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Ernst-Erich Doberkat

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Prof. Dr. Ernst-Erich Doberkat
Fachbereich Informatik
Universität Dortmund
Baroper Straße 301

D-44227 Dortmund

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Factoring Stochastic Relations

Ernst-Erich Doberkat*
Chair for Software Technology
University of Dortmund
doberkat@acm.org

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Abstract

When a system represented through a stochastic model is observed, the equivalence of behavior is described through the observation that equivalent inputs lead to equivalent outputs. This paper has a look at the systems that arise when the stochastic model is factored through the congruence. Congruences may refine each other, and we show that this refinement is reflected through factoring. We also show that factoring a factor does not give rise to any new constructions, since we are kept in the realm of factors for the original system. Thus we cannot have infinite long chains of factors, so that no new behavior can arise from the original system upon factoring (a system and its factors are bisimilar, after all).

1 Introduction

A stochastic relation $K : X \rightsquigarrow Y$ models the transformation of inputs to outputs with stochastic means; the special case of a state transition system is covered by assuming $X = Y = S$ with S as the state space. If the input to the system is $x \in X$, the subprobability measure $K(x)$ on Y yields the distribution of the outputs, since $K(x)(Y) < 1$ is not excluded, non-terminating processes are taken into account. A congruence for K models equivalent behavior. This is modelled through a pair $\mathsf{c} = (\alpha, \beta)$ of equivalence relations α on X and β on Y such that α-equivalent inputs are transformed into β-equivalent outputs. We usually cannot address the outputs directly, $(K(x)(\{y\})$ may always be 0 in an uncountable space), hence we characterize equivalent behavior through β-invariant output sets: $B \subseteq Y$ is β-invariant if β cannot distinguish the elements of B, formally, iff $y \in B$ and $y\beta y'$ together imply $y' \in B$. In a very natural way, this congruence induces a relation K_{c} on the set of equivalence classes. We will show that each congruence d on that system may be represented up to isomorphism through a congruence on the original one, since $(K_{\mathsf{c}})_{\mathsf{d}}$ is isomorphic to K_{e} for some congruence e which can be explicitly constructed from c and d. This is the stochastic analogon of Noether's Isomorphism Theorem for normal subgroups. Via the canonic projection it becomes clear that congruences are essentially the kernels of morphisms, so factoring of morphisms is investigated. It is shown that each morphism can be factored through a suitable isomorphism. This has as a consequence that the refinement of congruences can be represented through factoring.

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All this can only be done when some assumptions on the probability spaces are made. In our case we assume that the input and the output space are analytic spaces, thus images under Borel measurable maps from Polish spaces, which in turn are complete and separable metric spaces. Analycity may sound rather exotic, but these spaces have some measure theoretic properties which make the necessary constructions possible: this class of spaces is preserved for example under forming factors of equivalence relations which are spawned by Borel measurable maps. This is the kind of relations we are working with.
Congruences were investigated first in [2] under the name of *bisimulations* when investigating the approximation of labelled Markov transition processes. The main example for these bisimulations came from modal logic, where two states are called equivalent iff they satisfy exactly the same formulas. Originally only logics were investigated that have a countable number of diamonds, in [6] this is generalized to general modal logic, and a theorem of Hennessy-Milner type is proved. On the other hand, when transition systems are modelled through coalgebras, then congruences arise in a natural way from bisimulations (in Milner's original definition), and vice versa, and from morphisms for coalgebras, as is discussed at length in [10]. The connection between congruences and bisimulations is not as close for stochastic relations as for coalgebras. This is mainly due to the fact that congruences are set-based constructions, and bisimulations are spans of morphisms: these entities can be transformed into each other in the set-based case on which coalgebras nearly always rest, but there is a crevice not easily bridged for stochastic relations (as can be seen when dealing with elementary constructions like the converse for the stochastic case).
The paper is organized as follows: we first give the necessary definitions and carry out some helpful auxiliary constructions in Section 2, Section 3 establishes two isomorphisms and draws some conclusions, Section 4 wraps it up and proposes some further work.

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2 Stochastic Relations

This section collects some basic facts from topology and measure theory for the reader's convenience and for later reference. It defines stochastic relations.
A *Polish space* $(X, \mathcal{G})$ is a topological space which is second countable, i.e., which has a countable dense subset, and which is metrizable through a complete metric. A *measurable space* $(X, \mathcal{A})$ is a set X with a σ-algebra $\mathcal{A}$. The *Borel sets* $\mathcal{B}(X, \mathcal{G})$ for the topology $\mathcal{G}$ is the smallest σ-algebra on X which contains $\mathcal{G}$. Given two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, a map $f : X \to Y$ is *$\mathcal{A}$ - $\mathcal{B}$-measurable* whenever $f^{-1}[\mathcal{B}] \subseteq \mathcal{A}$ holds, where $f^{-1}[\mathcal{B}] := \{f^{-1}[B] | B \in \mathcal{B}\}$ is the set of inverse images $f^{-1}[B] := \{x \in X | f(x) \in B\}$ of elements of $\mathcal{B}$. Note that $f^{-1}[\mathcal{B}]$ is a σ-algebra, provided $\mathcal{B}$ is one. If f is $\mathcal{A}$ - $\mathcal{B}$ - measurable and a bijection such that $f^{-1} : Y \to X$ is also $\mathcal{B}$ - $\mathcal{A}$ - measurable, then f is called a *Borel isomorphism.* A measurable space which is Borel isomorphic to a Polish space is again a Polish space. If the σ-algebras are the Borel sets of some topologies on X and Y, resp., then a measurable map is called *Borel measurable* or simply a *Borel* map. The real numbers $\mathbb{R}$ carry always the Borel structure induced by the usual topology which will usually not be mentioned explicitly when talking about Borel maps.
If $(T, \mathcal{T})$ is a measurable space, and $f : T \to S$ is a map, then the *final σ-algebra with respect to f and $\mathcal{T}$* is the largest σ-algebra $\mathcal{S}$ on S such that f is $\mathcal{T}$-$\mathcal{S}$-measurable. If $(R, \mathcal{R})$ is a

measurable space, then a map $g : S \to R$ is $\mathcal{S}$-$\mathcal{R}$-measurable iff $g \circ f$ is $\mathcal{T}$-$\mathcal{R}$-measurable. This universal property characterizing the final σ-algebra will be helpful later.

An *analytic set* $X \subseteq Z$ for a Polish space Z is the image $f[Y]$ of a Polish space Y for some Borel measurable map $f : Y \to Z$. Endow X with the trace $\mathcal{A}$ of $\mathcal{B}(Z)$ on X, i.e., $\mathcal{A} := \{B \cap X | B \in \mathcal{B}(Z)\}$, the elements of which still being called Borel sets. A measurable space $(X', \mathcal{A}')$ which is Borel isomorphic to $(X, \mathcal{A})$ is called an *analytic space*. The elements of $\mathcal{A}'$ are still called the *Borel sets* of X', $\mathcal{A}'$ itself is denoted by $\mathcal{B}(X')$.

When the context is clear, we will write down topological or measurable spaces without their topologies and σ-algebras, resp., and the Borel sets are always understood with respect to the topology under consideration.

Denote for a measurable space $(T, \mathcal{T})$ by $\mathfrak{S}(T, \mathcal{T})$ the set of all subprobability measures on $(T, \mathcal{T})$ which is equipped with the *weak*-σ-algebra* for a measurable structure. The latter σ-algebra is the smallest σ-algebra on $\mathfrak{S}(X, \mathcal{A})$ which renders all maps $\mu \mapsto \mu(D)$ measurable, where $D \in \mathcal{T}$. It is well known that $\mathfrak{S}(T)$ with this σ-algebra is an analytic space, provided T is one.

Definition 1 *Given two analytic spaces X and Y, a* stochastic relation $K : X \rightsquigarrow Y$ between X and Y *is a Borel map from X to $\mathfrak{S}(Y)$.*

Hence $K : X \rightsquigarrow Y$ is a stochastic relation from X to Y iff

1. $K(x)$ is a subprobability measure on Y for all $x \in X$,
2. $x \mapsto K(x)(D)$ is a measurable map for each measurable set $D \subseteq Y$.

An $\mathcal{T}$ - $\mathcal{S}$- measurable map $f : T \to S$ between the measurable spaces $(T, \mathcal{T})$ and $(S, \mathcal{S})$ induces a map $\mathfrak{S}(f) : \mathfrak{S}(X, \mathcal{A}) \to \mathfrak{S}(Y, \mathcal{B})$ upon setting $\mathfrak{S}(f)(\mu)(D) := \mu(f^{-1}[D])$ ($\mu \in \mathfrak{S}(T, \mathcal{T}), D \in \mathcal{S}$). It is easy to see that $\mathfrak{S}(f)$ is measurable.

The category **Stoch** has as objects stochastic relations $\mathsf{K} = (X, Y, K)$ for analytic spaces X, Y and $K : X \rightsquigarrow Y$. A *morphism* $\mathsf{f} : \mathsf{K}_1 \to \mathsf{K}_2$ between the objects $\mathsf{K}_1 = (X_1, Y_1, K_1)$ and $\mathsf{K}_2 = (X_2, Y_2, K_2)$ is a pair $\mathsf{f} = (\phi, \psi)$ of surjective measurable maps $\phi : X_1 \to X_2$ and $\psi : Y_1 \to Y_2$ such that $K_1 \circ \phi = \mathfrak{S}(\psi) \circ K_2$ holds, i.e., such that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\ \phi\ } & X_2 \\ {\scriptstyle K_1}\downarrow & & \downarrow{\scriptstyle K_2} \\ \mathfrak{S}(Y_1) & \xrightarrow[\mathfrak{S}(\psi)]{} & \mathfrak{S}(Y_2) \end{array}$$

is commutative. Spelt out, this condition means that

$$K_2(\phi(x_1))(B_2) = K_1(x_1)(\psi^{-1}[B_2])$$

holds for each $x_1 \in X_1$ and each Borel set $B_2 \subseteq Y_2$.

Thus **Stoch** just is the comma category $1_{\mathbf{A}} \downarrow \mathfrak{S}$ [8, Section II.6] with $\mathbf{A}$ as the category of analytic spaces with surjective measurable maps as morphisms, and $\mathfrak{S} : \mathbf{A} \to \mathbf{A}$ as the subprobability functor. Alternatively, $\mathfrak{S}$ can be seen as the functorial part of a monad (for

this monad, the reader may wish to consult [7], for applications, see [3]). A subcategory of **Stoch** is the well-known category of *Markov state transition systems*. These systems may formally be described through a stochastic relation $K : S \rightsquigarrow S$ with S as a state space, and K modelling state transitions. A morphism $\phi : (S, K) \to (S', K')$ is a surjective Borel map $\phi : S \to S'$ such that $K' \circ \phi = \mathfrak{S}(\phi) \circ K$.
An equivalence relation ρ on a measurable space $(T, \mathcal{T})$ is said to be *smooth* iff there exists a sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{T}$ such that

$$t \, \rho \, t' \text{ iff } \forall n \in \mathbb{N} : \left[t \in A_n \Leftrightarrow t' \in A_n\right].$$

We say that the sequence $(A_n)_{n\in\mathbb{N}}$ determines relation ρ.
Denote by $\mathcal{I}(\mathcal{T}, \rho)$ the σ-algebra of ρ-invariant measurable sets, thus

$$\mathcal{I}(\mathcal{T}, \rho) := \{A \in \mathcal{T} | A \text{ is } \rho-\text{invariant}\},$$

where $A \subseteq T$ is called *ρ-invariant* iff $A = \bigcup\{[t]_\rho | t \in A\}$ holds, thus iff $t \in A$ and $t \, \rho \, t'$ together imply $t' \in A$. We will see that smooth equivalence relations are just the natural kind of equivalence relations compatible with the structure of stochastic relations.
Observing a stochastic system $K : X \rightsquigarrow Y$, pairs with equivalent behavior are identified. This leads to a pair (α, β) of equivalence relations on the inputs X resp. the outputs Y with the idea that equivalent inputs lead to equivalent outputs. While equivalent inputs can be described directly through α, the equivalence of outputs requires a description on the level of measurable sets. We argue that a set $B \subseteq Y$ does not distinguish between equivalent outputs iff it is invariant under β, i.e., if $y \in B$ and $y \, \beta \, y'$ together imply $y' \in B$. This leads then naturally to the notion of a congruence:

Definition 2 *A* congruence $\mathsf{c} = (\alpha, \beta)$ *for the stochastic relation* $\mathsf{K} = (X, Y, K)$ *is a pair of smooth equivalence relations α on X and β on Y such that $K(x)(D) = K(x')(D)$ holds whenever $x \, \alpha \, x'$ and D is an β-invariant measurable subset of Y.*

Denote for the equivalence relation ρ on the analytic space T by T/ρ the set of equivalence classes, and let $\eta_\rho : t \mapsto [t]_\rho$ assign to each t its class $[t]_\rho$; denote by $\mathcal{T}/\rho$ the final σ-algebra on T/ρ with respect to the Borel sets on T and the natural projection η_ρ.
Smooth equivalence relations arise in a natural fashion from kernels of measurable maps, as we will see in a moment. These relations enjoy the technically interesting property that the factor space $(T/\rho, \mathcal{B}(T)/\rho)$ for an analytic space T and a smooth relation ρ is an analytic space again, cf. [1, Corollary 3.3.5.2]. In particular, $\mathcal{B}(T/\rho) = \mathcal{B}(T)/\rho$ holds. It can be said a wee bit more. We define the *kernel* $\mathsf{ker}(f)$ of a map $f : M \to N$ as

$$\mathsf{ker}(f) := \{\langle m, m' \rangle \in M \times M | f(m) = f(m')\}.$$

It is clear that $\mathsf{ker}(f)$ is an equivalence relation on M.

Lemma 1 *Let S, T be analytic spaces, and assume that $f : S \to T$ is a surjective and Borel measurable map. Then $\mathsf{ker}(f)$ is smooth, and $f^{-1}[\mathcal{B}(T)] = \mathcal{I}(\mathcal{B}(S), \mathsf{ker}(f))$.*

Proof

0. Because T is an analytic space, its Borel sets possess a countable generator $(A_n)_{n\in\mathbb{N}}$ which separates points (thus for two distinct elements of T there exists A_n which contains one but not the other). Consequently,

$$\begin{aligned} s \ \mathsf{ker}(f) \ s' &\Leftrightarrow f(s) = f(s') \\ &\Leftrightarrow \forall n \in \mathbb{N} : \left[f(s) \in A_n \Leftrightarrow f(s') \in A_n\right] \\ &\Leftrightarrow \forall n \in \mathbb{N} : \left[s \in f^{-1}[A_n] \Leftrightarrow s' \in f^{-1}[A_n]\right] \end{aligned}$$

1. Given $B \in \mathcal{I}(\mathcal{B}(S), \mathsf{ker}(f))$, we show first that $f^{-1}[f[B]] = B$ holds. In fact, $B \subseteq f^{-1}[f[B]]$ is always true. Let $s \in f^{-1}[f[B]]$, thus $f(s) = f(s')$ for some $s' \in B$. Since B is $\mathsf{ker}(f)$-invariant, this implies $s \in B$, accounting for the other inclusion.
2. Let again $B \in \mathcal{I}(\mathcal{B}(S), \mathsf{ker}(f))$, then $f[B] \subseteq T$ is analytic. We claim that $f[S \setminus B] = T \setminus f[B]$ holds. For, if $t \in f[S \setminus B]$, we can find $s \notin B$ with $f(s) = t$. Assuming that $t = f(s')$ for some $s' \in B$, we would infer that $s \in B$ due to the $\mathsf{ker}(f)$-invariance of B, and since $\langle s, s'\rangle \in \mathsf{ker}(f)$. This is a contradiction. This settles the non-trivial inclusion. From the representation just established we see that $T \setminus f[B]$ is analytic, and from Souslin's Theorem [11, Theorem 4.4.3] we infer now that $f[B]$ is Borel in T.
3. It is clear that for each $C \in \mathcal{B}(T)$ its inverse image $f^{-1}[C]$ under f is a Borel set which is $\mathsf{ker}(f)$-invariant. On the other hand, if $B \in \mathcal{I}(\mathcal{B}(S), \mathsf{ker}(f))$, we write $B = f^{-1}[f[B]]$ by part 1, and $f[B] \in \mathcal{B}(T)$ by part 2. This implies the desired equality. □
As a by-product we obtain a characterization of ρ-invariant Borel sets in analytic spaces through the generating sequence $(A_n)_{n\in\mathbb{N}}$; this result is well-known for Polish spaces, cp. [11, Lemma 5.1.16]. As a consequence, we can characterize the ρ-invariant Borel set through the canonic projection.

Corollary 1 *Let T be an analytic space with a smooth equivalence relation ρ, then the ρ-invariant Borel sets of T are exactly the inverse images of the canonic projection η_ρ, viz., $\mathcal{I}(\mathcal{B}(T), \rho) = \eta_\rho^{-1}[\mathcal{B}(T/\rho)]$ holds. Moreover, if ρ is determined by the sequence $(A_n)_{n\in\mathbb{N}}$ of Borel sets $A_n \subseteq T$, then*

$$\mathcal{I}(\mathcal{B}(T), \rho) = \sigma(\{A_n | n \in \mathbb{N}\}).$$

Proof 1. T/ρ is an analytic space, and $\eta_\rho : T \to T/\rho$ is surjective and onto. Thus the first assertion follows from Lemma 1 upon observing that $\rho = \mathsf{ker}(\eta_\rho)$ holds.
2. Let $B \in \mathcal{B}(T/\rho)$ be a Borel set in T/ρ. Plainly,

$$B = \bigcup\{\{[t]_\rho\} | \, [t]_\rho \in B\},$$

so it is enough to show that each $\{[t]_\rho\}$ constitutes an atom in $\sigma(\{\eta_\rho[A_n] \,| n \in \mathbb{N}\})$.
Granted that, we can argue as follows: The Blackwell-Mackey-Theorem [11, Theorem 4.5.7] implies that

$$\mathcal{B}(T/\rho) = \sigma(\{\eta_\rho[A_n] \,| n \in \mathbb{N}\})$$

holds, thus $C \in \mathcal{I}(\mathcal{B}(T), \rho)$ iff $C = \eta_\rho^{-1}[B]$ for some

$$B \in \eta_\rho^{-1}[\sigma(\{\eta_\rho[A_n] \,| n \in \mathbb{N}\})] = \sigma(\{A_n | n \in \mathbb{N}\}).$$

3. It is easy to see that

$$\bigcap\{\eta_\rho[A_n] \,| t \in A_n\} \cap \bigcap\{T/\rho \setminus \eta_\rho[A_n] \,| t \notin A_n\},$$

contains the class $[t]_\rho$ as its only element, and that

$$T/\rho \setminus \eta_\rho [A_n] = \eta_\rho [T \setminus A_n],$$

because A_n is ρ-invariant, cp. part 2 of the proof of Lemma 1. Thus the atom $\{[t]_\rho\}$ is a member of $\sigma(\{\eta_\rho [A_n] \mid n \in \mathbb{N}\})$. □

The next Corollary shows that kernels of morphisms and congruences are basically the same thing. Denote for the morphism $\mathsf{f} : \mathsf{K}_1 \to \mathsf{K}_2$ with $\mathsf{f} = (\phi, \psi)$ its kernel $\ker(\mathsf{f})$ by the pair $(\ker(\phi), \ker(\psi))$.

Corollary 2 *If* $\mathsf{f} : \mathsf{K} \to \mathsf{K}'$ *is a morphism for the stochastic relations* K *and* K', *then* $\ker(\mathsf{f})$ *is a congruence for* K.

Proof Let $\mathsf{K} = (X, Y, K)$ and $\mathsf{K}' = (X', Y', K')$ with $\mathsf{f} = (\phi, \psi)$. Let $x \ker(\phi)\ x'$ and $D \subseteq Y$ be a $\ker(\psi)$-invariant Borel subset of Y. Lemma 1 shows that $D = \psi^{-1}[D']$ for some Borel set $D' \subseteq Y'$. Thus

$$\begin{aligned} K(x)(D) &= K(x)(\psi^{-1}[D']) \\ &= (\mathfrak{S}(\psi) \circ K)(x)(D') \\ &= (K' \circ \phi)(x)(D') \\ &= K(\phi(x))(D') \\ &= K(\phi(x'))(D') \\ &= K(x')(D), \end{aligned}$$

since $\mathsf{f} = (\phi, \psi)$ is a morphism. □

This construction permits introducing *factor objects*. Let $\mathsf{c} = (\alpha, \beta)$ be a congruence on the stochastic relation $\mathsf{K} = (X, Y, K)$, and define

$$K_{\alpha,\beta}([x]_\alpha)(D) := K(x)(\eta_\beta^{-1}[D])$$

for $x \in X, D \in \mathcal{B}(Y/\beta)$, then

$$\mathsf{K}/\mathsf{c} := (X/\alpha, Y/\beta, K_{\alpha,\beta})$$

is a stochastic relation, and

$$\eta_\mathsf{c} := (\eta_\alpha, \eta_\beta) : \mathsf{K} \to \mathsf{K}/\mathsf{c}$$

is a morphism [6, Proposition 3].

3 Two Isomorphisms

We will investigate the factor of a stochastic relation through a congruence. Two isomorphisms will be considered: factoring a factor space, and factoring a morphism. Before we tackle these questions, we investigate what happens on the level of the underlying analytic spaces.

Assume that ρ is a smooth equivalence relation on the analytic space T, and that τ is a smooth equivalence on T/ρ. Define for $t, t' \in T$

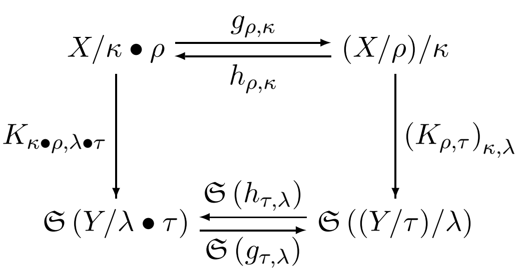
$$t\ (\tau \bullet \rho)\ t' \Leftrightarrow [t]_\rho\ \tau\ [t']_\rho$$

Proposition 1 *The equivalence relation $\tau \bullet \rho$ is smooth, and the analytic spaces $T/\tau \bullet \rho$ and $(T/\rho)/\tau$ are Borel isomorphic.*

Proof 0. Since τ is smooth, there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of Borel sets $A_n \subseteq T/\rho$ which determines it. Then $\left(\eta_\rho^{-1}\left[A_n\right]_{n\in\mathbb{N}}\right)$ determines $\tau \bullet \rho$. Its members are by construction Borel sets in T.
1. Define $g_{\rho,\tau}\left([t]_{\tau\bullet\rho}\right) := \left[[t]_\rho\right]_\tau$, then $g_{\rho,\tau} : T/\tau \bullet \rho \to (T/\rho)/\tau$ is well-defined and turns out to be a bijection. The construction shows that $g_{\rho,\tau} \circ \eta_{\tau\bullet\rho} = \eta_\tau \circ \eta_\rho$ holds, putting $h_{\rho,\tau} := g_{\rho,\tau}^{-1}$, we see that $\eta_{\tau\bullet\rho} = h_{\rho,\tau} \circ \eta_\tau \circ \eta_\rho$. This is also noted for later use.
2. Let $E \subseteq (T/\rho)/\tau$ be a Borel set, we need to show that $g_{\rho,\tau}^{-1}[E]$ is a Borel set in $T/\tau \bullet \rho$, equivalently, that $E_0 := \eta_{\tau\bullet\rho}^{-1}\left[g_{\rho,\tau}^{-1}[E]\right]$ is an $\tau \bullet \rho$-invariant Borel set in T by Lemma 1. But $E_0 = (\eta_\rho \circ \eta_\tau)^{-1}[E]$, so that E_0 is a Borel set by the measurability of the projections, and this set is clearly $\tau \bullet \rho$-invariant. Thus we get the measurability of E_0 again from Lemma 1.
3. Let $F \subseteq T/\tau \bullet \rho$ be a Borel set, hence $F_0 := \eta_{\tau\bullet\rho}^{-1}[F]$ is a Borel set in T, thus there exists a Borel set $F_1 \subseteq (T/\rho)/\tau$, such that $F_0 = \eta_\rho^{-1}\left[\eta_\tau^{-1}[F_1]\right]$ since F_0 is ρ-invariant. Hence $F_1 = h_{\rho,\tau}^{-1}[F]$, so $h_{\rho,\tau}$ is measurable, establishing the claim. □
Now fix a stochastic relation $\mathsf{K} = (X, Y, K)$, and let $\mathsf{c} = (\rho, \tau)$ be a congruence on K. Assume that $\mathsf{d} = (\kappa, \lambda)$ is a congruence of K/c. Define $\mathsf{d} \bullet \mathsf{c} := (\kappa \bullet \rho, \lambda \bullet \tau)$

Proposition 2 $\mathsf{d} \bullet \mathsf{c}$ *is a congruence on* K, *and* $\mathsf{K}/\mathsf{d} \bullet \mathsf{c}$ *is isomorphic to* $(\mathsf{K}/\mathsf{c})/\mathsf{d}$

Proof 1. The first assertion follows from Corollary 2 together with the observation that $(\kappa \bullet \rho, \lambda \bullet \tau) = (\mathsf{ker}\,(\eta_\kappa \circ \eta_\rho), \mathsf{ker}\,(\eta_\lambda \circ \eta_\tau))$ holds.
2. Construct the Borel isomorphisms $g_{\rho,\kappa} : X/\kappa \bullet \rho \to (X/\rho)/\kappa$ and $g_{\tau,\lambda} : Y/\lambda \bullet \tau \to (Y/\tau)/\lambda$ with their respective inverses $h_{\rho,\kappa}$ and $h_{\tau,\lambda}$ as in the proof of Proposition 1. We show that the inner and the outer diagram



both commute.
3. Let $B \in \mathcal{B}((Y/\tau)/\lambda)$, a Borel set in $(Y/\tau)/\lambda$, then

$$\begin{aligned} K_{\kappa\bullet\rho,\lambda\bullet\tau}([x]_{\kappa\bullet\rho})\left(g_{\tau,\lambda}^{-1}[B]\right) &= K(x)(\eta_{\lambda\bullet\tau}^{-1}\left[g_{\tau,\lambda}^{-1}[B]\right]) \\ &= K(x)(\eta_\tau^{-1}\left[\eta_\lambda^{-1}[B]\right]) \\ &= K_{\rho,\tau}([x]_\rho)(\eta_\lambda^{-1}[B]) \\ &= (K_{\rho,\tau})_{\kappa,\lambda}(g_{\rho,\kappa}([x]_\rho))(B), \end{aligned}$$

because $g_{\tau,\lambda} \circ \eta_{\lambda\bullet\tau} = \eta_\lambda \circ \eta_\tau$. Thus the outer diagram commutes. This implies that

$$\mathsf{g} := (g_{\rho,\kappa}, g_{\beta,\tau}) : \mathsf{K}/\mathsf{d} \bullet \mathsf{c} \to (\mathsf{K}/\mathsf{c})/\mathsf{d}$$

is a morphism.
4. Suppose that $G \in \mathcal{B}(Y/\lambda \bullet \tau)$ is a Borel set, then

$$\begin{aligned}
K_{\kappa\bullet\rho,\lambda\bullet\tau}(h_{\rho,\kappa}(\left[[x]_\rho\right]_\kappa)(G) &= K_{\kappa\bullet\rho,\lambda\bullet\tau}([x]_{\kappa\bullet\rho})(G) \\
&= K(x)(\eta^{-1}_{\lambda\bullet\tau}[G]) \\
&= K_{\rho,\tau}([x]_\rho)(\eta^{-1}_\lambda\left[h^{-1}_{\tau,\lambda}[G]\right]) \\
&= (K_{\rho,\tau})_{\kappa,\lambda}(\left[[x]_\rho\right]_\lambda)(h^{-1}_{\beta,\tau}[G])
\end{aligned}$$

This is so since $\eta_{\lambda\bullet\tau} = h_{\tau,\lambda} \circ \eta_\lambda \circ \eta_\tau$ holds (see the proof of Proposition 1). Thus the inner diagram commutes. This implies that

$$\mathsf{h} := (h_{\rho,\kappa}, h_{\beta,\tau}) : (\mathsf{K}/\mathsf{c})/\mathsf{d} \to \mathsf{K}/\mathsf{d} \bullet \mathsf{c}$$

is a morphism, and h is plainly left- and right inverse to g. □
Factoring a stochastic relation with a congruence entails identifying inputs resp. outputs that have been observed as representing identical behavior. Proposition 2 says then that identifying identical behavior in observing the factor system amounts to a system that can also be obtained through a single observational step from the original system. This means that there are no arbitrary long chains of factor systems which could not have been obtained directly from the original system, or, that factoring does not change the fundamental behavior of a system (after all, a system is bisimilar to its factor systems, bisimilarity requesting the existence of a span of morphisms, cp. [2, 4]).
Algebraically, this proposition is quite similar to the well known Second Isomorphism Theorem of Group Theory, cp. [9, § I.4]: Factoring the quotient of a normal subgroup gives a group isomorphic to a factor. A similar but slightly stronger construction for coalgebras is carried out by Rutten [10, Theorem 7.4] in the context of bisimulation relations for coalgebras. Proposition 2 and Rutten's Theorem are not comparable directly, however, since the functor underlying the coalgebra is assumed to have weak pullbacks (which is no realistic assumption for stochastic relations, see [4, Remark 2]), and since the relationship between bisimulations and congruences is slightly less involved in the coalgebraic case.
Let (α, β) and (α', β') be pairs of equivalence relations, and define $(\alpha, \beta) \preceq (\alpha', \beta')$ iff α refines α' and β refines β' simultaneously, formally:

$$(\alpha, \beta) \preceq (\alpha', \beta') \Leftrightarrow \alpha \subseteq \alpha' \text{ and } \beta \subseteq \beta'$$

It is clear that $\mathsf{c} \preceq \mathsf{d} \bullet \mathsf{c}$ for each congruence d.

Proposition 3 *Assume that* $\mathsf{f} : \mathsf{K} \to \mathsf{K}'$ *is a morphism, and let* c *be a congruence on* K *such that* $\mathsf{c} \preceq \ker(\mathsf{f})$. *Then there exists a unique morphism* $\mathsf{f}_\mathsf{c} : \mathsf{K}/\mathsf{c} \to \mathsf{K}'$ *with* $\mathsf{f} = \mathsf{f}_\mathsf{c} \circ \eta_\mathsf{c}$

Proof 1. Let $\mathsf{K} = (X, Y, K), \mathsf{K}' = (X', Y', K')$ with $\phi : X \to X', \psi : Y \to Y'$ constituting morphism f, and $\mathsf{c} = (\alpha, \beta)$. Because $\alpha \subseteq \ker(\phi), \beta \subseteq \ker(\psi)$, the maps

$$\begin{aligned}
\phi_\alpha([x]_\alpha) &:= \phi(x), \\
\psi_\beta([y]_\beta) &:= \psi(y)
\end{aligned}$$

are well defined. Since ϕ is $\mathcal{B}(X) - \mathcal{B}(X')$-measurable, and since $\mathcal{B}(X)/\alpha$ is the final σ-algebra on X/α with respect to η_α, $\mathcal{B}(X)/\alpha - \mathcal{B}(X')$-measurability of ϕ_α is inferred. A similar argument is used for ψ_β. Clearly, these maps are onto.
2. It remains to show that $\mathsf{f_c} := (\phi_\alpha, \psi_\beta)$ is a morphism. In fact, let $D' \subseteq Y'$ be a Borel set, then

$$\begin{aligned} K'(\phi_\alpha([x]_\alpha))(D') &= K'(\phi(x))(D') \\ &= K(x)(\psi^{-1}\left[D'\right]) \\ &= K_{\alpha,\beta}([x]_\alpha)(\psi_\beta^{-1}\left[D'\right]) \\ &= (\mathfrak{S}\left(\psi_\beta\right) \circ K_{\alpha,\beta})\left([x]_\alpha\right)(D') \end{aligned}$$

because $\psi^{-1}\left[D'\right] = \eta_\beta^{-1}\left[\psi_\beta^{-1}\left[D'\right]\right]$, and because $(\eta_\alpha, \eta_\beta)$ is a morphism. Consequently, the equality $K' \circ \phi_\alpha = \mathfrak{S}\left(\psi_\beta\right) \circ K_{\alpha,\beta}$ has been established. Uniqueness is obvious. □

Corollary 3 *Assume that* $\mathsf{f} : \mathsf{K} \to \mathsf{K}'$ *is a morphism. Then there exists a unique isomorphism* $\mathsf{f}^\sharp : \mathsf{K}/\mathsf{ker}\left(\mathsf{f}\right) \to \mathsf{K}'$ *with* $\mathsf{f} = \mathsf{f}^\sharp \circ \eta_{\mathsf{ker(f)}}$.

Proof Define $\mathsf{f}^\sharp := \mathsf{f}_{\mathsf{ker(f)}}$, then the maps constituting this morphism are bijective Borel maps, so by [11, Proposition 4.5.1] they are Borel isomorphisms. The equations establishing the morphism property for $\mathsf{f}_{\mathsf{ker(f)}}$ show that the inverses also constitute a morphism. □

Corollary 4 *Let* c *and* d *be congruences on* K*, then the following statement are equivalent:*

1. $\mathsf{c} \preceq \mathsf{d}$
2. $\mathsf{d} = \mathsf{e} \bullet \mathsf{c}$ *for some congruence* e *on* K.

Proof The implication (2) $\Rightarrow$ (1) is obvious. Assume that $\mathsf{c} \preceq \mathsf{d} = \mathsf{ker}\left(\eta_{\mathsf{d}}\right)$ holds. Then the assertion follows from Proposition 3 together with Corollary 2. □
This property is somewhat surprising in that it relates the refinement of congruences to factor spaces. If c is finer than d, then congruence d can be obtained through observing and factoring the behavior in the factor system for c (so that not the original system has to be observed but rather a simplified one).

4 Conclusion

Let's wrap things up by considering labelled Markov transition systems. Fix an analytic state space S and a countable set A of actions. Then $\mathbb{K} := (S, (k_a)_{a\in A})$ is called a *labelled Markov transition process* iff $k_a : S \rightsquigarrow S$ is a stochastic relation for each $a \in A$, see e.g. [2, 4]. The surjective Borel map $\phi : S \to S'$ constitutes a morphism $(S, (k_a)_{a\in A}) \to (S', (\ell_a)_{a\in A})$ iff $\ell_a \circ \phi = \mathfrak{S}\left(\phi\right) \circ k_a$ holds for each action $a \in A$, or, equivalently, if $(\phi, \phi) : (S, S, k_a) \to (S', S', \ell_a)$ holds in **Stoch** for each $a \in A$. Similarly, a smooth equivalence relation γ on S is said to be a *congruence on* $\mathbb{K}$ iff

$$s\,\gamma\,s' \Rightarrow \forall C \in \mathcal{I}\left(\mathcal{B}(S), \gamma\right) : k_a(s)(C) = k_a(s')(C)$$

holds for each $a \in A$. This generalizes the relation defined through the Hennessy-Milner equivalence for a simple negation free modal logic with $\langle a\rangle_q$ as diamonds for $a \in A$ and rational q that is investigated in [2, 4].

It is clear that a congruence γ gives rise to a factor system $\mathbb{K}/\gamma := (S, (k_{a,\gamma})_{a\in A})$, and that the kernel of a morphism is a congruence. We obtain then from the discussion above

1. If γ is a congruence on $\mathbb{K}$, and δ is a congruence on $\mathbb{K}/\gamma$, then $(\mathbb{K}/\gamma)/\delta$ is isomorphic to $\mathbb{K}/\delta \bullet \gamma$,
2. Each morphism $\phi : \mathbb{K} \to \mathbb{L}$ factors uniquely through $\mathbb{K}/\mathsf{ker}\,(\phi)$, and $\mathbb{K}/\mathsf{ker}\,(\phi)$ is isomorphic to $\mathbb{L}$.

In this way, investigating congruences for stochastic relations turns out to be a fruitful endeavor for labelled Markov transition systems.

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