Nonzero-sum stochastic differential games and backward stochastic differential equations

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Jeux Différentiels Stochastiques de Somme Non Nulle et Equations Différentielles Stochastiques Rétrogrades Multidimensionnelles

JURY

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CHAPTER 1

Abstract

This dissertation studies the multiple players nonzero-sum stochastic differential games (NZSDG) in the Markovian framework and their connection with multiple dimensional backward stochastic differential equations (BSDEs). There are three problems that we are focused on. Firstly, we consider a NZSDG where the drift coefficient is not bound but is of linear growth. Some particular case with the unbounded diffusion process is also considered. The existence of Nash equilibrium point is proved under the generalized Isaacs condition via the existence of the associated BSDE. The novelty is that the generator of the BSDE is of stochastic linear growth with respect to the volatility process. The second problem is a risk-sensitive case with the exponential type of payoff where the coefficients are unbounded. The associated BSDE is of multi-dimension whose generator is quadratic on the volatility. We show the existence of Nash equilibria. The last problem that we treat, is a bang-bang game where the payoff is not continuous. In this case, Nash equilibria exists and is of bang-bang type which is not continuous and the value of the control will jump between the border of the domain with respect to the sign of the derivative of the value function. The BSDE in this case is a coupled multi-dimensional system, whose generator is discontinuous on the volatility process.

Key Words: Nonzero-sum Stochastic Differential Games; Backward Stochastic Differential Equation; Nash Equilibrium Point.

CHAPTER 2

Résumé

Cette thèse traite les jeux différentiels stochastiques de somme non nulle (JDSNN) dans le cadre de Markovien et de leurs liens avec les équations différentielles stochastiques rétrogrades (EDSR) multidimensionnelles. Nous étudions trois problèmes différents. Tout d'abord, nous considérons un JDSNN où le coefficient de dérive n'est pas borné, mais supposé uniquement à croissance linéaire. Ensuite certains cas particuliers de coefficients de diffusion non bornés sont aussi considérés. Nous montrons que le jeu admet un point d'équilibre de Nash via la preuve de l'existence de la solution de l'EDSR associée et lorsque la condition d'Isaacs généralisée est satisfaite. La nouveauté est que le générateur de l'EDSR, qui est multidimensionnelle, est de croissance linéaire stochastique par rapport au processus de volatilité. Le deuxième problème est aussi relatif au JDSNN mais les payoffs ont des fonctions d'utilité exponentielles. Les EDSRs associées à ce jeu sont de type multidimensionnelles et quadratiques en la volatilité. Nous montrons de nouveau l'existence d'un équilibre de Nash. Le dernier problème que nous traitons, est un jeu bang-bang qui conduit à des hamiltoniens discontinus. Dans ce cas, nous reformulons le théorème de vérification et nous montrons l'existence d'un équilibre de Nash qui est du type bang-bang, i.e., prenant ses valeurs sur le bord du domaine en fonction du signe de la dérivée de la fonction valeur ou du processus de volatilité. L'EDSR dans ce cas est un système multidimensionnel couplé, dont le générateur est discontinu par rapport au processus de volatilité.

Mots Clés: Jeux Différentiels Stochastiques de Somme Non Nulle; Equations Différentielles Stochastiques Rétrogrades; Point d'équilibre de Nash.

CHAPTER 3

Introduction

3.1 Background of stochastic differential games

Differential game theory deals with the conflict or cooperate problems in a system which is influenced by different players. Each player imposes his own control to this system in order to gain some utilities or sometimes they should pay costs. We seek, in a game problem, the controls for all players such that each of them achieves his own goal, precisely speaking, those controls either maximum or minimum the concerned utility for the players.

3.1.1 Some classical examples

We would like to introduce the differential game theory by some classical examples in this subsection. We will not go deeply into the mathematical formulation for an intuitive understanding. Through Example 3.1.1, we expect to give a preliminary impression of basic elements in game theory, such as system, control and utility. Further on, Example 3.1.2 shows two different types, cooperative type and uncooperative type, of differential game.

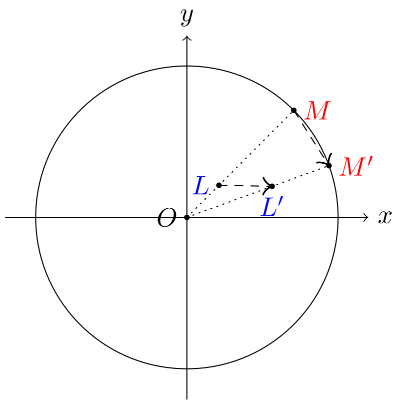
Example 3.1.1. ***The lion and the man***. This game is introduced as follows.

A lion and a man encounter in a closed circular arena
with the same maximum speeds. Can the lion catch the man?

This problem proposed by R. Rado (unpublished, see the historical record by Littlewood [76], p135 and Hajek's monograph [51]) in 1930s, attracts many researchers' attention.

For ease of discussion, both the lion and the man will be viewed as single points. From the game's perspective, apparently, the lion aims at minimizing the distance to the man, while, the man prefer to maximizing the distance between the two.

If we assume that the man stays on the boundary of the circle, then, it is not difficult to check that the lion does have a winning strategy. A candidate way is that the lion runs at the top speed directly towards the man, at the same time, he always stays on the radius vector from the center to the man. After a finite time, the man will be captured. This case can be demonstrated by Figure 3.1.



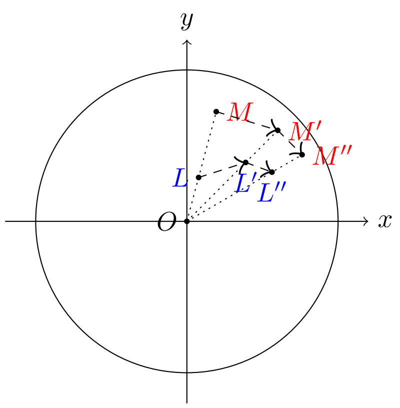
M: the man;
L: the lion.
Path: (L, M)→ (L', M').
Strategies: the man stays on the boundary of the circle; the lion runs directly towards the man and keep staying on the radius vector from center to the man.
Result: the man will be captured finally.

Figure 3.1: A bad strategy for the man

However, it is shown by A.S.Besicovitch in 1952 (see [76],p136 and [51] that it is not wise for the man to stay at the boundary of the domain. There exists a winning strategy for the man that he will not be captured forever. But, the distance of the lion and the man can be arbitrary small. In another word, the lion could get arbitrary close to the man but failed to catch him. The strategy for each player is as follows:

- For the lion: If we assume that he starts from the center of the circle. He follows the *curve of pursuit*, namely that the lion always running towards the man directly and keep his position in the same radius vector from the center to the man;
- For the man: He is assumed to stand initially in the interior of the circle. Then he is going to spiral up to the boundary. More precise, We split time into a sequence of intervals. For the first interval of time, the man runs in a straight line that is perpendicular to his radius vector at the start of the step. Meanwhile, he chooses the direction into the half plane such that it does not contain the lion (if the lion is on the radius then either direction will do). The man then repeats the same strategy in the next step until getting close to the boundary.

It will be clear if we illustrate Besicovitch's strategy by Figure 3.2.



M: the man;
L: the lion.
Path: (L, M)→ (L', M')→(L'', M'').
Strategies: the man runs in a straight line that is perpendicular to his radius vector; the lion runs directly towards the man and keep staying on the radius vector from center to the man.
Result: the lion could get arbitrary close to the man but failed to catch him.

Figure 3.2: A winning strategy for the man

We skip the proof of the result. Finally, it can be shown that the lion is failed to catch the man but can get close to the man as much as possible in this way. Interested readers are referred to the note by

P. Cardaliaguet ([89], pp.5-8) for the proof. There are some additional discussions on cases when the maximum speeds of the lion and the man are not the same. For this general problem, we also refer article [46].

Some further questions about this classical example are listed as follows:

(i) Is there exists a strategy such that at least one of the lion and the man will win?

(ii) What about the domain is not a closed disc, for example, replaced by a different metric space?

(iii) What would happen if those two players take turns to move?

(iv) The positions of players and directions which they plan to choose are heavily depend on the opponent's position. Therefore, a natural question arises, namely, will the time delay of each one's reaction influence the problem?

About the above generalized problems and even some open problems about this subject, see a recent work [13] and the references therein for more information.

Example 3.1.2. ***The rope pulling game.*** [1]

A point object (with identity mass) can move in a plane which is endowed with the standard (x, y)-coordinate system. Initially, at $t = 0$, the object is at the origin point. Two unit forces act on the point object with appropriate directions, where one is chosen by Player 1, the other by Player 2. The directions of these forces, measured by angles which are counter-clockwise with respect to the positive x-axis, are denoted by u_l and u_2 respectively. Therefore, this system can be described by the following differential equations:

$$\begin{cases} \ddot{x} = \cos(u_1) + \cos(u_2), \ \dot{x}(0) = x(0) = 0; \\ \ddot{y} = \sin(u_1) + \sin(u_2), \ \dot{y}(0) = y(0) = 0. \end{cases}$$

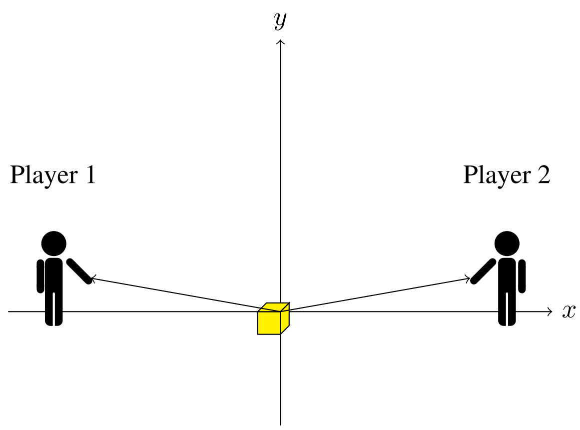


Figure 3.3: Rope pulling game

Case 1: Uncooperative case. At time $t = 1$, Player 1 wants to pull the object as far in the negative x-direction as possible, i.e., he wants to minimize $x(1)$, whereas Player 2 would like to pull it as far in the positive x-direction as possible, i.e., he wants to maximize $x(1)$.

The solution to this uncooperative game follows immediately. Each player pulls in his own favorite direction. The choice of their controls are apparently $(u_1, u_2) = (\pi, 0)$. Besides, the object remains at the origin point which is equivalent to say, the sum of the utility of each player is actually 0. Such a

[1] This example is borrowed from the book by T. Basar and G.J. Olsder (see [6], pp.3-5).

uncooperative game is called a *zero-sum differential game* and the solution is known as the *saddle point*. Later we will give the exact definition and explanation about those terms.

Case 2: Cooperative case. Let us now modify the problem slightly. The aim of Player 2, in the present time, is to pull the object as far as possible in the negative direction of y-axis. In another word, he prefer to minimize $y(1)$. Player 1 will maintain his original objective, to maximize $x(1)$.

In this case, the two players are not of relationship of conflict, alternatively, they are of cooperative relationship. The solution, which helps both of them win, is obvious the following one, $(u_1, u_2) = (\pi, -\pi/2)$. Such kind of game is usually called *Nonzero-sum differential game*. This kind of equilibrium behavior, where one player cannot improve his utility by altering his decision unilaterally, is called a *Nash equilibrium point* in game theory. □

In the previous part, we have an intuitive grasp of the concept of game by some simple examples. We also refer the reader to early works by Isaacs [65], Friedman [47] for the formulation and presentation of differential game theory.

3.1.2 Zero-sum stochastic differential game

As we observed in the examples presented in the previous part, the game associated is of deterministic type without any stochastic uncertainty elements in the system. In the following two subsection, we will introduce the stochastic differential game including the zero-sum case and the nonzero-sum case where the dynamic of the system contains some randomnesses. Let us now describe the stochastic differential game (SDG) in the Markovian framework.

In a zero-sum stochastic differential game (ZSDG), there are two players P_1 and P_2 who intervene in a system with opposing aims. Each player carries out an *admissible control* (or called *feasible control*) towards the system, namely $(u_t)_{t\leq T}, (v_t)_{t\leq T}$ for P_1 and P_2 respectively which is an adapted stochastic process. The set of (u, v) is denoted by $\mathcal{M}$. What we formulated here is a finite time game, i.e. $T > 0$ will be a fixed constant time. The dynamic of the controlled system is a process $(x_t)_{t\leq T}$, solution of the following standard stochastic differential equation (SDE),

$$dx_t = f(t, x_t, u_t, v_t)dt + \sigma(t, x_t)dB_t \text{ for } t \leq T \text{ and } x_0 = x. \tag{3.1}$$

The process $B := (B_t)_{t\leq T}$ is a Brownian motion. The control actions are not free and generate a payoff (or reward, utility) for P_1, contrarily, a cost (or lost) from the perspective of P_2, which denoted by $J(u, v)$. Certainly, there are several popular forms of payoff function, among which, we take the following payoff for example,

$$J(u, v) = \mathbf{E}[g(x_T) + \textstyle\int_0^T h(s, x_s, u_s, v_s)ds].$$

The function h and g represent the instantaneous (or running) payoff and the terminal payoff respectively. The ZSDG is a problem that the first player P_1 looks for maximize the payoff $J(u, v)$, while, P_2 aims at minimize the cost $J(u, v)$. We are concerned about the existent of the so-called saddle point, which is a joined control pair such that each of the player reaches their objective with this saddle point and no one can win more by unilaterally changing his now control. The definition of saddle point is given as follows,

Definition 3.1.1 (Saddle point). *An admissible control (u^*, v^*) in the control set is called a saddle point for a ZSDG if it satisfies:*

$$J(u^*, v) \geq J(u^*, v^*) \geq J(u, v^*), \textit{ for any } (u, v) \in \mathcal{M}.$$

Problem 3.1.1 (ZSDG). *Find a saddle point for the zero-sum stochastic differential game over $\mathcal{M}$.*

Remark 3.1.1. *The reason why this type of SDG is called zero-sum is that, the sum of the payoff of the players is 0 or equivalently, a constant. Since the payoff of one player is somehow the cost to his opponent.*

The zero-sum differential game is well documented in several works and from several points of view. See for example [11], [22], [41], [40], [45], [47], [60], [57], [65] etc. and the references inside. A detailed review on the literature and the methods to solve the differential games, we refer readers to Subsection 3.1.6.

3.1.3 Nonzero-sum stochastic differential game

For realistic applications to other fields, such as biology or economics, it is usually necessary to study games which are not zero-sum and which involve more than two players. Therefore, in this subsection we mainly discuss the formulation of nonzero-sum stochastic differential game (NZSDG). Hereafter, we will put our main emphasis on this subject.

Assume one has N players $P_1, ..., P_N$ which intervene on (or control) a system. Each one with the help of an admissible control which is an adapted stochastic process $u^i := (u_t^i)_{t\leq T}$ for P_i, $i = 1, \ldots, N$. The set of the admissible control $u := (u^1, ..., u^N)$ is denoted by $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2 \times ... \times \mathcal{M}_N$. When the N players make use of a control u, the dynamics of the controlled system is a process $(x_t^u)_{t\leq T}$, solution of the following standard SDE:

$$dx_t^u = f(t, x_t^u, u_t^1, ..., u_t^N)dt + \sigma(t, x_t^u)dB_t \text{ for } t \leq T \text{ and } x_0 = x\,; \tag{3.2}$$

The process $B := (B_t)_{t\leq T}$ is a Brownian motion. The control actions are not free and generate for each player P_i, $i = 1, ..., N$, a payoff which amounts to

$$J_i(u^1, ..., u^N) = \mathbf{E}[g^i(x_T^u) + \int_0^T h_i(s, x_s^u, u_s)ds \ ].$$

As we presented in the zero-sum case, the payoff is given by the accumulate of the instantaneous payoff h_i over time interval $[0, T]$ and the terminal payoff g^i for P_i. The unique difference is that the players will not share the same payoff, instead, each of them has his own payoff. All of the players have the same objective, to minimize (or maximize) their own payoff. A Nash equilibrium point (NEP for short) for the players is a strategy $u^* := (u^{1,*}, ..., u^{N,*})$ of control of the system which has the feature that each player P_i who takes unilaterally the decision to deviate from $u^{i,*}$, is penalized. The definition of NEP is given from the mathematical point of view as follows.

Definition 3.1.2 (Nash equilibrium point)**.** *For all $i = 1, ..., N$, for all control u^i of player P_i, a Nash equilibrium point is a joint control $u^* := (u^{1,*}, ..., u^{N,*})$, such that,*

$$J_i(u^*) \leq J_i([u^{*,-i}|u^i])$$

where $[u^{*,-i}|u^i] := (u^{1,*}, \ldots, u^{i-1,*}, u^i, u^{i+1,*}, \ldots, u^{N,*})$.

Problem 3.1.2 (NZSDG)**.** *A nonzero-sum stochastic differential game is a problem which concerns the existence of the Nash equilibrium point over $\mathcal{M}$.*

Remark 3.1.2. *The sum of the payoffs for all players is not zero any more, therefore this kind of SDG is called a nonzero-sum case.*

Remark 3.1.3. *In the main text of this thesis, we mainly formulated and discuss NZSDG in the framework of two-player for simplicity. However, all the results in this thesis hold for the multiple players case naturally.*

In this thesis, all the problem are built up based on this control against control framework. However, we should point out that the control against control type is not the unique mode considered in a NZSDG. Some of the literature consider strategies, such as memory strategy or feedback one, as control actions for the players (see [21], [75], [91]). A memory (or a nonanticipative) strategy is a strategy where this player takes into account the past controls applied by the other players, while, a feedback type is a strategy which only consider the present state of the system. We refer readers the survey paper by R.Buckdahn et al in 2004 [20] (pp.76-77 Sections 2.1 and 2.2) and the reference therein for more information. In the following, we give the definition of different strategies as an illustration.

Definition 3.1.3 (Nonanticipative strategy)**.** *This definition is borrowed from [21] (Definition 2.3, p.4). If the set of admissible controls for player P_1 (resp. P_2) on $[t, T]$ is denoted by $\mathcal{M}_1(t)$ (resp. $\mathcal{M}_2(t)$), then, a nonanticipative strategy for Player P_1 on $[t, T]$ is a mapping $\alpha : \mathcal{M}_2(t) \to \mathcal{M}_1(t)$ such that, for any $s \in [t, T]$ and for any $v_1,\ v_2 \in \mathcal{M}_2(t)$, if $v_1 \equiv v_2$ on $[t, s]$, then $\alpha(v_1) \equiv \alpha(v_2)$ on $[t, s]$. Nonanticipative strategies for Player P_2 are defined symmetrically.*

There are also more strict definition of nonanticipative strategy with delay. Take the work [21] as a reference. Those works usually formulate the SDG problem in two-player structure. For ease of generating to the multiple players case, we prefer the feedback type Nash equilibrium point as extensively explained in the later part of this thesis, which means that each player chooses at each time t its control as a function of t and of the current position of the system. The definition of feedback strategy is introduced as follows,

Definition 3.1.4 (Feedback strategy)**.** *Assume the control processes u and v for the two players take values from two compact sets U and V respectively. A feedback strategy for P_1 (resp. P_2) is a map $u^* = u^*(t, x_t) : [0, T] \times R^m \to U$ (resp.$v^* = v^*(t, x_t) : [0, T] \times R^m \to V$) where m is the dimension of the state process x.*

Comparative to the zero-sum differential game, the nonzero-sum case is so far less considered even though there are some works on the subject, including [21], [48], [58], [59], [54], [53], [73], [75], [79], [91], etc.). In these works, the objectives are various and so are the approaches, usually based on partial differential equations (PDEs) ([48, 79]) or backward SDEs ([58, 54, 53, 75, 73]). On the other hand, it should be pointed out that the frameworks in those papers are not the same. Some of them consider strategies as control actions for the players (e.g. [21], [75], [91]) while others deal with the control against control setting (e.g. [59], [58, 54, 53]). The first ones, formulated usually in the framework of two players, allow to study the case where the diffusion coefficient σ is controlled. In the latter ones, σ does not depend on the controls. However those papers do not reach the same objective. Note that for the control against control zero-sum game, Pham and Zhang [88] and M.Sirbu [92] have overcome this restriction related to the independence of σ on the controls.

A comprehensive review of literature is arranged in Subsection 3.1.6.

3.1.4 Risk-sensitive nonzero-sum stochastic differential game

In the field of NZSDG, there is a typical type, named risk-sensitive one. In this subsection, we introduce the framework of risk-sensitive NZSDG.

Assume that we have a system which is controlled by two players. Each one impose an admissible control which is an adapted stochastic process denoted by $u = (u_t)_{t \le T}$ (resp. $v = (v_t)_{t \le T}$) for player P_1 (resp. player P_2). The state of the system is described by a process $(x_t)_{t \le T}$ which is the solution of the following SDE in the same spirit of the previous subsetions:

$$dx_t = f(t, x_t, u_t, v_t)dt + \sigma(t, x_t)dB_t \text{ for } t \le T \text{ and } x_0 = x. \tag{3.3}$$

The above process B is a Brownian motion. We establish this game model in a two-player framework for an intuitive comprehension. The results in this thesis related to the risk-sensitive NZSDG are applicable to the multiple players case. Naturally, the control action is not free and has some risks. A *risk-sensitive nonzero-sum stochastic differential game* is a game model which takes into account the attitudes of the players toward risk. More precisely speaking, for player $i = 1, 2$, the utility (cost or payoff) is given by the following exponential form

$$J^i(u,v) = \mathbf{E}[e^{\theta\{\int_0^T h_i(s,X_s,u_s,v_s)ds + g^i(X_T)\}}]. \tag{3.4}$$

The parameter θ represents the attitude of the player with respect to risk. What we are concerned here is a NZSDG which means that the two players are of cooperate relationship. Both of them would like to minimize the cost and no one can cut more by unilaterally changing his own control. Therefore, the objective of the game problem is to find a NEP (u^*, v^*) such that,

$$J^1(u^*,v^*) \leq J^1(u,v^*) \text{ and } J^2(u^*,v^*) \leq J^2(u^*,v),$$

for any admissible control (u, v).

Let us illustrate now, why θ, in the cost function, can reflect the risk attitude of the controller. From the economic point of view, we denote by $G^i_{u,v} = \int_0^T h_i(s,X_s,u_s,v_s)ds + g^i(X_T)$ the wealth of each controller and for a smooth function $F(z)$, let $F(G^i_{u,v})$ be the cost might be brought from the wealth. The two participates would like to minimize the expected cost $\mathbf{E}[F(G^i_{u,v})]$. A notation *risk sensitivity* is proposed as follows:

$$\gamma = \frac{F''(G^i)}{F'(G^i)}.$$

It is a reasonable function to reflect the trend, more precise, the curvature of cost F with respect to the wealth G^i. See W.H. Fleming's work [42] for more details. In the present thesis, we choose a utility function $F(z)$ as an exponential form $F(z) = e^{\theta z}$. Both theory and practical experience have shown that it is often appropriate to use an exponential form of utility function. Therefore, the risk sensitivity γ is exactly the parameter θ.

We can also explain this specific case $\gamma = \theta$ in the following way. The expected utility $J^i(u,v) = \mathbf{E}[e^{\theta G^i_{u,v}}]$ is *certainty equivalent* to

$$\varrho^i_\theta(u,v) := \theta^{-1}\ln \mathbf{E}[e^{\theta G^i_{u,v}}].$$

By certainty equivalent, we refer to the minimum premium we are willing to pay to insure us against some risk (alternately in a payoff case, the maximum amount of money we are willing to pay for some gamble). Then, $\varrho^i_\theta(u,v) \sim \mathbf{E}[G^i_{u,v}] + \frac{\theta}{2}\text{Var}(G^i_{u,v})$ provided that $\theta\text{Var}(G^i_{u,v})$ is small (Var(.) is variance). Hence, minimizing $J^i(u,v)$ is equivalent to minimize $\varrho^i_\theta(u,v)$. The variance $\text{Var}(G^i_{u,v})$ of wealth reflects the risk of decision to a certain extent. Therefore, it is obvious that when $\theta > 0$, the less risk the better. Such a decision maker in economic market will have a *risk-averse* attitude. On the contrary, when $\theta < 0$, the optimizer is called *risk-seeking*. Finally, if $\theta = 0$, this situation corresponds to the risk-neutral controller. See Figure 3.4 for an illustration of the three risk attitudes. For ease of presentation, we consider in the main text the risk-averse case only, the risk-seeking case is treated similarly.

About the risk-sensitive stochastic differential game problem, including nonzero-sum, zero-sum and mean-field cases, there are some previous works. Readers are referred to [7, 36, 43, 44, 66, 93] for further acquaintance and Subsection 3.1.6 for details about the method.

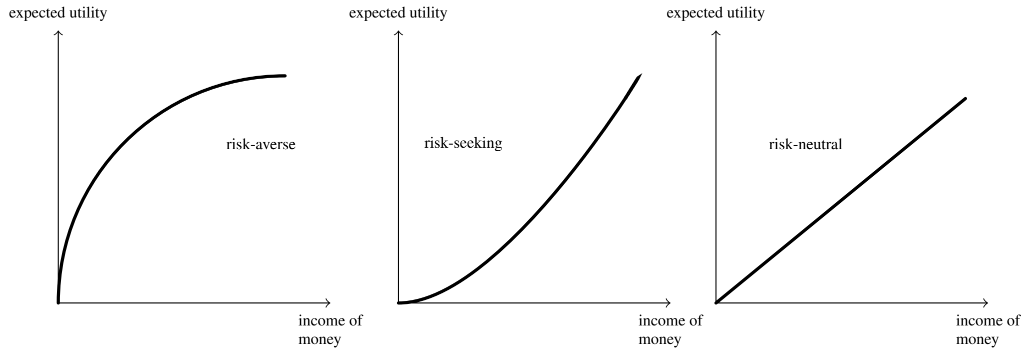


Figure 3.4: Risk sensitivity

3.1.5 Summarize and generalizations

Introductions and discussions in the previous sections yield an observation that, for ZSDG, the objectives of the players are opposite to each other, which is different with the NZSDG involving two players with cooperative relationship. However, on the other hand, a ZSDG can be viewed as a special case of NZSDG, since for example, for player P_2, to minimize $J(u, v)$ is actually to maximize $-J(u, v)$ which will be reduced to the Nonzero-sum case. The risk-sensitive NZSDG is associated with the exponential type of payoffs which may reflect the attitudes of the players toward risks. We summarize the properties of these different stochastic differential games by the following table.

	Payoff for P_1	Payoff of P_2	Describe of payoffs	Objective	optimal control (u^*, v^*)
Zero-sum SDG	$J(u,v)$	$J(u,v)$	Two players share the same payoff	P_1:maximize $J(u,v)$ P_2:minimize $J(u,v)$	saddle point $J(u^*,v) \geq J(u^*,v^*) \geq J(u,v^*)$ for any admissible control (u,v)
Nonzero-sum SDG	$J_1(u,v)$	$J_2(u,v)$	The payoffs for the two players are different	P_1:minimize $J_1(u,v)$ P_2:minimize $J_2(u,v)$	equilibrium point $J_1(u^*,v^*) \leq J_1(u,v^*)$ $J_2(u^*,v^*) \leq J_2(u^*,v)$, for any admissible control (u,v)
Risk-sensitive nonzero-sum SDG	$J^1(u,v)$	$J^2(u,v)$	The payoffs for the two players are different; Exponential type (3.4)	P_1:minimize $J^1(u,v)$ P_2:minimize $J^2(u,v)$	equilibrium point $J^1(u^*,v^*) \leq J^1(u,v^*)$ $J^2(u^*,v^*) \leq J^2(u^*,v)$ for any admissible control (u,v)

Table 3.1: Compare of ZSDG, NZSDG and Risk-sensitive NZSDG

For a NZSDG, Nash equilibrium point, as we demonstrated in Definition 3.1.2, is not the unique criteria. It can be slightly relaxed into the ϵ-equilibrium (or near-Nash) point which is a optimal control that approximately satisfies the condition of Nash equilibrium. In a Nash equilibrium, no player can benefit from modifying his own behavior while the other players keep theirs unchanged. In an ϵ-Nash equilibrium, this requirement is weakened to allow the possibility that a player may have a small hope to behave differently. This may still be considered an adequate solution concept under some appropriate assumptions. The definition about the ϵ-Nash equilibrium is given as follows,

Definition 3.1.5 (ϵ-Nash equilibrium)**.** *For players P_1 and P_2, for all admissible pair of controls (u, v), an ϵ-Nash equilibrium point is a joint control (u^*, v^*), such that,*

$$J_1(u^*, v^*) \leq J_1(u, v^*) + \epsilon \text{ and } J_2(u^*, v^*) \leq J_2(u^*, v) + \epsilon.$$

(Or alternatively, with the maximum-payoff objective,

$$J_1(u^*, v^*) \geq J_1(u, v^*) - \epsilon \text{ and } J_2(u^*, v^*) \geq J_2(u^*, v) - \epsilon. \quad)$$

We refer readers the works by R. Buckdahn et al [21] for this kind of construction. The mind of the concept of Nash equilibrium payoff in the above article is in the same spirit as here.

Problems in the field of stochastic differential games are various and are of different types. other interesting problems include the mean-field type with large population of small interacting individuals, Dykin type which concerning the optimal stopping and so on. See the survey by R. Buckdahn et al [20] for the literature introduction and some open problems.

As an applications to economics, S. Hamadène analysis the American game options in [55] (2006). More game application examples to economics are given in the book by S. Jørgensen, M. Quincampoix and T.L. Vincent [69] (2007, Part V).

3.1.6 Two approaches to stochastic differential game

There are various studies concerned in the domain of stochastic differential game. Generally speaking, a stochastic differential game is a competitive or cooperative game played by two or multiple players who intervene in a system. The state of the system is usually described by a diffusion process. Each player will choose their own control, or in some settings they may simultaneously choose a time to quite the game. The players will obtain some profits from dominating the system, or on the contrary, will pay some costs as a consequence of their controls. Their payoffs (or costs) will be represented by an expectation of their running payoff accumulated in the sustained duration of the game and ordinarily guaranteed a terminal payoff. The game is not necessarily terminated at a fixed time, in some circumstances, it will last for infinite time. The payoff criteria is manifold according to diversiform settings of games. In general, the objective of a stochastic differential game finally is to find an equilibrium point, which can be an optimal control or an optimal stopping time to quite, such that each of the players will behavior in a profitable way. We refer readers the survey paper by R. Buckdahn, P. Cardaliaguet and M. Quincampoix (2011) [20] for a full review of the development of differential game theory. Different types of settings for differential games and approaches respectively can be found there. Below, we will focus on zero-sum, nonzero-sum and typically risk-sensitive type of differential games.

There are typically, not restricted to, two popular approaches to solve a stochastic differential game: backward stochastic differential equation and partial differential equation.

Backward SDE is a young but efficient tool in optimal control field, specially for stochastic differential games (see Section 3.2 for the background of BSDE). It has been proved in many works that there is an equivalent relationship of a stochastic differential game problem and a corresponding BSDE system with Hamiltonian as its driver. The payoff associated in a game problem can be characterized by the initial value of the solution to the specific BSDE. In such a way, the existence of Nash equilibrium point is reduced to the existence of solutions for a BSDE system. This connection is observed by S. Hamadène and J.P. Lepeltier (1995) [57] [60] in a zero-sum case, followed by S. Hamadène, J.P. Lepeltier and S. Peng (1997) [58] in a nonzero-sum case. The latter solved the existence of Nash equilibrium point assuming that the dynamic coefficient of the forward state process is bounded and mainly by the L^2-domination technique. The restriction of the boundness of coefficients is relaxed by Hamadène and Mu in [63] (2014). More complicate case, such as mixed zero-sum stochastic differential game combining the control and stopping together, are investigated in [61] and [55]. Their main tool is the double barrier reflected backward SDE. The work [75] (2012) by Q. Lin investigated the Nash equilibrium payoffs (about Nash equilibrium payoffs, see works by Buckdahn and his coauthors [21] for more details) for nonzero-sum stochastic differential games where the cost function is nonlinear and not necessarily deterministic, with help of doubly controlled backward stochastic differential equations. [75] extends the earlier result by R. Buckdahn et al (2004) [21] where the cost function is deterministic.

In addition, there are also several special examples about stochastic differential games with various settings: not in a general form but with interesting features. A special case of linear-quadratic NZSDG is considered in [53, 54] by BSDE approach. A multiple players nonzero-sum stochastic differential game is solved under non-Markovian framework by BSDEs theory combined with Malliavin calculus techniques in [73] by J.P. Lepeltier et al (2009), where the associated BSDE has a coefficient which is of quadratic growth in the volatility process.

Differing from a traditional stochastic differential game, a risk-sensitive case involves an exponential type payoff. This kind of game is shown in several works that, it links with a backward SDE where the driver involves a quadratic term of the volatility process z. Under regular hypotheses, N. El-Karoui and S. Hamadène proved in [36] (2003) that the existence of saddle point for risk-sensitive zero-sum stochastic differential game and the equilibria for the nonzero-sum case by means of quadratic BSDE. The result for nonzero-sum case is extended by S. Hamadène and Mu in Chapter 5 of this memory to the situation when the dynamic coefficient of the forward equation is unbounded. A generalized mixed zero-sum stochastic differential game with exponential payoff is considered by S. Hamadène et al in [56] which is solved via a double barrier reflected BSDE with quadratic growth coefficient.

Another efficient tool to deal with stochastic differential game is a system of Hamilton-Jacobi equations and the related partial differential equation of elliptic or parabolic type. We refer readers to the classical book by A. Friedman [47] for a general introduction to the link between differential games and Hamilton-Jacobi equations. See also the early works by R. Isaacs (1965) [65], W.H. Fleming and P.E. Souganidis (1989) [45].

Recent literature related to the differential game via PDE technique concerns somehow the solutions of PDE in sense of viscosity or in sense of sobolev. A classical survey paper about viscosity solution to PDE is written by Crandall, Ishii and Lions (1992) [31]. R. Buckdahn and J. Li (2008) [22] studied a zero-sum stochastic differential game where the control action is allowed to depend on the past events occurring before the beginning of the game. Finally, the upper and the lower value are then shown to be the unique viscosity solutions of the upper and the lower Hamilton-Jacobi-Bellman-Isaacs equations, respectively.

The other sense of solution for PDE is the sobolev solution which consider the functions equipped with a norm that is a combination of L_p-norms of the function itself as well as its derivatives in a special sense up to a given order. For the relationship of classical stochastic control problem and the solutions in sobolev sense to the associated PDE, we refer readers to the book by A. Bensoussan (1982) [10]. A work about the application to a special nonzero-sum stochastic differential game is investigated by P. Mannucci (2004) [79]. With the help of existence of solutions in sobolev sense to the Hamilton-Jacobi equations system, [79] shows that a discontinuous Nash equilibria exits in type of Heaviside. However, in [79] the state process is restricted to a bounded domain. A generalized situation without this limitation is studied by S. Hamadène and Mu (2014) [62] through BSDEs method. More interesting and meaningful one-dimensional examples in field of deterministic nonzero-sum differential games can be found in a recent work by P. Cardaliaguet and P. Slawomir (2003) [24].

3.2 Background of BSDE

3.2.1 Birth of BSDE and classical results

Backward stochastic differential equations (BSDEs) was proposed firstly by J.M.Bismut in 1973 [12] in linear case to solve optimal control problems. Later this notion was generalized by Pardoux and Peng

[84] into the general nonlinear form and the existence and uniqueness results were proved under some appropriate conditions. In this subsection, we will recall the basic concept of BSDE and some useful properties without proofs.

A BSDE associated with a *driver* $f(t,\omega,y,z)$ and a *terminal value* ξ is presented as below,

$$Y_t = \xi + \int_t^T f(s,\omega,Y_s,Z_s)ds - \int_t^T Z_s dB_s, \quad t \leq T. \tag{3.5}$$

Definition 3.2.1 (Solution to BSDEs)**.** *A solution to BSDEs* (3.5) *is a pair of $\mathcal{F}_t$-adapted processes (Y,Z) such that $\boldsymbol{E}[\sup_{0\leq t\leq T}|Y_t|^2 + \int_0^T |Z_t|^2 dt] < \infty$ and* (3.5) *holds true a.s.*

For simplicity, hereafter, $f(s,\omega,y,z)$ will be denoted by $f(s,y,z)$. Processes (Y,Z) are named the *value process* and the *volatility process* respectively. The existence and uniqueness of BSDEs followed by the classical assumptions which are stated as follows,

Assumption 3.2.1 (Lipschitz; Pardoux and Peng [84])**.**

(i) $f(\cdot,0,0) \in L^2$; $\xi \in L^2$;

(ii) The function f is uniformly Lipschitz continuous with respect to (y,z). i.e. there exists a constant C s.t. $\forall (y,y',z,z'),\ |f(t,y,z) - f(t,y',z')| \leq C(|y-y'| + |z-z'|),\ dt \otimes d\boldsymbol{P} - a.e.$

Theorem 3.2.1 (Existence and Uniqueness; Pardoux and Peng [84])**.** *Under Assumption 3.2.1, BSDE* (3.5) *has a unique solution.*

The existence is proved with the help of fixed point theory. An equivalent proof by Picard iteration is show by El Karoui et al, 1997 [39].

By using ito's formula and some estimates techniques in those literature, we obtain a priori estimate as follows. This is applied in the same spirit in many later works associated to the generalized solution, such as L^p solution.

Proposition 3.2.1 (Priori estimate)**.** *Under Assumption 3.2.1, if (Y,Z) is a solution to BSDE* (3.5)*, then there exists a constant C, s.t.*

$$\boldsymbol{E}[\sup_{0\leq t\leq T}|Y_t|^2 + \int_0^T |Z_t|^2 dt] \leq C\boldsymbol{E}[|\xi|^2 + \int_0^T |f(t,0,0)|^2 dt].$$

The following linear example will help us to better understand BSDEs. An explicit solution to this example is worked out step by step. See more in [39].

Example 3.2.1 (Linear BSDEs)**.**

(i) Assume $\xi \in L^2$, consider BSDE: $Y_t = \xi - \int_t^T Z_s dB_s$.

The solutions to this equation is straightforward by martingale representation Theorem, i.e. there exists a stochastic process $\eta_t \in \mathcal{H}_T^2$, s.t. $\xi = \mathbf{E}[\xi] + \int_0^T \eta_t dB_t$. Therefore, the solutions to this BSDE are defined by:

$$Y_t := \mathbf{E}[\xi|\mathcal{F}_t] = \mathbf{E}[\xi] + \int_0^t \eta_s dB_s, \quad Z_t := \eta_t.$$

(ii) Assume additionally stochastic process $f \in \mathcal{H}_T^2$, consider BSDE: $Y_t = \xi + \int_t^T f_s ds - \int_t^T Z_s dB_s$.

The same approach as case (i) show that there exists a unique solution satisfying this BSDE.

(iii) Let (α,β) be a bounded $(\boldsymbol{R},\boldsymbol{R}^m)$-valued progressively measurable process, consider BSDE:
$Y_t=\xi+\int_t^T \alpha_s Y_s+\beta_s Z_s+f_s ds-\int_t^T Z_s dB_s.$

Let Γ be the solution to the following SDE: $d\Gamma_t = \Gamma_t(a_t dt + b_t dB_t)$, $\Gamma_0 = 1$ where a, b are progressively measurable process which will be specified later. Applying Itô's formula, we obtain,

$$d(\Gamma_t Y_t)=\Gamma_t[(a_t-\alpha_t)Y_t+(b_t-\beta_t)Z_t-f_t]dt+\Gamma_t(b_tY_t+Z_t)dB_t.$$

Set $a_t:=\alpha_t,\ b_t:=\beta_t$. Then we have,

$$\Gamma_t=e^{\int_0^t\beta_s dB_s+\int_0^t(\alpha_s-1/2\beta_s^2)ds} \text{ and } d(\Gamma_tY_t)=-\Gamma_tf_tdt+\Gamma_t(\beta_sY_t+Z_t)dB_t.$$

Let us now denote: $\tilde{Y}_t:=\Gamma_tY_t$; $\tilde{Z}_t:=\Gamma_t(\beta_tY_t+Z_t)$; $\tilde{\xi}:=\Gamma_T\xi$; $\tilde{f}_t:=\Gamma_tf_t$. Then, the original BSDE can be rewritten as:

$$\tilde{Y}_t=\tilde{\xi}+\int_t^T\tilde{f}_sds-\int_t^T\tilde{Z}_sdB_s$$

which has a solution $(\tilde{Y},\tilde{Z})$ according to case (ii). Therefore, the solution to the BSDE considered in case (iii) is

$$Y_t:=\Gamma_t^{-1}\tilde{Y}_t;\ Z_t:=\Gamma_T^{-1}\tilde{Z}_t-\beta_tY_t.$$

□

Specially in one-dimensional case, there is a well-known comparison result which tells that the value processes Y's can be compared once we know the comparison relationship of the associated drivers and terminal values. This property is introduced by S. Peng [86] (1992) and later generalized by N. El-Karoui et al [39] (1997) which is stated as follows:

Theorem 3.2.2 (Comparison Theorem; El-Karoui et al [39])**.** *Let processes (Y,Z) and (Y',Z') be solutions of two BSDEs associated with generators (f,ξ) and (f',ξ') which satisfy Assumption 3.2.1. Assume further that $\xi\geq\xi'$ **P**-a.s. and for $\forall(t,y,z)$, $f(t,y,z)\geq f'(t,y,z)$ **P**-a.s., then,*

$$Y_t\geq Y'_t,\quad \forall t\in[0,T]\quad \boldsymbol{P}\text{-a.s.}$$

A converse comparison result was investigated by P. Briand et al. [16] (2000) which reads, for each terminal value, if we can compare the initial values of BSDEs, then we can compare the associated generators.

3.2.2 Markovian framework and deterministic characterization

As we presented in Section 3.1, There is a state process associates to a stochastic differential game which is a diffusion process. In order to make apply of BSDE tool to stochastic differential games theory, we introduce in this subsection the Markovian framework of BSDE which is stated by the following decoupled forward BSDE (FBSDE):

$$\begin{cases} dX_s^{t,x}=b(s,X_s^{t,x})ds+\sigma(s,X_s^{t,x})dB_s, & t\leq s\leq T;\quad X_s^{t,x}=x,\quad 0\leq s\leq t;\\ -dY_s=\bar{f}(s,X_s,Y_s,Z_s)1_{s\geq t}ds-Z_sdB_s; & Y_T=\bar{g}(X_T^{t,x}).\end{cases}$$

For the general coupled FBSDE where the forward equation involving (Y,Z) as well, a four-step scheme is studied by J. Ma et al. in [77] (1994). Readers are also referred to the book by J. Ma and J. Yong [78] for a system presentation on this subject. Problems in FBSDE theory are introduced in a series of works by F. Delarue, for example, [32, 33, 28, 26, 30].

We impose some proper hypotheses on coefficients b and σ of the forward equation in order to guarantee the existence of solution X and a good estimate of it which we need in what followed. One of the popular hypotheses is the following one.

Assumption 3.2.2. *The applications b and σ are uniformly Lipschitz continuous with respect to x, besides, they are of linear growth with respect to x, i.e. there exists a constant C, s.t. for any (s,x), $|b(s,x)|+|\sigma(s,x)|\leq C(1+|x|)$.*

Except the classical Lipschitz condition on coefficients of BSDE, we assume additionally the polynomial growth condition to characterize the solutions of the Markovian BSDEs by some deterministic functions of s and $X_s^{t,x}$.

Assumption 3.2.3. *Let ℓ be an integer and let us consider $\bar{f}$ (resp. $\bar{g}$) a Borel measurable function from $[0,T] \times \boldsymbol{R}^{m+\ell+\ell\times m}$ (resp. $\boldsymbol{R}^m$) into $\boldsymbol{R}^\ell$ (resp. $\boldsymbol{R}^\ell$) such that:*

(a) For any fixed $(t,x) \in [0,T] \times \boldsymbol{R}^m$, the mapping $(y,z) \in \boldsymbol{R}^{\ell+\ell\times m} \longmapsto \bar{f}(t,x,y,z)$ is uniformly Lipschitz ;

(b) There exist real constants C and $p > 0$ such that

$$|\bar{f}(t,x,y,z)| + |\bar{g}(x)| \le C(1+|x|^p),\ \forall (t,x,y,z) \in [0,T] \times \boldsymbol{R}^{m+\ell+\ell\times m}.$$

Then we have the following result by El Karoui et al. [39] related to representation of solutions of BSDEs through deterministic functions in the Markovian framework of randomness.

Proposition 3.2.2 (Deterministic characterization; El Karoui et al [39])**.** *Assume that Assumptions 3.2.2 and 3.2.3 are fulfilled. Let $(t,x) \in [0,T] \times \boldsymbol{R}^m$ be fixed, then the markovian BSDE has solutions $(Y_s, Z_s)_{t\le s\le T} \in \mathcal{S}^2_{t,T,\ell} \times \mathcal{H}^2_{t,T,\ell\times m}$, besides, there exists a pair of measurable and deterministic applications ϖ: $[0,T] \times \boldsymbol{R}^m \to \boldsymbol{R}^\ell$ and υ: $[0,T] \times \boldsymbol{R}^m \to \boldsymbol{R}^{\ell\times d}$ such that,*

$$\boldsymbol{P} - a.s., \forall t \le s \le T, \quad Y_s = \varpi(s, X_s^{t,x})\ and\ Z_s = \upsilon(s, X_s^{t,x}).$$

Moreover,

(i) $\forall (t,x) \in [0,T] \times \boldsymbol{R}^m$, $\varpi(t,x) = \boldsymbol{E}[\int_t^T \bar{f}(r, X_r^{t,x}, Y_r, Z_r)dr + \bar{g}(X_T^{t,x})]$;

(ii) For any other $(t_1,x_1) \in [0,T] \times \boldsymbol{R}^m$, the process $(\varpi(s, X_s^{t_1,x_1}),\ \upsilon(s, X_s^{t_1,x_1}))_{t_1\le s\le T}$ is the unique solution in $\mathcal{S}^2_{t_1,T,\ell} \times \mathcal{H}^2_{t_1,T,\ell\times m}$ of the BSDE associated with the coefficients $(\bar{f}(s, X_s^{t_1,x_1}, y, z), \bar{g}(X_T^{t_1,x_1}))$ in the time interval $[t_1, T]$.

3.2.3 History of BSDEs

Backward stochastic differential equations (BSDEs) was proposed firstly by J.M. Bismut (1973) in [12] in linear case to solve the optimal control problems. Later this notion was generalized by E. Pardoux and S. Peng (1990) in [84] into the general nonlinear form and the existence and uniqueness results (see Theorem 3.2.1) were proved under the classical Lipschitz condition. A class of BSDE is also introduced by D. Duffie and L.G. Epstein (1992) [35] in point of view of recursive utility in economics. During the past twenty years, BSDEs theory attracts many researchers' interests and has been fully developed into various directions. Among the abundant literature, we refer readers the florilegium book edited by N. El-Karoui and L. Mazliark (1997) [38] for the early works before 1996. Surveys on BSDEs theory also includes [37] which is written by N. El-Karoui, S. Hamadène and A. Matoussi collected in book [25] (2009) (see Chapter 8) and the book by J. Yong and X. Zhou (1999) [94] (see Chapter 7). Some applications on optimization problems can be found in [37]. About Other applications such as in field of economics, we refer N. El-Kaoui, S. Peng and M.C. Quenez [39] (1997). Recently, a complete review on BSDEs theory as well as some new results on nonlinear expectation are introduced in a survey paper by S. Peng (2010) [87]. Below, we focus on the classical BSDE setting in one or multiple dimensions and review the main results including the quadratic case. About the other important results, such as BSDEs with reflections or barriers, numerical analysis, application in economics, we refer the books and articles mentioned above and the references therein.

One possible extension to the pioneer work of [84] is to relax as much as possible the uniform Lipschitz condition on the coefficient. A weaker hypothesis is presented by X. Mao (1995) [80] which we translate as follows according to the notion of BSDE (3.5). Mao's condition reads: for all $y, \bar{y}, z, \bar{z}$ and $t \in [0, T]$, $|f(t, y, z) - f(t, \bar{y}, \bar{z})|^2 \leq \kappa|y - \bar{y}|^2 + c|y - \bar{y}|^2$ a.s. where $c > 0$ and κ is a concave non-decreasing function from $\mathbf{R}^+$ to $\mathbf{R}^+$ such that $\kappa(0) = 0,\ \kappa(u) > 0$ for $u > 0$ and $\int_{0+} 1/\kappa(u)du = \infty$. An existence and uniqueness result is proved under such condition in [80]. S. Hamadène introduced in [52] (1996) a one-dimensional BSDE with locale Lipschitz generator. Later J.P. Lepeltier and J. San Martin provided an existence result of minimum solution in [72] (1997) of one dimensional BSDE (3.5) where function f is continuous and of linear growth in terms of (y, z). When f is uniformly continuous in z with respect to (ω, t) and independent of y, a uniqueness result was obtained by G. Jia [68]. BSDEs with polynomial growth generator is studied by Ph. Briand in [14]. The case of 1-dimensional BSDEs with coefficient which is monotonic in Y and non-Lipschitz on Z is shown in work [19].

There are plenty works on one-dimensional BSDE. However, limited results have been obtained about the multi-dimension case. We refer S. Hamadène, J-P. Lepeltier and Peng [58] for an existence result on BSDEs system of Markovian case where the driver is of linear growth on (y, z) and of polynomial growth on the state process. See K. Bahlali [4] [5] for high-dimension BSDE with local Lipschitz coefficient.

About the BSDE with continuous and quadratic growth driver, a classical research should be the one by M. Kobylanski [71] (2000) which investigated a one-dimensional BSDE with driver $|f(t, y, z) \leq C(1 + |y| + |z|^2)$ and bounded terminal value. This result was generated by P. Briand and Y. Hu into the unbounded terminal value case in [18] (2006).

In literature, the various approaches of BSDEs are more or less depend on the continuity, uniformly or in weak sense, of the generators. There are also works which related to the discontinuous generator. See G. Jia [67] (2008) for the case of one-dimensional BSDE with driver f: $f(t, \cdot, z)$ is left-continuous and $f(t, y, \cdot)$ is continuous. Recent works include also S. Hamadène and R. Mu [62] (2014) which presented an example of multi-dimensional BSDE whose driver is independent of y and discontinuous on z.

The other direction of the generalization of BSDEs is to consider solutions in $L^p(p > 1)$ space. See N. El-Karoui et al [39] (1997) and P. Briand et al [17] (2003) for examples.

3.2.4 Connection with NZSDG

In accordance with NZSDG built as in subsections 3.1.3 and 3.1.4, we still discuss in the Markovian framework. In this subsection, we will introduce the connection of BSDEs with NZSDG including the risk-sensitive case specially.

Before approaching, let us introduce the Hamiltonian system for a NZSDG. We consider a two-player NZSDG where the controls applied by the players are denoted by (u, v) take values from U and V respectively. Process $(x_t)_{t \leq T}$ is the solution to SDE (3.2) in two players case. For the other notations, we keep the ones in subsection 3.1.3. Then, for any admissible pair of controls (u, v), the Hamiltonians of the NZSDG is given by functions H_i, for player $i = 1, 2$, from $[0, T] \times \mathbf{R}^{2m} \times U \times V$ into $\mathbf{R}$ associate:

$$H_i(t, x, p, u, v) = p\sigma^{-1}(t, x)f(t, x, u, v) + h_i(t, x, u, v),$$

and we introduce the following assumption called the generalized *Isaacs condition* which has been already considered by A. Friedman in [47] for the same purpose as the ones in this thesis. But the treatment of the problem he used is the PDE approach. This assumption is well-posed due to the Benes' measure selection theorem, see [9] for details.

Assumption 3.2.4 (Generalized Isaacs condition)**.** *There exist two Borelian applications u^*, v^* defined on $[0, T] \times \boldsymbol{R}^{3m}$, with values in U and V, respectively, such that for any $(t, x, p, q, u, v) \in [0, T] \times \boldsymbol{R}^{3m} \times U \times V$,* *we have:*

$$\begin{cases} H_1^*(t,x,p,q) = H_1(t,x,p,u^*(t,x,p,q),v^*(t,x,p,q)) \leq H_1(t,x,p,u,v^*(t,x,p,q)); \\ H_2^*(t,x,p,q) = H_2(t,x,q,u^*(t,x,p,q),v^*(t,x,p,q)) \leq H_2(t,x,q,u^*(t,x,p,q),v). \end{cases}$$

We now give two examples when the generalized Isaacs condition is fulfilled.

Example 3.2.2. *Usually, the Hamiltonian (H_1^*, H_2^*) with optimal control are assumed to be continuous w.r.t. (p,q). But the Borelian applications u^* and v^* are not necessarily continuous. For example, in one dimensional case, we assume $\sigma = 1$ and the value sets of controls (u,v) are respectively $U = [-1,1]$ and $V = [0,1]$. if we take function $f = u$ and $h_i = u$, $i = 1,2$, then, $H_1(t,x,p,u,v) = pu + u$. Obviously, $u^*(p,\epsilon) = 1_{\{p>-1\}} - 1_{\{p<-1\}} + \epsilon_{\{p=-1\}}$ for arbitrary $\epsilon \in U$ is an application satisfying the generalized Isaacs condition and is not continuous. However, in this case, the Hamiltonian function $H_1^*(t,x,p) = (p+1)1_{\{p\geq -1\}} + (-p-1)1_{\{p<-1\}}$ is continuous on p.*

Example 3.2.3. *There are also examples which satisfy the generalized Isaacs condition with discontinuous Hamiltonian functions. For instance, in one dimensional case, the diffusion process $\sigma = 1$ and the value sets for the controls u and v are $U = [0,1]$ and $V = [-1,1]$. The dynamic function f takes form of $f(t,x,u,v) = u + v$ and $h_i = 0$. Now the Hamiltonian functions are $H_1(t,x,p,u,v) = p(u+v)$ and $H_2(t,x,q,u,v) = q(u+v)$. Therefore, we can check that $u^*(p,\epsilon) = 1_{\{p>0\}} + \epsilon 1_{\{p=0\}} + 0 \cdot 1_{\{p<0\}}$ and $v^*(q,\varepsilon) = 1_{\{q>0\}} + \varepsilon 1_{\{q=0\}} - 1_{\{q<0\}}$ with arbitrary $\epsilon \in U$ and $\varepsilon \in V$ are a pair of discontinuous controls which satisfy the generalized Isaacs conditon. Besides, note that the Hamiltonian is not continuous any more. This example has been solved in [62].*

Under the continuous assumption on Hamiltonians, we then have the following result which is given by Hamadène et al. [58] related to the existence of a NEP for the NZSDG.

Theorem 3.2.3 (Existence of NEP; Hamadène et al.1997 [58])**.** *Assume that Assumption 3.2.4 holds and applications f, h_i and g^i are bounded. The diffusion coefficient σ is non-degenerate, uniformly Lipschitz and in linear growth with respect to x. Besides, assume that the mapping $(p,q) \in \boldsymbol{R}^{2m} \longmapsto (H_1^*, H_2^*) \in \boldsymbol{R}$ is continuous for any fixed (t,x). Then, there exists a pair of processes $(Y_t^{i,*}, Z_t^{i,*})_{t\leq T}$ s.t.*

$$\begin{cases} Y^{1,*} \text{ and } Y^{2,*} \text{ are bounded, } \boldsymbol{E}[\int_0^T |Z_s^{1,*}\sigma(s,x_s)|^2 + |Z_s^{2,*}\sigma(s,x_s)|^2 ds] < \infty; \\ -dY_t^{1,*} = H_1(t,x_t,Z_t^{1,*},(u^*,v^*)(t,x_t,Z_t^{1,*},Z_t^{2,*})dt - Z_t^{1,*}dB_t, \quad Y_T^{1,*} = g^1(X_T); \\ -dY_t^{2,*} = H_2(t,x_t,Z_t^{2,*},(u^*,v^*)(t,x_t,Z_t^{1,*},Z_t^{2,*})dt - Z_t^{2,*}dB_t, \quad Y_T^{2,*} = g^2(X_T). \end{cases} \tag{3.6}$$

In addition, $J_i(u^*,v^*) = Y_0^{i,*}$, $i = 1,2$ and the pair of controls $(u^*,v^*) := ((u^*,v^*)(t,x_t,Z_t^{1,*},Z_t^{2,*}))_{t\leq T}$ is a NEP for the NZSDG. □

The above Theorem tells us that under the Isaacs condition and other appropriate hypothesis, the payoffs for players can be characterized by the initial values of a associated coupled BSDE involving a driver which is of Hamiltonian type. This BSDE finally provides a NEP for the NZSDG. Notice that the continuous property of Hamiltonian plays an crucial role for this result.

In particular, the risk-sensitive NZSDG (see Subsection 3.1.4) can be treated in the same way by a multiple-dimension BSDE. The difference is that the driver for this BSDE is of quadratic growth with respect to the volatility process Z. For more detail, readers are referred to paper by El Karoui and Hamadène [36] which includes additionally the cases of risk-sensitive control and ZSDG. The main result related to the duality of risk-sensitive NZSDG and BSDE and the existence of NEP is stated as follows.

Theorem 3.2.4 (NEP for Risk-sensitive case; El Karoui et al.2003[36])**.** *Under the same hypotheses of Theorem 3.2.3, there exists a process* (Y^1, Y^2, Z^1, Z^2) *solutions of the following BSDE:*

$$\begin{cases} Y^1 \text{ and } Y^2 \text{ are bounded, } Z^1, Z^2 \in \mathcal{H}^2_{T,m}; \\ -dY^1_t = \{H_1(t, x_t, Z^1_t, (u^*, v^*)(t, x_t, Z^1_t, Z^2_t)) + 1/2|Z^1_t|^2\}dt - Z^1_t dB_t,\ t \leq T;\ Y^1_T = g^1(x_T); \\ -dY^2_t = \{H_2(t, x_t, Z^2_t, (u^*, v^*)(t, x_t, Z^1_t, Z^2_t)) + 1/2|Z^2_t|^2\}dt - Z^2_t dB_t,\ t \leq T;\ Y^2_T = g^2(x_T). \end{cases} \tag{3.7}$$

In addition, the exponential type payoff functional (3.4) *coincides with the exponent of initial value of the above BSDE, i.e.* $J^1(u^*, v^*) = e^{Y^1_0}$ *and* $J^2(u^*, v^*) = e^{Y^2_0}$. *Besides,* $(u^*, v^*) = ((u^*, v^*)(t, x_t, Z^1_t, Z^2_t))_{t \leq T}$ *is a NEP for the risk-sensitive NZSDG.*

3.3 Main results

In this thesis, there are mainly three results, all related to NZSDG problem. We summarize, in this section briefly, those three different frameworks and their main results .

3.3.1 Nonzero-sum Stochastic Differential Games with Unbounded Coefficients

Chapter 4 in this thesis is a published cowork with Hamadène (ref.[63]).

In Chapter 4, we study the nonzero-sum stochastic differential game of type control against control with the diffusion process σ independent of controls, in the same line as in the paper by Hamadène et al. [58] in Markovian framework. The general formulation about NZSDG on multiple players framework is introduced in Subsection 3.1.3. We summarize the setting of this problem here in two-player case for simplicity. Notice that all those results and techniques in this Chapter can be generalized into the multiple players without any difficulties. Let us now recall the following:

Setting for a NZSDG:

$$\begin{cases} \text{State process:} & dx^{t,x}_s = \sigma(s, x^{t,x}_s)dB_s \text{ for } s \in [t, T] \text{ and } x^{t,x}_s = x \text{ for } s \in [0, t]; \\ \text{Girsanov transformation:} & \text{Let } \mathbf{P}^{u,v} \text{ be a new measure whose density is} \\ & d\mathbf{P}^{u,v}/d\mathbf{P} = \zeta_T(\sigma^{-1}(\cdot, x^{t,x}_\cdot)f(\cdot, x^{t,x}_\cdot, u_\cdot, v_\cdot)) \\ & \text{where } \zeta_s(\eta) := e^{\int_0^s \eta_r dB_r - 1/2\int_0^s |\eta_r|^2 dr} \text{ for any adapted process } \eta. \\ & \mathbf{P}^{u,v} \text{ is a new probability under appropriate assumptions. Then,} \\ & B^{u,v} := (B_s = \int_0^s \sigma^{-1}(r, x^{t,x}_r)f(r, x^{t,x}_r, u_r, v_r)dr)_{s \leq T} \text{ is a} \\ & \text{Brownian motion under probability } \mathbf{P}^{u,v}; \\ \text{Weak formulation:} & dx^{t,x}_s = f(s, x^{t,x}_s, u_s, v_s)ds + \sigma(s, x^{t,x}_s)dB^{u,v}_s, \text{ for } s \in [t, T] \\ & \text{and } x^{t,x}_s = x \text{ for } s < t; \\ \text{Payoffs:} & J_i(u, v) = \mathbf{E}^{u,v}[\int_0^T h_i(s, x^{0,x}_s, u_s, v_s)ds + g^i(x^{0,x}_T)] \text{ for player } i = 1, 2 \\ & \text{and a fixed point } (0, x) \text{ where } \mathbf{E}^{u,v} \text{is expectation under } \mathbf{P}^{u,v}; \\ \text{Objective:} & \text{to find a NEP } (u^*, v^*) \text{ s.t. } J_1(u^*, v^*) \leq J_1(u, v^*),\ J_2(u^*, v^*) \leq J_2(u^*, v) \\ & \text{for any admissible control } (u, v); \\ \text{Hamiltonian:} & H_i(t, x, p, u, v) = p\sigma^{-1}(t, x)f(t, x, u, v) + h_i(t, x, u, v) \text{ for player } i = 1, 2. \end{cases} \tag{3.8}$$

As shown by Theorem 3.2.3, in [58], the setting in literature concerns only the case when the coefficients f and σ of the diffusion in the weak formulation (3.8) are bounded. According to our knowledge the setting where those coefficients are not bounded and of linear growth is not considered yet. Therefore the main objective of Chapter 4 is to relax as much as possible the boundedness of the coefficients f (mainly) and σ (which is not bounded as stated in the final extension in Chapter 4).

The specific hypotheses that we impose are stated as follows.

Assumption 3.3.1.

$$\begin{cases} \textit{Diffusion } \sigma \textit{ is uniformly Lipschitz, invertible, bounded and its inverse is bounded}; \\ \textit{Drift function } f \textit{ is of linear growth w.r.t. } x; \\ \textit{Running payoff } h_i \textit{ is of polynomial growth w.r.t } x; \\ \textit{Terminal payoff } g^i \textit{ is of polynomial growth w.r.t } x; \\ \textit{Isaacs condition: there exist applications } u^*,\ v^* \textit{ s.t. for any } (t,x,p,q,u,v), \\ \quad H_1^*(t,x,p,q) = H_1(t,x,p,u_1^*(t,x,p,q),u_2^*(t,x,p,q)) \leq H_1(t,x,p,u,u_2^*(t,x,p,q)), \\ \quad H_2^*(t,x,p,q) = H_2(t,x,q,u_1^*(t,x,p,q),u_2^*(t,x,p,q)) \leq H_2(t,x,q,u_1^*(t,x,p,q),v); \\ \textit{Continuity: the mapping } (p,q) \longmapsto (H_1^*,H_2^*)(t,x,p,q) \textit{ is continuous for any fixed } (t,x). \end{cases}$$

The novelty of the results in Chapter 4 is that we show the existence of a Nash equilibrium point for the NZSDG when f is no longer bounded but only satisfies the linear growth condition. The formulation is analogous as in Hamadène, Lepeltier and Peng (1997) [58]. But in the framework of [58], the coefficient f is bounded. Since the work depends heavily on Girsanov's probability transformation: one has to deal with Doléans-Dade exponential of $\sigma^{-1}f$. When the coefficients σ^{-1} and f are bounded, obviously, Doléans-Dade exponential is a probability density. However, when f is of linear growth, this conclusion is not so straightforward, meanwhile, some good estimates and properties of solutions to the corresponding BSDE will be invalid. This is the main difficulty in our work.

An efficient tool that we applied to overcome this difficulty is a result by Haussmann (1986) (see Theorem 4.2.1 which related to the integrability of Doléans-Dade exponential with f in linear growth. Following from this assertion, we know that Doléans-Dade exponential corresponding to $\sigma^{-1}f$ belongs to L^p space with some constant p located between 1 and 2, even f is of linear growth. With this integrability property in hand, Girsanov's transformation can be carried out smoothly. Besides, a good link between the expectation under original probability and the one under the new probability is provided due to Haussmann's result. This plays an important role in latter techniques for our scheme.

As in [58] our approach is based on backward SDEs. As shown by Proposition 4.2.1, the payoffs for players are coincide with the initial values of solutions for the following BSDEs whose integrability is not standard.

$$\begin{cases} Y_t^{1,u,v} & = g^1(x_T^{0,x}) + \int_t^T H_1(s, x_s^{0,x}, Z_s^{1,u,v}, u_s, v_s)ds - \int_t^T Z_s^{1,u,v} dB_s; \\ Y_t^{2,u,v} & = g^2(x_T^{0,x}) + \int_t^T H_2(s, x_s^{0,x}, Z_s^{2,u,v}, u_s, v_s)ds - \int_t^T Z_s^{2,u,v} dB_s. \end{cases} \tag{3.9}$$

Then we are able to show that a NEP exists (see Theorem 4.2.1) with the help of BSDE (3.9) and the following BSDE (3.10) where the Hamiltonian type driver depend on Borel feedback controls . The proof is built by a localization scheme. Once we show that, there exist processes with proper integrabilities satisfying BSDE (3.10), then we can conclude that a NEP exists, which is exactly the pair of control processes $((u^*, v^*)(t, x_t^{0,x}, Z_s^1, Z_s^2))_{t\leq T}$, for this NZSDG. Therefore, basically the problem turns into studying the following BSDE:

$$\begin{cases} Y_t^1 & = g^1(x_T^{0,x}) + \int_t^T H_1(s, x_s^{0,x}, Z_s^1, u^*(s, x_s^{0,x}, Z_s^1, Z_s^2), v^*(s, x_s^{0,x}, Z_s^1, Z_s^2))ds - \int_t^T Z_s^1 dB_s; \\ Y_t^2 & = g^2(x_T^{0,x}) + \int_t^T H_2(s, x_s^{0,x}, Z_s^2, u^*(s, x_s^{0,x}, Z_s^1, Z_s^2), v^*(s, x_s^{0,x}, Z_s^1, Z_s^2))ds - \int_t^T Z_s^2 dB_s. \end{cases} \tag{3.10}$$

This specific BSDE is multiple dimensional and each dimension is coupled mutually by terms of volatility processes. The main difficulty to solve this BSDE consists in the fact that its driver involve the term $z^i\sigma^{-1}(t,x)f(t,x,(u^*,v^*)(t,x,z^1,z^2)),\ i = 1,2$, where f is not bounded but of linear growth in x. As a consequence of that, in Markovian framework, the driver of BSDE (3.10) is actually of linear growth in volatility term ω by ω. Alternatively, we may refer the driver is of stochastic linear growth or of stochastic Lipschitz (see [8]). In addition, the Hamiltonian with the feedback type controls, which plays a driver's role for this BSDE, is continuous in (z^1, z^2). Results of BSDEs with stochastic Lipschitz condition includes the one by Briand [15] for case of BSDE with irregular generator involving BMO martingale.

We finally show that this specific BSDE has a solution which then provides a NEP for the NZSDG when the generalized Isaacs condition is fulfilled and the laws of the dynamics of the non-controlled system satisfy the so-called L^q-domination condition. This latter is especially satisfied when the diffusion coefficient σ satisfies the well-known uniform ellipticity condition.

Our method is based on: (i) the introduction of an approximating scheme of BSDEs which is well-posed since the coefficients are Lipschitz. In this markovian framework of randomness, the solutions (Y^n, Z^n), $n \geq 1$, of this scheme can be represented via deterministic functions (ϖ^n, υ^n), $n \geq 1$, and the Markov process as well ; (ii) sharp estimates for (Y^n, Z^n) and (ϖ^n, υ^n) and the L^q-domination condition enable us to obtain the strong convergence of a subsequence $(\varpi_{n_k})_{k\geq 1}$ from a weak convergence in an appropriate space. This yields the strong convergence of the corresponding subsequences $(Y^{n_k})_{k\geq 1}$ and $(Z^{n_k})_{k\geq 1}$; (iii) we finally show that the limit of $(Y^{n_k}, Z^{n_k})_{k\geq 1}$ is a solution for the BSDE associated with the NZSDG.

To summarize, there are three main points that we require in our approach:

(i) Haussmann's result: there exists a $p \in (1, 2)$ such that for any admissible control (u, v), $\mathbf{E}[(\zeta_T(\sigma^{-1}(\cdot, x^{t,x}) f(\cdot, x^{t,x}_{\cdot}, u_{\cdot}, v_{\cdot})))^p] < \infty$ where ζ_T is defined as in (3.8);

(ii) The L^q-domination property or its adaptation ;

(iii) The generalized Isaacs condition.

At the end of this chapter we provide an example which illustrates our result. We also discuss possible extensions of our findings to the case when both the drift f and diffusion coefficient σ of the state process are not bounded.

3.3.2 Risk-sensitive Nonzero-sum Stochastic Differential Game with Unbounded Coefficients

Chapter 5 is a joint work with S. Hamadène.

Chapter 5 deals with the risk-sensitive NZSDG, as presented in Subsection 3.1.4, which is a game problem taking into account the attitudes of the players. For the details about the three different cases including risk averse, risk seeking and risk neutral, readers are referred to Subsection 3.1.4. In Chapter 5, we focus on the risk averse situation below. Besides, for notation's simplicity, we discuss under two-player framework. However, the generalization to multiple players case is formal and can be carried out in the same spirit.

The setting is analogous to the one for a standard NZSDG, see (3.8) for details. We also start with a state process which is a diffusion. We later set up our problem on the weak formulation, i.e. transform the original probability $\mathbf{P}$ into the new one $\mathbf{P}^{u,v}$ by Girsanov's transformation. Then the law of this state diffusion process under probability $\mathbf{P}$ maintains the same as the one under $\mathbf{P}^{u,v}$. The weak formulation of state process is given by:

$$dx_s^{t,x} = f(s, x_s^{t,x}, u_s, v_s)ds + \sigma(s, x_s^{t,x})dB_s^{u,v}, \text{ for } s \in [t, T] \text{ and } x_s^{t,x} = x \text{ for } s < t.$$

The unique distinguish is that the payoffs are more complicated and are of exponential types which are natural and popular especially in the economic field. The payoffs for a risk-sensitive NZSDG is stated as follows: for player $i = 1, 2$ and for each pair of admissible controls (u, v), the payoffs are

$$J^i(u, v) = \mathbf{E}^{u,v}[e^{\int_0^T h_i(s, x_s^{0,x}, u_s, v_s)ds + g^i(x_T^{0,x})}].$$

The objective of the risk-sensitive NZSDG is to find a NEP, which is a pair of admissible controls (u^*, v^*), such that $J^1(u^*, v^*) \leq J^1(u, v^*)$ and $J^2(u^*, v^*) \leq J^2(u^*, v)$.

We emphasis that, all the assumptions under Chapter 5 is the same as Assumption 3.3.1: (i) diffusion σ is uniformly Lipschitz, invertible, bounded and its inverse is bounded. It follows from above that such a σ satisfies uniform ellipticity condition; (ii) drift function f is of linear growth in x; (iii) both the running payoff h_i and the terminal payoff g^i are of polynomial growth in x for $i = 1, 2$; (iv) Isaacs condition is fulfilled; (v) the Hamiltonians with the feedback type controls are continuous.

About the risk-sensitive stochastic differential game problem, including nonzero-sum, zero-sum and mean-field cases, there are some previous works. Readers are referred to [7, 36, 43, 44, 66, 93] for further acquaintance. Among those results, a particular popular approach is partial differential equation, such as [7, 43, 44, 66, 93] with various objectives. Another method is through backward stochastic differential equation (BSDE) theory, see [36]. In Chapter 5, we also deal with this risk-sensitive game through BSDE tools in the same line as article by El-Karoui and Hamadène (2003) [36].

However in [36], the setting of game problem concerns only the case when the drift coefficient f in diffusion dynamic is bounded. This constrain is too strict to some extent. Therefore, our motivation is to relax as much as possible the boundedness of the coefficient f. We assume, like Subsection 3.3.1 (see Assumption 3.3.1) that f is not bounded any more but instead, it has a linear growth condition. It is the main novelty of this work. To our knowledge, this general case has not been studied in the literature for a risk-sensitive NZSDG.

As illustrated in the previous part for a standard NZSDG, the payoff is related to the initial value of a corresponding BSDE. For a risk-sensitive case, this fact is still true which tells us that the payoff coincides with the exponential of the initial value of some specific BSDE. The equivalence is proved in Chapter 5 in the main text (see Proposition 5.2.1). Therefore, to find a NEP for the risk-sensitive NZSDG is reduced to study the existence of solutions for the following BSDE:

$$\begin{cases} Y_t^1 &= g^1(x_T^{0,x}) &+ \int_t^T H_1(s, x_s^{0,x}, Z_s^1, u^*(s, x_s^{0,x}, Z_s^1, Z_s^2), v^*(s, x_s^{0,x}, Z_s^1, Z_s^2)) + 1/2|Z_s^1|^2 ds \\ & & - \int_t^T Z_s^1 dB_s; \\ Y_t^2 &= g^2(x_T^{0,x}) &+ \int_t^T H_2(s, x_s^{0,x}, Z_s^2, u^*(s, x_s^{0,x}, Z_s^1, Z_s^2), v^*(s, x_s^{0,x}, Z_s^1, Z_s^2)) + 1/2|Z_s^2|^2 ds \\ & & - \int_t^T Z_s^2 dB_s. \end{cases} \tag{3.11}$$

This BSDE is multiple-dimensional with continuous generator involving both linear and quadratic terms of z. The difficulties to solve this BSDE rely on two perspectives:

(i) The first difficulty is the quadratic term of z which is involved in the driver. In compare with the standard NZSDG as Subsection 3.3.1, we need to carry out some techniques to deal with this quadratic term specially.

(ii) The second one is the following: since the driver has two components, one is a linear part of the volatility process which is Hamiltonian including the feedback type controls, the other one is a quadratic term of the volatility process. It takes the form of
$H_i(s, x, z^i, (u^*, v^*)(t, x, z^1, z^2)) + 1/2|z^i|^2 = z^i f(s, x, z^1, z^2) + h_i(s, x, z^1, z^2) + 1/2|z^i|^2$ for $i = 1, 2$ where f is of linear growth in x. Similar as in Subsection 3.3.1, the first linear term of z is of linear growth ω by ω in Markovian framework due to the linear growth of f. The case when f is bounded has been dealt by [36].

Respect to these two difficulties, our strategies to overcome them are the following:

(i)' To deal with the quadratic term of z, we apply the classical exponential transform (see M.kobylanski et al (2000) [71]): $\bar{Y}^i = e^{Y^i}$; $\bar{Z}^i = \bar{Y}^i Z^i$ for $i = 1, 2$. By this technique, the quadratic term can be eliminated. However, as a cost, the value process get involved into the driver.

(ii)' The difficulty, specially in making apply the Girsanov's transformation, brought by the linear growth of f is overcome by a Haussmann's result. As illustrated in Subsection 3.3.1, the Doléans-Dade exponential local martingale on $\sigma^{-1}f$ is integrable in L^p with some $p \in (1, 2)$. This result enable us to carry out Girsanov's transformation in order to move out the volatility term from the driver, which then provide us an access to obtain the integrability of the value process.

Under the generalized Isaacs hypothesis and domination property related to the law of diffusion process, which holds when the uniform ellipticity condition on σ is satisfied, we finally show that the associated BSDE (3.11) has a solution which then provides the NEP for this risk-sensitive NZSDG.

The method is summarized as follows: (i) We firstly take an exponential transformation to eliminate the quadratic term of volatility process. The original BSDE is replaced by a new one which is multiple dimensional with a driver involving both parts of $\bar{Y}$ and $\bar{Z}$, being of linear growth ω by ω in these two components; (ii) We then introduce an approximate scheme which smooth the generator of BSDEs by the mollifier technique. The new sequence of BSDEs is well-posed since the coefficients are Lipschitz. Besides, under Markovian framework of randomness, the solutions $(\bar{Y}^n, \bar{Z}^n),\ n \geq 1$ of this scheme can be represented by deterministic functions $(\varsigma^n, \mathfrak{z}^n),\ n \geq 1$. An exponential type growth property of ς^n is provided which then yields the sharp estimates of $(\bar{Y}^n, \bar{Z}^n)$. (iii) We may subtract a subsequence $(\varsigma^{n_k})_{k\geq 1}$ which is proved to be a strong convergent sequence from a weak convergence in an appropriate space. This yields the strong convergence of the corresponding subsequences $(\bar{Y}^{n_k}, \bar{Z}^{n_k})_{k\geq 1}$; (iv) We finally show that the limit of subsequence $(\bar{Y}^{n_k}, \bar{Z}^{n_k})_{k\geq 1}$ is a solution for the transformed BSDE. By taking the inverse exponential transform, the solution for BSDE (3.11) exists.

3.3.3 Bang-Bang Type Equilibrium Point for Nonzero-sum Stochastic Differential Game

Chapter6 in this dissertation is a published work with Hamadène (ref. [62]).

The motivation of this work is the following. Notice that in the previous results about NZSDG, such as [58, 54, 53, 75], the authors concern only about the smooth feedback controls as well as the Hamiltonian functions. The same as our results in Chapter 4 and 5. The proofs rely heavily on the continuous assumption on Hamiltonian (see Assumption 3.3.1: Continuity). The case of discontinuous controls is not fully explored. Indeed, the discontinuous controls are naturally exist and reasonable, especially in economic and engineering fields.

Therefore, the main goal of this chapter is to study a special type of NZSDG in Markovian framework. We show the existence of Nash equilibrium point which is discontinuous and of bang-bang type under natural conditions. The main tool is the notion of BSDEs which, in our case, are multidimensional with discontinuous generator with respect to the volatility process z.

We now briefly introduce the game model in two players and one dimensional case for simplicity. The general multiple players and high dimensional situation is a straightforward adaption. The dynamic of this game system is given by a stochastic differential equation (SDE for short) as follows, for any fixed $(t, x) \in [0, T] \times \mathbf{R}$,

$$\forall s \leq T,\ X_s^{t,x} = x + (B_{s\vee t} - B_t). \tag{3.12}$$

Each player has his own control. Let us denote by U and V two bounded subsets on $\mathbf{R}$ and $\mathcal{M}_1$ (resp.$\mathcal{M}_2$) be the set of admissible controls which is the set of $\mathcal{P}$-measurable process $u = (u_t)_{t\leq T}$ (resp. $v = (v_t)_{t\leq T}$) on $[0, T] \times \Omega$ with value on U (resp. V). $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$. Let $\Gamma : (t, x, u, v) \in [0, T] \times \mathbf{R} \times U \times V \to \mathbf{R}$ be the dynamic function for the game problem. The precise assumptions on the coefficients are stated below.

For any admissible pair of controls $(u., v.) \in \mathcal{M}$, let $\mathbf{P}^{u,v}$ be the positive measure on $(\Omega, \mathcal{F})$ as follows, $d\mathbf{P}_{t,x}^{u,v} = \zeta_T(\Gamma(., X.^{t,x}, u., v.))d\mathbf{P}$ with $\zeta_t(\Theta) := 1 + \int_0^t \Theta_s \zeta_s dB_s,\ t \leq T$ for any measurable $\mathcal{F}_t$-adapted process $\Theta := (\Theta_t)_{t\leq T}$. Under appropriate assumptions on Γ, it follows that $\mathbf{P}_{t,x}^{u,v}$ is a probability on

$(\Omega, \mathcal{F})$. Then, the process $B^{u,v} = (B_s - \int_0^s \Gamma(r, X_r^{t,x}, u_r, v_r)dr)_{s\le T}$ is a $(\mathcal{F}_s, \mathbf{P}^{u,v})$-Brownian motion and $(X_s^{t,x})_{s\le T}$ satisfies the following SDE,

$$dX_s^{t,x} = \Gamma(s, X_s^{t,x}, u_s, v_s)ds + dB_s^{u,v},\ \forall s \in [t,T] \text{ and } X_s^{t,x} = x,\ s \in [0,t]. \tag{3.13}$$

We denote the terminal payoff function by $g_i : x \in \mathbf{R} \to \mathbf{R}$ for player $i = 1, 2$

For fixed $(0, x)$, Let us define the payoffs for players as following, for $(u, v) \in \mathcal{M}$,

$$J_1(u, v) := \mathbf{E}^{u,v}[g_1(X_T^{0,x})] \text{ and } J_2(u, v) := \mathbf{E}^{u,v}[g_2(X_T^{0,x})],$$

where $\mathbf{E}^{u,v}$ is the expectation under probability $\mathbf{P}^{u,v}$ for fixed $(0, x)$. We concern about the existence of Nash equilibrium point, i.e. a couple of controls $(u^*, v^*) \in \mathcal{M}$ satisfying, for all $(u, v) \in \mathcal{M}$,

$$J_1(u^*, v^*) \ge J_1(u, v^*) \text{ and } J_2(u^*, v^*) \ge J_2(u^*, v).$$

Our assumptions are listed below:

Assumption 3.3.2. *(i) The value sets for the admissible controls (u, v) are two bounded subsets U and V on **R** associate $U = [0, 1]$ and $V = [-1, 1]$;*

*(ii) The dynamic function Γ is an affine combination of controls which has form $\Gamma(t, x, u, v) = f(t, x) + u + v$ where $f : (t, x) \in [0, T] \times$ **R** $\to$ **R** be a Borelian function. The function f is of linear growth w.r.t. x, Therefore, Γ is also of linear growth on x uniformly w.r.t $(u, v) \in U \times V$.*

(iii) The terminal values $g_i,\ i = 1, 2$ are of polynomial growth on x.

There are several properties in the setting of this NZSDG that we would like to emphasis: (i) The dynamic function Γ is not bounded as the previous results related to NZSDG (see [58, 54, 53, 75]), instead, it is of linear growth with respect to x. As illustrated in Subsection 3.3.1 and 3.3.2, the difficulty brought by the linear growth property of Γ can be overcome by a result of Haussmann (see Lemma 6.1.1) which related to the integrability of Doléans-Dade exponential. Indeed, the Girsanov's transformation can be carried out smoothly which yields the weak formulation of the state process (3.13). (ii) The value sets U and V for the admissible controls are two specific bounded subsets on **R**. Besides, the dynamic Γ is also of specific affine form. In additional, there are only terminal values getting involved in the payoffs J_1 and J_2, however, without the instantaneous payoffs. Therefore, Nash equilibrium point, if exists, should be in general bang-bang type.

The notion of bang-bang type control comes from the classical stochastic control theory. Consider in our case, when Γ does not depend on v, the stochastic differential game problem will be reduced to a stochastic control problem. By bang-bang control, we refer to a discontinuous control which will jump at a certain point between the border of the domain depending on the sign of the gradient of the value function. A common form is expressed by Heaviside function as presented in our work. Let us explain in the following, the exact form of bang-bang type candidate Nash equilibrium point for this NZSDG.

Since the dynamic coefficient Γ is known as a specific function of u and v clearly, therefore, the candidate optimal control for us can be worked out directly by the generalized Isaacs' condition. Let H_1 and H_2 be the Hamiltonian functions of this game, i.e, the functions which are not depend on ω defined from $[0, T] \times \mathbf{R} \times \mathbf{R} \times U \times V$ into **R** by:

$$\begin{aligned} H_1(t, x, p, u, v) &:= p\Gamma(t, x, u, v) = p(f(t, x) + u + v); \\ H_2(t, x, q, u, v) &:= q\Gamma(t, x, u, v) = q(f(t, x) + u + v). \end{aligned}$$

Now, the controls $\bar{u}$ and $\bar{v}$ which defined on $\mathbf{R} \times U$ and $\mathbf{R} \times V$, valued on U and V respectively, as follows: $\forall p, q \in \mathbf{R},\ \epsilon \in U,\ \varepsilon \in V$,

$$\bar{u}(p,\epsilon) = \begin{cases} 1,\ p > 0, \\ \epsilon,\ p = 0, \\ 0,\ p < 0, \end{cases} \quad \text{and} \quad \bar{v}(q,\varepsilon) = \begin{cases} 1,\ q > 0, \\ \varepsilon,\ q = 0, \\ -1,\ q < 0 \end{cases} \tag{3.14}$$

will exactly satisfy the generalized Isaacs' condition as follows: For all $(t,x,p,q,u,v) \in [0,T] \times \mathbf{R} \times \mathbf{R} \times U \times V$ and $(\epsilon, \varepsilon) \in U \times V$, we have,

$$\begin{aligned} H_1^*(t,x,p,q,\varepsilon) &:= H_1(t,x,p,\bar{u}(p,\epsilon),\bar{v}(q,\varepsilon)) \geq H_1(t,x,p,u,\bar{v}(q,\varepsilon)), \\ H_2^*(t,x,p,q,\epsilon) &:= H_2(t,x,q,\bar{u}(p,\epsilon),\bar{v}(q,\varepsilon)) \geq H_2(t,x,q,\bar{u}(p,\epsilon),v). \end{aligned} \tag{3.15}$$

We should point out that, the function H_1^* (resp.H_2^*) does not depend on ϵ (resp.ε), since, $p\bar{u}(p,\epsilon) = p \vee 0$ (resp. $q\bar{v}(q,\varepsilon) = |q|$) does not depend on ϵ (resp. ε). Besides, the Hamiltonian function here is discontinuous w.r.t. (p,q). It follows from (3.14) that, the pair of control $(\bar{u},\bar{v})$ is of bang-bang type. However, it is not a feedback one since it is also depend on some constants. Similar nonzero-sum differential game of unsmooth type has been studied by G.J. Olsder [83] in the deterministic case. Recent works on this subject, also in deterministic case, include papers by P. Cardaliaguet and S. Plaskacz [24], P. Cardaliaguet [23] which show that there exists a unique Nash equilibrium payoff of feedback form. But this equilibrium payoff depend in a very unstable way on the terminal data. Besides, it is not obvious to generalize the result in [24] to higher dimensions. The stochastic case has been analyzed by P. Mannucci [79] with the help of a system of Hamilton-Jacobi equations and related parabolic PDE techniques. Notice that the state process in [79] belongs to a bounded domain. However, some techniques of PDE in the global domain are not so straightforward.

The main novelty of Chapter 6 is that we show the existence of Nash equilibrium point of bang-bang type to a nonzero-sum stochastic differential game in a global domain. Moreover, the results and the techniques can be generalized to the multiple dimensions directly. However, the existence of NEP of feedback form is still an open problem.

As in [58], we apply the BSDE approach. This game problem finally reduces to solving a multiple-dimensional BSDE with a discontinuous generator with respect to z component and of linear growth in z ω by ω. Under the generalized Isaacs' hypothesis, we show that the associated BSDE has a solution which then provides a bang-bang type NEP for the NZSDG. The main result is summarized as follows:

Theorem 3.3.1 (Existence of bang-bang type NEP)**.** *Let us suppose Assumption 3.3.2 and generalized Isaacs' condition* (3.15) *are fulfilled. Then, there exists* η^1, η^2, (Y^1,Z^1), (Y^2,Z^2) *and* θ, ϑ *such that:*

(i) η^1 *and* η^2 *are two deterministic measurable functions with polynomial growth from* $[0,T] \times \boldsymbol{R}$ *to* $\boldsymbol{R}$*;*

(ii) (Y^1,Z^1) *and* (Y^2,Z^2) *are two couples of* $\mathcal{P}$*-measurable processes with values on* $\boldsymbol{R}^{1+1}$*;*

(iii) θ *(resp.* ϑ*) is a* $\mathcal{P}$*-measurable process valued on* U *(resp.* V*),*

and satisfy:

(a) $\boldsymbol{P}$*-a.s.,* $\forall s \leq T$, $Y_s^i = \eta^i(s, X_s^{0,x})$ *and* $Z^i(\omega) := (Z_s^i(\omega))_{s \leq T}$ *is ds-square integrable;*

(b) For all $s \leq T$,

$$\begin{cases} -dY_s^1 = H_1^*(s, X_s^{0,x}, Z_s^1, Z_s^2, \vartheta_s)ds - Z_s^1 dB_s,\ Y_T^1 = g_1(X_T^{0,x}); \\ -dY_s^2 = H_2^*(s, X_s^{0,x}, Z_s^1, Z_s^2, \theta_s)ds - Z_s^2 dB_s,\ Y_T^2 = g_2(X_T^{0,x}). \end{cases} \tag{3.16}$$

Besides, $Y_0^i = J_i(\bar{u},\bar{v})$, $i = 1,2$ *and the pair of controls* $(\bar{u}(Z_s^1,\theta_s), \bar{v}(Z_s^2,\vartheta_s))_{s \leq T}$ *is a bang-bang type Nash equilibrium point of the nonzero-sum stochastic differential game.*

The link between NZSDG and BSDEs is obtained in a standard way as Subsection 3.3.1 which tells us that the initial value of the associated BSDE is coincide with the payoff for the game problem. In additional, once we show that BSDE (3.16) has solutions with appropriate properties, then, the existence of NEP naturally holds true which obtained by comparing the solutions of BSDEs after change probability and in using the fact that $(\bar{u}, \bar{v})$ verifies the generalized Isaacs' condition (3.15). Therefore, the main task of this work is thus dedicated to the proof of the solvability of the system of BSDE (3.16) which is multiple dimensional with a discontinuous generator with respect to the volatility z. Apparently, the discontinuity is a main difficulty of this work.

The way we solve BSDE (3.16) is the following: (i) We first construct an approximation scheme by smoothing the discontinuous functions $(\bar{u}, \bar{v})$ via Lipschitz continuous applications $(\bar{u}^n, \bar{u}^n)$, $n \geq 1$. The sequence of BSDEs is well-posed since the generators are uniformly Lipschitz which followed by the existence of solutions. Besides, the solutions (Y^n, Z^n) can be expressed by deterministic functions (η^n, ς^n). (ii) Sharp estimates on those solutions are provided in appropriated spaces, as well as the polynomial growth property of functions η^n. (iii) Sequence η^n is proved as a Cauchy sequence following from a result of weak convergence. This yields the strong convergence of (Y^n, Z^n) in some proper spaces. (iv) In order to verify that the limit process which comes from the strong convergence result is indeed the solution of the original BSDE, it remains to show that the sequence of the approximate drivers converge. We finally obtain a weak convergence of the subsequence of the drivers toward Hamiltonian when the arbitrary constants (ϵ, ε) are replaced by some process (θ, ϑ). The difficulty of the discontinuity for Hamiltonian is overcome in this step by the weak convergence arguments.

CHAPTER **4**

Nonzero-sum Stochastic Differential Games with Unbounded Coefficients

This chapter is a published joint work with Hamadène (ref.[63]).

In this work, we analyze a nonzero-sum stochastic differential game (NZSDG for short) which is described as follows. Assume one has N players $\pi_1, ..., \pi_N$ which intervene on (or control) a system. Each one with the help of an admissible control which is an adapted stochastic process $u^i := (u^i_t)_{t\leq T}$ for π_i, $i = 1, \ldots, N$. When the N players make use of a strategy $u := (u^1, ..., u^N)$, the dynamics of the controlled system is a process $(x^u_t)_{t\leq T}$ solution of the following standard stochastic differential equation (SDE for short):

$$dx^u_t = f(t, x^u_t, u^1_t, ..., u^N_t)dt + \sigma(t, x^u_t)dB_t \text{ for } t \leq T \text{ and } x_0 = x\,; \tag{4.1}$$

$B := (B_t)_{t\leq T}$ is a Brownian motion. The control actions are not free and generate for each player π_i, $i = 1, ..., N$, a payoff which amounts to

$$J_i(u^1, ..., u^N) := \mathbf{E}[\ g^i(x^u_T) + \textstyle\int_0^T h_i(s, x^u_s, u_s)ds\ ].$$

A Nash equilibrium point (NEP for short) for the players is a strategy $u^* := (u^{1,*}, ..., u^{N,*})$ of control of the system which has the feature that each player π_i who takes unilaterally the decision to deviate from $u^{i,*}$, is penalized: For all $i = 1, ..., N$, for all control u^i of player π_i,

$$J_i(u^*) \leq J_i([u^{*,-i}|u^i])$$

where $[u^{*,-i}|u^i] := (u^{1,*}, \ldots, u^{i-1,*}, u^i, u^{i+1,*}, \ldots, u^{N,*})$.

In the case when $N = 2$ and $J_1 + J_2 = 0$, this game reduces to the well known zero-sum differential game which is well documented in several works and from several points of view (see e.g. [11], [22], [41], [40], [45], [47], [60], [57], [65] etc. and the references inside).

Comparatively, the nonzero-sum differential game is so far less considered even though there are some works on the subject, including [21], [48], [58], [59], [54], [53], [73], [75], [79], [91], etc.). In these works, the objectives are various and so are the approaches, usually based on partial differential equations (PDEs) ([48, 79]) or backward SDEs ([58, 54, 53, 75, 73]). On the other hand, it should be pointed out that the frameworks in those papers are not the same. Some of them consider strategies as control actions for the players (e.g. [21], [75], [91]) while others deal with the control against control setting (e.g. [59], [58, 54, 53]). The first ones, formulated usually in the framework of two players, allow to study the case where the diffusion coefficient σ is controlled. In the latter ones, σ does not depend on the controls. However those papers do not reach the same objective. Note that for the control against control zero-sum game, Pham and Zhang [88] and M.Sirbu [92] have overcome this restriction related to the independence of σ on the controls.

In the present article, we study the nonzero-sum game of type control against control with the diffusion process σ independent of controls, in the same line as in the paper by Hamadène et al. [58]. But in [58], the setting concerns only the case when the coefficients f and σ of the diffusion (4.1) are bounded. According to our knowledge the setting where those coefficients are not bounded and of linear growth is not considered yet. Therefore the main objective of this work is to relax as much as possible the boundedness of the coefficients f (mainly) and σ(which is not bounded as stated in the final extension). The novelty of the paper is that we show the existence of a Nash equilibrium point for the NZSDG when f is no longer bounded but only satisfies the linear growth condition. As in [58] our approach is based on backward SDEs and basically the problem turns into studying its associated multi-dimensional BSDE which is of linear growth ω by ω. Under the generalized Isaacs hypothesis and the domination condition of laws of solutions of (4.1), which is satisfied when the uniform ellipticity condition on σ is satisfied, we show that the latter BSDE has a solution which then provides a NEP for the NZSDG.

The paper is organized as follows:

In Section 2 we fix the setting of the problem and recall some results which play an important role in our study. The formulation we adopt is of weak type. On the other hand for the sake of simplicity we have made the presentation for $N = 2$. The generalization to the situation where $N \geq 3$ is formal and can be carried out in the same spirit. Section 3 is devoted to the link between the game and BSDEs. We first express the payoffs of the game in using solutions of BSDEs whose integrability is not standard. Then we show that the existence of a NEP for the game turns into the existence of a solution of a specific BSDE which is of multi-dimensional type and linear growth ω by ω. It plays a role of a verification theorem for the NZSDG. In Section 4 we show that this specific BSDE has a solution when the generalized Isaacs condition is fulfilled and the laws of the dynamics of the non-controlled system satisfy the so-called L^q-domination condition. This latter is especially satisfied when the diffusion coefficient σ satisfies the well known uniform ellipticity condition. Our method is based on: (i) the introduction of an approximating scheme of BSDEs which is well-posed since the coefficients are Lipschitz. In this markovian framework of randomness, the solutions (Y^n, Z^n), $n \geq 1$, of this scheme can be represented via deterministic functions (ϖ^n, υ^n), $n \geq 1$, and the Markov process as well ; (ii) sharp estimates for (Y^n, Z^n) and (ϖ^n, υ^n) and the L^q-domination condition enable us to obtain the strong convergence of a subsequence $(\varpi_{n_k})_{k\geq 1}$ from a weak convergence in an appropriate space. This yields the strong convergence of the corresponding subsequences $(Y^{n_k})_{k\geq 1}$ and $(Z^{n_k})_{k\geq 1}$; (iii) we finally show that the limit of $(Y^{n_k}, Z^{n_k})_{k\geq 1}$ is a solution for the BSDE associated with the NZSDG. At the end of this section we provide an example which illustrates our result. We also discuss possible extensions of our findings to the case when both the drift f and diffusion coefficient σ of (4.1) are not bounded. □

4.1 Setting of the problem

Let $T > 0$ and let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space on which is defined an m-dimensional Brownian motion $B := (B_t)_{0\leq t\leq T}$. For $t \leq T$, let us denote by $(F_t := \sigma(B_u, u \leq t))_{t\leq T}$ the natural filtration of B and $(\mathcal{F}_t)_{t\leq T}$ the completion of $(F_t)_{t\leq T}$ with the **P**-null sets of $\mathcal{F}$, which then satisfies the usual conditions. Let $\mathcal{P}$ be the σ-algebra on $[0, T] \times \Omega$ of $\mathcal{F}_t$-progressively measurable sets. Let $p \geq 1$ be a real constant and $t \in [0, T]$ fixed. We then define the following spaces:

- $L^p_T(\mathbf{R}^m) = \{\xi : \mathcal{F}_T$-measurable and $\mathbf{R}^m$ -valued random variable s.t. $\mathbf{E}[|\xi|^p] < \infty\}$;

- $\mathcal{S}^p_{t,T}(\mathbf{R}^m) = \{\varphi = (\varphi_s)_{t\le s\le T} : \mathcal{P}$ -measurable and $\mathbf{R}^m$ -valued s.t. $\mathbf{E}[sup_{t\le s\le T}|\varphi_s|^p] < \infty\}$; $\mathcal{S}^p_{0,T}(\mathbf{R}^m)$ is simply denoted by $\mathcal{S}^p_T(\mathbf{R}^m)$;
- $\mathcal{H}^p_{t,T}(\mathbf{R}^m) = \{Z = (Z_s)_{t\le s\le T} : \mathcal{P}$ -measurable and $\mathbf{R}^m$ -valued s.t. $\mathbf{E}[(\int_t^T |Z_s|^2 ds)^{p/2}] < \infty\}$; $\mathcal{H}^p_{0,T}(\mathbf{R}^m)$ is simply denoted by $\mathcal{H}^p_T(\mathbf{R}^m)$.

Next let σ be a matrix function defined as:

$$\begin{aligned} \sigma : [0,T]\times\mathbf{R}^m &\longrightarrow \mathbf{R}^{m\times m} \\ (t,x) &\longmapsto \sigma(t,x) \end{aligned}$$

and which satisfies the following assumptions:

Assumptions (A1)

(i) σ is uniformly Lipschitz w.r.t x. *i.e.* there exists a constant C_1 such that,

$$\forall t\in[0,T], \forall\, x,x'\in\mathbf{R}^m, \quad |\sigma(t,x)-\sigma(t,x')|\le C_1|x-x'|.$$

(ii) σ is invertible and bounded and its inverse is bounded, *i.e.*, there exits a constant C_σ such that

$$\forall (t,x)\in[0,T]\times\mathbf{R}^m, \quad |\sigma(t,x)|+|\sigma^{-1}(t,x)|\le C_\sigma.$$

Remark 4.1.1. *Uniform ellipticity.*
Under (A1), there exists a real constant $\Upsilon>0$ such that for any $(t,x)\in[0,T]\times\mathbf{R}^m$,

$$\Upsilon.I\le\sigma(t,x).\sigma^\top(t,x)\le\Upsilon^{-1}.I \tag{4.2}$$

where I is the identity matrix of dimension m. □

Next let $(t,x)\in[0,T]\times\mathbf{R}^m$. Under the Assumptions (A1), (i)-(ii), we know that there exists a process $(X^{t,x}_s)_{s\le T}$ that satisfies the following stochastic differential equation (see e.g. Karatzas and Shreve, pp.289, 1991 [70]):

$$X^{t,x}_s = x+\int_t^s \sigma(r,X^{t,x}_r)dB_r, \quad \forall s\in[t,T] \text{ and } X^{t,x}_s = x \text{ for } s\in[0,t].$$

Hereafter for sake of simplicity we will deal with the setting of two players. However the generalization to the case of $N(\ge 3)$ players is formal and just a question of writing (see the comment of Remark 4.3.1). Also let us denote by U_1 and U_2 two compact metric spaces and let $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) be the set of $\mathcal{P}$-measurable processes $u=(u_t)_{t\le T}$ (resp. $v=(v_t)_{t\le T}$) with values in U_1 (resp. U_2). We denote by $\mathcal{M}$ the set $\mathcal{M}_1\times\mathcal{M}_2$ and call it the set of admissible controls for the players.

Let f be a Borelian function from $[0,T]\times\mathbf{R}^m\times U_1\times U_2$ into $\mathbf{R}^m$ and for $i=1,2$ let h_i and g^i be Borelian functions from $[0,T]\times\mathbf{R}^m\times U_1\times U_2$ (resp. $\mathbf{R}^m$) into $\mathbf{R}$ which satisfy:

Assumptions (A2)

(i) f is of linear growth w.r.t x, *i.e.* there exists a constant C_f such that $|f(t,x,u,v)|\le C_f(1+|x|)$, for any $(t,x,u,v)\in[0,T]\times\mathbf{R}^m\times U_1\times U_2$.

(ii) for $i=1,2$ h_i is of polynomial growth w.r.t x, *i.e.*, there exists a constant C_h and $\gamma\ge 0$ such that $|h_i(t,x,u,v)|\le C_h(1+|x|^\gamma)$ for any $(t,x,u,v)\in[0,T]\times\mathbf{R}^m\times U_1\times U_2$.

(iii) for $i = 1, 2$, g^i is of polynomial growth with respect to x, *i.e.* there exist constants C_g and $\gamma \geq 0$ such that $|g^i(x)| \leq C_g(1 + |x|^\gamma), \forall x \in \mathbf{R}^m$.□

For $(u, v) \in \mathcal{M}$, let $\mathbf{P}^{(u,v)}_{(t,x)}$ be the measure on $(\Omega, \mathcal{F})$ whose density function is defined as follows:

$$\frac{d\mathbf{P}^{(u,v)}_{(t,x)}}{d\mathbf{P}} = \zeta_T \left(\sigma^{-1}(\cdot, X^{t,x}_{\cdot})f(\cdot, X^{t,x}_{\cdot}, u_{\cdot}, v_{\cdot})\right), \tag{4.3}$$

where for any measurable $\mathcal{F}_t$-adapted process $\eta := (\eta_s)_{s\leq T}$ we define,

$$\zeta_s(\eta) := e^{\int_0^s \eta_r dB_r - \frac{1}{2}\int_0^s |\eta_r|^2 dr}, \ \forall s \leq T. \tag{4.4}$$

Thanks to the Assumptions (A1) and (A2)-(i) on σ and f, we can infer that $\mathbf{P}^{(u,v)}_{(t,x)}$ is a probability on $(\Omega, \mathcal{F})$ (see Appendix A of N. El-Karoui and S. Hamadène [36] or Karatzas-Shreve [70], pp.200). Then by Girsanov's theorem (Girsanov, [50]), the process $B^{(u,v)} := (B_s - \int_0^s \sigma^{-1}(r, X^{t,x}_r)f(r, X^{t,x}_r, u_r, v_r)dr)_{s\leq T}$ is a $(\mathcal{F}_s, \mathbf{P}^{(u,v)}_{(t,x)})$-Brownian motion and $(X^{t,x}_s)_{s\leq T}$ satisfies the following stochastic differential equation,

$$dX^{t,x}_s = f(s, X^{t,x}_s, u_s, v_s)ds + \sigma(s, X^{t,x}_s)dB^{(u,v)}_s, \ \text{for } s \in [t, T] \text{ and } X^{t,x}_s = x \text{ for } s < t. \tag{4.5}$$

In general, the process $(X^{t,x}_s)_{s\leq T}$ is not adapted with respect to the filtration generated by the Brownian motion $(B^{(u,v)}_s)_{s\leq T}$, therefore $(X^{t,x}_s)_{s\leq T}$ is called a weak solution for the SDE (4.5).

Next let us fix (t, x) to $(0, x_0)$ and for $i = 1, 2$, let us define the payoffs of the players by:

$$J^i(u, v) = \mathbf{E}^{(u,v)}_{(0,x_0)}[\int_0^T h_i(s, X^{0,x_0}_s, u_s, v_s)ds + g^i(X^{0,x_0}_T)], \tag{4.6}$$

where $\mathbf{E}^{(u,v)}_{(0,x_0)}(\cdot)$ is the expectation under the probability $\mathbf{P}^{(u,v)}_{(0,x_0)}$. Hereafter $\mathbf{E}^{(u,v)}_{(0,x_0)}(\cdot)$ (resp. $\mathbf{P}^{(u,v)}_{(0,x_0)}$) will be simply denoted by $\mathbf{E}^{(u,v)}(.)$ (resp. $\mathbf{P}^{(u,v)}$).

Our problem is to find an admissible control (u^*, v^*) such that

$$J^1(u^*, v^*) \leq J^1(u, v^*) \text{ and } J^2(u^*, v^*) \leq J^2(u^*, v) \ \text{ for any } (u, v) \in \mathcal{M}.$$

The control (u^*, v^*) is called a *Nash equilibrium point* for the nonzero-sum stochastic differential game.

Next we define the Hamiltonian functions H_i, $i = 1, 2$, of the game from $[0, T] \times \mathbf{R}^{2m} \times U_1 \times U_2$ into $\mathbf{R}$ by:

$$H_i(t, x, p, u, v) = p\sigma^{-1}(t, x)f(t, x, u, v) + h_i(t, x, u, v),$$

and we introduce the following assumption (A3) called the generalized *Isaacs condition*.

Assumption (A3): Generalized Isaacs condition.

(i) There exist two Borelian applications u_1^*, u_2^* defined on $[0, T] \times \mathbf{R}^{3m}$, with values in U_1 and U_2, respectively, such that for any $(t, x, p, q, u, v) \in [0, T] \times \mathbf{R}^{3m} \times U_1 \times U_2$, we have:

$$H_1^*(t, x, p, q) = H_1(t, x, p, u_1^*(t, x, p, q), u_2^*(t, x, p, q)) \leq H_1(t, x, p, u, u_2^*(t, x, p, q))$$

and

$$H_2^*(t, x, p, q) = H_2(t, x, q, u_1^*(t, x, p, q), u_2^*(t, x, p, q)) \leq H_2(t, x, q, u_1^*(t, x, p, q), v).$$

(ii) the mapping $(p, q) \in \mathbf{R}^{2m} \longmapsto (H_1^*, H_2^*)(t, x, p, q) \in \mathbf{R}$ is continuous for any fixed (t, x). □

Remark 4.1.2. *This condition has been already considered by A. Friedman in [47] for the same purpose as ours in this paper. But the treatment of the problem he used is the PDE approach.* □

In order to show that the game has a Nash equilibrium point, it is enough to show that its associated BSDE, which is multi-dimensional and with a continuous generator (see Theorem 4.2.1 below) has a solution. Therefore the main objective of the next section is to study the connection between NZSDGs and BSDEs.

4.2 Nonzero-sum differential game problem and BSDEs.

Let $(t,x) \in [0,T] \times \mathbf{R}^m$ and $(\theta_s^{t,x})_{s\le T}$ be the solution of the following forward stochastic differential equation:

$$\begin{cases} d\theta_s = b(s,\theta_s)ds + \sigma(s,\theta_s)dB_s, & s \in [t,T]; \\ \theta_s = x, & s \in [0,t] \end{cases} \tag{4.7}$$

where $\sigma : [0,T] \times \mathbf{R}^m \to \mathbf{R}^{m\times m}$ satisfies the Assumptions (A1) and b: $[0,T] \times \mathbf{R}^m \to \mathbf{R}^m$ is a measurable function which satisfies the following assumption:

Assumption (A4): The function b is uniformly Lipschitz w.r.t x and of linear growth, *i.e.*, there exist constants C_2 and C_b such that:

$$\forall t \in [0,T],\ \forall\, x,x' \in \mathbf{R}^m,\ |b(t,x) - b(t,x')| \le C_2|x - x'| \text{ and } |b(t,x)| \le C_b(1 + |x|).$$

It is well-known that, under (A1) and (A4), the stochastic process $(\theta_s^{t,x})_{s\le T}$ satisfies the following estimate, see for example (Karatzas, I. 1991 [70] pp.306):

$$\forall\, q \in [1,\infty), \qquad \mathbf{E}\Big[\big(\sup_{s\le T} |\theta_s^{t,x}|\big)^{2q}\Big] \le C(1 + |x|^{2q}). \tag{4.8}$$

As a particular case we have a similar estimate for the process $X^{t,x}$, i.e.,

$$\forall\, q \in [1,\infty), \qquad \mathbf{E}\Big[\big(\sup_{s\le T} |X_s^{t,x}|\big)^{2q}\Big] \le C(1 + |x|^{2q}). \tag{4.9}$$

Finally note that we have also a similar estimate for weak solutions of SDEs of types (4.5), i.e., if (u,v) belongs to $\mathcal{M}$ then

$$\forall\, q \in [1,\infty), \qquad \mathbf{E}_{(t,x)}^{(u,v)}\Big[\big(\sup_{s\le T} |X_s^{t,x}|\big)^{2q}\Big] \le C(1 + |x|^{2q}). \tag{4.10}$$

Next let us recall the following result by U.G.Haussmann [64] related to integrability of the exponential local martingale defined in (4.4).

Lemma 4.2.1. *Assume (A1)-(i),(ii) and (A4) and let $(\theta_s^{t,x})_{s\le T}$ be the solution of (4.7). Let φ be a $\mathcal{P} \otimes \mathcal{B}(\boldsymbol{R}^m)$-measurable application from $[0,T] \times \Omega \times \boldsymbol{R}^m$ to $\boldsymbol{R}^m$ which is uniformly of linear growth, that is, $\boldsymbol{P}$-a.s., $\forall (s,x) \in [0,T] \times \boldsymbol{R}^m$,*

$$|\varphi(s,\omega,x)| \le C_\varphi(1 + |x|).$$

Then, there exists $p \in (1,2)$ and a constant C, where p depends only on C_σ, C_b, C_φ, m while the constant C, depends only on m and p, but not on φ, such that:

$$\boldsymbol{E}\left[|\zeta_T(\varphi(s,\theta_s^{t,x}))|^p\right] \le C,$$

where the process $\zeta(\varphi(s,\theta_s^{t,x}))$ is the density function defined in (4.4).

Proof. See Appendix A, Theorem A.0.1. □

As a by-product we have:

Corollary 4.2.1. *Let (u,v) be an admissible control for the players and $(t,x) \in [0,T] \times \boldsymbol{R}^m$. Then there exists $p > 1$ such that*

$$\boldsymbol{E}\left[\left|\zeta_T(\sigma(s,X_s^{t,x})^{-1}f(s,X_s^{t,x},u_s,v_s))\right|^p\right] \leq C. \tag{4.11}$$

Next we give a preliminary result which characterizes $J(u,v)$ of (4.6) from its associated BSDE. This result generalizes the one by S. Hamadène, and J. P. Lepeltier, 1995b [57] (Theorem I.3). The main improvement is that the drift of the diffusion, weak solution of (4.5), is not bounded anymore but is instead of linear growth.

Proposition 4.2.1. *Assume that Assumptions (A1), and (A2) on f, h_i, g^i, $i = 1,2$, are fulfilled. Then for any pair $(u,v) \in \mathcal{M}$, there exists a pair of $\mathcal{P}$-measurable processes $(W^{i,(u,v)}, Z^{i,(u,v)})$, $i = 1,2$, with values in $\boldsymbol{R} \times \boldsymbol{R}^m$ such that:*

(i) For any $q \geq 1$ and $i = 1,2$, we have,

$$\boldsymbol{E}^{(u,v)}\Big[\sup_{0\leq s\leq T}|W_s^{i,(u,v)}|^q + \Big(\int_0^T |Z_s^{i,(u,v)}|^2 ds\Big)^{\frac{q}{2}}\Big] < \infty. \tag{4.12}$$

(ii) For $t \leq T$,

$$W_t^{i,(u,v)} = g^i(X_T^{0,x_0}) + \int_t^T H_i(s,X_s^{0,x_0},Z_s^{i,(u,v)},u_s,v_s)ds - \int_t^T Z_s^{i,(u,v)}dB_s. \tag{4.13}$$

The solution of BSDE (4.12)-(4.13) is unique, moreover, $W_0^{i,(u,v)} = J^i(u,v)$ for $i = 1,2$.

Proof. We will give the proof for $i = 1$ and of course it is similar for $i = 2$. So for $(u,v) \in \mathcal{M}$, the process $(X_s^{0,x_0})_{s\leq T}$ is a weak solution of the following stochastic differential equation:

$$dX_s^{0,x_0} = f(s,X_s^{0,x_0},u_s,v_s)ds + \sigma(s,X_s^{0,x_0})dB_s^{(u,v)}, \quad s \leq T \text{ and } X_0^{0,x_0} = x_0,$$

where $B^{(u,v)}$ is a Brownian motion under $\mathbf{P}^{(u,v)}$. Let us define the process $(W_t^{1,(u,v)})_{t\leq T}$ as follows:

$$\forall t \leq T,\ W_t^{1,(u,v)} \triangleq \mathbf{E}^{(u,v)}\Big[g^1(X_T^{0,x_0}) + \int_t^T h_1(s,X_s^{0,x_0},u_s,v_s)ds \mid \mathcal{F}_t\Big]. \tag{4.14}$$

Since the functions $g^1(x)$ and $h_1(s,x,u,v)$ are of polynomial growth w.r.t x, we have, for any $r \geq 1$,

$$\begin{aligned}
&\mathbf{E}^{(u,v)}\Big[|g^1(X_T^{0,x_0}) + \int_0^T |h_1(s,X_s^{0,x_0},u_s,v_s)|ds|^{2r}\Big] \\
&\leq \mathbf{E}^{(u,v)}\Big[C\big(1 + \sup_{s\leq T}|X_s^{0,x_0}|^{2\gamma r}\big)\Big] \\
&\leq C(1 + |x_0|^{2\gamma r}).
\end{aligned} \tag{4.15}$$

The last inequality is due to estimate (4.10). Then (4.15) implies that the process $(W_t^{1,(u,v)})_{t\leq T}$ is well defined.

Next for notation simplicity for any $t \leq T$, we denote by ζ_t the density function $\zeta_t\left(\sigma^{-1}(s, X_s^{0,x_0}) f(s, X_s^{0,x_0}, u_s, v_s)\right)$. Therefore,

$$\begin{aligned} W_t^{1,(u,v)} &= (\zeta_t)^{-1}\mathbf{E}\Big[\zeta_T \cdot \big(g^1(X_T^{0,x_0}) + \int_t^T h_1(s, X_s^{0,x_0}, u_s, v_s)ds\big) \mid \mathcal{F}_t\Big] \\ &= (\zeta_t)^{-1}\mathbf{E}\Big[\zeta_T \cdot \big(g^1(X_T^{0,x_0}) + \int_0^T h_1(s, X_s^{0,x_0}, u_s, v_s)ds\big) \mid \mathcal{F}_t\Big] \\ &\quad - \int_0^t h_1(s, X_s^{0,x_0}, u_s, v_s)ds. \end{aligned} \tag{4.16}$$

By Corollary 4.2.1, there exists some $1 < p_0 < 2$, such that $\zeta_T \in L^{p_0}(\mathbf{R})$. Therefore, from Young's inequality, we obtain, for any $1 < \bar{q} < p_0$,

$$\begin{aligned} &\mathbf{E}\Big[\Big|\zeta_T \cdot \big(g^1(X_T^{0,x_0}) + \int_0^T h_1(s, X_s^{0,x_0}, u_s, v_s)ds\big)\Big|^{\bar{q}}\Big] \\ &\leq \frac{\bar{q}}{p_0}\mathbf{E}\left[|\zeta_T|^{p_0}\right] + \frac{p_0 - \bar{q}}{p_0}\mathbf{E}\Big[|g^1(X_T^{0,x_0}) + \int_0^T h_1(s, X_s^{0,x_0}, u_s, v_s)ds|^{\bar{q}\cdot\frac{p_0}{p_0-\bar{q}}}\Big]. \end{aligned}$$

Since $\frac{p_0}{\bar{q}} < 2$, its conjugate $\frac{p_0}{p_0-\bar{q}} > 2$. Therefore, by the polynomial growth assumptions of g^1 and h_1 w.r.t x (Assumption (A2)-(ii),(iii)) and the estimate (4.9), we have

$$\mathbf{E}\Big[|g^1(X_T^{0,x_0}) + \int_0^T h_1(s, X_s^{0,x_0}, u_s, v_s)ds|^{\bar{q}\cdot\frac{p_0}{p_0-\bar{q}}}\Big] < \infty.$$

Then,

$$\zeta_T \cdot (g^1(X_T^{0,x_0}) + \int_0^T h_1(s, X_s^{0,x_0}, u_s, v_s)ds) \in L^{\bar{q}}.$$

Therefore thanks to the representation theorem, there exists a $\mathcal{P}$-measurable and $\mathbf{R}^m$-valued process, $(\bar{\theta}_s)_{s\leq T}$ which satisfies,

$$\mathbf{E}\Big[\big(\int_0^T \left|\bar{\theta}_s\right|^2 ds\big)^{\frac{\bar{q}}{2}}\Big] < \infty,$$

such that for any $t \leq T$,

$$\begin{aligned} W_t^{1,(u,v)} &= (\zeta_t)^{-1} \cdot \{\mathbf{E}\Big[\zeta_T \cdot \big(g^1(X_T^{0,x_0}) + \int_0^T h_1(s, X_s^{0,x_0}, u_s, v_s)ds\big)\Big] + \int_0^t \bar{\theta}_s dB_s\} \\ &\quad - \int_0^t h_1(s, X_s^{0,x_0}, u_s, v_s)ds \\ &\triangleq (\zeta_t)^{-1}R_t - \int_0^t h_1(s, X_s^{0,x_0}, u_s, v_s)ds, \end{aligned}$$

where

$$R_t \triangleq \mathbf{E}\Big[\zeta_T \cdot \big(g^1(X_T^{0,x_0}) + \int_0^T h_1(s, X_s^{0,x_0}, u_s, v_s)\big)\Big] + \int_0^t \bar{\theta}_s dB_s, \quad t \leq T.$$

But for any $s \leq T$,

$$d\zeta_s = \zeta_s \sigma^{-1}(s, X_s^{0,x_0}) f(s, X_s^{0,x_0}, u_s, v_s)dB_s.$$

Then by Itô's formula we have

$$\begin{aligned} d(\zeta_s)^{-1} = -(\zeta_s)^{-1} \cdot \Big\{&\sigma^{-1}\left(s, X_s^{0,x_0}\right) f\left(s, X_s^{0,x_0}, u_s, v_s\right) dB_s - \\ &|\sigma^{-1}\left(s, X_s^{0,x_0}\right) f\left(s, X_s^{0,x_0}, u_s, v_s\right)|^2 ds\Big\}, \quad s \leq T. \end{aligned}$$

Therefore, $\forall s \le T$,

$$\begin{aligned} dW_s^{1,(u,v)} &= -(\zeta_s)^{-1} \cdot \Big\{\sigma^{-1}(s, X_s^{0,x_0}) f(s, X_s^{0,x_0}, u_s, v_s) dB_s \\ &\quad - |\sigma^{-1}(s, X_s^{0,x_0}) f(s, X_s^{0,x_0}, u_s, v_s)|^2 ds\Big\} R_s \\ &\quad + (\zeta_s)^{-1} \bar{\theta}_s dB_s + \big(-(\zeta_s)^{-1}\sigma^{-1}(s, X_s^{0,x_0}) f(s, X_s^{0,x_0}, u_s, v_s)\big)\bar{\theta}_s ds \\ &\quad - h_1(s, X_s^{0,x_0}, u_s, v_s) ds. \end{aligned}$$

Next let us define

$$Z_s^{1,(u,v)} \triangleq -(\zeta_s)^{-1}\Big\{R_s \sigma^{-1}(s, X_s^{0,x_0}) f(s, X_s^{0,x_0}, u_s, v_s) - \bar{\theta}_s\Big\},\ s \le T. \tag{4.17}$$

Then it is easy to check that the pair of processes $(W_s^{1,(u,v)}, Z_s^{1,(u,v)})_{s\le T}$ of (4.14)-(4.17) satisfies the backward equation (4.13).

We now focus on the estimates for the processes $(W_s^{1,(u,v)}, Z_s^{1,(u,v)})_{s\le T}$. From (4.14), for any $s \le T$ and $q > 1$,

$$\sup_{0\le t\le T} |W_t^{1,(u,v)}|^q = \sup_{0\le t\le T} |\mathbf{E}^{(u,v)}\Big[g^1(X_T^{0,x_0}) + \int_t^T h_1\left(s, X_s^{0,x_0}, u_s, v_s\right) ds \Big| \mathcal{F}_t\Big]|^q.$$

Then by conditional Jensen's inequality we have,

$$\sup_{0\le t\le T} |W_t^{1,(u,v)}|^q \le \mathbf{E}^{(u,v)}\Big[\sup_{0\le t\le T} |g^1(X_T^{0,x_0}) + \int_t^T h_1\left(s, X_s^{0,x_0}, u_s, v_s\right) ds|^q \Big| \mathcal{F}_t\Big].$$

Therefore, since g^1 and h_1 are of polynomial growth, we have

$$\begin{aligned} \mathbf{E}^{(u,v)}\Big[\sup_{0\le t\le T} |W_t^{1,(u,v)}|^q\Big] &\le \mathbf{E}^{(u,v)}\Big[\sup_{0\le t\le T} |g^1(X_T^{0,x_0}) + \int_t^T h_1\left(s, X_s^{0,x_0}, u_s, v_s\right) ds|^q\Big] \\ &\le C\mathbf{E}^{(u,v)}\Big[\sup_{0\le t\le T}(1 + |X_t^{0,x_0}|^{\gamma q})\Big] \le C(1 + |x_0|^{\gamma q}) < \infty. \end{aligned} \tag{4.18}$$

Next for each integer k, let us define:

$$\tau_k = \inf\{s \ge 0, \int_0^s |Z_s^{1,(u,v)}|^2 ds \ge k\} \wedge T.$$

The sequence $(\tau_k)_k \ge 0$ is of stationary type and converges to T. By using Itô's formula with $(W_{t\wedge\tau_k}^{1,(u,v)})^2$ we obtain: $\forall t \le T$,

$$\begin{aligned} &|W_{t\wedge\tau_k}^{1,(u,v)}|^2 + \int_{t\wedge\tau_k}^{\tau_k} |Z_s^{1,(u,v)}|^2 ds \\ &= |W_{\tau_k}^{1,(u,v)}|^2 + 2\int_{t\wedge\tau_k}^{\tau_k} W_s^{1,(u,v)} h_1(s, X_s^{0,x_0}, u_s, v_s) ds - 2\int_{t\wedge\tau_k}^{\tau_k} W_s^{1,(u,v)} Z_s^{1,(u,v)} dB_s^{(u,v)}. \end{aligned}$$

Thus, for $q > 1$, taking the expectation of the power $\frac{q}{2}$ of the above equation on both sides and applying Young's inequality, we see that there exists a constant $\underline{C}$ such that,

$$\begin{aligned} \mathbf{E}^{(u,v)}\Big[\big(\int_0^{\tau_k} |Z_s^{1,(u,v)}|^2 ds\big)^{\frac{q}{2}}\Big] \quad \le \quad & \underline{C}\Big\{\mathbf{E}^{(u,v)}\Big[|W_{\tau_k}^{1,(u,v)}|^q\Big] + \mathbf{E}^{(u,v)}\Big[\big(\int_0^{\tau_k} |W_s^{1,(u,v)}|^2 ds\big)^{\frac{q}{2}}\Big] \\ & + \mathbf{E}^{(u,v)}\Big[\big(\int_0^{\tau_k} |h_1(s, X_s^{0,x_0}, u_s, v_s)|^2 ds\big)^{\frac{q}{2}}\Big] \\ & + \mathbf{E}^{(u,v)}\Big[\big|\int_0^{\tau_k} W_s^{1,(u,v)} Z_s^{1,(u,v)} dB_s^{(u,v)}\big|^{\frac{q}{2}}\Big]\Big\}. \end{aligned} \tag{4.19}$$

Next taking into account the Assumptions (A2)-(ii) and estimate (4.18), one can show that,

$$\begin{aligned}
&\mathbf{E}^{(u,v)}\Big[|W^{1,(u,v)}_{\tau_k}|^q\Big]+\mathbf{E}^{(u,v)}\Big[\big(\int_0^{\tau_k}|W^{1,(u,v)}_s|^2ds\big)^{\frac{q}{2}}\Big]\\
&\qquad\qquad\qquad\qquad+\mathbf{E}^{(u,v)}\Big[\big(\int_0^{\tau_k}|h_1(s,X^{0,x_0}_s,u_s,v_s)|^2ds\big)^{\frac{q}{2}}\Big]\\
&\le \bar{C}\Big\{\mathbf{E}^{(u,v)}\Big[\sup_{0\le s\le\tau_k}|W^{1,(u,v)}_s|^q\Big]+\mathbf{E}^{(u,v)}\Big[\sup_{0\le s\le\tau_k}\big(1+|X^{0,x_0}_s|^{2\gamma}\big)^{\frac{q}{2}}\Big]\Big\}\\
&\le \bar{C}\Big\{\mathbf{E}^{(u,v)}\Big[\sup_{0\le s\le T}|W^{1,(u,v)}_s|^q\Big]+\{\mathbf{E}^{(u,v)}\Big[\sup_{0\le s\le T}\big(1+|X^{0,x_0}_s|^{2\gamma q}\big)\Big]\}^{\frac{1}{2}}\Big\}<\infty.
\end{aligned}$$

Meanwhile, it follows from the Burkholder-Davis-Gundy (BDG for short) that there exists a constant C_q, depending on q, such that

$$\begin{aligned}
&\mathbf{E}^{(u,v)}\Big[\big|\int_0^{\tau_k}W^{1,(u,v)}_sZ^{1,(u,v)}_sdB^{(u,v)}_s\big|^{\frac{q}{2}}\Big]\\
&\quad\le C_q\mathbf{E}^{(u,v)}\Big[\big(\int_0^{\tau_k}|W^{1,(u,v)}_s|^2|Z^{1,(u,v)}_s|^2ds\big)^{\frac{q}{4}}\Big]\\
&\quad\le C_q\mathbf{E}^{(u,v)}\Big[\big(\sup_{0\le s\le\tau_k}|W^{1,(u,v)}_s|\big)^{\frac{q}{2}}\big(\int_0^{\tau_k}|Z^{1,(u,v)}_s|^2ds\big)^{\frac{q}{4}}\Big]\\
&\quad\le \frac{C_q^2\underline{\mathbf{C}}}{2}\mathbf{E}^{(u,v)}\Big[\big(\sup_{0\le s\le T}|W^{1,(u,v)}_s|\big)^q\Big]+\frac{1}{2\underline{\mathbf{C}}}\mathbf{E}^{(u,v)}\Big[\big(\int_0^{\tau_k}|Z^{1,(u,v)}_s|^2ds\big)^{\frac{q}{2}}\Big],
\end{aligned}$$

where $\underline{\mathbf{C}}$ is the one of (4.19). Going back now to (4.19) and using Fatou's Lemma, we conclude that for any $q>1$,

$$\mathbf{E}^{(u,v)}\Big[\big(\int_0^T|Z^{1,(u,v)}_s|^2ds\big)^{\frac{q}{2}}\Big]<\infty. \tag{4.20}$$

Estimates (4.18) and (4.20) yield to the conclusion (4.12).

Finally note that, taking $t=0$ in (4.14) we obtain $W^{1,(u,v)}_0=J^1(u,v)$ since $\mathcal{F}_0$ contains only **P** and $\mathbf{P}^{(u,v)}$ null sets since those probabilities are equivalent. □

Theorem 4.2.1. *Let us assume that:*

(i) The Assumptions (A1), (A2) and (A3) are fulfilled ;

(ii) There exist two deterministic functions $\varpi^i(t,x)$, $i=1,2$, with polynomial growth and two pairs of $\mathcal{P}$-measurable processes (W^i,Z^i), $i=1,2$, with values in $\boldsymbol{R}^{1+m}$ such that: For $i=1,2$,

*(a) **P**-a.s., $\forall s\le T$, $W^i_s=\varpi^i(s,X^{0,x}_s)$ and $Z^i(\omega):=(Z^i_t(\omega))_{t\le T}$ is dt-square integrable ;*

(b) For any $s\le T$,

$$\begin{cases}-dW^i_s=H_i\big(s,X^{0,x}_s,Z^i_s,u^*(s,X^{0,x}_s,Z^1_s,Z^2_s),v^*(s,X^{0,x}_s,Z^1_s,Z^2_s)\big)\,ds-Z^i_sdB_s,\\ \quad W^i_T=g^i(X^{0,x}_T).\end{cases} \tag{4.21}$$

Then the control $(u^*(s,X^{0,x}_s,Z^1_s,Z^2_s),v^*(s,X^{0,x}_s,Z^1_s,Z^2_s))_{s\le T}$ is admissible and a Nash equilibrium point for the NZSDG.

Proof. For $s\le T$, let us set $u^*_s=u^*(s,X^{0,x}_s,Z^1_s,Z^2_s)$ and $v^*_s=v^*(s,X^{0,x}_s,Z^1_s,Z^2_s)$, then $(u^*,v^*)\in\mathcal{M}$. On the other hand we obviously have, thanks to Proposition 4.2.1, $W^1_0=J^1(u^*,v^*)$.

Next let u be an arbitrary element of $\mathcal{M}_1$ and let us show that $W^1\le W^{1,(u,v^*)}$, which yields $W^1_0=J^1(u^*,v^*)\le W^{1,(u,v^*)}_0=J^1(u,v^*)$.

The control (u, v^*) is admissible and thanks to Proposition 4.2.1, there exists a pair of $\mathcal{P}$-measurable processes $(W^{1,(u,v^*)}, Z^{1,(u,v^*)})$ such that for any $q > 1$,

$$\begin{cases} \mathbf{E}^{(u,v^*)}\Big[\sup_{0\le t\le T}|W_t^{1,(u,v^*)}|^q + \big(\int_0^T |Z_s^{1,(u,v^*)}|^2 ds\big)^{\frac{q}{2}}\Big] < \infty \\ W_t^{1,(u,v^*)} = g^1(X_T^{0,x}) + \int_t^T H_1(s, X_s^{0,x}, Z_s^{1,(u,v^*)}, u_s, v_s^*)ds - \int_t^T Z_s^{1,(u,v^*)}dB_s,\ \forall t \le T. \end{cases} \tag{4.22}$$

Afterwards, we aim to compare W^1 and $W^{1,(u,v^*)}$. So let us denote by

$$\triangle W = W^1 - W^{1,(u,v^*)} \text{ and } \triangle Z = Z^1 - Z^{1,(u,v^*)}.$$

For $k \ge 0$, we define the stopping time τ_k as follows:

$$\tau_k := \inf\{s \ge 0, |\triangle W_s| + \int_0^s |\triangle Z_r|^2 dr \ge k\} \wedge T.$$

The sequence of stopping times $(\tau_k)_{k\ge 0}$ is of stationary type and converges to T. Next applying Itô-Meyer formula to $|(\triangle W)^+|^q$ $(q > 1)$ (see Theorem 71, P.Protter, [90], pp.221), between $t\wedge\tau_k$ and τ_k, we obtain: $\forall t \le T$,

$$\begin{aligned} &|(\triangle W_{t\wedge\tau_k})^+|^q + c(q)\int_{t\wedge\tau_k}^{\tau_k} |(\triangle W_s)^+|^{q-2}1_{\triangle W_s>0}|\triangle Z_s|^2 ds \\ &= |(\triangle W_{\tau_k})^+|^q + q\int_{t\wedge\tau_k}^{\tau_k} |(\triangle W_s)^+|^{q-1}1_{\triangle W_s>0}\Big(H_1(s, X_s^{0,x}, Z_s^1, u_s^*, v_s^*) - \\ &\qquad H_1(s, X_s^{0,x}, Z_s^{(u,v^*)}, u_s, v_s^*)\Big)ds - q\int_{t\wedge\tau_k}^{\tau_k} |(\triangle W_s)^+|^{q-1}1_{\triangle W_s>0}\triangle Z_s dB_s \\ &= |(\triangle W_{\tau_k})^+|^q + q\int_{t\wedge\tau_k}^{\tau_k} |(\triangle W_s)^+|^{q-1}1_{\triangle W_s>0}\Big(H_1(s, X_s^{0,x}, Z_s^1, u_s^*, v_s^*) - \\ &\quad H_1(s, X_s^{0,x}, Z_s^1, u_s, v_s^*) + H_1(s, X_s^{0,x}, Z_s^1, u_s, v_s^*) - H_1(s, X_s^{0,x}, Z_s^{(u,v^*)}, u_s, v_s^*)\Big)ds \\ &\quad - q\int_{t\wedge\tau_k}^{\tau_k} |(\triangle W_s)^+|^{q-1}1_{\triangle W_s>0}\triangle Z_s dB_s, \end{aligned}$$

where $c(q) = \frac{q(q-1)}{2}$. Considering now the generalized Isaacs' Assumption (A3), we have that,

$$H_1(s, X_s^{0,x}, Z_s^1, u_s^*, v_s^*) - H_1(s, X_s^{0,x}, Z_s^1, u_s, v_s^*) \le 0, \forall s \le T.$$

Therefore,

$$\begin{aligned} &\big|(\triangle W_{t\wedge\tau_k})^+\big|^q + c(q)\int_{t\wedge\tau_k}^{\tau_k} |(\triangle W_s)^+|^{q-2}1_{\triangle W_s>0}|\triangle Z_s|^2 ds \\ &\le |(\triangle W_{\tau_k})^+|^q + q\int_{t\wedge\tau_k}^{\tau_k} |(\triangle W_s)^+|^{q-1}1_{\triangle W_s>0}\triangle Z_s\sigma^{-1}(s, X_s^{0,x})f(s, X_s^{0,x}, u_s, v_s^*)ds \\ &\quad - q\int_{t\wedge\tau_k}^{\tau_k} |(\triangle W_s)^+|^{q-1}1_{\triangle W_s>0}\triangle Z_s dB_s \\ &= |(\triangle W_{\tau_k})^+|^q - q\int_{t\wedge\tau_k}^{\tau_k} |(\triangle W_s)^+|^{q-1}1_{\triangle W_s>0}\triangle Z_s dB_s^{(u,v^*)}, \end{aligned}$$

where $B^{(u,v^*)} = (B_t - \int_0^t \sigma^{-1}(s, X_s^{0,x})f(s, X_s^{0,x}, u_s, v_s^*)ds)_{t\le T}$ is an $(\mathcal{F}_t^0, \mathbf{P}^{(u,v^*)})$-Brownian motion. Then for any $t \le T$,

$$|(\triangle W_{t\wedge\tau_k})^+|^q \le |(\triangle W_{\tau_k})^+|^q - q\int_{t\wedge\tau_k}^{\tau_k} |(\triangle W_s)^+|^{q-1}1_{\triangle W_s>0}\triangle Z_s dB_s^{(u,v^*)}. \tag{4.23}$$

By definition of the stopping time τ_k, we have

$$\mathbf{E}^{(u,v^*)}\Big[\int_{t\wedge\tau_k}^{\tau_k} |(\triangle W_s)^+|^{q-1} 1_{\triangle W_s>0} \triangle Z_s dB_s^{(u,v^*)}\Big] = 0.$$

Then taking expectation on both sides of (4.23) we obtain:

$$\mathbf{E}^{(u,v^*)}\left[|(\triangle W_{t\wedge\tau_k})^+|^q\right] \leq \mathbf{E}^{(u,v^*)}\Big[|(W_{\tau_k}^1 - W_{\tau_k}^{1,(u,v^*)})^+|^q\Big]. \tag{4.24}$$

Next taking into account (4.22) and the fact that W^1 has a representation through ϖ^1 which is deterministic and of polynomial growth and finally (4.10), we deduce that

$$\mathbf{E}^{(u,v^*)}\Big[\sup_{s\leq T}(|W_s^{1,(u,v^*)}| + |W_s^1|)^q\Big] < \infty. \tag{4.25}$$

As the sequence $((W_{\tau_k}^1 - W_{\tau_k}^{1,(u,v^*)})^+)_k$ converges to 0 as $k\to\infty$, $\mathbf{P}^{(u,v^*)}$-a.s., then it converges also to 0 in $L^1(d\mathbf{P}^{(u,v^*)})$ since it is uniformly integrable thanks to (4.25). Taking now the limit w.r.t. k on both sides of (4.24) and finally by Fatou's Lemma we deduce that:

$$\mathbf{E}^{(u,v^*)}\left[\triangle W_t^+\right] = 0,\ \forall t\leq T,$$

which implies that $W^1 \leq W^{1,(u,v^*)}$, $\mathbf{P}$-a.s., since the probabilities $\mathbf{P}^{(u,v^*)}$ and $\mathbf{P}$ are equivalent. Thus $W_0^1 = J^1(u^*,v^*) \leq W_0^{1,(u,v^*)} = J^1(u,v^*)$.

In the same way one can show that if v is an arbitrary element of $\mathcal{M}_2$ then $W_0^2 = J^2(u^*,v^*) \leq W_0^{2,(u^*,v)} = J^2(u^*,v)$. Henceforth (u^*,v^*) is a Nash equilibrium point for the NZSDG. $\square$

Now, the main emphasis is placed on the existence of a solution for the BSDE (4.21) with its properties.

4.3 Existence of solutions for markovian BSDEs related to differential games

4.3.1 Deterministic representation

Let ℓ be an integer and let us consider $\bar{f}$ (resp. $\bar{g}$) a Borel measurable function from $[0,T]\times\mathbf{R}^{m+\ell+\ell\times m}$ (resp. $\mathbf{R}^m$) into $\mathbf{R}^\ell$ (resp. $\mathbf{R}^\ell$) such that:

(a) For any fixed $(t,x)\in[0,T]\times\mathbf{R}^m$, the mapping $(y,z)\in\mathbf{R}^{\ell+\ell\times m}\longmapsto\bar{f}(t,x,y,z)$ is uniformly Lipschitz ;

(b) There exist real constants C and $p>0$ such that

$$|\bar{f}(t,x,y,z)| + |\bar{g}(x)| \leq C(1+|x|^p),\ \forall(t,x,y,z)\in[0,T]\times\mathbf{R}^{m+\ell+\ell\times m}.$$

Then we have the following result by El Karoui et al. [39] related to representation of solutions of BSDEs through deterministic functions in the Markovian framework of randomness.

Proposition 4.3.1. *Assume that (A1), (i)-(ii) and (A4) are fulfilled. Let $(t,x)\in[0,T]\times\boldsymbol{R}^m$ be fixed and $(\theta_s^{t,x})_{t\leq s\leq T}$ be the solution of SDE (4.7). Let $(y_s^{t,x},z_s^{t,x})_{t\leq s\leq T}$ be the solution of the following BSDE:*

$$\begin{cases} y^{t,x}\in\mathcal{S}_{t,T}^2(\boldsymbol{R}^\ell), z^{t,x}\in\mathcal{H}_{t,T}^2(\boldsymbol{R}^{\ell\times m}); \\ -dy_s^{t,x} = \bar{f}(s,\theta_s^{t,x},y_s^{t,x},z_s^{t,x})ds - z_s^{t,x}dB_s,\ s\in[t,T]; \\ y_T = \bar{g}(\theta_T^{t,x}). \end{cases}$$

Then there exists a pair of measurable and deterministic applications ϖ: $[0,T]\times\boldsymbol{R}^m\to\boldsymbol{R}^\ell$ and υ: $[0,T]\times\boldsymbol{R}^m\to\boldsymbol{R}^{\ell\times d}$ such that,

$$\boldsymbol{P}-a.s., \forall t\leq s\leq T,\quad y_s^{t,x} = \varpi(s,\theta_s^{t,x})\ and\ z_s^{t,x} = \upsilon(s,\theta_s^{t,x}).$$

Moreover,
(i) $\forall(t,x) \in [0,T] \times \boldsymbol{R}^m$, $\varpi(t,x) = \boldsymbol{E}[\int_t^T \bar{f}(r, \theta_r^{t,x}, y_r^{t,x}, z_r^{t,x})dr + \bar{g}(\theta_T^{t,x})]$*;*
(ii) For any other $(t_1, x_1) \in [0,T] \times \boldsymbol{R}^m$*, the process* $(\varpi(s, \theta_s^{t_1,x_1}),\ \upsilon(s, \theta_s^{t_1,x_1}))_{t_1 \leq s \leq T}$ *is the unique solution in* $\mathcal{S}^2_{t_1,T}(\boldsymbol{R}^\ell) \times \mathcal{H}^2_{t_1,T}(\boldsymbol{R}^{\ell\times m})$ *of the BSDE associated with the coefficients* $(\bar{f}(s, \theta_s^{t_1,x_1}, y, z), \bar{g}(\theta_T^{t_1,x_1}))$ *in the time interval* $[t_1, T]$*.*□

We next recall the notion of domination which is important in order to show that equation (4.21) has a solution.

Definition 4.3.1. *:* L^q***-Domination condition***
Let $q \in]1,\infty[$ *be fixed. For a given* $t_1 \in [0,T]$*, a family of probability measures* $\{\nu_1(s,dx), s \in [t_1,T]\}$ *defined on* $\boldsymbol{R}^m$ *is said to be* L^q*- dominated by another family of probability measures* $\{\nu_0(s,dx), s \in [t_1,T]\}$*, if for any* $\delta \in (0, T-t_1]$*, there exists an application* $\phi_{t_1}^\delta : [t_1+\delta, T] \times \boldsymbol{R}^m \to \boldsymbol{R}^+$ *such that:*

(i) $\nu_1(s, dx)ds = \phi_{t_1}^\delta(s, x)\nu_0(s, dx)ds$ *on* $[t_1+\delta, T] \times \boldsymbol{R}^m$*.*
(ii) $\forall k \geq 1$, $\phi_{t_1}^\delta(s,x) \in L^q([t_1+\delta, T] \times [-k,k]^m; \nu_0(s,dx)ds)$*.*□

We then have:

Lemma 4.3.1. *Assume (A1) and (A4) fulfilled and the drift term* $b(t,x)$ *of SDE (4.7) is bounded. Let* $q \in]1,\infty[$ *be fixed,* $(t_0, x_0) \in [0,T] \times \boldsymbol{R}^m$ *and let* $(\theta_s^{t_0,x_0})_{t_0 \leq s \leq T}$ *be the solution of SDE (4.7). Then for any* $s \in (t_0, T]$*, the law* $\bar{\mu}(t_0, x_0; s, dx)$ *of* $\theta_s^{t_0,x_0}$ *has a density function* $\rho_{t_0,x_0}(s,x)$*, w.r.t. Lebesgue measure* dx*, which satisfies the following estimate:* $\forall (s,x) \in (t_0, T] \times \boldsymbol{R}^m$*,*

$$\varrho_1(s-t_0)^{-\frac{m}{2}} exp\left[-\frac{\Lambda|x-x_0|^2}{s-t_0}\right] \leq \rho_{t_0,x_0}(s,x) \leq \varrho_2(s-t_0)^{-\frac{m}{2}} exp\left[-\frac{\lambda|x-x_0|^2}{s-t_0}\right] \quad (4.26)$$

where ϱ_1*,* ϱ_2*,* Λ*,* λ *are real constants such that* $0 \leq \varrho_1 \leq \varrho_2$ *and* $0 \leq \lambda \leq \Lambda$*. Additionally for any* $(t_1, x_1) \in [t_0, T] \times \boldsymbol{R}^m$*, the family of laws* $\{\bar{\mu}(t_1, x_1; s, dx), s \in [t_1, T]\}$ *is* L^q*-dominated by* $\bar{\mu}(t_0, x_0; s, dx)$*.*

$Proof$: Since σ satisfies (4.2) and b is bounded, then by Aronson's result (see [1]), the law $\bar{\mu}(t_0, x_0; s, dx)$ of $\theta_s^{t_0,x_0}$, $s \in]t_0, T]$, has a density function $\rho_{t_0,x_0}(s,x)$ which satisfies estimate (4.26).

Let us focus on the second claim of the lemma. Let $(t_1, x_1) \in [t_0, T] \times \mathbf{R}^m$ and $s \in (t_1, T]$. Then,

$$\rho_{t_1,x_1}(s,x) = [\rho_{t_1,x_1}(s,x)\rho^{-1}_{t_0,x_0}(s,x)]\rho_{t_0,x_0}(s,x) = \phi_{t_1,x_1}(s,x)\rho_{t_0,x_0}(s,x)$$

with

$$\phi_{t_1,x_1}(s,x) = \left[\rho_{t_1,x_1}(s,x)\rho^{-1}_{t_0,x_0}(s,x)\right], (s,x) \in (t_1, T] \times \mathbf{R}^m.$$

For any $\delta \in (0, T-t_1]$, ϕ_{t_1,x_1} is defined on $[t_1+\delta, T]$. Moreover for any $(s,x) \in [t_1+\delta, T] \times \mathbf{R}^m$ it holds,

$$\begin{aligned}
\bar{\mu}(t_1, x_1; s, dx)ds &= \rho_{t_1,x_1}(s,x)dxds \\
&= \phi_{t_1,x_1}(s,x)\rho_{t_0,x_0}(s,x)dxds \\
&= \phi_{t_1,x_1}(s,x)\bar{\mu}(t_0, x_0; s, dx)ds.
\end{aligned}$$

Next by (4.26), for any $(s,x) \in [t_1+\delta, T] \times \mathbf{R}^m$,

$$0 \leq \phi_{t_1,x_1}(s,x) \leq \frac{\varrho_2(s-t_1)^{-\frac{m}{2}}}{\varrho_1(s-t_0)^{-\frac{m}{2}}} exp\left[\frac{\Lambda|x-x_0|^2}{s-t_0} - \frac{\lambda|x-x_1|^2}{s-t_1}\right] := \Phi_{t_1,x_1}(s,x).$$

It follows that for any $k \geq 0$, the function $\Phi_{t_1,x_1}(s,x)$ is bounded on $[t_1+\delta, T] \times [-k,k]^m$ by a constant κ which depends on t_0, t_1, δ, Λ, λ, k and x_0. Next let $q \in (1,\infty)$, then,

$$\begin{aligned}\int_{t_1+\delta}^{T}\int_{[-k,k]^m}\Phi_{t_1,x_1}(s,x)^q\bar{\mu}(t_0,x_0;s,dx)ds &\leq \kappa^q\int_{t_1+\delta}^{T}\int_{[-k,k]^m}\bar{\mu}(t_0,x_0;s,dx)ds \\ &= \kappa^q\int_{t_1+\delta}^{T}ds\mathbf{E}[1_{[-k,k]^m}(\theta_s^{t_0,x_0})] \leq \kappa^q T.\end{aligned}$$

Thus Φ and then ϕ belong to $L^q([t_1+\delta,T]\times[-k,k]^m;\nu_0(s,dx)ds)$. It follows that the family of measures $\{\bar{\mu}(t_1,x_1;s,dx), s\in[t_1,T]\}$ is L^q-dominated by $\bar{\mu}(t_0,x_0;s,dx)$. □

As a by-product we have:

Corollary 4.3.1. *Let $x_0 \in \boldsymbol{R}^m$, $(t,x)\in[0,T]\times\boldsymbol{R}^m$, $s\in(t,T]$ and $\mu(t,x;s,dy)$ the law of $X_s^{t,x}$, i.e.,*

$$\forall A \in \mathcal{B}(\boldsymbol{R}^m),\ \mu(t,x;s,A) = \boldsymbol{P}(X_s^{t,x}\in A).$$

Under (A1) on σ, for any $q\in(1,\infty)$, the family of laws $\{\mu(t,x;s,dy), s\in[t,T]\}$ is L^q-dominated by $\{\mu(0,x_0;s,dy), s\in[t,T]\}$.□

4.3.2 The main result

We are now ready to provide a solution for BSDE (4.21) which satisfies the representation property via deterministic functions with polynomial growth.

Theorem 4.3.1. *Let $x_0\in\boldsymbol{R}^m$ be fixed. Then under the Assumptions (A1), (A2) and (A3), there exist:*
(i) Two pairs of $\mathcal{P}$-measurable processes $(W_s^i, Z_s^i)_{s\leq T}$, $i=1,2$, such that: $\forall i\in\{1,2\}$,

$$\begin{cases}\boldsymbol{P}-a.s.,\ Z^i(\omega)=(Z_s^i(\omega))_{s\leq T} \text{ is } dt-\text{square integrable};\\ -dW_s^i = H_i\left(s,X_s^{0,x_0},Z_s^i,u_1^*(s,X_s^{0,x_0},Z_s^1,Z_s^2),u_2^*(s,X_s^{0,x_0},Z_s^1,Z_s^2)\right)ds - Z_s^i dB_s,\\ \hfill\forall s\leq T;\\ W_T^i = g^i(X_T^{0,x_0}).\end{cases} \tag{4.27}$$

(ii) Two measurable deterministic functions ϖ^i, $i=1,2$ with polynomial growth defined from $[0,T]\times\boldsymbol{R}^m$ into $\boldsymbol{R}$ such that:

$$\forall i=1,2,\ W_s^i=\varpi^i(s,X_s^{0,x_0}),\ \forall s\in[0,T].$$

Proof. It will be divided into five steps. We first construct an approximating sequence of BSDEs which have solutions according to Proposition 4.3.1, we then provide a priori estimates of the solutions of those BSDEs. Finally we prove that those solutions are convergent (at least for a subsequence) and the limit is a solution for the BSDE (4.27).

Step 1: Construction of the approximating sequence $(W_s^{in;(t,x)}, Z_s^{in;(t,x)})_{s\leq T}$, $n\geq 1$, $i=1,2$.

Let ξ be an element of $C^\infty(\mathbf{R}^{2m},\mathbf{R})$ with compact support and satisfying

$$\int_{\mathbf{R}^{2m}}\xi(y,z)dydz=1.$$

For $n\geq 1$, $i=1,2$ and $(t,x,z^1,z^2)\in[0,T]\times\mathbf{R}^{3m}$, we set

$$\begin{aligned}&\underline{\mathrm{H}}_i^n\left(t,x,z^1,(u_1^*,u_2^*)\left(t,x,z^1,z^2\right)\right)\\ &\quad=\int_{\mathbf{R}^{2d}}n^2 H_i\left(t,\varphi_n(x),y,(u_1^*,u_2^*)\left(t,\varphi_n(x),y,z\right)\right)\xi\left(n\left(z^1-y\right),n\left(z^2-z\right)\right)dydz,\end{aligned}$$

where for any $x=(x_j)_{1\le j\le m}\in\mathbf{R}^m$, $\varphi_n(x):=((x_j\vee(-n))\wedge n)_{1\le j\le m}$.

We next define $\psi\in C^\infty(\mathbf{R}^{2m},\mathbf{R})$ satisfying

$$\psi(y,z)=\begin{cases}1 \text{ if} & |y|^2+|z|^2\le 1,\\ 0 \text{ if} & |y|^2+|z|^2\ge 4.\end{cases}$$

Then, for $i=1,2$, the measurable functions H_i^n, $n\ge 1$, defined as follows:

$$H_i^n\left(t,x,z^1,z^2\right):=\psi\left(\frac{z^1}{n},\frac{z^2}{n}\right)\underline{\mathrm{H}}_i^n\left(t,x,z^1,(u_1^*,u_2^*)\left(t,x,z^1,z^2\right)\right),$$
$$(t,x,z^1,z^2)\in[0,T]\times\mathbf{R}^{3m},$$

satisfy the following properties:

(a) H_i^n is uniformly Lipschitz w.r.t (z^1,z^2) ;

(b) $|H_i^n\left(t,x,z^1,z^2\right)|\le C\left(1+|\varphi_n(x)|\right)|z^i|+C\left(1+|\varphi_n(x)|^\gamma\right)$.

(c) $|H_i^n\left(t,x,z^1,z^2\right)|\le c_n$, for any (t,x,z^1,z^2).

(d) For any $(t,x)\in[0,T]\times\mathbf{R}^m$ and K a compact subset of $\mathbf{R}^{2m}$,

$$\sup_{(z^1,z^2)\in K}|H_i^n\left(t,x,z^1,z^2\right)-H_i\left(t,x,z^1,(u_1^*,u_2^*)\left(t,x,z^1,z^2\right)\right)|\to 0, \text{ as } n\to\infty.$$

Let us notice that (b) is valid since u_1^* and u_2^* take their values in compact sets.

The constant γ, which we can choose greater than 1, is the one of polynomial growth of h_i, $i=1,2$. Therefore, from points (a) and (c) and Proposition 4.3.1, for each $n\ge 1$, $i=1,2$ and $(t,x)\in[0,T]\times\mathbf{R}^m$, there exist solutions $(W_s^{in;(t,x)},Z_s^{in;(t,x)})_{s\le T}$ in $\mathcal{S}^2_{t,T}(\mathbf{R})\times\mathcal{H}^2_{t,T}(\mathbf{R}^m)$ such that for any $s\in[t,T]$,

$$W_s^{in;(t,x)}=g^i(X_T^{t,x})+\int_s^T H_i^n(r,X_r^{t,x},Z_r^{1n;(t,x)},Z_r^{2n;(t,x)})dr-\int_s^T Z_r^{in;(t,x)}dB_r. \tag{4.28}$$

Then, by Proposition 4.3.1, for $i=1,2$, there exists a sequence of measurable deterministic functions ϖ^{in}: $[0,T]\times\mathbf{R}^m\to\mathbf{R}$ and υ^{in}: $[0,T]\times\mathbf{R}^m\to\mathbf{R}^m$ such that

$$\forall s\in[t,T],\ W_s^{in;(t,x)}=\varpi^{in}(s,X_s^{t,x})\quad\text{and}\quad Z_s^{in;(t,x)}=\upsilon^{in}(s,X_s^{t,x}). \tag{4.29}$$

Moreover, for $i=1,2$ and $n\ge 1$, ϖ^{in} satisfies

$$\varpi^{in}(t,x)=\mathbf{E}\Big[g^i(X_T^{t,x})+\int_t^T F^{in}(s,X_s^{t,x})ds\Big],\quad\forall(t,x)\in[0,T]\times\mathbf{R}^m,$$

with

$$F^{in}(t,x)=H_i^n\left(t,x,\upsilon^{1n}(t,x),\upsilon^{2n}(t,x)\right),\ (t,x)\in[0,T]\times\mathbf{R}^m.$$

Step 2: The deterministic functions ϖ^{in} are of polynomial growth uniformly w.r.t. n, i.e., there exist two constants C and λ such that for any $n\ge 1$, $i=1,2$,

$$\forall(t,x)\in[0,T]\times\mathbf{R}^m,\ |\varpi^{in}(t,x)|\le C(1+|x|^\lambda). \tag{4.30}$$

We will deal with the case of index $i=1$, the case of $i=2$ can be treated in the same way. For each $n\ge 1$, let us consider the following BSDE: $\forall s\in[t,T]$,

$$\left\{\begin{array}{l}(\bar W,\bar Z)\in\mathcal{S}^2_{t,T}(\mathbf{R})\times\mathcal{H}^2_{t,T}(\mathbf{R}^m)\,;\\ \bar W_s^{1n}=g^1(X_T^{t,x})+\int_s^T C(1+|\varphi_n(X_r^{t,x})|)|\bar Z_r^{1n}|+C(1+|\varphi_n(X_r^{t,x})|^\gamma)dr-\int_s^T\bar Z_r^{1n}dB_r.\end{array}\right. \tag{4.31}$$

For any $x \in \mathbf{R}^m$ and $n \geq 1$, the function
$z^1 \in \mathbf{R}^m \longmapsto C(1+|\varphi_n(X_s^{t,x})|)|z^1| + C(1+|\varphi_n(X_s^{t,x})|^\gamma)$ is Lipschitz continuous. Then the solution $(\bar{W}^{1n}, \bar{Z}^{1n})$ exists and is unique. Moreover through an adaptation of the result given in (El Karoui and al.,1997,[39]), we can infer the existence of deterministic measurable function $\bar{\varpi}^{1n}$: $[0,T] \times \mathbf{R}^m \to \mathbf{R}$ such that, for any $s \in [t,T]$,

$$\bar{W}_s^{1n} = \bar{\varpi}^{1n}(s, X_s^{t,x}). \tag{4.32}$$

Next let us consider the process

$$B_s^n = B_s - \int_0^s 1_{[t,T]}(r) C(1+|\varphi_n(X_r^{t,x})|)\text{sign}(\bar{Z}_r^{1n})dr,\ 0 \leq s \leq T,$$

which is, thanks to Girsanov's Theorem, a Brownian motion under the probability $\mathbf{P}^n$ on $(\Omega, \mathcal{F})$ whose density with respect to $\mathbf{P}$ is

$$\zeta_T\{C(1+|\varphi_n(X_s^{t,x})|)\text{sign}(\bar{Z}_s^{1n})1_{[t,T]}(s)\},$$

where for any $z = (z^i)_{i=1,\dots,d} \in \mathbf{R}^m$, $\text{sign}(z) = (1_{[|z^i| \neq 0]} \frac{z^i}{|z^i|})_{i=1,\dots,d}$ and $\zeta_T(\cdot)$ is defined by (4.4). Then (4.31) becomes

$$\bar{W}_s^{1n} = g^1(X_T^{t,x}) + \int_s^T C(1+|\varphi_n(X_r^{t,x})|^\gamma)dr - \int_s^T \bar{Z}_r^{1n} dB_r^n, \quad 0 \leq s \leq T.$$

Therefore, taking into account of (4.32), we deduce,

$$\bar{\varpi}^{1n}(t,x) = \mathbf{E}^n\Big[g^1(X_T^{t,x}) + \int_t^T C\left(1+|\varphi_n(X_s^{t,x})|^\gamma\right) ds|\mathcal{F}_t\Big],$$

where $\mathbf{E}^n$ is the expectation under probability $\mathbf{P}^n$. Taking the expectation on both sides under the probability $\mathbf{P}^n$ and considering $\bar{\varpi}^{1n}(t,x)$ is deterministic, one obtains,

$$\bar{\varpi}^{1n}(t,x) = \mathbf{E}^n\Big[g^1(X_T^{t,x}) + \int_t^T C\left(1+|\varphi_n(X_s^{t,x})|^\gamma\right) ds\Big].$$

Then by the Assumption (A2)-(iii) we have: $\forall (t,x) \in [0,T] \times \mathbf{R}^m$,

$$\begin{aligned}|\bar{\varpi}^{1n}(t,x)| &\leq C\mathbf{E}^n\Big[\sup_{0\leq s\leq T}\left(1+|X_s^{t,x}|^\gamma\right)\Big] \\ &= C\mathbf{E}\Big[\big(\sup_{0\leq s\leq T}\left(1+|X_s^{t,x}|^\gamma\right)\big)\Big(\zeta_T(C\left(1+\left|\varphi_n\left(X_s^{t,x}\right)\right|\right)\text{sign}(\bar{Z}_s^{1n}))\Big)\Big].\end{aligned}$$

By Lemma 4.2.1, there exists some $1 < p_0 < 2$ (which does not depend on (t,x)), such that,

$$\mathbf{E}\Big[\Big(\zeta_T(C\left(1+\left|\varphi_n\left(X_s^{t,x}\right)\right|\right)\text{sign}(\tilde{Z}_s^{1n}))\Big)^{p_0}\Big] < \infty. \tag{4.33}$$

Then thanks to Young's inequality, we obtain,

$$\begin{aligned}&|\bar{\varpi}^{1n}(t,x)| \\ &\leq C\{\mathbf{E}\Big[\sup_{0\leq s\leq T}\left(1+|X_s^{t,x}|^\gamma\right)^{\frac{p_0}{p_0-1}}\Big] + \mathbf{E}\Big[\left(\zeta_T(C\left(1+\left|\varphi_n\left(X_s^{t,x}\right)\right|\right)\text{sign}(\tilde{Z}_s^{1n}))\right)^{p_0}\Big]\}.\end{aligned}$$

Finally estimates (4.33) and (4.8) yield that,

$$|\bar{\varpi}^{1n}(t,x)| \leq C(1+|x|^\lambda),$$

where $\lambda = \frac{\gamma p_0}{p_0-1} > 2$. Next taking into account point (b) and using comparison of solutions of BSDEs ([39], pp.23) we obtain for any $s \in [t,T]$,

$$\bar{W}_s^{1n} = \bar{\varpi}^{1n}(s, X_s^{t,x}) \geq W_s^{1n;(t,x)} = \varpi^{1n}(s, X_s^{t,x}), \forall s \in [t,T],$$

and then, choosing $s = t$, we get that $\varpi^{1n}(t,x) \le C(1+|x|^\lambda)$, $(t,x) \in [0,T] \times \mathbf{R}^m$. But in a similar way one can show that for any $(t,x) \in [0,T] \times \mathbf{R}^m$,
$\varpi^{1n}(t,x) \ge -C(1+|x|^\lambda)$. Therefore ϖ^{1n} is of polynomial growth w.r.t. (t,x) uniformly in n, i.e., it satisfies (4.30).□

Step 3: There exists a constant C independent of n and t,x such that for any $t \le T$, for $i = 1,2$,

$$\mathbf{E}[\int_t^T |Z_s^{in;(t,x)}|^2 ds] \le C. \tag{4.34}$$

Actually, we obtain from estimates (4.30) and (4.9) that for any $\alpha \ge 1$, $i = 1,2$,

$$\mathbf{E}[\sup_{t\le s\le T} |W_s^{in;(t,x)}|^\alpha] \le C.$$

Going back to equation (4.28) and making use of Itô's formula with $(W_s^{1n;(t,x)})^2$, we obtain, in a standard way, the result (4.34). The proof is omitted for conciseness.□

Step 4: There exists a subsequence of $((W_s^{in;(0,x_0)}, Z_s^{in;(0,x_0)})_{s\in[0,T]})_{n\ge1}$, $i = 1,2$, which converges respectively to $(W_s^i, Z_s^i)_{0\le s\le T}$, $i = 1,2$, solution of the BSDE (4.27).
Actually for $i = 1,2$ and $n \ge 1$, by (4.29) we know that

$$W_s^{in;(0,x_0)} = \varpi^{in}(s, X_s^{0,x_0}), s \le T$$

where the deterministic functions ϖ^{in} verifies:

$$\forall (t,x) \in [0,T] \times \mathbf{R}^m,\ \varpi^{in}(t,x) = \mathbf{E}\Big[g^i(X_T^{t,x}) + \int_t^T F^{in}(s, X_s^{t,x}) ds\Big]. \tag{4.35}$$

Let us now fix $q \in (1,2)$. Taking account of point (b), we have:

$$\begin{aligned}\mathbf{E}\Big[\int_0^T |F^{in}\left(s, X_s^{0,x_0}\right)|^q ds\Big] &= \int_{[0,T]\times\mathbf{R}^m} |F^{in}(s,y)|^q \mu(0,x_0;s,dy) ds \\ &\le C\mathbf{E}\Big[\int_0^T |Z_s^{in;(0,x_0)}|^q \left(1+|X_s^{0,x_0}|^q\right) + \left(1+|X_s^{0,x_0}|^{\gamma q}\right) ds\Big].\end{aligned}$$

By Hölder and Young's inequalities, one can show that,

$$\begin{aligned}&\mathbf{E}\Big[\int_0^T |F^{in}\left(s, X_s^{0,x_0}\right)|^q ds\Big] \\ &\qquad \le C\mathbf{E}\Big[\big(\int_0^T |Z_s^{in;(0,x_0)}|^2 ds\big)^{\frac{q}{2}}\big(\int_0^T \left(1+|X_s^{0,x_0}|\right)^{\frac{2q}{2-q}} ds\big)^{\frac{2-q}{2}}\Big] \\ &\qquad\quad + C\mathbf{E}\Big[\int_0^T \left(1+|X_s^{0,x_0}|^{\gamma q}\right) ds\Big] \\ &\qquad \le C\{\mathbf{E}\Big[\int_0^T |Z_s^{in;(0,x_0)}|^2 ds\Big] + \mathbf{E}\Big[1+\sup_{0\le s\le T}|X_s^{0,x_0}|^\theta\Big]\}\end{aligned} \tag{4.36}$$

for constant $\theta = (\gamma q) \vee \frac{2q}{2-q}$ which is greater than 2 with $1 < q < 2$ and $\gamma > 1$. Taking now into account the estimates (4.34) and (4.9) we deduce that,

$$\mathbf{E}\Big[\int_0^T |F^{in}\left(s, X_s^{0,x_0}\right)|^q ds\Big] = \int_{[0,T]\times\mathbf{R}^m} |F^{in}(s,y)|^q \mu(0,x_0;s,dy) ds \le C.$$

As a result, there exists a subsequence $\{n_k\}$ (for notational simplification, we still denote it by $\{n\}$) and two $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbf{R}^m)$-measurable deterministic functions $F^i(s,x)$, $i = 1,2$, such that,

$$F^{in} \to F^i \quad \text{weakly in} \quad L^q([0,T]\times\mathbf{R}^m; \mu(0,x_0;s,dx)ds). \tag{4.37}$$

Next we aim to prove that $(\varpi^{in}(t,x))_{n\geq 1}$ is a Cauchy sequence for each $(t,x)\in[0,T]\times\mathbf{R}^m$, i=1,2. So let (t,x) be fixed, $\delta>0$, k, n and $m\geq 1$ be integers. From (4.35), we have,

$$\begin{aligned}|\varpi^{in}(t,x)-\varpi^{im}(t,x)| &= \Big|\mathbf{E}\Big[\int_t^T\big[F^{in}(s,X_s^{t,x})-F^{im}(s,X_s^{t,x})\big]\,ds\Big]\Big| \\ &\leq \Big|\mathbf{E}\Big[\int_t^{t+\delta}\big[F^{in}(s,X_s^{t,x})-F^{im}(s,X_s^{t,x})\big]\,ds\Big]\Big| \\ &\quad+\Big|\mathbf{E}\Big[\int_{t+\delta}^T\big(F^{in}(s,X_s^{t,x})-F^{im}(s,X_s^{t,x})\big).1_{\{|X_s^{t,x}|\leq k\}}ds\Big]\Big| \\ &\quad+\Big|\mathbf{E}\Big[\int_{t+\delta}^T\big(F^{in}(s,X_s^{t,x})-F^{im}(s,X_s^{t,x})\big).1_{\{|X_s^{t,x}|> k\}}ds\Big]\Big|.\end{aligned}$$

On the right side, according to (4.36), we have,

$$\begin{aligned}&\mathbf{E}\Big[\int_t^{t+\delta}|F^{in}(s,X_s^{t,x})-F^{im}(s,X_s^{t,x})|ds\Big] \\ &\qquad\leq \delta^{\frac{q-1}{q}}\Big\{\mathbf{E}\Big[\int_t^T|F^{in}(s,X_s^{t,x})-F^{im}(s,X_s^{t,x})|^q ds\Big]\Big\}^{\frac{1}{q}} \\ &\qquad\leq C\delta^{\frac{q-1}{q}}.\end{aligned}$$

At the same time, thanks to Corollary 4.3.1 associated to $L^{\frac{q}{q-1}}$-domination implies that:

$$\begin{aligned}&\big|\mathbf{E}\big[\int_{t+\delta}^T\big(F^{in}(s,X_s^{t,x})-F^{im}(s,X_s^{t,x})\big).1_{\{|X_s^{t,x}|\leq k\}}ds\big]\big| \\ &\qquad=\big|\int_{\mathbf{R}^m}\int_{t+\delta}^T(F^{in}(s,\eta)-F^{im}(s,\eta)).1_{\{|\eta|\leq k\}}\mu(t,x;s,d\eta)ds\big| \\ &\qquad=\big|\int_{\mathbf{R}^m}\int_{t+\delta}^T(F^{in}(s,\eta)-F^{im}(s,\eta)).1_{\{|\eta|\leq k\}}\phi_{t,x}(s,\eta)\mu(0,x_0;s,d\eta)ds\big|.\end{aligned}$$

Since $\phi_{t,x}(s,\eta)\in L^{\frac{q}{q-1}}([t+\delta,T]\times[-k,k]^m;\mu(0,x_0;s,d\eta)ds)$, for $k\geq 1$, it follows from (4.37) that for each $(t,x)\in[0,T]\times\mathbf{R}^m$, we have,

$$\mathbf{E}\Big[\int_{t+\delta}^T\big(F^{in}(s,X_s^{t,x})-F^{im}(s,X_s^{t,x})\big)\,1_{\{|X_s^{t,x}|\leq k\}}ds\Big]\to 0 \text{ as } n,m\to\infty.$$

Finally,

$$\begin{aligned}&\big|\mathbf{E}\big[\int_{t+\delta}^T\big(F^{in}(s,X_s^{t,x})-F^{im}(s,X_s^{t,x})\big).1_{\{|X_s^{t,x}|> k\}}ds\big]\big| \\ &\leq C\Big\{\mathbf{E}\Big[\int_{t+\delta}^T 1_{\{|X_s^{t,x}|>k\}}ds\Big]\Big\}^{\frac{q-1}{q}}\Big\{\mathbf{E}\Big[\int_{t+\delta}^T|F^{in}(s,X_s^{t,x})-F^{im}(s,X_s^{t,x})|^q ds\Big]\Big\}^{\frac{1}{q}} \\ &\leq Ck^{-\frac{q-1}{q}}\end{aligned}$$

Therefore, $(\varpi^{in}(t,x))_{n\geq 1}$ is a Cauchy sequence for each $(t,x)\in[0,T]\times\mathbf{R}^m$ and then there exists a measurable application ϖ^i on $[0,T]\times\mathbf{R}^m$, $i=1,2$, such that for each $(t,x)\in[0,T]\times\mathbf{R}^m$, $i=1,2$,

$$\lim_{n\to\infty}\varpi^{in}(t,x)=\varpi^i(t,x).$$

Additionally we obtain from estimate (4.30) that ϖ^i is of polynomial growth, i.e.,

$$\forall(t,x)\in[0,T]\times\mathbf{R}^m,\ |\varpi^i(t,x)|\leq C(1+|x|^\lambda). \tag{4.38}$$

Therefore for any $t \geq 0$.

$$\lim_{n\to\infty} W_t^{in;(0,x_0)}(\omega) = \varpi^i(t, X_t^{0,x_0}(\omega)) \,,\, |W_t^{in;(0,x_0)}(\omega)| \leq C(1 + |X_t^{0,x_0}(\omega)|^\lambda),\ \mathbf{P} - a.s.$$

By Lebesgue's dominated convergence theorem, $((W_t^{in;(0,x_0)})_{t\leq T})_{n\geq 1}$ converges to $W^i = (\varpi^i(t, X_t^{0,x_0}))_{t\leq T}$ in $\mathcal{H}_T^\kappa(\mathbf{R})$ for any $\kappa \geq 1$, that is, for $i = 1, 2$,

$$\mathbf{E}\Big[\int_0^T |W_s^{in;(0,x_0)} - W_s^i|^\kappa ds\Big] \to 0, \qquad \text{as } n \to \infty. \tag{4.39}$$

We next show that $(W^{in;(0,x_0)})_{n\geq 0}$ is convergent in $\mathcal{S}_T^2(\mathbf{R})$, $i = 1, 2$, as well. But first let us show that for $i = 1, 2$, the sequence $(Z^{in;(0,x_0)} = (\upsilon^{in}(t, X_t^{0,x_0}))_{t\leq T})_{n\geq 1}$ has a limit in $\mathcal{H}_T^2(\mathbf{R}^m)$. As usual, we only deal with the case $i = 1$. For $n, m \geq 1$ and $s \leq T$, using Itô's formula with $(W_s^{1n} - W_s^{1m})^2$ (we omit the subscript $(0, x_0)$ for convenience) and considering point (b), we get,

$$\begin{aligned}
&\left|W_s^{1n} - W_s^{1m}\right|^2 + \int_s^T |Z_r^{1n} - Z_r^{1m}|^2 dr \\
&= 2\int_s^T \left(W_r^{1n} - W_r^{1m}\right)\left(H_1^n\left(r, X_r^{0,x_0}, Z_r^{1n}, Z_r^{2n}\right) - H_1^m\left(r, X_r^{0,x_0}, Z_r^{1m}, Z_r^{2m}\right)\right) dr \\
&\quad - 2\int_s^T (W_r^{1n} - W_r^{1m})(Z_r^{1n} - Z_r^{1m}) dB_r \\
&\leq C\int_s^T |W_r^{1n} - W_r^{1m}| \left(\left(|Z_r^{1n}| + |Z_r^{1m}|\right)\left(1 + |X_r^{0,x_0}|\right) + \left(1 + |X_r^{0,x_0}|^\gamma\right)\right) dr \\
&\quad - 2\int_s^T (W_r^{1n} - W_r^{1m})(Z_r^{1n} - Z_r^{1m}) dB_r.
\end{aligned}$$

But for any $x, y, z \in \mathbf{R}$, $|xyz| \leq \frac{1}{p}|x|^p + \frac{1}{q}|y|^q + \frac{1}{r}|z|^r$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Then for any $\epsilon > 0$ we have,

$$\begin{aligned}
&\left|W_s^{1n} - W_s^{1m}\right|^2 + \textstyle\int_s^T |Z_r^{1n} - Z_r^{1m}|^2 dr \\
&\leq C\Big\{ \tfrac{\epsilon^2}{2}\textstyle\int_s^T (|Z_r^{1n}| + |Z_r^{1m}|)^2 dr + \tfrac{\epsilon^4}{4}\int_s^T \left(1 + |X_r^{0,x_0}|\right)^4 dr + \tfrac{1}{4\epsilon^8}\int_s^T |W_r^{1n} - W_r^{1m}|^4 dr + \\
&\quad \textstyle\int_s^T |W_r^{1n} - W_r^{1m}| \left(1 + |X_r^{0,x_0}|^\gamma\right) dr\Big\} - 2\int_s^T (W_r^{1n} - W_r^{1m})(Z_r^{1n} - Z_r^{1m}) dB_r.
\end{aligned} \tag{4.40}$$

Taking now $s = 0$, expectation on both sides and the limit w.r.t. n and m we deduce that,

$$\limsup_{n,m\to\infty} \mathbf{E}[\int_0^T |Z_r^{1n} - Z_r^{1m}|^2 dr] \leq C\{\frac{\epsilon^2}{2} + \frac{\epsilon^4}{4}\}$$

due to estimates (4.34), (4.9) and the convergence of (4.39). As ϵ is arbitrary then the sequence $(Z^{1n})_{n\geq 0}$ is convergent in $\mathcal{H}_T^2(\mathbf{R}^m)$ to a process Z^1.

Next once more going back to inequality (4.40), taking the supremum and using BDG's inequality we deduce that,

$$\begin{aligned}
&\mathbf{E}[\sup_{0\leq s\leq T} \left|W_s^{1n} - W_s^{1m}\right|^2 + \int_0^T |Z_r^{1n} - Z_r^{1m}|^2 dr] \\
&\leq C\mathbf{E}\{\frac{\epsilon^2}{2}\int_0^T (|Z_r^{1n}| + |Z_r^{1m}|)^2 dr + \frac{\epsilon^4}{4}\int_0^T \left(1 + |X_r^{0,x_0}|\right)^4 dr + \frac{1}{4\epsilon^8}\int_0^T |W_r^{1n} - W_r^{1m}|^4 dr \\
&\quad + \int_0^T |W_r^{1n} - W_r^{1m}| \left(1 + |X_r^{0,x_0}|^\gamma\right) ds\} + \frac{1}{4}\mathbf{E}[\sup_{0\leq s\leq T} \left|W_s^{1n} - W_s^{1m}\right|^2] \\
&\quad + 4\mathbf{E}[\int_0^T |Z_r^{1n} - Z_r^{1m}|^2 dr].
\end{aligned}$$

which implies that,

$$\limsup_{n,m\to\infty}\mathbf{E}[\sup_{0\le s\le T}\left|W_s^{1n}-W_s^{1m}\right|^2]=0$$

since ϵ is arbitrary. Thus the sequence of processes $(W^{1n})_{n\ge1}$ converges in $\mathcal{S}_T^2(\mathbf{R})$ to W^1 which is a continuous process.

Finally note that we can do the same for $i=2$, i.e., we have also the convergence of $(Z^{2n})_{n\ge0}$ (resp. $(W^{2n})_{n\ge0}$) in $\mathcal{H}_T^2(\mathbf{R}^m)$ (resp. $\mathcal{S}_T^2(\mathbf{R})$) to Z^2 (resp. $(W_t^2=\varpi^2(t,X_t^{0,x_0}))_{t\le T}$). $\Box$

Step 5: The limit processes $(W_s^i,Z_s^i)_{s\le T}$, $i=1,2$, are solutions of BSDE (4.27). Indeed, we need to prove that (for case $i=1$),

$$F^1(t,X_t^{0,x_0})=H_1\big(t,X_t^{0,x_0},Z_t^1,(u_1^*,u_2^*)\,(t,X_t^{0,x_0},Z_t^1,Z_t^2)\big)\quad dt\otimes d\mathbf{P}-a.s. \tag{4.41}$$

For $k\ge1$, we have,

$$\begin{aligned}
&\mathbf{E}\Big[\int_0^T\big|H_1^n(s,X_s^{0,x_0},Z_s^{1n},Z_s^{2n})-H_1(s,X_s^{0,x_0},Z_s^1,(u_1^*,u_2^*)(s,X_s^{0,x_0},Z_s^1,Z_s^2))\big|ds\Big]\\
=\;&\mathbf{E}\Big[\int_0^T\big|H_1^n(s,X_s^{0,x_0},Z_s^{1n},Z_s^{2n})\\
&\qquad -H_1(s,X_s^{0,x_0},Z_s^{1n},(u_1^*,u_2^*)\,(s,X_s^{0,x_0},Z_s^{1n},Z_s^{2n}))\big|\cdot 1_{\{|Z_s^{1n}|+|Z_s^{2n}|<k\}}ds\Big]\\
+\;&\mathbf{E}\Big[\int_0^T\big|H_1^n(s,X_s^{0,x_0},Z_s^{1n},Z_s^{2n})\\
&\qquad -H_1(s,X_s^{0,x_0},Z_s^{1n},(u_1^*,u_2^*)\,(s,X_s^{0,x_0},Z_s^{1n},Z_s^{2n}))\big|\cdot 1_{\{|Z_s^{1n}|+|Z_s^{2n}|\ge k\}}ds\Big]\\
+\;&\mathbf{E}\Big[\int_0^T\big|H_1(s,X_s^{0,x_0},Z_s^{1n},(u_1^*,u_2^*)\,(s,X_s^{0,x_0},Z_s^{1n},Z_s^{2n}))\\
&\qquad -H_1(s,X_s^{0,x_0},Z_s^1,(u_1^*,u_2^*)\,(s,X_s^{0,x_0},Z_s^1,Z_s^2))\big|ds\Big].
\end{aligned} \tag{4.42}$$

The first term converges to 0. Indeed, on one hand, for $n\ge1$, point (b) implies that,

$$\begin{aligned}
&|H_1^n(s,X_s^{0,x_0},Z_s^{1n},Z_s^{2n})-H_1(s,X_s^{0,x_0},Z_s^{1n},(u_1^*,u_2^*)\,(s,X_s^{0,x_0},Z_s^{1n},Z_s^{2n}))|\cdot 1_{\{|Z_s^{1n}|+|Z_s^{2n}|<k\}}\\
&\qquad\le C(1+|X_s^{0,x_0}|)k+C(1+|X_s^{0,x_0}|^\gamma).
\end{aligned}$$

On the other hand, considering point (d), we obtain that,

$$\begin{aligned}
&|H_1^n\left(s,X_s^{0,x_0},Z_s^{1n},Z_s^{2n}\right)-H_1\left(s,X_s^{0,x_0},Z_s^{1n},(u_1^*,u_2^*)\big(s,X_s^{0,x_0},Z_s^{1n},Z_s^{2n}\big)\right)|\cdot 1_{\{|Z_s^{1n}|+|Z_s^{2n}|<k\}}\\
&\qquad\le\sup_{\{(z_s^1,z_s^2),\ |z_s^1|+|z_s^2|\le k\}}|H_1^n\left(s,X_s^{0,x_0},z_s^1,z_s^2\right)\\
&\qquad\qquad -H_1\left(s,X_s^{0,x_0},z_s^1,(u_1^*,u_2^*)\left(s,X_s^{0,x_0},z_s^1,z_s^2\right)\right)|\to 0\text{ as }n\to\infty.
\end{aligned}$$

Then thanks to Lebesgue's dominated convergence theorem, the first term in (4.42) converges to 0 in $\mathcal{H}_T^1(\mathbf{R})$.

The second term is bounded by $\frac{C}{k^{2(q-1)/q}}$ with $q\in(1,2)$. Actually, from point (b) and Markov's inequality, for $1 < q < 2$ we have,

$$\begin{aligned}
&\mathbf{E}\Big[\int_0^T \big|H_1^n\left(s, X_s^{0,x_0}, Z_s^{1n}, Z_s^{2n}\right) \\
&\qquad -H_1\left(s, X_s^{0,x_0}, Z_s^{1n}, (u_1^*, u_2^*)\left(s, X_s^{0,x_0}, Z_s^{1n}, Z_s^{2n}\right)\right)\big| \cdot 1_{\{|Z_s^{1n}|+|Z_s^{2n}|\geq k\}} ds\Big] \\
\leq\ & C\Big\{\mathbf{E}\Big[\int_0^T (1+|X_s^{0,x_0}|)^q |Z_s^{1n}|^q + (1+|X_s^{0,x_0}|^\gamma)^q ds\Big]\Big\}^{\frac{1}{q}} \Big\{\mathbf{E}\Big[\int_0^T 1_{\{|Z_s^{1n}|+|Z_s^{2n}|\geq k\}} ds\Big]\Big\}^{\frac{q-1}{q}} \\
\leq\ & C\Big\{\mathbf{E}\Big[\int_0^T 1_{\{|Z_s^{1n}|+|Z_s^{2n}|\geq k\}} ds\Big]\Big\}^{\frac{q-1}{q}} \\
\leq\ & \frac{C}{k^{\frac{2(q-1)}{q}}}.
\end{aligned}$$

The second inequality is obtained by Young's inequality and estimates (4.34) and (4.9).

The third term in (4.42) also converges to 0, at least for a subsequence. Actually, since the sequence $(Z^{1n})_{n\geq 1}$ converges to Z^1 in $\mathcal{H}_T^2(\mathbf{R}^m)$, there exists a subsequence $(Z^{1n_k})_{k\geq 0}$ which converges to Z^1, $dt \otimes d\mathbf{P}$-a.e and such that $\sup_{k\geq 0} |Z_t^{1n_k}(\omega)|$ belongs to $\mathcal{H}_T^2(\mathbf{R})$. Therefore, taking the continuity Assumption (A3)-(ii) of $H_1(t, x, p, (u_1^*, u_2^*)(t, x, p, q))$ w.r.t (p, q), we obtain that,

$$\begin{aligned}
&H_1\left(s, X_s^{0,x_0}, Z_s^{1n_k}, (u_1^*, u_2^*)\left(s, X_s^{0,x_0}, Z_s^{1n_k}, Z_s^{2n_k}\right)\right) \\
&\qquad \longrightarrow_{k\to\infty} H_1\left(s, X_s^{0,x_0}, Z_s^1, (u_1^*, u_2^*)\left(s, X_s^{0,x_0}, Z_s^1, Z_s^2\right)\right)\ dt \otimes d\mathbf{P} - a.e.
\end{aligned}$$

and the process

$$\sup_{k\geq 0} |H_1\left(s, X_s^{0,x_0}, Z_s^{1n_k}, (u_1^*, u_2^*)\left(s, X_s^{0,x_0}, Z_s^{1n_k}, Z_s^{2n_k}\right)\right)| \in \mathcal{H}_T^q(\mathbf{R}).$$

Then once more by the dominated convergence theorem, we obtain,

$$\begin{aligned}
&H_1\left(s, X_s^{0,x_0}, Z_s^{1n_k}, (u_1^*, u_2^*)\left(s, X_s^{0,x_0}, Z_s^{1n_k}, Z_s^{2n_k}\right)\right) \\
&\qquad \longrightarrow_{k\to\infty} H_1\left(s, X_s^{0,x_0}, Z_s^1, (u_1^*, u_2^*)\left(s, X_s^{0,x_0}, Z_s^1, Z_s^2\right)\right) \text{ in } \mathcal{H}_T^q(\mathbf{R}),
\end{aligned}$$

which yields to the convergence of the third term in (4.42).

It follows that the sequence $((H_1^n(s, X_s^{0,x_0}, Z_s^{1n}, Z_s^{2n})_{s\leq T})_{n\geq 1}$ converges to $(H_1(s, X_s^{0,x_0}, Z_s^1, (u_1^*, u_2^*)(s, X_s^{0,x_0}, Z_s^1, Z_s^2)))_{s\leq T}$ in $L^1([0,T] \times \Omega, dt \otimes d\mathbf{P})$ and then

$$F^1(s, X_s^{0,x_0}) = H_1(s, X_s^{0,x_0}, Z_s^1, (u_1^*, u_2^*)(s, X_s^{0,x_0}, Z_s^1, Z_s^2)),\ dt \otimes d\mathbf{P} - a.e.$$

In the same way we have,

$$F^2(s, X_s^{0,x_0}) = H_2(s, X_s^{0,x_0}, Z_s^2, (u_1^*, u_2^*)(s, X_s^{0,x_0}, Z_s^1, Z_s^2)),\ dt \otimes d\mathbf{P} - a.s.$$

Thus the processes (W^i, Z^i), $i = 1, 2$, is solution of the backward equation (4.27). Finally taking into account of estimate (4.38) and the fact that Z^i, $i = 1, 2$, belong to $\mathcal{H}_T^2(\mathbf{R}^m)$ complete the proof. □

As a result of Theorems 4.2.1 and 4.3.1 we obtain the main result of this paper.

Theorem 4.3.2. *Assume that (A1), (A2) and (A3) are in force. Then the nonzero-sum differential game defined by (4.5) and (4.6) has a Nash equilibrium point.* □

Example: The linear quadratic case.

Let us take $m = 1$ and for $t \in [0, T]$, $X_t^{0,x_0} = x_0 + B_t$. Let $f(t, x, u, v) = ax + bu + cv$ and for $i = 1, 2$, $h_i(t, x, u, v) = \theta_i x^{p_i} + \gamma_i u^2 + \rho_i v^2$, $U = [-1, 1]$ and $V = [0, 1]$ where $a, b, c, p_i, \rho_i\ \gamma_i$ are real constants such that $p_i \geq 0$, $\gamma_1 > 0$ and $\rho_2 > 0$. Finally let g^i, $i = 1, 2$, be two Borel measurable functions with polynomial growth. The Hamiltonian functions H_i of the nonzero-sum differential game associated with X^{0,x_0}, f, h_i, g_i, $i = 1, 2$, and U, V are:

$$H_i(t, x, z_i, u, v) = z_i f(t, x, u, v) + h_i(t, x, u, v),\ i = 1, 2.$$

Next for $\eta \in \mathbf{R}$ let ψ and ϕ be functions from $\mathbf{R}$ to $\mathbf{R}$ defined by:

$$\psi(\eta) := -1_{[\eta<-1]} + \eta 1_{[-1\leq\eta\leq 1]} + 1_{[\eta>1]} \text{ and } \phi(\eta) := 1 \wedge \eta^+.$$

Thus the functions $u^*(t, x, z_1, z_2) := \psi(-\frac{bz_1}{2\gamma_1})$ and $v^*(t, x, z_1, z_2) := \phi(-\frac{cz_2}{2\rho_2})$ (which are continuous in (z_1, z_2)) verify the generalized Isaacs condition (A3). Therefore, according to Theorem 4.3.2, this game has a Nash equilibrium point $(u^*(t, X_t^{0,x_0}, Z_1(t), Z_2(t)), v^*(t, X_t^{0,x_0}, Z_1(t), Z_2(t)))_{t\leq T}$.

Remark 4.3.1. *For sake of simplicity we have dealt with the case of two players. However the method still work if we have more than two players. We just need a minor adaptation of the generalized Isaacs condition of (A3). Actually assume there are N players $P_1, ..., P_N$ ($N \geq 3$) and for $i = 1, ..., N$, let U_i be the compact set where the controls of player P_i take their values. Let $(H_i)_{i=1,...,N}$ be the Hamiltonian functions of the nonzero-sum differential game associated with $f(t, x, u_1, ..., u_N)$, $h_i(t, x, u_1, ..., u_N)$ and g^i, i.e.,*

$$H_i(t, x, z_i, u_1, ..., u_N) := z_i\sigma^{-1}(t, x).f(t, x, u_1, ..., u_N) + h_i(t, x, u_1, ..., u_N).$$

We assume that, uniformly w.r.t. x, f is of linear growth and h_i, g^i, $i = 1, ..., N$, are of polynomial growth. Next assume that generalized Isaacs condition, which reads as below, is satisfied:

(i) There exist N Borel functions $u_1^*, ..., u_N^*$ defined on $[0, T] \times \boldsymbol{R}^{(N+1)m}$, valued respectively in $U_1, ..., U_N$, such that for any $i = 1, ..., N$, $(t, x, z_1, ..., z_N) \in [0, T] \times \boldsymbol{R}^{(N+1)m}$, we have:

$$\begin{aligned}&H_i(t, x, z_i, (u_1^*, ..., u_N^*)(t, x, z_1, ..., z_N))\\ &\leq H_i(t, x, z_i, (u_1^*, ..., u_{i-1}^*)(t, x, z_1, ..., z_N), u_i, (u_{i+1}^*, ..., u_N^*)(t, x, z_1, ..., z_N)), \forall u_i \in U_i\ ;\end{aligned}$$

(ii) For any fixed (t, x) the mapping

$$(z_1, ..., z_N) \longmapsto H_i(t, x, z_i, (u_1^*, ..., u_N^*)(t, x, z_1, ..., z_N))$$

is continuous.

Then, if $\sigma(t, x)$ verifies the Assumption (A1), the differential game associated with the drift $f(t, x, u_1, ..., u_N)$, the instantaneous payoffs $h_i(t, x, u_1, ..., u_N)$ and the terminal payoffs $g^i(x)$, $i = 1, ..., N$, has a Nash equilibrium point.

Extension: In this study the main points we required are:

(i) the existence of $p > 1$ such that for any pair $(u, v) \in \mathcal{M}$, $\mathbf{E}[(\zeta_T(u, v))^p] < \infty$ where $\zeta_T(u, v)$ is defined as in (4.3) ;

(ii) the L^q-domination property or its adaptation ;

(iii) the generalized Isaacs condition.

As far as those points are in force, one can expect that the NZSDG has a NEP and, e.g., one can let drop the uniform ellipticity condition (4.2) on σ. Actually let us consider the following example where the matrix

σ is non longer bounded and does satisfy (4.2). Assume that $m = 1$ and for $(t, x) \in [0, T] \times \mathbf{R}$, $X^{t,x}$ is solution of:

$$dX_s^{t,x} = X_s^{t,x} dB_s,\ s \in [t, T] \text{ and } X_s^{t,x} = x > 0 \text{ for } s \in [0, t].$$

Note that

$$\forall\, s \leq T,\ X_s^{t,x} > 0. \tag{4.43}$$

Next let

$$f(s, x, u, v) := x(u_s + v_s),\ s \leq T,\ U = [-1, 1],\ V = [0, 1]$$

and assume that h_1 and h_2 are as in the previous example. Therefore the generalized Isaacs condition is satisfied with

$$u^*(t, x, z_1, z_2) := \psi(-\frac{z_1}{2\gamma_1}) \text{ and } v^*(t, x, z_1, z_2) := \phi(-\frac{z_2}{2\rho_2}),$$

where ψ and ϕ are the functions defined above.

Obviously for any $(u, v) \in \mathcal{M}$, $\zeta_T((u_s + v_s))$ belongs to $L^p(\mathbf{P})$ for any $p > 1$ since U and V are bounded sets. Next for any $s \in [t, T]$, we have:

$$X_s^{t,x} = x \exp\{B_s - B_t - \frac{1}{2}(s - t)\}.$$

Thus for $s \in]t, T]$, the law of $X_s^{t,x}$ has a density $p(t, x; s, y)$ given by

$$p(t, x; s, y) := \frac{1}{\sqrt{2\pi(s - t)}} \exp\{-\frac{1}{2(s - t)}[\ln(\frac{y}{x}) + \frac{1}{2}(s - t)]^2\} 1_{[y>0]},\ y \in \mathbf{R}.$$

So let $x_0 > 0$, $\delta > 0$ ($\delta + t < T$) and Φ defined by:

$$\Phi(s, y) := \frac{p(t, x; s, y)}{p(0, x_0; s, y)}, (s, y) \in [t + \delta, T] \times \mathbf{R}.$$

Therefore for any $\kappa > 0$, Φ belongs to $L^q([t + \delta, T] \times [\frac{1}{\kappa}, \kappa], p(0, x_0; s, y)dsdy)$ which is the adaptation of property (ii) of Definition 4.3.1 on L^q-domination, in taking into account (4.43). Now as the following estimate holds true:

$$\mathbf{E}\Big[\int_{t+\delta}^{T} \{1_{[X_s^{t,x} < \kappa^{-1}]} + 1_{[X_s^{t,x} > \kappa]}\} ds\Big] \leq \varrho(\kappa),$$

where the function ϱ is such that $\varrho(\kappa) \to 0$ as $\kappa \to \infty$. Then one can conclude that the NZSDG defined with those specific data σ, U, V, f, g_i, h_i, $i = 1, 2$, has a Nash equilibrium point. The proof can be established in the same way as we did in the previous sections. □

CHAPTER **5**

Risk-sensitive Nonzero-sum Stochastic Differential Games with Unbounded Coefficients

We consider, in this article, a risk-sensitive nonzero-sum stochastic differential game model. Assume that we have a system which is controlled by two players. Each one imposes a so-called admissible control which is an adapted stochastic process denoted by $u = (u_t)_{t\leq T}$ (resp. $v = (v_t)_{t\leq T}$) for player 1 (resp. player 2). The state of the system is described by a process $(x_t)_{t\leq T}$ which is the solution of the following stochastic differential equation:

$$dx_t = f(t, x_t, u_t, v_t)dt + \sigma(t, x_t)dB_t \text{ for } t \leq T \text{ and } x_0 = x. \tag{5.1}$$

The above process B is a Brownian motion. We establish this game model in a two-player framework for an intuitive comprehension. All results in this article are applicable to the multiple players case. Naturally, the control action is not free and has some risks. A *risk-sensitive nonzero-sum stochastic differential game* is a game model which takes into account the attitudes of the players toward risk. More precisely speaking, for player $i = 1, 2$, the utility (cost or payoff) is given by the following exponential form

$$J^i(u, v) = \mathbf{E}[e^{\theta\{\int_0^T h_i(s,X_s,u_s,v_s)ds+g^i(X_T)\}}].$$

The parameter θ represents the attitude of the player with respect to risk. What we are concerned here is a nonzero-sum stochastic differential game which means that the two players are of cooperate relationship. Both of them would like to minimize the cost and no one can cut more by unilaterally changing his own control. Therefore, the objective of the game problem is to find a *Nash equilibrium point* (u^*, v^*) such that,

$$J^1(u^*, v^*) \leq J^2(u, v^*) \text{ and } J^2(u^*, v^*) \leq J^2(u^*, v),$$

for any admissible control (u, v).

Let us illustrate now, why θ, in the cost function, can reflect the risk attitude of the controller. From the economic point of view, we denote by $G^i_{u,v} = \int_0^T h_i(s, X_s, u_s, v_s)ds + g^i(X_T)$ the wealth of each controller and for a smooth function $F(z)$, let $F(G^i_{u,v})$ be the cost might be brought from the wealth. The two participates would like to minimize the expected cost $\mathbf{E}[F(G^i_{u,v})]$. A notion of *risk sensitivity* is proposed as follows:

$$\gamma = \frac{F^{''}(G^i)}{F^{'}(G^i)}.$$

It is a reasonable function to reflect the trend, more precise, the curvature of cost F with respect to the wealth G^i. See W.H. Fleming's work [42] for more details. In the present paper, we choose utility function

$F(z)$ as an exponential form $F(z) = e^{\theta z}$. Both theory and practical experience have shown that it is often appropriate to use an exponential form of utility function. Therefore, the risk sensitivity γ is exactly the parameter θ.

We explain this specific case $\gamma = \theta$ in the following way. The expected utility $J^i(u, v) = \mathbf{E}[e^{\theta G^i_{u,v}}]$ is *certainty equivalent* to

$$\varrho^i_\theta(u, v) := \theta^{-1} \ln \mathbf{E}[e^{\theta G^i_{u,v}}].$$

By certainty equivalent, we refer to the minimum premium we are willing to pay to insure us against some risk (alternately in a payoff case, the maximum amount of money we are willing to pay for some gamble). Then, $\varrho^i_\theta(u, v) \sim \mathbf{E}[G^i_{u,v}] + \frac{\theta}{2}\mathrm{Var}(G^i_{u,v})$ provided that $\theta\mathrm{Var}(G^i_{u,v})$ is small (Var(.) is variance). Hence, minimizing $J^i(u, v)$ is equivalent to minimize $\varrho^i_\theta(u, v)$. The variance $\mathrm{Var}(G^i_{u,v})$ of wealth reflects the risk of decision to a certain extent. Therefore, it is obvious that when $\theta > 0$, the less risk the better. Such a decision maker in economic market will have a *risk-averse* attitude. On the contrary, when $\theta < 0$, the optimizer is called *risk-seeking*. Finally, if $\theta = 0$, this situation corresponds to the risk-neutral controller. For ease of presentation, we consider in the main text the risk-averse case only, the risk-seeking case is treated similarly.

About the risk-sensitive stochastic differential game problem, including nonzero-sum, zero-sum and mean-field cases, there are some previous works. Readers are referred to [7, 36, 43, 44, 66, 93] for further acquaintance. Among those results, a particular popular approach is partial differential equation, such as [7, 43, 44, 66, 93] with various objectives. Another method is through backward stochastic differential equation (BSDE) theory, see [36]. The nonlinear BSDE is introduced by Pardoux and Peng [84] and developed rapidly in the past two decades. The notion of BSDE is proofed as an efficient tool to deal with stochastic differential game. It has been used in the risk neutral case, see [58, 53]. About Other applications such as in field of mathematic finance, we refer the work by El-Kaoui et al. [39] (1997). A complete review on BSDEs theory as well as some new results on nonlinear expectation are introduced in a survey paper by Peng (2010) [87].

In the present paper, we study the risk-sensitive nonzero-sum stochastic differential game problem through BSDE in the same line as article by El-Karoui and Hamadène [36]. However in [36], the setting of game problem concerns only the case when the drift coefficient f in diffusion (5.1) is bounded. This constrain is too strict to some extent. Therefore, our motivation is to relax as much as possible the boundedness of the coefficient f. We assume that f is not bounded any more but instead, it has a linear growth condition. It is the main novelty of this work. To our knowledge, this general case has not been studied in the literature. Finally, we show the existence of Nash equilibrium point for this game. We provide a link between the game which we constructed and BSDE. The existence of the NEP is equivalent to the existence of solutions for a related BSDE, where this BSDE is multiple-dimensional with continuous generator involving both linear and quadratic terms of z. The difference with [36] is that the linear term of z is of linear growth ω by ω due to the linear growth of f. Under the generalized Isaacs hypothesis and domination property of solutions for (5.1), which holds when the uniform ellipticity condition on σ is satisfied, we finally show that the associated BSDE has a solution which then provides the NEP for our game.

The paper is organized as follows:

In Section 2, we present the precise model of risk-sensitive nonzero-sum stochastic differential game and necessary hypothesis on related coefficients. In Section 3, we firstly state some useful lemmas among which Lemma 5.2.2 and Corollary 5.2.1, which corresponding to integrability of the Doléan-Dade exponential local martingale, play a crucial role to overcome the difficulty that function f is not bounded. Then, the link beween game and BSDE is demonstrated by Proposition 5.2.1. The utility function is characterized by the initial value of a BSDE. Then, it turns out by Theorem 5.2.1 that the existence of the NEP for game problem is equivalent to the existence of some specific BSDE which is multiple dimensional, with continuous generator involving a quadratic term and a linear term of Z which is of linear growth ω by ω. Finally, we show, in Section 4, the solutions for this specific BSDE do exist when the generalized Isaacs condition is fulfilled and the law of the dynamics of the system satisfy the L^q-domination condition. The latter condition is naturally holds if the diffusion coefficient σ satisfies the well-known uniform ellipticity condition. Our method to deal with this BSDE with non-regular quadratic generator is that we firstly cancel the quadratic term by applying the exponential transform, then, smooth the new generator by mollifier technique in order to obtain the approximate solutions processes. Besides, in the Markovian framework, those approximate processes can be expressed via some deterministic functions. We then provide some good uniform estimates of the processes, as well as the growth properties of the corresponding deterministic functions. Then it followed by the convergence result which then gives a limit process finally proved as the solution of the BSDE after exponential transform. At the end, by taking the inverse transform, the proof for existence is finished.

5.1 Statement of the risk-sensitive game

In this section, we will give some basic notations, the preliminary assumptions throughout this paper, as well as the statement of the risk-sensitive nonzero-sum stochastic differential game. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space on which we define a d-dimensional Brownian motion $B = (B_t)_{0\leq t\leq T}$ with integer $d \geq 1$ and fixed $T > 0$. Let us denote by $\mathbf{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$, the natural filtration generated by process B and augmented by $\mathcal{N}_\mathbf{P}$ the $\mathbf{P}$-null sets, *i.e.* $\mathcal{F}_t = \sigma\{B_s, s \leq t\} \vee \mathcal{N}_\mathbf{P}$.

Let $\mathcal{P}$ be the σ-algebra on $[0, T] \times \Omega$ of $\mathcal{F}_t$-progressively measurable sets. Let $p \in [1, \infty)$ be real constant and $t \in [0, T]$ be fixed. We then define the following spaces:

- $\mathcal{L}^p$ = $\{\xi : \mathcal{F}_t$-measurable and $\mathbf{R}^m$-valued random variable such that $\mathbf{E}[|\xi|^p] < \infty\}$;
- $\mathcal{S}^p_{t,T}(\mathbf{R}^m)$ = $\{\varphi = (\varphi_s)_{t\leq s\leq T}$: $\mathcal{P}$-measurable, continuous and $\mathbf{R}^m$-valued such that $\mathbf{E}[\sup_{s\in[t,T]} |\varphi_s|^p] < \infty\}$;
- $\mathcal{H}^p_{t,T}(\mathbf{R}^m) = \{\varphi = (\varphi_s)_{t\leq s\leq T}$: $\mathcal{P}$-measurable and $\mathbf{R}^m$-valued such that $\mathbf{E}[(\int_t^T |\varphi_s|^2 ds)^{\frac{p}{2}}] < \infty\}$;
- $\mathcal{D}^p_{t,T}(\mathbf{R}^m) = \{\varphi = (\varphi_s)_{t\leq s\leq T}$: $\mathcal{P}$-measurable and $\mathbf{R}^m$-valued such that $\mathbf{E}[\sup_{s\in[t,T]} e^{p\varphi_s}] < \infty\}$.

Hereafter, $\mathcal{S}^p_{0,T}(\mathbf{R}^m)$, $\mathcal{H}^p_{0,T}(\mathbf{R}^m)$, $\mathcal{D}^p_{0,T}(\mathbf{R}^m)$ are simply denoted by $\mathcal{S}^p_T(\mathbf{R}^m)$, $\mathcal{H}^p_T(\mathbf{R}^m)$, $\mathcal{D}^p_T(\mathbf{R}^m)$. The following assumptions are in force throughout this paper. Let σ be the function defined as:

$$\begin{aligned} \sigma : [0,T] \times \mathbf{R}^m &\longrightarrow \mathbf{R}^{m\times m} \\ (t,x) &\longmapsto \sigma(t,x) \end{aligned}$$

which satisfies the following assumptions:

Assumptions (A1)

(i) σ is uniformly Lipschitz w.r.t x. *i.e.* there exists a constant C_1 such that,

$$\forall t \in [0,T], \forall\, x, x' \in \mathbf{R}^m, \quad |\sigma(t,x) - \sigma(t,x')| \leq C_1|x - x'|.$$

(ii) σ is invertible and bounded and its inverse is bounded, *i.e.*, there exits a constant C_σ such that

$$\forall (t,x) \in [0,T] \times \mathbf{R}^m, \quad |\sigma(t,x)| + |\sigma^{-1}(t,x)| \leq C_\sigma.$$

Remark 5.1.1. *Uniform ellipticity condition.*
Under Assumptions (A1), we can verify that, there exists a real constant $\epsilon > 0$ *such that for any* $(t,x) \in [0,T] \times \boldsymbol{R}^m$,

$$\epsilon.I \leq \sigma(t,x).\sigma^\top(t,x) \leq \epsilon^{-1}.I \tag{5.2}$$

where I *is the identity matrix of dimension* m.

We consider, in this article the 2-player case. The general multiple players game is a straightforward adaption.

For $(t,x) \in [0,T] \times \mathbf{R}^m$, let $X = (X_s^{t,x})_{s\leq T}$ be the solution of the following stochastic differential equation:

$$\begin{cases} X_s^{t,x} = x + \displaystyle\int_t^s \sigma(u, X_u^{t,x})dB_u, \ s \in [t,T]\,; \\ X_s^{t,x} = x, \ s \in [0,t]. \end{cases} \tag{5.3}$$

Under Assumptions (A1) above, we know such X exists and is unique (see Karatzas and Shreve, pp.289, 1991[70]). Let us now denote by U_1 and U_2 two compact metric spaces and let $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) be the set of $\mathcal{P}$-measurable processes $u = (u_t)_{t\leq T}$ (resp. $v = (v_t)_{t\leq T}$) with values in U_1 (resp. U_2). We denote by $\mathcal{M}$ the set $\mathcal{M}_1 \times \mathcal{M}_2$, hereafter $\mathcal{M}$ is called the set of admissible controls. We then introduce two Borelian functions

$$f : [0,T] \times \mathbf{R}^m \times U_1 \times U_2 \longrightarrow \mathbf{R}^m,$$
$$h_i \text{ (resp. } g^i) : [0,T] \times \mathbf{R}^m \times U_1 \times U_2 \text{ (resp. } \mathbf{R}^m) \longrightarrow \mathbf{R}, \ i = 1,2,$$

which satisfy:

Assumptions (A2)

(i) for any $(t,x) \in [0,T] \times \mathbf{R}^m$, $(u,v) \mapsto f(t,x,u,v)$ is continuous on $U_1 \times U_2$. Moreover f is of linear growth w.r.t x, *i.e.* there exists a constant C_f such that $|f(t,x,u,v)| \leq C_f(1+|x|), \forall (t,x,u,v) \in [0,T] \times \mathbf{R}^m \times U_1 \times U_2$.

(ii) for any $(t,x) \in [0,T] \times \mathbf{R}^m$, $(u,v) \mapsto h_i(t,x,u,v)$ is continuous on $U_1 \times U_2$, $i = 1,2$. Moreover, for $i = 1,2$, h_i is of sub-quadratic growth w.r.t x, *i.e.*, there exist constants C_h and $1 < \gamma < 2$ such that $|h_i(t,x,u,v)| \leq C_h(1+|x|^\gamma), \ \forall (t,x,u,v) \in [0,T] \times \mathbf{R}^m \times U_1 \times U_2$.

(iii) the functions g^i are of sub-quadratic growth with respect to x, *i.e.* there exist constants C_g and $1 < \gamma < 2$ such that $|g^i(x)| \leq C_g(1+|x|^\gamma), \forall x \in \mathbf{R}^m$, for i=1, 2.

For $(u,v) \in \mathcal{M}$, let $\mathbf{P}_{t,x}^{u,v}$ be the measure on $(\Omega, \mathcal{F})$ defined as follows:

$$d\mathbf{P}_{t,x}^{u,v} = \zeta_T\big(\int_0^\cdot \sigma^{-1}(s, X_s^{t,x}) f(s, X_s^{t,x}, u_s, v_s) dB_s\big) d\mathbf{P}, \tag{5.4}$$

where for any $(\mathcal{F}_t, \mathbf{P})$-continuous local martingale $M = (M_t)_{t\leq T}$,

$$\zeta(M) := \big(\exp\{M_t - \frac{1}{2}\langle M\rangle_t\}\big)_{t\leq T}, \tag{5.5}$$

where $\langle\,\rangle_\cdot$ denotes the quadratic variation process. We could deduce from Assumptions (A1), (A2)-(i) on σ and f that $\mathbf{P}_{t,x}^{u,v}$ is a probability on $(\Omega, \mathcal{F})$ (see Appendix A, [36] or [70] pp.200). By Girsanov's theorem (Girsanov, 1960 [50], pp.285-301), the process $B^{u,v} := (B_s - \int_0^s \sigma^{-1}(r, X_r^{t,x}) f(r, X_r^{t,x}, u_r, v_r) dr)_{s\leq T}$ is a $(\mathcal{F}_s, \mathbf{P}_{t,x}^{u,v})$-Brownian motion and $(X_s^{t,x})_{s\leq T}$ satisfies the following stochastic differential equation:

$$\begin{cases} dX_s^{t,x} = f(s, X_s^{t,x}, u_s, v_s)ds + \sigma(s, X_s^{t,x})dB_s^{u,v},\ s \in [t,T]\,; \\ X_s^{t,x} = x,\ s \in [0,t]. \end{cases} \quad (5.6)$$

As a matter of fact, the process $(X_s^{t,x})_{s\leq T}$ is not adapted with respect to the filtration generated by the Brownian motion $(B_s^{u,v})_{s\leq T}$ any more, therefore $(X_s^{t,x})_{s\leq T}$ is called a weak solution for the SDE (5.6). Now the system is controlled by player 1 (resp. Player 2) with u (resp. v).

Now, let us fix (t,x) to $(0,x_0)$, i.e., $(t,x) = (0,x_0)$. For a general risk preference coefficient θ, we define the *costs* (or *payoffs*) of the players for $(u,v) \in \mathcal{M}$ by:

$$J^i(u,v) = \mathbf{E}_{0,x_0}^{u,v}\left[e^{\theta\{\int_0^T h_i(s,X_s^{0,x_0},u_s,v_s)ds + g^i(X_T^{0,x_0})\}}\right],\ i = 1,2 \quad (5.7)$$

where $\mathbf{E}_{0,x_0}^{u,v}(.)$ is the expectation under the probability $\mathbf{P}_{0,x_0}^{u,v}$. Hereafter $\mathbf{E}_{0,x_0}^{u,v}$(resp. $\mathbf{P}_{0,x_0}^{u,v}$) will be simply denoted by $\mathbf{E}^{u,v}$(resp. $\mathbf{P}^{u,v}$). The functions h_1 and g^1 (resp. h_2 and g^2) are, respectively, the *instantaneous* and *terminal costs* for player 1 (resp. player 2). The player is called risk-averse (resp. risk-seeking) if $\theta > 0$ (resp. $\theta < 0$). Since the resolution of the problem is the same in all cases ($\theta > 0$, $\theta < 0$ or $\theta = 0$), without loss of generality, we assume $\theta = 1$ in (5.7) for simplicity below.

In this article, the quantity $J^i(u,v)$ is the cost that player i $(i = 1,2)$ has to pay for his control on the system. The problem is to find a pair of admissible controls (u^*,v^*) such that:

$$J^1(u^*,v^*) \leq J^1(u,v^*) \text{ and } J^2(u^*,v^*) \leq J^2(u^*,v),\ \forall (u,v) \in \mathcal{M}.$$

The control (u^*,v^*) is called a *Nash equilibrium point* for the risk-sensitive nonzero-sum stochastic differential game which means that each player chooses his best control, while, an equilibrium is a pair of controls, such that, when applied, no player will lower his/her cost by unilaterally changing his/her own control.

Let us introduce now the *Hamiltonian functions* for this game, for $i = 1,2$, by $H_i : [0,T] \times \mathbf{R}^{2m} \times U_1 \times U_2 \to \mathbf{R}$, associate:

$$H_i(t,x,p,u,v) = p\sigma^{-1}(t,x)f(t,x,u,v) + h_i(t,x,u,v). \quad (5.8)$$

Besides, we introduce the following assumptions which will play an important role in the proof of existence of equilibrium point.

Assumptions (A3)

(i) Generalized Isaacs condition: There exist two borelian applications u_1^*, u_2^* defined on $[0,T] \times \mathbf{R}^{3m}$, with values in U_1 and U_2 respectively, such that for any $(t,x,p,q,u,v) \in [0,T] \times \mathbf{R}^{3m} \times U_1 \times U_2$, we have:

$$H_1^*(t,x,p,q) = H_1(t,x,p,u_1^*(t,x,p,q),u_2^*(t,x,p,q)) \leq H_1(t,x,p,u,u_2^*(t,x,p,q))$$

and

$$H_2^*(t,x,p,q) = H_2(t,x,q,u_1^*(t,x,p,q),u_2^*(t,x,p,q)) \leq H_2(t,x,q,u_1^*(t,x,p,q),v).$$

(ii) The mapping $(p,q) \in \mathbf{R}^{2m} \longmapsto (H_1^*,H_2^*)(t,x,p,q) \in \mathbf{R}$ is continuous for any fixed $(t,x) \in [0,T] \times \mathbf{R}^m$. □

To solve this risk-sensitive stochastic differential game, we adopt the BSDE approach. Precisely speaking, to show the game has a Nash equilibrium point, it is enough to show that its associated BSDE, which is multi-dimensional and with a generator not standard, has a solution (see Theorem 5.2.1 below). Therefore the main objective of the next section is to study the connection between the risk-sensitive stochastic differential game and BSDEs.

5.2 Risk-sensitive nonzero-sum stochastic differential game and BSDEs

Let $(t,x) \in [0,T] \times \mathbf{R}^m$ and $(\theta_s^{t,x})_{s\le T}$ be the solution of the following forward stochastic differential equation:

$$\begin{cases} d\theta_s = b(s,\theta_s)ds + \sigma(s,\theta_s)dB_s, & s\in[t,T]; \\ \theta_s = x, & s \in [0,t], \end{cases} \tag{5.9}$$

where $\sigma : [0,T]\times \mathbf{R}^m \to \mathbf{R}^{m\times m}$ satisfies Assumptions (A1)(i)-(ii) and b: $[0,T]\times\mathbf{R}^m \to \mathbf{R}^m$ is a measurable function which verifies the following assumption:

Assumption (A4): The function b is uniformly Lipschitz and bounded, *i.e.*, there exist constants C_2 and C_b such that:

$$\forall t \in [0,T],\ \forall\, x, x' \in \mathbf{R}^m,\ |b(t,x) - b(t,x')| \le C_2|x-x'| \text{ and } |b(t,x)| \le C_b.$$

Before proceeding further, let us give some useful properties of stochastic process $(\theta_s^{t,x})_{s\le T}$.

Lemma 5.2.1. *Under Assumptions (A1) and (A4), we have*
(i) the stochastic process $(\theta_s^{t,x})_{s\le T}$ has moment of any order, i.e. there exists a constant $C_q \in \boldsymbol{R}$ such that: $\boldsymbol{P}$-a.s.

$$\forall\, q\in[1,\infty),\ \boldsymbol{E}\Big[\big(\sup_{s\le T}|\theta_s^{t,x}|\big)^{2q}\Big] \le C_q(1+|x|^{2q}); \tag{5.10}$$

(ii) additionally, it satisfies the following estimate: there exists a constant $C_{\lambda,l}\in\boldsymbol{R}$, such that $\boldsymbol{P}$-a.s.

$$\forall l\in[1,2),\ \lambda\in(0,\infty),\ \boldsymbol{E}\Big[e^{\lambda \sup\limits_{s\le T}|\theta_s^{t,x}|^l}\Big] \le e^{C_{\lambda,l}(1+|x|^l)}. \tag{5.11}$$

Apart from q, λ and l, the constants C_q and $C_{\lambda,l}$ in (5.10)(5.11) depend also on C_b and C_σ and T.

Proof. We refer readers [70] (pp.306) for the result (i). In the following, we only provide the proof of (ii). We denote $b(s,\theta_s^{t,x})$ and $\sigma(s,\theta_s^{t,x})$ simply by b_s and σ_s. Considering $(b_s)_{s\le T}$ is bounded and $\mathbf{E}[f] = \int_0^\infty \mathbf{P}\{f>u\}du$ for all positive function f, we obtain,

$$\begin{aligned}
&\mathbf{E}[e^{\lambda\sup_{s\le T}|\theta_s^{t,x}|^l}] \\
&= \mathbf{E}[e^{\lambda\sup_{s\le T}|x+\int_t^s b_r ds+\int_t^s\sigma_r dB_r|^l}] \\
&\le e^{C_{l,\lambda,b,T}\cdot(1+|x|^l)}\mathbf{E}[e^{C_{l,\lambda}\cdot\sup_{s\le T}|\int_0^s\sigma_r dB_r|^l}] \\
&= e^{C_{l,\lambda,b,T}\cdot(1+|x|^l)}\int_0^\infty \mathbf{P}\{e^{C_{l,\lambda}\cdot\sup_{s\le T}|\int_0^s\sigma_r dB_r|^l} > u\}du \\
&= e^{C_{l,\lambda,b,T}\cdot(1+|x|^l)}\left(1+\int_1^\infty \mathbf{P}\{e^{C_{l,\lambda}\cdot\sup_{s\le T}|\int_0^s\sigma_r dB_r|^l} > e^{C_{l,\lambda}\cdot u^l}\}de^{C_{l,\lambda}\cdot u^l}\right) \\
&= e^{C_{l,\lambda,b,T}\cdot(1+|x|^l)}\left(1+\int_0^\infty \mathbf{P}\{\sup_{s\le T}|\int_0^s\sigma_r dB_r| > u\}e^{C_{l,\lambda}\cdot u^l}C_{l,\lambda}lu^{l-1}du\right).
\end{aligned}$$

Apply Theorem 2 in [34] (pp.247), $\mathbf{P}\{\sup_{s\le T}|\int_0^s\sigma_r dB_r|>u\} \le e^{-\frac{u^2}{2TC_\sigma^2}}$. Therefore,

$$\begin{aligned}
&\mathbf{E}[e^{\lambda\sup_{s\le T}|\theta_s^{t,x}|^l}] \\
&\le e^{C_{l,\lambda,b,T}\cdot(1+|x|^l)}\left(1+\int_0^\infty e^{-\frac{u^2}{2TC_\sigma^2}}e^{C_{l,\lambda}\cdot u^l}C_{l,\lambda}lu^{l-1}du\right) \\
&\le e^{C_{l,\lambda,b,T,\sigma}\cdot(1+|x|^l)}.
\end{aligned}$$

The above inequality is finite since $1\le l<2$ and $u\le e^u$ for any $u>0$. □

Next let us recall the following result by Hausmann ([64], pp.14) related to integrability of the Doléan-Dade exponential local martingale defined by (5.5).

Lemma 5.2.2. *Assume (A1)-(i)(ii) and (A4), let $(\theta_s^{t,x})_{s\leq T}$ be the solution of (5.9) and φ be a $\mathcal{P}\otimes\mathcal{B}(\boldsymbol{R}^m)$-measurable application from $[0,T]\times\Omega\times\boldsymbol{R}^m$ to $\boldsymbol{R}^m$ which is of linear growth, that is, $\boldsymbol{P}$ -a.s., $\forall(s,x)\in[0,T]\times\boldsymbol{R}^m$,*

$$|\varphi(s,\omega,x)|\leq C_3(1+|x|).$$

Then, there exists some $p\in(1,2)$ and a constant C, where p depends only on C_σ, C_2, C_b, C_3, m while the constant C, depends only on m and p, but not on φ, such that:

$$\boldsymbol{E}\left[\left|\zeta_T(\int_0^\cdot \varphi(s,\theta_s^{t,x})dB_s)\right|^p\right]\leq C, \tag{5.12}$$

where the process $\zeta(\int_0^\cdot \varphi(s,\theta_s^{t,x})dB_s)$ is the density function defined in (5.5).

It follows from Lemma 5.2.2 that,

Corollary 5.2.1. *For an admissible control $(u,v)\in\mathcal{M}$ and $(t,x)\in[0,T]\times\boldsymbol{R}^m$, there exists some $p_0\in(1,2)$ and a constant C, such that*

$$\boldsymbol{E}\left[\left|\zeta_T(\int_0^\cdot \sigma(s,X_s^{t,x})^{-1}f(s,X_s^{t,x},u_s,v_s)dB_s)\right|^{p_0}\right]\leq C. \tag{5.13}$$

Remark 5.2.1. *Corollary 5.2.1 is needed for us in the proofs of Proposition 5.2.1 and Theorem 5.2.1 which is the main result of this work. Notice that the function f is no longer bounded as in the literature but is of linear growth in x.*

As a by-product of Lemma 5.2.1 and Lemma 5.2.2, we also have the similar estimates for the process $X^{t,x}$.

Lemma 5.2.3. *(i) There exist two constants $\bar{C}_q,\ \bar{C}_{\lambda,l}\in\boldsymbol{R}$, such that $\boldsymbol{P}$-a.s.*

$$\forall\, q\in[1,\infty),\ \boldsymbol{E}\Big[(\sup_{s\leq T}|X_s^{t,x}|)^{2q}\Big]\leq\bar{C}_q(1+|x|^{2q}), \tag{5.14}$$

and

$$\forall l\in[1,2),\ \lambda\in(0,\infty),\ \boldsymbol{E}\Big[e^{\lambda\sup_{s\leq T}|X_s^{t,x}|^l}\Big]\leq e^{\bar{C}_{\lambda,l}(1+|x|^l)}; \tag{5.15}$$

(ii) Moreover, for solutions of the weak formulation of SDEs (5.6), we have the similar results. Precisely speaking, for $(u,v)\in\mathcal{M}$, $\boldsymbol{E}_{t,x}^{u,v}$ is the expectation under the probability $\boldsymbol{P}_{t,x}^{u,v}$, then there exist constants $\tilde{C}_q$,$\tilde{C}_{\lambda,l}\in\boldsymbol{R}$, such that $\boldsymbol{P}$-a.s.

$$\forall\, q\in[1,\infty),\ \boldsymbol{E}_{t,x}^{u,v}\left[\Big(\sup_{s\leq T}|X_s^{t,x}|\Big)^{2q}\right]\leq\tilde{C}_q(1+|x|^{2q}), \tag{5.16}$$

and

$$\forall l\in[1,2),\ \lambda\in(0,\infty),\ \boldsymbol{E}_{t,x}^{u,v}\left[e^{\lambda\sup_{s\leq T}|X_s^{t,x}|^l}\right]\leq e^{\tilde{C}_{\lambda,l}(1+|x|^l)}. \tag{5.17}$$

Proof. We only prove (5.17). Since,

$$\mathbf{E}_{t,x}^{u,v}\left[e^{\lambda\sup_{s\leq T}|X_s^{t,x}|^l}\right]=\mathbf{E}\Big[e^{\lambda\sup_{s\leq T}|X_s^{t,x}|^l}\cdot\zeta_T\Big],$$

where ζ_T represents $\zeta_T(\int_0^\cdot \sigma(s,X_s^{t,x})^{-1}f(s,X_s^{t,x},u_s,v_s)dB_s)$. As a result of Corollary 5.2.1, there exists some $p_0 \in (1,2)$, such that, $\zeta_T \in L^{p_0}$. Therefore, by Young's inequality and (5.15), we obtain that,

$$\begin{aligned}\mathbf{E}_{t,x}^{u,v}\left[e^{\lambda \sup_{s\le T}|X_s^{t,x}|^l}\right] &\le \mathbf{E}\left[e^{\frac{p_0\lambda}{p_0-1}\sup_{s\le T}|X_s^{t,x}|^l}\right] + \mathbf{E}\left[|\zeta_T|^{p_0}\right]\\ &\le e^{\bar{C}_{\lambda,l,p_0}(1+|x|^l)} + C_{m,p_0}\\ &\le e^{\tilde{C}_{\lambda,l,m,p_0}(1+|x|^l)}.\end{aligned}$$

□

The next proposition characterizes the payoff function $J^i(u,v)$ for $i=1,2$ with form (5.7) by means of BSDEs. It turns out that the payoffs $J^i(u,v)$ can be expressed as the exponential of the initial value for a related BSDE. It is multidimensional, with a continuous generator involving a quadratic term of Z.

Proposition 5.2.1. *Under Assumptions (A1) and (A2), for any admissible control $(u,v)\in\mathcal{M}$, there exists a pair of adapted processes $(Y^{i,(u,v)},Z^{i,(u,v)})$, $i=1,2$, with values on $\boldsymbol{R}\times\boldsymbol{R}^m$ such that:*

(i) *For any $p>1$,*

$$\boldsymbol{E}^{u,v}\big[\sup_{0\le t\le T} e^{pY_t^{i,(u,v)}}\big]<\infty \text{ and } \boldsymbol{P}-a.s. \int_0^T |Z_t^{i,(u,v)}|^2dt<\infty. \tag{5.18}$$

(ii) *For $t\le T$,*

$$\begin{aligned}Y_t^{i,(u,v)} = g^i(X_T^{0,x_0}) &+ \int_t^T \big\{H_i(s,X_s^{0,x_0},Z_s^{i,(u,v)},u_s,v_s)+\frac{1}{2}|Z_s^{i,(u,v)}|^2\big\}ds\\ &- \int_t^T Z_s^{i,(u,v)}dB_s.\end{aligned} \tag{5.19}$$

The solution is unique for fixed $x_0\in\boldsymbol{R}^m$. Moreover, $J^i(u,v)=e^{Y_0^{i,(u,v)}}$.

Proof. Part I : Existence and uniqueness. We take the case of $i=1$ for example, and of course the case of $i=2$ can be solved in a similar way. The main method here is to define a reasonable form of the solution directly. We first eliminate the quadratic term in the generator by applying the classical exponential exchange. Then, the definition of Y component is closely related to Girsanov's transformation, and the process Z is given by the martingale representation theorem. Afterwards, we shall verify by Itô's formula that what we defined above is exactly the solution of the original BSDE.

As we stated in the previous section, the process $(X_s^{0,x_0})_{s\le T}$ satisfies SDE (5.6) by substituting $(0,x_0)$ for (t,x).

In order to remove the quadratic part in the generator of BSDE (5.19), we first take the classical exponential exchange as follows: $\forall t\le T$, let

$$\begin{cases}\bar{Y}_t^{1,(u,v)} = e^{Y_t^{1,(u,v)}};\\ \bar{Z}_t^{1,(u,v)} = \bar{Y}_t^{1,(u,v)}Z_t^{1,(u,v)}.\end{cases}$$

Therefore, the processes $(\bar{Y}_t^{1,(u,v)},\bar{Z}_t^{1,(u,v)})_{t\le T}$ solve the following BSDE:

$$\begin{aligned}\bar{Y}_t^{1,(u,v)} = e^{g^1(X_T^{0,x_0})} &+ \int_t^T \bar{Z}_s^{1,(u,v)}\sigma^{-1}(s,X_s^{0,x_0})f(s,X_s^{0,x_0},u_s,v_s)\\ &+ (\bar{Y}_s^{1,(u,v)})^+h(s,X_s^{0,x_0},u_s,v_s)ds - \int_t^T \bar{Z}_s^{1,(u,v)}dB_s,\ t\le T.\end{aligned} \tag{5.20}$$

Applying Girsanov's transformation as indicated by (5.4)-(5.5), the BSDE (5.20) then reduces to

$$\bar{Y}_t^{1,(u,v)} = e^{g^1(X_T^{0,x_0})} + \int_t^T (\bar{Y}_s^{1,(u,v)})^+ h(s, X_s^{0,x_0}, u_s, v_s)ds - \int_t^T \bar{Z}_s^{1,(u,v)} dB_s^{u,v},\ t \le T.$$

Let us now define the process $\bar{Y}^{1,(u,v)}$ explicitly by:

$$\bar{Y}_t^{1,(u,v)} := \mathbf{E}^{u,v}\Big[\exp\Big\{g^1(X_T^{0,x_0}) + \int_t^T h_1(s, X_s^{0,x_0}, u_s, v_s)ds\Big\}\Big|\mathcal{F}_t\Big],\ t \le T. \tag{5.21}$$

Considering the sub-quadratic growth Assumptions (A2)-(ii)(iii) on h_1 and g^1 and the estimate (5.17), we obtain,

$$\begin{aligned}
&\mathbf{E}^{u,v}\Big[\exp\Big\{g^1(X_T^{0,x_0}) + \int_0^T h_1(s, X_s^{0,x_0}, u_s, v_s)ds\Big\}\Big] \\
&\qquad \le \mathbf{E}^{u,v}\Big[\exp\Big\{C \sup_{0\le s\le T}\big(1 + |X_s^{0,x_0}|^\gamma\big)\Big\}\Big] < \infty,
\end{aligned}$$

with constant $C = C_g \vee (TC_h)$. Therefore, we claim that the process $(\bar{Y}_t^{1,(u,v)})_{t\le T}$ in (5.21) is well-defined.

We will give now the definition of process $(\bar{Z}_t^{1,(u,v)})_{t\le T}$. In the following, for notation convenience, we denote by ζ the following process $\zeta := (\zeta_t)_{t\le T} = (\zeta_t(\int_0^\cdot \sigma^{-1}(s, X_s^{0,x_0}) f(s, X_s^{0,x_0}, u_s, v_s)dB_s))_{t\le T}$. Then the definition (5.21) can be rewritten as:

$$\bar{Y}_t^{1,(u,v)} = \zeta_t^{-1} \cdot \mathbf{E}\Big[\zeta_T \cdot \exp\Big\{g^1(X_T^{0,x_0}) + \int_t^T h_1(s, X_s^{0,x_0}, u_s, v_s)ds\Big\}\Big|\mathcal{F}_t\Big],\ t \le T. \tag{5.22}$$

Thanks to Corollary 5.2.1, there exists some $p_0 \in (1,2)$, such that $\mathbf{E}[|\zeta_T|^{p_0}] < \infty$. Therefore, from Young's inequality, we get that for any constant $q \in (1, p_0)$,

$$\begin{aligned}
&\mathbf{E}\Big[\Big|\zeta_T \cdot \exp\Big\{g^1(X_T^{0,x_0}) + \int_0^T h_1(s, X_s^{0,x_0}, u_s, v_s)ds\Big\}\Big|^q\Big] \\
&\le \frac{q}{p_0}\mathbf{E}\left[|\zeta_T|^{p_0}\right] + \frac{p_0 - q}{p_0}\mathbf{E}\Big[\exp\Big\{\frac{qp_0}{p_0 - q} \cdot \big(g^1(X_T^{0,x_0}) + \int_0^T h_1(s, X_s^{0,x_0}, u_s, v_s)ds\big)\Big\}\Big].
\end{aligned}$$

As a consequence of Assumptions (A2)-(ii)(iii) and (5.11), the following expectation is finite, i.e.,

$$\frac{p_0 - q}{p_0}\mathbf{E}\Big[\exp\Big\{\frac{qp_0}{p_0 - q} \cdot \big(g^1(X_T^{0,x_0}) + \int_0^T h_1(s, X_s^{0,x_0}, u_s, v_s)ds\big)\Big\}\Big] < \infty$$

Then, we deduce that,

$$\zeta_T \cdot \exp\Big\{g^1(X_T^{0,x_0}) + \int_0^T h_1(s, X_s^{0,x_0}, u_s, v_s)ds\Big\} \in \mathcal{L}^q(d\mathbf{P}).$$

It follows from (5.22) and the representation theorem that, there exists a $\mathcal{P}$-measurable process $(\bar{\theta}_s)_{s\le T} \in \mathcal{H}_T^q(\mathbf{R}^m)$, such that for any $t \le T$,

$$\begin{aligned}
\bar{Y}_t^{1,(u,v)} &= \zeta_t^{-1} \exp\{-\int_0^t h_1(s, X_s^{0,x_0}, u_s, v_s)ds\} \times \\
&\qquad \times \{\mathbf{E}\Big[\zeta_T \exp\Big\{g^1(X_T^{0,x_0}) + \int_0^T h_1(s, X_s^{0,x_0}, u_s, v_s)ds\Big\}\Big] + \int_0^t \bar{\theta}_s dB_s\}
\end{aligned}$$

Let us denote by:

$$R_t := \mathbf{E}\Big[\zeta_T \exp\Big\{g^1(X_T^{0,x_0}) + \int_0^T h_1(s, X_s^{0,x_0}, u_s, v_s)ds\Big\}\Big] + \int_0^t \bar{\theta}_s dB_s, \quad t \le T.$$

Taking account of $d\zeta_t = \zeta_t\sigma^{-1}(t, X_t^{0,x_0})f(t, X_t^{0,x_0}, u_t, v_t)dB_t$ for $t \le T$, then by Itô's formula, we have $d\zeta_t^{-1} = -\zeta_t^{-1}\{\sigma^{-1}(t, X_t^{0,x_0})f(t, X_t^{0,x_0}, u_t, v_t)dB_t - |\sigma^{-1}(t, X_t^{0,x_0})f(t, X_t^{0,x_0}, u_t, v_t)|^2dt\},\ t \le T$. Moreover,

$$\begin{aligned} d\Big[\zeta_t^{-1}\exp\{-\int_0^t h_1(s, X_s^{0,x_0}, u_s, v_s)ds\}\Big] \\ &= -\zeta_t^{-1}\exp\{-\int_0^t h_1(s, X_s^{0,x_0,u_s,v_s})ds\}\Big\{\sigma^{-1}(t, X_t^{0,x_0})f(t, X_t^{0,x_0}, u_t, v_t)dB_t \\ &\quad + \big[-|\sigma^{-1}(t, X_t^{0,x_0})f(t, X_t^{0,x_0}, u_t, v_t)|^2 + h_1(t, X_t^{0,x_0}, u_t, v_t)\big]dt\Big\},\ t \le T. \end{aligned}$$

Hence, for $t \le T$,

$$\begin{aligned} d\bar{Y}_t^{1,(u,v)} =& -\zeta_t^{-1}\exp\{-\int_0^t h_1(s, X_s^{0,x_0}, u_s, v_s)ds\}\Big\{\sigma^{-1}(t, X_t^{0,x_0})f(t, X_t^{0,x_0}, u_t, v_t)dB_t \\ &\quad + \big[-|\sigma^{-1}(t, X_t^{0,x_0})f(t, X_t^{0,x_0}, u_t, v_t)|^2 + h_1(t, X_t^{0,x_0}, u_t, v_t)\big]dt\Big\}R_t \\ &+ \zeta_t^{-1}\exp\{-\int_0^t h_1(s, X_s^{0,x_0}, u_s, v_s)ds\}\bar{\theta}_t dB_t \\ &- \zeta_t^{-1}\exp\{-\int_0^t h_1(s, X_s^{0,x_0}, u_s, v_s)ds\}\sigma^{-1}(t, X_t^{0,x_0})f(t, X_t^{0,x_0}, u_t, v_t)\bar{\theta}_t dt, \end{aligned}$$

which allows us to define the process $\bar{Z}^{1,(u,v)}$ as the volatility coefficient of the above equation, *i.e.*, for $t \le T$,

$$\begin{aligned} \bar{Z}_t^{1,(u,v)} := -\zeta_t^{-1}\exp\{-\int_0^t h_1(s, X_s^{0,x_0}, u_s, v_s)ds\}\Big\{\sigma^{-1}(t, X_t^{0,x_0})f(t, X_t^{0,x_0}, u_t, v_t)R_t \\ - \bar{\theta}_t\Big\}. \end{aligned} \tag{5.23}$$

Then, it is not difficult to verify that the process $(\bar{Y}_t^{1,(u,v)}, \bar{Z}_t^{1,(u,v)})_{t\le T}$, as we defined by (5.21) (5.23) satisfies the BSDE (5.20). Moreover, it can be seen obviously from (5.21) that $\bar{Y}_t^{1,(u,v)} > 0$ for all $t \in [0, T]$. Therefore, we define the pair of processes $(Y^{1,(u,v)}, Z^{1,(u,v)})$ as follows:

$$\begin{cases} Y_t^{1,(u,v)} = \ln \bar{Y}_t^{1,(u,v)}; \\ Z_t^{1,(u,v)} = \dfrac{\bar{Z}_t^{1,(u,v)}}{\bar{Y}_t^{1,(u,v)}}, \quad t \le T. \end{cases}$$

which completes the proof of existence.

The uniqueness is natural by the above construction itself for fixed $x_0 \in \mathbf{R}^m$. Since, the solution of BSDE (5.20), if exists, will be of the form (5.21) and (5.23).

Part II : Norm estimates. Finally, let us focus on the estimate of $(Y_t^{1,(u,v)})_{t\le T}$ which is needed in the next theorem. First, as a consequence of the definition (5.21) that for any $p > 1$,

$$\begin{aligned} &\mathbf{E}^{u,v}\big[|\sup_{t\in[0,T]} \bar{Y}_t^{1,(u,v)}|^p\big] \\ &\le \mathbf{E}^{u,v}\Big[\Big|\sup_{t\in[0,T]}\Big\{\mathbf{E}^{u,v}\big[\exp\{g^1(X_T^{0,x_0}) + \int_0^T |h_1(s, X_s^{0,x_0}, u_s, v_s)|ds\}|\mathcal{F}_t\big]\Big\}\Big|^p\Big]. \end{aligned} \tag{5.24}$$

Noticing that the process $\mathbf{E}^{u,v}\big[\exp\{g^1(X_T^{0,x_0}) + \int_0^T |h_1(s, X_s^{0,x_0}, u_s, v_s)|ds)\}|\mathcal{F}_\cdot\big]$ is a $\mathcal{F}_t$-martingale, then Doob's maximal inequality(see [70] pp.14) implies that,

$$\begin{aligned} &\mathbf{E}^{u,v}\big[|\sup_{t\in[0,T]} \bar{Y}_t^{1,(u,v)}|^p\big] \\ &\le (\frac{p}{p-1})^p\mathbf{E}^{u,v}\Big[\Big|\mathbf{E}^{u,v}\big[\exp\{g^1(X_T^{0,x_0}) + \int_0^T |h_1(s, X_s^{0,x_0}, u_s, v_s)|ds\}|\mathcal{F}_T\big]\Big|^p\Big] \end{aligned} \tag{5.25}$$

Then, considering the Jensen's inequality and Assumption (A2)(ii)-(iii) on g^1 and h_1, it turns out that,

$$\begin{aligned}
&\mathbf{E}^{u,v}\big[|\sup_{t\in[0,T]}\bar{Y}_t^{1,(u,v)}|^p\big] \\
&\leq (\frac{p}{p-1})^p\mathbf{E}^{u,v}\big[\exp\{pg^1(X_T^{0,x_0})+p\int_0^T|h_1(s,X_s^{0,x_0},u_s,v_s)|ds\}\big] \\
&\leq (\frac{p}{p-1})^p\mathbf{E}^{u,v}\big[e^{\sup_{t\in[0,T]} C(1+|X_t^{0,x_0}|^\gamma)}\big]<\infty,
\end{aligned} \tag{5.26}$$

which is given by the estimate (5.17) with constant C depending on $p,\ C_g,\ C_h$, and T. Therefore,

$$\mathbf{E}^{u,v}\big[\sup_{t\in[0,T]}|\bar{Y}_t^{1,(u,v)}|^p\big]<\infty, \tag{5.27}$$

which gives,

$$\mathbf{E}^{u,v}\big[\sup_{t\in[0,T]}e^{pY_t^{1,(u,v)}}\big]<\infty, \qquad \forall p>1.$$

At last, note that in taking $t=0$ in (5.21) we obtain $J^1(u,v)=\bar{Y}_0^{1,(u,v)}=e^{Y_0^{1,(u,v)}}$ since $\mathcal{F}_0$ contains only **P** and $\mathbf{P}^{u,v}$ null sets. □

We are now ready to demonstrate the existence of Nash equilibrium point which is the main result of this article.

Theorem 5.2.1. *Let us assume that:*
(i) Assumptions (A1), (A2) and (A3) are fulfilled ;
(ii) There exist two pairs of $\mathcal{P}$-measurable processes (Y^i,Z^i) with values in $\mathbf{R}^{1+m}$, $i=1,2$, and two deterministic functions $\varpi^i(t,x)$ which are of subquadratic growth, i.e. $|\varpi^i(t,x)|\leq C(1+|x|^\gamma)$ with $1<\gamma<2$, $i=1,2$ such that:

$$\begin{cases}
\mathbf{P}\text{-a.s.}, \forall t\leq T, Y_t^i=\varpi^i(t,X_t^{0,x}) \text{ and } Z^i \text{is } dt\text{-square integrable } \mathbf{P}\text{-a.s.};\\
Y_t^i=g^i(X_T^{0,x})+\int_t^T\{H_i(s,X_s^{0,x},Z_s^i,(u^*,v^*)(s,X_s^{0,x},Z_s^1,Z_s^2))+\frac{1}{2}|Z_s^i|^2\}ds\\
\qquad -\int_t^T Z_s^i dB_s,\ \forall t\leq T.
\end{cases} \tag{5.28}$$

Then the pair of control $(u^*(s,X_s^{0,x},Z_s^1,Z_s^2),v^*(s,X_s^{0,x},Z_s^1,Z_s^2))_{s\leq T}$ is admissible and a Nash equilibrium point for the game.

Proof. For $s\leq T$, let us set $u_s^*=u^*(s,X_s^{0,x},Z_s^1,Z_s^2)$ and $v_s^*=v^*(s,X_s^{0,x},Z_s^1,Z_s^2)$. Then $(u^*,v^*)\in\mathcal{M}$. On the other hand, we obviously have $J^1(u^*,v^*)=e^{Y_0^1}$ by Proposition 5.2.1. Next for an arbitrary element $u\in\mathcal{M}_1$, let us show that $e^{Y^1}\leq e^{Y^{u,v^*}}$, which yields $e^{Y_0^1}=J^1(u^*,v^*)\leq J^1(u,v^*)=e^{Y_0^{1,(u,v^*)}}$. We focus on this point below. For the admissible control (u,v^*), thanks to Proposition 5.2.1, there exists a pair of $\mathcal{P}$-measurable processes $(Y_t^{i,(u,v^*)},Z_t^{i,(u,v^*)})_{t\leq T}$ for $i=1,2$, which satisfies: for any $p>1$,

$$\begin{cases}
Y^{i,(u,v^*)}\in\mathcal{D}_T^p(\mathbf{R},d\mathbf{P}^{u,v^*}),\ Z^{i,(u,v^*)} \text{ is } dt\text{-square integrable } \mathbf{P}\text{-a.s.}\\
Y_t^{i,(u,v^*)}=g^i(X_T^{0,x})+\int_t^T\{H_i(s,X_s^{0,x},Z_s^{i,(u,v^*)},u_s,v_s^*)+\frac{1}{2}|Z_s^{i,(u,v^*)}|^2\}dt\\
\qquad -\int_t^T Z_s^{i,(u,v^*)}dB_s,\ \forall t\leq T.
\end{cases} \tag{5.29}$$

Let us set: $\forall t\leq T$,

$$D_t^{u^*,v^*}:=e^{Y_t^1}, \qquad D_t^{u,v^*}:=e^{Y_t^{1,(u,v^*)}}.$$

Thus Itô-Meyer formula yields, for any $t \leq T$,

$$
\begin{aligned}
&-d(D_t^{u^*,v^*} - D_t^{u,v^*})^+ + dL_t^0(D^{u^*,v^*} - D^{u,v^*}) \\
&= \left[D_t^{u^*,v^*} H_1(t, X_t^{0,x}, Z_t^1, u_t^*, v_t^*) - D_t^{u,v^*} H_1(t, X_t^{0,x}, Z_t^{1,(u,v^*)}, u_t, v_t^*)\right] 1_{\{D_t^{u^*,v^*} - D_t^{u,v^*} > 0\}} dt \\
&\quad - (D_t^{u^*,v^*} Z_t^1 - D_t^{u,v^*} Z_t^{1,(u,v^*)}) 1_{\{D_t^{u^*,v^*} - D_t^{u,v^*} > 0\}} dB_t \\
&= \Big[D_t^{u^*,v^*} H_1(t, X_t^{0,x}, Z_t^1, u_t^*, v_t^*) - D_t^{u^*,v^*} H_1(t, X_t^{0,x}, Z_t^1, u_t, v_t^*) \\
&\quad + D_t^{u^*,v^*} H_1(t, X_t^{0,x}, Z_t^1, u_t, v_t^*) - D_t^{u,v^*} H_1(t, X_t^{0,x}, Z_t^{1,(u,v^*)}, u_t, v_t^*)\Big] 1_{\{D_t^{u^*,v^*} - D_t^{u,v^*} > 0\}} dt \\
&\quad - (D_t^{u^*,v^*} Z_t^1 - D_t^{u,v^*} Z_t^{1,(u,v^*)}) 1_{\{D_t^{u^*,v^*} - D_t^{u,v^*} > 0\}} dB_t \\
&= \Big[D_t^{u^*,v^*} \Big(H_1(t, X_t^{0,x}, Z_t^1, u_t^*, v_t^*) - H_1(t, X_t^{0,x}, Z_t^1, u_t, v_t^*)\Big) \\
&\quad + (D_t^{u^*,v^*} - D_t^{u,v^*})^+ h_1(t, X_t^{0,x}, u_t, v_t^*) \\
&\quad + (D_t^{u^*,v^*} Z_t^1 - D_t^{u,v^*} Z_t^{1,(u,v^*)}) \sigma^{-1}(t, X_t^{0,x}) f(t, X_t^{0,x}, u_t, v_t^*)\Big] 1_{\{D_t^{u^*,v^*} - D_t^{u,v^*} > 0\}} dt \\
&\quad - (D_t^{u^*,v^*} Z_t^1 - D_t^{u,v^*} Z_t^{1,(u,v^*)}) 1_{\{D_t^{u^*,v^*} - D_t^{u,v^*} > 0\}} dB_t,
\end{aligned}
\tag{5.30}
$$

where $L_t^0 = L_t^0(D^{u^*,v^*} - D^{u,v^*})$ is the local time of the continuous semimartingale $D^{u^*,v^*} - D^{u,v^*}$ at time 0. Next for $t \leq T$, let us give $B_t^{u,v^*} = (B_t - \int_0^t \sigma^{-1}(s, X_s^{0,x}) f(s, X_s^{0,x}, u_s, v_s^*) ds)_{t \leq T}$ which is an $\mathcal{F}_t$-Brownian motion under the probability $\mathbf{P}^{u,v^*}$, whose density w.r.t. $\mathbf{P}$ is defined by $\zeta_T := \zeta_T(\int_0^\cdot \sigma^{-1}(s, X_s^{0,x}) f(s, X_s^{0,x}, u_s, v_s^*) dB_s)$ as defined in (5.4). On the other hand, let us denote:

$$
\Gamma_t^1 := (D_t^{u^*,v^*} - D_t^{u,v^*})^+ \exp\{\int_0^t h_1(s, X_s^{0,x}, u_s, v_s^*) ds\}, \qquad t \leq T.
$$

Taking into account of (5.30), we then conclude by Itô's formula and Girsanov's transformation that, for $t \leq T$,

$$
\begin{aligned}
d\Gamma_t^1 = &- \exp\{\int_0^t h_1(s, X_s^{0,x}, u_s, v_s^*) ds\} \times \\
&\times \left[D_t^{u^*,v^*} \Delta_t^1 dt - (D_t^{u^*,v^*} Z_t^1 - D_t^{u,v^*} Z_t^{1,(u,v^*)}) dB_t^{u,v^*} - dL_t^0\right],
\end{aligned}
\tag{5.31}
$$

where

$$
\Delta_t^1 = H_1(t, X_t^{0,x}, Z_t^1, u_t^*, v_t^*) - H_1(t, X_t^{0,x}, Z_t^1, u_t, v_t^*) \leq 0,
$$

which is obtained by the generalized Isaacs' Assumption (A3)-(i). Next, let us define the stopping time τ_n as follows:

$$
\tau_n = \inf\{t \geq 0, |D_t^{u,v^*}| + |D_t^{u^*,v^*}| + \int_0^t (|Z_s^1|^2 + |Z_s^{1,(u,v^*)}|^2) ds \geq n\} \wedge T.
$$

The sequence of stopping times $(\tau_n)_{n \geq 0}$ is of stationary type and converges to T as $n \to \infty$. We then claim that, $\int_0^{t\wedge\tau_n} \exp\{\int_0^s h_1(r, X_r^{0,x}, u_r, v_r^*) dr\} 1_{\{D_s^{u^*,v^*} - D_s^{u,v^*} > 0\}} (D_s^{u^*,v^*} Z_s^1 - D_s^{u,v^*} Z_s^{1,(u,v^*)}) dB_s^{u,v^*}$ is a $\mathcal{F}_t$-martingale under the probability $\mathbf{P}^{u,v^*}$ as the following expectation

$$
\begin{aligned}
&\mathbf{E}^{u,v^*}\left[\int_0^{\tau_n} e^{2\int_0^s h_1(r, X_r^{0,x}, u_r, v_r^*) dr} (D_s^{u^*,v^*} Z_s^1 - D_s^{u,v^*} Z_s^{1,(u,v^*)})^2 ds\right] \\
&\leq \mathbf{E}^{u,v^*}\left[\int_0^{\tau_n} e^{2\int_0^s h_1(r, X_r^{0,x}, u_r, v_r^*) dr} \left(2|D_s^{u^*,v^*}|^2 |Z_s^1|^2 + 2|D_s^{u,v^*}|^2 |Z_s^{1,(u,v^*)}|^2\right) ds\right] \\
&\leq \mathbf{E}^{u,v^*}\left[\sup_{0 \leq s \leq \tau_n} \left\{2e^{2C_h(1+|X_s^{0,x}|^\gamma)} |D_s^{u^*,v^*}|^2\right\} \cdot \int_0^{\tau_n} |Z_s^1|^2\right] + \\
&\quad \mathbf{E}^{u,v^*}\left[\sup_{0 \leq s \leq \tau_n} \left\{2e^{2C_h(1+|X_s^{0,x}|^\gamma)} |D_s^{u,v^*}|^2\right\} \cdot \int_0^{\tau_n} |Z_s^{1,(u,v^*)}|^2\right]
\end{aligned}
\tag{5.32}
$$

is finite which is the consequence of the definition of τ_n and the estimate (5.17). Considering that L_t^0 is an increasing process, therefore, $\int_{t\wedge\tau_n}^{\tau_n}\exp\{\int_0^s h_1(r,X_r^{0,x},u_r,v_r^*)dr\}dL_s^0$ is positive. Now returning to equation (5.31), then taking integral on interval $(t\wedge\tau_n,\tau_n)$ and conditional expectation w.r.t. $\mathcal{F}_{t\wedge\tau_n}$ under the probability $\mathbf{P}^{u,v^*}$, yield that,

$$\Gamma^1_{t\wedge\tau_n}\le \mathbf{E}^{u,v^*}\left[\Gamma^1_{\tau_n}\middle|\mathcal{F}_{t\wedge\tau_n}\right],$$

i.e.,

$$\mathbf{E}^{u,v^*}\Gamma^1_{t\wedge\tau_n}\le \mathbf{E}^{u,v^*}\Gamma^1_{\tau_n}. \tag{5.33}$$

Indeed, for any $p>1$, $1<q<p$, and given $1<\gamma<2$, we have,

$$\begin{aligned}
&\mathbf{E}^{u,v^*}\left[\sup_{0\le t\le T}|\Gamma_t^1|^q\right]\\
&=\mathbf{E}^{u,v^*}\Big[\sup_{0\le t\le T}\Big\{|D_t^{u^*,v^*}-D_t^{u,v^*}|^q\exp\{q\int_0^t h_1(s,X_s^{0,x},u_s,v_s^*)ds\}\Big\}\Big]\\
&\le C\{\mathbf{E}^{u,v^*}\Big[\sup_{0\le t\le T}e^{pY_t^1}+\sup_{0\le t\le T}e^{pY_t^{1,(u,v^*)}}\Big]\\
&\quad+\mathbf{E}^{u,v^*}\Big[\sup_{0\le t\le T}e^{q\cdot\frac{p}{p-q}C_h(1+|X_t^{0,x}|^\gamma)}\Big]\}.
\end{aligned}\tag{5.34}$$

Indeed, for any $p>1$, $Y^1\in\mathcal{D}_T^p(\mathbf{R},d\mathbf{P}^{u,v^*})$, since we assume $Y_t^1=\varpi^1(t,X_t^{0,x})$ where ϖ^1 is deterministic and of subquadratic growth and finally (5.17). Meanwhile, $Y^{1,(u,v^*)}\in\mathcal{D}_T^p(\mathbf{R},d\mathbf{P}^{u,v^*})$ by (5.29). Therefore, (5.34) is finite. As the sequence $(\Gamma^1_{\tau_n})_{n\ge1}$ converges to $\Gamma_T^1=0$ as $n\to\infty$, $\mathbf{P}^{u,v^*}$-a.s., it then also converges to 0 in $\mathcal{L}^1(d\mathbf{P}^{u,v^*})$ since it is uniformly integral.

Next, by passing n to the limit on both sides of (5.33) and using the Fatou's lemma, we are able to show $\mathbf{E}^{u,v^*}[\Gamma_t^1]=0$, $\forall t\le T$, which implies $e^{Y_t^1}\le e^{Y_t^{u,v^*}}$, $\mathbf{P}$-a.s., since the probabilities $\mathbf{P}^{u,v^*}$ and $\mathbf{P}$ are equivalent. Thus, $e^{Y_0^1}=J^1(u^*,v^*)\le e^{Y_0^{1,(u,v^*)}}=J^1(u,v^*)$. In the same way, we can show that for arbitrary element $v\in\mathcal{M}_2$, then, $e^{Y_0^2}=J^2(u^*,v^*)\le e^{Y_0^{2,(u^*,v)}}=J^2(u^*,v)$, which indicate that, (u^*,v^*) is an equilibrium point of the game. □

5.3 Existence of solutions for markovian BSDE

In Section 5.2, we provide the existence of the Nash equilibrium point under appropriate conditions. It remains to show that the BSDEs (5.28) have solutions as desired in Theorem 5.2.1. Therefore, in this section, we focus on this objective.

We firstly recall the notion of domination.

5.3.1 Measure domination

Definition 5.3.1. ***: $\mathcal{L}^q$-Domination condition***
Let $q\in]1,\infty[$ be fixed. For a given $t_1\in[0,T]$, a family of probability measures $\{\nu_1(s,dx),s\in[t_1,T]\}$ defined on $\boldsymbol{R}^m$ is said to be $\mathcal{L}^q$- dominated by another family of probability measures $\{\nu_0(s,dx),s\in[t_1,T]\}$, if for any $\delta\in(0,T-t_1]$, there exists an application $\phi_{t_1}^\delta:[t_1+\delta,T]\times\boldsymbol{R}^m\to\boldsymbol{R}^+$ such that:

(i) $\nu_1(s,dx)ds=\phi_{t_1}^\delta(s,x)\nu_0(s,dx)ds$ on $[t_1+\delta,T]\times\boldsymbol{R}^m$.
(ii) $\forall k\ge1$, $\phi_{t_1}^\delta(s,x)\in\mathcal{L}^q([t_1+\delta,T]\times[-k,k]^m;\nu_0(s,dx)ds)$. □

We then have:

Lemma 5.3.1. *Let $q \in]1,\infty[$ be fixed, $(t_0,x_0) \in [0,T]\times \boldsymbol{R}^m$ and let $(\theta_s^{t_0,x_0})_{t_0\le s\le T}$ be the solution of SDE (5.9). If the diffusion coefficient function σ satisfies (5.2), then for any $s\in(t_0,T]$, the law $\bar\mu(t_0,x_0;s,dx)$ of $\theta_s^{t_0,x_0}$ has a density function $\rho_{t_0,x_0}(s,x)$, w.r.t. Lebesgue measure dx, which satisfies the following estimate: $\forall (s,x)\in(t_0,T]\times\boldsymbol{R}^m$,*

$$\varrho_1(s-t_0)^{-\frac{m}{2}}exp\left[-\frac{\Lambda|x-x_0|^2}{s-t_0}\right] \le \rho_{t_0,x_0}(s,x) \le \varrho_2(s-t_0)^{-\frac{m}{2}}exp\left[-\frac{\lambda|x-x_0|^2}{s-t_0}\right] \tag{5.35}$$

where ϱ_1, ϱ_2, Λ, λ are real constants such that $\varrho_1 \le \varrho_2$ and $\Lambda > \lambda$. Moreover for any $(t_1,x_1)\in[t_0,T]\times\boldsymbol{R}^m$, the family of laws $\{\bar\mu(t_1,x_1;s,dx), s\in[t_1,T]\}$ is L^q-dominated by $\bar\mu(t_0,x_0;s,dx)$.

Proof. Since σ satisfies (5.2) and b is bounded, then by Aronson's result (see [1]), the law $\bar\mu(t_0,x_0;s,dx)$ of $\theta_s^{t_0,x_0}$, $s\in]t_0,T]$, has a density function $\rho_{t_0,x_0}(s,x)$ which satisfies estimate (5.35).

Let us focus on the second claim of the lemma. Let $(t_1,x_1)\in[t_0,T]\times\mathbf{R}^m$ and $s\in(t_1,T]$. Then

$$\rho_{t_1,x_1}(s,x) = [\rho_{t_1,x_1}(s,x)\rho^{-1}_{t_0,x_0}(s,x)]\rho_{t_0,x_0}(s,x) = \phi_{t_1}(s,x)\rho_{t_0,x_0}(s,x)$$

with

$$\phi_{t_1,x_1}(s,x) = \left[\rho_{t_1,x_1}(s,x)\rho^{-1}_{t_0,x_0}(s,x)\right], (s,x)\in(t_1,T]\times\mathbf{R}^m.$$

For any $\delta\in(0,T-t_1]$, ϕ_{t_1,x_1} is defined on $[t_1+\delta,T]$. Moreover for any $(s,x)\in[t_1+\delta,T]$ it holds

$$\begin{aligned}\bar\mu(t_1,x_1;s,dx)ds &= \rho_{t_1,x_1}(s,x)dxds\\ &= \phi_{t_1,x_1}(s,x)\rho_{t_0,x_0}(s,x)dxds\\ &= \phi_{t_1,x_1}(s,x)\bar\mu(t_0,x_0;s,dx)ds.\end{aligned}$$

Next by (5.35), for any $(s,x)\in[t_1+\delta,T]\times\mathbf{R}^m$,

$$0\le\phi_{t_1,x_1}(s,x)\le\frac{\varrho_2(s-t_1)^{-\frac{m}{2}}}{\varrho_1(s-t_0)^{-\frac{m}{2}}}exp\left[\frac{\Lambda|x-x_0|^2}{s-t_0}-\frac{\lambda|x-x_1|^2}{s-t_1}\right]\equiv\Phi_{t_1,x_1}(s,x).$$

It follows that for any $k\ge0$, the function $\Phi_{t_1,x_1}(s,x)$ is bounded on $[t_1+\delta,T]\times[-k,k]^m$ by a constant κ which depends on t_1, δ, Λ, λ and k. Next let $q\in(1,\infty)$, then

$$\begin{aligned}\int_{t_1+\delta}^T\int_{[-k,k]^m}\Phi(s,x)^q\bar\mu(t_0,x_0;s,dx)ds &\le \kappa^q\int_{t_1+\delta}^T\int_{[-k,k]^m}\bar\mu(t_0,x_0;s,dx)ds\\ &= \kappa^q\int_{t_1+\delta}^T ds\mathbf{E}[1_{[-k,k]^m}(\theta_s^{t_0,x_0})]\le\kappa^qT.\end{aligned}$$

Thus Φ and then ϕ belong to $\mathcal{L}^q([t_1+\delta,T]\times[-k,k]^m;\nu_0(s,dx)ds)$. It follows that the family of measures $\{\bar\mu(t_1,x_1;s,dx), s\in[t_1,T]\}$ is $\mathcal{L}^q$-dominated by $\bar\mu(t_0,x_0;s,dx)$. □

As a by-product we have:

Corollary 5.3.1. *Let $x\in\boldsymbol{R}^m$ be fixed, $t\in[0,T]$, $s\in(t,T]$ and $\mu(t,x;s,dy)$ the law of $X_s^{t,x}$, i.e.,*

$$\forall A\in\mathcal{B}(\boldsymbol{R}^m),\ \mu(t,x;s,A)=\boldsymbol{P}(X_s^{t,x}\in A).$$

If σ satifies (5.2), then for any $q\in(1,\infty)$, the family of laws $\{\mu(t,x;s,dy), s\in[t,T]\}$ is $\mathcal{L}^q$-dominated by $\{\mu(0,x;s,dy), s\in[t,T]\}$. □

5.3.2 Existence of solutions for BSDE (5.28)

Now, we are well-prepared to provide the existence of solution for BSDE (5.28).

Theorem 5.3.1. *Let $x \in \boldsymbol{R}^m$ be fixed. Then under Assumptions (A1)-(A3), there exist two pairs of $\mathcal{P}$-measurable processes (Y^i, Z^i) with values in $\boldsymbol{R}^{1+m}$, $i = 1, 2$, and two deterministic functions $\varpi^i(t, x)$ which are of subquadratic growth, i.e. $|\varpi^i(t, x)| \leq C(1 + |x|^\gamma)$ with $1 < \gamma < 2$, $i = 1, 2$ such that,*

$$\begin{cases} \boldsymbol{P}\text{-a.s.}, \forall t \leq T, Y_t^i = \varpi^i(t, X_t^{0,x}) \text{ and } Z^i \text{is dt-square integrable } \boldsymbol{P}\text{-a.s.}; \\ Y_t^i = g^i(X_t^{0,x}) + \int_t^T \{H_i(s, X_s^{0,x}, Z_s^i, (u^*, v^*)(s, X_s^{0,x}, Z_s^1, Z_s^2)) + \frac{1}{2}|Z_s^i|^2\}ds \\ \qquad - \int_t^T Z_s^i dB_s, \quad \forall t \leq T. \end{cases} \tag{5.36}$$

Proof. We shall divide the proof into several steps. Our plan is the following. We apply the exponential exchange (see e.g. [71]) to eliminate the quadratic term in the generator. The pair of the solution processes (resp. the generator) is denoted by $(\bar{Y}, \bar{Z})$ (resp. G). We then approximate the new generator G by the Lipschitz continuous ones, which we denoted by G^n, such that the classical results about BSDE can be applied. It follows that, for each n, the BSDE with generator G being replaced by G^n, has a solution $(\bar{Y}^n, \bar{Z}^n)$. After that, we give the uniform estimates of the solutions, as well as the convergence property. In the convergence step, the measure domination property Corollary 5.3.1 plays a crucial role in passing from the weak limit to a strong sense one . Finally, we verify that the limits of the sequences are exactly the solutions of the BSDE.

Step 1. *Exponential exchange and approximation.*

For $t \in [0, T]$, and $i = 1, 2$, let us denote by:

$$\begin{cases} \bar{Y}_t^i = e^{Y_t^i}; \\ \bar{Z}_t^i = \bar{Y}_t^i Z_t^i. \end{cases} \tag{5.37}$$

Then, BSDE (5.36) reads, for $t \in [0, T]$ and $i = 1, 2$,

$$\bar{Y}_t^i = e^{g^i(X_T^{0,x})} + \int_t^T \mathbb{1}_{\bar{Y}_s^i > 0}\{\bar{Z}_s^i \sigma^{-1}(s, X_s^{0,x}) f(s, X_s^{0,x}, (u^*, v^*)(s, X_s^{0,x}, \frac{\bar{Z}_s^1}{\bar{Y}_s^1}, \frac{\bar{Z}_s^2}{\bar{Y}_s^2})) \\ + \bar{Y}_s^i h_i(s, X_s^{0,x}, (u^*, v^*)(s, X_s^{0,x}, \frac{\bar{Z}_s^1}{\bar{Y}_s^1}, \frac{\bar{Z}_s^2}{\bar{Y}_s^2}))\}ds - \int_t^T \bar{Z}_s^i dB_s. \tag{5.38}$$

Let us deal with the case $i = 1$ for example and the case $i = 2$ follows in the same way. Inspiring by the mollify technique in [58], we first denote here the generator of (5.38) by $G^1 : [0, T] \times \mathbf{R}^m \times \mathbf{R}^{+*} \times \mathbf{R}^{+*} \times \mathbf{R}^{2m} \longrightarrow \mathbf{R}$ (by $\mathbf{R}^{+*}$, we refer to $\mathbf{R}^+ \backslash \{0\}$), *i.e.*

$$\begin{aligned} G^1(s, x, y^1, y^2, z^1, z^2) &= \mathbb{1}_{y^1 > 0}\{z^1 \sigma^{-1}(s, x) f(s, x, (u^*, v^*)(s, x, \tfrac{z^1}{y^1}, \tfrac{z^2}{y^2})) \\ &\quad + y^1 h(s, x, (u^*, v^*)(s, x, \tfrac{z^1}{y^1}, \tfrac{z^2}{y^2}))\} \end{aligned}$$

which is still continuous w.r.t (y^1, y^2, z^1, z^2) considering the Assumption (A3)-(ii) and the transformation (5.37). Let ξ be an element of $C^\infty(\mathbf{R}^{+*} \times \mathbf{R}^{+*} \times \mathbf{R}^{2m}, \mathbf{R})$ with compact support and satisfying:

$$\int_{\mathbf{R}^{+*} \times \mathbf{R}^{+*} \times \mathbf{R}^{2m}} \xi(y^1, y^2, z^1, z^2) dy^1 dy^2 dz^1 dz^2 = 1.$$

For $(t, x, y^1, y^2, z^1, z^2) \in [0, T] \times \mathbf{R}^m \times \mathbf{R}^{+*} \times \mathbf{R}^{+*} \times \mathbf{R}^{2m}$, we set,

$$\tilde{G}^{1n}\left(t, x, y^1, y^2, z^1, z^2\right) = \int_{\mathbf{R}^{+*} \times \mathbf{R}^{+*} \times \mathbf{R}^{2m}} n^4 G^1(s, \varphi_n(x), p^1, p^2, q^1, q^2) \cdot \\ \xi\left(n(y^1 - p^1), n(y^2 - p^2), n(z^1 - q^1), n(z^2 - q^2)\right) dp^1 dp^2 dq^1 dq^2,$$

where $\varphi_n(x) = ((x_j \vee (-n)) \wedge n)_{j=1,2,\dots,m}$, for $x = (x_j)_{j=1,2,\dots,m} \in \mathbf{R}^m$. We next define $\psi \in C^\infty(\mathbf{R}^{2+2m}, \mathbf{R})$ by,

$$\psi(y^1, y^2, z^1, z^2) = \begin{cases} 1, & |y^1|^2 + |y^2|^2 + |z^1|^2 + |z^2|^2 \le 1, \\ 0, & |y^1|^2 + |y^2|^2 + |z^1|^2 + |z^2|^2 \ge 4. \end{cases}$$

Then, we define the measurable function sequence $(G^{1n})_{n\ge 1}$ as follows: $\forall (t, x, y^1, y^2, z^1, z^2) \in [0, T] \times \mathbf{R}^m \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}^{2m}$,

$$G^{1n}(t, x, y^1, y^2, z^1, z^2) = \psi(\frac{y^1}{n}, \frac{y^2}{n}, \frac{z^1}{n}, \frac{z^2}{n})\tilde{G}^{1n}(t, x, \psi_n(y^1), \psi_n(y^2), z^1, z^2),$$

where for each n, $\psi_n(y)$ is a continuous function for $y \in \mathbf{R}$, and $\psi_n(y) = 1/n$ if $y \le 0$; $\psi_n(y) = y$ if $y \ge 1/n$. We have the following properties:

$$\begin{cases} (a)\ G^{1n} \text{ is uniformly lipschitz w.r.t } (y^1, y^2, z^1, z^2); \\ (b)\ |G^{1n}(t, x, y^1, y^2, z^1, z^2)| \le C_f C_\sigma (1 + |\varphi_n(x)|)|z^1| + C_h(1 + |\varphi_n(x)|^\gamma)(y^1)^+; \\ (c)\ |G^{1n}(t, x, y^1, y^2, z^1, z^2)| \le c_n, \text{ for any } (t, x, y^1, y^2, z^1, z^2); \\ (d)\ \text{For any } (t, x) \in [0, T] \times \mathbf{R}^m, \varepsilon > 0 \text{ and } \mathbf{K} \text{ a compact subset of } [\varepsilon, \frac{1}{\varepsilon}]^2 \times \mathbf{R}^{2m}, \\ \qquad \sup_{(y^1, y^2, z^1, z^2) \in \mathbf{K}} |G^{1n}(t, x, y^1, y^2, z^1, z^2) - G^1(t, x, y^1, y^2, z^1, z^2)| \to 0, \text{ as } n \to \infty. \end{cases} \tag{5.39}$$

The same technique provides the sequence $(G^{2n})_{n\ge 1}$, which is indeed, the approximation of function G^2. For each $n \ge 1$ and $(t, x) \in [0, T] \times \mathbf{R}^m$, it is a direct result of (5.39)-(a) that (see [84]), there exist two pairs of processes $(\bar{Y}^{1n;(t,x)}_s, \bar{Z}^{1n;(t,x)}_s)_{t\le s\le T}$, $(\bar{Y}^{2n;(t,x)}_s, \bar{Z}^{2n;(t,x)}_s)_{t\le s\le T} \in \mathcal{S}^2_{t,T}(\mathbf{R}) \times \mathcal{H}^2_{t,T}(\mathbf{R}^m)$, which satisfy, for $s \in [t, T]$,

$$\begin{cases} \bar{Y}^{1n;(t,x)}_s = e^{g^1(X^{t,x}_T)} + \int_s^T G^{1n}(r, X^{t,x}_r, \bar{Y}^{1n;(t,x)}_r, \bar{Y}^{2n;(t,x)}_r, \bar{Z}^{1n;(t,x)}_r, \bar{Z}^{2n;(t,x)}_r)dr \\ \qquad\qquad - \int_s^T \bar{Z}^{1n;(t,x)}_r dB_r; \\ \bar{Y}^{2n;(t,x)}_s = e^{g^2(X^{t,x}_T)} + \int_s^T G^{2n}(r, X^{t,x}_r, \bar{Y}^{1n;(t,x)}_r, \bar{Y}^{2n;(t,x)}_r, \bar{Z}^{1n;(t,x)}_r, \bar{Z}^{2n;(t,x)}_r)dr \\ \qquad\qquad - \int_s^T \bar{Z}^{2n;(t,x)}_r dB_r. \end{cases} \tag{5.40}$$

Meanwhile, the properties (5.39)-(a),(c) and the result of El karoui et al. (ref. [39]) yield that, there exist two sequences of deterministic measurable applications $\varsigma^{1n}(resp.\ \varsigma^{2n}) : [0, T] \times \mathbf{R}^m \to \mathbf{R}$ and $\mathfrak{z}^{1n}(resp.\ \mathfrak{z}^{2n}) : [0, T] \times \mathbf{R}^m \to \mathbf{R}^m$ such that for any $s \in [t, T]$,

$$\bar{Y}^{1n;(t,x)}_s = \varsigma^{1n}(s, X^{t,x}_s) \quad (resp.\ \bar{Y}^{2n;(t,x)}_s = \varsigma^{2n}(s, X^{t,x}_s)) \tag{5.41}$$

and

$$\bar{Z}^{1n;(t,x)}_s = \mathfrak{z}^{1n}(s, X^{t,x}_s) \quad (resp.\ \bar{Z}^{2n;(t,x)}_s = \mathfrak{z}^{2n}(s, X^{t,x}_s)).$$

Besides, we have the following deterministic expression: for $i = 1, 2$, and $n \ge 1$,

$$\varsigma^{in}(t, x) = \mathbf{E}\Big[e^{g^i(X^{t,x}_T)} + \int_t^T F^{in}(s, X^{t,x}_s)ds\Big], \quad \forall (t, x) \in [0, T] \times \mathbf{R}^m, \tag{5.42}$$

where,

$$F^{in}(s, x) = G^{in}(s, x, \varsigma^{1n}(s, x), \varsigma^{2n}(s, x), \mathfrak{z}^{1n}(s, x), \mathfrak{z}^{2n}(s, x)).$$

Step 2. *Uniform integrability of* $(\bar{Y}^{1n;(t,x)})_{n\geq 1}$ *for fixed* $(t,x) \in [0,T] \times \boldsymbol{R}^m$.

In this step, we will deal with the case of $i=1$, the case of $i=2$ can be treated in a similar way. For each $n \geq 1$, let us consider BSDE as follows, for $s \in [t,T]$,

$$\tilde{Y}_s^{1n} = e^{g^1(X_T^{t,x})} + \int_s^T \Big\{ C_f C_\sigma (1 + |\varphi_n(X_r^{t,x})|)|\tilde{Z}_r^{1n}| + C_h(1 + |\varphi_n(X_r^{t,x})|^\gamma)(\tilde{Y}_r^{1n})^+ \Big\} dr \\ - \int_s^T \tilde{Z}_r^{1n} dB_r. \tag{5.43}$$

Obviously, for any $x \in \mathbf{R}^m$ and integer $n \geq 1$, the application which to $(y,z) \in \mathbf{R}^{1+m}$ associates $C_f C_\sigma(1+\varphi_n(x))|z| + C_h(1+\varphi_n(x))^\gamma y^+$ is Lipchitz continuous. Besides, $e^{g^1(X_T^{t,x})} \in \mathcal{L}^p(d\mathbf{P})$, $\forall p \geq 1$ which is the consequence of Assumption (A2)-(iii) and (5.15). Therefore, from the result of Pardoux and Peng [85], we know that a pair of solutions $(\tilde{Y}_s^{1n}, \tilde{Z}_s^{1n})_{t \leq s \leq T} \in \mathcal{S}_{t,T}^p(\mathbf{R}) \times \mathcal{H}_{t,T}^p(\mathbf{R}^m)$ exists for any $p > 1$. Moreover through an adaptation of the result given by El Karoui et al. (1997,[39]), we can infer the existence of deterministic measurable function $\tilde{\varsigma}^{1n}$: $[0,T] \times \mathbf{R}^m \to \mathbf{R}$ such that, for any $s \in [t,T]$,

$$\tilde{Y}_s^{1n} = \tilde{\varsigma}^{1n}(s, X_s^{t,x}). \tag{5.44}$$

Next let us consider the process

$$B_s^n = B_s - \int_0^s 1_{[t,T]}(r) C_f C_\sigma (1 + |\varphi_n(X_r^{t,x})|) \text{sign}(\tilde{Z}_r^{1n}) dr, \ 0 \leq s \leq T,$$

which is, thanks to Girsanov's Theorem, a Brownian motion under the probability $\mathbf{P}^n$ on $(\Omega, \mathcal{F})$ whose density with respect to $\mathbf{P}$ is

$$\zeta_T := \zeta_T\{C_f C_\sigma(1 + |\varphi_n(X_s^{t,x})|)\text{sign}(\tilde{Z}_s^{1n}) 1_{[t,T]}(s)\},$$

where for any $z = (z^i)_{i=1,\ldots,d} \in \mathbf{R}^m$, $\text{sign}(z) = (1_{[|z^i| \neq 0]} \frac{z^i}{|z^i|})_{i=1,\ldots,d}$ and $\zeta_T(\cdot)$ is defined by (5.5). Then (5.43) becomes

$$\tilde{Y}_s^{1n} = e^{g^1(X_T^{t,x})} + \int_s^T C_h(1 + |\varphi_n(X_r^{t,x})|^\gamma)(\tilde{Y}_r^{1n})^+ dr - \int_s^T \tilde{Z}_r^{1n} dB_r^n, \quad t \leq s \leq T.$$

Therefore, taking into account of (5.44), we deduce,

$$\tilde{\varsigma}^{1n}(t,x) = \mathbf{E}^n \Big[e^{g^1(X_T^{t,x}) + \int_t^T C_h \left(1 + |\varphi_n(X_s^{t,x})|^\gamma\right) ds} | \mathcal{F}_t \Big],$$

where $\mathbf{E}^n$ is the expectation under probability $\mathbf{P}^n$. Taking the expectation on both sides under the probability $\mathbf{P}^n$ and considering $\tilde{\varsigma}^{1n}(t,x)$ is deterministic, one obtains,

$$\tilde{\varsigma}^{1n}(t,x) = \mathbf{E}^n \Big[e^{g^1(X_T^{t,x}) + \int_t^T C_h \left(1 + |\varphi_n(X_s^{t,x})|^\gamma\right) ds} \Big].$$

Then by the Assumption (A2)-(iii) we have: $\forall (t,x) \in [0,T] \times \mathbf{R}^m$,

$$\begin{aligned} |\tilde{\varsigma}^{1n}(t,x)| &\leq \mathbf{E}^n \Big[e^{C \sup_{0 \leq s \leq T}(1 + |X_s^{t,x}|^\gamma)} \Big] \\ &= \mathbf{E} \Big[e^{C \sup_{0 \leq s \leq T}(1 + |X_s^{t,x}|^\gamma)} \cdot \zeta_T \Big]. \end{aligned}$$

By Lemma 5.2.2, there exists some $1 < p_0 < 2$ (which does not depend on (t,x)), such that $\mathbf{E}[|\zeta_T|^{p_0}] < \infty$. Applying Young's inequality, besides, considering (5.15) yield that,

$$\begin{aligned} |\tilde{\varsigma}^{1n}(t,x)| &\leq \mathbf{E}\big[e^{\frac{C p_0}{p_0 - 1} \sup_{0 \leq s \leq T}(1 + |X_s^{t,x}|^\gamma)} \big] + \mathbf{E}\big[|\zeta_T|^{p_0} \big] \\ &\leq e^{C(1 + |x|^\gamma)}. \end{aligned}$$

Next taking into account point (5.39)-(b) and using comparison Theorem of BSDEs, we obtain for any $s \in [t, T]$,

$$\tilde{Y}_s^{1n} = \tilde{\varsigma}^{1n}(s, X_s^{t,x}) \geq \bar{Y}_s^{1n;(t,x)} = \varsigma^{1n}(s, X_s^{t,x}).$$

Then, by choosing $s = t$, we get that $\varsigma^{1n}(t, x) \leq e^{C(1+|x|^\gamma)}$, $(t, x) \in [0, T] \times \mathbf{R}^m$. But in a similar way one can show that for any $(t, x) \in [0, T] \times \mathbf{R}^m$, $\varsigma^{1n}(t, x) \geq e^{-C(1+|x|^\gamma)}$. Therefore,

$$e^{-C(1+|x|^\gamma)} \leq \varsigma^{1n}(t, x) \leq e^{C(1+|x|^\gamma)},\ (t, x) \in [0, T] \times \mathbf{R}^m. \tag{5.45}$$

By (5.45),(5.41) and (5.15), we conclude, $\bar{Y}_s^{1n;(t,x)} \in \mathcal{S}_{t,T}^p(\mathbf{R}^m)$ holds, i.e., for any $p > 1$, we have,

$$\mathbf{E}\Big[\sup_{t\leq s\leq T} \big|\bar{Y}_s^{1n;(t,x)}\big|^p\Big] < \infty. \tag{5.46}$$

Step 3. *Uniform integrability of* $(\bar{Z}_s^{1n;(t,x)})_{t\leq s\leq T}$.

Recalling the equation (5.40) and making use of Itô's formula with $(\bar{Y}_s^{1n;(t,x)})^2$, we obtain, in a standard way, the following result.

There exists a constant C independent of n and t, x such that for any $t \leq T$, for $i = 1, 2$,

$$\mathbf{E}\big[\int_t^T |\bar{Z}_s^{1n;(t,x)}|^2 ds\big] \leq C. \tag{5.47}$$

The proof is omitted for conciseness.

Step 4. *There exists a subsequence of* $((\bar{Y}_s^{1n;(0,x)}, \bar{Z}_s^{1n;(0,x)})_{0\leq s\leq T})_{n\geq 1}$ *which converges respectively to* $(\bar{Y}_s^1, \bar{Z}_s^1)_{0\leq s\leq T}$*, solution of the BSDE* (5.38). Moreover, $\bar{Y}_s^1 > 0$, $\forall s \in [0, T]$, $\mathbf{P}$-a.s.

Let us recall the expression (5.42) for case $i = 1$,

$$\varsigma^{1n}(t, x) = \mathbf{E}\Big[e^{g^1(X_T^{t,x})} + \int_t^T F^{1n}(s, X_s^{t,x})ds\Big], \qquad \forall (t, x) \in [0, T] \times \mathbf{R}^m. \tag{5.48}$$

We now apply property (5.39)-(b) in Step 1 combined with the uniform estimates (5.46), (5.47) and the Young's inequality to show that, for $1 < q < 2$,

$$\begin{aligned}
&\mathbf{E}\Big[\int_0^T |F^{1n}(s, X_s^{0,x})|^q ds\Big] \\
&\quad\leq C\mathbf{E}\Big[\int_0^T (1 + |\varphi_n(X_s^{0,x})|)^q |\bar{Z}_s^{1n;(0,x)}|^q + (1 + |\varphi_n(X_s^{0,x})|)^{\gamma q} |\bar{Y}_s^{1n;(0,x)}|^q ds\Big] \\
&\quad\leq C\mathbf{E}\Big[\Big(\int_0^T |\bar{Z}_s^{1n;(0,x)}|^2 ds\Big)^{\frac{q}{2}} \Big(\int_0^T (1 + |X_s^{0,x}|)^{\frac{2q}{2-q}} ds\Big)^{\frac{2-q}{2}}\Big] \\
&\qquad + C\mathbf{E}\Big[\sup_{0\leq s\leq T} |\bar{Y}_s^{1n;(0,x)}|^q \cdot \int_0^T (1 + |X_s^{0,x}|)^{\gamma q} ds\Big] \\
&\quad\leq C\{\mathbf{E}\Big[\int_0^T |\bar{Z}_s^{1n;(0,x)}|^2 ds\Big] + \mathbf{E}\Big[\sup_{0\leq s\leq T} |\bar{Y}_s^{1n;(0,x)}|^2\Big] + 1\} \\
&\quad< \infty.
\end{aligned} \tag{5.49}$$

Therefore, there exists a sub-sequence $\{n_k\}$ (for notation simplification, we still denote it by $\{n\}$) and a $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbf{R}^m)$-measurable deterministic function $F^1(s, y)$ such that:

$$F^{1n} \to F^1 \text{ weakly in } \mathcal{L}^q([0, T] \times \mathbf{R}^m; \mu(0, x; s, dy)ds). \tag{5.50}$$

Next we aim to prove that $(\varsigma^{1n}(t,x))_{n\geq 1}$ is a Cauchy sequence for each $(t,x)\in[0,T]\times\mathbf{R}^m$. Now let (t,x) be fixed, $\eta>0$, k, n and $m\geq 1$ be integers. From (5.48), we have,

$$\begin{aligned}|\varsigma^{1n}(t,x)-\varsigma^{1m}(t,x)| &= \Big|\mathbf{E}\Big[\int_t^T F^{1n}(s,X_s^{t,x})-F^{1m}(s,X_s^{t,x})ds\Big]\Big|\\ &\leq \mathbf{E}\Big[\int_t^{t+\eta}\big|F^{1n}(s,X_s^{t,x})-F^{1m}(s,X_s^{t,x})\big|\,ds\Big]\\ &\quad+\Big|\mathbf{E}\Big[\int_{t+\eta}^T\big(F^{1n}(s,X_s^{t,x})-F^{1m}(s,X_s^{t,x})\big).\mathbb{1}_{\{|X_s^{t,x}|\leq k\}}ds\Big]\Big|\\ &\quad+\Big|\mathbf{E}\Big[\int_{t+\eta}^T\big(F^{1n}(s,X_s^{t,x})-F^{1m}(s,X_s^{t,x})\big).\mathbb{1}_{\{|X_s^{t,x}|>k\}}ds\Big]\Big|,\end{aligned}$$

where on the right side, noticing (5.49), we obtain,

$$\begin{aligned}&\mathbf{E}\Big[\int_t^{t+\eta}|F^{1n}(s,X_s^{t,x})-F^{1m}(s,X_s^{t,x})|ds\Big]\\ &\leq \eta^{\frac{q-1}{q}}\{\mathbf{E}\Big[\int_0^T|F^{1n}(s,X_s^{t,x})-F^{1m}(s,X_s^{t,x})|^q ds\Big]\}^{\frac{1}{q}}\leq C\eta^{\frac{q-1}{q}}.\end{aligned}$$

At the same time, Corollary 5.3.1 associates with the $\mathcal{L}^{\frac{q}{q-1}}$-domination property implies:

$$\begin{aligned}&\Big|\mathbf{E}\Big[\int_{t+\eta}^T\big(F^{1n}(s,X_s^{t,x})-F^{1m}(s,X_s^{t,x})\big).\mathbb{1}_{\{|X_s^{t,x}|\leq k\}}ds\Big]\Big|\\ &=\Big|\int_{\mathbf{R}^m}\int_{t+\eta}^T(F^{1n}(s,\eta)-F^{1m}(s,\eta)).\mathbb{1}_{\{|\eta|\leq k\}}\mu(t,x;s,d\eta)ds\Big|\\ &=\Big|\int_{\mathbf{R}^m}\int_{t+\eta}^T(F^{1n}(s,\eta)-F^{1m}(s,\eta)).\mathbb{1}_{\{|\eta|\leq k\}}\phi_{t,x}(s,\eta)\mu(0,x;s,d\eta)ds\Big|.\end{aligned}$$

Since $\phi_{t,x}(s,\eta)\in\mathcal{L}^{\frac{q}{q-1}}([t+\eta,T]\times[-k,k]^m;\mu(0,x;s,d\eta)ds)$, for $k\geq 1$, it follows from (5.50) that for each $(t,x)\in[0,T]\times\mathbf{R}^m$, we have,

$$\mathbf{E}\Big[\int_{t+\eta}^T\big(F^{1n}(s,X_s^{t,x})-F^{1m}(s,X_s^{t,x})\big)\,\mathbb{1}_{\{|X_s^{t,x}|\leq k\}}ds\Big]\to 0 \text{ as } n,m\to\infty.$$

Finally,

$$\begin{aligned}&\Big|\mathbf{E}\Big[\int_{t+\eta}^T\big(F^{1n}(s,X_s^{t,x})-F^{1m}(s,X_s^{t,x})\big).\mathbb{1}_{\{|X_s^{t,x}|>k\}}ds\Big]\Big|\\ &\leq C\{\mathbf{E}\Big[\int_{t+\eta}^T\mathbb{1}_{\{|X_s^{t,x}|>k\}}ds\Big]\}^{\frac{q-1}{q}}\{\mathbf{E}\Big[\int_{t+\eta}^T|F^{1n}(s,X_s^{t,x})-F^{1m}(s,X_s^{t,x})|^q ds\Big]\}^{\frac{1}{q}}\\ &\leq Ck^{-\frac{q-1}{q}}\end{aligned}$$

Therefore, for each $(t,x)\in[0,T]\times\mathbf{R}^m$, $(\varsigma^{1n}(t,x))_{n\geq 1}$ is a Cauthy sequence and then there exists a borelian application ς^1 on $[0,T]\times\mathbf{R}^m$, such that for each $(t,x)\in[0,T]\times\mathbf{R}^m$, $\lim_{n\to\infty}\varsigma^{1n}(t,x)=\varsigma^1(t,x)$, which indicates that for $t\in[0,T]$, $\lim_{n\to\infty}\bar{Y}_t^{1n;(0,x)}(\omega)=\varsigma^1(t,X_t^{0,x})$, $\mathbf{P}-a.s.$ Taking account of (5.46) and the Lebesgue dominated convergence theorem, we obtain the sequence $((\bar{Y}_t^{1n;(0,x)})_{0\leq t\leq T})_{n\geq 1}$ converges to $\bar{Y}^1=(\varsigma^1(t,X_t^{0,x}))_{0\leq t\leq T}$ in $\mathcal{L}^p([0,T]\times\mathbf{R}^m)$ for any $p>1$, that is:

$$\mathbf{E}\Big[\int_0^T|\bar{Y}_t^{1n;(0,x)}-\bar{Y}_t^1|^p dt\Big]\to 0,\quad \text{as } n\to\infty. \tag{5.51}$$

Next, we will show that for any $p > 1$, $\bar{Z}^{1n;(0,x)} = (\mathfrak{z}^{1n}(t, X_t^{0,x}))_{0\le t\le T})_{n\ge 1}$ has a limit in $\mathcal{H}_T^2(\mathbf{R}^m)$. Besides, $(\bar{Y}^{1n;(0,x)})_{n\ge 1}$ is convergent in $\mathcal{S}_T^2(\mathbf{R})$ as well.

We now focus on the first claim. For $n, m \ge 1$ and $0 \le t \le T$, using Itô's formula with $(\bar{Y}_t^{1n} - \bar{Y}_t^{1m})^2$ (we omit the subscript $(0,x)$ for convenience) and considering (5.39)-(b) in Step 1, we get,

$$\begin{aligned}
&\left|\bar{Y}_t^{1n} - \bar{Y}_t^{1m}\right|^2 + \int_t^T |\bar{Z}_s^{1n} - \bar{Z}_s^{1m}|^2 ds \\
&= 2\int_t^T (\bar{Y}_s^{1n} - \bar{Y}_s^{1m})(G^{1n}(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n}) - \\
&\quad - G^{1m}(s, X_s^{0,x}, \bar{Y}_s^{1m}, \bar{Y}_s^{2m}, \bar{Z}_s^{1m}, \bar{Z}_s^{2m}))ds - 2\int_t^T (\bar{Y}_s^{1n} - \bar{Y}_s^{1m})(\bar{Z}_s^{1n} - \bar{Z}_s^{1m})dB_s \\
&\le C\int_t^T |\bar{Y}_s^{1n} - \bar{Y}_s^{1m}|\Big[(|\bar{Z}_s^{1n}| + |\bar{Z}_s^{1m}|)(1 + |X_s^{0,x}|) + (|\bar{Y}_s^{1n}| + |\bar{Y}_s^{1m}|)(1 + |X_s^{0,x}|)^\gamma\Big]ds \\
&\quad - 2\int_t^T (\bar{Y}_s^{1n} - \bar{Y}_s^{1m})(\bar{Z}_s^{1n} - \bar{Z}_s^{1m})dB_s.
\end{aligned}$$

Since for any $x, y, z \in \mathbf{R}$, $|xyz| \le \frac{1}{a}|x|^a + \frac{1}{b}|y|^b + \frac{1}{c}|z|^c$ with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$, then, for any $\varepsilon > 0$, we have,

$$\begin{aligned}
&|\bar{Y}_t^{1n} - \bar{Y}_t^{1m}|^2 + \int_t^T |\bar{Z}_s^{1n} - \bar{Z}_s^{1m}|^2 ds \\
&\le C\Big\{\frac{\varepsilon^2}{2}\int_t^T (|\bar{Z}_s^{1n}| + |\bar{Z}_s^{1m}|)^2 ds + \frac{\varepsilon^4}{4}\int_t^T (1 + |X_s^{0,x}|)^4 ds + \frac{1}{4\varepsilon^8}\int_t^T |\bar{Y}_s^{1n} - \bar{Y}_s^{1m}|^4 ds \\
&\quad + \frac{\varepsilon^2}{2}\int_t^T (|\bar{Y}_s^{1n}| + |\bar{Y}_s^{1m}|)^2 ds + \frac{\varepsilon^4}{4}\int_t^T (1 + |X_s^{0,x}|)^{4\gamma} ds + \frac{1}{4\varepsilon^8}\int_t^T |\bar{Y}_s^{1n} - \bar{Y}_s^{1m}|^4 ds\Big\} \\
&\quad - 2\int_t^T (\bar{Y}_s^{1n} - \bar{Y}_s^{1m})(\bar{Z}_s^{1n} - \bar{Z}_s^{1m})dB_s.
\end{aligned} \tag{5.52}$$

Taking now $t = 0$ in (5.52), expectation on both sides and the limit w.r.t. n and m, we deduce that,

$$\limsup_{n,m\to\infty} \mathbf{E}\Big[\int_0^T |\bar{Z}_s^{1n} - \bar{Z}_s^{1m}|^2 ds\Big] \le C\{\frac{\varepsilon^2}{2} + \frac{\varepsilon^4}{4}\}, \tag{5.53}$$

due to (5.47), (5.14) and the convergence of (5.51). As ε is arbitrary, then the sequence $(\bar{Z}^{1n})_{n\ge 1}$ is convergent in $\mathcal{H}_T^2(\mathbf{R}^m)$ to a process Z^1.

Now, returning to inequality (5.52), taking the supremum over $[0, T]$ and using BDG's inequality, we obtain that,

$$\begin{aligned}
&\mathbf{E}\Big[\sup_{0\le t\le T} |\bar{Y}_t^{1n} - \bar{Y}_t^{1m}|^2 + \int_0^T |\bar{Z}_s^{1n} - \bar{Z}_s^{1m}|^2 ds\Big] \\
&\le C\{\frac{\varepsilon^2}{2} + \frac{\varepsilon^4}{4}\} + \frac{1}{4}\mathbf{E}\Big[\sup_{0\le t\le T} |\bar{Y}_t^{1n} - \bar{Y}_t^{1m}|^2\Big] + 4\mathbf{E}\Big[\int_0^T |\bar{Z}_s^{1n} - \bar{Z}_s^{1m}|^2 ds\Big]
\end{aligned}$$

which implies that

$$\limsup_{n,m\to\infty} \mathbf{E}\Big[\sup_{0\le t\le T} |\bar{Y}_t^{1n} - \bar{Y}_t^{1m}|^2\Big] = 0,$$

since ε is arbitrary and (5.53). Thus, the sequence of $(\bar{Y}^{1n})_{n\ge 1}$ converges to $\bar{Y}^1$ in $\mathcal{S}_T^2(\mathbf{R})$ which is a continuous process.

Next, note that since $\varsigma^{1n}(s, x) \ge e^{-C(1+|x|^\gamma)}$, then, $\bar{Y}_s^1 > 0, \forall s \le T$, **P**-a.s.

Finally, repeat the procedure for player $i = 2$, we have also the convergence of $(\bar{Z}^{2n})_{n\ge 1}$ (resp. $(\bar{Y}^{2n})_{n\ge 1}$) in $\mathcal{H}_T^2(\mathbf{R}^m)$ (resp. $\mathcal{S}_T^2(\mathbf{R})$) to $\bar{Z}^2$ (resp. $\bar{Y}^2 = \varsigma^2(., X^{0,x}.)$).

Step 5. *The limit process $(\bar{Y}_t^i, \bar{Z}_t^i)_{0\le t\le T}$ (i=1,2) is the solution of BSDE (5.38).*

Indeed, we need to show that (for case $i = 1$):

$$F^1(t, X_t^{0,x}) = G^1(t, X_t^{0,x}, \bar{Y}_t^1, \bar{Y}_t^2, \bar{Z}_t^1, \bar{Z}_t^2) \quad dt \otimes d\mathbf{P} - a.s.$$

For $k \ge 1$, we have,

$$\begin{aligned}
&\mathbf{E}\Big[\int_0^T |G^{1n}(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n}) - G^1(s, X_s^{0,x}, \bar{Y}_s^1, \bar{Y}_s^2, \bar{Z}_s^1, \bar{Z}_s^2)|ds\Big] \\
\le\ &\mathbf{E}\Big[\int_0^T |G^{1n}(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n}) - G^1(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n})|\cdot \\
&\quad \cdot \mathbb{1}_{\{\frac{1}{k}<|\bar{Y}_s^{1n}|+|\bar{Y}_s^{2n}|+|\bar{Z}_s^{1n}|+|\bar{Z}_s^{2n}|<k\}} ds\Big] \\
&+\mathbf{E}\Big[\int_0^T |G^{1n}(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n}) - G^1(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n})|\cdot \\
&\quad \cdot \mathbb{1}_{\{|\bar{Y}_s^{1n}|+|\bar{Y}_s^{2n}|+|\bar{Z}_s^{1n}|+|\bar{Z}_s^{2n}|\ge k\}} ds\Big] \\
&+\mathbf{E}\Big[\int_0^T |G^{1n}(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n}) - G^1(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n})|\cdot \\
&\quad \cdot \mathbb{1}_{\{|\bar{Y}_s^{1n}|+|\bar{Y}_s^{2n}|+|\bar{Z}_s^{1n}|+|\bar{Z}_s^{2n}|\le \frac{1}{k}\}} ds\Big] \\
&+\mathbf{E}\Big[\int_0^T |G^1(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n}) - G^1(s, X_s^{0,x}, \bar{Y}_s^1, \bar{Y}_s^2, \bar{Z}_s^1, \bar{Z}_s^2)|ds\Big] \\
:=\ &I_1^n + I_2^n + I_3^n + I_4^n,
\end{aligned} \tag{5.54}$$

where the sequence I_1^n, $n \ge 1$ converges to 0. On one hand, for $n \ge 1$, the point (5.39)-(b) in Step 1 implies that,

$$\begin{aligned}
&\big|G^{1n}(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n}) - G^1(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n})\big| \cdot \\
&\qquad \cdot \mathbb{1}_{\{\frac{1}{k}<|\bar{Y}_s^{1n}|+|\bar{Y}_s^{2n}|+|\bar{Z}_s^{1n}|+|\bar{Z}_s^{2n}|<k\}} \\
&< C_f C_\sigma (1 + |X_s^{0,x}|)k + C_h(1 + |X_s^{0,x}|^\gamma)k.
\end{aligned}$$

On the other hand, considering the point (5.39)-(d), we obtain that,

$$\begin{aligned}
&\big|G^{1n}(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n}) - G^1(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n})\big| \cdot \\
&\qquad \cdot \mathbb{1}_{\{\frac{1}{k}<|\bar{Y}_s^{1n}|+|\bar{Y}_s^{2n}|+|\bar{Z}_s^{1n}|+|\bar{Z}_s^{2n}|<k\}} \\
\le &\sup_{\substack{(y_s^1, y_s^2, z_s^1, z_s^2) \\ \frac{1}{k}<|y_s^1|+|y_s^2|+|z_s^1|+|z_s^1|<k}} \big|G^{1n}(s, X_s^{0,x}, y_s^1, y_s^2, z_s^1, z_s^2) - G^1(x, X_s^{0,x}, y_s^1, y_s^2, z_s^1, z_s^2)\big| \to 0,
\end{aligned}$$

as $n \to \infty$. Thanks to Lebesgue's dominated convergence theorem, the sequence I_1^n of (5.54) converges to 0 in $\mathcal{H}_T^1(\mathbf{R})$.

The sequence I_2^n in (5.54) is bounded by $\frac{C}{k^{2(q-1)/q}}$ with $q \in (1, 2)$. Actually, from point (5.39)-(b) and

Markov's inequality, for $q \in (1,2)$, we get,

$$\begin{aligned}
I_2^n &\leq C\Big\{\mathbf{E}\Big[\int_0^T (1+|X_s^{0,x}|)^q|\bar{Z}_s^{1n}|^q + (1+|X_s^{0,x}|^\gamma)^q|\bar{Y}_s^{1n}|^q ds\Big]\Big\}^{\frac{1}{q}} \times \\
&\qquad \times \Big\{\mathbf{E}\Big[\int_0^T \mathbb{1}_{\{|\bar{Y}_s^{1n}|+|\bar{Y}_s^{2n}|+|\bar{Z}_s^{1n}|+|\bar{Z}_s^{2n}|\geq k\}} ds\Big]\Big\}^{\frac{q-1}{q}} \\
&\leq C\Big\{\mathbf{E}\Big[\int_0^T |\bar{Z}_s^{1n}|^2 ds\Big] + \mathbf{E}\Big[\int_0^T (1+|X_s^{0,x}|)^{\frac{2q}{2-q}} ds\Big] \\
&\qquad + \mathbf{E}\Big[\int_0^T |\bar{Y}_s^{1n}|^2 ds\Big] + \mathbf{E}\Big[\int_0^T (1+|X_s^{0,x}|)^{\gamma\cdot\frac{2q}{2-q}} ds\Big]\Big\}^{\frac{1}{q}} \times \\
&\quad \times \frac{\Big\{\mathbf{E}\Big[\int_0^T |\bar{Y}_s^{1n}|^2 + |\bar{Y}_s^{2n}|^2 + |\bar{Z}_s^{1n}|^2 + |\bar{Z}_s^{2n}|^2 ds\Big]\Big\}^{\frac{q-1}{q}}}{(k^2)^{\frac{q-1}{q}}} \\
&\leq \frac{C}{k^{\frac{2(q-1)}{q}}}.
\end{aligned}$$

The last inequality is a straightforward result of the estimates (5.10)(5.46) and (5.47).

The third sequence I_3^n in (5.54) is bounded by C/k with constant C independent on k. Indeed, by (5.39)-(b) and (5.14),

$$I_3^n \leq \mathbf{E}\Big[\int_0^T C_f C_\sigma (1+|X_s^{0,x}|)\frac{1}{k} + C_h(1+|X_s^{0,x}|^\gamma)\frac{1}{k} ds\Big] \leq C/k.$$

The fourth sequence I_4^n, $n \geq 1$ in (5.54) also converges to 0, at least for a subsequence. Actually, since the sequence $(\bar{Z}^{1n})_{n\geq 1}$ converges to $\bar{Z}^1$ in $\mathcal{H}_T^2(\mathbf{R}^m)$, then there exists a subsequence $(\bar{Z}^{1n_k})_{k\geq 1}$ such that it converges to $\bar{Z}^1$, $dt \otimes d\mathbf{P}$-a.e., and furthermore, $\sup_{k\geq 1}|\bar{Z}_t^{1n_k}(\omega)| \in \mathcal{H}_T^2(\mathbf{R})$. On the other hand, $(\bar{Y}^{1n_k})_{k\geq 1}$ converges to $\bar{Y}^1 > 0$, $dt \otimes d\mathbf{P}$-a.e.. Thus, taking the continuity of function $G^1(t,x,y^1,y^2,z^1,z^2)$ w.r.t (y^1,y^2,z^1,z^2) into account, we obtain that

$$G^1(t,X_t^{0,x},\bar{Y}_t^{1n_k},\bar{Y}_t^{2n_k},\bar{Z}_t^{1n_k},\bar{Z}_t^{2n_k}) \rightarrow_{k\rightarrow\infty} G^1(t,X_t^{0,x},\bar{Y}_t^1,\bar{Y}_t^2,\bar{Z}_t^1,\bar{Z}_t^2) \quad dt\otimes d\mathbf{P}-a.e.$$

In addition, considering that

$$\sup_{k\geq 1}|G^1(t,X_t^{0,x},\bar{Y}_t^{1n_k},\bar{Y}_t^{2n_k},\bar{Z}_t^{1n_k},\bar{Z}_t^{1n_k})| \in \mathcal{H}_T^q(\mathbf{R}) \text{ for } 1<q<2,$$

which follows from (5.49). Finally, by the dominated convergence theorem, one can get that,

$$G^1(t,X_t^{0,x},\bar{Y}_t^{1n_k},\bar{Y}_t^{2n_k},\bar{Z}_t^{1n_k},\bar{Z}_t^{2n_k}) \rightarrow_{k\rightarrow\infty} G^1(t,X_t^{0,x},\bar{Y}_t^1,\bar{Y}_t^2,\bar{Z}_t^1,\bar{Z}_t^2) \quad \text{in } \mathcal{H}_T^q(\mathbf{R}),$$

which yields to the convergence of I_4^n in (5.54) to 0.

It follows that the sequence $(G^{1n}(t,X_t^{0,x},\bar{Y}_t^{1n},\bar{Y}_t^{2n},\bar{Z}_t^{1n},\bar{Z}_t^{2n})_{0\leq t\leq T})_{n\geq 1}$ converges to $(G^1(t,X_t^{0,x},\bar{Y}_t^1,\bar{Y}_t^2,\bar{Z}_t^1,\bar{Z}_t^2))_{0\leq t\leq T}$ in $\mathcal{L}^1([0,T]\times\Omega, dt\otimes d\mathbf{P})$ and then $F^1(t,X_t^{0,x}) = G^1(t,X_t^{0,x},\bar{Y}_t^1,\bar{Y}_t^2,\bar{Z}_t^1,\bar{Z}_t^2), dt\otimes d\mathbf{P}$-a.e. In the same way, we have, $F^2(t,X_t^{0,x}) = G^2(t,X_t^{0,x},\bar{Y}_t^1,\bar{Y}_t^2,\bar{Z}_t^1,\bar{Z}_t^2)$, $dt\otimes d\mathbf{P}$-a.e. Thus, the processes (Y^i,Z^i), $i=1,2$ are the solutions of the backward equation (5.38).

Step 6. *The solutions* (Y_t^i,Z_t^i), $i=1,2$ *for BSDE* (5.36) *exist.*

Obviously observed from (5.45) that $\bar{Y}_t^1$ is strict positive which enable us to obtain the solution of the original BSDE (5.36) by:

$$\begin{cases} Y_t^1 = \ln \bar{Y}_t^1; \\ Z_t^1 = \dfrac{\bar{Z}_t^1}{\bar{Y}_t^1}, \quad t\in[0,T]. \end{cases}$$

The same illustrate about the case $i = 2$ gives the existence of solution (Y^2, Z^2) for BSDE (5.36). Besides, it follows from the fact $\bar{Y}_t^i = \varsigma^i(t, X_t^{0,x})$ and for each $(t, x) \in [0, T] \times \mathbf{R}^m$, $e^{-C(1+|x|^\gamma)} \leq \varsigma^i(t, x) \leq e^{C(1+|x|^\gamma)}$ with $1 < \gamma < 2$, that Y^i also has a representation through a deterministic function $\varpi^i(t, x) = \ln \varsigma^i(t, x)$ which is of subquadratic growth, i.e. $|\varpi^i(t, x)| \leq C(1 + |x|^\gamma)$ with $1 < \gamma < 2$, $i = 1, 2$. The proof is completed. $\square$

CHAPTER 6

Bang-Bang Type Nash Equilibrium Point for Nonzero-sum Stochastic Differential Game

This chapter is a published joint work with Hamadène (ref. [62]).

In this work, we investigate a nonzero-sum stochastic differential game (NZSDG for short). Our main result is an existence theorem of the Nash equilibrium point (NEP for short) for this game.

Let us consider a two-player case. Actually, the multiple-player situation is a straightforward adaption. To be precise, let (u, v) denotes a pair of controls for the two players, B a Brownian motion and X be the state process of a system controlled by (u, v) as follows,

$$dX_t^{u,v} = \Gamma(t, X_t^{u,v}, u_t, v_t)dt + \sigma(t, X_t^{u,v})dB_t \text{ for } t \leq T, \text{ and } X_0 = x. \quad (6.1)$$

For player $i = 1, 2$, $J_i(u, v)$ are the corresponding payoff (or possible cost) functions which read,

$$J_i(u, v) = \mathbf{E}[g_i(X_T^{u,v})]. \quad (6.2)$$

Observing from (6.2) that, the choice of control for each player will affect the other one's utility through the terminal value of the state process. Therefore, a natural question will arise, is there an optimal pair of controls which give both of the players the maximum payoffs? The objective of this work is to find such optimal control (u^*, v^*) which is known as the Nash equilibrium point, i.e. under such NEP, no one can gain more by alternatively changing this own control. In another word,

$$J_1(u^*, v^*) \geq J_1(u, v^*) \text{ and } J_2(u^*, v^*) \geq J_2(u^*, v)$$

for any admissible control (u, v).

As we can see from the objective that, the two players are of cooperative relationship. This kind of game is well known as the nonzero-sum stochastic differential game. This subject has been studied in the literature, see eg. [21, 48, 58, 54, 53, 75, 79, 91, 29, 74, 27] , to name a few. There are typically two approaches. One method is related to the partial differential equation (PDE) theory. Some of the results show that the payoff function of the game is the unique viscosity solution of a related Bellman-Isaacs equation (see [74]) and there are also works which make use of the sobolev theory of PDE (see eg. [79]). Comparatively, the other popular way to deal with stochastic differential game is the backward stochastic differential equation (BSDE) approach ([58, 54, 53, 75]), which characterizes the payoff as the solution of an associated BSDE.

In the present article, we study the NZSDG via the BSDE arguments under the markovian setting. However, notice that in the previous results, such as [58, 54, 53, 75], the authors concern only about the smooth feedback controls as well as the Hamiltonian functions. The case of discontinuous controls is not fully explored. Indeed, the discontinuous controls are naturally exist and reasonable, especially in the economic and engineering fields. This is the main reason why we construct, in this article, a special game model with non-smooth controls by assuming that the dynamic function Γ in (6.1) is an affine combination of the controls, i.e.

$$\Gamma(t,x,u,v) = f(t,x) + h(u) + l(v).$$

The precise condition of functions f, h, l will be given in the main text section.

Taking account of the fact that the payoffs as shown in (6.2) have no instantaneous payoffs inside. Therefore, the NEP, if exists, should be of bang-bang form. Generally speaking, by bang-bang control, we mean it is not smooth and will jump between two bounds. This is the main novelty of this article. Another feather is that the function Γ is not bounded as most of the previous works, instead, is of linear growth with respect to the state process. As in [58], we apply the BSDE approach. This game problem finally reduces to solving a multiple-dimensional BSDE with a discontinuous generator with respect to z component and linear growth ω by ω. Under the generalized Isaacs' hypothesis, we show that the associated BSDE has a solution which then provides a bang-bang type NEP for the NZSDG.

This paper is organized as follows:

In Subsection 6.1.1, we introduce the game problem and some preliminaries. The formulation we adopt is of weak type. Besides, for intuitive understanding, we work on the framework of one-dimensional controls. The extension to the multiple-dimensional situation obviously holds following the same idea. The explicit form of discontinuous controls, namely, bang-bang controls are presented in Subsection 6.1.2 which heavily rely on the generalized Isaacs' condition. In Subsection 6.1.3, we give the main result (Theorem 6.1.2) of this work and some other related important results. We first provide a link between the game problem and Backward SDEs(see Proposition 6.1.1). The payoff of the game turns out to be the initial value of the solution for an associated BSDE. Then, by Proposition 6.1.2, we prove that the existence of a NEP for the game is equivalent to the existence of a solution of a BSDE which is of multiple-dimensional and with discontinuous generator with respect to z. Finally, under some reasonable assumption, we provide the solution of this special BSDE(see Theorem 6.1.1). All the proofs are stated in Subsection 6.1.4. The proofs of Propositions 6.1.1 and 6.1.2 are standard. For Theorem 6.1.1, the method is mainly based on an approximating scheme. In Section 6.2, we investigate some possible generalizations. The idea is the same with a bit modification which is given in the final remark.

6.1 Bang-bang type NZSDG and multi-dimensional BSDEs with discontinuous generators

In this section, we first establish a bang-bang type nonzero-sum stochastic differential game problem in 1-dimensional framework. A more general argument will be given in the next section.

6.1.1 Statement of the problem

For fixed $T > 0$, let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, on which we define a 1-dimensional standard Brownian motion $B := (B_t)_{t \le T}$. For $t \le T$, let us denote by $(F_t = \sigma(B_u, u \le t))_{t \le T}$ the natural filtration of B and $(\mathcal{F}_t)_{t\leq T}$ the completion of $(F_t)_{t\leq T}$ with the **P**-null sets of $\mathcal{F}$. Then, let $\mathcal{P}$ be the σ-algebra on $[0,T]\times\Omega$ of $\mathcal{F}_t$-progressively measurable sets. For a real constant $p\geq 1$, we introduce the following useful spaces:

- $L^p=\{\xi:\ \mathcal{F}_T$-measurable and **R**-valued random variable s.t.$\mathbf{E}[|\xi|^p]<\infty\}$;
- $\mathcal{S}_T^p(\mathbf{R})=\{Y=(Y_s)_{s\in[0,T]}:\ \mathcal{P}$-measurable, continuous and **R**-valued stochastic process s.t. $\mathbf{E}[\sup_{s\in[0,T]}|Y_s|^p]<\infty\}$;
- $\mathcal{H}_T^p(\mathbf{R})=\{Z=(Z_s)_{s\in[0,T]}:\ \mathcal{P}$-measurable and **R**-valued stochastic process s.t. $\mathbf{E}[(\int_0^T|Z_s|^2ds)^{p/2}]<\infty\}$.

We consider, in this article, the 2-player case. The general multiple players case is a straightforward adaptation. The dynamic of this game system is given by a stochastic differential equation (SDE for short) as follows, for any fixed $(t,x)\in[0,T]\times\mathbf{R}$,

$$\forall s\leq T,\ X_s^{t,x}=x+(B_{s\vee t}-B_t). \tag{6.3}$$

Obviously, the solution $X_s^{t,x}$, which is also called the state process of the game problem, exists.

Remark 6.1.1. *We consider a trivial situation for SDE* (6.3) *with an identity diffusion process, just for easy understanding. The trick of the technique in this article still valid for general diffusion process with appropriate properties. We would like to introduce this point in Section 6.2.*

Each player $i=1,2$ has his own control. Let us denote next by $U=[0,1]$, $V=[-1,1]$ two bounded subset on **R** and $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) be the set of $\mathcal{P}$-measurable process $u=(u_t)_{t\leq T}$ (resp. $v=(v_t)_{t\leq T}$) on $[0,T]\times\Omega$ with value on U (resp. V). Hereafter, we call $\mathcal{M}:=\mathcal{M}_1\times\mathcal{M}_2$ the set of admissible controls for the two players.

Let $f:(t,x)\in[0,T]\times\mathbf{R}\to\mathbf{R}$ be a Borelian function and $\Gamma:(t,x,u,v)$ be the dynamic function for the game problem, for any $(t,x,u,v)\in[0,T]\times\mathbf{R}\times U\times V$ associated $\Gamma(t,x,u,v)=f(t,x)+u+v$. The function Γ here is a kind of affine combination of controls which can be generalized as shown in the Section 6.2. Next, we impose our hypothesis on function f.

Assumption (A1) The function f is of linear growth w.r.t. x, *i.e.*, for any $(t,x)\in[0,T]\times\mathbf{R}$, there exists a constant C such that, $|f(t,x)|\leq C(1+|x|)$.

Therefore, Γ is also of linear growth on x uniformly w.r.t $(u,v)\in U\times V$. For $(u.,v.)\in\mathcal{M}$, let $\mathbf{P}_{t,x}^{u,v}$ be the positive measure on $(\Omega,\mathcal{F})$ as follows,

$$d\mathbf{P}_{t,x}^{u,v}=\zeta_T(\Gamma(.,X.^{t,x},u.,v.))d\mathbf{P}\text{ with }\zeta_t(\Theta):=1+\int_0^t\Theta_s\zeta_s dB_s,\ t\leq T, \tag{6.4}$$

for any measurable $\mathcal{F}_t$-adapted process $\Theta:=(\Theta_t)_{t\leq T}$. It follows from the uniformly linear growth property of Γ that $\mathbf{P}_{t,x}^{u,v}$ is a probability on $(\Omega,\mathcal{F})$ (see Appendix A of [36] or [70], pp.200). Then, by Girsanov's theorem ([50]), the process $B^{u,v}=(B_s-\int_0^s\Gamma(r,X_r^{t,x},u_r,v_r)dr)_{s\leq T}$ is a $(\mathcal{F}_s,\mathbf{P}_{t,x}^{u,v})$-Brownian motion and $(X_s^{t,x})_{s\leq T}$ satisfies the following SDE,

$$dX_s^{t,x}=\Gamma(s,X_s^{t,x},u_s,v_s)ds+dB_s^{u,v},\ \forall s\in[t,T]\text{ and }X_s^{t,x}=x,\ s\in[0,t]. \tag{6.5}$$

As a matter of fact, the process $X^{t,x}$ is not adapted with respect to the filtration generated by the Brownian motion $B^{u,v}$. Thereby, $X^{t,x}$ is a weak solution for the SDE (6.5). Now the system is controlled by player 1 (resp. player 2) with u (resp. v) and the law of the state process is the same as the one of $X^{t,x}$ under $\mathbf{P}_{t,x}^{u,v}$.

We introduce first the terminal payoff function $g_i : x \in \mathbf{R} \to \mathbf{R}$ for player $i = 1, 2$, which satisfies the following assumption.

Assumption (A2) The two functions $g_i, i = 1, 2$, are of polynomial growth w.r.t.x, *i.e.* there exists a constant C and a fixed constant $\gamma > 0$ such that $|g_1(x)| + |g_2(x)| \leq C(1 + |x|^\gamma),\ \forall x \in \mathbf{R}$.

Now, we are in a position to give the payoffs of the two players. For fixed $(0, x)$, Let us define for $(u, v) \in \mathcal{M}$ that,

$$\begin{aligned} J_1(u,v) &:= \mathbf{E}^{u,v}_{0,x}[g_1(X^{0,x}_T)]; \\ J_2(u,v) &:= \mathbf{E}^{u,v}_{0,x}[g_2(X^{0,x}_T)], \end{aligned} \tag{6.6}$$

where $\mathbf{E}^{u,v}_{0,x}$ is the expectation under probability $\mathbf{P}^{u,v}_{0,x}$, hereafter $\mathbf{E}^{u,v}_{0,x}(.)$ (resp. $\mathbf{P}^{u,v}_{0,x}$) will be simply denoted by $\mathbf{E}^{u,v}(.)$ (resp.$\mathbf{P}^{u,v}$).

As we can see from (6.5) and (6.6) that, the choice of control of each player has influence on the other one's payoff through the state process $X^{0,x}$. What we discussed here is a nonzero-sum stochastic differential game which means the two players are of cooperate relationship. Both of them want to reach the maximum payoff. Therefore, naturally, we concern about the existence of the *Nash equilibrium point*, which is a couple of controls $(u^*, v^*) \in \mathcal{M}$, such that, for all $(u, v) \in \mathcal{M}$,

$$J_1(u^*, v^*) \geq J_1(u, v^*) \text{ and } J_2(u^*, v^*) \geq J_2(u^*, v).$$

6.1.2 Bang-bang type control

As demonstrated in (6.6), there are no instantaneous payoffs in J_1 and J_2. Therefore, the equilibrium point of this game, if exists, should be of bang-bang type. That is to say, the optimal control u^* (resp.v^*) will jump between the two bounds of the value set U (resp.V). Let us illustrate it in detail.

Let H_1 and H_2 be the *Hamiltonian functions* of this game, *i.e.*, the functions which are not depend on ω defined from $[0, T] \times \mathbf{R} \times \mathbf{R} \times U \times V$ into $\mathbf{R}$ by:

$$\begin{aligned} H_1(t,x,p,u,v) &:= p\Gamma(t,x,u,v) = p(f(t,x) + u + v); \\ H_2(t,x,q,u,v) &:= q\Gamma(t,x,u,v) = q(f(t,x) + u + v). \end{aligned}$$

Now, we give firstly, the candidate optimal controls $\bar{u}$ and $\bar{v}$ which defined on $\mathbf{R} \times U$ and $\mathbf{R} \times V$, valued on U and V respectively, as follows: $\forall p, q \in \mathbf{R},\ \epsilon_1 \in U,\ \epsilon_2 \in V$,

$$\bar{u}(p,\epsilon_1) = \begin{cases} 1,\ p > 0, \\ \epsilon_1,\ p = 0, \\ 0,\ p < 0, \end{cases} \quad \text{and} \quad \bar{v}(q,\epsilon_2) = \begin{cases} 1,\ q > 0, \\ \epsilon_2,\ q = 0, \\ -1,\ q < 0. \end{cases} \tag{6.7}$$

Then, we can verify that such $\bar{u}$ and $\bar{v}$ will exactly satisfy the generalized *Isaacs' condition* as follows. For all $(t, x, p, q, u, v) \in [0, T] \times \mathbf{R} \times \mathbf{R} \times U \times V$ and $(\epsilon_1, \epsilon_2) \in U \times V$, we have,

$$\begin{cases} H_1^*(t,x,p,q,\epsilon_2) := H_1(t,x,p,\bar{u}(p,\epsilon_1),\bar{v}(q,\epsilon_2)) \geq H_1(t,x,p,u,\bar{v}(q,\epsilon_2)), \\ H_2^*(t,x,p,q,\epsilon_1) := H_2(t,x,q,\bar{u}(p,\epsilon_1),\bar{v}(q,\epsilon_2)) \geq H_2(t,x,q,\bar{u}(p,\epsilon_1),v). \end{cases} \tag{6.8}$$

Remark 6.1.2. *We should point out that, the function H_1^* (resp.H_2^*) does not depend on ϵ_1 (resp.ϵ_2), since, $p\bar{u}(p,\epsilon_1) = p \vee 0$ (resp. $q\bar{v}(q,\epsilon_2) = |q|$) does not depend on ϵ_1 (resp. ϵ_2). Besides, the Hamiltonian function here is discontinuous w.r.t. (p, q).*

We next give the main results of this article without proofs for intuitive understanding. All the proofs are given in the subsection 6.1.4.

6.1.3 Main results

For this particular nonzero-sum stochastic differential game, we still adopt the BSDE approach. We first state an useful result which characterize the payoffs by a multiple-dimensional BSDE.

Proposition 6.1.1. *Assume that the Assumptions (A1) and (A2) are fulfilled. For all $(u, v) \in \mathcal{M}$ and player $i = 1, 2$, there exists a couple of $\mathcal{P}$-measurable processes $(Y^{i;u,v}, Z^{i;u,v})$, with values on $\boldsymbol{R} \times \boldsymbol{R}$, such that:*

(i) *For all constant $q \geq 1$,*

$$\boldsymbol{E}^{u,v}\Big[\sup_{s\in[0,T]} |Y_s^{i;u,v}|^q + (\int_0^T |Z_s^{i;u,v}|^2 ds)^{\frac{q}{2}}\Big] < \infty. \tag{6.9}$$

(ii) $\forall t \leq T$,

$$- dY_t^{i;u,v} = H_i(s, X_s^{0,x}, Z_s^{i;u,v}, u_s, v_s)ds - Z_s^{i;u,v} dB_s,\ Y_T^{i;u,v} = g_i(X_T^{0,x}) \tag{6.10}$$

(iii) *The solution is unique, besides, $Y_0^{i;u,v} = J_i(u, v)$.*

Consequently, the equilibrium points exist, once provided that the BSDEs (6.11) have solutions. This result will be summarized as the following proposition.

Proposition 6.1.2. *Let us suppose the Assumptions (A1), (A2) are fulfilled. Besides, we suppose that there exists η^1, η^2, (Y^1, Z^1), (Y^2, Z^2) and θ, ϑ such that:*

(i) *η^1 and η^2 are two deterministic measurable functions with polynomial growth from $[0, T] \times \boldsymbol{R}$ to $\boldsymbol{R}$;*

(ii) *(Y^1, Z^1) and (Y^2, Z^2) are two couples of $\mathcal{P}$-measurable processes with values on $\boldsymbol{R}^{1+1}$;*

(iii) *θ(resp. ϑ) is a $\mathcal{P}$-measurable process valued on U(resp. V),*

and satisfy:

(a) *$\boldsymbol{P}$-a.s., $\forall s \leq T$, $Y_s^i = \eta^i(s, X_s^{0,x})$ and $Z^i(\omega) := (Z_s^i(\omega))_{s\leq T}$ is ds-square integrable;*

(b) *For all $s \leq T$,*

$$\begin{cases} -dY_s^1 = H_1^*(s, X_s^{0,x}, Z_s^1, Z_s^2, \vartheta_s)ds - Z_s^1 dB_s,\ Y_T^1 = g_1(X_T^{0,x}); \\ -dY_s^2 = H_2^*(s, X_s^{0,x}, Z_s^1, Z_s^2, \theta_s)ds - Z_s^2 dB_s,\ Y_T^2 = g_2(X_T^{0,x}). \end{cases} \tag{6.11}$$

Then, the pair of controls $(\bar{u}(Z_s^1, \theta_s), \bar{v}(Z_s^2, \vartheta_s))_{s\leq T}$ is a bang-bang type Nash equilibrium point of the nonzero-sum stochastic differential game.

It remains to show that, there exists a solution for BSDE (6.11) which is actually of multidimensional type. The main difficulty will rely on the discontinuity of the generator H_1^* (resp. H_2^*) w.r.t. (p, q) which comes from the discontinuity of $\bar{v}$ (resp. $\bar{u}$) on $q = 0$ (resp. $p = 0$). Fortunately, we obtain the positive result as follows:

Theorem 6.1.1. *Under the Assumptions (A1), (A2), there exist η^1, η^2, (Y^1, Z^1), (Y^2, Z^2) and θ, ϑ which satisfy (i)-(iii) and (a),(b) of Proposition 6.1.2.*

As a result of Theorem 6.1.1 and Proposition 6.1.2, we obtain the main result of this article.

Theorem 6.1.2. *The nonzero-sum stochastic differential game has a bang-bang type Nash equilibrium point.*

6.1.4 Proofs

Pre-results

We would like to introduce now two results about the state process $X^{t,x}$ as (6.3) which will be used in the following proofs. The first one is well-known that (see. Karatzas, I.1991 [70], pp.306), $X^{t,x}$ has moments of any order, i.e.,

$$\forall q \in [1, \infty), \quad \mathbf{E}\big[(\sup_{s\leq T} |X_s^{t,x}|)^{2q}\big] \leq C(1 + |x|^{2q}). \tag{6.12}$$

Note that under Assumption (A1), we have a similar result with the weak solution for SDE (6.5) which is stated as follows: for any admissible control $(u, v) \in \mathcal{M}$,

$$\forall q \in [1, \infty), \quad \mathbf{E}_{t,x}^{u,v}\big[(\sup_{s\leq T} |X_s^{t,x}|)^{2q}\big] \leq C(1 + |x|^{2q}). \tag{6.13}$$

Besides, we present the next important result by U.G.Haussmann (see Theorem 2.2, pp.14 [64]) related to the integrability of the exponential local martingale defined by (6.4).

Lemma 6.1.1. *(U.G.Haussmann) Let Θ be a $\mathcal{P} \otimes \mathcal{B}(\boldsymbol{R})$-measurable application from $[0, T] \times \Omega \times \boldsymbol{R}$ to $\boldsymbol{R}$ which is of uniformly linear growth, that is, $\boldsymbol{P}$-a.s. $\forall(s, x) \in [0, T] \times \boldsymbol{R}$, $|\Theta(s, \omega, x)| \leq C_0(1 + |x|)$. Then, there exists constants $p \in (1, 2)$ and C, where p depends only on C_0 while the constant C, depends only on p, but not on Θ, such that:*

$$\boldsymbol{E}\Big[\Big(\zeta_T\{\Theta(s, X_s^{t,x})\}\Big)^p\Big] \leq C,$$

where the process $\zeta_T(.)$ is the density function defined by (6.4).

As a by-product we have:

Corollary 6.1.1. *For any admissible control $(u, v) \in \mathcal{M}$ and $(t, x) \in [0, T] \times \boldsymbol{R}$, there exists a constant $p \in (1, 2)$ such that:*

$$\boldsymbol{E}\Big[\Big(\zeta_T\{\Gamma(s, X_s^{t,x}, u_s, v_s)\}\Big)^p\Big] \leq C.$$

Proof of Proposition 6.1.1

We will prove this Proposition by constructing the candidate solution of BSDE (6.10) directly. Then we check by Itô's formula that, the process defined is exactly the solution what we anticipate. In this proof, Corollary 6.1.1 plays an important role. Let us illustrate for player $i = 1$ and the same with player $i = 2$. For simplicity, only in this proof, we use the notation $(Y^{u,v}, Z^{u,v})$ instead of $(Y^{1;u,v}, Z^{1;u,v})$.

Let us fix $x \in \mathbf{R}$ and take $t = 0$ in (6.5). For any $(u, v) \in \mathcal{M}$, let us define the process $(Y_t^{u,v})_{t\leq T}$ as follows:

$$Y_t^{u,v} := \mathbf{E}^{u,v}[g_1(X_T^{0,x})|\mathcal{F}_t], \quad \forall t \leq T. \tag{6.14}$$

This process is well defined by noticing that, for any constant $r \geq 1$, we have $\mathbf{E}^{u,v}[|g_1(X_T^{0,x})|^{2r}] \leq C\mathbf{E}^{u,v}[C(1 + \sup_{s\leq T} |X_s^{0,x}|^{2r\gamma})] < \infty$ which is obtained by Assumption (A2) and (6.13). For writing convenience, we denote by ζ_t, the function $\zeta_t(\Gamma(., X_.^{0,x}, u_., v_.))$ as mentioned in (6.4). Therefore, (6.14) can be transformed into:

$$Y_t^{u,v} = \zeta_t^{-1}\mathbf{E}[\zeta_T \cdot g_1(X_T^{0,x})|\mathcal{F}_t], \quad \forall t \leq T.$$

In the following, we show that $\zeta_T \cdot g_1(X_T^{0,x}) \in L^{\bar{q}}$ for some $\bar{q} \in (1, 2)$. Indeed, according to Corollary 6.1.1, there exists some $p_0 \in (1, 2)$, such that $\zeta_T \in L^{p_0}(d\mathbf{P})$. Therefore, for any $\bar{q} \in (1, p_0)$, Young's inequality leads to:

$$\mathbf{E}[|\zeta_T \cdot g_1(X_T^{0,x})|^{\bar{q}}] \leq \frac{\bar{q}}{p_0}\mathbf{E}[|\zeta_T|^{p_0}] + \frac{p_0-\bar{q}}{p_0}\mathbf{E}[|g_1(X_T^{0,x})|^{\bar{q}\cdot\frac{p_0}{p_0-\bar{q}}}],$$

which is obviously finite by the polynomial growth of g_1 and (6.12). Actually, $Y^{u,v}$ can be understood as the solution of BSDE:

$$-dY_t^{u,v} = -Z_s^{u,v}dB_s^{u,v}, \quad \forall s \leq T; Y_T^{u,v} = g_1(X_T^{0,x}),$$

by the result of Briand et al. ([17]), since the terminal value $g_1(X_T^{0,x}) \in L^p$ for any $p > 1$.

Thanks to representation Theorem, there exists a $\mathcal{P}$-measurable and $\mathbf{R}$-valued process $(\Delta_s)_{s\leq T}$ which satisfies $\mathbf{E}[(\int_0^T |\Delta_s|^2 ds)^{\frac{\bar{q}}{2}}] < \infty$. Additionally,

$$Y_t^{u,v} = \zeta_t^{-1}\{\mathbf{E}[\zeta_T \cdot g_1(X_T^{0,x})] + \int_0^t \Delta_s dB_s\} := \zeta_t^{-1}R_t, \quad \forall t \leq T,$$

with $R_t := \mathbf{E}[\zeta_T \cdot g_1(X_T^{0,x})] + \int_0^t \Delta_s dB_s$, for any $t \leq T$. Next, noticing that $d\zeta_t = \zeta_t \cdot \Gamma(t, X_t^{0,x}, u_t, v_t)dB_t$, then, by using Itô's formula, we acquire,

$$d\zeta_t^{-1} = -\zeta_t^{-1} \cdot \{\Gamma(t, X_t^{0,x}, u_t, v_t)dB_t - |\Gamma(t, X_t^{0,x}, u_t, v_t)|^2 dt\}, \quad \forall t \leq T.$$

Therefore,

$$\begin{aligned} dY_t^{u,v} = -\zeta_t^{-1}\big\{\Gamma(t, X_t^{0,x}, u_t, v_t)dB_t - |\Gamma(t, X_t^{0,x}, u_t, v_t)|^2 dt\big\}R_t + \\ +\zeta_t^{-1} \cdot \Delta_t dB_t - \zeta_t^{-1}\Gamma(t, X_t^{0,x}, u_t, v_t)\Delta_t dt, \quad t \leq T. \end{aligned}$$

It is natural to define process $Z^{u,v}$ as the diffusion coefficient which is the following:

$$Z_t^{u,v} := -\zeta_t^{-1}\big\{R_t\Gamma(t, X_t^{0,x}, u_t, v_t) - \Delta_t\big\}, \quad t \leq T. \tag{6.15}$$

Finally, it is easy to check by Itô's formula that the pair of processes $(Y_t^{u,v}, Z_t^{u,v})_{t\leq T}$ of (6.14) and (6.15) satisfies the backward equation (6.10). Obviously, by the construct procedure, for fixed $x \in \mathbf{R}$, the solution is unique. The proof of existence and uniqueness is completed.

We now focus on the uniform estimate (6.9).

We begin from (6.14), using conditional Jensen's inequality, we show that, for any $q > 1$,

$$\sup_{t\in[0,T]} |Y_t^{u,v}|^q \leq \mathbf{E}^{u,v}\big[\sup_{t\in[0,T]} |g_1(X_T^{0,x})|^q|\mathcal{F}_t\big].$$

Taking the expectation on both sides under the probability $\mathbf{P}^{u,v}$ and using once again the polynomial growth of function g_1 and (6.13), we give, for any $q > 1$,

$$\begin{aligned} \mathbf{E}^{u,v}\big[\sup_{t\in[0,T]} |Y_t^{u,v}|^q\big] \leq \mathbf{E}^{u,v}\big[\sup_{t\in[0,T]} |g_1(X_T^{0,x})|^q\big] \leq C\mathbf{E}^{u,v}\big[\sup_{t\in[0,T]} (1 + |X_t^{0,x}|^{\gamma q})\big] \\ < \infty. \end{aligned} \tag{6.16}$$

Next for each integer k, let us define the following stopping time:

$$\tau_k = \inf\{s \geq 0, \int_0^s |Z_s^{u,v}|^2 ds \geq k\} \wedge T.$$

The sequence $(\tau_k)_{k\geq 0}$ is of stationary type and converges to T. By using Itô's formula with $(Y_{t\wedge\tau_k}^{u,v})^2$ we obtain: $\forall t \leq T$,

$$|Y_{t\wedge\tau_k}^{u,v}|^2 + \int_{t\wedge\tau_k}^{\tau_k} |Z_s^{u,v}|^2 ds = |Y_{\tau_k}^{u,v}|^2 - 2\int_{t\wedge\tau_k}^{\tau_k} Y_s^{u,v} Z_s^{u,v} dB_s^{u,v}.$$

Thus, for $q > 1$, taking the expectation of the $q/2$ power on both sides and applying Young's inequality, we see that there exists a constant $\underline{C}$ such that,

$$\mathbf{E}^{u,v}\big[\big(\int_0^{\tau_k} |Z_s^{u,v}|^2 ds\big)^{\frac{q}{2}}\big] \leq \underline{C}\{\mathbf{E}^{u,v}\,\big[|Y_{\tau_k}^{u,v}|^q\big] + \mathbf{E}^{u,v}\big[\big|\int_0^{\tau_k} Y_s^{u,v} Z_s^{u,v} dB_s^{u,v}\big|^{\frac{q}{2}}\big]\}. \tag{6.17}$$

Next taking into account the estimate (6.16), we deduce that,

$$\mathbf{E}^{u,v}\,\big[|Y_{\tau_k}^{u,v}|^q\big] \leq \mathbf{E}^{u,v}\big[\sup_{s\in[0,T]} |Y_s^{u,v}|^q\big] < \infty.$$

Meanwhile, it follows from the Burkholder-Davis-Gundy (BDG for short) inequality that there exists a constant C_q, depending on q, such that

$$\begin{aligned}
&\mathbf{E}^{u,v}\big[\big|\int_0^{\tau_k} Y_s^{u,v} Z_s^{u,v} dB_s^{u,v}\big|^{\frac{q}{2}}\big] \\
&\leq C_q \mathbf{E}^{u,v}\big[\big(\int_0^{\tau_k} |Y_s^{u,v}|^2 |Z_s^{u,v}|^2 ds\big)^{\frac{q}{4}}\big] \\
&\leq C_q \mathbf{E}^{u,v}\big[\big(\sup_{0\leq s\leq \tau_k} |Y_s^{u,v}|\big)^{\frac{q}{2}}\big(\int_0^{\tau_k} |Z_s^{u,v}|^2 ds\big)^{\frac{q}{4}}\big] \\
&\leq \frac{C_q^2 \underline{C}}{2} \mathbf{E}^{u,v}\big[\big(\sup_{0\leq s\leq T} |Y_s^{u,v}|\big)^q\big] + \frac{1}{2\underline{C}} \mathbf{E}^{u,v}\big[\big(\int_0^{\tau_k} |Z_s^{u,v}|^2 ds\big)^{\frac{q}{2}}\big],
\end{aligned}$$

where $\underline{C}$ is the one of (6.17). Going back now to (6.17) and using Fatou's Lemma, we conclude that for any $q > 1$,

$$\mathbf{E}^{u,v}\big[\big(\int_0^{T} |Z_s^{u,v}|^2 ds\big)^{\frac{q}{2}}\big] < \infty. \tag{6.18}$$

Estimates (6.16) and (6.18) yield to the conclusion (6.9).

Finally note that, taking $t = 0$ in (6.14), we obtain $Y_0^{u,v} = J_1(u, v)$ as a result of $\mathcal{F}_0$ contains only **P** and $\mathbf{P}^{u,v}$ null sets since those probabilities are equivalent. The proof of the Proposition 6.1.1 is completed. □

Proof of Proposition 6.1.2

For $s \leq T$, let us set $\bar{u}_s = \bar{u}(Z_s^1, \theta_s)$ and $\bar{v}_s = \bar{v}(Z_s^2, \vartheta_s)$, then $(\bar{u}_s, \bar{v}_s) \in \mathcal{M}$. On the other hand, thanks to Proposition 6.1.1, we obviously have, $Y_0^1 = J_1(\bar{u}, \bar{v})$.

Next let u be an arbitrary element of $\mathcal{M}_1$ and let us show that $Y^1 \geq Y^{1;u,\bar{v}}$, which yields $Y_0^1 = J_1(\bar{u}, \bar{v}) \geq Y_0^{1;u,\bar{v}} = J^1(u, \bar{v})$.

The control $(u, \bar{v})$ is admissible and thanks to Proposition 6.1.1, there exists a pair of $\mathcal{P}$-measurable processes $(Y^{1;u,\bar{v}}, Z^{1;u,\bar{v}})$ such that for any $q > 1$,

$$\begin{cases}
\mathbf{E}^{u,\bar{v}}\Big[\sup_{0\leq t\leq T} |Y_t^{1;u,\bar{v}}|^q + \big(\int_0^T |Z_s^{1;u,\bar{v}}|^2 ds\big)^{\frac{q}{2}}\Big] < \infty; \\
Y_t^{1;u,\bar{v}} = g_1(X_T^{0,x}) + \int_t^T H_1(s, X_s^{0,x}, Z_s^{1;u,\bar{v}}, u_s, \bar{v}_s) ds - \int_t^T Z_s^{1;u,\bar{v}} dB_s, \ \forall t \leq T.
\end{cases} \tag{6.19}$$

Afterwards, we aim to compare Y^1 in (6.11) and $Y^{1;u,\bar{v}}$ in (6.19). So let us denote by

$$\triangle Y = Y^{1;u,\bar{v}} - Y^1 \text{ and } \triangle Z = Z^{1;u,\bar{v}} - Z^1.$$

For $k \geq 0$, we define the stopping time τ_k as follows:

$$\tau_k := \inf\{s \geq 0, |\triangle Y_s| + \textstyle\int_0^s |\triangle Z_r|^2 dr \geq k\} \wedge T.$$

The sequence of stopping times $(\tau_k)_{k\geq 0}$ is of stationary type and converges to T. Next applying Itô-Meyer formula to $|(\triangle Y)^+|^q$ $(q>1)$ (see Theorem 71, P. Protter, [90], pp.221), between $t\wedge\tau_k$ and τ_k, we obtain: $\forall t\leq T$,

$$\begin{aligned}
&|(\triangle Y_{t\wedge\tau_k})^+|^q + c(q)\int_{t\wedge\tau_k}^{\tau_k} |(\triangle Y_s)^+|^{q-2}1_{\triangle Y_s>0}|\triangle Z_s|^2 ds \\
&= |(\triangle Y_{\tau_k})^+|^q + q\int_{t\wedge\tau_k}^{\tau_k} |(\triangle Y_s)^+|^{q-1}1_{\triangle Y_s>0}\big[H_1(s,X_s^{0,x},Z_s^{1;u,\bar v},u_s,\bar v_s)- \\
&\quad H_1(s,X_s^{0,x},Z_s^1,\bar u_s,\bar v_s)\big]ds - q\int_{t\wedge\tau_k}^{\tau_k} |(\triangle Y_s)^+|^{q-1}1_{\triangle Y_s>0}\triangle Z_s dB_s
\end{aligned} \tag{6.20}$$

where $c(q)=q(q-1)/2$. Besides,

$$\begin{aligned}
H_1(s,X_s^{0,x},Z_s^{1;u,\bar v},u_s,\bar v_s) - H_1(s,X_s^{0,x},Z_s^1,\bar u_s,\bar v_s) = \\
H_1(s,X_s^{0,x},Z_s^{1;u,\bar v},u_s,\bar v_s) - H_1(s,X_s^{0,x},Z_s^1,u_s,\bar v_s) \\
+ H_1(s,X_s^{0,x},Z_s^1,u_s,\bar v_s) - H_1(s,X_s^{0,x},Z_s^1,\bar u_s,\bar v_s)
\end{aligned}$$

Considering now Isaacs' condition (6.8), we have that, the distance of the last two terms $H_1(s,X_s^{0,x},Z_s^1,u_s,\bar v_s) - H_1(s,X_s^{0,x},Z_s^1,\bar u_s,\bar v_s)\leq 0,\quad \forall s\leq T$. Additionally, $H_1(s,X_s^{0,x},Z_s^{1;u,\bar v},u_s,\bar v_s) - H_1(s,X_s^{0,x},Z_s^1,u_s,\bar v_s) = \triangle Z_s\Gamma(s,X_s^{0,x},u_s,\bar v_s)$. Thus, equation (6.20) can be simplified into:

$$\begin{aligned}
&|(\triangle Y_{t\wedge\tau_k})^+|^q + c(q)\int_{t\wedge\tau_k}^{\tau_k} |(\triangle Y_s)^+|^{q-2}1_{\triangle Y_s>0}|\triangle Z_s|^2 ds \\
&\leq |(\triangle Y_{\tau_k})^+|^q + q\int_{t\wedge\tau_k}^{\tau_k} |(\triangle Y_s)^+|^{q-1}\mathbb{1}_{\triangle Y_s>0}\triangle Z_s\Gamma(s,X_s^{0,x},u_s,\bar v_s)ds \\
&\quad - q\int_{t\wedge\tau_k}^{\tau_k} |(\triangle Y_s)^+|^{q-1}1_{\triangle Y_s>0}\triangle Z_s dB_s \\
&= |(\triangle Y_{\tau_k})^+|^q - q\int_{t\wedge\tau_k}^{\tau_k} |(\triangle Y_s)^+|^{q-1}1_{\triangle Y_s>0}\triangle Z_s dB_s^{u,\bar v},
\end{aligned}$$

where $B^{u,\bar v} = (B_t - \int_0^t \Gamma(s,X_s^{0,x},u_s,\bar v_s)ds)_{t\leq T}$ is an $(\mathcal{F}_t,\mathbf{P}^{u,\bar v})$-Brownian motion. Then for any $t\leq T$,

$$|(\triangle Y_{t\wedge\tau_k})^+|^q \leq |(\triangle Y_{\tau_k})^+|^q - q\int_{t\wedge\tau_k}^{\tau_k} |(\triangle Y_s)^+|^{q-1}1_{\triangle Y_s>0}\triangle Z_s dB_s^{u,\bar v}.$$

By definition of the stopping time τ_k, we have

$$\mathbf{E}^{u,\bar v}\big[\textstyle\int_{t\wedge\tau_k}^{\tau_k} |(\triangle Y_s)^+|^{q-1}1_{\triangle Y_s>0}\triangle Z_s dB_s^{u,\bar v}\big] = 0.$$

Thus,

$$\mathbf{E}^{u,\bar v}\left[|(\triangle Y_{t\wedge\tau_k})^+|^q\right] \leq \mathbf{E}^{u,\bar v}\big[|(Y_{\tau_k}^{1;u,\bar v} - Y_{\tau_k}^1)^+|^q\big]. \tag{6.21}$$

Next taking into account (6.13) and the fact that Y^1 has a representation through η^1 which is deterministic and of polynomial growth, we obtain the following uniformly integral result, i.e.

$$\mathbf{E}^{u,\bar v}\Big[\sup_{s\leq T}(|Y_s^1| + |Y_s^{1;u,\bar v}|)^q\Big] < \infty \tag{6.22}$$

As the sequence $((Y_{\tau_k}^{1,u,\bar v} - Y_{\tau_k}^1)^+)_{k\geq 0}$ converges to 0 when $k\to\infty$, $\mathbf{P}^{u,\bar v}$-a.s., since $\lim_{k\to\infty} Y_{\tau_k}^{1;u,\bar v} = \lim_{k\to\infty} Y_{\tau_k}^1 = g_1(X_T^{0,x})$ $\mathbf{P}^{u,\bar v}$-a.s.. Then it converges also to 0 in $L^1(d\mathbf{P}^{u,\bar v})$ thanks to (6.22). As $k\to\infty$ on (6.21), it follows from Fatou's Lemma that:

$$\mathbf{E}^{u,\bar v}\left[\triangle Y_t^+\right] = 0,\ \forall t\leq T,$$

which implies that $Y^1 \geq Y^{1;u,\bar{v}}$, **P**-a.s., since the probabilities $\mathbf{P}^{u,\bar{v}}$ and $\mathbf{P}$ are equivalent. Thus $Y_0^1 = J^1(\bar{u},\bar{v}) \geq Y_0^{1,u,\bar{v}} = J^1(u,\bar{v})$.

Similarly, we can show that, $Y_0^2 = J^2(\bar{u},\bar{v}) \geq Y_0^{2,\bar{u},v} = J^2(\bar{u},v)$ for arbitrary $v \in \mathcal{M}_2$. Henceforth $(\bar{u},\bar{v})$ is a Nash equilibrium point for the NZSDG. □

Proof of Theorem 6.1.1

The proof will be split into several steps. Firstly, we construct an approximating sequence of BSDEs with continuous generators by smoothing the functions $\bar{u}$ and $\bar{v}$. The next thing to do is to introduce some uniform integral properties of the solutions. Another step is to prove the convergence of the sequences (at least for a subsequence). Finally, the most challenging part, is to verify the limit processes are indeed the solutions of the original BSDE.

Step 1: Approximation. At the beginning of this proof, we would like to clarify that the functions $p \in \mathbf{R} \mapsto p\bar{u}(p,\epsilon_1)$ and $q \in \mathbf{R} \mapsto q\bar{v}(q,\epsilon_2)$ are uniformly Lipschitz w.r.t. ϵ_1 and ϵ_2, since $p\bar{u}(p,\epsilon_1) = p\bar{u}(p,0) = \sup_{u\in U} pu$ and $q\bar{v}(q,\epsilon_2) = q\bar{v}(q,0) = \sup_{v\in V} qv$. Hereafter $\bar{u}(p,0)$ (resp. $\bar{v}(q,0)$) will be simply denoted by $\bar{u}(p)$ (resp. $\bar{v}(q)$).

Next for integer $n \geq 1$, let $\bar{u}^n$ and $\bar{v}^n$ be the functions defined as follows:

$$\bar{u}^n(p) = \begin{cases} 0 \text{ if } p \leq -1/n, \\ 1 \text{ if } p \geq 0, \\ np+1 \text{ if } p \in (-1/n, 0), \end{cases} \quad \text{and} \quad \bar{v}^n(q) = \begin{cases} -1 \text{ if } q \leq -1/n, \\ 1 \text{ if } q \geq 1/n, \\ nq \text{ if } q \in (-1/n, 1/n). \end{cases}$$

Note that $\bar{u}^n$ and $\bar{v}^n$ are Lipschitz in p and q respectively. Roughly speaking, they are the approximations of $\bar{u}$ and $\bar{v}$. Below, let Φ_n be the truncation function $x \in \mathbf{R} \mapsto \Phi_n(x) = (x \wedge n) \vee (-n) \in \mathbf{R}$, which is bounded by n. Now for $n \geq 1$, we establish the following BSDE of dimension two, with the generator which is Lipschitz. For $s \leq T$,

$$\begin{cases} -dY_s^{1,n;t,x} = \{\Phi_n(Z_r^{1,n;t,x})\Phi_n(f(r,X_r^{t,x})) + \Phi_n(Z_r^{1,n;t,x}\bar{u}(Z_r^{1,n;t,x}))+ \\ \qquad \Phi_n(Z_r^{1,n;t,x})\bar{v}^n(Z_r^{2,n;t,x})\}dr - Z_r^{1,n;t,x}dB_r,\ Y_T^{1,n;t,x} = g_1(X_T^{t,x}); \\ -dY_s^{2,n;t,x} = \{\Phi_n(Z_r^{2,n;t,x})\Phi_n(f(r,X_r^{t,x})) + \Phi_n(Z_r^{2,n;t,x}\bar{v}(Z_r^{2,n;t,x}))+ \\ \qquad \Phi_n(Z_r^{2,n;t,x})\bar{u}^n(Z_r^{1,n;t,x})\}dr - Z_r^{2,n;t,x}dB_r,\ Y_T^{2,n;t,x} = g_2(X_T^{t,x}). \end{cases} \tag{6.23}$$

From the Pardoux-Peng's result ([84]), this equation has a unique solution $(Y^{i,n;t,x}, Z^{i,n;t,x}) \in \mathcal{S}_T^2(\mathbf{R}) \times \mathcal{H}_T^2(\mathbf{R})$ for $n \geq 1$ and $i = 1,2$. Taking account of the result by El-Karoui et al.([39],pp.46, Theorem 4.1), we obtain that, there exist measurable deterministic functions $\eta^{i,n}$ and $\varsigma^{i,n}$ of $(s,x) \in [t,T] \times \mathbf{R}$, $i = 1,2$ and $n \geq 1$, such that:

$$Y_s^{i,n;t,x} = \eta^{i,n}(s, X_s^{t,x}) \quad \text{and} \quad Z_s^{i,n;t,x} = \varsigma^{i,n}(s, X_s^{t,x}). \tag{6.24}$$

Moreover, for $n \geq 1$ and $i = 1,2$, the functions $\eta^{i,n}$ verify: $\forall (t,x) \in [0,T] \times \mathbf{R}$,

$$\eta^{i,n}(t,x) = \mathbf{E}[g_i(X_T^{t,x})] + \int_t^T H_i^n(r, X_r^{t,x})dr] \tag{6.25}$$

with, for any $(s,x) \in [0,T] \times \mathbf{R}$,

$$\begin{cases} H_1^n(s,x) = \Phi_n(\varsigma^{1,n}(s,x))\Phi_n(f(s,x)) + \Phi_n\big(\varsigma^{1,n}(s,x)\bar{u}(\varsigma^{1,n}(s,x))\big)+ \\ \qquad + \Phi_n(\varsigma^{1,n}(s,x))\bar{v}^n(\varsigma^{2,n}(s,x)); \\ H_2^n(s,x) = \Phi_n(\varsigma^{2,n}(s,x))\Phi_n(f(s,x)) + \Phi_n\big(\varsigma^{2,n}(s,x)\bar{v}(\varsigma^{2,n}(s,x))\big)+ \\ \qquad + \Phi_n(\varsigma^{2,n}(s,x))\bar{u}^n(\varsigma^{1,n}(s,x)). \end{cases} \tag{6.26}$$

Step 2: Estimates for processes $(Y^{i,n;t,x}, Z^{i,n;t,x}), i = 1, 2.$ In order to show the uniform estimates for $Y^{i,n}$ of BSDE (6.23), we present the following comparative BSDE, for any $s \in [t, T]$ and $i = 1, 2$,

$$\bar{Y}_s^{i,n} = g_i(X_T^{t,x}) + \int_s^T \Phi_n(C(1 + |X_r^{t,x}|))|\bar{Z}_r^{i,n}| + C|\bar{Z}_r^{i,n}|dr - \int_s^T \bar{Z}_r^{i,n} dB_r, \qquad (6.27)$$

where the constant C is related to the bound of function f and the bound of sets U and V, which makes that the generators of (6.23) $H_i^n(s, X_s^{t,x})$ satisfy, $|H_i^n(s, X_s^{t,x})| \leq \Phi_n(C(1+|X_s^{t,x}|))|Z_s^{i,n;t,x}| + C|Z_s^{i,n;t,x}|$ for each $n \geq 1$ and $(t, x) \in [0, T] \times \mathbf{R}$. Observing that the application $z \in \mathbf{R} \mapsto \Phi_n(C(1 + |X_r^{t,x}|))|z| + C|z|$ is Lipschitz continuous, therefore the solution $(\bar{Y}^{i,n}, \bar{Z}^{i,n})$ of the above BSDE indeed exists on space $\mathcal{S}_T^2(\mathbf{R}) \times \mathcal{H}_T^2(\mathbf{R})$ and is unique. Provided that we show the uniform estimate for $\bar{Y}^{i,n}$ w.r.t. n, then, the estimate for $Y^{i,n}$ will be a straightforward consequence by the comparison Theorem of BSDEs ([39], pp.23). Below, we will focus on the property of $\bar{Y}^{i,n}$.

Using again the result by El Karoui et al. ([39], pp.46, Theorem 4.1) yields that, there exist deterministic measurable functions $\bar{\eta}^{i,n} : [0, T] \times \mathbf{R} \to \mathbf{R}$ such that, for any $s \in [t, T]$,

$$\bar{Y}_s^{i,n} = \bar{\eta}^{i,n}(s, X_s^{t,x}),\ i = 1, 2. \qquad (6.28)$$

Next let us consider the process
$B^{i,n} = (B_s - \int_0^s [\Phi_n(C(1 + |X_r^{t,x}|)) + C]\text{sign}(Z_r^{i,n})dr)_{t \leq s \leq T}$, $i = 1, 2$, which is, thanks to Girsanov's Theorem, a Brownian motion under the probability $\mathbf{P}^{i,n}$ on $(\Omega, \mathcal{F})$ whose density with respect to $\mathbf{P}$ is $\zeta_T\{[\Phi_n(C(1 + |X_s^{t,x}|)) + C]\text{sign}(Z_s^{i,n})\}$ where for any $z \in \mathbf{R}$, $\text{sign}(z) = 1_{\{|z| \neq 0\}} \frac{z}{|z|}$ and $\zeta_T(.)$ is defined by (6.4). Then the BSDE (6.27) will be simplified into,

$$\bar{Y}_s^{i,n} = g_i(X_T^{t,x}) - \int_s^T \bar{Z}_r^{i,n} dB_r^{i,n},\ s \in [t, T],\ i = 1, 2.$$

In view of (6.28), we obtain,

$$\bar{\eta}^{i,n}(t, x) = \mathbf{E}^{i,n}[g_i(X_T^{t,x})|\mathcal{F}_t],\ i = 1, 2,$$

where $\mathbf{E}^{i,n}$ is the expectation under probability $\mathbf{P}^{i,n}$. By taking the expectation on both sides of the above equation under the probability $\mathbf{P}^{i,n}$ and considering $\bar{\eta}^{i,n}(t, x)$ is deterministic, we arrive at,

$$\bar{\eta}^{i,n}(t, x) = \mathbf{E}^{i,n}[g_i(X_T^{t,x})],\ i = 1, 2.$$

The functions $g_i (i = 1, 2)$ are of polynomial growth as we stated in Assumption (A2), combining with the estimate (6.12) and Lemma 6.1.1, gives that there exists a constant $p_0 \in (1, 2)$ (which does not depend on (t, x)) such that:

$$\begin{aligned}
|\bar{\eta}^{i,n}(t, x)| &\leq C\mathbf{E}^{i,n}\big[\sup_{s \in [s,T]} \{1 + |X_s^{t,x}|^\gamma\}\big] \\
&= C\mathbf{E}\big[\big(\sup_{s \in [t,T]} \{1 + |X_s^{t,x}|^\gamma\}\big)\big(\zeta_T\{[\Phi_n(C(1 + |X_s^{t,x}|)) + C]\text{sign}(Z_s^{i,n})\}\big)\big] \\
&\leq C\mathbf{E}\big[\sup_{s \in [t,T]} (1 + |X_s^{t,x}|^\gamma)^{\frac{p_0}{p_0 - 1}}\big] + \\
&\quad +C\mathbf{E}\big[\big(\zeta_T\{[\Phi_n(C(1 + |X_s^{t,x}|)) + C]\text{sign}(Z_s^{i,n})\}\big)^{p_0}\big] \\
&\leq C(1 + |x|^\lambda),
\end{aligned}$$

where constant $\lambda = \frac{\gamma p_0}{p_0 - 1} > 2$. It follows from comparison Theorem that, for any $s \in [t, T]$ and $i = 1, 2$, $\bar{Y}_s^{i,n} = \bar{\eta}^{i,n}(s, X_s^{t,x}) \geq Y_s^{i,n} = \eta^{i,n}(s, X_s^{t,x})$. Choosing $s = t$ leads to $\eta^{i,n}(t, x) \leq C(1 + |x|^\lambda)$,

$(t,x) \in [0,T] \times \mathbf{R}$. In a similar way, we can show that $\eta^{i,n}(t,x) \geq -C(1+|x|^\lambda)$, $(t,x) \in [0,T] \times \mathbf{R}$. Therefore, $\eta^{i,n},\ i=1,2$ are of polynomial growth with respect to (t,x) uniformly in n.

To conclude this step, we have the following results. There exists a constant C independent of n and t,x such that, for $(t,x) \in [0,T] \times \mathbf{R}$, $i=1,2$,

$$\begin{cases} \text{(a) } |\eta^{i,n}(t,x)| \leq C(1+|x|^\lambda),\ \text{for any } \lambda > 2; \\ \text{(b) by the combination of (a), (6.12) with (6.24), we have, indeed for any } \alpha \geq 1, \\ \qquad \mathbf{E}[\sup_{s\in[t,T]} |Y_s^{i,n;t,x}|^\alpha] \leq C; \\ \text{(c) for any } (t,x),\ \mathbf{E}[\int_t^T |Z_s^{i,n;t,x}|^2 ds] \leq C \text{ which is a straightforward result by} \\ \qquad \text{using Itô's formula with process } (Y^{i,n;t,x})^2. \end{cases} \tag{6.29}$$

<u>*Step 3: Convergence of sequence* $(Y^{i,n;0,x}, Z^{i,n;0,x})_{n\geq 1}$, $i=1,2$.</u> Let us now fix the initial state to $(0,x)$, for $x \in \mathbf{R}$. We first show that
$H_i^n(s,y) \in L^q([0,T]\times\mathbf{R};\mu(0,x;s,dy)ds)$ [1] for fixed $q \in (1,2)$, $i=1,2$. H_i^n is defined by (6.26) which is the generator of BSDE (6.23). Actually,

$$\begin{aligned} \mathbf{E}[\int_0^T |H_i^n(s,X_s^{0,x})|^q ds] &= \int_{[0,T]\times\mathbf{R}} |H_i^n(s,y)|^q \mu(0,x;s,dy)ds \\ &\leq C\mathbf{E}[\int_0^T |Z_s^{i,n;0,x}|^q (1+|X_s^{0,x}|^q) ds] \\ &\leq C\{\mathbf{E}[\int_0^T |Z_s^{i,n;0,x}|^2 ds] + \mathbf{E}[1+\sup_{s\in[0,T]} |X_s^{0,x}|^{\frac{2q}{2-q}}]\} \\ &< \infty, \end{aligned} \tag{6.30}$$

which is obtained by the facts that $\mathbf{E}[\int_0^T |Z_s^{i,n;0,x}|^2 ds] \leq C$ and the estimate (6.12). As a result, there exists a subsequence $\{n_k\}$ (still denoted by $\{n\}$ for simplification) and two $\mathcal{B}([0,T]) \times \mathcal{B}(\mathbf{R})$-measurable deterministic functions $H_i(s,y)$, $i=1,2$, such that,

$$H_i^n \to H_i \text{ weakly in } L^q([0,T]\times\mathbf{R};\mu(0,x;s,dy)ds),\ \text{for } i=1,2,\ \text{fixed } q \in (1,2). \tag{6.31}$$

Next we focus on passing from the weak convergence to strong sense convergence by proving that $(\eta^{i,n}(t,x))_{n\geq 1}$ defined in (6.25) is a Cauchy sequence for each $(t,x) \in [0,T]\times\mathbf{R}$, $i=1,2$. Let (t,x) be fixed, $\delta > 0, k, n$ and $m \geq 1$ be integers. From (6.25), we have,

$$\begin{aligned} |\eta^{i,n}(t,x) - \eta^{i,m}(t,x)| &= \big|\mathbf{E}[\int_t^T H_i^n(s,X_s^{t,x}) - H_i^m(s,X_s^{t,x}) ds]\big| \\ &\leq \big|\mathbf{E}[\int_t^{t+\delta} H_i^n(s,X_s^{t,x}) - H_i^m(s,X_s^{t,x}) ds]\big| \\ &\quad + \big|\mathbf{E}[\int_{t+\delta}^T (H_i^n(s,X_s^{t,x}) - H_i^m(s,X_s^{t,x})) \cdot 1_{\{|X_s^{t,x}|\leq k\}} ds]\big| \\ &\quad + \big|\mathbf{E}[\int_{t+\delta}^T (H_i^n(s,X_s^{t,x}) - H_i^m(s,X_s^{t,x})) \cdot 1_{\{|X_s^{t,x}|> k\}} ds]\big|. \end{aligned} \tag{6.32}$$

We deal with the first term on the right side of inequality (6.32) by Young's inequality, i.e.
$\big|\mathbf{E}[\int_t^{t+\delta} H_i^n(s,X_s^{t,x}) - H_i^m(s,X_s^{t,x}) ds]\big| \leq \delta^{\frac{q-1}{q}} \{\mathbf{E}[\int_t^T \big|H_i^n(s,X_s^{t,x}) - H_i^m(s,X_s^{t,x})\big|^q ds]\}^{\frac{1}{q}}$. It follows from

[1] For $(t,x) \in [0,T]\times\mathbf{R},\ s \in [t,T],\ \mu(t,x;s,dy)$ is the law of $X_s^{t,x}$, i.e., $\forall A \in \mathcal{B}(\mathbf{R}),\ \mu(t,x;s,A) = \mathbf{P}(X_s^{t,x} \in A)$.

(6.30) that it is bounded by $C\delta^{\frac{q-1}{q}}$. The third part of the right side of inequality (6.32) is $\big|\mathbf{E}[\int_{t+\delta}^T (H_i^n(s,X_s^{t,x}) - H_i^m(s,X_s^{t,x}))\cdot 1_{\{|X_s^{t,x}|>k\}}ds]\big|$ $\leq C\{\mathbf{E}[\int_{t+\delta}^T 1_{\{|X_s^{t,x}|>k\}}ds]\}^{\frac{q-1}{q}}\{\mathbf{E}[\int_{t+\delta}^T |H_i^n(s,X_s^{t,x}) - H_i^m(s,X_s^{t,x})|^q ds]\}^{\frac{1}{q}}$, which is bounded by $Ck^{-\frac{q-1}{q}}$ as a result of Markov inequality and (6.30). The second component of (6.32) is exactly the following,

$$\begin{aligned}
&|\int_{\mathbf{R}}\int_{t+\delta}^T (H_i^n(s,y) - H_i^m(s,y))1_{\{|y|\leq k\}}\frac{1}{\sqrt{2\pi(s-t)}}e^{-\frac{(y-x)^2}{2(s-t)}}dsdy| \\
&= \big|\int_{\mathbf{R}}\int_{t+\delta}^T (H_i^n(s,y) - H_i^m(s,y))1_{\{|y|\leq k\}}\frac{\sqrt{s-t}}{\sqrt{s}}e^{-\frac{(y-x)^2 t}{2s(s-t)}}\frac{1}{\sqrt{2\pi s}}e^{-\frac{(y-x)^2}{2s}}dsdy\big|. \qquad (6.33)
\end{aligned}$$

In (6.33), notice that $\frac{1}{\sqrt{2\pi s}}e^{-\frac{(y-x)^2}{2s}}dy$ is the law of $X_s^{0,x}$. Besides, the function $\frac{\sqrt{s-t}}{\sqrt{s}}e^{-\frac{(y-x)^2 t}{2s(s-t)}}$ is bounded for any $(t,x)\in[0,T]\times\mathbf{R}$ and $(s,y)\in(t+\delta,T]\times\mathbf{R}$. Therefore, by the weak convergence result (6.31), we get that (6.33) converges to 0 as $n,m\to\infty$. Generally speaking, (6.32) will converges to 0 as $n,m,k\to\infty$ and $\delta\to 0$. Then, we conclude that $(\eta^{i,n}(t,x))_{n\geq 1}$ is a Cauchy sequence for each $(t,x)\in[0,T]\times\mathbf{R}$, $i=1,2$ and there exists a measurable application η^i on $[0,T]\times\mathbf{R}$, $i=1,2$ such that for each $(t,x)\in[0,T]\times\mathbf{R}$, $i=1,2$,

$$\lim_{n\to\infty}\eta^{i,n}(t,x) = \eta^i(t,x).$$

Next, we infer from the polynomial growth property of $\eta^{i,n}$ that η^i is of polynomial growth as well, *i.e.*, $\forall (t,x)\in[0,T]\times\mathbf{R}$ and $i=1,2$, $|\eta^i(t,x)|\leq C(1+|x|^\lambda)$, $\lambda>2$. It turns out that, for any $t\geq 0$,

$$\lim_{n\to\infty}Y_t^{i,n;0,x}(\omega) = \eta^i(t,X_t^{0,x}(\omega)), \quad |Y_t^{i,n;0,x}(\omega)|\leq C(1+|X_t^{0,x}(\omega)|^\lambda),\ \mathbf{P}-a.s.$$

By Lebesgue's dominated convergence theorem, the sequence $((Y_t^{i,n;0,x})_{t\leq T})_{n\geq 1}$ converges to $Y^i = (\eta^i(t,X_t^{0,x}))_{t\leq T}$ in $L^\alpha([0,T]\times\mathbf{R})$ for any $\alpha\geq 1$, $i=1,2$, *i.e.*,

$$\mathbf{E}[\int_0^T |Y_s^{i,n;0,x} - Y_s^i|^\alpha ds]\to 0, \quad \text{as } n\to\infty \quad \text{for any } \alpha\geq 1, i=1,2. \qquad (6.34)$$

It remains to show the convergence of sequence $((Z_s^{i,n;0,x})_{s\leq T})_{n\geq 1}$ for $i=1,2$. Taking Itô's formula to process $(Y^{i,n;0,x} - Y^{i,m;0,x})^2$ and considering Assumption (A1), we get,

$$\begin{aligned}
&|Y_t^{i,n;0,x} - Y_t^{i,m;0,x}|^2 + \int_t^T |Z_s^{i,n;0,x} - Z_s^{i,m;0,x}|^2 ds \\
&\leq 2\int_t^T C(Y_s^{i,n;0,x} - Y_s^{i,m;0,x})(1+|X_s^{0,x}|)(|Z_s^{i,n;0,x}| + |Z_s^{i,m;0,x}|)ds \\
&\quad - 2\int_t^T (Y_s^{i,n;0,x} - Y_s^{i,m;0,x})(Z_s^{i,n;0,x} - Z_s^{i,m;0,x})dB_s. \qquad (6.35)
\end{aligned}$$

Since for any $x,y,z\in\mathbf{R}$ and for any $\epsilon>0$, $|xyz|\leq\frac{\epsilon^2}{2}x^2 + \frac{\epsilon^4}{4}y^4 + \frac{1}{4\epsilon^8}z^4$. Then we have,

$$\begin{aligned}
&|Y_t^{i,n;0,x} - Y_t^{i,m;0,x}|^2 + \int_t^T |Z_s^{i,n;0,x} - Z_s^{i,m;0,x}|^2 ds \\
&\leq C\{\frac{\epsilon^2}{2}\int_t^T (|Z_s^{i,n;0,x}| + |Z_s^{i,m;0,x}|)^2 ds + \frac{\epsilon^4}{4}\int_t^T (1+|X_s^{0,x}|)^4 ds \\
&\quad + \frac{1}{4\epsilon^8}\int_t^T |Y_s^{i,n;0,x} - Y_s^{i,m;0,x}|^4 ds\} \\
&\quad - 2\int_t^T (Y_s^{i,n;0,x} - Y_s^{i,m;0,x})(Z_s^{i,n;0,x} - Z_s^{i,m;0,x})dB_s. \qquad (6.36)
\end{aligned}$$

Since ϵ is arbitrary, taking now $t=0$, expectation on both sides and the limit w.r.t. n,m, combining with (6.34), (6.12), (6.29)-(c) yields that,

$$\limsup_{n,m\to\infty}\mathbf{E}[\int_0^T|Z_s^{i,n;0,x}-Z_s^{i,n;0,x}|^2ds]\to 0,\quad i=1,2. \tag{6.37}$$

Consequently, the sequence $(Z^{i,n;0,x}=(\varsigma^{i,n}(t,X_t^{0,x}))_{t\leq T})_{n\geq 1}$ is convergent in $L^2([0,T]\times\mathbf{R})$ to a process Z^i, for $i=1,2$. Additionally, we shall substract a subsequence $\{n_k\}$ (denoted still by $\{n\}$) such that, $(Z^{i,n;0,x})_{n\geq 1}$ converges to Z^i, $dt\otimes d\mathbf{P}$-a.e. and $\sup_{n\geq 1}|Z_t^{i,n;0,x}(\omega)|\in L^2([0,T]\times\mathbf{R})$ for $i=1,2$.

Next going back to inequality (6.36), taking the supremum on interval $[0,T]$ and using BDG's inequality, we deduce that,

$$\begin{aligned}
&\mathbf{E}\big[\sup_{t\in[0,T]}|Y_t^{i,n;0,x}-Y_t^{i,m;0,x}|^2+\int_0^T|Z_s^{i,n;0,x}-Z_s^{i,m;0,x}|^2ds\big]\\
&\leq C\mathbf{E}\big\{\frac{\epsilon^2}{2}\int_0^T(|Z_s^{i,n;0,x}|+|Z_s^{i,m;0,x}|)^2ds+\frac{\epsilon^4}{4}\int_0^T(1+|X_s^{0,x}|)^4ds\\
&\qquad+\frac{1}{4\epsilon^8}\int_0^T|Y_s^{i,n;0,x}-Y_s^{i,m;0,x}|^4ds\big\}\\
&\quad+\frac{1}{2}\mathbf{E}\big[\sup_{t\in[0,T]}|Y_t^{i,n;0,x}-Y_t^{i,m;0,x}|^2\big]+2\mathbf{E}\big[\int_0^T|Z_s^{i,n;0,x}-Z_s^{i,m;0,x}|^2ds\big],
\end{aligned}$$

which implies,

$$\limsup_{n,m\to\infty}\mathbf{E}\big[\sup_{t\in[0,T]}|Y_t^{i,n;0,x}-Y_t^{i,m;0,x}|^2\big]=0,$$

since ϵ is arbitrary and the facts of (6.12), (6.37), (6.34) and (6.29)-(c). Thus the sequence of processes $(Y^{i,n;0,x})_{n\geq 1}$ converges in $\mathcal{S}_T^2(\mathbf{R})$ to Y^i for $i=1,2$ which are continuous processes.

To summarize this step, we have the following results: for $i=1,2$,

$$\begin{cases}
\text{(a) } H_i^n(s,X_s^{0,x})\in L^q([0,T]\times\mathbf{R};\ \mu(0,x;s,dy)ds) \text{ uniformly w.r.t. } n;\\
\text{(b) } Y^{i,n;0,x}\to_{n\to\infty}Y^i \text{ in } L^\alpha([0,T]\times\mathbf{R}) \text{ for any } \alpha\geq 1, \text{ besides,}\\
\qquad Y^{i,n;0,x}\to_{n\to\infty}Y^i \text{ in } \mathcal{S}_T^2(\mathbf{R});\\
(c)\ Z^{i,n;0,x}\to_{n\to\infty}Z^i \text{ in } L^2([0,T]\times\mathbf{R}), \text{ additionally, there exists a}\\
\qquad \text{subsequence } \{n\} \text{ s.t } Z^{i,n;0,x}\to_{n\to\infty}Z^i\ dt\otimes d\mathbf{P}-a.e. \text{ and}\\
\qquad \sup_{n\geq 1}|Z^{i,n;0,x}|\in L^2([0,T]\times\mathbf{R}).
\end{cases} \tag{6.38}$$

Step 4: Convergence of $(H_i^n)_{n\geq 1},\ i=1,2$. In this step, we verify that the limit processes $(Y_s^i,Z_s^i)_{s\leq T},\ i=1,2$ are the solutions of BSDE (6.11). Briefly speaking, we need to show there exists $\mathcal{P}$-measurable process θ (resp. ϑ) valued on U (resp. V), such that, Y^i,Z^i,η^i ($i=1,2$) and θ,ϑ verify (i)-(iii) and (a),(b) of Proposition 6.1.2. Hereafter, we delete the super script $(0,x)$ for convenience. Let us demonstrate first for player i=1, the case for player i=2 follows in a similar way.

First of all, we give a weak convergence result about the subsequence of H_1^n, i.e. there exists $\mathcal{P}$-measurable process ϑ valued on V such that, for integer $k\geq 0$,

$$H_1^{n_k}(s,X_s)\to_{k\to\infty}H_1^*(s,X_s,Z_s^1,Z_s^2,\vartheta)\quad\text{weakly in}\quad\mathcal{H}_T^2(\mathbf{R}). \tag{6.39}$$

Let us now prove (6.39). Recall (6.26) as following,

$$H_1^n(s,X_s)=\Phi_n(Z_s^{1,n})\Phi_n(f(s,X_s))+\Phi_n(Z_s^{1,n}\bar{u}(Z_s^{1,n})+\Phi_n(Z_s^{1,n})\bar{v}^n(Z_s^{2,n}). \tag{6.40}$$

Note that,

$$\Phi_n(Z_s^{1,n})\Phi_n(f(s,X_s)) + \Phi_n(Z_s^{1,n}\bar{u}(Z_s^{1,n})) \to_{n\to\infty} Z_s^1 f(s,X_s) + Z_s^1\bar{u}(Z_s^1),\ ds\otimes d\mathbf{P}\text{-a.e.}$$

since $Z^{1,n} \to_{n\to\infty} Z^1$, $ds\otimes d\mathbf{P}$-a.e. as stated in (6.38)-(c), for any $x\in\mathbf{R}$, $\Phi_n(x)\to_{n\to\infty} x$ and finally by the continuity of $p\in\mathbf{R}\mapsto p\bar{u}(p)$. The rest part in (6.40) is

$$\Phi_n(Z_s^{1,n})\bar{v}^n(Z_s^{2,n}) = \Phi_n(Z_s^{1,n})\bar{v}^n(Z_s^{2,n})1_{\{Z_s^2\neq 0\}} + \Phi_n(Z_s^{1,n})\bar{v}^n(Z_s^{2,n})1_{\{Z_s^2=0\}},$$

where

$$\Phi_n(Z_s^{1,n})\bar{v}^n(Z_s^{2,n})1_{\{Z_s^2\neq 0\}} \to_{n\to\infty} Z_s^1\bar{v}(Z_s^2)1_{\{Z_s^2\neq 0\}} \quad ds\otimes d\mathbf{P}\text{-a.e.}$$

since $\bar{v}$ is continuous at any point different from 0 and $Z_s^{2,n} \to_{n\to\infty} Z_s^2$ $ds\otimes d\mathbf{P}$-a.e. by (6.38)-(c). Let us next define an $\mathcal{P}$-measurable process $(\vartheta_s)_{s\le T}$ valued on V as the weak limit in $\mathcal{H}_T^2(\mathbf{R})$ of some subsequence $(\bar{v}^{n_k}(Z^{2,n_k})1_{\{Z^2=0\}})_{k\ge 0}$. The weak limit exists since $(\bar{v}^{n_k})_{k\ge 0}$ is bounded. Then, for $s\le T$,

$$\Phi_n(Z_s^{1,n_k})\bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}} \to_{k\to\infty} Z_s^1\vartheta_s 1_{\{Z_s^2=0\}} \text{ weakly in } \mathcal{H}_T^2(\mathbf{R}).$$

Therefore (6.39) holds.

In fact, we still need to show for any stopping time τ,

$$\int_0^\tau H_1^{n_k}(s,X_s)ds \to_{k\to\infty} \int_0^\tau H_1^*(s,X_s,Z_s^1,Z_s^2,\vartheta_s)ds \text{ weakly in } L^2(\Omega,d\mathbf{P}). \tag{6.41}$$

As explained before, taking account of the expression (6.40), we only need to prove the weak convergence of the following part,

$$\int_0^\tau \Phi_{n_k}(Z_s^{1,n_k})\bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}}ds \to_{k\to\infty} \int_0^\tau Z_s^1\vartheta_s 1_{\{Z_s^2=0\}}ds \text{ weakly in } L^2(\Omega,d\mathbf{P})$$

Obviously, we have,

$$\begin{aligned}\int_0^\tau \Phi_{n_k}(Z_s^{1,n_k})\bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}}ds &= \int_0^\tau (\Phi_{n_k}(Z_s^{1,n_k}) - Z_s^1)\bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}}ds \\ &\quad + \int_0^\tau Z_s^1\bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}}ds.\end{aligned}$$

On the right side, the first integral converges to 0 by Lebesgue's dominated convergence theorem since $\Phi_{n_k}(Z^{1,n_k}) \to Z^1$ $dt\otimes d\mathbf{P}$-a.e., $\sup_{k\ge 0}|Z_t^{1,n_k}| \in L^2([0,T]\times\mathbf{R})$ as shown in (6.38)-(c), $Z^1\in L^2([0,T]\times\mathbf{R})$ and the sequence $(\bar{v}^{n_k})_{k\ge 0}$ is bounded. Below, we will give the weak convergence in $L^2(\Omega,d\mathbf{P})$ of the integral $\int_0^\tau Z_s^1\bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}}ds$ to $\int_0^\tau Z_s^1\vartheta_s 1_{\{Z_s^2=0\}}ds$. That is, for any random variable $\xi\in L^2(\Omega,\mathcal{F}_T,d\mathbf{P})$, we need to show,

$$\mathbf{E}[\xi\int_0^\tau Z_s^1\bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}}ds] \to_{k\to\infty} \mathbf{E}[\xi\int_0^\tau Z_s^1\vartheta_s 1_{\{Z_s^2=0\}}ds]. \tag{6.42}$$

Thanks to martingale representation theorem, there exists a process $(\Lambda_s)_{s\le T}\in\mathcal{H}_T^2(\mathbf{R})$ such that, $\mathbf{E}[\xi|\mathcal{F}_\tau] = \mathbf{E}[\xi] + \int_0^\tau \Lambda_s dB_s$. Therefore,

$$\begin{aligned}&\mathbf{E}\Big[\xi\int_0^\tau Z_s^1\bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}}ds\Big] \\ &= \mathbf{E}\Big[\mathbf{E}[\xi\int_0^\tau Z_s^1\bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}}ds]\Big|\mathcal{F}_\tau\Big] \\ &= \mathbf{E}\Big[\mathbf{E}[\xi|\mathcal{F}_\tau]\cdot\int_0^\tau Z_s^1\bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}}ds\Big] \\ &= \mathbf{E}\Big[\mathbf{E}[\xi]\int_0^\tau Z_s^1\bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}}ds\Big] \\ &\quad + \mathbf{E}\Big[\int_0^\tau \Lambda_s dB_s\cdot\int_0^\tau Z_s^1\bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}}ds\Big].\end{aligned}$$

Notice that $\mathbf{E}[\xi]\mathbf{E}[\int_0^\tau Z_s^1 \bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}}ds] \to_{k\to\infty} \mathbf{E}[\xi]\mathbf{E}[\int_0^\tau Z_s^1 \vartheta_s 1_{\{Z_s^2=0\}}ds]$, since $(Z_s^1)_{s\le T} \in \mathcal{H}_T^2(\mathbf{R})$ and $\bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}} \to_{k\to\infty} \vartheta_s$ weakly in $\mathcal{H}_T^2(\mathbf{R})$. Next, by Itô's formula,

$$\begin{aligned}&\mathbf{E}[\int_0^\tau \Lambda_s dB_s \cdot \int_0^\tau Z_s^1 \bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}}ds]\\ &= \mathbf{E}[\int_0^\tau \big(\int_0^s \Lambda_u dB_u\big) Z_s^1 \bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}}ds]+\\ &\quad + \mathbf{E}[\int_0^\tau \big(\int_0^s Z_u^1 \bar{v}^{n_k}(Z_u^{2,n_k})1_{\{Z_u^2=0\}}du\big)\Lambda_s dB_s].\end{aligned}$$

The later one on the right side is 0, since $\int_0^\cdot (\int_0^s Z_u^1 \bar{v}^{n_k}(Z_u^{2,n_k})1_{\{Z_u^2=0\}}du)\Lambda_s dB_s$ is an $\mathcal{F}_t$-martingale which is followed by $(Z_s^1)_{s\le T} \in \mathcal{H}_T^2(\mathbf{R})$, $(\Lambda_s)_{s\le T} \in \mathcal{H}_T^2(\mathbf{R})$ and the boundness of $\bar{v}^{n_k}$. For the former part, let us denote $\int_0^s \Lambda_u dB_u$ by ψ_s for any $s \in [0,\tau]$. Then for any integer $\kappa > 0$, we have,

$$\begin{aligned}&|\mathbf{E}[\int_0^\tau \psi_s Z_s^1 \Big(\bar{v}^{n_k}(Z_s^{2,n_k}) - \vartheta_s\Big)1_{\{Z_s^2=0\}}ds]|\\ &= |\mathbf{E}[\int_0^\tau \psi_s Z_s^1 \Big(\bar{v}^{n_k}(Z_s^{2,n_k}) - \vartheta_s\Big)1_{\{|\psi_s Z_s^1|\le\kappa\}} \cdot 1_{\{Z_s^2=0\}}ds]|+\\ &\quad + |\mathbf{E}[\int_0^\tau \psi_s Z_s^1 \Big(\bar{v}^{n_k}(Z_s^{2,n_k}) - \vartheta_s\Big)1_{\{|\psi_s Z_s^1|\ge\kappa\}} \cdot 1_{\{Z_s^2=0\}}ds]|.\end{aligned}$$

On the right side of the above equation, the first component converges to 0 which is the consequence of $\bar{v}^{n_k}(Z_s^{2,n_k})1_{\{Z_s^2=0\}} \to_{k\to\infty} \vartheta_s$ weakly in $\mathcal{H}_T^2(\mathbf{R})$. For the second component, considering both $(\bar{v}^{n_k}(Z_s^{2,n_k}))_{s\le\tau}$ and $(\vartheta_s)_{s\le\tau}$ are bounded, it is smaller than $C|\mathbf{E}[\int_0^\tau |\psi_s Z_s^1|1_{\{|\psi_s Z_s^1|\ge\kappa\}}ds]|$ which obviously converges to 0 as $\kappa \to \infty$. Thus (6.42) holds, and so does (6.41).

Besides, we also have

$$\int_0^\tau Z_s^{1,n_k}dB_s \to_{k\to\infty} \int_0^\tau Z_s^1 dB_s \text{ in } L^2(\Omega, d\mathbf{P}), \tag{6.43}$$

which is obtained from the convergence of $(Z^{1,n_k})_{k\ge0}$ to Z^1 in $\mathcal{H}_T^2(\mathbf{R})$ and the isometric property, i.e.

$$\begin{aligned}&\mathbf{E}\big[\big(\int_0^\tau Z_s^{1,n_k}dB_s\big)^2\big] = \mathbf{E}\big[\int_0^\tau |Z_s^{1,n_k}|^2 ds\big]\\ &\longrightarrow_{k\to\infty} \mathbf{E}\big[\int_0^\tau |Z_s^1|^2 ds\big] = \mathbf{E}\big[\big(\int_0^\tau Z_s dB_s\big)^2\big]\end{aligned}$$

Then by observing the approximation BSDE (6.23) in a forward way, i.e. for any stopping time τ,

$$Y_\tau^{1,n_k} = Y_0^{1,n_k} - \int_0^\tau H_1^{n_k}(s, X_s)ds + \int_0^\tau Z_s^{1,n_k}dB_s,$$

combining with the convergence of $(Y^{1,n_k})_{k\ge0}$ to Y^1 in $\mathcal{S}_T^2(\mathbf{R})$, (6.41) and (6.43), we infer that

$$\mathbf{P}\text{-a.s., } Y_\tau^1 = Y_0^1 - \int_0^\tau H_1^*(s, X_s, Z_s^1, Z_s^2, \vartheta_s)ds + \int_0^\tau Z_s^1 dB_s \text{ for every stopping time } \tau.$$

As τ is arbitrary then the processes $Y_\cdot^1$ and $Y_0^1 - \int_0^\cdot H_1^*(s, X_s, Z_s^1, Z_s^2, \vartheta_s)ds + \int_0^\cdot Z_s^1 dB_s$ are indistinguishable, i.e., $\mathbf{P}$-a.s.

$$\forall t \le T, \quad Y_t^1 = Y_0^1 - \int_0^t H_1^*(s, X_s, Z_s^1, Z_s^2, \vartheta_s)ds + \int_0^t Z_s^1 dB_s.$$

On the other hand, $Y_T^1 = g_1(X_T)$, then,

$$\mathbf{P}\text{-a.s.,} \quad \forall t \le T, \quad Y_t^1 = g_1(X_T) + \int_t^T H_1^*(s, X_s, Z_s^1, Z_s^2, \vartheta_s)ds - \int_t^T Z_s^1 dB_s.$$

Similarly, for player π_2, there exists a $\mathcal{P}$-measurable process $(\theta_s)_{s\le T}$ valued on U, such that,

$$\mathbf{P}\text{-a.s.},\quad \forall t\le T,\quad Y_t^2 = g_2(X_T) + \int_t^T H_2^*(s, X_s, Z_s^1, Z_s^2, \theta_s)ds - \int_t^T Z_s^2 dB_s.$$

The proof is completed. □

6.2 Generalizations

In this Section, we introduce some generalizations of Theorem 6.1.2 in the following three aspects:

(i) For the drift term Γ in SDE (6.5) which reads,

$$\Gamma(t, x, u, v) = f(t, x) + u + v,$$

one can replace u (resp. v) by $h(u)$ (resp. $l(v)$) with continuous function:

$$h(u) : U \to U'(\text{ resp. } l(v) : V \to V').$$

Therefore, the value sets U' and V' are still bounded which followed by the continuity. Finally, the Nash equilibrium point $(\bar{u}, \bar{v})$ is still of bang-bang type. The unique difference is that, it will jump between the bound of set $U' \times V'$ instead of $U \times V$. □

(ii) In the same way one can deal with m-dimensional diffusion processes $X^{t,x}$ with integer $m \ge 2$. □

(iii) As we indicated in Remark 6.1.1, in the high-dimensional framework, the dynamics of the process $X^{t,x}$ of (6.3) may contain a diffusion term σ (see equation (6.44)) which is a matrix function defined as:

$$\sigma(t, x) : [0, T] \times \mathbf{R}^m \to \mathbf{R}^{m\times m},$$

with the following assumption:
Assumption (A3) The function $\sigma(t, x)$ is uniformly Lipschitz w.r.t. x; The function $\sigma(t, x)$ is invertible and bounded and its inverse is bounded.

Under Assumption (A3), we can verify that σ satisfies the *the uniform ellipticity condition*, i.e. there exists a constant $\Upsilon > 0$ such that for any $(t, x) \in [0, T] \times \mathbf{R}^m$, $\Upsilon I \le \sigma(t, x)\sigma^T(t, x) \le \Upsilon^{-1} I$ with I be the identity matrix of dimension m. Therefore, under Assumption (A3), the generalized SDE:

$$X_s^{t,x} = x + \int_t^s \sigma(r, X_r^{t,x})dB_r,\ \forall s \in [t, T] \text{ and } X_s^{t,x} = x \text{ for } s \in [0, t] \tag{6.44}$$

has solution (see e.g. Karatzas and Shreve, pp.289, [70]).

In this case, the Hamiltonian function will be defined from $[0, T] \times \mathbf{R}^m \times \mathbf{R}^m \times U \times V$ into $\mathbf{R}$ by:

$$\begin{aligned} H_1(t, x, p, u, v) &:= p\sigma^{-1}(t, x)\Gamma(t, x, u, v) = p\sigma^{-1}(t, x)(f(t, x) + u + v); \\ H_2(t, x, q, u, v) &:= q\sigma^{-1}(t, x)\Gamma(t, x, u, v) = q\sigma^{-1}(t, x)(f(t, x) + u + v). \end{aligned}$$

Noticing that σ^{-1} is bounded, it follows by the generalized Isaacs' condition (6.8) and the same approach in this article that, the Nash equilibrium point exists and is of bang-bang type. □

Remarks on Generalizations (ii) and (iii):

We should point out that, all the results in this article will hold by the same techniques in the cases of (ii) and (iii) with some regular adaptions except the second part on the right side of inequality (6.32). For the high-dimensional case, we could not deal with it in the same way as (6.33). In the following, we will treat it under the L^q-domination condition.

The objective is to prove: for fixed (t,x), k, n and m be integers,

$$|\mathbf{E}[\int_{t+\delta}^{T}(H_i^n(s,X_s^{t,x})-H_i^m(s,X_s^{t,x}))\cdot 1_{\{|X_s^{t,x}|\le k\}}ds]\big| \to_{n,m\to\infty} 0. \quad (6.45)$$

We first give the result of domination.

Lemma 6.2.1. *(L^q-Domination) Let $(t,x)\in[0,T]\times\boldsymbol{R}^m$, $s\in[t,T]$ and $\mu(t,x;s,dy)$ the law of $X_s^{t,x}$. Under Assumption (A3) on σ, for any $q\in(1,\infty)$, the family of laws $\{\mu(t,x;s,dy),\ s\in[t,T]\}$ is L^q-dominated by $\{\mu(0,x;s,dy),\ s\in[t,T]\}$, i.e., for any $\delta\in(0,T-t)$, there exists an application $\phi_{t,x}^{\delta}$: $[t+\delta,T]\times\boldsymbol{R}^m\to\boldsymbol{R}^+$ such that:*

- $\mu(t,x;s,dy)ds=\phi_{t,x}^{\delta}(s,y)\mu(0,x;s,dy)ds$ *for any* $(s,x)\in[t+\delta,T]\times\boldsymbol{R}^m$;
- $\forall k\ge 1,\ \phi_{t,x}^{\delta}(s,y)\in L^q([t+\delta,T]\times[-k,k]^m;\ \mu(0,x;s,dy)ds)$.

Proof. Readers are referred to [58] (Section 28, pp.123) and [63] (Lemma 4.3 and Corollary 4.4, pp.14-15) for the proof of this Lemma.

Proof of convergence (6.45): Thanks to Lemma 6.2.1, there exists a function $\phi_{t,x}^{\delta}:[t+\delta,T]\times\mathbf{R}^m\to\mathbf{R}^+$ such that:

$$\text{for } k\ge 1,\ \phi_{t,x}^{\delta}(s,y)\in L^{\frac{q}{q-1}}([t+\delta,T]\times[-k,k]^m;\ \mu(0,x;s,dy)ds) \quad (6.46)$$

and

$$\begin{aligned}&|\mathbf{E}[\int_{t+\delta}^{T}(H_i^n(s,X_s^{t,x})-H_i^m(s,X_s^{t,x}))\cdot 1_{\{|X_s^{t,x}|\le k\}}ds]\big|\\ &=|\int_{\mathbf{R}^m}\int_{t+\delta}^{T}(H_i^n(s,y)-H_i^m(s,y))\cdot 1_{\{|y|\le k\}}\mu(t,x;s,dy)ds|\\ &=|\int_{\mathbf{R}^m}\int_{t+\delta}^{T}(H_i^n(s,y)-H_i^m(s,y))\cdot 1_{\{|y|\le k\}}\phi_{t,x}^{\delta}(s,y)\mu(0,x;s,dy)ds|.\end{aligned}$$

The constant q in (6.46) is the same one as in (6.39) which makes that $H_i^n\to H_i$ weakly in $L^q([0,T]\times\mathbf{R};\ \mu(0,x;s,dy)ds)$ for $i=1,2$ and a fixed $q\in(1,2)$. Then combining the weak convergence result (6.39) and (6.46) yields (6.45). □

Appendices

APPENDIX **A**

Integrability of Doléans-Dade exponential [Haussmann 1986]

Let $(t,x) \in [0,T] \times \mathbf{R}^m$ and $(\theta_s^{t,x})_{s\le T}$ be the solution of the following forward stochastic differential equation:

$$\begin{cases} d\theta_s = b(s,\theta_s)ds + \sigma(s,\theta_s)dB_s, & s \in [t,T]; \\ \theta_s = x, & s \in [0,t]. \end{cases} \tag{A.1}$$

Process $(\theta_s^{0,x})_{s\le T}$ is simply denoted by $(\theta_s)_{s\le T}$. The coefficients $\sigma : [0,T] \times \mathbf{R}^m \to \mathbf{R}^{m\times m}$ and b: $[0,T] \times \mathbf{R}^m \to \mathbf{R}^m$ are measurable functions which satisfy the following assumption.

Assumption A.0.1. *(i) σ is uniformly Lipschitz w.r.t x and bounded and its inverse is bounded. i.e. there exist constants C_1 and C_σ such that, $\forall t \in [0,T], \forall\, x, x' \in \boldsymbol{R}^m, \quad |\sigma(t,x) - \sigma(t,x')| \le C_1|x - x'|$ and $|\sigma(t,x)| + |\sigma^{-1}(t,x)| \le C_\sigma$.*

(ii) The function b is uniformly Lipschitz w.r.t x and of linear growth, i.e., there exist constants C_2 and C_b such that: $\forall t \in [0,T],\ \forall\, x, x' \in \boldsymbol{R}^m,\ |b(t,x) - b(t,x')| \le C_2|x - x'|$ and $|b(t,x)| \le C_b(1 + |x|)$.

For any measurable $\mathcal{F}_t$-adapted process $\eta := (\eta_s)_{s\le T}$ we define the Doléans-Dade exponential as follows,

$$\zeta_s(\eta) := e^{\int_0^s \eta_r dB_r - \frac{1}{2}\int_0^s |\eta_r|^2 dr},\ \forall s \le T. \tag{A.2}$$

Next, we present a result which related to the integrability of Doléans-Dade exponential.

Theorem A.0.1 (Haussmann (1986) [64], p.14)**.** *Assume Assumption A.0.1 is fulfilled and let $(\theta_s)_{s\le T}$ be the solution of stochastic differential equation* (A.1) *with fixed initial point $(0,x)$. Let φ be a $\mathcal{P} \otimes \mathcal{B}(\boldsymbol{R}^m)$-measurable application from $[0,T] \times \Omega \times \boldsymbol{R}^m$ to $\boldsymbol{R}^m$ which is uniformly of linear growth, that is, $\boldsymbol{P}$-a.s., $\forall(s,x) \in [0,T] \times \boldsymbol{R}^m$,*

$$|\varphi(s,\omega,x)| \le C_\varphi(1 + |x|).$$

Then, there exists $p \in (1,2)$ and a constant C, where p depends only on C_σ, C_b, C_φ, m, T while the constant C, depends only on m and p, but not on φ, such that:

$$\boldsymbol{E}\left[\left|\zeta_T(\varphi(s,\theta_s))\right|^p\right] \le C,$$

Remark A.0.1. *In the same spirit, this theorem holds true with generalized process $(\theta_s^{t,x})_{s\le T}$ for fixed $(t,x) \in [0,T] \times \boldsymbol{R}^m$.*

Before proving this theorem, we state the following lemmas.

Lemma A.0.2. *Assume that Assumption A.0.1 holds and $M_t = \int_0^t \sigma(s, \theta_s) dB_s$ for each $t \leq T$, then for any $p > 1$, there exists a constant C_0 depending on C_σ, C_b, T and p, such that,*

$$|\theta_t|^p \leq C_0(1 + |x|^p + |M_t|^p) \text{ a.s.}$$

Proof. The proof follows from the boundness of σ, the linear growth of b and Gronwall's inequality. □

The following lemma is a time substitution scheme related to the stochastic integral on random interval.

Lemma A.0.3. *If $B := (B_t)_{t \leq T}$ is a $\boldsymbol{R}^m$-valued Brownian motion and $(\sigma_t)_{t \leq T}$ is a $\boldsymbol{R}^m$-valued stochastic process such that*

$$\boldsymbol{E}\Big[\int_0^T |\sigma_t|^2 dt\Big] < \infty,$$

then $I\big(S(t)\big)$ is a standard Brownian motion on $[0, R(T)]$ where

$$R(t) = \int_0^t |\sigma_r|^2 dr < \infty; \quad S(t) = \inf\{s > 0, R(s) = t\} \text{ and } I(t) = \int_0^t \sigma_s dB_s.$$

Proof. See Mckean (1969) [81] p.29. □

Lemma A.0.4. *Let $B = (B_t)_{t \geq 0}$ be a standard one dimensional Brownian motion. The law of $|B|$ has density*

$$\frac{2}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}, \quad t \geq 0.$$

If $2\lambda t < 1$ for a constant λ, then, $\boldsymbol{E}[e^{\lambda |B_t|^2}] < \infty$.

We are now ready to give the proof of Theorem A.0.1.

Proof of Theorem A.0.1. Let $\tau_N := \inf\{t \geq 0 : |\int_0^t \varphi(s, \theta_s) dB_s| \geq N$ or $\int_0^t |\varphi(s, \theta_s)|^2 ds \geq N\}$. For any $p > 1$, since

$$\begin{aligned}
&\mathbf{E}\Big[\int_0^T 1_{\tau_N}(t) \big|\zeta_t(p\varphi(s, \theta_s))\big|^2 |\varphi(t, \theta_t)|^2 dt\Big] \\
&\leq \mathbf{E}\Big[\int_0^{T \wedge \tau_N} e^{2\{\int_0^t p\varphi(s,\theta_s)dB_s - \frac{1}{2}\int_0^t p^2 |\varphi(s,\theta_s)|^2 ds\}} \cdot |\varphi(t, \theta_t)|^2 dt\Big] \\
&\leq N e^{2pN},
\end{aligned}$$

then process $(\int_0^{t \wedge \tau_N} \zeta_s(p\varphi(r, \theta_r)) \cdot \varphi(s, \theta_s) dB_s)_{t \leq T}$ is a $\mathcal{F}_t$-martingale. Therefore, by Itô's formula,

$$\mathbf{E}\Big[\zeta_{T \wedge \tau_N}(p\varphi(s, \theta_s))\Big] = 1 + \mathbf{E}\Big[\int_0^{T \wedge \tau_N} \zeta_{t \wedge \tau_N}(p\varphi(s, \theta_s)) \cdot p\varphi(t, \theta_t) dB_t\Big] = 1.$$

We now define $M_t := \int_0^t \sigma(s, \theta_s) dB_s$ for each $t \leq T$. Then we obtain from the linear growth of φ and Lemma A.0.2 that

$$\begin{aligned}
\big|\zeta_{T \wedge \tau_N}(\varphi(s, \theta_s))\big|^p &= \zeta_{T \wedge \tau_N}(p\varphi(s, \theta_s)) \cdot e^{\frac{1}{2}(p^2 - p)\int_0^{T \wedge \tau_N} |\varphi(s,\theta_s)|^2 dt} \\
&\leq \zeta_{T \wedge \tau_N}(p\varphi(s, \theta_s)) \cdot e^{\frac{1}{2}(p^2 - p)\underline{\mathrm{C}}(1 + |x|^2 + |M_{T \wedge \tau_N}|^2)} \qquad \text{(A.3)}
\end{aligned}$$

where the constant $\underline{\mathrm{C}}$ depends on T, C_0 and C_φ.

Let process $B^N = (B_t^N)_{t \leq T} := (B_t - \int_0^{t \wedge \tau_N} p\varphi(s, \theta_s) ds)_{t \leq T}$. Hence B^N is a Brownian motion under probability $\mathbf{P}^N$ which satisfies $d\mathbf{P}^N = \zeta_{T \wedge \tau_N}(p\varphi(s, \theta_s)) d\mathbf{P}$. Let us denote $M_t^N := \int_0^t \sigma(s, \theta_s) dB_s^N$ for each $t \leq T$. Then $M_t = M_t^N + \int_0^{t\wedge\tau_N} p\sigma(s,\theta_s)\varphi(s,\theta_s)ds$ and from Assumption A.0.1, the linear growth of φ and Lemma A.0.2, we know,

$$\begin{aligned}|M_t|^2 &\leq 2|M_t^N|^2 + \bar{C}p^2 C_\sigma C_\varphi^2 \int_0^{t\wedge\tau_N} \left(1+|\theta_s|^2\right) ds \\ &\leq \bar{C}\big(|M_t^N|^2 + 1 + |x|^2 + \int_0^t |M_s|^2 ds\big),\end{aligned}$$

where the constant $\bar{C}$ depends on C_0, C_σ, C_φ and p. Thanks to Gronwall's inequality, we have,

$$|M_T|^2 \leq \bar{C}\left(1 + |x|^2 + |M_T^N|^2\right). \tag{A.4}$$

Back to (A.3) and take expectation on both sides, we obtain, there exists a constant which we still denoted by $\underline{\mathbf{C}}$ depending on $C_0, C_\sigma, C_\varphi, p, m, T$, such that,

$$\begin{aligned}&\mathbf{E}\Big[\big|\zeta_{T\wedge\tau_N}(\varphi(s,\theta_s))\big|^p\Big] \\ &\quad\leq \mathbf{E}^N\left[e^{\frac{1}{2}(p^2-p)\underline{\mathbf{C}}\left(1+|x|^2+|M_T^N|^2\right)}\right] \\ &\quad\leq e^{\frac{1}{2}(p^2-p)\underline{\mathbf{C}}(1+|x|^2)} \cdot \mathbf{E}^N\left[e^{\frac{1}{2}(p^2-p)\underline{\mathbf{C}}|M_T^N|^2}\right]\end{aligned} \tag{A.5}$$

where $\mathbf{E}^N$ is the expectation under probability $\mathbf{P}^N$. If $\sigma_i(t)$ is the i^{th} row $(i = 1, 2, ..., m)$ of matrix $\sigma(t, X_t)$, then by a technique of splitting a stochastic integral into the integrals on random intervals, we get the following inequality,

$$|M_T^N|^2 \leq \sum_{i=1}^m \Big|\int_0^T \sigma_i(t)dB_t^N\Big|^2 \leq \sum_{i=1}^m \big|\beta_i^N\left(R_i(T)\right)\big|^2,$$

where $\beta_i^N(t) = \int_0^{S_i(t)} \sigma_i(s)dB_s^N$ and $S_i(t) = \inf\{s \geq 0 : R_i(s) = t\}$ with $R_i(s) = \int_0^s |\sigma_i(t)|^2 dt$.

It follows from Lemma A.0.3 that β_i^N is a Brownian motion on the random interval $[0, R_i(T)]$. Now Hölder's inequality implies for constant λ,

$$\begin{aligned}\mathbf{E}^N\left[e^{\lambda|M_T^N|^2}\right] &\leq \mathbf{E}^N\Big[\prod_i e^{\lambda|\beta_i^N(R_i(T))|^2}\Big] \\ &\leq \prod_i \mathbf{E}^N\Big[\left(e^{m\lambda|\beta_i^N(R_i(T))|^2}\right)^{\frac{1}{m}}\Big] \\ &= \prod_i \mathbf{E}^N\Big[\left(e^{m\lambda|\beta(R_i(T))|^2}\right)^{\frac{1}{m}}\Big]\end{aligned}$$

where β is a scalar Brownian motion on $(\Omega, \mathcal{F}, \mathbf{P}^N)$. Since $R_i(T) \leq T(C_\sigma)^2$, then by Lemma A.0.4, if $2\lambda m T(C_\sigma)^2 < 1$, we have,

$$\mathbf{E}^N\left[e^{\lambda|M_T^N|^2}\right] \leq \mathbf{E}^N\left[e^{m\lambda|\beta(T|C_\sigma|^2)|^2}\right] \equiv e_0 < \infty.$$

Now let $\lambda = \frac{1}{2}(p^2 - p)\underline{\mathbf{C}}$, the same $\underline{\mathbf{C}}$ as (A.5). Considering inequality (A.5), we can conclude that if $(p^2 - p)\underline{\mathbf{C}} < \varepsilon = (mT|C_\sigma|^2)^{-1}$ then

$$\mathbf{E}\big[\,|\zeta_{T\wedge\tau_N}(\varphi(s,\theta_s))|^p\,\big] \leq e^{\frac{1}{2}\varepsilon(1+|x|^2)} e_0^{\frac{1}{2}} \equiv C.$$

Then Fatou's lemma yields that there exists $p \in (1,2)$ which is sufficiently close to 1 depends on $C_0, C_\sigma, C_\varphi, p, m, T$, such that $\mathbf{E}\left[|\zeta_T(\varphi(s,\theta_s))|^p\right] \leq C$ with constant C depending only on p, m but not on φ. □

APPENDIX **B**

Bounds for the density of the law of a diffusion process [Aronson 1967]

Let $(t,x) \in [0,T] \times \mathbf{R}^m$ and $(\theta_s^{t,x})_{s\le T}$ be the solution of the following forward stochastic differential equation:

$$\begin{cases} d\theta_s = b(s,\theta_s)ds + \sigma(s,\theta_s)dB_s, & s \in [t,T]; \\ \theta_s = x, & s \in [0,t]. \end{cases} \tag{B.1}$$

The coefficients $\sigma : [0,T] \times \mathbf{R}^m \to \mathbf{R}^{m\times m}$ and b: $[0,T] \times \mathbf{R}^m \to \mathbf{R}^m$ are measurable functions which satisfy the following assumption.

Assumption B.0.2. *(i) σ is uniformly Lipschitz w.r.t x and bounded and its inverse is bounded. i.e. there exist constants C_1 and C_σ such that, $\forall t \in [0,T], \forall\, x, x' \in \boldsymbol{R}^m$, $\quad |\sigma(t,x) - \sigma(t,x')| \le C_1|x-x'|$ and $|\sigma(t,x)| + |\sigma^{-1}(t,x)| \le C_\sigma$.*

(ii) The function b is uniformly Lipschitz w.r.t x and bounded, i.e., there exist constants C_2 and C_b such that: $\forall t \in [0,T]$, $\forall\, x, x' \in \boldsymbol{R}^m$, $|b(t,x) - b(t,x')| \le C_2|x-x'|$ and $|b(t,x)| \le C_b$.

Remark B.0.2. *Under Assumption B.0.2-(i), σ satisfies the uniform ellipticity condition, i.e., there exists a real constant $\varpi > 0$ such that for any $(t,x) \in [0,T] \times \boldsymbol{R}^m$, $\varpi I \le \sigma(t,x)\sigma^T(t,x) \le \varpi^{-1} I$ where I is the identity matrix of dimension m.*

Theorem B.0.2 (Aronson 1967 [1])**.** *Under Assumption B.0.2, let $q \in]1,\infty[$ be fixed, $(t_0,x_0) \in [0,T] \times \boldsymbol{R}^m$ and let $(\theta_s^{t_0,x_0})_{t_0\le s\le T}$ be the solution of SDE* (B.1)*. Then for any $s \in (t_0,T]$, the law $\bar{\mu}(t_0,x_0;s,dx)$ of $\theta_s^{t_0,x_0}$ has a density function $\rho_{t_0,x_0}(s,x)$, w.r.t. Lebesgue measure dx, which satisfies the following estimate: $\forall (s,x) \in (t_0,T] \times \boldsymbol{R}^m$,*

$$\varrho_1 (s-t_0)^{-\frac{m}{2}} \exp\left[-\frac{\Lambda|x-x_0|^2}{s-t_0}\right] \le \rho_{t_0,x_0}(s,x) \le \varrho_2 (s-t_0)^{-\frac{m}{2}} \exp\left[-\frac{\lambda|x-x_0|^2}{s-t_0}\right] \tag{B.2}$$

where ϱ_1, ϱ_2, Λ, λ are real constants such that $0 \le \varrho_1 \le \varrho_2$ and $0 \le \lambda \le \Lambda$.

The main task of this appendix is to prove this theorem. It can be checked that the density function of the law of a diffusion process can be interpreted by the fundamental solution of some associated partial differential equation. Therefore, before approaching, we first give the definition of the fundamental solution of parabolic type PDE. Refer the book by Fridman [49] (Chapter 9) for more information.

Definition B.0.1 (Fundamental solution)**.** *A parabolic PDE system is introduced as follows,*

$$u_t - \{a_{ij}(t,x)u_{x_i} + a_j(t,x)u\}_{x_j} - b_j(t,x)u_{x_j} - c(t,x)u = 0. \quad i,j = 1,2,...,m. \tag{B.3}$$

By a fundamental solution (or a fundamental matrix) $\Gamma(t,x;\tau,\xi)$ *of* (B.3) *we mean an* $m \times m$ *matrix of functions defined for* $(t,x) \in [\tau,T] \times \bar{D}$, $(\tau,\xi) \in [0,T] \times \bar{D}$ *for any subset* $D \in \boldsymbol{R}^m$*, satisfies* (B.3) *and*

$$\lim_{t \searrow \tau} \int_\Omega \Gamma(t,x;\tau,\xi) f(\xi) d\xi = f(x)$$

for all $x \in D$ *and any continuous function* $f(\xi)$ *in* $\bar{D}$*. If* D *is unbounded,* f *is further assumed to satisfy the boundedness condition*

$$f(x) = o\{e^{k|x|^2}\} \text{ for some positive constant } k.$$

Therefore, to prove Theorem B.0.2 is reduced to study the global bound of the fundamental solution of the following PDE:

$$u_t - \{a_{ij}(t,x)u_{x_i} + a_j(t,x)u\}_{x_j} - b_j(t,x)u_{x_j} = 0. \quad i,j = 1,2,...,m.$$

with $a = (a_{ij}) = \frac{1}{2}\sigma\sigma^T$. The complete result related to the boundedness of the fundamental solution to the above equation can be found in Aronson (1968) [2]. Bellow, we will briefly introduce the proof in the same spirit by a simplified form (B.4) according to the paper by Aronson (1967) [1].

$$u_t - \{a_{ij}(t,x)u_{x_i}\}_{x_j} = 0. \tag{B.4}$$

We assume that the coefficients of (B.4) are smooth enough, say $a_{ij} \in C^\infty([0,T] \times \mathbf{R}^m)$. Moreover, to simplify the computations we assume $a_{ij} = a_{ji}$ and there exits a constant $\nu \geq 1$ such that $\nu^{-1}|\zeta|^2 \leq a_{ij}(t,x)\zeta_i\zeta_j \leq \nu|\zeta|^2$ for all $(t,x) \in [0,T] \times \mathbf{R}^m$ and $\zeta \in \mathbf{R}^m$. The latter uniform ellipticity condition is satisfied under Assumption B.0.2 (see Remark B.0.2).

Sketch of proof for Theorem B.0.2 (Aronson 1967 [1]). Under these conditions the fundamental solution $g(t,x;\tau,\xi)$ to PDE (B.4) exists and it is known that

$$\int_{\mathbf{R}^m} g(t,x;\tau,\xi)d\xi = \int_{\mathbf{R}^m} g(t,x;\tau,\xi)dx = 1. \tag{B.5}$$

In addition, Nash [82] has shown that

$$\int_{\mathbf{R}^m} g^2(t,x;\tau,\xi)d\xi \leq k(t-\tau)^{-\frac{m}{2}}, \quad \int_{\mathbf{R}^m} g^2(t,x;\tau,\xi)dx \leq k(t-\tau)^{-\frac{m}{2}} \tag{B.6}$$

and

$$g(t,x;\tau,\xi) \leq k(t-\tau)^{\frac{m}{2}}, \tag{B.7}$$

where k denotes a positive constant which depends only upon m and ν. In point of view of fundamental solution to PDE, we aim to show the following results:

$$g(t,x;\tau,\xi) \quad \geq \quad C_1(t-\tau)^{-\frac{m}{2}} \exp\left[-\frac{C_2|x-\xi|^2}{t-\tau}\right] \tag{B.8}$$

$$\text{and} \qquad g(t,x;\tau,\xi) \quad \leq \quad ke^{-\frac{1}{16\nu}}(t-\tau)^{\frac{m}{2}} \exp\left[-\frac{|x-\xi|^2}{64\nu(t-\tau)}\right] \tag{B.9}$$

for $(t,x),(\tau,\xi) \in [0,T] \times \mathbf{R}^m$ with $t > \tau$ and C_1, C_2, k are constants.

(I) Proof for the lower bound (B.8) depends on Theorem 7' of [3] which reads, if

$$\mathcal{M} = \inf_{0<t<T} \int_{|x-\xi|^2<\alpha(t-\tau)} g(t,x;\tau,\xi)dx > 0 \tag{B.10}$$

for some $\alpha > 0$, then (B.8) holds true for constant C_1 depending only on $\alpha, \mathcal{M}, m, \nu$ and C_2 depending only on m, ν. Therefore, it suffices to show (B.10)

Let $(\tau,\xi)\in[0,T]\times\mathbf{R}^m$ and $t\in(\tau,T]$ be fixed. Consider the function

$$v(s,y)=\int_{|x-\xi|^2<\alpha(t-\tau)} g(t,x;s,y)dx$$

for $s<t$, where $\alpha=16/T$. Note that v is a solution of the equation $v_s+(a_{ij}v_{y_i})_{y_j}=0$ for $(s,y)\in\mathbf{R}^n\times[0,t)$ with initial values

$$v(t,y)=\begin{cases}1, & |y-\xi|^2<\alpha(t-\tau);\\ 0, & |y-\xi|^2>\alpha(t-\tau).\end{cases}$$

Set

$$\tilde{a}_{ij}(s,y)=\begin{cases}a_{ij}(s,y), & s\le t;\\ \delta_{ij}, & s>t,\end{cases}\quad\text{and } \tilde{v}(s,y)=\begin{cases}v(s,y), & s\le t;\\ 1, & s>t.\end{cases}$$

Then, $\tilde{v}$ is a non-negative weak solution (in the sense of [3]) of the equation

$$\tilde{v}_s+\{\tilde{a}_{ij}(s,y)\tilde{v}_{y_i}\}_{y_j}=0 \tag{B.11}$$

in the cylinder $\{|y-\xi|^2<\alpha(t-\tau)\}\times[0,\infty)$. We now apply the Harnack inequality for weak solutions of (B.11) (see Theorem 5 in [3]). Then,

$$\int_{|x-\xi|^2<\alpha(t-\tau)} g(t,x;\tau,\xi)dx=\tilde{v}(\tau,\xi)\ge\tilde{v}(t,\xi)e^{-C(16/\alpha+1)}=e^{-C(T+1)}$$

where C depends only on m and ν.

(II) To obtain the upper bound (B.9), we shall use the following estimate for a solution of the Cauchy problem for (B.4) with data whose support lies outside a sphere in $\mathbf{R}^n$.

Lemma. *Let $u_0(x)$ be an $L^2(\boldsymbol{R}^m)$ function such that $u_0=0$ for $|x-y|<\sigma$, where $y\in\boldsymbol{R}^m$ and $\sigma>0$ are fixed. Suppose that u is a solution of* (B.4) *in $\boldsymbol{R}^m\times(\eta,T)$ with initial values $u(\eta,x)=u_0(x)$. Then if $u\in L^\infty((\eta,s)\times\boldsymbol{R}^m)$ for any s which satisfies $0<s-\eta\le\sigma^2$, we have*

$$|u(s,y)|\le k(s-\eta)^{-\frac{m}{4}}\exp\left[-\frac{\sigma^2}{32\nu(s-\eta)}\right]\|u_0\|_{L^2(\boldsymbol{R}^m)}$$

where k is a positive constant which depends only on n and ν.

See Theorem 2 in [1] for the proof.

In the following, we first prove that if $\sigma^2\ge s-\eta>0$ then

$$\int_{|y-\zeta|>\sigma} g^2(s,y;\eta,\zeta)d\zeta\le k(s-\eta)^{-m/2}\exp\left[-\frac{\sigma^2}{16\nu(s-\eta)}\right], \tag{B.12}$$

where $k>0$ is a constant depending only on n and ν. Set

$$u(t,x)=\int_{|y-\zeta|>\sigma} g(t,x;\chi,\zeta)g(s,y;\chi,\zeta)d\zeta.$$

Then u is a nonnegative solution of (B.4) for $t>\eta$ with initial values $u(\eta,x)=0$ if $|x-y|<\sigma$ and $u(\eta,x)=g(s,y;\eta,x)$ if $|x-y|>\sigma$. Moreover, in view of (B.5),(B.6) and (B.7),$u(\eta,x)\in L^2(\mathbf{R}^m)$ and $0\le u(t,x)\le k(s-\eta)^{-m/2}$. Thus by the above lemma, we obtain

$$\begin{aligned}0\le u(s,y)&=\int_{|y-\zeta|>\sigma} g^2(s,y;\eta,\zeta)d\zeta\\ &\le k(s-\eta)^{-m/4}\exp\left[-\frac{\sigma^2}{32\nu(s-\eta)}\right]\left\{\int_{|y-\zeta|>\sigma} g^2(s,y;\eta,\zeta)d\zeta\right\}^{\frac{1}{2}}\end{aligned}$$

and (B.12) follows easily. Note that a similar estimate holds if we integrate with respect to y instead of ζ in (B.12).

Let (t, x), (τ, ξ) be fixed points of $[0, T] \times \mathbf{R}^m$ with $t > \tau$. Set $\sigma = |x - \xi|/2$ and assume that $t - \tau \leq \sigma^2$. By the Kolmogorov identity

$$g(t, x; \tau, \xi) = \int_{\mathbf{R}^m} g(t, x; (t+\tau)/2, \zeta) g((t+\tau)/2, \zeta; \tau, \xi) d\zeta.$$

Split the integral over $\mathbf{R}^m$ into an integral J_1 over $|x - \xi| \geq \sigma$ and an integral J_2 over $|x - \xi| < \sigma$. By the Schwarz inequality

$$J_1 \leq \Big\{ \int_{|x-\xi| \geq \sigma} g^2(t, x; (t+\tau)/2, \zeta) d\zeta \Big\}^{\frac{1}{2}} \cdot \Big\{ \int_{|x-\xi| \geq \sigma} g^2((t+\tau)/2, \zeta; \tau, \xi) d\zeta \Big\}^{\frac{1}{2}}.$$

Now using (B.12) and (B.6), we obtain

$$J_1 \leq k(t-\tau)^{-m/2} \exp\left[-\frac{|x-\xi|^2}{64\nu(t-\tau)} \right], \tag{B.13}$$

where k depends only upon n and ν. The estimate (B.13) also holds for J_2. To show this we note that $|x - \zeta| < \sigma = |x - \xi|/2$ implies that $|\xi - \zeta| \geq \sigma$. Thus J_2 is dominated by the integral over $|\xi - \zeta| \geq \sigma$. The assertion now follows by the argument used above with the roles of (B.12) and (B.6) interchanged. Thus we have derived the required upper bound for g in case $|x - \xi|^2 \geq 4(t - \tau)$. If $|x - \xi|^2 < 4(t - \tau)$, then in view of (B.7), we have

$$g(t, x; \tau, \xi) \leq k(t-\tau)^{-n/2} \leq k e^{-1/16\nu} (t-\tau)^{-n/2} \exp\left[-\frac{|x-\xi|^2}{64\nu(t-\tau)} \right],$$

where k depends only on n and ν. This completes the proof of upper bound (B.9). □

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PÔLE DE RECHERCHE ET D'ENSEIGNEMENT SUPÉRIEUR

Université du Maine
Le Mans - Laval

Thèse de Doctorat

Rui MU

Jeux Différentiels Stochastiques de Somme Non Nulle et Equations Différentielles Stochastiques Rétrogrades Multidimensionnelles

Résumé

Cette thèse traite les jeux différentiels stochastiques de somme non nulle (JDSNN) dans le cadre de Markovien et de leurs liens avec les équations différentielles stochastiques rétrogrades (EDSR) multidimensionnelles. Nous étudions trois problèmes différents. Tout d'abord, nous considérons un JDSNN où le coefficient de dérive n'est pas borné, mais supposé uniquement à croissance linéaire. Ensuite certains cas particuliers de coefficients de diffusion non bornés sont aussi considérés. Nous montrons que le jeu admet un point d'équilibre de Nash via la preuve de l'existence de la solution de l'EDSR associée et lorsque la condition d'Isaacs généralisée est satisfaite. La nouveauté est que le générateur de l'EDSR, qui est multidimensionnelle, est de croissance linéaire stochastique par rapport au processus de volatilité. Le deuxième problème est aussi relatif au JDSNN mais les payoffs ont des fonctions d'utilité exponentielles. Les EDSRs associées à ce jeu sont de type multidimensionnelles et quadratiques en la volatilité. Nous montrons de nouveau l'existence d'un équilibre de Nash. Le dernier problème que nous traitons, est un jeu bang-bang qui conduit à des hamiltoniens discontinus. Dans ce cas, nous reformulons le théorème de vérification et nous montrons l'existence d'un équilibre de Nash qui est du type bang-bang, i.e., prenant ses valeurs sur le bord du domaine en fonction du signe de la dérivée de la fonction valeur ou du processus de volatilité. L'EDSR dans ce cas est un système multidimensionnel couplé, dont le générateur est discontinu par rapport au processus de volatilité.

Mots Clés: Jeux Différentiels Stochastiques de Somme Non Nulle; Equations Différentielles Stochastiques Rétrogrades; Point d'équilibre de Nash.

Abstract

This dissertation studies the multiple players nonzero-sum stochastic differential games (NZSDG) in the Markovian framework and their connections with multiple dimensional backward stochastic differential equations (BSDEs). There are three problems that we are focused on. Firstly, we consider a NZSDG where the drift coefficient is not bound but is of linear growth. Some particular cases of unbounded diffusion coefficient of the diffusion process are also considered. The existence of Nash equilibrium point is proved under the generalized Isaacs condition via the existence of the solution of the associated BSDE. The novelty is that the generator of the BSDE is multiple dimensional, continuous and of stochastic linear growth with respect to the volatility process. The second problem is of risk-sensitive type, i.e. the payoffs integrate utility exponential functions, and the drift of the diffusion is unbounded. The associated BSDE is of multi-dimension whose generator is quadratic on the volatility. Once again we show the existence of Nash equilibria via the solution of the BSDE. The last problem that we treat is a bang-bang game which leads to discontinuous Hamiltonians. We reformulate the verification theorem and we show the existence of a Nash point for the game which is of bang-bang type, i.e., it takes its values in the border of the domain according to the sign of the derivatives of the value function. The BSDE in this case is a coupled multi-dimensional system, whose generator is discontinuous on the volatility process.

Key Words: Nonzero-sum Stochastic Differential Games; Backward Stochastic Differential Equation; Nash Equilibrium Point.

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