THE SUSTAINABILITY OF GOVERNMENT FINANCIAL POLICIES IN OVERLAPPING-GENERATIONS MODELS

by

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SUMMARY

The objective of this thesis is to examine the implications of different government financial policies on the real sector of the economy. For this purpose we develop two overlapping-generations models. The first one allows us to evaluate the performance of the economy when debt is managed with different types of financial assets. A general result of the analysis is shown to be that an increase in the burden of debt leads to crowding out of the capital stock. A criterion for deriving endogenously *the maximum sustainable level of debt* within the model is also identified. The model turns out to be useful to provide an explanation of the *poverty trap* which is a very common phenomenon in some developing countries.

The second model is developed to discuss the effects on the real economic variables of two different government deficit financing policies. The framework is an overlapping-generations monetary economy with population growth. Firstly, we analyse the effects of public deficit financing policy by injection of money into the economy at an exogenous constant rate and we emphasise the *Mundell-Tobin effect* (or non-superneutrality of money). Secondly, we extend the previous financing policy to include an endogenised money growth rate and we succeed in providing a powerful framework to explain the conditions under which *dynamics of hyperinflation* may arise. The novelty and importance of the findings are highlighted throughout the thesis.

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I, alone, remain responsible for any errors.

INTRODUCTION

In recent years, government debt and its effects have received much attention from economists for several reasons. First of all, in many industrial countries there is an increasing concern about the possible unfavourable effects of a persistent large debt on the level of the capital stock. Secondly, the general increase in the real interest rates, due to the expansionist fiscal policy adopted in several countries, and the resulting increase in the burden of debt servicing may require a contractionary fiscal policy in the future in order to prevent debt from becoming excessively large relative to other macroeconomic variables. The aim of the present thesis is to investigate the effects of some government policies which can be used to finance both a certain level of debt, by using taxation, and a certain amount of public deficit (under the form of subsidies) by money printing. The study is carried out by using two overlapping-generations (OLG hereafter) models with perfect foresight, a framework conceived in the seminal articles of Samuelson (1958) and Diamond (1965). The two models we will develop differ both in some characteristics of the economy under investigation and in the kind of financial policy rule pursued by the government and are an extension of two models presented for the first time by Diamond (1965) and Stein (1971). The results which are highlighted in the thesis stand out as a proof of the potential of such a framework for analysing macroeconomic issues and for providing more relevant answers to questions which have already been dealt with in the literature adopting IS-LM models[1].

The first government policy discussed in the thesis is concerned with debt management under the form of different (non-money) financial assets issued by the government. Although there is no government spending inside the model, nevertheless we assume that the government has inherited a certain level of debt which is being financed by using taxation. Each type of asset implicitly generates a different government budget constraint. The

[1] For example, see Blinder and Solow (1973), Sargent and Wallace (1981) and Currie and Gazioglou (1983). A complete survey of this literature is contained in Currie (1978).

viability of each policy depends on whether it is sustainable and we derive the criterion for the maximum sustainable level of debt the government can issue.

The investigation of the second government deficit financing policy, through issue of money, is initially related to the issue of money non-superneutrality, i.e. to the effects of a change in the exogenous rate of money growth upon capital intensity. Subsequently, we endogenise the money growth rate by fixing the real government deficit exogenously. This allows us to generate hyperinflation dynamics in a context of endogenous real interest rates.

The models we will develop are similar in so far as they are both production OLG economies where individuals live for two periods. A government exists which is pursuing a programme of debt management or deficit financing. However, the related issues and the results obtained are different and, therefore, a separate analysis is required.

The thesis is divided into five chapters. Chapter 1 contains a brief overview of the literature on the issue of debt sustainability which has been developed over the years, and a discussion of the implications of unsustainable fiscal policies for the real economy. Our attention will be focused on some key papers which tackle in different ways the issue of debt sustainability; this will enable us to provide a general overview of the different approaches applied to investigate and to shed light on this topic. We also summarise the conclusions of some empirical studies which have been carried out for individual countries (especially Italy). The survey ends with a brief discussion about the Maastricht Treaty and the restrictions implicitly set on debt management policies for the European Community (EC) countries.

In chapter 2 we present the general model of debt management policy. The originality of this model consists in considering the existence of three types of bonds which are at disposal of a government and the different dynamic behaviour which is related to the instrument the government relies on. The main aim is to provide a criterion, which will be also tested by using real data, to determine whether a government debt management policy can be defined as sustainable. The model assumes an intertemporal optimising framework in which the capital stock is driven by utility and profit-maximising behaviour of private agents and a tax policy of the government. The presence of debt also provides an explanation why a poverty trap, which has been a topic of interest in the recent endogenous growth theory, may occur.

Chapter 3 illustrates further applications of the criterion of the maximum sustainable level of debt to other frameworks, to show under what conditions that criterion is applicable. These alternative frameworks are represented by a social security model with a pay-as-you-go system where contributions can be well compared to interest payments on debt and by a continuous time model with lifetime uncertainty as proposed by Blanchard (1984)-(1985). Some interesting findings related to a simple pure-exchange OLG economy are also put forward.

Chapters 4 and 5 deal with a re-formulation of Stein's (1971) model of a monetary OLG economy with government deficit financed through money printing. More precisely, in chapter 4 we illustrate the Mundell-Tobin effect of money non-superneutrality and consider the evolution of the system following an exogenous change in the rate of monetary growth. The recent incompletely successful efforts by economists to model hyperinflation in a perfect foresight context with certainty make more striking the results we obtain with our model in chapter 5 where money growth rate is made endogenous. In fact, in our new framework we find that there are some conditions which guarantee that hyperinflation may arise. The relaxation of some restrictive assumptions applied in the standard analysis is crucial for having an ever-increasing inflation rate rather than an ever-decreasing one, as more commonly found.

Appendices to chapters 2, 4 and 5 contain details on some theoretical specifications, mathematical analyses, studies of dynamics and simulations of the models.

A summary of the main findings is provided in the final conclusions, where we also highlight their contribution in explaining the real economic world.

Table 1.1

Year	Canada		France		Italy		United Kingdom	
	b	$\frac{b}{k}$	b	$\frac{b}{k}$	b	$\frac{b}{k}$	b	$\frac{b}{k}$
1986	348101	0.4685	1220.82	0.1894	736025.78	0.3680	249617.81	0.2887
1987	366577.84	0.4695	1292.58	0.1958	796624.67	0.3890	248750.62	0.2817
1988	382785.94	0.4630	1347.87	0.1987	849722.72	0.4037	226541.19	0.2497
1989	395177.37	0.4514	1405.56	0.2010	902557.74	0.4168	201292.95	0.2156
1990	410268.49	0.4462	1425.02	0.1973	946152.08	0.4247	187864.88	0.1967
1991	444635	0.4629	1473.69	0.1982	992811.23	0.4343	187813.20	0.1941
1992	490256.20	0.4934	1647.76	0.2162	1069268.23	0.4574	213891.72	0.2189
1993	529507.36	0.5170	1868.42	0.2410	1148238.75	0.4851	248016.72	0.2510
1994	573383.85	0.5400	2044.11	0.2585	1211114.17	0.5058	272164.60	0.2718

b = gross (including interest payments) debt-to-index price ratio of each country.

$\frac{b}{k}$= indexed debt-to-capital stock ratio.

As it is clear from table 1.1, there has been a steady increase in debt in all the countries considered while some of them, like Canada and Italy, display also a high debt-to-capital stock ratio. An immediate question is whether there is an upper limit to the level of debt beyond which an economy would collapse. In our thesis we, then, try to derive a working tool by which it is possible to define whether a debt-to capital stock ratio is too high and, therefore, unsustainable for the economy under consideration.

This chapter briefly introduces the main theoretical and empirical findings of the literature which concern debt management policies whereas a further overview of the literature of money financing policies is presented at the beginning of chapter 5 where we focus on seigniorage policy rules.

More specifically, this chapter is divided into three sections. The first one illustrates the concept of debt sustainability, gives a brief outlook on some models and their results and tries to underline the shortcomings of these studies. The second section deals with the empirical evidence about debt behaviour whereas the third one underlines the importance of the constraints of a maximum sustainable level of debt in the light of the Maastricht Treaty requirements.

1.2 The Concept of Debt Sustainability and Some Theoretical Explanations

In order to develop the debt sustainability issue, we start by resorting to the seminal model of debt management policy in an OLG economy proposed by Diamond (1965). In an economy with two production factors, capital and labour, the government is running a deficit and, in order to finance it, issues debt and levies lump sum taxes on the young generation. In a competitive solution the saving function is determined as a residual from the utility maximisation process and is a function of the rate of interest which, in equilibrium, is equal to the marginal product of capital. Population is assumed to grow at a constant exogenous rate n. The government issues a constant amount of one-period maturity debt[3] which can be held either domestically or externally and affects the consumer's income because it represents an asset to be held in the portfolio of the investors. The conclusions are different depending on the type of debt considered. If debt is held domestically, individuals' savings can be diversified between accumulating capital and purchase of debt[4]. A rise in taxes and investment in bonds affects negatively output, whereas the equilibrium rate of interest rises. Debt can reduce over-accumulation of capital and increases the steady state utility whenever the economy is dynamically inefficient.

Conversely, when debt is held externally, the interest payments on government debt would flow abroad. Increased taxes would alter the equilibrium on the capital market and factor prices, thus affecting directly utility. Increases in the stock of the per capita public debt become a burden, reducing both aggregate consumption and the supply of capital.

[3] In this thesis we intend to consider only the issue of a constant amount of domestic debt because, as far as we know, it is the case less studied in the literature.
[4] This is what Diamond calls the government's entering on the demand side of the capital market.

Diamond's analysis is concerned only with the comparative statics of the steady states of the model and with a welfare analysis of the effects on the individual consumer's utility following an exogenous increase in debt. However, it is intuitive that debt cannot reach a too high level because, if this happened, the burden of taxes would have to increase disproportionally in order to offset the widening of interest payments. This highlights the importance of analysing the issue of debt sustainability and of deriving a criterion to determine whether a certain level of debt is sustainable.

The term "unsustainability of a fiscal policy" has been frequently used in the literature and in recent multilateral political discussions with different connotations and in different circumstances. Generally speaking, it can be defined as a situation where a government is heading towards excessive accumulation of debt by following a particular fiscal policy, which must be reversed before the economy collapses. The term can refer equivalently to the level of debt, of public spending or to the tax rate. Other definitions have been suggested which can be summarised as follows.

1) Assuming that the intertemporal budget constraint for the public sector (see in the following pages for an accurate and mathematical description of this concept) must be satisfied, a fiscal policy is said to be sustainable when it is expected to generate sufficient net revenues in the future to finance the accumulated debt and interest payments (see over eq.(1.3)); conversely, it is unsustainable when the government pursues a policy of financing interest payments by issuing further debt and is unable to offset them by curbing other outlays or by raising other revenues (including seigniorage).

This assessment involves a projection of future taxes and spending and a forecast of some economic variables, such as the rate of growth of potential output and the interest rates. Moreover, it is related to the issue of solvency of government which can, ex post, be satisfied through either debt repudiation or monetization or revision of tax rates and spending plans[5].

2) Public debt sustainability implies that debt-to-GNP ratio does not exceed an upper limit beyond which the system would end onto an explosive path. This definition may pose some

[5] Obviously, fiscal sustainability does not necessarily implies optimality of public debt.

problems for an empirical evaluation, especially if this limit is to be set exogenously. In this respect, our attempt is to find a limit which can be derived endogenously within the model.

3) Sometimes debt sustainability is a synonymous for debt stability. A level of debt is sustainable if in the absence of unanticipated exogenous shocks the system converges to a stable steady state.

A simple but very enlightening theoretical work dealing with the analysis of the upper limit of debt sustainability (point 1) is by Blanchard et al.(1990), in which the authors define some indicators for an overall assessment of debt sustainability[6]. The model assumes that the primary deficit[7], transfers and interest payments are financed with the issue of new debt which is time-varying. The ratio debt-to-GNP, b, at time n is equal to its initial value at time $t=0$, b_o, accumulated at a rate equal to the difference between the interest rate and the growth rate, plus the accumulated value, at the same rate, of the primary deficit from time 0 to time n:

$$(1.1) \qquad b_n = b_o e^{(r-\theta)n} + \int_0^n d_s e^{(r-\theta)(n-s)} ds$$

where θ represents the real rate of growth of GNP, d the primary deficit-to-GNP ratio and r the real interest rate. It is usually assumed that the no-Ponzi-game condition holds, i.e.:

$$(1.2) \qquad \lim_{n\to\infty} b_n e^{-(r-\theta)n} = 0.$$

This condition, which will be referred to throughout this thesis, says that the discounted value of debt goes to zero and it will hold either if the ratio of debt-to-GNP, b_n, converges to zero or to a positive constant value (let us also, say, back to b_o) as n tends to infinity or if public debt grows but asymptotically at a rate smaller than the "growth-adjusted" interest rate. Therefore, the level of debt in the far future does not matter in the present time so long as it does not grow at a rate more than $(r-\theta)$. If condition (1.2) holds, eq.(1.1) can be transformed into the following:

[6] We choose to start off by illustrating Blanchard's paper disregarding the chronological order because it includes an important and complete analysis of the first interpretation of the term "debt sustainability".

[7] Deficits equal government spending net of tax and seigniorage revenues.

$$(1.3) \qquad -b_o = \int_0^\infty d_s e^{-(r-\theta)s} ds,$$

which implies that the existing debt must be financed in the future by either increased taxation or money printing in excess of public spending net of interest payments if debt is not to grow uncontrolled. A budget policy is sustainable if the present discounted value of the ratio of primary deficits-to-GNP is perfectly offset by a negative debt-to-GNP ratio.

Therefore, two scenarios are possible: if the real rate of interest is above the real growth rate of the economy ($r > \theta$), then an expansionary policy at present must imply a contractionary one in the future or the use of money creation for the fiscal policy to be sustainable. In the opposite case, when ($r < \theta$)[8], the government can continue to run indefinitely a constant primary deficit and service its debt by further borrowing[9].

The authors construct a generic indicator of sustainability which makes use of the "sustainable" tax rate defined as the tax rate which guarantees that eq.(1.3) is satisfied. The indicator is computed by subtracting from the sustainable rate, t^*, the current one, and is indicated as $(t^* - t)$. When the indicator $(t^* - t)$ is positive, taxes will have eventually to be increased[10] or spending (transfers) decreased by the size represented by the indicator itself. In other words, it is called for a fiscal retrenchment or a rise in the tax ratio to meet solvency. The indicator $(t^* - t)$ represents how much current policy should change if it is to become immediately sustainable[11].

[8] This corresponds to the case of a dynamically inefficient economy, where the issue of debt can correct the initial inefficiency represented by over-accumulation of capital: crowding out of private capital is a Pareto improvement and allows the present and future generations to increase their lifetime consumption.

[9] Horne (1991) states that the condition of the sign of $(r - \theta)$ imposes weak restrictions on fiscal policies for a number of reasons. Firstly, if output growth exceeds the real interest rate, no constraint needs to be imposed on government borrowing and on the ultimate size of debt because the debt ratio would ultimately reach a steady state level. Secondly, it does not take into account the possibility of shocks to the demand for bonds and, therefore, to the interest rates which could make an initially sustainable fiscal policy to become unsustainable. If interdependencies between interest rates, economic growth, public debt and risk premia are ignored, the solvency constraint fails in providing a warning signal about the size of debt. Finally, it does not include private sector's perception of government's commitment to meet the constraint and the lack of credibility following a postponement in the action of increasing the primary surplus to satisfy the budget constraint as time goes by.

[10] Of course, the possibility of increasing the tax rate is constrained by the rate already existing in the country.

[11] Namely, it measures the size of the necessary transfer of resources that would be needed to ensure that a given public sector expenditure meets the solvency constraint at the point in time when it is measured.

In case of delay in pursuing a tight policy, the size of t^* to be imposed would be influenced as well. The sustainable tax rate postponed to the far future, say period n, t^*_n, should be sufficient to offset the amount needed to keep the ratio of debt-to-GNP constant given a certain level of primary deficit and discounted value of spending and transfers between time 0 and n.

The shortcomings of this approach arise when we have to calculate the indicators $(t^*_n - t)$ for the medium and long run because it is necessary to use forecasts of the main economic variables on which the indicators depend. A lot of difficulties may arise: for instance, often those forecasts are hardly available[12] and there is a problem of how to estimate the net real interest rate for the medium and long term. However, there are also advantages connected to the long-run indicator, like the fact that it takes into account the ageing of the population, which influences government spending through public pensions and health expenditure programmes. Using the forecasts of non-interest spending, population growth rate, pension spending and public health care spending, a long-run analysis suggests that the gap $(r-\theta)$ has two contrasting implications for the sustainability of the policy itself. The first effect is that the higher $(r-\theta)$, the more likely the policy to be unsustainable because a primary surplus to be run, in order to keep the debt-to-GNP constant, should increase. The second effect is that it reduces the weights given to public spending in the distant future, so that the problem of ageing becomes less important. The higher the discount rate, the less valuable the present government spending.

Most of the simulations carried out by the authors with data referring to some OECD countries rely on projections of future spending and on the assumption that the real interest rate exceeds permanently the rate of economic growth. Generally speaking, evidence from the early 1990s suggests that the interest rates exceed the economy's growth rate, thus justifying the choice of dynamic efficiency within the model simulations.

[12] However, also the one-period indicator presents some drawbacks. Firstly, it assumes that present fiscal policy will remain unchanged in the future. Secondly, by ignoring budgetary projections and future government plans, it may give misleading signals about the underlining fiscal situation. Nevertheless, it requires minimal information, i.e. only about the present primary balance, the base year public debt-to-Gross Domestic Product (GDP) ratio and the trends for the real interest rates and output growth.

It is worth noting that, given the relationship between the variables within the government budget constraint, an indicator of sustainability of government spending or transfers could be derived, without altering the general interpretation of the concept and its implications.

Some criticisms of these indicators, which are used to judge solvency and sustainability of government debt, can be put forward: for instance, they seem to lack the properties of behavioural content, normative criteria and global perspective. Firstly, they do not take into account the possibility that some variables, like the real interest rates, may be affected by other factors within the economy and that any "adjustment policy" undertaken to achieve a partial sustainability may increase the gap in future periods. Secondly, it must be remembered that the target value of sustainable debt set by the government could significantly affect the speed and nature of the process to achieve debt stabilisation. Thirdly, they lack economic perspective since they do not take into consideration a shared macro policy adjustment arising from political interdependence. It is also important to note that, while public debt-to-GNP ratio is an ex post measure, indicators of sustainability are an ex ante measure of the magnitude of the necessary stabilisation to be carried out.

The paper just illustrated is an example of the stream of literature which deals with the issue of debt sustainability and focuses on a framework where debt is growing over time but interest rates remain constant. In contrast, Diamond's analysis considered a constant level of debt and endogenous interest rates, which is a framework not so frequently used to study the influence of debt on the economy, but which turns out to be interesting to detect dynamics of interest payments. However, the conclusions of both papers could be easily criticised on the basis that the first one does not take into account that the real interest rate cannot be considered constant over a period of time whereas the second one assumes a government policy which is sometimes hard to sustain.

On the other side, Masson (1985) developed an OLG model based on a time-varying interest rate and implying uncertainty about the sustainability of the fiscal policy in the future. The objective of his analysis is to study how this kind of uncertainty can affect the behaviour of the private sector. People assign probabilities to different government actions which are determined by the amount of government debt at any point in time (and, of course, by the size of current and prospective fiscal deficits) and the sustainability of government financing policy depends on the expectations of money holdings which are reduced if high rates of inflation are forecast. The government is allowed to "default" on the real value of its debt through monetization; however, uncertainty arises about whether the government will choose, as a financing policy, a money supply or a tax increase. The increasing risk of monetization and the possibility that consumption may be lower in the second period as a result of higher taxes shifts individuals' preference from the first to the second period consumption which favours "precautionary saving"[13], and, consequently, capital accumulation. However, if continued bond finance is used, the real return on bonds must be higher than the return on capital because bondholders must be compensated for the possible loss of value of their holdings which are fixed in nominal terms[14]. But the increase in interest rates aggravates the problem of debt management, making a slightly unsustainable financing policy more unsustainable. A wedge is driven between the return on capital and yields on bonds and this has some implications on savings and risk premia.

Generally speaking, the inflation tax can be an easier option if the government has to face an increasing debt. However, if people anticipate the possibility of future monetization, interest rates would rise and endanger debt stabilisation. On the other hand, the uncertainty characterising the economy allows for unexpected inflation which reduces the real cost of servicing debt so that the government can surprise investors[15].

Uncertainty is crucial in determining agents' demand for government bonds with respect to productive capital. The longer the deficit persists, the higher the level of government debt, the higher the bond interest rates and the higher the probability of fiscal stabilisation or monetization. This clearly affects agents' beliefs who expect a future monetization which

[13] This happens in the hypothesis that both first and second period consumption are normal goods and taking into account that both monetization and tax increases have a negative income effect.

[14] This implies that demand for bonds is subject to the inflation risk.

[15] Since it is likely that an upper limit on the tax rate exists, a policy of monetization seems the most likely. Subjective probabilities are influenced by the course of fiscal policy, but monetization would have some effects only if the increase in inflation was partly unexpected, namely if people would attribute a probability of less than one to the inflation tax when the government has decided to adopt that policy.

tends to increase interest rates. The author underlines that "*The uncertainty associated with a loss in the real value of bonds would be a net cost to the society and could be reflected in higher real interest rate. For the government, a consequence of a loss in credibility is that it may face higher financing costs [...]*"[16]. The important result of the model as pointed out by Masson is that a real interest rate (r) below the economy's growth rate (n) is not sufficient to ensure that a constant ratio of primary deficit to output is sustainable. This is also true for the case of perfect certainty. This result is justified as follows. Even though an interest rate below the real growth rate should hypothetically permit to finance a primary deficit indefinitely, however continuous shocks to government deficit would lead to endogenous changes in the bond stock. An increase in the supply of bonds for financing the additional deficit would require a higher interest rates to divert investors from their first period consumption towards saving. But higher interest rates are consistent with low levels of capital stock, which implies a low level of income available to be saved. If the latter effect dominates (which happens when the real interest rate becomes bigger than the population growth rate), bonds and interest rates would explode as the interest rates increase so much that the desired holdings of the capital stock fall to zero. A cumulative increase in debt and interest rates would lead the economy towards the collapse.

The conclusion that dynamic inefficiency (i.e. $r<n$) is not sufficient to guarantee a sustainable deficit in an economy, is in contrast with most studies which were trying to show that only under dynamic efficiency is there potential instability. Anyway, also the assumption of a time-varying debt, as assumed by most economists, is not a necessary condition for having an unsustainable fiscal policy. We will show that instability of the system may occur also in a constant government debt policy framework with endogenous interest rates.

As we anticipated in the introduction, one of our aims will be to derive a method for calculating the maximum sustainable level of debt which guarantees that interest rates are economically meaningful. A similar approach is followed by Soren Bo Nielsen (1992) who also investigates the sustainability of primary deficit financing policies in a non-monetary

[16] Masson (1985), p.591.

economy with capital accumulation, endogenous interest rate and no bequests motives. Population grows at a constant rate μ: individuals have infinite lifetimes and new generations appear at a birth rate of μ[17]. Each agent owns financial wealth, $a(t)$, consisting in ownership of capital stock per capita, $k(t)$, and public debt $d(t)$ (perfect substitutes in the investors' portfolio), on which he earns a positive return. Taxes (with the tax rate equal to τ,) are levied proportionally on labour income and are used to partly finance the fraction of production the public sector consumes. Each individual maximises his lifetime utility (assumed logarithmic for simplicity) and consumption turns out to be a fraction of his total wealth. The economy's productive sector operates competitively, the production function is of Cobb-Douglas form and in per capita terms equal to $y(t) = k(t)^{\alpha}$.

The model consists of three non-linear equations in the endogenous variables consumption, capital stock and debt: the stock of capital and public debt are interpreted as predetermined variables whereas consumption per capita is free to jump in response to an unanticipated shock.

By assuming that the public sector is in debt (i.e. $d(t) > 0$) and has a primary budget deficit, there exists a level beyond which primary deficits are no longer sustainable. This result is derived by calculating the steady state solutions for the real interest rate. In order to have economically meaningful values for the steady states (namely an acceptable solution for the equilibrium interest rate), it is necessary to set an upper limit to the primary budget deficit if public debt is positive[18].

Sustainability depends on the parameters in the economy, namely the population growth rate, the pure rate of time preference, the share of capital in production and the size of public sector. The conditions for sustainability are the following: the pure rate of time preference must be small relative to the rate of population growth (which implies that people must be "patient" and willing to save at a high rate) and the public sector must be not too large so that

[17] The framework is taken by Weil's (1987) paper, where the author considers a model of overlapping infinitely lived families model where Ricardian Equivalence does not hold.

[18] The condition for having an economically meaningful steady state represents a constraint on some economic variables and will be applied in this thesis to derive the maximum sustainable level of debt, although in our specific case we set public spending equal to zero.

not too many resources are diverted away from the production activity. If these conditions are satisfied, then two steady states exist. In the presence of a primary budget deficit and public debt, both steady states turn out to be dynamically inefficient; the fact that both steady states will be dynamically inefficient represents an innovation with respect to most of the models developed in the literature. In any case, the condition of dynamic inefficiency does not guarantee that the system, following an increase in the primary deficit, continues to display stable dynamics.

Generally speaking, an established method, which can be used to judge the sustainability of debt, has not been developed yet. Horne (1991) stresses the difficulty of capturing the essence of unsustainability only by looking at simple measures of solvency, where debt sustainability is identified with a stable long-run equilibrium path of the economy. The reasons put forward are "*...the weak restrictions imposed on the behaviour of the fiscal authorities [*and*] the assumed independence of real interest rates, economic growth and fiscal balances*"[19]. The author claims that policy makers may be worried about debt-to-GNP ratio because a rising stock of public debt is usually associated with a rising real interest rate through the portfolio effect or the risk premium arising from imperfect asset substitutability, and this might call for immediate intervention notwithstanding the sustainability[20] of the policy itself.

The sustainability analysis could be extended by including the policy choice between borrowing and repaying the debt through a future fiscal retrenchment, and resorting to the inflation tax. Sargent and Wallace (1981) have argued that debt management based on an initial stabilisation policy of disinflation and budget deficit financing through increased borrowing can be more inflationary than the alternative approach of financing through money printing. As a matter of fact, by following the first strategy the government loses a channel of revenues represented by the inflation tax and would roll over the amount of total debt which,

[19] Horne (1991), p.iii.
[20] The standard analysis of sustainability deals with the case of domestically financed deficits; but if the country issued external debt, then the condition of sustainability would be that the present value of trade surpluses must equal the stock of outstanding net external debt, all expressed as ratios of GDP.

at the end, must be repaid. If the interest rate is greater than the rate of growth of the economy, debt inclusive of interest payments would increase. Should the public at a certain date be no longer willing to buy the additional stock of bonds issued, then the government would have to switch to money financing, and here comes out the paradox that "*Although fighting current inflation with tight monetary policy works temporarily, it eventually leads to higher inflation*"[21]. As the stock of debt gets larger, higher revenues from seigniorage and a higher future inflation are required in order to hold constant future fiscal deficits.

In the next section we focus on the empirical studies which have approached the issue of government solvency and the problems which must still be solved to build up powerful and robust tests for debt sustainability.

1.3 Some Empirical Results about Debt Sustainability

As far as the empirical analysis is concerned, many studies have focused on the evidence of the conditions of sustainability as stated in eq.(1.3) and on the empirical testing of the satisfaction of the intertemporal government budget constraint which has recently acquired great importance for evaluating fiscal regimes and government financing policies in most countries. Among the most common tests, there is the cointegration analysis of the stock of public debt and deficit net of interest payments and the investigation of the intertemporal budget constraint satisfaction when the deficit is not generated by a stationary first-difference stochastic process or when the real interest rate cannot be assumed constant. This last approach coincides with studying whether the deficit gross of interest payments is stationary.

One of the most influential works on this topic is Hamilton and Flavin's (1986) paper[22]. The authors test the satisfaction of the budget constraint assuming particular stochastic

[21] Sargent and Wallace (1981), p.2.

[22] They compare the case of an ever-accumulating debt through perpetual deficit financing to the case of a continuous rise in prices in a self-fulfilling speculative bubble, so that their test resembles the one carried out to disentangle the presence of a bubble in the economy. "*We show that the proposition that the government can accumulate ever-growing debt through perpetual deficit financing has a mathematical parallel in the proposition that prices can rise continually in a self-fulfilling speculative bubble*" (Hamilton and Flavin, p.809).

characteristics of the processes which generate the variables considered. The test consists in estimating whether the deviation of debt from the sum of future surpluses is growing at a constant interest rate:

$$(1.4) \qquad b_t = a_o(1+r)^t + \sum_{j=1}^{\infty} E_t(1+r)^{-j} s_{t+j}$$

where s stands for public surplus, $s = T - G$ (T being the amount of revenues and G the real government expenditure exclusive of debt interest payments[23]). The borrowing constraint holds if $a_o = 0$, which implies that the sum of all current and expected future non-interest outlays in present value must not exceed the sum of all discounted revenues, including seigniorage[24]. The authors carry out three checks by using a data series for the US which spanned from 1960 to 1984 with different available information sets, i.e. current and lagged values of the surplus and lagged values of debt or expectations where surpluses are formed only on lagged values. In all cases, they find that a_o is not significantly different from zero so that the conclusion is the rejection of the null hypothesis of non-stationarity of the public debt series and deficit net of interests. Therefore, they conclude that, in that period, the fiscal policy in US has been consistent with the intertemporal budget constraint. However, their analysis has some drawbacks in so far as violations of the budget constraint are considered non-stochastic, which is equivalent to saying that the discounted debt series is stationary. Since their findings are not as expected, the authors consider it necessary to suggest an alternative measure of the government deficit which takes into account revenues from monetization and capital gains from gold. Using the new measure for deficit, they can conclude that "*...then the prevailing sentiment in Washington that current deficits can continue for ever is wrong; the adjusted deficit series must soon turn to surplus*"[25]. Their analysis is, naturally, limited by the assumption of a non stochastic real interest rate[26].

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23 This test is based on the hypothesis that the interest rate is constant, namely that given the information set at the beginning of period t, Ω_{t-1}, the real interest rate is such that $E\left(r_{t+i} \middle| \Omega_{t-1}\right) = r$.

24 This remarks what already expressed with eqs.(1.2)-(1.3).

25 Hamilton and Flavin (1986), p.818.

26 The authors take the average ex post real rate over the period 1960-1984.

In this respect, Wilcox (1989) argues that the conclusions of the tests by the previous authors are sensitive to the specification degree of the model adopted, especially to the number of lagged variables on the right-hand-side (RHS) of the equation for the Dickey-Fuller test used to check the stationarity of the process. Therefore, it is possible that a_o does not converge to zero but it is equal to a positive constant, which implies that a part of fiscal spending is not financed by an increase in taxes[27]. Wilcox's tests differ from Hamilton and Flavin's in so far as: "*First, [*he*] allows for stochastic real interest rate, whereas Hamilton and Flavin assumed a fixed real interest rate. Second, [*he*] allows for non-stationarity in the non-interest surplus, whereas Hamilton and Flavin require the surplus to be stationary*[28]. *Third, [*his tests have*] power against stochastic violations of the borrowing constraint, [*since the discounted debt is likely to be non-stationary*]*"[29]. The working tool is the following: eq.(1.4) can be rewritten in terms of discounted variables as:

$$(1.5) \qquad B_t = B_{t+N} + \sum_{j=1}^{N} S_{t+j},$$

where B_t is the discounted value of the debt and S_t is the discounted value of the surplus. According to Hamilton and Flavin's concept of sustainability, the expected value of the first term on the RHS approaches zero as N tends to infinity, so that the current value of the debt equals the sum of expected future non interest surpluses:

$$(1.6) \qquad B_t = \sum_{j=1}^{\infty} E_t S_{t+j}.$$

Wilcox argues that one has to test for the violations of the borrowing constraint as being either constant or stochastic[30]. To carry out his tests, the author applies the same data[31] series

[27] Generally speaking, as pointed out at the beginning of this chapter, a sustainable fiscal policy may be defined as a policy which is expected to generate a sequence of debt and deficits such that the present-value borrowing constraint holds, i.e. permanent non interest deficits are ruled out, and transitory deficits have to be offset in later periods.

[28] Debt stationarity implies that, according to the last two economists, a_o=0.

[29] Wilcox (1989), p.292.

[30] These violations $A_t = \lim_{N \to \infty} E_t B_{t+N} \neq 0$ would be stochastic if B_t is nonstationary.

[31] More precisely, the sequence of discount factors is derived from the real interest rates whose variation is attributed both to fluctuations in the real return on gold and to the shifts in the yield curve.

used by Hamilton and Flavin. The analysis shows that the hypothesis of a zero unconditional mean must be rejected[32] so that doubt about the sustainability of fiscal policy arises. However, there are still some problems in the tests for the detection of parameters instability which may be explained by the non inclusion of some variables affecting future values of debt.

Trehan and Walsh (1991) argue that, if the real interest rate is kept constant, the test applied by Wilcox based on the cointegration between B and S and the test based on the stationarity of budget deficit gross of interest payments should lead to the same result because, otherwise, either the hypothesis of constant interest rate would not be plausible or a structural break has occurred in the same period. Therefore, a sufficient condition for the sustainability of public debt is the stationarity of the gross interest payments deficit provided that the expected real interest rate can be assumed to be positive and independent of the stochastic characterisation of the process governing public debt stock and primary deficit. Because of the problems to measure the real interest rate to discount the series of debt, the estimation of the stationarity of gross interest payments deficit when the interest rate is not constant is a good alternative.

Some other empirical studies focus on debt management in some specific countries. An example is Baglioni and Cherubini's paper (1993) which deals with the sustainability of the Italian debt. The financial relaxation of the Italian public sector has recently been a source of concern with regard to its economic conditions: this is quite understandable if we consider that the proportion of the stock of debt-to-GNP is higher than 100% while the European Monetary Union (EMU) process entails much lower bounds on the public sector debt and deficit. The current Italian fiscal policy is judged unsustainable on the grounds that the intertemporal budget constraint for the public sector is violated, i.e. the present value of the sum of all current and expected future non-interest outlays exceeds the present value of all discounted revenues[33].

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[32] Moreover, a_0 in eq.(1.4) results to be significantly different from zero.

[33] Corsetti (1990) also concludes in favour of the insolvency for the Italian public sector. However, the economist is aware of the limitations of those tests in so far as they are conditional on the absence of major

In fact, evidence shows that debt non-stationarity started from 1983, when changes in monetary policy towards a more rigid discipline obliged the governmental authorities to rely more upon funds (public debt) rather than inflation tax revenues[34]. This change in monetary policy which gave back autonomy to the Bank of Italy did not only imply the possibility of having a separate policy programme for each institution, but it also allowed the Bank itself to get separated from the political power more concentrated on sterile debates. The switch of the Italian monetary policy has been mainly dictated by the attempt to re-equilibrate the structure of the economy in order to adhere more strictly to the commitments of the European Monetary System (EMS).

On the other side, some economists have argued that the required satisfaction of the intertemporal budget constraint is not always sufficient for judging the sustainability of the debt level. Although a converging limit is set, it might have the result that the fiscal pressure turns out to be politically unfeasible and socially unacceptable due to the radical impact on income and wealth distribution.

In conclusion, if interest rates are kept constant and deficit gross of interest payments is stationary, then public debt and deficit net of interest rates should be cointegrated and fiscal policy would be sustainable. It is a fact that the robustness of the empirical testing procedure requires estimating for potential structural breaks in subperiods, big and infrequent shocks or for the non-constancy of the rate of interest.

In the next section we focus on some interesting studies which have been carried out to illustrate the economic and fiscal situation in some European countries which can help us to understand why the conditions in the Maastricht Treaty can be interpreted as measures of virtue and rescue for some countries.

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changes in the relevant macroeconomic variables, like growth, interest rates and inflation which may actually prevent the intertemporal government budget constraint from being satisfied.

[34] Following the reasoning by Sargent and Wallace (1981), this change in policy is responsible of the high accumulation of a larger stock of public debt and, therefore, a more demanding policy of stabilisation.

1.4 Further Remarks about the Current European Economic Situation and the Maastricht Treaty Requirements

Italy, as most European and OECD countries, has faced a huge increase in debt, especially in the last few years as it was clear from the figures presented in table 1.1 at the beginning of this chapter. For the Italian case this situation is determined, as emphasised by Alesina-De Broeck-Prati-Tabellini (1992), by the high real interest rates currently paid on the Italian long-term debt which might represent a premium against a general risk of a financial crisis expected on the basis that in few years' time the government could be called on to roll over a large fraction of its outstanding debt. In theory, the government could be forced to repudiate its debt if investors refused to buy any public debt in the future.

The requirements of the Maastricht Treaty should prevent the EC countries (or at least the countries whose currency is still in the Exchange Rate Mechanism (ERM)) from running independent inflationary policies; so that inflating away debt is impossible. The only way of repudiating debt would be to cancel government debt obligations by law, which should raise the risk premium[35] since investors would require a higher compensation for the repudiation risk. A reason which is usually put forward to justify debt default is that it represents a non-distortionary lump sum tax which substitutes the distortionary taxes which could be levied to service the debt.

On the other hand, the policy of debt default involves some costs, like the loss in reputation, the difficulty of borrowing in the next period, income redistribution and the risk of bankruptcy in the financial sector. Especially for the OECD countries, the costs seem to outweigh the benefits so that repudiation is a very unlikely possibility.

To show how the risk premium on debt can affect the economic situation, Alesina-De Broeck-Prati-Tabellini (1992) analysed the default risk by comparing the returns from holding government debt with the returns from holding safe private debt of corresponding

[35] The presence of a risk premium is triggered by high levels of debt which make confidence crises more likely, but its size decreases in debt average maturity.

maturity in some OECD countries. A clear relationship between the risk premium and the level of government debt receives an empirical support only for Denmark, Italy and the Netherlands. It is possible that different institutional rules may affect this relation in the other countries. Moreover, for all the countries with a very high level of debt, the hypothesis of instability (unit root) of debt-to-GDP ratio cannot be rejected.

Generally speaking, data on the public debt in different countries have been used not only to estimate the solvency of the public sector, but also to carry out a comparison between the economic policy programmes in different countries and a general evaluation of their policies.

The general conclusion from the analysis of the data referred to some of the OECD countries for the period 1971-1989 is that the present path of fiscal policy is unsustainable in Belgium, Greece, Ireland, Italy and the Netherlands on the basis of a too high level of debt-to-GDP ratios, from which it follows that a serious fiscal retrenchment is required. In particular, Greece and Italy must carry out primary fiscal corrections to avoid complete insolvency, whereas Belgium and Ireland have to continue their policies of huge fiscal primary surpluses to embark on a path of decreasing debt-to-GDP ratio.

Before concluding our overview of the issue of debt sustainability, we would like to focus on illustrating the implications of the constraints set in the Maastricht Treaty, which can help us to visualise better the importance of the limits which must be set on some economic variables and which will be discussed more deeply throughout the presentation of our two main models. In fact, we will underline the importance of setting a maximum level of debt-to-capital stock ratio and of seigniorage in order for the economy to be well-behaved, an issue which is believed to be crucial for the monetary union.

A way of judging the sustainability of the current fiscal policy in the EC Member States (which is a necessary condition to allow a country to join the single currency) is measuring the size of the adjustment effort which is required to satisfy the fiscal convergence criteria by the deadline for the monetary union.

The concept of debt sustainability has been, therefore, invoked in relation to the convergence criteria as a *conditio sine qua non* for joining the monetary union as indicated in the Maastricht Treaty. These criteria require, among other things, that the government budget deficit is not higher than 3% of its GDP and its debt-to-GDP ratio moves significantly towards the target of 60%. The fiscal adjustment is very urgent for some countries like Italy as the future involvement in the process of European Monetary Integration prohibits monetization as a source of financing.

Since the fiscal constraints for joining the EMU are linked to the convergence criteria for inflation, interest rates and exchange rates, Buiter-Corsetti-Roubini (1993) point out that the convergence criterion on inflation[36] together with the limitation of the power of the central bank to finance national budget deficits created by the national governments, implies the impossibility of using a nationally differentiated strategy of debt management through seigniorage policies in order to reduce debt ratios. Therefore, there would exist a maximum level of seigniorage which could be run by each country individually once they have entered the EMU[37]. This can be interpreted as a sort of monetary counterpart of the maximum sustainable level of debt-to-GDP ratio[38]. In fact, this criterion of maximum deficit and debt-to-GDP ratio implicitly includes a limit on the size of the monetary base in a country as stated in the following sentence: "*Maastricht penalises countries that, for whatever reason, have a big monetary base*"[39]. This situation would penalise some countries like Italy whose most important instrument used over the years to repay its debt has been the inflation tax. In fact, "*...since the beginning of this century the Italian state has always repaid its debt either by resorting to new debt issues or to the printing press. When the outstanding stock had grown to an intolerable level so that a reduction was considered necessary, the state has never enacted the required transfer of real resources. It opted instead for implicit repudiation by curtailing the purchasing power of the Lira*"[40].

36 This criterion states that a country's inflation rate is not more than 1.5% higher than the average of the three lowest inflation rates in the EMS.

37 Namely, this condition deprives a country of an important source of financing.

38 Both levels of maximum sustainable seigniorage and debt will be the main topics of our thesis.

39 Buiter-Corsetti-Roubini (1993), p.72.

40 Canziani-Giavazzi-Manasse-Tabellini (1993), p.3. In their paper, the authors set up a model based on the experience of the French fund after the First World War. They wanted to show how the informational asymmetry about the strength of the government commitment to fight the debt and different structure of debt could influence the fiscal adjustment and help to build a strategy to redeem existing debt at lower costs.

The authors also suggest that an excessive size of government debt may be dangerous in so far as it creates a negative externality which may spread a financial crisis any time the market realises that a country's public debt is unsustainable while the other countries might be forced into a fiscal bail out of the insolvent government. If the Central Bank decided to partially finance such a deficit, then the EMU area would suffer a high inflation rate.

However, it must be noted that the fiscal criterion does not take into account the possibility that a government may own public enterprises whose value actually affects the total net value of the public sector liabilities. The process of privatisation could in fact provide enough receipts to be used for financing government spending and reducing government debt, although this does not mean that the fiscal retrenchment must be postponed. Moreover, it is also necessary, as underlined by Blanchard et al.(1990), to consider that the long-run fiscal position of a country is affected by its social security scheme and its policy of pensions financing. A natural process of homogenisation would be necessary to avoid big discrepancies in the tax and funding practice system.

The general feeling one can get from the codification of the Maastricht fiscal requirements is that if some retrenchments were inevitable, Maastricht has the power to tighten them, although it is not the only way by which fiscal discipline may be enforced. An example is represented by a situation where a government is facing a solvency crisis. The Treaty states that there should not be any bail-out by its Member States and the government should be the only responsible for its own debt. The general fear is that financial panic may spread around the EMU and lead to a general liquidity crisis[41]. This kind of macroeconomic spillovers must be taken into consideration in so far as they can affect the economic policy of the other countries. In the extreme case, the Central Bank should behave as a lender of last resort in order to safeguard the payment system. If only the fear that a crisis could occur were effective, then the European Central Bank (ECB) might be forced to monetise the budget deficits of some countries without discipline. But as one of the primary goals of this bank is

[41] It could also be argued that if the commitment was not credible, then on the point of an economic collapse other governments would intervene to avoid a complete financial disaster.

to maintain price stability, the independence of the ECB would help to safeguard it from outside intrusions in its own plans.

The social and political debates that have accompanied the agreement process to the Treaty itself help to understand how important Maastricht's fiscal constraints are and why in some countries, such as Italy and Greece, fiscal austerity is required [42].

The political aspect, i.e. the political instability and polarisation, has become also an important issue in studying debt growth. For political instability it is meant the probability that a government loses its power and for polarisation the degree of disagreement between alternating policy makers[43]. The intuitive result is that the more unstable and polarised the political systems, the more likely that they behave myopically and, therefore, public debt becomes larger. A support to this hypothesis is that the political instability and government weakness that prevail in Italy may be considered important factors for explaining the fiscal stalemate and inability to curtail fiscal deficits. On the other hand, an explanation of why similar countries experience different debt policies could be the differences in the degree of polarisation, political stability and flexibility in government decisions about public consumption.

The measures which can be taken as proxies for evaluating political instability are the rate of fractionisation between the parties in the Parliament whereas a measure of polarisation can be the inequality rate of personal income distribution represented by the Gini coefficient or the quantity of irregular government transfers. Evidence shows that the more unstable the government, the more likely that the same fiscal policy is unsustainable.

After having illustrated the main approaches in the literature to the issue of debt sustainability, in the next chapter we introduce our OLG model a-la-Diamond which is used

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[42] However, the question is still open on whether the fiscal criteria in the Maastricht Treaty should be considered as an expression of virtue. Frankel, in commenting on the paper of Buiter-Corsetti-Roubini (1993), suggests that the Treaty could be considered as a desirable goal to reach because of the difficulty of the task itself and it could represent a way of gaining reputation especially for those countries which are less virtuous. It could also be interpreted as a test of will in so far as it would test how far governments are disposed to go to support the agreement any time a shock occurs and jeopardises the process of unification. A sufficiently large shock would disrupt the currency union if the individual nationalities were not so serious in committing themselves to sustain the Treaty.

[43] Alesina and Tabellini (1990) focused and modelled this aspect.

to discuss a criterion of debt sustainability, endogenously derived within the model itself. The analysis of the maximum inflation tax which can be raised in a country and its implications for the economy is instead the topic of discussion in chapters 4 and 5.

CHAPTER 2

A MODEL OF DEBT MANAGEMENT POLICY RULES

2.1 Introduction

A key question which has been dealt with in the literature is whether issuing extra debt to finance a tax cut or an increase in government spending has any effect on the real economy. There are different approaches to investigate this subject. The first approach focuses on OLG models with flexible prices and instantaneously-clearing markets where, if Ricardian Equivalence fails because generations are not altruistically linked to future generations, crowding out of the capital stock takes place in the long run as a consequence of an increase in government debt. As shown by Diamond (1965), whenever the economy is dynamically inefficient (i.e. when the interest rate is lower than the population growth rate), this displacement of capital increases stationary utility. The second approach is headed by Blinder and Solow's paper (1973) where the two authors show that, in a dynamic IS-LM model with fixed prices and income-linked tax revenues rising faster than the additional burden of interest payments, bond financing of fiscal deficits is more expansionary in the long run than money financing.

In this chapter we focus on an OLG framework with debt and we show that the size of the crowding out of the long-run capital stock depends on the type of debt management rule followed by the government. In particular, our analysis will be concerned with three different kinds of financial instruments, each of which is associated with a particular budget constraint which has different implications for the repayment tasks to be faced by the government.

The basic framework of the model, presented in section 1, consists of two-period lived consumers and a production sector in which firms operate. We focus on the dynamics of capital in an economy where the government is absent. In section 2 we introduce a government which is running debt and we analyse three different debt management policies at its disposal[1]. We therefore use the same framework to study the impact of the issue of government bonds on the real side of the economy and the question of debt sustainability. Section 3 deals with a particular case of the general model by using Cobb-Douglas utility and production functions and highlights the properties of the steady states and the specific dynamics of the system.

Our study is well justified by the renewed interest in the analysis of the effects of debt on capital accumulation and welfare following the rapid growth of public indebtedness recently experienced by many industrialised countries. However, as we anticipated in the previous chapter, economic instability is not necessarily associated with an explosive path of debt: in fact, in this chapter we show that, in a context of endogenous interest rates and constant debt a-la-Diamond, the economy can still collapse.

Another interesting issue is also tackled by our analysis. As a consequence of the model, adopted to study debt policies, being characterised by multiple equilibria, we will show that the *poverty trap* situations could arise, provided that some particular conditions are satisfied. This issue had been overlooked by Diamond, since he assumed that a unique steady state existed[2]. We will show that in our OLG models the existence of a poverty trap is associated with the existence of debt or government deficit: the threshold level of the initial capital stock below which this phenomenon occurs turns out to be the amount of capital stock of the unstable equilibrium. If the initial capital stock is lower than that level, then the system would diverge from the equilibrium onto an unstable path characterised by decumulating capital stock.

In the following section we present the basic OLG model which will be extended in sections 2 and 3 to include government sector and debt management policies. Section 4

[1] It is worth remembering that permanent government deficits and debt can be visualised in economies in which the government is infinitely lived but households are not since constitutions and social agreements provide continuity for the government.

[2] Diamond (1965) admits that he is interested in comparing alternative Golden Age Paths to which the economy converges with different quantities of outstanding government debt. Therefore, he only examines the long-run implications of the issue of national debt.

instead contains a simulation exercise of the model which highlights the potential of this framework in evaluating the sustainability of fiscal policies in the G-7 countries.

SECTION 1.

A SIMPLE PRODUCTION ECONOMY WITH NO GOVERNMENT

2.1.1 Consumer's Behaviour and the Production Sector

Our basic OLG model resembles Diamond's (1965) model, with the simplification that we assume no population growth. In our economy L identical individuals are born in every period, namely the population growth rate is set equal to zero. Individuals live for two periods and they are identical within as well as across time. As far the **consumers' behaviour** is concerned, in the first period of their life they supply their unit-endowment of labour inelastically and earn the competitive wage rate w_t. They allocate their income from labour between the first period consumption, c_t^y and savings, s_t such that the first period budget constraint is:

(2.1) $w_t = c_t^y + s_t$.

Individuals' consumption in the first period must be weakly less than the wage they receive from old entrepreneurs. In the second period they retire and their consumption (c_{t+1}^o) is guaranteed by the returns earned on their first period saving[3] (in fact, we assume that consumers start life with no assets), which, in the absence of debt, has been entirely invested in the capital stock (the only asset available in the economy):

(2.2) $c_{t+1}^o = (1 + r_{t+1}) s_t$.

r_{t+1} is the rationally anticipated net return on next period's capital. We assume that agents can predict perfectly the future path of the economy and they use these predictions to form

[3] The rate of interest earned on the amount saved is net of the rate of depreciation of capital, set equal to zero for simplicity, whose source is saving itself.

their expectations about the future rate of return on their investment. Such "perfect foresight" expectations are independent of past observations and are self-fulfilling. However, it is not necessarily true that there exists a unique self-fulfilling expectations equilibrium.

The consumer's intertemporal allocation problem consists of maximising his intertemporal utility function whose arguments are the consumption in the two periods of life of a single, non storable physical commodity[4]:

(2.3) $\quad u(c_t^y, c_{t+1}^0)$

where $u: R_+^2 \rightarrow R$ is a well-behaved real-valued utility function, twice continuously differentiable, monotonically increasing, strictly quasi-concave and which does not change from one generation to the next[5]. From the maximisation process[6] we derive the following saving function:

(2.4) $\quad s_t = s(w_t, r_{t+1}) = \arg\max \; u[w_t - s_t, (1+r_{t+1})s_t]$.

Savings earn the return r_{t+1} in the following period, and enable the cohort to consume during retirement. Given (w_t, r_{t+1}), from the postulated properties of the utility function and under the perfect foresight assumption, an optimal level of savings exists and is uniquely determined. In addition, the saving function depends on each of its arguments in the following way:

(2.5) $\quad s_w = \dfrac{\partial s(\bullet)}{\partial(w_t)} \in (0,1)$, i.e. the marginal propensity to save with respect to net income is positive but less than 1.

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4 We assume no labour-leisure choice.

5 Future consumption is assumed to be a normal good and starvation is avoided in both periods, namely

$$\lim_{c_t^y \to 0} u_y(c_t^y, c_{t+1}^o) = \infty, \quad \lim_{c_{t+1}^o \to 0} u_o(c_t^y, c_{t+1}^o) = \infty.$$

6 The lifetime intertemporal budget constraint which states that the present discounted value of lifetime consumption cannot exceed that of labour income is given by:

$$c_t^y + \frac{c_{t+1}^o}{1+r_{t+1}} \leq w_t$$

and assuming an interior solution, eq.(2.4) follows.

(2.6) $s_r = \frac{\partial s(\bullet)}{\partial r_{t+1}}$ has instead an ambiguous sign. An increase in the interest rate in the following period decreases the relative price of second period consumption. Therefore, the individuals would tend to shift consumption from the first to the second period (the substitution effect). However, an increase in the interest rate also makes available a larger feasible consumption set so that consumption would tend to increase in both periods (the income effect). The result depends on which of the two effects dominates. If the substitution effect dominates, the relationship between s_t and r_{t+1} is positive, and conversely it is negative if the income effect dominates.

As far as the **production sector** is concerned, we assume that at any period the composite good consumed in the economy is produced through a production process whose structural components are capital and labour. We assume that the individuals work when they are young and retire when old and leave no bequests. The old, who serve as entrepreneurs in their retirement year, hire the young up to the point where profits are maximised given the existing capital stock. The supply of labour at any period of time is inelastic and equal to the number of young at that date (=*L*). The endowment of capital at time *t*, K_t, is the amount of resources which were not consumed in the period before. Thus, the expression for the change in capital is:

(2.7) $K_{t+1} - K_t = Ls(w_t, r_{t+1}) - K_t$.

The left-hand-side (LHS) is net investment and is represented by a change in the capital stock between period *t* and *t+1*. The RHS represents net saving which consists of saving of the young, $s(\bullet)$, net of dissaving of the old.

The technology employed in the production process is assumed to satisfy a constant returns to scale production function which is constant over time and is neoclassical in its specification:

(2.8) $Y_t = F(K_t, L_t) = L_t f(k_t)$

where $k_t = K_t / L_t$ is the capital-labour ratio. We assume that the production function $f: R_+ \to R_+$ is twice continuously differentiable, positive, strictly concave[7].

As we assume perfect competition in both capital and labour market, each factor is paid its marginal product. This yields the usual marginal conditions for factor rewards:

(2.9) $r_t = f'(k_t)$

(2.10) $w_t = f(k_t) - k_t f'(k_t)$.

From the properties of the production function it is immediate to see that real interest rates must be positive. As the growth rate of the population is assumed to be zero, this implies that the economy must be dynamically efficient. This is quite a realistic assumption especially for developed countries and has been tested empirically in studies on capital accumulation and economic growth[8].

2.1.2 Dynamic Equilibrium

Assuming that the only source of capital accumulation is savings, the system is characterised by the following relation:

(2.11) $k_{t+1} = s(w_t, r_{t+1})$

or, in a more detailed specification:

(2.12a) $k_{t+1} = s[f(k_t) - k_t f'(k_t), f'(k_{t+1})] = \theta(k_t)$

where only the young have an incentive to hold assets. Eq.(2.12a) represents a relationship between k_{t+1} and k_t which is called the *asset market equilibrium locus*[9]. This locus identifies equivalently a relationship between r_{t+1} and w_t as is clear from the following:

[7] Strict concavity implies the following sign of the derivatives in the Hessian matrix: $F_{ll} < 0,\ F_{kk} < 0,\ F_{ll}F_{kk} - (F_{lk})^2 > 0$.

[8] Dynamic efficiency (i.e. $r_t > n$) is quite a well-established result, although there are still some unresolved problems for its estimation. The empirical measures which have been usually adopted are both the marginal product of capital from observed accounting profit rates and the safe real interest rate, such as the return on treasury bills. Abel et al. (1989) provide empirical evidence for dynamic efficiency for the US whereas Corsetti's (1990) analysis presents the same conclusions for Italy.

[9] Blanchard and Fisher (1989), p.95, call the relationship in eq.(2.12a) the *saving locus*.

(2.12b) $f'^{-1}(r_{t+1}) = s(w_t, r_{t+1})$.

The equilibrium of the system is defined as follows:

Definition 2.1. *A dynamic equilibrium is defined as a sequence* $\{k_t\}_{t=0}^{\infty}$ *under which* $k_{t+1} = s[f(k_t) - f'(k_t)k_t, k_{t+1}]$ *and* k_o *is the exogenous initial condition*[10].

Galor and Ryder (1989) have proved this statement under the hypothesis that, if $\theta : R_+ \to R_+$ is a single valued function and exhibits certain properties[11], a dynamic equilibrium exists and is uniquely determined. A steady state equilibrium is defined as a stationary capital-labour ratio, $\bar{k}$, under which $\bar{k} = \theta(\bar{k})$. The stability condition of the steady state consists in the following constraint on the first derivative:

$$(2.13) \qquad \left|\frac{dk_{t+1}}{dk_t}\right| = \left|\frac{-s_w \bar{k} f''(\bar{k})}{1 - s_r f''(\bar{k})}\right| < 1.$$

If inequality (2.13) holds and the derivative is positive (negative), the endogenous variable will display a monotonic (oscillatory) convergence towards the steady state. The sign of the derivative in (2.13) is, however, ambiguous. Whilst the numerator is positive (as an increase in the capital stock in period t increases the wage and, therefore, savings), the sign of the denominator depends on the effect of the interest rate on saving itself. If s_r is positive the denominator is positive whereas if it is negative the sign of the denominator has an ambiguous sign. The 45 degree line in a (k_t, k_{t+1}) space represents the locus of all steady states.

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10 The analysis of the dynamics of the system by using a (k_t, k_{t+1}) diagram is quite popular (see Azariadis (1993) and Blanchard and Fischer (1989)) although for a welfare analysis it is more convenient to use the (r,w) diagram a-la-Diamond. The study of the effect of a change in debt on the interest rate which is presented in the next section does not include the analysis of the efficiency of the steady state; consequently, we use eq.(2.12a).

11 The conditions for the existence, uniqueness and stability of a non-trivial steady state equilibrium in an OLG model in which individuals live for two periods and a single good is produced using capital and labour are also listed and proved in Galor and Ryder (1989), p.372, proposition 5, where the production function is assumed to satisfy the Inada conditions: $\lim_{k\to 0} f(k) = 0$, $\lim_{k\to 0} f'(k) = \infty$, $\lim_{k\to\infty} f'(k) = 0$.

Moreover, in a specific interval of values more than one steady-state may exist so that the system may be characterised by a multiplicity of steady state equilibria, among which $\bar{k} = 0$ is to be included[12]. Two possible asset market equilibrium loci (phaselines *OS* and *OS'*) and the steady states (points *O* and *E*) are pictured in fig.2.1 below.

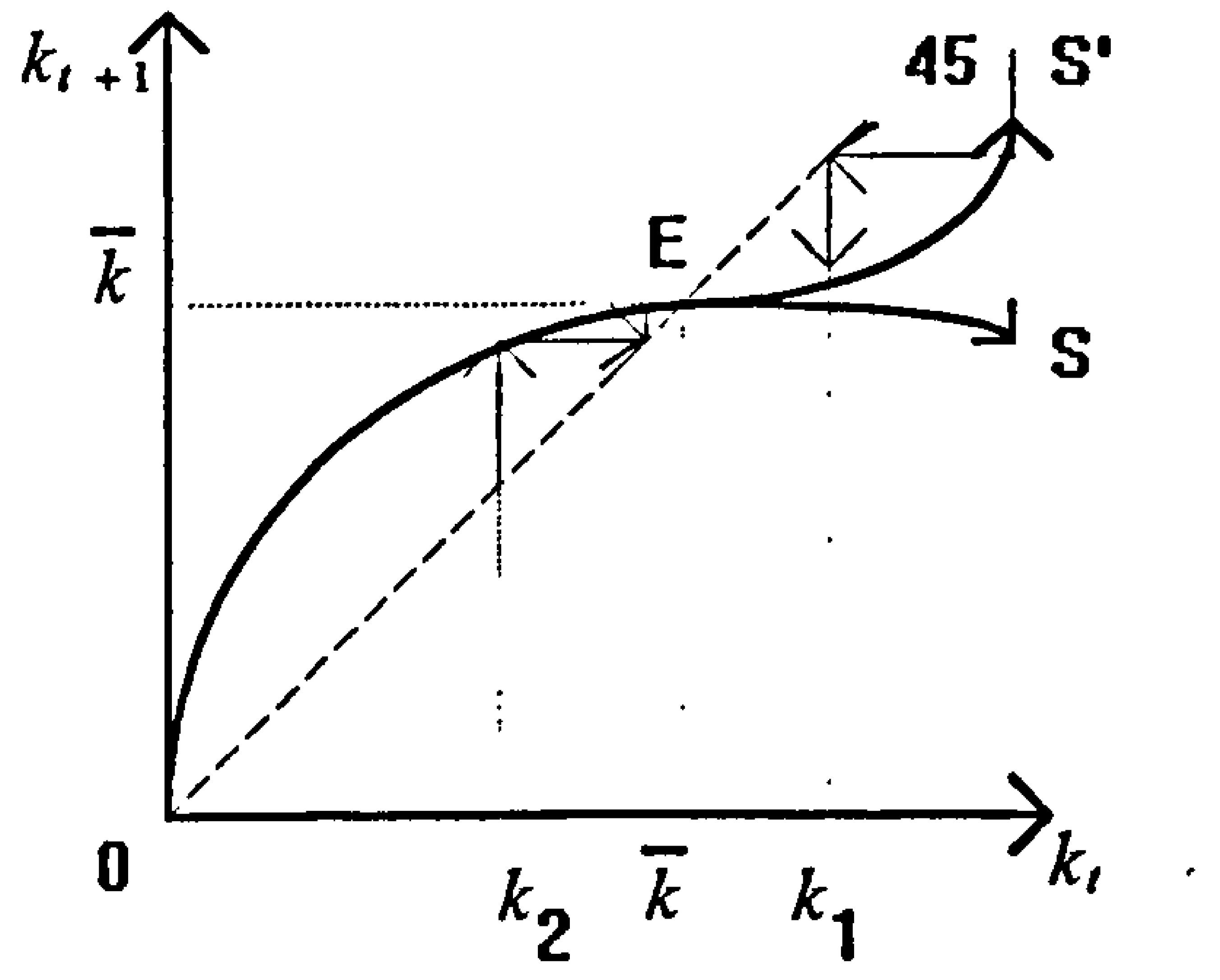

figure 2.1

Point *O* (*E*) is unstable (stable) because the slope of the phaseline is higher (lower) than 1, namely the phaseline intersects the 45 degree line from below (above). The dynamics of the figure above are the following. Starting from an arbitrary point in the neighbourhood of the steady state, say at k_2, we find the corresponding value of k_{t+1} on the phaseline. Then, we reflect that value on the 45 degree line and we find the corresponding value of k_{t+2} and so on. The dynamics will converge towards the steady state $\bar{k}$, which is stable. Moreover, among the multiple equilibria the Golden Rule level of capital-labour ratio, k^*, which is independent on the allocation of consumption between young and old, would be another equilibrium. As we underlined at the beginning, the equilibrium in our model will always be dynamically efficient since the population growth rate is zero.

We now focus on the dynamics of this economy when the government sector is introduced and its debt can be financed through different kinds of financial instruments.

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[12] See Galor and Ryder (1991), p.387 for a study of the efficiency of the equilibria.

SECTION 2.

INTRODUCING A GOVERNMENT

2.2.1 Introduction

We consider a decentralised economy in which the government has no direct control over the economy's resources and has inherited a certain level of debt. There are many reasons why this situation may arise. The government could, for instance, finance some windfall payments, such as veterans' bonuses, war debts or other kinds of debt which are not exactly a counterpart of an ongoing fiscal spending policy, through debt. We, therefore, investigate the dynamics of the change in capital stock when the government can manage debt by a variety of different financial instruments[13].

We assume that this governmental institution levies lump sum taxes on the young and it may trade in long-term assets as it is infinitely lived. Each of these assets is associated with a particular government budget constraint which defines different repayment tasks to be faced by the government and we address the question of what financing policy is more convenient for the government.

This section includes the description of each asset and the study of the effect of a change in debt on the steady state capital-labour ratio. In all cases debt per capita is held constant and the measure of the quantity of debt outstanding in any period is defined at the beginning of the period. The main conclusion is that, regardless of which kind of "policy rule" the government chooses to follow to manage its debt, the system always displays multiple equilibria. In the following Proposition 2.1 we state a quite well-known result which is represented by the negative relationship between capital intensity and national debt at the stable steady state and which has been already shown by Diamond for a specific policy.

[13] Government policies may affect the old and the young differently and, in particular, government bonds are a way of transferring wealth between the generations as the possibility of gifts and bequests is not considered here. In fact, as money does not exist, the only way of shifting purchasing power into the future is represented by those financial activities.

Proposition 2.1. *In an economy where government purchases are zero and a constant stock of per capita public debt is serviced by taxes on young individuals, an increase in the level of debt will exert a positive effect on the steady state interest rate. This will result in* *crowding out* *of the capital stock in the economy, independently of what kind of bonds the government is issuing. In fact, at the asymptotically stable stationary equilibrium, capital intensity is a decreasing function of per capita national debt.*

As a matter of fact, the originality of our analysis will be to investigate the validity of this proposition in the context of the other debt management policies which differ from the one considered by Diamond. The proof is carried out separately for each debt management policy (savings deposits, treasury bills and perpetuity bonds) and is discussed in the remaining part of this section.

2.2.2 1° Case: Savings Deposits

A "savings deposit" can be thought of as a loan of one unit of the good by the household in period t which will yield $(1+r_{t+1})$ in the next period. As a matter of fact, this kind of asset corresponds to a non-marketable form of government debt, such as National Savings accounts which are administered through the Post Office.

At time t the number of existing savings deposits which are made available from the households to the government are s_t. We assume that, in order to pay back the loans, the government raises lump sum taxes τ on the young and issues new bonds in the following period. The government budget constraint implies that the amount of taxes levied on the young and of the newly issued bonds are equal to the total payments on the bonds issued in the previous period:

(2.14) $\tau_t + b_{t+1} = b_t[1+r_t]$.

Past interest rates are important because the amount of saving that occurred in the previous period affects productivity, and hence the level of income available for saving today.

If the stock of existing bonds at the beginning of each period is constant, i.e. $b_{t+1} = b_t = \bar{b}$, taxes are endogenously determined in each period by $\tau_t = r_t\bar{b}$ and employed to finance interest costs.

This policy rule intuitively implies that the government is holding constant only the principal whereas the interest payments on the stock of debt itself can change; consequently, in order for the government budget to be balanced, in each period taxes must perfectly offset the interest payments which depend on the interest rate of that period[14].

The equilibrium on the capital market requires that the supply of capital by the households, s_t, must be sufficient to face the demand by the government to finance the fixed stock of bonds issued in each period, $\bar{b}$. As individuals can hold in their portfolio both debt (institutional asset) and physical capital, the intertemporal equilibrium for the entire economy is established when the asset market is cleared in every period:

(2.15) $\quad s_t = k_{t+1} + \bar{b}$.

Bonds and capital are assumed to be perfect substitutes and hence they have an identical rate of return, r. From eq.(2.15) it immediately follows that debt absorbs part of the households' wealth, thus reducing investment and output as less resources are invested into capital. The government can only affect the economy's capital intensity through the market mechanism by varying the amount of its outstanding debt.

The proof of what we claimed in Proposition 2.1 is the following. The asset market equilibrium in eq.(2.15) can be re-expressed as:

(2.16) $\quad k_{t+1} = s[f(k_t) - k_t f'(k_t) - f'(k_t)\bar{b}, f'(k_{t+1})] - \bar{b}$

where we have substituted the expressions for the wage rate from eq.(2.10) and for the lump sum tax. The properties of the asset market locus depend on the following derivative:

(2.17) $$\frac{dk_{t+1}}{dk_t} = \frac{-s_w f''(\bar{k} + \bar{b})}{1 - s_r f''(\bar{k})}$$

which must be less than one in absolute value for the stability condition to hold. The numerator is positive whereas the sign of the denominator depends on the effects of an

[14] This case strictly resembles Diamond's analysis of "internal" debt.

increase in the interest rate on saving. For the stability condition to hold, the effect of an increase in the interest rate on saving must be positive or, at least, not too negative. If this is satisfied, the dynamics imply a monotonic (oscillatory) convergence to the steady state depending on the positive (negative) sign of the denominator and, therefore, on the slope of the function at the steady state.

The representation of the phaselines in the diagram below (figure 2.2) assumes that the derivative in (2.17) is positive. The phaseline *OD* represents the asset market equilibrium locus in the simple case of $\bar{b} = 0$ where only one non-trivial steady state exists (point *E*). The phaseline *D'D'* instead has been drawn for a positive amount of debt in the economy: it is lower and flatter than *OD* because debt both drives part of the savings away from the capital sector (represented by a downward shift of the phaseline) and increases the level of taxation and lowers disposable income (i.e. flattening of the curve). Due to the presence of debt, there are two steady states, *E'* and *F*, the smaller of which is the unstable one since the phaseline intersects the 45 degree line from below.

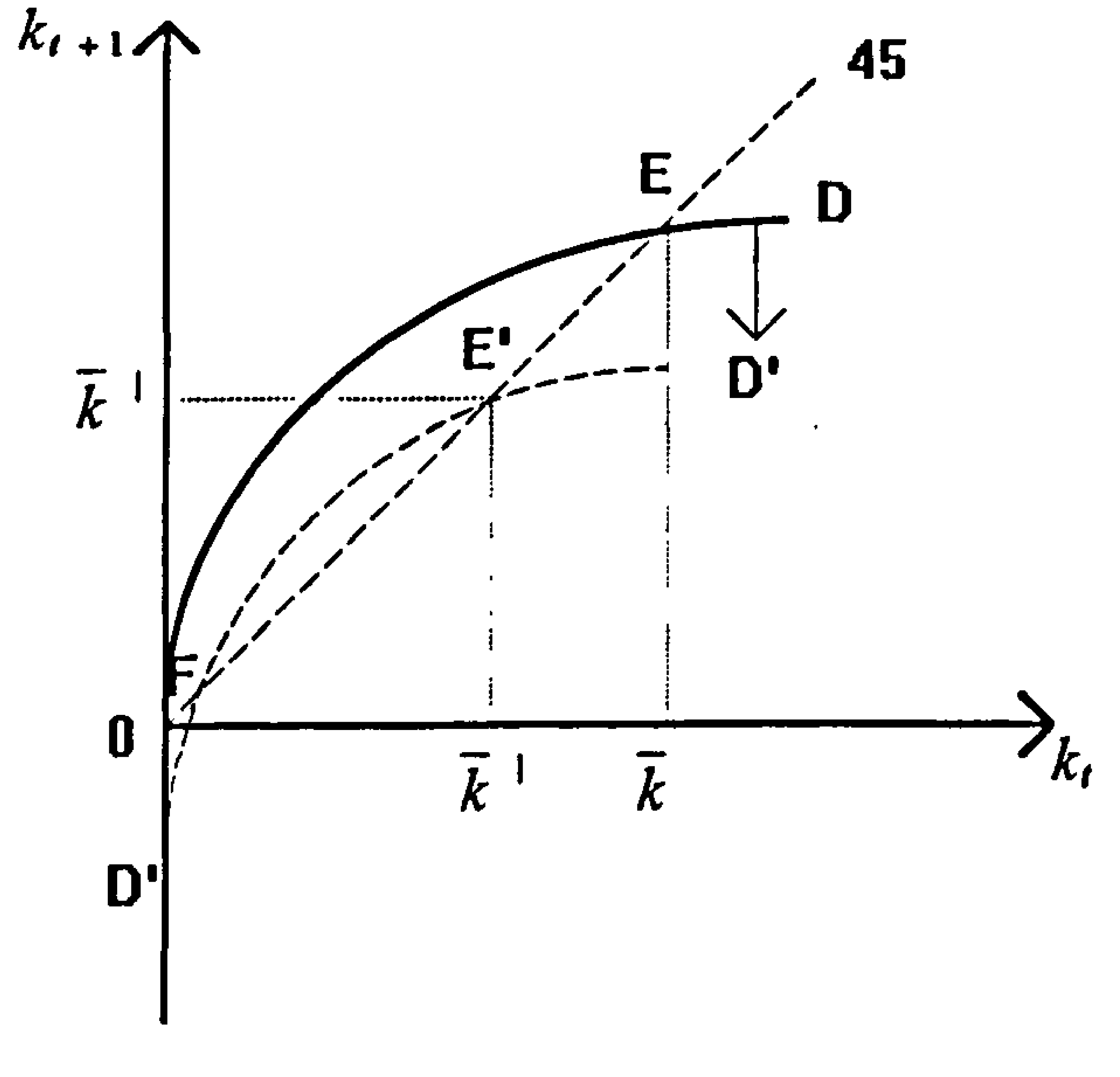

figure 2.2

The effect of an increase in debt on the capital stock is measured by the multiplier:

(2.18) $$\frac{d\bar{k}}{d\bar{b}}=\frac{-[s_w f'(\bar{k})+1]}{1+f''(\bar{k})[s_w(\bar{k}+\bar{b})-s_r]},$$

which is negative if the stability condition holds[15]. An increase in debt would, therefore, affect positively the rate of interest and negatively the stable steady state capital stock[16], so that crowding out of capital stock occurs[17]. Additional interesting aspects of this model and associated results will be illustrated in the next section where we consider the specific case of Cobb-Douglas production and utility functions.

2.2.3 2⁰ Case: Treasury Bills

A "treasury bill" is a one-period bond which yields a safe claim of one unit of the good in the next period. If the market price of such a bond in period *t-1* is q_{t-1}, then its rate of return in the next period will be $r_t=[(1-q_{t-1})/q_{t-1}]$. The government budget constraint requires that the payments at time t on the bonds issued at time *t-1*, b_t, are financed with the taxes levied on the young and with the present value of the revenues received from selling the bonds which will expire at time *t+1*. Algebraically, this relation is expressed by:

(2.19) $$b_t=\tau_t+[1+r_{t+1}]^{-1}b_{t+1}=\tau_t+q_t b_{t+1}.$$

When the stock of treasury bills issued is constant, the amount of taxes raised in each period is equal to $\tau_t=r_{t+1}[1+r_{t+1}]^{-1}\bar{b}$. In this case, by keeping constant the amount of total payments

15 This comes directly from the application of the Correspondence Principle which is commonly used in studying the comparative statics. The sign of the multiplier can be derived using the information provided by the stability condition $\left|\frac{dk_{t+1}}{dk_t}\right|<1$. In particular, the latter is satisfied if $(1-s_r f'')>-s_w f''(\bar{k}+\bar{b})$, from which it follows that the denominator in eq.(2.18) is positive.

16 It is interesting to note that the case of a steady decrease in the capital stock is associated with a too high initial level of the interest rate which reduces investments and crowds out the accumulation of capital.

17 It is worth noting that the analysis we present here is robust to the choice of a lump sum or income tax structure. The explanation is quite simple and is derived by examining the labour market structure. Since we assumed that prices and wages are perfectly flexible and the supply of labour is equal to the fixed total amount of the population (i.e. it is inelastic to the wage rate), the equilibrium on the market is not affected by an income tax. The wage earned by the workers decreases by the amount of the income tax; from the standard microeconomic theory, when the supply is inelastic to the price, the tax is fully passed onto the workers and the equilibrium levels of output and nominal wages do not change.

in each period, the government is implicitly holding constant the stock of debt inclusive of the interest payments and is paying them back by issuing new bonds and using tax revenues.

From the capital market equilibrium condition we derive the equation of motion of the system:

$$(2.20) \qquad s_t = k_{t+1} + q_t b_{t+1} = k_{t+1} + [1 + f'(k_{t+1})]^{-1} \bar{b}.$$

We can calculate the effect of an increase in debt on capital stock per capita by starting from the following relationship:

$$(2.21) \qquad k_{t+1} = s[f(k_t) - k_t f'(k_t) - f'(k_{t+1})(1 + f'(k_{t+1}))^{-1} \bar{b}, f'(k_{t+1})] - [1 + f'(k_{t+1})]^{-1} \bar{b}$$

which is a dynamic equation in the capital-labour ratio. The stability condition of the system which must be satisfied is:

$$(2.22) \qquad \left| \frac{dk_{t+1}}{dk_t} \right| = \left| \frac{-f''(\bar{k})\bar{k}s_w}{1 + \bar{b}f''(\bar{k})(s_w - 1)[1 + f'(\bar{k})]^{-2} - s_r f''(\bar{k})} \right| < 1.$$

The effect on the steady state level is given by the following multiplier:

$$(2.23) \qquad \frac{d\bar{k}}{d\bar{b}} = \frac{-[1 + f'(\bar{k})]^{-1}[s_w f'(\bar{k}) + 1]}{1 + f''(\bar{k})\bar{b}(s_w - 1)[1 + f'(\bar{k})]^{-2} - s_r f''(\bar{k}) + \bar{k}f''(\bar{k})s_w}.$$

The numerator is negative whereas the denominator is positive by direct application of the Correspondence Principle from (2.22).

In conclusion, also in the case of debt management through treasury bills, an increase in government debt would partially crowd out the capital stock in the economy. However, a direct comparison with eq.(2.18) is not immediate as it is necessary to specify functional forms both for the underlying utility and production functions[18].

2.2.4 3º Case: Perpetuity Bonds

This kind of bond entitles its buyer to receive one unit of the good in every period in perpetuity. If the market price of this bond in period *t-1* is q_{t-1}, then the rate of return will be $r_t = [1 + q_t - q_{t-1}] / q_{t-1}$. We assume that at time t the number of outstanding perpetuity bonds

[18] This exercise will be the topic of next section.

is b_t. The fact that these bonds are never repaid is analytically convenient although some complications, which were not present in the previous bonds examined, now arise. The government budget constraint is expressed as:

(2.24) $\quad b_t = \tau_t + q_t[b_{t+1} - b_t]$.

The government interest payments at time t are financed both by levying taxes on the young and by the revenue from selling more bonds between t and $t+1$. Unfortunately, it is not possible to derive an expression of q_t in terms of the interest rate only. However, a relation between q_t and r_t can be found by rewriting the expression for the bond price as follows:

$$(2.25) \quad q_{t-1} = \frac{1}{(1+r_t)} + \frac{q_t}{(1+r_t)}.$$

By substituting the same expression for q_t into eq.(2.25) we find the following:

$$(2.26) \quad q_{t-1} = \frac{1}{(1+r_t)} + \frac{1}{(1+r_t)(1+r_{t+1})} + \frac{q_{t+1}}{(1+r_t)(1+r_{t+1})}.$$

We can proceed in this way up to infinity and the equation which defines the relationship between the bond price and the rate of interest is:

$$(2.27) \quad q_{t-1} = \frac{1}{(1+r_t)} + \frac{1}{(1+r_t)(1+r_{t+1})} + ... + \frac{1}{(1+r_t)(1+r_{t+1})...(1+r_{t+i})} + \frac{q_{t+i}}{(1+r_t)...(1+r_{t+i})}.$$

Eq.(2.27) clearly indicates that the bond price in every period is the unit coupon discounted by a factor which depends on all the interest rates from the period when the bond was issued up to the time when the coupon is paid off. Therefore, the bond price can be interpreted as a forward-looking variable.

If the stock of perpetuity bonds issued is constant, from eq.(2.24) the amount of taxes raised on the young is equal to $\tau_t = \bar{b}$. Intuitively, as the government is holding constant the amount of interest payments, the amount of taxes to be raised does not change over time.

The capital market equilibrium requires that the amount of households' savings just equals the revenues the government obtains from selling bonds in each period. By substituting the expression for τ_t, the asset market equilibrium condition can be written as:

(2.28) $$\frac{1+q_{t+1}-q_t}{q_t}=f'\{s[f(k_t)-k_tf'(k_t)-\bar{b},\frac{1+q_{t+1}-q_t}{q_t}]-q_t\bar{b}\}.$$

This is a first-order difference equation in q; however, as the capital stock is an endogenous variable, we must find another equation to close the model. As we already know, the asset market equilibrium also implies that:

(2.29) $$k_{t+1}=s[f(k_t)-k_tf'(k_t)-\bar{b},f'(k_{t+1})]-q_t\bar{b}.$$

Equations (2.28)-(2.29) form a two first-order difference equation system which can be solved to find the steady state solutions. To evaluate the effect of an exogenous change in debt on the steady states, we substitute eq.(2.28):

(2.30) $$1/\bar{q}=\bar{r}=f'(\bar{k})$$

into eq.(2.29), both evaluated at the steady state, to get:

(2.31) $$\bar{k}=s[f(\bar{k})-\bar{k}f'(\bar{k})-\bar{b},f'(\bar{k})]-\frac{1}{f'(\bar{k})}\bar{b}.$$

The effect of a change in debt on the long-run level of capital stock is measured by the following multiplier:

(2.32) $$\frac{d\bar{k}}{d\bar{b}}=\frac{s_w+1/f'}{-1-f''(s_w\bar{k}-s_r)+\bar{b}f''/f'^2}.$$

Although the numerator is unambiguously positive, the sign of the denominator is ambiguous. An analysis of the comparative statics of the system is presented in appendix 1; a change in the level of debt turns out to decrease the saddle point steady state level of capital stock by direct application of the Correspondence Principle. The crowding out effect is, therefore, at work also when the government issues perpetuity bonds. In the same appendix we derive the overall effect on the steady state value of q, which is equal to:

(2.33) $$\frac{d\bar{q}}{d\bar{b}}=\frac{f''\bar{q}^2(s_w+\bar{q})}{-1-f''(s_w\bar{k}-s_r)+\bar{b}f''/f'^2}.$$

By the Correspondence Principle, this multiplier is negative at the saddle point equilibrium. This is a quite intuitive result since the relation between the perpetuity bonds price and the long-run interest rates is negative. In conclusion, an increase in debt has a crowding out effect on the steady state levels of capital and bond price. Proposition 2.1 is proved to hold also for this specific financing policy.

In the next section we study the dynamics of our model by specifying a particular form for the utility and production functions, which allows us to carry out interesting comparisons between the dynamics of the system in the presence of the different types of assets and to define a method to calculate the maximum sustainable debt level.

SECTION 3.
THE CASE OF COBB-DOUGLAS UTILITY AND PRODUCTION FUNCTIONS

2.3.1 Introduction

In this section we develop our model by appealing to Cobb-Douglas utility and production functions in order to evaluate the effects of a change in debt on the steady state capital stock and attain some definite results about the maximum sustainable level of debt-to-capital stock ratio. Each of the three financial assets analysed in the previous section are considered separately as a single specific case of our analysis.

We assume a Cobb-Douglas utility function so that the consumer's intertemporal allocation problem can be written as follows:

$$(2.34) \quad \textit{maximise} \quad \beta \ln c_t^y + (1-\beta)\ln c_{t+1}^o$$

$$\textit{subject to} \quad w_t - \tau_t = c_t^y + s_t$$

$$c_{t+1}^o = (1+r_{t+1})s_t.$$

Using the Lagrangean method, we find the saving function relevant to our model:

$$(2.35) \quad s_t = (1-\beta)(w_t - \tau_t).$$

At the same time we assume that the production function has the following Cobb-Douglas form in terms of per-capita variables:

$$(2.36) \quad y_t = k_t^\alpha \qquad 0 < \alpha < 1$$

so that we get the following expression for the wage:

(2.37) $w_t = (1-\alpha)k_t^{\alpha}$.

We now focus on the main results we obtained from our analysis which are summarised in the following proposition.

Proposition 2.2. *In our model a maximum sustainable level of debt exists. The latter is defined as the maximum level of debt which the economy can bear, consistently with a steady state existing*[19].

The concept expressed in Proposition 2.2 is strictly related, in our model, to the scenario that, if debt exceeds this maximum sustainable level, a steady state ceases to exist and the economy ends up on an unstable path. We will show that this proposition holds for all debt management policies, although dynamics differ in each particular case.

2.3.2 Savings Deposits

The asset-market locus (eq.(2.16)) is equal to:

(2.38) $k_{t+1} = (1-\beta)[(1-\alpha)k_t^{\alpha} - \alpha\bar{b}k_t^{\alpha-1}] - \bar{b}$.

The function is unambiguously concave whereas the number of steady states depends on the amount of debt. When debt is not too large, two steady states exist, and at the stable point the following condition holds:

$$(2.39)\qquad \left|\frac{dk_{t+1}}{dk_t}\right| = \left|(1-\beta)\alpha(1-\alpha)\bar{k}^{\alpha-1}\left[1+\frac{\bar{b}}{\bar{k}}\right]\right| < 1.$$

The higher $\bar{k}$, the more likely that the steady state is stable. Therefore, we can assert that the smaller of the two steady states is the unstable one, where the phaseline (*DD*) intersects the 45 degree line from below (see fig.2.3).

[19] The issue of debt sustainability is also discussed in subsection 2.3.5.

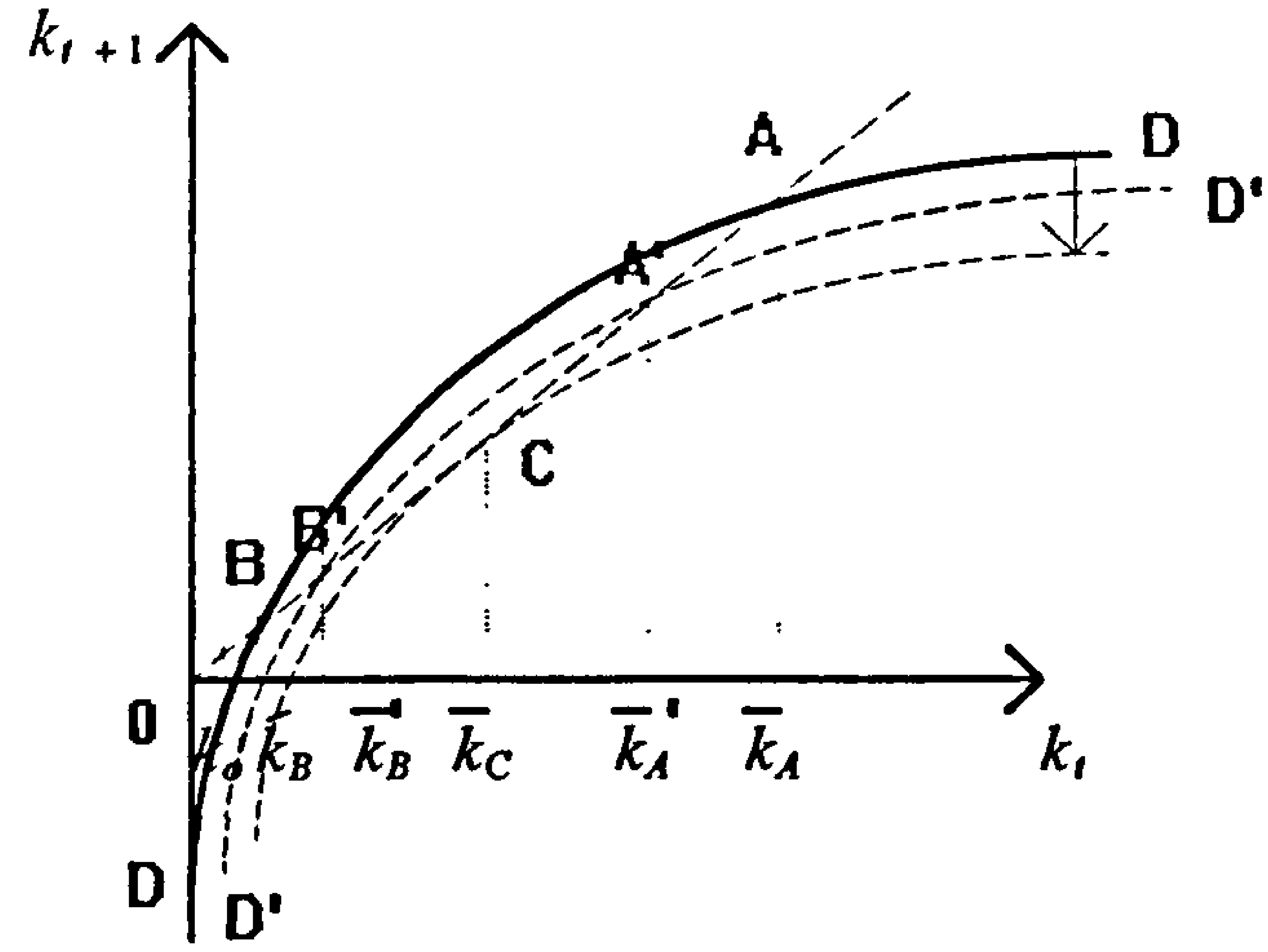

figure 2.3

A change in the level of debt affects capital stock by an amount represented by the following multiplier:

$$(2.40)\qquad \frac{d\bar{k}}{d\bar{b}} = -\frac{(1-\beta)\alpha\bar{k}^{\alpha-1}+1}{1-(1-\beta)(1-\alpha)\alpha\bar{k}^{\alpha-1}\left[1+\dfrac{\bar{b}}{\bar{k}}\right]}.$$

If the steady state is stable (point A), it is clear that an increase in debt would decrease the steady state level of capital stock (point A') by application of the Correspondence Principle[20], provided that the increase in debt is not too high. Conversely, if it starts off with a level of capital stock lower than the level corresponding to the unstable equilibrium (i.e. if the interest rate is too high or, namely, $k_o < \bar{k}_B$), the system would experience a steady decrease in capital stock until the latter becomes negative.

It is interesting to note that the instability is not associated with an explosive rise in the stock of debt - unlike in the most familiar scenario of an unstable fiscal policy - because, by assumption, the stock of debt is being held constant. Rather, the instability is a phenomenon which is based on an inherent possibility of instability in the private sector of the economy. In the model without government, we noticed that the origin is already an unstable steady state.

[20] In fig.2.3 the phaseline DD shifts to $D'D'$ following an increase in debt.

It is just because a negative stock of capital is ruled out that we cannot observe paths below this steady state where capital would follow an unstable path. When the government is introduced, debt just shifts the unstable steady state into the interior of the (k_t, k_{t+1}) space, so that paths below it become feasible.

The role of debt is, however, more important and subtle than it would seem. The simple dynamics of the model are quite intuitive. A fall in the current capital stock leads to a fall in the current wage, since less capital means lower demand for labour. A fall in the current wage leads to a fall in saving, and thereby to a fall in the future capital stock. The process then repeats itself. The instability must arise because a fall in current capital has a very large negative impact on current savings and thus on future capital (i.e. $dk_{t+1} / dk_t > 1$ at point B). With Cobb-Douglas preferences, savings are just a constant fraction, less than one, of the wage. This implies that the source of a strong negative impact of a fall in current capital on savings must lie in a large negative impact of a fall in capital on the wage. With Cobb-Douglas technology, the wage is just a constant fraction of per-capita output. A fall in capital has a negative impact on output. When capital is already close to zero, the impact of a fall in capital on the wage will be powerfully negative, and sufficient to make future capital fall more than one-for-one with current capital. Hence the source of the instability is the high marginal product of capital around the unstable steady state which occurs at low levels of initial capital and consequently a shrinkage of the current wage.

Debt risks exacerbating these unstable dynamics because it affects disposable income (which is already negatively affected by the reduction of capital) through the taxes. The latter are equal to $r_t\bar{b}$, and, remembering that the interest rate payable on debt tends to increase as current capital falls, the taxes which the government must levy are raised. So positive debt increases the likelihood of instability in the economy by worsening the negative effect which a fall in current capital has on the young's disposable income, and thereby on their savings.

In conclusion, instability may arise as a consequence of a too low initial level of capital stock irrespective of the size of debt (provided that it is non-zero). This result can be interpreted as an example of the *poverty trap* phenomenon which has been discovered to affect many poor countries. If a country cannot rely on external sources to increase its level of capital stock, then it would have to rely completely on its population's savings. However, if the country is temporarily at the unstable equilibrium with the lower level of capital stock, then any small shocks could endanger its future productive capacity. It is also interesting to note that the higher the level of debt, the larger the poverty trap. This conclusion can justify the fact that a high debt level is often interpreted as a signal of financial instability of an economy and it would discourage investments in that country. Savers would learn from experience and new bonds could be placed only at much higher real interest rates. Without additional financial resources, capital stock could not grow but, on the contrary, it would decrease as a higher number of investors would shift towards other investment opportunities elsewhere.

The analysis just presented is based on the assumption that debt does not exceed a certain level, as stated in Proposition 2.2. In fact, as debt rises, the two steady states get closer until for a certain level of $\bar{b}$ no steady state exists (i.e. beyond point C in fig.2.3). If the government decides to increase debt beyond the level for which the two steady states coincide, the system would head towards a collapse with decumulation of capital stock. The disappearance of the steady states is known as a *catastrophe*, and it is connected to debt being pushed slightly beyond the maximum sustainable level for which a steady state still exists. When this destructive phenomenon occurs, the economy leads towards a collapse.

This maximum level of debt which the economy can bear without ending on an unstable path is also called the *maximum sustainable level of debt.* At that level, the total amount of savings in the economy would be still sufficient to cover the new issue of debt and the requirements for new capital stock.

In this model the concepts of *sustainability* and *stability* are clearly distinct. While sustainability refers to an acceptable size of debt which can be an offsetting investment to productive capital, the intuition behind the stability property relates to the behaviour of the system when a temporary shock in the amount of debt run occurs. When the government issues additional bonds, individuals, by acquiring these bonds, would consider themselves wealthier (because bonds raise their lifetime income) and consume more, thus reducing their savings. Lower savings mean lower demand for bonds and, ceteris paribus, interest rates have to rise to balance the bonds market. As higher real interest rates are consistent with a lower level of capital stock, there would be lower income available to be devoted to savings in subsequent periods. However, if the effect of the interest rate on savings is not too strongly negative, the system would end up at the new lower steady state level of capital. The two concepts of sustainability and stability are interrelated since continuous positive shocks in debt may cause an initially sustainable policy to become unsustainable and bring about unstable dynamics[21].

2.3.3 Treasury Bills

The asset market equilibrium in this case is equal to:

$$(2.41)\qquad k_{t+1}=(1-\beta)(1-\alpha)k_t^{\alpha}+\frac{\alpha\beta\bar{b}\,k_{t+1}^{\alpha-1}}{1+\alpha k_{t+1}^{\alpha-1}}-\bar{b}$$

where r_{t+1} has been replaced by $\alpha k_{t+1}^{\alpha-1}$. The stability condition[22] is the following:

$$(2.42)\qquad \left|\frac{\alpha(1-\alpha)(1-\beta)\bar{k}^{\alpha-1}(1+\alpha\bar{k}^{\alpha-1})}{1+\alpha^2\bar{k}^{\alpha-1}+\alpha(1-\alpha)^2(1-\beta)\bar{k}^{2\alpha-2}-\alpha\bar{b}(1-\alpha)(1-\beta)\bar{k}^{\alpha-2}}\right|<1.$$

The shape of the function resembles the phaseline for the savings deposits, so that there exists a maximum sustainable level of debt beyond which the system heads towards the collapse (Proposition 2.2). The debt multiplier is equal to:

(2.43)

$$\frac{d\bar{k}}{d\bar{b}}=-\frac{1+\alpha(1-\beta)\bar{k}^{\alpha-1}}{1+\alpha^2\bar{k}^{\alpha-1}-\alpha(1-\alpha)(1-\beta)\bar{k}^{\alpha-1}-\alpha(1-\alpha)(1-\beta)(2\alpha-1)\bar{k}^{2\alpha-2}-\alpha\bar{b}(1-\alpha)(1-\beta)\bar{k}^{\alpha-2}}.$$

---

21 At the end of this chapter, we will point out that, in our model, the concept of maximum sustainable level of debt is associated with the occurrence of a catastrophe in the economy.

22 In order to calculate the derivatives, the asset market equilibrium has been rewritten as:

$$k_{t+1}+\alpha k_{t+1}^{\alpha}=(1-\alpha)(1-\beta)k_t^{\alpha}+\alpha(1-\alpha)(1-\beta)k_t^{\alpha}k_{t+1}^{\alpha}+\alpha\beta\bar{b}\,k_{t+1}^{\alpha-1}-\bar{b}-\alpha\bar{b}\,k_{t+1}^{\alpha-1}.$$

Again, by applying the Correspondence Principle, it follows that the effect of an increase in debt on the capital stock is negative, as previously analysed in Proposition 2.1.

2.3.4 Perpetuity Bonds

The existence of multiple equilibria can be demonstrated by solving the system made up by eqs.(2.28)-(2.29), evaluated at the steady state:

$$(2.44)\qquad \frac{1}{\bar{q}} = \alpha[(1-\alpha)(1-\beta)\bar{k}^{\alpha} - (1-\beta)\bar{b} - \bar{q}\bar{b}]^{\alpha-1}$$

$$(2.45)\qquad \bar{k} = (1-\beta)[(1-\alpha)\bar{k}^{\alpha} - \bar{b}] - \bar{q}\bar{b}.$$

By combining these last two equations, we obtain an equation either in $\bar{k}$:

$$(2.46a)\qquad \bar{b}(1-\beta)\alpha = \alpha(1-\alpha)(1-\beta)\bar{k}^{\alpha} - \alpha\bar{k} - \bar{b}\bar{k}^{1-\alpha}$$

or, equivalently, in $\bar{q}$:

$$(2.46b)\qquad \left[\frac{(\alpha\bar{q})^{\frac{1}{1-\alpha}} + \bar{b}(1-\beta) + \bar{b}\bar{q}}{(1-\alpha)(1-\beta)}\right]^{\frac{1}{\alpha}} - (\alpha\bar{q})^{\frac{1}{1-\alpha}} = 0.$$

The system has two solutions. This result seems to be robust to any values of the parameters (α,β), provided that the level of debt does not exceed a critical level beyond which no steady state exists.

The effect of a debt increase on the steady state level of capital stock is measured by the following multiplier:

$$(2.47)\qquad \frac{d\bar{k}}{d\bar{b}} = -\frac{\bar{k}^{1-\alpha} + \alpha(1-\beta)}{\bar{b}(1-\alpha)\bar{k}^{-\alpha} + \alpha - \alpha^2(1-\alpha)(1-\beta)\bar{k}^{\alpha-1}}.$$

The existence of multiple equilibria comes out clearly also from the analysis of the dynamic properties of the system.

Let us start from the first equation for the asset market stationary locus (eq.(2.28)[23]), where $\Delta q_t = 0$:

$$(2.48) \qquad k_t = \left[\frac{(\alpha q_t)^{1/(1-\alpha)} + (1-\beta)\bar{b} + q_t \bar{b}}{(1-\alpha)(1-\beta)} \right]^{1/\alpha}.$$

The phaseline QQ is illustrated in fig.2.4 where the intersection point with the abscissa axis is $\left(\frac{\bar{b}}{1-\alpha} \right)^{1/\alpha} > 0$. As debt rises, the stationary locus shifts down and outwards.

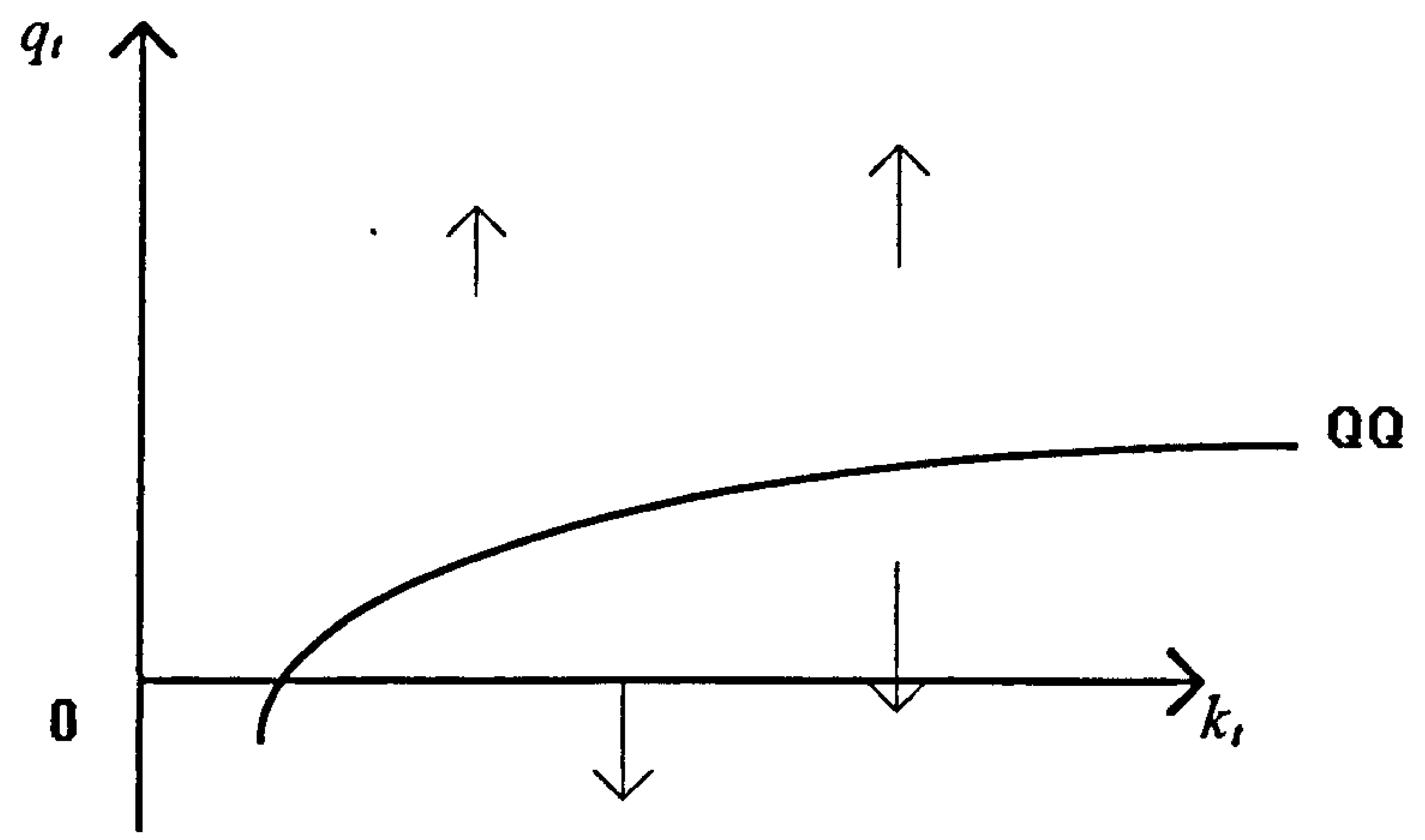

figure 2.4

The first derivative of the function is equal to:

$$(2.49) \qquad \frac{dq_t}{dk_t} = \frac{\alpha(1-\alpha)(1-\beta)\bar{k}^{\alpha-1}}{\frac{\alpha}{1-\alpha}(\alpha\bar{q})^{\frac{\alpha}{1-\alpha}} + \bar{b}}.$$

---

[23] The asset market difference equation, using the specific Cobb-Douglas utility and production functions, is:

$$\frac{1+q_{t+1}-q_t}{q_t} = \alpha\left[(1-\alpha)(1-\beta)k_t^{\alpha} - (1-\beta)\bar{b} - q_t\bar{b}\right]^{\alpha-1}.$$

Given that the acceptable values for the two endogenous variables must lie in the positive quadrant, the phaseline is upward sloping. For all the points above the phaseline the price of the bonds, q_t, increases and conversely for all the points below.

The second equation, which represents the capital stock stationary locus (eq.(2.29)[24]), $\Delta k_t = 0$, is:

$$(2.50)\qquad q_t = -\frac{k_t}{\bar{b}} + \frac{(1-\alpha)(1-\beta)k_t^{\alpha}}{\bar{b}} - (1-\beta).$$

The sign of the first derivative:

$$(2.51)\qquad \frac{dq_t}{dk_t} = -\frac{1}{\bar{b}} + \frac{\alpha(1-\alpha)(1-\beta)\bar{k}^{\alpha-1}}{\bar{b}}$$

depends on the size of the capital stock. The smaller is the capital stock, the more likely that the slope is positive. The function is concave as suggested by the second derivative:

$$(2.52)\qquad \frac{d^2 q_t}{dk_t^2} = -\frac{\alpha(1-\alpha)^2(1-\beta)}{\bar{b}}\bar{k}^{\alpha-2} < 0.$$

The phaseline, *KK*, in fig.2.5, has the shape of a parabola with the downward concavity. The capital stock increases below the function and decreases above it. As debt rises, the stationary locus shifts down at each level of k_t.

24 The capital stock difference equation using the specific Cobb-Douglas utility and production functions becomes:

$$k_{t+1} = (1-\alpha)(1-\beta)k_t^{\alpha} - q_t\bar{b} - (1-\beta)\bar{b}.$$

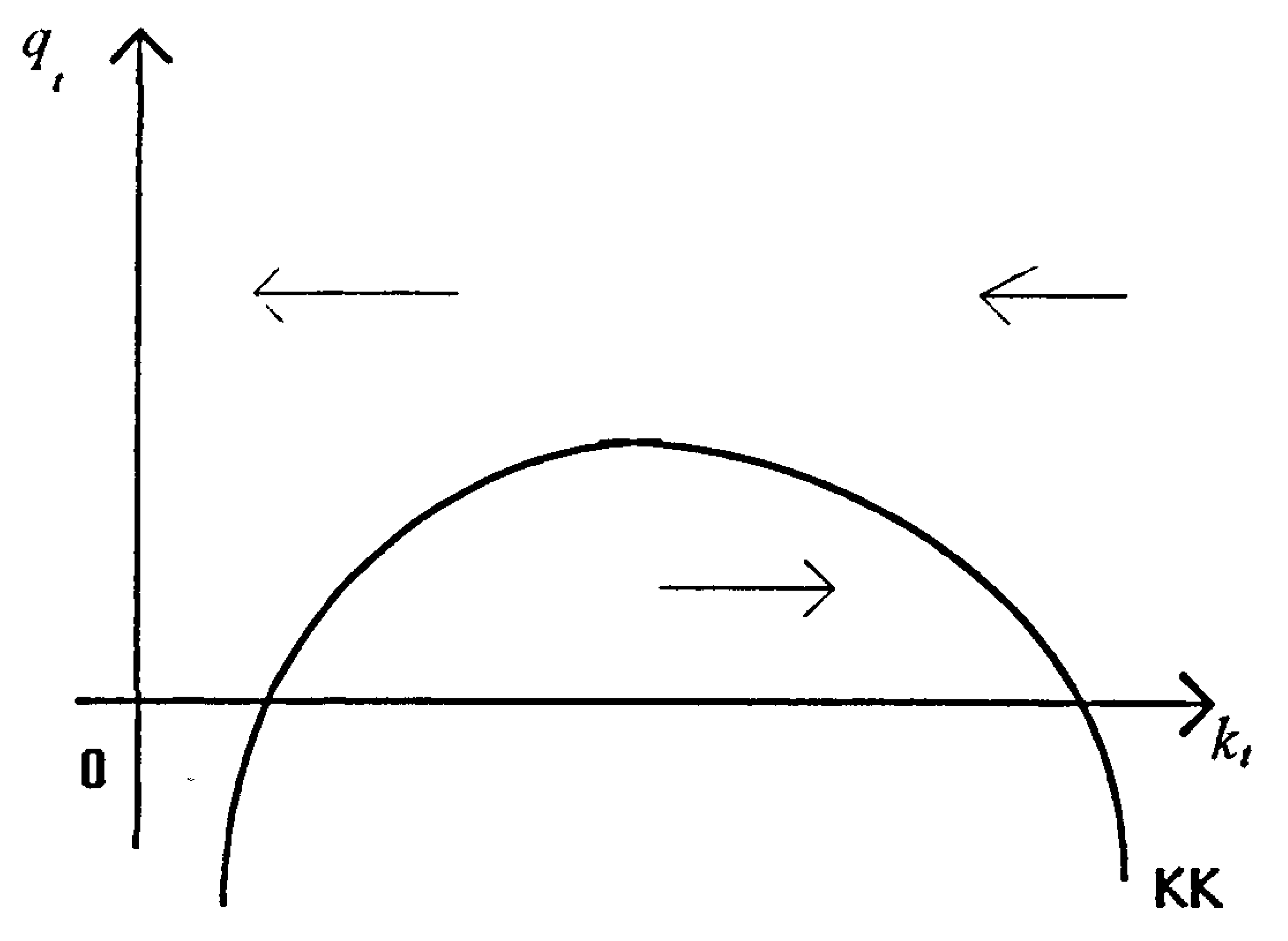

figure 2.5

The overall system and its dynamics are represented in fig.2.6.

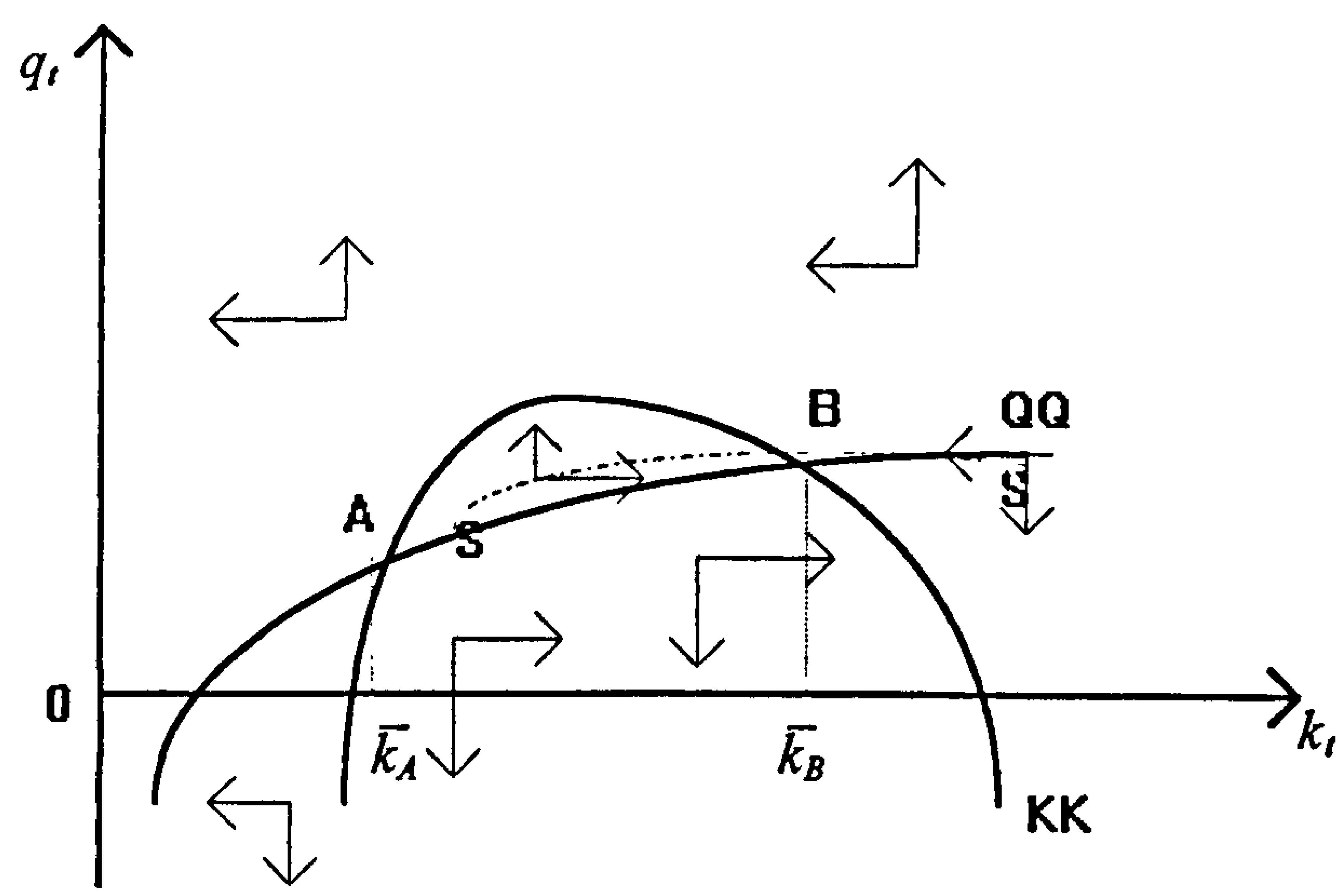

figure 2.6

The graph analysis suggests that point A is a source whereas point B is a saddle. The line SS represents the saddle path which is the unique locus of points which converges to the equilibrium in the neighbourhood of the saddle point steady state, B. It must be noted that the existence of the two steady states is conditional on the assumption that debt does not exceed a level, beyond which dynamics of capital stock decumulation would arise (Proposition 2.2).

Now the analysis proceeds with the investigation of the stability properties of the two steady states given some sets of parameter values and with the study of the effects of an increase in the size of debt on the steady state values of the capital stock and the bond price. The procedure followed is divided into three steps. Firstly, we find the Jacobian matrix of the system and we linearise the system around the steady state. Secondly, we provide the conditions which must be satisfied for the system to be a saddle and a source, respectively[25], and we apply the Correspondence Principle to determine the sign of the long-run debt multipliers. A brief description of the Correspondence Principle and its applications to a discrete time models is found in appendix 1 at the end of this chapter. Finally, we check that the eigenvalues of the system are consistent with the characterisation of the two stationary points as described before.

The local stability properties of the system are studied by using two particular solutions to the system found for some sets of parameters. These solutions are found by using "Mathematica" which has turned out to be very helpful in solving non-linear equations and depicting some of the functions analysed in this thesis. For instance, for the set of parameter values $\{\alpha = 0.5, \beta = 0.3, \bar{b} = 0.01\}$ we found the following solutions:

$$(2.53) \qquad A = \begin{cases} k_1 = 0.000519173 \\ q_1 = 0.0455707 \end{cases} \qquad B = \begin{cases} k_2 = 0.0943808 \\ q_2 = 0.614429 \end{cases},$$

whereas for $\{\alpha = 0.5, \beta = 0.7, \bar{b} = 0.01\}$ we found:

$$A = \begin{cases} k_1 = 0.0009 \\ q_1 = 0.06 \end{cases} \qquad B = \begin{cases} k_2 = 0.01 \\ q_2 = 0.2 \end{cases}.$$

These solutions will be useful to support our intuitions about the properties of the two steady states.

[25] See Azariadis (1993), pp.63-67.

STEP 1

The system can be rewritten in a matrix form as follows:

(2.54)

$$\begin{bmatrix} dq_{t+1} \\ dk_{t+1} \end{bmatrix} = \begin{bmatrix} 1+\frac{\alpha}{\bar{k}^{1-\alpha}}+\frac{(1-\alpha)\bar{b}}{\bar{k}} & -\alpha(1-\alpha)^2(1-\beta)\bar{k}^{\alpha-2} \\ -\bar{b} & \alpha(1-\alpha)(1-\beta)\bar{k}^{\alpha-1} \end{bmatrix} \begin{bmatrix} dq_t \\ dk_t \end{bmatrix} + \begin{bmatrix} \frac{1-\alpha}{\bar{k}}\left(1-\beta+\frac{\bar{k}^{1-\alpha}}{\alpha}\right) \\ -(1-\beta)-\bar{q} \end{bmatrix} d\bar{b}$$

where the Jacobian matrix is equal to:

$$(2.55) \qquad J(q,k) = \begin{bmatrix} 1+\frac{\alpha}{\bar{k}^{1-\alpha}}+\frac{(1-\alpha)\bar{b}}{\bar{k}} & -\alpha(1-\alpha)^2(1-\beta)\bar{k}^{\alpha-2} \\ -\bar{b} & \alpha(1-\alpha)(1-\beta)\bar{k}^{\alpha-1} \end{bmatrix}$$

with

$$T = 1+\frac{\alpha}{\bar{k}^{1-\alpha}}+\frac{(1-\alpha)\bar{b}}{\bar{k}}+\alpha(1-\alpha)(1-\beta)\bar{k}^{\alpha-1} > 0 \qquad (T\text{=Trace})$$

$$D = \alpha(1-\alpha)(1-\beta)\bar{k}^{\alpha-1}\left[1+\frac{\alpha}{\bar{k}^{1-\alpha}}\right] > 0 \qquad (D\text{=Determinant}).$$

If we denominate each term of the matrix above by a_{ij}, where $i,j = 1,2$ indicate the number of the rows and the columns respectively, and we subtract dq_t (dk_t) from the first (second) row, the system can be rewritten as:

$$(2.56) \qquad \begin{bmatrix} \Delta dq_{t+1} \\ \Delta k_{t+1} \end{bmatrix} = \begin{bmatrix} a_{11}-1 & a_{12} \\ a_{21} & a_{22}-1 \end{bmatrix} \begin{bmatrix} dq_t \\ dk_t \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} d\bar{b}.$$

On the two stationary loci $\Delta q_{t+1} = \Delta k_{t+1} = 0$, so that the system can be transformed as:

$$(2.57) \qquad \begin{bmatrix} a_{11}-1 & a_{12} \\ a_{21} & a_{22}-1 \end{bmatrix} \begin{bmatrix} dq_t \\ dk_t \end{bmatrix} = \begin{bmatrix} -b_1 \\ -b_2 \end{bmatrix} d\bar{b}.$$

where the matrix on the LHS is hereafter called coefficient matrix C.

STEP 2

i) If point B is a saddle, this implies that the eigenvalues are real and the dominant one is outside the unit circle while the other is inside. In a deterministic system, this is equivalent to saying that the system converges to the steady state with a positive debt only for an initial value of q_o, given some arbitrary k_o. This shows that, although the level of debt is kept constant, financial instability may, nevertheless, take place.

Considering the Jacobian matrix in (2.55), since $T > 0$, the condition which must be satisfied to be a saddle is: $-(T+1) < D < (T-1)$. By substituting the co-ordinates of point B for both sets of parameter values in (2.53), we find that these inequalities are satisfied.

ii) If point A is a source, this implies that both eigenvalues are real and outside the unit circle. The stationary state is an unstable node. Out of the three different sets of conditions for which, if satisfied, the system can be a source, the only set which applies with $T > 0$ is the following: $D > T-1, D > -(T+1), D > 1, T > 2$. Substituting the values of the co-ordinates of point A in (2.53) we find that all the inequalities are satisfied.

In order to determine the sign of the long-run debt multipliers, we apply the Correspondence Principle. As we explain in appendix 1 to this chapter, the sign of the determinant of the matrix on the LHS of (2.57) is related to the stability properties of the steady state. If the steady state is saddle point stable, the sign of the determinant is negative and conversely if it is unstable. The long-run multipliers are calculated from (2.57):

$$(2.58) \qquad \frac{d\overline{k}}{d\overline{b}} = \frac{(a_{11}-1)(-b_2)+b_1 a_{21}}{\det C}; \qquad \frac{d\overline{q}}{d\overline{b}} = \frac{-b_1(a_{22}-1)+b_2 a_{12}}{\det C}.$$

By substituting each term of the matrix and carrying out the calculations, we get:

$$(2.59) \qquad \frac{d\overline{k}}{d\overline{b}} = \frac{\left(1-\beta+\frac{\overline{k}^{1-\alpha}}{\alpha}\right)\left(\frac{\alpha}{\overline{k}^{1-\alpha}}\right)}{\det C}.$$

The numerator of the multiplier is positive whereas the sign of the denominator depends on the steady state at which $\det C$ is evaluated. The determinant of the coefficient matrix C is equal to:

$$\text{(2.60)} \qquad \det C = \frac{1}{(\bar{k}^{1-\alpha})^2} \cdot \left[\alpha^2(1-\alpha)(1-\beta) - \alpha\bar{k}^{1-\alpha} - (1-\alpha)\bar{b}\,\bar{k}^{1-2\alpha}\right].$$

The higher the steady state level of the capital stock, the more likely the determinant to be negative. As a matter of fact, when evaluated at the steady state B (in (2.53)), det C turns out to be negative whereas for the source A it is positive. By direct application of the Correspondence Principle[26], it follows that, as a consequence of a debt increase, there is crowding out of capital stock if the steady state is saddle point stable. Conversely for the unstable steady state.

The same reasoning can be applied to find the long-run debt multiplier of the bond price. The latter is equal to:

$$\text{(2.61)} \qquad \frac{d\bar{q}}{d\bar{b}} = \frac{\left(1-\beta+\dfrac{\bar{k}^{1-\alpha}}{\alpha}\right)\cdot\dfrac{1-\alpha}{\bar{k}}}{\det C}.$$

The numerator is positive, so that an increase in debt decreases the stable equilibrium value of the perpetuity bond, QED.

STEP 3

The eigenvalues, found by using "Mathematica" and substituting the steady state values of $\bar{k}$, satisfy the properties anticipated for each steady state[27]. In conclusion, there is a unique path which converges to the stationary equilibrium B. Therefore, if initially the system is off this path, i.e. there is a wrong combination of capital stock and bond price, it will never converge to a stationary point because dynamics would drive it away from the equilibrium. What distinguishes this analysis from that the two previous cases of financial assets is that a

[26] For a more complete explanation of the Correspondence Principle, see appendix 1.
[27] We omit to report the solutions for the eigenvalues for matter of simplicity.

completely stable (sink) equilibrium does not exist, so that the initial conditions are more stringent for the system to finally converge to an equilibrium: in fact, even if $k_o > \bar{k}_A$, we still have to assume away "bubble paths" to approach point B. On the other hand, if the initial condition for k implies that $k_o < \bar{k}_A$, then the poverty trap is still a possibility with this kind of financial instrument.

A final comment can be drawn from comparing the multipliers given in eqs.(2.40)-(2.43) and (2.47). After some calculations we can state that for $\bar{k} > [\alpha(1-\alpha)(1-\beta)]^{\frac{1}{1-\alpha}}$ (and setting $\bar{b} = 0$ for simplicity) the multiplier is bigger in the case of perpetuity bonds than in the other two cases, thus indicating a larger crowding out effect. This is a relevant result which has been derived by considering an initial framework which does not involve the presence of debt a government has suddenly to finance. Therefore, it implies that the financial instrument the government chooses to use can be relevant to the process of economic growth of a country[28].

2.3.5 The Issue of Debt Sustainability

The main findings of the model we have presented are twofold. The first one is that there is a positive likelihood that a system facing a positive amount of debt is unstable and the consequent **collapse of the economy** occurs if the latter exceeds a critical level[29]. This is a very important and interesting finding which denotes how "*...deficits may be simply unsustainable, a possibility never considered by the traditional view*"[30]. High levels of public-sector debt may simply push up interest rates, displacing private investment and creating the so called **debt trap.** Beyond a certain level, indebtedness can cause a vicious circle: rising debt boosts interest payments which in turns leads to extra borrowing. In addition, our model has shown that if a too high constant debt is accumulated, unsustainability can still arise. Only if the government runs a surplus on its primary budget it

[28] In appendix 2 we show that, conditional on debt equal to zero, the size of the multiplier for the treasury bills is the smallest one independently of the production function used.

[29] The concept of debt repudiation has been a major topic of discussion in these last few months given the possible debt default by the Mexican government.

[30] Blanchard (1984), p.7.

can lighten the burden on tomorrow's taxpayers. Looking at the real world, this issue is strictly related to the debate of whether debt may help the recovery of a country by increasing aggregate demand or, if instead, it will increase the long-run interest rates thus offsetting the expansionary effect.

The second finding is that it can explain the existence of the **poverty trap** if the capital stock is extremely low when the government increases the issue of debt. This tends to underline how a policy of debt management through the issue of bonds is sustainable only if the capital stock in the economy is high enough to be maintained in the long run[31].

However, the most probably distinctive feature of our model is that **a maximum sustainable level of debt** exists, which is crucial for determining the future path of the economy. When debt is pushed very slightly beyond this point, a **catastrophe** occurs in the economy. This catastrophe takes the form of a steady state with a strictly positive capital stock suddenly ceasing to exist, so that the economy is thrown into an abyss of uncontrolled capital decumulation, leading to complete annihilation. This catastrophic scenario can label a fiscal policy as destructive if a small shock makes debt bigger than the maximum sustainable level, which would provoke a massive collapse. This phenomenon will be underlined further on in this thesis and it turns out to be associated with the existence of two non-trivial steady states, a policy of increasing debt beyond the level for which no equilibrium exists.

Finally, we confirm the standard result that debt plays an important role in affecting the long-run level of the capital stock and, consequently, of aggregate demand and output.

Before considering a simulation exercise of our model, we briefly compare the results we have derived in this section with those reported in section 3 of Blanchard's (1984) paper, which focuses on the same issue of debt sustainability formalised in a standard macroeconomic continuous time but not microfounded set-up. Both the assumptions made

[31] Blanchard (1984) claims that the sustainable levels of debt are very large at the historical level of real rates of 1-2% but much smaller at the current 3-6%. His analysis is quite similar to ours in so far as his computations do not take into account GNP growth (which would increase the sustainable ratio of debt-to-GNP) and do not allow for monetization and the use of the inflation tax.

and the results achieved by Blanchard differ from ours, and his most interesting conclusions are here summarised.

First of all, he defines the maximum level of debt as a function of the difference between the maximum amount of taxes which can be collected by the government and the minimum socially acceptable amount of government outlays, and these measures are, noticeably, quite *ad hoc*. On the other side, the rule of deficit adjustment is criticizable. In fact, the budget surplus X (or deficit) is assumed to converge slowly to its maximum level until the gap between the current and the maximum level is eliminated. Debt D instead increases in order to repay the interest payments and net public spending. Only if D and X at time $t=0$ are such that the system is on the saddle path or above it on a trajectory which intersects the function $\dot{D}=0$, would the dynamics eventually lead to the equilibrium. Any time the starting point is off and below the saddle path, deficit is not sustainable: although it is reduced as fast as feasible, the maximum level of debt would be reached before deficits are totally eliminated. However, difficulties arise because the maximum socially acceptable net deficit is difficult to define empirically. With a different approach we derived the maximum level of debt endogenously in the model[32].

Secondly, in our model the interest rate paid on debt is no longer constant, and this permits to evaluate the effects of the fiscal policy on the real economy.

The concept of debt sustainability as introduced in our model seems to be very important in relation to the convergence criteria as the necessary condition for joining the monetary union according to the Maastricht Treaty. As underlined in the previous chapter, the fiscal adjustment is very urgent for some countries since the future involvement in the process of the European Monetary Integration prohibits monetization as a source of revenue. The transition to the final stage of the monetary union depends on the capability of a country to meet these requirements. The most likely forecast is that, in the future, countries will have to run high primary budget surpluses to prevent an explosive growth of debt.

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32 However, in Blanchard's analysis the requirement that the initial values of debt and deficit are on or above the saddle path may be broadly interpreted as the conditions for sustainability.

We continue our analysis of the sustainability of debt management policies by investigating whether our model fits in the reality and how reliable it can be as an instrument for evaluating current fiscal policies in the G-7 countries.

SECTION 4.
SOME CALIBRATION EXERCISES AND A REAL WORLD FISCAL POLICY EVALUATION

2.4.1 Introduction

From the model we have set up in this chapter, the issue of maximum sustainable debt-to-capital stock ratio has turned out to be very important for evaluating the fiscal policy and the future of a country. Now the purpose of our exercise is to highlight some of the aspects of fiscal policy sustainability. In the first subsection, we simulate our model to find the maximum sustainable level of debt for the three types of financing policies (for particular parameter values) and we compare one each other in order to determine under what policy the maximum sustainable debt-to-capital stock ratio is the biggest.

In the second subsection, the ratios found by the model simulations are compared with the debt-to-capital stock ratios which are calculated for some developed countries using the NIGEM[33] (National Global Econometric Model) data referring to the G-7 countries. A general evaluation of the sustainability of the policies pursued in those countries is then provided by taking the simulated maximum sustainable ratios as benchmarks (i.e. the maximum levels beyond which a debt management policy should be regarded as unsustainable according to our model).

[33] The data of this model were made available by the ESRC Macroeconomic Modelling Bureau at the University of Warwick.

This exercise aims at studying whether a very simple model like ours, when parameterised with plausible parameter values, provides values for the "maximum sustainable level of debt" which can be compared with the actual values. As a consequence, this is not intended as a precise "calibration exercise", but just a simple and rough attempt to see whether the model generates numbers which are reasonably consistent with the real-world numbers.

Before proceeding with our analysis, it is obvious to point out that this exercise presents some drawbacks due to the simplified structure of our model, which partially affect the final results. In the next pages we first illustrate the results of the simulations, and then we deal with their interpretation.

2.4.2 A Calibration Exercise

In this subsection we simulate our model with the aim of determining the maximum sustainable level of debt under the three different financing policies presented in this chapter. The simulations are carried out for different parameter values of the utility function, but in all of them the value of the share of the capital stock in the Cobb-Douglas production function (α) has been kept fixed at one-third as suggested by the empirical estimations in the literature.

The sets of parameters considered are $\{\alpha = 0.3, \beta = 0.3\}, \{\alpha = 0.3, \beta = 0.5\}, \{\alpha = 0.3, \beta = 0.7\}$ where β represents the pure time preference rate in the Cobb-Douglas utility function.

In the following table we report the values of the simulations for each financing policy marked with a subscript. The equations considered are (2.38)-(2.41)-{(2.48)-(2.50)} for savings deposits, treasury bills and perpetuity bonds, respectively.

Table 2.1

Param. values	Savings Deposits			Treasury Bills			Perpet. Bonds		
$\alpha=0.3$	$R^*_{SD} = \frac{b^*_{SD}}{k^*}$ (1)	$R^*_{TB} = R^*_{SD}(1+r^*_{SD})$ (2)	$R^*_{PB} = r^*_{SD} \cdot R^*_{SD}$ (3)	$R^*_{SD} = \frac{R^*_{TB}}{1+r^*_{TB}}$ (4)	$R^*_{TB} = \frac{b^*_{TB}}{k^*}$ (5)	$R^*_{PB} = R^*_{TB} \cdot \frac{r^*_{TB}}{1+r^*_{TB}}$ (6)	$R^*_{SD} = \frac{R^*_{PB}}{r^*_{PB}}$ (7)	$R_{TB} = R^*_{PB} \cdot \frac{1}{r^*_{PB}} \cdot (1+r^*_{PB})$ (8)	$R^*_{PB} = \frac{b^*_{PB}}{k^*}$ (9)
$\beta=0.3$	1.6113	4.5745	2.9632	1.0072	3.1727	2.1655	1.4823	7.6598	6.1754
$\beta=0.5$	0.5625	1.5689	0.9895	1.1469	5.2482	4.1014	1.2744	10.2845	8.7941
$\beta=0.7$	0.275	0.7584	0.0387	1.2228	9.3333	8.1105	1.4746	15.6531	14.1785

Table 2.1 reports the values of the maximum sustainable debt-to-capital stock ratios, $R^*_i = \frac{b^*_i}{k^*_i}$, where $i = SD, TB, PB$. Some interesting observations can be put forward. First of all, it seems to be a quite general result that the maximum sustainable debt-to-capital stock ratio is different according to the policy the government chooses to manage its debt.

Secondly, the comparison between the three different government financing policies depends on the value of the parameter β, so that the individual's pure time preference can influence the fiscal policy sustainability and the amount of debt which can be raised relatively to the capital stock within the economy. A careful investigation of the figures in table 2.1 suggests the following comments:

a) as far as R^{*}_{SD} is concerned, the comparison carried out between the figures in columns (1)-(4)-(7) allows to define the policy consistent with the highest sustainable debt-to-capital stock ratio:

- for $\beta = 0.3$, R^{*}_{SD} is the greatest under the savings deposits policy;

- for $\beta = 0.5$ and $\beta = 0.7$, this ratio is the greatest under the perpetuity bonds policy.

b) As far as R^{*}_{TB} and R^{*}_{PB} are concerned, the comparisons are carried out between figures in columns (2)-(5)-(8) for *TB* and in columns (3)-(6)-(9) for *PB*: both ratios are the greatest in the perpetuity bonds case independently of the value of β.

An explanation why the financing policy through perpetuity bonds yields the higher ratios for both the treasury bills and perpetuity bonds is that the interest payments which are included in the simulated amount of debt are very high since the interest rates are measured over a forty-year period (these values are here omitted for convenience). Of course, savings deposits are not affected by this problem since they are defined as net of interests. Consequently, when we compare real data (in which interest rates data are annual) with the present model simulations, it will be more appropriate to compare the ratios referring to the savings deposits case only, taking column (1) as the benchmark.

Before turning our attention to the real world, it is worth remembering that another measure usually adopted to evaluate the sustainability of a fiscal policy is the debt-to-GDP ratio. However, this measure within our model is meaningless. The reason is that the simulated values of the GDP (which is a flow variable) in our model would be measured over a period of 40 years; a big problem would then arise if we tried to divide a stock (debt) with a flow variable (although measured on the same period) and if the latter referred to a period of more than one year. Instead this problem does not arise if we divide two stocks (like debt and capital stock), no matter if measured over forty years like in our model, provided they are measured over the same time span.

In the next subsection we compare our model simulations with some actual figures for the ratios (R_{SD}, R_{TB}, R_{PB}) in some developed countries. These values will be important for evaluating government debt management policies and the trends of debt-to-capital stock ratio which reflect approximately the economic fiscal situation in each country.

2.4.3 An Evaluation of the Current Fiscal Policy in the G-7 Countries

As already explained, our main purpose is to resort to some data referring to the G-7 countries debt and capital stock in order to compare the debt-to-capital stock ratios so calculated with the simulated maximum sustainable ratios of our model and to give a general evaluation of the performance of the model itself in explaining the reality. For this purpose, we resort to the NIGEM data. Complete and detailed tables of the main economic variables data of our interest referring to the G-7 countries are contained in appendix 3, tables (C.1-C.7), where we also summarise briefly the fiscal situation in each individual country.

By comparing the actual ratios, R_{SD}, with the simulated ones, R^{*}_{SD}, we can notice that, generally, in some countries the former are just slightly higher than the latter and they have been increasing over the last decade: this is the reason why the fiscal policies in these countries, where the economic situation may jeopardise their future growth, should be reversed. Conversely, for those countries where the simulated ratios are higher than the actual ones, the fiscal policies are sustainable. Anyway, the consistency in size between the two ratios for all countries is quite reassuring in terms of the potentiality of our model as a tool for evaluating the sustainability of government fiscal policies.

As a consequence of the fact that some countries are running an unsustainable level of debt, it is important to underline that our model forecasts could be a bit pessimistic because it assumes that the government budget always balances and revenues from monetization are excluded, whereas in reality debt is also financed by seigniorage and inflation tax.

A further remark concerns a direct comparison between our R^{*}_{TB} (R^{*}_{PB}) and the actual ratios $\frac{b_{TB}}{k}$ ($\frac{b_{PB}}{k_{PB}}$), which must be carried out with a critical attitude, since simulated interest payments would tend to be overvalued. In fact, each generation is assumed to live for two periods of approximately forty years each, so that the simulated maximum ratios are calculated over this time span, whereas the actual ratios are yearly. As we mentioned in subsection 2.4.2, our simulations assume that the interests to be paid on debt are calculated over a forty-year period (which is half a lifetime, i.e. a high degree of temporal aggregation), whereas the empirical data on gross debt only include interests for one-year period.

On the base of the results achieved, by comparing the actual ratios R_{SD} with the simulated ones, R^*_{SD}, we can conclude that the fiscal policy in some countries, like Canada, Italy, Japan and the United States, should be soon reversed before debt starts rolling over on an unstable path which has already been reached.

Therefore, it follows that, notwithstanding the somehow simplistic structure of our model, nevertheless its simulations provide interesting results which can help to understand and investigate more deeply the reality.

We now continue our analysis of the sustainability of debt management policies in the next chapter where we apply the methodology developed in this chapter to other frameworks and we study the implications for other main real economic variables.

APPENDIX 1

The comparative statics analysis of the system composed by eqs.(2.28)-(2.29) can be carried out by linearising it about the steady state $(\bar{k},\bar{q})$, which allows us to rewrite:

$$\text{(A.1)}\quad \begin{bmatrix} dk_{t+1} \\ dq_{t+1} \end{bmatrix} = [J(k,q)]\begin{bmatrix} dk_t \\ dq_t \end{bmatrix} + \begin{bmatrix} -(s_w+\bar{q})/(1-s_r f'') \\ -f''\bar{q}(s_w+\bar{q})/(1-s_r f'') \end{bmatrix} d\bar{b},$$

where the Jacobian matrix of partial derivatives is equal to:

$$\text{(A.2)}\quad J(k,q) = \begin{pmatrix} -\bar{k}s_w f''/(1-s_r f'') & -\bar{b}/(1-s_r f'') \\ -f''^2 s_w \bar{k}\bar{q}/(1-s_r f'') & \{1/\bar{q}+1-f''[s_r(1/\bar{q}+1)+\bar{b}\bar{q}]\}/(1-s_r f'') \end{pmatrix}.$$

By subtracting dk_t and dq_t from the first and second row respectively and setting $\Delta dk_{t+1} = \Delta dq_{t+1} = 0$, we get:

$$\text{(A.3)}\quad \begin{bmatrix} -\dfrac{ks_w f''}{1-s_r f''}-1 & -\dfrac{\bar{b}}{1-s_r f''} \\ -\dfrac{ks_w f''^2\bar{q}}{1-s_r f''} & \dfrac{\dfrac{1}{\bar{q}}+1+f''\left(-\bar{b}\bar{q}-s_r\left(\dfrac{1}{\bar{q}}+1\right)\right)}{1-s_r f''}-1 \end{bmatrix}\begin{bmatrix} d\bar{k} \\ d\bar{q} \end{bmatrix} = \begin{bmatrix} (s_w+\bar{q})/(1-s_r f'') \\ f''\bar{q}(s_w+\bar{q})/(1-s_r f'') \end{bmatrix} d\bar{b}.$$

The long-run debt multiplier of the capital stock is determined by applying Cramer's rule:

$$\text{(A.4)}\quad \frac{d\bar{k}}{d\bar{b}} = \frac{\det\begin{bmatrix} \dfrac{s_w+\bar{q}}{1-s_r f''} & -\dfrac{\bar{b}}{1-s_r f''} \\ \dfrac{\bar{q}f''(s_w+\bar{q})}{1-s_r f''} & \dfrac{\dfrac{1}{\bar{q}}+f''\left(-\bar{b}\bar{q}-s_r\left(\dfrac{1}{\bar{q}}+1\right)\right)+s_r f''}{1-s_r f''} \end{bmatrix}}{\det C}$$

where $\det C$ stands for the determinant of the coefficients matrix on the LHS in (A.3). After some calculations we obtain the following expression:

$$\text{(A.5)}\quad \frac{d\bar{k}}{d\bar{b}}=\frac{\dfrac{(s_w+\bar{q})\frac{1}{\bar{q}}(1-s_rf'')}{(1-s_rf'')^2}}{\dfrac{\left(-\bar{k}s_wf''-1+s_rf''+f''\bar{b}\bar{q}^2\right)(1-s_rf'')\frac{1}{\bar{q}}}{(1-s_rf'')^2}},$$

which, after substituting $\bar{q}=\frac{1}{\bar{r}}=\frac{1}{f'(\bar{k})}$ and carrying out the necessary simplifications, turns out to be equal to the multiplier in eq.(2.32).

The application of the Correspondence Principle enables us to determine unambiguously the sign of the multiplier. This method will be used throughout the thesis in order to determine the sign of the expression measuring the effect of an exogenous shock in debt on the endogenous variable in the model (a rigorous analysis of the Correspondence Principle for a difference equations system was carried out by Rankin (1985a)). The system consists of two endogenous variables of which only one (the bond price) is a jump or forward-looking variable. Therefore, for the saddle point stability property to be satisfied, it is required that only one eigenvalue of the Jacobian matrix lies outside the unit circle, whereas the other must lie inside. An equivalent condition is that, assuming that the eigenvalues of the coefficient matrix in (A.3) are real, one should lie inside and the other outside the interval (-2;0). This means that their product could be positive or negative. Since we tend to exclude extravagant behaviour, i.e. oscillatory paths which are attributed to a negative sign of an eigenvalue[1], stability of this sort imposes that the determinant of the coefficient matrix is negative. In conclusion, the determinant for a saddle point stable steady state is negative whereas it is positive for an unstable steady state. On the other hand, since the condition $(1-s_rf'')>0$ is sufficient for the numerator to be positive, it

[1] For a study on the application of the Correspondence Principle to discrete time models, see Rankin (1985a). The author also underlines that a "perverse" case may occur when an odd number of unstable real eigenvalues lying below (-2) exists in a perfect foresight model with more than one state variable. However, in the author's words, "*Such eigenvalues give rise to "oscillations" along divergent paths, and since the latter do form part of the complete perfect foresight trajectory in cases in which, for example, there is a pre-announced policy change,* [...], *this case might conceivably be dismissed as involving too bizarre a form of behaviour to be of economic interest*" (Rankin, p.7). In conclusion, it is possible to exclude such perverse behaviour by the condition that all eigenvalues must be "non-oscillatory".

follows that there is a crowding out effect on the saddle point steady state level of capital stock, which is what we found for the other two cases of debt management policies.

The same reasoning can be applied to evaluate the overall effect on the steady state bond price following the same policy of an increase in government debt. The denominator is the same as in eq.(A.5) so that the Correspondence Principle still applies. After some calculations we find the following multiplier:

$$\text{(A.6)} \qquad \frac{d\overline{q}}{d\overline{b}} = \frac{\dfrac{-\overline{q}\overline{b}(s_w + \overline{q})(1 - s_r f'')}{(1 - s_r f'')^2}}{\dfrac{(-\overline{k}s_w f'' + s_r f'' + f''^2 \overline{b}\overline{q} - 1)\dfrac{1}{\overline{q}}}{(1 - s_r f'')^2}},$$

which is negative for the saddle point steady state provided that $(1 - s_r f'') > 0$, that is a condition usually assumed to hold and considered at the beginning of chapter 2. This is quite intuitive since there is an inverse relationship between the perpetuity bond price and the long-run interest rate (and, therefore, a positive relationship between $\overline{q}$ and $\overline{k}$).

APPENDIX 2

The purpose of this appendix is twofold. In the first part we rank by size the debt multipliers associated with the three financial assets which were presented in chapter 2. In the second part we further investigate the model by proving that the results obtained from our model simulations are not dependent on the "technology efficiency" parameter of the Cobb-Douglas production function. Moreover, we derive the equations which define the debt-to-capital stock ratios.

First of all, due to the fact that, following our analysis in chapter 2, $\bar{b}_{TB}$, $\bar{b}_{SD}$ and $\bar{b}_{PB}$ are not equivalent, since the first is inclusive of interest payments which are not included in the second, whereas the third consists only of interest payments, we have to find out the debt expression which holds for the three financing policies under investigation[2]. After doing so, we can compare rigorously the size of the multipliers $\left.\frac{d\bar{k}}{d\bar{b}}\right|_i$, where i=SD,TB,PB.

From the definitions given above, at the steady state the following relations hold:

(B.1) $\bar{b}_{SD}(1+\bar{r})=\bar{b}_{TB}$

(B.2) $\bar{b}_{SD}=\frac{\bar{b}_{PB}}{\bar{r}}.$

Equating (B.1) and (B.2), we derive the relationship between these different types of debt:

(B.3) $\frac{\bar{b}_{TB}}{1+\bar{r}}=\frac{\bar{b}_{PB}}{\bar{r}}.$

When we calculate the multipliers, the following equalities hold:

(B.4) $$\left.\frac{d\bar{k}}{d\bar{b}}\right|_{TB}=\left.\frac{d\bar{k}}{d\bar{b}}\right|_{PB}\cdot\frac{\bar{r}^2}{\left(\bar{r}+\bar{r}^2-\bar{b}_{PB}\cdot\left.\frac{d\bar{r}}{d\bar{b}}\right|_{PB}\right)}$$

[2] In all the three cases we assumed that the government budget constraint is balanced by endogenous taxes.

(B.5) $$\left.\frac{d\bar{k}}{d\bar{b}}\right|_{TB} = \left.\frac{d\bar{k}}{d\bar{b}}\right|_{SD} \cdot \frac{1}{\left(1+\bar{r}+b_{SD}\cdot\left.\frac{d\bar{r}}{d\bar{b}}\right|_{SD}\right)}$$

(B.6) $$\left.\frac{d\bar{k}}{d\bar{b}}\right|_{SD} = \left.\frac{d\bar{k}}{d\bar{b}}\right|_{PB} \cdot \frac{\bar{r}^2}{\left(\bar{r}-\bar{b}_{PB}\cdot\left.\frac{d\bar{r}}{d\bar{b}}\right|_{PB}\right)},$$

from which it is clear that, by setting $\bar{b}_t = 0$, the multiplier of the treasury bills is smaller than the multipliers of the savings deposits and the perpetuity bonds, whereas $\left.\frac{d\bar{k}}{d\bar{b}}\right|_{PB}$ is bigger than $\left.\frac{d\bar{k}}{d\bar{b}}\right|_{SD}$ if $\bar{r} < 1$.

We now briefly show that the ratios of debt-to-capital stock do not depend on the parameter of technological progress represented by the term A[3] of the Cobb-Douglas production function $y_t = Ak_t^{\alpha}$, and that a neat equation for calculating of these ratios can be found.

1) Savings Deposits.

The asset market equilibrium for which the simulations have been carried out is the following:

(B.7) $$k_{t+1} = (1-\beta)\left[(1-\alpha)k_t^{\alpha} - \alpha\bar{b}k_t^{\alpha-1}\right] - \bar{b}.$$

To show that the maximum ratio $R^{*}_{SD} = \left.\frac{b^*}{\bar{k}}\right|_{SD}$ (being b^* the maximum sustainable level of debt and $\bar{k}$ the steady state level of the capital stock) is independent on the parameter A, we first consider the tangency point condition which says that, at the maximum sustainable level of debt, the slope of the phaseline line must be equal to the slope of the 45 degree line, i.e.:

(B.8) $$1 = A(1-\beta)[\alpha(1-\alpha)\bar{k}^{\alpha} - \alpha b^*\bar{k}^{\alpha-1}] - b^*.$$

[3] This intuition is supported also by the results of the simulations which have been carried out for different values of A.

Taking into account that $R_{SD}^{*\ -1} = \left.\frac{\bar{k}}{b*}\right|_{SD}$, the previous relation can be rewritten after few rearrangements, as:

(B.9) $\quad 1 = \alpha(1-\alpha)(1-\beta)(Ab^{*\alpha-1})(R_{SD}^{*\ 1-\alpha} + R_{SD}^{*\ 2-\alpha}).$

The other condition which must be satisfied for a unique steady state to exist is that the steady state itself must lie on the 45 degree line. This implies that, from eq.(B.7), the following equation must be satisfied:

(B.10) $\quad \bar{k} = A(1-\beta)[(1-\alpha)\bar{k}^{\alpha} - \alpha b^* \bar{k}^{\alpha-1}] - b^*.$

Dividing it through b^* and doing the necessary rearrangements, we find:

(B.11) $\quad R_{SD}^{*\ -1} = (1-\beta)(Ab^{*\alpha-1})[(1-\alpha)R_{SD}^{*\ -\alpha} - \alpha R_{SD}^{*\ 1-\alpha}] - 1.$

Substituting the expression for $(Ab^{*\alpha-1})$, derived from eq.(B.11), into eq.(B.9), we find an equation where the only unknown is the ratio R_{SD}^{*}, whose value is clearly independent of the parameter A and the variable b^*, QED, and which allows us to calculate the capital stock-to-debt ratio given the parameter values.

2) Treasury Bills.

The asset market equilibrium is equal to:

(B.12) $$k_t = \left[\frac{k_{t+1} + \bar{b} - \dfrac{\alpha\beta\bar{b}\,k_{t+1}^{\alpha-1}}{1+\alpha k_{t+1}^{\alpha-1}}}{(1-\alpha)(1-\beta)}\right]^{\frac{1}{\alpha}}.$$

In order to show that the value of the ratio R_{TB}^{*} is independent of A we follow the same reasoning as in point 1 above. One of the two conditions for having a unique steady state is that the steady state itself must belong to the phaseline and the tangency condition must hold. Carrying out all the calculations, we obtain the final equation:

(B.13)

$$\frac{1}{\alpha(1-\alpha)(Ab^{*\alpha-1})^{\frac{1}{\alpha}}(1-\beta)}\left[\frac{R_{TB}^{*\ -1}+1-\dfrac{\alpha(Ab^{*\alpha-1})\beta R_{TB}^{*\ 1-\alpha}}{[1+\alpha(Ab^{*\alpha-1})R_{TB}^{*\ 1-\alpha}]^2}}{(1-\alpha)(1-\beta)}\right]^{\frac{1-\alpha}{\alpha}}\left[1+\frac{\alpha(1-\alpha)\beta(Ab^{*\alpha-1})R_{TB}^{*\ 2-\alpha}}{[1+\alpha(Ab^{*\alpha-1})R_{TB}^{*\ 1-\alpha}]^2}\right]=1$$

Substituting into eq.(B.13) the positive root of $(Ab^{*\alpha-1})$ which is derived from the steady state condition applied to eq.(B.12), we obtain an equation in the only unknown R_{TB}^{*}, from which it follows that the ratio debt-to-capital stock does not depend on the value of A and b^*.

3) Perpetuity Bonds.

The system consists of two equations in k and q:

$$\text{(B.14)}\qquad k_t=\left[\frac{(\alpha q)^{\frac{1}{1-\alpha}}+(1-\beta)\bar{b}+q\bar{b}}{(1-\alpha)(1-\beta)}\right]^{\frac{1}{\alpha}}$$

$$\text{(B.15)}\qquad q_t=-\frac{k}{\bar{b}}+\frac{(1-\alpha)(1-\beta)k^{\alpha}}{\bar{b}}-(1-\beta),$$

whose intersections give us the steady state values. Substituting eq.(B.14) into eq.(B.15) we get:

$$\text{(B.16)}\qquad \frac{(\alpha q)^{\frac{1}{1-\alpha}}}{\bar{b}}-\frac{\left[\dfrac{(\alpha q)^{\frac{1}{1-\alpha}}+(1-\beta)\bar{b}+q\bar{b}}{(1-\alpha)(1-\beta)}\right]^{\frac{1}{\alpha}}}{\bar{b}}=0.$$

The maximum level of debt corresponds to that value where there are two coincident solutions for $\bar{q}$ to eq.(B.16). We can prove that also in this case the ratio of debt-to-capital stock is independent on the parameter A. The procedure followed is the same as in the two previous cases. One condition is represented by eq.(B.16) evaluated at the steady state, and the other by the tangency of the two phaselines:

(B.17) $$\left.\frac{d\bar{q}}{d\bar{k}}\right|_{(B.13)}=\left.\frac{d\bar{q}}{d\bar{k}}\right|_{(B.14)}.$$

The final equation in R^{*}_{PB} and $(Ab^{*\alpha-1})$ is the following:

(B.18) $$\alpha(1-\alpha)(1-\beta)(Ab^{*\alpha-1})R^{*\,1-\alpha}_{PB}=[\alpha(1-\alpha)(1-\beta)(Ab^{*\alpha-1})R^{*\,1-\alpha}_{PB}-1]\cdot$$

$$\left[1+\alpha^{\frac{1}{1-\alpha}}\left(Ab^{*\alpha-1}\right)^{\frac{1}{1-\alpha}}\left(-R^{*\,-1}_{PB}-(1-\beta)+(1-\alpha)(1-\beta)(Ab^{*\alpha-1})R^{*\,-\alpha}_{PB}\right)^{\frac{\alpha}{1-\alpha}}\right].$$

Substituting into the positive root of $(Ab^{*\alpha-1})$ found from eq.(B.16) evaluated at the steady state, we derive an equation in the only unknown R^{*}_{PB}, which again suggests that the value of A does not affect the maximum sustainable ratio $\left.\frac{b^{*}}{\bar{k}}\right|_{PB}$, QED.

APPENDIX 3

In this appendix we report some data concerning the G-7 countries which are used for the exercise contained in chapter 2, section 4, and we briefly dwell upon the actual fiscal situation in each of them. The exact source of these real data is reported in the captions for each country separately. The series of the data available for the capital stock variable has been elaborated by the NIGEM from the following expression:

$KBV = (1-0.0125)\cdot KBV(-1)+IB$, where IB stands for fixed capital business investment, which depends on $IB(-1)$ and on the ratios $\frac{IB(-1)}{Y(-1)}, LR(-1), \frac{IB(-1)}{IB(-2)}$, where LR is the long-run interest rate and Y is the GNP.

Government debt under the form of treasury bills, B_{TB}, is in nominal terms. When it is deflated by the GDP deflator, real debt is marked by b_{TB}. The other two kinds of debt, namely b_{SD} and b_{PB}, are calculated with the formulas $b_{SD} = b_{TB}\cdot\frac{1}{(1+r_{TB})}$ and $b_{PB} = r_{PB}\cdot b_{SD}$, respectively.

The nominal interest rates (i) have been transformed in real terms (r) by taking into account the annual inflation rate. The year base is different for each country. In the following tables we report the actual situation elaborated for the G-7 countries. The underlined results and the related comments separately for each country are given at the end of the presentation of the tables.

CANADA

Table C.1

Year	B_{TB}	b_{TB}	k	r_{TB}	r_{PB}	R_{SD}	R_{TB}	R_{PB}
1984	264466	277800.42	678037.4	8.7984	11.5784	0.3766	0.4097	0.0436
1985	310374	317843.32	709957.9	6.1891	8.6191	0.4216	0.4477	0.0363
1986	348101	348101	743019.5	4.9774	6.4074	0.4463	0.4685	0.0286
1987	383807	366577.84	780524.7	4.3825	6.2525	0.4499	0.4695	0.0281
1988	419342	382785.94	826668.0	5.9131	6.4131	0.4372	0.4630	0.0280
1989	453960	395177.37	875368.1	8.7782	6.5182	0.4150	0.4514	0.0271
1990	486681	410268.49	919455.3	10.3635	8.5535	0.4043	0.4462	0.0346
1991	540565	444635	960782.1	8.035	9.255	0.4284	0.4629	0.0396
1992	602770	490256.20	993656.3	5.4279	7.7229	0.4680	0.4934	0.0361
1993	656192	529507.36	1024211	4.0122	7.0872	0.4965	0.5170	0.0352
1994	712975.3	573383.85	1061884	4.6972	7.6572	0.5157	0.5400	0.0395

Caption of table C.1: source of data.

- B_{TB} is the stock of government debt, expressed in million domestic currency, from the OECD Economic Outlook.

- k is the capital stock, 1986 prices, million domestic currency, from the OECD Business Sector database, and where unavailable, interpolate annual data from OECD Economic Outlook, gross real stock of business fixed capital. *ib* is business investment, 1986 domestic prices, million domestic currency.

- r_{TB} is the real 3-month interest rates, where i_{TB} is taken from MEI (OECD Main Economic Indicators), Canada Tables, National Accounts Section, 90 day deposit receipts.

- r_{PB} is the real long-term interest rates on government bonds, where i_{PB} is taken from the Bank of Canada.

FRANCE

Table C.2

Year	B_{TB}	b_{TB}	k	r_{TB}	r_{PB}	R_{SD}	R_{TB}	R_{PB}
1984	1618.7	1103.22	6173.597	7.6638	9.3938	0.1660	0.1787	0.0156
1985	1813.3	1168.55	6302.983	5.3935	7.3685	0.1759	0.1937	0.0130
1986	1993.3	1220.82	6444.847	5.5215	6.9165	0.1795	0.1894	0.0124
1987	2172.5	1292.58	6600.310	6.0351	8.1351	0.1847	0.1958	0.0150
1988	2329.8	1347.87	6784.242	5.6566	6.8366	0.1880	0.1987	0.0129
1989	2503.3	1405.56	6996.640	6.499	6.379	0.1886	0.2010	0.0120
1990	2614.2	1425.02	7222.365	8.1113	8.2713	0.1825	0.1973	0.0151
1991	2783.8	1473.69	7437.219	7.6363	7.4563	0.1841	0.1982	0.0137
1992	3184.3	1647.76	7623.175	9.0429	7.6229	0.1982	0.2162	0.0151
1993	3701.3	1868.42	7767.236	7.16	5.35	0.2245	0.2410	0.0120
1994	4109.9	2044.11	7908.218	4.4833	6.4733	0.2474	0.2585	0.0160

Caption of table C.2: source of data.

- B_{TB} is the stock of government debt, in billion domestic currency, from the OECD Economic Outlook.
- k is the capital stock, 1980 prices, in billion domestic currency, from the OECD Business Sector database. ib is taken from OECD Quarterly National Accounts, France section, table 5B, gross fixed capital formation, billion domestic currency.
- r_{TB} is the real short interest rates, where i_{TB} is taken from the International Monetary Fund (IMF), International Financial Statistics, French country page, interbank deposit rate, 60BS.
- r_{PB} is the real long term interest rates on government bonds, where i_{PB} is taken from the Banque de France, yield on government bonds on secondary market (over 7 years).

GERMANY

Table C.3

Year	B_{TB}	b_{TB}	k	r_{TB}	r_{PB}	R_{SD}	R_{TB}	R_{PB}
1984	734.6	878.38	4480.750	4.5885	6.4535	0.1874	0.1960	0.0121
1985	778.8	912.55	4582.380	3.272	4.807	0.1928	0.1991	0.0093
1986	824.2	935.90	4692.120	3.6251	5.1751	0.1925	0.1995	0.0100
1987	882.2	983.25	4810.230	3.2753	5.5703	0.1979	0.2044	0.0110
1988	938.1	1029.91	4933.610	2.7764	4.9414	0.2031	0.2088	0.0100
1989	975	1045.18	5068.300	5.1734	5.2134	0.1961	0.2062	0.0102
1990	1053.5	1094.86	5217.260	6.7716	7.2916	0.1965	0.2099	0.0143
1991	1173.9	1173.85	5393.300	5.7263	4.9713	0.2059	0.2176	0.0102
1992	1345.2	1288.27	5590.820	6.5987	4.9887	0.2162	0.2304	0.0108
1993	1509.2	1399.50	5778.110	5.1601	4.2801	0.2303	0.2422	0.0099
1994	1618.9	1463.02	5914.795	3.4862	5.3162	0.2390	0.2473	0.0127

Caption of table C.3: source of data.

- B_{TB} is the stock of government debt, billion domestic currency, from Monatsberichte der Deutschen Bundesbank, section VIII, public finance, table 7, col.1, verschuldung der offentlichen haushalte, total indebtedness of the public sector, millions domestic currency.

- k is the capital stock, 1991 prices, OECD Business Sector. *ib* is 1991 prices, billion domestic currency, is from Deutche Bundesbank Saisonbereinigte Wirtschaftszahlen, Statistisches Beiheft Zum Monatsbericht, table III-2, Ausrustungsinrvest plus bauinvest, taken away the proportion of total investment accounted for by housing investment over the last year, and government investment.

- r_{TB} is the real short-term interest rates, where i_{TB} is taken from the IMF International Financial Statistics, German country page, 60BS, interbank deposit rate.

- r_{PB} is the real long-term interest rate on government bonds, where i_{PB} is taken from MEI, summary table on long-term interest rates.

ITALY

Table C.4

Year	B_{TB}	b_{TB}	k	r_{TB}	r_{PB}	R_{SD}	R_{TB}	R_{PB}
1984	561489	611144.49	1923411	7.5955	8.9355	0.2953	0.3177	0.0264
1985	683044	683214.80	1959973	7.7719	8.7819	0.3234	0.3486	0.0284
1986	793583	736025.78	2000171	5.6682	6.2082	0.3482	0.3680	0.0216
1987	910542	796624.67	2048004	4.2442	5.3442	0.3731	0.3890	0.0199
1988	1035812	849722.72	2104600	3.7242	5.1842	0.3892	0.4037	0.0202
1989	1168361	902557.74	2165417	5.4367	6.7167	0.3953	0.4168	0.0266
1990	1318936	946152.08	2227822	3.9882	6.1652	0.4084	0.4247	0.0252
1991	1485742	992811.23	2285747	6.4257	6.6557	0.4081	0.4343	0.0272
1992	1675276	1069268.23	2337911	10.3097	9.4497	0.4146	0.4574	0.0392
1993	1862937	1148238.75	2367129	6.2317	6.6917	0.4566	0.4851	0.0310
1994	2023796	1211114.17	2394682	4.6642	6.5542	0.4832	0.5058	0.0317

Caption of table C.4: source of data.

- B_{TB} is the stock of gross government debt, billion domestic currency, from OECD Economic Outlook.
- k is the capital stock, from OECD Business Sector, 1985 prices and where unavailable, interpolate annual data from OECD Economic Outlook. *ib* is the total investment, 1985 prices, billion domestic currency, ISTAT gross fixed capital formation, seasonally adjusted.
- r_{TB} is the real short-term interest rates, where i_{TB} is taken from the IMF International Financial Statistics, Italian country page, money market rate, 60B.

- r_{PB} is the real long-term interest rates on government bonds, where i_{PB} is taken from MEI, summary table on long-term interest rates.

JAPAN

Table C.5

Year	B_{TB}	b_{TB}	k	r_{TB}	r_{PB}	R_{SD}	R_{TB}	R_{PB}
1984	204005.7	207237.57	545367.4	5.2675	5.9485	0.3610	0.3800	0.0215
1985	220176.7	220157.77	574096.8	5.4752	5.1142	0.3636	0.3835	0.0186
1986	241976.9	237783.12	606381.6	4.6057	4.5457	0.3749	0.3921	0.0170
1987	260814.9	256236.47	638685.2	4.1264	4.8004	0.3853	0.4012	0.0185
1988	270578.1	264883.80	671093.5	4.0713	4.6533	0.3793	0.3947	0.0176
1989	279729.9	268877.73	710823.2	2.8617	2.8017	0.3677	0.3783	0.0103
1990	296156.8	278612.04	757924.0	5.9223	5.2673	0.3470	0.3676	0.0183
1991	305456.2	281730.54	811589.1	5.9505	4.7985	0.3276	0.3471	0.0157
1992	329757.9	299237.47	865883.5	3.1886	3.8286	0.3349	0.3456	0.0128
1993	345466.9	310451.96	914576.9	2.2044	3.2184	0.3321	0.3394	0.0110
1994	374623.5	336985.59	952740.1	2.2659	4.3489	0.3459	0.3537	0.0150

Caption of table C.5: source of data.

- B_{TB} is the stock of gross government debt, OECD Economic Outlook, billion domestic currency.

- k is the capital stock, 1985 prices, billion domestic currency, from OECD Business Sector. *ib* is at 1985 prices, from the Economic Planning Agency, annual report on National Accounts, National Accounts section, machinery, equipment, etc.

- r_{TB} is the real short-term interest rates, where i_{TB} is taken from MEI, Japan section, certificates of deposit, % per annum.

- r_{PB} is the real long-term interest rates on government bonds, where i_{PB} is taken from MEI summary table on long-term interest rates.

UNITED KINGDOM

Table C.6

Year	B_{TB}	b_{TB}	k	r_{TB}	r_{PB}	R_{SD}	R_{TB}	R_{PB}
1984	177132.2	245240.45	830080.0	5.6045	6.3495	0.2798	0.2954	0.0178
1985	188167	246550.21	848480.0	7.7047	6.0697	0.2698	0.2906	0.0164
1986	196787	249617.81	864770.0	9.5153	8.4453	0.2636	0.2887	0.0223
1987	205804	248750.62	883030.0	5.9599	5.7299	0.2659	0.2817	0.0152
1988	198749.5	226541.19	907312.1	4.1321	3.1771	0.2398	0.2497	0.0076
1989	189406.5	201292.95	933650.1	8.4073	4.1023	0.1989	0.2156	0.0082
1990	190830.7	187864.88	954967.0	8.6307	4.9507	0.1811	0.1967	0.0090
1991	202808.2	187813.20	967858.9	7.8806	6.2756	0.1799	0.1941	0.0113
1992	240836.3	213891.72	977296.5	7.2821	6.7821	0.2040	0.2189	0.0138
1993	288581.1	248016.72	988296.5	2.906	4.831	0.2439	0.2510	0.0118
1994	326705.3	272164.60	1001312	3.4779	5.9979	0.2627	0.2718	0.0158

Caption of table C.6: source of data.

- B_{TB} is the stock of gross government debt, million domestic currency, OECD Economic Outlook.
- k is the capital stock, 1990 prices, from OECD Business Sector. *ib* is at 1990 prices, from CSO Economic Trends, Quarterly article on national income, table A.17, private sector less private investment in dwellings.
- r_{TB} is the short-term interest rates, where i_{TB} is taken from the Financial Statistic, table 13.15, 3 month inter-bank rate, CSO.

- r_{PB} is the long-term interest rates on government bonds, where i_{PB} is taken from the Financial Statistic, table 13.5, col.3, interest rate on 20 year UK government securities (average of monthly figures).

UNITED STATES

Table C.7

Year	B_{TB}	b_{TB}	k	r_{TB}	r_{PB}	R_{SD}	R_{TB}	R_{PB}
1984	1695.6	1862.47	6890	7.4848	9.1048	0.2515	0.2703	0.0229
1985	1941.2	2057.24	7130	5.6599	8.3699	0.2731	0.2885	0.0229
1986	2178.4	2247.79	7390	4.5026	6.1276	0.2911	0.3042	0.0178
1987	2362.9	2363.3	7620	4.4295	6.2045	0.2970	0.3101	0.0184
1988	2583.7	2488.15	7830	4.3342	5.5892	0.3046	0.3178	0.0170
1989	2794.4	2574.84	8050	6.123	5.633	0.3014	0.3199	0.0170
1990	3070.2	2710.95	8270	5.0001	5.5851	0.3122	0.3279	0.0174
1991	3368.5	2861.48	8490	3.6587	5.9887	0.3251	0.3370	0.0195
1992	3743.6	3091.46	8660	1.8719	5.1709	0.3504	0.3570	0.0200
1993	4075.6	3282.20	8810	1.8795	5.1595	0.3657	0.3726	0.0189
1994	4342.2	3406.72	9020	1.7593	4.5393	0.3712	0.3777	0.0168

Caption of table C.7: source of data.

- B_{TB} is the stock of gross government debt, billions domestic currency, OECD Economic Outlook.

- k is the capital stock, 1987 prices, from OECD Business Sector. *ib* is at 1987 prices, billion domestic currency, from Survey of Current Business (USA), nipa table 1.1, line 18, government purchases billions of dollars.

- r_{TB} is the real short-term interest rates, where i_{TB} is taken from the IMF International Financial Statistics, United States tables, certificate of deposit rate, series 60LC.

- r_{PB} is the real US long-term corporate bond rates (10-year Moody AAA), where i_{PB} is taken from MEI long-term interest rates, % per annum.

A brief comment of the situation in each country is reported here below.

Canada

The level of debt and of the debt-to-capital stock ratios have been increasing in the last decade and the latter turn out to be the highest among those of the G-7 countries. From a comparison with the simulated values, a reversed fiscal policy seems necessary to allow the country to head on a sustainable debt policy path.

France

France seems to be quite a virtuous country since all the ratios turn out to be among the lowest, although the interest rates have been quite high almost all over the time period considered. The trend for the ratios R_{SD} and R_{TB} has been increasing and only R_{PB} exhibits some constancy due to the fact that, in recent years, the long-run interest rates have been quite constant.

Germany

The same reasoning done for France applies to Germany, whose debt-to-capital stock ratios were between the French and the Canadian ones until 1993. The increase in the ratios during the 1990s is probably a result of the German unification and of the big increase in the interest rates determined by the need of the huge investments in the country.

Italy

The situation in this country is similar to Canada's since the debt-to-capital stock ratios are quite high. The first striking aspect to note is that the interest rate payments have been very huge during all the time span under investigation, and they have worsened the already precarious situation. The Italian debt management policy requires a urgent reversal since the ratios are on an increasing path and have already approached the maximum sustainable levels.

Japan

Notwithstanding the low values of the interest rates, it is interesting to note that the debt-to-capital stock ratios are just below those of Italy and Canada. However, the Japanese ratios seem to have reversed from the beginning of 1990 to a decreasing path while the other two countries are still struggling to contain their debt increase.

United Kingdom

The debt-to-capital stock ratios are quite constant, so that there does not seem to be particular problems of their sustainability, even though the absolute amount of debt is increasing.

United States

Notwithstanding that, quite recently, the real interest rates have been decreasing, the debt-to-capital stock ratios are quite large, and the American debt has been widening a lot. The situation in the US seems to be critical as it is for Canada and Italy.

The general result is that the debt-to-capital stock ratios are on an increasing path and in some countries they are already higher than the maximum sustainable ratios calculated with our model: current fiscal policies in developed countries like Canada, Italy, Japan and United States are quite likely to be unsustainable.

CHAPTER 3

SOME FURTHER APPLICATIONS OF THE CONCEPT OF MAXIMUM SUSTAINABLE LEVEL OF DEBT

3.1 Introduction

In this chapter we extend the methodology adopted in the previous chapter to derive the criterion of sustainability of government financing policies applied to other frameworks. By resorting to the model developed in chapter 2 we have been able to provide a mathematical and intuitive explanation of why the issuing of a certain amount of bonds might lead the economy to the collapse. Now we want to apply the same methodology for the analysis of systems and constraints different from those already examined. In this respect there are some models in the literature which have approached the issue of government financing policies without focusing on the sustainability of the policy itself. This is the reason why we decide to consider three other frameworks to which the procedure to find the maximum sustainable level of spending can be applied. These are:

1) a social security system which creates the need for the government to raise revenues;

2) a pure exchange economy based on the same assumptions for debt management policies as presented in chapter 2, which allows to investigate more deeply some results shadowed in the production economy analysis;

3) the perpetual youth economy a-la-Blanchard where the author had neglected the important aspect of sustainability of a fiscal policy.

This chapter is divided into three parts. We start considering the case of a social security spending policy. One of the two pension schemes currently applied in many countries, the pay-as-you-go system, can be represented by a model similar to the one developed in the previous chapter for the financing policy through savings deposits. Not surprisingly, due to its similarity to our original model, we can prove that the criterion to find the maximum sustainable level of debt still gives satisfactory results when applied to find the maximum sustainable level of social security contributions when using a Cobb-Douglas production function.

In the mid part of the chapter, we apply the concept of the maximum sustainable level of debt to the simple exchange version of the model illustrated in chapter 2. Notwithstanding the existence of only one endogenous variable (the real interest rate) and the simplicity of the economy, still neat and interesting results are generated, which highlight a particular intuition underlying that concept.

Finally, we investigate the issue of sustainability of a fiscal policy by using the "perpetual youth model" a-la-Blanchard (1985) in order to evaluate whether uncertainty and a positive level of government spending affect the applicability of this criterion. The investigation of these models leads to the conclusion that our criterion can be applied to different contexts, yielding interesting results.

3.2 Spending Sustainability within a Social Security Model

Generally speaking, social security programmes are introduced for providing old people with sufficient income during their retirement as they do not have any other source of income. This policy could be interpreted as a paternalistic attitude of the government towards the citizens of a country. In fact, people might behave myopically and not provide enough for their retirement. Moreover, another rationale for the emergence of social security is simply the desire to transfer resources to a particularly needy generation of elderly, albeit at the potential expense of subsequent generations. However, the proportion of elderly people, especially in developed countries, has been steadily increasing and this has raised the general fear that the burden of social security as a proportion of GDP would grow further over the coming decades. This situation reminds the increasing debt-to-GDP ratio which has been the topic investigated in the previous chapter.

The model assumes that individuals make their contributions when young and receive the payment foreseen by the social security scheme when they are old. Let us indicate with d_t the contribution of a young person at time t. There are two main different systems of social security which also differ in their method of finance and their redistribution effects. The first one is called the *fully funded* system where members' contributions over their working lives made at time t on an individually actuarially fair basis are invested in a trust fund which yields a market-determined rate of return r_t at time $t+1$ during retirement. The second system is called the *pay-as-you-go* system and consists in having at each point in time benefits to retirees to be paid directly out of the current contributions of workers living in the same period and using, for instance, current tax contributions to pay for current benefits. Assuming that the population grows at a constant rate n, this parameter represents also the rate of return on the contributions. If n is less than the economy's interest rate, workers contributing to social security receive a smaller return on their contributions than if they had been allowed to save these contributions and invest them in the economy. This system is also called unfunded system. Overlapping-generations models have been employed to consider the rationale for the existence of social security programmes (say Pareto-improving policies) and to study the macroeconomic effects upon the steady state per-capita capital stock. For our present purpose, and for consistency with our debt model, we consider the framework without bequests and gifts.

It is quite well-established that a fully funded social security system has no effect on total savings and capital accumulation in the economy. Intuitively, this can be explained by the fact that, investing in the fund this social security system provides for, it yields the same rate of return as private savings invested in productive capital so that the individual is indifferent between these two investment options at his disposal. Capital accumulation would not be affected because savings are always available to sustain the required path for capital.

The reasoning is different for the pay-as-you-go system because the rates of return on private savings and social contributions are not the same. While savings earn the rate of interest r, the rate of return paid on social contributions is instead n, that is the increase in the number of people alive in the next period who can pay for the contributions. We analyse the effects on capital accumulation from changes in the level of the contributions by using the model presented in Blanchard and Fischer (1989)[1]. Given that the social security is unfunded and the rate of return is n, the Lagrangean for the household's intertemporal utility maximisation problem in this case can be written as:

$$(3.1) \qquad L = \beta \ln c_t^y + (1-\beta)\ln c_{t+1}^o + \lambda\left[(w_t - d_t - c_t^y)(1+r_{t+1}) - c_{t+1}^o + (1+n)d_{t+1}\right].$$

Comparing the present optimisation with the Lagrangean in the previous chapter, we see that eq.(3.1) contains in the budget constraint the term of the social security contribution, d_t, which earns $(1+n)d_{t+1}$ to the investor in the next period. The intertemporal optimisation process yields the following expression for the saving function:

$$(3.2) \qquad s_t = (1-\beta)(w_t - d_t) - \frac{d_{t+1}(1+n)\beta}{1+r_{t+1}}.$$

The asset market clearing condition requires that the total amount of private savings available in the economy must be invested into capital accumulation:

$$(3.3) \qquad s_t\left[w_t(k_t), r_{t+1}(k_{t+1}), \bar{d}\right] = (1+n)k_{t+1},$$

where, for reason of consistency with the assumptions of the model in the previous chapter, we assume that the government is keeping constant the amount of contributions at a level of $d_t = d_{t+1} = \bar{d}$. From eq.(3.3) it is clear that the social security system is a pure transfer scheme from the young to the old if $\bar{d}$ is positive. This social contribution affects saving decisions but it is not a financial asset in itself which can offset investments in productive capital. By contrast, in the model of chapter 2, debt enters the asset market equilibrium because it represents a financial asset alternative to physical capital in which savings are preferably invested.

The asset market equilibrium is represented in the figure below by the phaseline KK, which intersects the abscissa axis at a positive value of k_t.

[1] The model developed in Blanchard and Fischer at pp.111-113 is here modified by resorting to the Cobb-Douglas utility and production functions used in chapter 2.

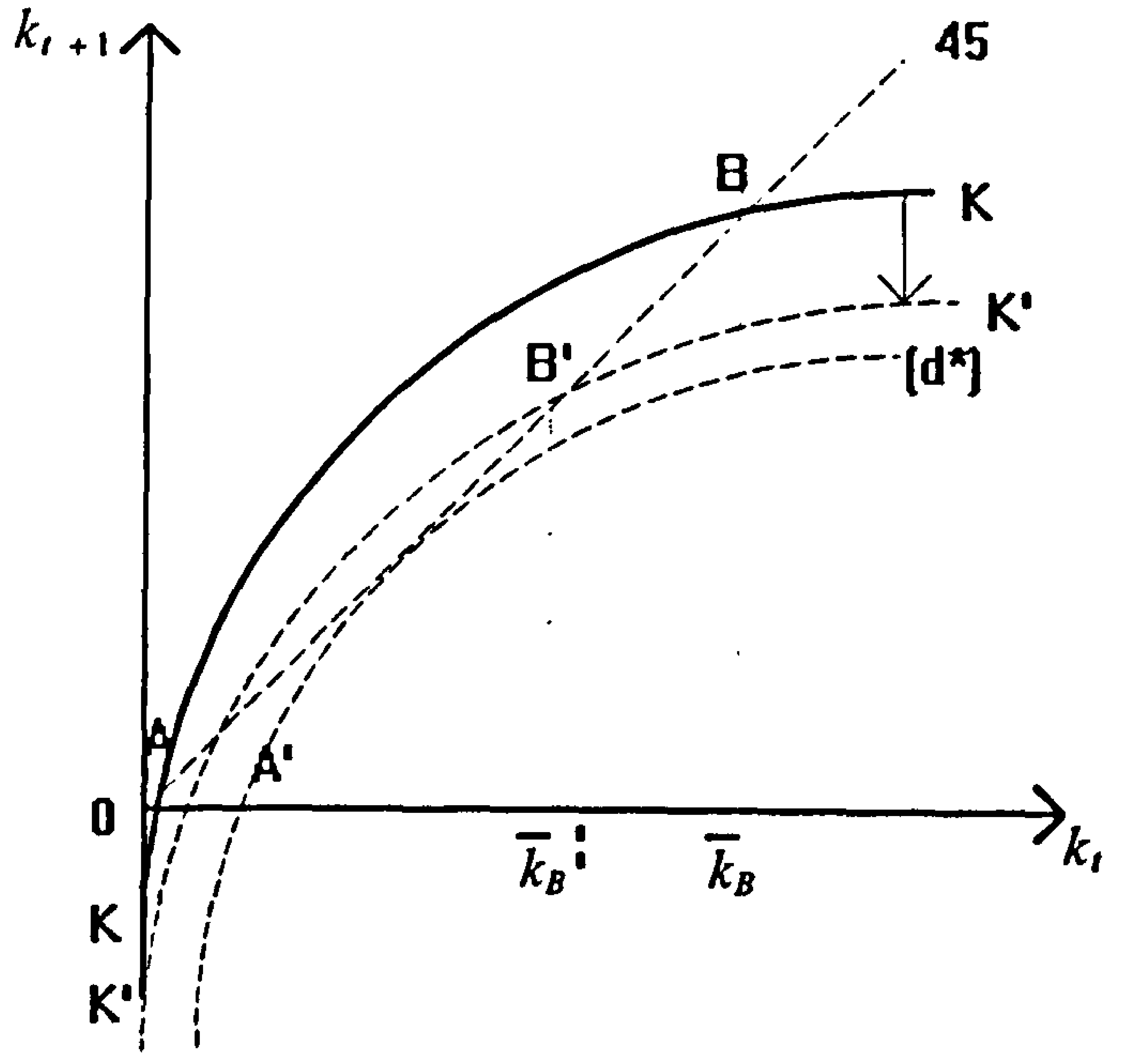

figure 3.1

The system displays two steady states, the unstable point A and the stable point B. The effects of an exogenous increase in social security contributions can be compared with the effects of an exogenous increase in debt in the model a-la-Diamond. The contributions multiplier which measures the change in the future capital stock, k_{t+1}, is equal to[2]:

$$(3.4) \quad \frac{dk_{t+1}}{d\bar{d}} = \frac{\frac{\partial s_t}{\partial \bar{d}}}{1+n-s_r f''(\bullet)}.$$

Substituting each of the arguments of the multiplier with their expressions derived from eq.(3.2), we can rewrite eq.(3.4) as follows:

$$(3.5) \quad \frac{dk_{t+1}}{d\bar{d}} = -\frac{(1-\beta)+\frac{\beta(1+n)}{1+\bar{r}}}{1+n+\frac{\bar{d}(1+n)\beta}{(1+\bar{r})^2}\cdot\frac{\alpha(1-\alpha)}{\bar{k}^{2-\alpha}}},$$

which is clearly negative on the assumptions that $0<\alpha,\beta<1$ and $n>0$. Therefore, an exogenous increase in the amount of social security contributions would shift down the

[2] This multiplier can be found in Blanchard and Fischer (1989), p.113.

phaseline KK to $K'K'$, so that, at each level of k_t, the value of the future capital stock decreases. The overall effect of an increase in $\bar{d}$ on the steady state $\bar{k}$ is measured by the following multiplier:

$$(3.6)\quad \frac{d\bar{k}}{d\bar{d}}=\frac{\left[-\left(1+\frac{\alpha}{\bar{k}^{1-\alpha}}\right)(1-\beta)-(1+n)\beta\right]\left(1+\frac{\alpha}{\bar{k}^{1-\alpha}}\right)}{(1+n)\left(1+\frac{\alpha}{\bar{k}^{1-\alpha}}\right)^2-\alpha(1-\alpha)(1-\beta)\bar{k}^{\alpha-1}\left(1+\frac{\alpha}{\bar{k}^{1-\alpha}}\right)^2+\bar{d}\alpha(1-\alpha)\beta(1+n)\bar{k}^{\alpha-2}}.$$

By applying the Correspondence Principle, it turns out that $\frac{d\bar{k}}{d\bar{d}}$ is negative when evaluated at the stable steady state, so that crowding out of the capital stock occurs. It is also worth noting that a well-defined level of $d=d^*$ corresponds to the *maximum sustainable level of social security contributions*. When d exceeds this threshold, the same scenario of a poverty trap evidenced in chapter 2 will take place.

Before summing up, it is important to note that this result has been derived by using a Cobb-Douglas production function which satisfies the Inada conditions. As already specified, we have chosen to use this particular production function for the sake of consistency, since we were interested in comparing the long-run effect (i.e. the multipliers) on the capital stock in this model with the one found in the debt model of the previous chapter in which we also adopted a Cobb-Douglas production function.

In conclusion, an increase in the burden of social security contributions negatively affects the rate of capital accumulation and the stable steady state level of capital stock is reduced. These findings can be intuitively explained by considering that the contribution represents only a transfer which, like a lump sum tax, affects the amount of disposable income which can be devoted to saving. If the burden of the contributions rose, this would reduce the amount of savings and, consequently, the amount of capital accumulation.

However, an important difference arises when comparing the mathematical structure of the social security model with the one of the debt model analysed in the previous chapter, although in both of them an expansionist fiscal policy has a crowding out effect on the capital stock. This difference consists in the fact that the former does not include a financial asset which can displace capital in the economy; nevertheless, an increase in social security contributions still may cause a poverty trap to occur. Moreover, the maximum sustainable level of social security is reached via an abrupt disappearance of all steady states as happened in the model with debt in chapter 2. This implies that also in this model the possibility of a catastrophe exists and social security contributions can be increased until the system reaches the point where productive resources will be rapidly exhausted.

Finally, eq.(3.3) also implies that the population growth rate can affect the steady state in the same way as the social security contributions. "*...faster population growth raises the per capita transfer to the old for a given lump sum tax on the young. The higher income leads to a shrinkage of saving at any given interest rate*"[3]. A high rate of population growth can lead to the adoption of a less-capital-intensive technology, to less savings per head and a higher interest rate.

3.3 The Interpretation of the Concept of Maximum Sustainable Level of Debt in a Simple Pure Exchange Economy

In the present section we simplify the model developed in section 2 of chapter 2 in order to apply it to a simple **pure exchange economy** where no productive capital exists. Although it is a simpler version of the previous model, our purpose is to use it to derive an explicit expression of the maximum sustainable level of debt and the conditions to be satisfied to find at least one equilibrium. We will show that some interesting conclusions can be reached through this approach.

The economy is again characterised by members of each generation living for two periods and a zero population growth rate. Since there is no production in this economy, the individual does not work but has an endowment made up of a composite perishable good, e^y, when young whereas when old he receives e^o. The interest rate is the only endogenous

[3] Azariadis (1993), p.296.

variable in the model. Households are willing to invest all their savings in financing government debt and dynamics in the endogenous real interest rate arise. The government financing policies are those explained in the previous chapter, and, briefly, the system exhibits the following characteristics for each type of debt management policy:

- In the savings deposits case, there exist two steady states (one stable and one unstable) provided that debt does not exceed a sustainable level beyond which the government would run a Ponzi-game scheme with a rolling over of interest payments. The existence of only one stable equilibrium (the smaller steady state interest rate) implies that, if the system starts at a too high real interest rate, it will be on an unstable path despite the fact that a stable path for the economy exists. Moreover, any attempt by the government to bring in a reform programme which permanently cuts the size of debt would not apply if prior to the reform r was at a too high level.

- In the treasury bills case, the steady state is always stable since there is a unique solution for the interest rate which is a function of the exogenous parameters within the economy.

- For the perpetuity bonds case, the reasoning done for the savings deposits still applies. An increase in the long-run debt is shown to decrease the bond price or, alternatively, increase the long-run interest rate which is a result consistent with what predicted by the standard macroeconomic theory.

The concept of a maximum sustainable level of debt applies also to this simple exchange economy. The consumer's intertemporal utility maximisation problem can be written as:

$$(3.7) \quad \text{maximise} \quad \beta \ln c_t^y + (1-\beta)\ln c_{t+1}^o$$
$$\text{subject to} \quad e^y - \tau_t = c_t^y + s_t$$
$$c_{t+1}^o = (1+r_{t+1})s_t + e^o.$$

Using the Lagrangean method, we find the saving function:

$$(3.8) \quad s_t = (1-\beta)(e^y - \tau_t) - \frac{e^o \beta}{(1+r_{t+1})}.$$

Savings are a function of net disposable income and real interest rate. The analysis which follows focuses on each bond financing system separately, as the implications for the asset market equilibrium and the interpretation of the results are different for each case.

Firstly, let us consider the **savings deposits** case. If the government holds the amount of debt constant, then taxes are equal to $\tau_t = r_t\bar{b}$; the asset market equilibrium condition can be written as[4]:

$$(3.9) \quad (1-\beta)(e^y - r_t\bar{b}) - \frac{\beta e^o}{(1+r_{t+1})} = \bar{b}$$

which, at the steady state, becomes:

$$(3.10) \quad (1-\beta)(e^y - \bar{r}\bar{b}) - \frac{\beta e^o}{1+\bar{r}} = \bar{b}.$$

The LHS represents savings calculated as the difference between the functions $(1-\beta)(e^y - \bar{r}\bar{b})$ and $\frac{\beta e^o}{1+\bar{r}}$, which are represented graphically in the upper part of fig.3.2 with a straight line (*YY*) and an hyperbola (*OO*) respectively. In the lower part of the figure, *SS* represents the saving function and the solutions to eq.(3.10) are the points where this function intersects the horizontal line *BB*. The hill shape of the saving locus is attributed to the conflicting effects of the interest rate on savings which can be described as follows:

a) the first effect consists in the fact that a higher interest rate lowers the present value of the future endowment e^o, thus decreasing the lifetime wealth. Since current consumption (c_t^y) is a positive function of total wealth, it decreases so that, as a result, savings rise;

b) the second effect, which operates in the opposite direction to the previous one, refers to the fact that since a higher interest rate raises the burden of taxation, current disposable income decreases and so do savings.

---

4 The asset market equilibrium condition is equal to: $s(e^y - r_t\bar{b}, e^o, r_{t+1}) = \bar{b}$, which is a first-order difference equation in r.

So long as the first effect dominates, which is more likely when the interest rate is low, savings are a positive function of $\bar{r}$; after a certain level, the effect is reversed so that savings are negatively related to $\bar{r}$.

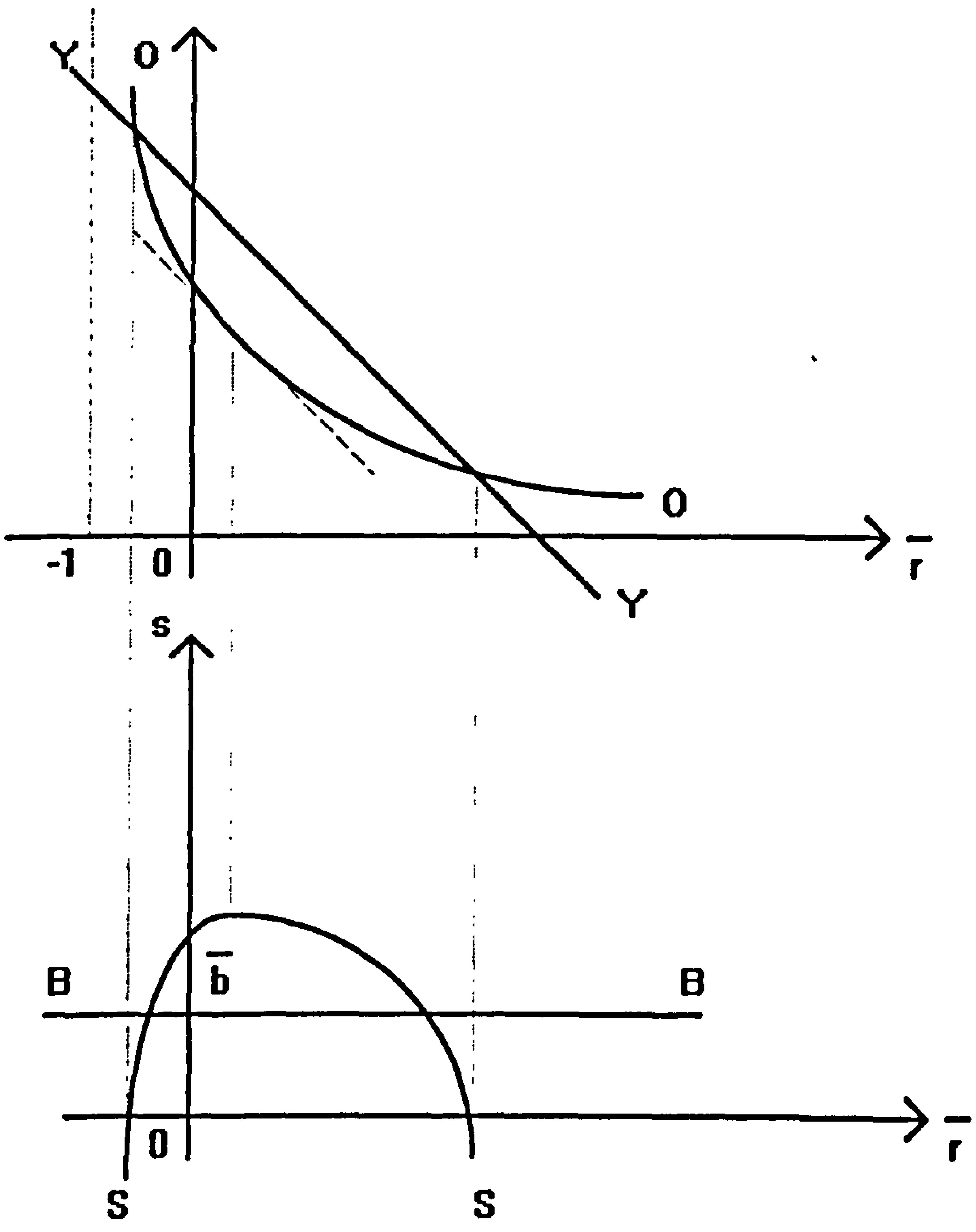

figure 3.2

Eq.(3.10) has two solutions, and the smaller root of $\bar{r}$ represents the stable steady state[5]. If debt rises, then the straight line *BB* shifts up whereas *SS* shifts down to the left. There exists a level of debt beyond which the maximum level of savings is not sufficient to absorb all debt

[5] This procedure of defining the maximum sustainable level of debt is similar to Soren Bo Nielsen's (1992) which has been briefly described in chapter 1.

issued so that no equilibrium point exists. At this maximum level of debt, called b^*, the scenario is the following.

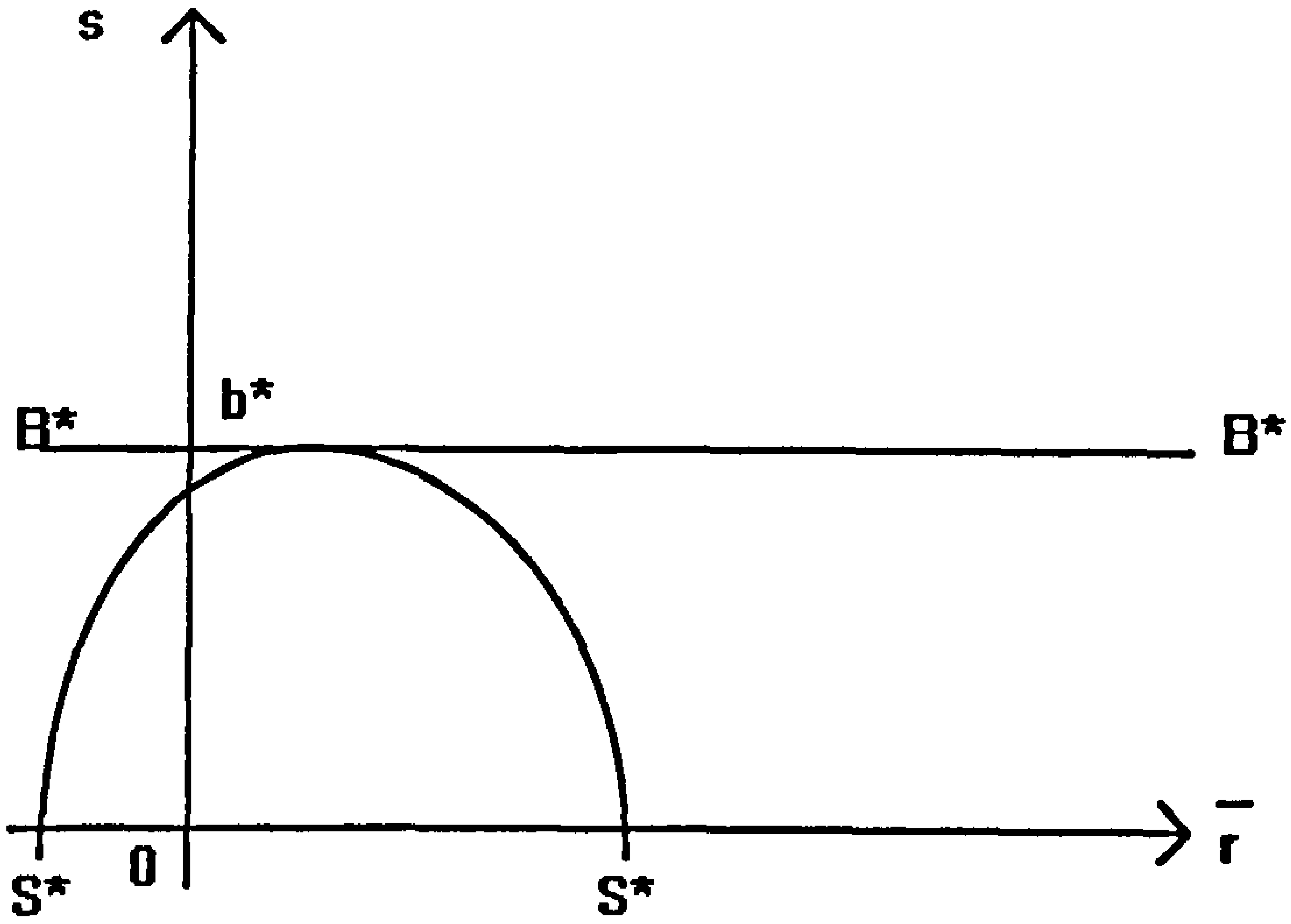

figure 3.3

The maximum sustainable level of debt is calculated by solving the quadratic in $\bar{r}$, eq.(3.10), and then setting the discriminant equal to zero, and we get two positive solutions for the debt which are:

$$(3.11) \qquad b^* = \frac{(1-\beta)\left[e^y + 2e^o \pm 2\sqrt{e^o(e^o + e^y)}\right]}{\beta}.$$

Unlike in the model with production, the simplicity of this model is crucial for solving explicitly for b^* and, consequently, for the steady state interest rate. The acceptable root b^* is the smaller one, since for the larger one the inequality $r^* > -1$ is not satisfied.

The unique solution for the steady state interest rate is equal to $r^* = \frac{e^y}{2b^*} - \frac{2-\beta}{2(1-\beta)}$. When we substitute the acceptable root for b^* and calculate, by assigning arbitrary values to the parameters, the value of the interest rate associated to b^*(i.e. r^*), the latter turns out to be positive. This confirms the quite well known result that instability of the system in the presence of debt is more likely to arise when there is dynamic efficiency, since the introduction of debt should, in principle, help to eliminate dynamic inefficiency.

In conclusion, for the case of savings deposits the maximum sustainable level of debt is associated with the existence of a maximum level of savings which the households are willing to make, given the constraint represented by their lifetime income.

Quite interestingly, it is possible to derive an explicit expression for the maximum sustainable level of debt for the other two cases of financing policies, although the implications are slightly different. Let us start with the case of **treasury bills.** The asset market equilibrium condition is satisfied by a unique value of the interest rate. However, an upper limit still exists and it coincides with the maximum level of debt for which the interest rate has still an economically meaningful value. Given that the asset market equilibrium is equal to[6]:

$$(3.12) \qquad (1-\beta)\left(e^{y} - \frac{r_{t+1}}{1+r_{t+1}}\bar{b}\right) - \frac{\beta e^{o}}{1+r_{t+1}} = \frac{\bar{b}}{1+r_{t+1}}$$

where the LHS represents the saving function, there is a unique steady state value of the interest rate:

$$(3.13) \qquad r_{t+1} = \bar{r} = \frac{\bar{b} + \beta e^{o} - (1-\beta)e^{y}}{(1-\beta)(e^{y} - \bar{b})},$$

which is the solution to eq.(3.12). A necessary condition for the steady state to be acceptable is[7] that $\bar{b} < e^{y}$. So long as this condition holds and provided that there is no upper limit to the value of the interest rate, then any amount of debt issued could, in principle, be absorbed by the market[8].

6 This follows from the condition: $s_t = q_t b_{t+1} = (1+r_{t+1})^{-1} b_{t+1} = (1+r_{t+1})^{-1}\bar{b}$ which is a static equation in r_{t+1} and satisfied by a unique value of the interest rate.

7 In fact, for $\lim_{\bar{b}\to e^{y-}} r_{t+1} = +\infty$, but, just beyond that level, $r_{t+1} \to -\infty$ and $\lim_{\bar{b}\to\infty} r_{t+1} = -\frac{1}{1-\beta} < -1$, which is not acceptable.

8 A comparison with the production economy suggests that the introduction of an additional endogenous variable would rise the order of the dynamics of the system so that the stability properties are no longer unambiguously satisfied.

When the government decides to issue **perpetuity bonds**, the equilibrium on the capital market becomes[9]:

$$(3.14) \qquad (1-\beta)(e^{y}-\bar{b})-\frac{\beta e^{o}}{1+r_{t+1}}=q_{t}\bar{b}$$

which, by using the information that $\bar{q}=\frac{1}{\bar{r}}$, can be written at the steady state as:

$$(3.15) \qquad (1-\beta)(e^{y}-\bar{b})-\frac{\beta e^{o}\bar{q}}{1+\bar{q}}=\bar{q}\bar{b},$$

which is a quadratic equation in $\bar{q}$. In order to have at least one solution, the condition $(1-\beta)(e^{y}-\bar{b})<\beta e^{o}$ must be satisfied. This condition can be represented graphically (see fig.3.4), and implies that the functions represented by the two expressions on the LHS intersect, which happens when the horizontal asymptote of the function $\frac{\beta e^{o}\bar{q}}{1+\bar{q}}$ *(OO)* is higher than the horizontal straight line *YY*, $(1-\beta)(e^{y}-\bar{b})$, which does not depend on $\bar{q}$.

[9] In general terms, it would be equal to: $s(e^{y}-\bar{b},e^{o},[1+q_{t+1}-q_{t}]/q_{t})=q_{t}b_{t+1}=q_{t}\bar{b}$.

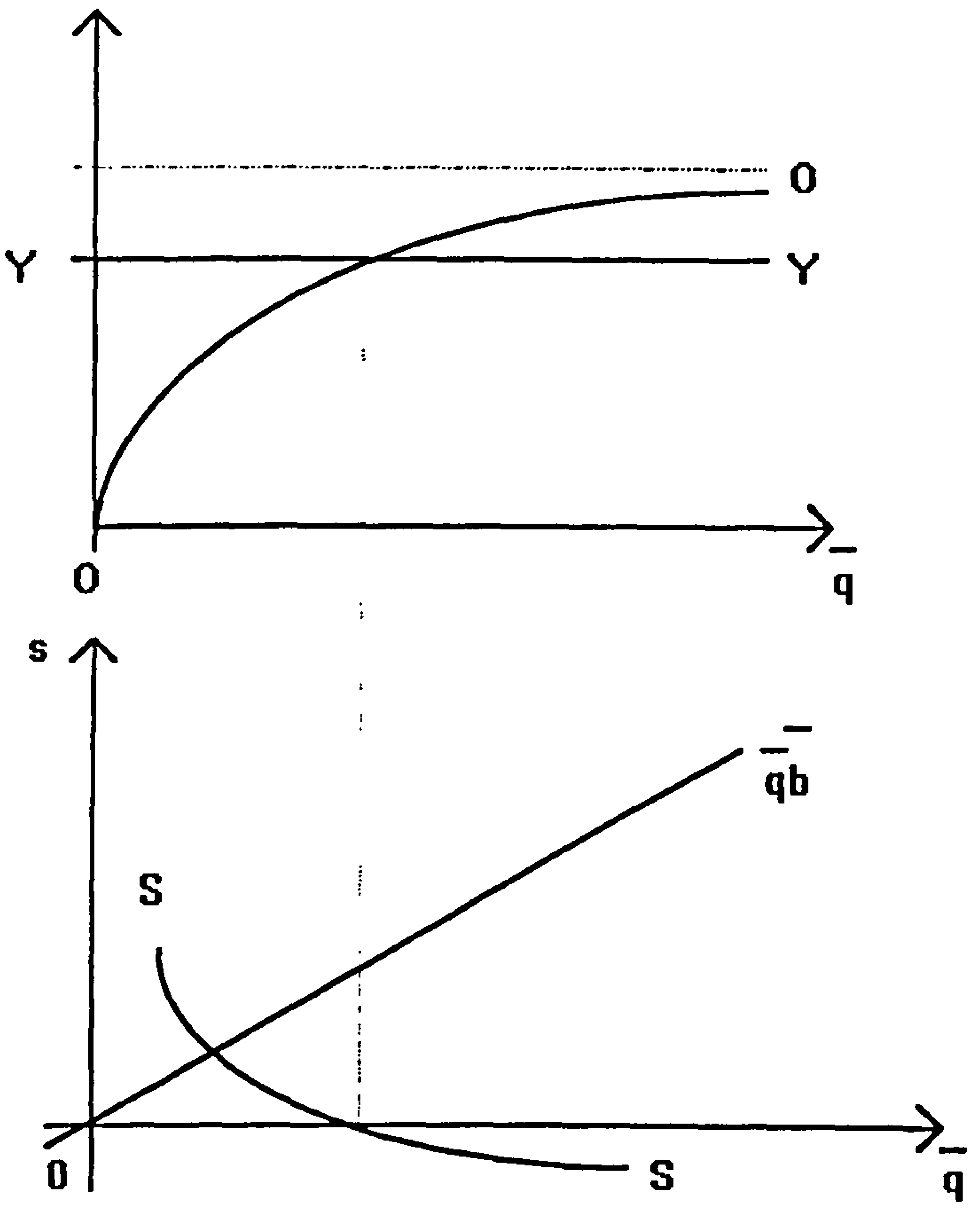

figure 3.4

As $\bar{b} \to e^y$, the line *SS* (i.e. the saving function) shifts to the left and the straight line pivots until they intersect only at the origin. Since negative values for the bond price are excluded, the value e^y (the same value found in the treasury bills case) represents the upper limit on the level of debt, and it is related to the existence of an acceptable steady state rather than to the existence of a steady state itself[10].

The conclusions derived from this analysis are very interesting in so far as they provide an explicit indication for some conditions which must be fulfilled for finding the equilibria of the system. The study of this pure exchange economy differs from the study of the production economy in so far as it succeeds to shed light on some specific aspects which have been left

[10] It is worth noting that the introduction of the capital stock variable would rise the order of the system although the saddle point property of one of the equilibria is consistent with the characterisation of the perpetuity bond price as a jump variable.

indefinite in chapter 2 and also allows to find an explicit mathematical specifications of the conditions which characterise the maximum sustainable level of debt. The very striking conclusion is that a constant issue of debt can be a source of instability also in a simple exchange economy with no capital stock and where only one financial asset and endogenous variable exists.

3.4 The Applicability of the Concept of Sustainability of Government Financing Policies in a Continuous Time Model a-la-Blanchard

The model we consider in this section is the "OLG perpetual youth model" a-la-Blanchard (1985), where the horizon of the agents is chosen arbitrarily. The lifetime period is uncertain and there exists an insurance company which individuals join to insure their future wealth; nevertheless, debt still plays an active role in smoothing out aggregate consumption when there are fluctuations in output and its size is important for determining the dynamics of the main macroeconomic variables in the economy.

We firstly present a brief description of the model[11] and then we investigate the issues of the maximum sustainable levels of government spending and debt applied to that framework. In this economy agents face a constant instantaneous probability of death, p, where $0 \leq p < 1$, so that their expected remaining lifetime is p^{-1}. The population is constant, normalised to one, so that at each instant a flow p of consumers die and p more are born. They contract with the insurance company to leave to it all their wealth (denoted by w) contingent on their death. In the event that they do not die, they will receive pw from the same company. Since consumers are large in number and with free entry to the insurance market in which insurance companies face no risks, an actuarially fair insurance is offered and companies make zero profits. Consumers choose to insure themselves completely since bequests do not

[11] Blanchard's (1985) model is quite well-known, and is investigated in details also in Blanchard and Fischer (1989), chapter 3. Therefore, we do not include a detailed description and an exhaustive derivation of all the formulas. We are more interested in underlining the most important features of the model in order to apply the criterion of the sustainability of a fiscal policy.

exist. Two other additional specifications are necessary: utility is assumed to be logarithmic with a positive subjective discount rate, $\theta + p$, and labour income is distributed equally across agents, thus allowing for aggregation. An agent maximises his utility:

$$(3.16) \qquad \int_t^{\infty} \log c(s,v) e^{(\theta+p)(t-v)} dv,$$

subject to the intertemporal budget constraint: $\dfrac{dw(s,t)}{dt} = [r(t)+p]w(s,t)+y(s,t)-c(s,t)$.

The latter states that the change in the lifetime non-human wealth is equal to the amount of non-human wealth (which at birth is assumed to be zero) on which the individual receives interest, $r(t)w(s,t)$, plus a sum from the insurance company, $pw(s,t)$, and the labour income, $y(s,t)$, derived from labour sold in return for wage income, net of consumption. The other type of wealth existing in the economy is the human wealth, $h(s,t)$, which is equal to the present value of expected future labour income:

$$(3.17) \qquad \int_t^{\infty} y(s,v) e^{-\int_t^v [r(\mu)+p]d\mu} dv.$$

Human wealth at time t is identical for all agents alive at t as non-interest income is, by assumption, age-independent, and all individuals are identical and have the same expected lifetime.

The discount rate in eq.(3.17) is equal to $(r+p)$, i.e. the same rate at which individual non-human wealth accumulates. Assuming that the no-Ponzi-game condition for the households holds[12], then aggregate consumption turns out to be a constant proportion $(p+\theta)$ of the sum of total human and non-human wealth[13]. Aggregation of consumption is possible because, as is clear from the expression for $c(s,v)$ obtainable from the optimisation of eq.(3.16), an individual's consumption depends only on his wealth and not on his age, so that it is possible

12 The no-Ponzi-game condition for the households can be defined as follows: if the individual is still alive at time v, then it is true that: $\lim_{v\to\infty} e^{-\int_t^v [r(\mu)+p]d\mu} w(s,v) = 0$, namely that in the infinite future when the household dies, the value of the non-human wealth has to be negligible.

13 The marginal propensity to consume out of total wealth is age-independent, reflecting the fact that the remaining lifetime is independent of the age of the individuals who then have the same remaining lifetime. Moreover, since preferences are logarithmic and separable in consumption and real balances, the marginal propensity to consume out of wealth is constant.

to sum over all individuals and get identical relationships between the corresponding aggregate variables. All the aggregate variables are denoted by upper case letters: consumption $C(t)$, human wealth, $H(t)$, which accumulates at a rate $[r(t)+p]$, and non-human wealth[14], $W(t)$, which accumulates at a rate $r(t)$, $pW(t)$ being just a transfer, through the insurance companies, from those who die to those who remain alive. The latter cannot, therefore, add to aggregate wealth because the p effects cancel out.

The system is formed by three differential equations in (C,H,W) which can be reduced to two[15]:

(3.18) $\dot{C}=(r-\theta)C-p(p+\theta)W$

and

(3.19) $\dot{W}=rW+Y-C.$

Any time $p>0$, the rate of change of consumption depends on non-human wealth. It can be noted that, even if we set $p=0$, for $r>\theta$ consumption would still exhibit an increasing dynamic behaviour. The wage rate ω and the rate of interest r are determined by capital accumulation. We assume that capital and labour are the two factors of production in the economy and that the production function $Y=F(K)$ is net of capital depreciation[16]. The profit maximisation condition yields the non-interest income (i.e. labour income) equal to the wage rate, $\omega(K)$, and $r(K)=F'(K)$, which can be positive or negative.

Lastly, we introduce into the economy a government which spends G on goods, from which private consumption does not gain any marginal utility, and finances it by lump sum taxes

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14 In aggregate terms, non-human wealth is equal to $W(t)=\int_{-\infty}^{t} w(s,t)pe^{p(s-t)}ds$, which, differentiated forward and taking into account that this financial wealth of the newly born is equal to zero (the assumption of $w(s,s)=0$ reflects the definition of a new cohort as being a family not linked to previous dynasties through operative bequests), yields eq.(3.19).

15 The three equations are:

$C=(p+\theta)(H+W)$, $\dot{H}=(r+p)H-Y$, $\dot{W}=rW+Y-C$; differentiating the first one and eliminating $\dot{H}$ and $\dot{W}$, it yields eqs.(3.18)-(3.19).

16 Namely, $F(K)\equiv\tilde{F}(K,1)-\delta k$ on the assumption that the size of the population can be normalised to one.

(transfer if negative), T, or by debt, D, any time spending plus interest payments over tax revenue is positive. The dynamic budget constraint is equal to:

(3.20) $\dot{D} = rD + G - T$.

Integrating the last equation forward yields:

(3.21) $D(t) = \int_t^{\infty} [T(s) - G(s)] e^{-\int_t^s r_v dv} ds,$

that is the current level of debt must be equal to the present discounted value of primary surpluses. Eq.(3.21) holds provided that the no-Ponzi-game condition for the government holds, which implies that debt will ultimately have to grow less rapidly than the interest rate, namely that $\lim_{t \to \infty} D(t) e^{-\int_t^s r_v dv} = 0$[17]. Non-human wealth is equal to the sum of capital and debt, $W = K + D$. Furthermore, taxes affect the level of human wealth which is equal to $H(t) = \int_t^{\infty} Y(s) e^{-\int_t^s (r_v + p) dv} ds - \int_t^{\infty} T(s) e^{-\int_t^s (r_v + p) dv} ds$, or, equivalently, to the present discounted value of non interest income minus taxes discounted at the rate $(r + p)$.

The government fiscal policy affects the macroeconomic system through the following mechanism:

a) debt is part of wealth and, as such, it affects consumption;

b) the level of G affects aggregate demand directly;

c) the sequence of taxes affects human wealth and, consequently, consumption.

The system is represented by the following three differential equations[18]:

(3.22) $\dot{C} = [r(K) - \theta] C - p(p + \theta)(D + K)$

(3.23) $\dot{K} = F(K) - C - G$

(3.24) $\dot{D} = r(K) D + G - T$.

If the government, at some time t_o, decides to issue new debt, for the budget to be balanced, taxes must be increased by the amount necessary to pay the additional interest

[17] This is the same condition we focused on in chapter 1 and which turned out to be important for evaluating debt sustainability.

[18] Naturally, there are some other implicit conditions which must be satisfied, such as that individual consumption must be non-negative at every instant.

payments, i.e. $dT_o = r(K_o)dD_o$. This financing policy adopted by the government corresponds in our model to the case of debt management through savings deposits: debt is held constant at the new increased level, whereas taxes change over time in order to cover interest payments. If we assume that the government budget is balanced[19], then the two stationary loci are:

$$(3.25) \qquad \dot{C}=0 \quad \Rightarrow \quad C=\frac{p(p+\theta)(K+D)}{r(K)-\theta}$$

and

$$(3.26) \qquad \dot{K}=0 \quad \Rightarrow \quad F(K)=C+G.$$

The system is represented in fig.3.5 below for a level of $G=0$; the capital stock is a predetermined variable whereas consumption is a jump variable[20]. The stationary locus $\dot{C}=0$ has been drawn for a value of debt which is higher than $-K^*$ and approaches the vertical asymptote from the left. The locus $\dot{K}=0$ has a concave shape. There are two equilibrium points: the origin O and point E. From the direction of the arrows, we can conclude that the steady state E is a saddle point and the convergent saddle path, SS, is upward sloping. All the other trajectories imply a negative level of C or K in finite time, which must be excluded.

19 This condition has some implications: for instance, if the government decides to increase the size of public spending at the steady state, for the budget to be balanced, it must reduce national debt or raise taxes.
20 Consumption is the outcome of forward-looking intertemporal choices made by individual consumers and therefore it is a non-predetermined variable.

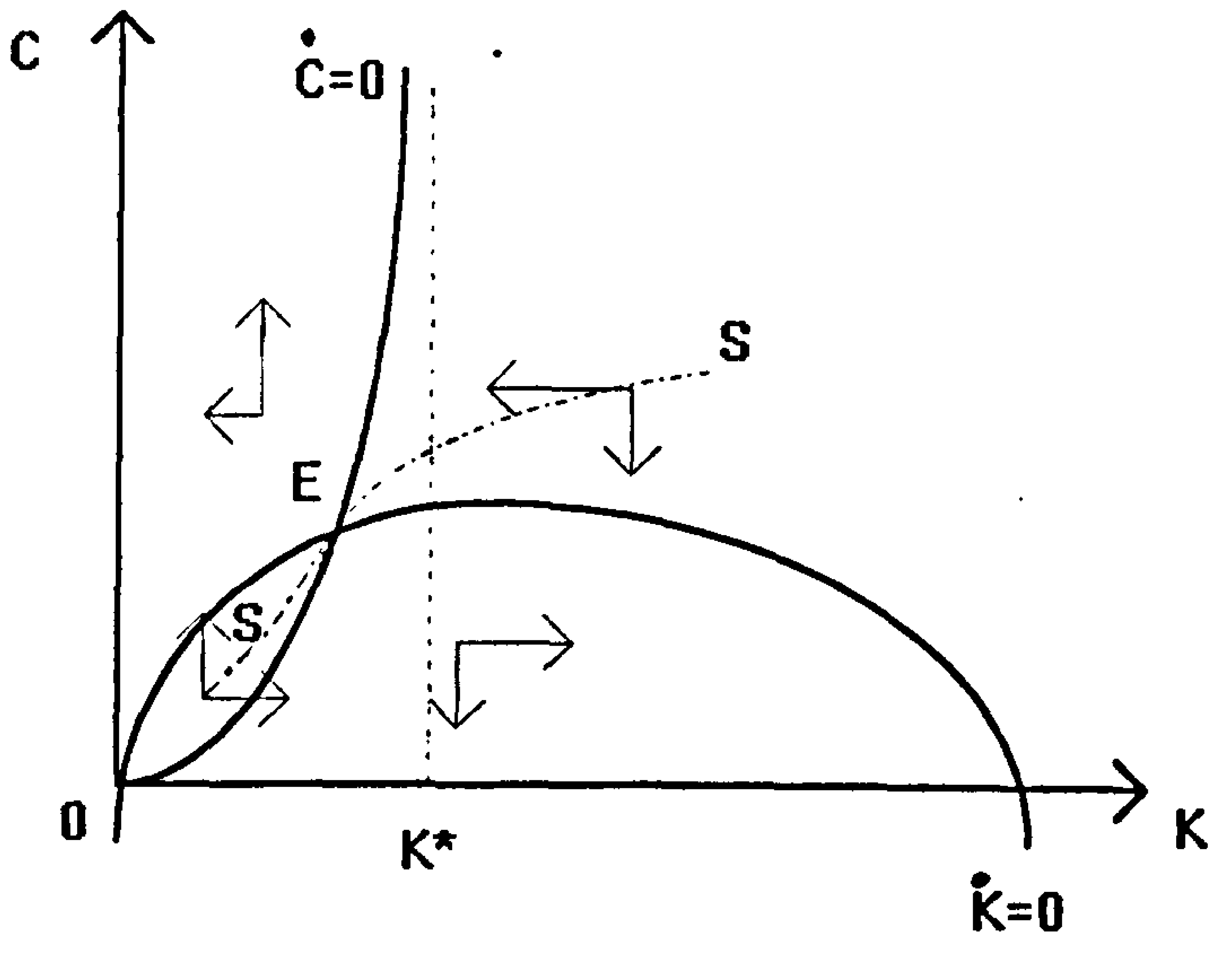

figure 3.5

The vertical asymptote is represented by the value of K^* for which $r(K^*) = \theta$. The steady state level of capital must satisfy the following conditions[21]:

(3.27) $\theta < F'(\overline{K}) < \theta + p$.

The proof is made by contradiction and setting $D=0$ for simplicity. If we assume that $\bar{r} > \theta + p$ (where $\bar{r}$ denotes the steady state interest rate), then $(\bar{r} - \theta)\overline{C} > p\overline{C}$, which implies that $\dfrac{\bar{r} - \theta}{p} > 1$ is true. As we know that for $\dot{C} = 0$, $\dfrac{(\bar{r} - \theta)\overline{C}}{p} = (p + \theta)\overline{K}$, then $(p + \theta)\overline{K} > \overline{C}$. Since for $\dot{K} = 0$, $F(\overline{K}) = C$, then using the condition stated at the beginning we would have $\bar{r}\overline{K} > (p + \theta)\overline{K} \Rightarrow r\overline{K} > F(\overline{K})$ which cannot hold, QED. The same reasoning can be followed to prove the other side of the inequality.

So long as $D > -K^*$, $\dot{C} = 0$ is upward sloping and positive, goes through the origin and approaches K^* from the left. An increase in debt would shift it up and thus decrease the steady state level of C and K. Therefore, the government can affect the level of capital stock by an appropriate choice of the level of debt. Each of the paths for C and K are associated with a particular fiscal policy. For a positive level of debt, we have that $r(\overline{K}) > \theta$, which

21 The value of K^{**} for which $F'(K^{**}) = p + \theta$ can be called the "Doubly Modified Golden Rule", since the "Modified Golden Rule" is represented by $F' = \theta + n$, where n is the population growth rate.

implies that, despite the fact that people have finite horizons, the equilibrium is necessarily dynamically efficient (i.e. $r > n = 0$).

In order to investigate how the concept of sustainability of a fiscal policy applies to this framework, let us focus on two different policies which the government may decide to adopt.

1) An increase in debt, *D*, given *G=0.*

An exogenous increase in D has an initial wealth effect on consumption; however, holding G constant at a zero level, taxes must be increased to balance the government dynamic intertemporal budget constraint. Therefore, the new steady state levels of both K and C will be lower. Graphically, the stationary locus $\dot{C} = 0$ gets steeper (although it always goes through the origin) whilst the stationary locus $\dot{K} = 0$ stays unchanged. Consequently, the positive steady state moves towards the origin. The adjustment is made possible because consumption is a jump variable and the mechanism is that agents base current demand on expected lifetime income. There are two contrasting effects on consumption: initially at time $t=0$ it would increase because households would feel wealthier due to an increased debt owned. Since $p \neq 0$, a smaller fraction of the anticipated higher future tax burden is expected to be borne by currently-alive consumers, so that a higher percentage of the current market value of debt is perceived as "net wealth". Spending would then temporarily increase. However, the increase in government debt and taxes displaces capital and causes a shortage of output produced. Consumption will gradually decline in the long run along the new saddle path which leads towards the new equilibrium point.

Since the positive effect on consumption due to the increased wealth from debt is more than offset, in the long run, by the negative effect on consumption due to the increased present value of future taxes, Ricardian Equivalence does not hold.

The effect of a change in debt on the capital stock can be calculated by differentiating eq.(3.22) evaluated, at the steady state, to obtain:

$$(3.28)\qquad \frac{d\overline{K}}{d\overline{D}}=\frac{p(p+\theta)}{F''\overline{C}-p(p+\theta)+(r-\theta)\dfrac{\partial\overline{C}}{\partial\overline{K}}}.$$

$\dfrac{\partial\overline{C}}{\partial\overline{K}}$ is equal to $\bar{r}$ from the goods market equilibrium: $F(\overline{K})=\overline{C}+G$. However, we can re-express $\bar{r}$ by considering that, when the government budget constraint is satisfied, $F'(\overline{K})\overline{D}=T-G$, so that the formula for consumption can be transformed as follows:

$$(3.29)\qquad \overline{C}=F(\overline{K})-G=F(\overline{K})-T+F'(\overline{K})\overline{D}=$$

$$=F(\overline{K})-\bar{r}\overline{K}+\bar{r}\overline{K}+F'(\overline{K})\overline{D}-T=\omega-T+\bar{r}(\overline{D}+\overline{K})$$

$$\therefore\quad \bar{r}=\frac{\overline{C}}{\overline{D}+\overline{K}}-\frac{\omega-T}{\overline{D}+\overline{K}}.$$

Substituting this expression, together with eq.(3.25), into eq.(3.28), we obtain the final expression for the multiplier:

$$(3.30)\qquad \frac{d\overline{K}}{d\overline{D}}=\frac{p(p+\theta)}{F''\overline{C}-(\bar{r}-\theta)\dfrac{(\omega-T)}{\overline{D}+\overline{K}}}.$$

A sufficient condition for the multiplier to be negative is that $\omega>T$. The level of government debt is crucial in so far as it displaces capital from the portfolio of savers. Taxes are endogenously set to balance the budget and, since aggregate demand is made up only by aggregate consumption, debt can be increased so long as the rise in the burden of taxation does not offset savings entirely. The maximum sustainable level of debt occurs when the two stationary loci are tangent, which happens to be at the origin.

Since there is a unique non-trivial steady state, the poverty trap phenomenon cannot arise and no positive critical level of debt exists beyond which a catastrophe would occur. More precisely, a maximum sustainable level of debt exists, but it just occurs where the steady-state level of the capital stock is driven to zero.

However, we now show that the introduction of a positive level of government spending can modify these conclusions.

2) The introduction of government spending.

The previous analysis was based on the assumption that government spending was negligible. However, if we introduce an active fiscal policy, new interesting results arise. If G rises from zero to a positive level, the locus $\dot{K}=0$ shifts down as the intersection point with the ordinate axis, from eq.(3.26), is equal to $-G$. In this case, there are two positive steady states (point A and B), the higher of which (point B) is saddle point stable (see figure below, where SS represents the saddle path).

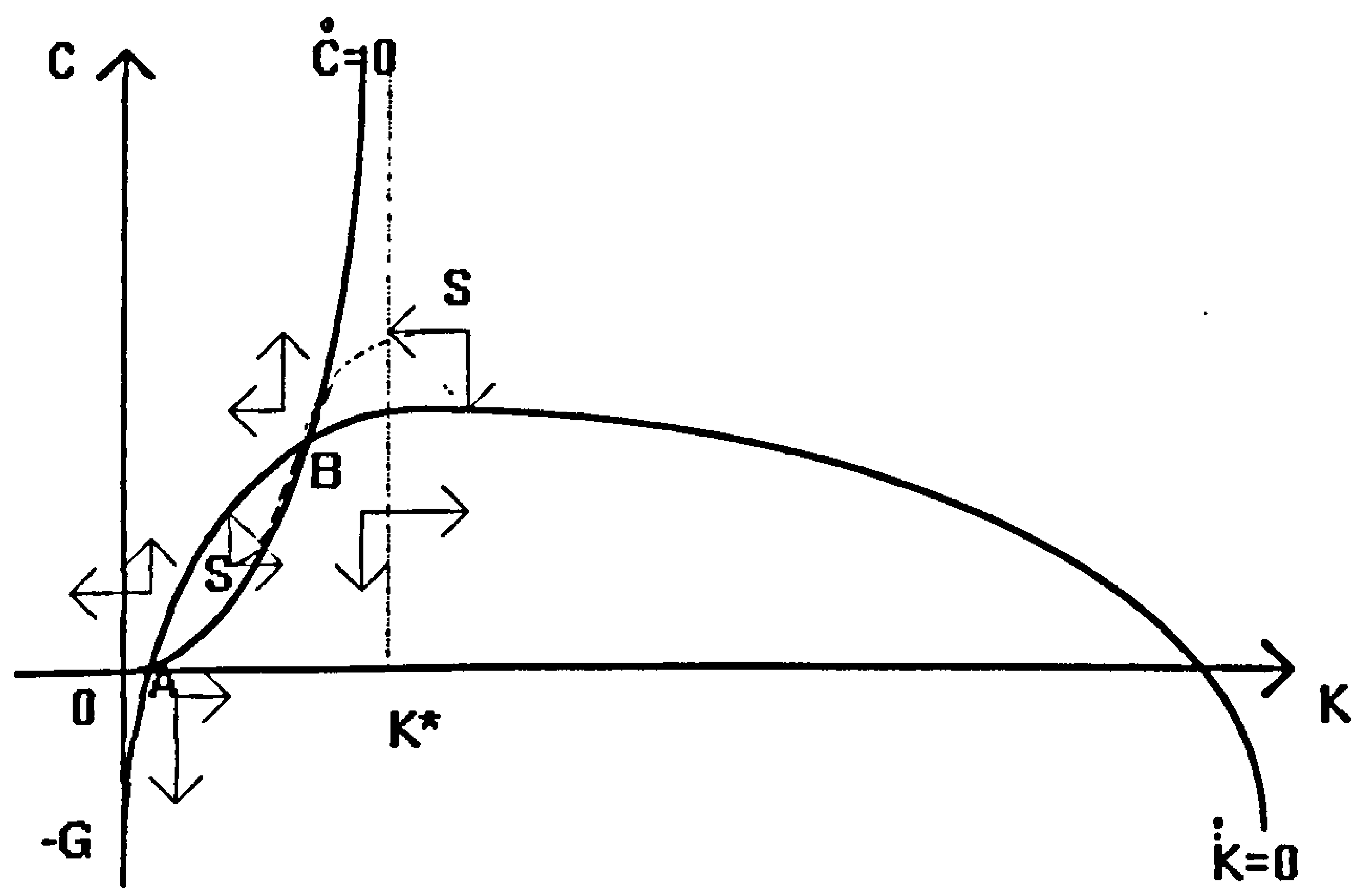

figure 3.6

The analysis proceeds by considering two different policies: an increase in debt, given a constant level of government spending, and an increase in spending plans financed by an increase in the taxes levied, holding constant the level of debt.

2.1) <u>An increase in debt,</u> given a constant positive level of G, would pivot the locus $\dot{C} = 0$, so that the two steady states would get closer. In this case it is possible to define again the *maximum sustainable level of* D^*, as the one related to the point where the two loci are tangent, i.e. point E in fig.3.7 below.

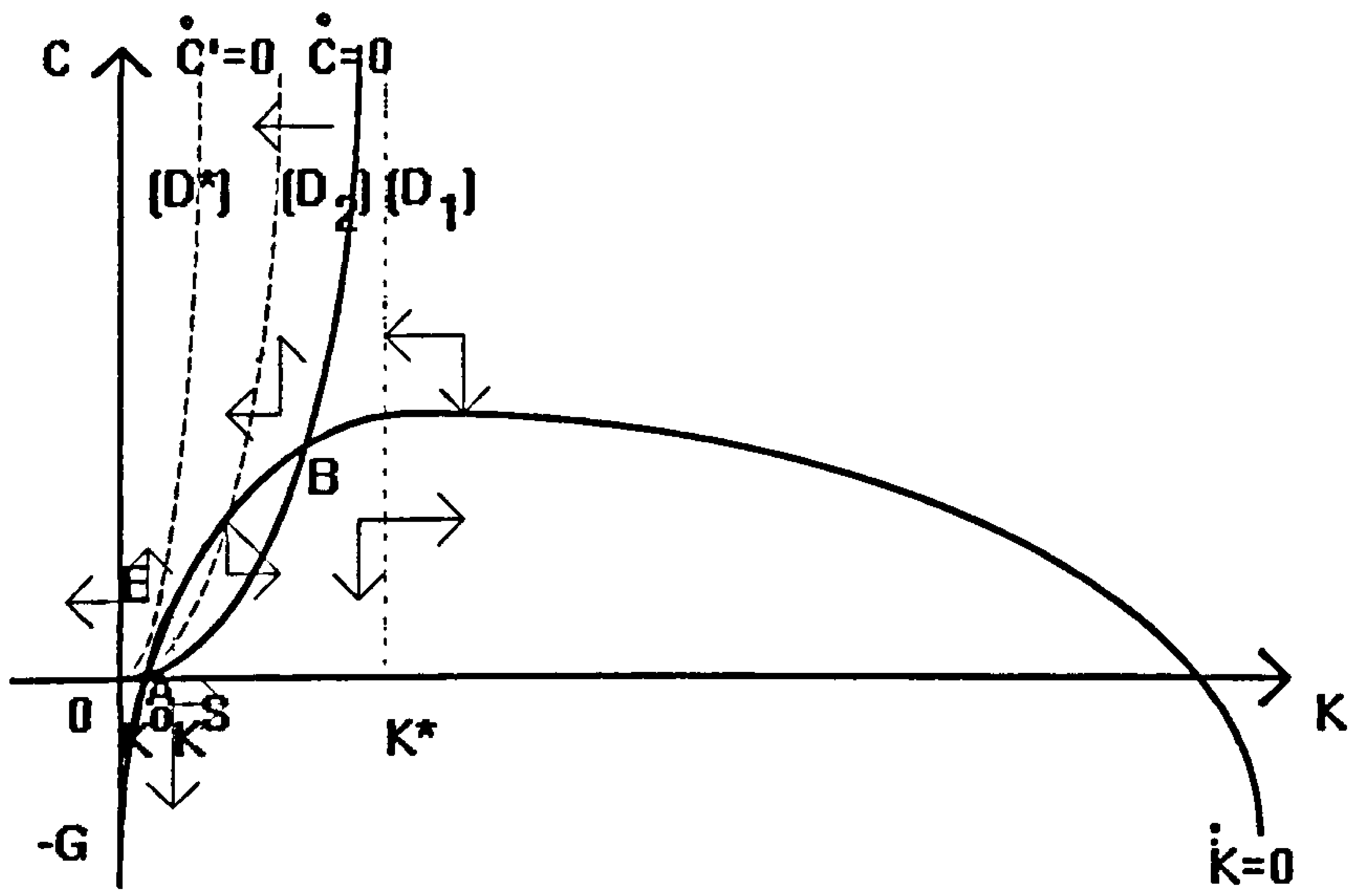

figure 3.7

All the stationary loci $\dot{C} = 0$ are drawn for different values of debt, $D_1, D_2, \ldots, D^*$: as debt rises, the loci become steeper. The maximum sustainable level of debt corresponds to an unambiguously strictly positive steady state level of the capital stock, whereas in the case of $G = 0$ (point 1 above) this level occurred at a zero value of the capital stock. As in the model presented in chapter 2, a further small increase in debt beyond D^* would produce a massive and discontinuous change in the capital stock towards a zero level, which is known as a catastrophe. What happens is that, beyond that point, taxes to be levied are so high that there are not enough resources to be invested in the capital market so that the system leads towards capital decumulation. The increase in debt has a negative effect on capital (as shown by the total multiplier in eq.(3.30)) and on consumption. In fact, the initial positive effect on consumption is represented by the increase in the total amount of wealth which depends positively on debt. However, the supply of resources in the market decreases and, since the amount of government spending, G, is kept constant, C starts declining. It is well established that the path followed by the system will depend on the initial conditions in the economy: also for $D > D^*$, dynamics below the locus $\dot{C} = 0$ would lead unambiguously to decumulation of both capital stock and consumption.

2.2) Let us assume that the government decides on a programme of an increase in government spending through an increase in the burden of taxation. The locus $\dot{K} = 0$ would shift down whereas $\dot{C} = 0$ would not be affected. The two steady states would get closer and the reasoning put forward in point 2.1 above would still apply. Increased taxes would decrease disposable income and the steady state consumption through both human and non-human wealth effects. In fig.3.8 below we represent the dynamics of the process just described. The set of loci $\dot{K} = 0$ is drawn in fig.3.8 for different values of G, namely $G_1, G_2, \ldots, G^*$.

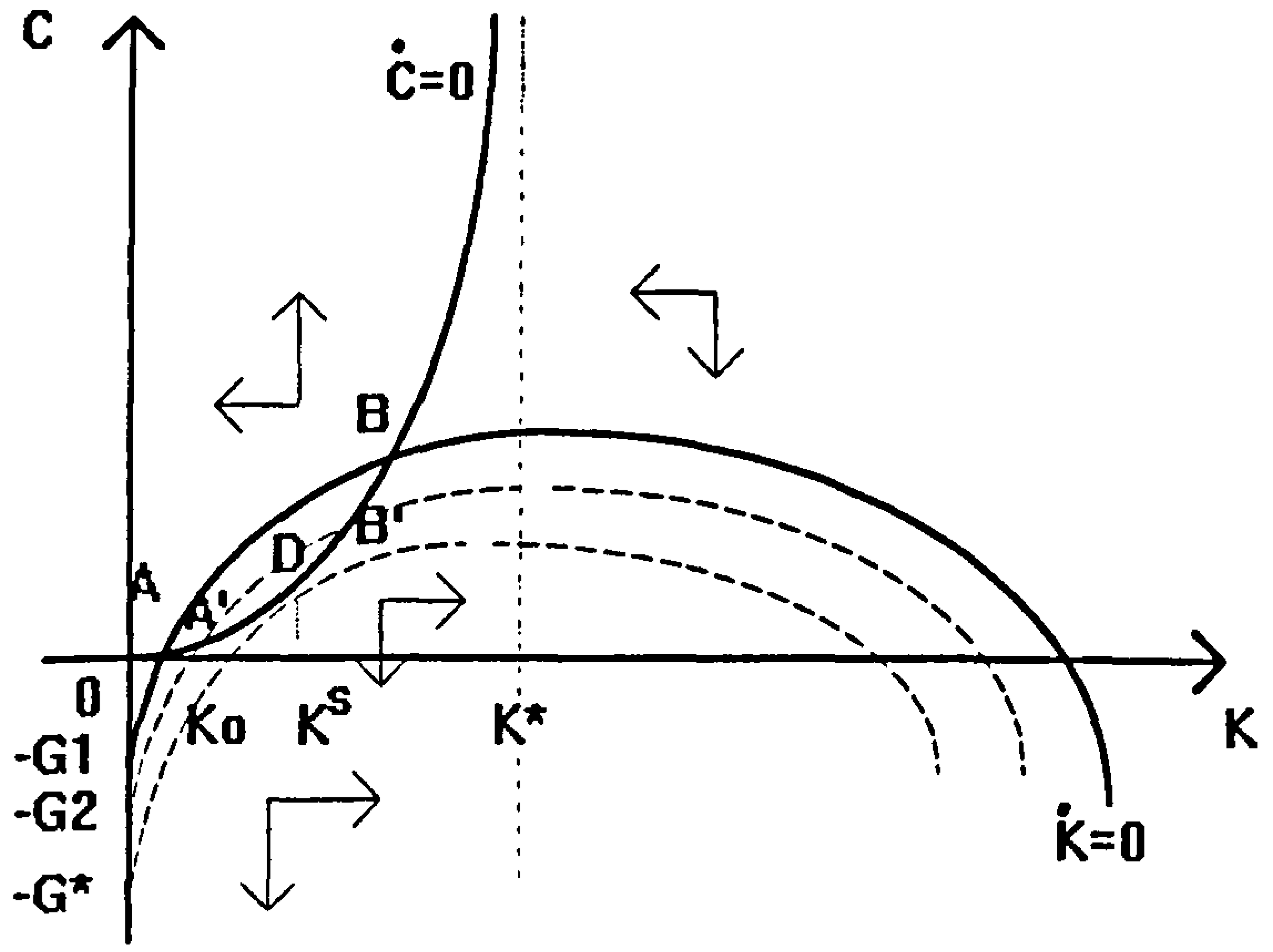

figure 3.8

As G rises, this locus shifts down until it is tangent to $\dot{C} = 0$ at point D for a particular level of government spending equal to G^*. The government must levy an increasing amount of taxes in order to finance a higher government demand. When government spending has increased so much that no intersection point exists, the economy would be brought to the collapse.

A final observation can be derived from fig.3.7. Depending on the initial conditions and given a positive level of government spending, two definitions of *the maximum sustainable level of debt, D**, can be proposed:

(i) D^* can be interpreted as the maximum capital-independent sustainable level of government debt, namely the level of government debt beyond which no equilibrium exists. It can also be referred to as the global maximum level of D^* since it does not depend on any initial conditions.

(ii) If the system initially is at a level of $K_o < K^S$, then D_2 can be interpreted as the maximum sustainable level of D conditional on the starting point K_o. In fact, when $D = D_2$, the equilibrium capital stock is exactly $\overline{K} = K_o$, so that the system is in equilibrium. This second concept of sustainable level of government debt is capital-dependent and thus weaker than the global D^*. If $K_{t_o} < K_o$, then the system would diverge from the equilibrium and D_2 could not be claimed to be the maximum sustainable level conditional on K_{t_o}; moreover, if $K_o > K^S$, then (i) and (ii) would coincide. The relationship between the maximum global level of D and the initial level K_o is represented in fig.3.9.

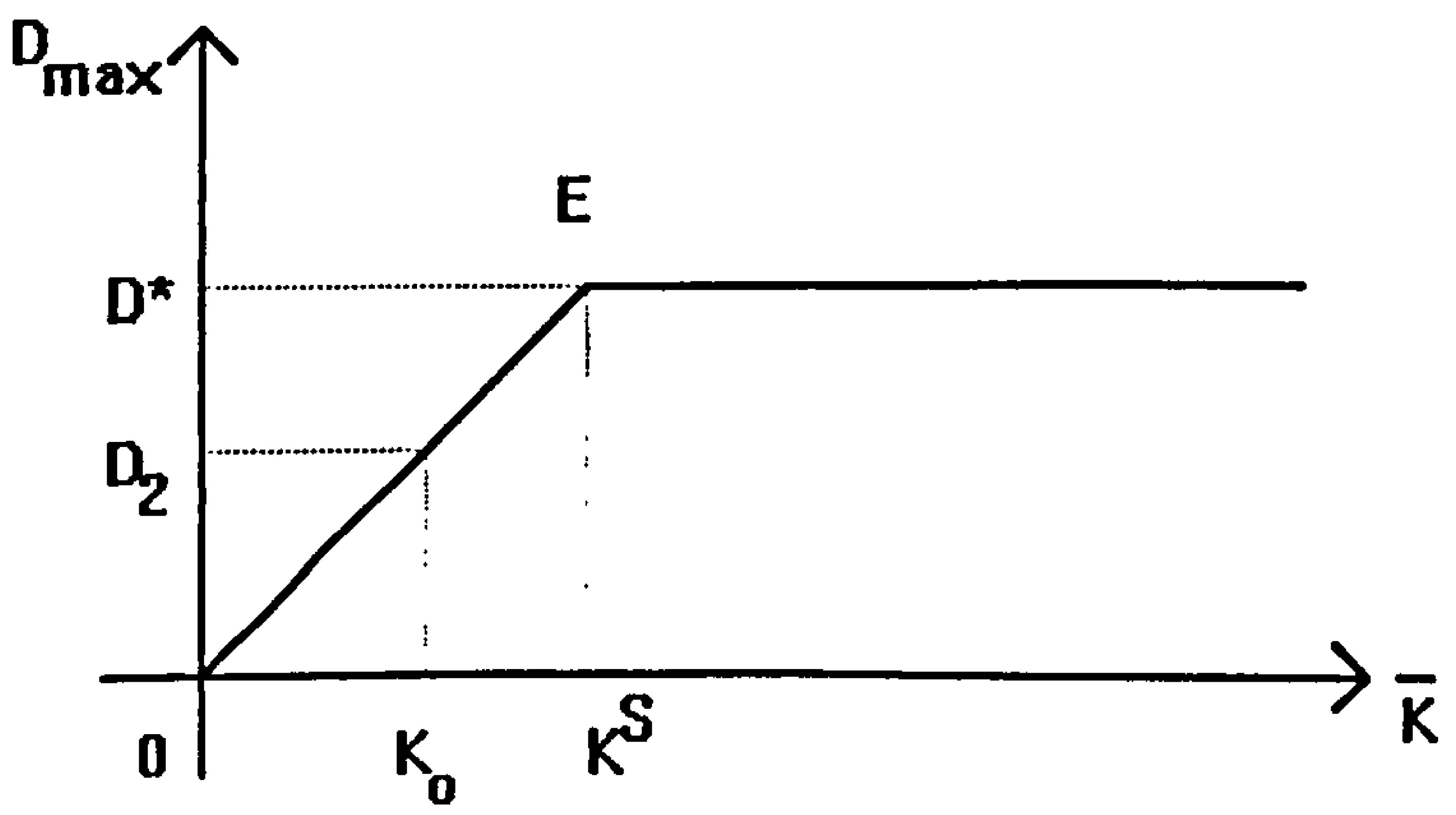

figure 3.9

The same analysis could be carried out for *the maximum level of government spending*, G^*, by following the same reasoning (as just done for debt) in fig.3.8.

To sum up, the technique applied in the previous chapter to derive the maximum level of debt can still be applied to the present framework where the catastrophe phenomenon occurs when we introduce government spending, which is interrelated with both debt (through the government budget constraint) and consumption (through the goods market equilibrium).

A last remark is worth being presented. The distinctive feature of this model, as analysed in point 2.2, which is also peculiar for the debt model presented in chapter 2 and for the social security model presented in this chapter, is not simply that a maximum sustainable level of debt (social security contributions) exists, but that when the latter is pushed very slightly beyond this threshold, a catastrophe occurs, and the economy is thrown into an abyss of uncontrolled capital decumulation, leading to a complete collapse. By contrast, Blanchard's perpetual youth model of debt with no government spending has still the property that a maximum sustainable level of debt exists, which, however, is reached smoothly at the point where the steady state capital stock has been driven to zero. An ignorant policy maker who raised debt by a bit, would allow the economy to reach zero capital step by step.

On the other hand, by examining our models and Blanchard's with positive government spending, such an ignorant policy maker would, at some point, receive a nasty shock finding that well before a zero capital steady state was reached, one more small increase in debt (or contributions) would provoke a massive collapse.

In the following chapters we investigate how the issue of sustainability of a particular policy can be extended to a monetary OLG framework and how this issue is interrelated to the issue of hyperinflation.

CHAPTER 4

NON-SUPERNEUTRALITY OF MONEY AND AN ENDOGENOUS SEIGNIORAGE POLICY

4.1 Introduction

The present and the following chapters contain a description of two different seigniorage policies and of their effects on the real variables in the economy. Our aim is to focus on the issue of sustainability of seigniorage policies which are used to finance the subsidies for the young. Therefore, for the rest of our thesis we extend our analysis of government financing by considering the policy of using money instead of the policy of issuing bonds, as previously investigated. The OLG framework adopted will guarantee consistency and continuity with the analysis carried out in the first part of the thesis.

In the present chapter, we deal with the investigation of the dynamics of real money balances and capital stock in an OLG monetary economy where money grows at a constant rate and is injected into the economy to finance government spending under the form of a subsidy given to the young generation. We also show that in this type of microfounded set-up superneutrality of money does not hold and the Mundell-Tobin effect is at work. An exogenous shock in the money growth rate affects capital intensity and the welfare of the consumers. The poverty trap issue and the effects of an increase in the money growth rate are investigated as well.

In the framework object of this investigation, the economy is populated by individuals who live for two periods and are identical over as well at a point in time. In period t a new generation of N_t individuals is born and, as population is assumed to grow at a constant rate n, the number of people living at time t will be $N_t = N_o(1+n)^t$. In such an economy consumers act to maximise their utility which depends on the amount of consumption over the two periods of their life and the quantity of money they hold. The framework of the general model consists of a series of behavioural equations for each sector of the economy, consumers and firms.

In the next chapter, instead, we extend our model and we endogenise the money growth rate variable. The purpose is to investigate how, in such a rational expectations context, a policy of financing a constant amount of public deficit through money printing (i.e. raising a constant level of seigniorage) may lead to hyperinflation. It is worth saying that the topic of hyperinflation has been a subject of many debates, especially during the last two decades, because of the difficulty of setting up a rational expectations theoretical framework in which this phenomenon could arise, and a conclusive result has not been reached yet. With our model, we hope to establish new findings to explain why economies with a high rate of money growth and a policy of deficit financing mainly through an inflation tax policy exhibit hyperinflation.

This chapter is divided into two main parts. In the first part, we start describing the basic framework which is an adaptation and a re-formulation of Stein's model (1971). We show that, in an intertemporal optimising set-up, superneutrality of money[1] may not hold so that, within a finite-horizon optimising model, the Mundell-Tobin effect is at work.

In the second part we extend the same model by using two different types of production functions. The conclusion is that, under some specific conditions, a poverty trap may arise, which enables us to draw a comparison with the results achieved with the debt model illustrated in chapter 2. As a matter of fact, with a production function which does not satisfy the Inada conditions and with initial conditions on the capital stock and real money balances which do not sustain the initial unstable equilibrium, the system could experience capital decumulation.

The whole model is described in the next section.

[1] This result was obtained by Sidrauski (1967) in an OLG model based on utility maximisation by an infinitely-lived individual. The issue of the superneutrality of money has already been a topic of research during the 60s although in that context an ad hoc saving rule had been used.

SECTION 1.
THE GENERAL MODEL

4.1.1 Consumer's behaviour

The intertemporal utility optimisation problem of the individual is the usual one, namely:

(4.1) *maximise* $u(c_t^y, m_t, c_{t+1}^o)$

subject to $w_t + g_t - c_t^y = m_t + a_t$

$$[1 + r_{t+1}]a_t + \frac{p_t}{p_{t+1}} m_t = c_{t+1}^o$$

where the two last equations are the budget constraints in the first and second period of the individual's life. w_t represents the wage the individual receives from supplying inelastically one unit of labour in the first period of his life; g_t is the subsidy the government gives to the individuals when young and which is financed by printing money. The total net amount of real wealth the consumer holds at the end of his first period of life, that is real wage plus transfers from the government, is both in the form of real money balances, $m_t = \frac{M_t}{p_t N_t}$, which are measured at the end of period t (and whose gross real return is equal to the rate of price deflation or minus the inflation rate), and in the form of real-value interest-bearing financial assets, a_t, which coincides with the expenditure on physical capital (i.e. shares), both defined at the end of period t. $\frac{p_{t+1}}{p_t} = 1 + \pi_{t+1}$ represents the gross rate of inflation.

Money is introduced in the utility function as it provides liquidity services and is a store of value[2]. This function attributed to real balances is associated to the services it can guarantee. One service is, for instance, the increased leisure derived from having larger inventories of

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[2] The view that money can intermediate trade, which is based on the idea that real money balances yield services to the holder like physical capital does (for instance in terms of economising on transaction costs), helps to explain why money is introduced as an argument of the individual's utility function. Of course, since money does not pay any interest, as an alternative it has to provide services for being a good store of value.

real balances: the consumer can save trips and shoe leather costs to go to the bank. The value of the service to the consumer is reflected by the product of the opportunity cost of real balances and the quantity of real balances held by the consumer. As agents have finite lives, real balances are acquired by the young in exchange of providing goods to the old. Finally, utility is always guaranteed by the value of additional leisure available by replacing barter with money.

The financial assets yield a gross real return of $(1+r_{t+1})$ which, together with the value of real money balances held in the portfolio, are used to finance consumption in the second period of life. The old instead consume the rental incomes received from the stock of capital in which they invested and exchange their money balances for goods as they do not leave any bequest. Both money and capital are stores of value.

The lifetime budget constraint can be written as:

$$(4.2) \qquad w_t + g_t = c_t^y + [1+r_{t+1}]^{-1} c_{t+1}^o + [1+r_{t+1}]^{-1}[1+r_{t+1} - \frac{p_t}{p_{t+1}}]m_t.$$

We define the nominal interest rate i_t between t and $t+1$ as the money yields derived from holding capital, that is the rent plus (minus) the expected capital gain (loss):

$$(4.3) \qquad [1+r_{t+1}]\frac{p_{t+1}}{p_t} = [1+i_t]$$

which can be rearranged to yield:

$$(4.4) \qquad [1+r_{t+1}]^{-1}[1+r_{t+1} - \frac{p_t}{p_{t+1}}] = i_t[1+i_t]^{-1}.$$

Substituting the last expression in eq.(4.2) we get:

$$(4.5) \qquad w_t + g_t = c_t^y + [1+r_{t+1}]^{-1} c_{t+1}^o + i_t[1+i_t]^{-1} m_t.$$

The LHS and RHS represent the lifetime real income and the present value of the lifetime consumption respectively. The intertemporal utility maximisation process yields a money and asset demand function which can be represented as follows:

$$(4.6) \qquad m_t = m(\overset{(+)}{w_t + g_t}, \overset{(+)}{r_{t+1}}, \overset{(-)}{i_t})$$

$$(4.7) \qquad a_t = a(\overset{(+)}{w_t + g_t}, \overset{(?)}{r_{t+1}}, \overset{(?)}{i_t}).$$

Demand for real balances per worker is a function of the lifetime real income and of the opportunity costs of holding real balances, r_{t+1} and i_t. A rise in the capital stock raises the transactions demand for real balances because it lowers the opportunity cost (the yield on real capital) of holding real balances. Investments in financial assets by the young depend positively on disposable income, but the effect of a change in the interest rates remains ambiguous[3].

4.1.2 Firm's behaviour

We assume that at any period of time the composite good consumed is produced according to a production process which employs two factors, capital and labour. We assume that the individuals work when they are young and retire when old and leave no bequests. Therefore, the supply of labour at any period of time is equal to the number of young at that date. The endowment of capital at the beginning of period t, K_t, is the amount of resources which were not consumed in the period before. Thus, the expression for capital is equal to:

(4.8) $\quad K_{t+1} - K_t = N_t a(w_t + g_t, r_{t+1}, i_t) - K_t$.

The LHS represents the net investment which is a change in the capital stock between period t and $t+1$. The RHS represents the net savings which consist of financial investment of the young ($a(\bullet)$) net of dissaving of the old. Stein (1971) assumes that savings of the young in the form of output, and which are not consumed, are rented to firms and constitute the amount of capital to be used in the next period. The only interest-bearing asset in the economy is represented by physical capital, so that the total real value of the interest-bearing financial assets is invested in building up capital stock. Eq.(4.8) can be written in per capita terms as follows:

(4.9) $\quad k_{t+1} = (1+n)^{-1} a(w_t + g_t, r_{t+1}, i_t)$

3 The relation between savings and the interest rate has already been discussed in chapter 2.

where $k_t = K_t / N_t$ is the capital-labour ratio. The asset market equilibrium in eq.(4.9) differs from the asset market equilibrium of the model in chapter 2 both because we assume $\bar{b} = 0$ and because we take into account population growth.

Output of the production process is governed by a constant returns to scale production function which is invariant over time, equal for each firm and neoclassical in its specification:

(4.10) $Y_t = F(K_t, N_t) = N_t f(k_t)$.

The production function $f: R_+ \rightarrow R_+$ is assumed to be twice continuously differentiable, positive and strictly concave[4]. Since we assume perfect competition in both capital and labour market, each factor is paid its marginal product. Henceforth, the rental rate to capital paid by firms to the owners of capital and the wage rate is given by:

(4.11) $r_t = f'(k_t)$

(4.12) $w_t = f(k_t) - k_t f'(k_t)$.

Therefore, eq.(4.9) can be rewritten as:

(4.13) $k_{t+1} = (1+n)^{-1} a[f(k_t) - k_t f'(k_t) + g_t, f'(k_{t+1}), i_t]$

which is an implicit first-order difference equation in k.

4.1.3 The Set-up of the Model and the Analysis of the Macroeconomic Equilibrium and of the Steady State Welfare

The set-up of the model consists in defining the relationships which characterise the remaining markets in the economy. **The goods market equilibrium** is represented by the resource constraint which implies that the quantity of output produced in each period is both consumed by the individuals during their lives and invested to increase the amount of the capital stock available in the economy in the next period. Therefore, at time t this relation can be written as follows:

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4. Strictly concavity implies the following sign of the cross derivative in the Hessian matrix: $F_{nn}F_{kk} - (F_{nk})^2 > 0$, where n stays for labour.

$$(4.14)\qquad Y_t = (N_t c_t^y + N_{t-1} c_t^o) + (K_{t+1} - K_t).$$

which, in per-capita terms, becomes:

$$(4.15)\qquad y_t = c_t^y + \frac{c_t^o}{1+n} + (1+n)k_{t+1} - k_t.$$

We assume that in this economy there is a government which is issuing national currency (i.e. injecting money) to finance a lump sum subsidy to the young generation. Let us suppose that M_t is the stock of money existing at the end of period t. The **government financing policy** consists in injecting money into the economy at a constant rate μ:

$$(4.16)\qquad \frac{M_t}{M_{t-1}} = 1 + \mu.$$

Money is used to finance the total value of the subsidy given to the young:

$$(4.17)\qquad N_t p_t g_t = M_t - M_{t-1},$$

so that, from the government budget constraint, the real transfers per worker are equal to:

$$(4.18)\qquad g_t = \frac{M_t}{N_t p_t} - \frac{M_t/(1+\mu)}{N_t p_t} = \frac{\mu}{1+\mu} m_t.$$

Therefore, real transfers per worker are proportional to the real balances per worker, m_t.

The **money market** always clears, which implies that the total supply of real balances injected into the economy is equal to the demand for money (which is derived from the individual's utility maximisation process). Substituting the expressions for some of the exogenous variables included as arguments in the demand function, eq.(4.6) can be rewritten as:

$$(4.19)\qquad m_t = m[f(k_t) - k_t f'(k_t) + \mu(1+\mu)^{-1} m_t, f'(k_{t+1}), i_t].$$

Recalling that the per-capita real money balances are given by the expression $m_{t+i} = \dfrac{M_{t+i}}{p_{t+i} N_{t+i}}, \forall i = 0,..,\infty$, the following relation holds:

$$(4.20)\qquad \frac{p_{t+1}}{p_t} = \frac{m_t}{m_{t+1}} \frac{M_{t+1}}{M_t} \frac{N_t}{N_{t+1}},$$

which, after some arrangements, becomes:

$$(4.21)\qquad m_{t+1}=\frac{1+\mu}{1+n}\frac{1+r_{t+1}}{1+i_t}m_t,$$

where the second term, $\frac{p_t}{p_{t+1}}=\frac{1+r_{t+1}}{1+i_t}$, represents the inverse of the gross inflation rate. The hypothesis of clearing markets is consistent with what follows. Since an individual can save either in the form of real capital or in the form of real balances, if he is saving the desired quantity of $\left(K_{t+1}+\frac{M_t}{p_t}\right)$ and holding the desired level of capital, then he must be holding also the desired stock of real balances. Therefore m_t represents the quantity demanded and supplied in the economy. Eq.(4.21) is a first-order difference equation in m and, together with eq.(4.13), forms a two difference equation system summarised as:

$$(4.22)\qquad k_{t+1}=\frac{1}{1+n}a[f(k_t)-k_tf'(k_t)+\mu(1+\mu)^{-1}m_t,f'(k_{t+1}),i_t]$$

$$(4.23)\qquad m_{t+1}=\frac{1+\mu}{1+n}\frac{1+f'(k_{t+1})}{1+i_t}m_t$$

where $i_t=i(k_t,k_{t+1},m_t;\mu)$ can be derived from eq.(4.19).

The study of the steady state properties is easily carried out by resorting to the First-Order-Conditions (FOCs); this is the reason why we turn back to considering the consumer's intertemporal optimisation problem through the relations which characterise the **steady state equilibrium**. In particular, we focus on the effects of a change in money growth rate on the steady state. We first solve the intertemporal allocation problem by using the Lagrangean:

$$(4.24)\qquad L=u(c_t^y,c_{t+1}^o,m_t)+\lambda\left[w_t+g_t-c_t^y-c_{t+1}^o(1+r_{t+1})^{-1}-i_t(1+i_t)^{-1}m_t\right].$$

The First-Order-Conditions are represented by the following equations:

$$(4.25)\qquad u_y(t)=(1+r_{t+1})u_o(t+1)$$

$$(4.26)\qquad u_m(t)=\frac{i_t}{1+i_t}(1+r_{t+1})u_o(t+1)=\frac{i_t}{1+i_t}u_y(t),$$

where the argument *time* in parenthesis stands for the time index of the variables which the marginal utility refers to, namely: $u_y(t)=\frac{\partial u(c_t^y,c_{t+1}^o,m_t)}{\partial c_t^y}$, $u_o(t+1)=\frac{\partial u(c_t^y,c_{t+1}^o,m_t)}{\partial c_{t+1}^o}$ and $u_m(t)=\frac{\partial u(c_t^y,c_{t+1}^o,m_t)}{\partial m_t}$.

From eq.(4.25) it is clear that savings, in the form of real capital, increase utility by $u_o(1+r_{t+1})$ in the following period, whereas, from eq.(4.26), savings in the form of real balances give a marginal utility of u_m as leisure gained by having more real balances to finance transactions (and which is also equal to $\frac{i_t u_o}{(1+\pi_{t+1})}$, i.e. the amount of goods provided for retirement).

At the steady state real balances per worker are constant, so that the equilibrium rate of the price change is equal to the growth rate of money supply per worker. The relation between the steady state nominal and real rate of interest is the following[5]:

$$(4.27)\qquad 1+\bar{i}=\frac{1+\mu}{1+n}\cdot[1+\bar{r}].$$

If the growth rate of money supply is negative, then the nominal interest rate is non-negative only if $r>n$. In conclusion, given that money supply changes to provide transfers to the young generation, it is not possible to have a dynamically inefficient system together with a positive nominal interest rate if the money growth rate is negative. A rearrangement of eq.(4.27), useful for our analysis further on, is the following:

$$(4.28)\qquad \frac{\bar{i}}{1+\bar{i}}=1-\frac{1+n}{1+\mu}\cdot\frac{1}{1+\bar{r}}.$$

At the steady state the resource constraint or goods market equilibrium (eq.(4.15)) can be written as:

$$(4.29)\qquad \bar{y}=f(\bar{k})=\bar{c}^y+\frac{1}{1+n}\bar{c}^o+n\bar{k}.$$

5 This relation clearly shows that the nominal interest rate is a function of the capital stock, money and population growth rate, $i_t=i[k_t;\mu,n]$, as anticipated above.

The assumption of additively separable utility allows us to rewrite the condition in eq.(4.25), evaluated at the steady state, as:

$$(4.30) \qquad u_y(\bar{c}^y) = [1 + f'(\bar{k})]u_o(\bar{c}^o).$$

Moreover, by combining the latter with eq.(4.26), we get:

$$(4.31) \qquad u_m(\bar{m}) = [1 + f'(\bar{k})]\frac{\bar{i}}{1+\bar{i}}u_o(\bar{c}^o).$$

By substituting eq.(4.28) into eq.(4.31) we obtain the relation:

$$(4.32) \qquad u_m(\bar{m}) = \left[1 - \frac{1+n}{1+\mu}\frac{1}{1+f'(\bar{k})}\right]u_y(\bar{c}^y) = \left[1 + f'(\bar{k}) - \frac{1+n}{1+\mu}\right]u_o(\bar{c}^o)$$

where $\bar{m}$ is a function of $\bar{k}, \bar{i}$ and $\bar{c}^o$. Since the endogenous variables in the system are the four $(\bar{c}^y, \bar{c}^o, \bar{k}, \bar{m})$, it is necessary to find one more equation in $\bar{m}$ and for this purpose we consider the household's second period budget constraint:

$$(4.33) \qquad [1 + r_{t+1}]a_t + \frac{p_t}{p_{t+1}}m_t = c^o_{t+1}.$$

As the market equilibrium implies that $a_t = [1+n]k_{t+1}$ and, at the steady state, $\frac{p_t}{p_{t+1}} = \frac{1+n}{1+\mu}$, real balances per worker become:

$$(4.34a) \qquad \bar{m} = (1+\mu)\left\{\frac{\bar{c}^o}{1+n} - [1 + f'(\bar{k})]\bar{k}\right\}$$

which implies, after a few transformations, that second period consumption is equal to:

$$(4.34b) \qquad \bar{c}^o = [1 + f'(\bar{k})][1+n]\bar{k} + \frac{1+n}{1+\mu}\bar{m}.$$

From equations (4.29)-(4.30)-(4.32)-(4.34b) we derive our model consisting of a two-equation system as follows:

$$(4.35) \qquad u_y\left(f(\bar{k}) - n\overset{(-)}{\bar{k}} - \frac{\bar{c}^o}{1+n}\right) = \left(1 + f'(\overset{(-)}{\bar{k}})\right)u_o(\bar{c}^o)$$

$$(4.36) \qquad u_m\left([1+\mu]\left[\frac{\bar{c}^o}{1+n} - [1 + f'(\bar{k})]\bar{k}\right]\right) = \left[1 + f'(\bar{k}) - \frac{1+n}{1+\mu}\right]u_o(\bar{c}^o).$$

The two equations (4.35) and (4.36) represent the goods market (*GG*) and the asset market (*AA)* equilibrium locus respectively, and are drawn in fig.4.1 in a $(\bar{k},\bar{c}^o)$ space, where point $E(\bar{k}_E,\bar{c}^o_E)$ represents the steady state equilibrium.

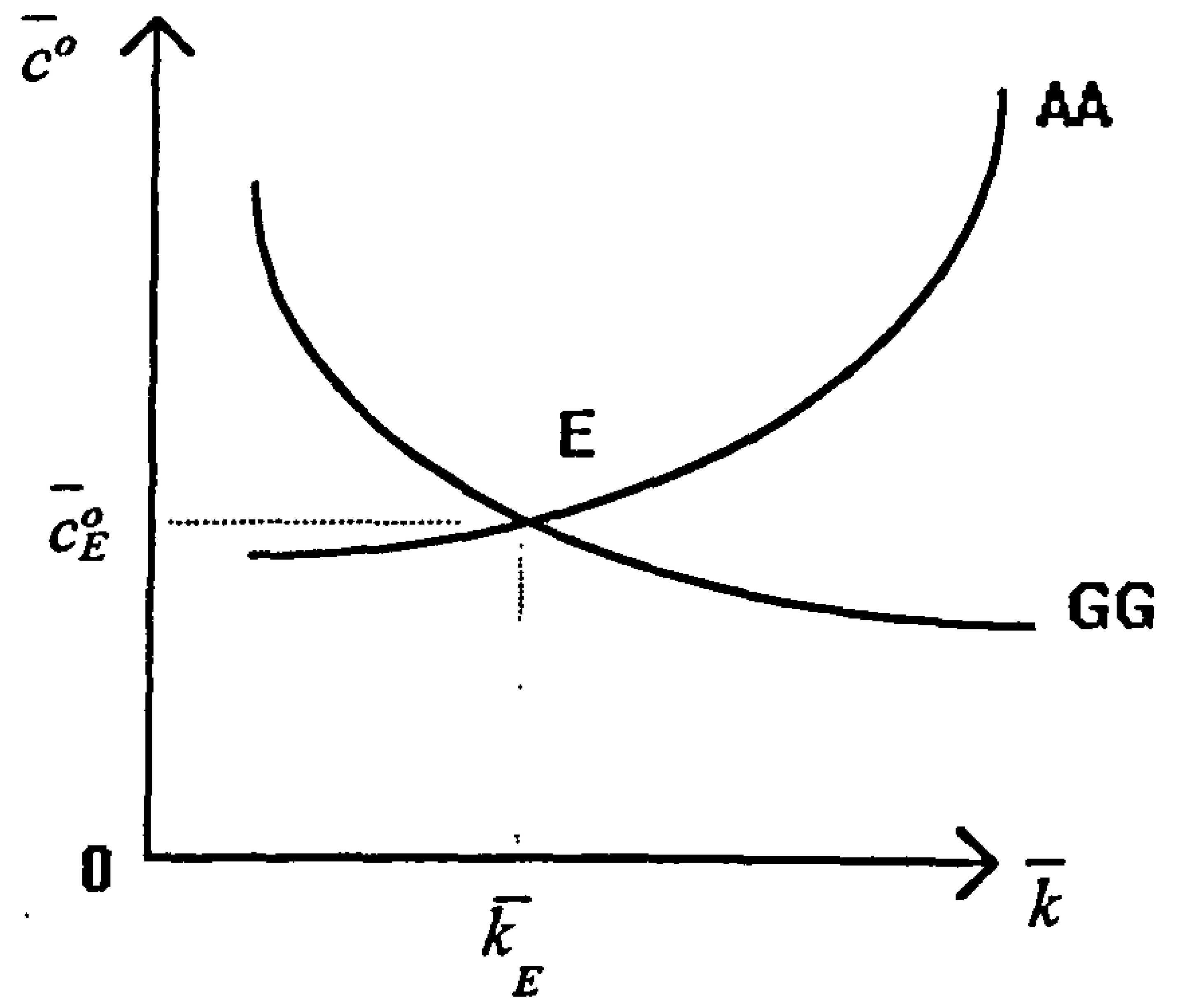

figure 4.1

The slopes of the two stationary loci and the effects of a change in μ on the two real variables $(\bar{k},\bar{c}^o)$ are found by studying the system in a matrix form as follows[6]:

$$(4.37)\quad \begin{bmatrix} u_{yy}[f'-\overset{(?)}{n}]-f''u_o & -u_{yy}/(1+\overset{(+)}{n})-(1+f')u_{oo} \\ -u_{mm}[1+\mu](1+\underset{(+)}{f'}+f''\bar{k})-f''u_o & u_{mm}(1+\mu)/(1+\underset{(?)}{n})-(u_m/u_o)u_{oo} \end{bmatrix}\begin{bmatrix} d\bar{k} \\ d\bar{c}^o \end{bmatrix}=$$

$$=\begin{bmatrix} 0 \\ -u_{mm}\overline{m}/(1+\mu)+\underset{(+)}{(}1+n)u_o/(1+\mu)^2 \end{bmatrix}d\mu$$

The slope of eq.(4.35) is equal to:

$$(4.38)\quad \left.\frac{d\bar{c}^o}{d\bar{k}}\right|_{GG}=-\frac{u_{yy}(f'-n)-f''u_o}{-\frac{1}{1+n}u_{yy}-(1+f')u_{oo}}$$

6 We assume that the marginal utilities with respect to consumption in the two periods of life are positive and decreasing in it.

and depends on the sign and size of the difference $(f'-n)$ (being $u_{yy}, u_{oo} < 0$). Let us suppose, initially, that $f' = n$. Consequently, the relation between $\bar{c}^o$ and $\bar{k}$ in eq.(4.35) is negative so that the GG schedule is negatively sloped. On the other hand, the slope of the relation expressed by eq.(4.36) is ambiguous even when $\mu = 0$, as is clear from the following:

$$(4.39) \quad \left.\frac{d\bar{c}^o}{d\bar{k}}\right|_{AA} = -\frac{\dfrac{u_{mm}(1+\mu)}{1+n} - \dfrac{u_m}{u_o} u_{oo}}{-u_{mm}(1+\mu)(1+f'+f''\bar{k}) - f''u_o}.$$

However, if we assume that $\bar{k}[1+f'(\bar{k})]$ is increasing in $\bar{k}$[7] and the numerator in eq.(4.39) is negative, then the slope of AA is positive. We can now prove the following proposition.

Proposition 4.1. *A positive change in the rate of money growth, μ, will affect the steady state equilibrium so that money is not superneutral.*

Proof. A positive change in the rate of money growth, μ, increases the RHS and decreases the LHS of the asset market equilibrium (eq.(4.36)). Therefore, at each level of $\bar{c}^o$, $\bar{k}$ must increase to re-establish the equilibrium and AA shifts to the right *(A'A')*, and the equilibrium point moves from E to E' (fig.4.2). The opposite trend holds for the consumption $\bar{c}^o$ which decreases. Money is not superneutral as capital intensity is increased by the rate of monetary growth[8].

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[7] It is worth remembering that as $\bar{k}$ increases, $f'(\bar{k})$ decreases in absolute value.

[8] The decrease in the long-run level of real money balances is based on the assumption that the expression $\bar{k}[1+f'(\bar{k})]$ is increasing in $\bar{k}$ as well.

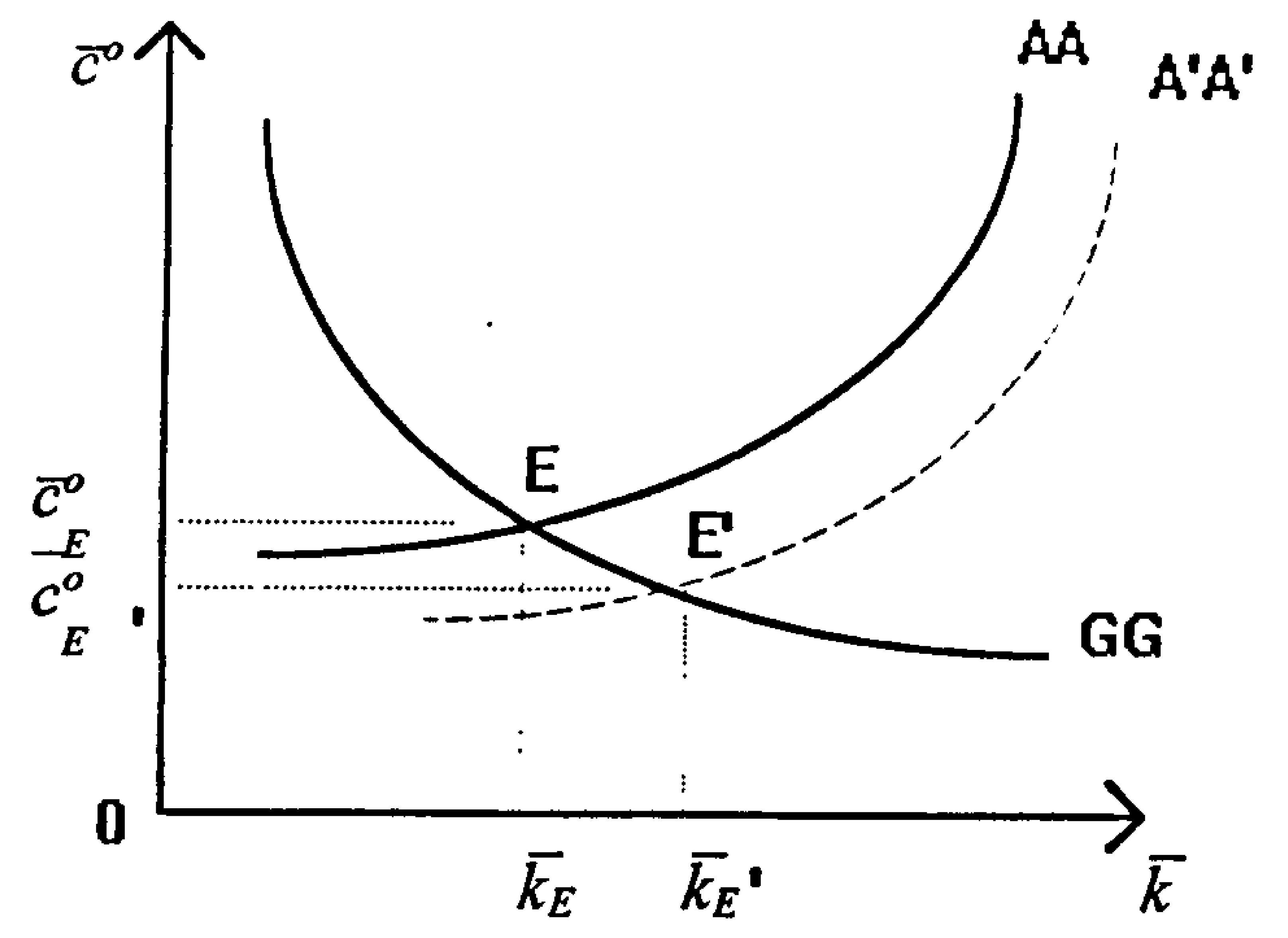

figure 4.2

Since the signs of the slope of the two loci are ambiguous, we consider also the case of $(f'-n) >> 0$ in fig.4.3 below. When the last condition holds, *GG* could be positively sloped whereas *AA* could be negatively sloped. In this case, an increase in μ would raise both the steady state capital intensity and the old generation's consumption.

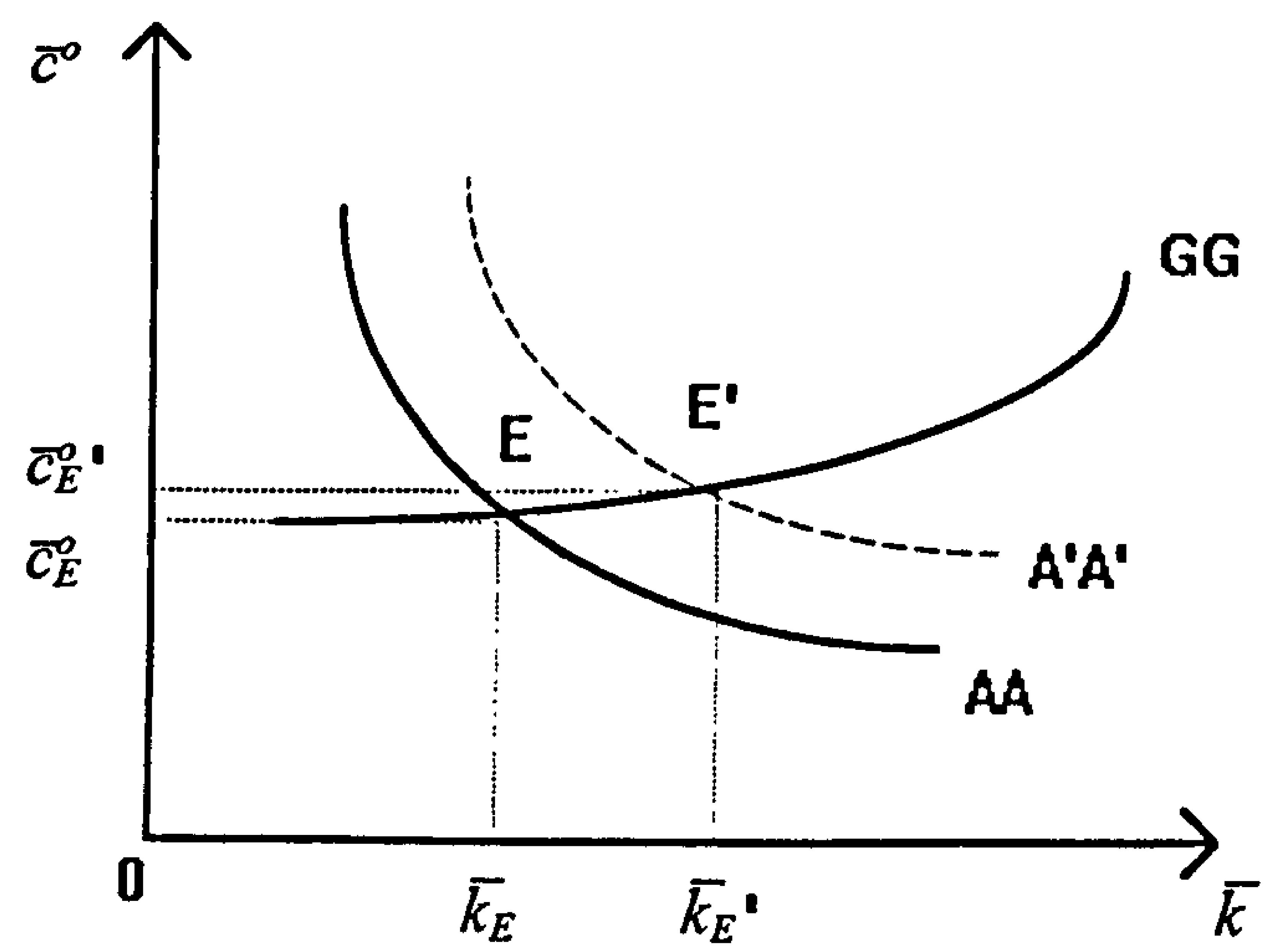

figure 4.3

Non-superneutrality of money implies that the "Mundell-Tobin" effect is at work: at the steady state, a higher rate of money growth per capita will bring about an increase in the expected inflation rate. A higher rate of inflation increases the opportunity cost of holding real money balances since the nominal interest rate will rise at any level of capital intensity. Agents will therefore change their portfolio's composition and will shift from money balances towards real physical capital so that the stock of productive capital per worker will rise. The fall in real balances demand by the young will make more resources available either for consumption in the first period or for investment in real capital. The level of capital intensity should increase. As underlined by Stein, *"[...] the crucial point to remember is that the older generation has absolutely no control over the rate of capital accumulation [...]. The relevant variable is simply the younger generation's desired holdings of real balances"*[9]. This result contradicts the hypothesis of super-neutrality of money put forward by Sidrauski (1967).

Succeeding studies confirmed the breakdown of the hypothesis of money superneutrality, although the assumptions relaxed every time were different. By considering a utility function which is not intertemporally separable in preferences, Epstein and Hynes (1983) showed that money super-neutrality does not hold. Moreover, if utility depends on leisure time but is not separable in it and the labour supply is endogenous, Brock (1974) showed that the rate of money growth could affect output.

To summarise the results obtained so far, we first rewrite the matrix in (4.37) as:

$$(4.40)\qquad \begin{bmatrix} \overset{(+;?)}{a_{11}} & \overset{(+)}{a_{12}} \\ \underset{(+)}{a_{21}} & \underset{(-;?)}{a_{22}} \end{bmatrix} \begin{bmatrix} d\bar{k} \\ d\bar{c}^o \end{bmatrix} = \begin{bmatrix} 0 \\ \underset{(+)}{b} \end{bmatrix} d\mu.$$

For a small $(f'-n)$ the sign of a_{11} is positive. Moreover, for big μ and small $\frac{u_m}{u_o}$ (which implies that the nominal interest rate is low) the sign of a_{22} is negative. By Cramer's rule, the multiplier of the steady state level of capital with respect to the money growth is:

$$(4.41)\qquad \frac{d\bar{k}}{d\mu} = \frac{-b\,a_{12}}{a_{11}a_{22} - a_{21}a_{12}} > 0.$$

[9] Stein (1971), p.48.

Therefore, $a_{11} > 0$ and $a_{22} < 0$ are sufficient, although not necessary, conditions for having a positive effect on the steady state level of capital stock. As we underlined above, the relation between $\bar{c}^o$ and $\bar{k}$ on each of the two stationary loci is ambiguous as the two following multipliers show:

$$\text{(4.42)} \quad \frac{d\bar{c}^o}{d\bar{k}}\Big|_{GG} = -\frac{a_{11}}{a_{12}},$$

which is negative if $a_{11} > 0$ (holding constant $\bar{c}^o$), and:

$$\text{(4.43)} \quad \frac{d\bar{c}^o}{d\bar{k}}\Big|_{AA} = -\frac{a_{21}}{a_{22}},$$

which is positive if $a_{22} < 0$ (holding constant $\bar{k}$). It must be noted that the term $\left[1 + f'(\bar{k}) - \frac{1+n}{1+\mu}\right]$ is positive because, from eq.(4.36), the marginal utility with respect to real balances must be positive. This implies that $1 + f' > \frac{1+n}{1+\mu} \Rightarrow f' > n$ if $\mu \leq 0$. However, the higher is μ, the more likely is for the equilibrium to be dynamically inefficient. This condition can be summarised in the following proposition:

Proposition 4.2. *If there is a constant nominal money stock, namely $\mu = 0$, $f' > n$ is always true and the system is characterised by dynamic efficiency. Overcapitalisation is then excluded by the positive rate of return of money.*

This result is claimed by both Stein and Weiss under the hypothesis that the monetary policy is assumed to control only the size of the nominal transfer to each young person.

By considering eq.(4.40), the scenario represented in fig.4.3 can be interpreted as follows. As the difference $(f' - n)$ becomes bigger, the positive value of a_{11} decreases and the curve GG gets flatter. When a_{11} is negative, GG becomes positively sloped. Conversely, as μ gets smaller, AA is more likely to be negatively sloped.

A last final consideration can be put forward. In the model just presented the Fisher effect[10] fails to hold: in fact, the effect of an increase in the money growth rate on $(1+\bar{i})$ is ambiguous since, in eq.(4.27), $[1+f'(\bar{k})]$ decreases.

Lastly, we are interested in calculating the effect of a change in μ on the **steady state welfare** of the representative individual when the authority can only control the size of the transfer to the young and it cannot acquire real capital. We claim that:

Proposition 4.3. *An increase in the rate of money growth starting from zero will both bring about a reallocation in the individual's portfolio composition and increase utility (since the initial equilibrium is dynamically efficient).*

Proof. By starting from the general specification of the utility function as in eq.(4.1), the overall effect on $u(\bullet)$ is derived:

$$(4.44)\qquad \frac{du(\bullet)}{d\mu}=u_y\frac{\partial\bar{c}^y}{\partial\mu}+u_o\frac{\partial\bar{c}^o}{\partial\mu}+u_m\frac{\partial\bar{m}}{\partial\mu}=u_y\left\{\frac{\partial\bar{c}^y}{\partial\mu}+\frac{1}{1+\bar{r}}\frac{\partial\bar{c}^o}{\partial\mu}+\left[1-\frac{1+n}{1+\mu}\frac{1}{1+\bar{r}}\right]\frac{\partial\bar{m}}{\partial\mu}\right\}.$$

Changes in the aggregate value of money must be accompanied by non-trivial changes in the real allocation of consumption over time. Eq.(4.29) is transformed as:

$$(4.45)\qquad \bar{c}^y=f(\bar{k})-n\bar{k}-\frac{\bar{c}^o}{1+n}$$

so that the multiplier with respect to the consumption in the first period is equal to:

$$(4.46)\qquad \frac{\partial\bar{c}^y}{\partial\mu}=[f'-n]\frac{\partial\bar{k}}{\partial\mu}-\frac{1}{1+n}\frac{\partial\bar{c}^o}{\partial\mu}.$$

We can follow the same procedure for eq.(4.34a) thus obtaining the following multiplier:

$$(4.47)\qquad \frac{\partial\bar{m}}{\partial\mu}=\frac{\bar{m}}{1+\mu}+(1+\mu)\left[\frac{1}{1+n}\frac{\partial\bar{c}^o}{\partial\mu}-(1+f'+f''\bar{k})\frac{\partial\bar{k}}{\partial\mu}\right].$$

---

10 The Fisher equation represents a link between the money growth rate (i.e. inflation) and the nominal rate of interest. In the long run, when expected and actual inflation rates coincide, an increase in the inflation rate is totally reflected on the nominal interest rate. This implies that the real interest rate is not influenced by money growth.

By substituting eqs.(4.46)-(4.47) into eq.(4.44) we derive an expression which, evaluated at $\mu = 0$, yields:

(4.48)

$$\frac{1}{u_y}\frac{du}{d\mu} = \left\{ f' - n - \left[1 - \frac{1+n}{1+f'}\right](1 + f' + f''\bar{k}) \right\} \frac{\partial \bar{k}}{\partial \mu} + \left\{ -\frac{1}{1+n} + \frac{1}{1+f'} + \left[1 - \frac{1+n}{1+f'}\right]\frac{1}{1+n} \right\} \frac{\partial \bar{c}^o}{\partial \mu} + \left[1 - \frac{1+n}{1+f'}\right]\overline{m}$$

Since we showed that $f' > n$ when $\mu = 0$, after some calculations we derive the overall effect on utility stemming from a change in the growth rate of the money supply:

$$\frac{1}{u_y}\frac{\partial u}{\partial \mu} = \left\{ -\frac{f''\bar{k}(f'-n)}{1+f'} \right\} \frac{\partial \bar{k}}{\partial \mu} + \frac{f'-n}{1+f'}\overline{m}. \qquad (4.49)$$

This expression shows that the steady state utility increases with money growth rate "*...so long as factor rewards are influenced by factor supplies ($f'' < 0$), [...] evaluated at a position of zero money growth*"[11] as already stated by Weiss (1980). Any decrease in the return on money would induce a shift towards capital accumulation. This would raise output which positively affect wages, thus lowering the return on capital. If the initial equilibrium is characterised by dynamic efficiency, then changes in factor payments induced by changes in factor supplies due to a portfolio reallocation would have a "*...first-order effect on the welfare of the typical individual*"[12].

In the next section we focus on the specific exercise of parameterising the general model investigated so far by adopting definite utility and production functions. This will help us to highlight some aspects of the model which have remained untreated in the present section.

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11 Weiss (1980), p.570.

12 Weiss (1980), p.569.

SECTION 2.

SOME PARAMETERISATIONS OF THE GENERAL MODEL

4.2.1 Introduction

In this section we parameterise the model presented in the previous section by using a specific log-linear utility function[13]. The individual's intertemporal utility maximisation problem can be written as:

$$(4.50)\qquad \textit{maximise}\quad u=\beta_1\ln c_t^y+\beta_2\ln c_{t+1}^o+\beta_3\ln m_t \qquad \text{where}\quad \beta_1+\beta_2+\beta_3=1$$

$$\textit{subject to}\quad w_t+g_t=c_t^y+(1+r_{t+1})^{-1}c_{t+1}^o+i_t(1+i_t)^{-1}m_t.$$

Log-linear utility over consumption and real balances has been widely used especially because of the completeness of the results obtained. The First-Order-Conditions are the following:

$$(4.51)\qquad \frac{\partial L}{\partial c_t^y}=\frac{\beta_1}{c_t^y}-\lambda=0$$

$$(4.52)\qquad \frac{\partial L}{\partial c_{t+1}^o}=\frac{\beta_2}{c_{t+1}^o}-\lambda(1+r_{t+1})^{-1}=0$$

$$(4.53)\qquad \frac{\partial L}{\partial m_t}=\frac{\beta_3}{m_t}-\lambda\frac{i_t}{1+i_t}=0.$$

By equating eqs.(4.51) and (4.52), we get:

$$(4.54)\qquad c_{t+1}^o=\frac{\beta_2}{\beta_1}(1+r_{t+1})c_t^y.$$

From eq.(4.53) real money balances are equal to:

$$(4.55)\qquad m_t=\frac{\beta_3}{\beta_1}\frac{1+i_t}{i_t}c_t^y,$$

and by substituting this expression into the budget constraint, we get the equations for the three endogenous variables of the system:

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13 Stein did not examine the sensitivity of his model to the specific form of the utility function used.

$$(4.56)\quad c_t^y = \beta_1(w_t + g_t)$$

$$(4.57)\quad c_{t+1}^o = \beta_2(1 + r_{t+1})(w_t + g_t)$$

$$(4.58)\quad m_t = \beta_3 \frac{1 + i_t}{i_t}(w_t + g_t).$$

Consumption in the first and second period, savings per worker and real money balances demand are a constant fraction of real disposable income per worker, $(w_t + g_t)$, because of the property of the logarithmic function: the income and substitution effects cancel out so that the demand function does not depend on the interest rates. However, although those variables are a constant proportion of the total amount of lifetime income, it is important to underline that they are derived from a utility maximisation process and are not represented by *ad hoc* expressions like those adopted in the standard IS-LM analysis. Positive quantities of real capital and real balances are held only if the nominal interest rate is not negative.

The expression for a_t is found by substituting the last three equations into the first period budget constraint:

$$(4.59)\quad a_t = \left[\beta_2 - \frac{\beta_3}{i_t}\right](w_t + g_t).$$

Introducing eq.(4.18) into eq.(4.58) we can derive the expression for the nominal rate of interest:

$$(4.60)\quad i_t = \frac{w(k_t) + \mu(1+\mu)^{-1} m_t}{\left[\frac{1}{\beta_3} - \frac{\mu}{1+\mu}\right] m_t - w(k_t)} = i(k_t, m_t; \mu),$$

which can be rewritten as:

$$(4.61)\quad \frac{1}{1 + i_t} = 1 - \frac{\mu}{1+\mu}\beta_3 - \beta_3 \frac{w(k_t)}{m_t}.$$

If we substitute eqs.(4.59) and (4.61) into eqs.(4.22) and (4.23) respectively, the two first-order difference equation system is the following:

$$(4.62)\qquad k_{t+1}=\frac{1}{1+n}\left\{(\beta_2+\beta_3)w(k_t)+\left[(\beta_2+\beta_3)\frac{\mu}{(1+\mu)}-1\right]m_t\right\}$$

$$(4.63)\qquad m_{t+1}=\frac{1+\mu}{1+n}[1+f'(k_{t+1})]\left[\left(1-\frac{\mu}{1+\mu}\beta_3\right)m_t-\beta_3 w(k_t)\right]=$$

$$=\frac{1+\mu}{1+n}\left[\left(1-\frac{\mu}{1+\mu}\beta_3\right)m_t-\beta_3 w(k_t)\right]\left\{1+f'\left[\frac{1}{1+n}\left((\beta_2+\beta_3)w(k_t)+[\frac{(\beta_2+\beta_3)\mu}{1+\mu}-1]m_t\right)\right]\right\}$$

where k_t is a predetermined variable established at the beginning of period t, whereas m_t is a non-predetermined variable and already deflated by p_t at the end of period t.

4.2.2 The Case of Cobb-Douglas and Logarithmic Production Functions

In order to find an explicit solution to the system formed by eqs.(4.62)-(4.63), we will resort to two different production functions, the first of which satisfies the Inada conditions.

A) Let us first consider the **Cobb-Douglas production function**:

$$(4.64)\qquad y=k^{\alpha}\qquad\qquad 0<\alpha<1,$$

for which the following relations are derived:

$$(4.65)\qquad f'(k)=\alpha k^{\alpha-1},\ \ kf'(k)=\alpha k^{\alpha},\ \ w(k)=f(k)-kf'(k)=(1-\alpha)k^{\alpha}.$$

A.1) In the simple case where $\mu=0$ the two stationary loci become:

$$(4.66)\quad \Delta k_t=0\ \Rightarrow\ m_t=(\beta_2+\beta_3)(1-\alpha)k_t^{\alpha}-(1+n)k_t$$

$$(4.67)\quad \Delta m_t=0\ \Rightarrow\ m_t=\frac{1}{1+n}\left[m_t-\beta_3(1-\alpha)k_t^{\alpha}\right]\left\{1+\alpha\left[\frac{1}{1+n}\left((\beta_2+\beta_3)(1-\alpha)k_t^{\alpha}-m_t\right)\right]^{\alpha-1}\right\}$$

which is a non-linear two equation system to be solved graphically to assess whether a steady state exists.

Eq.(4.66) is a concave function which intersects the horizontal axis at two points one of which is the origin. Its slope at the origin is infinity and it has a maximum in the positive quadrant (see fig.4.4 below, line *OA*). Capital stock increases for all the points below the function.

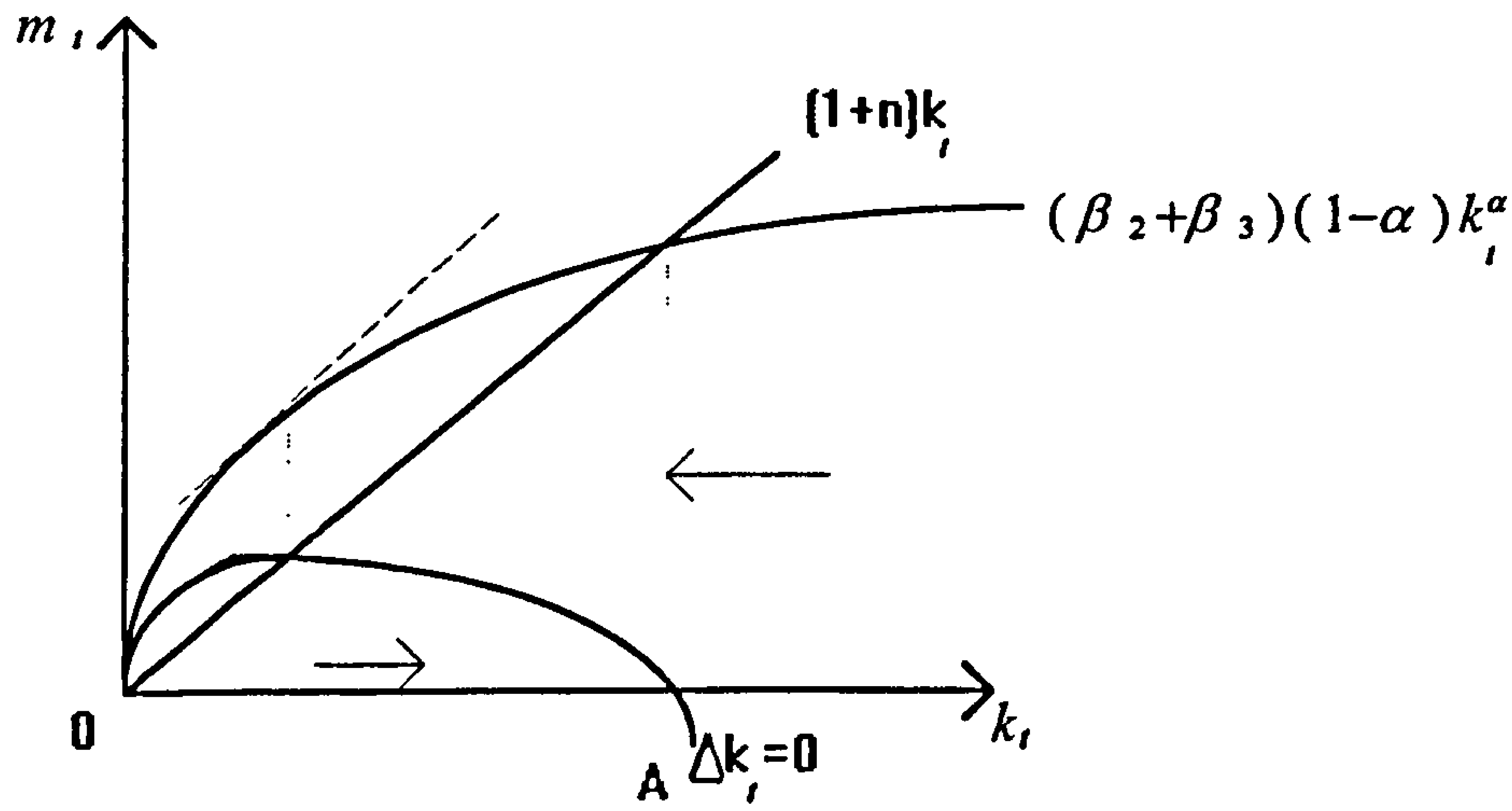

figure 4.4

Eq.(4.67) represents the stationary locus for the money market equilibrium and is pictured in fig.4.5. The RHS is unambiguously decreasing in k_t whereas both the LHS and the RHS are increasing in m_t. However, the condition $\frac{(1+f')}{(1+n)} \geq 1$[14] implies that an increase in money raises the RHS by more than the LHS so that the relationship between capital stock and money is positive. This function is concave, passes through the origin and reaches a maximum. Moreover, since the term in the second brackets on the RHS is decreasing in k_t, there will be a point where the relationship between m_t and k_t becomes negative and the function turns down to minus infinity for m_t. This part is not reported in the figure because the intersection with the function $\Delta k_t = 0$ would imply a negative steady state for real balances. Real money balances increase above the function.

14 This condition always holds at the steady state for $\mu = 0$ as we showed before, i.e. at the intersection point between $\Delta m_t = 0$ and $\Delta k_t = 0$.

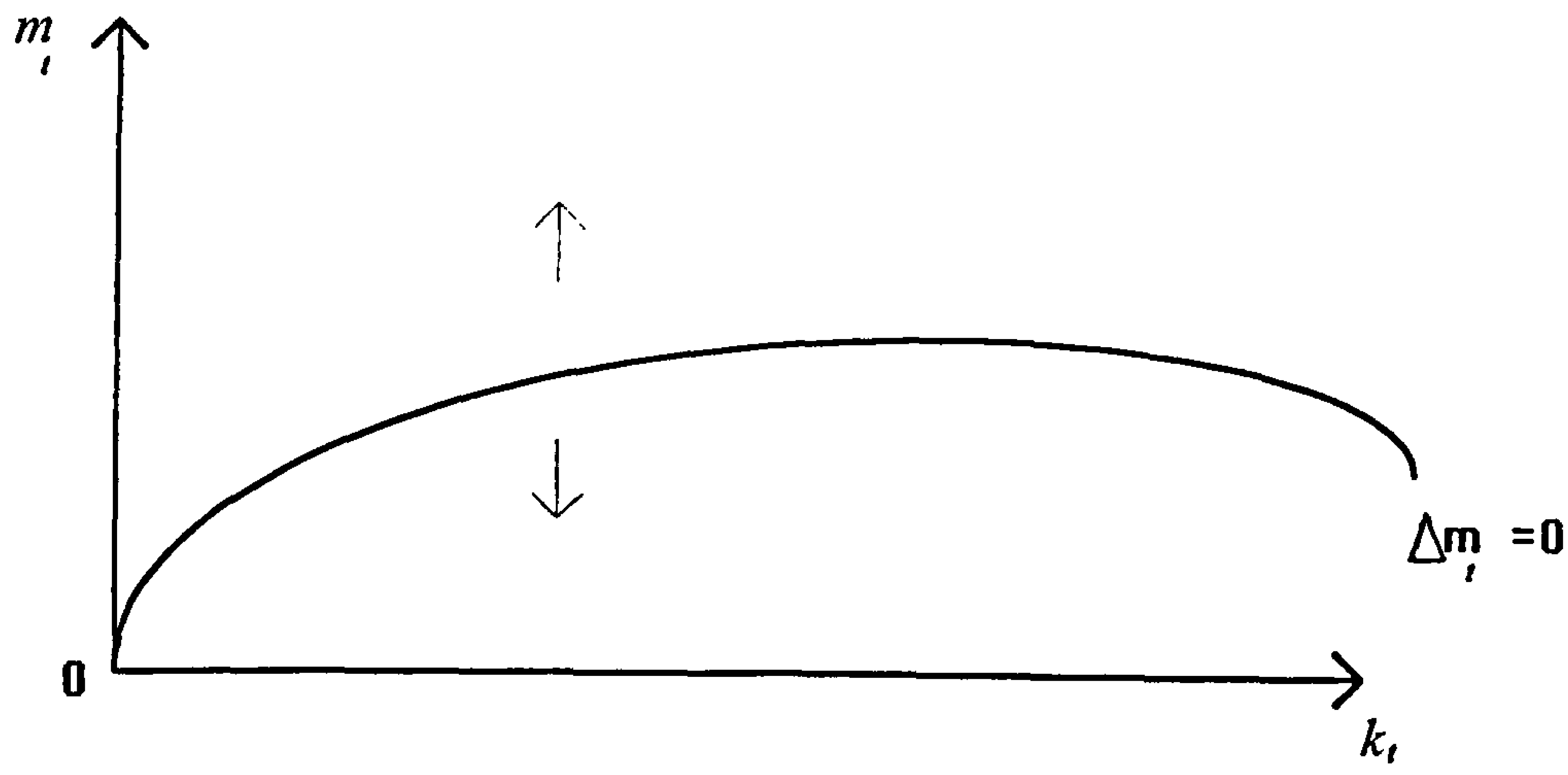

figure 4.5

As a matter of fact, the range of variation of m_t is further restricted by taking into account that both money and capital stock cannot be negative. Therefore, in order for the RHS of eq.(4.67) to be positive, we must have $m_t > \beta_3 w(k_t)$, whereas, from eq.(4.66), $m_t < (\beta_2 + \beta_3)w(k_t)$. Therefore, m_t must lie in the range:

(4.68) $\beta_3 w(k_t) < m_t < (\beta_2 + \beta_3)w(k_t)$.

Moreover, from eq.(4.67) evaluated at the steady state, the following equality holds:

(4.69) $$1 + f'(\bar{k}) = \frac{\bar{m}}{\bar{m} - \beta_3 w(\bar{k})}[1+n] > [1+n].$$

This condition implies that the steady state of the system will be dynamically efficient. This still confirms Stein's result that, if $\mu = 0$, then $f' > n$ at the steady state. This is a general result independent of the form of the production function adopted.

The solution of the system (eqs.(4.66)-(4.67)) is represented in fig.4.6 below and only one non-trivial positive and acceptable steady state exists which, from the directions of the arrows, turns out to be a saddle.

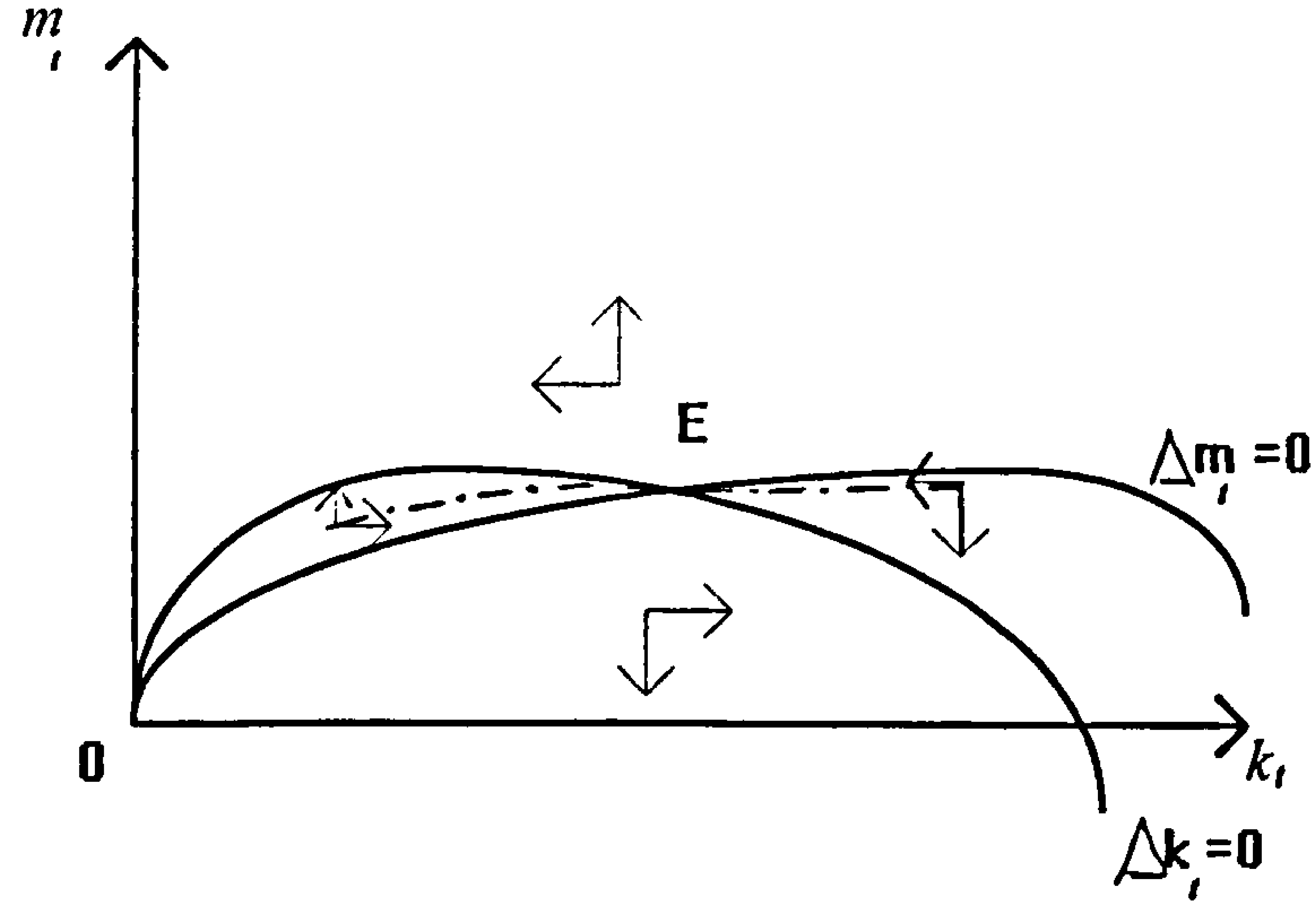

figure 4.6

A.2) Let us now consider the case of a positive money growth rate, i.e. $\mu > 0$. We illustrate this case since we are interested in analysing the effects on the capital stock following an increase in the rate of money growth. This exercise could be helpful to support the results already achieved with the general model in (4.37). To carry out this analysis, we rearrange eq.(4.62) evaluated at the steady state to get:

$$(4.70) \qquad \overline{m} = \frac{1}{1-(\beta_2+\beta_3)\frac{\mu}{1+\mu}}\left[(\beta_2+\beta_3)w(\overline{k})-(1+n)\overline{k}\right].$$

Substituting this expression for $\overline{m}$ into eq.(4.63) we obtain a single equation in $\overline{k}$:

$$(4.71) \qquad (\beta_2+\beta_3)w(\overline{k})-(1+n)\overline{k} = \frac{1+f'(\overline{k})}{1+n}\left\{[1+\mu]\beta_2 w(\overline{k})-[1+(1-\beta_3)\mu](1+n)\overline{k}\right\},$$

which can be rewritten as:

$$(4.72) \qquad \frac{1+f'(\overline{k})}{1+n} = \frac{1}{1+\mu-\mu\beta_3-\beta_3(1+\beta_1\mu)\frac{1}{(\beta_2+\beta_3)-(1+n)\overline{k}/w(\overline{k})}}.$$

The derivative of the RHS denominator with respect to μ is equal to:

(4.73) $$\frac{\partial RHS \;\; den.}{\partial \mu} = 1-\beta_3-\beta_3\beta_1\frac{1}{\beta_2+\beta_3-(1+n)\bar{k}/w(\bar{k})}$$

whose sign is ambiguous.

However, eq.(4.72) can be studied graphically by considering the LHS and the RHS separately. By using eq.(4.64) the LHS becomes:

(4.74) $$\frac{1+\alpha\bar{k}^{\alpha-1}}{1+n} = \frac{1+\dfrac{\alpha}{\bar{k}^{1-\alpha}}}{1+n}$$

which is clearly decreasing in $\bar{k}$. The horizontal asymptote is equal to $\left(\frac{1}{1+n}\right)$ whereas the vertical asymptote is represented by the ordinate axis. The function is pictured on the left-hand side of fig.4.7.

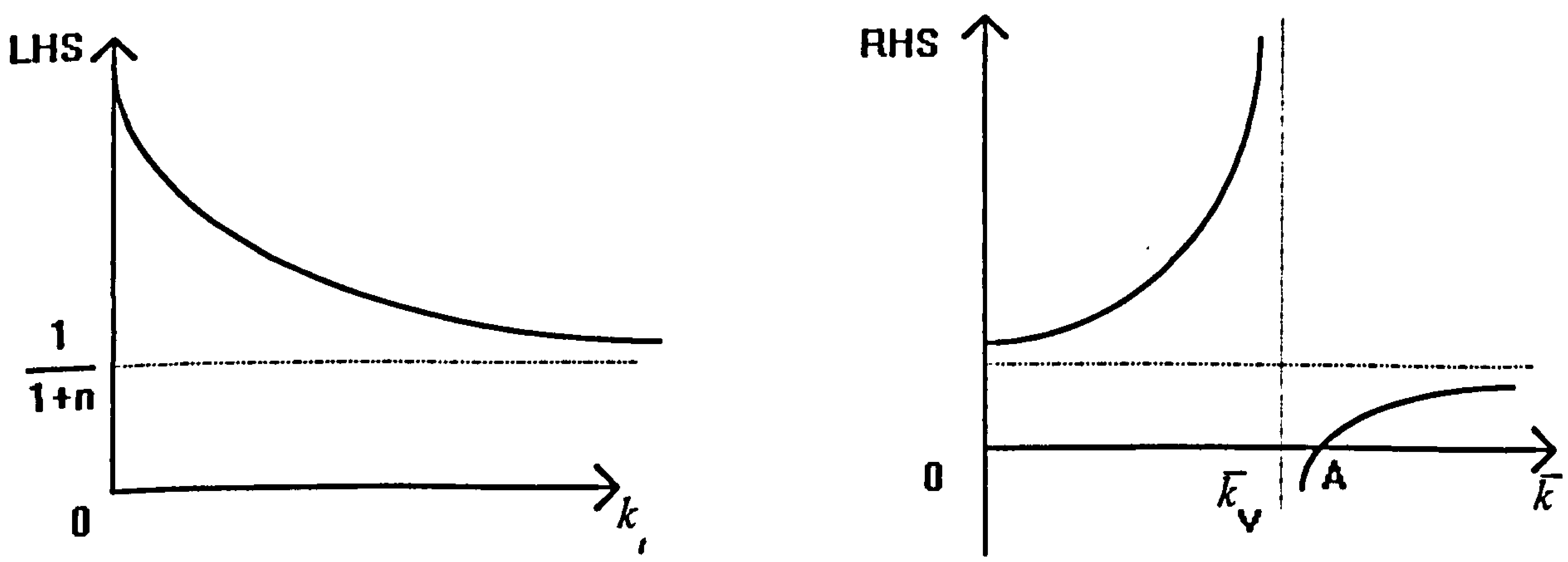

figure 4.7

On the right-hand side of fig.4.7 we represent the RHS of eq.(4.72) which is equal to:

(4.75) $$RHS-\frac{1}{[1+(1-\beta_3)\mu]} = -\frac{1}{[1+(1-\beta_3)\mu]}\cdot\frac{\left[(\beta_2+\beta_3)-\dfrac{\beta_2(1+\mu)}{1+(1-\beta_3)\mu}\right]}{\bar{x}-\dfrac{\beta_2(1+\mu)}{1+(1-\beta_3)\mu}}$$

where $\bar{x}=\frac{\bar{k}(1+n)}{w(\bar{k})}=\frac{\bar{k}^{1-\alpha}(1+n)}{(1-\alpha)}$ is monotonically increasing in $\bar{k}$. Eq.(4.75) represents an hyperbola. The vertical asymptote is equal to $\bar{k}_V=\left[\frac{(1-\alpha)\beta_2(1+\mu)}{[1+(1-\beta_3)\mu](1+n)}\right]^{\frac{1}{1-\alpha}}$. The horizontal asymptote instead is equal to $\frac{1}{1+(1-\beta_3)\mu}$. The numerator of eq.(4.75) is positive because the term $\frac{\beta_2(1+\mu)}{1+(1-\beta_3)\mu}$ can be written as $(\beta_2+\beta_3)-\frac{\beta_3(1+\beta_1\mu)}{1+(1-\beta_3)\mu}<(\beta_2+\beta_3)$. The hyperbola lies above the horizontal asymptote for $\bar{k}\in\left[0;\left(\frac{\beta_2(1+\mu)(1-\alpha)}{(1+\mu-\beta_3\mu)(1+n)}\right)^{\frac{1}{1-\alpha}}\right]$ and below it for higher values of $\bar{k}$.

All the sets of values of $\bar{k}$ to the right of $\bar{k}_A$, where $\bar{k}_A=\left[\frac{(\beta_2+\beta_3)(1-\alpha)}{(1+n)}\right]^{\frac{1}{1-\alpha}}$, are not acceptable because, since real money balances cannot be negative, from eq.(4.70) we have that $(\beta_2+\beta_3)w(\bar{k})-(1+n)\bar{k}>0$ or, identically, that the steady state level of $\bar{k}$ must satisfy the inequality: $\bar{k}<\left[\frac{(\beta_2+\beta_3)(1-\alpha)}{1+n}\right]^{\frac{1}{1-\alpha}}=\bar{k}_A$.

By merging the two figures representing the LHS and the RHS (fig.4.8), we find that only one positive steady state exists to the left of point *A*. A change in μ does not affect the LHS of eq.(4.72), so that in order to evaluate the effects of a change in money growth rate, we have to reconsider eq.(4.75). Some observations must be done:

1) for $k=0$, the $RHS=\frac{(\beta_2+\beta_3)}{\beta_2(1+\mu)}$ (i.e. the intersection point of the function with the ordinate axis) is decreasing in μ. As $\bar{k}\rightarrow\infty$, the RHS tends to $\frac{1}{1+(1-\beta_3)\mu}$;

2) by rearranging the RHS of eq.(4.72), the derivative of its denominator with respect to the money growth is equal to $\beta_2 w(\bar{k})-(1-\beta_3)(1+n)\bar{k}$ or, dividing by $w(\bar{k})$, to $[\beta_2-(1-\beta_3)\bar{x}]$, which, when evaluated at the vertical asymptote, becomes:

$$(4.76) \qquad \beta_2-\frac{(1-\beta_3)\beta_2(1+\mu)}{[1+(1-\beta_3)\mu]}=\frac{\beta_2\beta_3}{[1+(1-\beta_3)\mu]}>0.$$

From this last inequality, it is clear that a rise in μ positively affects the denominator of the RHS which shifts downward at every level of the steady state, $\bar{k}$. The horizontal asymptote shifts down whereas the vertical asymptote shifts to the right. The overall effect of an increase in μ is illustrated in fig.4.8 below, where the dashed lines are drawn for the increased money growth rate.

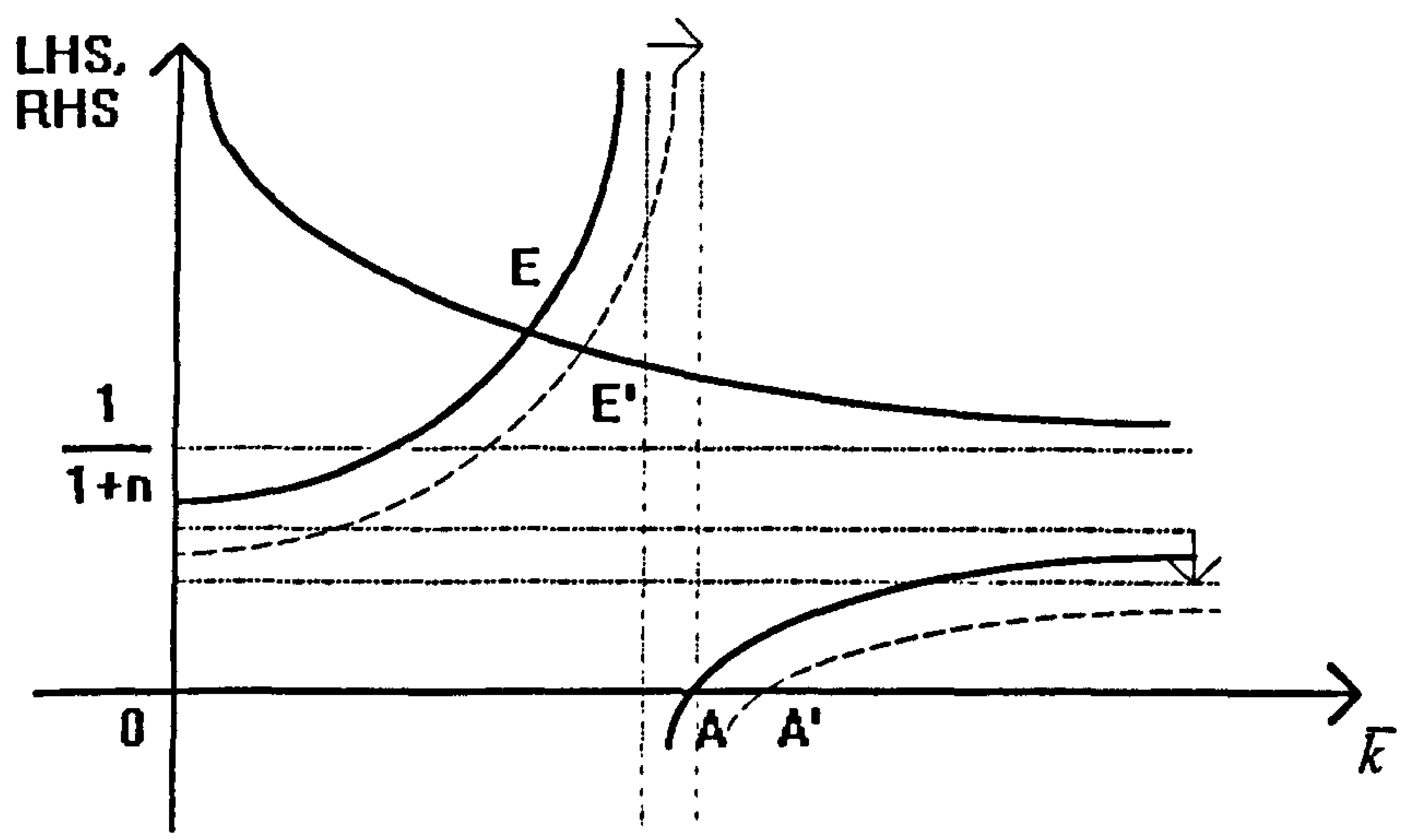

figure 4.8

It is clear that money is not superneutral and it affects capital intensity by leading to a portfolio reallocation once the growth rate of money has exogenously changed. Unlike Sidrauski's analysis, according to our model a change in the rate of money growth starts a process of intertemporal utility maximisation which leads to a change in the capital stock per worker.

The conclusion to be drawn from the mathematical analysis of the relationship between capital stock and money growth rate, is that we have been successful in deriving an explicit equations system useful for evaluating the effects of an expansionist monetary policy on the steady state capital stock.

We now continue our analysis by proving that the results of our model may depend on the production function we use and on the constraints applied to the system we investigate.

B) We now carry out the same analysis by using a **logarithmic production function** of the form:

(4.77) $\quad y = \alpha \ln[\gamma k + 1] \qquad\qquad 0 < \alpha < 1, \ \gamma > 0$

for which we have:

$$(4.78) \quad f'(k) = \frac{\alpha\gamma}{\gamma k + 1}, \ kf'(k) = \frac{\alpha\gamma k}{\gamma k + 1}, \ w(k) = \alpha \ln[\gamma k + 1] - \frac{\alpha\gamma k}{\gamma k + 1}.$$

B.1) The two stationary loci represented, in the case of $\mu = 0$, by equations (4.66) and (4.67), are now written as:

$$(4.79) \quad \Delta k_t = 0 \ \Rightarrow \ m_t = (\beta_2 + \beta_3)\left[\alpha \ln[\gamma k_t + 1] - \frac{\alpha\gamma k_t}{\gamma k_t + 1}\right] - (1+n)k_t$$

$$(4.80) \quad \Delta m_t = 0 \ \Rightarrow \ m_t = \frac{1}{1+n}\left[m_t - \beta_3 \alpha \ln[\gamma k_t + 1] + \beta_3 \frac{\alpha\gamma k_t}{\gamma k_t + 1}\right]$$

$$\left\{1 + \frac{\alpha\gamma}{1 + \gamma\left[\frac{1}{1+n}\left((\beta_2 + \beta_3)\left(\alpha \ln[\gamma k_t + 1] - \frac{\alpha\gamma k_t}{\gamma k_t + 1}\right) - m_t\right)\right]}\right\}.$$

The basic results are robust to the values we assign to the parameters. Eq.(4.79) has a similar shape of eq.(4.66) except for small values of the capital stock where the stationary locus corresponds to negative values of m_t. Eq.(4.80) is upward sloping for the initial interval of k_t like eq.(4.67), and then it turns down, so that there are three steady states: the origin and other two for positive values of m_t and k_t. In fig.4.9 we provide a sketch of the solutions found by simulating the model with different parameter values. The results of the simulations carried out to evaluate the effects of a change in the parameter values on the steady state level of the capital stock for both production functions are discussed in appendix 1 at the end of this chapter.

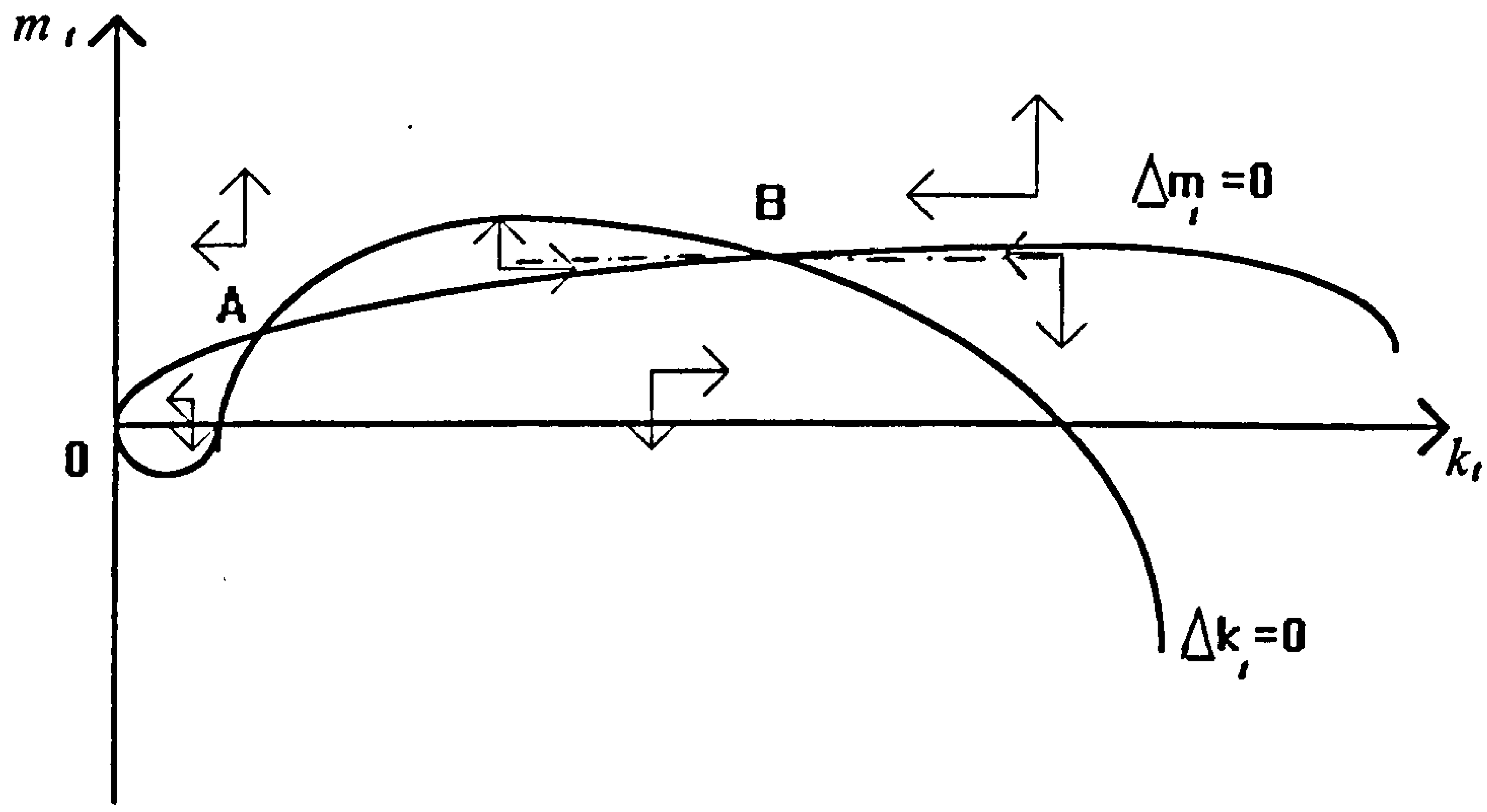

figure 4.9

From fig.4.9 the following remarks are straightforward. First of all, there are three steady states: points O and B, which are saddle point stable, and point A, which is unstable. Point O represents the trivial equilibrium of zero production and factor incomes in an economy which has shut down; this steady state is conditional on $f(0)=0$. On the contrary, point A represents the possibility that a poverty trap phenomenon can arise.

Secondly, if the economy starts above the stationary locus $\Delta m_t > 0$ and to the left of point A[15], so that the saddle path condition is not satisfied, then dynamics would lead towards decumulation of the capital stock. People would be consuming and investing in real money balances so that less resources would be available for capital investment. As time passes by, less capital would be available in the economy also due to the fall in the total income earned.

[15] We assume that to the right of point A, the saddle path condition of point B is satisfied.

This scenario implicitly tells us that if, initially, for a given level of capital stock, real money balances are too high (and the system starts off above the stationary locus $\Delta m_t = 0$), then the immediate consequences would be a decumulation of the capital stock and the collapse of the productive sector: the increasing demand for money would leave fewer and fewer resources for capital accumulation.

The existence of two non-trivial steady states is strictly associated to the kind of production function used. More precisely, the logarithmic production function does not satisfy the Inada conditions, so that when capital stock becomes negligible, $r_t \to \alpha\gamma$, whereas in the Cobb-Douglas production function case it goes to infinity. What happens is that the stationary locus does not have an infinite slope at the origin, and the increase in wage due to capital accumulation is smaller at every level of k_t with respect to the Cobb-Douglas case. Therefore, it seems that the switch in the technology process negatively affects the economy, since the amount of gross savings available to be invested in capital stock and money in the economy is now lower[16]. Provided that the amount of capital stock to be accumulated stays unchanged on the locus $\Delta k_t = 0$, the decrease in savings, at every level of k_t, which occurs when we consider the logarithmic production function instead of the Cobb-Douglas one, will be represented by a shortage of money in order to maintain a constant capital stock. This implies that money will reach negative values when savings decrease by a large amount. Such a behaviour is determined by the non satisfaction of the Inada conditions and helps to explain why, for small values of k_t, the stationary locus $\Delta k_t = 0$ lies in the negative quadrant for m_t.

B.2) We now carry out the analysis for the case of $\mu \neq 0$. The LHS of eq.(4.72) becomes:

$$(4.81) \qquad \frac{1+\dfrac{\alpha\gamma}{\gamma\bar{k}+1}}{1+n}.$$

16 The total amount of savings available in the economy is a proportion of the wage and is spent both on capital accumulation for next period generation and on money holdings, namely $s_t[w(k_t), A] = (1+n)k_{t+1} + m_t$, where A represents the parameter of the technological progress.

The intersection point with the ordinate axis is equal to $\frac{1+\alpha\gamma}{1+n}$, whereas the horizontal asymptote is $\frac{1}{1+n}$. The function is positive, decreasing and convex along the interval. The RHS in eq.(4.75) is equal to:

$$(4.82)\qquad RHS=\frac{1}{1+(1-\beta_3)\mu}\cdot\frac{(\beta_2+\beta_3)-\dfrac{(1+n)k}{\alpha\ln[\gamma k+1]-\dfrac{\alpha\gamma k}{\gamma k+1}}}{\left[\dfrac{\beta_2(1+\mu)}{1+(1-\beta_3)\mu}-\dfrac{(1+n)k}{\alpha\ln[\gamma k+1]-\dfrac{\alpha\gamma k}{\gamma k+1}}\right]}.$$

The general reasoning followed for eq.(4.75) still holds; however, due to the complexity of the expression for the wage, it is no longer possible to find a simple equation in $\bar{k}$ (eq.(4.75), for instance, was a simple hyperbola). Simulations of eq.(4.82), for some specific sets of parameter values, allow us to conclude that the shape of the RHS is the one reported in fig.4.10. Therefore, since the LHS and the RHS intersect twice, the equation (4.81)=(4.82) exhibits two non-trivial steady states (points A and B), as in the simplistic case of $\mu=0$.

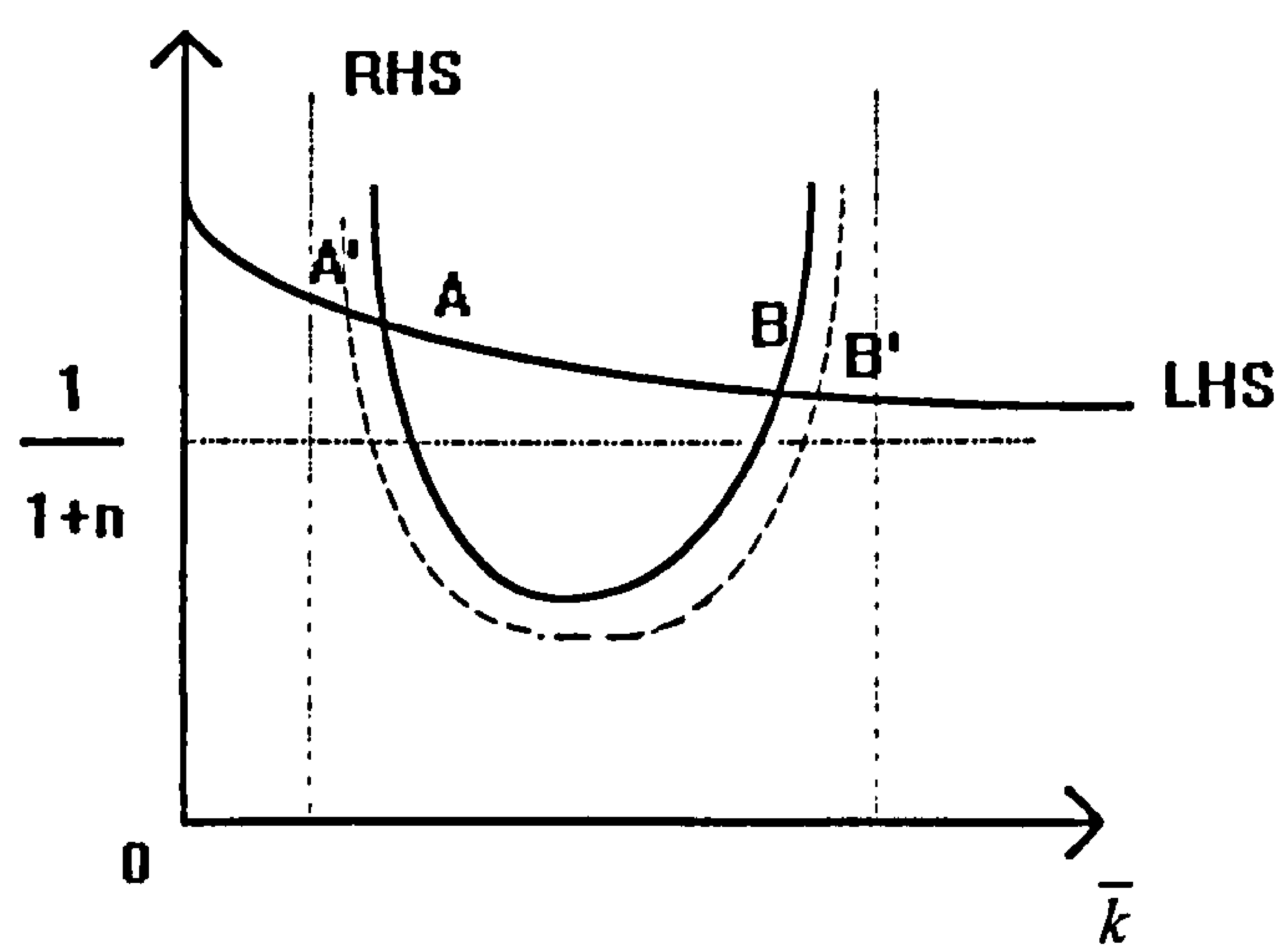

figure 4.10

Simulations also show that an increase in the growth rate of money shifts down the RHS while the LHS remains unchanged. This would imply the existence of the Mundell-Tobin effect. The saddle point steady state would then rise with μ (from point B to point B'), which is actually the same phenomenon occurring for the Cobb-Douglas production function case depicted in fig.4.8.

The results of the analysis done by parameterising the general model with particular utility and production functions support the validity of our model as a working tool for a more accurate investigation of the steady state properties.

4.2.3 Conclusions

To sum up, the analysis of this chapter has shed light on some important aspects overlooked by Stein, such as: 1) an evaluation of the effects on the individuals' welfare of the government monetary policy of issuing domestic currency at a constant rate and 2) a formal/graphic investigation of the issue of money non-superneutrality through a re-adaptation of the same model.

Our model has shown that the Mundell-Tobin effect holds whatever is the parameterisation of the model used for our investigation. The comparison between the present model and the debt model presented in chapter 2 has emphasised that a poverty trap is not associated exclusively with the presence of debt but to the existence of another investment opportunity available to the investors which looks more attractive than capital and, sometimes, also with the type of production function used.

In this chapter we have focused on a financing policy based on a source of revenue for the government represented by the right of creating fiat money (i.e. levying an inflation tax), and investigated how any change in the inflation rate would lead households to fluctuate between capital and money holdings. In the next chapter we will consider dynamics of hyperinflation and the real effects on the economy once the government decides to transfer a constant amount of resources to the young of future generations, so that the growth rate of money is no longer controlled by the government, but it is endogenously determined within the model.

APPENDIX 1

In the main text we have underlined the existence, for $\mu = 0$, of a multiplicity of equilibria which depends on the production function used. In this appendix we choose to analyse more deeply the system consisting of eqs.(4.66)-(4.67) (case of a Cobb-Douglas production function) and of eqs.(4.79)-(4.80) (case of a logarithmic production function). The purpose of this investigation is to illustrate what are the main results of the simulations of the model, by studying the effects of a change in each parameter on the steady state level of the capital stock for $\mu = 0$, while briefly commenting on the results for $\mu \neq 0$.

Let us, then, focus on the effects on the steady states following some changes in the parameters of the model when $\mu = 0$. Although the results from using the Cobb-Douglas production function are quite well established, nevertheless a brief investigation is provided before we focus more deeply on the logarithmic production function.

Cobb-Douglas production function.

From the analysis in section 2 of chapter 4, we were able to conclude that a unique acceptable steady state exists. This steady state strictly depends upon the value of the main parameters within the utility function, while all the simulations[1] have been carried out for a value of $\alpha = 0.3$, as suggested by most empirical estimations. A series of simulations have supported the following conclusions.

1) Holding β_1 constant, the lower β_2 with respect to β_3, the lower will be the steady state capital stock. In fact, notwithstanding that both β_2 and β_3 represents the individual's preference for the second period consumption, however only the amount actually saved goes to build up new capital stock, whereas money is just valueless paper which is held by the young in exchange of goods alienated to the old. Therefore, the lower β_2, the fewer resources are invested in capital accumulation and the lower the steady state capital stock.

1 As underlined at the beginning of the thesis, all these simulations have been carried out by using Mathematica

2) For a certain level of β_2, the higher β_1, the higher the steady state capital stock. An explanation might be that, from eq.(4.59), the amount a_t of investments in the financial asset (which coincides with the amount of capital stock to be built up in the next period) is negatively related to β_3, so that when the latter decreases, an unambiguous increase in the capital stock occurs.

3) For a given value of β_3, the higher β_1 with respect to β_2, the lower the steady state capital stock. This follows the same reasoning put forward in point 1): the lower the preference for the second period consumption and capital accumulation, the lower the population's savings which are invested and the lower the level of capital stock.

4) An increase in n decreases the value of the capital stock at the equilibrium.

Logarithmic production function.

1) We start off by considering a generic set of parameters which yields two steady states (for instance, $\{\alpha=0.3, \beta_1=0.1, \beta_2=0.8, \beta_3=0.1, n=0.01, \gamma=24\}$). Following an increase in n or a decrease in γ, the two steady states would get closer (namely, there would be crowding out of the saddle point stable capital stock and an increase in the unstable capital stock level). Moreover, if γ is below a critical value, no intersection point exists anymore, ceteris paribus.

2) For the same set of parameters ($\alpha=0.3, \beta_1=0.1, n=0.01, \gamma=24$), the lower β_2 with respect to β_3, the less likely to find two steady states with the value of $\gamma=24$. However, if we increase the parameter of "technology efficiency" to, for instance, $\gamma=50$, then we would find two steady states which, however, get closer and closer as β_2 decreases with respect to β_3. If β_2 is small, it is necessary that the parameter efficiency, γ, is high enough to guarantee an equilibrium. This is equivalent to saying that enough amount of resources must be invested in capital accumulation so that the technology adopted must be relatively high capital-intensive.

3) For a given set of parameters $(\alpha, \beta_2, n, \gamma)$, the higher β_1 with respect to β_3, the more distant the two steady states are for the same level of γ, so that it is more likely to find two steady states for a certain value of γ.

4) For a given set of parameters $(\alpha, \beta_3, n = 0.01, \gamma = 24)$, the lower β_2 with respect to β_1, the more likely that there are no steady states so that we have to increase again γ up to, say, 50, to find them. If the amount of resources invested in capital stock is a low proportion of income, then a high value of technology efficiency is necessary to have multiplicity of equilibria.

5) An increase in α has a positive effect because it widens the distance between the two steady states of the capital stock which get more distant. This means that it is more likely, for a given value of γ, to find two steady states as α rises.

6) The effect of an increase in n is the same as in the case of the Cobb-Douglas production function, namely it shifts $\Delta k_t = 0$ down and $\Delta m_t = 0$ up, so that beyond a certain population growth rate no steady state exists.

Some observations are noteworthy. Firstly, the whole model is based on the concept that domestic savings are a prerequisite for capital accumulation, so that policies which can promote high savings can facilitate the increase in the amount of capital stock in the economy. Of course, a high rate of saving is dependent on the availability of saving instruments and on high rates of return.

Secondly, a comparison between the results obtained with the two types of production functions suggests that the multiplicity of equilibria is partially correlated with the type of production function used and partially with the values assigned to the parameters. As underlined in the main text, when there is just one non trivial steady state, it is a saddle, whereas when two steady states exist, one of them is a saddle and the other is a source. The existence of an unstable equilibrium raises the issue of the poverty trap phenomenon and allows us to build a bridge between the analysis of debt management dealt with in chapter 2 and through money issue dealt with in chapter 4.

The considerations till now expressed refer to the particular case of $\mu = 0$. However, we have also carried out many simulations of the system in the case of $\mu > 0$ in order to check whether the number of the steady states associated to each production function is the same in the cases of zero and positive money growth rate. In order to determine the number of the equilibria, we have to plot the LHS against the RHS of eq.(4.72) for different set of parameter values. The conclusion is that there is a perfect correspondence between the number of equilibria in both cases.

This analysis has also turned out to be important to determine the effects of a change in the money growth rate on the steady state capital stock. An investigation of the results obtained by simulating eqs.(4.74)-(4.75) and eqs.(4.81)-(4.82) has suggested that the Mundell-Tobin effect still holds. More precisely, for the Cobb-Douglas production function, the unique steady state level of capital stock rises if the money growth rate increases. On the other hand, when considering the case of the logarithmic production function, the effect is the same apart from the existence of two steady state levels of the capital stock (one of which is unstable), so that the positive effect is only on the saddle point stable capital stock.

Below we provide an example of a simulation of eqs.(4.81)-(4.82) for the case of $\mu = 0$ (table A.1) and $\mu > 0$ (table A.2) for a particular set of parameters. The results of some other simulations with other parameters sets are omitted to avoid overweighing the analysis.

Table A.1

	$\alpha=0.3,\beta_1=0.1,\beta_2=0.8,\quad \beta_3=0.1,n=0.01,\gamma=24,\mu=0$	
	Steady State for m	
Steady State for k	eq.(4.79)	eq.(4.80)
0	0	0;-33.6199
0.01	-0.00427799	0.000748732;-30.3175
0.02	-0.0019162	0.0024266;-30.4452
0.03	0.0031043	0.00459078;-30.6003
0.05	0.0151108	0.00967959;-30.931
0.15	0.0492309	0.0376596;-32.3252
0.2	0.0491734	0.0506585;-32.8482
0.23	0.0453271	0.057993;-33.123
0.25	0.0414672	0.0626959;-33.2929
0.3	0.028043	0.0738433;-33.679
0.35	0.010215	0.0841998;-34.0197
0.4	-0.0110977	0.0938612;-34.3245

The steady states are the following: $0.03<\bar{k}_A<0.05$ and $0.15<\bar{k}_B<0.2$.

Table A.2

	$\alpha = 0.3, \beta_1 = 0.1, \beta_2 = 0.8,$	$\beta_3 = 0.1, n = 0.01, \gamma = 24, \mu = 0.02$
Steady State for k	LHS (eq.(4.81))	RHS (eq.(4.82))
0.032	5.02218	16.4377
0.035	4.8644	3.94415
0.05	4.23042	1.87059
0.1	3.08678	1.62051
0.16	2.46297	1.82985
0.2	2.21919	2.22559
0.25	2.00849	4.37176

The results in table A.2 confirm that two steady states also exist when $\mu > 0$, which are $0.032 < \bar{k}_{A'} < 0.035$ and $0.16 < \bar{k}_{B'} < 0.2$. Since in the case of an exponential Cobb-Douglas production function we found one non trivial steady state both for the case $\mu = 0$ and for the case $\mu > 0$, multiplicity of equilibria seems to be connected to the kind of production function used.

CHAPTER 5

A CONSTANT SEIGNIORAGE POLICY, HYPERINFLATION AND THE HIGH INFLATION TRAP PHENOMENON

5.1 Introduction

The model described in the previous chapter is now extended to investigate how hyperinflation can result when the government chooses to collect seigniorage to finance a too large budget deficit[1]. This issue has been recently dealt with by some authors, like Evans and Yarrow (1981), Buiter (1987), Blanchard and Fischer (1989) and Bruno and Fischer (1990) since, in extreme hyperinflationary situations, money printing becomes the only source of government revenues. However, the models developed in the literature do not seem to have been completely successful in dealing with this topic and their limitations are underlined later on in this section. On the other side, the framework we will develop is an improvement in so far as it incorporates the interactions between the monetary and real sectors of the economy; consequently, it becomes possible to analyse what are the effects on the capital stock in the economy when the government is trying to keep the amount of seigniorage constant. We also look for the existence of a maximum level of revenue a government can obtain from money creation.

The standard analysis developed in the last two decades (including the papers quoted above) has adopted the Cagan's type of money demand whereas real variables are kept fixed. Our approach is to apply the same spending policy rule, namely examining a constant government real deficit which is financed by printing money but in a context where real variables are no longer constant, and to consider the existence of possible multiple equilibria.

The duality of equilibria under pure deficit financing through money printing is a quite well-known result from the seminal contribution of Evans and Yarrow (1981) whose work

[1] It is realistic to believe that countries may inflate their way out of troubles, but intuitively this would lead investors to demand higher interests exacerbating debt burden and leading to an excessive increase in money growth rate. This is why hyperinflation would occur.

has been recently extended by Bruno and Fischer (1990). Both models are based on the Cagan assumption of money market equilibrium, and the core of Cagan's pioneering analysis of hyperinflation is a demand for money function relating desired real balances to the opportunity cost of holding money which, in period of rapid inflation, can be closely approximated by the expected rate of inflation, while the real interest rate is assumed to be constant over time. However, these previous analyses have some strong limitations which can be summarised as follows: (a) they do not take into account an explicit optimisation rule for the behavioural equations of the economy and (b) real variables are kept constant over time so that it is unclear why hyperinflation should matter if real variables are not affected by changes in the purchasing power. Conversely, the money demand function we use is derived through a process of lifetime utility maximisation and the real interest rate and output are no longer kept constant. This choice is perfectly consistent with the suggestion made by Evans (1983) who pointed out that the monetary models[2], which had been usually used to study the dynamic behaviour of real balances and the inflation rate, did not permit variations in output.

In the model formalised by Bruno and Fischer, the rate of money creation is assumed to be proportional to the price level and inversely related to the level of real balances. Two equilibria arise if deficit financing is sufficiently low. The stable steady state is associated with the higher inflation rate and lower real money balances and conversely for the unstable one. This implies that the economy may be stuck at a high inflation rate notwithstanding the existence of an unstable equilibrium with a lower inflation rate with guarantees the same amount of revenue[3]. This result is labelled as "high inflation trap". As the size of the deficit gets bigger, the two steady states shrink together until no steady state exists. The maximum seigniorage is defined as the greatest constant flow of real government expenditure that can be financed by the issue of additional currency at the steady state. This maximum seigniorage concept can be interpreted as the monetary counterpart of the maximum sustainable level of debt which we derived in chapter 2.

[2] Evans (1983) focuses on the analysis of different price adjustment mechanisms and their consequent effect on the short-run dynamics of real money balances.
[3] This clearly expresses the concept of the existence of the Laffer curve.

The main problem met by these models is that, in a rational expectations context (or very fast adaptive expectations) with exogenous seigniorage, dynamics do not lead to hyperinflation but to hyperdeflation. Only with adaptive expectations hyperinflation is found. The association of an increase in deficit with a decrease in the inflation rate is attributed to the fact that the economy is on the wrong side of the Laffer curve, so that a decrease in the inflation rate is needed to generate more revenue any time the deficit rises.

The conclusion is that "*[...] allowing monetary policy to accommodate fiscal pressures not only leaves the inflation rate to be determined by the fiscal authority, but - because of the possibility of multiple equilibria - also increases the likelihood that the economy will find itself operating at an inflation rate higher than it need to be*"[4].

Blanchard and Fischer (1989)[5] have analysed, in a simplified context, the same policy rule adopting the Cagan demand for money equation. When the government seeks to obtain a given amount of revenue from seigniorage, the high inflation trap is again associated with a rapid adjustment of expectations or a very interest rate elastic demand for money. Quite counterintuitively, if the expected inflation is low enough, the government can, by increasing the rate of money printing, obtain more revenue than it could at the steady state. An increase in seigniorage would then lower the stable steady state inflation rate. If the amount of seigniorage to be raised is higher than a critical level, then the equilibrium in the economy disappears, leading to hyperdeflation. Only if expectations adjust quite slowly to increasing inflation, the government can cause hyperinflation by increasing seigniorage.

Over the years, many economists have tried to overcome the problems faced to model hyperinflation within a perfect foresight context by introducing some particular devices to a general model like the one of Evans and Yarrow (1981). It is worth mentioning some of these attempts.

Kiguel (1989) considers the usual model of budget deficit financing through seigniorage, a Cagan-type demand for money function and perfect foresight. However, he assumes that the

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[4] Bruno and Fischer (1990), p.373.

[5] Blanchard and Fischer (1989), chapter 4, section 4.7.

money market does not clear instantaneously, and that the rate of inflation exceeds the actual money growth rate any time there is an excess supply of money. The system exhibits two equilibria for real money balances, one stable and one unstable[6]. If the government increases the size of its budget deficit to a level at which there is no steady state for the inflation rate, the economy will experience an explosive path of prices (i.e. hyperinflation). A larger deficit requires an increase in fiat money, which is expanded at a higher rate. The temporary excess supply of money leads to an increase in the rate of inflation, and, consequently, money demand is reduced. Of course, this phenomenon implies that the system starts off at (or to the left of) the unstable steady state, since only along the unstable path real money balances are falling as inflation accelerates.

The criticisms to these models concern the fact that they ignore the possibility that the inflationary process is set to an end and do not incorporate expectations of a stabilisation policy. On the other hand, there is another stream of the literature which deals with models set up to explain the hyperinflation mechanism by allowing agents to anticipate a change in government policies.

For instance, Bental and Eckstein (1990) have built up a model where the sudden end of a period of increasing inflation and falling real balances can be the result of an expected stabilisation programme. With expected stabilisation, the authors show that there is a unique equilibrium path, whose properties depend on the pre-stabilisation situation. Low pre-stabilisation deficits (i.e lower than the maximal steady state deficit) are a necessary condition for an increasing inflation towards the equilibrium together with the requirement that demand for real balances after stabilisation falls. Moreover, if the stabilisation date is delayed indefinitely, the lower of the two steady state inflation rates will be the stable one, so that hyperinflation based on the stability path converging to the high steady state inflation is inconsistent with expected stabilisation. On the contrary, in all the previous models which

[6] This result holds if specific constraints on the semielasticity of the demand for money with respect to the rate of inflation and the speed of adjustment of the money market are satisfied.

did not include the stabilisation expectations, the low inflation was dynamically unstable and, therefore, sustainable only under very specific model parameterisations.

Evans and Ramey (1995) have developed a new approach to modelling expectations formation in a hyperinflation context. Agents are assumed to know the correct structural model of the economy, and they have to use it to forecast the path of inflation. They follow an optimality criterion in so far as they balance the costs of revising their forecasts against the benefits of improved expectations. Equilibrium paths with accelerating inflation and final collapse are found to be associated with high budget deficits and a pattern of accelerating inflation which is sufficiently high to give the agents the incentive to revise inflation continuously.

Finally, Havrylyshyn-Miller-Perraudin (1994) have overcome the problem of finding hyperdeflation with unsustainable deficits and rational expectations by making seigniorage stochastic. As pointed out by Von Hagen "*...The important trick is to decouple actual from expected inflation. If this is done in the right way, high and even increasing inflation rates can persist for quite a while in a rational expectations equilibrium, although the deficit is unsustainably large*"[7].

We now prove that hyperinflation can occur in a perfect foresight and clearing markets context appealing to an extension of the model presented in chapter 4 based on the introduction of a constant seigniorage policy rule to finance a budget deficit.

5.2 The Model with Constant Seigniorage Policy and Endogenous Money Growth

In the analysis which follows we assume that the government is financing a constant level of real spending (or subsidy distributed to the young generation, g) by printing money. This policy can be expressed as follows:

(5.1) $M_t = (1+\mu_t)M_{t-1} = M_{t-1} + p_t g N_t$.

[7] Discussion by Von Hagen in Havrylyshyn-Miller-Perraudin (1994), p.378.

The choice of deficit financing entirely by printing high-powered money is justified by Bruno and Fischer (1990) by the absence of domestic capital market and of foreign sources of finance.

Since we have assumed that the money growth rate is no longer constant, then the equation for the inflation rate becomes:

$$(5.2) \qquad \frac{p_{t+1}}{p_t}=\frac{M_{t+1}}{M_t}\frac{m_t}{m_{t+1}}\frac{N_t}{N_{t+1}}=\frac{(1+\mu_{t+1})m_t}{m_{t+1}(1+n)}.$$

Since we know that the expression for real money balances, at a generic time $i=0,1...$, is equal to $m_{t+i}=\frac{1+\mu_{t+i}}{\mu_{t+i}}g$, we can substitute the expression for the ratio $\frac{m_t}{m_{t+1}}$ into eq.(5.2) and the expression found into eq.(4.4) to get:

$$(5.3) \qquad \frac{i_t}{1+i_t}=1-\frac{(1+n)\mu_t}{\mu_{t+1}(1+\mu_t)(1+r_{t+1})}.$$

The rate of monetary growth is now endogenous; then the expression for the subsidy paid to the young can be rewritten as a function of real money balances (using eq.(4.58)) as:

$$(5.4) \qquad g=\frac{\mu_t}{1+\mu_t}m_t=\frac{\mu_t}{1+\mu_t}\beta_3\frac{1+i_t}{i_t}(w_t+g).$$

Substituting the expression for $\frac{1+i_t}{i_t}$ derived in eq.(5.3) into eq.(5.4) yields:

$$(5.5) \qquad g=\frac{\mu_t}{1+\mu_t}\beta_3(w_t+g)\frac{\mu_{t+1}(1+\mu_t)(1+r_{t+1})}{\mu_{t+1}(1+\mu_t)(1+r_{t+1})-(1+n)\mu_t}.$$

Some rearrangements allow us to derive the first equation of the dynamic system:

$$(5.6) \qquad \mu_{t+1}=\frac{(1+n)g\mu_t}{[(1+\mu_t)g-\mu_t\beta_3(w_t+g)][1+f'(k_{t+1})]}.$$

The expression for k_{t+1} can be derived following the same procedure for eq.(4.62). The nominal interest rate is represented by eq.(4.60):

$$(5.7) \quad i_t = \frac{w_t + \frac{\mu_t}{1+\mu_t} m_t}{\frac{m_t}{\beta_3} - \frac{\mu_t}{1+\mu_t} m_t - w_t}.$$

Substituting eq.(5.7) into eq.(4.59) and the latter into eq.(4.22), we get:

$$(5.8) \quad k_{t+1} = \frac{1}{1+n}\left\{\beta_2 - \beta_3 \frac{\frac{m_t}{\beta_3} - \frac{\mu_t}{1+\mu_t} m_t - w_t}{w_t + \frac{\mu_t}{1+\mu_t} m_t}\right\}(w_t + g).$$

By rearranging eq.(5.8), we obtain:

$$(5.9) \quad k_{t+1} = \left(\frac{1}{1+n}\right)\left\{(\beta_2+\beta_3)w(k_t) + \left[(\beta_2+\beta_3)\frac{\mu_t}{1+\mu_t} - 1\right]m_t\right\} =$$

$$= \left(\frac{1}{1+n}\right)\left[(\beta_2+\beta_3)w(k_t) + (\beta_2+\beta_3)g - g\frac{1+\mu_t}{\mu_t}\right].$$

Eq.(5.6) can now be rewritten as:

(5.10)

$$\mu_{t+1} = \frac{g(1+n)\mu_t}{[(1+\mu_t)g - \mu_t\beta_3(w_t+g)]\left\{1+f'\left(\left(\frac{1}{1+n}\right)\left[(\beta_2+\beta_3)w(k_t)+(\beta_2+\beta_3)g - g\frac{1+\mu_t}{\mu_t}\right]\right)\right\}}.$$

After finding the dynamic relations which characterise the model, our analysis proceeds with the investigation of the dynamics of the system represented by eqs.(5.9)-(5.10).

At the steady state, eq.(5.9) becomes:

$$(5.11) \quad \Delta k_t = 0 \quad \Rightarrow \quad \mu_t = \frac{g}{(\beta_2+\beta_3)w_t - \beta_1 g - k_t - k_t n}.$$

The expression in the denominator has the same hill shape of the stationary locus $\Delta m_t = 0$ in chapter 4 (and pictured in fig.4.5), with the difference that, for any level of k_t, it is decreased by the term $\beta_1 g$. Since g has a positive value and μ_t is inversely related to the above expression, then the function in (5.11) lies in the positive quadrant and has a U-shape in an intermediate range of values of the capital stock (fig.5.1). Considering that, by assumption, seigniorage and real money balances are positive, then $\mu_t > 0$ must hold. The capital stock increases above the stationary locus for all the positive values of μ_t.

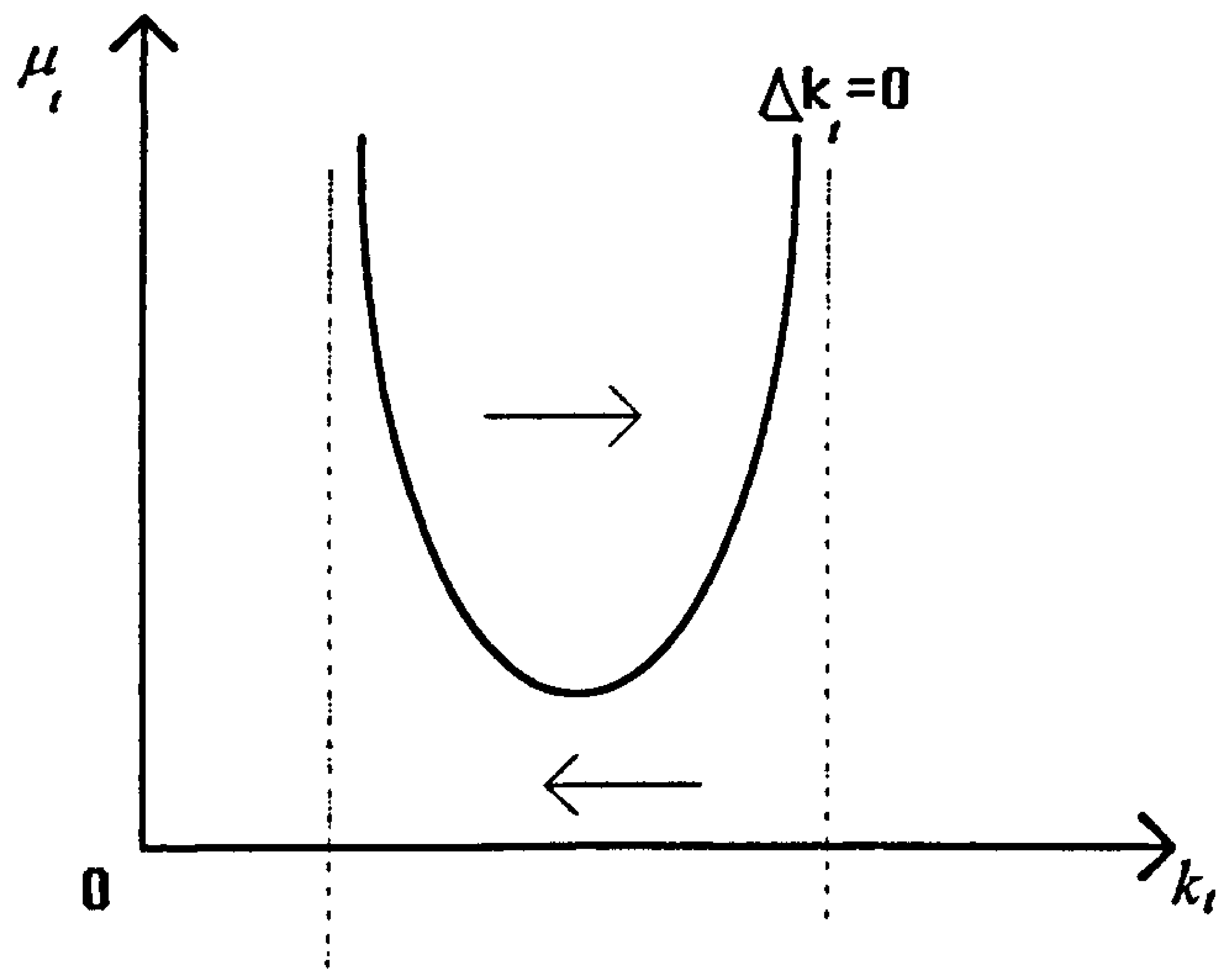

figure 5.1

What we expect to find is that a low level of capital stock is associated with a high inflation rate. This situation is equivalent to a scenario of high inflation trap only if the steady state were stable. Therefore, this finding could be useful to explain why hyperinflation matters so much in realistic terms.

To support our intuition, we consider eq.(5.10) which, at the steady state, is equal to:

(5.12)

$$\Delta\mu_t = 0 \;\Rightarrow\; \mu_t = \frac{(1+n)g\mu_t}{[(1+\mu_t)g - \mu_t\beta_3(w_t+g)]\left[1+f'\left(\frac{1}{1+n}\left[(\beta_2+\beta_3)(w_t+g) - g\frac{1+\mu_t}{\mu_t}\right]\right)\right]}.$$

After a few simplifications and rearrangements, we get the following equation:

$$(5.13)\quad \left[1+\mu_t-\beta_3\mu_t\left(\frac{w_t}{g}+1\right)\right]\left\{1+f'\left[\frac{1}{1+n}\left((\beta_2+\beta_3)(w_t+g)-g\frac{1+\mu_t}{\mu_t}\right)\right]\right\}=(1+n).$$

The two expressions on the LHS are both decreasing in μ_t and k_t and increasing in g for nearly all values of k_t[8]. For a given level of g, if k_t is higher than a critical value, the relationship between μ_t and k_t is negative throughout the range where μ_t is positive in order to hold the RHS constant (fig.5.2 below). It also comes out that, in the same interval, the growth rate of money increases above the stationary locus[9].

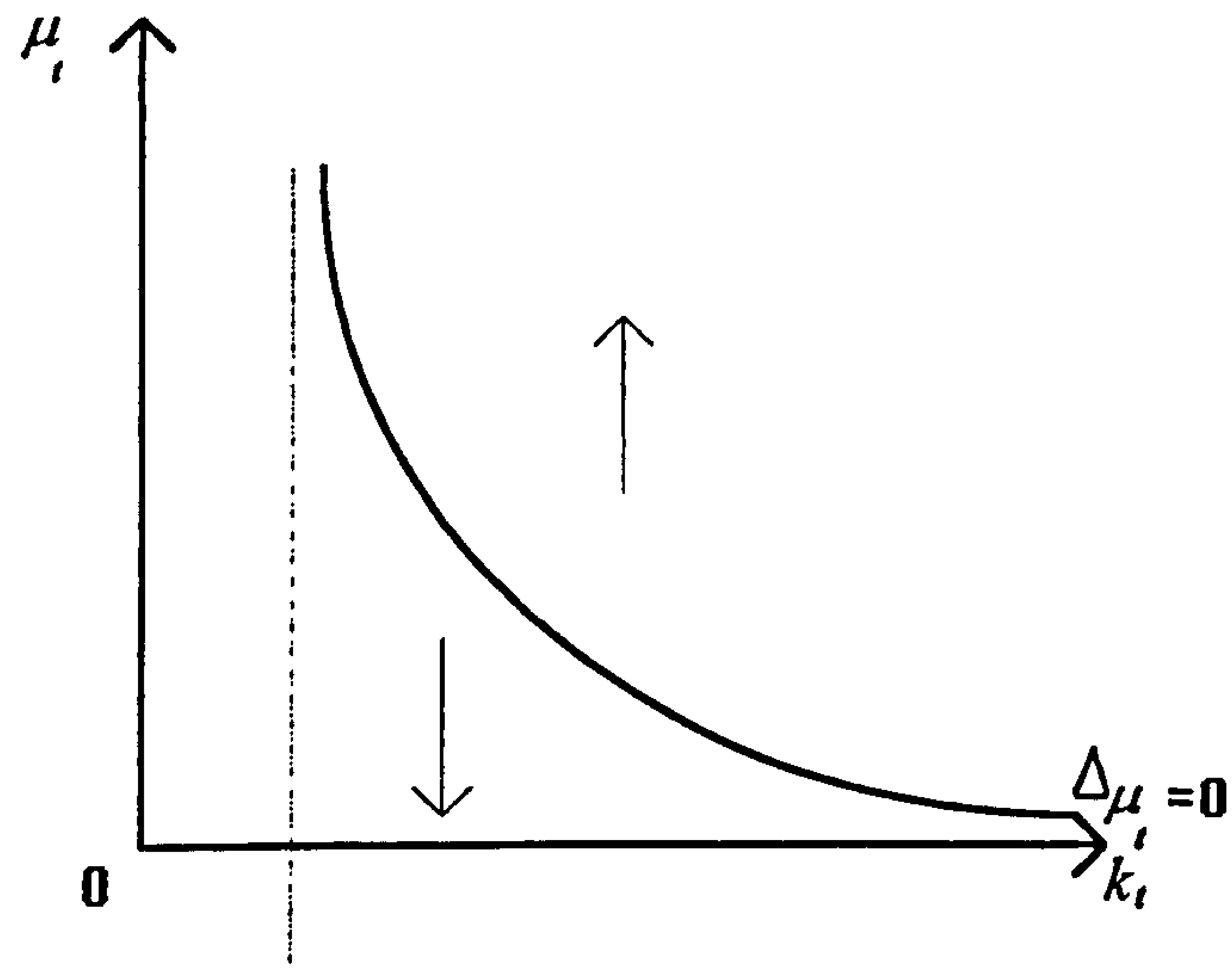

figure 5.2

[8] A sufficient condition for both parentheses on the LHS to decrease with μ_t is $w_t > \frac{(1-\beta_3)g}{\beta_3}$ (from the first parenthesis on the LHS of eq.(5.13) which can be rewritten as $\left[g+(g-\beta_3 w_t-\beta_3 g)\mu_t\right]$). Moreover, in order that $\mu_{t+1}>0$, it is necessary that $w_t < \frac{(1+\mu_t)g-\beta_3\mu_t g}{\beta_3\mu_t}$.

[9] This can be seen directly from eq.(5.10) where, as k_t increases, the denominator unambiguously decreases. Therefore, for a given μ_t, μ_{t+1} is increasing above the function.

The entire system is represented in fig.5.3 below. Two steady states exist: the unstable point A and the saddle point stable B. We have omitted the part of the two functions lying in the fourth quadrant because for a negative value for μ_t, we have an economically meaningless scenario.

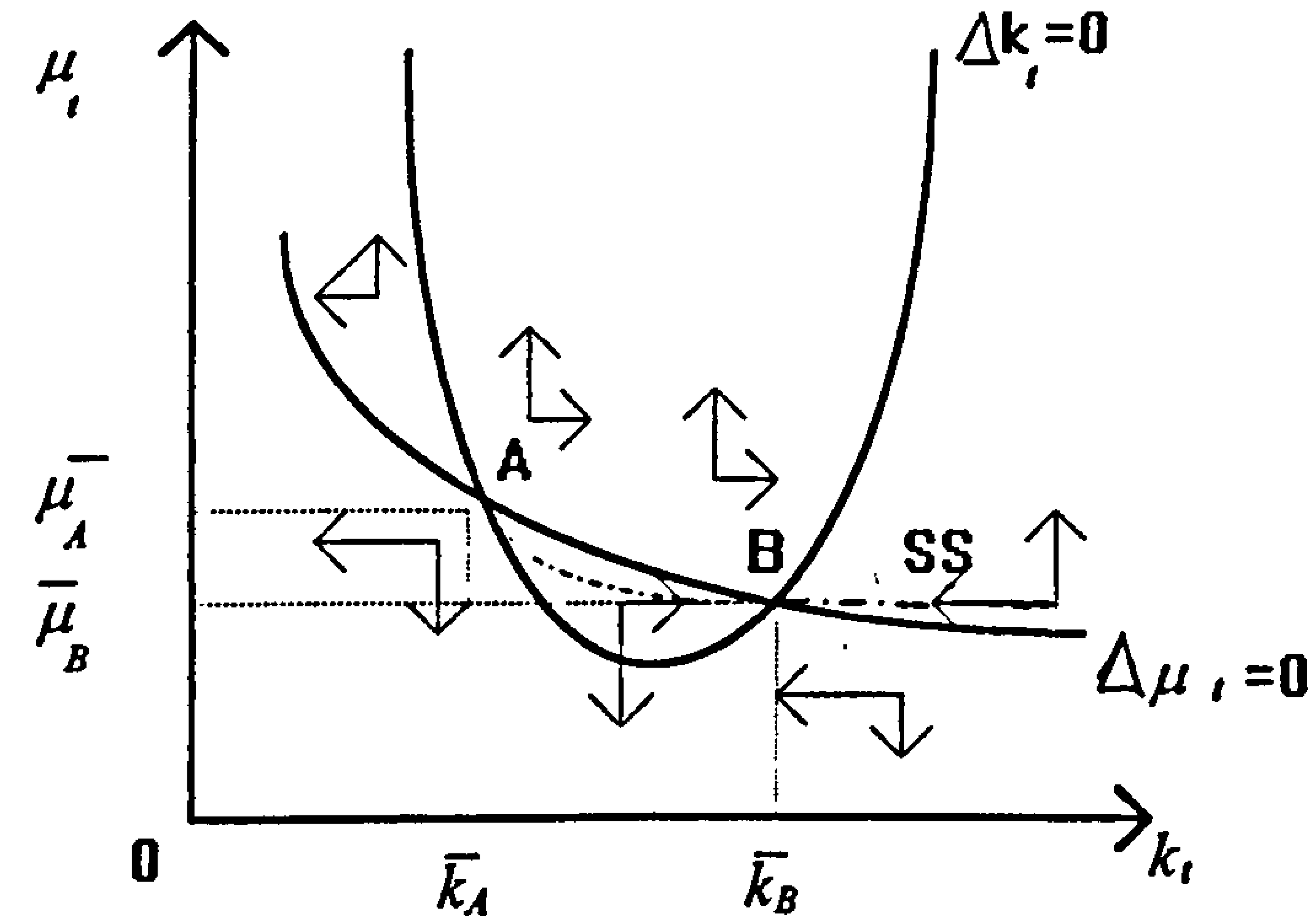

figure 5.3

Some comments are straightforward. First of all, point A exhibits a high inflation rate associated with a low level of capital intensity. The arrows suggest that this point is unstable, which implies that if the system starts off in a position above the locus $\Delta\mu_t = 0$ and to the left of A, the system would diverge towards hyperinflation. Conversely, point B is saddle point stable and there exists a saddle path which leads the system to a stationary equilibrium.

Following an increase in the amount of seigniorage, g, the locus $\Delta k_t = 0$ shifts up[10] whereas $\Delta\mu_t = 0$ shifts outwards[11]; two steady states still exist provided that the subsidy to be paid does not exceed a certain limit. The steady state level of money growth rate unambiguously rises whereas the effect on the steady state level of capital stock is ambiguous. However,

[10] The denominator in eq.(5.11) decreases at each level of k_t so that μ_t rises.

[11] The two terms on the LHS of eq.(5.13) are both increasing in g and decreasing in k_t, so that any increase in seigniorage will be offset by an increase in capital stock, at each level of μ_t, to hold the LHS constant.

from the results of the simulations reported in table B.1 in appendix 2 to this chapter it becomes clear that an increase in seigniorage would increase (decrease) the saddle point stable (unstable) steady state level of capital stock, *B* (*A*). This particular result is also confirmed by a formal analysis (included in appendix 2) of the FOCs of the system, expressed in dynamic terms, similar to those represented in eqs.(4.35)-(4.36), but taking into account the endogenisation of μ.

The shifts in the two loci following a small change in g are illustrated in fig.5.4, where the two stationary loci $\Delta k_t = 0$ and $\Delta \mu_t = 0$ are represented by the functions *KK* (which shifts to *K'K'*) and *MM* (which shifts to *M'M'*) respectively[12].

[12] By using the production function in eq.(4.77), we get the two loci:

$$\Delta k_t = 0 \qquad \mu_t = \frac{g}{-(1+n)k_t + (\beta_2+\beta_3)\left(\alpha \ln[\gamma k_t + 1] - \frac{\alpha\gamma k_t}{\gamma k_t + 1} + g\right) - g}$$

and

$$\Delta \mu_t = 0 \qquad 1 = \frac{g(1+n)}{\left[(1+\mu_t)g - \mu_t\beta_3\left(\alpha \ln[\gamma k_t + 1] - \frac{\alpha\gamma k_t}{\gamma k_t + 1}\right)\right]} \bullet$$

$$\bullet\left\{1 + \frac{\alpha\gamma}{1 + \frac{\gamma}{1+n}\left[(\beta_2+\beta_3)\left(\alpha \ln[\gamma k_t + 1] - \frac{\alpha\gamma k_t}{\gamma k_t + 1} + g\right) - g\frac{1+\mu_t}{\mu_t}\right]}\right\}.$$

Simulations for different values of the parameters $\{\beta_1\beta_2\beta_3\}$ confirm the graphical representation of the system as in fig.5.3.

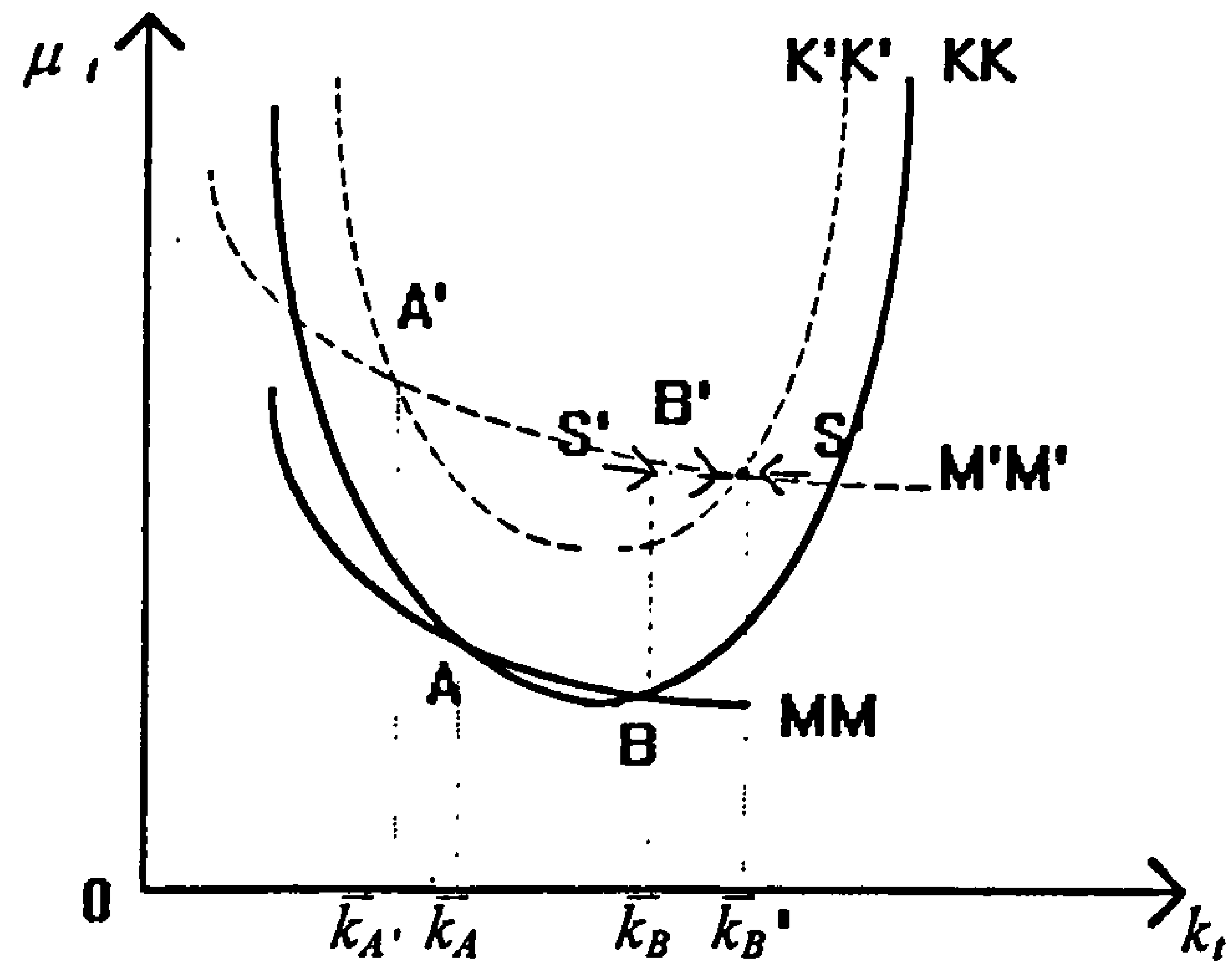

figure 5.4

Two different scenarios can be distinguished. If the system finds itself initially at the saddle point B, as soon as seigniorage increases, the system jumps onto the new saddle path $S'S'$ (the money growth rate being a jump variable). Subsequently, it converges to the new saddle point equilibrium B'. If, conversely, the system starts off at the unstable point A, then either it could jump immediately to the new equilibrium point A' (which is very unlikely since the capital stock is a predetermined variable) or it would be likely to jump onto the saddle path $S'S'$ and thus converge to B'. On the contrary, should the seigniorage decrease, with the system starting off at the unstable point A', then it would diverge towards an infinite rate of inflation and zero capital stock (i.e. a situation of a complete collapse of the economy). However, a more formal and complete analysis of the hyperinflation process is presented in the next concluding section.

As anticipated above, the existence of two stationary equilibria is conditional on the vertical asymptote of $\Delta\mu_t = 0$ being to the left of the asymptotes of $\Delta k_t = 0$. As g rises, the former continues shifting to the right. Beyond a certain level of g, there will be just one intersection point and, in the end, no steady state exists at all, as shown in fig.5.5 below.

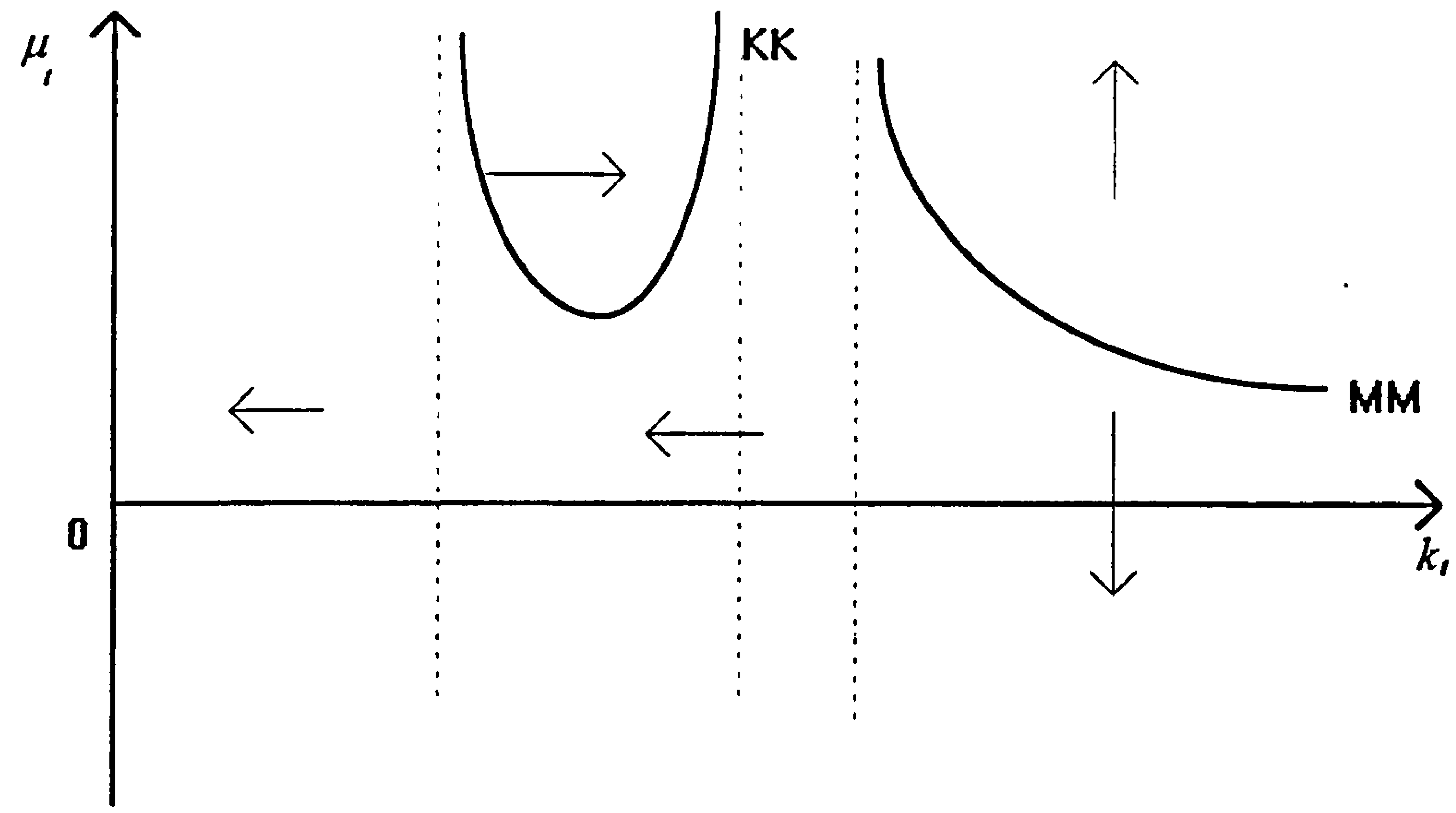

figure 5.5

Before focusing on the dynamics of hyperinflation we discuss some interesting characteristics of the comparative static properties of this extended framework.

First of all, the reduction of the capital stock together with the increase in the money growth rate at the unstable point A may appear, superficially, in contrast with the conclusions reached with the linearised Stein's model (4.37). However, the theory tells us that, when we apply the Correspondence Principle, the increase in the exogenous variable, g, should have a perverse effect on the unstable steady state. Therefore, the negative effect of an increase in the subsidy on the capital stock $\bar{k}_A$ accompanied by an increasing steady state money growth rate, is explained by the instability of the steady state itself.

On the other hand, the negative effect of an increase in g on $\bar{k}_A$ while money growth increases, would not be in contrast with the parameterised Stein's model consisting of eq.(4.81)=(4.82) when using the logarithmic production function. In fact, also in that case we have found two steady states, one saddle point stable and one unstable, and the increase in money growth rate was matched by a decrease in the capital stock for the unstable steady state (compare figures 5.5 and 4.10).

Secondly, the instability of point A, which is characterised by a high inflation steady state, is in contrast with the stability of the high inflation equilibrium in Bruno and Fischer's model. However, it can be argued that the full stability property possessed by the high inflation steady state in their model is also a source of problems. Given that the rate of inflation is a forward-looking variable, if a shock occurs, the necessary and sufficient condition for the system to converge to the new equilibrium is that it jumps on the saddle path, which implies that the inflation rate is the variable which initially absorbs the shock. However, since the high inflation steady state in Bruno and Fischer's model is stable, the initial conditions do not matter in terms of convergence to the new equilibrium. Therefore, the system is clearly overstable, which implies that there exist multiple non-divergent equilibrium sequences of which the steady state is only one; the economy can then flip in and out of it quite arbitrarily. Conversely, the saddle point property of B is more consistent with the specific properties of the endogenous variables in the system and the initial conditions regain their importance.

Thirdly, we have found that, if the amount of seigniorage exceeds a certain level, no steady state exists and, under certain conditions, the system may display hyperinflation, which is the expected and consistent result we were aiming to prove. Conversely, in Blanchard and Fischer (1989) and Bruno and Fischer's (1990) models the increase in the amount of seigniorage beyond a certain level generates dynamics of hyperdeflation, which had always represented a limitation of those models.

Simulations of the model for the case of a logarithmic production function[13] have shown that all steady states are dynamically efficient. However, before focusing on the dynamics of hyperinflation, we intend to present a mathematical proof that this property holds independently of the form of the production function considered. We start our proof from eq.(5.13): the first term on the LHS must be positive for the equation to be meaningful. This implies that:

[13] This simulations, as all the others in this thesis, have been carried out by using Mathematica. However, we avoid presenting the results for not overweighing the analysis.

(5.14) $\mu_t \leq \dfrac{g}{\beta_3(w_t + g) - g}$.

This function represents an hyperbola whose vertical asymptote is equal to: $w_A = \dfrac{g(1-\beta_3)}{\beta_3}$.

The acceptable solutions lie in the region below the function and to the right of the asymptote. However, we also need that the argument of $f'(k_t)$ is positive, namely that:

(5.15a) $(\beta_2 + \beta_3)(w_t + g) - g\dfrac{1+\mu_t}{\mu_t} \geq 0$,

or, equivalently, that:

(5.15b) $\mu_t \geq \dfrac{g}{(\beta_2 + \beta_3)(w_t + g) - g}$.

Again this function is an hyperbola with the vertical asymptote equal to: $w_B = \dfrac{(1-\beta_2-\beta_3)g}{(\beta_2+\beta_3)}$, where $w_B < w_A$. The solutions to the inequality (5.15b) lie in the area above the function and to the right of the vertical asymptote. It comes out that the values of (w_t, k_t) which satisfy both inequalities are included in the double dashed area represented in fig.5.6 below.

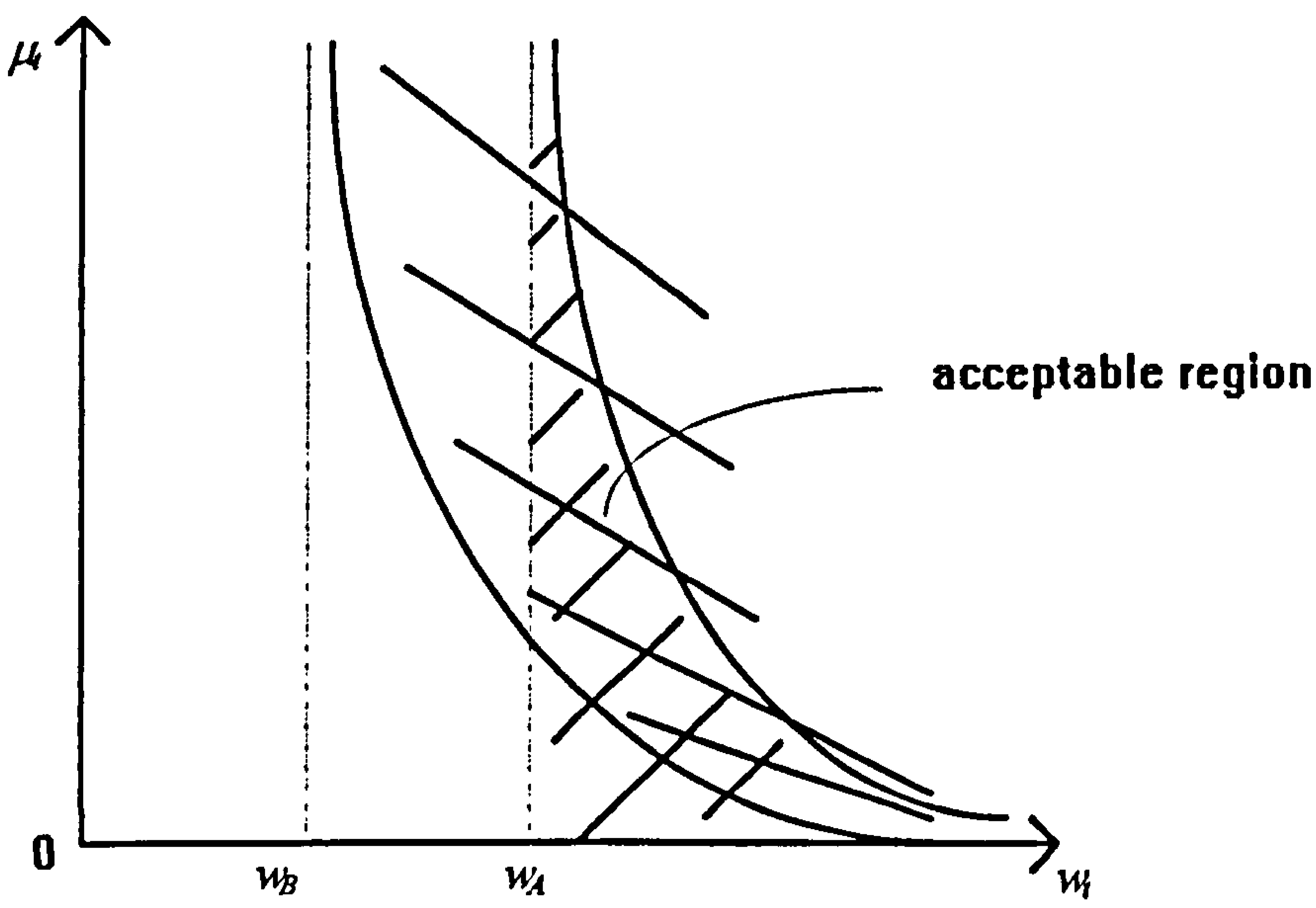

figure 5.6

Dynamic efficiency requires that $f'>n$. From eq.(5.13), which can be rewritten as:

$$(5.16) \qquad (1+f')=\frac{(1+n)}{\left[(1+\mu_t)-\mu_t\beta_3\left(\frac{w_t}{g}+1\right)\right]},$$

the denominator less than 1 is a necessary and sufficient condition for having the inequality $f'>n$ satisfied, and this is true when, for positive μ_t, the following holds:

$$(5.17) \qquad \left[1-\beta_3\left(\frac{w_t}{g}+1\right)\right]<0 \;\Rightarrow\; w_t>\frac{g(1-\beta_3)}{\beta_3}=w_A.$$

Since we have showed that the acceptability of w_t, for which the stationary locus $\Delta\mu_t=0$ is economically meaningful, implies that $w_t>w_A$, then from (5.17) is clear that inside the dashed region any existing steady state is dynamically efficient. This result can be summarised in the following proposition:

Proposition 5.1 *Irrespective of how many steady states exist, they are all dynamically efficient*[14].

Furthermore, it is interesting to note that the satisfaction of the inequality (5.17) represents a sufficient condition for having a downward sloping stationary locus $\Delta\mu_t=0$ (see eq.(5.13)), so that the analysis of the phase diagram is now complete.

In the next section we illustrate how hyperinflation can arise in our system and then we conclude our analysis by presenting the final results.

14 However, the solutions are not equivalent in terms of the amount of seigniorage which can be raised. Any time there is hyperinflation, in order to be guaranteed the same amount of revenue, the government should keep on increasing the inflation rate (i.e. the money printing rate) because the tax base would be decreasing.

5.3 Dynamics of Hyperinflation

In this section we first investigate the conditions which can produce hyperinflation. As a second step, we prove that the existence of two steady states in our model is not characterised by the Laffer curve property, so that the usual mechanism typical of the standard models, in which the same amount of seigniorage could be raised at two different rates of inflation, does not apply. Finally, we describe the dynamics through which hyperinflation arises and the behaviour of the main real and nominal variables.

From fig.5.3 it is clear that a rising hyperinflation is conditional on the existence of two steady states which is related to the fact that the stationary locus $\Delta k_t = 0$ has a U-shape, as shown in section 2. Any time the asymptote of $\Delta \mu_t = 0$ lies to the left of both asymptotes of $\Delta k_t = 0$, two equilibria exist. It is, therefore, interesting to show why the locus has this shape and what is the economic intuition behind it.

The most immediate explanation of why the stationary locus $\Delta k_t = 0$ has that particular shape can be derived by considering eq.(5.11). An increase in k_t exerts an effect on the money growth rate represented by the derivative:

$$(5.18) \qquad \frac{d\mu_t}{dk_t} = \frac{g(1+n) - g(\beta_2 + \beta_3)\dfrac{\partial w_t}{\partial k_t}}{\left[(\beta_2 + \beta_3)(\overline{w} + g) - g - (1+n)\overline{k}\right]^2},$$

which can be positive or negative depending on the sign of the numerator. The numerator is negative if $\dfrac{\partial w_t}{\partial k_t} > \dfrac{1+n}{(\beta_2 + \beta_3)} > 1$, which is more likely to hold the lower is k_t. Therefore, for small positive values of the capital stock the locus is downward sloping; as the capital stock gets bigger, the locus starts exhibiting an upward slope.

However, a deeper investigation of the model suggests that there is an important economic meaning behind the U-shape of the stationary locus $\Delta k_t = 0$. As we already mentioned in chapter 4, savings (which are a positive function of k_t) represent the resources which are either accumulated under the form of capital stock or invested in real money holdings. Any time k_t rises, there are two effects at work: the first one is the increase in wages, which positively affects savings, and the second one is the increase in the capital stock, k_{t+1}, which is needed to satisfy the relation $\Delta k_t = 0$, and which demands a higher level of savings, ceteris paribus[15]. Therefore, an increase in capital stock has two contrasting effects on savings which do not necessarily cancel out.

Any time k_t rises, if the increase in savings is sufficiently high to provide enough capital stock for the new generation, then the overall effect is positive, and real money holdings increase as well. Conversely, if the overall effect on savings is negative, real money holdings would decrease. Of course, since seigniorage is constant along the stationary locus, an increase in real money holdings implies a decrease in the money growth rate[16], and vice versa.

In order to evaluate the effect on savings following an increase in capital stock, we consider the wage expression consistent with the logarithmic production function, whose derivative $\frac{\partial w_t}{\partial k_t}$ is equal to $\frac{\alpha\gamma^2\bar{k}}{(\gamma\bar{k}+1)^2}$. When the derivative $\frac{\partial w_t}{\partial k_t}$ is quite high (which occurs for small values of k_t), then the increase in savings due to a big capital accumulation will be large enough to sustain the new level of capital intensity: as a consequence, real money holdings will increase and money growth rate will decrease. In contrast, when the sensitivity of wage with respect to capital stock starts getting smaller, which happens when capital stock becomes quite large, the net effect on savings will be negative: real money holdings will have to decrease and money growth rate will increase.

In conclusion, the slope of the stationary locus $\Delta k_t = 0$ is strictly associated with the size of the effect of the capital stock on wage (and, as a consequence, on savings). The importance

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15 It is worth remembering that $(1+n)k_{t+1} + m_t = s[w(k_t)]$ and that $k_{t+1} = k_t$ on the stationary locus $\Delta k_t = 0$.

16 From the expression for seigniorage, which is equal to: $g = \frac{\mu_t}{1+\mu_t} m_t$, μ_t and m_t are negatively related to keep seigniorage constant.

of the sensitivity of current wage to changes in current capital for determining the multiplicity of the equilibria has been already investigated and underlined in chapters 2 and 4.

Since two steady states exist for a certain level of seigniorage, we will investigate whether their presence in our model is consistent with the Laffer curve property. If this were true, it would be preferable for the country to be at the lowest level of inflation given a certain amount of seigniorage to be raised. Therefore, we consider both the seigniorage equation ($g=\frac{\overline{\mu}}{1+\overline{\mu}}\overline{m}$ in eq.(5.4)), and the demand for money equation, (eq.(4.58)) evaluated at the steady state, which, using eq.(5.3), can be transformed into:

$$(5.19) \qquad \overline{m}=\frac{(1+\overline{\mu})(1+\bar{r})}{(1+\overline{\mu})(1+\bar{r})-(1+n)}\beta_3(\overline{w}+g).$$

By equating (5.4) and (5.5), we find a unique solution for the money growth rate:

$$(5.20) \qquad \overline{\mu}=\frac{(\bar{r}-n)g}{(1+\bar{r})[\beta_3(\overline{w}+g)-g]}.$$

This implies that there is a unique steady state rate of inflation which guarantees a certain amount of seigniorage, treating *k*, *w* and *r* (i.e. the real economy) as given. An intuitive and formal explanation for why the Laffer curve phenomenon does not occur under these circumstances is provided by studying the shape of the seigniorage function. Substituting the equation for money demand in eq.(5.19) into the seigniorage equation, both evaluated at the steady state, we derive the following relation:

$$(5.21) \qquad g=\beta_3\frac{(1+\bar{r})\overline{w}}{\frac{(\bar{r}-n)}{\overline{\mu}}+(1+\bar{r})(1-\beta_3)}.$$

The RHS is always increasing (decreasing) in $\overline{\mu}$ if $\bar{r}>n$ $(\bar{r}<n)$.

Since we have found both mathematically (above) and by simulations that any steady state solution must be dynamically efficient, eq.(5.21) exhibits the shape as in fig.5.7. The line *OG* represents the seigniorage function in eq.(5.21).

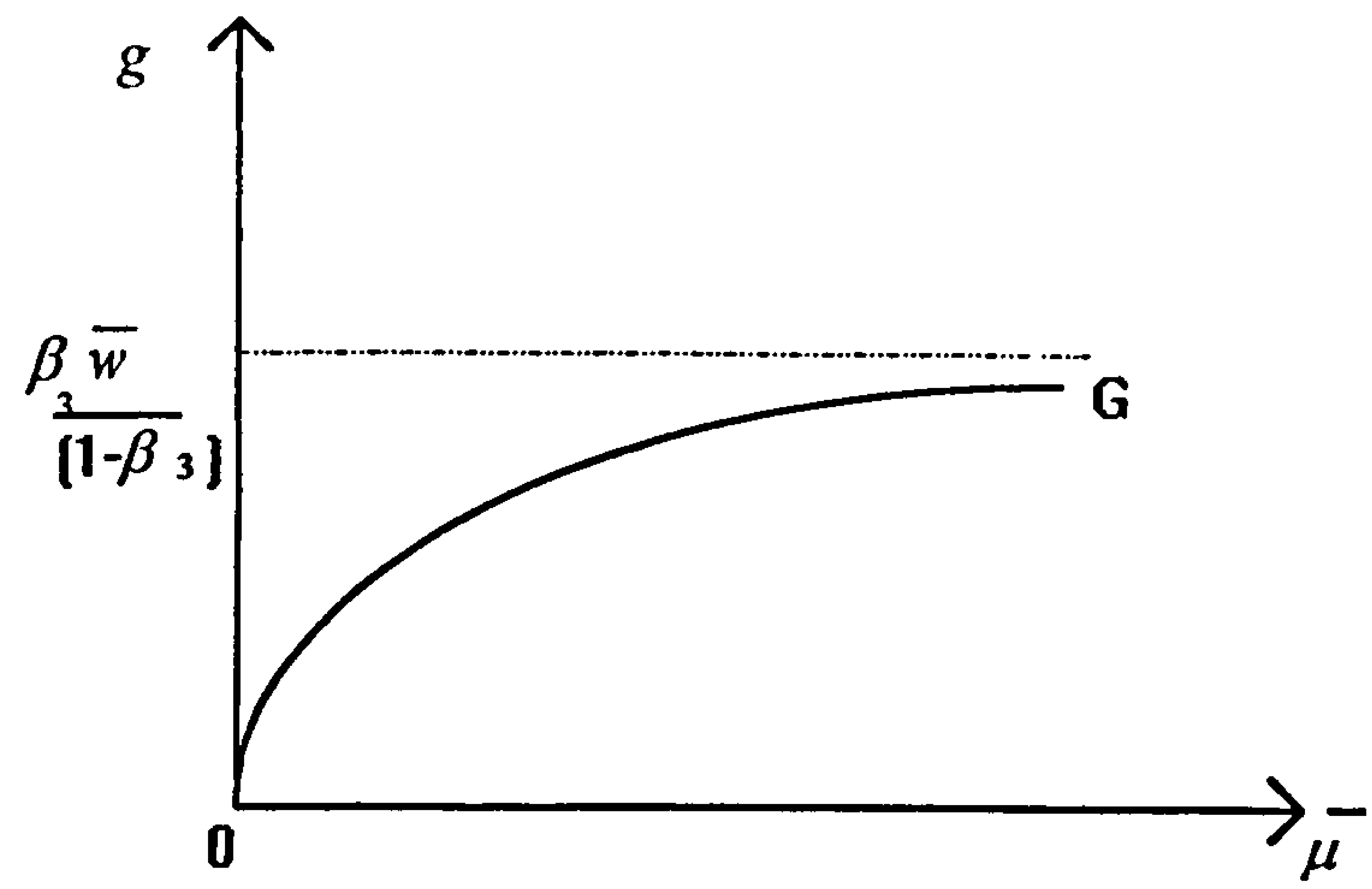

figure 5.7

The condition for having a positive solution for $\overline{\mu}$ to eq.(5.20) is that $\overline{w} > \frac{g(1-\beta_3)}{\beta_3}$, which is actually the same condition in (5.17) necessary to have a meaningful steady state. This leads us to conclude that:

Proposition 5.2 *Multiplicity of the steady states in our model is not associated with a hill-shaped seigniorage function and the Laffer curve phenomenon does not arise. A maximum level of seigniorage exists beyond which a steady state ceases to exist and the economy would be characterised by dynamics of hyperinflation.*

The lesson to be learned is that, if the initial stock of capital is too low, then a government must be very careful to set up a programme of subsidising the young through seigniorage, because this could have a negative effect on capital accumulation and on lifetime income of the consumers and make possible a persistent increase in the inflation rate.

Since we established that two intersection points exist, we can now start investigating the mechanism of hyperinflation in the system depicted in fig.5.3. First of all, it is interesting to note that dynamics of hyperinflation may arise if the system starts off above the stationary locus $\Delta\mu_t = 0$ and to the left of point *A*, since we follow the usual convention that, to the right of A, the conditions for the system to be on the saddle path are always satisfied. Using the information from fig.5.4 about the dynamics of the system and the instability of point A, we now list a set of conditions which are necessary to have hyperinflation.

a) When two steady states exist, hyperinflation arises if the initial level of capital stock is too low (i.e. $k < \bar{k}_A$) or if a shock occurs and the system is pushed away from the initial unstable equilibrium, A. Although a stable steady state exists, it cannot be reached because dynamics would bring the system away from any equilibrium point. However, it must be underlined that μ_t tends to plus infinity if the initial starting point is above the separation line (line *aa* in fig.5.8), which passes through point A and divides into two parts the space limited by the two stationary loci, $\Delta\mu_t = 0$ and $\Delta k_t = 0$. If the system starts below this separation line, the rate of money growth tends towards minus infinity.

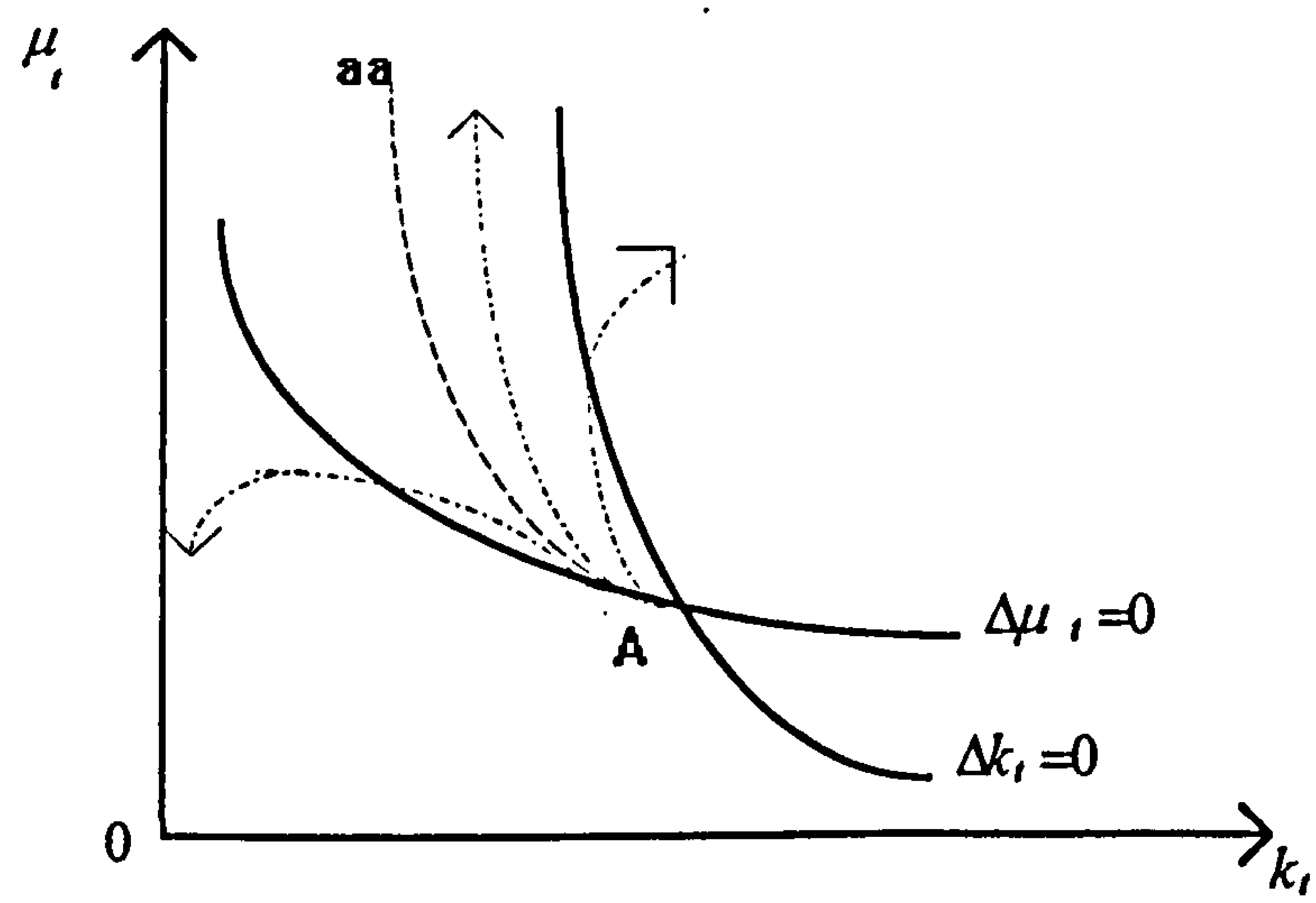

figure 5.8

On the other hand, when the system starts to the left of point A and below the separation line, a *poverty trap* arises. If the government subsidises the young by printing money and the amount of capital existing in the economy is too low, then this policy might lead to de-cumulation of capital stock to counterbalance the increased injection of money into the economy (always assuming that all markets clear).

Conversely, if the system starts to the left of point A and above the separation line, then as k_t starts decreasing, the real interest rate would rise. Since above the stationary locus $\Delta\mu_t = 0$ and above the separation line the inflation rate is also rising (this is because $\frac{p_{t+1}}{p_t} = \frac{(1+\mu_t)\mu_{t+1}}{(1+n)\mu_t}$), then the nominal interest rate would unambiguously increase together with μ_t. The decrease in k_t and the increase in i_t have a negative effect on the demand for real money balances. Therefore, to guarantee a constant level of seigniorage (eq.(5.4)), it is necessary that the growth rate of money increases. Inflation would be ever-increasing together with a decumulating capital stock until the economy reaches the collapse.

b) The amount of subsidy exceeds the maximum level above which a steady state fails to exist. In this case, for some initial points to the left of A and above the separation line, the dynamics lead to an ever increasing level of money growth rate.

These results can be summarised as in the following proposition.

Proposition 5.3 *In the modified Stein's model with endogenous money growth rate and constant seigniorage policy, hyperinflation may arise if either the initial level of capital stock is too low or if the amount of seigniorage raised is too high so that a steady state ceases to exist.*

This possibility that hyperinflation arises with perfect foresight constitutes an improvement with respect to some models, like Bruno and Fischer's, since in our model hyperinflation is associated with a collapse of the real economy and excessive deficit to be financed through seigniorage.

5.4 Concluding Remarks

In this chapter we set up a model of hyperinflation which is based on intertemporal utility optimisation by individuals and on the explicit introduction of the effects of inflation on the real sector of the economy. A policy of government deficit financing through printing money can influence the real side of the economy in so far as capital accumulation depends on the amount of resources people decide to invest out of their real lifetime income. Provided that the amount of seigniorage does not exceed a certain level, then there are two equilibrium points one of which is stable and one unstable. The latter is characterised by a high inflation rate and a low level of capital intensity. Although they are both dynamically efficient, if the initial level of capital stock is too low, then a poverty trap with decumulation of capital stock and an increase in inflation rate would arise.

The existence of the two steady states is not associated with the Laffer curve property so that there is not the possibility of raising the same amount of revenue at a lower rate of inflation; conversely, the stability properties of the equilibria determine the future behaviour of the economy any time an exogenous shock occurs.

Contrary to the analyses in the standard literature which adopted a Cagan demand for money equation, our demand for real balances is a function of real wage and stems from an explicit utility optimisation by the individuals. Since real variables are no longer kept constant, changes in prices can affect the process of capital accumulation and the production of goods in the economy. The real rate of return on investment is time-varying and depends on the dynamics of the inflation rate. This enforces a double-wedged mechanism on real balances and their growth rate through a policy of constant seigniorage. Any time the level of capital stock is too low to sustain that level of seigniorage which the government has targeted to raise at each inflation rate, the double effect of capital decumulation and of the increase in interest rates steers the dynamics of hyperinflation.

APPENDIX 1

In this appendix we study the system consisting of the two stationary loci represented by eqs.(5.11)-(5.12) and confined to the case of the logarithmic production function (eq.(4.77)):

$$\text{(A.1)} \quad \Delta k_t = 0 \Rightarrow \quad k_t = \frac{1}{1+n}\left[(\beta_2+\beta_3)\left(\alpha \ln(\gamma k_t+1) - \frac{\alpha\gamma k_t}{\gamma k_t+1}\right) + (\beta_2+\beta_3)g - g\frac{1+\mu_t}{\mu_t}\right]$$

$$\text{(A.2)} \quad \Delta\mu_t = 0 \Rightarrow \quad 1 = \frac{g(1+n)}{\left\{(1+\mu_t)g - \mu_t\beta_3\left[\alpha\ln(\gamma k_t+1) - \frac{\alpha\gamma k_t}{\gamma k_t+1} + g\right]\right\}} \cdot \frac{1}{\left\{1 + \frac{\alpha\gamma}{1+\frac{\gamma}{1+n}\left[(\beta_2+\beta_3)\left(\alpha\ln(\gamma k_t+1) - \frac{\alpha\gamma k_t}{\gamma k_t+1} + g\right) - g\frac{1+\mu_t}{\mu_t}\right]}\right\}}$$

The conclusions from the simulations we carried out is that for small values of g two steady states exist and the two stationary loci have the shape shown in fig.(5.3) in the text. Since the behaviour of the system depends on some parameter values, we now consider how a change in each parameter, ceteris paribus, can affect the behaviour of the system.

1) Holding $\{\alpha, \beta_1, n, \gamma\}$ constant, as β_2 decreases with respect to β_3, $\Delta\mu_t = 0$ shifts down, while $\Delta k_t = 0$ is not affected, so that for the same level of g, the two steady states get closer and it is necessary to increase seigniorage to sustain the same level of capital stock if the proportion of income invested is lower.

2) Holding $\{\alpha, \beta_2, n, \gamma\}$ constant, the bigger β_1 with respect to β_3, ceteris paribus, the more likely that an acceptable steady state exists, since more resources are invested in the financial assets.

3) Holding $\{\alpha, \beta_3, n, \gamma\}$ constant, the higher β_1 with respect to β_2, ceteris paribus, the higher the locus $\Delta\mu_t = 0$ and the lower $\Delta k_t = 0$, so that, if β_1 exceeds a certain value, a steady state will cease to exist. In fact, what happens is that the economy is not devoting enough resources to capital stock accumulation.

4) Ceteris paribus, an increase in α would shift down both loci $\Delta k_t = 0$ and $\Delta\mu_t = 0$. As a result, the two steady states tend to diverge.

5) A decrease in γ would shift up the locus $\Delta k_t = 0$ and up to the right the locus $\Delta\mu_t = 0$, so that, as γ decreases, the two steady states get closer till they disappear. Therefore, γ must be increased if we want two equilibria to exist, ceteris paribus.

6) When two steady states exist, an increase in the rate of population growth, n, would shift up the stationary locus $\Delta k_t = 0$ and slightly down $\Delta\mu_t = 0$ at any level of k_t, so that the two steady states would shrink and the stable steady state level of capital per head is reduced. Therefore, the existence of two equilibria is conditional upon a not too high rate of population growth, n.

This analysis implies that, given an initial level of per capita capital stock, k_o, the system is more likely to diverge towards capital decumulation the higher is the rate of population growth. This suggests that for the development of an economy it is necessary to have enough investments in physical capital to equip each new worker to produce more resources. A rapid rate of population growth would then deprive the young generation of most part of the resources which are necessary to make the economy sustainable.

7) As anticipated in the text, an increase in g would shift up $\Delta k_t = 0$ and up to the right $\Delta\mu_t = 0$ in such a way that the two steady states diverge; when the vertical asymptote for $\Delta\mu_t = 0$ lays to the right of both asymptotes for $\Delta k_t = 0$, the steady states would disappear.

APPENDIX 2

In this appendix we are interested in evaluating the effect of a small increase in the amount of seigniorage raised by the government on the steady states of the capital stock, $\bar{k}$, which are the solutions to the two first-order difference equation system (5.11)-(5.12). For this purpose, we follow the same procedure of chapter 4 and we calculate the overall multiplier $\frac{d\bar{k}}{dg}$ by using the First-Order-Conditions of the general model in dynamic terms represented by eqs.(4.35)-(4.36) (previously evaluated at the steady state). This allows us to derive results which are independent on the type of utility and production functions considered.

Using eqs.(4.25)-(4.26) and making the necessary substitutions, the FOCs can be rewritten as:

$$\text{(B.1)}\qquad u_y\left[f(k_t)-\frac{c_t^o}{1+n}-(1+n)k_{t+1}+k_t\right]=\left(1+f'(k_{t+1})\right)u_o(c_{t+1}^o)$$

$$\text{(B.2)}\qquad u_m(m_t)=\left[1+f'(k_{t+1})-\frac{(1+n)\mu_t}{\mu_{t+1}(1+\mu_t)}\right]u_o(c_{t+1}^o)$$

where the argument in the parenthesis of the LHS in eq.(B.1) is the resource constraint (eq.(4.15)), whereas the term in brackets on the RHS of eq.(B.2) is the expression for $\frac{i_t}{1+i_t}(1+r_{t+1})$ derived in the case of endogenous money growth rate (eq.(5.3)).

Real money balances are expressed as $m_t=\frac{1+\mu_t}{\mu_t}g$; however, further simplifications are necessary in order to eliminate one of the three endogenous variables (c^o,k,μ). The second-period budget constraint of the household, which is equal to:

$$\text{(B.3)}\qquad c_{t+1}^o=\left[1+r_{t+1}\right]a_t+\frac{p_t}{p_{t+1}}m_t,$$

can be written, by using eqs.(4.13) and (5.2) and the relation $m_{t+1}=\frac{1+\mu_{t+1}}{\mu_{t+1}}g$, as:

(B.4) $\quad c^o_{t+1}=(1+n)[1+f'(k_{t+1})]k_{t+1}+\frac{(1+n)g}{\mu_{t+1}}.$

The latter is a static equation since it relates only simultaneously-determined variables. The intuitive explanation of this expression is that the old consume all their wealth, which consists of accumulated physical capital, profits and real balances after the inflation tax. Therefore, c^o can be determined uniquely after finding the equilibrium values for k and μ as solutions to the system (B.1)-(B.2). If we substitute eq.(B.4) dated at time t into eq.(B.1) and again eq.(B.4) dated at time $t+1$ into eq.(B.2), the system becomes:

(B.5)

$$u_y\left[f(k_t)-f'(k_t)k_t-\frac{g}{\mu_t}-(1+n)k_{t+1}\right]=[1+f'(k_{t+1})]u_o\left[(1+f'(k_{t+1}))(1+n)k_{t+1}+\frac{1+n}{\mu_{t+1}}g\right]$$

(B.6) $$u_m\left(\frac{1+\mu_t}{\mu_t}g\right)=\left[1+f'(k_{t+1})-\frac{(1+n)\mu_t}{(1+\mu_t)\mu_{t+1}}\right]u_o\left[(1+f'(k_{t+1}))(1+n)k_{t+1}+\frac{1+n}{\mu_{t+1}}g\right].$$

The system consists of two difference equations and, for simplicity, can be summarised as:

(B.7) $\quad p(k_t,k_{t+1},\mu_t,\mu_{t+1};g)=0$

(B.8) $\quad q(k_t,k_{t+1},\mu_t,\mu_{t+1};g)=0.$

By indicating with a subscript index the argument with respect to which the derivative is calculated (i.e. $p_3=\frac{\partial p}{\partial \mu_t}$ and so on with the same order as they are written inside the brackets of p and q), we transform the system in a matrix form:

(B.9) $$\begin{bmatrix} p_2 & p_4 \\ q_2 & q_4 \end{bmatrix}\begin{bmatrix} dk_{t+1} \\ d\mu_{t+1} \end{bmatrix}=\begin{bmatrix} -p_1 & -p_3 \\ -q_1 & -q_3 \end{bmatrix}\begin{bmatrix} dk_t \\ d\mu_t \end{bmatrix}+\begin{bmatrix} -p_5 \\ -q_5 \end{bmatrix}dg.$$

Using the FOCs in eqs.(B.5) and (B.6) we calculate the partial derivatives ($p_i, q_i \quad i=1,\dots,5$) in (B.9) evaluated at the steady state which are listed below:

$p_1=-f''\bar{k}u_{yy}$

$p_2=-(1+n)u_{yy}-(1+f')u_{oo}(1+n)(1+f'+f''\bar{k})-f''u_o$

$p_3=\frac{u_{yy}g}{\bar{\mu}^2}$

$$p_4 = (1+f')u_{oo}\frac{(1+n)g}{\overline{\mu}^2}$$

$$p_5 = -\frac{u_{yy}}{\overline{\mu}} - (1+f')u_{oo}\frac{1+n}{\overline{\mu}}$$

$$q_1 = 0$$

$$q_2 = -f''u_o - \left(1+f' - \frac{1+n}{1+\overline{\mu}}\right)u_{oo}(1+n)(1+f'+f''\overline{k})$$

$$q_3 = -\frac{u_{mm}g}{\overline{\mu}^2} - \frac{u_o(1+n)}{(1+\overline{\mu})^2\overline{\mu}}$$

$$q_4 = -\frac{u_o(1+n)}{(1+\overline{\mu})\overline{\mu}} + \left(1+f' - \frac{1+n}{1+\overline{\mu}}\right)u_{oo}\frac{(1+n)g}{\overline{\mu}^2}$$

$$q_5 = u_{mm}\frac{1+\overline{\mu}}{\overline{\mu}} - \left(1+f' - \frac{1+n}{1+\overline{\mu}}\right)u_{oo}\frac{1+n}{\overline{\mu}}.$$

In order to get rid of the matrix on the LHS of (B.9), we find the partial derivatives by Cramer's rule, i.e. $\frac{dk_{t+1}}{dk_t} = \frac{-p_1q_4 + p_4q_1}{p_2q_4 - p_4q_2}$, and so on. In general terms, the system in (B.9) can be solved to find[1]:

$$\text{(B.10)} \quad \begin{bmatrix} \Delta dk_{t+1} \\ \Delta d\mu_{t+1} \end{bmatrix} = \begin{bmatrix} f_1 - 1 & f_2 \\ g_1 & g_2 - 1 \end{bmatrix} \begin{bmatrix} dk_t \\ d\mu_t \end{bmatrix} + \begin{bmatrix} f_3 \\ g_3 \end{bmatrix} dg.$$

The arguments of the matrix in (B.10) are related to the arguments of the matrices in (B.9) by the following expressions:

$$f_1 - 1 = \frac{-p_1q_4 + p_4q_1 - p_2q_4 + p_4q_2}{p_2q_4 - p_4q_2}$$

$$f_2 = \frac{-p_3q_4 + p_4q_3}{p_2q_4 - p_4q_2}$$

[1] The two difference equations can be written implicitly as:

$$\begin{cases} k_{t+1} = f(k_t, \mu_t; g) \\ \mu_{t+1} = g(k_t, \mu_t; g) \end{cases}.$$

$$f_3 = \frac{-p_5 q_4 + p_4 q_5}{p_2 q_4 - p_4 q_2}$$

$$g_1 = \frac{-p_2 q_1 + p_1 q_2}{p_2 q_4 - p_4 q_2}$$

$$g_2 = \frac{-p_2 q_3 + p_3 q_2 - p_2 q_4 + p_4 q_2}{p_2 q_4 - p_4 q_2}$$

$$g_3 = \frac{-p_2 q_5 + p_5 q_2}{p_2 q_4 - p_4 q_2}$$

The effect of an exogenous increase in seigniorage is measured by the total multiplier:

$$\text{(B.11)} \quad \frac{d\bar{k}}{dg} = \frac{-f_3(g_2 - 1) + g_3 f_2}{\det C},$$

where C is the matrix on the RHS of (B.10). As far as $\det C$ is concerned, we can apply the Correspondence Principle as we have done in chapter 2. Since capital is the only predetermined variable, for the system to be well-behaved, the steady states would be either unstable or saddle point stable. In the first case, the determinant of the Jacobian matrix is positive, whereas in the second case is negative. Therefore, the sign of the multiplier depends on the sign of the numerator (N) which is equal to:

$$\text{(B.12)} \quad N = -\frac{(-p_5 q_4 + p_4 q_5)}{(p_2 q_4 - p_4 q_2)} \cdot \frac{(-p_2 q_3 + p_3 q_2 - p_2 q_4 + p_4 q_2)}{(p_2 q_4 - p_4 q_2)} + \frac{(-p_3 q_4 + p_4 q_3)}{(p_2 q_4 - p_4 q_2)} \cdot \frac{(-p_2 q_5 + p_5 q_2)}{(p_2 q_4 - p_4 q_2)}$$

which, after some calculations, can be simplified to:

$$\text{(B.13)} \quad N = \frac{-p_5(q_3 + q_4) + q_5(p_3 + p_4)}{(p_2 q_4 - p_4 q_2)}.$$

After carrying out all the multiplications, the expression for N is found to be equal to:

$$\text{(B.14)} \quad \left[\frac{u_{yy}}{1+f'} + u_{oo}(1+n)\right]\left[\frac{u_o}{(1+\bar{\mu})^2} - \frac{u_{mm} g}{(1+n)\bar{\mu}}\right](1+f')\left(\frac{-\frac{1+n}{\bar{\mu}}}{D}\right).$$

The overall sign of N depends on the sign of the expression in the last parenthesis in (B.14), with D equal to:

(B.15)

$$D=\frac{(1+n)^2 u_{yy}u_o}{\overline{\mu}(1+\overline{\mu})}-\left(1+f'-\frac{1+n}{1+\overline{\mu}}\right)\frac{u_{yy}u_{oo}(1+n)^2 g}{\overline{\mu}^2}+\frac{(1+f')u_o u_{oo}(1+n)^2(1+f'+f''\overline{k})}{\overline{\mu}(1+\overline{\mu})}+$$

$$+\frac{f''u_o^2(1+n)}{\overline{\mu}(1+\overline{\mu})}+\frac{f''u_o u_{oo}(1+n)^2 g}{\overline{\mu}^2(1+\overline{\mu})},$$

whose sign is almost unambiguously negative (the only positive term is the last one, but for a small u_{oo} it is negligible[2]). In conclusion, using the information obtained about the sign of the denominator of the multiplier, we can state that an increase in the amount of seigniorage raised by the government would increase the saddle point stable steady state $\overline{k}$ and decrease the unstable steady state capital stock.

This conclusion is supported by some simulations of eqs.(A.1)-(A.2). Given the difficulties met with Mathematica to find the specific value for the steady states, we identified the range of values in which $\overline{k}$ must lie for different sets of parameters before and after the increase in g. The simulations yield the unambiguous result of a decrease (increase) in the capital stock at the unstable (saddle point stable) steady state. Some of the results that we have obtained are shown in the table B.1 below.

[2] Our reasoning is based on the assumption that $(1+f'+f''\overline{k})>0$.

Table B.1

	$\alpha=0.3, \beta_1=0.1, \beta_2=0.8,\quad \beta_3=0.1, n=0.01, \gamma=24$	
Steady State	g=0.001	g=0.003
point A	$0.03<\bar{k}<0.07$	$0.025<\bar{k}<0.028$
point B	$0.18<\bar{k}<0.2$	$0.2<\bar{k}<0.25$
	$\alpha=0.7, \beta_1=0.1, \beta_2=0.8,\quad \beta_3=0.1, n=0.01, \gamma=24$	
Steady State	g=0.0001	g=0.0005
point A	$0.0075<\bar{k}<0.008$	$0.007<\bar{k}<0.0075$
point B	$1<\bar{k}<1.05$	$1.05<\bar{k}<1.1$
	$\alpha=0.3, \beta_1=0.1, \beta_2=0.7,$	$\beta_3=0.2, n=0.01, \gamma=24$
Steady State	g=0.005	g=0.007
point A	$0.03<\bar{k}<0.05$	$0.028<\bar{k}<0.03$
point B	$0.08<\bar{k}<0.1$	$0.1<\bar{k}<0.14$

CONCLUSIONS

In this thesis we have focused on OLG economies with perfect foresight, certainty and two period-lived individuals and we have investigated the dynamics of the system when the government must follow a debt management policy. As a consequence of the framework adopted, the model exhibits solid microeconomic foundations. The main conclusions achieved with the two models considered can be summarised as follows.

The first model highlights how debt management policies in a dynamically efficient economy may hurt rather than help the recovery of an economy. Agents are assumed to invest their first period savings in one of three different kinds of bonds which the government can issue to manage an inherited amount of debt, namely *savings deposits, treasury bills and perpetuity bonds*. Each of these assets implies a particular financing policy rule and is related to a peculiar government budget constraint. The three types of bonds available to the government are simple enough that our model could withstand added complexity and realism without losing analytical tractability.

The dynamic analysis of the model suggests the possibility that multiple equilibria exist and a **poverty trap** arises if the initial level of the capital stock is too low (namely below the unstable steady state) and the government decides to increase the amount of debt. Dynamics of increasing interest rates and decumulating capital stock create a vicious circle where interest payments would feed upon themselves. The huge pressure coming from the need to pay the interests tends to divert savings towards financial investments and away from productive capital. Countries with a too low level of capital stock start experiencing piling up of interest payments as a consequence of an increase in debt.

We have also dealt with the concept of a **maximum sustainable level of debt**, defined as the maximum ratio of debt-to-capital stock which the economy can bear without heading towards a collapse. This upper limit on the debt-to-capital stock ratio has been endogenously derived within the model. When debt is pushed beyond this level, interest rates would be increasing and the endogenous response of private savings, which must absorb the new bonds issued, would start a process of long-run decumulation of capital stock. This phenomenon is also known as a **catastrophe**: a shock in debt makes a steady state fail to exist and the economy collapses. However, unstable dynamics might well arise as a consequence of an exogenous shock to the level of government debt when the system starts initially off the unstable equilibrium. This situation is a clear example of a poverty trap.

We have also presented some further applications of the issue of debt sustainability to other existing models, by underlining some interesting aspects, such as the existence of **a maximum sustainable level of social security contributions**, which had been overlooked in the papers of some authors concerned with government financing policies.

The second model describes a monetary economy, which is a re-formulation of Stein's (1971) OLG model, and concerns the issue of **money non-superneutrality**, i.e. the effects of changes in the rate of money growth on the steady state level of capital stock. The structure of the economy consists in an intertemporal finite-horizon optimising framework in which real balances are treated as another consumption good which yields utility to the consumer. We show that money superneutrality does not hold, contrary to what was found by Sidrauski (1967), who had considered an optimising framework but with infinitely lived individuals. The model is then used to investigate specific issues overlooked by Stein, such as multiplicity and properties of the equilibria with different production functions.

This model is also extended to incorporate the dynamics of money growth rate and inflation. We assume that the government has to finance a constant level of spending through seigniorage; so long as the system is in the neighbourhood of the saddle point equilibrium, it can converge to it, provided that seigniorage is not too large. On the contrary, if the system is at the unstable point, an increase in seigniorage might cause **hyperinflation** to arise. This result is very important since most recent studies on this topic have failed to model dynamics of hyperinflation in standard monetary frameworks. Endogenisation of the interest rate and output, microfoundations to all demand functions, including real money balances demand, and the failure of the Laffer curve property are the successful elements to overcome the limiting assumptions on which the standard models were based.

It is quite clear that, although the two models we presented in this thesis display some structural differences and deal with two separate issues, nevertheless they both have an empirical importance in relation to the constraints imposed by the Maastricht Treaty upon the macroeconomic policies of the EC countries. The upper limits set on the debt-to-GDP ratio and on the rate of inflation, codified in the Treaty, constrain the amount of debt which can be run by those countries which agreed to join the European Monetary Union and prevent them from resorting to inflation tax financing, which has always been an important source of revenues in some countries for spending financing. The discussion about debt sustainability and seigniorage financing policy presented in this thesis highlights the importance of these issues in terms of real-world macroeconomic policies. The models we presented turn out to be very reliable in forecasting the scenarios ahead of the countries which will not reverse their policies as requested by the agreed constraints.

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