

and on S_0 by

$$S_0(B(T)) = \exp(2z_t)(1 + \exp(-2\sigma)) + S_0(\tilde{B}(T-2).$$

Iterating this gives

$$\frac{S_0(B(T))}{1 + \exp(-2\sigma)} + \frac{S_1(B(T))}{(1 - \exp(-\sigma))^2} = O_{\mathbb{P}}(T \exp(\sigma \max\{0, z_T\})) = o_{\mathbb{P}}(S_0(T))$$

proving the theorem. □

The upper panel of Figure 1 shows the path behaviour of $z_t, 0 \leq t \leq T = 10^5$, with $\sigma = 0$ .01 and the ϵ_i being i.i.d. and taking the values 1 and -1 with probability 0.5. The middle

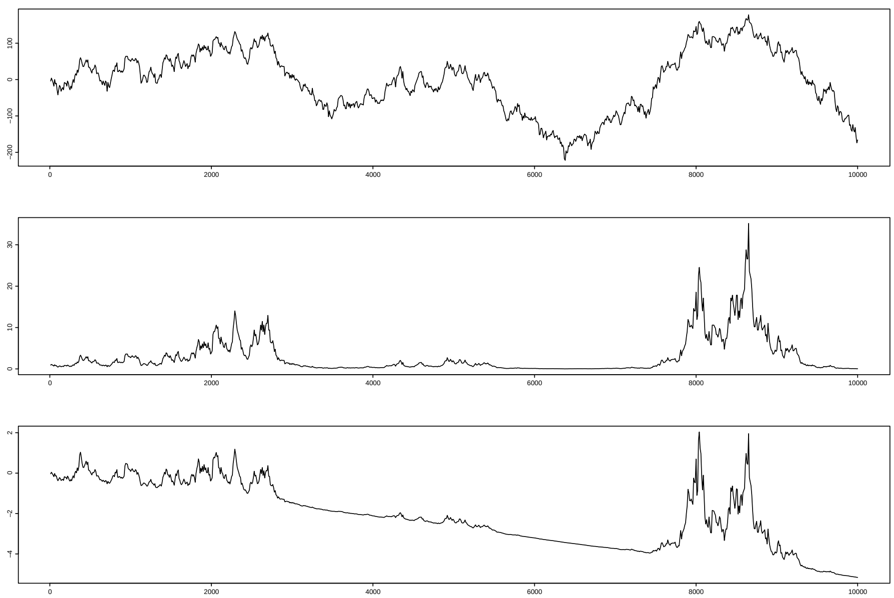

Figure 1: The upper panel shows the z_t process with $\sigma = 0$ .01 and$T = 100000$. The middle panel shows the corresponding y_t process. The lower panel shows the path behaviour of the Dickey-Fuller statistic.

panel shows the corresponding process y_t. The bottom panel shows the path behaviour of the Dickey-Fuller statistic. The final value of the statistics is $-5.163337e-05$ which compares well with the theoretical slope given by

$$-\frac{(\exp(0.01)-1)^2}{\exp(2\cdot 0.01)+1} = -4.999792e-05.$$

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