Purchasing Speculative Inventory for Price Sensitive Demand

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Abstract

The problem studied is one of buying and selling products cost efficiently over a number of periods in a finite horizon setting. Unit purchase costs vary across periods acording to some known distribution.and demand is deterministic but dependent on the price charged for the product. Thus, the problem becomes one of exploiting opportunities to "forward buy" and sell profitably in the face of costs for carrying product.

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1 Introduction

The vast majority of inventory models assume the purchase cost of an item is either constant and exogenous or the result of some type of bargaining agreement between supply chain partners. The common assumption of a constant purchase cost is valid as long as the possible change in the purchasing cost from one period to the next is less than the per-period inventory holding cost. Globalization, with its corresponding exposure to fluctuating exchange rates, along with the increasing volatility of commodity prices, have made such constant cost assumptions increasingly less realistic in practice however. Thus, many firms purchase inventory when the purchase cost is low, in anticipation (or speculation) the purchase cost will be higher in future periods. The problem studied is one of buying and selling inventory cost efficiently over a number of periods in a finite horizon setting. Unit purchase costs vary across periods according to some known distribution and demand is deterministic but dependent on the price charged for the product. Thus, the problem is one of exploiting opportunities to "forward buy" and sell profitably in the face of inventory carrying cost and stochastic purchase cost.

Magirou (1982) and Golabi (1985) were the first to study the problem of speculative buying under stochastic future purchasing cost. Both assume demand is known with certainty over an infinite horizon and develop critical cost policies that balance the cost of holding excess inventory with the opportunity cost of not taking advantage of a lower than average cost in the current period. The policies provide cost thresholds such that, if the realized cost in a given period falls between cost thresholds t and $t+1$, the firm is advised to purchase enough units to bring its inventory level up to the point where it can satisfy demand over the next t periods. Magirou (1987) shows the policies developed in Magirou (1982) and Golabi (1985) are equivalent under certain conditions. Krishna (1992) extends Golabi's model to the multi-brand case by collapsing a multivariate distribution of costs across a set of brands into a univariate distribution.

Li and Kouvelis (1999) consider buyer-supplier contracts under deterministic demand and stochastic future purchase costs. They use a geometric Brownian motion to model the purchase cost process and assume a deterministic and exogenous total demand quantity must be met by some known future period. Under these conditions, it is optimal for the firm to make a single purchase of the total quantity required at some period in the horizon before the demand becomes due. Gavirneni (2004) provides an exact algorithm for determining optimal order quantities when both demand and purchase costs are stochastic. He proves the optimality of an order-up-to policy for this situation and suggest a closed-form heuristic based on a myopic newsvendor equation.

We model the firm's problem as a finite horizon dynamic program where, in each period, the optimal purchase and selling quantity is determined by balancing the current period savings in purchase cost against the cost of holding inventory and the potential to purchase the product at a lower cost in future periods. As opposed to the papers cited above, in our model, demand occurs each period, is deterministic, but is price sensitive. Thus, the firm chooses a price and purchase quantity each period after observing the cost realization for that period. We prove the firm's optimal policy in this situation is to manage a set of "pipeline" inventory levels for each future period where inventory purchased in a period is immediately allocated to either the current period or some particular future period within the planning horizon. Surprisingly, it is never optimal for the firm to decrease its pipeline inventory levels, even when faced with an unusually high cost realization in the current given period. Thus, pipeline inventory levels only increase over time until the time period arrives when they are allocated to be sold. The use of pipeling inventory in our model results in a significant computation reduction over the full model.

The rest of this paper is organized as follows. In §2 we present our notation and formulate the model. In §3 we provide our main result regarding the optimality of pipeline inventories that never decrease over time. In §4 we present an upper bound on the planning horizon and show how much our proposed model reduces the computational complexity versus the original problem. In §5 we provide a numerical test to measure the impact of forward buying on expected profits and to measure the sensitivity of the model's parameter values. In §6 we conclude and in an Appendix we provide the details of the proof from §3.

2 Model Setup and Formulation

The state variables are the inventory, I, and the unit purchase price, c. Both variables are observed at the beginning of each period. After having observed the state, decisions are made on the amount purchased, z, and the amount sold, d. Inventory balance holds that $I + z = A + d$ where A is the amount of inventory at the end of the period and thus the inventory state in the next period. In the following it is tacitly assumed that all variables (states and decision variables) that relate to quantities of product are nonnegative. Thus, $I, z, A = I + z - d$, and $d \geq 0$ in all considerations, even when such is not stated explicitly

The total of the one-step costs-revenue, γ, consist of three components: the total purchase costs, $z\,c$, holding costs, $A\,h = (I + z - d)\,h$, and the revenue, $r(d)$, obtained from selling d units. So, the net costs are

$$\gamma(c, I; z, d) = z\,c + (I + z - d)\,h - r(d) \tag{1}$$

We will assume that $d \mapsto r(d)$ is concave. The one step costs-revenue function then becomes convex as a function of the variables that relate to quantities of product. Our restriction on the revenue function is very loose and includes the following common demand functions:

1. Linear demand function

One particular example for a concave revenue function is $r(d) = d\,\frac{a-d}{b}$, where a and b are positive constants. For this choice of revenue function, the unit selling price, p, is $p(d) = \frac{r(d)}{d} = \frac{a-d}{b}$, which leads to the demand function $d(p) = a - b\,p$.

2. Multiplicative demand function

Consider the demand function $d(p) = ap^{-b}$ where a and b are both positive constants. Then the inverse demand function is $p(d) = a^{\frac{1}{b}}\,d^{-\frac{1}{b}}$ and the revenue function becomes the following.

$$r(d) = d\,p(d) = a^{\frac{1}{b}}\,d^{1-\frac{1}{b}}$$

So $d \mapsto r(d)$ is concave provided $0 \leq 1 - \frac{1}{b} \leq 1$, that is, $b \geq 1$. When $b > 1$ we have to assume in addition that the purchase price is never zero (let alone negative) as we can otherwise achieve infinite revenue at no costs by letting $d \uparrow \infty$. When $0 < b < 1$ we would be in the 'perverse' situation that we can obtain infinite revenue by letting $p \uparrow \infty$. Put otherwise: $\lim_{d\downarrow 0} r(d) = \lim_{d\downarrow 0} a^{\frac{1}{b}}\,d^{1-\frac{1}{b}} = +\infty$ when $0 < b < 1$.

3. Exponential demand function

Consider the demand function $d(p) = ab^{-p}$ where a and b are positive constants. The inverse demand function is $p(d) = -\frac{\ln\frac{d}{a}}{\ln b}$ and the revenue function is

$$r(d) = d\,p(d) = -d\,\frac{\ln\frac{d}{a}}{\ln b} = d\,\frac{\ln\frac{a}{d}}{\ln b}$$

The revenue function is concave provided $b > 1$. Writing $b = \exp(\beta)$ (with $\beta > 0$), the demand function takes the form $d(p) = ae^{-\beta\,p}$.

The inventory state evolves according to the inventory balance equation $A = I - z - d$. The unit purchase price develops as an iid process and the distribution of the purchase price is given by the random variable C. Expectation with respect to C is written as E. In each period t, the sequence of events is as follows: 1) the firm starts the period with I units of inventory, 2) the current period cost is realized $C = c$, 3) the firm purchases z additional units with zero lead-time, and 4) the firm sets a selling price of p such that d units of demand are satisfied. Thus, the firm's decision variables each period starting with inventory level I are the number of additional units to purchase, z, given a realized per unit cost of c and the number of units to sell, d. The firm's value function (over periods $0, 1, \cdots, T$) is

$$J^{\text{tot}}_{t,T}(c, I) = \begin{cases} \min_{z,d: I+z-d\geq 0} \Big(\gamma(c, I; z, d) + E[J^{\text{tot}}_{t+1,T}(C, I + z - d)]\Big) & \text{if} \quad t < T \\ \min_{z,d: I+z-d=0} \gamma(c, I; z, d) & \text{if} \quad t = T \end{cases} \tag{2}$$

where T is the finite horizon. Note that for period T, we have $A = I + z - d = 0$ meaning that ending inventory is forced to be zero: no inventory is allowed to be carried past the horizon.

For small problems, (2) can be solved using backward induction. The solution to (2) provides an optimal inventory level to hold in each period t given every possible realization of C. Such a solution technique however, is impractical for most realistic problems. In addition, a firm may wish to know how far in the future to set its planning horizon, T, such that decisions in the current period are independent of the choice of T. In the next section, we provide an alternative formulation that addresses these issues.

3 The Pipeline Model

In this section we present the main result of this paper by establishing properties of the optimal policy that dramatically reduce the state space needed to be searched for each recursion. We do so by formulating the problem as a forward recursion and proving that for any future period, the amount of allocated inventory reserved for that period only increases as the period is approached in a rolling horizon. We call the inventory purchased either in or prior to period t that is reserved for period $\tau, \tau \geq t$, the Pipeline τ.

The value function at time $t \leq \tau$ for Pipeline τ is specified as:

$$Y_{t,\tau}(c, I) = \begin{cases} \min_z \left(\gamma(c, I; z, 0) + E[Y_{t+1,\tau}(C, I + z)]\right) & \text{if} \quad t < \tau \\ \min_{z,d: z+I=d} \gamma(c, I; z, d) & \text{if} \quad t = \tau \end{cases} \tag{3}$$

Note the Pipeline τ disallows demand (sales) prior to the Period τ, i.e. Pipeline τ only has a demand option in Period τ.

Define the aggregate of pipelines, Z, as $Z_{t,T}(c, i_t, \cdots, i_T) = \sum_{\tau=t}^{T} Y_{t,\tau}(c, i_\tau)$. The value i_τ is Pipeline τ's starting inventory in period t. We are particularly interested in the value of the aggregate under an optimal allocation of a stock I. Write the cost aggregate achieved by optimal allocation as Z^*:

$$Z^*_{t,T}(c, I) = \min_{\substack{i_t, \cdots, i_T: \\ \sum_{\tau=t}^{T} i_\tau = I}} Z_{t,T}(c, i_t, \cdots, i_T)$$

We will prove that

$$J_{t,T}^{\text{tot}}(c, I) = Z_{t,T}^*(c, I) \tag{4}$$

Equation (4) states that by optimal allocation of the total stock I, we can compute the optimal cost for the total model from aggregation of the optimal costs for pipelines. We will prove the relationship (4) by showing that $Z_{t,T}^*(c, I)$ solves the recursions (2) that define $J_{t,T}^{\text{tot}}(c, I)$. See Figure 1 for a pictorial interpretation of the pipelines.

Insert Figure 1 Here

3.1 Proof of the Main Result

In the remainder of this section, we present a proof for our main result given in (4). In doing so, we frequently encounter tuples of quantities indexed by values from $[t, \cdots, T]$. An example of this is $(i_t, \cdots, i_T)$. We will write $i_{[t,T]}$ for such quantities:

$$i_{[t,T]} = (i_t, \cdots, i_T)$$

We will also use the notation

$$(i + z)_{[t,T]} = (i_t + z_t, \cdots, i_T + z_T) \quad \text{and} \quad |i_{[t,T]}| = \sum_{\tau=t}^{T} i_\tau$$

3.1.1 Recursion for the aggregate

In this section we present the challenge involved in proving that $Z_{t,T}^* = J_{t,T}^{\text{tot}}$. To this end, we develop the recursion that holds for $Z_{t,T}^*$. We start by noting the simple but crucial result for the aggregation of the one-step cost-revenue shown in the following equation.

$$\gamma(c, i_t; z_t, d) + \sum_{\tau=t+1}^{T} \gamma(c, i_\tau; z_\tau, 0) = \gamma\left(c, \sum_{\tau=t}^{T} i_\tau; \sum_{\tau=t}^{T} z_\tau, d\right) \tag{5}$$

From the recursions (3), we therefore obtain,

$$Z_{t,T}(c, i_{[t,T]}) = \min_{\substack{z_{[t,T]}, d: \\ z_t + i_t = d}} \left(\gamma\left(c, |i_{[t,T]}|; |z_{[t,T]}|, d\right) + E[Z_{t+1,T}(C, (i+z)_{[t+1,T]}))\right]$$

with the understanding that $Z_{T+1,T}(.,.) = 0$. Here z_τ is the dedicated amount bought for Pipeline τ. For the aggregate under optimal allocation of $I \geq 0$ we obtain,

$$Z^*_{t,T}(c, I) = \min_{\substack{d, i_{[t,T]}, z_{[t,T]}: \\ z_t + i_t = d \\ |i_{[t,T]}| = I}} \left(\gamma\left(c, I; |z_{[t,T]}|, d\right) + E[Z_{t+1,T}(C, (i+z)_{[t+1,T]}))\right] \tag{6}$$

Note that $i_\tau + z_\tau$ represents the end-of-period inventory for Pipeline τ. In order to use induction reasoning we must put the second term in the minimization, $EZ_{t+1,T}(C, (i+z)_{[t+1,T]})$, into a form where we are allocating the total stock. To this end, we introduce $\alpha_\tau = i_\tau + z_\tau$. Then

$$|\alpha_{[t+1,T]}| = \sum_{\tau=t+1}^{T} \alpha_\tau = A = I + z - d$$

is the decomposition of the end-of-period inventory A (which is the beginning-of-period inventory for Period $t+1$). Here $z = |z_{[t,T]}|$ is the aggregate of the amounts bought for the pipelines. Using the newly introduced variables we can restate Equation (6) as

$$Z^*_{t,T}(c, I) = \min_{\substack{d, z, A, \alpha_{[t+1,T]}: \\ z + I = d + A \\ |\alpha_{[t+1,T]}| = A}} \left(\gamma(c, I; z, d) + E[Z_{t+1,T}(C, \alpha_{[t+1,T]}))\right] \tag{7}$$

$$= \min_{\substack{d, z, A: \\ z + I = d + A}} \left(\gamma(c, I; z, d) + \min_{\substack{\alpha_{[t+1,T]}: \\ |\alpha_{[t+1,T]}| = A}} E[Z_{t+1,T}(C, \alpha_{[t+1,T]})]\right) \tag{8}$$

Here the Equality in (7) derives from Equality (6) under the observation that, for any fixed I,

$$\left\{(|z_{[t,T]}|, d, (i+z)_{[t+1,T]}) \,\middle|\, z_t + i_t = d\,;\, |i_{[t,T]}| = I\right\} = \left\{(z, d, \alpha_{[t+1,T]}) \,\middle|\, z + I = d + A\,;\, |\alpha_{[t+1,T]}| = A\right\}$$

Note that using the variables $(|z_{[t,T]}|, d, (i+z)_{[t+1,T]})$ states the optimization problem in terms of

- dedicated amounts bought for pipeline $(z_{[t,T]})$,
- demand for Period t (d),
- amounts of anticipatory stock allocated to pipelines $((i+z)_{[t,T]})$ where the amounts are based on the specification of the allocation of initial stock $(i_{[t,T]})$ subject to $|i_{[t,T]}| = I$ and dedicated amounts bought for the pipelines $(z_{[t,T]})$.

Whereas use of the variables $(z, d, \alpha_{[t+1,T]})$ states the optimization problem in terms of

- an aggregate amount bought (z),
- demand for Period t (d),
- amounts of anticipatory inventory for pipelines, ($\alpha_{[t+1,T]}$) where this allocation is subject only to the aggregate constraint $|\alpha_{[t+1,T]}| = A = I + z - d$.

The equality of Expression (6) and Expression (7) states the equivalence of minimizing over $(|z_{[t,T]}|, d, (i+z)_{[t+1,T]})$ and $(z, d, \alpha_{[t+1,T]})$ under the appropriate constraints on aggregate inventory and the nonnegativity of variables. In Expression (8) the future costs are once more expressed through an optimal allocation of an aggregate amount, A:

$$\min_{\substack{\alpha_{[t+1,T]}: \\ |\alpha_{[t+1,T]}|=A}} E[Z_{t+1,T}(C, \alpha_{[t+1,T]})]$$

Suppose we already knew that, for any c, $\min_{\substack{\alpha_{[t+1,T]}: \\ |\alpha_{[t+1,T]}|=A}} Z_{t+1,T}(c, \alpha_{[t+1,T]}) = J^{\text{tot}}_{t+1,T}(c, A)$. If we were able to show that

$$\min_{\substack{\alpha_{[t+1,T]}: \\ |\alpha_{[t+1,T]}|=A}} E[Z_{t+1,T}(C, \alpha_{[t+1,T]})] = E[\min_{\substack{\alpha_{[t+1,T]}: \\ |\alpha_{[t+1,T]}|=A}} Z_{t+1,T}(C, \alpha_{[t+1,T]})], \tag{9}$$

then from (8) we would get $Z^*_{t,T}(c, I) = \min_{\substack{d,z,A: \\ z+I=d+A}} \left(\gamma(c, I; z, d) + E[Z^*_{t+1,T}(C, A)]\right)$ and we could conclude that $Z^*_{t,T}(C, I)$ satisfies (2). So, provided we have (9), an induction argument would yield that $Z^*_{t,T}(C, I)$ satisfies (2). The challenge in proving that $Z^*_{t,T}(C, I)$ solves (2) thus lies in establishing (9). Equation (9) requires that we push a minimum operation through an expectation operation. Generally

$$\min_y E_X H(X, y) \geq E_X \min_y H(X, y)$$

and equality only holds if $\text{argmin}_y H(x, y)$ is independent of x (for almost all x). So we need to prove that for each $I \geq 0$, there exist values $i^*_{[t,T]}$, independent of the purchase price c, such that

$$\min_{\substack{i_{[t,T]}: \\ |i_{[t,T]}|=I}} Z_{t,T}(c, i_{[t,T]}) = Z_{t,T}(c, i^*_{[t,T]}) \tag{10}$$

3.1.2 Independence of the purchase price

To show the pipeline allocation is independent of any realized purchase price, we start with a worst case scenario for a given purchase price realization in the current period. Let $i^*_{[t,T]}$ be the optimal allocation under the bad news (= extremely high purchase price) scenario. That is, let $i^*_{[t,T]}$ with

$|i^*_{[t,T]}| = I$ satisfy

$$\gamma\,(c, I; 0, i_t^*) + E[Z_{t+1,T}(C, i^*_{[t+1,T]})] = \min_{\substack{i_{[t,T]}: \\ |i_{[t,T]}|=I}} \left(\gamma\,(c, I; 0, i_t) + E[Z_{t+1,T}(C, i_{[t+1,T]})\right)] \tag{11}$$

As $\gamma(c, I; 0, i_t^*)$ and $\gamma(c, I; 0, i_t)$ do not depend on c (as nothing is bought), the values $i^*_{[t,T]}$ defined through (11) do not depend on c. We make the dependence of i^*_τ on I apparent by writing $i^*_\tau = i^*_\tau(I)$.

Given a high purchase price realization in the current period, one may conjecture it may be optimal to reallocate previously purchased units reserved for future periods back to the current period so as to minimize the number of units that must be purchased at the high cost realization. This conjecture is incorrect however; it is never optimal to subtract from a pipeline inventory, only to add to it. To show this, we need to show the allocation $I \mapsto i^*_\tau(I)$ is a good allocation of the stock I to Pipeline τ. Good here means that we will not regret this allocation after knowing the current purchase price. Suppose that, under the current purchase price, we wish to buy z units for the total of all the pipelines (or for the aggregate model). We will not regret our allocation of the initial stock I if an optimal allocation of $I + z$ is achieved by adding some fraction of the quantity purchased to the a priori allocated stock $i^*_\tau(I)$ for each Pipeline τ. That is, we ask that

$$i^*_\tau(I + z) \geq i^*_\tau(I) \quad \text{ for all } \tau \tag{12}$$

whenever $z \geq 0$. Indeed, whenever (12) holds we can allocate a *nonnegative* quantity z_τ to Pipeline τ where $z_\tau = i^*_\tau(I + z) - i^*_\tau(I)$. Note that $\sum_{\tau=t}^{T} z_\tau = z$. Condition (12) states that a minimizing allocation is monotonous in the total quantity being allocated: any amount purchased will only add to the previously allocated quantities by some amount. A well-established route for establishing monotonicity for minimizers is through submodularity. Indeed, this is the route taken here. Underlying the submodularity is convexity of the one-step cost function. The proof is rather involved so we present this argument in the appendix. The structure of the proof however, contains the two major steps below:

Step 1. Show the monotonicity property of Equation (12) implies the independence property of Equation (10). This requires some specifics of the model at hand. The crucial part is in the proof of Theorem 7.1.2 (provided in the appendix).

Step 2. Show that Inequality (12) is true. This requires some less specific features of the model at hand. Inequality (12) rests mainly on convexity properties of values functions. It holds when the one-step costs is convex and the constraints under which optimization takes place are ‘nice’ (such as linear for our case).

3.1.3 Main result

We are now ready to prove the result formulated in (4).

Theorem 1 $J^{tot}_{t,T}(c, I) = Z^*_{t,T}(c, I)$

Proof Corollary 3 (in the appendix) has established (9) and the theorem is now immediate from the remarks after (9). ■

Now that we have established our main result, we discuss how to calculate a minimum planning horizon.

4 Planning Horizon and Computational Effort

In this section we demonstrate how to solve for an upper bound on the number of periods the planning horizon must include and the reduction in the computational effort our algorithm provides over solving the full problem. We begin with an upper bound for the planning horizon. An upper bound on the length of the planning horizon, call it T_{bound}, is such that for any current period t, the optimal policy for period t does not change if the problem is solved for any horizon greater than $t + T_{\text{bound}}$.

We use the example case of a linear demand for our derivation of T_{bound}. Recall the linear demand function as $d(p) = a - bp$ so the inverse demand function is $p(d) = \frac{a-d}{b}$. This demand function is common in the literature and has been shown empirically to be a good representation of the demand/quantity relationship for small changes in the selling price (Lilien et al.; 1992). We assume that $a, b > 0$. We also assume the beginning inventory, I, is not excessively large such that the firm will wish to dispose of inventory. This ensures $0 \leq d \leq \frac{a}{2}$. We can now write the one-step costs expression (1) as:

$$\gamma(c, I; z, d) = \frac{1}{b}\left(z\, bc + (I + z - d)\, bh - (a - d)\, d\right) \tag{13}$$

The one-step costs expression is used in the value function (2). Consider the last cost in our one step cost expression,

$$-p(d)\, d = -\frac{a-d}{b} d \quad \text{for} \quad 0 \leq d \leq a.$$

Let the purchase price be c in the current period and suppose we do not have any starting inventory. How much product do we, a priori, consider to purchase for now and later periods?

1. For the current period we wish to buy an amount, d_0, up to the point where the marginal profit equals the marginal cost:

$$\frac{a - 2d_0}{b} = c$$

Thus, $d_0(c) = \frac{a-cb}{2}$.

2. For a bucket k periods into the future we wish to buy d_k where

$$\frac{a - 2d_k}{b} = c + kh$$

Thus, $d_k(c) = \frac{a-(c+kh)b}{2}$ (assuming this is nonnegative).

Therefore, the period furthest in the future for which we consider a buy is H_{spec} given as

$$H_{\text{spec}}(c) = \min\{T, \max\{k | a - (c + kh)b \geq 0\}\}.$$

An upper bound on the amount that we will order up to is

$$I' = \sum_{k=0}^{H_{\text{spec}}(c)} d_k = \sum_{k=0}^{H_{\text{spec}}(c)} \frac{a - (c + kh)b}{2} = (H_{\text{spec}}(c) + 1)\frac{a - cb}{2} - \frac{hb}{2} \sum_{k=1}^{H_{\text{spec}}(c)} k$$

or

$$I' = \frac{H_{\text{spec}}(c) + 1}{2} \left(a - cb - \frac{hbH_{\text{spec}}(c)}{2} \right).$$

As $a - cb - H_{\text{spec}}hb \geq 0$ we have $H_{\text{spec}}(c) \leq \frac{a-cb}{hb} \equiv H_{\max}(c)$ and from $a - cb - (H_{\text{spec}}(c) + 1)hb \leq 0$ we obtain $H_{\text{spec}}(c) \geq H_{\max}(c) - 1$. Therefore

$$\begin{aligned} I' &\leq \frac{\min\{T, H_{\max}(c)\} + 1}{2} \left(a - cb - \frac{hb(H_{\max}(c) - 1)}{2} \right) \\ &= \frac{hb}{4} \left(\min\{T, H_{\max}(c)\} + 1 \right) \left(H_{\max}(c) + 1 \right) \equiv I_{\max}(c, T). \end{aligned}$$

The inventory carried into the future is bounded by

$$\begin{aligned} I_{\max}(c, T) - d_0(c) &= \frac{hb}{4} \left(\left(\min\{T, H_{\max}(c)\} + 1 \right) \left(H_{\max}(c) + 1 \right) - \frac{H_{\max}}{2} \right) \\ &= \frac{hb}{4} \left(\left(\min\{T, H_{\max}(c)\} + \frac{1}{2} \right) \left(H_{\max}(c) + 1 \right) + \frac{1}{2} \right) \equiv A_{\max}(c, T). \end{aligned}$$

Let $\underline{c}$ be any lower bound on the purchase costs: $\underline{c} \leq C$. Then

$$H_{\max}(\underline{c}) = \frac{a - \underline{c}b}{hb} \geq \frac{a - Cb}{hb} = H_{\max}(C)$$

and $\min\{T, H_{\max}(\underline{c})\}$ is an upper bound on the terminal period in the planning horizon. While we have derived an expression for T_{bound} using the example case of linear demand, it is similarly straightforward to do so with other concave demand functions like multiplicative and exponential.

Next, we show the reduction in the computational complexity made possible by solving our pipeline model (4) as opposed to the full model (2).

Computational Effort

To estimate the effort needed in computing an optimal solution to the speculative inventory problem, we distinguish between two classes of problems, Class 0 and Class δ.

Class 0 consists of speculative inventory problems: with a linear demand function; with decisions restricted to nonnegative integer values for the amounts purchased and sold; with rational values for all parameters, positivity of a and b, nonnegativity of h as well as the purchasing costs C; and with starting inventory equal to 0. The integer amount to consider selling in any period is bounded by the smallest integer d that satisfies $p(d+1) < p(d)$. The integer amount that may be sold in any period is bounded from above by $\lfloor (a+1)/2 \rfloor \equiv n_{\max}$. Thus, in Period t the amount of units bought under an optimal policy is bounded by $(T-t+1)\, n_{\max}$ and the amount sold in Period t is bounded by $n_{\max}$. The number of decisions that need be evaluated in Period t is then bounded by $(T-t+1)\, \#^2_{\max}$. Summed over the periods $0, 1, \cdots, T$ this gives a bound on the number of minimizations needed for solving problems in Class 0 through the full model (2) in the order of $\sum_{t=0}^{T}(T-t+1)\, n^2_{\max} = (T+1)^2\, n^2_{\max}/2$.

Now consider using the decomposition scheme through the pipeline models (4) to solve problems in Class 0. The amount bought in any period is bounded by $n_{\max}$. The amount sold in any period, except for the last, is 0. The amount sold in the last period is bounded by $n_{\max}$. Consequently, the number of value functions to be evaluated over the periods $0, 1, \cdots, T$ is in the order of $T\, n_{\max} + n^2_{\max}$. For problems where there exist a uniform bound on the number of periods one will consider carrying inventory, the computational effort is slightly less.

Class δ : For fixed $\delta > 0$, Class δ consists of all Class 0 problems for which $\delta \leq hb$. For this class of problems, $H_{\max}(c) \leq \lceil a/\delta \rceil \equiv \overline{H}$. For a problem in Class δ, one will in, any period, never buy more than $(\overline{H}+1)\, n_{\max}$ and sell more than $n_{\max}$. Thus, an upper bound on the number of minimizations needed for solving the problem using the full model (2) over the periods $0, 1, \cdots, T$ is in the order of $T\, \overline{H}\, n^2_{\max}$. As $\overline{H} = \mathcal{O}(a) = \mathcal{O}(n_{\max})$, this upper bound is of the order $T\, n^3_{\max}$. Application of the pipeline model (4) for a problem from Class δ need not extend to periods before $T - (\overline{H}+1)$. Thus, the number of minimizations needed in the computation of the value functions using the pipeline model is $\mathcal{O}(\overline{H}\, n_{\max} + n^2_{\max}) = \mathcal{O}(n^2_{\max})$. Table 1 provides a summary of the computational effort required for each model under both cases.

We now proceed to show the significance of our policy on a firm's expected profits and to test the sensitivity of the expected profit to changes in the model's parameter values.

	full model	pipelines (aggregated)
Class 0	$\mathcal{O}(T^2\, n_{\max}^2)$	$\mathcal{O}(T\, n_{\max} + n_{\max}^2)$
Class δ	$\mathcal{O}(n_{\max}^3\, T)$	$\mathcal{O}(n_{\max}^2)$

Table 1: Computational Effort

5 Impact on Profit and Sensitivity Analysis

We evaluate the financial impact and parameter value sensitivity of our model through a numerical study. To measure the financial impact, we measure the percent improvement in a firm's total expected profits over the planning horizon when the firm uses the optimal buying and selling quantities from our model versus the myopic selling quantities when the product is purchased at the spot market price each period (no forward buying). For the myopic base-case, the firm purchases and sells the optimal quantity each period based only on that period's purchase price. Because demand is deterministic, no inventory is carried between periods in the myopic base-case. For convenience, we use the linear demand model ($d(p) = a - bp$) with a constant market potential $a = 50$ and our planning horizon, T, is held constant at 5 periods. All other parameters are varied over the following ranges:

Cost distribution type $\in$ (Uniform, Normal, Negative Binomial)
Mean of the purchase cost distribution $\in$ (20, 30, 40)
Standard deviation of the purchase cost distribution $\in$ (2, 4, 6)
Price sensitivity, $b \in$ (0.25, 0.5, 1)
Per-period holding cost as a percentage of the mean cost, $h \in$ (0.1, 0.2, 0.4)

Our numerical experiment consists of a full factorial design of the parameter values above for a total of 243 comparisons. A summary of the percent improvements in expected profits is provided in Figure 2.

Insert Figure 2 Here

Over all the experiments, we find an average expected profit improvement of 9.6% from using our model versus the "no forward buying" base-case policy. The results are rather insensitive to the type of distribution used to model the purchase cost with the Uniform, Normal, and Negative Binomial distributions resulting in overall average profit improvements of 9.9%, 9.8%, and 9.2% respectively. In terms of sensitivity to the parameter values tested, the price sensitivity shows the widest range followed by the standard deviation of the purchase cost, the mean of the purchase cost, and the holding cost showing the smallest range. The parameter values chosen for this study are intended to demonstrate possible impacts and sensitivities. They are not meant to be representative of any particular industry settings. Thus, firms should make their own evaluations based on their specific market conditions.

6 Discussion and Conclusion

In this paper, we provide an important state-space reduction policy for the finite-horizon problem of choosing purchase and selling quantities each period of the horizon when demand is deterministic and price sensitive but the purchase cost are random. We prove the optimal policy consist of allocated inventory levels for each future period in the horizon and that these allocations are non-decreasing over time. Our proof only requires concavity in the revenue function, making it applicable to a wide variety of demand functions. We also demonstrate the calculation of a minimum planning horizon for the special case of linear demand. We then show through a numerical study over a wide range of parameter values that the use of our proposed policy results in an average increase in expected profits of 9.6% compared to the case with no forward buying.

Possible limitations of our policy include our assumptions of deterministic, price-sensitive demand and a purely random distribution of the purchase cost. In reality, there may exist uncertainty over the parameters of the demand function. Also, purchase cost from one period to the next may be correlated and non-stationary over time. Our policy will still apply under this situation however, as long as the firm uses an unbiased forecasting technique to predict the direction of the future purchase cost realizations. The error of an unbiased forecasting method should then become the input for the purchase cost distribution parameters in our policy the same way the error from a demand forecasting model is used as an input for most inventory models. We leave further development of these two topics for future research.

7 Appendix

7.1 Proof that $i^*_\tau(I+z) \geq i^*_\tau(I)$ for all τ

To simplify the exposition, all global minimizers of functions are assumed unique in this section. (Otherwise we have to choose the least minimizer (in some partial ordering).) We also continue assuming, even when not stated as such, that all variables are nonnegative. When nonnegativity is pertinent to a result we will write nonnegativity constraints explicitly. The results all fall within 'generality theory' of convexity-submodularity and this is reflected in the notation. For example, the functions f appearing below will later be substituted by value functions for pipelines and the variable θ stands for an aggregate inventory.

7.1.1 Submodularity: a general minimizer-monotonicity result

In this subsection we turn to Step 2 of the 2-step framework of section 3.1.2: show that Inequality (12), the monotonicity property, is true. This result is contained in Theorem 2. Proposition 1 is a restatement of the monotonicity property which turns out to be the useful one for use in Subsection 7.1.2. The run up to Theorem 2 is made through four lemmas. Lemma 4 is Theorem 2 for the case of two functions (read: two pipelines). This lemma is then used in the induction argument in the proof of Theorem 2 which holds for an arbitrary number of functions (read pipelines).

All results, in this subsection are localized versions of results from general theory (except, perhaps, Theorem 2 which is an easy consequence of general theory).

Lemma 1 [Stability of convexity under convex minimization] Let $\{f_k\}_{k=1,\cdots K}$ be a collection of K convex functions. Define F through $F(\theta) = \min_{\substack{x_{[1,K]}: \\ |x_{[1,K]}|=\theta}} \sum_{k=1}^{K} f_k(x_k)$. Then F is convex.

Proof See Theorem 5.4 in Rockafellar (1970). ■

The following lemma is a version of Lemma 2.6.2.b in Topkis (1998).

Lemma 2 Let g be a convex function. Then $(x, \theta) \mapsto g(\theta - x)$ is submodular. That is, for any pair (x, θ) and (x', θ') it holds that[1]

$$g(\theta - x) + g(\theta' - x') \geq g(\theta \vee \theta' - (x \vee x')) + g(\theta \wedge \theta' - (x \wedge x')) \tag{14}$$

Proof Consider (x, θ) and (x', θ'). If either $x \leq x'$ and $\theta \leq \theta'$, or $x \geq x'$ and $\theta \geq \theta'$, then (14) is evidently true.

As (14) is symmetrical in (x, θ) and (x', θ') we can assume without loss of generality that $x \leq x'$ and $\theta \geq \theta'$. Then

$$\begin{aligned} g(\theta \vee \theta' - (x \vee x')) &+ g(\theta \wedge \theta' - (x \wedge x')) - g(\theta - x) - g(\theta' - x') \\ &= g(\theta - x') + g(\theta' - x) - g(\theta - x) - g(\theta' - x') \\ &= g(\theta' - x' + (x' - x)) - g(\theta' - x') - \big(g(\theta - x' + (x' - x)) - g(\theta - x')\big) \leq 0 \end{aligned}$$

Here the final inequality results from the convexity of g and the observations that $\theta - x' \geq \theta' - x'$ and $x' - x \geq 0$. ■

As any function on the set of real numbers is submodular, we have, in particular, that for any pair of convex functions f_1 and f_2 the function g defined through $g(x, \theta) \equiv f_1(x) + f_2(\theta - x)$ is submodular.

[1] We use the notation $x \vee y = \max(x, y)$ and $x \wedge y = \min(x, y)$.

The following lemma is a version[2] of Lemma 4.7.1 in Puterman (1994), see also Section 2.8 in Topkis (1998).

Lemma 3 Let $(x, \theta) \mapsto g(x, \theta)$ be submodular. Then $\theta \mapsto x^{(g)}(\theta) \equiv \arg\min_x g(x, \theta)$ is non-decreasing.

Proof [see Puterman (1994)] Consider $\theta^+ \geq \theta^-$ and choose $x \leq x^{(g)}(\theta^-)$. Submodularity of g states that

$$\begin{aligned} g(x, \theta^+) + g(x^{(g)}(\theta^-), \theta^-) &\geq g(x \vee x^{(g)}(\theta^-), \theta^+ \vee \theta^-) + g(x \wedge x^{(g)}(\theta^-), \theta^+ \wedge \theta^-) \\ &= g(x^{(g)}(\theta^-), \theta^+) + g(x, \theta^-) \end{aligned}$$

Rewriting this inequality results in

$$g(x, \theta^+) \geq g(x^{(g)}(\theta^-), \theta^+) + g(x, \theta^-) - g(x^{(g)}(\theta^-), \theta^-)$$

By definition of $x^{(g)}$ it holds that $g(x^{(g)}(\theta^-), \theta^-) \leq g(x, \theta^-)$ and the result is

$$g(x, \theta^+) \geq g(x^{(g)}(\theta^-), \theta^+) + g(x, \theta^-) - g(x, \theta^-) = g(x^{(g)}(\theta^-), \theta^+).$$

So $\arg\min_x g(x, \theta^+) \geq x^{(g)}(\theta^-)$. ■

Note. The notation employed shows the dependence of the minimizer on the function minimized explicitly through the superscript (g) as in $x^{(g)}$. Our main task lies in establishing an 'independence' result for minimizers and this is why we want to keep dependence of minimizers on functions and values explicit. Below notation will become more involved as we wish to keep track of the dependence for the minimizers of the sum of a collection of functions under a constraint. The dependence of the minimizers on the collection of functions will once more be made explicit through a superscript.

Lemma 4 Let f_1 and f_2 be two convex functions. Define $x_1^{(f_1,f_2)}(\theta)$ and $x_2^{(f_1,f_2)}(\theta)$ through the condition

$$f_1(x_1^{(f_1,f_2)}(\theta)) + f_2(x_2^{(f_1,f_2)}(\theta)) = \min_{x_1, x_2 : x_1 + x_2 = \theta} (f_1(x_1) + f_2(x_2))$$

Then, for $k = 1, 2$, we have that $\theta \mapsto x_k^{(f_1,f_2)}(\theta)$ is non-decreasing.

Proof Note that $x_2^{(f_1,f_2)}(\theta) = \arg\min_{x_1} (f_1(x_1) + f_2(\theta - x_1))$. Now $g(x, \theta) \equiv f_1(x) + f_2(\theta - x)$ is submodular by Lemma 7.1.1. It follows that $\theta \mapsto x_2^{(f_1,f_2)}(\theta)$ is non-decreasing from Lemma 3. The assertion that $\theta \mapsto x_1^{(f_1,f_2)}(\theta)$ is non-decreasing follows analogously. ■

Theorem 2 [Monotonicity in the aggregate of minimizing allocations] Let $\{f_k\}_{k=1,\cdots K}$ be a collec-

[2]The results in the references given are presented for maximization problems and supermodular functions.

tion of K convex functions. Define $x_k^{(f_{[1,K]})}(\theta)$ through

$$\sum_{k=1}^{K} f_k(x_k^{(f_{[1,K]})}(\theta)) = \min_{\substack{x_{[1,K]}: \\ |x_{[1,K]}|=\theta}} \sum_{k=1}^{K} f_k(x_k)$$

Then, for each $k \in \{1, \cdots, K\}$, we have that $\theta \mapsto x_k^{(f_{[1,K]})}(\theta)$ is non-decreasing.

Proof The proof uses an induction argument. The case $K = 1$ is evident and the case $K = 2$ is Lemma 4. Assume that the theorem holds for $K = L$ where L is some natural number larger than 2.

We can write

$$\min_{\substack{x_{[1,L+1]}: \\ |x_{[1,L+1]}|=\theta}} \sum_{k=1}^{L+1} f_k(x_k) = \min_{\substack{x_{L+1},y: \\ x_{L+1}+y=\theta}} (f_{L+1}(x_{L+1}) + F(y))$$

where $F(y) = \min_{\substack{x_{[1,L]}: \\ |x_{[1,L]}|=y}} \sum_{k=1}^{L} f_k(x_k)$. By Lemma 1, F is convex and we can therefore apply Lemma 4 to obtain

$$\min_{\substack{x_{[1,L+1]}: \\ |x_{[1,L+1]}|=\theta}} \sum_{k=1}^{L+1} f_k(x_k) = f_{L+1}(x_1^{(f_{L+1},F)}(\theta)) + F(x_2^{(f_{L+1},F)}(\theta))$$

and by applying the theorem for $K = L$, we obtain $F(y)) = f_1(x_1^{(f_{[1,L]})}(y)) + \cdots + f_L(x_L^{(f_{[1,L]})}(y))$. So

$$\min_{\substack{x_{[1,L+1]}: \\ |x_{[1,L+1]}|=\theta}} \sum_{k=1}^{L+1} f_k(x_k) = f_{L+1}(x_1^{(f_{L+1},F)}(\theta)) + f_1(x_1^{(f_{[1,L]})}(x_2^{(f_{L+1},F)}(\theta))) + \cdots + f_L(x_L^{(f_{[1,L]})}(x_2^{(f_{L+1},F)}(\theta)))$$

From

$$\begin{aligned}
x_1^{(f_{[1,L+1]})}(\theta) &= x_1^{(f_{[1,L]})}(x_2^{(f_{L+1},F)}(\theta)) \\
\cdots &= \cdots \\
x_\ell^{(f_{[1,L+1]})}(\theta) &= x_\ell^{(f_{[1,L]})}(x_2^{(f_{L+1},F)}(\theta)) \\
\cdots &= \cdots \\
x_L^{(f_{[1,L+1]})}(\theta) &= x_L^{(f_{[1,L]})}(x_2^{(f_{L+1},F)}(\theta)) \\
x_{L+1}^{(f_{[1,L+1]})}(\theta) &= x_1^{(f_{L+1},F)}(\theta)
\end{aligned}$$

we conclude that for $\ell \in \{1, \cdots, L\}$ the function $x_\ell^{(f_{[1,L+1]})}(.)$ is non-decreasing as it is the composition of two non-decreasing functions and that the function $x_{L+1}^{(f_{[1,L+1]})}(.)$ is non-decreasing by Lemma 4. ■

The following proposition is a restatement of the conclusion in Theorem 2.

Proposition 1 Under the conclusion of Theorem 2, i.e., $\theta \mapsto x_k^{(f_{[1,K]})}(\theta)$ is non-decreasing, we have

$$\min_{\substack{x_{[1,K]}: \\ |x_{[1,K]}|=\theta}} \min_{\substack{z_{[1,K]}: \\ |z_{[1,K]}|=z}} \sum_{k=1}^{K} f_k(x_k + z_k) = \min_{\substack{|z_{[1,K]}|: \\ |z_{[1,K]}|=z}} \sum_{k=1}^{K} f_k(x_k^{(f_{[1,K]})}(\theta) + z_k)$$

Proof We only need to establish that

$$\min_{\substack{x_{[1,K]}: \\ |x_{[1,K]}|=\theta}} \min_{\substack{z_{[1,K]}: \\ |z_{[1,K]}|=z}} \sum_{k=1}^{K} f_k(x_k + z_k) \geq \min_{\substack{z_{[1,K]}: \\ |z_{[1,K]}|=z}} \sum_{k=1}^{K} f_k(x_k^{(f_{[1,K]})}(\theta) + z_k) \tag{15}$$

as the reverse inequality is evident. Let $z_{[1,K]}$ with $|z_{[1,K]}| = z \geq 0$ be arbitrary. Now

$$\min_{\substack{|x_{[1,K]}|: \\ |x_{[1,K]}|=\theta}} \min_{\substack{z_{[1,K]} \\ |z_{[1,K]}|=z}} \sum_{k=1}^{K} f_k(x_k + z_k) = \min_{\substack{y_{[1,K]}: \\ |y_{[1,K]}|=\theta+z}} \sum_{k=1}^{K} f_k(y_k) = \sum_{k=1}^{K} f_k(x_k^{(f_{[1,K]})}(\theta + z))$$

Put $\omega_k^{(f_{[1,K]})}(\theta, z) = x_k^{(f_{[1,K]})}(\theta+z) - x_k^{(f_{[1,K]})}(\theta)$, then we can write $f_k(x_k^{(f_{[1,K]})}(\theta+z)) = f_k(x_k^{(f_{[1,K]})}(\theta) + \omega_k^{(f_{[1,K]})}(\theta, z))$. From Theorem 2 we have $\omega_k^{(f_{[1,K]})}(\theta, z) \geq 0$ and we conclude

$$\min_{\substack{x_{[1,K]}: \\ |x_{[1,K]}|=\theta}} \min_{\substack{z_{[1,K]}: \\ |z_{[1,K]}|=z}} \sum_{k=1}^{K} f_k(x_k + z_k) \geq \min_{\substack{z_{[1,K]}: \\ |z_{[1,K]}|=z}} \sum_{k=1}^{K} f_k(x_k^{(f_{[1,K]})}(\theta) + z_k)$$

as $\sum_{k=1}^{K} \omega_k^{(f_{[1,K]})}(\theta, z) = \theta + z - \theta = z$.

■

Proposition 1 is basically the result that we need to establish. It shows that we can first allocate $x_k^{(f_{[1,K]})}(\theta)$ to function k and then top up these quantities with z_k. The following subsection, Subsection 7.1.2, will re-establish Theorem 2 (in Corollary 2) and Proposition 1 (in Theorem 3) where the functions $\{f_k\}$ are substituted by the value functions for the pipelines.

7.1.2 Price-independence of allocations

In this subsection we turn to Step 1 of the 2-step framework of section 3.1.2: show that the monotonicity property of Equation (12) implies the independence property of Equation (10). In doing this we return to the context of the pipelines: the functions f of Subsection 7.1.1 get substituted by the value functions for the pipelines Z. We therefore first establish that the value functions are convex so that the convexity assumption on the functions f (read Z) of Subsection 7.1.1 applies and we can use the results of that subsection. The main result is obtained in Theorem 3.

Lemma 5 $I \mapsto Y_{t,\tau}(c, I)$ is convex

Proof First, note that $(z, d) \mapsto \gamma(c, I; z, d)$ is convex. Now

$$Y_{T,T}(c, I) = \min_{z,d:z+I=d} \gamma(c, I; z, d)$$

and application of Theorem 5.4 in Rockafellar (1970) shows that $I \mapsto Y_{T,T}(c, I)$ is convex. Second, assume that $Y_{t+1,T}(c, I)$ is convex in I where $t < T$. Note that $(I, z, d) \mapsto \gamma(c, I; z, d)$ is convex and consider the recursion

$$Y_{t,\tau}(c, I) = \min_z (\gamma(c, I; z, 0) + EY_{t+1,\tau}(C, I + z)) = \min_{(z,I'):I'=I} \left(\gamma(c, I'; z, 0) + E[Y_{t+1,\tau}(C, I' + z)\right)]$$

By the final representation of $Y_{t,\tau}(c, I)$ it is clear that application of Theorem 5.7 in Rockafellar (1970) yields that $I \mapsto Y_{t,\tau}(c, I)$ is convex. ■

Corollary 1 $\gamma(c, I; 0, i_t) + EZ_{t+1,T}(C, i_{t+1}, \cdots, i_T)$ depends convexly on $(i_t, i_{t+1}, \cdots, i_T)$.

Proof This is immediate from $Z_{t+1,T}(C, i_{t+1}, \cdots, i_T) = \sum_{\tau=t+1}^{T} Y_{t,\tau}(c, i_\tau)$ and Lemma 5. ■

The next corollary verifies (12) explicitly. We will not use it in the sequel, however, as the proof of the main result, Theorem 3, will refer to Proposition 1 directly.

Corollary 2 For each τ the function $I \mapsto i^*_\tau(I)$ is non-decreasing: $i^*_\tau(I) = \min_{\epsilon \geq 0} i^*_\tau(I + \epsilon)$.

Proof By definition of $i^*_\tau(I)$, see (11),

$$\gamma(c, I; 0, i^*_t) + EZ_{t+1,T}(C, i^*_{[t+1,T]}) = \min_{\substack{i_{[t,T]}: \\ |i_{[t,T]}|=I}} \left(\gamma(c, I; 0, i_t) + E[Z_{t+1,T}(C, i_{[t+1,T]})\right)] \quad (16)$$

$$= \min_{\substack{i_{[t,T]}: \\ |i_{[t,T]}|=I}} \left(-r(i_t) + \sum_{\tau=t+1}^{T} (hi_\tau + E[Y_{t+1,\tau}(C, i_\tau)])\right) \quad (17)$$

Because $i_t \mapsto -r(i_t)$ and $i_\tau \mapsto hi_\tau + EY_{t+1,\tau}(C, i_\tau)$ are convex the results now follows from Theorem 2. ■

The following theorem is a more explicit form of Corollary 2.

Theorem 3 [Price independence of allocation] There exist functions $\hat{x}_\tau(.)$ for $\tau = t, \cdots, T$, independent of c, such that

$$\min_{\substack{i_{[t,T]}: \\ |i_{[t,T]}|=I}} Z_{t,T}(c, i_{[t,T]}) = Z_{t,T}\left(c, \hat{x}_{[t,T]}(I)\right)$$

Proof The idea of the proof has been presented in Subsection 3.1.2. We only have to add the observation that the purchase costs do not depend on how a given purchase quantity z is distributed over pipelines: these costs only depend on the aggregate, z. The details are as follows.

$$\begin{aligned}
&\min_{\substack{i_{[t,T]}: \\ |i_{[t,T]}|=I}} Z_{t,T}(c, i_{[t,T]}) \\
=\quad & \min_{\substack{i_{[t,T]}: \\ |i_{[t,T]}|=I}} \sum_{\tau=t}^{T} Y_{t,\tau}(c, i_\tau) \\
=\quad & \min_{\substack{i_{[t,T]}: \\ |i_{[t,T]}|=I}} \min_{z} \min_{\substack{z_{[t,T]}: \\ |z_{[t,T]}|=z}} \left(\gamma(c, i_t; z_t, i_t + z_t) + \sum_{\tau=t+1}^{T} (\gamma(c, i_\tau | z_\tau, 0) + E[Y_{t+1,\tau}(C, i_\tau + z_\tau)]) \right) \\
=\quad & \min_{z} \left(z\, c + \min_{\substack{i_{[t,T]}: \\ |i_{[t,T]}|=I}} \min_{\substack{z_{[t,T]}: \\ |z_{[t,T]}|=z}} \left(-r(i_t + z_t) + \sum_{\tau=t+1}^{T} ((i_\tau + z_\tau)\, h + E[Y_{t+1,\tau}(C, i_\tau + z_\tau)]) \right) \right)
\end{aligned}$$

Because $i_t \mapsto -r(i_t)$ and $i_\tau \mapsto i_\tau\, h + EY_{t+1,\tau}(C, i_\tau)$ are convex by Corollary 1, we obtain from application of Theorem 2 and Proposition 1 that $\min_{\substack{i_{[t,T]}: \\ |i_{[t,T]}|=I}} Z_{t,T}(c, i_{[t,T]}) =$

$$\min_{z} \left(z\, c + \min_{\substack{z_{[t,T]}: \\ |z_{[t,T]}|=z}} \left(\begin{array}{c} -r(x_t^{(f_{[t,T]})}(I) + z_t) + \\ \sum_{\tau=t+1}^{T} \left(x_\tau^{(f_{[t,T]})}(I)\, h + E[Y_{t+1,\tau}(C, x_\tau^{(f_{[t,T]})}(I) + z_\tau)] \right) \end{array} \right) \right) \tag{18}$$

where $f_t(x) = -r(x)$, and $f_\tau(x) = xh + EY_{t+1,\tau}(C, x)$ for $t < \tau \leq T$. So (working backwards)

$$\begin{aligned}
&\min_{\substack{i_{[t,T]}: \\ |i_{[t,T]}|=I}} Z_{t,T}(c, i_{[t,T]}) \\
=\quad & \min_{\substack{z, z_{[t,T]}: \\ |z_{[t,T]}|=z}} \left(\begin{array}{c} \gamma(c, x_t^{(f_{[t,T]})}(I); z_t, x_t^{(f_{[t,T]})}(I) + z_t) + \\ \sum_{\tau=t+1}^{T} \left(\gamma(c, x_\tau^{(f_{[t,T]})}(I) | z_t, 0) + EY_{t+1,\tau}(C, x_\tau^{(f_{[t,T]})}(I) + z_\tau) \right) \end{array} \right) \\
=\quad & \sum_{\tau=t}^{T} Y_{t,\tau}(c, x_\tau^{(f_{[t,T]})}(I)) = Z_{t,T}(c, x_t^{(f_{[t,T]})}(I), \cdots, x_T^{(f_{[t,T]})}(I))
\end{aligned}$$

■

The following corollary gives us the sought result on interchanging minimization with expectation. It is a straightforward consequence of Theorem 3.

Corollary 3 [Interchange of expectation and minimization]

$$\min_{\substack{i_{[t,T]} \\ |i_{[t,T]}|=I}} E[Z_{t,T}(c, i_{[t,T]})] = E \min_{\substack{i_{[t,T]}: \\ |i_{[t,T]}|=I}} Z_{t,T}(C, i_{[t,T]})$$

Proof Note that $\min_{\substack{i_{[t,T]}:\\|i_{[t,T]}|=I}} Z_{t,T}(c, i_{[t,T]}) \leq Z_{t,T}(c, i'_{[t,T]})$ for any $i'_{[t,T]}$ with $|i'_{[t,T]}| = I$. So

$$E[\min_{\substack{i_{[t,T]}:\\|i_{[t,T]}|=I}} Z_{t,T}(C, i_{[t,T]})] \leq E[Z_{t,T}(C, i'_{[t,T]})]$$

and consequently $E[\min_{\substack{i_{[t,T]}:\\|i_{[t,T]}|=I}} Z_{t,T}(C, i_{[t,T]})] \leq \min_{\substack{i'_{[t,T]}:\\i'_{[t,T]}=I}} E[Z_{t,T}(C, i'_{[t,T]})]$. On the other hand, using Theorem 3,

$$E[\min_{\substack{i_{[t,T]}:\\|i_{[t,T]}|=I}} Z_{t,T}(C, i_{[t,T]})] = E[Z_{t,T}\left(C, \hat{x}_{[t,T]}(I)\right)] \geq \min_{\substack{i'_{[t,T]}:\\|i'_{[t,T]}|=I}} E[Z_{t,T}(C, i'_{[t,T]})]$$

as $\sum_{\tau=t}^{T} \hat{x}_\tau(I) = I$. ■

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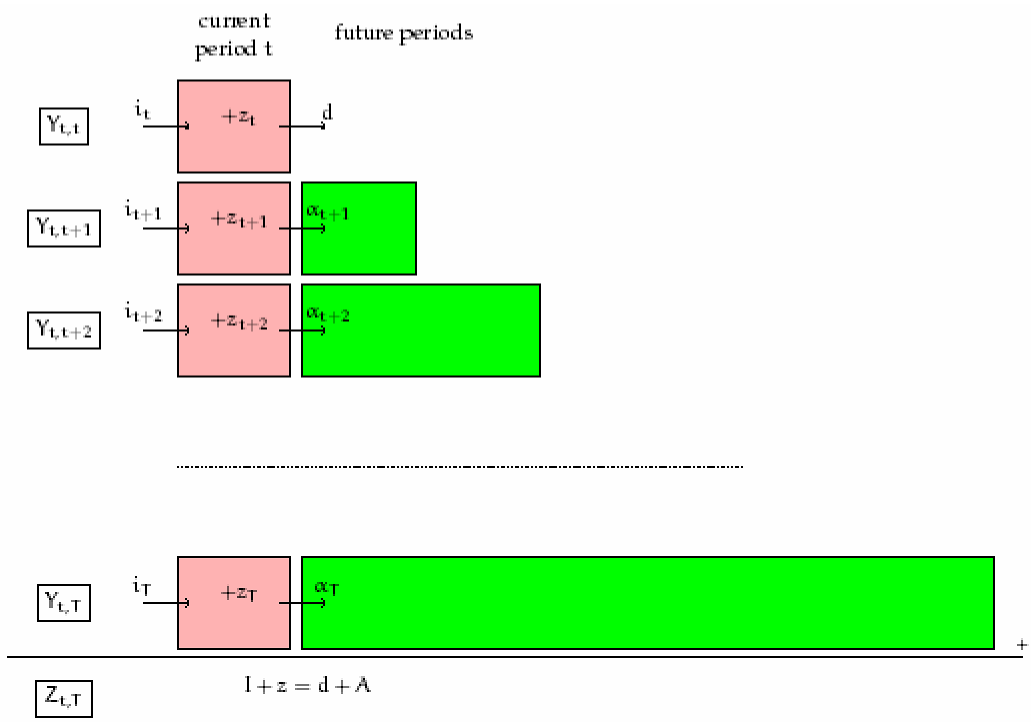

Figure 1: Aggregation

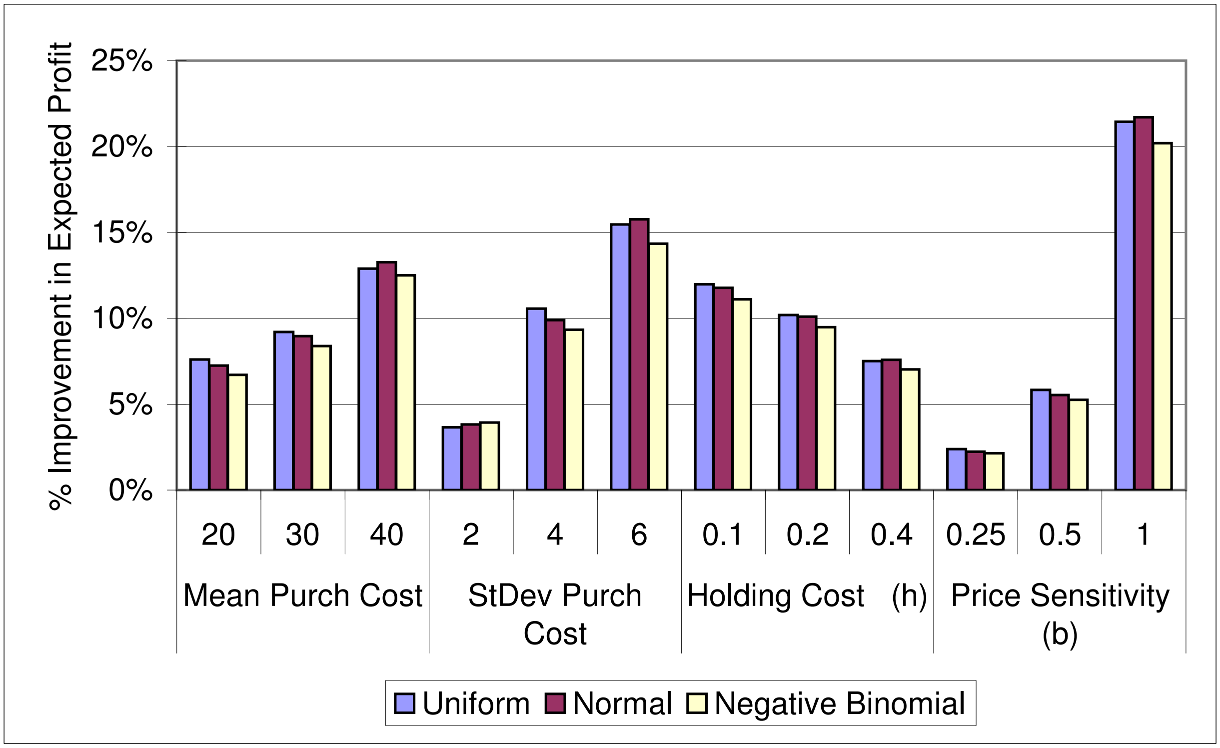

Figure 2: Percent Improvement from Forward Buying