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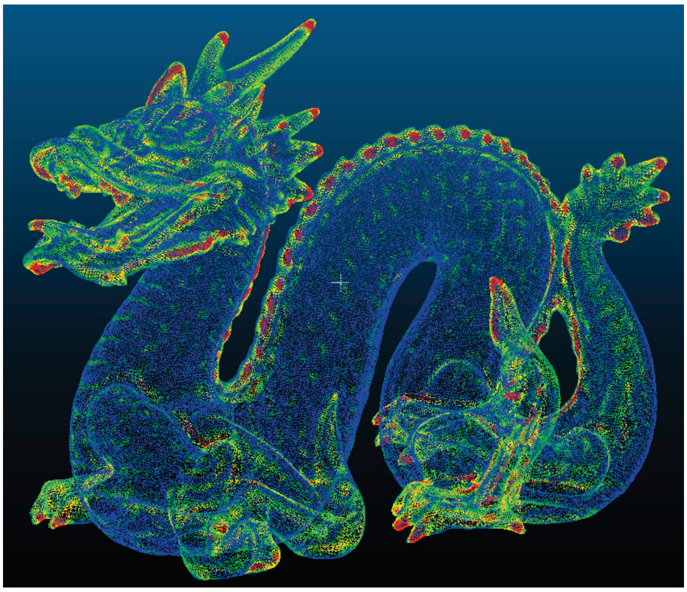

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Approximation de surfaces par des varifolds discrets : représentation, courbure, rectifiabilité

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Résumé - Abstract

Résumé : La motivation initiale de cette thèse est l'étude d'une discrétisation volumique de surface (introduite dans le Chapitre 2) naturellement liée à la structure de varifold. La théorie des varifolds a été développée par F. Almgren afin d'étudier les points critiques de la fonctionnelle d'aire. L'ensemble des varifolds rectifiables entiers fournit une notion de surface faible possédant de bonnes propriétés de compacité et munie d'une notion de courbure généralisée appelée variation première. Le point clé est qu'il est possible de munir d'une structure de varifold la plupart des objets utilisés pour représenter ou discrétiser des surfaces c'est-à-dire aussi bien des objets tels que les sous-variétés ou les ensembles rectifiables que des objets tels que des nuages de points ou encore la discrétisation volumique proposée, ce qui permet d'étudier dans un cadre unifié une surface et sa discrétisation.

Une difficulté essentielle est que, généralement, ces structures discrètes ne sont pas rectifiables, ce qui soulève la question suivante : comment assurer qu'un varifold, obtenu comme limite de discrétisations volumiques de la forme proposée, soit une surface, au moins en un sens faible ? De façon plus précise : quelles conditions sur une suite de varifolds quelconques assurent que le varifold limite est rectifiable (Chapitre 3) ou encore qu'il est à variation première bornée (Chapitre 5) ? Afin de tester la rectifiabilité d'un varifold, on peut étudier l'existence d'un plan tangent en presque tout point, mais la façon classique de le définir n'est pas adaptée (c'est-à-dire qu'elle ne se transfert pas aisément de la suite de varifolds à sa limite). Afin d'y remédier, on considère le plan tangent comme minimiseur d'une énergie liée aux nombres β de Jones, ce qui nous permet d'obtenir des conditions assurant la rectifiabilité d'une limite de varifolds.

On s'intéresse ensuite à la régularité du varifold limite en termes de courbure (variation première). Dans un premier temps, on a essayé de contrôler la variation première en observant qu'une certaine moyenne de la variation première sur des boules concentriques se réécrivait de façon à avoir un sens même pour un varifold à variation première non bornée. On a alors essayé de reconstruire par "packing" la variation première uniquement grâce à ces moyennes (Chapitre 4), mais cela n'a pas permis d'établir les conditions désirées. En revanche, cela nous a conduit à considérer une forme régularisée de la variation première d'un varifold, ce qui a permis d'établir des conditions assurant que la limite d'une suite de varifolds est à variation première bornée. Cette régularisation permet de définir des énergies de Willmore approchées qui Γ–convergent dans l'espace des varifolds vers l'énergie de Willmore classique ainsi qu'une approximation de la courbure qui est testée numériquement dans le Chapitre 6.

Abstract : The starting point of this work is the study of a volumetric surface discretization model naturally connected to the varifolds structure. Varifolds have proved to be useful when dealing with geometric variational problems in the continuous setting since they were introduced by F. Almgren as he was interested in finding critical points of the area functional in a broader class than parametrized surfaces. A sub-class of varifolds, called integral (rectifiable) varifolds provides a set of generalized surfaces with compactness properties and a consistent notion of generalized curvature. The point is that not only the discretization we propose can be endowed with a structure of varifold but also a great part of objects used for surface representation and discretization (triangulation, cloud points, level sets etc.) so that we can use varifolds tools to study in some unified setting different ways of discretizing surfaces.

An important point to overcome is that these structures are generally not rectifiable (i.e. not regular enough) so that we address the following question : how to ensure that the limit of a sequence of such discrete surfaces is regular ? Or in a more technical way, what conditions on a sequence of varifolds (not supposed rectifiable nor with bounded variation) ensure that the limit varifold is rectifiable (Chapter 3) or has bounded first variation (Chapter 5) ? In order to test the rectifiability of a varifold, we looked at the existence of approximate tangent plane but the problem is that the classical ways of defining them are not well-adapted in our case. Therefore, we first propose to consider the approximate tangent planes as the minimizer of some energy linked to Jones' β–numbers, allowing to give energetic conditions ensuring the rectifiability of a limit varifold.

We then address the question in terms of first variation (generalized curvature) of a limit varifold. We first tried a packing measure construction of the first variation of a varifold V (Chapter 4), based on the fact that in any ball, it is possible to define some kind of average value of the first variation that had sense even if V does not have bounded first variation. But it happens that this construction does not answer the question. Nevertheless, it leads us to define a regularized form of the classical first variation, allowing us to exhibit an energetic condition ensuring that a limit of a sequence of varifolds has bounded first variation. We use this regularized form to build an approximate Willmore energy Γ–converging in the class of varifolds to the Willmore energy. In the last chapter (Chapter 6), we test numerically a notion of approximate curvature derived from the regularized first variation.

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Je vous ai oublié ? Ou bien c'est volontaire (mais vous devez le savoir), ou bien j'ai été tête en

l'air et heureusement il vous reste de la place pour ajouter la phrase que vous méritez !

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Notations

We start by fixing some notations. From now on, we fix d, $n \in \mathbb{N}$ with $1 \leqslant d < n$ and an open set $\Omega \subset \mathbb{R}^n$. Then we adopt the following notations.

- $A \triangle B = (A \cup B) \setminus (A \cap B)$ is the symmetric difference.
- $B_r(x) = \{y \,|\, |y-x| < r\}$ is the open ball of center x and radius r **except** in Chapter 4 where $B_r(x) = \{y \,|\, |y-x| \leqslant r\}$ denotes the **closed ball**.
- Let ω and Ω be two open sets then $\omega \subset\subset \Omega$ means that ω is relatively compact in Ω.
- Let $A \subset \Omega$ then $A^c = \Omega \setminus A$ denotes the complementary of A in Ω.
- $G_{d,n} = \{P \subset \mathbb{R}^n \,|\, P \text{ is a vector subspace of dimension } d\}$.
- For $P \in G_{d,n}$, Π_P is the orthogonal projection onto P.
- Given a $\mathbb{R}^m$–valued function f defined in Ω, $\operatorname{supp} f$ is the closure in Ω of $\{y \in \Omega \,|\, f(y) \neq 0\}$.
- $\mathrm{C}_c^k(\Omega)$ is the space of real continuous compactly supported functions of class C^k ($k \in \mathbb{N}$) in Ω.
- $\mathrm{C}_o^0(\Omega)$ is the closure of $\mathrm{C}_c^0(\Omega)$ for the sup norm.
- $\mathrm{Lip}_k(\Omega)$ is the space of Lipschitz functions in Ω with Lipschitz constant less or equal to k.
- Let μ be a measure in some measurable topological space, then $\operatorname{supp} \mu$ denotes the topological support of μ.
- Given a measure μ, we denote by $|\mu|$ its total variation.
- $\mathcal{M}_{loc}(\Omega)^m$ is the space of $\mathbb{R}^m$–valued Radon measures and $\mathcal{M}(\Omega)^m$ is the space of R^m–valued finite Radon measures.
- $\mathcal{L}^n$ is the n–dimensional Lebesgue measure.
- $\mathcal{H}^d$ is the d–dimensional Hausdorff measure.
- $\omega_d = \mathcal{L}^d(B_1(0))$ is the d–volume of the unit ball in $\mathbb{R}^d$.

CHAPITRE 1

Généralités

Il existe de nombreuses façons de représenter et discrétiser les surfaces, tant en fonction du mode d'acquisition que des applications en vue. La question de la représentation des surfaces est au cœur de domaines divers et variées tels que l'animation 3D, la modélisation industrielle ou encore l'imagerie médicale. Ainsi, un scanner 3D va générer un nuages de points tandis qu'une acquisition IRM fournit des données de types volumiques et les animations 3D se font sur des surfaces triangulées.

Lorsqu'on a accès à une surface via sa discrétisation, un des enjeux est d'être en mesure de reconstruire un certain nombre d'informations concernant la surface initiale : des informations globales telles que la topologie ou l'orientation globale et des informations locales telles que le plan tangent ou la courbure. Se pose alors la question de la fiabilité de l'information reconstruite. L'erreur commise peut-elle être contrôlée ? Cette question sous-tend une autre question fondamentale : comment juger de la qualité de la discrétisation ? La surface continue et sa discrétisation ne vivant généralement pas dans le même espace, comment affirmer qu'une discrétisation est plus ou moins bonne ? Étant donnée la multiplicité des représentations et discrétisations existantes, un autre enjeu est la comparaison : comment estimer si deux discrétisations de type différents sont issues de deux surfaces proches ? C'est ainsi qu'ont été développées de nombreuses techniques permettant de passer d'un type de discrétisation à un autre, permettant d'exploiter les avantages liés aux différents modes de discrétisation.

Une approche consiste alors à essayer de donner un cadre commun à l'étude d'objets de nature continue régulière et d'objets de nature discrets. Le cadre des varifolds s'y prête assez bien, munissant objets de nature discrète ou continue d'une structure de mesure de Radon. On dispose alors pour mesurer l'erreur commise par discrétisation de la notion de convergence faible–$*$ et des différentes distances dont est pourvu l'espace des mesures de Radon. On dispose de plus d'une notion de courbure généralisée. Ces notions, prises telles quelles, ne sont pas adaptées à des objets de nature discrète, mais leur formulation générale est, comme on va le voir, adaptable à des objets discrétisés à une échelle donnée.

Le Chapitre 1 présente tout d'abord les notions de rectifiabilité et varifolds, en insistant sur la notion de variation première d'un varifold qui est une notion de courbure généralisée. Suit un exposé détaillé des questions qui ont guidé cette thèse et des différentes réponses qui ont pu être apportées.

Le Chapitre 2 étudie des structures de varifolds sur des objets discrets, en se concentrant particulièrement sur une discrétisation de nature volumique. On s'intéresse particulièrement aux propriétés d'approximations de ces espaces de varifolds de type discret, et au sens que prend la variation première de tels varifolds. On observe que la variation première, même si elle est définie pour n'importe quel varifold, est essentiellement adaptée aux varifolds rectifiables.

On cherche alors dans le Chapitre 3, des conditions permettant d'assurer que la limite faible–$*$ d'une suite de varifolds, a priori quelconques, soit rectifiable. On utilise pour cela une caractérisation de la rectifiabilité issue de l'étude de la rectifiabilité uniforme et de conditions quantitatives caractérisant l'uniforme rectifiabilité.

On revient alors dans le Chapitre 4 à la question de définir une notion de variation première adaptée à des varifolds discrets, étant donnée une certaine échelle de discrétisation. On essaie pour cela de reconstruire la variation première par des méthodes de construction de mesures, d'abord la construction métrique de Carathéodory, puis de type "packing". Mais ces constructions essentiellement métriques n'exploitent pas pleinement le cadre vectoriel ($\mathbb{R}^n$) dans lequel vivent nos objets.

Dans le Chapitre 5, on donne alors une notion de variation première approchée à une échelle ε en régularisant par convolution la variation première classique. Cette régularisation nous permet alors d'avoir accès à une notion de courbure moyenne approchée, qui nous permet dans un premier temps de définir des énergies de Willmore approchées Γ–convergeant dans l'espace des varifolds vers l'énergie de Willmore.

Puis dans le Chapitre 6, on teste numériquement cette approximation de la courbure sur des nuages de points, en étudiant l'influence des différents paramètres en jeu (nombre de points dans le nuage, échelle du noyau régularisant, forme du noyau régularisant).

1.1 Mesures de Radon et rectifiabilité

Avant d'introduire l'espace des varifolds, on va effectuer quelques rappels sur les mesures de Radon et les ensembles rectifiables.

1.1.1 Espace des mesures de Radon

On se place dans le cadre d'un ouvert $\Omega \subset \mathbb{R}^n$ muni de sa tribu borélienne $\mathcal{B}(\Omega)$, ces définitions et résultats demeurent dans le cas d'un espace métrique X localement compact. Une mesure de Radon est une mesure de Borel (régulière, ce qui est automatique pour une mesure de Borel sur Ω) et finie sur les compacts. Ce qui prend un sens un peu particulier dans le cas vectoriel puisque par définition, une mesure vectorielle est finie.

Définition 1.1 (Mesures de Radon).

1. *Une mesure positive μ sur $(\Omega, \mathcal{B}(\Omega))$ est appelée mesure de Borel. Si μ est une mesure de Borel positive et finie sur les compacts, on dit que μ est une mesure de Radon positive.*
2. *Mesures vectorielles. Soit $\mathcal{A}$ une tribu. On dit que $\mu : \mathcal{A} \to \mathbb{R}^m$ est une mesure vectorielle si $\mu(\emptyset) = 0$ et pour tout $\{E_i\}_{i\in\mathbb{N}}$ avec E_i éléments de $\mathcal{A}$ deux à deux disjoints*
$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$
On définit la variation totale de μ, pour tout borélien E,
$$|\mu|(E) = \sup\left\{\sum_{i=1}^{\infty} |\mu(E_i)| \text{ avec } E_i \text{ boréliens deux à deux disjoints tels que } E = \bigcup_{i=1}^{\infty} E_i\right\} \quad (1.1)$$
3. *Soit μ définie sur $\{$boréliens relativement compacts de $\Omega\} \subset \mathcal{B}(\Omega)$ et à valeurs dans $\mathbb{R}^m$. Si pour tout compact $K \subset \Omega$, $\mu : (K, \mathcal{B}(K)) \to \mathbb{R}^m$ est une mesure vectorielle, on dit que μ est une mesure de Radon.*

Proposition 1.1 (Propriétés de la variation totale, cf. 1.47 p.21 dans [AFP]). *Soit μ une mesure de Radon finie sur Ω à valeurs dans $\mathbb{R}^m$, alors $|\mu|$ est une mesure de Radon positive finie et pour tout ouvert $U \subset \Omega$,*

$$|\mu|(U) = \sup \left\{ \int u \cdot d\mu : u \in \mathrm{C}^0_c(U)^m, |u| \leqslant 1 \right\} .$$

De plus, il existe une unique fonction $\sigma : \Omega \to \mathbb{S}^{m-1}$ dans $\mathrm{L}^1(\Omega, |\mu|)$ telle que $\mu = \sigma\, |\mu|$.

Dans le point 2 de la Définition 1.1, on ne précise pas mesure vectorielle finie car il s'agit d'une conséquence de l'égalité (1.1) (plus particulièrement de l'absolue convergence de la série de droite puisque le membre de gauche ne dépend pas de l'ordre des ensembles considérés). Une mesure vectorielle sur $\mathcal{B}(\Omega)$ est une mesure de Radon finie, tandis qu'une mesure de Radon μ sur Ω n'est pas en général une mesure vectorielle sur Ω. En revanche, lorsque μ est une mesure de Radon sur Ω telle que $\sup \{|\mu|(K) : K \text{ compact } \subset \Omega\} < \infty$, on peut étendre μ en une mesure vectorielle sur Ω. Les mesures de Radon sont "localement" des mesures vectorielles, on va voir que l'espace des mesures de Radon sur Ω à valeurs dans $\mathbb{R}^m$, qu'on note $\mathcal{M}_{loc}(\Omega)^m$, apparaît comme le dual de l'espace $\mathrm{C}^0_c(\Omega, \mathbb{R}^m)$ des fonctions continues à support compact de même que l'espace des mesures de Radon finies sur Ω à valeurs dans $\mathbb{R}^m$, qu'on note $\mathcal{M}(\Omega)^m$, apparaît comme le dual de l'espace $\mathrm{C}^0_o(\Omega, \mathbb{R}^m)$. On va maintenant énoncer le théorème de Riesz afin de préciser les résultats de dualité que l'on vient d'évoquer.

Théorème 1.2 (Théorème de Riesz, cf. 1.54 p.25 dans [AFP]). *Soit $L : \mathrm{C}^0_o(\Omega, \mathbb{R}^m) \to \mathbb{R}$ une forme linéaire continue i.e.*

$$\sup \left\{ L(u) : u \in \mathrm{C}^0_o(\Omega, \mathbb{R}^m),\ \|u\|_\infty \leqslant 1 \right\} < +\infty .$$

Alors il existe une unique mesure de Radon finie $\mu = (\mu_1, \ldots, \mu_m)$ sur Ω telle que

$$L(u) = \sum_{j=1}^{n} \int u_j \, d\mu_j = \int u \cdot d\mu .$$

De plus $\|L\| = |\mu|(\Omega)$.

Théorème 1.3 (Théorème de Riesz, cf. 1.55 p.25 dans [AFP]). *Soit $L : \mathrm{C}^0_c(\Omega, \mathbb{R}^m) \to \mathbb{R}$ une forme linéaire continue i.e. pour tout compact $K \subset \Omega$,*

$$\sup \left\{ L(u) : u \in \mathrm{C}^0_c(\Omega, \mathbb{R}^m),\ |u| \leqslant 1, \text{ supp } u \subset K \right\} < +\infty .$$

Alors il existe une unique mesure de Radon vectorielle $\mu = (\mu_1, \ldots, \mu_n)$ sur Ω telle que

$$L(u) = \sum_{j=1}^{n} \int u_j \, d\mu_j = \int u \cdot d\mu .$$

Attention, la topologie considérée dans le cas de C^0_o est la topologie induite par $\| \cdot \|_\infty$ tandis que concernant C^0_c il ne s'agit pas d'une topologie induite par une norme, mais par une famille de semi-normes. On utilise ici la propriété que pour cette topologie, une forme linéaire L sur C^0_c est continue si et seulement si la restriction de L aux sous-espaces $\mathrm{C}_K = \left\{ \varphi \in \mathrm{C}^0_c : \mathrm{supp} \varphi \subset K \right\}$ est continue pour chaque compact K.
Ces théorèmes de dualité nous invitent à considérer la notion de convergence faible–$*$ sur $\mathcal{M}_{loc}(\Omega)^m$ et $\mathcal{M}(\Omega)^m$ en tant qu'espaces duaux.

Définition 1.2 (Convergence faible–$*$ dans $\mathcal{M}_{loc}(\Omega)^m$, cf. [AFP] définition 1.58 p. 26). *Soit μ et $(\mu_i)_i$ des mesures de Radon sur Ω à valeurs dans $\mathbb{R}^m$. On dit que μ_i converge faiblement–$*$ vers μ, noté $\mu_i \xrightarrow[i \to +\infty]{*} \mu$ si*

$$\int \varphi \cdot d\mu_i \xrightarrow[i \to +\infty]{} \int \varphi \cdot d\mu \quad \textit{pour tout } \varphi \in \mathrm{C}^0_c(\Omega, \mathbb{R}^m) .$$

Remarque 1.1. Si de plus les μ_i sont finies et telles que $\sup_i |\mu_i|(\Omega) < +\infty$ alors μ est finie et μ_i converge faiblement–$*$ vers μ au sens de la convergence faible–$*$ cette fois-ci dans $\mathcal{M}(\Omega)^m$ i.e.

$$\int \varphi \cdot d\mu_i \xrightarrow[i\to+\infty]{} \int \varphi \cdot d\mu \quad \text{pour tout } \varphi \in \mathrm{C}^0_o(\Omega, \mathbb{R}^m)\,.$$

On peut appliquer le théorème de Banach-Alaoglu concernant la compacité faible des suites bornées pour obtenir le théorème de compacité suivant :

Théorème 1.4 (Compacité faible–$*$, cf. [AFP] 1.59 et 1.60 p. 26)**.** *1. Soit $(\mu_i)_i$ une suite de mesures de Radon finies sur $(\Omega, \mathcal{B}(\Omega))$ telles que $\sup_i |\mu_i|(\Omega) < +\infty$, alors il existe une mesure de Radon finie μ et une sous-suite qui converge faiblement–$*$ vers μ.*

2. Soit $(\mu_i)_i$ une suite de mesures de Radon sur $(\Omega, \mathcal{B}(\Omega))$ telles que $\sup_i |\mu_i|(K) < +\infty$ pour tout compact $K \subset \Omega$, alors il existe une mesure de Radon μ et une sous-suite qui converge faiblement–$*$ vers μ.

On vient de voir que grâce au théorème de Riesz, on peut étudier l'espace des mesures de Radon comme un espace dual, si on revient maintenant au point de vue des mesures, comment se comporte la convergence faible–$*$ vis à vis des boréliens ? Commençons par le cas des mesures de Radon positives.

Proposition 1.5 (Cf. [EG92], theorem 1 p. 54)**.** *Soit $(\mu_k)_k$ une suite de mesures positives de Radon sur Ω convergeant faiblement–$*$ vers μ alors*

1. pour tout compact $K \subset \Omega$, $\limsup_k \mu_k(K) \leqslant \mu(K)$ et pour tout ouvert $U \subset \Omega$, $\mu(U) \leqslant \liminf_k \mu_k(U)$.

2. $\lim_k \mu_k(B) = \mu(B)$ pour tout borélien borné $B \subset \Omega$ tel que $\mu(\partial B) = 0$.

En réalité on a mieux, chacune de ces propositions équivaut à la convergence faible-$*$ ([EG92]). On a une propriété analogue dans le cas des mesures de Radon à valeurs vectorielles en ajoutant l'hypothèse de la convergence faible–$*$ de la suite des variations totales $|\mu_i|$. En effet comme le montre l'exemple suivant, la convergence faible–$*$ d'une suite de mesures n'entraîne pas la convergence des variations totales associées.

Exemple 1.1 (Cf. [AFP] exemple 1.63 p. 29). On définit μ_i sur $]0, \pi[$ par $d\mu_i = \sin(ix)\,dx$. On a alors que μ_i converge faiblement–$*$ vers 0 et $|\mu_i|$converge faiblement–$*$ vers $\frac{2}{\pi}\,dx$. En effet, on découpe $\int_{(0,\pi)} \varphi(x)|\sin(ix)|\,dx$ en i intervalles de longueur $\frac{\pi}{i}$ puis sur chaque intervalle $\left(\frac{k\pi}{i}, \frac{(k+1)\pi}{i}\right)$ pour $k = 0 \dots i-1$ on effectue le changement de variables $t = \frac{k\pi+u}{i}$. On obtient pour $\varphi \in \mathrm{C}^0_c(]0,\pi[, \mathbb{R})$,

$$\int_{(0,\pi)} \varphi(x)|\sin(ix)|\,dx = \frac{1}{i}\sum_{k=0}^{i-1}\int_0^{\pi} \varphi\left(\frac{k\pi+u}{i}\right)|\sin u|\,du.$$

par uniforme continuité $\varphi\left(\frac{k\pi+u}{i}\right) = \varphi\left(\frac{k\pi}{i}\right)$ à ε près si i est assez grand. On reconnait alors une somme de Riemann et

$$\int_{(0,\pi)} \varphi(x)|\sin(ix)|\,dx \longrightarrow \int_{(0,\pi)} |\sin(x)|\,dx \cdot \frac{1}{\pi}\int_0^{\pi} \varphi(x)dx = \frac{2}{\pi}\int_0^{\pi} \varphi(x)\,dx$$

Énonçons à présent la propriété correspondante dans le cas vectoriel :

Proposition 1.6 (Cf. [AFP] proposition 1.62(b) p. 27)**.** *Soit $(\mu_k)_k$ une suite de mesures de Radon à valeurs dans $\mathbb{R}^m$ sur Ω et convergeant faiblement–$*$ vers μ. On suppose de plus que $|\mu_i|$ converge faiblement–$*$ vers λ. On a alors $|\mu| \leqslant \lambda$ et pour tout borélien borné B tel que $\lambda(\partial B) = 0$ on a*

$\mu_i(B) \to \mu(B)$.
Un peu plus généralement, pour toute fonction mesurable bornée u dont l'ensemble des discontinuités est λ-négligeable,

$$\int u \, d\mu_i \longrightarrow \int u \, d\mu \, .$$

On verra que ces propriétés seront utiles lorsque l'on s'intéressera à la convergence de varifolds (qui est une convergence faible–$*$ de mesures de Radon).

1.1.2 Ensembles rectifiables

On effectue ici quelques rappels autour de la notion de rectifiabilité. Les ensembles rectifiables donnent un sens à la notion de "presque" régulier, prolongeant ainsi la notion de surface. On peut définir la notion de plan tangent approché à un ensemble rectifiable, et inversement, l'existence d'un plan tangent approché à un ensemble en presque tout point caractérise la rectifiabilité.

Définition 1.3 (Ensemble dénombrablement d–rectifiable)**.** *Un ensemble $M \subset \mathbb{R}^n$ est dénombrablement d-rectifable si*

$$M \subset M_0 \cup \bigcup_{j \in \mathbb{N}} f_j \left(\mathbb{R}^d \right)$$

où $\left\{ \begin{array}{l} \mathcal{H}^d(M_0) = 0 \\ \forall j \in \mathbb{N}, \ f_j : \mathbb{R}^d \to \mathbb{R}^n \textit{ est Lipschitz} \, . \end{array} \right.$
Si de plus $\mathcal{H}^d(M) < +\infty$, M est dit d–rectifiable.

Grâce au théorème d'extension de Whitney, on obtient la caractérisation suivante de la rectifiabilité :

Proposition 1.7 (cf. lemma 1.11, [Sim83])**.** *M est dénombrablement d-rectifiable si et seulement si*

$$M = M_0 \cup \left(\bigcup_{j \in \mathbb{N}} N_j \right)$$

où $\mathcal{H}^d(M_0) = 0$ et pour tout j, N_j est contenu dans une d–sous-variété C^1 S_j de $\mathbb{R}^n$.

Ainsi, lorsque M est un ensemble dénombrablement d-rectifiable, on peut définir un plan tangent $T_x M$ en $\mathcal{H}^d$–presque tout $x \in M$. En effet, avec les notations de la Proposition 1.7, on pose pour $x \in S_j$,

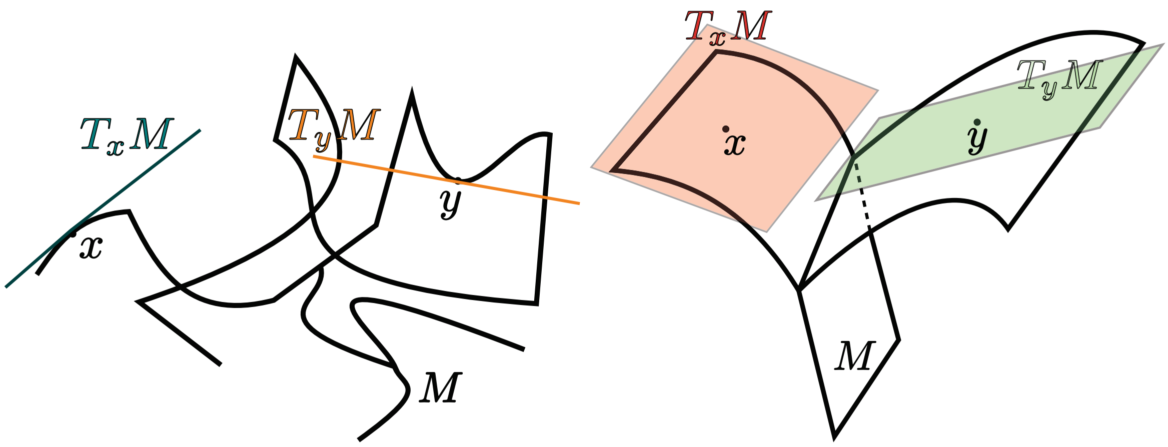
$$T_x M = T_x S_j \, .$$

Mais on aimerait définir le plan tangent de façon intrinsèque, et non liée à un choix particulier de décomposition en parties de sous-variétés. Avant cela, on définit la notion de mesure d–rectifiable.

Définition 1.4 (Mesure d–rectifiable)**.** *Soit μ une mesure de Radon sur $\mathbb{R}^n$. On dit que μ est d–rectifiable s'il existe un ensemble dénombrablement d–rectifiable M et une fonction borélienne positive θ tels que $\mu = \theta \mathcal{H}^d_{|M}$.*

Avec cette définition, un ensemble $M \subset \mathbb{R}^n$ est d–rectifiable si et seulement si la mesure $\mathcal{H}^d_{|M}$ est rectifiable. On définit maintenant de façon intrinsèque la notion de plan tangent approché, pour cela on introduit le changement d'échelle :

$$\varphi_{x,r}(y) = \frac{y - x}{r} \, .$$



(a) Ensemble 1–rectifiable (b) Ensemble 2–rectifiable

FIGURE 1.1 – Exemples d'ensembles rectifiables

On peut alors définir pour une mesure μ définie sur $\Omega \subset \mathbb{R}^n$, la mesure image par le changement d'échelle ${\varphi_{x,r}}_\#\mu$, c'est-à-dire pour tout borélien $A \subset \frac{1}{r}(\Omega - x)$,

$$
{\varphi_{x,r}}_\#\mu(A) = \mu\left({\varphi_{x,r}}^{-1}(A)\right) = \mu\left(x + rA\right) \ .
$$

Définition 1.5 (Plan tangent approché)**.** *Soit un ouvert $\Omega \subset \mathbb{R}^n$ et μ une mesure de Radon sur Ω. On dit que le sous-espace vectoriel P de dimension d est le plan tangent approché à μ avec multiplicité θ au point x si*

$$
\frac{1}{r^d}{\varphi_{x,r}}_\#\mu \xrightarrow[r\to 0_+]{*} \theta\mathcal{H}^d_{|P} \ \textit{dans } \mathbb{R}^n \tag{1.2}
$$

i.e. pour tout $\varphi \in \mathrm{C}^0_c(\mathbb{R}^n)$,

$$
\frac{1}{r^d}\int_\Omega \varphi\left(\frac{y-x}{r}\right)\, d\mu(y) \xrightarrow[r\to 0_+]{} \theta\int_P \varphi(y)\, d\mathcal{H}^d(y) \ .
$$

Soit $M \subset \mathbb{R}^n$ un ensemble de $\mathcal{H}^d$–mesure localement finie (de sorte que $\mathcal{H}^d_{|M}$ est une mesure de Radon), on dit que le sous-espace vectoriel P de dimension d est le plan tangent approché à M avec multiplicité θ au point x si P est le plan tangent approché à $\mu = \mathcal{H}^d_{|M}$ avec multiplicité θ au point x.

Remarque 1.2. Si $M \subset \mathbb{R}^n$ un ensemble de $\mathcal{H}^d$–mesure localement finie, (1.2) avec multiplicité $\theta = 1$ se réécrit : pour tout $\varphi \in \mathrm{C}^0_c(\mathbb{R}^n)$,

$$
\frac{1}{r^d}\int_{\frac{1}{r}(M-x)} \varphi(y)\, d\mathcal{H}^d(y) \xrightarrow[r\to 0_+]{} \int_P \varphi(y)\, d\mathcal{H}^d(y) \ .
$$

L'existence d'un plan tangent approché à M au point x traduit l'idée que les zooms successifs de M autour du point x se concentrent sur un d–plan commun avec une répartition de masse ressemblant à la mesure de Lebesgue d–dimensionnelle sur le d–plan.
Vérifions que dans le cas classique d'une sous-variété de $\mathbb{R}^n$, le plan tangent approché coïncide avec le plan tangent classique :

Exemple 1.2. Soit M sous-variété de dimension d, on suppose que $M = f(U)$ est paramétrée par un plongement $f : U \to \mathbb{R}^n$ de classe C^1. Montrons alors que le plan tangent approché en $x \in M$ coïncide

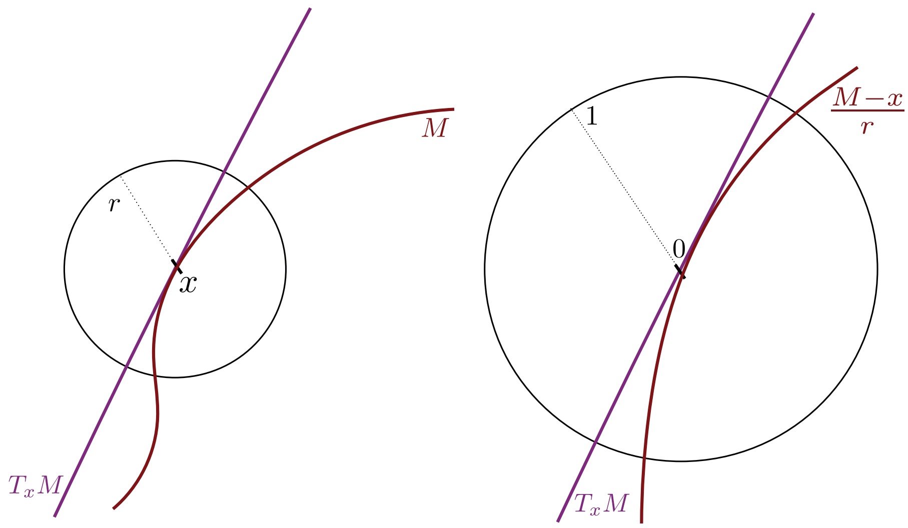


FIGURE 1.2 – Définition du plan tangent approché par “blow up”

avec le plan tangent classique $T_xM = Df(z_0)(\mathbb{R}^d)$ avec $x = f(z_0)$. Soit $\varphi \in \mathrm{C}_c^0(\mathbb{R}^n)$ on a alors par la formule de l'aire

$$\begin{aligned}\frac{1}{r^d}\int_M \varphi\left(\frac{x-y}{r}\right) d\mathcal{H}^d(y) &= \frac{1}{r^d}\int_{\mathbb{R}^d} \varphi\left(\frac{f(z_0)-f(z)}{r}\right) Jf(z)\,dz \\ &= \int_{\mathbb{R}^d} \varphi\left(\frac{f(z_0)-f(z_0+rh)}{r}\right) Jf(z_0+rh)\,dh\end{aligned}$$

qui tend vers

$$\int_{\mathbb{R}^d} \varphi\left(Df(z_0)\cdot h\right) Jf(z_0)dh = \int_{T_xM=\left\{Df(z_0)\cdot h : h\in\mathbb{R}^d\right\}} \varphi(z)\,d\mathcal{H}^d(z)\,,$$

quand $r \to 0$.

Voyons à présent ce qu'il en est pour les mesures rectifiables.

Proposition 1.8 (Cf. Theorem 2.83 et Proposition 2.85, [Sim83])**.** *Soit $\Omega \subset \mathbb{R}^n$ un ouvert et soit $\mu = \theta\mathcal{H}^d_{|M}$ une mesure d-rectifiable, alors*

1. *Pour $\mathcal{H}^d$–presque tout $x \in M$, μ admet un plan tangent approché $P(x)$ avec multiplicité $\theta(x)$.*
2. *Pour $\mathcal{H}^d$–presque tout $x \in M$,*

$$\theta(x) = \Theta(\mu, x) = \lim_{r\to 0}\frac{\mu(B_r(x))}{\omega_d r^d}\,.$$

3. *Si $\mu' = \theta'\mathcal{H}^d_{|M'}$ est une autre mesure d-rectifiable sur Ω avec pour plan tangent approché P' défini $\mathcal{H}^d$–presque partout, alors pour $\mathcal{H}^d$–presque tout $x \in M \cap M'$,*

$$P(x) = P'(x)\,.$$

Remarque 1.3. Si $M \subset \mathbb{R}^n$ est un ensemble rectifiable, on peut facilement construire une multiplicité θ strictement positive $\mathcal{H}^d$–presque partout sur M, localement $\mathcal{H}^d$-intégrable sur M, de sorte que $\mu = \theta\mathcal{H}^d_{|M}$ soit une mesure d–rectifiable et, par la Proposition 1.8, admette un plan tangent approché $\mathcal{H}^d$–presque partout. Le point 3 de la Proposition 1.8 assure que le plan tangent ne dépend pas de la multiplicité θ et on peut alors définir le plan tangent approché à M comme étant le plan tangent approché à $\theta\mathcal{H}^d_{|M}$.

On vient de voir qu'une mesure d-rectifiable possède un plan tangent approché en $\mathcal{H}^d$–presque tout point. La définition 1.5 ne s'applique pas uniquement aux mesures rectifiables, mais à toute mesure de Radon. On peut ainsi se demander quelle est la classe des mesures de Radon possédant un plan tangent approché (de dimension donnée d) en presque tout point. La réponse est donnée par le théorème suivant : ce sont exactement les mesures d–rectifiables. Autrement dit, l'existence d'un plan tangent approché $\mathcal{H}^d$–presque partout est équivalent à la rectifiabilité.

Théorème 1.9 (Cf. [AFP] Theorem 2.83 p.94)**.** *Soit $\Omega \subset \mathbb{R}^n$ un ouvert et soit μ une mesure de Radon sur Ω. Alors μ est d-rectifiable si et seulement si, pour μ–presque tout $x \in \Omega$, μ admet un plan tangent approché avec multiplicité $\theta(x) > 0$.*

Il existe aussi des caractérisations de la rectifiabilité uniquement en terme de densité ($\Theta^d(\mu, x) = \lim\limits_{r\to 0} \dfrac{\mu(B_r(x))}{\omega_d r^d}$), dont certaines sont très difficiles (voir notamment [Mat95] et [Pre87]). Ainsi, les ensembles rectifiables apparaissent comme un bon cadre pour étendre les notions de surface (sous-variété) et plan tangent.

1.2 Varifolds

On s'intéresse dans cette section à la notion de varifold en se concentrant plus particulièrement sur la notion de courbure généralisée d'un varifold. L'espace des varifolds peut être vu comme un espace de surfaces généralisées, en ce sens qu'il contient notamment les sous-variétés et les ensembles dénombrablement rectifiables ; il est de plus muni d'une notion de courbure généralisée. L'espace des varifolds est un espace de mesure de Radon comportant une information spatiale et une information tangentielle. Muni d'une notion de convergence faible (convergence faible–$*$ des mesures de Radon), il possède de bonnes propriétés de compacité vis-à-vis de cette convergence. Dans toute la suite, Ω est un ouvert de $\mathbb{R}^n$. On désigne par $G_{d,n}$ l'ensemble des sous espaces vectoriels de $\mathbb{R}^n$ de dimension d et on munit $G_{d,n}$ de la distance

$$d(T,S) = \left(\sum_{i,j=1}^{N} |P_T^{i,j} - P_S^{i,j}|^2 \right)^{\frac{1}{2}}$$

où $P_T \in \mathcal{M}_n(\mathbb{R})$ désigne la matrice de la projection orthogonale sur le sous espace T dans la base canonique de $\mathbb{R}^n$. Muni de cette métrique, $G_{d,n}$ est un espace compact et $G_d(\Omega) = \Omega \times G_{d,n}$ muni de la métrique produit est localement compact.

Définition 1.6 (Varifolds)**.** *On appelle d-varifold sur Ω une mesure de Radon V sur l'espace $G_d(\Omega)$.*

Remarque 1.4. On a défini les mesures de Radon sur un ouvert $\Omega \subset \mathbb{R}^n$, mais les définitions et propriétés énoncées sont valables plus généralement dans le cadre d'un espace métrique localement compact X (en remplaçant la tribu $\mathcal{B}(\Omega)$ par la tribu borélienne de X, $\mathcal{B}(X)$), ce qui est le cas de $G_d(\Omega)$.

À tout varifold V on associe une mesure de Radon positive, appelée masse, qui est l'image par la projection sur la composante spatiale de la mesure V :

Définition 1.7 (Masse)**.** *Soit V un d-varifold sur $G_d(\Omega)$, on définit sa mesure masse, notée $\|V\|$, par*

$$\|V\|(B) = \pi_\# V(B) = V(\pi^{-1}(B)) \quad \textit{pour tout borélien } B \subset \Omega\,,$$

où $\pi : G_d(\Omega) \to \Omega$, $(x,S) \mapsto x$ est la projection sur Ω.

On va voir que la structure de varifold est assez souple, en ce sens qu'elle peut munir aussi bien des objets très réguliers (surfaces, sous-variétés), faiblement réguliers (ensembles dénombrablement rectifiables) et des objets de nature plus "discrète" (nuages de points, approximation volumiques liées à un maillage ...). On va commencer par s'intéresser à la classe des varifolds rectifiables.

1.2.1 Varifolds rectifiables et autres exemples de varifolds

On donne ici un premier exemple de varifold, construit à partir d'une sous-variété.

Exemple 1.3 (Varifold associé à une sous-variété). Soit M une sous-variété de classe C^1 et de dimension d de $\mathbb{R}^n$ à laquelle on associe la mesure $\mu = \mathcal{H}^d_{|M}$ et l'application

$$\begin{array}{rccl} Id \times P & : & \Omega & \to & G_d(\Omega) \\ & & x & \mapsto & (x, P(x)) \, , \end{array}$$

avec $P(x) = T_x M$ pour $x \in M$ (la valeur de $P(x)$ pour $x \notin M$ n'a pas d'importance pour la suite). On définit le d-varifold $V = v(M)$ associé à la sous-variété M par, pour tout borélien $A \subset \Omega \times G_{d,n}$,

$$V(A) = (Id \times P)_{\#}\mu(A) = \mu(\pi(TM \cap A)) \, ,$$

où $TM = (Id \times P)(M) = \{(x, T_x M) \, : \, x \in M\}$. On a de façon équivalente, pour tout $\varphi \in C^0_c(\Omega \times G_{d,n})$,

$$\int_{(x,S) \in G_d(\Omega)} \varphi(x, S) \, dV(x, S) = \int_{x \in M} \varphi(x, T_x M) \, d\mathcal{H}^d(x) \, .$$

On utilisera la notation abusive :

$$v(M) = \mathcal{H}^d_{|M} \otimes \delta_{T_x M} \, .$$

Soit maintenant M un ensemble dénombrablement d-rectifiable de $\mathbb{R}^n$ et $\theta : M \to]0, +\infty[$ une fonction $\mathcal{H}^d$ intégrable sur M. En posant $\mu = \theta \mathcal{H}^d_{|M}$, on peut définir le d–varifold $V = v(M, \theta)$ associé à (M, θ) de la même façon par pour tout boréliens $A \subset \Omega \times G_{d,n}$,

$$V(A) = (Id \times P)_{\#}\mu(A)$$

où cette fois ci l'application $x \mapsto P(x)$ est l'application μ mesurable qui à x associe le plan tangent approché en x à μ. On notera souvent

$$v(M, \theta) = \theta \mathcal{H}^d \otimes \delta_{T_x M} \, .$$

On appelle un tel varifold $V = v(M, \theta)$ varifold rectifable.

Définition 1.8 (Varifold rectifiable). *Soit $\Omega \subset \mathbb{R}^n$ un ouvert. Un d–varifold rectifiable V sur Ω est une mesure de Radon sur $\Omega \times G_{d,n}$ de la forme $v(M, \theta) = \theta \mathcal{H}^d \otimes \delta_{T_x M}$, i.e. pour tout $\varphi \in C^0_c(\Omega \times G_{d,n})$,*

$$\int_{\Omega \times G_{d,n}} \varphi(x, S) \, dV(x, S) = \int_M \varphi(x, T_x M) \theta(x) \, d\mathcal{H}^d(x)$$

où

- *M est un ensemble dénombrablement d–rectifiable,*
- *$T_x M$ est le plan tangent approché à M au point $x \in M$, qui existe $\mathcal{H}^d$–presque partout,*
- *$\theta : M \to]0; +\infty[$ est borélienne.*

On a dans ce cas là $\|V\| = \mu = \theta \mathcal{H}^d_{|M}$. Lorsque la multiplicité θ est à valeurs entières, on parlera de varifold (rectifiable) entier.

Remarque 1.5. Le fait que la multiplicité θ soit localement $\mathcal{H}^d$–intégrable est une conséquence du fait que $V = v(M,\theta)$ est une mesure de Radon.

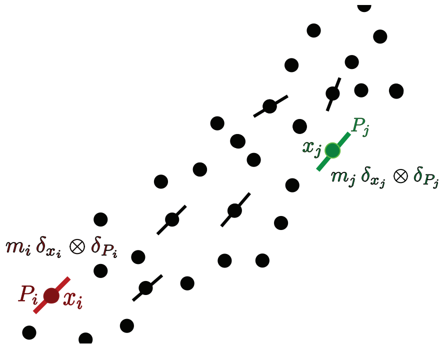
On s'intéresse donc à une classe d'objets qui contient sous-variétés et ensemble rectifiables. Donnons quelques autres exemples munis d'une structure de varifold.

Exemple 1.4 (Varifold associé à un nuage de points).
Soit $\{x_i\}_{i=1\ldots N} \subset \mathbb{R}^n$ un nuage de points, pondéré par les masses $\{m_i\}_{i=1\ldots N}$ et muni des directions $\{P_i\}_{i=1\ldots N} \subset G_{d,n}$. On peut alors définir sur $\mathbb{R}^n \times G_{d,n}$ le varifold

$$V = \sum_{i=1}^{N} m_i \, \delta_{x_i} \otimes \delta_{P_i} \,,$$

de sorte que pour $\varphi \in \mathrm{C}_c^0(\Omega \times G_{d,n})$,

$$\int \varphi \, dV = \sum_{i=1}^{N} \varphi(x_i, P_i) \,.$$



Dans l'exemple précédent d'un nuage de points, la dimension du support spatial est 0, différente de la dimension d de la structure de d–varifold définie sur le nuage de points. On donne à présent deux autres exemples de d–varifolds où la dimension d ne coïncide pas avec la dimension du support spatial du d–varifold considéré.

Exemple 1.5. (voir Figure 1.3) Considérons le plan affine $P = \{z = 0\}$ dans $\mathbb{R}^3$. Le varifold naturellement associé à P est le 2–varifold

$$v(P) = \mathcal{H}^2_{|P} \otimes \delta_{\overrightarrow{P}} \,.$$

Cependant, on peut aussi définir le 1–varifold

$$V_1 = \mathcal{H}^2_{|P} \otimes \delta_D \,,$$

pour une droite vectorielle $D \subset P$ fixée. D'une certaine façon, cela revient à considérer le plan affine P comme l'union des droites affines de direction D contenue dans P. Ces deux varifolds ont la même mesure masse $\mathcal{H}^2_{|P}$ qui est une mesure de Radon 2–rectifiable, pourtant $v(P)$ est un 2–varifold rectifiable au sens de la Définition 1.8 tandis que V_1 est 1–varifold mais pas un 1–varifold rectifiable.

Remarque 1.6. On remarque sur l'exemple précédent (Exemple 1.5) que la rectifiabilité d'un varifold V n'est pas équivalente à la rectifiabilité de la mesure de Radon $\|V\|$ mais est plus forte : non seulement $\|V\|$ doit être rectifiable mais en plus, la partie tangentielle du varifold V doit être cohérente avec la mesure $\|V\|$. Par exemple, si $D \subset \mathbb{R}^n$ est une droite affine, et $D' \in G_{1,n}$ est une direction fixée, le 1–varifold

$$V = \mathcal{H}^1_{|D} \otimes \delta_{D'}$$

est rectifiable si et seulement si D' est la direction de la droite affine D.

Ce type de d–varifold avec un support de dimension supérieure à d apparaît naturellement lorsqu'on veut considérer une représentation volumique d'une courbe ou d'une surface :

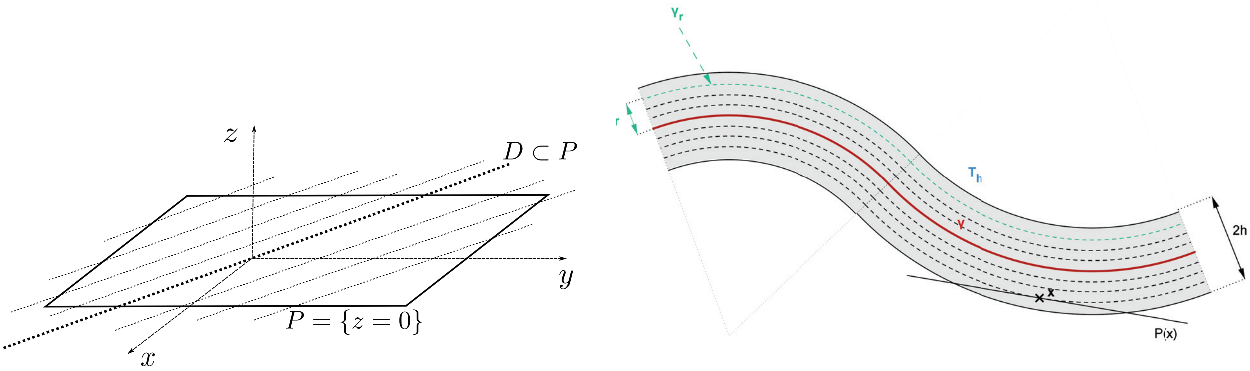
Exemple 1.6. (voir figure 1.3) On considère une courbe régulière $\Gamma \subset \mathbb{R}^2$ paramétrée par γ. On note δ la distance signée à Γ et γ_r la r–ligne de niveau de δ. Soit h tel que δ soit bien définie dans un voisinage h–tubulaire de Γ, $T_h = \{x \,|\, |\delta(x)| = d(x,\Gamma) \leqslant h\}$. On peut alors définir le varifold diffus v_Γ tel que pour tout $\varphi \in \mathrm{C}_c^0(\mathbb{R}^2 \times G_{1,2})$,

$$\int \varphi(x,S) \, dv_\Gamma(x,S) = \int_{T_h} \varphi(x, T_{\pi_\Gamma(x)}\Gamma) \, dx$$

où $\pi_\Gamma : T_h \to \Gamma$ désigne la projection sur Γ. Que l'on peut encore écrire

$$v_\Gamma = \mathcal{L}^2_{|T_h} \otimes \delta_{T_{\pi_\Gamma(x)}\Gamma} \, .$$

On reviendra sur ce type de varifolds dans le Chapitre 5, Section 5.4.



(a) (b)

FIGURE 1.3 – Exemples de varifolds

On va maintenant s'intéresser à la convergence au sens des varifolds.

1.2.2 Convergence au sens des varifolds

On munit l'espace des varifolds de la convergence faible–$*$ au sens des mesures de Radon sur $\Omega \times G_{d,n}$.

Définition 1.9 (Convergence au sens des varifolds)**.** *Soit $\Omega \subset \mathbb{R}^n$ un ouvert. On dit que la suite $(V_i)_i$ de d–varifolds converge dans Ω vers le d–varifold V si*

$$V_i \xrightarrow[i\to\infty]{*} V \, ,$$

i.e., pour tout $\varphi \in \mathrm{C}^0_c(\Omega \times G_{d,n})$,

$$\int_{\Omega \times G_{d,n}} \varphi(x,S) \, dV_i(x,S) \xrightarrow[i\to\infty]{} \int_{\Omega \times G_{d,n}} \varphi(x,S) \, dV(x,S) \, .$$

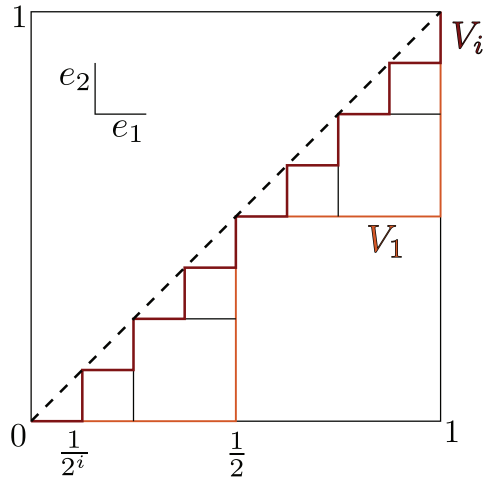
On insiste sur la différence entre la convergence au sens des varifolds et la convergence au sens de la masse

$$\|V_i\| \xrightarrow[i\to\infty]{*} \|V\| \, ,$$

qui est une conséquence de la convergence au sens des varifolds, mais qui n'est absolument pas équivalente. Reprenons à ce sujet l'exemple de [Mor09] p.110 :

Exemple 1.7. Soit $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$, e_1 la direction $\{x_1 = 0\}$, e_2 la direction $\{x_2 = 0\}$ et u la direction $\Delta = \{x_1 = x_2\}$. On définit sur Ω la suite de 1–varifolds $(V_i)_i$ par

$$V_i = \sum_{k=1}^{2^i} \mathcal{H}^1_{\big|\left[\frac{k-1}{2^i},\frac{k}{2^i}\right)\times\left\{\frac{k-1}{2^i}\right\}} \otimes \delta_{e_1} + \sum_{k=1}^{2^i} \mathcal{H}^1_{\big|\left\{\frac{k}{2^i}\right\}\times\left[\frac{k-1}{2^i},\frac{k}{2^i}\right)} \otimes \delta_{e_2} \,.$$



On peut facilement montrer que V_i converge faiblement–$*$ vers le 1–varifold

$$V = \sqrt{2}\mathcal{H}^1_{|\Delta} \otimes \left(\frac{1}{2}\delta_{e_1} + \frac{1}{2}\delta_{e_2} \right) \,,$$

et non vers le 1–varifold rectifiable associé à Δ,

$$v(\Delta) = \mathcal{H}^1_{|\Delta} \otimes \delta_u \,.$$

On peut dès maintenant énoncer un résultat de compacité qui est une simple conséquence du théorème de Banach-Alaoglu et du fait que pour un d–varifold V sur Ω, $V(\Omega \times G_{d,n}) = \|V\|(\Omega)$.

Théorème 1.10. *Soit $\Omega \subset \mathbb{R}^n$ un ouvert et soit $(V_i)_i$ une suite de d–varifolds sur Ω. Si*

$$\sup_{i<+\infty} \|V_i\|(\Omega) < +\infty \,,$$

alors il existe une sous-suite $(V_{i_k})_k$ et un d–varifold V tels que

$$V_{i_k} \xrightarrow[k\to+\infty]{*} V \,,$$

avec de plus $\|V\|(\Omega) \leqslant \liminf_k \|V_{i_k}\|(\Omega)$.

L'inconvénient de ce théorème est qu'on sait peu de choses sur le varifold limite. En particulier, même si les varifolds V_i sont rectifiables, V n'a aucune raison de l'être (comme on l'a vu dans l'Exemple 1.7). Or, si par exemple le varifold limite V est la solution d'un problème de minimisation dans l'ensemble des surfaces, on veut au moins assurer que V est rectifiable. On a ainsi besoin d'un théorème de compacité qui assure en plus la rectifiabilité et le caractère entier du varifold limite. Un tel théorème existe (cf. Théorème 1.13) et est dû à W. K. Allard ([All72]). Mais pour cela, il nous faut d'abord introduire la variation première qui est une notion de courbure généralisée dans l'espace des varifolds.

1.2.3 Variation première d'un varifold

Commençons par rappeler la définition du vecteur courbure moyenne dans le cas régulier et le théorème de la divergence :

Définition 1.10. *Soit M une sous variété de dimension d et de classe C^2 de $\mathbb{R}^n$. Le vecteur courbure moyenne H est défini localement par*

$$H(x) = -\sum_{j=1}^{n-d} (\mathrm{div}_{T_x M} \nu_j(x)) \nu_j(x)$$

où ν_j sont des champs de vecteurs réguliers définis sur M qui forment une base locale orthonormale de $N_x M = (T_x M)^\perp$ et l'opérateur div_P est défini par,

$$\mathrm{div}_P X(x) = \sum_{j=1}^{n} \langle \nabla^P X_j(x), e_j \rangle = \sum_{j=1}^{n} \langle \Pi_P(\nabla X_j(x)), e_j \rangle \,,$$

avec $P \in G_{d,n}$, $(e_1, \ldots, e_n)$ base canonique de $\mathbb{R}^n$ et $X = (X_1, \ldots, X_n) \in \mathrm{C}^1_c(\Omega, \mathbb{R}^n)$.

On peut réécrire le vecteur H comme $H = \text{tr } B_x$ où

$$\begin{array}{llll} B_x : & T_xM \times T_xM & \longrightarrow & N_xM \\ & (v,w) & \mapsto & B_x(v,w) = (d_xv(w))^\perp \quad \text{i.e. la projection orthogonale de } (d_xv(w)) \text{ sur } N_xM \end{array}$$

est la seconde forme fondamentale de M en x. On peut maintenant énoncer le théorème de la divergence

Théorème 1.11 (Théorème de la divergence)**.** *Soit M une sous variété de $\mathbb{R}^n$ de dimension d et de classe C^2, Ω un ouvert régulier et $X \in \mathrm{C}^1_c(\Omega, \mathbb{R}^n)$, on a alors*

$$\int_{M\cap\Omega} \mathrm{div}_M X \, d\mathcal{H}^d = -\int_{M\cap\Omega} < X, H > \, d\mathcal{H}^d \, .$$

Le Théorème de la divergence (Théorème 1.11) fournit une caractérisation variationnelle de la courbure qui permet de définir en un sens faible la courbure moyenne dans l'espace des varifolds :

Définition 1.11 (Variation première ou courbure généralisée)**.** *Soit V un d–varifold défini sur un ouvert $\Omega \subset \mathbb{R}^n$. La forme linéaire δV*

$$\begin{array}{llll} \delta V & : & \mathrm{C}^1_c(\Omega, \mathbb{R}^n) & \longrightarrow \mathbb{R} \\ & & X & \mapsto \int_{\Omega \times G_{d,n}} \mathrm{div}_P X(x) \, dV(x,P) \end{array}$$

est une forme linéaire continue sur $\mathrm{C}^1_c(\Omega, \mathbb{R}^n)$, c'est-à-dire une distribution d'ordre 1.

La variation première est définie pour un varifold quelconque. Lorsque la variation première δV d'un d–varifold V est plus régulière : par exemple lorsque δV s'étend en une forme linéaire continue sur $\mathrm{C}^0_c(\Omega, \mathbb{R}^n)$, on dit que V est à variation première localement bornée et on va voir dans le Théorème 1.12 que la régularité de la variation première est liée à la régularité du varifold.

Définition 1.12 (Variation première localement bornée)**.** *On dit qu'un varifold V est à variation première localement bornée lorsque pour tout compact $K \in \Omega$, il existe une constante c_K telle que pour tout $X \in \mathrm{C}^1_c(\Omega, \mathbb{R}^n)$, si* $\mathrm{supp}\, X \subset K$,

$$|\delta V(X)| \leqslant c_K \sup_K |X| \, .$$

Autrement dit, δV s'étend en une forme linéaire continue sur $\mathrm{C}^0_c(\Omega, \mathbb{R}^n)$.

Par le théorème de Riesz, si V est un d–varifold à variation première bornée, il existe une mesure de Radon (encore notée δV) telle que pour tout $X \in \mathrm{C}^0_c(\Omega, \mathbb{R}^n)$

$$\delta V(X) = \int_\Omega X \cdot \delta V \, .$$

Par le théorème de Radon Nikodym, il existe alors $H \in \mathrm{L}^1_{loc}(\Omega, \|V\|)$ et une mesure δV_s étrangère à $\|V\|$ tels que

$$\delta V = -H\|V\| + \delta V_s \, .$$

Définition 1.13 (Courbure moyenne généralisée et courbure singulière)**.** *Soit V un d–varifold à variation première localement bornée dans un ouvert $\Omega \subset \mathbb{R}^n$. Soit $H \in \mathrm{L}^1_{loc}(\Omega, \|V\|)$ et δV_s la mesure étrangère à $\|V\|$ donnés par la décomposition de Radon Nikodym de δV par rapport à $\|V\|$:*

$$\delta V = -H\|V\| + \delta V_s \, .$$

On appellera H courbure moyenne généralisée et δV_s courbure singulière.

Lorsque $V = v(M,1)$ est le d–varifold associée à une d–sous-variété M,

$$\delta V = -H\mathcal{H}^d_{|M} \, ,$$

et H est le vecteur courbure moyenne classique. On va voir quelques exemples où la variation première comporte une partie singulière.

1.2.4 Exemples : courbure singulière et influence de la multiplicité

Exemple 1.8 (Courbure singulière aux extrémités d'un segment). On va commencer avec un exemple très simple : un segment $M = [AB]$ dans $\mathbb{R}^2$ qu'on paramètre par $\gamma(t) = (1-t)a + tb$. On calcule la courbure de $V = v(M, 1)$. Pour $X \in \mathrm{C}_c^1(\mathbb{R}^2, \mathbb{R}^2)$, on note P la direction du segment (la direction de $\gamma'(t)$) et $\eta = \frac{\gamma'(t)}{|\gamma'(t)|}$.

$$\begin{aligned}
\delta V(X) &= \int_M \mathrm{div}_M X \, d\sigma = \int_0^1 \mathrm{div}_P X(\gamma(t)) |\gamma'(t)| \, dt \\
&= \int_0^1 < \nabla X(\gamma(t)), \eta > \eta |\gamma'(t)| \, dt = \int_0^1 \frac{d}{dt}(X(\gamma(t))) \, dt \eta \\
&= (X(b) - X(a))\eta \, .
\end{aligned}$$

On en déduit que $\delta V = (\delta_b - \delta_a)\eta$.

Exemple 1.9 (Courbure au niveau d'un nœud). On considère maintenant des demi-droites se rejoignant en un point, et on calcule la courbure du nœud obtenu c'est-à-dire la courbure du varifold $V_1 = v(N_1, 1)$. Pour cela, on va calculer la courbure d'une demi-droite partant de O et dirigée par un

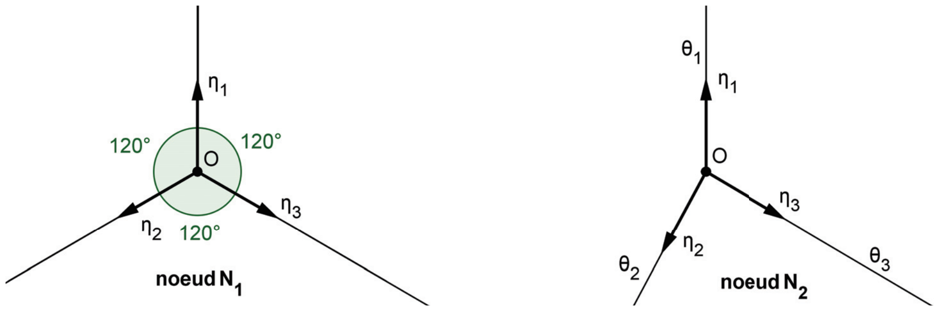


Figure 1.4 – Courbure au niveau d'un nœud

vecteur u unitaire, et de multiplicité constante θ_0. Le calcul est similaire à celui effectué dans le cas d'un segment. On paramètre la demi-droite par $\gamma(t) = tu$ (de sorte que $u = \dfrac{\gamma'(t)}{|\gamma'(t)|}$) et on calcule pour $X \in \mathrm{C}_c^1(\mathbb{R}^2, \mathbb{R}^2)$,

$$\begin{aligned}
\delta V(X) &= \int_0^\infty (< DX(\gamma(t)) \, u, u >) \, \theta_0 |\gamma'(t)| \, dt \\
&= \theta_0 \int_0^\infty \big(< DX(\gamma(t)) \, \gamma'(t), u >\big) \, dt \\
&= \theta_0 \left\langle \int_0^\infty \frac{d}{dt}(X(\gamma(t))) \, dt, u \right\rangle = -\theta_0 < X(0), u > = < \delta_0(X), \theta_0 u > \, .
\end{aligned}$$

On peut donc en déduire la courbure de $V_1 = v(N_1, 1)$, (Cf. Figure 1.4),

$$\delta V_1 = -\delta_0 \underbrace{(\eta_1 + \eta_2 + \eta_3)}_{0} = 0 \, .$$

De même on en déduit la courbure du varifold $V_2 = v(N_2, \theta)$ avec $\theta = \theta_i$ constante sur la demi-droite i (Cf. Figure 1.4)

$$\delta V_2 = -(\theta_1 \eta_1 + \theta_2 \eta_2 + \theta_3 \eta_3)\delta_0 \, .$$

On va maintenant détailler un dernier exemple qui illustre l'influence de la multiplicité sur la courbure. En effet, comme on vient de le voir avec le calcul de la courbure du nœud N_2, la multiplicité impacte la courbure. En effet, $\delta V_2 = -(\theta_1\eta_1 + \theta_2\eta_2 + \theta_3\eta_3)\delta_0$ est complètement dépendante de la multiplicité sur chaque demi-droite. On va maintenant voir que la multiplicité peut aussi avoir un impact sur la partie absolument continue de la courbure.

Exemple 1.10 (Droite à multiplicité variable). On considère la droite $D = \{y = 0\} \subset \mathbb{R}^2$, la fonction multiplicité $\theta(x, y) = x^2 + 1$ et on calcule la courbure du varifold $V = v(D, \theta)$. Pour $X = (X_1, X_2) \in \mathrm{C}_c^1(\mathbb{R}^2, \mathbb{R}^2)$:

$$\begin{aligned}
\delta V(X) &= \int_D \mathrm{div}_D X \, \theta d\mathcal{H}^1 = \int_{x\in\mathbb{R}} \mathrm{div}_D X(x,0)\,(x^2+1)\,dx \\
&= \int_{\mathbb{R}} \partial_1 X_1(x,0)\,(x^2+1)\,dx = \underbrace{\left[X_1\,(x^2+1)\right]}_{0 \text{ car } X\in \mathrm{C}_c^0} - \int_{\mathbb{R}} X_1(x,0)\,2x\,dx \\
&= -\int_D X(x,y)\cdot H(x,y)\theta(x,y)\,d\mathcal{H}^1(x,y)
\end{aligned}$$

avec $H(x, y) = (\frac{2x}{x^2+1}, 0)$.

La courbure n'est pas nulle ni constante, et pourtant ce varifold rectifiable est construit à partir d'une droite. Il faudra donc être attentif quand on étudiera des varifolds dont la courbure n'est pas constante. On peut de plus remarquer qu'on n'a pas $H \perp T_z D$ mais au contraire, $H \in T_z D$.

Attention, la mesure δV_s peut être bien plus compliquée que dans les quelques exemples présentés. Notamment δV_s n'est pas nécessairement portée par un ensemble de dimension un de moins que l'ensemble rectifiable, mais peut être portée par un ensemble de dimension intermédiaire par exemple. On peut trouver un tel exemple détaillé dans [Man93] p.33-34.

1.2.5 Variation première et rectifiabilité

Commençons avec un exemple très simple.

Exemple 1.11. Soit D une droite dans $\mathbb{R}^n$ et V le varifold porté par D avec pour direction D' : $V = \mathcal{H}^d_D \otimes \delta_{D'}$. Le varifold V est à variation première (localement) bornée si et seulement si $D = D'$.

Régularité de la variation première (de la courbure généralisée) et régularité du varifold sont fortement liées :

Théorème 1.12 (Théorème de rectifiabilité, cf. [Sim83] p. 243)**.** *Soit V un d–varifold dans un ouvert $\Omega \subset \mathbb{R}^n$. Supposons*

– *pour $\|V\|$–presque tout $x \in \Omega$,*

$$\liminf_{r\to 0_+} \frac{\|V\|(B_r(x))}{r^d} > 0\,,$$

– *la variation première δV de V est localement bornée ;*

alors V est un d–varifold rectifiable.

On va maintenant énoncer un théorème de compacité dans l'ensemble des varifolds, qui est une conséquence du théorème de rectifiabilité (Théorème 1.12), du théorème de compacité des mesures de Radon et de la semi-continuité inférieure de la variation totale par rapport à la convergence faible–∗, sauf pour ce qui concerne les varifolds entiers : le fait que le caractère entier est conservé est un théorème en soi (dû à W. K. Allard [All72]).

Théorème 1.13 (Théorème de compacité d'Allard)**.** *Soit $(V_j)_j$ une suite de varifolds d-rectifiables sur un ouvert Ω, à variations premières localement bornées dans Ω et tels que $\theta_j \geqslant 1$ $\|V_j\|$ presque partout. Supposons que pour tout ouvert $W \subset\subset \Omega$,*

$$\sup_j \{\|V_j(W)\| + |\delta V_j|(W)\} \leqslant c(W) < +\infty \,. \tag{1.3}$$

Alors il existe une sous suite $(V_{j_n})_n$ qui converge (faiblement–$*$) vers un varifold d–rectifiable V à variation première localement bornée dans Ω et vérifiant de plus $\theta \geqslant 1$ et

$$|\delta V|(W) \leqslant \liminf_{n\to\infty} |\delta V_{j_n}|(W) \quad \forall\, W \subset\subset \Omega \,.$$

Si de plus les varifolds V_j sont entiers alors V est entier lui aussi.

On va être amené dans cette thèse à chercher des versions alternatives à ces deux théorèmes (Théorèmes 1.12 et 1.13).

La suite de ce chapitre est dédiée à la description de l'articulation globale de la thèse et à l'énoncé des principaux résultats démontrés dans la thèse.

1.3 Varifolds discrets

Comme on l'a déjà mentionné, on veut munir les objets réguliers (surfaces, sous-variétés, ensembles rectifiables) ainsi que leurs discrétisations (nuages de points, triangulations, discrétisations de type volumique) d'une structure de varifold, afin de pouvoir les étudier dans un même espace, muni des propriétés de compacité et de la notion de courbure généralisée décrites dans la section précédente. Munir des objets réguliers, rectifiables, d'une structure de varifold se fait de façon naturelle (comme expliqué dans l'exemple 1.3) ; c'était d'ailleurs une des motivations de l'introduction des varifolds : définir une notion de surface généralisée.

Le cadre classique de la théorie géométrique de la mesure a déjà permis de définir une notion de "mesure de courbure" (curvature measure) unifiée pour les surfaces et leurs approximations discrètes, basée sur la notion de cycle normal [CSM06]. Valable tout d'abord pour des approximations de type triangulation [Mor08], cette notion a été étendue récemment [CCLT09] à des discrétisations plus générales (nuages de points par exemple). La proximité entre une surface et son approximation est mesurée en termes de distance de Hausdorff tandis que la proximité entre les mesures de courbure est estimée en termes de la "Bounded Lipschitz distance" qui est une notion de distance proche de la distance de Wasserstein.

Afin de mesurer la proximité entre deux objets géométriques donnés par des surfaces triangulées, A. Trouvé et N. Charon [CT13] munissent les triangulations d'une structure de varifold et définissent une distance dans l'espace des varifolds, à la fois calculable d'un point de vue numérique et adaptée à la comparaison de surfaces.

On définit dans le Chapitre 2 des structures de varifolds sur des objets discrets moins réguliers que les triangulations (qui sont des ensembles rectifiables et donc naturellement munis d'une structure de varifold rectifiable). On s'intéresse en particulier à une discrétisation volumique associée à une suite de maillages de l'espace ambiant, définie dans l'esprit de la théorie des varifolds. On étudie alors la capacité de cette structure discrète à approcher les objets réguliers : pour cela, on a besoin d'une notion de distance dans l'espace des varifolds. Cette distance apparaît naturellement dans les Chapitres 3 et 5, lorsque l'on s'intéresse à la convergence d'énergies définies sur les varifolds : la convergence d'une suite de varifolds en termes de la distance Bounded Lipschitz assure (sous certaines conditions liant les paramètres en jeu) la convergence de ces énergies. Une fois étudiée la capacité à approcher les varifolds rectifiables, on s'intéresse à la courbure généralisée de tels objets. Malheureusement, la notion classique de variation première s'avère inadaptée, la convergence des objets discrets vers une surface régulière n'entraîne pas en général la convergence des variations premières : la condition du théorème de compacité d'Allard (1.3) ne peut être satisfaite (à moins d'adapter le maillage à chaque varifold... ce qui n'est pas souhaitable d'un point de vue pratique). C'est ce qui motive les Chapitres 3 à 5 où on étudie une caractérisation de la rectifiabilité et une notion de courbure qui soient adaptées aux structures de varifolds non rectifiables (associées à des objets discrets) et cohérentes avec la convergence au sens des varifolds et/ou au sens de la distance bounded Lipschitz (ces deux convergences sont fortement liées).

1.3.1 La distance bounded Lipschitz

La distance bounded Lipschitz, aussi connue sous le nom de flat metric peut être vue comme une distance 1–Wasserstein modifiée.

Définition 1.14 (Distance 1–Wasserstein)**.** *Si μ et ν sont deux mesures de Radon sur un espace métrique localement compact X, on définit la distance 1–Wasserstein*

$$W_1(\mu,\nu) = \sup \left\{ \left| \int_X \varphi \, d\mu - \int_X \varphi \, d\nu \right| \; : \; \varphi \in \mathrm{Lip}(X) \; \textit{et} \; \mathrm{lip}(\varphi) \leqslant 1 \right\} .$$

Le problème est que si l'on considère deux mesures finies μ et ν telles que $\mu(X) \neq \nu(X)$, la distance 1–Wasserstein entre les deux mesures est infinie. C'est pourquoi on s'intéresse plutôt à la distance bounded Lipschitz :

Définition 1.15 (Bounded Lipschitz distance ou flat distance)**.** *Si μ et ν sont deux mesures de Radon sur un espace métrique localement compact X, on définit la distance*

$$\Delta^{1,1}(\mu,\nu) = \sup \left\{ \left| \int_X \varphi \, d\mu - \int_X \varphi \, d\nu \right| \; : \; \varphi \in \mathrm{Lip}(X), \; \|\varphi\|_\infty \leqslant 1 \; \textit{et} \; \mathrm{lip}(\varphi) \leqslant 1 \right\} .$$

Remarque 1.7. Tout comme les distances de Wasserstein (cf. [Vil09]), la flat distance a une formulation duale [PR14] (moins connue).

1.3.2 Varifolds volumiques discrets et approximations des varifolds rectifiables

Étant donné un maillage $\mathcal{K}$ d'un ouvert $\Omega \subset \mathbb{R}^n$, où $\mathcal{K}$ désigne l'ensemble des cellules du maillage, on appelle (d–)varifold volumique discret un varifold (i.e. une mesure de Radon sur $\Omega \times G_{d,n}$) de la forme

$$V = \sum_{K \in \mathcal{K}} \frac{m_k}{\mathcal{L}^n(K)} \mathcal{L}^n_{|K} \otimes \delta_{P_K} .$$

On a ainsi une bijection entre l'ensemble des varifolds volumiques discrets associés au maillage $\mathcal{K}$ et l'ensemble

$$\{(m_K, P_K) \in \mathbb{R}_+ \times G_{d,n} \mid K \in \mathcal{K}\} .$$

Question 1.1. Si on se donne une suite de maillages $(\mathcal{K}_i)_i$ dont le pas $\delta_i = \sup_{K \in \mathcal{K}_i} \mathrm{diam} K$ tend vers 0, et que l'on note $\mathcal{A}^{\delta_i}(\mathcal{K}_i)$ l'ensemble des varifolds volumiques discrets associés au maillage $\mathcal{K}_i$, les ensembles $\mathcal{A}^{\delta_i}(\mathcal{K}_i)$ permettent-ils d'approcher l'ensemble des d–varifolds rectifiables ?

À chaque d–varifold V, on peut associer une projection $V_{\mathcal{K}}$ de V sur l'ensemble des varifolds volumiques discrets en définissant $m_k = \|V\|(K)$ et $P_K = \operatorname*{arg\,min}_{P \in G_{d,n}} \int_{(x,S) \in K \times G_{d,n}} \|P - S\| \, dV(x,S)$ puis

$$\mathcal{V}_K = \sum_{K \in \mathcal{K}} \frac{m_k}{\mathcal{L}^n(K)} \mathcal{L}^n_{|K} \otimes \delta_{P_K} , \tag{1.4}$$

ce qui fournit un candidat naturel $(V_{\mathcal{K}_i})_i$ pour approcher un varifold rectifiable donné V. On a obtenu les résultats suivants d'approximation.

Théorème. 2.1. [cf. p.41] *Soit Ω un ouvert de $\mathbb{R}^n$ et $(\mathcal{K}_i)_i$ une famille de maillages de Ω dont le pas tend vers 0 :*

$$\delta_i = \sup_{K \in \mathcal{K}_i} \operatorname{diam}(K) \xrightarrow[i\to+\infty]{} 0 \, .$$

Alors,

- *pour tout d–varifold rectifiable V sur Ω, les projections successives $V_{\mathcal{K}_i}$ de V sur $\mathcal{A}_{\delta_i}(\mathcal{K}_i)$, définies par (1.4), convergent faiblement–$*$ vers V ;*
- *si $V = v(M,\theta)$ est un d–varifold rectifiable satisfaisant pour un choix de C et $\beta > 0$ et pour $\|V\|$–presque tout $x, y \in \Omega$,*
$$\|T_x M - T_y M\| \leqslant C|x-y|^\beta \, , \tag{1.5}$$
alors
$$\Delta^{1,1}(V, V_{\mathcal{K}_i}) \leqslant \left(\delta_i + 2C\delta_i^\beta\right) \|V\|(\Omega) \, ;$$
- *si $\mathcal{K}$ est un maillage de pas δ et si $\mathcal{A}_m^{C,\beta}$ désigne l'ensemble des d–varifolds rectifiables de masse inférieure à m et satisfaisant (1.5), on peut reformuler le point précédent en termes de distance de Hausdorff (dans l'espace des varifolds et associée à la métrique $\Delta^{1,1}$) :*
$$\begin{aligned} d_{\mathcal{H}}^{asym}(\mathcal{A}_m^{C,\beta}, \mathcal{A}_\delta(\mathcal{K})) &= \sup_{V \in \mathcal{A}_m^{C,\beta}} \inf_{W \in \mathcal{A}_\delta(\mathcal{K})} \Delta^{1,1}(V,W) \\ &\leqslant \left(\delta + 2C\delta^\beta\right) m \, . \end{aligned}$$
Il ne s'agit pas exactement de la distance de Hausdorff mais seulement d'une partie puisqu'on s'intéresse à l'approximation des varifolds rectifiables par les varifolds volumiques discrets et non l'inverse.

Remarque 1.8. Avec la définition de varifold volumique choisie ici, on ne peut pas espérer avoir de l'uniformité dans la qualité d'approximation des varifolds rectifiables sans une condition de régularité sur le plan tangent. En effet, l'espace Ω est discrétisé, mais la grassmannienne $G_{d,n}$ ne l'est pas avec cette discrétisation volumique, et la qualité d'approximation de la partie tangente du varifold rectifiable passe alors par la qualité de la discrétisation spatiale et la régularité de la partie tangente vis-à-vis de l'espace. Cependant, on a seulement besoin que la condition (1.5) soit satisfaite localement dans chaque cellule, et même seulement dans "presque toutes les cellules" au sens où

$$\sum_{\substack{K \in \mathcal{K}_i \\ \text{(1.5) n'est pas vérifiée dans } K}} \|V\|(K) \xrightarrow[i\to\infty]{} 0 \, .$$

Cette condition permet d'inclure des ensembles $\mathrm{C}^{1,\beta}$–rectifiables dont l'ensemble singulier n'est pas trop complexe, par exemple un nombre fini de de courbes $\mathrm{C}^{1,\beta}$ se joignant en un point. Ainsi l'hypothèse (1.5) peut être affaiblie.

1.3.3 Variation première d'un varifold volumique discret

On calcule ensuite explicitement la variation première d'un varifold volumique discret. Comme le plan tangent et la densité de masse sont constants à l'intérieur de chaque cellule, on s'attend à obtenir une courbure entièrement concentrée sur les faces du maillage. L'expression exacte est donnée dans la Proposition 2.2. On étudie alors cette quantité sur un exemple simple. On prend une droite dans $\mathbb{R}^2$ de direction $(1,1)$, et on projette le varifold associé V sur un maillage cartésien $\mathcal{K}$, obtenant ainsi le varifold volumique discret $V_{\mathcal{K}}$. Mais, lorsque le pas du maillage tend vers 0, la variation première $\delta V_{\mathcal{K}}$ de $V_{\mathcal{K}}$ explose (en variation totale), alors que $V_{\mathcal{K}}$ converge faiblement–$*$ vers V et $\delta V = 0$.

En réalité, le support d'un varifold volumique discret dans $\mathbb{R}^n$ est un objet n–dimensionnel et n–rectifiable. Ainsi, dans $\mathbb{R}^n$, un tel objet possède un bord $n-1$–dimensionnel, contrairement à la courbe ou la surface compacte dont il est la projection. Or la variation première "voit" ce bord, qui se trouve être hautement irrégulier (fractal) dans le cas d'un varifold volumique discret associé à un maillage cartésien (dans $\mathbb{R}^2$, on peut penser à la pixellisation d'une courbe). Ces considérations nous amènent à chercher de nouveaux outils, mieux adaptés au cas des varifolds non rectifiables associés à des objets discrets. Dans le Chapitre 3, on étudie le problème de la rectifiabilité d'un varifold obtenu comme limite de varifolds a priori non rectifiables (associés à des discrétisations par exemple) afin de répondre à la question suivante :

Question 1.2. Comment assurer qu'un varifold, obtenu comme limite de varifolds a priori non rectifiables, soit rectifiable ?

Dans les Chapitres 4 et 5, on étudie une question similaire mais du point de vue de la courbure :

Question 1.3. Comment assurer qu'un varifold, obtenu comme limite de varifolds quelconques, a priori à variations premières non bornée (varifolds associés à des nuages de points par exemple), soit à variation première bornée ?

1.4 Conditions quantitatives de rectifiabilité dans l'espace des varifolds

Comme on l'a dit, l'objet du Chapitre 3 est de répondre à la question 1.2 soulevée dans le paragraphe précédent :

Question. 1.2 Comment assurer qu'un varifold, obtenu comme limite de varifolds a priori non rectifiables, soit rectifiable ?

On cherche des conditions, portant sur une suite de d–varifolds $(V_i)_i$, convergeant faiblement–$*$ vers un d–varifold, qui assurent que V est d–rectifiable. On cherche des conditions suffisamment faibles pour être valides dans le cas où $(V_i)_i$ est la suite des varifolds volumiques discrets $(V_{\mathcal{K}_i})_i$ obtenue par projection d'un d–varifold rectifiable V sur une suite de maillages $(\mathcal{K}_i)_i$ dont le pas tend vers 0.

1.4.1 Conditions quantitatives de rectifiabilité

Il existe diverses façons de caractériser la rectifiabilité d'un ensemble M ou d'une mesure de Radon μ dans $\mathbb{R}^n$. En termes d'existence de plan tangent approché μ–presque partout, comme énoncé dans le Théorème 1.9. Il existe aussi une caractérisation en termes de d–densité, due à A. Besicovitch ([Bes28] [Bes38] [Bes39]) dans le cas $d = 1$ et P. Mattila [Mat75] dans le cas général :

Théorème 1.14. *Soit $E \subset \mathbb{R}^n$ un borélien de mesure de Hausdorff $\mathcal{H}^d(E)$ finie. Alors E est d–rectifiable si et seulement si pour $\mathcal{H}^d$–presque tout $x \in E$,*

$$\Theta^d(E, x) = \lim_{r \to 0} \frac{\mathcal{H}^d(E \cap B_r(x))}{\omega_d r^d} = 1 \, .$$

Ce résultat a été par la suite amélioré par D. Preiss [Pre87] (le fait que s ci-dessous est forcément un entier est dû à J. Marstrand) :

Théorème 1.15. *Soit μ une mesure de Radon positive dans $\mathbb{R}^n$. S'il existe $s > 0$ tel que pour μ–presque tout $x \in \mathbb{R}^n$,*

$$0 < \lim_{r \to 0} \frac{\mu(B_r(x))}{r^s} \text{ existe } < +\infty \,,$$

alors s est entier et μ est s–rectifiable.

On pourrait ajouter le théorème de structure de Besicovitch-Federer, qui caractérise les parties rectifiable et non rectifiable d'un ensemble en termes de projections sur les d–plans (cf. théorème 2.65 [AFP]). Cependant, ces caractérisations sont essentiellement qualitatives, tandis que pour le problème qu'on se pose, la suite de varifolds convergeant faiblement–$*$ est constituée de varifolds a priori non rectifiables, mais dont on voudrait en quelque sorte contrôler la non-rectifiabilité, afin d'obtenir la rectifiabilité du varifold limite. C'est pourquoi des conditions plus quantitatives de rectifiabilité sont plus adaptées à notre question. Une théorie quantitative de la rectifiabilité a été développée par G. David et S. Semmes ([DS91a] [DS93a]) dans le cas particulier des mesures d–régulières, c'est-à-dire les mesure de Radon vérifiant : il existe $C > 0$ tel que

$$\frac{1}{C} r^d \leqslant \mu(B_r(x)) \leqslant C r^d \quad \forall r > 0,\ \mu\text{–presque tout } x \,. \tag{1.6}$$

Rappelons qu'en toute généralité, une mesure d–rectifiable vérifie seulement pour μ–presque tout x,

$$0 < \liminf_{r \to 0} \frac{\mu(B_r(x))}{r^d} \leqslant \limsup_{r \to 0} \frac{\mu(B_r(x))}{r^d} < +\infty \,.$$

Parmi les différentes conditions quantitatives de rectifiabilité, on s'est plus particulièrement intéressée dans cette thèse à celle qu'on pourrait rapprocher de la caractérisation qualitative en termes d'existence d'un plan tangent (Théorème 1.9). Cette condition donne un sens quantitatif à la propriété de se concentrer localement autour d'un plan μ–presque partout et peut être mesurée par une généralisation des nombres β de Jones (cf. [Jon90]).

Définition 1.16 (Nombres β de Jones généralisés)**.** *Soit $1 \leqslant q < +\infty$, $r > 0$ et $E \subset \mathbb{R}^n$,*

$$\beta_q(x, r, E) = \inf_{P \text{ } d\text{–plan}} \left(\frac{1}{r^d} \int_{y \in E \cap B_r(x)} \left(\frac{d(y, P)}{r} \right)^q d\mathcal{H}^d(y) \right)^{\frac{1}{q}} \,.$$

En nous appuyant sur les conditions quantitatives de rectifiabilité (Theorem 3.2) énoncées par H. Pajot ([Paj97]), nous avons pu définir une énergie de type height excess dont le contrôle garantit la rectifiabilité d'un varifold d–régulier :

Théorème. 3.3. [Cf. p.53] *Soit $\Omega \subset \mathbb{R}^n$ un ouvert et V un d–varifold sur Ω de masse finie $\|V\|(\Omega) < +\infty$. Supposons que*

(i) il existe $0 < C_1 < C_2$ tels que, pour $\|V\|$–presque tout $x \in \Omega$ et pour tout $r > 0$,

$$C_1 r^d \leqslant \|V\|(B_r(x)) \leqslant C_2 r^d \,, \tag{1.7}$$

(ii) $\displaystyle\int_{\Omega \times G_{d,n}} E_0(x, P, V) \, dV(x, P) < +\infty$, où

$$E_0(x, P, V) = \int_{r=0}^1 \frac{1}{r^d} \int_{y \in B_r(x) \cap \Omega} \left(\frac{d(y - x, P)}{r} \right)^2 d\|V\|(y) \, \frac{dr}{r} \,.$$

Alors V est un d–varifold rectifiable.

1.4.2 Échelle et discrétisation

Il faut ensuite adapter l'énergie E_0 à des varifolds de type discret, c'est-à-dire qui ne sont pas rectifiables "et le sont d'autant moins qu'on les regarde de près". Tout objet discret vient avec une notion d'échelle. Un nuage de points a une courbure infinie partout, mais si on sait a priori qu'il s'agit de la discrétisation d'un objet régulier, on peut trianguler, régulariser à une échelle fixée ... et obtenir une courbure correspondant à cette échelle. On n'aura a priori pas de notion de courbure absolue, seulement des courbures associées à des échelles. Néanmoins, si on connaît le pas de discrétisation, on sait en général à quelle échelle on doit calculer la courbure (en fonction du pas de discrétisation). On va suivre cette logique avec l'énergie E_0. Dans l'intégrale sur les rayons, on va considérer des boules d'un rayon supérieur à une certaine échelle α. On définit donc

$$E_\alpha(x,P,V) = \int_{r=\alpha}^{1} \frac{1}{r^d} \int_{y\in B_r(x)\cap\Omega} \left(\frac{d(y-x,P)}{r} \right)^2 d\|V\|(y) \, \frac{dr}{r} \, . \tag{1.8}$$

On observe alors les propriétés suivantes.

1. Si $V_i \xrightarrow[i\to+\infty]{*} V$ est une suite de varifolds quelconques :
   (a) Il existe des échelles α_i adaptées, c'est-à-dire telles que pour $x \in \mathbb{R}^n$, $P \in G_{d,n}$, on a convergence ponctuelle des énergies
   $$E_0(x,P,V) = \lim_{i\to\infty} E_{\alpha_i}(x,P,V_i) \, .$$
   (b) On a une condition quantitative permettant de choisir l'échelle α_i, uniforme dans tout $\omega \subset\subset \mathbb{R}^n$, et dépendant d'une distance de type flat distance :
   $$\Delta_\omega(V,V_i) = \sup \left\{ \int_{r=0}^{1} \left| \int_{B_r(x)\cap\omega} \varphi \, d\|V_i\| - \int_{B_r(x)\cap\omega} \varphi \, d\|V\| \right| dr \;\middle|\; \begin{array}{l} \varphi \in \mathrm{Lip}_{2(\mathrm{diam}\omega)}(\omega), \\ \|\varphi\|_\infty \leqslant (\mathrm{diam}\omega)^2 \\ x \in \overline{\omega} \end{array} \right\}$$
   On choisit α_i tel que $\frac{\Delta_\omega(V,V_i)}{\alpha_i^{d+3}} \xrightarrow[i\to\infty]{} 0$, ce qui est possible car $\Delta_\omega(V,V_i) \to 0$ quand $V_i \xrightarrow[i\to+\infty]{*} V$.
   (c) On a alors un résultat de convergence uniforme par rapport à $(x,P) \in \omega \times G_{d,n}$ des énergies :
   $$\sup_{\substack{x\in\overline{\omega} \\ P\in G_{d,n}}} \left| E^\omega_{\alpha_i}(x,P,V_i) - E^\omega_{\alpha_i}(x,P,V) \right| \xrightarrow[i\to\infty]{} 0 \, .$$
   Ces trois points sont énoncés et démontrés dans les Propositions 3.23, 3.24 et 3.25.
2. Le cas où la suite V_i est une suite de discrétisations volumiques est traité dans le Théorème 3.29 : si V est un d–varifold rectifiable à support compact, vérifiant la condition de régularité (1.5) du théorème d'approximation par des varifolds volumiques discrets (Théorème 2.1), pour C, $\beta > 0$ et pour $\|V\|$–presque tout x, y,
   $$\|T_x M - T_y M\| \leqslant C|x-y|^\beta \, , \tag{1.9}$$
   et si les varifolds V_i sont les varifolds volumiques discrets obtenus par projection de V sur des maillages $\mathcal{K}_i$ de pas δ_i tendant vers 0 ; alors pour toute suite d'échelles α_i vérifiant
   $$\frac{\delta_i^\beta}{\alpha_i^{d+3}} \xrightarrow[i\to+\infty]{} 0 \, , \tag{1.10}$$
   on a
   $$\int_{\mathbb{R}^n\times G_{d,n}} E_0(x,P,V) \, dV(x,P) = \lim_{i\to+\infty} \int_{\mathbb{R}^n\times G_{d,n}} E_{\alpha_i}(x,P,V_i) \, dV_i(x,P) \, . \tag{1.11}$$

1.4.3 Condition quantitative assurant la rectifiabilité d'un varifold limite

On peut maintenant donner une réponse à la question initiale, c'est le théorème 3.4. De même qu'on doit tester la régularité d'un varifold quelconque à une échelle donnée, la condition de d–régularité sur la masse $\|V\|$, qui exprime le fait que le varifold est d–dimensionnel, doit être considérée à une échelle suffisamment grande. En effet, un varifold volumique discret V est n–dimensionnel et n'a donc aucune chance de vérifier

$$\frac{1}{C} r^d \leqslant \|V\|(B_r(x)) \leqslant C r^d$$

pour des petits rayons (inférieurs à la taille du maillage par exemple).

Théorème. 3.4. [Cf. p.53] *Soit $\Omega \subset \mathbb{R}^n$ un ouvert et $(V_i)_i$ une suite de d–varifolds dans Ω qui convergeant faiblement–$*$ vers un d–varifold V, et satisfaisant* $\sup_i \|V_i\|(\Omega) < +\infty$. *Soit $(\alpha_i)_i$ et $(\beta_i)_i$ deux suites strictement positives, décroissantes et tendant vers 0, fixées. Supposons que :*

(i) il existe $0 < C_1 < C_2$ tels que pour $\|V_i\|$–presque tout $x \in \Omega$ et pour $\beta_i < r < d(x, \Omega^c)$,

$$C_1 \omega_d r^d \leqslant \|V_i\|(B_r(x)) \leqslant C_2 \omega_d r^d \,, \tag{1.12}$$

(ii)

$$\sup_i \int_{\Omega \times G_{d,n}} E_{\alpha_i}(x, P, V_i)\, dV_i(x, P) < +\infty \,. \tag{1.13}$$

Alors V est un d–varifold rectifiable.

Ici, rien n'est supposé sur la suite d'échelles $(\alpha_i)_i$. En revanche, sachant qu'un varifold est rectifiable, on a vu dans le paragraphe précédent des conditions suffisantes ((1.9),(1.10),(1.11)) assurant que l'hypothèse (1.13) est satisfaite dans le cas de l'approximation par des verifolds volumiques discrets.

On va maintenant continuer à s'intéresser à la même problématique, non plus sous l'angle de la rectifiabilité mais celui de la variation première (courbure).

1.5 Une construction de mesure de type "packing" à partir de valeurs approchées sur les boules

On s'intéresse maintenant à la deuxième partie de la question soulevée à la fin de la Section 1.3, concernant la courbure, qui était :

Question. 1.3. Comment assurer qu'un varifold, obtenu comme limite de varifolds quelconques, a priori à variation première non bornée (varifolds associés à des nuages de points par exemple), soit à variation première bornée ?

Cette question est l'objet des Chapitres 4 et 5.

En s'appuyant sur la Proposition 3.2 de [LM09], qui exprime la courbure d'un d–varifold rectifiable à l'intérieur d'une boule en fonction de l'intégrale des vecteurs conormaux sur le bord de la boule, on observe le fait suivant : soit $V = v(M, \theta)$ un d–varifold rectifiable et soit $x \in M$, alors, pour presque tout $r > 0$,

$$\delta V(B_r(x)) = - \int_{\partial B_r(x) \cap M} \eta(y) \theta(y)\, d\mathcal{H}^{d-1}(y) \,, \tag{1.14}$$

où $\eta(y) = \dfrac{\Pi_{T_y M}(y-x)}{|\Pi_{T_y M}(y-x)|}$ est le vecteur conormal extérieur. Si on moyenne cette formule sur les rayons, on obtient par la formule de la co-aire :

$$\frac{1}{R} \int_{r=0}^{R} \delta V(B_r(x))\, dr = -\frac{1}{R} \int_{B_R(x) \times G_{d,n}} \frac{\Pi_S(y-x)}{|y-x|}\, dV(y, S) \,. \tag{1.15}$$

Remarque 1.9. En réalité, c'est plutôt la formule (1.14) qui est une conséquence de la formule (1.15) et de la formule de la co-aire, dans la preuve de la Proposition 3.2 dans [LM09].

Le fait remarquable de la formule (1.15), c'est que le membre de droite a un sens pour un d–varifold absolument quelconque (en tout (x, R) tel que $\|V\|(\{x\} \cup \partial B_R(x)) = 0$, ce qui est vérifié pour presque tout $R > 0$ et $\|V\|$–presque tout x par la Proposition 3.8). La stratégie qu'on adopte est la suivante, semblable à celle adoptée dans le chapitre précédent 3 :

- Étape 1 : À partir des valeurs moyennées de la variation première sur toutes les boules, construire un objet/une énergie $\mathcal{E}$, qui a un sens pour tout varifold et caractérise le fait d'être à variation première bornée (on a défini dans cette esprit l'énergie E_0 (Définition 3.12) au Chapitre 3 pour ce qui concerne la rectifiabilité).
- Étape 2 : Définir à partir de $\mathcal{E}$, des quantités $\mathcal{E}$ dépendant d'une échelle α et contrôlant $\mathcal{E}$ pour $V_i \xrightarrow[i \to 0]{*} V$, et une suite d'échelles $(\alpha_i)_i$ adaptées

$$\mathcal{E}(V) \leqslant \sup_i \mathcal{E}_{\alpha_i}(V_i) \, .$$

  En ce qui concerne la rectifiabilité, on avait défini les énergies E_α (1.8) et on avait montré qu'il existait des échelles α_i adaptées à une suite de varifolds $V_i \xrightarrow[i \to 0]{*} V$, le choix dépendant d'une distance de type flat distance $\Delta_\omega(V_i, V)$ (cf. Propositions 3.23 et 3.24). Les énergies E_α ont été construites pour ne pas tenir compte de ce qu'il se passe sur des boules de rayon $\leqslant \alpha$. Ce qui suggère ici que $\mathcal{E}_\alpha$ ne devrait être construite qu'à partir des valeurs (1.15) sur des boules de rayon $R \geqslant \alpha$.
- Étape 3 : Étudier le cas particulier des varifolds volumiques discrets.

Dans le Chapitre 4, on essaie de construire la quantité $\mathcal{E}$ en reconstruisant la mesure $|\delta V|$ par packing à partir de la fonction définie sur les boules :

$$\begin{array}{rrcl} q \ : & \mathcal{C} = \{\text{boules fermées } B_r(x) \subset X\} & \longrightarrow & \mathbb{R}_+ \cup \{+\infty\} \\ & B_r(x) & \longmapsto & \displaystyle\frac{1}{r}\int_{s=0}^{r} \delta V(B_s(x)) \, ds \\ & & & \displaystyle = - \int_{B_r(x) \times G_{d,n}} \frac{\Pi_S(y-x)}{|y-x|} \, dV(y,S) \, . \end{array} \tag{1.16}$$

Attention, dans tout le Chapitre 4, les constructions se font à partir des boules fermées et $\mathbf{B_r(x)}$ **désigne la boule fermée**.

1.5.1 Variation totale de la forme linéaire δV

On a dit vouloir construire un objet ou une énergie $\mathcal{E}(V)$ qui ait un sens que le varifold V soit à variation première bornée ou non, or δV n'a de sens en tant que mesure que si V est à variation première bornée. L'idée pour y remédier est de remarquer que, même lorsque δV est seulement une forme linéaire sur $\mathrm{C}_c^1(\mathbb{R}^n, \mathbb{R}^n)$, on peut définir sa variation totale $|\delta V|$ en tant que mesure positive. En effet, soit $U \subset \mathbb{R}^n$ un ouvert, on définit

$$\mu_{\delta V}^*(U) = \sup \left\{ |\delta V(X)| \ : \ X \in \mathrm{C}_c^1(\mathbb{R}^n, \mathbb{R}^n), \ |X| \leqslant 1, \ \mathrm{supp}\, X \subset U \right\} \leqslant +\infty \tag{1.17}$$

et pour $A \subset \mathbb{R}^n$,

$$\mu_{\delta V}^*(A) = \inf \left\{ \mu_{\delta V}^*(U) \,|\, U \text{ouvert} \supset A \right\} \, .$$

Lemme 1.16 (Cf. theorem 1 p. 49, [EG92])**.** *L'application $\mu_{\delta V}^*$ ainsi définie est une mesure extérieure métrique. En particulier, sa restriction $\mu_{\delta V}$ aux boréliens est une mesure borélienne positive.*

Preuve. C'est en fait une étape de la démonstration du théorème de représentation de Riesz. L'application $\mu^*_{\delta V}$ est monotone. Il reste à vérifier la sous-additivité et le caractère métrique. Soit $(U_i)_i$ une famille d'ouverts et $X \in \mathrm{C}^1_c(\mathbb{R}^n, \mathbb{R}^n)$ tel que $\operatorname{supp} X \subset U = \cup_i U_i$ et $|X| \leqslant 1$. Comme X est à support compact, il existe N tel que $\operatorname{supp} X \subset \cup_{i=1}^N U_i$. On considère alors une partition de l'unité $(\varphi_i)_{i \leqslant N} : \mathbb{R}^n \to [0,1]$ associée à ces ouverts telle que

$$\operatorname{supp} \varphi \subset U_i \text{ et } \sum_{i=1}^N \varphi_i = 1 \text{ sur } \operatorname{supp} X \, .$$

On a donc $X = \sum_{i=1}^N \varphi_i \, g$ et

$$|\delta V(X)| \leqslant \left| \sum_{i=1}^n \delta V(\varphi_i \, X) \right| \leqslant \sum_{i=1}^n |\delta V(\varphi_i \, X)| \leqslant \sum_{i=1}^N \mu^*_{\delta V}(U_i) \leqslant \sum_{i=1}^\infty \mu^*_{\delta V}(U_i) \, .$$

Et on conclut à la sous-additivité sur les ouverts en prenant le supremum sur X. La sous-additivité pour des ensemble $(A_i)_i$ découle alors de la définition de $\mu^*_{\delta V}$ par régularité extérieure. Soit $\varepsilon > 0$, pour chaque i, on choisit un ouvert $U_i \supset A_i$ et tel que $\mu^*_{\delta V}(U_i) \leqslant \mu^*_{\delta V}(A_i) + \dfrac{\varepsilon}{2^i}$ et on a

$$\mu^*_{\delta V}(\cup_i A_i) \leqslant \sum_i \mu^*_{\delta V}(A_i) + \varepsilon \, .$$

Il reste à montrer le caractère métrique. Si U_1, U_2 sont ouverts et $d(U_1, U_2) > 0$, alors par construction

$$\mu^*_{\delta V}(U_1 \cup U_2) = \mu^*_{\delta V}(U_1) + \mu^*_{\delta V}(U_2) \, .$$

Et le cas général $d(A_1, A_2) > 0 \Rightarrow \mu^*_{\delta V}(A_1 \cup A_2) = \mu^*_{\delta V}(A_1) + \mu^*_{\delta V}(A_2)$ en découle. □

Ainsi, on obtient :

Corollaire 1.17. *La mesure borélienne $\mu_{\delta V}$ associée au $\mu^*_{\delta V}$ de* (1.17) *est tout le temps bien définie, et si de plus $\mu_{\delta V}$ est finie sur les compacts, alors par définition, le varifold V est à variation première bornée et $\mu_{\delta V} = |\delta V|$.*

L'objectif est maintenant de contrôler $\mu_{\delta V}(\mathbb{R}^n)$ par une quantité $\mathcal{E}$ construite à partir de p définie par (1.16). On commence par essayer de reconstruire une mesure positive puis une mesure signée à partir de ses valeurs exactes sur les boules.

1.5.2 Reconstruction d'une mesure signée par la méthode de Carathéodory

Rappelons en quoi consiste la méthode de Carathéodory. Soit $\mathcal{C}$ l'ensemble des boules fermées d'un espace métrique (X, d) et $\mathcal{C}_\delta = \{B \in \mathcal{C} \mid \operatorname{diam} B \leqslant \delta\}$ (on pourrait travailler avec des ensembles de parties plus généraux). Soit $p : \mathcal{C} \to [0; +\infty]$ telle que $p(\emptyset) = 0$, on dit que p est une pré-mesure, attention, on insiste qu'ici, une pré-mesure ne vérifie a priori aucune propriété d'additivité ou sous-additivité.

Définition 1.17 (Méthode de Carathéodory métrique, cf. 3.3 p. 114, [BBT01])**.** *Soit (X, d) un espace métrique, pour tout $E \subset X$, on définit pour $\delta > 0$,*

$$\nu^p_\delta(E) = \inf \left\{ \sum_{i=0}^\infty p(B_i) \,\middle|\, E \subset \bigcup_{i \in \mathbb{N}} B_i, \forall i, \; B_i \in \mathcal{C}_\delta \right\} .$$

Or $\nu^p_\delta \geqslant \nu^p_{\delta'}$ si $\delta \leqslant \delta'$, et donc

$$\nu^{p,*}(E) = \lim_{\delta \to 0} \nu^p_\delta(E)$$

est bien défini (éventuellement ∞). Ainsi définie, $\nu^{p,*}$ est une mesure extérieure métrique sur X et sa restriction aux boréliens définit une mesure borélienne positive ν^p.

Le bon cadre pour reconstruire une mesure de Borel à partir de ses valeurs exactes sur les boules est un espace où on a un lemme de recouvrement de type Besicovitch, qui permet d'assurer l'existence de recouvrements suffisamment économiques, satisfaisant en particulier la propriété qu'il existe une constante ζ (dépendant de l'espace métrique uniquement) telle que tout point est couvert par au plus ζ boules du recouvrement. L'espace $\mathbb{R}^n$ euclidien vérifie cette propriété, c'est le cadre classique du lemme de recouvrement de Besicovitch :

Théorème 1.18 (Lemme de recouvrement de Besicovitch, Théorème 2 p.30 [EG92])**.** *Il existe une constante ζ_n dépendant uniquement de n telle que : si $\mathcal{F}$ est une famille de boules fermées non dégénérées de $\mathbb{R}^n$ telle que*

$$\sup\{\mathrm{diam} B \mid B \in \mathcal{F}\} < +\infty \, ,$$

et si $A = \{a \in \mathbb{R}^n \mid \exists r, B_r(a) \in \mathcal{F}\}$ est l'ensemble des centres des boules de $\mathcal{F}$, alors il existe ζ_n familles au plus dénombrables $\mathcal{G}_1, \ldots, \mathcal{G}_{\zeta_n} \subset \mathcal{F}$ de boules disjointes telles que

$$A \subset \bigcup_{j=1}^{\zeta_n} \bigsqcup_{B \in \mathcal{G}_j} B \, .$$

Une condition naturelle pour avoir une chance de généraliser ce théorème est que l'espace métrique soit séparable. H. Federer donne dans [Fed69] une condition géométrique (faisant intervenir la notion de distance directionnellement limitée, cf. Definition 4.5) assurant qu'un espace métrique séparable admette de tels recouvrements.

Dans ce cadre, on a tous les outils pour montrer que la méthode de Carathéodory permet de reconstruire une mesure borélienne positive à partir de ses valeurs exactes sur les boules, en termes de pré-mesure : $p(B) = \mu(B)$. Le cas d'une mesure signée est déjà plus délicat. En effet, afin d'appliquer la méthode de Carathéodory, on a besoin que la pré-mesure soit positive, sinon on n'obtient pas une mesure extérieure. On se ramène donc à construire des mesures de Borel positives via la décomposition de Hahn de la mesure signée $\mu = \mu^+ - \mu^-$, mais cela ne suffit pas à se ramener exactement au cas précédent : on veut reconstruire la mesure de Borel positive μ^+ (resp. μ^-), mais on n'a pas accès à la valeur exacte de μ^+ (resp. μ^-) sur une boule fermée B, on connaît seulement $p(B) = \mu(B)$. On essaie alors d'appliquer la méthode de Carathéodory à la pré-mesure définie par $p_+(B) = (p(B))_+ = (\mu(B))_+$ où $a_+ = max(0, a)$ désigne la partie positive de $a \in [-\infty, +\infty]$. Toujours dans le cadre d'un espace métrique séparable muni d'une métrique directionnellement limitée, on a montré que cette méthode permet effectivement de reconstruire μ^+ et μ^- et donc μ, c'est l'objet du théorème 4.10.

On étudie maintenant le cas où la pré-mesure q est de la forme (1.16).

1.5.3 Reconstruction d'une mesure à partir de valeurs approchées sur les boules et construction de type packing

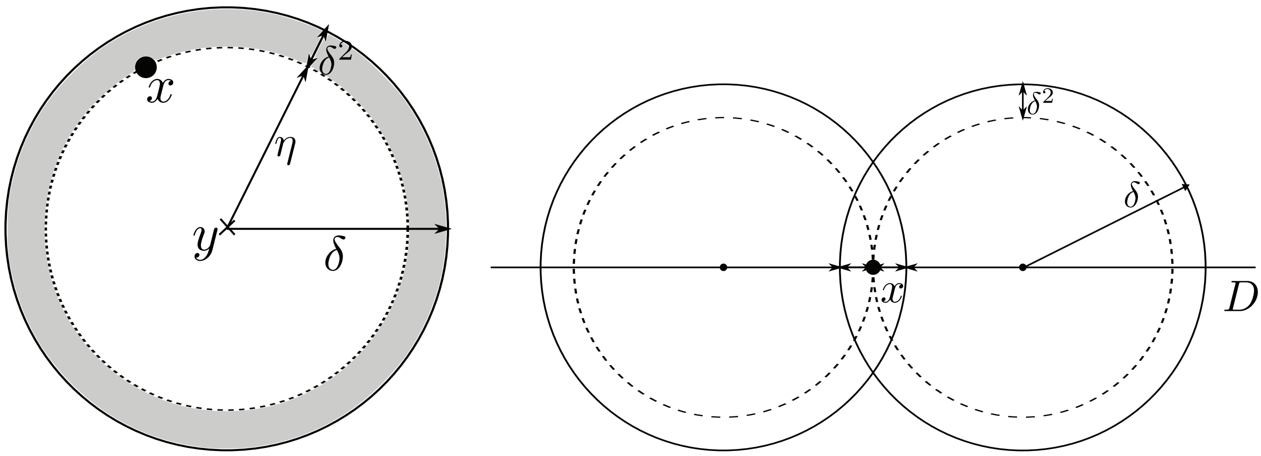
On considère maintenant une mesure borélienne positive μ sur un espace métrique (X, d), et on suppose qu'on a accès à la pré-mesure $q : \mathcal{C} \to [0, +\infty]$ définie pour $B = B_r(x) \in \mathcal{C}$ par

$$q(B_r(x)) = \frac{1}{r} \int_{s=0}^{r} \mu(B_s(x)) \, ds \, . \tag{1.18}$$

On pourrait essayer de reconstruire la mesure μ à partir de q par la méthode de Carathéodory. Cependant, en étudiant l'exemple où μ est une masse de Dirac $\mu = \delta_x$ dans $\mathbb{R}^n$, on se convainc rapidement que la méthode de Carathéodory ne fonctionne pas telle quelle. En effet, si on considère une boule fermée $B_\delta(y)$ recouvrant $\{x\}$ mais de telle façon que x soit près du bord, par exemple $d(x, \partial B_r(y)) = \delta^2$, on calcule

$$q(B_\delta(y)) = \frac{1}{\delta} \int_{s=\delta-\delta^2}^{\delta} \delta_x(B_s(y)) \, ds = \delta \xrightarrow[\delta \to 0]{} 0 \, .$$

On comprend alors que la masse de Dirac n'est pas un cas pathologique, mais plus généralement, dès qu'on peut recouvrir un ensemble par des boules décentrées, de sorte que la masse portée par la mesure soit proche du bord des boules du recouvrement, on se heurte au même problème. On pense alors à appliquer la construction de Carathéodory en centrant les boules sur le support de la mesure, mais l'exemple d'une droite D et de la mesure $\mu = \mathcal{H}^1_{|D} + \delta_x$ pour $x \in D$ nous montre que la position des centres des boules du recouvrement doit être optimisée plus finement (en centrant les boules du recouvrement sur le support de μ i.e. sur D, on perd la masse de Dirac exactement pour la même raison qu'on perd la masse de Dirac isolée).



(a) Recouvrement par une boule décentrée

(b) Recouvrement avec des boules centrées sur le support de la mesure

La pré-mesure sous-estime la mesure de la boule : pour toute boule fermée, $q(B) \leqslant \mu(B)$, et les quelques exemples précédents montrent que la méthode de Carathéodory reconstruit une mesure avec perte de masse. Au lieu de minimiser la mesure d'un recouvrement, on va plutôt essayer de maximiser la mesure d'un "remplissage" disjoint ou packing :

Définition 1.18. *Soit (X, d) un espace métrique séparable et q une pré-mesure sur $\mathcal{C}$. Pour $U \subset X$ ouvert et $\delta > 0$, un packing de U d'ordre δ est une union dénombrable disjointe de boules fermées de diamètre inférieur ou égal à δ incluses dans U et on définit*

$$\hat{\mu}^q_\delta(U) := \sup \left\{ \sum_{B \in \mathcal{F}} q(B) \; : \; \mathcal{F} \textit{ est un packing d'ordre } \delta \textit{ de } U \right\} .$$

Et comme dans la méthode de Carathéodory,

$$\hat{\mu}^q(U) = \lim_{\delta \to 0} \hat{\mu}^\delta_q(U) = \inf_{\delta > 0} \hat{\mu}^\delta_q(U) \,,$$

puisque $\delta' \leqslant \delta$ entraine $\hat{\mu}^{\delta'}_q(U) \leqslant \hat{\mu}^\delta_q(U)$. On définit ensuite $\hat{\mu}^q$ sur tout ensemble $A \subset X$ par

$$\hat{\mu}^q(A) = \inf \left\{ \hat{\mu}^q(U) \; : \; U \textit{ ouvert, } A \subset U \right\} .$$

Contrairement à la méthode de Carathéodory, cette construction ne donne pas systématiquement une mesure extérieure ($\hat{\mu}^q$ n'est pas nécessairement sous-additive), mais on va montrer que dans le cas où la pré-mesure q est de la forme (1.18) (ou plus généralement quand il existe une mesure borélienne ν qui domine la pré-mesure : $q(B) \leqslant \nu(B)$ pour tout $B \in \mathcal{C}$), la construction produit une mesure extérieure métrique (dans un espace métrique quelconque, on demande que μ soit finie, dans $\mathbb{R}^n$ une mesure de Borel suffit). Afin de rendre cette construction systématique, c'est-à-dire une construction par packing qui produise une mesure extérieure pour une pré-mesure quelconque, on pourrait appliquer la méthode de Carathéodory aussi connue sous le nom de méthode de Munroe I (pas la méthode de Carathéodory métrique, voir [BBT01]) à la pré-mesure $\hat{\mu}^q$, et cela revient alors (presque) à la définition de "packing measure" donnée par S. J. Taylor et C. Tricot dans [TT85]. Dans $\mathbb{R}^n$, on obtient par cette construction une mesure équivalente à la mesure de départ :

Théorème. 4.16. [Cf. p.106] *Soit $\Omega \subset \mathbb{R}^n$ un ouvert et μ une mesure de Borel positive dans Ω. Soit q la pré-mesure définie par* (1.18) *et $\hat{\mu}^q$ définie à partir de q comme dans la Définition 1.18. Alors,*

1. *$\hat{\mu}^q$ est une mesure métrique extérieure coïncidant avec la mesure obtenue par la construction par packing définie dans [TT85] (cf. Remarque 4.5).*
2. *il existe une constante dimensionnelle $C_n \geqslant 1$ telle que pour tout borélien $A \subset \Omega$,*
$$\frac{1}{C_n}\mu(A) \leqslant \hat{\mu}^q(A) \leqslant \inf\{\mu(U) \mid U \text{ ouvert }, A \subset U\} .$$
3. *si de plus μ est une mesure de Radon, alors μ et $\hat{\mu}^q$ sont équivalentes sur les boréliens : $\frac{1}{C_n}\mu \leqslant \hat{\mu}^q \leqslant \mu$.*

On en vient maintenant au cas où μ est une mesure signée.

1.5.4 Reconstruction par packing d'une mesure signée à partir de valeurs approchées sur les boules

Comme dans le cas où on essaie de reconstruire une mesure signée $\mu = \mu^+ - \mu^-$ à partir de ses valeurs exactes sur les boules, on n'a pas accès à $\frac{1}{r}\int_{s=0}^{r} \mu^+(B_s(x))\, ds$ mais seulement à $q(B_r(x)) = \frac{1}{r}\int_{s=0}^{r} \mu(B_s(x))\, ds$. On définit alors la pré-mesure q_+ sur les boules fermées par

$$q_+(B_r(x)) = \left(\frac{1}{r}\int_{s=0}^{r} \mu(B_s(x))\, ds\right)_+ ,$$

et on définit $\hat{\mu}^{q_+}$ par packing (Définition 1.18) associé à q_+. On construit de même $\hat{\mu}^{q_-}$ et on montre :

Théorème. 4.17. [Cf. p.106] *Soit $\Omega \subset \mathbb{R}^n$ un ouvert et $\mu = \mu^+ - \mu^-$ une mesure de Radon signée dans Ω. Alors,*

1. *$\hat{\mu}^{q_+}$, $\hat{\mu}^{q_-}$ sont des mesures extérieures métriques et les mesures de Borel associées $\hat{\mu}^{p_+}$ et $\hat{\mu}^{q_-}$ sont des mesures de Radon positives.*
2. *Il existe une constante $C_n \geqslant 1$ tel que pour tout borélien $A \subset \Omega$,*
$$\frac{1}{C_n}\mu^+(A) \leqslant \hat{\mu}^{q_+}(A) \leqslant \mu^+(A) \text{ et } \frac{1}{C_n}\mu^-(A) \leqslant \hat{\mu}^{q_-}(A) \leqslant \mu^-(A) .$$
3. *La mesure $\hat{\mu}^q = \hat{\mu}^{q_+} - \hat{\mu}^{q^-}$ est une mesure de Radon signée et pour tout borélien $A \subset \Omega$,*
$$\frac{1}{C_n}|\mu|(A) \leqslant |\hat{\mu}^q|(A) \leqslant |\mu|(A) .$$

Si on revient à notre stratégie initiale, pour $q_\pm(B_r(x)) = \left(\frac{1}{r}\int_{s=0}^{r} \delta V(B_s(x))\,ds\right)_\pm$, on aimerait définir :

$$\begin{aligned}\mathcal{E}(V) &= |\hat{\mu}^q|(\mathbb{R}^n) = \hat{\mu}^{q_+}(\mathbb{R}^n) + \hat{\mu}^{q_-}(\mathbb{R}^n)\\ &= \inf_{\delta>0}\left(\sup\left\{\sum_{B\in\mathcal{F}_\delta} q_+(B)\ :\ \mathcal{F}_\delta \text{ est un packing d'ordre } \delta \text{ de } U\right\}\right.\\ &\quad \left.+\sup\left\{\sum_{B\in\mathcal{F}_\delta} q_-(B)\ :\ \mathcal{F}_\delta \text{ est un packing d'ordre } \delta \text{ de } U\right\}\right)\ ,\end{aligned}$$

et peut-être

$$\begin{aligned}\mathcal{E}_\alpha(V) = \inf_{\delta>\alpha}&\left(\sup\left\{\sum_{B\in\mathcal{F}_\delta} q_+(B)\ :\ \mathcal{F}_\delta \text{ est un packing d'ordre } \delta \text{ de } U,\ \alpha < \operatorname{diam} B < \delta\right\}\right.\\ &\left.+\sup\left\{\sum_{B\in\mathcal{F}_\delta} q_-(B)\ :\ \mathcal{F}_\delta \text{ est un packing d'ordre } \delta \text{ de } U,\ \alpha < \operatorname{diam} B < \delta\right\}\right)\ .\end{aligned}$$

Mais pour un varifold V quelconque, quel sens donner à $q(B)$? Par exemple, projetons sur une direction $e \in \mathbb{S}^{n-1}$,

$$q(B_r(x))\cdot e = -\frac{1}{r}\int_{B_r(x)\times G_{d,n}} \frac{\Pi_S(y-x)}{|y-x|}\cdot e\,dV(y,S) = \delta V(T_{r,x}e) = (\delta V\cdot e)(T_{r,x})\ ,$$

avec

$$T_{r,x}(y) = 1 - \left|\frac{y-x}{r}\right| \text{ dans } B_r(x) \text{ et } 0 \text{ ailleurs.}$$

Comment décomposer la forme linéaire $\delta V\cdot e$ en $\mu_{\delta V\cdot e,+} - \mu_{\delta V\cdot e,-}$? Comme on l'a fait pour définir $\mu_{\delta V}$ par (1.17), on peut vérifier qu'en posant pour $U\subset\mathbb{R}^n$ ouvert

$$\mu_{\delta V\cdot e,+}(U) = \sup\{\delta V(\varphi\, e)\ |\ \varphi\in \mathrm{C}^1_c(U),\ 0\leqslant\varphi\leqslant 1\}$$

et

$$\mu_{\delta V\cdot e,-}(U) = -\inf\{\delta V(\varphi\, e)\ |\ \varphi\in \mathrm{C}^1_c(U),\ 0\leqslant\varphi\leqslant 1\}$$

et en étendant à tout $A\subset\mathbb{R}^n$ par $\mu_{\delta V\cdot e,+}(A) = \inf\{\mu_{\delta V\cdot e,+}(U)\,|\,A\subset U \text{ ouvert }\}$, on obtient bien deux mesures boréliennes positives, mais la différence $\mu_{\delta V\cdot e,+} - \mu_{\delta V\cdot e,-}$ n'est pas définie si les deux mesures sont infinies, ce qui ne permet pas d'obtenir (4.22) dans la preuve du Théorème 4.17.

On n'a pas poussé plus loin cette tentative parce qu'on a compris en manipulant ces objets, et en faisant des allers-retours entre formes linéaires et mesures, qu'on peut interpréter

$$q(B_r(x)) = -\frac{1}{r}\int_{B_r(x)\times G_{d,n}} \frac{\Pi_S(y-x)}{|y-x|}\,dV(y,S) = \delta V(T_{r,x}) = \delta V * T_r(x)$$

comme une convolution, et à renormalisation près, $q(B_r(x))\,\mathcal{L}^n(x)$ est une régularisation de la variation première δV. C'est l'objet du Chapitre 5.

1.6 Régularisation de la variation première par convolution et Γ−convergence d'énergies de Willmore approchées

On reconsidère donc la question soulevée à la fin de la Section 1.3 :

Question. 1.3 Comment assurer qu'un varifold, obtenu comme limite de varifolds quelconques, a priori à variation première non bornée (varifolds associés à des nuages de points par exemple), soit à variation première bornée ?

Nous allons répondre à cette question en ayant maintenant à l'esprit que la convolution de la variation première (en tant que distribution d'ordre 1) avec un noyau de type tente est justement donnée par

$$\delta V * T_\varepsilon = \delta V * T_\varepsilon(x)\, \mathcal{L}^n(x) \text{ avec } \delta V * T_\varepsilon(x) = -\frac{1}{\varepsilon^n}\frac{1}{\varepsilon}\int_{B_\varepsilon(x)\times G_{d,n}} \frac{\Pi_S(y-x)}{|y-x|}\, dV(y,S) \tag{1.19}$$

où $T : \mathbb{R}^n \to \mathbb{R}_+$ est la <u>fonction tente</u> renormalisée d'intégrale $\int_{\mathbb{R}^n} T\, d\mathcal{L}^n = 1$, à support dans la boule unité :

$$T(z) = \begin{cases} \frac{1}{\lambda_n}(1-|z|) & \text{si } |z| \leqslant 1 \\ 0 & \text{sinon} \end{cases} \quad \text{et} \quad T_\varepsilon(z) = \frac{1}{\varepsilon^n} T\left(\frac{z}{\varepsilon}\right) .$$

1.6.1 Régularisation de la variation première par convolution

L'expression (1.19) est donnée par le calcul direct de $\delta V(T_\varepsilon)$ (Proposition 5.1), qui peut être fait avec un noyau plus général $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right)$, où $\rho : \mathbb{R}^n \to \mathbb{R}_+ \in \mathrm{W}^{1,\infty}(\mathbb{R}^n)$ est positif symétrique et

$$\int_{\mathbb{R}^n} \rho\, d\mathcal{L}^n = 1 \text{ et } \operatorname{supp} \rho \subset B_1(0) . \tag{1.20}$$

On a juste besoin de vérifier que δV, qui est définie sur $\mathrm{C}_c^1(\Omega, \mathbb{R}^n)$, s'étend à $\mathrm{C}_c^1(\mathbb{R}^n, \mathbb{R}^n)$ (Proposition 5.1), ce qui est le cas pour un varifold de masse finie, et permet de définir le produit de convolution $\delta V * \rho_\varepsilon$. On montre alors facilement le Théorème 5.4 :

Théorème. 5.4. [Cf. p.117] *Soit V un d–varifold dans un ouvert $\Omega \subset \mathbb{R}^n$ de masse finie $\|V\|(\Omega) < +\infty$ et $(\rho_\varepsilon)_\varepsilon$ un noyau comme défini plus haut* (1.20). *Si on suppose*

$$\sup_{\varepsilon>0} \|\delta V * \rho_\varepsilon\|_{\mathrm{L}^1} = \sup_{\varepsilon>0} \frac{1}{\varepsilon^{n+1}} \int_{x\in\mathbb{R}^n} \left| \int_{B_\varepsilon(x)\times G_{d,n}} \nabla^S \rho \left(\frac{y-x}{\varepsilon}\right) dV(y,S) \right| d\mathcal{L}^n(x) \leqslant C < +\infty \, , \tag{1.21}$$

alors V est à variation première bornée et $|\delta V|(\Omega)$ est majorée par (1.21).
Dans le cas où $\rho_\varepsilon = T_\varepsilon$, on peut réécrire (1.21) *explicitement comme*

$$\sup_{\varepsilon>0} \frac{1}{\varepsilon^n} \int_{x\in\mathbb{R}^n} \frac{1}{\varepsilon} \left| \int_{y\in B_\varepsilon(x)\cap\Omega} \frac{\Pi_S(y-x)}{|y-x|}\, dV(y,S) \right| d\mathcal{L}^n(x) < +\infty \, . \tag{1.22}$$

Remarque 1.10. On peut comparer le théorème 5.4 avec le théorème 3.3 donnant des conditions quantitatives de rectifiabilité pour un varifold. Ici on peut voir (1.21) ou (1.22) comme des conditions quantitatives assurant que la variation première est bornée.

Remarque 1.11. Dans le Chapitre 4, on essayait de reconstruire δV à partir de p (1.18) par des méthodes de construction de mesures utilisant essentiellement la structure métrique, ici, on exploite, par le biais de la convolution, la structure vectorielle et les propriétés de la mesure de Lebesgue, et on construit ainsi les mesures $\delta V * \rho_\varepsilon = \frac{1}{\varepsilon^n} p(B_\varepsilon(x)) \, \mathcal{L}^n(x)$ qui convergent faiblement-$*$ vers δV sous l'hypothèse (1.21).

On est maintenant en mesure d'apporter une réponse à la Question 1.3 :

Théorème. 5.5. [Cf. p. 118] *Soit $(V_i)_i$ une suite de d–varifolds dans un ouvert $\Omega \subset \mathbb{R}^n$ et $(\rho_\varepsilon)_\varepsilon$ un noyau comme défini plus haut* (1.20). *Si on suppose qu'il existe une suite d'échelles $(\varepsilon_i)_i$ tendant vers 0 telles que*

$$\sup_i \left\{ \|V_i\|(\Omega) + \|\delta V_i * \rho_{\varepsilon_i}\|_{\mathrm{L}^1} \right\} < +\infty \, , \tag{1.23}$$

alors il existe une sous-suite $(V_{\varphi(i)})_i$ qui converge faiblement–$*$ dans Ω vers un d–varifold V qui est à variation première bornée et tel que $|\delta V|(\Omega)$ est majorée par (1.23).

La condition (1.23) se réécrit explicitement en fonction du noyau (comme en (1.21) et (1.22)), on peut comparer ce résultat donnant des conditions assurant que le varifold limite est à variation première bornée au Théorème 3.4 donnant des conditions assurant que le varifold limite est rectifiable.

Remarque 1.12. Si on ajoute à (1.23) une hypothèse de densité sur $\|V_i\|$ du type (1.12) (Théorème 3.4), on obtient des conditions impliquant en particulier la rectifiabilité du varifold limite grâce au Théorème 1.12 liant variation première et rectifiabilité.

Avec les notations adoptées lorsqu'on a décrit notre stratégie pour répondre à la Question 1.3, on pourrait poser

$$\mathcal{E}(V) = |\delta V|(\Omega) \text{ et } \mathcal{E}_\varepsilon(V) = \|\delta V * \rho_\varepsilon\|_{\mathrm{L}^1}$$

et on aurait bien pour $V_\varepsilon \xrightarrow[\varepsilon \to 0]{*} V$,

$$\mathcal{E}(V) \leqslant \liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon(V_\varepsilon) \, .$$

En réalité, on a même Γ–convergence de $\mathcal{E}_\varepsilon$ vers $\mathcal{E}$ dans l'espace des d–varifolds. On va voir que grâce à la régularisation de δV, on peut plus généralement construire des énergies de Willmore approchées qui vont Γ–converger vers l'énergie de Willmore dans l'espace des varifolds.

1.6.2 Γ–convergence d'énergies de Willmore approchées dans l'espace des d–varifolds

On a défini des régularisations de la variation première associées à un noyau $\delta V * \rho_\varepsilon$, peut-on en déduire des approximations de la courbure moyenne H définie, quand V est à variation première bornée, par $H \in \mathrm{L}^1(\|V\|) = -\frac{\delta V}{\|V\|}$ au sens de la dérivée de Radon Nikodym, ou encore,

$$\delta V = -H \, \|V\| + \delta V_s \, .$$

Une idée naturelle consiste à convoler la masse $\|V\|$ avec le même noyau que la variation première, et définir

$$H_\varepsilon(x) = \frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)} \, . \tag{1.24}$$

On montre dans la Proposition 5.6 que H_ε converge $\|V\|$–presque partout vers H, si V est un d–varifold rectifiable à variation première bornée et si ρ est un noyau (1.20) radial. La condition qui apparaît dans la preuve n'est pas ρ radial mais plutôt que ρ doit "voir" toutes les d–directions au sens où, pour tout $P \in G_{d,n}$,

$$\int_P \rho \, d\mathcal{H}^d > 0 \, .$$

Ce qui est en particulier le cas d'un noyau radial positif d'intégrale 1. Si de plus on considère les mesures $\mu_\varepsilon = \|V\| * \rho_\varepsilon$, alors μ_ε converge faiblement–$*$ vers $\|V\|$. On définit alors assez naturellement les énergies de Willmore approchées suivantes (Définition 5.2) : si V est un d–varifold dans Ω, pour $p \geqslant 1$,

$$\mathcal{W}_\varepsilon^p(V) = \int_{\mathbb{R}^n} |H_\varepsilon(x)|^p \, d\mu_\varepsilon(x) = \int_{x\in\mathbb{R}^n} \left| \frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)} \right|^p \|V\| * \rho_\varepsilon(x) \, d\mathcal{L}^n(x) \, . \tag{1.25}$$

Rappelons alors ce qu'on appelle énergie de Willmore d'ordre $p \geqslant 1$ d'un varifold (Définition 5.1) : si V est un d–varifold dans Ω à variation première bornée et courbure dans L^p c'est-à-dire que $\delta V = -H \, \|V\|$ avec $H \in \mathrm{L}^p(\|V\|)$,

$$\mathcal{W}^p(V) = \int_\Omega |H|^p \, d\|V\| \, ,$$

et sinon $\mathcal{W}^p(V) = +\infty$. On pose alors la question (Question 5.1) de la Γ–convergence des énergies approchées :

Question. 5.1 Est-ce que les énergies de Willmore approchées $\mathcal{W}_\varepsilon^p$ ainsi définies Γ–convergent dans l'espace des d–varifolds ? Et dans l'affirmative, est-ce que la Γ–limite est l'énergie de Willmore $\mathcal{W}^p$?

La réponse à cette question est l'objet des Théorèmes 5.8 et 5.10, que l'on peut résumer comme suit :

$$\mathcal{W}_\varepsilon^p \xrightarrow[\varepsilon\to 0]{\Gamma} \mathcal{W}^p \quad \text{for } 1 < p < +\infty$$
$$\mathcal{W}_\varepsilon^1 \xrightarrow[\varepsilon\to 0]{\Gamma} \text{ la variation totale de la variation première } \neq \mathcal{W}^1 \, .$$

On précise que la propriété de Γ–lim sup est en réalité une limite ponctuelle (Remarque 5.7).

À présent, de même qu'on l'avait fait pour les énergies E_α dans le Chapitre 3, on va étudier les énergies de Willmore approchées $\mathcal{W}_\varepsilon^p(V_{\mathcal{K}_\delta})$ de varifold volumétriques discrets $V_{\mathcal{K}_\delta}$ obtenus par projection d'un varifold rectifiable V sur un maillage $\mathcal{K}_\delta$ de pas δ (1.4). Comme on l'a déjà expliqué, on a à nouveau une question d'échelle à laquelle on regarde l'objet discret, et ici à chaque échelle ε, correspond une courbure H_ε (1.24). Quand on a un a priori sur le fait que l'objet discret qu'on considère a été discrétisé à une échelle donnée δ, se pose alors la question de l'échelle adaptée ε à laquelle calculer la courbure H_ε ou encore l'énergie de Willmore approchée $\mathcal{W}_\varepsilon^p$ en fonction de l'échelle de discrétisation δ. Dans le cas de discrétisation par varifold volumiques discrets, l'échelle de discrétisation est donnée par le pas du maillage. On obtient alors le résultat partiel suivant (concrètement, la Γ–lim inf est inchangée, c'est la propriété de Γ–lim sup qui doit être obtenue avec des varifolds volumiques discrets et non le varifold limite lui-même) :

Théorème. 5.13. [Cf. p. 127] *Soit $\Omega \subset \mathbb{R}^n$ un ouvert et $V = v(M,\theta)$ un d–varifold rectifiable défini sur Ω et de masse finie $\|V\|(\Omega) < +\infty$. On fixe un noyau $\rho \in \mathrm{W}^{2,\infty}$ et satisfaisant* (1.20). *Soit $\delta_i \downarrow 0$ une suite décroissante tendant vers 0 et $(\mathcal{K}_i)_i$ une famille de maillages de Ω tels que*

$$\sup_{K\in\mathcal{K}_i} \operatorname{diam}(K) \leqslant \delta_i \xrightarrow[i\to+\infty]{} 0 \, .$$

Si $(\mathcal{A}_{\delta_i}(\mathcal{K}_i))_i$ sont les espaces de varifolds volumiques discrets associés aux maillages $\mathcal{K}_i$, comme définis au Chapitre 2 (2.1) *et s'il existe $0 < \beta < 1$ et C tels que pour $\|V\|$–presque tout x, $y \in \Omega$,*

$$\|T_x M - T_y M\| \leqslant C|x-y|^\beta \, ,$$

alors, il existe une suite de varifolds volumiques discrets $(V_i)_i$ tels que

(i) pour tout i, $V_i \in \mathcal{A}_{\delta_i}(\mathcal{K}_i)$;

(ii) $V_i \xrightarrow[i\to+\infty]{*} V$,

(iii) *et pour toute suite* $\varepsilon_i \downarrow 0$ *vérifiant*

$$\frac{\delta_i^\beta}{\varepsilon_i^2} \xrightarrow[i\to+\infty]{} 0 \tag{1.26}$$

on a la convergence

$$\mathcal{W}^1_{\varepsilon_i}(V_i) \xrightarrow[i\to+\infty]{} |\delta V|(\Omega) = \Gamma - \lim \mathcal{W}_\varepsilon \, .$$

On retrouve une condition (1.26) similaire à celle du Théorème 3.29 qui étudiait justement le comportement des énergies $\int E_{\alpha_i}(x, P, V_i) \, dV_i(x, P)$ pour des suites $(V_i)_i$ de varifolds volumiques discrets. C'est une similarité naturelle, qui vient du fait que plus général qu'étant donnée une suite de varifolds quelconque V_i convergeant faiblement–$*$ vers un varifolds V, une façon de mesurer l'échelle δ_i à laquelle on peut considérer que V_i est une discrétisation de V est de considérer

$$\delta_i = \Delta^{1,1}(V, V_i) \, ,$$

où $\Delta^{1,1}$ est la métrique de la Définition 1.15. C'est la quantité qui apparaît lorsqu'on estime l'écart entre les énergies discrètes et les énergies continues (dans le Chapitre 3 comme dans le Chapitre 5), et dans le cas des varifolds volumiques discrets, le théorème d'approximation des varifolds rectifiables (Théorème 2.1) donne une estimation uniforme, ne dépendant que de la régularité du varifold limite (à travers β de la condition Hölder sur le plan tangent) et du pas du maillage.

Remarque 1.13. On remarque que le théorème énoncé ci-dessus ne concerne que le cas $p = 1$, c'est que pour $p > 1$, la question reste entière. La même technique de démonstration échoue essentiellement parce que pour $p > 1$, le quotient $\dfrac{\|V\| * \rho_\varepsilon}{(\|V\| * \rho_\varepsilon)^p}$ n'est plus borné. Cet aspect technique révèle-t-il une obstruction réelle à l'obtention de la Γ–lim sup ? Même en donnant plus de souplesse à nos objets discrets, en considérant par exemple des varifolds nuages de points, le problème demeure.

1.6.3 Identification de la régularisation $\delta V * \rho_\varepsilon$

Ceci nous a conduits à tenter de mieux comprendre le lien entre la convolution de la variation première et le varifold V.

Question. 5.3

- Peut-on réaliser la régularisation de la variation première $\delta V * \rho_\varepsilon$ comme la variation première $\delta\left(\widehat{V_\varepsilon}\right)$ d'un varifold $\widehat{V_\varepsilon}$?
- Dans ce cas, est-ce qu'on peut exprimer $\widehat{V_\varepsilon}$ comme la régularisation (en un sens à préciser) de V ?

La réponse est oui et la construction est explicite et détaillée dans le Théorème 5.14. On obtient que : pour toute fonction $\psi \in \mathrm{C}^0_c(\Omega \times G_{d,n})$,

$$\left\langle \widehat{V_\varepsilon}, \psi \right\rangle = \langle V, (y, S) \mapsto \psi(\cdot, S) * \rho_\varepsilon(y) \rangle \, ,$$

et avec cette définition, $\|\widehat{V_\varepsilon}\| = \|V\| * \rho_\varepsilon$ et $\delta\left(\widehat{V_\varepsilon}\right) = \delta V * \rho_\varepsilon$. Afin de mieux comprendre cette construction, on étudie dans la Proposition 5.15 la partie tangentielle $\widehat{\nu_x^\varepsilon}$ de

$$\widehat{V_\varepsilon} = \|\widehat{V_\varepsilon}\| \otimes \widehat{\nu_x^\varepsilon} \, .$$

On obtient une moyenne pondérée par le noyau ρ des contributions tangentielles dans une voisinage de taille ε : $\|\widehat{V_\varepsilon}\|$–presque $x \in \mathbb{R}^n$,

$$\widehat{\nu_x^\varepsilon}(A) = \frac{\int_{y\in\Omega} \nu_y(A)\, \rho_\varepsilon(y-x)\, d\|V\|(y)}{\int_{y\in\Omega} \rho_\varepsilon(y-x)\, d\|V\|(y)} \text{ pour un borélien } A \subset G_{d,n} \, .$$

On voit notamment à partir de cette expression, que même si on part d'un d–varifold rectifiable $V = v(M,\theta)$ avec une mesure tangentielle $\nu_y = \delta_{T_y M}$ concentrée en $\|V\|$–presque tout point y, le varifold $\widehat{V_\varepsilon}$ aura une partie tangentielle $\widehat{\nu_x^\varepsilon}$ généralement diffuse (à part dans des portions où la direction tangente serait constante). Si on considère par exemple le 1–varifold associé à une croix formée par deux droites $\{x_1 = 0\} \cup \{x_2 = 0\}$ dans $\mathbb{R}^2$, de directions respectivement notées T_1 et T_2, $\widehat{\nu_x^\varepsilon}$ est une combinaison convexe de δ_{T_1} et δ_{T_2} dont les coefficients dépendent de la position de x par rapport à $\{x_1 = 0\}$ et $\{x_2 = 0\}$ (c'est l'exemple 5.6).

1.7 Aspects numériques

Le calcul de la courbure (dans le cas d'une surface : courbure moyenne, courbure de Gauss, courbures principales) d'une surface discrétisée est un enjeu essentiel de l'étude des surfaces discrètes. La notion elle-même de courbure discrète n'est pas définie de façon universelle (contrairement à la notion de courbure classique en géométrie différentielle). Une définition de courbure discrète est bien souvent liée à la structure de la discrétisation de la surface. Ainsi, lorsque l'objet discret est une surface triangulée, on dispose des relations d'adjacences (et donc d'une paramétrisation locale) et il existe des notions de courbure discrètes exploitant cette structure, comme la *formule des cotangentes* (voir par exemple [PP93]) qui est obtenue en définissant la courbure comme un gradient discret de l'aire. Le lien entre courbure moyenne et variation première est d'ailleurs le point de départ de nombreuses approches pour définir une notion de courbure discrète. Par exemple, comme il est fréquent en géométrie discrète, le lien entre la courbure moyenne H au point x et le volume $V_r(x)$ enclos par la surface dans une petite boule $B_r(x)$,

$$H(x) = \frac{8}{3r} - \frac{4V_r(x)}{\pi r^4} + o(r) \, ,$$

est exploitée pour définir une courbure discrète (see [CLR12]). La théorie des mesures de courbure ("curvature measures") a permis de donner des notions de courbure moyenne et de directions principales unifiées pour les cadres continus et discrets (tout d'abord pour les surfaces triangulées [Mor08, CSM06] puis pour des discrétisations plus générales englobant les nuages de points [CCLT09]). La notion de courbure discrète à laquelle on s'intéresse dans ce chapitre tend aussi à unifier cadre continu et cadre discret dans le cadre de la théorie géométrique de la mesure, non pas cependant à l'aide des courants mais en utilisant leur pendant non orienté, les varifolds.

On a vu que la régularisation de la variation première nous donne la formule suivante pour approcher le vecteur courbure moyenne $H(x)$ d'un varifold à variation première bornée V au point x (la convergence $\|V\|$–presque partout dans le cas d'un varifold d–rectifiable à variation première bornée est établie dans la Proposition 5.6) :

$$-\frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)} \xrightarrow[\varepsilon \to 0]{} H(x) \, . \tag{1.27}$$

Mais cette formule se traduit aussi dans toute discrétisation qu'on munit d'une structure de varifold, et avec un noyau $\rho(z) = \zeta(|z|)$ par exemple radial. S'agissant d'un varifold discret $V_N = \sum_{j=1}^{N} m_j \delta_{x_j} \otimes \delta_{P_j}$

associé à un nuage de points, on obtient la formule (6.17) d'approximation de la courbure moyenne

$$H_\varepsilon^N(x) = -\frac{\delta V_N * \rho_\varepsilon(x)}{\|V_N\| * \rho_\varepsilon(x)} = -\frac{\displaystyle\sum_{j=1}^N m_j \zeta'\left(\frac{|x_j - x|}{\varepsilon}\right)\frac{\Pi_{P_j}(x_j - x)}{|x_j - x|}}{\displaystyle\sum_{j=1}^N m_j \varepsilon \zeta\left(\frac{|x_j - x|}{\varepsilon}\right)} .$$

La convergence de cette approximation est établie dans la proposition 6.1. Une particularité de cette approximation est qu'elle respecte la nullité de la courbure au niveau des croisements et plus généralement lorsque la courbure singulière $(\delta V)_s$ est nulle. On valide numériquement cette propriété sur une courbe en forme de huit, mais ce faisant, on observe un phénomène d'instabilité numérique, qui ne disparaît que lorsque le nombre de points dans la boule (dans laquelle s'effectue le calcul de la courbure) est suffisamment grand. On identifie un phénomène de compensation (au niveau continu) dans la formule (1.27). On explique dans l'exemple 6.1 comment se traduit ce phénomène au niveau discret. Afin d'y remédier, on décide de modifier (6.17) en remplaçant la projection Π_{P_j} sur l'espace tangent par une projection $\Pi_{P_j^\perp}$ sur la direction normale. On vérifie la validité de cette approche dans le cas des courbes régulières au niveau continu et l'on teste cette nouvelle formule numériquement. On remarque que le choix du noyau $\zeta(r) = r$ si $r \leqslant 1$ et 0 sinon, bien que peu régulier, semble le plus adapté à la lumière de nos tests numériques. Est-ce dû au fait que pour ce noyau, le dénominateur dans (6.17) se simplifie pour devenir indépendant de ε ? La réponse n'est pas claire pour l'instant. On teste ensuite cette approximation de la courbure moyenne (6.23) :

$$\boxed{\frac{\displaystyle\sum_{j=1}^N \mathbb{1}_{\{|x_j - x| < \varepsilon\}} m_j \frac{\Pi_{P_j^\perp}(x_j - x)}{|x_j - x|}}{\displaystyle\sum_{j=1}^N \mathbb{1}_{\{|x_j - x| < \varepsilon\}} m_j |x_j - x|}} ,$$

sur des discrétisations de courbes–test en $2D$ et on étudie au passage les liens entre les différents paramètres (nombre de points du nuage N, rayon de la boule ε, nombre de points dans la boule où le calcul s'effectue). On utilise alors cette approximation pour estimer la courbure moyenne sur des nuages de points $3D$ plus généraux.

1.8 Perspectives

Il existe de nombreuses perspectives à la modélisation de la courbure discrète que nous avons proposée et à son étude numérique :

- Il serait ainsi intéressant d'étudier les informations (autres que la courbure moyenne), auxquelles on peut avoir accès par le même type de stratégie (laplacien surfacique de la courbure, courbure anisotrope). Hutchinson [Hut86] a introduit une version généralisée de la variation première, qui permet de récupérer toute la seconde forme fondamentale et pas seulement la courbure moyenne, est-il possible d'en déduire une approximation de toute la seconde forme fondamentale (et notamment des courbures principales) ?
- D'autre part, l'influence du noyau choisi pour régulariser la variation première reste à étudier. Dans un premier temps il serait intéressant de tester des noyaux plus réguliers, de type gaussien par exemple. dans un second temps, il serait approprié de choisir des noyaux anisotropes, avec une anisotropie liée à l'orientation du plan tangent, peut-être que cela permettrait également d'améliorer l'approximation de la courbure moyenne actuelle.

- Afin de stabiliser le calcul numérique de la courbure moyenne, on pourra d'une part calculer la courbure dans une boule dont la taille est déterminée non pas par un rayon absolu mais par le nombre de point que doit contenir le voisinage dans le quel le calcul est effectué. On pourra aussi tester la formule modifiée (6.24) (proposée au chapitre 6), obtenue en calculant un équivalent de la masse $\|V\| * \rho_\varepsilon(x)$ pour un varifold rectifiable et à mettre en parallèle avec la formule (6.25) proposée dans [CRT04].
- Comme notre approximation de la courbure moyenne est valable dans un cadre unifié et se traduit dans toute discrétisation munie d'une structure de varifold, il est possible d'étudier et comparer les calculs de courbures, et d'effectuer des flots sur différentes discrétisations (volumiques, nuages de points, triangulations) de la même surface.

Par ailleurs, les conditions de rectifiabilité énoncées dans le chapitre 3 sont de nature quantitatives. Il serait donc naturel de vouloir conclure à des propriétés d'uniforme rectifiabilité. Est-il possible de modifier les énergies E_α introduites dans ce chapitre pour garantir l'uniforme rectifiabilité du varifold dans les Théorèmes 3.3 et 3.4 ?

On a également laissé en suspens la question de la Γ–convergence, dans l'espace des varifolds volumétriques discrets et pour $p > 1$, de l'énergie de p–Willmore approchée. Peut-on commencer par donner une réponse positive ou négative en considérant des ensembles limites simples ?

CHAPTER 2

Varifolds discrets

The space of varifolds has been introduced by F. Almgren in 1965 in Theory of Varifolds [Alm65] to study the existence of critical points of the area functional. An essential aspect of this theory is the definition of a notion of curvature in a very general context, allowing to consider curves, surfaces, rectifiable sets, but also "discrete" objects like triangulations and point clouds. In this part, we investigate a volumetric surface discretization model that aims at being both accurate and able to handle the presence of singularities (singularities like in soap films and bubbles for instance). We call these objects discrete volumetric varifolds. The idea to build them is simple: given a surface and some mesh of the space, each cell is associated with a non-negative number (the area in the cell) and a plane (a mean tangent plane). This is a natural way to discretize surfaces in the spirit of varifolds and it has the advantage to extend easily to any finite dimension or codimension. Moreover, not only the discretization we propose can be endowed with a structure of varifold, but also a great part of objects used for surface representation and discretization so that we can use varifolds tools (in particular the generalized curvature) to study in some unified setting different ways of discretizing surfaces.

– In the first section, we define discrete volumetric varifolds, raising the natural question:
  Question 2.1. Is it possible to approach any rectifiable d–varifold with sequences of discrete volumetric varifolds? And if so, is it possible to have something more quantitative, measuring the speed of convergence?
  We prove that discrete volumetric varifolds allow to approach the class of rectifiable varifolds in the sense of weak–$*$ convergence. Moreover, we obtain a control on the convergence with respect to the size of the mesh, assuming that some Hölder condition on the tangent plane of the rectifiable varifold holds.
– In the second section, we address the following question:
  Question 2.2. Is it possible to apply Allard's compactness theorem to sequences of discrete volumetric varifolds? Meaning, is the condition $\sup_i \|\delta V_i\| < +\infty$ reasonable for such varifolds?
  We compute the generalized curvature (i.e. the first variation) of discrete volumetric varifolds. Then, given a sequence of discrete volumetric varifolds $(V_i)_i$ weakly–$*$ converging to a varifold V, we explain why the condition

$$\sup_i \|\delta V_i\| < +\infty$$

  generally does not hold, even though V is very regular (associated with a smooth set with constant density for instance). This is the precise motivation of Chapters 3 to 5, motivating the introduction of Jones β numbers energies in Chapter 3, and the introduction of a more suitable notion of curvature in Chapter 5.

– In the third section, we endow point clouds with a varifold structure and we address the same questions (approximation of rectifiable varifolds and computation of the first variation).

2.1 Discrete volumetric varifolds

2.1.1 A family of volumetric approximations endowed with a varifold structure

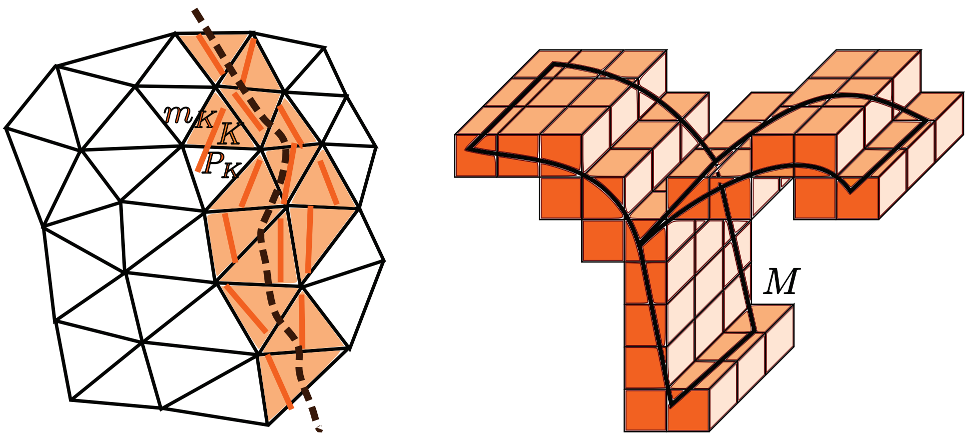
Let us explain what we mean by volumetric approximation. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $(\mathcal{K}, \mathcal{E})$ be a mesh of Ω, where $\mathcal{K}$ is the set of cells and $\mathcal{E}$ is the set of faces, $(\mathcal{K}, \mathcal{E})$ will be often shortened in $\mathcal{K}$ in the following. Given a d–rectifiable set $M \subset \mathbb{R}^n$ (a curve, a surface...), we can define for any cell $K \in \mathcal{K}$, a mass m_K (the length of the piece of curve in the cell, the area of the piece of surface in the cell...) and a mean tangent plane P_K as

$$m_K = \mathcal{H}^d(M \cap K) \text{ and } P_K \in \operatorname*{arg\,min}_{S \in G_{d,n}} \int_{M \cap K} |T_x M - S| \, d\mathcal{H}^d(x) \, ,$$

and similarly, given a rectifiable d–varifold V, defining

$$m_K = \|V\|(K) \text{ and } P_K \in \operatorname*{arg\,min}_{S \in G_{d,n}} \int_{K \times G_{d,n}} |P - S| \, dV(x, P) \, ,$$

gives what we call a volumetric approximation of V.



(a) Volumetric approximation of a curve

(b) Volumetric approximation of a 2–rectifiable set

We now introduce the family of varifolds of this form:

Definition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Consider $(\mathcal{K}, \mathcal{E})$ a mesh of Ω and a family $\{m_K, P_K\}_{K \in \mathcal{K}} \subset \mathbb{R}_+ \times G_{d,n}$. We can associate the d–varifold:*

$$V_{\mathcal{K}} = \sum_{K \in \mathcal{K}} \frac{m_K}{|K|} \mathcal{L}^n_{|K} \otimes \delta_{P_K} \text{ with } |K| = \mathcal{L}^n(K) \, .$$

This d–varifold is not rectifiable since its support is n–rectifiable but not d–rectifiable. We will refer to the set of d–varifolds of this special form as discrete volumetric varifolds*.*

Remark 2.1. We can consider different spaces of discrete volumetric varifolds:

– The space of discrete volumetric varifolds associated with a prescribed mesh $\mathcal{K}$ of prescribed size $\sup_{K\in\mathcal{K}} \operatorname{diam}(K) \leqslant \delta$:

$$\mathcal{A}_\delta(\mathcal{K}) = \left\{ V \ d\text{–varifold} \ : \ V = \sum_{K\in\mathcal{K}} \frac{m_K}{|K|} \mathcal{L}^n_{|K} \otimes \delta_{P_K} \right\} . \tag{2.1}$$

– The space of discrete volumetric varifolds V of prescribed size : for a fixed size $\delta > 0$, there exists a mesh $\mathcal{K}$ of size $\sup_{K\in\mathcal{K}} \operatorname{diam}(K) \leqslant \delta$ such that $V \in \mathcal{A}_\delta(\mathcal{K})$ i.e.

$$\begin{aligned} \mathcal{A}_\delta = & \bigcup_{\substack{\mathcal{K} \text{ mesh of size} \\ \sup_{K\in\mathcal{K}} \operatorname{diam}(K)\leqslant\delta}} \mathcal{A}_\delta(\mathcal{K}) \qquad (2.2) \\ = & \left\{ V \ d\text{–varifold} \ : \ \exists \text{ a mesh } \mathcal{K} \text{ such that } \sup_{K\in\mathcal{K}} \operatorname{diam}(K) \leqslant \delta \text{ and } V = \sum_{K\in\mathcal{K}} \frac{m_K}{|K|} \mathcal{L}^n_{|K} \otimes \delta_{P_K} \right\} . \end{aligned}$$

– The space of all discrete volumetric varifolds.

$$\mathcal{A} = \cup_{\delta>0} \mathcal{A}_\delta \, .$$

This definition of discrete volumetric varifolds raises a natural question:

Question. 1.1. Considering a sequence of meshes $(\mathcal{K}_i)_i$ whose size is tending to 0, what class of varifolds is it possible to approximate by discrete volumetric varifolds $(V_i)_i$ associated with these prescribed successive meshes?

2.1.2 Approximation of rectifiable varifolds by discrete volumetric varifolds

We now state and prove the following result which asserts that the family of discrete volumetric varifolds approximates well the space of rectifiable varifolds in the sense of weak–$*$ convergence. Moreover, we give a way of quantifying this convergence with respect to the size of the prescribed successive meshes.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $(\mathcal{K}_i)_i$ be a family of successive meshes of Ω such that*

$$\delta_i = \sup_{K\in\mathcal{K}_i} \operatorname{diam}(K) \xrightarrow[i\to+\infty]{} 0 \, .$$

Let $V = v(M,\theta)$ be a rectifiable d–varifold in Ω and for all i, define the discrete volumetric varifold V_i by

$$V_i = \sum_{K\in\mathcal{K}_i} \frac{m^i_K}{|K|} \mathcal{L}^n \otimes \delta_{P^i_K} \ \text{ with } m^i_K = \|V\|(K) \text{ and } P^i_K \in \operatorname*{arg\,min}_{P\in G_{d,n}} \int_{K\times G_{d,n}} \|P - S\| \, dV(x,S) \, . \tag{2.3}$$

Then,

$$V_i \xrightarrow[i\to+\infty]{*} V \ \text{ in } \Omega \, .$$

Moreover, let $\Pi : \Omega \times G_{d,n} \to \Omega$, $(y,T) \mapsto y$,

– for every Lipschitz function $\varphi \in \operatorname{Lip}(\Omega \times G_{d,n})$, with Lipschitz constant $\operatorname{lip}(\varphi)$, then

$$|\langle V_i, \varphi\rangle - \langle V, \varphi\rangle| \leqslant \operatorname{lip}(\varphi) \left(\delta_i \|V\| \left(\Pi(\operatorname{supp}\varphi) \cap \Omega\right) + \int_{(\Pi(\operatorname{supp}\varphi)\cap\Omega)\times G_{d,n}} \left\| P^i(y) - T \right\| \, dV(y,T) \right) \, , \tag{2.4}$$

where $P^i : \Omega \to G_{d,n}$ is cell-wise constant and for all $K \in \mathcal{K}_i$ and $y \in K$, $P^i(y) = P^i_K$.

– *If in addition there exist* $0 < \beta < 1$ *and* $C > 0$ *such that for* $\|V\|$*–almost* x, $y \in \Omega$,

$$\|T_x M - T_y M\| \leqslant C|x-y|^\beta \,, \tag{2.5}$$

then for all $\varphi \in \mathrm{Lip}(\Omega \times G_{d,n})$,

$$\int_{(\Pi(\mathrm{supp}\,\varphi)\cap\Omega)\times G_{d,n}} \left\|P^i(y) - T\right\| \, dV(y,T) \leqslant 2C\delta_i^\beta \|V\| \left(\Pi(\mathrm{supp}\,\varphi) \cap \Omega\right) \,,$$

and

$$|\langle V_i, \varphi\rangle - \langle V, \varphi\rangle| \leqslant \mathrm{lip}(\varphi)\|V\| \left(\Pi(\mathrm{supp}\,\varphi)\cap\Omega\right) \left(\delta_i + 2C\delta_i^\beta\right) \,. \tag{2.6}$$

Proof. – Step 1: Let $\varphi \in \mathrm{Lip}(\Omega \times G_{d,n})$ with Lipschitz constant $\mathrm{lip}(\varphi)$, then

$$|\langle V_i, \varphi\rangle - \langle V, \varphi\rangle| \leqslant \delta_i \mathrm{lip}(\varphi)\|V\|(\Omega) + \mathrm{lip}(\varphi) \int_{\Omega\times G_{d,n}} \left\|P^i(y) - T\right\| \, dV(y,T) \,,$$

Indeed,

$$\begin{aligned}
&|\langle V_i, \varphi\rangle - \langle V, \varphi\rangle| = \left| \int_{\Omega\times G_{d,n}} \varphi(x,S)\, dV_i(x,S) - \int_{\Omega\times G_{d,n}} \varphi(y,T)\, dV(y,T) \right| \\
&= \left| \sum_{K\in\mathcal{K}_i} \int_K \varphi(x, P_K^i) \frac{\|V\|(K)}{|K|} \, d\mathcal{L}^n(x) - \sum_{K\in\mathcal{K}_i} \int_{K\times G_{d,n}} \varphi(y,T)\, dV(y,T) \right| \\
&= \left| \sum_{K\in\mathcal{K}_i} \int_{x\in K} \int_{(y,T)\in K\times G_{d,n}} \varphi(x, P_K^i)\, dV(y,T) \, \frac{d\mathcal{L}^n(x)}{|K|} \right. \qquad (2.7) \\
&\qquad \left. - \sum_{K\in\mathcal{K}_i} \int_{x\in K} \int_{(y,T)\in K\times G_{d,n}} \varphi(y,T)\, dV(y,T) \, \frac{d\mathcal{L}^n(x)}{|K|} \right| \\
&\leqslant \sum_{K\in\mathcal{K}_i} \int_{x\in K} \int_{(y,T)\in K\times G_{d,n}} \underbrace{\left|\varphi(x,P_K^i) - \varphi(y,T)\right|}_{\leqslant \mathrm{lip}(\varphi)\left(|x-y| + \left\|P_K^i - T\right\|\right)} \, dV(y,T) \, \frac{d\mathcal{L}^n(x)}{|K|} \\
&\leqslant \mathrm{lip}(\varphi) \sum_{K\in\mathcal{K}_i} \frac{|\Pi(\mathrm{supp}\,\varphi)\cap K|}{|K|} \left(\delta_i \|V\|(\Pi(\mathrm{supp}\,\varphi)\cap K) + \int_{(K\cap\Pi(\mathrm{supp}(\varphi)))\times G_{d,n}} \left\|P^i(y) - T\right\| \, dV(y,T) \right) \\
&\leqslant \delta_i \mathrm{lip}(\varphi) \|V\|(\Omega \cap \Pi(\mathrm{supp}\,\varphi)) + \mathrm{lip}(\varphi) \int_{(\Omega\cap\Pi(\mathrm{supp}(\varphi)))\times G_{d,n}} \left\|P^i(y) - T\right\| \, dV(y,T) \\
&\leqslant \delta_i \mathrm{lip}(\varphi)\|V\|(\Omega) + \mathrm{lip}(\varphi) \int_{\Omega\times G_{d,n}} \left\|P^i(y) - T\right\| \, dV(y,T) \,.
\end{aligned}$$

We now study the convergence of the term $\int_{\Omega\times G_{d,n}} \left\|P^i(y) - T\right\| \, dV(y,T)$.

– Step 2: There exists $A^i : \Omega \to \mathcal{M}_n(\mathbb{R})$ constant in each cell $K \in \mathcal{K}_i$ such that

$$\int_{\Omega\times G_{d,n}} \left\|A^i(y) - T\right\| \, dV(y,T) = \int_{y\in\Omega} \left\|A^i(y) - T_y M\right\| \, d\|V\|(y) \xrightarrow[i\to+\infty]{} 0 \,.$$

Indeed, let $\varepsilon > 0$, as $x \mapsto T_x M \in \mathrm{L}^1(\Omega, \mathcal{M}^n(\mathbb{R}), \|V\|)$ then, there exists $A : \Omega \to \mathcal{M}^n(\mathbb{R}) \in \mathrm{Lip}(\Omega)$ such that

$$\int_{y\in\Omega} \|A(y) - T_y M\| \, d\|V\|(y) < \varepsilon \,.$$

For all i and $K \in \mathcal{K}_i$, define for $x \in K$,

$$A^i(x) = \frac{1}{\|V\|(K)} \int_K A(y)\, d\|V\|(y) \,.$$

Then

$$\begin{aligned}
\int_{y\in\Omega} \left\| A^i(y) - T_y M \right\| \, d\|V\|(y) &\leqslant \int_{y\in\Omega} \left\| A^i(y) - A(y) \right\| \, d\|V\|(y) + \int_{y\in\Omega} \| A(y) - T_y M \| \, d\|V\|(y) \\
&\leqslant \varepsilon + \sum_{K\in\mathcal{K}_i} \int_{y\in K} \left\| \frac{1}{\|V\|(K)} \int_K A(u)\, d\|V\|(u) - A(y) \right\| \, d\|V\|(y) \\
&\leqslant \varepsilon + \sum_{K\in\mathcal{K}_i} \frac{1}{\|V\|(K)} \int_{y\in K} \int_{u\in K} \| A(u) - A(y) \| \, d\|V\|(u)\, d\|V\|(y) \\
&\leqslant \varepsilon + \delta_i \mathrm{lip}(A) \|V\|(\Omega) \leqslant 2\varepsilon \text{ for } i \text{ large enough.}
\end{aligned}$$

– Step 3: There exists $T^i : \Omega \to G_{d,n}$ constant in each cell $K \in \mathcal{K}_i$ such that

$$\int_{\Omega\times G_{d,n}} \left\| T^i(y) - T \right\| \, dV(y,T) = \int_{y\in\Omega} \left\| T^i(y) - T_y M \right\| \, d\|V\|(y) \xrightarrow[i\to+\infty]{} 0 \,.$$

Indeed, let $\varepsilon > 0$, thanks to Step 2, fix i and $A^i : \Omega \to \mathcal{M}_n(\mathbb{R})$ such that

$$\sum_{K\in\mathcal{K}_i} \int_K \left\| A^i(y) - T_y M \right\| \, d\|V\|(y) < \varepsilon \,,$$

so that $\displaystyle\int_K \left\| A^i(y) - T_y M \right\| \, d\|V\|(y) = \varepsilon_K^i$ with $\displaystyle\sum_{K\in\mathcal{K}_i} \varepsilon_K^i < \varepsilon$. In particular, for all $K \in \mathcal{K}_i$, there exists $y_K \in K$ such that

$$\left\| A^i(y_K) - T_{y_K} M \right\| \leqslant \frac{\varepsilon_K^i}{\|V\|(K)} \,.$$

Define $T^i : \Omega \to G_{d,n}$, constant in each cell, by $T^i(y) = T_{y_K} M$ for $K \in \mathcal{K}_i$ and $y \in K$, and then,

$$\begin{aligned}
\int_{\Omega\times G_{d,n}} \left\| T^i(y) - T \right\| dV(y,T) &= \sum_{K\in\mathcal{K}_i} \int_K \| T_{y_K} M - T_y M \| \, d\|V\|(y) \\
&\leqslant \sum_{K\in\mathcal{K}_i} \int_K \| T_{y_K} M - \underbrace{A^i(y)}_{=A^i(y_K)} \| \, d\|V\|(y) + \int_{\Omega\times G_{d,n}} \left\| A^i(y) - T \right\| \, dV(y,T) \\
&\leqslant \sum_{K\in\mathcal{K}_i} \int_K \frac{\varepsilon_K^i}{\|V\|(K)} d\|V\|(y) + \varepsilon \\
&\leqslant 2\varepsilon \,.
\end{aligned}$$

– Step 4: $\displaystyle\int_{\Omega\times G_{d,n}} \left\| P^i(y) - T \right\| \, dV(y,T) \xrightarrow[i\to+\infty]{} 0$.

Indeed, thanks to Step 3, let $T^i : \Omega \to G_{d,n}$ constant in each cell $K \in \mathcal{K}_i$: for all $y \in K$, $T^i(y) = T_K^i$, and such that $\displaystyle\int_{\Omega\times G_{d,n}} \left\| T^i(y) - T \right\| \, dV(y,T) \xrightarrow[i\to+\infty]{} 0$. And remind that for all

$K \in \mathcal{K}_i$, $P_K^i \in \underset{P \in G_{d,n}}{\arg\min} \int_{K \times G_{d,n}} \|P - T\| \, dV(y,T)$ so that,

$$\begin{aligned}
\int_{\Omega \times G_{d,n}} \left\|P^i(y) - T\right\| \, dV(y,T) &= \sum_{K \in \mathcal{K}_i} \int_{K \times G_{d,n}} \left\|P_K^i - T\right\| \, dV(y,T) \\
&\leqslant \sum_{K \in \mathcal{K}_i} \int_{K \times G_{d,n}} \left\|T_K^i - T\right\| \, dV(y,T) \\
&= \int_{\Omega \times G_{d,n}} \left\|T^i(y) - T\right\| \, dV(y,T) \\
&\xrightarrow[i \to +\infty]{} 0 \, .
\end{aligned}$$

– Step 5: Thanks to Steps 1 to 4, we have proved that for any $\varphi \in \mathrm{Lip}(\Omega \times G_{d,n})$,

$$\langle V_i, \varphi \rangle \xrightarrow[i \to +\infty]{} \langle V, \varphi \rangle \ , \tag{2.8}$$

it remains to check the case $\varphi \in \mathrm{C}_c^0(\Omega \times G_{d,n})$. Let $\varphi \in \mathrm{C}_c^0(\Omega \times G_{d,n})$ and $\varepsilon > 0$. We can extend φ into $\overline{\varphi} \in \mathrm{C}_c^0(\Omega \times \mathcal{M}_n(\mathbb{R}))$ by Tietze-Urysohn theorem since $G_{d,n}$ is closed. Then, by density of $\mathrm{Lip}(\Omega \times \mathcal{M}_n(\mathbb{R}))$ in $\mathrm{C}_c^0(\Omega \times \mathcal{M}_n(\mathbb{R}))$ with respect to the uniform topology, there exists $\overline{\psi} \in \mathrm{Lip}(\Omega \times \mathcal{M}_n(\mathbb{R}))$ such that $\left\|\overline{\varphi} - \overline{\psi}\right\|_\infty < \varepsilon$. Let now $\psi \in \mathrm{Lip}(\Omega \times G_{d,n})$ be the restriction of $\overline{\psi}$ to $\Omega \times G_{d,n}$, then,

$$\begin{aligned}
|\langle V, \varphi \rangle - \langle V_i, \varphi \rangle| &\leqslant |\langle V, \varphi \rangle - \langle V, \psi \rangle| + |\langle V, \psi \rangle - \langle V_i, \psi \rangle| + |\langle V_i, \psi \rangle - \langle V_i, \varphi \rangle| \\
&\leqslant \|V\|(\Omega) \|\varphi - \psi\|_\infty + |\langle V, \psi \rangle - \langle V_i, \psi \rangle| + \|V_i\|(\Omega) \|\varphi - \psi\|_\infty \, .
\end{aligned}$$

As $\|V_i\|(\Omega) = \|V\|(\Omega)$ for all i by definition of V_i and $|\langle V, \psi \rangle - \langle V_i, \psi \rangle| \xrightarrow[i \to +\infty]{} 0$ by (2.8), there exists i large enough such that

$$|\langle V, \varphi \rangle - \langle V_i, \varphi \rangle| \leqslant (2\|V\|(\Omega) + 1) \, \varepsilon \, ,$$

which concludes the general case.

– Step 6: Assume now that the Hölder regularity of the tangent plane (2.5) holds, then in Step 2, directly define for all i and $K \in \mathcal{K}_i$,

$$A^i(x) = \frac{1}{\|V\|(K)} \int_K T_u M \, d\|V\|(u) \quad \forall x \in K \, .$$

Let $B \subset \Omega$,

$$\begin{aligned}
\int_{B \times G_{d,n}} \left\|A^i(y) - T\right\| \, dV(y,T) &= \sum_{K \in \mathcal{K}_i} \int_{K \cap B} \left\| \frac{1}{\|V\|(K)} \int_K T_u M \, d\|V\|(u) - T_y M \right\| \, d\|V\|(y) \\
&\leqslant \sum_{K \in \mathcal{K}_i} \int_{K \cap B} \frac{1}{\|V\|(K)} \int_K \underbrace{\|T_u M - T_y M\|}_{\leqslant C|u-y|^\beta \leqslant C\delta_i^\beta} \, d\|V\|(u) \, d\|V\|(y) \\
&\leqslant C \delta_i^\beta \|V\|(B) \, .
\end{aligned}$$

Then in Step 3, with the same definition of T^i with respect to A^i and $T_y M$,

$$\int_{B \times G_{d,n}} \left\|P^i(y) - T\right\| \, dV(y,T) \leqslant 2 \int_{B \times G_{d,n}} \left\|A^i(y) - T\right\| \, dV(y,T) \leqslant 2C\delta_i^\beta \|V\|(B) \, . \tag{2.9}$$

In particular,

$$\int_\Omega \left\| P^i(y) - T \right\| \, dV(y,T) \leqslant 2C\delta_i^\beta \|V\|(\Omega) \, .$$

Eventually, by (2.4) and (2.9),

$$| \langle V, \varphi \rangle - \langle V_i, \varphi \rangle | \leqslant \mathrm{lip}(\varphi) \left(\delta_i + 2C\delta_i^\beta \right) \|V\| \left(\Pi(\mathrm{supp}\, \varphi) \cap \Omega \right) \, .$$

□

Remark 2.2 ("Accuracy of the approximation spaces $\mathcal{A}_\delta(\mathcal{K})$"). The conclusion (2.6) can be reformulated in terms of an asymmetric quantity close to the Hausdorff distance between $\mathcal{A}_\delta(\mathcal{K})$ and $\mathcal{A}_m^\beta = \{$rectifiable d–varifolds of prescribed mass $\leqslant m$ satisfying (2.5) for some $\beta\}$:

$$\begin{aligned} d_{\mathcal{H}}^{asym}(\mathcal{A}_m^\beta, \mathcal{A}_\delta(\mathcal{K})) &= \sup_{V \in \mathcal{A}_m^\beta} \inf_{W \in \mathcal{A}_\delta(\mathcal{K})} \Delta^{1,1}(V,W) \\ &\leqslant \left(\delta + 2C\delta^\beta \right) m \, , \end{aligned}$$

where $\Delta^{1,1}$ is the flat distance defined in Chapter 1 in Definition 1.15 by

$$\Delta^{1,1}(V,W) = \sup \left\{ \left| \int \varphi \, dV - \int \varphi \, dW \right| \; : \; \varphi \in \mathrm{Lip}_1, \; \|\varphi\|_\infty \leqslant 1 \right\} \, .$$

Notice that $d_{\mathcal{H}}^{asym}$ is not exactly the Haussdorff distance $d_{\mathcal{H}}(\mathcal{A}_m^\beta, \mathcal{A}_\delta(K))$ since we care only of the approximation of $\mathcal{A}_m^\beta$ by $\mathcal{A}_\delta(\mathcal{K})$ and not the contrary:

$$d_{\mathcal{H}}(\mathcal{A}_m^\beta, \mathcal{A}_\delta(K)) = \max \left\{ d_{\mathcal{H}}^{asym}(\mathcal{A}_m^\beta, \mathcal{A}_\delta(K)), d_{\mathcal{H}}^{asym}(\mathcal{A}_\delta(K), \mathcal{A}_m^\beta) \right\} \, .$$

2.2 First variation of discrete volumetric varifolds

Before computing the first variation of a discrete volumetric varifold in the following result (Proposition 2.2), let us notice that by definition, the mass and the tangent plane are constant in each cell so that we expect the first variation to be concentrated on the faces of the mesh.

Proposition 2.2. *Let $(\mathcal{K}, \mathcal{E})$ be a mesh of $\mathbb{R}^n$. For K_+, $K_- \in \mathcal{K}$, we denote by $\sigma = K_+ | K_- \in \mathcal{E}$ the common face to K_+ and K_-, and $n_{K_+,\sigma}$ is then the outer-pointing normal to the face σ (pointing outside K_+). Decompose the set of faces into $\mathcal{E} = \mathcal{E}_{int} \cup \mathcal{E}_b \cup \mathcal{E}$ where*

- *$\mathcal{E}_{int}$ is the set of faces $\sigma = K_+ | K_-$ such that m_{K_+}, $m_{K_-} > 0$, called internal faces,*
- *$\mathcal{E}_0$ is the set of faces $\sigma = K_+ | K_-$ such that m_{K_+}, $m_{K_-} = 0$,*
- *$\mathcal{E}_b$ is the set of remaining faces $\sigma = K_+ | K_-$ such that $m_{K_+} > 0$ and $m_{K_-} = 0$ or conversely $m_{K_+} = 0$ and $m_{K_-} > 0$, called boundary faces. In this case, σ is denoted by $K_+ | \cdot$ with $m_{K_+} > 0$.*

For $\{m_K, P_K\}_{K \in \mathcal{K}} \subset \mathbb{R}_+ \times G_{d,n}$, define the d–varifold

$$V_{\mathcal{K}} = \sum_{K \in \mathcal{K}} \frac{m_K}{|K|} \mathcal{L}^n_{|K} \otimes \delta_{P_K} \, .$$

Then,

$$|\delta V_{\mathcal{K}}| = \sum_{\substack{\sigma \in \mathcal{E}_{int}, \\ \sigma = K_- | K_+}} \left| \left[\frac{m_{K_+}}{|K_+|} \Pi_{P_{K_+}} - \frac{m_{K_-}}{|K_-|} \Pi_{P_{K_-}} \right] (n_{K_+,\sigma}) \right| \mathcal{H}^{n-1}_{|\sigma} + \sum_{\substack{\sigma \in \mathcal{E}_b, \\ \sigma = K|\cdot}} \frac{m_K}{|K|} \left| \Pi_{P_K} n_{K,\sigma} \right| \mathcal{H}^{n-1}_{|\sigma} \, ,$$

where Π_P is the orthogonal projection onto the d-plane P.

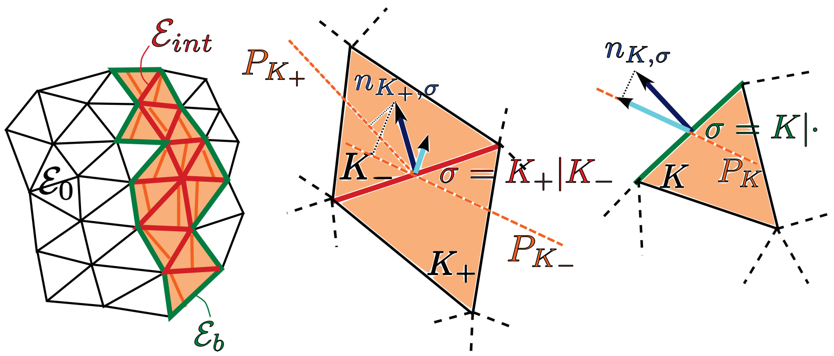


Figure 2.1: Contribution of the different faces to the first variation

We stress that the terms internal faces and boundary faces do not refer to the structure of the mesh $\mathcal{K}$ but to the structure of the support of $V_\mathcal{K}$.

Proof. Let $V_\mathcal{K} = \sum_{K\in\mathcal{K}} \frac{m_K}{|K|}\mathcal{L}^n_{|K} \otimes \delta_{P_K}$ be a discrete varifold associated with the mesh $\mathcal{K}$ and let $X \in \mathrm{C}^1_c(\Omega, \mathbb{R}^n)$. Then,

$$\delta V_\mathcal{K}(X) = \int_{\Omega\times G_{d,n}} \mathrm{div}_S X(x)\, dV_\mathcal{K}(x,S) = \sum_{K\in\mathcal{K}} \frac{m_K}{|K|} \int_K \mathrm{div}_{P_K} X(x)\, d\mathcal{L}^n(x)\,.$$

Let us compute this term. Fix $(\tau_1, \ldots, \tau_d)$ a basis of the tangent plane $P_\mathcal{K}$ so that

$$\int_K \mathrm{div}_{P_\mathcal{K}} X(x)\, d\mathcal{L}^n(x) = \sum_{j=1}^d \int_K DX(x)\tau_j \cdot \tau_j\, d\mathcal{L}^n(x)\,,$$

and $DX(x)\tau_j \cdot \tau_j = \sum_{k=1}^n (\nabla X_k(x)\cdot\tau_j)\tau_j^k$ so that

$$\begin{aligned}\int_K \mathrm{div}_{P_\mathcal{K}} X(x)\, d\mathcal{L}^n(x) &= \sum_{j=1}^d\sum_{k=1}^n \tau_j^k \int_K (\nabla X_k(x)\cdot\tau_j)\, d\mathcal{L}^n(x) = -\sum_{j=1}^d\sum_{k=1}^n \tau_j^k \int_{\partial K} X_k\tau_j\cdot n_{out}\, d\mathcal{H}^d \\ &= -\int_{\partial K}\sum_{j=1}^d(\tau_j\cdot n_{out})\sum_{k=1}^n X_k\tau_j^k\, d\mathcal{H}^d = -\int_{\partial K}\sum_{j=1}^d(\tau_j\cdot n_{out})(X\cdot\tau_j)\, d\mathcal{H}^d \\ &= -\int_{\partial K} X(x)\cdot(\Pi_{P_\mathcal{K}} n_{out})\, d\mathcal{H}^d(x)\,,\end{aligned}$$

where $\Pi_{P_\mathcal{K}}$ is the orthogonal projection onto $P_\mathcal{K}$ and n_{out} is the outward-pointing normal. Consequently

$$|\delta V_\mathcal{K}(X)| = \left|\sum_{K\in\mathcal{K}}\frac{m_K}{|K|}\int_{\partial K} X(x)\cdot(\Pi_{P_\mathcal{K}} n_{out})\, d\mathcal{H}^d(x)\right| \leqslant \|X\|_\infty \sum_{K\in\mathcal{K}}\frac{m_K}{|K|}\,|\Pi_{P_\mathcal{K}} n_{out}|\,\mathcal{H}^d(\partial K)\,.$$

For a fixed mesh, the sum is locally finite and then, $V_\mathcal{K}$ has locally bounded first variation. But what happens if the size of the mesh tends to 0? In order to compute the total variation of $\delta V_\mathcal{K}$ as a Radon measure, we just have to rewrite the sum as a sum on the faces $\mathcal{E}$ of the mesh. This is more natural since $\delta V_{\mathcal{K}}$ is concentrated on faces. Thus

$$\begin{aligned}\delta V_{\mathcal{K}} &= -\sum_{\substack{\sigma\in\mathcal{E}_{int},\\ \sigma=K_-|K_+}} \left[\frac{m_{K_+}}{|K_+|}\Pi_{P_{K_+}} n_{K_+,\sigma} + \frac{m_{K_-}}{|K_-|}\Pi_{P_{K_-}} n_{K_-,\sigma}\right] \mathcal{H}^{n-1}_{|\sigma} - \sum_{\substack{\sigma\in\mathcal{E}_b,\\ \sigma=K|\cdot}} \frac{m_K}{|K|}\Pi_{P_K} n_{K,\sigma}\,\mathcal{H}^{n-1}_{|\sigma} \\ &= -\sum_{\substack{\sigma\in\mathcal{E}_{int},\\ \sigma=K_-|K_+}} \left[\frac{m_{K_+}}{|K_+|}\Pi_{P_{K_+}} - \frac{m_{K_-}}{|K_-|}\Pi_{P_{K_-}}\right]\cdot (n_{K_+,\sigma})\,\mathcal{H}^{n-1}_{|\sigma} - \sum_{\substack{\sigma\in\mathcal{E}_b,\\ \sigma=K|\cdot}} \frac{m_K}{|K|}\Pi_{P_K} n_{K,\sigma}\,\mathcal{H}^{n-1}_{|\sigma}\,.\end{aligned}$$

Therefore,

$$|\delta V_{\mathcal{K}}| = \sum_{\substack{\sigma\in\mathcal{E}_{int},\\ \sigma=K_-|K_+}} \left| \left[\frac{m_{K_+}}{|K_+|}\Pi_{P_{K_+}} - \frac{m_{K_-}}{|K_-|}\Pi_{P_{K_-}}\right]\cdot (n_{K_+,\sigma})\right| \mathcal{H}^{n-1}_{|\sigma} + \sum_{\substack{\sigma\in\mathcal{E}_b,\\ \sigma=K|\cdot}} \frac{m_K}{|K|}\left|\Pi_{P_K} n_{K,\sigma}\right|\mathcal{H}^{n-1}_{|\sigma}\,.$$

□

Example 2.1. Let us estimate this first variation in a simple case. Let us assume that the mesh is a regular cartesian grid of $\Omega =]0,1[^2 \subset \mathbb{R}^2$ of size $h_{\mathcal{K}}$ so that for all $K \in \mathcal{K}$ and $\sigma \in \mathcal{E}$,

$$|K| = h_{\mathcal{K}}^2 \text{ and } \mathcal{H}^1(\sigma) = h_{\mathcal{K}}\,.$$

Consider the vector line D of direction given by the unit vector $\frac{1}{\sqrt{2}}(1,1)$. Let $V = \mathcal{H}^1_{|D} \otimes \delta_D$ be the canonical 1–varifold associated with D and $V_{\mathcal{K}}$ the volumetric approximation of V in the mesh $\mathcal{K}$, then

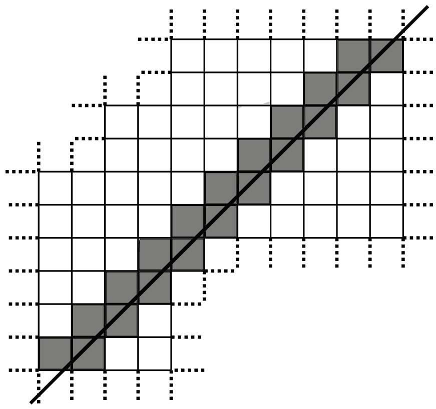
$$\begin{aligned}|\delta V_{\mathcal{K}}|(\Omega) &= \sum_{\substack{\sigma\in\mathcal{E}_{int},\\ \sigma=K_-|K_+}} \left| \left[\frac{m_{K_+}}{|K_+|}\Pi_{P_{K_+}} - \frac{m_{K_-}}{|K_-|}\Pi_{P_{K_-}}\right]\cdot (n_{K_+,\sigma})\right| \mathcal{H}^1(\sigma) + \sum_{\substack{\sigma\in\mathcal{E}_b,\\ \sigma=K|\cdot}} \frac{m_K}{|K|}\left|\Pi_{P_K} n_{K,\sigma}\right|\mathcal{H}^1(\sigma) \\ &= \frac{1}{h_{\mathcal{K}}} \sum_{\substack{\sigma\in\mathcal{E}_{int},\\ \sigma=K_-|K_+}} \left|m_{K_+} - m_{K_-}\right| \left|\Pi_D n_{K_+,\sigma}\right| + \frac{1}{h_{\mathcal{K}}} \sum_{\substack{\sigma\in\mathcal{E}_b,\\ \sigma=K|\cdot}} m_K \left|\Pi_D n_{K,\sigma}\right|\,.\end{aligned}$$

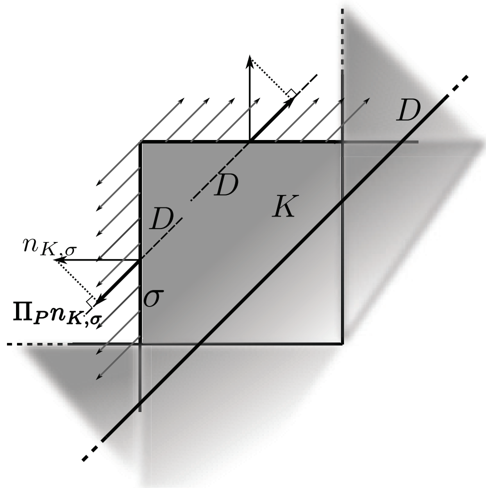
And $|\Pi_D n_{K,\sigma}| = \frac{\sqrt{2}}{2}$ (for any K, σ) so that

$$|\delta V_{\mathcal{K}}|(\Omega) = \frac{\sqrt{2}}{2h_{\mathcal{K}}} \sum_{\substack{\sigma\in\mathcal{E}_{int},\\ \sigma=K_-|K_+}} \left|m_{K_+} - m_{K_-}\right| + \frac{\sqrt{2}}{2h_{\mathcal{K}}} \underbrace{\sum_{\substack{\sigma\in\mathcal{E}_b,\\ \sigma=K|\cdot}} m_K}_{=\|V\|(\Omega)}\,.$$

So that if we now consider successive volumetric approximations $V_{\mathcal{K}_i}$ of V associated with successive meshes $\mathcal{K}_i$ whose size $h_{\mathcal{K}_i}$ tends to 0 when i tends to ∞,

$$|\delta V_{\mathcal{K}_i}|(\Omega) = \frac{\sqrt{2}}{2h_{\mathcal{K}_i}} \left(\sum_{\substack{\sigma\in\mathcal{E}_{int},\\ \sigma=K_-|K_+}} \left|m_{K_+} - m_{K_-}\right| + \|V\|(\Omega) \right) \geqslant \frac{\sqrt{2}}{2h_{\mathcal{K}_i}} \|V\|(\Omega) \xrightarrow[i\to\infty]{} +\infty\,.$$





More generally, the problem is that the tangential direction P_K and the direction of the face σ have no reason to be correlated so that the term $|\Pi_{P_K} n_{K,\sigma}|$ can be large (close to 1) and thus, if the mesh is not adapted to the tangential directions, $|\delta V_{\mathcal{K}_i}|(\Omega)$ may explode when the size of the mesh $h_{\mathcal{K}_i}$ tends to 0. Of course, we are not saying that $|\delta V_{\mathcal{K}_i}|(\Omega)$ always explodes when refining the mesh, but that it may happen and it is not something easy to control except by adapting the mesh to the tangential directions $P_{\mathcal{K}}$ in the boundary cells. This is clearly a problem showing that the classical notion of first variation is not well adapted to this kind of volumetric discretization.

2.3 Point cloud varifolds

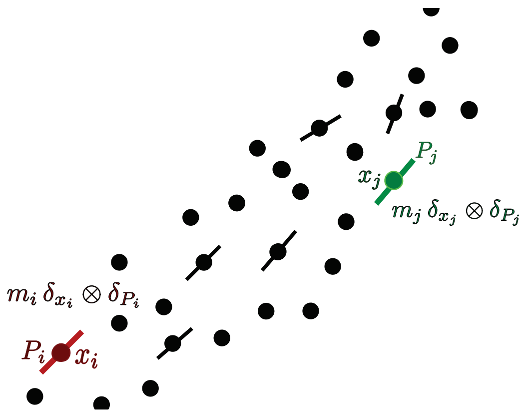
We already explained how it is possible to endow a point cloud with a d–varifold structure in Example 1.4. *We will now justify why any result we will prove on discrete volumetric varifolds easily transfers to point cloud varifolds.* Let first recall the definition of a point cloud varifold:

Definition 2.2 (Point cloud varifolds)**.** *Let $\{x_i\}_{i=1\ldots N} \subset \mathbb{R}^n$ be a point cloud, weighted by the masses $\{m_i\}_{i=1\ldots N}$ and provided with directions $\{P_i\}_{i=1\ldots N} \subset G_{d,n}$. We can thus associate a d–varifolds on $\mathbb{R}^n \times G_{d,n}$ with this point cloud:*

$$V = \sum_{i=1}^{N} m_i \, \delta_{x_i} \otimes \delta_{P_i} \, ,$$

so that for $\varphi \in \mathrm{C}_c^0(\Omega \times G_{d,n})$,

$$\int \varphi \, dV = \sum_{i=1}^{N} \varphi(x_i, P_i) \, .$$



Let us begin with the question of the approximation of rectifiable varifolds by point cloud varifolds. Notice that to each discrete volumetric varifold $V_{\mathcal{K}} = \displaystyle\sum_{K \in \mathcal{K}} \frac{m_K}{|K|} \mathcal{L}^n_{|K} \otimes \delta_{P_K}$ associated with a mesh $\mathcal{K}$ of size $\delta = \sup_{K \in \mathcal{K}} \mathrm{diam} K$, we can associate the following point cloud varifold:

$$V'_{\mathcal{K}} = \sum_{K \in \mathcal{K}} m_K \delta_{x_K} \otimes \delta_{P_K}$$

where, x_K is a point in the cell K, the center of mass for instance, but not necessarily. And with such a construction, for any $\varphi \in \mathrm{Lip}(\mathbb{R}^n \times G_{d,n})$,

$$\begin{aligned}
\left|\langle V_{\mathcal{K}}, \varphi\rangle - \langle V'_{\mathcal{K}}, \varphi\rangle\right| &= \left| \sum_{K\in\mathcal{K}} \frac{m_K}{|K|} \int_K \varphi(x, P_K) \mathcal{L}^n(x) - \sum_{K\in\mathcal{K}} m_K \varphi(x_K, P_K) \right| \\
&\leqslant \sum_{K\in\mathcal{K}} \frac{m_K}{|K|} \int_K |\varphi(x, P_K) - \varphi(x_K, P_K)| \, d\mathcal{L}^n(x) \\
&\leqslant \mathrm{lip}(\varphi) \sum_{K\in\mathcal{K}} m_K \mathrm{diam} K \\
&\leqslant \delta \mathrm{lip}(\varphi) \|V_{\mathcal{K}}\|(\mathbb{R}^n) \, .
\end{aligned}$$

Therefore (as in Step 5 of the proof of Theorem 2.1),

$$\left(V_{\mathcal{K}} - V'_{\mathcal{K}}\right) \xrightarrow[\delta \to 0]{} 0 \, .$$

And moreover,

$$\Delta^{1,1}(V_{\mathcal{K}}, V'_{\mathcal{K}}) \leqslant \delta \|V_{\mathcal{K}}\|(\mathbb{R}^n) \xrightarrow[\delta \to 0]{} 0 \, .$$

That is why in the following chapters, we focus only on discrete volumetric varifolds. Any result we will prove on discrete volumetric varifolds easily transfers to point cloud varifolds thanks to this correspondence.

First variation of a point cloud varifold

Point cloud varifolds never have bounded bounded first variation, independently of the directions P_K, since a Dirac mass does not. Indeed, let $V = \delta_0 \otimes \delta_D$ be a Dirac mass, take any $\varphi \in \mathrm{C}^1_c(B_1(0))$ with non-zero gradient $\nabla^D \varphi(0)$ in the direction D and define $\varphi_\varepsilon(y) = \varphi\left(\frac{y}{\varepsilon}\right)$. Then, for any $u \in \mathbb{R}^n$,

$$\delta V(\varphi_\varepsilon u) = \frac{1}{\varepsilon} \int_{\mathbb{R}^n \times G_{d,n}} \nabla^S \varphi\left(\frac{y}{\varepsilon}\right) \cdot u \, dV(y, S) = \frac{1}{\varepsilon} \nabla^D \varphi(0) \cdot u \, ;$$

and for instance, with $u = \nabla^D \varphi(0)$,

$$\delta V(\varphi_\varepsilon u) = \frac{1}{\varepsilon} \left|\nabla^D \varphi(0)\right|^2 \xrightarrow[\varepsilon \to 0]{} +\infty \, .$$

Conclusion. If we want to use varifold structures on discrete type objects (discrete volumetric varifolds, point cloud varifolds for instance) to handle in a general setting the minimization of functionals defined on surfaces as the area functional or the Willmore functional, there is however an important point to overcome. We just showed that point cloud varifolds do not have bounded first variation. As for discrete volumetric varifolds, we saw in Example 2.1 (where the limit object was a simple line) that there are weakly–$*$ converging sequences of discrete volumetric varifolds $V_{\mathcal{K}_i} \xrightarrow[i\to\infty]{*} V$, with individually (locally) bounded first variation, but such that the total variation of the first variation explodes

$$|\delta V_{\mathcal{K}_i}|(\Omega) \xrightarrow[i \to \infty]{} +\infty \, .$$

This means that the convergence of the sequence $V_{\mathcal{K}_i} \xrightarrow[i\to\infty]{*} V$ does not imply the weak–$*$ convergence of the first variations and the usual Allard's compactness theorem does not apply. We thus have to answer the following questions (**Questions 1.2** and **1.3**): what conditions on a weakly–$*$ converging sequence of varifolds (not supposed rectifiable) ensure that the limit varifold is rectifiable? has bounded first variation?

CHAPTER 3

Conditions quantitatives de rectifiabilité dans l'espace des varifolds

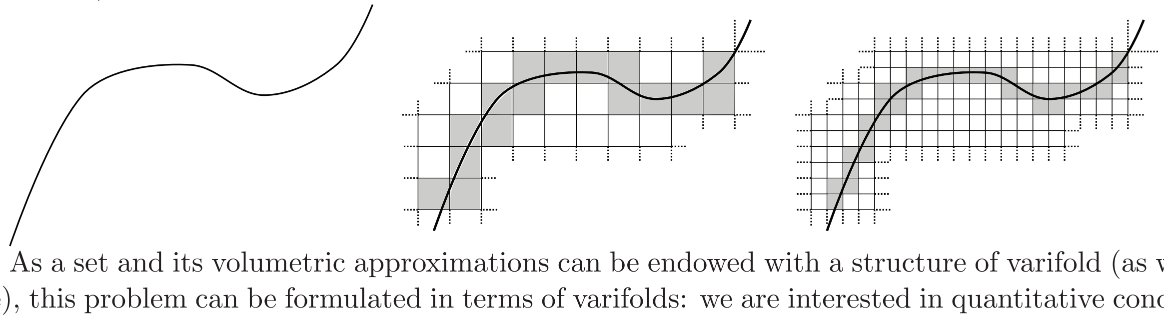
Ce chapitre constitue l'article [Bue14] Quantitative conditions of rectifiability for varifolds available at `http://adsabs.harvard.edu/abs/2014arXiv1409.4749B`.

In this chapter, we focus on Question 1.2:

Question. 1.2. What conditions on a weakly–$*$ converging sequence of varifolds (not supposed rectifiable) ensure that the limit varifold is rectifiable?

Introduction

The set of regular surfaces lacks compactness properties (for Hausdorff convergence for instance), which is a problem when minimizing geometric energies defined on surfaces. In order to gain compactness, the set of surfaces can be extended to the set of varifolds and endowed with a notion of convergence (weak–$*$ convergence of Radon measures). Nevertheless, the problem turns to be the following: how to ensure that a weak–$*$ limit of varifolds is regular (at least in the weak sense of rectifiability)? W. K. Allard (see [All72]) answered this question in the case where the weak–$*$ converging sequence is made of weakly regular surfaces (rectifiable varifolds to be precise). But what about the case when the weak–$*$ converging sequence is made of more general varifolds? Assume that we have a sequence of volumetric approximations of some set M, how can we know if M is regular (d–rectifiable for some d), knowing only its successive approximations ?



As a set and its volumetric approximations can be endowed with a structure of varifold (as we will see), this problem can be formulated in terms of varifolds: we are interested in quantitative conditions on a given sequence of d–varifolds ensuring that the limit (when it exists) is rectifiable. Before going into technical details, let us consider the problem of rectifiability in simplified settings.

- First, let $f : \mathbb{R} \to \mathbb{R}$. We are looking for conditions ensuring that f is differentiable (in some sense). The most simple answer is to impose that the difference quotient has a finite limit everywhere. But assume that moreover, we ask for something more quantitative, that is to say some condition that could be expressed through bounds on some well chosen quantities (for instance, from a numerical point of view, it is easier to deal with bounded quantities than with the existence of a limit). We will refer to this kind of condition as "quantitative conditions" (see also [DS93b]). There exists an answer by Dorronsoro [Dor85] (we give here a simplified version, see [DS93a]).

Theorem 3.1 (see [Dor85] and [DS93a])**.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be locally integrable and let $q \geqslant 1$ such that $q < \dfrac{2d}{d-2}$ if $d > 1$. Then, the distributional gradient of f is in L^2 if and only if*

$$\int_{\mathbb{R}^d} \int_0^1 \gamma_q(x,r)^2 \, \frac{dr}{r} \, dx < +\infty \quad \textit{with} \quad \gamma_q(x,r)^q = \inf_{\substack{a \text{ affine} \\ \text{function}}} \frac{1}{r^{d+1}} \int_{B_r(x)} |f(y) - a(y)|^q \, dy$$

The function γ_q penalizes the distance from f to its best affine approximation locally everywhere. This theorem characterizes the weak differentiability (in the sense of a L^2 gradient) quantitatively in terms of L^2–estimate on γ_q (with the singular weight $\frac{1}{r}$).

– Now, we take a set M in $\mathbb{R}^n$ and we ask the same question: how to ensure that this set is regular (meaning d–rectifiable for some d)? Of course, we are still looking for quantitative conditions. This problem has been studied by P.W. Jones (for 1–rectifiable sets) in connection with the travelling salesman problem ([Jon90]) then by K. Okikiolu ([Oki92]), by S. Semmes and G. David ([DS91b]) and by H. Pajot ([Paj97]). As one can see in the following result stated by H. Pajot in [Paj97], the exhibited conditions are not dissimilar to Dorronsoro's. We first introduce the L^q generalization of the so called Jones' β numbers, (see [Jon90] for Jones' β numbers and [Paj97] for the L^q generalization):

Definition 3.1. *Let $M \subset \mathbb{R}^n$ and $d \in \mathbb{N}$, $d \leqslant n$.*

$$\beta_\infty(x,r,M) = \inf_{P \text{ affine } d-\text{plane}} \sup_{y \in M \cap B_r(x)} \frac{d(y,P)}{r} \qquad \textit{if } B_r(x) \cap M \neq \emptyset\,,$$

$$\beta_\infty(x,r,M) = 0 \qquad \textit{if } B_r(x) \cap M = \emptyset\,,$$

$$\beta_q(x,r,M) = \inf_{P \text{ affine } d-\text{plane}} \left(\frac{1}{r^d} \int_{y \in B_r(x) \cap M} \left(\frac{d(y,P)}{r} \right)^q d\mathcal{H}^d(y) \right)^{\frac{1}{q}} \qquad \textit{if } 1 \leqslant q < +\infty\,.$$

The $\beta_q(x,r,M)$ measure the distance from the set M to its best affine approximation at a given point x and a given scale r.

Theorem 3.2 ([Paj97])**.** *Let $M \subset \mathbb{R}^n$ compact with $\mathcal{H}^d(M) < +\infty$. Let q be such that*

$$\begin{cases} 1 \leqslant q \leqslant \infty & \textit{if} \quad d = 1 \\ 1 \leqslant q < \dfrac{2d}{d-2} & \textit{if} \quad d \geqslant 2\,. \end{cases}$$

We assume that for $\mathcal{H}^d$–almost every $x \in M$, the following properties hold:

(i) $\theta_*^d(x,M) = \liminf_{r \downarrow 0} \dfrac{\mathcal{H}^d(M \cap B_r(x))}{\omega_d r^d} > 0$,

(ii) $\displaystyle\int_{r=0}^1 \beta_q(x,r,M)^2 \, \frac{dr}{r} < \infty$.

Then M is d–rectifiable.

– Let us get closer to our initial question: now we consider the same question in the context of varifolds. Recall that from a mathematical point of view, a d–varifold V in $\Omega \subset \mathbb{R}^n$ is a Radon measure on the product $\Omega \times G_{d,n}$, where

$$G_{d,n} = \{d\text{–dimensional subspaces of } \mathbb{R}^n \}\ .$$

Varifolds can be loosely seen as a set of generalized surfaces: let M be a d–submanifold (or a d–rectifiable set) in Ω and denote by $T_x M$ its tangent plane at x, then the Radon measure $V(x,P) = \mathcal{H}^d_{|M}(x) \otimes \delta_{T_x M}(P)$ is a d–varifold associated to M, involving both spatial and tangential information on M. The measure obtained by projecting V on the spatial part Ω is called the mass $\|V\|$. In the previous specific case where V comes from a d–rectifiable set M then the mass is $\|V\| = \mathcal{H}^d_{|M}$. See the next section for more details about varifolds. We can now state the first result that we obtain in this paper about quantitative conditions of rectifiability in the context of varifolds:

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let V be a d–varifold in Ω with finite mass $\|V\|(\Omega) < +\infty$. Assume that:*

(i) there exist $0 < C_1 < C_2$ such that for $\|V\|$–almost every $x \in \Omega$ and for every $r > 0$,

$$C_1 r^d \leqslant \|V\|(B_r(x)) \leqslant C_2 r^d \,, \tag{3.1}$$

(ii) $\displaystyle\int_{\Omega \times G_{d,n}} E_0(x,P,V)\, dV(x,P) < +\infty$, where

$$E_0(x,P,V) = \int_{r=0}^1 \frac{1}{r^d} \int_{y \in B_r(x) \cap \Omega} \left(\frac{d(y-x,P)}{r} \right)^2 d\|V\|(y) \, \frac{dr}{r}$$

defines the averaged height excess.

Then V is a rectifiable d–varifold.

The first assumption is called Ahlfors-regularity. It implies in particular that V is d–dimensional but with some uniform control on the d–density. Adding the second assumption both ensures that the support M of the mass measure $\|V\|$ is a d–rectifiable set and that the tangential part of V is coherent with M, that is to say $V = \|V\| \otimes \delta_{T_x M}$. We will refer to these two conditions as static quantitative conditions of rectifiability for a given d–varifold, by opposition to the next conditions, involving the limit of a sequence of d–varifolds, which we will refer to as the approximation case. These static conditions are not very difficult to derive from Pajot's theorem, the difficult part is the next one: the approximation case.

– Now we consider a sequence $(V_i)_i$ of d–varifolds (weakly–$*$) converging to a d–varifold V. The problem is to find quantitative conditions on $(V_i)_i$ that ensure the rectifiability of V? The idea is to consider the static conditions with uniform bounds and using a notion of scale encoded by the parameters α_i and β_i in the following result:

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $(V_i)_i$ be a sequence of d–varifolds in Ω weakly–$*$ converging to some d–varifold V of finite mass $\|V\|(\Omega) < +\infty$. Fix two decreasing and infinitesimal (tending to 0) sequences of positive numbers $(\alpha_i)_i$ and $(\beta_i)_i$ and assume that:*

(i) there exist $0 < C_1 < C_2$ such that for $\|V_i\|$–almost every $x \in \Omega$ and for every $\beta_i < r < d(x, \Omega^c)$,

$$C_1 r^d \leqslant \|V_i\|(B_r(x)) \leqslant C_2 r^d \,,$$

(ii) $\displaystyle\sup_i \int_{\Omega \times G_{d,n}} E_{\alpha_i}(x,P,V_i)\, dV_i(x,P) < +\infty$, where

$$E_\alpha(x,P,W) = \int_{r=\alpha_i}^1 \frac{1}{r^d} \int_{y \in B_r(x) \cap \Omega} \left(\frac{d(y-x,P)}{r} \right)^2 d\|W\|(y) \, \frac{dr}{r}$$

denotes the α–approximate averaged height excess.

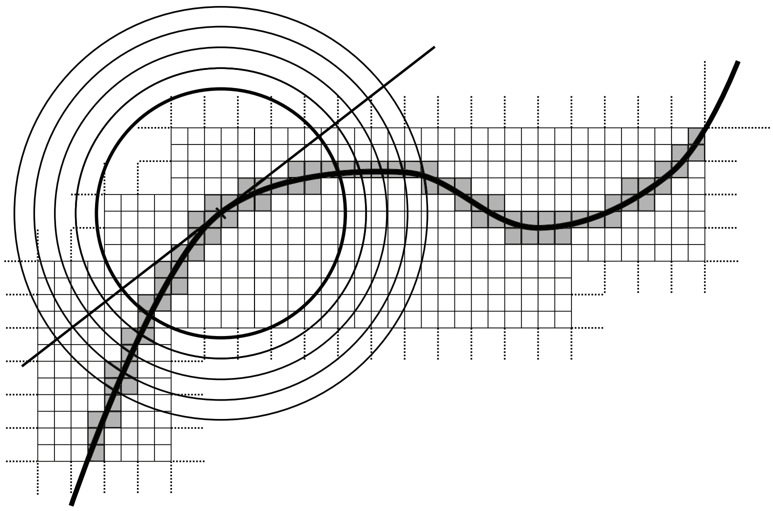
Then V is a rectifiable d–varifold.

We stress that the sequence $(V_i)_i$ in Theorem 3.4 is not necessarily made of rectifiable d–varifolds. The parameters α_i and β_i allow to study the varifolds at a large scale (from far away). The main difficulty in the proof of Theorem 3.4 is to understand the link between

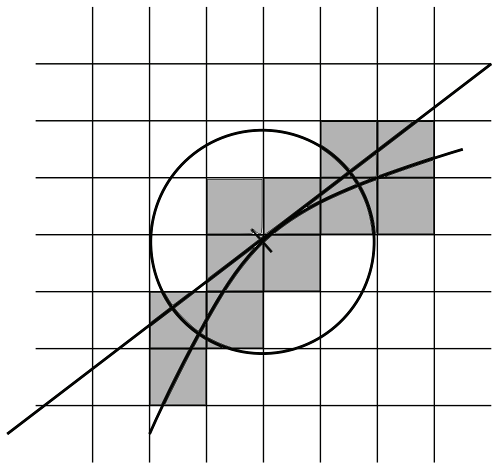
– the choice of α_i ensuring a good convergence of the successive approximate averaged height excess energies $E_{\alpha_i}(x, P, V_i)$ to the averaged height excess energy $E_0(x, P, V)$

– and a notion of convergence speed of the sequence $(V_i)_i$ obtained thanks to a strong characterization of weak–$*$ convergence.

In the following example, we can guess that the parameters α_i and β_i must be large with respect to the size of the mesh. Loosely speaking, in figure (a), even in the smallest ball, the grey approximation "looks" 1–dimensional. On the contrary, if we continue zooming like in figure (b), the grey approximation "looks" 2–dimensional. The issue is to give a correct sense to this intuitive fact.



(a)



(b)

The plan of the paper is the following: in section 3.1 we collect some basic facts about rectifiability and varifolds that we need thereafter. Then in section 3.2, we state and prove quantitative conditions of rectifiability for varifolds in the static case. In section 3.3, we first establish a result of uniform convergence for the pointwise averaged height excess energies E_α thanks to a strong characterization of weak–$*$ convergence. This allows us to state and prove quantitative conditions of rectifiability for varifolds in the approximation case. In the appendix, we consider some sequence of d–varifolds weakly–$*$ converging to some rectifiable d–varifold $V = \theta \mathcal{H}^d_{|M} \otimes \delta_{T_x M}$ (for some d–rectifiable set M) and we make a connection between the minimizers of $E_{\alpha_i}(x, \cdot, V_i)$, with respect to $P \in G_{d,n}$, and the tangent plane $T_x M$ to M at x.

3.1 Some facts about rectifiability and varifolds

This section contains basic definitions and facts about rectifiability and varifolds, which are already contained in Chapter 1 (but in French) and 2, except for Propositions 3.8 and 3.10.

From now on, we fix d, $n \in \mathbb{N}$ with $1 \leqslant d < n$ and an open set $\Omega \subset \mathbb{R}^n$. Then we recall that we adopted the following notations.

– $\mathcal{L}^n$ is the n–dimensional Lebesgue measure.

– $\mathcal{H}^d$ is the d–dimensional Hausdorff measure.

– $\mathrm{C}^k_c(\Omega)$ is the space of continuous compactly supported functions of class C^k in Ω.

- $B_r(x) = \{y \,|\, |y-x| < r\}$ is the open ball of center x and radius r.
- $G_{d,n} = \{P \subset \mathbb{R}^n \,|\, P$ is a vector subspace of dimension $d\}$.
- $A \triangle B = (A \cup B) \setminus (A \cap B)$ is the symmetric difference.
- $\mathrm{Lip}_k(\Omega)$ is the space of Lipschitz functions in Ω with Lipschitz constant less or equal to k.
- $\omega_d = \mathcal{L}^d(B_1(0))$ is the d–volume of the unit ball in $\mathbb{R}^d$.
- For $P \in G_{d,n}$, Π_P is the orthogonal projection onto P.
- Let ω and Ω be two open sets then $\omega \subset\subset \Omega$ means that ω is relatively compact in Ω.
- Let μ be a measure in some measurable topological space, then $\operatorname{supp} \mu$ denotes the topological support of μ.
- Let $A \subset \Omega$ then $A^c = \Omega \setminus A$ denotes the complementary of A in Ω.
- Given a measure μ, we denote by $|\mu|$ its total variation.

3.1.1 Radon measures and weak–∗ convergence

We recall here some useful properties concerning vector-valued Radon measures and weak–∗ convergence. See [EG92] and [AFP] for more details.

Definition 3.2 (weak–∗ convergence of Radon measures, see. [AFP] def. 1.58 p. 26)**.** *Let μ and $(\mu_i)_i$ be $\mathbb{R}^m$–vector valued Radon measures in $\Omega \subset \mathbb{R}^n$. We say that μ_i weakly–∗ converges to μ, denoted $\mu_i \xrightarrow[i\to\infty]{*} \mu$ if for every $\varphi \in \mathrm{C}_c(\Omega, \mathbb{R}^m)$,*

$$\int_\Omega \varphi \cdot d\mu_i \xrightarrow[i\to\infty]{} \int_\Omega \varphi \cdot d\mu \, .$$

Thanks to the Banach-Alaoglu weak compactness Theorem, we have the following result in the space of Radon measures.

Proposition 3.5 (Weak–∗ compactness, see [AFP] Theorem. 1.59 and 1.60 p. 26)**.** *Let $(\mu_i)_i$ be a sequence of Radon measures in some open set $\Omega \subset \mathbb{R}^n$ such that $\sup_i |\mu_i|(\Omega) < \infty$ then there exist a finite Radon measure μ and a subsequence $(\mu_{\varphi(i)})_i$ weakly–∗ converging to μ.*

Let us now study the consequences of weak–∗ convergence on Borel sets.

Proposition 3.6 (see 1.9 p.54 in [EG92])**.** *Let $(\mu_i)_i$ be a sequence of positive Radon measures weakly–∗ converging to μ in some open set $\Omega \subset \mathbb{R}^n$. Then,*

1. *for every compact set $K \subset \Omega$, $\limsup_i \mu_i(K) \leqslant \mu(K)$ and for every open set $U \subset \Omega$, $\mu(U) \leqslant \liminf_i \mu_i(U)$.*
2. *$\lim_i \mu_i(B) = \mu(B)$ for every Borel set $B \subset \Omega$ such that $\mu(\partial B) = 0$.*

Each one of the two properties in Proposition 3.6 is actually a characterization of weak–∗ convergence. Let us state a similar result in the vector case.

Proposition 3.7 (see [AFP] Prop. 1.62(b) p. 27)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $(\mu_i)_i$ be a sequence of $\mathbb{R}^m$–vector valued Radon measures weakly–∗ converging to μ. Assume in addition that the total variations $|\mu_i|$ weakly–∗ converge to some positive Radon measure λ. Then $|\mu| \leqslant \lambda$ and for every Borel set $B \subset \Omega$ such that $\lambda(\partial B) = 0$, $\mu_i(B) \to \mu(B)$. More generally,*

$$\int_\Omega u \cdot d\mu_i \longrightarrow \int_\Omega u \cdot d\mu$$

for every measurable bounded function u whose discontinuity set has zero λ–measure.

We end this part with a result saying that, for a given Radon measure μ, among all balls centred at a fixed point, at most a countable number of them have a boundary with non zero μ–measure.

Proposition 3.8. *Let μ be a Radon measure in some open set $\Omega \subset \mathbb{R}^n$. Then,*

(i) For a given $x \in \Omega$, the set of $r \in \mathbb{R}_+$ such that $\mu(\partial B_r(x)) > 0$ is at most countable. In particular,

$$\mathcal{L}^1\{r \in \mathbb{R}_+ \mid \mu(\partial B_r(x) \cap \Omega) > 0\} = 0 \,.$$

(ii) For almost every $r \in \mathbb{R}_+$,

$$\mu \{x \in \Omega \mid \mu(\partial B_r(x) \cap \Omega) > 0\} = 0 \,.$$

Proof. The first point is a classical property of Radon measures and comes from the fact that monotone functions have at most a countable set of discontinuities, applied to $r \mapsto \mu(B_r(x))$. For the second point, we use Fubini Theorem to get

$$\begin{aligned}\int_{r\in\mathbb{R}_+} \mu \{x \in \Omega \mid \mu(\partial B_r(x) \cap \Omega) > 0\} \, dr &= \int_{x\in\Omega}\int_{r\in\mathbb{R}_+} \mathbb{1}_{\{(x,r) \mid \mu(\partial B_r(x)\cap\Omega)>0\}}(x,r) \, d\mu(x) \, dr \\ &= \int_{x\in\Omega} \mathcal{L}^1\{r \in \mathbb{R}_+ \mid \mu(\partial B_r(x) \cap \Omega) > 0\} \, d\mu(x) = 0 \,,\end{aligned}$$

thanks to (i). □

These basic results will be widely used throughout this paper.

3.1.2 Rectifiability and approximate tangent space

Definition 3.3 (d–rectifiable sets, see definition 2.57 p.80 in [AFP])**.** *Let $M \subset \mathbb{R}^n$. M is said to be* <u>*countably d–rectifiable*</u> *if there exist countably many Lipschitz functions $f_i : \mathbb{R}^d \to \mathbb{R}^n$ such that*

$$M \subset M_0 \cap \bigcup_{i\in\mathbb{N}} f_i(\mathbb{R}^d) \text{ with } \mathcal{H}^d(M_0) = 0 \,.$$

If in addition $\mathcal{H}^d(M) < +\infty$ then M is said <u>*d–rectifiable*</u>*.*

Actually, it is equivalent to require that M can be covered by countably many Lipschitz d–graphs up to a $\mathcal{H}^d$–negligible set and thanks to Whitney extension Theorem (and thus Lusin's Theorem), one can ask for C^1 d–graphs. We can now define rectifiability for measures.

Definition 3.4 (d–rectifiable measures, see definition 2.59 p.81 in [AFP])**.** *Let μ be a positive Radon measure in $\mathbb{R}^n$. We say that μ is d–rectifiable if there exist a countably d–rectifiable set M and a Borel positive function θ such that $\mu = \theta \mathcal{H}^d_{|M}$.*

Thus, a set M is countably d–rectifiable if and only if $\mathcal{H}^d_{|M}$ is a d–rectifiable measure. When blowing up at a point, rectifiable measures have the property of concentrating on affine planes (at almost any point). This property leads to a characterization of rectifiable measures. Let us define $\psi_{x,r}$ as

$$\psi_{x,r}(y) = \frac{y-x}{r} \,.$$

Definition 3.5 (Approximate tangent space to a measure, see definition 2.79 p.92 in [AFP])**.** *Let μ be a positive Radon measure in $\mathbb{R}^n$. We say that μ has an* <u>*approximate tangent space*</u> *P with multiplicity $\theta \in \mathbb{R}_+$ at x if $P \in G_{d,n}$ is a d–plane such that*

$$\frac{1}{r^d} {\psi_{x,r}}_{\#} \mu \xrightarrow{\;*\;} \theta \mathcal{H}^d_{|P} \text{ as } r \downarrow 0.$$

That is,

$$\frac{1}{r^d} \int \varphi \left(\frac{y-x}{r} \right) d\mu(y) \underset{r \downarrow 0}{\longrightarrow} \theta \int_P \varphi(y) \, d\mathcal{H}^d(y) \quad \forall \varphi \in \mathrm{C}_c(\mathbb{R}^n) \, .$$

In the sequel the approximate tangent plane to M (resp. μ) at x is denoted by $T_x M$ (resp. $T_x \mu$). As we said, this provides a way to characterize rectifiability:

Theorem 3.9 (see theorem 2.83 p.94 in [AFP])**.** *Let μ be a positive Radon measure in $\mathbb{R}^n$.*

1. *If $\mu = \theta \mathcal{H}^d_{|M}$ with M countably d–rectifiable, then μ admits an approximate tangent plane with multiplicity $\theta(x)$ for $\mathcal{H}^d$–almost any $x \in M$.*
2. *If there exists a Borel set S such that $\mu(\mathbb{R}^n \setminus S) = 0$ and if μ admits an approximate tangent plane with multiplicity $\theta(x) > 0$ for μ–almost every $x \in S$ then S is countably d–rectifiable and $\mu = \theta \mathcal{H}^d_{|S}$.*

There are other characterizations of rectifiability in terms of density (see for instance [Mat95]). Let us point out an easy consequence of the existence of a tangent plane at a given point:

Proposition 3.10. *Let μ be a positive Radon measure in $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$, $P \in G_{d,n}$ and assume that μ has an approximate tangent space $T_x \mu$ with multiplicity $\theta(x) > 0$ at x. Then for all $\beta > 0$,*

$$\frac{1}{r^d} \mu \left\{ y \in B_r(x) \,|\, d(y-x, P) < \beta r \right\} \underset{r \to 0}{\longrightarrow} \theta(x) \mathcal{H}^d \left\{ y \in T_x \mu \cap B_1(0) \,|\, d(y, P) < \beta \right\} \, .$$

Proof. Indeed, let $\psi_{x,r} : y \mapsto \frac{y-x}{r}$, then $\frac{1}{r^d} {\psi_{x,r}}_{\#} \mu$ weakly star converges to $\theta(x) \mathcal{H}^d_{|_x \mu}$ so that for any Borel set A such that $\mathcal{H}^d_{|T_x \mu}(\partial A) = \mathcal{H}^d(\partial A \cap T_x \mu) = 0$, we have

$$\frac{1}{r^d} {\psi_{x,r}}_{\#} \mu(A) = \frac{1}{r^d} \mu \left(\psi_{x,r}^{-1}(A) \right) \underset{r \to 0_+}{\longrightarrow} \theta(x) \mathcal{H}^d \left(T_x \mu \cap A \right) \, . \tag{3.2}$$

The conclusion follows applying (3.2) with $A = \{ y \in B_1(0) \,|\, d(y, P) < \beta \}$ so that for any $0 < \beta < 1$,

$$\psi_{x,r}^{-1}(A) = \{ y \in B_r(x) \,|\, d(y-x, P) > \beta r \} \text{ and } \mathcal{H}^d(A \cap P) = 0 \, .$$

□

3.1.3 Some facts about varifolds

We recall here a few facts about varifolds, (for more details, see for instance [Sim83]). As we have already mentioned, the space of varifolds can be seen as a space of generalized surfaces. However, in this part we give examples showing that, not only rectifiable sets, but also objects like point clouds or volumetric approximations can be endowed with a varifold structure. Then we define the first variation of a varifold which is a generalized notion of mean curvature, and we recall the link between the boundedness of the first variation and the rectifiability of a varifold. We also introduce a family of volumetric discretizations endowed with a varifold structure. They will appear all along this paper in order to illustrate problems and strategies to solve them. We focus on this particular family of varifolds because they correspond to the volumetric approximations of sets that motivated us initially.

Definition of varifolds

We recall that $G_{d,n} = \{P \subset \mathbb{R}^n \,|\, P \text{ is a vector subspace of dimension } d\}$. Let us begin with the notion of rectifiable d–varifold.

Definition 3.6 (Rectifiable d–varifold)**.** *Given an open set $\Omega \subset \mathbb{R}^n$, let M be a countably d–rectifiable set and θ be a non negative function with $\theta > 0$ $\mathcal{H}^d$–almost everywhere in M. A rectifiable d–varifold $V = v(M, \theta)$ in Ω is a positive Radon measure on $\Omega \times G_{d,n}$ of the form $V = \theta \mathcal{H}^d_{|M} \otimes \delta_{T_x M}$ i.e.*

$$\int_{\Omega \times G_{d,n}} \varphi(x, T)\, dV(x, T) = \int_M \varphi(x, T_x M)\, \theta(x)\, d\mathcal{H}^d(x) \quad \forall \varphi \in \mathrm{C}_c(\Omega \times G_{d,n}, \mathbb{R})$$

where $T_x M$ is the approximative tangent space at x which exists $\mathcal{H}^d$–almost everywhere in M. The function θ is called the <u>multiplicity</u> of the rectifiable varifold.

Remark 3.1. We are dealing with measures on $\Omega \times G_{d,n}$, but we did not mention the σ–algebra we consider. We can equip $G_{d,n}$ with the metric

$$d(T, P) = \|\Pi_T - \Pi_P\|$$

where $\Pi_T \in M_n(\mathbb{R})$ is the matrix of the orthogonal projection onto T and $\|\cdot\|$ a norm on $M_n(\mathbb{R})$. We consider measures on $\Omega \times G_{d,n}$ with respect to the Borel algebra on $\Omega \times G_{d,n}$.

Let us turn to the general notion of varifold:

Definition 3.7 (Varifold)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set. A d–varifold in Ω is a positive Radon measure on $\Omega \times G_{d,n}$.*

Remark 3.2. As $\Omega \times G_{d,n}$ is locally compact, the Riesz theorem allows to identify Radon measures on $\Omega \times G_{d,n}$ and continuous linear forms on $\mathrm{C}^0_c(\Omega \times G_{d,n})$ (we used this fact in the definition of rectifiable d–varifolds) and the convergence in the sense of varifolds is then the weak–$*$ convergence.

Definition 3.8 (Convergence of varifolds)**.** *A sequence of d–varifolds $(V_i)_i$ weakly–$*$ converges to a d–varifolds V in Ω if, for all $\varphi \in \mathrm{C}_c(\Omega \times G_{d,n})$,*

$$\int_{\Omega \times G_{d,n}} \varphi(x, P)\, dV_i(x, P) \xrightarrow[i \to \infty]{} \int_{\Omega \times G_{d,n}} \varphi(x, P)\, dV(x, P)\,.$$

We now give some examples of varifolds:

Example 3.1. Consider a straight line $D \subset \mathbb{R}^3$, then the measure $v(D) = \mathcal{H}^1_{|D} \otimes \delta_D$ is the canonical 1–varifold associated to D.

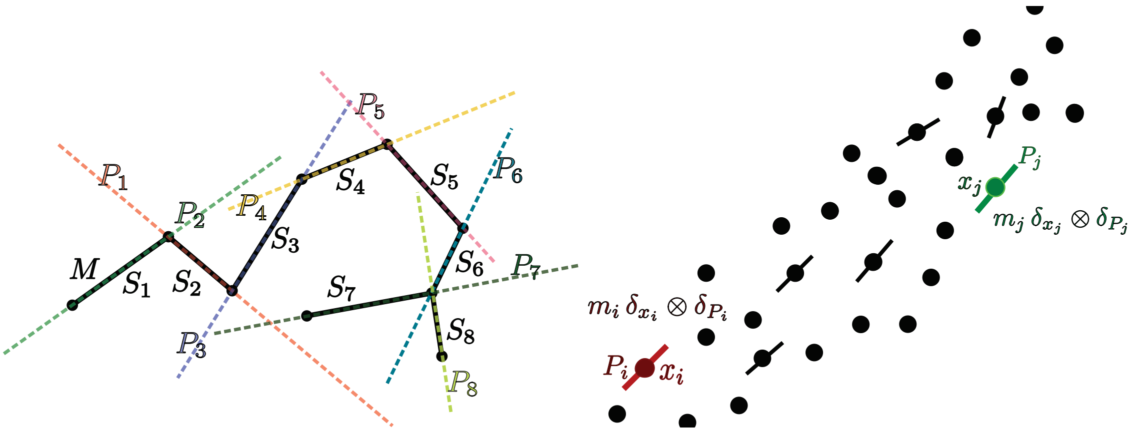
Example 3.2. Consider a polygonal curve $M \subset \mathbb{R}^2$ consisting of 8 line segments $S_1, \ldots, S_8$ of directions $P_1, \ldots, P_8 \in G_{1,2}$, then the measure $v(M) = \sum_{i=1}^8 \mathcal{H}^1_{|S_i} \otimes \delta_{P_i}$ is the canonical varifold associated to M.

Example 3.3. Consider a d–submanifold $M \subset \mathbb{R}^n$. According to the definition of rectifiable d–varifolds, the canonical d–varifold associated to M is $v(M) = \mathcal{H}^d \otimes \delta_{T_x M}$ or $v(M, \theta) = \theta \mathcal{H}^d \otimes \delta_{T_x M}$ adding some multiplicity $\theta : M \to \mathbb{R}_+$.

Example 3.4 (Point cloud). Consider a finite set of points $\{x_j\}_{j=1}^N \subset \mathbb{R}^n$ with additional information of masses $\{m_j\}_{j=1}^N \subset \mathbb{R}_+$ and tangent planes $\{P_j\}_{j=1 \ldots N} \subset G_{d,n}$ then the measure

$$\sum_{j=1}^N m_j \delta_{x_j} \otimes \delta_{P_j}$$

defines a d–varifolds associated with the point cloud.



(a) Polygonal curve (b) Point cloud

Definition 3.9 (Mass)**.** *If $V = v(M,\theta)$ is a d–rectifiable varifold, the measure $\theta\mathcal{H}^d_{|M}$ is called the* mass *of V and denoted by $\|V\|$. For a general varifold V, the mass of V is the positive Radon measure defined by $\|V\|(B) = V(\pi^{-1}(B))$ for every $B \subset \Omega$ Borel, with*

$$\left\{ \begin{array}{cccc} \pi : & \Omega \times G_{d,n} & \to & \Omega \\ & (x,S) & \mapsto & x \end{array} \right. .$$

For a curve, the mass is the length measure, for a surface, it is the area measure, for the previous point cloud, the mass is $\sum_j m_j \delta_{x_j}$. The mass loses the tangent information and keeps only the spatial part.

First variation of a varifold

The set of d–varifolds is endowed with a notion of generalized curvature called first variation. Let us recall the divergence theorem on a submanifold:

Theorem 3.11 (Divergence theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $M \subset \mathbb{R}^n$ be a d–dimensional* C^2– *submanifold. Then, for all $X \in \mathrm{C}^1_c(\Omega, \mathbb{R}^n)$,*

$$\int_{\Omega \cap M} \mathrm{div}_{T_x M} X(x)\, d\mathcal{H}^d(x) = - \int_{\Omega \cap M} H(x) \cdot X(x)\, d\mathcal{H}^d(x)\,,$$

where H is the mean curvature vector.

For $P \in G$ and $X = (X_1, \ldots, X_n) \in \mathrm{C}^1_c(\Omega, \mathbb{R}^n)$, the operator div_P is defined as

$$\mathrm{div}_P(x) = \sum_{j=1}^{n} \langle \nabla^P X_j(x), e_j \rangle = \sum_{j=1}^{n} \langle \Pi_P(\nabla X_j(x)), e_j \rangle \text{ whith } (e_1, \ldots, e_n) \text{ canonical basis of } \mathbb{R}^n.$$

This variational approach is actually a way to define mean curvature that can be extended to a larger class than C^2–manifolds: the class of varifolds with bounded first variation. We can now define the first variation of a varifold.

Definition 3.10 (First variation of a varifold)**.** *The first variation of a d–varifold in $\Omega \subset \mathbb{R}^n$ is the linear functional*

$$\begin{array}{cccc} \delta V : & \mathrm{C}^1_c(\Omega, \mathbb{R}^n) & \to & \mathbb{R} \\ & X & \mapsto & \int_{\Omega \times G_{d,n}} \mathrm{div}_P X(x)\, dV(x,P) \end{array}$$

This linear functional is generally not continuous with respect to the C_c^0 topology. When it is true, we say that the varifold has locally bounded first variation:

Definition 3.11. *We say that a d–varifold on Ω has locally bounded first variation when the linear form δV is continuous that is to say, for every compact set $K \subset \Omega$ there is a constant c_K such that for every $X \in C_c^1(\Omega, \mathbb{R}^n)$ with* $\operatorname{supp} X \subset K$,

$$|\delta V(X)| \leqslant c_K \sup_K |X| \, .$$

Now, if a d–varifold V has locally bounded first variation, the linear form δV can be extended into a continuous linear form on $C_c^0(\Omega, \mathbb{R}^n)$ and then by the Riesz theorem, there exists a Radon measure on Ω (still denoted by δV) such that

$$\delta V(X) = \int_\Omega X \cdot \delta V \quad \text{for every } X \in C_c(\Omega, \mathbb{R}^n)$$

Thanks to Radon-Nikodym Theorem, we can derive δV with respect to $\|V\|$ and there exist a function $H \in \left(L^1_{loc}(\Omega, \|V\|)\right)^n$ and a measure δV_s singular to $\|V\|$ such that

$$\delta V = -H\|V\| + \delta V_s \, .$$

The function H is called the generalized mean curvature vector. Thanks to the divergence theorem, it properly extends the classical notion of mean curvature for a C^2 submanifold.

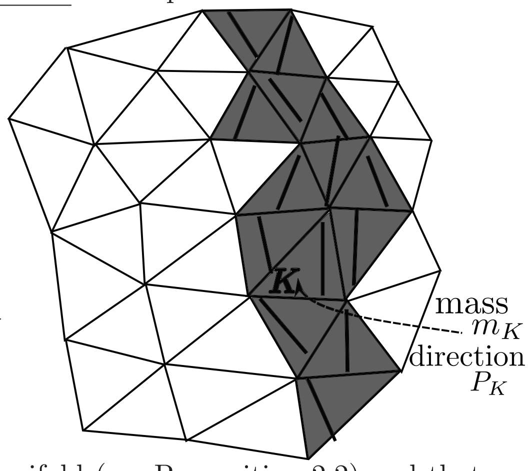
Another example: a family of volumetric approximations endowed with a varifold structure

Let us recall how we defined discrete volumetric varifolds in Chapter 2 Definition 2.1

Consider a mesh $\mathcal{K}$ and a family $\{m_K, P_K\}_{K \in \mathcal{K}} \subset \mathbb{R}_+ \times G_{d,n}$. We can associate the diffuse d–varifold:

$$V = \sum_{K \text{cell}} \frac{m_K}{|K|} \mathcal{L}^n_{|K} \otimes \delta_{P_K} \text{ with } |K| = \mathcal{L}^n(K) \, .$$

This d–varifold is not rectifiable since its support is n–rectifiable but not d–rectifiable.



Recall that we computed the first variation of such a varifold (see Proposition 2.2) and that we observed on a simple example (Example 2.1), considering the rectifiable 1–varifold V associated with a line D in $\mathbb{R}^2$ and the successive projections (as discrete volumetric varifolds) $V_{\mathcal{K}_i}$ onto a family of cartesian meshes $(\mathcal{K}_i)_i$ whose size δ_i tends to 0, that their first variation explodes:

$$|\delta V_{\mathcal{K}_i}|(\Omega) \geqslant \frac{\sqrt{2}}{2\delta_i} \|V\|(\Omega) \xrightarrow[i \to \infty]{} +\infty \, .$$

In particular, the first variation $\delta V_{\mathcal{K}_i}$ are not weakly–$*$ converging to $\delta V = 0$. We finally recall (see Section 2.2 for details) that it is not bad example but rather the general case. Of course, we are not saying that $|\delta V_{\mathcal{K}_i}|(\Omega)$ always explodes when refining the mesh, but that it may happen and it is not something easy to control except by adapting the mesh to the tangential directions $P_{\mathcal{K}}$ in the boundary cells. This is clearly a problem showing that the classical notion of first variation is not well adapted to this kind of volumetric discretization.

Control of the first variation and rectifiability

We will end these generalities about varifolds by linking the control of the first variation (generalized mean curvature) to the regularity of the varifolds. Let us begin with some property of the so called height excess proved by Brakke in [Bra78] (5.7 p. 153). There exist sharper estimates established by U. Menne in [Men12].

Theorem 3.12 (Height excess decay). *Let $V = v(M, \theta) = \theta \mathcal{H}^d_{|M} \otimes \delta_{T_x M}$ be a rectifiable d–varifold in some open set $\Omega \subset \mathbb{R}^n$. Assume that V is integral (that is $\theta(x) \in \mathbb{N}$ for $\|V\|$–almost every x) and assume that V has locally bounded first variation. Then for V–almost every $(x, P) \in \Omega \times G_{d,n}$,*

$$\mathrm{heightex}(x, P, V, r) := \frac{1}{r^d} \int_{B_r(x)} \left(\frac{d(y-x, P)}{r} \right)^2 d\|V\|(y) = o_x(r) \, .$$

Remark 3.3. Let us notice that

$$E_\alpha(x, P, V) = \int_{r=\alpha}^1 \mathrm{heightex}(x, P, V, r) \, \frac{dr}{r} \, .$$

That is why we called these quantities averaged height excess.

We now state a compactness result linking the rectifiability to the control of the first variation. It is exactly the kind of result we are interested in, with the exception that, in our setting, the approximating varifolds are generally not rectifiable and, moreover, the following control on the first variation is not satisfied.

Theorem 3.13 (Allard Compactness Theorem, see 42.7 in [Sim83]). *Let $(V_i)_i = (v(M_i, \theta_i))_i$ be a sequence of d–rectifiable varifolds with locally bounded first variation in an open set $\Omega \subset \mathbb{R}^n$ and such that $\theta_i \geqslant 1$ $\|V_i\|$–almost everywhere. If*

$$\sup_i \{ \|V_i(W)\| + |\delta V_i|(W) \} \leqslant c(W) < +\infty$$

for every open set $W \subset\subset \Omega$, then there exists a subsequence $(V_{i_n})_n$ weakly–$*$ converging to a rectifiable d–varifold V, with locally bounded first variation in Ω, such that $\theta \geqslant 1$, and moreover

$$|\delta V|(W) \leqslant \liminf_{n \to \infty} |\delta V_{i_n}|(W) \quad \forall \, W \subset\subset \Omega \, .$$

If for all i, V_i is an integral varifold then V is integral too.

The problem is that even if the limit d–varifold is rectifiable and has bounded first variation, it is not necessarily the case of an approximating sequence of varifolds. For instance, a point cloud varifold does not have bounded first variation. As for discrete volumetric varifolds, we have computed the first variation and seen that it is bounded for a fixed mesh, however, when the size of the mesh tends to zero, the total variation of the first variation is no longer bounded (in general) because of some boundary terms. We need some other way to ensure rectifiability. That is why we are looking for something more volumetric than the first variation, as defined in the introduction, in order to enforce rectifiability:

$$E_\alpha(x, P, V) = \int_{r=\alpha}^1 \frac{1}{r^d} \int_{y \in B_r(x) \cap \Omega} \left(\frac{d(y-x, P)}{r} \right)^2 d\|V\|(y) \, \frac{dr}{r} \, .$$

We now have two questions we want to answer:

1. Assume that $(V_i)_i$ is a sequence of d–varifolds weakly–$*$ converging to some d–varifold V with the following control
$$\sup_i \int_{\Omega \times G_{d,n}} E_{\alpha_i}(x, P, V_i) \, dV_i(x, P) < +\infty \, , \tag{3.3}$$
can we conclude that V is rectifiable ?

2. Is this condition better adapted to the case of (non-rectifiable) volumetric approximating varifolds (i.e. sequences of discrete volumetric varifolds)? We will prove that as soon as V_i weakly–$*$ converges to V, there exists a subsequence satisfying the control (3.3).

We begin with studying the static case.

3.2 Static quantitative conditions of rectifiability for varifolds

In this section, we begin with studying the averaged height excess $E_0(x, P, V)$ with respect to $P \in G_{d,n}$ (for a fixed d–varifold and a fixed $x \in \Omega$). We show that if V has bounded first variation then the approximate tangent plane at x is the only plane for which E_0 can be finite. Then we state and prove quantitative conditions of rectifiability for varifolds in the static case. Let us recall how we defined $E_0(x, P, V)$ in Theorem 3.3.

Definition 3.12 (Averaged height excess)**.** *Let V be a d–varifold in $\Omega \subset \mathbb{R}^n$ open subset. Then we define*

$$E_0(x, P, V) = \int_{r=0}^{1} \frac{1}{r^d} \int_{y \in B_r(x) \cap \Omega} \left(\frac{d(y-x, P)}{r} \right)^2 d\|V\|(y) \, \frac{dr}{r} \, .$$

We first study the averaged height excess $E_0(x, P, V)$ with respect to $P \in G_{d,n}$ for a fixed rectifable d–varifold.

3.2.1 The averaged height excess energy $E_0(x, P, V)$

Notice that if $\|V\| = \mathcal{H}^d_{|M}$ then for every d–vector plane $P \in G_{d,n}$,

$$\begin{aligned} \int_{r=0}^{1} \beta_2(x, r, M)^2 \, \frac{dr}{r} &= \int_{r=0}^{1} \inf_{S \in \{\text{affine } d-\text{plane}\}} \left(\frac{1}{r^d} \int_{y \in B_r(x) \cap M} \left(\frac{d(y, S)}{r} \right)^2 d\mathcal{H}^d(y) \right) \frac{dr}{r} \\ &\leqslant \int_{r=0}^{1} \frac{1}{r^d} \int_{y \in B_r(x) \cap M} \left(\frac{d(y-x, P)}{r} \right)^2 d\mathcal{H}^d(y) \, \frac{dr}{r} = E_0(x, P, V) \, . \end{aligned}$$

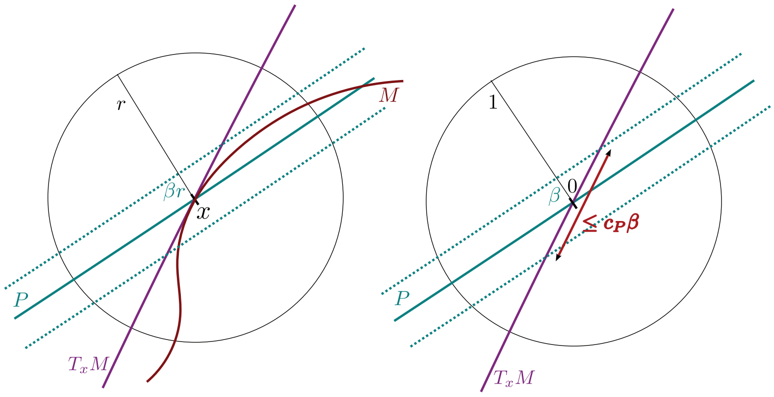
Thus, assume that for $\mathcal{H}^d$–almost every $x \in M$, $\theta^d_*(x, M) > 0$ holds and that there exists some $P_x \in G_{d,n}$ such that $E_0(x, P_x, \mathcal{H}^d_{|M}) < +\infty$. Then thanks to Pajot's Theorem 3.2, M is d–rectifiable. As we will see, the point is that for any $x \in M$ where the tangent plane $T_x M$ exists, then $P_x = T_x M$ is the best candidate, among all d–planes P, to satisfy $E_0(x, P_x, \mathcal{H}^d_{|M}) < +\infty$. Consequently, in order to test the rectifiability of a d–varifold V, it is natural to study $E_0(x, P, V)$ for (x, P) in $\operatorname{supp} V$ (which is more restrictive than for any $(x, P) \in \operatorname{supp} \|V\| \times G_{d,n}$). More concretely, we will study $\displaystyle\int_{\Omega \times G_{d,n}} E_0(x, P, V) \, dV(x, P)$ rather than $\displaystyle\int_{\Omega} \inf_{P \in G} E_0(x, P, V) \, d\|V\|(x)$.

In this whole part, we fix a rectifiable d–varifold in some open set $\Omega \subset \mathbb{R}^n$ and we study the behaviour of $E_0(x, P, V)$ with respect to $P \in G_{d,n}$. We are going to show that for a rectifiable d–varifold, this energy is critical: under some assumptions, it is finite if and only if P is the approximate tangent plane. More precisely:

Proposition 3.14. *Let $V = v(M, \theta)$ be a rectifiable d–varifold in an open set $\Omega \subset \mathbb{R}^n$. Then,*

1. *Let $x \in M$ such that the approximate tangent plane $T_x M$ to M at x exists and $\theta(x) > 0$ (thus for $\|V\|$–almost every x) then for all $P \in G_{d,n}$ such that $P \neq T_x M$,*

$$E_0(x, P, V) = +\infty \, .$$



2. *If in addition V is integral ($\theta \in \mathbb{N}$ $\|V\|$–almost everywhere) and has bounded first variation then for $\|V\|$–almost every x,*
$$E_0(x, T_xM, V) < +\infty \,.$$

Proof. We begin with the first assertion. Let $x \in M$ such that the approximate tangent plane T_xM to M at x exists. Let $P \in G_{d,n}$ such that $P \neq T_xM$. Thanks to Prop. 3.10, for all $\beta > 0$ we have

$$\frac{1}{r^d}\|V\|\left\{y \in B_r(x) \,|\, d(y-x, P) < \beta r\right\} \xrightarrow[r\to 0_+]{} \theta(x)\mathcal{H}^d\left(T_xM \cap \{y \in B_1(0) \,|\, d(y,P) < \beta\}\right) \,.$$

Now for all $\beta > 0$,

$$\begin{aligned} E_0(x,P,V) &= \int_{r=0}^1 \frac{dr}{r^{d+1}} \int_{B_r(x)} \left\{\frac{d(y-x,P)}{r}\right\}^2 d\|V\|(y) \\ &\geqslant \int_{r=0}^1 \frac{dr}{r}\frac{1}{r^d}\int_{\{y\in B_r(x)\,|\,d(y-x,P)\geqslant\beta r\}} \beta^2\, d\|V\|(y) \\ &= \beta^2 \int_{r=0}^1 \frac{dr}{r}\frac{1}{r^d}\|V\|\left\{y \in B_r(x)\,|\, d(y-x,P) \geqslant \beta r\right\} \,. \end{aligned}$$

Let us estimate

$$\frac{1}{r^d}\|V\|\left\{y \in B_r(x)\,|\,d(y-x,P)\geqslant \beta r\right\} = \underbrace{\frac{1}{r^d}\|V\|(B_r(x))}_{\xrightarrow[r\to 0]{}\theta(x)\omega_d} - \underbrace{\frac{1}{r^d}\|V\|\left\{y\in B_r(x)\,|\,d(y-x,P)<\beta r\right\}}_{\xrightarrow[r\to 0]{}\theta(x)\mathcal{H}^d(T_xM\cap\{y\in B_1(0)\,|\,d(y,P)<\beta\})} \,.$$

As $P \neq T_xM$, there exists some constant c_P depending on P and T_xM such that

$$\mathcal{H}^d(T_xM \cap \{y \in B_1(0)\,|\,d(y,P) < \beta\}) \leqslant c_P\beta \,.$$

Consequently,

$$\begin{aligned} \lim_{r\to 0}\frac{1}{r^d}\|V\|\left\{y\in B_r(x)\,|\,d(y-x,P)\geqslant\beta r\right\} &= \theta(x)\Big(\omega_d - \mathcal{H}^d(T_xM\cap\{y\in B_1(0)\,|\,d(y,P)<\beta\})\Big) \\ &\geqslant \theta(x)(\omega_d - c_P\beta) \\ &\geqslant \theta(x)\frac{\omega_d}{2} \text{ for } \beta \text{ small enough.} \end{aligned}$$

Eventually there exist $\beta > 0$ and $r_0 > 0$ such that for all $r \leqslant r_0$

$$\frac{1}{r^d}\|V\|\left\{y\in B_r(x)\,|\,d(y-x,P)\geqslant\beta r\right\} \geqslant \theta(x)\frac{\omega_d}{4} \,,$$

and thus

$$E_0(x,P,V) \geqslant \theta(x)\frac{\omega_d}{4}\beta^2 \int_{r=0}^{r_0} \frac{dr}{r} = +\infty \, .$$

The second assertion is a direct consequence of Brakke's estimate (see Proposition 3.12) for the height excess of an integral d–varifold with bounded first variation:

$$E_0(x, T_x M, V) = \int_{r=0}^{1} \underbrace{\frac{1}{r} \operatorname{heightex}(x,P,V,r)}_{=o_x(1)} \, dr < +\infty \, .$$

□

3.2.2 The static theorem

We begin with some lemmas before proving the static theorem (Theorem. 3.3). This first proposition recalls that the first assumption of the static theorem (Ahlfors regularity) implies that $\|V\|$ is equivalent to $\mathcal{H}^d_{|\operatorname{supp}\|V\|}$.

Proposition 3.15. *Let $\Omega \subset \mathbb{R}^n$ be an open set and μ be a positive Radon measure in Ω.*

(i) Let β_1, $\beta_2 : \Omega \to \mathbb{R}_+$ continuous and such that for all $x \in \Omega$, $\beta_1(x) < \beta_2(x)$, and let $C > 0$. Then the sets $A = \left\{ x \in \Omega \mid \forall r \in (\beta_1(x), \beta_2(x)) \, , \, \mu(B_r(x)) \geqslant Cr^d \right\}$ and $B = \left\{ x \in \Omega \mid \forall r \in (\beta_1(x), \beta_2(x)) \, , \, \mu(B_r(x)) \leqslant Cr^d \right\}$ are closed.

(ii) If there exist C_1, $C_2 > 0$ such that $C_1 \omega_d r^d \leqslant \mu(B_r(x)) \leqslant C_2 \omega_d r^d$ for μ–almost all $x \in \Omega$ and for all $0 < r < d(x, \Omega^c)$, then

$$C_1 \mathcal{H}^d_{|E} \leqslant \mu \leqslant 2^d C_2 \mathcal{H}^d_{|E} \quad \text{with } E = \operatorname{supp} \mu \, .$$

Proof. (i) Let us prove that $A = \left\{ x \in \Omega \mid \forall r \in (\beta_1(x), \beta_2(x)) \, , \, \mu(B_r(x)) \geqslant Cr^d \right\}$ is closed. Let $(x_k)_k \subset A$ such that $x_k \xrightarrow[k\infty]{} x \in \Omega$ and let $r > 0$ such that $\beta_1(x) < r < \beta_2(x)$. For k great enough, $\beta_1(x_k) < r < \beta_2(x_k)$ so that $Cr^d \leqslant \mu(B_r(x_k))$. If $\mu(\partial B_r(x)) = 0$ then $\mu(B_r(x_k)) \xrightarrow[k\to+\infty]{} \mu(B_r(x))$ and then $Cr^d \leqslant \mu(B_r(x))$ for almost every $r \in (\beta_1(x), \beta_2(x))$. But this is enough to obtain the property for all $r \in (\beta_1(x), \beta_2(x))$. Indeed, if $\mu(\partial B_r(x)) > 0$ then take $r_k^- < r$ such that for all k,

$$\mu(\partial B_{r_k^-}(x)) = 0 \text{ and } r_k^- \xrightarrow[k\to+\infty]{} r \, ,$$

and thus

$$\mu(B_r(x)) \geqslant \mu(B_{r_k^-}(x)) \geqslant C {r_k^-}^d \xrightarrow[k\to+\infty]{} Cr^d \, .$$

Eventually $x \in A$ and A is closed. We can prove that B is closed similarly.

(ii) As the set

$$E_1 = \left\{ x \in \Omega \mid \forall 0 < r < d(x, \Omega^c), \; \mu(B_r(x)) \geqslant C_1 \omega_d r^d \right\}$$

is closed (thanks to (i)) and of full μ–measure, then $E = \operatorname{supp} \mu \subset E_1$. Therefore, for every $x \in E$,

$$\theta^d_*(\mu, x) = \liminf_{r \to 0_+} \frac{\mu(B_r(x))}{\omega_d r^d} \geqslant C_1 \, .$$

So that (see Theorem 2.56 p.78 in [AFP]) $\mu \geqslant C_1 \mathcal{H}^d_{|E}$.

(iii) For the same reason,

$$E = \operatorname{supp}\mu \subset E_2 = \left\{ x \in \Omega \mid \forall 0 < r < d(x, \Omega^c),\ \mu(B_r(x)) \leqslant C_2 \omega_d r^d \right\} .$$

Therefore, for every $x \in E$,

$$\theta^{*\,d}(\mu, x) = \limsup_{r \to 0_+} \frac{\mu(B_r(x))}{\omega_d r^d} \leqslant C_2 .$$

So that (again by Theorem 2.56 p.78 in [AFP]) $\mu \leqslant 2^d C_2 \mathcal{H}^d_{|E}$.

□

The following lemma states that under some density assumption, the quantity $\min_{P \in G_{d,n}} E_0(x, P, V)$ controls the quantity linked to Jones' β numbers.

Lemma 3.16. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let V be a d–varifold in Ω. Assume that there is some constant $C > 0$ and a Borel set $E \subset \Omega$ such that $\mathcal{H}^d_{|E} \leqslant C \|V\|$ then for all $x \in \Omega$,*

$$\int_0^1 \beta_2(x, r, E)^2 \frac{dr}{r} \leqslant C \min_{P \in G_{d,n}} E_0(x, P, V) . \tag{3.4}$$

Proof. First notice that $G_{d,n} \subset \{\text{affine } d\text{–plane}\}$, therefore

$$\begin{aligned}
\int_{r=0}^1 \beta_2(x, r, E)^2 \frac{dr}{r^{d+1}} &= \int_{r=0}^1 \inf_{P \in \{\text{affine } d\text{–plane}\}} \left(\int_{E \cap B_r(x)} \left(\frac{d(y, P)}{r} \right)^2 d\mathcal{H}^d(y) \right) \frac{dr}{r^{d+1}} \\
&\leqslant \inf_{P \in \{\text{affine } d\text{–plane}\}} \int_{r=0}^1 \left(\int_{E \cap B_r(x)} \left(\frac{d(y, P)}{r} \right)^2 d\mathcal{H}^d(y) \right) \frac{dr}{r^{d+1}} \\
&\leqslant \min_{P \in G_{d,n}} \int_{r=0}^1 \left(\int_{E \cap B_r(x)} \left(\frac{d(y - x, P)}{r} \right)^2 d\mathcal{H}^d(y) \right) \frac{dr}{r^{d+1}} .
\end{aligned}$$

Then, the assumption $\mathcal{H}^d_{|E} \leqslant C \|V\|$ implies that for any positive function u, $\displaystyle\int_E u \, d\mathcal{H}^d \leqslant C \int_\Omega u \, d\|V\|$ so that

$$\min_{P \in G_{d,n}} \int_{r=0}^1 \left(\int_{B_r(x)} \left(\frac{d(y - x, P)}{r} \right)^2 d\mathcal{H}^d_{|E}(y) \right) \frac{dr}{r^{d+1}} \leqslant C \min_{P \in G_{d,n}} E_0(x, P, V) ,$$

which proves 3.4. □

We now state a lemma that will enable us to localise the property of rectifiability.

Lemma 3.17. *Let $\Omega \subset \mathbb{R}^n$ be an open set and μ be a positive Radon measure in Ω. Then there exists a countable family of open sets $(\omega_n)_n$ such that for all n, $\omega_n \subset\subset \omega_{n+1} \subset\subset \Omega$, $\mu(\partial \omega_n) = 0$ and $\Omega = \cup_n \omega_n$.*

Proof. For all $t > 0$, let us consider the family of open sets

$$\omega_t = B_t(0) \cap \{ x \in \Omega \mid d(y, \Omega^c) > 1/t \} .$$

The family $(\omega_t)_t$ is increasing so that $\mu(\omega_t)$ is increasing and has at most a countable number of jumps. Then for almost every t, $\mu(\omega_t) = 0$ and it is easy to conclude. □

The last step before proving Theorem 3.3 is to link the rectifiability of the mass $\|V\|$ and the rectifiability of the whole varifold. The key point is the coherence between the tangential part of the varifold and the approximate tangent plane to the spatial part $\|V\|$.

Lemma 3.18. *If V is a d–varifold in $\Omega \subset \mathbb{R}^n$ such that*
– *$\|V\|$ is d–rectifiable,*
– *$V\left(\{(x,P) \in \Omega \times G_{d,n} \,|\, E_0(x,P,V) = +\infty\}\right) = 0$,*
then V is a rectifiable d–varifold.

Proof. The mass $\|V\|$ is d–rectifiable so that $\|V\| = \theta \mathcal{H}^d_{|M}$ for some d-rectifiable set M. We have to show that $V = \|V\| \otimes \delta_{T_x M}$. Applying a disintegration theorem ([AFP] 2.28 p. 57), there exist finite Radon measures ν_x in $G_{d,n}$ such that for $\|V\|$–almost every $x \in \Omega$, $\nu_x(G_{d,n}) = 1$ and $V = \|V\| \otimes \nu_x$. We want to prove that for $\|V\|$–almost every x, $\nu_x = \delta_{T_x M}$ or equivalently,

$$\nu_x(\{P \in G_{d,n} \mid P \neq T_x M\}) = 0 \,.$$

For a d–rectifiable measure $\|V\| = \theta \mathcal{H}^d_{|M}$, we have shown in Proposition 3.14 that for $\|V\|$–almost every $x \in \Omega$,

$$P \neq T_x M \Longrightarrow E_0(x,P,V) = +\infty \,,$$

thus

$$\{(x,P) \in \Omega \times G_{d,n} \mid P \neq T_x M\} \subset A_0 \times G_{d,n} \cup \{(x,P) \in \Omega \times G_{d,n} \mid E_0(x,P,V) = +\infty\}$$

with $\|V\|(A_0) = 0$. Therefore $V(\{(x,P) \in \Omega \times G_{d,n} \mid P \neq T_x M\}) = 0$. Thus

$$\begin{aligned} V(\{(x,P) \in \Omega \times G_{d,n} \mid P \neq T_x M\}) &= \int_{\Omega \times G_{d,n}} \mathbb{1}_{\{P \neq T_x M\}}(x,P) \, dV(x,P) \\ &= \int_\Omega \left(\int_{G_{d,n}} \mathbb{1}_{\{P \neq T_x M\}}(x,P) \, d\nu_x(P) \right) d\|V\|(x) \\ &= \int_\Omega \nu_x(\{P \in G_{d,n} \mid P \neq T_x M\}) \, d\|V\|(x) \end{aligned}$$

which means that for $\|V\|$–almost every $x \in \Omega$, $\nu_x(\{P \in G_{d,n} \mid P \neq T_x M\}) = 0$ thus for $\|V\|$–almost every $x \in \Omega$, $\nu_x = \delta_{T_x M}$ and $V = \|V\| \otimes \delta_{T_x M}$ is a d–rectifiable varifold. □

Let us now prove the static theorem:

Theorem. 3.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let V be a d–varifold in Ω of finite mass $\|V\|(\Omega) < +\infty$. Assume that:*

(i) there exist $0 < C_1 < C_2$ such that for $\|V\|$–almost every $x \in \Omega$ and for all $0 < r < d(x, \Omega^c)$ such that $B_r(x) \subset \Omega$,

$$C_1 \omega_d r^d \leqslant \|V\|(B_r(x)) \leqslant C_2 \omega_d r^d \,,$$

(ii) $V\left(\{(x,P) \in \Omega \times G_{d,n} \,|\, E_0(x,P,V) = +\infty\}\right) = 0$.

Then V is a rectifiable d–varifold.

Remark 3.4. If in particular $\displaystyle\int_{\Omega \times G_{d,n}} E_0(x,P,V) \, dV(x,P) < +\infty$ then the assumption (ii) is satisfied.

Proof. Now we just have to gather the previous arguments and apply Pajot's Theorem (Theorem. 3.2).

– Step 1: the first hypothesis implies (thanks to Proposition 3.15) that, setting $C_3 = 2^d C_2 > 0$ and $E = \operatorname{supp} \|V\|$, we have

$$C_1 \mathcal{H}^d_{|E} \leqslant \|V\| \leqslant C_3 \mathcal{H}^d_{|E} \, .$$

Hence $C_1 \mathcal{H}^d(E) \leqslant \|V\|(\Omega) < +\infty$. Moreover, as $\|V\|$ and $\mathcal{H}^d_{|E}$ are Radon measures and $\|V\|$ is absolutely continuous with respect to $\mathcal{H}^d_E$, then by Radon-Nikodym Theorem there exists some function $\theta \in \mathrm{L}^1(\mathcal{H}^d_{|E})$ such that

$$\|V\| = \theta \mathcal{H}^d_{|E} \quad \text{with} \quad \theta(x) = \frac{d\|V\|}{d\mathcal{H}^d_{|E}}(x) = \lim_{r \to 0_+} \frac{\|V\|(B_r(x))}{\mathcal{H}^d(E \cap B_r(x))} \geqslant C_1 > 0 \text{ for } \mathcal{H}^d \text{ a.e. } x \in E \, .$$

– Step 2: Thus we can now apply Lemma 3.16 so that for any $x \in \Omega$,

$$\int_0^1 \beta_2(x,r,E)^2 \frac{dr}{r} \leqslant C_3 \min_{P \in G_{d,n}} E_0(x,P,V) \, ,$$

but thanks to the second assumption, $V\left(\{(x,P) \in \Omega \times G_{d,n} \,|\, E_0(x,P,V) = +\infty\}\right) = 0$. Let

$$B = \{x \in \Omega \,|\, \min_{P \in G_{d,n}} E_0(x,P,V) = +\infty\} = \{x \in \Omega \,|\, \forall P \in G_{d,n}, \; E_0(x,P,V) = +\infty\}$$

then

$$\begin{aligned} B \times G_{d,n} = & \{(x,P) \in \Omega \times G_{d,n} \,|\, \forall Q \in G_{d,n}, \; E_0(x,Q,V) = +\infty\} \\ \subset & \{(x,P) \in \Omega \times G_{d,n} \,|\, E_0(x,P,V) = +\infty\} \; . \end{aligned}$$

Therefore $\|V\|(B) = V(B \times G_{d,n}) \leqslant V\left(\{(x,P) \in \Omega \times G_{d,n} \,|\, E_0(x,P,V) = +\infty\}\right) = 0$. So that $\min_{P \in G_{d,n}} E_0(x,P,V)$ is finite for $\|V\|$–almost any $x \in \Omega$. And by step 1, $\|V\| = \theta \mathcal{H}^d_{|E}$ with $\theta \geqslant C_1$ for $\mathcal{H}^d$–almost every $x \in E$, thus for $\mathcal{H}^d$–almost every $x \in E$,

$$\int_0^1 \beta_2(x,r,E)^2 \frac{dr}{r} < +\infty \, , \tag{3.5}$$

and

$$\theta^d_*(x,E) = \liminf_{r \to 0_+} \frac{\mathcal{H}^d(E \cap B_r(x))}{\omega_d r^d} \geqslant \frac{1}{C_3} \frac{\|V\|(B_r(x))}{\omega_d r^d} \geqslant \frac{C_1}{C_3} > 0 \, . \tag{3.6}$$

– Step 3: We need to consider some compact subset of E to apply Pajot's Theorem. The set E being closed in Ω, thus for every compact set $K \subset \Omega$, $E \cap K$ is compact. Thanks to Lemma 3.17, let $(\omega_n)_n$ be an increasing sequence of relatively compact open sets such that $\Omega = \cup_n \omega_n$ and for all n, $\mathcal{H}^d(E \cap \partial \omega_n) = 0$. Let $K_n = \overline{\omega_n}$, then
  – for all $x \in (E \cap K_n) \setminus \partial K_n = E \cap \overline{\omega_n}$ we have $\theta^d_*(x, E \cap K_n) = \theta^d_*(x,E)$ and thus by (3.6) and since $\mathcal{H}^d(E \cap \partial K_n) = 0$,

$$\theta^d_*(x, E \cap K_n) > 0 \text{ for } \mathcal{H}^d\text{–almost every } x \in E \cap K_n \, , \tag{3.7}$$

  – thanks to (3.5), for $\mathcal{H}^d$–almost every $x \in E \cap K_n$,

$$\int_0^1 \beta_2(x,r,E \cap K_n)^2 \frac{dr}{r} \leqslant \int_0^1 \beta_2(x,r,E)^2 \frac{dr}{r} < +\infty \, . \tag{3.8}$$

According to (3.7) and (3.8), we can apply Pajot's theorem to get the d–rectifiability of $E \cap K_n$ for all n and hence the d–rectifiability of E and $\|V\| = \theta \mathcal{H}^d_{|E}$.

Eventually Lemma 3.18 leads the d–rectifiability of the whole varifold V. □

3.3 The approximation case

We will now study the approximation case. As we explained before, we introduce some scale parameters (denoted α_i and β_i) allowing us to consider the approximating objects "from far enough". The point is to check that we recover the static conditions (the assumptions (i) and (ii) of Theorem 3.3) in the limit. We begin with some technical lemmas concerning Radon measures. Then we prove a strong property of weak–$*$ convergence allowing us to gain some uniformity in the convergence. We end with the proof of the quantitative conditions of rectifiability for varifolds in the approximation case.

3.3.1 Some technical tools about Radon measures

Let us state two technical tools before starting to study the approximation case.

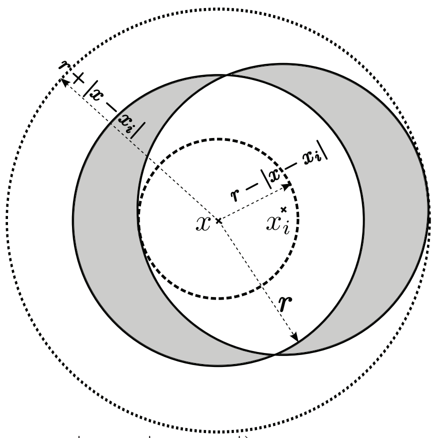
Lemma 3.19. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $(\mu_i)_i$ be a sequence of Radon measures weakly–$*$ converging to some Radon measure μ in Ω. Let $x \in \Omega$ and $x_i \xrightarrow[i\to\infty]{} x$. Then, for every $r > 0$,*

$$\limsup_i \mu_i(B_r(x) \triangle B_r(x_i)) \leqslant \mu(\partial B_r(x)) \, .$$

In particular, if $\mu(\partial B_r(x)) = 0$ then $\mu_i(B_r(x) \triangle B_r(x_i)) \xrightarrow[i\to\infty]{} 0$.

Proof. Let us define the ring of center x and radii $r_{\min}$ and $r_{\max}$:

$$R(x, r_{\min}, r_{\max}) := \{ y \in \Omega \,|\, r_{\min} \leqslant |y - x| \leqslant r_{\max} \} \, .$$



It is easy to check that for all i, $B_r(x_i) \triangle B_r(x)$ is included into the closed ring of center x and radii $r^i_{min} = r - |x - x_i|$ and $r^i_{max} = r + |x - x_i|$, that is

$$B_r(x_i) \triangle B_r(x) \subset R(x, r - |x - x_i|, r + |x - x_i|) \, .$$

Without loss of generality we can assume that $(|x - x_i|)_i$ is decreasing, then the sequence of rings $(R(x, r - |x - x_i|, r + |x - x_i|))_i$ is decreasing so that for all $p \leqslant i$,

$$\begin{aligned} \mu_i(B_r(x_i) \triangle B_r(x)) &\leqslant \mu_i(R(x, r - |x - x_i|, r + |x - x_i|) \\ &\leqslant \mu_i(R(x, r - |x - x_p|, r + |x - x_p|)) \, . \end{aligned}$$

Consequently, letting i tend to ∞ and using the fact that $R(x, r - |x - x_p|, r + |x - x_p|)$ is compact, we have for all p,

$$\limsup_{i \to +\infty} \mu_i(B_r(x_i) \triangle B_r(x)) \leqslant \mu(R(x, r - |x - x_p|, r + |x - x_p|)) \, ,$$

and thus by letting $p \to +\infty$ we finally have,

$$\limsup_{i \to +\infty} \mu_i(B_r(x_i) \triangle B_r(x)) \leqslant \mu(\partial B_r(x)) \, .$$

□

Proposition 3.20. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $(\mu_i)_i$ be a sequence of Radon measures weakly–$*$ converging to a Radon measure μ. Then, for every $x \in \operatorname{supp} \mu$, there exist $x_i \in \operatorname{supp} \mu_i$ such that $|x - x_i| \xrightarrow[i\to\infty]{} 0$.*

Proof. Let $x \in \operatorname{supp} \mu$, and choose $x_i \in \operatorname{supp} \mu_i$ such that $d(x, \operatorname{supp} \mu_i) = |x - x_i|$ (recall that $\operatorname{supp} \mu_i$ is closed). Let us check that $|x - x_i| \xrightarrow[i \to \infty]{} 0$. By contradiction, there exist $\eta > 0$ and a subsequence $(x_{\varphi(i)})_i$ such that for all i, $|x_{\varphi(i)} - x| \geqslant \eta$. Therefore, for all $y \in \operatorname{supp} \mu_{\varphi(i)}$, $|y - x| \geqslant |x_{\varphi(i)} - x| \geqslant \eta$ so that

$$\forall i,\ B_\eta(x) \cap \operatorname{supp} \mu_{\varphi(i)} = \emptyset \text{ and thus } \mu_{\varphi(i)}\left(B_\eta(x)\right) = 0\,.$$

Hence $\mu\left(B_\eta(x)\right) \leqslant \liminf_i \mu_{\varphi(i)}\left(B_\eta(x)\right) = 0$ and $x \notin \operatorname{supp} \mu$. □

3.3.2 Density estimates

We now look for density estimates for the limit varifold. Indeed, for sets of dimension greater than d, for instance $d+1$, the energy $E_0(x, P, V)$ does not convey information of rectifiability since

$$\frac{1}{r^{d+1}} \int_{B_r(x)} \left(\frac{d(y-x, P)}{r} \right)^2 d\|V\|(y) \leqslant \frac{\|V\|(B_r(x))}{r^{d+1}} \leqslant \theta^{*d+1}(\|V\|, x)$$

is finite for almost any x, not depending on the regularity of $\|V\|$. So that the first assumption in the static theorem (Ahlfors regularity (3.1) in Theorem 3.3) is quite natural. In this part, we link density estimates on V_i and density estimates on V and then recover the first assumption of the static theorem.

Proposition 3.21. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $(\mu_i)_i$ be a sequence of Radon measures in Ω, weakly–$*$ converging to some Radon measure μ. Assume that there exist $0 < C_1 < C_2$ and a positive decreasing sequence $(\beta_i)_i$ tending to 0 such that for μ_i–almost every $x \in \Omega$ and for every $r > 0$ such that $\beta_i < r < d(x, \Omega^c)$,*

$$C_1 r^d \leqslant \mu_i(B_r(x)) \leqslant C_2 r^d\,.$$

Then for μ–almost every $x \in \Omega$ and for every $0 < r < d(x, \Omega^c)$,

$$C_1 r^d \leqslant \mu(B_r(x)) \leqslant C_2 r^d\,.$$

Proof. Let $A_i = \left\{ x \in \Omega \mid \forall r \in]\beta_i, d(x, \Omega^c)[,\ C_1 r^d \leqslant \mu_i(B_r(x)) \leqslant C_2 r^d \right\}$.

(i) First notice that A_i is closed (thanks to Proposition 3.15 (*i*)) and $\mu_i(\Omega \setminus A_i) = 0$ so that $\operatorname{supp} \mu_i \subset A_i$.

(ii) Let $x \in \operatorname{supp} \mu$ and let $0 < r < d(x, \Omega^c)$. By Proposition 3.20, let $x_i \in \operatorname{supp} \mu_i$ such that $x_i \to x$ then

$$|\mu_i(B_r(x)) - \mu_i(B_r(x_i))| \leqslant \mu_i\left(B_r(x_i) \triangle B_r(x)\right) \leqslant \mu_i(R(x, r - |x - x_i|, r + |x - x_i|)\,,$$

so that by Proposition 3.19, $\limsup_i |\mu_i(B_r(x)) - \mu_i(B_r(x_i))| \leqslant \mu(B_r(x))$. Therefore, for almost every $0 < r < d(x, \Omega^c)$, $\mu_i(B_r(x_i)) \xrightarrow[i \to \infty]{} \mu(B_r(x))$. Eventually, as $x_i \in \operatorname{supp} \mu_i \subset A_i$ then for almost every $r < d(x, \Omega^c)$,

$$C_1 r^d \leqslant \mu(B_r(x)) = \lim_i \mu_i(B_r(x_i)) \leqslant C_2 r^d\,.$$

We can obtain this inequality for all r as in Proposition3.15, taking $r_k^- < r < r_k^+$ and $r_k^-, r_k^+ \to r$ and such that $\mu(\partial B_{r_k^+}(x)) = 0$, $\mu(\partial B_{r_k^-}(x)) = 0$.

□

3.3.3 Uniformity of weak–$*$ convergence in some class of functions

If we try to estimate $E_\alpha(x,P,V_\alpha) - E_\alpha(x,P,V)$, we can have the following:

$$\begin{aligned}&|E_\alpha(x,P,V_\alpha) - E_\alpha(x,P,V)| \\ &\leqslant \frac{1}{\alpha^{d+3}} \int_{r=0}^1 \left| \int_{B_r(x)} d(y-x,P)^2 d\|V_\alpha\|(y) - \int_{B_r(x)} d(y-x,P)^2 d\|V\|(y) \right| dr \,.\end{aligned}$$

We now prove that the integral term tends to 0 when $V_\alpha \xrightarrow{*} V$. For this purpose, we need a stronger way to write weak–$*$ convergence (with some uniformity) using the compactness of some subset of $\mathrm{C}^0_c(\Omega)$:

Proposition 3.22. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $(\mu_i)_i$ be a sequence of Radon measures in Ω weakly–$*$ converging to a Radon measure μ. Let $\omega \subset\subset \Omega$ such that $\mu(\partial\omega) = 0$, then for fixed $k, C \geqslant 0$,*

$$\sup \left\{ \left| \int_\omega \varphi \, d\mu_i - \int_\omega \varphi \, d\mu \right| \; : \; \varphi \in \mathrm{Lip}_k(\omega), \; \|\varphi\|_\infty \leqslant C \right\} \xrightarrow[i\to\infty]{} 0$$

Proof. As we already said, the idea is to make use of the compactness of the family

$$\{\varphi \in \mathrm{Lip}_k(\omega), \; \|\varphi\|_\infty \leqslant C\} \,.$$

By contradiction, there exists a sequence $(\varphi_i)_i$ with $\varphi_i \in \mathrm{Lip}_k(\omega)$ and $\|\varphi_i\|_\infty \leqslant C$ for all i and such that

$$\left| \int_\omega \varphi_i \, d\mu_i - \int_\omega \varphi_i \, d\mu \right| \text{ does not converge to } 0 \,.$$

So that, up to some extraction, there exists $\varepsilon > 0$ such that for all i,

$$\left| \int_\omega \varphi_i \, d\mu_i - \int_\omega \varphi_i \, d\mu \right| > \varepsilon \,.$$

Every φ_i can be extended to $\varphi_i \in \mathrm{C}(\overline{\omega}) \cap \mathrm{Lip}_k(\overline{\omega})$ and then

$$\begin{cases} (\varphi_i)_i \subset \mathrm{C}(\overline{\omega}) \cap \mathrm{Lip}_k(\overline{\omega}) \text{ is equilipschitz,} \\ \sup_i \|\varphi_i\|_\infty \leqslant C \,. \end{cases}$$

By Ascoli's theorem, up to a subsequence, there exists a function $\varphi \in \mathrm{C}(\overline{\omega}) \cap \mathrm{Lip}_k(\overline{\omega})$ with $\|\varphi\|_\infty \leqslant C$ such that

$$\varphi_i \longrightarrow \varphi \text{ uniformly in } \overline{\omega} \,.$$

We now estimate:

$$\begin{aligned}\varepsilon &< \left| \int_\omega \varphi_i \, d\mu_i - \int_\omega \varphi_i \, d\mu \right| \\ &\leqslant \left| \int_\omega \varphi_i \, d\mu_i - \int_\omega \varphi \, d\mu_i \right| + \left| \int_\omega \varphi \, d\mu_i - \int_\omega \varphi \, d\mu \right| + \left| \int_\omega \varphi \, d\mu - \int_\omega \varphi_i \, d\mu \right| \\ &\leqslant \|\varphi_i - \varphi\|_\infty \, \mu_i(\omega) + \left| \int_\omega \varphi \, d\mu_i - \int_\omega \varphi \, d\mu \right| + \|\varphi - \varphi_i\|_\infty \, \mu(\omega)\end{aligned}$$

As $\mu(\partial\omega) = 0$ then $\mu_i(\omega) \xrightarrow[i\to\infty]{} \mu(\omega) < +\infty$ (since $\mu(\omega) \leqslant \mu(\overline{\omega})$ and $\overline{\omega}$ is compact) so that the first and last terms tend to 0. Moreover, since $\mu(\partial\omega) = 0$ then for every $f \in \mathrm{C}^0(\omega)$ (not necessarily compactly supported),

$$\int f \, d\mu_i \xrightarrow[i\to\infty]{} \int f \, d\mu \,,$$

which allows to conclude that the second term also tends to 0 which leads to a contradiction. □

The following result is the key point of the proof of Theorem 3.4. Let us first define for two Radon measures μ and ν in Ω,

$$\Delta_\omega^{k,C}(\mu,\nu) := \sup\left\{ \int_{r=0}^{\frac{d(\overline{\omega},\Omega^c)}{2}} \left| \int_{B_r(x)\cap\omega} \varphi\, d\mu - \int_{B_r(x)\cap\omega} \varphi\, d\nu \right| dr \ :\ \varphi \in \mathrm{Lip}_k(\omega),\ \|\varphi\|_\infty \leqslant C,\ x\in\overline{\omega} \right\} . \tag{3.9}$$

Proposition 3.23. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $(\mu_i)_i$ be a sequence of Radon measures weakly–$*$ converging to a Radon measure μ in Ω and such that $\sup_i \mu_i(\Omega) < +\infty$. Let $\omega \subset\subset \Omega$ be open such that $\mu(\partial\omega) = 0$ then, for fixed $k, C \geqslant 0$,*

$$\Delta_\omega^{k,C}(\mu_i,\mu) \xrightarrow[i\to+\infty]{} 0 .$$

Proof. The upper bound on the radius r ensures that the closure of every considered ball, $B_r(x)$ for $x\in\Omega$, is included in Ω. We argue as in the proof of Proposition 3.22, assuming by contradiction that, after some extraction, there exist a sequence $(\varphi_i)_i$ with $\varphi_i \in \mathrm{Lip}_k(\omega)$ and $\|\varphi_i\|_\infty \leqslant C$ for all i, and a sequence $(x_i)_i$ with $x_i \in \overline{\omega}$ for all i, and $\varepsilon > 0$ such that for all i,

$$\int_{r=0}^{\frac{d(\overline{\omega},\Omega^c)}{2}} \left| \int_{B_r(x_i)\cap\omega} \varphi_i\, d\mu_i - \int_{B_r(x_i)\cap\omega} \varphi_i\, d\mu \right| dr > \varepsilon .$$

By Ascoli's theorem and up to an extraction, there exist a function $\varphi \in \mathrm{C}^0(\overline{\omega}) \cap \mathrm{Lip}_k(\overline{\omega})$ with $\|\varphi\|_\infty \leqslant C$ such that $\varphi_i \longrightarrow \varphi$ uniformly in $\overline{\omega}$. Moreover $\overline{\omega}$ is compact so that, up to another extraction, there exists $x\in\overline{\omega}$ such that $x_i \longrightarrow x$. We now estimate for every r,

$$\begin{aligned}
\left| \int_{B_r(x_i)\cap\omega} \varphi_i\, d\mu_i - \int_{B_r(x_i)\cap\omega} \varphi_i\, d\mu \right| \leqslant & \left| \int_{B_r(x_i)\cap\omega} \varphi_i\, d\mu_i - \int_{B_r(x_i)\cap\omega} \varphi\, d\mu_i \right| \\
& + \left| \int_{B_r(x_i)} \varphi\, d\mu_i - \int_{B_r(x)} \varphi\, d\mu_i \right| + \left| \int_{B_r(x)\cap\omega} \varphi\, d\mu_i - \int_{B_r(x)\cap\omega} \varphi\, d\mu \right| \\
& + \left| \int_{B_r(x)\cap\omega} \varphi\, d\mu - \int_{B_r(x_i)\cap\omega} \varphi\, d\mu \right| + \left| \int_{B_r(x_i)\cap\omega} \varphi\, d\mu - \int_{B_r(x_i)\cap\omega} \varphi_i\, d\mu \right| \\
\leqslant & \|\varphi_i - \varphi\|_\infty\, \mu_i\left(B_r(x_i)\right) + \|\varphi\|_\infty\, \mu_i\left(B_r(x_i)\triangle B_r(x)\right) + \left| \int_{B_r(x)\cap\omega} \varphi\, d\mu_i - \int_{B_r(x)\cap\omega} \varphi\, d\mu \right| \\
& + \|\varphi\|_\infty\, \mu\left(B_r(x_i)\triangle B_r(x)\right) + \|\varphi - \varphi_i\|_\infty\, \mu(B_r(x_i)) \\
\leqslant & \|\varphi_i - \varphi\|_\infty \left(\mu_i(\Omega) + \mu(\Omega)\right) + \left| \int_{B_r(x)\cap\omega} \varphi\, d\mu_i - \int_{B_r(x)\cap\omega} \varphi\, d\mu \right| \qquad (3.10) \\
& + \|\varphi\|_\infty \left(\mu_i\left(B_r(x_i)\triangle B_r(x)\right) + \mu\left(B_r(x_i)\triangle B_r(x)\right)\right) .
\end{aligned}$$

The first term in the right hand side of (3.10) tends to 0 since $\sup_i \mu_i(\Omega) < +\infty$ also implies $\mu(\Omega) < +\infty$. Concerning the second term, as $\mu(\partial\omega) = 0$ then for all $r \in (0, \frac{d(\overline{\omega},\Omega^c)}{2})$, $\mu(\partial(B_r(x)\cap\omega)) \leqslant \mu(\partial B_r(x))$ and therefore the second term tends to 0 for every r such that $\mu(\partial B_r(x)) = 0$, i.e. for almost every $r \in (0, \frac{d(\overline{\omega},\Omega^c)}{2})$. As for the last term, thanks to Proposition 3.19 we know that $\limsup_i \mu_i(B_r(x)\triangle B_r(x_i)) + \mu(B_r(x)\triangle B_r(x_i)) \leqslant 2\mu(\partial B_r(x)) = 0$ for almost every $r \in (0, \frac{d(\overline{\omega},\Omega^c)}{2})$. Moreover the whole quantity (3.10) is uniformly bounded by

$$5C\left(\mu(\Omega) + \sup_i \mu_i(\Omega)\right) .$$

Consequently the the right hand side of (3.10) tends to 0 for almost every $r \in (0, \frac{d(\overline{\omega},\Omega^c)}{2})$ (such that $\mu(\partial B_r(x)) = 0$) and is uniformly bounded by the constant $5C\left(\mu(\Omega) + \sup_j \mu_j(\Omega)\right)$, then by Lebesgue dominated theorem, we have

$$\varepsilon < \int_{r=0}^{\frac{d(\overline{\omega},\Omega^c)}{2}} \left| \int_{B_r(x_i)\cap\omega} \varphi_i \, d\mu_i - \int_{B_r(x_i)\cap\omega} \varphi_i \, d\mu \right| dr \xrightarrow[i\to\infty]{} 0$$

which concludes the proof. □

We can now study the convergence of $E_{\alpha_i}(x,P,V_i) - E_{\alpha_i}(x,P,V)$ uniformly with respect to P and locally uniformly with respect to x. Indeed, the previous result (Proposition 3.23) is given in some compact subset $\overline{\omega} \subset\subset \Omega$. Consequently, we define a local version of our energy:

Definition 3.13. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $\omega \subset\subset \Omega$ be a relatively compact open subset. For every d–varifold V in Ω and for every $x \in \omega$ and $P \in G_{d,n}$, we define*

$$E_\alpha^\omega(x,P,V) = \int_{r=\alpha}^{\min\left(1,\frac{d(\overline{\omega},\Omega^c)}{2}\right)} \frac{1}{r^d} \int_{B_r(x)\cap\omega} \left(\frac{d(y-x,P)}{r}\right)^2 d\|V\| \, \frac{dr}{r} \, .$$

Remark 3.5. Notice that

$$\begin{aligned} E_\alpha^\omega(x,P,V) &= \int_{r=\alpha}^{\min\left(1,\frac{d(\overline{\omega},\Omega^c)}{2}\right)} \frac{1}{r^d} \int_{B_r(x)\cap\omega} \left(\frac{d(y-x,P)}{r}\right)^2 d\|V\| \, \frac{dr}{r} \\ &\leqslant \int_{r=\alpha}^{1} \frac{1}{r^d} \int_{B_r(x)\cap\Omega} \left(\frac{d(y-x,P)}{r}\right)^2 d\|V\| \, \frac{dr}{r} = E_\alpha(x,P,V) \, . \end{aligned}$$

Proposition 3.24. *Let $(V_i)_i$ be a sequence of d–varifolds weakly—$*$ converging to a d–varifold V in some open set $\Omega \subset \mathbb{R}^n$ and such that $\sup_i \|V_i\|(\Omega) < +\infty$. For all open subsets $\omega \subset\subset \Omega$ such that $\|V\|(\partial\omega) = 0$, let us define*

$$\eta_i^\omega := \sup \left\{ \int_{r=0}^{\min\left(1,\frac{d(\overline{\omega},\Omega^c)}{2}\right)} \left| \int_{B_r(x)\cap\omega} \varphi \, d\|V_i\| - \int_{B_r(x)\cap\omega} \varphi \, d\|V\| \right| dr \;\middle|\; \begin{array}{l} \varphi \in \mathrm{Lip}_{2(\mathrm{diam}\omega)^2}(\omega), \\ \|\varphi\|_\infty \leqslant (\mathrm{diam}\omega)^2 \end{array} , \, x \in \overline{\omega} \right\}$$

Then,

1. *for every $0 < \alpha \leqslant 1$, $\displaystyle\sup_{\substack{x\in\overline{\omega} \\ P\in G_{d,n}}} |E_\alpha^\omega(x,P,V_i) - E_\alpha^\omega(x,P,V)| \leqslant \frac{\eta_i^\omega}{\alpha^{d+3}}$,*
2. *$\eta_i^\omega \xrightarrow[i\to\infty]{} 0$*

Proof. 1. is a direct application of Proposition 3.23, since $\|V_i\|$ weakly–$*$ converges to $\|V\|$. Now let us estimate

$$\begin{aligned} &|E_\alpha^\omega(x,P,V_i) - E_\alpha^\omega(x,P,V)| \\ &\quad\leqslant \frac{1}{\alpha^{d+3}} \int_{r=0}^{\min\left(1,\frac{d(\overline{\omega},\Omega^c)}{2}\right)} \left| \int_{B_r(x)\cap\omega} d(y-x,P)^2 d\|V_i\|(y) - \int_{B_r(x)\cap\omega} d(y-x,P)^2 d\|V\|(y) \right| dr \, . \end{aligned}$$

For all $x \in \overline{\omega}$, $P \in G_{d,n}$, let $\varphi_{x,P}(y) := d(y-x,P)^2$. One can check that

(1) $\varphi_{x,P}$ is bounded in ω by $(\mathrm{diam}\omega)^2$ indeed $\varphi_{x,P}(y) \leqslant |y-x|^2 \leqslant (\mathrm{diam}\omega)^2$,

(2) $\varphi_{x,P} \in \mathrm{Lip}_{2(\mathrm{diam}\omega)}(\omega)$ indeed

$$\begin{aligned}|\varphi_{x,P}(y) - \varphi_{x,P}(z)| &= \left| d(y-x,P)^2 - d(z-x,P)^2 \right| \\ &\leqslant 2(\mathrm{diam}\omega)\, |d(y-x,P) - d(z-x,P)| \\ &\leqslant 2(\mathrm{diam}\omega)\, d(y-z,P) \leqslant 2(\mathrm{diam}\omega)\, |y-z| \,.\end{aligned}$$

Consequently,

$$\sup_{\substack{x\in\overline{\omega} \\ P\in G_{d,n}}} \int_{r=0}^{\min\left(1,\frac{d(\overline{\omega},\Omega^c)}{2}\right)} \left| \int_{B_r(x)\cap\omega} d(y-x,P)^2 d\|V_i\|(y) - \int_{B_r(x)\cap\omega} d(y-x,P)^2 d\|V\|(y) \right| dr \leqslant \eta_i^\omega$$

and thus,

$$\sup_{\substack{x\in\overline{\omega} \\ P\in G_{d,n}}} |E_\alpha^\omega(x,P,V_i) - E_\alpha^\omega(x,P,V)| \leqslant \frac{\eta_i^\omega}{\alpha^{d+3}} \,.$$

□

It is now easy to deduce the following fact:

Proposition 3.25. *Let $(V_i)_i$ be a sequence of d–varifolds weakly–$*$ converging to a d–varifold V in some open set $\Omega \subset \mathbb{R}^n$, and let $\omega \subset\subset \Omega$ be such that $\|V\|(\partial\omega) = 0$. Assume that $\sup_i \|V_i\|(\Omega) < +\infty$, then, there exists a decreasing sequence $(\alpha_i)_i$ of positive numbers tending to 0 and such that*

$$\sup_{\substack{x\in\overline{\omega} \\ P\in G_{d,n}}} \left| E_{\alpha_i}^\omega(x,P,V_i) - E_{\alpha_i}^\omega(x,P,V) \right| \xrightarrow[i\to+\infty]{} 0 \,, \tag{3.11}$$

and for every $x \in \omega$, $P \in G_{d,n}$, the following pointwise limit holds

$$E_0^\omega(x,P,V) = \lim_{i\to\infty} E_{\alpha_i}^\omega(x,P,V_i) \,. \tag{3.12}$$

Conversely, given a decreasing sequence $(\alpha_i)_i$ of positive numbers tending to 0, there exists an extraction φ (depending on α_i, V_i but independent of $x \in \omega$ and $P \in G_{d,n}$) such that

$$\sup_{\substack{x\in\overline{\omega} \\ P\in G_{d,n}}} \left| E_{\alpha_i}^\omega(x,P,V_{\varphi(i)}) - E_{\alpha_i}^\omega(x,P,V) \right| \xrightarrow[i\to+\infty]{} 0 \,, \tag{3.13}$$

and again for every $x \in \omega$, $P \in G_{d,n}$, the following pointwise limit holds

$$E_0^\omega(x,P,V) = \lim_{i\to\infty} E_{\alpha_i}^\omega(x,P,V_{\varphi(i)}) \,. \tag{3.14}$$

Proof. Thanks to Proposition 3.24, for every $\alpha > 0$,

$$\sup_{\substack{x\in\overline{\omega} \\ P\in G_{d,n}}} |E_\alpha^\omega(x,P,V_i) - E_\alpha^\omega(x,P,V)| \leqslant \frac{\eta_i^\omega}{\alpha^{d+3}} \text{ and } \eta_i^\omega \xrightarrow[i\to\infty]{} 0 \,,$$

hence we can choose $(\alpha_i)_i$ such that $\dfrac{\eta_i^\omega}{\alpha_i^{d+3}} \xrightarrow[i\to\infty]{} 0$. Conversely, given the sequence $(\alpha_i)_i$ tending to 0, we can extract a subsequence $(\eta_{\varphi(i)}^\omega)_i$ such that $\dfrac{\eta_{\varphi(i)}^\omega}{\alpha_i^{d+3}} \xrightarrow[i\to\infty]{} 0$. For fixed $x \in \omega$ and $P \in G_{d,n}$, the pointwise convergences to the averaged height excess energy E_0^ω, (3.12) and (3.14), are a consequence of the previous convergence properties (3.11) and (3.13), and of the monotone convergence $E_\alpha^\omega(x,P,V) \xrightarrow[\alpha\to 0]{} E_0^\omega(x,P,V)$. □

Now, we can use this uniform convergence result in $\omega \times G_{d,n}$ to deduce the convergence of the integrated energies.

Proposition 3.26. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $(V_i)_i$ be a sequence of d–varifolds in Ω weakly–$*$ converging to some d–varifold V and such that $\sup_i \|V_i\|(\Omega) < +\infty$. Fix a decreasing sequence $(\alpha_i)_i$ of positive numbers tending to 0. Let $\omega \subset\subset \Omega$ with $\|V\|(\partial\omega) = 0$. Then there exists an extraction ψ such that*

$$\int_{\omega \times G_{d,n}} E_0^\omega(x,P,V)\, dV(x,P) = \lim_{i\to\infty} \int_{\omega\times G_{d,n}} E_{\alpha_i}^\omega (x,P,V_{\psi(i)})\, dV_{\psi(i)}(x,P) \,.$$

Proof. – Step 1: Let $(\alpha_i)_i \downarrow 0$ and $V_i \xrightarrow[i\to\infty]{*} V$. Thanks to Proposition 3.25), there exists an extraction φ such that

$$\sup_{\substack{x\in\overline{\omega}\\ P\in G_{d,n}}} \left| E_{\alpha_i}^\omega(x,P,V_{\varphi(i)}) - E_{\alpha_i}^\omega(x,P,V)\right| \xrightarrow[i\to\infty]{} 0\,.$$

But $\sup_i V_{\varphi(i)}(\omega\times G_{d,n}) \leqslant \sup_i \|V_i\|(\Omega) < +\infty$, hence

$$\left| \int_{\omega\times G_{d,n}} E_{\alpha_i}^\omega(x,P,V_{\varphi(i)})\, dV_{\varphi(i)}(x,P) - \int_{\omega\times G_{d,n}} E_{\alpha_i}^\omega(x,P,V)\, dV_{\varphi(i)}(x,P)\right| \xrightarrow[i\to\infty]{} 0\,. \tag{3.15}$$

– Step 2: Now, we estimate

$$\begin{aligned}
&\left| \int_{\omega\times G_{d,n}} E_{\alpha_i}^\omega(x,P,V)\, dV_{\varphi(i)}(x,P) - \int_{\omega\times G_{d,n}} E_{\alpha_i}^\omega(x,P,V)\, dV(x,P)\right| \\
&\leqslant \int_{r=\alpha_i}^{\min\left(1,\frac{d(\overline{\omega},\Omega^c)}{2}\right)} \frac{1}{r^{d+1}} \left| \int_{\omega\times G_{d,n}} \int_{B_r(x)\cap\omega} \left(\frac{d(y-x,P)}{r}\right)^2 d\|V\|(y)\, dV_{\varphi(i)}(x,P) \right.\\
&\left. - \int_{\omega\times G_{d,n}} \int_{B_r(x)\cap\omega} \left(\frac{d(y-x,P)}{r}\right)^2 d\|V\|(y)\, dV(x,P)\right| dr \\
&\leqslant \frac{1}{\alpha_i^{d+3}} \int_{r=\alpha_i}^{\min\left(1,\frac{d(\overline{\omega},\Omega^c)}{2}\right)} \left| \int_{\omega\times G_{d,n}} g_r(x,P) dV_{\varphi(i)}(x,P) - \int_{\omega\times G_{d,n}} g_r(x,P)\, dV(x,P)\right| dr\,,
\end{aligned}$$

with $g_r(x,P) = \int_{B_r(x)\cap\omega} d(y-x,P)^2\, d\|V\|(y)$. For every $r < \min\left(1, \frac{d(\overline{\omega},\Omega^c)}{2}\right)$, g_r is bounded by 1. Moreover the set of discontinuities of g_r, denoted by $\mathrm{disc}(g_r)$, satisfies

$$\begin{aligned}
\mathrm{disc}(g_r) &\subset \{(x,P)\in \omega\times G_{d,n} \,:\, \|V\|(\partial(B_r(x)\cap\omega)) > 0\} \\
&\subset \{(x,P)\in \omega\times G_{d,n} \,:\, \|V\|(\partial B_r(x)) > 0\}\,.
\end{aligned}$$

Hence $V(\mathrm{disc}(g_r)) \leqslant \|V\|\left(\{x\in\omega \,:\, \|V\|(\partial B_r(x))>0\}\right) = 0$ for almost every r by Proposition 3.8. Consequently,

$$\left| \int_{\omega\times G_{d,n}} g_r(x,P) dV_{\varphi(i)}(x,P) - \int_{\omega\times G_{d,n}} g_r(x,P)\, dV(x,P)\right| \xrightarrow[i\to\infty]{} 0 \quad \text{for a.e. } r\,,$$

and then by dominated convergence,

$$\int_{r=0}^{\min\left(1,\frac{d(\overline{\omega},\Omega^c)}{2}\right)} \left| \int_{\omega\times G_{d,n}} g_r(x,P) dV_{\varphi(i)}(x,P) - \int_{\omega\times G_{d,n}} g_r(x,P)\, dV(x,P)\right| dr \xrightarrow[i\to\infty]{} 0\,.$$

It is then possible to extract, again, a subsequence $(V_{\psi(i)})_i$ such that

$$\left| \int_{\omega \times G_{d,n}} E_{\alpha_i}^\omega (x, P, V) \, dV_{\psi(i)}(x, P) - \int_{\omega \times G_{d,n}} E_{\alpha_i}^\omega (x, P, V) \, dV(x, P) \right| \xrightarrow[i \to \infty]{} 0 \, . \tag{3.16}$$

– Step 3: Eventually by (3.15), (3.16) and monotone convergence, there exists an extraction ψ such that

$$\begin{aligned} \int_{\omega \times G_{d,n}} E_0^\omega (x, P, V) \, dV(x, P) &= \lim_{i \to \infty} \int_{\omega \times G_{d,n}} E_{\alpha_i}^\omega (x, P, V) \, dV(x, P) \\ &= \lim_{i \to \infty} \int_{\omega \times G_{d,n}} E_{\alpha_i}^\omega (x, P, V_{\psi(i)}) \, dV_{\psi(i)}(x, P) \, . \end{aligned}$$

□

3.3.4 Rectifiability theorem

We can now state the main result.

Theorem. 3.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $(V_i)_i$ be a sequence of d–varifolds in Ω weakly–$*$ converging to some d–varifold V and such that $\sup_i \|V_i\|(\Omega) < +\infty$. Fix $(\alpha_i)_i$ and $(\beta_i)_i$ decreasing sequences of positive numbers tending to 0 and assume that:*

(i) there exist $0 < C_1 < C_2$ such that for $\|V_i\|$–almost every $x \in \Omega$ and for every $\beta_i < r < d(x, \Omega^c)$,

$$C_1 \omega_d r^d \leqslant \|V_i\|(B_r(x)) \leqslant C_2 \omega_d r^d \, , \tag{3.17}$$

(ii)

$$\sup_i \int_{\Omega \times G_{d,n}} E_{\alpha_i}(x, P, V_i) \, dV_i(x, P) < +\infty \, . \tag{3.18}$$

Then V is a rectifiable d–varifold.

Proof. The point is to see that these two assumptions (3.17) and (3.18) actually imply the assumptions of the static theorem (Theorem 3.3) for the limit varifold V.

– Step 1: The first assumption (3.17) and Proposition 3.21 lead to the first assumption of the static theorem: there exist $0 < C_1 < C_2$ such that for $\|V\|$–almost every $x \in \Omega$ and for every $0 < r < d(x, \Omega^c)$,

$$C_1 \omega_d r^d \leqslant \|V\|(B_r(x)) \leqslant C_2 \omega_d r^d \, .$$

– Step 2: Let $\omega \subset\subset \Omega$ be a relatively compact open subset such that $\|V\|(\partial \omega) = 0$ then, thanks to Proposition 3.26, we know that there exists some extraction φ such that

$$\int_{\omega \times G_{d,n}} E_0^\omega (x, P, V) \, dV(x, P) = \lim_{i \to \infty} \int_{\omega \times G_{d,n}} E_{\alpha_i}^\omega (x, P, V_{\varphi(i)}) \, dV_{\varphi(i)}(x, P) \, . \tag{3.19}$$

But E_α^ω is decreasing in α and $\alpha_{\varphi(i)} \leqslant \alpha_i$, therefore for every $(x, P) \in \omega \times G_{d,n}$,

$$E_{\alpha_i}^\omega (x, P, V_{\varphi(i)}) \leqslant E_{\alpha_{\varphi(i)}}^\omega (x, P, V_{\varphi(i)}) \, ,$$

hence

$$\sup_i \int_{\omega \times G_{d,n}} E_{\alpha_i}^\omega (x, P, V_{\varphi(i)}) \, dV_{\varphi(i)}(x, P) \leqslant \sup_i \int_{\omega \times G_{d,n}} E_{\alpha_{\varphi(i)}}^\omega (x, P, V_{\varphi(i)}) \, dV_{\varphi(i)}(x, P) \, . \tag{3.20}$$

Moreover, recall that $E^{\omega}_{\alpha_i}(x,P,V_i) \leqslant E_{\alpha_i}(x,P,V_i)$ and thus

$$\sup_i \int_{\omega\times G_{d,n}} E^{\omega}_{\alpha_i}(x,P,V_i)\,dV_i(x,P) \leqslant \sup_i \int_{\Omega\times G_{d,n}} E_{\alpha_i}(x,P,V_i)\,dV_i(x,P) \leqslant C\,. \tag{3.21}$$

Eventually, by (3.19), (3.20) and (3.21),

$$\int_{\omega\times G_{d,n}} E^{\omega}_0(x,P,V)\,dV(x,P) \leqslant C\,. \tag{3.22}$$

– Step 3: By (3.22), for every $\omega \subset\subset \Omega$ such that $\|V\|(\partial\omega) = 0$ we get that

$$V\left(\{(x,P)\in\omega\times G_{d,n} \mid E^{\omega}_0(x,P,V) = +\infty\}\right) = 0\,.$$

At the same time, for $x\in\omega$ and $P\in G_{d,n}$,

$$\begin{aligned}
|E_0(x,P,V) - E^{\omega}_0(x,P,V)| &= \int_{r=\min\left(1,\frac{d(\overline{\omega},\Omega^c)}{2}\right)}^{1} \frac{1}{r^{d+1}} \int_{B_r(x)\cap\Omega} \left(\frac{d(y-x,P)}{r}\right)^2 d\|V\|(y)\,dr \\
&\quad + \int_{r=0}^{\min\left(1,\frac{d(\overline{\omega},\Omega^c)}{2}\right)} \frac{1}{r^{d+1}} \int_{B_r(x)\cap(\Omega\setminus\overline{\omega})} \left(\frac{d(y-x,P)}{r}\right)^2 d\|V\|(y)\,dr \\
&\leqslant \left(\frac{2}{d(\overline{\omega},\Omega^c)}\right)^{d+1} \|V\|(\Omega) + \int_{r=d(x,\omega^c)}^{\min\left(1,\frac{d(\overline{\omega},\Omega^c)}{2}\right)} \frac{1}{r^{d+1}} \int_{B_r(x)\cap\Omega} \left(\frac{d(y-x,P)}{r}\right)^2 d\|V\|(y)\,dr \\
&\leqslant \left(\left(\frac{2}{d(\overline{\omega},\Omega^c)}\right)^{d+1} + \left(\frac{1}{d(x,\omega^c)}\right)^{d+1}\right)\|V\|(\Omega) < +\infty\,.
\end{aligned}$$

Hence $E^{\omega}_0(x,P,V) = +\infty$ if and only if $E_0(x,P,V) = +\infty$, and consequently,

$$V\left(\{(x,P)\in\omega\times G_{d,n} \mid E_0(x,P,V) = +\infty\}\right) = V\left(\{(x,P)\in\omega\times G_{d,n} \mid E^{\omega}_0(x,P,V) = +\infty\}\right)\,.$$

Now, thanks to Lemma 3.17, we decompose Ω into $\Omega = \cup_k \omega_k$ with $\forall k$, $\omega_{k+1}\subset\subset\omega_k\subset\subset\Omega$ and $\|V\|(\partial\omega_k) = 0$. Then

$$\begin{aligned}
V\left(\{(x,P)\in\Omega\times G_{d,n} \mid E_0(x,P,V) = +\infty\}\right) &= \lim_k V\left(\{(x,P)\in\omega_k\times G_{d,n} \mid E_0(x,P,V) = +\infty\}\right) \\
&= \lim_k V\left(\{(x,P)\in\omega_k\times G_{d,n} \mid E^{\omega_k}_0(x,P,V) = +\infty\}\right) \\
&= 0\,.
\end{aligned}$$

Applying the static theorem (Theorem 3.3) allows us to conclude the proof. □

In Theorem 3.4, we have found conditions (3.17) and (3.18) ensuring the rectifiability of the weak–$*$ limit V of a sequence of d–varifolds $(V_i)_i$. Recall that the condition

$$\sup_i |\delta V_i|(\Omega) < +\infty \tag{3.23}$$

together with the condition (3.17) also ensure the rectifiability of the weak–$*$ limit V of $(V_i)_i$. But, in Proposition 2.2, we have computed the first variation of a discrete volumetric varifold and we have seen in Example 2.1 that even in the case where the limit varifold V is very simple (we considered a straight line), the natural approximations of V by discrete volumetric varifolds V_i generally do not satisfy (3.23) even though $|\delta V|(\Omega) = 0$.
We now check that the condition (3.18) in Theorem 3.4 is better adapted to general sequences of varifolds than the control of the first variation (3.23). Indeed, in the next Proposition, we prove that given a d–varifold V with some regularity property, and given any sequence of d–varifolds $V_i \xrightarrow[i\to\infty]{*} V$, there exists a subsequence of $(V_i)_i$ satisfying a local version of condition (3.18) in Theorem 3.4.

Proposition 3.27. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let V be a d–varifold in Ω such that*

$$\int_{\Omega \times G_{d,n}} E_0(x,P,V)\, dV(x,P) < +\infty \, .$$

Let $(V_i)_i$ be a sequence of d–varifolds weakly–$*$ converging to V with $\sup_i \|V_i\|(\Omega) < +\infty$. Then, given $\alpha_i \downarrow 0$, for every $\omega \subset\subset \Omega$ such that $\|V\|(\partial\omega) = 0$, there exists a subsequence $(W_i)_i = (V_{\varphi(i)})_i$ such that

$$\sup_i \int_{\omega \times G_{d,n}} E_{\alpha_i}^\omega (x,P,W_i)\, dW_i(x,P) < +\infty \, . \tag{3.24}$$

Proof. It is a direct consequence of Proposition 3.26. □

The condition (3.24) is expressed in terms of the local version E_α^ω of E_α. In the case where the varifolds are contained in the same compact set, then global condition (3.18) of Theorem 3.4 is satisfied by some subsequence.

Proposition 3.28. *Let $\alpha_i \downarrow 0$. Let V be a rectifiable d–varifold in $\mathbb{R}^n$ with compact support and such that*

$$\int_{\omega \times G_{d,n}} E_0(x,P,V)\, dV(x,P) < +\infty \, .$$

Assume moreover that there exists some sequence of d–varifolds $(V_i)_i$ weakly–$*$ converging to V with $\sup_i \|V_i\|(\mathbb{R}^n) < +\infty$. Then for any $\omega \subset\subset \mathbb{R}^n$ such that $\operatorname{supp}\|V\| + B_1(0) \subset \omega$ and for all i, $\operatorname{supp}\|V_i\| + B_1(0) \subset \omega$, there exists a subsequence $(V_{\varphi(i)})_i$ such that

$$\sup_i \int_{\omega \times G_{d,n}} E_{\alpha_i}(x,P,V_{\varphi(i)})\, dV_{\varphi(i)}(x,P) < +\infty \, .$$

Proof. It is again a direct consequence of Proposition 3.26 (since $\overline{\omega}$ is compact and $\|V\|(\partial\omega) = 0$) combined with the fact that $\operatorname{supp}\|V\| + B_1(0) \subset \omega$ implies

$$E_\alpha^\omega(x,P,V) = \int_{r=\alpha}^{\min\left(1, \frac{d(\overline{\omega}, (\mathbb{R}^n)^c)}{2}\right)} \frac{1}{r^d} \int_{B_r(x)\cap\omega} \left(\frac{d(y-x,P)}{r} \right)^2 d\|V\| \, \frac{dr}{r} = E_\alpha(x,P,V) \, .$$

□

Given $V_i \xrightarrow[i\to+\infty]{*} V$ and $\alpha_i \downarrow 0$, the previous propositions 3.27 and 3.28 give a subsequence $(V_{\varphi(i)})_i$ satisfying (3.18)

$$\sup_i \int E_{\alpha_i}(x,P,V_{\varphi(i)})\, dV_{\varphi(i)}(x,P) < +\infty$$

In the following proposition, we focus on sequences of discrete volumetric varifolds. Under some uniform regularity assumption on V, we give a sequence $(V_i)_i$ of discrete volumetric varifolds such that

$$V_i \xrightarrow[i\to+\infty]{*} V \, ,$$

and a condition linking the scale parameter α_i and the size δ_i of the mesh associated to the discrete volumetric varifold V_i, ensuring that (3.18) holds for V_i and not for a subsequence.

Theorem 3.29. *Let $V = v(M,\theta)$ be a rectifiable d–varifold in $\mathbb{R}^n$ with finite mass $\|V\|(\mathbb{R}^n) < +\infty$ and compact support. Let $\delta_i \downarrow 0$ be a sequence of infinitesimals and $(\mathcal{K}_i)_i$ a sequence of meshes satisfying*

$$\sup_{K\in\mathcal{K}_i} \operatorname{diam}(K) \leqslant \delta_i \xrightarrow[i\to+\infty]{} 0 \, .$$

Assume that there exists $0<\beta<1$ *and* $C>0$ *such that for* $\|V\|$*–almost every* $x,\, y\in\Omega$,

$$\|T_xM - T_yM\| \leqslant C|x-y|^\beta \, .$$

Define the sequence of discrete volumetric varifolds:

$$V_i = \sum_{K\in\mathcal{K}_i} \frac{m_K^i}{|K|}\mathcal{L}^n \otimes \delta_{P_K^i} \;\; \textit{with } m_K^i = \|V\|(K) \;\textit{and } P_K^i \in \operatorname*{arg\,min}_{P\in G_{d,n}} \int_{K\times G_{d,n}} \|P-S\| \, dV(x,S)\, .$$

Then,

(i) $V_i \xrightarrow[i\to+\infty]{*} V$,

(ii) *For any sequence of infinitesimals* $\alpha_i \downarrow 0$ *and such that for all* i,

$$\frac{\delta_i^\beta}{\alpha_i^{d+3}} \xrightarrow[i\to+\infty]{} 0\, , \tag{3.25}$$

we have,

$$\int_{\mathbb{R}^n\times G_{d,n}} E_0(x,P,V)\, dV(x,P) = \lim_{i\to+\infty} \int_{\mathbb{R}^n\times G_{d,n}} E_{\alpha_i}(x,P,V_i)\, dV_i(x,P) < +\infty\, .$$

Remark 3.6. We insist on the fact that the condition on the scale parameters α_i and the size of the mesh δ_i is not dependent on V_i but only on the regularity of V i.e. on β (and on the dimension d).

Proof. Thanks to Theorem 2.1, we have that

$$V_i \xrightarrow[i\to+\infty]{*} V \text{ in } \mathbb{R}^n\, ,$$

and moreover, for all $\varphi\in \operatorname{Lip}(\mathbb{R}^n\times G_{d,n})$,

$$\int_{(\Pi(\operatorname{supp}\varphi)\cap\mathbb{R}^n)\times G_{d,n}} \left\|P^i(y)-T\right\| \, dV(y,T) \leqslant 2C\delta_i^\beta \|V\| \left(\Pi(\operatorname{supp}\varphi)\cap\mathbb{R}^n\right)\, , \tag{3.26}$$

We now estimate,

$$\left| \int_{\mathbb{R}^n\times G_{d,n}} E_\alpha(x,P,V)\, dV(x,P) - \int_{\mathbb{R}^n\times G_{d,n}} E_\alpha(x,P,V_i)\, dV_i(x,P) \right| \tag{3.27}$$

$$\leqslant \left| \int_{\mathbb{R}^n\times G_{d,n}} E_\alpha(x,P,V)\, dV(x,P) - \int_{\mathbb{R}^n\times G_{d,n}} E_\alpha(x,P,V)\, dV_i(x,P) \right| \tag{3.28}$$

$$+ \left| \int_{\mathbb{R}^n\times G_{d,n}} E_\alpha(x,P,V)\, dV_i(x,P) - \int_{\mathbb{R}^n\times G_{d,n}} E_\alpha(x,P,V_i)\, dV_i(x,P) \right| \tag{3.29}$$

– Step 1: We begin with (3.28) and we prove that

$$\left| \int_{\mathbb{R}^n\times G_{d,n}} E_\alpha(x,P,V)\, dV_i(x,P) - \int_{\mathbb{R}^n\times G_{d,n}} E_\alpha(x,P,V)\, dV(x,P) \right| \leqslant \frac{1}{\alpha^{d+3}} \|V\|(\mathbb{R}^n)^2 \left[4\delta_i + 2C\delta_i^\beta\right]\, . \tag{3.30}$$

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n \times G_{d,n}} E_\alpha(x,P,V)\, dV_i(x,P) - \int_{\mathbb{R}^n \times G_{d,n}} E_\alpha(x,P,V)\, dV(x,P) \right| \\
&\qquad = \left| \int_{\mathbb{R}^n \times G_{d,n}} \int_{r=\alpha}^1 \frac{1}{r^{d+1}} \int_{y \in B_r(x)} \left(\frac{d(y-x,P)}{r} \right)^2 d\|V\|(y)\, dr\, dV_i(x,P) \right. \\
&\qquad \left. - \int_{\mathbb{R}^n \times G_{d,n}} \int_{r=\alpha}^1 \frac{1}{r^{d+1}} \int_{y \in B_r(x)} \left(\frac{d(y-x,P)}{r} \right)^2 d\|V\|(y)\, dr\, dV(x,P) \right| \\
&\qquad \leqslant \int_{r=\alpha}^1 \frac{1}{r^{d+3}} \int_{y \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n \times G_{d,n}} \mathbb{1}_{\{|y-x|<r\}}(x) \left(d(y-x,P)\right)^2 \, dV_i(x,P) \right. \\
&\qquad \left. - \int_{\mathbb{R}^n \times G_{d,n}} \mathbb{1}_{\{|y-x|<r\}}(x) \left(d(y-x,P)\right)^2 \, dV(x,P) \right| d\|V\|(y)\, dr
\end{aligned} \tag{3.31}$$

And by definition of V_i, for fixed y and $\alpha < r < 1$, we have:

$$\left| \int_{\mathbb{R}^n \times G_{d,n}} \mathbb{1}_{\{|y-x|<r\}}(x) \left(d(y-x,P)\right)^2 \, dV_i(x,P) - \int_{\mathbb{R}^n \times G_{d,n}} \mathbb{1}_{\{|y-x'|<r\}}(x') \left(d(y-x',P')\right)^2 \, dV(x',P') \right|$$

$$= \left| \sum_{K \in \mathcal{K}_i} \int_{x \in K} \mathbb{1}_{\{|y-x|<r\}}(x) \left(d(y-x,P_K^i)\right)^2 \frac{\|V\|(K)}{|K|}\, d\mathcal{L}^n(x) \right. \tag{3.32}$$

$$\left. - \int_{x \in \mathbb{R}^n} \int_{(x',P') \in \mathbb{R}^n \times G_{d,n}} \mathbb{1}_{\{|y-x'|<r\}}(x') \left(d(y-x',P')\right)^2 \, dV(x',P') \frac{d\mathcal{L}^n(x)}{|K|} \right| \tag{3.33}$$

$$\leqslant \sum_{K \in \mathcal{K}_i} \int_{x \in K} \int_{K \times G_{d,n}} \left| \mathbb{1}_{B_r(y)}(x) \left(d(y-x,P_K^i)\right)^2 - \mathbb{1}_{B_r(y)}(x') \left(d(y-x',P')\right)^2 \right| dV(x',P') \frac{d\mathcal{L}^n(x)}{|K|}\,, \tag{3.34}$$

writing $\|V\|(K) = \int_{(x',P') \in K \times G_{d,n}} d\|V\|(x',P')$. And in (3.34), either $x, x' \in B_r(y)$ and in this case

$$\begin{aligned}
\left| \mathbb{1}_{B_r(y)}(x) \left(d(y-x,P_K^i)\right)^2 - \mathbb{1}_{B_r(y)}(x') \left(d(y-x',P')\right)^2 \right| &\leqslant 2r \left| d(y-x,P_K^i) - d(y-x',P') \right| \\
&\leqslant 2r \left(|x-x'| + |y-x'| \|P_K^i - P'\| \right) \\
&\leqslant 2 \left(|x-x'| + \|P_K^i - P'\| \right)\,,
\end{aligned}$$

either $\left\{ \begin{array}{l} x \in B_r(y) \text{ and } x' \notin B_r(y) \text{ or,} \\ x' \in B_r(y) \text{ and } x \notin B_r(y), \end{array} \right.$ and in this case

$$\left| \mathbb{1}_{B_r(y)}(x) \left(d(y-x,P_K^i)\right)^2 - \mathbb{1}_{B_r(y)}(x') \left(d(y-x',P')\right)^2 \right| \leqslant r^2 \leqslant 1\,.$$

Notice that, as $|x - x'| \leqslant \delta_i$ this second case can only happen for $x,\, x' \in B_{r+\delta_i}(y) \setminus B_{r-\delta_i}(y)$.

Consequently,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n\times G_{d,n}} \mathbb{1}_{\{|y-x|<r\}}(x)\,(d(y-x,P))^2\,dV_i(x,P) - \int_{\mathbb{R}^n\times G_{d,n}} \mathbb{1}_{\{|y-x'|<r\}}(x')\,\left(d(y-x',P')\right)^2\,dV(x',P')\right|\\
&\leqslant \sum_{K\in\mathcal{K}_i}\int_{x\in K}\int_{K\times G_{d,n}} 2\,\left(|x-x'|+\|P^i_K-P'\|\right)\,dV(x',P')\,\frac{d\mathcal{L}^n(x)}{|K|}\\
&+\sum_{K\in\mathcal{K}_i} r^2\|V\|\left(K\cap B_{r+\delta_i}(y)\setminus B_{r-\delta_i}(y)\right)\underbrace{\frac{|K\cap B_{r+\delta_i}(y)\setminus B_{r-\delta_i}(y)|}{|K|}}_{\leqslant 1}\\
&\leqslant 2\delta_i\,\|V\|(\mathbb{R}^n)+\int_{\mathbb{R}^n\times G_{d,n}}\|P^i(x')-P'\|\,dV(x',P')+\|V\|\left(B_{r+\delta_i}(y)\setminus B_{r-\delta_i}(y)\right)\\
&\leqslant 2\left(\delta_i+C\delta_i^{\beta}\right)\|V\|(\mathbb{R}^n)+\|V\|\left(B_{r+\delta_i}(y)\setminus B_{r-\delta_i}(y)\right)\ \text{thanks to (3.26)}\ .
\end{aligned}$$

Notice that

$$\begin{aligned}
\int_{r=0}^1\|V\|\left(B_{r+\delta_i}(y)\setminus B_{r-\delta_i}(y)\right)\,dr &= \int_{r=0}^1\|V\|\left(B_{r+\delta_i}(y)\right)\,dr-\int_{r=\delta_i}^1\|V\|\left(B_{r-\delta_i}(y)\right)\,dr\\
&=\int_{r=\delta_i}^{1+\delta_i}\|V\|\left(B_r(y)\right)\,dr-\int_{r=0}^{1-\delta_i}\|V\|\left(B_r(y)\right)\,dr\\
&\leqslant\int_{r=1-\delta_i}^{1+\delta_i}\|V\|(B_r(y))\,dr\leqslant 2\delta_i\|V\|(\mathbb{R}^n)\,.
\end{aligned}$$

Eventually, by (3.31),

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n\times G_{d,n}} E_\alpha(x,P,V)\,dV_i(x,P)-\int_{\mathbb{R}^n\times G_{d,n}} E_\alpha(x,P,V)\,dV(x,P)\right|\\
&\qquad\leqslant\frac{1}{\alpha^{d+3}}\int_{r=0}^1\int_{\mathbb{R}^n} 2\left(\delta_i+C\delta_i^\beta\right)\|V\|(\mathbb{R}^n)+\|V\|\left(B_{r+\delta_i}(y)\setminus B_{r-\delta_i}(y)\right)\,d\|V\|(y)\,dr\\
&\qquad\leqslant\frac{1}{\alpha^{d+3}}\left[2\left(\delta_i+C\delta_i^\beta\right)\|V\|(\mathbb{R}^n)^2+\int_{\mathbb{R}^n}\int_{r=0}^1\|V\|\left(B_{r+\delta_i}(y)\setminus B_{r-\delta_i}(y)\right)\,dr\,d\|V\|(y)\right]\\
&\qquad\leqslant\frac{1}{\alpha^{d+3}}\|V\|(\mathbb{R}^n)^2\left[4\delta_i+2C\delta_i^\beta\right]\ .
\end{aligned}$$

– Step 2: It remains to estimate (3.29), we prove that

$$\left|\int_{\mathbb{R}^n\times G_{d,n}} E_\alpha(x,P,V)\,dV_i(x,P)-\int_{\mathbb{R}^n\times G_{d,n}} E_\alpha(x,P,V_i)\,dV_i(x,P)\right|\leqslant\frac{1}{\alpha^{d+3}}4\|V\|(\mathbb{R}^n)^2\delta_i. \tag{3.35}$$

Indeed, exactly as previously (but fixing x and integrating against $\|V_i\|$, $\|V\|$ instead of V_i, V, so that the term depending on P^i does not take part into this estimate), we have

$$\begin{aligned}
|E_\alpha(x,P,V_i)\ &-E_\alpha(x,P,V)|\\
&\leqslant\frac{1}{\alpha^{d+3}}\int_{r=0}^1\left|\int_{B_r(x)} d(y-x,P)^2\,d\|V_i\|(y)-\int_{B_r(x)} d(y'-x,P)^2\,d\|V\|(y')\right|\,dr\\
&\leqslant\frac{1}{\alpha^{d+3}}\int_{r=0}^1\left(2\delta_i\|V\|(\mathbb{R}^n)+\|V\|\left(B_{r+\delta_i}(y)\setminus B_{r-\delta_i}(y)\right)\right)\,dr\\
&\leqslant\frac{1}{\alpha^{d+3}}\|V\|(\mathbb{R}^n)4\delta_i\,.
\end{aligned}\tag{3.36}$$

We conclude this step by integrating against V_i, reminding that $V_i(\mathbb{R}^n \times G_{d,n}) = \|V_i\|(\mathbb{R}^n) = \|V\|(\mathbb{R}^n)$.

– Step 3: By (3.30) and (3.35),

$$\left| \int_{\mathbb{R}^n \times G_{d,n}} E_{\alpha_i}(x,P,V)\, dV(x,P) - \int_{\mathbb{R}^n \times G_{d,n}} E_{\alpha_i}(x,P,V_i)\, dV_i(x,P) \right| \leqslant \frac{1}{\alpha_i^{d+3}} \|V\|(\mathbb{R}^n)^2 \left(8\delta_i + 2C\delta_i^\beta \right) \tag{3.37}$$

$$\xrightarrow[i \to +\infty]{} 0$$

thanks to (3.25). Then, by monotone convergence and (3.37),

$$\begin{aligned} \int_{\mathbb{R}^n \times G_{d,n}} E_0(x,P,V)\, dV(x,P) &= \lim_{i \to +\infty} \int_{\mathbb{R}^n \times G_{d,n}} E_{\alpha_i}(x,P,V)\, dV(x,P) \\ &= \lim_{i \to +\infty} \int_{\mathbb{R}^n \times G_{d,n}} E_{\alpha_i}(x,P,V_i)\, dV_i(x,P)\,. \end{aligned}$$

□

3.4 Appendix: The approximate averaged height excess energy as a tangent plane estimator

Throughout this section, $(V_i)_i$ is a sequence of d–varifolds weakly–$*$ converging to some d–varifold V and $(\alpha_i)_i$ is a decreasing sequence of positive numbers tending to 0 and such that

$$\sup_{\substack{x \in \overline{\omega} \\ P \in G_{d,n}}} \left| E_{\alpha_i}^\omega(x,P,V_i) - E_{\alpha_i}^\omega(x,P,V) \right| \,. \tag{3.38}$$

The existence of such a sequence of $(\alpha_i)_i$ is given by Proposition 3.24 in general, and in the case of discrete volumetric varifolds associated to a varifold V, (3.38) holds as soon as

$$\frac{\delta_i}{\alpha_i^{d+3}} \xrightarrow[i \to +\infty]{} 0 \text{ thanks to (3.36)}\,.$$

We want to show that under this condition on the choice of $(\alpha_i)_i$, for fixed $x \in \Omega$, the minimizers of $P \mapsto E_{\alpha_i}^x(P) = E_{\alpha_i}^\omega(x,P,V_i)$ converge, up to some subsequence, to minimizers of $P \mapsto E_0^x(P) = E_0^\omega(x,P,V)$. In the proofs, we shorten $E_{\alpha_i}^x(P) = E_{\alpha_i}^\omega(x,P,V_i)$ and $E_0^x(P) = E_0^\omega(x,P,V)$. We begin with studying the pointwise approximate averaged height excess energy with respect to $P \in G$, for fixed $x \in \Omega$ and for a fixed d–varifold V.

3.4.1 The pointwise approximate averaged height excess energy

We now fix a d–varifold (not supposed rectifiable nor with bounded first variation) in some open set $\Omega \subset \mathbb{R}^n$ and we study the continuity of $E_\alpha(x,P,V)$ with respect to $P \in G_{d,n}$ and then $x \in \Omega$.

Proposition 3.30. *Let $0 < \alpha < 1$. Let V be a d–varifold in an open set $\Omega \subset \mathbb{R}^n$ such that $\|V\|(\Omega) < +\infty$. Then, for every P, $Q \in G_{d,n}$,*

$$|E_\alpha(x,P,V) - E_\alpha(x,Q,V)| \leqslant 2\|P - Q\| \int_{r=\alpha}^1 \frac{1}{r^{d+1}} \|V\|(B_r(x))\, dr$$

In particular, $P \mapsto E_\alpha(x,P,V)$ is Lipschitz with constant $K_\alpha \leqslant \dfrac{2}{\alpha^{d+1}}\|V\|(\Omega)$. If in addition $\forall \alpha < r < 1$, $\|V\|(B_r(x)) \leqslant Cr^d$ then $K_\alpha \leqslant C\|V\|(\Omega) \ln \frac{1}{\alpha}$.

Proof. Let P, $Q \in G_{d,n}$ then,

$$|E_\alpha(x,P,V) - E_\alpha(x,Q,V)| \leqslant \int_{r=\alpha}^1 \frac{1}{r^{d+1}} \int_{B_r(x)} \left| \left(\frac{d(y-x,P)}{r} \right)^2 - \left(\frac{d(y-x,Q)}{r} \right)^2 \right| d\|V\|(y) \, dr \, .$$

If π_P (respectively π_Q) denotes the orthogonal projection onto P (respectively Q), recall that $|d(y - x,P) - d(y-x,Q)| \leqslant \|P-Q\| |y-x|$. Indeed

$$\begin{aligned} d(y-x,P) &= |y-x-\pi_P(y-x)| \\ &\leqslant |y-x-\pi_Q(y-x)| + |\pi_Q(y-x) - \pi_P(y-x)| \\ &\leqslant d(y-x,Q) + \underbrace{\|\pi_Q - \pi_P\|_{op}}_{=\|P-Q\| \text{ by definition}} |y-x| \, . \end{aligned}$$

Moreover $y \in B_r(x)$ so that $\dfrac{d(y-x,P)}{r} \leqslant 1$ and thus

$$\begin{aligned} \left| \left(\frac{d(y-x,P)}{r} \right)^2 - \left(\frac{d(y-x,Q)}{r} \right)^2 \right| &\leqslant 2 \left| \frac{d(y-x,P)}{r} - \frac{d(y-x,Q)}{r} \right| \\ &\leqslant 2\|P-Q\| \frac{|y-x|}{r} \leqslant 2\|P-Q\| \, . \end{aligned}$$

Consequently,

$$|E_\alpha(x,P,V) - E_\alpha(x,Q,V)| \leqslant 2\|P-Q\| \int_{r=\alpha}^1 \frac{1}{r^{d+1}} \|V\|(B_r(x)) \, dr \, .$$

□

We now study the continuity of $x \mapsto E_\alpha(x,P,V)$.

Proposition 3.31. *Let $0 < \alpha < 1$. Let V be a d–varifold in an open set $\Omega \subset \mathbb{R}^n$ such that $\|V\|(\Omega) < +\infty$. Then,*

$$\sup_{P \in G_{d,n}} |E_\alpha(x,P,V) - E_\alpha(z,P,V)| \xrightarrow[z \to x]{} 0 \, .$$

Proof. First notice that for all x, y, $z \in \Omega$ and $P \in G_{d,n}$,

$$|d(y-x,P) - d(y-z,P)| = \big| |y-x-\pi_P(y-x)| - |y-z-\pi_P(y-z)| \big| \leqslant |z-x-\pi_P(z-x)| = d(z-x,P) .$$

We now split $B_r(x) \cup B_r(z)$ into $(B_r(x) \cap B_r(z))$ and $(B_r(x) \triangle B_r(z))$ so that

$$\left| \int_{B_r(x)} \left(\frac{d(y-x,P)}{r} \right)^2 d\|V\|(y) - \int_{B_r(z)} \left(\frac{d(y-z,P)}{r} \right)^2 d\|V\|(y) \right|$$

$$\leqslant \int_{B_r(x) \cap B_r(z)} \left| \left(\frac{d(y-x,P)}{r} \right)^2 - \left(\frac{d(y-z,P)}{r} \right)^2 \right| d\|V\|(y) \tag{3.39}$$

$$+ \int_{B_r(x) \setminus B_r(z)} \left(\frac{d(y-x,P)}{r} \right)^2 d\|V\|(y) + \int_{B_r(z) \setminus B_r(x)} \left(\frac{d(y-z,P)}{r} \right)^2 d\|V\|(y) \, . \tag{3.40}$$

We use the estimate linking $d(y-x,P)$ and $d(y-z,P)$ to control the first integral and then we show that the two other terms tend to 0.

Concerning the first integral (3.39):

$$
\begin{aligned}
&\int_{B_r(x)\cap B_r(z)} \left| \left(\frac{d(y-x,P)}{r} \right)^2 - \left(\frac{d(y-z,P)}{r} \right)^2 \right| d\|V\|(y) \\
\leqslant &\int_{B_r(x)\cap B_r(z)} 2 \left| \frac{d(y-x,P)}{r} - \frac{d(y-z,P)}{r} \right| d\|V\|(y) \\
\leqslant &2\frac{|z-x|}{r} \|V\| \left(B_r(x) \cap B_r(z) \right) .
\end{aligned}
$$

Concerning the two other integrals (3.40):

$$
\begin{aligned}
&\int_{B_r(x)\setminus B_r(z)} \left(\frac{d(y-x,P)}{r} \right)^2 d\|V\|(y) + \int_{B_r(z)\setminus B_r(x)} \left(\frac{d(y-z,P)}{r} \right)^2 d\|V\|(y) \\
&\qquad \leqslant \|V\| \left(B_r(x) \triangle B_r(z) \right) \leqslant R(x, r-|z-x|, r+|z-x|) ,
\end{aligned}
$$

where $R(x, r_{\min}, r_{\max}) := \{ y \in \Omega \,|\, r_{\min} \leqslant |y-x| \leqslant r_{\max} \}$.
Therefore,

$$
\begin{aligned}
&|E_\alpha(x,P,V) - E_\alpha(z,P,V)| \\
&\leqslant 2|z-x| \int_{r=\alpha}^1 \|V\|(B_r(x)\cap B_r(z)) \frac{dr}{r^{d+2}} + \int_{r=\alpha}^1 \|V\| \left(B_r(x) \triangle B_r(z) \right) \frac{dr}{r^{d+1}} \\
&\leqslant \frac{2}{d+1} |z-x| \frac{1}{\alpha^{d+1}} \|V\|(\Omega) + \frac{1}{\alpha^{d+1}} \int_{r=0}^1 \|V\| \left(R(x, r-|z-x|, r+|z-x|) \right) dr .
\end{aligned}
$$

The second term tends to 0 when $|z-x| \to 0$, by dominated convergence, since

$$
\lim_{z\to x} \|V\| \left(R(x, r-|z-x|, r+|z-x|) \right) = \|V\|(\partial B_r(x)) .
$$

□

3.4.2 Γ–convergence of $P \mapsto E^\omega_{\alpha_i}(x,P,V_i)$ to $P \mapsto E^\omega_0(x,P,V)$.

Proposition 3.32. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\omega \subset\subset \Omega$ be a relatively compact open subset such that $\|V\|(\partial\omega) = 0$. Let $(V_i)_i$ be a sequence of d–varifolds weakly–$*$ converging to V. Assume that $(\alpha_i)_i$ are chosen as explained in (3.38), uniformly in ω. For $(S_i)_i \subset G_{d,n}$ such that $S_i \xrightarrow[i\infty]{} S$ then, for all $x \in \omega$,*

$$
\lim_{i\to\infty} E^\omega_{\alpha_i}(x,S,V_i) = E^\omega_0(x,S,V) \leqslant \liminf_{i\to\infty} E^\omega_{\alpha_i}(x,S_i,V_i) .
$$

Proof. By monotone convergence, we already know that

$$
E^\omega_0(x,S,V) = \lim_{i\to\infty} E^\omega_{\alpha_i}(x,S,V) . \tag{3.41}
$$

So we now want to estimate $\left| E^\omega_{\alpha_i}(x,S,V) - E^\omega_{\alpha_i}(x,S_i,V) \right|$. Let us start with extracting some $(S_{\varphi(i)})_i$ such that

$$
\|S_{\varphi(i)} - S\| \frac{1}{\alpha_i^{d+1}} \xrightarrow[i\to\infty]{} 0
$$

so that we can now apply the regularity property (Proposition 3.30) of $E_\alpha(x,P,V)$ with respect to P:

$$
\left| E^\omega_{\alpha_i}(x,S,V) - E^\omega_{\alpha_i}(x,S_{\varphi(i)},V) \right| \leqslant \frac{2}{\alpha_i^{d+1}} \|V\|(\omega) \|S - S_{\varphi(i)}\| \xrightarrow[i\to\infty]{} 0.
$$

thus

$$E_0^\omega(x,S,V) = \lim_{i\to\infty} E_{\alpha_i}^\omega(x, S_{\varphi(i)}, V). \tag{3.42}$$

Notice that φ only depends on $(\alpha_i)_i$.
As the sequence $(\alpha_i)_i$ is decreasing, $\alpha_{\varphi(i)} \leqslant \alpha_i$ and then $E_{\alpha_i}^\omega(x,Q,V) \leqslant E_{\alpha_{\varphi(i)}}^\omega(x,Q,V)$ for all $Q \in G_{d,n}$, which implies in particular that

$$\lim_{i\to\infty} E_{\alpha_i}^\omega(x, S_{\varphi(i)}, V) \leqslant \liminf_{i\to\infty} E_{\alpha_{\varphi(i)}}^\omega(x, S_{\varphi(i)}, V) \, . \tag{3.43}$$

We now apply the uniform convergence of $\left| E_{\alpha_i}^\omega(\cdot,\cdot,V) - E_{\alpha_i}^\omega(\cdot,\cdot,V_i) \right|$ (3.38),

$$\left| E_{\alpha_{\varphi(i)}}^\omega(x, S_{\varphi(i)}, V) - E_{\alpha_{\varphi(i)}}^\omega(x, S_{\varphi(i)}, V_{\varphi(i)}) \right| \xrightarrow[i\to\infty]{} 0 \, , \tag{3.44}$$

so that by (3.42), (3.43) and (3.44)

$$E_0^\omega(x,S,V) \leqslant \liminf_{i\to\infty} E_{\alpha_{\varphi(i)}}^\omega(x, S_{\varphi(i)}, V) = \liminf_{i\to\infty} E_{\alpha_{\varphi(i)}}^\omega(x, S_{\varphi(i)}, V_{\varphi(i)}) \, . \tag{3.45}$$

As $\liminf_i E_{\alpha_i}^\omega(x, S_i, V_i) = \lim_i E_{\alpha_{\theta(i)}}^\omega(x, S_{\theta(i)}, V_{\theta(i)})$ for some extraction θ, we now apply (3.45) to these extracted sequences $(S_{\theta(i)})_i$ and $(V_{\theta(i)})_i$ so that there exists an extraction φ such that

$$\begin{aligned}
E_0^\omega(x,S,V) &\leqslant \liminf_{i\to\infty} E_{\alpha_{\theta(\varphi(i))}}^\omega(x, S_{\theta(\varphi(i))}, V_{\theta(\varphi(i))}) \\
&= \lim_i E_{\alpha_{\theta(i)}}^\omega(x, S_{\theta(i)}, V_{\theta(i)}) \text{ since the whole sequence } E_{\alpha_{\theta(i)}}^\omega(x, S_{\theta(i)}, V_{\theta(i)}) \text{ converges} \\
&= \liminf_i E_{\alpha_i}^\omega(x, S_i, V_i) \, .
\end{aligned}$$

□

We now turn to the consequences of this Γ–convergence property on the minimizers.

Proposition 3.33. *Let V_i be a sequence of d–varifolds weakly–$*$ converging to V in some open set $\Omega \subset \mathbb{R}^n$ and assume that $(\alpha_i)_i$ are chosen as explained in (3.38), uniformly in $\omega \subset\subset \Omega$ open subset such that $\|V\|(\partial\omega) = 0$. For $x \in \omega$ and $i \in \mathbb{N}$, let $T_i(x) \in \arg\min_{P\in G_{d,n}} E_{\alpha_i}^\omega(x,P,V_i)$. Then,*

1. *Any converging subsequence of $(T_i(x))_i$ tends to a minimizer of $E_0^\omega(x,\cdot,V)$.*
2. $\displaystyle\min_{P\in G_{d,n}} E_{\alpha_i}^\omega(x,P,V_i) \xrightarrow[i\to\infty]{} \min_{P\in G_{d,n}} E_0^\omega(x,P,V)$.
3. *If V is an integral rectifiable d–varifold with bounded first variation then*

$$\arg\min_{P\in G_{d,n}} E_0^\omega(x,P,V) = \{T_x M\} \ ,$$

hence for $\|V\|$–almost every x, $T_i(x) \xrightarrow[i\to\infty]{} T_x M$.

Proof. First, for fixed x and i, $P \mapsto E_{\alpha_i}^\omega(x,P,V_i)$ is continuous and $G_{d,n}$ is compact so that

$$\arg\min_{P\in G_{d,n}} E_{\alpha_i}^\omega(x,P,V_i) \neq \emptyset \, .$$

Let $T_i(x) \in \arg\min_{P\in G_{d,n}} E_{\alpha_i}^\omega(x,P,V_i)$ be a sequence of minimizers, as $G_{d,n}$ is compact, one can extract a subsequence converging to some $T_\infty(x)$. Now applying the previous result (Proposition 3.32), we get for every $P \in G_{d,n}$,

$$\begin{aligned} E_0^\omega(x, T_\infty(x), V) &\leqslant \liminf_{i\to\infty} E_{\alpha_i}^\omega(x, T_i(x), V_i) \\ &\leqslant \limsup_{i\to\infty} E_{\alpha_i}^\omega(x, T_i(x), V_i) \\ &\leqslant \limsup_{i\to\infty} E_{\alpha_i}^\omega(x, P, V_i) \\ &= \lim_i E_{\alpha_i}^\omega(x, P, V_i) = E_0^\omega(x, P, V) \\ &\leqslant E_0^\omega(x, T_\infty(x), V) \text{ for } P = T_\infty(x) \,. \end{aligned}$$

Therefore $T_\infty(x)$ minimizes $E_0^\omega(x, \cdot, V)$ which allows to conclude that the limit of any subsequence of minimizers of $E_{\alpha_i}^\omega(x, \cdot, V_i)$ is a minimizer of $E_0^\omega(x, \cdot, V)$. It also proves that

$$\lim_i \min_{P \in G_{d,n}} E_{\alpha_i}^\omega(x, P, V_i) = \lim_i E_{\alpha_i}^\omega(x, T_i(x), V_i) = E_0^\omega(x, T_\infty(x), V) = \min_{P \in G_{d,n}} E_0^\omega(x, P, V) \,.$$

Assume now that $E_0^\omega(x, \cdot, V)$ admits a unique minimizer $T(x)$. We have just shown that every subsequence of $(T_i(x))_i$ converges to $T(x)$. As $G_{d,n}$ is compact, it is enough to show that the whole sequence is converging to $T(x)$. Now if V is an integral d–rectifiable varifold with bounded first variation, for $\|V\|$–almost every x, $T_x M$ is the unique minimizer of $E_0^\omega(x, \cdot, V)$ (see Prop. 3.14) so that for $\|V\|$–almost every $x \in \omega$,

$$T_i(x) \xrightarrow[i\to\infty]{} T_x M \,.$$

□

Remark 3.7. Since $E_0^\omega(x, \cdot, V)$ has no continuity property, the existence of a minimizer of $E_0^\omega(x, \cdot, V)$ is not clear a priori. However, as $G_{d,n}$ is compact, every sequence of minimizers $(T_i(x))_i$ admits a converging subsequence so that $\arg\min_{P \in G_{d,n}} E_0^\omega(x, P, V)$ is not empty.

We end with studying the continuity of the minimum $\min_{P \in G_{d,n}} E_{\alpha_i}(x, P, V_i)$ with respect to x (for fixed i and V_i).

Proposition 3.34. *Assume that V_i weakly–$*$ converges to V in some open set $\Omega \subset \mathbb{R}^n$ and let $(\alpha_i)_i > 0$. Then for every fixed i and $\omega \subset\subset \Omega$, the function $x \mapsto \min\limits_{P \in G_{d,n}} E_{\alpha_i}^\omega(x, P, V_i)$ is continuous in ω.*
Moreover, every converging sequence of minimizers $\left(T_i(z_k) \in \arg\min_P E_{\alpha_i}^\omega(z_k, P, V_i)\right)_k$ tends to a minimizer of $E_{\alpha_i}^\omega(x, \cdot, V_i)$ when $z_k \to x$ and for a fixed i.

Remark 3.8. As i is fixed, meaning actually that a scale $\alpha = \alpha_i > 0$ and a d–varifold $V = V_i$ are fixed, we keep the notations V_i and α_i, with the explicit index i, only to be coherent with the whole context of this section and with the notations of the previous results. But that is why we do not assume anything on the choice of $\alpha_i > 0$ and $\omega \subset\subset \Omega$.

Proof. Let i be fixed. First we show that if $(z_k)_k \subset \omega$ is such that

$$\begin{cases} |z_k - x| \xrightarrow[k\to\infty]{} 0 \\ T_i(z_k) \xrightarrow[k\to\infty]{} T_i^\infty \text{ where } T_i(z_k) \in \arg\min_P E_{\alpha_i}^\omega(z_k, P, V_i) \,, \end{cases}$$

then,

$$\begin{cases} T_i^\infty \in \arg\min_P E_{\alpha_i}^\omega(x, P, V_i) \text{ and} \\ \min_P E_{\alpha_i}^\omega(z_k, P, V_i) = E_{\alpha_i}^\omega(z_k, T_i(z_k), V_i) \xrightarrow[k\to\infty]{} E_{\alpha_i}^\omega(x, T_i^\infty, V_i) = \min_P E_{\alpha_i}^\omega(x, P, V_i) \,. \end{cases} \tag{3.46}$$

Indeed,

$$\begin{aligned}\big|E^{\omega}_{\alpha_i}(x,T_i^{\infty},V_i) - E^{\omega}_{\alpha_i}(z_k,T_i(z_k),V_i)\big| \\ \leqslant \big|E^{\omega}_{\alpha_i}(x,T_i^{\infty},V_i) - E^{\omega}_{\alpha_i}(x,T_i(z_k),V_i)\big| + \big|E^{\omega}_{\alpha_i}(x,T_i(z_k),V_i) - E^{\omega}_{\alpha_i}(z_k,T_i(z_k),V_i)\big| \\ \leqslant K(\alpha_i)\|T_i^{\infty} - T_i(z_k)\| + \sup_P \big|E^{\omega}_{\alpha_i}(x,P,V_i) - E^{\omega}_{\alpha_i}(z_k,P,V_i)\big|\end{aligned}$$

applying Proposition 3.30 to the first term, $K(\alpha_i)$ is a constant depending only on α_i. Moreover, by Proposition 3.31, the second term tends to zero when k goes to ∞. Consequently,

$$E_{\alpha_i}(x,T_i^{\infty},V_i) = \lim_{k\to\infty} E_{\alpha_i}(z_k,T_i(z_k),V_i)\,.$$

And for every $P \in G_{d,n}$,

$$\begin{aligned}E^{\omega}_{\alpha_i}(x,T_i^{\infty},V_i) &= \lim_{k\to\infty} E^{\omega}_{\alpha_i}(z_k,T_i(z_k),V_i) \\ &\leqslant \lim_{k\to\infty} E^{\omega}_{\alpha_i}(z_k,P,V_i) \\ &= E^{\omega}_{\alpha_i}(x,P,V_i) \text{ by Proposition 3.31,}\end{aligned}$$

which yields (3.46).

It remains to prove the continuity of $x \mapsto \min_{P\in G_{d,n}} E^{\omega}_{\alpha_i}(x,P,V_i)$. Let x and $(z_k)_k \in \omega$ be such that $z_k \xrightarrow[k\to\infty]{} x$ and consider a subsequence $(z_{\varphi(k)})_k$ such that

$$\limsup_k E^{\omega}_{\alpha_i}(z_k,T_i(z_k),V_i) = \lim_k E^{\omega}_{\alpha_i}(z_{\varphi(k)},T_i(z_{\varphi(k)}),V_i)\,. \tag{3.47}$$

As $G_{d,n}$ is compact, there exists an extraction θ such that $(T_i(z_{\varphi(\theta(k))}))_k$ is converging and then applying the previous argument (3.46) to $(z_{\varphi(\theta(k))})_k$ and $(T_i(z_{\varphi(\theta(k))}))_k$,

$$\lim_{k\to+\infty} E^{\omega}_{\alpha_i}\left(z_{\varphi(\theta(k))},T_i(z_{\varphi(\theta(k))}),V_i\right) = \min_{P\in G_{d,n}} E^{\omega}_{\alpha_i}(x,P,V_i)\,. \tag{3.48}$$

Eventually, by (3.47) and (3.48),

$$\limsup_{k\to+\infty} E^{\omega}_{\alpha_i}(z_k,T_i(z_k),V_i) = \min_{P\in G_{d,n}} E^{\omega}_{\alpha_i}(x,P,V_i)\,.$$

Similarly $\liminf_k E^{\omega}_{\alpha_i}(z_k,T_i(z_k),V_i) = \min_P E^{\omega}_{\alpha_i}(x,P,V_i)$ which concludes the proof of the continuity. □

CHAPTER 4

Construction de mesures à partir de leurs valeurs approchées sur les boules par une méthode de type "packing"

In this chapter, we make an attempt to give an answer to Question 1.3 thanks to a "packing" measure construction.

Question. 1.3. What conditions on a weakly–$*$ converging sequence of varifolds (not supposed rectifiable) ensure that the limit varifold has bounded first variation?

As it is explained in the introduction of this thesis in chapter 1, section 1.5, this method does not allow us to answer in a satisfactory way to Question 1.3 and that is why we propose a better adapted approach in chapter 5.

Abstract. We consider the problem of reconstructing a Borel measure μ from its values on metric balls $B_r(x)$, or more generally from suitable approximations of $\mu(B_r(x))$ in the form of integral means of $\mu(B_\rho(x))$ over $\rho \in (0, r)$.

Introduction

Is a Borel measure μ on a metric space (X, d) fully determined by its values on balls? In the context of general Measure Theory, such a question appears to be of extremely basic nature. The answer (when it is known) strongly depends upon the interplay between the measure and the metric space. A clear overview on the subject is given in [JL01]. Let us mention some known facts about this issue. When $X = \mathbb{R}^n$, the answer to the above question is in the affirmative. The reason is the following: if two locally finite Borel measures μ and ν coincide on every ball $B_r(x) \subset \mathbb{R}^n$, then in particular they are mutually absolutely continuous, therefore by the Radon-Nikodym-Lebesgue Differentiation Theorem one has $\mu(A) = \int_A \eta \, d\nu = \nu(A)$ for any Borel set A, where

$$\eta(x) = \lim_{r \to 0} \frac{\mu(B_r(x))}{\nu(B_r(x))} = 1$$

is the Radon-Nikodym derivative of μ with respect to ν (defined for ν-almost all $x \in \mathbb{R}^n$). More generally, the same fact can be shown for any pair of Borel measures on a finite-dimensional Banach space X. Unfortunately, the Differentiation Theorem is valid on a Banach space X if and only if X is finite-dimensional. Of course, this does not prevent in general the possibility that Borel measures are uniquely determined by their values on balls. Indeed, Preiss and Tišer proved in [PT91] that in separable Banach spaces, two finite Borel measures coinciding on all balls also coincide on all Borel sets. Nevertheless, if we only take into account balls of radius bounded by 1, the question still stands. As for the case of separable metric spaces, Federer introduced in [Fed69] a geometrical condition on the distance (see Definition 4.5) implying a Besicovitch-type covering lemma that can be used to show the property above, i.e., that any finite Borel measure is uniquely identified by its values on closed balls. When this condition on the distance is dropped, some examples of measure spaces and of pairs of distinct Borel measures coinciding on balls of upper-bounded diameter are known (see [Dav71]).

Here we consider the case of a separable metric space (X, d) where Besicovitch covering lemma (or at least some generalized version of it) holds, and we ask the following questions:

Question 4.1. How can we concretely reconstruct a Borel measure from its values on balls, and especially, what about the case of signed measures?

A classical approach to construct a measure from a given pre-measure p defined on a family $\mathcal{C}$ of subsets of X (here the pre-measure p is defined on closed balls) is to apply Carathéodory constructions (Method I and Method II, see [BBT01]) to obtain an outer measure. We recall that a pre-measure p is a nonnegative function, defined on a given family $\mathcal{C}$ of subsets of X, such that $\emptyset \in \mathcal{C}$ and $p(\emptyset) = 0$. By Method I, an outer measure μ^* is defined starting from p as

$$\mu^*(A) = \inf \left\{ \sum_{k=1}^{\infty} p(B_k) \ : \ B_k \in \mathcal{C} \text{ and } A \subset \bigcup_{k=1}^{\infty} B_k \right\} ,$$

for any $A \subset X$. But, as it is explained in [BBT01] (Section 3.2), Method I does not take into account that X is a metric space, thus the resulting outer measure can be incompatible with the metric on X (in the sense that open sets are not necessarily μ^*-measurable). On the other hand, Method II is used to define Hausdorff measures (see Theorem 4.4) and it always produces a metric outer measure μ^*, for which Borel sets are μ^*-measurable.

As for a signed measure $\mu = \mu^+ - \mu^-$, the main problem is that, given a closed ball B, it is impossible to directly reconstruct $\mu^+(B)$ and $\mu^-(B)$ from $\mu(B)$. The idea is, then, to apply Carathéodory's construction to the pre-measure $p^+(B) = (\mu(B))_+$ (here a_+ denotes the positive part of $a \in \mathbb{R}$) and to check that the resulting outer measure is actually μ^+. Then, by a similar argument we recover μ^-.

Question 4.2. Given a positive Borel measure μ and defining the pre-measure

$$q(B_r(x)) = \frac{1}{r} \int_0^r \mu(B_s(x))\, ds \, , \tag{4.1}$$

is it possible to reconstruct μ from q? And what about the case when μ is a signed measure?

Some minimal explanations about the special form of $q(B_r(x))$ we are interested in are required. Indeed, our choice of $q(B_r(x))$ comes from the problem of approximating the first variation δV of a (rectifiable) d-varifold V in $\mathbb{R}^n$, which is the weak–$*$ limit of a sequence of more general d-varifolds $(V_k)_k$, by means of suitably defined "first variations" $\delta_{r_k}(V_k)$ depending upon scale parameters r_k that tend to 0 as $k \to \infty$. (See chapter 1 for more details and [Bue14, BLM] corresponding to chapters 3 and 5 in this thesis for alternative approaches).

We point out that, in order to address Question 4.2, Carathéodory's Method II is not the right choice. Indeed, considering the simple example of μ given by a Dirac delta, the measure reconstructed from the pre-measure q by means of Method II can be strictly smaller than μ. In other words, a loss of mass could happen in the recovery process. Indeed, if $\mu = \delta_y$, the closer to $\partial B_r(x)$ is the mass concentrated at y, the smaller is $q(B_r(x))$ (and indeed $p(B_r(x))$ vanishes when $y \in \partial B_r(x)$). Then for any $\varepsilon > 0$ one can consider x with $r(1-\varepsilon) < |x-y| < r$ and observe that $y \in B_r(x)$ and, at the same time, that $q(B_r(x))$ is small in terms of ε. This shows that the measure reconstructed by Method II is identically zero (see section 4.2.1 for more details).

In order to recover μ, or at least some measure equivalent or comparable to μ, the choice of centers of the balls in the collection used to cover the support of μ is crucial. Indeed they must be placed in some nearly-optimal positions, such that even the concentric balls with small radius have a significant overlapping with the support of μ. This has led us to considering a packing-type construction. Packing constructions are typically used to build the packing s-dimensional measure and its associated notion of packing dimension: these are in some sense dual to Hausdorff measure and dimension, and were introduced by C. Tricot in [Tri82]. Then Tricot and Taylor extended this notion to a general pre-measure in [TT85].

The chapter is organized as follows. In Section 4.1, we explain how to reconstruct a positive measure and then a signed measure (Theorem 4.9) from their values on balls, thanks to Carathéodory's construction, answering Question 4.1. Section 4.2 deals with Question 4.2, that is, the reconstruction of a measure starting from approximate values of the form (4.1). After explaining the limitations of Carathéodory's construction for this problem, we prove our main result, Theorem 4.16, saying that by suitable packing constructions one can reconstruct a signed measure equivalent to the initial one in $\mathbb{R}^n$.

Some notations

Let (X, d) be a metric space.

- $\mathcal{B}(X)$ denotes the σ–algebra of Borel subsets of X.
- Important note: in this chapter, $B_r(x) = \{y \in X \mid d(y, x) \leqslant r\}$ denotes the **closed** ball of radius $r > 0$ and center $x \in X$.
- $B_r^\circ(x) = \{y \in X \mid d(y, x) < r\}$ is the open ball of radius $r > 0$ and center $x \in X$.
- $\mathcal{C}$ denotes the collection of closed balls of X and for $\delta > 0$, $\mathcal{C}_\delta$ denotes the collection of closed balls of diameter $\leqslant \delta$.
- $\mathcal{L}^n$ is the Lebesgue measure in $\mathbb{R}^n$.
- $\mathcal{P}(X)$ is the set of all subsets of X.
- $\mathrm{card} A$ is the cardinality of the set A.

4.1 Carathéodory metric construction of outer measures

We recall here some standard definitions and well-known facts about general measures, focussing in particular on the construction of measures from pre-measures, in the sense of Carathéodory Method II [BBT01].

4.1.1 Outer measures and metric outer measures

Definition 4.1 (Outer measure)**.** *Let X be a set, and let $\mu^* : \mathcal{P}(X) \to [0; +\infty]$ satisfying*

(i) $\mu^*(\emptyset) = 0$.

(ii) μ^* *is monotone: if $A \subset B \subset X$, then $\mu^*(A) \leqslant \mu^*(B)$.*

(iii) μ^* *is countably subadditive: if $(A_k)_{k\in\mathbb{N}}$ is a sequence of subsets of X, then*

$$\mu^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leqslant \sum_{k=1}^{\infty} \mu^*(A_k)\,.$$

Then μ^* is called an outer measure on X.

In order to obtain a measure from an outer measure, we define the measurable sets with respect to μ^*.

Definition 4.2 (μ^*–measurable set)**.** *Let μ^* be an outer measure on X. A set $A \subset X$ is μ^*–measurable if for all sets $E \subset X$,*

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A) \, .$$

We can now define a measure associated with an outer measure. Thanks to the definition of μ^*–measurable sets, the additivity of μ^* among the measurable sets is straightforward, actually it happens that μ^* is σ–additive on μ^*–measurable sets.

Theorem 4.1 (Measure associated with an outer measure, see Theorem 2.32 in [BBT01])**.** *Let X be a set, μ^* an outer measure on X, and $\mathcal{M}$ the class of μ^*–measurable sets. Then $\mathcal{M}$ is a σ–algebra and μ^* is countably additive on $\mathcal{M}$. Thus the set function μ defined on $\mathcal{M}$ by $\mu(A) = \mu^*(A)$ for all $A \in \mathcal{M}$ is a measure.*

We now introduce metric outer measures.

Definition 4.3. *Let (X, d) be a metric space and μ^* be an outer measure on X. μ^* is called a metric outer measure if*

$$\nu(E \cup F) = \nu(E) + \nu(F)$$

for any $E, F \subset X$ such that $d(E, F) > 0$.

When μ^* is a metric outer measure, every Borel set is μ^*–measurable and thus the measure μ associated with μ^* is a Borel measure.

Theorem 4.2 (Carathéodory's Criterion, see Theorem 3.8 in [BBT01])**.** *Let μ^* be an outer measure on a metric space (X, d). Then every Borel set in X is μ^*-measurable if and only if μ^* is a metric outer measure. In particular, a metric outer measure is a Borel measure.*

We recall two approximation properties of Borel measures defined on metric spaces.

Theorem 4.3 (see Theorems 3.13 and 3.14 in [BBT01])**.** *Let (X, d) be a metric space and μ be a Borel measure on X.*

- *Approximation from inside: Let B be a Borel set such that $\mu(B) < +\infty$, then for any $\varepsilon > 0$, there exists a closed set $F_\varepsilon \subset B$ such that $\mu(B \setminus F_\varepsilon) < \varepsilon$.*
- *Approximation from outside: Assume that μ is finite on bounded sets and let B be a Borel set, then*

$$\mu(B) = \inf\{\mu(U) \ : \ B \subset U, U \text{ open set}\} \, .$$

We can now introduce Carathéodory's construction of metric outer measures (Method II).

Definition 4.4 (Pre-measure)**.** *Let X be a set and $\mathcal{C}$ be a family of subsets of X such that $\emptyset \in \mathcal{C}$. A nonnegative function p defined on $\mathcal{C}$ and such that $p(\emptyset) = 0$ is called a pre-measure.*

Theorem 4.4 (Carathéodory's construction, Method II)**.** *Suppose (X, d) is a metric space and $\mathcal{C}$ is a family of subsets of X which contains the empty set. Let p be a non-negative function on $\mathcal{C}$ which vanishes on the empty set. For each $\delta > 0$, let*

$$\mathcal{C}_\delta = \{A \in \mathcal{C} \mid \operatorname{diam}(A) \leqslant \delta\}$$

and for any $E \subset X$ define

$$\nu_\delta^p(E) = \inf \left\{ \sum_{i=0}^{\infty} p(A_i) \,\middle|\, E \subset \bigcup_{i\in\mathbb{N}} A_i, \forall i,\ A_i \in \mathcal{C}_\delta \right\}.$$

As $\nu_\delta^p \geqslant \nu_{\delta'}^p$ when $\delta \leqslant \delta'$,

$$\nu^{p,*}(E) = \lim_{\delta\to 0} \nu_\delta^p(E)$$

exists (possibly infinite). Then $\nu^{p,*}$ is a metric outer measure on X.

4.1.2 Effects of Carathéodory's construction on positive Borel measures

Let (X,d) be an open set and μ be a positive Borel σ–finite measure on X. Let $\mathcal{C}$ be the set of closed balls and let p be the pre-measure defined in $\mathcal{C}$ by,

$$\begin{array}{rcccl} p & : & \mathcal{C} & \to & [0,+\infty] \\ & & B & \mapsto & \mu(B) \end{array} \tag{4.2}$$

Let $\mu^{p,*}$ be the metric outer measure obtained by Carathéodory's metric construction applied to $(\mathcal{C}, p)$ and then μ^p the Borel measure associated with $\mu^{p,*}$. Then, the following question arises.

Question 4.3. *Do we have $\mu^p = \mu$? In other terms, can we recover the initial measure by Carathéodory's Method II?*

The following lemma shows one of the two inequalities needed to positively answer Question 4.3.

Lemma 4.5. *Let (X,d) be an open set and μ be a positive Borel measure on X. Then, in the same notations as above, we have $\mu \leqslant \mu^p$.*

Proof. Let $A \subset X$ be a Borel set, we have to show that $\mu(A) \leqslant \mu^p(A) = \mu^{p,*}(A)$. This inequality relies only on the definition of μ_δ^p as an infimum. Indeed, let $\delta > 0$, then for any $\eta > 0$ there exists a countable collection of closed balls $(B_j^\eta)_{j\in\mathbb{N}} \subset \mathcal{C}_\delta$ such that

$$A \subset \bigcup_j B_j^\eta \qquad \text{and} \qquad \mu_\delta^p(A) \geqslant \sum_{j=1}^{\infty} p(B_j^\eta) - \eta\,,$$

so that

$$\mu_\delta^p(A) + \eta \geqslant \sum_{j=1}^{\infty} p(B_j^\eta) = \sum_{j=1}^{\infty} \mu(B_j^\eta) \geqslant \mu\Big(\bigcup_j B_j^\eta\Big) \geqslant \mu(A)\,.$$

Letting $\eta \to 0$ and then $\delta \to 0$ leads to $\mu(A) \leqslant \mu^p(A)$. □

The other inequality is not true in general. We need extra assumptions on (X,d) ensuring that open sets are "well approximated" by closed balls, that is, we need some specific covering property. In $\mathbb{R}^n$ with the Euclidean norm, this approximation of open sets by disjoint unions of balls is provided by Besicovitch Theorem, which we recall here:

Theorem 4.6 (Besicovitch Theorem, see Corollary 1 p. 35 in [EG92])**.** *Let μ be a Borel measure on $\mathbb{R}^n$ and consider any collection $\mathcal{F}$ of non degenerated closed balls. Let A denote the set of centers of the balls in $\mathcal{F}$. Assume $\mu(A) < +\infty$ and that*

$$\inf\{r > 0 \mid B_r(a) \in \mathcal{F}\} = 0 \qquad \forall\, a \in A\,.$$

Then, for every open set $U \in \mathbb{R}^n$, there exists a countable collection $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that

$$\bigsqcup_{B\in\mathcal{G}} B \subset U \quad \textit{and} \quad \mu\left((A\cap U) - \bigsqcup_{B\in\mathcal{G}} B\right) = 0\,.$$

A generalization of Besicovitch Theorem for metric measure spaces is due to Federer, under a geometric assumption involving the distance function.

Definition 4.5 (Directionally limited distance, see 2.8.9 in [Fed69])**.** *Let (X, d) be a metric space, $A \subset X$ and $\xi > 0$, $0 < \eta \leqslant \frac{1}{3}$, $\zeta \in \mathbb{N}^*$. The distance d is said to be directionally (ξ, η, ζ)–limited at A if the following holds. Take any $a \in A$ and $B \subset A \cap \left(B^\circ_\xi(a) \setminus \{a\}\right)$, such that*

$$\frac{d(x, c)}{d(a, c)} \geqslant \eta \tag{4.3}$$

for all $b, c \in B$ and all $x \in X$, such that $b \neq c$, $d(a, x) = d(a, c)$, $d(b, x) = d(a, b) - d(a, c)$ and $d(a, b) \geqslant d(a, c)$. Then $\mathrm{card} B \leqslant \zeta$.

Let us say a few words about this definition. If $(X, |\cdot|)$ is a Banach space with strictly convex norm, then the above relations involving x imply that

$$x = a + \frac{|a - c|}{|a - b|}(b - a)\,,$$

hence in this case (4.3) is equivalent to

$$\frac{d(x, c)}{d(a, c)} = \left|\frac{c - a}{|c - a|} - \frac{b - a}{|b - a|}\right| \geqslant \eta\,.$$

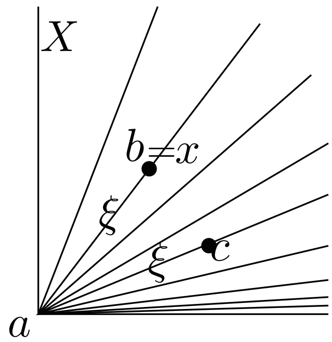
Consequently, if X is finite-dimensional, and thanks to the compactness of the unit sphere, for a given η there exists $\zeta \in \mathbb{N}$ such that $(X, |\cdot|)$ is directionally (ξ, η, ζ)-limited for all $\xi > 0$. Hereafter we provide two examples of metric spaces that are not directionally limited.

Example 4.1. Consider in $\mathbb{R}^2$ the union X of a countable number of half-lines, joining at the same point a. Then the geodesic metric d induced on X by the ambient metric is not directionally limited at $\{a\}$.

Indeed let $B = X \cap \{y \,:\, d(a, y) = \xi\}$, let b and $c \in B$ lying in two different lines, at the same distance $d(a, b) = d(a, c) = \xi$ of a. Then $x \in X$ such that $d(a, x) = d(a, c) = \xi$ and $d(b, x) = d(a, b) - d(a, c) = 0$ implies $x = b$ and thus

$$\frac{d(x, c)}{d(a, c)} = \frac{d(b, c)}{\xi} = \frac{2\xi}{\xi} = 2\,.$$

but $\mathrm{card}\, B$ is not finite.



Example 4.2. If X is a separable Hilbert space and $B = (e_k)_{k \in \mathbb{N}}$ a Hilbert basis, $a \in H$ and $b = a + e_j$, $c = a + e_k \in a + B$, the Hilbert norm is strictly convex so that $d(a, x) = d(a, c)$, $d(b, x) = d(a, b) - d(a, c)$ uniquely define x as

$$x = a + \frac{|e_k|}{|e_j|} e_j = b \text{ and } \frac{d(x, c)}{d(a, c)} = |e_k - e_j| = 2 \geqslant \eta$$

for all $\eta \leqslant \frac{1}{3}$ and $\mathrm{card}(a + B)$ is infinite. Therefore H is not directionally limited (anywhere).

We can now state the generalized versions of Besicovitch Covering Lemma and Besicovitch Theorem for directionally limited metric spaces.

Theorem 4.7 (Generalized Besicovitch Covering Lemma, see 2.8.14 in [Fed69]). *Let (X, d) be a separable metric space directionally (ξ, η, ζ)–limited at $A \subset X$. Let $0 < \delta < \frac{\xi}{2}$ and $\mathcal{F}$ be a family of closed balls of radii less than δ such that each point of A is the center of some ball of $\mathcal{F}$. Then, there exists $2\zeta + 1$ countable subfamilies of $\mathcal{F}$ of disjoint closed balls, $\mathcal{G}_1, \ldots, \mathcal{G}_{2\zeta+1}$ such that*

$$A \subset \bigcup_{j=1}^{2\zeta+1} \bigsqcup_{B \in \mathcal{G}_j} B \,.$$

Remark 4.1. In $\mathbb{R}^n$ endowed with the Euclidean norm it is possible to take $\xi = +\infty$ and ζ only dependent on η and n. If we fix $\eta = \frac{1}{3}$, then $\zeta = \zeta_n$ only depends on the dimension n.

Theorem 4.8 (Generalized Besicovitch Theorem, see 2.8.15 in [Fed69]). *Let (X, d) be a separable metric space directionally (ξ, η, ζ)–limited at $A \subset X$. Let $\mathcal{F}$ be a family of closed balls of X satisfying*

$$\inf \{r > 0 \mid B_r(a) \in \mathcal{F}\} = 0, \qquad \forall\, a \in A \,, \tag{4.4}$$

and let μ be a positive Borel measure on X, finite on bounded sets. Then, for any open set $U \subset X$ there exists a countable disjoint family $\mathcal{G}$ of $\mathcal{F}$ such that

$$\bigsqcup_{B \in \mathcal{G}} B \subset U \quad \textit{and} \quad \mu\left((A \cap U) - \bigsqcup_{B \in \mathcal{G}} B\right) = 0 \,.$$

We can now prove the coincidence of the initial measure and the reconstructed measure under the assumptions of Theorem 4.8.

Proposition 4.9. *Let (X, d) be a separable metric space directionally (ξ, η, ζ)–limited at X. Let μ be a positive Borel measure on X, finite on bounded sets. Let $\mathcal{C}$ be the family of closed balls in X and let p be the pre-measure defined in $\mathcal{C}$ by (4.2). Denote by $\mu^{p,*}$ the metric outer measure obtained by Carathéodory's metric construction applied to $(\mathcal{C}, p)$ and by μ^p the Borel measure associated with $\mu^{p,*}$. Then μ^p is finite on bounded sets and $\mu^p = \mu$.*

Proof. **Step one.** We prove that $\mu^{p,*}$ is finite on bounded sets. First we recall that by Theorem 4.4 $\mu^{p,*}$ is a metric outer measure, then thanks to Theorem 4.1 μ^p is a Borel measure. Moreover, μ is finite on bounded sets, let us prove that μ^p is finite on bounded sets. Let $A \subset X$ be a bounded Borel set and apply Besicovitch Covering Lemma (Theorem 4.7) with the family

$$\mathcal{F}_\delta = \{B = B_r(x) \text{ closed ball } : x \in A \text{ and } \operatorname{diam} B \leqslant \delta\} \,,$$

to get $2\zeta + 1$ countable subfamilies $\mathcal{G}_1^\delta \ldots, \mathcal{G}_{2\zeta+1}^\delta$ of disjoint balls in $\mathcal{F}$ such that

$$A \subset \bigcup_{j=1}^{2\zeta+1} \bigsqcup_{B \in \mathcal{G}_j^\delta} B \,.$$

Therefore,

$$\mu_\delta^p(A) \leqslant \sum_{j=1}^{2\zeta+1} \sum_{B \in \mathcal{G}_j^\delta} p(B) \leqslant \sum_{j=1}^{2\zeta+1} \mu\left(\bigsqcup_{B \in \mathcal{G}_j^\delta} B\right) \leqslant (2\zeta + 1)\mu(A + B_\delta(0)) \leqslant (2\zeta + 1)\mu(A + B_1(0)) \,,$$

where $A + B_1(0) = \bigcup_{x \in A} B_1(x)$ is bounded, thus $\mu(A + B_1(0)) < +\infty$ and hence for all $0 < \delta < 1$

$$\mu_\delta^p(A) \leqslant (2\zeta + 1)\mu(A + B_1(0)) < +\infty \,,$$

whence $\mu^{p,*}(A) < +\infty$.

Step two. We now prove that for any open set $U \subset X$ it holds $\mu^p(U) \leqslant \mu(U)$. Let $U \subset X$ be an open set and let $\delta > 0$ be fixed. Consider the collection of closed balls

$$\mathcal{C}_\delta = \{B_r(x) \mid x \in U,\ 0 < 2r \leqslant \delta\} .$$

The family $\mathcal{C}_\delta$ satisfies the assumption (4.4), thus we can apply Theorem 4.8 to μ^p and get a countable collection $\mathcal{G}^\delta$ of disjoint balls in $\mathcal{C}_\delta$ such that

$$\bigsqcup_{B \in \mathcal{G}^\delta} B \subset U \quad \text{and} \quad \mu^p(U) = \mu^p\left(\bigsqcup_{B \in \mathcal{G}^\delta} B\right) .$$

However we also have

$$\mu^p_\delta\left(\bigsqcup_{B \in \mathcal{G}^\delta} B\right) \leqslant \sum_{B \in \mathcal{G}^\delta} p(B) = \sum_{B \in \mathcal{G}^\delta} \mu(B) = \mu\left(\bigsqcup_{B \in \mathcal{G}^\delta} B\right) \leqslant \mu(U) . \tag{4.5}$$

By taking $A = \bigcap_{\substack{\text{countable} \\ \delta \downarrow 0}} \left(\bigsqcup_{B \in \mathcal{G}^\delta} B\right)$ we obtain $\mu^p(U) = \mu^p(A)$ and for any $\delta > 0$, $A \subset \bigsqcup_{B \in \mathcal{G}^\delta} B$. Thus, thanks to (4.5),

$$\mu^p_\delta(A) \leqslant \mu^p_\delta\left(\bigsqcup_{B \in \mathcal{G}^\delta} B\right) \leqslant \mu(U) \quad \Rightarrow \quad \mu^p(U) = \mu^p(A) \leqslant \mu(U) .$$

This shows that $\mu^p(U) \leqslant \mu(U)$, as wanted.

Step three. Since μ and μ^p are Borel measures, finite on bounded sets, they are also outer regular (see Theorem 4.3), then for any Borel set $B \subset X$, and owing to Step two, it holds

$$\begin{aligned} \mu^p(B) &= \inf\{\mu^p(U) \mid U \text{ open},\ B \subset U\} \\ &\leqslant \inf\{\mu(U) \mid U \text{ open},\ B \subset U\} = \mu(B) . \end{aligned}$$

Coupling this last inequality with Lemma 4.5 we obtain $\mu^p = \mu$. □

4.1.3 Carathéodory's construction for a signed measure

We recall that a Borel signed measure μ on (X, d) is an extended real-valued set function $\mu : \mathcal{B}(X) \to [-\infty, +\infty]$ such that $\mu(\emptyset) = 0$ and, for any sequence of disjoint Borel sets $(A_k)_k$, one has

$$\sum_{k=1}^{\infty} \mu(A_k) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right) . \tag{4.6}$$

Remark 4.2. Notice that when $\mu\left(\bigcup_{k=1}^{\infty} A_k\right)$ is finite, its value does not depend on the arrangement of the A_k, therefore the series on the right hand side of (4.6) is commutatively convergent, thus absolutely convergent. In particular, if we write the Hahn decomposition $\mu = \mu^+ - \mu^-$, with μ^+ and μ^- being two non-negative and mutually orthogonal measures, then $\mu^+(X)$ and $\mu^-(X)$ cannot be both $+\infty$.

The question is now the following:

Question 4.4. *Let (X, d) be a metric space, separable and directionally (ξ, η, ζ)–limited at X, and let μ be a Borel signed measure, finite on bounded sets. Is it possible to recover μ from its values on closed balls by some Carathéodory-type construction?*

The main difference with the case of a positive measure is that μ is not monotone and thus the previous construction is not directly applicable. A simple idea could be to rely on the Hahn decomposition of μ: indeed, μ^+ and μ^- are positive Borel measures, and since one of them is finite, both are finite on bounded sets (recall that μ is finite on bounded sets by assumption). Once again, we cannot directly apply Carathéodory's construction to μ^+ or μ^- since we cannot directly reconstruct $\mu^+(B)$ and $\mu^-(B)$ simply knowing $\mu(B)$ for any closed balls B. We thus try to apply Carathéodory's construction not with $\mu^+(B)$, but with $(\mu(B))_+$, where a_+ (resp. a_-) denote the positive part $\max(a, 0)$ (resp. the negative part $\max(-a, 0)$) for any $a \in \mathbb{R}$. To be more precise, we state the following definition.

Definition 4.6. *Let μ be a Borel signed Radon measure in X. We define*

$$\begin{array}{cccc} p_+ : & \mathcal{C} & \longrightarrow & \mathbb{R}_+ \\ & B & \longmapsto & (\mu(B))_+ \end{array} \quad \text{and} \quad \begin{array}{cccc} p_- : & \mathcal{C} & \longrightarrow & \mathbb{R}_+ \\ & B & \longmapsto & (\mu(B))_- \end{array} .$$

Then according to Carathéodory's construction, we define the metric outer measure $\mu^{p_+,*}$ such that for any $A \subset X$,

$$\mu^{p_+,*}(A) = \lim_{\delta \to 0} \mu_\delta^{p_+,*}(A) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=0}^{\infty} p_+(A_i) \,\middle|\, A \subset \bigcup_{i \in \mathbb{N}} A_i, \forall i,\ A_i \in \mathcal{C}_\delta \right\} .$$

Similarly we define $\mu^{p_-,*}$ and then call μ^{p_+} and μ^{p_-} the Borel measures associated with $\mu^{p_+,*}$ and $\mu^{p_-,*}$. Finally, we set $\mu^p = \mu^{p_+} - \mu^{p_-}$.

Theorem 4.10. *Let (X, d) be a metric space, separable and directionally (ξ, η, ζ)–limited at X and let $\mu = \mu^+ - \mu^-$ be a Borel signed measure on X, finite on bounded sets. Let $\mu^p = \mu^{p_+} - \mu^{p_-}$ be as in Definition 4.6. Then $\mu^p = \mu$.*

Proof. We observe that μ^{p_+} and μ^{p_-} are Borel measures: indeed, by construction they are metric outer measures and Carathéodory criterion implies then that these are Borel. Furthermore, for any closed ball $B \in \mathcal{C}$, if we set $p(\mu^+)(B) = \mu^+(B)$, then

$$p_+(B) = (\mu(B))_+ \leqslant \mu^+(B) = p(\mu^+)(B) \quad \text{and} \quad p_-(B) = (\mu(B))_- \leqslant \mu^-(B) ,$$

thus by construction, $\mu^{p_+,*} \leqslant \mu^{p(\mu^+),*}$ and then

$$\mu^{p_+} \leqslant \mu^{p(\mu^+)} = \mu^+ \text{ thanks to Proposition 4.9} .$$

In the same way we can show that $\mu^{p_-} \leqslant \mu^-$. In particular, μ^{p_+} and μ^{p_-} are finite on bounded sets, as it happens for μ^+ and μ^-.

Let now $A \subset X$ be a Borel set. It remains to prove that $\mu^{p_+}(A) = \mu^{p_+,*}(A) \geqslant \mu^+(A)$ (and the same for μ^{p_-}). We argue exactly as in the proof of Lemma 4.5. Let $\delta > 0$, then for any $\eta > 0$ there exists a countable collection of closed balls $(B_j^\eta)_{j \in \mathbb{N}} \subset \mathcal{C}_\delta$ such that $A \subset \bigcup_j B_j^\eta$ and $\mu_\delta^{p_+}(A) \geqslant \sum_{j=1}^{\infty} p_+(B_j^\eta) - \eta$ so that

$$\mu_\delta^{p_+,*}(A) + \eta \geqslant \sum_{j=1}^{\infty} p_+(B_j^\eta) = \sum_{j=1}^{\infty} \left(\mu(B_j^\eta)\right)_+ \geqslant \sum_{j=1}^{\infty} \mu(B_j^\eta) \geqslant \mu\left(\bigcup_j B_j^\eta\right) \geqslant \mu(A) .$$

Letting $\eta \to 0$ and then $\delta \to 0$ gives

$$\mu(A) \leqslant \mu^{p+,*}(A) = \mu^{p+}(A)\,. \tag{4.7}$$

Recall that in Hahn decomposition, μ^+ and μ^- are mutually singular so that there exists a Borel set $P \subset X$ such that, for any Borel set A,

$$\mu^+(A) = \mu(P \cap A) \quad \text{and} \quad \mu^-(A) = \mu(A \cap (X - P))\,.$$

Thanks to (4.7) we already know that $\mu \leqslant \mu^{p+}$, therefore we get $\mu^+(A) = \mu(P \cap A) \leqslant \mu^{p+}(P \cap A) \leqslant \mu^{p+}(A)$ for any Borel set A. We finally infer that $\mu^{p+} = \mu^+$, $\mu^{p-} = \mu^-$, i.e., that $\mu^p = \mu$. □

Remark 4.3. If μ is a vector-valued measure on X, with values in a finite vector space E, we can apply the same construction componentwise.

4.2 Recovering measures from approximate values on balls

We now want to reconstruct a measure μ, not from its exact values on balls, but from approximate values of the form

$$q(B_r(x)) = \frac{1}{r}\int_{s=0}^{r} \mu(B_s(x))\,ds\,. \tag{4.8}$$

More precisely:

Question 4.5. *Let (X, d) be a separable metric space, directionally (ξ, η, ζ)–limited at X and let μ be a positive Borel measure on X. Is it possible to reconstruct μ from q, possibly up to multiplicative constants? and what can be done in case μ is not positive?*

In section 4.2.1 below we explain with a simple example involving a Dirac mass why Carathéodory's construction does not allow to recover μ from q defined as in (4.8). Then we define a packing construction of a measure, that is in some sense dual to the one by Carathéodory, and we show that in $\mathbb{R}^n$ it produces a measure equivalent to the initial one.

4.2.1 Why Carathéodory's construction is not well-suited

Let us consider a Dirac mass $\mu = \delta_x$ in $\mathbb{R}^n$ and compute

$$q(B_r(y)) = \frac{1}{r}\int_{s=0}^{r} \mu(B_s(y))\,ds$$

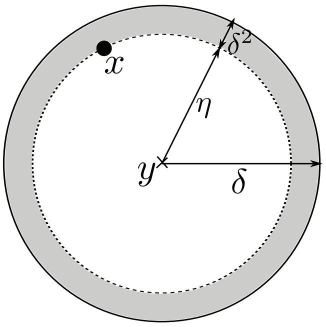
for a ball $B_r(y)$ containing x. First of all, for any $r > 0$,

$$q(B_r(x)) = \frac{1}{r}\int_{s=0}^{r} \delta_x(B_s(x))\,ds = \frac{1}{r}\int_{s=0}^{r} 1\,ds = 1\,.$$

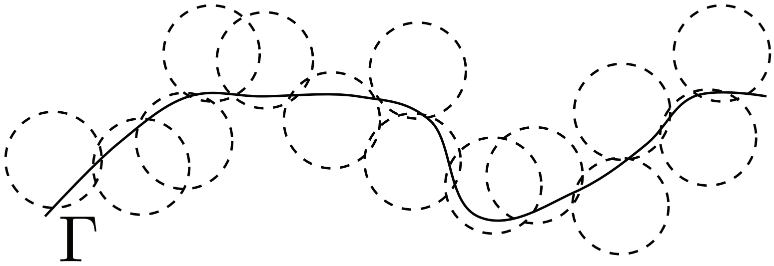
If now y is at distance η from x for some $0 < \eta < r$, we have

$$q(B_r(y)) = \frac{1}{r}\int_{s=0}^{r} \delta_x(B_s(y))\,ds = \frac{1}{r}\int_{s=\eta}^{r} 1\,ds = \frac{r-\eta}{r}\,.$$

Therefore, $q(B_r(y)) \to 0$ as $d(x, y) \to r$. We can thus find a covering made by a single ball of radius less than r for which $\mu_r^q(\{x\})$ is as small as we wish. This shows that Carathéodory's construction produces the zero measure.



(a) Bad covering of a Dirac mass



(b) Bad covering of a curve

More generally, as soon as it is possible to cover with small balls such that the mass of the measure inside each ball is close to the boundary, there is a loss of mass at the end of Carathéodory's construction. For instance, take $\mu = \mathcal{H}^1_{|\Gamma}$, where $\Gamma \subset \mathbb{R}^n$ is a curve of length L_Γ and $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure in $\mathbb{R}^n$, then cover Γ with a family of closed balls $\mathcal{B}_\delta$ of radii δ with centers at distance η from Γ. Assuming that no portion of the curve is covered more than twice, then

$$
\begin{aligned}
\sum_{B \in \mathcal{B}_\delta} q(B) = \sum_k \frac{1}{\delta} \int_{s=0}^{\delta} \mu(B_s(x_k)) &= \sum_k \frac{1}{\delta} \int_{s=\eta}^{\delta} \mu(B_s(x_k))\, ds \\
&\leqslant \frac{\delta - \eta}{\delta} \sum_k \mu(B_\delta(x_k)) \\
&\leqslant 2L_\Gamma \frac{\delta-\eta}{\delta} \xrightarrow[\delta \to 0]{} 0\,,
\end{aligned}
$$

with $\eta = \delta - \delta^2$ for instance.

The same phenomenon cannot be excluded by blindly centering balls on the support of the measure μ. Indeed, take a line D with a Dirac mass on it at a point x in $\mathbb{R}^2$, so that $\mu = \mathcal{H}^1_{|D} + \delta_x$. Then, by centering the balls on the support of μ, we may recover the line, but not the Dirac mass, for the same reason as before. We thus understand that the position of the balls should be optimized in order to avoid the problem. For this reason we consider an alternative method, based on a packing-type construction.

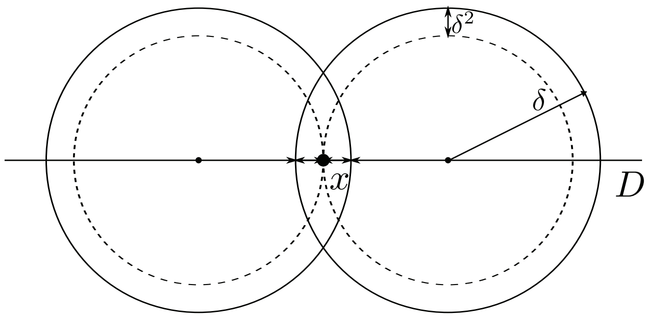


Figure 4.1: Bad covering with balls centered on the support of the measure

4.2.2 A packing-type construction

Because of the phenomenon appearing in the previous examples with Carathéodory's construction, we need to optimize the position of the centers of the balls. The idea is to consider a kind of dual construction, that is, to take a supremum over "fillings" rather than an infimum over coverings. To this aim we define the notion of packing.

Definition 4.7 (Admissible packing)**.** *Let (X,d) be a separable metric space and $U \subset X$ be an open set. We say that $\mathcal{F}$ is an admissible packing of U of order δ if $\mathcal{F}$ is a countable family of disjoint closed balls whose radius is less than δ and such that*

$$\bigsqcup_{B\in\mathcal{F}} B \subset U\,.$$

Definition 4.8 (Packing construction of measures)**.** *Let (X,d) be a separable metric space and let q be a non-negative set function defined on closed balls, such that $q(\emptyset)=0$. Let $U\subset X$ be an open set and fix $\delta>0$. We set*

$$\hat{\mu}^q_\delta(U) := \sup\left\{\sum_{B\in\mathcal{F}} q(B) \,:\, \mathcal{F} \textit{ is an admissible packing of order } \delta \textit{ of } U\right\}$$

and, in a similar way as in Carathéodory construction, define

$$\hat{\mu}^q(U) = \lim_{\delta\to 0}\hat{\mu}^q_\delta(U) = \inf_{\delta>0}\hat{\mu}^q_\delta(U)$$

and note that $\delta'\leqslant\delta$ implies $\hat{\mu}^q_{\delta'}(U)\leqslant\hat{\mu}^q_\delta(U)$. Then, $\hat{\mu}^q$ can be extended to all $A\subset X$ by setting

$$\hat{\mu}^q(A) = \inf\left\{\hat{\mu}^q(U) \,:\, A\subset U,\, U \textit{ open set}\right\}\,.$$

The main difference between Definition 4.8 and Carathéodory's construction is that the set function $\hat{\mu}^q$ is not automatically an outer measure: it is monotone but not sub-additive in general. In order to fix this problem we may apply the construction of outer measures, known as Munroe Method I, to the set function $\hat{\mu}^q$ restricted to the class of open sets. This amounts to setting, for any $A\subset X$,

$$\tilde{\mu}^q(A) = \inf\left\{\sum_{n\in\mathbb{N}}\hat{\mu}^q(U_n) \,:\, A\subset\bigcup_{n\in\mathbb{N}} U_n,\, U_n \text{ open set}\right\}\,.$$

One can check that $\tilde{\mu}^q$ is an outer measure.

Remark 4.4. The construction above is very similar to the one introduced in [TT85] for measures in $\mathbb{R}^n$. In that paper, starting from a given pre-measure q, a so-called packing pre-measure is defined for any $E\subset\mathbb{R}^n$ as

$$(q-P)(E) = \limsup_{\delta\to 0}\left\{\sum_{B\in\mathcal{B}} q(B) \,:\, \mathcal{B} \text{ is a packing of order } \delta \text{ of } E,\, \mathcal{B}\subset\{B^\circ_r(x) \,:\, x\in E,\, r>0\}\right\}\,.$$

$(q-P)(E)$ coincides with $\hat{\mu}^q$ on open sets (actually, open balls are considered in the definition, which is not important in the Euclidean $\mathbb{R}^n$ if one deals with Radon measures). Then, from this packing pre-measure, the authors define a packing measure μ^{q-P}, applying Carathéodory's construction, Method I, to $q-P$ on Borel sets. To be precise, for any $A\subset\mathbb{R}^n$,

$$\mu^{q-P}(A) = \inf\left\{\sum_{k=1}^{\infty}(q-P)(A_k) \,:\, A_k\in\mathcal{B}(\mathbb{R}^n),\, A\subset\bigcup_k A_k\right\}\,.$$

The outer measure $\tilde{\mu}^q$ is constructed in a very similar way to μ^{q-P}.

We will prove in Proposition 4.11 that, for the class of set functions q we are focusing on, $\hat{\mu}^q$ is already a Borel outer measure, that is, $\hat{\mu}^q = \tilde{\mu}^q$.

Remark 4.5. In order to show that $\hat{\mu}^q = \tilde{\mu}^q$, it is enough to prove the sub-additivity of $\hat{\mu}^q$ in the class of open sets. Indeed, the inequality $\tilde{\mu}^q(A) \leqslant \hat{\mu}^q(A)$ comes directly from the fact that minimizing $\hat{\mu}^q(U)$ over U open such that $A \subset U$ is a special case of minimizing $\sum_k \hat{\mu}^q(U_k)$ among countable families of open sets U_k such that $A \subset \bigcup_k U_k$. Assuming in addition that $\hat{\mu}^q$ is sub-additivite on open sets implies that for any countable family of open sets $(U_k)_k$ such that $A \subset \bigcup_k U_k$,

$$\hat{\mu}^q(A) \leqslant \hat{\mu}^q(\bigcup_k U_k) \leqslant \sum_k \hat{\mu}^q(U_k)\,.$$

By definition of $\tilde{\mu}^q$, taking the infimum over such families leads to $\hat{\mu}^q(A) \leqslant \tilde{\mu}^q(A)$.

Proposition 4.11. *Let (X, d) be a separable metric space and let μ be a Borel positive measure on X. Let q be the pre-measure associated with μ, defined on the class $\mathcal{C}$ of closed balls contained in X by*

$$q(B_r(x)) = \frac{1}{r}\int_{s=0}^{r} \mu(B_s(x))\,ds \qquad \forall\, B_r(x) \subset X\,.$$

Assume that μ is finite on bounded sets, then, for any countable family $(A_k)_k \subset X$ satisfying $\hat{\mu}^q\left(\bigcup_k A_k\right) < +\infty$, one has

$$\hat{\mu}^q\left(\bigcup_{k\in\mathbb{N}} A_k\right) \leqslant \sum_{k\in\mathbb{N}} \hat{\mu}^q(A_k) \tag{4.9}$$

In particular, if μ is finite, then $\hat{\mu}^q$ is an outer measure.

Proof. **Step 1.** We first prove (4.9) for open sets. Given a countable family $(U_k)_k$ of open subsets of X, such that $\sum_k \mu(U_k) < +\infty$, we show that

$$\hat{\mu}^q\left(\bigcup_{k\in\mathbb{N}} U_k\right) \leqslant \sum_{k\in\mathbb{N}} \hat{\mu}^q(U_k)\,. \tag{4.10}$$

Let $\varepsilon > 0$, then for all $k \in \mathbb{N}$ we define

$$U_k^\varepsilon = \{x \in U_k \,:\, d(x, X - U_k) > \varepsilon\}\,.$$

Let $0 < \delta < \frac{\varepsilon}{2}$ be fixed. If B is a closed ball such that $\mathrm{diam}B \leqslant 2\delta$ and $B \subset \bigcup_k U_k^\varepsilon$, then there exists k_0 such that $B \subset U_{k_0}$. Indeed, $B = B_\delta(x)$ and there exists k_0 such that $x \in U_{k_0}^\varepsilon$ and thus

$$B_\delta(x) \subset U_{k_0}^{\varepsilon-\delta} \subset U_{k_0}^{\frac{\varepsilon}{2}} \subset U_{k_0}\,.$$

Of course the inclusion $B \subset U_{k_0}$ remains true for any closed ball B with $\mathrm{diam}B \leqslant 2\delta$. Therefore any admissible packing $\mathcal{B}$ of $\displaystyle\bigcup_k U_k^\varepsilon$ of order $\delta \leqslant \frac{\varepsilon}{2}$ can be decomposed as the union of a countable family of admissible packings $\mathcal{B} = \bigsqcup_k \mathcal{B}_k$, where $\mathcal{B}_k$ is an admissible packing of U_k of order δ. Thus for any $\delta < \frac{\varepsilon}{2}$,

$$\sum_{B\in\mathcal{B}} q(B) = \sum_k \sum_{B\in\mathcal{B}_k} q(B)$$

and therefore, taking the supremum over all such packings $\mathcal{B}$ of $\displaystyle\bigcup_k U_k^\varepsilon$, we get

$$\hat{\mu}_\delta^q\left(\bigcup_k U_k^\varepsilon\right) \leqslant \sum_k \hat{\mu}_\delta^q(U_k)\,.$$

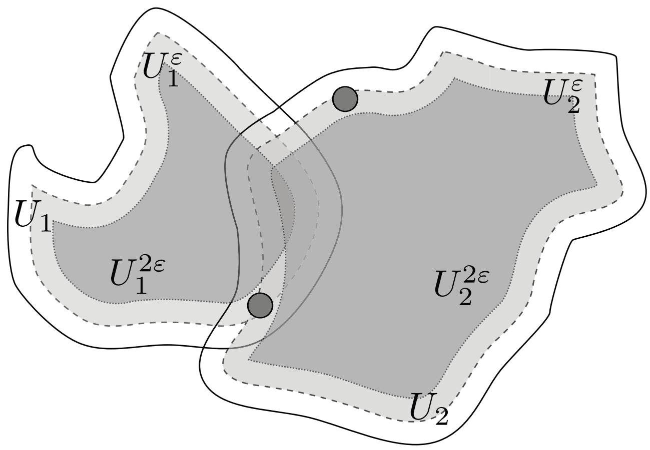


Figure 4.2: Sub-additivity for packing construction

Then, taking the infimum over $\delta > 0$ and then the supremum over $\varepsilon > 0$ gives

$$\sup_{\varepsilon>0} \hat{\mu}^q \left(\bigcup_k U_k^\varepsilon \right) \leqslant \inf_{\delta>0} \sum_{k\in\mathbb{N}} \hat{\mu}_\delta^q(U_k) \,. \tag{4.11}$$

We now want to prove that

$$\sup_{\varepsilon>0} \hat{\mu}^q \left(\bigcup_{k\in\mathbb{N}} U_k^\varepsilon \right) = \hat{\mu}^q \left(\bigcup_{k\in\mathbb{N}} U_k \right) \,. \tag{4.12}$$

Let $\mathcal{B}$ be an admissible packing of $\bigcup\limits_k U_k$ of order $\delta < \frac{\varepsilon}{2}$. We have

$$\sum_{B\in\mathcal{B}} q(B) = \sum_{\substack{B\in\mathcal{B} \\ B\subset\bigcup_k U_k^\varepsilon}} q(B) + \sum_{\substack{B\in\mathcal{B} \\ B\not\subset\bigcup_k U_k^\varepsilon}} q(B) \,. \tag{4.13}$$

Notice that since $2\delta < \varepsilon$, for any $B \in \mathcal{B}$, if $B \not\subset \bigcup\limits_k U_k^\varepsilon$ then $B \subset \bigcup\limits_k U_k \setminus \bigcup\limits_k U_k^{2\varepsilon}$. Since

$$q(B) = q(B_r(x)) = \frac{1}{r} \int_{s=0}^r \mu(B_s(x))\, ds \leqslant \mu(B_r(x)) = \mu(B) \,,$$

we get

$$\sum_{\substack{B\in\mathcal{B} \\ B\not\subset\bigcup_k U_k^\varepsilon}} q(B) \leqslant \sum_{\substack{B\in\mathcal{B} \\ B\not\subset\bigcup_k U_k^\varepsilon}} \mu(B) = \mu \left(\bigsqcup_{\substack{B\in\mathcal{B} \\ B\not\subset\bigcup_k U_k^\varepsilon}} B \right) \leqslant \mu \left(\bigcup_k U_k \setminus \bigcup_k U_k^{2\varepsilon} \right) \,. \tag{4.14}$$

Owing to the fact that $\bigcup\limits_k U_k = \bigcup\limits_{\substack{\text{countable} \\ \varepsilon>0}} \bigcup\limits_k U_k^{2\varepsilon}$ is decreasing in ε, we have that

$$\mu \left(\bigcup_k U_k \setminus \bigcup_k U_k^{2\varepsilon} \right) \xrightarrow[\varepsilon\to 0]{} 0 \tag{4.15}$$

as soon as $\mu(\bigcup_k U_k) < +\infty$, which is true under the assumption $\sum_k \mu(U_k) < +\infty$. Therefore, by(4.13), (4.14) and (4.15) we infer that

$$\sum_{B\in\mathcal{B}} q(B) \leqslant \sum_{\substack{B\in\mathcal{B}\\ B\subset\bigcup_k U_k^\varepsilon}} q(B) + \mu\left(\bigcup_k U_k \setminus \bigcup_k U_k^{2\varepsilon}\right) . \tag{4.16}$$

Taking the supremum in (4.16) over all admissible packings $\mathcal{B}$ of order δ of $\bigcup_k U_k$, we get

$$\begin{aligned}\hat{\mu}^q_\delta\left(\bigcup_k U_k\right) &\leqslant \sup\left\{\sum_{\substack{B\in\mathcal{B}\\ B\subset\bigcup_k U_k^\varepsilon}} q(B) \ : \ \mathcal{B} \text{ is a packing of } \bigcup_k U_k \text{ order } \delta\right\} + \mu\left(\bigcup_k U_k \setminus \bigcup_k U_k^{2\varepsilon}\right) \\ &\leqslant \hat{\mu}^q_\delta\left(\bigcup_k U_k^\varepsilon\right) + \mu\left(\bigcup_k U_k \setminus \bigcup_k U_k^{2\varepsilon}\right) ,\end{aligned}$$

Then taking the limit as $\delta \to 0$ we obtain

$$\hat{\mu}^q\left(\bigcup_k U_k\right) \leqslant \hat{\mu}^q\left(\bigcup_k U_k^\varepsilon\right) + \mu\left(\bigcup_k U_k \setminus \bigcup_k U_k^{2\varepsilon}\right)$$

and finally, letting $\varepsilon \to 0$, we prove that

$$\hat{\mu}^q\left(\bigcup_k U_k\right) = \lim_{\varepsilon\to 0} \hat{\mu}^q\left(\bigcup_k U_k^\varepsilon\right) , \tag{4.17}$$

that is, (4.12).

We now turn to the right hand side of (4.11). For fixed k, $\hat{\mu}^q_\delta(U_k)$ is decreasing when $\delta \downarrow 0$, therefore

$$\lim_{\delta\downarrow 0} \sum_k \hat{\mu}^q_\delta(U_k) = \sum_k \lim_{\delta\downarrow 0} \hat{\mu}^q_\delta(U_k) = \sum_k \hat{\mu}^q(U_k) \tag{4.18}$$

provided that $\sum_k \hat{\mu}^q_\delta(U_k)$ is finite for some $\delta > 0$. But, since $q(B) \leqslant \mu(B)$, $\hat{\mu}^q_\delta(U_k) \leqslant \mu(U_k)$ for all k so that $\sum_k \hat{\mu}^q_\delta(U_k) \leqslant \sum_k \mu(U_k) < +\infty$. Finally, thanks to (4.11) , (4.17) and (4.18) we obtain the countable sub-additivity for open sets (4.10).

Step 2. Let $(A_k)_k$ be a countable family of disjoint sets such that $\mu\left(\bigsqcup_k A_k\right) < +\infty$. We shall prove that

$$\hat{\mu}^q\left(\bigsqcup_k A_k\right) \leqslant \sum_k \hat{\mu}^q(A_k) . \tag{4.19}$$

Being μ a Borel measure, finite on bounded sets, let $(U_k)_k$ be a family of open set such that, by outer regularity (Theorem 4.3) and for any k,

$$A_k \subset U_k \quad \text{and} \quad \mu(U_k) \leqslant \mu(A_k) + \frac{1}{2^k} ,$$

so that $\sum \mu(U_k) \leqslant \sum \mu(A_k) + 2 < +\infty$. By (4.10) we thus find

$$\hat{\mu}^q\left(\bigsqcup_k A_k\right) \leqslant \hat{\mu}^q\left(\bigcup_k U_k\right) \leqslant \sum_k \hat{\mu}^q(U_k) .$$

Taking the infimum over such families of open sets $(U_k)_k$ leads to the required inequality (4.19).

Step 3. The case of a countable family $(A_k)_k$ such that $\mu\left(\bigcup_k A_k\right) < +\infty$ is obtained from Step 2, by the classical process to make the family disjoint, defining for all k, $B_k \subset A_k$ by

$$B_k = A_k - \bigcup_{i=1}^{k-1} A_i \,.$$

The family $(B_k)_k$ is disjoint and $\bigsqcup\limits_{k\in\mathbb{N}} B_k = \bigcup\limits_{k\in\mathbb{N}} A_k$ so that thanks to Step 2 (4.19),

$$\hat{\mu}^q\left(\bigcup_k A_k\right) = \hat{\mu}^q\left(\bigsqcup_k B_k\right) \leqslant \sum_k \hat{\mu}^q(B_k) \leqslant \sum_k \hat{\mu}^q(A_k) \,.$$

□

Remark 4.6. Notice that the fact that μ is finite on bounded sets is not used in Step 1, to get the sub-additivity on open sets, provided that $\sum\limits_k \mu(U_k) < +\infty$.

In order to have the countable sub-additivity of $\hat{\mu}^q$ (in the case where μ is not assumed to be finite), we want to show that $\sum_k \mu(U_k) = +\infty$ implies $\sum_k \hat{\mu}^q(U_k) = +\infty$. If so, either $\sum_k \mu(U_k) < +\infty$ and the sub-additivity is given by Proposition 4.11, or $\sum_k \hat{\mu}^q(U_k) = +\infty$ and the sub-additivity is clear. Thus we try to estimate $\hat{\mu}^q$ from above, comparing it to μ. The main problem is that

$$\frac{1}{r}\int_{s=0}^{r} \mu(B_s(x))\, ds \geqslant C\mu(B_r(x))$$

is generally false. Nevertheless, we still have this kind of lower control thanks to a smaller ball:

$$\frac{1}{r}\int_{s=0}^{r} \mu(B_s(x))\, ds \geqslant \frac{1}{2}\mu\left(B_{\frac{r}{2}}(x)\right) \,.$$

But once again, unless we know the measure μ is doubling, the following control $\mu(B) \geqslant C\mu(2B)$ does not hold for any ball B. Nevertheless, by comparing μ with a doubling measure, we will prove that it holds for enough balls, so that we can choose admissible packings among these balls.

Proposition 4.12. *Let μ be a positive Borel measure in an open set $\Omega \subset \mathbb{R}^n$. Let*

$$A_0 = \left\{ x \in \Omega \,:\, \liminf_{r\to 0} \frac{\mu(B_r(x))}{\mathcal{L}^n(B_r(x))} = 0 \right\} \quad \textit{and} \quad A_+ = \left\{ x \in \Omega \,:\, 0 < \liminf_{r\to 0} \frac{\mu(B_r(x))}{\mathcal{L}^n(B_r(x))} \leqslant +\infty \right\} .$$

Then

(i) For all $x \in A_+$, either $\mu(B_r(x)) = +\infty$ for all $r > 0$, or

$$\limsup_{r\to 0} \frac{\mu(B_r(x))}{\mu(B_{2r}(x))} \geqslant \frac{1}{2^n} \,.$$

(ii) $\mu(A_0) = 0$.

Consequently, for μ–almost any $x \in \Omega$, if $R > 0$ then there exists $r \leqslant R$ such that

$$\mu(B_r(x)) \geqslant \frac{1}{2^{n+1}}\mu(B_{2r}(x)) \,.$$

In some sense, the ratio $\dfrac{\mu(B_r(x))}{\mu(B_{2r}(x))}$ measures the "diffusion" of μ. In the proof, we formalize the idea that in $\mathbb{R}^n$, Lebesgue measure of a ball of radius is of order r^n,

$$\frac{\mathcal{L}^n(B_r(x))}{\mathcal{L}^n(B_{2r}(x))} = \frac{1}{2^n}\,,$$

and it is not possible for a positive Radon measure (finite on compact sets) to be substantially more diffused.

Proof. **Step 1.** Let $x \in A_+$. By monotonicity, either $\mu(B_r(x)) = +\infty$ for all $r > 0$ and then $\mu(B_r(x)) \geqslant 2^{-n-1}\mu(B_{2r}(x))$ is trivial, or there exists some R such that, for all $r \leqslant R$, $\mu(B_r(x)) < +\infty$. In this case the function defined by

$$f(r) = \frac{\mu(B_r(x))}{\mathcal{L}^n(B_r(x))}$$

is non-negative and finite for r small enough. Moreover, since $x \in A_+$, $\liminf_{r\to 0} f(r) > 0$. Let us prove that

$$\limsup_{r\to 0} \frac{f(r)}{f(2r)} \geqslant 1\,. \tag{4.20}$$

Assume by contradiction that $\limsup\limits_{r\to 0} \dfrac{f(r)}{f(2r)} < 1$, then there exists $r_0 > 0$ and $0 < \alpha < 1$ such that for all $r \leqslant r_0$, $f(r) \leqslant \alpha f(2r)$. Consider now the sequence $(r_k)_k$ defined by $r_k = 2^{-k}r_0$. Then $r_k \to 0$ and

$$f(r_k) \leqslant \alpha f(2r_k) = \alpha f(r_{k-1}) \leqslant \alpha^k f(r_0) \xrightarrow[k\to\infty]{} 0$$

which contradicts $\liminf_{r\to 0} f(r) > 0$. Let us then decompose

$$\frac{\mu(B_r(x))}{\mu(B_{2r}(x))} = \frac{f(r)}{f(2r)} \underbrace{\frac{\mathcal{L}^n(B_r(x))}{\mathcal{L}^n(B_{2r}(x))}}_{=2^{-n}}$$

so that

$$\limsup_{r\to 0} \frac{\mu(B_r(x))}{\mu(B_{2r}(x))} \geqslant \frac{1}{2^n} \limsup_{r\to 0} \frac{f(r)}{f(2r)} \geqslant \frac{1}{2^n}\,,$$

that is, a contradiction. This proves (4.20).

Step 2. Let us show that $\mu(A_0) = 0$. Assume that $\mathcal{L}^n(\Omega) < +\infty$ and let $\varepsilon > 0$. Consider

$$\mathcal{F}_\varepsilon = \{B \subset \Omega \mid B = B_r(a), a \in A_0 \text{ and } \mu(B) \leqslant \varepsilon\mathcal{L}^n(B)\}\,.$$

Let $a \in A_0$ be fixed. Since $\liminf\limits_{r\to 0} \dfrac{\mu(B_r(x))}{\mathcal{L}^n(B_r(x))} = 0$, there exists $r > 0$ such that $B_r(a) \in \mathcal{F}_\varepsilon$. Every point in A_0 is the center of some ball in $\mathcal{F}_\varepsilon$, so that we can apply Besicovitch Covering Lemma and obtain ζ_n countable families $\mathcal{G}_1, \dots \mathcal{G}_{\zeta_n}$ of disjoint balls in $\mathcal{F}_\varepsilon$, such that

$$A_0 \subset \bigcup_{j=1}^{\zeta_n} \bigsqcup_{B\in\mathcal{G}_j} B\,.$$

Therefore

$$\mu(A_0) \leqslant \sum_{j=1}^{\zeta_n} \sum_{B\in\mathcal{G}_j} \underbrace{\mu(B)}_{\leqslant\varepsilon\mathcal{L}^n(B)} \leqslant \varepsilon \sum_{j=1}^{\zeta_n} \mathcal{L}^n\left(\bigsqcup_{B\in\mathcal{G}_j} B\right) \leqslant \varepsilon\zeta_n\mathcal{L}^n(\Omega)\,.$$

Hence $\mu(A_0) = 0$ if $\mathcal{L}^n(\Omega) < +\infty$. Otherwise, replace Ω by $\Omega \cap B_k^\circ(0)$ to obtain that for any $k \in \mathbb{N}$, $\mu(A_0 \cap B_k^\circ(0)) = 0$, then let $k \to \infty$ to conclude that $\mu(A_0) = 0$. ☐

Remark 4.7. We make a couple of observations about Proposition 4.12. First, in the proof we make a systematic use of two properties of Lebesgue measure, i.e., that it is doubling ($\dfrac{\mathcal{L}^n(B_r(x))}{\mathcal{L}^n(B_{2r}(x))}$ is bounded from below by a universal constant) and that $\operatorname{supp}\nu \subset \operatorname{supp}\mathcal{L}^n = \mathbb{R}^n$. Therefore, the same argument could be applied with another measure satisfying the two properties above, even in a more general, separable and directionally limited metric space. Second, it is possible to replace $\dfrac{\mu(B_r(x))}{\mu(B_{2r}(x))}$ and $\dfrac{1}{2^n}$ by $\dfrac{\mu(B_{\theta r}(x))}{\mu(B_r(x))}$ and θ^n.

Corollary 4.13 (Besicovitch with doubling balls)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set, let μ be a positive Borel measure in Ω and for $\delta > 0$, let*

$$\mathcal{F}_\delta = \left\{ B \text{ closed ball } \subset \Omega \,:\, \mu(B) \geqslant \frac{1}{2^{n+1}}\mu(2B) \text{ and } \operatorname{diam} B \leqslant 2\delta \right\} .$$

Let $A \subset \Omega$ and $\mathcal{F}_\delta^A = \{B \in \mathcal{F}_\delta \,:\, B = B_r(a) \text{ with } a \in A\}$. Then there exist $A_0 \subset \Omega$ and $\zeta_n \in \mathbb{N}$ countable subfamilies of $\mathcal{F}_\delta^A$ of disjoint closed balls, $\mathcal{G}_1, \dots \mathcal{G}_{\zeta_n}$ such that

$$A \subset A_0 \cup \bigcup_{j=1}^{\zeta_n} \bigsqcup_{B \in \mathcal{G}_j} B \text{ and } \mu(A_0) = 0 .$$

Moreover, if $\mu(A) < +\infty$, then for any open set $U \subset \mathbb{R}^n$, there exists a countable collection $\mathcal{G}$ of disjoint balls in $\mathcal{F}_\delta^A$ such that

$$\bigsqcup_{B \in \mathcal{G}} B \subset U \quad \text{and} \quad \mu\left((A \cap U) - \bigsqcup_{B \in \mathcal{G}} B \right) = 0 .$$

Proof. Thanks to Proposition 4.12, we know that for μ–almost every $x \in \Omega$, for any $0 < R < \delta$, there exists $r \leqslant R$ such that $B_r(x) \in \mathcal{F}_\delta$ so that for μ–almost any $x \in \Omega$,

$$\inf \{r \mid B_r(x) \in \mathcal{F}_\delta\} = 0 .$$

The conclusion follows from Besicovitch Covering Lemma and Theorems 4.6 and 4.7. □

We can now prove that $\hat{\mu}^q$ and μ are equivalent on Borel sets.

Proposition 4.14. *Let $\Omega \subset \mathbb{R}^n$ be an open set, let μ be a positive Borel measure in Ω and let $\hat{\mu}^q$ be defined as in Definition 4.8 starting from the pre-measure q defined in (4.8). Then there exists a constant C_n only depending on the dimension such that for any open set $U \subset \Omega$,*

$$\frac{1}{C_n}\mu(U) \leqslant \hat{\mu}^q(U) \leqslant \mu(U) .$$

Consequently, for any Borel set $A \subset \Omega$ we have

$$\frac{1}{C_n}\mu(A) \leqslant \hat{\mu}^q(A) \leqslant \inf\{\mu(U) \mid A \subset U \text{ open set }\} .$$

If moreover μ is outer regular (for instance, if μ is bounded on open sets) then

$$\frac{1}{C_n}\mu(A) \leqslant \hat{\mu}^q(A) \leqslant \mu(A) .$$

Proof. Let $U \subset \Omega$ be an open set, the inequality $\hat{\mu}^q(U) \leqslant \mu(U)$ is just a consequence of the fact that for any closed ball B, $q(B) \leqslant \mu(B)$. Now let us prove the other inequality.

(i) Case $\mu(U) < +\infty$. Let $\delta > 0$, then we can apply Corollary 4.13 (Besicovitch with doubling balls) to get a countable family $\mathcal{G}_\delta$ of disjoint balls of

$$\mathcal{F}_\delta^U = \left\{ B = B_r(x) \text{ closed ball } \subset \Omega \,:\, x \in U,\, \mu(B) \geqslant \frac{1}{2^{n+1}}\mu(2B) \text{ and } \operatorname{diam} B \leqslant 2\delta \right\}$$

such that

$$\mu(U) = \mu\left(\bigsqcup_{B\in\mathcal{G}_\delta} B\right) \quad \text{and} \quad \bigsqcup_{B\in\mathcal{G}_\delta} B \subset U\,.$$

Therefore

$$\begin{aligned}\hat{\mu}_\delta^q(U) \geqslant \sum_{B\in\mathcal{G}_\delta} q(B) &= \sum_j \frac{1}{r_j}\int_{r=0}^{r_j} \mu(B_r(x_j))\,dr \geqslant \sum_j \frac{1}{2}\mu(B_{\frac{r_j}{2}}(x_j)) \\ &\geqslant \frac{1}{2}\sum_j \frac{1}{2^{n+1}}\mu(B_{r_j}(x_j)) = \frac{1}{2^{n+2}}\mu\left(\bigsqcup_{B\in\mathcal{G}_\delta} B\right) = \frac{1}{C_n}\mu(U)\,,\end{aligned}$$

with $C_n = 2^{n+2}$. Letting $\delta \to 0$ gives $\hat{\mu}^q(U) \geqslant \dfrac{1}{C_n}\mu(U)$.

(ii) If $\mu(U) = +\infty$. Let $\delta > 0$, then applying Corollary 4.13 (Besicovitch with doubling balls) with

$$\mathcal{F}_\delta^U \cap \{B \mid B \subset U\}\;,$$

gives ζ_n countable families $\mathcal{G}_\delta^1, \ldots, \mathcal{G}_\delta^{\zeta_n}$ of balls in $\mathcal{F}_\delta \cap \{B \mid B \subset U\}$ such that

$$U \subset U_0 \cup \bigcup_{j=1}^{\zeta_n} \bigsqcup_{B\in\mathcal{G}_\delta^j} B \text{ with } \mu(U_0) = 0\,.$$

So that

$$\sum_{j=1}^{\zeta_n} \mu\left(\bigsqcup_{B\in\mathcal{G}_\delta^j} B\right) \geqslant \mu(U) = +\infty\,.$$

Consequently there exists $j_0 \in \{1, \ldots, \zeta_n\}$ such that $\mu\left(\bigsqcup_{B\in\mathcal{G}_\delta^{j_0}} B\right) = +\infty$. Therefore we have the same estimate as in the case $\mu(U) < +\infty$:

$$\begin{aligned}\hat{\mu}_\delta^q(U) \geqslant \sum_{B\in\mathcal{G}_\delta^{j_0}} q(B) &= \sum_l \frac{1}{r_l}\int_{r=0}^{r_l} \mu(B_r(x_l))\,dr \geqslant \sum_l \frac{1}{2}\mu(B_{\frac{r_l}{2}}(x_l)) \\ &\geqslant \frac{1}{2}\sum_l \frac{1}{2^{n+1}}\mu(B_{r_l}(x_l)) = C_n\mu\left(\bigsqcup_{B\in\mathcal{G}_\delta^{j_0}} B\right) = +\infty\,.\end{aligned}$$

Hence $\hat{\mu}^q(U) = +\infty$.

□

Corollary 4.15. *Under the assumptions of Proposition 4.14, $\hat{\mu}^q$ is countably sub-additive.*

Proof. Let $(A_k)_k$ be a countable collection of subsets of Ω. If $\sum_k \mu(A_k) = +\infty$, by Proposition 4.14 we get $\mu(A_k) \leqslant C_n \hat{\mu}^q(A_k)$ for all k, therefore

$$\sum_k \hat{\mu}^q(A_k) \geqslant \frac{1}{C_n} \sum_k \mu(A_k) = +\infty\,,$$

whence the countable sub-additivity follows. Recall that if $\sum_k \mu(A_k) < +\infty$ and A_k are open sets then countable sub-additivity was proved in Proposition 4.11, without the assumption of finiteness on bounded sets. It remains to check the case $\sum_k \mu(A_k) < +\infty$, for any Borel sets A_k. For any family $(U_k)_k$ of open sets such that $A_k \subset U_k$ for all k, by sub-additivity on open sets we have

$$\hat{\mu}^q(\bigcup_k A_k) \leqslant \hat{\mu}^q(\bigcup_k U_k) \leqslant \sum_k \hat{\mu}^q(U_k)\,.$$

Taking the infimum over such families of open sets gives, by definition of $\hat{\mu}^q$,

$$\hat{\mu}^q(\bigcup_k A_k) \leqslant \inf\left\{\sum_k \hat{\mu}^q(U_k) \;:\; A_k \subset U_k \text{ open set}\right\} = \sum_k \hat{\mu}^q(A_k)\,.$$

□

Let us summarize the results contained in Proposition 4.11, Remark 4.5, Proposition 4.14 and Corollary 4.15:

Theorem 4.16. *Let $\Omega \subset \mathbb{R}^n$ be an open set, let μ be a positive Borel measure in Ω and let $\hat{\mu}^q$ defined as in Definition 4.8 starting from the pre-measure q defined as in* (4.8)*. Then, the following holds:*

1. *$\hat{\mu}^q$ is a metric outer measure, coinciding with the measure $\tilde{\mu}^q$.*
2. *there exists a dimensional constant $C_n \geqslant 1$ such that for any Borel set $A \subset \Omega$,*

$$\frac{1}{C_n}\mu(A) \leqslant \hat{\mu}^q(A) \leqslant \inf\{\mu(U) \mid A \subset U \text{ open set }\}\,.$$

3. *if moreover μ is outer regular, for instance if μ is finite on bounded sets (i.e., if μ is a Radon measure), then μ and the positive Borel measure associated with the outer measure $\hat{\mu}^q$ (still denoted as $\hat{\mu}^q$) are equivalent, that is, $\frac{1}{C_n}\mu \leqslant \hat{\mu}^q \leqslant \mu$.*

Remark 4.8. We stress that μ is not generally assumed to be finite on open sets, unless explicitly mentioned.

4.2.3 The case of a signed measure

Our aim is to prove that the packing-type reconstruction applied to a signed measure μ with pre-measures $q_\pm(B_r(x)) = \left(\frac{1}{r}\int_{s=0}^r \mu(B_s(x))\,ds\right)_\pm$ produces a signed measure $\hat{\mu}^p$ whose positive and negative parts are comparable with those of μ.

Theorem 4.17. *Let $\Omega \subset \mathbb{R}^n$ be an open set, let $\mu = \mu^+ - \mu^-$ be a signed Borel measure in Ω, finite on bounded sets. Let $\mathcal{C} = \{$closed balls $B_r(x) \subset X\}$ and take $\hat{\mu}^{q+}$ and $\hat{\mu}^{q-}$ as in Definition 4.8, corresponding to the pre-measures $q_\pm : \mathcal{C} \to \mathbb{R}_+ \cup \{+\infty\}$ defined by*

$$q_\pm(B_r(x)) = \left(\frac{1}{r}\int_{s=0}^r \mu(B_s(x))\,ds\right)_\pm\,.$$

Then the following holds.

(i) $\hat{\mu}^{q+}$, $\hat{\mu}^{q-}$ are metric outer measures, finite on bounded sets.

(ii) The measure $\hat{\mu}^q = \hat{\mu}^{q+} - \hat{\mu}^{q-}$ is a signed measure and there exists a dimensional constant $C_n \geqslant 1$ such that, for any Borel set $A \subset \Omega$,

$$\frac{1}{C_n}\mu^+(A) \leqslant \hat{\mu}^{q+}(A) \leqslant \mu^+(A) \quad \textit{and} \quad \frac{1}{C_n}\mu^-(A) \leqslant \hat{\mu}^{q-}(A) \leqslant \mu^-(A)\,,$$

whence in particular

$$\frac{1}{C_n}|\mu|(A) \leqslant |\hat{\mu}^q|(A) \leqslant |\mu|(A)\,.$$

Proof. Showing the countable sub-additivity of $\hat{\mu}^{q+}$ and $\hat{\mu}^{q-}$ under the assumption $\sum_k \mu(A_k) < +\infty$ (see Proposition 4.11) does not require the special form of the pre-measure q but only the fact that, for any closed ball B,

$$q(B) \leqslant \mu(B)\,. \tag{4.21}$$

In our particular case, for any such $B = B_r(x)$ we have

$$q_+(B) = \left(\frac{1}{r}\int_{s=0}^{r} \mu(B_s(x))\,ds\right)_+ \leqslant \frac{1}{r}\int_{s=0}^{r} (\mu(B_s(x)))_+\,ds \leqslant \mu^+(B)\,,$$

thus (4.21) is satisfied. This is sufficient to get the sub-additivity, under the assumption $\sum_k \mu(A_k) < +\infty$. It is also sufficient for concluding that, for any open set $U \subset \Omega$,

$$\hat{\mu}^{q+}(U) \leqslant \mu^+(U)\,.$$

This gives the proof of (i).

Let now $A \subset \Omega$ be a Borel set. If $\mu^+(A) < +\infty$, let $A \subset U$ open set, let $\delta > 0$ and apply Corollary 4.13 to μ_+ to get a family $\mathcal{G}_\delta$ of disjoint closed balls B of radius $\leqslant \delta$, with $\mu^+(B) \geqslant \frac{1}{2^{n+1}}\mu^+(2B)$ such that

$$\bigsqcup_{B\in\mathcal{G}_\delta} B \subset U \quad \text{and} \quad \mu^+(A) = \mu^+\left(\bigsqcup_{B\in\mathcal{G}_\delta} B\right)\,.$$

Hence,

$$\begin{aligned}
\hat{\mu}^{q+}_\delta(U) &\geqslant \sum_{B\in\mathcal{G}_\delta} q_+(B) = \sum_j \left(\frac{1}{r_j}\int_{s=1}^{r_j} \mu(B_s(x_j))\,ds\right)_+ \\
&\geqslant \sum_j \frac{1}{r_j}\int_{s=1}^{r_j} \mu(B_s(x_j))\,ds = \sum_j \frac{1}{r_j}\int_{s=1}^{r_j} (\mu(B_s(x_j)))_+\,ds - \sum_j \frac{1}{r_j}\int_{s=1}^{r_j} (\mu(B_s(x_j)))_-\,ds \\
&\geqslant C_n\mu^+\left(\bigsqcup_{B\in\mathcal{G}_\delta} B\right) - \mu^-\left(\bigsqcup_{B\in\mathcal{G}_\delta} B\right) \\
&\geqslant C_n\mu^+(A) - \mu^-(U)\,.
\end{aligned} \tag{4.22}$$

Letting $\delta \to 0$ we have

$$\hat{\mu}^{q+}(U) \geqslant C_n\mu^+(A) - \mu^-(U)\,. \tag{4.23}$$

By definition of $\hat{\mu}^{q+}(A)$, there exists a sequence of open sets $(U_k^1)_k$ such that, for all k, it holds $A \subset U_k^1$ and

$$\hat{\mu}^{q+}(U_k^1) \xrightarrow[k\to\infty]{} \hat{\mu}^{q+}(A)\,.$$

By outer regularity of μ^- (which is Borel and finite on bounded sets) there exists a sequence of open sets $(U_k^2)_k$ such that, for all k, we get $A \subset U_k^2$ and

$$\mu^-(U_k^2) \xrightarrow[k\to\infty]{} \mu^-(A)\,.$$

For all k, let $U_k = U_k^1 \cap U_k^2$, then U_k is an open set, $A \subset U_k$ and, by monotonicity,

$$\begin{aligned}\hat{\mu}^{q+}(A) \leqslant \hat{\mu}^{q+}(U_k) \leqslant \hat{\mu}^{q+}(U_k^1)\,,\\ \mu^-(A) \leqslant \mu^-(U_k) \leqslant \mu^-(U_k^2)\,,\end{aligned}$$

therefore

$$\hat{\mu}^{q+}(U_k) \xrightarrow[k\to\infty]{} \hat{\mu}^{q+}(A) \text{ and } \mu^-(U_k) \xrightarrow[k\to\infty]{} \mu^-(A)\,.$$

Evaluating (4.23) on the sequence $(U_k)_k$ and letting k go to $+\infty$, we eventually get

$$\hat{\mu}^{q+}(A) \geqslant C_n \mu^+(A) - \mu^-(A)\,. \tag{4.24}$$

Owing to Hahn decomposition of signed measures, we consider a Borel set $P \subset \Omega$ such that for all Borel $A \subset \Omega$ it holds

$$\mu^+(A) = \mu^+(A \cap P) = \mu(A \cap P) \text{ and } \mu^-(A) = \mu(A \cap (\Omega - P))\,.$$

Finally, let $A \subset \Omega$ be a Borel set, then by (4.24) applied to $A \cap P$ we find

$$\begin{aligned}\hat{\mu}^{q+}(A) \geqslant \hat{\mu}^{q+}(A \cap P) &\geqslant C_n \mu^+(A \cap P) - \mu^-(A \cap P)\\ &= C_n \mu^+(A)\end{aligned}$$

It remains to show that if $\mu^+(A) = +\infty$, then $\hat{\mu}^{q+}(A) = +\infty$. Let $A \subset \Omega$ be a Borel set such that $\mu^+(A) = +\infty$, and recall that, by definition of signed measure, μ^- must be finite. Let $U \subset \Omega$ be an open set containing A. The next argument is exactly the same as in the proof of Proposition 4.14 (positive case). Given $\delta > 0$, we apply Corollary 4.13 to μ^+ with

$$\mathcal{F}_\delta^U \cap \{B \mid B \subset U\}\ .$$

This gives ζ_n countable families $\mathcal{G}_\delta^1, \ldots, \mathcal{G}_\delta^{\zeta_n}$ of balls in $\mathcal{F}_\delta \cap \{B \mid B \subset U\}$ such that

$$U \subset U_0 \cup \bigcup_{j=1}^{\zeta_n} \bigsqcup_{B \in \mathcal{G}_\delta^j} B \text{ with } \mu^+(U_0) = 0\,,$$

hence

$$\sum_{j=1}^{\zeta_n} \mu^+ \left(\bigsqcup_{B \in \mathcal{G}_\delta^j} B \right) \geqslant \mu^+(U) = +\infty\,.$$

Consequently there exists $j_0 \in \{1, \dots, \zeta_n\}$ such that $\mu^+ \left(\bigsqcup_{B \in \mathcal{G}_\delta^{j_0}} B \right) = +\infty$ and

$$
\begin{aligned}
\hat{\mu}_\delta^{q+}(U) &\geqslant \sum_{B \in \mathcal{G}_\delta^{j_0}} q_+(B) = \sum_l \left(\frac{1}{r_l} \int_{r=0}^{r_l} \mu(B_r(x_l))\, dr \right)_+ \\
&\geqslant \sum_l \frac{1}{r_l} \int_{r=0}^{r_l} \mu(B_r(x_l))\, dr = \sum_l \frac{1}{r_l} \int_{r=0}^{r_l} \mu^+(B_r(x_l))\, dr - \sum_l \frac{1}{r_l} \int_{r=0}^{r_l} \mu^-(B_r(x_l))\, dr \\
&\geqslant \sum_l \frac{1}{2} \mu^+(B_{\frac{r_l}{2}}(x_l)) - \mu^- \left(\bigsqcup_{B \in \mathcal{G}_\delta^{j_0}} B \right) \\
&\geqslant \frac{1}{2} \sum_l \frac{1}{2^{n+1}} \mu^+(B_{r_l}(x_l)) - \mu^-(U) = C_n \mu^+ \left(\bigsqcup_{B \in \mathcal{G}_\delta^{j_0}} B \right) - \mu^-(U) \\
&= +\infty\,.
\end{aligned}
$$

Finally, letting $\delta \to 0$, we obtain that $\hat{\mu}^q(U) = +\infty$ for all open set $U \supset A$, hence that $\hat{\mu}^q(A) = +\infty$. This completes the proof of (ii) and thus of the theorem. $\square$

CHAPTER 5

Régularisation de la variation première

Introduction

Recall that our purpose is to study surface representation and discretization, and then geometric energies defined on surfaces, in the unified setting provided by varifolds. In Chapter 2 we introduced discrete volumetric varifolds, which are volumetric approximations associated with a family of meshes of the space, and point cloud varifolds. We stated an approximation result (Theorem 2.1) ensuring that rectifiable varifolds can be approximated by discrete varifolds in the sense of weak–$*$ convergence, and under some additional assumptions, in the sense of flat distance, with an estimate on the convergence rate depending on the size of the mesh and on the Hölder regularity of the tangent plane of the approximated varifold. The result raised the following question: given a sequence of approximating d–varifolds $(V_i)_i$ weakly–$*$ converging to some d–varifold V, is it possible to introduce a notion of approximate regularity for the approximating sequence (V_i) and to connect it with the regularity of the limit V? We have already studied this question from the point of view of rectifiability (**Question 1.2**) and proposed an answer in Chapter 3 (Theorem 3.4). We then asked the question from the point of view of curvature, let us recall it:

Question. 1.3 What conditions on a weakly–$*$ converging sequence of varifolds (not necessarily rectifiable) ensure that the limit varifold has bounded first variation?

In Chapter 4, we tried to answer this question by observing that when a d–varifold has bounded first variation, some averaged quantity (involving the first variation of balls centered at a same point) can be written in a way that makes sense for any varifold:

$$\frac{1}{R}\int_{r=0}^{R} \delta V(B_r(x))\, dr = -\frac{1}{R}\int_{B_R(x)\times G_{d,n}} \frac{\Pi_S(y-x)}{|y-x|}\, dV(y,S)\,. \tag{5.1}$$

We then used Carathéodory and packing–type constructions to recover the vector–valued measure δV using only the values (5.1) for any ball $B_R(x)$. We also noticed that (5.1) is simply the convolution of δV with a suitable kernel T_R: that is why the right hand side of (5.1) makes sense for any varifold. Indeed, even when a varifold does not have bounded variation, the first variation δV is by definition a linear form on C_c^1 or equivalently a distribution of order 1 which can thus always be convolved with a C^1 or Lipschitz function as is the kernel T_R (see (5.9)). And it led us to the approach developed in this chapter. We obtain the following results, giving an answer to Question 1.3, which can be compared with Theorems 3.3 and 3.4 of Chapter 3.

Theorem. 5.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set and V be a d–varifold in Ω with finite mass $\|V\|(\Omega)$. Let $\rho \in \mathrm{W}^{1,\infty}(\mathbb{R}^n)$ be a symmetric and positive function such that $\int_{\mathbb{R}^n} \rho = 1$ and $\operatorname{supp} \rho \subset B_1(0)$ and let $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right)$. Assume that*

$$\sup_{\varepsilon > 0} \|\delta V * \rho_\varepsilon\|_{\mathrm{L}^1} = \sup_{\varepsilon > 0} \frac{1}{\varepsilon^{n+1}} \int_{x \in \mathbb{R}^n} \left| \int_{(B_\varepsilon(x) \cap \Omega) \times G_{d,n}} \nabla^S \rho(y - x) \, dV(y, S) \right| d\mathcal{L}^n(x) < +\infty \, . \tag{5.2}$$

Then V has bounded first variation and $|\delta V|(\Omega)$ is bounded by the previous supremum.
In the particular case when $\rho_\varepsilon = T_\varepsilon$ is the tent kernel (5.9), the assumption (5.2) rewrites

$$\sup_{\varepsilon > 0} \frac{1}{\varepsilon^n} \int_{x \in \mathbb{R}^n} \frac{1}{\varepsilon} \left| \int_{y \in B_\varepsilon(x) \cap \Omega} \frac{\Pi_S(y - x)}{|y - x|} \, dV(y, S) \right| d\mathcal{L}^n(x) < +\infty \, .$$

Theorem. 5.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $(V_i)_i$ be a sequence of d–varifolds. Let $\rho \in \mathrm{W}^{1,\infty}(\mathbb{R}^n)$ be a symmetric non negative function such that $\int_{\mathbb{R}^n} \rho = 1$ and $\operatorname{supp} \rho \subset B_1(0)$ and let $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right)$. Assume that there exists a positive decreasing sequence $(\varepsilon_i)_i$, tending to 0, such that*

$$\sup_i \left\{ \|V_i\|(\Omega) + \frac{1}{\varepsilon_i^{n+1}} \int_{x \in \mathbb{R}^n} \left| \int_{(B_{\varepsilon_i}(x) \cap \Omega) \times G_{d,n}} \nabla^S \rho(y - x) \, dV_i(y, S) \right| d\mathcal{L}^n(x) \right\} < +\infty \, . \tag{5.3}$$

Then there exists a subsequence $(V_{\varphi(i)})_i$ weakly–$*$ converging in Ω to a d–varifold V, V has bounded first variation and $|\delta V|(\Omega)$ is bounded by the previous supremum.

This convolution of the first variation actually provides a notion of approximate curvature which is convenient to define approximate Willmore energies. More precisely, with the above notations, we define the approximate Willmore energie associated with ρ and $\varepsilon > 0$ as

$$\mathcal{W}_\varepsilon^p(V) = \int_{x \in \mathbb{R}^n} \left| \frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)} \right|^p \|V\| * \rho_\varepsilon(x) \, d\mathcal{L}^n(x) \, .$$

Question 5.1. Do the approximate Willmore energies $\mathcal{W}_\varepsilon^p$ Γ–converge in the space of d–varifolds? And if so, is the classical Willmore energy the Γ–limit?

Let $\mathcal{W}^p$ denote the classical p–Willmore energy. We obtained the following Γ–convergence results (the detailed statements are given in Theorems 5.8 and 5.10). Notice that in the case $p = 1$, the Γ–limit is not the 1–Willmore energy but the total variation of the first variation.

$$\begin{aligned} &\mathcal{W}_\varepsilon^p \xrightarrow[\varepsilon \to 0]{\Gamma} \mathcal{W}^p \quad \text{for } 1 < p < +\infty \\ &\mathcal{W}_\varepsilon^1 \xrightarrow[\varepsilon \to 0]{\Gamma} \text{the total variation of the first variation} \neq \mathcal{W}^1 \, . \end{aligned}$$

However, this Γ–convergence result is not fully satisfactory. Indeed, in a practical way, for a fixed ε, we want to minimize $\mathcal{W}_\varepsilon^p$, but not in the whole space of varifolds, we rather want to minimize in some subclass like the family of discrete volumetric varifolds. We thus need to make a link between the scale parameter ε of the energy $\mathcal{W}_\varepsilon^p$ and the scale of the discrete objects (with the size δ_i of the meshes $\mathcal{K}_i$ if we consider discrete volumetric varifolds $V_i \in \mathcal{A}_{\delta_i}(\mathcal{K}_i)$, defined in Chapter 2 (2.1)). More generally, in terms of Γ–convergence, this means that to each $\varepsilon > 0$ corresponds an approximation space $\mathcal{A}_\varepsilon$ and that the Γ–convergence must hold in these approximation spaces, denoted

$$\mathcal{W}_\varepsilon^p \xrightarrow[\mathcal{A}_\varepsilon, \, \varepsilon \to 0]{\Gamma} \mathcal{W}^p \, . \tag{5.4}$$

In other words:

– for any sequence $(V_\varepsilon)_\varepsilon$ of d–varifolds such that $V_\varepsilon \in \mathcal{A}_\varepsilon$ and $V_\varepsilon \xrightharpoonup[\varepsilon \to 0]{*} V$,

$$\mathcal{W}^p(V) \leqslant \liminf_{\varepsilon \to 0} \mathcal{W}^p_\varepsilon(V_\varepsilon) \, .$$

Notice that as the Γ–$\liminf$ holds for any sequence of d–varifolds $(V_\varepsilon)_\varepsilon$, it holds in particular when restricting the approximation spaces.

– for any d–varifold V, there exists a sequence $(V_\varepsilon)_\varepsilon$ of d–varifolds such that $V_\varepsilon \in \mathcal{A}_\varepsilon$, $V_\varepsilon \xrightharpoonup[\varepsilon \to 0]{*} V$ and

$$\limsup_{\varepsilon \to 0} \mathcal{W}^p_\varepsilon(V_\varepsilon) \leqslant \mathcal{W}^p(V) \, .$$

Conversely to the Γ–$\liminf$ property, the Γ–$\limsup$ property must be reconsidered when restricting the approximation spaces.

So that we have to study the following question:

Question 5.2. Given
– a subset of d–rectifiable varifolds $\mathcal{A}$;
– a family $(\mathcal{A}_\varepsilon)_\varepsilon$ where each $\mathcal{A}_\varepsilon$ is a prescribed subset of discrete volumetric varifolds.

For any rectifiable d–varifold $V \in \mathcal{A}$, is there a sequence $(V_\varepsilon)_\varepsilon$ of discrete volumetric varifolds such that
– for any $\varepsilon > 0$, $V_\varepsilon \in \mathcal{A}_\varepsilon$;
– $V_\varepsilon \xrightharpoonup[\varepsilon \to 0]{*} V$,
– and $\mathcal{W}^p_\varepsilon(V_\varepsilon) \xrightharpoonup[\varepsilon \to 0]{*} \mathcal{W}^p(V)$?

We study this question for $\mathcal{A}_\varepsilon$ of the form $\mathcal{A}_\delta(\mathcal{K})$ (the space of discrete volumetric varifolds associated to the mesh $\mathcal{K}$ of size $\sup_{K \in \mathcal{K}} \operatorname{diam}(K) \leqslant \delta$ defined in (2.1)): for $\mathcal{W}^1_\varepsilon$, we obtain a partial result, and for $\mathcal{W}^p_\varepsilon$, with $p > 1$, the question stands. One difficulty is to make a connection between the parameter ε and the size of the mesh δ. This is linked to what we called the "accuracy of the approximation spaces $\mathcal{A}_\delta(\mathcal{K})$" defined in Remark 2.2 by

$$d^{asym}_{\mathcal{H}}(V, W) := \sup_{V \in \mathcal{A}} \inf_{W \in \mathcal{A}_\delta(\mathcal{K})} \Delta^{1,1}(W, V) \, , \tag{5.5}$$

where $\Delta^{1,1}$ is the distance introduced in Definition 1.15 by:

$$\Delta(V, W) = \sup \left\{ \left| \int \varphi \, dV - \int \varphi \, dW \right| \; : \; \varphi \in \mathrm{Lip}_1, \; \|\varphi\|_\infty \leqslant 1 \right\} .$$

Recall that (5.5) is not the Hausdorff distance $d_{\mathcal{H}}(\mathcal{A}, \mathcal{A}_\delta(K))$ since we care only of the approximation of $\mathcal{A}$ by $\mathcal{A}_\delta(\mathcal{K})$ and not the contrary:

$$d_{\mathcal{H}}(\mathcal{A}, \mathcal{A}_\delta(K)) = \max \left\{ d^{asym}_{\mathcal{H}}(\mathcal{A}, \mathcal{A}_\delta(K)), d^{asym}_{\mathcal{H}}(\mathcal{A}_\delta(K), \mathcal{A}) \right\} .$$

In Remark 2.2, we state that for

$$\mathcal{A}^\beta = \{\text{rectifiable } d\text{–varifolds satisfying a } \beta\text{–Hölder condition on the tangent plane(2.5)}\} \, ,$$

we have the estimate

$$d^{asym}_{\mathcal{H}}(\mathcal{A}^\beta, \mathcal{A}_\delta(\mathcal{K})) \leqslant \left(\delta + 2C\delta^\beta \right) \|V\| \, (\Omega) \, ,$$

which can be controlled by the size of the mesh δ and the mass of the varifold $\|V\|(\Omega)$. Thanks to this estimate, we explicit a condition (5.6) linking ε and δ and ensuring the Γ–$\limsup$ property in the space of discrete volumetric varifolds:

Theorem. 5.13. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $V = v(M,\theta)$ be a rectifiable d–varifold in Ω. Let $\delta_i \downarrow 0$ be a sequence of infinitesimals and $(\mathcal{K}_i)_i$ a sequence of meshes satisfying*

$$\sup_{K\in\mathcal{K}_i} \operatorname{diam}(K) \leqslant \delta_i \xrightarrow[i\to+\infty]{} 0 \,.$$

Let the approximation spaces $\left(\mathcal{A}_{\delta_i}(\mathcal{K}_i)\right)_i$ be defined as in (2.1) *and let the kernel $\rho \in \mathrm{W}^{2,\infty}$ be as in* (5.13)*. Assume that there exist $0 < \beta < 1$ and C such that for $\|V\|$–almost every x, $y \in \Omega$,*

$$\|T_x M - T_y M\| \leqslant C|x-y|^\beta \,.$$

Then, there exists a sequence of discrete volumetric varifolds $(V_i)_i$ such that

(i) for all i, $V_i \in \mathcal{A}_{\delta_i}(\mathcal{K}_i)$,

(ii) $V_i \xrightarrow[i\to+\infty]{*} V$,

(iii) For any sequence of infinitesimals $\varepsilon_i \downarrow 0$ satisfying

$$\frac{\delta_i^\beta}{\varepsilon_i^2} \xrightarrow[i\to+\infty]{} 0 \,. \tag{5.6}$$

we have,

$$\mathcal{W}^1_{\varepsilon_i}(V_i) \xrightarrow[i\to+\infty]{} \mathcal{W}^1(V) \,.$$

In order to understand better the approximate first variation provided by the regularization $\delta V * \rho_\varepsilon$, we ask the following question:

Question 5.3.
– Given a d–varifold V, is the regularization $\delta V * \rho_\varepsilon$ of the first variation δV, the first variation $\delta\left(\widehat{V_\varepsilon}\right)$ of some varifold $\widehat{V_\varepsilon}$?
– And if so, is $\widehat{V_\varepsilon}$ the regularization (in a sense to be defined) of V?

The construction can be done explicitly:

Theorem. 5.14. *Let $\Omega \subset \mathbb{R}^n$ be an open set and V a d–varifold in Ω with finite mass $\|V\|(\Omega) < +\infty$. Let $\varepsilon > 0$ and ρ_ε as in* (5.13)*. Define the d–varifold $\widehat{V_\varepsilon}$ by: for every $\psi \in \mathrm{C}^0_c(\Omega \times G_{d,n})$,*

$$\left\langle \widehat{V_\varepsilon}, \psi \right\rangle = \langle V, (y,S) \mapsto \psi(\cdot, S) * \rho_\varepsilon(y) \rangle$$

or equivalently,

$$\int_{\Omega\times G_{d,n}} \psi(y,S)\, d\widehat{V_\varepsilon}(y,S) = \int_{(y,S)\in\Omega\times G_{d,n}} \int_{x\in\mathbb{R}^n} \psi(x,S)\rho_\varepsilon(y-x)\, d\mathcal{L}^n(x)\, dV(y,S) \,.$$

Then,

1. $\|\widehat{V_\varepsilon}\| = \|V\| * \rho_\varepsilon$,

2. $\delta\left(\widehat{V_\varepsilon}\right) = \delta V * \rho_\varepsilon$.

We observe that the mass $\|\widehat{V_\varepsilon}\|$ is the convolution of $\|V\|$ and in Proposition 5.15, we point out that the tangential part $\widehat{\nu^\varepsilon_x}$ of $\widehat{V_\varepsilon} = \|\widehat{V_\varepsilon}\| \otimes \widehat{\nu^\varepsilon_x}$ is generally not a Dirac mass nor a combination of Dirac masses.

Section 5.1 deals with answering to Question 1.3 thanks to a regularization of the first variation by convolution. In Section 5.2, we build approximate Willmore energies $\mathcal{W}^p_\varepsilon$ and study the Γ–convergence to the p–Willmore energy in the space of varifolds (Question 5.1). In Section 5.3, we address the Question 5.2 of a different Γ–convergence result: we want that the Γ–lim sup–approximation property holds for sequences of a prescribed type of varifolds, for instance for discrete volumetric varifolds. In section 5.4, we answer Question 5.3, giving a construction of a d–varifold $\widehat{V_\varepsilon}$ such that $\delta\left(\widehat{V_\varepsilon}\right) = \delta V * \rho_\varepsilon$.

5.1 Regularization of the first variation and quantitative conditions of rectifiability for sequences of varifolds

Given a sequence of approximating d–varifolds $(V_i)_i$ weakly–$*$ converging to some d–varifold, if we want to ensure that V has locally bounded first variation (i.e. δV is a Radon measure) we can impose that

$$\sup_i \|\delta V_i\| < +\infty \, . \tag{5.7}$$

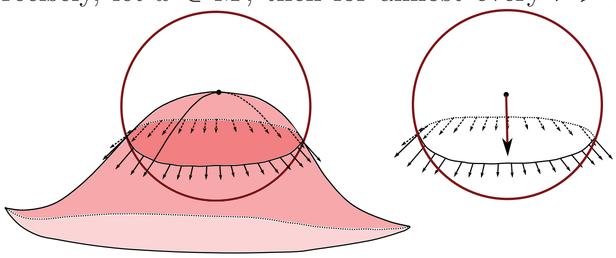
But this condition implies in particular that for fixed i, $\|\delta V_i\|$ is finite, that is V_i has bounded first variation. This is already a strong assumption for discrete volumetric varifolds, as we explained in Example 2.1. As for point cloud varifolds, they do not even have bounded first variation. Moreover, the computation of the first variation of discrete volumetric varifolds in Proposition 2.2 and Example 2.1 shows that condition (5.7) is generally not satisfied by sequences of weakly–$*$ converging discrete volumetric varifolds, even if the limit varifold is smooth. Then, with no additional assumption on V_i, it is not possible to consider the first variation δV_i of V_i as a measure. Nevertheless, it is a distribution of order 1 (Definition 1.11) so that we will rather ask for a control of a regularized form of the first variation of V_i.

5.1.1 Regularization of the first variation by convolution

Let us notice that for a d–rectifiable varifold $V = v(M, \theta)$ with bounded first variation, thanks to Proposition 3.2 in [LM09], the averaged generalized curvature of a ball can be expressed in terms of integrated conormals on the boundary. More precisely, let $x \in M$, then for almost every $r > 0$,

$$\delta V(B_r(x)) = -\int_{\partial B_r(x) \cap M} \eta(y)\theta(y)\, d\mathcal{H}^{d-1}(y) \, ,$$

where $\eta(y) = \dfrac{\Pi_{T_y M}(y-x)}{|\Pi_{T_y M}(y-x)|}$ is the outward conormal vector.



If we now write this relation in an averaged version, we have

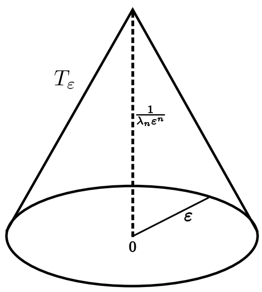
$$\begin{aligned}\frac{1}{R}\int_{r=0}^{R} \delta V(B_r(x))\, dr &= -\frac{1}{R}\int_{B_R(x)\cap M} \frac{\Pi_{T_y M}(y-x)}{|y-x|}\theta(y)\, d\mathcal{H}^d(y) \\ &= -\frac{1}{R}\int_{B_R(x)\times G_{d,n}} \frac{\Pi_S(y-x)}{|y-x|}\, dV(y,S) \, . \end{aligned} \tag{5.8}$$

On one hand, the first term of the previous equality involves δV as a Radon measure and thus can be defined only for a varifold with bounded first variation. On the other hand, the last term in (5.8) can be defined for any d–varifold (for x and R such that $V(\{x\} \cup \partial B_R(x)) = 0$). Consequently, given a d–varifold V, we want to give conditions ensuring that V has bounded first variation by controlling quantities of the type

$$-\frac{1}{R}\int_{B_R(x)\times G_{d,n}} \frac{\Pi_S(y-x)}{|y-x|}\, dV(y,S) \, .$$

A way to do so is to notice that (5.8) is, up to some constant and scale factor, the regularization of the distribution δV with a " tent kernel ". Let us first introduce the <u>tent kernel</u> $(T_\varepsilon)_\varepsilon : \mathbb{R}^n \to \mathbb{R}_+$:

$$\text{Let } T(z) = \begin{cases} \frac{1}{\lambda_n}(1-|z|) & \text{if } |z| \leqslant 1 \\ 0 & \text{otherwise} \end{cases}, \tag{5.9}$$



where λ_n is a constant only depending on the dimension n and such that $\int_{\mathbb{R}^n} T = 1$ (thus $\lambda_n = \int_{|z|\leqslant 1} (1-|z|)\, d\mathcal{L}^n(z)$). We can define (the associated approximate identity) $T_\varepsilon(z) = \frac{1}{\varepsilon^n} T\left(\frac{z}{\varepsilon}\right)$.
Then the regularized first variation with the tent kernel can be written explicitly.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and V be a d–varifold in Ω with finite mass $\|V\|(\Omega)$. Then,*

$$\begin{array}{ccc} \mathrm{C}^1_c(\mathbb{R}^n, \mathbb{R}^n) & \to & \mathbb{R} \\ X & \mapsto & \int_{\Omega \times G_{d,n}} \mathrm{div}_S X(x)\, dV(x,S) \end{array} \tag{5.10}$$

naturally extends δV into a linear continuous functional in $\mathrm{C}^1_c(\mathbb{R}^n, \mathbb{R}^n)$, again denoted as δV. Then $\delta V * T_\varepsilon \in \mathrm{L}^1(\Omega)$ is well defined and for $\mathcal{L}^n$–almost any $x \in \mathbb{R}^n$ we have

$$\delta V * T_\varepsilon(x) = -\frac{1}{\lambda_n \varepsilon^{n+1}} \int_{B_\varepsilon(x) \times G_{d,n}} \frac{\Pi_S(y-x)}{|y-x|}\, dV(y,S)\, .$$

More generally, if $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right)$ with $\rho \in \mathrm{W}^{1,\infty}(\mathbb{R}^n)$ a symmetric positive function such that

$$\rho \geqslant 0, \quad \int \rho = 1 \quad \textit{and} \quad \operatorname{supp} \rho \subset B_1(0)\, ,$$

then for $\mathcal{L}^n$–almost any $x \in \mathbb{R}^n$ we have,

$$\delta V * \rho_\varepsilon(x) = \int_{B_\varepsilon(x) \times G_{d,n}} \nabla^S \rho_\varepsilon(y-x)\, dV(y,S) = \frac{1}{\varepsilon^{n+1}} \int_{B_\varepsilon(x) \times G_{d,n}} \nabla^S \rho\left(\frac{y-x}{\varepsilon}\right) dV(y,S)\, . \tag{5.11}$$

The proof consists in direct computations:

Proof. First of all, notice that $(x,S) \mapsto \mathrm{div}_S X(x)$ is continuous and bounded and V is a finite Radon measure, thus (5.10) is well defined. Moreover,

$$\int_{\Omega \times G_{d,n}} \mathrm{div}_S X(x)\, dV(x,S) \leqslant \|V\|(\Omega)\, \|X\|_{\mathrm{C}^1}$$

leads to the continuity of the map defined in (5.10). δV now denotes this extended linear form (5.10). By definition, for any $X \in \mathrm{C}^1_c(\mathbb{R}^n, \mathbb{R}^n)$,

$$\langle \delta V * \rho_\varepsilon, X \rangle = \langle \delta V, X * \rho_\varepsilon \rangle = \langle V, (y,S) \mapsto \mathrm{div}_S (X * \rho_\varepsilon)(y) \rangle\, .$$

As $\rho_\varepsilon \in \mathrm{W}^{1,\infty}$ then $\mathrm{div}_S(X * \rho_\varepsilon) = X * \nabla^S \rho_\varepsilon$ and thus

$$\begin{aligned} \langle \delta V * \rho_\varepsilon, X \rangle &= \int_{\Omega \times G_{d,n}} (X * \nabla^S \rho_\varepsilon)(y)\, dV(y,S) \\ &= \int_{\Omega \times G_{d,n}} \int_{x \in \mathbb{R}^n} X(x) \nabla^S \rho_\varepsilon(y-x)\, d\mathcal{L}^n(x)\, dV(y,S) \\ &= \int_{x \in \mathbb{R}^n} X(x) \left(\int_{B_\varepsilon(x) \times G_{d,n}} \nabla^S \rho_\varepsilon(y-x)\, dV(y,S) \right) d\mathcal{L}^n(x)\, , \end{aligned} \tag{5.12}$$

which proves the case of a general kernel (5.11). To deduce the case of the tent kernel, we just have to compute $\nabla T_\varepsilon(z) = -\dfrac{1}{\lambda_n \varepsilon^{n+1}} \dfrac{z}{|z|}$. □

Remark 5.1. Of course, notice that $\delta V * \rho_\varepsilon$ is well-defined and is a L^1 function according to the right hand side of (5.12) even though V is not of bounded first variation.

5.1.2 Quantitative conditions for the first variation to be bounded

We fix a symmetric positive function $\rho \in \mathrm{W}^{1,\infty}$ such that

$$\int \rho = 1 \text{ and } \operatorname{supp} \rho \subset B_1(0) \,, \tag{5.13}$$

and we also fix $\rho_\varepsilon(x) = \dfrac{1}{\varepsilon^n} \rho\left(\dfrac{x}{\varepsilon}\right)$ (the associated approximate identity). We now check the connection between $\delta V * \rho_\varepsilon$ and δV.

Proposition 5.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set and V be a d–varifold in Ω with finite mass $\|V\|(\Omega) < +\infty$. Then for any $X \in \mathrm{C}_c^1(\mathbb{R}^n, \mathbb{R}^n)$,*

$$|\langle \delta V * \rho_\varepsilon , X \rangle - \langle \delta V , X \rangle| \leqslant \|V\|(\Omega \cap (\operatorname{supp} X + B_\varepsilon(0))) \, \|\rho_\varepsilon * X - X\|_{\mathrm{C}^1} \,.$$

Therefore for any $X \in \mathrm{C}_c^1(\Omega, \mathbb{R}^n)$, $\langle \delta V * \rho_\varepsilon , X \rangle \xrightarrow[\varepsilon \to 0]{} \langle \delta V , X \rangle$.

Proof. Let $X \in \mathrm{C}_c^1(\mathbb{R}^n, \mathbb{R}^n)$. As $\langle \delta V * \rho_\varepsilon , X \rangle = \langle \delta V , \rho_\varepsilon * X \rangle$ then

$$|\langle \delta V * \rho_\varepsilon , X \rangle - \langle \delta V , X \rangle| = |\langle \delta V , \rho_\varepsilon * X - X \rangle| \leqslant \|V\|(\Omega) \, \|\rho_\varepsilon * X - X\|_{\mathrm{C}^1} \,.$$

In order to complete the proof, we recall the following classical property.

Proposition 5.3. *Let $(\zeta_\varepsilon)_\varepsilon$ be a sequence of positive symmetric functions in $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} \zeta_\varepsilon(x)\,dx = 1$ and $\operatorname{supp} \zeta_\varepsilon \subset B_\varepsilon(0)$. For any function $f \in \mathrm{C}_c^k(\Omega)$, $\zeta_\varepsilon * f \in \mathrm{C}_c^k(\Omega)$ for ε small enough, and*

$$\zeta_\varepsilon * f \xrightarrow[\varepsilon \to 0]{} f \quad \text{in the } \mathrm{C}^k \text{ topology.}$$

Consequently $\|\rho_\varepsilon * X - X\|_{\mathrm{C}^1} \xrightarrow[\varepsilon \to 0]{} 0$. □

Let us now give quantitative conditions for a d–varifold to be of bounded first variation.

Theorem 5.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set and V be a d–varifold in Ω of finite mass $\|V\|(\Omega) < +\infty$. Assume that*

$$\sup_{\varepsilon > 0} \|\delta V * \rho_\varepsilon\|_{\mathrm{L}^1} = \sup_{\varepsilon > 0} \frac{1}{\varepsilon^{n+1}} \int_{x \in \mathbb{R}^n} \left| \int_{B_\varepsilon(x) \times G_{d,n}} \nabla^S \rho \left(\frac{y - x}{\varepsilon} \right) dV(y,S) \right| d\mathcal{L}^n(x) \leqslant C < +\infty \,. \tag{5.14}$$

Then V has bounded first variation and $|\delta V|(\Omega)$ is bounded by the previous supremum.
In the particular case when $\rho_\varepsilon = T_\varepsilon$ is the tent kernel, the assumption (5.2) rewrites

$$\sup_{\varepsilon > 0} \frac{1}{\varepsilon^n} \int_{x \in \mathbb{R}^n} \frac{1}{\varepsilon} \left| \int_{y \in (B_\varepsilon(x) \cap \Omega) \times G_{d,n}} \frac{\Pi_S(y - x)}{|y - x|} \, dV(y,S) \right| d\mathcal{L}^n(x) < +\infty \,.$$

Proof. By Proposition 5.2, for any $X \in \mathrm{C}_c^1(\Omega, \mathbb{R}^n)$,

$$|\langle \delta V , X \rangle| = \lim_{\varepsilon \to 0} |\langle \delta V * \rho_\varepsilon , X \rangle| \leqslant \lim_{\varepsilon \to 0} \|\delta V * \rho_\varepsilon\|_{\mathrm{L}^1} \, \|X\|_\infty \leqslant C \, \|X\|_\infty \,.$$

In other words δV is a linear on $\mathrm{C}_c^1(\Omega, \mathbb{R}^n)$ and continuous for the uniform topology. By density of $\mathrm{C}_c^1(\Omega, \mathbb{R}^n)$ in $\mathrm{C}_c^0(\Omega, \mathbb{R}^n)$, δV extends to a continuous linear form $\delta V : \mathrm{C}_c^0(\Omega, \mathbb{R}^n) \to \mathbb{R}$ with norm $\|\delta V\| = |\delta V|(\Omega) \leqslant C$. □

5.1.3 Consequences on sequences of varifolds

We now infer quantitative conditions for a weak–$*$ limit of d–varifolds to have bounded first variation.

Theorem 5.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $(V_i)_i$ be a sequence of d–varifolds. Assume that there exists a positive decreasing sequence $(\varepsilon_i)_i$, tending to 0, such that*

$$\sup_i \left\{ \|V_i\|(\Omega) + \frac{1}{\varepsilon_i^{n+1}} \int_{x \in \mathbb{R}^n} \left| \int_{y \in B_{\varepsilon_i}(x) \times G_{d,n}} \nabla^S \rho \left(\frac{y-x}{\varepsilon_i} \right) \, dV_i(y,S) \right| \, d\mathcal{L}^n(x) \right\} < +\infty \, . \tag{5.15}$$

Then there exists a subsequence $(V_{\varphi(i)})_i$ weakly–$*$ converging in Ω to a d–varifold V, V has bounded first variation and $|\delta V|(\Omega)$ is bounded by the previous supremum.

Remark 5.2. Of course the condition (5.15) can be written explicitly in the case of the tent kernel, as done in Theorem 5.4.

Proof. As $\sup\limits_i V_i(\Omega \times G_{d,n}) = \sup\limits_i \|V_i\|(\Omega) = C < +\infty$, there exists a subsequence $(V_{\varphi(i)})_i$ weakly–$*$ converging in Ω to a d–varifold V. As previously,

$$\frac{1}{\varepsilon_i^{n+1}} \int_{x \in \mathbb{R}^n} \left| \int_{y \in B_{\varepsilon_i}(x) \times G_{d,n}} \nabla^S \rho \left(\frac{y-x}{\varepsilon_i} \right) \, dV_i(y,S) \right| \, d\mathcal{L}^n(x) = \|\delta V_i * \rho_{\varepsilon_i}\|_{\mathrm{L}^1} \, .$$

Moreover, for any $X \in \mathrm{C}_c^1(\Omega, \mathbb{R}^n)$,

$$\begin{aligned} \left| \langle \delta V_{\varphi(i)} * \rho_{\varepsilon_{\varphi(i)}}, X \rangle - \langle \delta V, X \rangle \right| & \leqslant \left| \langle \delta V_{\varphi(i)} * \rho_{\varepsilon_{\varphi(i)}}, X \rangle - \langle \delta V_{\varphi(i)}, X \rangle \right| + \left| \langle \delta V_{\varphi(i)}, X \rangle - \langle \delta V, X \rangle \right| \\ & \leqslant \underbrace{\|V_i\|(\Omega)}_{\leqslant C < +\infty} \left\| X * \rho_{\varepsilon_{\varphi(i)}} - X \right\|_{\mathrm{C}^1} + \left| \langle \delta V_{\varphi(i)}, X \rangle - \langle \delta V, X \rangle \right| \\ & \xrightarrow[i \to \infty]{} 0 \text{ by Propositions 5.2 and 5.3} \, . \end{aligned}$$

Consequently, for any $X \in \mathrm{C}_c^1(\Omega, \mathbb{R}^n)$, $|\langle \delta V, X \rangle| \leqslant \sup\limits_i \|\delta V_i * \rho_{\varepsilon_i}\|_{\mathrm{L}^1} \, \|X\|_\infty$ and we conclude that δV extends into a continuous linear form in $\mathrm{C}_c^0(\Omega, \mathbb{R}^n)$ whose norm is bounded by $\sup_i \|\delta V_i * \rho_{\varepsilon_i}\|_{\mathrm{L}^1}$. □

Remark 5.3. This can be seen as a variant of Allard's compactness theorem (Theorem 1.13) but the varifolds V_i are not supposed to be rectifiable nor of bounded first variation, and moreover, we have no information about their multiplicity.

Remark 5.4. It is possible to obtain the rectifiability of the d–varifold V assuming the same uniform lower bound on the density as in (1.12) in Theorem 3.4 in Chapter 3.

5.2 A Willmore type energy in the set of d–varifolds

5.2.1 Approximate mean curvature

Let us first point out the simple following fact.

Proposition 5.6. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $V = v(M, \theta)$ be a rectifiable d–varifold with bounded first variation $\delta V = -H \, \|V\| + \delta V_s$ and assume that ρ is radial. Then, for $\|V\|$–almost any $x \in \Omega$,*

$$H_\varepsilon(x) = - \frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)} \xrightarrow[\varepsilon \to 0]{} H(x) \, .$$

Proof. For $x \in \Omega$,

$$\begin{aligned}\left| -\frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)} - H(x) \right| &= \frac{1}{\|V\| * \rho_\varepsilon(x)} \left| (-H\,\|V\| + \delta V_s) * \rho_\varepsilon(x) + H(x)\,(\|V\| * \rho_\varepsilon(x)) \right| \\ &\leqslant \frac{1}{\|V\| * \rho_\varepsilon(x)} \left| (-H\,\|V\|) * \rho_\varepsilon(x) + H(x)\,(\|V\| * \rho_\varepsilon(x)) \right| + \frac{|\delta V_s * \rho_\varepsilon(x)|}{\|V\| * \rho_\varepsilon(x)} \\ &\leqslant \frac{1}{\|V\| * \rho_\varepsilon(x)} \int_{y \in \mathbb{R}^n} |H(x) - H(y)|\, \rho_\varepsilon(x-y)\, d\|V\|(y) + \frac{|\delta V_s| * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)}\end{aligned}$$

And for $\|V\|$–almost every x, by definition of the approximate tangent plane and since ρ is continuous and radial,

$$\varepsilon^{n-d} \|V\| * \rho_\varepsilon(x) = \frac{1}{\varepsilon^d} \int_\Omega \rho\left(\frac{y-x}{\varepsilon}\right) d\|V\|(y) \xrightarrow[\varepsilon \to 0]{} \theta(x) \int_{T_x M} \rho(y)\, d\mathcal{H}^d(y) = C_\rho \theta(x) > 0\,.$$

Then, for $\|V\|$–almost any $x \in \Omega$ (i.e., at any Lebesgue point x of $H \in \mathrm{L}^1(\|V\|)$),

$$\begin{aligned}&\frac{1}{\|V\| * \rho_\varepsilon(x)} \int_{y \in \mathbb{R}^n} |H(x) - H(y)|\, \rho_\varepsilon(x-y)\, d\|V\|(y) \\ &\leqslant \frac{\|V\|(B_\varepsilon(x))}{\|V\| * \rho_\varepsilon(x)} \frac{1}{\|V\|(B_\varepsilon(x))} \frac{\|\rho\|_\infty}{\varepsilon^n} \int_{y \in \mathbb{R}^n} |H(x) - H(y)|\; d\|V\|(y) \\ &\leqslant \|\rho\|_\infty \underbrace{\frac{\varepsilon^{-d}\|V\|(B_\varepsilon(x))}{\varepsilon^{n-d}\|V\| * \rho_\varepsilon(x)}}_{\xrightarrow[\varepsilon \to 0]{} \frac{\theta(x)}{C_\rho \theta(x)}} \underbrace{\int_{y \in \mathbb{R}^n} |H(x) - H(y)|\; d\|V\|(y)}_{\xrightarrow[\varepsilon \to 0]{} 0} \\ &\xrightarrow[\varepsilon \to 0]{} 0\,.\end{aligned}$$

And similarly, for $\|V\|$–almost every x,

$$\frac{|\delta V_s| * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)} \leqslant \|\rho\|_\infty \underbrace{\frac{\varepsilon^{-d}\|V\|(B_\varepsilon(x))}{\varepsilon^{n-d}\|V\| * \rho_\varepsilon(x)}}_{\xrightarrow[\varepsilon \to 0]{} \frac{\theta(x)}{C_\rho \theta(x)}} \underbrace{\frac{|\delta V_s|(B_\varepsilon(x))}{\|V\|(B_\varepsilon(x))}}_{\xrightarrow[\varepsilon \to 0]{} 0} \xrightarrow[\varepsilon \to 0]{} 0\,.$$

□

Remark 5.5. The assumption for ρ to be radial can be weakened by requiring that

$$\int_P \rho(y)\, d\mathcal{H}^d(y) > 0, \quad \forall P \in G_{d,n}\,.$$

Let us study the quantities $\delta V * \rho_\varepsilon$ and $H_\varepsilon = -\dfrac{\delta V * \rho_\varepsilon}{\|V\| * \rho_\varepsilon}$ on some examples.

Example 5.1 (Regularization of the first variation of a circle). Let $V = v(\mathcal{C}, 1)$ be the rectifiable 1–varifold associated with a circle of radius 0.5. Then the mean curvature $H = -2\,n$ where n is the unit outward normal vector and, moreover

$$\delta V = -H\,\|V\| = 2\,n\,\|V\| \quad \text{and} \quad H_\varepsilon(x) = -\frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)} = -2 \frac{\int_{B_\varepsilon(x)} \rho_\varepsilon(x-y) n(y)\, d\|V\|(y)}{\int_{B_\varepsilon(x)} \rho_\varepsilon(x-y)\, d\|V\|(y)}\,.$$

Example 5.2 (Regularization of the first variation of a segment). Let $V = v([a,b],1)$ be the rectifiable 1–varifold associated with the segment $[a,b]$. Then, with $u = \frac{a-b}{|a-b|}$,

$$\delta V = u\,\delta_a - u\,\delta_b \,.$$

And thus,

$$\delta V * \rho_\varepsilon(x) = \int_{B_\varepsilon(x)} \rho_\varepsilon(y-x) u \, d\delta_a(y) - \int_{B_\varepsilon(x)} \rho_\varepsilon(y-x) u \, d\delta_b(y) = \rho_\varepsilon(a-x)\, u - \rho_\varepsilon(b-x)\, u \,.$$

Example 5.3 (Regularization of the first variation of a point cloud). Let $V = \sum_{j=1}^{N} m_j \delta_{x_j} \otimes \delta_{P_j}$ the varifold associated with a weighted point cloud. The first variation of V is not a Radon measure so that we need (5.11) to compute

$$\delta V * \rho_\varepsilon(x) = \int_{B_\varepsilon(x)} \nabla^S \rho_\varepsilon(y-x) \, dV(y,S) = \sum_{x_j \in B_\varepsilon} m_j \nabla^{P_j} \rho_\varepsilon(x_j - x) \,,$$

and

$$\frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)} = \frac{1}{\varepsilon} \frac{\sum_{x_j \in B_\varepsilon} m_j \nabla^{P_j} \rho(\frac{x_j - x}{\varepsilon})}{\sum_{x_j \in B_\varepsilon} m_j \rho(\frac{x_j - x}{\varepsilon})} \,.$$

In particular, if ρ is the tent kernel,

$$\frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)} = \frac{1}{\varepsilon} \frac{\sum_{x_j \in B_\varepsilon} m_j \Pi_{P_j} \frac{x_j - x}{|x_j - x|}}{\sum_{x_j \in B_\varepsilon} m_j \left(1 - \frac{|x_j - x|}{\varepsilon}\right)} \,.$$

Let us notice that the choice of the size ε is part of the problem. It is reasonable to expect that several points contribute to the regularized curvature at a given point. If not, the regularization of the first variation explodes at each point of the cloud since we look at them separately.

5.2.2 Approximate Willmore energies

We now build approximate Willmore energies in the space of varifolds and we study their Γ–convergence to the Willmore energy. Let us recall the definition of the p–Willmore energy in the space of d–varifolds:

Definition 5.1. *If V is a d–varifold with weak mean curvature in L^p, that is, V has bounded first variation δV and $\delta V = -H_V \|V\|$ with $H_V \in \mathrm{L}^p(\|V\|)$, then*

$$\mathcal{W}^p(V) = \int_\Omega |H_V|^p \, d\|V\| \,,$$

otherwise $\mathcal{W}^p(V) = +\infty$.

We define approximate Willmore energies associated to the kernel ρ. Notice that in the case where $\rho = T$ is the tent kernel, the following approximate energies rewrite in an explicit and simple way.

Definition 5.2 (Approximate Willmore energies)**.** *Let $p \geqslant 1$ and $\varepsilon > 0$. Let $\Omega \subset \mathbb{R}^n$ be an open set. For any d–varifold V in Ω, we define*

$$\mathcal{W}^p_\varepsilon(V) = \int_{x \in \mathbb{R}^n} \left| \frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)} \right|^p \|V\| * \rho_\varepsilon(x) \, d\mathcal{L}^n(x) \,.$$

Remark 5.6. The approximate Willmore energies depend on the chosen kernel ρ even though this dependence is not explicitly written. To be more precise, we may denote them as $\mathcal{W}^p_{\rho,\varepsilon}$ but we prefer avoiding too complicated notations.

Given a weakly–$*$ converging sequence of d–varifolds $(V_\varepsilon)_\varepsilon$, we now study the convergence of the regularized first variation $\delta V_\varepsilon * \rho_\varepsilon$ and regularized mass $\|V_\varepsilon\| * \rho_\varepsilon$, as $\varepsilon \downarrow 0$.

Proposition 5.7. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $(V_\varepsilon)_\varepsilon$ be a sequence of d–varifolds weakly–$*$ converging to a d–varifold V in Ω with finite mass $\|V\|(\Omega) < +\infty$. Then,*

(i) The sequence of measures $\mu_\varepsilon = (\|V_\varepsilon\| * \rho_\varepsilon)\, \mathcal{L}^n$ weakly–$*$ converges to the measure $\|V\|$.

(ii) If $\sup_{\varepsilon > 0} \|\delta V_\varepsilon * \rho_\varepsilon\|_{\mathrm{L}^1} \leqslant C < +\infty$ then $(\delta V_\varepsilon * \rho_\varepsilon)\, \mathcal{L}^n \xrightarrow{*} \delta V$.

Proof. (i) Let $\varphi \in \mathrm{C}^0_c(\Omega)$,

$$|\langle \|V_\varepsilon\| * \rho_\varepsilon, \varphi\rangle - \langle \|V\|, \varphi\rangle| \leqslant |\langle \|V_\varepsilon\| * \rho_\varepsilon, \varphi\rangle - \langle \|V_\varepsilon\|, \varphi\rangle| + \underbrace{|\langle \|V_\varepsilon\|, \varphi\rangle - \langle \|V\|, \varphi\rangle|}_{\xrightarrow[\varepsilon\to 0]{} 0}, \tag{5.16}$$

and

$$|\langle \|V_\varepsilon\| * \rho_\varepsilon, \varphi\rangle - \langle \|V_\varepsilon\|, \varphi\rangle| \leqslant \|V_\varepsilon\| \left(\operatorname{supp}\varphi + \overline{B}_\varepsilon(0)\right) \underbrace{\|\varphi * \rho_\varepsilon - \varphi\|_\infty}_{\xrightarrow[\varepsilon\to 0]{} 0} . \tag{5.17}$$

Moreover, there exists $\varepsilon_0 > 0$ small enough such that the compact set $\left(\operatorname{supp}\varphi + \overline{B}_{\varepsilon_0}(0)\right) \subset \Omega$ so that

$$\begin{aligned}\limsup_{\varepsilon\to 0} \|V_\varepsilon\| \left(\operatorname{supp}\varphi + \overline{B}_\varepsilon(0)\right) &\leqslant \limsup_{\varepsilon\to 0} \|V_\varepsilon\| \left(\operatorname{supp}\varphi + \overline{B}_{\varepsilon_0}(0)\right) \\ &\leqslant \|V\| \left(\operatorname{supp}\varphi + \overline{B}_{\varepsilon_0}(0)\right) \leqslant \|V\|(\Omega) \ ,\end{aligned} \tag{5.18}$$

and (i) follows from (5.16), (5.17) and (5.18).

(ii) Let us prove the second assertion (ii). Thanks to the assumption $\sup_\varepsilon \|\delta V_\varepsilon * \rho_\varepsilon\|_{\mathrm{L}^1} < +\infty$ and Theorem 5.5, V has bounded first variation and $\|\delta V\| \leqslant \sup_\varepsilon \|\delta V_\varepsilon * \rho_\varepsilon\|_{\mathrm{L}^1}$. Consequently δV is a Radon measure (and thus applies to continuous compactly supported vector fields). Let $X \in \mathrm{C}^0_c(\Omega, \mathbb{R}^n)$ and $X_k \in \mathrm{C}^1_c(\Omega, \mathbb{R}^n)$ such that $\|X_k - X\|_\infty \xrightarrow[k\to+\infty]{} 0$.

$$\begin{aligned}|\langle \delta V_\varepsilon * \rho_\varepsilon, X\rangle - \langle \delta V, X\rangle| &\leqslant |\langle \delta V_\varepsilon * \rho_\varepsilon, X\rangle - \langle \delta V_\varepsilon * \rho_\varepsilon, X_k\rangle| + |\langle \delta V_\varepsilon * \rho_\varepsilon, X_k\rangle - \langle \delta V, X_k\rangle| \\ &\quad + |\langle \delta V, X_k\rangle - \langle \delta V, X\rangle| \\ &\leqslant \|\delta V_\varepsilon * \rho_\varepsilon\|_{\mathrm{L}^1} \|X - X_k\|_\infty + |\langle \delta V_\varepsilon * \rho_\varepsilon, X_k\rangle - \langle \delta V_\varepsilon, X_k\rangle| \\ &\quad + |\langle \delta V_\varepsilon, X_k\rangle - \langle \delta V, X_k\rangle| + \|\delta V\| \|X_k - X\|_\infty \\ &\leqslant 2C\|X_k - X\|_\infty + |\langle \delta V_\varepsilon, X_k * \rho_\varepsilon - X_k\rangle| + |\langle \delta V_\varepsilon, X_k\rangle - \langle \delta V, X_k\rangle| \ .\end{aligned}$$

And for fixed k,

– $|\langle \delta V_\varepsilon, X_k\rangle - \langle \delta V, X_k\rangle| \xrightarrow[\varepsilon\to 0]{} 0$,

– $|\langle \delta V_\varepsilon, X_k * \rho_\varepsilon - X_k\rangle| \leqslant \|V_\varepsilon\| \left(\operatorname{supp} X_k + \overline{B}_\varepsilon(0)\right) \underbrace{\|X_k * \rho_\varepsilon - X_k\|_{\mathrm{C}^1}}_{\xrightarrow[\varepsilon\to 0]{} 0}$ with

$$\limsup_{\varepsilon\to 0} \|V_\varepsilon\| \left(\operatorname{supp} X_k + \overline{B}_\varepsilon(0)\right) \leqslant \|V\|(\Omega) \quad \text{as in (5.18)} .$$

Therefore $\delta V_\varepsilon * \rho_\varepsilon \xrightarrow[\varepsilon\to 0]{*} \delta V$.

□

We now want to study the Γ–convergence of $\mathcal{W}^p_\varepsilon$. We begin with $p = 1$.

5.2.3 The Γ–limit of the approximate Willmore energies $\mathcal{W}_\varepsilon^1$

We now study the Γ–convergence of the approximate Willmore energies $\mathcal{W}_\varepsilon^1$ (for $p=1$). We prove that $\mathcal{W}_\varepsilon^1$ Γ–converges to the total variation of the first variation $\|\delta\cdot\|$:

$$\|\delta V\| = \sup\left\{ \langle \delta V, X\rangle \,:\, X\in \mathrm{C}_c^1(\Omega,\mathbb{R}^n),\ \|X\|_\infty \leqslant 1\right\} = |\delta V|(\Omega)\,. \tag{5.19}$$

Theorem 5.8. *Let $\Omega\subset\mathbb{R}^n$ be an open set and ρ defined as in* (5.13)*. The approximate Willmore energies $\mathcal{W}_\varepsilon^1$ associated to ρ in Ω Γ–converge to $\|\delta\cdot\|$ defined above in* (5.19)*. That is to say:*

(i) *(Γ–liminf) For any sequence of d–varifolds $(V_\varepsilon)_\varepsilon$, such that $V_\varepsilon \xrightarrow[\varepsilon\to 0]{*} V$,*

$$\|\delta V\| = \|\delta V\| \leqslant \liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon^1(V_\varepsilon)\,.$$

(ii) *(Γ–limsup) For any d–varifold V, there exists a sequence of d–varifolds $(V_\varepsilon)_\varepsilon$ weakly–$*$ converging to V and such that*

$$\limsup_{\varepsilon\to 0} \mathcal{W}_\varepsilon^1(V_\varepsilon) \leqslant \|\delta V\|\,.$$

Moreover, for all $\varepsilon>0$, $\mathcal{W}_\varepsilon^1(V)\leqslant\|\delta V\|$.

Remark 5.7. In particular, for any d–varifold V, $\mathcal{W}_\varepsilon^1(V) \xrightarrow[\varepsilon\to 0]{} \|\delta V\|$.

Proof. We first prove the Γ–lim inf result. Let $(V_\varepsilon)_\varepsilon$ be a sequence of d–varifolds in Ω weakly–$*$ converging to a d–varifold V. If $\liminf_{\varepsilon\to 0}\mathcal{W}_\varepsilon^1(V_\varepsilon)=+\infty$, there is nothing to prove. We can thus assume that

$$\liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon^1(V_\varepsilon) < +\infty\,,$$

and choosing some subsequence $(\varepsilon_i)_i$ such that

$$\liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon^1(V_\varepsilon) = \lim_{i\to+\infty} \mathcal{W}_{\varepsilon_i}^1(V_{\varepsilon_i})\,, \tag{5.20}$$

we have that $\sup_i \|\delta V_{\varepsilon_i} * \rho_{\varepsilon_i}\|_{\mathrm{L}^1} = \sup_i \mathcal{W}_{\varepsilon_i}^1(V_{\varepsilon_i}) < +\infty$. By Proposition 5.7, V has bounded first variation and

$$\delta V_{\varepsilon_i} * \rho_{\varepsilon_i} \xrightarrow[i\to+\infty]{*} \delta V\,.$$

Consequently,

$$\|\delta V\| \leqslant \liminf_{i\to+\infty} |(\delta V_{\varepsilon_i} * \rho_{\varepsilon_i})\,\mathcal{L}^n|\,(\Omega) = \liminf_{i\to+\infty} \mathcal{W}_{\varepsilon_i}^1\,(V_{\varepsilon_i}) = \liminf_{\varepsilon\to 0}\mathcal{W}_\varepsilon^1(V_\varepsilon) \text{ by (5.20)}\,.$$

We now prove the Γ–lim sup result. Let V be a d–varifold in Ω. If V does not have bounded first variation, then $\|\delta V\|=+\infty$ and for all $\varepsilon>0$, $\mathcal{W}_\varepsilon^1(V)\leqslant\|\delta V\|$. Assume now that V has bounded first variation, then δV is a Radon measure and for all $\varepsilon>0$,

$$\begin{aligned}
\mathcal{W}_\varepsilon^1(V) = \|\delta V * \rho_\varepsilon\|_{\mathrm{L}^1} &= \int_{x\in\mathbb{R}^n} \left| \int_{y\in\Omega} \rho_\varepsilon(x-y)\, d\delta V(y)\right| d\mathcal{L}^n(x) \\
&\leqslant \int_{x\in\mathbb{R}^n}\int_{y\in\Omega} |\rho_\varepsilon(x-y)|\; d|\delta V|(y)\, d\mathcal{L}^n(x) \\
&\leqslant \int_{y\in\Omega} \underbrace{\int_{x\in\mathbb{R}^n} \rho_\varepsilon(x-y)\, d\mathcal{L}^n(x)}_{\leqslant 1}\; d|\delta V|(y) \\
&\leqslant |\delta V|(\Omega) = \|\delta V\|\,.
\end{aligned}$$

□

Remark 5.8. The approximate Willmore energy $\mathcal{W}_\varepsilon^1$ does not Γ–converge to the Willmore energy $\mathcal{W}^1$, but we now prove that $\mathcal{W}_\varepsilon^p$ Γ–converges to the Willmore energy $\mathcal{W}^p$ as soon as $p>1$.

5.2.4 The Γ–limit of the approximate Willmore energies $\mathcal{W}_\varepsilon^p$ for $p > 1$

We now prove that for $p > 1$, the approximate Willmore energies $\mathcal{W}_\varepsilon^p$ Γ–converge to the classical Willmore energy $\mathcal{W}^p$ in the space of d–varifolds. We first check that a control on $\mathcal{W}_\varepsilon^p(V)$ gives a control on $\mathcal{W}_\varepsilon^1(V)$ and thus on $\|\delta V_\varepsilon * \rho_\varepsilon\|_{\mathrm{L}^1}$.

Proposition 5.9. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $p > 1$ and $\varepsilon > 0$, for any d–varifold V in Ω,*

$$\mathcal{W}_\varepsilon^1(V) \leqslant \|V\|(\Omega)^{\frac{p-1}{p}} \, \mathcal{W}_\varepsilon^p(V)^{\frac{1}{p}} \, .$$

Proof. Let V be a d–varifold in Ω, then

$$\mathcal{W}_\varepsilon^1(V) = \int_{x \in \mathbb{R}^n} |\delta V * \rho_\varepsilon(x)| \, d\mathcal{L}^n(x) = \int_{x \in \mathbb{R}^n} |H_\varepsilon(x)| \, d\mu_\varepsilon(x) \, ,$$

with

$$H_\varepsilon(x) = -\frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon} \text{ and } \mu_\varepsilon = (\|V\| * \rho_\varepsilon) \, \mathcal{L}^n \, .$$

Moreover,

$$\begin{aligned} \mu_\varepsilon(\Omega) &= \int_{\mathbb{R}^n} \|V\| * \rho_\varepsilon(x) \, d\mathcal{L}^n(x) = \int_{x \in \mathbb{R}^n} \int_{y \in \Omega} \rho_\varepsilon(y-x) \, d\|V\|(y) \, d\mathcal{L}^n(x) \\ &= \int_{y \in \Omega} \underbrace{\int_{x \in \mathbb{R}^n} \rho_\varepsilon(y-x) \, d\mathcal{L}^n(x)}_{=1} \, d\|V\|(y) = \|V\|(\Omega) \, . \end{aligned}$$

Consequently, $\dfrac{\mu_\varepsilon}{\|V\|(\Omega)}$ is a probability measure and thanks to Jensen inequality,

$$\begin{aligned} \mathcal{W}_\varepsilon^1(V)^p &= \left[\|V\|(\Omega) \int_{\mathbb{R}^n} |H_\varepsilon(x)| \, \frac{d\mu_\varepsilon(x)}{\|V\|(\Omega)} \right]^p \leqslant \|V\|(\Omega)^p \int_{x \in \mathbb{R}^n} |H_\varepsilon(x)|^p \, \frac{d\mu_\varepsilon(x)}{\|V\|(\Omega)} \\ &\leqslant \|V\|(\Omega)^{p-1} \mathcal{W}_\varepsilon^p(V) \, . \end{aligned}$$

□

We can now state and prove the Γ–convergence of the approximate Willmore energies $\mathcal{W}_\varepsilon^p$ to the Willmore energy $\mathcal{W}^p$ for $p > 1$.

Theorem 5.10. *Let $\Omega \subset \mathbb{R}^n$ be an open set and take ρ defined as in* (5.13)*. For $p > 1$, the approximate Willmore energies $\mathcal{W}_\varepsilon^p$ associated with ρ in Ω Γ–converge to the Willmore energy $\mathcal{W}^p$:*

(i) *(Γ–liminf) For any sequence of d–varifolds $(V_\varepsilon)_\varepsilon$, such that $V_\varepsilon \xrightarrow[\varepsilon \to 0]{*} V$,*

$$\mathcal{W}^p(V) \leqslant \liminf_{\varepsilon \to 0} \mathcal{W}_\varepsilon^p(V_\varepsilon) \, ;$$

(ii) *(Pointwise convergence) For any d–varifold V,*

$$\mathcal{W}_\varepsilon^p(V) \xrightarrow[\varepsilon \to 0]{} \mathcal{W}^p(V) \, ;$$

(iii) *(Γ–limsup) In particular, for any d–varifold V, there exists a sequence $(V_\varepsilon)_\varepsilon \equiv V$ weakly–$*$ converging to V and such that*

$$\limsup_{\varepsilon \to 0} \mathcal{W}_\varepsilon^p(V_\varepsilon) \leqslant \mathcal{W}^p(V) \, .$$

Moreover, for all $\varepsilon > 0$, $\mathcal{W}_\varepsilon^p(V) \leqslant \mathcal{W}^p(V)$.

Proof. Let us begin with the Γ–lim inf assertion. Assume that $\liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon^p(V_\varepsilon) < +\infty$ (otherwise the inequality is trivial). We can extract a subsequence $(\varepsilon_i)_i$ such that $\liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon^p(V_\varepsilon) = \lim_{i\to+\infty} \mathcal{W}_{\varepsilon_i}^p(V_{\varepsilon_i})$ therefore $\sup_i \mathcal{W}_{\varepsilon_i}^p(V_{\varepsilon_i}) < +\infty$. Thanks to Proposition 5.9, for all $\varepsilon > 0$,

$$\mathcal{W}_\varepsilon^1(V_\varepsilon) \leqslant \|V\|(\Omega)^{\frac{p-1}{p}} \, \mathcal{W}_\varepsilon^p(V_\varepsilon)^{\frac{1}{p}} \, ,$$

and thus $\sup_i \mathcal{W}_{\varepsilon_i}^1(V_{\varepsilon_i}) < +\infty$ so that by Proposition 5.7

$$(\delta V_{\varepsilon_i} * \rho_{\varepsilon_i}) \, \mathcal{L}^n \xrightarrow[i\to+\infty]{*} \delta V \, .$$

Moreover (again thanks to Proposition 5.7),

$$\mu_\varepsilon = (\|V_\varepsilon\| * \rho_\varepsilon) \, \mathcal{L}^n \xrightarrow[\varepsilon\to 0]{*} \|V\| \, .$$

Let us write $(\delta V_\varepsilon * \rho_\varepsilon) \, \mathcal{L}^n = H_\varepsilon \, \mu_\varepsilon$ with $H_\varepsilon = \dfrac{\delta V_\varepsilon * \rho_\varepsilon}{\|V_\varepsilon\| * \rho_\varepsilon}$, then for all i, $(\delta V_{\varepsilon_i} * \rho_{\varepsilon_i}) \, \mathcal{L}^n << \mu_{\varepsilon_i}$ and we can apply Example 2.36 of [AFP] to conclude that $\delta V << \|V\|$ and

$$\int_{\mathbb{R}^n} \left| \frac{\delta V}{\|V\|} \right|^p d\|V\| = \mathcal{W}^p(V) \leqslant \liminf_{i\to+\infty} \int_{\mathbb{R}^n} |H_{\varepsilon_i}|^p \, d\mu_{\varepsilon_i} = \lim_{i\to+\infty} \mathcal{W}_{\varepsilon_i}^p(V_{\varepsilon_i}) = \liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon^p(V_\varepsilon) \, .$$

Let us now prove the Γ–lim sup assertion. Let V be d–varifold in Ω and assume that V has mean curvature in L^p, otherwise $\mathcal{W}^p(V) = +\infty$ and there is nothing to prove. Consequently, V has bounded first variation and moreover $\delta V = -H \, \|V\|$ with $H \in (\mathrm{L}^p(\Omega, \|V\|))^n$. We now show that for all $\varepsilon > 0$,

$$\mathcal{W}_\varepsilon^p(V) \leqslant \mathcal{W}^p(V) \, .$$

Indeed,

$$\mathcal{W}_\varepsilon^p(V) = \int_{x\in\mathbb{R}^n} \left| \frac{(H\|V\|) * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)} \right|^p (\|V\| * \rho_\varepsilon(x)) \, d\mathcal{L}^n(x) \, ,$$

and

$$(H\|V\|) * \rho_\varepsilon(x) = \int_{y\in\Omega} H(y) \rho_\varepsilon(x-y) \, d\|V\|(y) \, .$$

Consequently, for a fixed $x \in \Omega$, consider the measure $\nu_x(y) = \rho_\varepsilon(x-y) \, \|V\|(y)$ in Ω. Then, $\nu_x(\Omega) = \displaystyle\int_{y\in\Omega} \rho_\varepsilon(x-y) \, d\|V\|(y) = \|V\| * \rho_\varepsilon(x)$ so that $\dfrac{\nu_x}{\|V\| * \rho_\varepsilon(x)}$ is a probability measure. Therefore, we can apply Jensen inequality to obtain

$$\left| \frac{\int_{y\in\Omega} H(y)\rho_\varepsilon(x-y) \, d\|V\|(y)}{\|V\| * \rho_\varepsilon(x)} \right|^p \leqslant \frac{\int_{y\in\Omega} |H(y)|^p \rho_\varepsilon(x-y) \, d\|V\|(y)}{\|V\| * \rho_\varepsilon(x)} \, .$$

Thus,

$$\begin{aligned} \mathcal{W}_\varepsilon^p(V) &\leqslant \int_{x\in\mathbb{R}^n} \int_{y\in\Omega} |H(y)|^p \rho_\varepsilon(x-y) \, d\|V\|(y) \, d\mathcal{L}^n(x) \\ &\leqslant \int_{y\in\Omega} |H(y)|^p \int_{x\in\mathbb{R}^n} \rho_\varepsilon(x-y) \, d\mathcal{L}^n(x) \, d\|V\|(y) \leqslant \int_{y\in\Omega} |H(y)|^p \, d\|V\|(y) \\ &\leqslant \mathcal{W}^p(V) \, . \end{aligned}$$

□

To summarize, we have determined the Γ–limit of the approximate Willmore energies $\mathcal{W}_\varepsilon^p$ (introduced in Definition 5.2) in the space of d–varifolds. But if we now want to approximate the Willmore energy in some smaller class of varifolds (think of discrete volumetric varifolds, point cloud varifolds etc.), then the Γ–convergence must be studied in this class. In a practical way, this means that the Γ–lim sup must be obtained for a sequence of varifolds belonging to the prescribed class and thus, pointwise convergence of $\mathcal{W}_\varepsilon^p$ is not enough.

5.3 Γ–convergence of the approximate Willmore energies in different approximation spaces

We are now concerned with Question 5.2. As we said, if we want to study the Γ–convergence of $\mathcal{W}_\varepsilon^p$ in different approximation spaces, the Γ–lim inf property remains valid, but the Γ–lim sup property must be checked. We state a result for approximation spaces of discrete volumetric varifolds in the case of $\mathcal{W}_\varepsilon^1$. We study the case of $\mathcal{W}_\varepsilon^p$, $p > 1$, but without positive or negative answer concerning the Γ–convergence.

5.3.1 A Γ–convergence result in different approximation spaces for $\mathcal{W}_\varepsilon^1$

We now study the Γ–lim sup property for $\mathcal{W}_\varepsilon^1$ in different approximation spaces. We begin with a general result: Given $\varepsilon_i \downarrow 0$, from any weakly–∗ converging sequence of d–varifolds $V_i \xrightarrow[i\to+\infty]{*} V$, it is possible (stated in Proposition 5.11) to extract a subsequence $(V_{\varphi(i)})_i$ such that

$$\mathcal{W}_{\varepsilon_i}^1(V_{\varphi(i)}) \xrightarrow[i\to+\infty]{} \|\delta V\| \, .$$

Proposition 5.11. *Let $\Omega \subset \mathbb{R}^n$ be an open set with $|\Omega| < +\infty$ and let V be a d–varifold in Ω with finite mass $\|V\|(\Omega)$ and bounded first variation. Let $(\varepsilon_i)_i \downarrow 0$ be a positive and infinitesimal sequence. For any sequence $(V_i)_i$ weakly–∗ converging to V and such that $\sup_i \|V_i\|(\Omega) < +\infty$, there exists an extracted sequence $(V_{\varphi(i)})_i$ such that*

$$\limsup_i \mathcal{W}_{\varepsilon_i}^1(V_{\varphi(i)}) = \|\delta V\| \, .$$

Proof. First recall that,

$$\mathcal{W}_\varepsilon^1(V) \xrightarrow[\varepsilon\to0]{} \|\delta V\| \, .$$

Fix now $\varepsilon > 0$, then for $\mathcal{L}^n$–almost any x,

$$\delta V_i * \rho_\varepsilon(x) \xrightarrow[i\to+\infty]{} \delta V * \rho_\varepsilon(x) \, .$$

Consequently,

$$\begin{aligned} \left|\mathcal{W}_\varepsilon^1(V_i) - \mathcal{W}_\varepsilon^1(V)\right| &= \left| \int_{x\in\mathbb{R}^n} |\delta V_i * \rho_\varepsilon(x)| \, d\mathcal{L}^n(x) - \int_{x\in\mathbb{R}^n} |\delta V * \rho_\varepsilon(x)| \, d\mathcal{L}^n(x) \right| \\ &\leqslant \int_{x\in\mathbb{R}^n} |\delta V_i * \rho_\varepsilon(x) - \delta V * \rho_\varepsilon(x)| \, d\mathcal{L}^n(x) \, . \end{aligned} \tag{5.21}$$

Moreover, for $\mathcal{L}^n$–almost any x,

$$\begin{aligned} |\delta V_i * \rho_\varepsilon(x) - \delta V * \rho_\varepsilon(x)| &\leqslant \|\nabla \rho_\varepsilon\|_\infty \left(\|V_i\|(B_\varepsilon(x)) + \|V\|(B_\varepsilon(x)) \right) \\ &\leqslant \frac{1}{\varepsilon^n} \|\nabla \rho\|_\infty \left(\|V_i\|(\Omega) + \|V\|(\Omega) \right) \leqslant C < +\infty \end{aligned}$$

As $|\Omega| < +\infty$, $\left|\mathcal{W}_\varepsilon^1(V_i) - \mathcal{W}_\varepsilon^1(V)\right| \xrightarrow[i\to+\infty]{} 0$ by dominated convergence.
Consequently, fixing $\varepsilon_i \downarrow 0$, there exists an extracted sequence $(V_{\varphi(i)})_i$ such that

$$\begin{aligned} \left|\mathcal{W}_{\varepsilon_i}^1(V_{\varphi(i)}) - \|\delta V\|\right| &\leqslant \left|\mathcal{W}_{\varepsilon_i}^1(V_{\varphi(i)}) - \mathcal{W}_{\varepsilon_i}^1(V)\right| + \left|\mathcal{W}_{\varepsilon_i}^1(V) - \|\delta V\|\right| \\ &\xrightarrow[i\to+\infty]{} 0 \end{aligned}$$

□

Remark 5.9. The assumption $|\Omega| < +\infty$ is not necessary if we assume that there exists some constant $\delta > 0$ such that for any ball $B_r(x)$,

$$\|V_i\|(B_r(x)) \leqslant \|V\|(B_{r+\delta}(x)) \, .$$

Indeed, in this case we have the following domination: for all i and for $\mathcal{L}^n$–almost any x,

$$|\delta V_i * \rho_\varepsilon(x) - \delta V * \rho_\varepsilon(x)| \leqslant 2\frac{1}{\varepsilon^n} \|\nabla\rho\|_\infty \|V\|(B_{\varepsilon+\delta}(x)) \, ,$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} \|V\|(B_{\varepsilon+\delta}(x)) \, d\mathcal{L}^n(x) = \int_{y\in\Omega}\int_{x\in\mathbb{R}^n} \mathbb{1}_{\{|y-x|<\varepsilon+\delta\}}(x,y) \, d\mathcal{L}^n(x) \, d\|V\|(y) = \int_\Omega \mathcal{L}^n(B_{\varepsilon+\delta}(y)) \, dV(y) \\ \leqslant C\,(\varepsilon+\delta)^n \|V\|(\Omega) < +\infty \, .\end{aligned}$$

Let $\delta_i \downarrow 0$. Let us recall the approximation spaces defined in Remark 2.1, (2.2):

$$\mathcal{A}_{\delta_i} = \left\{ V_i \ d\text{–varifold} \, : \, \exists \text{ a mesh } \mathcal{K}_i \text{ such that } V_i = \sum_{K\in\mathcal{K}_i} \frac{m_K}{|K|} \mathcal{L}^n_{|K} \otimes \delta_{P^i_K}, \ \sup_{K\in\mathcal{K}_i} \operatorname{diam}(K) \leqslant \delta_i \right\}$$

that is without fixing a sequence of meshes but considering all possible meshes satisfying

$$\sup_{K\in\mathcal{K}_i} \operatorname{diam}(K) \leqslant \delta_i \, .$$

Then for any rectifiable d–varifold with bounded first variation, thanks to Theorem 2.1 and Proposition 5.11, there exists a sequence $V_i \in \mathcal{A}_{\delta_i}$ weakly–$*$ converging to V and an extracted sequence $V_{\varphi(i)} \in \mathcal{A}_{\delta_{\varphi(i)}}$ such that

$$\mathcal{W}^1_{\varepsilon_i}(V_{\varphi(i)}) \xrightarrow[i\to+\infty]{} \|\delta V\|$$

but $\mathcal{A}_{\delta_{\varphi(i)}} \subset \mathcal{A}_{\delta_i}$ (since δ_i is decreasing) so that we have the following result.

Proposition 5.12. *Let $\varepsilon_i \downarrow 0$ and $\mathcal{A}_{\delta_i}$ defined as in* (2.2)*. Then (see* (5.4) *for the notation below),*

$$\mathcal{W}_{\varepsilon_i} \xrightarrow[\mathcal{A}_{\delta_i},\, i\to+\infty]{\Gamma} \|\delta\cdot\| \, .$$

However, for numerical applications, the idea is rather to fix a sequence of meshes $(\mathcal{K}_i)_i$ satisfying $\sup_{K\in\mathcal{K}_i} \operatorname{diam}(K) \leqslant \delta_i$ and to consider the approximation spaces defined in Remark 2.1, (2.1):

$$\mathcal{A}_{\delta_i}(\mathcal{K}_i) = \left\{ V_i \ d\text{–varifold} \, : \, V_i = \sum_{K\in\mathcal{K}_i} \frac{m_K}{|K|} \mathcal{L}^n_{|K} \otimes \delta_{P^i_K} \right\} .$$

In this case, we do not have anymore the inclusion $\mathcal{A}_{\delta_{i+1}}(\mathcal{K}_{i+1}) \subset \mathcal{A}_{\delta_i}(\mathcal{K}_i)$ and thus an extracted sequence does not lie in the same spaces as the sequence itself. If we want to obtain a Γ–lim sup result, we need to approximate any rectifiable d–varifold with bounded first variation with a control on the convergence, in connection with the size of the mesh δ_i. More precisely, given $\delta_i \downarrow 0$ the size of the successive meshes, we search the scale $\varepsilon_i \downarrow 0$, depending only on δ_i and such that (see (5.4) for the notation below)

$$\mathcal{W}^1_{\varepsilon_i} \xrightarrow[\mathcal{A}_{\delta_i}(\mathcal{K}_i),\, i\to+\infty]{\Gamma} \|\delta\cdot\| \, .$$

The problem is that when defining discrete volumetric varifolds, the space Ω is discretized with a size δ_i going to 0, but the Grassmannian $G_{d,n}$ is not discretized. Therefore, given a rectifiable d–varifold $V = v(M, \theta)$, the accuracy of the approximation of the tangential part can be measured in terms of δ_i only if the tangential part is controlled by the spatial part, that is, only if we add some uniform regularity assumption on the tangent plane $x \mapsto T_x M$, as in the following result. (Actually, the assumption on the global Hölder regularity on the tangent plane (5.22) can be weakened as explained in remark 1.8).

Theorem 5.13. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $V = v(M, \theta)$ be a rectifiable d–varifold in Ω with finite mass $\|V\|(\Omega)$. Let $\delta_i \downarrow 0$ be a sequence of infinitesimals and $(\mathcal{K}_i)_i$ a sequence of meshes satisfying*

$$\sup_{K \in \mathcal{K}_i} \operatorname{diam}(K) \leqslant \delta_i \xrightarrow[i \to +\infty]{} 0 \, .$$

Let the approximation spaces $(\mathcal{A}_{\delta_i}(\mathcal{K}_i))_i$ defined as in (2.1) and let the kernel ρ be as in (5.13), assuming in addition that $\rho \in \mathrm{W}^{2,\infty}$. Assume that there exist $0 < \beta < 1$ and C such that for $\|V\|$–almost every x, $y \in \Omega$,

$$\|T_x M - T_y M\| \leqslant C |x - y|^\beta \, . \tag{5.22}$$

Then, there exists a sequence of discrete volumetric varifolds $(V_i)_i$ such that

(i) for all i, $V_i \in \mathcal{A}_{\delta_i}(\mathcal{K}_i)$,

(ii) $V_i \xrightarrow[i \to +\infty]{*} V$,

(iii) For any sequence of infinitesimals $\varepsilon_i \downarrow 0$ satisfying

$$\frac{\delta_i^\beta}{\varepsilon_i^2} \xrightarrow[i \to +\infty]{} 0 \, ,$$

one has

$$\mathcal{W}^1_{\varepsilon_i}(V_i) \xrightarrow[i \to +\infty]{} \|\delta V\| \, .$$

Proof. As in Theorem 2.1, we define the discrete volumetric varifolds $V_i \in \mathcal{A}_{\delta_i}(\mathcal{K}_i)$ by

$$V_i = \sum_{K \in \mathcal{K}_i} \frac{m_K^i}{|K|} \mathcal{L}^n \otimes \delta_{P_K^i} \text{ with } m_K^i = \|V\|(K) \text{ and } P_K^i \in \operatorname*{arg\,min}_{P \in G_{d,n}} \int_{K \times G_{d,n}} \|P - S\| \, dV(x, S) \, .$$

By Theorem 2.1, $V_i \xrightarrow[i \to +\infty]{*} V$ and by (2.6), for any $\varphi \in \operatorname{Lip}(\Omega \times G_{d,n})$ with Lipschitz constant $\operatorname{lip}(\varphi)$,

$$| \langle V, \varphi \rangle - \langle V_i, \varphi \rangle | \leqslant \operatorname{lip}(\varphi) \left(\delta_i + 2C\delta_i^\beta \right) \|V\| \left(\Pi(\operatorname{supp} \varphi) \cap \Omega \right) \, . \tag{5.23}$$

Then,

$$\begin{aligned} \left| \mathcal{W}^1_{\varepsilon_i}(V_i) - \mathcal{W}_{\varepsilon_i}(V) \right| &\leqslant \int_{x \in \mathbb{R}^n} |\delta V_i * \rho_\varepsilon(x) - \delta V * \rho_\varepsilon(x)| \, d\mathcal{L}^n(x) \\ &= \frac{1}{\varepsilon_i^{n+1}} \int_{x \in \mathbb{R}^n} \left| \int_{\Omega \times G_{d,n}} \nabla^S \rho \left(\frac{y - x}{\varepsilon_i} \right) dV_i(y, S) - \int_{\Omega \times G_{d,n}} \nabla^S \rho \left(\frac{y - x}{\varepsilon_i} \right) dV(y, S) \right| d\mathcal{L}^n(x) \, . \end{aligned}$$

As $\rho \in \mathrm{W}^{2,\infty}(\Omega)$, $(y,S) \mapsto \nabla^S \rho\left(\frac{y-x}{\varepsilon_i}\right) \in \mathrm{Lip}(\Omega \times G_{d,n})$ with Lipschitz constant $\leqslant \frac{1}{\varepsilon_i}\|\rho\|_{\mathrm{W}^{2,\infty}}$ and with support in $B_{\varepsilon_i}(x) \times G_{d,n}$. Therefore, thanks to (5.23),

$$\begin{aligned}
\left|\mathcal{W}^1_{\varepsilon_i}(V_i) - \mathcal{W}_{\varepsilon_i}(V)\right| &\leqslant \frac{1}{\varepsilon_i^{n+1}}\frac{1}{\varepsilon_i}\|\rho\|_{\mathrm{W}^{2,\infty}}\left(\delta_i + 2C\delta_i^\beta\right)\int_{x\in\mathbb{R}^n}\|V\|\left(B_{\varepsilon_i}(x)\cap\Omega\right)\,d\mathcal{L}^n(x) \\
&\leqslant \|\rho\|_{\mathrm{W}^{2,\infty}}\frac{\left(\delta_i + 2C\delta_i^\beta\right)}{\varepsilon_i^{n+2}}\int_{y\in\Omega}\mathcal{L}^n\left(B_{\varepsilon_i}(x)\right)\,d\|V\|(y) \\
&\leqslant \|\rho\|_{\mathrm{W}^{2,\infty}}\frac{\left(\delta_i + 2C\delta_i^\beta\right)}{\varepsilon_i^{n+2}}\omega_n\varepsilon_i^n\|V\|(\Omega) \\
&\leqslant \|\rho\|_{\mathrm{W}^{2,\infty}}\omega_n\|V\|(\Omega)\frac{\left(\delta_i + 2C\delta_i^\beta\right)}{\varepsilon_i^2}\,,
\end{aligned}$$

which leads to the conclusion. □

5.3.2 The case of $\mathcal{W}^p_\varepsilon$ for $p > 1$

As we said, the problem is not solved. Technically, this comes from the fact that in the proof of Proposition 5.11 (5.21) for $p = 1$,

$$\left|\mathcal{W}^1_\varepsilon(V_i) - \mathcal{W}^1_\varepsilon(V)\right| \leqslant \int_{x\in\mathbb{R}^n}\left|\delta V_i * \rho_\varepsilon(x) - \delta V * \rho_\varepsilon(x)\right|\,d\mathcal{L}^n(x)\,,$$

and for fixed $\varepsilon > 0$, $|\delta V_i * \rho_\varepsilon(x) - \delta V * \rho_\varepsilon(x)|$ is bounded. But in the case $p > 1$,

$$|\mathcal{W}^p_\varepsilon(V_i) - \mathcal{W}^p_\varepsilon(V)| \leqslant \int_{x\in\mathbb{R}^n}\left|\left|\frac{\delta V_i * \rho_\varepsilon(x)}{\|V_i\| * \rho_\varepsilon(x)}\right|^p (\|V_i\| * \rho_\varepsilon(x)) - \left|\frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)}\right|^p (\|V\| * \rho_\varepsilon(x))\right|\,d\mathcal{L}^n(x)\,,$$

but the ratio $\left|\frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)}\right|^p$ is not a priori bounded. However, nothing indicates whether it is a relevant obstacle or just a technical point. We need a better understanding of the regularisation $\delta V * \rho_\varepsilon$ and its connection with some suitable regularization of V.

5.4 A connection between the regularization of the first variation δV and the first variation of some appropriate regularization of V

In this section, we try to answer **Question 5.3**:
– Given a d–varifold V, is the regularization $\delta V * \rho_\varepsilon$ of the first variation δV, the first variation $\delta\left(\widehat{V_\varepsilon}\right)$ of some varifold $\widehat{V_\varepsilon}$?
– And if so, is $\widehat{V_\varepsilon}$ the regularization (in a sense to be defined) of V?
In short, is there a kind of convolution $\hat{*}$ such that the following formula makes sense

$$\delta V * \rho_\varepsilon = \delta\left(\widehat{V_\varepsilon}\right) = \delta\left(V \hat{*} \rho_\varepsilon\right)?$$

Let us first explain what $\widehat{V_\varepsilon}$ cannot be. As V is a Radon measure in $\Omega \times G_{d,n}$, notice that $V * \rho_\varepsilon$ does not have a canonical sense. A natural idea would be to:

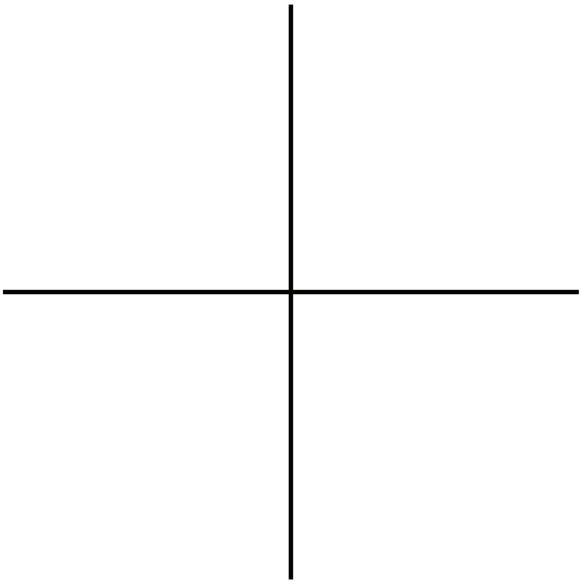
1. first regularize the mass $\|V\|$, defining $\|\widehat{V_\varepsilon}\| = (\|V\| * \rho_\varepsilon)\,d\mathcal{L}^n$,

2. then set $\widehat{V_\varepsilon} = (\|V\| * \rho_\varepsilon(x))\, d\mathcal{L}^n \otimes \delta_{T_\varepsilon(x)}$ and compute the tangential part $T_\varepsilon(x)$ from $\|\widehat{V_\varepsilon}\|$. For, instance, if $V = v(\Gamma, 1)$ is associated with a curve Γ in $\mathbb{R}^2$, set $u_\varepsilon(x) = d(x, \Gamma)$ and set $T_\varepsilon(x) = \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}^\perp$ (which gives the tangential direction to the level lines of u_ε) so that $\widehat{V_\varepsilon}$ would be:

$$\widehat{V_\varepsilon} = (\|V\| * \rho_\varepsilon(x))\, d\mathcal{L}^2 \otimes \delta_{T_\varepsilon(x)} = (\|V\| * \rho_\varepsilon(x))\, \mathcal{L}^2 \otimes \delta_{\frac{\nabla u_\varepsilon(x)}{|\nabla u_\varepsilon(x)|}^\perp} .$$

Let us consider a simple example to test this construction:

Example 5.4. Let $V = v(N, 1)$ where N is the cross constituted by the union of the lines $N_1 = \{x_1 = 0\}$ and $N_2 = \{x_2 = 0\}$ in $\mathbb{R}^2$, then $\delta V = 0$ and thus $\delta V * \rho_\varepsilon = 0$. But with the previous construction, we obtain $\widehat{V_\varepsilon} = (\|V\| * \rho_\varepsilon(x))\, d\mathcal{L}^2 \otimes \delta_{T_\varepsilon(x)}$ represented in Figure 5.1



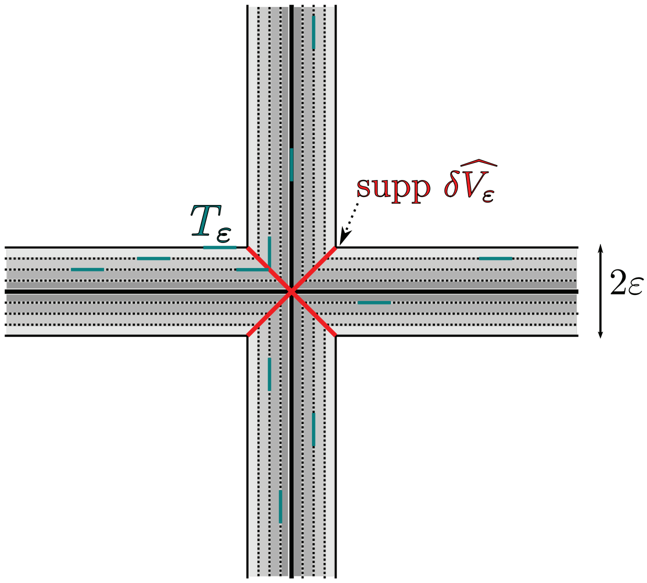


Figure 5.1:

Qualitatively, we observe that $\delta(\widehat{V_\varepsilon})$ is composed of a singular part concentrated on the red set in Figure 5.1 and an absolutely continuous part due to the fact that $\|V\| * \rho_\varepsilon(x)$ is not constant along the level-sets $\{d(x, \Gamma) = \lambda\}$. Exact computations can be done by dividing the cross along the red set into 4 parts and applying Fubini Theorem to integrate on the level-sets $\{d(x, \Gamma) = \lambda\}$, and then apply the divergence Theorem in each integral; but qualitatively, we can see that with this definition,

$$\delta\left(\widehat{V_\varepsilon}\right) \neq 0 = \delta V * \rho_\varepsilon .$$

The construction we proposed is not the right one, yet the idea of convolving the spatial part is reasonable, but the tangential part must be constructed from V and not from $\|\widehat{V_\varepsilon}\|$:

Theorem 5.14. *Let $\Omega \subset \mathbb{R}^n$ be an open set and V a d–varifold in Ω with finite mass $\|V\|(\Omega) < +\infty$. Let $\varepsilon > 0$ and ρ_ε as in* (5.13)*. Define the d–varifold $\widehat{V_\varepsilon}$ as:*

$$\left\langle \widehat{V_\varepsilon}, \psi \right\rangle = \langle V, (y, S) \mapsto \psi(\cdot, S) * \rho_\varepsilon(y) \rangle \quad \textit{for every } \psi \in \mathrm{C}^0_c(\Omega \times G_{d,n}) ;$$

or equivalently,

$$\int_{\Omega \times G_{d,n}} \psi(y, S)\, d\widehat{V_\varepsilon}(y, S) = \int_{(y,S) \in \Omega \times G_{d,n}} \int_{x \in \mathbb{R}^n} \psi(x, S) \rho_\varepsilon(y - x)\, d\mathcal{L}^n(x)\, dV(y, S) . \qquad (5.24)$$

Then,

1. $\|\widehat{V_\varepsilon}\| = \|V\| * \rho_\varepsilon$,
2. $\delta\left(\widehat{V_\varepsilon}\right) = \delta V * \rho_\varepsilon$.

Proof. Let us first compute $\|\widehat{V_\varepsilon}\|$, for $\varphi \in \mathrm{C}_c^0(\Omega)$,

$$\begin{aligned}\left\langle \|\widehat{V_\varepsilon}\|, \varphi \right\rangle = \left\langle \widehat{V_\varepsilon}, \varphi \right\rangle &= \int_{(y,S)\in\Omega\times G_{d,n}} \int_{x\in\mathbb{R}^n} \varphi(x)\rho_\varepsilon(y-x)\, d\mathcal{L}^n(x)\, dV(y,S) \\ &= \int_{y\in\Omega}\int_{x\in\mathbb{R}^n} \varphi(x)\rho_\varepsilon(y-x)\, d\mathcal{L}^n(x)\, d\|V\|(y) = \langle \|V\|, \varphi * \rho_\varepsilon\rangle \\ &= \langle \|V\| * \rho_\varepsilon\rangle \ .\end{aligned}$$

We now compute the first variation of $\widehat{V_\varepsilon}$. Let $X \in \mathrm{C}_c^1(\Omega, \mathbb{R}^n)$, then

$$\begin{aligned}\left\langle \delta\left(\widehat{V_\varepsilon}\right), X \right\rangle &= \left\langle \widehat{V_\varepsilon}, (y,S) \mapsto \mathrm{div}_S X(y) \right\rangle \\ &= \int_{(y,S)\in\Omega\times G_{d,n}} \int_{x\in\mathbb{R}^n} \mathrm{div}_S X(x) \rho_\varepsilon(y-x)\, d\mathcal{L}^n(x)\, dV(y,S)\ ,\end{aligned}$$

and for fixed $(y,S) \in \Omega \times G_{d,n}$, one has

$$\mathrm{div}_S\left[x \mapsto \rho_\varepsilon(y-x)X(x)\right] = \rho_\varepsilon(y-x)\mathrm{div}_S X(x) - \nabla^S \rho_\varepsilon(y-x)\cdot X(x)\ . \tag{5.25}$$

Moreover,

$$\begin{aligned}&\int_{(y,S)\in\Omega\times G_{d,n}} \int_{x\in\mathbb{R}^n} \nabla^S \rho_\varepsilon(y-x)\cdot X(x)\, d\mathcal{L}^n(x)\, dV(y,S) \\ &\qquad = \int_{x\in\mathbb{R}^n}\int_{(y,S)\in\Omega\times G_{d,n}} \nabla^S\rho_\varepsilon(y-x)\, dV(y,S)\cdot X(x)\, d\mathcal{L}^n(x) \\ &\qquad = \int_{x\in\mathbb{R}^n} \delta V * \rho_\varepsilon(x)\cdot X(x)\, d\mathcal{L}^n(x) \text{ thanks to (5.11)} \\ &\qquad = \langle \delta V * \rho_\varepsilon, X\rangle\ ,\end{aligned} \tag{5.26}$$

and since $x \mapsto \rho_\varepsilon(y-x)X(x)$ is compactly supported, for a fixed $S \in G_{d,n}$,

$$\int_{x\in\mathbb{R}^n} \mathrm{div}_S\left[x \mapsto \rho_\varepsilon(y-x)X(x)\right]\, d\mathcal{L}^n(x) = 0 \quad \text{so that}$$

$$\int_{(y,S)\in\Omega\times G_{d,n}}\int_{x\in\mathbb{R}^n} \mathrm{div}_S\left[x \mapsto \rho_\varepsilon(y-x)X(x)\right]\, d\mathcal{L}^n(x)\, dV(y,S) = 0\ . \tag{5.27}$$

Hence, thanks to (5.25), (5.26) and (5.27), we have,

$$\left\langle \delta\left(\widehat{V_\varepsilon}\right), X\right\rangle = \langle \delta V * \rho_\varepsilon, X\rangle\ .$$

□

Example 5.5. Let us come back to the example of the cross $V = v(N,1)$ in $\mathbb{R}^2$ with $N = N_1 \cup N_2$ and $N_1 = \{x_1 = 0\}$ and $N_2 = \{x_2 = 0\}$. Define the 2–varifolds $V_1 = v(N_1, 1)$ and $V_2 = v(N_2, 1)$ so that $V = V_1 + V_2$. Notice that the mapping $V \mapsto \widehat{V_\varepsilon}$ in (5.24) is linear. Hence $\delta(\widehat{V_\varepsilon}) = \delta(\widehat{V_{1\varepsilon}}) + \delta(\widehat{V_{2\varepsilon}})$ and the fact that

$$\delta(\widehat{V_{1\varepsilon}}) = \delta(\widehat{V_{2\varepsilon}}) = 0$$

can be easily proved. Let us check it by simple computations. Let $\psi \in \mathrm{C}_c^0(\mathbb{R}^2 \times G_{1,2})$,

$$\begin{aligned}\int_{\mathbb{R}^2\times G_{1,2}} \psi(y,S)\, d\widehat{V_\varepsilon}(y,S) &= \int_{(y,S)\in\mathbb{R}^2\times G_{1,2}} \int_{x\in\mathbb{R}^2} \psi(x,S)\rho_\varepsilon(y-x)\, d\mathcal{L}^2(x)\, dV_1(y,S)\\ &+ \int_{(y,S)\in\mathbb{R}^2\times G_{1,2}} \int_{x\in\mathbb{R}^n} \psi(x,S)\rho_\varepsilon(y-x)\, d\mathcal{L}^2(x)\, dV_2(y,S)\\ &= \int_{x\in\mathbb{R}^2} \psi(x,T_1) \int_{y\in\mathbb{R}^2} \rho_\varepsilon(y-x)\, d\|V_1\|(y)\, d\mathcal{L}^2(x)\\ &+ \int_{x\in\mathbb{R}^2} \psi(x,T_2) \int_{y\in\mathbb{R}^2} \rho_\varepsilon(y-x)\, d\|V_2\|(y)\, d\mathcal{L}^2(x)\,,\end{aligned}$$

where T_1, $T_2 \in G_{1,2}$ respectively denote the direction of N_1 and N_2. Thus, for $X \in \mathrm{C}_c^1(\mathbb{R}^2,\mathbb{R}^2)$,

$$\begin{aligned}\int_{\mathbb{R}^2\times G_{1,2}} \mathrm{div}_S X(y)\, d\widehat{V_\varepsilon}(y,S) &= \int_{x\in\mathbb{R}^2} \mathrm{div}_{T_1} X(x) \int_{y\in\mathbb{R}^2} \rho_\varepsilon(y-x)\, d\|V_1\|(y)\, d\mathcal{L}^2(x)\\ &+ \int_{x\in\mathbb{R}^2} \mathrm{div}_{T_2} X(x) \int_{y\in\mathbb{R}^2} \rho_\varepsilon(y-x)\, d\|V_2\|(y)\, d\mathcal{L}^2(x)\,.\end{aligned}$$

Moreover, in each set $\{d(x,N_1) = \lambda\}$, $\int_{y\in\mathbb{R}^2} \rho_\varepsilon(y-x)\, d\|V_1\|(y) = c_\lambda$ is constant. Then, thanks to Fubini Theorem and the divergence Theorem,

$$\begin{aligned}\int_{x\in\mathbb{R}^2} \mathrm{div}_{T_1} X(x) \int_{y\in\mathbb{R}^2} \rho_\varepsilon(y-x)\, d\|V_1\|(y)\, d\mathcal{L}^2(x) &= \int_{\lambda=-\varepsilon}^{\varepsilon} \int_{\{d(x,N_1)=\lambda\}} \mathrm{div}_{T_1} X(x) c_\lambda\, d\mathcal{H}^1(x)\, d\lambda\\ &= 0\,.\end{aligned}$$

Notice that the idea of convolving the spatial part was right so that the point was to build the right tangential part. In the following proposition, we study the tangential part of $\widehat{V_\varepsilon}$ defined in (5.24).

Proposition 5.15. *Let $\Omega \subset \mathbb{R}^n$ be an open set and V be a d–varifold in Ω with finite mass $\|V\|(\Omega) < +\infty$. Decompose V into $V = \|V\| \otimes \nu_x$ by disintegration with respect to $\|V\|$, ν_x is a probability measure for $\|V\|$–almost every x. Let $\varepsilon > 0$ and ρ_ε as in (5.13). Let $\widehat{V_\varepsilon}$ defined as in (5.24). Then, $\widehat{V_\varepsilon} = \|\widehat{V_\varepsilon}\| \otimes \widehat{\nu_x^\varepsilon}$ where, for $\|\widehat{V_\varepsilon}\|$–almost every $x \in \mathbb{R}^n$, $\widehat{\nu_x^\varepsilon}$ is a probability measure in $G_{d,n}$ and, for all $\psi \in \mathrm{C}^0(G_{d,n})$,*

$$\int_{G_{d,n}} \psi(S)\, d\widehat{\nu_x^\varepsilon}(S) = \frac{\int_{y\in\Omega}\int_{G_{d,n}} \psi(S)\, d\nu_y(S)\, \rho_\varepsilon(y-x)\, d\|V\|(y)}{\int_{y\in\Omega} \rho_\varepsilon(y-x)\, d\|V\|(y)}\,, \tag{5.28}$$

or equivalently, for any Borel set $A \in G_{d,n}$,

$$\widehat{\nu_x^\varepsilon}(A) = \frac{\int_{y\in\Omega} \nu_y(A)\, \rho_\varepsilon(y-x)\, d\|V\|(y)}{\int_{y\in\Omega} \rho_\varepsilon(y-x)\, d\|V\|(y)}\,. \tag{5.29}$$

Proof. Let $\varphi \in \mathrm{C}_c^0(\mathbb{R}^n)$ and $\psi \in \mathrm{C}^0(G_{d,n})$.

$$\begin{aligned}\left\langle \widehat{V_\varepsilon}, \varphi(x)\psi(S) \right\rangle &= \int_{x\in\mathbb{R}^n} \varphi(x) \int_{y\in\Omega} \int_{S\in G_{d,n}} \psi(S)\, d\nu_y(S) \rho_\varepsilon(y-x)\, d\|V\|(y)\, d\mathcal{L}^n(x)\\ &= \int_{x\in\mathbb{R}^n} \varphi(x) \int_{S\in G_{d,n}} d\widehat{\nu_x^\varepsilon}(S)\, d\|\widehat{V_\varepsilon}\|(x)\\ &= \int_{x\in\mathbb{R}^n} \varphi(x) \int_{S\in G_{d,n}} d\widehat{\nu_x^\varepsilon}(S) \int_{y\in\Omega} \rho_\varepsilon(y-x)\, d\|V\|(y)\, d\mathcal{L}^n(x)\,.\end{aligned}$$

Consequently, for $\mathcal{L}^n$–almost every x,

$$\int_{S\in G_{d,n}} \psi(S)\, d\widehat{\nu_x^\varepsilon}(S) = \frac{\int_{y\in\Omega}\int_{S\in G_{d,n}} \psi(S)\, d\nu_y(S)\rho_\varepsilon(y-x)\, d\|V\|(y)}{\int_{y\in\Omega}\rho_\varepsilon(y-x)\, d\|V\|(y)}\,.$$

$\square$

Example 5.6. Coming back again to the example of the cross $V = v(N,1)$ with $N = \{x_1 = 0\} \cup \{x_2 = 0\} \subset \mathbb{R}^2$, let $\widehat{V_\varepsilon}$ be the varifold associated with V by formula (5.24). We already know that $\|\widehat{V_\varepsilon}\| = \|V\| * \rho_\varepsilon$. We now want to identify the tangential part $\widehat{\nu_x^\varepsilon}$ in the decomposition $\widehat{V_\varepsilon} = \|\widehat{V_\varepsilon}\| \otimes \widehat{\nu_x^\varepsilon}$. Thanks to Proposition 5.15, for $\|\widehat{V_\varepsilon}\|$–almost every $x \in \mathbb{R}^2$ and for any Borel set $A \subset \mathbb{R}^2$,

$$\widehat{\nu_x^\varepsilon}(A) = \frac{\int_{y\in\Omega}\nu_y(A)\,\rho_\varepsilon(y-x)\, d\|V\|(y)}{\int_{y\in\Omega}\rho_\varepsilon(y-x)\, d\|V\|(y)}\,,$$

and applying it with $A = \{T_1\}$ and $A = \{T_2\}$ where $T_1,\, T_2 \in G_{1,2}$ respectively denote the direction of $N_1 = \{x_1 = 0\}$ and $N_2 = \{x_2 = 0\}$, we have for $i = 1, 2$,

$$\widehat{\nu_x^\varepsilon}(\{T_i\}) = \frac{\int_{\{y\in\Omega\,:\, y\in N_i\}}\rho_\varepsilon(y-x)\, d\|V\|(y)}{\int_{y\in\Omega}\rho_\varepsilon(y-x)\, d\|V\|(y)} \quad \text{and} \quad \widehat{\nu_x^\varepsilon}(\mathbb{R}^2 \setminus \{T_1,\, T_2\}) = 0\,.$$

Hence $\widehat{\nu_x^\varepsilon}$ is a convex combination of δ_{T_1} and δ_{T_2} whose coefficients depend on the distances $d(x, N_1)$ and $d(x, N_2)$. We try to represent it in Figure 5.2:

Remark 5.10. Notice that with this construction of $\widehat{V_\varepsilon}$ (5.24), $\widehat{\nu_x^\varepsilon}$ is generally not a sum of Dirac masses, unless the tangent plane to V is constant on a set of $\|V\|$–mass strictly positive.

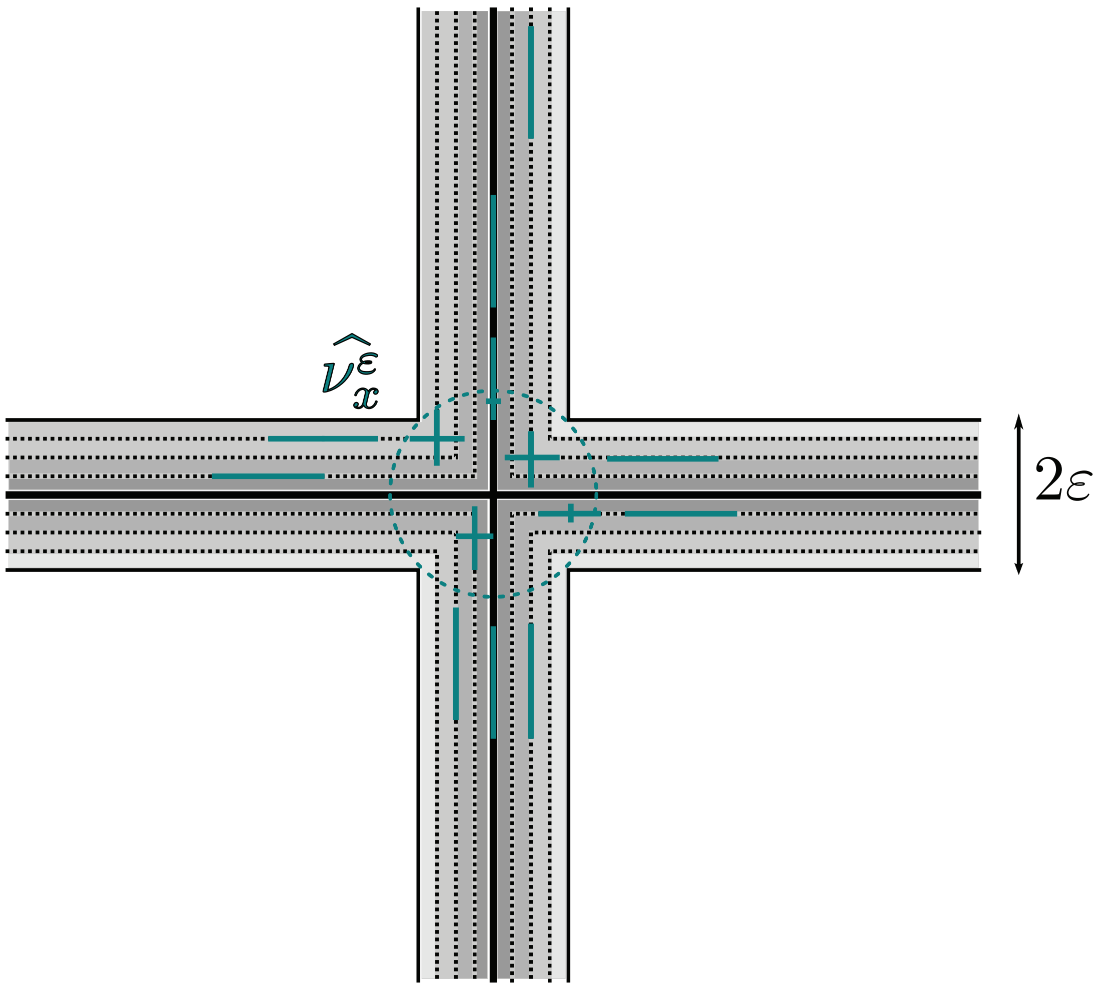


Figure 5.2:

CHAPTER 6

Aspects numériques

We study in this chapter the approximation of the mean curvature given by

$$\frac{\delta V_i * \rho_\varepsilon(x)}{\|V_i\| * \rho_\varepsilon(x)} \,, \tag{6.1}$$

on sequences of point cloud varifolds $(V_i)_i$. Let V be a rectifiable d–varifold with bounded first variation $\delta V = -H \, \|V\| + (\delta V)_s$ and $\rho \in \mathrm{W}^{1,\infty}$ be a radial kernel. We proved in Proposition 5.6 (in Chapter 5) that for $\|V\|$–almost any x,

$$H_\varepsilon(x) = -\frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)} \xrightarrow[\varepsilon \to 0]{} H(x) \,.$$

We will see now (Proposition 6.1) that under the assumptions of the theorem of approximation by discrete volumetric varifolds (Theorem 2.1), if $(V_i)_i$ is the sequence of discrete volumetric varifolds (given by Theorem 2.1) associated with a sequence of meshes $\mathcal{K}_i$ such that $\delta_i = \sup_{K \in \mathcal{K}_i} \operatorname{diam} K$ tends to 0 and if $\varepsilon_i \downarrow 0$ satisfies

$$\frac{\delta_i^\beta}{\varepsilon_i^2} \xrightarrow[i \to \infty]{} 0 \,,$$

then for $\|V\|$–almost any x,

$$-\frac{\delta V_i * \rho_{\varepsilon_i}(x)}{\|V_i\| * \rho_{\varepsilon_i}(x)} \xrightarrow[i \to \infty]{} H(x) \,.$$

(Recall that β is the Hölder–regularity of the tangent plane to V with the notations of Theorem 2.1.)

We then study this approximation on $2D$ point cloud varifolds, with what we call the "reversed tent kernel", which is simply $\rho(y) = |y|$ if $|y| < 1$ and 0 otherwise. We chose point cloud varifolds and not discrete volumetric varifolds for practical reasons. Anyway, as we have already pointed out, it is possible to associate a point cloud varifold with a discrete volumetric varifold by simply picking up any point x_K in each cell K and replacing $\displaystyle\sum_{K \in \mathcal{K}} \frac{m_K}{|K|} \mathcal{L}^n_{|K} \otimes \delta_{P_K}$ by $\displaystyle\sum_{K \in \mathcal{K}} m_K \delta_{x_K} \otimes \delta_{P_K}$, and with this construction, convergence properties of discrete volumetric varifolds transfer to the associated point cloud varifolds (see section 2.3). We will check (in Section 6.1.4) that the approximation of the mean curvature of point clouds given by (6.18) allows to recover a zero–curvature at crossing points, but it also presents instability as a result of the annihilation of large symmetric terms. In order to improve this aspect, we notice that for point clouds approximating regular curves or surfaces, Formula (6.18) can be turned into a more stable formula (6.23). We then test this formula in Section 6.1.5, on some discretizations of $2D$ regular shapes. Finally, we test in Section 6.2 on $3D$ point clouds our approximation of the mean curvature.

6.1 Approximation of the curvature on $2D$ point cloud varifolds

6.1.1 Pointwise convergence

Proposition 6.1 (Pointwise convergence)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $V = v(M, \theta)$ be a rectifiable d–varifold in Ω with finite mass $\|V\|(\Omega)$ and bounded first variation. Under the assumptions of Theorem 5.13, that is:*

– *$\delta_i \downarrow 0$ is a sequence of infinitesimals and $(\mathcal{K}_i)_i$ is a sequence of meshes satisfying*

$$\sup_{K \in \mathcal{K}_i} \operatorname{diam}(K) \leqslant \delta_i \xrightarrow[i \to +\infty]{} 0 \, ;$$

– *assume that there exist $0 < \beta < 1$ and C such that for $\|V\|$–almost every x, $y \in \Omega$,*

$$\|T_x M - T_y M\| \leqslant C |x - y|^\beta \, . \tag{6.2}$$

Assuming in addition that the kernel $\rho \in \mathrm{W}^{2,\infty}$ is as in (5.13) *and that moreover $\rho(x) = \zeta(|x|)$ is radial, with $\zeta \in \mathrm{W}^{2,\infty}(\mathbb{R}_+)$ non–increasing, we have the following result.*

If $\varepsilon_i \downarrow 0$ satisfies

$$\frac{\delta_i^\beta}{\varepsilon_i^2} \xrightarrow[i \to \infty]{} 0 \, ,$$

then, for $\|V\|$–almost any $x \in \Omega$,

$$-\frac{\delta V_i * \rho_{\varepsilon_i}(x)}{\|V_i\| * \rho_{\varepsilon_i}(x)} \xrightarrow[i \to \infty]{} H(x) \, .$$

Proof. Let $\varepsilon > 0$. First of all, thanks to Proposition 5.6, for $\|V\|$–almost any x,

$$\begin{aligned} \left| \frac{\delta V_i * \rho_\varepsilon(x)}{\|V_i\| * \rho_\varepsilon(x)} + H(x) \right| &\leqslant \left| \frac{\delta V_i * \rho_\varepsilon(x)}{\|V_i\| * \rho_\varepsilon(x)} - \frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)} \right| + \underbrace{\left| \frac{\delta V * \rho_\varepsilon(x)}{\|V\| * \rho_\varepsilon(x)} + H(x) \right|}_{\xrightarrow[\varepsilon \to 0]{} 0} \\ &\leqslant \frac{|\delta V_i * \rho_\varepsilon(x) - \delta V * \rho_\varepsilon(x)|}{\|V_i\| * \rho_\varepsilon(x)} + |\delta V * \rho_\varepsilon(x)| \left| \frac{1}{\|V_i\| * \rho_\varepsilon(x)} - \frac{1}{\|V\| * \rho_\varepsilon(x)} \right| + o(\varepsilon) \, . \end{aligned} \tag{6.3}$$

Step 1: We study the convergence of the first term in (6.3). Recall that, by (2.6) in Theorem 2.1, for all $\varphi \in \operatorname{Lip}(\Omega \times G_{d,n})$,

$$|\langle V_i, \varphi \rangle - \langle V, \varphi \rangle| \leqslant \operatorname{lip}(\varphi) \|V\| \left(\Pi(\operatorname{supp} \varphi) \cap \Omega \right) \left(\delta_i + 2C\delta_i^\beta \right) \, . \tag{6.4}$$

And as $\rho \in \mathrm{W}^{2,\infty}(\Omega)$, the function

$$(y, S) \in \Omega \times G_{d,n} \mapsto \nabla^S \rho \left(\frac{y - x}{\varepsilon} \right)$$

is Lipschitz with Lipschitz constant $\leqslant \dfrac{1}{\varepsilon}\|\rho\|_{\mathrm{W}^{2,\infty}}$ and with support in $B_\varepsilon(x)\times G_{d,n}$. By (6.4),

$$|\delta V_i * \rho_\varepsilon(x) - \delta V * \rho_\varepsilon(x)| = \frac{1}{\varepsilon^{n+1}}\left| \int_{\Omega\times G_{d,n}} \nabla^S \rho\left(\frac{y-x}{\varepsilon}\right) dV_i(y,S) - \int_{\Omega\times G_{d,n}} \nabla^S \rho\left(\frac{y-x}{\varepsilon}\right) dV(y,S)\right|$$
$$\leqslant \frac{1}{\varepsilon^{n+2}}\|\rho\|_{\mathrm{W}^{2,\infty}}\left(\delta_i + 2C\delta_i^\beta\right)\|V\|\left(B_\varepsilon(x)\cap\Omega\right) . \tag{6.5}$$

Let us now bound $\|V_i\| * \rho_\varepsilon(x)$ from below. As $\rho(x) = \zeta(|x|)$ for all x, with $\zeta \in \mathrm{W}^{2,\infty}(\mathbb{R}_+)$. In particular ζ is absolutely continuous, $\zeta(1) = 0$ and

$$\zeta(r) = -\int_{s=r}^1 \zeta'(s)\,ds .$$

Consequently,

$$\begin{aligned}\|V_i\| * \rho_\varepsilon(x) &= \int_{y\in B_\varepsilon(x)} \rho_\varepsilon(y-x)\,d\|V_i\|(y) = \frac{1}{\varepsilon^n}\int_{y\in B_\varepsilon(x)} \zeta\left(\frac{|y-x|}{\varepsilon}\right) d\|V_i\|(y)\\
&= -\frac{1}{\varepsilon^n}\int_{y\in B_\varepsilon(x)}\int_{s=\frac{|y-x|}{\varepsilon}}^1 \zeta'(s)\,ds\,d\|V_i\|(y) = -\frac{1}{\varepsilon^{n+1}}\int_{y\in B_\varepsilon(x)}\int_{u=|y-x|}^\varepsilon \zeta'(u)\,du\,d\|V_i\|(y)\\
&= -\frac{1}{\varepsilon^{n+1}}\int_{u=0}^\varepsilon \zeta'(u)\int_{y\in B_u(x)} d\|V_i\|(y)\,du = -\frac{1}{\varepsilon^{n+1}}\int_{u=0}^\varepsilon \zeta'(u)\|V_i\|(B_u(x))\,du .\end{aligned} \tag{6.6}$$

Recall that, by construction of discrete volumetric varifolds $(V_i)_i$ associated to V, for all $s > \delta_i$,

$$\|V\|(B_{s-\delta_i}(x)) \leqslant \|V_i\|(B_s(x)) \leqslant \|V\|(B_{s+\delta_i}(x)) .$$

So that, since $-\zeta' \geqslant 0$ and thanks to (6.6),

$$\begin{aligned}\|V_i\| * \rho_\varepsilon(x) &\geqslant \frac{1}{\varepsilon^{n+1}}\int_{u=\delta_i}^\varepsilon -\zeta'(u)\|V\|(B_{u-\delta_i}(x))\,du \geqslant \frac{1}{\varepsilon^{n+1}}\int_{u=0}^{\varepsilon-\delta_i} -\zeta'(u+\delta_i)\|V\|(B_u(x))\,du\\
&\geqslant \frac{1}{\varepsilon^{n+1}}\int_{u=0}^{\varepsilon-\delta_i} -\left(\zeta'(u) + \mathrm{lip}(\zeta')\delta_i\right)\|V\|(B_u(x))\,du \text{ since } \zeta' \in \mathrm{W}^{1,\infty} .\end{aligned} \tag{6.7}$$

Moreover, by (6.6) (applied with $\varepsilon - \delta_i$ instead of ε),

$$\frac{1}{\varepsilon-\delta_i}\int_{u=0}^{\varepsilon-\delta_i} -\zeta'(u)\|V\|(B_u(x))\,du = \int_{y\in B_{\varepsilon-\delta_i}(x)} \zeta\left(\frac{|y-x|}{\varepsilon-\delta_i}\right) d\|V\|(y) , \tag{6.8}$$

and

$$\frac{1}{\varepsilon}\int_{u=0}^{\varepsilon-\delta_i} \|V\|(B_u(x))\,du \leqslant \|V\|(B_\varepsilon(x)) . \tag{6.9}$$

By (6.7), (6.8) and (6.9), we have

$$\|V_i\| * \rho_\varepsilon(x) \geqslant \frac{1}{\varepsilon^n}\frac{\varepsilon-\delta_i}{\varepsilon}\int_{y\in B_{\varepsilon-\delta_i}(x)} \zeta\left(\frac{|y-x|}{\varepsilon-\delta_i}\right) d\|V\|(y) - \frac{1}{\varepsilon^n}\mathrm{lip}(\zeta')\delta_i\|V\|(B_\varepsilon(x)) \tag{6.10}$$

Remark 6.1. If ζ is supposed to be increasing, the same can be done, but using

$$-\|V_i\|(B_u(x)) \geqslant -\|V\|(B_{u+\delta_i}(x)) .$$

Let us consider a sequence $\varepsilon_i \downarrow 0$ and such that $\frac{\delta_i^\beta}{\varepsilon_i^2} \xrightarrow[i\to\infty]{} 0$. In particular, $\frac{\delta_i}{\varepsilon_i} \xrightarrow[i\to\infty]{} 0$ and $\varepsilon_i - \delta_i \xrightarrow[i\to\infty]{} 0$ with $\delta_i \leqslant \varepsilon_i$. Thanks to (6.10), we obtain

$$\begin{aligned}\frac{\|V\|(B_{\varepsilon_i}(x))}{\varepsilon_i^n \|V_i\| * \rho_{\varepsilon_i}(x)} &\leqslant \frac{\|V\|(B_{\varepsilon_i}(x))}{\frac{\varepsilon_i - \delta_i}{\varepsilon_i} \int_{y \in B_{\varepsilon_i - \delta_i}(x)} \zeta\left(\frac{|y-x|}{\varepsilon_i - \delta_i}\right) d\|V\|(y) - \mathrm{lip}(\zeta')\delta_i \|V\|(B_{\varepsilon_i}(x))} \\ &\leqslant \frac{1}{\frac{\varepsilon_i - \delta_i}{\varepsilon_i} \frac{1}{\|V\|(B_{\varepsilon_i}(x))} \int_{y \in B_{\varepsilon_i - \delta_i}(x)} \zeta\left(\frac{|y-x|}{\varepsilon_i - \delta_i}\right) d\|V\|(y) - \underbrace{\mathrm{lip}(\zeta')\delta_i}_{=o(\delta_i)}} \, . \end{aligned} \tag{6.11}$$

Moreover, as $\|V\| = v(M,\theta)$ is d–rectifiable, we have:

$$\|V\|(B_{\varepsilon_i}(x)) \sim_{i\to\infty} \theta(x)\varepsilon_i^d \, ; \tag{6.12}$$

and, thanks to the definition of approximate tangent plane (Definition 1.5),

$$\frac{1}{\theta(x)(\varepsilon_i - \delta_i)^d} \int_{y \in B_{\varepsilon_i - \delta_i}(x)} \zeta\left(\frac{|y-x|}{\varepsilon_i - \delta_i}\right) d\|V\|(y) \xrightarrow[i\to\infty]{} \int_{B_1(0)\cap T_x M} \zeta(|z|) \, d\mathcal{H}^d(z) \, . \tag{6.13}$$

By (6.12) and (6.13), we have

$$\begin{aligned} &\frac{\varepsilon_i - \delta_i}{\varepsilon_i} \frac{1}{\|V\|(B_{\varepsilon_i}(x))} \int_{y \in B_{\varepsilon_i - \delta_i}(x)} \zeta\left(\frac{|y-x|}{\varepsilon_i - \delta_i}\right) d\|V\|(y) \\ &= \left(1 - \frac{\delta_i}{\varepsilon_i}\right) \frac{\theta(x)(\varepsilon_i - \delta_i)^d}{\|V\|(B_{\varepsilon_i}(x))} \frac{1}{\theta(x)(\varepsilon_i - \delta_i)^d} \int_{y \in B_{\varepsilon_i - \delta_i}(x)} \zeta\left(\frac{|y-x|}{\varepsilon_i - \delta_i}\right) d\|V\|(y) \qquad (6.14) \\ &\sim_{i\to\infty} \left(1 - \frac{\delta_i}{\varepsilon_i}\right) \left(\frac{\varepsilon_i - \delta_i}{\varepsilon_i}\right)^d \int_{B_1(0)\cap T_x M} \zeta(|z|) \, d\mathcal{H}^d(z) \\ &\xrightarrow[i\to\infty]{} \int_{B_1(0)\cap T_x M} \rho(z) \, d\mathcal{H}^d(z) < +\infty \qquad (6.15) \end{aligned}$$

Finally, by (6.11) and (6.15), $\frac{\|V\|(B_{\varepsilon_i}(x))}{\varepsilon_i^n \|V_i\| * \rho_{\varepsilon_i}(x)}$ is bounded by some constant $C' > 0$ when $i \to +\infty$ and by (6.5)

$$\begin{aligned} \frac{|\delta V_i * \rho_{\varepsilon_i}(x) - \delta V * \rho_{\varepsilon_i}(x)|}{\|V_i\| * \rho_{\varepsilon_i}(x)} &\leqslant \frac{1}{\|V_i\| * \rho_{\varepsilon_i}(x)} \frac{1}{\varepsilon_i^{n+2}} \|\rho\|_{\mathrm{W}^{2,\infty}} \left(\delta_i + 2C\delta_i^\beta\right) \|V\| \left(B_{\varepsilon_i}(x)\right) \\ &\leqslant C' \|\rho\|_{\mathrm{W}^{2,\infty}} \frac{\delta_i + 2C\delta_i^\beta}{\varepsilon_i^2} \\ &\xrightarrow[i\to+\infty]{} 0 \, . \end{aligned}$$

Step 2: It remains to study the second term in (6.3). Applying again (6.4),

$$\big| \|V\| * \rho_{\varepsilon_i}(x) - \|V_i\| * \rho_{\varepsilon_i}(x) \big| \leqslant \mathrm{lip}(\rho_{\varepsilon_i}) \left(\delta_i + 2C\delta_i^\beta\right) \|V\|(B_{\varepsilon_{i(x)}}) \, , \tag{6.16}$$

and thus,

$$|\delta V * \rho_{\varepsilon_i}(x)| \left| \frac{1}{\|V_i\| * \rho_{\varepsilon_i}(x)} - \frac{1}{\|V\| * \rho_{\varepsilon_i}(x)} \right| = \underbrace{\frac{|\delta V * \rho_{\varepsilon_i}(x)|}{\|V\| * \rho_{\varepsilon_i}(x)}}_{\xrightarrow[i\to\infty]{} H(x)} \frac{1}{\|V_i\| * \rho_{\varepsilon_i}(x)} \left| \|V\| * \rho_{\varepsilon_i}(x) - \|V_i\| * \rho_{\varepsilon_i}(x) \right|$$

$$\leqslant C_0 \|\rho\|_{\mathrm{W}^{2,\infty}} \underbrace{\frac{1}{\|V_i\| * \rho_{\varepsilon_i}(x)} \frac{1}{\varepsilon_i^n} \|V\|(B_{\varepsilon_i}(x))}_{\leqslant C'} \left(\delta_i + 2C\delta_i^\beta \right)$$

$$\xrightarrow[i\to\infty]{} 0 \, .$$

Remark 6.2. The factor $\left(\delta_i + 2C\delta_i^\beta \right)$ in (6.16) can actually be replaced by δ_i (it comes from the proof of (6.4)).

Thanks to Step 1 and Step 2, we proved that for $\|V\|$–almost any x,

$$-\frac{\delta V_i * \rho_{\varepsilon_i}(x)}{\|V_i\| * \rho_{\varepsilon_i}(x)} \xrightarrow[i\to\infty]{} H(x) \, .$$

□

6.1.2 Formulation in terms of point cloud varifolds

For a d–varifold V_N associated with a point cloud and for a radial kernel $\rho(y) = \zeta(|y|)$,

$$V_N = \sum_{j=1}^N m_j \delta_{x_j} \otimes \delta_{P_j} \, ,$$

the ratio (6.1) rewrites

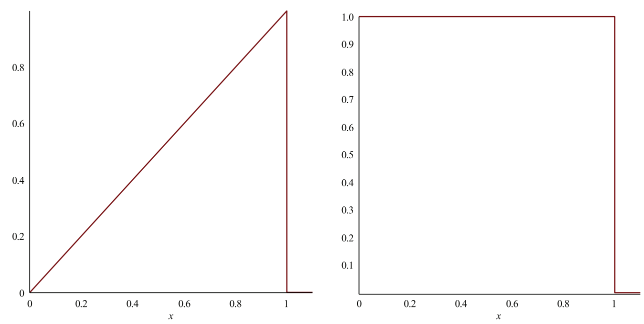
$$H_\varepsilon^N(x) = -\frac{\delta V_N * \rho_\varepsilon(x)}{\|V_N\| * \rho_\varepsilon(x)} = -\frac{\displaystyle\sum_{j=1}^N m_j \zeta' \left(\frac{|x_j - x|}{\varepsilon} \right) \frac{\Pi_{P_j}(x_j - x)}{|x_j - x|}}{\displaystyle\sum_{j=1}^N m_j \varepsilon \zeta \left(\frac{|x_j - x|}{\varepsilon} \right)} \, . \tag{6.17}$$

From now on, ζ is the "reversed tent kernel", $\zeta(|y|) = |y|$ if $|y| < 1$ and 0 otherwise.

In this case, the formula (6.17) rewrites:

$$H_\varepsilon^N(x) = -\frac{\displaystyle\sum_{j=1}^N \mathbb{1}_{\{|x_j - x| < \varepsilon\}} m_j \frac{\Pi_{P_j}(x_j - x)}{|x_j - x|}}{\displaystyle\sum_{j=1}^N \mathbb{1}_{\{|x_j - x| < \varepsilon\}} m_j |x_j - x|} \, . \tag{6.18}$$

Remark 6.3 (Choice of the kernel). Notice that the special form of the reversed tent kernel allows the simplification of $\varepsilon\zeta\left(\frac{|x_j - x|}{\varepsilon}\right) = |x_j - x|$ in (6.18), which makes the expression independent of ε (except for considering or not a point, of course). It appears that this special kernel, though not regular at all, behaves much better than the tent kernel or even more regular kernels, at least on the test shapes (circle, ellipse, flower, see the next subsection) given with their exact tangents. It remains to understand why, and see if this fact remains true on shapes given with approximate tangent or on noisy shapes.



(a) reversed tent kernel (b) derivative of the reversed tent kernel

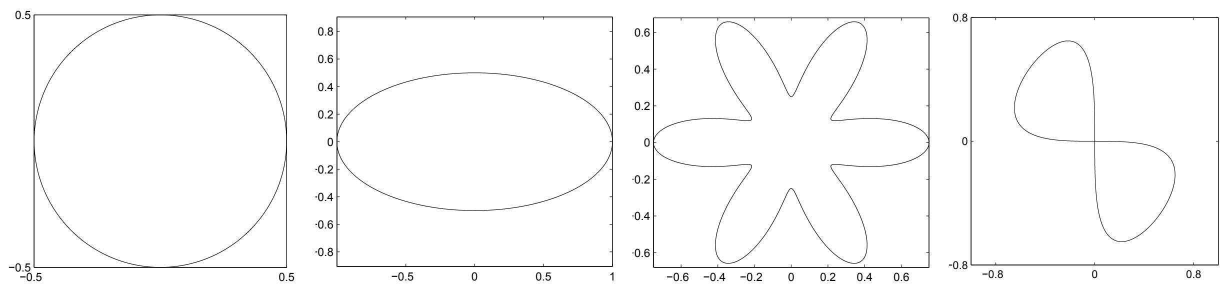
Figure 6.1: Profile of the reversed tent kernel and its derivative

6.1.3 Test shapes

We study the convergence of (6.18) on different shapes with respect to

- the number of points N in the point cloud,
- the radius ε of the ball supporting ρ_ε,
- the mean number of points N_{neigh} in a ball of radius ε centered in the point cloud.

Let us give the different shapes on which we will test (6.18):



(a) (b) (c) (d)

(a) A circle of radius 0.5

(b) An ellipse parametrized by $\begin{cases} x = a\cos(t) & t \in (0, 2\pi) \\ y = b\sin(t)\,, & \end{cases}$ with $a = 1$ and $b = 0.5$. In this case, the curvature vector is given by $H(t) = |H(t)|n(t)$ where $n(t)$ is the unit normal and

$$|H(t)| = \frac{a^2}{b}\left(1 - e^2\cos^2(t)\right)^{-\frac{3}{2}}\,, \quad e^2 = 1 - \left(\frac{b}{a}\right)^2 .$$

(c) A "flower" parametrized by $r(\theta) = 0.5(1 + 0.5\sin(6\theta + \frac{\pi}{2}))$.

(d) A "eight" parametrized by $\begin{cases} x = 0.5\,\sin(t)\,(\cos t + 1) & t \in (0, 2\pi) \\ y = 0.5\,\sin(t)\,(\cos t - 1)\,. & \end{cases}$

Remark 6.4. The specificity of our approximation of the mean curvature is that it is consistent with the 0–curvature of crossings, and more generally, it is consistent with the 0–curvature of singularities with $(\delta V)_s = 0$. We check this point on the "eight".

Unless another construction is given, the test point cloud varifolds are constructed from these parametrizations by computing the exact unit tangent vector $T(t)$, evaluating at the N points $\{0, h, 2h, \ldots, (N-1)h\}$ for $h = \frac{2\pi}{N}$, and setting

$$V_N = \sum_{j=1}^{N} m_j \delta_{(x(jh),y(jh))} \otimes \delta_{T(jh)} \,.$$

As this way of constructing point clouds is almost uniform, we consider that the weight m_j of each point is the same that is, for all j, $m_j = m$. And in this case, we do not need to compute m in (6.18). For all these shapes, the curvature vector $H(t)$ can be computed explicitly and evaluated at all $t_j = jh$, $j = 0 \ldots N-1$. To test the accuracy of the approximation (6.18), we compute the following average error on the curvature vector

$$E = \frac{1}{N} \sum_{j=1}^{N} |H_\varepsilon^N(x_j) - H(t_j)| \,. \tag{6.19}$$

6.1.4 Zero curvature of a crossing

Formula (6.18) is specially adapted to singularities of crossing-type, meaning singular curves with zero curvature in the sense of varifolds. We tested it on the "eight" figure with $N = 10^5$ points and the radius of the ball $\varepsilon = 0.01$. The computed curvature vector is represented in Figure 6.2, where the color corresponds to the norm of the computed curvature.

The advantages of Formula (6.18) is that it is very easy to compute, there is no need to know an approximation of the local length or area, it is *not depending on the orientation* (because it comes from varifolds setting and varifolds are not oriented) and it preserves the 0–singular curvature. But there is a major drawback, the preservation of 0–curvature at crossings is obtained thanks to a phenomenon of compensation. Indeed, the term

$$\mathbb{1}_{\{|x_j - x| < \varepsilon\}} \frac{\Pi_{P_j}(x_j - x)}{|x_j - x|}$$

is of order 1 and has to be compensated by a "symmetric point" (with respect to the normal at x) in the ball $B_\varepsilon(x)$ to produce a term of order ε with orientation given by the normal $n(x)$ to x. This is not particular to the discretized formula, it occurs also at the continuous level as represented in Figure 6.3. But this compensation phenomenon produces great instability at the discrete level. Let us consider a simple example:

Example 6.1. Let $S = [0,1] \times \{0\} \subset \mathbb{R}^2$ and discretize the segment S into a uniform point cloud, for instance

$$V_N = \sum_{j=1}^{N} \frac{1}{N} \delta_{\left(\frac{j}{N},0\right)} \otimes \delta_{e_1} \text{ with } e_1 \text{ the horizontal direction} \,.$$

Take a point x_0 and compute the approximated curvature at this point, in a ball of radius ε. Assume that for some reason, the discretization was not completely uniform and in the ball centered at x_0, there are n_+ points (in the cloud) on the right of x_0, and n_- points on the left, with $|n_+ - n_-| \geqslant 1$.

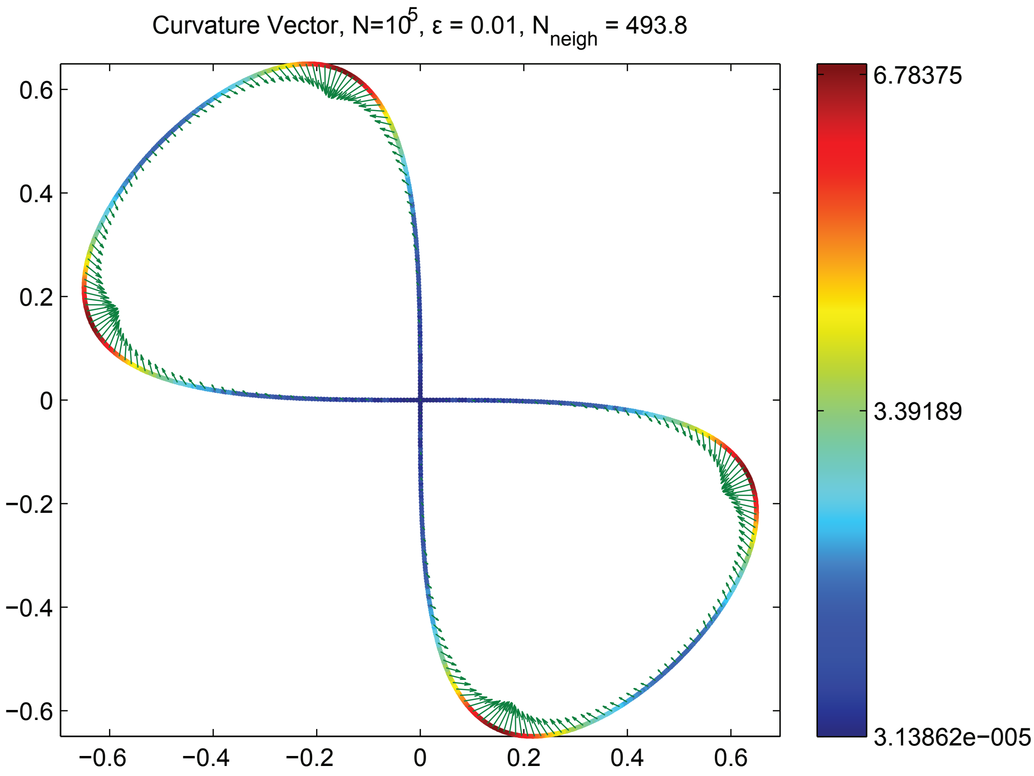


Figure 6.2: Zero curvature at a crossing point

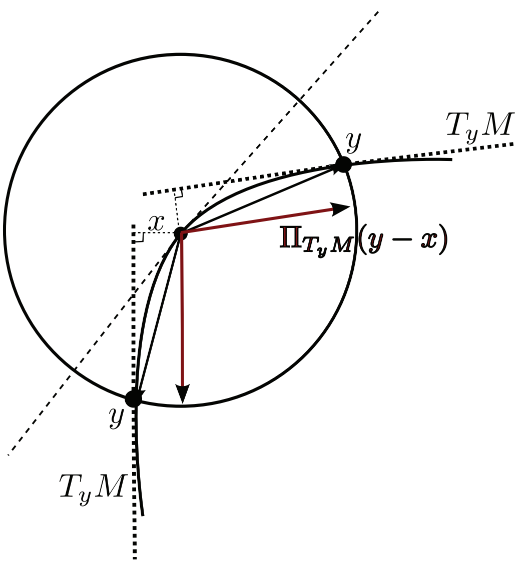


Figure 6.3: Compensation phenomenon

Then, formula (6.18) gives

$$\left|H_\varepsilon^N(x_0)\right| = \left|\frac{\displaystyle\sum_{j=1}^N \mathbb{1}_{\{|x_j-x_0|<\varepsilon\}} m_j \frac{\Pi_{e_1}(x_j-x_0)}{|x_j-x_0|}}{\displaystyle\sum_{j=1}^N \mathbb{1}_{\{|x_j-0|<\varepsilon\}} m_j |x_j-x_0|}\right| = \frac{m|n_+-n_-|}{\displaystyle\sum_{j=1}^N \mathbb{1}_{\{|x_j-0|<\varepsilon\}} m |x_j-x_0|} \simeq \frac{|n_+-n_-|}{|n_++n_-|\varepsilon}$$
$$\simeq \frac{1}{N_{neigh}\varepsilon}\,.$$

And imposing that $\frac{1}{N_{neigh}\varepsilon}$ is small forces to take large radii ε.

In order to avoid numerical instability linked to this compensation, it is possible to project the result onto the normal vector, but it becomes sensitive to the computation of the normal at a point. We thus prefer to project each term on the normal N_j at x_j. This allows to preserve a formula which is an average, and now the compensation is of smaller order. But of course, there is a priori no reason that this formula approximates the mean curvature vector, and it no longer preserves the 0–curvature at crossing points. We discuss this new formula in the next section.

6.1.5 Formula with projection onto the normal vector

As we just explained, we want to replace the projector onto the tangent Π_P by a projector onto the normal $\Pi_{P^\perp}$, but we do not know what is the limit of this new ratio (if it exists):

$$-\frac{\displaystyle\int_{B_\varepsilon(x)} \frac{\Pi_{P^\perp(y)}(y-x)}{|y-x|}\, d\|V\|(y)}{\displaystyle\int_{B_\varepsilon(x)} |y-x|\, d\|V\|(y)}\,, \tag{6.20}$$

neither how it is connected to the mean curvature. Let us then compute the limit of (6.20). For simplicity, we do the computation for curves in dimension 2, but it can be done for surfaces and hypersurfaces in the same way (locally parametrizing the sub-manifold on the tangent space). Let Γ be a C^2 curve and $x \in \Gamma$, for simplicity, let $x = 0$. In the adapted coordinates, Γ is locally parametrized by $(h, \gamma(h))$ such that

$$\gamma(h) = a\, h^2 + o(h^2) \text{ with } 2a = |H(0)|\,.$$

Therefore the tangent unit vector is given by

$$\frac{(1,\gamma'(h))}{\sqrt{1+\gamma'(h)^2}} = \frac{1}{\sqrt{1+4a^2h^2+o(h^2)}}(1, 2ah+o(h)) = \frac{1}{1+o(h)}(1,2ah+o(h)) = (1+o(h), 2ah+o(h))\,,$$

the normal unit vector is given by

$$(-2ah, 1) + o(h)$$

the radial vector is given by

$$\frac{y}{|y|} = \frac{(h,\gamma(h))}{|(h,\gamma(h))|} = \frac{1}{\sqrt{h^2+a^2h^4+o(h^4)}}(h, ah^2+o(h^2))$$
$$= \frac{1}{|h|}\frac{1}{\sqrt{1+ah^2+o(h^2)}}(h, ah^2+o(h^2)) = (1+o(h))\left(\frac{(h,ah^2)}{|h|}+o(h)\right) = \frac{(h,ah^2)}{|h|} + o(h)\,.$$

Therefore

$$\begin{aligned}\frac{\Pi_{P^\perp}(y)}{|y|} &= \left\langle \frac{(h, ah^2)}{|h|} + o(h), (-2ah, 1) + o(h) \right\rangle ((-2ah, 1) + o(h)) = \frac{-ah^2}{|h|}(-2ah, 1) + o(h) \\ &= -a|h|(0, 1) + o(h) \,,\end{aligned}$$

and

$$\begin{aligned}\int_{\Gamma \cap B_\varepsilon(0)} \frac{\Pi_{P^\perp}(y)}{|y|} \, d\mathcal{H}^1(y) &= \int_{h=-\varepsilon}^{\varepsilon} (-a|h|(0,1) + o(h)) \sqrt{1 + 4a^2h^2} \, dh + o(\varepsilon^2) = -2a \int_{h=0}^{\varepsilon} h \, dh + o(\varepsilon^2) \\ &= -a\varepsilon^2 + o(\varepsilon^2) \,. \end{aligned} \tag{6.21}$$

We have moreover

$$\int_{\Gamma \cap B_\varepsilon(0)} |y| \, d\mathcal{H}^1(y) = \int_{h=-\varepsilon}^{\varepsilon} (|h| + o(h)) \, dh + o(\varepsilon^2) = \varepsilon^2 + o(\varepsilon^2) \,. \tag{6.22}$$

Thanks to (6.21) and (6.22), we finally obtain that when V is a varifold associated with a regular curve,

$$-\frac{\displaystyle\int_{B_\varepsilon(x)} \frac{\Pi_{P^\perp(y)}(y-x)}{|y-x|} \, d\|V\|(y)}{\displaystyle\int_{B_\varepsilon(x)} |y-x| \, d\|V\|(y)} \xrightarrow[\varepsilon \to 0]{} -\frac{H(x)}{2} \,.$$

We now test the following formula, with projection onto the normal space, on point cloud discretizations of regular curves:

$$2\frac{\displaystyle\int_{B_\varepsilon(x)} \frac{\Pi_{P^\perp(y)}(y-x)}{|y-x|} \, d\|V\|(y)}{\displaystyle\int_{B_\varepsilon(x)} |y-x| \, d\|V\|(y)} \,,$$

which gives for a point cloud varifold $V_N = \sum_{j=1}^N m_j \delta_{x_j} \otimes \delta_{P_j}$,

$$\boxed{\frac{\displaystyle\sum_{j=1}^N \mathbb{1}_{\{|x_j - x| < \varepsilon\}} m_j \frac{\Pi_{P_j^\perp}(x_j - x)}{|x_j - x|}}{\displaystyle\sum_{j=1}^N \mathbb{1}_{\{|x_j - x| < \varepsilon\}} m_j |x_j - x|}} \,. \tag{6.23}$$

We first test this formula on the circle of radius 0.5 with exact given normals, and assuming that the weights m_j are all equal (since the discretization is uniform). We represent the result obtained for $N = 10^5$ points and $\varepsilon = 0.001$ in Figure 6.4. Color values represent again the norm of the numerical curvature, to be compared with the exact value $|H| = 2$.

As we already mentioned, there is another important parameter to study (apart from N and ε): the number N_{neigh} of points in a ball of radius ε, which is directly connected to the quantity εN. Therefore, we study the evolution of the averaged error on the curvature vector E (see (6.18)) with respect to the number of points N, for three different values of εN corresponding to the three curves in the following figure.

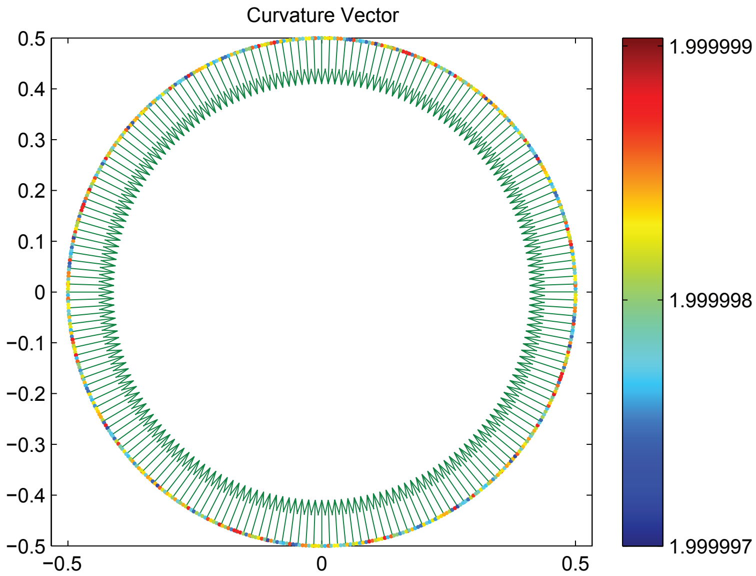
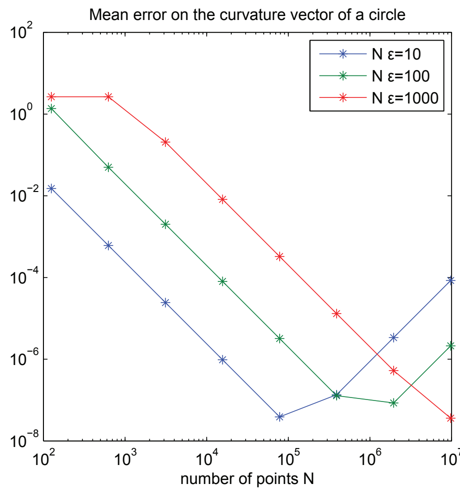


Figure 6.4: Approximate curvature of a circle

	circle		
$N\,\varepsilon$	10	100	1000
N_{neigh}	7	63	637
N	error E on the curvature		
5^3	$1.51.10^{-2}$	1.36	2.66
5^4	$6.06.10^{-4}$	$4.98.10^{-2}$	2.66
5^5	$2.42.10^{-5}$	$2.01.10^{-3}$	0.207
5^6	$9.70.10^{-7}$	$8.02.10^{-5}$	$8.19.10^{-3}$
5^7	$3.88.10^{-8}$	$3.21.10^{-6}$	$3.28.10^{-4}$
5^8	$1.35.10^{-7}$	$1.28.10^{-7}$	$1.31.10^{-5}$
5^9	$3.37.10^{-6}$	$8.51.10^{-8}$	$5.25.10^{-7}$
5^{10}	$8.43.10^{-5}$	$2.53.10^{-6}$	$3.60.10^{-8}$



Remark 6.5. We observe that for a given $\varepsilon\, N$, there is an optimal radius ε_{opt} (or equivalently N_{opt}) minimizing the error E. In other words, when the discretization scale is fixed (in our case, by the number of points N or rather by $\frac{1}{N}$), there is an optimal scale ε_{opt} to compute the curvature. Indeed, if ε is small, the irregularity, due to the fact that we consider a discrete object, distorts the computation of the curvature. Moreover, as our approximation consists in averaging in a ball of radius ε, if ε is large, our approximation of the curvature loses accuracy. In between, there is an optimal radius ε, that is an optimal scale, to compute the curvature. We thus study (see Figure 6.5) this optimal radius ε_{opt} for different numbers of points $N = 10^5$, $N = 10^6$ $N = 10^7$ and we observe that the optimal number of points ($N_{{neigh}_{opt}} = 5$, $N_{{neigh}_{opt}} = 67$, $N_{{neigh}_{opt}} = 763$) per ball increases with N. So that for great numbers of points, the optimal radii are too large.

We now make the same tests on two other shapes and what we observe is coherent with what was observed for the circle. We begin with an ellipse parametrized by $\left\{ \begin{array}{lcl} x & = & a\,\cos(t) \qquad t \in (0, 2\pi) \\ y & = & b\,\sin(t)\,, \end{array} \right.$

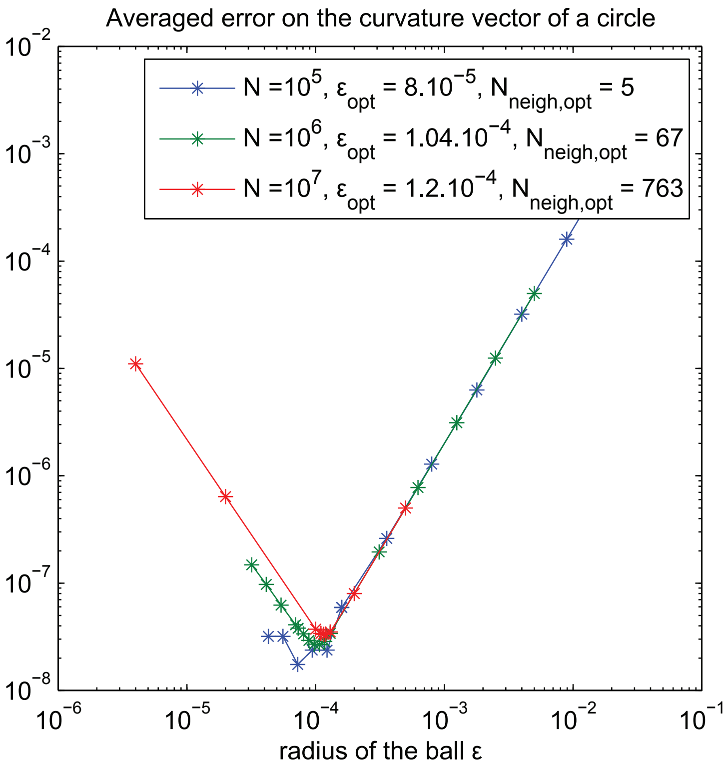
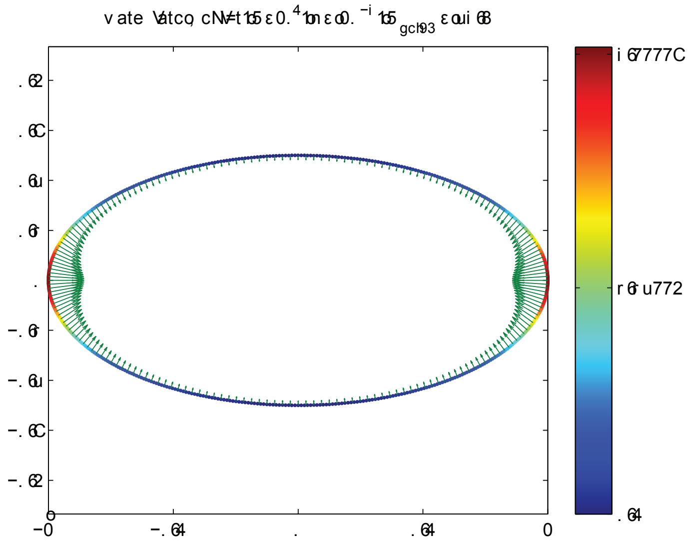
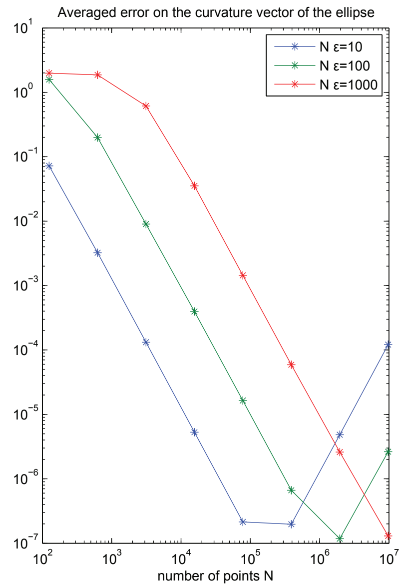


Figure 6.5: Optimal ε_{opt} and $N_{neigh_{opt}}$ for a given N

with $a = 1$ and $b = 0.5$, see Figure 6.6. Notice that for these values of a and b, the minimal and maximal values of the norm of curvature are 0.5 and 4.



(a) Approximate curvature of an ellipse



(b) Average curvature error for the ellipse

Figure 6.6: Curvature of an ellipse

We show in Figure 6.7 the same tests on the "flower" parametrized by $r(\theta) = 0.5(1+0.5\sin(6\theta+\frac{\pi}{2}))$. We observe that we have the same order of convergence on the error E and that the approximation is good even at points where the curvature is very high.

Many aspects need to be clarified concerning the link between the optimal parameters or the difference between the various kernels: is the reversed tent kernel the best choice? is it still the best choice for noisy shapes? and if it seems to be the best choice, why? For the moment, this computation is not very robust to noise, more precisely, the size of the ball needed to have a good curvature is too large with respect to the noise. Is it possible to stabilize the formula by replacing the convolution of the mass $\displaystyle\sum_{x_j \in B_\varepsilon(x)} \zeta\left(\frac{|x_j - x|}{\varepsilon}\right)$ by an equivalent of the continuous quantity (some computations are needed)

$$\int_{B_\varepsilon(x)} \zeta\left(\frac{|y-x|}{\varepsilon}\right) d\|V\|(y) \sim_{\varepsilon\to 0} \theta(x)\varepsilon^d \int_{r=0}^{1} \mathcal{H}^{d-1}(\mathbb{S}^{d-1}) r^{d-1}\zeta(r)dr\,.$$

And using $\displaystyle\sum_{x_j \in B_\varepsilon(x)} m_j = \|V\|(B_\varepsilon(x)) \sim_{\varepsilon\to 0} \theta(x)\omega_d\varepsilon^d$, we infer

$$m_j = m \sim_{\varepsilon\to 0} \frac{1}{\operatorname{card}\{j \,:\, x_j \in B_\varepsilon(x)\}} \omega_d \theta(x)\varepsilon^d\,,$$

so that (6.23) becomes (with some computations)

$$\frac{1}{\varepsilon}\frac{2d}{d+1}\frac{1}{\operatorname{card}\{j \,:\, x_j \in B_\varepsilon(x)\}} \sum_{x_j \in B_\varepsilon(x)} \frac{\Pi_{P_j^\perp}(x_j - x)}{|x_j - x|}\,. \tag{6.24}$$

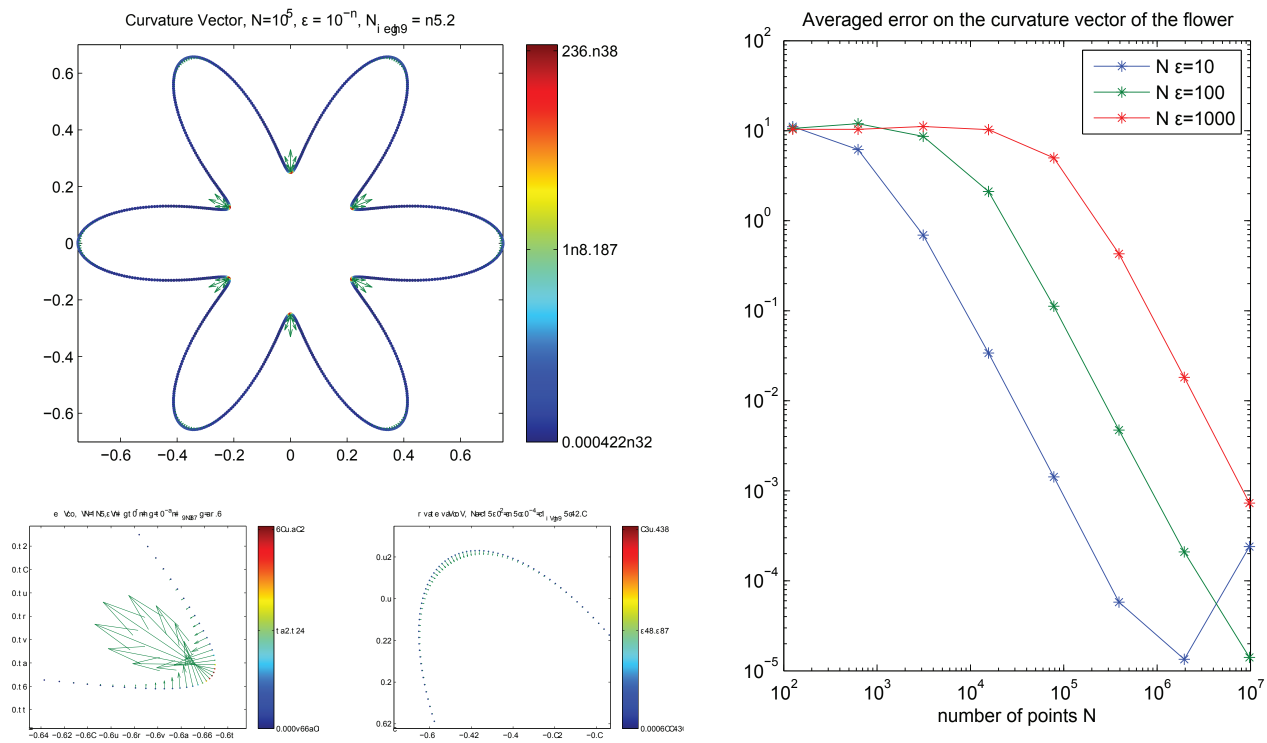


Figure 6.7: Approximating the curvature of the flower

We do not enter into more details since it is only prospective for now. It would be close to the formula obtained in [CRT04] which is (for a surface M and a dimensional constant $c(d)$)

$$\fint_{B_\varepsilon(x)\cap M} (y-x)\, d\mathcal{H}^2(y) = c(d)H(x) + o(\varepsilon^2)\,. \tag{6.25}$$

6.2 And for $3D$ point clouds ...

We now work with $3D$ point clouds. We still use the formula (6.23), that is with the reversed tent kernel and with projection onto the normal space. Notice that the computations we did to pass from the projector onto the tangent space to the projector onto the normal space can be done (exactly in the same way) by locally parametrizing a surface with its tangent space. We first test this formula on a ball of radius 1, parametrized with spherical coordinates. We could use the exact normal as we did for $2D$ point clouds, but we want to deal with more general point clouds (not given with their normal vectors) and we want to make point clouds evolve by curvature flows. For those reasons, we now compute the normal direction at each point thanks to a regression. In this section, computations are done using a C++ code and the libraries nanoflann and eigen, and the visualization uses the software CloudCompare.

6.2.1 Computation of the mean curvature on $3D$ point clouds

We first test it on a ball of radius 1 for different values of the number of points N and different radii ε. The computation of the normal vector is done by constructing the covariance matrix of centered coordinates and taking the eigenvector associated with the smallest eigenvalue (this computation is done in a ball of radius $\frac{\varepsilon}{2}$). Notice that the number of points in a ball of radius ε is now closely linked to the quantity $N\varepsilon^2$.

	ball	
$N\,\varepsilon^2$	127	507
N_{neigh}	33	128
N	error E on the curvature	
12684	$3.28.10^{-3}$	$1.21.10^{-2}$
79456	$7.13.10^{-4}$	$1.97.10^{-3}$
318062	$2.84.10^{-4}$	$5.21.10^{-4}$
715811	$1.76.10^{-4}$	$2.46.10^{-4}$
1988806	$7.4.10^{-5}$	$9.3.10^{-5}$
7956514	$2.5.10^{-5}$	$2.5.10^{-5}$

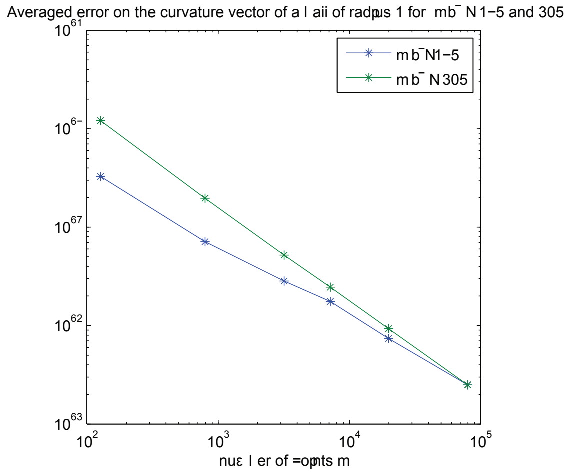


Figure 6.8: Averaged error on the mean curvature vector of a ball of radius 1: *evolution of the error with the number of points N, for two values of $N\varepsilon^2$.*

We then test this computation of mean curvature on a more complicated object: a point cloud representing a (surface) dragon and constituted of $N = 435000$ points. We represent the norm of the mean curvature in Figure 6.9.

6.2.2 Toward mean curvature flows for $3D$ point clouds

In this section, we test the evolution of a point cloud by the discrete mean curvature flow,

$$x_k^{n+1} = x_k^n + dtH(x_k^n)\,,$$

where $H(x_k^n)$ is the approximation of the mean curvature at the point x_k^n given by formula (6.23) and dt is a prescribed time step. Of course, as this scheme is explicit, we expect that instabilities appear. Let us begin with the flow of a ball of radius 1, with a large radius $\varepsilon = 0.6$ and a large enough time step $dt = 0.01$. After 40 iterations, we obtain in Figure 6.10 a smaller ball of radius around 0.6 (which is coherent with the time step and the curvature of the ball).

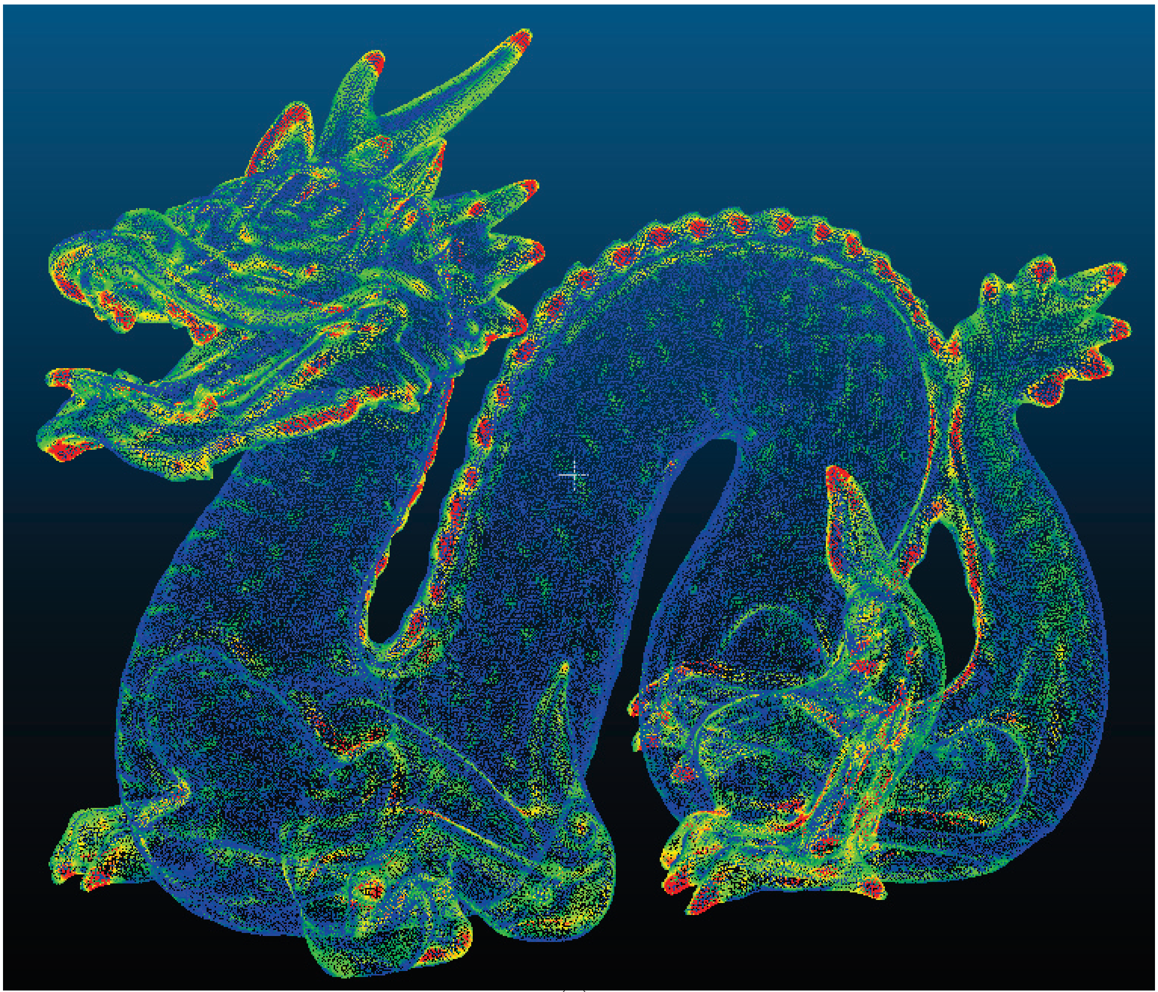
As expected, we can see instabilities appearing after 40 iterations, and the point cloud is no longer a "ball" after 50 iterations. This corresponds to the time when the radius of the ball used for computing curvature is the same as the radius of the ball itself.

We now observe the effects of this mean curvature flow on the bunny of diameter around 7 constituted of $N = 34835$ points. We take a radius $\varepsilon = 0.5$ and a step time $dt = 0.001$. We can observe that after 120 iterations, the body of the bunny has been smoothed (see the back of the bunny which is wavy before the flow in Figure 6.11). We also understand why the flow crashes at that moment: the ears are thin and collapse after 120 iterations. The color corresponds to the intensity of the computed curvature.

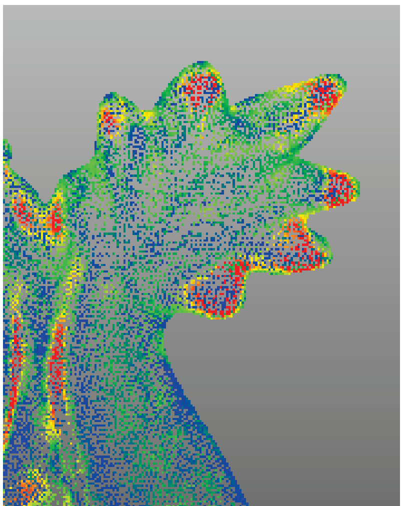
Let us now end with an entertaining experiment illustrated in Figure 6.12: we let the bunny evolve in the same conditions, but by the reverse mean curvature flow,

$$x_k^{n+1} = x_k^n - dtH(x_k^n)\,.$$

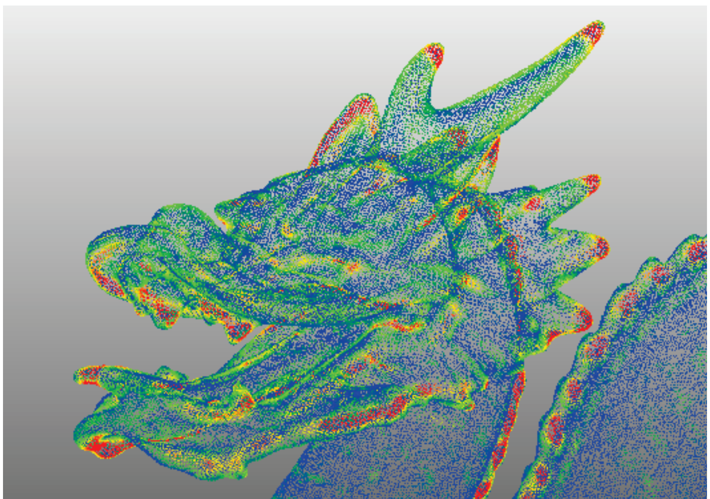
Let us conclude this chapter by mapping out some perspectives about the numerics related to the (direct) discrete mean curvature flow, whose stability issues have been mentioned above. A first aspect is to stabilize the approximation of the mean curvature itself, by changing our current approximation



(a)

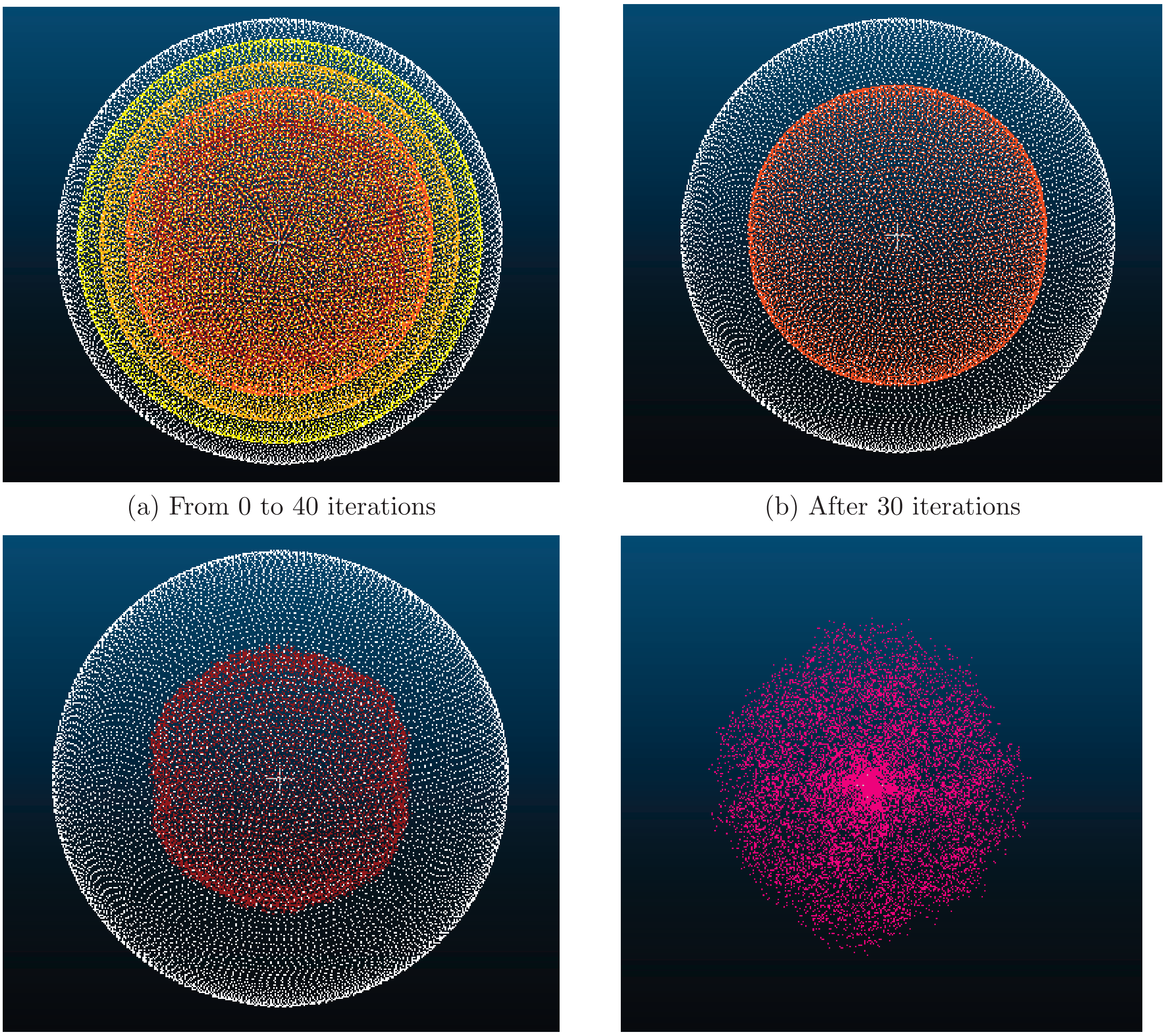


(b) Details on the tail of the dragon



(c) Details on the head of the dragon

Figure 6.9: Intensity of the mean curvature of a dragon, *the computations are done for* $\varepsilon = 0.02$ *for a dragon of diameter* 1



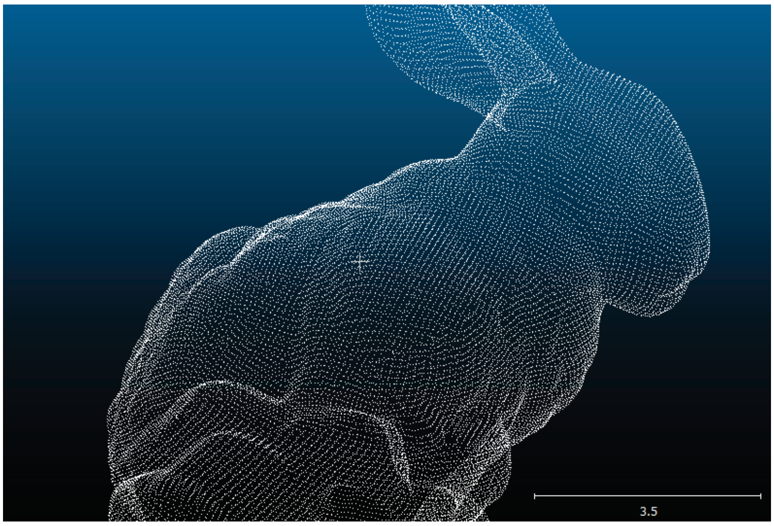
(a) From 0 to 40 iterations

(b) After 30 iterations

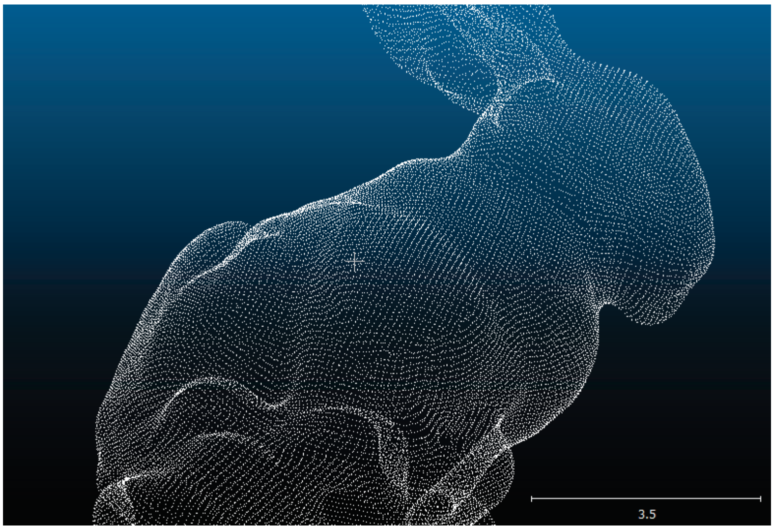
(c) After 40 iterations

(d) After 50 iterations

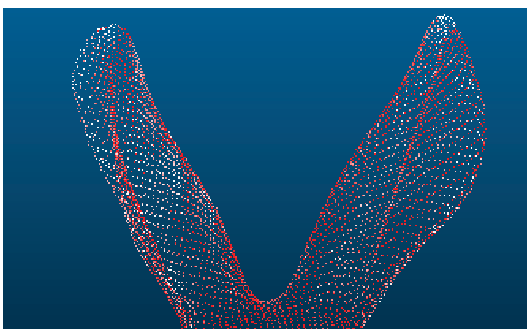
Figure 6.10: Balls evolving by mean curvature flow, *with radius* $\varepsilon = 0.6$ *and time step* $dt = 0.01$.



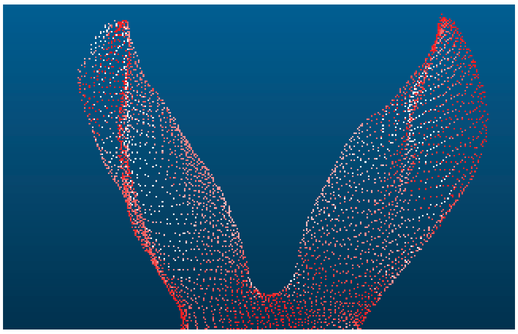
(a) Time 0



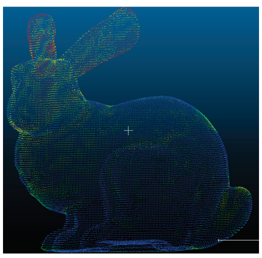
(b) After 120 iterations



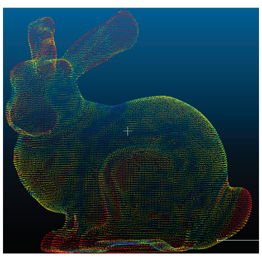
(c) Time 0



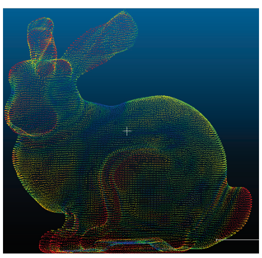
(d) After 120 iterations



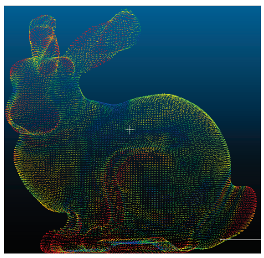
(e) Time 0



(f) Time 50



(g) Time 100



(h) Time 120

Figure 6.11: Bunny evolving by mean curvature flow: *global evolution and comparison after* 120 *iterations with* $dt = 0.001$ *and* $\varepsilon = 0.5$.

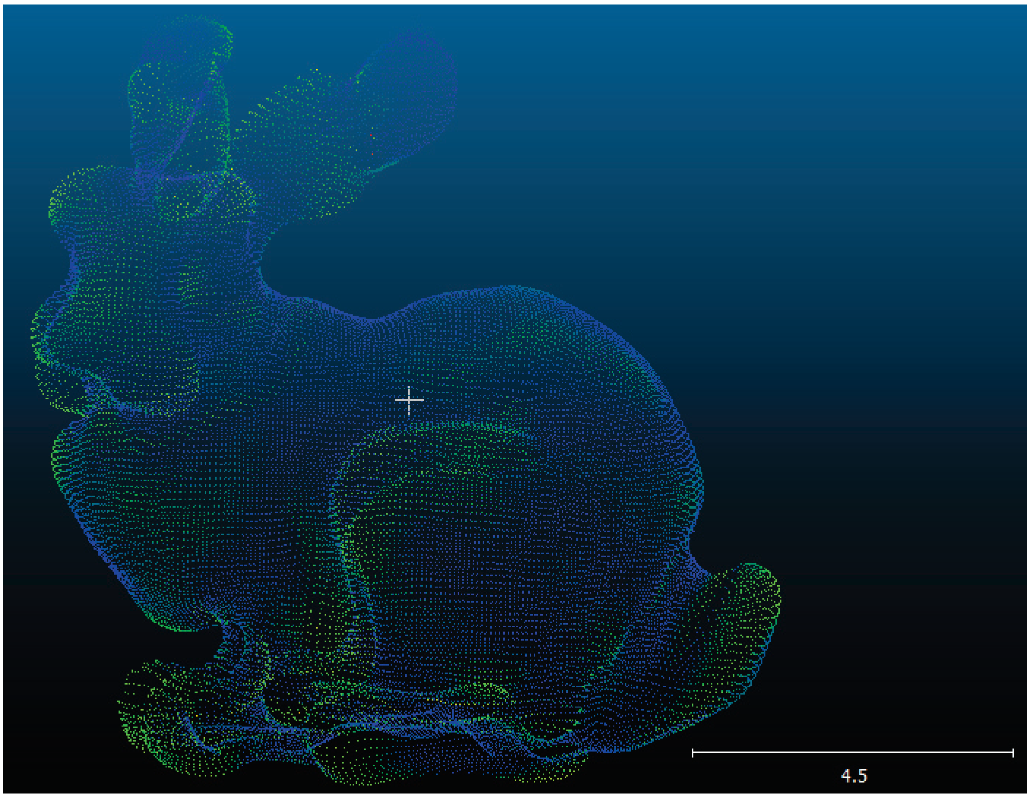


Figure 6.12: Bunny after a reverse main curvature flow: *After* 340 *iterations with a radius* $\varepsilon = 0.5$ *and a time step* $dt = 0.001$

for the one proposed in (6.24) and by fixing the number of points used to the computation instead of fixing the radius of the ball used. Another aspect is the instability due to the explicit discretization in time: is it possible to design a reasonable implicit or semi-implicit scheme with our approximation of the mean curvature? This will be the purpose of future work.

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Approximation de surfaces par des varifolds discrets : représentation, courbure, rectifiabilité

Résumé :
La motivation initiale de cette thèse est l'étude d'une discrétisation volumique de surface (introduite dans le Chapitre 2) naturellement liée la structure de varifold. La théorie des varifolds a été développée par F. Almgren afin d'étudier les points critiques de la fonctionnelle d'aire. L'ensemble des varifolds rectifiables entiers fournit une notion de surface faible possédant de bonnes propriétés de compacité et munie d'une notion de courbure généralisée appelée variation première. Le point clé est qu'il est possible de munir d'une structure de varifold la plupart des objets utilisés pour représenter ou discrétiser des surfaces c'est-à-dire aussi bien des objets tels que les sous-variétés ou les ensembles rectifiables que des objets tels que des nuages de points ou encore la discrétisation volumique proposée, ce qui permet d'étudier dans un cadre unifié une surface et sa discrétisation.
Une difficulté essentielle est que, généralement, ces structures discrètes ne sont pas rectifiables, ce qui soulève la question suivante : comment assurer qu'un varifold, obtenu comme limite de discrétisations volumiques de la forme proposée, soit une surface, au moins en un sens faible ? De façon plus précise : quelles conditions sur une suite de varifolds quelconques assurent que le varifold limite est rectifiable (Chapitre 3) ou encore qu'il est à variation première bornée (Chapitre 5) ? Afin de tester la rectifiabilité d'un varifold, on peut étudier l'existence d'un plan tangent en presque tout point, mais la faon classique de le définir n'est pas adaptée (c'est-à-dire qu'elle ne se transfère pas aisément de la suite de varifolds à sa limite). Afin d'y remédier, on considère le plan tangent comme minimiseur d'une énergie liée aux nombres β de Jones, ce qui nous permet d'obtenir des conditions assurant la rectifiabilité d'une limite de varifolds.
On s'intéresse ensuite la régularité du varifold limite en termes de courbure (variation première). Dans un premier temps, on a essayé de contrôler la variation première en observant qu'une certaine moyenne de la variation première sur des boules concentriques se réécrivait de façon à avoir un sens même pour un varifold à variation première non bornée. On a alors essayé de reconstruire par "packing" la variation première uniquement grâce à ces moyennes (Chapitre 4), mais cela n'a pas permis d'établir les conditions désirées. En revanche, cela nous a conduit considérer une forme régularisée de la variation première d'un varifold, ce qui a permis d'établir des conditions assurant que la limite d'une suite de varifolds est à variation première bornée. Cette régularisation permet de définir des énergies de Willmore approchées qui Γ–convergent dans l'espace des varifolds vers l'énergie de Willmore classique ainsi qu'une approximation de la courbure qui est testée numériquement dans le Chapitre 6.

Mots clés : Varifold ; Rectifiabilité ; Courbure ; Surfaces discrètes.

Discrete varifolds and surface approximation : representation, curvature, rectifiability

Keywords : Varifold ; Rectifiability ; Curvature ; Discrete surfaces.

Image en couverture : Intensité de la courbure moyenne calculée sur un nuage de points.

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INFOMATHS
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