Marches aléatoires sur Out(Fn) et sous-groupes d’automorphismes de produits libres

Camille Horbez

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Camille Horbez

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Institut de Recherche Mathématique de Rennes
U.F.R. Mathématiques

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devant le jury composé de :

Mladen BESTVINA
Professeur – University of Utah / *rapporteur*

Martin BRIDSON
Professeur – University of Oxford / *rapporteur*

Sébastien GOUËZEL
Professeur – Université de Rennes 1 / *examinateur*

Vincent GUIRARDEL
Professeur – Université de Rennes 1 / *directeur de thèse*

Anders KARLSSON
Professeur – Université de Genève / *examinateur*

Frédéric PAULIN
Professeur – Université Paris-Sud / *examinateur*

Karen VOGTMANN
Professeur – University of Warwick / *examinateur*

au vu des rapports de **M. BESTVINA**, **M. BRIDSON** et **P. HAÏSSINSKY**

Résumé

Soit G un groupe dénombrable, qui se scinde en un produit libre de la forme

$$G = G_1 * \cdots * G_k * F,$$

où F est un groupe libre de type fini, et les G_i sont librement indécomposables et non isomorphes à $\mathbb{Z}$. Nous montrons que le groupe $\mathrm{Out}(G)$ des automorphismes extérieurs de G satisfait l'alternative de Tits, dès lors que chacun des groupes G_i et $\mathrm{Out}(G_i)$ la satisfait. Par des méthodes similaires, nous montrons aussi l'alternative suivante pour tout sous-groupe H de $\mathrm{Out}(F_N)$, due à Handel et Mosher lorsque H est de type fini : soit H fixe virtuellement la classe de conjugaison d'un facteur libre propre de F_N, soit H contient un automorphisme complètement irréductible. Nos méthodes, géométriques, utilisent l'étude de la dynamique de l'action de certains sous-groupes de $\mathrm{Out}(G)$ sur des espaces hyperboliques. Nous décrivons notamment l'adhérence de l'outre-espace de G relatif aux G_i, et le bord de Gromov du complexe (hyperbolique) des scindements cycliques relatifs associé.

Nous étudions par ailleurs les marches aléatoires sur $\mathrm{Out}(F_N)$. Sous un certain nombre de conditions sur la mesure de probabilité μ, nous montrons que presque toute trajectoire de la marche aléatoire sur $(\mathrm{Out}(F_N), \mu)$ converge vers un point du bord de Gromov du complexe des facteurs libres de F_N, que nous identifions au bord de Poisson de $(\mathrm{Out}(F_N), \mu)$. Par ailleurs, nous décrivons l'horofrontière de l'outre-espace. Ceci a des applications à l'étude de la croissance des classes de conjugaison de F_N sous l'effet de produits aléatoires d'automorphismes extérieurs.

Abstract

Let G be a countable group that splits as a free product of the form

$$G = G_1 * \cdots * G_k * F,$$

where F is a finitely generated free group, and the groups G_i are freely indecomposable and not isomorphic to $\mathbb{Z}$. We show that $\mathrm{Out}(G)$ satisfies the Tits alternative, as soon as all the groups G_i and $\mathrm{Out}(G_i)$ do. Similar techniques also yield another alternative for subgroups H of $\mathrm{Out}(F_N)$, due to Handel and Mosher when H is finitely generated, namely : either H virtually fixes the conjugacy class of some proper free factor of F_N, or H contains a fully irreducible automorphism. Our methods are geometric, and require understanding the dynamics of the action of some subgroups of $\mathrm{Out}(G)$ on Gromov hyperbolic spaces. In particular, we determine the closure of the outer space of G relative to the G_i's, as well as the Gromov boundary of the (hyperbolic) complex of relative cyclic splittings of G.

We also study random walks on $\mathrm{Out}(F_N)$. Given a probability measure μ on $\mathrm{Out}(F_N)$ (satisfying some conditions), we prove that almost every sample path of the random walk on $(\mathrm{Out}(F_N), \mu)$ converges to a point of the Gromov boundary of the free factor complex of F_N, which we identify with the Poisson boundary of $(\mathrm{Out}(F_N), \mu)$. We also describe the horoboundary of outer space, and give applications to growth of conjugacy classes of F_N under random products of outer automorphisms.

Remerciements

La recherche en mathématiques ne peut être une activité solitaire. D'abord, ce n'est pas une activité linéaire : parfois les idées viennent, mais souvent aussi les choses semblent ne plus avancer. Discuter avec d'autres peut alors avoir un rôle salvateur, et aider à débloquer une situation que l'on était soi-même incapable de résoudre, à ouvrir ainsi de nouvelles pistes de recherche. Pour ne jamais se décourager devant l'obstacle. Pour essayer encore. Rater encore, mais rater mieux.

Attaquer un problème nécessite aussi de puiser des idées issues de thématiques de recherche voisine, et donc d'être à l'écoute des travaux d'autres chercheurs. Enfin, pour parvenir à une bonne compréhension d'un résultat – y compris d'un résultat issu de ses propres travaux – il n'est rien de plus profitable que d'en discuter avec d'autres, afin d'en questionner les diverses facettes pour pouvoir mieux le préciser.

Cette thèse n'échappe pas à la règle, elle n'aurait sans doute jamais vu le jour sans les multiples discussions que j'ai pu avoir au cours de ces trois dernières années.

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Introduction générale

Pour étudier un groupe G, il est souvent fructueux d'en étudier l'action sur un certain espace géométrique X qui lui est naturellement associé. Le *credo* général du géomètre des groupes est qu'il est possible de déduire des propriétés de nature algébrique de G, en étudiant d'une part les propriétés topologiques et géométriques de X, et d'autre part les propriétés dynamiques de l'action de G sur X.

Notre objet d'étude sera le groupe $\mathrm{Out}(F_N)$ des automorphismes extérieurs d'un groupe libre de type fini, et plus généralement le groupe $\mathrm{Out}(G)$ des automorphismes extérieurs d'un groupe dénombrable G qui se scinde en un produit libre de la forme

$$G = G_1 * \cdots * G_k * F,$$

où F désigne un groupe libre de type fini. Le point culminant de cette thèse est la démonstration de l'alternative de Tits pour le groupe $\mathrm{Out}(G)$, sous l'hypothèse que chacun des groupes G_i est librement indécomposable et non isomorphe à $\mathbb{Z}$, et que chacun des groupes G_i et $\mathrm{Out}(G_i)$ satisfait lui-même cette alternative. Un groupe G *satisfait l'alternative de Tits* si pour tout sous-groupe $H \subseteq G$, soit H est virtuellement résoluble, soit H contient un sous-groupe libre non abélien. Cette alternative a été montrée par Tits pour les sous-groupes de type fini des groupes linéaires [Tit72], puis généralisée à un certain nombre de classes de groupes au cours des dernières décennies. Nous mentionnerons en particulier le cas des groupes hyperboliques (Gromov [Gro87]), des groupes modulaires de surface (Ivanov [Iva84], McCarthy [McC85]), ou du groupe $\mathrm{Out}(F_N)$ (Bestvina, Feighn et Handel [BFH00, BFH05]).

Plus généralement, étant donné une collection $\mathcal{C}$ de groupes, nous disons que G *satisfait l'alternative de Tits relativement à* $\mathcal{C}$ si pour tout sous-groupe $H \subseteq G$, soit $H \in \mathcal{C}$, soit H contient un sous-groupe libre non abélien (le cas classique correspond au cas où $\mathcal{C}$ est la collection des groupes virtuellement résolubles). Notre résultat principal est le suivant.

Théorème 1.

Soit G un groupe dénombrable, qui se scinde en un produit libre de la forme

$$G = G_1 * \cdots * G_k * F,$$

où F est un groupe libre de type fini, et chacun des groupes G_i est librement indécomposable et non isomorphe à $\mathbb{Z}$. Soit $\mathcal{C}$ une collection de groupes, qui est stable par isomorphismes, contient $\mathbb{Z}$, et est stable par passage aux sous-groupes, aux extensions, et aux surgroupes d'indice fini. Supposons que pour tout $i \in \{1, \dots, k\}$, les groupes G_i et $\mathrm{Out}(G_i)$ satisfont l'alternative de Tits relativement à $\mathcal{C}$. Alors $\mathrm{Out}(G)$ et $\mathrm{Aut}(G)$ satisfont l'alternative de Tits relativement à $\mathcal{C}$.

La collection des groupes virtuellement résolubles satisfait les hypothèses du Théorème 1. Par conséquent, si chacun des groupes G_i et $\mathrm{Out}(G_i)$ satisfait l'alternative de Tits classique, il en est de même de $\mathrm{Out}(G)$ et $\mathrm{Aut}(G)$. En particulier, nous donnons une nouvelle démonstration de l'alternative de Tits pour le groupe $\mathrm{Out}(F_N)$. Le Théorème 1 permet de montrer l'alternative de Tits pour les groupes d'automorphismes de certaines classes intéressantes de groupes, comme les groupes d'Artin à angles droits, ou les groupes relativement hyperboliques toriques (ou plus généralement, les groupes sans torsion hyperboliques relativement à une famille finie $\mathcal{P}$ de groupes de type fini, telle que pour tout $H \in \mathcal{P}$, les groupes H et $\mathrm{Out}(H)$ satisfassent l'alternative de Tits).

En utilisant des techniques analogues, nous obtenons également une autre alternative pour les sous-groupes de $\mathrm{Out}(F_N)$, qui est due à Handel et Mosher dans le cas des sous-groupes de type fini. Un automorphisme $\Phi \in \mathrm{Out}(F_N)$ est *complètement irréductible* si aucune puissance non nulle de Φ ne fixe la classe de conjugaison d'un facteur libre propre de F_N.

Théorème 2.

Soit H un sous-groupe de $\mathrm{Out}(F_N)$ (non nécessairement de type fini). Alors soit

- le groupe H contient deux éléments complètement irréductibles qui engendrent un sous-groupe libre non abélien, soit
- le groupe H est virtuellement cyclique, virtuellement engendré par un automorphisme complètement irréductible, soit
- le groupe H fixe virtuellement la classe de conjugaison d'un facteur libre propre de F_N.

Soit $\mathrm{Out}(G, \{[G_1], \dots, [G_k]\})$ le sous-groupe de $\mathrm{Out}(G)$ formé des automorphismes qui préservent la classe de conjugaison de chacun des groupes G_i. Nos démonstrations des Théorèmes 1 et 2 reposent sur l'étude de l'action du groupe $\mathrm{Out}(F_N)$, et plus généralement du groupe $\mathrm{Out}(G, \{[G_1], \dots, [G_k]\})$, sur des espaces géométriques (en particulier sur certains complexes hyperboliques). Nous avons été amené à utiliser des techniques issues de la théorie des marches aléatoires sur les groupes, à travers l'étude des mesures harmoniques sur les bords de ces espaces.

La première partie de cette thèse est une introduction à l'étude géométrique de plusieurs espaces munis d'actions intéressantes du groupe $\mathrm{Out}(F_N)$. L'étude géométrique de $\mathrm{Out}(F_N)$ a été inaugurée avec la construction par Culler et Vogtmann de l'*outre-espace*, défini comme l'espace des classes d'homothéties d'actions simpliciales, libres, minimales et par isométries de F_N sur des arbres simpliciaux métriques. Le groupe $\mathrm{Out}(F_N)$ agit à droite sur l'outre-espace par précomposition des actions.

Au cours des dernières années, l'attention s'est portée sur la recherche de complexes hyperboliques munis d'actions intéressantes du groupe $\mathrm{Out}(F_N)$, analogues au complexe des courbes d'une surface compacte orientable. Plusieurs analogues ont été proposés, parmi lesquels nous citerons le graphe des facteurs libres, le graphe des scindements libres et le graphe des scindements cycliques. Une présentation des propriétés géométriques de ces complexes connues à ce jour est proposée au Chapitre 3.

Il existe des versions des espaces mentionnés ci-dessus dans le cadre plus général de produits libres de groupes. En vue de l'obtention de l'alternative de Tits pour le groupe des automorphismes extérieurs d'un produit libre de groupes, nous serons amenés à étendre l'étude de la géométrie des espaces associés à $\mathrm{Out}(F_N)$ à ce contexte plus général. Étant donné un groupe dénombrable G et un *système de facteurs libres* $\mathcal{F} := \{G_1, \dots, G_k\}$ comme ci-dessus, l'outre-espace $P\mathcal{O}(G, \mathcal{F})$, introduit par Guirardel et Levitt dans [GL07a], est l'espace des classes d'homothétie G-équivariante de G-arbres simpliciaux métriques minimaux, à stabilisateurs d'arcs triviaux, dont les stabilisateurs de sommets sont exactement les conjugués des groupes dans $\mathcal{F}$. L'outre-espace $P\mathcal{O}(G, \mathcal{F})$ se plonge dans l'espace projectif $\mathbb{PR}^G$ en associant à tout arbre $T \in P\mathcal{O}(G, \mathcal{F})$ la famille des longueurs de translation des éléments de G dans T. Nous étudions l'adhérence de $P\mathcal{O}(G, \mathcal{F})$ pour la topologie des axes, induite par ce plongement : nous décrivons les points de $\overline{P\mathcal{O}(G, \mathcal{F})}$, et en déterminons la dimension topologique. Nous disons qu'un $(G, \mathcal{F})$-arbre T (i.e. un arbre réel T, muni d'une action de G pour laquelle chacun des groupes dans $\mathcal{F}$ fixe un point) est *très petit* si les stabilisateurs d'arcs de T sont soit triviaux, soit maximalement cycliques et non conjugués à des sous-groupes des facteurs dans $\mathcal{F}$, et si les stabilisateurs de tripodes de T sont triviaux. En adaptant des arguments dus à Cohen et Lustig [CL95], Bestvina et Feighn [BF94], et Gaboriau et Levitt [GL95], nous obtenons le résultat suivant.

Théorème 3.

Soit G un groupe dénombrable, qui se scinde en un produit libre de la forme

$$G := G_1 * \cdots * G_k * F_N,$$

où F_N est un groupe libre de rang N. L'espace $\overline{P\mathcal{O}(G, \mathcal{F})}$ est l'espace des classes d'homothétie G-équivariante de $(G, \mathcal{F})$-arbres minimaux très petits non triviaux. Il est compact, de dimension topologique finie, égale à $3N + 2k - 4$. Sa frontière est de dimension topologique égale à $3N + 2k - 5$.

Suivant les arguments de Handel et Mosher [HM13a], Bestvina et Feighn [BF14c] et Mann [Man13] dans le cas classique où $G = F_N$, nous montrons également l'hyperbolicité du graphe des scindements libres $FS(G, \mathcal{F})$, du graphe des scindements cycliques $FZ(G, \mathcal{F})$, et du graphe des scindements maximalement cycliques $FZ^{max}(G, \mathcal{F})$ associés au couple $(G, \mathcal{F})$. L'hyperbolicité de $FS(G, \mathcal{F})$ a aussi été obtenue de manière indépendante par Handel et Mosher [HM14b].

Théorème 4.

Soit G un groupe dénombrable, et $\mathcal{F}$ un système de facteurs libres de G. Les graphes $FS(G, \mathcal{F})$, $FZ(G, \mathcal{F})$ et $FZ^{max}(G, \mathcal{F})$ sont hyperboliques au sens de Gromov.

Un ingrédient essentiel de notre démonstration de l'alternative de Tits pour le groupe des automorphismes d'un produit libre est la détermination du bord de Gromov des graphes $FZ(G, \mathcal{F})$ et $FZ^{max}(G, \mathcal{F})$. En particulier, nous déterminons le bord de Gromov du graphe FZ_N des scindements cycliques de F_N. Nous dirons qu'un arbre $T \in \overline{P\mathcal{O}(G, \mathcal{F})}$ est *$\mathcal{Z}$-étranger* s'il n'est compatible avec aucun arbre $T' \in \overline{P\mathcal{O}(G, \mathcal{F})}$ qui soit lui-même compatible avec un scindement cyclique de $(G, \mathcal{F})$ (deux arbres sont *compatibles* s'ils admettent un raffinement commun, nous renvoyons au Chapitre 3 pour une définition plus précise). Nous noterons $\mathcal{X}(G,\mathcal{F})$ le sous-espace de $\overline{\mathcal{PO}(G,\mathcal{F})}$ formé des arbres $\mathcal{Z}$-étrangers. Deux tels arbres T et T' sont *équivalents*, ce que nous notons $T \sim T'$, s'ils sont tous deux compatibles avec un même troisième arbre dans $\overline{\mathcal{PO}(G,\mathcal{F})}$. Il existe une application naturelle $\psi : \mathcal{PO}(G,\mathcal{F}) \to FZ(G,\mathcal{F})$.

Théorème 5.

Soit G un groupe dénombrable, et $\mathcal{F}$ un système de facteurs libres de G. Il existe un unique homéomorphisme $\text{Out}(G,\mathcal{F})$-équivariant

$$\partial\psi : \mathcal{X}(G,\mathcal{F})/{\sim} \to \partial FZ(G,\mathcal{F})$$

tel que pour tout $T \in \mathcal{X}(G,\mathcal{F})$, et toute suite $(T_n)_{n\in\mathbb{N}} \in \mathcal{PO}(G,\mathcal{F})^{\mathbb{N}}$ convergeant vers T, la suite $(\psi(T_n))_{n\in\mathbb{N}}$ converge vers $\partial\psi(T)$.

Nous établissons également un énoncé similaire pour le graphe $FZ^{max}(G,\mathcal{F})$. Afin de tirer parti de notre description de $\partial FZ(G,\mathcal{F})$ pour montrer l'alternative de Tits, il faut s'assurer que l'orbite d'un point de $FZ(G,\mathcal{F})$ sous l'action d'un sous-groupe H de $\text{Out}(G,\mathcal{F})$ qui ne préserve pas de facteur libre propre de $(G,\mathcal{F})$ (et donc en particulier ne fixe aucun point dans $FZ(G,\mathcal{F})$) est non bornée, et possède un point limite dans $\partial FZ(G,\mathcal{F})$ (auquel cas soit H contient un sous-groupe libre non abélien, soit H fixe virtuellement un point dans $\partial FZ(G,\mathcal{F})$, et nous concluons grâce à une description des stabilisateurs d'arbres dans $\overline{\mathcal{PO}(G,\mathcal{F})}$). La difficulté pour obtenir l'existence de ce point limite vient du défaut de compacité locale de $FZ(G,\mathcal{F})$. Notre argument pour contourner cette difficulté repose sur l'étude de mesures harmoniques sur l'espace compact $\overline{\mathcal{PO}(G,\mathcal{F})}$, que nous projetons ensuite sur $\partial FZ(G,\mathcal{F})$ au moyen de l'application $\partial\psi$. Lorsque H préserve un facteur libre propre de $(G,\mathcal{F})$, nous raisonnons par induction sur une notion de rang de ce facteur libre propre.

Cette étude des mesures harmoniques nous a amené à faire un détour par l'étude des marches aléatoires sur le groupe $\text{Out}(F_N)$, réalisées sur l'outre-espace ou sur le complexe des facteurs libres au moyen de l'action de $\text{Out}(F_N)$. Nous étudions en particulier deux notions de bords pour une marche aléatoire sur $\text{Out}(F_N)$, le *bord de Poisson* de $\text{Out}(F_N)$ et l'*horofrontière* de l'outre-espace CV_N.

Soit μ une loi de probabilité sur CV_N. La position au temps n de la *marche aléatoire à droite* sur $\text{Out}(F_N)$ est l'automorphisme (aléatoire) Φ_n obtenu par multiplications à droite successives d'incréments ϕ_i indépendants et tous distribués suivant la loi μ, autrement dit $\Phi_n = \phi_1 \dots \phi_n$. Lorsque le support de μ engendre un sous-groupe de $\text{Out}(F_N)$ suffisamment gros, nous montrons la convergence presque sûre d'une trajectoire typique de la marche aléatoire à droite sur $(\text{Out}(F_N),\mu)$, réalisée via l'action à gauche sur l'outre-espace CV_N, vers un simplexe de mesures associé à un arbre libre et arationnel (i.e. pour lequel tout facteur libre propre de F_N agit de manière simpliciale et libre sur son sous-arbre minimal) dans $\overline{CV_N}$. En utilisant la description de Bestvina et Reynolds [BR13] et Hamenstädt [Ham14a] du bord de Gromov ∂FF_N du complexe des facteurs libres de F_N, ceci donne une nouvelle démonstration d'un théorème de Calegari et Maher [CM12] qui affirme la convergence presque sûre de la marche aléatoire, réalisée sur FF_N, vers un point de ∂FF_N. Nous désignons par $\mathcal{FI}$ l'espace des classes d'arbres libres et arationnels de $\overline{CV_N}$, deux arbres étant équivalents s'ils appartiennent à un même simplexe. Nous identifions alors l'espace $\mathcal{FI}$ au bord de Poisson de $(\text{Out}(F_N),\mu)$, défini comme l'espace des composantes ergodiques du décalage dans l'espace des trajectoires $\mathrm{Out}(F_N)^{\mathbb{N}}$. Ce travail nous a été inspiré par les résultats analogues de Kaimanovich et Masur dans le cas des groupes modulaires de surfaces compactes orientables [KM96], et repose sur un critère dû à Kaimanovich permettant l'identification du bord de Poisson [Kai00]. Un sous-groupe de $\mathrm{Out}(F_N)$ est *non élémentaire* s'il ne fixe virtuellement aucun point de $FF_N \cup \partial FF_N$, et ne fixe virtuellement la classe de conjugaison d'aucun élément de F_N.

Théorème 6.

Soit μ une mesure de probabilité sur $\mathrm{Out}(F_N)$, dont le support engendre un sous-groupe non élémentaire de $\mathrm{Out}(F_N)$. Pour presque toute trajectoire $\mathbf{\Phi} := (\Phi_n)_{n\in\mathbb{N}}$ de la marche aléatoire sur $(\mathrm{Out}(F_N), \mu)$, il existe $\xi(\Phi) \in \mathcal{FI}$ tel que pour tout $T_0 \in CV_N$, la suite $(\Phi_n.T_0)_{n\in\mathbb{N}}$ converge vers $\xi(\Phi)$. La mesure de sortie ν est l'unique mesure μ-stationnaire sur $\mathcal{FI}$. Si de plus μ est de premier moment logarithmique fini pour la distance des mots sur $\mathrm{Out}(F_N)$, et d'entropie finie, alors $(\mathcal{FI}, \nu)$ est le bord de Poisson de $(\mathrm{Out}(F_N), \mu)$.

Nous déterminons également la *compactification par horofonctions* de l'outre-espace, pour la distance (asymétrique) étudiée par Francaviglia et Martino [FM11b] (la notion de compactification d'un espace métrique par horofonctions a été introduite par Gromov dans [Gro80]). Nous identifions la compactification par horofonctions avec la *compactification primitive* de CV_N, définie comme l'adhérence de l'image du plongement

$$\begin{array}{ccc} CV_N & \to & \mathbb{PR}^{\mathcal{P}_N} \\ T & \mapsto & \mathbb{R}^*(||g||_T)_{g\in\mathcal{P}_N} \end{array},$$

où $\mathcal{P}_N$ désigne l'ensemble des éléments primitifs de F_N (un élément de F_N est *primitif* s'il fait partie d'une base de F_N).

Théorème 7.

La compactification de CV_N par horofonctions est isomorphe à la compactification primitive de CV_N.

Afin de comprendre les points de cette compactification de CV_N, nous avons donc été amené à résoudre le problème de la rigidité spectrale pour l'ensemble $\mathcal{P}_N$ des éléments primitifs de F_N dans $\overline{CV_N}$, autrement dit à répondre à la question suivante : à quelle condition deux arbres $T, T' \in \overline{CV_N}$ ont-ils des fonctions longueurs de translation égales en restriction aux éléments primitifs de F_N ? Afin de donner un critère géométrique pour répondre à cette question, nous introduisons au Chapitre 2 une notion de *tiroirs-équivalence*, et montrons l'équivalence suivante. Dans l'énoncé suivant, nous désignons par cv_N l'*outre-espace non projectifié*, défini en considérant les actions à isométrie près plutôt qu'à homothétie près, et par $\overline{cv_N}$ l'adhérence de cet espace dans $\mathbb{R}^{F_N}$.

Théorème 8.

Deux arbres $T, T' \in \overline{cv_N}$ vérifient $||g||_{T'} = ||g||_T$ pour tout $g \in \mathcal{P}_N$ si et seulement s'ils sont tiroirs-équivalents.

L'étude de la compactification de $\mathrm{Out}(F_N)$ par horofonctions a des applications à l'étude de la marche aléatoire sur $\mathrm{Out}(F_N)$. Un théorème de Karlsson et Ledrappier [KL06] affirme que presque toute trajectoire de la marche aléatoire simple sur $\mathrm{Out}(F_N)$ est dirigée par une horofonction (aléatoire). Nous appliquons leur théorème à l'étude de la croissance des mots sous l'action de produits aléatoires d'automorphismes de F_N. Ceci nous permet de montrer un analogue pour $\mathrm{Out}(F_N)$ d'un théorème dû à Furstenberg et Kifer [FK83] et Hennion [Hen84], qui est une version du théorème multiplicatif d'Oseledets pour des produits indépendants de matrices aléatoires. Une *filtration* de F_N est un arbre enraciné, étiqueté par des sous-groupes (possiblement triviaux) de F_N, tel que l'étiquette de la racine soit F_N, et si H' est un fils de H, alors $H' \subseteq H$. Dans l'énoncé suivant, l'hypothèse sur la finitude du support de μ peut en fait être remplacée par une condition de moment.

Théorème 9.

Soit μ une loi de probabilité sur $\mathrm{Out}(F_N)$ à support fini. Il existe une filtration de F_N, et des *exposants de Lyapunov* (déterministes) $\lambda_H^\mu \geq 0$ associés aux sous-groupes H de la filtration, avec $\lambda_{H'}^\mu \leq \lambda_H^\mu$ lorsque H' est un fils de H, satisfaisant la propriété suivante.
Pour tout sous-groupe H de la filtration, tout $g \in H$ dont aucun conjugué n'appartient à un fils de H, et presque toute trajectoire $(\Phi_n)_{n\in\mathbb{N}}$ de la marche aléatoire sur $(\mathrm{Out}(F_N), \mu)$, nous avons

$$\lim_{n\to+\infty} \frac{1}{n} \log ||\Phi_n^{-1}(g)|| = \lambda_H^\mu.$$

De plus, le nombre d'exposants de Lyapunov de la mesure μ est borné par $\frac{3N-2}{4}$.

Dans le cas d'une marche aléatoire simple sur $\mathrm{Out}(F_N)$ (i.e. lorsque le support de μ engendre $\mathrm{Out}(F_N)$), nous obtenons l'unicité et la stricte positivité du coefficient de Lyapunov. Nous obtenons également une version affaiblie du Théorème 9 pour les cocycles intégrables d'éléments de $\mathrm{Out}(F_N)$.

Organisation de la thèse

La première partie est une introduction à la géométrie de plusieurs espaces munis d'actions du groupe $\mathrm{Out}(F_N)$. Dans le Chapitre 1, nous présentons l'outre-espace de Culler et Vogtmann, ainsi que l'espace des courants qui lui est dual. Nous redonnons en particulier une démonstration de l'identification de l'adhérence $\overline{CV_N}$ avec l'espace des actions minimales et très petites de F_N sur des arbres réels, afin de combler une lacune dans les arguments de Bestvina et Feighn. Le Chapitre 2 est consacré à la description de la compactification primitive de CV_N. Nous répondons notamment à la question de la rigidité spectrale de l'ensemble $\mathcal{P}_N$ des éléments primitifs de F_N. Dans le Chapitre 3, nous présentons un certain nombre de graphes hyperboliques munis d'actions de $\mathrm{Out}(F_N)$, et nous recensons les propriétés géométriques connues à ce jour de leur géométrie. Nous y décrivons le bord de Gromov du graphe des scindements (maximalement) cycliques de F_N. Nous présentons également le modèle des sphères du complexe des scindements libres de F_N, et donnons une esquisse d'une démonstration de l'hyperbolicité de ce complexe, obtenue en collaboration avec Arnaud Hilion. Nous présentons enfin une preuve, obtenue en collaboration avec Richard D. Wade, du fait que tout automorphisme simplicial du graphe des scindements cycliques (ou de certaines de ses variantes) est induit par un élément de $\mathrm{Out}(F_N)$.

La deuxième partie est consacrée à l'étude des marches aléatoires sur $\mathrm{Out}(F_N)$. Dans le Chapitre 4, nous introduisons les notions de base concernant les marches aléatoires sur des groupes discrets, ainsi que la notion du bord de Poisson. Nous esquissons alors notre démonstration de la convergence presque sûre d'une trajectoire typique de la marche aléatoire sur $\mathrm{Out}(F_N)$, réalisée sur CV_N, vers un simplexe d'arbres arationnels, et présentons un modèle du bord de Poisson de $\mathrm{Out}(F_N)$. Nous présentons également quelques autres propriétés de la marche aléatoire sur $\mathrm{Out}(F_N)$ (vitesse de fuite, automorphisme typique obtenu au temps n de la marche). Le Chapitre 5 introduit la notion d'horofrontière d'un espace métrique, et contient la démonstration de l'identification de la compactification par horofonctions de l'outre-espace avec la compactification primitive. Nous expliquons également l'intérêt de l'horofrontière pour l'étude des marches aléatoires sur $\mathrm{Out}(F_N)$, et établissons nos résultats sur la croissance des classes de conjugaison d'éléments de F_N le long des trajectoires de la marche aléatoire sur le groupe $\mathrm{Out}(F_N)$.

Enfin, la troisième partie de la thèse est consacrée à la présentation d'alternatives pour les sous-groupes de $\mathrm{Out}(F_N)$, et plus généralement du groupe $\mathrm{Out}(G)$ des automorphismes extérieurs d'un produit libre de groupes dénombrables. Le Chapitre 6 établit l'alternative d'Handel et Mosher pour les sous-groupes de $\mathrm{Out}(F_N)$. Au Chapitre 7, nous étudions la géométrie des versions adaptées aux groupes d'automorphismes de produits libres des espaces introduits en Partie I. En particulier, nous décrivons l'adhérence $\overline{P\mathcal{O}(G,\mathcal{F})}$, énonçons l'hyperbolicité des graphes de scindements correspondants, et décrivons le bord de Gromov du graphe des scindements (maximalement) cycliques de $(G,\mathcal{F})$. En utilisant des techniques similaires à celles introduites au Chapitre 6, et en utilisant ces complexes associés à des produits libres de groupes, nous présentons notre démonstration de l'alternative de Tits pour le groupe des automorphismes extérieurs d'un produit libre au Chapitre 8.

Les différentes annexes contiennent les démonstrations détaillées des résultats présentés dans cette thèse.

Nous avons essayé autant que possible de proposer une introduction didactique aux objets et méthodes qui ont fait l'objet de notre travail. Chacune des trois parties de cette thèse contient une présentation du contexte dans lequel s'inscrivent nos résultats, une introduction aux objets que nous avons considérés, et quelques éléments de preuve des théorèmes que nous avons obtenus, qui nous ont paru cruciaux dans notre démarche générale. Les points plus techniques des démonstrations de ces résultats ont été volontairement relégués en annexe. Les trois parties de la thèse peuvent se lire indépendamment des annexes, ou comme préambule à la lecture de ces annexes. Nous espérons ainsi pouvoir satisfaire tant le lecteur désireux de découvrir ou de s'initier à un domaine de recherche aussi passionnant que varié, celui qui voudrait avoir un aperçu des idées entrant en jeu dans les différents résultats que nous avons obtenus, que le lecteur désireux d'approfondir en détails tel point plus précis de notre travail.

Première partie

Actions de Out(F_N)

Introduction

Soit $N \geq 2$. Nous désignerons par F_N un groupe libre de rang N, et par $\mathrm{Out}(F_N)$ le groupe de ses automorphismes extérieurs, quotient du groupe $\mathrm{Aut}(F_N)$ des automorphismes de F_N par le sous-groupe distingué des automorphismes intérieurs, correspondant à la conjugaison par un élément de F_N. Notre approche à l'étude du groupe $\mathrm{Out}(F_N)$ sera géométrique. Une telle approche consiste à

1. construire un *joli* espace topologique X, muni d'une *jolie* action de $\mathrm{Out}(F_N)$, puis
2. étudier les propriétés topologiques, géométriques de l'espace X, ainsi que les propriétés dynamiques de l'action de $\mathrm{Out}(F_N)$ sur X, et enfin
3. déduire des propriétés algébriques du groupe $\mathrm{Out}(F_N)$ à partir des propriétés topologiques de X (il s'agira en quelque sorte d'établir un dictionnaire entre les propriétés algébriques du groupe et les propriétés topologiques de l'espace).

Il conviendra de garder à l'esprit ce fil conducteur tout au long de cette thèse. Par exemple, nous déduirons au Chapitre 6 un résultat de structure des sous-groupes de $\mathrm{Out}(F_N)$ à partir de l'étude de l'action de $\mathrm{Out}(F_N)$ sur un espace hyperbolique, et la détermination des mesures harmoniques sur le bord de Gromov de cet espace.

Dans cette première partie, nous introduisons donc un certain nombre d'espaces munis d'actions de $\mathrm{Out}(F_N)$, dont l'étude s'est révélée fructueuse au cours de ces dernières années. Nous commencerons par présenter au Chapitre 1 deux constructions classiques, à savoir

1. l'*outre-espace*, dont la construction par Culler et Vogtmann [CV86] par analogie avec les espaces de Teichmüller de surfaces marque le début de l'étude géométrique de $\mathrm{Out}(F_N)$,
2. l'espace des *courants géodésiques*, introduit par Kapovich dans [Kap05, Kap06].

Nous nous intéresserons particulièrement dans les deux premiers chapitres aux propriétés *à l'infini* de l'outre-espace. Culler et Morgan ont proposé une construction abstraite d'une compactification de l'outre-espace [CM87], dont les points ont été décrits concrètement par Cohen et Lustig [CL95] et Bestvina et Feighn [BF94]. Nous redonnons une preuve de cette description afin de combler une lacune dans l'argument de Bestvina et Feighn. Nous en décrirons une autre compactification, que nous appelons la *compactification primitive*. Notre motivation pour l'introduction de cette nouvelle compactification est le problème de la description de l'*horofrontière* de l'outre-espace : nous montrerons en effet au Chapitre 5 que la compactification de l'outre-espace par horofonctions, pour la distance asymétrique de Lipschitz étudiée notamment par Francaviglia et Martino [FM11b], est isomorphe à la compactification primitive. Afin de décrire la compactification primitve, nous répondrons au Chapitre 2 à la question de la rigidité spectrale de l'ensemble des éléments primitifs de F_N dans l'adhérence de l'outre-espace : à quelle condition deux F_N-arbres ont-ils même longueur de translation en restriction aux éléments primitifs de F_N (i.e. ceux qui font partie d'une base de F_N) ?

Plus récemment, la recherche s'est tournée vers la construction d'actions de $\mathrm{Out}(F_N)$ sur des complexes hyperboliques. Dans un troisième chapitre, nous présenterons trois de ces complexes, le *complexe des facteurs libres*, le *complexe des scindements libres* et le *complexe des scindements cycliques*, ainsi que leurs propriétés géométriques principales connues à ce jour. Notre apport consistera en

- une détermination du bord de Gromov du complexe des scindements (maximalement) cycliques.
- une nouvelle démonstration de l'hyperbolicité du complexe des scindements libres, obtenue en collaboration avec Arnaud Hilion, en travaillant dans un modèle plus géométrique du complexe, le *complexe des sphères*,
- une démonstration, obtenue en collaboration avec Richard D. Wade, du fait que toute isométrie du graphe des scindements cycliques (ou de certaines variantes de ce graphe) est induite par un élément de $\mathrm{Out}(F_N)$,

Une source d'inspiration permanente pour l'étude du groupe $\mathrm{Out}(F_N)$ des automorphismes extérieurs du groupe libre F_N vient des multiples analogies existant avec les groupes modulaires de surface, voir [Bes02, Vog02]. Le *groupe modulaire* $\mathrm{Mod}(S)$ d'une surface compacte connexe orientable S est défini comme le groupe des classes d'isotopie de difféomorphismes de S préservant l'orientation, et qui fixent chaque composante de bord point par point. Nous renvoyons notamment le lecteur à [Pau11] pour un exposé des analogies existant entre le groupe $\mathrm{Out}(F_N)$, le groupe modulaire $Mod(S)$ d'une surface compacte orientable S, et le groupe arithmétique $SL_N(\mathbb{Z})$. Ces analogies avec les groupes modulaires de surface nous serviront de *leitmotiv* tout au long de ce travail. Nous tâcherons autant que possible de mettre en parallèle les résultats que nous avons obtenus pour le groupe $\mathrm{Out}(F_N)$ avec les énoncés correspondants dans le cas des surfaces.

Chapitre 1

L'outre-espace de Culler et Vogtmann et l'espace des courants géodésiques

1.1 L'outre-espace

Définition

L'*outre-espace* a été introduit par Culler et Vogtmann dans [CV86] par analogie d'une part avec l'espace symétrique $SL(N,\mathbb{R})/SO(N)$, muni de l'action du groupe arithmétique $SL(N,\mathbb{Z})$, et d'autre part avec l'espace de Teichmüller d'une surface S, muni de l'action du groupe modulaire de S. Il en existe plusieurs modèles ; nous décrivons ici la construction originelle de Culler et Vogtmann de l'outre-espace comme espace de graphes métriques marqués, ainsi qu'un autre modèle en termes d'actions de F_N sur des arbres. Le lecteur est renvoyé à [Hat95, Appendice] pour une étude d'un troisième modèle de l'outre-espace en termes de systèmes de sphères dans la variété $M_N := \sharp^N S^1 \times S^2$. Nous renvoyons le lecteur à [Vog02] pour une excellente introduction à l'étude de l'outre-espace.

En guise de motivation, commençons par rappeler la définition de l'espace de Teichmüller d'une surface compacte connexe orientable S. Nous renvoyons le lecteur intéressé à [FM11a] pour une étude détaillée des groupes modulaires de surfaces. L'*espace de Teichmüller* de S est l'ensemble des classes d'équivalence de couples (X,ϕ), où X est une surface hyperbolique à bord totalement géodésique, et $\phi : S \to X$ est un difféomorphisme. Deux couples (X,ϕ) et (X',ϕ') sont *équivalents* s'il existe une isométrie $I : X \to X'$ telle que $I \circ \phi$ soit isotope à ϕ'. Autrement dit, l'espace de Teichmüller est l'espace des classes d'isotopie de *structures hyperboliques* sur S, que l'on peut également voir comme espace de déformation de métriques hyperboliques sur S.

Le modèle des graphes de l'outre-espace de Culler et Vogtmann. Un *graphe métrique* est un graphe fini G dont tous les sommets sont de valence supérieure à 3, et dont chaque arête e a une longueur $l(e) > 0$. Le graphe G est naturellement muni d'une distance géodésique pour laquelle chaque arête de longueur $l(e)$ est isométrique au segment $[0,l(e)] \subseteq \mathbb{R}$. Soit R_N le graphe (appelé une *rose*) ayant un sommet et N arêtes. Un *marquage* de G est une équivalence d'homotopie $\rho : R_N \to G$. Deux graphes métriques marqués (G,ρ) et (G',ρ') sont *équivalents* s'il existe une homothétie $f : G \to G'$ (i.e. une application multipliant toutes les longueurs par un même scalaire strictement positif) telle que $f \circ \rho$ soit homotope à ρ'. L'*outre-espace* CV_N est l'ensemble des classes d'équivalence

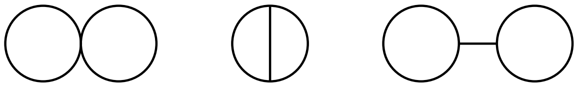


FIGURE 1.1 – Les trois graphes homotopiquement équivalents à R_2, à homéomorphisme près.

de graphes métriques marqués de groupe fondamental F_N. L'*outre-espace non projectifié* cv_N est défini de même en considérant les graphes métriques marqués à isométrie près, plutôt qu'à homothétie près.

La classe d'équivalence d'un graphe métrique marqué (G, ρ) dans CV_N définit un $(k-1)$-simplexe ouvert, où k désigne le nombre d'arêtes de G, obtenu en faisant varier les longueurs des arêtes de G, leur somme étant fixée égale à 1 (ce que nous pouvons toujours supposer puisque les graphes sont considérés à homothétie près). Étant donné deux graphes marqués (G, ρ) et (G', ρ'), le simplexe de G' s'identifie à une face du simplexe de G si G' peut être obtenu à partir de G en écrasant une forêt, de sorte que (quitte à modifier ρ et ρ' sans changer la classe d'équivalence des graphes métriques marqués (G, ρ) et (G', ρ')) l'application d'écrasement $\pi : G \to G'$ vérifie $\rho' = \pi \circ \rho$. L'outre-espace est alors la réunion de tous les simplexes de graphes marqués, modulo ces identifications de faces. La topologie ainsi définie sur l'outre-espace, pour laquelle un sous-ensemble est ouvert si son intersection avec chaque simplexe est ouverte, est appelée la *topologie faible*.

Lorsque $N = 2$, un argument de caractéristique d'Euler permet de montrer qu'il y a, à homéomorphisme près, exactement 3 graphes homotopes à la rose, représentés en Figure 1.1. En Figure 1.2, nous donnons une représentation de l'outre-espace CV_2, qui est obtenu en identifiant les simplexes associés à ces graphes comme prescrit ci-dessus. Sur cette représentation, nous avons fixé une base $\{a, b\}$ de F_2. Les étiquettes sur les arêtes des graphes définissent une équivalence d'homotopie f inverse au marquage correspondant : une arête étiquetée par un mot en a, b et leurs inverses est envoyée linéairement par f sur le chemin d'arêtes correspondant dans la rose dont les pétales représentent les éléments a et b.

Le groupe $\mathrm{Out}(F_N)$ agit à droite sur CV_N par précomposition des marquages. Plus précisément, tout automorphisme $\Phi \in \mathrm{Out}(F_N)$ est réalisable par une équivalence d'homotopie $f : R_N \to R_N$, et l'action est donnée par $[G, \rho].\Phi = [G, \rho \circ f]$ (où $[G, \rho]$ désigne la classe d'équivalence d'un graphe métrique marqué (G, ρ)). Le lecteur vérifiera aisément que ceci ne dépend pas du choix d'un représentant de la classe d'équivalence du graphe métrique marqué (G, ρ), ni du choix de l'équivalence d'homotopie f réalisant Φ, et que ceci définit bien une action à droite de $\mathrm{Out}(F_N)$ sur CV_N. Nous pouvons également définir une action à gauche en posant $\Phi.[G, \rho] := [G, \rho].\Phi^{-1}$.

Le modèle des arbres. Nous introduisons maintenant un autre modèle de l'outre-espace, défini en termes d'actions du groupe libre F_N sur des arbres simpliciaux métriques. L'*outre-espace non projectifié* cv_N est l'espace des classes d'isométries F_N-équivariantes d'actions libres, minimales, simpliciales et par isométries du groupe libre F_N sur des arbres simpliciaux métriques. L'*outre-espace* CV_N est l'espace des classes d'homothétie d'arbres de cv_N. L'action (à droite) de $\mathrm{Out}(F_N)$ sur CV_N (ou sur cv_N) est par précomposition des actions. Plus précisément, tout automorphisme $\phi \in \mathrm{Aut}(F_N)$ agit sur cv_N par précomposition des actions : si T est un arbre simplicial métrique muni d'une action $\rho : F_N \to \mathrm{Isom}(T)$, alors $(T, \rho).\phi = (T, \rho \circ \phi)$. L'action des automorphismes intérieurs de

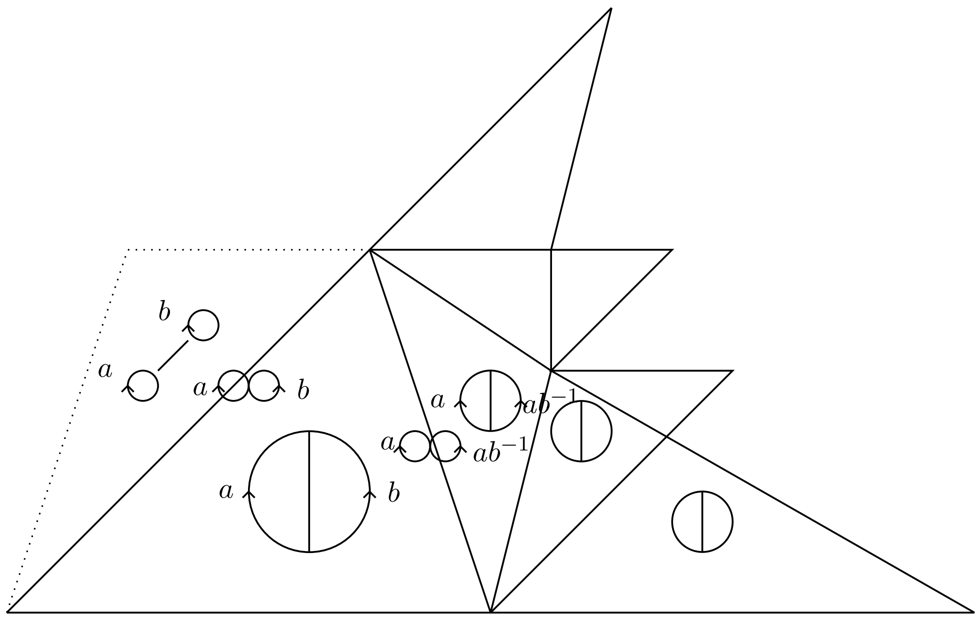


FIGURE 1.2 – Représentation de l'outre-espace CV_2.

F_N est par isométries F_N-équivariantes, donc nous obtenons au quotient une action de $\mathrm{Out}(F_N)$ sur cv_N.

L'espace ainsi défini est naturellement en bijection $\mathrm{Out}(F_N)$-équivariante avec le modèle des graphes introduit précédemment. En effet, si $[G, \rho]$ est la classe d'équivalence d'un graphe métrique marqué, alors le revêtement universel $\widetilde{G}$ de G est un arbre simplicial métrique, dont les arêtes sont naturellement munies de longueurs strictement positives, et $\widetilde{G}$ est muni d'une action du groupe fondamental de G, qui est identifié à F_N grâce au marquage ρ. Il est facile de vérifier que cette action est simpliciale et libre, minimale et par isométries. Réciproquement, soit $[T]$ la classe d'équivalence d'un arbre simplicial métrique. Choisissons un point base $y_0 \in T$. Soit T_0 un revêtement universel de la rose R_N, et x_0 un relevé du sommet de R_N dans T_0. Alors il existe une unique application F_N-équivariante de T_0 vers T envoyant x_0 sur y_0, et isométrique en restriction aux arêtes de T_0. Celle-ci passe au quotient en une équivalence d'homotopie de R_N vers le graphe quotient T/F_N, ce qui donne le graphe métrique marqué souhaité. Il est facile de vérifier que les applications de passage d'un modèle de l'outre-espace à l'autre sont $\mathrm{Out}(F_N)$-équivariantes, et inverses l'une de l'autre.

La topologie des axes et la compactification de Culler et Morgan de l'outre-espace.

Nous décrivons maintenant une compactification classique, dont la définition est due à Culler et Morgan [CM87], de l'outre-espace CV_N. La construction est analogue à celle de la compactification par Thurston [Thu88] de l'espace de Teichmüller d'une surface compacte S. Nous rappelons brièvement la construction de Thurston, et renvoyons à [FLP79, Chapitre 8] pour une étude détaillée. Étant donné une métrique hyperbolique x sur S, toute classe d'isotopie de courbes fermées simples sur S admet un unique représentant géodésique pour x. Ceci permet de définir une application

$$\begin{array}{rccc} i: & Teich(S) & \to & \mathbb{PR}^{\mathcal{C}} \\ & x & \mapsto & (l_x(c))_{c\in\mathcal{C}} \end{array},$$

où $\mathcal{C}$ désigne l'ensemble des classes d'isotopies de courbes fermées simples sur S, et $l_x(c)$ est la longueur de l'unique représentant géodésique d'une telle classe c, calculée par intégration de la métrique hyperbolique sur S. L'application i est injective, et son image est d'adhérence compacte dans $\mathbb{PR}^{\mathcal{C}}$ et définit une compactification de $Teich(S)$. Celle-ci est appelée la *compactification de Thurston* de $Teich(S)$. Thurston a identifié le bord de cette compactification avec l'espace $\mathcal{PMF}$ des feuilletages mesurés projectifs, voir [Thu88, FLP79].

Nous revenons maintenant au cas de l'outre-espace. Un *arbre réel* est un espace métrique (T, d_T) dans lequel deux points $x, y \in T$ sont toujours reliés par un unique arc topologique plongé, qui est alors isométrique à un segment de longueur $d_T(x, y)$. Par analogie avec la compactification de Thurston de l'espace de Teichmüller d'une surface compacte, Culler et Morgan [CM87] ont construit une compactification naturelle de CV_N. Étant donné un arbre réel T et un élément $g \in F_N$, la *longueur de translation* de g dans T est définie comme

$$||g||_T := \inf_{x\in T} d_T(x, gx).$$

L'application

$$\begin{array}{rccc} i: & cv_N & \to & \mathbb{R}^{F_N} \\ & T & \mapsto & (||g||_T)_{g\in F_N} \end{array}$$

est injective [CM87, Théorème 3.7], et c'est en fait un homéomorphisme sur son image. De plus, son image est d'adhérence projectivement compacte ([CM87, Théorème 4.5], voir aussi [Pau89]), et définit donc une compactification $\overline{CV_N}$ de CV_N. L'espace $\overline{CV_N}$ est muni de la topologie induite par celle de $\mathbb{PR}^{F_N}$, appelée la *topologie des axes*. Informellement, deux arbres T et T' sont proches au sens de cette topologie si tous les éléments d'un grand sous-ensemble fini de F_N ont des longueurs de translations proches dans T et T'.

Les arbres simpliciaux de $\overline{CV_N}$ ont été décrits par Cohen et Lustig [CL95], et Bestvina et Feighn ont ensuite donné une description complète des points de $\overline{CV_N}$ [BF94]. Ce sont les classes projectives d'actions minimales, isométriques, *très petites* de F_N sur des arbres réels. Une action de F_N sur un arbre réel est *très petite* si tous les stabilisateurs d'arcs sont triviaux ou maximalement cycliques, et tous les stabilisateurs de tripodes sont triviaux.

Théorème 1.1. *(Cohen–Lustig [CL95], Bestvina–Feighn [BF94]) L'espace $\overline{CV_N}$ est l'espace des classes d'homothétie équivariante d'actions minimales, non triviales, très petites et par isométries de F_N sur des arbres réels.*

Nous proposons ci-dessous une démonstration de ce résultat. En particulier, nous y corrigeons un défaut dans l'argument de Bestvina et Feighn.

Bestvina et Feighn ont montré que $\overline{CV_N}$ est de dimension topologique finie égale à $3N - 4$ [BF94, Corollaire 7.12]. Leur résultat a été amélioré par Gaboriau et Levitt, qui ont montré en outre que la frontière $\partial CV_N := \overline{CV_N} \smallsetminus CV_N$ est de dimension topologique égale à $3N-5$ [GL95]. L'espace $\overline{CV_N}$ est contractile (Steiner [Ste88], Skora [Sko89], White [Whi93], Guirardel–Levitt [GL07a]).

La topologie de Gromov–Hausdorff équivariante.

L'espace $\overline{CV_N}$ peut également être muni de la *topologie de Gromov–Hausdorff équivariante*, introduite par Paulin dans [Pau88], qui est équivalente à la topologie des axes

[Pau89]. Informellement, deux F_N-arbres T et T' seront proches pour cette topologie s'ils se ressemblent sur de grands sous-ensembles finis K et K', et les actions de grands sous-ensembles finis d'éléments de F_N sont presque les mêmes sur K et sur K'. Nous donnons maintenant une définition formelle de cette topologie. Soit T et T' deux arbres réels, soit $K \subset T$ et $K' \subset T'$ des sous-ensembles finis, soit $P \subset F_N$ un sous-ensemble fini, et soit $\epsilon > 0$. Une *ϵ-relation P-équivariante entre K et K'* est un sous-ensemble $R \subseteq K \times K'$, dont la projection sur chaque facteur est surjective, et telle que pour tous $(x, x'), (y, y') \in R$ et tous $g, h \in P$, nous ayons $|d_T(gx, hy) - d_{T'}(gx', hy')| < \epsilon$. Soit $O(T, K, P, \epsilon)$ l'ensemble des F_N-arbres T' tels qu'il existe un sous-ensemble fini $K' \subset T'$ et une ϵ-relation P-équivariante $R \subseteq K \times K'$. Paulin a montré que ces ensembles forment une base pour une topologie sur l'ensemble des classes d'isométries équivariantes de F_N-arbres minimaux [Pau88]. C'est la *topologie de Gromov–Hausdorff équivariante.*

Idée de la démonstration du Théorème 1.1.

Cohen et Lustig ont montré d'une part que le sous-espace de $\mathbb{PR}^{F_N}$ formé des actions très petites de F_N sur des arbres réels est fermé [CL95, Théorème I] (une autre démonstration de ce résultat, utilisant la topologie de Gromov–Hausdorff équivariante, est due à Paulin [Pau97, Théorème 2.2] ; nous renvoyons également à la Partie 3 de l'Annexe E). D'autre part, ils ont expliqué comment approximer toute action simpliciale, minimale, très petite et par isométries de F_N sur un arbre simplicial métrique, par une suite d'actions minimales, simpliciales et libres [CL95, Théorème II]. Leur argument repose sur l'étude de la dynamique des *twists de Dehn* sur l'espace des actions très petites.

Bestvina et Feighn ont ensuite expliqué comment approximer toute action minimale très petite (non nécessairement simpliciale) par des actions simpliciales. L'idée consiste en un premier temps à approximer toute action très petite par des actions très petites *géométriques*, duales à des 2-complexes feuilletés, puis à approximer les actions géométriques par des actions simpliciales. Toutefois, l'argument de Bestvina et Feighn nous paraît incomplet. Il nous semble en effet que le cas d'une action de F_N qui contient un arc à stabilisateur non trivial, et qui est duale à un 2-complexe feuilleté dont l'une des composantes minimales est un feuilletage mesuré sur une surface compacte non orientable, n'est pas pris en charge par l'argument proposé. Bestvina et Feighn n'expliquent pas comment approximer une action duale à un feuilletage mesuré sur une surface non orientable sans créer de feuille à un côté. Comme l'a noté Guirardel dans [Gui98], ceci serait possible si l'on savait répondre positivement à la question (plus forte) suivante.

Soit S une surface non orientable. Le sous-ensemble de l'espace des feuilletages mesurés projectifs sur S formé des feuilletages ne comportant pas de feuille compacte à un côté est un fermé nulle part dense [DN90]. L'action du groupe modulaire de S sur ce sous-ensemble est-elle minimale ?

Lorsque les stabilisateurs d'arcs de l'arbre T à approximer sont triviaux, cette difficulté peut être contournée par un argument de rabotage rendu possible par [BF94, Lemme 4.1]. Toutefois, cet argument semble être insuffisant lorsque T contient des stabilisateurs d'arcs non triviaux (en d'autres termes, lorque le complexe de bandes dual contient un anneau feuilleté par des cercles parallèles).

Nous proposons ici un argument complet, qui traite en même temps les cas d'actions simpliciales et non simpliciales. Celui-ci apparaît en Partie 5 de l'Annexe E, dans laquelle nous nous plaçons dans le cadre plus général d'actions de produits libres de groupes dénombrables sur des arbres réels. Notre argument repose essentiellement sur les idées de Cohen–Lustig et Bestvina–Feighn, et utilise des techniques d'approximations dues à Levitt et Paulin [LP97] et Guirardel [Gui98]. Avant de l'exposer, nous rappelons la notion d'*action géométrique* de F_N sur un arbre réel, introduite par Levitt et Paulin dans [LP97]. La présentation proposée ici repose sur des idées de Rips.

Actions géométriques. Un *système fini d'isométries* est une paire $\mathcal{K} = (K, A)$, où K est une forêt métrique finie, et A est un ensemble fini d'isométries ϕ entre deux sous-arbres fermés A_ϕ et B_ϕ de K, appelés les *bases* de ϕ. À tout système fini d'isométries $\mathcal{K}$ est associé un 2-complexe feuilleté Σ, appelé la *suspension* de $\mathcal{K}$. Le complexe Σ est obtenu à partir de K (feuilleté par les points) et d'une bande $A_\phi \times [0,1]$ associée à chaque isométrie $\phi \in A$ (feuilletée par les feuilles verticales $\{*\} \times [0,1]$), en effectuant les identifications suivantes. Pour toute isométrie $\phi \in A$, nous identifions tout point de la forme $(t,0) \in A_\phi \times \{0\}$ avec le point correspondant $t \in K$, et tout point de la forme $(t,1) \in A_\phi \times \{1\}$ avec le point $\phi(t) \in K$.

Un F_N*-système d'isométries* est un système fini d'isométries $\mathcal{K} = (K, A)$, dont la suspension Σ est connexe, muni d'un point $* \in \Sigma$ et d'un isomorphisme $\rho : \pi_1(\Sigma, *) \to F_N$. Nous notons $\widetilde{\Sigma}$ le revêtement universel de Σ associé à ρ, de sorte que F_N agit sur $\widetilde{\Sigma}$. Le complexe feuilleté $\widetilde{\Sigma}$ est naturellement muni d'une pseudodistance, obtenue par intégration de la mesure transverse. L'espace métrique $T_{\mathcal{K}}$ associé est un arbre réel [LP97, Proposition 1.7], muni d'une action naturelle de F_N. Un F_N-arbre réel T est *géométrique* s'il existe un F_N-système d'isométries $\mathcal{K}$ tel que T soit équivariamment isométrique à $T_{\mathcal{K}}$.

Décomposition dynamique des arbres géométriques, classification des composantes minimales. Soit $\mathcal{K} = (K, A)$ un système fini d'isométries, et Σ la suspension de $\mathcal{K}$. Soit S l'ensemble fini formé des sommets de K et des points de K qui sont extrémaux dans une base d'une isométrie dans A. Soit $\Sigma^* := \Sigma \smallsetminus S$, muni du feuilletage induit, et soit $C^* \subseteq \Sigma^*$ la réunion des feuilles de Σ^* qui sont fermées mais non compactes. Nous définissons le *lieu singulier* de Σ comme $C := C^* \cup S$.

Théorème 1.2. *(Imanishi [Ima79], Gaboriau–Levitt–Paulin [GLP94, Théorème 3.1]) Soit $\mathcal{K} = (K, A)$ un système fini d'isométries, et Σ la suspension de $\mathcal{K}$. Soit C le lieu singulier de Σ. Alors $\Sigma \smallsetminus C$ est une union finie d'ouverts $U_1, \dots, U_p$, qui sont des unions de feuilles de Σ^*. Pour tout $i \in \{1, \dots, p\}$, soit toutes les feuilles de U_i sont compactes, soit toutes les feuilles de U_i sont denses dans U_i.*

Nous dirons dans le premier cas que U_i est une *famille finie d'orbites*, et dans le second cas que U_i est une *composante minimale* de Σ. Les composantes minimales d'un F_N-système d'isométries $\mathcal{K}_0$ sont classées comme suit. Nous définissons un système d'isométries $\mathcal{K}_1 = (K_1, A_1)$, obtenu par *élagage* à partir de $\mathcal{K}_0 = (K_0, A_0)$, de la manière suivante. La forêt K_1 est l'ensemble des points de K_0 qui appartiennent à au moins deux bases distinctes de $\mathcal{K}_0$, et A_1 est l'ensemble des restrictions des isométries de A_0 à K_1. En itérant cette construction, nous définissons par récurrence une suite de F_N-systèmes d'isométries $(\mathcal{K}_i)_{i \in \mathbb{N}}$. Les arbres $T_{\mathcal{K}_i}$ sont deux à deux isométriques par une isométrie F_N-équivariante. Si $\mathcal{K}_{i+1} \neq \mathcal{K}_i$ pour tout $i \in \mathbb{N}$, le système d'isométries $\mathcal{K}_0$ est dit *exotique*. Sinon, la suspension Σ_0 de $\mathcal{K}_0$ est un feuilletage mesuré sur une surface compacte : en effet, dans le cas de F_N-arbres très petits, le cas de composantes minimales homogènes décrit dans [GLP94] est impossible, voir [BF94, Proposition 1.8]. Lorsque $\mathcal{K}_0$ ne contient pas de composante minimale exotique, nous dirons que $\mathcal{K}_0$ est *de type surface*.

Arbres géométriques de type surface. Soit T un F_N-arbre minimal, très petit, géométrique, sans composante minimale exotique. Bestvina et Feighn ont montré [BF94, Proposition 5.1] l'existence d'un système d'isométries $\mathcal{K}$ tel que T soit équivariamment isométrique à $T_{\mathcal{K}}$, et dont la suspension Σ est de la forme $\Sigma := (S \cup A \cup \Gamma) \cup_f G$, où

- S est une surface compacte (non nécessairement connexe, et non nécessairement orientable), dont chaque composante connexe est munie d'un feuilletage mesuré minimal, et
- A est une union finie d'anneaux, feuilletés par des cercles parallèles, et
- Γ est un graphe métrique fini, feuilleté par les points, et
- G est un graphe fini, sans sommet de valence 1, muni du feuilletage vide, et
- $f : \partial S \cup \partial A \cup F \to G$, où $F \subseteq S \cup A \cup \Gamma$ est un ensemble fini, est une application π_1-injective en restriction à chaque composante connexe de $\partial S \cup \partial A$.

De plus, nous pouvons supposer [BF94, Lemme 4.1] que G est de la forme $G = G' \vee S^1$, et que f induit un homéomorphisme entre l'une des composantes c de $\partial S \cup \partial A$ et le cercle S^1, et envoie toutes les autres composantes de $\partial S \cup \partial A$ dans G'. Une composante c pouvant être décrite de cette manière est dite *inutilisée.* Nous mettons en garde le lecteur que la définition proposée ici dans le contexte de F_N-arbres est légèrement différente de la définition proposée en Annexe E, puisque nous autorisons les composantes de bord inutilisées à faire partie des anneaux dans A.

Approximations des arbres réels très petits par des actions simpliciales et libres. Nous noterons VSL_N le sous-espace de $\mathbb{PR}^{F_N}$ formé des fonctions longueur de translation projectifiées des actions minimales, très petites et par isométries de F_N sur des arbres réels. Notre objectif est de montrer que toute action dans VSL_N est approximable par une suite d'actions minimales, libres et simpliciales.

A. Approximation des actions non géométriques par des actions géométriques.

Nous commençons par approximer les actions non géométriques dans VSL_N par des actions géométriques dans VSL_N. Ceci est possible grâce à un résultat de Levitt et Paulin [LP97].

Théorème 1.3. *(Levitt–Paulin [LP97], Gaboriau–Levitt [GL95]) Pour tout $T \in VSL_N$, il existe une suite $(T_n)_{n\in\mathbb{N}} \in VSL_N^{\mathbb{N}}$ formée d'arbres géométriques, qui converge vers T.*

Plus précisément, Levitt et Paulin ont montré que tout F_N-arbre T minimal et très petit peut être approximé par une suite d'arbres géométriques T_n minimaux et très petits, munis de *morphismes* d'arbres réels $f_n : T_n \to T$ (i.e. tout segment de T peut être subdivisé en un nombre fini de sous-segments, en restriction auxquels f_n est une isométrie). Leur idée est la suivante. Soit T un F_N-arbre minimal et très petit, et soit A une base de F_N. Soit $K \subseteq T$ un sous-arbre fini de T, choisi de sorte que $K \cap aK$ soit non vide pour tout $a \in A$. Nous définissons un F_N-système d'isométries $\mathcal{K}$ sur K, en associant à tout $a \in A$ l'isométrie ϕ_a donnée par la restriction de l'action de a à $A_a := a^{-1}K \cap K$. L'arbre $T_{\mathcal{K}}$ est alors naturellement muni d'un morphisme vers T, qui est une isométrie en restriction à chaque translaté de K dans $T_{\mathcal{K}}$. Lorsque $(K_n)_{n\in\mathbb{N}}$ est une exhaustion de T par des arbres finis, Levitt et Paulin montrent que la suite d'arbres géométriques $T_{\mathcal{K}_n}$ ainsi construite converge vers T. Il découle essentiellement des travaux de Gaboriau et Levitt [GL95, Corollaire I.6 et Proposition II.1] que l'exhaustion K_n peut être choisie de sorte que les arbres $T_{\mathcal{K}_n}$ soient minimaux et très petits (l'argument est donné en Partie 2.3 de l'Annexe E).

B. Approximation des actions géométriques par des actions libres et simpliciales.

Il nous reste donc à approximer tout arbre géométrique minimal et très petit par une suite d'actions libres et simpliciales dans cv_N. Nous argumentons par récurrence sur le rang N du groupe libre. Le cas où $N = 1$ est immédiat. Nous notons également que l'action triviale du groupe libre F_N sur un point est approximée par la suite $(T_n)_{n\in\mathbb{N}} \in cv_N^{\mathbb{N}}$ des arbres de Bass–Serre associés à la rose dont les pétales ont tous une longueur $\frac{1}{n}$, et sont étiquetés par une base de F_N. Soit $T \in VSL_N$ un arbre géométrique, soit $\mathcal{K}$ un F_N-système d'isométries tel que T soit équivariamment isométrique à $T_{\mathcal{K}}$, et soit Σ la suspension de $\mathcal{K}$.

B.1 Approximation des composantes exotiques par rabotage.

La première étape consiste à approximer les composantes exotiques de T par des composantes simpliciales comme dans [Gui98, Partie 7]. Nous présentons la construction de Guirardel, et envoyons le lecteur à [Gui98] pour des arguments détaillés. Après avoir effectué un nombre suffisant d'opérations d'élagage sur le système d'isométries $\mathcal{K}$, nous trouvons une bande B dans le 2-complexe feuilleté qui ne rencontre pas les feuilles compactes de $\mathcal{K}$. Nous définissons un nouveau système d'isométries $\mathcal{K}_\delta$ en *rabotant* cette bande B, c'est-à-dire en la raccourcissant d'une certaine largeur $\delta > 0$ à partir de l'une de ses feuilles de bord, comme cela est représenté sur la Figure 1.3. Soit T_δ l'arbre dual associé au système d'isométries $\mathcal{K}_\delta$, qui est très petit. Guirardel a montré que lorsque δ converge vers 0, les arbres T_δ convergent vers l'arbre T. De plus, il est possible de choisir $\delta > 0$ arbitrairement proche de 0, de sorte que dans l'arbre T_δ, la composante exotique dans laquelle nous avons effectué le rabotage soit remplacée par des composantes simpliciales et exotiques, et le nombre de bouts de feuilles *singulières* (i.e. qui rencontrent l'ensemble fini formé des sommets de K et des extrémités des bases des isométries partielles de $\mathcal{K}$) ait diminué strictement. En répétant cette construction un nombre fini de fois, nous obtenons donc une approximation de T par des arbres géométriques ne contenant aucune composante exotique. Nous remarquons que l'opération de rabotage ne crée pas d'arc à stabilisateur non trivial dans les arbres T_δ.

B.2 Rabotage à partir des courbes de bord inutilisées dans les composantes minimales de type surface.

En vue de l'argument ci-dessus d'approximation des composantes exotiques, il nous reste à comprendre le cas des arbres $T \in VSL_N$ de type surface. Supposons que Σ contient une composante minimale dont l'une des courbes de bord c est inutilisée. Comme précédemment, nous pouvons approximer S en la rabotant à partir de c, c'est-à-dire en coupant le feuilletage le long d'un petit arc transverse à la composante de bord c et au feuilletage, comme cela est représenté sur la Figure 1.4. Toutes les demi-feuilles d'un feuilletage mesuré sur une surface compacte sont denses, donc cette opération de rabotage permet d'approximer la composante minimale par des composantes simpliciales, sans introduire d'arc à stabilisateur non trivial.

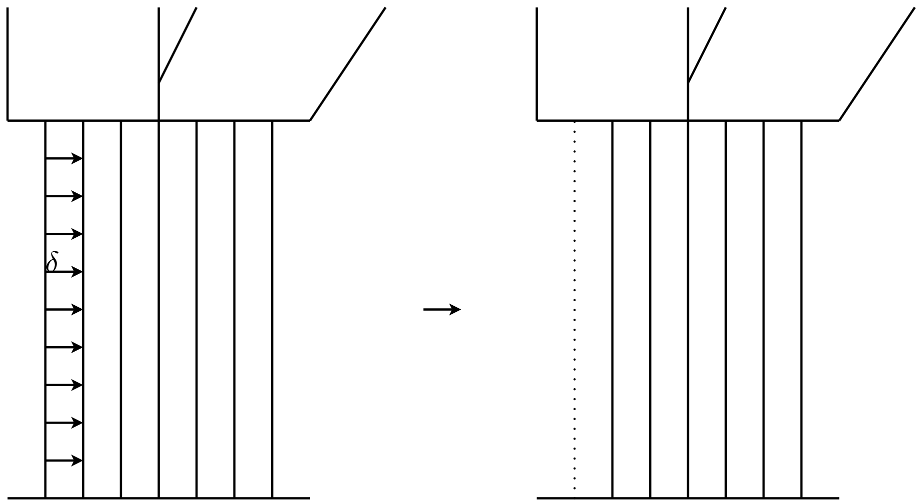


FIGURE 1.3 – Rabotage d'une bande dans une composante exotique.

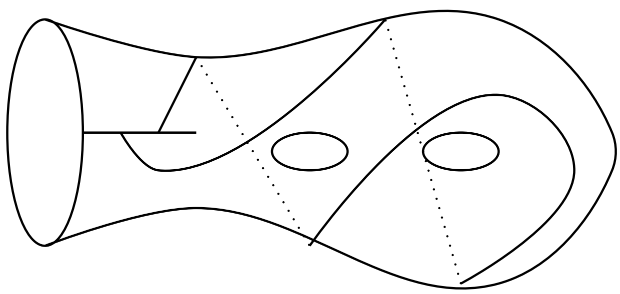


FIGURE 1.4 – Approximation d'une composante minimale de type surface par rabotage à partir d'une composante de bord inutilisée.

Intermède (Graphes d'actions).

Nous renvoyons le lecteur à [Ser77] pour une introduction à la théorie de Bass–Serre, notamment à la notion de *graphe de groupes* et à la terminologie associée. Un *graphe d'actions* est la donnée
– d'un graphe connexe métrique de groupes G, et
– d'une action isométrique du groupe de sommet G_v sur un arbre réel T_v (non nécessairement minimal, et possiblement réduit à un point) pour chaque sommet v de G, et
– d'un point d'attache $p_e \in T_{t(e)}$ fixé par le groupe $i_e(G_e) \subseteq G_{t(e)}$ pour chaque arête orientée e de G.

À tout graphe d'actions $\mathcal{G}$ pour lequel $\pi_1(G) = F_N$ est associé un F_N-arbre $T(\mathcal{G})$. De manière informelle, l'arbre $T(\mathcal{G})$ est obtenu à partir de l'arbre de Bass–Serre de G en attachant de manière équivariante l'arbre de sommet T_v au sommet v, les arêtes incidentes étant attachées au point d'attache prescrit. Un F_N-arbre T *se scinde en graphes d'actions* s'il existe un graphe d'actions $\mathcal{G}$ tel que $T = T(\mathcal{G})$. Levitt a montré que tout arbre $T \in \overline{cv_N}$ se scinde de manière unique en un graphe d'actions, pour lequel les arêtes de G sont de longueur strictement positive, et l'action de tout groupe de sommet G_v sur l'arbre T_v est à orbites denses (elle peut être triviale) [Lev94, Théorème 5]. Cette décomposition sera appelée la *décomposition canonique* de T en graphe d'actions à orbites denses.

B.3. L'argument inductif.

Nous nous sommes donc ramenés au cas d'un arbre $T \in VSL_N$ de type surface, tel qu'aucune composante minimale de Σ ne contienne de composante de bord inutilisée. En vue de la description de Bestvina et Feighn des systèmes d'isométries de type surface, soit T est un arbre simplicial dont toutes les arêtes ont un stabilisateur trivial, soit il existe une composante de bord inutilisée dans l'un des anneaux α de Σ. En rabotant l'anneau α à partir de cette composante de bord, nous obtenons une décomposition de l'arbre T en un graphe d'actions au-dessus d'un scindement libre de F_N, voir l'encadré en page 26. En argumentant par récurrence sur le rang du groupe libre, le Théorème 1.1 est alors une conséquence du lemme suivant.

Lemme 1.4. *Soit $T \in VSL_N$ un arbre qui se scinde en un graphe d'actions $\mathcal{G}$ au-dessus d'un scindement libre S à une arête de F_N. Si les sous-arbres minimaux de chacun des arbres de sommets de T admettent des approximations par des actions minimales, simpliciales et libres de leur stabilisateur, alors T admet une approximation par des actions minimales, simpliciales et libres de F_N.*

Le cas intéressant du Lemme 1.4 est en fait celui où les actions de sommets ne sont pas minimales, et ont des stabilisateurs d'arcs non triviaux. C'est le cas par exemple dans la situation décrite ci-dessus où la composante de bord inutilisée est contenue dans un anneau. Nous renvoyons également à la Figure 1.5 pour une illustration du cas d'actions de sommets non minimales. En particulier, le Lemme 1.4 est crucial pour traiter le cas d'arbres simpliciaux à stabilisateurs d'arcs non triviaux dans notre preuve du Théorème 1.1.

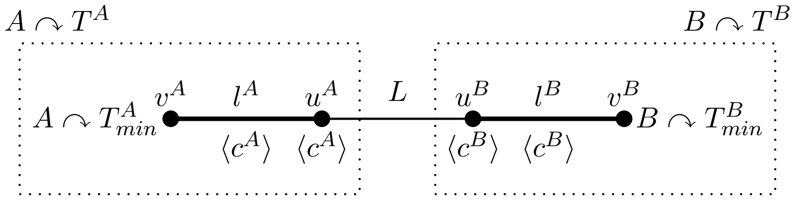


FIGURE 1.5 – Le graphe d'actions $\mathcal{G}$ dans la démonstration du Lemme 1.4.

Nous décrivons maintenant l'approximation de l'arbre T dans le cas où S est un scindement en produit libre de F_N, de la forme $F_N = A * B$. Nous renvoyons à la Partie 5 de l'Annexe E pour un argument plus détaillé, et un traitement du cas d'une extension HNN. Le graphe d'actions $\mathcal{G}$ admet la description suivante, illustrée en Figure 1.5. Soit T^A et T^B les arbres de sommets de $\mathcal{G}$, et u^A et u^B les points d'attache correspondants. Les arbres T^A et T^B ne sont pas nécessairement minimaux, nous notons T^A_{min} et T^B_{min} leurs sous-arbres minimaux respectifs. L'arbre $T^A \smallsetminus T^A_{min}$ contient au plus une orbite d'arcs, possiblement réduits à un point. Quitte à augmenter la longueur de l'arête de $\mathcal{G}$ si nécessaire, nous pouvons supposer que tout arc non dégénéré de $T^A \smallsetminus T^A_{min}$ a un stabilisateur non trivial, qui est alors maximalement cyclique.

Supposons que $T^A \smallsetminus T^A_{min}$ contient un arc non dégénéré $e^A = [v^A, u^A]$, et notons c^A son stabilisateur (c'est le cas intéressant de l'argument, nous renvoyons à la Partie 5 de l'Annexe E pour un traitement du cas où T^A s'obtient à partir de T^A_{min} en ajoutant uniquement des points dans l'adhérence de T^A_{min}). Comme les stabilisateurs de tripodes sont triviaux dans T, le point v^A est un point terminal du sous-arc de T^A_{min} fixé par c^A. Soit w^A l'autre extrémité de cet arc, dans le cas où celui-ci est non dégénéré. Si cet arc est réduit à un point, nous choisissons pour w^A un point quelconque de T distinct de v^A.

Soit $(T^A_{min,n})_{n\in\mathbb{N}}$ une approximation de T^A_{min} par des actions minimales, libres et simpliciales. Soit v^A_n et w^A_n des approximations respectives de v^A et w^A dans $T^A_{min,n}$. Nous pouvons supposer que v^A_n appartient à l'axe de translation de c^A dans $T^A_{min,n}$. Pour tout $n \in \mathbb{N}$, nous définissons un arbre pointé (T^A_n, u^A_n) de la manière suivante, illustrée en Figure 1.6. Partant de (T^A, u^A), nous commençons par remplacer l'arête e^A par une arête e^0 à stabilisateur trivial (c'est l'inverse de l'opération de pliage consistant à identifier de manière équivariante l'arête e^0 avec son translaté par c^A). Nous remplaçons alors (T^A_{min}, v^A) par son approximation $(T^A_{min,n}, v^A_n)$ pour obtenir un arbre $\widetilde{T_n}^A$, puis plions l'arête e^0 le long de l'axe de c^A dans $T^A_{min,n}$, dans une direction ne contenant pas w^A_n. Ce pliage identifie u^A avec un point u^A_n de l'axe de c^A dans $T^A_{min,n}$. Nous montrons en Partie 5 de l'Annexe E la convergence des arbres pointés (T^A_n, u^A_n) vers l'arbre pointé (T^A, u^A). En approximant de même l'arbre pointé (T^B, u^B) par une suite d'arbres pointés (T^B_n, u^B_n), et en remplaçant (T^A, u^A) et (T^B, u^B) par leurs approximations dans le graphe d'actions $\mathcal{G}$, nous obtenons alors une approximation de T (voir [Gui98, Partie 4]). $\square$

Approximations lipschitziennes.

Lorsque $T \in \overline{cv_N}$ est à stabilisateurs d'arcs triviaux, nous pouvons en fait être plus précis sur l'approximation obtenue par la construction présentée dans la partie précédente. Une *approximation lipschitzienne* de T est une suite $(T_n)_{n\in\mathbb{N}}$ de F_N-arbres T_n convergeant vers T, telle que pour tout $n \in \mathbb{N}$, il existe une application 1-lipschitzienne F_N-équivariante $f_n : T_n \to T$. Le théorème suivant fait l'objet de la Partie 3.2 de l'Annexe A, ainsi que

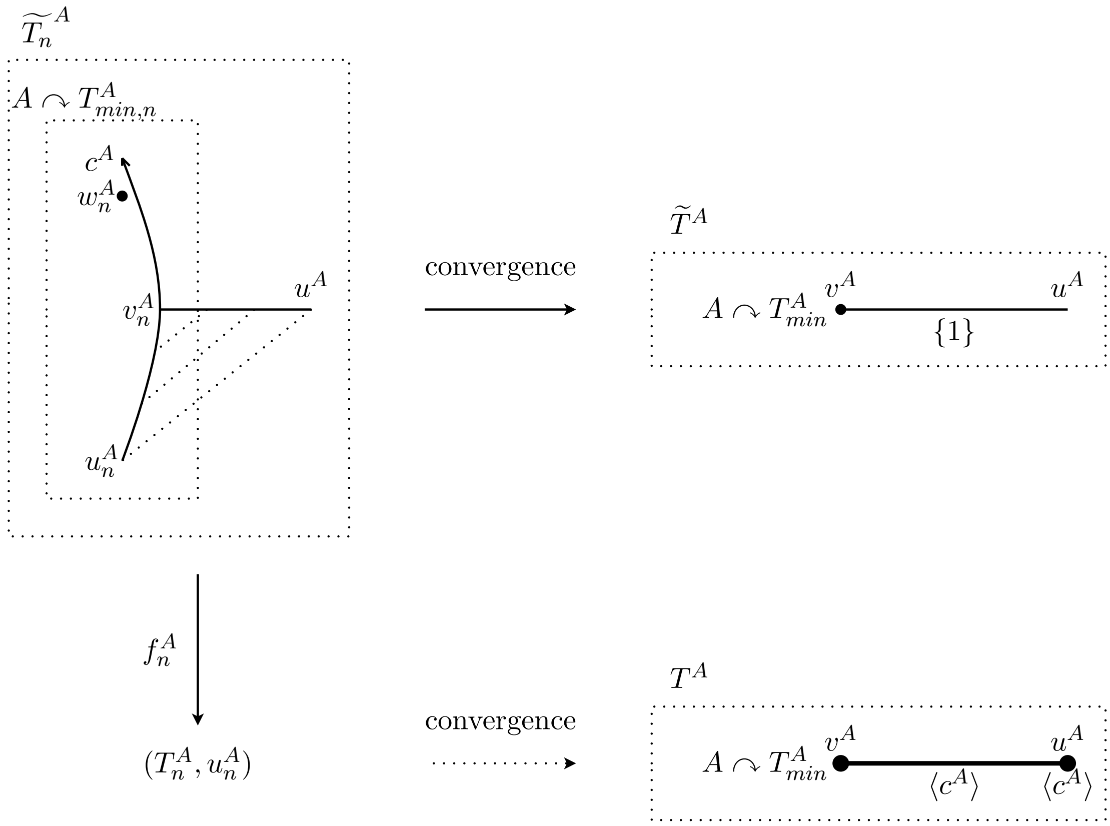


FIGURE 1.6 – Construction d'une approximation de (T^A, u^A) par des arbres pointés à stabilisateurs de points triviaux.

de la Partie 5 de l'Annexe E. Remarquons que puisqu'un morphisme d'arbres réels est 1-lipschitzien, il donne en particulier une obstruction à l'approximation forte par des éléments de cv_N.

Théorème 1.5.

Un arbre $T \in \overline{cv_N}$ admet une approximation lipschitzienne par des arbres dans cv_N si et seulement si T est à stabilisateurs d'arcs triviaux.

L'existence d'approximations lipschitziennes pour les arbres à stabilisateurs d'arcs triviaux s'est avérée cruciale à plusieurs reprises dans notre travail. Nous l'utilisons pour établir notre description du défaut de rigidité spectrale de l'ensemble des éléments primitifs de F_N dans $\overline{cv_N}$ (voir le Chapitre 2). Elle intervient également dans nos arguments menant à la description du bord de Gromov du graphe des scindements cycliques de F_N.

Propriétés métriques de l'outre-espace.

L'outre-espace est muni d'une distance asymétrique (i.e. elle vérifie l'axiome de séparation et l'inégalité triangulaire, mais pas l'axiome de symétrie) : la distance $d(T, T')$ entre deux arbres simpliciaux métriques est définie comme le logarithme de la plus petite constante de Lipschitz d'une application F_N-équivariante du représentant de covolume 1 de T vers le représentant de covolume 1 de T'. L'action de $\mathrm{Out}(F_N)$ sur CV_N se fait par isométries pour la distance d. L'étude systématique de la distance de Lipschitz sur CV_N (analogue à la distance asymétrique de Thurston [Thu98] sur les espaces de Teichmüller) a été initiée par Francaviglia et Martino dans [FM11b]. Mentionnons au passage les travaux de Meinert [Mei14] et de Francaviglia et Martino [FM14] qui généralisent cette distance asymétrique à des espaces de déformation plus généraux.

Un théorème attribué à White donne une caractérisation alternative de cette distance. Avant de l'énoncer, nous introduisons un peu de terminologie. Soit $T \in CV_N$. Un élément $g \in F_N$ est un *candidat* dans T si un domaine fondamental de son axe de translation dans T se projette en un lacet γ dans le graphe quotient $X := T/F_N$ qui est

- soit un cercle plongé dans X,
- soit un bouquet de deux cercles, i.e. γ est de la forme $\gamma = \gamma_1\gamma_2$, où γ_1 et γ_2 sont deux cercles plongés dans X qui se rencontrent en un unique point,
- soit un graphe en haltères, i.e. γ est de la forme $\gamma = \gamma_1 e \gamma_2 \overline{e}$, où γ_1 et γ_2 sont deux cercles plongés dans X qui ne se rencontrent pas, et e est un chemin plongé dans X qui rencontre γ_1 et γ_2 uniquement en leur origine (et $\overline{e}$ désigne le chemin e parcouru en sens inverse).

En particulier, les candidats dans T sont des éléments *primitifs* de F_N, i.e. ils font partie d'une base de F_N. Ceci peut se voir en remarquant qu'un lacet qui représente un candidat croise l'une des arêtes du graphe quotient au plus une fois, voir le Lemme 1.12 de l'Annexe A.

Théorème 1.6. *(White, voir [FM11b, Proposition 3.15] ou [AK11, Proposition 2.3]) Pour tous $T, T' \in CV_N$, nous avons*

$$d(T, T') = \log \sup_{g \in F_N \smallsetminus \{e\}} \frac{||g||_{T'}}{||g||_T},$$

où T et T' sont identifiés à leurs représentants de covolume 1*. De plus, la borne supérieure est atteinte en un élément (primitif) de F_N qui est un candidat dans T.*

En particulier, puisque l'ensemble des éléments de F_N qui sont des candidats dans T est fini, le théorème de White donne un moyen explicite de calculer la distance entre deux points de CV_N.

Mesures de longueurs.

Nous énonçons maintenant un théorème dû à Guirardel, qui donne un autre point de vue sur la finitude de la dimension de $\overline{CV_N}$. Soit $T \in \overline{cv_N}$ un arbre à orbites denses. Une *mesure de longueur invariante* sur T est la donnée pour tout segment $I \subseteq T$ d'une mesure borélienne finie μ_I sur I, de sorte que pour tous segments $J \subseteq I$, nous ayons $\mu_J = (\mu_I)_{|J}$, et pour tout segment $I \subseteq T$, et tout $g \in F_N$, nous ayons $\mu_{gI} = (g_{|I})_* \mu_I$, voir [Pau95]. Soit $\mathcal{M}_0(T)$ l'ensemble des mesures de longueur non atomiques sur T. Une mesure de longueur $\mu \in \mathcal{M}_0(T)$ est *ergodique* si tout ensemble mesurable F_N-invariant $E \subseteq T$ est soit de μ-mesure nulle, soit de μ-mesure pleine (i.e. pour tout segment $I \subseteq T$, nous avons soit $\mu_I(E \cap I) = 0$, soit $\mu_I({}^cE \cap I) = 0$).

Théorème 1.7. *(Guirardel [Gui00, Corollaire 5.3]) Pour tout $T \in \overline{cv_N}$ à orbites denses, l'ensemble $\mathcal{M}_0(T)$ est convexe et de dimension finie. De plus, l'arbre T possède au plus $3N-4$ mesures ergodiques non atomiques à homothétie près, et toute mesure dans $\mathcal{M}_0(T)$ est somme de mesures ergodiques.*

1.2 L'espace des courants et les laminations algébriques

L'espace des courants.

L'étude des courants géodésiques sur F_N, initiée par Bonahon [Bon91] et développée par Kapovich dans [Kap05, Kap06], est motivée par la notion analogue pour les espaces de Teichmüller, étudiée par Bonahon dans [Bon88].

Soit $\partial^2 F_N := \partial F_N \times \partial F_N \smallsetminus \Delta$, où Δ désigne la diagonale de $\partial F_N \times \partial F_N$. Soit $i : \partial F_N \times \partial F_N$ l'involution qui échange les facteurs. Un *courant géodésique* est une mesure borélienne non nulle sur $\partial^2 F_N$ qui est F_N-invariante, invariante par i, et finie sur les sous-ensembles compacts de $\partial^2 F_N$. Nous notons $Curr_N$ l'espace des courants géodésiques sur F_N, que nous munissons de la topologie faible-$*$, et $\mathbb{P}Curr_N$ l'espace des classes d'homothétie de courants géodésiques.

Le groupe $\mathrm{Out}(F_N)$ agit sur les espaces $Curr_N$ et $\mathbb{P}Curr_N$, de la manière suivante. Étant donné un borélien $S \subseteq \partial^2 F_N$, un élément $\Phi \in \mathrm{Out}(F_N)$, et un courant $\eta \in Curr_N$, l'action est donnée par $\Phi(\eta)(S) := \eta(\phi^{-1}(S))$, où $\phi \in \mathrm{Aut}(F_N)$ est un représentant de Φ. Là encore, le lecteur vérifiera que cette définition est indépendante des choix effectués, et définit une action (à gauche) du groupe $\mathrm{Out}(F_N)$ sur l'espace $Curr_N$, voir [Kap06, Proposition 2.15].

Étant donné $g \in F_N$, nous posons $g^{-\infty} := \lim_{n\to+\infty} g^{-n} \in \partial F_N$, et $g^{+\infty} := \lim_{n\to+\infty} g^n \in \partial F_N$. Un élément $g \in F_N$ est une *puissance* s'il existe $h \in F_N$ et $k \in \mathbb{Z} \smallsetminus \{-1,0,1\}$ tels que $g = h^k$. À tout élément $g \in F_N \smallsetminus \{e\}$ est associé un *courant rationnel* $[g]$, définit comme suit. Lorsque g n'est pas une puissance, nous définissons $[g](S)$ comme le nombre de translatés de $(g^{-\infty}, g^{+\infty})$ contenus dans S, pour tout borélien $S \subseteq \partial^2 F_N$, voir [Kap06, Définition 5.1]. Lorsque $h = g^k$, nous posons $[h] := k[g]$. L'ensemble des multiples scalaires de courants rationnels est dense dans $Curr_N$ [Kap05, Corollaire 3.5].

L'action de $\text{Out}(F_N)$ sur $\mathbb{P}Curr_N$ n'est pas minimale, mais il existe un unique sous-ensemble fermé et $\text{Out}(F_N)$-invariant minimal $\mathbb{P}M_N \subseteq \mathbb{P}Curr_N$, qui est l'adhérence de l'ensemble des courants rationnels associés à des éléments primitifs de F_N [KL07, Théorème B] (nous rappelons qu'un élément de F_N est *primitif* s'il fait partie d'une base de F_N). Nous noterons M_N le relevé à $Curr_N$ de l'espace $\mathbb{P}M_N$.

Kapovich et Lustig ont montré dans [KL09] l'existence d'une unique *forme d'intersection* continue $\langle .,. \rangle : \overline{cv_N} \times Curr_N \to \mathbb{R}_+$ qui soit $\mathbb{R}_+$-homogène en la première coordonnée et $\mathbb{R}_+$-linéaire en la seconde, et telle que pour tout $T \in \overline{cv_N}$ et tout $g \in F_N \smallsetminus \{e\}$, nous ayons $\langle T, [g] \rangle = ||g||_T$. Ceci permet de réinterpréter le Théorème 1.6 en termes de courants de la manière suivante.

Théorème 1.8. *Pour tous $T, T' \in CV_N$, nous avons*

$$d(T, T') = \log \sup_{\eta \in Curr_N} \frac{\langle T', \eta \rangle}{\langle T, \eta \rangle}.$$

Étant donné $T \in \overline{cv_N}$ et $\eta \in Curr_N$, nous dirons que η est *dual* à T si $\langle T, \eta \rangle = 0$. Coulbois et Hilion ont montré dans [CH14] que si l'action de F_N sur l'arbre T est libre et à orbites denses, alors l'ensemble des courants projectifs duaux à T est un simplexe de dimension inférieure à $3N - 6$. Ce résultat peut être vu comme une version duale du Théorème 1.7. Les courants *ergodiques* sont les points extrémaux de $Curr_N$.

Théorème 1.9. *(Coulbois–Hilion [CH14, Théorème 1.1]) Soit T un arbre réel muni d'une action libre, minimale et par isométries de F_N, qui est à orbites denses. Alors l'ensemble des courants projectifs duaux à T contient au plus $3N - 5$ classes projectives de courants ergodiques.*

Laminations algébriques.

Une *lamination algébrique* est un sous-ensemble non vide, fermé, F_N-invariant et i-invariant de $\partial^2 F_N$. Par exemple, le support d'un courant géodésique est une lamination algébrique. Soit $T \in \overline{cv_N}$. Pour tout $\epsilon > 0$, soit

$$L_\epsilon(T) := \overline{\{(g^{-\infty}, g^{+\infty}) | ||g||_T < \epsilon\}}.$$

Alors

$$L(T) := \bigcap_{\epsilon > 0} L_\epsilon(T)$$

est une lamination algébrique, appelée la *lamination algébrique duale* à l'arbre T. Nous renvoyons le lecteur à [CHL08a, CHL08b, CHL08c] pour une étude détaillée des laminations algébriques.

Chapitre 2

La compactification primitive de l'outre-espace

Dans ce chapitre, nous définissons et décrivons une autre compactification de CV_N, que nous appelons la *compactification primitive* de CV_N, obtenue en restreignant les fonctions longueurs de translation aux éléments primitifs de F_N dans la construction de Culler et Morgan. Notre motivation pour introduire cette nouvelle compactification vient de notre description de l'horofrontière de l'outre-espace, elle-même motivée par l'étude des marches aléatoires sur CV_N (voir le Chapitre 5). En effet, nous montrerons que la compactification de CV_N par horofonctions est isomorphe à la compactification primitive (Théorème 5.2).

Nous commençons par analyser le défaut de rigidité spectrale de l'ensemble $\mathcal{P}_N$ des éléments primitifs de F_N dans $\overline{cv_N}$. Autrement dit, nous allons répondre à la question suivante.

Question.

À quelle condition deux arbres $T, T' \in \overline{cv_N}$ ont-ils mêmes fonctions longueurs de translation en restriction aux éléments primitifs de F_N ?

Cette question apparaît dans [CFKM12, Problème 6.1]. Lorsque $N = 2$, Carette, Francaviglia, Kapovich et Martino donnent un exemple, qu'ils attribuent à Tao, d'arbres de $\overline{cv_2}$ avec les mêmes fonctions longueurs de translation en restriction à $\mathcal{P}_2$ [CFKM12, Exemple 6.2]. En conséquence du théorème de White rappelé au chapitre précédent (Théorème 1.6), Carette, Francaviglia, Kapovich et Martino montrent que l'ensemble $\mathcal{P}_N$ est spectralement rigide dans cv_N, i.e. deux arbres de cv_N qui ont les mêmes fonctions longueurs de translation en restriction aux éléments primitifs de F_N sont égaux.

Théorème 2.1. *(Carette–Francaviglia–Kapovich–Martino [CFKM12, Théorème 3.4]) Soit $T, T' \in CV_N$. Si les familles $(||g||_T)_{g\in\mathcal{P}_N}$ et $(||g||_{T'})_{g\in\mathcal{P}_N}$ sont projectivement égales, alors $T = T'$.*

L'idée de leur démonstration consiste à remarquer que, sous l'hypothèse que les familles $(||g||_{T'})_{g\in\mathcal{P}_N}$ et $(||g||_T)_{g\in\mathcal{P}_N}$ sont projectivement égales, le théorème de White entraîne que $d(T, T') = -d(T', T)$, puisque le rapport $\frac{||g||_{T'}}{||g||_T}$ est constant sur $\mathcal{P}_N$. Ceci entraîne que $T = T'$.

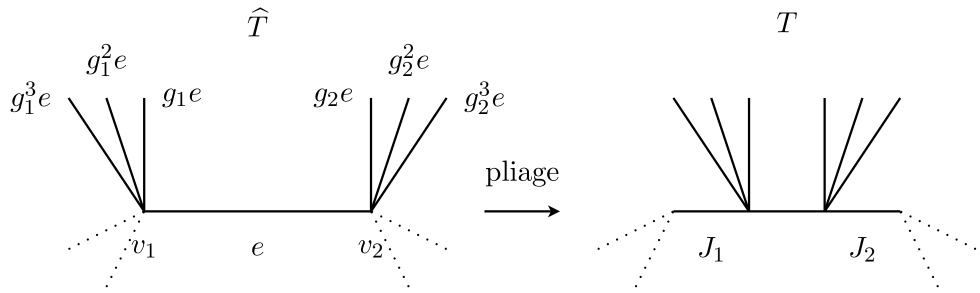


FIGURE 2.1 – L'arbre T est obtenu par tirage à partir de $\widehat{T}$.

2.1 Tiroirs-équivalence et défaut de rigidité spectrale de $\mathcal{P}_N$ dans $\overline{cv_N}$

Afin de décrire le défaut de rigidité spectrale de $\mathcal{P}_N$ dans $\overline{cv_N}$, nous introduisons le concept de *tiroirs-équivalence* entre arbres de $\overline{cv_N}$. Un *facteur libre* de F_N est un sous-groupe $A \subseteq F_N$ tel qu'il existe un sous-groupe $B \subseteq F_N$ pour lequel nous avons la décomposition en produit libre $F_N = A * B$. Nous rappelons qu'un élément $g \in F_N$ est *primitif* s'il fait partie d'une base de F_N. Il est *simple* s'il appartient à un facteur libre propre de F_N. Nous dirons que deux arbres $T, T' \in \overline{cv_N}$ sont *primitifs-équivalents* (resp. *simples-équivalents*) si pour tout élément primitif (resp. simple) $g \in F_N$, nous avons $||g||_{T'} = ||g||_T$. Nous introduisons maintenant une troisième relation d'équivalence sur $\overline{cv_N}$, que nous appelons la *tiroirs-équivalence*, qui va nous fournir la caractérisation souhaitée. Nous renvoyons à l'encadré en page 26 pour un rappel de la notion de *graphe d'actions*. La définition suivante est illustrée en Figure 2.1.

Définition 2.2. *Soit $T, \widehat{T} \in \overline{cv_N}$. L'arbre T est obtenu par* tirage *à partir de $\widehat{T}$ s'il existe*

- *une arête e à stabilisateur trivial dans $\widehat{T}$, dont nous appelons v_1 et v_2 les extrémités, et*
- *pour tout $i \in \{1, 2\}$, un sous-segment $J_i \subseteq e$ contenant v_i (qui peut être réduit à un point), de sorte que $J_1 \cap J_2$ contienne au plus un point, et*
- *pour tout $i \in \{1, 2\}$ pour lequel J_i est non dégénéré, un élément g_i appartenant au stabilisateur de v_i dans $\widehat{T}$, qui n'est pas une puissance,*

de sorte que T soit le quotient de $\widehat{T}$ obtenu en identifiant de manière F_N-équivariante J_i avec $g_i J_i$ pour tout $i \in \{1, 2\}$.

Nous dirons aussi que T est obtenu à partir de $\widehat{T}$ en tirant les éléments g_i le long de l'arête e à partir des extrémités v_i.

Définition 2.3. *Soit $T, \widehat{T} \in \overline{cv_N}$. L'arbre T est obtenu par* tirage non simple *à partir de $\widehat{T}$ si T est obtenu par tirage de $\widehat{T}$ et, avec les notations ci-dessus, l'arête e se projette sur une arête non séparante du graphe sous-jacent à la décomposition canonique de $\widehat{T}$ en graphe d'actions à orbites denses, et ni g_1 ni g_2 n'appartient à un facteur libre de rang $N-2$ de F_N.*

Définition 2.4. *Deux arbres $T, T' \in \overline{cv_N}$ sont* tiroirs-équivalents *si $T = T'$, ou s'il existe $\widehat{T} \in \overline{cv_N}$ tel que T et T' soient tous deux obtenus par tirage non simple à partir de $\widehat{T}$.*

Remarquons que l'arbre $\widehat{T}$ apparaissant dans la Définition 2.4 contient au plus une orbite d'arêtes à stabilisateur trivial, sinon l'élément tiré serait contenu dans un facteur

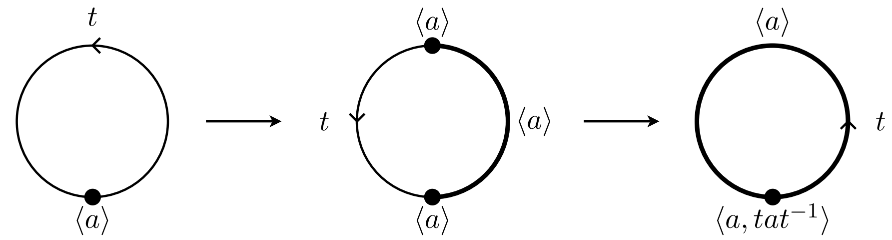


FIGURE 2.2 – Une famille d'arbres tiroirs-équivalents dans $\overline{cv_2}$.

libre de rang $N-2$ de F_N. Le fait que la tiroirs-équivalence est bien une relation d'équivalence sur $\overline{cv_N}$ est démontré au Lemme 2.2 de l'Annexe A. L'idée de notre démonstration consiste à remarquer que si T est obtenu par tirage non simple à partir de deux arbres $\widehat{T_1}$ et $\widehat{T_2}$, alors les arbres $\widehat{T_1}$ et $\widehat{T_2}$ sont tous deux obtenus par tirage non simple d'un arbre $\widehat{T}$, et les éléments tirés pour passer de $\widehat{T}$ aux arbres $\widehat{T_1}$, $\widehat{T_2}$ et T sont les mêmes. Cette remarque est une conséquence de la description donnée dans les deux parties ci-après de l'opération de tirage non simple. Il en découle alors facilement que la tiroirs-équivalence est une relation d'équivalence.

Théorème 2.5.

Soit $T, T' \in \overline{cv_N}$. Les affirmations suivantes sont équivalentes.
– Les arbres T et T' sont primitifs-équivalents.
– Les arbres T et T' sont simples-équivalents.
– Les arbres T et T' sont tiroirs-équivalents.

Remarquons que ces relations d'équivalence induisent aussi des relations d'équivalence sur le projectifié $\overline{CV_N}$: deux arbres $T, T' \in \overline{CV_N}$ sont *primitifs-équivalents* si les fonctions longueurs $(||g||_T)_{g\in\mathcal{P}_N}$ et $(||g||_{T'})_{g\in\mathcal{P}_N}$ sont projectivement égales. La démonstration du Théorème 2.5 fait l'objet de l'Annexe A. Dans les parties qui suivent, nous donnons une description de la tiroirs-équivalence, ainsi que quelques exemples. Nous donnons également une idée de la démonstration du Théorème 2.5 dans le cas où les arbres T et T' sont à stabilisateurs d'arcs triviaux. Ceci inclut le cas des actions libres et simpliciales, et le cas opposé des actions à orbites denses.

2.2 Description de la tiroirs-équivalence lorsque $N = 2$

Plaçons nous dans le cas où $N = 2$. La discussion ci-après est illustrée dans les Figures 2.2 à 2.4. Soit $T, \widehat{T} \in \overline{cv_2}$, de sorte que T soit obtenu à partir de $\widehat{T}$ par tirage non simple. Nous supposons que les arbres T et $\widehat{T}$ ne sont pas tous deux obtenus par tirage non simple à partir d'un arbre $\widetilde{T} \neq \widehat{T} \in \overline{cv_2}$. Dans ce cas, l'arbre $\widehat{T}$ est l'arbre de Bass–Serre d'un scindement libre de la forme $F_2 = \langle a\rangle *$, où a est un élément primitif de F_2, et T est obtenu par tirage de l'élément a, comme cela est représenté en Figure 2.2.

Culler et Vogtmann ont donné dans [CV91] une description explicite de $\overline{CV_2}$. Les arbres simpliciaux dans $\overline{CV_2}$ sont les arbres de Bass–Serre des scindements de F_2 représentés en Figure 2.3, et les arbres non simpliciaux sont duaux à des laminations mesurées arationnelles sur une surface compacte orientable de genre 1 ayant une composante de bord. En

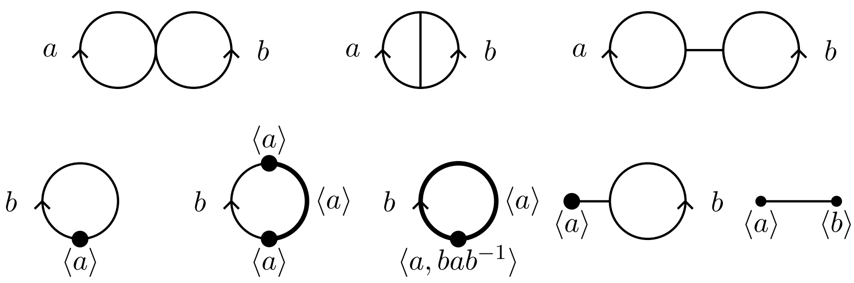


FIGURE 2.3 – Arbres simpliciaux dans $\overline{cv_2}$.

notant $\sim$ la relation de primitifs-équivalence sur $\overline{CV_2}$, le quotient $\overline{CV_2}/\sim$ est obtenu en écrasant dans $\overline{CV_2}$ les segments représentés en pointillés sur la Figure 2.4, qui correspondent aux pliages décrits ci-dessus et représentés en Figure 2.2. L'espace quotient ainsi obtenu, homéomorphe à un disque sur lequel se dressent des triangles, est représenté en Figure 2.5.

2.3 Description de la tiroirs-équivalence lorsque $N \geq 3$

Nous donnons une interprétation de la notion de tirage non simple en termes de graphes d'actions dans le cas où $N \geq 3$. Soit $\widehat{T} \in \overline{cv_N}$ un arbre ayant exactement une orbite d'arêtes e à stabilisateur trivial, et soit $T \neq \widehat{T}$ un arbre obtenu par tirage non simple de $\widehat{T}$. Nous supposons que les arbres T et $\widehat{T}$ ne sont pas tous les deux obtenus à partir d'un même arbre $\widetilde{T} \neq \widehat{T}$ par tirage non simple des mêmes éléments de F_N. L'arbre $\widehat{T}$ se scinde en un graphe d'actions ayant

- un unique sommet, pour lequel l'arbre de sommet correspondant est un A-arbre T_0 (pas forcément minimal), pour un certain facteur libre A de rang $N-1$ de F_N, et
- une unique arête, qui est non séparante et dont le stabilisateur est trivial.

Puisque $N \geq 3$, le groupe A n'est pas cyclique, donc il fixe au plus un point dans $\widehat{T}$. Le sous-arbre minimal T_0^{min} pour l'action de A sur T_0 est donc bien défini. L'arbre T_0 est obtenu à partir de T_0^{min} en ajoutant des points de la complétion métrique de T_0^{min}, ainsi qu'au plus deux orbites d'arêtes simpliciales. Les sommets de valence 1 de ces arêtes dans T_0 sont des points d'attache de e dans $\widehat{T}$. (En fait, il découle de la description ci-dessous que $T_0 \smallsetminus T_0^{min}$ contient au plus une orbite d'arêtes simpliciales).

Premier cas (représenté en Figure 2.6) : L'arbre T_0 est minimal, ou plus généralement T_0 est obtenu à partir de T_0^{min} en ajoutant des points dans le complété métrique de T_0, mais sans ajouter d'arête simpliciale.
Dans ce cas, l'arbre T est obtenu à partir de $\widehat{T}$ en tirant un élément de A qui n'est contenu dans aucun facteur libre propre de A, à chacune des extrémités de e.

Deuxième cas (représenté en Figure 2.7) : L'ensemble $T_0 \smallsetminus T_0^{min}$ contient une arête simpliciale e' dont le stabilisateur $\langle g\rangle$ n'est contenu dans aucun facteur libre propre de A.
Dans ce cas, le sommet de e' qui est de valence 1 dans T_0 se projette en un sommet de valence au moins 3 dans la décomposition de $\widehat{T}$ en graphe d'actions à orbites denses. En effet, sinon, l'arbre $\widehat{T}$ serait obtenu à partir d'un arbre $\widetilde{T}$ en tirant g le long de l'arête e', ce qui contredirait l'hypothèse faite sur $\widehat{T}$. Nous sommes donc dans la situation représentée sur la Figure 2.7, et T est obtenu par tirage de l'élément g le long de l'arête e. Les stabili-

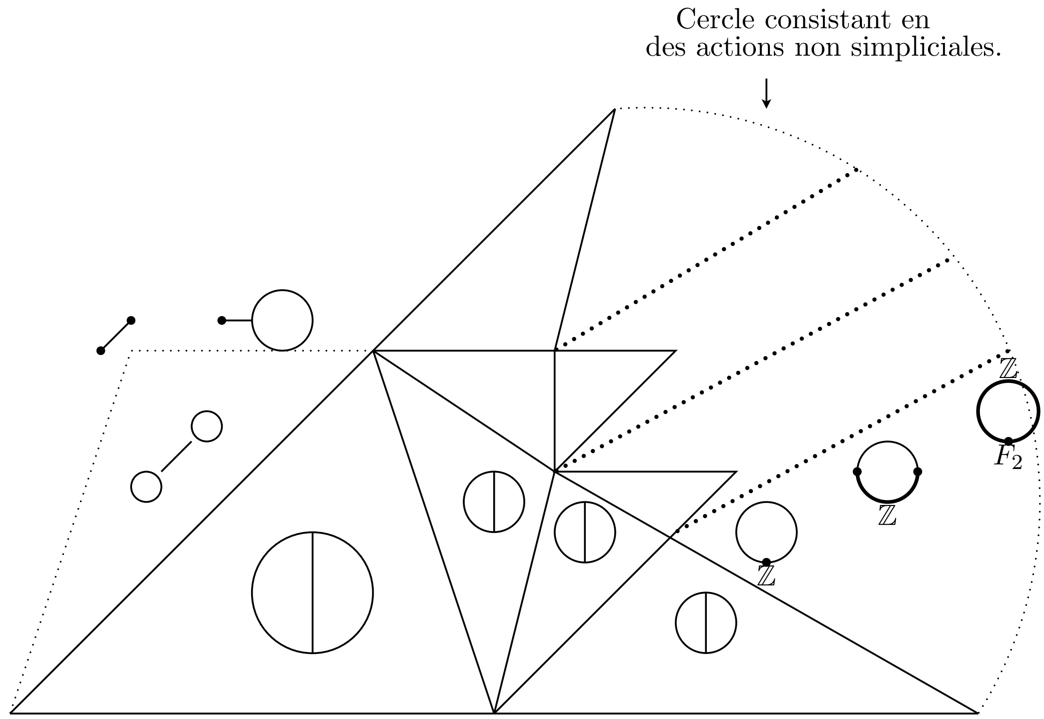


FIGURE 2.4 – Le quotient $\overline{CV_2}/\!\sim$ est obtenu en écrasant les segments en pointillés.

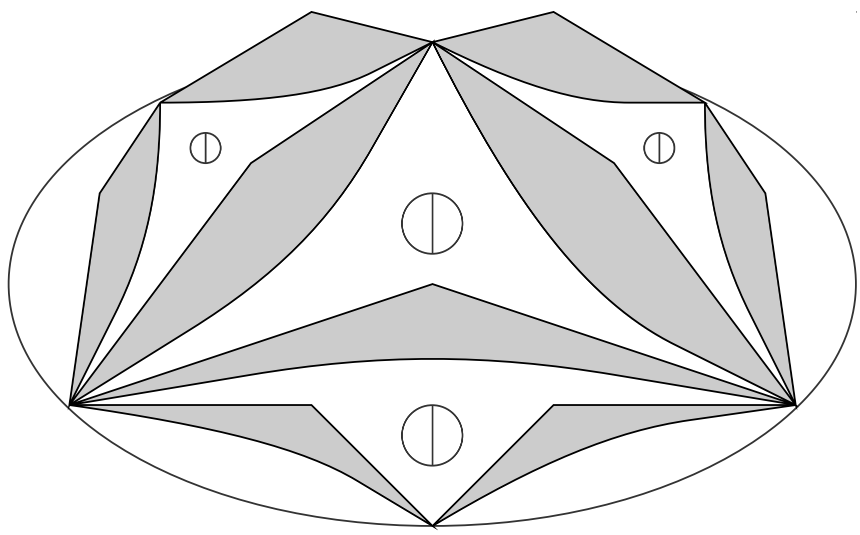


FIGURE 2.5 – Représentation de l'espace quotient $\overline{CV_2}/\!\sim$.

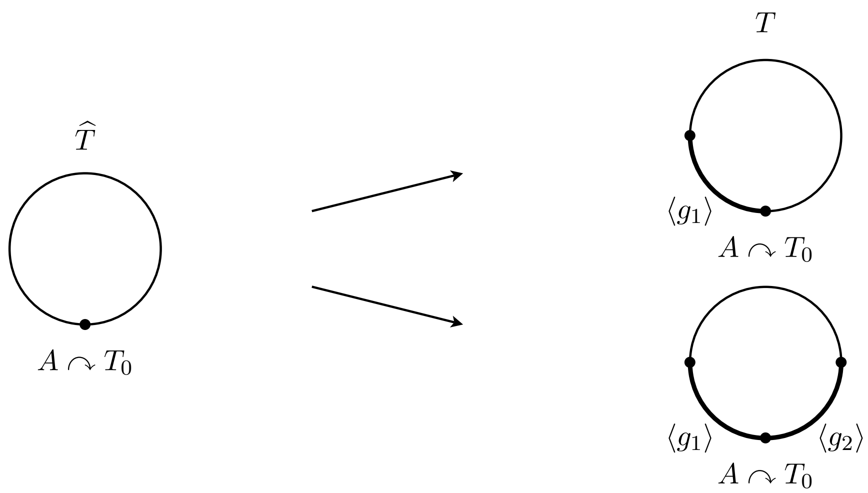


FIGURE 2.6 – La situation dans le premier cas.

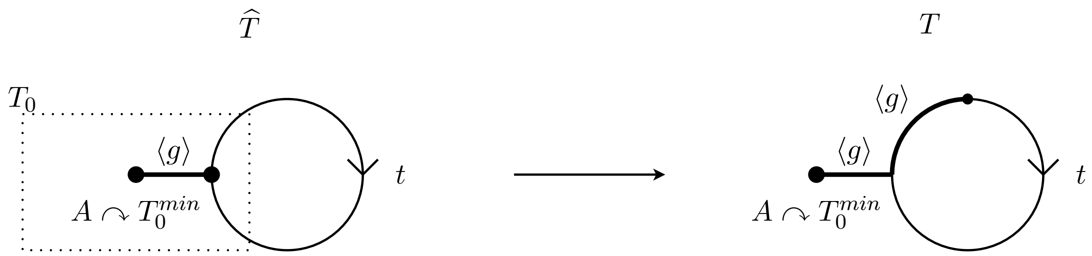


FIGURE 2.7 – La situation dans le deuxième cas.

sateurs de tripodes de T étant triviaux, ce tirage ne peut se faire qu'à partir de l'une des extrémités de l'arête e.

Troisième cas (représenté en Figure 2.8) : L'ensemble $T_0 \smallsetminus T_0^{min}$ contient une arête simpliciale, et tout stabilisateur d'une arête dans $T_0 \smallsetminus T_0^{min}$ est non trivial, et contenu dans un facteur libre propre d'un conjugué de A.
Dans ce cas, aucun tirage non simple ne peut se faire à partir d'un sommet de valence 1 de T_0. Il y a donc exactement une orbite d'arêtes simpliciales dans $T_0 \smallsetminus T_0^{min}$ dont le stabilisateur est engendré par un élément simple de A. L'arbre T est obtenu par tirage non simple à partir de l'extrémité de l'arête e qui n'est pas le sommet de valence 1 de T_0.

Quatrième cas : L'ensemble $T_0 \smallsetminus T_0^{min}$ contient une arête simpliciale e' à stabilisateur trivial.
Nous allons montrer que ce dernier cas est en fait impossible. En effet, comme dans le deuxième cas, l'une des extrémités de e' doit être de valence 3 dans $\widehat{T}$, mais alors aucun tirage non simple ne peut être réalisé dans $\widehat{T}$.

2.4 Quelques exemples

Donnons maintenant quelques exemples explicites d'arbres tiroirs-équivalents.

Exemple 2.6. (voir Figure 2.9) La Figure 2.9 donne des exemples d'arbres simpliciaux qui sont tiroirs-équivalents. Supposons que $N \geq 3$. Soit A un facteur libre de F_N de rang

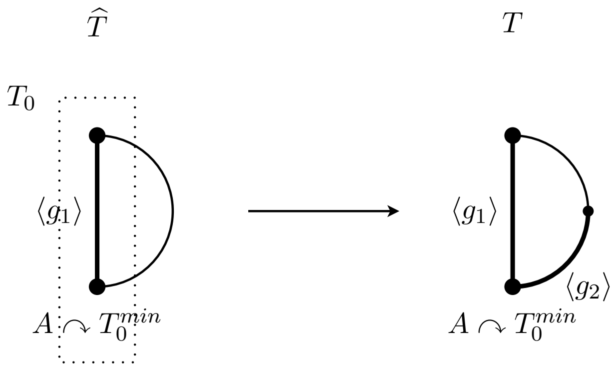


FIGURE 2.8 – La situation dans le troisième cas.

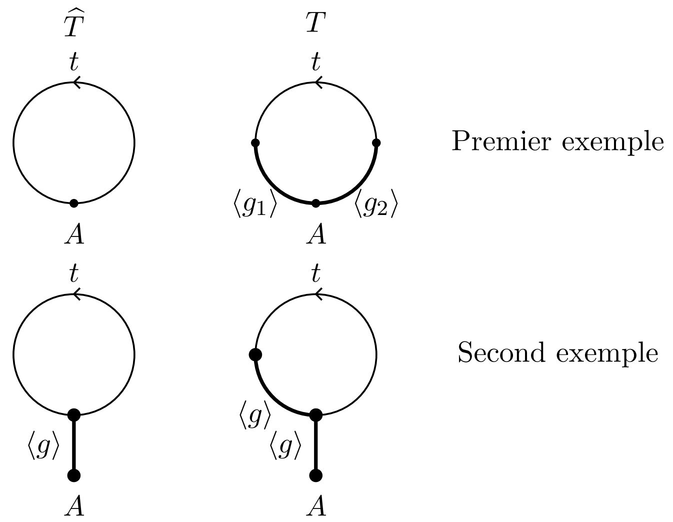


FIGURE 2.9 – Exemples d'arbres simpliciaux tiroirs-équivalents.

$N-1$.

Dans le premier exemple, l'arbre $\widehat{T}$ est l'arbre de Bass–Serre du scindement $F_N = A*$. Tous les arbres obtenus à partir de $\widehat{T}$ en tirant des éléments g_1 et g_2 qui ne sont contenus dans aucun facteur libre propre de A sont tiroirs-équivalents. Ainsi, nous avons construit une union de simplexes de dimension 2 dans $\overline{cv_N}$ qui contiennent tous l'arbre $\widehat{T}$ dans leur adhérence, formée d'arbres qui sont tiroirs-équivalents.

Dans le second exemple, l'arbre T est obtenu à partir de $\widehat{T}$ en tirant l'élément g, qui ne fait partie d'aucun facteur libre propre de A. On obtient ainsi un segment dans $\overline{cv_N}$ formé d'arbres tiroirs-équivalents.

Exemple 2.7. (voir Figure 2.10) Nous donnons maintenant des exemples d'arbres non simpliciaux qui sont tiroirs-équivalents. Soit S une surface compacte orientable ayant une unique composante de bord, de groupe fondamental F_{N-1}, et soit T_{N-1} un arbre dual à un feuilletage sur S. La composante de bord de S représente une classe de conjugaison g de F_{N-1} qui n'est contenue dans aucun facteur libre propre de F_{N-1}, et qui fixe l'orbite d'un point x dans T_{N-1}. Considérons le F_N-graphe d'actions dont l'unique arbre de sommet est T_{N-1}, et qui contient une unique arête e à stabilisateur trivial, attachée à l'arbre T_{N-1} au point x à l'une au moins de ses extrémités. Nous formons alors une famille d'arbres tiroirs-équivalents dans $\overline{cv_N}$ en pliant e le long du translaté ge de manière F_N-équivariante.

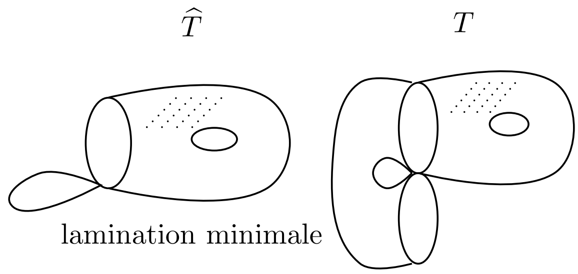


FIGURE 2.10 – Exemple d'arbres non simpliciaux tiroirs-équivalents.

Exemple 2.8. Nous présentons maintenant un procédé itératif qui permet de construire des familles d'arbres tiroirs-équivalents en tout rang. Soit Y un F_{N-1}-arbre minimal qui contient un point x dont le stabilisateur n'est contenu dans aucun facteur libre propre de F_{N-1}. Considérons l'arbre T dual à un graphe d'actions ayant Y pour unique arbre de sommet, et attachons une arête e à stabilisateur trivial, dont les deux extrémités sont attachées en des points de l'orbite de x dans Y. En pliant totalement l'arête e le long d'un translaté ge, où l'élément g n'est contenu dans aucun facteur libre propre de F_{N-1}, de manière équivariante, nous obtenons un arbre T', qui est tiroirs-équivalent à T. Cet arbre T' contient une orbite de points dont le stabilisateur n'est contenu dans aucun facteur libre propre de F_N. Il peut donc servir de nouvel arbre Y pour construire des arbres tiroirs-équivalents dans $\overline{cv_{N+1}}$.

Remarque 2.9. En fait, il résulte de notre preuve du Théorème 2.5 que si la classe de tiroirs-équivalence d'un arbre dans $\overline{cv_N}$ est non triviale, alors cet arbre est simplicial ou dual à un complexe feuilleté dont toutes les composantes minimales sont duales à des feuilletages sur des surfaces (voir la Remarque 3.12 de l'Annexe A).

2.5 La compactification primitive de CV_N

Le Théorème 2.5 donne une description d'une nouvelle compactification de CV_N, obtenue par restriction aux éléments primitifs de F_N des fonctions longueurs de translation dans la construction de Culler et Morgan. Nous noterons $\sim$ la relation de primitifs-équivalence introduite ci-dessus.

Théorème 2.10. *L'application $i_{prim} : CV_N \to \mathbb{PR}^{\mathcal{P}_N}$, obtenue par restriction des fonctions longueurs de translation aux éléments primitifs de F_N, est un homéomorphisme sur son image. L'adhérence de son image est compacte, et homéomorphe à $\overline{CV_N}/\sim$ par un unique homéomorphisme $Out(F_N)$-équivariant égal à l'identité en restriction à CV_N.*

Démonstration. La continuité de i_{prim} découle de celle de i, et son injectivité provient du Théorème 2.1. Pour montrer que i_{prim} est un homéomorphisme sur son image, il reste à vérifier que si une suite d'arbres $(T_n)_{n\in\mathbb{N}} \in \overline{CV_N}^{\mathbb{N}}$ sort de tout compact de CV_N, et si $T \in CV_N$, alors la suite $(i_{prim}(T_n))_{n\in\mathbb{N}}$ ne converge pas vers $i_{prim}(T)$. Par compacité de $\overline{CV_N}$, quitte à extraire, nous pouvons supposer que la suite $(T_n)_{n\in\mathbb{N}}$ converge vers un arbre $T_\infty \in \overline{CV_N}$, et $T_\infty \in \partial CV_N$ puisque la suite $(T_n)_{n\in\mathbb{N}}$ sort de tout compact de CV_N. Le Théorème 2.5 entraîne alors que $i_{prim}(T_\infty) \neq i_{prim}(T)$ (où nous étendons naturellement la définition de i_{prim} à T_∞), ce qui donne la conclusion souhaitée.

Pour montrer la compacité de $\overline{i_{prim}(CV_N)}$, nous remarquons dans un premier temps que si $\xi \in \overline{i_{prim}(CV_N)}$, alors par compacité de $\overline{i(CV_N)}$, il existe un arbre $T \in \overline{CV_N}$ dont les longueurs de translation d'éléments de $\mathcal{P}_N$ sont données par ξ. Il résulte de cette observation et de la compacité de $\overline{i(CV_N)}$ que toute suite $(T_n)_{n\in\mathbb{N}} \in \overline{i_{prim}(CV_N)}^{\mathbb{N}}$ possède une valeur d'adhérence, obtenue en restreignant à $\mathcal{P}_N$ la fonction longueurs de translation d'une valeur d'adhérence d'une suite de relevés $\widetilde{T_n} \in \overline{CV_N}$.

Enfin, l'application i_{prim} s'étend en une application continue et bijective entre les espaces compacts $\overline{CV_N}/\sim$ et $\overline{i_{prim}(CV_N)}$, et donc en un homéomorphisme entre ces espaces, qui est $\mathrm{Out}(F_N)$-équivariant et égal à l'identité sur CV_N par construction. $\square$

Notons $\overline{CV_N}^{prim}$ l'adhérence de l'image de i_{prim} dans $\mathbb{PR}^{\mathcal{P}_N}$. Le Théorème 2.10 assure que c'est une compactification de CV_N, que nous appelons la *compactification primitive*.

2.6 Idée de la démonstration du Théorème 2.5

Nous avons vu que la rigidité spectrale de $\mathcal{P}_N$ dans CV_N (Théorème 2.1) se déduit du théorème de White (Théorème 1.6). Afin d'étudier la question de la rigidité de $\mathcal{P}_N$ dans $\overline{CV_N}$, nous étendons le théorème de White au cas général d'arbres $T, T' \in \overline{cv_N}$. L'extension au cas où T est un arbre simplicial métrique est due à Algom-Kfir [AK13, Proposition 4.5] dans le cadre de son étude de la complétion métrique de l'outre-espace (Algom-Kfir énonce ce résultat dans le cas où T' est un arbre de la complétion métrique de CV_N). Étant donné $T, T' \in \overline{cv_N}$, nous notons $\mathrm{Lip}(T, T')$ la borne inférieure d'une constante de Lipschitz d'une application F_N-équivariante de T vers le complété métrique $\overline{T'}$ de T' (s'il n'existe pas de telle application, nous posons $\mathrm{Lip}(T, T') = +\infty$). Nous prenons comme conventions $\frac{0}{0} = 0$ et $\frac{1}{0} = +\infty$.

Théorème 2.11.

Pour tous $T, T' \in \overline{cv_N}$, nous avons $\mathrm{Lip}(T, T') = \sup_{g\in F_N\setminus\{e\}} \frac{||g||_{T'}}{||g||_T}$.

Le Théorème 2.11 répond à une question d'Algom-Kfir [AK13, Question 4.6]. Sa démonstration fait l'objet des Parties 5 et 6 de l'Annexe A. Remarquons qu'en général, il est nécessaire de passer à la complétion métrique de l'arbre T' dans la définition de $\mathrm{Lip}(T, T')$. Dès que $N \geq 3$, nous pouvons en effet trouver des arbres $T, T' \in \overline{cv_N}$ pour lesquels il existe une application lipschitzienne F_N-équivariante de T vers $\overline{T'}$, mais aucune telle application de T vers T'. Nous remarquons également que dans le cas général, la borne inférieure dans la définition de $\mathrm{Lip}(T, T')$ est toujours atteinte, mais la borne supérieure dans le membre de droite de l'égalité énoncée au Théorème 2.11 ne l'est pas forcément. Nous renvoyons à la Partie 6.1 de l'Annexe A pour des exemples et une discussion plus détaillée de ces faits.

Nous nous proposons maintenant d'esquisser une démonstration des Théorèmes 2.11 et 2.5 dans le où l'arbre T est à stabilisateurs d'arcs triviaux. Ceci couvre en particulier le cas des arbres à orbites denses. Nous commencerons par traiter le cas classique où $T \in cv_N$ est une action simpliciale et libre. Dans le cas général, l'arbre T se scinde en un graphe d'actions à orbites denses (voir l'encadré en page 26), et il s'agit alors d'étendre les techniques décrites ci-dessous pour pouvoir traiter du cas où T contient à la fois des parties à orbites denses et des parties simpliciales, avec des stabilisateurs d'arcs qui peuvent être non triviaux. Nous allons en fait montrer les résultats (plus précis) suivants.

Théorème 2.12. *Soit $T, T' \in \overline{cv_N}$. Si T est à stabilisateurs d'arcs triviaux, alors*

$$Lip(T,T') = \sup_{g\in F_N} \frac{||g||_{T'}}{||g||_T} = \sup_{g\in \mathcal{P}_N} \frac{||g||_{T'}}{||g||_T}.$$

Théorème 2.13. *Soit $T, T' \in \overline{cv_N}$. Supposons que les arbres T et T' soient à stabilisateurs d'arcs triviaux. Si $||g||_{T'} = ||g||_T$ pour tout $g \in \mathcal{P}_N$, alors $T = T'$.*

Schéma de démonstration du Théorème 2.12 dans le cas où $T \in cv_N$ est une action simpliciale et libre.

Dans ce cas, l'argument est classique et dû à White, voir aussi [AK13, Proposition 4.5]. Commençons par remarquer que l'inégalité $||g||_{T'} \leq \mathrm{Lip}(T,T')||g||_T$ est toujours satisfaite pour tout $g \in F_N$. Un argument de type Arzelà–Ascoli permet également de montrer l'existence d'une application F_N-équivariante $f : T \to T'$ dont la constante de Lipschitz est égale à $\mathrm{Lip}(T,T')$. Nous pouvons supposer en outre que cette application est linéaire en restriction à chaque arête de T, et que le sous-ensemble X des arêtes de T dont la longueur est multipliée par $\mathrm{Lip}(T,T')$ par l'application f est minimal pour l'inclusion (parmi les applications réalisant la constante de Lipschitz). Nous remarquons alors que de chaque sommet de X partent au moins deux arêtes contenues dans X, dont les images par f n'ont pas de segment initial commun. Sinon, en perturbant légèrement la définition de f en un tel sommet, nous obtiendrions une nouvelle application f' vérifiant soit $\mathrm{Lip}(f') < \mathrm{Lip}(f)$, soit $X' \subsetneq X$. Cette remarque permet de construire dans le graphe quotient T/F_N un lacet γ contenu dans X/F_N dont tous les tournants sont *légaux*, i.e. les images par f de deux arêtes consécutives de γ n'ont pas de segment initial commun. La longueur de translation de tout élément de F_N représenté par un tel lacet est multipliée par $\mathrm{Lip}(T,T')$ en appliquant f, ce qui termine la preuve du Théorème 2.11 dans le cas où $T \in cv_N$.

Le lacet γ peut en fait être choisi de sorte que l'une des arêtes du graphe T/F_N soit parcourue au plus une fois par γ, et l'élément $g \in F_N$ que nous obtenons est alors primitif. Le Théorème 2.11 se précise donc en le Théorème 2.12. L'argument présenté ci-dessus se généralise en fait au cas où T est simplicial, voir par exemple [AK13, Proposition 4.5].

Schéma de démonstration du Théorème 2.12 dans le cas général.

Nous proposons maintenant une esquisse de démonstration du Théorème 2.12. Soit $T, T' \in \overline{cv_N}$. Supposons que l'arbre T est à stabilisateurs d'arcs triviaux. Le Théorème 1.5 nous fournit une approximation lipschitzienne de T par des arbres simpliciaux libres T_n, et nous avons alors

$$\begin{aligned}\mathrm{Lip}(T_n,T') &= \sup_{g\in\mathcal{P}_N} \tfrac{||g||_{T'}}{||g||_{T_n}}\\ &\leq \sup_{g\in\mathcal{P}_N} \tfrac{||g||_{T'}}{||g||_{T}} \sup_{g\in\mathcal{P}_N} \tfrac{||g||_{T}}{||g||_{T_n}}\\ &\leq \sup_{g\in\mathcal{P}_N} \tfrac{||g||_{T'}}{||g||_{T}},\end{aligned}$$

où la première égalité découle du cas où $T_n \in cv_N$, et la dernière inégalité découle du caractère lipschitzien de l'approximation de T choisie. L'extension du théorème de White s'en déduit par passage à la limite en utilisant le théorème suivant, qui fait l'objet de la Partie 4 de l'Annexe A.

Théorème 2.14. *Soit T et T' deux F_N-arbres très petits. Soit $(T_n)_{n\in\mathbb{N}}$ (resp. $(T'_n)_{n\in\mathbb{N}}$) une suite de F_N-arbres convergeant vers T (resp. T'), et soit $(M_n)_{n\in\mathbb{N}}$ une suite de réels, telle que $M := \liminf_{n\to+\infty} M_n < +\infty$. Supposons que pour tout $n \in \mathbb{N}$, il existe une application M_n-lipschitzienne F_N-équivariante $f_n : T_n \to T'_n$. Alors il existe une application M-lipschitzienne F_N-équivariante $f : T \to \overline{T'}$, où $\overline{T'}$ désigne le complété métrique de T'.*

Notre preuve du Théorème 2.14 utilise la théorie des ultralimites d'espaces métriques. Étant donné un ultrafiltre non principal ω sur $\mathbb{N}$, l'idée est de trouver une suite $(p_n)_{n\in\mathbb{N}} \in \prod T_n$ pour laquelle la suite des espaces métriques pointés $(T_n, p_n)_{n\in\mathbb{N}}$ converge (au sens de ω) vers un F_N-arbre complet T_ω qui contient une copie isométrique plongée de T (et donc de $\overline{T}$). En prenant l'ultralimite des applications f_n, nous obtenons alors une application M-lipschitzienne F_N-équivariante $f_\omega : T_\omega \to T'_\omega$. L'application f recherchée est obtenue en précomposant f_ω par l'inclusion de T dans T_ω, et en la postcomposant par la projection de T'_ω vers $\overline{T'}$.

Remarque 2.15. Lorsqu'aucune hypothèse n'est faite sur les applications f_n, il est nécessaire en général de postcomposer par l'application de projection de T'_ω vers $\overline{T'}$. Nous ne savons pas si cela est nécessaire dans le cas où les arbres T'_n sont minimaux, et chacune des applications f_n est *optimale* au sens où en chaque point $x \in T$, il existe deux directions dans T issues de x dont les f_n-images sont distinctes.

Schéma de la démonstration du Théorème 2.13.

Par hypothèse, nous avons

$$\sup_{g\in\mathcal{P}_N} \frac{||g||_{T'}}{||g||_T} = \sup_{g\in\mathcal{P}_N} \frac{||g||_T}{||g||_{T'}} = 1,$$

et en utilisant le Théorème 2.12, nous avons donc $||g||_{T'} = ||g||_T$ pour tout $g \in F_N$. Le Théorème 2.13 est alors une conséquence du théorème de Culler et Morgan [CM87, Théorème 3.7].

Chapitre 3

Complexes hyperboliques

Nous présentons maintenant des actions de $\mathrm{Out}(F_N)$ sur des complexes simpliciaux hyperboliques, définis par analogie avec le complexe des courbes d'une surface compacte orientable, dont nous rappelons la définition en guise de motivation.

Soit S une surface connexe compacte orientable. Une *courbe fermée simple* sur S est l'image d'un plongement du cercle dans S. Elle est *essentielle* si elle ne borde pas de disque sur S, et n'est pas homotope à une composante de bord de S. Le *complexe des courbes* $\mathcal{C}(S)$, introduit par Harvey dans [Har80], est le complexe simplicial dont les sommets sont les classes d'isotopie de courbes fermées simples essentielles sur S, et dont les k-simplexes sont donnés plus généralement par les collections de $k+1$ classes d'isotopie de courbes fermées simples essentielles pouvant être réalisées de manière disjointe dans S.

Le complexe des courbes $\mathcal{C}(S)$, muni de la distance simpliciale, est un complexe simplicial connexe, de dimension finie, de diamètre infini [Kob88]. Il n'est en général pas localement fini, puisque dans le complémentaire d'une courbe sur S, on peut en général trouver une infinité de courbes deux à deux non isotopes. Le groupe modulaire $\mathrm{Mod}(S)$ agit naturellement sur $\mathcal{C}(S)$ par automorphismes simpliciaux, et Ivanov a montré dans [Iva97] que $\mathrm{Mod}(S)$ est un sous-groupe d'indice 2 du groupe des automorphismes simpliciaux de $\mathcal{C}(S)$. Harer a montré que le complexe des courbes a le type d'homotopie d'un bouquet de sphères dans son étude sur la dimension cohomologique virtuelle de $\mathrm{Mod}(S)$ [Har86]. Enfin, le résultat fondamental a été la preuve par Masur et Minsky de l'hyperbolicité (au sens de Gromov) du complexe des courbes $\mathcal{C}(S)$ [MM99], qui a été un point de départ pour une étude fructueuse de la géométrie de ce complexe, en vue d'étudier le groupe modulaire $\mathrm{Mod}(S)$. D'autres démonstrations de l'hyperbolicité de $\mathcal{C}(S)$ ont été proposées depuis [Bow06, Ham07], les plus récentes montrent que la constante d'hyperbolicité de $\mathcal{C}(S)$ est indépendante de la topologie de la surface S, voir [Aou13, Bow12, CRS13, HPW13]. Les éléments loxodromiques pour l'action de $\mathrm{Mod}(S)$ sur $\mathcal{C}(S)$ sont les difféomorphismes pseudo-Anosov de S, et tout élément qui n'est pas loxodromique a une orbite finie dans $\mathcal{C}(S)$ (ces éléments sont soit d'ordre fini dans $\mathrm{Mod}(S)$, ou préservent une famille finie de courbes deux à deux disjointes sur S). Les stabilisateurs des points fixes attractifs (ou répulsifs) d'isométries loxodromiques dans le bord de Gromov $\partial_\infty\mathcal{C}(S)$ sont virtuellement cycliques.

Au cours de ces dernières années, l'attention s'est portée sur la recherche d'un complexe simplicial hyperbolique muni d'une action de $\mathrm{Out}(F_N)$, analogue au complexe des courbes d'une surface compacte orientable. Plusieurs analogues ont été proposés, avec des géométries relativement différentes les unes des autres. Nous en présentons trois, ainsi que leurs propriétés géométriques connues à ce jour. Auparavant, nous effectuons quelques rappels d'ordre général sur les espaces hyperboliques au sens de Gromov.

3.1 Généralités sur les espaces hyperboliques

Définition. La notion d'espace métrique hyperbolique a été introduite par Gromov dans [Gro87], nous renvoyons également à [BH99, CDP90, GdlH90] pour une introduction détaillée à ce sujet. Soit X un espace métrique, et $p \in X$. Étant donné $x, y \in X$, le *produit de Gromov* de x et y par rapport à p est donné par

$$(x|y)_p := \frac{1}{2}(d(p,x) + d(p,y) - d(x,y)).$$

L'espace X est *hyperbolique au sens de Gromov* s'il existe $\delta \geq 0$ tel que pour tous $x, y, z, p \in X$, nous ayons

$$(x|y)_p \geq \max\{(x|z)_p, (y|z)_p\} - \delta.$$

Un exemple typique d'espace hyperbolique au sens de Gromov est celui d'un arbre réel : l'inégalité est vérifiée dans ce cas avec $\delta = 0$. L'espace hyperbolique $\mathbb{H}^n$ est un autre exemple d'espace hyperbolique au sens de Gromov. Lorsque X est un espace métrique géodésique, l'hyperbolicité de X est équivalente à l'existence d'un réel $\delta > 0$ tel que pour tous $x, y, z \in X$, et tous segments géodésiques $[x, y]$, $[y, z]$ et $[x, z]$, nous ayons $[x, z] \subseteq N_\delta([x, y]) \cup N_\delta([y, z])$ (où étant donné $Y \subseteq X$, nous désignons par $N_\delta(Y)$ le δ-voisinage de Y dans X). Autrement dit, les triangles géodésiques de X sont δ-fins. L'hyperbolicité au sens de Gromov est un invariant de quasi-isométrie sur la classe des espaces métriques géodésiques (nous renvoyons à l'encadré en page 47 pour la notion de quasi-isométrie entre espaces métriques).

Bord de Gromov d'un espace métrique hyperbolique. Soit X un espace métrique hyperbolique au sens de Gromov, et $p \in X$ un point base. Une suite $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ *converge à l'infini* si le produit de Gromov $(x_n|x_m)_p$ tend vers $+\infty$ lorsque n et m tendent vers $+\infty$. Deux suites $(x_n)_{n\in\mathbb{N}}$ et $(y_n)_{n\in\mathbb{N}}$ sont *équivalentes* si le produit de Gromov $(x_n|y_m)_p$ tend vers $+\infty$ lorsque n et m tendent vers $+\infty$. L'hyperbolicité de X assure que ceci définit bien une relation d'équivalence sur X. Le *bord de Gromov* ∂X de X est l'ensemble des classes d'équivalence de suites convergeant à l'infini.

Soit $a, b \in \partial X$. Leur *produit de Gromov* est donné par

$$(a|b)_p := \sup \liminf_{i,j\to+\infty} (x_i|y_j)_p,$$

où la borne supérieure est prise sur l'ensemble des suites $(x_i)_{i\in\mathbb{N}} \in X^{\mathbb{N}}$ convergeant vers a et $(y_j)_{j\in\mathbb{N}} \in X^{\mathbb{N}}$ convergeant vers b. L'espace ∂X est muni de la topologie pour laquelle un système fondamental de voisinages d'un point $a \in \partial X$ est donné par les ensembles $V_r(a) := \{b \in \partial X | (a|b)_p \geq r\}$. Là encore, ceci est bien défini grâce à l'hyperbolicité de X, voir [GdlH90, Chapitre 7]. Étant donné $a \in X$ et $b \in \partial X$, nous pouvons de même définir le produit de Gromov

$$(a|b)_p := \sup \liminf_{j\to+\infty} (a|y_j)_p,$$

où la borne supérieure est prise sur l'ensemble des suites $(y_j)_{j\in\mathbb{N}} \in X^{\mathbb{N}}$ convergeant vers b. Ceci permet de même de définir une topologie sur l'ensemble $\overline{X} := X \cup \partial X$. Lorsque X est géodésique, tout point $\xi \in \partial X$ est limite d'un rayon quasi-géodésique $\tau : \mathbb{R}_+ \to X$. Lorsque X est un espace métrique *propre*, i.e. dont les boules fermées sont compactes, l'espace $\overline{X}$ est compact, et constitue une compactification de X (en particulier, le bord de Gromov ∂X est lui aussi compact). Ceci n'est plus vrai en général lorsque l'espace

Intermède (Quelques notions de géométrie grossière)**.**

Nous rappelons dans cet encadré les notions de quasi-isométrie entre espaces métriques, et de quasi-géodésique (reparamétrée).
Soit (X, d) et (X', d') deux espaces métriques. Soit $K > 0$ et $L \geq 0$. Une (K, L)-*quasi-isométrie* de X vers X' est une application $f : X \to X'$ telle que

- pour tout $x' \in X'$, il existe $x \in X$ tel que $d'(f(x), x') \leq L$, et
- pour tous $x, y \in X$, nous ayons

$$\frac{1}{K}d(x, y) - L \leq d'(f(x), f(y)) \leq K d(x, y) + L.$$

Deux espaces métriques (X, d) et (X', d') sont *quasi isométriques* s'il existe $K > 0$ et $L \geq 0$, et une (K, L)-quasi-isométrie de X vers X'. Ceci définit une relation d'équivalence sur la classe des espaces métriques.

Soit (X, d) un espace métrique, et soit $x, y \in X$. Soit $K > 0$ et $L \geq 0$. Une (K, L)-*quasi-géodésique* entre x et y est une application $\gamma : [a, b] \to X$, où $[a, b] \subseteq \mathbb{R}$ est un segment, telle que $\gamma(a) = x$, $\gamma(b) = y$, et pour tous $s, t \in [a, b]$, nous ayons

$$\frac{1}{K}|t - s| - L \leq d(\gamma(s), \gamma(t)) \leq K|t - s| + L.$$

Une (K, L)-*quasi-géodésique reparamétrée* est une application $\gamma' : [a', b'] \to X$, où $[a', b'] \subseteq \mathbb{R}$ est un segment, telle qu'il existe un segment $[a, b] \subseteq \mathbb{R}$ et une application continue croissante $\theta : [a, b] \to [a', b']$ telle que $\gamma' \circ \theta : [a, b] \to X$ soit une (K, L)-quasi-géodésique. Autrement dit, quitte à modifier la paramétrisation de γ', nous obtenons une (K, L)-quasi-géodésique.

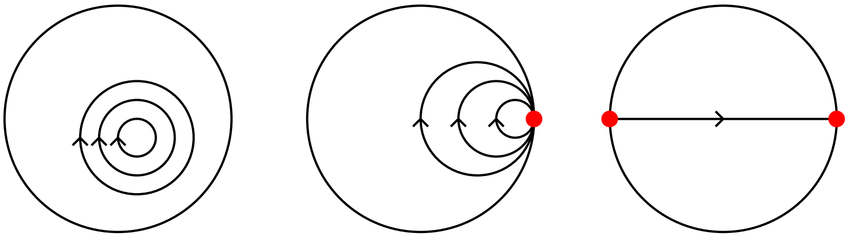


Figure 3.1 – Classification des isométries d'un espace hyperbolique. Les points limites apparaissent en rouge. Remarquons que dans le cas elliptique, les orbites ne sont pas forcément périodiques.

métrique X n'est pas supposé propre. L'exemple typique est celui d'un graphe étoilé X obtenu en attachant un ensemble infini dénombrable de copies de $\mathbb{R}_+$ en un point $*$. Dans ce cas, une suite $(x_n)_{n\in\mathbb{N}}$ obtenue en choisissant x_n dans la $n^{\text{ième}}$ copie de $\mathbb{R}_+$, de sorte que la distance $d(x_n, *)$ reste bornée inférieurement, n'a pas de valeur d'adhérence dans $\overline{X}$. De même, le bord de Gromov de X, qui contient un point au bout de chacune des copies de $\mathbb{R}_+$, n'est pas compact.

Classification des isométries d'un espace hyperbolique. Soit X un espace métrique hyperbolique au sens de Gromov, et ϕ une isométrie de X. Soit $x \in X$ un point base. L'action de ϕ sur X s'étend en une action par homéomorphismes sur ∂X. Nous dirons que ϕ est

- *elliptique* si les orbites de ϕ sont bornées,
- *parabolique* si ϕ a une orbite non bornée, et $\lim_{n\to+\infty} \frac{1}{n} d(x, \phi^n(x)) = 0$,
- *loxodromique* si $\lim_{n\to+\infty} \frac{1}{n} d(x, \phi^n(x)) > 0$.

Soit G un sous-groupe du groupe des isométries de X. L'*ensemble limite* de G dans X est

$$\Lambda_X G := \overline{G.x} \cap \partial X.$$

On peut vérifier [CDP90, Chapitre 9] qu'une isométrie de X est

- elliptique si et seulement si $\Lambda_X\langle\phi\rangle$ est vide,
- parabolique si et seulement si $\Lambda_X\langle\phi\rangle$ contient exactement un point,
- loxodromique si et seulement si $\Lambda_X\langle\phi\rangle$ contient exactement deux points.

Nous renvoyons à la Figure 3.1 pour une illustration de chacune de ces trois situations. Précisons qu'en général (en particulier pour des actions simpliciales sur des complexes hyperboliques qui ne sont pas propres), les orbites d'une isométrie elliptique ne sont pas nécessairement finies.

3.2 Trois complexes analogues au complexe des courbes, des géométries différentes

Nous présentons trois analogues possibles au complexe des courbes pour le groupe $\mathrm{Out}(F_N)$, et faisons un panorama des propriétés principales de ces complexes connues à ce jour.

3.2.1 Le complexe des facteurs libres

Nous rappelons qu'un *facteur libre* de F_N est un sous-groupe $A \subseteq F_N$ tel qu'il existe un sous-groupe $B \subseteq F_N$, de sorte que $F_N = A * B$. Pour $N \geq 3$, le *complexe des facteurs libres* FF_N, introduit par Hatcher et Vogtmann dans [HV98], est le complexe simplicial dont les sommets sont les classes de conjugaison de facteurs libres propres de F_N, les simplexes de dimension supérieure correspondant aux chaînes d'inclusion de telles classes de conjugaison. Lorsque $N = 2$, il faut modifier cette définition pour assurer la connexité de FF_2, en ajoutant une arête entre deux facteurs libres de F_N s'ils sont complémentaires. Le complexe FF_2 est alors isomorphe au graphe de Farey. Le complexe FF_N est naturellement muni d'une action à droite du groupe $\mathrm{Out}(F_N)$ par automorphismes simpliciaux.

Hyperbolicité. L'hyperbolicité de FF_N a été établie par Bestvina et Feighn [BF14b]. Une autre démonstration est due à Kapovich et Rafi [KR14].

Théorème 3.1. *(Bestvina–Feighn [BF14b]) Le complexe des facteurs libres FF_N est hyperbolique au sens de Gromov.*

Classification des isométries. Bestvina et Feighn ont également déterminé les éléments de $\mathrm{Out}(F_N)$ qui agissent de manière loxodromique sur FF_N. Un automorphisme $\Phi \in \mathrm{Out}(F_N)$ est *complètemement irréductible* si aucune puissance non nulle de Φ ne fixe la classe de conjugaison d'un facteur libre propre de F_N.

Théorème 3.2. *(Bestvina–Feighn [BF14b, Théorème 9.3]) Soit $\Phi \in Out(F_N)$. Alors Φ agit de manière loxodromique sur FF_N si et seulement si Φ est complètement irréductible.*

Remarquons que, comme dans le cas du complexe des courbes d'une surface compacte orientable, lorsque un automorphisme $\Phi \in \mathrm{Out}(F_N)$ n'agit pas de manière loxodromique sur FF_N, il admet une orbite finie dans FF_N (puisque l'une de ses puissances non triviales fixe la classe de conjugaison d'un facteur libre propre). Une autre propriété partagée avec le complexe des courbes est le fait que les stabilisateurs de points attractifs ou répulsifs d'automorphismes complètement irréductibles sont virtuellement cycliques, voir [BFH97] et [BF14b].

Bord de Gromov. Le bord de Gromov du complexe des facteurs libres a été déterminé par Bestvina et Reynolds [BR13], et indépendamment par Hamenstädt [Ham14a]. Un arbre réel $T \in \overline{CV_N}$ est *arationnel* si pour tout facteur libre propre $F \subsetneq F_N$, aucun élément de T n'est fixé par F, et le sous-arbre minimal T_F pour l'action de F sur T n'est pas à orbites denses (l'arbre T_F est défini comme le plus petit sous-arbre de T qui soit invariant pour l'action de F sur T obtenue par restriction de l'action de F_N ; il est aussi égal à la réunion des axes de translation des éléments hyperboliques dans F). Nous noterons $\widetilde{\mathcal{AT}}$ le sous-espace de $\overline{CV_N}$ formé des arbres arationnels. Deux arbres arationnels $T, T' \in \widetilde{\mathcal{AT}}$ sont *équivalents*, ce que nous noterons $T \sim T'$, s'ils ont la même lamination duale. Ceci revient à demander qu'ils aient mêmes arbres topologiques sous-jacents, ou autrement dit qu'ils déterminent le même simplexe de mesures dans $\overline{CV_N}$, au sens du Théorème 1.7. Nous noterons $\mathcal{AT}$ l'espace quotient $\mathcal{AT} := \widetilde{\mathcal{AT}}/\sim$. Nous définissons une application $\psi : CV_N \to FF_N$ en associant à chaque arbre $T \in CV_N$ le stabilisateur d'un sommet d'un arbre obtenu en écrasant en des points un sous-ensemble strict F_N-invariant d'arêtes de T, de sorte que l'arbre obtenu ne corresponde pas à une action libre de F_N. Cette définition requiert des choix, cependant les images d'un arbre $T \in CV_N$ par deux applications ψ correspondant à des choix différents seront toujours à distance bornée l'une de l'autre dans FF_N.

Théorème 3.3. *(Bestvina–Reynolds [BR13], Hamenstädt [Ham14a]) Il existe un unique homéomorphisme Out(F_N)-équivariant $\partial\psi : \mathcal{AT} \to \partial FF_N$ tel que pour tout $T \in \mathcal{AT}$, et toute suite $(T_n)_{n\in\mathbb{N}} \in \overline{CV_N}^{\mathbb{N}}$ convergeant vers T, la suite $(\psi(T_n))_{n\in\mathbb{N}}$ converge vers $\partial\psi(T)$.*

Nous renvoyons à l'encadré en page 51 pour une présentation des propriétés importantes des arbres arationnels qui jouent un rôle crucial pour construire l'application $\partial\psi$ dans la preuve du Théorème 3.3. La preuve du Théorème 3.3 de Bestvina–Reynolds et Hamenstädt requiert aussi de pouvoir associer à tout arbre $T \in \overline{CV_N} \smallsetminus \widetilde{\mathcal{AT}}$ un ensemble de *facteurs libres de réduction*, visibles depuis l'arbre T, de diamètre borné dans FF_N. Enfin, nous noterons que cette démonstration repose aussi de manière cruciale sur l'étude de la géométrie des *chemins de pliage* dans $\overline{CV_N}$.

Type d'homotopie. Hatcher et Vogtmann ont montré dans [HV98] que FF_N a le type d'homotopie d'un bouquet (infini) de sphères de dimension $N-2$.

3.2.2 Le complexe des scindements libres

Un *scindement libre* de F_N est un arbre simplicial muni d'une action minimale et simpliciale de F_N à stabilisateurs d'arêtes triviaux. Deux scindements libres T et T' sont *équivalents* s'il existe un homéomorphisme Out(F_N)-équivariant de T vers T'. Le *complexe des scindements libres* FS_N est le complexe simplicial dont les sommets sont les classes d'équivalence de scindements libres de F_N ayant exactement une F_N-orbite d'arêtes. Plus généralement, à chaque scindement libre de F_N à k orbites d'arêtes est associé un $k+1$-simplexe. Les faces du simplexe associé à un scindement T sont obtenues en écrasant chacune des arêtes d'un ensemble F_N-invariant d'arêtes de T en un point (un scindement libre T est déterminé par l'ensemble fini des scindements à une orbit d'arête dont il est un raffinement, et deux scindements correspondant à deux orbites d'arêtes distinctes dans T sont distincts [HM13a, Lemme 1.3]). Le complexe des scindements libres est lui aussi muni d'une action à droite naturelle du groupe Out(F_N) par automorphismes simpliciaux.

Hyperbolicité.

Théorème 3.8. *(Handel–Mosher [HM13a]) Le complexe FS_N est hyperbolique au sens de Gromov.*

La preuve de Handel et Mosher de l'hyperbolicité du complexe des scindements libres repose sur l'étude de *chemins de pliage* entre arbres simpliciaux, et utilise un critère dû à Masur et Minsky. Cette preuve montre en particulier que ces chemins de pliages sont des quasi-géodésiques reparamétrées de FS_N (voir l'encadré en page 47 pour la notion de quasi-géodésique reparamétrée). Dans un travail commun avec Arnaud Hilion [1], nous réinterprétons la preuve de Handel et Mosher dans un autre modèle plus géométrique du complexe, le *complexe des sphères* de Hatcher [Hat95]. Nous présenterons en Partie 3.3 cet autre modèle du complexe des scindements libres, ainsi qu'un schéma de notre preuve du Théorème 3.8. Plus récemment encore, Bestvina et Feighn ont simplifié la preuve originelle de Handel et Mosher [BF14c, Appendice].

En un sens, la preuve de l'hyperbolicité de FS_N par Handel et Mosher est plus naturelle que la preuve de l'hyperbolicité de FF_N de Bestvina et Feighn, car les arguments ont lieu directement dans le complexe simplicial. Au contraire, l'argument de Bestvina et Feighn

Intermède (Arbres arationnels et facteurs de réduction).

1. Arbres arationnels

Nous discutons quelques propriétés importantes des arbres arationnels. L'étude de cette classe importante d'arbres de ∂CV_N a été menée par Reynolds dans [Rey12], et poursuivie par Bestvina et Reynolds [BR13] et Hamenstädt [Ham14a]. Un arbre $T \in \overline{CV_N}$ est *indécomposable* [Gui08, Définition 1.17] si pour tous segments $I, J \subseteq T$, il existe un ensemble fini $\{g_1, \dots, g_r\} \subseteq F_N$ tel que $J = \cup_{i=1}^{r} g_i I$, et pour tout $i \in \{1, \dots, r-1\}$, l'intersection $g_i I \cap g_{i+1} I$ soit non dégénérée (i.e. non vide et non réduite à un point). Reynolds a donné la caractérisation suivante des arbres arationnels.

Théorème 3.4. *(Reynolds [Rey12, Théorème 1.1]) Soit $T \in \overline{CV_N}$. L'arbre T est arationnel si et seulement T est indécomposable, et soit T est libre, soit T est dual à une lamination géodésique mesurée sur une surface à une composante de bord, dont le support est minimal et remplissant.*

Nous noterons $\widetilde{\mathcal{FI}}$ le sous-espace de $\widetilde{\mathcal{AT}}$ formé des arbres arationnels libres, et $\mathcal{FI}$ l'espace quotient $\mathcal{FI} := \widetilde{\mathcal{FI}}/\sim$. Une propriété importante des arbres arationnels est la propriété d'unique dualité pour les courants qui leur sont duaux. Rappelons que l'ensemble minimal M_N est l'adhérence des courants rationnels associés à des éléments primitifs de F_N.

Théorème 3.5. *(Bestvina–Reynolds [BR13, Théorème 4.4], Hamenstädt [Ham14a]) Soit $T \in \widetilde{\mathcal{AT}}$, et $\eta \in Dual(T) \cap M_N$. Soit $T' \in \overline{CV_N}$. Si $\langle T', \eta \rangle = 0$, alors $T' \in \widetilde{\mathcal{AT}}$ et $T' \sim T$.*

Dans le cas où $T \in \widetilde{\mathcal{FI}}$, le résultat reste vrai même si l'on ne suppose plus que le courant η est dans l'ensemble minimal M_N, voir le Théorème 1.6 de l'Annexe C. Lorsque T est dual à une lamination arationnelle sur une surface ayant une composante de bord, le courant rationnel déterminé par la courbe de bord de la surface ne jouit pas de la propriété ci-dessus.
Nous terminons ce paragraphe en déterminant le stabilisateur dans $\mathrm{Out}(F_N)$ d'un arbre arationnel dans ∂CV_N.

Théorème 3.6. *Pour tout $T \in \widetilde{\mathcal{AT}}$, le stabilisateur de T dans $Out(F_N)$ est virtuellement cyclique.*

Démonstration. Lorsque l'action de F_N sur T est libre, ceci est dû à Kapovich et Lustig [KL11a, Théorème 1.1]. Lorsque T est dual à une lamination arationnelle sur une surface S ayant une composante de bord, le stabilisateur de T est un sous-groupe du groupe modulaire de S [BH92, Théorème 4.1]. Le Théorème 3.6 découle alors de [MP89, Proposition 2.2]. $\square$

Intermède (suite).

2. Facteurs de réduction pour des arbres non arationnels.

À tout arbre $T \in \partial CV_N$ qui n'est pas arationnel, on peut associer un ensemble canonique de *facteurs de réduction.*
L'ensemble $\mathrm{Dyn}(T)$ des classes de conjugaison de facteurs libres propres minimaux de F_N qui agissent avec orbites denses sur leur sous-arbre minimal dans T, mais ne sont pas elliptiques dans T, est fini [Rey12, Corollaire 7.4 et Proposition 9.2].
L'ensemble des classes de conjugaison de stabilisateurs de points dans T est fini [Jia91]. Tout stabilisateur H de point est contenu dans un unique facteur libre minimal $\mathrm{Fill}(H)$ de F_N, défini comme l'intersection de tous les facteurs libres de F_N qui contiennent H (nous pouvons avoir $\mathrm{Fill}(H) = F_N$). Nous notons $\mathrm{Per}(T)$ l'ensemble des classes de conjugaison de facteurs libres propres de F_N qui sont de la forme $\mathrm{Fill}(H)$, pour un stabilisateur de point H dans T.
Nous montrons l'alternative suivante pour les arbres de ∂CV_N (voir la Partie 2 de l'Annexe D), qui découle essentiellement des travaux de Reynolds [Rey12].

Théorème 3.7. *Soit $T \in \partial CV_N$. Si $T \notin \widetilde{\mathcal{AT}}$, alors $Dyn(T) \cup Per(T) \neq \emptyset$.*

requiert d'étudier des géodésiques dans l'outre-espace, avant de les projeter dans FF_N. Kapovich et Rafi ont montré dans [KR14] comment déduire l'hyperbolicité du complexe des facteurs libres à partir de l'hyperbolicité du complexe des scindements libres, via l'étude de l'application $\tau : FS_N \to FF_N$ (grossièrement définie) qui envoie un scindement S sur un groupe elliptique d'un scindement obtenu par écrasement d'arêtes dans S.

Classification des isométries. Récemment, Handel et Mosher ont également déterminé les isométries loxodromiques du complexe des scindements libres [HM14a]. Ce sont les éléments de $\mathrm{Out}(F_N)$ qui possèdent une lamination attractive qui n'est supportée par aucun facteur libre propre de F_N. L'ensemble de ces automorphismes contient strictement l'ensemble des automorphismes complètement irréductibles, voir [HM14a, Exemple 4.1] pour le caractère strict de cette inclusion. Nous renvoyons à [BFH00] pour une définition et une étude des laminations attractives associées à un automorphisme $\Phi \in \mathrm{Out}(F_N)$.

Handel et Mosher montrent également que toute isométrie qui n'est pas loxodromique a toutes ses orbites bornées dans FS_N. Cependant, contrairement au cas du complexe des courbes ou du complexe des facteurs libres, une telle isométrie n'a pas toujours d'orbite finie, voir [HM14a, Exemple 4.2]. De même, les stabilisateurs de points attractifs ou répulsifs d'isométries loxodromiques ne sont plus toujours virtuellement cycliques, voir [HM14a, Théorème 1.4].

Type d'homotopie. Le complexe des scindements libres FS_N est contractile (Hatcher [Hat95]).

Groupe d'automorphismes. Aramayona et Souto ont montré que le groupe des automorphismes simpliciaux de FS_N coïncide avec le groupe $\mathrm{Out}(F_N)$ [AS11].

3.2.3 Le graphe des scindements cycliques et ses variantes

Un *scindement cyclique* de F_N est un arbre simplicial muni d'une action minimale et simpliciale de F_N à stabilisateurs d'arêtes cycliques (ou triviaux). Le *graphe des scindements cycliques* FZ_N de F_N est le graphe dont les sommets sont les classes d'équivalence de scindements cycliques de F_N, deux sommets étant reliés par une arête si l'un des scindements raffine proprement l'autre, i.e. le second est obtenu à partir du premier en écrasant un ensemble F_N-invariant d'arêtes en des points. Nous définissons également deux variantes du graphe des scindements cycliques. Soit FZ_N^{max} le graphe dont les sommets sont les classes d'équivalence de scindements de F_N dont les stabilisateurs sont triviaux ou maximalement cycliques. Deux sommets sont reliés par une arête si l'un des scindements raffine proprement l'autre. Soit VS_N le sous-graphe dont les sommets sont les scindements très petits, i.e. pour lesquels les stabilisateurs de tripodes sont triviaux. On peut montrer que les complexes FZ_N^{max} et VS_N sont quasi-isométriques. Nous travaillerons parfois également avec des versions duales des graphes ci-dessus, dont les sommets sont les classes d'équivalence de scindements à une orbite d'arête, deux scindements étant reliés par une arête s'ils admettent un raffinement commun. Ces versions duales sont quasi-isométriques aux graphes définis ci-dessus, si bien que nous pourrons travailler avec l'une ou l'autre des versions pour en établir l'hyperbolicité et en décrire le bord de Gromov.

Hyperbolicité. En s'appuyant sur la méthode développée par Kapovich et Rafi dans [KR14], Mann déduit l'hyperbolicité de FZ_N de celle de FS_N.

Théorème 3.9. *(Mann [Man13]) Le complexe FZ_N est hyperbolique au sens de Gromov.*

La démonstration de Mann du Théorème 3.9 se traduit *verbatim* pour montrer l'hyperbolicité de FZ_N^{max} (et donc aussi de VS_N, qui lui est quasi-isométrique).

Bord de Gromov. Nous décrivons maintenant le bord de Gromov du graphe des scindements cycliques de F_N. Nous travaillons pour cela avec la version duale de FZ_N décrite ci-dessus, dont les sommets sont les scindements cycliques de F_N à une orbite d'arêtes. Soit $T \in \overline{CV_N}$. Nous noterons $\mathcal{R}^1(T)$ l'ensemble des scindements cycliques de F_N qui sont compatibles avec T. Nous noterons $\mathcal{R}^2(T)$ l'ensemble des scindements cycliques de F_N qui sont compatibles avec un arbre $T' \in \overline{CV_N}$, qui est lui-même compatible avec T. Nous dirons qu'un arbre $T \in \overline{CV_N}$ est *$\mathcal{Z}$-étranger* si $\mathcal{R}^2(T) = \emptyset$. Nous noterons $\mathcal{X}$ le sous-espace de $\overline{CV_N}$ formé des arbres $\mathcal{Z}$-étrangers. Deux arbres $\mathcal{Z}$-étrangers $T, T' \in \mathcal{X}$ sont *équivalents*, ce que nous notons $T \sim T'$, s'ils sont tous deux compatibles avec un arbre commun dans $\overline{CV_N}$. Nous montrons que ceci est équivalent à l'existence d'une suite finie $(T = T_0, \dots, T_k = T')$ d'arbres dans $\overline{CV_N}$ telle que pour tout $i \in \{1, \dots, k\}$, les arbres T_{i-1} et T_i soient compatibles. Ceci permet de justifier en particulier que $\sim$ est une relation d'équivalence sur $\mathcal{X}$. Nous définissons une application $\psi : CV_N \to FZ_N$ en envoyant tout arbre $T \in CV_N$ vers un scindement obtenu en écrasant toutes les arêtes de T en dehors d'une F_N-orbite en des points. Cette application n'est pas équivariante car elle requiert des choix, mais deux choix différents donnent des scindements à distance au plus 1 dans FZ_N.

Théorème 3.10.

Il existe un unique homéomorphisme $\mathrm{Out}(F_N)$-équivariant

$$\partial\psi : \mathcal{X}/\!\sim\, \to \partial FZ_N$$

tel que pour tout $T \in \mathcal{X}$, et toute suite $(T_n)_{n\in\mathbb{N}} \in CV_N^{\mathbb{N}}$ convergeant vers T, la suite $(\psi(T_n))_{n\in\mathbb{N}}$ converge vers $\partial\psi(T)$.

Lorsque $N \geq 3$, il existe des arbres $\mathcal{Z}$-étrangers dans $\overline{CV_N}$ qui ne sont pas arationnels. Par exemple, tout arbre $T \in \partial CV_N$ dual à une lamination minimale et remplissante sur une sphère à $N+1$ composantes de bord est indécomposable, et donc $\mathcal{Z}$-étranger. Cependant, un tel arbre n'est pas arationnel, puisque chacune des composantes de bord de S est un élément primitif du groupe fondamental de S qui agit de manière elliptique dans T. En comparant les descriptions des bords de Gromov ∂FF_N (Théorème 3.3) et ∂FZ_N, nous déduisons que l'application naturelle de FZ_N vers FF_N (qui envoie un scindement à une arête vers l'un des groupes de sommet qui est un facteur libre propre de F_N) n'est pas une quasi-isométrie.

En travaillant avec les scindements maximalement cycliques au lieu de scindements cycliques de F_N, nous définissons également la notion d'arbre $\mathcal{Z}^{max}$-étranger, et établissons l'énoncé analogue au Théorème 3.10 pour le graphe FZ_N^{max}. Soit $T \in \overline{CV_N}$. Soit $\mathcal{R}^{2,max}(T)$ l'ensemble des scindements cycliques de F_N qui sont compatibles avec un arbre $T' \in \overline{CV_N}$, qui est lui-même compatible avec un scindement maximalement cyclique de F_N. Un arbre $T \in \overline{CV_N}$ est $\mathcal{Z}^{max}$-étranger si $\mathcal{R}^{2,max}(T) = \emptyset$. Nous noterons $\mathcal{X}^{max}$ le sous-espace de $\overline{CV_N}$ formé des arbres $\mathcal{Z}^{max}$-étrangers. Deux arbres $\mathcal{Z}^{max}$-étrangers $T, T' \in \mathcal{X}$ sont *équivalents*, ce que nous notons $T \sim T'$, s'ils sont tous deux compatibles avec un arbre commun dans $\overline{CV_N}$.

Théorème 3.11.

Il existe un unique homéomorphisme $\mathrm{Out}(F_N)$-équivariant

$$\partial\psi : \mathcal{X}^{max}/\!\sim\, \to \partial FZ_N^{max}$$

tel que pour tout $T \in \mathcal{X}^{max}$, et toute suite $(T_n)_{n\in\mathbb{N}} \in CV_N^{\mathbb{N}}$ convergeant vers T, la suite $(\psi(T_n))_{n\in\mathbb{N}}$ converge vers $\partial\psi(T)$.

Remarque 3.12. Afin de justifier l'introduction des ensembles $\mathcal{R}^2(T)$ et $\mathcal{R}^{2,max}(T)$ dans nos arguments, nous donnons un exemple d'arbre qui n'est compatible avec aucun scindement cyclique de F_N, mais n'est pas pour autant $\mathcal{Z}$-étranger. Nous renvoyons à l'Exemple 5.30 de l'Annexe F pour les détails de l'argument. Soit T_1 un F_N-arbre indécomposable dans lequel un facteur libre propre $F_2 \subseteq F_N$ est elliptique (nous renvoyons par exemple à [Rey12, Partie 11.6] pour un exemple). Nous définissons un F_{2N-2}-arbre $T \in \overline{CV_{2N-2}}$ de la manière suivante : l'arbre T est obtenu comme graphe d'actions au-dessus du scindement $F_{2N-2} = F_N *_{F_2} F_N$, où chaque arbre de sommet est une copie de T_1, le point d'attache étant le point fixé par le sous-groupe F_2.

Remarquons tout d'abord que $\mathcal{R}^1(T) = \emptyset$. En effet, supposons par l'absurde que T soit compatible avec un scindement cyclique S de F_{2N-2}. En utilisant l'indécomposabilité de T, nous montrons que chacune des copies de F_N est elliptique dans S ; l'argument est détaillé au Lemme 5.10 de l'Annexe F. Les deux copies de F_N ont alors même point fixe dans S, sinon F_2 fixerait une arête de S. Autrement dit, le groupe F_{2N-2} est elliptique dans S, ce qui est absurde.

Cependant, en écrasant en un point chacun des sous-arbres de T dans l'orbite de T_1, nous obtenons un nouvel arbre $\overline{T} \in \overline{CV_{2N-2}}$ qui est compatible avec T, et vérifie $\mathcal{R}^1(\overline{T}) \neq \emptyset$. Donc $\mathcal{R}^2(T) \neq \emptyset$.

Idée de démonstration du Théorème 3.10. La démonstration des Théorèmes 3.10 et 3.11 fait l'objet des Parties 4 à 9 de l'Annexe F. Nous esquissons ici une idée de cette démonstration dans le cas du complexe FZ_N^{max}.

Étape 0 : Une définition équivalente des arbres $\mathcal{Z}^{max}$-étrangers.
Il est crucial pour notre démonstration d'établir l'équivalence suivante : un arbre $T \in \overline{CV_N}$ vérifie $\mathcal{R}^{2,max}(T) \neq \emptyset$ si et seulement s'il existe une suite finie $(T = T_0, \dots, T_k = S)$ d'arbres dans $\overline{CV_N}$, telle que pour tout $i \in \{1, \dots, k\}$, les arbres T_{i-1} et T_i soient compatibles, et S soit un scindement maximalement cyclique de F_N.

Un point clé pour montrer cette équivalence est le fait, dû à Guirardel et Levitt [GL], que tout arbre dans $\overline{CV_N}$ qui n'est pas $\mathcal{Z}^{max}$-compatible s'écrase sur un arbre *mélangeant*. Un arbre $T \in \overline{CV_N}$ est *mélangeant* (au sens de Morgan [Mor88]) si pour tous segments $I, J \subseteq T$, il existe un sous-ensemble fini $\{g_1, \dots, g_r\} \subseteq F_N$ tel que $J = \cup_{i=1}^r g_i I$, et pour tout $i \in \{1, \dots, r-1\}$, nous ayons $g_i I \cap g_{i+1} I \neq \emptyset$. Nous montrons également en Partie 6 de l'Annexe F que si $T_1, T_2 \in \overline{CV_N}$ sont deux arbres compatibles, et si T_1 est mélangeant et $\mathcal{Z}^{max}$-incompatible, alors il existe une application préservant l'alignement de T_2 vers T_1. Par conséquent, si $(T_0, \dots, T_k)$ est une suite d'arbres de $\overline{CV_N}$ telle que pour tout $i \in \{1, \dots, k\}$, les arbres T_{i-1} et T_i soient compatibles, alors soit chacun des T_i s'écrase sur un arbre $\mathcal{Z}^{max}$-compatible, soit ils s'écrasent tous sur un même arbre mélangeant et $\mathcal{Z}^{max}$-incompatible. L'équivalence entre les deux définitions proposées d'arbres $\mathcal{Z}^{max}$-étrangers découle de cette observation. Par ailleurs, tous les arbres $\mathcal{Z}^{max}$-étrangers dans une même classe d'équivalence de la relation $\sim$ s'écrasent sur un même arbre mélangeant, ce qui permet de définir un simplexe canonique de représentants dans chaque classe d'équivalence de la relation $\sim$. Deux arbres $T, T' \in \overline{CV_N}$ sont *faiblement homéomorphes* s'il existe deux applications F_N-équivariantes $f : T \to T'$ et $g : T' \to T$ inverses l'une de l'autre, et continues en restriction aux segments de T et T'. (Remarquons que de telles applications ne sont pas nécessairement continues globalement. Par exemple, si T et T' sont des actions duales à deux feuilletages mesurés arationnels sur une surface à bord non trivial qui ont même support topologique, mais sont munis de mesures transverses singulières l'une par rapport à l'autre, alors l'application naturelle de T vers T' est un homéomorphisme faible, mais n'est pas continue. L'existence de feuilletages mesurés arationnels admettant deux mesures transverses singulières est établie dans [KN76, Kea77]).

Proposition 3.13. *Pour tout $T \in \mathcal{X}^{max}$, il existe un arbre mélangeant $\overline{T} \in \mathcal{X}^{max}$ tel que $T \sim \overline{T}$. De plus, deux tels arbres mélangeants sont faiblement homéomorphes.*

Étape 1 : *Les arbres $\mathcal{Z}^{max}$-étrangers sont* à l'infini *de* FZ_N^{max}.
Le premier temps de notre démonstration consiste à montrer que si $T \in \mathcal{X}^{max}$, et si $(T_n)_{n \in \mathbb{N}} \in CV_N^{\mathbb{N}}$ est une suite d'arbres qui converge vers T, alors la suite $(\psi(T_n))_{n \in \mathbb{N}}$ converge vers un point du bord de Gromov ∂FZ_N^{max} qui ne dépend que de T (et en fait, ne dépend que de la classe d'équivalence de T). Nous montrons d'abord que toute telle

suite $(\psi(T_n))_{n\in\mathbb{N}}$ est non bornée. Ceci est *a priori* plus faible, puisque FZ_N^{max} n'est pas localement fini. L'argument ci-dessous est une variation autour de la preuve par Luo du caractère non borné du complexe des courbes d'une surface compacte orientable (hormis dans quelques cas sporadiques), qui apparaît dans le travail de Kobayashi [Kob88]. En particulier, nous obtenons une preuve du caractère non borné de FZ_N^{max}.

Théorème 3.14. *Soit $T \in \mathcal{X}^{max}$, et soit $(T_n)_{n\in\mathbb{N}} \in CV_N^{\mathbb{N}}$ une suite convergeant vers T. Alors la suite $(\psi(T_n))_{n\in\mathbb{N}}$ n'est pas bornée dans FZ_N^{max}.*

Démonstration. Supposons par l'absurde que la suite $(\psi(T_n))_{n\in\mathbb{N}}$ soit bornée dans FZ_N^{max}. Quitte à extraire, nous pouvons supposer qu'il existe $M \in \mathbb{N}$, et $* \in FZ_N^{max}$, de sorte que pour tout $n \in \mathbb{N}$, nous ayons $d_{FZ_N^{max}}(*, \psi(T_n)) = M$. Pour tout $n \in \mathbb{N}$, soit $(Z_n^k)_{0\le k\le M}$ un segment géodésique reliant $*$ à $\psi(T_n)$ dans FZ_N^{max}, et soit $T_n^k \in \overline{CV_N}$ tel que $\psi(T_n^k) = Z_n^k$. Quitte à extraire de nouveau et à renormaliser, nous pouvons supposer que pour tout $k \in \{0,\dots,M\}$, la suite $(T_n^k)_{n\in\mathbb{N}}$ converge (de manière non projective) vers un arbre $T_\infty^k \in \overline{cv_N}$. De plus, pour tout $k \in \{1,\dots,M\}$ et tout $n \in \mathbb{N}$, les arbres T_n^k et T_n^{k-1} sont compatibles, ce qui assure que T_∞^k et T_∞^{k-1} sont compatibles. Or $T_\infty^0 = *$ et T est compatible avec T_∞^k. Ceci contredit le fait que $T \in \mathcal{X}^{max}$. ☐

Remarque 3.15. Pour montrer l'énoncé analogue au Théorème 3.14 pour le graphe FZ_N, une difficulté supplémentaire intervient. En effet, les arbres T_n^k ne sont plus dans l'adhérence de l'outre-espace en général. Nous sommes donc amenés à travailler dans une classe plus large de F_N-arbres, que nous appelons la classe des arbres *tempérés*. Un F_N-arbre minimal T est *tempéré* si les stabilisateurs d'arcs de T sont cycliques, et T possède un nombre fini d'orbites de points de branchement, et d'orbites de directions en ces points de branchement. En général, une limite de F_N-arbres à stabilisateurs cycliques peut avoir un nombre infini d'orbites de points de branchement : c'est le cas par exemple d'un arbre obtenu comme limite d'arbres de Bass–Serre de scindements de $F_2 = \langle a,b\rangle$ donnés par $F_2 = \langle a\rangle *_{\langle a^2\rangle} \langle a^2\rangle *_{\langle a^4\rangle} \cdots *_{\langle a^{2^n}\rangle} \langle a^{2^n}\rangle * \langle b\rangle$. Toutefois, nous montrons qu'une limite de scindements cycliques de F_N à une arête est tempérée, ce qui nous permet d'adapter nos arguments. L'étude de la classe des arbres tempérés est menée en Partie 6 de l'Annexe E.

Comme FZ_N^{max} n'est pas localement fini, le Théorème 3.14 ne suffit pas à montrer que la suite $(\psi(T_n))_{n\in\mathbb{N}}$ converge vers un point du bord de Gromov ∂FZ_N^{max}. Les arguments supplémentaires que nous utilisons pour démontrer ce point viennent de l'étude de la géométrie des *chemins de pliage* dans FZ_N^{max}, dont nous savons grâce aux travaux de Mann [Man13] qu'ils restent à distance de Hausdorff bornée de toute géodésique reliant leurs extrémités dans FZ_N^{max}. Nous montrons qu'il existe $\xi \in \partial_\infty FZ_N^{max}$ tel que pour toute suite $(T_n)_{n\in\mathbb{N}} \in CV_N^{\mathbb{N}}$ convergeant vers T, la suite $(\psi(T_n))_{n\in\mathbb{N}}$ converge vers ξ. Nous pouvons alors définir une application $\partial\psi : \mathcal{X}^{max}/{\sim} \to \partial FZ_N^{max}$ de la manière suivante : pour tout $T \in \mathcal{X}^{max}$, nous définissons $\partial\psi(T)$ comme la limite commune de toutes les suites $(\psi(T_n))_{n\in\mathbb{N}}$, où $(T_n)_{n\in\mathbb{N}} \in CV_N^{\mathbb{N}}$ est une suite convergeant vers T. Nous montrons la continuité et l'injectivité de cette application. Il nous restera à montrer dans une dernière étape la surjectivité de $\partial\psi$.

Étape 2 : Bornitude des arbres dans $\overline{CV_N} \smallsetminus \mathcal{X}^{max}$, et surjectivité de $\partial\psi$.
Affirmation : Si $(T_n)_{n\in\mathbb{N}} \in \overline{CV_N}^{\mathbb{N}}$ converge vers un arbre $T \in \overline{CV_N} \smallsetminus \mathcal{X}^{max}$, alors la suite $(\psi(T_n))_{n\in\mathbb{N}}$ ne converge pas vers un point de ∂FZ_N^{max}.

Notons toutefois que la suite $(\psi(T_n))_{n\in\mathbb{N}}$ ne reste pas nécessairement dans une région bornée de FZ_N^{max}. Pour déduire la surjectivité de $\partial\psi$ à partir de cette affirmation, remarquons que si $\xi \in \partial FZ_N^{max}$, et $(X_n)_{n\in\mathbb{N}} \in (FZ_N^{max})^{\mathbb{N}}$ converge vers ξ, alors quitte à extraire, nous pouvons supposer que X_n tend vers un arbre T au sens de la convergence dans $\overline{CV_N}$. L'affirmation entraîne que $T \in \mathcal{X}^{max}$, et nous avons alors $\xi = \partial\psi(T)$.

Le point clé pour démontrer l'affirmation ci-dessus consiste à montrer la bornitude de $\mathcal{R}^{2,max}(T)$ dans FZ_N^{max} pour tout arbre $T \in \overline{CV_N} \smallsetminus \mathcal{X}^{max}$. En fait, nous montrons également que tout chemin de pliage dans $\overline{CV_N}$ ayant T pour point terminal reste asymptotiquement dans cette région bornée. La démonstration du théorème suivant fait l'objet de la Partie 8 de l'Annexe F.

Théorème 3.16. *Soit $T \in \overline{CV_N} \smallsetminus \mathcal{X}^{max}$. Alors l'ensemble $\mathcal{R}^{2,max}(T)$ est de diamètre borné dans FZ_N^{max}.*

Nous vérifions également par des arguments similaires le caractère fermé de $\partial\psi$. L'application $\partial\psi$ est donc un homéomorphisme de $\mathcal{X}^{max}/\!\sim$ vers ∂FZ_N^{max}, ce qui conclut la preuve du Théorème 3.11. ☐

Groupe d'automorphismes. Dans un travail commun avec Richard D. Wade, nous montrons que lorsque $N \geq 3$, tout automorphisme des graphes FZ_N, FZ_N^{max} ou VS_N est induit par un élément de $\mathrm{Out}(F_N)$.

Théorème 3.17 (Horbez–Wade [10])**.**

Pour tout $N \geq 3$, les applications naturelles de $\mathrm{Out}(F_N)$ vers les groupes des automorphismes simpliciaux de FZ_N, FZ_N^{max} et VS_N sont des isomorphismes.

Le théorème 3.17 est aussi valable pour les versions duales des graphes dont les sommets sont les scindements à une orbite d'arêtes, deux scindements étant reliés par une arête s'ils sont compatibles.

Remarque 3.18. Le théorème d'Ivanov, qui énonce que tout automorphisme de $\mathcal{C}(S)$ est induit par un élément du groupe modulaire étendu de S, permet de montrer que $\mathrm{Mod}(S)$ est son propre commensurateur abstrait [Iva97]. Le résultat analogue pour $\mathrm{Out}(F_N)$ a été montré par Farb et Handel dans [FH07]. Un problème naturel serait d'essayer de déduire le théorème de Farb et Handel du Théorème 3.17. La difficulté vient de la nécessité de caractériser les twists de Dehn parmi les automorphismes de $\mathrm{Out}(F_N)$. Une possibilité serait d'essayer de montrer une caractérisation en termes de la dimension cohomologique virtuelle de leurs centralisateurs.

Idée de démonstration du Théorème 3.17. Nous donnons une idée des différentes étapes de notre démonstration du Théorème 3.17. Soit $\mathcal{G}$ l'un des graphes FZ_N, FZ_N^{max} ou VS_N, et soit f une isométrie de $\mathcal{G}$. Nous voulons montrer que f est induite par un élément de $\mathrm{Out}(F_N)$.

1. Nous commençons par caractériser les scindements $\mathcal{G}$-maximaux de F_N, i.e. ceux qui n'admettent pas de raffinement propre par un scindement dans $\mathcal{G}$. Les scindements maximaux sont les scindements T de valence finie dans $\mathcal{G}$, qui ont la propriété suivante : il existe deux scindements $T_1, T_2 \in \mathcal{G}$, tous deux à distance 1 de T, pour

lesquels l'unique chemin de longueur 2 entre T_1 et T_2 passe par T. Deux tels scindements sont donnés par une partition des arêtes de T en deux sous-ensembles propres F_N-invariants E_1 et E_2, le scindement T_i étant obtenu par écrasement des arêtes dans E_i. (Un scindement cyclique de F_N est déterminé par la famille des scindements cycliques à une orbite d'arêtes sur lesquels il s'écrase.)

2. Nous caractérisons ensuite les scindements maximaux de $\mathcal{G}$ à stabilisateurs d'arêtes triviaux. Lorsque $\mathcal{G} = FZ_N$, nous montrons que tout scindement maximal est à stabilisateurs d'arêtes triviaux. Ceci n'est plus vrai lorsque $\mathcal{G} = FZ_N^{max}$ ou $\mathcal{G} = VS_N$. Parmi les scindements VS_N-maximaux, nous distinguons les scindements libres des scindements contenant une arête à stabilisateur non trivial par la propriété suivante. Lorsqu'on écrase une orbite d'arêtes dans un scindement libre maximal, le scindement obtenu reste de valence finie dans VS_N. Ceci n'est plus vrai pour les scindements non libres : dans un scindement VS_N-maximal non libre, nous pouvons toujours déplier une arête à stabilisateur non trivial (voir [BF94, Lemme 4.1]) pour obtenir un nouveau scindement de valence infinie dans VS_N. (Ceci n'est plus vrai si l'on autorise les stabilisateurs de tripodes. Dans ce cas, il faut préciser un peu l'argument pour caractériser les scindements libres maximaux dans FZ_N^{max}.)

3. Un scindement est libre si et seulement s'il est à distance au plus 1 d'un scindement maximal libre. Par conséquent, toute isométrie de $\mathcal{G}$ préserve le sous-graphe induit par les scindements libres.

4. En utilisant le théorème d'Aramayona et Souto sur les automorphismes de FS_N [AS11], nous pouvons donc supposer, quitte à composer f par un élément de $\mathrm{Out}(F_N)$, que f est l'identité en restriction au sous-graphe de $\mathcal{G}$ engendré par les scindements libres.

5. Nous disons qu'un scindement à une arête est *bon* s'il n'est pas de la forme $(F_{N-1} * \langle g^t \rangle)*_{\langle g \rangle}$, où $t \in F_N$, et $g \in F_{N-1}$ n'est contenu dans aucun facteur libre propre de F_{N-1}. Par une analyse de cas, nous montrons qu'étant donné deux bons scindements à une arête, il existe toujours un scindement libre qui est compatible avec l'un seulement de ces deux scindements. Puisque f fixe tous les scindements libres, ceci entraîne que f fixe tous les bons scindements à une arête. (L'argument n'est plus valable si les scindements considérés ne sont pas bons. En effet, le seul scindement libre compatible avec le scindement $(F_{N-1} * \langle g^t \rangle)*_{\langle g \rangle}$ est l'extension HNN donnée par $F_N = F_{N-1}*$).

6. Étant donné deux mauvais scindements à une arête, nous montrons qu'il existe un bon scindement qui est compatible avec l'un seulement de ces deux scindements. Ceci entraîne alors que f fixe tous les mauvais scindements à une arête.

7. L'isométrie f fixe donc tous les scindements à une arête, et donc tous les scindements puisqu'un scindement est déterminé par les scindements à une arête qu'il raffine, grâce à un théorème de Scott et Swarup [SS00, Théorème 2.5].

3.3 Hyperbolicité du complexe des sphères

Cette partie a pour but de présenter notre démonstration alternative de l'hyperbolicité du complexe des scindements libres, obtenue avec Arnaud Hilion, dans le modèle des sphères de ce complexe.

Le complexe des sphères. Le complexe des sphères est un autre modèle du complexe des scindements libres, introduit par Hatcher dans [Hat95] dans son travail sur la stabilité homologique pour les groupes $\mathrm{Out}(F_N)$ et $\mathrm{Aut}(F_N)$.

Soit $M_N := \sharp^N S^1 \times S^2$ la somme connexe de N copies de $S^1 \times S^2$, dont le groupe fondamental est libre de rang N. La variété M_N est homéomorphe à un *double corps à anses*, obtenu par recollement de deux corps à anses de genre N le long de leur frontière commune par l'application identité. Nous fixons une identification entre le groupe fondamental de M_N et le groupe libre F_N. Une sphère plongée dans M_N est *essentielle* si elle ne borde pas de boule. Un *système de sphères* est une collection de sphères essentielles plongées dans M_N, deux à deux disjointes et non isotopes. Le *complexe des sphères* $\mathcal{S}_N$ est le complexe simplicial dont les k-simplexes sont les classes d'isotopies de systèmes de $k+1$ sphères, avec les relations de faces évidentes.

Nous décrivons maintenant l'action à droite de $\mathrm{Out}(F_N)$ sur le complexe $\mathcal{S}_N$. Le *groupe modulaire* $\mathrm{Mod}(M_N)$ est le groupe des classes d'isotopie de difféomorphismes de M_N préservant l'orientation. Tout comme le groupe modulaire d'une surface agit sur le complexe des courbes, le groupe $\mathrm{Mod}(M_N)$ agit naturellement sur $\mathcal{S}_N$. Tout difféomorphisme de M_N induit un automorphisme du groupe fondamental de M_N, ce qui fournit un morphisme de $\mathrm{Mod}(M_N)$ dans $\mathrm{Out}(F_N)$. La surjectivité de ce morphisme se vérifie en réalisant chacun des générateurs de Nielsen de $\mathrm{Out}(F_N)$ (voir [KMS66]) comme un difféomorphisme de M_N, voir [Lau74, Lemme III.4.3.1]. Laudenbach a montré que son noyau est le sous-groupe fini de $\mathrm{Mod}(S)$ engendré par les *twists de Dehn* le long de N sphères disjointes [Lau74, Théorème III.4.3], définis comme suit. Soit S une sphère plongée dans M_N, et U un voisinage collier de S (i.e. U est difféomorphe à $S^2 \times [0,1]$, avec $S^2 \times \{0\} = S$). Soit $\alpha : ([0,1], \partial[0,1]) \to (SO(3), \mathrm{Id})$ l'élément non trivial du groupe fondamental de $SO(3)$. Le *twist de Dehn* le long de S est le difféomorphisme δ de M_N, à support dans U, donné pour $(x,t) \in U$ par $\delta(x,t) = (\alpha(t)x, t)$. Les twists de Dehn agissent de manière triviale sur $\mathcal{S}_N$. Par conséquent, l'action naturelle de $\mathrm{Mod}(M_N)$ sur $\mathcal{S}_N$ se factorise en une action de $\mathrm{Out}(F_N)$.

Les complexes FS_N et $\mathcal{S}_N$ sont naturellement isomorphes, via un isomorphisme $\mathrm{Out}(F_N)$-équivariant, voir [AS11, Lemme 2]. En effet, il résulte du théorème de van Kampen que tout système de sphères dans M_N définit un scindement de son groupe fondamental F_N au-dessus du groupe trivial (une sphère étant simplement connexe). Deux sphères isotopes définissent le même scindement de F_N. De plus, l'arbre dual à la réunion de deux sphères S et S' dans M_N est l'arbre de Bass–Serre d'un scindement de F_N qui raffine les arbres duaux à S et à S'. Ceci montre l'existence d'une application $\Phi : \mathcal{S}_N \to FS_N$, qui conjugue les actions de $\mathrm{Out}(F_N)$. L'injectivité de Φ découle des travaux de Laudenbach [Lau74, Théorème IV.3.1], et la surjectivité de Φ est due à Stallings [Sta71, Théorème 2.B.3].

Chemins de chirurgie. Nous décrivons maintenant une famille de chemins dans $\mathcal{S}_N$ entre deux systèmes de sphères quelconques. Ces chemins, utilisés par Hatcher dans [Hat95] pour montrer la contractibilité de $\mathcal{S}_N$, sont définis à l'aide d'un procédé de chirurgie entre systèmes de sphères. Soit S et Σ deux systèmes de sphères. Nous supposerons que S et Σ sont représentés par des systèmes de sphères plongés dans M_N, de sorte à minimiser leur nombre d'intersection (une description explicite de cette position minimale est due à Hatcher [Hat95]). L'ensemble des cercles d'intersection entre S et Σ définit une collection de cercles sur les sphères de Σ, et chacun de ces cercles borde deux disques sur l'une des sphères de Σ. Nous obtenons ainsi une collection $\mathcal{D}$ de disques sur les sphères de Σ. Soit $D \in \mathcal{D}$ un disque qui ne contient aucun autre disque de $\mathcal{D}$, soit C son bord, et soit $s \in S$ la sphère de S qui contient C. Le cercle C divise la sphère S en deux disques D_1 et D_2. Une

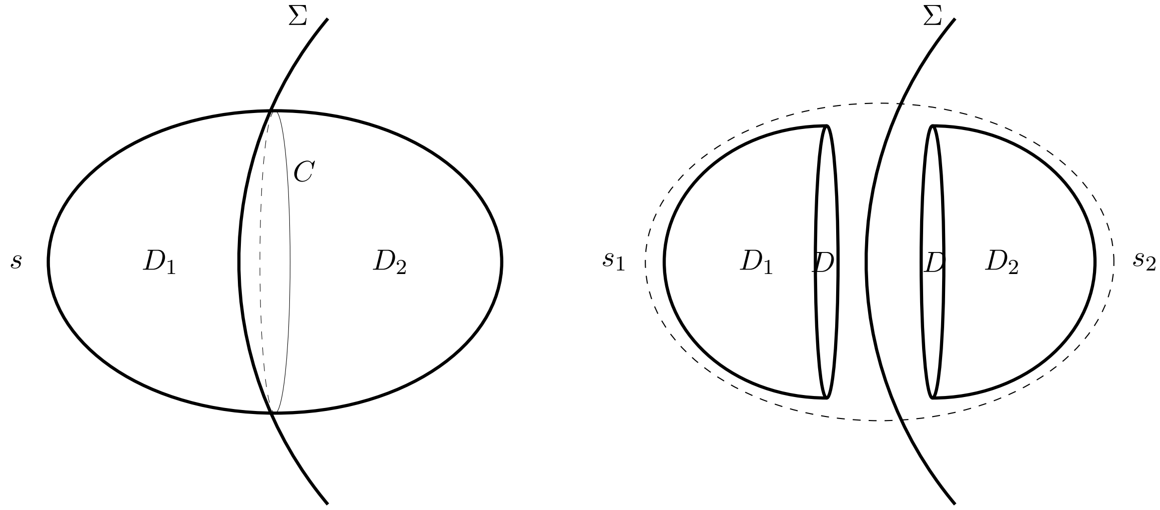


FIGURE 3.2 – Une chirurgie élémentaire sur S dans la direction de Σ.

chirurgie élémentaire sur S dans la direction de Σ consiste à remplacer la sphère s (dans le système S) par deux nouvelles sphères s_1 et s_2, où pour tout $i \in \{1, 2\}$, la sphère s_i est la réunion d'une copie parallèle de D_i et d'une copie parallèle de D (voir la Figure 3.2). Le choix du disque D assure que les sphères s_1 et s_2 n'intersectent aucune des sphères de S. Après suppression des sphères inessentielles et identification des sphères isotopes dans $S \setminus \{s\} \cup \{s_1, s_2\}$, nous obtenons un nouveau système de sphères S', dont nous disons qu'il est obtenu par *chirurgie élémentaire* sur S dans la direction de Σ. Étant donné deux systèmes de sphères S et Σ dans M_N, un *chemin de chirurgie* de S vers Σ est une suite finie $S = S_0, \ldots, S_K = \Sigma$ telle que pour tout $i \in \{0, \ldots, K-2\}$, le système de sphères S_{i+1} soit obtenu par chirurgie élémentaire sur S_i dans la direction de Σ, et K soit le plus petit entier tel que S_{K-1} n'intersecte pas Σ.

Théorème 3.19 (Hilion–Horbez [1])**.**

Le complexe $\mathcal{S}_N$ est hyperbolique au sens de Gromov. De plus, les chemins de chirurgie sont des quasi-géodésiques reparamétrées uniformes.

Nous présentons un schéma de notre preuve du Théorème 3.19. La notion de quasi-géodésique reparamétrée est rappelée dans l'encadré en page 47.

Schéma de démonstration du Théorème 3.19. Notre preuve, tout comme la preuve originelle d'Handel et Mosher, repose sur un critère dû à Masur et Minsky [MM99, Théorème 2.3]. L'idée consiste à vérifier qu'un ensemble de chemins reliant les points de $\mathcal{S}_N$ (en l'occurrence ici, les chemins de chirurgie) vérifie un certain ensemble de propriétés partagées par les géodésiques d'un espace hyperbolique. Avant d'énoncer précisément le critère de Masur et Minsky, nous introduisons quelques définitions. Soit $\mathcal{X}$ un complexe simplicial connexe, muni de la distance simpliciale $d_{\mathcal{X}}$. Dans l'énoncé suivant, un *chemin* désignera une suite finie de sommets $\gamma(0), \ldots, \gamma(K)$ de $\mathcal{X}$ telle que pour tout $i \in \{1, \ldots, K\}$, nous ayons $d_{\mathcal{X}}(\gamma(i), \gamma(i+1)) \leq 2$. Une collection Γ de chemins dans $\mathcal{X}$ est *transitive* si pour tous

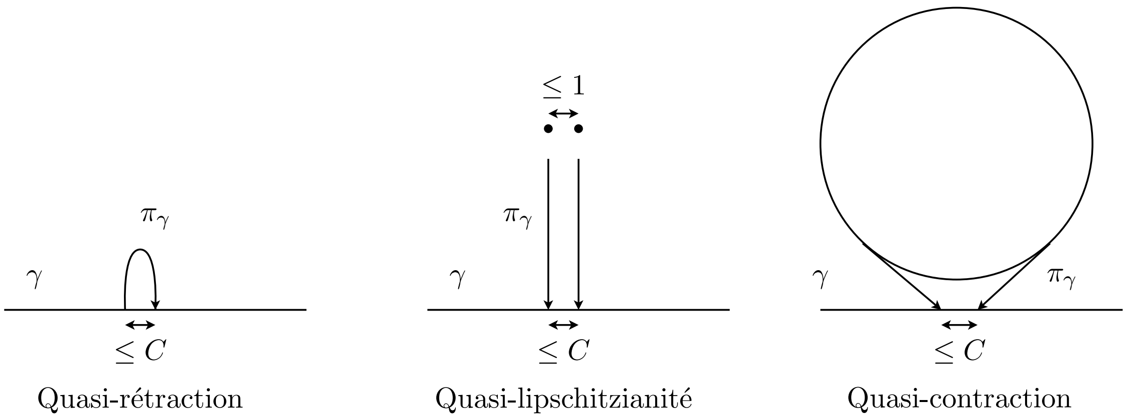


FIGURE 3.3 – Illustration des propriétés de quasi-rétraction, quasi-lipschitzianité et quasi-contraction.

sommets $v, w \in \mathcal{X}$, il existe $\gamma \in \Gamma$ tel que $\gamma(0) = v$ et $\gamma(K) = w$. Soit $\gamma : \{0, \ldots, K\} \to \mathcal{X}$ un chemin, et $\pi : \mathcal{X} \to \{0, \ldots, K\}$ une application. Nous définissons trois propriétés du couple (γ, π), que nous illustrons en Figure 3.3. Nous dirons que le couple (γ, π) est

- C-*quasi rétractant* si pour tout $k \in \{0, \ldots, K\}$, le diamètre de $\gamma([k, \pi(\gamma(k))])$ est inférieur à C ;
- C-*quasi lipschitzien* si pour tous sommets $v, w \in \mathcal{X}$ vérifiant $d_{\mathcal{X}}(v, w) \leq 1$, le diamètre de $\gamma([\pi(v), \pi(w)])$ est inférieur à C ;
- (A, B, C)-*quasi contractant* si pour tous sommets $v, w \in \mathcal{X}$, si $d_{\mathcal{X}}(v, \gamma([0, K])) \geq A$ et $d_{\mathcal{X}}(v, w) \leq B.d_{\mathcal{X}}(v, \gamma([0, K]))$, alors le diamètre de $\gamma([\pi(v), \pi(w)])$ est inférieur à C.

Remarquons que la propriété de contraction est une propriété qu'ont les géodésiques dans les espaces hyperboliques au sens de Gromov, lorsque π est l'application de projection. L'exemple typique est le cas des arbres, pour lesquels toute boule disjointe d'une géodésique γ se projette en un unique point de γ par l'application de projection naturelle. Dans le théorème suivant, on pourra penser à l'application π_γ comme à une projection sur le chemin γ, quoique nous n'imposions rien sur la définition de γ *a priori*.

Théorème 3.20. *(Masur–Minsky [MM99, Théorème 2.3]) Soit $\mathcal{X}$ un complexe simplicial connexe, muni de la distance simpliciale. Supposons qu'il existe des constantes $A \geq 0$, $B > 0$ et $C \geq 0$, ainsi qu'une collection transitive Γ de chemins dans $\mathcal{X}$, et pour tout $\gamma \in \Gamma$ de longueur K, une application $\pi_\gamma : \mathcal{X} \to \{0, \ldots, K\}$ telle que le couple (γ, π_γ) soit C-quasi rétractant, C-quasi lipschitzien, et (A, B, C)-quasi contractant. Alors $\mathcal{X}$ est hyperbolique au sens de Gromov, et il existe des constantes $K, L > 0$ (ne dépendant que de A, B et C) telles que tous les chemins de la collection Γ soient des (K, L)-quasi-géodésiques reparamétrées.*

La projection sur un chemin de chirurgie est définie comme suit. Soit S et Σ deux systèmes de sphères, et γ un chemin de chirurgie de S vers Σ. Soit S' un système de sphères. La *projection* $\pi_\gamma(S')$ est le plus petit entier k tel qu'il existe

- un chemin de chirurgie $S' = S'_0, \ldots, S'_{K'} = \Sigma$ de S' vers Σ (dans la paramétrisation duquel nous autorisons l'ajout de temps d'attente), et
- un chemin de chirurgie $S = S_0, \ldots, S_K = \Sigma$ de S vers Σ, obtenu à partir de Σ en ajoutant (si nécessaire) des temps d'attente dans la paramétrisation, et

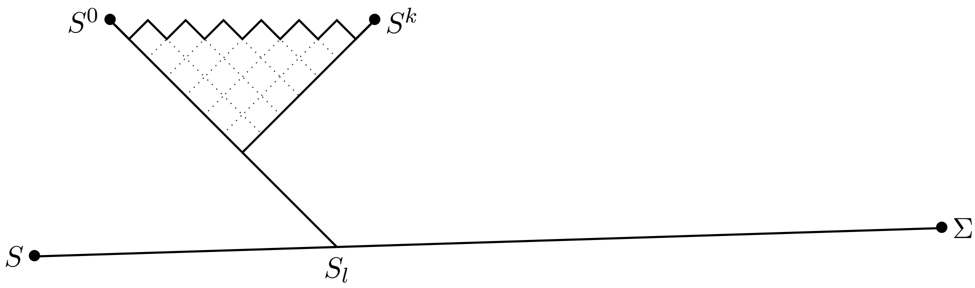


FIGURE 3.4 – La propriété de contraction pour les chemins de chirurgie.

- un entier t tel que $S_t = \gamma(k)$, et pour tout $i \geq t$, les systèmes S_i et S'_i aient une sphère commune.

La propriété de quasi-rétraction est vérifiée dans [1, Proposition 5.1] par un argument de décroissance des nombres d'intersection le long des chemins de chirurgie. Nous donnons maintenant une idée de la démonstration de la propriété de contraction (l'argument pour la quasi-lipschitzianité est similaire). Soit γ un chemin de chirurgie entre deux systèmes de sphères S et Σ, et soit S^0 et S^k deux sphères contenues dans une boule $\mathcal{B}$ de $\mathcal{S}_N$, disjointe de γ. Nous voulons montrer que les projections $\pi_\gamma(S^0)$ et $\pi_\gamma(S^k)$ sont proches. Pour cela, étant donné un chemin de chirurgie γ_0 entre S^0 et Σ, nous allons construire un chemin de chirurgie de S^k vers Σ qui se rapproche de γ_0 avant de quitter la boule $\mathcal{B}$. La construction se fait en deux étapes, illustrées sur la Figure 3.4.

1. Nous relions S^0 à S^k par un segment géodésique $(S^i)_{i\in\{0,\dots,k\}}$ (représenté par un zigzag sur la Figure 3.4). Nous construisons alors pour tout $i \in \{1, \dots, k\}$ un chemin de chirurgie de S^i vers Σ, ces chemins vérifiant une certaine propriété de compatibilité (l'argument est détaillé en Partie 4 de [1]).
2. L'ingrédient clé de la preuve consiste à montrer que si le chemin de chirurgie de S^i vers Σ progresse dans $\mathcal{S}_N$, alors deux sous-systèmes quelconques S^i_1 et S^i_2 de S^i doivent rapidement avoir un descendant commun, i.e. nous devons rapidement trouver une sphère obtenue par chirurgies successives à partir de S^i_1, et une sphère obtenue par chirurgies successives à partir de S^i_2, qui soient isotopes. La Proposition 6.2 de [1] donne une version quantitative de ce fait. Nous utilisons dans sa preuve une notion de *complexité* d'une partition d'un système de sphères en deux sous-systèmes, qui mesure l'intrication de ces sous-sytèmes ; nous donnons une borne (qui dépend de N) sur cette complexité, et montrons que la complexité croît le long des chemins de chirurgie. Cette propriété de rapide descendant commun nous permet de contracter le diagramme construit au point précédent, voir la Figure 3.4.

Applications. Nous terminons cette partie en mentionnant quelques applications de notre démonstration de l'hyperbolicité du complexe des sphères. La première application porte sur l'étude du complexe des arcs d'une surface compacte orientable à bord non trivial.

Soit S une surface compacte orientable à bord non trivial, qui n'est pas un cylindre. Un *arc* dans S est l'image de l'intervalle $[0, 1]$ par un plongement. Un arc α dans S est *essentiel* si aucune composante connexe de $S \smallsetminus \alpha$ n'a pour adhérence un disque. Le *complexe des arcs* $\mathcal{A}(S)$, introduit par Harer dans [Har85], est le complexe simplicial dont les sommets sont les classes d'isotopies d'arcs essentiels dans S. Plus généralement, un k-simplexe est défini par une collection de $k+1$ arcs qui peuvent être réalisés de manière disjointe dans S.

Notre démonstration de l'hyperbolicité du complexe des sphères donne une nouvelle preuve d'un théorème dû à Masur et Schleimer, affirmant l'hyperbolicité du complexe des arcs associé à une surface à bord [MS13]. Mentionnons également la preuve due à Hensel, Przytycki et Webb qui montre en outre que la constante d'hyperbolicité est indépendante de la topologie de la surface [HPW13]. L'idée de notre démonstration consiste à réaliser le complexe des arcs d'une surface compacte orientable $S_{g,s}$ de genre g ayant s composantes de bord comme sous-complexe de $\mathcal{S}_{2g+s-1}$. Pour cela, nous remarquons que la variété M_{2g+s-1} est homéomorphe à la variété obtenue en recollant deux copies de $S_{g,s} \times [0,1]$ le long de leur frontière commune, par l'application identité. L'image d'un arc de $S_{g,s}$ par cette opération est une sphère dans M_{2g+s-1}, ce qui définit une application injective $i : \mathcal{A}(S_{g,s}) \to \mathcal{S}_{2g+s-1}$. De plus, tout chemin de chirurgie entre deux sphères de l'image de i reste dans l'image de i, et s'interprète comme un chemin de chirurgie entre les arcs correspondants. Nous déduisons de ces observations l'hyperbolicité du complexe des arcs de la surface $S_{g,s}$, voir [1, Partie 8.2].

Corollaire 3.21. *(Masur–Schleimer [MS13, Théorème 20.3]) Soit S une surface compacte orientable à bord non trivial. Le complexe des arcs $\mathcal{A}(S)$ est hyperbolique au sens de Gromov. Les chemins de chirurgie sont des quasi-géodésiques reparamétrées uniformes.*

Notre approche de l'hyperbolicité du complexe des sphères permet également de réinterpréter géométriquement l'argument de Kapovich et Rafi remontrant l'hyperbolicité du complexe des facteurs libres, voir le Théorème 8.3 de [1].

Deuxième partie

Marches aléatoires sur $\mathbf{Out}(F_N)$

Introduction

Cette deuxième partie est consacrée à l'étude des marches aléatoires sur le groupe $\mathrm{Out}(F_N)$. Nous nous intéressons au comportement à l'infini d'une suite aléatoire d'automorphismes $\Phi_n \in \mathrm{Out}(F_N)$, obtenus par multiplications successives d'incréments indépendants distribués selon une même loi de probabilité μ sur $\mathrm{Out}(F_N)$, et étudions deux notions de bords à l'infini pour la marche aléatoire.

Dans un premier temps, nous identifions (sous un certain nombre de conditions sur la mesure μ) le bord de Poisson de $(\mathrm{Out}(F_N), \mu)$ avec le bord de Gromov du graphe des facteurs libres de F_N. Le *bord de Poisson* est un espace mesuré qui, en un sens, décrit complètement le comportement à l'infini d'une trajectoire typique de la marche aléatoire. Nous en donnons une définition précise au Chapitre 4. Nous montrons que presque toute trajectoire de la marche aléatoire à droite sur $(\mathrm{Out}(F_N), \mu)$, réalisée via l'action à gauche sur $\mathrm{Out}(F_N)$, converge vers le simplexe dans $\overline{CV_N}$ d'un arbre libre et arationnel. La mesure de sortie ν est l'unique mesure stationnaire sur l'espace $\mathcal{FI}$, et $(\mathcal{FI}, \nu)$ est le bord de Poisson de $(\mathrm{Out}(F_N), \mu)$. Le Chapitre 4 peut être lu comme un préambule à la lecture de l'Annexe C : nous y rappelons les constructions et résultats classiques concernant les marches aléatoires sur les groupes et le bord de Poisson, avant de donner une idée de nos techniques de preuve dans le cas du groupe $\mathrm{Out}(F_N)$. Nous terminons ce chapitre par la question de la vitesse de fuite d'une marche aléatoire sur $\mathrm{Out}(F_N)$, et des propriétés typiques de l'automorphisme obtenu au temps n de la marche.

Dans un second temps, nous identifions l'*horofrontière* de l'outre-espace CV_N pour la distance de Lipschitz au bord de la compactification primitive de CV_N. Grâce à un théorème de Karlsson et Ledrappier, ceci a des applications à l'étude de la marche aléatoire sur $\mathrm{Out}(F_N)$: toute trajectoire typique de la marche aléatoire est dirigée par une certaine horofonction (aléatoire). Nous utilisons notre description de l'horofrontière de CV_N pour étudier la croissance des éléments du groupe libre F_N sous l'action d'un produit aléatoire d'automorphismes, et nous établissons un analogue à un théorème de Furstenberg et Kifer [FK83] et Hennion [Hen84] pour le groupe $\mathrm{Out}(F_N)$. Dans le Chapitre 5, nous présentons la construction de l'horofrontière d'un espace métrique (possiblement asymétrique), que nous illustrons à l'aide de quelques exemples simples, et nous donnons un aperçu de résultats connus à ce jour sur les horofrontières de certains espaces métriques. Nous établissons alors notre description de l'horofrontière de l'outre-espace pour la distance (asymétrique) de Lipschitz, et nous discutons également quelques propriétés de l'horofrontière de CV_N pour la distance inversée, qui est différente de l'horofrontière précédente : en particulier, nous donnons une description explicite de l'horofrontière pour la distance inversée lorsque $N = 2$, et nous montrons que celle-ci est de dimension topologique infinie lorsque $N \geq 3$. Enfin, nous présentons le théorème de Karlsson et Ledrappier, et son application à l'étude de la marche aléatoire sur $\mathrm{Out}(F_N)$ et à la croissance des classes de conjugaison d'éléments de $\mathrm{Out}(F_N)$ sous l'action de produits aléatoires d'automorphismes.

Chapitre 4

Le bord de Poisson de Out(F_N)

4.1 Marches aléatoires sur des groupes discrets et bord de Poisson

Généralités.

Soit G un groupe dénombrable, et soit μ une loi de probabilité sur G. Soit λ une mesure (quelconque) sur G. La *marche aléatoire à droite* sur (G,μ) de distribution initiale λ est la chaîne de Markov sur G dont la distribution initiale est λ, et dont les probabilités de transition sont données par $p(x,y) := \mu(x^{-1}y)$ pour tous $x,y \in G$. L'espace $\Omega := (G^{\mathbb{N}^*}, \mu^{\otimes \mathbb{N}^*})$ est appelé l'*espace des incréments*. La position de la marche aléatoire au temps n est obtenue à partir de sa position initiale g_0 par multiplications successives à droite par une suite $(s_i)_{i\in\mathbb{N}^*} \in \Omega$ d'incréments indépendants, tous distribués suivant la loi μ, i.e. $g_n = g_0 s_1 \dots s_n$. Sa distribution est donnée par la convolution $\lambda * \mu^{*n}$. (Nous rappelons que la *convolution* de deux mesures μ et μ' est la mesure sur G définie par

$$\mu * \mu'(g) := \sum_{h\in G} \mu(h)\mu'(h^{-1}g)$$

pour tout $g \in G$). L'*espace des trajectoires* $\mathcal{T} := G \times G^{\mathbb{N}^*}$ est muni de la σ-algèbre $\mathcal{A}$ engendrée par les cylindres $C_{i,g} := \{(g_n) \in \mathcal{T} | g_i = g\}$ pour $i \in \mathbb{N}$ et $g \in G$. Nous noterons $\mathbb{P}_\lambda$ la distribution sur l'espace $\mathcal{T}$, image de la mesure $\lambda \otimes \mu^{\otimes \mathbb{N}^*}$ par l'application

$$\begin{array}{ccc} G \times \Omega & \to & \mathcal{T} \\ (g,(s_i)_{i\in\mathbb{N}}) & \mapsto & (g, gs_1, gs_1s_2, \dots) \end{array}.$$

Lorsque λ est la mesure de Dirac en l'identité de G, nous noterons simplement $\mathbb{P}$ la mesure associée sur l'espace des trajectoires. Soit m la mesure de comptage sur G, et $\overline{\mathcal{A}}$ la complétion de la σ-algèbre $\mathcal{A}$ par rapport à la mesure $\mathbb{P}_m$. L'espace mesuré $(\mathcal{T}, \overline{\mathcal{A}}, \mathbb{P}_m)$ est un *espace de Lebesgue* (voir l'encadré en page 70). Le groupe G agit diagonalement à gauche sur l'espace des trajectoires. Le *décalage temporel* sur l'espace des trajectoires est l'application

$$\begin{array}{cccc} T: & \mathcal{T} & \to & \mathcal{T} \\ & (g_n)_{n\in\mathbb{N}} & \mapsto & (g_{n+1})_{n\in\mathbb{N}} \end{array}.$$

Bord de Poisson et μ-frontières.

La notion du *bord de Poisson* d'un opérateur markovien a été introduite par Furstenberg dans l'article [Fur71], point de départ d'une vaste littérature à ce sujet. Nous

Intermède (Espaces de Lebesgue, partitions mesurables, et tutti quanti).

Nous introduisons les notions d'espaces de Lebesgue et de leurs partitions mesurables. Nous renvoyons à l'article originel de Rokhlin [Rok49] pour une étude détaillée. Nous renvoyons également à [Cou02] pour une démonstration courte de la correspondance de Rokhlin.

1. Espaces de Lebesgue

Soit $(\Omega, \mathcal{A}, \theta)$ un espace mesuré *complet*, i.e. si $A \in \mathcal{A}$ vérifie $\theta(A) = 0$, et $B \subseteq A$, alors $B \in \mathcal{A}$. L'espace $(\Omega, \mathcal{A}, \theta)$ est un *espace de Lebesgue* s'il existe un sous-ensemble $\Omega' \subseteq \Omega$ tel que $\theta(\Omega \smallsetminus \Omega') = 0$, et un ensemble dénombrable $\{B_n\}_{n\in\mathbb{N}}$ de sous-ensembles mesurables de Ω', tel que pour tous points distincts $x_1 \neq x_2 \in \Omega'$, il existe $n \in \mathbb{N}$ tel que soit $x_1 \in B_n$ et $x_2 \notin B_n$, soit $x_1 \notin B_n$ et $x_2 \in B_n$.

2. Partitions mesurables

Soit $(\Omega, \mathcal{A}, \theta)$ un espace de Lebesgue. Une *partition (mod* 0*)* de $(\Omega, \mathcal{A}, \theta)$ est un ensemble ξ de sous-ensembles disjoints de Ω tel que

$$\theta(\Omega \smallsetminus \bigcup_{C\in\xi} C) = 0.$$

Une partition ξ de $(\Omega, \mathcal{A}, \theta)$ est *mesurable* s'il existe un ensemble dénombrable $\{B_n\}_{n\in\mathbb{N}}$ de sous-ensembles mesurables de Ω, qui sont réunions (*a priori* non dénombrables) de sous-ensembles dans la partition ξ, tel que pour tous $C_1, C_2 \in \xi$, il existe $n \in \mathbb{N}$ tel que soit $C_1 \subseteq B_n$ et $C_2 \subseteq \Omega \smallsetminus B_n$, soit $C_1 \subseteq \Omega \smallsetminus B_n$ et $C_2 \subseteq B_n$. En particulier, tout ensemble de la partition est mesurable.

Proposition 4.1. *(Rokhlin [Rok49]) Soit $(\Omega, \mathcal{A}, \theta)$ un espace de Lebesgue, et ξ une partition mesurable de $(\Omega, \mathcal{A}, \theta)$. Alors l'espace mesuré quotient Ω/ξ est un espace de Lebesgue.*

La *correspondance de Rokhlin* établit une bijection entre les partitions mesurables de $(\Omega, \mathcal{A}, \theta)$ et les sous-σ-algèbres complètes de $\mathcal{B}$, définie comme suit. À toute partition mesurable ξ, nous associons la complétion de la sous-σ-algèbre $\mathcal{A}_\xi$ de $\mathcal{A}$ formée des ensembles de $\mathcal{A}$ qui sont unions d'éléments de la partition. Notons que $\mathcal{A}_\xi$ n'est pas toujours égale à la σ-algèbre engendrée par les atomes de la partition ξ : par exemple, la sous-σ-algèbre de $[0, 1]$ engendrée par les atomes de la partition ξ en points ne contient que des ensembles négligeables et leurs complémentaires, tandis que $\mathcal{A}_\xi = \mathcal{A}$. Réciproquement, si $\mathcal{B}$ est une sous-σ-algèbre complète de $\mathcal{A}$, alors il existe un ensemble dénombrable $\{B_n\}$ de parties de Ω tel que $\mathcal{A}$ soit engendrée par les ensembles B_n et les ensembles négligeables. Nous associons alors à $\mathcal{B}$ la partition $\xi_\mathcal{B}$ dont les atomes sont donnés par les ensembles

$$\xi_\mathcal{B}(x) = \bigcap_{x\in B_n} B_n \cap \bigcap_{x\notin B_n} (\Omega \smallsetminus B_n),$$

dont on vérifie qu'elle est indépendante du choix des B_n. La correspondance de Rokhlin établit également une isométrie entre l'espace des fonctions θ-essentiellement bornées qui sont constantes sur chaque atome de la partition, et $L^\infty(\Omega/\xi)$.

Intermède (suite).

3. Un exemple de partition mesurable

Soit $(\Omega, \mathcal{A}, \theta)$ un espace de Lebesgue, et soit X un espace métrique séparable, muni de la tribu borélienne. Soit $f : \Omega \to X$ une application mesurable. Alors la partition ξ de Ω donnée par les préimages de points par l'application f est une partition mesurable de Ω. En effet, soit $(x_n)_{n\in\mathbb{N}}$ une suite dense dans X, et pour tout $N \in \mathbb{N}$, soit $\mathcal{P}_N$ la partition de Voronoi de X relative aux points $\{x_0, \ldots, x_N\}$ (pour tout $i \in \{0, \ldots, N\}$, l'élément de la partition associé à x_i est l'ensemble des points $x \in X$ pour lesquels $d(x, x_i) = \inf_{j\in\{0,\ldots,N\}} d(x, x_j)$, et $d(x, x_i) < d(x, x_j)$ pour tout $j < i$). Alors les préimages par f des partitions $\mathcal{P}_N, N \in \mathbb{N}$ séparent les atomes de ξ.

renvoyons par exemple le lecteur à [KV83, Kai96, Kai00, Bab06, Ers10, GM12] pour des introductions à ce sujet.

Soit G un groupe dénombrable, et μ une mesure de probabilité sur G. Soit T l'opérateur de décalage temporel dans l'espace des trajectoires. Une *μ-frontière* est un espace mesuré (B, ν) qui est un quotient de l'espace des trajectoires $(\mathcal{T}, \overline{\mathcal{A}}, \mathbb{P})$ par une *partition mesurable* (voir l'encadré en page 70) G-invariante et T-invariante. Remarquons que la σ-algèbre sur $\mathcal{T}$ est définie par complétion de $\mathcal{A}$ pour la mesure $\mathbb{P}_m$, mais la mesure dont nous munissons B est la mesure image de $\mathbb{P}$ par l'application de projection naturelle. En fait, à toute distribution initiale λ sur G correspond une mesure ν_λ sur B, image de la mesure $\mathbb{P}_\lambda$ par l'application de projection. Lorsque $\lambda = m$ est la mesure de comptage sur G, l'espace (B, ν_m) est un espace de Lebesgue (voir la Proposition 4.1 de l'encadré en page 70). Lorsque λ est la mesure de Dirac en l'identité de G, nous notons simplement ν la mesure associée sur B. C'est une mesure *μ-stationnaire* sur B, i.e. pour tout borélien $S \subseteq B$, nous avons

$$\nu(S) = \sum_{g\in G} \mu(g)\nu(g^{-1}S).$$

Pour toute distribution initiale λ sur G, nous avons $\nu_\lambda = \lambda * \nu$. En particulier, l'espace B peut être muni de chacune des mesures de probabilité $\nu_g := g_*\nu$ correspondant à la marche aléatoire issue de $g \in G$, et nous avons

$$\nu_m = \sum_{g\in G} \nu_g.$$

En un sens, la mesure ν porte donc toute l'information sur l'espace B. Remarquons au passage que les mesures ν_g ne sont pas nécessairement absolument continues les unes par rapport aux autres, lorsque le support de μ n'engendre pas le groupe G en entier.

Un exemple typique de μ-frontière est donné par la situation suivante. Soit X un espace métrique séparable muni d'une action de G, et $*$ un point base dans X. Supposons que pour $\mathbb{P}$-presque toute trajectoire $\mathbf{g} := (g_n)_{n\in\mathbb{N}}$ de la marche aléatoire sur (G, μ), la suite $(g_n*)_{n\in\mathbb{N}}$ converge vers un point $\mathrm{bnd}(\mathbf{g}) \in X$. Alors l'application $\mathrm{bnd} : \mathcal{T} \to X$ est une application mesurable à valeurs dans un espace métrique séparable, qui est T-invariante et G-équivariante. La partition ξ de l'espace des trajectoires $\mathcal{T}$ donnée par les bnd-préimages de points est donc une partition mesurable T-invariante et G-invariante de $\mathcal{T}$ (voir l'encadré en page 71). L'espace $(X, \mathrm{bnd}_*\mathbb{P})$, quotient de $\mathcal{T}$ par ξ, est donc une μ-frontière.

Une μ-frontière (B,ν) est un *bord de Poisson* de (G,μ) si elle est maximale, i.e. si toute μ-frontière (B',ν') est quotient de (B,ν) pour une certaine partition mesurable G-invariante de (B,ν). Un bord de Poisson (B,ν) de (G,μ) est unique, au sens où si (B,ν) et (B',ν') sont deux bords de Poisson de (G,μ), alors il existe un isomorphisme mesurable entre un sous-ensemble de ν-mesure pleine de B et un sous-ensemble de ν'-mesure pleine de B'.

Une réalisation abstraite du bord de Poisson de (G,μ) est donnée par la construction suivante. Soit $\mathcal{A}_T$ la σ-algèbre formée par les sous-ensembles mesurables et T-invariants de $\mathcal{T}$. Nous rappelons que m est la mesure de comptage sur G, et $\mathbb{P}_m$ la mesure associée sur l'espace des trajectoires $G\times G^{\mathbb{N}*}$, qui est G-invariante. Soit $\overline{\mathcal{A}_T}$ la complétion de la σ-algèbre $\mathcal{A}_T$ pour la mesure $\mathbb{P}_m$. L'espace $(\mathcal{T}, \overline{\mathcal{A}}, \mathbb{P}_m)$ est un espace de Lebesgue. La correspondance de Rokhlin (voir l'encadré en page 70) associe à la sous-σ-algèbre complète $\overline{\mathcal{A}_T}$ de $\overline{\mathcal{A}}$ une unique partition mesurable η de $G^{\mathbb{N}}$, bien définie aux ensembles $\mathbb{P}_m$-négligeables près. En tant qu'espace mesurable, le bord de Poisson de (G,μ) est l'espace quotient $\Gamma := \mathcal{T}/\eta$, muni de la σ-algèbre quotient (c'est donc aussi l'espace des composantes ergodiques de l'opérateur de décalage temporel dans l'espace des trajectoires). Comme précédemment, l'espace Γ peut être muni de chaque mesure ν_λ, image par l'application quotient $\mathcal{T} \to \Gamma$ de la mesure $\mathbb{P}_\lambda$ associée à une distribution initiale λ. La mesure ν associée à la distribution initiale donnée par la mesure de Dirac en l'identité de G est appelée la *mesure harmonique* sur Γ. Le *bord de Poisson* de (G,μ) est l'espace mesuré (Γ,ν).

Fonctions harmoniques.

Soit G un groupe dénombrable, et μ une mesure de probabilité sur G. Une des motivations pour l'étude du bord de Poisson de la marche aléatoire sur (G,μ) est l'étude des fonctions harmoniques sur G. Une fonction $f : G \to \mathbb{R}$ est *μ-harmonique* si pour tout $g \in G$, nous avons

$$f(g) = \sum_{h\in G} \mu(h) f(gh).$$

Nous noterons $H^\infty(G,\mu)$ l'espace de Banach des fonctions μ-harmoniques μ-essentiellement bornées sur G, muni de la norme du supremum essentiel. Une manière de construire des fonctions μ-harmoniques sur G est donnée par le procédé suivant. Soit (X,ν) un espace de probabilité muni d'une action de G. Supposons que la mesure ν soit μ-stationnaire. Posons

$$\nu_m := \sum_{g\in G} g_*\nu.$$

Soit $F \in L^\infty(X,\nu_m)$. Alors l'application $f : G \to \mathbb{R}$ définie par

$$f(g) = \int_X F(x) dg_*\nu(x)$$

est une fonction μ-harmonique μ-essentiellement bornée sur G. Nous définissons ainsi une application linéaire

$$\begin{array}{ccc} L^\infty(X,\nu_m) & \to & H^\infty(G,\mu) \\ F & \mapsto & f \end{array},$$

de norme inférieure à 1. Lorsque (X,ν) est le bord de Poisson de (G,μ), cette application est une isométrie. En effet, si f est une fonction harmonique μ-essentiellement bornée sur G, et $\mathbf{g} := (g_n)_{n\in\mathbb{N}}$ est une trajectoire de la marche aléatoire sur (G,μ), alors la suite $(f(g_n))_{n\in\mathbb{N}}$ est une martingale bornée, et admet donc $\mathbb{P}$-presque sûrement une limite $\widehat{F}(\mathbf{g})$. Comme la fonction $\widehat{F}$ est T-invariante et $\mathcal{A}_T$-mesurable, il existe $F \in L^\infty(X,\nu_m)$ telle que $F(\mathrm{bnd}(\mathbf{g})) = \lim f(g_n)$, où $\mathrm{bnd} : \mathcal{T} \to X$ désigne l'application quotient (voir l'encadré en page 70). Nous avons ainsi construit une application linéaire

$$\begin{array}{ccc} H^\infty(G,\mu) & \to & L^\infty(X,\nu_m) \\ f & \mapsto & F \end{array}$$

de norme inférieure à 1, dont on vérifie qu'elle est inverse de l'application définie ci-dessus. Ainsi, le bord de Poisson permet de décrire les fonctions μ-harmoniques μ-essentiellement bornées sur G. En particulier, le bord de Poisson est trivial si et seulement si toute fonction μ-harmonique μ-essentiellement bornée sur G est constante. C'est le cas par exemple si G est abélien [Bla55, CD60] ou plus généralement nilpotent [DM61]. C'est aussi le cas si G est à croissance sous-exponentielle et μ est à support fini [Ave74].

Il existe également un espace, le *bord de Martin*, qui permet de décrire l'ensemble des fonctions harmoniques positives sur G. Tandis que le bord de Poisson est un espace mesuré, le bord de Martin est un espace topologique, souvent plus difficile à décrire que le bord de Poisson. Nous renvoyons le lecteur intéressé à [Anc90] pour une étude détaillée de cet espace.

Mentionnons pour conclure d'autres applications possibles de la description du bord de Poisson, par exemple à la question de la croissance des groupes [BE11], ou à des problèmes de rigidité [Fur63b, BF14a].

Le critère de Kaimanovich pour vérifier la maximalité d'une μ-frontière.

Étant donné un groupe G, et une mesure de probabilité μ sur G, une question récurrente est celle de trouver un modèle du bord de Poisson de (G,μ), c'est-à-dire d'identifier ce bord de Poisson avec un espace concret X sur lequel le groupe G agit. Quoique nous insistions sur le fait que le bord de Poisson de (G,μ) est défini comme espace mesuré, et n'admet pas de topologie intrinsèque (contrairement au bord de Martin mentionné ci-dessus), l'espace X sera le plus souvent un espace topologique. Par exemple, l'espace X pourra être un bord topologique ∂G du groupe G (l'exemple typique étant celui du bord de Gromov d'un groupe hyperbolique). Il s'agira alors de munir X d'une mesure de probabilité ν, avant de montrer que l'espace (X,ν) obtenu est bien le bord de Poisson de G. La stratégie consistera souvent à montrer dans un premier temps que $\mathbb{P}$-presque toute trajectoire de la marche aléatoire sur (G,μ) converge vers un point du bord ∂G. Lorsque ∂G est un espace métrisable séparable, en notant ν la mesure de sortie sur ∂G, l'espace $(\partial G,\nu)$ est alors une μ-frontière. Dans un deuxième temps, on cherchera à montrer la maximalité de cette μ-frontière.

Kaimanovich a donné dans [Kai00] des critères géométriques, issus de la théorie de l'entropie des marches aléatoires, permettant de vérifier la maximalité d'une μ-frontière. Nous présentons l'un d'entre eux, le *critère des bandes*. Nous rappelons que $\check{\mu}$ est la mesure de probabilité sur G définie par $\check{\mu}(g) := \mu(g^{-1})$ pour tout $g \in G$. Étant donné une $\check{\mu}$-frontière (B_-,ν_-) et une μ-frontière (B_+,ν_+), l'idée consiste à associer à toute paire d'éléments $(b_-,b_+) \in B_- \times B_+$ une *bande* $S(b_-,b_+)$ dans G qui soit suffisamment fine, au sens où l'on a un contrôle sur le cardinal de son intersection avec les boules pour une distance des mots sur G. Typiquement, lorsque B_- et B_+ sont des bords topologiques de G, on pourra penser (lorsque cela existe !) à un ensemble de géodésiques reliant les points b_- et b_+.

Nous donnons maintenant une présentation formelle du critère de Kaimanovich. Soit G un groupe de type fini, et S une partie génératrice finie de G. Munissons G de la *distance des mots* relative à S : la distance entre deux éléments $g, h \in G$ est la longueur minimale d'un mot représentant $g^{-1}h$ dans la base S (deux telles distances sont équivalentes). Soit μ une mesure de probabilité sur G. Le *premier moment logarithmique* de μ pour la distance d est défini comme

$$|\mu| := \sum_{g\in G} \log d(e,g)\mu(g).$$

L'*entropie* de μ est

$$H(\mu) := \sum_{g\in G} -\mu(g)\log \mu(g).$$

Théorème 4.2. *(Kaimanovich [Kai00, Théorème 6.5]) Soit G un groupe de type fini, soit d une distance des mots sur G, et soit μ une mesure de probabilité sur G ayant un premier moment logarithmique fini pour la distance d, et d'entropie finie. Soit (B_-, ν_-) une $\check{\mu}$-frontière, et (B_+, ν_+) une μ-frontière. Supposons qu'il existe une application mesurable G-équivariante*

$$\begin{array}{cccc} S: & B_- \times B_+ & \to & 2^G \\ & (b_-, b_+) & \mapsto & S(b_-, b_+) \end{array}$$

telle que pour $\nu_- \otimes \nu_+$-presque tout $(b_-, b_+) \in B_- \times B_+$, l'ensemble $S(b_-, b_+)$ soit non vide, et

$$\sup_{k\in\mathbb{N}\smallsetminus\{0\}} \frac{1}{\log k} \log card(S(b_-, b_+) \cap \mathcal{B}_k) < +\infty,$$

où $\mathcal{B}_k$ est la boule de centre e et de rayon k pour la distance d. Alors (B_-, ν_-) (resp. (B_+, ν_+)) est le bord de Poisson de $(G, \check{\mu})$ (resp. (G, μ)).

Ce théorème de Kaimanovich donne un critère géométrique pour vérifier qu'une μ-frontière est un bord de Poisson. En utilisant ce critère, Kaimanovich a énoncé dans [Kai00, Théorèmes 2.4 et 6.6] un certain nombre de conditions topologiques sous lesquelles une compactification $\overline{G}$ de G vérifie que $\mathbb{P}$-presque toute trajectoire de la marche aléatoire sur (G, μ) converge vers un point de $\overline{G}$, et sous lesquelles l'espace $(\overline{G}, \nu)$, où ν désigne la mesure de sortie sur $\overline{G}$, est le bord de Poisson de (G, μ).

Exemples connus de bords de Poisson de groupes discrets.

Nous présentons un aperçu d'un certain nombre de résultats connus portant sur le bord de Poisson de certaines classes de groupes dénombrables de type fini. Nous ne prétendons pas à l'exhaustivité des résultats mentionnés, tant la littérature est riche à ce sujet, mais nous espérons que les résultats suivants donneront un bon aperçu de l'état de l'art. Par souci de simplicité, nous supposerons à chaque fois, sauf mention explicite du contraire, que le support de la mesure μ est une partie génératrice finie du groupe G. Nous renvoyons le lecteur aux références proposées pour des énoncés plus précis dans chacun des cas.

Étant donné un groupe G, une première question que nous pouvons nous poser est celle de la (non-)trivialité du bord de Poisson de G. Le bord de Poisson d'un groupe abélien est trivial (Blackwell [Bla55], Choquet et Deny [CD60]). C'est le cas plus généralement pour les groupes nilpotents (Dynkin et Maliutov [DM61]), ou pour les groupes à croissance sous-exponentielle (Avez [Ave74]). L'hypothèse de finitude du support de μ n'est en fait nécessaire que dans ce dernier cas. Erschler a montré la trivialité du bord de Poisson pour des marches aléatoires sur des produits en couronne itérés de $\mathbb{Z}$ et $\mathbb{Z}^2$ [Ers01].

Pour certains groupes, on est en fait capable de déterminer la frontière de Martin, ce qui permet de retrouver le bord de Poisson. C'est le cas par exemple pour les groupes hyperboliques, dont le bord de Poisson s'identifie au bord de Gromov (Ancona [Anc87]). De même, Series a identifié le bord de Poisson de certains groupes fuchsiens avec leur ensemble limite [Ser83].

Nous donnons maintenant un certain nombre d'exemples pour lesquels le critère des bandes de Kaimanovich a permis de déterminer le bord de Poisson. Dans chacun des cas ci-dessous, les trajectoires de la marche aléatoire simple sur G convergent presque sûrement vers un point du bord, et nous munissons ce bord de la mesure de sortie.

- Le bord de Poisson d'un groupe hyperbolique s'identifie à son bord de Gromov (Kaimanovich [Kai00, Partie 7]). Remarquons que l'argument de Kaimanovich permet d'affaiblir les conditions sur la mesure μ dans le résultat d'Ancona [Anc87]. Le critère des bandes de Kaimanovich s'applique ici à la réunion de toutes les géodésiques joignant deux points quelconques du bord de Gromov. Ceci s'applique plus généralement aux groupes d'isométries d'espaces hyperboliques propres.
- Le bord de Poisson d'un groupe ayant une infinité de bouts s'identifie avec l'espace des bouts (Kaimanovich [Kai00, Partie 8]). Le critère des bandes s'applique, celles-ci sont données par les unions de boules de rayon minimal qui séparent deux bouts.
- Le bord de Poisson d'un réseau cocompact d'une variété de Cartan–Hadamard (i.e. une variété riemannienne complète, simplement connexe et de courbure sectionnelle négative ou nulle) s'identifie au bord visuel de la variété (Kaimanovich [Kai00, Partie 9]).
- Le bord de Poisson du groupe modulaire d'une surface compacte s'identifie au bord de Thurston de l'espace de Teichmüller associé (Kaimanovich–Masur [KM96]). Kaimanovich et Masur appliquent le critère des bandes à des lignes bi-infinies données par les géodésiques de Teichmüller.
- Le cas des sous-groupes discrets de $SL(d,\mathbb{R})$ a été traité par Furstenberg [Fur67, Fur71] : le bord de Poisson s'identifie à l'espace des drapeaux de $\mathbb{R}^d$. Ses résultats ont des applications à l'étude des réseaux de $SL(d,\mathbb{R})$: Furstenberg montre que si $d \geq 3$, alors aucun réseau de $SL(2,\mathbb{R})$ ne peut être réalisé comme réseau de $SL(d,\mathbb{R})$. Plus généralement, pour une étude de bords de Poisson de sous-groupes discrets de groupes de Lie semi-simples, nous renvoyons à [Led85, Kai00, Bro06, Sch09, BS11].
- Gautero et Mathéus ont déterminé le bord de Poisson de certaines extensions de groupes libres et de groupes hyperboliques [GM12]. En particulier, le bord de Poisson d'une extension d'un groupe libre F par un groupe cyclique coïncide avec le bord topologique de F.
- Karlsson et Woess ont déterminé le bord de Poisson du produit en couronne d'un groupe libre avec un groupe fini [KW07].

Citons pour conclure d'autres exemples de groupes pour lesquels le bord de Poisson a été déterminé. Le bord de Poisson d'un groupe d'isométries d'un espace métrique complet, uniformément convexe, à courbure de Busemann négative s'identifie avec le bord visuel (Karlsson et Margulis [KM99]). Ceci s'applique en particulier aux groupes d'isométries d'espaces $CAT(0)$ propres, et en particulier aux groupes de Coxeter agissant sur leur complexe de Moussong, voir [KL06]. Deroin a déterminé le bord de Poisson de groupes localement discrets agissant par difféomorphismes sur le cercle [Der13].

4.2 Convergence au bord de la marche aléatoire sur Out(F_N)

Dans cette partie, nous étudions le comportement typique d'une marche aléatoire sur Out(F_N), réalisée soit sur l'outre-espace CV_N, soit sur le complexe des facteurs libres FF_N, via l'action de Out(F_N) sur ces espaces. Dans toute la suite de ce chapitre, nous ne considérerons que des marches aléatoires dont la distribution initiale est donnée par la mesure de Dirac en l'identité.

Définition 4.3. *Un sous-groupe $H \subseteq Out(F_N)$ est* non élémentaire *si*

1. *la H-orbite de toute classe de conjugaison de facteurs libres propres de F_N est infinie, et*
2. *la H-orbite de tout simplexe d'arbres arationnels dans ∂CV_N est infinie, et*
3. *la H-orbite de toute classe de conjugaison d'éléments de F_N est infinie.*

Nous rappelons que $\mathcal{FI}$ désigne l'espace des classes d'équivalence d'arbres arationnels libres, deux arbres étant équivalents s'ils font partie d'un même simplexe de mesures de longueur dans $\overline{CV_N}$ (voir la Partie 3.2.1). La démonstration du théorème suivant fait l'objet de la Partie 2 de l'Annexe C.

Théorème 4.4.

Soit μ une loi de probabilité sur Out(F_N), dont le support engendre un sous-groupe non élémentaire de Out(F_N). Alors pour $\mathbb{P}$-presque toute trajectoire $\mathbf{\Phi} := (\Phi_n)_{n\in\mathbb{N}} \in \mathrm{Out}(F_N)^{\mathbb{N}}$ de la marche aléatoire sur $(\mathrm{Out}(F_N), \mu)$, il existe $\xi(\mathbf{\Phi}) \in \mathcal{FI}$ tel que pour tout $T_0 \in CV_N$, la suite $(\Phi_n.T_0)_{n\in\mathbb{N}}$ converge vers $\xi(\mathbf{\Phi})$.

En vue de la description par Bestvina–Reynolds et Hamenstädt du bord de Gromov du complexe des facteurs libres de F_N (Théorème 3.3), nous déduisons la convergence de la marche aléatoire réalisée sur FF_N.

Théorème 4.5. *Soit μ une loi de probabilité sur Out(F_N), dont le support engendre un sous-groupe non élémentaire de Out(F_N). Alors pour $\mathbb{P}$-presque toute trajectoire $\mathbf{\Phi} := (\Phi_n)_{n\in\mathbb{N}}$ de la marche aléatoire sur $(Out(F_N), \mu)$, il existe $\xi(\mathbf{\Phi}) \in \partial FF_N$ tel que pour tout $x \in FF_N$, la suite $(\Phi_n.x)_{n\in\mathbb{N}}$ converge vers $\xi(\mathbf{\Phi})$.*

Les deux énoncés ci-dessus nous ont été inspirés par l'énoncé analogue dû à Kaimanovich et Masur [KM96] dans le cadre des groupes modulaires de surfaces fermées orientables. Soit S une surface fermée orientable. Un sous-groupe $H \subseteq Mod(S)$ est *non élémentaire* s'il contient deux homéomorphismes pseudo-Anosov qui engendrent un sous-groupe libre de rang 2. De manière équivalente, le sous-groupe H est non élémentaire s'il n'a pas d'orbite finie dans $\mathcal{C}(S) \cup \partial\mathcal{C}(S)$. Nous renvoyons à [KM96, Lemme 1.2.1] pour une démonstration de cette équivalence, qui repose sur une classification des sous-groupes de Mod(S) par McCarthy et Papadopoulos [MP89, Théorème 4.6] ; une discussion analogue pour le groupe Out(F_N) sera proposée au Chapitre 6 de cette thèse. Nous rappelons que $\mathcal{PMF}$ est l'espace des feuilletages mesurés projectifs, et $\mathcal{UE} \subset \mathcal{PMF}$ désigne le sous-espace de $\mathcal{PMF}$ formé des feuilletages *uniquement ergodiques*, i.e. qui admettent une unique mesure transverse.

Théorème 4.6. *(Kaimanovich–Masur [KM96, Théorème 2.2.4]) Soit S une surface fermée orientable, et μ une mesure de probabilité sur le groupe modulaire $Mod(S)$, dont le support engendre un sous-groupe non élémentaire de $Mod(S)$. Alors pour $\mathbb{P}$-presque toute trajectoire $\mathbf{g} = (g_n)_{n\in\mathbb{N}}$ de la marche aléatoire sur $(Mod(S), \mu)$, et tout $x \in Teich(S)$, la suite $(g_n.x)_{n\in\mathbb{N}}$ converge dans $\mathcal{PMF}$ vers une limite $F = F(\mathbf{g}) \in \mathcal{UE}$.*

En utilisant la description de Klarreich du bord de Gromov du complexe des courbes [Kla99], le Théorème 4.6 entraîne de même la convergence d'une trajectoire typique de la marche aléatoire sur $Mod(S)$ vers un point du bord de Gromov du complexe $\mathcal{C}(S)$ (voir [Mah10, Théorème 5.1] où cet énoncé apparaît). Il est intéressant de remarquer que dans le cas des groupes modulaires de surfaces, la convergence se fait au niveau de l'espace de Teichmüller. Nous conjecturons que dans le contexte des automorphismes de F_N, la convergence a lieu dans $\overline{CV_N}$. Ceci serait vérifié si nous savions que les points limites de la marche aléatoire sont *uniquement ergométriques* (i.e. le simplexe de mesures de longueur associé consiste en un unique point). La preuve de Kaimanovich et Masur de l'unique ergodicité du point limite repose sur un théorème de Masur [Mas92] qui énonce que toute géodésique de Teichmüller dont le feuilletage vertical associé est minimal, mais pas uniquement ergodique, doit quitter la partie épaisse de l'espace de Teichmüller $Teich(S)$. Toutefois, nous ne connaissons pas d'analogue au théorème de Masur pour l'action de Out(F_N) sur CV_N. Dans le cas de Out(F_N), nous pouvons aussi poser la question duale de l'unique ergodicité des points limites, i.e. y a-t-il unicité d'un courant géodésique dual aux arbres limites d'une trajectoire typique de la marche aléatoire sur (G, μ) ?

Le Théorème 4.5 a été montré également par Calegari et Maher [CM12, Théorème 5.34]. Calegari et Maher se placent dans le cadre plus général de la marche aléatoire sur un groupe G d'isométries d'un complexe simplicial X hyperbolique au sens de Gromov, qui n'est pas nécessairement localement fini. Deux éléments de G qui agissent de manière loxodromique sur X sont *indépendants* s'ils n'ont pas de point fixe commun dans le bord de Gromov ∂X. Notre démonstration du Théorème 4.4 n'utilise pas l'action de Out(F_N) sur le complexe des facteurs libres de F_N.

Théorème 4.7. *(Calegari–Maher [CM12, Théorème 5.34]) Soit X un complexe simplicial hyperbolique au sens de Gromov, et soit G son groupe d'isométries. Soit μ une mesure de probabilité sur G, dont le support engendre un sous-groupe de G contenant deux automorphismes loxodromiques indépendants. Alors pour $\mathbb{P}$-presque toute trajectoire $\boldsymbol{g} := (g_n)_{n\in\mathbb{N}}$ de la marche aléatoire sur (G, μ), et pour tout $x \in X$, la suite $(g_n.x)_{n\in\mathbb{N}}$ converge vers un point $\xi(\mathbf{g}) \in \partial X$.*

Dans le cas où X est localement fini, le Théorème 4.7 est dû à Kaimanovich [Kai00, Théorème 2.4]. La difficulté supplémentaire vient donc de l'absence de compacité locale pour le complexe des facteurs libres propres de F_N (et donc la non compacité de son bord de Gromov ∂FF_N). En particulier, l'existence d'une mesure μ-stationnaire sur ∂FF_N n'est pas garantie *a priori* (tandis qu'on peut toujours construire une mesure μ-stationnaire sur un G-espace compact en considérant un point d'accumulation faible-$*$ des moyennes de Cesàro de la suite des convolutions $(\mu^{*n} * \lambda)_{n\in\mathbb{N}}$, où λ est n'importe quelle mesure de probabilité sur X). Calegari et Maher contournent cette difficulté en construisant la mesure stationnaire dans l'espace compact obtenu en ajoutant à X l'ensemble des *horofonctions* sur X (voir la construction au Chapitre 5), puis en montrant que cette mesure se projette en une mesure sur le bord de Gromov de X. Notre stratégie consiste à construire une mesure stationnaire dans l'espace compact ∂CV_N. En montrant que celle-ci est concentrée sur le sous-espace formé des arbres libres et arationnels, nous pouvons alors projeter cette mesure en une mesure stationnaire sur $\mathcal{FI}$.

Éléments de démonstration du Théorème 4.4.

Première étape : Construction d'une mesure μ-stationnaire sur $\mathcal{FI}$. Soit μ une mesure de probabilité sur $\mathrm{Out}(F_N)$, dont le support engendre un sous-groupe non élémentaire de $\mathrm{Out}(F_N)$. Comme ∂CV_N est compact, il existe une mesure ν sur ∂CV_N qui est μ-stationnaire, que nous obtenons comme point d'accumulation faible-$*$ des moyennes de Cesàro de la suite de mesures $(\mu^{*n} * \delta_{x_0})_{n\in\mathbb{N}}$, où δ_{x_0} est une mesure de Dirac supportée en un point $x_0 \in \overline{CV_N}$. Il s'agit alors de voir que $\nu(\widetilde{\mathcal{FI}}) = 1$. L'argument, proposé dans l'encadré en page 79, sera aussi un ingrédient clé de notre démonstration de l'alternative d'Handel et Mosher pour les sous-groupes de $\mathrm{Out}(F_N)$ au Chapitre 6.

Plus précisément, nous montrons dans cet encadré que la première condition dans la définition des sous-groupes non élémentaires de $\mathrm{Out}(F_N)$ entraîne que ν est concentrée sur l'ensemble des arbres arationnels. Par un argument similaire, nous montrons que la troisième condition dans cette définition entraîne que ν est concentrée sur l'espace des actions libres. La deuxième condition entraîne quant à elle que ν est non atomique.

Deuxième étape : convergence au bord. Soit ν une mesure μ-stationnaire sur le compact ∂CV_N. Par un théorème de Furstenberg [Fur73], pour presque toute trajectoire $\mathbf{\Phi} := (\Phi_n)_{n\in\mathbb{N}}$ de la marche aléatoire sur $(\mathrm{Out}(F_N), \mu)$, les translatés $(\Phi_n)_*\nu$ convergent faiblement vers une mesure de probabilité $\lambda(\mathbf{\Phi})$ sur ∂CV_N, et nous avons

$$\nu = \int_{\mathcal{T}} \lambda(\mathbf{\Phi}) d\mathbb{P}(\mathbf{\Phi}),$$

où $\mathcal{T}$ désigne l'espace des trajectoires. L'argument de Furstenberg utilise de manière cruciale la compacité de ∂CV_N. Il consiste à remarquer que pour toute fonction f continue et bornée sur ∂CV_N, la suite $((\Phi_n)_*\nu)(f)$ est une martingale bornée, et admet donc presque sûrement une limite. En utilisant la séparabilité de l'espace des fonctions continues sur ∂CV_N, et l'identification du dual des fonctions continues sur ∂CV_N avec l'espace des mesures de probabilité sur ∂CV_N, nous déduisons la convergence faible des mesures $(\Phi_n)_*\nu$ vers une mesure $\lambda(\mathbf{\Phi})$.

La décomposition de ν donnée ci-dessus entraîne que pour presque toute trajectoire de la marche aléatoire, la mesure $\lambda(\mathbf{\Phi})$ est à support dans $\mathcal{FI}$. Il s'agit alors de voir que pour presque toute trajectoire $\mathbf{\Phi}$ de la marche aléatoire, le support de la mesure $\lambda(\mathbf{\Phi})$ est contenu dans un simplexe de mesures correspondant à un arbre arationnel libre, et que pour tout $T_0 \in CV_N$, tous les points d'accumulation de la suite $(\Phi_n.T_0)_{n\in\mathbb{N}}$ appartiennent à ce simplexe. Notre démonstration repose de manière cruciale sur la propriété d'unique dualité pour les courants duaux à un arbre arationnel libre : nous montrons que si T est un point d'accumulation arationnel et libre de la suite $(\Phi_n.T_0)_{n\in\mathbb{N}}$, dual à un courant η, alors tous les points limites de suites $(\Phi_n.T'_0)$ pour $T'_0 \in CV_N$ (et aussi pour T'_0 dans un ensemble de ν-mesure pleine de ∂CV_N) sont duaux à η, et font donc partie du simplexe de T. L'argument est détaillé en Partie 2 de l'Annexe C. Nous concluons alors en faisant appel à un résultat dû à Kaimanovich et Masur [KM96, Lemme 1.5.5].

Unicité de la mesure stationnaire sur $\mathcal{FI}$.

Soit μ une mesure de probabilité sur $\mathrm{Out}(F_N)$, dont le support engendre un sous-groupe non élémentaire de $\mathrm{Out}(F_N)$. La *mesure de sortie* sur $\mathcal{FI}$ est la mesure μ-stationnaire définie par

$$\nu(S) := \mathbb{P}(\xi(\mathbf{\Phi}) \in S)$$

Intermède (L'argument des mesures stationnaires).

Proposition 4.8. *Soit μ une loi de probabilité sur Out(F_N), dont le support engendre un sous-groupe de Out(F_N) qui ne fixe virtuellement la classe de conjugaison d'aucun facteur libre propre de Out(F_N). Alors toute mesure stationnaire sur ∂CV_N est concentrée sur le sous-espace $\widetilde{\mathcal{AT}}$ formé des arbres arationnels.*

Nous noterons $gr(\mu)$ le sous-groupe de $\mathrm{Out}(F_N)$ engendré par le support de la mesure μ. Notre démonstration de la Proposition 4.8 repose sur l'existence d'une partition dénombrable de ∂CV_N en
- le sous-ensemble $\mathcal{AT}$ formé des arbres arationnels, et
- chacun des sous-ensembles formés des arbres T dont l'ensemble (fini) des facteurs de réduction $\mathrm{Per}(T) \cup \mathrm{Dyn}(T)$ est donné (nous renvoyons à l'encadré en page 52 pour une définition des facteurs de réduction d'un arbre $T \in \partial CV_N$).

L'hypothèse faite sur le support de la mesure μ entraîne que la $\mathrm{gr}(\mu)$-orbite de chacun des ensembles du second type est une réunion infinie dénombrable d'ensembles de ce type. Si l'un de ces ensembles était de ν-mesure non nulle, il y aurait dans sa $gr(\mu)$-orbite un sous-ensemble de ν-mesure maximale, ce qui contredirait le principe du maximum du fait de la μ-stationnarité de ν. Plus précisément, nous faisons appel au résultat suivant.

Lemme 4.9. *(Ballmann [Bal89], Woess [Woe89, Lemme 3.4], Kaimanovich–Masur [KM96, Lemme 2.2.2]) Soit G un groupe dénombrable, soit μ une mesure de probabilité sur G, et soit ν une mesure de probabilité μ-stationnaire sur un G-espace X. Soit D un G-ensemble dénombrable, et $\Theta : X \to D$ une application mesurable G-équivariante. Si $E \subseteq X$ est un sous-ensemble mesurable G-invariant de X vérifiant $\nu(E) > 0$, alors $\Theta(E)$ contient une $gr(\mu)$-orbite finie.*

Démonstration. Soit $\widetilde{\nu} := \Theta_* \nu$, alors $\widetilde{\nu}$ est une mesure de probabilité μ-stationnaire sur D. Soit $M \subseteq \Theta(E)$ le sous-ensemble fini non vide formé des éléments $x \in \Theta(E)$ pour lesquels $\widetilde{\nu}(x)$ est maximal. Pour tout $x \in M$, nous avons

$$\widetilde{\nu}(x) = \sum_{g \in G} \mu(g) \widetilde{\nu}(g^{-1}x) \leq \widetilde{\nu}(x) \sum_{g \in G} \mu(g) = \widetilde{\nu}(x).$$

Ceci entraîne que pour tout $g \in gr(\mu)$, nous avons $\widetilde{\nu}(g^{-1}x) = \widetilde{\nu}(x)$. L'ensemble M est donc invariant par le semi-groupe engendré par $\check{\mu}$. Étant fini, il est $gr(\mu)$-invariant, et contient donc une $gr(\mu)$-orbite finie. ☐

Démonstration de la Proposition 4.8. Soit D l'ensemble dénombrable des familles finies de classes de conjugaison de facteurs libres propres de F_N. La Proposition 4.8 découle du Lemme 4.9 appliqué à la fonction $\Theta : \partial CV_N \to D$ définie par

$$\Theta(T) := \begin{cases} \emptyset & \text{si } T \in \widetilde{\mathcal{AT}} \\ \mathrm{Dyn}(T) \cup \mathrm{Per}(T) & \text{si } T \in \partial CV_N \smallsetminus \widetilde{\mathcal{AT}} \end{cases}.$$

En effet, le Théorème 3.7 montre que $\Theta(T) = \emptyset$ si et seulement si $T \in \widetilde{\mathcal{AT}}$, et l'hypothèse faite sur $\mathrm{gr}(\mu)$ entraîne que la seule $gr(\mu)$-orbite finie dans l'image de Θ est celle de l'ensemble vide. ☐

pour tout borélien $S \subseteq \mathcal{FI}$, où $\xi(\mathbf{\Phi}) \in \mathcal{FI}$ désigne la limite de la suite $(\Phi_n.x)_{n\in\mathbb{N}}$ pour n'importe quel $x \in CV_N$, qui est $\mathbb{P}$-presque sûrement bien définie grâce au Théorème 4.4. Nous montrons l'unicité de la mesure μ-stationnaire sur $\mathcal{FI}$ en Partie 2.3 de l'Annexe C.

Théorème 4.10. *Soit μ une mesure de probabilité sur Out(F_N), dont le support engendre un sous-groupe non élémentaire de Out(F_N). Alors la mesure de sortie est l'unique mesure μ-stationnaire sur $\mathcal{FI}$, et elle est non atomique.*

4.3 Le bord de Poisson de Out(F_N)

Sous un certain nombre d'hypothèses sur la mesure μ, nous identifions le bord de Poisson de $(\mathrm{Out}(F_N), \mu)$ avec l'espace mesuré $(\mathcal{FI}, \nu)$, où ν désigne la mesure de sortie sur $\mathcal{FI}$. Ceci fait l'objet de la Partie 3 de l'Annexe C.

Théorème 4.11.

Soit μ une loi de probabilité sur $\mathrm{Out}(F_N)$ dont le support engendre un sous-groupe non élémentaire de $\mathrm{Out}(F_N)$, de premier moment logarithmique fini pour la distance des mots, et d'entropie finie. Soit ν la mesure de sortie sur $\mathcal{FI}$. Alors $(\mathcal{FI}, \nu)$ est le bord de Poisson de $(\mathrm{Out}(F_N), \mu)$.

Grâce aux travaux de Bestvina et Reynolds [BR13] et Hamenstädt [Ham14a], le Théorème 4.11 montre que le bord de Gromov de FF_N est un modèle pour le bord de Poisson de $(\mathrm{Out}(F_N), \mu)$.

Nous présentons un schéma de notre preuve du Théorème 4.11. Le Théorème 4.5 montre que $(\mathcal{FI}, \nu)$ est une μ-frontière (la métrisabilité de $\mathcal{FI}$, et donc sa séparabilité, a été montrée par Bestvina et Reynolds [BR13, Corollaire 7.2]). Pour montrer le Théorème 4.11, il reste donc à en montrer la maximalité. Nous allons pour cela appliquer le critère des bandes de Kaimanovich (Théorème 4.2). Il s'agit donc d'associer à toute paire d'arbres libres et arationnels distincts une bande suffisamment fine dans $\mathrm{Out}(F_N)$, de manière mesurable et $\mathrm{Out}(F_N)$-équivariante. Notre construction de ces bandes s'inspire de la construction par Hamenstädt de *lignes de minima* [Ham14b]. Elle utilise de manière cruciale la finitude de la dimension du simplexe des courants duaux à un arbre dans $\overline{CV_N}$ muni d'une action libre de F_N, rappelée au Théorème 1.9, et la propriété d'unique dualité pour les courants duaux à des arbres arationnels, rappelée au Théorème 3.5.

Soit $L \geq 1$. Une paire de courants $(\eta, \eta') \in Curr_N^2$ est *positive* si pour tout $T \in \overline{cv_N}$, nous avons $\langle T, \eta\rangle + \langle T, \eta'\rangle > 0$. À toute paire positive de courants $(\eta, \eta') \in Curr_N^2$, nous associons un *$L$-axe* dans CV_N. Celui-ci est défini comme l'ensemble des arbres T pour lesquels le facteur de dilatation $\Lambda_{[\eta],[\eta']}(T, T')$ (introduit en Partie 1.2) donne une bonne estimation de la distorsion $\mathrm{Lip}(T, T')$ pour tout $T' \in \overline{CV_N}$ (et L mesure la qualité de cette estimation). Plus précisément, un arbre $T \in CV_N$ est dans le L-axe de la paire (η, η') si pour tout $T' \in \overline{CV_N}$, nous avons

$$1 \leq \frac{\mathrm{Lip}(T, T')}{\Lambda_{[\eta],[\eta']}(T, T')} \leq L.$$

Cette définition est modelée de sorte que le L-axe d'une paire de courants soit proche d'être une géodésique pour la distance symétrisée sur CV_N, à ceci près qu'il peut contenir des trous (si L est trop petit, le L-axe de la paire (η, η') peut même être vide). Cette propriété constitue le point clé pour vérifier la condition de croissance des bandes dans le critère de Kaimanovich.

Étant donné deux arbres $T, T' \in \mathcal{FI}$, nous définissons alors le *L-axe* de (T, T') comme la réunion (finie) des axes de (η, η'), pour (η, η') variant dans l'ensemble fini $\mathrm{Erg}(T) \times \mathrm{Erg}(T')$ (où $\mathrm{Erg}(T)$ désigne l'ensemble des courants ergodiques duaux à T). Ceci permet d'associer une *L-bande* dans $\mathrm{Out}(F_N)$ à toute paire (T, T') d'arbres arationnels libres : elle est définie comme l'ensemble des automorphismes $\Phi \in \mathrm{Out}(F_N)$ pour lesquels $\Phi *_{CV_N}$ appartient au L-axe de (T, T'), où $*_{CV_N}$ est un point base fixé une fois pour toutes. En utilisant l'ergodicité du décalage de Bernoulli dans l'espace des trajectoires bi-infinies de la marche aléatoire, nous montrons alors qu'il est possible de choisir L de manière uniforme, de sorte que les bandes soient presque sûrement non vides.

4.4 Autres propriétés de la marche aléatoire sur Out(F_N)

Nous terminons ce chapitre en présentant deux autres propriétés des marches aléatoires sur $\mathrm{Out}(F_N)$, concernant d'une part la vitesse de fuite de la marche aléatoire, et d'autre part les propriétés typiques de l'automorphisme obtenu au temps n de la marche lorsque n tend vers $+\infty$.

Vitesse de fuite.

Soit G un groupe dénombrable, soit d une distance (pas nécessairement symétrique) sur G, et soit μ une loi de probabilité sur G, de premier moment fini pour la distance d. Il découle du théorème ergodique sous-additif de Kingman [Kin68] que la limite

$$\lim_{n \to +\infty} \frac{1}{n} d(e, g_n)$$

existe pour presque toute trajectoire $(g_n)_{n \in \mathbb{N}}$ de la marche aléatoire sur (G, μ). Elle est appelée la *vitesse de fuite* de la marche aléatoire. Nous étudions maintenant la vitesse de fuite d'une marche aléatoire sur le groupe $\mathrm{Out}(F_N)$, réalisée sur CV_N ou sur FF_N. L'étude de la vitesse de fuite de la marche aléatoire sur un groupe d'isométries d'un espace hyperbolique au sens de Gromov est due à Maher [Mah12] (l'énoncé de Maher est donné dans le contexte de l'action du groupe modulaire $\mathrm{Mod}(S)$ d'une surface compacte orientable S, toutefois la démonstration s'adapte au cadre plus général d'un groupe d'isométries d'un espace hyperbolique).

Théorème 4.12. *(Maher [Mah12]) Soit G un groupe agissant par isométries sur un complexe simplicial X hyperbolique au sens de Gromov, et soit $x \in X$. Soit μ une mesure de probabilité à support fini sur G. Supposons que $\mathbb{P}$-presque toute trajectoire de la marche aléatoire sur (G, μ) converge vers un point de ∂X, et que la mesure de sortie soit non atomique. Alors il existe des constantes $L > 0$ et $c < 1$ telles que $\mathbb{P}(d(x, g_n x) \leq Ln) = O(c^n)$. En particulier, la marche aléatoire a une vitesse de fuite strictement positive.*

Il résulte alors du Théorème 4.5 que la marche aléatoire a une vitesse de fuite strictement positive pour la distance simpliciale sur FF_N. Soit $*_{FF_N}$ un point base dans FF_N.

Théorème 4.13. *(Calegari–Maher [CM12]) Soit μ une mesure de probabilité sur $\mathrm{Out}(F_N)$, dont le support est fini et engendre un sous-groupe non élémentaire de $\mathrm{Out}(F_N)$. Alors il existe des constantes $L > 0$ et $c < 1$ telles que $\mathbb{P}(d_{FF_N}(*_{FF_N}, \Phi_n *_{FF_N}) \leq Ln) = O(c^n)$. En particulier, la marche aléatoire a une vitesse de fuite strictement positive.*

Nous déduisons alors un résultat analogue pour la marche aléatoire réalisée sur CV_N. Soit $*_{CV_N}$ un point base dans CV_N.

Théorème 4.14. *Soit μ une mesure de probabilité sur Out(F_N), dont le support est fini et engendre un sous-groupe non élémentaire de Out(F_N). Alors il existe des constantes $L > 0$ et $c < 1$ telles que $\mathbb{P}(d_{CV_N}(*_{CV_N}, \Phi_n *_{CV_N}) \leq Ln) = O(c^n)$. En particulier, la marche aléatoire a une vitesse de fuite strictement positive.*

Le passage du Théorème 4.13 au Théorème 4.14 utilise l'estimation suivante, qui relie les distances sur CV_N et sur FF_N. Nous en donnons une démonstration en Partie 5.3 de l'Annexe B. Celle-ci repose sur une estimation reliant la distance asymétrique sur CV_N à une notion de nombres d'intersection sur CV_N [2, Proposition 2.8].

Proposition 4.15. *Il existe $K, L \in \mathbb{R}$ tels que pour tous $\Phi, \Psi \in$ Out(F_N), nous ayons*

$$d_{FF_N}(\Phi *_{FF_N}, \Psi *_{FF_N}) \leq K d_{CV_N}(\Phi *_{CV_N}, \Psi *_{CV_N}) + L.$$

Généricité des automorphismes complètement irréductibles.

Nous mentionnons enfin le résultat suivant, montré par Sisto [Sis13] et par Calegari et Maher [CM12] par des méthodes différentes.

Théorème 4.16. *(Sisto [Sis13], Calegari–Maher [CM12, Théorème 5.35]) Soit μ une mesure de probabilité sur Out(F_N), dont le support est fini et engendre un sous-groupe non élémentaire de Out(F_N). Alors la probabilité que l'élément obtenu au temps n de la marche aléatoire sur (Out(F_N), μ) soit un automorphisme complètement irréductible de Out(F_N) décroît exponentiellement en n.*

Remarquons que Calegari et Maher montrent en outre que la longueur de translation de l'automorphisme Φ obtenu au temps n de la marche aléatoire, réalisé comme isométrie de FF_N, est bornée inférieurement par une fonction linéaire de n, avec probabilité exponentiellement proche de 1 [CM12, Théorème 5.35].

Chapitre 5

L'horofrontière de l'outre-espace

Ce chapitre est consacré à la description d'un autre bord utile à l'étude des marches aléatoires sur $\mathrm{Out}(F_N)$, l'*horofrontière* de CV_N. Là encore, nous commencerons par une présentation générale de la construction de l'horofrontière d'un espace métrique (non nécessairement symétrique), et un aperçu des résultats connus à ce jour, avant de décrire l'horofrontière de CV_N. Nous expliquerons ensuite comment l'horofrontière d'un espace métrique intervient dans l'étude de la marche aléatoire sur son groupe d'isométries, et appliquerons ces principes à l'étude de la croissance des classes de conjugaisons d'éléments de F_N sous l'action de produits aléatoires d'automorphismes extérieurs de F_N. Mentionnons au passage d'autres applications possibles de l'étude de l'horofrontière d'un espace métrique, d'une part à l'étude de son groupe d'isométries [KN09, LW11, Wal11], d'autre part à la théorie de Patterson–Sullivan [BM96].

5.1 L'horofrontière d'un espace métrique

Définition.

Soit (X, d) un espace métrique, que nous ne supposons pas nécessairement symétrique. Sous un certain nombre d'hypothèses sur d, nous construisons une compactification de (X, d) au moyen d'*horofonctions*. La notion d'*horofrontière* d'un espace métrique a été introduite par Gromov [Gro80], le concept d'*horofonction* étant une généralisation de la notion de *fonction de Busemann* [Bus55]. Intuitivement, il s'agit d'étendre à l'infini la distance sur l'espace X. La généralisation au cas d'une distance asymétrique est traitée par Walsh dans [Wal11, Partie 2]. Nous suivons la présentation de Walsh de cette construction.

Fixons un point base $b \in X$. A tout point $z \in X$, nous associons une fonction continue

$$\begin{array}{rccc} \psi_z : & X & \to & \mathbb{R} \\ & x & \mapsto & d(x,z) - d(b,z) \end{array} .$$

Soit $\mathcal{C}(X)$ l'espace des fonctions continues à valeurs réelles sur X. Nous munissons $\mathcal{C}(X)$ de la topologie de la convergence uniforme sur les compacts de (X, d_{sym}), où d_{sym} est la distance symétrisée sur X définie par $d_{sym}(x, y) := d(x, y) + d(y, x)$. Nous avons donc une application

$$\begin{array}{rccc} \psi : & X & \to & \mathcal{C}(X) \\ & z & \mapsto & \psi_z \end{array} .$$

Cette application ψ est continue et injective. En effet, pour tous $z, z' \in X$ et tout $x \in X$, nous avons

$$|\psi_{z'}(x) - \psi_z(x)| = |d(x, z') - d(b, z') - d(x, z) + d(b, z)| \leq d_{sym}(z, z'),$$

ce qui montre la continuité de ψ. Par ailleurs, si $z \neq z'$, en supposant sans perte de généralité que $d(b, z) \geq d(b, z')$, nous obtenons

$$\psi_{z'}(z) - \psi_z(z) = d(z, z') - d(b, z') + d(b, z) > 0,$$

donc $\psi_{z'} \neq \psi_z$. Ceci montre l'injectivité de ψ. La proposition suivante donne des conditions sous lesquelles l'application ψ est en fait un homéomorphisme sur son image, et permet de définir une compactification de X. Un espace métrique est *propre* si les boules fermées sont compactes.

Proposition 5.1. *(Ballmann [Bal95, Chapitre II.1], Walsh [Wal11, Proposition 2.2]) Soit (X, d) un espace métrique, non nécessairement symétrique. Supposons que*

- *l'espace (X, d) est géodésique, et*
- *l'espace (X, d_{sym}) est propre, et*
- *pour tout $x \in X$ et toute suite $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$, la distance $d(x_n, x)$ tend vers 0 si et seulement si $d(x, x_n)$ tend vers 0.*

Alors ψ est un homéomorphisme de X sur son image, et l'adhérence de $\psi(X)$ dans $\mathcal{C}(X)$ est compacte.

L'espace $\overline{\psi(X)}$ est appelé la *compactification par horofonctions* de (X, d), les éléments de $X(\infty) := \overline{\psi(X)} \smallsetminus \psi(X)$ sont des *horofonctions.*

Démonstration. (Walsh) Pour tous $x, y, z \in X$, nous avons

$$|\psi_z(x) - \psi_z(y)| = |d(x, z) - d(b, z) - d(y, z) + d(b, z)| \leq 2d_{sym}(x, y).$$

Par conséquent, toute fonction dans $\overline{\psi(X)}$ est 1-lipschitzienne pour la distance d_{sym}. Par ailleurs, nous avons $|\psi_z(x)| \leq d_{sym}(x, b)$. Comme d_{sym} est propre, la compacité de $\overline{\psi(X)}$ découle du théorème d'Arzelà–Ascoli.

Pour vérifier que ψ est un homéomorphisme de X sur son image, nous montrons que si $(z_n)_{n \in \mathbb{N}}$ est une suite d'éléments de X qui sort de tout compact de X, alors tout point limite $\xi \in \mathcal{C}(X)$ de la suite $(\psi_{z_n})_{n \in \mathbb{N}}$ est contenu dans $\overline{\psi(X)} \smallsetminus \psi(X)$.

Soit $y \in X$. Nous allons montrer que $\xi \neq \psi_y$. Comme d_{sym} est propre, la distance $d_{sym}(y, z_n)$ tend vers $+\infty$. Soit $r > d(b, y) + \xi(y)$. Pour tout $n \in \mathbb{N}$, choisissons un point x_n sur une d-géodésique de y à z_n tel que $d_{sym}(y, x_n) = r$. Comme d_{sym} est propre, quitte à extraire, nous pouvons supposer que $(x_n)_{n \in \mathbb{N}}$ converge vers un point $x \in X$.

Pour tout $n \in \mathbb{N}$, nous avons $\psi_{z_n}(x_n) = \psi_{z_n}(y) - d(y, x_n)$. Comme chacune des applications ψ_{z_n} est 1-lipschitzienne, nous pouvons passer à la limite pour obtenir $\xi(x) = \xi(y) - d(y, x)$. Par ailleurs, nous avons $\psi_y(x) = d(x, y) - d(b, y)$, et donc $\psi_y(x) - \xi(x) = d_{sym}(x, y) - \xi(y) - d(b, y) > 0$. Ceci montre que $\xi \neq \psi_y$. □

Soit G un groupe d'isométries d'un espace métrique (X, d) (non nécessairement symétrique) satisfaisant les hypothèses de la Proposition 5.1. Alors l'action de G s'étend en une action par homéomorphismes sur $X(\infty)$, donnée par

$$g.\xi(x) = \xi(g^{-1}x) - \xi(g^{-1}b)$$

pour tout $g \in G$, toute horofonction $\xi \in X(\infty)$ et tout $x \in X$ [Wal11, Proposition 2.4].

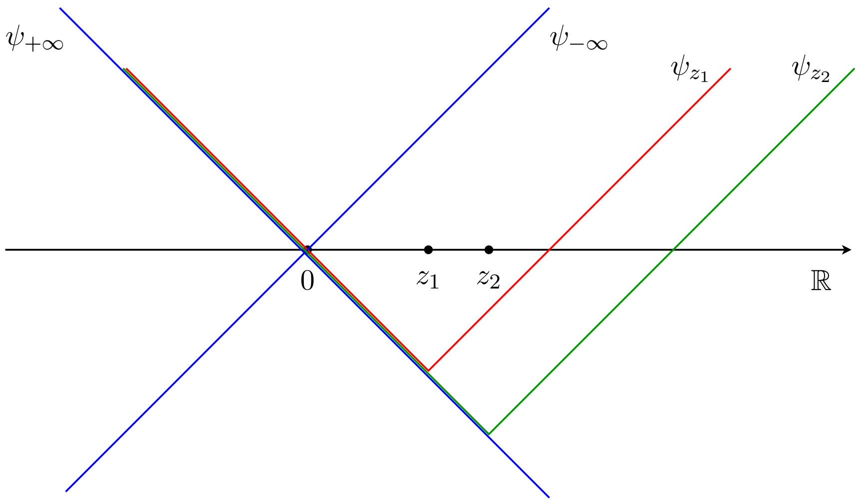


FIGURE 5.1 – La compactification par horofonctions de $(\mathbb{R}, |.|)$.

Quelques exemples simples de compactifications par horofonctions.

Afin d'illustrer la notion ci-dessus, nous donnons quelques exemples simples d'horofrontières d'espaces métriques usuels.

Exemple 1 (voir la Figure 5.1) : Soit $X = (\mathbb{R}, |.|)$ la droite réelle, munie de la distance usuelle (nous choisissons 0 comme point base). Pour tout $z \in X$, la fonction ψ_z (représentée en Figure 5.1) est donnée par

$$\psi_z(x) = |x - z| - |z|$$

pour tout $x \in X$. Ainsi, la fonction ψ_z converge lorsque z tend vers $+\infty$ vers la fonction $\psi_{+\infty} : x \mapsto -x$, tandis qu'elle converge vers $\psi_{-\infty} : x \mapsto x$ lorsque z tend vers $-\infty$. La compactification par horofonctions de X est donc isomorphe à la compactification à deux points de la droite réelle.

Exemple 2 (voir la Figure 5.2) : Soit $X = (\mathbb{R}^2, ||.||_2)$ le plan muni de la norme euclidienne (nous choisissons l'origine $(0, 0)$ comme point base). Alors pour tout $z \in X$, la fonction ψ_z est la fonction dont les lignes de niveau sont les cercles centrés en z (en particulier, la ligne de niveau 0 de ψ_z, représentée sur la Figure 5.2, est le cercle centré en z et passant par 0). Soit $(z_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ une suite d'éléments de X. Quitte à extraire, nous pouvons supposer que soit

- la suite $(z_n)_{n\in\mathbb{N}}$ converge vers un point $z \in X$, soit
- la suite $(z_n)_{n\in\mathbb{N}}$ sort de tout compact, et l'angle θ_n entre l'axe des abscisses et la demi-droite $[Oz_n)$ converge vers un angle $\theta \in [0, 2\pi)$.

Dans le second cas, en notant v_θ le vecteur unitaire sur la demi-droite formant un angle θ avec la demi-droite horizontale, nous vérifions que la fonction ψ_{z_n} converge lorsque n tend vers $+\infty$ vers la fonction $\psi_\theta = \langle ., v_\theta\rangle$, dont les lignes de niveau sont les droites orthogonales au vecteur v_θ. Ainsi, la compactification de X par horofonctions est homéomorphe au disque, avec une horofonction au bout de chaque direction angulaire.

Exemple 3 (voir la Figure 5.3) : Soit $X = (\mathbb{R}^2, ||.||_1)$ (nous choisissons de nouveau l'origine $(0, 0)$ comme point base). Nous représentons en Figure 5.3 la ligne de niveau 0 d'une

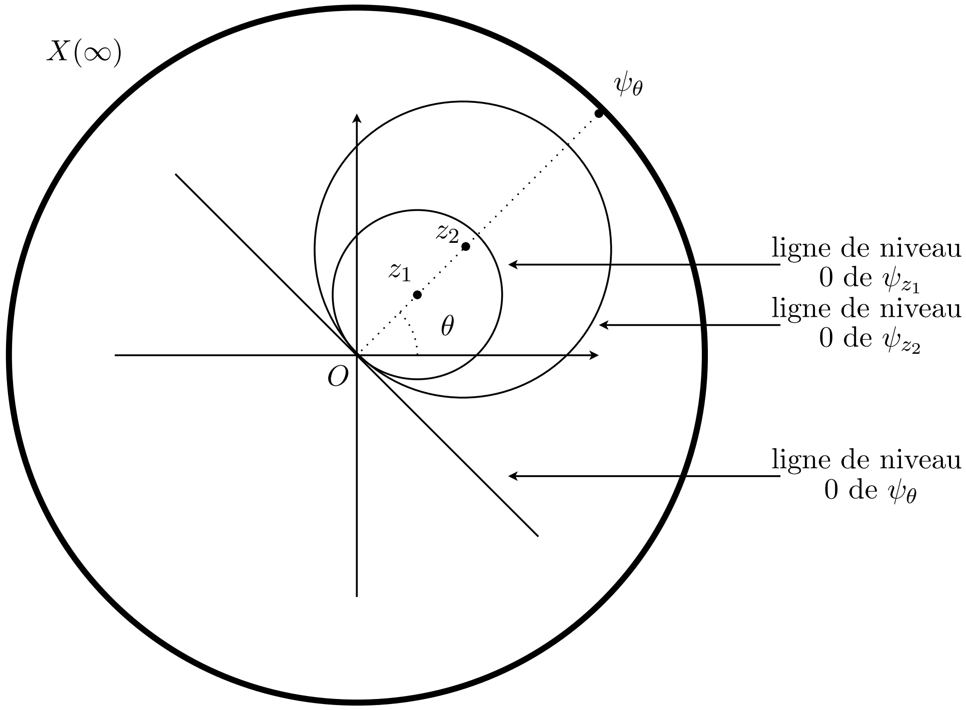


FIGURE 5.2 – La compactification par horofonctions de $(\mathbb{R}^2, ||.||_2)$.

fonction ψ_z avec $z \in X$, qui est un carré centré en z. Nous voyons alors qu'il y a une horofonction limite au bout de chaque demi-droite horizontale et une horofonction limite au bout de chaque demi-droite verticale. À celles-ci s'ajoutent quatre autres horofonctions, vers lesquelles une suite s'accumule lorsque ses deux coordonnées divergent. Ainsi, la compactification par horofonctions de $(\mathbb{R}^2, ||.||_1)$ est elle aussi homéomorphe au disque, mais elle n'est pas isomorphe (en tant que compactification) à la compactification par horofonctions de $(\mathbb{R}^2, ||.||_2)$ décrite ci-dessus. Cet exemple illustre le fait que la compactification par horofonctions dépend vraiment de la distance sur X, et pas seulement de la topologie sur X (et pas seulement non plus de la classe de quasi-isométrie de la distance).

Exemple 4 (voir la Figure 5.4) : La compactification de l'espace hyperbolique $\mathbb{H}^2$ par horofonctions est isomorphe à la compactification de Gromov, qui est homéomorphe au disque. Les lignes de niveau des horofonctions sont les *horosphères* représentées en Figure 5.4.

Exemple 5 (voir la Figure 5.5) : Soit X l'échelle bi-infinie représentée en Figure 5.5, munie de la distance géodésique qui rend chaque arête du graphe X isométrique au segment $[0, 1]$. La compactification par horofonctions de X est obtenue en ajoutant un segment vertical de longueur 1 à chaque bout de l'échelle. Remarquons que dans ce cas, l'espace X est hyperbolique au sens de Gromov, et son bord de Gromov (qui consiste en un point à chacun des deux bouts de l'échelle) est un quotient strict de son horofrontière. En général, le bord de Gromov d'un espace hyperbolique (au sens de Gromov) géodésique et propre est toujours un quotient de son horofrontière, voir [CP01, Théorème 3.10] ou [WW05, Théorème 4.5] (lorsque l'espace X considéré n'est pas propre, on peut définir de même l'horofrontière de X, qui n'est plus nécessairement compacte ; le bord de Gromov de X

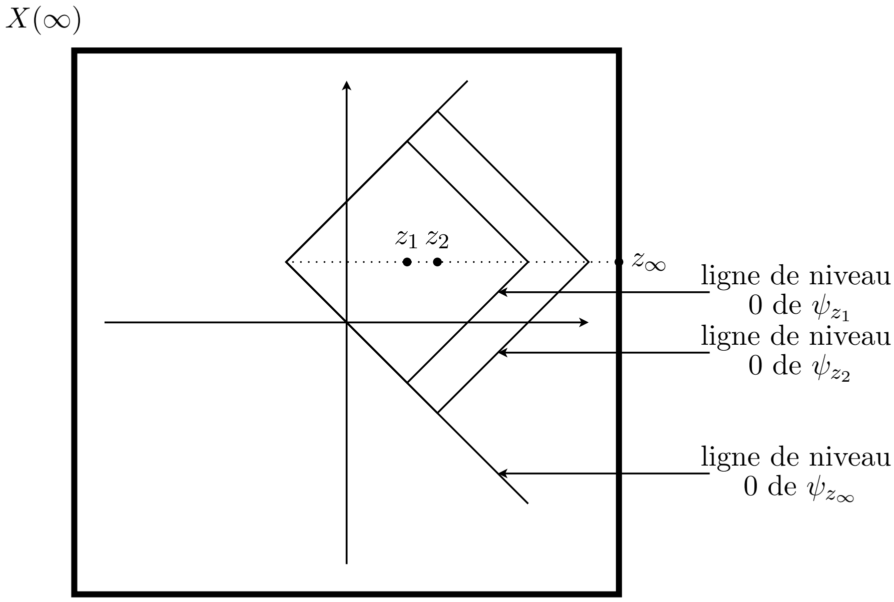


FIGURE 5.3 – La compactification par horofonctions de $(\mathbb{R}^2, ||.||_1)$.

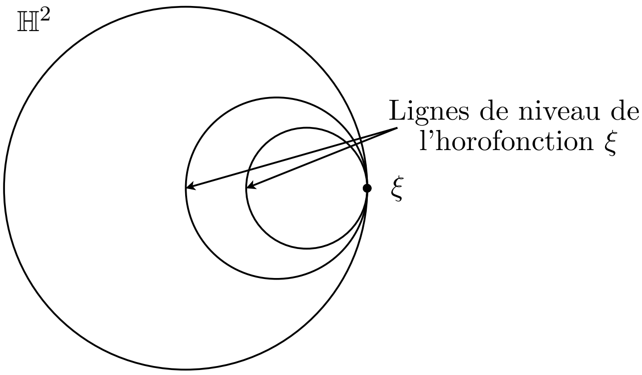


FIGURE 5.4 – La compactification par horofonctions de l'espace hyperbolique $\mathbb{H}^2$.

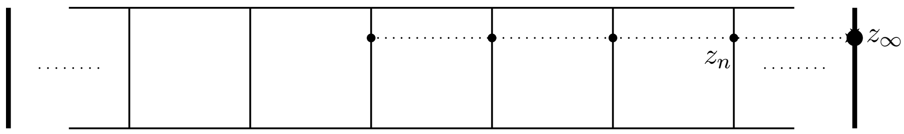


FIGURE 5.5 – La compactification par horofonctions de l'échelle bi-infinie.

s'exprime alors comme quotient d'un sous-ensemble de l'horofrontière, voir [MT14, Partie 3]).

Points de Busemann. Soit (X, d) un espace métrique (qui peut être asymétrique), qui satisfait les hypothèses de la Proposition 5.1, de sorte que la compactification de X par horofonctions est bien définie. Suivant la terminologie de Rieffel [Rie02], nous dirons qu'un chemin $\gamma : \mathbb{R}_+ \to X$ est un *rayon presque géodésique* si pour tout $\epsilon > 0$, il existe $t_0 \in \mathbb{R}_+$ tel que pour tous $s, t \geq t_0$, nous ayons $|d(\gamma(0), \gamma(s)) + d(\gamma(s), \gamma(t)) - t| \leq \epsilon$. Rieffel a montré que tout rayon presque géodésique dans X converge vers un point de $X(\infty)$. Une horofonction $\xi \in X(\infty)$ est un *point de Busemann* s'il existe un rayon presque géodésique qui converge vers ξ. Dans les quatre premiers exemples ci-dessus, toutes les horofonctions sont des points de Busemann. Le cinquième exemple fournit un exemple d'espace métrique pour lequel il existe des horofonctions qui ne sont pas des points de Busemann.

Exemples connus d'horofrontières.

Il y assez peu d'espaces métriques dont les compactifications par horofonctions sont connues explicitement. Nous mentionnons quelques résultats connus à ce jour.

- L'horofrontière d'un espace CAT(0) coincide avec le bord visuel, défini en termes de classes d'équivalences de rayons géodésiques [BH99, Théorème 8.13].
- Lorsque S est une surface hyperbolique compacte orientable, Walsh a identifié l'horocompactification de $Teich(S)$ pour la distance asymétrique de Thurston à la compactification de Thurston $\mathcal{PMF}$ [Wal11]. Tous les points de $\mathcal{PMF}$ sont des points de Busemann [Wal11, Théorème 4.1], obtenus comme limites de *lignes de stretch*, qui sont des géodésiques pour la distance de Thurston.
- Liu et Su ont identifié l'horocompactification de $Teich(S)$ pour la distance de Teichmüller avec la compactification de Gardiner–Masur [LS12]. Nous renvoyons également à [Wal12] pour une autre démonstration de leur résultat et une étude de la géométrie de cette compactification. Certains points de l'horofrontière ne sont pas des points de Busemann [Miy14].
- Klein et Nicas ont déterminé l'horofrontière du groupe de Heisenberg pour la distance de Carnot et Carathéodory [KN09].
- Il est souvent plus facile d'identifier les points de Busemann d'un espace métrique, plutôt que de décrire l'ensemble de son horofrontière. Webster et Winchester ont donné un critère sous lequel toutes les horofonctions d'un groupe de type fini sont des points de Busemann [WW06]. Mentionnons les travaux de Walsh portant sur l'horofrontière des espaces vectoriels normés de dimension finie [Wal07], des groupes d'Artin diédraux [Wal09], de la géométrie de Hilbert [Wal08]. Dans chacun de ces cas, Walsh a déterminé l'ensemble des points de Busemann, et donné une condition nécessaire et suffisante sous laquelle tous les points de l'horofrontière sont des points de Busemann.

- Blachère, Haïssinsky et Mathieu ont montré que l'horofrontière d'un groupe muni de la distance de Green associée à une marche aléatoire est homéomorphe à la frontière de Martin ; lorsque la distance de Green est hyperbolique au sens de Gromov, l'horofrontière est aussi le bord de Gromov associé à cette distance [BHM08]

5.2 L'horofrontière de l'outre-espace

Nous renvoyons au Chapitre 2 pour une définition et une description de la *compactification primitive* $\overline{CV_N}^{prim}$ de l'outre-espace. Le théorème suivant fait l'objet de la Partie 3 de l'Annexe B.

Théorème 5.2.

Il existe un unique homéomorphisme $\mathrm{Out}(F_N)$-équivariant de $\overline{CV_N}^{prim}$ vers l'horocompactification de CV_N égal à l'identité en restriction à CV_N. Pour tout $z \in \overline{CV_N}^{prim}$, l'horofonction associée à z est donnée par

$$\psi_z(x) = \log \sup_{g \in \mathcal{P}_N} \frac{||g||_z}{||g||_x} - \log \sup_{g \in \mathcal{P}_N} \frac{||g||_z}{||g||_b} \tag{5.1}$$

pour tout $x \in CV_N$ (identifié avec son représentant de covolume 1).

Remarquons que les suprema qui interviennent dans l'expression des horofonctions peuvent être pris sur l'ensemble fini des éléments de $\mathcal{P}_N$ qui sont des candidats dans x et dans b. Il est également possible de remplacer $z \in \overline{CV_N}^{prim}$ par l'un quelconque de ses représentants dans $\overline{cv_N}$, et de prendre les suprema sur F_N. L'horofonction ψ_z peut alors également s'écrire comme

$$\psi_z(x) = \log \mathrm{Lip}(x, z) - \log \mathrm{Lip}(b, z).$$

Idée de la démonstration. L'unicité vient de la densité de CV_N dans $\overline{CV_N}^{prim}$, et l'équivariance est immédiate. Pour tout $z \in \overline{CV_N}^{prim}$ et tout $x \in CV_N$, posons

$$\psi'_z(x) = \log \sup_{g \in \mathcal{P}_N} \frac{||g||_z}{||g||_x} - \log \sup_{g \in \mathcal{P}_N} \frac{||g||_z}{||g||_b}$$

(si bien que nous avons $\psi_z = \psi'_z$ pour tout $z \in CV_N$). Montrons que pour tout $z \in \overline{CV_N}^{prim}$, la fonction ψ'_z est continue sur CV_N. Soit $z \in \overline{CV_N}^{prim}$, et soit $(z_n)_{n\in\mathbb{N}} \in CV_N^{\mathbb{N}}$ une suite d'éléments de CV_N qui converge vers z. Nous fixons un représentant dans les classes d'homothétie de z et de chacun des arbres z_n. Pour tout $n \in \mathbb{N}$, nous avons

$$\psi'_z(x) = \log \sup_{g \in \mathcal{F}(x)} \frac{||g||_z}{||g||_x} - \log \sup_{g \in \mathcal{F}(b)} \frac{||g||_z}{||g||_b},$$

où $\mathcal{F}(x)$ (resp. $\mathcal{F}(b)$) est l'ensemble fini des candidats dans x (resp. dans b). Par définition de la topologie sur $\mathbb{PR}^{\mathcal{P}_N}$, il existe une suite $(\lambda_n)_{n\in\mathbb{N}}$ de réels tels que pour tout $g \in \mathcal{P}_N$, la suite $(\lambda_n ||g||_{z_n})_{n\in\mathbb{N}}$ converge vers $||g||_z$, si bien que $\psi'_{z_n}(x)$ converge vers $\psi'_z(x)$. L'application ψ'_z est donc la limite ponctuelle des applications 1-lipschitziennes ψ'_{z_n}, elle est donc continue.

La fonction $\psi : CV_N \to \mathcal{C}(CV_N)$ s'étend donc en une fonction $\psi : \overline{CV_N}^{prim} \to \mathcal{C}(CV_N)$. La continuité de cette extension s'obtient par un argument similaire à celui présenté ci-dessus pour montrer la continuité de chacune des applications ψ'_z.

Nous montrons maintenant l'injectivité de $\psi : \overline{CV_N}^{prim} \to \mathcal{C}(CV_N)$. Soit $z, z' \in \overline{CV_N}^{prim}$ tels que $\psi_z = \psi_{z'}$. Nous cherchons à montrer que le rapport $\frac{||g||_{z'}}{||g||_z}$ est constant sur $\mathcal{P}_N$. Soit $g \in \mathcal{P}_N$, et soit $x \in CV_N$ une rose dont l'un des pétales représente g. Pour tout $\epsilon > 0$, soit x_ϵ la rose dans CV_N obtenue à partir de x en attribuant au pétale représentant g la longueur ϵ, tandis que tous les autres pétales sont de même longueur.

Si $||g||_z \neq 0$, alors pour $\epsilon > 0$ suffisamment petit, nous avons $\psi_z(x_\epsilon) = \log \frac{||g||_z}{\epsilon C(z)}$, où $C(z) := \sup_{g \in \mathcal{P}_N} \frac{||g||_z}{||g||_b}$. En particulier $\psi_z(x_\epsilon)$ tend vers $+\infty$ lorsque ϵ tend vers 0. À l'inverse, si $||g||_z = 0$, alors $\psi_z(x_\epsilon)$ est borné, indépendamment de $\epsilon > 0$. Comme $\psi_z = \psi_{z'}$, ceci montre que $||g||_{z'} = 0$ si et seulement si $||g||_z = 0$, et dans le cas contraire le rapport $\frac{||g||_{z'}}{||g||_z} = \frac{C(z)}{C(z')}$ ne dépend pas de $g \in \mathcal{P}_N$. Donc $z = z'$.

L'application $\psi : \overline{CV_N}^{prim} \to \mathcal{C}(CV_N)$ est donc une injection continue, et comme $\overline{CV_N}^{prim}$ est compact, c'est un homéomorphisme sur son image (en particulier, cette image est fermée dans $\mathcal{C}(CV_N)$). La continuité de ψ montre que $\psi(CV_N) \subseteq \psi(\overline{CV_N}^{prim}) \subseteq \overline{\psi(CV_N)}$, ce qui montre le résultat souhaité. ☐

Nous déterminons également les points de Busemann de la compactification de CV_N par horofonctions (Partie 3.5 de l'Annexe B).

Théorème 5.3. *Pour tout $T \in \overline{CV_N}^{prim}$, l'horofonction ψ_T est un point de Busemann si et seulement si T est à orbites denses.*

5.3 L'horofrontière inverse de l'outre-espace

Comme la distance sur CV_N est asymétrique, il est naturel de se poser la question de la description de l'horofrontière correspondant la distance d^{back} donnée par $d^{back}(T, T') = d(T', T)$. Nous notons $\overline{CV_N}^{back}$ la compactification de CV_N par horofonctions correspondante. Étant donné un sous-ensemble $S \subset M_N$ de courants dans l'ensemble minimal M_N, nous définissons une application

$$\begin{array}{cccc} f_S : & CV_N & \to & \mathbb{R} \\ & T & \mapsto & f_S(T) = \log \sup_{\eta \in S} \langle T, \eta \rangle - \log \sup_{\eta \in S} \langle b, \eta \rangle \end{array},$$

où nous identifions les arbres b et T à leurs représentants de covolume 1.

Théorème 5.4. *Pour tout $\xi \in \overline{CV_N}^{back}$, il existe un sous-ensemble $S \subseteq M_N$ tel que $\xi = f_S$.*

Le Théorème 5.4 est démontré en Partie 4 de l'Annexe B, qui est consacrée à une étude de quelques propriétés de l'horofrontière inverse de CV_N. En particulier, nous y montrons que la dimension topologique de $\overline{CV_N}^{back}$ est infinie dès que $N \geq 3$. Soit $\sim$ la relation d'équivalence sur M_N qui identifie deux courants μ et μ' si $\langle T, \mu \rangle = \langle T, \mu' \rangle$ pour tout $T \in CV_N$ (cette relation a été étudiée dans [KLSS07]). Nous montrons l'existence d'un plongement topologique de l'espace $\mathbb{P}M_N/\sim$, qui est de dimension topologique infinie, dans $\overline{CV_N}^{back}$. Il découle au contraire des travaux de Gaboriau et Levitt [GL95] et de notre description de l'horofrontière de CV_N (Théorèmes 2.5 et 5.2) que $\overline{CV_N}^{prim}$ est de dimension topologique finie, égale à $3N-4$ (le bord étant de dimension égale à $3N-5$). Ainsi, les deux compactifications obtenues sont de nature topologique différente. Il semble en fait qu'il y ait une sorte de dualité entre ces deux compactifications. Ainsi, si $(T_n)_{n\in\mathbb{N}} \in CV_N^{\mathbb{N}}$ est une suite qui converge vers $T \in \overline{CV_N}$, alors tout point d'accumulation de la suite $(\psi_{T_n}^{back})_{n\in\mathbb{N}}$ est de la forme f_S avec $S \subseteq \mathrm{Dual}(T)$. Toutefois, le problème de la description des points de $\overline{CV_N}^{back}$ reste ouvert (en particulier, quels sont les ensembles $S \subseteq M_N$ pour lesquels f_S est une horofonction ?). Reiner Martin a aussi construit une compactification de CV_N de dimension infinie en termes de courants [Mar95]. Les deux compactifications obtenues sont-elles isomorphes ? Notons que ce problème de la description de l'horofrontière inverse est aussi ouvert dans le cadre des espaces de Teichmüller, munis de la distance asymétrique de Thurston.

Afin d'illustrer la différence entre $\overline{CV_N}^{prim}$ et $\overline{CV_N}^{back}$, nous donnons des exemples de deux suites d'éléments de CV_N convergeant vers le même point de l'une de ces compactifications, mais vers des points différents de l'autre compactification.

Exemple 5.5. Nous donnons un exemple de deux suites $(T_n)_{n\in\mathbb{N}}, (T'_n)_{n\in\mathbb{N}} \in CV_3^{\mathbb{N}}$ qui convergent vers le même point dans $\overline{CV_3}^{prim}$, mais vers des points distincts dans $\overline{CV_3}^{back}$. Pour tout $n \in \mathbb{N}$, soit T_n (resp. T'_n) la rose associée à une base $\{a, b, c\}$ de F_3, ayant des pétales de longueurs 1, $\frac{1}{n}$ et $\frac{1}{n^2}$ (resp. 1, $\frac{1}{n^2}$ et $\frac{1}{n}$). Alors les suites $(T_n)_{n\in\mathbb{N}}$ et $(T'_n)_{n\in\mathbb{N}}$ convergent toutes deux dans $\overline{CV_3}^{prim}$ vers la rose dans laquelle les pétales étiquetés par b et c ont été écrasés, mais $(T_n)_{n\in\mathbb{N}}$ (resp. $(T'_n)_{n\in\mathbb{N}}$) converge vers $f_{\{[c]\}}$ (resp. $f_{\{[b]\}}$) dans $\overline{CV_3}^{back}$.

Exemple 5.6. Nous donnons maintenant un exemple de deux suites $(T_n)_{n\in\mathbb{N}}, (T'_n)_{n\in\mathbb{N}} \in CV_3^{\mathbb{N}}$ qui convergent vers le même point dans $\overline{CV_3}^{back}$, mais vers des points distincts dans $\overline{CV_3}^{prim}$. Pour tout $n \in \mathbb{N}$, soit T_n (resp. T'_n) la rose associée à une base $\{a, b, c\}$ de F_3, ayant des pétales de longueurs 1, 2 et $\frac{1}{n}$ (resp. 2, 1 et $\frac{1}{n}$). Alors les suites $(T_n)_{n\in\mathbb{N}}$ et $(T'_n)_{n\in\mathbb{N}}$ convergent toutes deux vers $f_{\{[c]\}}$ dans $\overline{CV_3}^{back}$, mais elles convergent vers des arbres distincts dans $\overline{CV_3}^{prim}$.

Description de l'horofrontière inverse de CV_2. De même que nous avons donné une description explicite de l'horofrontière de CV_2 en Partie 2.2, nous donnons maintenant une description explicite de l'horofrontière inverse de CV_2. La compactification $\overline{CV_2}^{back}$ est représentée en Figure 5.6. Elle est de dimension topologique égale à 2. Elle n'est pas isomorphe à la compactification $\overline{CV_2}^{prim}$, toutefois les *parties réduites* de ces compactifications (obtenues en ne considérant pas les graphes en haltères dans l'outre-espace) sont isomorphes. Nous renvoyons à la Partie 4.3 de l'Annexe B pour des arguments plus détaillés.

Soit $(T_n)_{n\in\mathbb{N}} \in CV_2^{\mathbb{N}}$ une suite d'arbres dans CV_2 qui converge vers une horofonction ξ dans $CV_2^{back}(\infty)$. Quitte à extraire, nous pouvons supposer que la suite $(T_n)_{n\in\mathbb{N}}$ converge également vers un arbre T dans $\overline{CV_2}$.

Si l'arbre T est simplicial, et si le graphe quotient T/F_2 a l'une des quatre premières formes représentées sur la deuxième ligne de la Figure 2.3, alors $\xi = f_{\{[a]\}}$. En particulier, les simplexes de $\overline{CV_2}$ écrasés dans $\overline{CV_2}^{prim}$ le sont aussi dans $\overline{CV_2}^{back}$. Par ailleurs, les simplexes associés à des graphes en haltères semi-dégénérés (correspondant à la quatrième forme présentée en Figure 2.3) sont aussi écrasés dans $\overline{CV_2}^{back}$ (alors qu'ils ne le sont pas dans $\overline{CV_2}^{prim}$).

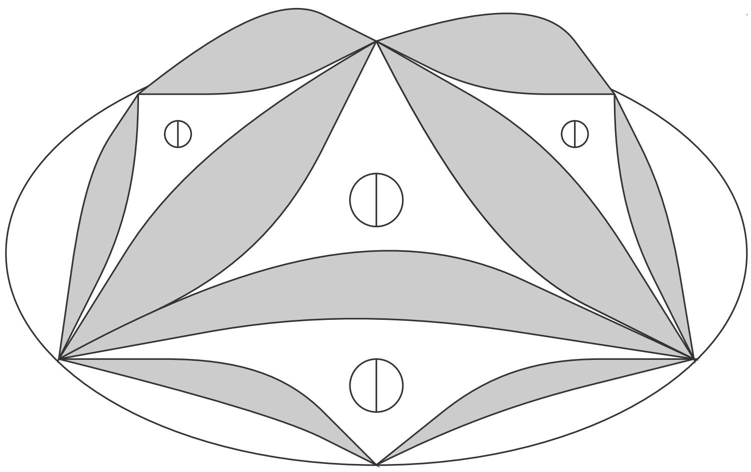


FIGURE 5.6 – La compactification $\overline{CV_2}^{back}$.

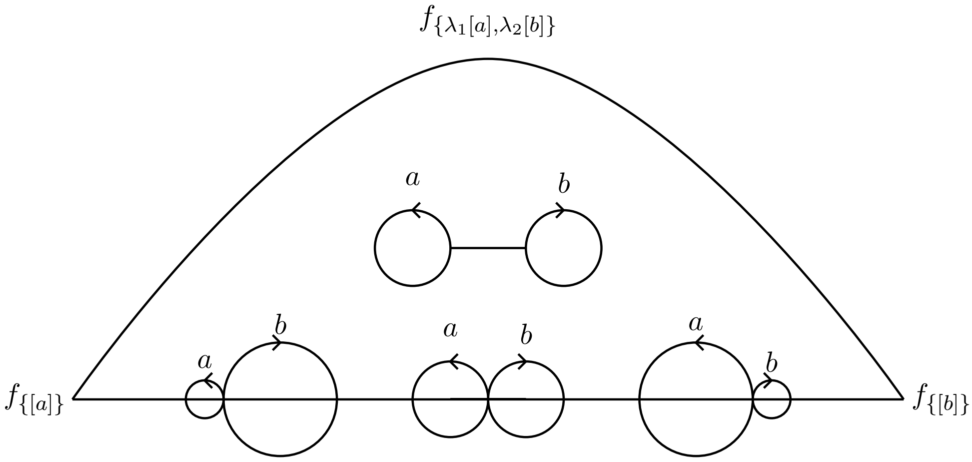


FIGURE 5.7 – L'adhérence du simplexe d'un graphe en haltères dans $\overline{CV_2}^{back}$.

Si l'arbre T est un arbre de Bass–Serre associé au scindement libre $F_2 = \langle a\rangle * \langle b\rangle$, alors ξ est de la forme $f_{\{\lambda_1[a],\lambda_2[b]\}}$, avec $\lambda_1 + \lambda_2 = 1$. Toutes ces fonctions s'obtiennent comme limites de suites $(T_n)_{n\in\mathbb{N}} \in CV_2^{\mathbb{N}}$, en définissant T_n comme le graphe en haltères dont le lacet étiqueté par a (resp. par b) a pour longueur $\frac{1}{\lambda_1 n}$ (resp. $\frac{1}{\lambda_2 n}$). Ainsi, le point de $\overline{CV_2}$ correspondant au scindement $F_2 = \langle a\rangle * \langle b\rangle$ est éclaté en un simplexe de dimension 1 dans $\overline{CV_2}^{back}$. Nous représentons sur la Figure 5.7 l'adhérence du simplexe d'un graphe en haltères dans $\overline{CV_2}^{back}$.

Enfin, lorsque l'arbre T est dual à une lamination minimale arationnelle sur un tore ayant une composante de bord, il existe un unique courant $\eta \in M_2$ dual à T, et ce courant η n'est dual à aucun autre arbre $T' \in \overline{CV_2}$. Nous avons alors $\xi = f_{\{\eta\}}$.

5.4 Horofrontières et marches aléatoires sur les groupes

Soit (X, d) un espace métrique (possiblement asymétrique). Nous supposerons que (X, d) satisfait les hypothèses de la Proposition 5.1, de sorte à pouvoir définir la compactification de X par horofonctions. Soit G un groupe localement compact agissant sur X par isométries. L'étude de l'horofrontière de l'espace métrique (X, d) a des applications à l'étude des marches aléatoires sur le groupe G. Typiquement, une trajectoire de la marche aléatoire sur (G, μ) est dirigée asymptotiquement par une horofonction (aléatoire). C'est l'objet du Théorème 5.7, dû à Karlsson et Ledrappier, qui s'énonce dans le cadre plus général de *cocycles intégrables*. Notons que l'extension du théorème de Karlsson et Ledrappier au cas d'une distance asymétrique est due à Karlsson [Kar14]. Nous rappelons au préalable la notion de cocycle intégrable.

Soit (Ω, θ) un espace de probabilité lebesguien standard, et soit $T : \Omega \to \Omega$ une transformation ergodique de Ω, qui préserve la mesure θ. Soit G un groupe dénombrable, et $g : \Omega \to G$ une application mesurable. Pour tout $n \in \mathbb{N}$ et tout $\omega \in \Omega$, nous poserons

$$g_n(\omega) := g(\omega) \dots g(T^{n-1}\omega),$$

et nous dirons que $(g_n)_{n\in\mathbb{N}}$ est un *cocycle ergodique*. Soit d une distance (que nous ne supposons pas symétrique *a priori*) invariante à gauche sur G. Le cocycle $(g_n)_{n\in\mathbb{N}}$ est *intégrable* pour d si

$$\int_\Omega d_{sym}(e, g(\omega))d\theta(\omega) < +\infty.$$

Dans le cas où μ est une loi de probabilité sur G, où (Ω, θ) est l'espace de probabilité produit $(G^{\mathbb{N}^*}, \mu^{\otimes\mathbb{N}^*})$, et où

$$T : \begin{array}{ccc} \Omega & \to & \Omega \\ (s_n)_{n\in\mathbb{N}} & \mapsto & (s_{n+1})_{n\in\mathbb{N}} \end{array}$$

est l'opération de décalage, l'élément $g_n(\omega)$ est l'élément obtenu au temps n de la marche aléatoire à droite sur (G, μ).

Soit $(g_n)_{n\in\mathbb{N}}$ un cocycle intégrable. L'inégalité triangulaire et l'invariance à gauche de la distance d assurent que pour tous $m, n \in \mathbb{N}$, nous avons

$$\begin{aligned} d(e, g_{n+m}(\omega)) &\leq d(e, g_m(\omega)) + d(g_m(\omega), g_{n+m}(\omega)) \\ &= d(e, g_m(\omega)) + d(e, g_n(T^m(\omega))). \end{aligned}$$

Le théorème ergodique sous-additif de Kingman [Kin68] assure alors l'existence de la limite

$$\lim_{n\to+\infty} d(e, g_n(\omega))$$

pour θ-presque tout $\omega \in \Omega$. Cette limite est appelée la *vitesse de fuite* du cocycle g_n.

Théorème 5.7. *(Karlsson–Ledrappier [KL06]) Soit (Ω, θ) un espace de probabilité, soit $T : \Omega \to \Omega$ une transformation ergodique qui préserve la mesure θ. Soit G un groupe localement compact agissant par isométries sur un espace métrique asymétrique (X, d) vérifiant les hypothèses de la Proposition 5.1, et soit $b \in X$. Soit $(g_n)_{n\in\mathbb{N}}$ un cocycle ergodique, intégrable pour la distance $d(g, g') := d(g.b, g'.b)$. Alors pour θ-presque tout $\omega \in \Omega$, il existe une horofonction (aléatoire) $h_\omega \in X(\infty)$, qui dépend de manière mesurable de ω, telle que*

$$\lim_{n\to+\infty} -\frac{1}{n}h_\omega(g_n(\omega)b) = \lim_{n\to+\infty} \frac{1}{n}d(b, g_n(\omega)b).$$

De manière informelle, le Théorème 5.7 exprime l'idée que presque toute trajectoire de la marche aléatoire sur (G, μ), réalisée sur l'espace X via l'action de G, a une direction asymptotique déterminée par une certaine horofonction (aléatoire) h_ω : la vitesse à laquelle une trajectoire typique de la marche aléatoire fuit l'origine b est aussi égale à la vitesse à laquelle cette trajectoire se rapproche de l'horofonction h_ω.

Nous mentionnons quelques applications du Théorème 5.7, et renvoyons à [KL06, KL11a] pour une démonstration de ce théorème, et une discussion plus précise de ses applications. Dans le cas où $G = \mathbb{R}$ et (X, d) est la droite réelle munie de la distance usuelle, le Théorème 5.7 se spécifie en la loi des grands nombres. Le Théorème 5.7 peut donc être vu comme une généralisation de la loi des grands nombres au cadre non commutatif. Appliqué à l'action du groupe $GL(N, \mathbb{R})$ sur l'espace symétrique $X = Pos(N, \mathbb{R})$, le Théorème 5.7 permet de retrouver le théorème d'Oseledets [Ose68]. Il a aussi des applications intéressantes lorsque $G = X = \mathbb{R}$, mais la distance d sur $\mathbb{R}$ n'est pas la distance usuelle.

Sous certaines hypothèses de courbure négative pour l'espace X (par exemple lorsque X est un espace hyperbolique au sens de Gromov ou un espace $CAT(0)$), l'égalité fournie par le Théorème 5.7 se traduit en un résultat d'approximation des trajectoires de la marche aléatoire par des rayons géodésiques (aléatoires).

Karlsson a appliqué le Théorème 5.7 à l'action du groupe modulaire d'une surface fermée S de genre $g \geq 2$ sur l'espace de Teichmüller $Teich(S)$, en utilisant la description par Walsh de l'horofrontière de $Teich(S)$ pour la distance asymétrique de Thurston. Ceci permet de déduire des résultats sur la croissance de la longueur des classes d'isotopies de courbes fermées simples (i.e., la longueur de l'unique représentant géodésique dans cette classe d'isotopie, pour une métrique hyperbolique sur S choisie une fois pour toutes), sous l'action de produits aléatoires d'éléments du groupe modulaire de la surface. Nous nous sommes inspirés des travaux de Karlsson pour établir des résultats analogues pour le groupe $\text{Out}(F_N)$.

5.5 Application à l'étude de marches aléatoires sur Out(F_N)

Croissance des éléments de F_N sous l'action d'un cocycle d'automorphismes.

Pour tout $g \in F_N$, nous désignerons par $||g||$ la longueur d'un mot cycliquement réduit qui représente la classe de conjugaison de g dans une base de F_N. Le groupe $\text{Out}(F_N)$ agit sur l'ensemble des classes de conjugaison d'éléments de F_N. En utilisant notre description de l'horofrontière de CV_N (Théorème 5.2), nous appliquons le théorème de Karlsson et Ledrappier pour obtenir le résultat suivant. Cette étude est menée en Partie 5.2 de l'Annexe B.

Théorème 5.8.

Soit $(\Phi_n)_{n\in\mathbb{N}}$ un cocycle intégrable d'éléments de $\mathrm{Out}(F_N)$, et soit l sa vitesse de fuite. Alors pour θ-presque tout $\omega \in \Omega$, il existe un arbre (aléatoire) $T(\omega) \in \overline{CV_N}$ tel que

1. pour tout $g \in F_N$ hyperbolique dans $T(\omega)$, nous ayons
$$\lim_{n\to+\infty} \frac{1}{n} \log ||\Phi_n^{-1}(g)|| = l,$$
2. pour tout $g \in F_N$ elliptique dans $T(\omega)$, nous ayons
$$\limsup_{n\to+\infty} \frac{1}{n} \log ||\Phi_n^{-1}(g)|| \leq l.$$

Démonstration. Soit $\Phi_n = \phi_1 \dots \phi_n$ un cocycle intégrable d'éléments de $\mathrm{Out}(F_N)$. Soit $b \in CV_N$ un arbre de Cayley de F_N, de sorte que $||g|| = ||g||_b$. Soit

$$l := \lim_{n\to+\infty} \frac{1}{n} d(b, \Phi_n.b)$$

sa vitesse de fuite. Pour tout $g \in F_N$, nous avons

$$\log \frac{||\Phi_n^{-1}(g)||}{||g||} = \log \frac{||g||_{\Phi_n.b}}{||g||_b} \leq d(b, \Phi_n.b).$$

Par conséquent, si $l = 0$, la croissance de $||\Phi_n^{-1}(g)||$ est sous-exponentielle. Nous supposons donc maintenant que $l > 0$. Le Théorème 5.7 donne l'existence, pour θ-presque tout ω, d'une horofonction h_ω, associée à un arbre $T(\omega) \in CV_N(\infty)$, telle que

$$\lim_{n\to+\infty} -\frac{1}{n} h_\omega(\Phi_n.b) = l.$$

Pour tout $\epsilon > 0$, il existe $n_0 \in \mathbb{N}$ tel que pour tout $n > n_0$, nous ayons

$$\log \sup_{g\in F_N} \frac{||g||_{T(\omega)}}{||g||_{\Phi_n.b}} - \log \sup_{g\in F_N} \frac{||g||_{T(\omega)}}{||g||_b} \leq -(l-\epsilon)n.$$

Posons

$$C(\omega)^{-1} := \sup_{g\in F_N} \frac{||g||_{T(\omega)}}{||g||_b},$$

alors

$$\sup_{g\in F_N} \frac{||g||_{T(\omega)}}{||g||_{\Phi_n.b}} \leq C(\omega)^{-1} e^{-(l-\epsilon)n},$$

et pour tout $g \in F_N$, nous avons donc

$$||\Phi_n(g)||_b \geq C(\omega) ||g||_{T(\omega)} e^{(l-\epsilon)n}.$$

Par ailleurs, il découle de la définition de l et de la distance sur CV_N que pour tout $n \in \mathbb{N}$ suffisamment grand, et tout $g \in F_N$, nous avons également

$$||\Phi_n(g)||_b \leq ||g||_b e^{(l+\epsilon)n}.$$

Le Théorème 5.8 est une conséquence des deux inégalités précédentes. $\square$

Le cas d'une marche aléatoire sur un sous-groupe non élémentaire de Out(F_N).

Nous précisons maintenant le Théorème 5.8 dans le cas où les incréments sont indépendants, i.e. dans le cas de la marche aléatoire sur (G, μ), où μ est une loi de probabilité sur G. Nous supposons dans un premier temps que le sous-groupe de $\mathrm{Out}(F_N)$ engendré par le support de μ est non élémentaire. Dans ce cas, nous montrons que tous les éléments de F_N ont le même taux exponentiel de croissance le long d'une trajectoire typique de la marche aléatoire. Ce taux de croissance est déterministe (i.e. il ne dépend pas de la trajectoire considérée), il est égal à la vitesse de fuite de la marche aléatoire. Nous montrons un analogue pour le groupe $\mathrm{Out}(F_N)$ d'un résultat de Furstenberg pour les produits aléatoires de matrices inversibles [Fur63a], et d'un résultat de Karlsson pour les produits aléatoires dans le groupe modulaire d'une surface fermée orientable, que nous mentionnons ci-dessous afin de mettre en valeur l'analogie entre ces différents cas. Un sous-groupe $G \subseteq SL(N, \mathbb{R})$ est *irréductible* s'il ne fixe virtuellement aucun sous-espace vectoriel propre (i.e. différent de $\mathbb{R}^N$ et de $\{0\}$) de $\mathbb{R}^N$.

Théorème 5.9. *(Furstenberg [Fur63b, Théorèmes 8.5 et 8.6]) Soit μ une loi de probabilité sur $SL(N, \mathbb{R})$, dont le support engendre un sous-groupe irréductible de $SL(N, \mathbb{R})$, et telle que*

$$\int_{SL(N,\mathbb{R})} \log ||g|| d\mu(g) < +\infty.$$

Alors il existe $\lambda > 0$ tel que pour tout $v \in \mathbb{R}^N \setminus \{0\}$ et $\mathbb{P}$-presque toute trajectoire $(A_n)_{n\in\mathbb{N}}$ de la marche aléatoire sur $(SL(N, \mathbb{R}), \mu)$, nous ayons

$$\lim_{n\to+\infty} \frac{1}{n} \log ||A_n^{-1} v|| = \lambda.$$

Si de plus, le sous-groupe engendré par le support de μ est non compact, alors $\lambda > 0$.

Soit S une surface fermée orientée de genre $g \geq 2$, et ρ une métrique hyperbolique sur S. Pour toute courbe fermée simple α sur S, nous désignerons par $l_\rho(\alpha)$ la longueur de l'unique représentant géodésique dans la classe d'isotopie de α pour la métrique ρ.

Théorème 5.10. *(Karlsson [Kar14, Corollaire 4]) Soit S une surface fermée orientée de genre $g \geq 2$. Soit μ une mesure de probabilité sur le groupe modulaire $Mod(S)$, de premier moment fini pour la distance de Thurston sur $Teich(S)$, dont le support engendre un sous-groupe de $Mod(S)$ qui contient un sous-groupe libre engendré par deux homéomorphismes de S. Alors il existe un réel $\lambda > 1$ tel que pour toute courbe fermée simple α sur S, toute métrique riemannienne ρ sur S, et $\mathbb{P}$-presque toute trajectoire $(\Phi_n)_{n\in\mathbb{N}}$ de la marche aléatoire sur $(Mod(S), \mu)$, nous ayons*

$$\lim_{n\to+\infty} l_\rho(\Phi_n^{-1}\alpha)^{\frac{1}{n}} = \lambda.$$

Théorème 5.11.

Soit μ une mesure de probabilité sur $\mathrm{Out}(F_N)$, dont le support est fini et engendre un sous-groupe non élémentaire de $\mathrm{Out}(F_N)$. Alors il existe $\lambda > 0$ tel que pour tout $g \in F_N$, et pour $\mathbb{P}$-presque toute trajectoire $(\Phi_n)_{n\in\mathbb{N}} \in \mathrm{Out}(F_N)^{\mathbb{N}}$ de la marche aléatoire sur $(\mathrm{Out}(F_N), \mu)$, nous ayons

$$\lim_{n\to+\infty} \frac{1}{n} \log ||\Phi_n^{-1}(g)|| = \lambda.$$

Ici, le taux de croissance λ est égal à la vitesse de fuite de la marche aléatoire sur $(\mathrm{Out}(F_N), \mu)$ pour la distance asymétrique sur CV_N.

Pour déduire le Théorème 5.11 à partir du Théorème 5.8, nous remarquons que lorsque $gr(\mu)$ est non élémentaire, l'action $T(\omega)$ associée à l'horofonction donnée par le théorème de Karlsson et Ledrappier peut être supposée libre. En effet, il découle des travaux de Karlsson et Ledrappier [KL11b] qu'il existe une mesure ν sur $CV_N(\infty)$ qui est μ-stationnaire, et telle que l'arbre $T(\omega)$ puisse être choisi dans un ensemble de ν-mesure pleine. En utilisant la non-élémentarité du sous-groupe engendré par le support de μ, nous montrons alors par un argument similaire à celui présenté dans l'encadré en page 79 que toute mesure μ-stationnaire sur $CV(\infty)$ est concentrée sur l'ensemble des actions libres. Le fait que $\lambda > 0$ vient du fait que la vitesse de fuite de la marche aléatoire est strictement positive (Théorème 4.14).

Le cas d'une marche aléatoire quelconque sur Out(F_N).

Nous continuons maintenant de supposer que les incréments sont indépendants, mais ne faisons plus d'hypothèse de non-élémentarité du sous-groupe engendré par le support de μ. Dans ce cas, plusieurs taux de croissance peuvent apparaître. Nous montrons un analogue pour $\mathrm{Out}(F_N)$ d'un théorème dû à Furstenberg et Kifer [FK83] et Hennion [Hen84] sur les produits aléatoires de matrices, dont nous rappelons l'énoncé.

Théorème 5.12. *(Furstenberg–Kifer [FK83, Théorème 3.9], Hennion [Hen84]) Soit μ une loi de probabilité sur $GL(N,\mathbb{R})$, telle que*

$$\int_{GL(N,\mathbb{R})} (\log^+ ||g|| + \log^+ ||g^{-1}||) d\mu(g) < +\infty.$$

Alors il existe des sous-espaces vectoriels (déterministes) $\{0\} = L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_{r-1} \subsetneq L_r = \mathbb{R}^N$, et une suite de réels (déterministes) $0 \le \beta^1(\mu) < \cdots < \beta^r(\mu)$, tels que pour tout $i \in \{1,\dots,r\}$, tout $v \in L_i \smallsetminus L_{i-1}$, et $\mathbb{P}$-presque toute trajectoire $(A_n)_{n\in\mathbb{N}}$ de la marche aléatoire sur $(GL(N,\mathbb{R}),\mu)$, nous ayons

$$\lim_{n\to+\infty} \frac{1}{n} \log ||A_n^{-1} v|| = \beta^i(\mu).$$

Le théorème de Furstenberg–Kifer et Hennion est une version du théorème multiplicatif d'Oseledets [Ose68] dans le cas d'incréments indépendants. La différence avec le théorème d'Oseledets est que la filtration obtenue est déterministe, i.e. elle ne dépend pas de l'aléa ω. Nous renvoyons à [FK83, Partie 5] pour une comparaison plus détaillée de la différence entre ces deux théorèmes.

Dans le cas d'un produit aléatoire d'automorphismes extérieurs d'un groupe libre F_N de type fini, l'analogue à la suite des sous-espaces emboîtés L_i est donnée par la notion suivante d'une *filtration* de F_N. Une *filtration* de F_N est un arbre enraciné, étiqueté par des sous-groupes (possiblement triviaux) de F_N, tel que l'étiquette de la racine soit F_N, et si H' est un fils de H, alors $H' \subseteq H$.

Théorème 5.13.

Soit μ une mesure de probabilité sur $\mathrm{Out}(F_N)$, de premier moment fini pour la distance $d^{sym}_{CV_N}$. Alors il existe une filtration (déterministe) τ^μ de F_N, et des *exposants de Lyapunov* (déterministes) $\lambda^\mu_H \geq 0$ associés à tous les sommets de la filtration, avec $\lambda_{H'} \leq \lambda_H$ si H' est un fils de H, et vérifiant la propriété suivante. Pour tout $g \in F_N$ ayant un conjugué dans un sous-groupe $H \subseteq F_N$ de la filtration, mais n'ayant aucun conjugué dans un descendant de H, et $\mathbb{P}$-presque toute trajectoire de la marche aléatoire sur $(\mathrm{Out}(F_N), \mu)$, nous avons

$$\lim_{n\to+\infty} \frac{1}{n} \log ||\Phi_n^{-1}(g)|| = \lambda^\mu_H.$$

De plus, le nombre d'exposants de Lyapunov non nuls est borné par $\frac{3N-2}{4}$.

Nos arguments donnent des précisions sur la filtration qui apparaît dans le Théorème 5.13 : à chaque étape, les descendants de H forment une famille de représentants des classes de conjugaison de stabilisateurs de points d'un H-arbre très petit. De plus, la $gr(\mu)$-orbite de la classe de conjugaison de tout sous-groupe de F_N associé à un nœud de la filtration est finie.

Dans le cas où la mesure μ est une mesure de Dirac supportée sur un automorphisme $\Phi \in \mathrm{Out}(F_N)$, le Théorème 5.13 se spécifie en un théorème dû à Levitt [Lev09] sur les taux exponentiels de croissance possibles d'un élément de F_N sous l'itération d'un automorphisme de F_N (dans ce cas, Levitt étudie également les degrés de croissance polynomiale possibles). La borne obtenue sur le nombre d'exposants de Lyapunov non nuls est optimale. En effet, Levitt a donné un exemple d'automorphisme Φ pour lequel il y a $\frac{3N-2}{4}$ taux de croissance distincts : l'automorphisme Φ est induit par un homéomorphisme d'une surface obtenue par recollement de $\frac{3N-2}{4}$ sous-surfaces Σ_i qui sont des tores épointés et des sphères à 4 composantes de bord, et la restriction de Φ à chacune des sous-surfaces Σ_i est un homéomorphisme pseudo-Anosov de la sous-surface. Cet exemple est présenté dans l'Exemple 6.26 de l'Annexe B. Au contraire, dans le cas opposé où le sous-groupe de $\mathrm{Out}(F_N)$ engendré par le support de la mesure μ est non élémentaire, nous avons vu que tous les éléments de F_N ont le même taux de croissance.

Nos techniques s'appliquent également au cas de produits aléatoires d'éléments du groupe modulaire d'une surface compacte orientable. Soit S une surface compacte orientable de genre g, ayant s composantes de bord. Cette fois, l'analogue de la filtration sera une décomposition de S en sous-surfaces. La *complexité* de S est $\xi(S) := 3g + s - 3$. Étant donné une métrique hyperbolique ρ sur S, pour laquelle S est à bord totalement géodésique, et une courbe fermée simple α sur S, nous désignerons par $l_\rho(\alpha)$ la longueur de l'unique représentant géodésique dans la classe d'isotopie de α pour la métrique ρ.

Théorème 5.14.

Soit S une surface hyperbolique compacte orientable à bord totalement géodésique (ou sans bord) pour une métrique hyperbolique ρ. Soit μ une loi de probabilité sur $\mathrm{Mod}(S)$, de premier moment fini pour la distance de Thurston symétrisée sur $Teich(S)$. Il existe une décomposition (déterministe) de S en sous-surfaces $S_1^\mu, \dots, S_k^\mu$, et pour tout $i \in \{1, \dots, k\}$, un *exposant de Lyapunov* (déterministe) λ_i^μ, de sorte que pour presque toute trajectoire $(\Phi_n)_{n\in\mathbb{N}}$ de la marche aléatoire sur $(\mathrm{Mod}(S), \mu)$, toute courbe fermée simple α sur S, la limite

$$\lim_{n\to+\infty} l_\rho(\Phi_n^{-1}(\alpha))^{\frac{1}{n}}$$

existe, et soit égale au plus grand exposant de Lyapunov d'une sous-surface S_i croisée par α (lorsque α est l'une des courbes intervenant dans la décomposition de S, cette limite vaut 1). De plus, le nombre d'exposants de Lyapunov est borné par la complexité $\xi(S)$.

Idée de la démonstration du Théorème 5.13. La démonstration des Théorèmes 5.13 et 5.14 fait l'objet de la Partie 6.4 de l'Annexe B. Dans le cas du groupe $\mathrm{Out}(F_N)$, l'argument se fait par induction sur le rang N du groupe libre. Nous expliquons ici comment construire une filtration aléatoire de F_N (i.e. qui dépend de l'aléa ω) qui vérifie les conditions souhaitées. Nous renvoyons le lecteur à l'Annexe B pour un argument permettant de montrer que celle-ci peut en fait être choisie déterministe.

Soit $(\Phi_n)_{n\in\mathbb{N}}$ une trajectoire typique de la marche aléatoire sur $(\mathrm{Out}(F_N), \mu)$. Nous savons déjà, par le Théorème 5.8, qu'il existe un arbre (aléatoire) $T(\omega)$ tel que tout élément de F_N qui est hyperbolique dans T croisse avec une vitesse (maximale) donnée par la vitesse de fuite, sous l'action des automorphismes Φ_n. Il s'agit de comprendre la croissance des éléments elliptiques dans $T(\omega)$. Il résulte des travaux de Karlsson et Ledrappier [KL11b] qu'il existe une mesure ν sur $CV_N(\infty)$ qui est μ-stationnaire, et telle que l'arbre $T(\omega)$ puisse être choisi dans un ensemble de ν-mesure pleine. Par un argument analogue à celui présenté dans l'encadré en page 79, nous montrons que la mesure ν est concentrée sur l'ensemble des arbres T pour lesquels la $gr(\mu)$-orbite de toute classe de conjugaison de stabilisateur de point dans T est finie. Soit $A \subseteq gr(\mu)$ le stabilisateur de l'une de ces classes de conjugaison H. Comme A est d'indice fini dans $gr(\mu)$, nous pouvons considérer la mesure μ^A déterminée par le premier retour en A d'une marche aléatoire sur $gr(\mu)$. Le rang de H étant strictement inférieur à N par un théorème dû à Gaboriau et Levitt [GL95, Corollaire III.4], nous obtenons par récurrence une filtration τ^H de H pour la mesure μ^A. La filtration $\tau(\omega)$ recherchée a pour racine l'arbre $\tau(\omega)$, et ses fils sont les racines des filtrations τ^H.

La borne sur le nombre d'exposants de Lyapunov utilise un argument de comptage qui nous a été inspiré par les travaux de Levitt [Lev09]. □

Troisième partie

L'alternative de Tits pour les groupes d'automorphismes de produits libres

Introduction

Dans un article de 1972 devenu célèbre [Tit72], Jacques Tits a démontré une conjecture due à Bass et Serre, qui affirme que tout sous-groupe d'un groupe linéaire de type fini (sur un corps arbitraire) contient soit un sous-groupe résoluble d'indice fini, soit un sous-groupe libre non abélien. L'alternative de Tits est un résultat profond, qui a eu de nombreuses applications à l'étude de la structure des sous-groupes des groupes linéaires. Elle a notamment été utilisée par Gromov pour établir son célèbre théorème selon lequel tout groupe de type fini à croissance polynomiale est virtuellement nilpotent [Gro81]. Maints travaux ont porté depuis sur diverses généralisations de cette alternative.

En particulier, cette même alternative a été montrée pour les sous-groupes d'autres classes de groupes, parmi lesquels nous mentionnerons les groupes hyperboliques (Gromov [Gro87]), les groupes modulaires de surfaces compactes (Ivanov [Iva84], McCarthy [McC85]), le groupe $\text{Out}(F_N)$ (Bestvina, Feighn et Handel [BFH00, BFH05]), les groupes agissant librement et proprement sur un complexe cubique $CAT(0)$ (Sageev et Wise [SW05]), le groupe $\text{Aut}[\mathbb{C}^2]$ des automorphismes polynomiaux du plan complexe (Lamy [Lam01]), les groupes d'automorphismes biméromorphes de variétés complexes compactes kähleriennes (Oguiso [Ogu06]), les groupes de transformations birationnelles de surfaces complexes compactes kähleriennes (Cantat [Can11]).

Étant donné une collection $\mathcal{C}$ de groupes, nous dirons plus généralement qu'un groupe G *satisfait l'alternative de Tits relativement à* $\mathcal{C}$ si pour tout sous-groupe $H \subseteq G$, soit $H \in \mathcal{C}$, soit H contient un sous-groupe libre non abélien. L'alternative de Tits classique correspond au cas où $\mathcal{C}$ est la collection des groupes virtuellement résolubles.

Il est également intéressant de montrer des propriétés de stabilité de l'alternative de Tits, permettant d'établir l'alternative de Tits pour un groupe G construit à partir de groupes plus simples pour lesquels celle-ci est connue. Des résultats de stabilité de l'alternative de Tits pour les produits graphés de groupes ont été montrés par Antolín et Minasyan [AM13].

Dans cet esprit, le résultat principal de cette thèse, que nous exposons au Chapitre 8, et dont la démonstration détaillée fait l'objet de l'Annexe G, est le suivant. Un groupe G est *librement indécomposable* s'il ne se scinde pas en un produit libre de la forme $G = A * B$, où A et B sont tous deux non triviaux.

Théorème.

Soit $\{G_1, \ldots, G_k\}$ un ensemble fini de groupes dénombrables, librement indécomposables et non isomorphes à $\mathbb{Z}$, soit F un groupe libre de type fini, et soit

$$G := G_1 * \cdots * G_k * F.$$

Soit $\mathcal{C}$ une collection de groupes qui est stable par isomorphismes, contient $\mathbb{Z}$, et est stable par passage aux sous-groupes, aux extensions, et aux surgroupes d'indice fini. Supposons que pour tout $i \in \{1, \ldots, k\}$, les groupes G_i et $\mathrm{Out}(G_i)$ satisfassent l'alternative de Tits relativement à $\mathcal{C}$. Alors $\mathrm{Out}(G)$ et $\mathrm{Aut}(G)$ satisfont l'alternative de Tits relativement à $\mathcal{C}$.

Ce théorème s'applique en particulier à la collection $\mathcal{C}$ des groupes virtuellement résolubles, qui est le cas classique de l'alternative de Tits. En particulier, nous donnons une nouvelle démonstration, plus courte, de l'alternative de Tits pour le groupe $\mathrm{Out}(F_N)$. Notre théorème permet de déduire l'alternative de Tits pour les groupes d'automorphismes (extérieurs) de certaines classes intéressantes de groupes, comme les groupes d'Artin à angles droits ou les groupes relativement hyperboliques toriques.

Par des techniques similaires, nous montrons également une autre alternative pour les sous-groupes de $\mathrm{Out}(F_N)$, qui est due à Handel et Mosher dans le cas des sous-groupes de type fini de $\mathrm{Out}(F_N)$. Un élément de $\mathrm{Out}(F_N)$ est *complètement irréductible* s'il ne fixe virtuellement la classe de conjugaison d'aucun facteur libre propre de F_N.

Théorème.

Pour tout sous-groupe $H \subseteq \mathrm{Out}(F_N)$ (non nécessairement de type fini), soit

- le groupe H contient deux automorphismes complètement irréductibles qui engendrent un sous-groupe libre non abélien de H, soit
- le groupe H est virtuellement cyclique, virtuellement engendré par un automorphisme complètement irréductible de F_N, soit
- le groupe H fixe virtuellement la classe de conjugaison d'un facteur libre propre de F_N.

La démonstration de ce théorème est exposée au Chapitre 6, et fait l'objet de l'Annexe D. Cette démonstration pourra être lue de manière indépendante, ou comme un préambule à notre démonstration de l'alternative de Tits pour les groupes d'automorphismes de produits libres.

Afin de passer du cadre du groupe $\mathrm{Out}(F_N)$ au cadre plus général du groupe des automorphismes extérieurs d'un produit libre de groupes dénombrables, nous serons amené à généraliser à ce cadre certains des espaces introduits dans la première partie de cette thèse, ainsi que des résultats portant sur leur géométrie. Ces divers résultats, dont les démonstrations font l'objet des Annexes E et F, sont exposés au Chapitre 7. Nous y étudions en particulier l'adhérence de l'outre-espace associé à un produit libre de groupes (introduit par Guirardel et Levitt dans [GL07b]). Nous définissons également les analogues des graphes des scindements libres et des scindements (maximalement)-cycliques pour les produits libres. Nous en montrons l'hyperbolicité, et déterminons le bord de Gromov des graphes de scindements cycliques.

Chapitre 6

Une alternative pour les sous-groupes de Out(F_N)

En préambule à la démonstration de l'alternative de Tits pour les groupes d'automorphismes de produits libres, nous démontrons une autre alternative pour les sous-groupes de Out(F_N), par des techniques similaires. Ceci fait l'objet de l'Annexe D. Nous rappelons qu'un automorphisme $\Phi \in \mathrm{Out}(F_N)$ est *complètement irréductible* si aucune puissance non nulle de Φ ne fixe la classe de conjugaison d'un facteur libre propre de F_N. Dans le cas où H est un sous-groupe de type fini de Out(F_N), le résultat suivant a été obtenu par Handel et Mosher [HM09] par des méthodes différentes.

Théorème 6.1.

Soit $H \subseteq \mathrm{Out}(F_N)$ un sous-groupe (pas nécessairement de type fini) de Out(F_N). Alors soit H contient un automorphisme complètement irréductible de F_N, soit H fixe virtuellement la classe de conjugaison d'un facteur libre propre de F_N.

Dans le cas où le sous-groupe $H \subseteq \mathrm{Out}(F_N)$ contient un élément complètement irréductible, le Théorème 6.1 se précise de la manière suivante. Le résultat suivant est dû à Bestvina, Feighn et Handel [BFH97] et à Kapovich et Lustig [KL11a] (une autre démonstration, reposant sur des arguments de marches aléatoires sur le complexe FF_N, est due à Sisto [Sis13]).

Théorème 6.2. *(Bestvina–Feighn–Handel [BFH97], Kapovich–Lustig [KL11a, Corollaire 1.3], Sisto [Sis13, Théorème 6.8]) Soit $H \subseteq$ Out(F_N) un sous-groupe (pas nécessairement de type fini) de Out(F_N) qui contient un automorphisme complètement irréductible de F_N. Alors soit H contient deux automorphismes complètement irréductibles qui engendrent un sous-groupe libre de rang 2, soit H est virtuellement cyclique.*

Le Théorème 6.1 a trouvé plusieurs applications. Par exemple, Bridson et Wade l'utilisent pour montrer que si Γ est un réseau irréductible d'un groupe de Lie semi-simple de rang réel supérieur ou égal à 2, alors tout morphisme de Γ dans Out(F_N) est d'image finie [BW11]. Carette, Francaviglia, Kapovich et Martino ont appliqué le Théorème 6.1 à des questions de rigidité spectrale [CFKM12].

La démonstration par Handel et Mosher du Théorème 6.1 dans le cas où H est de type fini fait appel à la théorie des *réalisations ferroviaires*, initiée par Bestvina et Handel [BH92] et développée dans [BFH97, BFH00, BFH05], ainsi qu'à la théorie de l'*attraction faible*, initiée par Bestvina, Feighn et Handel dans [BFH00]. L'argument de Handel et Mosher pour construire un élément complètement irréductible dès lors que H ne fixe virtuellement la classe de conjugaison d'aucun facteur libre propre de F_N, fait appel à des arguments de ping-pong sur un espace de lignes géodésiques.

Notons que récemment, en développant des versions relatives des techniques de [HM09], Handel et Mosher ont montré un résultat de classification plus précis pour les sous-groupes de type fini de $\text{Out}(F_N)$ [HM13b, HM13c, HM13d, HM13e, HM13f], analogue au théorème d'Ivanov de classification des sous-groupes du groupe modulaire d'une surface compacte orientable [Iva92].

Schéma de notre démonstration des Théorèmes 6.1 et 6.2. Notre démonstration des Théorèmes 6.1 et 6.2 repose sur l'étude de la dynamique de l'action du groupe $\text{Out}(F_N)$ sur le complexe (hyperbolique) des facteurs libres propres de F_N. Nous en donnons un aperçu.

Supposons dans un premier temps connue la propriété suivante : étant donné un sous-groupe $H \subseteq \text{Out}(F_N)$, soit H possède une orbite finie dans FF_N, soit l'ensemble limite de H dans ∂FF_N est non vide. Nous affirmons que les Théorèmes 6.1 et 6.2 se déduisent de cette propriété, en utilisant le théorème de classification des groupes d'isométries d'un espace hyperbolique présenté dans l'encadré ci-après, et illustré en Figure 6.1.

En effet, dans le cas borné, le groupe H admet en fait une orbite finie, i.e. H fixe virtuellement la classe de conjugaison d'un facteur libre propre de F_N.

Dans les cas horocyclique, linéal ou focal, le groupe H a une orbite finie dans ∂FF_N, et fixe donc virtuellement la classe d'équivalence d'un arbre arationnel dans $\overline{CV_N}$. Or, nous savons qu'une telle classe d'équivalence est un simplexe de mesures dans $\overline{CV_N}$, et H fixe donc virtuellement l'un de ses points extrémaux. Nous concluons alors grâce au Théorème 3.6 que H est virtuellement cyclique, virtuellement engendré par un automorphisme complètement irréductible (sinon les H-orbites seraient bornées). Remarquons en particulier que, puisque tout automorphisme complètement irréductible de F_N agit sur FF_N avec une dynamique nord-sud, les cas horocyclique et focal sont exclus.

Enfin, dans le cas où H est de type général, le groupe H contient deux isométries loxodromiques qui engendrent un groupe libre non abélien, qui sont deux automorphismes complètement irréductibles par le Théorème 3.2.

Il nous reste donc à montrer la propriété annoncée ci-dessus, à savoir : si les orbites de H sont bornées, alors H possède une orbite finie. La difficulté provient de l'absence de finitude locale du complexe FF_N.

Si X est un complexe hyperbolique localement fini, et G un groupe d'isométries de X, la propriété recherchée est toujours vérifiée. Dans ce cas en effet, l'espace $\overline{X} := X \cup \partial X$ est compact, donc soit les orbites de G sont finies, soit l'ensemble limite de G dans ∂X est non vide. Remarquons que la propriété recherchée est également vérifiée pour l'action du groupe modulaire sur le complexe des courbes $\mathcal{C}(S)$: c'est l'objet du théorème de classification des sous-groupes de $\text{Mod}(S)$ de McCarthy et Papadopoulos [MP89, Théorème 4.6]. En revanche, il existe des complexes X pour lesquelles la propriété ci-dessus est mise en défaut. Ainsi, nous avons vu en Partie 3.2.2 qu'une isométrie du complexe des scindements libres FS_N peut avoir toutes ses orbites bornées sans avoir aucune orbite finie, voir [HM14a, Exemple 4.2].

Notre argument pour contourner le défaut de compacité locale de FF_N, et montrer

Intermède (Groupes d'isométries d'espaces hyperboliques).

Le lecteur trouvera une preuve du théorème suivant, qui donne une classification des groupes d'isométries d'un espace hyperbolique, dans [CdCMT13] ou [Ham13], où la terminologie est introduite. Les différentes possibilités sont illustrées en Figure 6.1.

Théorème 6.3. *(voir [CdCMT13, Proposition 3.1] ou [Ham13, Théorème 2.7]) Soit X un espace métrique hyperbolique géodésique, et G un groupe agissant par isométries sur X. Alors G est soit*

- borné, *i.e. toutes les G-orbites sont bornées dans X, soit*
- horocyclique, *i.e. G n'est pas borné et ne contient pas d'élément loxodromique ; dans ce cas $\Lambda_X G$ est réduit à un point, soit*
- linéal, *i.e. G contient un élément loxodromique, et deux tels éléments ont les mêmes points fixes dans ∂X ; dans ce cas $\Lambda_X G$ consiste en ces deux points, soit*
- focal, *i.e. G n'est pas linéal, et G contient un élément loxodromique, et deux éléments loxodromiques quelconques ont un point fixe commun dans ∂X ; dans ce cas $\Lambda_X G$ est non dénombrable, et G a un point fixe dans ∂X, soit*
- de type général, *i.e. G contient deux éléments loxodromiques sans point fixe commun ; dans ce cas $\Lambda_X G$ est non dénombrable, et G n'a pas d'orbite finie dans ∂X. De plus, le groupe G contient deux isométries loxodromiques qui engendrent un sous-groupe libre de rang 2.*

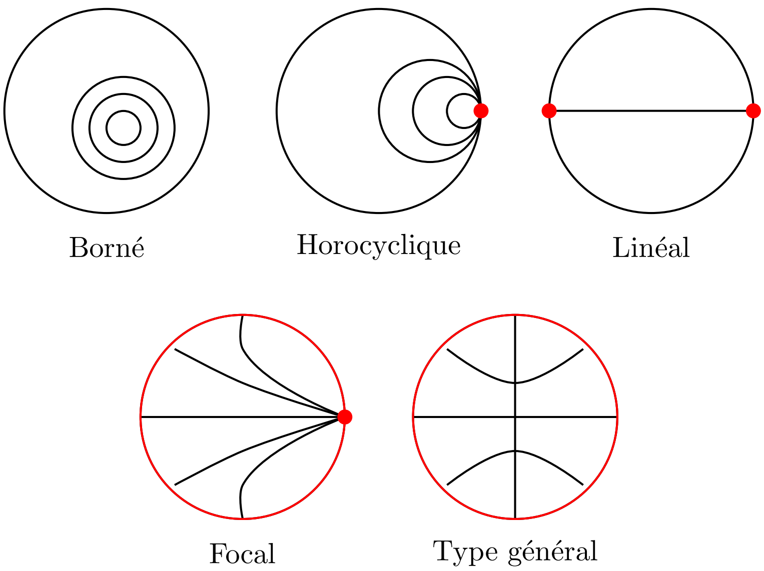


FIGURE 6.1 – Classification des groupes d'isométries d'un espace hyperbolique. Dans chacun des cas, nous avons représenté l'ensemble limite en rouge.

l'existence d'un point limite dans ∂FF_N dès lors que H ne fixe virtuellement la classe de conjugaison d'aucun facteur libre propre de F_N, passe par l'étude des mesures stationnaires sur $\overline{CV_N}$ effectuée dans l'encadré en page 79. Cette étude nous permet de montrer que toute H-orbite dans $\overline{CV_N}$ a un point limite qui est un arbre arationnel. Grâce à la description de Bestvina–Reynolds et Hamenstädt du bord de Gromov de FF_N, ceci montre l'existence d'un point limite dans ∂FF_N. Plus précisément, soit H un sous-groupe de $\mathrm{Out}(F_N)$ ne fixant virtuellement la classe de conjugaison d'aucun facteur libre propre de F_N, et μ une mesure de probabilité sur $\mathrm{Out}(F_N)$, dont le support engendre H. La Proposition 4.8 montre que toute mesure μ-stationnaire sur $\overline{CV_N}$ est concentrée sur l'ensemble $\widetilde{\mathcal{AT}}$ des arbres arationnels. Soit $x_0 \in CV_N$. Comme $\overline{CV_N}$ est compact, la suite des moyennes de Cesàro des mesures $\mu^{*n} * \delta_{x_0}$ a un point d'accumulation ν, qui est une mesure μ-stationnaire sur $\overline{CV_N}$ vérifiant $\nu(\overline{Hx_0}) = 1$. Ceci entraîne que la H-orbite de tout point $x_0 \in CV_N$ a un point d'accumulation dans $\widetilde{\mathcal{AT}}$. En utilisant le Théorème 3.3, nous en déduisons que la H-orbite de tout point $x \in FF_N$ a un point limite dans ∂FF_N. $\square$

En préparation à notre démonstration de l'alternative de Tits pour le groupe des automorphismes d'un produit libre de groupes, il nous paraît intéressant de rassembler ci-dessous les arguments qui nous ont servi dans notre démonstration des Théorèmes 6.1 et 6.2, que nous serons amenés à généraliser. Nous avons utilisé de manière cruciale

1. la description des mesures μ-stationnaires sur l'espace compact $\overline{CV_N}$, lorsque μ est une mesure de probabilité engendrant un sous-groupe non élémentaire de $\mathrm{Out}(F_N)$, effectuée dans l'encadré en page 79. Cette description repose elle-même de manière cruciale sur la possibilité d'associer à tout arbre non arationnel un ensemble fini canonique de facteurs de réduction (voir l'encadré en page 52).
2. l'hyperbolicité du complexe FF_N des facteurs libres de F_N, et la détermination de son bord de Gromov, ainsi que le lien entre la topologie de CV_N et la topologie de FF_N (Théorème 3.3).
3. le théorème de classification des sous-groupes d'isométries d'un espace hyperbolique au sens de Gromov (Théorème 6.3).
4. l'étude des stabilisateurs d'arbres arationnels dans $\overline{CV_N}$ (Théorème 3.6).
5. l'existence d'un ensemble fini de représentants canoniques dans la classe d'équivalence d'un arbre arationnel (à savoir les points extrémaux du simplexe de longueurs associé).
6. l'identification des éléments de $\mathrm{Out}(F_N)$ agissant de manière loxodromique sur le complexe FF_N avec les éléments complètement irréductibles (Théorème 3.2).

Nous pouvons préciser encore un peu le résultat de classification des sous-groupes de $\mathrm{Out}(F_N)$ énoncé ci-dessus. Un automorphisme $\Phi \in \mathrm{Out}(F_N)$ est *atoroïdal* s'il ne fixe la classe de conjugaison d'aucun élément de F_N. Les éléments irréductibles qui ne sont pas atoroïdaux sont représentés par des homéomorphismes d'une surface ayant une composante de bord, et fixent cette composante de bord, voir [BH92, Théorème 4.1]. Un résultat d'Uyanik affirme que tout groupe qui contient un élément complètement irréductible, et n'est pas réalisable comme un sous-groupe du groupe modulaire d'une surface, contient en fait un élément complètement hyperbolique atoroïdal [Uya14, Théorème 5.4]. En utilisant en outre un théorème dû à Kapovich et Lustig [KL11a, Théorème 5.6], nous obtenons le résultat suivant. Les sous-groupes de groupes modulaires de surfaces sont compris grâce au résultat de McCarthy et Papadopoulos [MP89].

Théorème 6.4. *Soit H un sous-groupe de Out(F_N) (pas nécessairement de type fini). Supposons que le groupe H n'est pas un sous-groupe du groupe modulaire d'une surface compacte ayant une composante de bord. Alors soit*

- *le groupe H contient un sous-groupe libre non abélien, engendré par deux automorphismes complètement irréductibles atoroïdaux, soit*
- *le groupe H est virtuellement cyclique, virtuellement engendré par un automorphisme complètement irréductible atoroïdal, soit*
- *le groupe H fixe virtuellement la classe de conjugaison d'un facteur libre propre de F_N.*

Remarque 6.5. Le Théorème 6.2 établit l'alternative de Tits pour un sous-groupe de $\mathrm{Out}(F_N)$ qui contient un automorphisme complètement irréductible de F_N. Afin d'établir l'alternative complète pour $\mathrm{Out}(F_N)$, il est naturel de vouloir raisonner par récurrence sur N. Ainsi, si $H \subseteq \mathrm{Out}(F_N)$ est un sous-groupe préservant la classe de conjugaison d'un facteur libre propre $F_k \subsetneq F_N$, comme F_k est égal à son propre normalisateur dans F_N, tout élément de H induit un élément bien défini de $\mathrm{Out}(F_k)$. Ceci permet de définir par restriction un morphisme $\theta : H \to \mathrm{Out}(F_k)$. La difficulté provient du fait que F_k ne possède pas de facteur libre supplémentaire canonique dans F_N (à l'inverse du cas des surfaces où toute sous-surface possède une sous-surface complémentaire). Par conséquent, le noyau de θ, contenu dans le groupe $\mathrm{Out}(F_N, F_k^{(t)})$ des automorphismes qui induisent une conjugaison sur F_k, ne s'identifie pas à un sous-groupe de $\mathrm{Out}(F_{N-k})$. Nous allons donc être amenés à étudier des versions relatives de l'outre-espace et des complexes hyperboliques correspondants pour le groupe $\mathrm{Out}(F_N, F_k^{(t)})$. Ceci fait l'objet du chapitre suivant, dans lequel nous nous plaçons dans le contexte plus général des automorphismes relatifs d'un produit libre de groupes dénombrables.

Chapitre 7

Produits libres, systèmes de facteurs libres, complexes relatifs

L'objectif de ce chapitre est d'étendre la définition d'espaces topologiques munis d'action de $\mathrm{Out}(F_N)$ que nous avons rencontrés jusqu'ici (l'outre-espace, le graphe des scindements libres, le graphe des scindements cycliques et ses variantes) au cadre de produits libres de groupes, et d'étendre certains résultats concernant la géométrie des espaces liés à $\mathrm{Out}(F_N)$ à ce contexte plus général. Ceci nous fournira les outils nécessaires pour établir l'alternative de Tits pour le groupe des automorphismes extérieurs d'un produit libre au chapitre suivant.

7.1 Produits libres de groupes

Décomposition de Grushko d'un groupe de type fini. Une motivation pour l'étude des produits libres de groupes vient de l'existence d'une *décomposition de Grushko* associée à tout groupe de type fini. Un groupe G est *librement indécomposable* s'il ne se scinde pas en un produit libre $G = A * B$, avec A et B tous deux non triviaux. Le théorème suivant est dû à Grushko ; Stallings en a proposé une preuve topologique dans [Sta65].

Théorème 7.1. *(Grushko [Gru40]) Soit G un groupe de type fini. Il existe des groupes $G_1, \dots, G_k$ non triviaux, librement indécomposables, et non isomorphes à $\mathbb{Z}$, et un groupe libre F de type fini, tels que*

$$G = G_1 * \cdots * G_k * F.$$

De plus, l'entier k et le rang de F sont uniquement déterminés par G, et les classes de conjugaison des groupes G_i sont uniquement déterminées à permutation près.

Systèmes de facteurs libres. Soit G un groupe dénombrable, qui se scinde en un produit libre de la forme

$$G = G_1 * \cdots * G_k * F,$$

où F est un groupe libre de type fini. Nous notons $\mathcal{F} := \{[G_1], \dots, [G_k]\}$ la collection (finie) des classes de conjugaison de sous-groupes G_i intervenant dans cette décomposition de G. Cette collection est appelée un *système de facteurs libres* de G. Le rang du groupe libre F apparaissant dans la décomposition de G ne dépend que de G et de $\mathcal{F}$, nous l'appelons le *rang libre* de $(G, \mathcal{F})$ et le notons $\mathrm{rk}_f(G, \mathcal{F})$. Le *rang de Kurosh* de $(G, \mathcal{F})$ est par définition

$$\mathrm{rk}_K(G, \mathcal{F}) := \mathrm{rk}_f(G, \mathcal{F}) + |\mathcal{F}|.$$

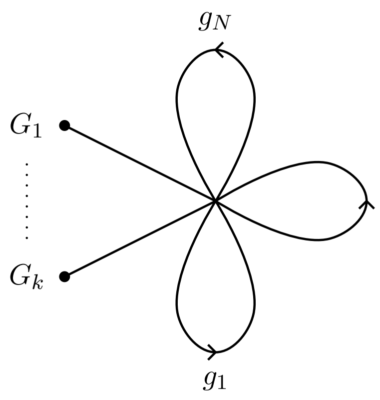


FIGURE 7.1 – L'arbre T^{def} est l'arbre de Bass–Serre du graphe de groupes représenté ci-dessus.

Les éléments de G (et plus généralement les sous-groupes de G) qui sont conjugués à un élément (ou un sous-groupe) de l'un des éléments de $\mathcal{F}$ sont dits *périphériques*. Lorsque $G = G_1 * G_2$ et $\mathcal{F} = \{[G_1], [G_2]\}$, ou $G = G_1*$ et $\mathcal{F} = \{[G_1]\}$, nous dirons que $(G, \mathcal{F})$ est *sporadique*.

Le scindement de G écrit ci-dessus assure l'existence d'un arbre simplicial T, muni d'une action simpliciale et minimale de G, dont les stabilisateurs de points sont exactement les conjugués des sous-groupes G_i, et à stabilisateurs d'arêtes triviaux. L'exemple typique d'un tel arbre est donné par l'arbre de Bass–Serre T^{def} du graphe de groupes représenté en Figure 7.1, où $\{g_1, \dots, g_N\}$ désigne une base de F.

Sous-groupes d'un produit libre de groupes. Soit G un groupe dénombrable, comme au paragraphe ci-dessus. La structure des sous-groupes de G a été étudiée par Kurosh [Kur34]. Soit $H \subseteq G$ un sous-groupe. En considérant le sous-arbre H-minimal de l'arbre T^{def}, nous obtenons l'existence d'un ensemble J, et pour tout $j \in J$, d'un entier $i_j \in \{1, \dots, k\}$, d'un sous-groupe non trivial $H_j \subseteq G_{i_j}$ et d'un élément $g_j \in G$, ainsi que d'un groupe libre (non nécessairement de type fini) F', tels que

$$H = *_{j \in J} g_j H_j g_j^{-1} * F'.$$

Le *rang de Kurosh* de H est défini comme $\mathrm{rk}_K(H) := \mathrm{rk}(F') + |J|$. Il peut être infini en général. Nous notons $\mathcal{F}_H$ l'ensemble des H-classes de conjugaison des sous-groupes de H de la forme $g_j H_j g_j^{-1}$ avec $j \in J$.

Automorphismes relatifs. L'étude algébrique des groupes d'automorphismes de produits libres remonte aux travaux de Fouxe-Rabinovitch [FR40, FR41] et Gilbert [Gil87], qui ont déterminé des présentations de ces groupes.

Soit G un groupe dénombrable, et $\mathcal{F}$ un système de facteurs libres de G. Nous notons $\mathrm{Out}(G, \mathcal{F})$ le sous-groupe de $\mathrm{Out}(G)$ formé des automorphismes qui préservent la G-classe de conjugaison de chacun des sous-groupes périphériques. Nous notons $\mathrm{Out}(G, \mathcal{F}^{(t)})$ le sous-groupe de $\mathrm{Out}(G, \mathcal{F})$ formé des automorphismes qui agissent par conjugaison (par un élément de G) sur chacun des sous-groupes périphériques.

Tout sous-groupe périphérique $H \subseteq G$ est égal à son propre normalisateur dans G. Tout élément de $\mathrm{Out}(G)$ qui préserve la classe de conjugaison de H induit donc par restriction un automorphisme extérieur de H bien défini. Il existe donc un morphisme (surjectif) de $\mathrm{Out}(G, \mathcal{F})$ vers le produit direct des groupes $\mathrm{Out}(H)$, le produit étant pris sur un ensemble de représentants des classes de conjugaison dans $\mathcal{F}$. Par définition, le noyau de ce morphisme est le sous-groupe $\mathrm{Out}(G, \mathcal{F}^{(t)})$.

7.2 L'outre-espace relatif

Un $(G, \mathcal{F})$-*arbre* est un arbre réel muni d'une action de G, dans lequel tous les sous-groupes périphériques de G sont elliptiques. Un $(G, \mathcal{F})$-*arbre de Grushko* est un $(G, \mathcal{F})$-arbre simplicial métrique minimal, à stabilisateurs d'arcs triviaux, dans lequel tous les stabilisateurs de sommets non triviaux ont leur classe de conjugaison dans $\mathcal{F}$. L'exemple typique d'un $(G, \mathcal{F})$-arbre de Grushko est l'arbre T^{def} représenté en Figure 7.1, muni d'une distance simpliciale. L'outre-espace $P\mathcal{O}(G, \mathcal{F})$ est l'espace des classes d'homothétie G-équivariante de $(G, \mathcal{F})$-arbres de Grushko. Cet espace a été introduit par Guirardel et Levitt [GL07a], en vue de montrer des propriétés de finitude pour les groupes d'automorphismes de produits libres. De nouveau, l'outre-espace peut être muni de plusieurs topologies, dont la topologie faible, la topologie des axes (i.e. la topologie issue du plongement de $P\mathcal{O}(G, \mathcal{F})$ dans $\mathbb{PR}^G$ via les fonctions longueurs de translation), et la topologie de Gromov–Hausdorff équivariante. Ces deux dernières topologies sont équivalentes [Pau89]. Cependant, comme $P\mathcal{O}(G, \mathcal{F})$ n'est pas localement fini, la topologie faible n'est pas équivalente aux deux autres topologies [GL07b, Partie 5]. Nous munirons dans la suite l'outre-espace de la topologie des axes, et nous définirons l'adhérence $\overline{P\mathcal{O}(G, \mathcal{F})}$ pour cette topologie. Nous nous proposons désormais de décrire les points de cette adhérence.

Définition 7.2. *Un $(G, \mathcal{F})$-arbre T est* très petit *si tous les stabilisateurs d'arcs de T sont triviaux, ou maximalement cycliques et non périphériques, et tous les stabilisateurs de tripodes de T sont triviaux.*

Rappelons que dans le cas où $G = F_N$ et $\mathcal{F} = \emptyset$, la compactification de Culler et Morgan de l'outre-espace CV_N s'identifie à l'espace des classes d'homothétie équivariante d'actions minimales et très petites de F_N sur des arbres réels (voir la Partie 1.1). L'adhérence $\overline{CV_N}$ est de dimension topologique finie, égale à $3N-4$, et la frontière $\partial CV_N := \overline{CV_N} \smallsetminus CV_N$ est de dimension topologique égale à $3N-5$. Nous étendons ces résultats au cas de l'outre-espace $P\mathcal{O}(G, \mathcal{F})$. La démonstration du théorème suivant fait l'objet de l'Annexe E. Elle est inspirée des travaux analogues de Bestvina et Feighn [BF94] et Gaboriau et Levitt [GL95] dans le cas d'un groupe libre de type fini. Nous étudions en particulier une notion de $(G, \mathcal{F})$-arbres *géométriques*.

Théorème 7.3.

Soit G un groupe dénombrable, et $\mathcal{F}$ un système de facteurs libres de G. L'adhérence $\overline{P\mathcal{O}(G, \mathcal{F})}$ est l'espace des classes d'homothétie G-équivariante de $(G, \mathcal{F})$-arbres minimaux très petits. Elle est compacte, de dimension topologique finie égale à $3\mathrm{rk}_f(G, \mathcal{F}) + 2|\mathcal{F}| - 4$, et la frontière $\partial P\mathcal{O}(G, \mathcal{F})$ est de dimension topologique égale à $3\mathrm{rk}_f(G, \mathcal{F}) + 2|\mathcal{F}| - 5$.

Au cours de la démonstration du Théorème 7.3, nous établissons une borne sur le nombre d'orbites de points de branchement et de centres d'inversions dans un $(G, \mathcal{F})$-arbre minimal et très petit, ainsi que sur le rang de Kurosh des stabilisateurs de points.

Proposition 7.4. *Soit T un $(G,\mathcal{F})$-arbre minimal, très petit. Alors T contient au plus $2rk_K(G,\mathcal{F})-2$ orbites de points de branchement et de centres d'inversion. Tout stabilisateur de point dans T a un rang de Kurosh inférieur ou égal à $rk_K(G,\mathcal{F})$.*

Lorsque T est à stabilisateurs d'arcs triviaux, la borne sur les rangs de Kurosh des stabilisateurs de points dans T est même stricte.

7.3 Graphes de scindements relatifs

7.3.1 Hyperbolicité

Un $(G,\mathcal{F})$-*scindement libre* est un $(G,\mathcal{F})$-arbre simplicial minimal dont tous les stabilisateurs d'arêtes sont triviaux. Notons $\mathcal{Z}$ la collection des sous-groupes de G qui sont soit triviaux, soit cycliques et non périphériques. Notons $\mathcal{Z}^{max}$ la collection des sous-groupes de G qui sont soit triviaux, soit cycliques, non périphériques et stables par racine. Nous utiliserons la notation $\mathcal{Z}^{(max)}$ pour désigner soit la classe $\mathcal{Z}$, soit la classe $\mathcal{Z}^{max}$. Un $\mathcal{Z}^{(max)}$-*scindement* de $(G,\mathcal{F})$ est un $(G,\mathcal{F})$-arbre simplicial minimal dont tous les stabilisateurs d'arêtes sont dans $\mathcal{Z}^{(max)}$. Deux $(G,\mathcal{F})$-scindements sont *équivalents* s'il existe un homéomorphisme G-équivariant entre eux. Étant donné deux $(G,\mathcal{F})$-scindements T et T', nous disons que T' est un *raffinement* de T si T s'obtient à partir de T' en écrasant en un point chaque composante connexe d'un sous-ensemble G-invariant d'arêtes de T'. Le *graphe des scindements libres* $FS(G,\mathcal{F})$ (resp. le *graphe des* $\mathcal{Z}^{(max)}$*-scindements* $FZ^{(max)}(G,\mathcal{F})$) est le graphe dont les sommets sont les classes d'équivalence de $(G,\mathcal{F})$-scindements libres (resp. de $\mathcal{Z}^{(max)}$-scindements de $(G,\mathcal{F})$) à une orbite d'arêtes, deux sommets étant reliés par une arête si les scindements correspondants ont un raffinement commun. Le groupe $\mathrm{Out}(G,\mathcal{F})$ agit naturellement à droite sur chacun de ces complexes par précomposition des actions. La démonstration du théorème suivant fait l'objet des Parties 2 et 3 de l'Annexe F. Nous suivons les arguments de Bestvina et Feighn [BF14c, Appendice] pour la démonstration de l'hyperbolicité du complexe des $(G,\mathcal{F})$-scindements libres, et ceux de Mann [Man13] pour la démonstration de l'hyperbolicité de $FZ(G,\mathcal{F})$ et $FZ^{max}(G,\mathcal{F})$. L'hyperbolicité de $FS(G,\mathcal{F})$ a été obtenue de manière indépendante par Handel et Mosher [HM14b], qui ont aussi établi l'hyperbolicité du graphe des facteurs libres relatifs.

Théorème 7.5.

Soit G un groupe dénombrable, et $\mathcal{F}$ un système de facteurs libres de G. Alors les graphes $FS(G,\mathcal{F})$, $FZ(G,\mathcal{F})$ et $FZ^{max}(G,\mathcal{F})$ sont hyperboliques au sens de Gromov.

7.3.2 Bord de Gromov du graphe des $\mathcal{Z}$-scindements

Nous déterminons également le bord de Gromov des graphes $FZ(G,\mathcal{F})$ et $FZ^{max}(G,\mathcal{F})$. Une idée des techniques employées pour la description du bord de Gromov de FZ_N et FZ_N^{max} a été donnée au Chapitre 3. Nos techniques s'étendent au contexte plus général de $(G,\mathcal{F})$-arbres.

Un arbre $T\in\overline{\mathcal{O}(G,\mathcal{F})}$ est $\mathcal{Z}$-*compatible* s'il est compatible avec un $\mathcal{Z}$-scindement de $(G,\mathcal{F})$, et $\mathcal{Z}$-*incompatible* sinon. Il est $\mathcal{Z}$-*étranger* s'il n'est compatible avec aucun arbre $\mathcal{Z}$-compatible dans $\overline{\mathcal{O}(G,\mathcal{F})}$ (là encore, deux $(G,\mathcal{F})$-arbres T,T' sont *compatibles* s'il existe un $(G,\mathcal{F})$-arbre $\widehat{T}$ qui admet des applications G-équivariantes préservant l'alignement vers T et T'). Nous noterons $\mathcal{X}(G,\mathcal{F})$ le sous-espace de $\overline{\mathcal{O}(G,\mathcal{F})}$ formé des arbres $\mathcal{Z}$-étrangers. Deux arbres $T,T' \in \mathcal{X}(G,\mathcal{F})$ sont *équivalents*, ce que nous noterons $T \sim T'$, s'ils sont tous deux compatibles avec un même troisième arbre. Ici encore, il y a une application naturelle (grossièrement définie) $\psi : \mathcal{O}(G,\mathcal{F}) \to FZ(G,\mathcal{F})$.

Théorème 7.6.

Soit G un groupe dénombrable, et $\mathcal{F}$ un système de facteurs libres de G. Alors il existe un unique homéomorphisme $\mathrm{Out}(G,\mathcal{F})$-équivariant

$$\partial\psi : \mathcal{X}(G,\mathcal{F})/\!\sim\; \to \partial FZ(G,\mathcal{F})$$

tel que pour tout $T \in \mathcal{X}(G,\mathcal{F})$, et toute suite $(T_n)_{n\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$ qui converge vers T, la suite $(\psi(T_n))_{n\in\mathbb{N}}$ converge vers $\partial\psi(T)$.

En travaillant avec la classe $\mathcal{Z}^{max}$ au lieu de la classe $\mathcal{Z}$, nous définissons de la même manière la notion d'arbre $\mathcal{Z}^{max}$*-étranger*. L'analogue du Théorème 7.6 pour cette classe de groupes est également satisfait. L'inclusion naturelle de $FZ^{max}(G,\mathcal{F})$ dans $FZ(G,\mathcal{F})$ n'est pas une quasi-isométrie lorsque $\mathrm{rk}_f(G,\mathcal{F}) \geq 1$ et $\mathrm{rk}_K(G,\mathcal{F}) \geq 3$. Par contre, lorsque $\mathrm{rk}_f(G,\mathcal{F}) = 0$, les complexes $FZ(G,\mathcal{F})$ et $FZ^{max}(G,\mathcal{F})$ sont quasi isométriques. Nous renvoyons pour ces faits à la Partie 5.6.1 de l'Annexe F.

Nous montrons également que chaque classe d'équivalence d'arbres $\mathcal{Z}$-étrangers contient un simplexe de représentants mélangeants. Rappelons qu'un arbre $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ est *mélangeant* si pour tous segments $I,J \subseteq T$, il existe un ensemble fini $\{g_1,\dots,g_r\} \subseteq G$ tel que $J \subseteq g_1I \cup \dots \cup g_kI$ et pour tout $i \in \{1,\dots,r-1\}$, nous ayons $g_iI \cap g_{i+1}I \neq \emptyset$. Étant donné deux arbres réels T et T', une application $f : T \to T'$ *préserve l'alignement* si elle envoie les segments de T sur des segments de T'. Deux $(G,\mathcal{F})$-arbres $T,T' \in \overline{\mathcal{O}(G,\mathcal{F})}$ sont *faiblement homéomorphes* s'il existe des applications G-équivariantes $f : T \to T'$ et $g : T' \to T$, continues en restriction aux segments de T et T', et inverses l'une de l'autre.

Proposition 7.7. *Pour tout $T \in \mathcal{X}(G,\mathcal{F})$, il existe un arbre $\overline{T} \in \mathcal{X}(G,\mathcal{F})$ mélangeant tel que tout arbre $T' \in \mathcal{X}(G,\mathcal{F})$ équivalent à T admette une application G-équivariante préservant l'alignement vers $\overline{T}$. De plus, deux tels arbres mélangeants sont faiblement homéomorphes.*

Lorsque $G = G_1 * G_2$ et $\mathcal{F} = \{[G_1],[G_2]\}$, les graphes $FZ(G,\mathcal{F})$ et $FZ^{max}(G,\mathcal{F})$ sont réduits à un point. Lorsque $G = G_1*$ et $\mathcal{F} = \{[G_1]\}$, le graphe $FZ^{max}(G,\mathcal{F})$ est une étoile de diamètre 2 : le centre de cette étoile est le scindement $G = G_1*$, et ses sommets extrémaux sont les scindements de la forme $G = G_1 * \langle g_1t\rangle$, où $t \in G \smallsetminus G_1$ est fixé, et g_1 varie dans G_1. Le graphe $FZ(G,\mathcal{F})$ est lui aussi borné dans ce cas, de diamètre 4. Dans tous les autres cas, nous construisons des exemples d'arbres dans $\mathcal{X}(G,\mathcal{F})$. Le Théorème 7.6 montre alors que les bords de Gromov des complexes $FZ(G,\mathcal{F})$ et $FZ^{max}(G,\mathcal{F})$ sont non vides, et en particulier les graphes $FZ(G,\mathcal{F})$ et $FZ^{max}(G,\mathcal{F})$ sont non bornés.

Proposition 7.8. *Soit G un groupe dénombrable, et $\mathcal{F}$ un système de facteurs libres de G. Les conditions suivantes sont équivalentes.*

1. *Nous avons $G = F_2$ et $\mathcal{F} \neq \emptyset$, ou $rk_K(G,\mathcal{F}) \geq 3$.*

2. *L'ensemble $\mathcal{X}(G,\mathcal{F})$ est non vide.*
3. *Le complexe $FZ(G,\mathcal{F})$ est non borné.*
4. *Le bord de Gromov $\partial FZ(G,\mathcal{F})$ est non vide.*

Nous présentons maintenant notre construction d'arbres $\mathcal{Z}^{max}$-étrangers. Cette construction est détaillée à la Proposition 5.6 de l'Annexe F. Lorsque $G = F_2$ et $\mathcal{F} = \emptyset$, tout arbre dual à une lamination mesurée arationnelle sur un tore ayant une composante de bord est arationnel, et donc $\mathcal{Z}^{max}$-étranger.

Supposons maintenant que $\text{rk}_K(G,\mathcal{F}) \geq 3$. Soit $N := \text{rk}_f(G,\mathcal{F})$, et soit $\{G_1,\dots,G_k\}$ un système de représentants des classes de conjugaison dans $\mathcal{F}$, de sorte que

$$G = G_1 * \dots * G_k * F_N.$$

Pour tout $i \in \{1,\dots,k\}$, soit $g_i \in G_i \smallsetminus \{e\}$. Nous notons p_i l'ordre de g_i dans G_i, et notons $i_1,\dots,i_l$ les indices pour lesquels $p_i < +\infty$. Soit S l'orbisurface obtenue à partir d'une sphère avec $N + (k-l) + 1$ composantes de bord, à laquelle nous ajoutons une singularité conique d'ordre p_{i_j} pour tout $j \in \{1,\dots,l\}$. Remarquons que lorsqu'aucun des G_i n'est un groupe de torsion, la construction peut être faite de sorte que S soit une surface compacte. L'hypothèse sur le rang de Kurosh de $(G,\mathcal{F})$ assure l'existence d'un feuilletage arationnel $\mathcal{F}$ sur l'orbisurface S. Nous définissons un arbre T comme un graphe d'actions $\mathcal{G}$ de la manière suivante (voir la Figure 7.2). L'un des arbres de sommets de $\mathcal{G}$ est l'arbre Y dual au feuilletage $\mathcal{F}$. Nous attachons pour tout $i \in \{1,\dots,k\}$ une copie d'un G_i-arbre trivial en le point x_i correspondant dans Y, qui est associé à une composante de bord de S si $p_i = +\infty$, ou à un point conique d'ordre p_i sinon. Le générateur du stabilisateur de x_i dans Y est identifié à g_i dans le graphe de groupes sous-jacent. L'arbre T dual au complexe mesuré ainsi défini est alors un $(G,\mathcal{F})$-arbre, dont nous montrons qu'il est $\mathcal{Z}^{max}$-étranger à la Proposition 5.6 de l'Annexe F.

À partir de cette construction, nous donnons également des exemples d'arbres $\mathcal{Z}^{max}$-étrangers qui ne sont pas $\mathcal{Z}$-étrangers, lorsque $\text{rk}_f(G,\mathcal{F}) \geq 1$ et $\text{rk}_K(G,\mathcal{F}) \geq 3$. Ceci montre en particulier que l'inclusion naturelle de $FZ^{max}(G,\mathcal{F})$ dans $FZ(G,\mathcal{F})$ n'est pas une quasi-isométrie. À l'inverse, lorsque $\text{rk}_f(G,\mathcal{F}) = 0$ et $\text{rk}_K(G,\mathcal{F}) \geq 2$, les graphes $FZ(G,\mathcal{F})$ et $FZ^{max}(G,\mathcal{F})$ sont quasi isométriques.

Arbres arationnels. Soit G un groupe dénombrable, et $\mathcal{F}$ un système de facteurs libres propres de G. Un arbre $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ est *arationnel* si pour tout $(G,\mathcal{F})$-facteur libre propre $H \subseteq G$, le facteur H ne fixe pas de point dans T, et le sous-arbre minimal de H dans T est un $(H,\mathcal{F}_H)$-arbre de Grushko. En particulier, l'action de H sur son sous-arbre minimal dans T est simpliciale. Nous notons $\mathcal{AT}(G,\mathcal{F})$ le sous-espace de $\overline{\mathcal{O}(G,\mathcal{F})}$ formé des arbres arationnels. Un arbre arationnel n'est jamais compatible avec un $(G,\mathcal{F})$-scindement maximalement cyclique. Nous montrons aussi que tout arbre arationnel est mélangeant (ceci a été montré par Reynolds [Rey12, Proposition 8.3] dans le cas des groupes libres). En particulier, nous avons $\mathcal{AT}(G,\mathcal{F}) \subseteq \mathcal{X}(G,\mathcal{F})$.

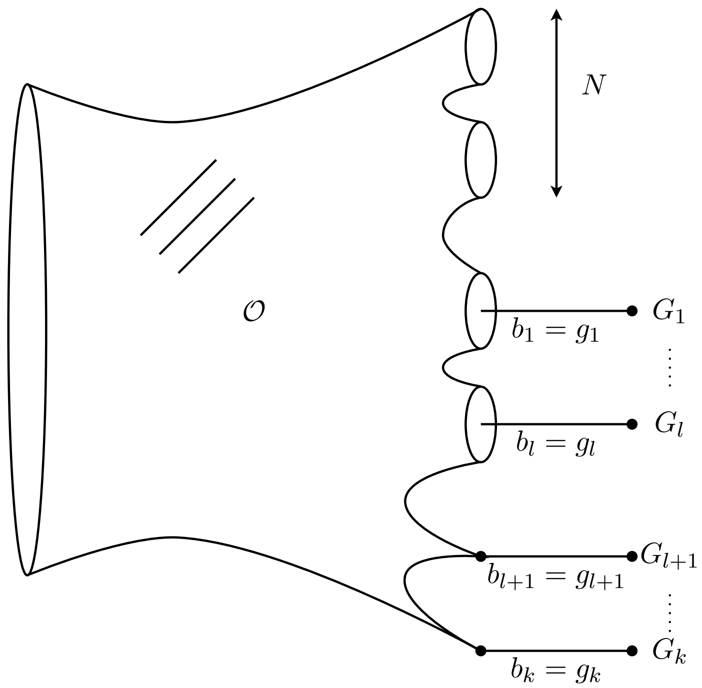


FIGURE 7.2 – Un exemple d'arbre $\mathcal{Z}^{max}$-étranger.

Chapitre 8

L'alternative de Tits pour le groupe des automorphismes extérieurs d'un produit libre

Un groupe G *satisfait l'alternative de Tits* si pour tout sous-groupe $H \subseteq G$, soit

- le groupe H est virtuellement résoluble, soit
- le groupe H contient un sous-groupe libre de rang 2.

Plus généralement, étant donné une collection $\mathcal{C}$ de groupes, nous disons que G *satisfait l'alternative de Tits relativement à* $\mathcal{C}$ si pour tout sous-groupe $H \subseteq G$, soit $H \in \mathcal{C}$, soit H contient un sous-groupe libre non abélien. L'alternative de Tits classique correspond au cas où $\mathcal{C}$ est la collection des groupes virtuellement résolubles. Nous renvoyons à l'introduction de cette partie pour un bref historique de cette alternative, et des exemples de groupes pour lesquels cette alternative a été démontrée. Notre résultat principal est le suivant.

Théorème 8.1.

Soit $\{G_1, \dots, G_k\}$ un ensemble de groupes dénombrables, librement indécomposables et non isomorphes à $\mathbb{Z}$, soit F un groupe libre de type fini, et soit

$$G := G_1 * \dots * G_k * F.$$

Soit $\mathcal{C}$ une collection de groupes qui est stable par isomorphismes, contient $\mathbb{Z}$, et est stable par passage aux sous-groupes, aux extensions, et aux surgroupes d'indice fini. Supposons que pour tout $i \in \{1, \dots, k\}$, les groupes G_i et $\mathrm{Out}(G_i)$ satisfont l'alternative de Tits relativement à $\mathcal{C}$. Alors $\mathrm{Out}(G)$ et $\mathrm{Aut}(G)$ satisfont l'alternative de Tits relativement à $\mathcal{C}$.

Remarquons que l'alternative de Tits pour $\mathrm{Aut}(G)$ est une conséquence de l'alternative de Tits pour $\mathrm{Out}(G)$. En effet, sous les hypothèses du Théorème 8.1, les groupes G et $\mathrm{Out}(G)$ satisfont tous deux l'alternative de Tits relativement à $\mathcal{C}$, et cette propriété est stable par extensions [Can11, Proposition 6.3].

Le Théorème 8.1 s'applique en particulier au cas où $\mathcal{C}$ est la collection des groupes virtuellement résolubles (nous renvoyons par exemple à [Can11, Lemme 6.1] pour une démonstration de la stabilité par extensions de la collection des groupes virtuellement résolubles). Ainsi, si chacun des groupes G_i et $\mathrm{Out}(G_i)$ satisfait l'alternative de Tits classique, il en est de même de $\mathrm{Out}(G)$ et $\mathrm{Aut}(G)$. Le Théorème 8.1 s'applique également au cas où $\mathcal{C}$ est la collection des groupes virtuellement polycycliques.

En particulier, nous obtenons une nouvelle démonstration, plus courte, de l'alternative de Tits pour le groupe $\mathrm{Out}(F_N)$, dont la preuve originelle est due à Bestvina, Feighn et Handel [BFH00, BFH05]. La démonstration originelle de Bestvina, Feighn et Handel repose fortement sur la théorie des réalisations ferroviaires [BH92], et sur un développement de la théorie de l'attraction faible. Par des arguments de ping-pong sur un certain *espace de lignes*, Bestvina, Feighn et Handel commencent par montrer l'alternative de Tits pour les sous-groupes de $\mathrm{Out}(F_N)$ qui contiennent un automorphisme à croissance exponentielle [BFH00]. Le cas des sous-groupes de $\mathrm{Out}(F_N)$ ne contenant que des automorphismes à croissance polynomiale est traité dans [BFH05], où Bestvina, Feighn et Handel montrent un théorème de Kolchin pour $\mathrm{Out}(F_N)$. Notre démonstration repose sur l'étude de la dynamique de l'action des sous-groupes de $\mathrm{Out}(F_N)$ sur le graphe (hyperbolique) des scindements cycliques de F_N.

Une question ouverte est de savoir si ce théorème reste valable pour la classe $\mathcal{C}$ des groupes virtuellement abéliens : si chacun des groupes G_i et $\mathrm{Out}(G_i)$ satisfait l'alternative de Tits relativement à la collection des groupes virtuellement abéliens, en est-il de même de $\mathrm{Out}(G)$? Ceci ne peut pas se déduire du Théorème 8.1 puisque la collection des groupes virtuellement abéliens n'est pas stable par extensions. La question est toutefois assez naturelle, dans la mesure où l'alternative de Tits relative à la classe des groupes virtuellement abéliens est connue pour un certain nombre de groupes (pour lesquels tout sous-groupe résoluble est en fait virtuellement abélien). Elle est vérifiée notamment par le groupe $\mathrm{Out}(F_N)$ des automorphismes extérieurs d'un groupe libre de type fini (Bestvina, Feighn et Handel [BFH04], Alibegovic [Ali02, Corollaire 1.2]). Elle est également vérifiée par les groupes hyperboliques (Gromov [Gro87]), les groupes modulaires de surfaces (Birman, Lubotzky et McCarthy [BLM83]), ou les groupes agissant librement et proprement sur un complexe cubique $CAT(0)$ [BH99]. Elle n'est en revanche pas vérifiée pour les groupes linéaires en général : le groupe de Heisenberg est en effet un exemple de groupe linéaire résoluble qui n'est pas virtuellement abélien.

À l'opposé, un autre problème naturel serait de remplacer dans la définition de l'alternative de Tits, l'existence d'un sous-groupe libre non abélien par d'autres notions de groupe *large*. On pourrait par exemple demander l'existence d'un morphisme surjectif d'un sous-groupe d'indice fini de H vers un groupe libre non-abélien, ou la propriété de *SQ-universalité*, i.e. tout groupe dénombrable est sous-groupe d'un quotient de H (cette propriété est vérifiée par les groupes libres non abéliens). Ainsi, Dahmani, Guirardel et Osin ont montré que tout sous-groupe du groupe modulaire d'une surface fermée orientable est SQ-universel ou virtuellement abélien [DGO14, Corollaire 8.4].

Le Théorème 8.1 est une conséquence du théorème suivant.

Théorème 8.2.

Soit G un groupe dénombrable, et $\mathcal{F}$ un système de facteurs libres de G. Si tout sous-groupe périphérique de G satisfait l'alternative de Tits, alors $\mathrm{Out}(G, \mathcal{F}^{(t)})$ satisfait l'alternative de Tits.

Le Théorème 8.1 se déduit à partir du Théorème 8.2 de la manière suivante. Puisque chacun des groupes G_i est librement indécomposable et non isomorphe à $\mathbb{Z}$, tout élément de $\mathrm{Out}(G)$ permute les classes de conjugaison des G_i. Il existe donc un sous-groupe d'indice fini $\mathrm{Out}^0(G)$ de $\mathrm{Out}(G)$ qui préserve la classe de conjugaison de chacun des G_i. Pour tout $i \in \{1,\dots,k\}$, le groupe G_i est égal à son propre normalisateur dans G. Par conséquent, tout automorphisme dans $\mathrm{Out}^0(G)$ induit un automorphisme bien défini dans chacun des groupes $\mathrm{Out}(G_i)$. Ceci définit un morphisme

$$\Phi : \mathrm{Out}^0(G) \to \prod_{i=1}^{k} \mathrm{Out}(G_i),$$

dont le noyau est un sous-groupe de $\mathrm{Out}(G,\mathcal{F}^{(t)})$. L'alternative de Tits étant une propriété stable par extension, le Théorème 8.1 se déduit du Théorème 8.2.

8.1 Stabilisateurs d'arbres dans $\overline{\mathcal{O}(G,\mathcal{F})}$

De même que nous avons eu besoin de comprendre les stabilisateurs d'arbres arationnels dans $\overline{CV_N}$ pour démontrer l'alternative de Handel et Mosher pour les sous-groupes de $\mathrm{Out}(F_N)$ au Chapitre 6, un ingrédient important de notre démonstration de l'alternative de Tits pour $\mathrm{Out}(G)$ consiste à comprendre les stabilisateurs d'arbres dans $\overline{P\mathcal{O}(G,\mathcal{F})}$. Si $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ est un arbre à stabilisateurs d'arcs triviaux, tout stabilisateur de point G_v de T est égal à son propre normalisateur dans G. Ainsi, tout élément de $\mathrm{Out}(G,\mathcal{F})$ induit par restriction un automorphisme extérieur de G_v. Nous notons $\mathrm{Out}(T,\{[G_v]\}^{(t)})$ le sous-groupe de $\mathrm{Out}(G,\mathcal{F})$ formé des automorphismes qui fixent l'arbre T, et induisent une conjugaison en restriction à chacun des groupes G_v.

Théorème 8.3. *(Guirardel–Levitt [GL]) Soit $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ un arbre à stabilisateurs d'arcs triviaux. Soit V l'ensemble des points de T dont le stabilisateur est non trivial, et soit $\{G_v\}_{v\in V}$ l'ensemble des stabilisateurs de points dans T. Il existe un sous-groupe d'indice fini $Out^0(T,\{[G_v]\}^{(t)})$ de $Out(T,\{[G_v]\}^{(t)})$ tel que*

$$Out^0(T,\{[G_v]\}^{(t)}) \subseteq \prod_{v\in V} G_v^{d_v}/Z(G_v),$$

où d_v est le degré de v dans T, et $Z(G_v)$ est le centre de G_v, qui est plongé de manière diagonale dans le produit $G_v^{d_v}$.

Afin de comprendre les stabilisateurs d'arbres projectifs dans $\overline{P\mathcal{O}(G,\mathcal{F})}$, nous faisons appel au résultat suivant.

Théorème 8.4. *(Guirardel–Levitt [GL]) Soit $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ un arbre à stabilisateur d'arcs triviaux, et $[T]$ sa classe projective dans $\overline{P\mathcal{O}(G,\mathcal{F})}$. Alors le sous-groupe $Out([T])$ de $Out(G,\mathcal{F})$ formé des automorphismes qui fixent $[T]$ est une extension cyclique du groupe $Out(T)$ formé des automorphismes qui préservent T.*

8.2 Schéma de démonstration du Théorème 8.2

Le cas d'un sous-groupe sporadique. Lorsque $(G,\mathcal{F})$ est sporadique, la description du groupe $\mathrm{Out}(G,\mathcal{F}^{(t)})$ est une conséquence des travaux de Levitt sur les automorphismes de graphes de groupes [Lev04].

Proposition 8.5. *Soit G_1 et G_2 deux groupes dénombrables non triviaux, et $G = G_1 * G_2$. Soit $\mathcal{F} := \{[G_1],[G_2]\}$. Alors $Out(G,\mathcal{F}^{(t)})$ est isomorphe à $G_1/Z(G_1) \times G_2/Z(G_2)$.*

Proposition 8.6. *Soit G_1 un groupe dénombrable, et $G = G_1*$. Soit $\mathcal{F} := \{[G_1]\}$. Alors $Out(G, \mathcal{F}^{(t)})$ contient un sous-groupe d'indice 2 isomorphe à $(G_1 \times G_1)/Z(G_1)$, où $Z(G_1)$ s'identifie à un sous-groupe de $G_1 \times G_1$ par le plongement diagonal.*

Le cas d'un sous-groupe non élémentaire. Un sous-groupe $H \subseteq \mathrm{Out}(G, \mathcal{F}^{(t)})$ est *non élémentaire* si la H-orbite de toute classe de conjugaison de $(G, \mathcal{F})$-facteur libre propre, et de toute classe d'équivalence d'arbres $\mathcal{Z}$-étrangers dans $\overline{PO(G, \mathcal{F})}$, est infinie. Nous adaptons l'argument utilisé dans la preuve de l'alternative de Handel et Mosher pour les sous-groupes de $\mathrm{Out}(F_N)$, afin de montrer que tout sous-groupe non élémentaire de $\mathrm{Out}(G, \mathcal{F}^{(t)})$ contient un sous-groupe libre non abélien. Par un argument similaire à celui présenté dans l'encadré en page 79, nous montrons le résultat suivant.

Proposition 8.7. *Soit G un groupe dénombrable, et soit $\mathcal{F}$ un système de facteurs libres propres de G, tels que $(G, \mathcal{F})$ soit non sporadique. Soit μ une mesure de probabilité sur $Out(G, \mathcal{F}^{(t)})$, dont le support engendre un sous-groupe non élémentaire de $Out(G, \mathcal{F}^{(t)})$. Alors toute mesure μ-stationnaire sur $\overline{PO(G, \mathcal{F})}$ est concentrée sur l'ensemble des arbres arationnels.*

Comme dans la démonstration de l'alternative de Handel et Mosher, l'argument nécessite d'associer à tout arbre non arationnel un ensemble fini canonique de facteurs de réduction, comme dans l'encadré en page 52. Soit H un sous-groupe non élémentaire de $\mathrm{Out}(G, \mathcal{F}^{(t)})$. Nous déduisons de la Proposition 8.7 que la H-orbite de tout point de $PO(G, \mathcal{F})$ possède des points limites arationnels, et donc $\mathcal{Z}$-étrangers, dans $\partial PO(G, \mathcal{F})$. Notre description du bord de Gromov de $FZ(G, \mathcal{F})$ donnée au Théorème 3.10, et du lien entre la convergence dans $\overline{PO(G, \mathcal{F})}$ et dans $FZ(G, \mathcal{F}) \cup \partial FZ(G, \mathcal{F})$, montre alors que toute H-orbite dans $FZ(G, \mathcal{F})$ possède des points limites dans $\partial FZ(G, \mathcal{F})$. Par ailleurs, le groupe H ne préserve aucun ensemble fini d'éléments de $\partial FZ(G, \mathcal{F})$. Le théorème de classification des groupes d'isométries d'un espace hyperbolique rappelé dans l'encadré en page 109 permet alors d'obtenir le résultat suivant.

Proposition 8.8. *Soit G un groupe dénombrable, et soit $\mathcal{F}$ un système de facteurs libres propres de G, tels que $(G, \mathcal{F})$ soit non sporadique. Alors tout sous-groupe non élémentaire de $Out(G, \mathcal{F}^{(t)})$ contient un sous-groupe libre non abélien.*

L'argument inductif. Soit $\mathcal{F}$ un système de facteurs libres propres de G. Nous raisonnons par récurrence sur la paire $\mathrm{rk}(G, \mathcal{F}) := (\mathrm{rk}_K(G, \mathcal{F}), \mathrm{rk}_f(G, \mathcal{F}))$, pour l'ordre lexicographique. Si $\mathrm{rk}_K(G, \mathcal{F}) = 1$, alors G est isomorphe soit à $\mathbb{Z}$, soit à un sous-groupe dans $\mathcal{F}$, et le Théorème 8.2 est vérifié. Nous supposons donc désormais que $\mathrm{rk}_K(G, \mathcal{F}) \geq 2$, et nous montrons que soit H contient un sous-groupe libre non abélien, soit H est virtuellement résoluble. Le cas où $(G, \mathcal{F})$ est sporadique est traité dans les Propositions 8.5 et 8.6 ; nous supposons donc que $(G, \mathcal{F})$ est non sporadique. Le cas où H est un sous-groupe non élémentaire de $\mathrm{Out}(G, \mathcal{F}^{(t)})$ est traité dans la Proposition 8.8. Quitte à passer à un sous-groupe d'indice fini, nous pouvons donc supposer que soit H préserve la classe de conjugaison d'un $(G, \mathcal{F})$-facteur libre propre, soit H préserve la classe d'équivalence d'un arbre $\mathcal{Z}$-étranger dans $\overline{PO(G, \mathcal{F})}$.

Supposons dans un premier temps que H préserve la classe de conjugaison d'un $(G, \mathcal{F})$-facteur libre propre G'. Le groupe G' étant égal à son propre normalisateur dans G, tout automorphisme dans $\mathrm{Out}(G, \mathcal{F}^{(t)})$ induit par restriction un automorphisme dans $\mathrm{Out}(G')$, qui est une conjugaison par un élément $g \in G$ en restriction à chacun des facteurs dans $\mathcal{F}_{G'}$. Comme G' est malnormal dans G, nous avons en fait $g \in G'$. Nous obtenons donc un morphisme de $\mathrm{Out}(G, \mathcal{F}^{(t)})$ dans $\mathrm{Out}(G', \mathcal{F}_{G'}^{(t)})$, dont le noyau est un sous-groupe de $\mathrm{Out}(G, \mathcal{F}'^{(t)})$, où $\mathcal{F}' := (\mathcal{F} \smallsetminus \mathcal{F}_{G'}) \cup \{[G']\}$ (avec un léger abus de notation, puisqu'ici $\mathcal{F}_{G'}$ désigne en fait l'ensemble des G-classes de conjugaison d'éléments de $\mathcal{F}_{G'}$). Nous vérifions alors que $\mathrm{rk}(G', \mathcal{F}_{G'}), \mathrm{rk}(G, \mathcal{F}') < \mathrm{rk}(G, \mathcal{F})$. Le résultat s'obtient donc par induction, l'alternative de Tits étant une propriété stable par extension.

Si H préserve la classe d'équivalence d'un arbre $\mathcal{Z}$-étranger dans $\overline{\mathcal{O}(G, \mathcal{F})}$, alors H préserve le simplexe de représentants mélangeants associé (Théorème 7.7), qui est de dimension finie. Le groupe H fixe donc virtuellement les points extrémaux de ce simplexe. Donc H fixe virtuellement la classe projective d'un arbre à stabilisateurs d'arcs triviaux $[T] \in \overline{P\mathcal{O}(G, \mathcal{F})}$, et c'est alors une extension cyclique du stabilisateur H' d'un représentant non projectif T de $[T]$ par le Théorème 8.4.

Quitte à passer à un sous-groupe d'indice fini, nous pouvons supposer que H' préserve la classe de conjugaison de chaque stabilisateur de point G_v dans T. Chacun des groupes G_v est égal à son normalisateur dans G, donc nous obtenons comme ci-dessus un morphisme de H' dans le produit direct des $\mathrm{Out}(G_v, \mathcal{F}_{G_v}^{(t)})$, dont le noyau est contenu dans $\mathrm{Out}(T, \{G_v\}^{(t)})$. Le Théorème 8.3 montre alors que $\mathrm{Out}(T, \{G_v\}^{(t)})$ satisfait l'alternative de Tits. De plus, puisque les stabilisateurs d'arcs dans T sont triviaux, nous avons $\mathrm{rk}_K(G_v, \mathcal{F}_{G_v}) < \mathrm{rk}_K(G, \mathcal{F})$ d'après la remarque suivant la Proposition 7.4. Nous pouvons de nouveau conclure par induction. $\square$

8.3 Remarques et questions

Notre démonstration de l'alternative de Tits pour un produit libre de groupes montre en particulier la trichotomie suivante pour les sous-groupes de $\mathrm{Out}(G, \mathcal{F})$.

Théorème 8.9. *Soit G un groupe dénombrable, et $\mathcal{F}$ un système de facteurs libres de G. Soit $H \subseteq \mathrm{Out}(G, \mathcal{F})$. Alors*

- *soit H contient un sous-groupe libre de rang 2, engendré par deux isométries hyperboliques de $FZ(G, \mathcal{F})$,*
- *soit H fixe virtuellement un arbre $\mathcal{Z}$-étranger dans $\overline{P\mathcal{O}(G, \mathcal{F})}$,*
- *soit H fixe virtuellement la classe de conjugaison d'un $(G, \mathcal{F})$-facteur libre propre.*

Il serait sans doute plus naturel de travailler dans le complexe hyperbolique des $(G, \mathcal{F})$-facteurs libres (nous renvoyons à [HM14b] pour une démonstration de l'hyperbolicité de ce complexe), dont nous conjecturons que le bord de Gromov s'identifie à l'espace des simplexes de $(G, \mathcal{F})$-arbres arationnels dans $\overline{P\mathcal{O}(G, \mathcal{F})}$. La détermination du bord de Gromov de FF_N par Bestvina et Reynolds [BR13] et Hamenstädt [Ham14a] repose de manière cruciale sur la dualité entre l'outre-espace et l'espace des courants géodésiques. Il s'agirait donc de développer une théorie des courants géodésiques pour les produits libres de groupes.

8.4 Applications

Le Théorème 8.1 permet de montrer l'alternative de Tits pour deux classes intéressantes de groupes, les groupes d'automorphismes extérieurs de groupes d'Artin à angles droits, et les groupes d'automorphismes extérieurs de groupes relativement hyperboliques toriques.

Groupes des automorphismes extérieurs d'un groupe d'Artin à angles droits.

Soit Γ un graphe fini. Le *groupe d'Artin à angles droits* A_Γ est le groupe donné par la présentation suivante. Les générateurs de A_Γ sont les sommets de Γ, et il y a une relation de commutation entre deux générateurs v et w si les sommets correspondants de Γ sont reliés par une arête de Γ. Lorsque Γ est un graphe complet, le groupe A_Γ est abélien libre. Lorsque Γ est un graphe totalement disconnexe, le groupe A_Γ est un groupe libre. Plus généralement, en notant N le nombre de composantes connexes de Γ réduites à un point, et $\Gamma_1, \dots, \Gamma_k$ les autres composantes connexes de Γ, le groupe A_Γ se scinde en un produit libre

$$A_\Gamma = A_{\Gamma_1} * \cdots * A_{\Gamma_k} * F_N.$$

Chacun des groupes A_{Γ_i} est librement indécomposable et non isomorphe à $\mathbb{Z}$. Le scindement écrit ci-dessus est donc la décomposition de Grushko de A_Γ.

Théorème 8.10. *Soit A un groupe d'Artin à angles droits. Alors Out(A) satisfait l'alternative de Tits.*

Le Théorème 8.10 a été montré par Charney, Crisp et Vogtmann dans le cas où le graphe Γ est connexe et ne contient pas de triangle [CCV07], puis étendu par Charney et Vogtmann au cas de graphes satisfaisant une condition d'homogénéité [CV09]. Charney et Vogtmann remarquent que le Théorème 8.10 découle du Théorème 8.1. C'est d'ailleurs cette remarque qui a amené Charney et Vogtmann à poser la question de l'alternative de Tits pour les automorphismes de produits libres. Nous renvoyons à la Partie 7 de l'Annexe G pour une présentation de l'argument, et aux articles mentionnés ci-dessus pour plus de détails.

Groupes d'automorphismes de groupes relativement hyperboliques.

Soit G un groupe, et $\mathcal{P}$ une famille finie de sous-groupes de G. Le groupe G est *hyperbolique relativement à* $\mathcal{P}$ si G agit sur un graphe connexe $\mathcal{K}$ hyperbolique au sens de Gromov tel que

- pour tout $n \in \mathbb{N}$, chaque arête de $\mathcal{K}$ est contenue dans un nombre fini de circuits simples de longueur n, et
- il y a un nombre fini d'orbites d'arêtes dans $\mathcal{K}$, et
- les stabilisateurs d'arêtes pour l'action de G sur $\mathcal{K}$ sont finis, et
- l'ensemble $\mathcal{P}$ est un ensemble de représentants des classes de conjugaison des stabilisateurs infinis de sommets.

Nous montrons le résultat suivant.

Théorème 8.11. *Soit G un groupe sans torsion, hyperbolique relativement à une famille finie $\mathcal{P}$ de sous-groupes de type fini. Soit $\mathcal{C}$ une collection de groupes qui contient $\mathbb{Z}$, stable par isomorphisme, par passage aux sous-groupes, aux extensions, et aux surgroupes d'indice fini. Supposons que pour tout $H \in \mathcal{P}$, les groupes H et Out(H) satisfont tous deux l'alternative de Tits relativement à $\mathcal{C}$. Alors Out$(G, \mathcal{P})$ satisfait l'alternative de Tits relativement à $\mathcal{C}$.*

Le Théorème 8.11 se déduit de notre théorème principal (Théorème 8.1), et de la description suivante du groupe $\mathrm{Out}(G, \mathcal{P})$ dans le cas où G est *librement indécomposable relativement à* $\mathcal{P}$, i.e. G ne se scinde pas en un produit libre $G = A * B$ de sorte que chaque sous-groupe dans $\mathcal{P}$ soit conjugué à un sous-groupe de A ou de B.

Théorème 8.12. *(Guirardel-Levitt [GL14, Théorème 1.4]) Soit G un groupe sans torsion à un bout, hyperbolique relativement à une famille finie $\mathcal{P}$ de groupes de type fini, et librement indécomposable relativement à $\mathcal{P}$. Il existe un sous-groupe d'indice fini $Out^0(G, \mathcal{P})$ de $Out(G, \mathcal{P})$ pour lequel nous avons une suite exacte de la forme*

$$1 \to \mathcal{T} \to Out^0(G, \mathcal{P}) \to \prod_{i=1}^{p} Mod(S_i) \times \prod_{H \in \mathcal{H}} Out(H),$$

où $\mathcal{T}$ est un groupe abélien libre de type fini, et $Mod(S_i)$ est le groupe modulaire d'une surface compacte S_i.

Un groupe G est *relativement hyperbolique VPC* s'il est sans torsion, et hyperbolique relativement à une famille finie de sous-groupes virtuellement polycycliques. Le résultat suivant s'obtient comme conséquence du Théorème 8.11 ; nous renvoyons à la Partie 7 de l'Annexe G pour un argument détaillé.

Théorème 8.13. *Soit G un groupe relativement hyperbolique VPC. Alors $Out(G)$ satisfait l'alternative de Tits.*

En particulier, ce théorème s'applique lorsque G est un groupe *relativement hyperbolique torique*, i.e. G est sans torsion, et est hyperbolique relativement à une famille finie de sous-groupes abéliens de type fini.

Questions ouvertes et perspectives

Nous regroupons pour conclure un certain nombre de questions ouvertes, rencontrées au fil de cette thèse, qui ouvrent un certain nombre de perspectives pour un travail futur.

Questions portant sur la géométrie des complexes hyperboliques associés à Out(F_N).

1. Déterminer le groupe des automorphismes simpliciaux du complexe des facteurs libres. Est-il vrai que tout automorphisme est induit par un élément de $\text{Out}(F_N)$?
2. Le complexe des scindements libres est-il uniformément hyperbolique, i.e. la constante d'hyperbolicité dépend-elle du rang N du groupe libre ? (La constante d'hyperbolicité du complexe des courbes [ou des arcs] d'une surface compacte orientable est indépendante de la topologie de la surface). La même question se pose également pour les complexes des facteurs libres et des scindements cycliques. Plus généralement, la constante d'hyperbolicité des versions de ces complexes associées à des produits libres de groupes dépend-elle du rang de Kurosh ? Une stratégie envisageable pour répondre à ces questions serait de définir des analogues aux chemins de licorne utilisés par Hensel, Przytycki et Webb dans [HPW13].
3. Peut-on donner une nouvelle démonstration du théorème de Farb et Handel affirmant que le groupe $\text{Out}(F_N)$ est égal à son propre commensurateur abstrait [FH07], à partir de la détermination du groupe des isométries du graphe des scindements cycliques de F_N ? Il semblerait utile à cette fin de pouvoir caractériser les twists de Dehn parmi les éléments de $\text{Out}(F_N)$, par exemple en termes de leurs centralisateurs.
4. Peut-on proposer une autre interprétation du bord de Gromov du complexe des scindements cycliques en termes de laminations ?
5. Établir une classification des isométries du complexe des scindements cycliques. En particulier, caractériser les éléments de $\text{Out}(F_N)$ agissant de manière loxodromique sur ce complexe.
6. Déterminer le bord de Gromov du complexe des scindements libres.
7. Poursuivre l'investigation de la géométrie des différents complexes hyperboliques associés à $\text{Out}(F_N)$. Par exemple, l'action de $\text{Out}(F_N)$ sur le graphe des facteurs libres, ou sur le graphe des scindements cycliques, est-elle acylindrique au sens de Bowditch [Bow06] ? L'action sur le graphe des scindements cycliques satisfait-elle la condition WPD de Bestvina et Fujiwara [BF02] ?

Questions portant sur la compactification primitive et l'horofrontière de CV_N.

1. Montrer que la compactification primitive de l'outre-espace est contractile.
2. Décrire l'horofrontière de l'outre-espace pour la distance inversée. La même question se pose pour les espaces de Teichmüller munis de la distance asymétrique de Thurston.

Questions portant sur les marches aléatoires sur Out(F_N).

1. Soit μ une loi de probabilité sur $\mathrm{Out}(F_N)$, dont le support engendre un sous-groupe non élémentaire de $\mathrm{Out}(F_N)$. Est-il vrai que $\mathbb{P}$-presque toute trajectoire de la marche aléatoire sur $(\mathrm{Out}(F_N), \mu)$ converge vers un arbre uniquement ergométrique ? uniquement ergodique ? Peut-on donner un analogue dans l'outre-espace au critère de Masur pour les géodésiques de Teichmüller [Mas92] ?
2. Définir des géodésiques bi-infinies dans CV_N. A-t-on une propriété d'approximation de la trajectoire typique d'une marche aléatoire par un rayon géodésique, comme dans le cas des espaces de Teichmüller [Tio14] ?
3. Peut-on donner un analogue au théorème d'Oseledets pour les produits aléatoires d'automorphismes de F_N, dans le cas de cocycles mesurables (i.e. sans hypothèse d'indépendance des incréments successifs) ?
4. Étudier la dépendance des coefficients de Lyapunov pour la marche aléatoire sur $(\mathrm{Out}(F_N), \mu)$ en fonction de la mesure μ.
5. Que peut-on dire de la croissance des éléments de F_N sous l'action d'un produit aléatoire d'automorphismes obtenus par multiplication successive d'incréments à droite ?
6. Peut-on établir un théorème central limite pour la marche aléatoire sur $\mathrm{Out}(F_N)$?

Questions portant sur les complexes associés à un produit libre.

1. Le complexe des facteurs libres associé à un produit libre est hyperbolique (Handel–Mosher [HM14b]). Le bord de Gromov s'identifie-t-il à l'espace des arbres arationnels ? Il semblerait intéressant pour cela d'initier une théorie des courants géodésiques pour les produits libres.
2. Quels sont les éléments de $\mathrm{Out}(G, \mathcal{F})$ qui agissent de manière loxodromique sur chacun des différents complexes étudiés ?

Questions relatives à l'alternative de Tits, portant sur les sous-groupes du groupe des automorphismes d'un produit libre.

1. Les sous-groupes résolubles de $\mathrm{Out}(G, \mathcal{F}^{(t)})$ sont-ils virtuellement abéliens dès lors que chacun des facteurs périphériques vérifie cette propriété ?
2. Soit Γ un réseau irréductible d'un groupe de Lie de rang réel supérieur ou égal à 2. Que peut-on dire des morphismes de Γ à valeurs dans $\mathrm{Out}(G, \mathcal{F}^{(t)})$?
3. Peut-on montrer un résultat analogue à l'alternative d'Handel et Mosher pour le groupe $\mathrm{Out}(G, \mathcal{F})$? Pour obtenir un tel résultat, il serait intéressant d'essayer de faire fonctionner l'argument proposé pour établir l'alternative de Tits pour le groupe des automorphismes extérieurs d'un produit libre, en considérant l'action de $\mathrm{Out}(G, \mathcal{F})$ sur le complexe des facteurs libres relatifs.
4. Peut-on montrer des versions plus fortes de l'alternative de Tits ? Par exemple, peut-on renforcer la condition consistant à contenir un sous-groupe libre non abélien en d'autres conditions de largeur, comme la SQ-universalité ?

Annexes

Présentation des annexes

Le lecteur trouvera dans les annexes suivantes des démonstrations détaillées des résultats présentés dans cette thèse.

L'Annexe A résout le problème de la rigidité spectrale de l'ensemble des éléments primitifs de F_N dans $\overline{cv_N}$ présenté au Chapitre 2. Nous obtenons donc une description de la compactifictaion primitive de CV_N, qui est motivée par son identification avec la compactification par horofonctions de CV_N.

Cette identification est faite en Annexe B. Le lecteur y trouvera également une description plus détaillée de la géométrie de l'horofrontière de CV_N (avec notamment la description des points de Busemann, et le lien avec la complétion métrique de CV_N), ainsi qu'une discussion de l'horofrontière inverse de CV_N. On y trouvera également l'application de notre description de l'horofrontière de l'outre-espace à l'étude de la croissance des éléments de F_N sous l'effet d'un produit aléatoire d'automorphismes du groupe libre. L'Annexe B complète le Chapitre 5 de cette thèse.

Dans l'Annexe C, nous poursuivons notre étude des marches aléatoires sur $\mathrm{Out}(F_N)$, avec la détermination du bord de Poisson de $\mathrm{Out}(F_N)$. En particulier, nous y donnons une démonstration de la convergence presque sûre de la marche aléatoire sur $\mathrm{Out}(F_N)$ vers un simplexe d'arbres arationnels et libres. L'Annexe C complète le Chapitre 4 de cette thèse.

L'Annexe D est consacrée à la démonstration de l'alternative pour les sous-groupes de $\mathrm{Out}(F_N)$ présentée au Chapitre 6. Elle peut être lue de manière indépendante, ou comme préambule à la lecture de la preuve de l'alternative de Tits pour le groupe des automorphismes d'un produit libre, qui fait l'objet de l'Annexe G.

Les Annexes E et F préparent la démonstration de cette alternative de Tits, en étudiant la géométrie des espaces associés à un produit libre de groupes. Dans l'Annexe E, nous étudions l'adhérence de l'outre-espace associé à un produit libre. Nous l'identifions avec l'espace des classes d'homothétie équivariante d'actions minimales très petites, et en déterminons la dimension topologique.

Dans l'Annexe F, nous montrons l'hyperbolicité des graphes des scindements libres et (maximalement)cycliques pour un produit libre. Surtout, nous décrivons le bord de Gromov du graphe des scindements cycliques.

L'Annexe G contient la démonstration de l'alternative de Tits pour le groupe des automorphismes extérieurs d'un produit libre de groupes.

Annexe A

Spectral rigidity for primitive elements of F_N

Abstract

Two trees in the boundary of outer space are said to be *primitive-equivalent* whenever their translation length functions are equal in restriction to the set of primitive elements of F_N. We give an explicit description of this equivalence relation, showing in particular that it is nontrivial. This question is motivated by our description of the horoboundary of outer space for the Lipschitz metric in [4]. Along the proof, we extend a theorem due to White about the Lipschitz metric on outer space to trees in the boundary, showing that the infimal Lipschitz constant of an F_N-equivariant map between the metric completion of any two minimal, very small F_N-trees is equal to the supremal ratio between the translation lengths of the elements of F_N in these trees. We also provide approximation results for trees in the boundary of outer space.

Contents

Introduction

Outer space CV_N was introduced by Culler and Vogtmann in [CV86] with a view to studying the group $\mathrm{Out}(F_N)$ of outer automorphisms of a finitely generated free group. The space CV_N (or its unprojectivized version cv_N) is the space of equivariant homothety (isometry) classes of simplicial free, minimal, isometric actions of F_N on simplicial metric trees. The *translation length* of an element g of a group G acting on an $\mathbb{R}$-tree T is defined as $||g||_T := \inf_{x\in T} d_T(x, gx)$. Translation lengths provide an embedding of cv_N into $\mathbb{R}^{F_N}$, whose image has projectively compact closure, as was proved by Culler and Morgan [CM87]. This compactification $\overline{CV_N}$ of outer space was described by Cohen and Lustig [CL95] and Bestvina and Feighn [BF94] as the space of homothety classes of minimal, very small actions of F_N on $\mathbb{R}$-trees.

Instead of considering the translation lengths of all elements of F_N, one might only look at the subset $\mathcal{P}_N$ of primitive elements of F_N, i.e. those elements that belong to some free basis of F_N, and get another compactification of CV_N as a subspace of $\mathbb{PR}^{\mathcal{P}_N}$, which we call the *primitive compactification* of outer space. Our original motivation for describing this alternative compactification comes from the desire to get a description of the horoboundary of outer space with respect to the so-called Lipschitz metric on CV_N, whose systematic study was initiated by Francaviglia and Martino in [FM11b] (the distance between $T, T' \in CV_N$ is defined as the logarithm of the infimal Lipschitz constant of an F_N-equivariant map from the covolume 1 representative of T to the covolume 1 representative of T'). This in turn is motivated by the question of describing the behaviour of random walks on $\text{Out}(F_N)$: in [4], we derive an Oseledets-like theorem about possible growth rates of elements of F_N under iteration of random automorphisms of the free group from a description of the horoboundary of outer space. It turns out that the horocompactification of outer space is isomorphic to the primitive compactification [4].

Describing the primitive compactification of outer space requires understanding the lack of rigidity of the set $\mathcal{P}_N$ in $\overline{cv_N}$, i.e. giving a description of the equivalence relation that identifies $T, T' \in \overline{cv_N}$ whenever their translation length functions are equal in restriction to $\mathcal{P}_N$. This question may also be considered of independent interest, as part of a much wider class of problems arising in several contexts. The *marked length spectrum rigidity conjecture* is still open for Riemannian manifolds : it is not known whether the isometry type of a negatively curved closed Riemannian manifold M is determined by the length of the geodesic representatives of the free homotopy classes of curves in M (this was proven to be true in the case of surfaces by both Croke [Cro90] and Otal [Ota90]). Culler and Morgan's result states that an analogue of the marked length spectrum rigidity conjecture holds for free groups. It is then natural to ask, given a subset $S \subseteq F_N$, whether it is spectrally rigid in cv_N (or in the closure $\overline{cv_N}$), i.e. whether the restriction to S of the marked length spectrum of a tree $T \in cv_N$ (or $T \in \overline{cv_N}$) determines T up to F_N-equivariant isometry. Several results have already been obtained for some classes of subsets of F_N. Smillie and Vogtmann have shown that no finite subset of F_N is spectrally rigid in $\overline{cv_N}$ for $N \geq 3$ [SV92]. Kapovich has proved that almost every trajectory of the simple nonbacktracking random walk on F_N with respect to any free basis yields a subset of F_N that is spectrally rigid in cv_N [Kap12]. Ray has proved that for all $\phi \in \text{Aut}(F_N)$ and all $g \in F_N$, the ϕ-orbit of g is not spectrally rigid in cv_N [Ray12]. Finally, Carette, Francaviglia, Kapovich and Martino have shown that the set $\mathcal{P}_N$ (and more generally the H-orbit of any $g \in F_N$ for $N \geq 3$, where H is any subgroup of $\text{Aut}(F_N)$ that projects to a nontrivial normal subgroup of $\text{Out}(F_N)$) is spectrally rigid in cv_N [CFKM12], and they raise the question of whether $\mathcal{P}_N$ is spectrally rigid in $\overline{cv_N}$ for $N \geq 3$ (for $N = 2$, they provide a counterexample, attributed to Tao).

An element of F_N is *simple* if it belongs to some proper free factor of F_N. One can define another equivalence relation on $\overline{cv_N}$, by saying that two trees are *simple-equivalent* if they have the same translation length functions in restriction to the set of *simple* elements of F_N. One easily checks that this equivalence relation is the same as the one define above using primitive elements (this is the content of Proposition A.2.1 of the present paper), and it turns out that it is sometimes easier to work with the collection of simple elements of F_N rather than primitive elements in our arguments.

Generalizing Tao's counterexample to higher ranks, we show that the set $\mathcal{P}_N$ is not spectrally rigid in $\overline{cv_N}$, yet we also show that this class of examples is the only obstruction to spectral rigidity of $\mathcal{P}_N$ in $\overline{cv_N}$. Our construction is the following. Let T_0 be a (non necessarily minimal) F_{N-1}-tree in which some point is fixed by an element of F_{N-1} not

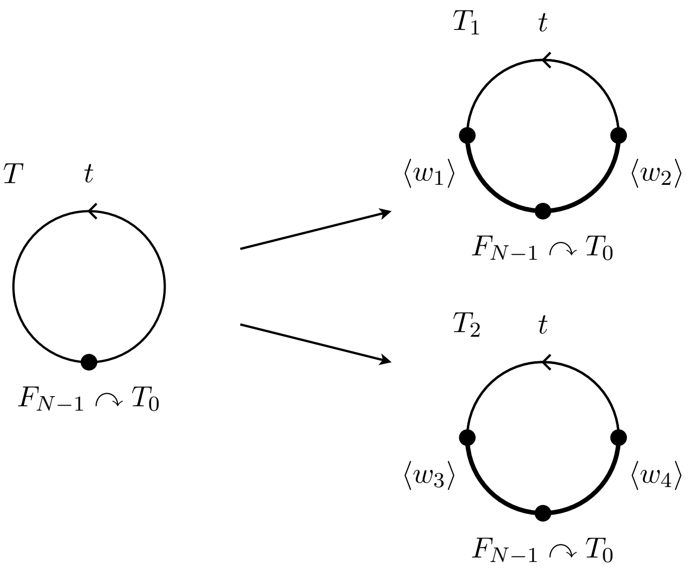


Figure A.1: The trees T_1 and T_2 are *special-pull-equivalent* if w_1, w_2, w_3, w_4 do not belong to any proper free factor of F_{N-1}.

contained in any proper free factor (the simplest example is to consider a tree T_0 reduced to a point, but one can also find more elaborated simplicial examples, as well as nonsimplicial examples by considering trees dual to a measured foliation on a surface with one single boundary component). Let $T \in \overline{cv_N}$ be a tree given by a graph of actions having T_0 as its only vertex tree, and having a single edge e with trivial edge group. Two trees are said to be *special-pull-equivalent* if they are both obtained from such a tree by partially or totally folding the edge e from one or both of its extremities along translates of the form ge, where $g \in F_{N-1}$ does not belong to any proper free factor of F_{N-1}, see Figure A.1, see also Section A.2.2 for precise definitions. The lack of spectral rigidity of the set $\mathcal{P}_N$ in $\overline{cv_N}$ is precisely given by this equivalence relation on trees.

Theorem A.0.1. *For all $T, T' \in \overline{cv_N}$, the following assertions are equivalent.*

- *For all $g \in \mathcal{P}_N$, we have $||g||_T = ||g||_{T'}$.*
- *For all simple elements $g \in F_N$, we have $||g||_T = ||g||_{T'}$.*
- *The trees T and T' are special-pull-equivalent.*

Carette, Francaviglia, Kapovich and Martino's result, which states that the set of primitive elements is spectrally rigid in cv_N, is derived from Francaviglia and Martino's work [FM11b] about extremal Lipschitz distortion between trees $T, T' \in cv_N$. The key point, due to White, is that the minimal Lipschitz constant of an F_N-equivariant map from T to T' is also equal to $\Lambda(T,T') := \sup_{g\in F_N} \frac{||g||_{T'}}{||g||_T}$, and this supremum is achieved on a finite set of primitive elements called *candidates*, represented in the quotient graph T/F_N by a special class of loops.

In order to study rigidity of $\mathcal{P}_N$ in $\overline{cv_N}$, we extend White's result to trees in the boundary of the unprojectivized outer space. Interested in the metric completion of outer space, Algom-Kfir extended it to the case where T is a simplicial tree (possibly with nontrivial stabilizers) [AK13, Proposition 4.5]. We generalize it to arbitrary trees in $\overline{cv_N}$, thus answering Algom-Kfir's question [AK13, Question 4.6]. Given $T, T' \in \overline{cv_N}$, we define $\mathrm{Lip}(T,T')$ to be the infimum of a Lipschitz constant of an F_N-equivariant map $f : T \to \overline{T'}$, where $\overline{T'}$ denotes the metric completion of T', if such a map exists, and $\mathrm{Lip}(T,T') = +\infty$ otherwise. We define $\Lambda(T,T') := \sup_{g\in F_N} \frac{||g||_{T'}}{||g||_T}$ (where we take the conventions $\frac{0}{0}=0$ and $\frac{1}{0}=+\infty$). We prove equality between these two notions of stretching.

Theorem A.0.2. *For all $T,T' \in \overline{cv_N}$, we have $Lip(T,T') = \Lambda(T,T')$.*

Our proof of Theorem A.0.2 relies on a structure theory of trees in the boundary of outer space. Levitt showed in [Lev94] that any tree $T \in \overline{cv_N}$ splits as a graph of actions whose vertex trees have dense orbits (they can be reduced to a point). The case of trees with dense orbits, considered in Section A.5, relies on two side results that provide us some approximation techniques. The first of these results extends work by Bestvina and Feighn [BF94] and Guirardel [Gui98], and gives a way of approximating trees with dense orbits by free actions on simplicial metric trees.

Theorem A.0.3. *For all $T \in \overline{cv_N}$, the following assertions are equivalent.*

- *There exists a sequence $(T_n)_{n\in\mathbb{N}}$ of trees in cv_N converging to T, such that for all $n \in \mathbb{N}$, there exists a 1-Lipschitz F_N-equivariant map $f_n : T_n \to T$.*
- *All arc stabilizers in T are trivial.*

The second side result we use enables us to build Lipschitz F_N-equivariant maps between trees in $\overline{cv_N}$ by a limiting process.

Theorem A.0.4. *Let T and T' be two very small F_N-trees, let $(T_n)_{n\in\mathbb{N}}$ (resp. $(T'_n)_{n\in\mathbb{N}}$) be a sequence of trees converging to T (resp. T') in the equivariant Gromov-Hausdorff topology, and let $(M_n)_{n\in\mathbb{N}}$ be a sequence of real numbers, satisfying $M := \liminf_{n\to+\infty} M_n < +\infty$. Assume that for all $n \in \mathbb{N}$, there exists an M_n-Lipschitz F_N-equivariant map $f_n : T_n \to T'_n$. Then there exists an M-Lipschitz F_N-equivariant map $f : T \to \overline{T'}$.*

The rest of our proof of Theorem A.0.2, which is carried out in Section A.6, consists in extending the standard techniques in the proof of White's theorem to be able to deal with graphs of actions.

We also extend the notion of candidates to trees in the boundary of outer space. However, the supremum in the definition of $\Lambda(T,T')$ can no longer be taken over the set $\mathcal{P}_N$ in general. This property nevertheless holds true for trees that contain at least two orbits of edges with trivial stabilizer in their simplicial part. Further building on the work of Bestvina and Feighn [BF94] and Guirardel [Gui98], we obtain an approximation result by such trees, which is used in Section A.7 to derive Theorem A.0.1.

The paper is organized as follows. In Section A.1, we review basic facts about outer space and $\mathbb{R}$-trees. In Section A.2, we prove that two trees in $\overline{cv_N}$ are primitive-equivalent if and only if they are simple-equivalent (Section A.2.1). We also give (and discuss) the precise definition of the special-pull-equivalence relation (Section A.2.2), and we show that special-pull-equivalent trees are simple-equivalent, using the Whitehead algorithm (Section A.2.3). We also define the primitive compactification of outer space (Section A.2.4). Section A.3 is dedicated to the proof of Theorem A.0.3, by using techniques of approximations by geometric trees. We also prove an approximation result by trees having at least two orbits of edges with trivial stabilizers (Theorem A.3.11). Section A.4 is devoted to the proof of Theorem A.0.4. In the next two sections, we prove our extension of White's theorem to trees in $\overline{cv_N}$. The case of trees with dense orbits is treated in Section A.5, where we also prove that simple-equivalent trees with dense orbits are equal. We complete the proof of Theorem A.0.2 in Section A.6. We also generalize the notion of candidates

(Section A.6.5), and give more precise statements in the case of trees that have two distinct orbits of edges with trivial stabilizers in their simplicial parts (Section A.6.6). In Section A.7, we complete the proof of Theorem A.0.1, by proving that simple-equivalent trees are special-pull-equivalent.

Acknowledgments

It is a pleasure to thank my advisor Vincent Guirardel for his helpful advice, and his patience in reading through first drafts of the present paper and suggesting many improvements and simplifications.

A.1 Outer space and its closure

We start by fixing a few notations and recalling standard facts about outer space and F_N-actions on $\mathbb{R}$-trees in its closure.

A.1.1 Outer space and its closure

Outer space CV_N was defined by Culler and Vogtmann in [CV86] to be the space of simplicial, free, minimal, isometric actions of F_N on simplicial metric trees, up to equivariant homothety (an action of F_N on a tree is said to be *minimal* if there is no proper invariant subtree). We denote by cv_N the *unprojectivized outer space*, in which trees are considered up to isometry, instead of homothety. The reader is referred to [Vog02] for an excellent survey and reference article about outer space.

An $\mathbb{R}$-*tree* is a metric space (T, d_T) in which any two points x and y are joined by a unique arc, which is isometric to a segment of length $d_T(x, y)$ (the reader is referred to [CM87] for an introduction to $\mathbb{R}$-trees). Let T be an F_N-*tree*, i.e. an $\mathbb{R}$-tree equipped with an isometric action of F_N. For $g \in F_N$, the *translation length* of g in T is defined to be

$$||g||_T := \inf_{x \in T} d_T(x, gx).$$

Culler and Morgan have shown in [CM87, Theorem 3.7] that the map

$$\begin{array}{cccc} i: & cv_N & \to & \mathbb{R}^{F_N} \\ & T & \mapsto & (||g||_T)_{g \in F_N} \end{array}$$

is injective, and actually a homeomorphism onto its image. More precisely, the following holds.

Theorem A.1.1. *(Culler-Morgan [CM87, Theorem 3.7]) Let T, T' be two minimal F_N-trees. If $||g||_T = ||g||_{T'}$ for all $g \in F_N$, then there is a unique F_N-equivariant isometry from T to T'.*

Taking the quotient by equivariant homotheties, we get an embedding of CV_N into the projective space $\mathbb{PR}^{F_N}$, whose image has compact closure $\overline{CV_N}$ [CM87, Theorem 4.5]. Hence $\overline{CV_N}$ is a compactification of CV_N. Bestvina and Feighn [BF94], extending results by Cohen and Lustig [CL95], have identified the compactification $\overline{CV_N}$ as the space of homothety classes of minimal, *very small* F_N-trees, i.e. trees with trivial or maximally cyclic arc stabilizers and trivial tripod stabilizers. We also denote by $\overline{cv_N}$ the lift of $\overline{CV_N}$ to $\mathbb{R}^{F_N}$. We call the topology induced by this embedding on each of the spaces $CV_N, \overline{CV_N}, cv_N$ and $\overline{cv_N}$ the *axes topology*, it is equivalent to the *weak topology* on CV_N introduced by Culler and Vogtmann in [CV86].

A.1.2 A metric on outer space

There is a natural asymmetric metric on outer space, whose systematic study was initiated by Francaviglia and Martino in [FM11b] : given $T, T' \in cv_N$, the distance $d(T,T')$ is defined as the logarithm of the infimal Lipschitz constant of an F_N-equivariant map from T to T' (see also [AK13, Section 2.4]). An easy Arzelà-Ascoli argument shows that this infimal Lipschitz constant is actually achieved [FM11b, Lemma 3.4]. This defines a topology on outer space, which is equivalent to the usual one [FM11b, Theorems 4.11 and 4.18]. An element $g \in F_N$ is a *candidate* in T if it is represented in the quotient graph $X := T/F_N$ by a loop which is either

- an embedded circle in X, or
- an embedded bouquet of two circles in X, i.e. $\gamma = \gamma_1\gamma_2$, where γ_1 and γ_2 are embedded circles in X which meet in a single point, or
- a barbell graph, i.e. $\gamma = \gamma_1\eta\gamma_2\overline{\eta}$, where γ_1 and γ_2 are embedded circles in X that do not meet, and η is an embedded path in X that meets γ_1 and γ_2 only at their origin (and $\overline{\eta}$ denotes the path η crossed in the opposite direction). We call η the *central path* of γ.

The following result, due to White, gives an alternative description of the metric on outer space. A proof can be found in [FM11b, Proposition 3.15], it was simplified by Algom-Kfir in [AK11, Proposition 2.3].

Theorem A.1.2. *(White, see [FM11b, Proposition 3.15] or [AK11, Proposition 2.3]) For all F_N-trees T,$T' \in CV_N$, we have*

$$d(T,T') = \log \sup_{g \in F_N \smallsetminus \{e\}} \frac{||g||_{T'}}{||g||_T}.$$

Furthermore, the supremum is achieved for an element $g \in F_N$ which is a candidate in $X := T/F_N$.

Notice in particular that candidates in X are *primitive* elements of F_N, i.e. they belong to some free basis of F_N (see Lemma A.1.12, for instance). White's theorem has been extended by Algom-Kfir to the case of two trees $T, T' \in \overline{cv_N}$ when T is assumed to be simplicial (in [AK13, Proposition 4.5], Algom-Kfir states her result when T' is a tree in the metric completion of outer space, but it actually holds true with the same proof for all trees $T' \in \overline{cv_N}$). We denote by $\mathrm{Lip}(T,T')$ the infimal Lipschitz constant of an F_N-equivariant map from T to T'.

Theorem A.1.3. *(Algom-Kfir [AK13, Proposition 4.5]) Let $T, T' \in \overline{cv_N}$. If T is simplicial, then*

$$Lip(T,T') = \sup_{g \in F_N \smallsetminus \{e\}} \frac{||g||_{T'}}{||g||_T}.$$

Furthermore, the supremum is achieved for an element $g \in F_N$ which is a candidate in $X := T/F_N$.

A.1.3 Decomposing actions in $\overline{CV_N}$

We now recall a result due to Levitt [Lev94] which allows to decompose any F_N-tree into simpler actions. The reader is referred to [Ser77] for an introduction to graphs of groups and related terminology. An *F_N-graph of actions* consists of

- a marked metric graph of groups, whose edges all have positive length, with fundamental group F_N, with vertex groups G_v, edge groups G_e, and for every oriented edge e with terminal vertex $t(e)$, an injective morphism $i_e : G_e \to G_{t(e)}$, and
- an isometric action of every vertex group G_v on an $\mathbb{R}$-tree T_v (possibly reduced to a point), and
- a point $p_e \in T_{t(e)}$ fixed by $i_e(G_e) \subseteq G_{t(e)}$ for every oriented edge e.

Associated to any F_N-graph of actions $\mathcal{G}$ is an F_N-tree $T(\mathcal{G})$. Informally, the tree $T(\mathcal{G})$ is obtained from the Bass-Serre tree of the underlying graph of groups by equivariantly attaching the vertex trees T_v at the vertices v, an incoming edge being attached to T_v at the prescribed attaching point. The reader is referred to [Gui98, Proposition 3.1] for a precise description of the tree $T(\mathcal{G})$. We say that an F_N-tree T *splits as a graph of actions* if there exists a graph of actions $\mathcal{G}$ such that $T = T(\mathcal{G})$. An F_N-tree T has *dense orbits* if the F_N-orbit of one (and hence every) point of T is dense in T.

Theorem A.1.4. *(Levitt [Lev94, Theorem 5]) Every $T \in \overline{cv_N}$ splits uniquely as a graph of actions with vertex trees having dense orbits (possibly reduced to a point).*

We denote by T^{simpl} the corresponding simplicial tree, obtained by collapsing all the vertex trees to points. An *edge* in T is a segment in the simplicial part of T that projects to an edge in T^{simpl}.

A.1.4 Trees with dense orbits

In this head, we collect a few facts about F_N-trees with dense orbits.

Lemma A.1.5. *(Bestvina-Feighn [BF94, Remark 1.9], Gaboriau-Levitt [GL95, Proposition I.10], Sela [Sel96, Proposition 1.4], Levitt-Lustig [LL03, Lemma 4.2]) Every very small F_N-tree with dense orbits has trivial arc stabilizers.*

Given a tree $T \in \overline{cv_N}$, a subset $X \subseteq T$, and $M \in \mathbb{R}$, we denote by $\mathcal{N}_M(X)$ the M-neighborhood of X in T. The *bridge* between two closed subtrees $X, Y \subseteq T$ which do not intersect is the unique segment in T which meets $X \cup Y$ only at its endpoints. Given a closed subtree $X \subseteq T$ and $x \in T$, we denote by $\pi_X(x)$ the closest point projection of x to the subtree X.

Lemma A.1.6. *Let T be an $\mathbb{R}$-tree, let $X, Y \subseteq T$ be closed subtrees, and let $M \in \mathbb{R}$. If $X \cap Y \neq \emptyset$, then $\mathcal{N}_M(X) \cap \mathcal{N}_M(Y) = \mathcal{N}_M(X \cap Y)$. If $X \cap Y = \emptyset$, then for any point y in the bridge between X and Y in T, we have $\mathcal{N}_M(X) \cap \mathcal{N}_M(Y) \subseteq \mathcal{N}_M(\{y\})$.*

Proof. Let $x \in \mathcal{N}_M(X) \cap \mathcal{N}_M(Y)$. Let J denote the subtree $X \cap Y$, or the bridge between X and Y in case $X \cap Y = \emptyset$. Assume that $\pi_X(x) \notin J$. Then one checks that x and $\pi_X(x)$ must belong to the same component of $T \smallsetminus J$, and hence that $\pi_Y(x) \in J$. So either $\pi_X(x) \in J$, or $\pi_Y(x) \in J$, and the claim follows. $\square$

In the following statement, notice that whenever $T \in \overline{cv_N}$ is a tree with dense orbits, then the F_N-action on T uniquely extends to an isometric action on its metric completion $\overline{T}$, and $\overline{T}$ again has dense orbits. Recall that an F_N-tree is *minimal* if it contains no proper F_N-invariant subtree. An F_N-tree T which is not minimal has a unique minimal proper F_N-invariant subtree $T^{\min}$, which is also the union of all axes of hyperbolic elements in T. In particular, for all $g \in F_N$, we have $||g||_T = ||g||_{T^{\min}}$. When T has dense orbits, we have $T^{\min} \subseteq T \subseteq \overline{T^{\min}}$, since the orbit of any point $x \in T^{\min}$ is dense in T and contained in

$T^{\min}$. For all F_N-trees T and all $g \in F_N$, either $||g||_T = 0$ (we say that g is *elliptic* in T), and in this case g has a fixed point in T, or $||g||_T > 0$ (then g is said to be *hyperbolic* in T), and in this case g has an axis in T, i.e. there exists a subspace of T homeomorphic to the real line on which g acts by translation, with translation length $||g||_T$. In both cases, we define the *characteristic set* of g to be $C_T(g) := \{x \in T | d(x, gx) = ||g||_T\}$ (see [CM87, 1.3] for a description of the action of elements of F_N on F_N-trees).

Proposition A.1.7. *Let T, T' be very small F_N-trees with dense orbits. Then there exists at most one Lipschitz F_N-equivariant map from T to $\overline{T'}$.*

Proof. It is enough to show that for all $M \in \mathbb{R}$, there exists at most one M-Lipschitz F_N-equivariant map from T to $\overline{T'}$, so we fix $M \in \mathbb{R}$. Let $x \in T$, and let $\epsilon > 0$. We claim that we can find a subset $X_\epsilon \subset \overline{T'}$ whose diameter is bounded above by $4M\epsilon$, with the property that $f(x) \in X_\epsilon$ for all M-Lipschitz F_N-equivariant maps $f : T \to \overline{T'}$. Indeed, let $f : T \to \overline{T'}$ be M-Lipschitz and F_N-equivariant. As T has dense orbits, there exists $g \in F_N$ such that $d_T(x, gx) \leq \epsilon$, and the set of all such elements of F_N is not contained in any cyclic subgroup of F_N. As f is M-Lipschitz and F_N-equivariant, any such $g \in F_N$ satisfies $d_{\overline{T'}}(f(x), gf(x)) \leq M\epsilon$. We also have $d_{\overline{T'}}(f(x), gf(x)) = ||g||_{\overline{T'}} + 2d_{\overline{T'}}(f(x), C_{\overline{T'}}(g))$ (see [CM87, 1.3]), so $f(x) \in \mathcal{N}_{M\epsilon}(C_{\overline{T'}}(g))$ and $||g||_{T'} \leq M\epsilon$. Let $g, g' \in F_N$ be two elements satisfying $d_T(x, gx) \leq \epsilon$ and $d_T(x, g'x) \leq \epsilon$, which do not generate a cyclic subgroup of F_N (in particular, the commutator $[g, g']$ is nontrivial). We have $f(x) \in \mathcal{N}_{M\epsilon}(C_{\overline{T'}}(g)) \cap \mathcal{N}_{M\epsilon}(C_{\overline{T'}}(g'))$. As $\overline{T'}$ is very small and has dense orbits, it follows from Lemma A.1.5 that the commutator $[g, g']$ does not fix any arc in $\overline{T'}$, so $C_{\overline{T'}}(g) \cap C_{\overline{T'}}(g')$ is a (possibly empty) segment of length at most $2M\epsilon$ (see [CM87, 1.10]). By Lemma A.1.6, the set $\mathcal{N}_{M\epsilon}(C_{\overline{T'}}(g)) \cap \mathcal{N}_{M\epsilon}(C_{\overline{T'}}(g')) = \mathcal{N}_{M\epsilon}(C_{\overline{T'}}(g) \cap C_{\overline{T'}}(g'))$ has diameter at most $4M\epsilon$. We set $X_\epsilon := \mathcal{N}_{M\epsilon}(C_{\overline{T'}}(g)) \cap \mathcal{N}_{M\epsilon}(C_{\overline{T'}}(g'))$.

If $f, f' : T \to \overline{T'}$ are two M-Lipschitz, F_N-equivariant maps, then for all $\epsilon > 0$, we have $f(x), f'(x) \in X_\epsilon$, hence $d_{\overline{T'}}(f(x), f'(x)) \leq 4M\epsilon$. This implies that $f(x) = f'(x)$. As this is true for all $x \in T$, we get that $f = f'$. ☐

A.1.5 Morphisms between F_N-trees

A *morphism* between two $\mathbb{R}$-trees T and T' is a map $f : T \to T'$ such that every segment $J \subset T$ can be subdivided into finitely many subsegments, in restriction to which f is an isometry (in particular, any morphism between two $\mathbb{R}$-trees is 1-Lipschitz). We say that two arcs in T are *folded* by f if they have initial subsegments whose f-images are equal.

Let T be an F_N-tree containing an edge e with trivial stabilizer. A *U-turn* over e is a pair of distinct adjacent edges in T of the form (e, ge), where g belongs to the stabilizer of one of the extremities v of e and is not a proper power, such that either the stabilizer of the image of v in T^{simpl} has rank at least 2, or v does not project to a valence one vertex of the quotient graph of actions. In the following lemmas, we collect a few facts about F_N-equivariant morphisms between F_N-trees.

Lemma A.1.8. *Let T and T' be two very small F_N-trees. If T' has trivial arc stabilizers, then an F_N-equivariant morphism from T to T' cannot fold any U-turn in T.*

Proof. Assume that an F_N-equivariant morphism $f : T \to T'$ folds a U-turn (e, e') in T. Then there exists an initial segment I of e such that $f(I)$ has nontrivial stabilizer. The hypothesis made on T' implies that $f(I)$ is a point, contradicting the definition of a morphism. ☐

Lemma A.1.9. *Let T and T' be two very small F_N-trees. An F_N-equivariant morphism from T to T' cannot identify nontrivial initial segments of edges in T having distinct nontrivial stabilizers. It cannot either identify a nontrivial initial segment of an edge in T with nontrivial stabilizer with one of its translates.*

Proof. Otherwise, as edge stabilizers in T are maximally cyclic, the stabilizer of the image of these segments would have rank at least 2. As T' is very small, this image would be a point, contradicting the definition of a morphism. ☐

Lemma A.1.10. *Let T and T' be two very small F_N-trees. An F_N-equivariant morphism from T to T' cannot identify a nontrivial initial segment of an edge in T with nontrivial stabilizer with an arc lying in a vertex tree of T with dense orbits.*

Proof. Otherwise, the image of this segment would be an arc with nontrivial stabilizer lying in a vertex tree of T' with dense orbits. By Lemma A.1.5, it would thus be reduced to a point, contradicting the definition of a morphism. ☐

A.1.6 The quotient volume of F_N-trees

Let $T \in \overline{cv_N}$. The *volume* of a finite subtree $K \subset T$ (i.e. the convex hull of a finite number of points, which is a finite union of segments) is the sum of the lengths of the segments in K. The *quotient volume* of T is defined to be the infimal volume of a finite subtree of T whose F_N-translates cover T. We collect a few facts which were observed by Algom-Kfir in [AK13, Section 3.3].

Proposition A.1.11. *(Algom-Kfir [AK13])*

- *For all F_N-trees T, all minimal F_N-trees T' and all $L \in \mathbb{R}$, if there exists an L-Lipschitz F_N-equivariant map from T to T', then $qvol(T') \leq Lqvol(T)$.*
- *Let $T \in \overline{cv_N}$, and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of trees in cv_N converging to T. Then $qvol(T) \geq \limsup_{n\to+\infty} qvol(T_n)$. If in addition T^{simpl} contains an orbit of edges with nontrivial stabilizers, then $qvol(T) > \limsup_{n\to+\infty} qvol(T_n)$.*

A.1.7 Finding primitive elements in graphs of groups decompositions of F_N

We now state a lemma that will be useful for detecting primitive elements in graphs of groups decompositions of F_N.

Lemma A.1.12. *Let X be a minimal graph of groups decomposition of F_N containing an edge e with trivial stabilizer. Let $g \in F_N$. If g is elliptic in the Bass-Serre tree of X, or if any fundamental domain of its axis crosses the orbit of a lift of e at most once, then g is simple. For all vertex groups G in X, there exists a proper free factor of F_N that contains G.*

Proof. Let e be an edge in X with trivial stabilizer. Collapsing each component of the complement of e in X to a point yields a free splitting of F_N either of the form $F_N = F_{N-1}*$ if e is nonseparating (in which case we denote by t a stable letter), or of the form $F_N = F_k * F_{N-k}$ if e is separating. All vertex groups in X are contained in a free factor provided by the splitting. If the axis of a hyperbolic element $g \in F_N$ does not cross any lift of e, then g belongs to a proper free factor provided by the splitting. If any fundamental domain of the axis of g in the Bass-Serre tree T of X crosses the orbit of a lift of e

exactly once, then e is nonseparating and g is conjugated to an element of the form tg' with $g' \in F_{N-1}$, so g is primitive (because if $\{x_1, \dots, x_{N-1}\}$ is a free basis of F_{N-1}, then $\{x_1, \dots, x_{N-1}, tg'\}$ is a free basis of F_N). $\square$

A.2 Some equivalence relations on $\overline{cv_N}$

A.2.1 Primitive-equivalence versus simple-equivalence

Recall that an element $g \in F_N$ is *primitive* if it belongs to some free basis of F_N (we denote by $\mathcal{P}_N$ the set of primitive elements of F_N). It is *simple* if it belongs to some proper free factor of F_N. Two trees $T, T' \in \overline{cv_N}$ are *primitive-equivalent* if for all $g \in \mathcal{P}_N$, we have $||g||_T = ||g||_{T'}$. They are *simple-equivalent* if for all simple elements $g \in F_N$, we have $||g||_T = ||g||_{T'}$.

Proposition A.2.1. *Two elements $T, T' \in \overline{cv_N}$ are primitive-equivalent if and only if they are simple-equivalent.*

Proof. Simple-equivalent trees are obviously primitive-equivalent, as primitive elements are simple. Assume that $||g||_{T'} = ||g||_T$ for all $g \in \mathcal{P}_N$. Let $w \in F_N$ be simple, i.e there exists a free basis $\{a_1, \dots, a_N\}$ of F_N such that w belongs to the free factor of F_N generated by $a_1, \dots, a_{N-1}$. Then for all $k \in \mathbb{N}$, we have $a_N w^k \in \mathcal{P}_N$, because $\{a_1, \dots, a_{N-1}, a_N w^k\}$ is again a free basis of F_N. So

$$\begin{aligned} ||w||_{T'} &= \lim_{k \to +\infty} \tfrac{||a_N w^k||_{T'}}{k} \\ &= \lim_{k \to +\infty} \tfrac{||a_N w^k||_T}{k} \\ &= ||w||_T. \end{aligned}$$

This shows that T and T' are simple-equivalent. $\square$

A.2.2 Special-pull-equivalent trees

The following notion is illustrated in Figure A.2. The *corank* of a free factor F of F_N is the rank of any complementary free factor of F_N, i.e. it is equal to N minus the rank of F.

Definition A.2.2. *Let $T, \widehat{T} \in \overline{cv_N}$. The tree T is a* pull *of $\widehat{T}$ if there exist*

- *an edge e in $\widehat{T}$ with trivial stabilizer, and extremal vertices v_1 and v_2, and*
- *for each $i \in \{1, 2\}$, a (possibly degenerate) subsegment $J_i \subseteq e$ that contains v_i, such that $J_1 \cap J_2$ contains at most one point, and*
- *for each $i \in \{1, 2\}$ such that J_i is nondegenerate, an element g_i in the stabilizer of v_i, which is not a proper power,*

so that T is obtained as a quotient of $\widehat{T}$ by equivariantly identifying J_i with $g_i J_i$ for each $i \in \{1, 2\}$.

We will also say that T is obtained from $\widehat{T}$ by pulling the edge e, with pulling elements g_1 and g_2.

Definition A.2.3. *Let $T, \widehat{T} \in \overline{cv_N}$. The tree T is a* special pull *of $\widehat{T}$ if T is a pull of $\widehat{T}$ and, with the above notations, the edge e projects to a nonseparating edge in the underlying graph of the decomposition of $\widehat{T}$ as a graph of actions with dense orbits, and neither g_1 nor g_2 belongs to a corank 2 free factor of F_N.*

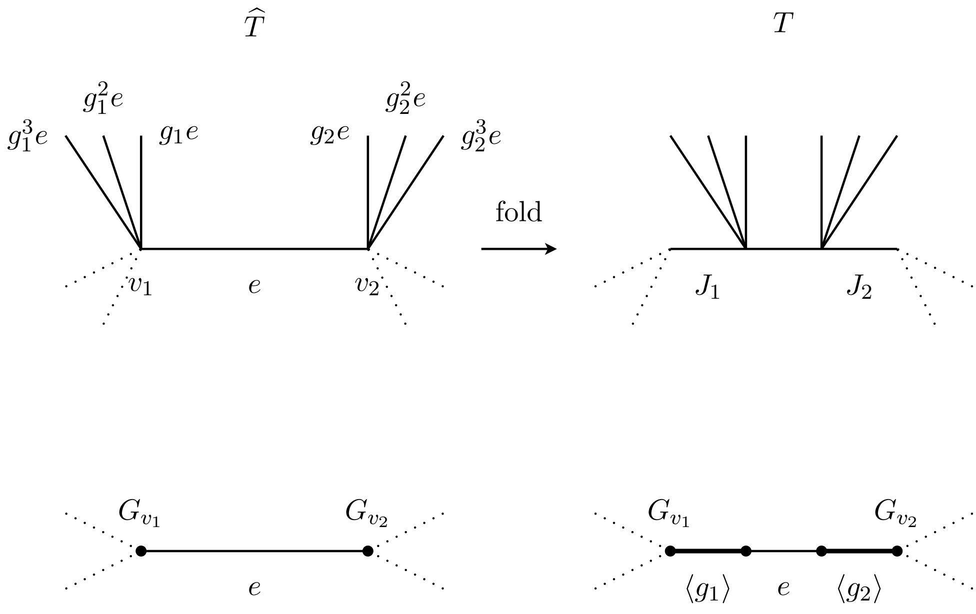


Figure A.2: The tree T is obtained by pulling $\widehat{T}$.

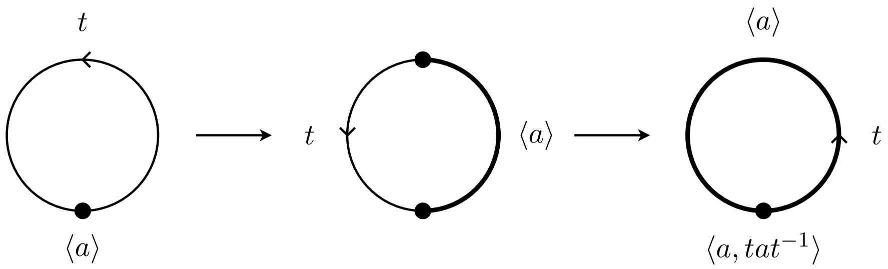


Figure A.3: Pulls in $\overline{cv_2}$.

Definition A.2.4. *Two trees $T, T' \in \overline{cv_N}$ are* special-pull-equivalent *if either $T = T'$, or there exists a tree $\widehat{T} \in \overline{cv_N}$ such that T and T' are special pulls of $\widehat{T}$.*

In the second case of Definition A.2.4, the tree $\widehat{T}$ has a single orbit of edges with trivial stabilizer, otherwise the pulling element would be contained in a corank 2 free factor of F_N. That special-pull-equivalence is indeed an equivalence relation will be proved in Lemma A.2.5.

Description of special-pull-equivalence in the case $N = 2$. If $N = 2$, then any pull is special. Let $T, \widehat{T} \in \overline{cv_2}$ be such that T is a special pull of $\widehat{T}$. We assume that there does not exist any tree $\widetilde{T} \neq \widehat{T} \in \overline{cv_2}$ such that both T and $\widehat{T}$ are special pulls of $\widetilde{T}$ with same pulling elements. As displayed on Figure A.3, the tree $\widehat{T}$ is then the Bass-Serre tree of a splitting of the form $F_2 = \langle a\rangle *$, where a is a primitive element of F_2, and T is obtained by (partially or totally) pulling the edge of $\widehat{T}$, with a as a pulling element.

An exhaustive description of the boundary of CV_2 was given by Culler and Vogtmann in [CV91]. With their terminology, the quotient $\overline{CV_2}/\sim$ is given by collapsing all spikes in $\overline{CV_2}$, and is thus homeomorphic to a disk with fins attached on top, see Figure A.4.

Description of special-pull-equivalence in the case $N \geq 3$. We now assume that $N \geq 3$, and we give a description of special-pull-equivalence in terms of graphs of actions.

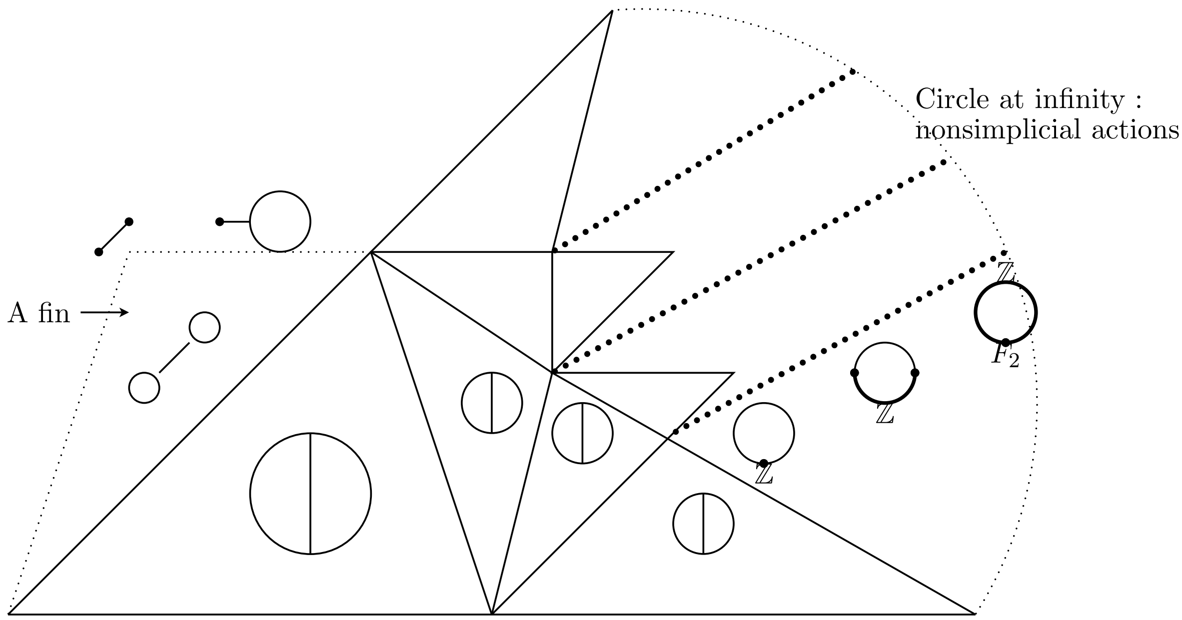


Figure A.4: The quotient space $\overline{CV_2}/\sim$ is obtained by collapsing all peaks in bold dotted lines to points, and hence is homeomorphic to a disk with fins attached on top.

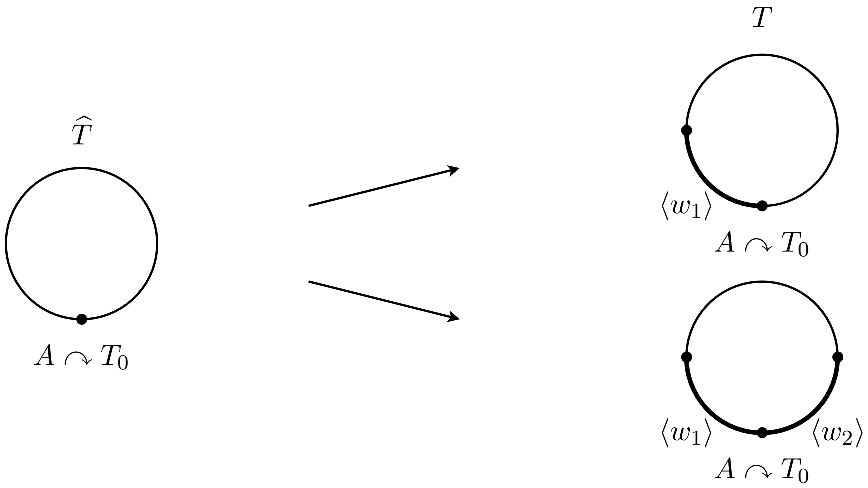


Figure A.5: The situation in Case 1.

The discussion below is illustrated in Figures A.5 to A.7. Let $\widehat{T} \in \overline{cv_N}$ be a tree with exactly one orbit of edges with trivial stabilizer e, and let $T \neq \widehat{T}$ be a special pull of $\widehat{T}$. We assume that there does not exist any tree $\widetilde{T} \neq \widehat{T} \in cv_N$ such that both T and $\widehat{T}$ are special pulls of $\widetilde{T}$ with same pulling elements. The tree $\widehat{T}$ splits as a graph of actions having

- a single vertex, whose corresponding vertex tree is a (non necessarily minimal) A-tree T_0, where A is a corank one free factor of F_N, and
- a single loop-edge with trivial stabilizer.

As $N \geq 3$, the group A is not cyclic, so it has at most one fixed point in $\widehat{T}$, and the A-minimal subtree T_0^{min} of T_0 is well-defined. Minimality of T implies that T_0 is obtained from T_0^{min} by possibly adding some completion points, and attaching at most two A-orbits of edges (the discussion below will show that we can actually attach at most one A-orbit of edges when passing from T_0^{min} to T_0). The valence one extremities of these edges are attaching points for e in $\widehat{T}$. One of the following situations occurs.

Case 1 (see Figure A.5): The tree T_0 is minimal (or more generally, we have T_0 is the closure of T_0^{min}, i.e. T_0 is obtained from T_0^{min} by adding completion points, or in other words $T_0 \smallsetminus T_0^{min}$ does not contain any simplicial edge).
Then T is obtained from $\widehat{T}$ by pulling e, either at one of its extremities or at both of its extremities. (Notice that we cannot perform any pull from a completion point).

Case 2 (see Figure A.6): The tree T_0 is not minimal, and $T_0 \smallsetminus T_0^{min}$ contains a simplicial edge e' whose stabilizer $\langle w \rangle$ is not contained in any proper free factor of A.
Then the valence one extremity of e' in the decomposition of T_0 as a graph of actions has valence at least 3 in the decomposition of $\widehat{T}$ as a graph of actions. Otherwise, the tree $\widehat{T}$ would be obtained from a tree $\widetilde{T}$ by pulling this edge, contradicting the assumption made on $\widehat{T}$. This implies in particular that T_0 is obtained from T_0^{min} by attaching a single orbit of edges. When passing from $\widehat{T}$ to T, the edge e is pulled at only one of its extremity, otherwise this would create a tripod stabilizer.

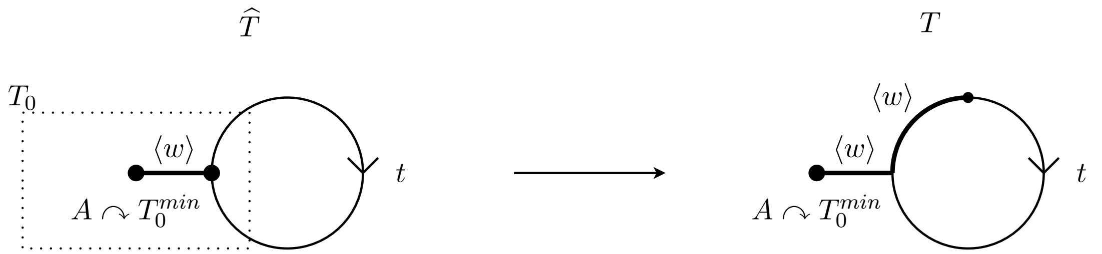


Figure A.6: The situation in Case 2.

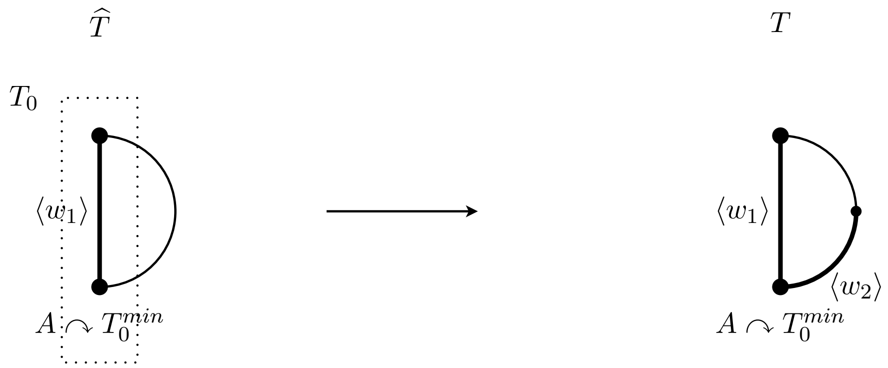


Figure A.7: The situation in Case 3.

Case 3 (see Figure A.7): The tree T_0 is not minimal, and $T_0 \smallsetminus T_0^{min}$ contains a simplicial edge whose stabilizer is nontrivial, and contained in some proper free factor of A.
If there were two such edges, or if e projected to a loop-edge in the decomposition of $\widehat{T}$ as a graph of action, then no special pulling operation could be performed on $\widehat{T}$, so we are in the situation displayed on Figure A.7.

Case 4 : The tree T_0 is not minimal, and $T_0 \smallsetminus T_0^{min}$ contains a simplicial edge with trivial stabilizer.
We will show that this case never happens. Indeed, the valence one extremity of this edge in the decomposition of T_0 as a graph of actions has valence at least 3 in the decomposition of $\widehat{T}$ as a graph of actions, and it has trivial stabilizer. In this situation, no special pulling operation can be performed on $\widehat{T}$, a contradiction.

Notice that in all cases, the simplicial part of $T_0 \smallsetminus T_0^{min}$ contains at most one orbit of edges.

Special-pull-equivalence is an equivalence relation.

Lemma A.2.5. *Special-pull-equivalence is an equivalence relation on $\overline{cv_N}$.*

Proof. Let $T \in \overline{cv_N}$, and assume that there exists a tree $\widehat{T} \in \overline{cv_N}$ so that T is a special pull of $\widehat{T}$. Then the corank one free factor A (with the notations from the above paragraph) is uniquely determined as being the smallest free factor of F_N containing all arc stabilizers in T, and the minimal A-tree T_0^{min} is determined by the restriction to A of the translation length function of T. It then follows from the description given in the previous paragraph of the relationship between T_0^{min} and $\widehat{T}$ that $\widehat{T}$ is uniquely determined. Lemma A.2.5 follows from this observation. □

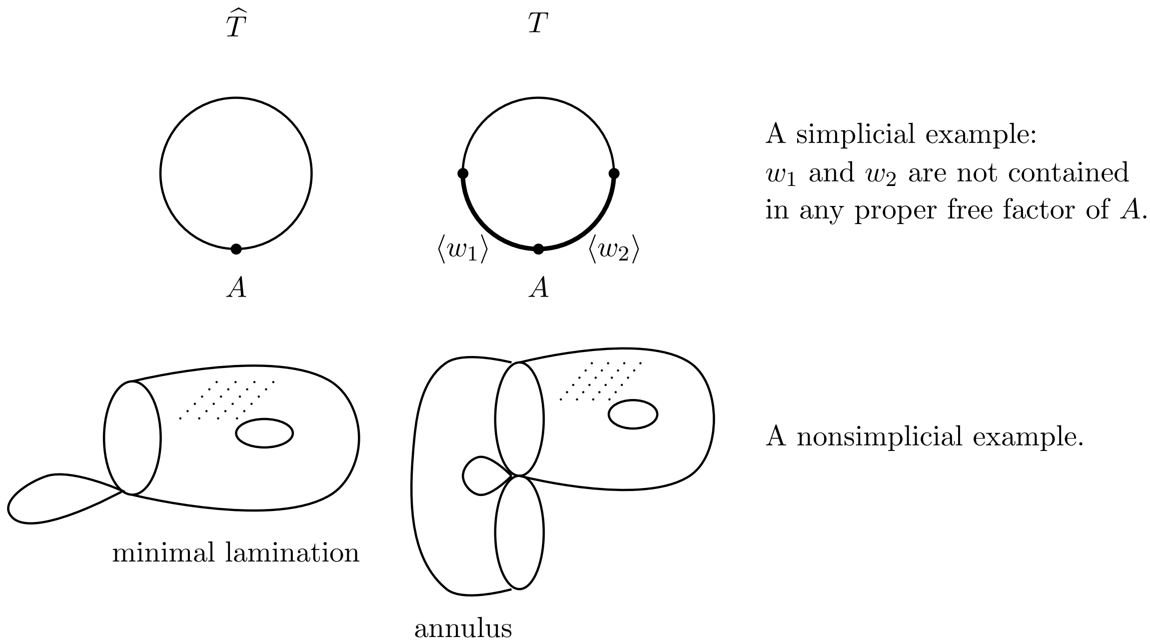


Figure A.8: Examples of NS-pull-equivalent trees.

We denote by $\sim$ the special-pull-equivalence relation on $\overline{cv_N}$. The *standard element* of a nontrivial class of special-pull-equivalence is the unique element of the class in which the length of the edge with trivial stabilizer (if any) is maximal. Each equivalence class is star-shaped and contractible, and consists of a union of simplices of dimension at most 2.

A few examples. The simplest examples of special-pull-equivalent trees arise by pulling the Bass-Serre tree of a splitting of F_N of the form $F_N = A*$ (where A is a corank one free factor of F_N), with any nonsimple elements of F_{N-1} as pulling elements, see Figure A.8.

A more elaborate class of examples arises by letting T_{N-1} be any geometric F_{N-1}-tree dual to a foliation on a surface S with a single boundary component, and forming a graph of actions whose underlying graph of groups represents the splitting $F_N = F_{N-1}*$, with attaching point x given by the boundary curve of S, and pulling elements stabilizing x in T_{N-1} (see Figure A.8, see also Section A.3 for a more detailed account of this construction).

There is a way of building new examples by an iterative process. Start from a minimal F_{N-1}-tree Y that contains a point x whose stabilizer is not contained in any proper free factor of F_{N-1}, form an HNN-extension $F_{N-1}*$, and the corresponding graph of actions with attaching point x, and fold the corresponding edge e totally over a translate ge, where g does not belong to any proper free factor of F_{N-1}. This gives a tree T' having a point stabilizer which is not contained in any proper free factor of F_N. Hence it can serve as the vertex tree of an F_{N+1}-tree whose NS-pull-equivalence class is nontrivial. Iterating this process creates a class of NS-pull-equivalent trees.

A.2.3 Special-pull-equivalent trees are simple-equivalent.

This section is devoted to the proof of the following implication.

Proposition A.2.6. *Any two special-pull-equivalent trees $T, T' \in \overline{cv_N}$ are simple-equivalent.*

In order to prove Proposition A.2.6, we start by checking that a certain class of elements of F_N are nonsimple, using methods due to Whitehead [Whi36] and further developed by

Stallings [Sta99]. The *Whitehead graph* of an element $w \in F_N$ with respect to a free basis B of F_N, denoted by $Wh_B(w)$, is the graph whose vertices are the elements of $B^{\pm 1}$, two vertices a and b being joined by an edge if ab^{-1} occurs as a subword of the cyclic word that represents w in the basis B. A *cutpoint* in a connected graph X is a point $p \in X$ such that $X \setminus \{p\}$ is disconnected.

Proposition A.2.7. *(Whitehead [Whi36], Stallings [Sta99]) An element $w \in F_N$ is simple if and only if its Whitehead graph with respect to any free basis of F_N is either disconnected or contains a cutpoint.*

Proposition A.2.8. *Let A be a corank 1 free factor of F_N. For all $w \in A$, the following assertions are equivalent.*

- *The element w is contained in some proper free factor of A.*
- *There exist a basis $B = \{x_1, \dots, x_N\}$ of F_N, such that A is the subgroup generated by $x_1, \dots, x_{N-1}$, and an element $v \in F_N$, such that $vx_Nwx_N^{-1}$ is primitive, and the product $vx_Nwx_N^{-1}$ is cyclically reduced when v and w are written as reduced words in the basis B.*

Proof. First assume that w is contained in a proper free factor of A, and let $\{x_1, \dots, x_k\}$ denote a basis of this free factor. Let $\{x_{k+1}, \dots, x_{N-1}\}$ be a basis of a complementary free factor of A, let $x_N \in F_N$ be such that $F_N = A * \langle x_N \rangle$, and let $v := x_{k+1}$. Then $\{vx_Nwx_N^{-1}, x_1, \dots, x_k, x_{k+2}, \dots, x_{N-1}, x_N\}$ is a free basis of F_N, so $vx_Nwx_N^{-1}$ is primitive. In addition, the product $vx_Nwx_N^{-1}$ is cyclically reduced when written as a reduced word in the basis $\{x_1, \dots, x_N\}$ of F_N.

Assume now that w is not contained in any proper free factor of A. Assume by contradiction that there exists a basis $B = \{x_1, \dots, x_N\}$ of F_N such that $A = \langle x_1, \dots, x_{N-1} \rangle$, and an element $v \in F_N$ such that the product $vx_Nwx_N^{-1}$ is cyclically reduced when v and w are written as reduced words in the basis B, and $vx_Nwx_N^{-1}$ is primitive. By Proposition A.2.7, we can choose $x_1, \dots, x_{N-1}$ such that the Whitehead graph of w is connected without cutpoint in the basis $\{x_1, \dots, x_{N-1}\}$ of A. We denote by a the first letter of w in B, by b its last letter, and by c_1 the last letter of v. The Whitehead graph of $W := vx_Nwx_N^{-1}$ in B contains $Wh_B(w)$, in which an edge joining b to a^{-1} is replaced by an edge joining b to x_N and an edge joining a^{-1} to x_N, and $Wh_B(W)$ also contains an edge joining x_N^{-1} to c_1, see Figure A.9. In particular, it is connected, and its only possible cutpoint is c_1, provided there is no edge joining x_N^{-1} to a vertex different from c_1. This implies that c_1^{-1} (resp. c_1) is the first (resp. last) letter of the reduced word that represents v in the basis B, i.e. there exists a subword $\widetilde{v}$ of v so that $vx_Nwx_N^{-1} = (c_1^{-1}\widetilde{v}c_1)x_Nwx_N^{-1}$.

First observe that $c_1 \neq x_N$, otherwise all occurrences of x_N^{-1} in the cyclic word that represents W in the basis B should be followed by another occurrence of x_N^{-1}, and W would be a power of x_N, a contradiction. As c_1 is a cutpoint of $Wh_B(w)$, all occurrences of the letter x_N in the reduced word representing v in B are preceded by an occurrence of c_1, and all occurrences of x_N^{-1} are followed by an occurrence of c_1^{-1}. Let $x_N^{(1)} := c_1x_N$. In the basis $B_1 := \{x_1, \dots, x_{N-1}, x_N^{(1)}\}$, the element W is represented by a reduced cyclic word of the form $v_1x_N^{(1)}wx_N^{(1)^{-1}}$, and the length of v_1 in B_1 is strictly smaller than the length of v in B. In addition, the element $w \in F_N$ is represented by the same reduced word in B and in B'. Repeating the above argument shows that there exists $c_2 \in B_1$ such that the first letter of v_1 is c_2^{-1} and its last letter is c_2. Letting $x_N^{(2)} := c_2x_N^{(1)}$, the element W is represented by a reduced cyclic word of the form $v_2x_N^{(2)}wx_N^{(2)^{-1}}$ in the basis $B_2 := \{x_1, \dots, x_{N-1}, x_N^{(2)}\}$, and the length of v_2 in B_2 is strictly smaller than the length of

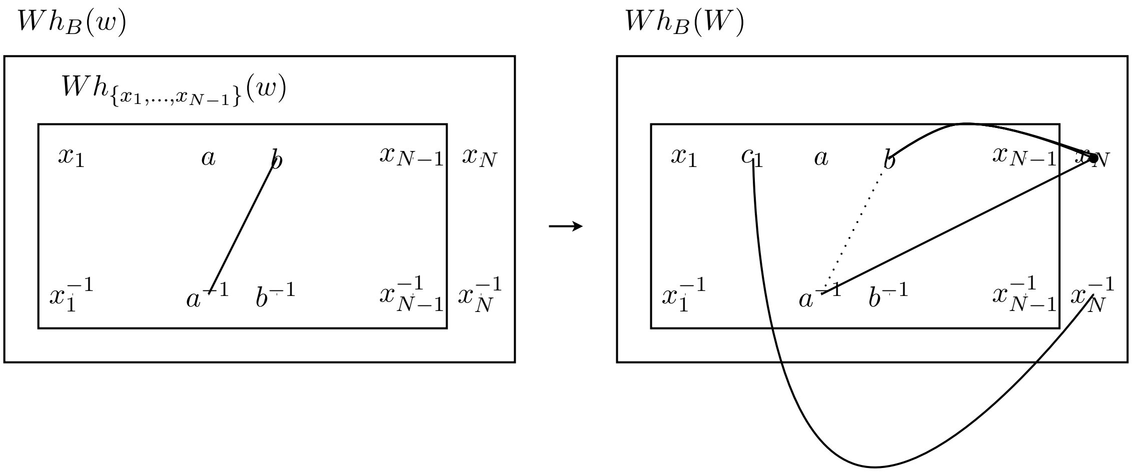


Figure A.9: The Whitehead graphs $Wh_B(w)$ and $Wh_B(W)$ in the proof of Proposition A.2.8.

v_1 in B_1. One can then repeat this process infinitely often, contradicting the fact that the lengths of the words representing W in the bases we get along the process form a strictly decreasing sequence of positive integers. □

Proof of Proposition A.2.6. Let $T, T' \in \overline{cv_N}$ be special-pull-equivalent. Assume that $T \neq T'$, and let $\widehat{T} \in \overline{cv_N}$ be a tree with a single orbit of edges with trivial stabilizer e, such that T and T' are both pulls of $\widehat{T}$, with pulling elements g_1, g_2, g'_1 and g'_2. Equivariantly collapsing the complement of e to a point in $\widehat{T}$ yields a splitting $F_N = A*$ (we denote by t a stable letter). Any element $w \in F_N$ either belongs to a conjugate of A, or of the cyclic subgroup of F_N generated by t, or is conjugated to an element of the form $w_1 t^{\alpha_1} w_2 t^{\alpha_2} \dots w_k t^{\alpha_k}$, with $\alpha_i \in \mathbb{Z} \smallsetminus \{0\}$ and $w_i \in A \smallsetminus \{e\}$ for all $i \in \{1, \dots, k\}$. Such an element has the same translation length in T and T', unless it is of the form $t^\epsilon g_i^k t^{-\epsilon} w$ or $t^\epsilon {g'_i}^k t^{-\epsilon} w$ for some $\epsilon = \pm 1$, some $k \in \mathbb{Z} \smallsetminus \{0\}$, and some $i \in \{1, 2\}$ with g_i (or g'_i) nonsimple in $A \smallsetminus \{e\}$. As any element of F_N of this form is nonsimple by Proposition A.2.8, all simple elements of F_N have the same translation length in T and T'. □

A.2.4 The primitive compactification of outer space

Our main result gives a description of a new compactification of outer space, which we call the *primitive compactification*, defined by restricting translation lengths functions to the set $\mathcal{P}_N$ of primitive elements of F_N in Culler and Morgan's construction. Our motivation for introducing this compactification comes from our description of the compactification of outer space by horofunctions, which is itself motivated by the desire to study random walks on $\mathrm{Out}(F_N)$. In [4], we will prove that the compactification of outer space by horofunctions is isomorphic to the primitive compactification. Let

$$i_{prim} : CV_N \to \mathbb{PR}^{\mathcal{P}_N}$$

be the map obtained from the map i defined in Section A.1.1 by only considering translation lengths of primitive elements of F_N. The relation $\sim$ again denotes the primitive-equivalence relation defined above.

Theorem A.2.9. *The map i_{prim} is a homeomorphism onto its image. The closure $\overline{i_{prim}(CV_N)}$ is compact, and homeomorphic to $\overline{CV_N}/\sim$.*

This means that $\overline{i_{prim}(CV_N)}$ is indeed a compactification of CV_N.

Proof. Continuity of i_{prim} follows from the continuity of i, and injectivity of i_{prim} was proved in [CFKM12, Theorem 3.4] as a consequence of White's theorem (this is a particular case of our main result). To show that i_{prim} is an embedding, we let $(T_n)_{n\in\mathbb{N}} \in CV_N^{\mathbb{N}}$ be a sequence that leaves every compact subspace of CV_N, and let $T \in CV_N$. By compactness of $\overline{CV_N}$, some subsequence of $(T_n)_{n\in\mathbb{N}}$ converges to a tree $T_\infty \in \overline{CV_N} \smallsetminus CV_N$, and Theorem A.7.1 implies that T_∞ is not primitive-equivalent to T (this actually only uses the particular case of Theorem A.7.1 where one of the trees belongs to CV_N, which can easily be deduced from Theorem A.1.3). Therefore, the sequence $(i_{prim}(T_n))_{n\in\mathbb{N}}$ does not converge to $i_{prim}(T)$, showing that i_{prim} is an embedding. Compactness of $\overline{i_{prim}(CV_N)}$ follows from compactness of $\overline{i(CV_N)}$. By definition, the map i_{prim} extends to a bijective continuous map, and hence a homeomorphism, from the compact space $\overline{CV_N}/\sim$ to the Hausdorff space $\overline{i_{prim}(CV_N)}$. ☐

A.3 Approximations of trees

The next two sections aim at developing techniques that will turn out to be useful for extending White's theorem to trees in the boundary of outer space, and describing the lack of spectral rigidity of the set $\mathcal{P}_N$ of primitive elements of F_N in $\overline{cv_N}$. In the present section, building on ideas of Bestvina and Feighn [BF94] and Guirardel [Gui98], we provide nice approximations for a wide class of trees in $\overline{cv_N}$ by nicer trees, see Theorems A.3.6 and A.3.11.

A.3.1 Geometric trees

Of particular interest are trees in $\overline{cv_N}$ which are dual to measured foliations on some 2-complexes, which are called *geometric* trees. Geometric trees can be decomposed in a nice and controlled way, and can be used to approximate every tree $T \in \overline{cv_N}$. We recall a few facts about this class of trees, and refer the reader to [BF95] or [GLP94] for details.

A *system of partial isometries* X of a finite tree or multi-interval K is a finite collection of isometries $\phi_j : A_j \to B_j$ between nonempty finite subtrees of K. The subtrees A_j and B_j are called the *bases* of X, and ϕ_j is called a *singleton* if its bases are reduced to points. The *suspension* of X is the foliated 2-complex Σ built in the following way. Start with the union of K (foliated by points) and *bands* $A_j \times [0,1]$ (foliated by $\{*\} \times [0,1]$). For all $t \in A_j$, glue $(t,0) \in A_j \times \{0\}$ with $t \in A_j$ and $(t,1) \in A_j \times \{1\}$ with $\phi_j(t) \in B_j$. There is a natural transverse measure on each band given by the metric on the base A_j. This induces a transverse measure on Σ. We will denote by $(\Sigma, \mathcal{F}, \mu)$ (or simply by Σ if the context is clear) the band complex Σ equipped with its foliation $\mathcal{F}$ and its transverse measure μ.

Associated to a system of k partial isometries X (or its corresponding measured foliated band complex $(\Sigma, \mathcal{F}, \mu)$), together with a set $\mathcal{C}$ of closed curves contained in leaves of Σ, is a dual $\mathbb{R}$-tree, constructed as follows. Choose a basepoint $*$ on Σ. When K is a finite tree (or a multi-interval in which the extremities of the intervals are joined by singletons, in such a way that collapsing the subsegments of the leaves determined by these singletons to points yields a finite tree), the fundamental group of Σ is naturally identified with the free group F_k having one generator for each partial isometry in X. Let N denote the subgroup of F_k normally generated by the free homotopy classes of the curves in $\mathcal{C}$, and let $G(X) := F_k/N$. There is a canonical epimorphism $\rho : F_k \to G(X)$. We denote by $\overline{\Sigma}$ the covering space of Σ corresponding to ρ. The measured foliation on Σ lifts to a measured foliation on $\overline{\Sigma}$, we denote by $\overline{\mu}$ the transverse measure on $\overline{\Sigma}$. Define a pseudo-metric on $\overline{\Sigma}$ by $\delta(x,y) := \inf_\gamma \overline{\mu}(\gamma)$, where the infimum is taken over all paths joining x to y in $\overline{\Sigma}$ (and $\overline{\mu}(\gamma)$ is obtained by integrating the measure $\overline{\mu}$ along the path γ). The metric space obtained by making this pseudo-distance Hausdorff (sometimes called the *leaf space made Hausdorff*) is an $\mathbb{R}$-tree [LP97, Proposition 1.7], which we denote by $T(X,\mathcal{C})$ (or equivalently $T(\Sigma,\mathcal{C})$). It is naturally equipped with an isometric action of $G(X)$. An $\mathbb{R}$-tree equipped with an action of a finitely presented group G is called *geometric* if there exists a system of partial isometries X, and a set of curves $\mathcal{C}$ contained in leaves of the associated measured foliated band complex, such that $G = G(X)$ and $T = T(X,\mathcal{C})$. Otherwise it is called *nongeometric*. Let Σ, Σ' be two measured foliated band complexes, together with sets of curves $\mathcal{C}$ and $\mathcal{C}'$. We call $(\Sigma,\mathcal{C})$ and $(\Sigma',\mathcal{C}')$ *equivalent* if $T(\Sigma,\mathcal{C}) = T(\Sigma',\mathcal{C}')$. Let Σ^* denote Σ minus its singletons. We say that Σ has *pure components* if K is a multi-interval, and in each component of Σ^*, each finite singular $\mathring{X}$-orbit (i.e. the orbit of each point under the restrictions of the partial isometries in X, or their inverses, to the interior of their bases) is reduced to one point in ∂K (an orbit is *singular* if it contains a point in the boundary of some base).

Given a geometric F_N-tree T, there is a way of producing a system of isometries X on a finite tree K, so that $T = T(X,\emptyset)$. Fix a free basis $\{g_1,\dots,g_N\}$ of F_N, and let K be a finite subtree of T. For all $i \in \{1,\dots,N\}$, the generator g_i defines a partial isometry of K, with domain $g_i^{-1}(K) \cap K$ and image $K \cap g_i(K)$, and we may assume K to be sufficiently big, so that these bases are nondegenerate. If T is geometric, then K can be chosen so that the associated geometric tree is equal to T [GL95, Proposition II.1]. The following theorem provides a normal form for systems of partial isometries dual to a given geometric F_N-tree.

Theorem A.3.1. *(Imanishi [Ima79], Gaboriau-Levitt-Paulin [GLP94]) Let T be a geometric F_N-tree. Then there exist a system of partial isometries X having pure components, and a set of curves $\mathcal{C}$ contained in leaves of Σ, such that $T = T(X,\mathcal{C})$. The subcomplex Σ^* is a disjoint union of finitely many open $\mathring{X}$-invariant sets, and if U is one of these sets, then either every leaf contained in U is compact (in which case U is called a* family of finite orbits*), or else every leaf contained in U is dense in U (in which case U is called* minimal*). Furthermore, the system X may be chosen in such a way that all families of finite orbits are orientable (i.e. no $\mathring{X}$-word fixes a point in an orbit and reverses orientation).*

One can give the following classification of minimal components. Starting from a foliated band complex Σ_0 associated to a minimal system of partial isometries X_0 on a finite tree or multi-interval K_0, we define a new band complex Σ_1 in the following way. Let K_1 denote the set of points in K_0 which belong to at least two bases of Σ_0. Let X_1 be the system of partial isometries of K_1 obtained by restricting the elements of X_0 to K_1. We define Σ_1 to be the suspension of X_1. Starting from Σ_0 and iterating this process, we build a sequence of foliated band complexes Σ_i. If for all $i \in \mathbb{N}$ we have $\Sigma_{i+1} \neq \Sigma_i$, we say that Σ_0 is *exotic* (or *Levitt*, or *thin*), otherwise Σ_0 is a measured foliation on a compact surface [GLP94]. (In the case of F_N-trees, the homogeneous case described in [GLP94, Section 4] cannot occur, see [BF94, Proposition 1.8]). A band $B = b \times [-1,1]$ of a band complex is *very naked* if $b \times (-1,1)$ does not meet the curves in $\mathcal{C}$. Exotic components have the following property.

Proposition A.3.2. *(Bestvina-Feighn [BF95], Gaboriau-Levitt-Paulin [GLP94], see also [Gui98, Section 7.1]) If $T \in \overline{cv_N}$ contains an exotic minimal component, then there exist a band complex X satisfying the conclusions of Theorem A.3.1 and a collection of curves $\mathcal{C}$ in X such that $T = T(X, \mathcal{C})$, and X contains a very naked band (contained in an exotic component of X).*

The structure of band complexes which only have simplicial and surface components is also well-understood, thanks to the following results of Bestvina and Feighn.

Proposition A.3.3. *(Bestvina-Feighn [BF94, Proposition 5.1]) Let X be a band complex with only simplicial and surface components dual to an F_N-tree $T \in \overline{cv_N}$. Then there exists another band complex X' dual to T of the form $X' = (S \cup A \cup \Gamma) \cup_f G$ such that*

- *S is a (possibly disconnected) compact surface, none of whose components is homeomorphic to an annulus or a Möbius band, and each connected component of S is equipped with a minimal foliation, and*
- *Γ is a finite metric graph, and*
- *G is a finite graph with no valence 1 vertices and empty foliation, and*
- *A is a finite disjoint union of annuli foliated by essential loops, and*
- *$f : \partial S \cup \partial A \cup F \to G$, where F is a finite subset of $S \cup A \cup \Gamma$ and f is essential on each component of $\partial S \cup \partial A$.*

Proposition A.3.4. *(Bestvina-Feighn [BF94, Lemma 4.1]) Let Y be a finite graph, and S a compact (possibly disconnected) surface. Let $f : \partial S \to Y$ be a map that is essential on each boundary component. Assume that $X := S \cup_f Y$ has free fundamental group. Then there exist a finite graph Y' and a homotopy equivalence $\psi : Y \to S^1 \vee Y'$ such that $\psi \circ f : \partial S \to S^1 \vee Y'$ is homotopic to a map that sends one boundary component of S homeomorphically onto S^1, and sends all other boundary components of S into Y'.*

We call the boundary component of S that is sent homeomorphically to S^1 a *distinguished circle*.

We finish this section by explaining how geometric trees can be used to approximate all actions in $\overline{cv_N}$, and give a characterization of geometric trees due to Levitt and Paulin [LP97]. Let $T \in \overline{cv_N}$. Following [GS90], we say that a sequence $(T_n)_{n\in\mathbb{N}}$ of trees in $\overline{cv_N}$ *converges strongly* to T if there exist surjective F_N-equivariant morphisms $f_{np} : T_n \to T_p$ for all $n < p$, and $f_n : T_n \to T$ for all $n \in \mathbb{N}$ such that

- for all $n < p$, we have $f_p \circ f_{np} = f_n$, and
- for all $n \in \mathbb{N}$ and all $x, y \in T_n$, there exists $p \geq n$ such that $d_{T_p}(f_{np}(x), f_{np}(y)) = d_T(f_n(x), f_n(y))$.

The following result is due to Levitt and Paulin [LP97, Theorem 2.6], see also [GL95, Proposition II.1] where the minimality statement appears. The fact that the trees T_n can be chosen to belong to $\overline{cv_N}$ follows from [BF94, Proposition 1.8] and [GL95, Corollary I.6].

Proposition A.3.5. *(Levitt-Paulin [LP97, Theorem 2.6], Gaboriau-Levitt [GL95, Proposition II.1]) For all $T \in \overline{cv_N}$, there exists a sequence $(T_n)_{n\in\mathbb{N}} \in \overline{cv_N}$ of minimal geometric F_N-trees which converges strongly to T. A tree is geometric if and only if it cannot occur as such a strong limit in a nonstationary way.*

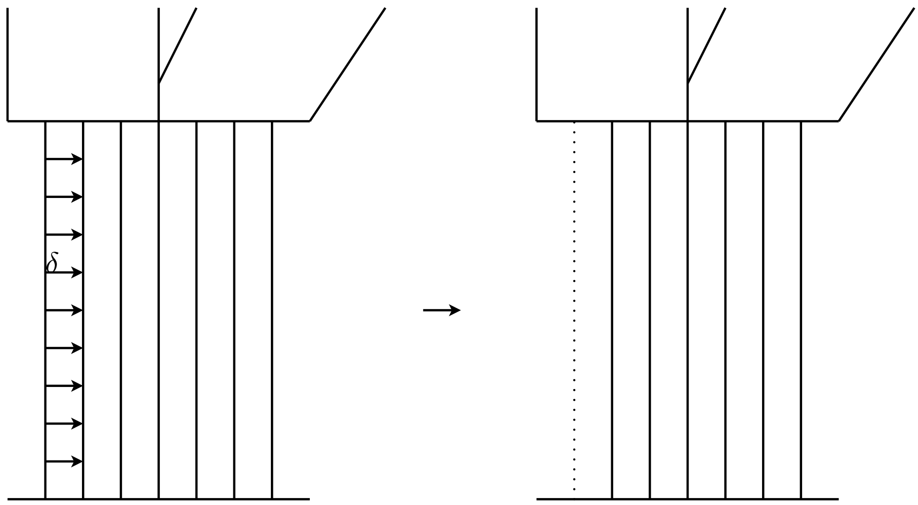


Figure A.10: Narrowing a band.

The trees T_n can be constructed from T by applying the construction preceding Theorem A.3.1 to a well-chosen exhaustion of T by finite trees K_n. In particular, the morphisms f_{np} and f_n can be chosen to be injective in restriction to every segment of K_n which has a translate in K_n. By choosing K_n to contain an edge in each orbit of edges in T, we can thus assume f_{np} and f_n to be injective on segments with nontrivial stabilizers of T_n.

A.3.2 Approximations of F_N-trees with dense orbits by free and simplicial actions

A *Lipschitz approximation* of a tree $T \in \overline{cv_N}$ is a sequence of trees $(T_n)_{n\in\mathbb{N}} \in \overline{cv_N}^{\mathbb{N}}$ converging (non-projectively) to T, together with 1-Lipschitz F_N-equivariant maps $f_n : T_n \to T$ for all $n \in \mathbb{N}$. We give a characterization of trees in $\overline{cv_N}$ that admit a Lipschitz approximation by free, simplicial actions.

Theorem A.3.6. *A tree $T \in \overline{cv_N}$ admits a Lipschitz approximation by elements of cv_N if and only if all arc stabilizers in T are trivial.*

Let $T \in \overline{cv_N}$ be a geometric tree, and let X be a system of partial isometries associated to T given by Theorem A.3.1, together with a set of curves $\mathcal{C}$ contained in the leaves of Σ. Assume that some band B of Σ is very naked. For small $\delta > 0$, let Σ_δ be a band complex obtained by narrowing B of width δ from one of its boundary leaves, see Figure A.10. The inclusion $\Sigma_\delta \subset \Sigma$ is a homotopy equivalence, so there is an epimorphism $\rho_\delta : \pi_1(\Sigma_\delta) \to F_N$, whose kernel is normally generated by the free homotopy classes of the curves in $\mathcal{C}$, which are still contained in leaves of Σ_δ. Denote by $\overline{\Sigma}$ and $\overline{\Sigma_\delta}$ the covering spaces corresponding to ρ and ρ_δ, respectively. Let T_δ be the minimal subtree of the F_N-tree obtained by making the leaf space of $\overline{\Sigma_\delta}$ Hausdorff. There is a natural F_N-equivariant morphism of $\mathbb{R}$-trees from T_δ to T induced by the inclusion $\overline{\Sigma_\delta} \subset \overline{\Sigma}$.

Lemma A.3.7. *(Guirardel [Gui98, Section 7.2]) The trees T_δ converge to T as δ goes to* 0.

Proof of Theorem A.3.6. First assume that T admits a Lipschitz approximation by a sequence $(T_n)_{n\in\mathbb{N}}$ of trees in cv_N. As there exist 1-Lipschitz F_N-equivariant maps $f_n : T_n \to$

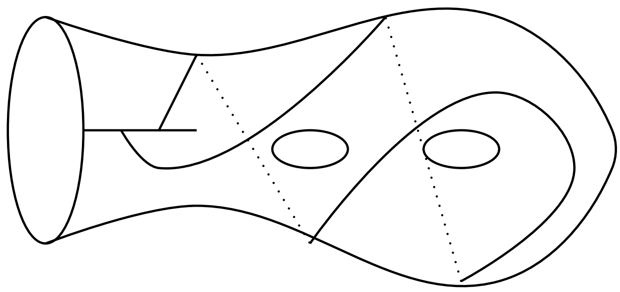


Figure A.11: Narrowing a surface component creates compact leaves.

T, by Proposition A.1.11, we have $qvol(T) \leq qvol(T_n)$ for all $n \in \mathbb{N}$. However, if T has a nontrivial arc stabilizer, then $qvol(T) > \limsup_{n\to+\infty} qvol(T_n)$. Hence T has trivial arc stabilizers.

Conversely, let T be a tree in $\overline{cv_N}$ with trivial arc stabilizers. First assume that T contains an exotic component. Then Proposition A.3.2 yields an equivalent band complex which contains a very naked band B, to which we can apply the narrowing process. Guirardel shows in [Gui98, Section 7] that we can choose $\delta > 0$ arbitrarily small and get a tree T_δ, in which the exotic component of T has been replaced by new simplicial and exotic components, and the number $E(\Sigma_\delta)$ of ends of singular leaves satisfies $E(\Sigma_\delta) < E(\Sigma)$. Iterating the construction a finite number of times yields an approximation of T in which the minimal component T_v has been replaced by a simplicial part with trivial edge stabilizers. Iterating this process, we can approximate all exotic components in T without creating arc stabilizers. Hence we are left with a band complex which can be assumed to have the form prescribed by Proposition A.3.3.

As T has trivial arc stabilizers, this band complex contains no annulus. Assume that it contains some surface component, and let C be a distinguished circle provided by Proposition A.3.4. One can narrow the surface that contains C from its boundary along width $\delta > 0$ to either create compact leaves, or leaves having a single end (except for at most finitely many of them), see Figure A.11. However, in a minimal surface component, all half-leaves are dense, so in the new band complex Σ_δ created in this way, the surface containing C has been replaced by a simplicial component, with trivial arc stabilizers. As in Lemma A.3.7, the trees T_δ dual to the band complex Σ_δ converge to T as δ tends to 0, and they come with F_N-equivariant morphisms from T_δ to T. Iterating this process, we successively approximate all the surface components by simplicial components with trivial edge stabilizers. Finally, we can approximate all vertices with nontrivial stabilizer in the quotient graph by roses having arbitrarily small petals to get a Lipschitz approximation of T by elements of cv_N. $\square$

As a consequence of Theorem A.3.6, we show that any Lipschitz F_N-equivariant map between F_N-trees with dense orbits preserves alignment. In particular, any F_N-equivariant morphism between minimal F_N-trees with dense orbits is an isometry. Let $T, T' \in \overline{cv_N}$, and $f : T \to T'$ be an F_N-equivariant map. The *bounded cancellation constant* of f, denoted by $BCC(f)$, is defined to be the supremum of all real numbers B with the property that there exist $a, b, c \in T$ with $b \in [a, c]$, such that $d_{T'}(f(b), [f(a), f(c)]) = B$. Notice that an F_N-equivariant map $f : T \to T'$ preserves alignment if and only if $BCC(f) = 0$. We denote by $\mathrm{Lip}(f)$ the Lipschitz constant of f.

Proposition A.3.8. *(Bestvina-Feighn-Handel [BFH97, Lemma 3.1]) Let $T \in cv_N$ and $T' \in \overline{cv_N}$, and let $f : T \to T'$ be an F_N-equivariant map. Then $BCC(f) \leq Lip(f)qvol(T)$.*

Corollary A.3.9. *Let $T, T' \in \overline{cv_N}$ have dense orbits, and let $f : T \to \overline{T'}$ be a Lipschitz F_N-equivariant map. Then f preserves alignment.*

Proof. Let $a, b, c \in T$ with $b \in [a, c]$, and let $C := d_{\overline{T'}}(f(b), [f(a), f(c)])$. Assume by contradiction that $C > 0$. As T has dense orbits, all arc stabilizers in T are trivial (Lemma A.1.5), so Theorem A.3.6 provides a Lipschitz approximation $(T_n)_{n\in\mathbb{N}}$ of T by free and simplicial F_N-trees. By Proposition A.1.11, the quotient volume of T_n converges to 0 as n goes to infinity. By definition of a Lipschitz approximation, for all $n \in \mathbb{N}$, there exists a 1-Lipschitz F_N-equivariant map $f_n : T_n \to T$. Minimality of T implies that f_n is surjective for all $n \in \mathbb{N}$. Composing f_n with f yields a Lip(f)-Lipschitz F_N-equivariant map $f'_n : T_n \to \overline{T'}$. Tightening f'_n on edges if necessary (which does not increase its Lipschitz constant), we can assume that f'_n is linear on edges. Slightly perturbing f'_n on the vertices of T_n, and extending it linearly on the edges of T_n again if necessary, we get the existence of a Lip(f)-Lipschitz F_N-equivariant map $f''_n : T_n \to T'$, with $d_{\overline{T'}}(f'_n(x), f''_n(x)) \leq \frac{C}{4}$ for all $x \in T_n$. By Proposition A.3.8, the bounded cancellation constant $BCC(f''_n)$ tends to 0 as n goes to infinity. For all $n \in \mathbb{N}$, let a_n (resp. c_n) be a preimage of a (resp. c) by f_n in T_n. Then there exists $b_n \in [a_n, c_n]$ such that $f_n(b_n) = b$. We have $d_{\overline{T'}}(f'_n(b_n), [f'_n(a_n), f'_n(c_n)]) = C$, so $d_{T'}(f''_n(b_n), [f''_n(a_n), f''_n(c_n)]) \geq \frac{C}{2}$. This implies that $BCC(f''_n) \geq \frac{C}{2}$ for all $n \in \mathbb{N}$, a contradiction. Hence f preserves alignment. ☐

Corollary A.3.10. *Let $T, T' \in \overline{cv_N}$ have dense orbits. Then any F_N-equivariant morphism from T to T' is an isometry.* ☐

A.3.3 Approximations by trees having two edges with trivial stabilizers

An F_N-tree $T \in \overline{cv_N}$ is *good* if there exists a Lipschitz approximation $(T_n)_{n\in\mathbb{N}} \in \overline{cv_N}^{\mathbb{N}}$ of T such that for all $n \in \mathbb{N}$, the tree T_n^{simpl} contains at least two F_N-orbits of edges with trivial stabilizers. The following statement will be used in Section A.7 to describe the lack of rigidity of the set $\mathcal{P}_N$ in $\overline{cv_N}$. We recall the definition of a pull from Section A.2.2.

Theorem A.3.11. *Every tree $T \in \overline{cv_N}$ is a pull of a good tree. More precisely, for all $T \in \overline{cv_N}$, either T is good, or there exists a good tree $T' \in \overline{cv_N}$ which has exactly one orbit of edges with trivial stabilizer, such that T is a pull of T'.*

Proof. We argue differently depending on whether T is geometric or not.

Case 1 : The tree T is geometric.

Case 1.1 : The tree T contains an exotic component.
Applying the same narrowing process as in the proof of Theorem A.3.6 to this exotic component yields a Lipschitz approximation $(T_n)_{n\in\mathbb{N}}$ of T, in which the exotic component – dual to some subtree T_v of T – is replaced by a family of finite orbits, dual to some tree T^1 with trivial edge stabilizers. If for some $n \in \mathbb{N}$, the tree T_n^{simpl} contains at most one orbit of edges with trivial stabilizer, then T^1 is the Bass-Serre tree of a one-edge free splitting, and Lemmas A.1.5 and A.1.8 imply that a morphism $f : T^1 \to T_v$ cannot fold any U-turn. If T^1 is the Bass-Serre tree of a splitting of the form $F_i * F_{k-i}$, then f might only reduce the length of the unique orbit of edges in T^1, and T_n cannot converge to T. If T^1 is the Bass-Serre tree of a splitting of the form $F_{k-1}*$, then f can either reduce the length of the

unique orbit of edges in T^1, or create a second orbit of edges with trivial stabilizers, in which case we can assume T_n^{simpl} to contain two F_N-orbits of edges with trivial stabilizers.

Case 1.2 : The tree T is dual to a band complex Σ which has the structure prescribed by Proposition A.3.3.
If Σ contains no surface component and no annulus, then T is simplicial and has trivial edge stabilizers. So either T contains two F_N-orbits of edges with trivial stabilizers, or T is the Bass-Serre tree of a one-edge free splitting of F_N, in which case T can be approximated by blowing up its vertex groups, adding a small loop with trivial stabilizer of length going to 0. Otherwise, let C be a distinguished circle provided by Proposition A.3.4. If C belongs to a surface component, then as in Case 1.1 we get a Lipschitz approximation of T by trees having at least two orbits of edges with trivial stabilizers. We now assume that C belongs to an annulus A. In this case, narrowing a band corresponds to unfolding an edge in the dual tree, see Figure A.12, and this operation creates an orbit of edges e with trivial stabilizer. This operation does not affect minimality of the dual tree. If there is another simplicial orbit of edges with trivial stabilizer in the tree dual to Σ, then T is good. It may happen that some extremity of e has cyclic stabilizer, and is such that there are exactly two F_N-orbits of edges coming out of it, one of which has nontrivial stabilizer. We let T' be the tree obtained from T by totally unfolding the edges with nontrivial stabilizers coming out of such extremities of e. This operation does not create obtrusive powers or tripod stabilizers, so the tree T' is again very small, and by definition T is a pull of T'. In addition, if we equivariantly remove the edge with trivial stabilizer of T' we have just constructed, we get (at least) one tree, to which we can apply the above argument (this tree might not be minimal for the action of its stabilizer, if e projects to a loop-edge in the associated graph of actions, but the above argument still works in this case). If all distinguished circles of T' are contained in surface components, then the above argument shows that T' is good. Otherwise, one can again unfold an annulus. This operation creates a second edge with trivial stabilizer, again showing that T' is good.

Case 2 : The tree T is nongeometric.

Let $(T_n)_{n\in\mathbb{N}} \in \overline{cv_N}^{\mathbb{N}}$ be a sequence of minimal geometric F_N-trees converging strongly to T, given by Proposition A.3.5. Denote by $f_n : T_n \to T$ and $f_{n,p} : T_n \to T_p$ the corresponding morphisms, which might be assumed to be injective on the edges in T_n with nontrivial stabilizers. In particular, the sequence $(T_n)_{n\in\mathbb{N}}$ is a Lipschitz approximation of T, so it is enough to show that T_n can be assumed to contain two F_N-orbits of edges with trivial stabilizers for all $n \in \mathbb{N}$. Assume by contradiction that for some $n \in \mathbb{N}$, the tree T_n contains at most one edge e_n with trivial stabilizer in its simplicial part. The morphism $f_{n,n+1}$ cannot identify

- two initial subsegments of edges in the simplicial part of T_n with distinct nontrivial stabilizer (Lemma A.1.9), nor
- a nontrivial subsegment of an edge with nontrivial stabilizer with a path lying in a vertex tree of T_n with dense orbits (Lemma A.1.10), nor
- two arcs lying in a vertex tree of T_n with dense orbits (Corollary A.3.10).

If $f_{n,n+1}$ identifies a subsegment J of e_n with a subsegment J' of one of the translates of e_n, such that J' meets the F_N-orbit of J in a single point (this might happen if e_n projects to a loop-edge in the associated graph of actions), then we can replace T_n by a tree with two F_N-orbits of edges with trivial stabilizers. So up to reducing the length of e_n in T_n,

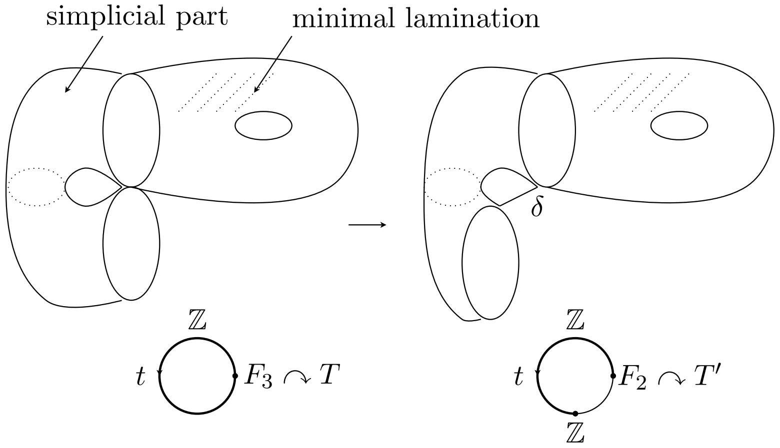


Figure A.12: Narrowing a band in a simplicial component.

we might assume that to pass from T_n to T_{n+1}, we need only fold a subsegment of the edge e_n along some path in T_n, at each of its extremities, and iterating the same argument shows that for all $k \geq n$, the tree T_k has a unique edge with trivial stabilizer, and in order to pass from T_k to T_{k+1}, one has to fold a subsegment of e_k equivariantly along some path in T_k. Assume that the sequence $(T_k)_{k\in\mathbb{N}}$ is nonstationary. Then we can find $x \in e_n$ and a nonstationary sequence $(x_k)_{k\in\mathbb{N}}$ of elements of e_n converging to x such that the subsegment $[a, x_k]$ of e_n is folded when passing from T_n to T_k (where a denotes one of the extremities of e_n). For all $k \in \mathbb{N}$, let y_k be a point in T_n that is identified with x_k during the folding process. In particular, the sequence $(y_k)_{k\in\mathbb{N}}$ is bounded, and the segments $[x, y_k]$ form an increasing sequence of segments in T_n. So $(y_k)_{k\in\mathbb{N}}$ converges to some point $y \in T_n$, and $f_n(x) = f_n(y)$. However, for all $k \in \mathbb{N}$, we have $d(f_{n,k}(x), f_{n,k}(y)) > 0$, contradicting strong convergence of the sequence $(T_k)_{k\in\mathbb{N}}$ to T. $\square$

Remark A.3.12. The proof of Theorem A.3.11 shows that if T is either nongeometric, or contains an exotic component, then T is good. The constructions made in Section A.2.2 provide examples of trees with simplicial and surface components which are obtained as pulls of good trees but are not good, as will follow from Proposition A.6.22.

A.4 Limits of Lipschitz maps between very small F_N-trees

The goal of this section is to explain how to construct Lipschitz F_N-equivariant maps between very small F_N-trees, by using a limiting process (Theorem A.4.3). We start by recalling some facts about the equivariant Gromov-Hausdorff topology on the space of F_N-trees.

A.4.1 Equivariant Gromov-Hausdorff topology

In [Pau88], Paulin introduced yet another topology on $\overline{cv_N}$. Let T and T' be two F_N-trees, let $K \subset T$ and $K' \subset T'$ be finite subsets, let $P \subset F_N$ be a finite subset of F_N, and let $\epsilon > 0$. A *P-equivariant ϵ-relation* between K and K' is a subset $R \subseteq K \times K'$ whose projection to each factor is surjective, such that for all $(x, x'), (y, y') \in R$ and all $g, h \in P$, we have $|d_T(gx, hy) - d_{T'}(gx', hy')| < \epsilon$. We denote by $O(T, K, P, \epsilon)$ the set of F_N-trees T' for which there exists a finite subset $K' \subset T'$ and a P-equivariant ϵ-relation

$R \subseteq K \times K'$. Paulin showed that these sets define a basis of open sets for a topology on the set of F_N-trees, called the *equivariant Gromov-Hausdorff topology* [Pau88]. This topology is equivalent to the axes topology on $\overline{cv_N}$ [Pau89].

Let T be an F_N-tree, and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of F_N-trees that converges to T in the equivariant Gromov-Hausdorff topology. Let $x \in T$. Let $(K^k)_{k\in\mathbb{N}}$ be an increasing sequence of finite subsets of T containing x, such that the finite trees spanned by the subsets K^k yield an exhaustion of T, and let $F_N = \bigcup_{k\in\mathbb{N}} P^k$ be an exhaustion of F_N by finite subsets. For all $k \in \mathbb{N}$, let n_k be the smallest integer such that $T_{n_k} \in O(T, K^k, P^k, \frac{1}{k})$. For all $n \in \{n_k, \dots, n_{k+1}-1\}$, we can find a finite subset $K_n \subset T_n$ and a P^k-equivariant $\frac{1}{k}$-relation $R_n \subseteq K^k \times K_n$. Choose $x_n \in K_n$ such that $(x, x_n) \in R_n$. We say that the sequence $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} T_n$ is an *approximation* of x in the trees T_n, relative to the exhaustions determined by K^k and P^k.

Lemma A.4.1. *Let T be an F_N-tree, and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of F_N-trees that converges to T in the equivariant Gromov-Hausdorff topology. Let $x, y \in T$, let $g \in F_N$, let $M \in \mathbb{R}$. Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} T_n$ be approximations of x, y and gx relative to the same exhaustions. Then*

- *the distance $d_{T_n}(x_n, y_n)$ converges to $d_T(x, y)$, and*
- *the distance $d_{T_n}(gx_n, z_n)$ converges to 0, and*
- *if $x \in \mathcal{N}_M(C_T(g))$, then for sufficiently large $n \in \mathbb{N}$, we have $x_n \in \mathcal{N}_{M+1}(C_{T_n}(g))$.*

Proof. The first two assertions follow from the definition of the equivariant Gromov-Hausdorff topology. To prove the third assertion, one uses the fact that in an F_N-tree T, we have $d_T(x, gx) = 2d_T(x, C_T(g)) + ||g||_T$ (see [CM87, 1.3]), and the continuity of translation lengths in the equivariant Gromov-Hausdorff topology. □

Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of F_N-trees. A sequence $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} T_n$ is *bounded* if for all $g \in F_N$, the distance $d_{T_n}(x_n, gx_n)$ is bounded.

Proposition A.4.2. *Let T be a very small F_N-tree, and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of F_N-trees that converges to T in the equivariant Gromov-Hausdorff topology. A sequence $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} T_n$ is bounded if and only if there exist $x \in T$, exhaustions of T and F_N, and an approximation $(x'_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} T_n$ of x relative to these exhaustions, such that $d_{T_n}(x_n, x'_n)$ is bounded.*

In particular, Proposition A.4.2 shows the existence of bounded sequences in any converging sequence of very small F_N-trees. Note that its proof is not specific to the case of F_N-trees, and only requires the tree T to be *irreducible*, i.e. there exist two hyperbolic isometries in T whose commutator is also hyperbolic in T.

Proof. First assume that there exists $x \in T$, exhaustions of T and F_N, and an approximation $(x'_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} T_n$ of x relative to these exhaustions, such that $d_{T_n}(x_n, x'_n)$ is bounded. It follows from the first two assertions of Lemma A.4.1 that for all $g \in F_N$, the distance $d_{T_n}(x'_n, gx'_n)$ is bounded. The triangular inequality, together with the fact that the F_N-action on T_n is isometric for all $n \in \mathbb{N}$, implies that $d_{T_n}(x_n, gx_n)$ is bounded.

Conversely, assume that for all $g \in F_N$, the distance $d_{T_n}(x_n, gx_n)$ is bounded. Let $x \in T$, and let $(x'_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} T_n$ be an approximation of x in the trees T_n relative to some exhaustions (without loss of generality, we can assume that for all $g \in F_N$, there exists $k \in \mathbb{N}$ such that $gx \in K^k$). Let $a, b \in F_N$ be such that the commutator $[a, b]$ is hyperbolic in T. Using [CM87, 1.3], we can find $M \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, we have $x_n \in \mathcal{N}_M(C_{T_n}(a)) \cap \mathcal{N}_M(C_{T_n}(b))$. Let $M' \in \mathbb{R}$ be such that $x \in \mathcal{N}_{M'}(C_T(a)) \cap \mathcal{N}_{M'}(C_T(b))$.

For $n \in \mathbb{N}$ sufficiently large, we have $x'_n \in \mathcal{N}_{M'+1}(C_{T_n}(a)) \cap \mathcal{N}_{M'+1}(C_{T_n}(b))$ (Lemma A.4.1). As $[a,b]$ is hyperbolic in T, it is also hyperbolic in T_n for all n sufficiently large, and as $||a||_{T_n}$ and $||b||_{T_n}$ are bounded, this implies that the intersection $C_{T_n}(a) \cap C_{T_n}(b)$ has bounded length. By Proposition A.1.6, both x_n and x'_n lie in a neighborhood of $C_{T_n}(a) \cap C_{T_n}(b)$ (or of any point in the bridge between $C_{T_n}(a)$ and $C_{T_n}(b)$) in T_n of bounded diameter, so $d_{T_n}(x_n, x'_n)$ is bounded. ☐

A.4.2 Limits of Lipschitz F_N-equivariant maps between very small F_N-trees

Given an $\mathbb{R}$-tree T, recall that $\overline{T}$ denotes the metric completion of T. We aim at showing the following result.

Theorem A.4.3. *Let T and T' be two very small F_N-trees, let $(T_n)_{n\in\mathbb{N}}$ (resp. $(T'_n)_{n\in\mathbb{N}}$) be a sequence of F_N-trees converging to T (resp. T') in the equivariant Gromov-Hausdorff topology, and let $(M_n)_{n\in\mathbb{N}}$ be a sequence of real numbers, satisfying $M := \liminf_{n\to+\infty} M_n < +\infty$. Assume that for all $n \in \mathbb{N}$, there exists an M_n-Lipschitz F_N-equivariant map $f_n : T_n \to T'_n$. Then there exists an M-Lipschitz F_N-equivariant map $f : T \to \overline{T'}$.*

Again, Theorem A.4.3 can be generalized to more general contexts. We only need to require the existence of hyperbolic isometries whose commutator is again hyperbolic in the trees T and T'.

Remark A.4.4. It is not true in general that we can find an M-Lipschitz F_N-equivariant map $f : T \to T'$ without passing to the completion, see Example A.6.3. However, this is possible in some particular cases, for example if the tree T is simplicial. Indeed, in this case, one can always slightly move the f-image of a vertex in T to make it lie in T' without increasing the Lipschitz constant of f (no element of F_N fixes a point in $\overline{T'} \smallsetminus T'$), and tighten f on the edges of T to make the image $f(T)$ entirely lie in T' (which again does not increase the Lipschitz constant of f).

Our proof of Theorem A.4.3 uses the theory of ultralimits of metric spaces. Given a nonprincipal ultrafilter ω on $\mathbb{N}$, we first show that if $(T_n)_{n\in\mathbb{N}}$ is a sequence of very small F_N-trees converging to T, and $(p_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} T_n$ is a bounded sequence, then the ω-limit of the pointed metric spaces (T_n, p_n) is a complete F_N-tree T_ω, which contains an isometrically embedded copy of T (hence of $\overline{T}$) as an F_N-invariant subtree (Proposition A.4.5). Taking the ultralimit of the maps f_n provides a Lipschitz F_N-equivariant map $f_\omega : T_\omega \to T'_\omega$. We get the desired map $f : T \to \overline{T'}$ by precomposing f_ω with the embedding $T \hookrightarrow T_\omega$, and postcomposing it with the projection $T'_\omega \to \overline{T'}$.

We start by recalling the construction of ultralimits of metric spaces and maps between them. We refer the reader to [Kap09, Chapter 9] for an introduction to this topic. A *nonprincipal ultrafilter* on the set $\mathbb{N}$ of natural numbers is a map $\omega : 2^{\mathbb{N}} \to \{0,1\}$ such that

- for all $A, B \subseteq \mathbb{N}$, we have $\omega(A \cup B) = \omega(A) + \omega(B) - \omega(A \cap B)$, and
- we have $\omega(\emptyset) = 0$ and $\omega(\mathbb{N}) = 1$, and
- for all finite sets $A \subseteq \mathbb{N}$, we have $\omega(A) = 0$.

The existence of nonprincipal ultrafilters follows from the axiom of choice. We fix once and for all such a nonprincipal ultrafilter ω on $\mathbb{N}$. Given a sequence $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, there exists a unique $x_\omega \in \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ such that for every neighborhood U of x_ω in $\overline{\mathbb{R}}$, we have $\omega(\{n \in \mathbb{N} | x_n \in U\}) = 1$. We call x_ω the *ω-limit* of the sequence $(x_n)_{n\in\mathbb{N}}$, and denote it by $\lim_\omega x_n$.

Let $((X_n, d_n, p_n))_{n\in\mathbb{N}}$ be a sequence of pointed metric spaces, and let

$$X := \{(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n | \lim_\omega d_n(x_n, p_n) < +\infty\}.$$

Define a pseudo-metric on X by $d_\omega((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) := \lim_\omega d_n(x_n, y_n) \in [0, +\infty)$. The *$\omega$-limit* of the pointed metric spaces (X_n, p_n), denoted by $\lim_\omega(X_n, p_n)$, is defined to be the Hausdorff quotient of X for this pseudo-metric. It is a well-known fact that the ω-limit of any sequence of pointed metric spaces is complete.

The class of $\mathbb{R}$-trees is closed under taking ultralimits (see [Sta07, Lemma 4.6], for instance). Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of F_N-trees converging to a very small F_N-tree T in the equivariant Gromov-Hausdorff topology, and let $(p_n)_{n\in\mathbb{N}}$ be a bounded sequence. Whenever a sequence $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} T_n$ is such that the distance $d_{T_n}(x_n, p_n)$ is bounded, then the distance $d_{T_n}(gx_n, p_n) \leq d_{T_n}(gx_n, gp_n) + d_{T_n}(gp_n, p_n)$ is also bounded. Hence there is a natural isometric F_N-action on T_ω defined by $g(x_n)_{n\in\mathbb{N}} = (gx_n)_{n\in\mathbb{N}}$. From now on, whenever an $\mathbb{R}$-tree T_ω is obtained as an ultralimit of a converging sequence of F_N-trees (in the equivariant Gromov-Hausdorff topology) with respect to a bounded sequence, we will equip it with the F_N-action described above.

Proposition A.4.5. *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of F_N-trees, converging in the equivariant Gromov-Hausdorff topology to a very small F_N-tree T. Let $(p_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} T_n$ be a bounded sequence, and denote by T_ω the ω-limit of $(T_n, p_n)_{n\in\mathbb{N}}$. Then $\overline{T}$ isometrically embeds into T_ω as a closed F_N-invariant subtree.*

Proof. Using Proposition A.4.2, we can find an approximation $(p'_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} T_n$ of some point $p \in T$ such that $d_{T_n}(p_n, p'_n)$ is bounded. For $x \in T$, let $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} T_n$ be an approximation of x with respect to the same exhaustions as those used to define the approximation $(p'_n)_{n\in\mathbb{N}}$ of p (we can assume that $x \in K^k$ for all $k \in \mathbb{N}$). By Lemma A.4.1, the distance $d_{T_n}(x_n, p'_n)$ is bounded. The triangle inequality then implies that $d_{T_n}(x_n, p_n)$ is bounded, so we get a map

$$\begin{array}{cccc} \psi : & T & \to & T_\omega \\ & x & \mapsto & (x_n)_{n\in\mathbb{N}} \end{array}.$$

The first assertion of Lemma A.4.1 shows the map ψ is an isometric embedding, and the second shows that ψ is F_N-equivariant. In particular, the tree T isometrically embeds as an F_N-invariant subtree in T_ω. The $\mathbb{R}$-tree T_ω is complete, so the completion $\overline{T}$ also isometrically embeds as a (closed) F_N-invariant subtree of T_ω. $\square$

Let $M \in \mathbb{R}$, and let (X_n, d_n, p_n) and (X'_n, d'_n, p'_n) be two sequences of pointed metric spaces, together with M-Lipschitz maps $f_n : X_n \to X'_n$. Assume that for all sequences $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$ such that $d_n(p_n, x_n)$ is bounded, we have $\lim_\omega d'_n(p'_n, f_n(x_n)) < +\infty$. Then we can define a map $f_\omega : X_\omega \to X'_\omega$ by setting $f_\omega((x_n)_{n\in\mathbb{N}}) := (f_n(x_n))_{n\in\mathbb{N}}$. This applies for example to the case where $p'_n = f_n(p_n)$ for all $n \in \mathbb{N}$. The map f_ω is also M-Lipschitz (this applies more generally to the case where the maps f_n are M_n-Lipschitz with $\lim_\omega M_n = M$).

Proof of Theorem A.4.3. Up to passing to a subsequence, we may assume that the sequence $(M_n)_{n\in\mathbb{N}}$ converges to M. Let $(p_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} T_n$ be a bounded sequence, and for all $n \in \mathbb{N}$, let $q_n := f_n(p_n)$, then $(q_n)_{n\in\mathbb{N}}$ is bounded. Letting $T_\omega := \lim_\omega(T_n, p_n)$ and

$T'_\omega := \lim_\omega(T'_n, q_n)$, we thus get an F_N-equivariant M-Lipschitz map $f_\omega : T_\omega \to T'_\omega$ by setting $f_\omega((x_n)_{n\in\mathbb{N}}) := (f_n(x_n))_{n\in\mathbb{N}}$ for all $(x_n)_{n\in\mathbb{N}} \in T_\omega$. By Proposition A.4.5, the tree T (resp. $\overline{T'}$) isometrically embeds in T_ω (resp. T'_ω) as an F_N-invariant subtree. Denote by $i : T \hookrightarrow T_\omega$ the inclusion map (which is obviously F_N-equivariant and 1-Lipschitz), and by $\pi : T'_\omega \to \overline{T'}$ the closest point projection, which is also easily seen to be F_N-equivariant and 1-Lipschitz. The map $\pi \circ f_\omega \circ i : T \to \overline{T'}$ is the desired M-Lipschitz F_N-equivariant map from T to $\overline{T'}$. ☐

A.5 The case of trees with dense orbits

In this section, we prove our two main results (Theorems A.0.1 and A.0.2) in the case of trees with dense orbits.

A.5.1 An easy inequality in the extension of White's theorem

Given $T, T' \in \overline{cv_N}$, we define $\mathrm{Lip}(T,T')$ to be the infimum of a Lipschitz constant of an F_N-equivariant map $f : T \to \overline{T'}$ if such a map exists, and $\mathrm{Lip}(T,T') = +\infty$ otherwise. We define $\Lambda(T,T') := \sup_{g\in F_N} \frac{||g||_{T'}}{||g||_T}$ (where we take the conventions $\frac{0}{0} = 0$ and $\frac{1}{0} = +\infty$). Given a subset $\mathcal{C} \subseteq F_N$, we define $\Lambda_{\mathcal{C}}(T,T') := \sup_{g\in\mathcal{C}} \frac{||g||_{T'}}{||g||_T}$ (in particular, we have $\Lambda_{F_N}(T,T') = \Lambda(T,T')$). Given a map f between $\mathbb{R}$-trees, we denote by $\mathrm{Lip}(f)$ the Lipschitz constant of f. We start by recalling the proof of the following inequality, which shows in particular that strong domination implies weak domination.

Proposition A.5.1. *For all $T, T' \in \overline{cv_N}$, we have $\Lambda(T,T') \leq Lip(T,T')$. In particular, for all $\mathcal{C} \subseteq F_N$, we have $\Lambda_{\mathcal{C}}(T,T') \leq Lip(T,T')$.*

Proof. Let $T, T' \in \overline{cv_N}$. Assume that $\mathrm{Lip}(T,T') < +\infty$ (otherwise the claim is obvious), and let $f : T \to \overline{T'}$ be a Lipschitz F_N-equivariant map. Let $g \in F_N$, and let $x \in C_T(g)$. Then

$$\begin{aligned} ||g||_{T'} &\leq d_{\overline{T'}}(f(x), gf(x)) \\ &\leq \mathrm{Lip}(f) d_T(x, gx) \\ &= \mathrm{Lip}(f)||g||_T, \end{aligned}$$

so for all $g \in F_N$, we have $\frac{||g||_{T'}}{||g||_T} \leq \mathrm{Lip}(f)$. The claim follows. ☐

A.5.2 Extending White's theorem to trees with dense orbits

Given $\mathcal{C} \subseteq F_N$, we say that a tree $T \in \overline{cv_N}$ *satisfies White's theorem relatively to* $\mathcal{C}$ if for all $T' \in \overline{cv_N}$, we have $\mathrm{Lip}(T,T') = \Lambda_{\mathcal{C}}(T,T')$.

Proposition A.5.2. *Let $\mathcal{C} \subseteq F_N$, and $T \in \overline{cv_N}$. If T admits a Lipschitz approximation by a sequence of trees $T_n \in \overline{cv_N}$ which all satisfy White's theorem relatively to $\mathcal{C}$, then T satisfies White's theorem relatively to $\mathcal{C}$.*

Proof. Proposition A.5.1 shows that $\Lambda_{\mathcal{C}}(T,T') \leq \mathrm{Lip}(T,T')$, and if $\Lambda_{\mathcal{C}}(T,T') = +\infty$, then the reverse inequality is obvious. We can thus assume that $\Lambda_{\mathcal{C}}(T,T') < +\infty$. As the trees T_n satisfy White's theorem relatively to $\mathcal{C}$, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \mathrm{Lip}(T_n,T') &= \Lambda_{\mathcal{C}}(T_n,T') \\ &\leq \Lambda_{\mathcal{C}}(T_n,T)\Lambda_{\mathcal{C}}(T,T') \\ &\leq \Lambda_{\mathcal{C}}(T,T'), \end{aligned}$$

since it follows from Proposition A.5.1 and the definition of a Lipschitz approximation that $\Lambda_{\mathcal{C}}(T_n, T) \leq 1$. As $\Lambda_{\mathcal{C}}(T, T') < +\infty$, Theorem A.4.3 thus shows the existence of a $\Lambda_{\mathcal{C}}(T, T')$-Lipschitz F_N-equivariant map from T to $\overline{T'}$, hence $\text{Lip}(T, T') \leq \Lambda_{\mathcal{C}}(T, T')$. $\square$

Corollary A.5.3. *Let $T, T' \in \overline{cv_N}$, and assume that T has dense orbits. Then*

$$Lip(T, T') = \Lambda(T, T') = \Lambda_{\mathcal{P}_N}(T, T').$$

Proof. Theorem A.3.6 and Lemma A.1.5 show that T admits a Lipschitz approximation by trees in cv_N, and Theorem A.1.3 shows that trees in cv_N satisfy White's theorem relatively to $\mathcal{P}_N$ (this actually only uses the particular case of Theorem A.1.3 where the simplicial tree belongs to CV_N). Corollary A.5.3 thus follows from Proposition A.5.2. $\square$

A.5.3 Simple-equivalent trees with dense orbits are equal.

Proposition A.5.4. *Let $T, T' \in \overline{cv_N}$ be two trees that both satisfy White's theorem relatively to $\mathcal{P}_N$. If T and T' are simple-equivalent, then $T = T'$.*

Proof. The hypotheses ensure the existence of 1-Lipschitz F_N-equivariant maps from T to $\overline{T'}$ and from T' to $\overline{T}$, so Proposition A.5.1 implies that $||g||_T = ||g||_{T'}$ for all $g \in F_N$. Theorem A.1.1 thus implies that $T = T'$. $\square$

From Corollary A.5.3 and Proposition A.5.4, we deduce the following corollary.

Corollary A.5.5. *Let $T, T' \in \overline{cv_N}$ be two trees with dense orbits. If T and T' are simple-equivalent, then $T = T'$.* $\square$

A.5.4 Computing stretching factors between trees with dense orbits

We give a formula for $\text{Lip}(T, T')$ for trees $T, T' \in \overline{cv_N}$ having dense orbits, in terms of length measures on T and T'. This notion was introduced by Paulin in [Pau95], and developed by Guirardel in [Gui00, Section 5]. An *invariant length measure* μ on T is a collection of finite Borel measures μ_I for all segments $I \subseteq T$ such that

- for all segments $J \subseteq I$, we have $\mu_J = (\mu_I)_{|J}$, and
- for all segments $I \subseteq T$ and all $g \in F_N$, we have $\mu_{gI} = (g_{|I})_* \mu_I$.

Given a segment $I \subseteq T$, we will simply write $\mu(I)$ to denote $\mu_I(I)$. We denote by μ^T the *Lebesgue measure* on T given by $\mu^T([x, y]) := d_T(x, y)$ for all $x, y \in T$. A subset $E \subseteq T$ is *measurable* if each intersection of E with an arc of T is measurable. A measurable subset E has *μ-measure* 0 if for every arc $I \subseteq T$, we have $\mu_I(E \cap I) = 0$. It has *full μ-measure* if $T \smallsetminus E$ has μ-measure 0. A measure μ on an F_N-tree T is *ergodic* if every F_N-invariant measurable subset of T has either zero or full μ-measure. We denote by $\mathcal{M}_0(T)$ the space of nonatomic measures on T. The following theorem, due to Guirardel, states that any tree in $\overline{cv_N}$ with dense orbits is finite-dimensional from the measure-theoretic viewpoint.

Theorem A.5.6. *(Guirardel [Gui00, Corollary 5.4]) For all very small F_N-trees T with dense orbits, the set $\mathcal{M}_0(T)$ is a finite-dimensional convex set. Furthermore, the tree T has at most $3N - 4$ nonatomic ergodic measures up to homothety, and every measure in $\mathcal{M}_0(T)$ is a sum of these ergodic measures.*

Let $T, T' \in \overline{cv_N}$ be two trees with dense orbits. Denote by k the dimension of $\mathcal{M}_0(T)$, and let $\{\mu_i\}_{i=1,\dots,k}$ be a collection of ergodic measures on T given by Theorem A.5.6. The measures μ_i are pairwise mutually singular, and there exist disjoint measurable sets $E_1, \dots, E_k$ that cover T such that for all $i \in \{1, \dots, k\}$, the set E_i has full μ_i-measure. As the Lebesgue measure μ^T is nonatomic, it decomposes as $\mu^T = \sum_{i=1}^k \lambda_i \mu_i$, with $\lambda_i \geq 0$ for all $i \in \{1, \dots, k\}$. The measures μ_i for which $\lambda_i > 0$ are absolutely continuous with respect to μ^T. In particular, they are *regular*, i.e. for all Borel subsets $X \subseteq T$ and all $\epsilon > 0$, there exists an open subset $U \subseteq T$ containing X such that for all segments $I \subseteq T$, we have $\mu_i(X \cap I) \geq \mu_i(U \cap I) - \epsilon$. If there exists a Lipschitz F_N-equivariant, alignment-preserving map $f : T \to T'$, then the measure μ^f defined on T by $\mu^f([x,y]) := d_{T'}(f(x), f(y))$ is absolutely continuous with respect to μ^T. In particular, it decomposes as $\mu^f = \sum_{i=1}^k \lambda'_i \mu_i$, where $\lambda'_i \geq 0$ for all $i \in \{1, \dots, k\}$ and $\lambda'_i = 0$ whenever $\lambda_i = 0$. It follows from Proposition A.1.7 that there exists at most one Lipschitz, F_N-equivariant, alignment-preserving map $f : T \to T'$. If such a map exists, and if μ^T denotes the Lebesgue measure on T, we denote by $\mu^{T \to T'}$ the measure μ^f on T.

Proposition A.5.7. *Let $T, T' \in \overline{cv_N}$ be two F_N-trees with dense orbits. Then $Lip(T, T') < +\infty$ if and only if there exists a Lipschitz F_N-equivariant, alignment-preserving map from T to T'. In this case, denote by k the dimension of $\mathcal{M}_0(T)$, let $\mu^T = \sum_{i=1}^k \lambda_i \mu_i$ be the decomposition of the Lebesgue measure on T on its ergodic components, and let $\mu^{T \to T'} = \sum_{i=1}^k \lambda'_i \mu_i$ be the decomposition of $\mu^{T \to T'}$. Then*

$$Lip(T, T') = \max_{i \in \{1, \dots, k\}} \frac{\lambda'_i}{\lambda_i}.$$

Proof. If $\mathrm{Lip}(T, T') < +\infty$, then there exists a Lipschitz F_N-equivariant map $f : T \to \overline{T'}$, which is unique by Proposition A.1.7, and preserves alignment by Proposition A.3.9. This implies that $f(T) \subseteq T'$, otherwise we would find $x \in T$ with $f(x) \in \overline{T'} \smallsetminus T'$, and some component of $T \smallsetminus \{x\}$ would be entirely mapped to $f(x)$. However, as T has dense orbits, this would imply that $f(T) \subseteq \overline{T'} \smallsetminus T'$, a contradiction. Let $C := \max_{i \in \{1, \dots, k\}} \frac{\lambda'_i}{\lambda_i}$. By definition of $\mu^{T \to T'}$, for all $x, y \in T$, we have

$$\begin{aligned} d_{T'}(f(x), f(y)) &= \mu^{T \to T'}([x,y]) \\ &= \textstyle\sum_{i=1}^k \lambda'_i \mu_i([x,y]) \\ &\leq C \mu^T([x,y]) \\ &= C d_T(x,y), \end{aligned}$$

so $\mathrm{Lip}(f) \leq C$. Let $i \in \{1, \dots, k\}$ be such that $\lambda_i > 0$, and let $I \subseteq T$ be an arc such that $\mu_i(I) > 0$. We denote by μ_i^c the measure $\mu^T - \lambda_i \mu_i$. The measures $\mu_1, \dots, \mu_k$ are pairwise mutually singular, so there exists a Borel subset X of I such that $\mu_i(X) = \mu_i(I)$ and $\mu_j(X) = 0$ for all $j \neq i$. As μ_i is regular, for all $\epsilon > 0$, there exists an open set $U \subseteq I$ that contains X, such that $\mu_i^c(U) < \epsilon \mu_i(I)$. Since $U \subseteq I$ is open, it is the disjoint union of a countable collection of open intervals. At least one of these intervals I' must satisfy $\mu_i^c(I') < \epsilon \mu_i(I')$. By definition of $\mu^{T \to T'}$, we have

$$\begin{aligned} \mathrm{Lip}(f) &\geq \tfrac{\mu^{T \to T'}(I')}{\mu^T(I')} \\ &= \tfrac{\lambda'_i \mu_i(I') + \sum_{j \neq i} \lambda'_j \mu_j(I')}{\lambda_i \mu_i(I') + \mu_i^c(I')} \\ &\geq \tfrac{\lambda'_i \mu_i(I')}{\lambda_i \mu_i(I') + \mu_i^c(I')} \\ &\geq \tfrac{\lambda'_i}{\lambda_i + \epsilon}. \end{aligned}$$

By choosing $\epsilon > 0$ arbitrarily small, we thus get that $\frac{\lambda'_i}{\lambda_i} \leq \mathrm{Lip}(f)$. This holds for all $i \in \{1, \dots, k\}$ for which $\lambda_i > 0$, and in addition we have $\lambda'_i = 0$ whenever $\lambda_i = 0$. This shows that $\mathrm{Lip}(f) \geq C$, and hence $\mathrm{Lip}(f) = \mathrm{Lip}(T, T') = C$. □

A.6 Generalization of White's theorem

We now generalize Theorems A.1.2 and A.1.3 to arbitrary actions in the boundary of outer space. This answers a question by Algom-Kfir [AK13, Question 4.6].

Theorem A.6.1. *For all $T, T' \in \overline{cv_N}$, we have Lip$(T, T') = \Lambda(T, T')$.*

The proof of Theorem A.6.1 will be carried out in Sections A.6.2 to A.6.4. White's theorem for trees in cv_N is actually a bit stronger, as it provides a finite set (depending on T but not on T') of (conjugacy classes of) primitive elements of F_N called candidates, represented by loops in T/F_N having a particular shape, on which the supremum in the definition of $\Lambda(T, T')$ is achieved for all $T' \in cv_N$. In particular, this gives an explicit procedure for computing $\mathrm{Lip}(T, T')$ for all $T, T' \in cv_N$. In Section A.6.5, we will give a generalization of the notion of candidates in a tree $T \in \overline{cv_N}$, and show that for all $T' \in \overline{cv_N}$, the supremum in the definition of $\Lambda(T, T')$ can be taken over the set of candidates in T (Theorem A.6.17). In Section A.6.6, we show that in the case of good trees, this supremum can be taken over the set of primitive elements of F_N. This will turn out to be a crucial tool for tackling the problem of spectral rigidity of the set of primitive elements of F_N in $\overline{cv_N}$.

A.6.1 A few examples

When $T, T' \in cv_N$, both the infimum in the definition of $\mathrm{Lip}(T, T')$ and the supremum in the definition of $\Lambda(T, T')$ are achieved. This remains true more generally when T is simplicial, and in this case we can replace $\overline{T'}$ by T' in the definition of $\mathrm{Lip}(T, T')$, see [AK13, Proposition 4.5] and Remark A.4.4 of the present paper. When $T, T' \in \overline{cv_N}$ are arbitrary trees, the infimum in the definition of $\mathrm{Lip}(T, T')$ is still realized as long as there exists a Lipschitz F_N-equivariant map $f : T \to \overline{T'}$ (Proposition A.6.4). However the supremum in the definition of $\Lambda(T, T')$ may not be realized (even if it is finite), as shown in the following example.

Example A.6.2. (see Figure A.13). We provide an example of a pair of trees $T_1, T_2 \in \overline{cv_N}$ for which the supremum in the definition of $\Lambda(T_1, T_2)$ is not achieved. For all $i \in \{1, 2\}$, let T_i be the tree associated to a graph of actions with a single edge of length i having trivial stabilizer, and two vertices, one having cyclic stabilizer generated by an element $t \in F_N$, and the other being a nontrivial G-tree T_0 with dense orbits (where G is a complementary free factor in F_N of the cyclic group generated by t), whose attaching point p is not fixed by any element of F_N (the existence of such a point p follows from [Jia91]).

There is an obvious 2-Lipschitz F_N-equivariant map from T_1 to T_2 which stretches the edges in the simplicial part of T by a factor of 2, hence $\mathrm{Lip}(T_1, T_2) \leq 2$. As T_0 has dense orbits, for all $\epsilon > 0$, there exists $g \in G$ such that $d_{T_0}(p, gp) < \epsilon$. So for all $i \in \{1, 2\}$, we have $2i < ||tg||_{T_i} < 2i + \epsilon$, hence $\frac{||tg||_{T_2}}{||tg||_{T_1}} \geq \frac{4}{2+\epsilon}$, which becomes arbitrary close to 2 as ϵ goes to 0. So $\Lambda(T_1, T_2) \geq 2$, and hence by Proposition A.5.1 we have $\mathrm{Lip}(T_1, T_2) = \Lambda(T_1, T_2) = 2$. However, any element $g \in F_N$ either belongs to a conjugate of G, or of the cyclic group generated by t (in which case $||g||_{T_1} = ||g||_{T_2}$), or is conjugated to an element represented by a reduced word of the form $t^{\alpha_1} g_1 t^{\alpha_2} \dots t^{\alpha_k} g_k$, with $\alpha_i \neq 0$

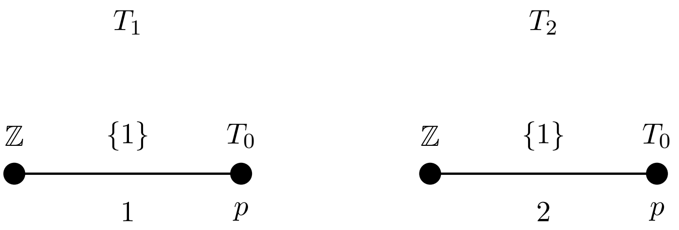


Figure A.13: The trees T_i in Example A.6.2.

Figure A.14: The trees T and T' in Example A.6.3.

and $g_i \in G \smallsetminus \{e\}$ for all $i \in \{1, \dots, k\}$. In this last case, we have $l_i := d_{T_0}(p, g_i p) > 0$ for all $i \in \{1, \dots, k\}$, because p is not fixed by any element of G, and

$$||g||_{T_1} = 2k + \sum_{i=1}^{k} l_i,$$

and similarly

$$||g||_{T_2} = 4k + \sum_{i=1}^{k} l_i < 2||g||_{T_1}.$$

So no element in F_N is stretched exactly by an amount of 2 from T_1 to T_2.

Example A.6.3. (see Figure A.14). We give an example of a pair of trees $T, T' \in \overline{cv_N}$ for which $\overline{T'}$ cannot be replaced by T' in the definition of $\mathrm{Lip}(T, T')$. More precisely, we give an example of a pair of trees $T, T' \in \overline{cv_N}$ for which there exists an F_N-equivariant Lipschitz map from T to $\overline{T'}$, but no such map from T to T'.

Let T_0 be a minimal nontrivial F_{N-1}-tree with dense orbits, then T_0 is strictly contained in its metric completion (see [GL95, Example II.6]). Let $p \in \overline{T_0} \smallsetminus T_0$, and let T be the tree associated to a graph of actions having

- two vertices v_1 and v_2, with v_1 having nontrivial cyclic vertex group, and v_2 having $\overline{T_0}$ as its vertex tree, and
- one single edge e of length 2 with trivial edge group, whose origin is v_1, and whose terminal vertex is v_2, with attaching point p.

Let $p' \in T_0$ be such that $d_T(p, p') = 1$, and let T' be the F_N-tree obtained by equivariantly folding half of the edge e along the segment $[p, p']$, and passing to a minimal subtree.

The definition of T' provides a 1-Lipschitz F_N-equivariant morphism from T to $\overline{T'}$. However, we claim that for all $M \in \mathbb{R}$, there is no M-Lipschitz F_N-equivariant map from T to T'. Indeed, suppose $f : T \to T'$ is F_N-equivariant and Lipschitz. Then $f(T_0)$ is an F_{N-1}-tree with dense orbits contained in T', so $f(T_0) \subseteq T_0$, and by Proposition A.1.7, the map f restricts to the identity on T_0. As p is the limit of a sequence $(p_n)_{n \in \mathbb{N}}$ of elements in

T_0, its image $f(p)$ should be the limit of $f(p_n)=p_n$ in T'. However, the sequence $(p_n)_{n\in\mathbb{N}}$ does not converge in T'.

A.6.2 Optimal maps and legal turns

Let $T,T'\in\overline{cv_N}$. A map $f:T\to\overline{T'}$ is *piecewise-linear* if it is Lipschitz, and linear in restriction to the edges in the simplicial part of T. Let X denote the underlying graph of the canonical decomposition of T as a graph of actions with vertex trees having dense orbits (Proposition A.1.4). The length of a segment $\gamma\subset T$ is denoted by $l_T(\gamma)$, and similarly the length of a path γ in X is denoted by $l_X(\gamma)$. We define T_f to be the (possibly empty) subset of T consisting of the edges e in the simplicial part of T for which $\frac{l_{\overline{T'}}(f(e))}{l_T(e)}=\mathrm{Lip}(f)$. We denote by T_f^{simpl} the projection of T_f to T^{simpl}, and by X_f its projection to X. An *optimal map* $f:T\to\overline{T'}$ is an F_N-equivariant, piecewise-linear map such that $\mathrm{Lip}(f)=\mathrm{Lip}(T,T')$, and X_f is minimal for the inclusion among all F_N-equivariant, piecewise-linear $\mathrm{Lip}(f)$-Lipschitz maps. Note that in the case where T has dense orbits, this last condition is empty.

Proposition A.6.4. *Let $T,T'\in\overline{cv_N}$. If $Lip(T,T')<+\infty$, then there exists an optimal map $f:T\to\overline{T'}$.*

Remark A.6.5. Again, this extends to more general contexts than F_N-actions. For example, this is true if T and T' are two trees belonging to an irreducible deformation space, see [Mei14, Theorem 20].

Proof. Applying Theorem A.4.3 to a sequence of F_N-equivariant maps $f_n:T\to\overline{T'}$ with $\mathrm{Lip}(f_n)\le\mathrm{Lip}(T,T')+\frac{1}{n}$ gives the existence of a $\mathrm{Lip}(T,T')$-Lipschitz F_N-equivariant map $f:T\to\overline{T'}$. Tightening f on the edges in the simplicial part of T cannot increase its Lipschitz constant, hence we may choose $f:T\to\overline{T'}$ to be piecewise-linear. As X is a finite graph, we can also choose f so that X_f is minimal. $\square$

We fix once and for all two trees $T,T'\in\overline{cv_N}$ such that $\mathrm{Lip}(T,T')<+\infty$, together with an optimal map $f:T\to\overline{T'}$. A *turn* at a vertex v of T^{simpl} is a pair $(e=[a,b],e'=[c,d])$ of distinct edges in the simplicial part of T such that $[b,c]$ projects to v (in other terms, the projections of e and e' to T^{simpl} share a common vertex). Let v be a vertex of T_f^{simpl} such that T_v is reduced to a point. A turn (e,e') at v is *legal* for f if $e,e'\subseteq T_f$, and $f(e)\cap f(e')=\{f(b)\}$, and *illegal* otherwise. It is *legal up to G_v* for f if there exists $g\in G_v$ such that (e,ge') is legal for f. The following proposition, already used by Algom-Kfir in her proof of [AK13, Proposition 4.5], gives control over legal turns at a vertex v of T_f^{simpl} for which T_v is reduced to a point. We provide a proof for completeness.

Proposition A.6.6. *Let $f:T\to\overline{T'}$ be an optimal map, and assume that $T_f\neq\emptyset$. Let v be a vertex in T_f^{simpl} such that T_v is reduced to a point. Then there exists a turn at v which is legal for f, and if G_v has rank at least 2, then all turns at v are legal up to G_v for f. In addition, for all edges e,e',e'' in T_f adjacent to v, if (e,e') and (e',e'') are both illegal for f, then (e,e'') is also illegal for f.*

Proof. If the f-images of all edges in T_f adjacent to v have a common initial germ, and have a common initial segment with their g-translate for all $g\in G_v$, then all f-images of edges in T_f adjacent to v in a single F_N-orbit have a common initial segment, which is an arc fixed by G_v in $\overline{T'}$. As there are finitely many such orbits, this implies that all f-images of edges in T_f adjacent to v have a common initial segment, which is an arc fixed by G_v

in $\overline{T'}$. One can then slightly homotope f to either decrease Lip(f) or X_f, contradicting optimality of f (see the proof of [FM11b, Proposition 3.15] or [AK11, Proposition 2.3] for details). So we can find a turn at v which is legal for f. Assume in addition that G_v has rank at least 2. For all edges e, e' in T_f adjacent to v (possibly with $e = e'$), the f-images of e and e' are not reduced to points. As T' is very small, the subgroup G'_v of elements $g \in G_v$ such that $f(e)$ and $gf(e')$ share a nondegenerate initial segment is at most cyclic, and for all $g \in G_v \smallsetminus G'_v$, the turn (e, ge') is legal for f. The assertion stating that illegality at v is a transitive relation follows from the definition of illegal turns. ☐

One has to be slightly more careful when defining legality of turns at vertices v of T_f^{simpl} for which T_v is not reduced to a point. Let v be such a vertex. For $\epsilon > 0$, a turn $(e = [a,b], e' = [c,d])$ at v is said to be *ϵ-legal* for f if $e, e' \subseteq T_f$, and $d_T(b,c) < \epsilon$, and $l_{\overline{T'}}(f(e) \cap f(e')) < \epsilon$. It is *legal up to G_v* for f if for all $\epsilon > 0$, there exists $g \in G_v$ such that (e, ge') is ϵ-legal for f. We aim at giving an analogue of Proposition A.6.6 in this situation. The following lemma, illustrated in Figure A.15, will turn out to be useful.

Lemma A.6.7. *Let T be an $\mathbb{R}$-tree, let $l, \epsilon \in \mathbb{R}$ with $\epsilon < \frac{l}{10}$, and let $a, a', a'', b, b', b'' \in T$. Assume that $l_T([a,b] \cap [a',b']), l_T([a,b] \cap [a'',b'']) \geq l$ and $l_T([a,a']), l_T([a,a'']) \leq \epsilon$. Then $l_T([a',b'] \cap [a'',b'']) \geq l - \epsilon$.*

Proof. As $\epsilon < \frac{l}{10}$, one can check that the tripods $\{a, a', a''\}$ and $\{b, b', b''\}$ do not intersect. One then argue depending on whether the intersection of the bridge between them with the tripod $\{a, a', a''\}$ (resp. $\{b, b', b''\}$) is in the direction of a, a' or a'' (resp. b, b' or b''). The various possibilities are displayed on Figure A.15. ☐

Proposition A.6.8. *Let $f : T \to \overline{T'}$ be an optimal map, and assume that $T_f \neq \emptyset$. Let v be a vertex in T_f^{simpl} such that T_v is not reduced to a point. Then all turns (e, e') at v with $e, e' \subseteq T_f$ are legal up to G_v for f.*

Proof. We denote by b (resp. c) the attaching point of e (resp. e') to T_v. Assume towards a contradiction that there exists $\epsilon > 0$ such that for all $g \in G_v$, the turn (e, ge') is not ϵ-legal. Let $\epsilon' := \frac{\epsilon}{100 \max\{M,1\}}$, where $M := \text{Lip}(f)$. Let $g \in G_v$ be a hyperbolic element in T_v such that $d_T(b, gb) < \epsilon'$. As T_v has dense orbits, there exists $g_0 \in G_v$ such that $d_T(b, g_0 c) < \epsilon'$, and hyperbolic elements $g_1, g_2 \in G_v$ which generate a rank 2 subgroup of G_v, such that for all $i \in \{1,2\}$, we have $d_T(c, g_i c) < \epsilon'$ (in particular $||g_i||_T < \epsilon'$ and $||g_i||_{T'} < \frac{\epsilon}{100}$). By the triangle inequality and the fact that the F_N-action is isometric, we also have $d_T(b, g_0 g_i c) < 2\epsilon'$ for all $i \in \{1,2\}$. The hypothesis thus implies that $l_{\overline{T'}}(f(e) \cap g_0 f(e')) \geq \epsilon$ and $l_{\overline{T'}}(f(e) \cap g_0 g_i f(e')) \geq \epsilon$ for all $i \in \{1,2\}$. In addition, as f is M-Lipschitz, we have $d_{\overline{T'}}(f(b), f(g_0 c)) < \frac{\epsilon}{100}$, and $d_{\overline{T'}}(f(b), f(g_0 g_i c)) < \frac{\epsilon}{50}$, so Lemma A.6.7 implies that the segments $f(e'), f(g_1 e')$ and $f(g_2 e')$ pairwise intersect along a subsegment of length greater than $\frac{49\epsilon}{50}$. However, as $||g_i||_{T'} < \frac{\epsilon}{100}$ for all $i \in \{1,2\}$, this implies that the axes of g_1 and g_2 in T' have a nontrivial overlap, of length greater than $||g_1||_{T'} + ||g_2||_{T'}$. Hence some nontrivial element in the rank 2 subgroup generated by g_1 and g_2 fixes a nondegenerate subsegment of these axes, contradicting Lemma A.1.5. ☐

A.6.3 Case where Lip$(T, T') < +\infty$

We first prove Theorem A.6.1 in the case where Lip$(T, T') < +\infty$.

Proposition A.6.9. *Let $T, T' \in \overline{cv_N}$. If Lip$(T, T') < +\infty$, then Lip$(T, T') \leq \Lambda(T, T')$.*

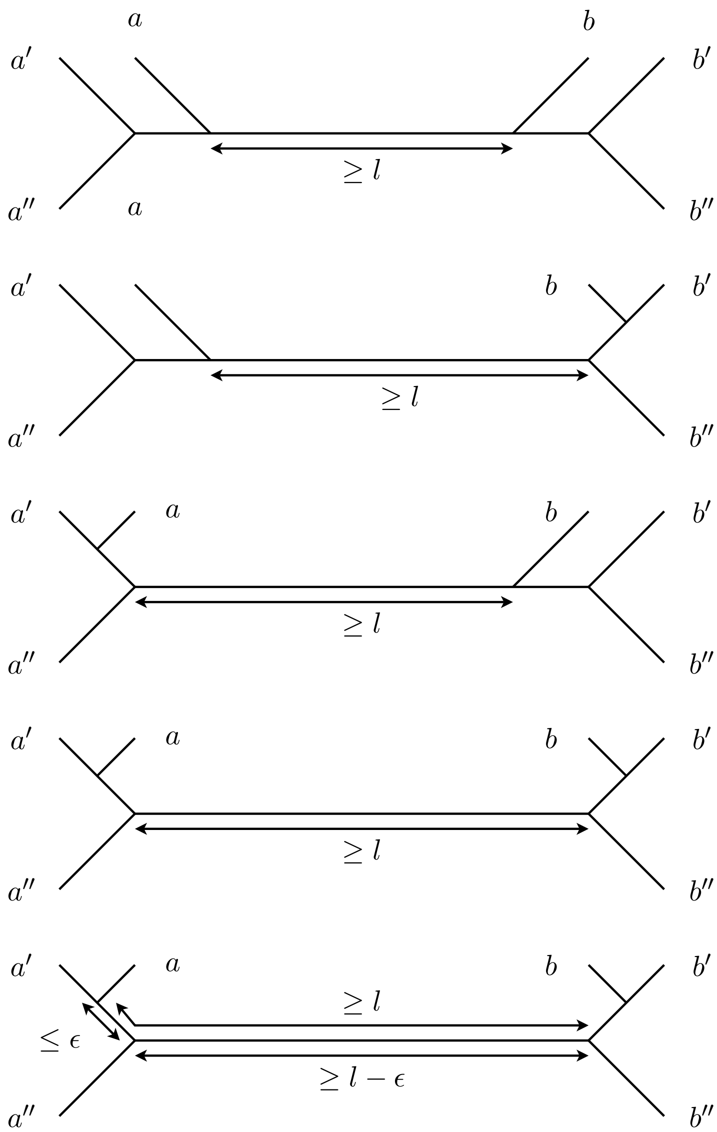


Figure A.15: The situation in Lemma A.6.7.

Let $T \in \overline{cv_N}$ and $g \in F_N$. The *combinatorial length* of g in T, denoted by $l_T^{comb}(g)$, is defined as the length of g in the simplicial tree obtained from T^{simpl} by making all edge lengths equal to 1. As there are finitely many orbits of branch points in T by [GL95, Corollary III.3], the number of orbits of edges in T^{simpl} is finite. An element $g \in F_N$ is *ϵ-legal* for f if

- its axis $C_T(g)$ crosses an edge in the simplicial part of T, and
- whenever $C_T(g)$ crosses a turn at a vertex v of T^{simpl} whose corresponding vertex tree in T is reduced to a point, then this turn is legal, and
- whenever $C_T(g)$ crosses a turn at a vertex v of T^{simpl} whose corresponding vertex tree in T is not reduced to a point, then this turn is ϵ-legal.

Lemma A.6.10. *Let $T, T' \in \overline{cv_N}$ be such that $Lip(T, T') < +\infty$, and let $f : T \to \overline{T'}$ be an optimal map. If $T_f \neq \emptyset$, then there exists $K > 0$ such that for all $\epsilon > 0$, there exists an element $g \in F_N$ with $l_T^{comb}(g) \leq K$, which is ϵ-legal for f.*

Proof. Let K be the (finite) number of orbits of oriented edges in T^{simpl}, let $\epsilon > 0$, and let $x \in T_f$. Starting from x and using Propositions A.6.6 and A.6.8, we construct a path in T by only crossing ϵ-legal turns for f (legal turns at vertices whose corresponding vertex tree is reduced to a point). After crossing at most K turns, we have necessarily crossed the same orbit of oriented edges twice, so we have constructed a segment of the form $[v, gv]$ for some $g \in F_N$. In particular, we have $l_T^{comb}(g) \leq K$, and g is ϵ-legal for f. □

An element $g \in F_N$ is called an *ϵ-witness* for the pair (T, T') if $\frac{||g||_{T'}}{||g||_T} \geq \mathrm{Lip}(T, T') - \epsilon$.

Lemma A.6.11. *Let $T, T' \in \overline{cv_N}$ be such that $Lip(T, T') < +\infty$, and let $f : T \to \overline{T'}$ be an optimal map. Assume that $T_f \neq \emptyset$. For all $\epsilon > 0$ and all $K \in \mathbb{N}$, there exists $\epsilon' > 0$ such that any element $g \in F_N$ with $l_T^{comb}(g) \leq K$ and which is ϵ'-legal for f, is an ϵ-witness for the pair (T, T').*

Proof. Let $\lambda > 0$ be the smallest length of an edge in T^{simpl}, and let $\epsilon' > 0$ be smaller than $\frac{M}{3}\lambda$, where $M := \mathrm{Lip}(f)$. Let $g \in F_N$ be ϵ'-legal for f and such that $l_T^{comb}(g) \leq K$. Let $v \in C_T(g)$, and let γ be the projection of $[v, gv]$ to X. Then $||g||_T \leq l_X(\gamma) + K\epsilon'$. In addition, every edge in T_f is mapped by f to a segment of length at least $M\lambda$. As $\epsilon' \leq \frac{M}{3}\lambda$, the control we have over cancellation for ϵ'-legal turns ensures that after tightening, the length of any fundamental domain of the f-image of $C_T(g)$ is at least $\mathrm{Lip}(T, T')l_X(\gamma) - 2K\epsilon'$, and we have $||g||_{T'} \geq \mathrm{Lip}(T, T')l_X(\gamma) - 2K\epsilon'$, see Figure A.16. Hence

$$\frac{||g||_{T'}}{||g||_T} \geq \frac{\mathrm{Lip}(T, T')l_X(\gamma) - 2K\epsilon'}{l_X(\gamma) + K\epsilon'}.$$

By making ϵ' arbitrarily small, we can make $\frac{||g||_{T'}}{||g||_T}$ arbitrarily close to $\mathrm{Lip}(T, T')$ (we can assume that $l_X(\gamma)$ is bounded below because X is a finite graph). □

Proposition A.6.12. *Let $T, T' \in \overline{cv_N}$ be such that $Lip(T, T') < +\infty$. For all $\epsilon > 0$, there exists an ϵ-witness g for the pair (T, T'). If $f : T \to \overline{T'}$ is an optimal map, and $T_f = \emptyset$, then we can choose g to be contained in a vertex stabilizer of T^{simpl}.*

Proof. Let $f : T \to \overline{T'}$ be an optimal map (whose existence is provided by Proposition A.6.4). If $T_f \neq \emptyset$, the claim follows from Lemmas A.6.10 and A.6.11. Otherwise, as X is a finite graph, there exists a vertex v in T^{simpl} corresponding to a tree T_v with dense

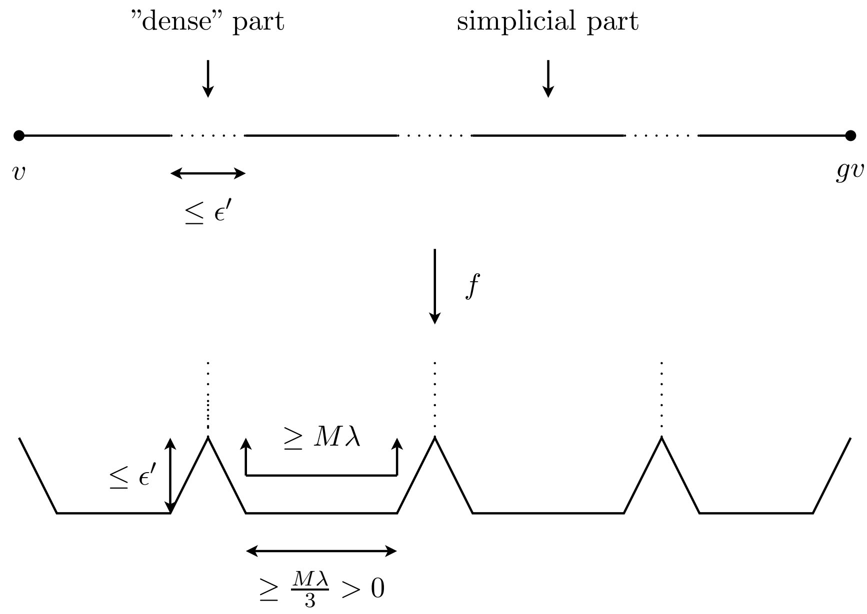


Figure A.16: The control over cancellation in an ϵ'-legal path.

orbits such that $f_{|T_v} : T_v \to f(T_v)$ has Lipschitz constant $\mathrm{Lip}(T, T')$. Denoting by $T_v^{\min}$ the minimal G_v-subtree of T_v, we get a G_v-equivariant map $f_{|T_v^{\min}} : T_v^{\min} \to f(T_v^{\min})$, and $\mathrm{Lip}(f_{|T_v^{\min}}) = \mathrm{Lip}(f_{|T_v})$. But $T_v^{\min}$ is a tree with dense orbits, and so is $f(T_v^{\min})$. Hence by Proposition A.1.7, there exists a unique $\mathrm{Lip}(T, T')$-Lipschitz G_v-equivariant map from $T_v^{\min}$ to $f(T_v^{\min})$, and this map is equal to $f_{|T_v^{\min}}$. Hence $\mathrm{Lip}(T_v^{\min}, f(T_v^{\min})) = \mathrm{Lip}(f_{|T_v}) = \mathrm{Lip}(T, T')$. By Corollary A.5.3, we have $\mathrm{Lip}(T_v^{\min}, f(T_v^{\min})) = \sup_{g \in G_v} \frac{||g||_{T'}}{||g||_T} \leq \sup_{g \in F_N} \frac{||g||_{T'}}{||g||_T}$, whence $\mathrm{Lip}(T, T') \leq \Lambda(T, T')$, and the claim follows. $\square$

Proof of Proposition A.6.9. Let $T, T' \in \overline{cv_N}$ be such that $\mathrm{Lip}(T, T') < +\infty$. By Proposition A.6.12, for all $\epsilon > 0$, there exists an ϵ-witness for the pair (T, T'), so $\Lambda(T, T') \geq \mathrm{Lip}(T, T')$. The reverse inequality follows from Proposition A.5.1. $\square$

A.6.4 End of the proof of Theorem A.6.1

In this section, we finish the proof of Theorem A.6.1. Proposition A.6.15 will be used in the following sections in various contexts to get refinements of Theorem A.6.1 (see Theorem A.6.17 and Corollaries A.6.19 and A.6.21).

Proposition A.6.13. *Let $T \in \overline{cv_N}$, and let $\mathcal{C}(T) \subseteq F_N$ be a subset that contains all vertex stabilizers of T^{simpl}. For all $T' \in \overline{cv_N}$, if $\Lambda_{\mathcal{C}(T)}(T, T') < +\infty$, then Lip$(T, T') < +\infty$.*

Proof. Let T_v be a vertex tree of T^{simpl} whose stabilizer G_v is nontrivial, and let $T_v^{\min}$ be the minimal G_v-invariant subtree of T_v. Let T'_v be the minimal G_v-invariant subtree of T'. As $G_v \subseteq \mathcal{C}(T)$, we have $\Lambda_{G_v}(T_v^{\min}, T'_v) < +\infty$, so by Corollary A.5.3, there exists a Lipschitz F_N-equivariant map from $T_v^{\min}$ to $\overline{T'_v}$, and hence from T_v to $\overline{T'_v}$. Notice that if an attaching point $p \in T_v$ is fixed by $g \in G_v$, then $f(p)$ is also fixed by g. Hence we can define a Lipschitz F_N-equivariant map from T to $\overline{T'}$ by sending every vertex tree T_v with dense orbits into the corresponding tree $\overline{T'_v} \subseteq \overline{T'}$ (in particular, every vertex tree which is reduced to a point with nontrivial vertex group G_v is sent to a point fixed by G_v), sending the points in T projecting to vertices in T^{simpl} with trivial stabilizer arbitrarily in an F_N-equivariant way, and extending linearly on edges. The map we get is Lipschitz because there is a finite number of orbits of vertices and of orbits of edges in T^{simpl}. $\square$

In particular, Proposition A.6.13 applied to $\mathcal{C}(T) := F_N$, together with Proposition A.5.1, implies the following corollary.

Corollary A.6.14. *For all $T, T' \in \overline{cv_N}$, we have Lip$(T, T') < +\infty$ if and only $\Lambda(T, T') < +\infty$.* $\square$

The following proposition will be applied in the sequel to various choices of the set $\mathcal{C}(T)$ to get refinements of Theorem A.6.1.

Proposition A.6.15. *Let $T, T' \in \overline{cv_N}$. Let $\mathcal{C}(T)$ be a subset of F_N that contains all vertex stabilizers of T^{simpl}. Assume in addition that either Lip$(T, T') = +\infty$, or that for all $\epsilon > 0$, there exists $g \in \mathcal{C}(T)$ which is an ϵ-witness for the pair (T, T'). Then Lip$(T, T') = \Lambda_{\mathcal{C}(T)}(T, T')$.*

Proof. Let $T, T' \in \overline{cv_N}$. Proposition A.5.1 shows that $\Lambda_{\mathcal{C}(T)}(T, T') \leq \text{Lip}(T, T')$, and if $\Lambda_{\mathcal{C}(T)}(T, T') = +\infty$, then the reverse inequality is obvious. So we may assume that $\Lambda_{\mathcal{C}(T)}(T, T') < +\infty$. Proposition A.6.13 then shows that $\text{Lip}(T, T') < +\infty$, and the conclusion follows from the assumption made on $\mathcal{C}(T)$. $\square$

Proof of Theorem A.6.1. Theorem A.6.1 follows from Proposition A.6.15 applied to $\mathcal{C}(T) := F_N$ and Proposition A.6.12. $\square$

A.6.5 Candidates

We extend the notion of candidates from Section A.1.2 to arbitrary trees in $\overline{cv_N}$, compare with [AK13, Definition 4.4]. An element $g \in F_N$ is a *candidate* in T if there exists $v \in C_T(g)$ such that the segment $[v, gv]$ projects to a loop γ in X which is either

- an embedded loop, or
- an embedded bouquet of two circles, or
- a barbell graph, or
- a *simply-degenerate barbell*, i.e. γ is of the form $u\eta\overline{\eta}$, where u is an embedded loop in X and η is an embedded path in X with two distinct endpoints which meets u only at its origin, and whose terminal endpoint is a vertex in X with nontrivial stabilizer, or
- a *doubly-degenerate barbell*, i.e. γ is of the form $\eta\overline{\eta}$, where η is an embedded path in X whose two distinct endpoints have nontrivial stabilizers, or
- a vertex in X.

We display the possible shapes of the loop γ on Figure A.17. In the case of (possibly simply- or doubly-degenerate) barbells, we call η the *central path* of γ. By a more careful analysis of the path built in the proof of Lemma A.6.10, we show the following result, which was already noticed by Algom-Kfir [AK13, Proposition 4.5]. Our strategy of proof follows [AK11, Proposition 2.3].

Proposition A.6.16. *Let $T, T' \in \overline{cv_N}$ be such that Lip$(T, T') < +\infty$. Then for all $\epsilon > 0$, there exists an element $g \in F_N$ which is a candidate in T and is an ϵ-witness for the pair (T, T'). More precisely, let $f : T \to \overline{T'}$ be an optimal map.*

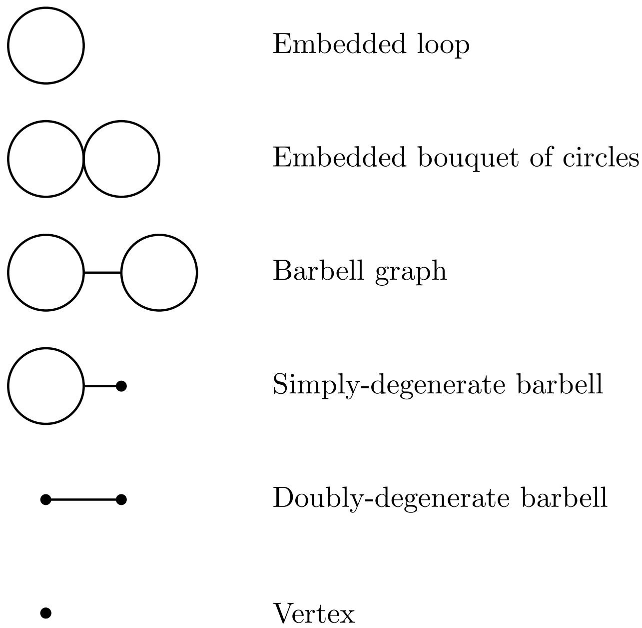


Figure A.17: The shape of loops in X that represent candidates in T.

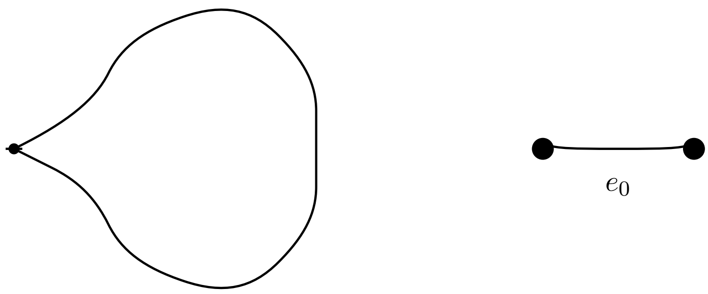
- *If $X_f = \emptyset$, then there exists $g \in F_N$ whose characteristic set in T projects to a point in X, and which is an ϵ-witness for the pair (T, T').*
- *If $X_f \neq \emptyset$, then there exists $g \in F_N$ which is a candidate in T, and which is ϵ-legal for f.*

Proof. Let $\epsilon > 0$. If $X_f = \emptyset$, the claim follows from Proposition A.6.12, so we assume that $X_f \neq \emptyset$. Choose a vertex $v_0 \in T_f^{simpl}$, and an edge e_0 in T_f whose projection to T^{simpl} is adjacent to v_0. Propositions A.6.6 and A.6.8 enable us to construct a path in T of the form $e_0\gamma_0 e_1\gamma_1 e_2 \dots$, where for all integers i,

- the subpath e_i is an edge in T_f, with origin x_i and terminal endpoint x_i', and
- the subpath $\gamma_i = [x_i', x_{i+1}]$ lies in a vertex tree of T (it projects to a vertex $v_{i+1} \in T^{simpl}$), and
- the turn (e_i, e_{i+1}) is ϵ-legal for f (and legal for f when $T_{v_{i+1}}$ is reduced to a point).

As the number of orbits of vertices in the simplicial part of T is finite, there exist integers $i, k \in \mathbb{N}$ and an element $g \in F_N$, such that $v_{i+k} = gv_i$. After possibly renumbering the edges, we get a path in T of the form $e_0\gamma_0 e_1 \dots e_{k-1}$, such that all the turns (e_i, e_{i+1}) are ϵ-legal for f (legal at vertices whose vertex tree is reduced to a point), and $v_i \neq v_j$ for all $i \neq j \in \{1, \dots, k\}$, but (e_{k-1}, ge_0) might not be ϵ-legal (or legal) for f. This path projects to a loop γ in X_f which is either embedded, or consists of a single edge crossed successively in both directions, in which case we say it is *degenerate*, see Figure A.18 (the degenerate case occurs when $k = 2$ and the edges e_0 and e_1 belong to the same orbit of edges).

If there exists $g_k \in G_{v_k}$ so that the turn $(e_{k-1}, g_k g e_0)$ is ϵ-legal (or legal) for f (which happens for instance as soon as G_{v_k} has rank at least 2 by Propositions A.6.6 and A.6.8), then $g_k g$ is a candidate in T which is ϵ-legal for f. From now on, we assume that for all $g_k \in G_{v_k}$, the turn $(e_{k-1}, g_k g e_0)$ is not ϵ-legal for f, so in particular the vertex group G_{v_k} is at most cyclic. Proposition A.6.6 shows that for all $g_k \in G_{v_k}$, the turn $(e_{k-1}, g_k e_{k-1})$ is



Embedded loop Degenerate loop

Figure A.18: The projection to X of the path $e_0\gamma_1 e_1 \dots e_{k-1}$.

not legal for f, but ensures the existence of an edge e_k in T_f adjacent to v_k (not in the same F_N-orbit as e_{k-1}), such that the turn (e_{k-1}, e_k) is legal for f. Take this direction, and continue crossing turns which are ϵ-legal for f (legal for f at vertices of T^{simpl} with trivial vertex trees) till you reach a vertex v_l whose orbit has already been visited (i.e. $v_l = g'v_j$ for some $j \in \{0, \dots, l-1\}$ and some $g' \in F_N$). Discussing on the rank of G_{v_l}, Propositions A.6.6 and A.6.8 ensure the existence of $g_l \in G_{v_l}$ such that one of the turns $(e_{l-1}, g_l g' e_{j-1})$ or $(e_{l-1}, g_l g' e_j)$ is ϵ-legal for f (and legal for f if T_{v_l} is reduced to a point). As above, the path $e_j \dots e_{l-1}$ projects to a loop in X_f which is either embedded or degenerate. Also notice that for all $g \in G_{v_k}$, the turn (e_0, ge_k) is legal for f, otherwise Proposition A.6.6 would imply that (e_{k-1}, e_k) is not legal for f, a contradiction. We give a description of all possible situations, see Figure A.19 where we display the projection to X of the path we have constructed. For simplicity of notations, we will denote a path in T by the sequence of the simplicial edges it crosses.

Case 1 : The turn $(e_{l-1}, g_l g' e_j)$ is ϵ-legal for some $g_l \in G_{v_l}$, and the path $e_j \dots e_{l-1}$ projects to an embedded loop.
Then the path $e_j \dots e_{l-1}$ is a fundamental domain for the axis of an element $g \in F_N$ which is ϵ-legal in T, and it projects to an embedded loop.

Case 2 : The turn $(e_{l-1}, g_l g' e_j)$ is ϵ-legal for some $g_l \in G_{v_l}$, and the path $e_j \dots e_{l-1}$ projects to a degenerate loop.
Then the path $e_j \dots e_{l-1}$ is a fundamental domain for the axis of an element $g \in F_N$ which is ϵ-legal in T, and it projects to a doubly-degenerate barbell.

Case 3 : We have $j \in \{1, \dots, k-1\}$, and the turn $(e_{l-1}, g_l g' e_{j-1})$ is ϵ-legal for some $g_l \in G_{v_l}$.
Then the path $e_0 \dots e_{j-1}(g_l g')^{-1}(\overline{e_{l-1}} \dots \overline{e_k})$ is a fundamental domain for the axis of an element $g \in F_N$ which is ϵ-legal in T, and it projects to an embedded loop.

Case 4 : We have $j = k$, the path $e_0 \dots e_{k-1}$ projects to an embedded loop, the path $e_j \dots e_{l-1}$ projects to an embedded loop, and the turn $(e_{l-1}, g_l g' e_{j-1})$ is ϵ-legal for some $g_l \in G_{v_l}$.
Then the path $e_0 \dots e_{j-1}(g_l g')^{-1}(\overline{e_{l-1}} \dots \overline{e_k})$ is a fundamental domain for the axis of an element $g \in F_N$ which is ϵ-legal in T, and it projects to a bouquet of two circles.

Case 5 : We have $j \in \{k, \dots, l-1\}$, the path $e_0 \dots e_{k-1}$ projects to a degenerate loop, the path $e_j \dots e_{l-1}$ projects to an embedded loop, and the turn $(e_{l-1}, g_l g' e_{j-1})$ is ϵ-legal for some $g_l \in G_{v_l}$.
Then the path $e_0 \dots e_{j-1}(g_l g')^{-1}(\overline{e_{l-1}} \dots \overline{e_k})$ is a fundamental domain for the axis of an element $g \in F_N$ which is ϵ-legal in T, and it projects to a simply-degenerate barbell.

Case 6 : We have $j \in \{k+1, \dots, l-1\}$, the path $e_0 \dots e_{k-1}$ projects to an embedded loop, the path $e_j \dots e_{l-1}$ projects to an embedded loop, and the turn $(e_{l-1}, g_l g' e_{j-1})$ is ϵ-legal for some $g_l \in G_{v_l}$.
Then the path $e_0 \dots e_{k-1} e_k \dots e_{j-1} e_j \dots e_{l-1} g_l g'(\overline{e_{j-1}} \dots \overline{e_k})$ is a fundamental domain for the axis of an element $g \in F_N$ which is ϵ-legal in T, and it projects to a barbell.

Case 7 : We have $j \in \{k, \dots, l-1\}$, the path $e_0 \dots e_{k-1}$ projects to an embedded loop, the path $e_j \dots e_{l-1}$ projects to a degenerate loop, and the turn $(e_{l-1}, g_l g' e_{j-1})$ is ϵ-legal for some $g_l \in G_{v_l}$.
Then the path $e_0 \dots e_{k-1} e_k \dots e_{j-1} e_j \dots e_{l-1} g_l g'(\overline{e_{j-1}} \dots \overline{e_k})$ is a fundamental domain for the axis of an element $g \in F_N$ which is ϵ-legal in T, and it projects to a simply-degenerate barbell.

Case 8 : We have $j \in \{k, \dots, l-1\}$, the path $e_0 \dots e_{k-1}$ projects to a degenerate loop, the path $e_j \dots e_{l-1}$ projects to a degenerate loop, and the turn $(e_{l-1}, g_l g' e_{j-1})$ is ϵ-legal for some $g_l \in G_{v_l}$.
Then the path $e_0 \dots e_{k-1} e_k \dots e_{j-1} e_j \dots e_{l-1} g_l g'(\overline{e_{j-1}} \dots \overline{e_k})$ is a fundamental domain for the axis of an element $g \in F_N$ which is ϵ-legal in T, and it projects to a doubly-degenerate barbell.

In all cases, we have found an element $g \in F_N$ which is a candidate in T, and which is ϵ-legal for f. In addition, there exists $K \in \mathbb{N}$ such that for all $g \in F_N$, if g is a candidate in T, then $l_T^{comb}(g) \leq K$ (we recall the notation l_T^{comb} from Section A.6.3). The conclusion thus follows from Lemma A.6.11. $\square$

Theorem A.6.17. *For all $T, T' \in \overline{cv_N}$, we have*

$$Lip(T, T') = \sup_{g \text{ candidate in } T} \frac{||g||_{T'}}{||g||_T}.$$

Proof. Let $T, T' \in \overline{cv_N}$, and let $\mathcal{C}(T)$ be the set of elements of F_N which are candidates in T. By definition, the set $\mathcal{C}(T)$ contains all vertex groups of T^{simpl}, and Proposition A.6.16 shows that $\mathcal{C}(T)$ satisfies the assumption of Proposition A.6.15. The conclusion thus follows from Proposition A.6.15. $\square$

A.6.6 The case of good trees

Let $T \in \overline{cv_N}$. We now carry on some further analysis on the set of candidates to show that when T^{simpl} contains at least two F_N-orbits of edges with trivial stabilizers, for all $\epsilon > 0$, we can find an element of F_N which is simple, is a candidate in T, and is an ϵ-witness for the pair (T, T').

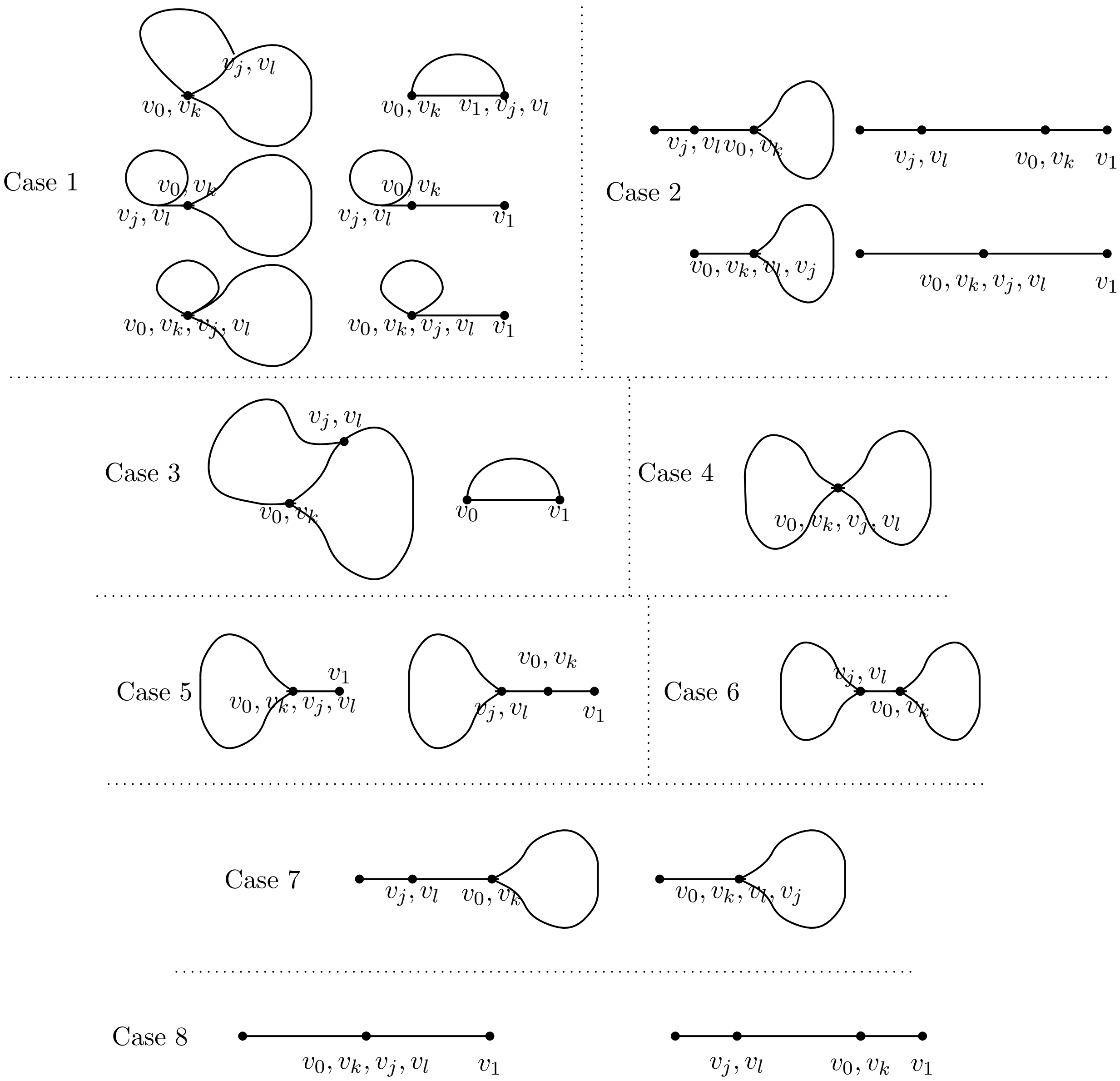


Figure A.19: The projection to X of the path constructed in the different cases of the proof of Proposition A.6.16.

Proposition A.6.18. *Let $T, T' \in \overline{cv_N}$ be such that $Lip(T, T') < +\infty$. Assume that T^{simpl} contains at least two orbits of edges with trivial stabilizers. Then for all $\epsilon > 0$, there exists $g \in F_N$ which is simple, is a candidate in T, and is an ϵ-witness for the pair (T, T').*

Proof. Let $f : T \to \overline{T'}$ be an optimal map (which exists by Proposition A.6.4), and let $\epsilon > 0$. If $X_f = \emptyset$, the claim follows from Proposition A.6.16 and Lemma A.1.12, so we assume that $X_f \neq \emptyset$. By Proposition A.6.16, there exists $g \in F_N$ which is a candidate in T and is ϵ-legal for f. Let $v \in C_T(g)$ be such that the projection γ of $[v, gv]$ to X has one of the forms prescribed by the definition of a candidate. If γ is either an embedded loop or an embedded bouquet of two circles, then γ crosses each edge of X at most once. As T contains an edge with trivial stabilizer in its simplicial part, Lemma A.1.12 ensures that g is simple. The same argument also shows that g is simple in the case where γ does not cross some edge with trivial stabilizer of X, or when γ is a (possibly simply-degenerate) barbell, one of whose loops crosses an edge of X with trivial stabilizer. Hence we can assume that γ is a (possibly simply- or doubly-degenerate) barbell, and that all edges in X with trivial stabilizer belong to the central path γ' of γ.

Assume that γ' contains a vertex v whose stabilizer has rank at least 2, and such that if v is an endpoint of γ', then v is adjacent to a loop of the barbell. Then v separates γ into two shorter simply- or doubly-degenerate barbells or embedded loops, at least one of which, which we denote by γ'', avoids an edge of X with trivial stabilizer. Propositions A.6.6 and A.6.8 show that there exists $g \in F_N$ whose axis in T projects to γ'' and which is ϵ-legal for f, and g is simple by Lemma A.1.12.

We now restrict to the case where all nonextremal vertices of γ' have vertex group at most cyclic. Assume that two edges in γ' with nontrivial cyclic stabilizers have a common vertex v (whose vertex group is cyclic). Then there exists an edge e' in $X \smallsetminus \gamma$ adjacent to v, and e' has trivial stabilizer because T is very small, and the vertex group G_v is cyclic. Lemma A.1.12 ensures that g is simple. From now on, we assume that γ does not contain two consecutive edges with nontrivial cyclic stabilizers.

Now assume that γ' contains two edges with trivial edge groups having a common vertex v (whose vertex group is at most cyclic). If G_v is trivial, then any third edge coming out of v has trivial stabilizer, and Lemma A.1.12 shows that g is simple. We now assume that G_v is infinite cyclic. Let e, e' be two consecutive edges in $C_T(g)$ (not in the same F_N-orbit) adjacent to a vertex $\widetilde{v} \in T$ that projects to v. Denote by t a generator of the cyclic group $G_{\widetilde{v}}$. If (e, te) is legal for f, then again we can replace g by another candidate g' which is ϵ-legal for f, and is represented by a loop which does not cross the orbit of e'. We now assume that for all $t \in G_{\widetilde{v}}$, the turn (e, te) is not legal for f. If for some $t' \in G_{\widetilde{v}}$, the turn $(e, t'e')$ were not legal for f, then by Proposition A.6.6, the turn $(t'e, t'e')$ would not be legal for f, contradicting the fact that g is ϵ-legal for f. So for all $t \in G_{\widetilde{v}}$, the turn (e, te') is legal for f. Let $\widetilde{\gamma}$ be a fundamental domain of the axis of g that projects to γ and crosses twice a turn at a vertex in the orbit of $\widetilde{v}$. The previous argument shows that up to replacing g by another candidate g' which is also ϵ-legal for f, we can assume that these two turns belong to the same F_N-orbit (of the form $(e, t^k e')$ for some $k \in \mathbb{Z}$). We claim that g' is simple. Indeed, by equivariantly folding small initial segments of the edges e and $t^k e'$, one constructs a new F_N-tree that projects to a graph of groups in which g' is represented by a loop that avoids an edge with trivial stabilizer, see Figure A.20. By Lemma A.1.12, this shows that g' is simple.

We are thus left with the case where γ' contains an edge e with nontrivial cyclic edge group, which is surrounded in γ' by two edges with trivial edge groups. Denote by e_1 and e_2 the two edges in a lift of γ to T that are adjacent to a lift of e. By the same argument as above, we can assume that for all $t \in G_e$, the turns (e_1, te_1) and (e_2, te_2) are not legal

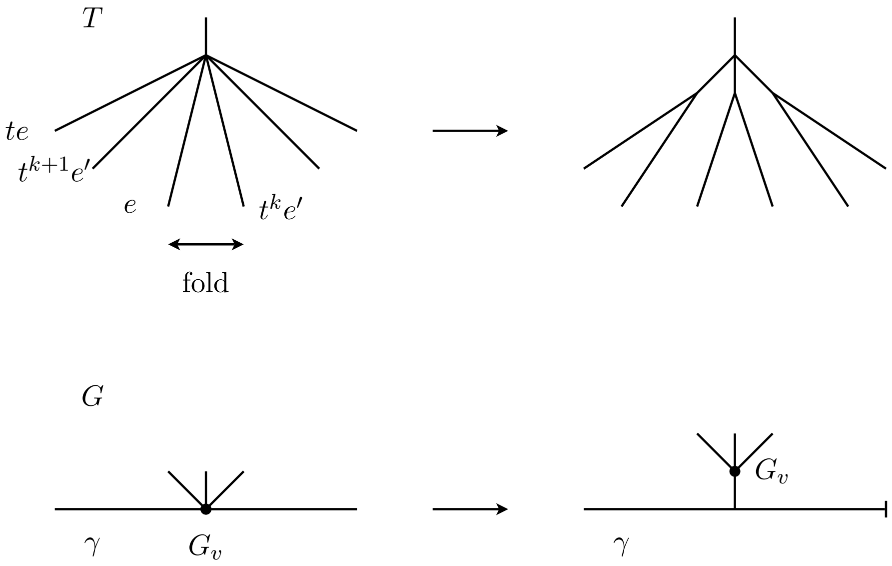


Figure A.20: Sliding the vertex v to detect simple elements.

for f. So for all $t \in G_e$, the turns (te_1, e) and (e, te_2) are legal for f, otherwise Proposition A.6.6 would imply that (e, e_1) or (e, e_2) is not legal for f, contradicting the fact that g is ϵ-legal for f. As above we can construct a candidate g' which is ϵ-legal for f, and is simple. Indeed, we can choose g' such that when equivariantly collapsing e to a vertex, and applying the same folding argument as above, we get a new graph of groups in which g' is represented by a loop that avoids an edge with trivial edge group. This again shows that g' is simple.

Hence we have found an element $g \in F_N$ which is simple, is a candidate in T, and is ϵ-legal for f. The conclusion thus follows from Lemma A.6.11 since there is a bound on the combinatorial length of a candidate in T. $\square$

Corollary A.6.19. *Let $T, T' \in \overline{cv_N}$. If T^{simpl} contains two distinct orbits of edges with trivial stabilizers, then*

$$Lip(T, T') = \sup_{g \text{ simple}} \frac{||g||_{T'}}{||g||_T}.$$

Proof. Let $\mathcal{C}$ be the set of simple elements of F_N. Lemma A.1.12 shows that $\mathcal{C}$ contains all the vertex groups of T^{simpl}, and Proposition A.6.18 shows that $\mathcal{C}$ satisfies the assumption of Proposition A.6.15. Hence Corollary A.6.19 follows from Proposition A.6.15. $\square$

Corollary A.6.20. *Let $T, T' \in \overline{cv_N}$. If T is good, then*

$$Lip(T, T') = \sup_{g \text{ simple}} \frac{||g||_{T'}}{||g||_T}.$$

Proof. This follows from Corollary A.6.19 and Proposition A.5.2 applied to the set of simple elements of F_N. $\square$

Corollary A.6.21. *Let $T, T' \in \overline{cv_N}$. If T is good, then*

$$Lip(T, T') = \sup_{g \in \mathcal{P}_N} \frac{||g||_{T'}}{||g||_T}.$$

Proof. Let $w \in F_N$ be a simple element of F_N (contained in some proper free factor of F_N), and let w' be a primitive element contained in a complementary free factor. Then $w'w^k$ is primitive for all $k \in \mathbb{N}$, and $||w||_T = \lim_{k\to+\infty} \frac{||w'w^k||_T}{k}$. If $||w||_T \neq 0$, we thus get that

$$\frac{||w||_{T'}}{||w||_T} = \lim_{k\to+\infty} \frac{||w'w^k||_{T'}}{||w'w^k||_T},$$

so

$$\frac{||w||_{T'}}{||w||_T} \leq \sup_{g\in\mathcal{P}_N} \frac{||g||_{T'}}{||g||_T}.$$

If $||w||_T = 0$ and $||w||_{T'} > 0$, then $\frac{||w'w^k||_{T'}}{||w'w^k||_T}$ tends to $+\infty$ as k tends to $+\infty$, so the above inequality still holds. It also holds when $||w||_T = ||w||_{T'} = 0$, because in this case we have $\frac{||w||_{T'}}{||w||_T} = 0$ by convention. Hence

$$\sup_{w \text{ simple}} \frac{||w||_{T'}}{||w||_T} = \sup_{g\in\mathcal{P}_N} \frac{||g||_{T'}}{||g||_T},$$

and the claim follows from Corollary A.6.20. □

From Proposition A.5.4 and Corollary A.6.21, we deduce the following statement.

Proposition A.6.22. *Let $T, T' \in \overline{cv_N}$ be two good F_N-trees. If T and T' are primitive-equivalent, then $T = T'$.* □

Remark A.6.23. The condition on T cannot be removed in Corollary A.6.21, otherwise the simple-equivalence relation would be trivial on $\overline{cv_N}$, contradicting our analysis in Section A.2. If the translation length functions of two distinct trees $T, T' \in \overline{cv_N}$ are equal in restriction to $\mathcal{P}_N$, then either T or T' is not good. Applying Theorem A.3.11 to the F_{N-1}-tree appearing in the definition of pull-equivalence classes, we see that the standard element of the class is thus the only good tree in its class. It follows from Remark A.3.12 that trees whose pull-equivalence class is nontrivial are geometric and contain no exotic components.

A.7 End of the proof of the main theorem

We now finish the proof of the main theorem of the paper.

Theorem A.7.1. *For all $T, T' \in \overline{cv_N}$, the following assertions are equivalent.*

- *For all $g \in \mathcal{P}_N$, we have $||g||_T = ||g||_{T'}$.*
- *For all simple elements $g \in F_N$, we have $||g||_T = ||g||_{T'}$.*
- *The trees T and T' are special-pull-equivalent.*

In view of Propositions A.2.1 and A.2.6, we are left showing that simple-equivalent trees are special-pull-equivalent. Given an $\mathbb{R}$-tree T and $x \in T$, a *direction* based at x is a germ of nondegenerate segments $[x, y]$ with $y \neq x$. In particular, any U-turn in an F_N-tree T is defined by a pair of directions based at some point in T. Any direction based at x defines an open half-tree of T, which is the set of all $y \in T \smallsetminus \{x\}$ such that $[x, y]$ contains d. The axis of an element $g \in F_N$ which is hyperbolic in T *crosses* a direction d based at a point $x \in T$ if $x \in C_T(g)$ and there exists $y \in C_T(g)$ such that d is the germ of $[x, y]$

(we also say that $C_T(g)$ crosses a pair of distinct directions $\{d, d'\}$ based at a point in T if it crosses both d and d'). A *full family of U-turns* over e is the collection of all turns of the form $(e, g^k e)$ for $k \in \mathbb{Z}$, where (e, ge) is some given U-turn over e (in particular g fixes an endpoint of e and is not a proper power). A *compatible set of U-turns* over e is a set consisting of at most one full family of U-turns at each extremity of e.

Proposition A.7.2. *Let T be an F_N-tree which contains an orbit of edges with trivial stabilizer e^0. Let Y be a compatible set of U-turns over e^0. Let $\{d, d'\}$ be a pair of distinct directions based at a point $x_0 \in T$, which do not define a U-turn over e^0, and do not define a U-turn over an edge in T with nontrivial stabilizer. Then there exists a simple element $g \in F_N$ which is hyperbolic in T and whose axis in T crosses $\{d, d'\}$ but does not cross any of the orbits of the turns in Y.*

Proof. Let v_1 and v_2 denote the extremities of e^0. We think of e^0 as an open subset of T.

Case 1 : The edge e^0 projects to a nonseparating edge in the quotient graph of actions. Let T' be a connected component of $T \smallsetminus F_N.e^0$. One can always find a path γ in T' which joins a point in the orbit of v_1 to a point in the orbit of v_2, and let γ' be the concatenation of γ and of an edge in the orbit of e^0. We will show that if $x_0 \in T'$, then γ can be chosen so that γ' crosses a turn in the orbit of $\{d, d'\}$ (this property is automatic if x_0 belongs to the interior of e^0). Then γ' is a fundamental domain for the axis of an element $g \in F_N$ which has the desired properties (in particular, it is primitive because γ' crosses the orbit of e^0 exactly once).

Assume that $x_0 \in T'$. Let A be the stabilizer of T'. If A stabilizes an edge in T, then A is cyclic, and T' consists of a single edge e with cyclic stabilizer. In this case, we choose γ to be equal to this edge. Otherwise, the A-minimal subtree of T' is well-defined, we denote it by T'_{min}.

If $T' \smallsetminus T'_{min}$ contains a simplicial edge with trivial stabilizer, then e^0 projects to a loop-edge in the quotient graph of actions, which has a valence 3 vertex with trivial stabilizer, attached to a separating edge. In this case, each of the directions d and d' is either contained in e^0, or determines a half-tree of T' that contains a point in the orbit of v_1, which is equal to the orbit of v_2. Therefore, we can find a path γ in T' which joins two points in the orbit of v_1, so that the concatenation γ' of γ and an edge in the orbit of e^0 crosses a turn in the orbit of $\{d, d'\}$.

Otherwise, all edges in $T' \smallsetminus T'_{min}$ (if any) have nontrivial stabilizer, equal to the stabilizer of the extremity of the edge in the orbit of e^0 to which they are attached. If none of the open half-trees of T determined by the directions d and d' intersects T'_{min}, then the assumption made on the turn $\{d, d'\}$ implies that either

- the vertices v_1 and v_2 belong to the same F_N-orbit, and d and d' are contained in translates of e^0, in which case we can choose γ to be reduced to a point, or
- one of the directions, say d, is contained in an edge e with nontrivial stabilizer, and d' is contained in e^0, in which case we choose γ to be equal to e, or
- the directions d and d' belong to two distinct orbits of edges with nontrivial stabilizer whose concatenation forms the desired path γ.

Otherwise, up to exchanging d and d', the open half-tree of T' determined by d contains both a point in the orbit of v_1 and a point in the orbit of v_2. In addition, either x_0 belongs to the orbit of v_1 or v_2, or the open half-tree determined by d' contains a point in the orbit of v_1 or v_2. So we can find a path γ in T' joining a point in the orbit of v_1 to a point in the orbit of v_2, so that γ' crosses a turn in the orbit of $\{d, d'\}$.

Case 2 : The edge e^0 projects to a separating edge in the quotient graph of actions. Let T_1 (resp. T_2) be the connected component of $T \smallsetminus F_N.e^0$ that contains v_1 (resp. v_2). For all $i \in \{1,2\}$, let A_i be the stabilizer of T_i, and let T_i^{min} be the A_i-minimal subtree of T_i. Up to interchanging the roles of T_1 and T_2, we can assume that $x_0 \in T_1$, or $x_0 \in e^0$.

If $x_0 \in T_1$, then the open half-tree determined by one of the directions d or d' intersects T_1^{min}, and in both cases we can choose a reduced path γ_1 in T_1 that crosses a turn in the orbit of $\{d, d'\}$ and joins v_1 to a translate $g_1 v_1$. If A_2 is not elliptic in T_2, then we can choose a primitive element $g_2 \in A_2$ that is hyperbolic in T_2. If A_2 is elliptic in T_2 and has rank at least 2, then we can choose a primitive element $g_2 \in A_2$ such that $(e^0, g_2 e^0) \notin Y$. If A_2 is cyclic and elliptic in T_2, generated by an element g_2, then our definition of U-turns implies that $(e^0, g_2 e^0) \notin Y$. Then $g_1^{-1} g_2$ satisfies the required conditions (in particular it is primitive, because if $\{a_1, \dots, a_k\}$ is a free basis of A_1, and $\{a_{k+1}, \dots, a_{N-1}, g_2\}$ is a free basis of A_2, then $\{a_1, \dots, a_{N-1}, g_1^{-1} g_2\}$ is a free basis of F_N).

If $x_0 \in e^0$, then we can find two primitive elements $g_1 \in A_1$ and $g_2 \in A_2$ as above and let $g := g_1^{-1} g_2$. ☐

Proposition A.7.2 can also be restated in the following way.

Corollary A.7.3. *Let $T, \widehat{T} \in \overline{cv_N}$. Assume that $\widehat{T}$ contains exactly one orbit of edges e^0 with trivial stabilizer, and that T is a pull of $\widehat{T}$. Let $\{d, d'\}$ be a pair of distinct directions based at the same point in $\widehat{T}$, which does not define a U-turn over e^0, and does not define a U-turn over an edge with nontrivial stabilizer. Then there exists a simple element $g \in F_N$, which is hyperbolic in $\widehat{T}$, whose axis in $\widehat{T}$ crosses $\{d, d'\}$, and such that $||g||_T = ||g||_{\widehat{T}}$.*

Proof. As T is obtained from $\widehat{T}$ by equivariantly folding a collection Y of U-turns, we have $||g||_T \leq ||g||_{\widehat{T}}$ for all $g \in F_N$, with equality as long as $C_{\widehat{T}}(g)$ does not cross any turn in the orbit of a turn in Y. Corollary A.7.3 thus follows from Proposition A.7.2. ☐

Proposition A.7.4. *Let $T, T' \in \overline{cv_N}$. If T and T' are simple-equivalent, then there exists a good tree $\widehat{T} \in \overline{cv_N}$ such that either $T = T' = \widehat{T}$, or there exists an edge e^0 in $\widehat{T}$ with trivial stabilizer such that T and T' are both obtained from $\widehat{T}$ by pulling e^0.*

Proof. If both trees T and T' are good, then by Proposition A.6.22 we have $T = T'$. We can thus assume that T is not good, hence there exists a good tree $\widehat{T}$ having exactly one orbit of edges with trivial stabilizer such that T is a pull of $\widehat{T}$ (Theorem A.3.11).

We first show the existence of an F_N-equivariant morphism from $\widehat{T}$ to $\overline{T'}$, which is isometric on edges. As T is a pull of $\widehat{T}$, we have $||g||_{\widehat{T}} \geq ||g||_T$ for all $g \in F_N$. As T and T' are simple-equivalent, we thus have $||g||_{\widehat{T}} \geq ||g||_{T'}$ for all simple elements $g \in F_N$. As $\widehat{T}$ is good, Corollary A.6.20 provides a 1-Lipschitz F_N-equivariant map $f : \widehat{T} \to \overline{T'}$, which we may choose to be linear on edges in the simplicial part of $\widehat{T}$. As $\widehat{T}$ contains an edge with trivial stabilizer, Lemma A.1.12 ensures that all vertex stabilizers of $\widehat{T}^{simpl}$ lie in some proper free factor of F_N, so for all $g \in F_N$ belonging to one of these vertex stabilizers, we have $||g||_{\widehat{T}} = ||g||_T = ||g||_{T'}$. Let v be a vertex in $\widehat{T}^{simpl}$ whose stabilizer G_v has rank at least 2. Then the G_v-minimal subtree of T' has the same translation length function as the G_v-minimal subtree T_v^{min} of $\widehat{T}$, so by Theorem A.1.1, these two trees are F_N-equivariantly isometric. So T_v^{min} isometrically embeds as an F_N-invariant subtree in T', and by Proposition A.1.7, the map f restricts to a G_v-equivariant isometry on T_v^{min}, and hence on T_v. Hence we can write $f = f_2 \circ f_1$, where f_1 reduces the length of some edges in the simplicial part of $\widehat{T}$, and f_2 is a morphism which is isometric on edges. If f_1 is not equal to the identity map, then f_1 strictly reduces the length of an edge e' in the simplicial part of $\widehat{T}$. Corollary A.7.3, applied to a pair of opposite directions in the edge e', gives the existence of a simple element $g \in F_N$, whose axis in T crosses e' (so that $||g||_{T'} < ||g||_{\widehat{T}}$), and such that $||g||_T = ||g||_{\widehat{T}}$. This is impossible as T and T' are simple-equivalent. So f_1 is equal to the identity map, and hence f is a morphism.

Assume that f identifies a pair $\{d, d'\}$ of directions in T. Since U-turns over edges with nontrivial stabilizers cannot be folded by f (Lemma A.1.9), Corollary A.7.3 ensures that the pair $\{d, d'\}$ defines a U-turn over e^0, otherwise we would find a simple element $g \in F_N$ with $||g||_{T'} < ||g||_T$. In other words, all turns in T, except possibly U-turns over e^0, are legal for the morphism f. So f factors through a tree T_1 obtained by equivariantly identifying maximal subsegments of the unique edge of $\widehat{T}$ with trivial stabilizer along some translate at each of its extremities. If T_1 also contains an edge with trivial stabilizer, then the maximality condition in the definition of T_1 ensures that all turns in T_1 are legal, so $T' = T_1$. If all edges in the simplicial part of the tree T_1 have nontrivial stabilizer, then no more folding can occur (Lemmas A.1.8, A.1.9 and A.1.10), so again $T' = T_1$. The claim follows. ☐

In order to complete the proof of Theorem A.7.1, we are thus left showing the following.

Proposition A.7.5. *Let $T, T' \in \overline{cv_N}$ be simple-equivalent. Assume that there exists a good tree $\widehat{T}$ and an edge e^0 in $\widehat{T}$ with trivial stabilizer, such that T and T' are obtained from $\widehat{T}$ by pulling e^0. Then T and T' are special-pull-equivalent.*

Remark A.7.6. The content of Proposition A.7.5 is to show that the pull operation performed when passing from T to T' is of the particular form prescribed by the definition of special-pull-equivalent trees (Definition A.2.3).

Proof. Let X denote the underlying graph of the canonical graph of actions associated to $\widehat{T}$ provided by Proposition A.1.4. We again denote by e^0 the image of e^0 in X.

Case 1 : The edge e^0 is separating in X.
Then there exists a free splitting of F_N of the form $F_N = A_1 * A_2$ (for which we denote by k_i the rank of A_i for all $i \in \{1, 2\}$, with $k_1 + k_2 = N$), an A_i-tree $T_i \in \overline{cv_{k_i}}$ together with an attaching point $p_i \in \overline{T_i}$ for all $i \in \{1, 2\}$, nonnegative real numbers $l, l_1, l_2, l'_1, l'_2 \in \mathbb{R}_+$ satisfying $l_1 + l_2 \leq l$ and $l'_1 + l'_2 \leq l$, and elements $g_i, g'_i \in A_i$ which are elliptic in T_i and fix p_i for all $i \in \{1, 2\}$ such that T and T' are the trees dual to the graphs of actions displayed on Figure A.21. (Notice that up to changing the values of l_i and l'_i, we can always assume the trees T_i to be minimal). If $N = 2$, then $\widehat{T}$ is dual to a (possibly simply- or doubly-degenerate) barbell graph, and no pull can be performed on $\widehat{T}$, whence $T = T'$.

We now assume that $N \geq 3$. Up to interchanging the roles of A_1 and A_2, we may assume that $k_2 \geq 2$, so that there exists a primitive element $g''_2 \in A_2 \smallsetminus (\langle g_2 \rangle \cup \langle g'_2 \rangle)$. The element $g_1 g''_2$ is primitive in F_N, because if $\{a_1, \dots, a_{k_1}\}$ is a free basis of A_1, and $\{g''_2, a'_2, \dots, a'_{k_2}\}$ is a free basis of A_2, then $\{a_1, \dots, a_{k_1}, g_1 g''_2, a'_2, \dots, a'_{k_2}\}$ is a free basis of F_N. We have

$$||g_1 g''_2||_T = 2(l - l_1) + d_{\overline{T_2}}(p_2, g''_2 p_2),$$

and as $d_{\overline{T_1}}(p_1, g_1 p_1) = 0$, we have

$$||g_1 g''_2||_{T'} = \begin{cases} 2(l - l'_1) + d_{\overline{T_2}}(p_2, g''_2 p_2) & \text{if } g'_1 = g_1^{\pm 1} \\ 2l + d_{\overline{T_2}}(p_2, g''_2 p_2) & \text{if } g'_1 \neq g_1^{\pm 1} \end{cases}.$$

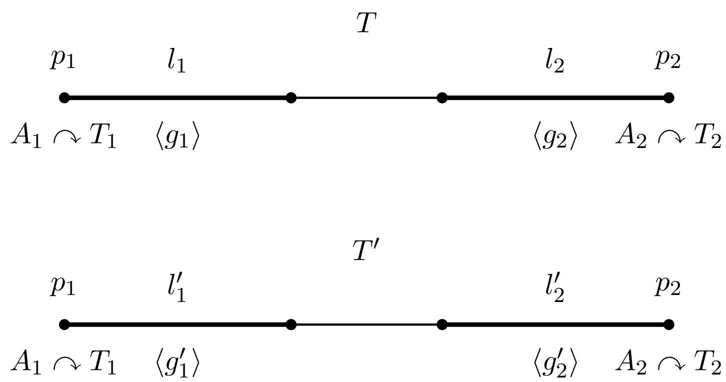


Figure A.21: The trees T and T' in Case 1 of the proof of Proposition A.7.5.

As T_1 and T_2 are simple-equivalent, as soon as $l_1 > 0$, we have $l_1 = l'_1$ and $g_1 = {g'_1}^{\pm 1}$. The same argument shows that as soon as $l'_1 > 0$, we have $l_1 = l'_1$ and $g_1 = {g'_1}^{\pm 1}$. If $k_1 \geq 2$, we argue similarly to show that either $l_2 = l'_2 = 0$, or $l_2 = l'_2$ and $g_2 = {g'_2}^{\pm 1}$, hence $T = T'$. If $k_1 = 1$, then $l_1 = l'_1 = 0$. Let $g''_1 \in A_1$ be a generator of A_1. Comparing the translation lengths of $g''_1 g_2$ in T and in T' also shows that either $l_2 = l'_2 = 0$, or $l_2 = l'_2$ and $g_2 = {g'_2}^{\pm 1}$, whence $T = T'$.

Case 2 : The edge e^0 is nonseparating in X.
Then there exists a corank one free factor A of F_N, with a choice of a stable letter t, a (not necessarily minimal) very small A-tree T_{N-1}, attaching points $p_1, p_2 \in \overline{T_{N-1}}$, nonnegative real numbers $l, l_1, l_2, l'_1, l'_2 \in \mathbb{R}_+$ satisfying $l_1 + l_2 \leq l$ and $l'_1 + l'_2 \leq l$, and elements $g_i, g'_i \in A$ which are elliptic in T_{N-1} and fix p_i for all $i \in \{1, 2\}$, such that T and T' are the trees dual to the graphs of actions displayed on Figure A.22. If $N = 2$, then A contains no proper free factor, so T and T' are special-pull-equivalent. We now assume that $N \geq 3$. Assume that $l_1 > 0$ and that g_1 is contained in a corank one free factor B of A. As $N \geq 3$, we can find $g' \in A \smallsetminus (\langle g_2 \rangle \cup \langle g'_2 \rangle)$ such that $A = B * \langle g' \rangle$. Then $g' t^{-1} g_1 t$ is primitive in F_N (because if $\{b_1, \dots, b_{N-2}\}$ is a free basis of B, then $\{b_1, \dots, b_{N-2}, t, g' t^{-1} g t\}$ is a free basis of F_N). We have

$$||g' t^{-1} g_1 t||_T = 2(l - l_1) + d_{\overline{T_{N-1}}}(p_2, g' p_2),$$

and as $d_{\overline{T_{N-1}}}(p_1, g_1 p_1) = 0$, we also have

$$||g' t^{-1} g_1 t||_{T'} = \left\{ \begin{array}{rl} 2(l - l'_1) + d_{\overline{T_{N-1}}}(p_2, g' p_2) & \text{if } g'_1 = g_1^{\pm 1} \\ 2l + d_{\overline{T_{N-1}}}(p_2, g' p_2) & \text{if } g'_1 \neq g_1^{\pm 1} \end{array} \right. .$$

As T and T' are simple-equivalent, as soon as $l_1 > 0$, we have $g_1 = {g'_1}^{\pm 1}$ and $l_1 = l'_1$. Similarly, as soon as $l'_1 > 0$ and g'_1 is contained in a proper free factor of A, we have $g_1 = {g'_1}^{\pm 1}$ and $l_1 = l'_1$. A similar argument also shows that if $l_2 > 0$ (resp. $l'_2 > 0$) and if g_2 (resp. g'_2) is contained in a proper free factor of A, then $g_2 = {g'_2}^{\pm 1}$ and $l_2 = l'_2$. Hence T and T' are NS-pull-equivalent. $\square$

Proof of Theorem A.7.1. Theorem A.7.1 follows from Propositions A.2.1, A.2.6, A.7.4 and A.7.5. $\square$

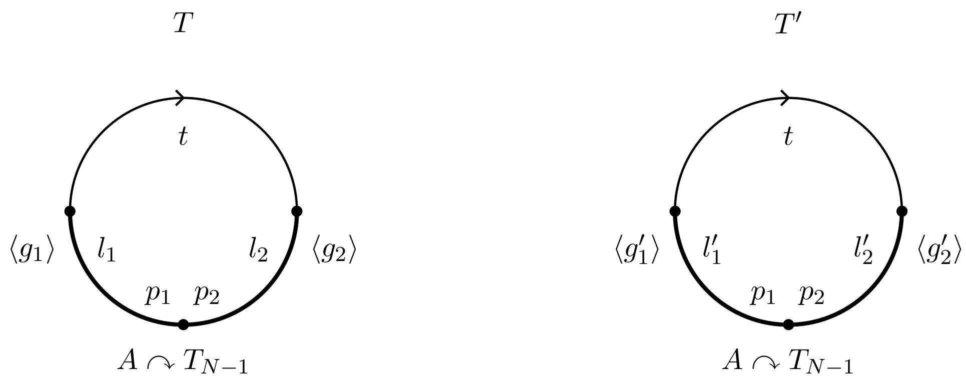


Figure A.22: The trees T and T' in Case 2 of the proof of Proposition A.7.5.

Annexe B

The horoboundary of outer space, and the growth of elements of F_N under random products of automorphisms

Abstract

We show that the horoboundary of outer space for the Lipschitz metric is a quotient of Culler and Morgan's classical boundary, two trees being identified whenever their translation length functions are homothetic in restriction to the set of primitive elements of F_N. We identify the set of Busemann points with the set of trees with dense orbits. We also investigate a few properties of the horoboundary of outer space for the backward Lipschitz metric, and show in particular that it is infinite-dimensional when $N \geq 3$. We then use our description of the horoboundary of outer space to derive an analogue of a theorem of Furstenberg and Kifer [FK83] and Hennion [Hen84] for random products of outer automorphisms of F_N, that estimates possible growth rates of conjugacy classes of elements of F_N under such products.

Contents

Introduction

Over the past decades, the study of the group $\mathrm{Out}(F_N)$ of outer automorphisms of a free group of rank N has benefited a lot from the study of its action on some geometric complexes, among which stands Culler and Vogtmann's outer space [CV86]. A main source of inspiration in this study comes from analogies with arithmetic groups acting on symmetric spaces, and mapping class groups of surfaces acting on Teichmüller spaces. *Outer space* CV_N (or its unprojectivized version cv_N) is the space of equivariant homothety (isometry) classes of simplicial free, minimal, isometric actions of F_N on simplicial metric trees. It is naturally equipped with an asymmetric metric d (i.e. d satisfies the separation axiom and the triangle inequality, but we can have $d(x,y) \neq d(y,x)$). This metric is defined in analogy with Thurston's asymmetric metric on $Teich(S)$. The distance between two trees $T, T' \in CV_N$ is the logarithm of the infimal Lipschitz constant of an F_N-equivariant map from the covolume one representative of T to the covolume one representative of T' [FM11b]. We aim at giving a description of the horoboundary of outer space, which we then use to derive a statement about the growth of elements of F_N under random products of automorphisms, analogous to a theorem of Furstenberg and Kifer [FK83] and Hennion [Hen84] about random products of matrices.

The horoboundary of a metric space was introduced by Gromov in [Gro80]. Let (X,d) be a metric space, and b be a basepoint in X. Associated to any $z \in X$ is a continuous map

$$\begin{array}{cccc} \psi_z : & X & \to & \mathbb{R} \\ & x & \mapsto & d(x,z) - d(b,z) \end{array}.$$

Let $\mathcal{C}(X)$ be the space of real-valued continuous functions on X, equipped with the topology of uniform convergence on compact sets. Under some geometric assumptions on X, the map

$$\begin{array}{cccc} \psi : & X & \to & \mathcal{C}(X) \\ & z & \mapsto & \psi_z \end{array}$$

is an embedding, and taking the closure of its image yields a compactification of X, called the *horofunction compactification*. The space $\overline{\psi(X)} \smallsetminus \psi(X)$ is called the *horoboundary* of X. In [Wal11], Walsh extended this notion to the case of asymmetric metric spaces.

Walsh identified the horofunction compactification of the Teichmüller space of a compact surface, with respect to Thurston's asymmetric metric, with Thurston's compactification, defined as follows (see [FLP79]). Let $\mathcal{C}(S)$ denote the set of free homotopy classes of simple closed curves on S. The space Teich(S) embeds into $\mathbb{PR}^{\mathcal{C}(S)}$ by sending any element to the collection of all lengths of geodesic representatives of homotopy classes of simple closed curves, and the image of this embedding has compact closure. Thurston has identified the boundary with the space of projectivized measured laminations on S.

In the context of group actions on trees, lengths of curves are replaced by translation lengths of elements of the group. The *translation length* of an element g of a group G acting by isometries on an $\mathbb{R}$-tree T is defined as $||g||_T := \inf_{x \in T} d_T(x, gx)$. Looking at the translation lengths of all elements of F_N yields an embedding of cv_N into $\mathbb{R}^{F_N}$, whose image has projectively compact closure, as was proved by Culler and Morgan [CM87]. This compactification $\overline{CV_N}$ of outer space was described by Cohen and Lustig [CL95] and Bestvina and Feighn [BF94] as the space of homothety classes of minimal, very small, isometric actions of F_N on $\mathbb{R}$-trees, see also [7].

We prove that Culler and Morgan's compactification of outer space is not isomorphic to the horofunction compactification. To get the horocompactification of outer space, one has to restrict translation length functions to the set $\mathcal{P}_N$ of primitive elements of F_N, i.e. those elements that belong to some free basis of F_N. This yields an embedding of CV_N into $\mathbb{PR}^{\mathcal{P}_N}$, whose image has compact closure $\overline{CV_N}^{prim}$, called the *primitive compactification* [3]. Alternatively, the space $\overline{CV_N}^{prim}$ is the quotient of $\overline{CV_N}$ obtained by identifying two trees whenever their translation length functions are equal in restriction to $\mathcal{P}_N$. An explicit description of this equivalence relation in terms of trees was given in [3, Theorem 0.2]. The equivalence class of a tree with dense F_N-orbits consists of a single point. The typical example of a nontrivial equivalence class is obtained by equivariantly folding an edge e of the Bass–Serre tree of a splitting of the form $F_N = F_{N-1}*$ along some translate ge, where $g \in F_{N-1}$ is not contained in any proper free factor of F_{N-1}.

Theorem B.0.1. *There exists a unique Out(F_N)-equivariant homeomorphism from* $\overline{CV_N}^{prim}$ *to the horocompactification of* CV_N *which restricts to the identity on* CV_N. *For all* $z \in \overline{CV_N}^{prim}$*, the horofunction associated to* z *is given by*

$$\psi_z(x) = \log \sup_{g \in \mathcal{P}_N} \frac{||g||_z}{||g||_x} - \log \sup_{g \in \mathcal{P}_N} \frac{||g||_z}{||g||_b}$$

for all $x \in CV_N$ *(identified with its covolume* 1 *representative).*

Both suprema in the above formula can be taken over a finite set of elements that only depends on x and b. We could also choose any representative of z in $\overline{cv_N}$, and take the supremum over all elements of F_N. Denoting by $\text{Lip}(x, z)$ the infimal Lipschitz constant of an F_N-equivariant map from x to a fixed representative of z in $\overline{cv_N}$, we also have

$$\psi_z(x) = \log \text{Lip}(x, z) - \log \text{Lip}(b, z).$$

A special class of horofunctions in the horoboundary of a metric space X comes from points arising as limits of infinite almost-geodesic rays in X, called *Busemann points* [Rie02]. Walsh proved that all points in the horoboundary of the Teichmüller space of a compact surface are Busemann. This is no longer true in outer space, one obstruction coming from the noncompleteness of outer space, see [AK13]: some points in the boundary are reached in finite time along geodesic intervals. We show that Busemann points in the horoboundary of outer space coincide with trees having dense orbits under the F_N-action.

As the Lipschitz metric on outer space is not symmetric, one can also consider the horoboundary of outer space for the backward metric. We investigate some of its properties, but we only give a complete description when $N = 2$. There seems to be some kind of duality between the two boundaries we get, the horofunctions for the backward metric being expressed in terms of dual currents. Topologically though, both boundaries are of rather different nature. For example, we show that the backward horocompactification has infinite topological dimension when $N \geq 3$, while the forward horocompactification of outer space has dimension $3N - 4$.

Our motivation for understanding the horoboundary of outer space comes from the question of describing the behaviour of random walks on $\text{Out}(F_N)$. Karlsson and Ledrappier proved that a typical trajectory of the random walk on a locally compact group G acting by isometries on a proper metric space X follows a (random) direction, given by a point in the horoboundary of X, see [KL06, KL11b].

Given a probability measure μ on a group G, the *left random walk* on (G, μ) is the Markov chain on G whose initial distribution is given by the Dirac measure at the identity element of G, with transition probabilities given by $p(x, y) := \mu(yx^{-1})$. In other words, the position Φ_n of the random walk at time n is given from its position e at time 0 by multiplying successively n independent increments ϕ_i of law μ on the left, i.e. $\Phi_n = \phi_n \dots \phi_1$.

Random walks on linear groups were first considered by Furstenberg and Kesten [FK60], who studied the asymptotic behaviour of the norm $||X_n \dots X_1||$ of a product of independent matrix-valued random variables. Furstenberg then studied the growth of the vector norms $||X_n \dots X_1 v||$, where $v \in \mathbb{R}^N$, along typical sample paths of the random walk on $(SL(N, \mathbb{R}), \mu)$, where μ is a probability measure. He showed [Fur63a, Theorems

8.5 and 8.6] that if μ satisfies some moment condition, and if the support of μ generates a noncompact *irreducible* subgroup of $SL(N,\mathbb{R})$, then all vectors in $\mathbb{R}^N \setminus \{0\}$ have the same positive exponential growth rate along typical sample paths of the random walk. Here, a subgroup is *irreducible* if it does not virtually preserve any proper linear subspace of $\mathbb{R}^N$. An analogue of Furstenberg's result for random products in the mapping class group of a compact surface S, was provided by Karlsson [Kar14, Corollary 4]. More precisely, Karlsson showed (again under some moment and irreducibility condition) that the lengths of all isotopy classes of essential simple closed curves on S have the same exponential growth rate. Karlsson derived this statement from Karlsson and Ledrappier's ergodic theorem, by using Walsh's description of the horoboundary of the Teichmüller space of S, equipped with Thurston's asymmetric metric.

We use our description of the horoboundary of outer space to study the growth of conjugacy classes of elements of F_N under random products of outer automorphisms, and prove an analogue of Furstenberg's and Karlsson's results in this context. Let μ be a probability measure on $\text{Out}(F_N)$. In the (generic) case where the support of μ generates a *nonelementary* subgroup of $\text{Out}(F_N)$, we show that all elements of F_N have the same positive exponential growth rate along typical sample paths of the random walk on $(\text{Out}(F_N),\mu)$. Here, by *nonelementary*, we mean a subgroup of $\text{Out}(F_N)$ which does not virtually fix the conjugacy class of any finitely generated subgroup of F_N of infinite index (this is the analogue of Furstenberg's irreducibility condition). The length $||g||$ of an element $g \in F_N$ (or more precisely of its conjugacy class) is defined as the word length of the cyclically reduced representative of g in the standard basis of F_N. The group $\text{Out}(F_N)$ acts on the set of conjugacy classes of elements of F_N. In the following statement, we denote by $d^{sym}_{CV_N}$ the symmetrized version of the Lipschitz metric on CV_N, defined by setting $d^{sym}_{CV_N}(T,T') := d_{CV_N}(T,T') + d_{CV_N}(T',T)$.

Theorem B.0.2. *Let μ be a probability measure on Out(F_N), whose support is finite and generates a non virtually cyclic, nonelementary subgroup of Out(F_N). Then there exists a (deterministic) constant $\lambda > 0$ such that for all $g \in F_N$, and almost every sample path $(\Phi_n)_{n\in\mathbb{N}}$ of the random walk on $(Out(F_N),\mu)$, we have*

$$\lim_{n\to+\infty} \frac{1}{n} \log ||\Phi_n(g)|| = \lambda.$$

The growth rate λ is equal to the drift of the random walk on CV_N for the asymmetric Lipschitz metric.

Even if we no longer assume the subgroup generated by the support of μ to be nonelementary, we can still provide information about possible growth rates of elements of F_N under random automorphisms. This time, several growth rates can arise, and we give a bound on their number. Our main result is an analogue of a version of Oseledets' multiplicative theorem, that is due to Furstenberg and Kifer [FK83, Theorem 3.9] and Hennion [Hen84, Théorème 1] in the case of random products of matrices. Given a probability measure μ on the linear group $GL(N,\mathbb{R})$, Furstenberg–Kifer and Hennion's theorem states that there exists a (deterministic) filtration of $\mathbb{R}^N$ by linear subspaces $\{0\} = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_r = \mathbb{R}^N$, and (deterministic) *Lyapunov exponents* $0 \leq \lambda_1 < \cdots < \lambda_r$, so that for all $i \in \{1,\dots,r\}$, all $v \in L_i \setminus L_{i-1}$, and almost every sample path $(A_n)_{n\in\mathbb{N}}$ of the left random walk on $(GL(N,\mathbb{R}),\mu)$, we have

$$\lim_{n\to+\infty} \frac{1}{n} \log ||A_n v|| = \lambda_i.$$

In the case of free groups, the filtration of $\mathbb{R}^N$ is replaced by the following notion. A *filtration* of F_N is a finite rooted tree, whose nodes are labelled by subgroups of F_N, in which the label of the root is F_N, and the children of a node labelled by H are labelled by subgroups of H.

Theorem B.0.3. *Let μ be a probability measure on Out(F_N), with finite first moment with respect to $d^{sym}_{CV_N}$. Then there exists a (deterministic) filtration of F_N, and a (deterministic)* Lyapunov exponent $\lambda_H \geq 0$ *associated to each subgroup H of the filtration, with $\lambda_{H'} \leq \lambda_H$ as soon as H' is a child of H, such that the following holds.*

For all $g \in F_N$ conjugate into a subgroup H of the filtration, but not conjugate into any of the children of H, and almost every sample path $(\Phi_n)_{n\in\mathbb{N}}$ of the random walk on $(Out(F_N), \mu)$, we have

$$\lim_{n\to+\infty} \frac{1}{n} \log ||\Phi_n(g)|| = \lambda_H.$$

In addition, there are at most $\frac{3N-2}{4}$ positive Lyapunov exponents.

We can actually be a bit more precise about the filtration that arises in Theorem B.0.3, namely: at each level of the filtration, the children of H coincide with a set of representatives of the conjugacy classes of point stabilizers of some very small H-tree with dense orbits. In addition, the conjugacy class of a subgroup of F_N arising in the filtration has finite orbit under the subgroup of $\mathrm{Out}(F_N)$ generated by the support of μ.

To prove Theorem B.0.3, we start by associating to almost every sample path of the random walk on $(\mathrm{Out}(F_N), \mu)$ a (random) filtration, before showing that this filtration can actually be chosen to be deterministic (i.e. independent from the sample path). By Karlsson and Ledrappier's theorem, almost every sample path of the random walk is directed by a (random) horofunction. It follows from our description of the horoboundary of outer space that this horofunction is associated to a (random) tree T. We show that we may choose T to have dense F_N-orbits as soon as the drift of the random walk is positive (if not, then all elements of F_N have subexponential growth). The children of the root will be representatives of the conjugacy classes of the point stabilizers of T. We then argue by induction on the rank to construct the whole filtration. One ingredient of the proof is the study of stationary measures on the horoboundary of outer space.

A consequence of Theorem B.0.3 is that given an element $g \in F_N$, either g grows subexponentially along almost every sample path of the random walk, or g grows exponentially at speed λ_H (independent of the sample path). The number of possible growth rates is uniformly bounded in the rank N of the free group.

The bound on the number of Lyapunov exponents was inspired by a result of Levitt about possible growth rates of elements of F_N under iteration of a single automorphism [Lev09]. In the case where μ is a Dirac measure supported on some element of $\mathrm{Out}(F_N)$, Theorem B.0.3 specifies to Levitt's statement. In a sense, the bound on the number of Lyapunov exponents is optimal: in [Lev09], Levitt gave an example of a single automorphism of F_N with exactly $\frac{3N-2}{4}$ exponential growth rates, see Example B.5.26 of the present paper. We saw however that in the opposite (and generic) case where the support of μ generates a sufficiently big subgroup of $\mathrm{Out}(F_N)$, all conjugacy classes in F_N have the same positive growth rate.

By using Walsh's description of the horoboundary of the Teichmüller space for Thurston's metric, and building on Karlsson's ideas in [Kar14], our methods also yield the analogous result about growth of curves under random products of elements of the mapping class group of a compact surface. In this case, the appropriate analogue of the filtration is given by a decomposition of the surface into subsurfaces.

We also provide a version of Theorem B.0.2 in the case where increments are no longer assumed to be independent, in analogy to Karlsson's in [Kar14, Theorem 2], see Section B.5.2. It would be interesting to know whether our methods can generalize to give a full version of an Oseledets-like theorem for ergodic cocycles of automorphisms of free groups.

The paper is organised as follows. In Section B.1, we recall basic facts about outer space, and present two ways of compactifying it (namely, Culler and Morgan's compactification, and the primitive compactification), as well as the Lipschitz (asymmetric) metric on outer space. Section B.2 is devoted to the identifiaction of the horofunction compactification of outer space with the primitive compactification. In Section B.3, we investigate the geometry of the horoboundary of outer space. In particular, we discuss the link between the horoboundary and the metric completion of outer space, which was identified by Algom-Kfir in [AK13] with the space of trees in $\overline{CV_N}$ having a nontrivial simplicial part, and with trivial arc stabilizers; we also identify the set of Busemann points with the set of trees with dense F_N-orbits in $\overline{CV_N}$. In Section B.4, we discuss some properties of the horoboundary of outer space for the backward metric, and give a description when $N = 2$. The last section of the paper is devoted to the study of the growth of elements of F_N under random products of elements of $\mathrm{Out}(F_N)$. We start with the case of ergodic cocycles of automorphisms (Section B.5.2), before turning to the case of random walks on $\mathrm{Out}(F_N)$. The case of a random walk on a nonelementary subgroup is treated in Section B.5.3. Finally, Section B.5.4 is devoted to the proof of our $\mathrm{Out}(F_N)$-version of Furstenberg–Kifer and Hennion's theorem.

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B.1 Background on outer space

B.1.1 Outer space

Let $N \geq 2$. Denote by R_N the graph having one vertex and N edges, whose petals are identified with some free basis of F_N. A *marked metric graph* is a pair (X, ρ), where X is a compact graph, all of whose vertices have valence at least 3, equipped with a path metric (each edge being assigned a positive length that makes it isometric to a segment), and $\rho : R_N \to X$ is a homotopy equivalence. *Outer space* CV_N was defined by Culler and Vogtmann in [CV86] to be the space of equivalence classes of marked metric graphs, two graphs (X, ρ) and (X', ρ') being equivalent if there exists a homothety $h : X \to X'$ such that ρ' is homotopic to $h \circ \rho$. Passing to the universal cover, one can alternatively define outer space as the space of simplicial free, minimal, isometric actions of F_N on simplicial metric trees, up to equivariant homothety (an action of F_N on a tree is said to be *minimal* if there is no proper invariant subtree). It is possible to normalize all the graphs in CV_N to have volume 1. We denote by cv_N the *unprojectivized outer space*, in which graphs (or equivalently trees) are considered up to isometry, instead of homothety. The group $\mathrm{Out}(F_N)$ acts on CV_N on the right by precomposing the markings (we may also want to consider a left action by setting $\Phi.X := X.\Phi^{-1}$ for $\Phi \in \mathrm{Out}(F_N)$ and $X \in CV_N$). This action is proper but not cocompact, however outer space has a *spine* K_N, which is a deformation retract of CV_N on which $\mathrm{Out}(F_N)$ acts cocompactly. For $\epsilon > 0$, we also define the *ϵ-thick part* CV_N^ϵ of outer space to be the subspace of CV_N consisting of (normalized) graphs having no loop of length smaller than ϵ, on which $\mathrm{Out}(F_N)$ acts cocompactly. The reader is referred to [Vog02] for an excellent survey and reference article about outer space.

B.1.2 Culler and Morgan's compactification of outer space

An *$\mathbb{R}$-tree* is a metric space (T, d_T) in which any two points x and y are joined by a unique embedded topological arc, which is isometric to a segment of length $d_T(x, y)$. Let T be an *F_N-tree*, i.e. an $\mathbb{R}$-tree equipped with an isometric action of F_N. For $g \in F_N$, the *translation length* of g in T is defined to be

$$||g||_T := \inf_{x \in T} d_T(x, gx).$$

Culler and Morgan have shown in [CM87, Theorem 3.7] that the map

$$\begin{array}{rccc} i: & cv_N & \to & \mathbb{R}^{F_N} \\ & T & \mapsto & (||g||_T)_{g \in F_N} \end{array}$$

is injective (and actually a homeomorphism onto its image for the *weak topology* on outer space introduced in [CV86]), whose image has projectively compact closure $\overline{CV_N}$ [CM87, Theorem 4.5]. Bestvina and Feighn [BF94], extending results by Cohen and Lustig [CL95], have characterized the points of this compactification, see also [7]. They showed that $\overline{CV_N}$ is the space of homothety classes of minimal, *very small* F_N-trees, i.e. trees with trivial or maximally cyclic arc stabilizers and trivial tripod stabilizers. We also denote by $\overline{cv_N}$ the lift of $\overline{CV_N}$ to $\mathbb{R}^{F_N}$. We call the topology induced by this embedding on each of the spaces $CV_N, \overline{CV_N}, cv_N$ and $\overline{cv_N}$ the *axes topology*, it is equivalent to the weak topology on CV_N. Bestvina and Feighn showed that $\overline{CV_N}$ has topological dimension $3N-4$, their result was improved by Gaboriau and Levitt who computed the dimension of the boundary ∂CV_N.

Theorem B.1.1. *(Bestvina–Feighn [BF94, Corollary 7.12], Gaboriau–Levitt [GL95, Theorem V.2]) The closure $\overline{CV_N}$ of outer space has dimension $3N-4$. The boundary ∂CV_N has dimension $3N-5$.*

B.1.3 Primitive compactification of outer space

Let $\mathcal{P}_N$ denote the set of primitive elements of F_N, i.e. elements that belong to some free basis of F_N. In [3, Section 2.4], we defined another compactification of outer space, called the *primitive compactification*, by only looking at translation lengths of primitive elements of F_N. We get a continuous injective map

$$i_{prim} : CV_N \to \mathbb{PR}^{\mathcal{P}_N}$$

which is a homeomorphism onto its image, and whose image has compact closure $\overline{CV_N}^{prim}$ [3, Theorem 2.9]. This compactification is isomorphic to $\overline{CV_N}/\sim$, where $\sim$ denotes the *primitive-equivalence* relation, that identifies two trees whose translation length functions are projectively equal in restriction to $\mathcal{P}_N$. The $\sim$-relation was explicitely described in [3]. In particular, we showed that the $\sim$-class of every tree with dense F_N-orbits is reduced to a point. We also proved that every $\sim$-class contains a *canonical* representative T, so that for all trees $T' \sim T$, there is an F_N-equivariant morphism from T to T' (in particular, all elliptic elements in T are also elliptic in T').

The computation of the topological dimension of the closure and the boundary of outer space in [GL95, Theorem V.2] adapts to compute the topological dimension of $\overline{CV_N}^{prim}$ and the boundary $\partial CV_N^{prim} := \overline{CV_N}^{prim} \smallsetminus CV_N$.

Corollary B.1.2. *The space $\overline{CV_N}^{prim}$ has dimension $3N-4$. The boundary ∂CV_N^{prim} has dimension $3N-5$.*

Proof. For all $T \in \overline{CV_N}^{prim}$, let $L(T)$ be the subgroup of $\mathbb{R}$ generated by the translation lengths in T of all primitive elements of F_N. The $\mathbb{Q}$-*rank* $r_{\mathbb{Q}}(T)$ is the dimension of the $\mathbb{Q}$-vector space $L(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then [GL95, Theorem IV.4] shows that for all $T \in \overline{CV_N}^{prim}$, we have $r_{\mathbb{Q}}(T) \leq 3N-3$, and that equality may hold only if $T \in CV_N$. In addition, we get as in [GL95, Proposition V.1] that the space $\mathbb{PR}^{\mathcal{P}_N}_{\leq k}$ of all projectivized length functions with $\mathbb{Q}$-rank smaller than or equal to k has topological dimension smaller than or equal to $k-1$. Since we can find a $(3N-4)$-simplex in CV_N, and a $(3N-5)$-simplex consisting of simplicial actions in ∂CV_N^{prim}, the claim follows. ☐

B.1.4 Metric properties of outer space

There is a natural asymmetric metric on outer space, whose systematic study was initiated by Francaviglia and Martino in [FM11b]: given normalized marked metric graphs (X,ρ) and (X',ρ') in CV_N, the distance $d(X,X')$ is defined to be the logarithm of the infimal (in fact minimal by an easy Arzelà–Ascoli argument, see [FM11b, Lemma 3.4]) Lipschitz constant of a map $f : X \to X'$ such that ρ' is homotopic to $f \circ \rho$. This may also be defined as the logarithm of the infimal Lipschitz constant of an F_N-equivariant map between the corresponding trees (see the discussion in [AK13, Sections 2.3 and 2.4]). This defines a topology on outer space, which is equivalent to the classical one (see [FM11b, Theorems 4.11 and 4.18]). The metric on outer space is not symmetric. One can define a symmetric metric by setting $d_{sym}(X,X') := d(X,X') + d(X',X)$. Elements of $\mathrm{Out}(F_N)$ act by isometries on CV_N with respect to d or d_{sym}. Given an F_N-tree $T \in CV_N$, an element $g \in F_N$ is a *candidate* in T if it is represented in the quotient graph $X := T/F_N$ by a loop which is either

- an embedded circle in X, or
- a bouquet of two circles in X, i.e. $\gamma = \gamma_1\gamma_2$, where γ_1 and γ_2 are embedded circles in X which meet in a single point, or
- a barbell graph, i.e. $\gamma = \gamma_1 e \gamma_2 \overline{e}$, where γ_1 and γ_2 are embedded circles in X that do not meet, and e is an embedded path in X that meets γ_1 and γ_2 only at their origin (and $\overline{e}$ denotes the path e crossed in the opposite direction).

The following result, due to White, gives an alternative description of the metric on outer space. A proof can be found in [FM11b, Proposition 3.15], it was simplified by Algom-Kfir in [AK11, Proposition 2.3].

Theorem B.1.3. *(White, see [FM11b, Proposition 3.15] or [AK11, Proposition 2.3]) For all T,$T' \in CV_N$, we have*

$$d(T,T') = \log \sup_{g \in F_N \smallsetminus \{e\}} \frac{||g||_{T'}}{||g||_T}.$$

The supremum is achieved for an element $g \in F_N$ which is a candidate in T.

Notice in particular that candidates in T are primitive elements of F_N (see [3, Lemma 1.12], for instance). White's theorem has been extended by Algom-Kfir in [AK13, Proposition 4.5] to the case where $T \in \overline{cv_N}$ is a simplicial tree, and $T' \in \overline{cv_N}$ is arbitrary

(Algom-Kfir actually states her result for trees in the metric completion of outer space). The extension to all trees in $\overline{cv_N}$ was made in [3, Theorem 0.3]. Given $T, T' \in \overline{cv_N}$, we denote by $\mathrm{Lip}(T, T')$ the infimal Lipschitz constant of an F_N-equivariant map from T to the metric completion $\overline{T'}$ if such a map exists, and $+\infty$ otherwise. In the following statement, we take the conventions $\frac{1}{0} = +\infty$ and $\frac{0}{0} = 0$.

Theorem B.1.4. *(Horbez [3, Theorem 0.3]) For all $T, T' \in \overline{cv_N}$, we have*

$$Lip(T, T') = \sup_{g \in F_N} \frac{||g||_{T'}}{||g||_T}.$$

B.2 The horocompactification of outer space

In this section, we define the horocompactification of outer space and show that it is isomorphic to the primitive compactification (in particular, it has finite topological dimension). Our approach is motivated by Walsh's analogous statements in the case of the Teichmüller space of a surface, equipped with Thurston's asymmetric metric [Wal11].

We start by recalling the construction of a compactification of an (asymmetric) metric space by horofunctions, under some geometric assumptions. This notion was first introduced in the symmetric case by Gromov in [Gro80], we refer the reader to [Wal11, Section 2] for the case of an asymmetric metric.

Let (X, d) be a (possibly asymmetric) metric space, and let $b \in X$ be some fixed basepoint. For all $z \in X$, we define a continuous map

$$\begin{array}{cccc} \psi_z : & X & \to & \mathbb{R} \\ & x & \mapsto & d(x,z) - d(b,z) \end{array}.$$

Let $\mathcal{C}(X)$ be the space of continuous real-valued functions on X, equipped with the topology of uniform convergence on compact sets of (X, d_{sym}) (where we recall that $d_{sym}(x, y) := d(x, y) + d(y, x)$). We get a map

$$\begin{array}{cccc} \psi : & X & \to & \mathcal{C}(X) \\ & z & \mapsto & \psi_z \end{array}$$

which is continuous and injective, see [Bal95, Chapter II.1] or [Wal11, Lemma 2.1]. We say that an asymmetric metric space is *quasi-proper* if

- the space (X, d) is geodesic, and
- the space (X, d_{sym}) is proper (i.e. closed balls are compact), and
- for all $x \in X$ and all sequences $(x_n)_{n \in \mathbb{N}}$ of elements of X, the distance $d(x_n, x)$ converges to 0 if and only if $d(x, x_n)$ does.

Proposition B.2.1. *(Ballmann [Bal95, Chapter II.1], Walsh [Wal11, Proposition 2.2]) Let (X, d) be a (possibly asymmetric) quasi-proper metric space. Then ψ defines a homeomorphism from X to its image in $\mathcal{C}(X)$, and the closure $\overline{\psi(X)}$ in $\mathcal{C}(X)$ is compact.*

We call $\overline{\psi(X)}$ the *horocompactification* of X, the elements in $X(\infty) := \overline{\psi(X)} \smallsetminus \psi(X)$ being *horofunctions*. As noted in [Wal11, Section 2], all the functions in $\overline{\psi(X)}$ are 1-Lipschitz with respect to d_{sym}, so uniform convergence on compact sets of (X, d_{sym}) is equivalent to pointwise convergence. By the work of Francaviglia and Martino [FM11b, Theorems 5.5, 4.12 and 4.18], outer space is quasi-proper, so we can define its horocompactification.

Theorem B.2.2. *There exists a unique Out(F_N)-equivariant homeomorphism from* $\overline{CV_N}^{prim}$ *to the horocompactification of* CV_N *which restricts to the identity on* CV_N*. For all* $z \in \overline{CV_N}^{prim}$*, the horofunction associated to z is given by*

$$\psi_z(x) = \log \sup_{g \in \mathcal{P}_N} \frac{||g||_z}{||g||_x} - \log \sup_{g \in \mathcal{P}_N} \frac{||g||_z}{||g||_b}$$

for all $x \in CV_N$ *(identified with its covolume* 1 *representative).*

Proof. Uniqueness follows from the density of CV_N in $\overline{CV_N}^{prim}$. Let $x \in CV_N$, which we identify with its covolume 1 representative. For all $z \in \overline{CV_N}^{prim}$, we let

$$\psi'_z(x) := \log \sup_{g \in \mathcal{P}_N} \frac{||g||_z}{||g||_x} - \log \sup_{g \in \mathcal{P}_N} \frac{||g||_z}{||g||_b}.$$

This is well-defined, because ψ'_z only depends on the projective class of $||.||_z$. By definition of the metric on CV_N, we have $\psi'_z = \psi_z$ for all $z \in CV_N$. In addition, the suprema arising in the expression of $\psi'_z(x)$ are achieved on finite sets $\mathcal{F}(x)$ (resp. $\mathcal{F}(b)$) consisting of candidates in x (resp. in b) by Theorem B.1.3.

We claim that for all $z \in \overline{CV_N}^{prim}$, the map ψ'_z is continuous. Indeed, let $z \in \overline{CV_N}^{prim}$, and let $(z_n)_{n\in\mathbb{N}} \in CV_N^{\mathbb{N}}$ be a sequence that converges to z. For all $n \in \mathbb{N}$, we have

$$\psi'_{z_n}(x) = \log \sup_{\mathcal{F}(x)} \frac{||g||_{z_n}}{||g||_x} - \log \sup_{\mathcal{F}(b)} \frac{||g||_{z_n}}{||g||_b}.$$

By definition of the topology on $\mathbb{PR}^{\mathcal{P}_N}$, there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of real numbers such that for all $g \in \mathcal{P}_N$, the sequence $(\lambda_n ||g||_{z_n})_{n\in\mathbb{N}}$ converges to $||g||_z$. So $\psi'_{z_n}(x)$ converges to $\psi'_z(x)$. Therefore, the map ψ'_z is the pointwise limit of the 1-Lipschitz maps ψ'_{z_n}, so ψ'_z is continuous.

We can thus extend the map ψ to a map from $\overline{CV_N}^{prim}$ to $\mathcal{C}(CV_N)$, which we still denote by ψ. We claim that this extension is continuous. Indeed, if a sequence $(z_n)_{n\in\mathbb{N}} \in (\overline{CV_N}^{prim})^{\mathbb{N}}$ converges to $z \in \overline{CV_N}^{prim}$, then the maps ψ_{z_n} converge pointwise to ψ_z, and hence they converge uniformly on compact sets of (X, d_{sym}) because all maps ψ_{z_n} are 1-Lipschitz.

We now prove that the map $\psi : \overline{CV_N}^{prim} \to \mathcal{C}(CV_N)$ is injective. Let $z,z' \in \overline{CV_N}^{prim}$ be such that $\psi_z = \psi_{z'}$. Let $g \in \mathcal{P}_N$. Let $x \in CV_N$ be a rose, one of whose petals is labelled by g. Denote by x_ϵ the rose in CV_N with same underlying graph as x, in which the petal labelled by g has length $\epsilon > 0$, while the other petals all have the same length. As ϵ tends to 0, the length $||g||_{x_\epsilon}$ tends to 0, while $||g'||_{x_\epsilon}$ is bounded below for all $\epsilon > 0$ and all $g' \neq g^{\pm 1} \in \mathcal{F}(x_\epsilon)$, and $\mathcal{F}(x_\epsilon)$ does not depend on ϵ. Hence for $\epsilon > 0$ sufficiently small, we have the following dichotomy (we fix representatives of z and z' in their projective classes).

- If $||g||_z \neq 0$, then $\psi_z(x_\epsilon) = \log \frac{||g||_z}{\epsilon C(z)}$ (where $C(z) := \sup_{\mathcal{F}(b)} \frac{||g||_z}{||g||_b}$) tends to $+\infty$ as ϵ goes to 0.
- If $||g||_z = 0$, then $\psi_z(x_\epsilon)$ is bounded above independently of $\epsilon > 0$.

As $\psi_z = \psi_{z'}$, an element $g \in \mathcal{P}_N$ is elliptic in z if and only if it is elliptic in z', and in addition, we have $\frac{||g||_{z'}}{||g||_z} = \frac{C(z)}{C(z')}$ for all $g \in \mathcal{P}_N$ which are not elliptic in z. Hence $z = z'$.

We have shown that $\psi : \overline{CV_N}^{prim} \to \mathcal{C}(CV_N)$ is a continuous injection. As $\overline{CV_N}^{prim}$ is compact, the map ψ is a homeomorphism from $\overline{CV_N}^{prim}$ to its image in $\mathcal{C}(CV_N)$. In particular, the image $\psi(\overline{CV_N}^{prim})$ is compact, and hence closed in $\mathcal{C}(CV_N)$. By continuity of ψ, we also have $\psi(CV_N) \subseteq \psi(\overline{CV_N}^{prim}) \subseteq \overline{\psi(CV_N)}$, so $\psi(\overline{CV_N}^{prim}) = \overline{\psi(CV_N)}$, i.e. $\overline{CV_N}^{prim}$ is isomorphic to the horocompactification of CV_N. That ψ is $\mathrm{Out}(F_N)$-equivariant follows from its construction. □

Remark B.2.3. In order to prove the injectivity of the map $\psi : \overline{CV_N}^{prim} \to \mathcal{C}(CV_N)$, we "select" the primitive element g in the rose x by making the length of the corresponding petal tend to 0. There is another way of "selecting" the primitive element g which does not require leaving the thick part of outer space, and will therefore enable us to prove the corresponding statement for the spine or the thick part of outer space. The idea is to replace the rose x whose petals all have the same length, and are labelled by a basis $(g, g_2, \dots, g_N)$ of F_N, by a rose x_k whose petals are labelled by $(g, g_2, \dots, g_{N-1}, g_N g^k)$, for k sufficiently large. Unless $||g||_z = 0$, the translation length in z of a word represented by a candidate in x_k containing the petal labelled by $g_N g^k$ becomes arbitrarily large as k tends to $+\infty$, and the translation lengths of such a word in two (unprojectivized) trees z and z' may be equal for arbitrarily large k only if $||g||_z = ||g||_{z'}$.

The spine K_N (considered as a subspace of CV_N) and the ϵ-thick part CV_N^ϵ, equipped with the restriction of the Lipschitz metric, are not geodesic metric spaces. However, we show that we can still define their horocompactification. We recall that for all metric spaces X, we have defined an embedding $\psi : X \to \mathcal{C}(X)$. We define $\overline{K_N}^{prim}$ and $\overline{CV_N^\epsilon}^{prim}$ in the same way as we defined $\overline{CV_N}^{prim}$.

Proposition B.2.4. *The map ψ defines a homeomorphism from K_N to its image in $\mathcal{C}(K_N)$, and the closure $\overline{\psi(K_N)}$ in $\mathcal{C}(K_N)$ is compact, and isomorphic to $\overline{K_N}^{prim}$. For all $\epsilon > 0$, the map ψ defines a homeomorphism from CV_N^ϵ to its image in $\mathcal{C}(CV_N^\epsilon)$, and the closure $\overline{\psi(CV_N^\epsilon)}$ in $\mathcal{C}(CV_N^\epsilon)$ is compact, and isomorphic to $\overline{CV_N^\epsilon}^{prim}$.*

Proof. In the proof of Proposition B.2.1, the assumption that (X, d) is geodesic is only used to show that if $(z_n)_{n\in\mathbb{N}}$ is a sequence in X escaping to infinity (i.e. eventually leaving and never returning to every compact set), then no subsequence of $(\psi_{z_n})_{n\in\mathbb{N}}$ converges to a function ψ_y with $y \in X$.

Assume that there exists a sequence $(z_n)_{n\in\mathbb{N}}$ of elements of K_N escaping to infinity such that some subsequence of $(\psi_{z_n})_{n\in\mathbb{N}}$ converges to ψ_y, with $y \in K_N$. Up to passing to a subsequence again, we may assume that $(z_n)_{n\in\mathbb{N}}$ converges to an element z in $\overline{CV_N}^{prim}$ (and actually $z \in \partial CV_N^{prim}$), so by Theorem B.2.2 we have $\psi_z = \psi_y$. However, in this case, the argument in Remark B.2.3 shows that $z = y$, a contradiction. So ψ defines a homeomorphism from K_N to its image in $\mathcal{C}(K_N)$, and the closure $\overline{\psi(K_N)}$ in $\mathcal{C}(K_N)$ is compact. The argument then goes as in the proof of Theorem B.2.2, by using Remark B.2.3, to show that $\overline{\psi(K_N)}$ is isomorphic to $\overline{K_N}^{prim}$. The same argument also yields the result for the ϵ-thick part of outer space. □

B.3 Completion and Busemann points

B.3.1 The metric completion of outer space

We follow Algom-Kfir's exposition in [AK13, Section 1] of the construction of a completion of an asymmetric metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ of elements in a (possibly asymmetric) metric space X is *forward admissible* if for all $\epsilon > 0$, there exists $N(\epsilon) \in \mathbb{N}$ such that for all $n \geq N(\epsilon)$, there exists $K(n, \epsilon) \in \mathbb{N}$ such that $d(x_n, x_k) \leq \epsilon$ for all $k \geq K(n, \epsilon)$. Two forward admissible sequences are *equivalent* if their *interlace* (i.e. the sequence $(z_n)_{n\in\mathbb{N}}$ defined by $z_{2n} = x_n$ and $z_{2n+1} = y_n$ for all $n \in \mathbb{N}$) is forward admissible. The *forward metric completion* $\widehat{X}$ of X is defined to be the set of equivalence classes of forward admissible sequences. The reader is referred to [AK13, Section 1] for a detailed account of this construction.

Lemma B.3.1. *(Algom-Kfir [AK13, Lemma 1.8]) Let X be a (possibly asymmetric) metric space, and let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two forward admissible sequences of elements in X, then either*

- *for all $r \geq 0$, there exists $N(r) \in \mathbb{N}$ such that for all $n \geq N(r)$, there exists $K(n,r) \in \mathbb{N}$ such that for all $k \geq K(n,r)$, we have $d(x_n, y_k) \geq r$, or*
- *there exists $c \geq 0$ such that for all $\epsilon > 0$, there exists $N(\epsilon) \in \mathbb{N}$ such that for all $n \geq N(\epsilon)$, there exists $K(n,\epsilon) \in \mathbb{N}$ such that for all $k \geq K(n,\epsilon)$, we have $|d(x_n, y_k) - c| \leq \epsilon$.*

Given two forward admissible sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of elements in X, we denote by $c((x_n),(y_n))$ the number provided by Lemma B.3.1 (in the first case, we set $c((x_n),(y_n)) := +\infty$). In the particular case where $(x_n)_{n\in\mathbb{N}}$ is constant, Algom-Kfir's proof of Lemma B.3.1 actually shows that the second case occurs.

Lemma B.3.2. *For all $b \in X$, and all forward admissible sequences $(z_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$, we have $c(b,(z_n)) < +\infty$. If $(z_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ and $(z'_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ are two equivalent forward admissible sequences, then $c(b,(z_n)) = c(b,(z'_n))$.*

Proof. As $(z_n)_{n\in\mathbb{N}}$ is forward admissible, the sequence $(d(b,z_n))_{n\in\mathbb{N}}$ is almost monotonically decreasing in the sense of [AK13, Definition 1.9]. Hence by [AK13, Proposition 1.10], it converges to a limit $c(b,(z_n))$. If $(z_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ and $(z'_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ are two equivalent forward admissible sequences, then $c(b,(z_n)) = c(b,(z'_n)) = c(b,(z''_n))$, where $(z''_n)_{n\in\mathbb{N}}$ is the interlace of $(z_n)_{n\in\mathbb{N}}$ and $(z'_n)_{n\in\mathbb{N}}$. ☐

Algom-Kfir shows that two forward admissible sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of elements in X are equivalent if and only if $c((x_n),(y_n)) = c((y_n),(x_n)) = 0$ [AK13, Lemma 1.12]. The metric on X extends to an asymmetric metric $\widehat{d}$ on $\widehat{X}$ (which might not satisfy the separation axiom, and might be ∞-valued) by setting $\widehat{d}((x_n),(y_n)) := c((x_n),(y_n))$ [AK13, Proposition 1.16]. The collection of balls $B(x,r) := \{y \in \widehat{X} | \widehat{d}(y,x) < r\}$ for $x \in \widehat{X}$ and $r \in \mathbb{R}_+^*$ is a basis for a topology on $\widehat{X}$. One can also consider the symmetrized metric $\widehat{d}_{sym}$ on $\widehat{X}$, which defines another topology on $\widehat{X}$.

Algom-Kfir has determined the metric completion of outer space in [AK13].

Theorem B.3.3. *(Algom-Kfir [AK13, Theorem B]) Let $T \in \overline{CV_N}$. Then $T \in \widehat{CV_N}$ if and only if T does not have dense orbits, and T has trivial arc stabilizers. In addition, for all $T, T' \in \widehat{CV_N}$, we have $\widehat{d}(T,T') = \log Lip(\widetilde{T}, \widetilde{T}')$, where $\widetilde{T}$ (resp. $\widetilde{T}'$) denotes the covolume one representative of T (resp. T') in $\overline{cv_N}$.*

B.3.2 The metric completion as a subspace of the horocompactification

Throughout the section, we assume that X is a quasi-proper metric space, so that the horocompactification of X is well-defined. We recall that associated to any $z \in X$ is a function $\psi_z \in \mathcal{C}(X)$.

Proposition B.3.4. *For all forward admissible sequences $(z_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$, the sequence $(\psi_{z_n})_{n\in\mathbb{N}}$ has a limit in $\mathcal{C}(X)$. If $(z_n)_{n\in\mathbb{N}}$ and $(z'_n)_{n\in\mathbb{N}}$ are two equivalent forward admissible sequences, then the sequences $(\psi_{z_n})_{n\in\mathbb{N}}$ and $(\psi_{z'_n})_{n\in\mathbb{N}}$ converge to the same limit in $\mathcal{C}(X)$.*

Proof. Let $(z_n)_{n\in\mathbb{N}}$ and $(z'_n)_{n\in\mathbb{N}}$ be two equivalent forward admissible sequences. Let $(z_{\sigma(n)})_{n\in\mathbb{N}}$ (resp. $(z'_{\sigma'(n)})_{n\in\mathbb{N}}$) be a subsequence of $(z_n)_{n\in\mathbb{N}}$ (resp. $(z'_n)_{n\in\mathbb{N}}$) that converges to some function ψ (resp. ψ') in $\mathcal{C}(X)$. Let $\epsilon > 0$. By definition of c, there exists an integer $N(\epsilon) \in \mathbb{N}$ such that for all $n \geq N(\epsilon)$, there exists $K(n,\epsilon) \in \mathbb{N}$ such that for all $k \geq K(n,\epsilon)$, we have $d(z_{\sigma(n)}, z'_{\sigma'(k)}) \leq \epsilon$. In addition, Lemma B.3.2 shows the existence of $N'(\epsilon) \in \mathbb{N}$ and $c \in \mathbb{R}$ such that for all $n \geq N'(\epsilon)$, we have $|d(b, z_{\sigma(n)}) - c| \leq \epsilon$ and $|d(b, z'_{\sigma'(n)}) - c| \leq \epsilon$. For all $n \geq \max\{N(\epsilon), N'(\epsilon)\}$, all $k \geq \max\{K(n,\epsilon), N'(\epsilon)\}$ and all $x \in X$, we have

$$\begin{aligned} \psi_{z'_{\sigma'(k)}}(x) - \psi_{z_{\sigma(n)}}(x) &= d(x, z'_{\sigma'(k)}) - d(x, z_{\sigma(n)}) + d(b, z_{\sigma(n)}) - d(b, z'_{\sigma'(k)}) \\ &\leq d(z_{\sigma(n)}, z'_{\sigma'(k)}) + d(b, z_{\sigma(n)}) - d(b, z'_{\sigma'(k)}) \\ &\leq 3\epsilon. \end{aligned}$$

By making $\epsilon > 0$ arbitrarily small, and letting n and k tend to infinity, we get that $\psi'(x) \leq \psi(x)$ for all $x \in X$. Symmetrizing the argument, we also get that $\psi(x) \leq \psi'(x)$ for all $x \in X$, whence $\psi = \psi'$. In particular, the sequence $(\psi_{z_n})_{n\in\mathbb{N}}$ associated to any forward admissible sequence $(z_n)_{n\in\mathbb{N}}$ has at most one limit point, and hence it converges in the horocompactification of X. Two equivalent sequences give rise to the same limit. $\square$

Proposition B.3.4 yields a map i from the metric completion $\widehat{X}$ of X to the horocompactification of X, which is the identity map in restriction to X, by setting

$$\begin{array}{rccc} i: & \widehat{X} & \to & X \cup X(\infty) \\ & (z_n)_{n\in\mathbb{N}} & \mapsto & \lim_{n\to+\infty} \psi_{z_n} \end{array}.$$

Proposition B.3.5. *The map $i : \widehat{X} \to X \cup X(\infty)$ is injective.*

Proof. Let $(z_n)_{n\in\mathbb{N}}, (z'_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ be two forward admissible sequences. Assume that $i((z_n)_{n\in\mathbb{N}}) = i((z'_n)_{n\in\mathbb{N}}) = \psi \in \mathcal{C}(X)$. Let $\epsilon > 0$. For all $p \in \mathbb{N}$, there exists $K_0(p,\epsilon)$ such that for all $n, q \geq K_0(p,\epsilon)$, we have $|\psi_{z_n}(z_p) - \psi_{z'_q}(z_p)| \leq \epsilon$ and $|\psi_{z_n}(z'_p) - \psi_{z'_q}(z'_p)| \leq \epsilon$. As $(z_n)_{n\in\mathbb{N}}$ is forward admissible, there exists $N_1(\epsilon) \in \mathbb{N}$ such that for all $n \geq N_1(\epsilon)$, there exists $K_1(n,\epsilon) \in \mathbb{N}$ such that for all $k \geq K_1(n,\epsilon)$, we have $d(z_n, z_k) \leq \epsilon$. As $(z'_n)_{n\in\mathbb{N}}$ is forward admissible, there exists $N_2(\epsilon) \in \mathbb{N}$ such that for all $n \geq N_2(\epsilon)$, there exists $K_2(n,\epsilon) \in \mathbb{N}$ such that for all $k \geq K_2(n,\epsilon)$, we have $d(z'_n, z'_k) \leq \epsilon$. By Lemma B.3.2, there also exist $N_3(\epsilon) \in \mathbb{N}$ and $c, c' \in \mathbb{R}$ such that for all $n \geq N_3(\epsilon)$, we have $|d(b, z_n) - c| \leq \epsilon$ and $|d(b, z'_n) - c'| \leq \epsilon$. For all $p \geq N_1(\epsilon)$, all $n \geq \max\{K_0(p,\epsilon), K_1(p,\epsilon), N_3(\epsilon)\}$ and all $q \geq \max\{K_0(p,\epsilon), N_3(\epsilon)\}$, as

$$\psi_{z_n}(z_p) - \psi_{z'_q}(z_p) = d(z_p, z_n) - d(z_p, z'_q) + d(b, z'_q) - d(b, z_n),$$

we have $d(z_p, z'_q) + c - c' \leq 4\epsilon$. In particular, for all $\epsilon > 0$, we have $c - c' \leq 4\epsilon$, whence $c \leq c'$. Similarly, for all $p \geq N_2(\epsilon)$, all $n \geq \max\{K_0(p,\epsilon), N_3(\epsilon)\}$ and all $q \geq \max\{K_0(p,\epsilon), K_2(p,\epsilon), N_3(\epsilon)\}$, as

$$\psi_{z_n}(z'_p) - \psi_{z'_q}(z'_p) = d(z'_p, z_n) - d(z'_p, z'_q) + d(b, z'_q) - d(b, z_n),$$

we have $d(z'_p, z_n) + c' - c \leq 4\epsilon$, and in particular this implies that $c' \leq c$. So $c = c'$, and the inequalities we have established thus imply that $c((z_n), (z'_n)) = c((z'_n), (z_n)) = 0$. It then follows from [AK13, Lemma 1.12] that the sequences $(z_n)_{n\in\mathbb{N}}$ and $(z'_n)_{n\in\mathbb{N}}$ are equivalent, thus showing that i is injective. $\square$

In particular, the space $\widehat{X}$ inherits a (metrizable) topology induced by the topology on $\mathcal{C}(X)$.

B.3.3 Comparing the topologies on $\widehat{X}$

Let X be a quasi-proper (possibly asymmetric) metric space. We now compare the three topologies on $\widehat{X}$ we have introduced in the previous two sections, namely the topology defined by $\widehat{d}_{sym}$, the topology defined by $\widehat{d}$, and the topology coming from $\mathcal{C}(X)$. The topology defined by $\widehat{d}_{sym}$ dominates the topology defined by $\widehat{d}$. The following proposition shows that the topology defined by $\widehat{d}_{sym}$ also dominates the topology induced by the topology coming from $\mathcal{C}(X)$ (all these topologies are second-countable, which justifies the use of sequential arguments).

Proposition B.3.6. *Let $z = (z_n)_{n\in\mathbb{N}} \in \widehat{X}$, and let $(z^k)_{k\in\mathbb{N}} = ((z_n^k)_{n\in\mathbb{N}})_{k\in\mathbb{N}}$ be a sequence of elements of $\widehat{X}$. If $\widehat{d}_{sym}(z^k, z)$ converges to 0, then ψ_{z^k} converges to ψ_z in $\mathcal{C}(X)$.*

Proof. Assume that $\widehat{d}_{sym}(z^k, z)$ converges to 0, i.e. $c(z^k, z)$ and $c(z, z^k)$ both converge to 0. Let $\epsilon > 0$. There exists $K_0 \in \mathbb{N}$ such that for all $k \geq K_0$, we have $c(z^k, z) \leq \epsilon$ and $c(z, z^k) \leq \epsilon$. We fix $k \geq K_0$. As $c(z^k, z) \leq \epsilon$, there exists an integer $N_1(\epsilon) \in \mathbb{N}$ such that for all $n \geq N_1(\epsilon)$, there exists $K_1(n, \epsilon) \in \mathbb{N}$ such that for all $m \geq K_1(n, \epsilon)$, we have $d(z_n^k, z_m) \leq 2\epsilon$. Similarly, as $c(z, z^k) \leq \epsilon$, there exists $N_2(\epsilon) \in \mathbb{N}$ such that for all $n \geq N_2(\epsilon)$, there exists $K_2(n, \epsilon) \in \mathbb{N}$ such that for all $m \geq K_2(n, \epsilon)$, we have $d(z_n, z_m^k) \leq 2\epsilon$. By Lemma B.3.2, there also exists $N'(\epsilon) \in \mathbb{N}$, and $c^k, c \in \mathbb{R}$ such that for all $n \geq N'(\epsilon)$, we have $|d(b, z_n^k) - c^k| \leq \epsilon$ and $|d(b, z_n) - c| \leq \epsilon$. Choosing $n \geq \max\{N_1(\epsilon), N'(\epsilon)\}$ and $m \geq \max\{K_1(n, \epsilon), N'(\epsilon)\}$, we get that

$$\begin{aligned} c - c^k &\leq d(b, z_m) - d(b, z_n^k) + 2\epsilon \\ &\leq d(z_n^k, z_m) + 2\epsilon \\ &\leq 4\epsilon. \end{aligned}$$

So for all $n \geq \max\{N_2(\epsilon), N'(\epsilon)\}$, all $m \geq \max\{K_2(n, \epsilon), N'(\epsilon)\}$ and all $x \in X$, we have

$$\begin{aligned} \psi_{z_m^k}(x) - \psi_{z_n}(x) &= d(x, z_m^k) - d(x, z_n) + d(b, z_n) - d(b, z_m^k) \\ &\leq d(z_n, z_m^k) + c - c^k + 2\epsilon \\ &\leq 8\epsilon. \end{aligned}$$

Letting m go to infinity, we get that $\psi_{z^k}(x) - \psi_{z_n}(x) \leq 8\epsilon$, and letting n go to infinity, we get $\psi_{z^k}(x) - \psi_z(x) \leq 8\epsilon$. Similarly, choosing $n \geq \max\{N_2(\epsilon), N'(\epsilon)\}$ and $m \geq \max\{K_2(n, \epsilon), N'(\epsilon)\}$, we get that

$$\begin{aligned} c^k - c &\leq d(b, z_m^k) - d(b, z_n) + 2\epsilon \\ &\leq d(z_n, z_m^k) + 2\epsilon \\ &\leq 4\epsilon. \end{aligned}$$

So for all $n \geq \max\{N_1(\epsilon), N'(\epsilon)\}$, all $m \geq \max\{K_1(n, \epsilon), N'(\epsilon)\}$ and all $x \in X$, we have

$$\begin{aligned} \psi_{z_m}(x) - \psi_{z_n^k}(x) &= d(x, z_m) - d(x, z_n^k) + d(b, z_n^k) - d(b, z_m) \\ &\leq d(z_n^k, z_m) + c^k - c + 2\epsilon \\ &\leq 8\epsilon. \end{aligned}$$

Again letting m and then n tend to infinity, we get that $\psi_z(x) - \psi_{z^k}(x) \leq 8\epsilon$ for all $x \in X$. So $|\psi_{z^k}(x) - \psi_z(x)| \leq 8\epsilon$ for all $x \in X$ and all $k \geq K_0$. Hence $(\psi_{z^k})_{k\in\mathbb{N}}$ converges uniformly (and in particular uniformly on compact sets) to ψ_z. $\square$

However, the examples below show that no two of the three topologies we have defined on $\widehat{X}$ are equivalent when $X = CV_N$. In this case, the topology induced by the topology on $\mathcal{C}(X)$ is the *primitive axes topology*, given by the embedding of $\widehat{X}$ into $\mathbb{PR}^{\mathcal{P}_N}$. In the case of outer space, there is a fourth natural topology on $\widehat{CV_N}$, called the *axes topology*, given by the embedding into $\mathbb{PR}^{F_N}$. The axes topology dominates the primitive axes topology. The examples below show that no two of the four topologies on $\widehat{CV_N}$ are equivalent.

Example B.3.7. *The topology defined by $\widehat{d}_{sym}$ is not dominated by any of the other three topologies.*
Let $T \in \widehat{CV_N}$ be a nonsimplicial tree. Let $T^{simpl} \in \widehat{CV_N}$ be the tree obtained by collapsing all vertex trees in the Levitt decomposition of T as a graph of actions [Lev94] to points. For $n \in \mathbb{N}$, let T_n be the tree obtained from T by applying a homothety with factor $\frac{1}{n}$ to all vertex trees of T. Then the sequence $(T_n)_{n\in\mathbb{N}}$ converges to $T^{simpl} \in \widehat{CV_N}$ in the axes topology (and hence also in the primitive axes topology). For all $n \in \mathbb{N}$, there is an obvious F_N-equivariant 1-Lipschitz map from T_n to T^{simpl} given by collapsing all components of the complement of the simplicial part of T to points, so $\widehat{d}(T_n, T^{simpl}) = 0$, while $\widehat{d}(T^{simpl}, T_n) = +\infty$. So the sequence $(T_n)_{n\in\mathbb{N}}$ also converges to T^{simpl} in the topology defined by $\widehat{d}$, but not in the topology defined by $\widehat{d}_{sym}$.

Example B.3.8. *The topology defined by $\widehat{d}$ does not dominate the primitive axes topology.*
Let $T \in \widehat{CV_N}$ be a nonsimplicial tree. As in the previous example, we have $\widehat{d}(T, T^{simpl}) = 0$. Hence the space $(\widehat{CV_N}, \widehat{d})$ is not separated, while $\mathcal{C}(CV_N)$ is.

Example B.3.9. *The axes topology does not dominate the topology defined by $\widehat{d}$.*
Let $T' \in \overline{cv_N}$ be a tree with dense orbits, and let $p \in T'$. Let $(T'_n, p_n)_{n\in\mathbb{N}}$ be a sequence of pointed trees with dense orbits in $\overline{cv_N}$ that converges (non projectively) to (T', p), and such that for all $n \in \mathbb{N}$, the trees T' and T'_n do not belong to a common closed simplex of length measures in $\overline{cv_N}$ (in the sense of [Gui00, Section 5]). Let $T \in \widehat{CV_{N+1}}$ (resp. $T_n \in \widehat{CV_{N+1}}$) be the tree associated to the graph of actions having

- two vertices v_1 and v_2, where the vertex tree T_{v_1} is equal to T (resp. T'_n), with attaching point p (resp. p_n) and vertex group generated by $x_1, \dots, x_N$, and T_{v_2} is reduced to a point, and G_{v_2} is the cyclic subgroup of F_{N+1} generated by x_{N+1}, and
- a single edge of length 1 joining v_1 and v_2, with trivial stabilizer.

The sequence $(T_n)_{n\in\mathbb{N}}$ converges in the axes topology to T by Guirardel's Reduction Lemma [Gui98, Section 4]. However, for all $n \in \mathbb{N}$, we have $\widehat{d}(T^n, T) = +\infty$ by [3, Proposition 5.7].

Remark B.3.10. However, Algom-Kfir has shown that the axes topology is strictly finer than the topology defined by $\widehat{d}$ in restriction to the simplicial part of $\widehat{CV_N}$ [AK13, Theorem 5.12].

Example B.3.11. *The primitive axes topology does not dominate the axes topology.*
Let $T \in \widehat{CV_N}$ be the Bass–Serre tree of an HNN-extension of the form $F_N = F_{N-1}*$. Let $g \in F_{N-1}$ be an element that does not belong to any proper free factor of F_{N-1}. Let $T' \in \overline{CV_N}$ be the tree obtained from T by equivariantly folding an edge $e \subseteq T$ along ge. We have shown in [3] that the trees T and T' have the same translation length functions in restriction to $\mathcal{P}_N$. This implies that any sequence of trees $(T_n)_{n\in\mathbb{N}}$ that converges to T' in $\overline{CV_N}$ for the axes topology, does not converge in $\widehat{CV_N}$ for the axes topology. However, such a sequence converges to $T \in \widehat{CV_N}$ for the primitive axes topology.

B.3.4 Folding paths and geodesics

Let $T \in CV_N$, and $T' \in \overline{CV_N}$ be a tree with dense orbits. A *$\widehat{d}$-geodesic ray* from T to T' is a path $\gamma : \mathbb{R}_+ \to \widehat{CV_N}$ such that for all $s \leq t \in \mathbb{R}_+$, we have

$$\widehat{d}(\gamma(s), \gamma(t)) = t - s,$$

and the trees $\gamma(t)$ converge to T' for the axes topology on $\overline{CV_N}$ as t goes to $+\infty$. Using the classical construction of folding paths (see [FM11b, FM14, GL07b, Mei14]), one shows the following fact. We sketch a proof for completeness.

Proposition B.3.12. *For all $T \in CV_N$ and all $T' \in \overline{CV_N}$ having dense orbits, there exists a $\widehat{d}$-geodesic ray in $\widehat{CV_N}$ from T to T'.*

Proof. Let $f : T \to T'$ be an optimal map, and $g \in F_N$ be a legal element for f in T, whose axis in T is contained in the tension graph of f, i.e. the subgraph made of those edges in T that are maximally stretched by f (the reader is referred to [3, Section 6.2] for definitions and a proof of the existence of such an element $g \in F_N$). We fix representatives of T and T' in $\overline{cv_N}$, again denoted by T and T', slightly abusing notations.

We define a simplicial tree $\overline{T} \in \overline{cv_N}$ that belongs to the same closed simplex as T, in the following way. We first collapse all edges in T which are mapped to a point by f. We then shrink all edges outside of the tension graph of f, so that all edges in $\overline{T}$ are stretched by a factor of $M := \mathrm{Lip}(T, T')$ under the map $\overline{f} : \overline{T} \to T'$ induced by f. Denote by K the distance (for $\widehat{d}$) from the covolume 1 representative of T to the covolume 1 representative of $\overline{T}$ in $\widehat{CV_N}$. Let $(\gamma_1(t))_{t\in[0,K]}$ be a straight segment of length K (staying in a closed simplex of $\widehat{CV_N}$) joining T to $\overline{T}$, parameterized by arc length.

Let $M\overline{T}$ be the tree obtained from T by multiplying all edge lengths by M. There exists a morphism $f : M\overline{T} \to T'$. Let $(T_t)_{t\in\mathbb{R}_+}$ be the folding path guided by f constructed in [GL07b, Section 3]. Notice that for all $t \in \mathbb{R}_+$, the tree T_t has trivial arc stabilizers, because T has trivial arc stabilizers. If the tree T_t had dense orbits for some $t \in \mathbb{R}_+$, then we would have $T_t = T'$, since no folding can occur starting from a tree with dense orbits [3, Corollary 3.10]. Denoting by t_0 the smallest such $t \in \mathbb{R}_+$, the sequence $(T_{t_0-\frac{1}{n}})_{n\in\mathbb{N}}$ would then be a Cauchy sequence converging to T', contradicting Theorem B.3.3. For all $t \in \mathbb{R}_+$, we denote by $\gamma_2(t)$ the projection of T_t to $\widehat{CV_N}$.

Let γ be the path in $\widehat{CV_N}$ defined as the concatenation of the paths γ_1 and γ_2. As the axis of g is contained in the tension graph of T, it does not get shortened when passing from T to $\overline{T}$ (and lengths do not increase when passing from T to $\overline{T}$). Legality of g implies that its axis never gets folded along the path γ_2. Therefore, for all $s \leq t \in \mathbb{R}_+$, we have

$$\widehat{d}(\gamma(s), \gamma(t)) = \log \frac{||g||_{\gamma(t)}}{||g||_{\gamma(s)}}.$$

This shows that for all $s \leq t \leq u \in \mathbb{R}_+$, we have $\widehat{d}(\gamma(s), \gamma(u)) = \widehat{d}(\gamma(s), \gamma(t)) + \widehat{d}(\gamma(t), \gamma(u))$. Therefore, up to reparameterization, the path γ is a $\widehat{d}$-geodesic ray that converges to T'. ☐

B.3.5 Busemann points

Let X be a (possibly asymmetric) quasi-proper metric space. A path $\gamma : \mathbb{R}_+ \to X$ is an *almost geodesic ray* if for all $\epsilon > 0$, there exists $t_0 \in \mathbb{R}_+$ such that for all $s, t \geq t_0$, we have $|d(\gamma(0), \gamma(s)) + d(\gamma(s), \gamma(t)) - t| \leq \epsilon$. Rieffel proved that every almost geodesic ray converges to a point in $X(\infty)$ [Rie02, Theorem 4.7]. A horofunction is called a *Busemann point* if there exists an almost geodesic converging to it. We denote by $X_B(\infty)$ the subspace of $X(\infty)$ consisting of Busemann points.

Walsh showed that in the case of the Teichmüller space of a surface, equipped with Thurston's asymmetric metric, all horofunctions are Busemann points, since they are limits of *stretch lines*, which are geodesics for Thurston's metric, see [Wal11, Theorem 4.1]. We prove the following characterization of Busemann points in the horoboundary of outer space. Given a tree $T \in \overline{CV_N}^{prim}$, we denote by ψ_T the corresponding horofunction.

Theorem B.3.13. *For all $T \in \overline{CV_N}^{prim}$, the following assertions are equivalent.*

- *The tree T has dense orbits.*
- *The horofunction ψ_T is a Busemann point.*
- *The horofunction ψ_T is the limit of a $\widehat{d}$-geodesic ray in $\widehat{CV_N}$.*
- *The horofunction ψ_T is unbounded from below.*

Proof. It follows from [Wal11, Lemma 5.2] that horofunctions corresponding to Busemann points are unbounded from below.

Let $b \in CV_N$, and let $T \in \overline{CV_N}^{prim}$ be a tree with dense orbits. Theorem B.3.12 gives the existence of a $\widehat{d}$-geodesic γ starting at b and converging to T in $\widehat{CV_N}$. By slightly perturbing γ, we will construct an almost geodesic ray staying in CV_N and converging to T.

We define by induction a sequence $(\gamma'(n))_{n\in\mathbb{N}} \in CV_N^{\mathbb{N}}$ satisfying $\widehat{d}(\gamma'(n), \gamma(n)) \leq \frac{1}{n}$ for all $n \in \mathbb{N}$, and $n - k - \frac{2}{k} \leq d(\gamma'(k), \gamma'(n)) \leq n - k + \frac{2}{k}$ for all $k < n$, in the following way. We let $\gamma'(0) := \gamma(0)$. Let now $n \in \mathbb{N}$, and assume that $\gamma'(k)$ has already been defined for all $k < n$. Since $\widehat{d}(\gamma'(k), \gamma(k)) \leq \frac{1}{k}$ for all $k < n$, and as γ is a $\widehat{d}$-geodesic ray in $\widehat{CV_N}$, by the triangle inequality, we have $\widehat{d}(\gamma'(k), \gamma(n)) \leq n - k + \frac{1}{k}$. By definition of $\widehat{d}$, we can choose $\gamma'(n) \in CV_N$ so that

- we have $\widehat{d}(\gamma'(n), \gamma(n)) \leq \frac{1}{n}$, and
- for all $k < n$, we have $d(\gamma'(k), \gamma'(n)) \leq n - k + \frac{2}{k}$, and
- we have $n - \frac{1}{n} \leq d(\gamma'(0), \gamma'(n)) \leq n + \frac{1}{n}$.

The triangle inequality then ensures that for all $k \leq n$, we have

$$\begin{aligned} d(\gamma'(k), \gamma'(n)) &\geq d(\gamma'(0), \gamma'(n)) - d(\gamma'(0), \gamma'(k)) \\ &\geq n - k - \tfrac{2}{k}. \end{aligned}$$

We then extend γ' to a piecewise-geodesic ray $\gamma' : \mathbb{R}_+ \to CV_N$ by adding a geodesic segment joining $\gamma'(n)$ to $\gamma'(n+1)$ for all $n \in \mathbb{N}$. Let $t_0 \in \mathbb{R}$ be such that $\frac{7}{\lfloor t_0 \rfloor} \leq \epsilon$. For all $t_0 \leq s \leq t$, letting $n := \lfloor s \rfloor$ and $m := \lfloor t \rfloor$, the sum $d(\gamma'(0), \gamma'(s)) + d(\gamma'(s), \gamma'(t))$ is bounded above by

$$d(\gamma'(0), \gamma'(n)) + d(\gamma'(n), \gamma'(n+1)) + d(\gamma'(n+1), \gamma'(m)) + d(\gamma'(m), \gamma'(t)) \leq t + \epsilon,$$

and on the other hand we have

$$\begin{aligned} d(\gamma'(0), \gamma'(s)) + d(\gamma'(s), \gamma'(t)) &\geq d(\gamma'(0), \gamma'(t)) \\ &\geq d(\gamma'(0), \gamma'(m+1)) - d(\gamma'(t), \gamma'(m+1)) \\ &\geq t - \epsilon. \end{aligned}$$

Hence γ' is an almost geodesic ray. In particular it converges to some $\xi \in \overline{CV_N}^{prim}$, and $\xi = T$ by construction. Hence T is a Busemann point.

If T does not have dense orbits, then we can choose a representative $\widetilde{T} \in \overline{cv_N}$ of quotient volume 1. As T is minimal, for all $x \in CV_N$ (which we identify with its covolume 1 representative), any F_N-equivariant map from x to $\widetilde{T}$ has Lipschitz constant at least 1. Hence for all $x \in CV_N$, we have

$$\xi_T(x) \geq \log \frac{1}{\mathrm{Lip}(b, \widetilde{T})},$$

so ξ_T is bounded below. □

Hence the horoboundary of outer space is naturally partitioned into three subsets, namely

- trees having dense orbits, which coincide with the set of Busemann points, i.e. those points that are limits of almost geodesic rays (or of geodesic rays in the completion $\widehat{CV_N}$), and
- trees without dense orbits and with trivial arc stabilizers, which coincide with completion points, i.e. those points that are limits of Cauchy sequences, and
- trees having nontrivial arc stabilizers.

B.4 Geodesic currents and the backward horoboundary of outer space

As d is not symmetric, we can also consider the horocompactification of outer space for the metric d^{back} defined by $d^{back}(X, Y) = d(Y, X)$ for all $X, Y \in CV_N$, which satisfies the hypotheses of Proposition B.2.1 as d does. We denote by $\overline{CV_N}^{back}$ this compactification of outer space. In this section, we investigate some properties of $\overline{CV_N}^{back}$, which show that $\overline{CV_N}^{prim}$ and $\overline{CV_N}^{back}$ are rather different in nature. It seems that there is some kind of duality between the two compactifications. Having a more explicit description of this duality and of the backward horocompactification would be of interest. For example, is the backward horocompactification isomorphic to Reiner Martin's compactification of outer space [Mar95, Section 6.3] ? The same question is also still open in the context of Teichmüller spaces equipped with Thurston's asymmetric metric. We start by recalling the notion of geodesic currents on F_N.

B.4.1 Geodesic currents

Let $\partial^2 F_N := \partial F_N \times \partial F_N \smallsetminus \Delta$, where Δ is the diagonal, and denote by $i : \partial^2 F_N \to \partial^2 F_N$ the involution that exchanges the factors. A *current* on F_N is an F_N-invariant Borel measure ν on $\partial^2 F_N$ that is finite on compact subsets of $\partial^2 F_N$, see [Kap05, Kap06]. We denote by $Curr_N$ the space of currents on F_N, equipped with the weak-$*$ topology, and by $\mathbb{P}Curr_N$ the space of projective classes (i.e. homothety classes) of currents.

To every $g \in F_N$ which is not of the form h^k for any $h \in F_N$ and $k > 1$ (we say that g is not a *proper power*), one associates a *rational current* $[g]$ by letting $[g](S)$ be the number of translates of $(g^{-\infty}, g^{+\infty})$ that belong to S (where $g^{-\infty} := \lim_{n\to+\infty} g^{-n}$ and $g^{+\infty} := \lim_{n\to+\infty} g^n$) for all clopen subsets $S \subseteq \partial^2 F_N$, see [Kap06, Definition 5.1]. For the case of proper powers, one may set $[h^k] := k[h]$. The group $\mathrm{Out}(F_N)$ acts on $Curr_N$ on the left in the following way [Kap06, Proposition 2.15]. Given a compact set $K \subseteq \partial^2 F_N$, an element $\Phi \in \mathrm{Out}(F_N)$, and a current $\nu \in Curr_N$, we set $\Phi(\nu)(K) := \nu(\phi^{-1}(K))$, where $\phi \in \mathrm{Aut}(F_N)$ is any representative of Φ. The action of $\mathrm{Out}(F_N)$ on $\mathbb{P}Curr_N$ is not minimal, but there is a unique closed (hence compact), minimal, $\mathrm{Out}(F_N)$-invariant subset $\mathbb{P}M_N \subseteq \mathbb{P}Curr_N$, which is the closure of rational currents associated to primitive conjugacy classes of F_N, see [KL09, Theorem B]. We denote by M_N the lift of $\mathbb{P}M_N$ to $Curr_N$. In [Kap06, Section 5], Kapovich defined an intersection form between elements of cv_N and currents, which was then extended by Kapovich and Lustig to trees in $\overline{cv_N}$ [KL09].

Theorem B.4.1. *(Kapovich–Lustig [KL09, Theorem A]) There exists a unique Out(F_N)-invariant continuous function*

$$\langle .,. \rangle : \overline{cv_N} \times Curr_N \to \mathbb{R}_+$$

which is $\mathbb{R}_+$-homogeneous in the first coordinate and $\mathbb{R}_+$-linear in the second, and such that for all $T \in \overline{cv_N}$, and all $g \in F_N \smallsetminus \{e\}$, we have $\langle T, [g] \rangle = ||g||_T$.

Two currents $\mu, \mu' \in \mathrm{Curr}_N$ are *translation-equivalent* if for all $T \in cv_N$, we have $\langle T, \mu \rangle = \langle T, \mu' \rangle$. This descends to an equivalence relation $\sim$ on $\mathbb{P}\mathrm{Curr}_N$ by setting $[\mu] \sim [\nu]$ if there exist representatives μ and ν which are translation-equivalent.

Let A be a free basis of F_N. Let w be a nontrivial cyclically reduced word written in the basis A. The *Whitehead graph* of w in the basis A is the labelled undirected graph $Wh_A(w)$ defined as follows. The vertex set of $Wh_A(w)$ is $A^{\pm 1}$. For all $x \neq y \in A^{\pm 1}$, there is an edge between x and y labelled by the number of occurrences of the word xy^{-1} in the cyclic word w, i.e. the number of those $i \in \{0, \dots, |w|-1\}$ such that the infinite word $www\dots$ begins with $w_i xy^{-1}$, where w_i is the initial segment of w of length i. The *Whitehead graph* of a nontrivial conjugacy class $[g]$ of elements of F_N in the basis A is $Wh_A([g]) := Wh_A(w)$, where w is any cyclically reduced word representing $[g]$ in A. More generally, given a linear combination $\eta = \lambda_1[g_1] + \dots + \lambda_k[g_k]$ of rational currents, the *Whitehead graph* $Wh_A(\eta)$ is the labelled undirected graph defined as follows. The vertex set of $Wh_A(\eta)$ is $A^{\pm 1}$. For all $x \neq y \in A^{\pm 1}$, there is an edge between x and y labelled by $\lambda_1\alpha_1 + \dots + \lambda_k\alpha_k$, where α_i is the label of the edge between x and y in $Wh_A([g_i])$. The following proposition was proven in [KLSS07] in the case of conjugacy classes of F_N (i.e. rational currents), however its proof still works in the case of linear combinations of rational currents.

Proposition B.4.2. *(Kapovich–Levitt–Schupp–Shpilrain [KLSS07, Theorem A]) Let $\eta, \eta' \in Curr_N$ be two linear combinations of rational currents. Then η and η' are translation-equivalent if and only if for all free bases A of F_N, we have $Wh_A(\eta) = Wh_A(\eta')$.*

Proposition B.4.3. *Let $x_1, x_2 \in F_N$ be two elements that belong to a common free basis. Let $\eta, \eta' \in Span\{[x_1x_2^i]\}_{i\in\mathbb{N}}$. If η and η' are translation-equivalent, then $\eta = \eta'$.*

Proof. Let $\{x_1, x_2, \dots, x_N\}$ be a free basis of F_N that contains x_1 and x_2. There exists $k \in \mathbb{N}$ and real numbers $\lambda_1, \dots, \lambda_k, \lambda'_1, \dots, \lambda'_k$ such that $\eta = \sum_{i=1}^k \lambda_i[x_1x_2^i]$ and $\eta' = \sum_{i=1}^k \lambda'_i[x_1x_2^i]$. Assume that $\eta \neq \eta'$, and let $i \in \{1, \dots, k\}$ be such that $\lambda_i \neq \lambda'_i$. The set $B := \{x_1x_2^i, x_2, \dots, x_N\}$ is a free basis of F_N. The edge joining $x_1x_2^i$ to $(x_1x_2^i)^{-1}$ has label λ_i in $Wh_B(\eta)$ and λ'_i in $Wh_B(\eta')$, so Proposition B.4.2 implies η and η' are not translation-equivalent, a contradiction. Hence $\eta = \eta'$. ☐

We notice the following property of the currents we considered in Proposition B.4.3.

Proposition B.4.4. *Let $N \geq 3$, and let $x_1, x_2 \in F_N$ be two elements that belong to a common free basis of F_N. Then $Span\{[x_1 x_2^i]\}_{i\in\mathbb{N}} \subseteq M_N$.*

Proof. Let $F < F_N$ be the free factor generated by x_1 and x_2. By [Kap06, Proposition 12.1], there is a linear topological embedding $\iota : Curr(F) \to Curr_N$ such that for all $g \in F$, we have $\iota([g]) = [g]$. In particular, the subspace $\mathrm{Span}_{Curr_N}\{[x_1 x_2^i]\}_{i\in\mathbb{N}}$ identifies with the image $\iota(\mathrm{Span}_{Curr(F)}\{[x_1 x_2^i]\}_{i\in\mathbb{N}})$. The proof of [KL07, Proposition 4.3] thus shows that $\mathrm{Span}_{Curr_N}\{[x_1 x_2^i]\}_{i\in\mathbb{N}} \subseteq M_N$. □

B.4.2 The backward horoboundary of outer space

We recall that b denotes some fixed basepoint in CV_N. For all $z \in CV_N$, we define the function

$$\begin{array}{cccc} \psi_z^{back}: & CV_N & \to & \mathbb{R} \\ & x & \mapsto & d(z,x) - d(z,b). \end{array}$$

Given a finite set $S \subseteq M_N$, we define a function f_S on CV_N by setting

$$f_S(T) := \log \frac{\sup_S \langle T, \mu\rangle}{\sup_S \langle b, \mu\rangle}$$

for all $T \in CV_N$.

Proposition B.4.5. *For all $\xi \in \overline{CV_N}^{back}$, there exists a finite set $S \subseteq M_N$ such that $\xi = f_S$.*

Proof. Let $\xi \in \overline{CV_N}^{back}$, and let $(T_n)_{n\in\mathbb{N}} \in CV_N^{\mathbb{N}}$ be a sequence of elements of CV_N that converges to ξ. For all $n \in \mathbb{N}$, let $\mathcal{F}_n$ denote the set of candidates in T_n. There is a uniform bound on the cardinality of $\mathcal{F}_n$. Up to passing to a subsequence, we can thus assume that there exists $k \in \mathbb{N}$, currents $\eta^1, \dots, \eta^k \in M_N$, and sequences $(g_n^i)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathcal{F}_n$ and $(\lambda_n^i)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ for all $i \in \{1, \dots, k\}$, such that the sequence $(\lambda_n^i [g_n^i])_{n\in\mathbb{N}}$ converges (non-projectively) to the current η^i, and for all $n \in \mathbb{N}$, we have $\mathcal{F}_n = \{g_n^i\}_{1\leq i\leq k}$. For all $n \in \mathbb{N}$ and all $i, j \in \{1, \dots, k\}$, we set

$$\alpha_n^{i,j} := \frac{\lambda_n^i \langle T_n, [g_n^i]\rangle}{\lambda_n^j \langle T_n, [g_n^j]\rangle}.$$

Up to passing to a subsequence again, we may assume that for all $i, j \in \{1, \dots, k\}$, the sequence $(\alpha_n^{i,j})_{n\in\mathbb{N}}$ converges in $\mathbb{R} \cup \{+\infty\}$. Denoting its limit by $\alpha^{i,j}$, we can find $i_0 \in \{1, \dots, k\}$ such that for all $j \in \{1, \dots, k\}$, we have $\alpha^{i_0,j} < +\infty$. We set

$$S := \{\alpha^{i_0,j}\eta^j | \alpha^{i_0,j} \neq 0\}.$$

For all $T \in CV_N$ and all $n \in \mathbb{N}$, we have

$$\begin{aligned} \psi_{T_n}^{back}(T) &= \log(\sup_j \tfrac{\langle T,[g_n^j]\rangle}{\langle T_n,[g_n^j]\rangle}) - \log(\sup_j \tfrac{\langle b,[g_n^j]\rangle}{\langle T_n,[g_n^j]\rangle}) \\ &= \log(\sup_j \tfrac{\lambda_n^{i_0}\langle T_n,[g_n^{i_0}]\rangle}{\lambda_n^j\langle T_n,[g_n^j]\rangle} \tfrac{\lambda_n^j\langle T,[g_n^j]\rangle}{\lambda_n^{i_0}\langle T_n,[g_n^{i_0}]\rangle}) - \log(\sup_j \tfrac{\lambda_n^{i_0}\langle T_n,[g_n^{i_0}]\rangle}{\lambda_n^j\langle T_n,[g_n^j]\rangle} \tfrac{\lambda_n^j\langle b,[g_n^j]\rangle}{\lambda_n^{i_0}\langle T_n,[g_n^{i_0}]\rangle}) \\ &= \log(\sup_j \alpha_n^{i_0,j}\lambda_n^j\langle T,[g_n^j]\rangle) - \log(\sup_j \alpha_n^{i_0,j}\lambda_n^j\langle b,[g_n^j]\rangle), \end{aligned}$$

which tends to $f_S(T)$ as n goes to $+\infty$. Hence $\xi = f_S$. □

Remark B.4.6. It follows from the proof of Proposition B.4.5 that if a sequence $(T_n)_{n\in\mathbb{N}} \in CV_N^{\mathbb{N}}$ converges to a horofunction f_S in the backward horoboundary of CV_N, then all currents in S are dual to all limit points of $(T_n)_{n\in\mathbb{N}}$ in $\overline{CV_N}$.

Proposition B.4.7. *There exists a topological embedding from* $\mathbb{P}M_N/\sim$ *to* $\overline{CV_N}^{back}$.

Proof. Let $\eta \in M_N$, and let $(g_n)_{n\in\mathbb{N}} \in \mathcal{P}_N^{\mathbb{N}}$ be a sequence of primitive elements so that the rational currents $[g_n]$ projectively converge to η. For all $n \in \mathbb{N}$, let $T_n \in CV_N$ be a rose, one of whose petals is labelled by g_n. By making the length of this petal arbitrarily small, we can ensure, with the notations from the proof of Proposition B.4.5 (where we assume that $g_n^1 := g_n$), that $\alpha^{1,j} = 0$ for all $j > 1$. This implies that the functions $\psi_{T_n}^{back}$ converge pointwise (and hence uniformly on compact sets of d_{sym}) to $f_{\{\eta\}}$. Therefore, we get an injective map from the compact space $\mathbb{P}M_N/\sim$ to the Hausdorff space $\overline{CV_N}^{back}$, which is continuous by continuity of the intersection form (Theorem B.4.1). ☐

Corollary B.4.8. *For all* $N \geq 3$*, the space* $\overline{CV_N}^{back}$ *has infinite topological dimension.*

Proof. Let $\{x_1, \dots, x_N\}$ be a free basis of F_N. Propositions B.4.3, B.4.4 and B.4.7 show that $\overline{CV_N}^{back}$ contains an embedded copy of the infinite-dimensional projective space spanned by all currents of the form $[x_1x_2^i]$ for $i \in \mathbb{N}$. The claim follows. ☐

B.4.3 The backward horocompactification of CV_2

We finish this section by giving a description of the backward horocompactification of CV_2. Culler and Vogtmann gave in [CV91] an explicit description of $\overline{CV_2}$, and an explicit description of the primitive compactification $\overline{CV_2}^{prim}$ was given in [3, Section 2.2]. We will show that the backward horocompactification of CV_2 is 2-dimensional, homeomorphic to a disk with fins attached, see Figure B.3. The reduced part of this compactification, obtained by collapsing the fins, is isomorphic to the reduced part of the forward horocompactification. However, when we include the fins, there are examples of sequences of trees that converge to a point in the forward horoboundary, but not in the backward horoboundary, and vice versa.

Every outer automorphism of F_2 can be realized by a mapping class of a torus with one boundary component. The set of rational geodesic currents associated to essential simple closed curves on the surface that are not isotopic to the boundary curve is $\mathrm{Out}(F_2)$-invariant, and its closure consists of currents associated to measured laminations on the surface. Every such lamination is either minimal and filling, or a simple closed curve on the surface. The minimal set of currents M_2 consists of currents of this form. Currents η associated to filling laminations are dual to a unique tree T, and by unique ergodicity the current η is then the unique current in M_2 dual to T, up to homothety. This implies in particular that currents in M_2 that are dual to simplicial trees are rational currents corresponding to primitive elements of F_2.

Let $(T_n)_{n\in\mathbb{N}} \in CV_2^{\mathbb{N}}$ be a sequence that converges to a horofunction f_S in $\overline{CV_2}^{back}$. Up to passing to a subsequence, we can assume that $(T_n)_{n\in\mathbb{N}}$ also converges to a tree $T \in \overline{CV_2}$. By [CV91], the tree T is either simplicial, or dual to an arational measured foliation on a torus with one boundary component. The list of simplicial trees in ∂CV_2 is displayed on Figure B.1.

First assume that T is a simplicial metric tree, whose quotient graph T/F_2 has one of the first four shapes displayed on Figure B.1. It then follows from the above description of M_2 that there is a unique projectivized current $\eta \in M_2$ that is dual to T (which is a

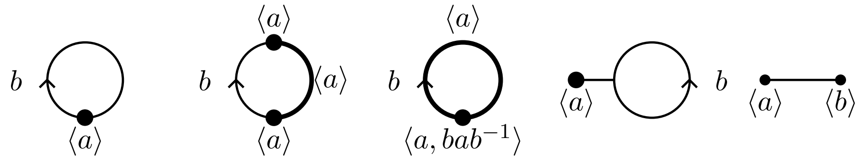

Figure B.1: The simplicial trees in ∂CV_2.

rational current, associated to a). This is clear except in the case where the quotient graph T/F_2 has the third shape displayed on Figure B.1. In this case, since T is simplicial, all currents in M_2 dual to T are rational. In addition, it follows from [3, Proposition 2.8] that the only primitive elements of F_2 contained in the subgroup $\langle a, bab^{-1}\rangle$ are conjugate to a. Remark B.4.6 implies that $S = \{[a]\}$.

Notice in particular that all sequences of trees that converge to a simplicial metric tree T of the fourth type in $\overline{CV_2}$ converge to the same horofunction $f_{\{[a]\}}$ in $\overline{CV_2}^{back}$, regardless of the ratio between the lengths of the separating edge and the nonseparating edge in the quotient graph T/F_2. Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of Bass–Serre trees of barbell graphs whose petals are labelled by a and b, where the length of the petal labelled by a (respectively by b) converges to 0 (resp. to $l \in (0,1]$) and whose separating edge has a length converging to $1-l$. Then $(T_n)_{n\in\mathbb{N}}$ converges to $f_{\{[a]\}}$ in $\overline{CV_2}^{back}$, and it converges in $\overline{CV_2}^{prim}$ to the Bass–Serre tree of a graph having the fourth shape displayed on Figure B.1, whose separating edge has length $1-l$, and whose nonseparating edge has length l. By making l vary, we get examples of sequences of trees that converge to the same point in $\overline{CV_2}^{back}$, but to distinct trees in $\overline{CV_2}^{prim}$.

Also notice that all sequences of trees that converge in $\overline{CV_2}$ to a simplicial tree T having one of the first three shapes displayed, converge to the same horofunction $f_{\{[a]\}}$ in $\overline{CV_2}^{back}$. All these trees are also identified by the quotient map $\overline{CV_2} \to \overline{CV_2}^{prim}$.

Now assume that T is dual to an arational measured foliation on a torus with a single boundary component. As noticed above, there is a unique projectivized current $\eta \in \mathbb{P}M_2$ that is dual to T. Remark B.4.6 implies that $S = \{\eta\}$.

In the remaining case where T is the Bass–Serre tree of a splitting of the form $F_2 = \langle a\rangle * \langle b\rangle$, there are exactly 2 projectivized currents in $\mathbb{P}M_2$ that are dual to T: these are the currents whose lifts to M_2 are $[a]$ and $[b]$. We claim that the set S may consist of any pair of the form $\{\lambda_1[a], \lambda_2[b]\}$, where we may assume that $\lambda_1+\lambda_2=1$ because multiplying all currents by a same factor does not change the map f_S. Indeed, first assume that $\lambda_1, \lambda_2 > 0$. For all $n \in \mathbb{N}$, we let T_n be the Bass–Serre tree of a barbell graph, whose central edge has length 1, and whose loops are labelled by a and b and have respective lengths $\frac{1}{\lambda_1 n}$ and $\frac{1}{\lambda_2 n}$. Then $(T_n)_{n\in\mathbb{N}}$ converges to $f_{\{\lambda_1[a],\lambda_2[b]\}}$ in $\overline{CV_2}^{back}$. If $\lambda_1 = 0$ and $\lambda_2 = 1$, then we let T_n be a barbell whose loops are labelled by a and b and have respective lengths 1 and $\frac{1}{n}$ to get the desired convergence in $\overline{CV_2}^{back}$. In all cases, the sequence $(T_n)_{n\in\mathbb{N}}$ converges in $\overline{CV_2}^{prim}$ to the Bass–Serre tree of the splitting $F_2 = \langle a\rangle * \langle b\rangle$. This provides examples of sequences of trees that converge to the same point in $\overline{CV_2}^{prim}$, but to distinct points in $\overline{CV_2}^{back}$. The closure of the simplex of a barbell graph in $\overline{CV_2}^{back}$ is displayed on Figure B.2.

We claim that the backward horoboundary $\overline{CV_2}^{back}$ is isomorphic to the forward horoboundary, in which the closures of the simplices of barbell graphs have been replaced

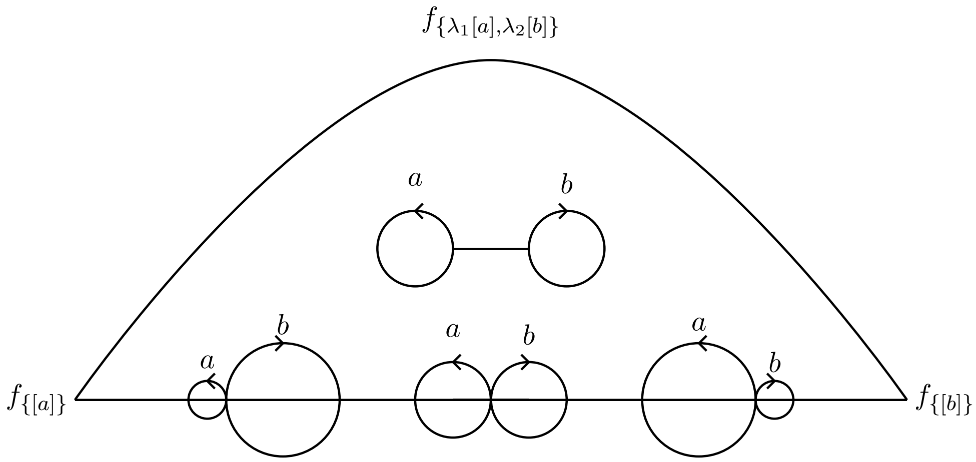

Figure B.2: The closure of the simplex of a barbell graph in $\overline{CV_2}^{back}$.

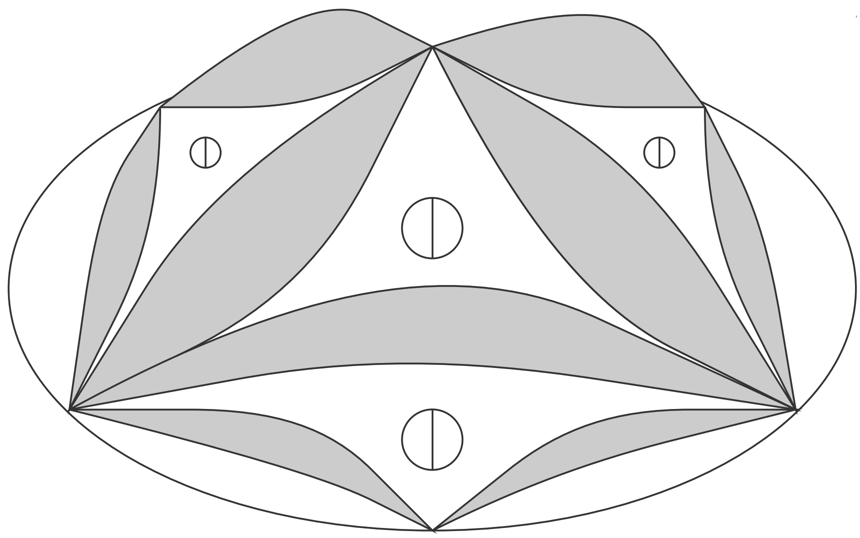

Figure B.3: The backward horocompactification of CV_2.

by simplices having the shape displayed on Figure B.2. Indeed, there is a bijection between $\mathbb{P}M_2$ and the set of simplicial trees in $\overline{CV_2}^{prim}$ that do not contain any separating edge. It thus follows from the above that a sequence $(T_n)_{n\in\mathbb{N}} \in CV_2^{\mathbb{N}}$ converges to a horofunction $f_{\{\eta\}}$ with $\eta \in M_2$ if and only if all its limit points in $\overline{CV_2}^{prim}$ are dual to η. From this observation, one deduces that the *reduced* parts of $\overline{CV_2}^{back}$ and $\overline{CV_2}^{prim}$ (obtained by forgetting trees in CV_2 whose quotient graphs contain a separating edge) coincide. A sequence in $\overline{CV_2}^{back}$ can converge to a horofunction of the form $f_{\{\lambda_1[a],\lambda_2[b]\}}$ only if it eventually stays in the corresponding simplex. We note that a sequence $(T_n)_{n\in\mathbb{N}} \in CV_2^{\mathbb{N}}$ of barbell graphs with petals labelled by a and b converges to $f_{\{\lambda_1[a],\lambda_2[b]\}}$ if and only if the ratio $\frac{||b||_{T_n}}{||a||_{T_n}}$ converges to $\frac{\lambda_1}{\lambda_2}$. One also checks that all horofunctions described above are pairwise distinct.

B.5 Growth of elements of F_N under random products of automorphisms

In this section, we will use our description of the horoboundary of outer space to derive results about random products of outer automorphisms of a finitely generated free group, through the study of the possible growth rates of elements of F_N under such products. This is inspired from Karlsson's analogous work for random products of mapping classes of a surface [Kar14].

B.5.1 Background on ergodic cocycles

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a standard probability Lebesgue space, and $T : \Omega \to \Omega$ an ergodic measure-preserving transformation. Let $\phi : \Omega \to \mathrm{Out}(F_N)$ be a measurable map. We call

$$\Phi_n(\omega) := \phi(T^{n-1}\omega)\dots\phi(\omega)$$

an *ergodic cocycle.* We say that it is *integrable* if

$$\int_\Omega d^{sym}_{CV_N}(\phi(\omega)b, b) < +\infty,$$

where we recall that b is any basepoint in CV_N. The case where $\Omega = (\mathrm{Out}(F_N)^{\mathbb{N}}, \mu^{\otimes\mathbb{N}})$ is a product probability space (here μ denotes a probability measure on $\mathrm{Out}(F_N)$), and T is the shift operator, corresponds to the *left random walk* on $(\mathrm{Out}(F_N), \mu)$. This is the Markov chain on $\mathrm{Out}(F_N)$ whose initial distribution is the Dirac measure at the identity, and with transition probabilities $p(x, y) := \mu(yx^{-1})$. In other words, the position of the random walk at time n is given from its initial position $\Phi_0 = \mathrm{id}$ by successive multiplications on the left of independent μ-distributed increments ϕ_i, i.e. $\Phi_n = \phi_n \dots \phi_1$.

The *drift* of an integrable ergodic cocycle Φ_n is defined as

$$l := \lim_{n\to+\infty} \frac{1}{n} d(b, \Phi_n(\omega)^{-1}b),$$

which almost surely exists by Kingman's subadditive ergodic theorem [Kin68], and is independent of ω by ergodicity of T.

B.5.2 Growth of elements of F_N under cocycles of automorphisms

Given an element $g \in F_N$, we denote by $||g||$ the length of the cyclically reduced word that represents the conjugacy class of g in some fixed basis of F_N (word lengths with respect to two different bases are bi-Lipschitz equivalent). We will show the following theorem.

Theorem B.5.1. *Let $\Phi_n = \phi_n \dots \phi_1$ be an integrable ergodic cocycle of elements of Out(F_N), and let l be its drift. For $\mathbb{P}$-a.e. ω, there exists a (random) tree $T(\omega) \in \overline{CV_N}$ such that*

- *for all $g \in F_N$ which are hyperbolic in $T(\omega)$, we have*

$$\lim_{n\to+\infty} \frac{1}{n} \log ||\Phi_n(\omega)(g)|| = l;$$

- *for all $g \in F_N$ which are elliptic in $T(\omega)$, we have*

$$\limsup_{n\to+\infty} \frac{1}{n} \log ||\Phi_n(\omega)(g)|| \leq l.$$

As in [Kar14], Theorem B.5.1 follows from the following (more precise) quantitative version.

Theorem B.5.2. *Let $\Phi_n = \phi_n \dots \phi_1$ be an integrable ergodic cocycle of elements of $Out(F_N)$, and let l be its drift. For $\mathbb{P}$-a.e. ω, there exist a (random) constant $C(\omega) > 0$ and a (random) tree $T(\omega) \in \overline{CV_N}$ such that for all $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $g \in F_N$, we have*

$$C(\omega)||g||_{T(\omega)}e^{n(l-\epsilon)} \leq ||\Phi_n(\omega)(g)|| \leq ||g||e^{n(l+\epsilon)}.$$

Our proof of Theorems B.5.1 and B.5.2 relies on the following theorem of Karlsson and Ledrappier [KL06], which was originally stated for symmetric metric spaces. The extension to the case of an asymmetric metric is due to Karlsson [Kar14].

Theorem B.5.3. *(Karlsson–Ledrappier [KL06]) Let T be a measure-preserving transformation of a Lebesgue probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let G be a locally compact group acting by isometries on a (possibly asymmetric) quasi-proper metric space X, let $b \in X$, and let $\phi : \Omega \to G$ be a measurable map satisfying*

$$\int_\Omega d^{sym}(\phi(\omega)b, b)d\mathbb{P}(\omega) < +\infty.$$

Let Φ_n be the associated integrable ergodic cocycle. Then, for $\mathbb{P}$-almost every ω, there exists $\xi_\omega \in X(\infty)$ such that

$$\lim_{n\to+\infty} -\frac{1}{n}\xi_\omega(\Phi_n(\omega)^{-1}b) = l.$$

Proof of Theorems B.5.1 and B.5.2. We may choose as a basepoint $b \in CV_N$ a Cayley tree of F_N with respect to our fixed free basis of F_N, so that for all $g \in F_N$, we have $||g||_b = ||g||$. Let $\Phi_n = \phi_n \dots \phi_1$ be an integrable ergodic cocycle of elements of $\mathrm{Out}(F_N)$. Theorem B.5.3 ensures that for almost every ω, there exists $\xi = \xi(\omega)$, associated to a tree $T = T(\omega) \in \overline{CV_N}$, such that

$$\lim_{n\to+\infty} -\frac{1}{n}\xi(\Phi_n(\omega)^{-1}b) = l$$

(if $l = 0$ then we can choose $T(\omega) \in CV_N$, while if $l > 0$, then ξ is unbounded, and hence $T(\omega) \in CV_N(\infty)$ is a tree with dense orbits by Theorem B.3.13, in particular the $\sim$-class of T in $\overline{CV_N}$ is reduced to a point). Using the expression of horofunctions given by Theorem B.2.2, for all $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ one has

$$\log \sup_{g\in F_N \smallsetminus \{e\}} \frac{||g||_T}{||g||_{\Phi_n^{-1}b}} - \log \sup_{g\in F_N \smallsetminus \{e\}} \frac{||g||_T}{||g||_b} \leq -(l-\epsilon)n.$$

Letting

$$C(\omega)^{-1} := \sup_{g\in F_N \smallsetminus \{e\}} \frac{||g||_T}{||g||_b},$$

we obtain

$$\sup_{g \in F_N} \frac{||g||_T}{||g||_{\Phi_n^{-1}b}} \leq C(\omega)^{-1} e^{-(l-\epsilon)n},$$

so for all $g \in F_N$ we have

$$||\Phi_n(g)||_b = ||g||_{\Phi_n^{-1}b} \geq C(\omega)||g||_T e^{(l-\epsilon)n}.$$

As

$$\lim_{n \to +\infty} \frac{1}{n} d(b, \Phi_n^{-1}b) = l,$$

for all sufficiently large $n \in \mathbb{N}$ and all $g \in F_N$, we also have

$$||\Phi_n(g)||_b \leq ||g||_b e^{(l+\epsilon)n}.$$

This shows Theorem B.5.2. In view of these two inequalities we have

$$\lim_{n \to +\infty} \frac{1}{n} \log ||\Phi_n(g)|| = l$$

for all $g \in F_N$ which are hyperbolic in T, showing Theorem B.5.1. □

B.5.3 The case of a random walk on a nonelementary subgroup of Out(F_N)

A subgroup of Out(F_N) is *nonelementary* if it does not virtually fix the conjugacy class of any finitely generated subgroup of F_N of infinite index. In the case of independent increments (i.e. Ω is a product probability space, and T is the shift operator), Theorem B.5.1 specifies as follows. The following corollary is analogous to a theorem of Furstenberg for random products of matrices [Fur63a], and to Karlsson's theorem for random products of elements of the mapping class group of a surface [Kar14, Corollary 4].

Corollary B.5.4. *Let μ be a probability measure on Out(F_N), whose support generates a nonelementary subgroup of Out(F_N). For all $g \in F_N$, and almost every sample path $(\Phi_n)_{n \in \mathbb{N}}$ of the random walk on (Out(F_N), μ), we have*

$$\lim_{n \to +\infty} \frac{1}{n} \log ||\Phi_n(g)|| = l,$$

where l is the drift of the random walk.

Proof. In view of Theorem B.5.2, it is enough to show that the tree $T(\omega)$ associated to the horofunction provided by Karlsson and Ledrappier's theorem, can almost surely be chosen to be free. This will be a consequence of Propositions B.5.6 and B.5.8. □

Remark B.5.5. When the random walk on (Out(F_N), μ) has positive drift with respect to the Lipschitz distance on Out(F_N), we therefore get that all elements of F_N almost surely have exponential growth along the sample path of the random walk, with the same exponential growth rate. Positivity of the drift is discussed in Section B.5.3.

Our proof of Corollary B.5.4 relies on the following refinement of Karlsson and Ledrappier's theorem in the case of independent increments. The following statement was noticed by Karlsson in [Kar14, Section 2], and follows from the proof of [KL11b, Theorem 18].

Proposition B.5.6. *(Karlsson [Kar14, Section 2]) Let G be a locally compact group acting by isometries on a (possibly asymmetric) quasi-proper metric space, and let μ be a probability measure on G with finite first moment with respect to d^{sym}. Let $E \subseteq X(\infty)$ be a measurable subset such that for all μ-stationary measures ν on $X \cup X(\infty)$, we have $\nu(E) = 1$. Then for μ-almost every ω, the horofunction ξ_ω from Theorem B.5.3 may be chosen to belong to E.*

Stationary measures in the horoboundary of outer space

Let μ be a probability measure on $\mathrm{Out}(F_N)$. We now aim at understanding some properties of μ-stationary measures on $CV_N(\infty)$. Given a probability measure μ on a countable group G acting on a compact space X, there always exists a μ-stationary Borel probability measure on X, obtained as a weak limit of the Cesàro averages of the convolution of μ^{*n} and any Borel probability measure on X (see [Fur73] or [KM96, Lemma 2.2.1]). Compactness of $CV_N(\infty)$ thus yields the following lemma.

Lemma B.5.7. *Let μ be a probability measure on $Out(F_N)$. Then there exists a μ-stationary measure on $CV_N(\infty)$.* □

The following statement is essentially proved in [6, Proposition 2.4], we sketch a proof for completeness. We recall that we have associated a *canonical* representative to every class of primitive-equivalence in Section B.1.3.

Proposition B.5.8. *(Horbez [6, Proposition 2.4]) Let μ be a probability measure on $Out(F_N)$, and let ν be a μ-stationary measure on $CV_N(\infty)$. Then ν is concentrated on the set of trees $T \in CV_N(\infty)$ such that all conjugacy classes of point stabilizers in the canonical lift of T to $\overline{CV_N}$ have finite $gr(\mu)$-orbits. In particular, if $gr(\mu)$ is nonelementary, then every μ-stationary measure on $CV_N(\infty)$ is concentrated on the set of free F_N-actions.*

Our proof of Proposition B.5.8 makes use of the following classical lemma, whose proof relies on a maximum principle argument.

Lemma B.5.9. *(Ballmann [Bal89], Woess [Woe89, Lemma 3.4], Kaimanovich–Masur [KM96, Lemma 2.2.2], Horbez [5, Lemma 3.3]) Let μ be a probability measure on a countable group G, and let ν be a μ-stationary probability measure on a G-space X. Let D be a countable G-set, and let $\Theta : X \to D$ be a measurable G-equivariant map. If $E \subseteq X$ is a G-invariant measurable subset of X satisfying $\nu(E) > 0$, then $\Theta(E)$ contains a finite $gr(\mu)$-orbit.*

Proof of Proposition B.5.8. Let D be the countable set of all finite collections of conjugacy classes of finitely generated subgroups of F_N. For all $T \in CV_N(\infty)$, we let $\Theta(T)$ be the collection of conjugacy classes of point stabilizers in the canonical lift of T to $\overline{CV_N}$. This set is finite [Jia91] and belongs to D by [GL95, Corollary III.4]. We have $\Theta(T) \neq \emptyset$ as soon as some representative of T in $\overline{CV_N}$ is not free. We now prove that Θ is measurable, which will be enough to conclude by applying Lemma B.5.9 to Θ. We denote by ∂CV_N the boundary of Culler and Morgan's compactification of CV_N. The projection map $\pi : \partial CV_N \to CV_N(\infty)$ is closed, so [Cas67, Theorem V.3] and [CV77, Corollary III.3] imply that there exist countably many measurable maps $f_n : CV_N(\infty) \to \partial CV_N$, so that for all $T \in CV_N(\infty)$, we have $\pi^{-1}(T) = \overline{\{f_n(T) | n \in \mathbb{N}\}}$. Given a conjugacy class $H \in D$, we have $H \in \Theta(T)$ if and only if

- for all $g \in F_N$ which is conjugate into H, and all $n \in \mathbb{N}$, we have $||g||_{f_n(T)} = 0$, and
- for all $g \in F_N$ which is not conjugate into H, there exists $n \in \mathbb{N}$ such that $||g||_{f_n(T)} \neq 0$.

Measurability of Θ follows. □

Drift of a random walk on a nonelementary subgroup of Out(F_N)

The *free factor complex* FF_N, introduced by Hatcher and Vogtmann in [HV98], is defined when $N \geq 3$ as the simplicial complex whose vertices are the conjugacy classes of nontrivial proper free factors of F_N, and higher dimensional simplices correspond to chains of inclusions of free factors. (When $N = 2$, one has to modify this definition by adding an edge between any two complementary free factors to ensure that FF_2 remains connected, and FF_2 is isomorphic to the Farey graph). We equip FF_N with the simplicial metric d_{FF_N}, and we fix a basepoint $*_{FF_N} \in FF_N$ for measuring the drift of the random walk. The following theorem, due to Calegari and Maher [CM12, Section 5.10], relies on work of Maher [Mah11] and on the convergence of almost every sample path of the random walk on $(\mathrm{Out}(F_N), \mu)$ to the Gromov boundary ∂FF_N (which is also established in [6, Theorem 4.2] by other methods).

Theorem B.5.10. *(Calegari–Maher [CM12, Theorem 5.34]) Let μ be a probability measure on Out(F_N), whose support is finite and generates a nonelementary subgroup of Out(F_N) which is not virtually cyclic. Then the random walk on $(Out(F_N), \mu)$ has positive drift with respect to d_{FF_N}.*

Corollary B.5.11. *Let μ be a probability measure on Out(F_N), whose support is finite and generates a nonelementary subgroup of Out(F_N) which is not virtually cyclic. Then the random walk on $(Out(F_N), \mu)$ has positive drift with respect to d_{CV_N}.*

Proof. Corollary B.5.11 follows from Theorem B.5.10 and from the following estimate relating the distances d_{FF_N} and d_{CV_N}. ☐

Proposition B.5.12. *There exist $K, L \in \mathbb{R}$ such that for all $\Phi, \Psi \in Out(F_N)$, we have*

$$d_{FF_N}(\Phi *_{FF_N}, \Psi *_{FF_N}) \leq K d_{CV_N}(\Phi b, \Psi b) + L.$$

Proposition B.5.12 will follow from several distance estimates between various Out(F_N)-complexes, provided by Lemmas B.5.13 and B.5.14 and Proposition B.5.15. We will first introduce yet another Out(F_N)-complex. Let $M_N := \#^N S^1 \times S^2$ be the connected sum of N copies of $S^1 \times S^2$, whose fundamental group is free of rank N. A *sphere system* is a collection of disjoint, embedded 2-spheres in M_N, none of which bounds a ball, and no two of which are isotopic. The *sphere complex* $\mathcal{S}_N$, introduced by Hatcher in [Hat95], is the simplicial complex whose k-simplices are the isotopy classes of systems of $k+1$ spheres in M_N (a $(k-1)$-dimensional face of a k-simplex Δ is obtained by removing one sphere from the sphere system corresponding to Δ). We denote by $\mathcal{S}'_N$ the one-skeleton of the first barycentric subdivision of $\mathcal{S}_N$, which we equip with the simplicial metric $d_{\mathcal{S}'_N}$. Again, we fix a basepoint $*_{\mathcal{S}'_N} \in \mathcal{S}'_N$. There is a coarsely well-defined, coarsely equivariant map $\tau : \mathcal{S}'_N \to FF_N$, that maps a sphere system S to the conjugacy class of the fundamental group of a complementary component in M_N of a sphere in S. The map τ is Lipschitz, so we get the following estimate.

Lemma B.5.13. *There exists $C > 0$ such that for all $\Phi, \Psi \in Out(F_N)$, we have*

$$d_{FF_N}(\Phi *_{FF_N}, \Psi *_{FF_N}) \leq C d_{\mathcal{S}'_N}(\Phi *_{\mathcal{S}'_N}, \Psi *_{\mathcal{S}'_N}).$$

☐

Given two sphere systems S and S' in M_N, the *intersection number* $i(S, S')$ is the minimal number of intersection circles between representatives of the isotopy classes of

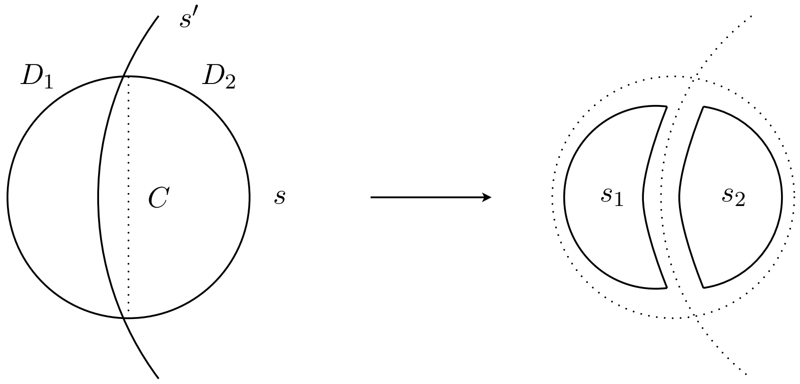

Figure B.4: The surgery procedure.

S and S'. Assume that S and S' have been isotoped so as to minimize their number of intersection circles. There is a classical surgery procedure [Hat95] that creates from a sphere s in S two spheres s_1 and s_2 that are both disjoint from s, and have fewer intersection circles with S'. Pick an innermost disk on a sphere $s' \in S'$, bounded by a circle C of intersection with s. The circle C splits s into two disks D_1 and D_2. For all $i \in \{1, 2\}$, the sphere s_i consists of a parallel copy of D_i attached to a parallel copy of D, see Figure B.4. Notice that all other intersection circles between s and S' are distributed over s_1 and s_2. In particular, there exists $j \in \{1, 2\}$ such that $i(s_j, S') \leq \frac{i(s,S')}{2}$ (isotoping s_j to minimize the number of intersection circles with S' can only decrease this number). An iterated application of this argument yields the following distance estimate in $\mathcal{S}'_N$ in terms of intersection numbers. (In the following statement, we take the convention $\log 0 = 0$.)

Lemma B.5.14. *There exist $K, L \in \mathbb{R}$ such that for all sphere systems $S, S' \in \mathcal{S}'_N$, we have*

$$d_{\mathcal{S}'_N}(S, S') \leq K \log i(S, S') + L.$$

Proof. Let $s \in S$ be a sphere. Iterating the above argument yields a sequence $s = s_0, s_1, \ldots, s_N$ of spheres, with $i(s_j, s_{j+1}) = 0$ for all $j \in \{1, \ldots, N-1\}$ and $i(s_j, S') \leq \frac{i(s,S')}{2^j}$. In particular, for $N := \lceil \log_2 i(S, S') \rceil$, we have $i(s_N, S') = 0$. Hence the sequence $S, s, s \cup s_1, s_1, s_1 \cup s_2, \ldots, s_N, s_N \cup S', S'$ is a path of length $2\lceil \log_2 i(S, S') \rceil + 4$ joining S to S' in $\mathcal{S}'_N$. The lemma follows. $\square$

One can finally relate the Lipschitz distance on CV_N to intersection numbers.

Proposition B.5.15. *(Horbez [2, Proposition 2.8]) There exist $K, L \in \mathbb{R}$ such that for all $\Phi, \Psi \in Out(F_N)$, we have*

$$\frac{1}{K} \log i(\Phi *_{\mathcal{S}'_N}, \Psi *_{\mathcal{S}'_N}) - L \leq d_{CV_N}(\Phi b, \Psi b) \leq K \log i(\Phi *_{\mathcal{S}'_N}, \Psi *_{\mathcal{S}'_N}) + L.$$

B.5.4 An Oseledets-like theorem for random products of outer automorphisms of F_N

When $gr(\mu)$ is elementary, we can no longer expect all elements of F_N to grow exponentially fast along a typical sample path of the random walk, with the same exponential growth rate. A typical situation is the case where the support of μ only contains automorphisms that act as the identity on some proper subgroup of F_N: in this case, elements of F_N belonging to this subgroup will not grow along any sample path of the random walk.

However, in the case where $gr(\mu)$ is elementary, we can still provide information about the possible growth rates of elements of F_N under random products of automorphisms of F_N. In this case, we give an analogue of a theorem due to Furstenberg and Kifer [FK83] and Hennion [Hen84] in the case of random products of matrices (which may be seen as a version of Oseledets' theorem). Several growth rates may arise, and we give a bound on their number.

Filtrations of F_N

A *filtration* of F_N is a finite rooted tree τ such that

- associated to every node of τ is a (possibly trivial) subgroup $H \subseteq F_N$, and
- the subgroup associated to the root of τ is F_N, and
- we have $H' \subseteq H$ whenever H' is a child of H.

A *system of Lyapunov exponents* for the filtration τ is a set of real numbers $\lambda_H \geq 0$ associated to the nodes of τ, such that $\lambda_{H'} \leq \lambda_H$ whenever H' is a descendant of H, and $\lambda_H = 0$ if and only if H is a leaf of τ. A particular case of filtrations of F_N is given by the following construction. We say that a group action on a tree is *trivial* if the tree is reduced to a point. An F_N*-chain of actions* is a finite rooted tree τ such that

- associated to every node of τ is a pair (H, T_H), where H is a subgroup of F_N (the subgroup associated to the root of τ is F_N), and T_H is a minimal, very small H-tree with dense orbits (the group H might be equal to $\{e\}$, and the tree T_H might be reduced to a point), and
- all nodes whose associated action is trivial are leaves of τ, and
- for all nodes whose associated action (H, T_H) is nontrivial, the collection of subgroups $H' \subseteq F_N$ associated to the children of (H, T_H) is a set of representatives of the conjugacy classes of point stabilizers in T_H (in particular, the group $\{e\}$ is one of the children of H as soon as the action on T_H is nontrivial).

In particular, leaves of τ are in one-to-one correspondence with trivial actions. We might have preferred not to add the trivial group to the collection of descendants of a nontrivial action, which would have led to some leaves of τ corresponding to free actions. However, it will turn out that including the trivial group in this collection is more natural for our purpose, because e always has zero growth along any sample path of a random walk. Associated to any F_N-chain of actions is a filtration of F_N, obtained by forgetting the actions. The following theorem, whose proof we postpone to Section B.5.4, gives a bound on the size of any F_N-chain of actions.

Theorem B.5.16. *Any F_N-chain of actions has at most $N-1$ non-leaf nodes.*

Oseledets filtrations of probability measures on Out(F_N)

Definition B.5.17. *Let μ be a probability measure on Out(F_N). Let F be a finitely generated subgroup of F_N. A filtration τ of F is an* Oseledets filtration *for μ if there exists a system of Lyapunov exponents $\{\lambda_H^\mu\}_{H\in V(\tau)}$ for τ, such that for $\mathbb{P}$-almost every sample path of the random walk on $(Out(F_N),\mu)$, all nodes H of τ, and all elements $g\in H$ that are not conjugate into any child of H, we have*

$$\lim_{n\to+\infty}\frac{1}{n}\log||\Phi_n(g)||=\lambda_H^\mu.$$

For all $g\in F$, the growth rate

$$\lim_{n\to+\infty}\frac{1}{n}\log||\Phi_n(g)||$$

is called the Lyapunov exponent *of g for the measure μ, and denoted by $\lambda^\mu(g)$.*

Theorem B.5.18. *Let μ be a probability measure on Out(F_N), having finite first moment with respect to $d_{CV_N}^{sym}$. Then there exists an Oseledets filtration for μ, associated to an F_N-chain of actions. Moreover, for all nodes H of the filtration, the conjugacy class of H has finite $gr(\mu)$-orbit.*

As a consequence of Theorems B.5.16 and B.5.18, we deduce that for all probability measures μ on Out(F_N) having finite first moment with respect to $d_{CV_N}^{sym}$, there exists a finite collection of (deterministic) exponents $\lambda_1,\dots,\lambda_p>0$ such that for $\mathbb{P}$-almost every sample path of the random walk on $(\mathrm{Out}(F_N),\mu)$, and all $g\in F_N\smallsetminus\{e\}$, the limit

$$\lim_{n\to+\infty}\frac{1}{n}\log||\Phi_n(g)||$$

exists and belongs to $\{0\}\cup\{\lambda_1,\dots,\lambda_p\}$. Theorem B.5.16 implies that $p\le N-1$, we will improve this bound in Section B.5.4, see Corollary B.5.25.

Existence of Oseledets filtrations

First return measures. Let μ be a probability measure on Out(F_N), and let A be a finite index subgroup of $gr(\mu)$. The subgroup A is positively recurrent for the random walk on $(\mathrm{Out}(F_N),\mu)$. The *first return measure* on A, denoted by μ^A, is the probability measure defined as the distribution of the point where the random walk issued from the identity of Out(F_N) returns for the first time to A. Given a sample path $(\Phi_n(\omega))_{n\in\mathbb{N}}$ of the random walk on $(\mathrm{Out}(F_N),\mu)$, and $m\in\mathbb{N}$, we let $\tau_m^A(\omega)$ be the $(m+1)^{st}$ time $n\in\mathbb{N}$ at which we have $\Phi_n(\omega)\in A$. Notice in particular that $\tau_0^A(\omega)=0$, and $\tau_1^A(\omega)$ is the first (positive) time at which the sample path returns to the recurrent subgroup A. We let

$$C_A:=\lim_{n\to+\infty}\frac{\tau_n^A(\omega)}{n},$$

which almost surely exists, is independent of ω, and $C_A>0$ by positive recurrence of the random walk on the finite set $gr(\mu)/A$. The following proposition is a variation in our context of a classical fact about first return measures, see for example [Kai91, Lemma 2.3] or [BQ13, Lemma 6.10] where it appears in other contexts.

Proposition B.5.19. *Let μ be a probability measure on $Out(F_N)$ which has finite first moment with respect to $d^{sym}_{CV_N}$. Let A be a finite index subgroup of $gr(\mu)$ which fixes the conjugacy class of a finitely generated malnormal subgroup $H \subseteq F_N$ of rank k. Then μ^A has finite first moment with respect to $d^{sym}_{CV_k}$.*

Proof. Since H is malnormal, all elements of A induce a well-defined element of $\mathrm{Out}(H)$. We choose a basepoint $*_{CV_N}$ in CV_N, and let its H-minimal subtree be a basepoint $*_{CV_k}$ for CV_k. Then there exists $C > 0$ such that for all $\Phi \in A$, we have

$$d^{sym}_{CV_k}(\Phi *_{CV_k}, *_{CV_k}) \leq C d^{sym}_{CV_N}(\Phi *_{CV_N}, *_{CV_N}).$$

Indeed, as Φ fixes the conjugacy class of H, the H-minimal subtrees of $*_{CV_N}$ and $\Phi *_{CV_N}$ have the same quotient volumes, and the translation length of any $g \in H$ is stretched by the same amount from $\Phi *_{CV_N}$ to $*_{CV_N}$ and from $\Phi *_{CV_k}$ to $*_{CV_k}$. Denoting by L the (finite) first moment of μ with respect to $d^{sym}_{CV_N}$, we have

$$\begin{aligned}
\int_A d^{sym}_{CV_k}(\Phi *_{CV_k}, *_{CV_k}) d\mu^A(\Phi) &\leq C \int_A d^{sym}_{CV_N}(\Phi *_{CV_N}, *_{CV_N}) d\mu^A(\Phi) \\
&= C \int_\Omega d^{sym}_{CV_N}(\Phi_{\tau_1^A(\omega)}(\omega) *_{CV_N}, *_{CV_N}) d\mathbb{P}(\omega) \\
&\leq C \int_\Omega \sum_{i=1}^{\tau_1^A(\omega)} d^{sym}_{CV_N}(\phi_i(\omega) *_{CV_N}, *_{CV_N}) d\mathbb{P}(\omega) \\
&= C \sum_{i=1}^{+\infty} \int_{\{\tau_1^A(\omega) \geq i\}} d^{sym}_{CV_N}(\phi_i(\omega) *_{CV_N}, *_{CV_N}) d\mathbb{P}(\omega) \\
&= CL \sum_{i=1}^{+\infty} \mathbb{P}(\tau_1^A(\omega) \geq i),
\end{aligned}$$

where the last equality follows from independence of $\{\tau_1^A \geq i\}$ and the increments ϕ_j's for $j \geq i$. We thus get

$$\int_A d^{sym}_{CV_k}(\Phi *_{CV_k}, *_{CV_k}) d\mu^A(\Phi) \leq CL \sum_{i=1}^{+\infty} i \mathbb{P}(\tau_1^A(\omega) = i),$$

which is finite by positive recurrence of the random walk on the finite set $gr(\mu)/A$. ☐

Proposition B.5.20. *Let μ be a probability measure on $Out(F_N)$, with finite first moment with respect to $d^{sym}_{CV_N}$. Let H be a finitely generated malnormal subgroup of F_N of rank k, whose conjugacy class $[H]$ has finite $gr(\mu)$-orbit. Let $A := Stab([H])$.*

Assume that for all probability measures μ' on A with finite first moment with respect to $d^{sym}_{CV_k}$, there exists an Oseledets filtration of H for μ'. Then any Oseledets filtration of H for the measure μ^A is an Oseledets filtration of H for the measure μ, and for all $g \in H$, we have $\lambda^\mu(g) = \frac{1}{C_A} \lambda^{\mu^A}(g)$.

Proof. Let $\{[H] = [H_1], \dots, [H_p]\}$ be the $\mathrm{gr}(\mu)$-orbit of the conjugacy class of H, and for all $i \in \{1, \dots, p\}$, let $A_i := \mathrm{Stab}([H_i])$. We start by showing the existence, for all $i \in \{1, \dots, p\}$, of an Oseledets filtration of H_i for the measure μ^{A_i}. Let $i \in \{1, \dots, p\}$. We choose an automorphism $\alpha_i \in gr(\mu)$ such that $\alpha_i([H]) = [H_i]$ (with $\alpha_1 = \mathrm{id}$). Let μ_i be the measure on A defined as $\mu_i := (\mathrm{ad}_{\alpha_i})_* \mu^{A_i}$ (where ad_{α_i} denotes the conjugation by α_i). Using Proposition B.5.19, we get that μ_i has finite first moment with respect to $d^{sym}_{CV_k}$, so by hypothesis, there exists an Oseledets filtration $\mathcal{F}$ of H for the measure μ_i. For almost every sample path $(\Psi^i_n)_{n \in \mathbb{N}}$ of the random walk on (A, μ_i), and all $g \in H$, the limit

$$\lim_{n \to +\infty} \frac{1}{n} \log ||\Psi^i_n(g)|| = \lambda^{\mu_i}(g)$$

only depends on the node of $\mathcal{F}$ to which g belongs. Therefore, for all $g' \in H_i$, the limit

$$\lim_{n \to +\infty} \frac{1}{n} \log ||\alpha_i \Psi^i_n \alpha_i^{-1}(g')|| = \lambda^{\mu_i}(\alpha_i^{-1}(g'))$$

only depends on the node of the filtration $\alpha_i(\mathcal{F})$ to which g' belongs. By definition of the measure μ_i, this implies the existence of an Oseledets filtration of H_i for the measure μ^{A_i}, which is the α_i-image of the Oseledets filtration of H for μ_i. The Lyapunov exponents λ^{A_i} of the measure μ^{A_i} satisfy $\lambda^{A_i}(g') = \lambda^{\mu_i}(\alpha_i^{-1}(g'))$ for all $g' \in H_i$.

Let now $(\Phi_n)_{n\in\mathbb{N}}$ be a sample path of the random walk on $(\mathrm{Out}(F_N), \mu)$, and let $g \in H$. For all $n \in \mathbb{N}$, we set $g_n := \Phi_n(g)$. For all $i \in \{1, \dots, p\}$, we let $I_i \subseteq \mathbb{N}$ be the set of all integers n such that $\Phi_n([H]) = [H_i]$, and we let $\tau_i(n)$ be the n^{th} integer in I_i. The limit

$$C_i := \lim_{n\to+\infty} \frac{\tau_i(n)}{n}$$

almost surely exists and is constant almost everywhere (it only depends on the finite Markov chain on $\{[H_1], \dots, [H_p]\}$), and $C_i > 0$ by positive recurrence of this finite Markov chain. For all $n \in \mathbb{N}$, we have $\Phi_n(g) = \Psi_n^i(g_{\tau_i(1)})$, where $\Psi_n^i := \phi_n \dots \phi_{\tau_i(1)+1}$. The sequence $(\Psi_n^i)_{n\in I_i}$ is a sample path of the random walk on (A_i, μ^{A_i}), and therefore we have

$$\lim_{\substack{n\to+\infty\\ n\in I_i}} \frac{1}{n} \log ||\Phi_n(g)|| = \frac{1}{C_i} \lambda^{A_i}(g_{\tau_i(1)}).$$

We will now prove that the limit

$$\lim_{n\to+\infty} \frac{1}{n} \log ||\Phi_n(g)||$$

almost surely exists, i.e. that $\frac{1}{C_i}\lambda^{A_i}(g_{\tau_i(1)})$ does not depend on i. In particular, by choosing $i = 1$, this will imply that

$$\lim_{n\to+\infty} \frac{1}{n} \log ||\Phi_n(g)|| = \frac{1}{C_A} \lambda^A(g),$$

and this limit only depends on the node of the filtration $\mathcal{F}$ to which g belongs (it is independent from the sample path). So any Oseledets filtration for μ^A is an Oseledets filtration for μ.

Assume towards a contradiction that there is a subset $Y \subseteq \Omega$ of positive measure so that for all $\omega \in Y$, the above limit does not exist. Let $\omega \in Y$ be such that the above claim fails for the corresponding sample path. For all $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ (depending on ω) such that for all $i \in \{1, \dots, p\}$, all $n \in I_i \cap [n_0, +\infty)$, and all $g \in H$, we have

$$\left| \frac{1}{n} \log ||\Phi_n(\omega)(g)|| - \frac{1}{C_i} \lambda^{A_i}(g) \right| \leq \epsilon.$$

Assume towards a contradiction that $\frac{1}{C_i}\lambda^{A_i}(g) \neq \frac{1}{C_{i'}}\lambda^{A_{i'}}(g)$ for some $i, i' \in \{1, \dots, p\}$. Then there exists $\alpha > 0$, independent of ω (it only depends on the Markov chain on the finite set $\{[H_1], \dots, [H_p]\}$), and an infinite set of integers X (depending on ω) with density at least α, such that for all $n \in X$, the integers n and $n+1$ belong to two different sets $I_i, I_{i'}$ of the partition, with $\frac{1}{C_i}\lambda^{A_i}(g) \neq \frac{1}{C_{i'}}\lambda^{A_{i'}}(g)$. Let

$$\delta := \frac{1}{3} \left| \frac{1}{C_i} \lambda^{A_i}(g) - \frac{1}{C_{i'}} \lambda^{A_{i'}}(g) \right|,$$

which is independent on ω. If $\epsilon > 0$ has been chosen small enough, then there exists an infinite set of integers X' (depending on ω) with density at least α, such that for all $n \in X'$, we have

$$\left| \frac{1}{n+1} \log ||\Phi_{n+1}(g)|| - \frac{1}{n} \log ||\Phi_n(g)|| \right| \geq \delta,$$

and therefore
$$\frac{||\Phi_{n+1}(g)||}{||\Phi_n(g)||} \geq e^{n\delta}.$$
Let $k \in \mathbb{N}$. There exists $n_k \in \mathbb{N}$ such that $n_k\delta \geq k$. For all $n \in X' \cap [n_k, +\infty)$, we have

$$d_{CV_N}(\Phi_n^{-1}b, \Phi_{n+1}^{-1}b) \geq k,$$

or in other words $d_{CV_N}(\phi_{n+1}b, b) \geq k$. We also notice that for all $\epsilon > 0$, if $\mu(\{\phi \in \mathrm{Out}(F_N) | d_{CV_N}(\phi b, b) \geq k\}) < \epsilon$, then almost surely, the density of times such that $d_{CV_N}(\phi_{n+1}b, b) \geq k$ should be at most ϵ. Therefore, it follows from the above that the measure $\mu(\{\phi \in \mathrm{Out}(F_N) | d_{CV_N}(\phi b, b) \geq k\})$ is bounded below independently of $k \in \mathbb{N}$, which is impossible. So for all $i, i' \in \{1, \dots, p\}$, we have $\frac{1}{C_i}\lambda^{A_i}(g) = \frac{1}{C_{i'}}\lambda^{A_{i'}}(g)$, as claimed. $\square$

Proof of Theorem B.5.18. We argue by induction on the rank N of the free group. The claim holds true for $N = 1$, so we assume that $N \geq 2$. We will first show that for almost every sample path $\mathbf{\Phi}$ of the random walk on $(\mathrm{Out}(F_N), \mu)$, there exists an (*a priori* random) filtration $\tau(\mathbf{\Phi})$ of F_N, together with an (*a priori* random) system of Lyapunov exponents $\{\lambda^{\mathbf{\Phi}}(H)\}_{H \in V(\tau(\mathbf{\Phi}))}$, such that for all nodes H of the filtration, and all $g \in H$ that are not conjugate into any child of H, we have

$$\lim_{n \to +\infty} \frac{1}{n} \log ||\Phi_n(g)|| = \lambda^{\mathbf{\Phi}}(H).$$

The fact the Lyapunov exponents are deterministic, and that the filtration can be chosen not to depend on the sample path, will be shown in the last paragraph of the proof. We keep the notations introduced in the proof of Theorems B.5.1 and B.5.2 in Section B.5.2. Recall that we have shown that

$$\lim_{n \to +\infty} \frac{1}{n} \log ||\Phi_n(g)|| = l$$

for all $g \in F_N$ which are hyperbolic in T, where l is the drift of the random walk for the Lipschitz metric on CV_N. We are left understanding possible growth rates of elements of F_N that are elliptic in T. If $l = 0$, then all elements $g \in F_N$ grow subexponentially along the random walk, and we can choose T to be trivial. Otherwise, the horofunction ξ provided by Theorem B.5.3 is unbounded from below, so Theorem B.3.13 implies that for almost every ω, the tree $T(\omega)$ has dense orbits. Propositions B.5.6 and B.5.8 show that we may have chosen T so that all conjugacy classes of point stabilizers in T have finite $\mathrm{gr}(\mu)$-orbit.

Let $\mathcal{C}$ be the collection of conjugacy classes of point stabilizers of T. All subgroups in $\mathcal{C}$ are malnormal, and they have rank at most $N - 1$ by [GL95, Theorem III.2] (see Proposition B.5.21 below). Therefore, our induction hypothesis implies that for all $H \in \mathcal{C}$ of rank k, and all measures μ' on $\mathrm{Out}(H)$ with finite first moment with respect to CV_k, there exists an Oseledets filtration of H for the measure μ', which is associated to an H-chain of actions. Proposition B.5.20 then shows the existence of an Oseledets filtration of H for the measure μ, which is equal to the Oseledets filtration for μ^A, where $A := \mathrm{Stab}([H])$. The conjugacy class of every node $H' \subseteq H$ of the filtration has finite $gr(\mu^A)$-orbit, and hence it has finite $gr(\mu)$-orbit.

To get the desired filtration $\tau(\mathbf{\Phi})$ of F_N, notice that all elements of F_N that do not belong to any subgroup in $\mathcal{C}$ have a Lyapunov exponent, which is greater than or equal to the Lyapunov exponent of any other element of F_N (Theorem B.5.1). We then let

$\tau(\mathbf{\Phi})$ be the filtration of F_N associated to the F_N-chain of actions whose root is the action $(F_N, T(\omega))$, to which we attach the H-chains of actions associated to the elliptic subgroups H of $T(\omega)$ which were provided by the induction hypothesis.

We now show that the filtration $\tau(\mathbf{\Phi})$ is actually a (deterministic) Oseledets filtration for the measure μ (i.e. it is adapted to almost every sample path of the random walk on $(\mathrm{Out}(F_N), \mu)$). It is enough to show that for all $g \in F_N$, the growth rate $\lambda^{\mathbf{\Phi}(\omega)}(g)$ of g along the sample paths of the random walk on $(\mathrm{Out}(F_N), \mu)$ is $\mathbb{P}$-essentially constant. Let $g \in F_N$. If $\lambda^{\mathbf{\Phi}(\omega)}(g)$ is not $\mathbb{P}$-essentially constant, then in particular $\mathbb{P}(\lambda^{\mathbf{\Phi}(\omega)}(g) < l) > 0$ (where we recall that l is the drift of the random walk on $(\mathrm{Out}(F_N), \mu)$). Hence g belongs to some subgroup $H \subseteq F_N$, whose conjugacy class has finite $\mathrm{gr}(\mu)$-orbit. Let $A := \mathrm{Stab}([H])$. The induction hypothesis implies that the growth rate of g along the sample paths of the random walk on (A, μ^A) is essentially constant, equal to $\lambda^{\mu^A}(g)$. Proposition B.5.20 therefore implies that $\lambda^{\mathbf{\Phi}}(g)$ is $\mathbb{P}$-essentially constant, equal to $\frac{1}{C_A}\lambda^{\mu^A}(g)$. This proves the claim. $\square$

Bounding the size of F_N-chains of actions

We will now prove Theorem B.5.16, which bounds the size of any F_N-chain of actions. For all $T \in \overline{CV_N}$, and all $x \in T$, we define the *index* $i(x) := 2\mathrm{rk}(\mathrm{Stab}(x)) + v_1(x) - 2$, where $v_1(x)$ denotes the number of $\mathrm{Stab}(x)$-orbits of directions with trivial stabilizer at x. This only depends on the F_N-orbit of x in T. The *index* $i(T)$ is then defined to be the sum of the indices of x over all F_N-orbits of points $x \in T$. We will appeal to the following result of Gaboriau and Levitt.

Proposition B.5.21. *(Gaboriau–Levitt [GL95, Theorem III.2]) For all trees $T \in \overline{CV_N}$, we have $i(T) \leq 2N - 2$. In particular, if T has trivial arc stabilizers, then for all $x \in T$, we have $rk(Stab(x)) \leq N - 1$.*

Proof of Theorem B.5.16. We argue by induction on N. Every $\mathbb{Z}$-action on a tree with dense orbits is trivial, so we can assume that $N \geq 2$. Let τ be an F_N-chain of actions, and let T be the action corresponding to the root of τ. We denote by $p(\tau)$ the number of non-leaf nodes in τ. Let V be the collection of nodes of depth 1 in τ, which correspond to a set of representatives of the conjugacy classes of point stabilizers in T. For all $v \in V$, let G_v be the associated subgroup of F_N, and let τ_v be the corresponding G_v-chain of actions. As T has dense orbits, it follows from Proposition B.5.21 that for all $v \in V$, we have $\mathrm{rk}(G_v) < N$. The induction hypothesis implies that for all $v \in V$, we have $p(\tau_v) \leq \mathrm{rk}(G_v) - 1$, which implies that

$$\begin{array}{ll} p(\tau) & \leq 1 + \sum_{v \in V}(\mathrm{rk}(G_v) - 1) \\ & < 1 + \frac{1}{2}\sum_{v \in V}(2\mathrm{rk}(G_v) - 1). \end{array}$$

As arc stabilizers in T are trivial, Proposition B.5.21 implies that $p(\tau) \leq N - 1$. $\square$

Example B.5.22. We construct an example of an F_N-chain of actions with $N-1$ nontrivial nodes, thus showing the optimality of the bound provided by Theorem B.5.16 in general. Let T be a geometric F_N-tree with dense orbits, whose skeleton (see [Gui04, Definition 4.8] or [Gui08, Section 1]) consists of

- one vertex corresponding to a minimal action with dense orbits of a subgroup of F_N of rank 2, dual to a measured lamination on a torus with a single boundary component, and
- one vertex corresponding to a trivial action of a subgroup of F_N of rank $N-1$, and

- an edge of length 0 joining them, whose stabilizer is cyclic, represented by the boundary curve of the torus.

This defines a nontrivial, minimal, very small F_N-tree, in which a subgroup of F_N of rank $N-1$ is elliptic. Repeating this construction, we get a sequence of subgroups $F_N = H_N \supseteq \cdots \supseteq H_1 = \mathbb{Z}$, in which the subgroup H_i has rank i, together with minimal, very small H_i-trees with dense orbits, which are nontrivial as soon as $i \geq 2$, and such that for all $i \in \{2, \dots, N\}$, the subgroup H_{i-1} is elliptic in T_i. This defines an F_N-chain of actions with $N-1$ non-leaf nodes, in which each node (H_i, T_{H_i}) with $i \geq 2$ has two children, namely the action (H_{i-1}, T_{i-1}), and the trivial action of the trivial group.

Good F_N-chains of actions

Example B.5.22 shows that the bound on the size of an F_N-chain of actions provided by Theorem B.5.16 is optimal in general. We will now define a special class of *good* F_N-chains of actions for which this bound can actually be improved. We will show that all Oseledets filtrations constructed in the proof of Theorem B.5.18 are good, which will lead to a better bound on the number of possible growth rates of elements of F_N under random products of automorphisms.

We refer to [LP97] for a definition of geometric trees, see also [Gui08, Section 1.7]. Any geometric tree with dense orbits has a decomposition into a graph of actions where each nondegenerate vertex action is indecomposable [Gui08, Proposition 1.25] (the reader is referred to [Gui08, Section 1] for background material). We say that a tree in $\overline{CV_N}$ is *of surface type* if it is geometric, and all its indecomposable subtrees are dual to laminations on surfaces. Let τ be an F_N-chain of actions. An element $g \in F_N$ is a *special curve* for τ if there exists a node (H, T_H) of τ corresponding to an action of surface type, such that g is conjugate to an element that represents a boundary curve of a surface dual to one of the indecomposable subtrees of T_H. An F_N-chain of actions τ is *good* if all special curves g of τ are elliptic in all nodes (H, T_H) such that $g \in H$ (up to conjugacy). In other words, an F_N-chain of actions τ is good if and only if all special curves of τ are conjugate into some leaf of τ.

Theorem B.5.23. *Any good F_N-chain of actions has at most $\frac{3N-2}{4}$ non-leaf nodes.*

Theorem B.5.24. *Let μ be a probability measure on $Out(F_N)$, having finite first moment with respect to $d^{sym}_{CV_N}$. Then there exists an Oseledets filtration for μ, which is associated to a good F_N-chain of actions.*

The proof of Theorem B.5.23 is given in Section B.5.4, and the proof of Theorem B.5.24 is given in Section B.5.4. As a consequence of Theorems B.5.23 and B.5.24, we get the following result.

Corollary B.5.25. *Let μ be a probability measure on $Out(F_N)$, having finite first moment with respect to $d^{sym}_{CV_N}$. Then there exist (deterministic) $\lambda_1, \dots, \lambda_p > 0$ such that for $\mathbb{P}$-almost every sample path of the random walk on $(Out(F_N), \mu)$, and all $g \in F_N \smallsetminus \{e\}$, the limit*

$$\lim_{n \to +\infty} \frac{1}{n} \log ||\Phi_n(g)||$$

exists and belongs to $\{0\} \cup \{\lambda_1, \dots, \lambda_p\}$. In addition, we have $p \leq \frac{3N-2}{4}$.

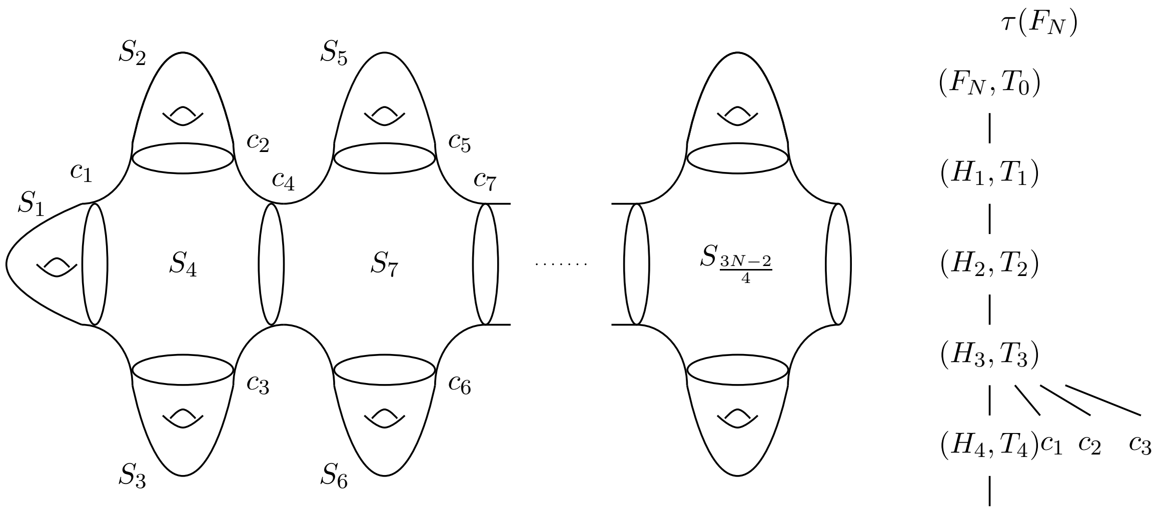

Figure B.5: The surface in Example B.5.26.

Example B.5.26. We give an example, due to Levitt [Lev09], of a good chain of actions with $\frac{3N-2}{4}$ non-leaf nodes, thus showing that the bound in Theorem B.5.23 is optimal. Let S be the compact, oriented surface of rank N displayed on Figure B.5, decomposed into $\frac{3N-2}{4}$ subsurfaces S_i that are either tori with one boundary component, or spheres with 4 boundary components. Let $H_0 := F_N$, and for all $i \in \{1, \dots, \frac{3N-2}{4}\}$, let H_i be the fundamental group of the subsurface Σ_i of S obtained by removing $S_1, \dots, S_i$ from S (we let $H_{\frac{3N-2}{4}}$ be the cyclic group generated by the rightmost boundary curve of the surface S displayed on Figure B.5). Let T_i be a nontrivial H_i-tree with dense orbits, dual to a measured lamination on Σ_i that is supported on S_{i+1} (in particular $T_{\frac{3N-2}{4}}$ is trivial). Then the F_N-chain of actions displayed on Figure B.5 is good, because the boundary curves of Σ_i are elliptic in all the descendants of H_i that contain them. In addition, this F_N-chain of actions contains $\frac{3N-2}{4}$ nontrivial nodes.

The same example also shows that the bound in Corollary B.5.25 is sharp, by letting μ be a Dirac measure supported on a diffeomorphism of S that restricts to a pseudo-Anosov diffeomorphism of each surface S_i, with $\frac{3N-2}{4}$ different growth rates.

Existence of good Oseledets filtrations

The goal of this section is to prove Theorem B.5.24, by showing that the F_N-chain of actions constructed in the proof of Theorem B.5.18 is good.

More on stationary measures on $CV_N(\infty)$. Let $\mathcal{Y}$ denote the collection of (finitely generated) subgroups of F_N that are maximally elliptic in some simplicial tree in $\overline{CV_N}$. Given $T \in CV_N(\infty)$, we denote by $\mathrm{Dyn}(T)$ the collection of all conjugacy classes of minimal subgroups in $\mathcal{Y}$ which act nontrivially with dense orbits on their minimal subtree in T. This definition makes sense by the descending chain condition satisfied by groups in $\mathcal{Y}$ [HM09, Proposition 4.1]. Subgroups whose conjugacy classes belong to $\mathrm{Dyn}(T)$ are called *dynamical subgroups* of T. It follows from our description of $\overline{CV_N}^{prim}$ in [3] that all lifts to $\overline{CV_N}$ of a tree in $CV_N(\infty)$ have the same dynamical subgroups. We let

$$\Theta(T) := \begin{cases} \mathrm{Dyn}(T) & \text{if } \mathrm{Dyn}(T) \text{ is finite} \\ \emptyset & \text{otherwise} \end{cases}.$$

Measurability of Θ comes from upper semicontinuity of the quotient volume (see [AK13, Section 3.3], where it is proved that the quotient volume of a tree $T \in \overline{cv_N}$ is equal to 0 if and only if T has dense orbits), and continuity of translation lengths. Applying Lemma B.5.9 to the map Θ yields the following fact.

Proposition B.5.27. *Let μ be a probability measure on $Out(F_N)$. Then every μ-stationary measure on $CV_N(\infty)$ is concentrated on the set of trees which either have infinitely many dynamical subgroups, or all of whose dynamical subgroups have finite $gr(\mu)$-orbits.* ☐

We now determine $\mathrm{Dyn}(T)$ in the case where $T \in CV_N(\infty)$ is of surface type.

Lemma B.5.28. *Let $T \in CV_N(\infty)$ be a tree of surface type with dense orbits. Then $Dyn(T)$ is equal to the set of stabilizers of the indecomposable subtrees of T.*

Proof. The tree T admits a transverse covering $\mathcal{Z}$ by indecomposable subtrees (see [Gui08, Section 1] for definitions), whose skeleton has cyclic (or trivial) edge groups, and each tree in $\mathcal{Z}$ is dual to a minimal lamination on a surface [Gui08, Proposition 1.25]. Let $H \subseteq F_N$ be the stabilizer of one of these indecomposable subtrees $T_H \in \mathcal{Z}$. Let $F \in \mathrm{Dyn}(T)$. The F-minimal subtree T_F of T inherits a transverse covering, given by the intersections of T_F with the subtrees in $\mathcal{Z}$. As F acts with dense orbits on T_F by assumption, the intersection $T_F \cap T_H$ is either empty, or has dense $F \cap H$-orbits. As T_H is indecomposable, this implies by [Rey11b, Theorem 4.4] that either $T_F \cap T_H$ contains at most one point, or else that $T_F \cap T_H = T_H$, and $F \cap H$ has finite index in H. By minimality of F, we have $T_F \cap T_H = T_H$ for exactly one of the subtrees T_H in the family $\mathcal{Z}$. As groups in $\mathcal{Y}$ do not have proper finite index extensions, this implies that $F \cap H = H$, and F is the stabilizer of one of the indecomposable subtrees of T. Therefore, the set $\mathrm{Dyn}(T)$ consists of the conjugacy classes of these stabilizers. ☐

As a consequence of Proposition B.5.27 and Lemma B.5.28, we get the following fact.

Corollary B.5.29. *Let μ be a probability measure on $Out(F_N)$. Then every μ-stationary measure on $CV_N(\infty)$ is concentrated on the set of trees $T \in CV_N(\infty)$ such that either*

- *the tree T is not of surface type, or*
- *the tree T is of surface type, and all conjugacy classes of the stabilizers of its indecomposable components have finite $gr(\mu)$-orbits.* ☐

Proof of Theorem B.5.24. We prove that the F_N-chain of actions τ constructed in the proof of Theorem B.5.18 is good. We keep the notations from this proof. Arguing by induction again, we can assume that all filtrations τ_H are associated to good H-chains of actions. Proposition B.5.6, together with Corollary B.5.29, shows that if T is of surface type, then we might assume that the conjugacy classes of all stabilizers of the indecomposable components of T (which are dual to minimal foliations on surfaces) have finite $gr(\mu)$-orbit. Let $c \in F_N$ represent a boundary curve of a surface dual to one of the indecomposable components of T. Then c is the intersection of a point stabilizer of T with a dynamical subgroup of T, which implies that the $gr(\mu)$-orbit of the conjugacy class of c is finite, and therefore c grows subexponentially along the random walk. In particular, the element c belongs to one of the leaves of τ, thus showing that τ is good. ☐

Bounding the size of good F_N-chains of actions

The aim of this section is to prove Theorem B.5.23, which provides a bound on the size of good F_N-chains of actions. Our proof is inspired from Levitt's similar statement in [Lev09] for counting growth rates of a single automorphism of F_N.

A Euler characteristic formula for small graph of groups decompositions.

Lemma B.5.30. *Let $\mathcal{G}$ be a graph of groups decomposition of F_N, whose edge groups are (at most) cyclic. Denote by V the number of vertices of $\mathcal{G}$, by E_0 the number of edges with trivial stabilizer, and by R the sum of the ranks of the vertex stabilizers of $\mathcal{G}$. Then $N = R + E_0 - V + 1$.*

Our proof of Lemma B.5.30 relies on the following classical result.

Lemma B.5.31. *(Shenitzer [She55], Swarup [Swa86], Stallings [Sta91], Bestvina–Feighn [BF94, Lemma 4.1]) Let $\mathcal{G}$ be a graph of groups decomposition of F_N, whose edge groups are (at most) cyclic. Then there exists an edge e in $\mathcal{G}$ with nontrivial stabilizer G_e, adjacent to a vertex v, and a free splitting of G_v of the form $G_v = G_e * A$, so that if $e' \neq e$ is another edge adjacent to v in $\mathcal{G}$, then $G_{e'}$ is conjugate into A.*

Lemma B.5.31 shows that we can "unfold" the edge e and get another graph of groups decomposition of F_N having fewer edges with nontrivial stabilizer, in which the vertex v is replaced by a vertex with stabilizer equal to A, which has corank 1 in G_v.

Proof of Lemma B.5.30. Using Lemma B.5.31, and arguing by downward induction on the number of edges with nontrivial stabilizer, we reduce to the case where all edges in $\mathcal{G}$ have trivial stabilizer (each unfolding operation decreases R by 1, and increases E_0 by 1). By iteratively collapsing all edges in a maximal subtree of $\mathcal{G}$ (such a collapse decreases both E_0 and V by 1), we reduce to the case where the underlying graph of $\mathcal{G}$ is a rose, in which case Lemma B.5.30 clearly holds. ☐

Proof of Theorem B.5.23. Let τ be a good F_N-chain of actions. Let $k(\tau)$ be the rank of the subgroup of the abelianization of F_N generated by the leaf groups of τ. We will show by induction on N that the number $p(\tau)$ of non-leaf nodes of τ satisfies

$$p(\tau) \leq \frac{3N - k(\tau) - 2}{4}.$$

We let T be the F_N-tree associated to the root of τ. We denote by $\{G_v\}$ the collection of all subgroups associated to the children of the root in τ, whose conjugates are the point stabilizers in T by definition.

Case 1 : The tree T is not of surface type.

We refer the reader to [GL10a] for an introduction to (relative) JSJ decompositions (and Grushko decompositions in particular). By [3, Theorem 3.11], there exists a two-edge free splitting S of F_N in which all G_v's are elliptic. This implies that any Grushko decomposition of F_N relative to the collection $\{G_v\}$ has at least two edges with trivial stabilizer. We denote by $\mathcal{C}$ the collection of all conjugacy classes of subgroups of F_N that are elliptic in all Grushko decompositions of F_N relative to the collection $\{G_v\}$. Notice that for all $H \in \mathcal{C}$, the point stabilizers of the action of H on its minimal subtree T_H are conjugates of the G_v's. We let τ_H be the H-chain of actions whose root corresponds to either

- the action (H, T_H), where T_H denotes the H-minimal subtree in T, if this action is nontrivial, or
- the action associated to G_v in τ if $H = G_v$ for some $v \in V$,

to which we attach the trees τ_v corresponding to the subgroups G_v conjugate into H. We have

$$p(\tau) \leq 1 + \sum_{H \in \mathcal{C}} p(\tau_H).$$

As all subgroups $H \in \mathcal{C}$ have rank at most $N-1$, our induction hypothesis shows that

$$p(\tau_H) \leq \frac{3\mathrm{rk}(H) - k(\tau_H) - 2}{4}.$$

Let G be a Grushko decomposition of F_N relative to $\mathcal{C}$, and denote by V (resp. E) the number of vertices (resp. of edges) in the graph of groups G. By collapsing edges to points if necessary, we can assume that no vertex of G has trivial stabilizer. Therefore, we have

$$\begin{aligned}
\textstyle\sum_{H \in \mathcal{C}} (3\mathrm{rk}(H) - 2) \quad &= 3 \textstyle\sum_{H \in \mathcal{C}} \mathrm{rk}(H) - 2V \\
&= 3(N - E + V - 1) - 2V \\
&= 3N - 3E + V - 3 \\
&\leq 3N - 2E - 2 \\
&\leq 3N - 6
\end{aligned}$$

because $V \leq E + 1$ by connectedness of G, and $E \geq 2$.

In addition, any relation between elements in the subgroup generated by the leaves of τ_H still holds true when viewing these elements as elements of F_N, so

$$k(\tau) \leq \sum_{H \in \mathcal{C}} k(\tau_H).$$

Combining the above inequalities, we get that

$$1 + \sum_{H \in \mathcal{C}} p(\tau_H) \leq \frac{3N - k(\tau) - 2}{4},$$

and we are done in this case.

Case 2 : The tree T is geometric, and only contains minimal surface components.

Then T is dual to a graph of actions $\mathcal{G}$ having a vertex associated to each orbit of indecomposable subtrees Y of T, a vertex associated to each conjugacy class of elliptic subgroup H of T, and an edge joining Y to H whenever the fixed point of H belongs to Y. All edges in $\mathcal{G}$ have cyclic (possibly trivial) stabilizer. Notice that $\mathcal{G}$ might not be minimal, in the case where some point stabilizer in T is cyclic (and corresponds to a boundary curve of one of the surfaces dual to an indecomposable subtree of T) and extremal. We denote by $\mathcal{C}$ the set of conjugacy classes of point stabilizers in T, and by $\mathcal{C}_1$ (resp. $\mathcal{C}_{\geq 2}$) the set of conjugacy classes in $\mathcal{C}$ which have valence 1 (resp. valence at least 2) in $\mathcal{G}$. It follows from Proposition B.5.21 that

$$\sum_{H\in\mathcal{C}_1}(2\mathrm{rk}(H)-1)+\sum_{H\in\mathcal{C}_{\geq 2}}(2\mathrm{rk}(H))\leq 2N-2,$$

from which we deduce that

$$\sum_{H\in\mathcal{C}_1}(3\mathrm{rk}(H)-2)+\frac{1}{2}|\mathcal{C}_1|+\sum_{H\in\mathcal{C}_{\geq 2}}(3\mathrm{rk}(H)-2)+2|\mathcal{C}_{\geq 2}|\leq 3N-3,$$

or in other words

$$\sum_{H\in\mathcal{C}}(3\mathrm{rk}(H)-2)\leq 3N-3-(\frac{1}{2}|\mathcal{C}_1|+2|\mathcal{C}_{\geq 2}|).$$

If

$$\sum_{H\in\mathcal{C}}(3\mathrm{rk}(H)-2)\leq 3N-6,$$

then we are done as in Case 1. Otherwise, we either have $|\mathcal{C}_{\geq 2}|=1$ and $|\mathcal{C}_1|=0$, or $|\mathcal{C}_{\geq 2}|=0$ and $|\mathcal{C}_1|\leq 4$.

We now assume that $|\mathcal{C}_{\geq 2}|=1$ and $|\mathcal{C}_1|=0$. In this case, the graph of actions $\mathcal{G}$ consists of a central vertex corresponding to an elliptic subgroup $H\in\mathcal{C}_{\geq 2}$, which is attached to k indecomposable subtrees (dual to laminations on surfaces) by edges with trivial or cyclic stabilizers. We denote by $\sigma_1,\dots,\sigma_k$ the ranks of the stabilizers of these minimal components, and by E_0 the number of edges with trivial stabilizer in $\mathcal{G}$. For all $i\in\{1,\dots,k\}$, we have $\sigma_i\geq 2$, and Lemma B.5.30 implies that

$$N=\sum_{i=1}^{k}(\sigma_i-1)+\mathrm{rk}(H)+E_0.$$

Again, we get that $3\mathrm{rk}(H)-2\leq 3N-6$, except possibly if $k=1$ and $\sigma_1=2$ (and $E_0=0$). In this case, the corresponding surface is a torus having a single boundary component (there are no minimal foliations on spheres having at most 3 boundary components, nor on projective planes with at most 2 boundary components, nor on a Klein bottle with one boundary component [CV91]). This contradicts the fact that $H\in\mathcal{C}_{\geq 2}$.

We now assume that $|\mathcal{C}_{\geq 2}|=0$, and $|\mathcal{C}_1|\leq 4$. In this case, the graph of actions $\mathcal{G}$ is a tree that consists of a single vertex v_0 corresponding to a connected surface S, attached to vertices corresponding to subgroups in $\mathcal{C}$ by edges with trivial or cyclic stabilizer. Denoting by m the number of boundary components of S (which is also equal to the number of edges with nontrivial stabilizer in $\mathcal{G}$), and by s the rank of the fundamental group of S, we get from Lemma B.5.30 that

$$N=\sum_{H\in\mathcal{C}}\mathrm{rk}(H)+s-m.$$

This implies that

$$\sum_{H\in\mathcal{C}}(3\mathrm{rk}(H)-2)\leq 3N+m-3s,$$

which is bounded by $3N-6$ as soon as $3s-m \geq 6$ (in which case we conclude as in Case 1).

If S is a nonorientable surface of genus $g \geq 1$, then $s = g+m-1$, and the condition $3s-m \geq 6$ is equivalent to $3g+2m \geq 9$. This condition is satisfied, except in the cases where either $g = 1$ and $m \leq 2$, or $g = 2$ and $m = 1$. However, as we have already mentioned, there is no minimal measured lamination on a projective plane having at most 2 boundary components, nor on a Klein bottle with one boundary component.

If S is an orientable surface of genus g, then $s = 2g+m-1$, and the condition $3s-m \geq 6$ is equivalent to $6g+2m \geq 9$. This condition is satisfied, except in the cases where either $g=1$ and $m=1$, or $g=0$ and $m \leq 4$.

If $g = m = 1$, then S is a torus with a single boundary component, whose fundamental group F_2 is amalgamated in the corresponding splitting of F_N to a group G_v along its boundary curve c. The curve c is trivial in the abelianization of F_N (it is a commutator), while it is not in the abelianization of G_v (it represents a primitive element in G_v). As τ is good, the element c belongs to a leaf of the subtree τ_v of τ, whose root subgroup is G_v. Hence $k(\tau) < k(\tau_v)$, and as $3\mathrm{rk}(\tau_v) - 2 = 3N-5$, we deduce that

$$3\mathrm{rk}(G_v) - k(\tau_v) - 2 \leq 3N - k(\tau) - 6,$$

which is enough to conclude.

If $g = 0$ and $m \leq 4$, then S is a sphere with 4 boundary components (there is no minimal lamination on a sphere having at most 3 boundary components). Using goodness of τ, and the fact that the product of the elements corresponding to its boundary curves is equal to 1, we get that

$$k(\tau) < \sum_{H \in \mathcal{C}} k(\tau_H)$$

(where τ_H denotes the subtree of τ whose root subgroup is H), and we conclude similarly. ☐

Random products of mapping classes of surfaces

In the case where $gr(\mu)$ is contained in the mapping class group $\mathrm{Mod}(S)$ of a compact, orientable, hyperbolic surface S with nonempty totally geodesic boundary, the length of the isotopy class of any simple closed curve on S, measured in any hyperbolic metric on S, is bi-Lipschitz equivalent to the length of the corresponding element of the (free) fundamental group of S. Given an oriented compact surface S with genus g and s boundary components, the *complexity* of S is defined as $\xi(S) := 3g+s-3$. In the case where $s \geq 1$, the rank N of the fundamental group of S satisfies $\xi(S) \geq \frac{3N-2}{4}$. Corollary B.5.25 therefore yields the following statement, which refines Karlsson's results in [Kar14].

Corollary B.5.32. *Let S be a compact hyperbolic oriented surface with nonempty totally geodesic boundary. Let μ be a probability measure on $Mod(S)$, with finite first moment with respect to Thurston's asymmetric metric on the Teichmüller space of S. Then there exist (deterministic) $\lambda_1, \dots, \lambda_p > 0$ such that for almost every sample path $(\Phi_n)_{n\in\mathbb{N}}$ of the random walk on $(Mod(S), \mu)$, all simple closed curves α on S, and all hyperbolic metrics ρ on S, either $l_\rho(\Phi_n(\alpha))$ grows subexponentially, or there exists $i \in \{1, \dots, p\}$ such that*

$$\lim_{n\to+\infty} \frac{1}{n} \log l_\rho(\Phi_n(\alpha)) = \lambda_i.$$

In addition, we have $p \leq \xi(S)$.

By combining our arguments with Karlsson's [Kar14], Corollary B.5.32 can also be proved in the case of a closed orientable surface. Proper subsurfaces play the role of proper free factors of F_N, and the filtration of F_N provided by Theorem B.5.25 is replaced by a decomposition of the surface into subsurfaces.

Theorem B.5.33. *Let S be a compact hyperbolic oriented surface with (possibly empty) totally geodesic boundary. Let μ be a probability measure on $Mod(S)$ having finite first moment with respect to Thurston's asymmetric metric on the Teichmüller space of S. Then there exists a decomposition of S into subsurfaces $\{S_i\}_{1\leq i\leq k}$, and for all $i\in\{1,\dots,k\}$, a* Lyapunov exponent *$\lambda_i\geq 0$, so that for almost every sample path $(\Phi_n)_{n\in\mathbb{N}}$ of the random walk on $(Mod(S),\mu)$, all simple closed curves α on S, and all hyperbolic metrics ρ on S, the limit*

$$\lim_{n\to+\infty}\frac{1}{n}\log l_\rho(\Phi_n(\alpha))$$

exists, and is equal to the maximum of the Lyapunov exponents of a subsurface S_i crossed by α (in the case where α is one of the curves defining the decomposition of S, the limit is equal to 0). The number of positive Lyapunov exponents is bounded by $\xi(S)$.

We call such a decomposition an *Oseledets decomposition* of S for the measure μ.

Sketch of proof of Theorem B.5.33. The horoboundary of the Teichmüller space $Teich(S)$ of S, equipped with Thurston's asymmetric metric, has been identified by Walsh with the space $\mathcal{PMF}$ of projectified measured foliations. Applying Lemma B.5.9 to the map Θ that sends a measured foliation to its support (which is a disjoint union of subsurfaces of S), we see that all μ-stationary measures on $\mathcal{PMF}$ are concentrated on the set of measured foliations whose supports have finite $gr(\mu)$-orbit. Following Karlsson's argument in [Kar14], we get for almost every sample path $(\Phi_n)_{n\in\mathbb{N}}$ of the random walk on $(\mathrm{Mod}(S),\mu)$ the existence of $\eta\in\mathcal{PMF}$ such that for all simple closed curves α on S such that $i(\eta,\alpha)>0$, and all hyperbolic metrics ρ on S, we have

$$\lim_{n\to+\infty}\frac{1}{n}\log l_\rho(\Phi_n(\alpha))=l,$$

where l denotes the drift of the random walk on $(\mathrm{Mod}(S),\mu)$. In addition, we can assume that the support of η has finite $gr(\mu)$-orbit. The condition $i(\eta,\alpha)=0$ is equivalent to α lying in the complement S' of the support of η in S (or α being one of the boundary curves of this support). Arguing by induction on the complexity of the surface, we get the existence of a decomposition of S', which is an Oseledets decomposition for the first return measure on the stabilizer of S'. Arguing as in Proposition B.5.20, we get that the decomposition of S obtained by adding the boundary curves of S' to this decomposition of S' is an Oseledets decomposition for μ. ∎

Annexe C

The Poisson boundary of Out(F_N)

Abstract

Let μ be a probability measure on $\text{Out}(F_N)$ with finite first logarithmic moment with respect to the word metric, finite entropy, and whose support generates a nonelementary subgroup of $\text{Out}(F_N)$. We show that almost every sample path of the random walk on $(\text{Out}(F_N), \mu)$, when realized in Culler and Vogtmann's outer space, converges to the simplex of a free, arational tree. We then prove that the space $\mathcal{FI}$ of simplices of free and arational trees, equipped with the hitting measure, is the Poisson boundary of $(\text{Out}(F_N), \mu)$. Using Bestvina–Reynolds' and Hamenstädt's description of the Gromov boundary of the complex $\mathcal{FF}_N$ of free factors of F_N, this gives a new proof of the fact, due to Calegari and Maher, that the realization in $\mathcal{FF}_N$ of almost every sample path of the random walk converges to a boundary point. We get in addition that $\partial\mathcal{FF}_N$, equipped with the hitting measure, is the Poisson boundary of $(\text{Out}(F_N), \mu)$.

Contents

Introduction

Over the past decades, the study of the group $\text{Out}(F_N)$ of outer automorphisms of a finitely generated free group has greatly benefited from the study of its action on some geometric complexes, among which stand Culler and Vogtmann's outer space CV_N, which is the space of homothety classes of free, minimal, isometric and simplicial actions of F_N on simplicial metric trees [CV86], and Hatcher and Vogtmann's complex of free factors [HV98]. A main source of inspiration in this study comes from analogies with mapping class groups of surfaces, and their actions on the associated Teichmüller spaces and curve complexes. We aim at understanding the behaviour of random walks on $\text{Out}(F_N)$-complexes.

Given a countable group G and a probability measure μ on G, the *(right) random walk* on (G, μ) is the Markov chain on G whose initial distribution is given by the Dirac measure at the identity element, and whose transition probabilities are given by $p(g, h) := \mu(g^{-1}h)$. In other words, the position of the random walk on (G, μ) at time n is given from its position $g_0 = e$ by successive multiplications on the right by independent μ-distributed increments s_i, i.e. $g_n = s_1 \dots s_n$, and the distribution of this position is given by the n-fold convolution of μ. We equip the *path space* $\mathcal{T} := G^{\mathbb{N}}$ with the measure $\mathbb{P}$ defined as the image of the product measure $\mu^{\otimes \mathbb{N}}$ under the map $(s_i)_{i\in\mathbb{N}} \mapsto (g_i)_{i\in\mathbb{N}}$.

Random walks on mapping class groups have first been studied by Kaimanovich and Masur, whose seminal paper [KM96] has been a main source of inspiration for our work. Given a probability measure μ on the mapping class group $Mod(S)$ of a surface S, whose support generates a nonelementary subgroup of $Mod(S)$, Kaimanovich and Masur have shown that $\mathbb{P}$-almost every sample path of the random walk on $(Mod(S), \mu)$ converges to a uniquely ergodic minimal measured foliation in the Thurston boundary $\mathcal{PMF}$ of $Teich(S)$, and that the hitting measure ν is the only μ-stationary measure on $\mathcal{PMF}$. Using Reynolds' study of *arational trees* in the boundary of outer space [Rey12] as an analogue for minimal foliations in the boundary of Teichmüller spaces, we partly translate Kaimanovich and Masur's work to the Out(F_N) case. A tree $T \in \partial CV_N$ is *arational* if every proper free factor of F_N acts freely and discretely on its minimal subtree in T. Arational trees are either free (and indecomposable) actions of F_N, or they are dual to an arational lamination on a surface having a single boundary component [Rey12].

Associated to any $T \in \overline{CV_N}$ with dense orbits is a *simplex* of length measures [Gui00], which describes the collection of all possible metrics on the topological tree underlying T. We denote by $\mathcal{AT}$ the space of equivalence classes of arational trees, and by $\mathcal{FI}$ the space of equivalence classes of free arational trees, under the equivalence relation that identifies two trees that belong to the same simplex. A tree is *uniquely ergometric* if its simplex is reduced to a point. Uniquely ergometric trees provide an analogue of uniquely ergodic foliations on surfaces. We don't know whether sample paths of random walks on Out(F_N) almost surely converge to uniquely ergometric trees. However, we shall prove the following statement. We define a subgroup $H \subseteq \text{Out}(F_N)$ to be *nonelementary* if the H-orbits of all proper free factors of F_N, of all projective arational trees, and of all conjugacy classes of elements of F_N, are infinite.

Theorem C.0.1. *Let μ be a probability measure on Out(F_N), whose support generates a nonelementary subgroup of Out(F_N). For $\mathbb{P}$-a.e. sample path $\mathbf{g} := (g_n)_{n\in\mathbb{N}}$ of the random walk on (Out(F_N), μ), there exists a simplex $\xi(\mathbf{g}) \in \mathcal{FI}$ such that for all $T_0 \in CV_N$, the sequence $(g_nT_0)_{n\in\mathbb{N}}$ converges to $\xi(\mathbf{g})$. The hitting measure is nonatomic, and it is the only μ-stationary measure on $\mathcal{FI}$.*

We then show (under some further assumptions on the measure μ) that $\mathcal{FI}$, equipped with the hitting measure ν, is the Poisson boundary of $(\text{Out}(F_N), \mu)$. Theorem C.0.1 ensures that $(\mathcal{FI}, \nu)$ is the typical example of a μ-boundary. A *μ-boundary* is a probability space (B, ν), which is the quotient of the path space $(\mathcal{T}, \mathbb{P})$ with respect to some shift-invariant and G-invariant measurable partition (in particular $\nu = bnd_*\mathbb{P}$, where $bnd : \mathcal{T} \to B$ is the projection map).

A μ-boundary (B, ν) is a *Poisson boundary* if it is maximal, i.e. every μ-boundary is the quotient of (B, ν) under some G-invariant measurable partition. If we equip the path space $\mathcal{T}$ with the measure $\mathbb{P}_m$ corresponding to an initial distribution for the random walk given by the counting measure on G, then the space of ergodic components of the shift in $\mathcal{T}$ is an abstract realization of the Poisson boundary of (G, μ). Given a group G equipped with a probability measure μ, a natural question is that of identifying the Poisson boundary of (G, μ) with some concrete G-space (which will usually be a topological space, although the Poisson boundary does not carry any intrinsic topology, and is only defined as a measure space). One motivation for this question comes from the problem of understanding bounded μ-harmonic functions on G. Indeed, when (B, ν) is the Poisson boundary of (G, μ), the formula

$$f(g) = \int_B \widehat{f}(x) dg_* \nu(x)$$

gives an isometry between the Banach space of μ-essentially bounded μ-harmonic functions on X, and $L^\infty(B)$. Our main result is the following.

Theorem C.0.2. *Let μ be a probability measure on $Out(F_N)$, whose support is finite and generates a nonelementary subgroup of $Out(F_N)$, and let ν be the hitting measure on $\mathcal{FI}$. Then the measure space $(\mathcal{FI}, \nu)$ is the Poisson boundary of $(Out(F_N), \mu)$.*

Theorem C.0.2 is actually true under more general assumptions on the measure μ (finiteness of the support can be replaced by a finite first logarithmic moment condition with respect to the word metric on $\mathrm{Out}(F_N)$, and a finite entropy condition, see Theorem C.3.3).

In [Kai00], Kaimanovich has developed tools coming from entropy theory to prove that a μ-boundary is the Poisson boundary. In particular, he provides a *strip criterion* which requires considering a μ-boundary B_+ simultaneously with a $\check{\mu}$-boundary B_- (where $\check{\mu}$ is the probability measure on G defined by $\check{\mu}(g) := \mu(g^{-1})$ for all $g \in G$), and assigning to almost every pair of points in $B_- \times B_+$ a strip contained in G, which is sufficiently thin in the sense that its intersection with balls for a word metric on G grows subexponentially with the radius of the ball. Given a probability measure μ on the mapping class group $Mod(S)$ of a surface S, satisfying some finiteness conditions, and whose support generates a subgroup of $Mod(S)$ that contains two independent pseudo-Anosov homeomorphisms, Kaimanovich and Masur have shown that $(\mathcal{PMF}, \nu)$ is the Poisson boundary of $(Mod(S), \mu)$, by using strips coming from Teichmüller geodesics [KM96, Theorem 2.3.1].

Our definition of the strips is based on a simplified version of Hamenstädt's construction of *lines of minima* in outer space [Ham14b]. We now provide an outline of this construction. There is a natural length pairing between trees in CV_N and elements in F_N, defined by letting $\langle T, g\rangle$ be the translation length of g in T. Kapovich and Lustig have shown [KL09] that this length pairing extends to an intersection form between trees and *geodesic currents*, which were introduced by Kapovich in [Kap05, Kap06]. Given trees $T \in cv_N$ and $T' \in \overline{cv_N}$ (in $\overline{cv_N}$, trees are considered up to isometry, instead of homothety), and a pair (η, η') of geodesic currents, we define

$$\Lambda_{\eta,\eta'}(T, T') := \max\{\frac{\langle T', \eta\rangle}{\langle T, \eta\rangle}, \frac{\langle T', \eta'\rangle}{\langle T, \eta'\rangle}\}.$$

This measures the maximal stretch of η and η' from T to T'. Denoting by $\mathrm{Lip}(T, T')$ the smallest Lipschitz constant of an F_N-equivariant map from T to T', we always have

$$\Lambda_{\eta,\eta'}(T, T') \leq \mathrm{Lip}(T, T'),$$

and White has shown that we can always find a candidate element $g \in F_N$ whose stretch from T to T' is equal to $\mathrm{Lip}(T, T')$ (and we can even choose g among a finite set of elements of F_N that only depends on the tree T), see [FM11b].

For generic pairs (η, η') of currents, we have $\Lambda_{\eta,\eta'}(T, T') > 0$, and for all $L \geq 1$, we define the *L-axis* of the pair (η, η') as the set of all trees in CV_N for which

$$1 \leq \frac{\mathrm{Lip}(T, T')}{\Lambda_{\eta,\eta'}(T, T')} \leq L$$

for all $T' \in \overline{CV_N}$. In other words, a tree $T \in CV_N$ is in the L-axis of (η, η') if the stretch of either η or η' gives a good estimate of the Lipschitz distortion from T to any $T' \in \overline{CV_N}$, up to an error controlled by L (or informally, if the pair (η, η') is a fairly good pair of candidates for the tree T). Following Hamenstädt's arguments [Ham14b], we show that these axes are close to being geodesics in CV_N for the symmetric Lipschitz metric (see [FM11b] for an introduction to this metric), although they may contain holes (notice that the L-axis of a pair (η, η') can even be empty if L is too small). This will be the key point for checking the growth condition on the strips.

Associated to any arational tree T is a finite collection of ergodic currents $\text{Erg}(T)$. This enables us to associate an L-strip in $\text{Out}(F_N)$ to almost every pair of trees $(T_-, T_+) \in \mathcal{FI} \times \mathcal{FI}$. We then show that we can choose L in a uniform way to ensure that the strips are almost surely nonempty.

Using recent work of Bestvina and Reynolds [BR13] and Hamenstädt [Ham14a], our results can be interpreted in terms of the free factor complex and its Gromov boundary. When $N \geq 3$, the *free factor complex* $\mathcal{FF}_N$ is the simplicial complex whose vertices are the conjugacy classes of proper free factors of F_N, and higher dimensional simplices correspond to chains of inclusion of free factors (one has to slightly modify the definition when $N = 2$ to ensure that $\mathcal{FF}_2$ is connected). It was proven to be Gromov hyperbolic by Bestvina and Feighn [BF14b], see also [KR14] for an alternative proof. Its Gromov boundary was identified by Bestvina and Reynolds [BR13] and Hamenstädt [Ham14a] with the space of simplices of arational trees in ∂CV_N. Using their work, Theorems C.0.1 and C.0.2 lead to the following statement.

Theorem C.0.3. *Let μ be a probability measure on Out(F_N), whose support generates a nonelementary subgroup of Out(F_N). Then for $\mathbb{P}$-almost every sample path $\mathbf{g} := (g_n)_{n \in \mathbb{N}}$ of the random walk on $(\mathit{Out}(F_N), \mu)$, there exists $\xi(\mathbf{g}) \in \partial\mathcal{FF}_N$, such that for all $x \in \mathcal{FF}_N$, the sequence $(g_n x)_{n \in \mathbb{N}}$ converges to $\xi(\mathbf{g})$. The hitting measure ν on $\partial\mathcal{FF}_N$ is the unique μ-stationary measure on $\partial\mathcal{FF}_N$. If in addition, the measure μ has finite support, then $(\partial\mathcal{FF}_N, \nu)$ is the Poisson boundary of $(\mathit{Out}(F_N), \mu)$.*

The convergence statement was obtained with different methods by Calegari and Maher, in the more general context of groups acting on (possibly nonproper) Gromov hyperbolic spaces [CM12, Theorem 5.34].

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C.1 Preliminaries on Out(F_N) and related complexes

C.1.1 Outer space

Let $N \geq 2$. *Outer space* CV_N is defined to be the space of simplicial free, minimal, isometric actions of F_N on simplicial metric trees, up to F_N-equivariant homotheties [CV86] (an F_N-action on a tree is *minimal* if there is no proper invariant subtree). We denote by cv_N the *unprojectivized outer space*, in which trees are considered up to F_N-equivariant isometries, instead of homotheties. The group $\text{Out}(F_N)$ acts on CV_N and on cv_N on the left by setting $\Phi(T,\rho) = (T, \rho\circ\phi^{-1})$ for all $\Phi \in \mathrm{Out}(F_N)$, where $\rho : F_N \to \mathrm{Isom}(T)$ denotes the action, and $\phi \in \mathrm{Aut}(F_N)$ is any lift of Φ to $\mathrm{Aut}(F_N)$.

An $\mathbb{R}$-*tree* is a metric space (T, d_T) in which any two points x and y are joined by a unique arc, which is isometric to a segment of length $d_T(x,y)$. Let T be an F_N-*tree*, i.e. an $\mathbb{R}$-tree equipped with an isometric action of F_N. For $g \in F_N$, the *translation length* of g in T is defined to be

$$||g||_T := \inf_{x\in T} d_T(x, gx).$$

Culler and Morgan have shown in [CM87, Theorem 3.7] that the map

$$\begin{array}{rccc} i: & cv_N & \to & \mathbb{R}^{F_N} \\ & T & \mapsto & (||g||_T)_{g\in F_N} \end{array}$$

is an embedding, whose image has projectively compact closure $\overline{CV_N}$ [CM87, Theorem 4.5]. Bestvina and Feighn [BF94], extending results by Cohen and Lustig [CL95], have characterized the points of this compactification as being the minimal, *very small* F_N-trees, i.e. the F_N-trees with trivial or maximally cyclic arc stabilizers and trivial tripod stabilizers. We denote by $\overline{cv_N}$ the lift of $\overline{CV_N}$ to $\mathbb{R}^{F_N}$.

C.1.2 Algebraic laminations and currents

Let $\partial^2 F_N := \partial F_N \times \partial F_N \smallsetminus \Delta$, where Δ is the diagonal. Denote by $i : \partial^2 F_N \to \partial^2 F_N$ the involution that exchanges the factors. An *algebraic lamination* is a nonempty, closed, F_N-invariant and i-invariant subset of $\partial^2 F_N$. Any nontrivial element $g \in F_N$ determines an element $(g^{-\infty}, g^{+\infty}) \in \partial^2 F_N$ by setting $g^{-\infty} := \lim_{n\to+\infty} g^{-n}$ and $g^{+\infty} := \lim_{n\to+\infty} g^n$. Let $T \in \overline{cv_N}$. For $\epsilon > 0$, let

$$L_\epsilon(T) := \overline{\{(g^{-\infty}, g^{+\infty}) | ||g||_T < \epsilon\}}.$$

Then

$$L(T) := \bigcap_{\epsilon>0} L_\epsilon(T)$$

is an algebraic lamination, called the lamination *dual* to the tree T (see [CHL08a, CHL08b] for an extended study of algebraic laminations). Notice that $L(T)$ only depends on the projective class of the tree T, and hence can be defined for $T \in \overline{CV_N}$. We have $L(T) \neq \emptyset$ if and only if $T \in \partial CV_N := \overline{CV_N} \smallsetminus CV_N$.

A *current* on F_N is an F_N-invariant Borel measure on $\partial^2 F_N$ that is finite on compact subsets of $\partial^2 F_N$. The systematic study of currents on F_N was initiated by Kapovich [Kap05, Kap06]. We denote by $Curr_N$ the set of currents on F_N, equipped with the weak-$*$ topology, and by $\mathbb{P}Curr_N$ the space of projective classes (i.e. homothety classes) of currents. The space $\mathbb{P}Curr_N$ is compact [Kap06, Proposition 2.5].

To every $g \in F_N$ which is not of the form h^k for any $h \in F_N$ and $k > 1$ (we say that g is not a *proper power*), one associates a *rational current* η_g by letting $\eta_g(S)$ be the number of translates of $(g^{-\infty}, g^{+\infty})$ that belong to S for all closed-open subsets $S \subseteq \partial^2 F_N$, see [Kap06, Definition 5.1] (for the case of proper powers one may set $\eta_{h^k} := k\eta_h$). The group $\mathrm{Out}(F_N)$ acts on $Curr_N$ on the left in the following way [Kap06, Proposition 2.15]. Given a compact set $K \subseteq \partial^2 F_N$, an element $\Phi \in \mathrm{Out}(F_N)$, and a current $\eta \in Curr_N$, we set $\Phi(\eta)(K) := \eta(\phi^{-1}(K))$, where $\phi \in \mathrm{Aut}(F_N)$ is any representative of Φ. In [Kap06, Section 5], Kapovich defined an intersection form between elements of cv_N and currents, which was then extended by Kapovich and Lustig to trees in $\overline{cv_N}$ [KL09].

Theorem C.1.1. *(Kapovich–Lustig [KL09, Theorem A]) There exists a unique Out(F_N)-invariant continuous function*

$$\langle .,. \rangle : \overline{cv_N} \times Curr_N \to \mathbb{R}_+$$

which is $\mathbb{R}_+$-homogeneous in the first coordinate and $\mathbb{R}_+$-linear in the second coordinate, and such that for all $T \in \overline{cv_N}$, and all $g \in F_N \smallsetminus \{e\}$, we have $\langle T, \eta_g \rangle = ||g||_T$.

Kapovich and Lustig give the following characterization of zero intersection.

Theorem C.1.2. *(Kapovich–Lustig [KL10, Theorem 1.1]) For all $T \in \overline{cv_N}$ and all $\eta \in Curr_N$, we have $\langle T, \eta \rangle = 0$ if and only if Supp(η) $\subseteq L(T)$. In particular, for all $T \in cv_N$ and all $\eta \in Curr_N$, we have $\langle T, \eta \rangle \neq 0$, while for all $T \in \partial cv_N$, there exists $\eta \in Curr_N$ such that $\langle T, \eta \rangle = 0$.*

A projective current $[\eta] \in \mathbb{P}Curr_N$ is *ergodic* if for every F_N-invariant measurable subset $S \subseteq \partial^2 F_N$, we either have $\eta(S) = 0$ or $\eta(\partial^2 F_N \smallsetminus S) = 0$. We denote by Erg_N the space of ergodic currents, which coincides with the set of extreme points of the compact convex space $\mathbb{P}Curr_N$. Given an F_N-tree T, we denote by Dual(T) the space of all projective currents $[\eta] \in \mathbb{P}Curr_N$ such that $\langle T, \eta \rangle = 0$ (this makes sense since nullity of $\langle T, \eta \rangle$ only depends on the projective class of η). For all $T \in \overline{cv_N}$, the space Dual(T) is a compact convex subspace of $\mathbb{P}Curr_N$. Equivariance of the intersection form implies that for all $T \in \overline{cv_N}$ and $\Phi \in \text{Out}(F_N)$, we have $\Phi\text{Dual}(T) = \text{Dual}(\Phi T)$. The extreme points of Dual(T) are the ergodic currents which are dual to T. Denoting by Erg(T) the set of such ergodic currents, for all $\Phi \in \text{Out}(F_N)$, we have $\text{Erg}(\Phi T) = \Phi\text{Erg}(T)$. Coulbois and Hilion have shown that Dual(T) is finite-dimensional as soon as the F_N-action on T is free and has dense orbits [CH14].

Theorem C.1.3. *(Coulbois–Hilion [CH14, Theorem 1.1]) Let T be an $\mathbb{R}$-tree with a free, minimal action of F_N by isometries with dense orbits. Then Dual(T) contains at most $3N-5$ projective classes of ergodic currents.*

C.1.3 The Lipschitz metric on outer space

Outer space is equipped with an asymmetric metric: the distance $d(T, T')$ between two trees $T, T' \in CV_N$ is defined as the logarithm of the infimal Lipschitz constant $\text{Lip}(T, T')$ of an F_N-equivariant map from the covolume 1 representative of T to the covolume 1 representative of T', see [FM11b]. One can symmetrize the metric on CV_N by setting $d_{sym}(T, T') := d(T, T') + d(T', T)$. The Lipschitz metric on CV_N can be interpreted in terms of the intersection pairing between trees and currents. Given a subset $X \subseteq \mathbb{P}Curr_N$, we let

$$\Lambda_X(T, T') := \sup_{[\eta] \in X} \frac{\langle T', \eta \rangle}{\langle T, \eta \rangle}.$$

Theorem C.1.4. *(White, see [AK11, Proposition 2.3], [FM11b, Proposition 3.15] or [Ham14b, Lemma 4.1]) For all $T, T' \in CV_N$, we have $Lip(T, T') = \Lambda_{\mathbb{P}Curr_N}(T, T')$.*

Algom-Kfir has shown in [AK13, Proposition 4.5] that the equality stated in Theorem C.1.4 also holds when $T' \in \overline{CV_N}$ (she actually states her result when T' belongs to the metric completion of CV_N). Notice that equality between $\text{Lip}(T, T')$ and $\Lambda_{\mathbb{P}Curr_N}(T, T')$ does not depend on the choice of a representative of the projective classes of T and T'.

C.1.4 Arational trees

Let $H \leq F_N$ be a finitely generated subgroup of F_N. The boundary ∂H naturally embeds in ∂F_N. We say that H *carries a leaf* of an algebraic lamination L if $L \cap \partial^2 H \neq \emptyset$. A tree $T \in \partial CV_N$ is *arational* if no proper free factor of F_N carries a leaf of $L(T)$. We denote by $\widetilde{\mathcal{AT}}$ the subspace of ∂CV_N consisting of arational trees. Arational trees have dense orbits, and Reynolds has shown that arational trees are either free (and indecomposable) or dual to an arational measured lamination on a surface with one boundary component [Rey12, Theorem 1.1]. We denote by $\widetilde{\mathcal{FI}}$ (standing for *free indecomposable*) the subspace of $\widetilde{\mathcal{AT}}$ consisting of free actions of F_N. Let $\sim$ be the equivalence relation on $\widetilde{\mathcal{AT}}$ defined by $T \sim T'$ if $L(T) = L(T')$. Two trees $T, T' \in \widetilde{\mathcal{AT}}$ are equivalent if and only if they belong to the same *simplex* of length measures in ∂CV_N (see [Gui98, Section 5] for definitions), i.e. they have the same underlying topological tree, see [CHL07]. Let $\mathcal{AT} := \widetilde{\mathcal{AT}}/\sim$ and $\mathcal{FI} := \widetilde{\mathcal{FI}}/\sim$. Classes of the relation $\sim$ are compact subspaces of ∂CV_N [BR13, Lemma 7.1]. By definition of the relation $\sim$, and thanks to Theorem C.1.2, it makes sense to define Dual(T) and Erg(T) for $T \in \mathcal{FI}$. Theorem C.1.3 therefore implies the following fact.

Proposition C.1.5. *For all $T \in \mathcal{FI}$, the set Erg(T) is finite, of cardinality at most $3N-5$.* $\square$

The following unique duality statement is a version of a theorem due Bestvina and Reynolds [BR13, Theorem 4.4] and Hamenstädt [Ham14a, Corollary 10.6].

Theorem C.1.6. *Let $T_1 \in \widetilde{\mathcal{FI}}$, and let $\eta \in Curr_N$ be such that $\langle T_1, \eta \rangle = 0$. If $T_2 \in \partial CV_N$ also satisfies $\langle T_2, \eta \rangle = 0$, then $T_2 \in \widetilde{\mathcal{FI}}$ and $T_1 \sim T_2$.*

Proof. By Theorem C.1.2, as $\langle T_1, \eta \rangle = 0$, we have Supp($\eta$) $\subseteq L(T_1)$. If Supp(η) carried a periodic leaf (whose F_N-translates form the support of a rational current η_g for some $g \in F_N$), then we would have $||g||_{T_1} = 0$, contradicting freeness of the F_N-action on T_1. In addition, the support of a current cannot carry isolated nonperiodic leaves, since translates of such leaves have accumulation points, and currents are Radon measures. This implies that Supp(η) does not carry any isolated leaf. Therefore Supp(η) is contained in the derived lamination $L'(T_1)$ (i.e. the sublamination of $L(T_1)$ consisting of non-isolated leaves). Since $T_1 \in \widetilde{\mathcal{FI}}$, by [BR13, Proposition 4.2], the lamination $L'(T_1)$ is minimal (i.e. it does not contain any proper sublamination), so Supp(η) $= L'(T_1)$. Since we also have $\langle T_2, \eta \rangle = 0$, Theorem C.1.2 implies that $L'(T_1) \subseteq L(T_2)$.

If T_2 does not have dense orbits, then all leaves of $L(T_2)$ are carried by a vertex group of the canonical decomposition of T_2 as a graph of actions with dense orbits (see [KL09]). Such vertex groups have infinite index in F_N. However, as T_1 is free and indecomposable, a theorem of Reynolds [Rey11a] shows that no leaf of $L(T_1)$ is carried by an infinite index subgroup of F_N. This yields a contradiction.

Therefore, the tree T_2 has dense orbits, and it follows from [CHL08b, Section 8] that $L(T_2)$ is diagonally closed. By [BR13, Proposition 4.2], the lamination $L(T_1)$ is the diagonal closure of $L'(T_1)$. Hence we have $L(T_1) \subseteq L(T_2)$. Since T_1 is indecomposable, this implies that $L(T_1) = L(T_2)$ by [BR13, Proposition 3.1], and $T_2 \in \widetilde{\mathcal{FI}}$. $\square$

Following Hamenstädt [Ham14b, Section 3], we say that a pair $(\eta, \eta') \in Curr_N^2$ is *positive* if for all $T \in \overline{cv_N}$, we have $\langle T, \eta + \eta' \rangle > 0$ (this again makes sense when $[\eta], [\eta'] \in \mathbb{P}Curr_N$). Denote by Δ the diagonal in $\mathcal{FI} \times \mathcal{FI}$. As a consequence of Proposition C.1.5 and Theorem C.1.6, we get the following fact.

Corollary C.1.7. *For all pairs $(T,T') \in \mathcal{FI} \times \mathcal{FI} \smallsetminus \Delta$, and all $(\eta,\eta') \in Dual(T) \times Dual(T')$, the pair (η,η') is positive. In particular, the set $Erg(T) \times Erg(T')$ is a finite set of positive pairs of projective currents.* ☐

C.2 Random walks in Out(F_N)

In this section, all topological spaces are equipped with their Borel σ-algebra. Let G be a countable group, and μ a probability measure on G. We denote by $gr(\mu)$ the subgroup of G generated by the support of the measure μ. The *random walk* on G with respect to the measure μ is the Markov chain on G with transition probabilities $p(x,y) := \mu(x^{-1}y)$. The *step space* for the random walk is the product probability space $(G^{\mathbb{N}}, \mu^{\otimes \mathbb{N}})$. The position of the random walk at time n is given from its position $g_0 = e$ at time 0 by multiplying on the right by a sequence of independent μ-distributed increments s_i, i.e. $g_n = s_1 \dots s_n$. So the distribution of the location of the random walk at time n is given by the n-fold convolution of the measure μ, which we denote by μ^{*n}. We equip the *path space* $\mathcal{T} := G^{\mathbb{N}}$ with the σ-algebra $\mathcal{A}$ generated by the cylinders $\{\mathbf{g} \in \mathcal{T} | g_i = g\}$ for all $i \in \mathbb{N}$ and all $g \in G$. We denote by $\mathbb{P}$ the probability measure on $\mathcal{T}$ induced by the map $(s_1, s_2, \dots) \mapsto (g_1, g_2, \dots)$.

Let μ be a probability measure on $\mathrm{Out}(F_N)$. We aim at understanding the asymptotic behaviour of the random walk on $\mathrm{Out}(F_N)$ with respect to the measure μ. A subgroup $H \subseteq \mathrm{Out}(F_N)$ is *nonelementary* if the H-orbits of all conjugacy classes of proper free factors of F_N, of all projective arational trees, and of all conjugacy classes of elements of F_N, are infinite. Combining [5] (see also [HM09]) with [Uya14, Theorem 5.4] and [KL11a, Theorem 5.6], one can show that this is equivalent to H containing two independent atoroidal fully irreducible elements (we will not use this fact in the sequel). The following result is a consequence of Propositions C.2.6, C.2.8 and C.2.15.

Theorem C.2.1. *Let μ be a probability measure on Out(F_N), whose support generates a nonelementary subgroup of Out(F_N). For $\mathbb{P}$-a.e. sample path $\mathbf{g} := (g_n)_{n\in\mathbb{N}}$ of the random walk on (Out(F_N), μ), there exists a simplex $\xi(\mathbf{g}) \in \mathcal{FI}$ such that for all $T_0 \in CV_N$, the sequence $(g_nT_0)_{n\in\mathbb{N}}$ converges to $\xi(\mathbf{g})$. The hitting measure is nonatomic, and it is the only μ-stationary measure on $\mathcal{FI}$.*

Question C.2.2. Is it true that $\mathbb{P}$-a.e. sample path of the random walk on $(\mathrm{Out}(F_N), \mu)$ converges to a *uniquely ergometric* tree, i.e. a tree whose corresponding simplex consists of a single element, as in the case of mapping class groups [KM96, Theorem 2.2.4] ? One could also ask the dual question of *unique ergodicity*, in the sense that there exists a unique current dual to the tree T, for limit points of sample paths of the random walk. It is known that the attracting tree in ∂CV_N of a nongeometric fully irreducible element of $\mathrm{Out}(F_N)$ is uniquely ergodic [CHL08c, Proposition 5.6]. As generic elements of $\mathrm{Out}(F_N)$ are fully irreducible and nongeometric, it seems reasonable to hope for such a result. However, Kaimanovich and Masur's argument in the case of mapping class groups relies on a theorem of Masur stating that any Teichmüller geodesic whose vertical foliation is minimal but not uniquely ergodic has to leave the thick part of the Teichmüller space of the associated surface [Mas92], and we do not know any good analogue of this theorem for outer space.

Remark C.2.3. If we remove the condition on orbits of conjugacy classes of elements of F_N in the definition of nonelementary subgroups, we still get convergence of almost every sample path to an element of $\mathcal{AT}$. However, if $gr(\mu)$ is nonelementary in this new sense, and virtually fixes the conjugacy class of an element in F_N, then it is virtually a subgroup of the mapping class group of a compact surface with a single boundary component. This case is already covered by Kaimanovich and Masur's work [KM96].

C.2.1 Stationary measures on ∂CV_N

The following proposition was essentially proved in [5, Proposition 3.2], without the assumption that $gr(\mu)$ does not preserve any finite set of conjugacy classes of elements of F_N. By the same reasoning as in the proof in [5], we will show this extra assumption implies that every μ-stationary measure is concentrated on the set of free actions. Measurability of $\widetilde{\mathcal{AT}}$ was proved in [5, Lemma 3.4], and measurability of $\widetilde{\mathcal{FI}}$ follows since freeness of the action is a measurable condition.

Proposition C.2.4. *Let μ be a probability measure on $Out(F_N)$, such that $gr(\mu)$ is nonelementary. Then every μ-stationary Borel probability measure on ∂CV_N is purely nonatomic and concentrated on $\widetilde{\mathcal{FI}}$.*

The proof of Proposition C.2.4 is based on the following classical statement, whose proof relies on a maximum principle argument.

Lemma C.2.5. *(Ballmann [Bal89], Kaimanovich–Masur [KM96, Lemma 2.2.2], Woess [Woe89, Lemma 3.4]) Let μ be a probability measure on a countable group G, and let ν be a μ-stationary measure on a G-space X. Suppose $E \subset X$ is a measurable subset such that for all $g \in gr(\mu)$, either $gE = E$ or $gE \cap E = \emptyset$, and the $gr(\mu)$-orbit of E is infinite. Then $\nu(E) = 0$.*

Proof of Proposition C.2.4. Let ν be a μ-stationary measure on ∂CV_N. The fact that $\nu(\widetilde{\mathcal{AT}}) = 1$ was proved in [5, Lemma 3.4]. Nonatomicity of ν follows from Lemma C.2.5 applied to the singleton $E = \{T\}$, where $T \in \widetilde{\mathcal{AT}}$, since nonelementarity of $gr(\mu)$ implies that the $gr(\mu)$-orbit of T is infinite.

Let X be a finite set of conjugacy classes of elements of F_N. The set E_X of trees in ∂CV_N whose collection of cyclic point stabilizers is equal to X is measurable, see [5, Lemma 3.5], and nonelementarity of $gr(\mu)$ implies that the $gr(\mu)$-orbit of E_X is infinite. As arational trees which are not free are dual to an arational measured foliation on a surface with one boundary component [Rey12, Theorem 1.1], for which the boundary curve is the only point stabilizer, Proposition C.2.4 follows from Lemma C.2.5 applied to the sets E_X. ☐

As a consequence of Proposition C.2.4, we get the following result.

Proposition C.2.6. *Let μ be a probability measure on $Out(F_N)$, whose support generates a nonelementary subgroup of $Out(F_N)$. Then there exists a μ-stationary probability measure on $\mathcal{FI}$, and all such measures are nonatomic.*

Proof. The first part of the statement is a consequence of Proposition C.2.4. Nonatomicity is proved as above, by noticing that if $gr(\mu)$ virtually fixes a simplex in $\mathcal{FI}$, then it also preserves virtually the set of extremal points of this simplex in $\overline{CV_N}$, which is finite by [Gui00, Corollary 5.4]. ☐

As any μ-stationary measure on ∂CV_N projects to a nonatomic μ-stationary measure on $\mathcal{FI}$, we get the following result. Notice that $\sim$-classes are compact subsets of ∂CV_N, in particular they are measurable.

Proposition C.2.7. *Let μ be a probability measure on $Out(F_N)$, whose support generates a nonelementary subgroup of $Out(F_N)$, and let ν be a μ-stationary probability measure on ∂CV_N. Then every class of the relation $\sim$ on $\widetilde{\mathcal{AT}}$ has ν-measure 0.* ☐

C.2.2 Limit points of random walks on CV_N, and convergence to $\mathcal{FI}$

Adapting Kaimanovich and Masur's argument from [KM96, Section 1.5] to the Out(F_N) case, we now study the possible limit points of sequences $(g_nT)_{n\in\mathbb{N}}$, where $(g_n)_{n\in\mathbb{N}}$ is a sequence of elements of Out(F_N) which tends to infinity, and $T \in \overline{CV_N}$ is an F_N-tree. We recall that whenever X is a Borel space, a sequence of measures $(\nu_n)_{n\in\mathbb{N}}$ on X *weakly converges* to a measure ν if $(\nu_n(f))_{n\in\mathbb{N}}$ converges to $\nu(f)$ for every bounded continuous real-valued function on X. The goal of the present section is to prove the following result. The convergence statement in Theorem C.2.1 is a consequence of Proposition C.2.8.

Proposition C.2.8. *Let μ be a probability measure on Out(F_N), whose support generates a nonelementary subgroup of Out(F_N), and let ν be a μ-stationary measure on ∂CV_N. Then for $\mathbb{P}$-a.e. sample path $\mathbf{g} := (g_n)_{n\in\mathbb{N}}$ of the random walk on $(Out(F_N), \mu)$, there is a simplex $\Delta(\mathbf{g}) \subseteq \widetilde{\mathcal{FI}}$, such that*

- *the translates $g_n\nu$ weakly converge to a measure $\lambda(\mathbf{g})$ supported on $\Delta(\mathbf{g})$, and*
- *for all $T \in CV_N$, all limit points of the sequence $(g_nT)_{n\in\mathbb{N}}$ belong to $\Delta(\mathbf{g})$.*

Our proof of Proposition C.2.8 relies on the following general statement about random walks on countable groups. We recall that $\mathcal{T}$ denotes the path space of the random walk.

Lemma C.2.9. *(Furstenberg [Fur73], Kaimanovich–Masur [KM96, Lemma 2.2.3]) Let μ be a probability measure on a countable group G, and let ν be a μ-stationary measure on a compact separable G-space. Then for $\mathbb{P}$-a.e. sample path $\mathbf{g} = (g_n)_{n\in\mathbb{N}}$ of the random walk on (G, μ), the translates $g_n\nu$ converge weakly to a limit $\lambda(\mathbf{g})$, and*

$$\nu = \int_{\mathcal{T}} \lambda(\mathbf{g}) d\mathbb{P}(\mathbf{g}).$$

We will show the following statement.

Proposition C.2.10. *Let μ be a probability measure on Out(F_N), whose support generates a nonelementary subgroup of Out(F_N). Let ν be a μ-stationary probability measure on ∂CV_N, and let $(g_n)_{n\in\mathbb{N}}$ be an unbounded sequence of elements of Out(F_N) such that $g_n\nu$ converges weakly to a measure λ on ∂CV_N. Then either λ is concentrated on $\partial CV_N \smallsetminus \widetilde{\mathcal{FI}}$, or it is concentrated on $\widetilde{\mathcal{FI}}$, on a single class of the relation $\sim$. In the first case, all limit points of sequences $(g_nT)_{n\in\mathbb{N}}$ for $T \in CV_N$ are contained in $\partial CV_N \smallsetminus \widetilde{\mathcal{FI}}$, and in the second case they are all contained in the same class of the relation $\sim$, on which λ is concentrated.*

We first explain how to deduce Proposition C.2.8 from Lemma C.2.9 and Proposition C.2.10.

Proof of Proposition C.2.8. Let ν be a μ-stationary measure on ∂CV_N. As $gr(\mu)$ is nonelementary, the measure ν is concentrated on $\widetilde{\mathcal{FI}}$ (Proposition C.2.4). Lemma C.2.9 thus implies that for $\mathbb{P}$-a.e. sample path $\mathbf{g}$ of the random walk on (Out(F_N), μ), the limit measure $\lambda(\mathbf{g})$ exists and is concentrated on $\widetilde{\mathcal{FI}}$. As gr(μ) is nonelementary, it is unbounded, so $\mathbb{P}$-a.e. sample path of the random walk is unbounded. Proposition C.2.10 implies that for $\mathbb{P}$-a.e. sample path $\mathbf{g} = (g_n)_{n\in\mathbb{N}}$ of the random walk, the measure $\lambda(\mathbf{g})$ is concentrated on a single $\sim$-class $\Delta(\mathbf{g})$, and for all $T \in CV_N$, all limit points of the sequence $(g_nT)_{n\in\mathbb{N}}$ belong to $\Delta(\mathbf{g})$. ☐

We are left showing Proposition C.2.10. We will appeal to another general statement due to Kaimanovich and Masur. We provide a proof for completeness.

Lemma C.2.11. *(Kaimanovich–Masur [KM96, Lemma 1.5.5]) Let ν be a Borel probability measure on ∂CV_N. Let $(g_n)_{n\in\mathbb{N}} \in Out(F_N)^{\mathbb{N}}$ be a sequence of elements in $Out(F_N)$ such that $g_n\nu$ converges weakly to a probability measure λ on ∂CV_N. If there is a Borel subset $E \subseteq \partial CV_N$ with $\nu(E) = 1$ and a closed subset $\Omega \subseteq \partial CV_N$ that contains all limit points of sequences $(g_nT)_{n\in\mathbb{N}}$ for $T \in E$, then the measure λ is supported on Ω.*

Proof. Let $U \subseteq \partial CV_N$ be an open subset that contains Ω. Compactness of ∂CV_N implies the existence for all $T \in E$ of an integer $n(T)$ such that for all $n \geq n(T)$, we have $g_nT \in U$. Let $\epsilon > 0$. As $\nu(E) = 1$, there exists an integer $N \in \mathbb{N}$ such that $\nu(\{T|n(T) \leq N\}) \geq 1-\epsilon$. This implies that for all $n \geq N$, we have $g_n\nu(U) \geq 1-\epsilon$, and therefore $g_n\nu(\overline{U}) \geq 1-\epsilon$. As $\overline{U}$ is a closed set, weak convergence of the measures $g_n\nu$ implies that $\lambda(\overline{U}) \geq 1-\epsilon$, see [Bil68, Theorem 2.1]. Therefore, we get that for all open neighborhoods U of Ω, we have $\lambda(\overline{U}) = 1$. By letting U_n be the $\frac{1}{n}$-regular neighborhood of Ω for any metric on ∂CV_N, as Ω is a closed set, we have
$$\Omega = \bigcap_{n\in\mathbb{N}} \overline{U_n},$$
which implies that $\lambda(\Omega) = 1$. $\square$

To prove Proposition C.2.10, we need to understand possible limit points of sequences in $\overline{CV_N}$. Let $*_{CV_N} \in CV_N$. Let $\eta_0 \in \mathbb{P}Curr_N$ be such that for all $T \in \overline{CV_N}$, we have $\langle T, \eta_0\rangle > 0$ (take for example a basis $\{x_1, \dots, x_N\}$ of F_N, and let $\eta_0 := [x_1] + \cdots + [x_N] + [x_1x_2] + \cdots + [x_1x_N]$). Let $(h_n)_{n\in\mathbb{N}} \in \mathrm{Out}(F_N)^{\mathbb{N}}$, and let $\eta \in \mathbb{P}Curr_N$. The pair $((h_n)_{n\in\mathbb{N}}, \eta)$ is *universally converging* if

- the sequence $(h_n*_{CV_N})_{n\in\mathbb{N}}$ converges (projectively) to a tree $T_\infty \in \overline{CV_N}$ such that $\langle T_\infty, \eta\rangle = 0$, and
- the sequence $(h_n^{-1}\eta_0)_{n\in\mathbb{N}}$ converges (projectively) to a current $\eta_0^\infty \in \mathbb{P}Curr_N$, and
- the sequence $(h_n^{-1}\eta)_{n\in\mathbb{N}}$ converges (projectively) to a current $\eta^\infty \in \mathbb{P}Curr_N$.

The following lemma follows from compactness of $\mathbb{P}Curr_N$.

Lemma C.2.12. *Let $(g_n)_{n\in\mathbb{N}}$ be an unbounded sequence of elements of $Out(F_N)$, let $T_\infty \in \partial CV_N$ be a limit point of $(g_n*_{CV_N})_{n\in\mathbb{N}}$, and $\eta \in Dual(T_\infty)$. Then there exists a subsequence $(h_n)_{n\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$ such that the pair $((h_n)_{n\in\mathbb{N}}, \eta)$ is universally converging.* $\square$

Given two projective currents $\eta_1, \eta_2 \in \mathbb{P}Curr_N$, we define
$$E(\eta_1, \eta_2) := \{T \in \widetilde{\mathcal{FI}} | \langle T, \eta_1\rangle \neq 0 \text{ and } \langle T, \eta_2\rangle \neq 0\}.$$

Lemma C.2.13. *Let μ be a probability measure on $Out(F_N)$, whose support generates a nonelementary subgroup of $Out(F_N)$. For all $\eta_1, \eta_2 \in \mathbb{P}Curr_N$ and all μ-stationary measures ν on ∂CV_N, the set $E(\eta_1, \eta_2)$ is measurable and has full ν-measure.*

Proof. Measurability of $E(\eta_1, \eta_2)$ comes from measurability of $\widetilde{\mathcal{FI}}$ and continuity of the intersection form (Theorem C.1.1). By Proposition C.2.4, we have $\nu(\widetilde{\mathcal{FI}}) = 1$. Theorem C.1.6 implies that $\widetilde{\mathcal{FI}} \smallsetminus E(\eta_1, \eta_2)$ consists of at most two classes of the relation $\sim$. As each of these classes has ν-measure 0 (Proposition C.2.7), we get that $\nu(E(\eta_1, \eta_2)) = 1$. $\square$

Lemma C.2.14. *Let $((h_n)_{n\in\mathbb{N}}, \eta)$ be a universally converging pair, and $T' \in CV_N \cup E(\eta^\infty, \eta_0^\infty)$. If $(h_nT')_{n\in\mathbb{N}}$ converges to a tree $T'_\infty \in \overline{CV_N}$, then $\langle T'_\infty, \eta\rangle = 0$.*

Proof. Let T_∞ be the limit of the sequence $(h_n *_{CV_N})_{n\in\mathbb{N}}$. We choose lifts of T_∞, T'_∞ and T' to the unprojectivized outer space $\overline{cv_N}$, which we again denote by T_∞, T'_∞ and T', slightly abusing notations, and we denote by $*_{cv_N}$ a lift of $*_{CV_N}$ to cv_N. There exist sequences $(t_n)_{n\in\mathbb{N}}$ and $(t'_n)_{n\in\mathbb{N}}$ of positive real numbers such that $t_n h_n *_{cv_N}$ converges to T_∞, and $t'_n h_n T'$ converges to T'_∞. Similarly, as $((h_n)_{n\in\mathbb{N}},\eta)$ is universally converging, there exist sequences $(\lambda_n^0)_{n\in\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lambda_n^0 h_n^{-1}\eta_0$ converges (non-projectively) to η_0^∞, and $\lambda_n h_n^{-1}\eta$ converges (non-projectively) to η^∞. We have $\langle *_{cv_N},\eta_0^\infty\rangle \neq 0$ (Theorem C.1.2). By continuity and Out(F_N)-invariance of the intersection form (Theorem C.1.1), we have

$$\frac{\langle T_\infty,\eta_0\rangle}{\langle *_{cv_N},\eta_0^\infty\rangle} = \frac{\lim_{n\to+\infty} t_n\langle h_n *_{cv_N},\eta_0\rangle}{\lim_{n\to+\infty}\lambda_n^0\langle *_{cv_N},h_n^{-1}\eta_0\rangle} = \lim_{n\to+\infty}\frac{t_n}{\lambda_n^0},$$

and similarly, as $T'\in CV_N\cup E(\eta^\infty,\eta_0^\infty)$, we have

$$\frac{\langle T'_\infty,\eta_0\rangle}{\langle T',\eta_0^\infty\rangle} = \frac{\lim_{n\to+\infty} t'_n\langle h_n T',\eta_0\rangle}{\lim_{n\to+\infty}\lambda_n^0\langle T',h_n^{-1}\eta_0\rangle} = \lim_{n\to+\infty}\frac{t'_n}{\lambda_n^0}.$$

We have $\langle T_\infty,\eta_0\rangle > 0$ and $\langle T'_\infty,\eta_0\rangle > 0$, so the limits $\lim_{n\to+\infty}\frac{t_n}{\lambda_n^0}$ and $\lim_{n\to+\infty}\frac{t'_n}{\lambda_n^0}$ are both finite and non-zero, and hence there exists a non-zero limit $\lim_{n\to+\infty}\frac{t_n}{t'_n}$. As $\langle *_{cv_N},\eta^\infty\rangle\neq 0$ and $T'\in CV_N\cup E(\eta^\infty,\eta_0^\infty)$, the same argument as above also shows that

$$0 = \frac{\langle T_\infty,\eta\rangle}{\langle *_{cv_N},\eta^\infty\rangle} = \lim_{n\to+\infty}\frac{t_n}{\lambda_n},$$

and

$$\frac{\langle T'_\infty,\eta\rangle}{\langle T',\eta^\infty\rangle} = \lim_{n\to+\infty}\frac{t'_n}{\lambda_n},$$

thus proving that $\langle T'_\infty,\eta\rangle = 0$. □

Proof of Proposition C.2.10. As $(g_n)_{n\in\mathbb{N}}$ is unbounded, the sequence $(g_n *_{CV_N})_{n\in\mathbb{N}}$ has a limit point $T_\infty\in\partial CV_N$. Theorem C.1.2 and Lemma C.2.12 provide a current $\eta\in$ Dual(T_∞), and a subsequence $(h_n)_{n\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$ such that the pair $((h_n)_{n\in\mathbb{N}},\eta)$ is universally converging. By Lemma C.2.13, the set $E(\eta^\infty,\eta_0^\infty)$ is measurable and $\nu(E(\eta^\infty,\eta_0^\infty)) = 1$, and by Lemma C.2.14, all limit points of sequences $(h_nT')_{n\in\mathbb{N}}$ for $T'\in E(\eta^\infty,\eta_0^\infty)$ belong to the closed set $\widetilde{\mathrm{Dual}}(\eta) := \{T\in\overline{CV_N} | \langle T,\eta\rangle = 0\}$. Lemma C.2.11 shows that λ is concentrated on $\widetilde{\mathrm{Dual}}(\eta)$. If $T_\infty\in\widetilde{\mathcal{FI}}$, Theorem C.1.6 implies that $\widetilde{\mathrm{Dual}}(\eta)$ is contained in $\widetilde{\mathcal{FI}}$, in a single class of the relation $\sim$. If $T_\infty\in\partial CV_N\smallsetminus\widetilde{\mathcal{FI}}$, then Theorem C.1.6 implies that for all $T\in\widetilde{\mathcal{FI}}$, we have $\langle T,\eta\rangle\neq 0$. Hence $\widetilde{\mathrm{Dual}}(\eta)\subseteq\partial CV_N\smallsetminus\widetilde{\mathcal{FI}}$, so λ is concentrated on $\partial CV_N\smallsetminus\widetilde{\mathcal{FI}}$.

Now let $T\in CV_N$, and let T'_∞ be a limit point of the sequence $(g_nT)_{n\in\mathbb{N}}$. In other words, there exists an unbounded subsequence $(h'_n)_{n\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$ such that the sequence $(h'_nT)_{n\in\mathbb{N}}$ converges to T'_∞, and up to passing to a subsequence again, we may assume that the sequence $(h'_n *_{CV_N})_{n\in\mathbb{N}}$ converges to a tree $T''_\infty\in\partial CV_N$. Notice that $T''_\infty\in\widetilde{\mathcal{FI}}$ if and only if $T_\infty\in\widetilde{\mathcal{FI}}$, and in this case they belong to the same $\sim$-class, otherwise the above argument applied to both T_∞ and T''_∞ would imply that λ is simultaneously supported on two disjoint measurable sets. The last part of the claim then follows from Lemma C.2.14. □

C.2.3 Uniqueness of the stationary measure on $\mathcal{FI}$

Let μ be a probability measure on $\mathrm{Out}(F_N)$, whose support generates a nonelementary subgroup of $\mathrm{Out}(F_N)$. Given a sample path $(g_n)_{n\in\mathbb{N}}$ of the random walk on $(\mathrm{Out}(F_N),\mu)$, we denote by $\xi(\mathbf{g})\in\mathcal{FI}$ the limit of any sequence $(g_nT_0)_{n\in\mathbb{N}}$ (with $T_0\in CV_N$), which $\mathbb{P}$-almost surely exists and is independent from T_0 by Proposition C.2.8. The *hitting measure* ν on $\mathcal{FI}$ is the μ-stationary measure defined by letting

$$\nu(X):=\mathbb{P}(\xi(\mathbf{g})\in X)$$

for all Borel subsets $X\subseteq\mathcal{FI}$.

Proposition C.2.15. *Let μ be a probability measure on Out(F_N), whose support generates a nonelementary subgroup of Out(F_N). Then the hitting measure is the unique μ-stationary measure on $\mathcal{FI}$.*

Proof. Let ν be a μ-stationary measure on $\mathcal{FI}$. For $\mathbf{g}\in\mathcal{T}$, let $\lambda(\mathbf{g})$ be the Dirac measure on $\xi(\mathbf{g})$. As ν is purely nonatomic (Proposition C.2.6), Lemma C.2.14 shows that for $\mathbb{P}$-a.e. sample path $\mathbf{g}\in\mathcal{T}$ of the random walk, and ν-a.e. $x\in\mathcal{FI}$, the sequence $(g_nx)_{n\in\mathbb{N}}$ converges to $\xi(\mathbf{g})$. So for all bounded continuous functions F on $\mathcal{FI}$, the integrals

$$\int_{\mathcal{FI}}F(g_nx)d\nu(x)$$

converge to $F(\xi(\mathbf{g}))$ as n goes to $+\infty$, thus showing that the measures $g_n\nu$ weakly converge to $\lambda(\mathbf{g})$. In other words, for $\mathbb{P}$-a.e. sample path $\mathbf{g}$ of the random walk, and all open subsets $U\subseteq\mathcal{FI}$, we have

$$\liminf_{n\to+\infty}g_n\nu(U)\geq\lambda(\mathbf{g})(U)$$

by the Portmanteau Theorem (see [Bil68, Theorem 2.1]). For all $n\in\mathbb{N}$, the measure ν is μ^{*n}-stationary, so for all $n\in\mathbb{N}$ and all open subsets $U\subseteq\mathcal{FI}$, we have

$$\nu(U)=\int_{\mathcal{T}}g_n\nu(U)d\mathbb{P}(\mathbf{g}).$$

Hence

$$\nu(U)\geq\left(\int_{G^{\mathbb{N}}}\lambda(\mathbf{g})d\mathbb{P}(\mathbf{g})\right)(U),$$

and $\int_{\mathcal{T}}\lambda(\mathbf{g})d\mathbb{P}(\mathbf{g})$ is the hitting measure on $\mathcal{FI}$. Regularity of ν and of the hitting measure [Bil68, Theorem 1.1] implies that the inequality holds true for all Borel subsets of $\mathcal{FI}$. As both ν and the hitting measure are probability measures on $\mathcal{FI}$, they are equal. $\square$

C.3 The Poisson boundary of Out(F_N)

This section is devoted to the description of the Poisson boundary of $\mathrm{Out}(F_N)$. We start by recalling the construction of the Poisson boundary of a finitely generated group equipped with a probability measure.

C.3.1 Generalities on Poisson boundaries and statement of the main result

Let G be a finitely generated group, and μ be a probability measure on G. The *Poisson boundary* of (G,μ) is the space of ergodic components of the *time shift* T, defined in the path space of the random walk on (G,μ) by $(T\mathbf{g})_n = g_{n+1}$. More precisely, let $\mathcal{A}_T$ be the σ-algebra of all T-invariant measurable subsets of the path space $\mathcal{T}$, and let $\overline{\mathcal{A}_T}$ be its completion with respect to the measure $\mathbb{P}_m = \sum_{g\in G} g\mathbb{P}$ corresponding to the distribution of the sample paths of a random walk whose initial distribution is the counting measure on G. We recall that $\mathcal{A}$ denotes the σ-algebra on $\mathcal{T}$ generated by the cylinder subsets. Let $\overline{\mathcal{A}}$ be its completion with respect to the measure $\mathbb{P}_m$. Since $(\mathcal{T},\overline{\mathcal{A}},\mathbb{P}_m)$ is a Lebesgue space, the Rokhlin correspondence associates to $\overline{\mathcal{A}_T}$ a measurable partition η of $\mathcal{T}$, see [Rok49] (we recall that a partition of a measurable space into measurable subsets is *measurable* if it is countably separated). This partition is unique in the sense that if η and η' are two such partitions, then there exists a subset of $\mathcal{T}$ of full $\mathbb{P}_m$-measure on which they coincide. We call it the *Poisson partition* of $\mathcal{T}$. The quotient space $\Gamma := (\mathcal{T},\overline{\mathcal{A}})/\eta$ carries several measures. On the one hand, it can be equipped with the image ν_m of the measure $\mathbb{P}_m$ under the quotient map, and (Γ,ν_m) is a Lebesgue space. On the other hand, it can be equipped with the *harmonic measure* ν, which is the image of $\mathbb{P}$ under the quotient map. It can also be equipped with all translates $g\nu$ with $g\in \mathrm{Out}(F_N)$, which are not necessarily absolutely continuous with respect to ν if the support of μ does not generate $\mathrm{Out}(F_N)$. The measure ν is μ-stationary, and the measure ν_m can be recovered from ν by the formula

$$\nu_m = \sum_{g\in G} g\nu.$$

We call (Γ,ν) the *Poisson boundary* of (G,μ).

A *μ-boundary* is a probability space (B,λ), which is the quotient of the path space $(\mathcal{T},\mathbb{P})$ with respect to some shift-invariant and G-invariant measurable partition. Equivalently, a μ-boundary is a probability space which is the quotient of the Poisson boundary with respect to some G-invariant measurable partition. So the Poisson boundary is itself a μ-boundary, and it is maximal with respect to this property. Typical examples of μ-boundaries arise when G is embedded into a metric separable G-space, and $\mathbb{P}$-a.e. sample path $\mathbf{g}$ converges to a limit $bnd(\mathbf{g})$.

In [Kai00], Kaimanovich gave a criterion for checking that a μ-boundary is maximal. Let d be the word metric on G with respect to some finite generating set – any two such metrics are bi-Lipschitz equivalent. The *first logarithmic moment* of μ with respect to d is defined (with the convention that $\log 0 = 0$) as

$$|\mu| := \sum_{g\in G} \log d(e,g)\mu(g).$$

The *entropy* of μ is defined as

$$H(\mu) := \sum_{g\in G} -\mu(g)\log\mu(g).$$

Given a measure μ on a countable group G, we denote by $\check{\mu}$ the *reflected measure* on G defined by $\check{\mu}(g) := \mu(g^{-1})$ for all $g\in G$. In the following statement, we take the convention that $\log 0 = 0$.

Theorem C.3.1. *(Kaimanovich [Kai00, Theorem 6.5]) Let G be a finitely generated group, let d be a word metric on G, and let μ be a probability measure on G which has finite first logarithmic moment with respect to d, and finite entropy. Let (B_-,ν_-) and (B_+,ν_+) be $\check{\mu}$- and μ-boundaries, respectively, and assume there exists a measurable G-equivariant map*

$$\begin{array}{ccc} B_-\times B_+ & \to & 2^G \\ (b_-,b_+) & \mapsto & S(b_-,b_+) \end{array}$$

such that for $\nu_-\otimes\nu_+$-a.e. $(b_-,b_+)\in B_-\times B_+$, the set $S(b_-,b_+)$ is nonempty, and

$$\sup_{k\in\mathbb{N}\smallsetminus\{0\}} \frac{1}{\log k}\log card[S(b_-,b_+)\cap\mathcal{B}_k]<+\infty,$$

where $\mathcal{B}_k$ denotes the d-ball of radius k centered at e. Then the boundaries (B_-,ν_-) and (B_+,ν_+) are Poisson boundaries.

In terms of μ-boundaries, Theorem C.2.1 can be restated as follows.

Theorem C.3.2. *Let μ be a probability measure on $Out(F_N)$, whose support generates a nonelementary subgroup of $Out(F_N)$, and let ν be the hitting measure on $\mathcal{FI}$. Then $(\mathcal{FI},\nu)$ is a μ-boundary.*

Proof. Theorem C.2.1 provides an (almost-surely well-defined) measurable map

$$bnd:\mathcal{T}\to\mathcal{FI},$$

that sends a sample path $(g_n)_{n\in\mathbb{N}}$ to the limit of the sequence $(g_nT_0)_{n\in\mathbb{N}}$ for any $T_0\in CV_N$. The space $\mathcal{FI}$ is metrizable [BR13, Corollary 7.2] and separable (it is a quotient of a subspace of a separable metric space), so its Borel σ-algebra is countably separated. This implies that the *bnd*-preimage of the point partition of $\mathcal{FI}$ is a measurable partition of the path space $\mathcal{T}$, and therefore $(\mathcal{FI},\nu)$ is a μ-boundary. □

Under some further hypotheses on the measure μ, we will show that the μ-boundary $(\mathcal{FI},\nu)$ is the Poisson boundary of $(\mathrm{Out}(F_N),\mu)$.

Theorem C.3.3. *Let μ be a probability measure on $Out(F_N)$ such that $gr(\mu)$ is nonelementary, which has finite first logarithmic moment with respect to the word metric, and finite entropy. Let ν be the hitting measure on $\mathcal{FI}$. Then the measure space $(\mathcal{FI},\nu)$ is the Poisson boundary of $(Out(F_N),\mu)$.*

We will use Kaimanovich's criterion (Theorem C.3.1) to prove the maximality of the μ-boundary provided by Theorem C.3.2. Our construction of the strips is inspired from Hamenstädt's construction of *lines of minima* in outer space [Ham14b].

C.3.2 Axes in outer space

The following construction is inspired from Hamenstädt's construction of *lines of minima* [Ham14b]. We recall from Section C.1.4 that a pair $([\eta],[\eta'])\in\mathbb{P}Curr_N^2$ is *positive* if $\langle T,\eta+\eta'\rangle>0$ for all $T\in\overline{CV_N}$. Given a positive pair of projective currents $([\eta],[\eta'])\in\mathbb{P}Curr_N^2$, we want to define an axis in outer space which will roughly consist of trees for which either η or η' can serve as a fairly good candidate for computing the infimal Lipschitz distortion to any other tree in the closure of outer space. We first define

$$\begin{array}{cccc} l_{[\eta],[\eta']}: & CV_N\times\overline{CV_N} & \to & \mathbb{R} \\ & (T,T') & \mapsto & \frac{\mathrm{Lip}(T,T')}{\Lambda_{\{[\eta],[\eta']\}}(T,T')}, \end{array}$$

where we recall the notations from Section C.1.3. This measures to which extent the stretch of either η or η' gives a good estimate of the Lipschitz distortion $\mathrm{Lip}(T,T')$. We always have $l_{[\eta],[\eta']}(T,T') \geq 1$ (the closer to 1 it is, the better the estimate will be), and positivity of the pair $([\eta],[\eta'])$ ensures that $l_{[\eta],[\eta']}(T,T') < +\infty$. Notice that this only depends on the projective classes of the trees T and T' and the currents $[\eta]$ and $[\eta']$. So for all $T \in CV_N$, the map $l_{[\eta],[\eta']}(T,.)$ is a continuous function on a compact set, so we can let

$$L_{[\eta],[\eta']}(T) := \max_{T'\in\overline{CV_N}} l_{[\eta],[\eta']}(T,T') < +\infty.$$

Given $L \geq 1$, a tree $T \in CV_N$ satisfies $L_{[\eta],[\eta']}(T) \leq L$ if for all $T' \in \overline{CV_N}$, the stretch of either η or η' from T to T' gives a good estimate of the Lipschitz distortion $\mathrm{Lip}(T,T')$, up to an error controlled by L. We define the *L-axis* $A_L([\eta],[\eta'])$ of a positive pair of projective currents as the set of all $T \in CV_N$ such that $L_{[\eta],[\eta']}(T) \leq L$. For all $\Psi \in \mathrm{Out}(F_N)$, all positive pairs $([\eta],[\eta']) \in \mathbb{P}Curr_N^2$ and all $T \in CV_N$, we have $L_{\Psi[\eta],\Psi[\eta']}(\Psi T) = L_{[\eta],[\eta']}(T)$, so the L-axis $A_L([\eta],[\eta'])$ depends $\mathrm{Out}(F_N)$-equivariantly on the positive pair $([\eta],[\eta']) \in \mathbb{P}Curr_N^2$.

We now associate to any pair $(T_-,T_+) \in \mathcal{FI}\times\mathcal{FI}\smallsetminus\Delta$ (where Δ denotes the diagonal) an axis in CV_N. The key point is that associated to any free and arational tree is a finite set of ergodic currents (Proposition C.1.5), and given $(T_-,T_+) \in \mathcal{FI}\times\mathcal{FI}\smallsetminus\Delta$, any pair of currents $([\eta_-],[\eta_+]) \in \mathrm{Erg}(T_-)\times\mathrm{Erg}(T_+)$ is positive (Corollary C.1.7). Given $L \geq 1$, we define the *L-axis* $A_L(T_-,T_+)$ as the union of all $A_L([\eta_-],[\eta_+])$, with $([\eta_-],[\eta_+])$ varying in the finite set $\mathrm{Erg}(T_-)\times\mathrm{Erg}(T_+)$. For $T \in CV_N$, letting

$$L_{T_-,T_+}(T) := \min_{\substack{[\eta_-]\in\mathrm{Erg}(T_-)\\ [\eta_+]\in\mathrm{Erg}(T_+)}} L_{[\eta_-],[\eta_+]}(T),$$

the L-axis $A_L(T_-,T_+)$ is also equal to the set of all $T \in CV_N$ such that $L_{T_-,T_+}(T) \leq L$.

Remark C.3.4. Hamenstädt has shown in [Ham14b, Proposition 4.9] that all accumulation points of the L-axis of a pair $(T_-,T_+) \in \mathcal{FI}\times\mathcal{FI}\smallsetminus\Delta$ are free and arational, and project to either T_- or T_+ in $\mathcal{FI}$.

C.3.3 Definition of the strips

In order to use Kaimanovich's criterion for proving Theorem C.3.3, we need to associate to every pair $(T_-,T_+) \in \mathcal{FI}\times\mathcal{FI}$ a strip in $\mathrm{Out}(F_N)$. We fix once and for all a basepoint $*_{CV_N} \in CV_N$.

Definition C.3.5. *Let $(T_-,T_+) \in \mathcal{FI}\times\mathcal{FI}$, and let $L \geq 1$. If $T_- \neq T_+$, the L-strip $S_L(T_-,T_+)$ is defined to be the set of all $\Phi \in Out(F_N)$ such that $\Phi *_{CV_N} \in A_L(T_-,T_+)$. If $T_- = T_+$, we let $S_L(T_-,T_+) = \emptyset$.*

For all $\Psi \in \mathrm{Out}(F_N)$ and all $L \geq 1$, we have $S_L(\Psi T_-,\Psi T_+) = \Psi S_L(T_-,T_+)$. In view of Theorems C.3.1 and C.3.2, Theorem C.3.3 will be a consequence of the following three facts.

Proposition C.3.6. *There exists $L_1 > 1$ such that for $\nu_-\otimes\nu_+$-a.e. $(T_-,T_+) \in \mathcal{FI}\times\mathcal{FI}$, we have $S_{L_1}(T_-,T_+) \neq \emptyset$.*

In the next two statements, we fix the constant L_1 provided by Proposition C.3.6. For all $k \in \mathbb{N}$, let $\mathcal{B}_k$ be the ball of radius k in $\mathrm{Out}(F_N)$ for the word metric.

Proposition C.3.7. *For $\nu_- \otimes \nu_+$-a.e. $(T_-, T_+) \in \mathcal{FI} \times \mathcal{FI}$, there exists $\lambda \in \mathbb{R}$ such that for all $k \in \mathbb{N}$, we have*

$$card(S_{L_1}(T_-, T_+) \cap \mathcal{B}_k) \leq \lambda k.$$

Proposition C.3.8. *The map*

$$\begin{array}{ccc} \mathcal{FI} \times \mathcal{FI} & \to & 2^{Out(F_N)} \\ (T_-, T_+) & \mapsto & S_{L_1}(T_-, T_+) \end{array}$$

is measurable.

C.3.4 Choosing L_1 to ensure nonemptiness of the strips

Our proof of Proposition C.3.6 is inspired from Kaimanovich and Masur's analogous argument in the case of mapping class groups of surfaces [KM96, Theorem 2.3.1].

Proof of Proposition C.3.6. Consider the measure space $(\mathrm{Out}(F_N)^{\mathbb{Z}}, \overline{\mathbb{P}})$ of *bilateral paths* $\overline{\mathbf{g}} = (g_n)_{n\in\mathbb{Z}}$ satisfying $g_0 = e$, corresponding to bilateral sequences of independent μ-distributed increments $(s_n)_{n\in\mathbb{Z}}$ by the formula $g_n = g_{n-1}s_n$. The unilateral paths $\mathbf{g} = (g_n)_{n\geq 0}$ and $\check{\mathbf{g}} = (g_{-n})_{n\geq 0}$ are independent, and correspond to sample paths of the random walks on $(\mathrm{Out}(F_N), \mu)$ and $(\mathrm{Out}(F_N), \check{\mu})$, respectively. The *Bernoulli shift* U is the transformation defined in the space $(\mathrm{Out}(F_N)^{\mathbb{Z}}, \mu^{\otimes\mathbb{Z}})$ of increments $s = (s_n)_{n\in\mathbb{Z}}$ by $(Us)_n = s_{n+1}$ for all $n \in \mathbb{Z}$. We again denote by U the measure-preserving, ergodic transformation induced by the Bernoulli shift in the space of bilateral paths $(\mathrm{Out}(F_N)^{\mathbb{Z}}, \overline{\mathbb{P}})$, defined by

$$(U^k\overline{\mathbf{g}})_n = g_k^{-1}g_{n+k}.$$

Let $(\mathcal{FI}, \nu_-)$ and $(\mathcal{FI}, \nu_+)$ be the boundaries corresponding to $(\mathrm{Out}(F_N), \check{\mu})$ and $(\mathrm{Out}(F_N), \mu)$ provided by Theorem C.3.2. We let $bnd_-(\overline{\mathbf{g}}) \in \mathcal{FI}$ (resp. $bnd_+(\overline{\mathbf{g}}) \in \mathcal{FI}$) be the limit as n goes to $+\infty$ of the sequence $(g_{-n}*_{CV_N})_{n\in\mathbb{N}}$ (resp. $(g_n*_{CV_N})_{n\in\mathbb{N}}$), which is $\overline{\mathbb{P}}$-almost surely well-defined by Theorem C.2.1. Then for all $k \in \mathbb{Z}$, we have

$$bnd_-(U^k\overline{\mathbf{g}}) = g_k^{-1}bnd_-(\overline{\mathbf{g}}),$$

and similarly

$$bnd_+(U^k\overline{\mathbf{g}}) = g_k^{-1}bnd_+(\overline{\mathbf{g}}).$$

Let

$$\psi(\overline{\mathbf{g}}) := L_{bnd_-(\overline{\mathbf{g}}),bnd_+(\overline{\mathbf{g}})}(*_{CV_N})$$

(when $bnd_-(\overline{\mathbf{g}}) = bnd_+(\overline{\mathbf{g}})$, we let $\psi(\overline{\mathbf{g}}) := +\infty$). Measurability of ψ will follow from the proof of Proposition C.3.8 in Section C.3.6. For $\overline{\mathbb{P}}$-a.e $\overline{\mathbf{g}} := (g_n)_{n\in\mathbb{Z}}$, we have $\psi(\overline{\mathbf{g}}) < +\infty$. Hence there exists $L_1 > 1$ such that $\overline{\mathbb{P}}[\psi(\overline{\mathbf{g}}) \leq L_1] > 0$. For all $k \in \mathbb{N}$, we have

$$\begin{aligned} \psi(U^k\overline{\mathbf{g}}) &= L_{g_k^{-1}bnd_-(\overline{\mathbf{g}}),g_k^{-1}bnd_+(\overline{\mathbf{g}})}(*_{CV_N}) \\ &= L_{bnd_-(\overline{\mathbf{g}}),bnd_+(\overline{\mathbf{g}})}(g_k*_{CV_N}). \end{aligned}$$

Applying Birkhoff's ergodic theorem [Bir31] to the ergodic transformation U, we get that for $\overline{\mathbb{P}}$-a.e. bilateral path $\overline{\mathbf{g}}$, the density of times $k \geq 0$ such that $L_{bnd_-(\overline{\mathbf{g}}),bnd_+(\overline{\mathbf{g}})}(g_k*_{CV_N}) \leq L_1$ is positive. Therefore, for $\overline{\mathbb{P}}$-a.e. bilateral path $\overline{\mathbf{g}}$, the L_1-strip $S_{L_1}(bnd_-(\overline{\mathbf{g}}), bnd_+(\overline{\mathbf{g}}))$ is nonempty, so $\nu_- \otimes \nu_+$-a.e. the set $S_{L_1}(T_-, T_+)$ is nonempty. $\square$

Remark C.3.9. The proof of Proposition C.3.6 actually shows that for $\overline{\mathbb{P}}$-a.e. bilateral path $\overline{\mathbf{g}} := (g_n)_{n\in\mathbb{Z}}$ of the random walk on $(\mathrm{Out}(F_N), \mu)$, the density of times $k \in \mathbb{N}$ such that $g_k \in S_{L_1}(bnd_-(\overline{\mathbf{g}}), bnd_+(\overline{\mathbf{g}}))$ is positive.

C.3.5 Thinness of the strips

In this head, we will prove Proposition C.3.7. Our argument is inspired from Hamenstädt's estimates in [Ham14b, Proposition 4.4]. From now on, we fix a (non-projective) positive pair $(\eta_-, \eta_+) \in Curr_N^2$. Given $T \in CV_N$, we let $\sigma(T) \in \mathbb{R}$ be such that $\langle T, \eta_+\rangle = e^{\sigma(T)}\langle T, \eta_-\rangle$. This defines a height function on the axis $A_{L_1}([\eta_-], [\eta_+])$. We will show that $A_{L_1}([\eta_-], [\eta_+])$ is close to being a d_{sym}-geodesic with holes. Proposition C.3.7 will then follow from proper discontinuity of the action of $\mathrm{Out}(F_N)$ on CV_N.

Proposition C.3.10. *For all $(\eta_-, \eta_+) \in Curr_N^2$, and all $S, T \in A_{L_1}([\eta_-], [\eta_+])$, we have*

$$|\sigma(S) - \sigma(T)| \leq d_{sym}(S, T) \leq |\sigma(S) - \sigma(T)| + 2\log L_1.$$

Proof. We have $\Lambda_{[\eta_+]}(S, T) = e^{\sigma(T)-\sigma(S)}\Lambda_{[\eta_-]}(S, T)$. Assume without loss of generality that $\sigma(S) \leq \sigma(T)$. Then $\Lambda_{\{[\eta_-],[\eta_+]\}}(S, T) = \Lambda_{[\eta_+]}(S, T)$, and we get from the definition of $A_{L_1}([\eta_-], [\eta_+])$ that

$$\frac{1}{L_1}\mathrm{Lip}(S, T) \leq \Lambda_{[\eta_+]}(S, T) \leq \mathrm{Lip}(S, T).$$

Taking logarithms, we get

$$d(S, T) - \log L_1 \leq \log \Lambda_{[\eta_+]}(S, T) \leq d(S, T).$$

Reversing the roles of S and T, we also have

$$d(T, S) - \log L_1 \leq \log \Lambda_{[\eta_-]}(T, S) \leq d(T, S).$$

By summing the above inequalities, we obtain

$$d_{sym}(S, T) - 2\log L_1 \leq \sigma(T) - \sigma(S) \leq d_{sym}(S, T),$$

which is the desired inequality. □

Proof of Proposition C.3.7. Let $T_- \neq T_+ \in \mathcal{FI}$. As $\mathrm{Erg}(T_-)$ and $\mathrm{Erg}(T_+)$ are finite, it is enough to show that for all pairs $([\eta_-], [\eta_+]) \in \mathrm{Erg}(T_-) \times \mathrm{Erg}(T_+)$, the cardinality of the set $\{\Phi \in \mathcal{B}_k | \Phi *_{CV_N} \in A_{L_1}([\eta_-], [\eta_+])\}$ grows linearly with k. For all $k \in \mathbb{N}$, we denote by $\mathcal{B}_k^{sym}$ the d_{sym}-ball centered at $*_{CV_N}$ in CV_N. There exists $C \in \mathbb{R}$ such that for all $k \in \mathbb{N}$, and all $\Phi \in \mathcal{B}_k$, we have $\Phi *_{CV_N} \in \mathcal{B}_{Ck}^{sym}$. Therefore, it is enough to check that for all pairs $([\eta_-], [\eta_+]) \in \mathrm{Erg}(T_-) \times \mathrm{Erg}(T_+)$, the cardinality of the set $\{\Phi \in \mathrm{Out}(F_N) | \Phi *_{CV_N} \in A_{L_1}([\eta_-], [\eta_+]) \cap \mathcal{B}_k^{sym}\}$ grows linearly with k.

Let $([\eta_-], [\eta_+]) \in \mathrm{Erg}(T_-) \times \mathrm{Erg}(T_+)$. We fix a representative η_- (resp. η_+) of $[\eta_-]$ (resp. $[\eta_+]$) in $Curr_N$. For all $T \in A_{L_1}([\eta_-], [\eta_+])$, let $f(T) := \lfloor \sigma(T) \rfloor$. Denote by M the maximal cardinality of the intersection of a d_{sym}-ball of radius $1 + 2\log L_1$ with the $\mathrm{Out}(F_N)$-orbit of $*_{CV_N}$ (which is finite by proper discontinuity of the action). Proposition C.3.10 shows that

- the f-preimage of any integer has diameter at most $1 + 2\log L_1$, so its intersection with the $\mathrm{Out}(F_N)$-orbit of $*_{CV_N}$ has cardinality at most M, and
- for all $\Phi, \Psi \in \mathrm{Out}(F_N)$, if $\Phi *_{CV_N}$ and $\Psi *_{CV_N}$ both belong to $A_{L_1}([\eta_-], [\eta_+]) \cap \mathcal{B}_k^{sym}$, then $|\sigma(\Phi *_{CV_N}) - \sigma(\Psi *_{CV_N})| \leq 2k$.

The cardinality of $\{\Phi \in \mathrm{Out}(F_N) | \Phi *_{CV_N} \in A_{L_1}([\eta_-], [\eta_+]) \cap \mathcal{B}_k^{sym}\}$ is therefore bounded above by $(2k + 1)M$. □

C.3.6 Measurable dependence of the strips on the pair $(T_-, T_+) \in \mathcal{FI} \times \mathcal{FI}$

Proof of Proposition C.3.8. Since $\text{Out}(F_N)$ is countable, we only need to check that for all $\Phi \in \text{Out}(F_N)$, the set

$$S_{L_1}^{-1}(\Phi) := \{(T_-, T_+) \in \mathcal{FI} \times \mathcal{FI} | \Phi \in S_{L_1}(T_-, T_+)\}$$

is measurable. So we only need to check that $L_{T_-,T_+}(\Phi *_{CV_N})$ depends measurably on (T_-, T_+) for all $\Phi \in \text{Out}(F_N)$. As Erg_N is a Borel subset of $\mathbb{P}Curr_N$ by [Phe66, Proposition 1.3], finiteness of $\text{Erg}(T)$ for all $T \in \mathcal{FI}$ (Proposition C.1.5) and continuity of the intersection form imply the existence of countably many measurable maps $f_k : \mathcal{FI} \to \mathbb{P}Curr_N$ so that for all $T \in \mathcal{FI}$, we have $\text{Erg}(T) = \{f_k(T) | k \in \mathbb{N}\}$, see [Cas67]. Using again continuity of the intersection form, this ensures that $L_{T_-,T_+}(\Phi *_{CV_N})$ depends measurably on (T_-, T_+) (notice that for any positive pair $([\eta_-], [\eta_+])$ of currents, the supremum in the definition of $L_{[\eta_-],[\eta_+]}(\Phi *_{CV_N})$ can be taken on a dense countable subset of $\overline{CV_N}$). ☐

C.4 The free factor complex

We now give another interpretation of our results, by realizing the random walk on $\text{Out}(F_N)$ on the complex of free factors of F_N, instead of realizing it on CV_N. The *free factor complex* $\mathcal{FF}_N$, introduced by Hatcher and Vogtmann in [HV98], is defined when $N \geq 3$ as the simplicial complex whose vertices are the conjugacy classes of nontrivial proper free factors of F_N, and higher dimensional simplices correspond to chains of inclusions of free factors. (When $N = 2$, one has to modify this definition by adding an edge between any two complementary free factors to ensure that $\mathcal{FF}_2$ remains connected, and $\mathcal{FF}_2$ is isomorphic to the Farey graph). There is a natural, coarsely well-defined map $\psi : CV_N \to \mathcal{FF}_N$, that maps any tree $T \in CV_N$ to one of the conjugacy classes of the cyclic free factors of F_N generated by an element of F_N whose axis in T projects to an embedded simple loop in the quotient graph T/F_N. When equipped with the simplicial metric, the free factor complex is Gromov hyperbolic ([BF14b], see also [KR14]). Bestvina and Reynolds, and independently Hamenstädt, have determined its Gromov boundary $\partial\mathcal{FF}_N$.

Theorem C.4.1. *(Bestvina–Reynolds [BR13], Hamenstädt [Ham14a]) There exists a unique $Out(F_N)$-equivariant homeomorphism $\partial\psi : \mathcal{AT} \to \partial\mathcal{FF}_N$, so that for all $T \in \mathcal{AT}$ and all sequences $(T_n)_{n\in\mathbb{N}} \in CV_N^{\mathbb{N}}$ that converge to T, the sequence $(\psi(T_n))_{n\in\mathbb{N}}$ converges to $\partial\psi(T)$.*

As a consequence of Theorems C.2.1, C.3.3 and C.4.1, we therefore get the following result. The first part of the statement was obtained with different methods by Calegari and Maher [CM12, Theorem 5.34], who worked in the more general context of isometry groups of (possibly nonproper) hyperbolic metric spaces.

Theorem C.4.2. *Let μ be a probability measure on $Out(F_N)$, such that $gr(\mu)$ is nonelementary. Then for $\mathbb{P}$-almost every sample path $\mathbf{g} := (g_n)_{n\in\mathbb{N}}$ of the random walk on $(Out(F_N), \mu)$, there exists $\xi(\mathbf{g}) \in \partial\mathcal{FF}_N$, such that for all $x \in \mathcal{FF}_N$, the sequence $(g_n x)_{n\in\mathbb{N}}$ converges to $\xi(\mathbf{g})$. The hitting measure ν on $\partial\mathcal{FF}_N$ is the unique μ-stationary measure on $\partial\mathcal{FF}_N$. If in addition, the measure μ has finite first logarithmic moment with respect to the word metric on $Out(F_N)$, and finite entropy, then $(\partial\mathcal{FF}_N, \nu)$ is the Poisson boundary of $(Out(F_N), \mu)$.* ☐

Annexe D

A short proof of Handel and Mosher's alternative for subgroups of Out(F_N)

Abstract

We give a short proof of a theorem of Handel and Mosher [HM09] stating that any finitely generated subgroup of $\mathrm{Out}(F_N)$ either contains a fully irreducible automorphism, or virtually fixes the conjugacy class of a proper free factor of F_N, and we extend their result to non finitely generated subgroups of $\mathrm{Out}(F_N)$.

Contents

Introduction

Let $N \geq 2$, and let F_N denote a finitely generated free group of rank N. A *free factor* of F_N is a subgroup A of F_N such that F_N splits as a free product of the form $F_N = A * B$, for some subgroup $B \subseteq F_N$. An automorphism $\Phi \in \mathrm{Out}(F_N)$ is *fully irreducible* if no power of Φ preserves the conjugacy class of any proper free factor of F_N. The goal of this paper is to give a short proof of the following classification theorem for subgroups of $\mathrm{Out}(F_N)$, which was shown by Handel and Mosher in the case of finitely generated subgroups of F_N in [HM09].

Theorem D.0.1. *Every (possibly non finitely generated) subgroup of Out(F_N) either*

- *contains two fully irreducible elements that generate a rank two free subgroup, or*
- *is virtually cyclic, generated by a fully irreducible automorphism, or*
- *virtually fixes the conjugacy class of a proper free factor of F_N.*

Our proof of Theorem D.0.1 involves studying the action of subgroups of $\mathrm{Out}(F_N)$ on the free factor complex, whose hyperbolicity was proved by Bestvina and Feighn in [BF14b] (see also [KR14] for an alternative proof) and whose Gromov boundary was described by

Bestvina and Reynolds [BR13] and Hamenstädt [Ham14a]. We also use elementary tools that originally arose in the study of random walks on groups, by studying stationary measures on the boundaries of outer space and of the free factor complex.

Theorem D.0.1 has already found various applications, for example to the study of morphisms from lattices to Out(F_N) [BW11] or to spectral rigidity questions [CFKM12].

Handel and Mosher have generalized Theorem D.0.1 in a recent series of papers [HM13b, HM13c, HM13d, HM13e, HM13f] to give a complete classification of finitely generated subgroups of Out(F_N), analogous to Ivanov's classification of subgroups of the mapping class group of a finite type oriented surface [Iva92].

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D.1 Review

D.1.1 Gromov hyperbolic spaces

A geodesic metric space (X, d) is *Gromov hyperbolic* if there exists $\delta > 0$ such that for all $x, y, z \in X$, and all geodesic segments $[x, y], [y, z]$ and $[x, z]$, we have $N_\delta([x, z]) \subseteq N_\delta([x, y]) \cup N_\delta([y, z])$ (where given a subset $Y \subseteq X$ and $r \in \mathbb{R}_+$, we denote by $N_r(Y)$ the r-neighborhood of Y in X). The *Gromov boundary* ∂X of X is the space of equivalence classes of quasi-geodesic rays in X, two rays being equivalent if their images lie at bounded Hausdorff distance.

Isometry groups of Gromov hyperbolic spaces. Let X be a hyperbolic geodesic metric space. An isometry ϕ of X is *loxodromic* if for all $x \in X$, we have

$$\lim_{n \to +\infty} \frac{1}{n} d(x, \phi^n x) > 0.$$

Given a group G acting by isometries on X, we denote by $\partial_X G$ the *limit set* of G in ∂X, which is defined as the intersection of ∂X with the closure of the orbit of any point in X under the G-action. The following theorem, due to Gromov, gives a classification of isometry groups of (possibly nonproper) Gromov hyperbolic spaces. The interested reader will find a sketch of proof in [CdCMT13, Proposition 3.1].

Theorem D.1.1. *(Gromov [Gro87, Section 8.2]) Let X be a hyperbolic geodesic metric space, and let G be a group acting by isometries on X. Then G is either*

- bounded*, i.e. all G-orbits in X are bounded; in this case $\partial_X G = \emptyset$, or*
- horocyclic*, i.e. G is not bounded and contains no loxodromic element; in this case $\partial_X G$ is reduced to one point, or*
- lineal*, i.e. G contains a loxodromic element, and any two loxodromic elements have the same fixed points in ∂X; in this case $\partial_X G$ consists of these two points, or*
- focal*, i.e. G is not lineal, contains a loxodromic element, and any two loxodromic elements have a common fixed point in ∂X; in this case $\partial_X G$ is uncountable and G has a fixed point in $\partial_X G$, or*

- of general type*, i.e. G contains two loxodromic elements with no common endpoints; in this case $\partial_X G$ is uncountable and G has no finite orbit in ∂X. In addition, the group G contains two loxodromic isometries that generate a rank two free subgroup.*

In particular, we have the following result.

Theorem D.1.2. *(Gromov [Gro87, Section 8.2]) Let X be a hyperbolic geodesic metric space, and let G be a group acting by isometries on X. If $\partial_X G \neq \emptyset$, and G has no finite orbit in ∂X, then G contains a rank two free subgroup generated by two loxodromic isometries.*

D.1.2 Outer space

Let $N \geq 2$. *Outer space* CV_N is defined to be the space of simplicial free, minimal, isometric actions of F_N on simplicial metric trees, up to F_N-equivariant homotheties [CV86] (an action of F_N on a tree is *minimal* if there is no proper invariant subtree). We denote by cv_N the *unprojectivized outer space*, in which trees are considered up to equivariant isometries, instead of homotheties. The group $\mathrm{Out}(F_N)$ acts on CV_N and on cv_N on the right by precomposing the actions (one can also consider the $\mathrm{Out}(F_N)$-action on the left by setting $\Phi(T,\rho) = (T, \rho \circ \phi^{-1})$ for all $\Phi \in \mathrm{Out}(F_N)$, where $\rho : F_N \to \mathrm{Isom}(T)$ denotes the action, and $\phi \in \mathrm{Aut}(F_N)$ is any lift of Φ to $\mathrm{Aut}(F_N)$).

An $\mathbb{R}$-*tree* is a metric space (T, d_T) in which any two points x and y are joined by a unique arc, which is isometric to a segment of length $d_T(x,y)$. Let T be an F_N-*tree*, i.e. an $\mathbb{R}$-tree equipped with an isometric action of F_N. For $g \in F_N$, the *translation length* of g in T is defined to be

$$||g||_T := \inf_{x \in T} d_T(x, gx).$$

Culler and Morgan have shown in [CM87, Theorem 3.7] that the map

$$\begin{array}{rccc} i: & cv_N & \to & \mathbb{R}^{F_N} \\ & T & \mapsto & (||g||_T)_{g \in F_N} \end{array}$$

is an embedding, whose image projects to a subspace of $\mathbb{PR}^{F_N}$ with compact closure $\overline{CV_N}$ [CM87, Theorem 4.5]. Bestvina and Feighn [BF94], extending results by Cohen and Lustig [CL95], have characterized the points of this compactification as being the minimal F_N-trees with trivial or maximally cyclic arc stabilizers and trivial tripod stabilizers.

D.1.3 The free factor complex

The *free factor complex* $\mathcal{FF}_N$, introduced by Hatcher and Vogtmann in [HV98], is defined when $N \geq 3$ as the simplicial complex whose vertices are the conjugacy classes of nontrivial proper free factors of F_N, and higher dimensional simplices correspond to chains of inclusions of free factors. (When $N = 2$, one has to modify this definition by adding an edge between any two complementary free factors to ensure that $\mathcal{FF}_2$ remains connected, and $\mathcal{FF}_2$ is isomorphic to the Farey graph). Gromov hyperbolicity of $\mathcal{FF}_N$ was proved by Bestvina and Feighn [BF14b] (see also [KR14] for an alternative proof). There is a natural, coarsely well-defined map $\psi : CV_N \to \mathcal{FF}_N$, that maps any tree $T \in CV_N$ to one of the conjugacy classes of the cyclic free factors of F_N generated by an element of F_N whose axis in T projects to an embedded simple loop in the quotient graph T/F_N. The Gromov boundary of $\mathcal{FF}_N$ was determined independently by Bestvina and Reynolds

[BR13] and by Hamenstädt [Ham14a]. A tree $T \in \partial CV_N$ is *arational* if no proper free factor of F_N acts with dense orbits on its minimal subtree in T (in particular, no proper free factor of F_N is elliptic in T). We denote by $\mathcal{AT}$ the subspace of ∂CV_N consisting of arational trees. We define an equivalence relation $\sim$ on $\mathcal{AT}$ by setting $T \sim T'$ whenever T and T' have the same underlying topological tree.

Theorem D.1.3. *(Bestvina–Reynolds [BR13], Hamenstädt [Ham14a]) There is a unique homeomorphism $\partial\psi : \mathcal{AT}/\sim \to \partial\mathcal{FF}_N$, so that for all $T \in \mathcal{AT}$ and all sequences $(T_n)_{n\in\mathbb{N}} \in CV_N^{\mathbb{N}}$ that converge to T, the sequence $(\psi(T_n))_{n\in\mathbb{N}}$ converges to $\partial\psi(T)$.*

Recall from the introduction that an automorphism $\Phi \in \mathrm{Out}(F_N)$ is *fully irreducible* if no nonzero power of Φ preserves the conjugacy class of any proper free factor of F_N. Bestvina and Feighn have characterized elements of $\mathrm{Out}(F_N)$ which act as loxodromic isometries of $\mathcal{FF}_N$.

Theorem D.1.4. *(Bestvina–Feighn [BF14b, Theorem 9.3]) An outer automorphism $\Phi \in Out(F_N)$ acts loxodromically on $\mathcal{FF}_N$ if and only if it is fully irreducible.*

D.2 An alternative for trees in the boundary of outer space

Given $T \in \partial CV_N \smallsetminus \mathcal{AT}$, the set of conjugacy classes of minimal (with respect to inclusion) proper free factors of F_N which act with dense orbits on their minimal subtree in T, but are not elliptic in T, is finite [Rey12, Corollary 7.4 and Proposition 9.2], and depends $\mathrm{Out}(F_N)$-equivariantly on T. We denote it by $\mathrm{Dyn}(T)$. The following proposition essentially follows from Reynolds' arguments in his proof of [Rey12, Theorem 1.1], we provide a sketch for completeness.

Proposition D.2.1. *For all $T \in \partial CV_N \smallsetminus \mathcal{AT}$, either $Dyn(T) \neq \emptyset$, or there is a nontrivial point stabilizer in T which is contained in a proper free factor of F_N.*

In the proof of Proposition D.2.1, we will make use of the following well-known fact.

Lemma D.2.2. *(see [Rey12, Corollary 11.2]) Let T be a simplicial F_N-tree, all of whose edge stabilizers are (at most) cyclic. Then every edge stabilizer in T is contained in a proper free factor of F_N, and there is at most one conjugacy class of vertex stabilizers in T that is not contained in any proper free factor of F_N.*

Proof of Proposition D.2.1. Let $T \in \partial CV_N \smallsetminus \mathcal{AT}$, and assume that $\mathrm{Dyn}(T) = \emptyset$. First assume that T contains an edge with nontrivial stabilizer. Let S be the simplicial tree obtained by collapsing all vertex trees to points in the decomposition of T as a graph of actions defined in [Lev94]. Then edge stabilizers in T are also edge stabilizers in S, and the conclusion follows from Lemma D.2.2.

Otherwise, as in the proof of [Rey12, Proposition 10.3], we get that if some point stabilizer in T is not contained in any proper free factor of F_N, then T is geometric, has dense orbits, and all its minimal components are surfaces (the reader is referred to [BF95, GLP94] for background on geometric F_N-trees). Dual to the decomposition of T into its minimal components is a bipartite simplicial F_N-tree S called the *skeleton* of T, defined as follows [Gui08, Section 1.3]. Vertices of S are of two kinds: some correspond to minimal components Y of T, and the others correspond to points $x \in T$ belonging to the intersection of two distinct minimal components. There is an edge from the vertex associated to x to the vertex associated to Y whenever $x \in Y$. In particular, point stabilizers in S are either point stabilizers in T, or groups acting with dense orbits on their minimal subtree in T. In a minimal surface component, all point stabilizers are cyclic, so S is a simplicial F_N-tree with (at most) cyclic edge stabilizers. If S is nontrivial, Lemma D.2.2 implies that either a point stabilizer in T is contained in a proper free factor, or some subgroup of F_N is contained in a proper free factor F of F_N and acts with dense orbits on its minimal subtree in T. In the latter case, by decomposing the F-action on the F-minimal subtree of T as a graph of actions with trivial arc stabilizers [Lev94], we get that $\mathrm{Dyn}(T) \neq \emptyset$, which has been excluded. If S is reduced to a point, then T is minimal and dual to a surface with at least two boundary curves (otherwise T would be arational by [Rey12, Theorem 1.1]). Any of these curves yields the desired point stabilizer in T. ☐

D.3 Nonelementary subgroups of Out(F_N)

A subgroup $H \subseteq \mathrm{Out}(F_N)$ is *nonelementary* if it does not preserve any finite set of $\mathcal{FF}_N \cup \partial\mathcal{FF}_N$. In this section, we will prove Theorem D.0.1 for nonelementary subgroups of $\mathrm{Out}(F_N)$.

Theorem D.3.1. *Every nonelementary subgroup of Out(F_N) contains a rank two free subgroup, generated by two fully irreducible automorphisms.*

Stationary measures on ∂CV_N. Our proof of Theorem D.3.1 is based on techniques that originally arose in the study of random walks on groups. All topological spaces will be equipped with their Borel σ-algebra. Let μ be a probability measure on $\mathrm{Out}(F_N)$. A probability measure ν on $\overline{CV_N}$ is *μ-stationary* if $\mu * \nu = \nu$, i.e. for all ν-measurable subsets $E \subseteq \overline{CV_N}$, we have

$$\nu(E) = \sum_{\Phi \in \mathrm{Out}(F_N)} \mu(\Phi)\nu(\Phi^{-1}E).$$

Our first goal will be to prove the following fact.

Proposition D.3.2. *Let μ be a probability measure on Out(F_N), whose support generates a nonelementary subgroup of Out(F_N). Then every μ-stationary probability measure on $\overline{CV_N}$ is supported on $\mathcal{AT}$.*

We will make use of the following classical lemma, whose proof is based on a maximum principle argument (we provide a sketch for completeness). We denote by $gr(\mu)$ the subgroup of $\mathrm{Out}(F_N)$ generated by the support of the measure μ.

Lemma D.3.3. *(Ballmann [Bal89], Woess [Woe89, Lemma 3.4], Kaimanovich–Masur [KM96, Lemma 2.2.2]) Let μ be a probability measure on a countable group G, and let ν be a μ-stationary probability measure on a G-space X. Let D be a countable G-set, and let $\Theta : X \to D$ be a measurable G-equivariant map. If $E \subseteq X$ is a G-invariant measurable subset of X satisfying $\nu(E) > 0$, then $\Theta(E)$ contains a finite $gr(\mu)$-orbit.*

Proof. Let $\widetilde{\nu}$ be the probability measure on D defined by setting $\widetilde{\nu}(Y) := \nu(\Theta^{-1}(Y))$ for all subsets $Y \subseteq D$. It follows from μ-stationarity of ν and G-equivariance of Θ that $\widetilde{\nu}$ is μ-stationary. Let $M \subseteq \Theta(E)$ denote the set consisting of all $x \in \Theta(E)$ such that $\widetilde{\nu}(x)$ is maximal (and in particular positive). Since $\widetilde{\nu}$ is a probability measure, the set M is finite and nonempty. For all $x \in M$, we have

$$\widetilde{\nu}(x) = \sum_{g\in G} \mu(g)\widetilde{\nu}(g^{-1}x) \leq \widetilde{\nu}(x) \sum_{g\in G} \mu(g) = \widetilde{\nu}(x),$$

which implies that for all $g \in G$ belonging to the support of μ, we have $\widetilde{\nu}(g^{-1}x) = \widetilde{\nu}(x)$. Therefore, the set M is invariant under the semigroup generated by the support of $\check{\mu}$ (where $\check{\mu}(g) := \mu(g^{-1})$). As M is finite, this implies that M is $\mathrm{gr}(\mu)$-invariant, so it contains a finite $\mathrm{gr}(\mu)$-orbit. ☐

We now define an $\mathrm{Out}(F_N)$-equivariant map Θ from $\overline{CV_N}$ to the (countable) set D of finite collections of conjugacy classes of proper free factors of F_N. Given a tree $T \in CV_N$, we define $\mathrm{Loop}(T)$ to be the finite collection of conjugacy classes of elements of F_N whose axes in T project to an embedded simple loop in the quotient graph T/F_N (these may be viewed as cyclic free factors of F_N). Given $T \in \overline{CV_N}$, the set of conjugacy classes of point stabilizers in T is finite [Jia91]. Every point stabilizer is contained in a unique minimal (possibly non proper) free factor of F_N, defined as the intersection of all free factors of F_N containing it (the intersection of a family of free factors of F_N is again a free factor). We let $\mathrm{Per}(T)$ be the (possibly empty) finite set of conjugacy classes of proper free factors of F_N that arise in this way, and we set

$$\Theta(T) := \begin{cases} \emptyset & \text{if } T \in \mathcal{AT} \\ \mathrm{Loop}(T) & \text{if } T \in CV_N \\ \mathrm{Dyn}(T) \cup \mathrm{Per}(T) & \text{if } T \in \partial CV_N \smallsetminus \mathcal{AT} \end{cases}.$$

Proposition D.2.1 implies that $\Theta(T) = \emptyset$ if and only if $T \in \mathcal{AT}$.

Lemma D.3.4. *The set $\mathcal{AT}$ is measurable, and Θ is measurable.*

We postpone the proof of Lemma D.3.4 to the next paragraph and first explain how to deduce Proposition D.3.2.

Proof of Proposition D.3.2. Nonelementarity of $gr(\mu)$ implies that the only finite $gr(\mu)$-orbit in D is the orbit of the empty set. Therefore, since $\Theta(T) \neq \emptyset$ as soon as $T \in \overline{CV_N} \smallsetminus \mathcal{AT}$ (Proposition D.2.1), the set $\Theta(\overline{CV_N} \smallsetminus \mathcal{AT})$ contains no finite $gr(\mu)$-orbit. Proposition D.3.2 then follows from Proposition D.3.3. ☐

Measurability of Θ. Given a finitely generated subgroup F of F_N, we denote by $\mathcal{P}(F)$ the set of trees $T \in \overline{CV_N}$ in which F is elliptic, by $\mathcal{E}(F)$ the set of trees $T \in \overline{CV_N}$ in which F fixes an edge, and by $\mathcal{D}(F)$ the set of trees $T \in \overline{CV_N}$ whose F-minimal subtree is a nontrivial F-tree with dense orbits.

Lemma D.3.5. *For all finitely generated subgroups $F \subseteq F_N$, the sets $\mathcal{P}(F)$, $\mathcal{E}(F)$ and $\mathcal{D}(F)$ are measurable.*

Proof. Let F be a finitely generated subgroup of F_N. Let $s : \overline{CV_N} \to \overline{cv_N}$ be a continuous section. We have

$$\mathcal{P}(F) = \bigcap_{w\in F} \{T \in \overline{CV_N} | ||w||_{s(T)} = 0\},$$

so $\mathcal{P}(F)$ is measurable. An element $g \in F_N$ fixes an arc in a tree $T \in \overline{CV_N}$ if and only if it is elliptic, and there exist two hyperbolic isometries h and h' of T whose translation axes both meet the fixed point set of g but are disjoint from each other. These conditions can be expressed in terms of translation length functions: they amount to requiring that $||gh||_{s(T)} \leq ||h||_{s(T)}$ and $||gh'||_{s(T)} \leq ||h'||_{s(T)}$ and $||hh'||_{s(T)} > ||h||_{s(T)} + ||h'||_{s(T)}$, see [CM87, 1.5]. So $\mathcal{E}(F)$ is a measurable set, too.

The F-minimal subtree of a tree $T \in \overline{CV_N}$ has dense orbits if and only if for all $n \in \mathbb{N}$, there exists a free basis $\{s_1, \dots, s_k\}$ of F so that for all $i, j \in \{1, \dots, k\}$, we have $||s_i||_{s(T)} \leq \frac{1}{n}$ and $||s_is_j||_{s(T)} \leq \frac{1}{n}$. This implies that the set $\mathrm{Dense}(F)$ consisting of those trees in $\overline{CV_N}$ whose F-minimal subtree has dense orbits is measurable. Therefore $\mathcal{D}(F) = \mathrm{Dense}(F) \cap^c \mathcal{P}(F)$ is also measurable. □

Proof of Lemma D.3.4. Measurability of $\mathcal{AT}$ follows from Lemma D.3.5. For all $T \in \overline{CV_N}$, the set $\mathrm{Dyn}(T)$ consists of conjugacy classes of minimal free factors of F_N that act with dense orbits on their minimal subtree in T but are not elliptic, so measurability of the map $T \mapsto \mathrm{Dyn}(T)$ follows from measurability of $\mathcal{D}(F)$ for all finitely generated subgroups F of F_N. Point stabilizers in T are either maximal among elliptic subgroups, or fix an arc in T. Therefore, since $\mathcal{P}(F)$ and $\mathcal{E}(F)$ are measurable for all finitely generated subgroups F of F_N, the set of conjugacy classes of point stabilizers in a tree $T \in \overline{CV_N}$ depends measurably on T. Measurability of $T \mapsto \mathrm{Per}(T)$ follows from this observation. As open simplices in CV_N are also measurable, measurability of Θ follows. □

End of the proof of Theorem D.3.1.

Proposition D.3.6. *Let $H \subseteq Out(F_N)$ be a nonelementary subgroup of $Out(F_N)$. Then the H-orbit of any point $x_0 \in CV_N$ has a limit point in $\mathcal{AT}$.*

Proof. Let μ be a probability measure on $\mathrm{Out}(F_N)$ whose support generates H. Since $\overline{CV_N}$ is compact, the sequence of Cesàro averages of the convolutions $(\mu^{*n} * \delta_{x_0})_{n\in\mathbb{N}}$ has a weak-$*$ limit point ν, which is a μ-stationary measure on $\overline{CV_N}$. We have $\nu(\overline{Hx_0}) = 1$, where Hx_0 denotes the H-orbit of x_0 in CV_N, and Proposition D.3.2 implies that $\nu(\mathcal{AT}) = 1$. This shows that $\overline{Hx_0} \cap \mathcal{AT}$ is nonempty. □

As a consequence of Theorem D.1.3 and Proposition D.3.6, we get the following fact.

Corollary D.3.7. *Let $H \subseteq Out(F_N)$ be a nonelementary subgroup of $Out(F_N)$. Then the H-orbit of any point in $\mathcal{FF}_N$ has a limit point in $\partial\mathcal{FF}_N$.* □

Proof of Theorem D.3.1. Let H be a nonelementary subgroup of $\mathrm{Out}(F_N)$. Corollary D.3.7 shows that the H-orbit of any point in $\mathcal{FF}_N$ has a limit point in $\partial\mathcal{FF}_N$. As H has no finite orbit in $\partial\mathcal{FF}_N$, Theorem D.1.2 shows that H contains two loxodromic isometries which generate a free group of rank two. Theorem D.3.1 then follows from the fact that elements of $\mathrm{Out}(F_N)$ that act loxodromically on $\mathcal{FF}_N$ are fully irreducible (Theorem D.1.4). □

D.4 Proof of Theorem D.0.3

Let H be a subgroup of $\mathrm{Out}(F_N)$. If H is nonelementary, then the claim follows from Theorem D.3.1. Otherwise, either H fixes a finite subset of conjugacy classes of proper free factors (in which case a finite index subgroup of H fixes the conjugacy class of a proper free factor of F_N), or H virtually fixes a point in $\partial\mathcal{FF}_N$. The set of trees in ∂CV_N that project to this point is a finite-dimensional simplex in ∂CV_N by [Gui00, Corollary 5.4], and H fixes the finite subset of extremal points of this simplex. Up to passing to a finite index subgroup again, we can assume that H fixes an arational tree $T \in \partial CV_N$.

By Reynolds' characterization of arational trees [Rey12, Theorem 1.1], either T is free, or else T is dual to an arational measured lamination on a surface S with one boundary component. In the first case, it follows from [KL11a, Theorem 1.1] that H is virtually cyclic, virtually generated by an automorphism $\Phi \in \text{Out}(F_N)$, and in this case Φ is fully irreducible, otherwise H would virtually fix the conjugacy class of a proper free factor of F_N. In the second case, all automorphisms in H can be realized as diffeomorphisms of S [BH92, Theorem 4.1]. So H is a subgroup of the mapping class group of S, and the claim follows from the analogous classical statement that stabilizers of arational measured foliations are virtually cyclic [MP89, Proposition 2.2]. ☐

Annexe E

The boundary of the outer space of a free product

Abstract

Let G be a countable group that splits as a free product of groups of the form $G = G_1 * \cdots * G_k * F_N$, where F_N is a finitely generated free group. We identify the closure of the outer space $P\mathcal{O}(G, \{G_1, \dots, G_k\})$ for the axes topology with the space of projective minimal, *very small* $(G, \{G_1, \dots, G_k\})$-trees, i.e. trees whose arc stabilizers are either trivial, or cyclic, closed under taking roots, and not conjugate into any of the G_i's, and whose tripod stabilizers are trivial. Its topological dimension is equal to $3N+2k-4$, and the boundary has dimension $3N+2k-5$. We also prove that any very small $(G, \{G_1, \dots, G_k\})$-tree has at most $2N + 2k - 2$ orbits of branch points.

Contents

Introduction

Let G be a countable group that splits as a free product

$$G = G_1 * \cdots * G_k * F_N,$$

where F_N denotes a finitely generated free group of rank N. We assume that $N + k \geq 2$. A natural group of automorphisms associated to such a splitting is the group $\mathrm{Out}(G, \{G_1, \dots, G_k\})$ consisting of those outer automorphisms of G that preserve the conjugacy classes of each of the groups G_i.

The study of the group $\mathrm{Out}(F_N)$ of outer automorphisms of a finitely generated free group has greatly benefited from the study of its action on Culler and Vogtmann's outer space [CV86], as well as some hyperbolic complexes. The present paper is a starting point of a work in which we extend results about the geometry of these $\mathrm{Out}(F_N)$-spaces to analogues for free products, with a view to establishing a Tits alternative for the group of outer automorphisms of a free product [9]. The second main step towards this will be to define hyperbolic complexes equipped with $\text{Out}(G, \{G_1, \dots, G_k\})$-actions, and compute the Gromov boundary of the graph of cyclic splittings of G relative to the G_i's [8].

The group $\text{Out}(G, \{G_1, \dots, G_k\})$ acts on a space $P\mathcal{O}(G, \{G_1, \dots, G_k\})$ called *outer space*. This was introduced by Guirardel and Levitt in [GL07b], who generalized Culler and Vogtmann's construction [CV86] of an outer space CV_N associated to a finitely generated free group of rank N, with a view to proving finiteness properties of the group $\text{Out}(G, \{G_1, \dots, G_k\})$. The outer space $P\mathcal{O}(G, \{G_1, \dots, G_k\})$ is defined as the space of all G-equivariant homothety classes of minimal *Grushko* $(G, \{G_1, \dots, G_k\})$*-trees*, i.e. metric simplicial G-trees in which nontrivial point stabilizers coincide with the conjugates of the G_i's, and edge stabilizers are trivial.

Outer space can be embedded into the projective space $\mathbb{PR}^G$ by mapping any tree in $P\mathcal{O}(G, \{G_1, \dots, G_k\})$ to the collection of all translation lengths of elements in G. The goal of the present paper is to describe the closure of the image of this embedding.

The closure of Culler and Vogtmann's classical outer space has been identified by Bestvina and Feighn [BF94] and Cohen and Lustig [CL95], with the space of projective length functions of minimal, *very small* actions of F_N on $\mathbb{R}$-trees. An F_N-tree is *very small* if arc stabilizers are cyclic (possibly trivial) and closed under taking roots, and tripod stabilizers are trivial.

More precisely, Cohen and Lustig have first proved [CL95, Theorem I] that $\overline{CV_N}$ is contained in the space of projective length functions of very small F_N-actions on $\mathbb{R}$-trees. In addition, they have shown that every simplicial, very small F_N-tree is a limit of free and simplicial actions [CL95, Theorem II]. Bestvina and Feighn have shown that this remains true of every very small (possibly nonsimplicial) F_N-action on an $\mathbb{R}$-tree. However, their proof does not seem to handle the case of geometric actions that are dual to foliated band complexes, one of whose minimal components is a measured foliation on a nonorientable surface, and in which some arc stabilizer is nontrivial. Indeed, in this case, it is not clear how to approximate the foliation by rational ones without creating any one-sided compact leaf, and one-sided compact leaves are an obstruction for the dual action to be very small (arc stabilizers are not closed under taking roots). Building on Cohen and Lustig's and Bestvina and Feighn's arguments, and using approximation techniques due to Levitt and Paulin [LP97] and Guirardel [Gui98], we reprove the fact that $\overline{CV_N}$ identifies with the space of minimal, very small projective F_N-trees. Our proof tackles both cases of simplicial and nonsimplicial trees at the same time (it gives a new interpretation of Cohen and Lustig's argument in the simplicial case). We work in the more general framework of free products of groups. A $(G, \{G_1, \dots, G_k\})$*-tree* is an $\mathbb{R}$-tree, equipped with a G-action, in which all G_i's fix a point. A $(G, \{G_1, \dots, G_k\})$-tree will be termed *very small* if arc stabilizers are either trivial, or cyclic, closed under taking roots, and not conjugate into any of the G_i's, and tripod stabilizers are trivial. We prove the following theorem.

Theorem E.0.1. *The closure* $\overline{P\mathcal{O}(G, \{G_1, \dots, G_k\})}$ *in* $\mathbb{PR}^G$ *is compact, and it is the space of projective length functions of minimal, very small* $(G, \{G_1, \dots, G_k\})$*-trees.*

When T is a $(G, \{G_1, \dots, G_k\})$-tree with trivial arc stabilizers, we can be a bit more precise about the approximations we get, and show that T is an unprojectivized limit of Grushko $(G, \{G_1, \dots, G_k\})$-trees T_n, that come with G-equivariant 1-Lipschitz maps from T_n to T, see Theorem E.5.3.

We then compute the topological dimension of the closure and the boundary of the outer space of a free product of groups. In the case of free groups, Bestvina and Feighn have shown in [BF94] that $\overline{CV_N}$ has dimension $3N - 4$, their result was extended by

Gaboriau and Levitt who proved in addition that ∂CV_N has dimension $3N-5$ in [GL95]. Following Gaboriau and Levitt's arguments, we show the following.

Theorem E.0.2. *The space $\overline{P\mathcal{O}(G,\{G_1,\dots,G_k\})}$ has topological dimension $3N+2k-4$, and $\partial P\mathcal{O}(G,\{G_1,\dots,G_k\})$ has dimension $3N+2k-5$.*

Along the proof, we provide a bound on the number of orbits of branch points and centers of inversion in a very small $(G,\{G_1,\dots,G_k\})$-tree, and on the possible Kurosh ranks of point stabilizers.

We also introduce the slightly larger class of *tame* $(G,\{G_1,\dots,G_k\})$-trees, defined as those trees whose arc stabilizers are either trivial, or cyclic and nonperipheral, and with a finite number of orbits of directions at branch points and inversion points. We study some properties of this class, and provide some conditions under which a limit of tame $(G,\mathcal{F})$-trees is tame. This class will turn out to be the right class of trees for carrying out our arguments to describe the Gromov boundary of the graph of cyclic splittings of G relative to the G_i's in [8].

The paper is organized as follows. In Section E.1, we review basic facts about free products of groups, and the associated outer spaces. In Section E.2, we introduce a notion of *geometric* $(G,\{G_1,\dots,G_k\})$-trees, and explain in particular how to approximate every very small $(G,\{G_1,\dots,G_k\})$-tree by a sequence of geometric $(G,\{G_1,\dots,G_k\})$-trees. We then prove compactness of the space of projective, minimal, very small $(G,\{G_1,\dots,G_k\})$-trees in Section E.3, and compute its topological dimension in Section E.4. In Section E.5, we identify the closure of outer space with the space of minimal, very small projective $(G,\{G_1,\dots,G_k\})$-trees. We finally introduce the class of tame $(G,\{G_1,\dots,G_k\})$-trees, and discuss some of its properties, in Section E.6.

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E.1 Background

E.1.1 Free products of groups and free factors

Let G be a countable group which splits as a free product of groups of the form

$$G = G_1 * \dots * G_k * F,$$

where F is a finitely generated free group. We let $\mathcal{F} := \{[G_1],\dots,[G_k]\}$ be the finite collection of the G-conjugacy classes of the G_i's, which we call a *free factor system* of G. The rank of the free group F arising in such a splitting only depends on $\mathcal{F}$. We call it the *free rank* of $(G,\mathcal{F})$ and denote it by $\text{rk}_f(G,\mathcal{F})$. The *Kurosh rank* of $(G,\mathcal{F})$ is defined as $\text{rk}_K(G,\mathcal{F}) := \text{rk}_f(G,\mathcal{F}) + |\mathcal{F}|$. Subgroups of G that are conjugate into one of the subgroups in $\mathcal{F}$ will be called *peripheral*.

A $(G,\mathcal{F})$*-free splitting* is a free splitting of G in which all subgroups in $\mathcal{F}$ are elliptic. A $(G,\mathcal{F})$*-free factor* is a subgroup of G which is a vertex stabilizer in some $(G,\mathcal{F})$-free splitting.

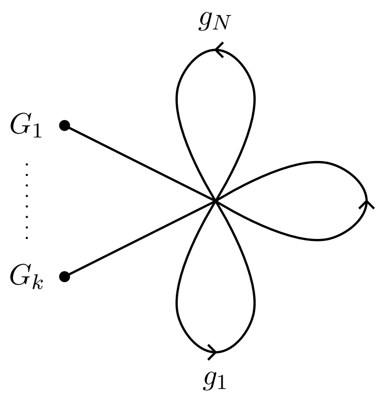

Figure E.1: A standard $(G,\mathcal{F})$-free splitting.

Subgroups of free products have been studied by Kurosh in [Kur34]. Let H be a subgroup of G. Let T be the Bass–Serre tree of the graph of groups decomposition of G represented in Figure E.1. By considering the H-minimal subtree in T, we get the existence of a (possibly infinite) set J, together with an integer $i_j \in \{1,\dots,k\}$, a nontrivial subgroup $H_j \subseteq G_{i_j}$ and an element $g_j \in G$ for each $j \in J$, and a (not necessarily finitely generated) free subgroup $F' \subseteq G$, so that

$$H = *_{j\in J}\; g_j H_j g_j^{-1} * F'.$$

This decomposition is called a *Kurosh decomposition* of H. The *Kurosh rank* of H (which can be infinite in general) is defined as $\mathrm{rk}_K(H) := \mathrm{rk}(F') + |J|$, it does not depend on a Kurosh decomposition of H. We let $\mathcal{F}_{|H}$ be the set of all H-conjugacy classes of the subgroups $g_j H_j g_j^{-1}$, for $j \in J$, which does not depend on a Kurosh decomposition of H either.

When H is a $(G,\mathcal{F})$-free factor, we have $H_j = G_{i_j}$ for all $j \in J$, and all integers i_j are distinct (in particular J is finite). In this case, the free group F' is finitely generated. Hence the Kurosh rank of H is finite.

E.1.2 Outer space and its closure

An $\mathbb{R}$-tree is a metric space (T, d_T) in which any two points $x, y \in T$ are joined by a unique embedded topological arc, which is isometric to a segment of length $d_T(x,y)$.

Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. A $(G,\mathcal{F})$-tree is an $\mathbb{R}$-tree T equipped with an isometric action of G, in which all peripheral subgroups are elliptic. A *Grushko $(G,\mathcal{F})$-tree* is a minimal, simplicial metric $(G,\mathcal{F})$-tree with trivial edge stabilizers, whose collection of point stabilizers coincides with the conjugates of the subgroups in $\mathcal{F}$. Two $(G,\mathcal{F})$-trees are *equivalent* if there exists a G-equivariant isometry between them.

The *unprojectivized outer space* $\mathcal{O}(G,\mathcal{F})$ is defined to be the space of all equivalence classes of Grushko $(G,\mathcal{F})$-trees. *Outer space* $P\mathcal{O}(G,\mathcal{F})$ is defined as the space of homothety classes of trees in $\mathcal{O}(G,\mathcal{F})$. We note that in the case where $\mathcal{F} = \{G\}$, outer space is reduced to a single point, corresponding to the trivial action of G on a point.

For all $(G,\mathcal{F})$-trees T and all $g \in G$, the *translation length* of g in T is defined to be

$$||g||_T := \inf_{x\in T} d_T(x, gx).$$

Theorem E.1.1. *(Culler–Morgan [CM87]) The map*

$$\begin{array}{rccc} i: & \mathcal{O}(G,\mathcal{F}) & \to & \mathbb{R}^G \\ & T & \mapsto & (||g||_T)_{g\in G} \end{array}$$

is injective.

We equip $\mathcal{O}(G,\mathcal{F})$ with the topology induced by this embedding, which is called the *axes topology.* Culler and Morgan have shown in [CM87, Theorem 4.5] that if G is finitely generated, then the subspace of $\mathbb{PR}^G$ made of projective classes of translation length functions of minimal G-trees is compact, so in particular the embedding of $P\mathcal{O}(G,\mathcal{F})$ into the projective space $\mathbb{PR}^G$ provided by Theorem E.1.1 has compact closure. However, the hypothesis that G be finitely generated is only used to ensure the existence of a finite set D of elements of G such that for all G-trees T, there exists $d \in D$ with $||d||_T > 0$ (see the proof of [CM87, Theorem 4.2]). We claim that in the context of $(G,\mathcal{F})$-trees, the above fact is still satisfied. Indeed, letting $G = G_1 * \cdots * G_k * F_N$, we choose a free basis $\{f_1,\dots,f_N\}$ of F_N, and an element $g_i \in G_i \smallsetminus \{e\}$ for every $i \in \{1,\dots,k\}$. Then the set $D := \{f_1,\dots,f_N, f_1f_2,\dots,f_1f_n, f_1g_1, g_1g_2,\dots,g_1g_k\}$ satisfies the required condition. Indeed, if T is a $(G,\mathcal{F})$-tree, and $||d||_T = 0$ for all $d \in D$, then $f_1,\dots,f_n$ all have a common fixed point in T, and this fixed point is also a fixed point for all the G_i's. We summarize the above discussion in the following proposition.

Proposition E.1.2. *The image $i(\mathcal{O}(G,\mathcal{F}))$ in $\mathbb{R}^G$ has projectively compact closure.* $\square$

The goal of the present paper is to give a concrete description of this closure in terms of $(G,\mathcal{F})$-trees. This closure will be identified in Section E.5 with the space of projective classes of minimal, *very small* $(G,\mathcal{F})$-trees, which are defined in the following way.

Definition E.1.3. *A $(G,\mathcal{F})$-tree T is* small *if arc stabilizers in T are either trivial, or cyclic and non-peripheral. A $(G,\mathcal{F})$-tree T is* very small *if it is small, and in addition nontrivial arc stabilizers in T are closed under taking roots, and tripod stabilizers in T are trivial.*

We note that the trivial action of G on a point is very small in the above sense. We will denote by $VSL(G,\mathcal{F})$ the subspace of $\mathbb{PR}^G$ made of very small $(G,\mathcal{F})$-trees.

E.1.3 The equivariant Gromov–Hausdorff topology

The equivariant Gromov–Hausdorff topology on the space of $(G,\mathcal{F})$-trees. The space $\mathcal{O}(G,\mathcal{F})$ can also be equipped with the *equivariant Gromov–Hausdorff topology* [Pau88], which is equivalent to the axes topology [Pau89]. We now recall the definition of the equivariant Gromov–Hausdorff topology on the space of $(G,\mathcal{F})$-trees. Let T and T' be two $(G,\mathcal{F})$-trees, let $K \subset T$ and $K' \subset T'$ be finite subsets, let $P \subset G$ be a finite subset of G, and let $\epsilon > 0$. A *P-equivariant ϵ-relation* between K and K' is a subset $R \subseteq K \times K'$ whose projection to each factor is surjective, such that for all $(x,x'),(y,y') \in R$ and all $g,h \in P$, we have $|d_T(gx,hy) - d_{T'}(gx',hy')| < \epsilon$. A basis of open sets for the equivariant Gromov–Hausdorff topology is given by the sets $O(T,K,P,\epsilon)$ of all $(G,\mathcal{F})$-trees T' for which there exist a finite subset $K' \subset T'$ and a P-equivariant ϵ-relation $R \subseteq K \times K'$ [Pau88].

The equivariant Gromov–Hausdorff topology on the space of pointed $(G,\mathcal{F})$-trees. The equivariant Gromov–Hausdorff topology can also be defined on the space of pointed $(G,\mathcal{F})$-trees. Let T be a $(G,\mathcal{F})$-tree, and let $(x_1,\dots,x_l)\in T^l$. Let $K\subset T$ and $P\subset G$ be finite subsets, and let $\epsilon>0$. A basis of open sets for the equivariant Gromov–Hausdorff topology is given by the sets $O'((T,(x_1,\dots,x_l)),K,P,\epsilon)$ of all pointed $(G,\mathcal{F})$-trees $(T',(x'_1,\dots,x'_l))$ for which there exist a finite subset $K'\subset T'$ and a P-equivariant ϵ-relation $R\subseteq(K\cup\{x_1,\dots,x_l\})\times(K'\cup\{x'_1,\dots,x'_l\})$ with $(x_i,x'_i)\in R$ for all $i\in\{1,\dots,l\}$.

Let T be a $(G,\mathcal{F})$-tree, let $x\in T$, and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of $(G,\mathcal{F})$-trees that converges to T in the equivariant Gromov–Hausdorff topology. A sequence $(x_n)_{n\in\mathbb{N}}\in\prod_{n\in\mathbb{N}}T_n$ is an *approximation* of x if the sequence $((T_n,x_n))_{n\in\mathbb{N}}$ of pointed $(G,\mathcal{F})$-trees converges to (T,x).

Proposition E.1.4. *(Horbez [3, Theorem 4.3]) Let (T,u) (resp. (T',u')) be a pointed very small $(G,\mathcal{F})$-tree, and let $((T_n,u_n))_{n\in\mathbb{N}}$ (resp. $((T'_n,u'_n))_{n\in\mathbb{N}}$) be a sequence of pointed very small $(G,\mathcal{F})$-trees that converges to (T,u) (resp. (T',u')) in the equivariant Gromov–Hausdorff topology. Assume that for all $n\in\mathbb{N}$, there exists a 1-Lipschitz G-equivariant map $f_n:T_n\to T'_n$, such that $f_n(u_n)=u'_n$. Then there exists a 1-Lipschitz G-equivariant map $f:T\to\overline{T'}$, such that $f(u)=u'$, where $\overline{T'}$ denotes the metric completion of T'.*

E.1.4 Graphs of actions and transverse coverings

Let G be a countable group, and $\mathcal{F}$ be a free factor system of G. A $(G,\mathcal{F})$-*graph of actions* consists of

- a marked metric graph of groups $\mathcal{G}$ (in which we allow some edges to have length 0), whose fundamental group is isomorphic to G, such that all subgroups in $\mathcal{F}$ are conjugate into vertex groups of $\mathcal{G}$, and
- an isometric action of every vertex group G_v on a G_v-tree T_v (possibly reduced to a point), in which all intersections of G_v with peripheral subgroups of G are elliptic, and
- a point $p_e\in T_{t(e)}$ fixed by $i_e(G_e)\subseteq G_{t(e)}$ for every oriented edge e.

A $(G,\mathcal{F})$-graph of actions is *nontrivial* if the associated graph of groups is not reduced to a point. Associated to any $(G,\mathcal{F})$-graph of actions $\mathcal{G}$ is a G-tree $T(\mathcal{G})$. Informally, the tree $T(\mathcal{G})$ is obtained from the Bass–Serre tree of the underlying graph of groups by equivariantly attaching each vertex tree T_v at the corresponding vertex v, an incoming edge being attached to T_v at the prescribed attaching point. The reader is referred to [Gui98, Proposition 3.1] for a precise description of the tree $T(\mathcal{G})$. We say that a $(G,\mathcal{F})$-tree T *splits as a* $(G,\mathcal{F})$-*graph of actions* if there exists a $(G,\mathcal{F})$-graph of actions $\mathcal{G}$ such that $T=T(\mathcal{G})$.

A *transverse covering* of an $\mathbb{R}$-tree T is a family $\mathcal{Y}$ of nondegenerate closed subtrees of T such that every arc in T is covered by finitely many subtrees in $\mathcal{Y}$, and for all $Y\neq Y'\in\mathcal{Y}$, the intersection $Y\cap Y'$ contains at most one point. It is *trivial* if $\mathcal{Y}=\{T\}$, and *nontrivial* otherwise. The *skeleton* of $\mathcal{Y}$ is the simplicial tree S defined as follows. The vertex set of S is the set $\mathcal{Y}\cup V_0(S)$, where $V_0(S)$ is the set of all intersection points between distinct subtrees in $\mathcal{Y}$. There is an edge between $Y\in\mathcal{Y}$ and $y\in V_0(S)$ whenever $y\in Y$.

Proposition E.1.5. *(Guirardel [Gui08, Lemma 1.5]) A $(G,\mathcal{F})$-tree splits as a nontrivial $(G,\mathcal{F})$-graph of actions if and only if it admits a nontrivial transverse covering.*

E.2 Geometric $(G, \mathcal{F})$-trees

E.2.1 From systems of isometries to $\mathbb{R}$-trees

Systems of isometries and their suspensions. A *system of isometries* is a pair $\mathcal{K} = (K, \Phi)$, where K is a finite forest, and $\Phi = (\phi_j)_{j \in J}$ is a family of isometries between nonempty closed subtrees A_{ϕ_j} and B_{ϕ_j} of K. The trees A_{ϕ_j} and B_{ϕ_j} are called the *bases* of the isometry ϕ_j. A *singleton* is an isometry whose bases are reduced to points. Given an isometry ϕ, we denote by ϕ^{-1} its inverse, which is a partial isometry from B_ϕ to A_ϕ. Given partial isometries $\phi_1, \dots, \phi_n$, we denote by $\phi_1 \circ \cdots \circ \phi_n$ the composition of the ϕ_i's, which is a partial isometry whose domain is the set of all $x \in A_{\phi_n}$ such that for all $i \in \{2, \dots, n\}$, we have $\phi_i \circ \cdots \circ \phi_n(x) \in A_{\phi_{i-1}}$. A word in the partial isometries in the family Φ and their inverses is *reduced* if it does not contain any subword of the form $\phi \circ \phi^{-1}$ or $\phi^{-1} \circ \phi$.

The *suspension* of a system of isometries $\mathcal{K}$ is the foliated 2-complex Σ, called a *band complex*, defined in the following way. Start with the disjoint union of K (foliated by points), and a copy of the *band* $A_\phi \times [0,1]$ (foliated by *leaf segments* $\{*\} \times [0,1]$) for each $\phi \in \Phi$. We get Σ by gluing each $A_\phi \times [0,1]$ to K, identifying each $(t,0) \in A_\phi \times \{0\}$ with $t \in A_\phi \subseteq K$, and each $(t,1) \in A_\phi \times \{1\}$ with $\phi(t) \in B_\phi \subseteq K$.

A *relatively finite system of isometries* is a triple $\mathcal{K} = (K, (x_1, \dots, x_k), \Phi)$, where (K, Φ) is a system of isometries, and $\{x_1, \dots, x_k\} \subset K$ is a finite subset of K (possibly with repetitions), so that all but finitely many isometries in Φ are singletons with both bases equal, and equal to one of the x_i's. The suspension of a relatively finite system of isometries is a finite foliated 2-complex, with a set of (possibly infinite) bouquets of circles attached at the x_i's. The *singular set* Sing of a relatively finite system of isometries is the finite set consisting of the x_i's, of all vertices of the finite forest K, and of all extremities of the bases of the isometries in Φ.

A system of isometries $\mathcal{K}$ has *independent generators* if no reduced word in the isometries in Φ and their inverses represents a partial isometry of K that fixes some nondegenerate arc. Equivalently, a system of isometries $\mathcal{K}$ has independent generators if their is no cycle of regular leaves in the suspension of $\mathcal{K}$.

From $(G, \mathcal{F})$-systems of isometries to $(G, \mathcal{F})$-trees. Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. The group G splits as a free product of the form $G = G_1 * \cdots * G_k * F_N$, where the G_i's are representatives of the conjugacy classes in $\mathcal{F}$, and F_N is a finitely generated free group. For all $i \in \{1, \dots, k\}$, we fix once and for all a set $\mathcal{C}_i \subseteq G_i$ such that $\langle G_i | \mathcal{C}_i \rangle$ is a presentation of G_i.

Definition E.2.1. *A* $(G, \mathcal{F})$-system of isometries *is a relatively finite system of isometries* $\mathcal{K} = (K, (x_1, \dots, x_k), \Phi = ((\phi_g)_{g \in (G_1 \cup \cdots \cup G_k) \smallsetminus \{e\}}, (\phi_j)_{j \in J}))$ *(where we allow to have $x_i = x_{i'}$ with $i \neq i'$), whose suspension Σ is connected, such that*

- *for all $i \in \{1, \dots, k\}$ and all $g \in G_i \smallsetminus \{e\}$, we have $x_i \in A_{\phi_g}$ and $\phi_g(x_i) = x_i$; the leaf segments passing through x_i of the associated bands of Σ are called* special loops*, they form a (possibly infinite) bouquet of circles R_i based at x_i called a* special rose*, and*
- *there is a point $* \in \Sigma$, and for all $i \in \{1, \dots, k\}$, a path e_i from $*$ to x_i, and*
- *there is a surjective morphism $\rho : \pi_1(\Sigma, *) \to G$, such that for all $i \in \{1, \dots, k\}$ and all $g \in G_i \smallsetminus \{e\}$, if γ is the leaf segment of R_i associated to the isometry ϕ_g, then $\rho(e_i \gamma \overline{e_i}) = g^{-1}$, and whose kernel is normally generated by $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k$, where the elements of $\mathcal{C}_i$ are represented by curves of the form $e_i \gamma \overline{e_i}$.*

We denote by $\widetilde{\Sigma_\rho}$ the covering of Σ corresponding to ρ, so that G acts on $\widetilde{\Sigma_\rho}$. The foliated 2-complex $\widetilde{\Sigma_\rho}$ is naturally equipped with a pseudo-metric, obtained by integration of the transverse measure. As $\ker\rho$ is normally generated by loops contained in leaves of $\widetilde{\Sigma_\rho}$, it follows from [LP97, Proposition 1.7] that the metric space obtained by making this pseudo-metric Hausdorff is an $\mathbb{R}$-tree $T_{\mathcal{K}}$, equipped with a natural G-action. All subgroups G_i are elliptic in $T_{\mathcal{K}}$, so $T_{\mathcal{K}}$ is a $(G,\mathcal{F})$-tree.

For all $i\in\{1,\dots,k\}$, we let $G_i^{\pm}:=(G_i\smallsetminus\{e\})\times\{\pm 1\}=G_i^+\amalg G_i^-$. All elements $g\in G_i$ come in two flavours in $G_i^{\pm}$, both as a positive letter $g^+:=(g,+1)$ and as a negative letter $g^-:=(g,-1)$. The map

$$\begin{array}{ccc} G_i\times\{\pm 1\} & \to & G \\ (g,\epsilon) & \mapsto & g^{\epsilon} \end{array}$$

gives a way of associating an element of the group G to any word in the letters in $G_i^{\pm}$. A $G_i^{\pm}$-word is *freely reduced* if it does not contain any subword of the form g^+g^- or g^-g^+ with $g\in G_i$. For all $g\in G_i$, we let $\phi_g^+:=\phi_g$, and we let ϕ_g^- be the isometry from B_{ϕ_g} to A_{ϕ_g} defined as the inverse of ϕ_g. Given a reduced $G_i^{\pm}$-word $w:=g_1^{\epsilon_1}\dots g_k^{\epsilon_k}$ (with $\epsilon_i\in\{\pm\}$ for all $i\in\{1,\dots,k\}$), we let $\phi_w:=\phi_{g_1}^{\epsilon_1}\circ\dots\circ\phi_{g_k}^{\epsilon_k}$, which is an isometry whose domain is the set of all $x\in K$ such that for all $i\in\{1,\dots,k-1\}$, we have $\phi_{i+1}^{\epsilon_{i+1}}\circ\dots\circ\phi_k^{\epsilon_k}(x)\in A_{\phi_i^{\epsilon_i}}$.

Definition E.2.2. *A $(G,\mathcal{F})$-system of isometries $\mathcal{K}=(K,(x_1,\dots,x_k),\Phi)$ has* relatively independent generators *if for all $i\in\{1,\dots,k\}$, and all nontrivial freely reduced $G_i^{\pm}$-words w, the point x_i is the only element fixed by the isometry ϕ_w.*

E.2.2 Standard $(G,\mathcal{F})$-systems of isometries

We will be mainly interested in trees $T_{\mathcal{K}}$ associated to systems of isometries for which K is a finite tree (i.e. K is connected). We fix once and for all a free basis X of F_N, and we let $B:=(X\cup G_1\cup\dots\cup G_k)\smallsetminus\{e\}$. We also let $B^{\pm}:=B\times\{\pm 1\}=B^+\amalg B^-$. Every $B^{\pm}$-word w can be decomposed as a concatenation of *syllables*, defined as maximal subwords of w that either consist in a single letter in $X^{\pm}$, or consist of a succesion of letters in a single $G_i^{\pm}$. A *B-syllable word* is a word in the letters in $X^{\pm}\amalg(G_1\amalg\dots\amalg G_k)$ (notice that there is a copy 1_i of the identity element of G in each of the G_i's), where no two consecutive letters belong to the same G_i. It is *reduced* if it does not contain any subword of the form x^+x^- or x^-x^+ with $x\in X$, and does not contain any letter 1_i. A B-syllable word W that does not represent a peripheral element of G is *cyclically reduced* if in addition, the first and last letters of W are not of the form x^+ and x^- for any $x\in X$, and they do not belong to the same G_i. Every element of G can be represented in a unique way as a reduced B-syllable word. A nonperipheral element $g\in G$ is *cyclically reduced* if the reduced B-syllable word that represents g is cyclically reduced.

Definition E.2.3. *A* standard $(G,\mathcal{F})$-system of isometries *is a system of isometries $\mathcal{K}=(K,(x_1,\dots,x_k),\Phi=(\phi_g)_{g\in B})$ such that*

- *the forest K is a tree (i.e. it is connected), and $*\in K$, and*
- *for all $i\in\{1,\dots,k\}$ and all $g\in G_i$, the point x_i belongs to the domain of ϕ_g, and $\phi_g(x_i)=x_i$, and*
- *the natural map*

$$\begin{array}{cccc} \theta: & B & \to & \pi_1(\Sigma)/\langle\langle\mathcal{C}_1\cup\dots\cup\mathcal{C}_k\rangle\rangle \\ & g & \mapsto & [*,*_g]\cdot\gamma\cdot[\phi_{(g,-1)}(*_g),*] \end{array},$$

  where $*_g\in B_{\phi_g}$ for all $g\in B$ (with $*_g=x_i$ for all $g\in G_i$), and $[*,*_g]$ is a segment contained in K, and γ is the leaf segment based at $*_g$ in the band associated to $\phi_{(g,-1)}$, induces the inverse of the isomorphism induced by ρ, and

- *for all $i \in \{1,\dots,k\}$, if w is a $G_i^{\pm}$-word that represents the identity of G, then ϕ_w is a restriction of the identity on K.*

Again, we can define the isometry ϕ_w for all $B^{\pm}$-words w as above. For all $g \in X^{\pm}$, we let $\psi_g := \phi_g$. Let $i \in \{1,\dots,k\}$, and let $g \in G_i$. The last property in Definition E.2.3 implies that if w and w' are two $G_i^{\pm}$-words that both represent g, then the isometries ϕ_w and $\phi_{w'}$ are equal in restriction to the intersection of their domains. We can thus define ψ_g to be the partial isometry of K defined on the union of the domains of all ϕ_w, where w ranges over the collection of $G_i^{\pm}$-words that represent g. We claim that the domain of ψ_g is a closed subtree of K. Indeed, for all $B^{\pm}$-words w that represent g, the domain of ϕ_w is a closed subtree of K that contains x_i. In addition, the set of possible distances between x_i and the extremity of the domain of some ϕ_w is finite. The claim follows from these two observations.

The following theorem extends a theorem of Gaboriau and Levitt [GL95, Theorem I.1] to the context of $(G,\mathcal{F})$-trees.

Theorem E.2.4. *Let $\mathcal{K} = (K,(x_1,\dots,x_k),\Phi)$ be a standard $(G,\mathcal{F})$-system of isometries. Then $T_{\mathcal{K}}$ is the unique $(G,\mathcal{F})$-tree such that*
- *the tree K isometrically embeds into $T_{\mathcal{K}}$, and*
- *for all $g \in B$ and all $x \in A_{\phi_g}$, we have $g.x = \phi_g(x)$, and*
- *every segment of $T_{\mathcal{K}}$ is contained in a finite union of translates $g.K$ with $g \in G$, and*
- *if T' is any $(G,\mathcal{F})$-tree satisfying the first two above properties, then there exists a unique G-equivariant morphism $j : T_{\mathcal{K}} \to T'$ such that $j(x) = x$ for all $x \in K$.*

If T' is a tree satisfying the first two bullets of Theorem E.2.4, the morphism j provided by Theorem E.2.4 is called a *resolution* of T'. The following proposition extends [GL95, Proposition I.4].

Proposition E.2.5. *Let $\mathcal{K} = (K,(x_1,\dots,x_k),\Phi)$ be a standard $(G,\mathcal{F})$-system of isometries. For all $x,y \in K$, and all $g \in G$, we have $y = g.x$ in $T_{\mathcal{K}}$ if and only if there exists a $B^{\pm}$-word w representing g, whose associated B-syllable word is reduced, such that $y = \phi_w(x)$.*

Proof of Theorem E.2.4 and Proposition E.2.5. For all $g \in G$, and all $x,y \in K$, let

$$\delta(x,y,g) := \inf\{d_K(x,z_p) + d_K(\phi_p(z_p),z_{p-1}) + \cdots + d_K(\phi_2(z_2),z_1) + d_K(\phi_1(z_1),y)\},$$

where the infimum is taken over all $B^{\pm}$-words $g_1^{\epsilon_1}\dots g_p^{\epsilon_p}$ representing g, and all points z_j in the domain of $\phi_j := \phi_{g_j}^{\epsilon_j}$. We define a pseudometric on $K \times G$ by letting $\delta((x,g),(y,h)) := \delta(x,y,h^{-1}g)$. By definition, the tree $T_{\mathcal{K}}$ is the metric space obtained by making $(K\times G,\delta)$ Hausdorff. The group G acts on this space by $g'.(x,g) = (x,g'g)$. The second property in Theorem E.2.4 follows from the definition, and the third property follows from connectedness of Σ. The last is obtained by letting $\delta'((x,g),(y,h)) := d_{T'}(gx,hy)$, and observing that as soon as T' satisfies the first two bullets of Theorem E.2.4, then $\delta' \leq \delta$. We claim that the infimum defining δ is achieved, and that it is achieved on a $B^{\pm}$-word whose associated B-syllable word is reduced. As the only reduced B-syllable word representing the identity of g is the empty word, the first property of $T_{\mathcal{K}}$ will follow from our claim. Proposition E.2.5 also follows from this claim. We now prove the above claim.

We first introduce some notations. Given a $B^{\pm}$-word w that represents g, we denote by $\delta_w(x,y,g)$ the infimum in the definition of δ, taken over all points z_j in the domains of the isometries ϕ_j associated to the word w. We will also need to work with B-syllable words. We recall the definition of the isometries ψ_j from the paragraph below Definition E.2.3. For all $g \in G$ and all $x, y \in K$, we let

$$\delta^{syl}(x,y,g) := \inf\{d_K(x,z_l) + d_K(\psi_l(z_l), z_{l-1}) + \cdots + d_K(\psi_2(z_2), z_1) + d_K(\psi_1(z_1), y)\},$$

where the infimum is taken over all B-syllable words $W_1 \dots W_l$ representing g, and all points z_j in the domain of $\psi_j := \psi_{W_j}$.

Step 1: For all $g \in G$ and all $x, y \in K$, the infimum in the definition of $\delta^{syl}(x,y,g)$ is achieved by the unique reduced B-syllable word W^{red} that represents g.

Given a B-syllable word W that represents g, we let $\delta^{syl}_W(x,y,g)$ be the infimum in the definition of δ^{syl}, taken over all points z_j in the domains of the isometries ψ_j associated to the word W. It follows from the triangle inequality that $\delta_W(x,y,g)$ is achieved by letting z_l be the point in the domain of ψ_l that is the closest to x, and inductively defining z_{i-1} as the point in the domain of ψ_{i-1} that is the closest to z_i, for all $i \in \{2,\dots,l\}$ (see the proof of [GL95, Theorem I.1] for details of the computation). Note that if W contains a subword of the form x^+x^- or x^-x^+ with $x \in X$, then the word $\widetilde{W}$ obtained from W by removing this subword also represents g, and $\delta^{syl}_{\widetilde{W}}(x,y,g) \leq \delta^{syl}_W(x,y,g)$. If W contains an identity letter 1_i, then ψ_{1_i} is a restriction of the identity of K, so this letter can also be removed without increasing the value of $\delta^{syl}_W(x,y,g)$. This implies that $\delta^{syl}(x,y,g)$ is achieved by the unique reduced B-syllable word W^{red} that represents g.

Step 2: There exists a $B^{\pm}$-word $\widetilde{w}$ representing g, such that $\delta_{\widetilde{w}}(x,y,g) \leq \delta^{syl}_{W^{red}}(x,y,g)$.

The word W^{red} reads as $W^{red} := W_1 \dots W_l$, where for all $j \in \{1,\dots,l\}$, the letter W_j is either an $X^{\pm}$-letter, or a G_{i_j}-letter for some $i_j \in \{1,\dots,k\}$. There are points $z_1,\dots,z_l$ defined as in Step 1 (in particular z_j belongs to the domain of ψ_j) such that

$$\begin{aligned}\delta^{syl}(x,y,g) &= \delta^{syl}_{W^{red}}(x,y,g)\\ &= d_K(x,z_l) + d_K(\psi_l(z_l), z_{l-1}) + \cdots + d_K(\psi_1(z_1), y).\end{aligned}$$

For all j such that $W_j \in X^{\pm}$, we let $\widetilde{w}_j := W_j$. By definition of ψ_j, for all j such that $W_j \in G_{i_j}$, there exists a $G_{i_j}^{\pm}$-word $\widetilde{w}_j := w_1 \dots w_s$ representing W_j, such that the points $z_j, \phi_{w_s}(z_j), \dots, \phi_{w_1} \circ \cdots \circ \phi_{w_s}(z_j) = \psi_j(z_j)$ are all contained in a leaf of the foliation. By inserting these points between z_j and z_{j-1} in the above expression of $\delta^{syl}(x,y,g)$, we see that the $B^{\pm}$-word $\widetilde{w}$ obtained by concatenating the $\widetilde{w}_j$'s satisfies $\delta_{\widetilde{w}}(x,y,g) \leq \delta^{syl}_{W^{red}}(x,y,g)$.

Step 3: For all $B^{\pm}$-words w representing g, the associated B-syllable word W satisfies $\delta^{syl}_W(x,y,g) \leq \delta_w(x,y,g)$.

Let $w := w_1 \dots w_p$ be a $B^{\pm}$-word that represents g, and let $W := W_1 \dots W_l$ be the corresponding B-syllable word. For all $j \in \{1,\dots,l\}$, we denote by g_j the element of G represented by W_j. Arguing as in Step 1, we see that $\delta_w(x,y,g)$ is achieved by letting z_p be the point in the domain of ϕ_{w_p} that is the closest to x, and inductively defining z_{i-1} as the point in the domain of $\phi_{w_{i-1}}$ that is the closest to z_i, for all $i \in \{2,\dots,p\}$. We let $x = z_{i_{l+1}}, z_{i_l}, \dots, z_{i_1}, z_{i_0} = y$ be those points along the sequence corresponding to first letters of the syllables of w.

Let $j \in \{1,\dots,l\}$ be such that g_j belongs to some G_{l_j}. The above argument gives points $u_1,\dots,u_s \in K$ such that

$$\delta_{w_{i_j}\dots w_{i_{j+1}-1}}(z_{i_{j+1}}, z_{i_j}, g_j) = d_K(z_{i_{j+1}}, u_s) + d_K(\phi_s(u_s), u_{s-1}) + \dots + d_K(\phi_1(u_1), z_{i_j}),$$

where $\phi_1 \circ \dots \circ \phi_s$ is the partial isometry associated to the word $w_{i_j}\dots w_{i_{j+1}-1}$. As u_m is the projection of u_{m+1} to the domain of ϕ_m, and as x_{l_j} belongs to this domain, the distance $d_K(x_{l_j}, u_m)$ cannot increase from u_{m+1} to u_m. This implies that $\phi_1(u_1)$ belongs to the range of ψ_{g_j}, and has a preimage u'_s in the segment $[x_{l_j}, z_{i_{j+1}}]$. Hence we have $\delta^{syl}_{W_j}(z_{i_{j+1}}, z_{i_j}, g_j) \leq d_K(z_{i_{j+1}}, u'_s) + d_K(\phi_1(u_1), z_{i_j})$. On the other hand, we have $\delta_{w_{i_j}\dots w_{i_{j+1}-1}}(z_{i_{j+1}}, z_{i_j}, g_j) = d_K(z_{i_{j+1}}, u'_s) + d_K(\phi_1(u_1), z_{i_j})$. This shows that $\delta^{syl}_{W_j}(z_{i_{j+1}}, z_{i_j}, g_j) \leq \delta_{w_{i_j}\dots w_{i_{j+1}-1}}(z_{i_{j+1}}, z_{i_j}, g_j)$. This inequality also holds (and is in fact an equality) when W_j is an $X^{\pm}$-letter. As W reads as the concatenation of all W_j's, the triangle inequality then implies that $\delta^{syl}_W(x,y,g) \leq \delta_w(x,y,g)$.

Step 4: The infimum defining δ is achieved on the $B^{\pm}$-word $\widetilde{w}$ defined in Step 2.

Let w be a $B^{\pm}$-word that represents g. By combining Step 1 and Step 3, we get that $\delta^{syl}_{W^{red}}(x,y,g) \leq \delta_w(x,y,g)$. Using Step 2, we then get that $\delta_{\widetilde{w}}(x,y,g) \leq \delta_w(x,y,g)$. This concludes Step 4, and proves the claim. ☐

E.2.3 Construction of standard $(G,\mathcal{F})$-systems of isometries, geometric trees

An important example of standard $(G,\mathcal{F})$-systems of isometries with relatively independent generators comes from the following construction. Let T be a $(G,\mathcal{F})$-tree. For all $i \in \{1,\dots,k\}$, let x_i be the point in T which is fixed by the subgroup G_i. Let $K_0 \subseteq T$ be a finite subtree such that $K_0 \cap gK_0 \neq \emptyset$ for all $g \in X$, and $x_i \in K_0$ for all $i \in \{1,\dots,k\}$. For all $i \in \{1,\dots,k\}$, let D_i be a set of representatives of the G_i-orbits of the connected components of $T \smallsetminus \{x_i\}$ that meet K_0. For all $d \in D_i$, let K_d be the set of all connected components C of $K_0 \smallsetminus \{x_i\}$ such that there exists $g_C \in G_i$ with $g_C C \subseteq d$. We let

$$\tau^d := \bigcup_{C \in K_d} g_C C.$$

The closure of τ^d is a finite subtree of T that contains x_i. We then let

$$\tau_i := \bigcup_{d \in D_i} \tau^d,$$

and let

$$K := K_0 \cup \left(\bigcup_{i=1}^{k} \tau_i \right),$$

which is a finite closed subtree of T. Let Φ be the family of partial isometries of K made of

- one isometry ϕ_g associated to each element $g \in X$, with bases $g^{-1}K_0 \cap K_0$ and $K_0 \cap gK_0$, and
- for all $i \in \{1,\dots,k\}$ and all $g \in G_i$, one isometry ϕ_g with bases $g^{-1}\tau_i \cap K_0$ and $\tau_i \cap gK_0$.

The system $\mathcal{K} := (K, (x_1, \dots, x_k), \Phi)$ is a standard $(G, \mathcal{F})$-system of isometries. The tree T satisfies the first two bullets of Theorem E.2.4, so there is a morphism $j : T_{\mathcal{K}} \to T$.

Lemma E.2.6. *Let T be a $(G, \mathcal{F})$-tree in which no peripheral element of G fixes an arc. Then the system $\mathcal{K}$ has relatively independent generators. If all arc stabilizers in T are trivial, then $\mathcal{K}$ has independent generators.*

Proof. Let $i \in \{1, \dots, k\}$, and let $(g, +1)$ and (g', ϵ') be two distinct elements in $G_i^{\pm}$. We claim that the domain of $\phi_{(g,+1)}$ intersects the domain of $\phi_{(g',\epsilon')}$ only at x_i. Indeed, since g does not fix any arc in T, the domain of $\phi_{(g,+1)}$ is not contained in a direction in τ_i (while its range is by construction). Therefore, if the domains of $\phi_{(g,+1)}$ and $\phi_{(g',\epsilon')}$ intersected nondegenerately, we would have $\epsilon' = +1$, and the ranges of $\phi_{(g,+1)}$ and $\phi_{(g',+1)}$ would also intersect nontrivially by construction. The element $g'g^{-1}$ would then fix an arc in T, so $g = g'$, a contradiction. Therefore, the only $G_i^{\pm}$-words of length 2 representing isometries with nondegenerate domain are of the form $(g, +1)(g', -1)$ or $(g, -1)(g', +1)$. Relative independence of the generators follows from this observation.

Now assume that some reduced word in the partial isometries in $\mathcal{K}$ and their inverses represents an isometry that fixes a degenerate arc of K. This implies that there is a freely reduced $B^{\pm}$-word w, in which all syllables are either of length 1, or of length 2 and of the form $(g, +1)(g', -1)$, that labels a partial isometry of $\mathcal{K}$ which fixes a nondegenerate arc of K. In particular, the B-syllable word associated to w is reduced. Therefore, the associated element of G is nontrivial, and it fixes an arc in T by Proposition E.2.5. If arc stabilizers in T are trivial, we reach a contradiction. ☐

Definition E.2.7. *A $(G, \mathcal{F})$-tree T is* geometric *if there exists a finite subtree $K \subseteq T$ such that the above construction yields $T = T_{\mathcal{K}}$.*

Let T be a $(G, \mathcal{F})$-tree, and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of $(G, \mathcal{F})$-trees. The sequence $(T_n)_{n\in\mathbb{N}}$ *strongly converges* towards T (in the sense of Gillet and Shalen [GS90]) if for all integers $n \leq n'$, there exist morphisms $j_{n,n'} : T_n \to T_{n'}$ and $j_n : T_n \to T$ such that for all $n \leq n'$ and all segments $I \subseteq T_n$, the morphism $j_{n'}$ is an isometry in restriction to $j_{n,n'}(I)$. Strong convergence implies in particular that for all $g \in G$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $||g||_{T_n} = ||g||_T$. The following theorem essentially follows from work by Levitt and Paulin [LP97, Theorem 2.2] and Gaboriau and Levitt [GL95, Proposition II.1].

Theorem E.2.8. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. Let T be a minimal $(G, \mathcal{F})$-tree. Then there exists a sequence $(T_n)_{n\in\mathbb{N}}$ of minimal geometric $(G, \mathcal{F})$-trees that strongly converges towards T. If in addition H is a subgroup of G with finite Kurosh rank that is elliptic in T, then the approximation can be chosen so that H is elliptic in T_n.*

Proof. Let T be a minimal, very small $(G, \mathcal{F})$-tree. For all $i \in \{1, \dots, k\}$, let $\{g_i^1, g_i^2, \dots\}$ be an enumeration of $G_i \smallsetminus \{e\}$. For all $n \in \mathbb{N}$, let $G_i^n := \{g_i^1, \dots, g_i^n\}$, let $B_n := X \cup G_1^n \cup \dots \cup G_k^n$, and let G^n be the set of elements of G that can be written as $B_n^{\pm}$-words of length smaller than or equal to n. Let $x_0 \in T$, and for all $n \geq 1$, let K_n^0 be the convex hull of $\{gx_0 | g \in G_n\}$ in T. Let K_n be the finite subtree of T constructed from K_n^0 as above. Notice that for all $n \geq 1$ and all $g \in X$, we have $K_n \cap gK_n \neq \emptyset$, and for all $i \in \{1, \dots, k\}$, we have $x_i \in K_n$. All extreme points of K_n belong to the orbit of x_0. By minimality, the tree T is the increasing union of the trees K_n, and there exists $g_0 \in G$ such that x_0 belongs to the translation axis of g_0 in T. Let $n_0 \in \mathbb{N}$ be large enough so that K_{n_0} contains g_0x_0.

For all $n \geq n_0$, we let T_n be the tree obtained from K_n by the above construction. The distance between x_0 and g_0x_0 is the same in T_n and in T, which implies that x_0 belongs to the axis of g_0 in T_n. Being the convex hull of the orbit of x_0, the tree T_n is minimal.

We claim that the sequence $(T_n)_{n\in\mathbb{N}}$ strongly converges towards T. Theorem E.2.4 provides morphisms $j_{n,n'} : T_n \to T_{n'}$ for all $n \leq n'$, and morphisms $j_n : T_n \to T$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$, and $I \subseteq T_n$. Theorem E.2.4 enables us to choose a finite set Y of elements of G so that I is covered by the translates of K_n under elements in Y. We then let n' be large enough so that $K_{n'}$ contains all these translates of K_n. Then $j_{n'}$ is an isometry in restriction to $j_{n,n'}(I)$. This shows strong convergence of the trees T_n towards T.

By choosing n large enough so that all elements in a basis of the free part of the Kurosh decomposition of H, as well as all conjugators arising in this decomposition, can be written as $B_n^{\pm}$-words of length smaller than or equal to n, and K_n contains a fixed point of H, we can ensure that the last property of Theorem E.2.8 is satisfied. $\square$

Remark E.2.9. Notice that if branch points are dense in T, then the tree $K_n \cap K_0$ has edges of arbitrarily small length as n tends to $+\infty$, from which it follows that simplicial edges in the geometric approximation of T have lengths going to 0.

E.2.4 Stabilizers in geometric $(G,\mathcal{F})$-trees

We now list a few other useful properties of the tree $T_{\mathcal{K}}$, which were proved by Gaboriau and Levitt in [GL95] in the case of F_N-trees.

Proposition E.2.10. *Let $\mathcal{K}$ be a standard $(G,\mathcal{F})$-system of isometries, and let $g \in G$ be nonperipheral and cyclically reduced. Then the fixed point set of g in $T_{\mathcal{K}}$ is contained in K.*

Proof. The proof goes as in [GL95, Proposition I.5]. Let $a \in T_{\mathcal{K}}$ be a fixed point of g. Choose a representative $(x,h) \in K \times G$ of a (with the notations from the proof of Theorem E.2.4), such that the length of the unique B-syllable word $U = U_1 \dots U_k$ representing h is minimal, and assume that $h \neq e$. Since $(x,h) = (x,gh)$, it follows from Proposition E.2.5 that there exists a $B^{\pm}$-word $w = (w_1,\epsilon_1)\dots(w_n,\epsilon_n)$ representing $h^{-1}gh$, such that $x = \phi_w(x)$. We let $W_1 \dots W_l$ be the associated B-syllable word (which is not a single letter in some G_i because g is nonperipheral). Then $(\psi^{-}_{W_1}(x), U_1 \dots U_k W_1)$ and $(\psi_{W_l}(x), U_1 \dots U_k W_l^{-1})$ also represent a. Minimality of the length of U implies that $U_kW_1 \neq 1$ and $U_kW_l^{-1} \neq 1$. Therefore, the word $U_1 \dots U_k W_1 \dots W_l U_k^{-1} \dots U_1^{-1}$ (in which we might have to replace U_kW_1 and $W_nU_k^{-1}$ by a single letter in some $G_i \smallsetminus \{1_i\}$) is a reduced B-syllable word that represents g. Therefore g is not cyclically reduced, a contradiction. $\square$

Corollary E.2.11. *Let T be a small $(G,\mathcal{F})$-tree, and let K be a finite subtree of T that contains all fixed points of the G_i's, and such that $K \cap sK \neq \emptyset$ for all $s \in X$. For all $g \neq 1 \in G$, the restriction of the resolution map $j : T_{\mathcal{K}} \to T$ to the fixed point set of g is an isometry. In particular, arc stabilizers in $T_{\mathcal{K}}$ are either trivial, or cyclic and non-peripheral. If T is very small, then tripod stabilizers in $T_{\mathcal{K}}$ are trivial.* $\square$

Let $k \in \mathbb{N}$. A $(G,\mathcal{F})$-tree T is *k-tame* if T is small, and in addition, we have $\mathrm{Fix}(g^{kl}) = \mathrm{Fix}(g^k)$ for all $l \geq 1$. We refer to Section E.6 for details and equivalent definitions.

Corollary E.2.12. *Let T be a small $(G,\mathcal{F})$-tree, and let K be a finite subtree of T that contains all fixed points of the G_i's, and such that $K \cap sK \neq \emptyset$ for all $s \in X$. For all $k \in \mathbb{N}$, if T is k-tame, then $T_{\mathcal{K}}$ is k-tame.*

Proof. Let $g \in G$ be nonperipheral, and let $k \in \mathbb{N}$. Up to passing to a conjugate, we can assume that g is cyclically reduced, and in this case the fixed point set of g in $T_{\mathcal{K}}$ is contained in K by Proposition E.2.10. Let $x \in K$ be such that there exists $l \in \mathbb{N}$ such that $g^{kl}(x) = x$. We will show that $g^k(x) = x$. By Proposition E.2.5, there exist partial isometries $\phi_1, \dots, \phi_n$ associated to a $B^{\pm}$-word representing g^{kl}, such that $\phi_w(x) = x$, whose associated B-syllable word $W = W_1 \dots W_p$ is reduced. As g is cyclically reduced, there exists a subword $W_s \dots W_p$ of W that represents g^k, and we have $g^k.x = \psi_{W_s} \dots \psi_{W_p}(x)$. As $g^k x = x$ in T, we have $\psi_{W_s} \dots \psi_{W_p}(x) = x$, so $g^k x = x$ in $T_{\mathcal{K}}$. □

As a consequence of Theorem E.2.8 and the above two corollaries, we get the following approximation result. The statement about relative independence of the generators comes from Lemma E.2.6.

Theorem E.2.13. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. Let T be a minimal, small $(G,\mathcal{F})$-tree. Then there exists a sequence $(T_n)_{n\in\mathbb{N}}$ of minimal, small, geometric $(G,\mathcal{F})$-trees with relatively independent generators, that strongly converges towards T. If T is very small (resp. k-tame for some $k \in \mathbb{N}$), the approximation can be chosen very small (resp. k-tame).* □

E.3 Compactness of the space of projective very small $(G,\mathcal{F})$-trees

We recall that $VSL(G,\mathcal{F})$ denotes the space of projective classes of nontrivial, minimal, very small $(G,\mathcal{F})$-trees, equipped with the axes topology. We will prove that this space is compact. As the space of all minimal $(G,\mathcal{F})$-trees is compact, this amounts to showing that every limit point of a sequence of very small $(G,\mathcal{F})$-trees is very small. This was proved by Cohen and Lustig for actions of finitely generated groups on $\mathbb{R}$-trees [CL95, Theorem I]: by working in the axes topology, they proved closedness of the conditions that nontrivial arc stabilizers are cyclic and root-closed, and that tripod stabilizers are trivial. We will provide a shorter proof of these facts by working in the equivariant Gromov–Hausdorff topology. The Gromov–Hausdorff topology is equivalent to the axes topology on the space of minimal, irreducible $(G,\mathcal{F})$-trees [Pau89], and small $(G,\mathcal{F})$-trees are irreducible (i.e. they have no global fixed end). A proof of the fact that being small is a closed condition (in the equivariant Gromov–Hausdorff topology) also appears in [Pau88, Lemme 5.7]. In our setting, we also need to check closedness of the condition that arc stabilizers are not peripheral. We will make use of classical theory of group actions on $\mathbb{R}$-trees, and refer the reader to [CM87] for an introduction to this theory.

Proposition E.3.1. *For all countable groups G and all free factor systems $\mathcal{F}$ of G, the space $VSL(G,\mathcal{F})$ is compact in the equivariant Gromov–Hausdorff topology.*

Lemma E.3.2. *Let T be a minimal $(G,\mathcal{F})$-tree, and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of minimal $(G,\mathcal{F})$-trees that converges (non-projectively) to T. Let $g \in G$ be an element that fixes a nondegenerate arc in T. Then for all $n \in \mathbb{N}$ sufficiently large, either g fixes a nondegenerate arc in T_n, or g is hyperbolic in T_n.*

Proof. Otherwise, up to passing to a subsequence, we can assume that for all $n \in \mathbb{N}$, the element g fixes a single point x_n in T_n. Let $[a,b]$ be a nondegenerate arc fixed by g in T. Let a_n (resp. b_n) be an approximation of a (resp. b) in the tree T_n. As $d_{T_n}(a_n, ga_n)$ and $d_{T_n}(b_n, gb_n)$ both tend to 0, the points a_n and b_n are both arbitrarily close to x_n. Therefore, the distance $d_{T_n}(a_n, b_n)$ converges to 0, and $a = b$, a contradiction. □

Lemma E.3.3. *Let T be a minimal $(G,\mathcal{F})$-tree, and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of minimal $(G,\mathcal{F})$-trees that converges to T. Let $g \in G$. Assume that some power g^k of g fixes a nondegenerate arc I in T. If for all sufficiently large $n \in \mathbb{N}$, the element g is hyperbolic in T_n, then g fixes I.*

Proof. Let $I := [a,b]$. Let a_n (resp. b_n) be an approximation of a (resp. b) in T_n. Since $d_{T_n}(a_n, g^k a_n)$ and $d_{T_n}(b_n, g^k b_n)$ both converge to 0, the points a_n and b_n are arbitrarily close to the axis of g in T_n, and $||g||_{T_n}$ converges to 0. Hence both $d_{T_n}(a_n, ga_n)$ and $d_{T_n}(b_n, gb_n)$ converge to 0, so g fixes $[a,b]$. □

Proof of Proposition E.3.1. As the space of all translation length functions of minimal, irreducible $(G,\mathcal{F})$-trees is compact, we only need to prove that being very small is a closed condition in the space of minimal, irreducible $(G,\mathcal{F})$-trees. Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of very small $(G,\mathcal{F})$-trees that converges to a $(G,\mathcal{F})$-tree T.

Let $g \in G$ be a peripheral element. Then for all $n \in \mathbb{N}$, the element g fixes a single point in T_n. Lemma E.3.2 implies that g fixes a single point in T.

Let now $g, h \in G$ be two elements that fix a common nondegenerate arc $[a,b] \subseteq T$. We will show that the group $\langle g, h\rangle$ is abelian, and hence cyclic because g and h are nonperipheral. Let a_n (resp. b_n) be an approximation of a (resp. b) in T_n. Let $\epsilon > 0$, chosen to be small compared to $d_T(a,b)$. Since $d_{T_n}(a_n, ga_n), d_{T_n}(b_n, gb_n) \leq \epsilon$ for n large enough, while $d_{T_n}(a_n, b_n) \geq d_T(a,b) - \epsilon$, the characteristic sets of g and h in T_n (i.e. either their translation axes or their fixed point sets) have an overlap of length greater than 3ϵ. On the other hand, we have $||g||_{T_n}, ||h||_{T_n} \leq \epsilon$. This implies that the elements $[g,h]$, $g[g,h]g^{-1}$ and $h[g,h]h^{-1}$ all fix a common nondegenerate arc in T_n. As T_n is very small, the group generated by these elements is (at most) cyclic, and in addition $[g,h]$ is nonperipheral. This implies that $[g,h]$ is hyperbolic in any Grushko $(G,\mathcal{F})$-tree. Both g and h preserve the axis of $[g,h]$ in a Grushko $(G,\mathcal{F})$-tree, and hence g and h commute.

Let now $g \in G$ be an element, one of whose proper power g^k fixes a nondegenerate arc $[a,b] \subseteq T$.

We first assume that g fixes a nondegenerate arc in T_n for all $n \in \mathbb{N}$, and let I_n denote the fixed point set of g in T_n. Since T_n is very small, the element g also fixes I_n for all $n \in \mathbb{N}$. Let a_n (resp. b_n) be an approximation of a (resp. b) in T_n. Since $d_{T_n}(g^k a_n, a_n)$ and $d_{T_n}(g^k b_n, b_n)$ both converge to 0, the arc I_n comes arbitrarily close to both a_n and b_n. This implies that both $d_{T_n}(ga_n, a_n)$ and $d_{T_n}(gb_n, b_n)$ converge to 0, and therefore g fixes $[a,b]$ in T.

Otherwise, up to passing to a subsequence, we can assume that g^k, and hence g, is hyperbolic in T_n for all $n \in \mathbb{N}$. It then follows from Lemma E.3.3 that g fixes $[a,b]$.

We finally assume that g fixes a nondegenerate tripod in T, whose extremities we denote by a, b and c. Let m be the center of this tripod, and $L > 0$ be the shortest distance in T between m and one of the points a, b or c. Let a_n (resp. b_n, c_n, m_n) be an approximation of a (resp. b, c, m) in T_n, and let $\epsilon > 0$ be such that $\epsilon < \frac{L}{2}$. For n sufficiently large, the point m_n lies at distance at most ϵ from the center m'_n of the tripod formed by a_n, b_n and c_n in T_n. In addition, as a_n, b_n and c_n all lie at distance at most ϵ from $C_{T_n}(g)$, the distance from m'_n to one of the points a_n, b_n or c_n is at most ϵ. This leads to a contradiction. ☐

E.4 Dimension of the space of very small $(G, \mathcal{F})$-trees

Bestvina and Feighn have shown in [BF94, Corollary 7.12] that the space of very small F_N-trees has dimension $3N-4$. Their result was improved by Gaboriau and Levitt in [GL95, Theorem V.2], who showed in addition that $VSL(F_N) \smallsetminus CV_N$ has dimension $3N-5$. Following Gaboriau and Levitt's proof, we extend their computation to the general case of $(G, \mathcal{F})$-trees. We recall the notion of the free rank of $(G, \mathcal{F})$ from Section E.1.1.

Theorem E.4.1. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G, such that $rk_K(G, \mathcal{F}) \geq 2$. Then $VSL(G, \mathcal{F})$ has topological dimension $3rk_f(G, \mathcal{F}) + 2|\mathcal{F}| - 4$.*

Theorem E.4.2. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G, such that $rk_K(G, \mathcal{F}) \geq 2$. Then $VSL(G, \mathcal{F}) \smallsetminus P\mathcal{O}(G, \mathcal{F})$ has topological dimension $3rk_f(G, \mathcal{F}) + 2|\mathcal{F}| - 5$.*

E.4.1 The index of a small $(G, \mathcal{F})$-tree

Let T be a small $(G, \mathcal{F})$-tree, and let $x \in T$. The Kurosh decomposition of the stabilizer $\mathrm{Stab}(x)$ reads as
$$\mathrm{Stab}(x) = g_1 G_{i_1} g_1^{-1} * \dots * g_r G_{i_r} g_r^{-1} * F.$$
We claim that the groups $G_{i_1}, \dots, G_{i_r}$ are pairwise non conjugate in G, which implies in particular that there are only finitely many free factors arising in the Kurosh decomposition of $\mathrm{Stab}(x)$. Indeed, otherwise, we could find $i \in \{1, \dots, k\}$, and $g \in G$, such that both G_i and gG_ig^{-1} fix x. This would imply that G_i fixes both x and $g^{-1}x$, and therefore $g^{-1}x = x$ because no arc of T is fixed by a peripheral element. Hence G_i and gG_ig^{-1} are conjugate in $\mathrm{Stab}(x)$, a contradiction.

Notice that the free group F might *a priori* not be finitely generated (though it will actually follow from Corollary E.4.5 that it is). We define the *index* of x as
$$i(x) = 2\ \mathrm{rk}_K(\mathrm{Stab}(x)) + v_1(x) - 2,$$
where $v_1(x)$ denotes the number of $\mathrm{Stab}(x)$-orbits of directions from x in T with trivial stabilizer. A point $x \in T$ is a *branch point* if $T \smallsetminus \{x\}$ has at least 3 connected components. It is an *inversion point* if $T \smallsetminus \{x\}$ has 2 connected components, and some element $g \in G$ fixes x and permutes the two directions at x. The following proposition is a generalization of [GL95, Proposition III.1].

Proposition E.4.3. *For all small minimal $(G, \mathcal{F})$-trees T and all $x \in T$, we have $i(x) \geq 0$. If T is very small, then $i(x) > 0$ if and only if x is a branch point or an inversion point.*

Proof. If $\mathrm{rk}_K(\mathrm{Stab}(x)) \geq 2$, then we have $i(x) \geq 2$, and in this case x is a branch point. If $\mathrm{Stab}(x)$ is trivial, then $i(x) = v_1(x) - 2$, where $v_1(x)$ is the number of connected components of $T \smallsetminus \{x\}$, which is nonnegative because T is minimal, and $i(x) > 0$ if and only if x is a branch point. Finally, if $\mathrm{rk}_K(\mathrm{Stab}(x)) = 1$, then $i(x) = v_1(x) \geq 0$. If $i(x) > 0$, then either x is a branch point as in the first case, or x has valence 2 and is therefore an inversion point. If $i(x)=0$, and T is very small, the stabilizer of any direction from x is isomorphic to $\mathrm{Stab}(x)$. As tripod stabilizers are trivial in T, this implies that x is not a branch point. $\square$

Let T be a small $(G,\mathcal{F})$-tree, and let $x,x'\in T$. If x and x' belong to the same G-orbit, then $i(x)=i(x')$. Given a G-orbit $\mathcal{O}$ of points in T, we can thus define $i(\mathcal{O})$ to be equal to $i(x)$ for any $x\in\mathcal{O}$. We then let

$$i(T):=\sum_{\mathcal{O}\in T/G} i(\mathcal{O}).$$

We now extend [GL95, Theorem III.2] and its corollaries [GL95, Corollaries III.3 and III.4] to the context of $(G,\mathcal{F})$-trees.

Proposition E.4.4. *For all small $(G,\mathcal{F})$-trees T, we have $i(T)\leq 2rk_K(G,\mathcal{F})-2$. If T is geometric, then $i(T)=2rk_K(G,\mathcal{F})-2$.*

Using Proposition E.4.3, we get the following result as a corollary of Proposition E.4.4.

Corollary E.4.5. *Any very small $(G,\mathcal{F})$-tree has at most $2rk_K(G,\mathcal{F})-2$ orbits of branch or inversion points, and the Kurosh rank of the stabilizer of any $x\in T$ is at most equal to $rk_K(G,\mathcal{F})$.* $\square$

Given a finite tree K and $x\in K$, we have

$$\sum_{x\in K}(v_K(x)-2)=-2, \tag{E.1}$$

where $v_K(x)$ denotes the valence of x in K.

Proof of Proposition E.4.4. First assume that T is geometric. For all $i\in\{1,\dots,k\}$, we let $x_i\in T$ be the fixed point of G_i. Let K be a finite subtree of T, constructed as in Section E.2.3, such that $T=T_{\mathcal{K}}$. We fix a G-orbit of points $\mathcal{O}\subset T$. We will associate to $\mathcal{O}$ two graphs $\mathcal{S}$ and $\mathcal{S}'$ in the following way.

Vertices of $\mathcal{S}$ are the points in $\mathcal{O}\cap K$. There is an edge e labelled by g from z to $\phi_g(z)$ whenever z belongs to the basis of the isometry $\phi_g\in\Phi$. We note that Proposition E.2.5 implies that $\mathcal{S}$ is connected. The graph $\mathcal{S}$ is actually equal to the leaf in Σ passing through any point of $\mathcal{O}\cap K$. We then denote by $v_g(e)$ the valence of z in the domain of ϕ_g. The *multiplicity* $m(v)$ of a vertex $v\in V(\mathcal{S})$ is the number of indices $i\in\{1,\dots,k\}$ such that $v=x_i$, and the *weight* of v is defined as $w(v):=1-m(v)$. The *weight* $w(e)$ of an edge $e\in E(\mathcal{S})$ is defined to be 0 if there exists $i\in\{1,\dots,k\}$ such that e is labelled by an element $g\in G_i$ and joins x_i to itself, and to 1 otherwise. Notice that edges of weight 0 are contained in one of the special roses of $\mathcal{K}$. We define the *Kurosh rank* of $\mathcal{S}$ to be

$$\mathrm{rk}_K(\mathcal{S})=1+\sum_{e\in E(\mathcal{S})}w(e)-\sum_{v\in V(\mathcal{S})}w(v). \tag{E.2}$$

We similarly define the Kurosh rank of any subgraph $\mathcal{G}\subseteq\mathcal{S}$ by only summing over vertices and edges in $\mathcal{G}$. Let $\mathcal{S}^{(2)}$ be the 2-complex obtained from $\mathcal{S}$ by gluing a disk along each curve in the roses R_i that is labelled by an element of $\mathcal{C}_i$, with the notations from Definition E.2.1.

Lemma E.4.6. *We have $rk_K(\mathcal{S})=rk_K(Stab(x))$.*

Proof. The natural morphism $\rho : \pi_1(\mathcal{S}) \to G$, induced by the inclusion of $\mathcal{S}$ as a leaf of Σ, takes its values in $\mathrm{Stab}(x)$. Surjectivity of $\rho : \pi_1(\mathcal{S}) \to \mathrm{Stab}(x)$ follows from Proposition E.2.5. We claim that the kernel of ρ is normally generated by the loops in $\mathcal{C}_i$ contained in the special roses at the points x_i that belong to $\mathcal{O}$. Indeed, let $\gamma \subseteq \mathcal{S}$ be a loop whose label w represents the identity of G. We can assume w to be freely reduced. If w is not the empty word, since w represents the identity, the B-syllable word associated to w is a nonreduced B-syllable word, i.e. there exists a syllable in w which represents the identity element of some G_i. By relative independence of the generators, this can only happen if it is a path contained in the special rose, and hence nullhomotopic in $\mathcal{S}^{(2)}$. Therefore, any loop contained in $\mathcal{S}$, and whose label represents the identity of G, is homotopic to the trivial loop in $\mathcal{S}^{(2)}$. Therefore, the group $\pi_1(\mathcal{S}^{(2)})$ is isomorphic to $\mathrm{Stab}(x)$. The Kurosh rank $\mathrm{rk}_K(\mathcal{S})$ is equal by definition to the rank of the fundamental group of the graph obtained from $\mathcal{S}$ by collapsing the special roses at the points x_i, to which we add 1 for each $i \in \{1, \dots, k\}$ such that $x_i \in \mathcal{O}$. We also recall that the factors arising in the Kurosh decomposition of $\mathrm{Stab}(x)$ are pairwise nonconjugate. By collapsing the special roses in $\mathcal{S}$, we get a graph whose fundamental group has rank $\mathrm{rk}_f(\mathrm{Stab}(x))$, and the number of collapsed roses is equal to $|\mathcal{F}_{\mathrm{Stab}(x)}|$. Therefore, we have $\mathrm{rk}_K(\mathcal{S}) = \mathrm{rk}_K(\mathrm{Stab}(x))$. ☐

We now define a graph $\mathcal{S}'$ by considering orbits of directions instead of orbits of points. Vertices of $\mathcal{S}'$ are the directions from points in $\mathcal{O} \cap K$, and there is an edge labelled by g from d to $\phi_g(d)$ whenever d belongs to the domain of ϕ_g. In this way, every vertex of $\mathcal{S}$ is replaced by $v_K(x)$ vertices in $\mathcal{S}'$, and every edge e in $\mathcal{S}$ labelled by g is replaced by $v_g(e)$ edges in $\mathcal{S}'$. There is a natural map $\pi : \mathcal{S}' \to \mathcal{S}$ that sends vertices to vertices and edges to edges.

Lemma E.4.7. *The set of components of $\mathcal{S}'$ is in one-to-one correspondence with the set of Stab(x)-orbits of directions at x in T. For all directions $d \in V(\mathcal{S}')$, the fundamental group of the component of $\mathcal{S}'$ that contains d is isomorphic to the stabilizer of d, hence to $\{1\}$ or $\mathbb{Z}$.*

Proof. Let $\mathcal{S}'_1$ be a component of $\mathcal{S}'$, and let $d_0 \in V(\mathcal{S}'_1)$ be a direction based at a point $y \in \mathcal{O} \cap K$. Applying any $g \in G$ taking y to x, we get a direction $d := gd_0$ from x in T. The $\mathrm{Stab}(x)$-orbit of d only depends on the component $\mathcal{S}'_1$ (and not on the choices of d_0 and g). This defines a map Ψ from the set of connected components of $\mathcal{S}'$ to the set of $\mathrm{Stab}(x)$-orbits of directions at x in T.

We now prove injectivity of the map Ψ. Let d_0 and d'_0 be two directions in K having the same Ψ-image, then there exists $g \in G$ mapping d_0 to d'_0. Proposition E.2.5, applied to two nearby points defining the direction d_0, implies that d_0 and d'_0 belong to the same component of $\mathcal{S}'$, showing injectivity of Ψ.

We now show surjectivity of Ψ. Let d be a $\mathrm{Stab}(x)$-orbit of directions at x in T. There exists a segment $[x, x_1]$, such that gd is contained in $[x, x_1]$. Then $[x, x_1]$ is contained in some translate wK with $w \in G$, and $w^{-1}gd \subseteq K$. This shows that d belongs to the image of Ψ.

The proof of the second statement of the lemma is similar to the proof of Lemma E.4.6. For all $d \in V(\mathcal{S}'_1)$, there is a morphism $\rho' : \pi_1(\mathcal{S}'_1) \to \mathrm{Stab}(d)$. Surjectivity of ρ' follows from Proposition E.2.5, applied to two nearby points defining d. Injectivity of ρ again comes from the fact that G is a free product, and from relative independence of the generators. ☐

We say that a $(G, \mathcal{F})$-tree T has *finitely many orbits of directions* if there are finitely many orbits of directions based at branch or inversion points in T.

Corollary E.4.8. *For all $x \in K$, there are only finitely many $Stab(x)$-orbits of directions at x in T. In addition, there are finitely many orbits of branch or inversion points in T (and hence finitely many orbits of directions in T).*

Proof. Let d be a direction at x in T. Then the $\mathrm{Stab}(x)$-orbit of d contains a direction in K based at a point $y \in \mathcal{O} \cap K$. Since $\mathcal{S}$ is connected, there is a path γ contained in a leaf of Σ joining y to x. By lifting d along γ, we get that either d is a direction at x that is contained in K, or d is in the G-orbit of a direction in K at a point of the singular set Sing (we recall that the *singular set* of $\mathcal{K}$ is the finite subset of K made of all branch points, all endpoints of the bases of the partial isometries of $\mathcal{K}$, and all points x_i). The claim follows because Sing is a finite set. Using the fact that the orbit of any point of T meets K, the above argument also shows that the orbit of any branch or inversion point in T meets the singular set Sing. ☐

Let $\mathcal{G}$ be a finite connected subgraph of $\mathcal{S}$ containing all vertices in $\mathcal{O}\cap$Sing (where Sing denotes the singular set) and all edges $e \in E(\mathcal{S})$ with $v_g(e) \neq 2w(e)$. This is well-defined, because all but finitely many edges in the special roses satisfy $v_g(e) = 2w(e) = 0$, and all edges based at points of A_{ϕ_g} that do not belong to the finite set Sing satisfy $v_g(e) = 2$ and $w(e) = 1$. We add to $\mathcal{G}$ the special roses at the points x_i that belong to $\mathcal{O}$. Let $\mathcal{G}' \subseteq \mathcal{S}'$ be the π-preimage of $\mathcal{G}$ in $\mathcal{S}'$. Lemma E.4.7, together with the fact that there are only finitely many $\mathrm{Stab}(x)$-orbits of directions at any point $x \in T$ (Corollary E.4.8), shows that up to enlarging $\mathcal{G}$ if necessary, we may assume that $\pi_1(\mathcal{G}')$ generates the fundamental group of every component of $\mathcal{S}'$. Denote by $\mathcal{G}'_j$ the components of $\mathcal{G}'$. As the fundamental group of any finite graph X satisfies

$$1 - \mathrm{rk}(\pi_1(X)) = |V(X)| - |E(X)|,$$

we have

$$\sum_j (1 - \mathrm{rk}(\pi_1(\mathcal{G}'_j))) = \sum_{x \in V(\mathcal{G})} v_K(x) - \sum_{e \in E(\mathcal{G})} v_g(e).$$

Moreover, the definition of the Kurosh rank of $\mathcal{G}$ given in Equation (E.2) gives

$$2\mathrm{rk}_K(\mathcal{G}) - 2 = \sum_{x \in V(\mathcal{G})} (-2 + 2m(v)) + 2 \sum_{e \in E(\mathcal{G})} w(e).$$

Summing the above two equalities, we get

$$2\mathrm{rk}_K(\mathcal{G}) - 2 + \sum_j (1 - \mathrm{rk}(\pi_1(\mathcal{G}'_j))) = \sum_{x \in V(\mathcal{G})} (v_K(x) - 2) + 2|\mathcal{F}_\mathcal{O}| + \sum_{e \in E(\mathcal{G})} (2w(e) - v_g(e)),$$

where $|\mathcal{F}_\mathcal{O}|$ is the common value of $|\mathcal{F}_{\mathrm{Stab}(x)}|$ for all $x \in \mathcal{O}$. We claim that $\mathrm{rk}_K(\mathcal{G})$ is bounded independently of the choice of the finite graph $\mathcal{G}$, which implies that $\mathrm{rk}_K(\mathcal{S})$ is finite. Indeed, Lemma E.4.7 implies that $1 - \mathrm{rk}(\pi_1(\mathcal{G}'_j))$ cannot be negative. In addition, the right-hand side of the equality does not depend on $\mathcal{G}$, because $v_K(x) = 2$ as soon as $x \notin$ Sing, and $v_g(e) = 2w(e)$ for all edges of $\mathcal{S}$ that do not belong to $\mathcal{G}$. Up to enlarging $\mathcal{G}$ if necessary, we can thus assume that $\pi_1(\mathcal{G}) = \pi_1(\mathcal{S})$. This implies that $\mathcal{G}$ contains any embedded path in $\mathcal{S}$ with endpoints in $\mathcal{G}$, and therefore each component of $\mathcal{S}'$ contains only one component of $\mathcal{G}'$. Lemmas E.4.6 and E.4.7 then imply that the left-hand side of the above inequality is equal to $i(\mathcal{O})$, so

$$i(\mathcal{O}) = \sum_{x \in V(\mathcal{S})} (v_K(x) - 2) + 2|\mathcal{F}_\mathcal{O}| + \sum_{e \in E(\mathcal{S})} (2w(e) - v_g(e)). \tag{E.3}$$

We will now sum up the above equality over all orbits of points in K to get an expression of the index of T. Equation (E.1) implies that

$$\sum_{\mathcal{O}\in T/G}\sum_{x\in V(\mathcal{S})}(v_K(x)-2)=-2.$$

We also have

$$\sum_{\mathcal{O}\in T/G}2|\mathcal{F}_{\mathcal{O}}|=2|\mathcal{F}|,$$

and we are left understanding the rightmost sum in Equation (E.3). We will compute the contribution of each element of $B=(X\cup G_1\cup\dots\cup G_k)\smallsetminus\{e\}$ to this sum.

Let $g\in X$. Then all edges $e\in E(\mathcal{S})$ labelled by g have weight $w(e)=1$, so their contributions to the rightmost sum of Equation (E.3) sum up to

$$\sum_{x\in\mathcal{O}\cap A_{\phi_g}}(2-v_{A_{\phi_g}}(x)).$$

Using Equation (E.1), we get

$$\sum_{\mathcal{O}\in T/G}\sum_{x\in\mathcal{O}\cap A_{\phi_g}}(2-v_{A_{\phi_g}}(x))=2.$$

Let now $i\in\{1,\dots,k\}$, and $g\in G_i$. Then all edges $e\in E(\mathcal{G})$ labelled by g have weight $w(e)=1$, except the one at the point x_i, which has weight 0, in case $x_i\in\mathcal{O}$. Therefore, their contributions to the rightmost sum of Equation (E.3) sum up to

$$\sum_{x\in\mathcal{O}\cap A_{\phi_g}}(2-v_{A_{\phi_g}}(x))-2\epsilon_i(\mathcal{O}),$$

where $\epsilon_i(\mathcal{O})$ is equal to 1 if $x_i\in\mathcal{O}$, and to 0 otherwise. Using Equation (E.1) again, we get that

$$\sum_{\mathcal{O}\in T/G}\sum_{x\in\mathcal{O}\cap A_{\phi_g}}((2-v_{A_{\phi_g}}(x))-2\epsilon_i(\mathcal{O}))=0.$$

By combining all the above equalities, we thus get that

$$i(T)=-2+2|\mathcal{F}|+2\mathrm{rk}_f(G,\mathcal{F})=2\mathrm{rk}_K(G,\mathcal{F})-2,$$

and we are done in the case where T is geometric.

We now turn to the general case, where T need no longer be geometric. Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of finite subtrees of T constructed as in Section E.2.3, such that the corresponding geometric $(G,\mathcal{F})$-trees T_n strongly converge to T. Let $x\in T$ be a branch or inversion point, and let $s\leq i(x)$ be an integer. As T is a $(G,\mathcal{F})$-tree, the Kurosh decomposition of $\mathrm{Stab}(x)$ reads as

$$\mathrm{Stab}(x)=g_1G_{i_1}g_1^{-1}*\dots*g_rG_{i_r}g_r^{-1}*F,$$

where $G_{i_1},\dots,G_{i_r}$ are pairwise non conjugate in G, and F is a free group (which might *a priori* not be finitely generated). Let Y be a finite subset of $\mathrm{Stab}(x)$ made of elements from a free basis of F and one nontrivial element in each of the subgroups $g_jG_{i_j}g_j^{-1}$. Let $d_1,\dots,d_q$ be directions at x in T with trivial stabilizers, in distinct $\mathrm{Stab}(x)$-orbits. We make these choices in such a way that $2|Y|+q-2=s$. Because of strong convergence, it is possible for n large enough to lift x to an element $x_n\in T_n$ in such a way that all elements in Y fix x_n, and we can similarly lift all directions d_i to a direction from x_n in T_n. We have $v_1(x_n)\geq q$, and the resolution morphism from T_n to T provided by Theorem E.2.4 induces an injective morphism from $\mathrm{Stab}(x_n)$ to $\mathrm{Stab}(x)$, whose image contains all elements in Y, and all subgroups $g_jG_{i_j}g_j^{-1}$ because T_n is a $(G,\mathcal{F})$-tree. As the subgroup generated by Y and the collection of subgroups of the form $g_jG_{i_j}g_j^{-1}$ is a free factor, we get that $\mathrm{rk}_K(\mathrm{Stab}(x))\geq|Y|$. Hence $i(x_n)\geq s$. As this is true for all $s\leq i(x)$, we get that $i(x)\leq i(x_n)$. Since lifts to T_n of branch or inversion points in distinct G-orbits in T belong to distinct G-orbits of T_n, it follows from the first part of the argument that $i(T)\leq 2\ \mathrm{rk}_K(G,\mathcal{F})-2$. ☐

E.4.2 Bounding $\mathbb{Q}$-ranks, and the dimension of $VSL(G,\mathcal{F})$

We now compute the dimension of $VSL(G,\mathcal{F})$, following the arguments in [GL95, Sections IV and V]. Let T be a minimal, small $(G,\mathcal{F})$-tree. We denote by L the additive subgroup of $\mathbb{R}$ generated by the values of the translation lengths $||g||_T$, for g varying in G. The $\mathbb{Z}$-*rank* $r_{\mathbb{Z}}(T)$ is the rank of the abelian group L, i.e. the minimal number of elements in a generating set of L (it is infinite if L is not finitely generated). The $\mathbb{Q}$-*rank* $r_{\mathbb{Q}}(T)$ is defined to be the dimension of the $\mathbb{Q}$-vector space $L\otimes_{\mathbb{Z}}\mathbb{Q}$. Notice that we always have $r_{\mathbb{Q}}(T)\leq r_{\mathbb{Z}}(T)$. Let Y be the set of points in T which are either branch points or inversion points. We define Λ as the subgroup of $\mathbb{R}$ generated by distances between points in Y. We have $2\Lambda\subseteq L\subseteq\Lambda$, see [GL95, Section IV]. We recall our notation $G=G_1*\cdots*G_k*F_N$. The following two propositions were stated by Gaboriau and Levitt in the case of nonabelian actions of finitely generated groups on $\mathbb{R}$-trees without inversions. Their proofs adapt to our framework.

Proposition E.4.9. *(Gaboriau–Levitt [GL95, Proposition IV.1]) Let T be a small $(G,\mathcal{F})$-tree, and let $\{g_1,\dots,g_N\}$ be a free basis of F_N. Then the set $\{||g_i||_T\}_{i\in\{1,\dots,N\}}$ generates $L/2\Lambda$.*

Proposition E.4.10. *(Gaboriau–Levitt [GL95, Proposition IV.1]) Let T be a small $(G,\mathcal{F})$-tree, and let $\{p_j\}_{j\in J}$ be a set of representatives of the G-orbits of branch and inversion points in T. Then for all $j_0\in J$, the set $\{d_T(p_{j_0},p_j)\}_{j\in J\smallsetminus\{j_0\}}$ generates Λ/L.*

We refer the reader to [GL95, Proposition IV.1] for a proof of the above two facts. We mention that these proofs are based on the following lemma, which follows from standard theory of group actions on $\mathbb{R}$-trees.

Lemma E.4.11. *(Gaboriau–Levitt [GL95, Proposition IV.1]) Let T be a small $(G,\mathcal{F})$-tree.*
- *For all branch or inversion points $p,q,r\in T$, we have $d(p,r)=d(p,q)+d(q,r)$ mod 2Λ.*
- *For all branch or inversion points $p\in T$ and all $g\in G$, we have $d(p,gp)=||g||_T$ mod 2Λ.*
- *For all $g,h\in G$, we have $||gh||_T=||g||_T+||h||_T$ mod 2Λ.*

Proposition E.4.12. *Let $T\in VSL(G,\mathcal{F})$ be a geometric tree, and let b be the number of orbits of branch or inversion points in T. Then $r_{\mathbb{Z}}(T)\leq rk_f(G,\mathcal{F})+b-1$.*

Proof of Proposition E.4.12. It follows from the proof of Theorem E.2.4 that Λ is generated by distances between points in the finite singular set Sing. So Λ is finitely generated, and therefore L is finitely generated (recall that $2\Lambda\subseteq L\subseteq\Lambda$). Hence $\Lambda/2\Lambda$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{r_{\mathbb{Z}}(T)}$, and the upper bound on $r_{\mathbb{Z}}(T)$ follows from Propositions E.4.9 and E.4.10. ☐

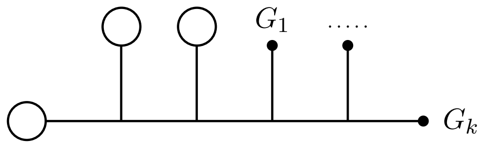

Figure E.2: A $(3\mathrm{rk}_f(G,\mathcal{F})+2|\mathcal{F}|-4)$-simplex in $P\mathcal{O}(G,\mathcal{F})$.

We also recall the following result from [GL95, Proposition IV.2].

Proposition E.4.13. *(Gaboriau–Levitt [GL95, Proposition IV.2]) Let $T \in VSL(G,\mathcal{F})$ be a nongeometric tree obtained as the strong limit of a system $T_{\mathcal{K}(t)}$ of geometric trees. Then*

$$r_{\mathbb{Q}}(T) \leq \liminf_{t\to+\infty} r_{\mathbb{Z}}(T_{\mathcal{K}(t)}),$$

and

$$r_{\mathbb{Q}}(T) < \limsup_{t\to+\infty} r_{\mathbb{Z}}(T_{\mathcal{K}(t)}).$$

Proposition E.4.14. *For all $T \in VSL(G,\mathcal{F})$, we have $r_{\mathbb{Q}}(T) \leq 3rk_f(G,\mathcal{F})+2|\mathcal{F}|-3$.*

Proof. When T is geometric, Proposition E.4.12 implies that $r_{\mathbb{Q}}(T) \leq r_{\mathbb{Z}}(T) \leq \mathrm{rk}_f(G,\mathcal{F})+b-1$. Corollary E.4.5 shows that $b \leq 2\mathrm{rk}_f(G,\mathcal{F})+2|\mathcal{F}|-2$, and the claim follows. When T is nongeometric, it is a strong limit of a system of geometric trees, and the claim follows from Proposition E.4.13. ☐

Proposition E.4.15. *(Gaboriau–Levitt [GL95, Proposition V.1]) Let G be a countable group, let $\mathcal{F}$ be a free factor system of G, and let $k \geq 1$ be an integer. The space of projectivized length functions of $(G,\mathcal{F})$-trees with $\mathbb{Q}$-rank smaller than or equal to k has dimension smaller than or equal to $k-1$.*

Proof of Theorem E.4.1. Theorem E.4.1 follows from Propositions E.4.14 and E.4.15, since outer space $P\mathcal{O}(G,\mathcal{F})$ contains $(3\mathrm{rk}_f(G,\mathcal{F})+2|\mathcal{F}|-4)$-simplices, obtained for instance by varying the edge lengths of a graph of groups that has the shape displayed on Figure E.2. ☐

E.4.3 Very small graphs of actions

In this section, we mention a decomposition result which was proved by Levitt for actions of finitely generated groups on $\mathbb{R}$-trees having finitely many orbits of branch points [Lev94, Theorem 1]. The proof uses the fact that every such action on a tree T is *finitely supported*, i.e. there exists a finite tree $K \subset T$ such that every arc $I \subset T$ is covered by finitely many translates of K. The fact that minimal $(G,\mathcal{F})$-actions on $\mathbb{R}$-trees are finitely supported was noticed by Guirardel in [Gui08, Lemma 1.14]. Using finiteness of the number of orbits of branch and inversion points in a very small $(G,\mathcal{F})$-tree, Levitt's theorem adapts to our more general framework.

Theorem E.4.16. *(Levitt [Lev94, Theorem 1]) Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. Then every tree $T \in VSL(G,\mathcal{F})$ splits uniquely as a graph of actions, all of whose vertex trees have dense orbits, such that the Bass–Serre tree of the underlying graph of groups is very small, and all its edges have positive length.*

E.4.4 Additional arguments for computing the dimension of $VSL(G,\mathcal{F})\smallsetminus PO(G,\mathcal{F})$

We start by recalling the following well-known fact. We recall that a $(G,\mathcal{F})$-tree T has *finitely many orbits of directions* if they are finitely many orbits of directions at branch or inversion points in T.

Proposition E.4.17. *Let T be a $(G,\mathcal{F})$-tree with dense orbits. If T is small, and has finitely many orbits of directions (in particular, if T is very small, or small and geometric), then all stabilizers of nondegenerate arcs in T are trivial.*

Proof. Let $e \subseteq T$ be a nondegenerate arc in T, and assume there exists a nontrivial element $g \in G$ such that $ge = e$. We can find two distinct directions d, d' in e based at branch or inversion points of T (oriented in the same way), and an element $h \in G$ so that $d' = hd$. Notice in particular h is hyperbolic in T, so h is nonperipheral, and $\langle g, h\rangle$ is not cyclic. The points at which these directions are based can be chosen to be both arbitrarily close to the midpoint of e, and in this case g and hgh^{-1} fix a common nondegenerate subarc of e. As T is small, this implies that g and hgh^{-1} commute. Hence h preserves the axis of g in any Grushko $(G,\mathcal{F})$-tree, which implies that g and h generate a cyclic subgroup of G, a contradiction. ☐

The following proposition extends [GL95, Theorem III.2].

Proposition E.4.18. *Let T be a small $(G,\mathcal{F})$-tree. If T is nongeometric, then $i(T) < 2rk_K(G,\mathcal{F}) - 2$.*

Proof. We know from Proposition E.4.4 that $i(T) \leq 2\mathrm{rk}_K(G,\mathcal{F}) - 2$. Assume towards a contradiction that $i(T) = 2\mathrm{rk}_K(G,\mathcal{F}) - 2$.

Let $Y \subset T$ be a finite set that contains one point from each G-orbit with positive index, and let $x \in Y$. The Kurosh decomposition of $\mathrm{Stab}(x)$ reads as

$$\mathrm{Stab}(x) = H_{i_1} * \cdots * H_{i_k} * F,$$

where F is a finitely generated free group, and H_{i_k} is G-conjugate to G_{i_k} for all $j \in \{1,\dots,k\}$. Let $X := \{f_1,\dots,f_q\}$ be a free basis of F, and let $B := H_{i_1} \cup \cdots \cup H_{i_k} \cup X$. Let $(T_n)_{n\in\mathbb{N}}$ be an approximation of T constructed as in the proof of Theorem E.2.8. We can assume that K_n has been chosen so that the extremities of K_n are branch points or inversion points in T (this can be achieved by choosing for x_0 a branch point or inversion point of T, with the notations from the proof of Theorem E.2.8). As in the proof of Proposition E.4.4 in the nongeometric case, we choose directions $d_1,\dots,d_r$, so that $2k + 2q + r - 2 = i(x)$, and $n \in \mathbb{N}$ so that we can associate a point $x_n \in T_n$ to each $x \in Y$.

As $i(T_n) = i(T)$, the orbit of every branch or inversion point of T_n with positive index contains some x_n. Furthermore, every direction from x_n with trivial stabilizer belongs to the $\mathrm{Stab}(x_n)$-orbit of the lift d'_β of one of the directions d_β to T_n.

The morphism $j_n : T_n \to T$ is not an isometry, otherwise T would be geometric. Hence there exist $y \in T_n$, and two adjacent arcs e_1 and e_2 at y whose j_n-images have a common initial segment. If y is a branch point or an inversion point with positive index, it follows from the above paragraph that both e_1 and e_2 have nontrivial stabilizer (otherwise we would have $i(T) < i(T_n)$). As T is small, the stabilizers of e_1 and e_2 generate a cyclic subgroup of G, so there exists $g \in G$ that fixes both e_1 and e_2 in T_n. This contradicts injectivity of j in restriction to the fixed point set of g (Proposition E.2.11). If y is a branch or inversion point with index 0, then y has cyclic stabilizer, and there exists $g \in G$

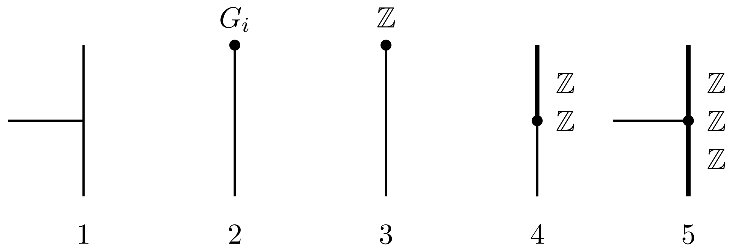

Figure E.3: Vertices of index 1 in a very small simplicial $(G, \mathcal{F})$-tree.

that stabilizes all adjacent edges. Again, this contradicts injectivity of j in restriction to the fixed point set of g. If y is neither a branch point nor an inversion point, then Theorem E.2.4 implies that e_1 and e_2 are contained in the interior of a common G-translate of K_n, because extremal points of K_n have been chosen to be branch or inversion points in T. This again leads to a contradiction, since the restriction of j_n to this translate of K_n is an isometry. $\square$

The following proposition is an extension of [GL95, Theorem IV.1].

Proposition E.4.19. *For all very small $(G, \mathcal{F})$-tree $T \notin \mathcal{O}(G, \mathcal{F})$, we have $r_{\mathbb{Q}}(T) < 3rk_f(G, \mathcal{F}) + 2|\mathcal{F}| - 3$.*

Proof. When T is nongeometric, the claim follows from Propositions E.4.13 and E.4.14. We will assume that T is geometric and show that $r_{\mathbb{Z}}(T) < 3\text{rk}_f(G, \mathcal{F}) + 2|\mathcal{F}| - 3$. We have $r_{\mathbb{Z}}(T) < +\infty$ (Proposition E.4.12), and $\Lambda/2\Lambda$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{r_{\mathbb{Z}}(T)}$.

If the number of distinct orbits of branch or inversion points in T is strictly smaller than $2\text{rk}_K(G, \mathcal{F}) - 2$, then $r_{\mathbb{Z}}(T) < 3\text{rk}_f(G, \mathcal{F}) + 2|\mathcal{F}| - 3$ by Proposition E.4.12, and we are done. Otherwise, let $p_1, \dots, p_{2\text{rk}_K(G,\mathcal{F})-2}$ be a set of representatives in K of the orbits of branch or inversion points in T. Proposition E.4.4 implies that for all $j \in \{1, \dots, 2\text{rk}_K(G, \mathcal{F}) - 2\}$, we have $i(p_j) \leq 1$, and hence $i(p_j) = 1$ by Proposition E.4.3.

If T is a simplicial tree, then Λ is generated by the lengths of the edges of the quotient graph of groups. In particular, Proposition E.4.14 implies that the maximal number of edges of a simplicial tree in $VSL(G, \mathcal{F})$ is $3\text{rk}_f(G, \mathcal{F}) + 2|\mathcal{F}| - 3$. All vertices of T have index 1. Therefore, if $x \in T$ is a vertex, we either have $\text{Stab}(x) = \{e\}$ and $v_1(x) = 3$, or $\text{rk}_K(\text{Stab}(x)) = 1$ and $v_1(x) = 1$. Using the fact that T is very small, we get that every vertex v of T satisfies one of the following possibilities, displayed on Figure E.3: either v

- has valence 3, and trivial stabilizer, or
- projects in the quotient graph of groups to a valence 1 vertex whose stabilizer is peripheral, or
- projects in the quotient graph of groups to a valence 1 vertex whose stabilizer is isomorphic to $\mathbb{Z}$, and not peripheral, or
- projects in the quotient graph of groups to a valence 2 vertex with stabilizer isomorphic to $\mathbb{Z}$ and not peripheral, adjacent to both an edge with trivial stabilizer and an edge with $\mathbb{Z}$ stabilizer, or
- projects in the quotient graph of groups to a valence 3 vertex with stabilizer isomorphic to $\mathbb{Z}$ and not peripheral, adjacent to one edge with trivial stabilizer, and two edges with $\mathbb{Z}$ stabilizers.

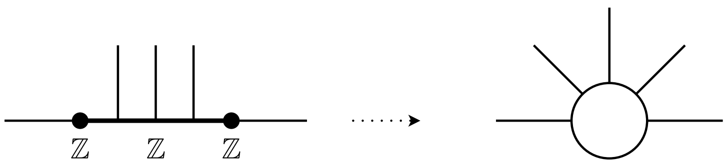

Figure E.4: Simplicial trees in $VSL(G,\mathcal{F}) \smallsetminus P\mathcal{O}(G,\mathcal{F})$ do not have maximal $\mathbb{Z}$-rank.

As $T \notin \mathcal{O}(G,\mathcal{F})$, some vertex in T satisfies one of the last three possibilities. If some vertex in T satisfies the third possibility, then one can split its stabilizer by adding a loop-edge to the quotient graph of groups. This operation yields a new minimal, very small simplicial tree T' for which $r_{\mathbb{Z}}(T') > r_{\mathbb{Z}}(T)$, so $r_{\mathbb{Z}}(T) < 3\text{rk}_f(G,\mathcal{F}) + 2|\mathcal{F}| - 3$. Otherwise, the graph of groups T/G contains a concatenation of edges that all have the same $\mathbb{Z}$ stabilizer, whose two extremal vertices have valence 2, and are adjacent to an edge with trivial stabilizer, and whose interior vertices have valence 3, and are adjacent to a single edge with trivial stabilizer, see Figure E.4. Figure E.4 illustrates how to construct a tree T' with strictly more orbits of edges than T, so that $r_{\mathbb{Z}}(T') > r_{\mathbb{Z}}(T)$. Again, we have $r_{\mathbb{Z}}(T) < 3\text{rk}_f(G,\mathcal{F}) + 2|\mathcal{F}| - 3$.

Assume now that T has dense orbits. Notice that $\Lambda/2\Lambda = \Lambda/L + L/2\Lambda$, so by Propositions E.4.9 and E.4.10, it suffices to prove that the rank of Λ/L is strictly less than $b-1$, where b denotes the number of orbits of branch or inversion points in T. Let $K \subseteq T$ be a finite subtree defined as in Section E.2.3, chosen in such a way that every terminal vertex of K is either a branch point or an inversion point in T. Let $\mathcal{K} = (K, (x_0, \dots, x_k), \Phi)$ be the system of isometries on K constructed in Section E.2.3, so that we have $T = T_{\mathcal{K}}$. The $(G,\mathcal{F})$-tree T has trivial arc stabilizers by Proposition E.4.17, so the generators of $\mathcal{K}$ are independent by Lemma E.2.6. Using [GLP94, Proposition 6.1], we get that

$$|K| = \sum_{\phi\in\Phi} |A_\phi|, \tag{E.4}$$

where $|K|$ (resp. $|A_\phi|$) denotes the total length of K (resp. of A_ϕ). Our hypothesis on the extremal vertices of K implies that

$$|K| = \sum_{e=[q,r]\in E(K)} d_T(q,r),$$

where the sum is taken over all edges in K, after subdividing K at the x_i's. Lemma E.4.11 implies that for all $q, r \in K$, the length of $[q,r]$ is equal modulo L to the sum $d_T(p_1,p_i) + d_T(p_1,p_j)$, where p_i (resp. p_j) belongs to the G-orbit of q (resp. r). Denoting by $\mathcal{O}(p_i)$ the orbit of p_i for all $i \in \{1, \dots, 2\text{rk}_K(G,\mathcal{F}) - 2\}$, we have

$$|K| = \sum_{i=1}^{2\text{rk}_K(G,\mathcal{F})-2} d_T(p_1,p_i) \times \left(\sum_{x\in K\cap\mathcal{O}(p_i)} v_K(x) \right) \mod L,$$

and similarly, for all $\phi \in \Phi$, we have

$$|A_\phi| = \sum_{i=1}^{2\text{rk}_K(G,\mathcal{F})-2} d_T(p_1,p_i) \times \left(\sum_{x\in A_\phi\cap\mathcal{O}(p_i)} v_g(x) \right) \mod L.$$

Using the above two equalities, Equation (E.4) gives a linear relation in Λ/L between the numbers $d_T(p_1,p_i)$, where the coefficient of $d_T(p_1,p_i)$ is equal to

$$\sum_{x\in K\cap\mathcal{O}(p_i)}\left(v_K(x)-\sum_{\phi\in\Phi}v_g(x)\right).$$

For all $i\in\{1,\dots,2\mathrm{rk}_K(G,\mathcal{F})-2\}$, the index of p_i is equal to 1. Therefore, Equation (E.3) from the proof of Proposition E.4.4 (and the analysis below) implies that

$$\sum_{x\in K\cap\mathcal{O}(p_i)}\left(v_K(x)-\sum_{\phi\in\Phi}v_g(x)\right)=1 \bmod L.$$

Equation (E.4) thus leads to the nontrivial relation

$$\sum_{j=2}^{2\mathrm{rk}_K(G,\mathcal{F})-2} d_T(p_1,p_j)=0 \bmod L$$

between the generators of Λ/L, so $r_{\mathbb{Z}}(T)<3\mathrm{rk}_f(G,\mathcal{F})+2|\mathcal{F}|-3$.

In general, let $\mathcal{G}$ be the decomposition of T as a graph of actions provided by Theorem E.4.16. We assume that T is not simplicial, and let T_v be a nontrivial vertex tree of this decomposition. Then T_v is a very small $(G_v,\mathcal{F}_{G_v})$-tree with dense G_v-orbits. Let T' be the very small $(G,\mathcal{F})$-tree obtained from T by collapsing all vertex trees in the G-orbit of T_v to points. By definition of the index, we have

$$i(T)-i(T')=i(T_v)-(2\mathrm{rk}_K(G_v)-2).$$

As T is geometric, Proposition E.4.4 implies that the left-hand side of the above equality is nonnegative, while the right-hand side is nonpositive. This implies that $i(T_v)=2\mathrm{rk}_K(G_v)-2$. Using Proposition E.4.4 again, this shows that the tree T_v is geometric. Assume that the number of distinct G_v-orbits of branch or inversion points in the minimal subtree of T_v is strictly smaller than $2\mathrm{rk}_K(G_v)-2$. Then one of these orbits index at least 2 in T_v, and hence in T. This implies that the number of distinct G-orbits of branch or inversion points in T is strictly smaller than $2\mathrm{rk}_K(G,\mathcal{F})-2$, and we are done by Proposition E.4.12. We are thus left with the case where the number of distinct G_v-orbits of branch or inversion points in T_v is equal to $2\mathrm{rk}_K(G_v)-2$. As distinct G-translates of T_v are disjoint in T, all these G_v-orbits of branch or inversion points are distinct when viewed as G-orbits of points in T. We denote by $p_1,\dots,p_{2\mathrm{rk}_K(G_v)-2}$ a set of representatives of the G_v-orbits of branch or inversion points of T_v. In this case, as T_v has dense orbits, the analysis from the above paragraph provides a nontrivial relation between the generators $d_{T_v}(p_1,p_i)$ of $\Lambda(T_v)/L(T_v)$. The numbers $d_{T_v}(p_1,p_i)$ may also be viewed as part of a generating set of $\Lambda(T)/L(T)$, and we have a nontrivial relation between these generators. Again, this implies that $r_{\mathbb{Z}}(T)<3\mathrm{rk}_K(G,\mathcal{F})+2|\mathcal{F}|-3$. ☐

Proof of Theorem E.4.2. Theorem E.4.2 follows from Propositions E.4.15 and E.4.19, because $VSL(G,\mathcal{F})\smallsetminus PO(G,\mathcal{F})$ contains a $(3\mathrm{rk}_f(G,\mathcal{F})+2|\mathcal{F}|-5)$-simplex made of simplicial $(G,\mathcal{F})$-trees (except in the case where $G=G_1*G_2$ and $\mathcal{F}=\{G_1,G_2\}$, for which $PO(G,\mathcal{F})$ is reduced to a point and $VSL(G,\mathcal{F})\smallsetminus PO(G,\mathcal{F})$ is empty). An example of such a simplex is given by varying edge lengths in a graph of groups obtained from the graph of groups displayed on Figure E.2 by collapsing a loop, or merging two points corresponding to subgroups G_1 and G_2, and adding an edge with nontrivial cyclic stabilizer generated by a nonperipheral element in G_1*G_2, for instance. ☐

E.5 Very small actions are in the closure of outer space.

In the classical case where $G = F_N$ is a finitely generated free group of rank N, and $\mathcal{F} = \emptyset$, Cohen and Lustig have shown that a minimal, simplicial F_N-tree lies in the closure $\overline{cv_N}$ if and only if it is very small [CL95]. Bestvina and Feighn [BF94] have extended their result to all minimal F_N-actions on $\mathbb{R}$-trees. However, it seems that their proof does not handle the case of actions that contain both nontrivial arc stabilizers, and minimal components dual to measured foliations on compact, nonorientable surfaces. Indeed, for such actions, it is not clear how to approximate the foliation by rational ones without creating any one-sided leaf (in which case the action we get is not very small). If the action has trivial arc stabilizers (i.e. if the dual band complex contains no annulus), then the argument in [BF94, Lemma 4.1] still enables to get an approximation by very small, simplicial F_N-trees, by using the narrowing process described in [Gui98, Section 7]. However, this argument does not seem to handle the case of trees having nontrivial arc stabilizers. We will give a proof of the fact that $\overline{cv_N}$ is the space of very small, minimal, isometric actions of F_N on $\mathbb{R}$-trees that does not rely on train-track arguments for approximating measured foliations on surfaces by rational ones. Our proof also gives an interpretation of Cohen and Lustig's for simplicial trees. We again work in our more general framework of $(G,\mathcal{F})$-trees, and show the following result.

Theorem E.5.1. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. The closure $\overline{\mathcal{O}(G,\mathcal{F})}$ (resp. $\overline{P\mathcal{O}(G,\mathcal{F})}$) is the space of (projective) length functions of very small $(G,\mathcal{F})$-trees.*

In particular, Theorem E.5.1 states that every minimal, very small $(G,\mathcal{F})$-tree T can be approximated by a sequence of Grushko $(G,\mathcal{F})$-trees. When T has trivial arc stabilizers, we can be a bit more precise about the nature of the approximation we get.

Definition E.5.2. *Let T be a $(G,\mathcal{F})$-tree. A* Lipschitz approximation *of T is a sequence $(T_n)_{n\in\mathbb{N}}$ of $(G,\mathcal{F})$-trees that converges to T, and such that for all $n \in \mathbb{N}$, there exists a 1-Lipschitz G-equivariant map from T_n to T.*

Lipschitz approximations seem to be useful: they were a crucial ingredient in [3] for tackling the question of spectral rigidity of the set of primitive elements of F_N in $\overline{cv_N}$. They also turn out to be a useful ingredient for describing the Gromov boundary of the (hyperbolic) graph of $(G,\mathcal{F})$-cyclic splittings in [8]. The *quotient volume* of a very small $(G,\mathcal{F})$-tree T is defined as the sum of the edge lengths in the Levitt decomposition of T.

Theorem E.5.3. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. Then every minimal $(G,\mathcal{F})$-tree T with trivial arc stabilizers admits a Lipschitz approximation by (unprojectivized) Grushko $(G,\mathcal{F})$-trees, whose quotient volumes converge to the quotient volume of T.*

E.5.1 Reduction lemmas

To prove Theorem E.5.1, we are left showing that every very small minimal $(G,\mathcal{F})$-tree T can be approximated by a sequence of Grushko $(G,\mathcal{F})$-trees. By Theorem E.2.13, we can approximate every minimal, very small $(G,\mathcal{F})$-tree by a sequence of minimal, very small, geometric $(G,\mathcal{F})$-trees. This approximation is a Lipschitz approximation, and Remark E.2.9 implies that the quotient volumes of the trees in the approximation converge to the quotient volume of T. To complete the proof of Theorems E.5.1 and E.5.3, we are left understanding how to approximate minimal, very small, geometric $(G,\mathcal{F})$-trees by minimal Grushko $(G,\mathcal{F})$-trees.

Our proof of Theorems E.5.1 and E.5.3 will make use of the following lemmas, which enable us to approximate very small $(G,\mathcal{F})$-trees that split as graphs of actions, as soon as we are able to approximate the vertex actions. Lemma E.5.5 is a version of Guirardel's Reduction Lemma in [Gui98, Section 4], where we keep track of the fact that the approximations of the trees are Lipschitz approximations. In Lemma E.5.6, we tackle the problem of approximating trees with nontrivial arc stabilizers by Grushko $(G,\mathcal{F})$-trees. Our argument may be seen as an interpretation of Cohen and Lustig's twisting argument for approximating such trees [CL95]. We consider graphs of actions, instead of restricting ourselves to simplicial trees. We first recall Guirardel's Reduction Lemma from [Gui98, Section 4]. In the statements below, all limits are nonprojective.

Lemma E.5.4. *(Guirardel [Gui98, Section 4]) Let T be a very small $(G,\mathcal{F})$-tree that splits as a graph of actions $\mathcal{G}$. Assume that all pointed vertex actions $(T^v,(u_1^v,\dots,u_k^v))$ admit an approximation by a sequence of pointed $(G^v,\mathcal{F}_{|G^v})$-actions $((T_n^v,(u_{1,n}^v,\dots,u_{k,n}^v)))_{n\in\mathbb{N}}$, in which the approximation points are fixed by the adjacent edge stabilizers. For all $n\in\mathbb{N}$, let T_n be the $(G,\mathcal{F})$-tree obtained by replacing all vertex actions $(T^v,(u_1^v,\dots,u_k^v))$ by their approximation $(T_n^v,(u_{1,n}^v,\dots,u_{k,n}^v))$ in $\mathcal{G}$. Then $(T_n)_{n\in\mathbb{N}}$ converges to T.*

We say that a sequence $((T_n,(u_n^1,\dots,u_n^k)))_{n\in\mathbb{N}}$ of pointed $(G,\mathcal{F})$-trees is a *Lipschitz approximation* of a pointed $(G,\mathcal{F})$-tree $(T,(u^1,\dots,u^k))$ if $((T_n,(u_n^1,\dots,u_n^k)))_{n\in\mathbb{N}}$ converges to $(T,(u^1,\dots,u^k))$, and for all $n\in\mathbb{N}$, there exists a 1-Lipschitz G-equivariant map $f_n:T_n\to T$ such that for all $i\in\{1,\dots,k\}$, we have $f_n(u_n^i)=u^i$. Guirardel's Reduction Lemma can be refined in the following way.

Lemma E.5.5. *Let T be a very small $(G,\mathcal{F})$-tree with trivial arc stabilizers, that splits as a $(G,\mathcal{F})$-graph of actions $\mathcal{G}$. If all pointed vertex trees $(T^v,(u_1^v,\dots,u_k^v))$ of $\mathcal{G}$ admit Lipschitz approximations by pointed Grushko $(G^v,\mathcal{F}_{|G^v})$-trees, in which the approximation points $u_{1,n}^v,\dots,u_{k_n}^v$ are fixed by the adjacent edge stabilizers, then T admits a Lipschitz approximation by Grushko $(G,\mathcal{F})$-trees.* ☐

Lemma E.5.6. *Let T be a minimal, very small $(G,\mathcal{F})$-tree, that splits as a $(G,\mathcal{F})$-graph of actions over a one-edge $(G,\mathcal{F})$-free splitting (where the vertex actions need not be minimal). If the minimal subtrees of all vertex trees of $\mathcal{G}$ (with respect to the action of their stabilizer G^v) admit approximations by minimal Grushko $(G^v,\mathcal{F}_{|G^v})$-trees, then T admits an approximation by minimal Grushko $(G,\mathcal{F})$-trees.*

Figures E.5, E.7, and E.8 provide examples of trees for which the vertex actions of the splitting are not minimal (but they are minimal in the sense of pointed trees when we keep track of the attaching points). These are the crucial cases of Lemma E.5.6, in which we deal with the problem of approximating trees with nontrivial arc stabilizers. Considering nonminimal vertex actions is crucial to deal with the simplicial case in Theorem E.5.1 (when there are edges with nontrivial stabilizers), and Lemma E.5.6 provides a new interpretation of Cohen and Lustig's argument for dealing with this case. Lemma E.5.6 will also be crucial for dealing with the case of geometric actions of surface type containing nontrivial arc stabilizers.

Proof. We will provide a detailed argument in the case where the $(G,\mathcal{F})$-free splitting S is a free product, and explain how to adapt the argument to the case of an HNN extension.

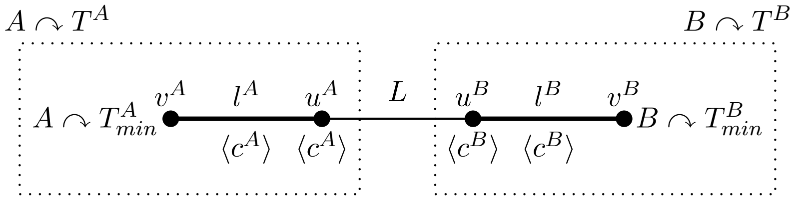

Figure E.5: The splitting of T as a graph of actions in Case 1 of the proof of Lemma E.5.6.

Case 1 : The splitting S is of the form $G = A * B$.

The following description of $\mathcal{G}$ is illustrated in Figure E.5. We denote by L the length of the edge of $\mathcal{G}$, which might be equal to 0. Denote by T^A and T^B the vertex trees of $\mathcal{G}$, and by $u^A \in T^A$ and $u^B \in T^B$ the corresponding attaching points. The trees T^A and T^B may fail to be minimal, we denote by T^A_{min} and T^B_{min} their minimal subtrees. Up to enlarging L if necessary, we can assume that the set $T^A \smallsetminus T^A_{min}$ is either empty (in the case where T^A is minimal), or consists of the orbit of a single point in the closure of T^A_{min}, or consists of the orbit of a nondegenerate half-open arc with nontrivial stabilizer.

We will explain how to approximate the tree (T^A, u^A) by a sequence of pointed Grushko $(A, \mathcal{F}_{|A})$-trees. By approximating the pointed tree (T^B, u^B) in the same way, our claim then follows from Guirardel's Reduction Lemma (Lemma E.5.4).

If T^A is minimal, we can approximate (T^A, u^A) by a sequence of pointed Grushko $(A, \mathcal{F}_{|A})$-trees (T^A_n, u^A_n) by assumption (by choosing u^A_n to be an approximation of u^A in the tree T^A_n, provided by the definition of the equivariant Gromov–Hausdorff topology). This also remains true in the case where $T^A \smallsetminus T^A_{min}$ consists of the orbit of a single point u^A in the closure of T^A_{min}. Indeed, in this case, we can first approximate u^A by a sequence of points $(u'_n)_{n \in \mathbb{N}} \in (T^A_{min})^{\mathbb{N}}$, and then choose for each $n \in \mathbb{N}$ an approximation u^A_n of u'_n in an approximation of T^A_{min}.

We now assume that $\overline{T^A \smallsetminus T^A_{min}}$ consists of the orbit of a nondegenerate arc $[u^A, v^A]$ with nontrivial stabilizer, whose length we denote by l^A. We will also assume that T^A_{min} is not reduced to a point. As T is very small, the stabilizer $\langle c^A \rangle$ of the arc $[u^A, v^A]$ is cyclic, closed under taking roots, and non-peripheral. As tripod stabilizers are trivial in T, the point v^A is an endpoint of the subarc of T^A_{min} fixed by c^A. If this arc is nondegenerate, then we let w^A be its other endpoint. Otherwise, we let w^A be any point that is not equal to v^A.

Let $(T^A_{min,n})_{n \in \mathbb{N}}$ be an approximation of T^A_{min} by minimal Grushko $(A, \mathcal{F}_{|A})$-trees. Denote by v^A_n (respectively w^A_n) an approximation of v^A (resp. w^A) in the tree $T^A_{min,n}$, provided by the definition of convergence in the equivariant Gromov–Hausdorff topology. We can assume that for all $n \in \mathbb{N}$, the point v^A_n belongs to the axis of c^A in $T^A_{min,n}$.

We refer to Figure E.6 for an illustration of the following construction. For all $n \in \mathbb{N}$, let (T^A_n, u^A_n) be the pointed tree obtained from (T^A, u^A) in the following way. We start by equivariantly unfolding the arc $[u^A, v^A]$ to obtain a tree $\widetilde{T}^A$ that contains an edge e^0 of length l^A with trivial stabilizer. We then equivariantly replace the pointed tree (T^A_{min}, v^A) in the graph of actions defining $\widetilde{T}^A$ by its approximation $(T^A_{min,n}, v^A_n)$, to get a tree $\widetilde{T_n}^A$. Finally, we define the tree (T^A_n, u^A_n) in the following way: the tree (T^A_n, u^A_n) is obtained from $(\widetilde{T_n}^A, u^A)$ by fully folding the edge e^0 along the axis of c^A in $T^A_{min,n}$, in a direction that does not contain w^A_n. We denote by $f^A_n : (\widetilde{T_n}^A, u^A) \to (T^A_n, u^A_n)$ the folding map.

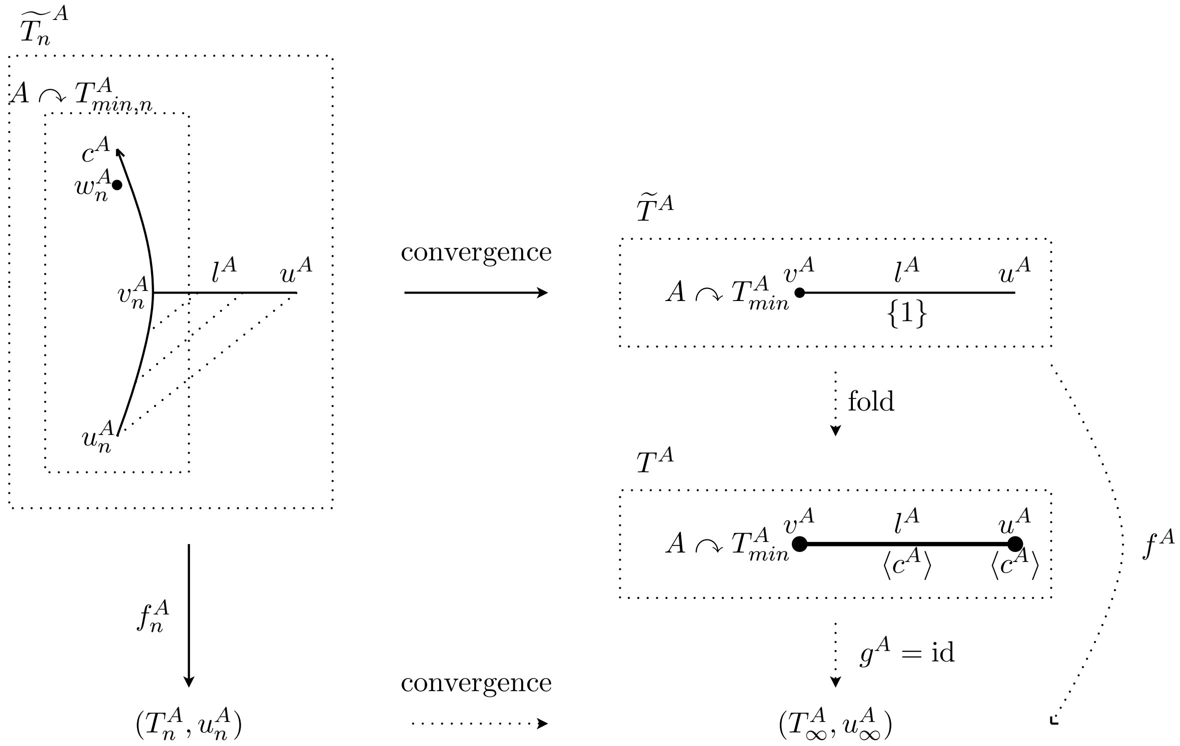

Figure E.6: The situation in Case 1 of the proof of Lemma E.5.6.

We now prove that the pointed trees (T^A_n, u^A_n) converge to (T^A, u^A). Lemma E.5.4 implies that the trees $(\widetilde{T_n}^A, u^A)$ converge to $(\widetilde{T}^A, u^A)$. For all $n \in \mathbb{N}$, there is a 1-Lipschitz G-equivariant map $f^A_n : (\widetilde{T_n}^A, u^A) \to (T^A_n, u^A_n)$. This implies that for all $g \in G$, we have $d_{T^A_n}(u^A_n, gu^A_n) \leq d_{\widetilde{T_n}^A}(u^A, gu^A)$. Therefore, up to possibly passing to a subsequence, the pointed trees (T^A_n, u^A_n) converge to a pointed tree (T^A_∞, u^A_∞) in the Gromov–Hausdorff equivariant topology, that is minimal in the sense of pointed G-trees. Proposition E.1.4 shows that there exists a 1-Lipschitz map $f^A : (\widetilde{T}^A, u^A) \to (\overline{T^A_\infty}, u^A_\infty)$, where $\overline{T^A_\infty}$ denotes the metric completion of T^A_∞. We will show that f^A factors through a map $g^A : (T^A, u^A) \to (\overline{T^A_\infty}, u^A_\infty)$, and that g^A is an isometry between (T^A, u^A) and (T^A_∞, u^A_∞). This will imply that the pointed trees (T^A_n, u^A_n) converge to (T^A, u^A).

We first notice that for all $n \in \mathbb{N}$, the map f^A_n is an isometry in restriction to $T^A_{min,n}$. By taking limits, this implies that the A-minimal subtree T^A_{min} of T^A isometrically embeds into T^A_∞. In addition, for all $n \in \mathbb{N}$, the point u^A_n belongs to the axis of c^A in T_n. By definition of the equivariant Gromov–Hausdorff topology on the set of pointed $(G, \mathcal{F})$-trees, this implies that c^A fixes u^A_∞ in T^A_∞. Similarly, the element c^A fixes all points of the image of $[u^A, v^A]$ in T^A. Therefore, the map f^A factors through a map $g^A : (T^A, u^A) \to (\overline{T^A_\infty}, u^A_\infty)$. As T^A_{min} isometrically embeds into T^A_∞, the map g^A can only decrease the length of the segment $[u^A, v^A]$, and fold this segment over a subarc of $[v^A, w^A]$.

Let $g \in A$ be an element that is hyperbolic in T^A_{min} (we recall that we have assumed T^A_{min} not to be reduced to a point), such that $d_{T^A}(v^A, gv^A) = d_{T^A}(w^A, gw^A) + 2d_{T^A}(v^A, w^A)$. In particular, we have $d_{T^A}(u^A, gu^A) = d_{T^A}(w^A, gw^A) + 2l^A + 2d_{T^A}(v^A, w^A)$. Using the definition of the equivariant Gromov–Hausdorff topology, we get that the distance $d_{T^A_n}(u^A_n, gu^A_n)$ gets arbitrarily close to $d_{T^A_n}(w^A_n, gw^A_n) + 2l^A + 2d_{T^A}(v^A, w^A)$ as n tends to

$+\infty$, so $d_{T^A_\infty}(u^A_\infty, gu^A_\infty) = d_{T^A}(w^A, gw^A) + 2l^A + 2d_{T^A}(v^A, w^A)$. This implies that g^A is an isometry from (T^A, u^A) to (T^A_∞, u^A_∞), and we are done.

If T^A_{min} is reduced to a point, then it can be approximated by a sequence $(T^A_{min,n})_{n\in\mathbb{N}}$ of Grushko $(A, \mathcal{F}_{|A})$-trees, where all edge lengths are equal to $\frac{1}{n}$. We also choose two distinct constant sequences v^A_n and w^A_n in the trees $T^A_{min,n}$, and construct the trees T^A_n as above. Let $g \in A$ be any element such that $d_{T^A_n}(v^A_n, gv^A_n) = d_{T^A_n}(w^A_n, gw^A_n) + 2d_{T^A_n}(v^A_n, w^A_n)$ for all $n \in \mathbb{N}$. Arguing similarly as above, we get that $d_{T^A_\infty}(u^A_\infty, gu^A_\infty) = 2l^A$. This again implies that the map g^A defined as above is an isometry.

Case 2: The splitting S is of the form $G = C*$.

The vertex tree T^C of $\mathcal{G}$ may fail to be minimal. We denote by u_1 and u_2 two points in T^C in the orbits of the attaching points (the points u_1 and u_2 may belong to the same G-orbit). We denote by v_1 and v_2 their projections to the closure $\overline{T^C_{min}}$ of the C-minimal subtree of T. One of the following cases occurs.

Case 2.1: The segments $[u_1, v_1]$ and $[u_2, v_2]$ are nondegenerate, and their stabilizers are nontrivial and nonconjugate in C.
In other words, the tree T splits as a graph of actions that has the shape displayed on Figure E.7, where $l_1, l_2 > 0$, and the stabilizers $\langle c_1\rangle$ and $\langle c_2\rangle$ are nonconjugate. We allow the case where v_1 and v_2 belong to the same G-orbit. For all $i \in \{1, 2\}$, we let w_i be such that $[v_i, w_i]$ is the maximal arc fixed by c_i in T^C_{min}, if this arc is nondegenerate, and we let w_i be any point distinct from v_i otherwise (as in Case 1, one has to slightly adapt the argument when T^C_{min} is reduced to a point). Let $\widetilde{T}^C$ be the tree obtained from T^C by replacing the edges $[u_1, v_1]$ and $[u_2, v_2]$ by edges of the same length with trivial stabilizer. For all $i \in \{1, 2\}$, let $v_{n,i}$ (resp. $w_{n,i}$) be an approximation of v_i (resp. w_i) in an approximation of T^C_{min}. We can assume $v_{n,i}$ to belong to the translation axis of c_i. Let $(\widetilde{T_n}^C, u_1, u_2)$ be the approximation of $(\widetilde{T}^C, u_1, u_2)$ obtained from an approximation of T^C_{min} by adding an edge of length l_1 (resp. l_2) with trivial stabilizer at $v_{n,1}$ (resp. $v_{n,2}$). Let T^C_n be the tree obtained from $\widetilde{T_n}^C$ by G-equivariantly fully folding the edge $[u_i, v_{n,i}]$ along the axis of c_i, in a direction that does not contain $w_{n,i}$, for all $i \in \{1, 2\}$. We denote by $f^C_n : \widetilde{T_n}^C \to T^C_n$ the corresponding morphism. Arguing as in Case 1, one shows that the trees $(T^C_n, f^C_n(u_1), f^C_n(u_2))$ converge to (T^C, u_1, u_2). Let now T_n be the tree obtained by replacing (T^C, u_1, u_2) by its approximation $(T^C_n, f^C_n(u_1), f^C_n(u_2))$ in the graph of actions $\mathcal{G}$. Lemma E.5.4 implies that the trees T_n converge to T.

Case 2.2: The segments $[u_1, v_1]$ and $[u_2, v_2]$ are nondegenerate and have nontrivial stabilizers that are conjugate in C, and no two nondegenerate subsegments of $[u_1, v_1]$ and $[u_2, v_2]$ belong to the same G-orbit.
Again, the tree T splits as a graph of actions that has the shape displayed on Figure E.7, where this time the groups $\langle c_1\rangle$ and $\langle c_2\rangle$ are conjugate. Up to a good choice of the stable letter t, we can assume that $\langle c_1\rangle = \langle c_2\rangle$. As tripod stabilizers are trivial in T, the segment $[v_1, v_2]$ is the maximal arc fixed by c_1 in T^C_{min}. Again, we let $\widetilde{T}^C$ be the tree obtained from T^C by replacing the edges $[u_1, v_1]$ and $[u_2, v_2]$ by edges of the same length with trivial stabilizer. For all $i \in \{1, 2\}$, let $v_{n,i}$ be an approximation of v_i in an approximation of $\overline{T^C_{min}}$ by minimal Grushko $(C, \mathcal{F}_{|C})$-trees, which we can assume to belong to the translation axis of c_1. Let $(\widetilde{T_n}^C, u_1, u_2)$ be the approximation of $(\widetilde{T}^C, u_1, u_2)$ obtained from an approximation

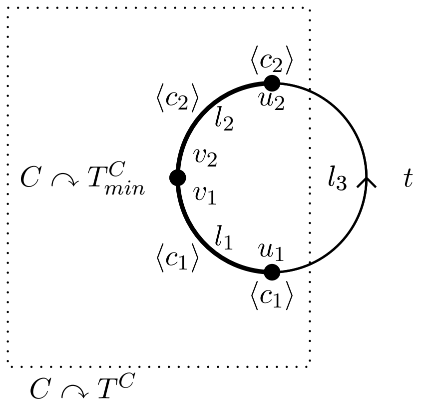

Figure E.7: The splitting of T as a graph of actions in Cases 2.1, 2.2 and 2.5 of the proof of Lemma E.5.6.

of T^C_{min} by adding an edge of length l_1 (resp. l_2) with trivial stabilizer at $v_{n,1}$ (resp. $v_{n,2}$). The tree T_n^C is then obtained from $\widetilde{T_n}^C$ by G-equivariantly fully folding the edges $[u_1, v_{n,1}]$ and $[u_2, v_{n,2}]$ along the axis of c_1 in opposite directions. The folding directions should not contain the segment $[v_{n,1}, v_{n,2}]$, in case this segment is nondegenerate. Again, denoting by $f_n^C : \widetilde{T_n}^C \to T_n^C$ the corresponding morphism, the trees $(T_n^C, f_n^C(u_1), f_n^C(u_2))$ converge to (T^C, u_1, u_2). The trees T_n obtained by replacing (T^C, u_1, u_2) by $(T_n^C, f_n^C(u_1), f_n^C(u_2))$ in $\mathcal{G}$ converge to T.

Case 2.3: Some nondegenerate subsegments of $[u_1, v_1]$ and $[u_2, v_2]$ belong to the same G-orbit, and their common stabilizer is nontrivial.
Using the fact that tripod stabilizers in T are trivial, we can assume that $v_1 = v_2$ (and we let $v := v_1 = v_2$), and that $[u_1, v] \subseteq [u_2, v]$. The tree T splits as a graph of actions that has the form displayed on Figure E.8. We let w be such that $[v, w]$ is the maximal arc fixed by c in T^C_{min} if this arc is nondegenerate, and choose any $w \neq v$ otherwise (as in Case 1, one has to slightly adapt the argument if T^C_{min} is reduced to a point). Let $\widetilde{T}^C$ be the tree obtained from T by replacing the segment $[u_2, v]$ by a segment of same length $l_1 + l_2$ with trivial stabilizer. Let v_n (resp. w_n) be an approximation of v (resp. w) in an approximation Y_n of $\overline{T^C_{min}}$. We can assume v_n and w_n to belong to the translation axis of c in Y_n. Let $(\widetilde{T_n}^C, u_2)$ be the approximation of $(\widetilde{T}^C, u)$ obtained from Y_n by adding an edge of length $l_1 + l_2$ with trivial stabilizer at v_n. The tree T_n^C is then obtained from $\widetilde{T_n}^C$ by G-equivariantly fully folding the edge $[u_2, v_n]$ along the axis of c, in a direction that does not contain w_n. Denoting by $f_n^C : \widetilde{T_n}^C \to T_n^C$ the corresponding morphism, the trees $(T_n^C, f_n^C(u_1), f_n^C(u_2))$ converge to (T^C, u_1, u_2). Again, the trees T_n obtained by replacing (T^C, u_1, u_2) by $(T_n^C, f_n^C(u_1), f_n^C(u_2))$ in $\mathcal{G}$ converge to T.

Case 2.4: Some nondegenerate subsegments of $[u_1, v_1]$ and $[u_2, v_2]$ belong to the same G-orbit, and they have trivial stabilizer.
Then T splits as a graph of actions of the form displayed on Figure E.9. This case may be viewed as a particular case of Case 1.

Case 2.5: One of the subsegments $[u_1, v_1]$ or $[u_2, v_2]$ is degenerate.
This case is treated in a similar way as Case 2.1, and left to the reader. $\square$

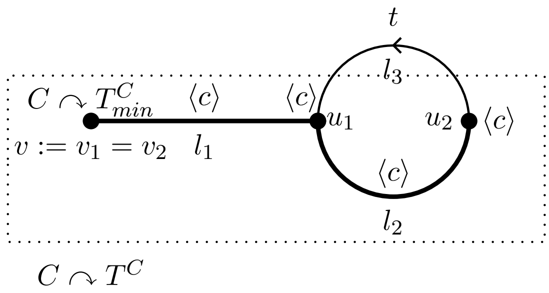

Figure E.8: The splitting of T as a graph of actions in Case 2.3 of the proof of Lemma E.5.6.

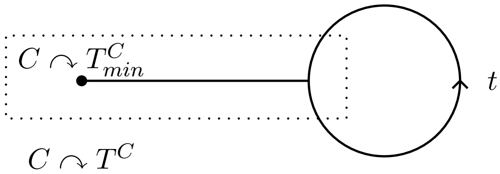

Figure E.9: The splitting of T as a graph of actions in Case 2.4 of the proof of Lemma E.5.6.

E.5.2 Dynamical decomposition of a geometric very small $(G, \mathcal{F})$-tree

Every geometric very small $(G, \mathcal{F})$-tree splits as a graph of actions, which has the following description.

Proposition E.5.7. *Any very small geometric $(G, \mathcal{F})$-tree T splits as a graph of actions $\mathcal{G}$, where for each nondegenerate vertex action Y, with vertex group G_Y, and attaching points $v^1, \dots, v^k$ fixed by subgroups $H^1, \dots, H^k$, either*

- *the tree Y is an arc containing no branch point of T except at its endpoints, or*
- *the group G_Y is the fundamental group of a 2-orbifold with boundary Σ holding an arational measured foliation, and Y is dual to $\widetilde{\Sigma}$, or*
- *there exists a Lipschitz approximation of Y by pointed Grushko $(G_Y, \mathcal{F}_{|G_Y})$-trees $(Y_n, (v_n^1, \dots, v_n^k))$, whose quotient volumes converge to 0, such that for all $n \in \mathbb{N}$, there exists a morphism $f_n : Y_n \to Y$, with $f_n(v_n^i) = v^i$ for all $i \in \{1, \dots, k\}$, and v_n^i is fixed by H^i for all $i \in \{1, \dots, k\}$ and all $n \in \mathbb{N}$.*

We call $\mathcal{G}$ the *dynamical decomposition* of T, it is determined by T. Vertex actions of the third type are called *exotic*. In case no vertex action is exotic, we say that T is *of surface type*.

The proof of Proposition E.5.7 goes as follows. Let T be a geometric $(G, \mathcal{F})$-tree, and let $\mathcal{K} = (K, (x_1, \dots, x_k), \Phi)$ be a $(G, \mathcal{F})$-system of isometries such that $T = T_{\mathcal{K}}$. As T is a small $(G, \mathcal{F})$-tree, we can assume that $\mathcal{K}$ has relatively independent generators (Lemma E.2.6). Let Σ be the suspension of $\mathcal{K}$. Let Σ^* be the complement of the singular set in Σ, endowed with the restriction of the foliation of Σ. Let $C^* \subseteq \Sigma^*$ be the union of the leaves of Σ^* which are closed but not compact. The *cut locus* of Σ is defined as $C := C^* \cup \mathrm{Sing}$. The set $\Sigma \setminus C$ is a union of finitely many open sets $U_1, \dots, U_p$, which are unions of leaves of Σ^*. By a classical result of Imanishi [Ima79], see also [GLP94, Section 3], for all $i \in \{1, \dots, p\}$, either every leaf of U_i is compact, or else every leaf of U_i is dense in U_i. The proof in [GLP94, Section 3] is actually presented for finite systems of isometries (i.e. when Φ is finite), however it adapts to relatively finite systems of isometries because all leaves in Σ

associated to singletons in Φ are contained in the cut locus. It also follows from this last observation that the fundamental group of each component U_i is finitely generated, and dual to a finite system of isometries.

As noticed in [Gui08, Propositions 1.25 and 1.31], Imanishi's theorem provides a transverse covering of T in the following way. Denote by $\rho : \pi_1(\Sigma) \to G$ the natural morphism, and let $p : \widetilde{\Sigma_\rho} \to \Sigma$ be the corresponding covering of Σ. Let $\widetilde{C}$ be the lift of the cut locus to $\widetilde{\Sigma_\rho}$. Given a component U of $\widetilde{\Sigma_\rho} \smallsetminus \widetilde{C}$, we let T_U be the tree dual to the foliated 2-complex $\overline{U}$, i.e. the leaf space made Hausdorff of $\overline{U}$. Then the family $\{T_U\}_U$ is a transverse covering of T. Each T_U is either an arc (in the case where every leaf of $p(U)$ is compact) or has dense orbits (in the case where all leaves of $p(U)$ are dense in $p(U)$). Associated to this transverse covering of T is a graph of actions, whose vertex groups are finitely generated, and whose vertex actions are dual to foliated 2-complexes. In addition, arc stabilizers in the vertex actions with dense orbits are trivial (Proposition E.4.17). Therefore, we can apply [Gui08, Proposition A.6] to each of the vertex actions with dense orbits. This provides a classification of vertex actions with dense orbits into three types (*axial*, *surface* and *exotic*). As we started from a system of isometries $\mathcal{K}$ with relatively independent generators, we can also assume that all vertex actions with dense orbits of the decomposition are dual to finite systems of isometries with independent generators (Lemma E.2.6). This excludes the axial case, see [Gab97, Proposition 3.4]. The existence of the Lipschitz approximation with the required properties in the exotic case was proved by Guirardel in [Gui98, Proposition 7.2], using a pruning and narrowing argument.

We finish this section by mentioning a consequence of Proposition E.5.7, that will turn out to be useful in [9].

Lemma E.5.8. *Let T be a small, minimal $(G,\mathcal{F})$-tree. If there exists a subgroup $H \subseteq G$ that is elliptic in T, and not contained in any proper $(G,\mathcal{F})$-free factor, then T is geometric of surface type.*

Proof. If T is geometric, this follows from Proposition E.5.7, so we assume that T is nongeometric. Up to replacing H by the point stabilizer of T in which it is contained, we can assume that $\mathrm{rk}_K(H) < +\infty$. Theorems E.2.8 and E.2.13 let us approximate T by a sequence $(T_n)_{n\in\mathbb{N}}$ of small, minimal geometric $(G,\mathcal{F})$-trees, in which H is elliptic. The trees T_n come with morphisms onto T. As T is nongeometric, Corollary E.2.11 ensures that the trees T_n contain an edge with trivial stabilizer. This implies that H is contained in a proper $(G,\mathcal{F})$-free factor, a contradiction. □

E.5.3 Trees of surface type

Let T be a very small geometric $(G,\mathcal{F})$-tree of surface type (where we recall the definition from the paragraph below Proposition E.5.7). Let $\mathcal{G}$ be the dynamical decomposition of T, and let S be the skeleton of the corresponding transverse covering. It follows from Proposition E.5.7 that there are three types of vertices in S, namely: vertices of *surface type*, of *arc type*, and vertices associated to nontrivial intersections between the trees of the transverse covering. All edge stabilizers in S are cyclic (possibly finite or peripheral). Indeed, stabilizers of edges adjacent to vertices of surface type are either trivial, or they are cyclic, and represent boundary curves or conical points of the associated orbifold. Both edges adjacent to vertices of arc type have the same stabilizer, equal to the stabilizer of the corresponding arc in T, which is cyclic because T is very small.

Definition E.5.9. *Let T be a very small geometric $(G,\mathcal{F})$-tree of surface type, and let σ be a compact 2-orbifold arising in the dynamical decomposition $\mathcal{G}$ of T. Let $g \in G$ be an element represented by a boundary curve of σ. We say that g is* used *in T if either g is peripheral, or g is conjugate into some adjacent edge group of $\mathcal{G}$. Otherwise g is* unused *in T.*

Proposition E.5.10. *Let T be a minimal, very small, geometric $(G,\mathcal{F})$-tree of surface type. Then either there exists an unused element in T, or T splits as a $(G,\mathcal{F})$-graph of actions over a one-edge $(G,\mathcal{F})$-free splitting.*

Our proof of Proposition E.5.10 is based on the following lemma. Given a group G, and a family Y of subgroups of G, we say that G *splits as a free product relatively to Y* if there exists a splitting of the form $G = A * B$, such that every subgroup in Y is contained in either A or B.

Lemma E.5.11. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. Let T be a minimal, simplicial $(G,\mathcal{F})$-tree, whose edge stabilizers are all cyclic and nontrivial (they may be finite or peripheral). Then there exists a vertex v in $\mathcal{G}$, such that G_v splits as a free product relative to incident edge groups and subgroups in $\mathcal{F}_{|G_v}$.*

Let T, T' be two simplicial $(G,\mathcal{F})$-trees. A map $f: T \to T'$ is a *G-equivariant edge fold* (or simply a *fold*) if there exist two edges $e_1, e_2 \subseteq T$, incident to a common vertex in T, such that T' is obtained from T by G-equivariantly identifying e_1 and e_2, and $f: T \to T'$ is the quotient map. A fold $f: T \to T'$ is determined by the orbit of the pair of edges (e_1, e_2) identified by f. We say that f is

- of *type* 1 if e_1 and e_2 belong to distinct G-orbits of oriented edges in T, and both e_1 and e_2 have nontrivial stabilizer, and
- of *type* 2 if e_1 and e_2 belong to distinct G-orbits of oriented edges in T, and either e_1 or e_2 (or both) has trivial stabilizer in T, and
- of *type* 3 if e_1 and e_2 belong to the same G-orbit of oriented edges in T.

Assume that $f: T \to T'$ is a fold. We note that if $H \subseteq G$ is a subgroup of G that fixes an edge $e_1 \subseteq T'$, and $\widetilde{e_1}$ is an edge in the f-preimage of e_1 in T, then H fixes an extremity of $\widetilde{e_1}$. We start by making the following observation.

Lemma E.5.12. *Let T and T' be two simplicial $(G,\mathcal{F})$-trees with cyclic stabilizers. Assume that T' is obtained from T by performing a fold f of type 2 or 3. If e_1 and e_2 are two edges of T that are identified by f, then either e_1 or e_2 (or both) has trivial stabilizer.* ☐

Proof of Lemma E.5.11. Let T_0 be any Grushko $(G,\mathcal{F})$-tree. All point stabilizers in T_0 are elliptic in T, so up to possibly collapsing some edges in T_0, and subdividing edges of T_0, there exists a simplicial morphism $f: T_0 \to T$ (i.e. sending vertices to vertices and edges to edges). By [Sta83, 3.3], the morphism f can be decomposed as a sequence of G-equivariant edge folds $f_i: T_i \to T_{i+1}$.

We can assume that along the folding sequence, we always perform edge folds of type 1 before performing edge folds of type 2 and 3, and we always perform edge folds of type 2 before edge folds of type 3, for as long as possible. This is possible because the number of orbits of edges decreases when performing a fold of type 1 or 2.

We claim that we can also assume that folds are maximal in the following sense: if g fixes an edge e in T, then we never identify a preimage e' of e with a translate of the form $g^k e'$ without identifying it with ge'. Assume otherwise, and let i be the first time at which a non-maximal fold occurs. Let e' be a preimage of e in T_i, having a vertex y

stabilized by g^k, so that we fold e' and $g^k e'$ when passing from T_i to T_{i+1}. By our choice of i and the fact that edge stabilizers in T are cyclic, the edge e' has trivial stabilizer in T_i (in particular, the fold performed from T_i to T_{i+1} is not of type 1). The element g is also elliptic in T_i. We let x be the point closest to y that is fixed by g in T_i. If $x = y$, then we could choose to identify e' and ge' when passing from T_i to y. Otherwise, all edges in the segment $[x, y] \subseteq T_i$ are stabilized by g^k. Our choice of i implies that the stabilizer of their images in T is equal to g^k (and not to any proper root of g^k). Since the image of e' in T is stabilized by g, this implies that we can find two consecutive edges on the segment $[x, y]$ that are identified in T. This shows that one could perform a fold of type 1 on the tree T_i, contradicting our choice of folding path.

Let T_k be the last tree along the folding sequence that contains an edge with nontrivial stabilizer. The fold f_k is either of type 2 or of type 3. It identifies an edge e_k of T_k with trivial stabilizer with some translate ge_k with $g \in G$ (although the pair (e_k, ge_k) might not be the defining pair of the edge fold). We can assume $\langle g\rangle$ to be maximal in the following sense: if g is of the form h^l with $h \in G$ and $l > 1$, then e_k is not identified with he_k. We claim that $T_{k+1} = T$.

We postpone the proof of the claim to the next paragraph, and first explain how to derive the lemma from this claim. Let v_k be the vertex of e_k such that $gv_k = v_k$, and v'_k be the other vertex of e_k. If f_k is a fold of type 3, defined by the pair (e_k, ge_k), then the vertex $f_k(v'_k)$ satisfies the conclusion of the lemma. Indeed, we have $G_{f_k(v'_k)} = G_{v'_k} * \langle g\rangle$. If f_k is a fold of type 2, then f_k identifies e_k with an edge e'_k with nontrivial stabilizer, because otherwise T_{k+1} would contain an edge with nontrivial stabilizer. Denote by v''_k the vertex of e'_k distinct from v_k. If v'_k and v''_k do not belong to the same G-orbit, then we have a splitting $G_{f_k(v'_k)} = G_{v'_k} * G_{v''_k}$. This splitting is nontrivial: indeed, if $G_{v''_k}$ were trivial, then all edges in T_k adjacent to v''_k would have trivial stabilizer, and there would be at least three distinct G-orbits of such edges. Since the fold f_k involves at most two orbits of edges, there would be an edge with trivial stabilizer in T_{k+1}, contradicting the definition of T_{k+1}. If v'_k and v''_k belong to the same G-orbit, then we have a splitting $G_{f_k(v'_k)} = G_{v''_k} * \mathbb{Z}$, which is again nontrivial. In both cases, this splitting is relative to incident edge groups and to $\mathcal{F}_{|G_v}$, because all trees along the folding sequence are $(G, \mathcal{F})$-trees.

We now prove the above claim that $T_{k+1} = T$. Assume towards a contradiction that $T_{k+1} \neq T$. It follows from our choice of folding path, and the fact that T has cyclic edge stabilizers, that all possible folds in T_{k+1} identify two edges e_1 and e_2 at $f_k(v'_k)$ in distinct G-orbits, and e_1 and e_2 have the same nontrivial stabilizer H in T_{k+1}. Let $\widetilde{e_1}$ (resp. $\widetilde{e_2}$) be an edge in T_k in the f_k-preimage of e_1 (resp. e_2). The group H fixes an extremity of both $\widetilde{e_1}$ and $\widetilde{e_2}$. If $\widetilde{e_1}$ and $\widetilde{e_2}$ were disjoint, then H would fix the segment between them, a contradiction (Lemma E.5.12). Therefore, the edges $\widetilde{e_1}$ and $\widetilde{e_2}$ are adjacent in T_k. By our choice of folding path, at least one of them, say $\widetilde{e_1}$, has trivial stabilizer (otherwise we could perform a fold of type 1 in T_k identifying $\widetilde{e_1}$ and $\widetilde{e_2}$), and hence belongs to the G-orbit of e_k. Since it is possible to fold $\widetilde{e_1}$ and $\widetilde{e_2}$ in T_k, the fold f_k is of type 2. Hence f_k identifies $\widetilde{e_1}$ with an edge e_3 whose stabilizer is equal to H. Then e_3 is adjacent to e_2, and is identified with e_2 in T, so we could have performed a fold of type 1 in T_k, a contradiction. $\square$

Proof of Proposition E.5.10. Let S be the skeleton of the dynamical decomposition of T. If S is reduced to a point, then T is dual to a minimal measured foliation on a compact 2-orbifold σ. Some boundary component c of σ represents a nonperipheral conjugacy class. Indeed, all boundary components of a compact 2-orbifold cannot be elliptic in a common free splitting of the fundamental group of the orbifold. Then c is an unused element in T.

Now assume that S is a nontrivial minimal $(G,\mathcal{F})$-tree. If S contains an edge with trivial stabilizer, then this edge defines a $(G,\mathcal{F})$-free splitting, and T splits as a graph of actions over this splitting by [Gui04, Lemma 4.7]. Otherwise, let v be a vertex of S provided by Lemma E.5.11.

We note that the vertex v cannot be of arc type, because a vertex of arc type has a stabilizer equal to the stabilizer of the incident edges. If v is of surface type, it is associated to a compact 2-orbifold σ, equipped with a minimal measured foliation. The fundamental group of σ splits as a free product relatively to incident edge groups and to subgroups in $\mathcal{F}_{|\pi_1(\sigma)}$. This implies the existence of an unused element in T, otherwise $\pi_1(\sigma)$ would split as a free product in which all boundary components of σ are elliptic.

If v is of trivial type, then we get a one-edge $(G,\mathcal{F})$-free splitting S_0 refining S by splitting the vertex group G_v. Associated to each vertex of S_0 with vertex group $G_{v'}$ is a geometric (possibly trivial) $G_{v'}$-action $T_{v'}$. The tree T splits as a graph of actions over S_0, with the trees $T_{v'}$ as vertex actions. □

E.5.4 Approximating very small geometric $(G,\mathcal{F})$-trees by Grushko $(G,\mathcal{F})$-trees

Proof of Theorems E.5.1 and E.5.3. Let T be a very small, minimal $(G,\mathcal{F})$-tree. We want to show that we can approximate T by a sequence of minimal Grushko $(G,\mathcal{F})$-trees, and that the approximation can be chosen to be a Lipschitz approximation, by trees whose quotient volumes converge to the quotient volume of T, if T has trivial arc stabilizers. We will argue by induction on $\mathrm{rk}_K(G,\mathcal{F})$. The claim holds true when $\mathrm{rk}_K(G,\mathcal{F})=1$, so we assume that $\mathrm{rk}_K(G,\mathcal{F})\geq 2$. The claim also holds true if T is reduced to a point. Thanks to Theorem E.2.8, we can assume T to be geometric. (For the statement about the volume, notice that branch points are dense in trees with dense orbits, so the lengths of the simplicial edges in the geometric approximation converge to 0 by Remark E.2.9). By Proposition E.5.7, the tree T decomposes as a graph of actions whose vertex actions are either arcs, or of surface or exotic type. Proposition E.5.7 also enables us to approximate all exotic vertex actions. Using Lemmas E.5.4 and E.5.5, we can therefore reduce to the case where T is a tree of surface type (notice that all edges of the decomposition as a graph of actions whose stabilizer is noncyclic, or nontrivial and peripheral, have length 0).

First assume that there exists an unused element c in T, corresponding to a boundary curve in a minimal orbifold σ of Σ. One can then narrow the band complex by width $\delta>0$ from c to get a Lipschitz approximation of T: this is done by cutting a segment on Σ of length δ (arbitrarily small) transverse to the boundary curve c and to the foliation. In this way, all leaves of the foliations become segments (half-leaves of the original foliation on Σ are dense), so the tree dual to the foliated complex by which the minimal foliation on σ has been replaced is simplicial. By choosing $\delta>0$ arbitrarily close to 0, we can ensure the volume of this tree to be arbitrarily small. In the new band complex obtained in this way, the orbifold σ has therefore been replaced by a simplicial component, with trivial edge stabilizers.

We thus reduce to the case where no element of G is unused in T. Lemma E.5.10 thus ensures that T splits as a $(G,\mathcal{F})$-graph of actions over a one-edge $(G,\mathcal{F})$-free splitting, and we can conclude by induction, using Lemmas E.5.5 and E.5.6. □

E.6 Tame $(G,\mathcal{F})$-trees

We finish this paper by introducing another class of $(G,\mathcal{F})$-trees, larger than the class of very small $(G,\mathcal{F})$-trees, which we call *tame* $(G,\mathcal{F})$-trees. This class will turn out to provide the right setting for carrying out our arguments in [8] to describe the Gromov boundary of the graph of cyclic splittings of $(G,\mathcal{F})$.

Definition E.6.1. *A minimal $(G,\mathcal{F})$-tree is* tame *if it is small, and has finitely many orbits of directions.*

There exist small $(G,\mathcal{F})$-actions that are not tame. A typical example is the following: a sequence of splittings of $F_2=\langle a,b\rangle$ of the form $F_2=(\langle a\rangle *_{\langle a^2\rangle}\langle a^2\rangle *_{\langle a^4\rangle}\cdots *_{\langle a^{2^n}\rangle}\langle a^{2^n}\rangle)*\langle b\rangle$, in which the edge with stabilizer generated by a^{2^k} has length $\frac{1}{2^k}$, converges to a small F_2-tree with infinitely many orbits of branch points.

By the discussion in Section E.4.3, the class of tame $(G,\mathcal{F})$-trees is the right class of trees in which Levitt's decomposition makes sense.

Theorem E.6.2. *(Levitt [Lev94, Theorem 1]) Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. Then every tame $(G,\mathcal{F})$-tree splits uniquely as a graph of actions, all of whose vertex trees have dense orbits for the action of their stabilizer, such that the Bass–Serre tree of the underlying graph of groups is small, and all its edges have positive length.*

However, the above example of a small $(G,\mathcal{F})$-tree that is not tame shows that the space of tame $(G,\mathcal{F})$-trees is not closed. We will describe conditions under which a limit of tame trees is tame. It will be of interest to introduce yet another class of $(G,\mathcal{F})$-trees. The equivalences in the definition below are straightforward.

Definition E.6.3. *Let $k\in\mathbb{N}$. A small minimal $(G,\mathcal{F})$-tree is k-*tame *if one of the following equivalent conditions occurs.*

- *For all nonperipheral $g\in G$ and all arcs $I\subseteq T$, if $\langle g\rangle\cap Stab(I)$ is nontrivial, then its index in $\langle g\rangle$ divides k.*
- *For all nonperipheral $g\in G$, and all $l\geq 1$, we have $Fix(g^l)\subseteq Fix(g^k)$.*
- *For all nonperipheral $g\in G$, and all $l\geq 1$, we have $Fix(g^{kl})=Fix(g^k)$.*

Notice in particular that 1-tame $(G,\mathcal{F})$-trees are those $(G,\mathcal{F})$-trees in which all arc stabilizers are either trivial, or maximally-cyclic and nonperipheral.

Proposition E.6.4. *For all $k\in\mathbb{N}$, the space of k-tame $(G,\mathcal{F})$-trees is closed in the space of small minimal $(G,\mathcal{F})$-trees.*

Proof. Let T be a small, minimal $(G,\mathcal{F})$-tree, and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of k-tame $(G,\mathcal{F})$-trees that converges to T. Let $g\in G$, and assume that there exists $l\geq 1$ such that g^l fixes a nondegenerate arc $[a,b]\subseteq T$. If g is hyperbolic in T_n for infinitely many $n\in\mathbb{N}$, then Lemma E.3.3 implies that g fixes $[a,b]$. We can therefore assume that for all $n\in\mathbb{N}$, the fixed point set I_n of g^l is nonempty. Let a_n (resp. b_n) be an approximation of a (resp. b) in T_n. Since $d_{T_n}(g^la_n,a_n)$ and $d_{T_n}(g^lb_n,b_n)$ both converge to 0, the distances $d_{T_n}(a_n,I_n)$ and $d_{T_n}(b_n,I_n)$ both converge to 0. As T_n is k-tame, this implies that the distances of both a_n and b_n to the fixed point set of g^k in T_n converge to 0. So both $d_{T_n}(g^ka_n,a_n)$ and $d_{T_n}(g^kb_n,b_n)$ converge to 0, and therefore g^k fixes $[a,b]$ in T. This shows that T is k-tame. ☐

Proposition E.6.5. *A minimal $(G,\mathcal{F})$-tree is tame if and only if there exists $k \in \mathbb{N}$ such that T is k-tame.*

Proof. Let T be a tame minimal $(G,\mathcal{F})$-tree. By Theorem E.6.2, the tree T splits as a graph of actions, all of whose vertex actions have dense orbits for the action of their stabilizer. As tame $(G,\mathcal{F})$-trees with dense orbits have trivial arc stabilizers by Lemma E.4.17, we do not modify the collection of arc stabilizers of T if we collapse all vertex trees to points. We can therefore reduce to the case where T is simplicial. In this case, minimality implies that the G-action on T has finitely many orbits of edges, from which it follows that T is k-tame for some $k \in \mathbb{N}$. The converse statement will follow from the following proposition. ☐

Proposition E.6.6. *For all $k \in \mathbb{N}$, there exists $\gamma(k) \in \mathbb{N}$ such that any k-tame minimal $(G,\mathcal{F})$-tree has at most $\gamma(k)$ orbits of directions.*

Proof. Let T be a k-tame minimal $(G,\mathcal{F})$-tree. We first assume that T is geometric. Let $\mathcal{G}$ be the dynamical decomposition of T. We recall that all vertex trees of $\mathcal{G}$ are either arcs or have dense orbits. As T is geometric, there are finitely many orbits of directions in T (see Corollary E.4.8). This implies that arc stabilizers are trivial in the vertex trees of $\mathcal{G}$ with dense orbits. If x is a branch or inversion point of T contained in one of the vertex trees with dense orbits T_v of $\mathcal{G}$, then all directions at x contained in T_v have a positive contribution to the index $i(T)$. It follows from Proposition E.4.4 that there is a bound on the number of such orbits of directions. Therefore, we can collapse all the vertex trees with dense orbits to points, and reduce to the case where T is simplicial.

In this case, we argue by induction on $\mathrm{rk}_K(G,\mathcal{F})$, and show that there is a bound $\gamma(k,l)$ on the number of orbits of directions in any k-tame minimal simplicial $(G,\mathcal{F})$-tree with $\mathrm{rk}_K(G,\mathcal{F}) \leq l$. By splitting one of the vertex stabilizers of the splitting relatively to incident edge stabilizers if needed (which is made possible by Lemma E.5.11, and can only increase the number of orbits of directions), we can assume that T contains an edge e with trivial stabilizer. By removing from T the interior of the edges in the orbit of e in T, we get one or two orbits of trees, whose stabilizers have a strictly smaller Kurosh rank. Let T' be one of the trees obtained in this way, whose stabilizer we denote by G'. Then T' is k-tame. If T' is minimal, then we are done by induction. However, the tree T' may fail to be minimal. The quotient graph T'/G' consists of a minimal graph of groups T'_{min}/G', with a segment $I = e_1 \cup \cdots \cup e_n$ attached to T'_{min}/G' at one of its extremities (where we denote by e_i the edges in I). All edge groups are cyclic, and they satisfy $G_{e_i} = G_{o(e_i)} \subseteq G_{e_{i+1}}$ for all $i \in \{1,\dots,n-1\}$. Since T is k-tame, we have $n \leq k$. By induction, there are at most $\gamma(k,l-1)$ orbits of directions in T'_{min}, so we get a uniform bound on the possible number of orbits of directions in T.

We have thus shown that there is a uniform bound $\gamma(k)$ on the possible number of orbits of directions in a minimal k-tame geometric $(G,\mathcal{F})$-tree. Let now T be a minimal nongeometric $(G,\mathcal{F})$-tree, and assume that T has strictly more than $\gamma(k)$ orbits of directions. Arguing as in the proof of Proposition E.4.4 in the nongeometric case, we can find a geometric $(G,\mathcal{F})$-tree T' in which we can lift at least $\gamma(k)+1$ orbits of directions. Lemma E.2.12 shows that the approximation T' can be chosen to be k-tame. This is a contradiction. ☐

We finally establish one more condition under which a limit of tame trees is tame, which will be used in [8].

Proposition E.6.7. *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of simplicial metric small $(G,\mathcal{F})$-trees that converges to a minimal $(G,\mathcal{F})$-tree T. If all trees T_n contain a single orbit of edges, then T is tame.*

We will make use of the following lemma.

Lemma E.6.8. *Let T be a simplicial small $(G,\mathcal{F})$-tree, with one orbit of edges. Then the fixed point set of any element of G is a star of diameter at most 2 for the simplicial metric on T.*

Proof. Let e_1 and e_2 be two edges of T stabilized by a common element $g \in G$. As T has only one orbit of edges, there exists $h \in G$ such that $he_1 = e_2$. By choosing an orientation on e_1, this relation defines an orientation on e_2. Then $hge_1 = ghe_1$, which implies that h commutes with g because T is small. Hence h is elliptic in T. This implies that e_1 and e_2 point in opposite directions in T. As this is true of any pair of edges stabilized by g, the fixed point set of g has the desired description. ☐

Proof of Proposition E.6.7. Up to possibly passing to a subsequence, one of the following situations occurs.

Case 1: The length of the unique orbit of edges in T_n converges to 0.

In this case, we will show that T is very small. This implies that T is 1-tame, and hence tame by Proposition E.6.5. We have seen in the proof of Proposition E.3.1 that limits of small $(G,\mathcal{F})$-trees are small, and limits of trees with trivial tripod stabilizers have trivial tripod stabilizers. Let $g \in G$ be a nonperipheral element, and assume that there exists $l \geq 1$ such that g^l fixes a nondegenerate arc $[a,b] \subseteq T$. Let a_n (resp. b_n) be an approximation of a (resp. b) in T_n. If g were elliptic in T_n, then both a_n and b_n would be arbitrarily close to the fixed point set X_n of g in T_n, as n goes to $+\infty$. It follows from Lemma E.6.8 that the diameter of X_n in T_n converges to 0 as n tends to $+\infty$. This contradicts the fact that $d_{T_n}(a_n,b_n)$ is bounded below (because $a \neq b$). Therefore, for n large enough, the element g is hyperbolic in T_n. The distances $d_{T_n}(a_n,g^la_n)$ and $d_{T_n}(b_n,g^lb_n)$ converge to 0, so the points a_n and b_n are arbitrarily close to the axis of g in T_n, and $||g||_{T_n}$ converges to 0. This implies that $d_{T_n}(a_n,ga_n)$ and $d_{T_n}(b_n,gb_n)$ both tend to 0, so g fixes $[a,b]$.

Case 2: There is a positive lower bound on the length of the unique orbit of edges in T_n.

Up to passing to a subsequence and rescaling T_n by a factor $\lambda_n > 0$ converging to some $\lambda > 0$, we can assume that all trees T_n have edge lengths equal to 1. This implies that all translation lengths in T_n belong to $\mathbb{Z}$, so all translation lengths in T belong to $\mathbb{Z}$. It follows that T is a simplicial metric tree (see [Mor92, Theorem 10]), so it has finitely many orbits of directions. Since a limit of small $(G,\mathcal{F})$-trees is small, the tree T is tame. ☐

Annexe F

Hyperbolic graphs for free products, and the Gromov boundary of the graph of cyclic splittings

Abstract

We define analogues of the graphs of free splittings, of cyclic splittings, and of maximally-cyclic splittings of F_N for free products of groups, and show their hyperbolicity. Given a countable group G which splits as $G = G_1 * \cdots * G_k * F$, where F denotes a finitely generated free group, we identify the Gromov boundary of the graph of relative cyclic splittings with the space of equivalence classes of $\mathcal{Z}$-averse trees in the boundary of the corresponding outer space. A tree is *$\mathcal{Z}$-averse* if it is not compatible with any tree T', that is itself compatible with a relative cyclic splitting. Two $\mathcal{Z}$-averse trees are *equivalent* if they are both compatible with a common tree in the boundary of the corresponding outer space. We give a similar description of the Gromov boundary of the graph of maximally-cyclic splittings.

Contents

Introduction

A celebrated theorem of Grushko [Gru40] states that any finitely generated group G splits as a free product of the form

$$G = G_1 * \cdots * G_k * F_N,$$

where F_N is a free group of rank N, and each group G_i is freely indecomposable, nontrivial, and not isomorphic to $\mathbb{Z}$. In this paper, we introduce relative versions of the graphs of free splittings, of cyclic splittings and of maximally-cyclic splittings of a finitely generated free group for free products of groups, and prove their Gromov hyperbolicity. We then give a description of the Gromov boundary of the graph of relative (maximally)-cyclic splittings. Such complexes are useful for studying the outer automorphism group of a free product of groups: in [9], we will use our description of the Gromov boundary of the graph of relative cyclic splittings to study the Tits alternative for $\mathrm{Out}(G)$.

Hyperbolicity of graphs of splittings.

In recent years, the quest for a good $\mathrm{Out}(F_N)$-analogue of the curve graph of a compact surface has found much interest among geometric group theorists. Several analogues have been proposed, and proven to be Gromov hyperbolic. Among them stand

- the *free factor graph* FF_N, whose vertices are the conjugacy classes of proper free factors of F_N, two factors being joined by an edge if one properly contains the other. Hyperbolicity of FF_N was first proved by Bestvina and Feighn [BF14b], an alternative proof is due to Kapovich and Rafi [KR14].
- the *free splitting graph* FS_N, whose vertices are the *free splittings* of F_N with one orbit of edges (up to equivariant homeomorphism), two distinct splittings being joined by an edge if they have a common refinement. We recall that a *free splitting* of F_N is a minimal, simplicial F_N-tree with trivial edge stabilizers (so that a free splitting of F_N with one orbit of edges is the Bass–Serre tree of a decomposition of F_N either as a free product or as an HNN extension). Hyperbolicity of FS_N was proved by Handel and Mosher [HM13a], alternative proofs are due to Hilion and Horbez [1] and Bestvina and Feighn [BF14c].
- the *cyclic splitting graph* FZ_N, whose vertices are the *cyclic splittings* of F_N with one orbit of edges (up to equivariant homeomorphism), two distinct splittings being joined by an edge if they have a common refinement. A *cyclic splitting* of F_N is a minimal, simplicial F_N-tree with cyclic (possibly trivial) edge stabilizers. Hyperbolicity of FZ_N was proved by Mann [Man13].

We will also consider the graph FZ_N^{max} of maximally-cyclic splittings of F_N, which is defined similarly as the graph FZ_N, except that we also require edge stabilizers to be closed under taking roots.

We define relative versions of the graphs FS_N, FZ_N and FZ_N^{max} for free products. Let $\{G_1,\dots,G_k\}$ be a finite collection of nontrivial countable groups, let F_N be a free group of rank N, and let

$$G := G_1 * \dots * G_k * F_N.$$

We do not assume here that this decomposition is the Grushko decomposition of G, i.e. the groups G_i might be freely decomposable or isomorphic to $\mathbb{Z}$ (they can even be infinitely generated). We denote by $\mathcal{F} := \{[G_1],\dots,[G_k]\}$ the finite collection of the G-conjugacy classes of the subgroups $G_1,\dots,G_k$. We denote by $\mathcal{Z}$ the collection of subgroups of G that are either trivial, or cyclic and *nonperipheral*, i.e. not conjugate into any of the G_i's. We denote by $\mathcal{Z}^{max}$ the collection of subgroups in $\mathcal{Z}$ that are either trivial, or closed under taking roots.

A $(G,\mathcal{F})$-splitting is a minimal, simplicial G-tree, in which all subgroups in $\mathcal{F}$ have a fixed point. It is a *free splitting* (resp. a $\mathcal{Z}$-splitting, resp. a $\mathcal{Z}^{max}$-splitting) if all edge stabilizers are trivial (resp. belong to $\mathcal{Z}$, resp. belong to $\mathcal{Z}^{max}$). It is a *one-edge splitting* if it contains a single orbit of edges.

Vertices of the graph $FS(G,\mathcal{F})$ of $(G,\mathcal{F})$*-free splittings* are the equivalence classes of $(G,\mathcal{F})$-free splittings. Vertices of the graph $FZ(G,\mathcal{F})$ of $\mathcal{Z}$*-splittings* are the equivalence classes of one-edge $\mathcal{Z}$-splittings. Vertices of the graph $FZ^{max}(G,\mathcal{F})$ of $\mathcal{Z}^{max}$*-splittings* are the equivalence classes of one-edge $\mathcal{Z}^{max}$-splittings. In all cases, two distinct splittings are joined by an edge if they admit a common refinement.

Let $\mathrm{Out}(G,\mathcal{F})$ be the subgroup of $\mathrm{Out}(G)$ made of those automorphisms that preserve the conjugacy class of each G_i. The graphs $FS(G,\mathcal{F})$, $FZ(G,\mathcal{F})$ and $FZ^{max}(G,\mathcal{F})$ all come equipped with a natural right action of $\mathrm{Out}(G,\mathcal{F})$, given by precomposition of the G-actions on the splittings.

Theorem F.0.1. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. Then the graphs $FS(G,\mathcal{F})$, $FZ(G,\mathcal{F})$ and $FZ^{max}(G,\mathcal{F})$ are Gromov hyperbolic.*

Hyperbolicity of $FS(G,\mathcal{F})$ is proved in Section F.2, following Bestvina and Feighn's arguments in [BF14c, Appendix]. Hyperbolicity of $FZ(G,\mathcal{F})$ and $FZ^{max}(G,\mathcal{F})$ are proved in Section F.3, following Mann's arguments [Man13]. We mention in particular that hyperbolicity of FZ_N^{max} seems to be new. Hyperbolicity of $FS(G,\mathcal{F})$ was also obtained independently by Handel and Mosher [HM14b], who also showed the hyperbolicity of the relative free factor graph.

The Gromov boundary of $FZ(G,\mathcal{F})$.

In what follows, we will use the notation $FZ^{(max)}(G,\mathcal{F})$ as a shortcut to denote either $FZ(G,\mathcal{F})$ or $FZ^{max}(G,\mathcal{F})$. In [Kla99], Klarreich has described the Gromov boundary of the complex of curves of a compact surface as a quotient of a subspace of Thurston's compactification of the associated Teichmüller space. In analogy with Klarreich's work, we now give a description of the Gromov boundary $\partial_\infty FZ^{(max)}(G,\mathcal{F})$, as a quotient of a subspace of the compactification $\overline{P\mathcal{O}(G,\mathcal{F})}$ of the relative outer space $P\mathcal{O}(G,\mathcal{F})$. This compactification was described in [7], as follows.

A $(G,\mathcal{F})$*-tree* is an $\mathbb{R}$-tree equipped with a G-action by isometries, in which all subgroups in $\mathcal{F}$ are elliptic. The *unprojectivized outer space* $\mathcal{O}(G,\mathcal{F})$ of a free product is the space of isometry classes of *Grushko* $(G,\mathcal{F})$*-trees*, i.e. minimal, simplicial, metric $(G,\mathcal{F})$-trees, in which nontrivial vertex stabilizers coincide with the subgroups in $\mathcal{F}$, and edge stabilizers are trivial. *Outer space* $P\mathcal{O}(G,\mathcal{F})$ is defined as the space of homothety classes of trees in $\mathcal{O}(G,\mathcal{F})$. These were introduced by Guirardel and Levitt in [GL07b], in analogy to Culler and Vogtmann's outer space for a finitely generated free group [CV86], with a view to proving finiteness properties of the group $\mathrm{Out}(G,\mathcal{F})$. The geometry of $\mathcal{O}(G,\mathcal{F})$ was further investigated in [FM14, GL07b] and [7]. In particular, we identified in [7] the closure $\overline{\mathcal{O}(G,\mathcal{F})}$ with the space of minimal, *very small* $(G,\mathcal{F})$-trees, i.e. trees whose arc stabilizers belong to the class $\mathcal{Z}^{max}$, and whose tripod stabilizers are trivial.

We describe $\partial_\infty FZ^{(max)}(G,\mathcal{F})$ as $\mathcal{X}^{(max)}(G,\mathcal{F})/\sim$ as follows. Again, we warn the reader that our notation $\mathcal{X}^{(max)}(G,\mathcal{F})$ is just a shortcut for denoting either $\mathcal{X}(G,\mathcal{F})$ or $\mathcal{X}^{max}(G,\mathcal{F})$. Let $\mathcal{X}^{(max)}(G,\mathcal{F})$ be the subspace of $\overline{\mathcal{O}(G,\mathcal{F})}$ consisting of $\mathcal{Z}^{(max)}$*-averse* trees, defined as those trees that are not compatible with any tree in $\overline{\mathcal{O}(G,\mathcal{F})}$ that is itself compatible with some $\mathcal{Z}^{(max)}$-splitting (two $(G,\mathcal{F})$-trees T and T' are *compatible* if there exists a $(G,\mathcal{F})$-tree $\widehat{T}$ which admits G-equivariant alignment-preserving maps onto both T and T'). In Theorems F.5.1 and F.5.25, we give several equivalent definitions of $\mathcal{Z}^{(max)}$-averse trees. In particular, it is proved that if two trees are compatible, and one is

$\mathcal{Z}^{(max)}$-averse, then so is the other. We show that being compatible with a common tree in $\overline{\mathcal{O}(G,\mathcal{F})}$ defines an equivalence relation on $\mathcal{X}^{(max)}(G,\mathcal{F})$, which we denote by $\sim$. We prove that the Gromov boundary $\partial_\infty FZ^{(max)}(G,\mathcal{F})$ is isomorphic to $\mathcal{X}^{(max)}(G,\mathcal{F})/\sim$, see Theorem F.0.2 below.

It is also of interest to relate the topologies on the spaces $\overline{\mathcal{O}(G,\mathcal{F})}$ and $FZ(G,\mathcal{F}) \cup \partial_\infty FZ(G,\mathcal{F})$ – this turns out to be crucial in our work on the Tits alternative for the automorphism group of a free product, for instance. To this means, we introduce a map $\psi : \mathcal{O}(G,\mathcal{F}) \to FZ(G,\mathcal{F})$, defined by sending any tree $T \in \mathcal{O}(G,\mathcal{F})$ to a $(G,\mathcal{F})$-splitting obtained by collapsing all but one orbit of edges to points in T. Our main result extends the map ψ to $\mathcal{X}(G,\mathcal{F})$, and show that this extension is continuous at every point of $\mathcal{X}(G,\mathcal{F})$, and induces an isomorphism from $\mathcal{X}(G,\mathcal{F})/\sim$ to $\partial_\infty FZ(G,\mathcal{F})$.

Theorem F.0.2. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. Then there exists a unique Out$(G,\mathcal{F})$-equivariant homeomorphism*

$$\partial\psi : \mathcal{X}(G,\mathcal{F})/\!\sim\, \to \partial_\infty FZ(G,\mathcal{F}),$$

so that for all $T \in \mathcal{X}(G,\mathcal{F})$, and all sequences $(T_n)_{n\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$ converging to T, the sequence $(\psi(T_n))_{n\in\mathbb{N}}$ converges to $\partial\psi(T)$.

When taking the quotient by equivariant homotheties, the relation $\sim$ induces an equivalence relation on $P\mathcal{X}(G,\mathcal{F})$, and $\partial_\infty FZ(G,\mathcal{F})$ is also homeomorphic to the quotient $P\mathcal{X}(G,\mathcal{F})/\sim$. The analogous statement holds true for the Gromov boundary $\partial_\infty FZ^{max}(G,\mathcal{F})$, with $\mathcal{X}^{max}(G,\mathcal{F})$ instead of $\mathcal{X}(G,\mathcal{F})$. We also provide information on the fibers of the equivalence relation $\sim$: every $\sim$-class of $\mathcal{Z}^{(max)}$-averse trees contains *mixing* representatives (in the sense of Morgan [Mor88]), and any two such representatives belong to the same simplex of length measures in $\partial\mathcal{O}(G,\mathcal{F})$. We refer to Propositions F.5.3 and F.5.26 for precise statements.

As an application, Theorem F.0.2 implies that the complex $FZ^{(max)}(G,\mathcal{F})$ is unbounded, except in the two sporadic cases where either $G = G_1 * G_2$ and $\mathcal{F} = \{[G_1],[G_2]\}$, or $G = G_1 * \mathbb{Z}$ and $\mathcal{F} = \{[G_1]\}$. For these two sporadic cases, an explicit description of the graphs $FS(G,\mathcal{F})$, $FZ(G,\mathcal{F})$ and $FZ^{max}(G,\mathcal{F})$ is given in Remarks F.2.2 and F.3.2.

Another application is the fact that the inclusion map from $FZ^{max}(G,\mathcal{F})$ to $FZ(G,\mathcal{F})$ is not a quasi-isometry as soon as the free rank N of the decomposition of G is at least 1, and its Kurosh rank $k+N$ is at least 3. However, the two graphs are quasi-isometric when F is trivial. We refer to Section F.5.6 for details.

There is also a natural map from $FZ_N^{(max)}$ to the free factor graph FF_N, defined by mapping any free splitting to one of its vertex groups, and mapping any $\mathcal{Z}^{(max)}$-splitting with nontrivial edge group to the smallest free factor of F_N containing the edge group. We deduce from Theorem F.0.2 that this map is not a quasi-isometry as soon as $N \geq 3$.

We now say a word of our proof of Theorem F.0.2, and its relation to descriptions of Gromov boundaries of other related complexes. For simplicity, we will only describe the boundary $\partial_\infty FZ^{max}(G,\mathcal{F})$. Klarreich has described the Gromov boundary of the curve graph of a compact surface S as the space of ending laminations on S [Kla99]. Using similar techniques, Bestvina and Reynolds [BR13], and independently Hamenstädt [Ham14a], have identified the Gromov boundary of the free factor graph of F_N with a space of equivalence classes of arational F_N-trees in the boundary ∂cv_N of Culler and Vogtmann's unprojectivized outer space. A tree $T \in \partial cv_N$ is *arational* if no proper free factor of F_N acts trivially or with dense orbits on its minimal subtree in T. Hamenstädt also gives a description of the Gromov boundaries of the graphs FZ_N and FS_N. Her description of the Gromov boundary of FZ_N is somewhat different from ours.

We follow the same strategy of proof. Arational trees should be thought of as trees in ∂cv_N in which no proper free factor of F_N is visible. Similarly, we need a notion of trees in $\overline{\mathcal{O}(G,\mathcal{F})}$ in which no $\mathcal{Z}^{max}$-splitting is visible. One could want to work with trees that are not compatible with any $\mathcal{Z}^{max}$-splitting. It turns out that the right notion is the (more restrictive) notion of $\mathcal{Z}^{max}$-averse trees.

We show that a tree $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ is $\mathcal{Z}^{max}$-averse if and only if there is no finite sequence $(T = T_0, \dots, T_k)$ of trees in $\overline{\mathcal{O}(G,\mathcal{F})}$ such that T_i is compatible with T_{i+1} for all $i \in \{0, \dots, k-1\}$, and T_k is simplicial. This characterization of $\mathcal{Z}^{max}$-averse trees is used in Section F.5.2 for proving that any sequence of trees in $\mathcal{O}(G,\mathcal{F})$ that converges to a $\mathcal{Z}^{max}$-averse tree $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ has unbounded image in $FZ^{max}(G,\mathcal{F})$. By refining our arguments, we then prove that any such sequence actually converges to a point in the Gromov boundary (Proposition F.8.5). This defines the map from $\mathcal{X}^{max}(G,\mathcal{F})$ to $\partial_\infty FZ^{max}(G,\mathcal{F})$.

On the other hand, we associate to every tree $T \in \overline{\mathcal{O}(G,\mathcal{F})} \smallsetminus \mathcal{X}^{max}(G,\mathcal{F})$ a set of *reducing splittings*. These are defined as those $\mathcal{Z}^{max}$-splittings S such that there exists a tree $T' \in \overline{\mathcal{O}(G,\mathcal{F})}$ that is compatible with both T and S. We prove boundedness of the diameter in $FZ^{max}(G,\mathcal{F})$ of the set of reducing splittings of any tree in $\overline{\mathcal{O}(G,\mathcal{F})} \smallsetminus \mathcal{X}^{max}(G,\mathcal{F})$ (Section F.7). This is used to prove that the projection of any sequence of trees in $\mathcal{O}(G,\mathcal{F})$ that converges to a tree in $\overline{\mathcal{O}(G,\mathcal{F})} \smallsetminus \mathcal{X}^{max}(G,\mathcal{F})$ does not converge to any point of the Gromov boundary $\partial_\infty FZ^{max}(G,\mathcal{F})$ (Proposition F.8.10).

Structure of the paper.

The paper is organized as follows. In Section F.1, we review some basic facts about free products of groups, outer spaces, folding paths, and Gromov hyperbolic spaces. We then prove the hyperbolicity of the graph of free splittings (Section F.2) and of the graph of $\mathcal{Z}^{(max)}$-splittings (Section F.3). Our proofs closely follow the arguments of Bestvina and Feighn [BF14c, Appendix] and Mann [Man13]. The rest of the paper is devoted to the description of the Gromov boundary $\partial_\infty FZ^{(max)}(G,\mathcal{F})$. In Section F.4, we introduce some more material about the geometry of folding paths in $\mathcal{O}(G,\mathcal{F})$, and the description of trees in $\overline{\mathcal{O}(G,\mathcal{F})}$. All this material will be used in the proof of our main theorem. In Section F.5, we study the properties of $\mathcal{Z}^{(max)}$-averse trees, and explain why these trees lie in some sense *at infinity* of the complex $FZ^{(max)}(G,\mathcal{F})$. We then describe constructions of collapses and pullbacks of folding sequences in Section F.6, which are then used in Section F.7 to associate to every tree in $\overline{\mathcal{O}(G,\mathcal{F})} \smallsetminus \mathcal{X}^{(max)}(G,\mathcal{F})$ a bounded set of *reducing splittings* in $FZ^{(max)}(G,\mathcal{F})$. We eventually complete the proof of Theorem F.0.2 in Section F.8.

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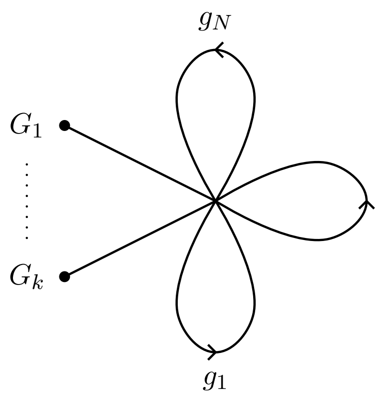

Figure F.1: A standard $(G,\mathcal{F})$-free splitting.

F.1 Background

We start by collecting general facts about free products of groups, outer spaces, $\mathbb{R}$-trees and Gromov hyperbolic spaces.

F.1.1 Free products and free factors

Let G be a countable group which splits as a free product of the form

$$G = G_1 * \cdots * G_k * F,$$

where F is a finitely generated free group. We let $\mathcal{F} := \{[G_1],\dots,[G_k]\}$ be the finite collection of the G-conjugacy classes of the G_i's, which we call a *free factor system* of G. The rank of the free group F arising in such a splitting only depends on $\mathcal{F}$. We call it the *free rank* of $(G,\mathcal{F})$ and denote it by $\mathrm{rk}_f(G,\mathcal{F})$. The *Kurosh rank* of $(G,\mathcal{F})$ is defined as $\mathrm{rk}_K(G,\mathcal{F}) := \mathrm{rk}_f(G,\mathcal{F}) + |\mathcal{F}|$. Subgroups of G which are conjugate into some subgroup in $\mathcal{F}$ are called *peripheral* subgroups. A $(G,\mathcal{F})$-*free splitting* is a minimal simplicial G-tree in which all subgroups in $\mathcal{F}$ are elliptic, and all of whose edge stabilizers are trivial. A $(G,\mathcal{F})$-*free factor* is a subgroup of G which is a point stabilizer in some $(G,\mathcal{F})$-free splitting. A $(G,\mathcal{F})$-free factor is *proper* if it is nonperipheral (and in particular nontrivial), and not equal to G.

Subgroups of free products were studied by Kurosh in [Kur34]. Let H be a subgroup of G. Let T be the Bass–Serre tree of the decomposition of G as a graph of groups represented in Figure F.1 (on which $\{g_1,\dots,g_N\}$ denotes a free basis of F). By considering the H-minimal subtree in T, we get the existence of a (possibly infinite) set J, together with an integer $i_j \in \{1,\dots,k\}$, a nontrivial subgroup $H_j \subseteq G_{i_j}$ and an element $g_j \in G$ for each $j \in J$, and a (not necessarily finitely generated) free subgroup $F' \subseteq G$, so that

$$H = *_{j\in J}\ g_j H_j g_j^{-1} * F'.$$

This is called a *Kurosh decomposition* of H. The *Kurosh rank* of H is defined as $\mathrm{rk}_K(H) := |J| + \mathrm{rk}(F')$ (this does not depend on a Kurosh decomposition of H). We note that $\mathrm{rk}_K(H)$ may be infinite in general. We let $\mathcal{F}_H$ be the collection of all H-conjugacy classes of the subgroups $g_j H_j g_j^{-1}$. If H is a $(G,\mathcal{F})$-free factor, then for all $j \in J$, we have $H_j = G_{i_j}$ by definition. In addition, the integers i_j are pairwise distinct, because two distinct G-conjugates of the subgroup G_{i_j} cannot have a common fixed point in a splitting of G

without fixing an arc in the splitting. In this case, we also get that F' is finitely generated. So any $(G,\mathcal{F})$-free factor has finite Kurosh rank, smaller than $\mathrm{rk}_K(G,\mathcal{F})$.

We denote by $\mathrm{Out}(G,\mathcal{F})$ the subgroup of the outer automorphism group $\mathrm{Out}(G)$ of G made of those automorphisms which preserve all conjugacy classes in $\mathcal{F}$.

We denote by $\mathcal{Z}$ the collection of all subgroups of G that are either trivial, or cyclic and nonperipheral. We denote by $\mathcal{Z}^{max}$ the collection of subgroups of G that are either trivial, or cyclic, nonperipheral, and closed under taking roots.

F.1.2 Outer space and its closure

An $\mathbb{R}$-*tree* is a metric space (T, d_T) in which any two points $x, y \in T$ are joined by a unique arc, which is isometric to a segment of length $d_T(x,y)$. A $(G,\mathcal{F})$-*tree* is an $\mathbb{R}$-tree equipped with an isometric action of G, in which all peripheral subgroups are elliptic. A *Grushko* $(G,\mathcal{F})$-*tree* is a metric simplicial minimal $(G,\mathcal{F})$-tree with trivial arc stabilizers, whose vertex stabilizers coincide with the subgroups in G whose G-conjugacy class belongs to $\mathcal{F}$. Two $(G,\mathcal{F})$-trees are *equivalent* if there exists a G-equivariant isometry between them. The *unprojectivized outer space* $\mathcal{O}(G,\mathcal{F})$, introduced by Guirardel and Levitt in [GL07b], is the space of all equivalence classes of Grushko $(G,\mathcal{F})$-trees. *Outer space* $P\mathcal{O}(G,\mathcal{F})$ is defined as the space of homothety classes of trees in $\mathcal{O}(G,\mathcal{F})$. The group $\mathrm{Out}(G,\mathcal{F})$ acts on both $\mathcal{O}(G,\mathcal{F})$ and $P\mathcal{O}(G,\mathcal{F})$ on the right, by precomposing the actions. Let $T \in \mathcal{O}(G,\mathcal{F})$. For all $g \in G$, the *translation length* of g in T is defined to be

$$||g||_T := \inf_{x \in T} d_T(x, gx).$$

Culler and Morgan have shown in [CM87, Theorem 3.7] that the map

$$\begin{array}{rccc} i: & \mathcal{O}(G,\mathcal{F}) & \to & \mathbb{R}^G \\ & T & \mapsto & (||g||_T)_{g \in G} \end{array}$$

is injective. We equip $\mathcal{O}(G,\mathcal{F})$ with the topology induced by this embedding, which is called the *axes topology*. Taking the quotient by G-equivariant homotheties yields an embedding of $P\mathcal{O}(G,\mathcal{F})$ into the projective space $\mathbb{PR}^G$, whose image has compact closure $\overline{P\mathcal{O}(G,\mathcal{F})}$, see [CM87, Theorem 4.5] and [7, Proposition 1.2]. We denote by $\overline{\mathcal{O}(G,\mathcal{F})}$ the lift of $\overline{P\mathcal{O}(G,\mathcal{F})}$ to $\mathbb{R}^G$. A $(G,\mathcal{F})$-tree T is *very small* if arc stabilizers in T belong to the class $\mathcal{Z}^{max}$, and tripod stabilizers are trivial. We identified the space $\overline{\mathcal{O}(G,\mathcal{F})}$ with the space of nontrivial, minimal, very small $(G,\mathcal{F})$-trees [7, Theorem 0.1].

F.1.3 Liberal folding paths

From now on, all maps between G-trees will be G-equivariant. Let T and T' be two $\mathbb{R}$-trees. A *direction* at a point $x \in T$ is the germ at x of a connected component of $T \smallsetminus \{x\}$. A *train track structure* on T is a partition of the set of directions at each point $x \in T$. Elements of the partition are called *gates* at x. A pair (d, d') of directions at x is *legal* if d and d' do not belong to the same gate. A path in T is *legal* if it only crosses legal pairs of directions. A *morphism* $f : T \to T'$ is a map such that every segment of T can be subdivided into finitely many subsegments, in restriction to which f is an isometry. Any morphism $f : T \to T'$ defines a train track structure on T, two directions at a point in T being in the same class of the partition if they have the same f-image. A morphism is *optimal* if there are at least two gates at every point in T. Let T and T' be two $(G,\mathcal{F})$-trees, and $f : T \to T'$ be a morphism. Following [BF14c, Appendix A.1], we define a *liberal folding path guided by* f to be a continuous family $(T_t)_{t \in \mathbb{R}_+}$, together with a collection of morphisms $f_{t_1,t_2} : T_{t_1} \to T_{t_2}$ for all $0 \le t_1 < t_2$, such that

- there exists $L \in \mathbb{R}$ such that for all $t \geq L$, we have $T_t = T'$, and
- we have $f_{0,L} = f$, and
- for all $0 \leq t_1 < t_2 < t_3$, we have $f_{t_1,t_3} = f_{t_2,t_3} \circ f_{t_1,t_2}$.

Given two $(G,\mathcal{F})$-trees T and T', a *liberal folding path* from T to T' is a folding path guided by some morphism $f : T \to T'$. A liberal folding path guided by a morphism f is *optimal* if f is optimal. Notice that in this case, all morphisms $f_{t,t'}$ with $t < t'$ are also optimal.

Given two $(G,\mathcal{F})$-trees T and T', we say that T' is obtained from T by a *fold* if there exist arcs e and e' in T having a common endpoint, such that T' is obtained from T by G-equivariantly identifying e with e'. Given a morphism $f : T \to T'$, a way of constructing liberal folding paths from T to T' is by folding pairs of directions that have the same f-image. We refer to [BF14b, Section 2] and references therein for various constructions of liberal folding paths between $(G,\mathcal{F})$-trees.

Given two $(G,\mathcal{F})$-trees T and T', and an optimal morphism $f : T \to T'$, Guirardel and Levitt described in [GL07b, Section 3] a construction of a *canonical optimal folding path* $(T_t)_{t\in\mathbb{R}}$ guided by f. As a set, the tree T_t is defined in the following way. Given $a, b \in T$, we define the *identification time* between a and b as $\tau(a,b) := \sup_{x\in[a,b]} d_{T'}(f(a), f(x))$. We define equivalence relations $\sim_t$ on T for all $t \in \mathbb{R}_+$ by letting $a \sim_t b$ if $f(a) = f(b)$ and $\tau(a,b) \leq t$. The tree T_t is the quotient of T by the equivalence relation $\sim_t$. In [GL07b, Section 3.1], Guirardel and Levitt defined a metric on $(T_t)_{t\in\mathbb{R}}$ that turns it into an $\mathbb{R}$-tree.

We now establish one more property of optimal liberal folding paths.

Proposition F.1.1. *Let T and T' be simplicial $(G,\mathcal{F})$-trees with trivial edge stabilizers, and let $(T_t)_{t\in[0,L]}$ be an optimal liberal folding path from T to T'. Then for all $t \in [0,L]$, the tree T_t is simplicial and has trivial edge stabilizers.*

Proof. Arc stabilizers in T_t are trivial, otherwise the $f_{t,L}$-image of an arc with nontrivial stabilizer in T_t would be an arc with nontrivial stabilizer in T'. Therefore, the tree T_t splits as a graph of actions, whose vertex actions have dense orbits for their stabilizers (see Proposition F.4.18). The morphism $f_{s,L}$ is an isometry in restriction to the vertex trees of this decomposition (see Corollary F.4.13). As T' is simplicial, this implies that T_t is simplicial. ☐

We will also work with the following discrete version of optimal liberal folding paths. Let T be a $(G,\mathcal{F})$-tree. A *folding sequence ending at T* is a sequence $(T_p)_{p\in\mathbb{N}}$ of $(G,\mathcal{F})$-trees that converges to T in a nonstationary way, such that for all integers $p < q$, there are morphisms $f_p : T_p \to T$ and $f_{p,q} : T_p \to T_q$ such that $f_p = f_q \circ f_{p,q}$ for all $p < q$, and $f_{p,r} = f_{q,r} \circ f_{p,q}$ for all $p < q < r$.

F.1.4 Coarse geometry notions

We now recall the notions of quasi-isometries between metric spaces, and of (reparameterized) quasi-geodesics.

Two metric spaces (X,d) and (X',d') are *quasi-isometric* if there exist $K, L \geq 0$ and a map $f : X \to X'$ such that

- for all $x' \in X'$, there exists $x \in X$ such that $d'(x', f(x)) \leq L$, and
- for all $x, y \in X$, we have

$$\frac{1}{K}d(x,y) - L \leq d'(f(x), f(y)) \leq K d(x,y) + L.$$

Let (X,d) be a metric space, let $x,y \in X$, and let $K,L \geq 0$. A (K,L)-*quasi-geodesic* from x to y is a map $\gamma : [a,b] \to X$, where $[a,b] \subseteq \mathbb{R}$ is a segment, such that $\gamma(a) = x$ and $\gamma(b) = y$, and for all $s,t \in [a,b]$, we have

$$\frac{1}{K}|t-s| - L \leq d(\gamma(s),\gamma(t)) \leq K|t-s| + L.$$

A *reparameterized quasi-geodesic* is a map $\gamma' : [a',b'] \to X$, where $[a',b'] \subseteq \mathbb{R}$ is a segment, so that there exists a segment $[a,b] \subseteq \mathbb{R}$ and a continuous nondecreasing map $\theta : [a,b] \to [a',b']$, such that $\gamma' \circ \theta$ is a (K,L)-quasi-geodesic.

F.1.5 Gromov hyperbolic spaces

We give a very brief account on hyperbolic spaces, which were defined by Gromov [Gro87]. The reader is referred to [BH99, CDP90, GdlH90] for a detailed introduction. Let (X,d) be a metric space. Let $p \in X$ be some basepoint. For all $x,y \in X$, the *Gromov product* of x and y with respect to p is defined as

$$(x|y)_p := \frac{1}{2}(d(p,x) + d(p,y) - d(x,y)).$$

A metric space X is *Gromov hyperbolic* if there exists a constant $\delta > 0$ such that for all $x,y,z,p \in X$, we have

$$(x|y)_p \geq \max\{(x|z)_p, (y|z)_p\} - \delta.$$

(When X is geodesic, hyperbolicity of X is equivalent to a *thin triangles* condition, see the aforementioned references). If (X,d) is Gromov hyperbolic, we say that a sequence $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ *converges to infinity* if the Gromov product $(x_n|x_m)_p$ goes to $+\infty$ as n and m both go to $+\infty$. Two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ that both converge to infinity are *equivalent* if the Gromov product $(x_n|y_m)_p$ goes to $+\infty$ as n and m go to $+\infty$. It follows from the hyperbolicity of (X,d) that this is indeed an equivalence relation. The *Gromov boundary* $\partial_\infty X$ of X is defined to be the collection of equivalence classes of sequences that converge to infinity. For all $a,b \in \partial_\infty X$, the *Gromov product* of a and b with respect to p is defined as

$$(a|b)_p = \sup \liminf_{i,j\to+\infty} (x_i|y_j)_p,$$

where the supremum is taken over all sequences $(x_i)_{i\in\mathbb{N}}$ converging to a and all sequences $(y_j)_{j\in\mathbb{N}}$ converging to b. The set $\partial_\infty X$ is equipped with the topology for which every point $a \in \partial_\infty X$ has a basis of open neighborhoods made of the sets of the form $N_r(a) := \{b \in \partial_\infty X | (a|b)_p \geq r\}$. One can also define the Gromov product between an element in X and an element in $\partial_\infty X$ similarly, and get a topology on $X \cup \partial_\infty X$. Given any $\xi \in \partial_\infty X$, there exists a quasi-geodesic ray $\tau : \mathbb{R}_+ \to X$ such that $\tau(t)$ converges to ξ as t goes to $+\infty$.

F.2 Hyperbolicity of the free splitting graph

We recall that a $(G,\mathcal{F})$-*free splitting* is a minimal, simplicial $(G,\mathcal{F})$-tree, all of whose edge stabilizers are trivial. Two $(G,\mathcal{F})$-free splittings are *equivalent* if they are G-equivariantly homeomorphic. Given two $(G,\mathcal{F})$-free splittings T and T', we say that T' is a *refinement* of T if T is obtained from T' by collapsing a G-invariant set of edges of T to points. Two $(G,\mathcal{F})$-splittings are *compatible* if they have a common refinement. The *free splitting graph* $FS(G,\mathcal{F})$ is the graph whose vertices are the equivalence classes of one-edge $(G,\mathcal{F})$-free splittings, two distinct splittings being joined by an edge if they are compatible. Alternatively, one can define $FS(G,\mathcal{F})$ to be the graph whose vertices are the equivalence classes of all $(G,\mathcal{F})$-free splittings, two splittings being joined by an edge if one properly refines the other. The two versions of the complex are quasi-isometric to each other. We will rather use the second version in our proof of its Gromov hyperbolicity. A *natural vertex* of a $(G,\mathcal{F})$-free splitting is a vertex which either has valence at least 3, or is the center of an inversion. A *natural edge* is a complementary component of the set of natural vertices. We note that a $(G,\mathcal{F})$-free splitting with k orbits of natural edges has exactly k distinct one-edge collapses, and is determined by the set of these collapses, see [SS00, Theorem 2.5] or [HM13a, Lemma 1.3]. There is map $\phi : \mathcal{O}(G,\mathcal{F}) \to FS(G,\mathcal{F})$, which extends to the set of simplicial trees in $\overline{\mathcal{O}(G,\mathcal{F})}$ with trivial edge stabilizers, defined by choosing a one-edge collapse of every simplicial tree in $\overline{\mathcal{O}(G,\mathcal{F})}$ (as we have to make choices, this map is not equivariant, but making any other choice can only change distances by at most 2). Proposition F.1.1 implies that the ϕ-image of any folding path between simplicial trees in $\overline{\mathcal{O}(G,\mathcal{F})}$ with trivial edge stabilizers is well-defined. The graph $FS(G,\mathcal{F})$ comes with a right action of $\mathrm{Out}(G,\mathcal{F})$, by precomposition of the actions.

Hyperbolicity of the free splitting graph of a finitely generated free group was shown by Handel and Mosher [HM13a], whose proof involves studying folding paths between simplicial F_N-trees with trivial edge stabilizers. An alternative proof in the sphere model of the free splitting graph, based on the study of surgery paths, was given by Hilion and Horbez [1]; Bestvina and Feighn also gave a simplified proof in Handel and Mosher's setting [BF14c, Appendix]. We will adapt Bestvina and Feighn's viewpoint to generalize Handel and Mosher's result to the case of $(G,\mathcal{F})$-splittings. As in [BF14c], the proof of the hyperbolicity of $FS(G,\mathcal{F})$ will also give information about projections to $FS(G,\mathcal{F})$ of optimal liberal folding paths between simplicial trees with trivial edge stabilizers in $\overline{\mathcal{O}(G,\mathcal{F})}$ (under the map ϕ). The following theorem was recently obtained independently by Handel and Mosher [HM14b].

Theorem F.2.1. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. Then the graph $FS(G,\mathcal{F})$ is Gromov hyperbolic. Images in $FS(G,\mathcal{F})$ of optimal liberal folding paths are uniform reparameterized quasi-geodesics.*

Remark F.2.2. If $G = G_1 * G_2$ and $\mathcal{F} = \{[G_1],[G_2]\}$, then $FS(G,\mathcal{F})$ is reduced to a point. If $G = G_1 * \mathbb{Z}$ and $\mathcal{F} = \{[G_1]\}$, then $FS(G,\mathcal{F})$ is a star of diameter equal to 2, whose central vertex corresponds to the HNN extension $G = G_1*$, and whose extremal vertices correspond to all splittings of the form $G = G_1 * \langle g_1 t\rangle$, where t is a stable letter of the HNN extension, and g_1 varies in G_1.

F.2.1 Distance estimates in $FS(G,\mathcal{F})$

Let $T, T' \in FS(G,\mathcal{F})$. Assume that both T and T' have been equipped with a simplicial metric, and let $f : T \to T'$ be an optimal morphism. Let R and B be two disjoint G-invariant sets of points in T', both disjoint from the set of natural vertices, which project to finite sets in the quotient graph T'/G. A *mixed region* in T' is a component of the complement of $R \cup B$ in T' whose frontier intersects both R and B. Assuming in addition that $f^{-1}(R)$ and $f^{-1}(B)$ are disjoint from the set of natural vertices, we also define mixed regions in T. The following proposition is an adaptation of [BF14c, Lemma A.4].

Proposition F.2.3. *There exists a constant $C_1 > 0$ such that the following holds.*

Let $T, T' \in FS(G,\mathcal{F})$. Assume that T and T' are equipped with simplicial metrics, and let $f : T \to T'$ be an optimal morphism. Let R and B be nonempty disjoint G-invariant *sets in T', both disjoint from the set of natural vertices in T', whose projections to the quotient graph T'/G are finite, and such that $f^{-1}(R)$ and $f^{-1}(B)$ are disjoint from the set of natural vertices in T. Let N_0 denote the number of G-orbits of mixed regions in T. Then $d_{FS(G,\mathcal{F})}(T,T') \leq 2N_0 + 2rk_K(G,\mathcal{F}) + 4$.*

Proof. There exist G-equivariant subdivisions of the edges of T and T' such that f maps edgelets to edgelets, and each edgelet of the subdivisions of T and T' contains at most one point in $R \cup B$. (To see this, one starts by subdividing T' so that each edgelet of the subdivision of T' contains at most one point in $R \cup B$, then add all f-images of natural vertices in T to this subdivision of T', and finally pull back the subdivision of T' by f to get the desired subdivision of T). We call the edgelets that contain a point in R (resp. in B) *red* (resp. *blue*) edgelets, the other edgelets are *neutral.* By construction, the morphism f maps neutral edgelets of T to neutral edgelets of T'. Therefore, we can collapse all neutral edgelets to points in both T and T', and assume that all edgelets are either red or blue. In particular, mixed regions contain a unique vertex, adjacent to both red and blue edgelets. Such vertices are called *mixed vertices.* We choose an optimal folding path $(T_t)_{t\in[0,L]}$ guided by f (in particular it joins T to T') that is obtained by first only folding red edgelets for as long as possible, and then only folding blue edgelets for as long as possible, etc. Thus the folding path is broken up into segments, called *phases*, and we talk about red and blue phases, depending on which color is allowed to get folded. The Kurosh decomposition of any vertex stabilizer $H \subseteq G$ in any T_t reads as

$$H = g_1 G_{i_1} g_1^{-1} * \cdots * g_s G_{i_s} g_s^{-1} * F',$$

where $G_{i_1}, \ldots, G_{i_k} \in \mathcal{F}$ are pairwise non conjugate in G (or in other words, the indices i_j are pairwise distinct), and F' is a finitely generated free group, see Section F.1.1. Let N_0 be the number of orbits of mixed regions in T, and let r be the sum of the Kurosh ranks $\mathrm{rk}_K(\mathrm{Stab}(x))$, where x ranges over a set of representatives of the orbits of mixed vertices in T. We define the *complexity* of the morphism $f : T \to T'$ as $C(f) := N_0 - r$. We make the following observations.

- The sum of the Kurosh ranks of the stabilizers of a subset of the orbits of vertices in T may not exceed the Kurosh rank of $(G, \mathcal{F})$ (because all subgroups in $\mathcal{F}$ are contained in exactly one vertex stabilizer in T).
- Under folding, the complexity $C(f_{s,L})$ never increases with s (where $f_{s,L} : T_s \to T'$ is the induced morphism). Indeed, the integer N_0 cannot increase because a blue vertex (i.e. a vertex that is only adjacent to blue edges) and a red vertex cannot be identified by $f_{s,L}$. The integer r cannot decrease because mixed vertices are mapped to mixed vertices by $f_{s,L}$. Complexity changes exactly in the event that either two mixed vertices merge into one (in which case N_0 decreases if these vertices belong to distinct G-orbits, while r increases if they belong to the same G-orbit), or a mixed vertex gets identified with a non-mixed vertex whose stabilizer has positive Kurosh rank, in which case r increases.
- If T_t and $T_{t'}$ belong to the same phase, then $d_{FS(G,\mathcal{F})}(T_t, T_{t'}) \leq 2$. Indeed, all splittings in a phase refine a common splitting, obtained by equivariantly collapsing all but one G-orbits of edges to points, the uncollapsed orbit having the color not corresponding to the phase.
- The complexity decreases at least once during each phase, except possibly first and last. Indeed, consider a phase, say red, starting with T_t, which is a tree decomposed into maximal red and blue subtrees. If our red phase is not the first phase, each

component of the blue subforest of T_t embeds in T_L. If our red phase is not the last, then there is a time during the phase at which a red fold identifies points in distinct blue subtrees, so the complexity decreases.

The first, second and last bullets imply that there are at most $N_0+\mathrm{rk}_K(G,\mathcal{F})+2$ phases in the process, and the third bullet implies that $d_{FS(G,\mathcal{F})}(T,T') \leq 2N_0+2\mathrm{rk}_K(G,\mathcal{F})+4$. $\square$

By choosing R and B to consist of the G-orbits of two nearby points in the interior of an edge of T', Proposition F.2.3 yields the following distance estimate in $FS(G,\mathcal{F})$ (see [BF14c, Lemma A.3]).

Corollary F.2.4. *Let $T,T' \in FS(G,\mathcal{F})$. Assume that T and T' are equipped with simplicial metrics, and let $f : T \to T'$ be an optimal morphism. Let $y \in T'$ be a point that belongs to the interior of an edge, such that $f^{-1}(y)$ does not intersect the set of natural vertices in T. Then $d_{FS(G,\mathcal{F})}(T,T')$ is bounded by a linear function of the cardinality of $f^{-1}(y)$.*

Proof. Let R and B be the orbits of two points in T' that are so close to y that their f-preimages consist of two sets of $|f^{-1}(y)|$ points that are pairwise close to each other (and close to the f-preimages of y). Each point in $f^{-1}(R\cup B)$ belongs to at most 2 mixed regions, so the number of orbits of mixed regions in T is bounded above by $4|f^{-1}(y)|$. The claim then follows from Proposition F.2.3. $\square$

F.2.2 Folding paths joining close splittings

We now adapt [BF14c, Lemma A.7], which is used in Bestvina and Feighn's proof to show the retraction property from Masur and Minsky's axioms for folding paths (see Section F.2.3 of the present paper). Let T be a $(G,\mathcal{F})$-free splitting, equipped with a train track structure. A *hanging tree* in T (see Figure F.2) is a triple (H,l,v), where

- there exists an element $g_0 \in G$ that is hyperbolic in T, whose axis in T is equal to l and is legal, and
- the subtree $H \subseteq T$ is a finite (not necessarily closed) subtree of T, such that
    - we have $gH \cap H = \emptyset$ for all $g \in G \smallsetminus \{e\}$, and
    - the intersection $H \cap l$ is a finite segment of the form $[v,v']$ (possibly reduced to a point), and
    - the vertex v has exactly two gates in T, and v has exactly one gate in H, and
    - every other vertex in H has two gates in T, with the direction towards v being its own gate.

We call v the *top vertex* of the hanging tree.

The second situation of the following proposition is illustrated in Figure F.3. In the statement of this proposition, note that any simplicial edge e in a $(G,\mathcal{F})$-tree defines a $(G,\mathcal{F})$-splitting by collapsing all edges that do not belong to the G-orbit of e to points.

Proposition F.2.5. *There exists $C_2 > 0$ such that for all $T,T' \in FS(G,\mathcal{F})$, equipped with simplicial metrics, and all optimal morphisms $f : T \to T'$, if there exist edges $e \subset T$ and $e' \subset T'$ that define the same $(G,\mathcal{F})$-free splitting, then either*

- *there exists a point in the interior of e' whose f-preimage has cardinality at most C_2, or*

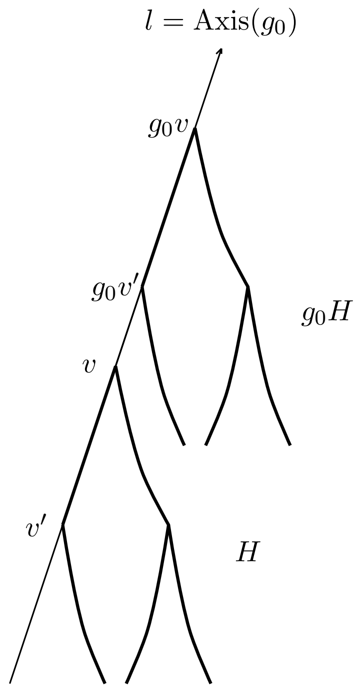

Figure F.2: A hanging tree and one of its G-translates in T.

- *there is a hanging tree (H, l, v) in T (for the train track structure determined by f), such that the f-preimage of any interior point of e' is contained in the union of l and of the Stab(l)-translates of H, and contains at most one point in l.*

Proof. We assume that the edges of T and T' have been G-equivariantly subdivided, so that f maps edgelets to edgelets. Let $(T_t)_{t\in[0,L]}$ be a liberal folding path guided by f, chosen so that on an initial segment $[0, s]$, we perform all folds which do not involve the image of e, and the edge $e_s := f_{0,s}(e)$ is involved in all illegal turns in T_s. We denote by $\widehat{e_s}$ the natural edge in T_s that contains e_s. If $T_s = T'$, then $\widehat{e_s} = e'$, and the first conclusion of the proposition holds, so we assume otherwise. Note that since all folds in T_s involve e_s, and f is optimal, at least one endpoint of e_s is a natural vertex (but the other might not be natural).

Let G' be one of the elliptic groups of the $(G, \mathcal{F})$-free splitting determined by $\widehat{e_s}$. Let $\widetilde{T_s}$ (resp. $\widetilde{T'}$) denote the G'-tree which is one of the components of $T_s \smallsetminus G.\widehat{e_s}$ (resp. $T' \smallsetminus G.e'$). By construction, the splitting determined by $\widehat{e_s}$ in T_s is the same as the splitting determined by e in T, and it is also the same as the splitting determined by e' in T' by assumption.

Case 1: The tree $\widetilde{T_s}$ is a minimal G'-tree.
In this case, we will show that the first conclusion of Proposition F.2.5 holds. The morphism $f_{s,L}$ restricts to an isometric embedding from $\widetilde{T_s}$ to the minimal G'-subtree of T', which is contained in $\widetilde{T'}$ because e and e' define the same $(G, \mathcal{F})$-splitting. In other words, the image $f_{s,L}(\widetilde{T_s})$ does not meet the interior of e'.

Let $\widehat{T_s}$ (resp. $\widehat{T'}$) be the tree obtained by collapsing all edges in $T_s \smallsetminus G.\widehat{e_s}$ (resp. $T' \smallsetminus G.e'$) to points in T_s (resp. T'). The optimal morphism $f_{s,L}$ induces a G-equivariant simplicial map $\widehat{f_{s,L}} : \widehat{T_s} \to \widehat{T'}$ which preserves alignment in restriction to each natural edge of $\widehat{T_s}$. The trees $\widehat{T_s}$ and $\widehat{T'}$ are G-equivariantly homeomorphic by construction, so

$\widehat{f}_{s,L}$ can be viewed as a map from $\widehat{T}_s$ to itself. As vertices of $\widehat{T}_s$ have nontrivial stablizer, the map $\widehat{f}_{s,L}$ maps vertices to themselves. Optimality of $f_{s,L}$ then implies that $\widehat{f}_{s,L}$ also maps edges to themselves. This implies that $\widehat{f}_{s,L}(\widehat{e_s}) = ge'$ for some $g \in G$, so $f_{s,L}(\widehat{e_s})$ crosses the G-orbit of e' exactly once. Therefore, the $f_{s,L}$-preimage of any point y in the interior of e' is a single point y_s in some G-translate of $\widehat{e_s}$. As the $f_{0,s}$-image of any edge in $T \smallsetminus G.e$ is disjoint from e_s, we can choose y in the interior of e' so that the $f_{0,s}$-preimage of y_s has cardinality 1, and the first conclusion of the proposition follows.

Case 2: The tree $\widetilde{T}_s$ is not minimal.
This implies that the splitting of G defined by e (and $\widehat{e_s}$) is an HNN extension, and the edge $\widehat{e_s}$ projects in the quotient graph T_s/G to a loop-edge, whose extremal vertex lifts to a valence 3 vertex in T_s (with trivial stabilizer). In particular, there exists $g_0 \in G$ such that $(\widehat{e_s}, g_0\widehat{e_s})$ forms a turn in T_s. The proof then runs as in [BF14c, Lemma A.7]. As $T_s \neq T'$ by assumption, either $(\widehat{e_s}, g_0\widehat{e_s})$ is illegal, or else $\widehat{e_s}$ forms an illegal turn with an edge that is not in the G-orbit of $\widehat{e_s}$.

Case 2.1: The turn $(\widehat{e_s}, g_0\widehat{e_s})$ is illegal.
Optimality of f, together with our choice of folding path, implies that this is the only illegal turn in T_s. Therefore, the tree T' is obtained from T_s by equivariantly folding $\widehat{e_s}$ along one of its translates. Then for all $y \in f_{s,L}(\widehat{e_s})$, the $f_{s,L}$-preimage of y contains at most two points in T_s, and is contained in $\widehat{e_s} \cup g_0^{\pm 1}\widehat{e_s}$. As in the case where $\widetilde{T}_s$ is minimal, the $f_{0,s}$-image of an edge in T that is distinct from e cannot cross e_s by construction, so the first conclusion of the proposition again holds.

Case 2.2 (see Figure F.3): The turn $(\widehat{e_s}, g_0\widehat{e_s})$ is legal.
The edge $\widehat{e_s}$ contains a vertex a of valence 3, with trivial stabilizer, and there are two gates at a. Let l_s be the axis of g_0 in T_s, which is legal. Let v_s be the extremity of e_s such that $e_s = [a, v_s]$. There is an edge z in T_s incident to a, such that z and $g_0\widehat{e_s}$ form an illegal turn, and the G'-tree we get by equivariantly removing $\widehat{e_s} \cup \mathring{z}$ in T_s is minimal, and hence isometrically embeds into $T' \smallsetminus G.e'$. All points in $\widehat{e_s} \cup \mathring{z}$ have trivial stabilizer. The $f_{s,L}$-preimage of any point y in the interior of e' is contained in the G-orbit of $\widehat{e_s} \cup \mathring{z}$. We claim that it meets the orbit of $\widehat{e_s}$ in a single point. Indeed, legality of l_s implies that y has at most one preimage in each G-translate of l_s. As $\widehat{e_s}$ and e' define the same $(G,\mathcal{F})$-free splitting, the image $f_{s,L}(l_s)$ collapses to a legal axis in $\widehat{T}'$ that meets its translates degenerately. This implies that y has a preimage in at most one of the translates of l_s, and proves the claim. However, the point y can have an arbitrarily large number of preimages in translates of $\mathring{z}$.

Let $H \subset T$ be the $f_{0,s}$-preimage of the half-open segment $I := \mathring{z} \cup \widehat{e_s} \smallsetminus \mathring{e_s}$. Denote by $v \in T$ the extremity of e such that $f_{0,s}(v) = v_s$, and by $m_e \in T$ the midpoint of the edge e. We claim that for all $x \in H$, the segment $[x, m_e]$ is legal, and $[x, v] \subseteq H$. Indeed, legality of f enables us to find a legal half-line starting at x in T, whose $f_{0,s}$-image contains e_s. As $f_{0,s}^{-1}(e_s) = e$, the point m_e belongs to this half-line, and v is the unique point on $[x, m_e]$ that is mapped to v_s, which proves the above claim. This claim implies that H is connected, that v is the unique $f_{0,s}$-preimage of v_s (because if v' were another preimage of v_s, then the segment $[v', v]$ would have to be a legal segment mapped to a point by $f_{0,s}$). Since $f_{0,s}(H) = I$, and $f_{0,s}$ is legal, there are exactly two gates at every point of $H \smallsetminus \{v\}$. Our above claim implies that the direction towards v is its own gate. Indeed, assume by contradiction that there exists another direction d at a point $x \in H \smallsetminus \{v\}$ in the same gate as the direction towards v. Then $f_{0,s}(d)$ is contained in I, so d is contained in H. If

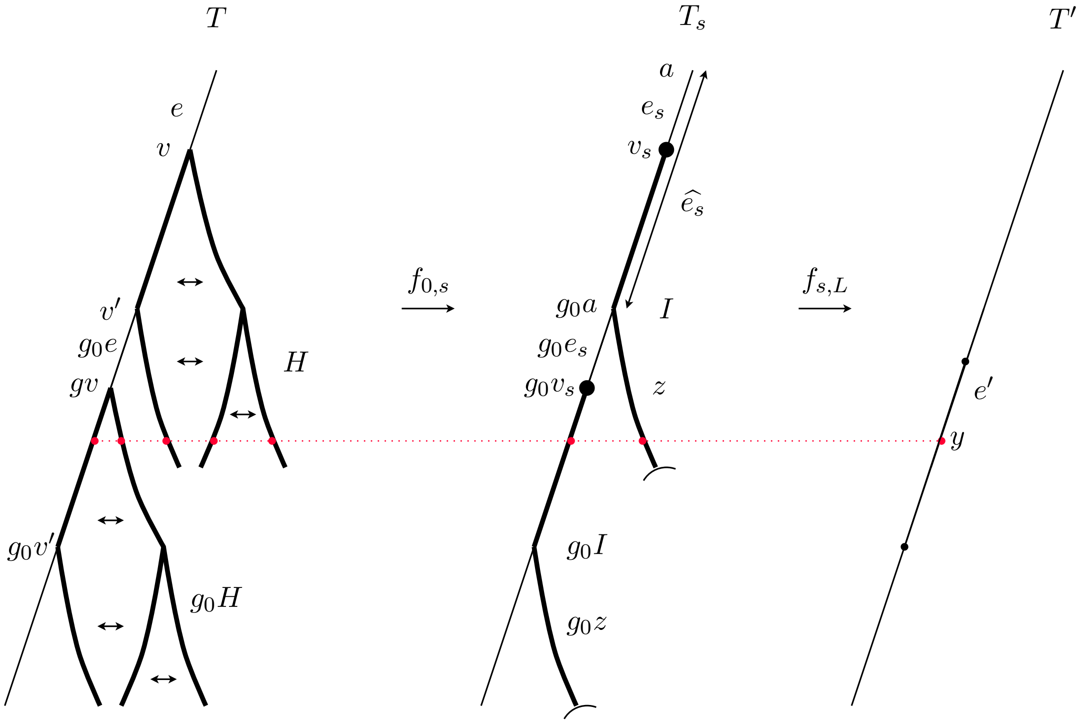

Figure F.3: The situation in Case 2.2 of the proof of Proposition F.2.5.

$v' \in H$ is a point in this direction, then the segment $[v', v]$ is not legal, a contradiction. Finally, we have $gH \cap H = \emptyset$ for all $g \in G \setminus \{e\}$, because the same holds true for I.

Let v' be the $f_{0,s}$-preimage of $g_0 a$ in $g_0 e$, and let l be the axis of g_0 in T. A fundamental domain for this axis is given by $e \cup [v, v']$, and l is legal because it maps injectively onto l_s, which is legal for $f_{s,L}$. Then the f-preimage of any interior point of e' has the desired description. □

As in [BF14c, Proposition A.9], we get as a corollary the following distance estimate in $FS(G, \mathcal{F})$.

Corollary F.2.6. *There exists $C_3 > 0$ so that for all $T, T' \in FS(G, \mathcal{F})$, and all optimal morphisms $f : T \to T'$, if T and T' contain edges which determine the same $(G, \mathcal{F})$-free splitting, then the image in $FS(G, \mathcal{F})$ of any liberal folding path guided by f has diameter at most C_3.*

Proof. If the first conclusion of Proposition F.2.5 holds, this follows from Corollary F.2.4. Otherwise, denote by e and e' the edges in T and T' that determine the same $(G, \mathcal{F})$-free splitting, let y be a point in the interior of e', and let t_1 be the first time at which the preimage of y in T_{t_1} consists of a single point. We claim that for any small $\epsilon > 0$, the paths (T_t) for $t \in [0, t_1 - \epsilon]$, $t \in [t_1 - \epsilon, t_1]$ and $t \in [t_1, L]$ all map to bounded sets in $FS(G, \mathcal{F})$. This is true for the last by Corollary F.2.4, for the second if ϵ is small enough. We will prove that it also holds for $(T_t)_{t \in [0, t_1 - \epsilon]}$.

Let (H, l, v) be a hanging tree for $f : T \to T'$, provided by Proposition F.2.5. Let $l' := f(l)$. For all $t \in [0, L]$, the $f_{t,L}$-preimage of any interior point y of e' in T_t is equal to $f_{0,t}(f^{-1}(y))$, and $f^{-1}(y)$ is described by Proposition F.2.5. We get that $f_{t,L}^{-1}(y)$ is contained in the union of the legal line $l_t = f_{0,t}(l)$ and the $\mathrm{Stab}(l_t)$-orbit of the finite subtree $H_t = f_{0,t}(H)$, and it contains a single point in l_t. We orient $l_t = (-\infty, +\infty)$ so that all paths from points in the $\mathrm{Stab}(l_t)$-orbit of H_t to $+\infty$ are legal. Let $y_1 \in T_{t_1-\epsilon}$ be the preimage of y in $l_{t_1-\epsilon}$, and $x_1 \in T_{t_1-\epsilon}$ be another preimage of y. Then all preimages of x_1 in T belong to translates of H.

We claim that no two preimages of x_1 in T belong to the same orbit of edges in T. This will imply that the point $x_1 \in T_1$ has a bounded number of preimages in T. It then follows from Corollary F.2.4 that the path $(T_t)_{t\in[0,t_1-\epsilon]}$ maps to a set of bounded diameter in $FS(G,\mathcal{F})$.

We now prove the above claim. For all $t \in [0, L]$, we say that two points $u, v \in l_t \cup \mathrm{Stab}(l_t).H_t$ are *comparable* if we either have $u \in [v, +\infty)$ or $v \in [u, +\infty)$, and *incomparable* otherwise. Notice that any segment in T_t joining two comparable points is legal. In particular, comparable points remain comparable under folding.

We observe that $G.x_1 \cap H_{t_1-\epsilon}$ is a set of pairwise incomparable points. Indeed, if two points $u, v \in G.x_1 \cap H_{t_1-\epsilon}$ were comparable, then the amount of time needed to identify u with y_1 in a folding path would be distinct from the amount of time needed to identify v with y_1, which contradicts the definition of t_1 if ϵ has been chosen small enough. This observation implies that the G-orbit of the set of all preimages of x_1 in T is a set of pairwise incomparable points, and our claim follows. ☐

F.2.3 End of the proof

Masur and Minsky's criterion for checking the hyperbolicity of a graph. The proof of the hyperbolicity of $FS(G,\mathcal{F})$ relies on a set of axioms that is due to Masur and Minsky. Before stating Masur and Minsky's theorem, we recall a few definitions.

Let $\mathcal{X}$ be a connected graph, equipped with the simplicial metric. A collection Γ of paths in $\mathcal{X}$ is *coarsely transitive* if there exists $M' > 0$ so that for all vertices $v, w \in \mathcal{X}$, there exist vertices $v', w' \in \mathcal{X}$ which are joined by some path in Γ, with $d_{\mathcal{X}}(v, v'), d_{\mathcal{X}}(w, w') \leq M'$. Let $a, b \in \mathbb{R}$, let $\gamma : [a, b] \to \mathcal{X}$ be a path in $\mathcal{X}$, and let $\pi : \mathcal{X} \to [a, b]$ be a map. Let $A \geq 0$, $B > 0$ and $C \geq 0$. We say that (γ, π) is

- *C-coarsely retracting* if for all $t \in [a, b]$, the diameter of the set $\gamma([t, \pi(\gamma(t))])$ is smaller than C,
- *C-coarsely Lipschitz* if for all vertices $v, w \in \mathcal{X}$ satisfying $d_{\mathcal{X}}(v, w) \leq 1$, the diameter of the set $\gamma([\pi(v), \pi(w)])$ is smaller than C,
- *(A, B, C)-strongly contracting* if for all vertices $v, w \in \mathcal{X}$ satisfying $d_{\mathcal{X}}(v, \gamma([a, b])) \geq A$ and $d_{\mathcal{X}}(v, w) \leq B d_{\mathcal{X}}(v, \gamma([a, b]))$, the diameter of the set $\gamma([\pi(v), \pi(w)])$ is smaller than C.

Theorem F.2.7. *(Masur–Minsky [MM99, Theorem 2.3]) Let $\mathcal{X}$ be a connected graph, equipped with the simplicial metric. Assume that there exist constants $A, B, C > 0$, a coarsely transitive collection of paths Γ in $\mathcal{X}$, and for each path $\gamma : [a, b] \to \mathcal{X}$ in Γ, a map $\pi_\gamma : \mathcal{X} \to [a, b]$, such that all (γ, π_γ) are C-coarsely retracting, C-coarsely Lipschitz and (A, B, C)-strongly contracting. Then $\mathcal{X}$ is Gromov hyperbolic, and all the paths $\gamma \in \Gamma$ are reparameterized quasi-geodesics with uniform constants.*

The proof of Theorem F.2.1 goes by checking Masur and Minsky's axioms for the set of ϕ-images in $FS(G,\mathcal{F})$ of optimal liberal folding paths between simplicial trees with trivial edge stabilizers in $\overline{\mathcal{O}(G,\mathcal{F})}$, which is coarsely transitive (the existence of optimal morphisms between splittings in $\mathcal{O}(G,\mathcal{F})$ follows from [FM14, Corollary 6.8], and there is a canonical way to build a folding path from a morphism, as recalled in Section F.1.3).

We will define the projection to the image in $FS(G,\mathcal{F})$ of an optimal liberal folding path. Before doing so, we recall two constructions of liberal folding paths from Bestvina and Feighn's paper [BF14c, Section A.2]. These contructions will be carried out in a more detailed way, and in a more general setting, in Section F.6, where we do not restrict to the case of simplicial trees.

Collapses. Let $T, T' \in FS(G,\mathcal{F})$. Assume that T and T' have been equipped with simplicial metrics, and let $(T_t)_{t\in[0,L]}$ be an optimal liberal folding path joining T to T'. Let $F_L \subset T'$ be a proper G-invariant subforest. For all $t \in [0,L]$, let $\overline{T_t}$ be the tree obtained by G-equivariantly collapsing each connected component of the $f_{t,L}$-preimage of F_L in T_t to a point. For all $t < t' \in [0,L]$ and all $x \in \overline{T_t}$, the preimage of x in T_t is a subtree of T_t whose $f_{t,t'}$-image is contained in a subtree of $T_{t'}$ which is collapsed to a point x' in $\overline{T_{t'}}$. Hence one can define maps $\overline{f}_{t,t'} : \overline{T_t} \to \overline{T_{t'}}$ by setting $\overline{f}_{t,t'}(x) := x'$. These maps are morphisms. Indeed, every segment in T_t can be subdivided into finitely many subsegments, so that the interior of each of these subsegments is either entirely contained in the collapsed forest, or is entirely contained outside of this forest, such that f is an isometry in restriction to each of these subsegments. Then $\overline{f}_{t,t'}$ is an isometry in restriction to each subsegment that does not get collapsed, and $(\overline{T_t})_{t\in[0,L]}$ is an optimal liberal folding path. Continuity will be proved in Proposition F.6.1. For optimality, notice that every edge in $\overline{T_t}$ lifts to an edge of T_t that is contained in some legal axis. We sum up the above discussion in the following statement.

Proposition F.2.8. *Let $T, T', \overline{T'} \in FS(G,\mathcal{F})$, such that $\overline{T'}$ is a collapse of T'. Assume that T and T' have been equipped with simplicial metrics, and let $(T_t)_{t\in[0,L]}$ be an optimal liberal folding path from T to T'. Then there exists an optimal liberal folding path $(\overline{T_t})_{t\in[0,L]}$ that ends at $\overline{T'}$, such that for all $t \in [0,L]$, the tree $\overline{T_t}$ is a collapse of T_t.* ☐

We say that the path $(\overline{T_t})_{t\in[0,L]}$ is a *collapse* of the path $(T_t)_{t\in[0,L]}$.

Pullbacks. Let $T, T' \in FS(G,\mathcal{F})$. Assume that T and T' have been equipped with simplicial metrics, and let $(T_t)_{t\in[0,L]}$ be an optimal liberal folding path joining T to T'. Let $\widehat{T'}$ be a $(G,\mathcal{F})$-free splitting, with a collapse map $\pi : \widehat{T'} \to T'$. For all $t \in [0,L]$, let

$$\mathcal{C}'(T_t, \widehat{T'}) := \{(x,y) \in T_t \times \widehat{T'} | f_{t,L}(x) = \pi(y)\}$$

be the fiber product of the maps $f_{t,L}$ and π, which naturally comes with a structure of $(G,\mathcal{F})$-free splitting, and let $\widehat{T_t}$ be the G-minimal subtree of $\mathcal{C}'(T_t, \widehat{T'})$. Again, we refer to Section F.6.2 of the present paper for a more detailed construction. We also refer to [HM13a, Section 4.3] or [BF14c, Section A.2], where it is presented for F_N-splittings. For all $t, s \in [0,L]$, there is an optimal morphism

$$\begin{array}{cccc} \widehat{f}_{t,s} : & \widehat{T_t} & \to & \widehat{T_s} \\ & (x,y) & \mapsto & (f_{t,s}(x), y) \end{array}.$$

The path $(\widehat{T_t})_{t\in[0,L]}$ is not an optimal liberal folding path in general, because it might not be continuous. As length functions can only decrease along the path, there are (at most) countably many times at which T_t is discontinuous. At those times t, there exists an optimal morphism from the limit T_{t^-} of $(T_s)_{s<t}$, to the limit T_{t^+} of $(T_s)_{s>t}$ (see Section F.6.2 for details). One can insert a (continuous) optimal liberal folding path between T_{t^-} and T_{t^+} at each of these times, all of whose intermediate trees collapse to T_t. By doing

$$\begin{tikzcd}
T_0 \arrow[r] \arrow[d, two heads] & \cdots \arrow[r] \arrow[d, two heads] & T_{t'_0} \arrow[r] \arrow[d, two heads] & \cdots \arrow[r] & T \\
T'_0 \arrow[r] & \cdots \arrow[r] & T'_{t'_0} & & \\
\gamma'(0) \arrow[r] \arrow[u, two heads] & \cdots \arrow[r] \arrow[u, two heads] & \gamma'(t'_0) \arrow[r] \arrow[u, two heads] \arrow[d, equal] & \cdots \arrow[r] & \gamma(L) \\
& & \gamma(t_0) & &
\end{tikzcd}$$

Figure F.4: The projection to an optimal liberal folding path. The bottom horizontal arrows represent the folding path γ, the top horizontal arrows represent a folding path that ends at T, and the vertical arrows represent collapse maps.

so, we get a (continuous) optimal liberal folding path from T'_0 to T'_L. We summarize the above discussion in the following statement.

Proposition F.2.9. *Let $T, T', \widehat{T'} \in FS(G,\mathcal{F})$, equipped with simplicial metrics, such that $\widehat{T'}$ collapses to T'. Let $(T_t)_{t\in[0,L]}$ be an optimal liberal folding path from T to T'. Then there exists a reparameterization $(\gamma(t))_{t\in[0,L']}$ of $(T_t)_{t\in[0,L]}$, and an optimal liberal folding path $(\widehat{\gamma}(t))_{t\in[0,L']}$, such that for all $t \in [0,L']$, the tree $\widehat{\gamma}(t)$ collapses to $\gamma(t)$.*

The path $(\widehat{T}_t)_{t\in[0,L]}$ is called a *pullback* of the path $(T_t)_{t\in[0,L]}$.

Defining the projection. The *projection* $\pi_\gamma(T)$ (see Figure F.4) is the supremum of all times $t_0 \in [0,L]$ such that there exist

- an optimal liberal folding path (T_t) ending at T, and
- an optimal liberal folding path (T'_t), and
- a reparameterization γ' of γ, and
- a real number $t'_0 \in \mathbb{R}$,

such that $\gamma(t_0) = \gamma'(t'_0)$, and $(T'_t)_{t\leq t'_0}$ is a collapse of $(\gamma'(t))_{t\leq t'_0}$, and $(T_t)_{t\leq t'_0}$ is a pullback of $(T'_t)_{t\leq t'_0}$.

End of the argument. Coarse retraction is a consequence of Corollary F.2.6, see [BF14c, Section A.6.3]. Indeed, given a continuous optimal liberal folding path γ, and $t_0 \in [0,L]$, we clearly have $\pi_\gamma(\gamma(t_0)) \geq t_0$ by definition of the projection. If $\pi_\gamma(\gamma(t_0)) = t'_0 > t_0$, then $[\gamma(t_0), \gamma(t'_0)]$ is close to a folding path from a splitting T to $\gamma(t_0)$, where T contains an edge that defines the same splitting as an edge in $\gamma(t_0)$. Corollary F.2.6 then shows that $[\gamma(t_0), \gamma(t'_0)]$ has bounded diameter.

The Lipschitz and strong contraction properties both follow from the distance estimates obtained in Proposition F.2.3, as in [BF14c, Sections A.6.4 and A.6.5]. Basically, given $T_0, T_1, T_2 \in FS(G,\mathcal{F})$, an optimal liberal folding path from T_0 to T_1, and a geodesic segment α joining T_1 to T_2 in $FS(G,\mathcal{F})$, one starts by inductively defining folding paths from T_0 to $\alpha(i)$ for all values of the integer i by using the collapse and pullback constructions. This yields a *big diagram*, which we then progressively contract: thanks to Proposition F.2.3, we might replace successive pullbacks by successive collapses, see [BF14c, Theorem

A.10]. Since this part of the proof is in no way specific to the $(G = F_N, \mathcal{F} = \emptyset)$-case, we refer the reader to Bestvina and Feighn's paper for a verification of Masur and Minsky's axioms for optimal liberal folding paths.

F.3 Hyperbolicity of the graph of $\mathcal{Z}$-splittings

Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. We recall from Section F.1.1 that $\mathcal{Z}^{(max)}$ is the collection of all subgroups of G that are either trivial, or (maximally-)cyclic and nonperipheral. A *$\mathcal{Z}^{(max)}$-splitting* is a minimal, simplicial (hence cocompact) $(G, \mathcal{F})$-tree, all of whose edge stabilizers belong to the collection $\mathcal{Z}^{(max)}$. The *graph of $\mathcal{Z}^{(max)}$-splittings* $FZ^{(max)}(G, \mathcal{F})$ is the graph whose vertices are the equivalence classes of one-edge $\mathcal{Z}^{(max)}$-splittings, two distinct vertices being joined by an edge if the corresponding splittings are compatible (note that if two $\mathcal{Z}^{(max)}$-splittings have a common refinement, then they have a common refinement which is a $\mathcal{Z}^{(max)}$-splitting). Again, there are natural maps $\psi^{(max)} : \mathcal{O}(G, \mathcal{F}) \to FZ^{(max)}(G, \mathcal{F})$, which extend to the set of trees in $\overline{\mathcal{O}(G, \mathcal{F})}$ having a nontrivial simplicial part. The graphs $FZ(G, \mathcal{F})$ and $FZ^{max}(G, \mathcal{F})$ both come with right $\mathrm{Out}(G, \mathcal{F})$-actions, given by precomposition of the actions.

In the case where G is a finitely generated free group and $\mathcal{F} = \emptyset$, hyperbolicity of the graph of $\mathcal{Z}$-splittings was proved by Mann [Man13]. We generalize Mann's proof to the case of free products of groups. We mention in particular that Mann's proof adapts to show the hyperbolicity of the graph of $\mathcal{Z}^{max}$-splittings of a finitely generated free group.

Theorem F.3.1. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. Then $FZ(G, \mathcal{F})$ and $FZ^{max}(G, \mathcal{F})$ are Gromov hyperbolic. Images in $FZ(G, \mathcal{F})$ and in $FZ^{max}(G, \mathcal{F})$ of optimal liberal folding paths between simplicial trees in $\overline{\mathcal{O}(G, \mathcal{F})}$ with trivial edge stabilizers are uniformly Hausdorff close to any geodesic segment joining their endpoints.*

Remark F.3.2. When $G = G_1 * G_2$ and $\mathcal{F} = \{[G_1], [G_2]\}$, all graphs $FZ(G, \mathcal{F})$, $FZ^{max}(G, \mathcal{F})$ and $FS(G, \mathcal{F})$ are equal, and reduced to a point. When $G = G_1 * \mathbb{Z}$ and $\mathcal{F} = \{[G_1]\}$, the graphs $FZ^{max}(G, \mathcal{F})$ and $FS(G, \mathcal{F})$ are equal: all one-edge $\mathcal{Z}^{max}$-splittings are free splittings. The graph $FZ(G, \mathcal{F})$ is also star-shaped: the splitting $G = G_1 *$ is the central vertex, it is joined by an edge to all free splittings of the form $G = G_1 * \langle g_1 t \rangle$, where t denotes the stable letter of the HNN extension, and g_1 varies in G_1. These free splittings are also joined by edges to one-edge $\mathcal{Z}$-splittings of the form $G = (G_1 * \langle (g_1 t)^k \rangle) *_{\langle (g_1 t)^k \rangle} \langle g_1 t \rangle$, with $k \geq 2$. Therefore $FZ(G, \mathcal{F})$ is bounded, of diameter 4.

We denote by $FZ'(G, \mathcal{F})$ (resp. $(FZ^{max})'(G, \mathcal{F})$) the graph whose vertices are one-edge $(G, \mathcal{F})$-free splittings, two splittings being joined by an edge if they are both compatible with a common $\mathcal{Z}$-splitting (resp. $\mathcal{Z}^{max}$-splitting). Since every one-edge $\mathcal{Z}$-splitting is compatible with a one-edge $(G, \mathcal{F})$-free splitting [7, Lemma 5.11], the graphs $FZ'(G, \mathcal{F})$ and $FZ(G, \mathcal{F})$ are quasi-isometric to each other, and similarly $(FZ^{max})'(G, \mathcal{F})$ and $FZ^{max}(G, \mathcal{F})$ are quasi-isometric. Following Mann's proof, we will show hyperbolicity of $FZ'(G, \mathcal{F})$ and $(FZ^{max})'(G, \mathcal{F})$. This will follow from the hyperbolicity of $FS(G, \mathcal{F})$ by applying a criterion due to Kapovich and Rafi, which we now recall, to the natural inclusion maps from $FS(G, \mathcal{F})$ to these graphs.

Proposition F.3.3. *(Kapovich–Rafi [KR14, Proposition 2.5]) For all $\delta_0, M > 0$, there exist $\delta_1, H > 0$ such that the following holds.*

Let X and Y be connected graphs, such that X is δ_0-hyperbolic. Let $f : X \to Y$ be a map sending $V(X)$ onto $V(Y)$, and sending edges to edges. Assume that for all $x, y \in V(X)$, *if $d_Y(f(x), f(y)) \leq 1$, then the f-image of any geodesic segment joining x to y in X has diameter bounded by M in Y. Then Y is δ_1-hyperbolic, and for any $x, y \in V(X)$, the f-image of any geodesic segment joining x to y in X is H-Hausdorff close to any geodesic segment joining $f(x)$ to $f(y)$ in Y.*

Proof of Theorem F.3.1. Let T_1 and T_2 be two one-edge $(G,\mathcal{F})$-free splittings, both compatible with a one-edge $\mathcal{Z}^{(max)}$-splitting T. For simplicity of notations, we consider the case where the quotient graphs T_1/G, T_2/G and T/G are segments. The case of loop-edges is left to the reader, as the argument is similar, and similar to that in the proof of [Man13, Theorem 5]. Then T is of the form $A *_{\langle w\rangle} B$. Without loss of generality, we can assume that there exist two free splittings of B of the form $B = B_1 * B_1'$ and $B = B_2 * B_2'$, such that for all $i \in \{1,2\}$, the splitting $T + T_i$ (with the notation from Section F.4.6 below) is of the form $A *_{\langle w\rangle} B_i * B_i'$. Indeed, otherwise, the trees T_1 and T_2 are compatible, in which case they are already at distance 1 in $FS(G,\mathcal{F})$.

By blowing up the vertex groups of the splitting $T + T_1$, using their action on $T + T_2$ (which is possible because $T + T_1$ and $T + T_2$ have the same edge stabilizers), we get a tree $\widehat{T_1}$ that collapses to $T + T_1$, and comes with a morphism $f : \widehat{T_1} \to T + T_2$. We denote by $\widehat{p_1} : \widehat{T_1} \to T$ and $p_2 : T + T_2 \to T$ the natural alignement-preserving maps. The B-minimal subtree of $\widehat{T_1}$ is mapped by f to the B-minimal subtree of $T + T_2$, so $\widehat{p_1} = p_2 \circ f$. Using Lemma F.6.9 below, we see that all trees T_t on an optimal liberal folding path guided by f collapse to T. By equivariantly collapsing the edge with stabilizer $\langle w\rangle$ to a point in T_t, we get an optimal liberal folding path γ from T_1 to T_2, whose ψ-image stays at bounded distance from T in $FZ^{(max)}(G,\mathcal{F})$.

Recall that $FS(G,\mathcal{F})$ is Gromov hyperbolic, and ϕ-images of optimal liberal folding paths between simplicial trees with trivial edge groups are uniform reparameterized quasi-geodesics (Theorem F.2.1). Therefore, any geodesic from T_1 to T_2 in $FS(G,\mathcal{F})$ is uniformly close to the folding path γ, with constants depending only on the hyperbolicity constant of $FS(G,\mathcal{F})$. Hence there is a constant M such that the diameter of the f-image of any geodesic segment joining T_1 to T_2 in $FS(G,\mathcal{F})$ is bounded by M in $FZ'(G,\mathcal{F})$. By choosing for T a $\mathcal{Z}^{max}$-splitting, the same holds true for $(FZ^{max})'(G,\mathcal{F})$. □

F.4 More material

The following sections of the paper aim at describing the Gromov boundaries of the graph $FZ(G,\mathcal{F})$ and $FZ^{max}(G,\mathcal{F})$. We first introduce more background material that will be used in the proof of our main theorem.

F.4.1 Tame $(G,\mathcal{F})$-trees

In this section, we review the definition and the properties of the class of tame $(G,\mathcal{F})$-trees, introduced in [7, Section 6], in which we will carry out most of our arguments.

Definition and properties. Let T be a $(G,\mathcal{F})$-tree. A point $x \in T$ is a *branch point* if $T \smallsetminus \{x\}$ has at least three connected components. It is an *inversion point* if $T \smallsetminus \{x\}$ has exactly two connected components, and there exists $g \in G$ that exchanges these two components.

Definition F.4.1. *A minimal $(G,\mathcal{F})$-tree is* small *if its arc stabilizers belong to the class $\mathcal{Z}$. It is* tame *if in addition, it has finitely many orbits of directions at branch or inversion* *points. A* $\mathcal{Z}^{max}$-tame tree *is a tame* $(G,\mathcal{F})$*-tree whose arc stabilizers belong to the class* $\mathcal{Z}^{max}$*.*

Tame $(G,\mathcal{F})$-trees also have the following alternative description. For all $k \in \mathbb{N}$, we say that a small $(G,\mathcal{F})$-tree T is *k-tame* if for all nonperipheral $g \in G$, all arcs $I \subseteq T$, and all $l \geq 1$, if $g^l I = I$, then $g^k I = I$. Equivalently, a small $(G,\mathcal{F})$-tree is k-tame if for all nonperipheral $g \in G$, and all $l \in \mathbb{N}$, we have $\mathrm{Fix}(g^k) = \mathrm{Fix}(g^{kl})$. Notice that being 1-tame is equivalent to being $\mathcal{Z}^{max}$-tame. Notice also that if a $(G,\mathcal{F})$-tree T is k-tame, then it is also kl-tame for all $l \geq 1$. In particular, in view of the proposition below, if T and T' are two tame $(G,\mathcal{F})$-trees, then there exists $k \in \mathbb{N}$ such that both T and T' are k-tame.

Proposition F.4.2. *(Horbez [7, Proposition 6.5]) Let T be a minimal $(G,\mathcal{F})$-tree. Then T is tame if and only if there exists $k \in \mathbb{N}$ such that T is k-tame.*

In [7, Corollary 4.5], we proved that trees in $\overline{\mathcal{O}(G,\mathcal{F})}$ have finitely many G-orbits of directions at branch or inversion points, so trees in $\overline{\mathcal{O}(G,\mathcal{F})}$ are $\mathcal{Z}^{max}$-tame trees. The converse is not true in general, because $\mathcal{Z}^{max}$-tame trees are not required to have trivial tripod stabilizers. However, tame $(G,\mathcal{F})$-trees with dense G-orbits have trivial arc stabilizers [7, Proposition 4.17]. So tame $(G,\mathcal{F})$-trees with dense orbits belong to $\overline{\mathcal{O}(G,\mathcal{F})}$.

Recall that the space of minimal $(G,\mathcal{F})$-trees is equipped with the axes topology. The subspace consisting of small $(G,\mathcal{F})$-trees is closed ([CM87, 5.3], [Pau88, Lemme 5.7], [7, Proposition 3.1]), and for all $k \in \mathbb{N}$, the subspace consisting of k-tame $(G,\mathcal{F})$-trees is closed [7, Proposition 6.4]. However, the subspace consisting of all tame $(G,\mathcal{F})$-trees is not: for example, a sequence of splittings of $F_2 = \langle a,b\rangle$ of the form $F_2 = (\langle a\rangle *_{\langle a^2\rangle} \langle a^2\rangle *_{\langle a^4\rangle} \cdots *_{\langle a^{2^n}\rangle} \langle a^{2^n}\rangle) * \langle b\rangle$, in which the edge with stabilizer generated by a^{2^k} has length $\frac{1}{2^k}$, does not converge to a tame F_2-tree. The following proposition gives a condition under which a limit of tame trees is tame.

Proposition F.4.3. *(Horbez [7, Proposition 6.7]) Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of one-edge $\mathcal{Z}$-splittings that converges (projectively) in the axes topology to a minimal $(G,\mathcal{F})$-tree T. Then T is tame.*

Tame optimal folding paths and sequences. An optimal liberal folding path $(T_t)_{t\in\mathbb{R}_+}$ is *tame* (resp. *k-tame*) if for all $t \in \mathbb{R}_+$, the tree T_t is tame (resp. k-tame). Similarly, an optimal folding sequence $(T_n)_{n\in\mathbb{N}}$ is *tame* if T_n is tame for all $n \in \mathbb{N}$. We recall from Section F.1.3 the existence of a canonical optimal liberal folding path associated to any optimal morphism between two $(G,\mathcal{F})$-trees T and T'.

Proposition F.4.4. *Let T and T' be two tame $(G,\mathcal{F})$-trees, and let $f : T \to T'$ be an optimal morphism. Then the canonical optimal folding path γ guided by f is tame. More precisely, for all $k \in \mathbb{N}$, if T and T' are k-tame, then γ is k-tame.*

Proof. Let $g \in G$ be a nonperipheral element, let $l \geq 1$, and let K_t be the fixed point set of g^{kl} in T_t. We want to show that g^k also fixes K_t. Let $a_t \in K_t$, and let a be a preimage of a_t in T. By definition of $\sim_t$ (see Section F.1.3 for notations), we have $g^{kl}f(a) = f(a)$ and $\tau(a, g^{kl}a) \leq t$. As T' is k-tame, this implies that $g^k f(a) = f(a)$. We claim that $\tau(a, g^k a) = \tau(a, g^{kl}a)$. This will imply that $g^k a_t = a_t$, and therefore g^k fixes K_t pointwise.

First assume that g is hyperbolic in T. The segment $[a, g^{kl}a] \subseteq T$ decomposes as $[a,a'] \cup [a', g^k a'] \cup g^k[a', g^k a'] \cup \cdots \cup g^{k(l-1)}[a', g^k a'] \cup g^{kl}[a', a]$. As $g^k f(a) = f(a)$, by equivariance of f, the supremum of the distance between $f(a)$ and a point in the f-image of either $[a, g^{kl}a]$ or $[a, g^k a]$ is achieved by a point in the f-image of $[a, g^k a']$. This implies that $\tau(a, g^k a) = \tau(a, g^{kl}a)$.

Now assume that g is elliptic in T. Then $[a, g^{kl}a]$ decomposes as $[a, g^{kl}a] = [a, a'] \cup [a', g^{kl}a]$, where a' is the point in $\mathrm{Fix}_T(g^k) = \mathrm{Fix}_T(g^{kl})$ closest to a. Since $g^{kl}f(a) = f(a)$, the supremum of the distance between $f(a)$ and a point in the f-image of $[a, g^{kl}a]$ is achieved by a point in the f-image of $[a, a']$. This implies that $\tau(a, g^k a) = \tau(a, g^{kl}a)$. $\square$

Using the existence of an optimal morphism from a point in the closed cone of T_0 (i.e. the set of trees in $\overline{\mathcal{O}(G,\mathcal{F})}$ obtained from T_0 by varying some of the edge lengths, and possiby collapsing some edges to points) to T (see the arguments in [FM14], for instance), we deduce the following fact.

Proposition F.4.5. *Let T be a tame $(G,\mathcal{F})$-tree, and let $T_0 \in \mathcal{O}(G,\mathcal{F})$. Then there exists a tame optimal liberal folding path from a point in the closed cone of T_0 to T. In particular, there exists a tame optimal folding sequence ending at T.* $\square$

F.4.2 Metric properties of $\mathcal{O}(G,\mathcal{F})$

We review work by Francaviglia and Martino [FM14]. Let G be a countable group, and $\mathcal{F}$ be a free factor system of G. For all $T, T' \in \mathcal{O}(G,\mathcal{F})$, we denote by $\mathrm{Lip}(T,T')$ the infimal Lipschitz constant of an equivariant map from T to T'. Let $T \in \mathcal{O}(G,\mathcal{F})$. An element $g \in G$ is a *candidate* in T if it is hyperbolic in T and, denoting by $C_T(g)$ its translation axis in T, there exists $v \in C_T(g)$ such that the segment $[v, gv]$ projects to a loop γ in the quotient graph $X := T/G$ which is either

- an embedded loop, or
- a bouquet of two circles in X, i.e. $\gamma = \gamma_1\gamma_2$, where γ_1 and γ_2 are embedded circles in X which meet in a single point, or
- a barbell graph, i.e. $\gamma = \gamma_1\eta\gamma_2\overline{\eta}$, where γ_1 and γ_2 are embedded circles in X that do not meet, and η is an embedded path in X that meets γ_1 and γ_2 only at their origin (and $\overline{\eta}$ denotes the path η crossed in the opposite direction), or
- a *simply-degenerate barbell*, i.e. γ is of the form $u\eta\overline{\eta}$, where u is an embedded loop in X and η is an embedded path in X, with two distinct endpoints, which meets u only at its origin, and whose terminal endpoint is a vertex in X with nontrivial stabilizer, or
- a *doubly-degenerate barbell*, i.e. γ is of the form $\eta\overline{\eta}$, where η is an embedded path in X whose two distinct endpoints have nontrivial stabilizer,

see Figure F.5 for a representation of the possible shapes of candidate loops. Given $T \in \mathcal{O}(G,\mathcal{F})$, we denote by $\mathrm{Cand}(T)$ the (infinite) set of all elements in G which are candidates in T.

Theorem F.4.6. *(Francaviglia–Martino [FM14, Theorem 9.10]) For all $T, T' \in \mathcal{O}(G,\mathcal{F})$, we have*

$$Lip(T,T') = \sup_{g \in Cand(T)} \frac{||g||_{T'}}{||g||_T}.$$

In addition, there exists a tree $\overline{T} \in \mathcal{O}(G,\mathcal{F})$ onto which T admits a $Lip(T,T')$-Lipschitz alignment-preserving map, together with an optimal morphism from $\overline{T}$ to T'.

Building on Francaviglia and Martino's theorem, we show the following result.

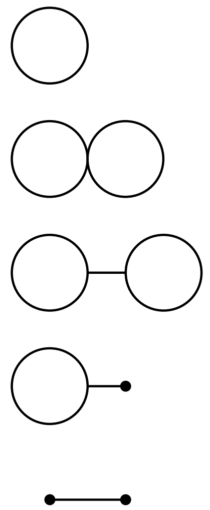

Figure F.5: Shapes of candidate loops.

Theorem F.4.7. *For all $T \in \mathcal{O}(G,\mathcal{F})$, there exists a finite set $X(T) \subseteq Cand(T)$ such that for all $T' \in \mathcal{O}(G,\mathcal{F})$, we have*

$$Lip(T,T') = \sup_{g \in X(T)} \frac{||g||_{T'}}{||g||_T}.$$

Let $T, T' \in \mathcal{O}(G,\mathcal{F})$, and $f : T \to T'$ be an optimal morphism. An element $g \in G$ is *legal* for f if it is hyperbolic in T, and if its axis is legal for f. The *tension graph* of f is the set of edges of T that are maximally strecthed by f. Francaviglia and Martino's proof of Theorem F.4.6 shows that there exists an optimal morphism $f : T \to T'$, and there exists $g \in G$ which is legal for f, and whose axis is contained in the tension graph of f. In addition, such an element $g \in G$ can be chosen to be a candidate in T. Theorem F.4.6 follows, because every such element maximizes the stretch factor from T to T'.

Proof of Theorem F.4.7. The set $Y(T)$ of possible projections of axes of candidates in T to the quotient graph T/G is finite. By Theorem F.4.6, it is thus enough to show that from the set of all candidates in T whose projections are equal to some $\gamma \in Y(T)$, we can extract a finite subset $X(\gamma)$ so that for all $T' \in \mathcal{O}(G,\mathcal{F})$, and all optimal morphisms $f : T \to T'$, at least one element in $X(\gamma)$ is legal for f. This follows from the observation that for every pair (e, e') of adjacent vertices in T whose common vertex v has nontrivial stabilizer G_v, and any two distinct elements $g, g' \in G_v$, either (e, ge') or $(e, g'e')$ is legal for f, since otherwise $g'g^{-1}$ would fix a nondegenerate arc in T'. In addition, any loop in $Y(T)$ crosses boundedly many vertices in T/G. ☐

F.4.3 Lipschitz approximations of trees

Let $T \in \overline{\mathcal{O}(G,\mathcal{F})}$. A *Lipschitz approximation* of T is a sequence $(T_n)_{n\in\mathbb{N}} \in \overline{\mathcal{O}(G,\mathcal{F})}^{\mathbb{N}}$ converging (non-projectively) to T such that for all $n \in \mathbb{N}$, there exists a 1-Lipschitz map from T_n to T. The following proposition follows from [7, Theorem 5.3].

Proposition F.4.8. *(Horbez [7, Theorem 5.3]) Every tree $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ with dense orbits admits a Lipschitz approximation by Grushko $(G,\mathcal{F})$-trees.*

Proposition F.4.9. *Let $S,T \in \overline{\mathcal{O}(G,\mathcal{F})}$, let $(S_i)_{i\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$ be a Lipschitz approximation of S, and let $(T_j)_{j\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$ be a sequence that converges (non-projectively) to T. Assume in addition that there exists a 1-Lipschitz map from S to T. Then for all $i \in \mathbb{N}$, there exists $J_i \in \mathbb{N}$ so that for all $j \geq J_i$, we have $Lip(S_i,T_j) \leq 1+\frac{1}{i}$.*

Proof. Let $i \in \mathbb{N}$. As $(T_j)_{j\in\mathbb{N}}$ converges non-projectively to T, there exists $J_i \in \mathbb{N}$ so that for all $j \geq J_i$, all elements g in the finite set $X(S_i)$ provided by Theorem F.4.7 have translation length at most $(1+\frac{1}{i})||g||_T \leq (1+\frac{1}{i})||g||_S \leq (1+\frac{1}{i})||g||_{S_i}$ in T_j. The claim then follows from Theorem F.4.7. ☐

F.4.4 Alignment-preserving maps

A map $f : T \to T'$ is *alignment-preserving* if the f-image of every segment in T is a segment in T'. We note that alignment-preserving maps are not assumed to be continuous. However, any alignment-preserving map is continuous in restriction to every segment of T, and more generally in restriction to every finite subtree of T. If there exists an alignment-preserving map from T to T', we say that T *collapses* to T'. The following lemma states a few basic topological properties of alignment-preserving maps. Its proof is left to the reader.

Lemma F.4.10. *Let T and $\widehat{T}$ be two $(G,\mathcal{F})$-trees. Let $p : \widehat{T} \to T$ be a surjective alignment-preserving map. Then the p-preimage of every closed subtree in T is a closed subtree of $\widehat{T}$. The p-image of every closed subtree of $\widehat{T}$ is a closed subtree of T.* ☐

F.4.5 Limits of folding paths

The goal of this section is to prove Proposition F.4.14, which will be used in Section F.8, and gives information about possible limit points in $\overline{\mathcal{O}(G,\mathcal{F})}$ of some folding paths in $\mathcal{O}(G,\mathcal{F})$. In the following statement, the last assertion about alignment-preserving maps is an immediate consequence of our description in [3] of the map f, which is obtained from an ultralimit of the maps f_n by projecting to the minimal subtree.

Proposition F.4.11. *(Horbez [3, Theorem 4.4]) Let T and T' be tame $(G,\mathcal{F})$-trees, let $(T_n)_{n\in\mathbb{N}}$ (resp. $(T'_n)_{n\in\mathbb{N}}$) be a sequence of tame $(G,\mathcal{F})$-trees that converges to T (resp. T'), and let $M \in \mathbb{R}$. Assume that for all $n \in \mathbb{N}$, there exists an M-Lipschitz map from T_n to T'_n. Then there exists an M-Lipschitz map from T to the metric completion of T'. If in addition, all maps f_n are alignment-preserving, then f can be chosen to be alignment-preserving.*

Let $T,T' \in \overline{\mathcal{O}(G,\mathcal{F})}$, and $f : T \to T'$ be a map. The *bounded cancellation constant* of f, denoted by $BCC(f)$, is defined to be the supremum of all real numbers B with the property that there exist $a,b,c \in T$ with $b \in [a,c]$, such that $d_{T'}(f(b),[f(a),f(c)]) = B$. Note that a map $f : T \to T'$ is alignment-preserving if and only if $BCC(f) = 0$. We denote by $\mathrm{Lip}(f)$ the Lipschitz constant of f, and by $qvol(T)$ the quotient volume of T, defined as the infimal volume of a finite subtree of T whose G-translates cover T (the existence of such a tree was proved by Guirardel in [Gui08, Lemma 1.14]). The following proposition is a generalization of [BFH97, Lemma 3.1].

Proposition F.4.12. *Let $T \in \mathcal{O}(G,\mathcal{F})$, let $T' \in \overline{\mathcal{O}(G,\mathcal{F})}$, and let $f : T \to T'$ be a Lipschitz map. Then $BCC(f) \leq Lip(f)qvol(T)$.*

Sketch of proof. In the case where T' is a Grushko $(G,\mathcal{F})$-tree, the statement follows by decomposing f into Stallings' folds (see the proof of [Gui98, Proposition 9.6]). The claim is then proved by approximating T' by trees in $\mathcal{O}(G,\mathcal{F})$. $\square$

As in [3, Corollary 3.9], the following fact is a corollary of Propositions F.4.8 and F.4.12.

Corollary F.4.13. *(Horbez [3, Corollary 3.9]) Let $T,T' \in \overline{\mathcal{O}(G,\mathcal{F})}$ have dense orbits. Then any Lipschitz map from T to the metric completion of T' preserves alignment (and hence takes its values in T'). In particular, every morphism from T to T' is an isometry.* $\square$

Proposition F.4.14. *Let $S,T \in \overline{\mathcal{O}(G,\mathcal{F})}$ be two trees with dense orbits. Let $(S_i)_{i\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$ (resp. $(T_i)_{i\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$) be a sequence that converges (non-projectively) to S (resp. to T). Assume that S admits a 1-Lipschitz alignment-preserving map onto T, and that for all $i \in \mathbb{N}$, we have $Lip(S_i,T_i) \leq 1+\frac{1}{i}$. Then there exists an optimal liberal folding path γ_i from the open cone of S_i to T_i for all $i \in \mathbb{N}$, so that all sequences $(Z_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} Im(\gamma_i)$ have nontrivial accumulation points in $\overline{\mathcal{O}(G,\mathcal{F})}$, and all such accumulation points Z come with 1-Lipschitz alignment-preserving maps from S to Z and from Z to T.*

Proof. Theorem F.4.6 yields the existence for all $i \in \mathbb{N}$ of a tree $S_i' \in \mathcal{O}(G,\mathcal{F})$, obtained from S_i by rescaling the lengths of the edges by a factor bounded above by $1+\frac{1}{i}$, such that there exist optimal liberal folding paths from S_i' to T_i. For all $i \in \mathbb{N}$, let $Z_i \in \mathcal{O}(G,\mathcal{F})$ be a tree that lies on a liberal folding path from S_i' to T_i. There are 1-Lipschitz maps from $(1+\frac{1}{i})S_i$ to Z_i and from Z_i to T_i. In particular, for any accumulation point Z of the sequence $(Z_i)_{i\in\mathbb{N}}$, Proposition F.4.11 yields the existence of 1-Lipschitz maps from S to the metric completion of Z and from Z to the metric completion of T (in particular, the set of accumulation points contains nontrivial G-trees). Corollary F.4.13 implies that these maps are alignment-preserving, and take values in Z and T (without passing to the completion). $\square$

F.4.6 Refinements of metric trees

Let T_1 and T_2 be two *compatible* $(G,\mathcal{F})$-trees, i.e. there exists a $(G,\mathcal{F})$-tree $\widehat{T}$ and alignment-preserving maps $g_i : \widehat{T} \to T_i$ for all $i \in \{1,2\}$. Then T_1 and T_2 have a *standard common refinement*, defined as follows [GL10b, Section 3.2]. For all $i \in \{1,2\}$, denote by d_i the metric on T_i, and by l_i the associated length function, and for all $x,y \in \widehat{T}$, let

$$\delta(x,y) := d_1(g_1(x),g_1(y)) + d_2(g_2(x),g_2(y)).$$

This defines a pseudometric on $\widehat{T}$, which satisfies $\delta(x,y) = \delta(x,z) + \delta(z,y)$ whenever $z \in [x,y]$. The metric space T_s obtained by making this pseudometric Hausdorff is a $(G,\mathcal{F})$-tree, which admits a 1-Lipschitz alignment-preserving map $f_i : T_s \to T_i$ for all $i \in \{1,2\}$, such that

$$d_{T_s}(x,y) = d_1(f_1(x),f_1(y)) + d_2(f_2(x),f_2(y)).$$

Arc stabilizers of T_s fix an arc in either T_1 or T_2, and if T_1 and T_2 are k-tame, then so is T_s. Since

$$||g||_{T_s} = \lim_{n\to+\infty} \frac{1}{n} d_{T_s}(x,g^n x)$$

for all $x \in T_s$, it follows that the length function of T_s is the sum of the length functions of T_1 and T_2. We will denote $T_s =: T_1 + T_2$.

Lemma F.4.15. *(Guirardel–Levitt [GL10b, Corollary 3.9]) Let S and T be two $(G,\mathcal{F})$-trees. Let $(S_i)_{i\in\mathbb{N}}$ (resp. $(T_i)_{i\in\mathbb{N}}$) be a sequence of trees that converges in the axes topology to S (resp. to T). If S_i is compatible with T_i for all $i\in\mathbb{N}$, then S is compatible with T.*

F.4.7 Transverse families, transverse coverings and graphs of actions

Let T be a $(G,\mathcal{F})$-tree. A *transverse family* in T is a G-invariant collection $\mathcal{Y}$ of nondegenerate subtrees of T such that for all $Y\neq Y'\in\mathcal{Y}$, the intersection $Y\cap Y'$ contains at most one point. A *transverse covering* of T is a transverse family $\mathcal{Y}$ in T, all of whose elements are closed subtrees of T, such that every finite arc in T can be covered by finitely many elements of $\mathcal{Y}$. A transverse covering $\mathcal{Y}$ of T is *trivial* if $\mathcal{Y}=\{T\}$. The *skeleton* of a transverse covering $\mathcal{Y}$ is the bipartite simplicial tree S, whose vertex set is $V(S)=V_0(S)\cup\mathcal{Y}$, where $V_0(S)$ is the set of points of T which belong to at least two distinct trees in $\mathcal{Y}$, with an edge between $x\in V_0(S)$ and $Y\in\mathcal{Y}$ whenever $x\in Y$ [Gui04, Definition 4.8].

Lemma F.4.16. *(Guirardel [Gui08, Lemmas 1.5 and 1.15]) Let T be a minimal $(G,\mathcal{F})$-tree, and let $\mathcal{Y}$ be a transverse covering of T. Then the skeleton of $\mathcal{Y}$ is a minimal $(G,\mathcal{F})$-tree which is compatible with T.*

A $(G,\mathcal{F})$*-graph of actions* consists of

- a marked metric graph of groups $\mathcal{G}$ (in which we might allow some edges to have length 0), whose fundamental group is isomorphic to G, such that all subgroups in $\mathcal{F}$ are conjugate into vertex groups of $\mathcal{G}$, and
- an isometric action of every vertex group G_v on a G_v-tree T_v (possibly reduced to a point), in which all intersections of G_v with peripheral subgroups of G are elliptic, and
- a point $p_e\in T_{t(e)}$ fixed by $i_e(G_e)\subseteq G_{t(e)}$ for every oriented edge e.

It is *nontrivial* if $\mathcal{G}$ is not reduced to a point. Associated to any $(G,\mathcal{F})$-graph of actions $\mathcal{G}$ is a G-tree $T(\mathcal{G})$. Informally, the tree $T(\mathcal{G})$ is obtained from the Bass–Serre tree of the underlying graph of groups by equivariantly attaching each vertex tree T_v at the corresponding vertex v, an incoming edge being attached to T_v at the prescribed attaching point. The reader is referred to [Gui98, Proposition 3.1] for a precise description of the tree $T(\mathcal{G})$. We say that a $(G,\mathcal{F})$-tree T *splits as a* $(G,\mathcal{F})$*-graph of actions* if there exists a $(G,\mathcal{F})$-graph of actions $\mathcal{G}$ such that $T=T(\mathcal{G})$.

Proposition F.4.17. *(Guirardel [Gui08, Lemma 1.5]) A $(G,\mathcal{F})$-tree splits as a nontrivial $(G,\mathcal{F})$-graph of actions if and only if it admits a nontrivial transverse covering. The skeleton of any transverse covering of T is compatible with T.*

Knowing that a $(G,\mathcal{F})$-tree T is compatible with a simplicial $(G,\mathcal{F})$-tree S gives a way of splitting T as a $(G,\mathcal{F})$-graph of actions, in the following way. Let $\pi_T:T+S\to T$ and $\pi_S:T+S\to S$ be the natural alignment-preserving maps.

We first claim that the family $\mathcal{Y}$ made of all nondegenerate π_S-preimages of vertices of S, and of the closures of π_S-preimages of open edges of S, is a transverse covering of $T+S$. Indeed, these are closed subtrees of $T+S$ (see Lemma F.4.10), whose pairwise intersections are degenerate (i.e either empty, or reduced to a point). In addition, let $I\subseteq T+S$ be a segment. Then $\pi_S(I)$ is a segment in S, so it is covered by a finite set of open edges and vertices of S. The π_S-preimages in $T+S$ of these edges and vertices cover I, which proves that I is covered by finitely many elements of the family $\mathcal{Y}$.

We claim that the family consisting of nondegenerate subtrees in $\pi_T(\mathcal{Y})$ is a nontrivial transverse covering of T. Indeed, since π_T is alignment-preserving, this is a transverse family made of closed subtrees of T (Lemma F.4.10). If $I \subseteq T$ is a segment, then there is a segment $J \subseteq T + S$ with $\pi_T(J) = I$. Then J is covered by finitely many elements of $\mathcal{Y}$, and I is covered by the π_T-images of these elements. The family $\pi_T(\mathcal{Y})$ is nontrivial, because otherwise $\mathcal{Y}$ would also be trivial, and hence $\pi_S(T + S)$ would be contained in a vertex or a closed edge of S, a contradiction.

We finish this section by mentioning a result due to Levitt [Lev94], which gives a canonical way of splitting any tame $(G, \mathcal{F})$-tree as a graph of actions, whose vertex actions have dense orbits. The key point in the proof of Proposition F.4.18 is finiteness of the number of orbits of branch points in any tame $(G, \mathcal{F})$-tree.

Proposition F.4.18. *(Levitt [Lev94]) Every tame $(G, \mathcal{F})$-tree T splits uniquely as a graph of actions, all of whose vertex trees have dense orbits for the action of their stabilizer (they might be reduced to points), and all of whose edges have positive length, and have a stabilizer that belongs to the class $\mathcal{Z}$.*

We call this decomposition the *Levitt decomposition* of T as a graph of actions. Proposition F.4.18 gives a natural way of extending the map $\psi : \mathcal{O}(G, \mathcal{F}) \to FZ(G, \mathcal{F})$ to the set of tame $(G, \mathcal{F})$-trees without dense orbits.

F.5 $\mathcal{Z}$-averse trees

We now introduce the notion of *$\mathcal{Z}$-averse* trees. These will be the trees lying *at infinity* of the complex $FZ(G, \mathcal{F})$. Most arguments work exactly the same way when working with $\mathcal{Z}^{max}$-splittings instead of $\mathcal{Z}$-splittings, we will mention the places where some slight adaptations are required. The case of $\mathcal{Z}^{max}$-splittings will be treated in Section F.5.5.

F.5.1 Definition

Given a tame $(G, \mathcal{F})$-tree T, we denote by $\mathcal{R}^1(T)$ the set of $\mathcal{Z}$-splittings that are compatible with T, and by $\mathcal{R}^2(T)$ the set of $\mathcal{Z}$-splittings that are compatible with a tame $(G, \mathcal{F})$-tree T', which is itself compatible with T. We say that T is *$\mathcal{Z}$-incompatible* if $\mathcal{R}^1(T) = \emptyset$, and *$\mathcal{Z}$-compatible* otherwise.

Theorem F.5.1. *For all tame $(G, \mathcal{F})$-trees T, the following assertions are equivalent.*

- *There exists a finite sequence $(T = T_0, T_1, \dots, T_k = S)$ of tame $(G, \mathcal{F})$-trees, such that S is simplicial, and for all $i \in \{0, \dots, k-1\}$, the trees T_i and T_{i+1} are compatible.*
- *We have $\mathcal{R}^2(T) \neq \emptyset$.*
- *The tree T collapses to a tame $\mathcal{Z}$-compatible $(G, \mathcal{F})$-tree.*

The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are obvious. The proof that (1) implies (3) is postponed to Section F.5.3. We note that the conditions in Theorem F.5.1 are not equivalent to being $\mathcal{Z}$-incompatible, see Example F.5.30, which is why we really need to introduce the set $\mathcal{R}^2(T)$ in our arguments. We will see however in Proposition F.5.3 that mixing $\mathcal{Z}$-incompatible trees satisfy $\mathcal{R}^2(T) = \emptyset$.

Definition F.5.2.

- *A tree $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ is* $\mathcal{Z}$-averse *if any of the conditions in Theorem F.5.1 fails. We denote by $\mathcal{X}(G,\mathcal{F})$ the subspace of $\overline{\mathcal{O}(G,\mathcal{F})}$ consisting of $\mathcal{Z}$-averse trees.*
- *Two $\mathcal{Z}$-averse trees $T,T' \in \mathcal{X}(G,\mathcal{F})$ are* equivalent*, which we denote by $T \sim T'$, if there exists a finite sequence $(T = T_0, T_1, \dots, T_k = T')$ of trees in $\overline{\mathcal{O}(G,\mathcal{F})}$ such that for all $i \in \{0,\dots,k-1\}$, the trees T_i and T_{i+1} are compatible.*
- *Given a tame $(G,\mathcal{F})$-tree T, the set $\mathcal{R}^2(T)$ is called the set of* reducing splittings *of T.*

Note that $\mathcal{Z}$-averse trees are $\mathcal{Z}$-incompatible, so in particular they have dense orbits. A tree $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ is *mixing* [Mor88] if for all finite subarcs $I,J \subseteq T$, there exist $g_1,\dots,g_k \in G$ such that $J \subseteq g_1I \cup \dots \cup g_kI$ and for all $i \in \{1,\dots,k-1\}$, we have $g_iI \cap g_{i+1}I \neq \emptyset$. We will show the existence of a canonical simplex of mixing representatives in any equivalence class of $\mathcal{Z}$-averse trees. Two $\mathbb{R}$-trees T and T' are *weakly homeomorphic* if there exist maps $f : T \to T'$ and $g : T' \to T$ that are continuous in restriction to segments, and inverse of each other. Again, the proof of Proposition F.5.3 is postponed to Section F.5.3.

Proposition F.5.3. *For all $T \in \mathcal{X}(G,\mathcal{F})$, there exists a mixing tree $\overline{T} \in \mathcal{X}(G,\mathcal{F})$ onto which all trees $T' \in \mathcal{X}(G,\mathcal{F})$ that are equivalent to T collapse. In addition, any two such trees are G-equivariantly weakly homeomorphic. Any tree $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ that is both mixing and $\mathcal{Z}$-incompatible is $\mathcal{Z}$-averse.*

F.5.2 Unboundedness of $FZ(G,\mathcal{F})$

Luo's argument

We now explain that $\mathcal{Z}$-averse trees lie *at infinity* of $FZ(G,\mathcal{F})$ in some sense. In particular, we show that $\mathcal{X}(G,\mathcal{F})$ is unbounded (except in the two sporadic cases mentioned in Remark F.3.2, for which we have $\mathcal{X}(G,\mathcal{F}) = \emptyset$). The following theorem is a variation over an argument due to Luo to prove unboundedness of the curve complex of a compact surface, which first appeared in [Kob88]. We recall our notation ψ for the map from $\mathcal{O}(G,\mathcal{F})$ to $FZ(G,\mathcal{F})$.

Theorem F.5.4. *Let $T \in \mathcal{X}(G,\mathcal{F})$, and let $(T_i)_{i\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$ be a sequence that converges to T. Then $\psi(T_i)$ is unbounded in $FZ(G,\mathcal{F})$.*

Proof. Assume towards a contradiction that the sequence $(\psi(T_i))_{i\in\mathbb{N}}$ lies in a bounded region of $FZ(G,\mathcal{F})$. Up to passing to a subsequence, there exist $M \in \mathbb{N}$ and $* \in FZ(G,\mathcal{F})$ such that for all $i \in \mathbb{N}$, we have $d_{FZ(G,\mathcal{F})}(*,\psi(T_i)) = M$. For all $i \in \mathbb{N}$, let $(T_i^k)_{0\leq k\leq M}$ be a geodesic segment joining $*$ to $\psi(T_i)$ in $FZ(G,\mathcal{F})$. Up to passing to a subsequence again and rescaling, we may assume that for all $k \in \{0,\dots,M\}$, the sequence $(T_i^k)_{i\in\mathbb{N}}$ of one-edge splittings converges (non-projectively) to a tame $(G,\mathcal{F})$-tree T_∞^k (Proposition F.4.3). In addition, for all $k \in \{1,\dots,M\}$ and all $i \in \mathbb{N}$, the trees T_i^k and T_i^{k-1} are compatible. Lemma F.4.15 implies that for all $k \in \{1,\dots,M\}$, the trees T_∞^k and T_∞^{k-1} are compatible, and T is compatible with T_∞^M. As $T_\infty^0 = *$, the tree T does not satisfy the first definition of $\mathcal{Z}$-averse trees, a contradiction. ☐

Remark F.5.5. In the case of $\mathcal{Z}^{max}$-splittings, the argument is even a bit simpler, because in this case, we know that all trees T_i^k, and hence all limits T_∞^k, belong to the closure of outer space (i.e. they are very small). Therefore, we can avoid to appeal to our analysis of tame $(G,\mathcal{F})$-trees (in particular Proposition F.4.3) in Section F.4.1.

Examples of $\mathcal{Z}$-averse trees, and unboundedness of $FZ(G,\mathcal{F})$

We now give examples of $\mathcal{Z}$-averse trees in $\overline{\mathcal{O}(G,\mathcal{F})}$ when either $G = F_2$ and $\mathcal{F} = \emptyset$, or $\mathrm{rk}_K(G,\mathcal{F}) \geq 3$. In view of Theorem F.5.4, this will prove unboundedness of the graph $FZ(G,\mathcal{F})$ in these cases. Recall from Remark F.3.2 that if either $G = G_1 * G_2$ and $\mathcal{F} = \{[G_1],[G_2]\}$, or $G = G_1*$ and $\mathcal{F} = \{[G_1]\}$, then $FZ(G,\mathcal{F})$ is bounded. In other words, we have the following.

Proposition F.5.6. *Let G be a countable group, and $\mathcal{F}$ be a free factor system of G. Assume that either $G = F_2$ and $\mathcal{F} = \emptyset$, or that $rk_K(G,\mathcal{F}) \geq 3$. Then $\mathcal{X}(G,\mathcal{F}) \neq \emptyset$.*

Corollary F.5.7. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. Then $FZ(G,\mathcal{F})$ has unbounded diameter if and only if either $G = F_2$ and $\mathcal{F} = \emptyset$, or $rk_K(G,\mathcal{F}) \geq 3$.* ☐

Remark F.5.8. The examples we provide belong to $\mathcal{X}^{max}(G,\mathcal{F})$. We refer to Section F.5.6 for examples of $\mathcal{Z}^{max}$-averse trees that are not $\mathcal{Z}$-averse.

Remark F.5.9. It will actually follow from our main result (Theorem F.8.1) that, except in the sporadic cases, the Gromov boundary $\partial_\infty FZ(G,\mathcal{F})$ is nonempty.

A tree $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ is *indecomposable* [Gui08, Definition 1.17] if for all segments $I, J \subseteq T$, there exist $g_1,\dots,g_r \in G$ such that $J = \cup_{i=1}^r g_i I$, and for all $i \in \{1,\dots,r-1\}$, the intersection $g_i I \cap g_{i+1} I$ is a nondegenerate arc (i.e. it is nonempty, and not reduced to a point). The following lemma follows from [Gui08, Lemma 1.18] and the description of the transverse covering of T provided by a simplicial tree S that is compatible with T (Section F.4.7).

Lemma F.5.10. *(Guirardel [Gui08, Lemma 1.18]) Let $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ be a tree that is compatible with a $\mathcal{Z}$-splitting S. Let $H \subseteq G$ be a subgroup, such that the H-minimal subtree T_H of T is indecomposable. Then H is elliptic in S.*

Proof of Proposition F.5.6. If $G = F_2$ and $\mathcal{F} = \emptyset$, then any tree dual to an arational measured lamination on a compact surface of genus 1 having exactly one boundary component is arational in the sense of [Rey12]. Hence it is mixing and $\mathcal{Z}$-incompatible, so it belongs to $\mathcal{X}(G,\mathcal{F})$ by Proposition F.5.3.

We now assume that $\mathrm{rk}_K(G,\mathcal{F}) \geq 3$. Let $N := \mathrm{rk}_f(G,\mathcal{F})$, and let $\{G_1,\dots,G_k\}$ be a set of representatives of the conjugacy classes in $\mathcal{F}$, such that

$$G = G_1 * \dots * G_k * F_N.$$

For all $i \in \{1,\dots,k\}$, we choose an element $g_i \in G_i \smallsetminus \{e\}$, whose order we denote by $p_i \in \mathbb{N} \cup \{+\infty\}$. We denote by l be the number of indices i so that $p_i = +\infty$. Up to reordering the g_i's, we can assume that $p_1,\dots,p_l = +\infty$, and $p_{l+1},\dots,p_k < +\infty$.

Let $\mathcal{O}$ be the orbifold obtained from a sphere with $N+l+1$ boundary components by adding a conical point of order p_i for each $i \in \{l+1,\dots,k\}$. For all $i \in \{1,\dots,l\}$, we denote by b_i a generator of the i^{th} boundary curve in $\pi_1(\mathcal{O})$, and for all $i \in \{l+1,\dots,k\}$, we denote by b_i a generator of the subgroup of $\pi_1(\mathcal{O})$ associated to the corresponding conical point. The group G is isomorphic to the group obtained by amalgamating $\pi_1(\mathcal{O})$ with the groups G_i, identifying b_i with g_i for all $i \in \{1,\dots,k\}$, see Figure F.6. We denote by S the corresponding splitting of G.

As $\mathrm{rk}_K(G,\mathcal{F}) \geq 3$, we can equip $\mathcal{O}$ with a minimal and filling measured foliation. Dual to this foliation is an indecomposable $\pi_1(\mathcal{O})$-tree Y (indecomposability is shown in [Gui08,

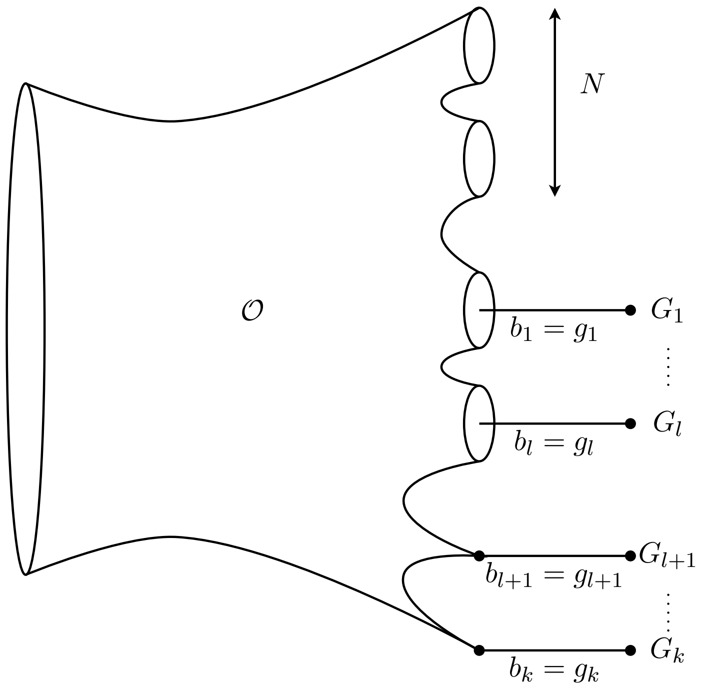

Figure F.6: The decomposition of G as an amalgam of $\pi_1(\mathcal{O})$ and the G_i's.

Proposition 1.25]). We then form a graph of actions $\mathcal{G}$ over the splitting S: vertex trees are the $\pi_1(\mathcal{O})$-tree Y, and a trivial G_i-tree for all $i \in \{1, \dots, k\}$, and edges have length 0. We denote by T the $(G, \mathcal{F})$-tree defined in this way.

We claim that $T \in \mathcal{X}(G, \mathcal{F})$. Indeed, the tree T admits a transverse covering by translates of Y, so T is mixing. We claim that T is also $\mathcal{Z}$-incompatible, which implies that T is $\mathcal{Z}$-averse by the last assertion of Proposition F.5.3. If T were compatible with a $\mathcal{Z}$-splitting S', then Lemma F.5.10 would imply that the stabilizer $\pi_1(\mathcal{O})$ of the indecomposable subtree Y fixes a vertex v in S'. Therefore, for all $i \in \{1, \dots, k\}$, the element $b_i = g_i$ fixes v. As S' is a $(G, \mathcal{F})$-splitting, the subgroup G_i fixes a vertex v_i in S', and in particular g_i fixes v_i. As nontrivial edge stabilizers in S' are nonperipheral, the element g_i does not fix any arc in S', so $v_i = v$. So all subgroups G_i fix the same vertex v of S'. Hence G is elliptic in S', a contradiction. $\square$

Remark F.5.11. When $G = F_N$ with $N \geq 3$, the trees we get are $\mathcal{Z}$-averse trees whose $\sim$-class does not contain any arational tree in the sense of [Rey12]. By comparing our description of the Gromov boundary of $\partial_\infty FZ_N$ with Bestvina–Reynolds' and Hamenstädt's description of the Gromov boundary of the free factor graph FF_N as the space of equivalence classes of arational F_N-trees [BR13, Ham14a], we get that the natural map from FZ_N to FF_N is not a quasi-isometry (this map is defined by mapping any one-edge free splitting of F_N to one of its vertex groups, and mapping any $\mathcal{Z}$-splitting with nontrivial edge stabilizers to the smallest free factor of F_N that contains the edge group, which is proper by [7, Lemma 5.11]). When $N = 2$, it is known that all trees with dense orbits in the boundary ∂cv_2 are dual to arational measured foliations on a once-punctured torus, and are therefore arational. So the Gromov boundaries $\partial_\infty FF_2$, $\partial_\infty FZ_2^{max}$ and $\partial_\infty FZ_2$ are all isomorphic.

$$\begin{tikzcd}
& T+T_1 \arrow[dl] \arrow[dr] & & T_1+T_2 \arrow[dl] \arrow[dr] & & \cdots & & T_{k-1}+S \arrow[dl] \arrow[dr] & \\
T \arrow[d] & & T_1 \arrow[dll, dotted] & & T_2 \arrow[dllll, dotted] & \cdots & T_{k-1} \arrow[dllllll, dotted] & & S \arrow[dllllllll, dotted] \\
\overline{T} & & & & & & & &
\end{tikzcd}$$

Figure F.7: The situation in the proof of Theorem F.5.1.

F.5.3 Proof of the equivalences in the definition of $\mathcal{Z}$-averse trees (Theorem F.5.1)

Our proofs of Theorem F.5.1 and Proposition F.5.3 are based on the following two propositions.

Proposition F.5.12. *Every tame $(G,\mathcal{F})$-tree is either $\mathcal{Z}$-compatible, or collapses to a mixing tree in $\overline{\mathcal{O}(G,\mathcal{F})}$.*

Proposition F.5.13. *Let T_1 and T_2 be tame $(G,\mathcal{F})$-trees. If T_1 and T_2 are compatible, and if T_1 is mixing and $\mathcal{Z}$-incompatible, then there is an alignment-preserving map from T_2 to T_1.*

Remark F.5.14. The analogues of Propositions F.5.12 and F.5.13 for $\mathcal{Z}^{max}$-tame trees also hold. The proof of Proposition F.5.12 is the same. For Proposition F.5.13, we will explain how one has to slightly adapt the argument in the proof of Proposition F.5.20 to handle the case of $\mathcal{Z}^{max}$-splittings.

We first explain how to deduce Theorem F.5.1 and Proposition F.5.3 from Propositions F.5.12 and F.5.13, before proving these two propositions.

Proof of Theorem F.5.1. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are obvious, so we need only show that (1) implies (3). Let T be a tame $(G,\mathcal{F})$-tree. Assume that there exists a finite sequence $(T=T_0,T_1,\dots,T_k=S)$ of tame $(G,\mathcal{F})$-trees, where S is simplicial, and for all $i\in\{0,\dots,k-1\}$, the trees T_i and T_{i+1} are compatible. If T did not collapse onto a tame $\mathcal{Z}$-compatible tree, then by Proposition F.5.12, the tree T would collapse onto a mixing $\mathcal{Z}$-incompatible tree $\overline{T}\in\overline{\mathcal{O}(G,\mathcal{F})}$. Notice in particular that T_1 is compatible with $\overline{T}$. An iterative application of Proposition F.5.13 then implies that all T_i's collapse onto $\overline{T}$ (see Figure F.7, where all arrows represent collapse maps). In particular, the $\mathcal{Z}$-splitting S collapses to $\overline{T}$, a contradiction. ☐

Proof of Proposition F.5.3. The argument is similar to the proof of Theorem F.5.1. Let $T,T'\in\mathcal{X}(G,\mathcal{F})$ be two equivalent trees. As T is $\mathcal{Z}$-incompatible, by Proposition F.5.12, it collapses onto a mixing tree $\overline{T}\in\overline{\mathcal{O}(G,\mathcal{F})}$, and $\overline{T}\in\mathcal{X}(G,\mathcal{F})$ because $T\in\mathcal{X}(G,\mathcal{F})$. As $T\sim T'$, there exists a finite sequence $(T=T_0,T_1,\dots,T_{k-1},T_k=T')$ of trees in $\overline{\mathcal{O}(G,\mathcal{F})}$ such that for all $i\in\{0,\dots,k-1\}$, the trees T_i and T_{i+1} are compatible. In particular T_1 is compatible with $\overline{T}$. An iterative application of Proposition F.5.13 shows that all T_i's collapse to $\overline{T}$, which proves the first assertion of Proposition F.5.3.

If $\overline{T}_1$ and $\overline{T}_2$ are two mixing trees in $\overline{\mathcal{O}(G,\mathcal{F})}$ that both satisfy the conclusion of Proposition F.5.3, then there is an alignment-preserving map from T_1 to T_2, and an alignment-preserving map from T_2 to T_1. As any alignment-preserving map from a tree in $\overline{\mathcal{O}(G,\mathcal{F})}$ with dense orbits to itself is an isometry, this implies that $\overline{T_1}$ and $\overline{T_2}$ are weakly homeomorphic.

To prove the last assertion of Proposition F.5.3, let $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ be a mixing tree that is not $\mathcal{Z}$-averse. Then there exists a finite sequence $(T = T_0, T_1, \dots, T_k = S)$ of trees in $\overline{\mathcal{O}(G,\mathcal{F})}$, where S is simplicial, and for all $i \in \{0, \dots, k-1\}$, the trees T_i and T_{i+1} are compatible. If T were $\mathcal{Z}$-incompatible, an iterative application of Proposition F.5.13 would imply that S collapses to T, a contradiction. So T is $\mathcal{Z}$-compatible. □

Proof of Proposition F.5.12

A *topological* $(G,\mathcal{F})$*-tree* is a topological space T which is homeomorphic to an $\mathbb{R}$-tree, together with a minimal, bijective, non-nesting (i.e. for all $g \in G$ and all segments $I \subseteq T$, we have $gI \nsubseteq I$), alignment-preserving G-action with trivial arc stabilizers and no simplicial arc, with a finite number of orbits of branch points, such that there exists a tree $\widehat{T} \in \overline{\mathcal{O}(G,\mathcal{F})}$ which admits an alignment-preserving map onto T (we recall that a map is *alignment-preserving* if it sends segments onto segments). A topological $(G,\mathcal{F})$-tree T *splits over terminal points* if there exists a subtree $Y \subsetneq T$ such that for all $g \in G$, we either have $gY = Y$, or $gY \cap Y = \emptyset$, and $\{\overline{gY}\}_{g\in G}$ is a transverse covering of T. Proposition F.5.12 is a consequence of the following three propositions.

Proposition F.5.15. *(Guirardel–Levitt [GL]) Every tree $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ with trivial arc stabilizers collapses onto a topological $(G,\mathcal{F})$-tree that is either mixing, or splits over terminal points.*

Proposition F.5.16. *(Guirardel–Levitt [GL]) Every mixing topological $(G,\mathcal{F})$-tree admits a G-invariant metric that turns it into an element of $\overline{\mathcal{O}(G,\mathcal{F})}$.*

Proposition F.5.17. *Let $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ be a $(G,\mathcal{F})$-tree with trivial arc stabilizers. If T collapses onto a topological $(G,\mathcal{F})$-tree which splits over terminal points, then T is compatible with a $(G,\mathcal{F})$-free splitting.*

Proof of Proposition F.5.17. Let T' be a topological $(G,\mathcal{F})$-tree which splits over terminal points, and $\pi : T \to T'$ be an alignment-preserving map. Let $Y \subsetneq T'$ be a subtree of T', such that for all $g \in G$, we either have $gY = Y$ or $gY \cap Y = \emptyset$, and $\{\overline{gY}\}_{g\in G}$ is a transverse covering of T'. The tree Y is not closed, since otherwise, any segment in T' would be covered by finitely many closed disjoint subtrees, which would imply that $Y = T'$. We denote by H the stabilizer of Y in T'. Denote by $\{x_1, \dots, x_k\}$ a set of representatives of the orbits of points in $\overline{Y} \smallsetminus Y$. Finiteness of this set comes from the fact that these points are vertices of the skeleton S of the transverse covering $\{\overline{gY}\}_{g\in G}$, and S is a minimal simplicial $(G,\mathcal{F})$-tree by Lemma F.4.16.

Let $x_{i_1}, \dots, x_{i_s}$ be those of the x_i's that do not belong to any G-translate of Y (there might not be any such x_{i_j}). We claim that the family $\mathcal{Y}$ made of $\{\overline{g\pi^{-1}(Y)}\}_{g\in G}$ and the sets $\{\overline{g\pi^{-1}(x_{i_j})}\}_{g\in G}$ for $j \in \{1, \dots, s\}$ is a transverse covering of T. Indeed, this is a transverse family made of closed subtrees of T. Let now $I \subseteq T$ be a segment. Then $\pi(I)$ is a segment in T', which is covered by a finite number of translates of Y and of the points x_{i_j}. Their π-preimages provide a covering of I by finitely many subtrees in $\mathcal{Y}$.

We now claim that the skeleton of $\mathcal{Y}$ contains an edge with trivial stabilizer. This will conclude the proof of Proposition F.5.17, since the skeleton of any transverse covering of T is compatible with T (Lemma F.4.16).

To check the above claim, we first notice that the preimage $\pi^{-1}(Y)$ is not closed (Lemma F.4.10). Let $y \in \overline{\pi^{-1}(Y)} \smallsetminus \pi^{-1}(Y)$. There is only one direction at y in $\overline{\pi^{-1}(Y)}$. As T is minimal, there exists a subtree $Y' \neq \overline{\pi^{-1}(Y)}$ in $\mathcal{Y}$ such that $y \in Y'$. The point y is a vertex of the skeleton of $\mathcal{Y}$, and there is an edge e in this skeleton associated to the pair $(\overline{\pi^{-1}(Y)}, y)$. We claim that e has trivial stabilizer. Indeed, if $g \in G$ stabilizes e, then as y has valence 1 in $\overline{\pi^{-1}(Y)}$, the element g stabilizes an arc in $\overline{\pi^{-1}(Y)}$. As T has trivial arc stabilizers, this implies that g is the identity of G. ☐

Proof of Proposition F.5.12. Let T be a tame $(G,\mathcal{F})$-tree. If T has trivial arc stabilizers, then the conclusion of Proposition F.5.12 is a consequence of Propositions F.5.15, F.5.16 and F.5.17. If T contains an arc with nontrivial stabilizer, then T does not have dense orbits. Proposition F.4.18 implies that T projects to a simplicial tree S with cyclic, nonperipheral arc stabilizers, so T is $\mathcal{Z}$-compatible. ☐

Proof of Proposition F.5.13

The following proposition gives control over the possible point stabilizers in a tree in $\overline{\mathcal{O}(G,\mathcal{F})}$. It can be deduced from [Gui08, Proposition 4.4] by noticing that any simple closed curve on a closed 2-orbifold with boundary provides a $\mathcal{Z}^{max}$-splitting of its fundamental group.

Proposition F.5.18. *(Bestvina–Feighn [BF95], Guirardel [Gui08, Proposition 4.4], Guirardel–Levitt [GL14]) Let T be a tame $(G,\mathcal{F})$-tree, and let $X \subset T$ be a finite subset of T. Then there exists a $\mathcal{Z}^{max}$-splitting in which $Stab_T(x)$ is elliptic for all $x \in X$.*

Remark F.5.19. Knowing the existence of a $\mathcal{Z}$-splitting would be enough if we were only interested in proving the $\mathcal{Z}$-version of Proposition F.5.13.

Proof of Proposition F.5.13. Let $\widehat{T} := T_1 + T_2$. As T_1 is $\mathcal{Z}$-incompatible, the tree $\widehat{T}$ has dense orbits. Let $p_1 : \widehat{T} \to T_1$ and $p_2 : \widehat{T} \to T_2$ be the associated 1-Lipschitz alignment-preserving maps. Assuming that p_2 is not a bijection (otherwise the map $p_1 \circ {p_2}^{-1}$ satisfies the conclusion of Proposition F.5.13 and we are done), we can find a point $x \in T_2$ whose p_2-preimage in $\widehat{T}$ is a nondegenerate closed subtree Y of $\widehat{T}$. The set $\{gY\}_{g\in G}$ is a transverse family in $\widehat{T}$.

First assume that $p_1(Y)$ is reduced to a point for all $x \in T_2$, and let f be the map from T_2 to T_1 that sends any $x \in T_2$ to $p_1(Y)$, with the above notations. We claim that f preserves alignment. Indeed, let $x, z \in T_2$, and $y \in [x,z]$. Then ${p_2}^{-1}(\{x\})$ and ${p_2}^{-1}(\{z\})$ are closed subtrees of $\widehat{T}$, and the bridge in $\widehat{T}$ between them meets ${p_2}^{-1}(\{y\})$. Since p_1 preserves alignment, this implies that f preserves alignment, and we are done in this case.

We now choose $x \in T_2$ so that $p_1(Y)$ is not reduced to a point. The family $\{gp_1(Y)\}_{g\in G}$ is a transverse family made of closed subtrees of T_1 (Lemma F.4.10). As T_1 is mixing, it is a transverse covering of T_1. The stabilizer of $p_1(Y)$ in T_1 is equal to the stabilizer of Y in $\widehat{T}$, which in turn is also equal to the stabilizer of x in T'. Proposition F.5.18 shows that there exists a $\mathcal{Z}$-splitting in which $\mathrm{Stab}_{T'}(x)$, and hence $\mathrm{Stab}_T(p_1(Y))$, is elliptic. This contradicts the following proposition. ☐

Proposition F.5.20. *Let $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ be mixing and $\mathcal{Z}$-incompatible, and let $\mathcal{Y}$ be a transverse covering of T. Then for all $Y \in \mathcal{Y}$, the stabilizer $Stab_T(Y)$ is not elliptic in any $\mathcal{Z}$-splitting.*

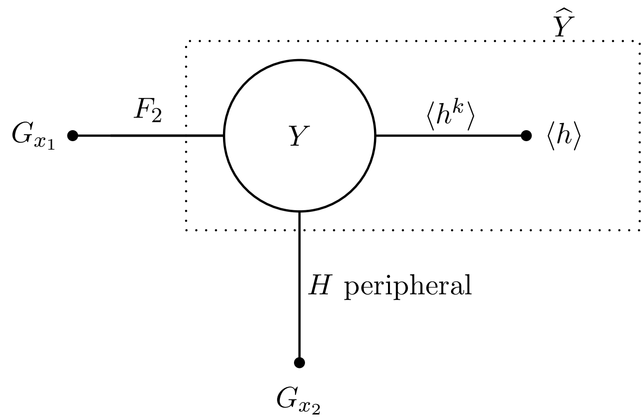

Figure F.8: The graph of actions in the proof of Proposition F.5.22.

Remark F.5.21. We warn the reader that the argument in the following proof has to be slightly adapted in the case of $\mathcal{Z}^{max}$-splittings. This will be done in Proposition F.5.22 below. The difficulty comes from edges with nonperipheral cyclic stabilizers (not belonging to $\mathcal{Z}^{max}$) in $\mathcal{G}$.

Proof of Proposition F.5.20. As T is mixing, any transverse covering of T contains at most one orbit of subtrees (otherwise a segment contained in one of these orbits could not be covered by translates of a segment contained in another subtree). We denote by S the skeleton of $\mathcal{Y}$, and by $\Gamma := S/G$ the quotient graph of groups. The vertex set of Γ consists of a vertex associated to Y, with vertex group $\mathrm{Stab}_T(Y)$, together with a finite collection of points $x_1, \dots, x_l$. Each x_i is joined to Y by a finite set of edges, whose stabilizers do not belong to the class $\mathcal{Z}$ because T is $\mathcal{Z}$-incompatible. We denote by G_Y and G_{x_i} the corresponding stabilizers. Assume towards a contradiction that G_Y fixes a vertex v in a $\mathcal{Z}$-splitting S_0.

Suppose first that all vertex groups of S are elliptic in S_0. As edge stabilizers of S do not belong to $\mathcal{Z}$, and as all vertex stabilizers of S fix a point in S_0, two adjacent vertex stabilizers of S must have the same fixed point in S_0. This implies that G is elliptic in S_0, a contradiction.

Hence one of the G_{x_i}'s acts nontrivially on S_0. Edge groups of S are elliptic in S_0 because $\mathrm{Stab}_T(Y)$ is. By blowing up S at the vertex x_i, using the action of G_{x_i} on its minimal subtree in S_0, we get a splitting S', which contains an edge whose stabilizer belongs to the class $\mathcal{Z}$. The tree T splits as a graph of actions over S' (by the discussion following Proposition F.4.17 in Section F.4.7). This contradicts $\mathcal{Z}$-incompatibility of T. $\square$

Proposition F.5.22. *Let $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ be mixing and $\mathcal{Z}^{max}$-incompatible, and let $\mathcal{Y}$ be a transverse covering of T. Then for all $Y \in \mathcal{Y}$, the stabilizer $\mathrm{Stab}_T(Y)$ is not elliptic in any $\mathcal{Z}^{max}$-splitting.*

Proof. We keep the notations from the proof of Proposition F.5.20, where this time S_0 is a $\mathcal{Z}^{max}$-splitting. We denote by $\mathcal{G}$ the graph of actions corresponding to $\mathcal{Y}$, which is represented on Figure F.8. Note that for all $i \in \{1, \dots, l\}$, and all nonperipheral elements $g \in G$, if $g^p \in G_{x_i}$ for some $p \geq 1$, then $g \in G_{x_i}$. Up to reordering the x_i's, we can assume that for all $i \in \{1, \dots, k\}$, no edge joining Y to x_i has $\mathcal{Z}$-stabilizer, and for all $i \in \{k+1, \dots, l\}$, there is an edge e_i with $\mathcal{Z} \setminus \mathcal{Z}^{max}$-stabilizer $\langle g_i \rangle$ joining x_i to Y. Then g_i is a proper power of the form h_i^k, with $h_i \in G_{x_i}$. Subdivide the edge $[x_i, Y]$ into

$[x_i, m_i] \cup [m_i, Y]$, and fold $[x_i, m_i]$ with its image under h_i. We get a refinement S' of S that is still compatible with T. By collapsing the orbit of the edge with stabilizer equal to h_i^k, we get a new splitting of T as a graph of actions $\mathcal{G}'$. The stabilizer of the edge e_i' joining Y' to x_i in $\mathcal{G}'$ is equal to $\langle h_i \rangle$, and hence it belongs to the class $\mathcal{Z}^{max}$. The splitting of G dual to the edge e_i' is not minimal, otherwise T would be $\mathcal{Z}^{max}$-compatible. Hence $G_{x_i} = \langle h_i \rangle$, and x_i is joined to Y by a single edge in $\mathcal{G}$. As G_Y fixes the vertex v of S_0, so does g_i. As S_0 is a $\mathcal{Z}^{max}$-splitting, the element h_i, and hence G_{x_i}, also fixes v in S_0.

Therefore, by replacing G_Y by $G_{\widehat{Y}} := \langle G_Y, G_{x_{k+1}}, \dots, G_{x_l} \rangle$, we build a new splitting of T as a graph of actions $\widehat{\mathcal{G}}$, which has the following description. The graph of actions $\widehat{\mathcal{G}}$ consists of a new vertex tree $\widehat{Y}$ with dense orbits, whose stabilizer $G_{\widehat{Y}}$ is elliptic in S_0, attached to $x_1, \dots, x_k$, and all its edges have either peripheral or noncyclic stabilizer. The proof then goes as in the case of $\mathcal{Z}$-splittings, by working with the graph of actions $\widehat{\mathcal{G}}$ instead of $\mathcal{G}$. □

Using Proposition F.5.22, we deduce the following $\mathcal{Z}^{max}$-analogue of Proposition F.5.13.

Proposition F.5.23. *Let T_1 and T_2 be $\mathcal{Z}^{max}$-tame $(G, \mathcal{F})$-trees. If T_1 and T_2 are compatible, and if T_1 is mixing and $\mathcal{Z}^{max}$-incompatible, then there is an alignment-preserving map from T_2 to T_1.*

F.5.4 Folding paths ending at mixing and $\mathcal{Z}$-incompatible trees

We now prove the following property for folding paths ending at mixing $\mathcal{Z}$-incompatible trees.

Proposition F.5.24. *Let $T \in \overline{\mathcal{O}(G, \mathcal{F})}$ be mixing and $\mathcal{Z}$-incompatible, and let $\gamma : [0, L] \to \overline{\mathcal{O}(G, \mathcal{F})}$ be an optimal liberal folding path ending at T. Then for all $t < L$, the tree $\gamma(t)$ is simplicial and has trivial edge stabilizers unless $\gamma(t) = T$.*

Proof. As T has dense orbits, all arc stabilizers in T are trivial, hence all arc stabilizers in trees lying on optimal liberal folding paths ending at T are trivial. Assume towards a contradiction that there exists $t_0 < L$, such that $\gamma(t_0) \neq T$ is nonsimplicial. Notice that $\gamma(t_0)$ contains a nontrivial simplicial part, otherwise it would be equal to T, as any morphism between two $(G, \mathcal{F})$-trees with dense orbits is an isometry (Corollary F.4.13). By Proposition F.4.18, the tree $\gamma(t_0)$ contains a subtree T_0 which has dense orbits for the action of its stabilizer H. Moreover, the group H is a proper $(G, \mathcal{F})$-free factor.

Let $Y := f_{t_0,L}(T_0)$. We claim that for all $g \in G \smallsetminus H$, the intersection $gY \cap Y$ contains at most one point. Otherwise, there exist nondegenerate segments $I \subset T_0$ and $J \subset gT_0$ such that $f_{t_0,L}(I) = f_{t_0,L}(J)$. So there exist $h \in H$ hyperbolic in T_0 (whose axis intersects I nondegenerately), and $h' \in H^g$ hyperbolic in gT_0 (whose axis intersects J nondegenerately), such that the axes of h and h' have nondegenerate intersection in T. We thus have $||hh'||_T \leq ||h||_T + ||h'||_T$ (see [CM87, 1.8]). Let t_1 be the smallest real number for which this inequality holds, so that for all $t < t_1$, the axes of h and h' are disjoint in $\gamma(t)$. By continuity of γ, we deduce that both $||hh'||_{\gamma(t_1)}$ and $||hh'^{-1}||_{\gamma(t_1)}$ are greater than or equal to $||h||_{\gamma(t_1)} + ||h'||_{\gamma(t_1)}$, so the intersection of the axes of h and h' in $f_{t_0,t_1}(T_0)$ is reduced to a point. The image $f_{t_0,t_1}(I \cup J)$ is contained in a subtree with dense orbits of the Levitt decomposition of $\gamma(t_1)$ as a graph of actions given by Proposition F.4.18. The morphism $f_{t_1,L}$ is injective in restriction to this subtree (Corollary F.4.13). This implies that $gY \cap Y$ is reduced to a point.

Hence the collection $\{gY\}_{g \in G}$ is a transverse family in T, and so is the collection $\{g\overline{Y}\}_{g \in G}$. As T is mixing, the collection $\{g\overline{Y}\}_{g \in G}$ is a transverse covering of T. In addition, the stabilizer of $\overline{Y}$ in T is equal to H, and hence is elliptic in a $\mathcal{Z}$-splitting (it is even a $(G,\mathcal{F})$-free factor). This contradicts Proposition F.5.20. □

F.5.5 The case of $\mathcal{Z}^{max}$-splittings

By only considering $\mathcal{Z}^{max}$-splittings, we similarly define the space $\mathcal{X}^{max}(G,\mathcal{F})$ of $\mathcal{Z}^{max}$*-averse trees* in the following way. For all $\mathcal{Z}^{max}$-tame trees T, we denote by $\mathcal{R}^{1,max}(T)$ the set of $\mathcal{Z}^{max}$-splittings that are compatible with T, and by $\mathcal{R}^{2,max}(T)$ the set of $\mathcal{Z}^{max}$-splittings that are compatible with a $\mathcal{Z}^{max}$-tame tree T', which is compatible with T. A $\mathcal{Z}^{max}$-tame tree T is $\mathcal{Z}^{max}$*-averse* if $\mathcal{R}^{2,max}(T)=\emptyset$. Two $\mathcal{Z}^{max}$-averse trees $T,T'\in\mathcal{X}^{max}(G,\mathcal{F})$ are *equivalent* if there exists a finite sequence $(T=T_0,T_1,\dots,T_k=T')$ of tame $(G,\mathcal{F})$-trees such that for all $i\in\{1,\dots,k\}$, the trees T_i and T_{i+1} are compatible.

The analogues of Theorem F.5.1, Proposition F.5.3, and Theorem F.5.4 also hold true in this setting. The proofs are the same, the only difference is in the proof of Proposition F.5.23, as explained above.

Theorem F.5.25. *($\mathcal{Z}^{max}$-analogue of Theorem F.5.1) For all $\mathcal{Z}^{max}$-tame $(G,\mathcal{F})$-trees T, the following assertions are equivalent.*

- *There exists a finite sequence $(T=T_0,T_1,\dots,T_k=S)$ of $\mathcal{Z}^{max}$-tame $(G,\mathcal{F})$-trees, such that S is simplicial, and for all $i\in\{0,\dots,k-1\}$, the trees T_i and T_{i+1} are compatible.*
- *We have $\mathcal{R}^{2,max}(T)\neq\emptyset$.*
- *The tree T collapses to a $\mathcal{Z}^{max}$-tame $\mathcal{Z}^{max}$-compatible $(G,\mathcal{F})$-tree.*

Proposition F.5.26. *($\mathcal{Z}^{max}$-analogue of Proposition F.5.3) For all $T\in\mathcal{X}^{max}(G,\mathcal{F})$, there exists a mixing tree in $\mathcal{X}^{max}(G,\mathcal{F})$ onto which all trees $T'\in\mathcal{X}^{max}(G,\mathcal{F})$ that are equivalent to T collapse. In addition, any two such trees are G-equivariantly weakly homeomorphic. Any tree $T\in\overline{\mathcal{O}(G,\mathcal{F})}$ that is both mixing and $\mathcal{Z}^{max}$-incompatible is $\mathcal{Z}^{max}$-averse.*

Theorem F.5.27. *($\mathcal{Z}^{max}$-analogue of Theorem F.5.4) Let $T\in\mathcal{X}^{max}(G,\mathcal{F})$, and let $(T_i)_{i\in\mathbb{N}}\in\mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$ be a sequence that converges to T. Then $\psi^{max}(T_i)$ is unbounded in $FZ^{max}(G,\mathcal{F})$.*

F.5.6 A few remarks and examples

$\mathcal{Z}$-averse trees versus $\mathcal{Z}^{max}$-averse trees

Building on our construction from the proof of Proposition F.5.6, we give examples of $\mathcal{Z}^{max}$-averse trees that are not $\mathcal{Z}$-averse as soon as $\mathrm{rk}_f(G,\mathcal{F})\geq 1$ and $\mathrm{rk}_K(G,\mathcal{F})\geq 3$. Together with our main results (Theorem F.8.1 and F.8.2), this implies that the inclusion map from $FZ^{max}(G,\mathcal{F})$ into $FZ(G,\mathcal{F})$ is not a quasi-isometry in these cases.

Proposition F.5.28. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. Assume that $rk_f(G,\mathcal{F})\geq 1$ and $rk_K(G,\mathcal{F})\geq 3$. Then $\mathcal{X}^{max}(G,\mathcal{F})\neq\mathcal{X}(G,\mathcal{F})$, so the inclusion map from $FZ^{max}(G,\mathcal{F})$ into $FZ(G,\mathcal{F})$ is not a quasi-isometry.*

Proof. For all $i\in\{1,\dots,k\}$, we choose an element $g_i\in G_i\smallsetminus\{e\}$, whose order we denote by $p_i\in\mathbb{N}\cup\{+\infty\}$. We denote by l the number of indices such that $p_i=+\infty$. Up to reindexing the g_i's, we can assume that $p_1,\dots,p_l=+\infty$, and $p_{l+1},\dots,p_k<+\infty$.

Let $\mathcal{O}$ be the orbifold obtained from a sphere with $N+l+1$ boundary components, where $N:=\mathrm{rk}_f(G,\mathcal{F})\geq 1$, by adding a conical point of order p_i for each $i\in\{l+$

$1,\dots,k\}$. As $\mathrm{rk}_K(G,\mathcal{F})\geq 3$, we can equip $\mathcal{O}$ with an arational measured foliation. For all $i\in\{1,\dots,l\}$, we denote by b_i a generator of the i^{th} boundary curve in $\pi_1(\mathcal{O})$, and for all $i\in\{l+1,\dots,k\}$, we denote by b_i a generator of the subgroup associated to the corresponding conical point. We denote by b_0 a generator of one of the other boundary curves. The group G is isomorphic to the group obtained by amalgamating $\pi_1(\mathcal{O})$ with the groups G_i and $\mathbb{Z}=\langle a_0\rangle$, identifying b_i with g_i for all $i\in\{1,\dots,k\}$, and identifying b_0 with a_0^2.

We then form a graph of actions $\mathcal{G}$ over this splitting of G: vertex trees are the $\pi_1(\mathcal{O})$-tree Y dual to the foliation on $\mathcal{O}$, a trivial G_i-tree for all $i\in\{1,\dots,k\}$, and a trivial $\langle a_0\rangle$-tree, and edges have length 0.

This construction yields a G-tree T which is not $\mathcal{Z}$-averse, because it splits as a graph of actions, one of whose edge groups belongs to $\mathcal{Z}$. We claim that T is $\mathcal{Z}^{max}$-averse. Indeed, as T is mixing, it is enough to show that T is $\mathcal{Z}^{max}$-incompatible (Proposition F.5.26). Assume towards a contradiction that T is compatible with a $\mathcal{Z}^{max}$-splitting S_0. The $\pi_1(\mathcal{O})$-minimal subtree of T is indecomposable, so $\pi_1(\mathcal{O})$ has to be elliptic in S_0. We denote by S the skeleton of $\mathcal{G}$, and by x_0 the vertex of S with vertex group $\langle a_0\rangle$. Arguing as in the proof of Proposition F.5.6, we then get that for any two adjacent vertices $u,u'\in S$ with $u,u'\notin G.x_0$, the vertex groups G_u and $G_{u'}$ fix a common vertex v of S_0. This is still true if $u'\in G.x_0$ because S_0 is a $\mathcal{Z}^{max}$-splitting. ☐

On the other hand, we show that if $\mathrm{rk}_f(G,\mathcal{F})=0$, then the graphs $FZ(G,\mathcal{F})$ and $FZ^{max}(G,\mathcal{F})$ are quasi-isometric to each other.

Proposition F.5.29. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. Assume that $\mathrm{rk}_f(G,\mathcal{F})=0$, and $\mathrm{rk}_K(G,\mathcal{F})\geq 3$. Then the inclusion from $FZ^{max}(G,\mathcal{F})$ into $FZ(G,\mathcal{F})$ is a quasi-isometry.*

Proof. We will define an inverse map $\tau: FZ(G,\mathcal{F})\to FZ^{max}(G,\mathcal{F})$. Let S be a one-edge $\mathcal{Z}$-splitting, of the form $A*_{\langle g^k\rangle}B$ (where $\langle g\rangle\in\mathcal{Z}^{max}$). As g^k is elliptic in S, so is g. We assume without loss of generality that $g\in A$ (and $g\notin B$), and we let S^{max} be the $\mathcal{Z}^{max}$-splitting $A*_{\langle g\rangle}\langle B,g\rangle$. In the case where S is an HNN extension of the form $A*_{\langle g^k\rangle}$, we let $S^{max}:=\langle A,g,g^t\rangle *_{\langle g\rangle}$, where t is a stable letter. We claim that the G-minimal subtree of S^{max} is nontrivial. The map τ is then defined by letting $\tau(S):=S^{max}$. In addition, if S_1 and S_2 are two compatible one-edge $\mathcal{Z}$-splittings, one checks that S_1^{max} and S_2^{max} are also compatible. This shows that τ is Lipschitz, and proves that $FZ(G,\mathcal{F})$ and $FZ^{max}(G,\mathcal{F})$ are quasi-isometric to each other.

Assume towards a contradiction that S^{max} is trivial. Then S is of the form $\langle g\rangle *_{\langle g^k\rangle}B$, so $A=\langle g\rangle$ is cyclic and nonperipheral. We claim that A is a proper $(G,\mathcal{F})$-free factor. This is a contradiction because G has no free factor in $\mathcal{Z}$, since $\mathrm{rk}_f(G,\mathcal{F})=0$.

By [7, Lemma 5.11], the splitting S is compatible with a one-edge $(G,\mathcal{F})$-free splitting S_0. Since A is cyclic and $g^k\in A$ is elliptic in S, the splitting $S+S_0$ can only be obtained by splitting the vertex group B in S. Some proper $(G,\mathcal{F})$-free factor B' of B is elliptic in $S+S_0$ and contains g^k. Repeating the above argument, we can split B' further. Arguing by induction on the Kurosh rank of B, we end up with a $\mathcal{Z}$-splitting S' in which the edge with nontrivial stabilizer $\langle g^k\rangle$ is attached to a vertex whose stabilizer has Kurosh rank equal to 1, and is therefore equal to $\langle g^k\rangle$. The splitting S' collapses to a $(G,\mathcal{F})$-free splitting in which A is elliptic. ☐

Why working with $\mathcal{R}^2(T)$ instead of $\mathcal{R}^1(T)$?

Example F.5.30. We give an example of a tree $T \in \overline{cv_N}$ that is $\mathcal{Z}$-incompatible but is not $\mathcal{Z}$-averse. In other words, we have $\mathcal{R}^1(T) = \emptyset$, while $\mathcal{R}^2(T) \neq \emptyset$. This justifies the introduction of the set $\mathcal{R}^2(T)$ in our arguments.

Let T_1 be an indecomposable F_N-tree in which some free factor $F_2 \subseteq F_N$ of rank 2 fixes a point x_1. Examples of such trees were given in [Rey12, Part 11.6]. Form a graph of actions over the splitting $F_{2N-2} = F_N *_{F_2} F_N$, where the vertex trees are two copies of T_1, and the attaching points are the copies of x_1. In this way, we get a tree $T \in \overline{cv_{2N-2}}$.

We claim that T is $\mathcal{Z}$-incompatible. Indeed, assume towards a contradiction that T is compatible with a $\mathcal{Z}$-splitting S of F_{2N-2}. Lemma F.5.10 implies that both copies of F_N are elliptic in S. Therefore, the subgroup F_2 is also elliptic in S. As edge stabilizers in S are cyclic, this implies that F_{2N-2} is elliptic in S, a contradiction.

However, the tree T is not $\mathcal{Z}$-averse. Indeed, let $\overline{T}$ be the tree obtained by equivariantly collapsing to a point one of the copies of T_1 in T (but not the other). Then $\overline{T}$ is $\mathcal{Z}$-compatible, because one can blow up the copy of F_N that got collapsed by using a splitting in which the free factor F_2 is elliptic.

The importance of working with cyclic splittings rather than free splittings

Example F.5.31. We now give an example of two mixing compatible F_N-trees $T_1, T_2 \in \overline{cv_N}$, such that T_2 is compatible with a free splitting of F_N, while T_1 is not. This shows that it is crucial to work with cyclic splittings rather than free splittings in Theorem F.5.1. The following situation is illustrated on Figure F.9. Let S be a compact orientable surface of genus 2, with one boundary component. Let c be a simple closed curve that splits the surface S into two subsurfaces S_1 and S_2, where S_1 has genus 1 and two boundary components, and S_2 has genus 1 and one boundary component.

For all $i \in \{1,2\}$, let L_i be an arational measured lamination on the surface S_i. Let T_1 be the tree dual to the measured lamination on S obtained by equipping S_1 with the lamination L_1, and equipping S_2 with the empty lamination. Let T_2 be the tree dual to the measured lamination on S obtained by equipping S_1 with the empty lamination, and equipping S_2 with the lamination L_2. Both trees T_1 and T_2 are mixing. The trees T_1 and T_2 are compatible, as they are both refined by the tree T dual to the lamination obtained by equipping S_1 with L_1 and S_2 with L_2. The tree T_2 is compatible with any free splitting of F_4 determined by an essential arc of S that lies on the subsurface S_1. However, the tree T_1 is not compatible with any free splitting of S. Indeed, otherwise, Lemma F.5.10 would imply that the boundary curve of S is elliptic in this splitting, which is impossible. Notice however that both trees T_1 and T_2 are compatible with the $\mathcal{Z}^{max}$-splitting determined by the simple closed curve c.

Non-mixing $\mathcal{Z}$-averse trees

Example F.5.32. We have seen (Corollary F.5.3) that any equivalence class in $\mathcal{X}(G,\mathcal{F})$ contains mixing representatives. We now give an example of a tree $T \in \overline{cv_N}$ that is $\mathcal{Z}$-averse but not mixing. We refer to [Rey11a, Example 10.10] for details. Let $\Phi \in \mathrm{Out}(F_N)$ be an automorphism with two strata, and assume that the Perron–Frobenius eigenvalue of the lower stratum is strictly greater than the Perron–Frobenius eigenvalue of the upper stratum. Then the attractive tree of Φ is not mixing, however it collapses onto a tree which is mixing and $\mathcal{Z}$-incompatible, and hence it is $\mathcal{Z}$-averse.

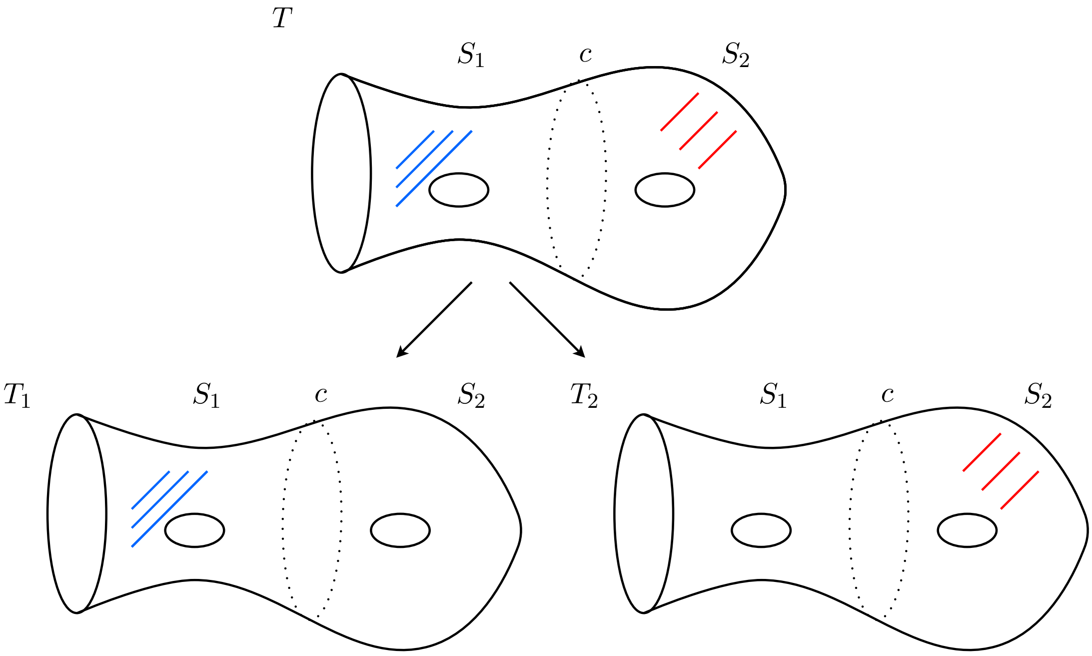

Figure F.9: The laminations dual to the trees in Example F.5.31.

F.6 Collapses and pullbacks of folding paths and folding sequences

We now describe two constructions that will turn out to be useful in the next section, for the proof of Theorem F.7.1. These constructions are inspired from the analogous constructions in [HM13a, Section 4.2] or [BF14c, Section A.2] in the case of folding paths between simplicial F_N-trees with trivial edge stabilizers (see also Section F.2.3 of the present paper for a brief account of these constructions in the simplicial case).

F.6.1 Collapses

Let S, T and $\overline{T}$ be tame $(G, \mathcal{F})$-trees. Let γ be a tame optimal liberal folding path from S to T. Assume that there exists an alignment-preserving map $\pi : T \to \overline{T}$. Let $L \in \mathbb{R}_+$ be such that $\gamma(L) = T$, and let $t \leq L$. Recall from Proposition F.4.18 that $\gamma(t)$ splits as a graph of actions $\mathcal{G}$, all of whose vertex trees have dense orbits for the action of their stabilizer (some vertex trees may be trivial). If $Y \subseteq \gamma(t)$ is one of the vertex trees of this splitting, then the morphism $f_{t,L} : \gamma(t) \to T$ provided by the definition of a liberal folding path is an isometry in restriction to Y. Optimality of f implies that the morphism $f_{t,L}$ is an isometry in restriction to each edge in the simplicial part of $\gamma(t)$ (i.e. each edge of the Levitt decomposition of $\gamma(t)$ as a graph of actions). We define $\overline{\gamma}(t)$ from $\gamma(t)$ by replacing in $\mathcal{G}$ each vertex subtree Y by its image $\pi \circ f_{t,L}(Y)$ in $\overline{T}$, replacing each attaching point x in $\mathcal{G}$ by its image $\pi \circ f_{t,L}(x)$, and modifying the metric on each edge e in the simplicial part of $\gamma(t)$, so that e becomes isometric to the segment $\pi(f_{t,L}(e)) \subseteq \overline{T}$ (this may collapse some subsegments of e). We denote by $\pi_t : \gamma(t) \to \overline{\gamma}(t)$ the natural alignment-preserving map. The π_t-preimage of any point $x \in \overline{\gamma}(t)$ is a subtree of $\gamma(t)$ whose f-image in T is collapsed to a point by π. Therefore, for all $t' > t$, the $f_{t,t'}$-image of the subtree $\pi_t^{-1}(x)$ collapses to a point x' in $\overline{\gamma}(t')$. Therefore, the optimal morphism $f_{t,t'}$ induces a map $\overline{f}_{t,t'}$ (sending

x to x', with the above notations) for all $t < t'$, and this map is again a morphism by construction. We claim that the morphism $\overline{f}_{t,t'}$ is optimal. Indeed, if $x \in \overline{\gamma}(t)$ is a point contained in a simplicial edge of $\overline{\gamma}(t)$, then any π_t-preimage of x is contained in a legal axis in $\gamma(t)$ that does not collapse to a point under π_t. If $x \in \overline{\gamma}(t)$ belongs to the interior of one of the nondegenerate subtrees with dense orbits T_v of the Levitt decomposition of T as a graph of actions, then any line passing through x and contained in T_v lifts to a legal line in $\gamma(t)$. These morphisms again satisfy $\overline{f}_{t,t''} = \overline{f}_{t',t''} \circ \overline{f}_{t,t'}$ for all $t < t' < t''$ by construction. We call the folding path constructed in this way the *collapse* of γ induced by π.

Proposition F.6.1. *Let S, T and $\overline{T}$ be tame $(G,\mathcal{F})$-trees, and $\pi : T \to \overline{T}$ be a 1-Lipschitz alignment-preserving map. Let γ be a tame optimal liberal folding path from S to T, and let $\overline{\gamma}$ be the collapse of γ induced by π. Then $t \mapsto \overline{\gamma}(t)$ is continuous.*

Proof. Since π is 1-Lipschitz, all alignment-preserving maps $\pi_t : \gamma(t) \to \overline{\gamma}(t)$ are 1-Lipschitz. Let $g \in G$, and let $\epsilon > 0$. For t close enough to t_0, the total length in a fundamental domain of g that gets folded under the morphism f_{t,t_0} (or $f_{t_0,t}$) between time t and time t_0 is at most ϵ. As π_t is 1-Lipschitz, this implies that the total length in a fundamental domain of g that gets collapsed under π_t is close to the total length in a fundamental domain of g that gets collapsed under π_{t_0}. More precisely, for t close enough to t_0, we have $|(||g||_{\gamma(t)} - ||g||_{\overline{\gamma}(t)}) - (||g||_{\gamma(t_0)} - ||g||_{\overline{\gamma}(t_0)})| \leq 2\epsilon$. This implies that $||g||_{\overline{\gamma}(t)}$ converges to $||g||_{\overline{\gamma}(t_0)}$ as t tends to t_0. As this is true for all $g \in G$, the collapse $t \mapsto \overline{\gamma}(t)$ is continuous at t_0. □

One can also give a discrete version of the above construction. We recall the definition of a tame optimal folding sequence from Section F.4.1. The following proposition follows from the above analysis.

Proposition F.6.2. *Let T and $\overline{T}$ be tame $(G,\mathcal{F})$-trees, and $\pi : T \to \overline{T}$ be a 1-Lipschitz alignment-preserving map. Let $(\gamma(n))_{n\in\mathbb{N}}$ be a tame optimal folding sequence ending at T. Then there exists a tame optimal folding sequence $(\overline{\gamma}(n))_{n\in\mathbb{N}}$ ending at $\overline{T}$, such that for all $n \in \mathbb{N}$, there exists an alignment-preserving map from $\gamma(n)$ to $\overline{\gamma}(n)$.* □

A sequence $(\overline{\gamma}(n))_{n\in\mathbb{N}}$ satisfying the conclusions of Proposition F.6.2 is called a *collapse* of $(\gamma(n))_{n\in\mathbb{N}}$ induced by π.

F.6.2 Pullbacks

The following construction is inspired from the construction of pullbacks of simplicial metric trees, as it appears in [HM13a, Proposition 4.4] or [BF14c, Lemma A.3]. Let S, T and $\widehat{T}$ be tame $(G,\mathcal{F})$-trees, such that there exists a 1-Lipschitz alignment-preserving map $p : \widehat{T} \to T$. Let $f : S \to T$ be an optimal morphism. Let

$$\mathcal{C}'(S,\widehat{T}) := \{(x,y) \in S \times \widehat{T} | f(x) = p(y)\}$$

be the fiber product of S and $\widehat{T}$. The space $\mathcal{C}'(S,\widehat{T})$ is naturally equipped with a G-action induced by the diagonal action on $S \times \widehat{T}$.

Let $(x,y), (x',y') \in \mathcal{C}'(S,\widehat{T})$. A finite sequence $((x,y) = (x_0,y_0), \dots, (x_k,y_k) = (x',y'))$ of elements of $\mathcal{C}'(S,\widehat{T})$ is *admissible* if for all $i \in \{0,\dots,k-1\}$, the morphism f is injective in restriction to $[x_i, x_{i+1}]$. The existence of admissible sequences between any two points of

$\mathcal{C}'(S,\widehat{T})$ comes from the fact that f is a morphism and p is surjective. Given an admissible sequence $\sigma := ((x,y) = (x_0,y_0),\dots,(x_k,y_k) = (x',y'))$, we let

$$l(\sigma) := \sum_{i=0}^{k-1} d_S(x_i,x_{i+1}) + d_{\widehat{T}}(y_i,y_{i+1}).$$

For all $((x,x'),(y,y')) \in \mathcal{C}'(S,\widehat{T})^2$, we then let

$$d((x,y),(x',y')) := \inf_{\sigma} l(\sigma),$$

where the infimum is taken over all admissible sequences between (x,y) and (x',y'). The map d defines a metric on $\mathcal{C}'(S,\widehat{T})$: the triangle inequality follows from the fact that the concatenation of two admissible sequences is again admissible, and the separation axiom follows from the observation that $d((x,y),(x',y')) \geq d_S(x,x') + d_{\widehat{T}}(y,y')$. We first make the following observation.

Lemma F.6.3. *Let $(x,y),(x',y') \in \mathcal{C}'(S,\widehat{T})$, and let $\sigma := ((x,y) = (x_0,y_0),\dots,(x_k,y_k) = (x',y'))$ be an admissible sequence between (x,y) and (x',y'). Let $i \in \{0,\dots,k-1\}$, let $x'_i \in [x_i,x_{i+1}]$, and let y'_i be the projection of y_i to $p^{-1}(x'_i)$. Let σ' be the sequence obtained by inserting (x'_i,y'_i) between (x_i,y_i) and (x_{i+1},y_{i+1}) in σ. Then σ' is admissible, and $l(\sigma') = l(\sigma)$.*

Proof. As σ is admissible, the morphism f is injective in restriction to $[x_i,x_{i+1}]$. Since $x'_i \in [x_i,x_{i+1}]$, it is injective in restriction to both $[x_i,x'_i]$ and $[x'_i,x_{i+1}]$, so σ' is admissible. We also have $d_S(x_i,x_{i+1}) = d_S(x_i,x'_i) + d_S(x'_i,x_{i+1})$, and $f(x'_i) \in [f(x_i),f(x_{i+1})]$. Since p preserves alignment, the bridge between the (disjoint) closed subtrees $p^{-1}(f(x_i))$ and $p^{-1}(f(x_{i+1}))$ meets $p^{-1}(f(x'_i))$, and therefore it contains the projection y'_i of y_i to $p^{-1}(x'_i)$. As $y_i \in p^{-1}(f(x_i))$ and $y_{i+1} \in p^{-1}(f(x_{i+1}))$, this implies that $d_{\widehat{T}}(y_i,y_{i+1}) = d_{\widehat{T}}(y_i,y'_i) + d_{\widehat{T}}(y'_i,y_{i+1})$, from which Lemma F.6.3 follows. $\square$

Lemma F.6.4. *The metric space $(\mathcal{C}'(S,\widehat{T}),d)$ is a $(G,\mathcal{F})$-tree.*

Proof. We start by proving that the topological space $\mathcal{C}'(S,\widehat{T})$ is path-connected. Let $(x,y),(x',y') \in \mathcal{C}'(S,\widehat{T})$. Since f is a morphism, the collection $\{x_1,\dots,x_{k-1}\}$ of points in $[x,x']$ at which $f_{|[x,x']}$ is not locally injective is finite. We let $x_0 := x$ and $x_k := x'$, and let $y_0 := y$. As p preserves alignment, for all $i \in \{0,\dots,k-1\}$, the preimage $p^{-1}(f(x_i))$ is a closed subtree of $\widehat{T}$ (Lemma F.4.10). We inductively define y_{i+1} as the projection of y_i to $p^{-1}(f(x_{i+1}))$, for $i \in \{0,\dots,k-2\}$, and we let $y_k := y'$. We then let $\gamma_i : [0,1] \to \widehat{T}$ be the straight path joining y_i to y_{i+1} in $\widehat{T}$. Since p is alignment-preserving, the path $p \circ \gamma_i$ is a continuous path joining $f(x_i)$ to $f(x_{i+1})$ in T, whose image is contained in the segment $[f(x_i),f(x_{i+1})]$. Therefore, composing with the inverse of $f_{|[x_i,x_{i+1}]}$, we get a path $\gamma_i^S : [0,1] \to S$ that joins x_i to x_{i+1}. By construction, for all $t \in [0,1]$, we have $f(\gamma_i^S(t)) = p(\gamma_i(t))$. The concatenation of all paths (γ_i^S,γ_i) is a continuous path joining (x,y) to (x',y') in $\mathcal{C}'(S,\widehat{T})$. This proves that $\mathcal{C}'(S,\widehat{T})$ is path-connected.

We will now prove that the path we have constructed has length $d((x,y),(x',y'))$, which shows that the metric space $(\mathcal{C}'(S,\widehat{T}),d)$ is geodesic. Notice that the sequence $\sigma := ((x_0,y_0),\dots,(x_k,y_k))$ constructed above is admissible. We first claim that it realizes the infimum in the definition of $d((x,y),(x',y'))$. Let $\sigma' := ((x'_0,y'_0),\dots,(x'_{k'},y'_{k'}))$ be another admissible sequence. For all $i \in \{1,\dots,k'-1\}$, let x''_i be the projection of x'_i to the segment $[x,x']$, and define inductively y''_i as the projection of x'_i to the closed subtree

$\pi^{-1}(x_i'')$. By Lemma F.6.3, the sequence σ'' we get by inserting (x_i'', y_i'') between (x_i', y_i') and (x_{i+1}', y_{i+1}') in σ' for all $i \in \{1, \dots, k'-1\}$ such that x_i' and x_{i+1}' do not project to the same point of $[x, x']$ is admissible, and $l(\sigma'') = l(\sigma')$. By construction, the sequence σ'' is a refinement of σ, so $l(\sigma) \leq l(\sigma'')$. The claim follows.

In addition, if we identify $[x_i, x_{i+1}]$ (respectively $[y_i, y_{i+1}]$) with $[0, d_S(x_i, x_{i+1})]$ (resp. $[0, d_{\widehat{T}}(y_i, y_{i+1})]$), the path (γ_i, γ_i^S) is the graph of a continuous non-decreasing map, whose length is thus equal to $d_S(x_i, x_{i+1}) + d_{\widehat{T}}(y_i, y_{i+1})$. This follows from the fact that f is isometric in restriction to $[x_i, x_{i+1}]$. Together with the claim from the above paragraph, this shows that the arc we have built from (x, y) to (x', y') has length $d((x, y), (x', y'))$.

We finally show that $\mathcal{C}'(S, \widehat{T})$ is uniquely path-connected. Assume that there exists a topological embedding $\gamma = (\gamma_S, \gamma_{\widehat{T}}) : S^1 \to \mathcal{C}'(S, \widehat{T})$ from the circle into $\mathcal{C}'(S, \widehat{T})$. The map γ_S cannot be constant, because the fiber of every point in S (under the projection map from $\mathcal{C}'(S, \widehat{T})$ to S) is a tree. As S is an $\mathbb{R}$-tree, there exists $u \in S^1$ whose γ_S-image is extremal in $\gamma_S(S^1)$. Then $\gamma_S(S^1)$ contains a segment $I \subseteq S$ whose extremity is equal to $\gamma_S(u)$. There is a subsegment $I' \subseteq I$, one of whose endpoints is equal to $\gamma_S(u)$, such that all points in the interior of I' have at least two γ_S-preimages in S^1 (one on each side of u). Hence there exists an uncountable set J of elements $s \in S^1$, with $\gamma_S(s) \neq \gamma_S(t)$ for all $s \neq t \in J$, and such that for all $s \in J$, there exists $s' \in S^1 \setminus \{s\}$ satisfying $\gamma_S(s) = \gamma_S(s')$. Injectivity of γ implies that for all $s \in S$, the segment $[\gamma_{\widehat{T}}(s), \gamma_{\widehat{T}}(s')]$ is nondegenerate. In addition, for all $s \neq t \in J$, the segments $[\gamma_{\widehat{T}}(s), \gamma_{\widehat{T}}(s')]$ and $[\gamma_{\widehat{T}}(t), \gamma_{\widehat{T}}(t')]$ are disjoint, because they project to distinct points in T. We have thus found an uncountable collection of pairwise disjoint nondegenerate segments in $\widehat{T}$, which contradicts separability of $\widehat{T}$ (which follows from minimality). Therefore, there is no topological embedding from the circle into $\mathcal{C}'(S, \widehat{T})$. We have thus proved that any two points $(x, y), (x', y') \in \mathcal{C}'(S, \widehat{T})$ are joined by a unique embedded topological arc, and this arc has length $d((x, y), (x', y'))$. This shows that $\mathcal{C}'(S, \widehat{T})$ is an $\mathbb{R}$-tree.

For all $g \in G$, the image of an admissible sequence under the action of g is again admissible (by equivariance of f and p). Therefore, as G acts by isometries on each of the trees S and $\widehat{T}$, it also acts by isometries on $\mathcal{C}'(S, \widehat{T})$. As all peripheral subgroups of G act elliptically in both S and $\widehat{T}$, they also act elliptically in $\mathcal{C}'(S, \widehat{T})$. Hence $\mathcal{C}'(S, \widehat{T})$ is a $(G, \mathcal{F})$-tree. □

Definition F.6.5. *Let S, T and $\widehat{T}$ be tame $(G, \mathcal{F})$-trees, such that there exists a 1-Lipschitz alignment-preserving map $p : \widehat{T} \to T$ and a morphism $f : S \to T$. The* pullback $\mathcal{C}(S, \widehat{T})$ *induced by f and p is defined to be the G-minimal subtree of $\mathcal{C}'(S, \widehat{T})$.*

Lemma F.6.6. *Let S, T and $\widehat{T}$ be $(G, \mathcal{F})$-trees, let $p : \widehat{T} \to T$ be an alignment-preserving map, and let $f : S \to T$ be a morphism. Assume that there exists $k \in \mathbb{N}$ such that S and $\widehat{T}$ are k-tame. Then $\mathcal{C}(S, \widehat{T})$ is k-tame.*

Proof. Let $g \in G$, and let $I := [(x, y), (x', y')]$ be an arc in $\mathcal{C}(S, \widehat{T})$. Assume that there exists $l \in \mathbb{N}$ such that $g^l I = I$. Then both $[x, x'] \subseteq S$ and $[y, y'] \subseteq \widehat{T}$ are fixed by g^l. Since S and $\widehat{T}$ are k-tame, we have $g^k[x, x'] = [x, x']$ and $g^k[y, y'] = [y, y']$, so $g^k I = I$. This implies that $\mathcal{C}(S, \widehat{T})$ is k-tame. □

Let now S, T and $\widehat{T}$ be k-tame $(G, \mathcal{F})$-trees, such that there exists a 1-Lipschitz alignment-preserving map $p : \widehat{T} \to T$. Let γ be a k-tame optimal liberal folding path from S to T, guided by an optimal morphism $f : S \to T$. For all $t \in \mathbb{R}_+$, let $\widehat{\gamma}(t)$ denote the pullback $\mathcal{C}(\gamma(t), \widehat{T})$. Lemma F.6.6 implies that for all $t \in \mathbb{R}_+$, the $(G, \mathcal{F})$-tree $\widehat{\gamma}(t)$ is k-tame. For all $t \in \mathbb{R}_+$, there is an alignment-preserving map $p_t : \widehat{\gamma}(t) \to \gamma(t)$.

Let $t < t' \in \mathbb{R}_+$. We can define a G-equivariant map $\widetilde{f}_{t,t'} : \mathcal{C}'(\gamma(t), \widehat{T}) \to \mathcal{C}'(\gamma'(t), \widehat{T})$ by setting $\widetilde{f}_{t,t'}(x,y) := (f_{t,t'}(x), y)$. Indeed, if $(x,y) \in \mathcal{C}'(\gamma(t), \widehat{T})$, then $f_{t,L}(x) = p(y)$, hence $f_{t',L}(f_{t,t'}(x)) = p(y)$. We claim that the map $\widetilde{f}_{t,t'}$ is a morphism. Indeed, let $((x,y),(x',y')) \in \mathcal{C}'(\gamma(t), \widehat{T})^2$. As $f_{t,t'}$ is a morphism, the segment $[x,x'] \subseteq \gamma(t)$ can be subdivided into finitely many subsegments $[x_i, x_{i+1}]$, in restriction to which $f_{t,t'}$ is an isometry. Using the arguments from the proof of Lemma F.6.4, we see that the segment $[(x,y),(x',y')]$ can be subdivided into finitely many subsegments that are either of the form $[(x_i,y_i),(x_{i+1},y_{i+1})]$ or $[(x_i,y_i),(x_i,y_i')]$, in restriction to which $\widetilde{f}_{t,t'}$ is an isometry.

We now prove that $\widetilde{f}_{t,t'}$ induces an optimal morphism $\widehat{f}_{t,t'} : \widehat{\gamma}(t) \to \widehat{\gamma}(t')$. For all $t \in \mathbb{R}$, the map π_t preserves alignment and is surjective by minimality of $\gamma(t)$, so every arc in $\gamma(t)$ lifts to an arc in $\widehat{\gamma}(t)$. We also notice that for all $t \in \mathbb{R}$, the fibers of the map π_t isometrically embed into $\widehat{T}$, and hence into $\widehat{\gamma}(t')$ for all $t' > t$. Let $\widehat{x} \in \widehat{\gamma}(t)$. Let $x := \pi_t(\widehat{x})$. If $\pi_t^{-1}(x)$ is not reduced to a point, then we can find a direction d at $\widehat{x}$ in $\widehat{\gamma}(t)$ that is contained in $\pi_t^{-1}(x)$. If there exists another direction d' at $\widehat{x}$ contained in $\pi_t^{-1}(x)$, then the turn (d,d') is legal. Otherwise, minimality of $\widehat{\gamma}(t)$ shows the existence of a direction d' at x that is not contained in $\pi_t^{-1}(x)$. We claim that the directions d and d' cannot be identified by $\widehat{f}_{t,t'}$. Indeed, otherwise, any small nondegenerate arc I contained in the direction d' would be mapped to a point by $\pi_{t'} \circ \widehat{f}_{t,t'}$, and hence by $f_{t,t'} \circ \pi_t$, contradicting the fact that $f_{t,t'}$ is a morphism and $\pi_t(I)$ is a nondegenerate arc. If $\pi_t^{-1}(x)$ is reduced to a point, then every legal turn at x lifts to a legal turn at $\widehat{x}$, and optimality of $f_{t,t'}$ ensures the existence of such turns.

For all $t \in \mathbb{R}_+$, we denote by $\widehat{\gamma}(t^-)$ (resp. $\widehat{\gamma}(t^+)$) the limit of the trees $\widehat{\gamma}(s)$ as s converges to t from below (resp. from above). This exists by monotonicity of length functions along the path $\widehat{\gamma}$, which comes from the existence of the morphisms $\widehat{f}_{t,t'}$ for all $t,t' \in \mathbb{R}$. As the space of k-tame $(G,\mathcal{F})$-trees is closed, the trees $\widehat{\gamma}(t^-)$ and $\widehat{\gamma}(t^+)$ are k-tame. It follows from Proposition F.4.11 that there are alignment-preserving maps $\pi_{t^-} : \widehat{\gamma}(t^-) \to \gamma(t)$ and $\pi_{t^+} : \widehat{\gamma}(t^+) \to \gamma(t)$. Proposition F.4.11 also implies that for all $s < t$, there are 1-Lipschitz maps $\widehat{f}_{s,t^-}$ and $\widehat{f}_{s,t^+}$ from $\widehat{\gamma}(s)$ to the metric completions of both $\widehat{\gamma}(t^-)$ and $\widehat{\gamma}(t^+)$. For all $s > t$, there are 1-Lipschitz maps $\widehat{f}_{t^-,s}$ and $\widehat{f}_{t^+,s}$ from both $\widehat{\gamma}(t^-)$ and $\widehat{\gamma}(t^+)$ to the metric completion of $\widehat{\gamma}(s)$. There is also a 1-Lipschitz map $\widehat{f}_{t^-,t^+}$ from $\widehat{\gamma}(t^-)$ to the metric completion of $\widehat{\gamma}(t^+)$. We will show that all these maps are morphisms. We will make use of the following easy lemma, that was noticed by Guirardel and Levitt in [GL07b, Lemma 3.3].

Lemma F.6.7. *Let T_1, T_2, T_3 be $\mathbb{R}$-trees. Let $f : T_1 \to T_3$ be a morphism, and let $\phi : T_1 \to T_2$ and $\psi : T_2 \to T_3$ be 1-Lipschitz surjective maps, such that $f = \psi \circ \phi$. Then ϕ and ψ are morphisms.* □

Lemma F.6.8. *For all $t \in \mathbb{R}$, the map $\widehat{f}_{t^-,t^+}$ is an optimal morphism, and $\pi_{t^+} \circ \widehat{f}_{t^-,t^+} = \pi_{t^-}$. For all $s < t$, the maps $\widehat{f}_{s,t^-}$ and $\widehat{f}_{s,t^+}$ are optimal morphisms. For all $s > t$, the maps $\widehat{f}_{t^-,s}$ and $\widehat{f}_{t^+,s}$ are optimal morphisms.*

Proof. We refer to [3, Section 4] for notations and definitions. Let $\widehat{\gamma}^\omega(t^-)$ (resp. $\widehat{\gamma}^\omega(t^+)$) be an ultralimit of a sequence of trees $\widehat{\gamma}(s)$, with s converging to t^- (resp. to t^+). Then $\widehat{\gamma}^\omega(t^-)$ (resp. $\widehat{\gamma}^\omega(t^+)$) contains $\widehat{\gamma}(t^-)$ (resp. $\widehat{\gamma}(t^+)$) as its G-minimal subtree. In [3, Theorem 4.3], the map $\widehat{f}_{t^-,t^+}$ is constructed from the ultralimit $\widehat{f}^\omega_{t^-,t^+} : \widehat{\gamma}^\omega(t^-) \to \widehat{\gamma}^\omega(t^+)$ of the maps $\widehat{f}_{s,s'}$, by restricting to the minimal subtree $\widehat{\gamma}(t^-)$, and projecting to the closure of the minimal subtree $\widehat{\gamma}(t^+)$. For all $s < t < s'$, we have $\widehat{f}_{s,s'} = \widehat{f}^\omega_{t^+,s'} \circ \widehat{f}^\omega_{t^-,t^+} \circ \widehat{f}^\omega_{s,t^-}$ (where the maps $\widehat{f}^\omega_{t^+,s'}$ and $\widehat{f}^\omega_{s,t^-}$ are defined similarly as ultralimits). The difficulty might come from projection to minimal subtrees in the definition of $\widehat{f}_{t^-,t^+}$. However, optimality of $\widehat{f}_{s,s'}$ implies that minimal subtrees are mapped to minimal subtrees, so we get $\widehat{f}_{s,s'} = \widehat{f}_{t^+,s'} \circ \widehat{f}_{t^-,t^+} \circ \widehat{f}_{s,t^-}$. Using Lemma F.6.7, this implies that all maps $\widehat{f}_{t^+,s'}$, $\widehat{f}_{t^-,t^+}$ and $\widehat{f}_{s,t^-}$ are morphisms. We can similarly prove that $\widehat{f}_{s,t^+}$ and $\widehat{f}_{t^-,s'}$ are morphisms. Any ultralimit of alignment-preserving maps is again alignment-preserving, and hence maps minimal subtrees to minimal subtrees. We similarly deduce that $\pi_{t^+} \circ \widehat{f}_{t^-,t^+} = \pi_{t^-}$. $\square$

Lemma F.6.9. *Let T, T' and $\overline{T}$ be minimal $(G,\mathcal{F})$-trees, let $f : T \to T'$ be an optimal morphism, and let $(T_t)_{t\in[0,L]}$ be a folding path guided by f. Let $\pi : T \to \overline{T}$ and $\pi' : T' \to \overline{T}$ be alignment-preserving maps, such that $\pi' \circ f = \pi$. Then for all $t \in [0, L]$, there is an alignment-preserving map from T_t to $\overline{T}$.*

Proof. Let $y \in \overline{T}$, and let $x \in T_t$ be a preimage of y under the map $\pi' \circ f_{t,L}$. As $\pi' \circ f = \pi$, all $f_{0,t}$-preimages of x in T map to y under π, so $(\pi' \circ f_{t,L})^{-1}(y) = f_{0,t}(\pi^{-1}(y))$. As π preserves alignment, the preimage $\pi^{-1}(y)$ is connected, and therefore $(\pi' \circ f_{t,L})^{-1}(y)$ is connected. This implies that $\pi' \circ f_{t,L}$ preserves alignment. $\square$

The path $\widehat{\gamma}$, which we call the *pullback* of γ induced by p, is not an optimal liberal folding path, because it may be discontinuous. There is a way of turning $\widehat{\gamma}$ into a (continuous) optimal liberal folding path. As length functions can only decrease along the path $\widehat{\gamma}$, there are (at most) countably many times t at which $\widehat{\gamma}(t^-) \neq \widehat{\gamma}(t^+)$. Let t be one of these times. Consider an optimal morphism from $\widehat{\gamma}(t^-)$ to $\widehat{\gamma}(t^+)$ guided by $\widehat{f}_{t^-,t^+}$, and let $\gamma'(t)$ be a tree lying on this path. Since $\pi_{t^+} \circ \widehat{f}_{t^-,t^+} = \pi_{t^-}$ (Lemma F.6.8), Lemma F.6.9 implies that $\gamma'(t)$ collapses to $\gamma(t)$. By inserting these optimal liberal folding paths at all discontinuity times, we get a continuous path $\widehat{\gamma}^{cont}$, called a *continuous pullback* of γ induced by p.

Again, there is a discrete version of the above construction. The following proposition follows from the above analysis.

Proposition F.6.10. *Let T and $\widehat{T}$ be tame $(G,\mathcal{F})$-trees, and let $p : \widehat{T} \to T$ be a 1-Lipschitz alignment-preserving map. Let $(\gamma(n))_{n\in\mathbb{N}}$ be a tame optimal folding sequence ending at T. Then there exists a tame $(G,\mathcal{F})$-tree $\widehat{\gamma}(\infty)$, and a tame optimal folding sequence $(\widehat{\gamma}(n))_{n\in\mathbb{N}}$ ending at $\widehat{\gamma}(\infty)$, such that*

- *for all $n \in \mathbb{N}$, the tree $\widehat{\gamma}(n)$ collapses to $\gamma(n)$, and*
- *there is an alignment-preserving map $p_\infty : \widehat{\gamma}(\infty) \to T$, and a morphism $\widehat{f}_\infty : \widehat{\gamma}(\infty) \to \widehat{T}$, such that $p_\infty = p \circ \widehat{f}_\infty$.*

$\square$

A sequence $(\widehat{\gamma}(n))_{n\in\mathbb{N}}$ satisfying the conclusions of Proposition F.6.10 will be called a *pullback* of $(\gamma(n))_{n\in\mathbb{N}}$ induced by p. We note that in general, the morphism $\widehat{f}_\infty : \widehat{\gamma}(\infty) \to \widehat{T}$ given by Proposition F.6.10 need not be an isometry. In other words, the sequence $(\widehat{\gamma}(n))_{n\in\mathbb{N}}$ need not end at $\widehat{T}$. The following proposition provides a condition under which it does.

Proposition F.6.11. *Let T and $\widehat{T}$ be tame $(G,\mathcal{F})$-trees, and let $p : \widehat{T} \to T$ be an alignment-preserving map. Let $(\gamma(n))_{n\in\mathbb{N}}$ be a tame optimal folding sequence ending at T. If T is $\mathcal{Z}$-incompatible, then any pullback of $(\gamma(n))_{n\in\mathbb{N}}$ induced by p ends at $\widehat{T}$.*

Proof. There exists a 1-Lipschitz alignment-preserving map from $\widehat{\gamma}(\infty)$ to T. As T is not compatible with any $\mathcal{Z}$-splitting, this implies that $\widehat{\gamma}(\infty)$ has dense orbits. We also know that there exists a morphism from $\widehat{\gamma}(\infty)$ to $\widehat{T}$. Corollary F.4.13 implies that this morphism is an isometry. $\square$

F.7 Boundedness of the set of reducing splittings of a tame $(G, \mathcal{F})$-tree

We will now prove the following result that bounds the diameter in $FZ(G, \mathcal{F})$ of the set of reducing splittings of a tame $(G, \mathcal{F})$-tree $T \notin \mathcal{X}(G, \mathcal{F})$. We recall there is a map $\psi : \mathcal{O}(G, \mathcal{F}) \to FZ(G, \mathcal{F})$, which naturally extends to the set of tame $(G, \mathcal{F})$-trees having a nontrivial simplicial part. By definition, any tame optimal sequence ending at T comes with morphisms $f_n : T_n \to T$ and is nonstationary. By Corollary F.4.13, this implies that for all $n \in \mathbb{N}$, the tree T_n does not have dense orbits. The existence of the Levitt decomposition of T_n as a graph of actions (Proposition F.4.18) implies that $\psi(T_n)$ is well-defined for all $n \in \mathbb{N}$. The goal of the present section is to prove the following theorem.

Theorem F.7.1. *There exists $C_1 \in \mathbb{R}$ so that for all tame $(G, \mathcal{F})$-trees $T \notin \mathcal{X}(G, \mathcal{F})$, the diameter of $\mathcal{R}^2(T)$ in $FZ(G, \mathcal{F})$ is at most C_1. Furthermore, there exists $C_2 \in \mathbb{R}$ such that for all tame $(G, \mathcal{F})$-trees $T \notin \mathcal{X}(G, \mathcal{F})$, all ψ-images of tame optimal folding sequences ending at T eventually stay at distance at most C_2 from $\mathcal{R}^2(T)$ in $FZ(G, \mathcal{F})$.*

Again, the proof of Theorem F.7.1 adapts without change to the case of $\mathcal{Z}^{max}$-splittings to give the following statement.

Theorem F.7.2. *($\mathcal{Z}^{max}$-analogue of Theorem F.7.1) There exists $C_1 \in \mathbb{R}$ so that for all $\mathcal{Z}^{max}$-tame trees $T \notin \mathcal{X}^{max}(G, \mathcal{F})$, the diameter of $\mathcal{R}^{2,max}(T)$ in $FZ^{max}(G, \mathcal{F})$ is at most C_1. Furthermore, there exists $C_2 \in \mathbb{R}$ such that for all $\mathcal{Z}^{max}$-tame trees $T \notin \mathcal{X}^{max}(G, \mathcal{F})$, all ψ^{max}-images of tame optimal folding sequences ending at T eventually stay at distance at most C_2 from $\mathcal{R}^{2,max}(T)$ in $FZ^{max}(G, \mathcal{F})$.*

Remark F.7.3. Here again, it is crucial to work with cyclic splittings rather than free splittings. Indeed, it is possible to find a tree $T \in \overline{cv_N}$ that is compatible with infinitely many free splittings of F_N that do not lie in a region of finite diameter of the free splitting graph FS_N. Here is an example. The tree T_2 from Example F.5.31 is compatible with all the free splittings of F_N that are determined by arcs on S that lie in the subsurface S_1. These arcs form an unbounded subset of the arc graph of S [MS13]. In addition, it is known that the arc graph of S embeds quasi-isometrically into the free splitting graph of F_N [HH14, Lemma 4.17 and Proposition 4.18].

F.7.1 The case where T has a nontrivial simplicial part

Proposition F.7.4. *Let T be a tame $(G, \mathcal{F})$-tree having a nontrivial simplicial part. Let $(T_n)_{n \in \mathbb{N}}$ be a tame optimal folding sequence ending at T. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, the tree T_n has a nontrivial simplicial part, and we have $d_{FZ(G,\mathcal{F})}(\psi(T_n), \psi(T)) \leq 2$.*

Proof. Let e be a simplicial edge in T. If T contains an edge with nontrivial stabilizer, then we choose e to be an edge whose stabilizer $\langle g_0 \rangle$ is nontrivial, such that no proper root of g_0 fixes an arc in T. The tree T splits as graph of actions $\mathcal{G}$, dual to the edge e. We will prove that for $n \in \mathbb{N}$ sufficiently large, the tree T_n splits as a graph of actions over the skeleton of $\mathcal{G}$. This will imply that both T_n and T are compatible with this skeleton, and therefore $d_{FZ(G,\mathcal{F})}(\psi(T_n), \psi(T)) \leq 2$. We denote by $f_n : T_n \to T$ the morphism given by the definition of a tame optimal folding sequence.

We first assume that the edge e projects to a separating edge in $\mathcal{G}$. We denote by T^A and T^B the adjacent vertex trees, with nontrivial stabilizers A and B. Let K denote the length of e in T, and let $\epsilon > 0$, chosen to be very small compared to K. For all $n \in \mathbb{N}$, let T_n^A be the closure of the A-minimal subtree in T_n. We will first show that for $n \in \mathbb{N}$ large enough, the f_n-image of T_n^A is contained in an ϵ-neighborhood of T^A in T. The analogous statement will also hold for the B-minimal subtrees.

Assume that A is not elliptic in T. Let $s \in A$ be hyperbolic in T. The Kurosh decomposition of A reads as

$$A := g_1 G_{i_1} g_1^{-1} * \cdots * g_s G_{i_s} g_s^{-1} * F.$$

We let X_A be a finite set made of a free basis of F, and a nontrivial element in each of the peripheral subgroups $g_j G_{i_j} g_j^{-1}$. Let X'_A be the finite set consisting of s, and of the elements of the form $s.s^a$ for all $a \in X_A$ such that the axes of s and s^a do not intersect in T. Notice that all elements in X'_A are hyperbolic in T, and hence in T_n for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$ the translates of the axes in T_n of elements in X'_A cover T_n^A, because the same holds true in T. For all $g \in X'_A$, the f_n-image of the axis of g in T_n is contained in the ϵ-neighborhood of the axis of g in T for n large enough. Our claim follows in this case.

Assume now that A is elliptic in T, and let $s \in A$ be an element that fixes a unique point p in T (this exists by minimality of T). If s is hyperbolic in T_n, then the axis of s is mapped to the ϵ-neighborhood of p for n large enough. If s is elliptic in T_n, then s fixes a unique point $p_n \in T_n$ (otherwise, by optimality, the f_n-image of a nondegenerate arc fixed by s would be a nondegenerate arc fixed by s in T), and $f_n(p_n) = p$. In both cases, we denote by Y_n the union of the characteristic sets in T_n of all elements s^a with $a \in X_A$. As $||s.s^a||_{T_n}$ converges to 0 for all $a \in X_A$, the convex hull of Y_n in T_n is contained in the ϵ-neighborhood of Y_n, for n large enough. The translates of the convex hull of Y_n cover T_n^A. Hence the f_n-image of T_n^A is contained in an ϵ-neighborhood of p, and the claim follows.

Therefore, for n large enough, all translates of T_n^A and T_n^B are disjoint, and the stabilizer of T_n^A (resp. T_n^B) is equal to A (resp. B). For n large enough, the bridge between T_n^A and T_n^B has length $l_n \geq K - \epsilon$. Since $g_0 \in A \cap B$, the element g_0 has to be elliptic in both T_n^A and T_n^B, and therefore g_0 fixes the bridge between T_n^A and T_n^B. Hence we can form a graph of actions S_n over the skeleton of $\mathcal{G}$, with vertex trees T_n^A and T_n^B, whose attaching points are given by the extremities of the bridge between T_n^A and T_n^B. The unique orbit of edges e_n of this splitting is assigned length l_n. We claim that S_n is isometric to T_n, which will prove that T_n splits as a graph of actions over $\mathcal{G}$. By construction, there is a morphism $f_n : S_n \to T_n$, which is an isometry in restriction to both T_n^A and T_n^B. The morphism f_n cannot identify a subarc of e_n with a subarc in either T_n^A or T_n^B by definition of l_n. It cannot either identify a subarc of e_n with one of its translates by our choice of the edge e (because otherwise T would contain an arc fixed by a proper subgroup of $\langle g_0 \rangle$). This implies that f_n is an isometry, and proves the claim.

In the case of an HNN extension, we denote by C the vertex group and by t a stable letter. The same argument as above yields $l_n \geq K - \epsilon$, where this time l_n denotes the distance between the closure T_n^C of the C-minimal subtree of T_n, and its t-translate (one has to be slightly careful if $G = F_2$, because in this case C is cyclic, so the C-minimal subtree of T_n is not well-defined; we leave the argument to the reader in this case). We can similarly define the graph of actions S_n over the skeleton of $\mathcal{G}$, with vertex tree T_n^C, with attaching points the extremities of the bridge between T_n^C and its t-translate. The unique orbit of edges e_n is assigned length l. In this case, the morphism $f_n : S_n \to T_n$ may fail to be an isometry, however it can only fold e_n with $t'\overline{e_n}$, where t' is a stable letter of the HNN extension. As t' is hyperbolic in T, and hence in T_n, the edge e_n is not entirely folded. Again, we get that T_n splits as a graph of actions over $\mathcal{G}$. $\square$

F.7.2 The general case

Proposition F.7.5. *Let T_1, T_2 and $\overline{T}$ be tame $(G,\mathcal{F})$-trees. Assume that there exist alignment-preserving maps $p_1 : T_1 \to \overline{T}$ and $p_2 : T_2 \to \overline{T}$, and a morphism $f : T_1 \to T_2$, such that $p_1 = p_2 \circ f$. If T_1 and T_2 both have a nontrivial simplicial part, then $\mathcal{R}^1(T_1) \cup \mathcal{R}^1(T_2)$ has diameter at most 4 in $FZ(G,\mathcal{F})$.*

Proof. If $\overline{T}$ does not have dense orbits, then any splitting in $\mathcal{R}^1(T_1) \cup \mathcal{R}^1(T_2)$ is at distance 1 from a splitting defined by a simplicial edge in $\overline{T}$. From now on, we assume that $\overline{T}$ has dense orbits. For all $i \in \{1,2\}$, let $\mathcal{Y}_i$ be the collection of all nondegenerate vertex subtrees with dense orbits of the Levitt decomposition of T_i as a graph of actions (Proposition F.4.18). Let $\mathcal{Z}_i$ be the collection of all connected components of the union of the closures of the edges in the simplicial part of T_i. Then $\mathcal{Y}_i \cup \mathcal{Z}_i$ is a transverse covering of T_i. By definition, the vertex set of its skeleton is equal to $\mathcal{Y}_i \cup \mathcal{Z}_i \cup V_i$, where V_i is the set of intersection points between distinct trees in $\mathcal{Y}_i \cup \mathcal{Z}_i$. Notice that trees in $\mathcal{Y}_i$ are pairwise disjoint, and similarly trees in $\mathcal{Z}_i$ are pairwise disjoint. Therefore, any vertex of V_i is joined by an edge to exactly one vertex in $\mathcal{Y}_i$, and one vertex in $\mathcal{Z}_i$.

We first explain how to reduce to the case where for all $i \in \{1,2\}$, the map p_i is isometric in restriction to the dense orbits part of T_i. For all $i \in \{1,2\}$, let $\overline{T_i}$ be the tree obtained from the decomposition of T_i as a graph of actions by replacing each subtree $Y \in \mathcal{Y}_i$ by $p_i(Y)$, each $v \in V_i$ by $p_i(v)$, and leaving each Z_i unchanged. Let $\overline{\mathcal{Y}_i}$ be the collection of all subtrees of $\overline{T_i}$ defined by the trees $p_i(Y)$ with $Y \in \mathcal{Y}_i$. All f-images of trees $Y \in \mathcal{Y}_1$ are contained in the dense orbits part of T_2, and $p_2(f(Y)) = p_1(Y)$. Hence f induces a 1-Lipschitz map $\overline{f} : \overline{T_1} \to \overline{T_2}$, which is an isometry in restriction to each of the subtrees in $\overline{\mathcal{Y}_1}$. By modifying the lengths of the edges in the simplicial part of T_1 if needed, we may turn f into a morphism. Some of these edges may have to be assigned length 0, however some edge with positive length must survive because T_2 has a nontrivial simplicial part. Denoting by $\overline{p_1} : \overline{T_1} \to \overline{T}$ and by $\overline{p_2} : \overline{T_2} \to \overline{T}$ the induced alignment-preserving maps, we still have $\overline{p_1} = \overline{p_2} \circ \overline{f}$. In this way, as $\mathcal{R}^1(T_i) \subseteq \mathcal{R}^1(\overline{T_i})$ for all $i \in \{1,2\}$, we have reduced the proof of Proposition F.7.5 to the case where all maps f, p_1 and p_2 are isometric in restriction to trees in the dense orbits part of their origin. From now on, we assume that we are in this case.

We now prove that $f(\mathcal{Y}_1) \cup f(\mathcal{Z}_1)$ is a transverse covering of T_2. This follows from the following observations.

- If $Y_1 \neq Y_2 \in \mathcal{Y}_1$, then $f(Y_1) \cap f(Y_2)$ contains at most one point. Indeed, the map p_2 is isometric on both $f(Y_1)$ and $f(Y_2)$. If $f(Y_1) \cap f(Y_2)$ contained a nondegenerate arc I, then $p_2 \circ f(Y_1) \cap p_2 \circ f(Y_2)$ would contain the nondegenerate arc $p_2(I)$. In other words, the intersection $p_1(Y_1) \cap p_1(Y_2)$ would be nondegenerate. This is impossible because p_1 preserves alignment and $Y_1 \cap Y_2 = \emptyset$.
- If $Z_1 \neq Z_2 \in \mathcal{Z}_1$, then $f(Z_1) \cap f(Z_2) = \emptyset$. Indeed, since $\overline{T}$ has dense orbits, the image $p_1(Z_i)$ is a point for all $i \in \{1,2\}$, so $p_2(f(Z_i))$ is a point. If $f(Z_1) \cap f(Z_2) \neq \emptyset$, then the subtrees $f(Z_1)$ and $f(Z_2)$ would be mapped to the same point $z \in \overline{T}$ under p_2. Let $z_1 \in Z_1$ and $z_2 \in Z_2$. Then $p_1(z_1) = p_1(z_2)$, so p_1 collapses the segment $[z_1, z_2]$ to a point. Since $Z_1 \neq Z_2$, it follows from the description of the skeleton $\mathcal{G}_1$ that the segment $[z_1, z_2]$ intersects some tree $Y \in \mathcal{Y}_1$ along a nondegenerate segment I, and $p_1(I)$ is nondegenerate, a contradiction.
- Similarly, the f-images of two trees $Y_1 \in \mathcal{Y}_1$ and $Z_1 \in \mathcal{Z}_1$ intersect nontrivially if and only if $Y_1 \cap Z_1 \neq \emptyset$ (otherwise the bridge between Y_1 and Z_1 would be collapsed by p_1), and in this case their intersection is the f-image of the intersection point between Y_1 and Z_1.

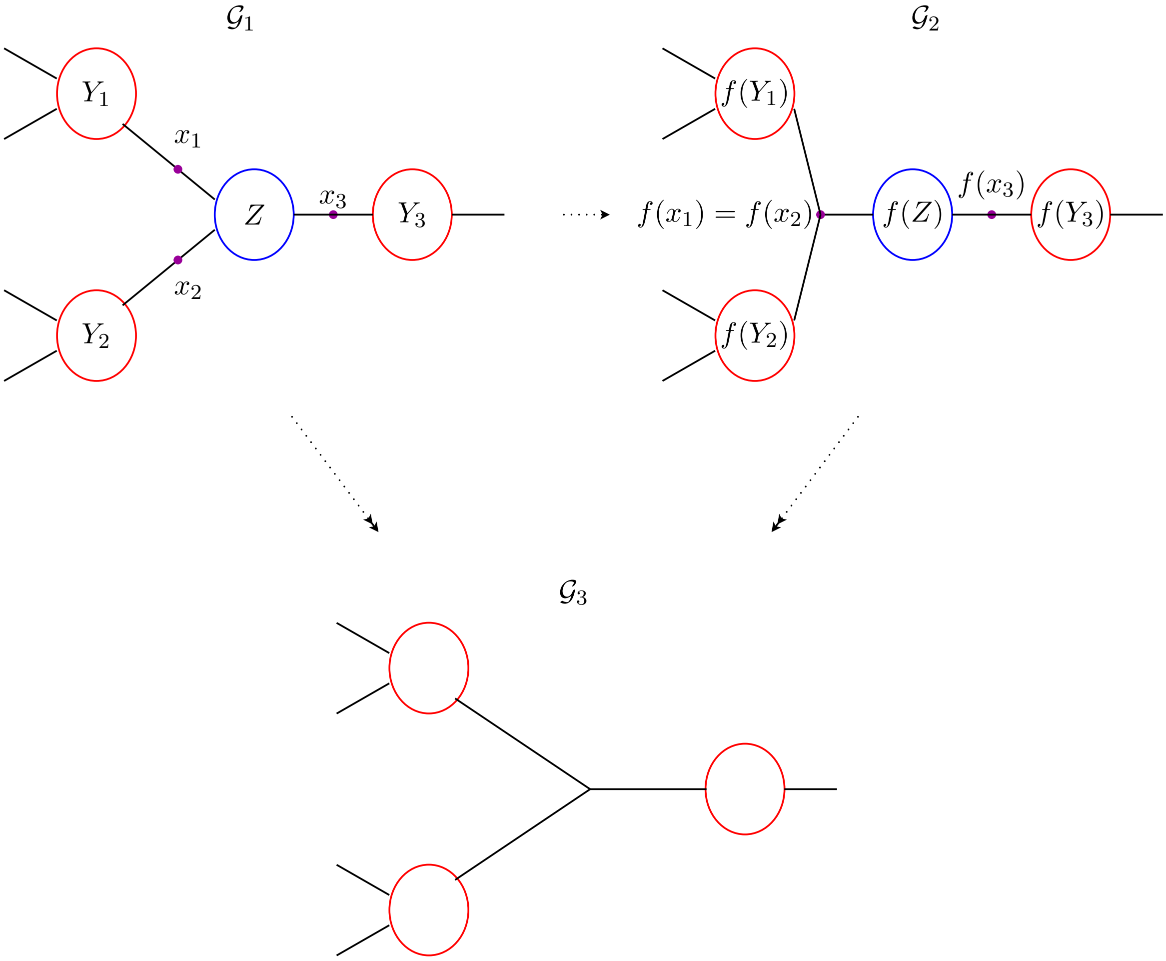

Figure F.10: The skeletons $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ in the proof of Proposition F.7.5.

This implies that the union $f(\mathcal{Y}_1) \cup f(\mathcal{Z}_1)$ is a transverse covering of T_2. We denote by $\mathcal{G}_2$ its skeleton, which is depicted on Figure F.10. The above observations imply that the map f induces a map from $\mathcal{G}_1$ to $\mathcal{G}_2$, that sends edges to edges, and is injective in a neighborhood of any vertex in $\mathcal{Y}_1$. The map f can fold several edges attached to a vertex in $\mathcal{Z}_1$. In particular, the map f induces an isomorphism between the graphs $\mathcal{G}'_1$ and $\mathcal{G}'_2$, obtained by equivariantly collapsing the 1-neighborhood of all vertices in $\mathcal{Z}_1$ (resp. $f(\mathcal{Z}_1)$). We let $\mathcal{G}_3 := \mathcal{G}'_1 = \mathcal{G}'_2$, see Figure F.10.

Let Y be one of the nontrivial subtrees in $\mathcal{Y}_1$ (this exists because $\overline{T}$ has dense orbits). We denote by G_Y the stabilizer of Y. The subgroup G_Y is a vertex stabilizer in a $\mathcal{Z}$-splitting, obtained from T_1 by collapsing all vertex trees of the Levitt decomposition to points. Therefore, we have $\mathrm{rk}_K(G_Y) < +\infty$ (see [7, Corollary 4.5]). Let S' be a $\mathcal{Z}$-splitting of $(G_Y, \mathcal{F}_{|G_Y})$, such that the stabilizers of all attaching points in Y are elliptic in S' (this exists by Proposition F.5.18). Let e be an edge of S'.

For all $i \in \{1,2,3\}$, let T'_i be the $(G,\mathcal{F})$-tree obtained by replacing Y by S' in $\mathcal{G}_i$. This is well-defined because attaching points and edge stabilizers are the same in neighborhoods of Y and $f(Y)$. Then T'_1 and T'_2 both collapse to T'_3, and for all $i \in \{1,2\}$, the tree T'_i collapses to $\psi(T_i)$. This implies that for all $i \in \{1,2\}$, we have $d_{FZ(G,\mathcal{F})}(\psi(T_i), T'_3) \leq 1$, and hence $d_{FZ(G,\mathcal{F})}(\psi(T_1), \psi(T_2)) \leq 2$. Since T_i has a nontrivial simplicial part, the set $\mathcal{R}^1(T_i)$ is contained in the 1-neighborhood of $\psi(T_i)$ in $FZ(G,\mathcal{F})$. Hence $\mathcal{R}^1(T_1) \cup \mathcal{R}^1(T_2)$ has diameter at most 4 in $FZ(G,\mathcal{F})$. $\square$

We now finish the proof of Theorem F.7.1. In the case where $\mathcal{R}^1(T) \neq \emptyset$, we start by proving the following lemma.

Lemma F.7.6. *There exists a constant $C > 0$ such that the following holds. Let T be a tame $(G,\mathcal{F})$-tree, such that $\mathcal{R}^1(T) \neq \emptyset$. Let $(\gamma(n))_{n\in\mathbb{N}}$ be a tame optimal folding sequence ending at T. Then there exists $n_0 \in \mathbb{N}$ such that the diameter in $FZ(G,\mathcal{F})$ of the set $\psi(\gamma([n_0,+\infty))) \cup \mathcal{R}^1(T)$ is at most C.*

Proof. The case of trees without dense orbits has been dealt with in Proposition F.7.4, hence we can assume T to have dense orbits. Let $S \in \mathcal{R}^1(T)$, let $\widehat{T} := T + S$, and let $p : T + S \to T$ be the corresponding 1-Lipschitz alignment-preserving map. Let $(\widehat{\gamma}(n))_{n\in\mathbb{N}}$ be a pullback of $(\gamma(n))_{n\in\mathbb{N}}$ induced by p, provided by Proposition F.6.10: the sequence $(\widehat{\gamma}(n))_{n\in\mathbb{N}}$ is a tame optimal folding sequence that ends at a tame tree $\widehat{\gamma}(\infty)$, and there is a 1-Lipschitz alignment-preserving map $p_\infty : \widehat{\gamma}(\infty) \to T$, and a morphism $\widehat{f}_\infty : \widehat{\gamma}(\infty) \to T + S$, such that $p_\infty = p \circ \widehat{f}_\infty$. As $T + S$ has a nontrivial simplicial part, so does $\widehat{\gamma}(\infty)$. Proposition F.7.4 implies that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $d_{FZ(G,\mathcal{F})}(\psi(\widehat{\gamma}(n)), \psi(\widehat{\gamma}(\infty))) \leq 4$. By applying Proposition F.7.5 to $T_1 = \widehat{\gamma}(\infty)$ and $T_2 = \widehat{T}$, we get that $d_{FZ(G,\mathcal{F})}(\psi(\widehat{\gamma}(\infty)), \psi(\widehat{T})) \leq 2$. So for all $n \geq n_0$, we have $d_{FZ(G,\mathcal{F})}(\psi(\widehat{\gamma}(n)), \psi(\widehat{T})) \leq 6$. As $\widehat{\gamma}(n)$ collapses to $\gamma(n)$, and $\widehat{T}$ collapses to S, this implies that for all $n \geq n_0$, we have $d_{FZ(G,\mathcal{F})}(\psi(\gamma(n)), S) \leq 8$. The simplicial tree S has been chosen independently from the sequence $(\gamma(n))_{n\in\mathbb{N}}$, so the above inequality holds for all $S \in \mathcal{R}^1(T)$. This implies that the distance in $FZ(G,\mathcal{F})$ between any two splittings $S, S' \in \mathcal{R}^1(T)$ is at most 16, and proves the lemma. $\square$

Lemma F.7.7. *There exists a constant $C > 0$ such that the following holds. Let T and T' be tame $(G,\mathcal{F})$-trees. Assume that T admits a 1-Lipschitz alignment-preserving map onto T', and that $\mathcal{R}^1(T') \neq \emptyset$. Let γ be a tame optimal liberal folding sequence ending at T. Then there exists $n_0 \in \mathbb{N}$ such that the diameter in $FZ(G,\mathcal{F})$ of the set $\psi(\gamma([n_0,+\infty))) \cup \mathcal{R}^1(T')$ is at most C.*

Proof. Let $p : T \to T'$ be a 1-Lipschitz alignment-preserving map, and let $(\overline{\gamma}(n))_{n\in\mathbb{N}}$ be a collapse of $(\gamma(n))_{n\in\mathbb{N}}$ induced by p, provided by Proposition F.6.2: the sequence $(\overline{\gamma}(n))_{n\in\mathbb{N}}$ is a tame optimal folding sequence ending at T'. Lemma F.7.6 applied to T' ensures that the diameter in $FZ(G,\mathcal{F})$ of $\psi(\overline{\gamma}([n_0,+\infty)) \cup \mathcal{R}^1(T')$ is bounded. As $\gamma(n)$ collapses to $\overline{\gamma}(n)$ for all $n \in \mathbb{N}$, the diameter of $\psi(\gamma([n_0,+\infty))) \cup \mathcal{R}^1(T')$ is also bounded. □

Proof of Theorem F.7.1. As $T \notin \mathcal{X}(G,\mathcal{F})$, there exists a tame $(G,\mathcal{F})$-tree T' that is compatible with T, such that $\mathcal{R}^1(T') \neq \emptyset$. Let $(\gamma(n))_{n\in\mathbb{N}}$ be a tame optimal folding sequence ending at T, whose existence follows from Proposition F.4.5. We will show that there exists $n_0 \in \mathbb{N}$ such that the diameter in $FZ(G,\mathcal{F})$ of $\psi(\gamma([n_0,+\infty))) \cup \mathcal{R}^1(T')$ is bounded. As this is true for any tree T' which is compatible with T and satisfies $\mathcal{R}^1(T') \neq \emptyset$, and as γ is chosen independently from T', all trees in $\mathcal{R}^2(T)$ will be close to each other (and close to the end of γ), and Theorem F.7.1 will follow.

Let $\widehat{T} := T + T'$. If $\widehat{T}$ has a nontrivial simplicial part, then any splitting determined by this simplicial part belongs to both $\mathcal{R}^1(T)$ and $\mathcal{R}^1(T')$. Lemma F.7.6 implies that both $\mathcal{R}^1(T)$ and $\mathcal{R}^1(T')$ have bounded diameter, and since $\mathcal{R}^1(T) \cap \mathcal{R}^1(T') \neq \emptyset$, the union $\mathcal{R}^1(T) \cup \mathcal{R}^1(T')$ also has bounded diameter. Lemma F.7.6 applied to T and the sequence $(\gamma(n))_{n\in\mathbb{N}}$ shows the existence of $n_0 \in \mathbb{N}$ such that the diameter of $\psi(\gamma([n_0,+\infty))) \cup \mathcal{R}^1(T)$ is bounded. The claim follows in this case.

From now on, we assume that $\widehat{T}$ has dense orbits. Consider 1-Lipschitz alignment-preserving maps $p : \widehat{T} \to T$ and $p' : \widehat{T} \to T'$. First suppose that $\mathcal{R}^1(T) \neq \emptyset$, and let $(\widetilde{\gamma}(n))_{n\in\mathbb{N}}$ be any tame optimal folding sequence ending at $\widehat{T}$. Lemma F.7.7 applied to both p and p' implies the existence of $n_0 \in \mathbb{N}$ such that $\psi(\widetilde{\gamma}([n_0,+\infty))) \cup \mathcal{R}^1(T)$ and $\psi(\widetilde{\gamma}([n_0,+\infty))) \cup \mathcal{R}^1(T')$ are bounded. In addition, Lemma F.7.6 ensures that we can choose n_0 so that the diameter of $\psi(\gamma([n_0,+\infty))) \cup \mathcal{R}^1(T)$ is also bounded. The claim follows.

Suppose now that $\mathcal{R}^1(T) = \emptyset$. Let $(\widehat{\gamma}(n))_{n\in\mathbb{N}}$ be a pullback of $(\gamma(n))_{n\in\mathbb{N}}$ induced by p (provided by Proposition F.6.10). Then $(\widehat{\gamma}(n))_{n\in\mathbb{N}}$ ends at $\widehat{T}$ by Proposition F.6.11. Lemma F.7.7 applied to p' shows the existence of $n_0 \in \mathbb{N}$ such that the diameter of $\psi(\widehat{\gamma}([n_0,+\infty))) \cup \mathcal{R}^1(T')$ is bounded. As $\widehat{\gamma}(n)$ collapses to $\gamma(n)$ for all $n \in \mathbb{N}$, the diameter of $\psi(\gamma([n_0,+\infty))) \cup \mathcal{R}^1(T')$ is also bounded, and we are done. □

Corollary F.7.8. *There exists $C \in \mathbb{R}$ such that for all tame $(G,\mathcal{F})$-trees $T_1, T_2 \notin \mathcal{X}(G,\mathcal{F})$, if T_1 and T_2 are both refined by a common tame $(G,\mathcal{F})$-tree, then the diameter of $\mathcal{R}^2(T_1) \cup \mathcal{R}^2(T_2)$ in $FZ(G,\mathcal{F})$ is bounded by C.*

Proof. Denoting by T the common refinement of T_1 and T_2, Corollary F.7.8 follows from Theorem F.7.1 and the fact that for all $i \in \{1,2\}$, we have $\mathcal{R}^2(T) \subseteq \mathcal{R}^2(T_i)$, and $\mathcal{R}^2(T) \neq \emptyset$ by Proposition F.5.1. □

F.8 The Gromov boundary of the graph of cyclic splittings

We now turn to the proof of our main theorem, which gives a description of the Gromov boundary of $FZ(G,\mathcal{F})$. We recall that there is a map $\psi : \mathcal{O}(G,\mathcal{F}) \to FZ(G,\mathcal{F})$ that sends every tree $T \in \mathcal{O}(G,\mathcal{F})$ to a one-edge collapse of T (any two choices of such a map are at bounded distance in $FZ(G,\mathcal{F})$). We will extend ψ to a map $\partial\psi : \mathcal{X}(G,\mathcal{F})/{\sim} \to \partial_\infty FZ(G,\mathcal{F})$, and show that this extension is an $\mathrm{Out}(G,\mathcal{F})$-equivariant homeomorphism.

Theorem F.8.1. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. There exists a unique $Out(G,\mathcal{F})$-equivariant homeomorphism*

$$\partial\psi : \mathcal{X}(G,\mathcal{F})/\sim \to \partial_\infty FZ(G,\mathcal{F}),$$

so that for all $T \in \mathcal{X}(G,\mathcal{F})$ and all sequences $(T_n)_{n\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$ converging to T, the sequence $(\psi(T_n))_{n\in\mathbb{N}}$ converges to $\partial\psi(T)$.

Theorem F.8.1 also holds true (with the same proof) for the graph $FZ^{max}(G,\mathcal{F})$.

Theorem F.8.2. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. There exists a unique $Out(G,\mathcal{F})$-equivariant homeomorphism*

$$\partial\psi : \mathcal{X}^{max}(G,\mathcal{F})/\sim \to \partial_\infty FZ^{max}(G,\mathcal{F}),$$

so that for all $T \in \mathcal{X}^{max}(G,\mathcal{F})$ and all sequences $(T_n)_{n\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$ converging to T, the sequence $(\psi(T_n))_{n\in\mathbb{N}}$ converges to $\partial\psi(T)$.

Definition of $\partial\psi$. The following lemma may be viewed as a kind of Cauchy criterion for Gromov products.

Lemma F.8.3. *Let $S,T \in \mathcal{X}(G,\mathcal{F})$, such that there exists a 1-Lipschitz alignment-preserving map from S to T. Let $(S_i)_{i\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$ (resp. $(T_i)_{i\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$) be a sequence of trees that converges (non-projectively) to S (resp. to T). Assume that for all $i \in \mathbb{N}$, there exists $J_i \in \mathbb{N}$ so that for all $j \geq J_i$, we have $Lip(S_i,T_j) \leq 1+\frac{1}{i}$. Then*

$$\forall C \geq 0, \exists I_C \in \mathbb{N}, \forall i \geq I_C, \exists J_{i,C} \in \mathbb{N}, \forall j \geq J_{i,C}, (\psi(S_i)|\psi(T_j)) \geq C.$$

Proof. Otherwise, there would exist $C \geq 0$ and increasing sequences $(i_k)_{k\in\mathbb{N}}$ and $(j_k)_{k\in\mathbb{N}}$ of integers, so that for all $k \in \mathbb{N}$, we have $j_k \geq J_{i_k}$ and $(\psi(S_{i_k})|\psi(T_{j_k})) \leq C$. For all $k \in \mathbb{N}$, let γ_k be an optimal liberal folding path from a point in the open cone of S_{i_k} to T_{j_k} given by Proposition F.4.14. As ψ-images of optimal liberal folding paths are uniformly Hausdorff close to geodesics (Theorem F.3.1), for all $k \in \mathbb{N}$, we can find Z_k in the image of γ_k, so that the sequence $(\psi(Z_k))_{k\in\mathbb{N}}$ is bounded in $FZ(G,\mathcal{F})$. Proposition F.4.14 implies that $(Z_k)_{k\in\mathbb{N}}$ has an accumulation point $Z \in \overline{\mathcal{O}(G,\mathcal{F})}$ that comes with alignment-preserving maps from S to Z and from Z to T. In particular, we have $Z \in \mathcal{X}(G,\mathcal{F})$. By Theorem F.5.4, the sequence $(\psi(Z_k))_{k\in\mathbb{N}}$ should be unbounded, a contradiction. □

We will also make use of the following general statement about Gromov hyperbolic metric spaces.

Lemma F.8.4. *Let X be a Gromov hyperbolic metric space. Let $(X_i)_{i\in\mathbb{N}} \in X^{\mathbb{N}}$ and $(Y_i)_{i\in\mathbb{N}} \in X^{\mathbb{N}}$ be two sequences in X. Assume that*

$$\forall C \geq 0, \exists I_C \in \mathbb{N}, \forall i \geq I_C, \exists J_{i,C} \in \mathbb{N}, \forall j \geq J_{i,C}, (X_i|Y_j) \geq C.$$

Then $(X_i)_{i\in\mathbb{N}}$ and $(Y_i)_{i\in\mathbb{N}}$ both converge to the same point of the Gromov boundary $\partial_\infty X$.

Proof. Let δ be the hyperbolicity constant of X. The assumption implies that for all $C \geq 0$, and all $j,j' \geq J_{I_C,C}$, we have $(X_{I_C}|Y_j) \geq C$ and $(X_{I_C}|Y_{j'}) \geq C$, whence $(Y_j|Y_{j'}) \geq C-\delta$. Therefore, the sequence $(Y_j)_{j\in\mathbb{N}}$ converges to some point $\xi \in \partial_\infty X$. Then for all $C \geq 0$, there exists $I_C \in \mathbb{N}$ such that for all $i \geq I_C$, we have $(X_i|\xi) \geq C-\delta$. This implies that $(X_i)_{i\in\mathbb{N}}$ also converges to ξ. □

Proposition F.8.5. *There exists a unique map $\partial\psi : \mathcal{X}(G,\mathcal{F}) \to \partial_\infty FZ(G,\mathcal{F})$ such that for all $T \in \mathcal{X}(G,\mathcal{F})$ and all sequences $(T_j)_{j\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$ converging to T, the sequence $(\psi(T_j))_{j\in\mathbb{N}}$ converges to $\partial\psi(T)$. In addition, if $S,T \in \mathcal{X}(G,\mathcal{F})$ satisfy $S \sim T$, then $\partial\psi(S) = \partial\psi(T)$.*

Proof. Let $T \in \mathcal{X}(G,\mathcal{F})$. We will prove that for all sequences $(T_j)_{j\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$ that converge (non-projectively) to T, the sequence $(\psi(T_j))_{j\in\mathbb{N}}$ converges to a point in $\partial_\infty FZ(G,\mathcal{F})$. This implies that all such sequences have the same limit, which we call $\partial\psi(T)$. This will define $\partial\psi$ (and show its uniqueness).

Let $(T_j)_{j\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$ be a sequence that converges (non-projectively) to T. Let $T' \in \mathcal{X}(G,\mathcal{F})$ be any tree that is compatible with T, and let $\widehat{T} := T + T'$. The tree $\widehat{T}$ has dense orbits, otherwise T would be $\mathcal{Z}$-compatible, so Proposition F.4.8 shows the existence of a Lipschitz approximation $(\widehat{T}_i)_{i\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$ of $\widehat{T}$. Proposition F.4.9 ensures that for all $i \in \mathbb{N}$, there exists $J_i \in \mathbb{N}$ such that for all $j \geq J_i$, we have $\mathrm{Lip}(\widehat{T}_i, T_j) \leq 1 + \frac{1}{i}$. Lemma F.8.3 then shows that

$$\forall C \in \mathbb{N}, \exists I_C \in \mathbb{N}, \forall i \geq I_C, \exists J_{i,C} \in \mathbb{N}, \forall j \geq J_{i,C}, (\psi(\widehat{T}_i)|\psi(T_j)) \geq C.$$

Together with Lemma F.8.4, this implies that $(\psi(T_j))_{j\in\mathbb{N}}$ converges to some point $\xi \in \partial_\infty FZ(G,\mathcal{F})$. This defines $\partial\psi$.

Furthermore, the sequence $(\psi(\widehat{T}_i))_{i\in\mathbb{N}}$ also converges to ξ (Lemma F.8.4). Therefore, we have proved that for all $T,T' \in \mathcal{X}(G,\mathcal{F})$, if T is compatible with T', then $\partial\psi(T) = \partial\psi(T')$. Therefore, if $T,T' \in \mathcal{X}(G,\mathcal{F})$ satisfy $T \sim T'$, then $\partial\psi(T) = \partial\psi(T')$. ☐

Proposition F.8.5 shows that $\partial\psi$ descends to a map $\partial\psi : \mathcal{X}(G,\mathcal{F})/\sim \to \partial_\infty FZ(G,\mathcal{F})$.

Continuity of $\partial\psi$.

Proposition F.8.6. *The map $\partial\psi : \mathcal{X}(G,\mathcal{F})/\sim \to \partial_\infty FZ(G,\mathcal{F})$ is continuous.*

Proof. By definition of the quotient topology, it is enough to check that $\partial\psi : \mathcal{X}(G,\mathcal{F}) \to \partial_\infty FZ(G,\mathcal{F})$ is continuous. Let $T \in \mathcal{X}(G,\mathcal{F})$, and let $(T_i)_{i\in\mathbb{N}} \in \mathcal{X}(G,\mathcal{F})^{\mathbb{N}}$ be a sequence that converges non-projectively to T. We want to show that $(\partial\psi(T_i))_{i\in\mathbb{N}}$ converges to $\partial\psi(T)$. Let $(S_k)_{k\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$ be a sequence that converges to T. Proposition F.8.5 implies that the sequence $(\psi(S_k))_{k\in\mathbb{N}}$ converges to $\partial\psi(T)$. Therefore, up to replacing $(S_k)_{k\in\mathbb{N}}$ by a subsequence, we can assume that for all $k \in \mathbb{N}$, we have

$$(\psi(S_k)|\partial\psi(T_k)) \leq (\partial\psi(T)|\partial\psi(T_k)) + \delta. \tag{F.1}$$

Recall that for all $k \in \mathbb{N}$, we have $T_k \in \mathcal{X}(G,\mathcal{F})$. Therefore, using Proposition F.8.5, we can find a sequence $(S'_k)_{k\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$ (where we choose S'_k sufficiently close to T_k) such that

- the sequence $(S'_k)_{k\in\mathbb{N}}$ converges to T in $\overline{\mathcal{O}(G,\mathcal{F})}$, and
- for all $k \in \mathbb{N}$, we have

$$(\psi(S_k)|\psi(S'_k)) \leq (\psi(S_k)|\partial\psi(T_k)) + \delta. \tag{F.2}$$

Combining Equations (F.1) and (F.2), we then get that for all $k \in \mathbb{N}$, we have

$$(\psi(S_k)|\psi(S'_k)) \leq (\partial\psi(T)|\partial\psi(T_k)) + 2\delta. \tag{F.3}$$

As both $(\psi(S_k))_{k\in\mathbb{N}}$ and $(\psi(S'_k))_{k\in\mathbb{N}}$ converge to $\partial\psi(T)$ (Proposition F.8.5), the Gromov product $(\psi(S_k)|\psi(S'_k))$ tends to $+\infty$, and hence $(\partial\psi(T)|\partial\psi(T_k))$ tends to $+\infty$. This implies that $(\partial\psi(T_k))_{k\in\mathbb{N}}$ converges to $\partial\psi(T)$. ☐

Injectivity of $\partial\psi$.

Proposition F.8.7. *The map $\partial\psi : \mathcal{X}(G,\mathcal{F})/\!\sim\, \to \partial_\infty FZ(G,\mathcal{F})$ is injective.*

Proof. Let $T, T' \in \mathcal{X}(G,\mathcal{F})$ be such that $\partial\psi(T) = \partial\psi(T')$. We choose T and T' to be mixing and $\mathcal{Z}$-incompatible representatives in their equivalence classes (Proposition F.5.3). Let $* \in \mathcal{O}(G,\mathcal{F})$, and consider an optimal liberal folding path γ (resp. γ') from the closed cone of $*$ to T (resp. to T'). By Proposition F.5.24, all trees along the paths γ and γ' are simplicial and have trivial arc stabilizers. As $\partial\psi(T) = \partial\psi(T')$, it follows from Theorem F.3.1 that the images $\psi(\gamma)$ and $\psi(\gamma')$ are at finite Hausdorff distance M from each other, where M is bounded independently from the paths γ and γ'. Let $(\psi(T_i))_{i\in\mathbb{N}}$ be a sequence of points lying on $\psi(\gamma)$ and converging to $\partial\psi(T)$, and let $(\psi(T'_i))_{i\in\mathbb{N}}$ be a sequence of points lying on $\psi(\gamma')$ and converging to $\partial\psi(T) = \partial\psi(T')$, so that for all $i \in \mathbb{N}$, we have $d_{FZ(G,\mathcal{F})}(\psi(T_i), \psi(T'_i)) \le M$. Up to passing to a subsequence, we may assume that the distance between $\psi(T_i)$ and $\psi(T'_i)$ is constant, we denote it by C. For all $i \in \mathbb{N}$, let $\psi(T_i) = \psi(T_i^0), \dots, \psi(T_i^C) = \psi(T'_i)$ be a geodesic segment in $FZ(G,\mathcal{F})$ joining $\psi(T_i)$ to $\psi(T'_i)$. Up to rescaling and passing to a subsequence, we may assume that for all $k \in \{0, \dots, C\}$, the sequence of one-edge simplicial $(G,\mathcal{F})$-trees $(T_i^k)_{i\in\mathbb{N}}$ converges non-projectively to a tame $(G,\mathcal{F})$-tree T_∞^k (Proposition F.4.3). For all $i \in \mathbb{N}$, the splitting $\psi(T_i)$ (resp. $\psi(T'_i)$) is a collapse of T_i (resp. T'_i), so T (resp. T') collapses to T_∞^0 (resp. T_∞^C). For all $k \in \{0, \dots, C-1\}$ and all $i \in \mathbb{N}$, the trees T_i^k and T_i^{k+1} are compatible, so Lemma F.4.15 implies that T_∞^k and T_∞^{k+1} are compatible. Therefore $T \sim T'$. ☐

Surjectivity of $\partial\psi$.

Proposition F.8.8. *For all $M \in \mathbb{R}$, there exists $C_M \in \mathbb{R}$ such that the following holds. Let $T \in \overline{\mathcal{O}(G,\mathcal{F})} \smallsetminus \mathcal{X}(G,\mathcal{F})$, and let $(T_n)_{n\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$ be a sequence that converges to T, such that the sequence $(\psi(T_n))_{n\in\mathbb{N}}$ lies in a region of $FZ(G,\mathcal{F})$ of diameter bounded by M. Then $d_{FZ(G,\mathcal{F})}(\psi(T_n), \mathcal{R}^2(T)) \le C_M$ for all $n \in \mathbb{N}$.*

Proof. It is enough to prove the desired bound for a subsequence of $(\psi(T_n))_{n\in\mathbb{N}}$, since the bound for the whole sequence follows by replacing C_M by $C_M + M$. Up to passing to a subsequence, we can assume that there exists $* \in FZ(G,\mathcal{F})$ and $M' \le M$ such that for all $n \in \mathbb{N}$, we have $d_{FZ(G,\mathcal{F})}(*, \psi(T_n)) = M'$. For all $n \in \mathbb{N}$, let $(\psi(T_n^k))_{k=0,\dots,M'}$ be a geodesic segment joining $*$ to $\psi(T_n)$ in $FZ(G,\mathcal{F})$. We may rescale the one-edge simplicial trees T_n^k so that up to passing to a subsequence, for all $k \in \{0, \dots, M'\}$, the sequence $(T_n^k)_{n\in\mathbb{N}}$ converges non-projectively to a tame $(G,\mathcal{F})$-tree T^k (Proposition F.4.3). For all $k \in \{0, \dots, M'-1\}$ and all $n \in \mathbb{N}$, the trees T_n^k and T_n^{k+1} are compatible, so Lemma F.4.15 implies that T^k and T^{k+1} are compatible. None of the trees T^k is $\mathcal{Z}$-averse, and Corollary F.7.8 shows the existence of $C' \in \mathbb{R}$ such that for all $k \in \{0, \dots, M'-1\}$, the diameter of $\mathcal{R}^2(T^k) \cup \mathcal{R}^2(T^{k+1})$ is bounded by C'. Since $T^0 = *$ and $T^{M'}$ is compatible with T, the distance between $*$ and $\mathcal{R}^2(T)$ is at most $(M+1)C'$. In addition, we have $d_{FZ(G,\mathcal{F})}(\psi(T_n), *) \le M$ for all $n \in \mathbb{N}$. It then follows from the triangular inequality that for all $n \in \mathbb{N}$, the distance in $FZ(G,\mathcal{F})$ between $\psi(T_n)$ and $\mathcal{R}^2(T)$ is at most $(M+1)C' + M$. ☐

The following statement follows from classical arguments about Gromov hyperbolic spaces. We leave its proof to the reader.

Proposition F.8.9. *Let $\delta > 0$, let X be a δ-hyperbolic geodesic metric space. There exists $M \in \mathbb{R}$ only depending on δ such that the following holds. Let $\xi \in \partial_\infty X$, let $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$*

be a sequence that converges to ξ, and for all $n \in \mathbb{N}$, let γ_n be a geodesic segment from x_0 to x_n. Let $R > 0$, and for all $n \in \mathbb{N}$, let $y_n \in \gamma_n$ be a point at distance exactly R from x_0. Then there exists $n_0 \in \mathbb{N}$, such that $\{y_n\}_{n \geq n_0}$ is contained in a region of X of diameter at most M. ☐

Proposition F.8.10. *Let $T \in \overline{\mathcal{O}(G,\mathcal{F})} \setminus \mathcal{X}(G,\mathcal{F})$, and let $(T_n)_{n\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$ be a sequence that converges to T. Then $(\psi(T_n))_{n\in\mathbb{N}}$ does not converge to any point in $\partial_\infty FZ(G,\mathcal{F})$.*

Proof. Assume towards a contradiction that the sequence $(\psi(T_n))_{n\in\mathbb{N}}$ converges to some $\xi \in \partial_\infty FZ(G,\mathcal{F})$. Let $M \in \mathbb{R}$ be the constant provided by Proposition F.8.9, and let $C_M \in \mathbb{R}$ be the constant provided by Proposition F.8.8, which we can choose to be greater than the constants C_1 and C_2 from Theorem F.7.1. Using the arguments of Guirardel and Levitt in [GL07a, Section 6], we can find trees T_0' in the closed cone of T_0, and T_0^m in the open cone of T_0 for all $m \in \mathbb{N}$, together with optimal morphisms $f : T_0' \to T$ and $f_m : T_0^m \to T_m$, so that up to passing to a subsequence, the sequence $(f_m)_{m\in\mathbb{N}}$ converges to f. Let γ (resp. γ_m) be the canonical folding path directed by f (resp. f_m) constructed in [GL07b, Section 3] (see Section F.1.3 of the present paper for a brief review of Guirardel and Levitt's construction). By [GL07b, Proposition 3.4], for all $t \in \mathbb{R}_+$, the trees $\gamma_m(t)$ converge to $\gamma(t)$ as m tends to $+\infty$. In addition, by Proposition F.4.4, all trees in the image of γ are tame $(G,\mathcal{F})$-trees.

By Theorem F.7.1, the ψ-image of γ is a bounded region of $FZ(G,\mathcal{F})$. We apply Proposition F.8.9 to the collection of paths $\psi(\gamma_n)$, which are all uniformly Hausdorff close to geodesic segments in $FZ(G,\mathcal{F})$ by Theorem F.3.1. We choose R to be large enough compared to the diameter of the ψ-image of γ. This provides an integer $m_0 \in \mathbb{N}$, and a sequence $(t_m)_{m\geq m_0} \in \mathbb{R}^{\mathbb{N}}$, so that $(\psi(\gamma_m(t_m)))_{m\geq m_0}$ lies in a region of diameter bounded by M in $FZ(G,\mathcal{F})$, and for all $m \in \mathbb{N}$, the distance between $\psi(\gamma_m(t_m))$ and the ψ-image of γ is at least $4C_M$. Up to passing to a subsequence, we can assume that $(t_m)_{m\in\mathbb{N}}$ converges to some $t_\infty \in \mathbb{R} \cup \{+\infty\}$, and hence $(\gamma_m(t_m))_{m\in\mathbb{N}}$ converges to $\gamma(t_\infty)$. We have $\mathcal{R}^2(\gamma(t_\infty)) \neq \emptyset$, and

- for all $m \in \mathbb{N}$, we have $d_{FZ(G,\mathcal{F})}(\psi(\gamma_m(t_m)), \mathcal{R}^2(\gamma(t_\infty))) \leq C_M$ (Proposition F.8.8), and
- the diameter of $\mathcal{R}^2(\gamma(t_\infty))$ is at most C_M (Theorem F.7.1), and
- there exists $t_0 \in \mathbb{R}$ so that $d_{FZ(G,\mathcal{F})}(\mathcal{R}^2(\gamma(t_\infty)), \psi(\gamma(t_0))) \leq C_M$ (Theorem F.7.1).

Therefore, for all $m \in \mathbb{N}$, we have $d_{FZ(G,\mathcal{F})}(\psi(\gamma_m(t_m)), \psi(\gamma(t_0))) \leq 3C_M$, which is a contradiction. ☐

Proposition F.8.11. *The map $\partial\psi : \mathcal{X}(G,\mathcal{F})/\!\sim \to \partial_\infty FZ(G,\mathcal{F})$ is surjective.*

Proof. Let $\xi \in \partial_\infty FZ(G,\mathcal{F})$, and let $(Z_n)_{n\in\mathbb{N}} \in FZ(G,\mathcal{F})^{\mathbb{N}}$ be a sequence that converges to ξ. For all $n \in \mathbb{N}$, let $T_n \in \mathcal{O}(G,\mathcal{F})$ be a simplicial tree such that $\psi(T_n)$ is at bounded distance from Z_n. Up to passing to a subsequence and rescaling, we can assume that $(T_n)_{n\in\mathbb{N}}$ converges non-projectively to some tree $T \in \overline{\mathcal{O}(G,\mathcal{F})}$. Proposition F.8.10 ensures that $T \in \mathcal{X}(G,\mathcal{F})$, and Proposition F.8.5 ensures that $\xi = \partial\psi(T)$. ☐

Closedness of $\partial\psi$.

Proposition F.8.12. *The map $\partial\psi : \mathcal{X}(G,\mathcal{F}) \to \partial_\infty FZ(G,\mathcal{F})$ is closed.*

Proof. Let $\xi \in \partial_\infty FZ(G,\mathcal{F})$, and let $(T_n)_{n\in\mathbb{N}} \in \mathcal{X}(G,\mathcal{F})^{\mathbb{N}}$ be such that $(\partial\psi(T_n))_{n\in\mathbb{N}}$ converges to ξ. We will show that for all limit points $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ of the sequence $(T_n)_{n\in\mathbb{N}}$, we have $T \in \mathcal{X}(G,\mathcal{F})$ and $\partial\psi(T) = \xi$. For all $n \in \mathbb{N}$, we have $T_n \in \mathcal{X}(G,\mathcal{F})$, so it follows from Proposition F.8.5 that there exists a sequence $(X_n)_{n\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$ that converges to T, so that $(\psi(X_n))_{n\in\mathbb{N}}$ converges to ξ. Proposition F.8.10 ensures that $T \in \mathcal{X}(G,\mathcal{F})$, and Proposition F.8.6 then ensures that $\partial\psi(T) = \xi$. ☐

End of the proof of the main theorem.

Proof of Theorem F.8.1. The map $\partial\psi : \mathcal{X}(G,\mathcal{F})/\!\sim\, \to \partial_\infty FZ(G,\mathcal{F})$ is a continuous, bijective, closed map (Propositions F.8.6, F.8.7, F.8.11 and F.8.12), and hence a homeomorphism. That $\partial\psi$ is $\mathrm{Out}(G,\mathcal{F})$-equivariant follows from its construction. ☐

Annexe G

The Tits alternative for the automorphism group of a free product

Abstract

Let $G = G_1 * \cdots * G_k * F$ be a countable group which splits as a free product, where all groups G_i are freely indecomposable and not isomorphic to $\mathbb{Z}$, and F is a finitely generated free group. If for all $i \in \{1, \dots, k\}$, both G_i and its outer automorphism group $\mathrm{Out}(G_i)$ satisfy the Tits alternative, then $\mathrm{Out}(G)$ satisfies the Tits alternative. As an application, we prove that the Tits alternative holds for outer automorphism groups of right-angled Artin groups, and of torsion-free groups that are hyperbolic relative to a finite family of virtually polycyclic groups.

Contents

Introduction

In his celebrated 1972 paper [Tit72], Tits proved that any subgroup of a finitely generated linear group over an arbitrary field is either virtually solvable, or contains a rank two free subgroup. This dichotomy has since been shown to hold for various classes of groups, such as hyperbolic groups (Gromov [Gro87]), mapping class groups of compact surfaces (Ivanov [Iva84], McCarthy [McC85]), outer automorphism groups $\mathrm{Out}(F_N)$ of finitely generated free groups (Bestvina, Feighn and Handel [BFH00, BFH05]), groups acting freely and properly on a CAT(0) cube complex (Sageev and Wise [SW05]), the group of polynomial automorphisms of $\mathbb{C}^2$ (Lamy [Lam01]), groups of bimeromorphic automorphisms of compact complex Kähler manifolds (Oguiso [Ogu06]), groups of birational transformations of compact complex Kähler surfaces (Cantat [Can11]).

For the four first classes of groups mentioned above, as well as in Oguiso's theorem, a slightly stronger result than Tits' actually holds, since virtually solvable subgroups can be shown to be finitely generated and virtually abelian, with a bound on the index of the abelian subgroup (see [BLM83] for the mapping class group case, see [Ali02, BFH04] for the $\text{Out}(F_N)$ case, see [BH99] for the case of groups acting on a CAT(0) cube complex).

Definition G.0.1. *A group G* satisfies the Tits alternative *if every subgroup of G (finitely generated or not) is either virtually solvable, or contains a rank two free subgroup.*

More generally, we will make the following definition. The classical Tits alternative corresponds to the case where $\mathcal{C}$ is the class of virtually solvable groups.

Definition G.0.2. *Let $\mathcal{C}$ be a collection of groups. A group G* satisfies the Tits alternative relative to $\mathcal{C}$ *if every subgroup of G either belongs to $\mathcal{C}$, or contains a rank two free subgroup.*

It is often interesting to show stability results for the Tits alternative: when a group G is built in some way out of simpler subgroups G_i, it is worth knowing that one can deduce the Tits alternative for G from the Tits alternative for the G_i's. The Tits alternative is known to be stable under some basic group-theoretic constructions, such as passing to subgroups or to finite index supergroups; it is also stable under extensions – we insist that it is important here to allow for subgroups of G that are not finitely generated in the definition of the Tits alternative. Antolín and Minasyan established stability results of the Tits alternative for graph products of groups [AM13].

Our main result is about deducing the Tits alternative for the outer automorphism group of a free product of groups G_i, under the assumption that all groups G_i and $\text{Out}(G_i)$ satisfy it. A celebrated theorem of Grushko [Gru40] states that any finitely generated group G splits as a free product of the form

$$G = G_1 * \dots * G_k * F,$$

where for all $i \in \{1, \dots, k\}$, the group G_i is nontrivial, not isomorphic to $\mathbb{Z}$, and freely indecomposable, and F is a finitely generated free group. This *Grushko decomposition* is unique in the sense that both the number k of indecomposable factors, and the rank of the free group F, are uniquely determined by G, and the conjugacy classes of the freely indecomposable factors are also uniquely determined, up to permutation.

Our main result reduces the study of the Tits alternative of the outer automorphism group of any finitely generated group to that of its indecomposable pieces. It answers a question of Charney and Vogtmann, who were interested in the Tits alternative for outer automorphisms of right-angled Artin groups.

Theorem G.0.3. *Let G be a finitely generated group, and let*

$$G := G_1 * \dots * G_k * F$$

be the Grushko decomposition of G. Assume that for all $i \in \{1, \dots, k\}$, both G_i and $Out(G_i)$ satisfy the Tits alternative.
Then $Out(G)$ satisfies the Tits alternative.

Again, we insist on the fact that when we assume that the groups G_i and $\text{Out}(G_i)$ satisfy the Tits alternative, it is important to consider all their subgroups (finitely generated or not) in the definition of the Tits alternative, even if we are only interested in establishing this alternative for finitely generated subgroups of $\text{Out}(G)$.

Under the assumptions of Theorem G.0.3, since the Tits alternative is stable under extensions, the full automorphism group $\mathrm{Aut}(G)$ also satisfies the Tits alternative. When $k = 0$, we get a new, shorter proof of the Tits alternative for the outer automorphism group $\mathrm{Out}(F_N)$ of a finitely generated free group, that was originally established by Bestvina, Feighn and Handel [BFH00, BFH05]. In particular, this gives a new proof of the Tits alternative for the mapping class group of a compact surface with nonempty boundary.

More generally, if $\mathcal{C}$ is a collection of groups that is stable under isomorphisms, contains $\mathbb{Z}$, and is stable under passing to subgroups, to extensions, and to finite index supergroups, we show that $\mathrm{Out}(G)$ satisfies the Tits alternative relative to $\mathcal{C}$, as soon as all G_i and $\mathrm{Out}(G_i)$ do, see Theorem G.6.1. This applies for example to the class of virtually polycyclic groups. Bestvina, Feighn and Handel actually proved the Tits alternative for $\mathrm{Out}(F_N)$ relative to the collection of all abelian groups [BFH04], which does not follow from our main result. More generally, it would be of interest to know whether the version of Theorem G.0.3 relative to the class of abelian groups holds.

Theorem G.0.3 can be applied to prove the Tits alternative for outer automorphism groups of various interesting classes of groups. In [CV11], Charney and Vogtmann proved the Tits alternative for the outer automorphism group of a right-angled Artin group A_Γ associated to a finite simplicial graph Γ, under a homogeneity assumption on Γ. As noticed in [CV11, Section 7], Theorem G.0.3 enables us to remove this assumption. This was Charney and Vogtmann's original motivation for asking the question about the Tits alternative for the outer automorphism group of a free product. Basically, when Γ is disconnected, the group A_Γ splits as a free product of the subgroups A_{Γ_i} associated to its connected components, and Theorem G.0.3 enables us to argue by induction on the number of vertices of Γ, using Charney and Vogtmann's results from [CV11].

Theorem G.0.4. *For all finite simplicial graphs Γ, the group Out(A_Γ) satisfies the Tits alternative.*

Theorem G.0.3 also applies to the outer automorphism group of a torsion-free group G that is hyperbolic relative to a finite collection $\mathcal{P}$ of virtually polycyclic subgroups. Indeed, it enables to restrict to the case where G is freely indecomposable relative to $\mathcal{P}$, i.e. G does not split as a free product of the form $G = A * B$, where all subgroups in $\mathcal{P}$ are conjugate into either A or B. In the freely indecomposable case, the group of outer automorphisms of G was described by Guirardel and Levitt as being built out of mapping class groups and subgroups of linear groups [GL14].

Theorem G.0.5. *Let G be a torsion-free group that is hyperbolic relative to a finite collection of virtually polycyclic subgroups. Then Out(G) satisfies the Tits alternative.*

More generally, if G is a torsion-free group that is hyperbolic relative to a finite family of finitely generated parabolic subgroups, we show that if all parabolic subgroups, as well as their outer automorphism groups, satisfy the Tits alternative, then the subgroup of $\mathrm{Out}(G)$ made of those automorphisms that preserve the conjugacy classes of all parabolic subgroups also satisfies the Tits alternative. We refer to Theorem G.7.4 for a precise statement.

We now describe the main ideas in our proof of Theorem G.0.3. In the case of the mapping class group $\mathrm{Mod}(S)$ of a compact surface S, one way of proving the Tits alternative is to start by proving the following trichotomy: every subgroup $H \subseteq \mathrm{Mod}(S)$ either

- contains two pseudo-Anosov diffeomorphisms of S that generate a rank two free subgroup of H, or
- is virtually cyclic, virtually generated by a pseudo-Anosov diffeomorphism, or
- virtually fixes the isotopy class of a simple closed curve on S.

This trichotomy was proved by Ivanov in [Iva92], and independently by McCarthy and Papadopoulos in [MP89]. They started by proving that every subgroup of $\mathrm{Mod}(S)$ either contains a pseudo-Anosov diffeomorphism, or virtually fixes the isotopy class of a simple closed curve on S, before studying subgroups of $\mathrm{Mod}(S)$ that contain a pseudo-Anosov diffeomorphism. Once the above trichotomy is established, a second step in the proof of the Tits alternative consists in arguing by induction, in the case where H preserves the isotopy class of a simple closed curve γ. In this case, by cutting S along γ, we get a collection of subsurfaces. The Tits alternative is proved by induction, by considering the restrictions of the diffeomorphisms in H to these subsurfaces.

Our proof of Theorem G.0.3 follows the same strategy. For the inductive step, we will need to work with decompositions of G into free products that are not necessarily equal to the Grushko decomposition. From now on, we let G be a countable group that splits as a free product of the form

$$G := G_1 * \cdots * G_k * F,$$

where F is a finitely generated free group, and all G_i are nontrivial. We do not require this decomposition to be the Grushko decomposition of G: some factors G_i can be equal to $\mathbb{Z}$, or be freely decomposable. We actually do not even require G to be finitely generated: some G_i might be infinitely generated (however the number k of factors arising in the splitting is finite, and F is finitely generated). We denote by $\mathcal{F} := \{[G_1], \dots, [G_k]\}$ the finite set of all G-conjugacy classes of the G_i's, which we call a *free factor system* of G. We denote by $\mathrm{Out}(G, \mathcal{F})$ the subgroup of $\mathrm{Out}(G)$ made of those outer automorphisms of G that send each G_i to a conjugate. Theorem G.0.3 is a particular case of the following version, which is suitable for our inductive arguments.

Theorem G.0.6. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. Assume that for all $i \in \{1, \dots, k\}$, both G_i and Out(G_i) satisfy the Tits alternative relative to $\mathcal{C}$, where $\mathcal{C}$ is a collection of groups that is stable under isomorphisms, contains $\mathbb{Z}$, and is stable under subgroups, extensions, and passing to finite index supergroups.*
Then Out($G, \mathcal{F}$) satisfies the Tits alternative relative to $\mathcal{C}$.

As mentioned above, our proof of Theorem G.0.6 will consist in two steps: establishing a trichotomy for subgroups $H \subseteq \mathrm{Out}(G, \mathcal{F})$, and applying an inductive argument. The induction step consists in dealing with the case where H virtually preserves the conjugacy class of a proper $(G, \mathcal{F})$*-free factor*. A $(G, \mathcal{F})$*-free factor* is a subgroup $A \subseteq G$ such that G splits as a free product of the form $G = A * B$, and for all $i \in \{1, \dots, k\}$, the group G_i is conjugate into either A or B. A $(G, \mathcal{F})$-free factor is *proper* if it is nontrivial, not conjugate to any of the G_i's, and not equal to G. When H preserves the conjugacy class of a proper free factor A, the group H is contained in $\mathrm{Out}(G, \mathcal{F}')$, where $\mathcal{F}'$ is the free factor system of G obtained from $\mathcal{F}$ by removing all subgroups in $\mathcal{F}$ that are conjugate into A, and replacing them by the G-conjugacy class of the factor A. When passing from $(G, \mathcal{F})$ to $(G, \mathcal{F}')$, some measure of complexity decreases, which enables us to argue by induction.

We now describe our analogue of Ivanov's trichotomy for subgroups of $\mathrm{Out}(G,\mathcal{F})$. We first state an analogous trichotomy for subgroups of $\mathrm{Out}(F_N)$. We recall that an automorphism $\Phi \in \mathrm{Out}(F_N)$ is *fully irreducible* if no nontrivial power of Φ preserves the conjugacy class of a proper free factor of F_N. Every subgroup of $\mathrm{Out}(F_N)$ (finitely generated or not) either

- contains two fully irreducible automorphisms that generate a rank two free subgroup, or
- is virtually cyclic, virtually generated by a fully irreducible automorphism, or
- virtually fixes the conjugacy class of a proper free factor of F_N.

In [HM09], Handel and Mosher proved that any finitely generated subgroup of $\mathrm{Out}(F_N)$ either contains a fully irreducible automorphism, or virtually fixes the conjugacy class of a proper free factor. Their proof uses the same kinds of techniques as Bestvina, Feighn and Handel's proof of the Tits alternative [BFH00], so it cannot be used to get a new proof of the Tits alternative for $\mathrm{Out}(F_N)$. The study of subgroups of $\mathrm{Out}(F_N)$ that contain a fully irreducible element is due to Bestvina, Feighn and Handel [BFH97], another approach is due to Kapovich and Lustig [KL11a]. In [5], we gave a new, shorter proof of the above trichotomy, independent from the work in [BFH00], that also works for non finitely generated subgroups of $\mathrm{Out}(F_N)$.

Our proof of this statement uses the action of $\mathrm{Out}(F_N)$ on the *free factor complex* FF_N, whose hyperbolicity was originally proved by Bestvina and Feighn [BF14b]. Bestvina and Feighn also proved that an automorphism $\Phi \in \mathrm{Out}(F_N)$ acts loxodromically on FF_N if and only if Φ is fully irreducible. In terms of the action of $\mathrm{Out}(F_N)$ on FF_N, the above trichotomy can be restated as follows: every subgroup of $\mathrm{Out}(F_N)$ either

- contains a rank two free subgroup generated by two loxodromic isometries of FF_N, or
- is virtually cyclic, virtually generated by a loxodromic isometry of FF_N, or
- has a finite orbit in FF_N.

More generally, given a group G acting by isometries on a (possibly non-proper) hyperbolic space X, it follows from a classification of groups of isometries of hyperbolic spaces due to Gromov [Gro87] that either G

- contains a rank two free subgroup, generated by two loxodromic isometries of X, or
- has a fixed point in the Gromov boundary $\partial_\infty X$, or
- has a bounded orbit in X.

The key point for deducing the above trichotomy statement for subgroups of $\mathrm{Out}(F_N)$ from Gromov's statement consists in showing that if H has a bounded orbit in FF_N, then H has a finite orbit in FF_N. This is not obvious because FF_N is not locally finite. To bypass this difficulty, we studied stationary measures on the compact closure of Culler and Vogtmann's outer space CV_N, and projected them to the Gromov boundary of the complex of free factors. In our proof of the above trichotomy, we also need to understand stabilizers of points in $\partial_\infty FF_N$ for dealing with the second case in Gromov's theorem.

We prove a similar trichotomy for subgroups of $\mathrm{Out}(G,\mathcal{F})$, with $(G,\mathcal{F})$ as above. To this means, we work with relative outer space $P\mathcal{O}(G,\mathcal{F})$, and the complex of relative cyclic splittings $FZ(G,\mathcal{F})$. The geometry of these complexes was investigated in a series of previous papers [7, 8]. In [7], we described a compactification $\overline{P\mathcal{O}(G,\mathcal{F})}$ of the relative outer space in terms of very small actions of G on $\mathbb{R}$-trees. In [8], we proved the hyperbolicity of the complex of relative cyclic splittings, and described its Gromov boundary as a quotient of a subspace $P\mathcal{X}(G,\mathcal{F})$ of $\overline{P\mathcal{O}(G,\mathcal{F})}$. Assume that the pair $(G,\mathcal{F})$ is *non-sporadic*, i.e. we do not have $G = G_1 * G_2$ and $\mathcal{F} = \{[G_1],[G_2]\}$, or $G = G_1 * \mathbb{Z}$ and $\mathcal{F} = \{[G_1]\}$. The trichotomy that we prove for subgroups of $\mathrm{Out}(G,\mathcal{F})$ is the following: every subgroup $H \subseteq \mathrm{Out}(G,\mathcal{F})$ (finitely generated or not) either

- contains a rank two free subgroup, generated by two loxodromic isometries of $FZ(G,\mathcal{F})$, or
- virtually fixes a tree with trivial arc stabilizers in $\partial P\mathcal{O}(G,\mathcal{F})$, or
- virtually preserves the conjugacy class of a proper $(G,\mathcal{F})$-free factor.

Again, the key point is to understand subgroups of $\mathrm{Out}(G,\mathcal{F})$ with bounded orbits in $FZ(G,\mathcal{F})$. We show that if a subgroup $H \subseteq \mathrm{Out}(G,\mathcal{F})$ does not virtually preserve the conjugacy class of any proper $(G,\mathcal{F})$-free factor, then the H-orbit of any point of $FZ(G,\mathcal{F})$ has a limit point in the Gromov boundary.

Our argument relies on techniques coming from the theory of random walks on groups. Given a probability measure μ on $\mathrm{Out}(F_N)$ whose support generates the subgroup H, we consider *μ-stationary* measures ν on $\overline{P\mathcal{O}(G,\mathcal{F})}$, i.e. probability measures that satisfy

$$\nu(E) = \sum_{\Phi \in \mathrm{Out}(G,\mathcal{F})} \mu(\Phi)\nu(\Phi^{-1}E)$$

for all ν-measurable subsets $E \subseteq \overline{P\mathcal{O}(G,\mathcal{F})}$. Compactness of $\overline{P\mathcal{O}(G,\mathcal{F})}$ yields the existence of a μ-stationary measure on $\overline{P\mathcal{O}(G,\mathcal{F})}$ that describes the repartition of accumulation points of sample paths of the random walk on $\mathrm{Out}(G,\mathcal{F})$, realized on $P\mathcal{O}(G,\mathcal{F})$ via the action. This is the Markov chain whose position at time n is obtained by successive multiplications on the right of n independent automorphisms, all distributed with law μ. We prove that any μ-stationary measure ν on $\overline{P\mathcal{O}(G,\mathcal{F})}$ is supported on the subspace $P\mathcal{X}(G,\mathcal{F})$. The measure ν therefore projects to a μ-stationary measure on the Gromov boundary of $FZ(G,\mathcal{F})$. The closure of the H-orbit of any point in $FZ(G,\mathcal{F})$ meets the support of ν, which shows the existence of a limit point in the Gromov boundary.

To prove the Tits alternative for $\mathrm{Out}(G,\mathcal{F})$, we also need to understand subgroups of $\mathrm{Out}(G,\mathcal{F})$ that stabilize a tree with trivial arc stabilizers in $\partial P\mathcal{O}(G,\mathcal{F})$, which is made possible by work of Guirardel and Levitt [GL]. When H fixes the conjugacy class of a proper free factor, we argue by induction, as explained above.

As we are considering invariant free factors (and not invariant splittings) for the inductive step, it could seem to be more natural to work directly in the complex of proper $(G,\mathcal{F})$-free factors, whose hyperbolicity was recently proved by Handel and Mosher [HM14b], and try to prove that every subgroup of $\mathrm{Out}(G,\mathcal{F})$ either has a finite orbit, or has a limit point in the Gromov boundary. However, describing the Gromov boundary of the complex of proper $(G,\mathcal{F})$-free factors is still an open problem. We bypass this difficulty by working in the complex $FZ(G,\mathcal{F})$, whose Gromov boundary was described in [8].

The paper is organized as follows. In Section G.1, we review basic facts about Gromov hyperbolic spaces, free products of groups, and relative spaces associated to them. In Section G.2, we deal with the *sporadic* cases where either $G = G_1 * G_2$ and $\mathcal{F} = \{[G_1],[G_2]\}$, or $G = G_1 * \mathbb{Z}$ and $\mathcal{F} = \{[G_1]\}$. In Section G.3, we state Guirardel and Levitt's theorem about stabilizers of trees in $\overline{P\mathcal{O}(G,\mathcal{F})}$ that is needed in our proof of Theorem G.0.6. Section

G.4 contains a study of *arational* $(G,\mathcal{F})$-trees, which is used in Section G.5 to establish the trichotomy for subgroups of $\mathrm{Out}(G,\mathcal{F})$. Theorem G.6 is devoted to the inductive arguments. The reader will also find complete versions of our various statements of the Tits alternative in this section. Finally, in Section G.7, we give applications of our main result to automorphism groups of right-angled Artin groups, and of relatively hyperbolic groups.

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G.1 Review

G.1.1 Gromov hyperbolic spaces

A geodesic metric space (X,d) is *Gromov hyperbolic* if there exists $\delta>0$ such that for all $x,y,z\in X$, and all geodesic segments $[x,y],[y,z]$ and $[x,z]$, we have $N_\delta([x,z])\subseteq N_\delta([x,y])\cup N_\delta([y,z])$ (where given $Y\subseteq X$, we denote by $N_\delta(Y)$ the δ-neighborhood of Y in X). The *Gromov boundary* $\partial_\infty X$ of X is the space of equivalence classes of quasi-geodesic rays in X, two rays being equivalent if their images lie at bounded Hausdorff distance (we recall that a *quasi-geodesic ray* is a map $\gamma:\mathbb{R}_+\to X$, so that there exist $K,L>0$ such that for all $s,t\in\mathbb{R}_+$, we have $\frac{1}{K}|t-s|-L\leq d(\gamma(s),\gamma(t))\leq K|t-s|+L$). An isometry ϕ of X is *loxodromic* if for all $x\in X$, we have

$$\lim_{n\to+\infty}\frac{1}{n}d(x,\phi^n x)>0.$$

Given a group G acting by isometries on X, we denote by $\Lambda_X G$ the *limit set* of G in $\partial_\infty X$, which is defined as the intersection of $\partial_\infty X$ with the closure of the orbit of any point in X under the G-action. The following theorem, essentially due to Gromov, gives a classification of isometry groups of (possibly nonproper) Gromov hyperbolic spaces. A sketch of proof can be found in [CdCMT13, Proposition 3.1], see also [Ham13, Theorem 2.7].

Theorem G.1.1. *(Gromov [Gro87, Section 8.2]) Let X be a hyperbolic geodesic metric space, and let G be a group acting by isometries on X. Then G is either*

- bounded*, i.e. all G-orbits in X are bounded; in this case $\Lambda_X G=\emptyset$, or*
- horocyclic*, i.e. G is not bounded and contains no loxodromic element; in this case $\Lambda_X G$ is reduced to one point, or*
- lineal*, i.e. G contains a loxodromic element, and any two loxodromic elements have the same fixed points in $\partial_\infty X$; in this case $\Lambda_X G$ consists of these two points, or*
- focal*, i.e. G is not lineal, contains a loxodromic element, and any two loxodromic elements have a common fixed point in $\partial_\infty X$; in this case $\Lambda_X G$ is uncountable and G has a fixed point in $\Lambda_X G$, or*
- of general type*, i.e. G contains two loxodromic elements with no common endpoints; in this case $\Lambda_X G$ is uncountable and G has no finite orbit in $\partial_\infty X$. In addition, the group G contains two loxodromic isometries that generate a rank two free subgroup.*

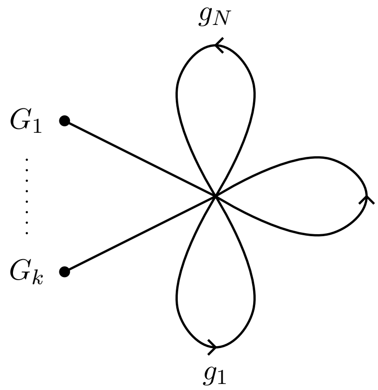

Figure G.1: The tree T^{def} is the Bass–Serre tree of the above graph of groups decomposition of G.

In particular, we have the following result.

Theorem G.1.2. *(Gromov [Gro87, Section 8.2]) Let X be a hyperbolic geodesic metric space, and let G be a group acting by isometries on X. If $\Lambda_X G \neq \emptyset$, and G has no finite orbit in $\partial_\infty X$, then G contains a rank two free subgroup generated by two loxodromic isometries.*

G.1.2 Free factor systems and relative complexes

Free factor systems. Let G be a countable group that splits as a free product of the form

$$G := G_1 * \cdots * G_k * F,$$

where F is a finitely generated free group. We let $\mathcal{F} := \{[G_1], \dots, [G_k]\}$ be the finite collection of all G-conjugacy classes of the G_i's. We fix a free basis $\{g_1, \dots, g_N\}$ of F, and we let T^{def} be the G-tree defined as the Bass–Serre tree of the graph of group decomposition of G depicted on Figure G.1. The rank of the free group F arising in the splitting of G only depends on $\mathcal{F}$. We call it the *free rank* of $(G, \mathcal{F})$ and denote it by $\mathrm{rk}_f(G, \mathcal{F})$. The *Kurosh rank* of $(G, \mathcal{F})$ is defined as $\mathrm{rk}_K(G, \mathcal{F}) := |\mathcal{F}| + \mathrm{rk}_f(G, \mathcal{F})$.

Subgroups of G which are conjugate into one of the subgroups of $\mathcal{F}$ will be called *peripheral* subgroups. A *$(G, \mathcal{F})$-free splitting* is a minimal, simplicial G-tree T in which all peripheral subgroups are *elliptic* (i.e. they fix a point in T), and edge stabilizers are trivial.

Subgroups of free products. Subgroups of free products were studied by Kurosh in [Kur34]. Let H be a subgroup of G. By considering the H-minimal subtree in the tree T^{def} (see the definition in Section G.1.3 below), we get the existence of a (possibly infinite) set J, together with an integer $i_j \in \{1, \dots, k\}$, a nontrivial subgroup $H_j \subseteq G_{i_j}$ and an element $g_j \in G$ for each $j \in J$, and a (not necessarily finitely generated) free subgroup $F' \subseteq G$, so that

$$H = *_{j \in J}\; g_j H_j g_j^{-1} * F'.$$

This splitting will be called the *Kurosh decomposition* of H. The *Kurosh rank* of H is equal to $\mathrm{rk}_K(H) := |J| + \mathrm{rk}(F')$, its *free rank* is $\mathrm{rk}_f(H) := \mathrm{rk}(F')$. They can be infinite in general. We also let $\mathcal{F}_H$ denote the set of H-conjugacy classes of the subgroups $g_j H_j g_j^{-1}$, which might also be infinite in general. We note that $\mathrm{rk}_f(G,\mathcal{F})$ and $\mathcal{F}_H$ (and hence $\mathrm{rk}_K(G,\mathcal{F})$) only depend on H and $\mathcal{F}$, and not of our initial choice of T^{def}.

Free factors. A $(G,\mathcal{F})$-*free factor* is a subgroup of G that is a point stabilizer in some $(G,\mathcal{F})$-free splitting. A $(G,\mathcal{F})$-free factor is *proper* if it is nonperipheral (in particular nontrivial), and not equal to G. The Kurosh decomposition of a proper $(G,\mathcal{F})$-free factor reads as

$$H = G'_{i_1} * \cdots * G'_{i_r} * F',$$

where each of the subgroups G'_{i_j} is conjugate in G to one of the factors in $\mathcal{F}$ (with no repetition in the indices, i.e. the G'_{i_j}'s are pairwise non conjugate in G), and F' is a finitely generated free group. In particular, the Kurosh rank of H is finite. The group G then splits as

$$G = H * G'_{i_{r+1}} * \cdots * G'_{i_k} * F'',$$

where F'' is a finitely generated free subgroup of G, and the G'_{i_j}'s are conjugate to the factors in $\mathcal{F}$ that do not arise in the Kurosh decomposition of H. The finite collection $\mathcal{F}' := \{[H],[G_{i_{r+1}}],\dots,[G_{i_k}]\}$ (where we consider G-conjugacy classes) is a free factor system of G, and we have

$$|\mathcal{F}'| + |\mathcal{F}_H| = |\mathcal{F}| + 1, \tag{G.1}$$

and

$$\mathrm{rk}_f(G,\mathcal{F}') + \mathrm{rk}_f(H) = \mathrm{rk}_f(G,\mathcal{F}), \tag{G.2}$$

whence

$$\mathrm{rk}_K(G,\mathcal{F}') + \mathrm{rk}_K(H) = \mathrm{rk}_K(G,\mathcal{F}) + 1. \tag{G.3}$$

Let H and H' be two $(G,\mathcal{F})$-free factors, and let T be a $(G,\mathcal{F})$-free splitting, one of whose elliptic subgroups is equal to H. By looking at the H'-minimal subtree of T, we see that $H \cap H'$ is an $(H',\mathcal{F}_{H'})$-free factor, so it is a $(G,\mathcal{F})$-free factor. This implies that the intersection of any family of $(G,\mathcal{F})$-free factors is again a free factor. In particular, any subgroup $A \subseteq G$ is contained in a smallest $(G,\mathcal{F})$-free factor, obtained as the intersection of all $(G,\mathcal{F})$-free factors that contain A. We denote it by $\mathrm{Fill}(A)$.

Relative automorphisms. Let G be a countable group, and $\mathcal{F}$ be a free factor system of G. We denote by $\mathrm{Out}(G,\mathcal{F})$ the subgroup of $\mathrm{Out}(G)$ made of those automorphisms that preserve the conjugacy classes in $\mathcal{F}$. We denote by $\mathrm{Out}(G,\mathcal{F}^{(t)})$ the subgroup of $\mathrm{Out}(G)$ made of those automorphisms that act as a conjugation by an element of G on each peripheral subgroup.

For all $i \in \{1,\dots,k\}$, the group G_i is equal to its normalizer in G. Therefore, any element of $\mathrm{Out}(G)$ that preserves the conjugacy class of G_i induces a well-defined outer automorphism of G_i. In other words, there is a morphism

$$\mathrm{Out}(G,\{[G_i]\}) \to \mathrm{Out}(G_i).$$

By taking the product over all groups G_i, we thus get a (surjective) morphism

$$\mathrm{Out}(G,\mathcal{F}) \to \prod_{i=1}^{k} \mathrm{Out}(G_i),$$

whose kernel is equal to $\mathrm{Out}(G,\mathcal{F}^{(t)})$.

More generally, suppose that we are given a collection of subgroups $A_i \subseteq \mathrm{Out}(G_i)$ for all $i \in \{1, \dots, k\}$, and let $\mathcal{A} = \{A_1, \dots, A_k\}$. We can define the subgroup $\mathrm{Out}(G, \mathcal{F}^{\mathcal{A}})$ of $\mathrm{Out}(G)$ made of those automorphisms that preserve all conjugacy classes in $\mathcal{F}$, and which induce an element of A_i in restriction to G_i for all $i \in \{1, \dots, k\}$. As above, there is a (surjective) morphism

$$\mathrm{Out}(G, \mathcal{F}^{\mathcal{A}}) \to \prod_{i=1}^{k} A_i,$$

whose kernel is equal to $\mathrm{Out}(G, \mathcal{F}^{(t)})$.

G.1.3 Relative outer spaces

An $\mathbb{R}$-*tree* is a metric space (T, d_T) in which any two points $x, y \in T$ are joined by a unique embedded topological arc, which is isometric to a segment of length $d_T(x, y)$. A $(G, \mathcal{F})$-tree is an $\mathbb{R}$-tree equipped with a minimal, isometric action of G, in which all peripheral subgroups of G are *elliptic*. We recall that an action on a tree is termed *minimal* if there is no proper and nontrivial invariant subtree. Whenever a group G acts on an $\mathbb{R}$-tree T, and some element of G does not fix any point in T, there is a unique subtree of T on which the G-action is minimal. In particular, whenever H is a subgroup of G that contains a hyperbolic element, we can consider the minimal subtree for the induced action of H on T, which we call the H-*minimal subtree* of T. The action of H on T is *simplicial* if the H-minimal subtree is homeomorphic (when equipped with the topology defined by the metric) to a simplicial tree. We say that the action of H on T is *relatively free* if all point stabilizers of the H-minimal subtree of T are conjugate into $\mathcal{F}_H$.

A *Grushko* $(G, \mathcal{F})$-*tree* is a simplicial $(G, \mathcal{F})$-tree with trivial edge stabilizers, all of whose elliptic subgroups are peripheral. Two $(G, \mathcal{F})$-trees are *equivalent* if there exists a G-equivariant isometry between them.

The *unprojectivized outer space* $\mathcal{O}(G, \mathcal{F})$, introduced by Guirardel and Levitt in [GL07b], is defined to be the space of all equivalence classes of Grushko $(G, \mathcal{F})$-trees. *Outer space* $P\mathcal{O}(G, \mathcal{F})$ is defined as the space of homothety classes of trees in $\mathcal{O}(G, \mathcal{F})$. Outer space, as well as its unprojectivized version, comes equipped with a right action of $\mathrm{Out}(G, \mathcal{F})$, given by precomposing the actions (this can be turned into a left action by letting $\Phi.T := T.\Phi^{-1}$ for all $T \in \mathcal{O}(G, \mathcal{F})$ and all $\Phi \in \mathrm{Out}(G, \mathcal{F})$).

For all $g \in G$ and all $T \in \mathcal{O}(G, \mathcal{F})$, the *translation length* of g in T is defined to be

$$||g||_T := \inf_{x \in T} d_T(x, gx).$$

Culler and Morgan have shown in [CM87] that the map

$$\begin{array}{cccc} i: & \mathcal{O}(G, \mathcal{F}) & \to & \mathbb{R}^G \\ & T & \mapsto & (||g||_T)_{g \in G} \end{array}$$

is injective. We equip $\mathcal{O}(G, \mathcal{F})$ with the topology induced by this embedding, which is called the *axes topology*. Outer space is then embedded as a subspace of the projective space $\mathbb{PR}^G$, and is equipped with the quotient topology. Its closure $\overline{P\mathcal{O}(G, \mathcal{F})}$, whose lift to $\mathbb{R}^G$ we denote by $\overline{\mathcal{O}(G, \mathcal{F})}$, is compact (see [CM87, Theorem 4.2] and [7, Proposition 1.2]). We let $\partial P\mathcal{O}(G, \mathcal{F}) := \overline{P\mathcal{O}(G, \mathcal{F})} \smallsetminus P\mathcal{O}(G, \mathcal{F})$, and similarly $\partial \mathcal{O}(G, \mathcal{F}) := \overline{\mathcal{O}(G, \mathcal{F})} \smallsetminus \mathcal{O}(G, \mathcal{F})$. A $(G, \mathcal{F})$-tree T is *very small* if its arc stabilizers are either trivial, or maximally-cyclic and nonperipheral, and its tripod stabilizers are trivial. In [7, Theorem 0.1], we identified the space $\overline{P\mathcal{O}(G, \mathcal{F})}$ with the space of very small, minimal, projective $(G, \mathcal{F})$-trees. We also proved that it has finite topological dimension equal to $3\mathrm{rk}_f(G, \mathcal{F}) + 2|\mathcal{F}| - 4$.

G.1.4 The cyclic splitting graph

Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. A *$\mathcal{Z}$-splitting* of $(G,\mathcal{F})$ is a minimal, simplicial $(G,\mathcal{F})$-tree, all of whose edge stabilizers are either trivial, or cyclic and nonperipheral. It is a *one-edge* splitting if it has exactly one G-orbit of edges. Two $\mathcal{Z}$-splittings are *equivalent* if there exists a G-equivariant homeomorphism between them. Given two $(G,\mathcal{F})$-trees T and T', a map $f: T \to T'$ is *alignment-preserving* if the f-image of every segment in T is a segment in T'. If there exists a G-equivariant alignment-preserving map from T to T', we say that T is a *refinement* of T'. The *cyclic splitting graph* $FZ(G,\mathcal{F})$ is the graph whose vertices are the equivalence classes of one-edge $\mathcal{Z}$-splittings of $(G,\mathcal{F})$, two distinct vertices being joined by an edge if the corresponding splittings admit a common refinement. The graph $FZ(G,\mathcal{F})$ admits a natural right action of $\mathrm{Out}(G,\mathcal{F})$, by precomposition of the actions. In [8], we proved hyperbolicity of the graph $FZ(G,\mathcal{F})$.

Theorem G.1.3. *(Horbez [8, Theorem 3.1]) Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. Then the graph $FZ(G,\mathcal{F})$ is Gromov hyperbolic.*

We also described the Gromov boundary of $FZ(G,\mathcal{F})$. A tree $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ is *$\mathcal{Z}$-compatible* if it is compatible with some $\mathcal{Z}$-splitting of $(G,\mathcal{F})$, and *$\mathcal{Z}$-incompatible* otherwise. It is *$\mathcal{Z}$-averse* if it is not compatible with any $\mathcal{Z}$-compatible tree $T' \in \overline{\mathcal{O}(G,\mathcal{F})}$ (see [8, Section 5.6.1] for examples of $\mathcal{Z}$-incompatible trees that are not $\mathcal{Z}$-averse). We denote by $\mathcal{X}(G,\mathcal{F})$ the subspace of $\overline{\mathcal{O}(G,\mathcal{F})}$ consisting of $\mathcal{Z}$-averse trees. Two trees $T, T' \in \mathcal{X}(G,\mathcal{F})$ are *equivalent*, which we denote by $T \sim T'$, if they are both compatible with a common tree in $\overline{\mathcal{O}(G,\mathcal{F})}$. There is a natural, coarsely well-defined map $\psi : \mathcal{O}(G,\mathcal{F}) \to FZ(G,\mathcal{F})$.

Theorem G.1.4. *(Horbez [8, Theorem 0.2]) Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. Then there exists a unique $\mathrm{Out}(G,\mathcal{F})$-equivariant homeomorphism*

$$\partial\psi : \mathcal{X}(G,\mathcal{F})/\!\sim \to \partial_\infty FZ(G,\mathcal{F}),$$

so that for all $T \in \mathcal{X}(G,\mathcal{F})$, and all sequences $(T_i)_{i\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$ converging to T, the sequence $(\psi(T_i))_{i\in\mathbb{N}}$ converges to $\partial\psi(T)$.

We also proved that every $\sim$-class of $\mathcal{Z}$-averse trees contains a unique simplex of mixing representatives. A tree $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ is *mixing* if for all finite subarcs $I, J \subseteq T$, there exist $g_1,\dots,g_k \in G$ such that $J \subseteq g_1 I \cup \dots \cup g_k I$, and for all $i \in \{1,\dots,k-1\}$, we have $g_i I \cap g_{i+1} I \neq \emptyset$. Two $\mathbb{R}$-trees T and T' are *weakly homeomorphic* if there exist maps $f: T \to T'$ and $g: T' \to T$ that are continuous in restriction to segments, and inverse of each other.

Proposition G.1.5. *(Horbez [8, Proposition 5.3]) For all $T \in \mathcal{X}(G,\mathcal{F})$, there exists a mixing tree $\overline{T} \in \mathcal{X}(G,\mathcal{F})$ onto which all trees $T' \in \mathcal{X}(G,\mathcal{F})$ that are equivalent to T collapse. In addition, any two such trees are G-equivariantly weakly homeomorphic. Any tree $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ that is both $\mathcal{Z}$-incompatible and mixing, is $\mathcal{Z}$-averse.*

We also mention the following fact about $\mathcal{Z}$-splittings of $(G,\mathcal{F})$.

Lemma G.1.6. *(Horbez [7, Lemma 5.11]) Let S be a $\mathcal{Z}$-splitting of $(G,\mathcal{F})$. Then every edge stabilizer in S is trivial, or contained in a proper $(G,\mathcal{F})$-free factor.*

G.1.5 Transverse families, transverse coverings, graphs of actions

Let T be a $(G,\mathcal{F})$-tree. A *transverse family* in T is a G-invariant collection $\mathcal{Y}$ of nondegenerate (i.e. nonempty and not reduced to a point) subtrees of T, such that for all $Y \neq Y' \in \mathcal{Y}$, the intersection $Y \cap Y'$ contains at most one point.

A *transverse covering* of T is a transverse family $\mathcal{Y}$ in T, all of whose elements are closed subtrees of T, such that every finite arc in T can be covered by finitely many elements of $\mathcal{Y}$. A transverse covering $\mathcal{Y}$ of T is *trivial* if $\mathcal{Y} = \{T\}$. The *skeleton* of a transverse covering $\mathcal{Y}$ is the bipartite simplicial tree S, whose vertex set is $V(S) = V_0(S) \cup \mathcal{Y}$, where $V_0(S)$ is the set of points of T which belong to at least two distinct trees in $\mathcal{Y}$, with an edge between $x \in V_0(S)$ and $Y \in \mathcal{Y}$ whenever $x \in Y$ [Gui04, Definition 4.8].

Let G be a countable group, and $\mathcal{F}$ be a free factor system of G. A $(G,\mathcal{F})$*-graph of actions* consists of

- a metric graph of groups $\mathcal{G}$ (in which we allow some edges to have length 0), with an isomorphism from G to the fundamental group of $\mathcal{G}$, such that all peripheral subgroups are conjugate into vertex groups of $\mathcal{G}$, and
- an isometric action of every vertex group G_v on a G_v-tree T_v (possibly reduced to a point), in which all intersections of G_v with peripheral subgroups of G are elliptic, and
- a point $p_e \in T_{t(e)}$ fixed by $i_e(G_e) \subseteq G_{t(e)}$ for every oriented edge e, where $i_e : G_e \to G_{t(e)}$ denotes the inclusion morphism from the edge group G_e into the adjacent vertex group $G_{t(e)}$.

A $(G,\mathcal{F})$-graph of actions is *nontrivial* if $\mathcal{G}$ is not reduced to a point. Associated to any $(G,\mathcal{F})$-graph of actions $\mathcal{G}$ is a $(G,\mathcal{F})$-tree $T(\mathcal{G})$. Informally, the tree $T(\mathcal{G})$ is obtained from the Bass–Serre tree of the underlying graph of groups by equivariantly attaching each vertex tree T_v at the corresponding vertex v, an incoming edge being attached to T_v at the prescribed attaching point. The reader is referred to [Gui98, Proposition 3.1] for a precise description of the tree $T(\mathcal{G})$. We say that a $(G,\mathcal{F})$-tree T *splits as a* $(G,\mathcal{F})$*-graph of actions* if there exists a $(G,\mathcal{F})$-graph of actions $\mathcal{G}$ such that $T = T(\mathcal{G})$.

Proposition G.1.7. *(Guirardel [Gui08, Lemma 1.5]) A $(G,\mathcal{F})$-tree splits as a nontrivial $(G,\mathcal{F})$-graph of actions if and only if it admits a nontrivial transverse covering.*

Knowing that a $(G,\mathcal{F})$-tree T is compatible with a simplicial $(G,\mathcal{F})$-tree S provides a nontrivial transverse covering of T, defined in the following way (see the discussion in [8, Section 4.7]). Since T and S are compatible, their length functions sum up to the length function of a $(G,\mathcal{F})$-tree, denoted by $T+S$, which comes with 1-Lipschitz alignment-preserving maps $\pi_T : T+S \to T$ and $\pi_S : T+S \to S$, see [GL10b, Section 3.2]. Then the family $\mathcal{Y}$ made of all nondegenerate π_S-preimages of vertices of S, and of the closures of π_S-preimages of open edges of S, is a transverse covering of $T+S$. Its image $\pi_T(\mathcal{Y})$ is a nontrivial transverse covering of T.

We now mention a result, due to Levitt [Lev94], which gives a canonical way of splitting any very small $(G,\mathcal{F})$-tree as a $(G,\mathcal{F})$-graph of actions, whose vertex actions have dense orbits.

Proposition G.1.8. *(Levitt [Lev94]) Every $(G,\mathcal{F})$-tree $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ splits uniquely as a $(G,\mathcal{F})$-graph of actions, all of whose vertex trees have dense orbits for the action of their stabilizer (they might be reduced to points), and all of whose edges have positive length, and have either trivial, or maximally-cyclic and nonperipheral stabilizer.*

We call this splitting the *Levitt decomposition* of T as a graph of actions. We note in particular that if $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ is a very small $(G,\mathcal{F})$-tree, and $H \subseteq G$ is a subgroup of G of finite Kurosh rank, then the H-minimal subtree of T admits a Levitt decomposition.

Lemma G.1.9. *Let $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ be a tree with dense orbits. Let $\mathcal{Y}$ be a transverse family in T, and let $Y \in \mathcal{Y}$. If $rk_K(Stab(Y)) < +\infty$, then the action of $Stab(Y)$ on Y has dense orbits. If $Stab(Y)$ is contained in a proper $(G,\mathcal{F})$-free factor H, then the H-minimal subtree of T is not a Grushko $(H,\mathcal{F}_H)$-tree.*

Proof. Assume that one of the conclusions of the lemma fails. Then Y has a nontrivial simplicial part, which contains a simplicial edge e. There is a finite number of G-orbits of directions at branch points in T [7, Corollary 4.8]. As T has dense orbits, the arc e contains two distinct branch points x and x' of T, and two directions d (resp. d') at x (resp. x'), such that there exists $g \in G \setminus \{1\}$ with $gd = d'$. In particular, the intersection $gY \cap Y$ is nondegenerate (i.e. nonempty and not reduced to a point). As $\mathcal{Y}$ is a transverse family, this implies that $g \in \mathrm{Stab}(Y)$. So ge is a simplicial edge of Y that meets e, and therefore $ge = e$. This implies that T contains an arc with nontrivial stabilizer, which is impossible because T has dense orbits [7, Proposition 4.17]. □

G.1.6 Trees of surface type

Definition G.1.10. *A tree $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ is* of surface type *if it admits a transverse covering by trees that are either simplicial arcs, or are dual to arational measured foliations on compact 2-orbifolds.*

Proposition G.1.11. *(Horbez [7, Proposition 5.10]) Let T be a minimal, very small $(G,\mathcal{F})$-tree of surface type, and let $\mathcal{Y}$ be the associated transverse covering of T. Then either*

- *there exists an element of G, represented by a boundary curve of one of the orbifolds dual to a tree in $\mathcal{Y}$, that is nonperipheral, and not conjugate into any edge group of the skeleton of $\mathcal{Y}$, or*
- *the tree T splits as a $(G,\mathcal{F})$-graph of actions over a one-edge $(G,\mathcal{F})$-free splitting S, such that all stabilizers of subtrees in $\mathcal{Y}$ dual to arational foliations on compact 2-orbifolds are elliptic in S.*

Proposition G.1.12. *(Horbez [7, Lemma 5.8]) Let $T \in \overline{\mathcal{O}(G,\mathcal{F})}$. If there exists a subgroup $H \subseteq G$ that is elliptic in T, and not contained in any proper $(G,\mathcal{F})$-free factor, then T is of surface type.*

G.2 Sporadic cases

Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. We say that $(G,\mathcal{F})$ is *sporadic* if either $G = G_1 * G_2$ and $\mathcal{F} = \{[G_1],[G_2]\}$, or $G = G_1*$ and $\mathcal{F} = \{[G_1]\}$. Otherwise $(G,\mathcal{F})$ is *nonsporadic.* We noticed in [8, Corollary 5.8] that the graph $FZ(G,\mathcal{F})$ is unbounded if and only if $(G,\mathcal{F})$ is nonsporadic. Given a group A, we denote by $Z(A)$ its center. The following propositions, which describe $\mathrm{Out}(G,\mathcal{F}^{(t)})$ when $(G,\mathcal{F})$ is sporadic, are particular cases of Levitt's work about automorphisms of graphs of groups [Lev04].

Proposition G.2.1. *Let G_1 and G_2 be nontrivial countable groups. Then $Out(G_1 * G_2, \{[G_1],[G_2]\}^{(t)})$ is isomorphic to $G_1/Z(G_1) \times G_2/Z(G_2)$.*

Proposition G.2.2. *Let G_1 be a countable group. Then $Out(G_1*, \{[G_1]\}^{(t)})$ has a subgroup of index 2 that is isomorphic to $(G_1 \times G_1)/Z(G_1)$, where $Z(G_1)$ sits as a subgroup of $G_1 \times G_1$ via the diagonal inclusion map.*

G.3 Stabilizers of trees in $\overline{\mathcal{O}(G, \mathcal{F})}$

Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. Given $T \in \overline{\mathcal{O}(G, \mathcal{F})}$ (resp. $[T] \in \overline{P\mathcal{O}(G, \mathcal{F})}$), we denote by $\text{Out}(T)$ (resp. $\text{Out}([T])$) the subgroup of $\text{Out}(G, \mathcal{F}^{(t)})$ consisting of those automorphisms that fix T (resp. $[T]$). Notice that $\text{Out}(T)$ sits inside $\text{Out}([T])$ as a normal subgroup. There is a natural morphism

$$\lambda : \text{Out}([T]) \to \mathbb{R}_+^*,$$

where $\lambda(\Phi)$ is defined as the unique real number such that $T.\Phi = \lambda(\Phi)T$. The kernel of λ is equal to $\text{Out}(T)$, so $\text{Out}([T])$ is an abelian extension of $\text{Out}(T)$. One can actually show that the image of λ is a cyclic subgroup of $\mathbb{R}_+^*$ [GL].

In [7, Corollary 3.5], we proved the following about point stabilizers of trees in $\overline{\mathcal{O}(G, \mathcal{F})}$.

Proposition G.3.1. *(Horbez [7, Corollary 3.5]) Let $T \in \overline{\mathcal{O}(G, \mathcal{F})}$ be a tree with trivial arc stabilizers. Then there are finitely many orbits of points in T with nontrivial stabilizer. For all $v \in T$, we have $rk_K(Stab(v)) < rk_K(G, \mathcal{F})$.*

Let $T \in \overline{\mathcal{O}(G, \mathcal{F})}$ be a tree with trivial arc stabilizers. Let V be the collection of G-orbits of points with nontrivial stabilizer in T. Let $\{G_v\}_{v \in V}$ be a set of representatives of the G-conjugacy classes of point stabilizers in T. We define $\text{Out}(T, \{[G_v]\}_{v \in V}^{(t)})$ to be the subgroup of $\text{Out}(T)$ made of those automorphisms that are a conjugation by an element of G in restriction to every point stabilizer of T.

Theorem G.3.2. *(Guirardel–Levitt [GL]) Let $T \in \overline{\mathcal{O}(G, \mathcal{F})}$ be a tree with trivial arc stabilizers. Let V be the collection of orbits of points in T with nontrivial stabilizer, and let $\{G_v\}_{v \in V}$ be the collection of point stabilizers in T. Then $Out(T, \{[G_v]\}^{(t)})$ has a finite index subgroup $Out^0(T, \{[G_v]\}^{(t)})$ which admits an injective morphism*

$$Out^0(T, \{[G_v]\}^{(t)}) \hookrightarrow \prod_{v \in V} G_v^{d_v}/Z(G_v),$$

where d_v denotes the degree of v in T, and $Z(G_v)$ denotes the center of G_v, and $Z(G_v)$ sits as a diagonal subgroup of $G_v^{d_v}$ via the diagonal inclusion map.

A consequence of Guirardel and Levitt's theorem is the following fact.

Corollary G.3.3. *Let $T \in \overline{\mathcal{O}(G, \mathcal{F})}$ be a tree with trivial arc stabilizers. Let V be the collection of orbits of points in T with nontrivial stabilizer, and let $\{G_v\}_{v \in V}$ be the collection of point stabilizers in T. If G satisfies the Tits alternative, then $Out(T, \{[G_v]\}^{(t)})$ satisfies the Tits alternative.*

G.4 Arational $(G, \mathcal{F})$-trees

Let G be a countable group, and let $\mathcal{F} := \{[G_1], \dots, [G_k]\}$ be a free factor system of G. We recall that a $(G, \mathcal{F})$-free factor is *proper* if it is nonperipheral (in particular nontrivial), and not equal to G.

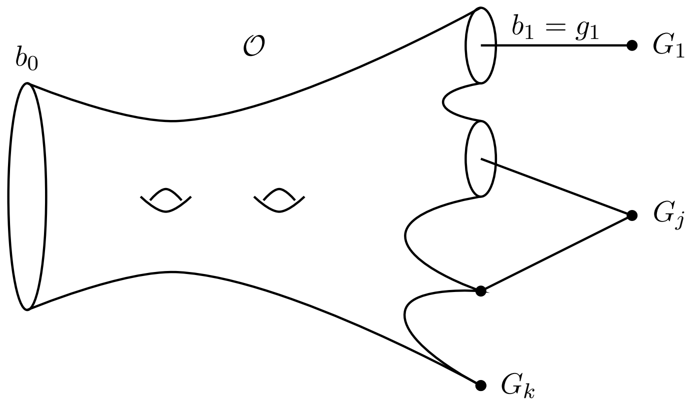

Figure G.2: An arational surface $(G,\mathcal{F})$-tree.

Definition G.4.1. *A $(G,\mathcal{F})$-tree $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ is* arational *if $T \in \partial\mathcal{O}(G,\mathcal{F})$ and for every proper $(G,\mathcal{F})$-free factor $H \subset G$, the factor H is not elliptic in T, and the H-minimal subtree T_H of T is a Grushko $(H,\mathcal{F}_H)$-tree, i.e. the action of H on T_H is simplicial and relatively free.*

We denote by $\mathcal{AT}(G,\mathcal{F})$ the subspace of $\overline{\mathcal{O}(G,\mathcal{F})}$ consisting of arational $(G,\mathcal{F})$-trees.

G.4.1 Arational surface $(G,\mathcal{F})$-trees

We describe a way of constructing arational $(G,\mathcal{F})$-trees, illustrated in Figure G.2. We first need the following fact.

Proposition G.4.2. *Let T be a tree dual to an arational measured foliation on a compact 2-orbifold $\mathcal{O}$ with conical singularities, and let $H \subseteq \pi_1(\mathcal{O})$ be a finitely generated subgroup of $\pi_1(\mathcal{O})$ of infinite index. Then the H-minimal subtree of T is simplicial.*

A proof of Proposition G.4.2 appears in [Rey11b] in the case where $\mathcal{O}$ is a compact surface, and it adapts to the case where $\mathcal{O}$ is a 2-orbifold. Proposition G.4.2 can also be deduced from the surface case by using Selberg's Lemma, which states that $\pi_1(\mathcal{O})$ has a finite-index subgroup which is the fundamental group of a compact surface.

Let $\mathcal{O}$ be a compact 2-orbifold of genus g with conical singularities, having $s+1$ boundary curves $b_0, b_1, \dots, b_s$, and q conical points $b_{s+1}, \dots, b_{s+q}$, equipped with an arational measured foliation. We build a graph of groups $\mathcal{G}'$ in the following way. One of the vertex groups of $\mathcal{G}'$ is the fundamental group of the orbifold $\mathcal{O}$, and the others are the peripheral subgroups G_i. For all $i \in \{1, \dots, s+q\}$, we choose $j_i \in \{1, \dots, k\}$, and an element $g_i \in G_{j_i}$, of same order as b_i. We put an edge between the vertex of $\mathcal{G}'$ associated to $\mathcal{O}$ and the vertex associated to G_{j_i}, and we amalgamate b_i with g_i. Choices are made in such a way that the graph $\mathcal{G}'$ we get is connected. We then define a graph of groups $\mathcal{G}$ as the minimal subgraph of groups of $\mathcal{G}'$, i.e. $\mathcal{G}$ is obtained from $\mathcal{G}'$ by removing vertices G_j with exactly one incident edge, and such that G_i is cyclic and generated by b_i. Notice that the element of $\pi_1(\mathcal{O})$ corresponding to the boundary curve b_0 does not fix any edge in $\mathcal{G}$. The fundamental group of $\mathcal{G}$ is isomorphic to $G := G_1 * \cdots * G_k * F_N$, where $N = 2g + b_1(\mathcal{G})$ if $\mathcal{O}$ is orientable, and $N = g + b_1(\mathcal{G})$ if $\mathcal{O}$ is nonorientable.

Dual to the foliation on $\mathcal{O}$ is a $\pi_1(\mathcal{O})$-tree Y. We form a graph of actions over $\mathcal{G}$: vertex trees are the $\pi_1(\mathcal{O})$-tree Y, and a trivial G_i-tree for all $i \in \{1, \dots, k\}$, attaching points in Y are the points fixed by the b_i's, and edges have length 0. We denote by T the $(G,\mathcal{F})$-tree defined in this way.

Definition G.4.3. *A $(G,\mathcal{F})$-tree obtained by the above construction is called an* arational surface *$(G,\mathcal{F})$-tree.*

We claim that the $(G,\mathcal{F})$-tree T we have built is an arational $(G,\mathcal{F})$-tree, which justifies our terminology.

We start by making the following remarks: all point stabilizers in Y are peripheral, except b_0. The element b_0 is not contained in any proper $(G,\mathcal{F})$-free factor. Indeed, otherwise, there would exist a $(G,\mathcal{F})$-free splitting S in which b_0 is elliptic, and all other boundary components of $\mathcal{O}$ would also be elliptic in S because they are peripheral. The splitting S would then restrict to a free splitting of $\pi_1(\mathcal{O})$ in which all boundary components are elliptic. Such a splitting does not exist, so we have reached a contradiction.

Let now H be a proper $(G,\mathcal{F})$-free factor. Assume towards a contradiction that the H-minimal subtree of T is not a Grushko $(H,\mathcal{F}_H)$-tree. The action of H on T is relatively free because b_0 is not contained in any proper $(G,\mathcal{F})$-free factor, so the action of H is not discrete. The transverse covering of T made of the translates of the $\pi_1(\mathcal{O})$-minimal subtree of T induces a transverse covering of the H-minimal subtree of T, whose nontrivial elements are $H \cap \pi_1(\mathcal{O})^g$-trees, for some $g \in G$. Therefore, there exists a conjugate H^g of H so that $H^g \cap \pi_1(\mathcal{O}) \neq \{e\}$, and the action of $H^g \cap \pi_1(\mathcal{O})$ on its minimal subtree is non-simplicial. By Proposition G.4.2, this implies that $H^g \cap \pi_1(\mathcal{O})$ has finite index in $\pi_1(\mathcal{O})$. As H is elliptic in a $(G,\mathcal{F})$-free splitting S, so is $\pi_1(\mathcal{O})$: the group $\pi_1(\mathcal{O})$ fixes a unique point in S. All other vertex stabilizers of the Bass–Serre tree S_0 of $\mathcal{G}$ are peripheral, so each of them fixes a unique point in S. Since edge stabilizers of S_0 are peripheral, the stabilizers of any two adjacent vertices in S_0 contain a common peripheral element. This implies that they have the same fixed point in S, because no peripheral element fixes an arc in S. Therefore, all vertex groups of S_0 fix the same point in S. Hence G is elliptic in S, a contradiction.

G.4.2 A classification result

The goal of this section is to provide a classification result for trees in $\overline{\mathcal{O}(G,\mathcal{F})}$. When $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ is not arational, a proper $(G,\mathcal{F})$-free factor is a *dynamical free factor* for T if it acts with dense orbits on its minimal subtree but does not fix any point in T. The following proposition is an extension of [6, Proposition 2.1] to the context of $(G,\mathcal{F})$-trees.

Proposition G.4.4. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. Then for all $(G,\mathcal{F})$-trees $T \in \overline{\mathcal{O}(G,\mathcal{F})}$, either*

- *we have $T \in \mathcal{O}(G,\mathcal{F})$, or*
- *the tree T is arational, or*
- *the tree T has a dynamical free factor, or*
- *the tree T has no dynamical free factor, and there exists $x \in T$ whose stabilizer is nonperipheral, and is contained in a proper $(G,\mathcal{F})$-free factor.*

Lemma G.4.5. *Let T be a $(G,\mathcal{F})$-tree with trivial arc stabilizers. Let $H \subseteq G$ be a nonperipheral subgroup of G that is contained in a proper $(G,\mathcal{F})$-free factor. If H fixes a point in T, then T is not arational. If the H-minimal subtree of T is not simplicial, then T has a dynamical proper free factor.*

Proof. Let F be a proper $(G,\mathcal{F})$-free factor that contains H. If H fixes a point in T, then the action of F is not relatively free, which implies that T is not arational.

By Proposition G.1.8, the F-minimal subtree T_F of T splits as a graph of actions $\mathcal{G}$ with trivial edge stabilizers, in which all vertex actions have dense orbits (they may be trivial). Vertex groups of $\mathcal{G}$ are $(G,\mathcal{F})$-free factors. If the H-minimal subtree of T is non-simplicial, then T_F is non-simplicial, so one of the vertex groups of $\mathcal{G}$ is a dynamical proper $(G,\mathcal{F})$-free factor of T. ☐

Lemma G.4.6. *Let T be a $(G,\mathcal{F})$-tree with trivial arc stabilizers. Assume that T is not relatively free. Then either*

- *the tree T is an arational surface tree (in particular, all elliptic subgroups in T are either cyclic or peripheral), or*
- *the tree T has a dynamical proper free factor, or*
- *there exists a nonperipheral point stabilizer in T that is contained in a proper $(G,\mathcal{F})$-free factor, and all noncyclic, nonperipheral point stabilizers in T are contained in proper $(G,\mathcal{F})$-free factors.*

Proof. If all elliptic subgroups of T are contained in proper $(G,\mathcal{F})$-free factors, then the last assertion holds. Otherwise, Lemma G.1.12 implies that T is a tree of surface type. Let $\mathcal{Y}$ be the transverse covering of T provided by the definition of trees of surface type.

If the stabilizer of a tree in $\mathcal{Y}$ dual to an arational measured foliation on a compact 2-orbifold is contained in a proper $(G,\mathcal{F})$-free factor, then the second assertion holds by Lemma G.4.5. This occurs in particular if the skeleton of $\mathcal{Y}$ contains an edge with trivial stabilizer, so we can assume that this is not the case.

Otherwise, Proposition G.1.11 implies that there exists an element of G, represented by a boundary curve c of an orbifold Σ dual to a tree in $\mathcal{Y}$, that is nonperipheral, and not conjugate into any edge group of the skeleton of $\mathcal{Y}$. If the transverse covering $\mathcal{Y}$ contains at least two orbits of nondegenerate trees, then an arc on Σ whose endpoints lie on c determines a $(G,\mathcal{F})$-free splitting, in which the other orbifold groups are elliptic, and hence contained in a proper $(G,\mathcal{F})$-free factor. Again, the second assertion of the lemma holds. Similarly, if there exists a point in T, whose stabilizer is nonperipheral and not conjugate to c, then the third conclusion of the lemma holds.

In the remaining case, the skeleton of $\mathcal{Y}$ contains a single orbit of vertices v associated to a tree T_0 dual to an arational lamination on a 2-orbifold $\mathcal{O}$. All vertices v' adjacent to v have stabilizer isomorphic to some G_i. The edge joining v' to v has nontrivial stabilizer, so it is attached in T_0 to a point corresponding to a boundary curve or a conical point of $\mathcal{O}$. In addition, all boundary curves (and conical points) of Σ distinct from c are peripheral. This implies that T is an arational surface $(G,\mathcal{F})$-tree. ☐

Proof of Proposition G.4.4. Let $T \in \partial\mathcal{O}(G,\mathcal{F})$ be a tree which is not arational, and has no dynamical proper $(G,\mathcal{F})$-free factor. Then the G-action on T is not relatively free. If T has trivial arc stabilizers, then the conclusion follows from Lemma G.4.6.

We now assume that T contains an arc e with nontrivial stabilizer, and let S be the very small simplicial $(G,\mathcal{F})$-tree obtained by collapsing to points all vertex trees in the Levitt decomposition of T as a graph of actions (Proposition G.1.8). The stabilizer G_e of e in T also stabilizes an edge in S. By Lemma G.1.6, the group G_e is contained in a proper $(G,\mathcal{F})$-free factor, and in addition G_e is nonperipheral because T is very small. We can thus choose for x some interior point of e. ☐

G.4.3 Arational $(G,\mathcal{F})$-trees are $\mathcal{Z}$-averse.

Proposition G.4.7. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. Then $\mathcal{AT}(G,\mathcal{F}) \subseteq \mathcal{X}(G,\mathcal{F})$.*

Proof. In view of Proposition G.1.5, it is enough to show that any tree $T \in \mathcal{AT}(G,\mathcal{F})$ is both $\mathcal{Z}$-incompatible and mixing. This will be done in Lemmas G.4.8 and G.4.9. □

Lemma G.4.8. *Every arational $(G,\mathcal{F})$-tree is $\mathcal{Z}$-incompatible.*

Proof of Lemma G.4.8. Let $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ be a $\mathcal{Z}$-compatible tree. It follows from the discussion below Proposition G.1.7 that T splits as a $(G,\mathcal{F})$-graph of actions $\mathcal{G}$, whose edge groups are either trivial, or cyclic and nonperipheral. If $\mathcal{G}$ contains a nontrivial edge group G_e, then G_e must be elliptic in T. The group G_e is contained in a proper $(G,\mathcal{F})$-free factor F (Lemma G.1.6), and it is nonperipheral because T is very small. By Lemma G.4.5, the tree T is not arational.

If all edge groups of $\mathcal{G}$ are trivial, then all vertex groups of $\mathcal{G}$ are proper $(G,\mathcal{F})$-free factors. If all vertex actions of $\mathcal{G}$ are Grushko $(G_v,\mathcal{F}_{G_v})$-trees, then T is simplicial, with trivial edge stabilizers. So either T is a Grushko $(G,\mathcal{F})$-tree, or some vertex stabilizer of T is a proper free factor that acts elliptically on T. In both cases, the tree T is not arational. □

The following lemma was proved by Reynolds in [Rey12, Proposition 8.3] in the case of F_N-trees in the closure of Culler and Vogtmann's outer space.

Lemma G.4.9. *Every arational $(G,\mathcal{F})$-tree is mixing.*

Let $T,\overline{T} \in \overline{\mathcal{O}(G,\mathcal{F})}$. We say that T *collapses* onto $\overline{T}$ if there exists a G-equivariant map $p : T \to \overline{T}$ that sends segments of T onto segments of $\overline{T}$. The following lemma follows from work by Guirardel and Levitt [GL], together with [8, Proposition 5.17].

Lemma G.4.10. *Let $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ be a tree with dense G-orbits, and let $Y \subsetneq T$ be a proper subtree, such that for all $g \in G$, either $gY = Y$, or $gY \cap Y = \emptyset$. Then either T is compatible with a $(G,\mathcal{F})$-free splitting, or else T collapses onto a mixing tree $\overline{T} \in \overline{\mathcal{O}(G,\mathcal{F})}$ in which Stab(Y) is elliptic.* □

Proof of Lemma G.4.9. Let $T \in \mathcal{AT}(G,\mathcal{F})$. Then T has dense orbits, otherwise any simplicial edge in T would be dual to a $\mathcal{Z}$-splitting that is compatible with T, contradicting Lemma G.4.8. Assume towards a contradiction that T is not mixing, and let $I \subset T$ be a segment. Define Y_I to be the subtree of T consisting of all points $x \in T$ such that there exists a finite set of elements $\{g_0 = e, g_1, \dots, g_r\} \subset G$, with $x \in g_r I$, and $g_i I \cap g_{i+1} I \neq \emptyset$ for all $i \in \{0,\dots,r-1\}$. Then for all $g \in G$, we either have $gY_I = Y_I$, or $gY_I \cap Y_I = \emptyset$.

As T is not mixing, there exists a nondegenerate arc $I \subset T$ such that Y_I is a proper subtree of T. By Lemma G.4.10, either T is compatible with a $(G,\mathcal{F})$-free splitting, or else T collapses onto a mixing tree $\overline{T} \in \overline{\mathcal{O}(G,\mathcal{F})}$, in which $\mathrm{Stab}(Y_I)$ is elliptic. The first case is excluded by Lemma G.4.8, so we assume that we are in the second case. As T has dense orbits, the stabilizer $\mathrm{Stab}(Y_I)$ is not cyclic by Lemma G.1.9. It thus follows from Lemma G.4.6 that either $\overline{T}$ has a dynamical proper $(G,\mathcal{F})$-free factor F (if the second situation of Lemma G.4.6 occurs), or else $\mathrm{Stab}(Y_I)$ is contained in a proper $(G,\mathcal{F})$-free factor (if the third situation of this lemma occurs). In the first case, the F-minimal subtree T_F of T cannot be a Grushko $(F,\mathcal{F}_F)$-tree, because T_F collapses to a nontrivial tree with dense orbits in $\overline{T}$. This contradicts arationality of T. Hence the second case occurs, i.e. $\mathrm{Stab}(Y_I)$ is contained in a proper $(G,\mathcal{F})$-free factor F. By Lemma G.1.9, the F-minimal subtree of T is not a Grushko $(F,\mathcal{F}_F)$-tree, again contradicting arationality of T. □

G.4.4 Finite sets of reducing factors associated to non-arational $(G,\mathcal{F})$-trees

Given a $(G,\mathcal{F})$-tree $T \in \overline{\mathcal{PO}(G,\mathcal{F})}$, we denote by $\mathrm{Dyn}(T)$ the set of minimal (with respect to inclusion) conjugacy classes of dynamical proper $(G,\mathcal{F})$-free factors for T. We denote by $\mathrm{Ell}(T)$ the set of nonperipheral conjugacy classes of point stabilizers in T. Recall that given a subgroup $H \subseteq G$, we denote by $\mathrm{Fill}(H)$ the smallest $(G,\mathcal{F})$-free factor that contains H. For all $\Phi \in \mathrm{Out}(G,\mathcal{F}^{(t)})$, we have $\Phi\mathrm{Dyn}(T) = \mathrm{Dyn}(\Phi T)$, and $\Phi\mathrm{Fill}(\mathrm{Ell}(T)) = \mathrm{Fill}(\mathrm{Ell}(\Phi T))$. It follows from Proposition G.3.1 that $\mathrm{Ell}(T)$ is finite, we will now show that $\mathrm{Dyn}(T)$ is also finite.

Proposition G.4.11. *For all $T \in \overline{\mathcal{PO}(G,\mathcal{F})}$, the set $\mathrm{Dyn}(T)$ is finite.*

Let $T \in \overline{\mathcal{O}(G,\mathcal{F})}$. A finite subtree $K \subseteq T$ (i.e. the convex hull of a finite set of points) is a *supporting subtree* of T if for all segments $J \subseteq T$, there exists a finite subset $\{g_1, \dots g_r\} \subseteq G$ such that $J \subseteq g_1K \cup \dots \cup g_rK$.

Lemma G.4.12. *Let $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ be a tree with dense orbits. For all $\epsilon > 0$, there exists a finite supporting subtree $K \subseteq T$ whose volume is at most ϵ.*

Proof. As T has dense orbits, it follows from [7, Theorem 5.3] that there exists a sequence $(T_n)_{n\in\mathbb{N}} \in \mathcal{O}(G,\mathcal{F})^{\mathbb{N}}$, such that the volume of the quotient graph T_n/G converges to 0, and for all $n \in \mathbb{N}$, there exists a 1-Lipschitz G-equivariant map $f_n : T_n \to T$. Letting K_n be a finite supporting subtree of T_n, with volume converging to 0 as n goes to $+\infty$, the images $f_n(K_n)$ are finite supporting subtrees of T whose volumes converge to 0. ☐

Given a finite system $S = (F, A)$ of partial isometries of a finite forest F, we define $m(S)$ as the volume of F, and $d(S)$ as the sum of the volumes of the domains of the partial isometries in A. We say that S has *independent generators* if no reduced word in the partial isometries in A and their inverses defines a partial isometry of F that fixes a nondegenerate arc. Gaboriau, Levitt and Paulin have shown in [GLP94, Proposition 6.1] that if S has independent generators, then $m(S) - d(S) \geq 0$. The following proposition is a generalization of [Rey11b, Lemma 3.10] to the context of $(G,\mathcal{F})$-trees.

Proposition G.4.13. *Let $T \in \overline{\mathcal{O}(G,\mathcal{F})}$ be a tree with dense orbits, and let H be a $(G,\mathcal{F})$-free factor. Assume that H acts with dense orbits on its minimal subtree T_H in T. Then $\mathrm{Stab}(T_H) = H$, and $\{gT_H | g \in G\}$ is a transverse family in T.*

Proof. Let $g \in G \setminus H$. Assume towards a contradiction that $gT_H \cap T_H$ contains a nondegenerate arc I of length $L > 0$. Let $\epsilon > 0$, with $\epsilon < \frac{L}{2}$. Lemma G.4.12 applied to the $(H,\mathcal{F}_H)$-tree T_H ensures the existence of a finite tree $F_\epsilon \subseteq T_H$ of volume smaller than ϵ, such that I is covered by finitely many translates of F_ϵ, and we can choose F_ϵ to be disjoint from I. We can therefore subdivide I into finitely many subsegments $I_1, \dots, I_k$ such that for all $i \in \{1, \dots, k\}$, there exists $g_i \in H$ with $g_iI_i \subseteq F_\epsilon$. Similarly, there exists a finite forest $F'_\epsilon \subseteq gT_H$ of volume smaller than ϵ, such that I is covered by finitely many translates of F'_ϵ, and again we can choose F'_ϵ to be disjoint from both I and F_ϵ in T. We similarly have a subdivision $I'_1, \dots, I'_l$ of I, and an element $g'_j \in H^g$ for each $j \in \{1, \dots, l\}$, so that $g'_jI'_j \subseteq F'_\epsilon$. We build a system of partial isometries S on the forest $I \cup F_\epsilon \cup F'_\epsilon$, with an isometry ϕ_i from I_i to F_ϵ corresponding to the action of g_i for all $i \in \{1, \dots, k\}$, and an isometry ϕ'_j from I'_j to F'_ϵ corresponding to the action of g'_j for all $j \in \{1, \dots, l\}$. Then $m(S) \leq L + 2\epsilon$, while $d(S) = 2L$. Therefore $m(S) - d(S) < 0$, and hence the system of isometries S does not have independent generators [GLP94, Proposition 6.1]. This means that there exists a reduced word w in the partial isometries ϕ_i, ϕ'_j and their inverses, associated to an element $g \in G$ which fixes an arc in T. It follows from the construction of the system of isometries that up to cyclic conjugation, the word w is a concatenation of 2-letter words of the form $\phi_{i_1} \circ \phi_{i_2}^{-1}$ and $\phi'_{j_1} \circ {\phi'_{j_2}}^{-1}$, with $i_1 \neq i_2$ and $j_1 \neq j_2$, and these two types of subwords alternate in w. So g is of the form $h_1 h_2^g \dots h_{s-1} h_s^g$, where $h_i \in H$ is a nontrivial element for all $i \in \{1, \dots, s\}$. Since H is a proper $(G, \mathcal{F})$-free factor, and $g \in G \smallsetminus H$, we have $\langle H, H^g \rangle = H * H^g$, so $g \neq e$. This contradicts the fact that T has dense orbits, and hence trivial arc stabilizers [7, Proposition 4.17]. Therefore, for all $g \in G \smallsetminus H$, the intersection $gT_H \cap T_H$ consists in at most one point. This implies that $\mathrm{Stab}(T_H) = H$, and that $\{gT_H\}_{g \in G}$ is a transverse family in T. ☐

Proposition G.4.14. *Let $T \in \overline{\mathcal{O}(G, \mathcal{F})}$ be a tree with dense orbits. Then the collection $\{gT_H | H \in Dyn(T), g \in G\}$ is a transverse family in T.*

Proof. Let $H, H' \in \mathrm{Dyn}(T)$, and assume that $T_H \cap T_{H'}$ contains a nondegenerate arc. By Proposition G.4.13, since H and H' are proper $(G, \mathcal{F})$-free factors, we have $\mathrm{Stab}(T_H) = H$ and $\mathrm{Stab}(T_{H'}) = H'$. The collections $\{gT_H\}_{g \in G}$ and $\{gT_{H'}\}_{g \in G}$ are transverse families in T (Proposition G.4.13), hence so is the collection of nondegenerate intersections of the form $gT_H \cap g'T_{H'}$ for $g, g' \in G$. If $g \in G$ stabilizes $T_H \cap T_{H'}$, then $gT_H \cap T_H$ and $gT_{H'} \cap T_{H'}$ both contain a nondegenerate arc, and hence $gT_H = T_H$ and $gT_{H'} = T_{H'}$. So we have $\mathrm{Stab}(T_H \cap T_{H'}) = \mathrm{Stab}(T_H) \cap \mathrm{Stab}(T_{H'}) = H \cap H'$. By Lemma G.1.9, the $(G, \mathcal{F})$-free factor $H \cap H'$ acts with dense orbits on the minimal subtree of $T_H \cap T_{H'}$. By minimality of the factors in $\mathrm{Dyn}(T)$, this implies that $H = H'$ and $T_H = T_{H'}$. So $\{gT_H | H \in \mathrm{Dyn}(T), g \in G\}$ is a transverse family in T. ☐

Proof of Proposition G.4.11. Finiteness of $\mathrm{Dyn}(T)$ for all trees $T \in \overline{P\mathcal{O}(G, \mathcal{F})}$ follows from Proposition G.4.14, since every transverse family in a tree with dense orbits contains boundedly many orbits of trees (where the bound is given by the number of orbits of directions at branch points in T). ☐

G.5 Nonelementary subgroups of Out$(G, \mathcal{F})$, and a trichotomy for subgroups of Out$(G, \mathcal{F})$

Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G, such that $(G, \mathcal{F})$ is nonsporadic.

Definition G.5.1. *A subgroup $H \subseteq Out(G, \mathcal{F})$ is* nonelementary *if*

- *it does not preserve any finite set of proper $(G, \mathcal{F})$-free factors, and*
- *it does not preserve any finite set of points in $\partial_\infty FZ(G, \mathcal{F})$.*

We now aim at showing that any nonelementary subgroup of $\mathrm{Out}(G, \mathcal{F})$ contains a rank two free subgroup.

Theorem G.5.2. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G, so that $(G, \mathcal{F})$ is nonsporadic. Then any nonelementary subgroup of $Out(G, \mathcal{F})$ contains a free subgroup of rank two, generated by two loxodromic isometries of $FZ(G, \mathcal{F})$.*

As a consequence of Theorem G.5.2 and of our description of the Gromov boundary of $\partial_\infty FZ(G, \mathcal{F})$, we get the following trichotomy for subgroups of $\mathrm{Out}(G, \mathcal{F})$.

Theorem G.5.3. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G, so that $(G, \mathcal{F})$ is nonsporadic. Then every subgroup of Out$(G, \mathcal{F})$ either*

- *contains a rank two free subgroup generated by two loxodromic isometries of $FZ(G, \mathcal{F})$, or*
- *virtually fixes a tree with trivial arc stabilizers in $\partial\mathcal{PO}(G, \mathcal{F})$, or*
- *virtually fixes the conjugacy class of a proper $(G, \mathcal{F})$-free factor.*

Proof. Let H be a subgroup of $\mathrm{Out}(G, \mathcal{F})$. If H is nonelementary, Theorem G.5.3 follows from Theorem G.5.2. Otherwise, either H virtually fixes the conjugacy class of a proper $(G, \mathcal{F})$-free factor, or H virtually fixes a point $\xi \in \partial_\infty FZ(G, \mathcal{F})$. In the latter case, the group H preserves the simplex of length measures in $\overline{\mathcal{PO}(G, \mathcal{F})}$ corresponding to a mixing representative of ξ, provided by Proposition G.1.5, and this simplex has finite dimension by [Gui00, Corollary 5.4] (the extension of Guirardel's result concerning finite dimensionality of this simplex to the case of free products is made possible by the fact that $\overline{\mathcal{PO}(G, \mathcal{F})}$ has finite topological dimension [7, Theorem 0.2]). So H virtually fixes any extremal point of this simplex, which is a tree with trivial arc stabilizers. □

Our proof of Theorem G.5.2 uses techniques coming from the theory of random walks on groups. These were already used in [5] for giving a new proof of a result of Handel and Mosher [HM09], which establishes a dichotomy for subgroups of $\mathrm{Out}(F_N)$, namely: every subgroup of $\mathrm{Out}(F_N)$ (finitely generated or not) either contains a fully irreducible automorphism, or virtually fixes the conjugacy class of a proper free factor of F_N. All topological spaces will be equipped with their Borel σ-algebra. Let μ be a probability measure on $\mathrm{Out}(G, \mathcal{F})$. A probability measure ν on $\overline{\mathcal{PO}(G, \mathcal{F})}$ is *μ-stationary* if $\mu * \nu = \nu$, i.e. for all ν-measurable subsets $E \subseteq \overline{\mathcal{PO}(G, \mathcal{F})}$, we have

$$\nu(E) = \sum_{\Phi \in \mathrm{Out}(G, \mathcal{F})} \mu(\Phi)\nu(\Phi^{-1}E).$$

We denote by $\mathcal{PAT}(G, \mathcal{F})$ the image of $\mathcal{AT}(G, \mathcal{F})$ in $\overline{\mathcal{PO}(G, \mathcal{F})}$. Our first goal will be to show that given a probability measure μ on $\mathrm{Out}(G, \mathcal{F})$, any μ-stationary measure on $\overline{\mathcal{PO}(G, \mathcal{F})}$ is supported on $\mathcal{PAT}(G, \mathcal{F})$. Since $\mathcal{AT}(G, \mathcal{F}) \subseteq \mathcal{X}(G, \mathcal{F})$ (Proposition G.4.7), it follows that any μ-stationary measure on $\overline{\mathcal{PO}(G, \mathcal{F})}$ pushes to a μ-stationary measure on $\partial_\infty FZ(G, \mathcal{F})$ via the map $\partial\psi$ provided by Theorem G.1.4 (this map factors through $\overline{\mathcal{PO}(G, \mathcal{F})}$). We will make use of the following classical lemma, whose proof is based on a maximum principle argument. The following version of the statement appears in [5, Lemma 3.3]. We denote by $gr(\mu)$ the subgroup of $\mathrm{Out}(G, \mathcal{F})$ generated by the support of the measure μ.

Lemma G.5.4. *(Ballmann [Bal89]) Let μ be a probability measure on a countable group G, and let ν be a μ-stationary probability measure on a G-space X. Let D be a countable G-set, and let $\Theta : X \to D$ be a measurable G-equivariant map. If $E \subseteq X$ is a G-invariant measurable subset of X satisfying $\nu(E) > 0$, then $\Theta(E)$ contains a finite $gr(\mu)$-orbit.*

We now define a G-equivariant map Θ from $\overline{\mathcal{PO}(G, \mathcal{F})}$ to the (countable) set D of finite collections of conjugacy classes of proper $(G, \mathcal{F})$-free factors. Given a tree $T \in \mathcal{PO}(G, \mathcal{F})$, we define $\mathrm{Red}(T)$ to be the finite collection of proper $(G, \mathcal{F})$-free factors that occur as vertex groups of trees obtained by equivariantly collapsing some of the edges of T to points. The collection $\mathrm{Red}(T)$ is nonempty because $(G, \mathcal{F})$ is nonsporadic. Given $T \in \partial\mathcal{PO}(G, \mathcal{F})$, the set of conjugacy classes of point stabilizers in T is finite [Jia91]. Every point stabilizer

G_v is contained in a unique minimal (possibly non proper) $(G,\mathcal{F})$-free factor $\mathrm{Fill}(G_v)$. We let $\mathrm{Per}(T)$ be the (possibly empty) finite set of conjugacy classes of proper $(G,\mathcal{F})$-free factors that arise in this way, and we set

$$\Theta(T) := \begin{cases} \emptyset & \text{if } T \in \mathcal{PAT}(G,\mathcal{F}) \\ \mathrm{Red}(T) & \text{if } T \in \mathcal{PO}(G,\mathcal{F}) \\ \mathrm{Dyn}(T) \cup \mathrm{Per}(T) & \text{if } T \in \partial\mathcal{PO}(G,\mathcal{F}) \smallsetminus \mathcal{PAT}(G,\mathcal{F}) \end{cases}.$$

Proposition G.4.4 implies that $\Theta(T) = \emptyset$ if and only if $T \in \mathcal{PAT}(G,\mathcal{F})$. The following lemma was proved in [5, Lemma 3.4]. Its proof adapts to the context of $(G,\mathcal{F})$-trees.

Lemma G.5.5. *The set $\mathcal{PAT}(G,\mathcal{F})$ is measurable, and Θ is measurable.* □

Proposition G.5.6. *Let G be a countable group, and $\mathcal{F}$ be a free factor system of G. Let μ be a probability measure on Out$(G,\mathcal{F})$, whose support generates a nonelementary subgroup of Out$(G,\mathcal{F})$. Then every μ-stationary measure on $\overline{\mathcal{PO}(G,\mathcal{F})}$ is concentrated on $\mathcal{PAT}(G,\mathcal{F})$.*

Proof of Proposition G.5.6. Let ν be a μ-stationary measure on $\overline{\mathcal{PO}(G,\mathcal{F})}$. Let $E := \overline{\mathcal{PO}(G,\mathcal{F})} \smallsetminus \mathcal{PAT}(G,\mathcal{F})$. By Proposition G.4.4, the image $\Theta(E)$ does not contain the empty set. However, nonelementarity of $gr(\mu)$ implies that the only finite $gr(\mu)$-orbit in D is the orbit of the empty set. Lemma G.5.4 thus implies that $\nu(E) = 0$, or in other words ν is concentrated on $\mathcal{PAT}(G,\mathcal{F})$. □

Corollary G.5.7. *Let $H \subseteq$ Out$(G,\mathcal{F})$ be a nonelementary subgroup of Out$(G,\mathcal{F})$. Then the H-orbit of any point $x_0 \in \mathcal{PO}(G,\mathcal{F})$ has a limit point in $\mathcal{PAT}(G,\mathcal{F})$.*

Proof. Let μ be a probability measure on $\mathrm{Out}(G,\mathcal{F})$ such that $\mathrm{gr}(\mu) = H$. An example of such a measure is obtained by giving a positive weight $\mu(h) > 0$ to every element $h \in H$, in such a way that

$$\sum_{h \in H} \mu(h) = 1$$

(and $\mu(g) = 0$ if $g \in G \smallsetminus H$). Let δ_{x_0} be the Dirac measure at x_0. Since $\overline{\mathcal{PO}(G,\mathcal{F})}$ is compact [7, Proposition 3.1], the sequence of the Cesàro averages of the convolutions $\mu^{*n} * \delta_{x_0}$ has a weak-$*$ limit point ν, which is a μ-stationary measure on $\overline{\mathcal{PO}(G,\mathcal{F})}$, see [KM96, Lemma 2.2.1]. We have $\nu(\overline{Hx_0}) = 1$, where Hx_0 denotes the H-orbit of x_0 in $\overline{\mathcal{PO}(G,\mathcal{F})}$, and Proposition G.5.6 implies that $\nu(\mathcal{PAT}(G,\mathcal{F})) = 1$. This implies that $\overline{Hx_0} \cap \mathcal{PAT}(G,\mathcal{F})$ is nonempty. □

As a consequence of Theorem G.1.4 and Corollary G.5.7, we get the following fact.

Corollary G.5.8. *Let $H \subseteq$ Out$(G,\mathcal{F})$ be a nonelementary subgroup of Out$(G,\mathcal{F})$. Then the H-orbit of any point in $FZ(G,\mathcal{F})$ has a limit point in $\partial_\infty FZ(G,\mathcal{F})$.* □

Proof of Theorem G.5.2. Let $\mathcal{F}$ be a free factor system of G, and let H be a nonelementary subgroup of $\mathrm{Out}(G,\mathcal{F})$. Corollary G.5.7 shows that the H-orbit of any point in $FZ(G,\mathcal{F})$ has a limit point in $\partial_\infty FZ(G,\mathcal{F})$. As H does not fix any element in $\partial_\infty FZ(G,\mathcal{F})$, the conclusion follows from the classification of subgroups of isometries of Gromov hyperbolic spaces (Theorem G.1.2). □

G.6 The inductive argument

G.6.1 Variations over the Tits alternative

We recall from the introduction that a group G is said to satisfy the Tits alternative relative to a class $\mathcal{C}$ of groups if every subgroup of G either belongs to $\mathcal{C}$, or contains a rank two free subgroup. Our main result is the following. A group H is *freely indecomposable* if it does not split as a free product of the form $H = A * B$, where both A and B are nontrivial.

Theorem G.6.1. *Let $\{G_1, \dots, G_k\}$ be a finite collection of freely indecomposable countable groups, not isomorphic to $\mathbb{Z}$, let F be a finitely generated free group, and let*

$$G := G_1 * \dots * G_k * F.$$

Let $\mathcal{C}$ be a collection of groups that is stable under isomorphisms, contains $\mathbb{Z}$, and is stable under subgroups, extensions, and passing to finite index supergroups. Assume that for all $i \in \{1, \dots, k\}$, both G_i and $Out(G_i)$ satisfy the Tits alternative relative to $\mathcal{C}$.
Then $Out(G)$ and $Aut(G)$ satisfy the Tits alternative relative to $\mathcal{C}$.

In particular, Theorem G.6.1 applies to the case where $\mathcal{C}$ is either the class of virtually solvable groups (see [Can11, Lemme 6.11] for stability of $\mathcal{C}$ under extensions), or the class of virtually polycyclic groups.

Theorem G.6.1 will be a consequence of the following relative version. For all $i \in \{1, \dots, k\}$, let $A_i \subseteq \mathrm{Out}(G_i)$ be a subgroup of $\mathrm{Out}(G_i)$, and let $\mathcal{A} := (A_1, \dots, A_k)$. We recall from Section G.1.2 that $\mathrm{Out}(G, \mathcal{F}^{\mathcal{A}})$ denotes the subgroup of $\mathrm{Out}(G)$ consisting of those automorphisms that preserve the conjugacy classes of all subgroups G_i, and induce an outer automorphism in A_i in restriction to each G_i.

Theorem G.6.2. *Let G be a countable group, let $\mathcal{F}$ be a free factor system of G, and let $\mathcal{A}$ be as above. Let $\mathcal{C}$ be a collection of groups that is stable under isomorphisms, contains $\mathbb{Z}$, and is stable under subgroups and extensions, and passing to finite index supergroups. Assume that for all $i \in \{1, \dots, k\}$, both G_i and A_i satisfy the Tits alternative relative to $\mathcal{C}$.*
Then $Out(G, \mathcal{F}^{\mathcal{A}})$ satisfies the Tits alternative relative to $\mathcal{C}$.

When all subgroups in $\mathcal{A}$ are trivial, Theorem G.6.2 specifies as follows.

Theorem G.6.3. *Let G be a countable group, and let $\mathcal{F}$ be a free factor system of G. Let $\mathcal{C}$ be a collection of groups that is stable under isomorphisms, contains $\mathbb{Z}$, and is stable under subgroups and extensions, and passing to finite index supergroups. Assume that all peripheral subgroups of G satisfy the Tits alternative relative to $\mathcal{C}$.*
Then $Out(G, \mathcal{F}^{(t)})$ satisfies the Tits alternative relative to $\mathcal{C}$.

In the classical case where $\mathcal{C}$ is the class of virtually solvable groups, we also mention that our proof of Theorem G.6.1 also provides a bound on the degree of solvability of the finite-index solvable subgroup arising in the statement.

Question G.6.4. If all groups G_i and $\mathrm{Out}(G_i)$ satisfy the Tits alternative relative to the class of virtually abelian subgroups, does $\mathrm{Out}(G)$ also satisfy the Tits alternative relative to this class ? Similarly, if all groups G_i satisfy the Tits alternative relative to the class of virtually abelian subgroups, does $\mathrm{Out}(G, \mathcal{F}^{(t)})$ also satisfy the Tits alternative relative to this class ? The issue here is that this class is not stable under extensions. Our question is motivated by the classical case of finitely generated free groups, for which Bestvina, Feighn and Handel have proved that every virtually solvable subgroup of $\mathrm{Out}(F_N)$ is actually virtually abelian and finitely generated, with a bound on the index of the abelian subgroup that only depends on N ([BFH05], see also [Ali02]).

We first explain how to derive Theorems G.6.2 and G.6.1 from Theorem G.6.3, before proving Theorem G.6.3 in the next section.

Proof of Theorem G.6.2. There is a morphism from $\mathrm{Out}(G, \mathcal{F}^{\mathcal{A}})$ to the direct product $A_1 \times \cdots \times A_k$, whose kernel is equal to $\mathrm{Out}(G, \mathcal{F}^{(t)})$. Since $\mathcal{C}$ is stable under extensions, the class of groups satisfying the Tits alternative relative to $\mathcal{C}$ is stable under extensions, so Theorem G.6.2 follows from Theorem G.6.3. □

Proof of Theorem G.6.1. Let $\mathcal{F} := \{[G_1], \dots, [G_k]\}$. As all G_i's are freely indecomposable, the group $\mathrm{Out}(G)$ permutes the conjugacy classes in $\mathcal{F}$. Therefore, there exists a finite-index subgroup $\mathrm{Out}^0(G)$ of $\mathrm{Out}(G)$ which preserves all conjugacy classes in $\mathcal{F}$. For all $i \in \{1, \dots, k\}$, the group G_i is equal to its own normalizer in G, so every element $\Phi \in \mathrm{Out}^0(G)$ induces a well-defined element of $\mathrm{Out}(G_i)$. In other words, the subgroup $\mathrm{Out}^0(G)$ is a subgroup of $\mathrm{Out}(G, \mathcal{F}^{\mathcal{A}})$, with $A_i = \mathrm{Out}(G_i)$ for all $i \in \{1, \dots, k\}$. Theorem G.6.1 thus follows from Theorem G.6.2 (the statement for the group $\mathrm{Aut}(G)$ also follows, because if both G and $\mathrm{Out}(G)$ satisfy the Tits alternative relative to $\mathcal{C}$, then so does $\mathrm{Aut}(G)$). □

G.6.2 Proof of Theorem G.6.3

The proof is by induction on the pair $(\mathrm{rk}_K(G, \mathcal{F}), \mathrm{rk}_f(G, \mathcal{F}))$, for the lexicographic order. Let $\mathcal{F}$ be a free factor system of G. The conclusion holds if $\mathrm{rk}_K(G, \mathcal{F}) = 1$: in this case, the group G is either peripheral, or isomorphic to $\mathbb{Z}$. It also holds in the sporadic cases by Propositions G.2.1 and G.2.2. We now assume that $(G, \mathcal{F})$ is nonsporadic, and let H be a subgroup of $\mathrm{Out}(G, \mathcal{F}^{(t)})$. We will show that either H contains a rank two free subgroup, or $H \in \mathcal{C}$. Using Theorem G.5.3, we can assume that either H preserves a finite set of conjugacy classes of proper $(G, \mathcal{F})$-free factors, or that H virtually fixes a tree with trivial arc stabilizers in $\partial P\mathcal{O}(G, \mathcal{F})$.

We first assume that H has a finite index subgroup H^0 which preserves the conjugacy class of a proper $(G, \mathcal{F})$-free factor G'. We denote by $\mathrm{Out}(G, \mathcal{F}^{(t)}, G')$ the subgroup of $\mathrm{Out}(G, \mathcal{F}^{(t)})$ made of those elements that preserve the conjugacy class of G' (so H^0 is a subgroup of $\mathrm{Out}(G, \mathcal{F}^{(t)}, G')$). Since G' is equal to its own normalizer in G, every element $\Phi \in \mathrm{Out}(G, \mathcal{F}^{(t)}, G')$ induces by restriction a well-defined outer automorphism $\Phi_{G'}$ of G'. The automorphism $\Phi_{G'}$ coincides with a conjugation by an element $g \in G$ in restriction to every factor in $\mathcal{F}_{G'}$ (where we recall that $\mathcal{F}_{G'}$ is the collection of G'-conjugacy classes of subgroups in $\mathcal{F}$ that are contained in G'). Since G' is malnormal, we have $g \in G'$. In other words, there is a restriction morphism

$$\Psi : \mathrm{Out}(G, \mathcal{F}^{(t)}, G') \to \mathrm{Out}(G', \mathcal{F}_{G'}^{(t)}).$$

Since G' is a $(G, \mathcal{F})$-free factor, there exist $i_1 < \cdots < i_s$ such that G splits as

$$G = G' * G'_{i_1} * \cdots * G'_{i_s} * F',$$

where G'_{i_j} is conjugate to G_{i_j} for all $j \in \{1, \dots, s\}$, and F' is a finitely generated free group. We let $\mathcal{F}' := \{[G'], [G'_{i_1}], \dots, [G'_{i_s}]\}$. Then the kernel of Ψ is equal to $\mathrm{Out}(G, \mathcal{F}'^{(t)})$.

Recall from Equations (G.2) and (G.3) in Section G.1.2 that $\mathrm{rk}_f(G',\mathcal{F}_{G'})+\mathrm{rk}_f(G,\mathcal{F}')=\mathrm{rk}_f(G,\mathcal{F})$, and $\mathrm{rk}_K(G',\mathcal{F}_{G'})+\mathrm{rk}_K(G,\mathcal{F}')=\mathrm{rk}_K(G,\mathcal{F})+1$. Since G' is a proper $(G,\mathcal{F})$-free factor, we either have $\mathrm{rk}_K(G',\mathcal{F}_{G'})\geq 2$, in which case $\mathrm{rk}_K(G,\mathcal{F}')<\mathrm{rk}_K(G,\mathcal{F})$, or else $\mathrm{rk}_K(G',\mathcal{F}_{G'})=\mathrm{rk}_f(G',\mathcal{F}_{G'})=1$, in which case $\mathrm{rk}_K(G,\mathcal{F}')=\mathrm{rk}_K(G,\mathcal{F})$ and $\mathrm{rk}_f(G,\mathcal{F}')<\mathrm{rk}_f(G,\mathcal{F})$. Since G' is a proper $(G,\mathcal{F})$-free factor, we also have $\mathrm{rk}_K(G',\mathcal{F}_{G'})<\mathrm{rk}_K(G,\mathcal{F})$. Our induction hypothesis therefore implies that both $\mathrm{Out}(G',\mathcal{F}_{G'}^{(t)})$ and $\mathrm{Out}(G,\mathcal{F}'^{(t)})$ satisfy the Tits alternative relative to $\mathcal{C}$. Since $\mathcal{C}$ is stable under extensions, the class of groups satisfying the Tits alternative relative to $\mathcal{C}$ is stable under extensions. So H^0, and hence H, satisfies the Tits alternative relative to $\mathcal{C}$.

We now assume that H has a finite index subgroup H^0 which fixes the projective class of a tree $[T]\in\overline{\mathbb{P}\mathcal{O}(G,\mathcal{F})}$ with trivial arc stabilizers. Then H^0 is a cyclic extension of a subgroup H' that fixes a nonprojective tree $T\in\overline{\mathcal{O}(G,\mathcal{F})}$ [GL]. It is enough to show that $\mathrm{Out}(T)$ satisfies the Tits alternative relative to $\mathcal{C}$.

Denote by V the finite set of G-orbits of points with nontrivial stabilizer in T, and by $\{G_v\}_{v\in V}$ the collection of point stabilizers in T. As any element of $\mathrm{Out}(T)$ induces a permutation of the finite set V, some finite index subgroup $\mathrm{Out}^0(T)$ of $\mathrm{Out}(T)$ preserves the conjugacy class of all groups G_v with $v\in V$. As T has trivial arc stabilizers, all point stabilizers in T are equal to their normalizer in G. As above, there is a morphism from $\mathrm{Out}^0(T)$ to the direct product of all $\mathrm{Out}(G_v,\mathcal{F}_{G_v}^{(t)})$, whose kernel is contained in $\mathrm{Out}(T,\{[G_v]\}^{(t)})$.

Corollary G.3.3 shows that $\mathrm{Out}(T,\{[G_v]\}^{(t)})$ satisfies the Tits alternative relative to $\mathcal{C}$. Since T has trivial arc stabilizers, Proposition G.3.1 implies that $\mathrm{rk}_K(G_v,\mathcal{F}_{G_v})\leq\mathrm{rk}_K(G,\mathcal{F})-1$ for all $v\in V$. Therefore, our induction hypothesis implies that $\mathrm{Out}(G_v,\mathcal{F}_{G_v}^{(t)})$ satisfies the Tits alternative relative to $\mathcal{C}$. As the Tits alternative is stable under extensions, we deduce that $\mathrm{Out}(T)$, and hence H, satisfies the Tits alternative relative to $\mathcal{C}$. ☐

G.7 Applications

G.7.1 Outer automorphisms of right-angled Artin groups

Given a finite simplicial graph Γ, the *right-angled Artin group* A_Γ is the group defined by the following presentation. Generators of A_Γ are the vertices of Γ, and relations are given by commutation of any two generators that are joined by an edge in Γ. As a consequence of Theorem G.6.1 and of work by Charney and Vogtmann [CV11], we show that the outer automorphism group of any right-angled Artin group satisfies the Tits alternative.

Theorem G.7.1. *For all finite simplicial graphs Γ, the group $Out(A_\Gamma)$ satisfies the Tits alternative.*

Let N be the number of components of Γ consisting of a single point, and let $\Gamma_1,\dots,\Gamma_k$ be the connected components of Γ consisting of more than one point. Then we have $A_\Gamma=A_{\Gamma_1}*\dots*A_{\Gamma_k}*F_N$. All subgroups A_{Γ_i} of this decomposition are freely indecomposable and not isomorphic to $\mathbb{Z}$: it is the Grushko decomposition of A_Γ.

Theorem G.7.1 was first proven by Charney, Crisp and Vogtmann in the case where Γ is connected and triangle-free [CCV07], then extended by Charney and Vogtmann in [CV11] to the case of graphs satisfying some homogeneity condition, where it was noticed that the full version would follow from Theorem G.6.1. We now explain how to make this deduction. The reader is referred to [Cha07] for a survey paper on right-angled Artin groups, and to [CCV07, CV09, CV11] for a study of their automorphism groups.

Let Γ be a finite simplicial connected graph. Let $v \in \Gamma$ be a vertex of Γ. The *link* of v, denoted by $lk(v)$, is the full subgraph of Γ spanned by all vertices adjacent to v. The *star* of v, denoted by $st(v)$, is the full subgraph of Γ spanned by v and $lk(v)$. The relation $\leq$ defined on the set of vertices of Γ by setting $v \leq w$ whenever $lk(v) \subseteq st(w)$ is transitive, and induces a partial ordering on the set of equivalence classes of vertices $[v]$, where $w \in [v]$ if and only if $v \leq w$ and $w \leq v$ [CV09, Lemma 2.2]. A vertex v of Γ is *maximal* if its equivalence class is maximal for this relation. The *link* $lk(\Theta)$ of a subgraph Θ of Γ is the intersection of the links of all vertices in Θ. The *star* $st(\Theta)$ of Θ is the full subgraph of Γ spanned by both Θ and its link. Given a full subgraph Θ of Γ, the group A_Θ embeds as a subgroup of A_Γ.

Laurence [Lau95], extending work of Servatius [Ser89], gave a finite generating set of $\mathrm{Out}(A_\Gamma)$, consisting of graph automorphisms, inversions of a single generator, transvections $v \mapsto vw$ with $v \leq w$, and partial conjugations by a generator v on one component of $\Gamma \smallsetminus st(v)$.

The subgroup $\mathrm{Out}^0(A_\Gamma)$ of $\mathrm{Out}(A_\Gamma)$ generated by inversions, transvections and partial conjugations, has finite index in $\mathrm{Out}(A_\Gamma)$. Assume that Γ is connected, and let v be a maximal vertex. Then any element of $\mathrm{Out}^0(A_\Gamma)$ has a representative f_v which preserves both $A_{[v]}$ and $A_{st[v]}$ [CV09, Proposition 3.2]. Restricting f_v to $A_{st[v]}$ gives a *restriction morphism*

$$R_v : \mathrm{Out}^0(A_\Gamma) \to \mathrm{Out}^0(A_{st[v]})$$

[CV09, Corollary 3.3]. The map from A_Γ to $A_{\Gamma \smallsetminus [v]}$ that sends each generator in $[v]$ to the identity induces an *exclusion morphism*

$$E_v : \mathrm{Out}^0(A_\Gamma) \to \mathrm{Out}^0(A_{\Gamma \smallsetminus [v]}).$$

Since v is a maximal vertex for the subgraph $st[v]$, and since $lk[v] = st[v] \smallsetminus [v]$, we can compose the restriction morphism on A_Γ with the exclusion morphism on $A_{st[v]}$ to get a *projection morphism*

$$P_v : \mathrm{Out}^0(A_\Gamma) \to \mathrm{Out}^0(A_{lk[v]}).$$

[CV09, Corollary 3.3]. By combining the projection morphisms for all maximal equivalence classes of vertices of Γ, we get a morphism

$$P : \mathrm{Out}^0(A_\Gamma) \to \prod \mathrm{Out}^0(A_{lk[v]}),$$

where the product is taken over the set of maximal equivalence classes of vertices of Γ.

Proposition G.7.2. *(Charney–Vogtmann [CV09, Theorem 4.2]) If Γ is a connected graph that contains at least two equivalence classes of maximal vertices, then the kernel of P is a free abelian subgroup of $Out^0(A_\Gamma)$.*

Proposition G.7.3. *(Charney–Vogtmann [CV09, Proposition 4.4]) If Γ is a connected graph that contains a single equivalence class $[v]$ of maximal vertices, then $A_{[v]}$ is abelian, and there is a surjective morphism*

$$Out(A_\Gamma) \to GL(A_{[v]}) \times Out(A_{lk([v])}),$$

whose kernel is a free abelian subgroup of $Out(A_\Gamma)$.

Proof of Theorem G.7.1. The proof is by induction on the number of vertices of Γ. The case of a graph having a single vertex is obvious. Thanks to Theorem G.6.1 and the description of the Grushko decomopsition of A_Γ, we can assume that Γ is connected. Let v be a maximal vertex of Γ. As $lk_{[v]}$ has stricly fewer vertices than Γ, it follows from the induction hypothesis that $\mathrm{Out}(A_{lk[v]})$ satisfies the Tits alternative, and so does $\mathrm{Out}^0(A_{lk[v]})$. If Γ contains a single equivalence class of maximal vertices, then it follows from Proposition G.7.3, and from Tits' original version of the alternative for linear groups [Tit72], that $\mathrm{Out}(A_\Gamma)$ satisfies the Tits alternative. If Γ contains at least two equivalence classes of maximal vertices, then it follows from Proposition G.7.2 that $\mathrm{Out}(A_\Gamma)$ satisfies the Tits alternative. □

G.7.2 Outer automorphisms of relatively hyperbolic groups

Let G be a group, and $\mathcal{P}$ be a finite collection of subgroups of G. Following Bowditch [Bow12] (see [Hru10, Osi06] for equivalent definitions), we say that G is *hyperbolic relative to $\mathcal{P}$* if G admits a simplicial action on a connected graph $\mathcal{K}$ such that

- the graph $\mathcal{K}$ is Gromov hyperbolic, and for all $n \in \mathbb{N}$, every edge of $\mathcal{K}$ is contained in finitely many simple circuits of length n, and
- the edge stabilizers for the action of G on $\mathcal{K}$ are finite, and there are finitely many orbits of edges, and
- the set $\mathcal{P}$ is a set of representatives of the conjugacy classes of the infinite vertex stabilizers.

Theorem G.7.4. *Let G be a torsion-free group, which is hyperbolic relative to a finite family $\mathcal{P}$ of finitely generated subgroups. Let $\mathcal{C}$ be a collection of groups that is stable under isomorphisms, contains $\mathbb{Z}$, and is stable under subgroups, extensions, and passing to finite index supergroups. Assume that for all $H \in \mathcal{P}$, both H and $Out(H)$ satisfy the Tits alternative relative to $\mathcal{C}$.*
Then $Out(G, \mathcal{P})$ satisfies the Tits alternative relative to $\mathcal{C}$.

Proof. The peripheral subgroups G_i arising in the Grushko decomposition of G relative to $\mathcal{P}$ (see [GL10a] for a definition of the relative Grushko decomposition) are torsion-free, freely indecomposable relative to $\mathcal{P}_{G_i}$ (i.e. they do not split as a free product in which all subgroups in $\mathcal{P}_{G_i}$ are conjugate into one of the factors), and hyperbolic relative to $\mathcal{P}_{G_i}$. Each subgroup G_i satisfies the Tits alternative relative to $\mathcal{C}$ as soon as all groups in $\mathcal{P}$ do (this follows from [Gro87]). Our main result (Theorem G.6.1) therefore enables us to reduce to the case where G is freely indecomposable relative to $\mathcal{P}$. In this case, we can use the description of $\mathrm{Out}(G, \mathcal{P})$ stated below, which is due to Guirardel and Levitt. Since the Tits alternative holds for mapping class groups of compact surfaces (Ivanov [Iva84], McCarthy [McC85]), we deduce the Tits alternative for $\mathrm{Out}(G, \mathcal{P})$. □

Theorem G.7.5. *(Guirardel–Levitt [GL14, Theorem 1.4]) Let G be a torsion-free group, which is hyperbolic relative to a finite family $\mathcal{P}$ of finitely generated subgroups, and freely indecomposable relative to $\mathcal{P}$. Then some finite index subgroup $Out^0(G, \mathcal{P})$ of $Out(G, \mathcal{P})$ fits in an exact sequence*

$$1 \to \mathcal{T} \to Out^0(G, \mathcal{P}) \to \prod_{i=1}^{p} MCG(\Sigma_i) \times \prod_{H \in \mathcal{P}} Out(H),$$

where $\mathcal{T}$ is finitely generated free abelian, and $MCG(\Sigma_i)$ is the mapping class group of a compact surface Σ_i.

When the parabolic subgroups are virtually polycyclic, we get the following result.

Theorem G.7.6. *Let G be a torsion-free group, which is hyperbolic relative to a finite family of virtually polycyclic subgroups. Then Out(G) satisfies the Tits alternative relative to the class of virtually polycyclic groups.*

Proof. We first recall that the outer automorphism group $\mathrm{Out}(P)$ of a virtually polycyclic group P satisfies the Tits alternative relative to the class of virtually polycyclic groups. Indeed, a theorem of Auslander [Aus67] establishes that $\mathrm{Out}(P)$ embeds as a subgroup of $SL_N(\mathbb{Z})$ for some $N \in \mathbb{N}$. Tits' original statement of the Tits alternative [Tit72] implies that $\mathrm{Out}(P)$ satisfies the Tits alternative relative to the class of virtually solvable groups (every linear group over a field of characteristic 0, finitely generated or not, satisfies the Tits alternative). In addition, a theorem of Mal'cev states that solvable subgroups of $SL_N(\mathbb{Z})$ are polycyclic [Mal51]. Hence $\mathrm{Out}(P)$ satisfies the Tits alternative relative to the class of virtually polycyclic groups.

Denote by $\mathcal{P}$ the collection of parabolic subgroups. We can assume that $\mathcal{P}$ does not contain any virtually cyclic subgroup. Then every element of $\mathrm{Out}(G)$ induces a permutation of the conjugacy classes of the subgroups in $\mathcal{P}$. Indeed, subgroups in $\mathcal{P}$ can be characterized as the maximal subgroups which do not contain a free subgroup of rank 2, and are not virtually cyclic. Therefore, the group $\mathrm{Out}(G, \mathcal{P})$ is a finite index subgroup of $\mathrm{Out}(G)$. Theorem G.7.6 thus follows from Theorem G.7.4. ☐

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