Surface Homeomorphisms: the interplay between Topology, Geometry and Dynamics

by

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Declarations

I declare that this thesis is my own original work, except where stated otherwise, conducted under the supervision of Professor Sebastian van Strien. To the best of my knowledge, this thesis contains no material previously published or written by another person, except where due reference has been made. This thesis has not been submitted for a degree at another university.

The results in chapters 2, 3 and 4 of this thesis have resulted in the following three papers respectively.

- *Minimal sets of non-resonant torus homeomorphisms*, by F. Kwakkel (to be submitted),
- *Topological entropy and diffeomorphisms of surfaces with wandering domains*, by F. Kwakkel and V. Markovic, to appear in Annales Academiae Scientiarum Fennicae.
- *Quasiconformal homogeneity of genus zero surfaces*, by F. Kwakkel and V. Markovic, to appear in Journal d'Analyse Mathématique.

Further, during my appointment at CODY I worked, jointly with Mauricio Peixoto at IMPA, on problems related to the *focal decomposition of surfaces*, extending the results in [30]. This is work in progress.

Abstract

In this thesis we study certain classes of surface homeomorphisms and in particular the interplay between the topology of the underlying surface and topological, geometrical and dynamical properties of the homeomorphisms. We study three problems in three independent chapters:

The first problem is to describe the minimal sets of non-resonant torus homeomorphisms, i.e. those homeomorphisms which are in a sense close to a minimal translation of the torus. We study the possible minimal sets that such a homeomorphism can admit, uniqueness of minimal sets and their relation with other limit sets. Further, we give examples of homeomorphisms to illustrate the possible dynamics. In a sense, this study is a two-dimensional analogue of H. Poincaré's study of orbit structures of orientation preserving circle homeomorphisms without periodic points.

The second problem concerns the interplay between smoothness of surface diffeomorphisms, entropy and the existence of wandering domains. Every surface admits homeomorphisms with positive entropy that permutes a dense collection of domains that have bounded geometry. However, we show that at a certain level of differentiability it becomes impossible for a diffeomorphism of a surface to have positive entropy and permute a dense collection of domains that has bounded geometry.

The third problem concerns quasiconformal homogeneity of surfaces; i.e., whether a surface admits a transitive family of quasiconformal homeomorphisms, with an upper bound on the maximal distortion of these homeomorphisms. In the setting of hyperbolic surfaces, this turns out to be a very intriguing question. Our main result states that there exists a universal lower bound on the maximal dilatation of elements of a transitive family of quasiconformal homeomorphisms on a hyperbolic surface of genus zero.

Chapter 1

Introduction

The main theme of this thesis is the study of certain classes of surface homeomorphisms and in particular the interplay between the topology of the underlying surface and topological, geometrical and dynamical properties of the homeomorphisms. We will study three independent problems concerning three classes of homeomorphisms in the chapters 2, 3 and 4 below. In this introductory chapter, we give a description of the problems we study in later chapters. We refer to the relevant chapters for the precise definitions, results and references.

The first problem is to describe the *minimal sets of non-resonant torus homeomorphisms*, i.e. homeomorphisms of the torus, isotopic to the identity, for which the translation set is a single point with rationally independent irrational coordinates. The translations of the torus with this property are exactly those for which every orbit is dense in the torus. This class of torus homeomorphisms is a natural two-dimensional analogue of the class of orientation preserving circle homeomorphisms of the circle without periodic points, which were first systematically studied and their (topological) behaviour classified by H. Poincaré around 1880. We classify the possible minimal sets these non-resonant homeomorphism can admit, uniqueness of minimal sets and their relation with other limit sets. We show that the minimal sets come in three different types. Roughly speaking, these are: (I) minimal sets for which the components of the complement in the torus are all open topological disks, (II) minimal sets for which the components of the complement of the minimal set contain essential annuli and (III) the minimal sets that are in a sense a topological extension of a Cantor set. Using this classification, we prove that the only locally connected minimal sets are sets closely resembling, but not necessarily homeomorphic to, a Sierpiński set of the torus.

Further, we construct homeomorphisms that admit minimal sets of all above types, including some rather exotic minimal sets. These results can be found in chapter 2.

In chapter 3, we study the *interplay between smoothness of surface diffeomorphisms, topological entropy and the existence of wandering domains.* A wandering domain is a domain in the surface such that the iterates of this domain under the diffeomorphism are mutually disjoint. We say a diffeomorphism permutes a dense collection of domains if there exists a dense collection of domains, with disjoint closures, that are wandering. Further, a collection of domains in the surface is said to have bounded geometry if every domain can be contained in a ball and in turn contains a ball, such that the ratio of the radii of these balls is bounded from above.

It is not difficult to construct examples of homeomorphisms with positive entropy that permute a dense collection of domains with bounded geometry. When one requires the domains to have bounded geometry, then the more regular the homeomorphism, the more difficult it becomes for it to simultaneously have positive entropy and retain the bounded geometry property for the domains it permutes. We show that at a certain level of differentiability, it becomes impossible for a diffeomorphism to have positive entropy and permute a dense collection of domains with bounded geometry. The idea of the proof is to show that the geometry of the domains puts strong bounds on the maximal dilatation of the diffeomorphism on the complement of the permuted domains. Using the differentiability assumptions, and geometrical estimates that relate to the topological entropy, we then show that the maximal dilatation grows at a rate that is slow enough to ensure the topological entropy is zero.

The third problem, see chapter 4, concerns *quasiconformal homogeneity of Riemann surfaces*. Given a Riemann surface M, it is said to be K-quasiconformally homogeneous if the exist a transitive family of K-quasi-conformal homeomorphisms of M, where K is the smallest such constant $K \geq 1$. The notion of quasiconformal homogeneity of surfaces was introduced by Gehring and Palka in 1976. It is easy to see that the surfaces $\mathbb{C}$, $\mathbb{C}^*$ and $\mathbb{D}^2$ are 1-quasiconformally (i.e. conformally) homogeneous, and so are the surfaces $\mathbb{T}^2$, the torus, and $\mathbb{P}^1$, the Riemann sphere. It can be shown that these are the only conformally homogeneous Riemann surfaces. In other words, any other surface is K-quasiconformally homogeneous, where $1 < K \leq \infty$.

The following problem naturally presents itself: given a class of surfaces, are the quasiconformality constants of these surfaces uniformly bounded away from 1? Natural classes to consider are genus zero surfaces, i.e. those surfaces that can be embedded in the Riemann sphere and closed surfaces of genus $g \geq 2$. In chapter 4, we focus our attention to the former class, i.e. genus zero surfaces. Our main result states that there exists a universal lower bound $\mathcal{K} > 1$ such that if M is any hyperbolic genus zero surface, then $K \geq \mathcal{K}$. The proof of this result makes essential use of the fact that the genus of the surface is zero and the idea of the proof is as follows.

If M is K-quasiconformally homogeneous, then the lengths of the (homotopically non-trivial) simple closed geodesics on M is uniformly bounded from below. On a planar surface, any two simple closed geodesics that intersect do so in an even number of intersection points (which clearly fails to be true on a surface of higher genus). Using the transitivity of the family of K-quasiconformal homeomorphisms of M, we construct configurations of simple closed geodesics that intersect in a particular way. In the near conformal limit, i.e. if K is sufficiently close to 1, we then show that these configurations contain essential closed curves whose length is strictly less than that of the shortest closed curve the surface allows and thus these configurations can not exist.

It is the fact that the genus of the surface M is zero, and thus geodesics intersect in an even number, that makes that this argument works well; it is difficult to see how to construct similar configurations of simple closed curves on closed surfaces of genus $g \geq 2$. Nevertheless, the (as yet) unanswered case of the closed surfaces is very interesting, see the open problem section at the end of chapter 4.

Chapter 2

Minimal Sets of Non-Resonant Torus Homeomorphisms

As was known to H. Poincaré, an orientation preserving circle homeomorphism without periodic points is either minimal or has no dense orbits, and every orbit accumulates on the unique minimal set. In the first case the minimal set is the circle, in the latter case a Cantor set. In this chapter we study a two-dimensional analogue of this classical result: we classify the minimal sets of non-resonant torus homeomorphisms; that is, torus homeomorphisms isotopic to the identity for which the rotation set is a point with rationally independent irrational coordinates.

2.1 Definitions and statement of results

Let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ and $f\colon \mathbb{T}^1 \to \mathbb{T}^1$ an orientation preserving circle homeomorphism. A lift $F\colon \mathbb{R} \to \mathbb{R}$ of f satisfies $f \circ p_1 = p_1 \circ F$, with $p_1\colon \mathbb{R} \to \mathbb{T}^1$ the canonical projection. The number

$$\rho(F,x) := \lim_{n\to\infty} \frac{\Phi^n(x) - x}{n}, \tag{2.1}$$

exists for all $x \in \mathbb{R}$, is independent of x and well defined up to an integer; that is, if F and $\widehat{F}$ are two lifts of f then $\rho(F) - \rho(\widehat{F}) \in \mathbb{Z}$. The number $\rho(f) := \rho(F,x) \mod \mathbb{Z}$

is called the *rotation number* of f and $\rho(f) \in \mathbb{Q}$ if and only if f has periodic points. Denote $r_\theta\colon \mathbb{T}^1 \to \mathbb{T}^1$ the rigid rotation of the circle with rotation number θ. The following classical result classifies the possible topological dynamics of orientation preserving homeomorphisms of the circle without periodic points [38, 39, 40].

Poincaré Classification Theorem. *Let $f\colon \mathbb{T}^1 \to \mathbb{T}^1$ be an orientation preserving homeomorphism such that $\rho(f) \in \mathbb{R} \setminus \mathbb{Q}$. Then*

(i) if f is transitive then f is conjugate to the rigid rotation $r_{\rho(f)}$, and

(ii) if f is not transitive then f is semi-conjugate to the rotation $r_{\rho(f)}$ via a non-invertible continuous monotone map.

Moreover, f has a unique minimal set $\mathcal{M}$, which is the circle $\mathbb{T}^1$ in case (i)*, or a Cantor set in case* (ii) *and $\mathcal{M} = \Omega(f) = \omega(x) = \alpha(x) =$ for all $x \in \mathbb{T}^1$.*

Every connected component I of the complement of the Cantor minimal set is a wandering interval, i.e. $f^n(I) \cap I = \emptyset$, for all $n \neq 0$. Given a Cantor set in the circle, there exists a circle homeomorphism with any given irrational rotation number that has this Cantor set as its minimal set $\mathcal{M}$. This fact was first explicitly mentioned by Denjoy [14], but essentially known already by Bohl [7] and Kneser [26]. Denjoy [14] proved that an orientation preserving circle diffeomorphism $f \in \mathrm{Diff}^2(\mathbb{T}^1)$ with irrational rotation number is necessarily transitive and hence can not have a wandering interval, see also [20].

The key feature of orientation preserving circle homeomorphisms without periodic points is that it has an irrational rotation number which is independent of the basepoint, where the rotation with the corresponding rotation number is minimal. A natural generalization to dimension two is as follows. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, where $p\colon \mathbb{R}^2 \to \mathbb{T}^2$ is the canonical projection mapping. We denote $\mathrm{Homeo}(\mathbb{T}^2)$ the class of homeomorphisms of the torus. Given an element $f \in \mathrm{Homeo}(\mathbb{T}^2)$, we denote $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ a lift to the cover. Any two different lifts $F, \widehat{F}$ of f differ by an integer translation, that is, $F(z) = \widehat{F}(z) + (n, m)$ where $(n, m) \in \mathbb{Z}^2$. Let $\mathrm{Homeo}_0(\mathbb{T}^2) \subset \mathrm{Homeo}(\mathbb{T}^2)$ the subclass of homeomorphisms isotopic to the identity, i.e. those homeomorphisms whose lifts commute with integer translations. We denote $\rho(f)$ the rotation set corresponding to $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ (see section 2.2.2 below for precise definitions). We define

$$\mathrm{Homeo}_*(\mathbb{T}^2) \subset \mathrm{Homeo}_0(\mathbb{T}^2), \tag{2.2}$$

the class of homeomorphisms isotopic to the identity for which the rotation set $\rho(f) = (\alpha, \beta) \mod \mathbb{Z}^2$ is a single point for which the numbers $1, \alpha, \beta$ are rationally independent. These homeomorphisms are said to be *non-resonant torus homeomorphisms.*

Generalizations of Poincaré's Theorem and Denjoy's Theorem have recently attracted much attention, perhaps most notably the work of F. Béguin, S. Crovisier, T. Jäger, G. Keller, F. le Roux and J. Stark [3, 22, 23], where one considers an analogous class of torus homeomorphisms, namely *quasiperiodically forced circle homeomorphisms*, which are torus homeomorphisms of the form $(x, \theta) \mapsto (x + \alpha, g_\theta(x)) \mod \mathbb{Z}^2$, with $(x, \theta) \in \mathbb{T}^2$ and $g_\theta \colon \mathbb{T}^1 \to \mathbb{T}^1$ a family of circle homeomorphisms and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. In [3, 22, 23], the appropriate analogues of the results of Poincaré and Denjoy in the class of quasiperiodically forced circle homeomorphisms are developed. Analogues of Poincaré's Theorem in the setting of conservative torus homeomorphisms are developed by T. Jäger in [21].

To state our results, we need the following definitions. A connected set $X \subset \mathbb{T}^2$ is said to be *(un)bounded* according to whether a lift $\widetilde{X} \subset \mathbb{R}^2$ is (un)bounded as a subset of $\mathbb{R}^2$, where a lift $\widetilde{X}$ of X is a connected component of $p^{-1}(X)$. A compact and connected set is called a *continuum.* A continuum $\mathcal{C}$ in $\mathbb{T}^2$ is called *non-separating* if the complement in $\mathbb{T}^2$ is connected. Given a bounded continuum $\mathcal{C} \subset \mathbb{T}^2$, we define $\mathrm{Fill}(\mathcal{C}) \subset \mathbb{T}^2$, the *filled continuum*, the smallest (with respect to inclusion) non-separating bounded continuum containing $\mathcal{C}$. A bounded non-separating continuum in $\mathbb{T}^2$ is called *acyclic.*

Definition 2.1 (Extension of a Cantor set)**.** Let $\mathcal{M}$ be a minimal set for $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$, with $\{\Lambda_i\}_{i\in I}$ be the collection of connected components of $\mathcal{M}$. If $\mathrm{Fill}(\Lambda_i)$ is acyclic for every $i \in I$, $\mathrm{Fill}(\Lambda_i) \cap \mathrm{Fill}(\Lambda_j) = \emptyset$ if $i \neq j$ and there exists a continuous $\phi : \mathbb{T}^2 \to \mathbb{T}^2$, homotopic to the identity, and an $\widehat{f} \in \mathrm{Homeo}_*(\mathbb{T}^2)$, such that

(i) $\phi \circ f = \widehat{f} \circ \phi$, i.e. f is semi-conjugate to $\widehat{f}$, and

(ii) $\widehat{\mathcal{M}} := \phi(\widehat{\mathcal{Q}}) \subset \mathbb{T}^2$ is a Cantor minimal set for $\widehat{f}$,

where $\widehat{\mathcal{Q}} = \bigcup_{i\in I} \mathrm{Fill}(\Lambda_i)$, then we say $\mathcal{M}$ is an *extension of a Cantor set.*

Put in words, $\mathcal{M}$ is an extension of a Cantor set, if the semi-conjugacy ϕ between f and $\widehat{f}$ sends the collection of filled in components of $\mathcal{M}$ to points in a one-to-one fashion, and the corresponding totally disconnected set $\widehat{\mathcal{M}}$ is a Cantor minimal set of the factor $\widehat{f}$. An extension of a Cantor set is called *non-trivial*, if there exist components of $\mathcal{M}$ that are not singletons.

Further, we define the following. A *disk* $D \subset \mathbb{T}^2$ in the torus is an injection by a homeomorphism of the open unit disk $\mathbb{D}^2 \subset \mathbb{R}^2$ into the torus. An *annulus* $A \subset \mathbb{T}^2$ is an injection by a homeomorphism of the open annulus $\mathbb{S}^1 \times (0,1)$ into the torus, where A is said to be *essential* if the inclusion $A \hookrightarrow \mathbb{T}^2$ induces an injection of $\pi_1(A)$ into $\pi_1(\mathbb{T}^2)$. Let us now state our main results. Our first result gives a classification of the possible minimal sets of homeomorphisms in our class $\mathrm{Homeo}_*(\mathbb{T}^2)$.

Theorem 2.A (Classification of minimal sets)**.** *Let $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ and let $\mathcal{M}$ be a minimal set for f. Let $\{\Sigma_k\}_{k\in\mathbb{Z}}$ be the connected components of the complement $\mathcal{M}$ in $\mathbb{T}^2$. If $\mathcal{M} \neq \mathbb{T}^2$, then either*

(I) *$\{\Sigma_k\}$ is a collection of bounded and unbounded disks,*

(II) *$\{\Sigma_k\}$ is a collection of essential annuli and bounded disks,*

(III) *$\mathcal{M}$ is an extension of a Cantor set.*

This result is proved in sections 2.3.1-2.3.2. It follows from the proof of Theorem 2.A that

Corollary 2.1 (Structure of orbits; type I and II)**.** *Let* $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ *with a minimal set* $\mathcal{M}$ *of type I or II. Then*

$$\mathcal{M} = \Omega(f) = \omega(z) = \alpha(z), \tag{2.3}$$

for all $z \in \mathbb{T}^2$*. In particular,* $\mathcal{M}$ *is unique.*

In [3, Thm 1.2], F. Béguin, S. Crovisier, T. Jäger and F. le Roux construct a counterexample to Corollary 2.1 in the case where $\mathcal{M}$ is of type III in the setting of quasiperiodically forced circle homeomorphisms. Formulated in our terminology, this result reads

Counterexample 2.2 (Structure of orbits; type III [3])**.** *There exist homeomorphisms* $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ *which have a unique Cantor minimal set* $\mathcal{M}$ *(and are thus of type III), but are transitive.*

In other words, $\mathcal{M} \neq \mathbb{T}^2$ is the unique Cantor minimal set, but $\Omega(f) = \mathbb{T}^2$. Uniqueness of minimal sets of type III homeomorphisms has not been settled, see the open problem section at the end of this chapter. Further, we have that

Corollary 2.3 (Connected minimal sets)**.** *Let* $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$*. Then* $\mathcal{M}$ *is connected if and only if* $\mathcal{M}$ *is of type I.*

See section 2.3.2 for the proofs of the above two corollaries. To state our second result, we need the following. Recall that a *null-sequence* is a sequence of positive real numbers for which for every given $\epsilon > 0$ there exist only finitely many elements of the sequence that are greater than ϵ.

Definition 2.2 (quasi-Sierpiński set)**.** A *quasi-Sierpiński set* is a continuum $S = \mathbb{T}^2 \setminus \bigcup_{k\in\mathbb{Z}} D_k$ with $\{D_k\}_{k\in\mathbb{Z}}$ a family of disks such that $\bigcup_{k\in\mathbb{Z}} D_k$ is dense in $\mathbb{T}^2$, and

a) D_k is the interior of a closed embedded disk, for every $k \in \mathbb{Z}$,

b) $\mathrm{Cl}(D_k) \cap \mathrm{Cl}(D_{k'})$ is at most a single point if $k \neq k'$, and

c) $\mathrm{diam}(D_k)$, $k \in \mathbb{Z}$, is a null-sequence.

If property b) above is replaced by the condition that $\mathrm{Cl}(D_k) \cap \mathrm{Cl}(D_{k'}) = \emptyset$, if $k \neq k'$, then we refer to S as a *Sierpiński set*.

A closed subset of a topological space is *locally connected*, if every of its points has arbitrarily small connected neighbourhoods. Requiring a minimal set to be locally connected, reduces the list of Theorem 2.A to one type of non-trivial minimal set, see section 2.3.3 for the proof.

Theorem 2.B (Locally connected minimal sets)**.** *Let* $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ *and suppose that the minimal set* $\mathcal{M}$ *of* f *is locally connected. Then either* $\mathcal{M} = \mathbb{T}^2$ *or* $\mathcal{M}$ *is a quasi-Sierpiński set.*

This result was (essentially) proved by A. Biś, H. Nakayama and P. Walczak in [4] where locally connected minimal sets of general homeomorphisms of closed surfaces are classified; it is shown that any locally connected minimal set of a homeomorphism of a surface other than the torus $\mathbb{T}^2$ is either a finite set of points or a finite union of disjoint simple closed curves. In the case of the torus, it is shown that, in addition to these locally connected minimal sets, any other locally connected minimal set is a quasi-Sierpiński set (in our terminology). We show how this result, for our class of homeomorphisms, can be recovered from Theorem 2.A above, and our line of approach is different. Rather than assuming the minimal set is locally connected from the start as in [4], we start with the list of minimal sets of Theorem 2.A and show that most of these minimal sets are not locally connected, ultimately arriving at the only possible locally connected minimal set, a quasi-Sierpiński set.

Our final result says that the classification of Theorem 2.A. is sharp in the following sense.

Theorem 2.C (Existence of minimal sets)**.** *Every type of minimal set Theorem 2.A. allows is realized by homeomorphisms in* $\mathrm{Homeo}_*(\mathbb{T}^2)$.

This result is proved by a number of examples in section 2.4. Let us briefly discuss these. It is well-known there exist homeomorphisms $f \in \text{Homeo}_*(\mathbb{T}^2)$ for which the minimal set is a Sierpiński set. The first example given, a locally connected quasi-Sierpiński minimal set that is not a Sierpiński set, is known [4]. The examples constructed in section 2.4 are a minimal set for which the complement is a single unbounded disk (type I), a minimal set for which the complement components are essential annuli and bounded disks (type II) and examples of rather exotic non-trivial extensions of Cantor sets (type III), where the minimal sets constructed are homeomorphic to Cantor dust interspersed with various continua.

In section 2.5, we discuss some open problems related to the results obtained.

2.2 Preliminary results

Let us first introduce some background results and set notation used throughout the remainder of this chapter.

2.2.1 Limit sets

A *minimal set* $\mathcal{M}$ of a homeomorphism $f \in \text{Homeo}(\mathbb{T}^2)$ is a non-empty closed and f-invariant set $\mathcal{M} \subseteq \mathbb{T}^2$ that is minimal (relative to inclusion) with respect to the properties of being non-empty, closed and invariant. As $\mathbb{T}^2$ is compact, every $f \in \text{Homeo}(\mathbb{T}^2)$ admits at least one minimal set (see e.g. [47, Thm 5.2]).

The *non-wandering set* $\Omega(f)$ is defined as

$$\Omega(f) = \{z \in \mathbb{T}^2 \mid \forall U \ni z \ \exists n \neq 0 \text{ with } f^n(U) \cap U \neq \emptyset\}. \tag{2.4}$$

and, for $z \in \mathbb{T}^2$, the *omega limit set* $\omega(z)$ and *alpha limit set* $\alpha(z)$ are defined by

$$\begin{aligned} \omega(z) &= \{w \in \mathbb{T}^2 \mid \exists\, n_k \text{ such that } f^{n_k}(z) \to w, \text{ for } k \to \infty\} \\ \alpha(z) &= \{w \in \mathbb{T}^2 \mid \exists\, n_k \text{ such that } f^{-n_k}(z) \to w, \text{ for } k \to \infty\} \end{aligned}$$

respectively. The sets $\Omega(f)$, $\omega(z)$ and $\alpha(z)$ are closed and f-invariant and the following inclusions hold

$$\mathcal{M} \subseteq \Omega(f) \text{ and } \omega(z), \alpha(z) \subseteq \Omega(f), \tag{2.5}$$

for every $z \in \mathbb{T}^2$.

2.2.2 Rotation sets

The notion of rotation number for orientation preserving homeomorphisms of the circle is generalized in [34] to homeomorphisms $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$, as follows.

Definition 2.3. Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ and $F : \mathbb{R}^2 \to \mathbb{R}^2$ a lift of f. Define $\rho(F)$ as

$$\rho(F) = \bigcap_{m=1}^{\infty} \mathrm{Cl}\left(\bigcup_{n=m}^{\infty} \left\{ \frac{F^n(\widetilde{z}) - \widetilde{z}}{n} \mid \widetilde{z} \in \mathbb{R}^2 \right\} \right) \subset \mathbb{R}^2. \tag{2.6}$$

Relative to a single basepoint $\widetilde{z} \in \mathbb{R}^2$, this reads

$$\rho(F, \widetilde{z}) = \bigcap_{m=1}^{\infty} \mathrm{Cl}\left(\frac{F^n(\widetilde{z}) - \widetilde{z}}{n} \mid n > m \right). \tag{2.7}$$

The *rotation set* is defined as $\rho(f) = \rho(F) \mod \mathbb{Z}^2$ and (the pointwise rotation set) $\rho(f, z) = \rho(F, \widetilde{z}) \mod \mathbb{Z}^2$, where $z = p(\widetilde{z})$.

In words, $\rho(F)$ collects all limit points of the sequences

$$\frac{F^{n_k}(\widetilde{z}_k) - \widetilde{z}_k}{n_k}$$

with $n_k \to \infty$ for $k \to \infty$ and $\widetilde{z}_k \in \mathbb{R}^2$. The rotation set for an $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is in general no longer a single point, but a convex compact and connected set (see again [34]).

We say the vector $(\alpha, \beta) \in \mathbb{R}^2$ is *irrational*, if the numbers $1, \alpha, \beta$ are rationally independent; that is, if the only solution over the integers of

$$N_1 + N_2\alpha + N_3\beta = 0 \tag{2.8}$$

is $N_1 = N_2 = N_3 = 0$. The translation $\tau\colon \mathbb{T}^2 \to \mathbb{T}^2$ corresponding to (α, β),

$$\tau\colon (x, y) \mapsto (x + \alpha, y + \beta) \mod \mathbb{Z}^2, \tag{2.9}$$

is minimal if and only if the vector (α, β) is irrational. The class of homeomorphisms of the torus $\mathbb{T}^2$ isotopic to the identity with rotation set consisting of a single irrational vector will be denoted by $\mathrm{Homeo}_*(\mathbb{T}^2)$. It is easy to see that a homeomorphism $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ has no periodic points.

Lemma 2.4. *Let $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$. If $X \subset \mathbb{T}^2$ is a bounded connected set, then $f^n(X) \neq X$, for all $n \neq 0$.*

Proof. If $X \subset \mathbb{T}^2$ is bounded and $f^N(X) = X$, for some $N \neq 0$, we can take a lift F of f and a lift $\widetilde{X}$ of X such that $F^N(\widetilde{X}) = \widetilde{X}$. Let $\widetilde{z} \in \widetilde{X}$. As $\widetilde{X}$ is bounded, we must have that $\rho(F, \widetilde{z}) = (0, 0)$ and thus $\rho(f, z) = (0, 0) \mod \mathbb{Z}^2$, where $z = p(\widetilde{z})$, contrary to our assumption on the rotation set. $\square$

In other words, if $X \subset \mathbb{T}^2$ is a connected and f-invariant set, then X is necessarily unbounded.

Lemma 2.5. *Let $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ and F a lift of f. Let $D \subset \mathbb{R}^2$ be a closed topological disk. Then there exists no $N \neq 0$, and $(p, q) \in \mathbb{Z}^2$, such that*

$$F^N(D) \subseteq T_{p,q}(D) \quad \text{or} \quad T_{p,q}(D) \subseteq F^N(D).$$

Proof. Suppose that there exists an $N \neq 0$ and $(p, q) \in \mathbb{Z}^2$ such that $F^N(D) \subseteq T_{p,q}(D)$. Choosing a different lift $\widehat{F}$ if necessary, we may assume that $\widehat{F}^N(D) \subseteq D$. By the Brouwer Fixed Point Theorem, $\widehat{F}^N$ has a fixed point on D, and thus f has a periodic point, contrary to our assumptions. The case where $T_{p,q}(D) \subseteq F^N(D)$ follows by considering the inverse F^{-1}. $\square$

2.2.3 Topology of torus domains

Next, we turn to the topology of domains in the torus. In the subsequent proof, the various topological types of domains on the torus play an important role. In what follows, a *domain* is an open connected set. Let $\gamma \subset \mathbb{T}^2$ be an essential simple closed curve. We say the curve γ has homotopy type (p,q) if γ lifts to a curve $\widetilde{\gamma} \in \mathbb{R}^2$ such that, up to a suitable translation, $\widetilde{\gamma}$ connects the lattice points $(0,0) \in \mathbb{Z}^2$ and $(p,q) \in \mathbb{Z}^2$ with p and q coprime. If we define

$$T_{p,q} \colon \mathbb{R}^2 \to \mathbb{R}^2, \quad T_{p,q}(x,y) = (x+p, y+q), \tag{2.10}$$

then $\widetilde{\gamma}$ is periodic in the sense that

$$\widetilde{\gamma} = \bigcup_{n \in \mathbb{Z}} T^n_{p,q}(\eta), \tag{2.11}$$

where $\eta \subset \widetilde{\gamma}$ is the arc connecting $(0,0)$ and (p,q). Let $D \subset \mathbb{T}^2$ be a domain. The inclusion $D \hookrightarrow \mathbb{T}^2$ naturally induces an injection of $\pi_1(D)$ into $\pi_1(\mathbb{T}^2)$. This gives rise to the following

Definition 2.4 (Types of domains)**.** A domain $D \subset \mathbb{T}^2$ is said to be *trivial*, *essential* or *doubly essential* according to whether the inclusion of $\pi_1(D)$ into $\pi_1(\mathbb{T}^2)$ is isomorphic to $0, \mathbb{Z}$ or $\mathbb{Z}^2$ respectively.

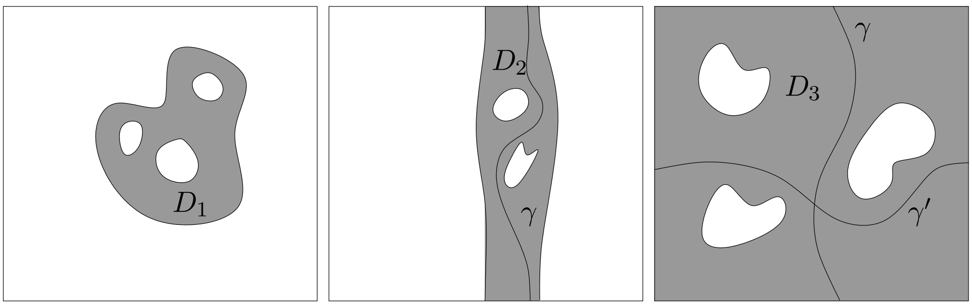

Figure 2.1: A trivial, essential and doubly essential domain D_1, D_2 and D_3 in $\mathbb{T}^2$ respectively; D_1 contains no essential simple closed curves, D_2 contains the essential curve γ, and D_3 contains two non-homotopic essential curves γ, γ'.

Definition 2.5. An essential domain $D \subset \mathbb{T}^2$ has characteristic (p,q) if an essential closed curve $\gamma \subset D$ has homotopy type (p,q).

Note that definition 2.5 is well-defined, in the sense that every other essential simple closed curve in D must have the same homotopy type (as otherwise the domain D would be doubly essential). The following lemma relates the notion of a trivial and essential domain to that of a disk and essential annulus in the torus respectively.

Lemma 2.6. *A domain $D \subset \mathbb{T}^2$, such that $\widetilde{D}$ is simply connected, is trivial (resp. essential) if and only if it is a disk (resp. essential annulus) in the torus.*

Proof. As the *if* part is evident, we need only prove the *only if* part. First suppose that D is trivial and let $\widetilde{D}$ a lift of D. By the Riemann mapping theorem, there exists a biholomorphism $\phi\colon \mathbb{D}^2 \to \widetilde{D}$. As D is trivial, no two points in $\widetilde{D}$ are identified under the action of the translation group $\mathbb{Z}^2 \subset \mathbb{R}^2$ and thus $p|_{\widetilde{D}}$ is injective. Therefore, we have that $p \circ \phi\colon \mathbb{D}^2 \to \mathbb{T}^2$ with $p \circ \phi(\mathbb{D}^2) = D$, and thus $D \subset \mathbb{T}^2$ is a disk.

Next, suppose that D is essential. Then there exists a unique pair $(p, q) \in \mathbb{Z}^2$, with $\gcd(p, q) = 1$, such that the translation $T_{p,q}$ leaves $\widetilde{D}$ invariant, i.e. $T_{p,q}(\widetilde{D}) = \widetilde{D}$. Further, as $\widetilde{D}$ is simply connected, again by the Riemann mapping theorem, there exists a biholomorphism $\phi\colon \mathbb{D}^2 \to \widetilde{D}$. As $T_{p,q}\colon \widetilde{D} \to \widetilde{D}$ is a biholomorphism, the map

$$\mu\colon \mathbb{D}^2 \longrightarrow \mathbb{D}^2,\ \ \mu = \phi^{-1} \circ T_{p,q} \circ \phi$$

is itself a biholomorphism and thus a Möbius transformation. Moreover, as $T_{p,q}$ does not fix any point in $\widetilde{D}$, μ does not fix any point in $\mathbb{D}^2$ and thus μ is either a hyperbolic or a parabolic Möbius transformation. It is well-known that $\mathbb{D}^2/\langle\mu\rangle$ is then topologically equivalent to an annulus $\mathbb{S}^1 \times (0, 1)$, therefore so is $\widetilde{D}/\langle T_{p,q}\rangle$. As $\widetilde{D}$ admits no translations other than (multiples of) $T_{p,q}$ that leave $\widetilde{D}$ invariant, the continuous projection p restricted to $\widetilde{D}/\langle T_{p,q}\rangle$ into the torus $\mathbb{T}^2$ is an injection and thus D is indeed topologically equivalent to the annulus $\mathbb{S}^1 \times (0, 1)$. $\square$

2.2.4 Decomposition theory

We recall some standard results from decomposition theory, to be used in the proof of Theorem 2.A. In the following statements, let M be a closed surface.

Definition 2.6. A collection $\mathcal{U} = \{\mathcal{U}_i\}_{i\in I}$ of continua in a surface M is said to be *upper semi-continuous* if the following holds:

(1) If $\mathcal{U}_i, \mathcal{U}_j \in \mathcal{U}$, then $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$.

(2) If $\mathcal{U}_i \in \mathcal{U}$, then $\mathcal{U}_i$ is non-separating.

(3) We have that $M = \bigcup_{i\in I} \mathcal{U}_i$.

(4) If $\mathcal{U}_{i_k}$ with $k \in \mathbb{N}$ is a sequence that has the Hausdorff limit $\mathcal{C}$, then there exists $\mathcal{U}_j \in \mathcal{U}$ such that $\mathcal{C} \subset \mathcal{U}_j$.

In a compact metric space, every Hausdorff limit of continua is again a continuum. We have the following classical result, see for example [48].

Moore's Theorem. *Let $\mathcal{U}$ be an upper semi-continuous decomposition of a surface M so that every element of $\mathcal{U}$ is acyclic. Then there is a continuous map $\phi : M \to M$ that is homotopic to the identity and such that for every $z \in M$, we have that $\phi^{-1}(z) = \mathcal{U}_i$ for some element $\mathcal{U}_i \in \mathcal{U}$.*

The following result is easily proved with Moore's Theorem.

Lemma 2.7. *Given an upper semicontinuous decomposition $\mathcal{U}$ of a surface M into acyclic elements and a $f \in \mathrm{Homeo}(M)$, with the property that f sends elements of $\mathcal{U}$ into elements of $\mathcal{U}$. Then the natural quotient map $\widehat{f} \in \mathrm{Homeo}(M)$ defined by $\phi \circ f(z) = \widehat{f} \circ \phi(z)$, for every $z \in M$, is a homeomorphism. In other words, f is semi-conjugate to $\widehat{f}$ through ϕ.*

2.3 Classification of the minimal sets

This section deals with the proof of Theorems 2.A. and 2.B. In what follows, let $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ with $\mathcal{M}$ a minimal set of f. The outline of the proof is as follows. The main part of the proof of Theorem 2.A. is the study of the topology of the connected components $\{\Sigma_k\}$ of the complement of $\mathcal{M}$. Using that a component Σ_k is either trivial, essential or doubly essential, we show that a minimal set comes in either one of the three different types in the statement of Theorem 2.A.

Using the classification of Theorem 2.A, we show that the only locally connected minimal sets are those of type I and, moreover, the components Σ_k are bounded disks, which are the interiors of closed embedded topological disks intersecting pairwise in at most one point, leading to Theorem 2.B.

2.3.1 Topology of the domains Σ_k

In what follows, let $\Sigma = \Sigma_k$ be any element of $\{\Sigma_k\}$, the collection of connected components of the complement of $\mathcal{M}$, a minimal set of an element $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$. Further, let $\widetilde{d}(\cdot,\cdot)$ be the standard Euclidean metric on $\mathbb{R}^2$ and let $d(\cdot,\cdot)$ the (induced) metric on $\mathbb{T}^2$.

Lemma 2.8. *If $f^n(\Sigma) \cap \Sigma = \emptyset$ for all $n \neq 0$, then Σ is a disk or an essential annulus.*

Proof. First, suppose that Σ is doubly essential and let $\gamma, \gamma' \subset \Sigma$ be two non-homotopic essential simple closed curves. As f is homotopic to the identity, the homotopy classes of $f(\gamma)$ and γ are equal. As every two non-homotopic simple closed curves on the torus intersect, we have that $f(\gamma) \cap \gamma' \neq \emptyset$. Therefore $f(\Sigma) \cap \Sigma \neq \emptyset$ and thus Σ has to be either trivial or essential.

Thus let Σ be a trivial or essential domain and let $\widetilde{\Sigma}$ be a lift of Σ. In order to show that Σ is a disk or essential annulus respectively, by Lemma 2.6, it suffices to show that $\widetilde{\Sigma}$ is simply connected. To prove this, suppose to the contrary that $\widetilde{\Sigma}$

is not simply connected. Then there exists a simple closed curve $\gamma \subset \widetilde{\Sigma}$ such that the open disk D_γ with boundary curve γ has the property that

$$D_\gamma \cap p^{-1}(\mathcal{M}) \neq \emptyset. \tag{2.12}$$

Let F be a lift of f. As every point in $\mathcal{M}$ is recurrent, there exists a subsequence n_k such that $f^{n_k}(z) \to z$ for $k \to \infty$. Therefore, by passing to a subsequence if necessary, we may assume that for all $k \geq 1$, we have that

$$F^{n_k}(D_\gamma) \cap T_{p_k,q_k}(D_\gamma) \neq \emptyset, \tag{2.13}$$

for certain $(p_k, q_k) \in \mathbb{Z}^2$. Given (2.13), there are two possibilities. For a given $k \geq 1$, we have that either

(a) $F^{n_k}(D_\gamma) \subset T_{p_k,q_k}(D_\gamma)$ or $T_{p_k,q_k}(D_\gamma) \subset F^{n_k}(D_\gamma)$, or

(b) $F^{n_k}(\gamma) \cap T_{p_k,q_k}(\gamma) \neq \emptyset$.

Case (a) can be excluded as, by Lemma 2.5, this yields periodic points for f. Furthermore, case (b) is ruled out as this implies that

$$F^{n_k}(\widetilde{\Sigma}) \cap T_{p_k,q_k}(\widetilde{\Sigma}) \neq \emptyset, \tag{2.14}$$

implying that $f^{n_k}(\Sigma) \cap \Sigma \neq \emptyset$, contrary to our assumption. Therefore, $\widetilde{\Sigma}$ must be simply connected indeed. $\square$

In what follows, a fundamental domain of $\mathbb{T}^2$ is defined as the standard square $[0,1] \times [0,1] \subset \mathbb{R}^2$ and the integer translates thereof.

Lemma 2.9. *If Σ is trivial, then Σ is a disk. Moreover, if Σ is bounded, then $f^n(\Sigma) \cap \Sigma = \emptyset$ for all $n \neq 0$.*

Proof. Because $\mathcal{M}$ is invariant, we have that either: (a) $f^n(\Sigma) \cap \Sigma = \emptyset$ for all $n \neq 0$ or (b) $f^N(\Sigma) = \Sigma$ for some $N \neq 0$. In case (a), Σ is a disk by Lemma 2.8. In case (b), it follows from Lemma 2.4 that Σ is necessarily unbounded.

Thus we need to show that for unbounded Σ, we have that $\widetilde{\Sigma}$ is simply connected, if $f^N(\Sigma) = \Sigma$ for some $N \neq 0$. We may as well assume that $N = 1$. Take a lift F of f such that $F(\widetilde{\Sigma}) = \widetilde{\Sigma}$. If $\widetilde{\Sigma}$ is not simply connected, then there exists a simple closed curve $\gamma \subset \widetilde{\Sigma}$ such that the disk $D_\gamma \subset \mathbb{R}^2$ with boundary curve γ has the property that $D_\gamma \cap p^{-1}(\mathcal{M}) \neq \emptyset$. Similarly to Lemma 2.8, there exists a subsequence n_k such that, for $k \geq 1$, we have that

$$F^{n_k}(D_\gamma) \cap T_{p_k,q_k}(D_\gamma) \neq \emptyset, \tag{2.15}$$

for certain $(p_k, q_k) \in \mathbb{Z}^2$. As

$$\rho(f) = \rho(f, z) = (\alpha, \beta) \mod \mathbb{Z}^2 \neq (0,0) \mod \mathbb{Z}^2,$$

for every $z \in \mathbb{T}^2$, it follows that $\widetilde{d}(F^{n_k}(\widetilde{z}), \widetilde{z}) \to \infty$, for $k \to \infty$. In particular, passing to a subsequence once again, we may assume that $F^{n_k}(\widetilde{z})$ is contained in a fundamental domain different from that of $\widetilde{z}$, for all $k \geq 1$. Condition (2.15) gives again the two possiblities (a) and (b) of Lemma 2.8 and we can exclude case (a) as this would yield periodic points for f. Therefore, for all $k \geq 1$, (2.15) reduces to the condition that

$$F^{n_k}(\gamma) \cap T_{p_k,q_k}(\gamma) \neq \emptyset, \tag{2.16}$$

for some $(p_k, q_k) \in \mathbb{Z}^2$. Thus for every $k \geq 1$, we have that

(i) $F^{n_k}(\gamma) \cap T_{p_k,q_k}(\gamma) \neq \emptyset$,

(ii) $F^{n_k}(\gamma)$ lies in a fundamental domain different from that of γ, and

(iii) $F^{n_k}(\gamma) \subset \widetilde{\Sigma}$.

Condition (iii) follows simply from the fact that $\widetilde{\Sigma}$ is F-invariant and $\gamma \subset \widetilde{\Sigma}$. Fix any $k \geq 1$ and choose $\widetilde{w} \in F^{n_k}(\gamma) \cap T_{p_k,q_k}(\gamma)$ and let $\widetilde{w}' = T^{-1}_{p_k,q_k}(\widetilde{w}) \in \gamma$. As $\widetilde{\Sigma}$ is a domain, it is path-connected and thus there exists an arc $\eta \subset \widetilde{\Sigma}$ connecting $\widetilde{w}$ and $\widetilde{w}'$. As these endpoints lie in different fundamental domains of $\mathbb{T}^2$, η projects under p to an essential closed curve, as its endpoints differ by an integer translate. However, this contradicts our assumption that Σ is trivial (and thus does not contain any essential simple closed curves). This contradiction shows that Σ must be simply connected and this completes the proof. $\square$

Using the irrationality of the rotation vector (α, β), we now deduce the following.

Lemma 2.10. *If Σ is essential, then Σ is an essential annulus and $f^n(\Sigma) \cap \Sigma = \emptyset$ for all $n \neq 0$.*

Proof. It suffices to show that, if Σ is essential, then $f^n(\Sigma) \cap \Sigma \neq \emptyset$ for all $n \neq 0$. It then follows from Lemma 2.8 that Σ is an essential annulus. Assume that Σ has characteristic (p, q). We will show that, by our choice of translation number, f^N can not fix an essential domain, for all $N \neq 0$. To derive a contradiction, suppose there exist an $N \neq 0$ such that $f^N(\Sigma) = \Sigma$. Without loss of generality, we may assume that $N = 1$, i.e. that $f(\Sigma) = \Sigma$. Let $\gamma \subset \Sigma$ be an essential simple closed curve and let $\widetilde{\gamma}$ be a lift of γ. We may assume that $\widetilde{\gamma}$ intersects $(0, 0) \in \mathbb{R}^2$, and by definition it also intersects $(p, q) \in \mathbb{R}^2$, where $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$. The arc $\eta \subset \widetilde{\gamma}$ connecting $(0, 0)$ and (p, q) is compact and therefore bounded. Therefore, the curve $\widetilde{\gamma}$ divides $\mathbb{R}^2$ into two unbounded connected components $\mathcal{H}_l$ and $\mathcal{H}_r$, homeomorphic to half-planes, so that $\mathbb{R}^2 \backslash \widetilde{\gamma} = \mathcal{H}_l \cup \mathcal{H}_r$ and $\mathcal{H}_l \cap \mathcal{H}_r = \emptyset$. Further, as γ is a simple closed curve, any integer translate $\widetilde{\gamma}' = T_{p',q'}(\widetilde{\gamma})$, where (p', q') is not an integer multiple of (p, q), has the property that $\widetilde{\gamma}' \cap \widetilde{\gamma} = \emptyset$. This follows from the fact that Σ is essential, but not doubly essential; if $\widetilde{\gamma}' \neq \widetilde{\gamma}$, then there exists an arc $\zeta \subset \widetilde{\Sigma}$ connecting $(0, 0)$ to a point $(p', q') = T_{p',q'}(0, 0)$, which is not a multiple of (p, q), the projection of ζ under p would lie in a homotopy class other than that of γ, implying that Σ would be doubly essential, contrary to our assumption. Therefore, we can choose integer translates $\widetilde{\gamma}_l$ and $\widetilde{\gamma}_r$ of $\widetilde{\gamma}$ contained in $\mathcal{H}_l$ and $\mathcal{H}_r$ respectively and we can define $\Gamma \subset \mathbb{R}^2$ to be the infinite strip bounded by $\widetilde{\gamma}_l \cup \widetilde{\gamma}_r$.

We claim that $\widetilde{\Sigma} \subset \Gamma$. Indeed, if $\widetilde{\Sigma} \cap \Gamma^c \neq \emptyset$, then $\widetilde{\Sigma} \cap (\widetilde{\gamma}_l \cup \widetilde{\gamma}_r) \neq \emptyset$. Suppose that $\widetilde{\Sigma} \cap \widetilde{\gamma}_l \neq \emptyset$. The case where $\widetilde{\Sigma} \cap \widetilde{\gamma}_l \neq \emptyset$ (or both) is similar. Let $\widetilde{z}' \in \widetilde{\Sigma} \cap \widetilde{\gamma}_l$.

Because $\widetilde{\gamma} \subset \widetilde{\Sigma}$ and $\widetilde{\Sigma}$ is path-connected, there exists an arc $\zeta \subset \widetilde{\Sigma}$ connecting $\widetilde{z}'$ to a point $\widetilde{z} \in \widetilde{\gamma}_l$ such that $z = p(\widetilde{z}') = p(\widetilde{z})$. But this again implies that $\widetilde{\Sigma}$ has to be doubly essential, contrary to our assumption. Thus $\widetilde{\Sigma} \subset \Gamma$.

To finish the proof, choose a lift F of f such that $F(\widetilde{\Sigma}) = \widetilde{\Sigma}$. As Γ is invariant under the translation $T_{p,q}$, and $\widetilde{\Sigma} \subset \Gamma$, we thus must have that

$$\rho(F, \widetilde{z}) = \lim_{n\to\infty} \frac{F^n(\widetilde{z}) - \widetilde{z}}{n} = (a, b), \text{ where } \frac{b}{a} = \frac{q}{p}, \tag{2.17}$$

for every $\widetilde{z} \in \widetilde{\Sigma}$. As $(a, b) = (\alpha + s, \beta + t)$ for certain $s, t \in \mathbb{Z}$ and $\alpha, \beta \notin \mathbb{Q}$, we have that $a = \alpha + s \neq 0$ and $b = \beta + t \neq 0$, and we obtain

$$\frac{\alpha + s}{\beta + t} = \frac{a}{b} = \frac{q}{p}. \tag{2.18}$$

Rewriting (2.18) gives that

$$p\alpha - q\beta - (ps - qt) = 0.$$

As $p, q, ps - qt \in \mathbb{Z}$, with $(p, q) \neq (0, 0)$, this gives a non-trivial solution of (2.8), which contradicts the irrationality of (α, β). $\square$

The following lemma shows that not all combinations of types of domains can occur.

Lemma 2.11. *The collection $\{\Sigma_k\}$ can not contain both an essential annulus and an unbounded disk.*

Proof. Suppose, to derive a contradiction, that the collection of domains $\{\Sigma_k\}$ contains both an essential annulus and an unbounded disk. By Lemma 2.10, the collection $\{\Sigma_k\}$ contains infinitely many essential annuli; let us denote these by $\{\Sigma_k^a\}$. Note further that, as all these annuli are disjoint, these all have the same characteristic, which we assume to be $(0, 1)$; the proof in case of any other characteristic is entirely similar. Denote Σ an element of $\{\Sigma_k\}$ homeomorphic to an unbounded disk.

Let $\widetilde{\Sigma} \subset \mathbb{R}^2$ and $\widetilde{\Sigma}^a_k \subset \mathbb{R}^2$ be lifts of Σ and Σ^a_k respectively. Take $\widetilde{z} \in \widetilde{\Sigma}$ and let $\ell_{\widetilde{z}}$ be the horizontal (Euclidean) line through $\widetilde{z}$. Let $I \subset \ell_{\widetilde{z}} \cap \widetilde{\Sigma}$ be the connected component containing $\widetilde{z}$. As the line $\ell_{\widetilde{z}}$ is horizontal and the characteristic of the essential annuli Σ^a_k is $(0, 1)$, the length of the interval I is finite. Let $\widetilde{z}^-, \widetilde{z}^+ \in \partial I$ be the left and right endpoint of the interval I respectively. As $\widetilde{z}^-, \widetilde{z}^+ \in \partial\widetilde{\Sigma}$ we have that

$$z^{\pm} := p(\widetilde{z}^{\pm}) \in \partial\Sigma \subset \mathcal{M}. \tag{2.19}$$

Define $I^{\pm 1} = T^{\pm 1}_{0,1}(I)$. Let $\gamma_k \subset \Sigma^a_k$ be a simple closed curve and $\widetilde{\gamma}_k$ a lift of γ_k. Certainly, we have that $\widetilde{\gamma}_k \cap T^n_{0,1}(I)$ for all $n \in \mathbb{Z}$.

As every orbit in $\mathcal{M}$ is dense, we can take a point $z' \in \partial\Sigma^a_k$, for some $k \in \mathbb{Z}$, and find subsequences k_t and k'_t such that $f^{k_t}(z') \to z^+$ and $f^{k'_t}(z) \to z^-$ for $t \to \infty$. After appropriately labeling the annuli if necessary, we find points $z_{k_t} \in \gamma_{k_t} \subset \Sigma^a_{k_t}$ and $z_{k'_t} \in \gamma_{k'_t} \subset \Sigma^a_{k'_t}$ such that $z_{k_t} \to z^+$ and $z_{k_t} \to z^-$ for $t \to \infty$. Thus we can find lifts $\widetilde{\gamma}_{k_t}$ and $\widetilde{\gamma}_{k'_t}$ and points $\widetilde{z}_{k_t} \subset \widetilde{\gamma}_{k_t}$ and $\widetilde{z}_{k'_t} \subset \widetilde{\gamma}_{k'_t}$, such that $\widetilde{z}_{k_t} \to \widetilde{z}^+$ and $\widetilde{z}_{k'_t} \to \widetilde{z}^-$, for $t \to \infty$. As the curves $\widetilde{\gamma}_{k_t}, \widetilde{\gamma}_{k'_t}$ are periodic (in the sense of (2.11)), they define an infinite strip Γ_t that contain the line segments $T^n_{0,1}(I)$, for all $n \in \mathbb{Z}$. Further, after a relabeling if neccesary, we may assume that $\Gamma_{t'} \subset \Gamma_t$ if $t' > t$. We now have that $\widetilde{\Sigma} \subset \Gamma_t$ for every $t \geq 1$ and $I \subset \widetilde{\Sigma}$.

By periodicity, $T^{\pm 1}_{0,1}(\widetilde{z}_{k_t})$ and $T^{\pm 1}_{0,1}(\widetilde{z}_{k'_t})$ limit to $T^{\pm 1}(\widetilde{z}^+)$ and $T^{\pm 1}(\widetilde{z}^-)$ respectively. Therefore, as $\widetilde{\Sigma}$ is unbounded, we must have that either $\widetilde{\Sigma} \cap T_{0,1}(I) \neq \emptyset$ or $\widetilde{\Sigma} \cap T^{-1}_{0,1}(I) \neq \emptyset$, or both. As $\widetilde{\Sigma}$ is path-connected, we can find an arc $\eta \subset \widetilde{\Sigma}$ that projects under p to an essential closed curve $\gamma \subset \Sigma$, which is the desired contradiction; Σ is a disk and thus does not contain any essential closed curves. $\square$

We recall that a Cantor set can be characterized topologically as being compact, perfect and totally-disconnected.

Lemma 2.12. *If Σ is doubly essential, then $\mathcal{M}$ is an extension of a Cantor set.*

Proof. Denote $\{\Lambda_i\}_{i \in I}$ the collection of connected components of $\mathcal{M}$. Because Σ

is doubly essential, $f(\Sigma) \cap \Sigma \neq \emptyset$, hence $f(\Sigma) = \Sigma$. As $f(\Sigma) = \Sigma$, we have that $f(\partial\Sigma) = \partial\Sigma$. Further, as $\partial\Sigma \subset \mathcal{M}$ and $\partial\Sigma$ is closed, $\partial\Sigma = \mathcal{M}$ by minimality of $\mathcal{M}$. Let $\Lambda := \Lambda_i$ be any connected component of $\mathcal{M}$, for some $i \in I$. As Λ is closed in $\mathcal{M}$ and $\mathcal{M}$ is closed in $\mathbb{T}^2$, Λ is closed, and thus compact, in $\mathbb{T}^2$. Therefore, Λ is a continuum. Further, as $\mathcal{M}$ is nowhere dense (as $\mathcal{M} \neq \mathbb{T}^2$), it follows that Λ is nowhere dense.

We need to show that a lift $\widetilde{\Lambda}$ of Λ is bounded. As Σ is doubly essential, there exist two non-homotopic essential simple closed curves $\gamma, \gamma' \subset \Sigma$. As these curves are non-homotopic, the respective lifts $\widetilde{\gamma}, \widetilde{\gamma}' \subset \widetilde{\Sigma}$ of γ and γ' and the integer translates of these curves tile $\mathbb{R}^2$ into bounded disks. As $\widetilde{\Lambda} \cap \widetilde{\Sigma} = \emptyset$, it follows that $\widetilde{\Lambda}$ has to be contained in one of these bounded disks, implying $\widetilde{\Lambda}$ itself is bounded.

As Λ is a connected component of $\mathcal{M}$, we must either have that $f^n(\Lambda) \cap \Lambda = \emptyset$ for all $n \neq 0$, or $f^N(\Lambda) = \Lambda$ for some finite $N \neq 0$. However, the latter is excluded by Lemma 2.4 as Λ is bounded and thus $f^n(\Lambda) \cap \Lambda = \emptyset$ for all $n \neq 0$. As Λ is a bounded continuum, the connected components of $\mathbb{T}^2 \setminus \Lambda$ consists of a unique unbounded component and every other component is a bounded disk. Let D be one such disk. Then Σ is contained in this unbounded component; indeed, if this would not be the case, then we can take a point $z \in D \cap \Sigma$ and an essential simple closed curve $\gamma \subset \Sigma$ passing through z. As $\mathcal{M} \cap \Sigma = \emptyset$, and $z \in D$, this implies that $\gamma \subset D$, contradicting that D is a disk. We thus conclude that $D \cap \Sigma = \emptyset$. Further, if $D \cap \mathcal{M} \neq \emptyset$, then this implies that $\Sigma \cap D \neq \emptyset$ as $\partial\Sigma = \mathcal{M}$, which contradicts our earlier conclusion. In other words, to a component Λ, we can uniquely adjoin the open disks which, apart from the unique doubly essential component containing Σ, form the connected components of $\mathbb{T}^2 \setminus \Lambda$. This proves that $\mathrm{Fill}(\Lambda_i) \cap \mathrm{Fill}(\Lambda_j) = \emptyset$ if $i \neq j$, with $\mathrm{Fill}(\Lambda_i)$ a bounded non-separating continuum, for every $i \in I$.

Let again Λ_i be any component of $\mathcal{M}$ and define $\widehat{\mathcal{Q}} = \bigcup_{i \in I} \mathrm{Fill}(\Lambda_i)$. Define the decomposition $\mathcal{U}$ of $\mathbb{T}^2$ into the continua $\{\mathrm{Fill}(\Lambda_i)\}_{i \in I}$ and singletons in the complement of these continua. In order to show that $\mathcal{M}$ is an extension of a Cantor set, we first show that the decomposition $\mathcal{U}$ is upper semi-continuous. By Moore's Theorem (see section 2.2.4), this implies there exists a continuous $\phi\colon \mathbb{T}^2 \to \mathbb{T}^2$ such that $\phi^{-1}(z) = \mathcal{U}_j$ for every $z \in \mathbb{T}^2$. We have already shown that the decomposition $\mathcal{U}$ satisfies conditions (1), (2) and (3) of definition 2.6. To prove it satisfies condition (4), we need to show that if a sequence of continua $\mathcal{U}_{j_k}$, with $k \in \mathbb{Z}$, has Hausdorff limit $\mathcal{C}$, then $\mathcal{C} \subset \mathcal{U}_j$ for some $j \in J$. As the statement is obvious if $\mathcal{C}$ is a singleton, assume $\mathcal{C}$ to be a non-trivial continuum. Note that every element non-degenerate element $\mathcal{U}_{j_k} \in \mathcal{U}$ has the property that $\partial\mathcal{U}_{j_k} \subset \partial\Sigma = \mathcal{M}$. Further, without loss of generality, we may assume that no element $\partial\mathcal{U}_{j_k}$ is a singleton and that the elements are mutually disjoint. We first claim that the interior of $\mathcal{C}$ has to be empty. Indeed, if not, there would exist a subsequence of elements for which the largest open disk contained in the interior of $\mathcal{U}_{j_k}$ would be bounded from below, contradicting that the torus is compact and the elements mutually disjoint. Therefore, as $\partial\mathcal{U}_{j_k} \subset \mathcal{M}$, every point of the Hausdorff limit $\mathcal{C}$ is the limit point of a sequence of points of $\mathcal{M}$. As $\mathcal{M}$ is closed, this implies $\mathcal{C}$ is itself contained in $\mathcal{M}$. In particular, as $\mathcal{C}$ is connected, $\mathcal{C}$ is contained in a connected component of $\mathcal{M}$, i.e. $\mathcal{C} \subset \Lambda_i \subset \mathcal{U}_j$ for some $j \in J$. So $\mathcal{U}$ is upper semi-continuous indeed. We have already shown that all non-trivial elements of $\mathcal{U}$ are non-separating and bounded, and thus acyclic.

Thus, by Moore's Theorem, there exists a continuous $\phi\colon \mathbb{T}^2 \to \mathbb{T}^2$, homotopic to the identity, such that for every $z \in \mathbb{T}^2$, $\phi^{-1}(z)$ is a unique element of $\mathcal{U}$. By Lemma 2.7, as $\mathcal{U}$ is upper semi-continuous and f sends elements of $\mathcal{U}$ into elements of $\mathcal{U}$, the mapping $\widehat{f}$ defined by $\phi \circ f = \widehat{f} \circ \phi$ is a homeomorphism. As $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ and ϕ is homotopic to the identity, we have that $\widehat{f}$ is isotopic to the identity and by a standard argument, $\rho(f) = \rho(\widehat{f}) \mod \mathbb{Z}^2$, thus $\widehat{f} \in \mathrm{Homeo}_*(\mathbb{T}^2)$. This proves condition (i) of definition 2.1.

To prove condition (ii) of definition 2.1, we need to show that $\widehat{\mathcal{M}} := \phi(\widehat{\mathcal{Q}}) \subset \mathbb{T}^2$ is a Cantor minimal set for $\widehat{f}$. As $\mathcal{M}$ is a minimal set for f, $\widehat{\mathcal{M}}$ is a minimal set for $\widehat{f}$. Further, as $\widehat{\mathcal{M}}$ is totally disconnected by construction, it suffices to show that $\widehat{\mathcal{M}}$ is compact and perfect. First, $\widehat{\mathcal{Q}}$ is compact as the complement Σ is open. Because ϕ is continuous, $\widehat{\mathcal{M}}$ is compact. To show $\widehat{\mathcal{M}}$ is perfect, we observe that, because $\mathrm{Fill}(\Lambda_i) \cap \mathrm{Fill}(\Lambda_j) = \emptyset$ if $i \neq j$, no element $\mathrm{Fill}(\Lambda_i)$ is isolated, as this would imply that a component Λ_i is isolated. Therefore, by continuity of ϕ, no point of $\widehat{\mathcal{M}}$ is isolated, and thus $\widehat{\mathcal{M}}$ is perfect. $\square$

2.3.2 Proof of Theorem 2.A

Proof of Theorem 2.A. To show that the minimal set $\mathcal{M}$ of f is either of type I, II or III as given above, assume that $\mathcal{M} \neq \mathbb{T}^2$ and let $\{\Sigma_k\}$ be the collection of connected components of the complement of $\mathcal{M}$. If no element of $\{\Sigma_k\}$ is doubly essential, then Σ_k is either trivial or essential, for all $k \in \mathbb{Z}$. By Lemma 2.9 and 2.10, $\{\Sigma_k\}$ are all disks and/or essential annuli; however, by Lemma 2.11, $\{\Sigma_k\}$ can not both contain an essential annulus and an unbounded disk. In case no element Σ_k is essential, we have a type I minimal set. In case at least one, and therefore infinitely many, connected components are essential, we have a type II minimal set. If for some k, Σ_k is doubly essential, then $\mathcal{M}$ is an extension of a Cantor set by Lemma 2.12 and these correspond to a type III minimal sets. This concludes the proof. $\square$

We finish this section with the proofs of the corollaries stated above.

Proof of Corollary 2.1. Let $\mathcal{M}$ be a minimal set of f of type I. It suffices to show that $\mathcal{M} = \Omega(f)$. Indeed, if this is shown, then by minimality of $\mathcal{M}$ and the inclusions $\alpha(z), \omega(z) \subseteq \Omega(f)$, with $\omega(z), \alpha(z)$ closed and f-invariant sets for every $z \in \mathbb{T}^2$, we obtain (2.3). Uniqueness then also follows, as any other minimal set $\mathcal{M}'$ of f has to be contained in the complement of $\Omega(f)$, which is clearly impossible; if $z \in \mathcal{M}'$ then z is both recurrent and wandering, which are incompatible conditions to hold simultaneously.

First, suppose that $\mathcal{M}$ is of type I. Fix a component $\Sigma := \Sigma_k$. Then Σ is a disk. If Σ is bounded, then by Lemma 2.9 we have that $f^n(\Sigma) \cap \Sigma = \emptyset$ for all $n \neq 0$, and thus $\Sigma \cap \Omega(f) = \emptyset$. So it remains to prove the case where Σ is an unbounded disk. We may assume that there exists an $N \neq 0$ such that $f^N(\Sigma) = \Sigma$, as otherwise we are done by the previous argument. Further, we may assume that $N = 1$. A modification of the argument of Lemma 2.9 shows that $\Omega(f) \cap \Sigma = \emptyset$ in this case as well. Indeed, let F be a lift of f such that $F(\widetilde{\Sigma}) = \widetilde{\Sigma}$ and suppose that $\Omega(f) \cap \Sigma \neq \emptyset$. Take a $\widetilde{z} \in p^{-1}(\Omega(f)) \cap \widetilde{\Sigma}$. Let $D_\rho \subset \widetilde{\Sigma}$ be a small (Euclidean) disk centered at $\widetilde{z}$ with radius $\rho > 0$. As $z = p(\widetilde{z}) \in \Omega(f)$, there exists a subsequence n_k, where $n_k \to \infty$ for $k \to \infty$, and a sequence of real numbers $\rho_k > 0$, where $\rho_k \to 0$ for $k \to 0$, such that $F^{n_k}(D_{\rho_k}) \cap T_{p_k,q_k}(D_{\rho_k}) \neq \emptyset$ for some $(p_k, q_k) \in \mathbb{Z}^2$. By choosing k large enough, we can find a n_k so that $F^{n_k}(D_{\rho_k})$ is contained in a fundamental domain other than that of D_{ρ_k}. If we take any intersection point of $F^{n_k}(D_{\rho_k}) \cap T_{p_k,q_k}(D_{\rho_k})$, translate it back by $T^{-1}_{p_k,q_k}$ and connect the two points by a simple arc η, then η projects under p to an essential closed curve contained in Σ, contradicting that Σ is a disk.

If $\mathcal{M}$ is of type II, then all elements of $\{\Sigma_k\}$ are essential annuli or bounded disks. We need only show that if Σ_k is an essential annulus, then $\Omega(f) \cap \Sigma_k = \emptyset$. This follows from Lemma 2.10 stating that $f^n(\Sigma_k) \cap \Sigma_k = \emptyset$ for all $n \neq 0$, and this finishes the proof. $\square$

Proof of Corollary 2.3. First, it is clear that no minimal set of type III is connected. Thus we have to show that no element of the collection of connected components of the complement of $\mathcal{M}$ can contain an essential annulus. Indeed, if one such component would be an essential annulus, then by Lemma 2.10, there would in fact be infinitely many disjoint essential annuli. Taking any two essential annuli, taking an essential simple closed curve in each and deleting these two curves, separates the torus into two disjoint essential annuli A_1 and A_2. Each of these two annuli A_1 and A_2 has to contain points of $\mathcal{M}$; if A_1 does not contain points of $\mathcal{M}$, then A_1 is contained in a connected component of the complement of $\mathcal{M}$, contrary to our assumption. Similarly for A_2. But this gives a separation of $\mathcal{M}$.

Conversely, let $\mathcal{M}$ be of type I, so that Σ_k is an open topological disk for every

$k \in \mathbb{Z}$. In each disk Σ_k, we can find a sequence D_k^t of nested disks, i.e. $D_k^t \subset D_k^{t+1}$, embedded in Σ_k, such that $\mathrm{Cl}(D_k^t)$ is a closed disk and such that $\bigcup_{t\geq 1} D_k^t = \Sigma_k$. We can accomplish this by uniformizing each disk Σ_k to the unit disk $\mathbb{D}^2$, taking nested such disks centered at the origin in $\mathbb{D}^2$, and pulling these back to Σ_k. Define $\Gamma_t = \mathbb{T}^2 \setminus \bigcup_{k\in\mathbb{Z}} D_k^t$. We claim that Γ_t is connected. Indeed, define the compact sets $\Gamma_t^s = \mathbb{T}^2 \setminus \bigcup_{k=-s}^{s} D_k^t$. Clearly, Γ_t^s, as the torus with finitely many disjoint disks whose closures are disjoint deleted, is connected. As $\Gamma_t^{s+1} \subset \Gamma_t^s$, we have that $\Gamma_t = \bigcap_{s\geq 1} \Gamma_t^s$ is connected as well. By the same token, as $\Gamma_{t+1} \subset \Gamma_t$ with Γ_t compact and connected for every $t \geq 1$, we have that $\mathcal{M} = \bigcap_{t\geq 1} \Gamma_t$ is connected. □

2.3.3 Locally connected minimal sets

We proceed with the proof of Theorem 2.B. It what follows, let again $\mathcal{M}$ be a minimal set of an element $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$.

Lemma 2.13. *If $\mathcal{M} \neq \mathbb{T}^2$ is locally connected, then $\mathcal{M}$ is of type I.*

Proof. First we show that if $\mathcal{M}$ is locally connected, it can not be of type II. Indeed, if $\mathcal{M}$ is of type II, then there exists an infinite number of essential annuli among its complement component $\{\Sigma_k\}_{k\in\mathbb{Z}}$ and suppose $\mathcal{M}$ is locally connected. Let $\Sigma := \Sigma_k$ be an essential annulus. Then $f^n(\Sigma) \cap \Sigma = \emptyset$ for all $n \neq 0$. Denote $\Sigma_n = f^n(\Sigma)$. Let $\gamma_n \subset \Sigma_n$ be an essential simple closed curve and choose $z_n \in \gamma_n$. Passing to a subsequence n_k, where $n_k \to \infty$ for $k \to \infty$, we may assume that $z_{n_k} \to z$. By Corollary 2.1 we must have that $z \in \mathcal{M}$. Let $U \ni z$ be an open neighbourhood. By locally connectivity of $\mathcal{M}$, there exists a connected open neighbourhood $V \subset U$, with $V \ni z$, such that $V \cap \mathcal{M}$ is connected. There exists a small open (Euclidean) disk $D \subset V$ centered at z. By passing to a subsequence, and upon relabeling if necessary, we may assume that $z_{n_k} \subset V$ for all $k \geq 1$ and that $d(z, \gamma_{n_k}) < d(z, \gamma_{n_{k'}})$ if $k > k'$. The two curves γ_{n_k} and $\gamma_{n_{k+1}}$ bound an essential annulus A_k, for every $k \geq 1$. Define $V_k = A_k \cap V$. Then it holds that $V_k \cap \mathcal{M} \neq \emptyset$. Indeed, otherwise we can take any connected component V_k' of V_k and connect γ_{n_k} and $\gamma_{n_{k+1}}$ by an arc

$\eta \subset V_k'$ disjoint from $\mathcal{M}$, which is impossible as these two curves are contained in two disjoint essential annuli. As all A_k are mutually disjoint, up to the boundary curves γ_{n_k} which are disjoint from $\mathcal{M}$, we see that $V \cap \mathcal{M}$ consists in fact of infinitely many connected components, contrary to our assumption. Thus $\mathcal{M}$ can not be locally connected.

Secondly, if $\mathcal{M}$ is locally connected, then $\mathcal{M}$ can not be of type III. To show this, let $\{\Lambda_i\}_{i\in I}$ be the collection of connected components of $\mathcal{M}$ and take $z \in \mathcal{M}$. Then $z \in \Lambda_i$ for some $i \in I$. As Λ_i is a bounded continuum, by Lemma 2.4, we have that $f^n(\Lambda_i) \cap \Lambda_i = \emptyset$ for all $n \neq 0$. Since $z \in \mathcal{M}$, there exist a subsequence n_k, with $n_k \to \infty$ for $k \to \infty$, such that $f^{n_k}(z) \to z$. Therefore, every open neighbourhood U of z meets infinitely many components Λ_k, all of which are mutually disjoint, hence $\mathcal{M}$ is not locally connected. $\square$

It follows from Lemma 2.13 that if $\mathcal{M}$ is locally connected, it is also connected by Corollary 2.3.

Lemma 2.14. *If $\mathcal{M} \neq \mathbb{T}^2$ is locally connected, then $f^n(\Sigma_k) \cap \Sigma_k = \emptyset$, for all $n \neq 0$.*

Proof. As Σ_k is a disk for every $k \in \mathbb{Z}$, by Lemma 2.9, if $f^N(\Sigma_k) = \Sigma_k$ for some $N \neq 0$, then Σ_k is an unbounded disk. Denote $\Sigma := \Sigma_k$ and let $\widetilde{\Sigma}$ be its lift. Fix $\widetilde{z}_0 \in \widetilde{\Sigma}$ and let $\widetilde{z}_t \in \widetilde{\Sigma}$ be a sequence of points such that $\widetilde{d}(\widetilde{z}_0, \widetilde{z}_t) \to \infty$, for $t \to \infty$, and that any two such points $\widetilde{z}_t$ and $\widetilde{z}_{t'}$, with $t \neq t'$, are contained in two different fundamental domains. This can be done as $\widetilde{\Sigma}$ is unbounded. Let $\eta \subset \widetilde{\Sigma}$ be a simple arc passing through each of these points $\widetilde{z}_t$, $t \geq 1$.

For every $t \geq 1$, define $\widetilde{D}_t \subset \widetilde{\Sigma}$ the largest Euclidean disk centered at $\widetilde{z}_t$ contained in $\widetilde{\Sigma}$. By passing to a subsequence if necessary, we may assume that all these disks are mutually disjoint. Therefore, the disks $D_t := p(\widetilde{D}_t) \subset \mathbb{T}^2$ are mutually disjoint and thus the collection of disks D_t is a null-sequence as $\mathbb{T}^2$ is compact. In particular, there exists a sequence of points $w_t \in \mathcal{M}$, contained in ∂D_t, with $d(w_t, z_t) \to 0$, where $z_t = p(\widetilde{z}_t) \in \Sigma$. Passing to a convergent subsequence such that $w_t \to w \in \mathcal{M}$ for $t \to \infty$, we obtain a sequence $z_t \to w$, for $t \to \infty$.

Let $\widetilde{w}$ be a lift of w. We claim that $\widetilde{\mathcal{M}}$ is not locally connected at $\widetilde{w}$. If this is shown, then it follows that $\mathcal{M}$ is not locally connected at w as p is a local homeomorphism, contradicting our assumption. To prove the claim, let D be an open Euclidean disk centered at $\widetilde{w}$ and let $U \subset D$ any neighbourhood of $\widetilde{w}$. As the collection of points $\widetilde{z}_t$, $t \in \mathbb{Z}$, are contained in different fundamental domains and $p(\widetilde{z}_t) \to w$, there exist $(p_t, q_t) \in \mathbb{Z}^2$ such that, if we define $\eta_t := T_{p_t,q_t}(\eta)$ and $\widetilde{z}'_t = T_{p_t,q_t}(\widetilde{z}_t)$, we have that $\widetilde{z}'_t \to \widetilde{w} \in D$ and $\eta_t \cap \eta_{t'} = \emptyset$, if $t \neq t'$. Indeed, by passing to a subsequence, we may assume that for all $t \geq 1$, we have that $\widetilde{z}'_t \in D$. If $\eta_t \cap \eta_{t'} \neq \emptyset$, then two points of η are identified under a translation $T_{p,q}$, where $(p, q) \neq (0, 0)$, thus yielding an essential closed curve under projection of $\eta \subset \widetilde{\Sigma}$ under p, contrary to our assumption.

Define $\zeta_t \subset \eta_t \cap D$ the smallest (relative to inclusion) simple arc passing through $\widetilde{z}'_t$ such that the endpoints of ζ_t are contained in ∂D. As $\eta_t \cap \eta_{t'} = \emptyset$ if $t \neq t'$, we have that $\zeta_t \cap \zeta_{t'} = \emptyset$ if $t \neq t'$. Any two such different arcs ζ_t and $\zeta_{t'}$, joined by the two arcs in ∂D, form a disk $D(t, t') \subset D$. We claim that every such disk $D(t, t')$ is such that $D(t, t') \cap \widetilde{\mathcal{M}} \neq \emptyset$. Indeed, if this was not the case, then there exists an arc $\xi \subset D(t, t')$ joining $\widetilde{z}'_t$ and $\widetilde{z}'_{t'}$ with $t \neq t'$, such that $\xi \cap \widetilde{\mathcal{M}} = \emptyset$. It follows that the arc $\eta' \subset \eta$ joining $\widetilde{z}_t$ and $T_{\bar{p},\bar{q}}(\widetilde{z}_t)$ concatenated with the arc $T_{\bar{p},\bar{q}}(\xi)$, where $(\bar{p}, \bar{q}) = (p_t, q_t) - (p_{t'}, q_{t'}) \neq (0, 0)$, joins two different lattice points and is contained in the complement of $\widetilde{\mathcal{M}}$, which again yields an essential closed curve under projection of p, contrary to our assumptions.

Therefore, as a neighbourhood $U \ni \widetilde{w}$ contains infinitely many points $\widetilde{z}'_t$ for t sufficiently large, $U \cap \widetilde{\mathcal{M}}$ consists of infinitely many connected components and thus $\widetilde{\mathcal{M}}$ is not locally connected at $\widetilde{w}$. $\square$

So any locally connected minimal set $\mathcal{M}$ has to be of type I, and $f^n(\Sigma_k) \cap \Sigma_k = \emptyset$ for all $n \neq 0$ and $k \in \mathbb{Z}$. To finish the proof of Theorem 2.B, we argue in a way similar to that in [4]. The proof of the following lemma follows from [4, Lemma 7].

Lemma 2.15. *If $\mathcal{M} \neq \mathbb{T}^2$ is locally connected, such that $f^n(\Sigma_k) \cap \Sigma_k = \emptyset$, for all $n \neq 0$ and every $k \in \mathbb{Z}$, then* $\operatorname{diam}(\Sigma_k)$*, $k \in \mathbb{Z}$, is a null sequence.*

Combining Lemma 2.14 with Lemma 2.15, it thus follows that $\operatorname{diam}(\Sigma_k)$ is a null-sequence. A *cut point* of a minimal set $\mathcal{M} \subset \mathbb{T}^2$ is a point $z \in \mathcal{M}$ that separates $\mathcal{M}$, i.e. $\mathcal{M} \setminus \{z\}$ consists of at least two connected components. We have the following general property for minimal sets of homeomorphisms of compact metric spaces, see [4, Lemma 2].

Lemma 2.16. *Let $\mathcal{M}$ be a connected minimal set of a homeomorphism of a compact metric space X. Then $\mathcal{M}$ has no cut points.*

The following result, see e.g. [29, Thm 61-4], will be used in the subsequent lemma to prove that the disks Σ_k in the complement of $\mathcal{M}$ are indeed interiors of closed embedded disks.

Lemma 2.17. *Let $B \subset \mathbb{R}^2$ be a closed topological disk and $D \subset B$ an open disk. If $B \setminus D$ has no cut points, then D is the interior of a closed topological disk.*

Lemma 2.18. *If $\mathcal{M}$ is locally connected, then Σ_k is the interior of a closed embedded disk, for every $k \in \mathbb{Z}$.*

Proof. By Lemma 2.15, Σ_k is a bounded open disk for every $k \in \mathbb{Z}$ and the sequence $\operatorname{diam}(\Sigma_k)$ is a null-sequence. Therefore, $\operatorname{Cl}(\Sigma_k) \subset \mathbb{T}^2$ is embedded. Define $\Gamma_k = \mathbb{T}^2 \setminus \Sigma_k$, for some $k \in \mathbb{Z}$. We claim that Γ_k is locally connected. Let $z \in \partial\Sigma_k$. As $\mathcal{M}$ is locally connected, there exists an open connected neighbourhood $U \ni z$ such that $U \cap \mathcal{M}$ is connected. Take a smaller disk V contained in U. Then the union of $U \cap \mathcal{M}$ and $\Sigma_{k'} \cap V$, for all $k' \neq k$, is the required connected neighbourhood.

Secondly, by Lemma 2.16, $\mathcal{M}$ has no cut points. We claim that Γ_k has no cut points either. Indeed, if $z \in \mathcal{M}$, then $\mathcal{M} \setminus \{z\}$ is connected and so is Γ_k as every disk $\Sigma_{k'}$, with $k' \neq k$, shares boundary points with $\mathcal{M} \setminus \{z\}$. Clearly, no point $z \in \Sigma_{k'}$, for some $k' \neq k$, can cut Γ_k. Thus Γ_k is free of cut points. As $\operatorname{Cl}(\Sigma_k)$ is embedded in $\mathbb{T}^2$, we can take a simple closed curve $\gamma \subset \Gamma_k$ bounding an embedded disk in $\mathbb{T}^2$ that properly contains $\mathrm{Cl}(\Sigma_k)$. Now we apply Lemma 2.17 to conclude that Σ_k is the interior of a closed embedded topological disk. $\square$

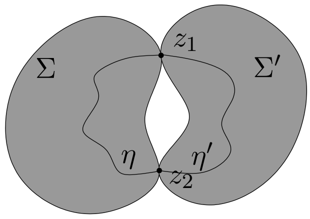

Figure 2.2: Proof of Theorem 2.B.

Proof of Theorem 2.B. By Lemma 2.13, $\mathcal{M}$ has to be of type I and is thus connected. Furthermore, by Lemma 2.15, the disks $\{\Sigma_k\}$ are bounded and their diameter form a null sequence. By Lemma 2.18, Σ_k is the interior of a closed embedded topological disk, for every $k \in \mathbb{Z}$. To finish the proof, we need to show that $\mathrm{Cl}(\Sigma) \cap \mathrm{Cl}(\Sigma')$ consists of at most a single point, where Σ and Σ' are two distinct elements of $\{\Sigma_k\}$. Denote $\gamma = \partial\Sigma$ and $\gamma' = \partial\Sigma'$, both simple closed (trivial) curves. Assume, to the contrary, that $\gamma \cap \gamma'$ contains at least two points z_1, z_2. Take exists an arc $\eta \subset \mathrm{Cl}(\Sigma)$ starting at z_1 and ending at z_2 such that $\eta \cap \partial\Sigma = \{z_1, z_2\}$. Similarly, there exists an arc $\eta' \subset \mathrm{Cl}(\Sigma')$ starting at z_1 and ending at z_2 such that $\eta' \cap \partial\Sigma' = \{z_1, z_2\}$. Then $\eta \cup \eta'$ forms a simple closed curve that bounds a disk D. As the diameters of Σ and Σ' tend to zero, $\mathrm{diam}(f^n(D)) \to 0$ for $|n| \to \infty$. Furthermore, as $z_1 \neq z_2$, D contains arcs contained in $\partial\Sigma$ and $\partial\Sigma'$ joining z_1 and z_2 in its interior, we have that $D \cap \mathcal{M} \neq \emptyset$. As $\mathrm{diam}(f^n(D))$ is a null sequence, for sufficiently large N, we have that $f^N(D) \subset D$, which by Lemma 2.5 implies f has periodic points. Therefore, $\mathrm{Cl}(\Sigma) \cap \mathrm{Cl}(\Sigma')$ can consist of at most a single point and thus $\mathcal{M}$ is indeed a quasi-Sierpiński set. This finishes the proof. $\square$

2.4 Examples

Having given a classification of the possible minimal sets a homeomorphism $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ allows in Theorem 2.A, this section is aimed at constructing homeomorphisms admitting a minimal set of every type of minimal set Theorem 2.A. allows. More precisely, there exist $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ such that its minimal set $\mathcal{M}$

1. is a quasi-Sierpiński set, but not a Sierpiński set,
2. is such that the complement consists of a single unbounded disk,
3. is such that the complement consists of essential annuli and disks,
4. is a non-trivial extension of a Cantor set.

It is well-known there exist $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ for which the minimal set is a Sierpiński set. Example 1, constructed in [4], is derived from the Sierpiński set, see also section 2.4.3 below for a discussion of this example. The remainder of this section is devoted to the construction of Example 2 (type I), Example 3 (type II) and Example 4 (type III). Combined these examples prove Theorem 2.C.

2.4.1 Homeomorphisms semi-conjugate to an irrational translation

There is a natural subclass of $\mathrm{Homeo}_*(\mathbb{T}^2)$, namely those homeomorphisms that are semi-conjugate to an irrational translation of the torus. Indeed, a standard argument shows that an element $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ semi-conjugate to a translation τ, through a continuous map homotopic to the identity, has the property that $\rho(f) = \rho(\tau) \mod \mathbb{Z}^2$. Given a continuous map $\pi\colon \mathbb{T}^2 \to \mathbb{T}^2$, we call the set of points

$$\mathcal{R}_\pi = \left\{ z \in \mathbb{T}^2 \mid \# \left(\pi^{-1}(\pi(z))\right) = 1 \right\} \subset \mathbb{T}^2, \tag{2.20}$$

the *regular set* of π.

Definition 2.7. We define the class $\mathrm{Homeo}_\#(\mathbb{T}^2) \subset \mathrm{Homeo}_*(\mathbb{T}^2)$ the class of homeomorphisms which satisfy the following:

(i) f is isotopic to the identity, i.e. $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$,

(ii) there exists a monotone[1] and continuous $\pi : \mathbb{T}^2 \to \mathbb{T}^2$, homotopic to the identity, and an irrational translation τ such that $\pi \circ f = \tau \circ \pi$, where τ is an irrational translation (cf. (2.9)), and

(iii) the regular set $\mathcal{R}_\pi$ contains uncountably many elements.

The following simple but important observation plays a crucial role in the constructions below.

Lemma 2.19. *Let $f \in \mathrm{Homeo}_\#(\mathbb{T}^2)$, with π the corresponding semi-conjugacy. Then f has a unique minimal set $\mathcal{M}$ and*

$$\mathcal{M} = \mathrm{Cl}(\mathcal{R}_\pi) = \mathrm{Cl}(\mathcal{O}_f(z)), \tag{2.21}$$

for any $z \in \mathcal{R}_\pi$.

Proof. Let us first prove $\mathcal{M}$ is the unique minimal set. Let $\mathcal{M}$ and $\mathcal{M}'$ be two minimal sets for f. Because $\mathcal{M}$ is closed and f-invariant, $\pi(\mathcal{M})$ is closed and τ-invariant. In particular, $\pi(\mathcal{M})$ contains the complete orbit of every point. Since every orbit of τ is dense, we have that $\pi(\mathcal{M}) = \mathbb{T}^2$. Similarly, $\pi(\mathcal{M}') = \mathbb{T}^2$. Take $z \in \mathcal{R}_\pi \neq \emptyset$. Then $\{z\} = \pi^{-1}(\pi(z))$ is contained in both $\mathcal{M}$ and $\mathcal{M}'$. As two minimal sets are either identical or disjoint, this implies that $\mathcal{M} = \mathcal{M}'$. Thus $\mathcal{M}$ is unique.

Next we prove that $\mathcal{M} = \mathrm{Cl}(\mathcal{O}_f(z))$, for every $z \in \mathcal{R}_\pi$. Take any $z \in \mathcal{R}_\pi$ and consider $\mathcal{O}_f(z)$. Then $\mathrm{Cl}(\mathcal{O}_f(z))$ is closed and invariant, hence it contains the (unique) minimal set $\mathcal{M}$ of f, i.e. $\mathcal{M} \subseteq \mathrm{Cl}(\mathcal{O}_f(z))$. We need to show that $\mathrm{Cl}(\mathcal{O}_f(z)) \subseteq \mathcal{M}$. If $\mathcal{M} \cap \mathcal{O}_f(z) = \emptyset$, then $\pi(\mathcal{M}) \neq \mathbb{T}^2$, therefore $\mathcal{M} \cap \mathcal{O}_f(z) \neq \emptyset$. Let $z' \in \mathcal{M} \cap \mathcal{O}_f(z)$, then $\mathcal{O}_f(z') = \mathcal{O}_f(z) \subseteq \mathcal{M}$, since $\mathcal{M}$ is invariant. But since $\mathcal{M}$ is also closed, $\mathrm{Cl}(\mathcal{O}_f(z)) \subseteq \mathcal{M}$. Hence $\mathcal{M} = \mathrm{Cl}(\mathcal{O}_f(z))$, for any $z \in \mathcal{R}_\pi$ and, consequently, $\mathcal{M} = \mathrm{Cl}(\mathcal{R}_\pi)$. ☐

[1]A map is said to be monotone if every point-inverse is connected.

Let us further introduce the following notation, to be used in the proofs of examples 2, 3 and 4 below. A non-transitive orientation preserving circle homeomorphism with irrational rotation number will be referred to as a *Denjoy counterexample*. Moreover, given a Cantor set in the circle $\mathcal{Q} = \mathbb{T}^1 \setminus \bigcup_{k\in\mathbb{Z}} I_k$, we denote $\mathcal{Q}_{\mathrm{rat}} \subset \mathcal{Q}$ and $\mathcal{Q}_{\mathrm{irr}} = \mathcal{Q} \setminus \mathcal{Q}_{\mathrm{rat}}$ the *rational* and *irrational* part of $\mathcal{Q}$, comprised of all the endpoints of the deleted intervals and the complement in $\mathcal{Q}$ of these endpoints respectively. It is readily verified, using Poincaré's Theorem (see section 2.1), that

(1) a product of a Denjoy counterexample and an irrational rotation, and

(2) a product of two Denjoy counterexamples,

provided the factors are chosen such that the corresponding rotation numbers are rationally independent, are elements of $\mathrm{Homeo}_{\#}(\mathbb{T}^2)$.

2.4.2 A topological blow-up procedure

In order to construct the examples, we need a tool with which to construct homeomorphisms exhibiting the desired behaviour. Below we devise such a tool that enables us to *blow up* an orbit of a point under a homeomorphism to a collection of disks. A. Biś, H. Nakayama and P. Walczak in [5] define such a blow-up procedure that works for (groups of) diffeomorphisms. J. Aarts and L. Oversteegen in [1] defined a similar blowup construction for a homeomorphism that has the property that it sends straight rays emanating from a point again to straight rays. In both constructions, this allows for the mapping to be extended to the disks glued to the surface by the infinitesimal behaviour of the mapping. As this would not work for a general homeomorphism, we circumvent this by inductively blowing up punctures to disks, by pulling back points near a puncture along leafs of a dynamically defined foliation emanating from the puncture. We use the continuity of the foliation to define an extension of the mapping to the disks.

Let $f \in \mathrm{Homeo}_{\#}(\mathbb{T}^2)$, with $\pi\colon \mathbb{T}^2 \to \mathbb{T}^2$ the semi-conjugacy between f and an irrational translation τ. Take a point $z_0 \in \mathcal{R}_\pi$ and consider its orbit $\mathcal{O}_f(z_0)$; we do not require f to be minimal, so $\mathcal{O}_f(z_0)$ may or may not be dense. Define $\Gamma = \mathbb{T}^2 \backslash \mathcal{O}_f(z_0)$. Clearly, $f(\Gamma) = \Gamma$ and $f|_\Gamma$ is a homeomorphism. Let $B_\delta := B(z_0, \delta) \subset \mathbb{T}^2$ be the embedded closed Euclidean disk of radius $0 < \delta \leq 1/4$ centered at z_0. Choose $0 < \delta_0 \leq 1/4$ and let $\mathcal{F}_0$ be the foliation of B_{δ_0} by straight rays emanating from z_0, see Figure 2.3. The leaves $\rho_\theta \in \mathcal{F}_0$ are parametrized by $\theta \in [0, 2\pi)$. Fix $0 < \epsilon_0 < 1$. To blow up the punctures to disks, we define the following auxiliary planar map, which reads in polar coordinates,

$$g_\epsilon \colon \widetilde{B}_1 \setminus \{0\} \to \widetilde{B}_1 \setminus \widetilde{B}_\epsilon, \quad g_\epsilon(r, \theta) = \left(\frac{r + \epsilon}{1 + r\epsilon}, \theta \right) \tag{2.22}$$

where $\widetilde{B}_\rho \subset \mathbb{R}^2$ is the closed Euclidean disk centered at $0 \in \mathbb{R}^2$ of radius $0 < \rho \leq 1$. Conjugating g_{ϵ_0} with a linear injection (into the torus) $\lambda_0 \colon \widetilde{B}_1 \hookrightarrow B_{\delta_0}$ yields a homeomorphism

$$h_0 = \lambda_0 \circ g_{\epsilon_0} \circ \lambda_0^{-1} \colon A_0 \longrightarrow h_0(A_0), \tag{2.23}$$

where $A_0 = B_{\delta_0} \setminus \{z_0\}$ and $h_0(A_0) = B_{\delta_0} \setminus B_{\epsilon_0 \delta_0}$ the corresponding annuli. We can extend h_0 to $\mathbb{T}^2 \setminus \{z_0\}$ by declaring it to be the identity off A_0, the homeomorphism we denote again by h_0, and it naturally acts on $\Gamma \subset \mathbb{T}^2 \setminus \{z_0\}$ by restriction. Note that h_0 acts on $B_{\delta_0} \setminus \{z_0\}$ along the foliation $\mathcal{F}_0$.

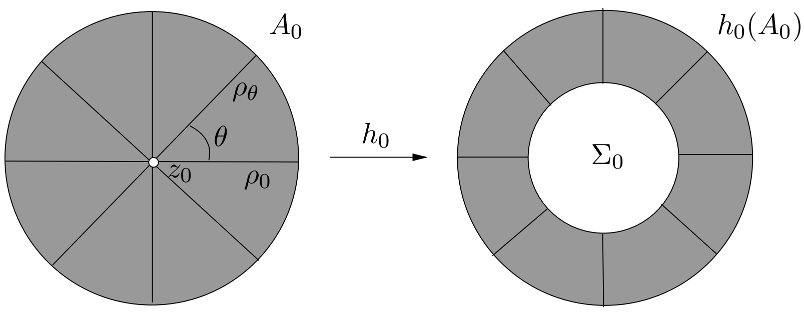

Figure 2.3: Radial blow up of a puncture to a disk.

Define $\Gamma_0 = h_0(\Gamma)$ and define the homeomorphism

$$f_0 \colon \Gamma_0 \to \Gamma_0, \quad f_0 = h_0 \circ f|_\Gamma \circ h_0^{-1} \tag{2.24}$$

and define the continuous $\phi_0 \colon \Gamma_0 \to \Gamma$ where $\phi_0 = h_0^{-1}$. Note that, by construction, $f_0 = \phi_0^{-1} \circ f|_\Gamma \circ \phi_0$. Define $\Sigma_0 := \mathrm{Int}(B_{\epsilon_0 \delta_0})$ and $\gamma_0 := \partial \Sigma_0$, see again Figure 2.3. Consider the points $z_{\pm 1} := \phi_0^{-1}(f^{\pm 1}(z_0))$ and define

$$d_1 = \frac{1}{4} \min \left\{ (1/4)^2, d(z_{-1}, z_1), d(z_{-1}, \Sigma_0), d(z_1, \Sigma_0) \right\} > 0. \tag{2.25}$$

Given $0 < \epsilon_0 < \epsilon < 1$, define $\epsilon' = \frac{\epsilon + \epsilon_0}{2}$ and define the second auxiliary planar map

$$q_\epsilon \colon \widetilde{B}_\epsilon \setminus \widetilde{B}_{\epsilon_0} \to \widetilde{B}_\epsilon \setminus \widetilde{B}_{\epsilon'}, \quad q_\epsilon = \hat{g}_{\epsilon'/\epsilon} \circ \hat{g}_{\epsilon_0/\epsilon}^{-1}, \tag{2.26}$$

where $r_\epsilon : \widetilde{B}_1 \to \widetilde{B}_\epsilon$ is a linear (planar) rescaling, and

$$\hat{g}_{\delta/\epsilon} := r_\epsilon \circ g_{\delta/\epsilon} \circ r_\epsilon^{-1}, \tag{2.27}$$

for $\epsilon_0 \leq \delta < \epsilon$. Let $\lambda_\epsilon \colon \widetilde{B}_\epsilon \hookrightarrow B_{\epsilon \delta_0}$ be the linear injection of the disk $\widetilde{B}_\epsilon \subset \mathbb{R}^2$ onto the disk $B_{\epsilon \delta_0} \subset \mathbb{T}^2$. Define $A_{\epsilon, \epsilon'} = B_{\epsilon' \delta_0} \setminus B_{\epsilon \delta_0}$ and

$$\hat{q}_\epsilon \colon A_{\epsilon, \epsilon_0} \to A_{\epsilon, \epsilon'}, \quad \hat{q}_\epsilon = \lambda_\epsilon \circ q_\epsilon \circ \lambda_\epsilon^{-1}. \tag{2.28}$$

In words, $\hat{q}_\epsilon$ has the effect of mapping the annulus A_{ϵ, ϵ_0} radially, i.e. along (part of) the foliation $\mathcal{F}_0$, to the annulus $A_{\epsilon, \epsilon'}$ with the same outer boundary curve, but larger inner boundary curve, so as to half the modulus of the annulus. There exist $0 < \epsilon_0 < \epsilon_{\pm 1} < 1$, such that, if we denote $A_{\pm 1} := f_0^{\pm 1}(A_{\epsilon_{\pm 1}, \epsilon_0})$, then $\mathrm{diam}(A_{\pm 1}) \leq d_1$. Define

$$h_{\pm 1} \colon A_{\pm 1} \to h_{\pm 1}(A_{\pm 1}), \quad h_{\pm 1} = f_0^{\mp 1} \circ \hat{q}_{\epsilon_{\pm 1}} \circ f_0^{\pm 1}, \tag{2.29}$$

defined on $A_{-1} \cup A_1$ and we extend $h_{\pm 1}$ to $\mathbb{T}^2$ by declaring it to be the identity off $A_{\pm 1}$. The annuli $A_{\pm 1}$ are foliated by $\mathcal{F}_{\pm 1} = f_0|_{A_{\epsilon_{\pm 1}, \epsilon_0}}(\mathcal{F}_0)$. The maps $h_{\pm 1}$ have the effect of blowing up the puncture $z_{\pm 1}$ along the foliation $\mathcal{F}_{\pm 1}$ to a disk $\Sigma_{\pm 1}$, see Figure 2.4. Denote $\Sigma_{\pm 1}$ the open disks obtained by blowing up the corresponding

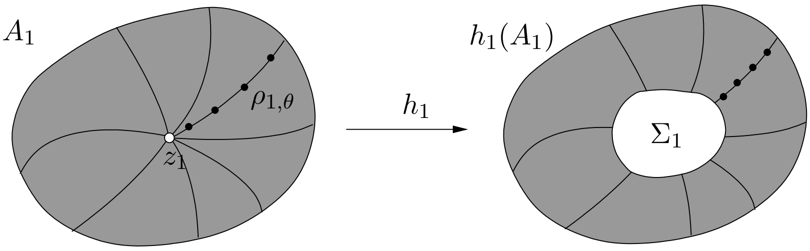

Figure 2.4: Blowing up the puncture z_1 to a disk; $\rho_{1,\theta}$ is a leaf of the foliation $\mathcal{F}_1$ emanating from z_1 and h_1 has the effect of pulling back points on $\rho_{1,\theta} \in \mathcal{F}_1$ along this leaf, for every $\theta \in [0, 2\pi)$.

puncture $z_{\pm 1}$. As $\partial \Sigma_{\pm 1} = f_0(C_{\epsilon'_{\pm 1}})$, where $C_{\epsilon'_{\pm 1}}$ is the Euclidean circle centered at z_0 of radius $\epsilon_0 < \epsilon'_{\pm 1} < \epsilon_{\pm 1}$, $\gamma_{\pm 1} := \partial \Sigma_{\pm 1}$ is a simple closed curve, as $f_0^{\pm 1}|_{A_{\epsilon_{\pm 1}, \epsilon_0}}$ is a homeomorphism.

Define $\hat{h}_1 := h_{-1} \circ h_1$ on $A_{-1} \cup A_1$, define $\Gamma_1 = \hat{h}_1(\Gamma_0)$ and define the homeomorphism

$$f_1 \colon \Gamma_1 \to \Gamma_1, \quad f_1 := \hat{h}_1 \circ f_0 \circ \hat{h}_1^{-1}. \tag{2.30}$$

Further, define the continuous $\phi_1 \colon \Gamma_1 \to \Gamma$ where $\phi_1 = \phi_0 \circ \hat{h}_1^{-1}$.

We proceed by induction. Assume we have blown up the punctures z_k to disks Σ_k, where $-n+1 \leq k \leq n-1$, and consider the points $z_{\pm n} := \phi_{n-1}^{-1}(f^{\pm n}(z_0))$. Define $\Delta_{n-1} = \bigcup_{k=-n+1}^{n-1} \Sigma_k$ and define

$$d_n = \frac{1}{4} \min \left\{ (1/4)^{n+1}, d(z_{-n}, z_n), d(z_{-n}, \Delta_{n-1}), d(z_n, \Delta_{n-1}) \right\} > 0. \tag{2.31}$$

There exist $0 < \epsilon_0 < \epsilon_{\pm n} < 1$, such that $\operatorname{diam}(A_{\pm n}) \leq d_n$. Define

$$h_{\pm n} \colon A_{\pm n} \to h_{\pm n}(A_{\pm n}), \quad h_{\pm n} = f_{n-1}^{\mp n} \circ \hat{q}_{\epsilon_{\pm n}} \circ f_{n-1}^{\pm n}, \tag{2.32}$$

defined on $A_{\pm n}$, where we can extend $h_{\pm n}$ to $\mathbb{T}^2$ by declaring it to be the identity off $A_{\pm n}$. The annuli $A_{\pm 1}$ are foliated by $\mathcal{F}_{\pm n} = f_{n-1}^{n}|_{A_{\epsilon_{\pm n}, \epsilon_0}}(\mathcal{F}_0)$. The maps $h_{\pm n}$ blow up the puncture $z_{\pm n}$ along the foliation $\mathcal{F}_{\pm n}$ to a disk $\Sigma_{\pm n}$. The boundaries $\gamma_{\pm n}$ are again simple closed curves, as $\gamma_{\pm n} = f_{n-1}^{\pm n}(C_{\epsilon'_{\pm n}})$, where $C_{\epsilon'_{\pm n}}$ is the Euclidean circle centered at z_0 of radius $\epsilon_0 < \epsilon'_{\pm n} < \epsilon_{\pm n}$ and $f_{n-1}^{\pm n}|_{A_{\epsilon_{\pm n}, \epsilon_0}}$ is a homeomorphism. Define $\hat{h}_n := h_{-n} \circ h_n$ on $A_{-n} \cup A_n$, define $\Gamma_n = \hat{h}_n(\Gamma_{n-1})$ and define the homeomorphism

$$f_n \colon \Gamma_n \to \Gamma_n, \quad f_n := \hat{h}_n \circ f_{n-1} \circ \hat{h}_n^{-1}. \tag{2.33}$$

Further, define the continuous $\phi_n \colon \Gamma_n \to \Gamma$ where $\phi_n = \phi_{n-1} \circ \hat{h}_n^{-1}$.

Next, we show that the above sequences of maps and homeomorphisms converge and have the desired properties. First, by (2.31) combined with (2.32), it holds that $\Gamma_n \subset \Gamma_{n-1}$ and that $\Gamma_\infty = \lim_{n\to\infty} \Gamma_n$ converges in the Hausdorff sense, as $\sum_{n\geq 0} 1/4^{n+1} < 1 < \infty$. Denote $\mathcal{N} = \mathrm{Cl}(\Gamma_\infty)$. Notice that $\mathcal{N} = \Gamma_\infty \cup \bigcup_{k\in\mathbb{Z}} \gamma_k$, since no point in Σ_k can be the limit point of points in Γ_∞ as $\Sigma_k \cap \Gamma_\infty = \emptyset$. Furthermore, note that, as the boundary curves γ_k are simple closed curves, the extension $\bar{\phi}_n \colon \mathrm{Cl}(\Gamma_n) \to \mathbb{T}^2$ of ϕ_n is continuous.

Lemma 2.20. *The homeomorphisms $f_n \colon \Gamma_n \to \Gamma_n$ converge to a homeomorphism $f_\infty \colon \Gamma_\infty \to \Gamma_\infty$, and extends to a homeomorphism $f' \in \mathrm{Homeo}_*(\mathbb{T}^2)$ with $f'(\mathcal{N}) = \mathcal{N}$. Further, the disks $\{\Sigma_k\}$ in the complement of $\mathcal{N}$ are interiors of closed topological disks. Similarly, the continuous maps ϕ_n converge to a continuous map $\phi_\infty \colon \Gamma_\infty \to \Gamma$, and extends to a continuous $\phi \colon \mathbb{T}^2 \to \mathbb{T}^2$ for which $\phi(\mathcal{N}) = \mathbb{T}^2$. Furthermore, f' is semi-conjugate to f through ϕ.*

Proof. First, we show that $f_n \to f_\infty$ converges to a homeomorphism of Γ_∞. Indeed, for every $n \geq 0$, $f_n \colon \Gamma_n \to \Gamma_n$ is a homeomorphism and we observed above that $\Gamma_n \to \Gamma_\infty$ converges. As $\hat{h}_n$ moves points by no more than the distance of $d_n \leq 1/4^{n+1}$, and $\sum_{n\geq 0} 1/4^{n+1} < 1 < \infty$, $f_n \to f_\infty$ converges uniformly and thus the limit f_∞ is a homeomorphism. Further, we observed that γ_k is a simple closed curve, for every $k \in \mathbb{Z}$ and thus Σ_k is the interior of the closed topological disk $\mathrm{Cl}(\Sigma_k) = \Sigma_k \cup \gamma_k$.

Next, we show that f_∞ extends to a homeomorphism f' of $\mathcal{N}$. To this end, we first show that f_∞ induces a homeomorphism from γ_k to γ_{k+1}, for every $k \in \mathbb{Z}$. To prove this, we note that the disks Σ_k, for $-n \leq k \leq n$, which have been constructed after n steps, are left unmoved by future perturbations by virtue of our choice of d_n. Moreover, again by our choice of d_n, it holds that $f_n|_{\gamma_k} = f_\infty|_{\gamma_k} : \gamma_k \to \gamma_{k+1}$, for $-n \le k \le n-1$, where $f_n|_{\gamma_k}$ and $f_\infty|_{\gamma_k}$ are the extensions of f_n and f_∞ to γ_k. To prove that $f_{n|_{\gamma_k}}$ is a homeomorphism, it suffices to show that $f_{n|_{\gamma_k}}$ is one-to-one and continuous. We prove this by induction, where we consider $0 \le k \le n$, the case for negative k being handled by considering the inverse.

Assume that after step $n-1$, we have shown that $f_{n-1}|_{\gamma_k} : \gamma_k \to \gamma_{k+1}$, for $0 \le k \le n-2$ are homeomorphisms and consider step n, where we have to show that $f_n|_{\gamma_{n-1}} : \gamma_{n-1} \to \gamma_n$ is a homeomorphism. By choice of ϵ_n, A_n is disjoint from the previously constructed disks and disjoint from A_{-n}. Restricting to a smaller neighbourhood of A_{n-1} if necessary, we may as well assume that $A_{n-1} \cap A_n = \emptyset$. As $h_n = \hat{h}_n|_{A_n}$ (as defined by (2.29)) acts along the foliation $\mathcal{F}_n$, f_n sends leafs of $\mathcal{F}_{n-1}$ to leafs of $\mathcal{F}_n$ which foliate A_{n-1} and A_n respectively. To each $\theta \in [0, 2\pi)$ corresponds a unique point $z(\theta) \in \gamma_{n-1}$ lying on $\rho_{n-1,\theta} \in \mathcal{F}_{n-1}$, which is by f_n mapped to a unique point $z'(\theta) = f_n(z(\theta)) \in \gamma_n$ lying on $\rho_{n,\theta} = f_n(\rho_{n-1,\theta})$. As these foliations are continuous, and as the curves γ_{n-1} and γ_n are (continuous) simple closed curves, the points $z'(\theta)$ vary continuously as θ varies, and thus continuously as $z(\theta)$ varies and this is what we needed to show.

By induction, f_∞ extends homeomorphically to every boundary curve γ_k, $k \in \mathbb{Z}$. It thus follows that the extension f' to $\mathcal{N}$ is one-to-one, as $\mathcal{N} = \Gamma_\infty \cup \bigcup_{k \in \mathbb{Z}} \gamma_k$. To show f' is continuous, we distinguish between two cases. First, let $z \in \Gamma_\infty$. As f_∞ is a homeomorphism, given a neighbourhood $V \subset \mathcal{N}$ containing z', we can find a small neighbourhood $U \subset \mathcal{N}$, containing the point z for which $f'(z) = z'$, such that $\mathrm{Cl}(f_\infty(W)) \subset V$, where $W = U \cap \Gamma_\infty$. As $\mathrm{Cl}(\Gamma_\infty) = \mathcal{N}$ and f' extends homeomorphically to $\mathcal{N}$, we have that $f'(U) = f'(\mathrm{Cl}(W)) = \mathrm{Cl}(f_\infty(W)) \subset V$. Secondly, suppose that $z \in \gamma_k$ for some $k \in \mathbb{Z}$. For $N \ge k+1$, it holds that $f_N|_{\gamma_k} = f_\infty|_{\gamma_k} : \gamma_k \to \gamma_{k+1}$ is a homeomorphism. Therefore, given a neighbourhood $V \subset \mathcal{N}$ containing $z' = f_N(z) = f'(z)$, there exists a small neighbourhood $U \ni z$, such that $f_N(U) \subset V$. Choosing N larger, and a smaller neighbourhood $U' \subset U$ containing z, if necessary, as $\sum_{n \geq N} d_n \to 0$ for $N \to \infty$, we have that $f'(U') \subset V$ as well. Thus f' is continuous, and therefore a homeomorphism, being one-to-one as well. We can extend the homeomorphism $f' \colon \mathcal{N} \to \mathcal{N}$ to a homeomorphism of $\mathbb{T}^2$ by extending, e.g. by Alexander's trick, the induced homeomorphisms of the boundary curves γ_k to homeomorphisms of the corresponding closed disks $\mathrm{Cl}(\Sigma_k) = \Sigma_k \cup \gamma_k$. As the disks Σ_k are disjoint, and $\mathrm{diam}(\Sigma_k)$ forms a null-sequence, the extension of $f' \colon \mathcal{N} \to \mathcal{N}$ to $\mathbb{T}^2$ is a homeomorphism, which we denote again by f'.

To show that $\phi \colon \mathcal{N} \to \mathbb{T}^2$ is continuous, we recall that $\mathrm{Cl}(\Gamma_n) = \mathbb{T}^2 \setminus \Delta_n$, where $n \geq 0$. As we observed, for every $n \geq 0$, $\phi_n \colon \Gamma_n \to \Gamma$ is continuous and it extends to a continuous $\bar{\phi}_n \colon \mathrm{Cl}(\Gamma_n) \to \mathbb{T}^2$. As $\phi_n = \phi_{n-1} \circ \hat{h}_n^{-1}$, with $\hat{h}_n$ as in (2.32), whose norm is bounded by d_n, $\phi = \lim_{n \to \infty} \bar{\phi}_n \colon \mathcal{N} \to \mathbb{T}^2$ is continuous as a limit of uniformly converging continuous maps $\bar{\phi}_n$. By declaring $\phi(\Sigma_k) = f^k(z_0)$, ϕ extends to a continuous map defined on $\mathbb{T}^2$.

Finally, we show that $f' \in \mathrm{Homeo}_{\#}(\mathbb{T}^2)$. First, we observe that, as $\mathcal{R}_\pi$ is uncountable, and a countable number of points of $\mathcal{R}_\pi$ is blown up to disks, we have that $\mathcal{R}_{\pi'}$ is uncountable. Further, the ϕ thus constructed is homotopic to the identity. Thus it suffices to show that $\phi \circ f' = f \circ \phi$. For this, we note that for every $n \geq 0$ we have that $\phi_n \circ f_n = f|_\Gamma \circ \phi_n$, where $f_n \colon \Gamma_n \to \Gamma_n$ and $\phi_n \colon \Gamma_n \to \Gamma$. As both f_n and ϕ_n converge uniformly and extend continuously to $\mathcal{N}$, it follows that $\phi \circ f' = f \circ \phi$, where $f' \colon \mathcal{N} \to \mathcal{N}$. Further, as Σ_k, along with γ_k, is mapped to a single point by ϕ, it thus also holds that $\phi \circ f' = f \circ \phi$ when f' is extended to $\mathbb{T}^2$. $\square$

The following lemma, which combines Lemma 2.19 and Lemma 2.20 is the key ingredient in the construction of Examples 3 and 4. Let $B_{\delta_0} \setminus \{z_0\} \subset \mathbb{T}^2$ be an embedded punctured disk centered at $z_0 \in \mathcal{R}_\pi$ with $\delta_0 \leq 1/4$ and $\mathcal{F}_0$ the corresponding foliation of $B_\delta \setminus \{z_0\}$ by straight rays emanating from z_0, in the notation of the construction above. A *wedge* $\mathcal{W}(r, \theta_1, \theta_2) \subset B_{r\delta_0} \setminus \{z_0\}$ is the region bounded by two leaves $\rho_{\theta_1}, \rho_{\theta_2} \in \mathcal{F}_0$, where $0 < |\theta_1 - \theta_2| < \pi$ and $0 < r \leq 1$.

Lemma 2.21. *In the construction above, let f' be semi-conjugate to f through ϕ by blowing up the orbit $\mathcal{O}_f(z_0)$, with $z_0 \in \mathcal{R}_\pi$, to disks whose interiors are Σ_k, and $\gamma_k = \partial\Sigma_k$, where $k \in \mathbb{Z}$. Let $\mathcal{M}'$ be the minimal set of f' and define $\pi' = \pi \circ \phi$. Then*

(1) $\mathcal{M}' = \mathrm{Cl}(\mathcal{R}_{\pi'}) = \mathrm{Cl}(\phi^{-1}(\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)))$,

(2) $\gamma_k \subset \mathcal{M}'$, *for all* $k \in \mathbb{Z}$, *if for every* $0 < r \leq 1$ *and every* $\theta_1, \theta_2 \in [0, 2\pi)$, *with* $0 < |\theta_1 - \theta_2| < \pi$, *we have that* $\mathcal{W}(r, \theta_1, \theta_2) \cap (\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)) \neq \emptyset$.

Proof. To prove (1), as $\mathcal{R}_\pi$ is uncountable, $R^0_\pi \neq \emptyset$. As the points $f^k(z_0)$ are blown up to disks, i.e. $\phi^{-1}(f^k(z_0)) = \mathrm{Cl}(\Sigma_k)$, where $\mathrm{Cl}(\Sigma_k)$ is a closed topological disk, we have that $\mathcal{R}_{\pi'} = \phi^{-1}(\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0))$. By Lemma 2.19, we have that $\mathcal{M}' = \mathrm{Cl}(\mathcal{R}_{\pi'})$, and this proves (1).

To prove (2), define $\mathcal{R}^n_\pi := \phi_n^{-1}(\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0))$. First assume that $\gamma_0 \subset \mathrm{Cl}(R^0_\pi)$. As the size of the perturbations $\hat{h}_n$, by virtue of our choice of d_n, converge to zero as the perturbations approach γ_0, it holds that $\gamma_0 \subset \mathrm{Cl}(R^n_\pi)$ for every $n \geq 0$. As the maps ϕ_n converge, it thus holds that $\gamma_0 \subset \mathrm{Cl}(\mathcal{R}_{\pi'}) = \mathcal{M}'$, by (1). As $\mathcal{M}'$ is f'-invariant, and $f'(\gamma_k) = \gamma_{k+1}$, we have that $\gamma_k \subset \mathcal{M}'$, for every $k \in \mathbb{Z}$. To finish the proof, suppose that $\mathcal{W}(r, \theta_1, \theta_2) \cap (\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)) \neq \emptyset$ for every $0 < r \leq 1$ and every $\theta_1, \theta_2 \in [0, 2\pi)$ for which $0 < |\theta_1 - \theta_2| < \pi$. We have to show that $\gamma_0 \subset \mathrm{Cl}(R^0_\pi)$. Suppose, to derive a contradiction, that $\gamma_0 \cap \mathrm{Cl}(R^0_\pi) \neq \gamma_0$. As $\gamma_0 \cap \mathrm{Cl}(R^0_\pi)$ is closed, this implies there exists an open subarc $\eta \subset \gamma_0$ such that $\eta \cap \mathrm{Cl}(R^0_\pi) = \emptyset$. Let $z \in \eta$ be the midpoint of η and let $\eta' \subset \eta$ be a closed subsegment properly contained in η, and containing $z \in \eta$, with endpoints $\{z^-, z^+\} = \partial\eta'$. Let ρ_{θ_1} and ρ_{θ_2} be the two rays passing through z^- and z^+ and $\mathcal{W}(1, \theta_1, \theta_2)$ the corresponding wedge. As $\mathrm{Cl}(R^0_\pi)$ is closed, there exists an open neighbourhood $U \supset \eta'$ such that $U \cap \mathrm{Cl}(R^0_\pi) = \emptyset$. However, this implies that $\mathcal{W}(r, \theta_1, \theta_2) \cap (\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)) = \emptyset$ for $r > 0$ sufficiently small, contrary to our assumption. ∎

2.4.3 Minimal sets of type I

It is well-known that, given any Sierpiński set $S \subset \mathbb{T}^2$, there exist a homeomorphism $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ for which the minimal set $\mathcal{M} = S$. The following example can be found in [4, Thm 3]. We will only sketch the proof.

Example 1 (Type I : a quasi-Sierpiński set)**.** There exist homeomorphisms $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ for which the minimal set $\mathcal{M}$ is a quasi-Sierpiński set, but not a Sierpiński set.

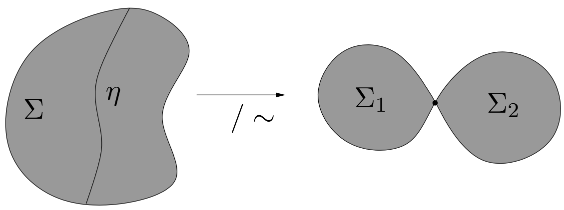

Figure 2.5: Construction of a quasi-Sierpiński set: collapsing arcs to points.

Sketch of the proof. Let $\mathcal{M}$ be a Sierpiński minimal set of an element $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ and let Σ be a component of the complement of $\mathcal{M}$. Denote $\Sigma_n = f^n(\Sigma)$ and $\gamma_n = \partial\Sigma_n$. Take an arc $\eta \subset \mathrm{Cl}(\Sigma)$ such that only the endpoints of η intersect $\gamma = \gamma_0$, see Figure 2.5. Let $\eta_n := f^n(\eta) \subset \mathrm{Cl}(\Sigma_n)$ the corresponding arcs in the image disks. Using techniques from decomposition theory, it can be shown that $\mathbb{T}^2/\sim$, where $z \sim z'$ if and only if $z, z' \in \eta_n$ (i.e. collapsing the arcs η_n to points), yields a well-defined quotient space homeomorphic to $\mathbb{T}^2$ and that $\mathcal{M}$ quotients to a quasi-Sierpiński set $\mathcal{M}' = \mathcal{M}/\sim$, which is not a Sierpiński set. The corresponding quotient homeomorphism $f' \in \mathrm{Homeo}_*(\mathbb{T}^2)$ has $\mathcal{M}'$ as its minimal set and this minimal set $\mathcal{M}'$ is locally connected. $\square$

Next, we give an example of a minimal set which is of type I, but not locally connected. It shows the existence of homeomorphisms $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ for which the minimal set $\mathcal{M}$ is such that the complement consists of a single unbounded disk Σ, which is f-invariant. The desired homeomorphism is derived from the derived-from-Anosov construction, see e.g. [25], and is similar to McSwiggen's construction [33].

Example 2 (Type I : unbounded disks)**.** There exist minimal sets $\mathcal{M}$ of homeomorphisms $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ of type I such that the complement of $\mathcal{M}$ in $\mathbb{T}^2$ is a single unbounded disk.

Proof. Take $A \in \mathrm{SL}(2,\mathbb{Z})$ and let $L := L_A \colon \mathbb{T}^2 \to \mathbb{T}^2$ be the induced linear toral automorphism. Let $z_0 \in \mathbb{T}^2$, with $z_0 = p(0)$ where $0 \in \mathbb{R}^2$ is the origin and $p\colon \mathbb{R}^2 \to \mathbb{T}^2$ is the canonical projection, be the fixed point of L and v^u its unstable eigenvector. Let $\mathcal{F}^u_{\mathrm{lin}}$ the unstable foliation of $\mathbb{T}^2$ by parallel lines of this linear Anosov and let $\ell_0 \subset \mathbb{T}^2$ be the unstable leaf passing through the saddle fixed point z_0. Relative to the standard basis, the eigenvector v^u has an irrational slope, and, consequently, every leaf $\ell \in \mathcal{F}^u_{\mathrm{lin}}$ is an isometric immersion of a Euclidean line in $\mathbb{R}^2$. Define $\mathcal{F}_{\mathrm{hor}}$ be the foliation of $\mathbb{T}^2$ by horizontal (relative to the standard basis) simple closed curves, parametrized by $y \in \mathbb{T}^1$; i.e. $\mathcal{F}_{\mathrm{hor}} = \{C_y\}_{y\in\mathbb{T}^1}$ with $C_y \subset \mathbb{T}^2$ the curve of height $y \in \mathbb{T}^1$.

A small smooth perturbation of L around the saddle fixed point $z_0 \in \mathbb{T}^2$ turnes z_0 into three fixed points $z_{-1}, z_0, z_1 \in \mathbb{T}^2$, two of which are saddles and the original fixed point is turned into a repeller. Denote $g \in \mathrm{Diff}^\infty(\mathbb{T}^2)$ the diffeomorphism obtained by perturbing L. There exists a strong unstable g-invariant foliation $\mathcal{F}^u_{\mathrm{DA}}$ of $\mathbb{T}^2$ of the perturbed system. All elements of $\mathcal{F}^u_{\mathrm{DA}}$ are (smooth) immersed copies of $\mathbb{R}$ and the two elements $W^u_{\pm 1} := W^u_{z_{\pm 1}} \in \mathcal{F}^u_{\mathrm{DA}}$ bound an unbounded disk $\Sigma \subset \mathbb{T}^2$ which is dense in the torus, i.e. $\mathrm{Cl}(\Sigma) = \mathbb{T}^2$. The perturbation can be chosen small enough so that the angle of every unstable leaf of $\mathcal{F}^u_{\mathrm{DA}}$ with a leaf of $\mathcal{F}_{\mathrm{hor}}$ is uniformly bounded from below and above, see [25, 33] for these facts.

As $\Sigma \subset \mathbb{T}^2$ is dense, it follows that $C_y \cap \Sigma$ is dense and that $\mathcal{Q}_y = C_y \setminus (C_y \cap \Sigma)$, being the circle minus a dense union of disjoint intervals, is a Cantor set. Note that the endpoints $\mathcal{Q}_{y,\mathrm{rat}}$ of $\mathcal{Q}_y$ are exactly the set of points $C_y \cap \partial\Sigma$ and that

$\partial\Sigma = \bigcup_{y\in\mathbb{T}^1} \mathcal{Q}_{y,\mathrm{rat}}$. The complement in $\mathcal{Q}_y$ of these endpoints are the irrational points, i.e. $\mathcal{Q}_{y,\mathrm{irr}} = \mathcal{Q}_y \setminus \mathcal{Q}_{y,\mathrm{rat}}$. Further, as the slope of v^u is irrational, there exists a suitable $\nu \in \mathbb{R}$ such that $\nu v^u = (\alpha, \beta)$ with $1, \alpha, \beta$ rationally independent. For example, if we take $A \in \mathrm{SL}(2,\mathbb{Z})$ to be Arnold's cat map, then $v^u = (1, \frac{1+\sqrt{5}}{2})$ and choosing $\nu = e$, we have that (α,β) is irrational as $1, \sqrt{5}, e$ are rationally independent. Let $\tau := \tau_{\alpha,\beta}$ be the corresponding irrational translation of the torus where, by construction, $\tau(\ell_0) = \ell_0$.

Choose compatible orientations on the foliations $\mathcal{F}^u_{\mathrm{DA}}$ and $\mathcal{F}^u_{\mathrm{lin}}$. Given a $y \in \mathbb{T}^1$, consider the holonomy homeomorphism $h_y, h'_y \colon C_y \to C_y$, defined as the return map to C_y under the unstable foliation of the linear and perturbed system respectively. It is proved in [33] that, for every $y \in \mathbb{T}^1$, there exists a semi-conjugacy $\pi_y \colon C_y \to C_y$ such that $\pi_y \circ h'_y = h_y \circ \pi_y$. As the continuous foliations $\mathcal{F}^u_{\mathrm{DA}}$ and $\mathcal{F}^u_{\mathrm{lin}}$ are everywhere transversal to the horizontal foliation, the holonomy homeomorphisms h'_y, and consequently the maps π_y, depend continuously on $y \in \mathbb{T}^1$. In other words, as $\mathbb{T}^2 = \bigcup_{y\in\mathbb{T}^1} C_y$, this defines a continuous

$$\pi \colon \mathbb{T}^2 \to \mathbb{T}^2, \quad \pi(x,y) = (\pi_y(x), y), \tag{2.34}$$

that has the property that $\pi(\Sigma \cup W^u_- \cup W^u_+) = \ell_0$. For every $y \in \mathbb{T}^1$, there exists a homeomorphism $f_y \colon C_y \to C_{r_\beta(y)}$, defined by mapping the point $z \in C_y$ to the first intersection point of the unique leaf of $\mathcal{F}^u_{\mathrm{DA}}$ through z with $C_{r_\beta(y)}$ (along the positive direction of the leaf), where $r_\beta \colon \mathbb{T}^1 \to \mathbb{T}^1$ is the irrational rotation with rotation number $\rho(r) = \beta \mod \mathbb{Z}$, see Figure 2.6.

As $\mathcal{F}^u_{\mathrm{DA}}$ is a foliation which is transversal to $\mathcal{F}^u_{\mathrm{hor}}$, f_y is one-to-one. Further, as $\mathcal{F}^u_{\mathrm{DA}}$ is continuous, f_y is continuous as well and thus f_y is a homeomorphism, for every $y \in \mathbb{T}^1$. Further, it follows from the definitions that $\pi_{r_\beta(y)} \circ f_y = \tau \circ \pi_y$. Define

$$f \colon \mathbb{T}^2 \to \mathbb{T}^2, \quad f(x,y) = (f_y(x), r_\beta(y)), \tag{2.35}$$

As the maps f_y are homeomorphisms for every $y \in \mathbb{T}^1$ and depend continuously on $y \in \mathbb{T}^1$ (by virtue again of the foliation $\mathcal{F}^u_{\mathrm{DA}}$ being tranversal and continuous),

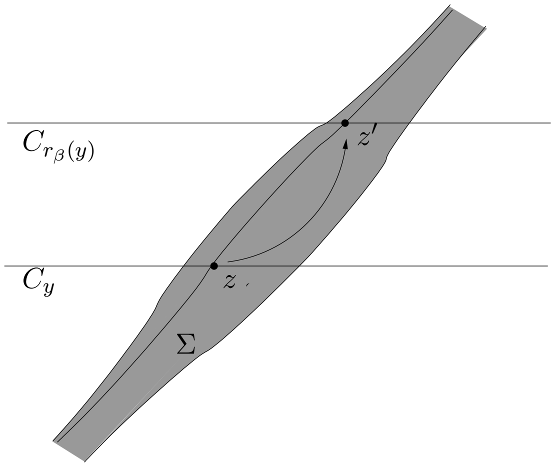

Figure 2.6: f_y maps the point $z \in C_y$ along a leaf of the foliation $\mathcal{F}^u_{\mathrm{DA}}$ to a point $z' = f_y(z) \in C_{r_\beta(y)}$.

it holds that f as defined by (2.35) is a homeomorphism. It is clear that this f is isotopic to the identity, π is homotopic to the identity and, by construction, $\pi \circ f = \tau \circ \pi$ and $f(\Sigma) = \Sigma$ with Σ the unbounded disk bounded by smooth unstable leaves $W^u_{\pm 1} \in \mathcal{F}^u_{\mathrm{DA}}$, as $\tau(\ell_0) = \ell_0$. Let $\mathcal{M}$ be the minimal set of f. As π is one-to-one, except on $\Sigma \cup W^u_- \cup W^u_+$, it holds that

$$\mathcal{R}_\pi = \pi^{-1}(\mathbb{T}^2 \setminus \ell_0) = \mathbb{T}^2 \setminus (\Sigma \cup W^u_- \cup W^u_+) = \bigcup_{y \in \mathbb{T}^1} \mathcal{Q}_{y,\mathrm{irr}}. \tag{2.36}$$

As $\mathcal{Q}_y = \mathrm{Cl}(\mathcal{Q}_{y,\mathrm{irr}})$, combining (2.36) with Lemma 2.19, it follows that

$$\mathcal{M} = \mathrm{Cl}(\bigcup_{y \in \mathbb{T}^1} \mathcal{Q}_{y,\mathrm{irr}}) = \bigcup_{y \in \mathbb{T}^1} \mathrm{Cl}\,(\mathcal{Q}_{y,\mathrm{irr}}) = \bigcup_{y \in \mathbb{T}^1} \mathcal{Q}_y = \mathbb{T}^2 \setminus \Sigma, \tag{2.37}$$

where $\mathrm{Cl}(\bigcup_{y \in \mathbb{T}^1} \mathcal{Q}_{y,\mathrm{irr}}) = \bigcup_{y \in \mathbb{T}^1} \mathrm{Cl}\,(\mathcal{Q}_{y,\mathrm{irr}})$ holds as the Cantor sets $\mathcal{Q}_y$ (and therefore their irrational parts $\mathcal{Q}_{y,\mathrm{irr}}$) depend continuously on $y \in \mathbb{T}^1$. Thus $\mathcal{M} = \mathbb{T}^2 \setminus \Sigma$, with Σ an unbounded and f-invariant disk, and $f \in \mathrm{Homeo}_\#(\mathbb{T}^2)$, as required. This finishes the proof. $\square$

2.4.4 Examples of type II

Let us next give examples of homeomorphisms for which the connected components $\{\Sigma_k\}$ of the complement of $\mathcal{M}$ are essential annuli and disks.

Example 3 (Type II : essential annuli and disks)**.** There exist $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ with minimal set $\mathcal{M}$ of the form $\mathcal{M} = \mathbb{T}^2 \setminus \bigcup_{n\in\mathbb{Z}} \Sigma_k$ with $\{\Sigma_k\}_{k\in\mathbb{Z}}$ a collection of essential annuli and disks. Furthermore, the essential annuli can be constructed to have any characteristic (p, q), where $\gcd(p, q) = 1$.

The proof of Example 3 uses the following.

Lemma 2.22. *Let* $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ *and* $f' = L_A^{-1} \circ f \circ L_A$ *with* $L_A \colon \mathbb{T}^2 \to \mathbb{T}^2$ *induced by a linear* $A \in \mathrm{SL}(2, \mathbb{Z})$*. Then* $f' \in \mathrm{Homeo}_*(\mathbb{T}^2)$*.*

Proof. Let L_A be a linear conjugation induced by an element $A \in \mathrm{SL}(2, \mathbb{Z})$. As $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$, $f' \in \mathrm{Homeo}_0(\mathbb{T}^2)$ as well. Let F, F' a lift of f, f' respectively. By [27, Lemma 2.4], we have that

$$\rho(L_A^{-1} \circ F \circ L_A) = L_A^{-1}\rho(F) \mod \mathbb{Z}^2.$$

Therefore, $\rho(f') = (\alpha', \beta') \mod \mathbb{Z}^2$, where

$$\alpha' = a\alpha + b\beta \text{ and } \beta' = c\alpha + d\beta,$$

with $a, b, c, d \in \mathbb{Z}$. The condition $N_1 + N_2\alpha' + N_3\beta' = 0$ implies that $N_1 = N_2a + N_3c = N_2b + N_3d = 0$, as $1, \alpha, \beta$ are rationally independent. Multiplying $N_2a + N_3c$ by b and $N_2b + N_3d$ by a and subtracting yields that $N_2(ad - bc) = 0$, which yields that $N_2 = 0$ as $A \in \mathrm{SL}(2, \mathbb{Z})$ and thus $ad - bc = 1$. Similarly, it holds that $N_3 = 0$ and it thus follows that $1, \alpha', \beta'$ are rationally independent as well. Therefore, $f' \in \mathrm{Homeo}_*(\mathbb{T}^2)$. □

Proof of Example 3. First, we construct a homeomorphism $f \in \mathrm{Homeo}_\#(\mathbb{T}^2)$ for which the complement of $\mathcal{M}$ is a collection of essential annuli of a given characteristic. Let (p, q) with $\gcd(p, q) = 1$ be given. Let $f \in \mathrm{Homeo}_\#(\mathbb{T}^2)$ be a product of a Denjoy counterexample $\varphi \in \text{Homeo}(\mathbb{T}^1)$ with irrational rotation number $\alpha \notin \mathbb{Q}$ and an irrational rotation r_β, with $\beta \notin \mathbb{Q}$ chosen so that $1, \alpha, \beta$ are rationally independent. The corresponding semi-conjugacy π is of the form $\pi = (\pi_1, \text{Id})$, where π_1 is the semi-conjugacy of φ to the irrational rotation r_α. As $\mathcal{R}_\pi = \mathcal{Q}_{\text{irr}} \times \mathbb{T}^1$, the minimal set $\mathcal{M}$ of f is $\mathcal{M} = \text{Cl}(\mathcal{Q}_{\text{irr}} \times \mathbb{T}^1) = \mathcal{Q} \times \mathbb{T}^1$, where $\mathcal{Q} \subset \mathbb{T}^1$ is the Cantor minimal set of φ. The characteristic of the corresponding essential annuli is $(0, 1)$. For later reference, denote $\{\Sigma_t^a\}$ the collection of essential annuli in the complement of $\mathcal{M}$. Given any pair $(p, q) \in \mathbb{Z}^2$ such that $\gcd(p, q) = 1$, there exists an element $A \in \text{SL}(2, \mathbb{Z})$ such that the (linear) $L_A \in \text{Homeo}(\mathbb{T}^2)$ induced by A has the property that an essential simple closed curve of characteristic $(0, 1)$ is mapped to an essential simple closed curve of characteristic (p, q). By Lemma 2.22, conjugating f with L_A gives a homeomorphism $f' \in \text{Homeo}_*(\mathbb{T}^2)$ and the components of the complement of the minimal set $\mathcal{M}' = L_A^{-1}(\mathcal{M})$ now consists of essential annuli of characteristic (p, q).

To obtain an example of a homeomorphism $f' \in \text{Homeo}_*(\mathbb{T}^2)$ with a minimal set $\mathcal{M}'$ for which the complementary components contain both essential annuli and disks, we modify the above example as follows. Let f again be the product homeomorphism given above. Choose $z_0 \in \mathcal{R}_\pi$ and blow up the orbit $\mathcal{O}_f(z_0)$ to disks by the procedure in section 2.4.2. This gives a homeomorphism $f' \in \text{Homeo}_{\#}(\mathbb{T}^2)$ and a continuous $\phi\colon \mathbb{T}^2 \to \mathbb{T}^2$ such that $\phi \circ f' = f \circ \phi$ and we define $\pi' = \pi \circ \phi$. We have that $\phi^{-1}(f^k(z_0)) = \text{Cl}(\Sigma_k)$, where Σ_k is the interior of the closed disk $\text{Cl}(\Sigma_k)$ and $\gamma_k = \partial \Sigma_k$ a simple closed curve. Denote $\mathcal{M}'$ the corresponding minimal set of f'.

In order to show that the complement of $\mathcal{M}'$ consists of essential annuli and disks, it suffices to show that $\gamma_k \subset \mathcal{M}'$. Indeed, as ϕ is one-to-one on $\mathbb{T}^2 \setminus \bigcup_{k \in \mathbb{Z}} \text{Cl}(\Sigma_k)$, $\phi^{-1}(\Sigma_t^a)$ is again an essential annulus, for every $t \in \mathbb{Z}$, where $\phi^{-1}(\Sigma_t^a) \cap \mathcal{M}' = \emptyset$. Thus to show that $\gamma_k \subset \mathcal{M}'$, for every $k \in \mathbb{Z}$, by Lemma 2.21 (2), it suffices to show that for every $0 < r \leq 1$ and every $\theta_1, \theta_2 \in [0, 2\pi)$, with $0 < |\theta_1 - \theta_2| < \pi$, we have that $\mathcal{W}(r, \theta_1, \theta_2) \cap (\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)) \neq \emptyset$. This is proved as follows: as

$z_0 \in \mathcal{R}_\pi = \mathcal{Q}_{\mathrm{irr}} \times \mathbb{T}^1$, through every wedge $\mathcal{W}(r, \theta_1, \theta_2)$ pass infinitely many vertical simple closed curves (i.e. the connected components of $\mathcal{Q}_{\mathrm{irr}} \times \mathbb{T}^1$), arbitrarily close to z_0. As only countably many of these points are deleted from these curves, every wedge $\mathcal{W}(r, \theta_1, \theta_2)$ contains points of $\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)$, for any $r > 0$. Therefore, $\gamma_k \subset \mathcal{M}'$ for every $k \in \mathbb{Z}$ indeed, where $\mathcal{M}' = \mathrm{Cl}(\mathcal{R}_{\pi'})$ by Lemma 2.21 (1). This finishes the proof. $\square$

2.4.5 Examples of type III

The most simple example of an extension of a Cantor set is of course a Cantor set itself. A homeomorphism $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ admitting such a Cantor minimal set is obtained by taking a product of two Denjoy-counterexamples with rationally independent rotation numbers. Its minimal set is the product of the Cantor minimal sets of its factors, and thus itself a Cantor set. Recall that an extension of a Cantor set $\mathcal{M}$ is said to be non-trivial if not all connected components of $\mathcal{M}$ are singletons. Below we give examples of non-trivial extensions of a Cantor set. Recently, F. Béguin, S. Crovisier, T. Jäger and F. le Roux in [3, Thm 1.2] also constructed an example of a homeomorphism $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ for which the minimal set $\mathcal{M}$ is a non-trivial extension of a Cantor set (in our terminology). This homeomorphism is constructed by adapting a quasiperiodically forced circle homeomorphism (see [3, Thm 1.2], cf. Counterexample 2.2) with a Cantor minimal set, and blowing up an orbit of points to arcs, using a construction due to M. Rees [43, 44]. The minimal set thus constructed has a countable number of arcs among its connected components. The examples below show that in our class $\mathrm{Homeo}_*(\mathbb{T}^2)$ there exist minimal sets, which are non-trivial extensions of Cantor sets, which have separating connected components among its connected components.

Example 4 (Type III : extensions of Cantor sets)**.** There exist $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ for which $\mathcal{M}$ is a non-trivial extension of a Cantor set for which the non-degenerate components are simple closed curves.

Remark 1. *In a way similar to [4, Thm 3], cf. Example 1, by defining a suitable family of arcs in the disks enclosed by the non-degenerate components of the above minimal set, if we pass to the quotient by collapsing these arcs to points, this gives new quotient homeomorphisms of type III for which the corresponding minimal set is a again a non-trivial extension of a Cantor set, possible connected components of which include flowers and dendrites.*

The proof of the above example needs two further lemmas. Let $\mathcal{Q} \subset \mathbb{T}^1$ be a Cantor set and denote $\{I_k\}_{k\in\mathbb{Z}}$ the collection of the connected components of $\mathbb{T}^1 \setminus \mathcal{Q}$. In what follows, we denote $|I|$ the length (relative to the Haar measure on the circle) of an interval $I \subset \mathbb{T}^1$ and denote $\widetilde{d}_1$ the Euclidean metric on $\mathbb{R}$.

Lemma 2.23. *There exist Cantor sets $\mathcal{Q} \subset \mathbb{T}^1$ with the following property: there exists a point $x_0 \in \mathcal{Q}_{\mathrm{irr}}$, an interval $J \subset \mathbb{T}^1$, with $\partial J \subset \mathcal{Q}_{\mathrm{irr}}$ and x_0 as midpoint of J, and a constant $C > 0$, such that for every interval $I_k \subset J \setminus \mathcal{Q}$, $k \geq 1$, it holds that $|I_k| \leq C(d(x_0, x_k))^2$, where x_k is the midpoint of the interval I_k.*

Proof. First, let $[-1, 1] \subset \mathbb{R}$ with midpoint $0 \in [-1, 1]$. Inductively delete intervals from $[-1, 1]$: at each step $t \geq 1$, choose a point $x_t \in (-1, 1) \setminus \bigcup_{s=0}^{t-1} I'_s$, with $x_t \neq 0$, and delete an interval $I'_t \subset (-1, 1) \setminus \bigcup_{s=0}^{t-1} I'_s$, centered at x_t and not overlapping 0, of length at most $(\widetilde{d}_1(0, x_t))^2$. Repeating this ad infinitum produces a Cantor set $\mathcal{Q}' \subset [-1, 1]$. Given a Cantor set $\mathcal{Q} \subset \mathbb{T}^1$, take a small (closed) interval $J \subset \mathbb{T}^1$ for which $\partial J \subset \mathcal{Q}_{\mathrm{irr}}$. Replacing $J \cap \mathcal{Q} \subset \mathbb{T}^1$ with a rescaled copy of $\mathcal{Q}'$ into J yields the desired Cantor set in $\mathbb{T}^1$, with $C = |J|/2$. $\square$

Lemma 2.24. *Let $f \in \mathrm{Homeo}_{\#}(\mathbb{T}^2)$ be the product of two Denjoy counterexamples $\varphi, \psi \in \mathrm{Homeo}(\mathbb{T}^1)$, semi-conjugate to an irrational translation τ through π. Let $\mathcal{Q}_1, \mathcal{Q}_2 \subset \mathbb{T}^1$ be two Cantor minimal sets of φ and ψ respectively, with $x_0 \in \mathcal{Q}_{1,\mathrm{irr}}$ and $y_0 \in \mathcal{Q}_{2,\mathrm{irr}}$ points and corresponding intervals J_1 and J_2 satisfying the conditions of Lemma 2.23. Set $z_0 := (x_0, y_0) \in \mathcal{R}_\pi$. For every $0 < r \leq 1$ and $\theta_1, \theta_2 \in [0, 2\pi)$, with $0 < |\theta_1 - \theta_2| < \pi$, we have that $\mathcal{W}(r, \theta_1, \theta_2) \cap (\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)) \neq \emptyset$.*

Proof. First, we observe that $\mathcal{R}_\pi = \mathcal{Q}_{1,\mathrm{irr}} \times \mathcal{Q}_{2,\mathrm{irr}}$, so the minimal set $\mathcal{M}$ of f is $\mathcal{M} = \mathrm{Cl}(\mathcal{Q}_{1,\mathrm{irr}} \times \mathcal{Q}_{2,\mathrm{irr}}) = \mathcal{Q}_1 \times \mathcal{Q}_2$, the product of the Cantor sets of the factors φ and ψ. Let $B_{\delta_0} \subset \mathbb{T}^2$ be the closed embedded disk centered at z_0, where δ_0 is small enough so that it is contained in the rectangle of width $|J_1|$ and height $|J_2|$ centered at z_0.

Relative to the disk B_{δ_0}, let $\mathcal{W}(r, \theta_1, \theta_2)$ be any wedge, denoted $\mathcal{W}$ for brevity from now on. Let $0 < \nu < \pi$, where $\nu = |\theta_1 - \theta_2|$ is the angle between the two rays $\rho_{\theta_1}, \rho_{\theta_2}$ that bound the wedge, and define $\bar{\rho} := \rho_{(\theta_1+\theta_2)/2}$ the bisector of the two rays. Further, let $\nu' \in [0, 2\pi)$ be the angle between $\bar{\rho}$ and the positive horizontal line through z_0. As the vertical and horizontal lines through z_0 contain points of $\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)$ arbitrarily close to z_0, by symmetry, we may assume without loss of generality that $0 < \nu' < \pi/2$. Given a point $w = (x, y) \in \bar{\rho} \cap \mathcal{W}$, let $\ell_h(w), \ell_v(w)$ be the horizontal and vertical straight line through w respectively and define the intercepts $\ell'_h(w) = \ell_h(w) \cap \mathcal{W}$ and $\ell'_v(w) = \ell_v(w) \cap \mathcal{W}$, which for w sufficiently close to z_0 only pass through ρ_{θ_1} and ρ_{θ_2} (and not through the circular arc that cuts off the wedge).

There exist constants $K_h, K_v > 0$, depending only on θ_1 and θ_2, such that $\ell'_h(w) = K_h d(y, y_0)$ and $\ell'_v(w) = K_v d(x, x_0)$. Given any $0 < r \leq 1$, choose $w \in \bar{\rho}$ such that $d(w, z_0) < \delta_0 r$. As the lengths $\ell'_h(w)$, $\ell'_v(w)$ behave linearly, and the lengths of the intervals $|I_{1,k}| \leq C_1(d(x_0, x_k))^2$ and $|I_{2,t}| \leq C_2(d(y_0, y_t))^2$ behave quadratically, with $C_1, C_2 > 0$ uniform constants, if we choose w sufficiently close to z_0, then $w \in R := I_{1,k} \times I_{2,t}$, for some $k, t \in \mathbb{Z}$, with $\mathrm{Cl}(R)$ properly contained in $\mathcal{W}$. The cornerpoints of the rectangle R are limit points of $\mathcal{R}_\pi$, and thus also of $\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)$, as a neighbourhood of any point $z \in \mathcal{R}_\pi$ contains uncountably many points and only a countable orbit is deleted. Therefore, as $\mathrm{Cl}(R) \subset \mathcal{W}$, we have that $\mathcal{W} \cap (\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)) \neq \emptyset$, as required. $\square$

Proof of Example 4. We start with the homeomorphism $f \in \mathrm{Homeo}_\#(\mathbb{T}^2)$, where f is the product of two Denjoy-counterexamples $\varphi, \psi \in \mathrm{Homeo}(\mathbb{T}^1)$, with a Cantor minimal set $\mathcal{M}_1 = \mathcal{Q}_1$ and $\mathcal{M}_2 = \mathcal{Q}_2$ respectively, semi-conjugate to an irrational translation τ through π. As every Cantor set in $\mathbb{T}^1$ can be realized as a minimal set of a Denjoy counterexample, we can choose φ and ψ such that the Cantor sets $\mathcal{Q}_i$, with $i = 1, 2$, with $x_0 \in \mathcal{Q}_{1,\mathrm{irr}}$ and $y_0 \in \mathcal{Q}_{2,\mathrm{irr}}$ points and corresponding intervals J_1 and J_2 satisfy the conditions of Lemma 2.23. Let $z_0 = (x_0, y_0) \in \mathcal{R}_\pi$ and let $B_{\delta_0} \subset \mathbb{T}^2$ be the closed embedded Euclidean disk with radius $\delta_0 > 0$ small enough so that B_{δ_0} is contained in the rectangle of width $|J_1|$ and height $|J_2|$ centered at z_0. Through the procedure in section 2.4.2, we blow up the orbit $\mathcal{O}_f(z_0)$ to a collection of disks to obtain a homeomorphism $f' \in \mathrm{Homeo}_*(\mathbb{T}^2)$ and a continuous $\phi \colon \mathbb{T}^2 \to \mathbb{T}^2$, such that $\phi \circ f' = f \circ \phi$ and $\phi^{-1}(f^k(z_0)) = \mathrm{Cl}(\Sigma_k)$ is a closed topological disk, and $\mathrm{Cl}(\Sigma_k) = \Sigma_k \cup \gamma_k$ with γ_k a simple closed curve, for every $k \in \mathbb{Z}$.

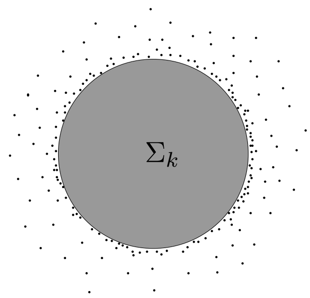

Figure 2.7: A non-trivial extension of a Cantor set $\mathcal{M}'$; Cantor dust accumulating on the boundary $\gamma_k \subset \mathcal{M}'$ of each disk Σ_k and, conversely, every point of the Cantor dust is a limit point of increasingly small disks Σ_k.

If we denote Σ the doubly essential component of the complement of the minimal set $\mathcal{M}$ of f, then $\Sigma' := \phi^{-1}(\Sigma) \subset \mathbb{T}^2$ is open (as ϕ is continuous) and doubly essential, thus $\mathcal{M}'$ is of type III. The other connected components of the complement of $\mathcal{M}'$ are, by construction, the disks Σ_k for $k \in \mathbb{Z}$.

Now, $\phi^{-1}(\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)) \subset \mathcal{M}'$ are all singletons, which by Lemma 2.21 (2) combined with Lemma 2.24, accumulate on the boundaries of the disks Σ_k to form the non-trivial connected components γ_k, see Figure 2.7. Thus, as $\mathcal{M}' = \mathrm{Cl}(\phi^{-1}(\mathcal{R}_\pi \setminus \mathcal{O}_f(z_0)))$ by Lemma 2.21 (1), $\gamma_k \subset \mathcal{M}$ for every $k \in \mathbb{Z}$, which are the desired non-degenerate components of $\mathcal{M}'$. $\square$

Remark 2. *In the proof of example 4, we explicitly constructed a semi-conjugacy between the extension of the Cantor set $\mathcal{M}'$ and the original Cantor set $\mathcal{M}$. Theorem 2.A. in essence says that all extensions of Cantor sets are of this form.*

2.5 Open problems

Let us finish by addressing some open problems that arise from the results obtained in this chapter.

Open problem 1 (Uniqueness of type III minimal sets)**.** *Let $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ with a minimal set $\mathcal{M}$ of type III. Is the minimal set $\mathcal{M}$ unique?*

Further, it would be interesting to get a completer description of the possible topology of extensions of Cantor sets.

Open problem 2 (Topology of type III minimal sets)**.** *Let $f \in \mathrm{Homeo}_*(\mathbb{T}^2)$ with a minimal set $\mathcal{M}$ of type III.*

(i) *Exactly what continua can arise as a connected component of $\mathcal{M}$?*

(ii) *Do there exist extensions of Cantor sets with uncountably many non-degenerate connected components? Can all components be non-degenerate?*

For example, a classical result by R. Moore [35] implies that not all components of $\mathcal{M}$ can be triodic continua, as the number of connected components of $\mathcal{M}$ is uncountable.[2] However, it is not entirely clear whether uncountably many components could be for example an arc.

[2] A planar continuum $\mathcal{C}$ is said to be *triodic* if there exists a connected closed set $\mathcal{C}_0 \subset \mathcal{C}$ such that $\mathcal{C} \setminus \mathcal{C}_0$ has at least three connected components.

Chapter 3

Topological Entropy and Diffeomorphisms with Wandering Domains

Let M be a closed surface and f a diffeomorphism of M. A diffeomorphism is said to permute a dense collection of domains, if the union of the domains are dense and the iterates of any one domain are mutually disjoint. In this chapter, we study the interplay between topological entropy of f, the smoothness of f and the geometry of the domains that are permuted by it. We show that if $f \in \mathrm{Diff}^{1+\alpha}(M)$, with $\alpha > 0$, and permutes a dense collection of domains with bounded geometry, then f has zero topological entropy.

3.1 Definitions and statement of results

A result of A. Norton and D. Sullivan [36] states that a diffeomorphism $f \in \mathrm{Diff}_0^3(\mathbb{T}^2)$ having *Denjoy-type* can not have a wandering disk whose iterates have the same *generic shape*. By diffeomorphisms of Denjoy-type are meant diffeomorphisms of the two-torus, isotopic to the identity, that are obtained as an extension of an irrational translation of the torus, for which the semi-conjugacy has countably many non-trivial fibers. If these fibers have non-empty interior, then the corresponding diffeomorphism has a wandering disk. Further, by generic shape is meant that the only elements of $\mathrm{SL}(2,\mathbb{Z})$ preserving the shape are elements of $\mathrm{SO}(2,\mathbb{Z})$, such as round disks and squares. In a similar spirit, C. Bonatti, J.M. Gambaudo, J.M. Lion and C. Tresser in [8] show that certain infinitely renormalizable diffeomorphisms of the two-disk that are sufficiently smooth, can not have wandering domains if these domains have a certain boundedness of geometry.

In this chapter, we study an analogous problem, namely the interplay between the geometry of iterates of domains under a diffeomorphism and its topological entropy. To state the precise result, we first need some definitions. Let (M,g) be a closed surface, that is, a smooth, closed, oriented Riemannian two-manifold, equipped with the canonical metric g induced from the standard conformal metric of the universal cover $\mathbb{P}^1, \mathbb{C}$ or $\mathbb{D}^2$. We denote by $d(\cdot,\cdot)$ the distance function relative to the metric g. Let $\mathrm{Diff}^r(M)$ be the group of diffeomorphisms of M, where for $r \geq 0$ finite, f is said to be of class C^r if f is continuously differentiable up to order $[r]$ and the $[r]$-th derivative is (r)-Hölder, with $[r]$ and (r) the integral and fractional part of r respectively. We identity $\mathrm{Diff}^0(M)$ with $\mathrm{Homeo}(M)$, the group of homeomorphisms of M.

Given $f \in \mathrm{Homeo}(M)$, for each $n \geq 1$, define the metric d_n on M given by $d_n(x,y) = \max_{1\leq i\leq n}\{d(f^i(x), f^i(y))\}$. Given $\epsilon > 0$, a subset $U \subset M$ is said to be (n,ϵ) separated if $d_n(x,y) \geq \epsilon$ for every $x, y \in U$ with $x \neq y$. Let $N(n,\epsilon)$ be the maximum cardinality of an (n,ϵ) separated set. The topological entropy is defined as

$$h_{\mathrm{top}}(f) = \lim_{\epsilon\to 0}\left(\lim_{n\to\infty}\sup\frac{1}{n}\log N(n,\epsilon)\right).$$

Next, we make precise the notion of a homeomorphism of a surface permuting a dense collection of domains.

Definition 3.1. Let $S \subset M$ be compact and $\mathcal{D} := \{D_k\}_{k\in\mathbb{Z}}$ the collection of connected components of the complement of S, with the property that $\mathrm{Int}(\mathrm{Cl}(D_k)) = D_k$, where $\mathrm{Cl}(D)$ is the closure of D in M. We say $f \in \mathrm{Homeo}(M)$ *permutes a dense collection of domains* if

(1) $f(S) = S$ and $\mathrm{Cl}(D_k) \cap \mathrm{Cl}(D_{k'}) = \emptyset$ if $k \neq k'$,

(2) for every $k \in \mathbb{Z}$, $f^n(D_k) \cap D_k = \emptyset$ for all $n \neq 0$, and

(3) $\bigcup_{k\in\mathbb{Z}} D_k$ is dense in M.

Note that we do not assume a domain to be recurrent, nor do we assume the orbit of a single domain to be dense. A *wandering domain* is a domain with mutually disjoint iterates under f such that the orbit of the domain is recurrent. Thus a diffeomorphism with a wandering domain with dense orbit is a special case of definition 3.1. Denote $\exp_p\colon T_pM \to M$ the exponential mapping at $p \in M$. The *injectivity radius* at a point $p \in M$ is defined as the largest radius for which $\exp_p$ is a diffeomorphism. The injectivity radius $\iota(M)$ of M is the infimum of the injectivity radii over all points $p \in M$. As M is compact, $\iota(M)$ is positive.

Definition 3.2 (Bounded geometry)**.** A collection of domains $\{D_k\}_{n\in\mathbb{Z}}$ on a surface M is said to have *bounded geometry* if the following holds: $\mathrm{Cl}(D_k)$ is contractible in M and there exists a constant $\beta \geq 1$ such that for every domain D_k in the collection, there exist $p_k \in D_k$ and $0 < r_k \leq R_k$ such that

$$B(p_k, r_k) \subseteq D_k \subseteq B(p_k, R_k), \text{ with } R_k/r_k \leq \beta, \tag{3.1}$$

where $B(p, r) \subset M$ is the ball centered at $p \in M$ with radius $r > 0$. If no such β exists, then the collection is said to have *unbounded geometry.*

By $\mathrm{Cl}(D_k)$ being contractible in M we mean that $\mathrm{Cl}(D_k)$ is contained in an embedded topological disk in M. Our definition of bounded geometry is equivalent to the notion of bounded geometry in the theory of Kleinian groups and complex dynamics. It is not difficult, given a surface of any genus, to construct homeomorphisms of that surface with positive entropy that permute a dense collection of domains. We show that producing examples that have a certain amount of smoothness is possible only to a limited degree.

Theorem 3.A (Topological entropy versus bounded geometry)**.** *Let M be a closed surface and $f \in \mathrm{Diff}^{1+\alpha}(M)$, with $\alpha > 0$. If f permutes a dense collection of domains with bounded geometry, then f has zero topological entropy.*

The outline of the proof of Theorem 3.A is as follows. First we show that the bounded geometry of the permuted domains, combined with their density in the surface, give bounds on the dilatation of f on the complement of the union of the permuted domains. The differentiability assumptions on f allow us to estimate the rate of growth of the dilatation on the whole surface M. Using a result by Przytycki [41], we show this rate of growth is slow enough so as to ensure the topological entropy of f is zero.

Remark 3. *Oleg Kozlovski and Jean-Marc Gambaudo pointed out that Theorem 3.A. can also be derived from A. Katok's results in [24] about the existence of saddle fixed points for $C^{1+\alpha}$ diffeomorphisms with positive entropy. However, our proof is completely independent from the techniques in [24]; moreover, it is likely that our result can be generalized to higher dimensions, whereas the techniques in [24] do not appear to allow for a straightforward generalization to higher dimensions.*

3.2 Entropy and diffeomorphisms with wandering domains

First, we study the relation between geometry of domains and the complex dilatation of a diffeomorphism.

3.2.1 Geometry of domains and complex dilatation

We denote λ the measure associated to g and $d\lambda$ the Riemannian volume form. By compactness of M, there exists a constant $\kappa > 0$ such that

$$\lambda(B(p,r)) = \int_{B(p,r)} d\lambda \geq \kappa r^2. \tag{3.2}$$

A sequence of positive real numbers x_k is called a *null-sequence*, if for every given $\epsilon > 0$ there exist only finitely many elements of the sequence for which $x_k \geq \epsilon$. Henceforth, we denote $\ell_k := \operatorname{diam}(D_k)$, the diameter of D_k measured in g, with $D_k \in \mathcal{D}$.

Lemma 3.1. *Let M be a closed surface and let $\{D_k\}_{k\in\mathbb{Z}}$ be a collection of mutually disjoint domains with bounded geometry. Then the sequence ℓ_k is a null-sequence.*

Proof. Suppose, to the contrary, that $\{D_k\}_{k\in\mathbb{Z}}$ is not a null-sequence. Then there exist an $\epsilon > 0$ and an infinite subsequence k_t such that $\operatorname{diam}(D_{k_t}) \geq \epsilon$. By the bounded geometry property, we have that $\operatorname{diam}(D_{k_t}) \leq 2R_{k_t} \leq 2\beta r_{k_t}$ and therefore $r_{k_t} \geq \epsilon/2\beta$. Therefore, by (3.2),

$$\lambda(D_{k_t}) \geq \kappa r_{k_t}^2 \geq \frac{\kappa\epsilon^2}{4\beta^2},$$

for every $t \in \mathbb{Z}$. But this yields that

$$\sum_{t\in\mathbb{Z}} \lambda(D_{k_t}) = \infty,$$

contradicting the fact that $\lambda(M) < \infty$ as M is compact. $\square$

Recall that S is the complement of the union of the permuted domains, i.e. $S = M \setminus \bigcup_{k\in\mathbb{Z}} D_k$.

Lemma 3.2. *Let M be a closed surface and let $f \in \operatorname{Homeo}(M)$ permute a dense collection $\mathcal{D}$ of domains with bounded geometry. For every $p \in S$, there exists a sequence of domains D_{k_t} with $\operatorname{diam}(D_{k_t}) \to 0$ for $t \to \infty$ such that $D_{k_t} \to p$.*

Proof. Fix $p \in S$ and let $U \subset M$ be an open (connected) neighbourhood of p. First assume that $p \in S \backslash \bigcup_{k \in \mathbb{Z}} \partial D_k$. This set in non-empty, as otherwise the surface M is a union of countably many mutually disjoint continua; but this contradicts Sierpiński's Theorem [45], which states that no countable union of disjoint continua is connected. We claim that U intersects infinitely many different elements of $\mathcal{D}$. Indeed, if U intersects only finitely many elements $D_{k_1}, ..., D_{k_m}$, then $\Omega := \bigcup_{i=1}^{m} \mathrm{Cl}(D_{k_i})$ is closed. This implies that $U \setminus \Omega$ is open and non-empty, as otherwise M would be a finite union of disjoint continua, which is impossible. However, as the union of the elements of $\mathcal{D}$ is dense, $U \backslash \Omega$ can not be open. Thus, there are infinitely many distinct elements $D_{k_1}, D_{k_2}, ...$ of $\mathcal{D}$ that intersect U. Taking a sequence of nested open connected neighbourhoods U_t containing p, we can find elements $D_{k_t} \subset U_t \setminus U_{t+1}$ for every $t \geq 1$. By Lemma 3.1, $\mathrm{diam}(D_{k_t})$ is a null-sequence and thus we obtain a sequence of domains D_{k_t} with $\mathrm{diam}(D_{k_t}) \to 0$ for $t \to \infty$ such that $D_{k_t} \to p$.

As $\mathrm{Int}(\mathrm{Cl}(D_k)) = D_k$, given $p \in \partial D_k$ and given any neighbourhood $U \ni p$, U has non-empty intersection with $M \setminus \mathrm{Cl}(D_k)$. By the same reasoning as above, p is again is a limit point of arbitrarily small domains in the collection $\mathcal{D}$. Thus we have proved the claim for all points $p \in S$ and this concludes the proof. $\square$

Next, we turn to the *complex dilatation* of a diffeomorphism $f \in \mathrm{Diff}(M)$ and its behaviour under compositions of diffeomorphisms, see for example [16] or [31]. We first consider the case where $f \in \mathrm{Diff}(\mathbb{C})$. The complex dilatation μ_f of f is defined by

$$\mu_f \colon \mathbb{C} \to \mathbb{D}^2, \ \mu_f(p) = \frac{f_{\bar{z}}}{f_z}(p), \tag{3.3}$$

and the corresponding differential

$$\mu_f(p) \frac{d\bar{z}}{dz}, \tag{3.4}$$

is the *Beltrami differential* of f. The *dilatation* of f is defined by

$$K_f(p) = \frac{1 + |\mu_f(p)|}{1 - |\mu_f(p)|}, \tag{3.5}$$

which equals

$$K_f(p) = \frac{\max_v |Df_p(v)|}{\min_v |Df_p(v)|}, \tag{3.6}$$

where v ranges over the unit circle in $T_p\mathbb{C}$ and the norm $|\cdot|$ is induced by the standard (conformal) Euclidean metric g on $\mathbb{C}$. Denote $[\cdot,\cdot]$ be the hyperbolic distance in $\mathbb{D}^2$, i.e. the distance induced by the Poincaré metric on $\mathbb{D}^2$. When one composes two diffeomorphisms $f, g\colon \mathbb{C}\to\mathbb{C}$, then

$$\mu_{g\circ f}(p) = \frac{\mu_f(p) + \theta_f(p)\mu_g(f(p))}{1 + \overline{\mu_f(p)}\theta_f(p)\mu_g(f(p))}, \tag{3.7}$$

where $\theta_f(p) = \frac{\overline{f_z}}{f_z}(p)$. It follows that

$$\mu_{f^{n+1}}(p) = \frac{\mu_f(p) + \theta_f(p)\mu_{f^n}(f(p))}{1 + \overline{\mu_f(p)}\theta_f(p)\mu_{f^n}(f(p))}. \tag{3.8}$$

We can rewrite (3.7) as

$$\mu_{g\circ f}(p) = T_{\mu_f(p)}(\theta_f(p)\mu_g(f(p))) \tag{3.9}$$

where

$$T_a(z) = \frac{a+z}{1+\bar{a}z} \in \text{Möb}(\mathbb{D}^2) \tag{3.10}$$

is an isometry relative to the Poincaré metric, for a given $a \in \mathbb{D}^2$. Further, the following relation holds

$$\log(K_{g\circ f^{-1}}(f(p))) = [\mu_g(p), \mu_f(p)]\,. \tag{3.11}$$

To define the complex (and maximal) dilatation of a diffeomorphism of a surface M, we first lift $f\colon M\to M$ to the universal cover $\widetilde{f}\colon \widetilde{M}\to\widetilde{M}$ and denote $\pi\colon \widetilde{M}\to M$ be the corresponding canonical projection mapping, where $M = \widetilde{M}/\Gamma$, with Γ a Fuchsian group. We assume here that $\widetilde{M}$ is either $\mathbb{C}$ or $\mathbb{D}^2$, the trivial case of the sphere $\mathbb{P}^1$ is excluded here. As π is an analytic local diffeomorphism, $\widetilde{f}$ is a diffeomorphism. Further, as M is compact, f is K-quasiconformal on M for some

$K \geq 1$ and thus $\widetilde{f}$ is K-quasiconformal on $\widetilde{M}$. Since $\widetilde{f} \circ h \circ \widetilde{f}^{-1}$ is conformal for every $h \in \Gamma$, it follows from (3.7) that

$$\mu_{\widetilde{f}}(p) = \mu_{\widetilde{f}}(h(p)) \frac{\overline{h_z}}{h_z}(p). \tag{3.12}$$

In other words, $\mu_{\widetilde{f}}$ defines a Beltrami differential on $\widetilde{M}$ for the group Γ, or equivalently, it defines a Beltrami differential for f on the surface M. Furthermore, the same formulas (3.5) and (3.6), defined relative to the canonical (conformal) metric defined on M, hold for the dilatation K_f of f on M.

The following lemma shows that the bounded geometry assumption of the domains has a strong effect on the dilatation of iterates of f on S. We say f has *uniformly bounded dilatation* on $S \subset M$, if $K_{f^n}(p)$ is bounded by a constant independent of $n \in \mathbb{Z}$ and $p \in S$.

Lemma 3.3 (Bounded dilatation)**.** *Let M be a closed surface and let $f \in \mathrm{Diff}^1(M)$ permute a dense collection of domains $\mathcal{D}$. If the collection $\mathcal{D}$ has β-bounded geometry, then f has uniformly bounded dilatation on S.*

Proof. Suppose the collection of domains $\mathcal{D} = \{D_k\}_{k \in \mathbb{Z}}$ has β-bounded geometry for some $\beta \geq 1$. Fix $N \in \mathbb{Z}$ and $p \in S$ and take a small open neigbhourhood $U \subset M$ containing p. By Lemma 3.2, there exists a subsequence of domains D_{k_t}, where $|k_t| \to \infty$ and $\mathrm{diam}(D_{k_t}) \to 0$ for $t \to \infty$ and such that $D_{k_t} \to p$. Denote $q = f^N(p) \in S$. We may as well assume that for all $t \geq 1$ the domains D_{k_t} are contained in U. Define $D'_{k_t} := f^N(D_{k_t})$. If we denote $U' = f^N(U)$, then the sequence D'_{k_t} converges to q and $D'_{k_t} \subset U'$. By the bounded geometry assumption, for every $t \geq 1$, there exists $p_t \in D_{k_t}$ and $0 < r_t \leq R_t$ such that

$$B(p_t, r_t) \subseteq D_{k_t} \subseteq B(p_t, R_t)$$

with $R_t / r_t \leq \beta$. As $f \in \mathrm{Diff}^1(M)$, the local behaviour of f^N around q converges to the behaviour of the linear map Df_p^N. In particular, if we take $p_t \in D_{k_t}$, then $p_t \to p$

and thus $q_t := f^N(p_t) \to q$, and in order for all D'_{k_t} to have β-bounded geometry, it is required that

$$K_{f^N}(p) \leq \frac{R_t \beta}{r_t},$$

for t sufficiently large. Indeed, this is easily seen to hold if the map acts locally by a linear map and is thus sufficient as $f \in \mathrm{Diff}^1(M)$ and the increasingly smaller domains approach p. As $R_t/r_t \leq \beta$, we must therefore have $K_{f^N}(p) \leq \beta^2$. As this argument holds for every (fixed) $N \in \mathbb{Z}$ and every $p \in S$, we find β^2 the uniform bound on the dilatation on S. □

Our smoothness assumptions on f allow us to give bounds on the (complex) dilatation of iterates of f on M in terms of the diameters of the permuted domains.

Lemma 3.4 (Sum of diameters)**.** *Let M be a closed surface and let $f \in \mathrm{Diff}^{1+\alpha}(M)$, with $\alpha > 0$, which permutes a collection of domains $\mathcal{D} = \{D_k\}_{k\in\mathbb{Z}}$ with β-bounded geometry. Then there exists a constant $C = C(\beta) > 0$ such that, if $p \in D_t$ (for some $t \in \mathbb{Z}$) and $q \in \partial D_t$, then*

$$\left[\mu_{f^{n+1}}(p), \mu_{f^{n+1}}(q)\right] \leq C \cdot \sum_{s=t}^{t+n} \ell_s^\alpha, \tag{3.13}$$

where the domains are labeled such that $f^s(D_t) = D_{t+s}$.

To prove Lemma 3.4, we use the following.

Lemma 3.5. *Let $f \in \mathrm{Diff}^1(M)$ and $p_0, q_0 \in M$. Then*

$$\left[\mu_{f^{n+1}}(p_0), \mu_{f^{n+1}}(q_0)\right] \leq \sum_{s=0}^{n} \left[T_{\mu_f(p_s)}(\theta_f(p_s)\mu_{f^{n-s}}(q_{s+1})), T_{\mu_f(q_s)}(\theta_f(q_s)\mu_{f^{n-s}}(q_{s+1}))\right], \tag{3.14}$$

where $p_s = f^s(p_0)$ and $q_s = f^s(q_0)$.

Proof. Using (3.9), we write

$$\left[\mu_{f^{n+1}}(p_0), \mu_{f^{n+1}}(q_0)\right] = \left[T_{\mu_f(p_0)}(\theta_f(p_0)\mu_{f^n}(p_1)), T_{\mu_f(q_0)}(\theta_f(q_0)\mu_{f^n}(q_1))\right].$$

By the triangle inequality, we thus have the following inequality

$$\begin{aligned}\left[\mu_{f^{n+1}}(p_0), \mu_{f^{n+1}}(q_0)\right] &\leq \left[T_{\mu_f(p_0)}(\theta_f(p_0)\mu_{f^n}(p_1)), T_{\mu_f(p_0)}(\theta_f(p_0)\mu_{f^n}(q_1))\right] \\ &+ \left[T_{\mu_f(p_0)}(\theta_f(p_0)\mu_{f^n}(q_1)), T_{\mu_f(q_0)}(\theta_f(q_0)\mu_{f^n}(q_1))\right].\end{aligned}$$

As both T_a (as defined by (3.10)) and rotations are isometries in the Poincaré disk, we have that

$$\left[T_{\mu_f(p_0)}(\theta_f(p_0)\mu_{f^n}(p_1)), T_{\mu_f(p_0)}(\theta_f(p_0)\mu_{f^n}(q_1))\right] = \left[\mu_{f^n}(p_1), \mu_{f^n}(q_1)\right].$$

Inequality (3.14) now follows by induction. □

As $\partial D_t \subset S$, by Lemma 3.3, $\mu_{f^{n-s}}(q_{s+1}) \in B_\delta$, with $B_\delta \subset \mathbb{D}^2$ the compact disk centered at $0 \in \mathbb{D}^2$ with radius

$$\delta = \frac{\beta^2 - 1}{\beta^2 + 1}. \tag{3.15}$$

Further, define

$$\delta' = \sup_{p \in M} |\mu_f(p)| < 1, \tag{3.16}$$

and let $B_{\delta'} \subset \mathbb{D}^2$ be the compact disk centered at $0 \in \mathbb{D}^2$ and radius δ'.

Lemma 3.6. *There exists a constant $C_1(\delta, \delta')$ such that*

$$[T_a(z), T_b(z)] \leq C_1 [a, b], \tag{3.17}$$

for given $a, b \in B_{\delta'}$ and $z \in B_\delta$.

Proof. First we observe that there exists a constant $0 < \delta'' < 1$ (depending only on δ and δ'), such that $[T_a(z), 0] \leq \delta''$, for every $a \in B_{\delta'}$ and every $z \in B_\delta$, as the disks $B_\delta, B_{\delta'} \subset \mathbb{D}^2$ are compact. Define $\bar{\delta} = \max\{\delta, \delta', \delta''\}$ and $B_{\bar{\delta}} \subset \mathbb{D}^2$ the compact disk with center $0 \in \mathbb{D}^2$ and radius $\bar{\delta}$.

As the Euclidean metric and the hyperbolic metric are equivalent on the compact disk $B_{\bar{\delta}}$, it suffices to show that there exists a constant $C_1'(\bar{\delta})$ such that

$$|T_a(z) - T_b(z)| \leq C_1' |a - b|, \tag{3.18}$$

where $|z-w|$ denotes the Euclidean distance between two points $z, w \in \mathbb{D}^2$. Indeed, if this is shown then (3.17) follows for a constant C_1 which differs from C_1' by a uniform constant depending only on $\bar{\delta}$. To prove (3.18), we compute that

$$|T_a(z) - T_b(z)| = \left| \frac{(a-b) + (a\bar{b} - \bar{a}b)z + (\bar{b} - \bar{a})z^2}{(1+\bar{a}z)(1+\bar{b}z)} \right|. \tag{3.19}$$

As $a, b \in B_{\delta'}$ and $z \in B_\delta$, there exists a constant $Q_1(\delta, \delta') > 0$ so that

$$|(1+\bar{a}z)(1+\bar{b}z)| \geq Q_1^{-1}.$$

Therefore, it holds that

$$|T_a(z) - T_b(z)| \leq Q_1 \left(|a-b| + \delta|a\bar{b} - \bar{a}b| + \delta^2|a-b| \right). \tag{3.20}$$

In order to prove (3.18), we show there exists a constant $Q_2(\delta') > 0$ such that

$$|a\bar{b} - \bar{a}b| \leq Q_2|a-b|. \tag{3.21}$$

To this end, write $a = re^{i\phi}$ and $b = r'e^{i\phi'}$ and $x = a\bar{b}$, so that $x = rr'e^{i\nu}$ with $\nu = \phi - \phi'$. We may assume that $\nu \in [0, \pi)$. It follows that $a\bar{b} - \bar{a}b = x - \bar{x} = 2irr'\sin(\nu)$. Therefore,

$$|a\bar{b} - \bar{a}b| = |x - \bar{x}| = 2rr'|\sin(\nu)| \leq 2\delta' r|\sin(\nu)|, \tag{3.22}$$

as $r' \leq \delta'$. As the angle between the vectors $a, b \in B_{\delta'}$ is ν, it is easily seen that $r|\sin(\nu)| \leq |a-b|$. Combining this estimate with (3.22), we obtain that

$$|a\bar{b} - \bar{a}b| \leq 2\delta' r|\sin(\nu)| \leq 2\delta'|a-b|. \tag{3.23}$$

Setting $Q_2 = 2\delta'$ yields (3.21). If we now combine (3.23) in turn with (3.20), we obtain a uniform constant

$$C_1'(\delta, \delta') = Q_1(1 + \delta Q_2 + \delta^2)$$

for which (3.18) holds, as required. $\square$

Proof of Lemma 3.4. As $f \in \mathrm{Diff}^{1+\alpha}(M)$, we have that $\mu_f(p) \in C^\alpha(M, \mathbb{D}^2)$ and $\theta_f \in C^\alpha(M, \mathbb{C})$, are uniformly Hölder continuous by compactness of M. By the triangle inequality, we can estimate the summand in the right-hand side of (3.14) of Lemma 3.5 as

$$\Big[T_{\mu_f(p_s)}(\theta_f(p_s)\mu_{f^{n-s}}(q_{s+1})), T_{\mu_f(q_s)}(\theta_f(q_s)\mu_{f^{n-s}}(q_{s+1}))\Big] \leq \tag{3.24}$$
$$\Big[T_{\mu_f(p_s)}(\theta_f(p_s)\mu_{f^{n-s}}(q_{s+1})), T_{\mu_f(q_s)}(\theta_f(p_s)\mu_{f^{n-s}}(q_{s+1}))\Big] + \tag{3.25}$$
$$\Big[T_{\mu_f(q_s)}(\theta_f(p_s)\mu_{f^{n-s}}(q_{s+1})), T_{\mu_f(q_s)}(\theta_f(q_s)\mu_{f^{n-s}}(q_{s+1}))\Big]. \tag{3.26}$$

To estimate (3.25), define

$$z_s := \theta_f(p_s)\mu_{f^{n-s}}(q_{s+1}) \in B_\delta \text{ and } a_s = \mu_f(p_s), b_s = \mu_f(q_s) \in B_{\delta'} \subset \mathbb{D}^2.$$

Then (3.25) reads

$$\Big[T_{\mu_f(p_s)}(\theta_f(p_s)\mu_{f^{n-s}}(q_{s+1})), T_{\mu_f(q_s)}(\theta_f(p_s)\mu_{f^{n-s}}(q_{s+1}))\Big] = [T_{a_s}(z_s), T_{b_s}(z_s)]. \tag{3.27}$$

By Lemma 3.6, there exists a constant $C_1 > 0$ such that

$$[T_{a_s}(z_s), T_{b_s}(z_s)] \leq C_1[a_s, b_s]. \tag{3.28}$$

By Hölder continuity of μ_f, there exists a constant $\widehat{C}_1$ such that

$$[a_s, b_s] \leq \widehat{C}_1(d(p_s, q_s))^\alpha. \tag{3.29}$$

Therefore, combining equations (3.27), (3.28) and (3.29), we obtain that

$$\Big[T_{\mu_f(p_s)}(\theta_f(p_s)\mu_{f^{n-s}}(q_{s+1})), T_{\mu_f(q_s)}(\theta_f(p_s)\mu_{f^{n-s}}(q_{s+1}))\Big] \leq \widetilde{C}_1\ell_{t+s}^\alpha, \tag{3.30}$$

as $d(p_s, q_s) \leq \ell_{t+s}$, with $\widetilde{C}_1 := C_1\widehat{C}_1$.

To estimate (3.26), we note that the hyperbolic distance and the Euclidean distance are equivalent on the compact disk B_δ. Therefore, as the (Euclidean) distance between a point $z \in B_\delta$ and $e^{i\phi}z$ is bounded from above by a constant (depending only on δ) multiplied by the angle $|\phi|$, by Hölder continuity of θ_f there exists a constant $\widetilde{C}_2(\delta)$, such that

$$\left[\theta_f(p)z, \theta_f(p')z\right] \le \widetilde{C}_2(d(p,p'))^\alpha,$$

for all $z \in B_\delta$ and $p, p' \in M$, using the local equivalence of the hyperbolic and Euclidean metric. Hence, (3.26) reduces to

$$\left[\theta_f(p_s)\mu_{f^{n-s}}(q_{s+1}), \theta_f(q_s)\mu_{f^{n-s}}(q_{s+1})\right] \le \widetilde{C}_2 d(p_s,q_s))^\alpha \le \widetilde{C}_2 \ell_{t+s}^\alpha, \tag{3.31}$$

as $d(p_s, q_s) \le \ell_{t+s}$. Therefore, if we set $C := \widetilde{C}_1 + \widetilde{C}_2$, then (3.13) follows. ☐

3.2.2 Upper bounds on the entropy of a surface diffeomorphism

Next, we relate the topological entropy of a diffeomorphism to its dilatation.

Lemma 3.7 (Entropy and dilatation)**.** *Let M be a closed surface and let $f \in \mathrm{Diff}^{1+\alpha}(M)$ with $\alpha > 0$. Then*

$$h_{\mathrm{top}}(f) \le \lim_{n\to\infty} \sup \frac{1}{2n} \log \int_M K_{f^n}(p) d\lambda(p), \tag{3.32}$$

with K_f the dilatation of f.

To prove this we use a result of F. Przytycki [41]. We need the following notation. Let $L : \mathbb{R}^m \to \mathbb{R}^m$ be a linear map and $L^{k\wedge} : \mathbb{R}^{m\wedge k} \to \mathbb{R}^{m\wedge k}$ the induced map on the k-th exterior algebra of $\mathbb{R}^m$. $L^\wedge$ denotes the induced map on the full exterior algebra. The norm $\|L^{k\wedge}\|$ of L^k has the following geometrical meaning. Let $\mathrm{Vol}_k(v_1, ..., v_k)$ be the k-dimensional volume of a parallelepiped spanned by the vectors $v_1, ..., v_k$, where $v_i \in \mathbb{R}^m$ with $1 \le i \le k$. Then

$$\|L^{k\wedge}\| = \sup_{v_i \in \mathbb{R}^m} \frac{\mathrm{Vol}_k(L(v_1), ..., L(v_k))}{\mathrm{Vol}_k(v_1, ..., v_k)}, \tag{3.33}$$

$$\|L^\wedge\| = \max_{1\le k\le m} \|L^{k\wedge}\|. \tag{3.34}$$

Further, let

$$\|L\| = \sup_{|v|=1} |L(v)|, \tag{3.35}$$

the standard norm on operators, with $v \in \mathbb{R}^m$ and $|\cdot|$ induced by the corresponding inner product on $\mathbb{R}^m$. The following result is due to F. Przytycki [41] (see also [28]).

Theorem 3.8. *Given a smooth, closed Riemannian manifold M and $f \in \mathrm{Diff}^{1+\alpha}(M)$ with $\alpha > 0$. Then*

$$h_{\text{top}}(f) \leq \lim_{n\to\infty} \sup \frac{1}{n} \log \int_M \|(Df^n)^\wedge\| d\lambda(p). \tag{3.36}$$

where $h_{\text{top}}(f)$ is the topological entropy of f, λ is a Riemannian measure on M induced by a given Riemannian metric, $(Df^n)^\wedge$ is a mapping between exterior algebras of the tangent spaces T_pM and $T_{f^n(p)}M$, induced by the Df_p^n and $\|\cdot\|$ is the norm on operators, induced from the Riemannian metric.

Proof of Lemma 3.7. Fix $p \in M$ and let $Df_p^n : T_pM \to T_{f^n(p)}M$. Then

$$\|Df_p^n\|^2 = K_{f^n}(p) J_{f^n}(p).$$

Thus

$$\|(Df_p^n)^{1\wedge}\| = \sqrt{K_{f^n}(p)J_{f^n}(p)}, \text{ and } \|(Df_p^n)^{2\wedge}\| = J_{f^n}(p). \tag{3.37}$$

It follows that

$$\|(Df_p^n)^\wedge\| = \max\left\{\sqrt{K_{f^n}(p)J_{f^n}(p)}, J_{f^n}(p)\right\}. \tag{3.38}$$

As

$$\max\left\{\sqrt{K_{f^n}(p)J_{f^n}(p)}, J_{f^n}(p)\right\} \leq \sqrt{K_{f^n}(p)J_{f^n}(p)} + J_{f^n}(p),$$

we have that

$$\begin{aligned} \int_M \|(Df_p^n)^\wedge\| d\lambda(p) &\leq \int_M \left(\sqrt{K_{f^n}J_{f^n}} + J_{f^n}\right) d\lambda \\ &= \lambda(M) + \int_M \sqrt{K_{f^n}J_{f^n}} d\lambda \end{aligned}$$

as $\lambda(M) = \int_M J_{f^n} d\lambda$, for every $n \in \mathbb{Z}$. Either $\int_M \sqrt{K_{f^n}J_{f^n}} d\lambda$ is bounded as a sequence in n, in which case (3.32) holds trivially, or the sequence is unbounded in n, in which case it is readily verified that

$$\lim_{n\to\infty} \sup \frac{1}{n} \log \left(\lambda(M) + \int_M \sqrt{K_{f^n}J_{f^n}} d\lambda\right) = \lim_{n\to\infty} \sup \frac{1}{n} \log \int_M \sqrt{K_{f^n}J_{f^n}} d\lambda.$$

By the Cauchy-Schwartz inequality, we have that

$$\int_M \sqrt{K_{f^n} J_{f^n}} d\lambda \leq \sqrt{\lambda(M)} \cdot \sqrt{\int_M K_{f^n} d\lambda}.$$

and thus,

$$\log \int_M \sqrt{K_{f^n} J_{f^n}} d\lambda \leq \frac{1}{2} \log \lambda(M) + \frac{1}{2} \log \int_M K_{f^n} d\lambda.$$

It now follows that

$$\lim_{n\to\infty} \sup \frac{1}{n} \log \int_M \|(Df^n)^\wedge\| d\lambda \leq \lim_{n\to\infty} \sup \frac{1}{2n} \log \int_M K_{f^n} d\lambda.$$

and this proves (3.32). □

3.2.3 Proof of Theorem 3.A

Let us now complete the proof. Let $f \in \mathrm{Diff}_A^{1+\alpha}(M)$, with $\alpha > 0$, and suppose that f permutes a dense collection of domains $\{D_k\}_{k\in\mathbb{Z}}$ with bounded geometry. By Lemma 3.1, the sequence ℓ_k is a null-sequence. Therefore, ℓ_k^α is a null-sequence as well, for every $\alpha > 0$. Let $p \in D_t$ for some $t \in \mathbb{Z}$ and $q \in \partial D_t$ and label the domains such that $f^s(D_t) = D_{t+s}$. By (3.11),

$$\log K_{f^n}(f(p)) = [\mu_{f^{n+1}}(p), \mu_f(p)]$$

and thus, by the triangle inequality,

$$\log K_{f^n}(f(p)) \leq \left[\mu_{f^{n+1}}(p), \mu_{f^{n+1}}(q)\right] + \left[\mu_{f^{n+1}}(q), \mu_f(p)\right] \tag{3.39}$$

As the second term in the right hand side of (3.39) stays uniformly bounded, we have that

$$\log K_{f^n}(f(p)) \leq \left[\mu_{f^{n+1}}(p), \mu_{f^{n+1}}(q)\right] + C' \tag{3.40}$$

for some constant $C' > 0$, independent of $p \in M$ and $n \in \mathbb{Z}$. Define

$$\xi(n) = \max \sum_{i=0}^{n} \ell_{k_i}^\alpha$$

where the maximum is taken over all collections of $n+1$ distinct elements $\{D_{k_0}, ..., D_{k_n}\}$ of $\mathcal{D}$. As ℓ_k^α is a null-sequence, we have that

$$\lim_{n\to\infty} \sup \frac{\xi(n)}{n} = 0. \tag{3.41}$$

By Lemma 3.4, we have that

$$\left[\mu_{f^{n+1}}(p), \mu_{f^{n+1}}(q)\right] \leq C \cdot \sum_{s=t}^{t+n} \ell_s^\alpha,$$

for some constant $C > 0$. Combined with (3.40), we obtain the following uniform estimate

$$\log K_{f^n}(f(p)) \leq C\xi(n) + C', \tag{3.42}$$

for every $p \in M$ and $n \in \mathbb{Z}$. Therefore

$$\begin{aligned} \log \int_M K_{f^n} d\lambda &\leq \log \int_M \exp(C\xi(n) + C')d\lambda && (3.43) \\ &= \log\big((\exp(C\xi(n) + C')\lambda(M)\big) && (3.44) \\ &= C\xi(n) + C' + \log(\lambda(M)). && (3.45) \end{aligned}$$

Combining (3.45) in turn with Lemma 3.7 yields

$$h_{\text{top}}(f) \leq \lim_{n\to\infty} \sup \frac{1}{2n} \log \int_M K_{f^n} d\lambda \leq C \lim_{n\to\infty} \sup \frac{\xi(n)}{2n} = 0, \tag{3.46}$$

by (3.41). This proves Theorem 3.A.

3.3 Open problems

Our first problem is a natural question arising from the main result in this chapter.

Open problem 3 (Differentiable counterexamples)**.** *Let M be a closed surface. Do there exist diffeomorphisms $f \in \mathrm{Diff}^1(M)$ with positive entropy that permute a dense collection of domains with bounded geometry?*

Remark 4. *An anonymous referee pointed out that, if* $f \in \mathrm{Diff}^1(M)$*, then* μ_f *has a modulus of continuity* η*; that is*

$$[\mu_f(p), \mu_f(q)] \leq \eta(d(p,q)), \tag{3.47}$$

where $\eta(\ell) \to 0$ *if* $\ell \to 0$*. It follows that, if* $f \in \mathrm{Diff}^1(M)$*, by adapting the proof of Lemma 3.4,*

$$\frac{\left[\mu_{f^{n+1}}(p), \mu_{f^{n+1}}(q)\right]}{n} \tag{3.48}$$

is still a null-sequence. However, it is not known whether Przytycki's Theorem holds in the class of $\mathrm{Diff}^1(M)$ *that would guarantee zero entropy.*

As mentioned in section 3.1, it was shown in [36] that diffeomorphisms $f \in \mathrm{Diff}^3_0(\mathbb{T}^2)$ having Denjoy-type can not have a wandering disk for which the iterates have the same generic shape. Conversely, P. McSwiggen in [33] constructed examples of diffeomorphisms $f \in \mathrm{Diff}^{3-\epsilon}_0(\mathbb{T}^2)$, for every $\epsilon > 0$, that have Denjoy-type. It is not known whether the orbit of the wandering domain for these diffeomorphisms has bounded geometry. The first examples of C^2 diffeomorphisms of a surface with a wandering domain were constructed by J. Harrison, see [18, 19]. The following problem remains.

Open problem 4 (Smoothness versus wandering domains)**.** *Let* $f \in \mathrm{Diff}^r(\mathbb{T}^2)$ *have Denjoy-type and suppose that* f *has a wandering disk with bounded geometry. Is it possible that* $r = 3$*? What if the wandering disk has unbounded geometry?*

The Denjoy-type diffeomorphisms are typically modeled on a specific minimal set, namely a Sierpiński set. The topological classification of the more general class of non-resonant torus homeomorphisms of chapter 2 in a sense gives a topological foundation on which to layer more geometrical questions of the above kind; it would be interesting to understand the interplay between the topology of the wandering domains (i.e. bounded disk, unbounded disk or essential annulus) and geometrical behaviour, such as the complex dilatation, of the diffeomorphisms exhibiting these wandering domains.

Chapter 4

Quasiconformal Homogeneity of Genus Zero Surfaces

Let M be a Riemann surface. Then M is said to be K-quasiconformally homogeneous if for every two points $p, q \in M$, there exists a K-quasiconformal homeomorphism $f\colon M \to M$ such that $f(p) = q$. In other words, M is K-quasiconformally homogeneous if there exists a transitive family of K-quasi-conformal homeomorphisms of M to itself.

In this chapter, we study quasiconformal homogeneity of genus zero surfaces. Our main result establishes the existence of a universal constant $\mathcal{K} > 1$ such that if M is a K-quasiconformally homogeneous hyperbolic genus zero surface other than $\mathbb{D}^2$, then $K \geq \mathcal{K}$.

4.1 Definitions and statement of results

In 1976, the notion of quasiconformal homogeneity of Riemann surfaces was introduced by Gehring and Palka [17] in the setting of genus zero surfaces (and higher dimensional analogues). It was shown in [17] that the only genus zero surfaces admitting a transitive conformal family are exactly those which are not hyperbolic, i.e. the surfaces conformally equivalent to either $\mathbb{P}^1$, $\mathbb{C}$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ or $\mathbb{D}^2$. It was also shown, by means of examples, that there exist hyperbolic genus zero surfaces, homeomorphic to $\mathbb{P}^1$ minus a Cantor set, that are K-quasiconformally homogeneous, for some finite $K > 1$. Recently, this problem has received renewed interest. Using Sullivan's Rigidity Theorem, it was shown in [10] by Bonfert-Taylor, Canary, Martin and Taylor, that in dimension $n \geq 3$, there exists a universal constant $\mathcal{K}_n > 1$ such that for every K-quasiconformally homogeneous hyperbolic n-manifold other than $\mathbb{D}^n$, it must hold that $K \geq \mathcal{K}_n$. In [9], by Bonfert-Taylor, Bridgeman and Canary, a partial result was obtained in dimension two for a certain subclass of closed hyperbolic surfaces satisfying a fixed-point condition. In the setting of genus zero surfaces, notions similar to (but stronger than) quasiconformal homogeneity have been studied by MacManus, Näkki and Palka in [32] and further developed in [11] and [12] by Bonfert-Taylor, Canary, Martin, Taylor and Wolf.

To state our results, let us first define precisely the notions involved. Let M be a hyperbolic Riemann surface of genus $g \geq 0$. By uniformizing M, we may assume that $M = \mathbb{D}^2/\Gamma$ is covered by the Poincaré disk $\mathbb{D}^2$. We endow M with the metric $d(\cdot,\cdot)$, induced from the canonical hyperbolic metric $\widetilde{d}(\cdot,\cdot)$ on $\mathbb{D}^2$. Let $\mathcal{F}_K(M)$ be the family of all K-quasiconformal homeomorphisms of M. Then M is said to be K-quasiconformally homogeneous if the family $\mathcal{F}_K(M)$ is transitive; that is, given any two points $p, q \in M$, there exists an element $f \in \mathcal{F}_K(M)$ such that $f(p) = q$.

Let us also recall the following. An open Riemann surface M is a said to be *extentable* or *non-maximal* if it can be embedded in another Riemann surface M_0 as a proper subregion; that is, if there exists a conformal mapping of M onto a proper subregion of M_0. If M is not extentable, then it is called *maximal.* Every open Riemann surface is contained in a maximal Riemann surface (including infinite genus surfaces), see [6]. A non-maximal Riemann surface of genus 0, i.e. embedded in the Riemann sphere, is also called a *planar domain.* Our first result is the following

Theorem 4.A (Non-maximal surfaces of positive genus)**.** *Let M be a non-maximal surface of genus $1 \leq g \leq \infty$. Then M is not K-quasiconformally homogeneous for any $K \geq 1$.*

Therefore, as far as quasiconformal homogeneity of hyperbolic surfaces is concerned, one can further restrict to either:

(i) hyperbolic genus zero surfaces, or

(ii) maximal surfaces of genus $2 \leq g \leq \infty$.

Our main result solves the case of *hyperbolic genus zero surfaces*.

Theorem 4.B (Genus zero surfaces)**.** *There exists a constant $\mathcal{K} > 1$, such that if M is a K-quasiconformally homogeneous hyperbolic genus zero surface other than $\mathbb{D}^2$, then $K \geq \mathcal{K}$.*

The outline of the proof of Theorem 4.B is as follows. First, we restrict our attention to *short geodesics*, that is, simple closed geodesics which are close in length to the infimum of the lengths of all simple closed geodesics on our surface M. For $K > 1$ small enough, using K-quasiconformal homogeneity, we show there exist intersections of short simple closed geodesics in a small neighbourhood of any preassigned point. Using this information, we construct a configuration of three intersecting short simple closed geodesics, see the three-circle Lemma below. By a combinatorial argument, we show that if M is near conformally homogeneous, these configurations can not exist, leading to the desired contradiction.

As the only genus zero surfaces homogeneous with respect to a conformal family are conformally equivalent to either $\mathbb{P}^1, \mathbb{C}, \mathbb{C}^*$ or $\mathbb{D}^2$, we thus have the following corollary of Theorem 4.B.

Corollary 4.1. *There exists a constant $\mathcal{K} > 1$ such that if M is a K-quasiconformally homogeneous genus zero surface with $K < \mathcal{K}$, then M is conformally equivalent to either $\mathbb{P}^1, \mathbb{C}$, $\mathbb{C}^*$ or $\mathbb{D}^2$.*

4.2 Geometrical estimates

The *injectivity radius* $\iota(M)$ of M is the infimum over all $p \in M$ of the largest radius for which the exponential map at p is a injective. Define $\lambda(M)$ to be the infimum of the lengths of all simple closed geodesics on M. We have that $\lambda(M) \geq 2\iota(M)$. We denote by $D(p, \rho) \subset M$ the closed hyperbolic disk with center p and radius ρ. Given a closed curve $\gamma \in M$, we denote by $[\gamma]$ the homotopy class of γ in M. The *geometric intersection number* of isotopy classes of two closed curves $\alpha, \beta \in \pi_1(M)$ is defined by

$$i(\alpha, \beta) = \min \#\{\gamma \cap \gamma'\}, \tag{4.1}$$

where the minimum is taken over all closed curves $\gamma, \gamma' \subset M$ with $[\gamma] = \alpha$ and $[\gamma'] = \beta$. In other words, the geometric intersection number is the least number of intersections between curves representing the two homotopy classes. Let us recall some standard facts about simple closed curves, and in particular simple closed geodesics, on genus zero surfaces, see e.g. [15].

Lemma 4.2 (Curves on genus zero surfaces)**.** *Let M be a genus zero surface.*

(i) *Every simple closed curve $\gamma \subset M$ separates M into exactly two connected components.*

(ii) *If $\gamma, \gamma' \subset M$ are two simple closed curves, then $i(\gamma, \gamma')$ is even.*

(iii) *If $\gamma, \gamma' \subset M$ are non-homotopic closed geodesics, then the closed curve $\alpha \subset M$ formed by any two subarcs $\eta \subset \gamma$ and $\eta' \subset \gamma'$ connecting two points of $\gamma \cap \gamma'$ is homotopically non-trivial.*

The following lemma describes the asymptotic behaviour of the injectivity radius $\iota(M)$ of a K-quasiconformally homogeneous hyperbolic surface in terms of K, see [10].

Lemma 4.3. *Let M be a K-quasiconformally homogeneous hyperbolic surface and $\iota(M)$ its injectivity radius. Then $\iota(M)$ is uniformly bounded from below (for K bounded from above) and $\iota(M)\to\infty$ for $K\to 1$.*

Consequently, $\lambda(M)$ is uniformly bounded from below and $\lambda(M)\to\infty$ for $K\to 1$. We fix $K_0 > 1$ such that $\lambda(M) \geq 10$ for every K-quasiconformally homogeneous hyperbolic surface M with $K \leq K_0$.

Remark 5. *If M is a K-quasiconformally homogeneous hyperbolic surface, then the fact that $\lambda(M) > 0$ implies M has no cusps and therefore, any essential simple closed curve $\alpha \subset M$ has a unique simple closed geodesic representative $\gamma \subset M$, see e.g. [46]. In particular, if $\gamma \subset M$ is a simple closed geodesic and $f\colon M\to M$ a homeomorphism, then the closed geodesic homotopic to $f(\gamma)$ exists and is simple.*

A *pair of pants* is a surface homeomorphic to the sphere $\mathbb{P}^1$ with the interior of three mutually disjoint closed topological disks removed. Geometrically, it is the surface obtained by gluing two hyperbolic hexagons along their seams.

In what follows, we denote by $|\gamma|$ the hyperbolic length of a piecewise geodesic curve $\gamma \subset M$. Here by piecewise geodesic curve we mean a finite concatenation of geodesics arcs. We have the following uniform estimate on lengths of simple closed geodesics, see [16, Theorem 4.3.3].

Lemma 4.4. *Let M be a hyperbolic surface and γ a simple closed geodesic. Let $f\colon M\to M$ a K-quasiconformal homeomorphism and γ' the simple closed geodesic homotopic to $f(\gamma)$. Then*

$$\frac{1}{K}|\gamma| \leq |\gamma'| \leq K|\gamma|. \tag{4.2}$$

Further, we use the following classical result in the geometry of hyperbolic surfaces.

Collar Lemma. *Set $m(\ell) = \arcsin(1/(\sinh(\ell/2)))$. For a simple closed geodesic $\gamma \subset M$ of length $\ell = |\gamma|$, the set*

$$A(\gamma) = \{p \in M : d(p,\gamma) < m(\ell)\} \tag{4.3}$$

is an embedded annular neighbourhood of γ*.*

In the next two lemma's we recollect uniform approximation estimates of K-quasiconformal homeomorphisms, in particular the behaviour when $K \to 1$, see e.g. [31, 16].

Lemma 4.5. *For every* $K \geq 1$ *and* $0 < \rho < 1$*, there exists a constant* $C_1(K, \rho)$*, depending only on* K *and* ρ*, such that if* $f \colon \mathbb{D}^2 \to \mathbb{D}^2$ *is a* K*-quasiconformal homeomorphism and* $D(p, \rho) \subset \mathbb{D}^2$ *the closed hyperbolic disk of radius* ρ *centered at* $p \in \mathbb{D}^2$*, there exists a Möbius transformation* $\mu \in \text{Möb}(\mathbb{D}^2)$ *such that*

$$\widetilde{d}(f(q), \mu(q)) \leq C_1(K, \rho) \tag{4.4}$$

for all $q \in D(p, \rho)$*. For fixed* $\rho > 0$*, we have that* $C_1(K, \rho) \to 0$ *for* $K \to 1$*.*

Proof. By normalizing with suitable Möbius transformations, we may assume that $f(0) = 0$ and $f(1) = 1$. As the family of K-quasiconformal homeomorphisms of $\mathbb{D}^2$ onto itself fixing $0, 1 \in \mathbb{D}^2$ is a normal family, see e.g. [2, p. 32], by a standard argument of uniform convergence on compact subsets, there exists a function $C_1(K, \rho)$ with $C_1(K, \rho) \to 0$ if $K \to 1$ such that $\widetilde{d}(f(z), z) \leq C_1(K, \rho)$, as the only conformal mapping of $\mathbb{D}^2$ onto itself fixing $0, 1 \in \mathbb{D}^2$ is the identity. Thus (4.4) follows. $\square$

Further, we will utilize the following, see e.g. [13, Lemma 2].

Lemma 4.6. *For every* $K \geq 1$*, there exists a constant* $C_2(K)$ *depending only on* K *with the following property. Let* M *be a hyperbolic surface and* $\gamma \subset M$ *a simple closed geodesic and* $p \in \gamma$*. If* $f \colon M \to M$ *is a* K*-quasiconformal homeomorphism, then the geodesic* γ' *homotopic to* $f(\gamma)$ *has the property that*

$$d(f(\gamma), \gamma') \leq C_2(K). \tag{4.5}$$

Furthermore, $C_2(K) \to 0$ *as* $K \to 1$*.*

Proof. In [13, Lemma 2], the above lemma is proved under the assumption that $f\colon M \to M$ is a quasi-isometry for the hyperbolic metric, that is, f satisfies

$$\frac{1}{L} d(z,w) \leq d(f(z), f(w)) \leq L d(z,w), \tag{4.6}$$

for some $L \geq 1$, for every $z, w \in \mathbb{D}^2$. By lifting the quasi-isometry f, which is a quasi-isometry of $\mathbb{D}^2$, and the geodesic γ to the cover $\mathbb{D}^2$ (which we denote again by f and γ), one needs to find an upper bound on the maximal hyperbolic distance of the curve $f(\gamma)$ and the geodesic γ' with the same endpoints on $\partial\mathbb{D}^2$ as $f(\gamma)$. Using that f is a quasi-isometry, by a polygonal approximation, it is shown in [13, Lemma 2] that there exists a finite upper bound $\phi(L)$, depending only on L, on the distance between $f(\gamma)$ and γ' in $\mathbb{D}^2$, where $\phi(L) \to 0$ if $L \to 1$.

By the following approximation argument, we show that this result holds equally well for quasiconformal homeomorphisms, as follows. As a K-quasiconformal homemorphism of M lifts to a K-quasiconformal homeomorphism of $\mathbb{D}^2$, it suffices to show that for every $K \geq 1$, there exist constants $\psi(K)$ and $\varphi(K)$ depending only on K, where $\psi(K) \to 0$ and $\varphi(K) \to 1$ for $K \to 1$, such that if $f\colon \mathbb{D}^2 \to \mathbb{D}^2$ is K-quasiconformal, then there exists a $\varphi(K)$-quasi-isometry $g\colon \mathbb{D}^2 \to \mathbb{D}^2$ such that

$$\widetilde{d}(f(z), g(z)) \leq \psi(K). \tag{4.7}$$

To prove this, let $f\colon \mathbb{D}^2 \to \mathbb{D}^2$ be a K-quasiconformal homeomorphism and transport it to $\mathbb{H}^2$ (which we denote again by f). Then f induces a homeomorphism $h\colon \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ of the boundary $\overline{\mathbb{R}} = \partial\mathbb{H}^2$. Without loss of generality we may assume that h is normalized so that $h(\infty) = \infty$. Then h is k-quasisymmetric, where $k(K)$ depends only on K and $k \to 1$ as $K \to 1$ (see [31, Thm 5.1]). Let g be the Ahlfors-Beurling extension of h. Then g is K_1 quasiconformal, where $K_1(K)$ depends only on k and thus only on K and $K_1(K) \to 1$ if $K \to 1$. Moreover, g is $\varphi(K)$-biLipschitz, relative to the hyperbolic metric, where $\varphi(K)$ again only depends only on K and $\varphi(K) \to 1$ as $K \to 1$ (see [31, Thm 5.2]).

Define $K_2(K) := KK_1(K)$ and consider the K_2-quasiconformal homeomorphism $f^{-1} \circ g \colon \mathbb{H}^2 \to \mathbb{H}^2$, which extends to the identity on $\partial\mathbb{H}^2$. As the family of K-quasiconformal homeomorphisms that extend to the identity on $\partial\mathbb{H}^2$ is a compact family, by a normal family argument, see e.g. [10, Lemma 4.1], there exists a decreasing function $\widetilde{\psi}(t)$ with $\widetilde{\psi}(t) \to 0$ if $t \to 1$, such that $\widetilde{d}(z, f^{-1} \circ g(z)) \leq \widetilde{\psi}(K_2)$, for all $z \in \mathbb{H}^2$. It follows from Mori's Theorem (see [2, p. 30]), that on every compact disk $D(p, 1) \subset \mathbb{H}^2$, centered at $p \in \mathbb{H}^2$ and hyperbolic radius (say) 1, there exists a constant C, such that

$$\widetilde{d}(f(z), f(z')) \leq C(d(z, z'))^{1/K}. \tag{4.8}$$

Therefore, for all $z \in \mathbb{H}^2$, we have that

$$\widetilde{d}(f(z), g(z)) \leq \psi(K), \tag{4.9}$$

where $\psi(K) := C\widetilde{\psi}(K_2(K))^{1/K_2(K)}$. As $K_2(K) \to 1$ if $K \to 1$, we have that $\psi(K) \to 0$ if $K \to 1$, as desired. This concludes the proof. $\square$

4.3 Non-maximal surfaces of positive genus

Using the geometrical estimates derived in the previous section, we are now in the position to prove Theorem 4.A, which we have stated again below for the reader's convenience.

Theorem 4.A (Non-maximal surfaces of positive genus)**.** *Let M be a non-maximal surface of genus $1 \leq g \leq \infty$. Then M is not K-quasiconformally homogeneous for any $K \geq 1$.*

Proof. To derive a contradiction, assume that M is K-quasiconformally homogeneous for some finite $K \geq 1$. As M is non-maximal, M is embedded in a maximal hyperbolic surface M_0 of genus $g \geq 1$. Let $\bar{p} \in M_0$ be an ideal boundary point of M and let $D \subset M_0$ be a small closed disk embedded in M_0 and centered at $\bar{p}$. There exists a sequence of points $p_n \in M \cap D$ so that $d_0(p_n, \bar{p}) \to 0$ (where d_0 is the metric on M_0) and thus $d(p_n, \partial D) \to \infty$, as $\bar{p}$ is in the ideal boundary of M. On the other hand, as $1 \leq g \leq \infty$, there exists a non-separating simple closed geodesic $\gamma \subset M$. Mark a point $p \in \gamma$. By transitivity of the family $\mathcal{F}_K(M)$, for every $n \geq 1$, there exists an element $f_n \in \mathcal{F}_K(M)$ such that $f_n(p) = p_n$. Therefore the simple closed curve $f_n(\gamma)$ is non-separating for every $n \geq 1$ and thus

$$f_n(\gamma) \cap \partial D \neq \emptyset. \tag{4.10}$$

Indeed, otherwise we have that $f_n(\gamma) \subset M \cap D$, implying that $f_n(\gamma)$ is separating, as D is an embedded disk in the maximal surface M_0 and thus the connected components of $M \cap D$ are planar subsurfaces, contradicting that γ is non-separating. By Lemma 4.6, the geodesic γ_n homotopic to $f^n(\gamma)$ has to stay within a bounded distance of $f_n(\gamma)$ and therefore, for n large enough, the geodesic γ_n has the property that

$$\gamma_n \cap \partial D \neq \emptyset \tag{4.11}$$

by (4.10). As $d(p_n, \partial D) \to \infty$, combined with (4.11), we have

$$|\gamma_n| \geq 2(d(p_n, \partial D) - C_2(K)) \tag{4.12}$$

with $C_2(K)$ the uniform constant of Lemma 4.6; put in words, the geodesic γ_n has to enter $D \cap M$, pass close to p_n, and leave $D \cap M$ again. It follows that $|\gamma_n| \to \infty$ for $n \to \infty$, contradicting Lemma 4.4. Thus M can not be K-quasiconformally homogeneous for any finite $K \geq 1$. □

4.4 Quasiconformal homogeneity of genus zero surfaces

In the sections 4.4.2 - 4.4.5 below, we present our proof of Theorem 4.B. But before we proceed to the proof, let us first consider examples of quasiconformally homogeneous genus zero surfaces.

4.4.1 Examples of a quasiconformally homogenous genus zero surfaces

The following example is taken from [17, Example 4.4], we refer to this paper for the proofs of the statements in this example. First we recall some notation and terminology. A group Γ of Möbius transformations acting on $\mathbb{P}^1$ is *discontinuous* at a point $p \in \mathbb{P}^1$ provided there exists a neighbourhood U of p such that $\mu(U) \cap U = \emptyset$ for all but finitely many $\mu \in \Gamma$. The *region of discontinuity* of Γ, denoted $O(\Gamma)$, is the set of all $p \in \mathbb{P}^1$ at which Γ is discontinuous. It follows that $O(\Gamma)$ is an open set, possibly empty. The complement $\mathbb{P}^1 \setminus O(\Gamma)$ of $O(\Gamma)$ is called the *limit set* of Γ and is denoted by $L(\Gamma)$. Both $O(\Gamma)$ and $L(\Gamma)$ are invariant under Γ. The group Γ is said to be *discontinuous group* if $O(\Gamma) \neq \emptyset$.

Now, let $\{B_i\}_{i=1}^m$ be a collection of $m \geq 3$ pairwise disjoint closed disks in the sphere $\mathbb{P}^1$. Denote by $\mu_i \in \text{Möb}(\mathbb{P}^1)$ the inversion in $\mathbb{P}^1$ of ∂B_i. Then the mappings $\mu_1, ..., \mu_m$ generate a discontinuous group Γ. The limit set $L(\Gamma)$ is a totally disconnected perfect set with positive Hausdorff dimension, and

$$L(\Gamma) \subset \bigcup_{i=1}^{m} \text{Int}(B_i). \tag{4.13}$$

For these reflection groups, the open set $M = O(\Gamma)$ is connected and M is a genus zero surface. For example, if we take $m = 3$ in the above construction, we obtain a surface M conformally equivalent to a repeated gluing of pairs of pants to the cuffs of the pairs of pants, see Figure 4.1.

One can show that the surface M is indeed K-quasiconformally homogeneous for a finite $K > 1$. In fact, it can be shown, see [17, Remark 4.5], that the following upper bound

$$K \leq \left(\frac{e}{s}\right)^4, \text{ where } s \approx 0.483, \tag{4.14}$$

holds for the quasiconformality constant K of the surface M.

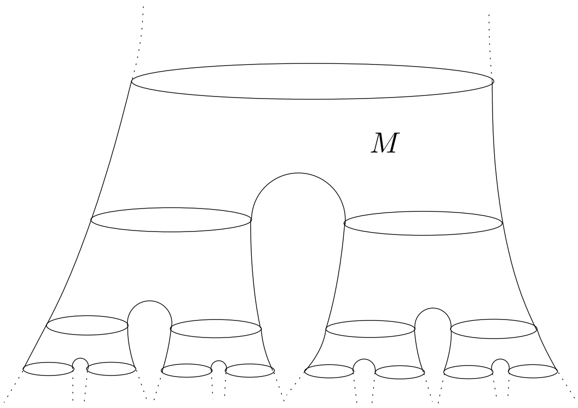

Figure 4.1: A quasiconformally homogeneous genus zero surface M; the surface is homeomorphic to the Riemann sphere minus a Cantor set.

4.4.2 The two-circle Lemma

Let us now proceed with the proof of Theorem 4.B. In the remainder, let $K_0 > 1$ as defined in section 4.2 and let M be a K-quasiconformally homogeneous hyperbolic genus zero surface, with $1 < K \leq K_0$, and $\mathcal{F}_K(M)$ the family of all K-quasiconformal homeomorphisms of M, which is transitive by homogeneity of M.

In what follows, we focus on *short geodesics*, in the following sense.

Definition 4.1 (δ-short geodesics)**.** Given $\delta > 0$, a simple closed geodesic $\gamma \subset M$ is said to be δ-*short* if $|\gamma| \leq (1+\delta)\lambda(M)$.

By Lemma 4.3, and the remark following it, for every $K \leq K_0$, there is a uniform lower bound on the length of simple closed geodesics on M. Fix

$$\delta_0 = \frac{1}{378}. \tag{4.15}$$

Definition 4.2 (Two-circle configuration)**.** A *two-circle configuration* is a union of two δ_0-short geodesics $\gamma_1, \gamma_2 \in M$ such that γ_1 and γ_2 intersect in exactly two points $p_1, p_2 \in M$.

Topologically a two-circle configuration is a union of two simple closed curves γ_1, γ_2 in the surface M, intersecting transversely in exactly two points and

$$M \setminus (\gamma_1 \cup \gamma_2)$$

consists of four connected components and the boundary of each component consists of two arcs. For future reference, let us label the four arcs $\eta_1, \eta_2 \subset \gamma_1$ and $\eta_3, \eta_4 \subset \gamma_2$ connecting the two intersection points p_1 and p_2, see Figure 4.2.

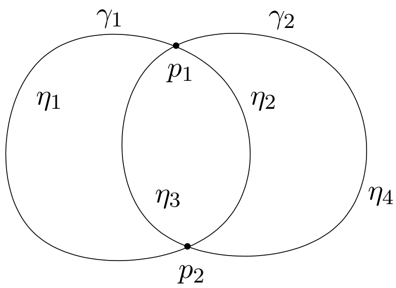

Figure 4.2: A two-circle configuration with labeling.

First, we observe the following, see also [37, Proposition 4.6].

Lemma 4.7. *There exists a uniform constant $r_0 > 0$ such that for a pair of pants $P \subset M$, there exists a $p \in P$ such that $D(p, r_0) \subset P$.*

Proof. Each pair of pants decomposes into two ideal triangles. As every ideal triangle contains a disk of radius $\frac{1}{2}\log 3$, every pair of pants therefore contains a disk of (at least) that radius. Thus we may take $r_0 = \frac{1}{2}\log 3$. $\square$

We first prove the existence of intersecting δ_0-short geodesics on M for sufficiently small $K > 1$, as we build forth upon this result in the remainder of the proof.

Lemma 4.8 (Intersections of short geodesics)**.** *There exists a constant $1 < K_1 \leq K_0$, such that if M is K-quasiconformally homogeneous with $1 < K \leq K_1$, then there exist δ_0-short geodesics that intersect.*

Proof. To prove there exist intersecting δ_0-short geodesics on M, we argue by contradiction. That is, suppose all δ_0-short geodesics on M are mutually disjoint. Choose $1 < K_1 \leq K_0$ so that

$$K_1 \leq \frac{1+\delta_0}{1+\delta_0/2} \text{ and } C_2(K_1) \leq \frac{r_0}{2}, \tag{4.16}$$

with $C_2(K)$ the constant of Lemma 4.6 and r_0 the constant of Lemma 4.7.

Let us first observe that there exist infinitely many distinct δ_0-short geodesics on M. Indeed, as $\lambda(M) > 0$, there exists a simple closed geodesic $\gamma_0 \subset M$ such that $|\gamma_0| \leq (1+\delta_0/2)\lambda(M)$. Mark a point $p_0 \in \gamma_0$ and choose $f \in \mathcal{F}_K(M)$ such that $f(p_0) = q$, for a certain $q \in M$. By our choice of K_1, cf. (4.16), combined with Lemma 4.4, the geodesic γ homotopic to $f(\gamma_0)$ is δ_0-short, for every $f \in \mathcal{F}_K(M)$. As the surface is unbounded (in the hyperbolic metric), by transporting the geodesic γ_0 by different elements of $\mathcal{F}_K(M)$ sufficiently far apart, by Lemma 4.6, we see that there must indeed exist infinitely many different δ_0-short curves. Denote Γ_0 the (countable) family of all δ_0-short geodesics on M.

As all elements of Γ_0 lie in different homotopy classes, and all elements are mutually disjoint, we claim that the elements of Γ_0 are locally finite, in the sense that a compact subset of M only intersects finitely many distinct elements of Γ_0. Indeed, suppose that a compact subset of M intersects infinitely many elements of Γ_0. Label these geodesics γ_n, $n \in \mathbb{Z}$. By compactness, there exists an element $\gamma := \gamma_n$, for some $n \in \mathbb{Z}$, and a subsequence γ_{n_k}, with $n_k \neq n$, such that $d(\gamma, \gamma_{n_k}) \to 0$ for $k \to \infty$. As all these elements are mutually disjoint, we can find points $p_k \in \gamma_{n_k}$ such that $p_k \to p \in \gamma$, where, moreover, the vectors $v_k \in T_{p_k}M$ tangent to γ_{n_k} at p_k converge to the tangent vector $v \in T_pM$ of γ at p. As the lengths of the geodesics γ_{n_k} are uniformly bounded from above, by the Collar Lemma (see section 4.2), every curve γ_{n_k} is contained in a uniformly thick embedded annulus $A_k := A(\gamma_{n_k}) \subset M$. Conversely, as the lengths of the geodesics are uniformly bounded from above, and as the initial data (p_k, v_k) of γ_{n_k} converges to the initial data (p, v) of γ, the geodesics γ_{n_k} converge uniformly to γ. In particular, for sufficiently large k, γ is entirely contained in A_k. However, this implies that γ is homotopic to γ_{n_k}, a contradiction as these were all assumed to be mutually disjoint and thus non-homotopic.

Choose an element $\gamma_1 \in \Gamma_0$. As the elements of Γ_0 are locally finite, and as the distance between any two elements of Γ_0 is finite, the distance between γ_1 and the union of the elements $\Gamma_0 \setminus \{\gamma_1\}$ is therefore bounded from below and above. In particular, there exists a δ_0-short geodesic with shortest distance to γ_1 (though this geodesic need not be unique). Denote one such geodesic by γ_2. There exists a geodesic arc $\eta \subset M$ connecting γ_1 and γ_2, with $|\eta| = d(\gamma_1, \gamma_2)$. Take a simple closed curve $\alpha \subset M$ homotopic to $\gamma_1 \cup \eta \cup \gamma_2$ and let γ' be the (not necessarily δ_0-short) geodesic homotopic to α. As γ_1 and γ_2 are disjoint, and therefore in distinct homotopy classes, γ' is non-trivial. By Lemma 4.2 (iii), we have that

$$\gamma' \cap (\gamma_1 \cup \gamma_2) = \emptyset.$$

Let $P \subset M$ be the pair of pants bounded by the simple closed geodesics γ', γ_1 and γ_2. As every pair of pants contains a unique simple geodesic arc connecting each pair of boundary geodesics of the pair of pants, $\eta \subset P$ is the unique geodesic arc in P joining γ_1 and γ_2 such that $|\eta| = d(\gamma_1, \gamma_2)$, see Figure 4.3.

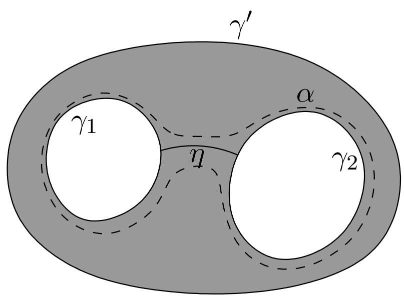

Figure 4.3: Proof of Lemma 4.8

Next, we claim that the interior of P is disjoint from any δ_0-short geodesic. Indeed, let

$$\gamma_3 \in \Gamma_0 \setminus \{\gamma_1, \gamma_2\},$$

and suppose that $\gamma_3 \cap \mathrm{Int}(P) \neq \emptyset$. As $\gamma_1, \gamma_2 \in \Gamma_0$, by assumption γ_3 can not intersect γ_1 or γ_2. Thus, if $\gamma_3 \cap \mathrm{Int}(P) \neq \emptyset$, then we necessarily have that $\gamma_3 \cap \gamma' \neq \emptyset$. Consider any two consecutive intersection points $q_1, q_2 \in \gamma'$ of γ' and γ_3 and denote $\eta' \subset \gamma_3 \cap P$ the corresponding simple arc. We first show that we must have that $\eta' \cap \eta \neq \emptyset$. To show this, suppose that $\eta' \cap \eta = \emptyset$. Then $\eta' \cap (\gamma_1 \cup \eta \cup \gamma_2) = \emptyset$. Let $\gamma'_{1,2} \subset \gamma'$ be the two simple arcs connecting q_1 with q_2. Consider the subsurfaces $M_1, M_2 \subset M$ bounded by $\gamma'_1 \cup \eta'$ and $\gamma'_2 \cup \eta'$ respectively and intersecting $\mathrm{Int}(P)$. As $\eta' \cap (\gamma_1 \cup \eta \cup \gamma_2) = \emptyset$, $\gamma_1 \cup \eta \cup \gamma_2$ is contained in either M_1 or M_2. However, this implies that one of the subsurfaces M_1 or M_2 is a topological disk, which contradicts Lemma 4.2 (iii). Therefore, we must have that $\eta' \cap \eta \neq \emptyset$. This in turn implies that

$$d(\gamma_3, \gamma_1) < d(\gamma_1, \gamma_2) \tag{4.17}$$

which contradicts the assumption that γ_2 is the closest δ_0-short geodesic to γ_1. Thus the interior of P is disjoint from any δ_0-short geodesic.

By Lemma 4.7, there exists a point $p \in P$ such that $D(p, r_0) \subset P$. By the previous paragraph, the disk $D(p, r_0)$ is disjoint from any δ_0-short geodesic. Take $f \in \mathcal{F}_K(M)$ such that $f(p_0) = p$. The geodesic γ'' homotopic to $f(\gamma_0)$ is δ_0-short and, again by our choice of K_1, combined with Lemma 4.6, the geodesic γ'' has the property that $\gamma'' \cap D(p, r_0) \neq \emptyset$. This contradicts our earlier conclusion that the interior of P is disjoint from δ_0-short geodesics and thus there must exist δ_0-short geodesics that intersect. $\square$

Lemma 4.9 (Two-circle Lemma)**.** *Let M be K-quasiconformally homogeneous with $1 < K \leq K_1$ and let γ_1 and γ_2 be two intersecting δ-short geodesics, where $\delta < 1/3$. Then $\gamma_1 \cup \gamma_2$ is a two-circle configuration and the four arcs η_i, $1 \leq i \leq 4$, connecting the intersection points p_1 and p_2 have lengths*

$$\frac{\lambda(M)}{2} - \frac{\delta}{2}\lambda(M) \leq |\eta_i| \leq \frac{\lambda(M)}{2} + \frac{3\delta}{2}\lambda(M). \tag{4.18}$$

Proof. Let $\gamma_1, \gamma_2 \subset M$ be two δ-short geodesics that intersect. By Lemma 4.2 (ii), γ_1 and γ_2 intersect in an even number of points. To prove there can be no more than two intersection points, suppose that there are $2k$ intersection points, with $k \geq 2$. Label these points $p_1, ..., p_{2k}$ according to their cyclic ordering on γ_1, relative to an orientation on γ_1 and an initial point. Define the arcs α_i, with $1 \leq i \leq 2k$, to be the connected components of

$$\gamma_1 \setminus \bigcup_{i=1}^{2k} p_i.$$

As $k \geq 2$ by assumption, at least one of these arcs has length at most $(1+\delta)\lambda(M)/4$. Without loss of generality, we may suppose that this is the case for α_1. Then the endpoints of α_1, p_1 and p_2, cut the geodesic γ_2 into two connected components β_1 and β_2. One of these components, say β_1, has length at most $(1+\delta)\lambda(M)/2$. By Lemma 4.2 (iii), $\alpha_1 \cup \beta_1$ is a non-trivial closed curve. However, we have that

$$|\alpha_1 \cup \beta_1| \leq \frac{3(1+\delta)\lambda(M)}{4} < \lambda(M),$$

as $(1+\delta) < 4/3$, which is impossible. Thus γ_1 and γ_2 intersect in exactly two points.

To prove (4.18), we adopt the labeling in Figure 4.2. As $\gamma_1 = \eta_1 \cup \eta_2$ and $|\gamma_1| \leq (1+\delta)\lambda(M)$, one of the arcs η_1 or η_2 has length at most $(1+\delta)\lambda(M)/2$. We may assume this is the case for η_1. As the closed curve $\eta_3 \cup \eta_1$ is homotopically non-trivial, we must have that

$$\frac{(1+\delta)\lambda(M)}{2} + |\eta_3| \geq |\eta_1| + |\eta_3| \geq \lambda(M),$$

and thus

$$|\eta_3| \geq \frac{(1-\delta)\lambda(M)}{2}. \tag{4.19}$$

Conversely, in order that $|\eta_3| + |\eta_4| \leq (1+\delta)\lambda(M)$, by (4.19), we must have that

$$|\eta_4| \leq \left(\frac{1}{2} + \frac{3}{2}\delta\right)\lambda(M). \tag{4.20}$$

The other cases follow by symmetry. This finishes the proof. $\square$

In particular, the two-circle Lemma holds for all $\delta \leq 6\delta_0 < 4/3$. For future reference, we introduce the following.

Definition 4.3 (Tight pair of pants)**.** A *tight pair of pants* is a pair of pants $P \subset M$ such that the three boundary curves are $3\delta_0$-short geodesics.

We have the following corollary of the two-circle Lemma.

Corollary 4.10. *Let M be K-quasiconformally homogeneous with $1 < K \leq K_1$. Then there exists a tight pair of pants.*

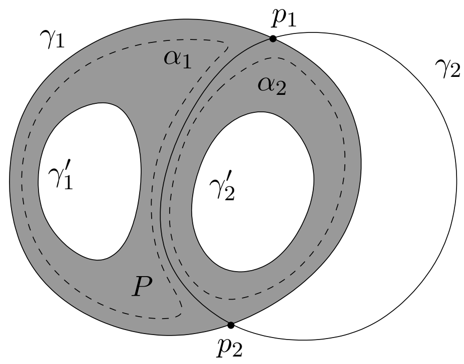

Figure 4.4: Proof of Corollary 4.10.

Proof. By the two-circle Lemma, for $K \leq K_1$, there exists δ_0-short geodesics γ_1 and γ_2 that intersect in exactly two points. In the labeling of Figure 4.2, let α_1 be the simple closed curve $\eta_1 \cup \eta_3$ and γ_1' be the simple closed geodesic homotopic to α_1. Similarly, let α_2 the simple closed curve $\eta_2 \cup \eta_3$ and γ_2' be the simple closed geodesic homotopic to α_2. By Lemma 4.2 (iii), the three geodesics γ_1', γ_2' and γ_1 are disjoint and thus the region bounded by these three simple closed geodesics is a pair of pants P, see Figure 4.4.

To prove P is a tight pair of pants, it suffices to show that γ_i' is $3\delta_0$-short, for $i = 1, 2$, as γ_1 is δ_0-short. It follows from (4.18) of the two-circle Lemma that

$$|\eta_1| + |\eta_3| \leq 2\left(\frac{\lambda(M)}{2} + \frac{3\delta_0}{2}\lambda(M)\right) = (1 + 3\delta_0)\lambda(M). \tag{4.21}$$

Therefore, the length of γ_1' is bounded by the length of $\eta_1 \cup \eta_3$, which is at most $(1+3\delta_0)\lambda(M)$. Similarly, considering the length of $\eta_2 \cup \eta_4$, we obtain that γ_2' is $3\delta_0$-short. Therefore, P is a tight pair of pants. $\square$

4.4.3 Definite angles of intersection

In what follows, we use the following notation. Let $\gamma_1, \gamma_2 \subset M$ be two simple closed geodesics that intersect at a point $p \in M$. The angle between the two geodesics at $p \in M$, denoted $\angle(\gamma_1, \gamma_2)_p$, is defined to be the minimum of $\angle(v_1, v_2)_p$ and $\angle(v_1, -v_2)_p$ where $v_1, v_2 \in T_pM$ is a tangent vector to γ_1, γ_2 respectively. In order to produce certain configurations of intersecting simple closed geodesics, we show that for all $K > 1$ sufficiently small, there exist $3\delta_0$-short geodesics intersecting at a uniformly large angle. More precisely,

Lemma 4.11 (Definite angles of intersection)**.** *There exists a constant $1 < K_2 \leq K_1$, such that if M is K-quasiconformally homogeneous with $1 < K \leq K_2$, then there exist two $3\delta_0$-short geodesics $\gamma_1, \gamma_2 \subset M$, intersecting at a point $q \in M$, such that $\angle(\gamma_1, \gamma_2)_q \geq \pi/4$.*

The proof of Lemma 4.11 uses the following two auxiliary lemma's.

Lemma 4.12. *Let $H \subset \mathbb{D}^2$ be a right-angled hyperbolic hexagon. Let a, b, c be the sides of the alternate edges and a', b', c' the sides of the opposite edges. Suppose that $|a| = (1+\epsilon_1)\ell, |b| = (1+\epsilon_2)\ell$ and $|c| = (1+\epsilon_3)\ell$ with $0 \leq \epsilon_i < 1/2$, $1 \leq i \leq 3$. For every $\epsilon > 0$, there exists $\ell_\epsilon > 0$ such that, if $\ell \geq \ell_\epsilon$, then the lengths of the sides a', b' and c' are at most ϵ.*

Proof. By the hyperbolic cosine law for right-angled hexagons (see [42]), we have that

$$\cosh(|c'|) = \frac{\cosh(|a|)\cosh(|b|) + \cosh(|c|)}{\sinh(|a|)\sinh(|b|)}, \tag{4.22}$$

which we can write as

$$\cosh(|c'|) = \frac{1}{\tanh(|a|)\tanh(|b|)} + \frac{\cosh(|c|)}{\sinh(|a|)\sinh(|b|)} \tag{4.23}$$

We have that

$$\frac{\cosh(|c|)}{\sinh(|a|)\sinh(|b|)} \asymp \frac{e^{(1+\epsilon_3)\ell}}{e^{(2+\epsilon_1+\epsilon_2)\ell}} \to 0, \quad \text{for} \quad \ell \to \infty, \tag{4.24}$$

as $0 \leq \epsilon_i < 1/2$ for $1 \leq i \leq 3$. Further, as $\tanh(r) \to 1$ for $r \to \infty$, given any $\epsilon' > 0$, there exists an $\ell_{\epsilon'}$ such that, if $\ell \geq \ell_{\epsilon'}$, then

$$\cosh(|c'|) \leq 1 + \epsilon'. \tag{4.25}$$

Thus given an $\epsilon > 0$, choose $\ell_{\epsilon'}$ such that (4.25) is satisfied with $\epsilon := \cosh^{-1}(1+\epsilon')$. For this $\ell_{\epsilon'}$ (with ϵ' depending on ϵ only), we have that

$$|c'| \leq \cosh^{-1}(1 + \epsilon') = \epsilon \tag{4.26}$$

Cyclically permuting a, b, c and a', b', c' gives a similar estimate for a' and b' and this finishes the proof. $\square$

Lemma 4.13. *Let $T \subset \mathbb{D}^2$ be an ideal triangle with boundary $\partial T = \gamma_1 \cup \gamma_2 \cup \gamma_3$ and barycenter $0 \in \mathbb{D}^2$. Let $\gamma \subset \mathbb{D}^2$ be a geodesic passing through $0 \in \mathbb{D}^2$. Then for an $i \in \{1, 2, 3\}$, γ intersects ∂T at a point $p \in \gamma_i$, such that $\pi/4 + \epsilon_0 \leq \angle(\gamma, \gamma_i)_p \leq \pi/2$, where $\epsilon_0 \approx 0.24$.*

Proof. Let us label the boundary geodesics $\gamma_1, \gamma_2, \gamma_3$ of the ideal triangle T as in Figure 4.5. Let $l_0 \subset \mathbb{D}^2$ be the axis of symmetry of T, relative to the symmetry that exchanges γ_1 and γ_2. The distance of $0 \in \mathbb{D}^2$ to the point of intersection of ℓ_0 with γ_3 is $\frac{1}{2}\log 3$. Let l_1 be the axis perpendicular to the axis of symmetry relative to the symmetry that exchanges γ_2 and γ_3, see Figure 4.5. Let θ_1 be the angle between the axis l_1 and the geodesic γ_3. The angle between the axes l_0 and l_1 is $\pi/6$. Therefore, by the hyperbolic cosine law, the angle θ_1 between γ_3 with l_1 is given by

$$\cos(\theta_1) = \cosh\left(\frac{1}{2}\log 3\right)\sin\left(\frac{\pi}{6}\right).$$

It is readily verified that $\theta_1 = \pi/4 + \epsilon_0$, with $\epsilon_0 \approx 0.24$.

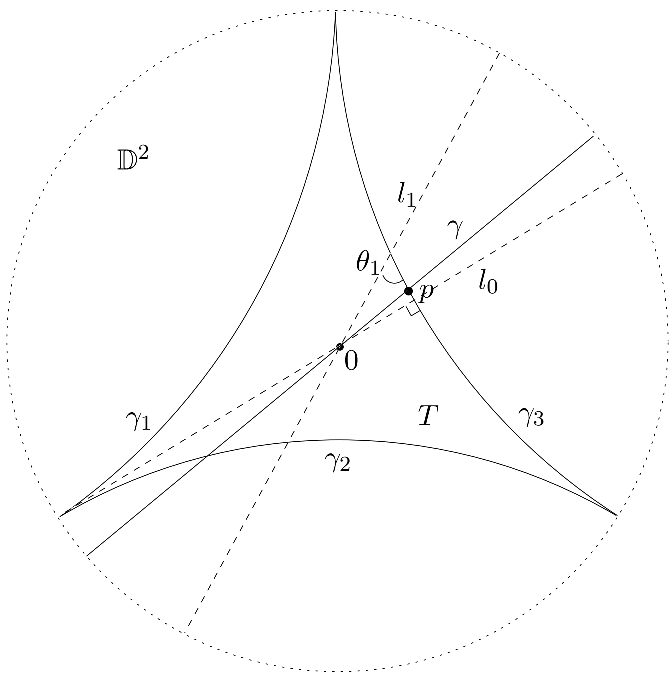

Figure 4.5: Proof of Lemma 4.13.

Suppose that the geodesic γ that passes through $0 \in \mathbb{D}^2$ is such that $\gamma \cap \gamma_3 \neq \emptyset$. Denote p the point of intersection of γ and γ_3. By symmetry of the configuration, we may assume that p is contained in the arc $\eta \subset \gamma_3$ cut out by the two intersection points of l_0, l_1 and γ_3, which, up to symmetry, represents the extremal case. Thus we have that $\pi/4 + \epsilon_0 \leq \angle(\gamma, \gamma_i)_p \leq \pi/2$, with $\epsilon_0 \approx 0.24$. This finishes the proof of the Lemma. $\square$

Proof of Lemma 4.11. By Corollary 4.10, there exists a tight pair of pants $P \subset M$, i.e. a pair of pants P with $\partial P = \gamma_1 \cup \gamma_2 \cup \gamma_3$, with the property that the three boundary geodesics $\gamma_1, \gamma_2, \gamma_3$ are $3\delta_0$-short. Lifting the pair of pants P to the cover $\mathbb{D}^2$, it unfolds to two right-angled hexagons $H, H' \subset \mathbb{D}^2$, each of which contains exactly a half of the component of the lift $\widetilde{\gamma}_i$ of γ_i to $\mathbb{D}^2$ as its alternate boundary arcs, with $1 \leq i \leq 3$. Restrict to H and denote η_i with $1 \leq i \leq 3$ the alternating boundary arcs. As the length of η_i is exactly half of that of γ_i, with $1 \le i \le 3$, and these are $3\delta_0$-short, the lengths η_i satisfy the length requirement of Lemma 4.12, so that, given any $\epsilon > 0$ there exists a $K > 1$ such that the lengths of the three sides of the hexagon H opposite to η_i are of length at most ϵ, as $\lambda(M)\to\infty$ for $K\to 1$ by Lemma 4.3.

Therefore, by normalizing by a suitable element of $\text{Möb}(\mathbb{D}^2)$ if necessary, the hexagon $H \subset \mathbb{D}^2$ converges on a compact disk $D(0,10)$ to an ideal triangle with barycenter $0 \in \mathbb{D}^2$, for $K\to 1$. In particular, by Lemma 4.13, there exists $1 < K_2 \le K_1$, such that any geodesic $\gamma' \subset \mathbb{D}^2$ passing through $0 \in \mathbb{D}^2$ intersects one of the arcs η_i of H under an angle at least $\pi/4 + \epsilon_0/2$, with ϵ_0 as in Lemma 4.13. As geodesics in $\mathbb{D}^2$ passing near $0 \in \mathbb{D}^2$ are almost straight lines, there exists an $\epsilon_1 > 0$ such that any geodesic $\gamma' \subset \mathbb{D}^2$ passing through the disk $D(0,\epsilon_1) \subset \mathbb{D}^2$ intersect one of the arcs η_i at an angle at least $\pi/4$. By choosing $K_2 > 1$ smaller if necessary, we can be sure that $C_2(K_2) \le \epsilon_1$, where $C_2(K)$ is the constant of Lemma 4.6.

Let $\gamma_0 \subset M$ be a δ_0-short geodesic and let $p_0 \in \gamma_0$. Choose the point $p \in P \subset M$ in the tight pair of pants, which without loss of generality we may assume to correspond to $0 \in \mathbb{D}^2$ in the lift. Choose an element $f \in \mathcal{F}_K(M)$ such that $f(p_0) = p$ and denote $\gamma \subset M$ the geodesic homotopic to $f(\gamma_0)$. By our choice of K_3, the lift $\widetilde{\gamma}$ of γ will intersect at least one of the three arcs η_1, η_2 or η_3 at an angle $\angle(\widetilde{\gamma}, \eta_i)_q \ge \pi/4$. In other words, γ intersects one of the three boundary geodesics γ_1, γ_2 or γ_3 at an angle at least $\pi/4$. Further, as $K_2 \le K_1$, we have that

$$K_2(1+\delta_0) \le K_1(1+\delta_0) \le 1 + 3\delta_0,$$

as $K_1(1+\delta_0/2) \le 1 + \delta_0$, and thus γ is $3\delta_0$-short. This proves the Lemma. □

4.4.4 The three-circle Lemma

A *triangle* $T \subset M$ is a subsurface of M such that its boundary consists of a simple closed curve comprised of three geodesic arcs. The triangle T is said to be *trivial* if the simple closed curve ∂T is homotopically trivial and *non-trivial* otherwise.

Definition 4.4 (Three-circle configuration)**.** A *three-circle configuration* is a union of three $6\delta_0$-short geodesics $\gamma_1, \gamma_2, \gamma_3$, such that each pair of geodesics intersect in exactly two points and the connected components $M \setminus \bigcup_{j=1}^{3} \gamma_j$ consists of exactly 8 triangles, see Figure 4.6.

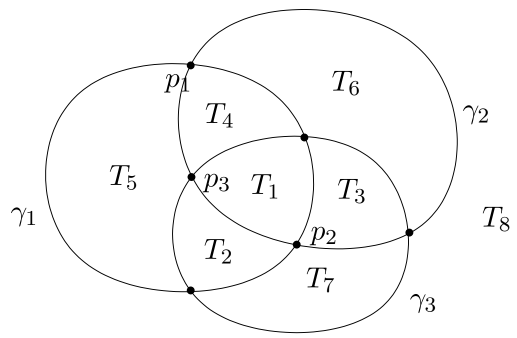

Figure 4.6: Three-circle configuration.

Lemma 4.14. *Let $\gamma_1, \gamma_2, \gamma_3 \subset M$ comprise a three-circle configuration and let $\{T_j\}_{j=1}^{8}$ be the collection of triangles associated to the configuration. If a triangle T_j for some $1 \leq j \leq 8$ is trivial, then the length of any of the three geodesic arcs that comprise ∂T_j is at most $\lambda(M)/7$.*

Proof. In what follows, we write the juxtaposition of arcs to denote the closed curve comprised by concatenating the arcs in counterclockwise direction. Suppose that a triangle $T := T_j$ for some $1 \leq j \leq 8$ is trivial and let the labeling be as given in Figure 4.7, where $\partial T = x_2 z_2 y_2$. The geodesics $\gamma_1, \gamma_2, \gamma_3$ are labeled $\gamma_s, \gamma_t, \gamma_k$ with $1, 2, 3$ some permutation of the letters s, t, k, where arcs $x_i \subset \gamma_s, y_i \subset \gamma_t$ and $z_i \subset \gamma_k$ with $1 \leq i \leq 3$.

Define the simple closed curves

$$\alpha_1 = x_2 x_3 y_3 y_2, \ \alpha_2 = x_1 x_2 z_2 z_1 \text{ and } \alpha_3 = y_2 y_1 z_3 z_2. \tag{4.27}$$

By Lemma 4.2 (iii), the simple closed curves α_i, with $1 \leq i \leq 3$, are non-trivial. Further, as each of the three geodesics γ_s, γ_t and γ_k is $6\delta_0$-short (by definition 4.4),

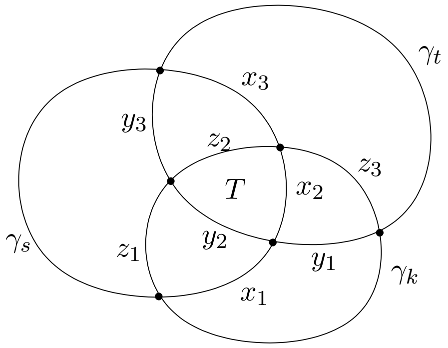

Figure 4.7: Proof of Lemma 4.14.

by the two-circle Lemma we thus have that

$$\lambda(M) \leq |\alpha_i| \leq (1 + 18\delta_0)\lambda(M), \tag{4.28}$$

for $1 \leq i \leq 3$. Next, consider the simple closed curves

$$\beta_1 = x_3y_3z_2,\ \beta_2 = x_1y_2z_1 \text{ and } \beta_3 = y_1z_3x_2. \tag{4.29}$$

As the curves α_i are non-trivial, and the triangle T is trivial, the curve β_i is homotopic to α_i, for $1 \leq i \leq 3$, and therefore non-trivial. Moreover, as T is trivial, by the triangle-inequality applied to ∂T, we have that

$$|x_2| \leq |y_2| + |z_2|,\ |y_2| \leq |x_2| + |z_2| \text{ and } |z_2| \leq |x_2| + |y_2|. \tag{4.30}$$

Combining (4.28) with (4.30), it is readily verified that

$$\lambda(M) \leq |\beta_j| \leq (1 + 18\delta_0)\lambda(M), \tag{4.31}$$

for $1 \leq j \leq 3$. Summing up the lengths of the arcs that constitute the closed curves β_1, β_2 and β_3, cf. (4.29), and reordering the terms, it follows that

$$3\lambda(M) \leq |x_1|+|x_2|+|x_3|+|y_1|+|y_2|+|y_3|+|z_1|+|z_2|+|z_3| \leq 3(1+18\delta_0)\lambda(M). \tag{4.32}$$

The length of α_1 can by (4.27) be expressed as the sum of the lengths of its constituent arcs and estimated by

$$\lambda(M) \leq |x_2| + |x_3| + |y_2| + |y_3| \leq (1 + 18\delta_0)\lambda(M). \tag{4.33}$$

Subtracting (4.33) from (4.32), we obtain the estimate

$$2\lambda(M) - 18\delta_0\lambda(M) \leq |x_1| + |y_1| + |z_1| + |z_2| + |z_3| \leq 2\lambda(M) + 54\delta_0\lambda(M) \quad (4.34)$$

However, adding up the lengths of β_2 and β_3, we must have that

$$2\lambda(M) \leq |x_1| + |x_2| + |y_1| + |y_2| + |z_1| + |z_3| \leq 2(1 + 18\delta_0)\lambda(M) \quad (4.35)$$

Therefore, subtracting (4.35) from (4.34), we obtain

$$-54\delta_0\lambda(M) \leq |z_2| - (|x_2| + |y_2|) \leq 54\delta_0\lambda(M). \quad (4.36)$$

Repeating the same argument for α_2 and α_3, one obtains

$$-54\delta_0\lambda(M) \leq |y_2| - (|x_2| + |z_2|) \leq 54\delta_0\lambda(M). \quad (4.37)$$

$$-54\delta_0\lambda(M) \leq |x_2| - (|y_2| + |z_2|) \leq 54\delta_0\lambda(M). \quad (4.38)$$

It then follows from (4.36), (4.37) and (4.38) that

$$|x_2| \leq 54\delta_0\lambda(M),\ \ |y_2| \leq 54\delta_0\lambda(M) \text{ and } |z_2| \leq 54\delta_0\lambda(M). \quad (4.39)$$

As $\delta_0 = 1/378$, it thus follows that the lengths of the arcs x_2, y_2 and z_2 have to be at most $\lambda(M)/7$. □

In Lemma 4.16 below, we prove the existence of certain three-circle configurations satisfying additional geometrical properties. The proof uses the following geometric estimate.

Lemma 4.15. *There exists a constant $1 < K_3 \leq K_2$, such that if M is K-quasiconformally homogeneous with $1 < K \leq K_3$, then the following holds. Let $\gamma_1, \gamma_2 \subset M$ be two $3\delta_0$-short geodesics intersecting at a point $p \in \gamma_1 \cap \gamma_2$, such that $\angle(\gamma_1, \gamma_2)_p \geq \pi/4$. Let $\gamma_3 \subset M$ be a geodesic and let $q_0 \in \gamma_3$. Let $f \in \mathcal{F}_K(M)$ with $f(p) = q_0$ and let γ_1', γ_2' be the geodesic homotopic to $f(\gamma_1)$ and $f(\gamma_2)$ respectively. Then both γ_1' and γ_2' are $6\delta_0$-short and at least one of γ_1' or γ_2' intersects γ_3 transversely at a point $q \in \gamma_3$ with $d(q, q_0) \leq 1/20$.*

Proof. First, by choosing $1 < K_3 \leq K_2$, we have that

$$K_3(1 + 3\delta_0) \leq 1 + 6\delta_0,$$

as $K_2 \leq K_1$ and $K_1(1 + 3\delta_0) \leq 1 + 6\delta_0$. Let $\gamma_1, \gamma_2 \subset M$ be two $3\delta_0$-short geodesics intersecting at a point $p \in \gamma_1 \cap \gamma_2$, such that $\angle(\gamma_1, \gamma_2)_p \geq \pi/4$. Let $\gamma_3 \subset M$ be a geodesic and let $q_0 \in \gamma_3$, where $f(p) = q_0$. Then the geodesics γ_1', γ_2' homotopic to $f(\gamma_1)$ and $f(\gamma_2)$ respectively are $6\delta_0$-short, as γ_1 and γ_2 are $3\delta_0$-short. By Lemma 4.5, f is approximated by a Möbius transformation on a compact disk. Further, by Lemma 4.6, the geodesic γ_i' stays close to $f(\gamma_i)$, for $i = 1, 2$. Therefore, by choosing K_3 small enough, we have that

(i) γ_1' and γ_2' intersect at a point $q_1 \in M$ with $d(q_1, q_0) \leq 1/100$, where $q_0 \in \gamma_3$, and

(ii) $\theta := \angle(\gamma_1', \gamma_2')_{q_1} \geq \frac{\pi}{5}$.

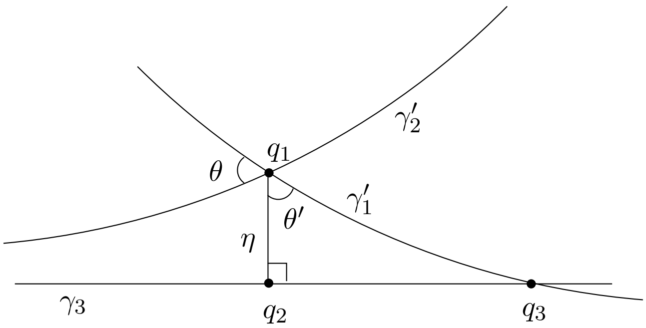

Figure 4.8: Proof of Lemma 4.15.

Let $\eta \subset M$ be the arc emanating from q_1 projecting perpendicularly onto γ_3 at the point $q_2 \in \gamma_3$. As $d(q_1, q_0) \leq 1/100$ and $q_2 \in \gamma_3$, we have that $|\eta| \leq 1/100$. Furthermore, as $\theta \geq \pi/5$, at least one of the geodesics γ_i' with $i = 1, 2$ intersects the arc η at an angle at most $2\pi/5$. Without loss of generality, we may suppose this is the case for γ_1', i.e. that

$$\theta' := \angle(\eta, \gamma_1')_{q_1} \leq \frac{2\pi}{5},$$

see Figure 4.8. If we consider the (embedded) geodesic triangle with vertices q_1, q_2 and q_3, combined with $\theta' \leq 2\pi/5$ and $|\eta| \leq 1/100$, it follows from the hyperbolic sine law that

$$d(q_1, q_3) \leq 4/100.$$

As $d(q_1, q_0) \leq 1/100$, we have that $d(q_0, q_2) \leq 1/100$. Further, we have that $d(q_2, q_3) \leq d(q_1, q_3)$ and thus

$$d(q_0, q_3) \leq d(q_0, q_2) + d(q_2, q_3) \leq 1/100 + 4/100 = 1/20.$$

Thus setting $q := q_3$ finishes the proof. $\square$

Lemma 4.16 (Three-circle Lemma)**.** *If M is K-quasiconformally homogeneous for $1 < K \leq K_3$, then there exists a three-circle configuration consisting of simple closed geodesics $\gamma_1, \gamma_2, \gamma_3 \subset M$, such that*

(i) *$\gamma_1 \cup \gamma_2$ is a two-circle configuration, and*

(ii) *γ_3 is a $6\delta_0$-short geodesic intersecting the arc $\eta_3 \subset \gamma_2$ at a point p_3 for which*

$$\left(\frac{1}{4} - \frac{1}{20}\right) \lambda(M) \leq d(p_j, p_3) \leq \left(\frac{1}{4} + \frac{1}{20}\right) \lambda(M),$$

with $j = 1, 2$, in the labeling of Figure 4.2, and

(iii) *γ_3 intersects the interior of the arc η_i in exactly one point for every $1 \leq i \leq 4$.*

Proof. As $K_3 \leq K_1$, by Lemma 4.9, there exists a two-circle configuration, comprised of two δ_0-short geodesics $\gamma_1, \gamma_2 \subset M$. Label the configuration according to Figure 4.2. Mark a point $q \in \eta_3 \subset \gamma_2$ such that $d(q, p_1) = d(q, p_2)$. As $K_3 \leq K_2$, by Lemma 4.11, there exist two $3\delta_0$-short geodesics γ_3, γ_4 intersecting at a point $p \in M$, such that

$$\angle(\gamma_3, \gamma_4)_p \geq \pi/4.$$

Applying Lemma 4.15 to γ_3 and γ_4 and the target point $q \in \eta_3 \subset \gamma_2$, there exists a $6\delta_0$-short geodesic γ' intersecting γ_2 transversely at a point $q' \in \gamma_2$ such that

$d(q,q') \leq 1/20$. Therefore, setting $p_3 := q'$ and $\gamma_3 := \gamma'$, the conditions (i) and (ii) of Lemma 4.16 are satisfied.

We are left with showing that condition (iii) of Lemma 4.16 is satisfied. That is, we need to show that γ_3 intersects the interior of the arc η_i in exactly one point for every $1 \leq i \leq 4$. To this end, we first show that γ_3 can not intersect the arc η_3 (including the boundary points p_1 and p_2) more than once. To show this is indeed impossible, suppose that γ_3 intersects the arc η_3 in a point $p' \subset \eta_3$ other than p_3. Let $\alpha_1, \alpha_2 \subset \eta_3$ be the connected components of $\eta_3 \setminus \{p_3\}$. We may assume that $p' \subset \alpha_1$, the case when $p' \in \alpha_2$ is similar. Therefore, if we let $\alpha' \subset \alpha_1$ the subarc with endpoints p_3 and p', where we include the case that $p' = p_1$, then it follows that

$$|\alpha'| \leq \left(\frac{1}{4} + \frac{1}{20}\right)\lambda(M).$$

The two points p_3 and p' cut γ_3 into two component arcs β_1 and β_2, one component of which is of length at most $(1+6\delta_0)\lambda(M)/2$; without loss of generality, we may suppose this is the case for β_1. Then the closed curve $\alpha' \cup \beta_1$ is homotopically nontrivial and

$$|\alpha' \cup \beta_1| \leq \left(\frac{1}{4} + \frac{1}{20}\right)\lambda(M) + \frac{(1+6\delta_0)\lambda(M)}{2} = \left(\frac{3}{4} + \frac{1}{20} + 3\delta_0\right)\lambda(M) < \lambda(M),$$

as $\delta_0 = 1/378$, which is a contradiction. Therefore, γ_3 intersects γ_2 at the point p_3, but does not intersects the arc η_3 in any point other than p_3, and γ_3 does not pass through p_1 or p_2.

As γ_3 intersects γ_1, by Lemma 4.2 (ii), there has to exist at least one more intersection point of γ_3 with γ_2. By the above argument, all other intersection points are contained in the interior of the arc η_4. By the two-circle Lemma, applied to $\delta = 6\delta_0$, γ_3 intersects γ_2 only twice, and therefore γ_3 intersects the interior of η_4 exactly once. Similarly, as γ_3 intersects the arc η_3 exactly once, γ_3 has to intersect the interior of the arc η_1 and η_2 at least once. Again by the two-circle Lemma, applied to $\delta = 6\delta_0$, as the total number of intersection points of γ_3 with γ_1 is exactly two, γ_3 has to intersect the interior of the arc η_1 and η_2 exactly once. Thus condition (iii) is indeed satisfied and this proves the Lemma. $\square$

4.4.5 Proof of Theorem 4.B.

The endgame of the proof of Theorem 4.B. is a combinatorial argument layered on the three-circle configuration of the Three-Circle Lemma. Thus, let $1 < K \leq K_3$ with K the quasiconformal homogeneity constant of M, with K_3 the constant which we obtained in the Three-Circle Lemma in order to ensure the existence of a three-circle configuration.

Lemma 4.17. *Let $\gamma_1, \gamma_2, \gamma_3$ be the three-circle configuration of Lemma 4.16. Then the triangle T_j is non-trivial, for every $1 \leq j \leq 8$.*

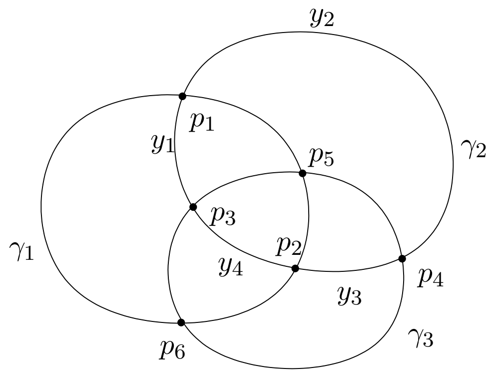

Figure 4.9: Proof of Lemma 4.17.

Proof. By Lemma 4.16, the $6\delta_0$-short geodesic γ_3 intersects γ_2 at a point $p_3 \in \eta_3 \subset \gamma_1$ and

$$\left(\frac{1}{4} - \frac{1}{20}\right) \lambda(M) \leq d(p_k, p_3) \leq \left(\frac{1}{4} + \frac{1}{20}\right) \lambda(M), \tag{4.40}$$

with $k = 1, 2$, as given in Figure 4.9. Let y_i with $1 \leq i \leq 4$ be the connected components of $\gamma_2 \setminus \{p_1, p_2, p_3, p_4\}$. It suffices to show that

$$|y_i| > \frac{\lambda(M)}{6}, \text{ for } 1 \leq i \leq 4. \tag{4.41}$$

To prove sufficiency, note that every triangle ∂T_j, with $1 \leq j \leq 8$, contains exactly one edge y_i for some $1 \leq i \leq 4$. Now, if (4.41) holds, then by Lemma 4.14, ∂T_j has to be non-trivial, as otherwise all three edges of ∂T_j have to be of length less than $\lambda/7 < \lambda/6$.

To prove (4.41), first note that, by (4.40), the arcs y_1 and y_4 satisfy requirement (4.41). Therefore, we are left with proving the estimate for y_2 and y_3. As both γ_1 and γ_2 are δ_0-short, and γ_3 is $6\delta_0$-short, the two-circle Lemma applied to the pairs of geodesics γ_1, γ_2 and γ_2, γ_3 gives respectively

$$\frac{\lambda(M)}{2} - \frac{\delta_0 \lambda(M)}{2} \leq |y_2| + |y_3| \leq \frac{\lambda(M)}{2} + \frac{3\delta_0 \lambda(M)}{2}, \tag{4.42}$$

and

$$\frac{\lambda(M)}{2} - \frac{6\delta_0 \lambda(M)}{2} \leq |y_3| + |y_4| \leq \frac{\lambda(M)}{2} + \frac{18\delta_0 \lambda(M)}{2}. \tag{4.43}$$

Combining (4.42) and (4.43), it follows that

$$-\frac{19\delta_0 \lambda(M)}{2} \leq |y_2| - |y_4| \leq \frac{19\delta_0 \lambda(M)}{2}. \tag{4.44}$$

As $|y_4| = d(p_3, p_2)$, combining (4.44) with (4.40), one obtains

$$|y_2| \geq \lambda(M) \left(\frac{1}{4} - \frac{1}{20} - \frac{19\delta_0}{2} \right) > \frac{\lambda(M)}{6}, \tag{4.45}$$

as $\delta_0 = 1/378$. By symmetry, the same estimate holds for the arc $|y_3|$. This concludes the proof. $\square$

Let us now conclude the proof.

Proof of Theorem 4.B. Let M be K-quasiconformally homogeneous with $1 < K \leq K_3$ and let $\gamma_1, \gamma_2, \gamma_3$ be the three-circle configuration of Lemma 4.16. By Lemma 4.17, all triangles T_j with $1 \leq j \leq 8$ have to be non-trivial. Therefore, the length of ∂T_j has to be at least $\lambda(M)$ for every $1 \leq j \leq 8$. Adding up the lengths of all ∂T_j, $1 \leq j \leq 8$, means we count every boundary arc of a triangle T_j with multiplicity two and thus

$$2 \sum_{i=1}^{3} |\gamma_i| = \sum_{j=1}^{8} |\partial T_j| \geq 8\lambda(M). \tag{4.46}$$

However, as the geodesics γ_i, with $1 \leq i \leq 3$, are (at most) $6\delta_0$-short by construction, the total length of these three geodesics counted with multiplicity two is bounded by

$$2\sum_{i=1}^{3} |\gamma_i| \leq 2 \cdot 3(1+6\delta_0)\lambda(M) = 6(1+6\delta_0)\lambda(M) < 7\lambda(M), \tag{4.47}$$

as $36\delta_0 < 1$. The contradictory claims (4.46) and (4.47) finish the proof. $\square$

4.5 Open problems

The first open problem relates to our proof of Theorem 4.B. We proved the existence of a universal lower bound $\mathcal{K}$ on the quasiconformality constant of a hyperbolic genus zero surface. However, the proof, as it stands, does not give precise estimates of the numerical value $\mathcal{K}$ (compare the example in section 4.4.1).

Open problem 5 (An explicit lower bound)**.** *Determine the numerical value of the universal constant $\mathcal{K}$ whose existence was proved in Theorem 4.B.*

Beyond this problem, there is the case of closed Riemann surfaces of genus $\geq$ 2. Bonfert-Taylor, Bridgeman and Canary obtained a partial result in this direction stating that a closed hyperbolic surface admitting a conformal automorphism with "many" fixed points is quasiconformally homogeneous, with the quasiconformality constant being uniformly bounded away from 1. For example, all hyperelliptic surfaces, admitting involutions with $2(g+1)$ fixed points, satisfy this fixed-point condition, see [9] for the details. However, the case of general closed surfaces of genus ≥ 2 remains unanswered.

Open problem 6 (Quasiconformal homogeneity of closed surfaces)**.** *Does there exist a universal constant $\mathcal{K}' > 1$ such that if M is a K-quasiconformally homogeneous closed surface of genus ≥ 2, then $K \geq \mathcal{K}'$?*

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