Poissonian ensembles of Markovian loops

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Résumé

L'objet d'étude de cette thèse est une mesure infinie sur les boucles (lacets) naturellement associée à une large classe de processus de Markov et les processus ponctuels de Poisson d'intensité proportionnelle à cette mesure (paramètre d'intensité $\alpha > 0$). Ces processus ponctuels de Poisson portent le nom d'ensembles poissoniens de boucles markoviennes ou de soupes de boucles. La mesure sur les boucles est covariante par un certain nombre de transformations sur les processus de Markov, par exemple le changement de temps.

Dans le cadre de soupe de boucles brownienne à l'intérieur d'un sous-domaine ouvert propre simplement connexe de $\mathbb{C}$, il a été montré que les contours extérieurs des amas extérieurs de boucles sont, pour $\alpha \leqslant \frac{1}{2}$, des Conformal Loop Ensembles CLE_κ, $\kappa \in (\frac{8}{3}, 4]$. D'autre part il a été montré pour une large classe de processus de Markov symétriques que lorsque $\alpha = \frac{1}{2}$, le champ d'occupation d'une soupe de boucle (somme des temps passés par les boucles aux dessus des points) est le carré du champ libre gaussien.

J'ai étudié d'abord les soupes de boucles associés aux processus de diffusion unidimensionnels, notamment leur champ d'occupation dont les zéros délimitent dans ce cas les amas de boucles. Puis j'ai étudié les soupes de boucles sur graphe discret ainsi que sur graphe métrique (arêtes remplacés par des fils continus). Sur graphe métrique on a d'une part une géométrie non triviale pour les boucles et d'autre part on a comme dans le cas unidimensionnel continu la propriété que les zéros du champ d'occupation délimitent les amas des boucles. En combinant les graphes métriques et l'isomorphisme avec le champ libre gaussien j'ai montré que $\alpha = \frac{1}{2}$ est le paramètre d'intensité critique pour la percolation par soupe de boucles de marche aléatoire sur le demi plan discret $\mathbb{Z} \times \mathbb{N}$ (existence ou non d'un amas infini) et que pour $\alpha \leqslant \frac{1}{2}$ la limite d'échelle des contours extérieurs des amas extérieurs sur $\mathbb{Z} \times \mathbb{N}$ est un CLE_κ dans le demi-plan continu.

Abstract

In this thesis I study an infinite measure on loops naturally associated to a wide range of Markovian processes and the Poisson point processes of intensity proportional to this measure (intensity parameter $\alpha > 0$). This Poissson point processes are called Poisson ensembles of Markov loops or loop soups. The measure on loops is covariant with some transformation on Markovian processes, for instance the change of time.

In the setting of Brownian loop soups inside a proper open simply connected domain of $\mathbb{C}$ it was shown that the outer boundaries of outermost clusters of loops are, for $\alpha \leqslant \frac{1}{2}$, Conformal Loop Ensembles CLE_{κ}, $\kappa \in (\frac{8}{3}, 4]$. Besides, it was shown for a wide range of symmetric Markovian processes that for $\alpha = \frac{1}{2}$ the occupation field of a loop soup (the sum of times spent by loops over points) is the square of the Gaussian free field.

First I studied the loop soups associated to one-dimensional diffusions, and particularly the occupation field and its zeros that delimit in this case the clusters of loops. Then I studied the loop soups on discrete graphs and metric graphs (edges replaced by continuous lines). On a metric graph on one hand the loops have a non-trivial geometry and on the other hand one has the same property as in the setting of one-dimensional diffusions that the zeros of the occupation field delimit the clusters of loops. By combing metric graphs and the isomorphism with the Gaussian free field I have shown that $\alpha = \frac{1}{2}$ is the critical parameter for random walk loop soup percolation on the discrete half-plane $\mathbb{Z} \times \mathbb{N}$ (existence or not of an infinite cluster of loops) and that for $\alpha \leqslant \frac{1}{2}$ the scaling limit of outer boundaries of outermost clusters on $\mathbb{Z} \times \mathbb{N}$ is a CLE_{κ} on the continuum half plane.

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CHAPITRE 1

Introduction

1.1. Définitions

L'objet d'étude de cette thèse est une mesure infinie sur les boucles (lacets) naturellement associée à une large classe de processus de Markov et les processus ponctuels de Poisson d'intensité proportionnelle à cette mesure. Ces processus ponctuels de Poisson portent le nom d'ensembles poissoniens de boucles markoviennes ou de soupes de boucles. Dans [**LMR15**] et [**FR14**] apparaissent des cadres assez larges dans lesquels cette mesure et ces processus ponctuels de Poissons sont définis. Dans cette thèse, le cadre sera plus restreint et en particulier tous les processus de Markov seront symétriques. Soit S un espace localement compact à base dénombrable, muni de sa tribu borélienne. Soit $(X_t)_{0\leqslant t<\zeta}$ un processus de Feller sur S à trajectoires càdlàg défini jusqu'à un temps de mort $\zeta \in (0,+\infty]$. On suppose que X a des densités de transition $p_t(x,y)$ par rapport une mesure σ-finie m sur S, continues en le triplet (t,x,y) et symétriques ($p_t(x,y) = p_t(y,x)$). On note $\mathbb{P}_x$ et $\mathbb{E}_x$ les probabilités et espérances relatives à des chemins issus de $x \in S$. On note $\mu^{x,y}$ la mesure sur les chemins de durée finie $(x_t)_{0\leqslant t<\xi}$ joignant x à y définie par

$$\mu^{x,y}(1_{\xi>t}F((x_s)_{0\leqslant s\leqslant t})f(\xi)) = \int_t^{+\infty} \mathbb{E}_x[1_{\zeta>t}F((X_s)_{0\leqslant s\leqslant t})f(u)p_{u-t}(X_t,y)]du$$

La mesure sur les boucles μ associée au processus de Markov X est définie comme

$$\mu(d\gamma) = \frac{1}{t_\gamma}\int_{x\in S}\mu^{x,x}(d\gamma)m(dx) \tag{1.1.1}$$

où t_γ désigne la durée totale de la boucle γ. Dans cette thèse les processus de Markov sous-jacents seront soit des diffusions unidimensionnelles, soit le mouvement brownien planaire, soit des processus à sauts symétriques sur des graphes non-orientés, soit des diffusions sur des graphes métriques, c'est-à-dire des graphes où les arêtes sont remplacées par des "fils" continus.

Le plus souvent on considérera le boucles définies à translation de paramétrisation près. Étant donnés $(\gamma(s))_{0\leqslant s\leqslant t_\gamma}$ et $T \in (0,t_\gamma)$, on identifiera à γ la boucle $\tilde{\gamma}$ définie par

$$\begin{cases} \tilde{\gamma}(s) = \gamma(s+t_\gamma-T) & \text{si } s\in[0,T] \\ \tilde{\gamma}(s) = \gamma(s-T) & \text{si } s\in[T,t_\gamma] \end{cases}$$

La raison d'être de ceci est qu'après cette identification la mesure est covariante par changement du temps du processus de Markov sous-jacent par l'inverse d'une fonctionnelle additive continue. Ceci est montré dans un cadre assez large dans [**FR14**] et apparaîtra dans le cadre des diffusions unidimensionnelles dans le chapitre 2 de cette thèse. Par exemple, étant donnée v une fonction strictement positive continue sur l'espace S, on peut introduire une famille $(\tau_s^v)_{s\geqslant 0}$ de temps d'arrêts pour X :

$$\tau_s^v := \inf\left\{t\geqslant 0 \,\middle|\, \int_0^t v(X_u)du > s\right\}$$

Le processus changé de temps $(X_{\tau_s^v})_{0\leqslant s<\int_0^\zeta v(X_u)du}$ est encore markovien et la mesure sur les boucles quotientées par les translations de paramétrisation associée à ce processus changé de temps est l'image de μ par le même changement du temps. C'est là une propriété qui explique la particularité et l'intérêt de la mesure μ sur les boucles ; elle sera utilisée à plusieurs reprises dans cette thèse. On peut aussi noter que si le processus X est un mouvement Brownien sur $\mathbb{R}^d$ alors la mesure μ est invariante par tout changement d'échelle Brownien de l'espace-temps.

On étudiera les processus ponctuels de Poisson d'intensité $\alpha\mu$, notés $\mathcal{L}_\alpha$, où α est une constante positive. $\mathcal{L}_\alpha$ est appelé *Ensemble poissonien de boucles markoviennes* ou bien *soupe de boucles*. Dans les notations qu'on utilisera on verra $\mathcal{L}_\alpha$ comme un ensemble aléatoire dénombrable de boucles. On s'intéressera également à la mesure d'occupation de $\mathcal{L}_\alpha$, qui à un borélien A de S associe la masse

$$\sum_{\gamma\in\mathcal{L}_\alpha}\int_0^{t_\gamma} 1_{\gamma(s)\in A}ds \tag{1.1.2}$$

Dans les différents cadres qui apparaîtront dans cette thèse, la mesure d'occupation sera bien définie pour les processus de Markov transients. De plus, pour des processus ayant des temps locaux la mesure d'occupation d'ensembles poissoniens de boucles Markoviennes aura elle-même des densités, notées $(\widehat{\mathcal{L}}_\alpha^x)_{x\in S}$ et on parlera de *champ d'occupation*.

1.2. Boucles browniennes bidimensionnelles, CLE, SLE et champ libre gaussien

La mesure sur les boucles a été étudiée par Lawler et Werner dans [**LW04**] pour le mouvement Brownien planaire sur $\mathbb{C}$ tout entier ou dans un domaine ouvert simplement connexe, le mouvement Brownien étant tué en atteignant le bord. Si on considère les boucles simplement comme des lieux géométriques en oubliant la paramétrisation par le temps, alors la mesure sur les boucles est invariante par transformation conforme du domaine. Ceci découle directement de la covariance de la mesure sur les boucles par changement du temps du processus de Markov sous-jacent.

Dans l'article [**SW12**], Sheffield et Werner utilisent les ensembles poissoniens de boucles browniennes planaires pour donner une construction des Conformal Loop Ensembles (CLE). Étant donné un domaine ouvert simplement connexe $D \neq \mathbb{C}$, un ensemble CLE dans D est une famille aléatoire dénombrable de boucles simples (lacets de Jordan qu'on voit comme des lieux géométriques) dans D qui vérifie les propriétés suivantes :

- Les boucles ne s'intersectent pas deux à deux.
- Les boucles ne s'entourent pas l'une l'autre.
- Finitude locale : Pour tout $\varepsilon > 0$ il y a un nombre fini de boucles de diamètre plus grand que ε.
- Invariance conforme : La loi des boucles est invariante par transformation conforme de D.
- Restriction : Les boucles vérifient la propriété de restriction à un sous-domaine de D suivante. Soit $\widetilde{D}$ un sous-domaine ouvert simplement connexe de D. On prive $\widetilde{D}$ de toutes les boucles CLE dans D qui ne sont pas contenues dans $\widetilde{D}$, ainsi que de leurs intérieurs. On obtient un ouvert (voir finitude locale) $\widetilde{D}^*$ dont toutes les composantes connexes sont simplement connexe. Sachant les boucles dont on a privé $\widetilde{D}$, la loi conditionnelle des boucles dans chacune des composantes connexes de $\widetilde{D}^*$ est un CLE indépendant.

Il y a une famille d'ensembles vérifiant ces propriétés, paramétrée par $\kappa \in (\frac{8}{3}, 4]$ (CLE_κ). Dans [**SW12**], les auteurs considèrent, pour $\alpha > 0$, l'ensemble poissonien de boucles browniennes contenues dans D, $\mathcal{L}_\alpha$, ainsi que les amas de $\mathcal{L}_\alpha$. Deux boucles dans $\mathcal{L}_\alpha$ sont dans un même amas s'il existe une chaîne de boucles dans $\mathcal{L}_\alpha$, dont les éléments extrêmes sont les deux boucles précédentes, et tel que deux boucles consécutives visitent un point de D en commun. Les auteurs utilisent la notion de *charge centrale*, notée c, issue de la Théorie

Conforme des Champs. Mais contrairement à ce qu'affirment les auteurs, la charge centrale c n'est pas le paramètre d'intensité α, mais en réalité $\alpha = \frac{c}{2}$. L'existence d'un facteur $\frac{1}{2}$ m'a été communiqué par Werner. Dans l'article [**Law09**] de Lawler, le facteur $\frac{1}{2}$ apparaît également. Sheffield et Werner montrent que si $c > 1$ (et donc $\alpha > \frac{1}{2}$) il n'y a qu'un seul amas et si $c \in (0, 1]$, il y a une infinité d'amas, tous bornés et à distance strictement positive du bord de D. De plus, dans le dernier cas les contours extérieurs des amas extérieurs (non-entourés par un autre amas) forment un ensemble CLE_κ avec la correspondance

$$2\alpha = c = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa} \tag{1.2.1}$$

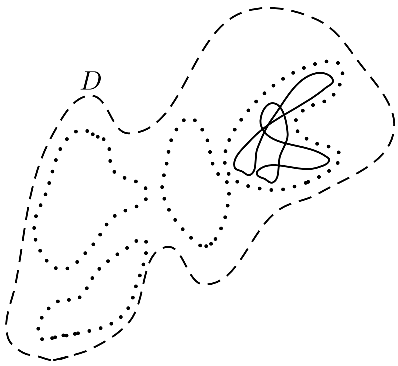

Fig. 1.1. : Représentation de quatre boucles CLE_κ en pointillés
et des boucles browniennes en traits pleins à l'intérieur d'une d'elle.

Les boucles CLE sont reliées aux courbes SLE (Stochastic Loewner Evolution). Étant donné un domaine ouvert simplement connexe $D \neq \mathbb{C}$ et deux points du bord (au sens frontière de Carathéodory, images du bord du disque unité par transformation conforme) $a \neq b$, un SLE chordal dans D joignant a à b est une courbe ξ dans $\bar{D}$ allant de a vers b telle que :

- La loi de ξ est invariante par les transformations conformes de D qui laissent a et b fixes.
- Conditionnellement à une partie de la courbe $\xi([0, t])$, la loi du reste est un SLE chordal dans $D\backslash\xi([0, t])$ joignant ξ_t à b (la loi est l'image de celle dans D par transformation conforme).

Les courbes SLE sont classées par un paramètre $\kappa \in (0, 8)$. Si $D = \mathbb{H} = \{z \in \mathbb{C} | \Im(z) > 0\}$, $a = 0$ et $b = \infty$ on décrit SLE_κ à l'aide du flot de Loewner suivant. Il y a une unique transformation conforme g_t de $\mathbb{H}\backslash\xi([0, t])$ vers $\mathbb{H}$ telle que à l'infini

$$g_t(z) - z = O\left(\frac{1}{z}\right)$$

g_t vérifie l'équation différentielle suivante

$$\frac{\partial g_t}{\partial t}(z) = \frac{2}{g_t(z) - \sqrt{\kappa} W_t}$$

où $(W_t)_{t \geqslant 0}$ est un mouvement brownien standard unidimensionnel. Si $\kappa \in (0, 4]$, SLE_κ est une courbe qui ne s'autointersecte pas et qui ne touche pas le bord du domaine sauf aux deux extrémités. Pour plus de détails sur les processus SLE, voir [**RS05**].

Pour $\kappa \in (\frac{8}{3}, 4]$, les tronçons de boucles CLE_κ "ressemblent" aux tronçons de courbes SLE_κ. On peut donner à cela un sens plus précis suivant. Soit z_0 un point dans le domaine D et a un point sur le bord (au sens frontière de Carathéodory). On considère la boucle du CLE_κ qui entoure z_0 (qui existe p.s.) et on la conditionne à passer dans un ε-voisinage du point a. En faisant tendre ε vers 0 on a une convergence en loi vers une boucle "épinglée", allant de a vers a et entourant z_0. Cette loi de boucle épinglée peut être obtenue d'une autre manière, à partir de processus SLE_κ. a_ε va désigner un point du bord du domaine situé dans un ε-voisinage de a, différent de a. On considère un SLE_κ chordal allant de a vers a_ε et conditionné à "entourer" z_0, cet-à-dire à laisser z_0 du même côté de la courbe SLE_κ que le segment ε-petit du bord du domaine. Lorsque ε tend vers 0, on a une convergence en loi vers la même boucle "épinglée" que précédemment (voir [**SW12**]).

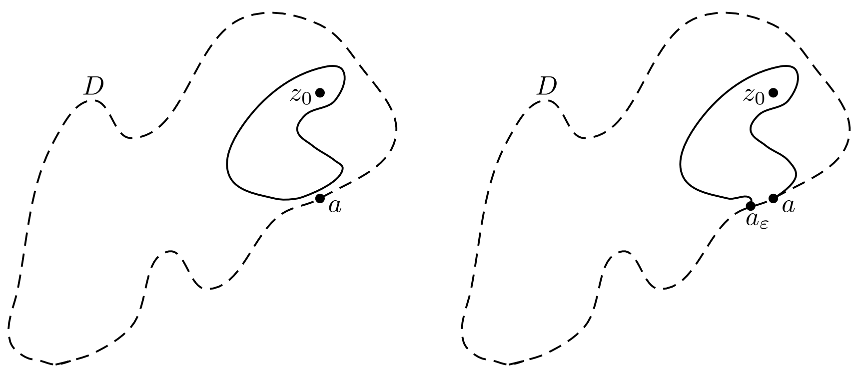

Fig. 1.2. : Deux manières d'obtenir une boucle épinglée au bord,
à partir d'une boucle CLE_κ à gauche et à partir d'une boucle SLE_κ à droite.

Dans [**Law09**], Lawler établit la relation suivante entre les courbes SLE_κ, $\kappa \in (0, 4]$, et la mesure μ sur les boucles browniennes dans D. On considère deux points distincts sur le bord (frontière de Carathéodory) a et b, ainsi qu'un sous domaine ouvert simplement connexe D' de D dont le bord contient des voisinages du bord de D autour de a et b. Soit $\mathbb{P}^D_{\kappa,a\to b}$ la loi du SLE_κ chordal de a vers b dans D, et $\mathbb{P}^{D'}_{\kappa,a\to b}$ l'analogue dans D'. $\mathbb{P}^{D'}_{\kappa,a\to b}$ est absolument continu par rapport à $\mathbb{P}^D_{\kappa,a\to b}$. Soit c la quantité appelée *charge centrale*, reliée à κ par (1.2.1). Lorsque $\kappa \leqslant \frac{8}{3}$, $c \leqslant 0$. Étant donnée une courbe ξ, soit $\mu(\xi \leftrightarrow D\backslash D')$ la μ-masse des boucles browniennes qui visitent à la fois ξ et $D\backslash D'$. Alors

$$\frac{d\mathbb{P}^D_{\kappa,a\to b}}{d\mathbb{P}^{D'}_{\kappa,a\to b}}(\xi)1_{\xi\subset\bar{D}'} = \frac{1}{Z}1_{\xi\subset\bar{D}'}\exp\left(-\frac{c}{2}\mu(\xi\leftrightarrow D\backslash D')\right)$$

A partir de ce type de relations, on obtient l'expression (1.2.1) de la valeur de κ pour laquelle CLE_κ est le contour des amas de l'ensemble poissonien de boucles browniennes d'intensité α. C'est-à-dire que pour $\kappa \in (\frac{8}{3}, 4]$, $\frac{c}{2}$ est l'intensité α.

On présente la relation entre CLE_4 (correspondant au paramètre d'intensité $\alpha = \frac{1}{2}$ pour les boucles browniennes) et le champ libre gaussien. Dans le domaine ouvert simplement connexe $D \neq \mathbb{C}$, on considère le champ libre gaussien ϕ^D ayant pour forme de Dirichlet

$$\frac{1}{2}\int_D (\nabla f)^2$$

avec des conditions nulles sur ∂D. On peut coupler ϕ^D et CLE_4 de manière suivante :

(i) On échantillonne d'abord les boucles CLE_4 dans D.
(ii) A l'intérieur de l'intérieur de chaque boucle $\Gamma \in CLE_4$ (noté $\mathrm{Int}(\Gamma)$) on échantillonne un champ libre $\phi^{\mathrm{Int}(\Gamma)}$ avec des conditions nulles sur Γ, de manière indépendante conditionnellement à CLE_4.
(iii) Pour chaque boucle $\Gamma \in CLE_4$, on choisit de manière uniforme un signe $\sigma_\Gamma \in \{-1, +1\}$, de manière indépendante et indépendamment des champs libres précédents conditionnellement à CLE_4.

(iv) On a alors l'identité en loi

$$\phi^D \overset{(d)}{=} \sum_{\Gamma \in CLE_4} 1_{\mathrm{Int}(\Gamma)}(\phi^{\mathrm{Int}(\Gamma)} + \sigma_\Gamma \sqrt{\pi})$$

Pour plus de détails voir [**WW14**]. La normalisation du champ libre dans l'article cité est différente.

Dans le couplage précédent, le complémentaire des intérieurs des boucles CLE_4 apparait en quelque sorte comme une ligne de niveau 0 du champ libre ϕ^D. On pense en plus que dans ce couplage les boucles CLE_4 sont fonction déterministe de ϕ^D.

1.3. Boucles de marches aléatoires et approximation

Lawler et Trujillo Ferreras dans [**LF07**] étudient des processus de Poisson de boucles discrètes sur $\mathbb{Z}^2$ qu'ils appellent *soupes de boucles de marche aléatoire* ("random walk loop soup"). Les auteurs considèrent la mesure que l'on va noter $\mu^\sharp$ sur les chemins à temps discret, de longueur finie, aux plus proches voisins sur $\mathbb{Z}^2$, dont le point d'arrivée est le point de départ, et qui à tout tel chemin de longueur n paire associe la masse

$$\frac{4^{-n}}{n}$$

On va noter par $\mathcal{L}_\alpha{}^\sharp$ le processus ponctuel de Poisson d'intensité $\alpha\mu^\sharp$.

Les auteurs cherchent à approximer par $\mathcal{L}_\alpha{}^\sharp$ les ensembles poissoniens de boucles browniennes (associés au mouvement brownien standard) dans $\mathbb{C}$. On va noter ici par $\mathcal{L}_\alpha$ les ensembles poissoniens de boucles browniennes et considérer que les boucles ont un instant de départ. Pour $N \in \mathbb{N}^*$, on considère l'application Φ_N de changement d'échelle sur les boucles discrètes suivante. Étant donné $\gamma^\sharp = (z_0, \ldots, z_{n-1}, z_0)$ dans $\mathbb{Z}^2$ une boucle aux plus proches voisins, $\Phi_N\gamma^\sharp$ est une boucle à temps continu à espace continu dans $\mathbb{C}$ qui vérifie :

- La durée de $\Phi_N\gamma^\sharp$ est $\frac{n}{2N^2}$.
- Pour $j \in \{0, \ldots, n-1\}$, $\Phi_N\gamma^\sharp(\frac{j}{2N^2}) = \frac{z_j}{N}$.
- $\Phi_N\gamma^\sharp(\frac{n}{2N^2}) = \Phi_N\gamma^\sharp(0) = \frac{z_0}{N}$.
- Entre les temps $\frac{j}{2N^2}$, $j \in \{0, \ldots, n\}$, $\Phi_N\gamma^\sharp$ interpole linéairement.

Soit $s_{\gamma^\sharp}$ le nombre de sauts n d'une boucle discrète $\gamma^\sharp$. Soit $\theta \in (\frac{2}{3}, 2)$ et $r \geqslant 1$. Dans [**LF07**] il est montré qu'il y a un couplage entre $\mathcal{L}_\alpha^\sharp$ and $\mathcal{L}_\alpha$ tel qu'à l'exception d'un événement de probabilité au plus $cste \cdot (\alpha+1) r^2 N^{2-3\theta}$ il y a une bijection entre les deux ensembles

- $\{\gamma^\sharp \in \mathcal{L}_\alpha^\sharp | s_{\gamma^\sharp} > 2N^\theta, |\gamma^\sharp(0)| < Nr\}$
- $\{\gamma \in \mathcal{L}_\alpha | t_\gamma > N^{\theta-2}, |\gamma(0)| < r\}$

tel que, étant données une boucle discrète $\gamma^\sharp$ et une boucle continue γ qui lui est associée,

$$\left|\frac{s_{\gamma^\sharp}}{2N^2} - t_\gamma\right| \leqslant \frac{5}{8}N^{-2} \qquad \text{et} \qquad \sup_{0 \leqslant u \leqslant 1}\left|\Phi_N\gamma\Big(u\frac{s_{\gamma^\sharp}}{2N^2}\Big) - \gamma(ut_\gamma)\right| \leqslant cste \cdot N^{-1}\log(N)$$

Dans l'article [**dBCL14**], van den Brug, Camia et Lis utilisent le résultat d'approximation de Lawler et Trujillo Ferreras pour montrer le résultat de convergence vers le CLE suivant. Ils considèrent un domaine ouvert simplement connexe et borné D, ainsi qu'un coefficient $\theta \in (\frac{16}{9}, 2)$. Parmi les boucles discrètes $\gamma^\sharp \in \mathcal{L}_\alpha^\sharp$, ils considèrent celles qui sont contenues dans ND et qui font au moins N^θ sauts (sans les plus petites boucles donc). Ils montrent que pour $\alpha \in (0, \frac{1}{2}]$, la limite d'échelle (lorsque N tend vers l'infini) des contours extérieurs des amas formés par ces boucles est un CLE_κ, où α et κ sont reliés par (1.2.1). Toutefois, on peut se demander si la convergence en loi tient toujours si on prend en compte toutes les boucles discrètes à l'intérieur de ND et non seulement celles qui sont assez grandes. Dans cette thèse, il sera démontré que c'est la cas.

1.4. Boucles de processus à sauts et champ libre gaussien

Dans [**Jan11**], Le Jan considère les ensembles poissoniens de boucles associés à des processus de Markov à sauts sur des réseaux électriques. Le cadre est le suivant. $\mathcal{G} = (V, E)$ est un graphe connexe non-orienté. L'ensemble des sommets V est au plus dénombrable. Chaque sommet a un degré fini. Les arêtes sont munies de conductances strictement positives $(C(e))_{e \in E}$ et les sommets d'une mesure de meurtre positive ou nulle $(k(x))_{x \in V}$. Le processus de Markov à sauts $(X_t)_{0 \leqslant t < \zeta}$ saute d'un sommet x vers un de ses voisins y avec un taux égal à la conductance $C(x, y)$. Si jamais $k(x) > 0$, X_t saute de x vers un état puits qu'on appellera cimetière avec un taux $k(x)$. Le processus est alors tué. ζ est soit l'infini, soit le temps où le processus est tué par la mesure de meurtre, soit le temps où il explose (sort de tout compact) en temps fini. Si $(X_t)_{0 \leqslant t < \zeta}$ est transient (meurtre non nul ou graphe infini avec conductances idoines) alors on notera par $(G(x, y))_{x,y \in V}$ la fonction de Green du processus :

$$G(x, y) = \mathbb{E}_x \left[\int_0^{\zeta} 1_{X=y} dt \right]$$

G est symétrique.

Le Jan considère la mesure μ sur les boucles associée à $(X_t)_{0 \leqslant t < \zeta}$ qui est donnée par (1.1.1) ainsi que les ensembles poissoniens de boucles $\mathcal{L}_\alpha$. Les boucles sont à espace discret mais à temps continu. Notons que dans ce cadre-là $\mathcal{L}_\alpha$ contient des boucles non-triviales qui visitent plusieurs sommets mais aussi, au dessus de chaque sommet, une infinité de "boucles" qui ne visitent qu'un sommet. Si le graphe est $\mathbb{Z}^2$, les conductances uniformes et il n'y a pas de mesure de meurtre, alors les soupes de boucles de marche aléatoire de Lawler et Trujillo Ferreras sont les boucles de processus à sauts qui ne sont pas réduites à un sommet où l'on remplace le temps continu par un temps discret. En particulier les amas de boucles sont exactement les mêmes.

Le Jan étudie la champ d'occupation $(\widehat{\mathcal{L}}^x_\alpha)_{x \in V}$ de $\mathcal{L}_\alpha$. $\widehat{\mathcal{L}}^x_\alpha$ est la somme sur les boucles de $\mathcal{L}^x_\alpha$ du temps total qu'elles passent en x. Si le processus à sauts est récurrent, le champ d'occupation est infini en tout point. Si le processus à sauts est transient, le champ d'occupation est fini. Le Jan établit un "isomorphisme" entre le champ d'occupation $\widehat{\mathcal{L}}^x_{\frac{1}{2}}$ et le champ libre gaussien discret ϕ. ϕ est un champ gaussien centré donc la fonction de variance-covariance est la fonction de Green G. Le Jan établit l'égalité en loi

$$(\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{x \in V} \overset{(d)}{=} \left(\frac{1}{2} \phi_x^2 \right)_{x \in V} \tag{1.4.1}$$

Le paramètre d'intensité $\alpha = \frac{1}{2}$ est le même que celui qui dans le cas bidimensionnel brownien est relié au CLE_4 qui a son tour est relié au champ libre en continu. L'isomorphisme de Le Jan fait partie d'une famille d'isomorphismes reliant le champ d'occupation de trajectoires markoviennes au carré d'un champ libre gaussien au même titre que l'isomorphisme de Dynkin, l'isomorphisme d'Eisenbaum, l'isomorphisme de Sznitman pour les entrelacements aléatoires et les théorèmes de Ray-Knight généralisés (voir [**Szn12b**]). L'isomorphisme de Le Jan ne relie pas le signe du champ libre à l'ensemble Poissonien de boucles $\mathcal{L}_{\frac{1}{2}}$. Ce sera fait dans cette thèse.

Dans le cas où l'on considère un processus de diffusion unidimensionnel transient en dimension 1 au lieu d'un processus à sauts, le champ d'occupation est encore défini ponctuellement et l'isomorphisme de Le Jan tient. Ce sera étudié dans cette thèse. Dans le cas des boucles browniennes en dimension supérieure, le champ d'occupation n'est pas défini ponctuellement et au lieu de cela on a une mesure d'occupation (1.1.2). Mais elle est localement infinie même si le processus est transient. Toutefois en dimension deux et trois on peut définir une mesure d'occupation centrée.

Le cadre est celui des boucles browniennes dans un sous-domaine strict ouvert de $\mathbb{C}$ ou dans $\mathbb{R}^3$. Soit $\widehat{\mathcal{L}}_{\alpha,\varepsilon}$ le champ d'occupation des boucles de $\mathcal{L}_\alpha$ dont la durée est supérieure à ε. Alors

$$\widehat{\mathcal{L}}_{\alpha,\varepsilon} - \mathbb{E}[\widehat{\mathcal{L}}_{\alpha,\varepsilon}]$$

a une limite en loi quand ε tend vers 0, qui est une distribution aléatoire et qu'on notera $\widehat{\mathcal{L}}_{\alpha,\text{cent}}$. Pour l'intensité $\alpha = \frac{1}{2}$, $\widehat{\mathcal{L}}_{\frac{1}{2},\text{cent}}$ a même loi que la moitié du carré de Wick du champ libre continu, qu'on note

$$\frac{1}{2} : \phi^2 :$$

Le carré de Wick est défini ainsi. Soit ϕ_ε le champ libre moyenné sur des boules de rayon ε de manière à avoir un champ défini ponctuellement. Alors

$$: \phi^2 := \lim_{\varepsilon \to 0} (\phi_\varepsilon^2 - \mathbb{E}[\phi_\varepsilon^2])$$

Dans le cas d'un sous-domaine strict ouvert simplement connexe de $\mathbb{C}$, on a le diagramme suivant qui relie $\mathcal{L}_{\frac{1}{2}}$, CLE_4, le champ libre ϕ et $\frac{1}{2} : \phi^2 :$.

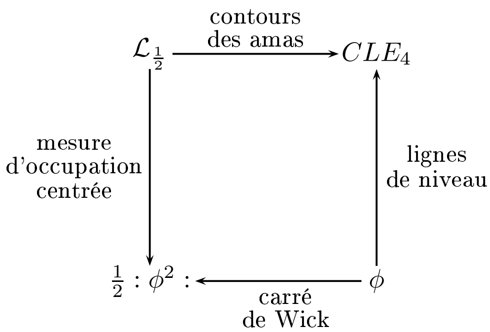

Fig. 1.3 : Relations entre $\mathcal{L}_{\frac{1}{2}}$, CLE_4 et le champ libre ϕ

1.5. Algorithme de Wilson à effacement de boucles

Une autre propriété mise en évidence par Le Jan dans [**Jan11**] est le lien entre l'ensemble poissonien de boucles $\mathcal{L}_1$ et l'algorithme de Wilson d'effacement de boucles. Le cadre est d'un réseau électrique fini connexe $\mathcal{G} = (V, E)$ muni de conductances mais sans mesure de meurtre. Un sommet particulier est la racine †. On considère le processus à sauts sur V tué en atteignant † ainsi que les ensembles poissoniens de boucles correspondants.

Le Jan part de la remarque suivante. Soit $x_1 \in V\backslash\{\dagger\}$ et $(X_t)_{0\leqslant t<\zeta}$ la trajectoire du processus à sauts issue de x_1. Soit $\widehat{T}_{x_1}$ le dernier temps de passage de X_t en x_1 avant ζ. Alors le processus

$$(X_t)_{0\leqslant t<\widehat{T}_{x_1}}$$

a même loi que si on prenait toutes les boucles de $\mathcal{L}_1$ qui passent par x_1, les enracinait en x_1 et les concaténait bout à bout. Plus précisément, en partageant le temps que $(X_t)_{0\leqslant t<\widehat{T}_{x_1}}$ passe en x_1 suivant une partition aléatoire de Poisson-Dirichlet $PD(0,1)$ on obtient un infinité dénombrable de boucles de x_1 à x_1 qui ont même loi que le sous-ensemble de boucles de $\mathcal{L}_1$ passant par x_1.

S'en suit le lien entre les boucles $\mathcal{L}_1$ et l'algorithme de Wilson ([**Wil96**]) d'échantillonnage de l'arbre couvrant uniforme pondéré par le produit des conductances des arêtes de l'arbre. V_i respectivement Υ_i seront des ensembles de sommets respectivement arêtes aléatoires. L'algorithme de Wilson est le suivant :

- Fixer $(x_1, \ldots, x_n)$ une numérotation de $V\backslash\{\dagger\}$.

- Fixer $V_0 := \{\dagger\}$ et $\Upsilon_0 := \emptyset$.
- Pour i allant de 1 à n faire :
  - Si $x_i \in V_{i-1}$ poser $V_i = V_{i-1}$ et $\Upsilon_i = \Upsilon_{i-1}$.
  - Sinon lancer un processus de Markov à sauts partant de x_i et l'arrêter lorsqu'il atteint V_{i-1}. Effacer les boucles du processus à saut au fur et à mesure qu'elles apparaissent. On obtient ainsi un chemin simple joignant x_i à V_{i-1}. Ajouter les sommets visités par ce chemin à V_{i-1} pour obtenir V_i. Ajouter le arêtes empruntées par ce chemin à Υ_{i-1} pour obtenir Υ_i.

Υ_n est alors un arbre couvrant enraciné en $\dagger$ dont la loi est celle d'un arbre couvrant uniforme pondéré par le produit des conductances des arêtes. Ses arêtes sont un processus déterminantal. Les boucles effacées au cours de l'exécution de l'algorithme sont en loi les boucles $\mathcal{L}_1$ où certaines ont été recollées entre elles. En particulier les champs d'occupation sont les mêmes.

La propriété mise en évidence par Le Jan pour les boucles des processus à sauts, a un analogue pour dans le cas brownien bidimensionnel. Soit D un sous-domaine strict de $\mathbb{C}$, ouvert simplement connexe et a et b deux points distincts de son bord. Étant donné l'excursion brownienne dans D joignant a à b, on peut définir un chemin simple joignant a à b obtenu en effaçant les boucles de l'excursion. Il a pour loi un processus de SLE_2 chordal ([**Zha12**]). Il a été conjecturé que si l'on prend un SLE_2 chordal de a à b ainsi qu'un ensemble poissonien de boucles browniennes $\mathcal{L}_1$ indépendant dans D, et qu'on attache à SLE_2 toutes les boucles qu'il rencontre, on reconstitue en loi l'excursion brownienne([**LW04**]).

1.6. Percolation par boucles sur réseaux

Dans [**JL13**], Le Jan et Lemaire initient l'étude des amas formés par les ensembles poissoniens de boucles des processus de Markov à sauts symétriques. On s'y intéresse notamment sous l'angle de la percolation : existence d'un amas infini de boucles sur $\mathbb{Z}^d$. Cette étude est poursuivie par Chang et Sapozhnikov dans [**CS14**]. Le cadre considéré est celui du processus à sauts symétriques aux plus proches voisins sur $\mathbb{Z}^d$, avec une mesure de meurtre uniforme ($d \geqslant 2$) ou sans ($d \geqslant 3$). Dans le cas deux-dimensionnel sans mesure de meurtre il n'y qu'un seul amas, ce qui est relié à la récurrence du processus à sauts sous-jacent. Dans les autres cas, il a été montré l'existence d'une transition de phase non triviale lorsque le paramètre d'intensité α augmente, l'unicité de l'amas infini, certaines estimées quantitatives des probabilités de connexion et un équivalent asymptotique en grande dimension du paramètre α critique. Dans cette thèse, cette étude sera poursuivie.

1.7. Organisation de la thèse et résultats

Les chapitres 2 et 3 de cette thèse sont tirés de mon mémoire [**Lup13**]. Le cadre étudié est celui des diffusions unidimensionnelles sur un intervalle, tuées si éventuellement elles atteignent le bord, avec ou sans mesure de meurtre à l'intérieur de l'intervalle. Dans le chapitre 2, je montre qu'on peut écrire la mesure sur les boucles de diffusions unidimensionnelles non seulement comme une somme pondérée de probabilités de ponts ((1.1.1)) mais aussi comme une somme de mesures d'excursions positives avec des minima balayant tout l'intervalle. Dans le cas du mouvement Brownien sur $\mathbb{R}$, il s'agit juste d'une application de la transformation de Vervaat. Quant au cas général, il me conduit à mettre en évidence une relation, valable pour une diffusion quelconque avec ou sans mesure de meurtre, entre un pont de x à x conditionné à atteindre son minimum en a et l'excursion positive issue de a. La loi du premier objet, après permutation de tronçons avant et après le minimum, est absolument continue par rapport à la loi du second et la densité est proportionnelle au temps local en x. On peut y voir une généralisation de la transformation de Vervaat. Toujours dans le chapitre 2 j'étudie le champ d'occupation des ensembles poissoniens de boucles de diffusions unidimensionnelles. Si on considère la variable d'espace comme un paramètre d'évolution, les champs d'occupation sont des processus de branchement avec immigration non-homogènes à état continu. Enfin dans le chapitre 2 je montre aussi comment obtenir tout ou en partie les ensembles poissoniens de boucles de paramètre $\alpha = 1$ en découpant de manière déterministe ou aléatoire les trajectoires de diffusions. Ceci est l'analogue du lien avec les boucles effacées mis en évidence par Le Jan dans [**Jan11**] dans le cadre de processus de Markov à sauts.

Dans le chapitre 3, j'étudie ce qu'on obtient si l'on applique l'algorithme d'effacement de boucles, utilisé pour échantillonner des arbres couvrants aléatoires, à un mouvement brownien sur $\mathbb{R}$ avec une mesure de meurtre non nulle. On peut imaginer un "graphe" constitué de $\mathbb{R}$ et d'une "racine" située à l'extérieur. Chaque point $x \in \mathbb{R}$ est relié à ses deux voisins infinitésimaux $x - dx$ et $x + dx$. De plus, chaque point dans le support de la mesure de meurtre est relié à la racine. Un "arbre couvrant" de cet objet peut être décrit par deux processus ponctuels sur $\mathbb{R}$. Le premier correspond aux sommets dans "l'arbre" reliés à la "racine" à l'extérieur de $\mathbb{R}$. Le second aux points $x \in \mathbb{R}$ tels que $x - dx$ et $x + dx$ ne soient pas reliés par une "arête" infinitésimale dans "l'arbre". Les deux processus ponctuels sont entrelacés : entre deux points consécutifs de l'un, il y a un point de l'autre. J'identifie la loi jointe de ces deux processus ponctuels. Les deux sont des processus déterminantaux. Il y a ici analogie avec l'arbre couvrant uniforme habituel dont les arêtes sont également un processus déterminantal. Je décris aussi comment, à partir de bouts de trajectoires browniennes utilisées dans l'algorithme, on peut reconstruire l'ensemble poissonien de boucles $\mathcal{L}_1$ associé au mouvement brownien avec mesure de meurtre. C'est l'analogue de ce que Le Jan décrit dans [**Jan11**] pour les processus à sauts. Enfin je montre que, si l'on augmente par endroits la mesure de meurtre sur $\mathbb{R}$, on peut obtenir des couplages monotones explicites des processus déterminantaux-"arbres couvrants" correspondants à l'ancienne et à la nouvelle mesure de meurtre.

Le chapitre 4 est tiré de mon article [**Lup14**], augmenté d'une section supplémentaire. L'article a été accepté à *Annals of Probability*. Le cadre est celui de processus markoviens à sauts symétriques transients sur un graphe (V, E). D'après l'isomorphisme de Le Jan (1.4.1), le module du champ libre gaussien $(|\phi_x|)_{x \in V}$ est donné par le champ d'occupation $(\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{x \in V}$. Je montre qu'en rajoutant de l'aléa supplémentaire, on peut créer un couplage entre l'ensemble poissonien de boucles $\mathcal{L}_{\frac{1}{2}}$ et le champ libre gaussien ϕ tel qu'en plus le signe de ϕ soit constant sur chaque amas de $\mathcal{L}_{\frac{1}{2}}$. Pour construire le couplage, j'interpole le graphe par un graphe métrique où chaque arête est remplacée par un fil "continu" d'une longueur égale à la moitié de la résistance (inverse de la conductance) et je considère le mouvement brownien sur cet objet. L'ensemble poissonien de boucles du processus à sauts peut être obtenu à partir de l'ensemble poissonien de boucles browniennes sur le graphe métrique en prenant la restriction de ces dernières au sommets. Ainsi les amas de boucles des processus à sauts sont contenus dans les amas de boucles browniennes sur le graphe métrique. L'isomorphisme de Le Jan fonctionne également sur le graphe métrique. De plus au paramètre $\alpha = \frac{1}{2}$, les amas des boucles sur le graphe métrique sont exactement des composantes connexes du signe du champ libre sur le graphe métrique. Donc le signe du champ libre est aussi constant sur les amas de boucles du processus à sauts. De ce couplage je déduis que sur $\mathbb{Z}^d$ ($d \geqslant 3$) les boucles de $\mathcal{L}_{\frac{1}{2}}$ ne percolent pas, ne forment pas d'amas infini. Cette approche utilisant les graphes métriques a été inspirée par les résultats portant sur les ensembles poissoniens de boucles de diffusions unidimensionnelles présentés dans le chapitre 2.

Dans le chapitre 4, je considère également les boucles de processus à sauts aux plus proches voisins sur le demi-plan discret $\mathtt{H} = \mathbb{Z} \times \mathbb{N}$ tués en atteignant la frontière $\mathbb{Z} \times \{0\}$. J'y étudie l'existence d'un amas infini et le paramètre d'intensité critique $\alpha_*^{\mathtt{H}}$. La motivation est de comparer avec le modèle des boucles browniennes sur le demi-plan de $\mathbb{C}$ où le paramètre d'intensité critique est $\frac{1}{2}$. Le résultat de non-percolation des boucles pour $\alpha = \frac{1}{2}$ s'applique aussi au demi-plan discret et on a $\alpha_*^{\mathbb{H}} \geqslant \frac{1}{2}$. Je montre en plus que $\frac{1}{2}$ est le paramètre d'intensité critique comme dans le cas brownien bidimensionnel. J'utilise les résultats d'approximation des ensembles poissoniens de boucles browniennes par les ensembles poissoniens de boucles de marches aléatoires obtenus par Lawler et Trujillo Ferreras dans [**LF07**].

Dans le chapitre 4, j'applique aussi l'idée d'interpolation du graphe discret par un graphe métrique au modèle d'entrelacements aléatoires de Sznitman. Un entrelacement aléatoire sur $\mathbb{Z}^d$, $d \geqslant 3$, est un processus ponctuel de Poisson fait de chemins aux plus proches voisins bi-infinis (paramétrés par un temps infini dans les deux sens) se comportant comme des trajectoires de marches aléatoires. Étant donné un paramètre d'intensité $u > 0$ et $\mathcal{I}^u$ un entrelacement aléatoire de ce niveau d'intensité, l'ensemble vacant $\mathcal{V}^u$ est l'ensemble des sites non visités par aucun des chemins dans $\mathcal{I}^u$. A un niveau critique $u_* \in (0, +\infty)$ se produit une transition de phase au-delà de laquelle $\mathcal{V}^u$ n'a que des composantes connexes bornées ([**Szn10**], [**SS09**]). On peut paramétrer les chemins dans $\mathcal{I}_u$ par un temps continu et dans ce cas, il y un champ d'occupation de $\mathcal{I}^u$, $(L^x(\mathcal{I}^u))_{x\in\mathbb{Z}^d}$. Sznitman établit un isomorphisme avec le champ libre gaussien ϕ pour ce modèle là. Si $\mathcal{I}^u$ et ϕ sont indépendants, alors

$$\left(L^x(\mathcal{I}^u) + \frac{1}{2}\phi_x^2\right)_{x\in\mathbb{Z}^d} \overset{(d)}{=} \left(\frac{1}{2}(\phi_x - \sqrt{2u})^2\right)_{x\in\mathbb{Z}^d}$$

A partir de cet isomorphisme, en interpolant $\mathbb{Z}^d$ par un graphe métrique, je montre qu'il y a un couplage entre $\mathcal{I}^u$ et ϕ tel que

$$\{x \in \mathbb{Z}^d | \phi_x > \sqrt{2u}\} \subseteq \mathcal{V}^u$$

Il existe un paramètre $h_* \geqslant 0$ critique à partir duquel l'ensemble de niveau

$$\{x \in \mathbb{Z}^d | \phi_x > h_*\}$$

n'a que des composantes connexes bornées ([**RS13**]). Mon couplage entre $\mathcal{I}^u$ et ϕ implique que $h_* \leqslant \sqrt{2u_*}$.

Le chapitre 5 est tiré de mon article [**Lup15**]. J'y considère les demi-plans discrets $(\frac{1}{n}\mathbb{Z}) \times (\frac{1}{n}\mathbb{N})$ interpolés par des graphes métriques (arêtes remplacées par des fils continus de même longueur) et j'y étudie les amas d'ensembles poissoniens de boucles sur des graphes métriques de paramètre d'intensité $\alpha = \frac{1}{2}$. Je m'interroge si la limite d'échelle des frontières extérieures des amas extérieurs quand n tend vers $+\infty$ est CLE_4. En effet, $\kappa = 4$ correspond selon la formule (1.2.1) au paramètre d'intensité $\alpha = \frac{1}{2}$ pour les boucles browniennes planaires sur un domaine simplement connexe de $\mathbb{C}$. La réponse est positive. Brug, Camia et Lis on montré dans [**dBCL14**] que si au lieu de prendre toutes les boucles on prends seulement les boucles pas trop petites, alors leurs amas vérifient cette convergence vers le CLE. Pour montrer la convergence en général, je calcule certaines probabilités sur les frontières extérieures qui, quand n tend vers $+\infty$, convergent vers des probabilités analogues pour CLE_4. Plus précisément, sur le demi-plan "continu" $\mathbb{H} = \{\Im(z) > 0\}$, je considère deux familles poissoniennes indépendantes d'excursions browniennes allant du bord vers le bord et indépendantes de l'ensemble CLE_κ à l'intérieur de $\mathbb{H}$. Les extrémités des excursions de la première famille sont contenues dans l'intervalle $(-\infty, 0) \times \{0\}$. Les extrémités de la deuxième famille sont contenues dans l'intervalle $(1, q) \times \{0\}$ où $q > 1$. Je considère l'événement que les deux familles d'excursions soient reliées soit en s'intersectant soit en intersectant une boucle de CLE_κ commune. On peut écrire une équation différentielle pour cette probabilité en faisant varier q. Ceci découle des résultats de Lawler, Schramm et Werner dans [**LSW03**] et de Werner et Wu dans [**WW13**]. Ensuite, sur les graphes métriques associés à $(\frac{1}{n}\mathbb{Z}) \times (\frac{1}{n}\mathbb{N})$, je considère l'ensemble poissonien de boucles $\mathcal{L}_{\frac{1}{2}}$ et deux familles poissoniennes indépendantes entre elles et de $\mathcal{L}_{\frac{1}{2}}$ d'excursions du mouvement brownien sur graphe métrique allant du bord $\frac{1}{n}\mathbb{Z} \times \{0\}$ vers lui-même. Ces deux familles d'excursions convergent en loi vers les familles d'excursions browniennes dans $\mathbb{H}$ décrites précédemment, quand n tend vers $+\infty$. Je considère la probabilité que les deux familles d'excursions sur le graphe métrique soient reliés soit en s'intersectant, soit en intersectant une boucle commune de $\mathcal{L}_{\frac{1}{2}}$. Cette probabilité peut être calculée explicitement en utilisant l'isomorphisme avec le champ libre gaussien. J'obtiens une convergence vers une probabilité analogue à celle obtenue avec CLE_4. En combinant avec le résultat de Brug, Camia et Lis de [**dBCL14**] je montre sur le graphe métrique que la limite d'échelle des amas des boucles de paramètre d'intensité $\alpha = \frac{1}{2}$ est bien CLE_4. Plus loin, je montre qu'il y a aussi convergence des amas pour toutes les valeurs de α dans $(0, \frac{1}{2}]$, à la fois sur le graphe métrique et sur le graphe discret, vers CLE_κ où α et κ sont liées par (1.2.1). La méthode consiste à montrer que s'il n'y avait pas convergence pour une valeur de α dans $(0, \frac{1}{2})$, il n'y aurait pas non plus de convergence pour $\alpha = \frac{1}{2}$. Dans le chapitre 5, certaines preuves de convergence manquent de détails mais toutes les idées principales y figurent.

Ainsi, en ce qui concerne la comparaison de la percolation par boucles sur le demi-plan discret et le demi-plan continu, on obtient le tableau suivant. Dans les deux cas, le paramètre d'intensité critique est $\frac{1}{2}$ et, pour α dans $(0, \frac{1}{2}]$, la limite d'échelle des frontières extérieures des amas extérieurs des boucles discrètes a la même loi que les frontières extérieures des amas extérieurs des boucles deux-dimensionnelles browniennes, c'est-à-dire des CLE_κ. De plus, sur le graphe métrique, on a une analogie complète avec le diagramme de la figure 1.3.

Les chapitres 4 et 5 sont auto-contenus en termes de notations. Dans le chapitre 3, j'utilise les notations introduites dans le chapitre 2. Dans les chapitre 2, 3 et 4, la lettre κ va désigner une mesure de meurtre alors que dans le chapitre 5, ce sera le paramètre des SLE_κ et CLE_κ.

1.8. Perspectives

Dans le chapitre 4, nous avons obtenu un couplage naturel entre le champ libre gaussien et l'ensemble poissonien de boucles des processus à sauts $\mathcal{L}_{\frac{1}{2}}$. Le point départ y est $\mathcal{L}_{\frac{1}{2}}$, à partir duquel on construit un champ libre en rajoutant de l'aléa supplémentaire. Un piste d'étude est d'avoir au contraire la loi conditionnelle de $\mathcal{L}_{\frac{1}{2}}$ par rapport au champ libre.

Une autre piste d'étude future concerne les boucles browniennes dans un domaine simplement connexe en dimension 2. Le diagramme de la figure 1.3 présente la relation entre $\mathcal{L}_{\frac{1}{2}}$, CLE_4, le champ libre ϕ et $\frac{1}{2} : \phi^2 :$. On a des couplages entre les paires de ces objets qui sont reliées par des flèches dans le diagramme. Mais, la question de coupler les quatre objets ensemble de manière naturelle et consistante avec les couplages par paires se pose. Ici aussi l'approximation du domaine par des graphes métriques peut être utile, étant donné que sur le graphe métrique, les analogues des quatre objets sont couplés naturellement ensemble, et on peut donc essayer de passer à la limite leur couplage.

Enfin, il sera intéressant d'étudier les amas des boucles browniennes en dimension 3. En effet, la trajectoire du mouvement brownien tridimensionnel est non polaire et les ensembles poissoniens de boucles browniennes en dimension 3 forment des amas non-triviaux. Il y a une transition de phase, lorsque le paramètre d'intensité α augmente, entre l'absence et la présence d'un amas non-borné. Il est vraisemblable que lorsque $\alpha = \frac{1}{2}$, il n'y a pas d'amas non-borné tout comme c'est le cas pour les amas de boucles de la marche aléatoire sur $\mathbb{Z}^3$. Il se pose aussi la question du sens qu'on peut donner au signe du champ libre sur $\mathbb{R}^3$ et ce que pourrait signifier d'avoir un signe constant sur les amas des boucles browniennes, c'est-à-dire transposer, dans le continu sur $\mathbb{R}^3$, les résultats vrais en discret sur $\mathbb{Z}^3$.

CHAPTER 2

Poisson ensembles of loops of one-dimensional diffusions

2.1. Introduction

Lawler and Werner introduced in [**LW04**] the notion of Poisson ensemble of Markov loops ("loop soup") for planar Brownian motion. In [**SW12**] it was used by Sheffield and Werner to construct the Conformal Loops Ensemble (CLE). Le Jan studied in [**Jan11**] the analogue of the Poissonian ensembles of Markov loops in the setting of a symmetric Markov jump process on a finite graph. In both cases one defines an infinite measure μ^* on time-parametrizes unrooted loops (i.e. loops parametrized by a circle where it is not specified when the cut between the beginning and the end occurs) and considers the Poisson point ensemble of intensity $\alpha\mu^*$, $\alpha > 0$, denoted here $\mathcal{L}_\alpha$. In both cases the ensemble $\mathcal{L}_1$ (where $\alpha = 1$) is related to the loops erased during the loop-erasure procedure applied to Markovian sample paths.In particular in the discrete setting Wilson's algorithm ([**Wil96**]) leads to a duality between $\mathcal{L}_1$ and the Uniform Spanning Trees. In [**Jan11**] Le Jan also studied the occupation field of $\mathcal{L}_\alpha$, that is the sum of the occupation times in a given vertex of the graph of individual loops. In case $\alpha = \frac{1}{2}$ he found that it the square of a Gaussian Free Field and related it to the Dynkin's Isomorphism ([**Dyn84b**]).

The analogue of the measure μ^* can be defined for a much larger class of Markov processes ([**LMR15**], [**FR14**]). The aim of this essay is on one hand to study the measure μ^* and the Poisson ensembles of Markov loops $\mathcal{L}_\alpha$ in the setting of one-dimensional, not necessarily conservative, diffusion processes, and on the other hand to define and study some determinatal point processes on $\mathbb{R}$ that are analogous to Uniform Spanning Tress and dual to $\mathcal{L}_1$. The diffusion processes we consider take values on a subinterval I of $\mathbb{R}$, are always killed at hitting a boundary point of I, and may be killed by a killing measure on the interior of I. One can transform a diffusion process into an other applying a change of scale, a random change of time, a restriction to a subinterval, an increase of the killing measure or a conjugation of the semi-group by a positive sufficiently regular function ($P_t \mapsto h^{-1}P_t h$). The measure μ^* is covariant with all these transformations on Markov processes. In other words the map diffusion to measure on loops is a covariant functor. Moreover we will show that μ^* is invariant by conjugation on underlying diffusions. We will also extend the scope of our study by associating a measure on loops to "generators" which contain a creation of mass term: If $L = L^{(0)} + \nu$ where $L^{(0)}$ is a second order differential operator on I and ν is a signed measure, and if one sets zero Dirichlet boundary conditions for L, one can define in a consistent way a measure on loops related to L even in case the semi-group $(e^{tL})_{t\geqslant 0}$ does not make sense. This extended definition of μ^* will be particularly handy for computing the exponential moments of the Poissonian ensemble of Markov loops.

The layout of this paper is the following: In chapter 2.2 we will recall some facts on one-dimensional diffusions and set the important notations. We will further consider "generators" with creation of mass term and characterize a class of such operators which up to a conjugation are equivalent to the generators of diffusions. In chapter 2.3 we will define the measure μ^* and point out different covariance and invariance properties. Further we will make a connection between the Brownian measure on loops and the Levy-Itô measure on Brownian excursion using the Vervaat's bridge-to-excursion transformation. This in turn will lead us to a conditioned version of Vervaat's transformation that holds for any one-dimensional diffusion process, that is an absolute continuity relation between the bridge conditioned to have a given minimum and an excursion of the same duration above this minimum. The Vervaat's transformation is deeply related to the measure on loops μ^*: The loops are unrooted, so one can freely chose a moment separating the end from the start. If one chooses this moment uniformly over the life-time of the loop, then the loop under the measure μ^* looks in some sense like a bridge. If one chooses this moment when the loop hits its minimum, then it looks like an excursion. In chapter 2.4 we will study the occupation field of the Poisson ensemble of Markov loops. Each loop is endowed with a family of local times. The occupation field is the sum of local times over the loops. We will identify its law as an non-homogeneous continuous state branching process with immigration parametrised by the position points in I. In case $\alpha = \frac{1}{2}$ we will identify it as the square of a Gaussian Free Field and show how it is possible to derive particular versions of the Dynkin's Isomorphism using this fact and Palm's identity for Poissonian ensembles. In chapter 2.5 we will root each loop in $\mathcal{L}_\alpha$ at its minimum and obtain this way a collection of positive excursions. Then we will order this excursions in the decreasing sense of their minima and glue them together. We will obtain this way a continuous path which can be described using two-dimensional Markov processes. This is a way to sample $\mathcal{L}_\alpha$. In the particular case $\alpha = 1$ the path we obtain is the sample path of an one-dimensional diffusion. This is the analogue in our setting of the relation between $\mathcal{L}_1$ and the loop-erasure procedure observed in the setting of the two-dimensional Brownian motion or of the symmetric Markov jump processes on graphs. In chapter 3.1 we will apply an extension of Wilson's algorithm to transient one-dimensional diffusions and obtain a couple of interwoven determinantal point processes on $\mathbb{R}$ which is dual to $\mathcal{L}_1$. In chapter 3.2 we will prove some monotone coupling properties for the determinantal point processes introduced in chapter 3.1.

The author thanks Yves Le Jan for fruitful discussions and its helpful advice in relation with this work.

2.2. Preliminaries on generators and semi-groups

2.2.1. A second order ODE. In this chapter we will introduce the one-dimensional diffusions we will consider throughout this work (section 2.2.2). In the section 2.2.3 we will extend the framework to the "generators" containing a mass-creation term. In the section 2.2.1 we will prove or recall some facts on the functions harmonic for these generators.

Let I be an open interval of $\mathbb{R}$ and ν a signed measure on I. By signed measure we mean that the total variation $|\nu|$ is a positive Radon measure, but not necessarily finite, and $\nu(dx) = \epsilon(x)|\nu|(dx)$ where ϵ takes values in $\{\pm 1\}$. We look for the solutions of the linear second order differential equation on I:

$$\frac{d^2u}{dx^2} + u\nu = 0 \tag{2.2.1}$$

Given a solution u of (2.2.1) we will write $\frac{du}{dx}(x^+)$ and $\frac{du}{dx}(x^-)$ for the right-hand side respectively left-hand side derivative of u at x. The two are related by

$$\frac{du}{dx}(x^+) - \frac{du}{dx}(x^-) = -u(x)\nu(\{x\})$$

Using a standard fixed point argument one can show that (2.2.1) satisfies a Cauchy-Lipschitz principle: if $x_0 \in I$ and $u_0, v_0 \in \mathbb{R}$, there is a unique solution u of (2.2.1), continuous on I, satisfying $u(x_0) = u_0$ and $\frac{du}{dx}(x_0^+) = v_0$. Let $x_1 \in I \cap (x_0, +\infty)$. A continuous function u on $[x_0, x_1]$ is solution of (2.2.1) with previous initial conditions at x_0 if and only if it is a fixed point of the affine operator $\mathfrak{I}$ on $\mathcal{C}([x_0, x_1])$ defined as

$$(\mathfrak{I}u)(x) := u_0 + (x - x_0)v_0 - \int_{(x_0,x]} (x-y)u(y)\nu(dy)$$

The Lipschitz norm of $\mathfrak{I}^n$ is smaller or equal to $\frac{|\nu|([x_0,x_1])^n(x_1-x_0)^n}{n!}$. So for n large enough $\mathfrak{I}^n$ is contracting and thus $\mathfrak{I}$ has a unique fixed point in $\mathcal{C}([x_0, x_1])$.

Let $W(u_1, u_2)(x)$ be the Wronskian of two functions u_1, u_2:

$$W(u_1, u_2)(x) := u_1(x)\frac{du_2}{dx}(x^+) - u_2(x)\frac{du_1}{dx}(x^+)$$

If u_1, u_2 are both solutions of (2.2.1), $W(u_1, u_2)$ is constant on I. Using this fact we get a results which is similar to Sturm's separation theorem for the case of a measure ν with a continuous density with respect to the Lebesgue measure (see theorem 7, section 2.6 in [**BR89**]):

PROPERTY 2.2.1. *Given $x_0 < x_1$ be two points in I:*

(i) *Let u_1 be a solution of (2.2.1) satisfying $u_1(x_0) = 0$, $\frac{du_1}{dx}(x_0^+) > 0$, and u_2 a solution such that $u_2(x_0) > 0$. Assume that $u_2 \geqslant 0$ on $[x_0, x_1]$. Then $u_1 > 0$ on $(x_0, x_1]$.*
(ii) *Let u_1, u_2 be two solutions such that $u_1(x_0) = u_2(x_0) > 0$ and $\frac{du_1}{dx}(x_0^+) > \frac{du_2}{dx}(x_0^+)$. Assume that $u_2 \geqslant 0$ on $[x_0, x_1]$. Then $u_1 > u_2$ on $(x_0, x_1]$.*
(iii) *If there is a solution u to (2.2.1) positive on (x_0, x_1) and zero at x_0 and x_1 then any other linearly independent solution of (2.2.1) has exactly one zero in (x_0, x_1).*

Next we prove a lemma that will be useful in the section 2.2.3.

LEMMA 2.2.2. *Let ν_+ be the positive part of ν. Let $x_0 < x_1 \in I$. Let f be a continuous positive function on $[x_0, x_1]$ such that $\min_{[x_0,x_1]} f > \nu_+([x_0, x_1])^2$. Then the equation*

$$\frac{d^2u}{dx^2} + u\nu - uf = 0 \qquad (2.2.2)$$

has a positive solution that is non-decreasing on $[x_0, x_1]$.

PROOF. Set $a := \min_{[x_0,x_1]} f$. Let u be the solution to (2.2.2) with the initial values $u(x_0) = 1$, $\frac{du}{dx}(x_0^+) = \sqrt{a}$. We will show that u is non-decreasing on $[x_0, x_1]$. Assume that this is not the case. This means that $\frac{du}{dx}(x^+)$ takes negative values somewhere in $[x_0, x_1]$. Let

$$x_2 := \inf\left\{x \in [x_0, x_1] \middle| \frac{du}{dx}(x^+) \leqslant 0\right\}$$

Since $\frac{du}{dx}(x^+)$ is right-continuous, $\frac{du}{dx}(x_2^+) \leqslant 0$. Let $r(x) := \frac{1}{u(x)}\frac{du}{dx}(x^+)$. u is positive on $[x_0, x_2]$ hence r is defined $[x_0, x_2]$. $r(x_0) = \sqrt{a}$. r is cadlag and satisfies the equation

$$dr = (f - r^2)dx - d\nu$$

Let $x_3 := \sup\{x \in [x_0, x_2] | r(x) \geqslant \sqrt{a}\}$. We have

$$r(x_2) = r(x_3^-) + \int_{x_3}^{x_2} (f(x) - r^2(x))dx - \nu([x_3, x_2])$$

By construction $r(x_3^-) \geqslant \sqrt{a}$. By definition $f - r^2 \geqslant 0$ on $(x_3, x_2]$. Thus

$$r(x_2) \geqslant \sqrt{a} - \nu([x_3, x_2]) > 0$$

It follows that $r(x_2) > 0$, which is absurd. □

In the case $\nu = -2\kappa$ where κ is a non-zero positive Radon measure, the equation (2.2.1) becomes:

$$\frac{1}{2}\frac{d^2u}{dx^2} - u\kappa = 0 \tag{2.2.3}$$

It commonly appears when studying the Brownian motion with a killing measure κ. In this case the two-dimensional linear space of solutions is spanned by two convex positive solutions $u_\uparrow$ and $u_\downarrow$, $u_\uparrow$ being non-decreasing and $u_\downarrow$ non-increasing. Given $x_0 \in I$, we can construct $u_\uparrow$ as the limit when $x_1 \to \inf I$ of the unique solution which equals 0 in x_1 and 1 in x_0. For $u_\downarrow$ we take the limit as $x_1 \to \sup I$. $u_\uparrow$ and $u_\downarrow$ are defined up to a positive multiplicative constant. See [**Bre92**], section 16.11, or [**RY99**], Appendix 8, for more details. Next we give equivalent conditions on the asymptotic behaviour of $u_\uparrow$ and $u_\downarrow$ that will be used in chapter 3.1.

PROPOSITION 2.2.3. *In case $[0, +\infty) \subseteq I$, the following four conditions are equivalent:*

(i) $\int_{(0,+\infty)} x\kappa(dx) < +\infty$
(ii) $u_\downarrow(+\infty) > 0$
(iii) *There is $C > 0$ such that for all $x \geqslant 1$, $u_\uparrow(x) \leqslant Cx$*
(iv) $\int_{(0,+\infty)} u_\uparrow(x)u_\downarrow(x)\kappa(dx) < +\infty$

PROOF. We will prove in order that (ii) implies (i), (iii) implies (i), (i) implies (ii), (i) implies (iii) and (iv) implies (ii). (iv) is obviously implied by the combination of (i), (ii) and (iii).

(ii) implies (i): For all $x \in [0, +\infty)$:

$$-\frac{du_\downarrow}{dx}(x^+) = 2\int_{(x,+\infty)} u_\downarrow(y)\kappa(dy) \leqslant 2u_\downarrow(+\infty)\kappa((x, +\infty))$$

$-\frac{du_\downarrow}{dx}(x^+)$ is integrable on $(0, +\infty)$. Since $u_\downarrow(+\infty) > 0$, this implies that:

$$\int_{(0,+\infty)} \kappa((x, +\infty))dx < +\infty$$

But

$$\int_{(0,+\infty)} \kappa((x, +\infty))dx = \int_{(0,+\infty)} y\kappa(dy)$$

and hence (i).

(iii) implies (i): If (iii) holds then for all $x \in [0, +\infty)$, $\frac{du_\uparrow}{dx}(x^+) \leqslant C$. But

$$\frac{du_\uparrow}{dx}(x^+) = \frac{du_\uparrow}{dx}(0^+) + 2\int_{(0,x]} u_\uparrow(y)\kappa(dy)$$

This implies that

$$\int_{(0,+\infty)} u_\uparrow(y)\kappa(dy) < +\infty$$

Since $u_\uparrow$ is convex, $u_\uparrow(y) \geqslant u_\uparrow(0) + \frac{du_\uparrow}{dy}(0^+)y$. So (i) is satisfied.

(i) implies (ii): For all $y \in [0, +\infty)$:

$$u_\downarrow(y) - u_\downarrow(+\infty) = 2\int_y^{+\infty}\int_{(z,+\infty)} u_\downarrow(x)\kappa(dx)dz \leqslant 2u_\downarrow(y)\int_{(y,+\infty)} (x - y)\kappa(dx)$$

Condition (i) implies that:

$$\lim_{y\to+\infty} 2\int_{(y,+\infty)} (x - y)\kappa(dx) = 0$$

So for y large enough, $u_\downarrow(y) - u_\downarrow(+\infty) < u_\downarrow(y)$. Necessarily $u_\downarrow(+\infty) > 0$.

(i) implies (iii): For all $y < x \in [0, +\infty)$:

$$\frac{du_\uparrow}{dx}(x^+) = \frac{du_\uparrow}{dy}(y^+) + 2u_\uparrow(y)\kappa((y,x]) + 2\int_{(y,x]}(u_\uparrow(z) - u_\uparrow(y))\kappa(dz) \tag{2.2.4}$$

Let y be large enough such that:

$$2\int_{(y,+\infty)}(z-y)\kappa(dz) < 1$$

Then there is $C > 0$ large enough such that:

$$C > \frac{du_\uparrow}{dy}(y^+) + 2u_\uparrow(y)\kappa((y,+\infty)) + 2C\int_{(y,+\infty)}(z-y)\kappa(dz) \tag{2.2.5}$$

Assume that there is $x \in [0,+\infty)$ such that $\frac{du_\uparrow}{dx}(x^+) \geqslant C$. Let

$$x_0 := \inf\left\{x \geqslant y \middle| \frac{du_\uparrow}{dx}(x^+) \geqslant C\right\}$$

$x \mapsto \frac{du_\uparrow}{dx}(x^+)$ is right-continuous. Thus $\frac{du_\uparrow}{dx}(x_0^+) \geqslant C$. By definition, for all $z \in [y, x_0]$, $\frac{du_\uparrow}{dz}(z^+) \leqslant C$ and hence $u_\uparrow(z) - u_\uparrow(y) \leqslant C(z-y)$. But then (2.2.4) and (2.2.5) imply that $\frac{du_\uparrow}{dx}(x_0^+) < C$ which is contradictory. It follows that $\frac{du_\uparrow}{dx}(x^+)$ is bounded by C, which implies property (iii).

(iv) implies (ii): Applying integration by parts we get that for all $x > 0$:

$$\begin{aligned}2\int_{(0,x]}u_\uparrow(y)u_\downarrow(y)\kappa(dy) &= \int_{(0,x]}u_\downarrow(y)d\Big(\frac{du_\uparrow}{dy}\Big)(dy)\\ &= \frac{du_\uparrow}{dx}(x^+)u_\downarrow(x) - \frac{du_\uparrow}{dx}(0^+)u_\downarrow(0) - \int_0^x \frac{du_\downarrow}{dy}(y^+)\frac{du_\uparrow}{dy}(y^+)dy\end{aligned}$$

$\frac{du_\uparrow}{dx}(x^+)u_\downarrow(x)$ is positive. We get that:

$$-\int_0^{+\infty}\frac{du_\downarrow}{dy}(y^+)\frac{du_\uparrow}{dy}(y^+)dy \leqslant 2\int_{(0,+\infty)}u_\uparrow(y)u_\downarrow(y)\kappa(dy) + \frac{du_\uparrow}{dx}(0^+)u_\downarrow(0) < +\infty \tag{2.2.6}$$

Next

$$\begin{aligned}\frac{du_\uparrow}{dx}(x^+)(u_\downarrow(x) - u_\downarrow(+\infty)) &= -\frac{du_\uparrow}{dx}(x^+)\int_x^{+\infty}\frac{du_\downarrow}{dy}(y^+)dy\\ &\leqslant -\int_x^{+\infty}\frac{du_\downarrow}{dy}(y^+)\frac{du_\uparrow}{dy}(y^+)dy\end{aligned} \tag{2.2.7}$$

Assume that $u_\downarrow(+\infty) = 0$. Then (2.2.7) implies that:

$$\lim_{x\to+\infty}\frac{du_\uparrow}{dx}(x^+)u_\downarrow(x) = 0$$

and

$$\lim_{x\to+\infty} -\frac{du_\downarrow}{dx}(x^+)u_\uparrow(x) = W(u_\downarrow, u_\uparrow) - \lim_{x\to+\infty}\frac{du_\uparrow}{dx}(x^+)u_\downarrow(x) = W(u_\downarrow, u_\uparrow) \tag{2.2.8}$$

(2.2.6) together with (2.2.8) imply that

$$\int_0^{+\infty}\frac{1}{u_\uparrow(y)}\frac{du_\uparrow}{dy}(y^+)dy < +\infty$$

But this is impossible because $\log(u_\uparrow(+\infty)) = +\infty$. Thus $u_\downarrow(+\infty) > 0$. □

Next we deal with the continuity of $u_\uparrow$ and $u_\downarrow$ with respect the measure κ. We will write $u_{\kappa,\uparrow}$ and $u_{\kappa,\downarrow}$ to denote the dependence on κ.

LEMMA 2.2.4. *Let $x_0 \in I$. Let $(\kappa_n)_{n\geqslant 0}$ be a sequence of non-zero positive Radon measures on I converging vaguely (i.e. against functions with compact support) to κ. Then $\frac{u_{\kappa_n,\uparrow}}{u_{\kappa_n,\uparrow}(x_0)}$ converges to $\frac{u_{\kappa,\uparrow}}{u_{\kappa,\uparrow}(x_0)}$, $\frac{u_{\kappa_n,\downarrow}}{u_{\kappa_n,\downarrow}(x_0)}$ converges to $\frac{u_{\kappa,\downarrow}}{u_{\kappa,\downarrow}(x_0)}$ and the convergences are uniform on compact subsets of I.*

PROOF. We will deal with the convergence of $\frac{u_{\kappa_n,\downarrow}}{u_{\kappa_n,\downarrow}(x_0)}$, the other one being similar. To simplify notations we will chose the normalization $u_{\kappa,\downarrow}(x_0) = u_{\kappa_n,\downarrow}(x_0) = 1$. Without loss of generality we will also assume that $\kappa(\{x_0\}) = 0$. The proof will be made of two parts. First we will show that if u is the solution of (2.2.3) and u_n solution of

$$\frac{1}{2}\frac{d^2u}{dx^2} - u\kappa_n = 0 \tag{2.2.9}$$

and if $u_n(x_0) = u(x_0) = 1$ and $\frac{du}{dx}(x_0^+) = \lim_{n\to+\infty} \frac{du_n}{dx}(x_0^+)$ then u_n converges to u uniformly on compact subsets of I. After that we will show that $\frac{du_{\kappa_n,\downarrow}}{dx}(x_0^+)$ converges to $\frac{du_{\kappa,\downarrow}}{dx}(x_0^+)$.

Let $x_1 \in I \cap (x_0, +\infty)$. Let $(v_n)_{n\geqslant 0}$ be a sequence in $\mathbb{R}$ converging to v. Let $\mathfrak{I}_n$ respectively $\mathfrak{I}$ be the following affine operators on $\mathcal{C}([x_0, x_1])$:

$$(\mathfrak{I}_n f)(x) := 1 + (x - x_0)v_n + 2\int_{(x_0,x]} (x - y)f(y)\kappa_n(dy)$$

$$(\mathfrak{I} f)(x) := 1 + (x - x_0)v + 2\int_{(x_0,x]} (x - y)f(y)\kappa(dy)$$

Let u_n respectively u be the fixed points of $\mathfrak{I}_n$ respectively $\mathfrak{I}$. Let $\varepsilon \in (0, 1)$. The Lipschitz norm of $\mathfrak{I}_n^j$ is bounded by $\frac{2^j}{j!}\kappa_n([x_0, x_1])^j(x_1 - x_0)^j$. For $j \geqslant j_\varepsilon$, for all $n \in \mathbb{N}$, this norm is less then ε. Then

$$\begin{aligned}\max_{[x_0,x_1]} |u_n - u| &= \max_{[x_0,x_1]} |\mathfrak{I}_n^{j_\varepsilon} u_n - \mathfrak{I}^{j_\varepsilon} u| \leqslant \max_{[x_0,x_1]} |\mathfrak{I}_n^{j_\varepsilon} u - \mathfrak{I}^{j_\varepsilon} u| + \max_{[x_0,x_1]} |\mathfrak{I}_n^{j_\varepsilon} u_n - \mathfrak{I}_n^{j_\varepsilon} u| \\ &\leqslant \max_{[x_0,x_1]} |\mathfrak{I}_n^{j_\varepsilon} u - \mathfrak{I}^{j_\varepsilon} u| + \varepsilon \max_{[x_0,x_1]} |u_n - u|\end{aligned}$$

Hence

$$\max_{[x_0,x_1]} |u_n - u| \leqslant \frac{1}{1-\varepsilon} \max_{[x_0,x_1]} |\mathfrak{I}_n^{j_\varepsilon} u - \mathfrak{I}^{j_\varepsilon} u| \tag{2.2.10}$$

For $y < x \in I$ and $i \in \mathbb{N}^*$ let

$$f_{n,i}(y, x) := \int_{y<y_1<\cdots<y_{i-1}<x} (x - y_{i-1}) \dots (y_2 - y_1)(y_1 - y)\kappa_n(dy_1) \dots \kappa_n(dy_{i-1})$$

$$f_i(y, x) := \int_{y<y_1<\cdots<y_{i-1}<x} (x - y_{i-1}) \dots (y_2 - y_1)(y_1 - y)\kappa(dy_1) \dots \kappa(dy_{i-1})$$

and $f_{0,i}(y, x) = f_0(y, x) = x - y$. $f_{n,i}$ and f_i are continuous functions. Moreover the vague convergence of κ_n to κ ensures that if $(y_n, x_n)_{n\geqslant 0}$ is a sequence converging to (y, x) then $f_{n,i}(y_n, x_n)$ converges to $f_i(y, x)$.

$$\begin{aligned}(\mathfrak{I}_n^{j_\varepsilon} u)(x) =& 1 + (x - x_0)v_n + \sum_{i=0}^{j_\varepsilon-2} \int_{x_0}^{x} (1 + (y - x_0)v_n)f_{n,i}(y, x)\kappa_n(dy) \\ &+ \int_{x_0}^{x} u(y)f_{n,j_\varepsilon-1}(y, x)\kappa_n(dy) \\ (\mathfrak{I}^{j_\varepsilon} u)(x) =& 1 + (x - x_0)v + \sum_{i=0}^{j_\varepsilon-2} \int_{x_0}^{x} (1 + (y - x_0)v)f_i(y, x)\kappa(dy) \\ &+ \int_{x_0}^{x} u(y)f_{j_\varepsilon-1}(y, x)\kappa(dy)\end{aligned}$$

For fixed x, the functions $y \mapsto 1_{x_0<y<x} f_{n,i}(y,x)$ and $y \mapsto 1_{x_0<y<x} f_i(y,x)$ have a compact support but are discontinuous at x_0. If $(z_n)_{n\geqslant 0}$ is a sequence in $[x_0,x_1]$ converging to z, then the convergence of v_n to v, the weak convergence of κ_n to κ and the condition $\kappa(\{x_0\}) = 0$ ensure that $(\mathfrak{I}_n^{j_\varepsilon} u)(z_n)$ converges to $(\mathfrak{I}^{j_\varepsilon} u)(z)$. This implies the uniform convergence of $\mathfrak{I}_n^{j_\varepsilon} u$ to $\mathfrak{I}^{j_\varepsilon} u$ on $[x_0,x_1]$. From (2.2.10) follows that u_n converges uniformly to u on $[x_0,x_1]$. The situation is similar for $x_1 < x_0$ and we get the uniform convergence on compact sets of u_n to u.

Let

$$\underline{v} := \liminf_{n\to+\infty} \frac{du_{\kappa_n,\downarrow}}{dx}(x_0^+) \qquad \overline{v} := \limsup_{n\to+\infty} \frac{du_{\kappa_n,\downarrow}}{dx}(x_0^+)$$

Let $v < \frac{du_{k,\downarrow}}{dx}(x_0^+)$. There is $x_1 \in I \cap (x_0,+\infty)$ such that the solution of (2.2.3) with initial conditions $u(x_0) = 1$, $\frac{du}{dx}(x^+) = v$ is zero at x_1 since $u_{\kappa_n,\downarrow}$ converges to $u_{\kappa,\downarrow}$ uniformly on $[x_0,x_1]$ and $u_{\kappa,\downarrow}$ is positive on $[x_0,x_1]$, we get that for n large enough, $u_{\kappa_n,\downarrow}$ is positive on $[x_0,x_1]$ and $\frac{du_{\kappa_n,\downarrow}}{dx}(x_0^+) > v$. Thus $\underline{v} \geqslant \frac{du_{k,\downarrow}}{dx}(x_0^+)$.

Conversely, let $v < \overline{v}$. Let u_n be the solution of (2.2.9) with initial conditions $u_n(x_0) = 1$, $\frac{du_n}{dx}(x_0^+) = v$. If $\frac{du_{\kappa_n,\downarrow}}{dx}(x_0^+) > v$, then for any $x \in I \cap [x_0,+\infty)$

$$\frac{du_n}{dx}(x^+) \leqslant \frac{du_{\kappa_n,\downarrow}}{dx}(x^+) - \Big(\frac{du_{\kappa_n,\downarrow}}{dx}(x_0^+) - v\Big) \leqslant -\Big(\frac{du_{\kappa_n,\downarrow}}{dx}(x_0^+) - v\Big)$$

$$u_n(x) \leqslant u_{\kappa_n,\downarrow} - \Big(\frac{du_{\kappa_n,\downarrow}}{dx}(x_0^+) - v\Big)(x - x_0)$$

If $\sup I < +\infty$ then by convexity of $u_{\kappa_n,\downarrow}$:

$$u_n(x) \leqslant \frac{\sup I - x}{\sup I - x_0} - \Big(\frac{du_{\kappa_n,\downarrow}}{dx}(x_0^+) - v\Big)(x - x_0)$$

and $u_n(z_n) \leqslant 0$ where

$$z_n := \frac{\sup I + \Big(\frac{du_{\kappa_n,\downarrow}}{dx}(x_0^+) - v\Big)x_0(\sup I - x_0)}{1 + \Big(\frac{du_{\kappa_n,\downarrow}}{dx}(x_0^+) - v\Big)(\sup I - x_0)}$$

This is also true if $\sup I = +\infty$ and in this case $z_n = x_0 + \Big(\frac{du_{\kappa_n,\downarrow}}{dx}(x_0^+) - v\Big)^{-1}$. Let u be the solution of of (2.2.3) with initial conditions $u(x_0) = 1$, $\frac{du}{dx}(x^+) = v$ and

$$z_\infty := \frac{\sup I + (\overline{v} - v)x_0(\sup I - x_0)}{1 + (\overline{v} - v)(\sup I - x_0)}$$

Considering a subsequence along which $\frac{du_{\kappa_n,\downarrow}}{dx}(x_0^+)$ converges to $\overline{v}$, we get by uniform convergence of u_n tu u on compact sets that $u(z_\infty) \geqslant 0$. It follows that $\frac{du_{\kappa,\downarrow}}{dx}(x_0^+) \geqslant v$. Hence $\frac{du_{\kappa,\downarrow}}{dx}(x_0^+) \geqslant \overline{v}$.

Finally $\underline{v} = \overline{v} = \frac{du_{\kappa,\downarrow}}{dx}(x_0^+)$ and this implies the uniform convergence on compact subsets of $u_{\kappa_n,\downarrow}$ to $u_{\kappa,\downarrow}$. □

2.2.2. One-dimensional diffusions. In this subsection we will describe the kind of linear diffusion we are interested in, recall some facts and introduce notations that will be used subsequently. For a detailed presentation of one-dimensional diffusions see [**IM74**] and [**Bre92**], chapter 16.

Let I be an open interval of $\mathbb{R}$, m and w continuous positive functions on I. We consider a diffusion $(X_t)_{0\leqslant t<\zeta^{(0)}}$ on I with generator

$$L^{(0)} := \frac{1}{m(x)}\frac{d}{dx}\left(\frac{1}{w(x)}\frac{d}{dx}\right)$$

and killed as it hits the boundary of I. In case I is unbounded, we also allow for X to blow up to infinity in finite time. $\zeta^{(0)}$ is the first time X either hits the boundary or explodes. To avoid some technicalities we will assume that $\frac{dw}{dx}$ is locally bounded, although this condition is not essential. Given such a diffusion, the speed measure $m(x)dx$ and the scale measure $w(x)dx$ are defined up to a positive multiplicative constant, but the product mw is uniquely defined. A primitive S of w is a natural scale function of X. Consider the random time change $d\tilde{t} = \frac{1}{m(X_t)}dt$. Then $(\frac{1}{2}S(X_{\tilde{t}}))_{0\leqslant\tilde{t}<\tilde{\zeta}^{(0)}}$ is a standard Brownian motion on $S(I)$ killed when it first hits the boundary of $S(I)$. For all f, g smooth, compactly supported in I,

$$\int_I (L^{(0)}f)(x)g(x)m(x)dx = \int_I f(x)(L^{(0)}g)(x)m(x)dx$$

The diffusion X has a family of local times $(\ell_t^x(X))_{x\in I,t\geqslant 0}$ with respect to the measure $m(x)dx$ such that $(x,t)\mapsto \ell_t^x(X)$ is continuous. We can further consider diffusions with killing measures. Let κ be a non-negative Radon measure on I. We kill X as soon as $\int_I \ell_t^x(X)m(x)\kappa(dx)$ hits an independent exponential time with parameter 1. The corresponding generator is

$$L = \frac{1}{m(x)}\frac{d}{dx}\left(\frac{1}{w(x)}\frac{d}{dx}\right) - \kappa \tag{2.2.11}$$

Let $(X_t)_{0\leqslant t<\zeta}$ be the diffusion of generator (2.2.11), which is killed either by hitting ∂I, or by exploding, or by the killing measure k. For $x \in I$ let $\eta_{exc}^{>x}$ and $\eta_{exc}^{<x}$ be the excursion measures of X above and below the level x up to the last time X visits x. The behaviour of X from the first to the last time it visits x is a Poisson point process with intensity $\eta_{exc}^{>x} + \eta_{exc}^{<x}$, parametrized by the local time at x up to the value $\ell_t^{\zeta}(X)$. $\eta_{exc}^{>x}$ and $\eta_{exc}^{<x}$ are obtained from the Levy-Itô measure on Brownian excursions through scale change, time change and multiplication by a density function accounting for the killing. See [**SVY07**] for details on excursion measures in case of recurrent diffusions.

If X is transient the Green's function of L,

$$G(x,y) := \mathbb{E}_x[\ell_t^{\zeta}(X)]$$

is finite, continuous and symmetric. For $x \leqslant y$ it can be written

$$G(x,y) = u_{\uparrow}(x)u_{\downarrow}(y)$$

where $u_{\uparrow}(x)$ and $u_{\downarrow}(y)$ are positive , respectively non-decreasing and non-increasing solutions to the equation $Lu = 0$, which through a change of scale reduces to an equation of form (2.2.3). If S is bounded from below, $u_{\uparrow}(\inf I^+) = 0$. If S is bounded from above, $u_{\downarrow}(\sup I^-) = 0$. $u_{\uparrow}(x)$ and $u_{\downarrow}(y)$ are each determined up to a multiplication by a positive constant, but when entering the expression of G, the two constants are related. For $x \leqslant y \in I$:

$$\frac{u_{\uparrow}(x)}{u_{\uparrow}(y)} = \mathbb{P}_y(X \text{ hits } x \text{ before time } \zeta) \qquad \frac{u_{\downarrow}(y)}{u_{\downarrow}(x)} = \mathbb{P}_x(X \text{ hits } y \text{ before time } \zeta)$$

See [**IM74**] or [**Bre92**], chapter 16, for details. Let $W(u_{\downarrow}, u_{\uparrow})$ be the Wronskian of $u_{\downarrow}$ and $u_{\uparrow}$:

$$W(u_{\downarrow}, u_{\uparrow})(x) := u_{\downarrow}(x)\frac{du_{\uparrow}}{dx}(x^+) - u_{\uparrow}(x)\frac{du_{\downarrow}}{dx}(x^+)$$

This Wronskian is actually the density of the scale measure: $W(u_{\downarrow}, u_{\uparrow}) \equiv w$. We may write G_L when there is an ambiguity on L.

If the killing measure κ is non zero, then the probability that X, starting from x, gets killed by κ before reaching a boundary of I or exploding equals $\int_I G(x,y)m(y)\kappa(dy)$.

Conditional on this event, the distribution of $X_{\zeta-}$ is:

$$\frac{1_{z\in I}G(x,z)m(z)\kappa(dz)}{\int_I G(x,y)m(y)\kappa(dy)}$$

Indeed, let f be a non-negative compactly supported measurable function on I and

$$\tau_l := \inf\left\{t \in [0,\zeta^{(0)}\Big| \int_I \ell_t^y(X)m(y)\kappa(dy) > l\right\}$$

Then by definition

$$\mathbb{E}_x\big[f(X_{\zeta-})\big] = \int_0^{+\infty} e^{-l}\mathbb{E}_x\big[f(X_{\tau_l\wedge\zeta^{(0)}})\big]dl = \int_0^{+\infty} dv e^{-v}\mathbb{E}_x\Big[\int_0^v f(X_{\tau_l\wedge\zeta^{(0)}})dl\Big]$$

But

$$\int_0^v f(X_{\tau_l\wedge\zeta^{(0)}})dl = \int_I \ell^y_{\tau_v\wedge\zeta^{(0)}}(X)m(y)\kappa(dy)$$

(see corollary 2.13, chapter X in [**RY99**]). It follows that

$$\mathbb{E}\big[f(X_{\zeta-})\big] = \int_I f(y)\Big[\int_0^{+\infty} e^{-v}\ell^y_{\tau_v\wedge\zeta^{(0)}}(X)dv\Big]m(y)\kappa(dy) = \int_I f(y)G(x,y)m(y)\kappa(dy)$$

The semi-group of L has positive transition densities $p_t(x,y)$ with respect to the speed measure $m(y)dy$ and $(t,x,y)\mapsto p_t(x,y)$ is continuous on $(0,+\infty)\times I\times I$. McKean gives a proof of this in [**McK56**] in case when the killing measure k has a continuous density with respect to the Lebesgue measure. If this is not the case, we can take u a positive continuous solution to $Lu = 0$ and consider the conjugation of L by u: $u^{-1}Lu$. The latter is the generator of a diffusion without killing measure and by [**McK56**] this diffusion has continuous transition densities $\tilde{p}_t(x,y)$ with respect to $m(y)dy$. Then $u(x)\tilde{p}_t(x,y)\frac{1}{u(y)}$ are the transition densities of the semi-group of L. Transition densities with respect to the speed measure are symmetric: $p_t(x,y) = p_t(y,x)$. For all $x,y\in I$ and $t\geqslant 0$ the following equality holds:

$$\mathbb{E}_x\big[\ell^y_{t\wedge\zeta}(X)\big] = \int_0^t p_s(x,y)ds \tag{2.2.12}$$

Next we deal with bridge probability measures.

PROPOSITION 2.2.5. *The bridge probability measures $\mathbb{P}^t_{x,y}(\cdot)$ (bridge of X from x to y in time t conditioned neither to die nor to explode in the interval) satisfy: for all $x\in I$ the map $(x,y,t)\mapsto\mathbb{P}^t_{x,y}(\cdot)$ is continuous for the weak topology on probability measures on continuous paths.*

PROOF. Our proof mainly relies on absolute continuity arguments of [**PFY93**] and [**CB11**], and the time reversal argument of [**PFY93**]. [**CB11**] gives a proof of weak continuity of bridges for conservative Feller cadlag processes on second countable locally compact spaces. But since the proof contains an error and we do not restrict to conservative diffusions, we give here accurate arguments for the weak continuity.

First we can restrict to the case $\kappa = 0$. Otherwise consider u a solution to $Lu = 0$, positive on I. The generator of the conjugate of L by u is

$$\frac{1}{u(x)^2m(x)}\frac{d}{dx}\left(\frac{u(x)^2}{w(x)}\frac{d}{dx}\right)$$

and does not contain any killing measure. The conjugation preserves the bridge measures and changes the density functions relative to $m(y)dy$ to $\frac{1}{u(x)}p_t(x,y)u(y)$, and thus preserves their continuity.

Then we normalise the length of bridges: if $(X_s^{(x,y,t)})_{0\leqslant s\leqslant t}$ is a path under the law $\mathbb{P}^t_{x,y}(\cdot)$, let $\widetilde{\mathbb{P}}^t_{x,y}(\cdot)$ be the law of $(X_{rt}^{(x,y,t)})_{0\leqslant r\leqslant 1}$. It is sufficient to prove that $(x,y,t)\mapsto\widetilde{\mathbb{P}}^t_{x,y}(\cdot)$ is continuous. For $v\in[0,1]$, let $\widetilde{\mathbb{P}}^{t,v}_{x,y}(\cdot)$ be the law of $(X_{rt}^{(x,y,t)})_{0\leqslant r\leqslant v}$. Let $\widetilde{\mathbb{P}}^{t,v}_{x}(\cdot)$ be the law of the Markovian path $(X_{rt})_{0\leqslant r\leqslant v}$ starting from x. For $v\in[0,1)$ we have the following absolute continuity relationship:

$$d\widetilde{\mathbb{P}}^{t,v}_{x,y}=1_{vt<\zeta}\frac{p_{(1-v)t}(X_{vt},y)}{p_t(x,y)}d\widetilde{\mathbb{P}}^{t,v}_{x} \tag{2.2.13}$$

Let $(J_n)_{n\geqslant 0}$ be an increasing sequence of compact subintervals of I such that $I=\bigcup_{n\geqslant 0}J_n$. Let T_n be the first exit time from J_n. Let f_n be continuous compactly supported function on I such that $0\leqslant f_n\leqslant 1$ and $f_{n|J_n}\equiv 1$. We can further assume that the sequence $(f_n)_{n\geqslant 0}$ is non-decreasing. The map

$$(x,y,t)\mapsto f_n(\sup_{[0,vt]}X)f_n(\inf_{[0,vt]}X)d\widetilde{\mathbb{P}}^{t,v}_{x}$$

is weakly continuous. Let $(x_j,y_j,t_j)_{j\geqslant 0}$ be a sequence converging to (x,y,t). Let F be a continuous bounded functional on $\mathcal{C}([0,v])$. Then applying (2.2.13) we get:

$$\widetilde{\mathbb{P}}^{t_j,v}_{x_j,y_j}(f_n(\sup_{[0,v]}\gamma)f_n(\inf_{[0,v]}\gamma)F(\gamma))-\widetilde{\mathbb{P}}^{t,v}_{x,y}(f_n(\sup_{[0,v]}\gamma)f_n(\inf_{[0,v]}\gamma)F(\gamma))= \tag{2.2.14}$$

$$\widetilde{\mathbb{P}}^{t_j,v}_{x_j}\left(\frac{p_{(1-v)t}(\gamma(v),y)}{p_t(x,y)}f_n(\sup_{[0,v]}\gamma)f_n(\inf_{[0,v]}\gamma)F(\gamma)\right) \tag{2.2.15}$$

$$-\widetilde{\mathbb{P}}^{t,v}_{x}\left(\frac{p_{(1-v)t}(\gamma(v),y)}{p_t(x,y)}f_n(\sup_{[0,v]}\gamma)f_n(\inf_{[0,v]}\gamma)F(\gamma)\right) \tag{2.2.16}$$

$$+\widetilde{\mathbb{P}}^{t_j,v}_{x_j}\left(\frac{p_{(1-v)t_j}(\gamma(v),y_j)}{p_{t_j}(x_j,y_j)}f_n(\sup_{[0,v]}\gamma)f_n(\inf_{[0,v]}\gamma)F(\gamma)\right) \tag{2.2.17}$$

$$-\widetilde{\mathbb{P}}^{t_j,v}_{x_j}\left(\frac{p_{(1-v)t}(\gamma(v),y)}{p_t(x,y)}f_n(\sup_{[0,v]}\gamma)f_n(\inf_{[0,v]}\gamma)F(\gamma)\right) \tag{2.2.18}$$

Since $\frac{p_{(1-v)t}(\cdot,y)}{p_t(x,y)}$ is continuous and bounded on J_n, (2.2.15)−(2.2.16) converges to 0. Moreover for j large enough, $\frac{p_{(1-v)t_j}(\cdot,y_j)}{p_{t_j}(x_j,y_j)}$ is uniformly close on J_n to $\frac{p_{(1-v)t}(\cdot,y)}{p_t(x,y)}$. Thus the difference (2.2.17)−(2.2.18) converges to 0 and finally (2.2.14) converges to 0. Let $n_0\in\mathbb{N}$ and $n\geqslant n_0$. Then

$$\begin{aligned}\widetilde{\mathbb{P}}^{t_j,v}_{x_j,y_j}(1-f_n(\sup_{[0,v]}\gamma)f_n(\inf_{[0,v]}\gamma))&=1-\widetilde{\mathbb{P}}^{t_j,v}_{x_j,y_j}(f_n(\sup_{[0,v]}\gamma)f_n(\inf_{[0,v]}\gamma))\\&\leqslant 1-\widetilde{\mathbb{P}}^{t_j,v}_{x_j,y_j}(f_{n_0}(\sup_{[0,v]}\gamma)f_{n_0}(\inf_{[0,v]}\gamma))\to 1-\widetilde{\mathbb{P}}^{t,v}_{x,y}(f_{n_0}(\sup_{[0,v]}\gamma)f_{n_0}(\inf_{[0,v]}\gamma))\end{aligned}$$

and consequently

$$\lim_{n\to+\infty}\limsup_{j\to+\infty}\widetilde{\mathbb{P}}^{t_j,v}_{x_j,y_j}(1-f_n(\sup_{[0,v]}\gamma)f_n(\inf_{[0,v]}\gamma))=0$$

It follows that

$$\lim_{j\to+\infty}\widetilde{\mathbb{P}}^{t_j,v}_{x_j,y_j}(F(\gamma))=\widetilde{\mathbb{P}}^{t,v}_{x,y}(F(\gamma))$$

From this we get that the law of any finite-dimensional family of marginals of $\widetilde{\mathbb{P}}^t_{x,y}(\cdot)$ depends continuously on (x,y,t). To conclude we need a tightness result for $(x,y,t)\mapsto\widetilde{\mathbb{P}}^t_{x,y}(\cdot)$. We have already tightness for $(x, y, t) \mapsto \widetilde{\mathbb{P}}^{t,v}_{x,y}(\cdot)$. The image of $\widetilde{\mathbb{P}}^{t}_{x,y}(\cdot)$ through time reversal is $\widetilde{\mathbb{P}}^{t}_{y,x}(\cdot)$. So we also have tightness on intervals $[1 - v', 1]$ where $0 < v' < 1$. But if $v + v' > 1$, tightness on $[0, v]$ and on $[1 - v', 1]$ implies tightness on $[0, 1]$. This concludes. The article [**CB11**] contains an error in the proof of the tightness of bridge measures in the neighbourhood of the endpoint. □

2.2.3. "Generators" with creation of mass. In this section we consider more general operators

$$L = \frac{1}{m(x)}\frac{d}{dx}\left(\frac{1}{w(x)}\frac{d}{dx}\right) + \nu \tag{2.2.19}$$

with zero Dirichlet boundary conditions on ∂I, where ν is a signed measure on I which is no longer assumed to be negative. We set

$$L^{(0)} := L - \nu$$

In the sequel we may call L "generator" even in case the semi-group $(e^{tL})_{t\geqslant 0}$ does not make sense. Our main goal in this subsection is to characterize through a positivity condition the subclass of operators of form (2.2.19) that are equivalent up to a conjugation to the generator of a diffusion of form (2.2.11).

We will consider several kinds of transformations on operators of the form (2.2.19). First, the conjugation: Let h be a positive continuous function on I such that $\frac{d^2h}{dx^2}$ is a signed measure. We call $Conj(h, L)$ the operator

$$Conj(h, L) = \frac{1}{h(x)^2 m(x)}\frac{d}{dx}\left(\frac{h(x)^2}{w(x)}\frac{d}{dx}\right) + \nu + \frac{1}{h}L^{(0)}h$$

If f is smooth function compactly supported in I then

$$Conj(h, L)f = h^{-1}L(hf)$$

We will call $Conj(h, L)$ the conjugation of L by h. h may not be harmonic ($Lh = 0$) or superharmonic ($Lh \leqslant 0$) and L is not necessarily the generator of a diffusion.

Second, the change of scale: If A is a $\mathcal{C}^1$ function on I such that $\frac{dA}{dx} > 0$ and $\frac{d^2A}{dx^2} \in \mathbb{L}^{\infty}_{loc}(I)$ and $(\gamma(t))_{0\leqslant t\leqslant T}$ a continuous path in I, then we will set $Scale_A(\gamma)$ to be the continuous path $(A(\gamma(s)))_{0\leqslant t\leqslant T}$ in $A(I)$. Let $Scale^{gen}_A(L)$ be the operator on functions on $A(I)$ with zero Dirichlet boundary conditions induced by this change of scale:

$$Scale^{gen}_A(L) = \frac{1}{m\circ A^{-1}(a)}\frac{d}{da}\left(\frac{1}{w\circ A^{-1}(a)}\frac{d}{da}\right) + A_*\nu$$

where $A_*\nu$ is the push-forward of the measure ν by A.

Third, the change of time: If V is positive continuous on I then we can consider the change of time $ds = V(\gamma(t))dt$. Let $Speed_V$ be the corresponding transformation on paths. The corresponding "generator" is $\frac{1}{V}L$.

Finally, the restriction: if $\widetilde{I}$ is an open subinterval of I then set $L_{|\widetilde{I}}$ to be the operator L acting on functions supported in $\widetilde{I}$ and with zero Dirichlet conditions on $\partial\widetilde{I}$.

For the analysis of L we will use a bit of spectral theory: If $[x_0, x_1]$ is a compact interval of $\mathbb{R}$ and $\tilde{m}$, $\tilde{w}$ are positive continuous functions on $[x_0, x_1]$, then the operator $\frac{1}{\tilde{m}(x)}\frac{d}{dx}\left(\frac{1}{\tilde{w}(x)}\frac{d}{dx}\right)$ with zero Dirichlet boundary conditions has a discrete spectrum of negative eigenvalues. Let $-\tilde{\lambda}_1$ be the first eigenvalue. It is simple. According to Sturm-Liouville theory (see for instance [**Tes12**], section 5.5) we have the following picture:

PROPERTY 2.2.6. *Let $\lambda > 0$ and u a solution to*

$$\frac{1}{\tilde{m}}\frac{d}{dx}\left(\frac{1}{\tilde{w}}\frac{d}{dx}\right) + \lambda u = 0$$

with initial conditions $u(x_0) = 0, \dfrac{du}{dx}(x_0) > 0$.

(i) *If u is positive on (x_0, x_1) and $u(x_1) = 0$ then $\lambda = \tilde{\lambda}_1$ and u is the fundamental eigenfunction.*
(ii) *If u is positive on $(x_0, x_1]$ then $\lambda < \tilde{\lambda}_1$*
(iii) *If u changes sign on (x_0, x_1) then $\lambda > \tilde{\lambda}_1$*

Next we state and prove the main result of this section.

PROPOSITION 2.2.7. *The following two conditions are equivalent:*

(i) *There is a positive continuous function u on I satisfying $Lu = 0$.*
(ii) *For any f smooth compactly supported in I*

$$\int_I (L^{(0)}f)(x)f(x)m(x)dx + \int_I f(x)^2 m(x)\nu(dx) \leqslant 0 \tag{2.2.20}$$

PROOF. (i) implies (ii): First observe that the equation $Lu = 0$ reduces through a change of scale to an equation of the form (2.2.1). Let u be given by condition (i). Let $\widetilde{L} := Conj(u, L)$. Since $Lu = 0$, $\widetilde{L}$ is a generator of a diffusion without killing measure. Let $\tilde{m}(x) := u^2(x)m(x)$. Then for all g smooth compactly supported in I, $\int_I (\widetilde{L}g)(x)g(x)\tilde{m}(x)dx \leqslant 0$. But

$$\int_I (\widetilde{L}g)(x)g(x)\tilde{m}(x)dx = \int_I (L^{(0)}(ug))(x)(ug)(x)m(x)dx + \int_I (ug)(x)^2 m(x)\nu(dx)$$

Thus (2.2.20) holds for all f positive compactly supported in I such that $u^{-1}f$ is smooth. By density arguments, this holds for general smooth f.

(ii) implies (i): First we will show that for every compact subinterval J of I there is a positive continuous function u_J on $\mathring{J}$ satisfying $Lu_J = 0$ on $\mathring{J}$. Let J be such an interval. According to lemma 2.2.2 there is $\lambda > 0$ and u_λ positive continuous on J satisfying $Lu_\lambda - \lambda u_\lambda = 0$ on J. Let $L_\lambda := Conj(u_\lambda, L_{|\mathring{J}})$. Then

$$L_\lambda = \frac{1}{u_\lambda^2 m}\frac{d}{dx}\left(\frac{u_\lambda^2}{w}\frac{d}{dx}\right) + \lambda$$

Let $L_\lambda^{(0)} := L_\lambda - \lambda$. $L^{(0)}$ is the generator of a diffusion on $\mathring{J}$. We can apply the standard spectral theorem to $L_\lambda^{(0)}$. Let $-\lambda_1$ be its fundamental eigenvalue. $L_\lambda^{(0)} + \lambda = L_\lambda$ is a non-positive operator because it is a conjugate of $L_{|J}$ which satisfies condition (ii). This implies that $\lambda \leqslant \lambda_1$. Let $\tilde{u}$ be a solution of $L_\lambda^{(0)}\tilde{u} + \lambda\tilde{u} = 0$ with initial conditions $\tilde{u}(\min J) = 0$ and $\frac{d\tilde{u}}{dx}(\min J) > 0$. Since $\lambda \leqslant \lambda_1$, according to property 2.2.6, $\tilde{u}$ is positive on $\mathring{J}$. We set $u_J := u_\lambda \tilde{u}$. Then u_J is positive continuous on $\mathring{J}$ and satisfies $Lu_J = 0$. This finishes the proof of the first step.

Now consider a fixed point x_0 in I and $(J_n)_{n\geqslant 0}$ an increasing sequence of compact subintervals of I such that $x_0 \in \mathring{J}_0$ and $\bigcup_{n\geqslant 0} J_n = I$. Let u_{J_n} be a positive L-harmonic function on $\mathring{J}_n$. We may assume that $u_{J_n}(x_0) = 1$. The sequence $\left(\frac{du_{J_n}}{dx}(x_0^+)\right)_{n\geqslant 0}$ is bounded from below. Otherwise some of the u_{J_n} would change sign on $I \cap (x_0, +\infty)$. Similarly, since none of the u_{J_n} changes sign on $I \cap (-\infty, x_0)$, $\left(\frac{du_{J_n}}{dx}(x_0^+)\right)_{n\geqslant 0}$ is bounded from above. Let

v be an accumulation value of the sequence $\left(\frac{du_{J_n}}{dx}(x_0^+)\right)_{n\geqslant 0}$. Then the L-harmonic function satisfying the initial conditions $u(x_0) = 1$ and $\frac{du}{dx}(x_0^+) = v$ is positive on I.

We will divide the operators of the form (2.2.20) in two sets: $\mathfrak{D}^{0,-}$ for those that satisfies the constraints of the proposition 2.2.7 and $\mathfrak{D}^+$ for those that don't. $\mathfrak{D}^{0,-}$ is made exactly of operators that are equivalent up to a conjugation to the generator of a diffusion. We will subdivide the set $\mathfrak{D}^{0,-}$ in two: $\mathfrak{D}^-$ for the operators that are a conjugate of the generator of a transient diffusion and $\mathfrak{D}^0$ for those that are a conjugate of the generator of a recurrent diffusion. These two subclasses are well defined since a transient diffusion can not be a conjugate of a recurrent one. Observe that each of $L \in \mathfrak{D}^-$, $\mathfrak{D}^0$ and $\mathfrak{D}^+$ is stable under conjugations, changes of scale and of speed. Operators in $\mathfrak{D}^-$ and $\mathfrak{D}^0$ do not need to be generators of transient or recurrent diffusions themselves. For instance consider on $\mathbb{R}$

$$L = \frac{1}{2}\frac{d^2}{dx^2} + a_+\delta_1 - a_-\delta_{-1}$$

where $a_+, a_- > 0$. If $3a_+ - a_- > 0$ then $L \in \mathfrak{D}^+$, if $3a_+ - a_- = 0$ then $L \in \mathfrak{D}^0$, if $3a_+ - a_- < 0$ then $L \in \mathfrak{D}^-$. □

If $L \in \mathfrak{D}^{0,-}$, the semi-group $(e^{tL})_{t\geqslant 0}$ is well defined. Indeed, let X be the diffusion on I of generator $L^{(0)}$ and ζ the first time it hits the boundary of I or blows up to infinity. Let u be a positive L-harmonic function and $\widetilde{L} := Conj(u, L)$. $\widetilde{L}$ is the generator of a diffusion $\widetilde{X}$ on I without killing measure. Let $\tilde{\zeta}$ be the first time $\widetilde{X}$ hits the boundary of I or blows up to infinity. Using Girsanov's theorem, one can show that for any F positive measurable functional on paths, $x \in I$ and $t > 0$ the following equality holds:

$$\mathbb{E}_x\left[1_{t<\zeta}\exp\left(\int_I \ell_t^y(X)m(y)\nu(dy)\right)F((X_s)_{0\leqslant s\leqslant t})\right] = \frac{1}{u(x)}\mathbb{E}_x\left[1_{t<\tilde{\zeta}}u(\widetilde{X}_t)F((\widetilde{X}_s)_{0\leqslant s\leqslant t})\right]$$

In case $L \in \mathfrak{D}^-$, let $(G_{\widetilde{L}}(x,y))_{x,y\in I}$ be the Green's function of $\widetilde{L}$ relative to the measure $u(x)^2m(x)dx$. Then L has a Green's function $(G_L(x,y))_{x,y\in I}$ that equals

$$G_L(x,y) = \mathbb{E}_x\left[\int_0^\zeta \exp\left(\int_I \ell_t^z(X)m(z)\nu(dz)\right)d_t\ell_t^y(X)\right] = u(x)u(y)G_{\widetilde{L}}(x,y)$$

For $L \in \mathfrak{D}^-$, the Green's functions G_L satisfy the following resolvent identities

LEMMA 2.2.8. *If $L \in \mathfrak{D}^-$ and $\tilde{\nu}$ is a signed measure with compact support on I such that $L + \tilde{\nu} \in \mathfrak{D}^-$, then for all $x, y \in I$*

$$\begin{aligned} G_{L+\tilde{\nu}}(x,y) - G_L(x,y) &= \int_I G_{L+\tilde{\nu}}(x,z)G_L(z,y)m(z)\tilde{\nu}(dz) \\ &= \int_I G_L(x,z)G_{L+\tilde{\nu}}(z,y)m(z)\tilde{\nu}(dz) \end{aligned}$$

PROOF. We decompose L as $L = L^{(0)} + \nu$ where $L^{(0)}$ does not contain measures and ν is a signed measure on I. Let $(X_t)_{0\leqslant t<\zeta}$ be the diffusion of generator $L^{(0)}$. Then

$$G_L(x,y) = \mathbb{E}_x\left[\int_0^\xi \exp\left(\int_I \ell_t^a(X)m(a)\nu(da)\right)d_t\ell_t^y(X)\right]$$

$$G_{L+\tilde{\nu}}(x,y) = \mathbb{E}_x\left[\int_0^\xi \exp\left(\int_I \ell_t^a(X)m(a)(\nu+\tilde{\nu})(da)\right)d_t\ell_t^y(X)\right]$$

and

$$\exp\left(\int_I \ell_t^a(X)m(a)(\nu+\tilde{\nu})(da)\right) - \exp\left(\int_I \ell_t^a(X)m(a)\nu(da)\right)$$
$$= \exp\left(\int_I \ell_t^a(X)m(a)\nu(da)\right) \times \left(\exp\left(\int_I \ell_t^a(X)m(a)\tilde{\nu}(da)\right) - 1\right)$$
$$= \exp\left(\int_I \ell_t^a(X)m(a)\nu(da)\right) \int_I \int_0^t \exp\left(\int_I \ell_s^a(X)m(a)\tilde{\nu}(da)\right) d_s\ell_s^z(X)m(z)\tilde{\nu}(dz)$$

Thus $G_{L+\tilde{\nu}}(x,y) - G_L(x,y)$ equals

(2.2.21)
$$\mathbb{E}_x\left[\int_I \int_0^\xi \int_0^t \exp\left(\int_I m(a)(\ell_t^a(X)\nu(da) + \ell_s^a(X)\tilde{\nu}(da))\right) d_s\ell_s^z(X) d_s\ell_t^y(X)m(z)\tilde{\nu}(dz)\right]$$

We would like to interchange $\mathbb{E}_x[\cdot]$ and $\int_I(\cdot)m(z)\tilde{\nu}(dz)$. Let $z \in I$ and $(X_t^{(x)})_{0\leqslant t<\zeta_x}$, $(X_t^{(z)})_{0\leqslant t<\zeta_z}$ be two independent diffusions of generator $L^{(0)}$ starting in x respectively z. Applying Markov property, we get

$$\mathbb{E}_x\left[\int_0^\xi \int_0^t \exp\left(\int_I m(a)(\ell_t^a(X)\nu(da) + \ell_s^a(X)\tilde{\nu}(da))\right) d_s\ell_s^z(X) d_s\ell_t^y(X)\right]$$
$$= \mathbb{E}\left[\int_0^{\zeta_x} \int_0^{\zeta_z} \exp\left(\int_I m(a)(\ell_s^a(X^{(x)})(\nu+\tilde{\nu})(da)\right)\right.$$
$$\left.\times \exp\left(\int_I m(a)(\ell_u^a(X^{(z)})\nu(da)\right) d_u\ell_u^z(X^{(z)}) d_s\ell_s^x(X^{(x)})\right]$$
$$= G_{L+\tilde{\nu}}(x,z)G_L(z,y)$$

Since $\tilde{\nu}$ has compact support

$$\mathbb{E}_x\left[\int_I \int_0^\xi \int_0^t \exp\left(\int_I m(a)(\ell_t^a(X)\nu(da) + \ell_s^a(X)\tilde{\nu}(da))\right) d_s\ell_s^z(X) d_s\ell_t^y(X)m(z)|\tilde{\nu}|(dz)\right]$$
$$= \int_I \mathbb{E}_x\left[\int_0^\xi \int_0^t \exp\left(\int_I m(a)(\ell_t^a(X)\nu(da) + \ell_s^a(X)\tilde{\nu}(da))\right)\right.$$
$$\left. d_s\ell_s^z(X) d_s\ell_t^y(X)\right] m(z)|\tilde{\nu}|(dz)$$
$$= \int_I G_{L+\tilde{\nu}}(x,z)G_L(z,y)m(z)|\tilde{\nu}|(dz) < +\infty$$

Thus in (2.2.21) we can interchange $\mathbb{E}_x[\cdot]$ and $\int_I(\cdot)m(z)\tilde{\nu}(dz)$ and get

$$G_{L+\tilde{\nu}}(x,y) - G_L(x,y) = \int_I G_{L+\tilde{\nu}}(x,z)G_L(z,y)m(z)\tilde{\nu}(dz)$$

Since L and $L+\tilde{\nu}$ play symmetric roles, we also have

$$G_L(x,y) - G_{L+\tilde{\nu}}(x,y) = \int_I G_L(x,z)G_{L+\tilde{\nu}}(z,y)m(z)(-\tilde{\nu})(dz)$$

□

The discrete analogue of the sets $\mathfrak{D}^-$, $\mathfrak{D}^0$ and $\mathfrak{D}^+$ are symmetric matrices with non-negative off-diagonal coefficients inducing a connected transition graph, with the highest eigenvalue that is respectively negative, zero and positive. However in continuous case the sets $L \in \mathfrak{D}^-$, $\mathfrak{D}^0$ and $\mathfrak{D}^+$ can not be defined spectrally because for operators from $L \in \mathfrak{D}^-$ and $\mathfrak{D}^+$ the maximum of the spectrum can also equal zero. However the next result shows that the sets $\mathfrak{D}^-$ and $\mathfrak{D}^+$ are stable under small perturbations of the measure ν and that $\mathfrak{D}^0$ is not.

PROPOSITION 2.2.9. (i) *If $L \in \mathfrak{D}^0$ and κ is a non-zero positive Radon measure on I then $L - \kappa \in \mathfrak{D}^-$ and $L + \kappa \in \mathfrak{D}^+$.*

(ii) *If $L \in \mathfrak{D}^-$ and J is a compact subinterval of I then there is $K > 0$ such that for any positive measure κ supported in J satisfying $\kappa(J) < K$ we have $L + \kappa \in \mathfrak{D}^-$.*

(iii) *If $L \in \mathfrak{D}^+$ then there is $K > 0$ such that for any positive finite measure κ satisfying $\kappa(I) < K$ we have $L - \kappa \in \mathfrak{D}^+$.*

(iv) *If $L \in \mathfrak{D}^+$, there is a positive Radon measure κ on I such that $L - \kappa \in \mathfrak{D}^0$.*

(v) *Let $L \in \mathfrak{D}^+$ and $x_0 < x_1 \in I$. Then $L_{|(x_0,x_1)} \in \mathfrak{D}^0$ if and only if there is an L-harmonic function u positive on (x_0, x_1) and zero in x_0 and x_1.*

PROOF. (i): Consider h positive continuous on I such that $Conj(h, L)$ is the generator of a recurrent diffusion. Since $Conj(h, L - \kappa) = Conj(h, L) - \kappa$, $Conj(h, L - \kappa)$ is the generator of a diffusion killed at rate κ and thus $L - \kappa \in \mathfrak{D}^-$. Similarly we can not have $L + \kappa \in \mathfrak{D}^{0,-}$ because this would mean $L = (L + \kappa) - \kappa \in \mathfrak{D}^-$.

(ii): Without loss of generality we may assume that L is the generator of a transient diffusion and that it is at natural scale, that is $L = \frac{1}{m(x)}\frac{d^2}{dx^2}$. Since the diffusion is transient, $I \neq \mathbb{R}$. We may assume that $x_0 := \inf I > -\infty$. Write $J = [x_1, x_2]$. Let κ be a positive measure supported in $[x_1, x_2]$. Let u be the solution to $Lu + u\kappa = 0$ with the initial conditions $u(x_0) = 0$, $\frac{du}{dx}(x_0^+) = 1$. u is affine on $[x_0, x_1]$ and on $[x_2, \sup I)$. On $[x_1, x_2]$ u is bounded from above by $x_2 - x_0$. Thus, if

$$\kappa([x_1, x_2]) \leqslant \frac{\min_{[x_1,x_2]} m}{(x_2 - x_0)}$$

then u is non-decreasing on I and hence positive. This implies that $L + \kappa \in \mathfrak{D}^{0,-}$. By the point (i) of current proposition, if $\kappa([x_1, x_2]) < \frac{\min_{[x_1,x_2]} m}{(x_2 - x_0)}$ then $L + \kappa \in \mathfrak{D}^-$.

(iii): By definition there is f smooth compactly supported in I such that (2.2.20) does not hold for f. Let U be the value of the left-hand side in (2.2.20). $U > 0$. If κ is a positive finite measure on I satisfying

$$\kappa(I) < \frac{U}{\|f\|_\infty^2 \max_{Supp f} m}$$

then if we replace ν by $\nu - \kappa$ in (2.2.20), keeping the same function f, we still get something positive. Thus $L - \kappa \in \mathfrak{D}^+$.

(iv): Let f be a smooth function compactly supported in I such that (2.2.20) does not hold for f. Let J be a compact subinterval of I containing the support of f. The set

$$\{s \in [0, 1] | L - \nu_+ + s\, 1_J \nu_+ \in \mathfrak{D}^-\}$$

is not empty because it contains 0, and open by proposition 2.2.9 (ii). Let s_{max} by its supremum. Then $s_{max} < 1$ and $L - \nu_+ + s_{max} 1_J \nu_+ \in \mathfrak{D}^0$. Then

$$\kappa := 1_{I \setminus J} \nu_+ + (1 - s_{max}) 1_J \nu_+$$

is appropriate.

(v): First assume that there is such a function u. Then by definition $L_{|(x_0,x_1)} \in \mathfrak{D}^{0,-}$. $Conj(u, L_{|(x_0,x_1)})$ does not have any killing measure and the derivative of its natural scale function is $\frac{w}{u^2}$. It is not integrable in the neighbourhood of x_0 or x_1. Thus the corresponding diffusion never hits x_0 or x_1. This means that it is recurrent. Conversely, assume that $L_{|(x_0,x_2)} \in \mathfrak{D}^0$. Let u be a solution to $Lu = 0$ satisfying $u(x_0) = 0$ and $\frac{du}{dx}(x_0^+) > 0$. If u

changed its sign on (x_0, x_1) then according to the preceding we would have $L_{|(x_0,x_1)} \in \mathfrak{D}^+$. If u were positive on an interval larger that (x_0, x_1) we would have $L_{|(x_0,x_1)} \in \mathfrak{D}^-$. The only possibility is that u is positive on (x_0, x_1) and zero in x_1. □

2.3. Measure on loops and its basic properties

2.3.1. Spaces of loops. In this chapter, in the section 2.3.3, we will introduce the infinite measure μ^* on loops which is at the center of this work. Prior to this, in the section 2.3.2 we will introduce measures $\mu^{x,y}$ on finite life-time paths which will be instrumental for defining μ^*. In the sections 2.3.4, 2.3.5, 2.3.7, 2.3.8 will be explored different aspects of μ^*. In the section 2.3.6 we will extend the Vervaat's Brownian bridge to Brownian excursion transformation to general diffusions. This generalisation can be easily interpreted in terms of measure μ^* and is related to the results of section 2.3.5. In the section 2.3.1 we will introduce the spaces of paths and loops on witch will be defined the measures we will consider throughout the paper.

First we will consider continuous, time parametrized, paths on $\mathbb{R}$, $(\gamma(t))_{0 \leqslant t \leqslant T(\gamma)}$, with finite life-time $T(\gamma) \in (0, +\infty)$. Given two such paths $(\gamma(t))_{0 \leqslant t \leqslant T(\gamma)}$ and $(\gamma'(t))_{0 \leqslant t \leqslant T(\gamma')}$, a natural distance between them is

$$d_{paths}(\gamma, \gamma) := |\log(T(\gamma)) - \log(T(\gamma'))| + \max_{v \in [0,1]} |\gamma(vT(\gamma)) - \gamma'(vT(\gamma'))|$$

A rooted loop in $\mathbb{R}$ will be a continuous finite life-time path $(\gamma(t))_{0 \leqslant t \leqslant T(\gamma)}$ such that $\gamma(T(\gamma)) = \gamma(0)$ and $\mathfrak{L}$ will stand for the space of such loops. $\mathfrak{L}$ endowed with the metric d_{paths} is a Polish space. In the sequel we will use the corresponding Borel σ-algebra, $\mathcal{B}_{\mathfrak{L}}$, for the definition of measures on $\mathfrak{L}$. For $v \in [0, 1]$ we define a parametrisation shift transformation $shift_v$ on $\mathfrak{L}$: $shift_v(\gamma) = \tilde{\gamma}$ where $T(\tilde{\gamma}) = T(\gamma)$ and

$$\tilde{\gamma}(t) = \begin{cases} \gamma(vT(\gamma) + t) & \text{if } t \leqslant (1-v)T(\gamma) \\ \gamma(t - (1-v)T(\gamma)) & \text{if } t \geqslant (1-v)T(\gamma) \end{cases}$$

We introduce an equivalence relation on $\mathfrak{L}$: $\gamma \sim \gamma$ if $T(\gamma') = T(\gamma)$ and there is $v \in [0, 1]$ such that $\gamma' = shift_v(\gamma)$. We call the quotient space $\mathfrak{L} \diagup \sim$ the space of unrooted loops, or just loops, and denote it $\mathfrak{L}^*$. Let π be the projection $\pi : \mathfrak{L} \to \mathfrak{L}^*$. There is a natural metric $\delta_{\mathfrak{L}^*}$ on $\mathfrak{L}^*$:

$$d_{\mathfrak{L}^*}(\pi(\gamma), \pi(\gamma')) := \min_{v \in [0,1]} d_{paths}(shift_v(\gamma), \gamma')$$

$(\mathfrak{L}^*, d_{\mathfrak{L}^*})$ is a Polish space and π is continuous. For defining measures on $\mathfrak{L}^*$ we will use its Borel σ-algebra, $\mathcal{B}_{\mathfrak{L}^*}$. $\pi^{-1}(\mathcal{B}_{\mathfrak{L}^*})$, the inverse image of $\mathcal{B}_{\mathfrak{L}^*}$ by π, is a sub-algebra of $\mathcal{B}_{\mathfrak{L}}$.

In the sequel we will consider paths and loops that have a continuous family of local times $(\ell_t^x(\gamma))_{x \in \mathbb{R}, 0 \leqslant t \leqslant T(\gamma)}$ relative to a measure $m(x)dx$ such that for any positive measurable function f on $\mathbb{R}$ and any $t \in [0, T(\gamma)]$

$$\int_0^t f(\gamma(s))ds = \int_I \ell_t^x(\gamma) m(x) dx$$

We will simply write $\ell^x(\gamma)$ for $\ell_{T(\gamma)}^x(\gamma)$.

In the sequel we will also consider transformations on paths and loops and the images of different measures by these transformation. We will use everywhere the following notation: If $\mathcal{E}$ and $\mathcal{E}'$ are two measurable spaces, $\varphi : \mathcal{E} \mapsto \mathcal{E}'$ a measurable map and η a positive measure on $\mathcal{E}$, $\varphi_* \eta$ will be the measure on $\mathcal{E}'$ obtained as the image of η trough φ.

2.3.2. Measures $\mu^{x,y}$ on finite life-time paths. First we recall the framework that Le Jan used in [**Jan11**]: $\mathcal{G} = (V, E)$ is a finite connected undirected graph. $L_{\mathcal{G}}$ is the generator of a symmetric Markov jump process with killing on $\mathcal{G}$. $m_{\mathcal{G}}$ is the duality measure for $L_{\mathcal{G}}$. $(p_t^{\mathcal{G}}(x,y))_{x,y\in V,t\geqslant 0}$ is the family of transition densities of the jump process and $(\mathbb{P}_{x,y}^{\mathcal{G},t}(\cdot))_{x,y\in V,t\geqslant 0}$ the family of bridge probability measures. The measure on rooted loops associated with $L_{\mathcal{G}}$ is

$$\mu_{L_{\mathcal{G}}}(\cdot) = \int_{t>0} \sum_{x\in V} \mathbb{P}_{x,x}^{\mathcal{G},t}(\cdot) p_t^{\mathcal{G}}(x,x) m_{\mathcal{G}}(x)\frac{dt}{t} \tag{2.3.1}$$

$\mu^*_{L_{\mathcal{G}}}$ is the image of $\mu_{L_{\mathcal{G}}}$ by the projection on unrooted loops. The definition of $\mu^*_{L_{\mathcal{G}}}$ is the exact formal analogue of the definition used in [**LW04**] for the loops of the two-dimensional Brownian motion. In [**Jan11**] also appear variable life-time bridge measures $(\mu^{x,y}_{L_{\mathcal{G}}})_{x,y\in V}$ which are related to $\mu^*_{L_{\mathcal{G}}}$:

$$\mu^{x,y}_{L_{\mathcal{G}}}(\cdot) = \int_0^{+\infty} \mathbb{P}_{x,y}^{\mathcal{G},t}(\cdot) p_t^{\mathcal{G}}(x,y)dt \tag{2.3.2}$$

In this subsection we will define and give the important properties of the formal analogue of the measures $\mu^{x,y}_{L_{\mathcal{G}}}$ in case of one-dimensional diffusions. In the next section 2.3.2 we will do the same with the measure on loops $\mu^*_{L_{\mathcal{G}}}$.

I is an open interval of $\mathbb{R}$. $(X_t)_{0\leqslant t<\zeta}$ is a diffusion on I with a generator L of the form (2.2.11). We use the notations of the section 2.2.1. Let $x, y \in I$. Following the pattern of (2.3.2) we define:

DEFINITION 2.1.

$$\mu^{x,y}_{L}(\cdot) := \int_0^{+\infty} \mathbb{P}_{x,y}^{t}(\cdot) p_t(x,y)dt$$

We will write $\mu^{x,y}$ instead of $\mu^{x,y}_L$ whenever there is no ambiguity on L. The definition of $\mu^{x,y}$ depends on the choice of m, but $m(y)\mu^{x,y}$ does not. Measures $\mu^{x,y}$ were first introduced by Dynkin in [**Dyn84a**] and enter the expression of Dynkin's isomorphism between the Gaussian Free Field and the local times of random paths. Pitman and Yor studied this measures in [**PY96**] in the setting of one-dimensional diffusions without killing measure ($\kappa = 0$). Next we give a handy representation of $\mu^{x,y}$ in the setting of one-dimensional diffusions. It was observed and proved by Pitman and Yor in case $\kappa = 0$. We consider the general case.

PROPOSITION 2.3.1. *Let F be a non-negative measurable functional on the space of variable life-time paths starting from x. Then*

$$\mu^{x,y}(F(\gamma)) = \mathbb{E}_x\left[\int_0^{\zeta} F((X_s)_{0\leqslant s\leqslant t}) d_t\ell_t^y(X)\right] \tag{2.3.3}$$

Equivalently

$$\mu^{x,y}(F(\gamma)) = \mathbb{E}_x\left[\int_0^{\ell_\zeta^y(X)} F((X_s)_{0\leqslant s\leqslant \tau_l^y})dl\right]$$

where $\tau_l^y := \inf\{t \geqslant 0 | \ell_t^z(X) > l\}$.

PROOF. It is enough to prove this for F non-negative continuous bounded functional witch takes value 0 if either the life-time of the paths exceeds some value $t_{max} < +\infty$ or of it is inferior to some value t_{min} or if the endpoint of the path lies out of a compact subinterval

$[z_1, z_2]$ of I. For $j \leqslant n \in \mathbb{N}$, set $t_{j,n} := t_{min} + \frac{j(t_{max}-t_{min})}{n}$ and $\Delta t_n := \frac{t_{max}-t_{min}}{n}$. Almost surely $\int_0^\zeta F((X_s)_{0\leqslant s\leqslant t})d_t l_t^y$ is a limit as $n \to +\infty$ of

$$\sum_{j=0}^{n-1} F((X_s)_{0\leqslant s\leqslant t_{j,n}})(\ell^y_{t_{j+1,n}\wedge\zeta}(X) - \ell^y_{t_{j,n}\wedge\zeta}(X)) \tag{2.3.4}$$

Moreover (2.3.4) is dominated by $\|F\|_\infty l^y_{t_{max}\wedge\zeta}$. It follows that the expectations converge too. Using the Markov property and (2.2.12), we get that the expectation of (2.3.4) equals

$$\sum_{j=0}^{n-1} \int_{z\in I}\int_0^{\Delta t_n} \mathbb{P}^{t_{j,n}}_{x,z}\left(F((X_s)_{0\leqslant s\leqslant t_{j,n}})\right) p_{t_{j,n}}(x,z)p_r(z,y)drm(z)dz \tag{2.3.5}$$

Using the fact that $p_r(\cdot,\cdot)$ is symmetric, we can rewrite (2.3.5) as

$$\int_{z_1}^{z_2}\Big(\sum_{j=0}^{n-1}\Delta t_n \mathbb{P}^{t_{j,n}}_{x,z}\left(F((X_s)_{0\leqslant s\leqslant t_{j,n}})\right)p_{t_{j,n}}(x,z)\Big)\frac{1}{\Delta t_n}\int_0^{\Delta t_n} p_r(y,z)drm(z)dz \tag{2.3.6}$$

As $n \to +\infty$ the measure $\frac{1}{\Delta t_n}\int_0^{\Delta t_n} p_r(y,z)drm(z)dz$ converges weakly to δ_y. Using the weak continuity of bridge probabilities (proposition 2.2.5) we get that (2.3.6) converges to

$$\int_{t_{min}}^{t_{max}} \mathbb{P}^t_{x,y}\left(F((X_s)_{0\leqslant s\leqslant t})\right)p_t(x,y)dt$$

□

Proposition 2.3.1 also holds in case of a Markov jump processes on a graph, where the local time is replaced by the occupation time in a vertex dived by its weight. Proposition 2.3.1 shows that we can consider $\mu^{x,y}$ as a measure on paths $(\gamma(t))_{0\leqslant t\leqslant T(\gamma)}$ endowed with continuous occupation densities $(\ell^z_t(\gamma))_{z\in I, 0\leqslant t\leqslant T(\gamma)}$. Next we state several properties that either follow almost immediately from the definition 2.1 and proposition 2.3.1 or are already known.

PROPERTY 2.3.2. (i) *The total mass of the measure $\mu^{x,y}$ is finite if and only if X is transient and then it equals $G(x,y)$. If it is the case, $\frac{1}{G(x,x)}\mu^{x,x}$ is the law of X, starting from $X(0)=x$, up to the last time it visits x. $\frac{1}{G(x,y)}\mu^{x,y}$ is the law of X, starting from $X(0)=x$, conditioned to visit y before ζ, up to the last time it visits y.*

(ii) *The measure $\mu^{y,x}$ is image of the measure $\mu^{x,y}$ by time reversal.*

(iii) *If $\tilde{I}$ is an open subinterval of I then*

$$\mu^{x,y}_{L_{|\tilde{I}}}(d\gamma) = 1_{\gamma \text{ contained in } \tilde{I}}\mu^{x,y}_L(d\gamma)$$

(iv) *If $\tilde{\kappa}$ is a positive Radon measure on I then*

$$\mu^{x,y}_{L-\tilde{\kappa}}(d\gamma) = \exp\left(-\int_I \ell^z(\gamma)m(z)\tilde{\kappa}(dz)\right)\mu^{x,y}_L(d\gamma)$$

(v) *If A is a change of scale function then*

$$\mu^{A(x),A(y)}_{Scale^{gen}_A L} = Scale_{A*}\mu^{x,y}_L$$

(vi) *If V is a positive continuous function on I then for the time changed diffusion of generator $\frac{1}{V}L$:*

$$\mu^{x,y}_{\frac{1}{V}L} = Speed_{V*}\mu^{x,y}_L$$

(vii) *If h is a positive continuous function on I such that $\frac{d^2h}{dx^2}$ is a signed measure and Lu is a negative measure then*

$$\mu^{x,y}_{Conj(h,L)} = \frac{1}{h(x)h(y)}\mu^{x,y}_{L}$$

(viii) *Let X and $\widetilde{X}$ be two independent Markovian paths of generator L starting from $X(0) = x$ and $\widetilde{X}(0) = y$. For $a \leqslant x \wedge y$, we introduce T_a and $\widetilde{T}_a$ the first time X respectively $\widetilde{X}$ hits a. Let $\mathbb{P}^{T_a}_x$ be the first passage bridge of X from x to a, conditioned by the event $T_a < \zeta$. Let $\widetilde{\mathbb{P}}^{\widetilde{T}_a}_y$ be the analogue for $\widetilde{X}$. Let $\widetilde{\mathbb{P}}^{\widetilde{T}_a\wedge}_y$ be the image of $\widetilde{\mathbb{P}}^{\widetilde{T}_a}_y$ through time reversal and $\mathbb{P}^{T_a}_x \lhd \widetilde{\mathbb{P}}^{\widetilde{T}_a\wedge}_y$ the image of $\mathbb{P}^{T_a}_x \otimes \widetilde{\mathbb{P}}^{\widetilde{T}_a\wedge}_y$ through concatenation at a of two paths, one ending and the other starting in a. Then*

$$\mu^{x,y}(\cdot) = \int_{a\in I, a\leqslant x\wedge y} \mathbb{P}_x(T_a < \zeta)\mathbb{P}_y(\widetilde{T}_a < \tilde{\zeta})\left(\mathbb{P}^{T_a}_x \lhd \widetilde{\mathbb{P}}^{\widetilde{T}_a\wedge}_y\right)(\cdot)w(a)da$$

Previous equalities depend on a particular choice of the speed measure for the modified generator. For (iv) we keep the measure $m(y)dy$. For (iii) we restrict $m(y)dy$ to $\widetilde{I}$. For (v) we choose $\left(\frac{dA}{dx}\circ A^{-1}\right)^{-1} m \circ A^{-1}da$. For (vi) we choose $\frac{1}{V(y)}m(y)dy$. For (vii) we choose $h(y)^2m(y)dy$. Property (ii) follows from that $p_t(x,y) = p_t(y,x)$ and $\mathbb{P}^t_{y,x}(\cdot)$ is the image of $\mathbb{P}^t_{x,y}(\cdot)$ by time reversal. Property (vi) is not immediate from definition 1 because fixed times are transformed by time change in random times, but follows from proposition 2.3.1. Property (vii) follows from the fact that a conjugation does not change bridge probability measures and changes the semi-group $(p_t(x,y)m(y)dy)_{t\geqslant 0, x\in I}$ to $(\frac{1}{h(x)}p_t(x,y)h(y)m(y)dy)_{t\geqslant 0,x\in I}$. Properties (ii) and (viii) were proved by Pitman and Yor in case $\kappa = 0$. See [**PY96**]. The case $\kappa \neq 0$ can be obtained through conjugations.

Next property was given without proof by Dynkin in [**Dyn84a**].

LEMMA 2.3.3. *Assume $\kappa \neq 0$. Let $\mathbb{P}_x(\cdot)$ be the law of $(X_t)_{0\leqslant t<\zeta}$ where $X(0) = x$. Then*

$$\int_{y\in I} \mu^{x,y}(\cdot)m(y)\kappa(dy) = 1_{X \textit{ killed by } \kappa}\mathbb{P}_x(\cdot)$$

PROOF. Let $0 < t_1 < t_2 < \cdots < t_n$ and let $A_1, A_2, \dots A_n, A_{n+1}$ be Borel subsets of I. The measure $\mu^{x,y}$ satisfies the following Markov property

$$\begin{aligned}&\mu^{x,y}(T(\gamma) > t_n, \gamma(t_1) \in A_1, \dots \gamma(t_n) \in A_n, \gamma(T(\gamma)) \in A_{n+1}) = \\ &\int_{A_1\times\dots\times A_n} p_{t_1}(x,x_1)m(x_1)\dots p_{t_n-t_{n-1}}(x_{n-1},x_n)m(x_n)\mu^{x_n,y}(T(\gamma)\in A_{n+1})dx_1\dots dx_n \\ &= 1_{y\in A_{n+1}}\int_{A_1\times\dots\times A_n} p_{t_1}(x,x_1)m(x_1)\dots p_{t_n-t_{n-1}}(x_{n-1},x_n)m(x_n)G(x_n,y)dx_1\dots dx_n\end{aligned}$$

Hence

$$\begin{aligned}(2.3.7)\quad &\int_{y\in I}\mu^{x,y}(T(\gamma) > t_n, \gamma(t_1) \in A_1, \dots \gamma(t_n) \in A_n, \gamma(T(\gamma)) \in A_{n+1})m(y)\kappa(dy) = \\ &\int_{A_1\times\dots\times A_{n+1}} p_{t_1}(x,x_1)m(x_1)\dots p_{t_n-t_{n-1}}(x_{n-1},x_n)m(x_n)G(x_n,y)m(y)dx_1\dots dx_n\kappa(dy)\end{aligned}$$

From Markov property of X follows

$$\begin{aligned}&\mathbb{P}_x(\zeta > t_n, X_{t_1} \in A_1, \dots, X_{t_n} \in A_n, X_{\zeta-} \in A_{n+1}) = \\ &\int_{A_1\times\dots\times A_n} p_{t_1}(x,x_1)m(x_1)\dots p_{t_n-t_{n-1}}(x_{n-1},x_n)m(x_n)P_{x_n}(X_{\zeta-}\in A_{n+1})dx_1\dots dx_n\end{aligned}$$

Since the distribution of X_{ζ^-} on the event of X killed by κ is $1_{y\in I}G(X_0,y)m(y)\kappa(dy)$, we get

(2.3.8) $\mathbb{P}_x(\zeta > t_n, X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n, X_{\zeta^-} \in A_{n+1}) =$

$$\int_{A_1\times\cdots\times A_{n+1}} p_{t_1}(x,x_1)m(x_1)\ldots p_{t_n-t_{n-1}}(x_{n-1},x_n)m(x_n)G(x_n,y)m(y)dx_1\ldots dx_n\kappa(dy)$$

The equality between (2.3.7) and (2.3.8) implies the lemma. □

Next we study the continuity of $(x,y)\mapsto \mu^{x,y}$.

LEMMA 2.3.4. *Let J be a compact subinterval of I. Then the family of local times of X satisfies: for every $\varepsilon > 0$*

$$\lim_{t\to 0^+}\sup_{x\in J}\mathbb{P}_x\left(\sup_{y\in I}\ell^y_{t\wedge\zeta}(X) > \varepsilon\right) = 0$$

PROOF. It is enough to prove it in case the killing measure κ is zero because adding a killing measure only lowers $\ell^y_{t\wedge\zeta}(X)$. Without loss of generality we may also assume that the diffusion is on its natural scale, that is to say $w\equiv 2$. Then X is just a time changed Brownian motion on some open subinterval of $\mathbb{R}$. For a Brownian motion $(B_t)_{t\geqslant 0}$ the statement is clear. In this case $\mathbb{P}_x\left(\sup_{y\in\mathbb{R}}\ell^y_{t\wedge\zeta}(B) > \varepsilon\right)$ does not depend on x and for a given x

$$\lim_{t\to 0^+}\mathbb{P}_x\left(\sup_{y\in\mathbb{R}}\ell^y_{t\wedge\zeta}(B) > \varepsilon\right) = 0$$

Otherwise let

$$\mathcal{I}_t := \int_0^t m(X_s)ds$$

Then given the time change that transforms X into a Brownian motion B, we have

$$\ell^y_t(X) = \ell^y_{\mathcal{I}_t}(B)$$

Let $J = [x_0, x_1]$. Let $x_{min}\in I$, $x_{min} < x_0$ and $x_{max}\in I$, $x_{max} > x_1$. Let $T_{x_{min},x_{max}}$ the first time X hits either x_{min} or x_{max}. Let $s > 0$, $\varepsilon > 0$ and $x\in J$. If $t \leqslant \frac{s}{\max_{[x_{min},x_{max}]} m}$ then on the event $T_{x_{min},x_{max}} \geqslant t$, $\mathcal{I}_t$ is less or equal to s. So for t small enough

$$\mathbb{P}_x\left(\sup_{y\in I}\ell^y_{t\wedge\zeta}(X) > \varepsilon\right) \leqslant \mathbb{P}_x\left(\sup_{y\in\mathbb{R}}\ell^y_s(B) > \varepsilon\right) + \mathbb{P}_x\left(T_{x_{min},x_{max}} < t\right)$$

But

$$\mathbb{P}_x\left(T_{x_{min},x_{max}} < t\right) = \mathbb{P}_{x_0}\left(T_{x_{min},x_{max}} < t\right) + \mathbb{P}_{x_1}\left(T_{x_{min},x_{max}} < t\right)$$

and

$$\lim_{t\to 0^+}\sup_{x\in J}\mathbb{P}_x\left(T_{x_{min},x_{max}} < t\right) = 0$$

Thus

$$\limsup_{t\to 0^+}\sup_{x\in J}\mathbb{P}_x\left(\sup_{y\in I}\ell^y_{t\wedge\zeta}(X) > \varepsilon\right) \leqslant \mathbb{P}_x\left(\sup_{y\in\mathbb{R}}\ell^y_s(B) > \varepsilon\right)$$

Letting s go to 0 we get the statement of the lemma. □

PROPOSITION 2.3.5. *Let $t_{max} > 0$. Let F be a bounded functional on finite life-time paths endowed with continuous local times that depends continuously on the path $(\gamma_t)_{0\leqslant t\leqslant T(\gamma)}$ and on $(l^x_{T(\gamma)}(\gamma))_{x\in I}$ where we take the topology of uniform convergence for the occupation densities on I. On top of that we assume that F is zero if $T(\gamma) > t_{max}$. Then the function $(x,y)\mapsto\mu^{x,y}(F(\gamma))$ is continuous on $I\times I$.*

PROOF. If we had assumed that F does only depend on the path regardless to its occupation field then the continuity of $(x,y) \mapsto \mu^{x,y}(F(\gamma))$ would just be a consequence of the continuity of transition densities and of the weak continuity of bridge probability measures. For our proof we further assume that L does not contain any killing measure. If this is not the case, then we can consider a continuous positive L-harmonic function u. Then $Conj(u,L)$ does not contain any killing measure and up to a continuous factor $u(x)u(y)$ gives the same measure $\mu^{x,y}$ (property 2.3.2 (vii)). We will mainly rely on the representation given by proposition 2.3.1.

Let $x, y \in I$ and $(x_j, y_j)_{j\geqslant 0}$ a sequence in $I \times I$ converging to (x,y). Without loss of generality we assume that $(x_j)_{j\geqslant 0}$ is increasing. We consider sample paths $(X_t)_{0\leqslant t<\zeta}$ and $(X_t^{(j)})_{0\leqslant t<\zeta_j}$ of the diffusion of generator L starting from x and each of x_j, coupled on a same probability space in the following way: First we sample X starting from x. Then we sample $X^{(0)}$ starting from x_0. It starts independently from X until the first time $X_t^{(0)} = X_t$. After that time $X^{(0)}$ sticks to X. This two paths may never meet if one of them dies to early. If $X, X^{(0)}, \dots, X^{(j)}$ are already sampled, we start $X^{(j+1)}$ from x_{j+1} independently from the preceding sample paths until it meets one of them. After that time $X^{(j+1)}$ sticks to the path it has met. Let

$$T^{(j)} := \inf\{t \geqslant 0 | X_t^{(j)} = X_t\}$$

If $X^{(j)}$ does not meet X, we set $T^{(j)} = +\infty$. By construction, $(T^{(j)})_{j\geqslant 0}$ is a non-increasing sequence. Here we use that there is no killing measure. $T^{(j)}$ is equal in law to the first time two independent sample paths of the diffusion, one starting from x and the other from x_j, meet. Thus the sequence $(T^{(j)})_{j\geqslant 0}$ converges to 0 in probability. Since it is decreasing, it converges almost surely to 0.

We use reduction to absurdity. The sequence $(\mu^{x_j,y_j}(F(\gamma)))_{j\geqslant 0}$ is bounded because F is bounded and zero on paths with life-time greater then t_{max}. Assume that it does not converge to $\mu^{x,y}(F(\gamma))$. Then there is a subsequence that converges to a value other than $\mu^{x,y}(F(\gamma))$. We may as well assume that the whole sequence $(\mu^{x_j,y_j}(F(\gamma)))_{j\geqslant 0}$ converges to a value $v \neq \mu^{x,y}(F(\gamma))$. According to lemma 2.3.4, the sequence $((\ell^z_{T^{(j)}}(X^{(j)}))_{z\in I})_{j\geqslant 0}$ of occupation density functions converges in probability to the null function. Thus there is an extracted subsequence $((\ell^z_{T^{(j_n)}}(X^{(j_n)}))_{z\in I})_{n\geqslant 0}$ that converges almost surely uniformly to the null function. We will show that $(\mu^{x_{j_n},y_{j_n}}(F(\gamma)))_{n\geqslant 0}$ converges to $\mu^{x,y}(F(\gamma))$ and obtain a contradiction.

For $z \in I$ and $l > 0$ let

$$\tau_l^z := \inf\{t \geqslant 0 | \ell_t^z(X) > l\}$$

and

$$\tau_{j,l}^z := \inf\{t \geqslant 0 | \ell_t^z(X^{(j)}) > l\}$$

Then according to proposition 2.3.1

$$\mu^{x,y}(F(\gamma)) = \mathbb{E}\Bigg[\int_0^{\ell^y_{t_{max}\wedge\zeta}(X)} F((X_s)_{0\leqslant s\leqslant \tau_l^y}) dl\Bigg]$$

$$\mu^{x_j,y_j}(F(\gamma)) = \mathbb{E}\Bigg[\int_0^{\ell^{y_j}_{t_{max}\wedge\zeta_j}(X^{(j)})} F((X_s^{(j)})_{0\leqslant s\leqslant \tau_{j,l}^{y_j}}) dl\Bigg]$$

For any $z \in I$, if $\tau_{j,l}^z \in [T^{(j)}, \zeta_j)$ then $\tau_{j,l}^z = \tau_{l'}^z$ where

$$l' = l + \ell^z_{T^{(j)}}(X) - \ell^z_{T^{(j)}}(X^{(j)})$$

Along the subset of indices $(j_n)_{n\geqslant 0}$, $\tau_{j_n,l}^{y_{j_n}}$ converges to τ_l^y for every $l \in (0, l_\zeta^y(X))$ except possibly the countable set of values of l where $l \mapsto \tau_{j,l}^y$ jumps. For any l such that $\tau_{j_n,l}^{y_{j_n}}$

converges to τ_l^y, the path $(X_s^{(j)})_{0\leqslant s\leqslant \tau^{y_{j_n}}_{j_n,l}}$ converges to the path $(X_s)_{0\leqslant s\leqslant \tau_l^y}$. Moreover for such l the occupation densities $(l^z_{\tau^{y_{j_n}}_{j_n,l}}(X^{(j_n)}))_{z\in I}$ converge uniformly to $(l^z_{\tau_l^y}(X))_{z\in I}$. Indeed

$$\ell^z_{\tau^{y_{j_n}}_{j_n,l}}(X^{(j_n)}) = \ell^z_{\tau^{y_{j_n}}_{j_n,l}}(X) - \ell^z_{T^{(j)}}(X) + \ell^z_{T^{(j)}}(X^{(j_n)})$$

Thus for all $l \in (0, \ell^y_\zeta(X))$, except possibly countably many,

$$\lim_{n\to+\infty} F((X_s^{(j_n)})_{0\leqslant s\leqslant \tau^{y_{j_n}}_{j_n,l}}) = F((X_s)_{0\leqslant s\leqslant \tau_l^y})$$

For n large enough, $\zeta_j = \zeta$ and $\ell^{y_{j_n}}_{t_{max}\wedge\zeta_{j_n}}(X^{(j_n)})$ converges to $\ell^y_{t_{max}\wedge\zeta}(X)$. It follows that the following almost sure convergence holds

$$\lim_{n\to+\infty}\int_0^{\ell^{y_{j_n}}_{t_{max}\wedge\zeta_{j_n}}(X^{(j_n)})} F((X_s^{(j_n)})_{0\leqslant s\leqslant \tau^{y_{j_n}}_{j_n,l}})dl = \int_0^{l^y_{t_{max}\wedge\zeta}(X)} F((X_s)_{0\leqslant s\leqslant \tau_l^y})dl \tag{2.3.9}$$

The left-hand side of (2.3.9) is dominated by $\|F\|_{+\infty}\ell^{y_{j_n}}_{t_{max}\wedge\zeta_{j_n}}(X^{(j_n)})$. In order to conclude that the almost sure convergence (2.3.9) is also an $\mathbb{L}^1$ convergence we need only to show that

$$\mathbb{E}\left[|\ell^{y_{j_n}}_{t_{max}\wedge\zeta_{j_n}}(X^{(j_n)}) - \ell^y_{t_{max}\wedge\zeta}(X)|\right] = 0 \tag{2.3.10}$$

We already know that $\ell^{y_{j_n}}_{t_{max}\wedge\zeta_{j_n}}(X^{(j_n)})$ converges almost surely to $\ell^y_{t_{max}\wedge\zeta}(X)$. Moreover

$$\mathbb{E}\left[\ell^{y_{j_n}}_{t_{max}\wedge\zeta_{j_n}}(X^{(j_n)})\right] = \int_0^{t_{max}} p_t(x_{j_n}, y_{j_n})$$

and

$$\mathbb{E}\left[\ell^y_{t_{max}\wedge\zeta}(X)\right] = \int_0^{t_{max}} p_t(x, y)$$

It follows that the expectations converge. By Scheffe's lemma, the $\mathbb{L}^1$ convergence (2.3.10) holds.

We have shown that there is always a subsequence $(\mu^{x_{j_n},y_{j_n}}(F(\gamma)))_{n\geqslant 0}$ that converges to $\mu^{x,y}(F(\gamma))$ which contradict the convergence of $(\mu^{x_j,y_j}(F(\gamma)))_{j\geqslant 0}$ to a different value. □

2.3.3. The measure μ^* on unrooted loops.

The measure $\mu^{x,x}$ can be seen as a measure on the space of rooted loops $\mathfrak{L}$. Next we define a natural measure μ^*_L on $\mathfrak{L}^*$ following the pattern (2.3.1)

DEFINITION 2.2. *Let μ_L be the following measure on $\mathfrak{L}$:*

$$\mu_L(d\gamma) := \int_{t>0}\int_{x\in I} \mathbb{P}^t_{x,x}(d\gamma)p_t(x,x)m(x)dx\frac{dt}{t} = \frac{1}{T(\gamma)}\int_{x\in I}\mu_L^{x,x}(d\gamma)m(x)dx$$

$\mu^*_L := \pi_*\mu_L$ is a measure on $\mathfrak{L}^*$.

We will drop the subscript L whenever there is no ambiguity on L. The definition 2 does not depend on the choice of the speed measure $m(x)\,dx$. The measures μ and μ^* are σ-finite but not finite. They satisfy the following elementary properties:

PROPERTY 2.3.6. (i) *μ is invariant by time reversal.*

(ii) *If $\tilde{I}$ is an open subinterval of I then*

$$\mu_{L_{|\tilde{I}}}(d\gamma) = 1_{\gamma \textit{ contained in } \tilde{I}}\, \mu_L(d\gamma)$$

(iii) *If $\tilde{\kappa}$ is a positive Radon measure on I then*

$$\mu_{L-\tilde{\kappa}}(d\gamma) = \exp\left(-\int_I \ell^z(\gamma)m(z)\tilde{\kappa}(dz)\right)\mu_L(d\gamma)$$

(iv) *If A is a change of scale function then*

$$\mu_{Scale_A^{gen} L} = Scale_{A*}\mu_L$$

(v) *If h is a positive continuous function on I such that $\frac{d^2h}{dx^2}$ is a signed measure and Lu is a negative measure then*

$$\mu_{Conj(h,L)} = \mu_L$$

Same properties hold for μ^*.

The measures μ and μ^* contain some information on the diffusion X but the invariance by conjugations (property 2.3.6 (v)) shows that they do not capture its asymptotic behaviour. In the section 2.3.4 we will prove a converse to the property property 2.3.6 (v). In our setting, most important examples of conjugations are:

- The Bessel 3 process on $(0, +\infty)$ is a conjugate of the Brownian motion on $(0, +\infty)$, killed when hitting 0, through the function $x \mapsto x$.
- The Brownian motion on $\mathbb{R}$ killed with uniform rate κdx (i.e. κ constant) is a conjugate of the drifted Brownian motion on $\mathbb{R}$ with constant drift $\sqrt{2\kappa}$, through the function $x \mapsto e^{-\sqrt{2\kappa}x}$.

In the sequel we will be interested mostly in μ^* and not μ. As it will be clear from the next propositions, the measure μ^* has some nice features that μ does not.

PROPOSITION 2.3.7. *Let $v \in [0,1]$. Then $shift_{v*}\mu = \mu$. In particular*

$$\mu(\cdot) = \int_{v\in[0,1]} shift_{v*}\mu(\cdot)dv \tag{2.3.11}$$

PROOF. For a rooted loop γ of life-time $T(\gamma)$ we will introduce γ_1 the path restricted to time interval $[0, vT(\gamma)]$ and γ_2 the path restricted to $[vT(\gamma), T(\gamma)]$. By bridge decomposition property, the measure $\mu(d\gamma_1, d\gamma_2)$ equals

$$\int_{t>0}\int_I\int_I \mathbb{P}^{vt}_{x,y}(d\gamma_1)\mathbb{P}^{(1-v)t}_{y,x}(d\gamma_2)p_{vt}(x,y)p_{(1-v)t}(y,x)m(y)\,dy\,m(x)\,dx\,\frac{dt}{t}$$

Since γ_1 and γ_2 play symmetric roles, changing the order of γ_1 and γ_2 does not change the measure μ. □

Formula (2.3.11) shows that we can get back to the measure μ from the measure μ^* by cutting the circle parametrizing a loop in $\mathfrak{L}^*$ in a point chosen uniformly on this circle, in order to separate the start from the end.

COROLLARY 2.3.8. *Let F be a positive measurable functional on $\mathfrak{L}$. Then the map $\gamma \mapsto \int_0^1 F(shift_v(\gamma))dv$ is $\pi^{-1}(\mathcal{B}_{\mathfrak{L}^*})$-measurable and*

$$\frac{d(F(\gamma)\mu)}{d\mu}_{|\pi^{-1}(\mathcal{B}_{\mathfrak{L}^*})} = \int_0^1 F(shift_v(\gamma))dv$$

PROOF. We need only to show that for every F' measurable functional on $\mathfrak{L}^*$:

$$\int_{\mathfrak{L}} F(\gamma)F'(\pi(\gamma))\mu(d\gamma) = \int_0^1\int_{\mathfrak{L}} F(shift_v(\gamma))F'(\pi(\gamma))\mu(d\gamma)dv \tag{2.3.12}$$

From proposition 2.3.7 follows that for every $v \in [0,1]$:

$$\int_{\mathfrak{L}} F(\gamma)F'(\pi(\gamma))\mu(d\gamma) = \int_{\mathfrak{L}} F(shift_v(\gamma))F'(\pi(\gamma))\mu(d\gamma) \tag{2.3.13}$$

Integrating (2.3.13) on $[0,1]$ leads to (2.3.12). □

The next identity appears in [**Jan11**] in the setting of Markov jump processes on graphs. It can be generalized to a wider class of Markov processes admitting local times (see lemma 2.2 in [**FR14**]). We will give a short proof that suits our framework.

COROLLARY 2.3.9. *Let $x \in I$. Then*

$$\ell^x(\gamma)\mu^*(d\gamma) = \pi_*\mu^{x,x}(d\gamma) \tag{2.3.14}$$

For $l > 0$, let $\mathbb{P}_x^{\tau_l^x}(\cdot)$ be the law of the sample paths of a diffusion X of generator L, started from x, until the time τ_l^x when $\ell_t^x(X)$ hits l, conditioned by $\tau_l^x < \zeta$. Then

$$1_{\gamma \text{ visits } x}\mu^*(d\gamma) = \int_0^{+\infty} \pi_*\mathbb{P}_x^{\tau_l^x}(d\gamma)e^{-\frac{l}{G(x,x)}}\frac{dl}{l} \tag{2.3.15}$$

Conventionally we set $G(x,x) = +\infty$ if X is recurrent.

PROOF. Let $\varepsilon > 0$ such that $[x-\varepsilon, x+\varepsilon] \subseteq I$. Let $T_{[x-\varepsilon,x+\varepsilon]}(\gamma)$ be the time a loop γ spends in $[x-\varepsilon, x+\varepsilon]$. From the identity (2.3.11) follows that

$$\frac{T_{[x-\varepsilon,x+\varepsilon]}(\gamma)}{T(\gamma)}\mu^*(d\gamma) = \frac{1}{T(\gamma)}\int_{x-\varepsilon}^{x+\varepsilon}\pi_*\mu^{z,z}(d\gamma)m(z)dz$$

and simplifying $T(\gamma)$:

$$T_{[x-\varepsilon,x+\varepsilon]}(\gamma)\mu^*(d\gamma) = \int_{x-\varepsilon}^{x+\varepsilon}\pi_*\mu^{z,z}(d\gamma)m(z)dz$$

Using local times we rewrite the previous expression as

$$\frac{\int_{x-\varepsilon}^{x+\varepsilon}\ell^z(\gamma)m(z)dz}{\int_{x-\varepsilon}^{x+\varepsilon}m(z)dz}\mu^*(d\gamma) = \frac{1}{\int_{x-\varepsilon}^{x+\varepsilon}m(z)dz}\int_{x-\varepsilon}^{x+\varepsilon}\pi_*\mu^{z,z}(d\gamma)m(z)dz \tag{2.3.16}$$

Let $\varepsilon_0 > 0$ such that $[x-\varepsilon_0, x+\varepsilon_0] \subseteq I$. Let F be a continuous bounded functional on loops endowed with continuous local times such that F is zero if the life-time of the loop exceeds $t_{max} > 0$ and if $\sup_{z\in[x-\varepsilon_0,x+\varepsilon_0]} l^z(\gamma)$ exceeds l_{max}. According to the proposition 2.2.5, the right-hand side of (2.3.16) applied to F converges as $\varepsilon \to 0$ to $(\pi_*\mu^{x,x})(F(\gamma))$. By dominated convergence it follows that the left-hand side of (2.3.16) applied to F converges as $\varepsilon \to 0$ to

$$\int_{\mathfrak{L}*}\ell^x(\gamma)F(\gamma)\mu^*(d\gamma)$$

Thus we have the equality

$$\int_{\mathfrak{L}*}\ell^x(\gamma)F(\gamma)\mu^*(d\gamma) = (\pi_*\mu^{x,x})(F(\gamma)) \tag{2.3.17}$$

The set of test functionals F that satisfy (2.3.17) is large enough to deduce the equality (2.3.14) between measures.

From proposition 2.3.1 follows that

$$\mu^{x,x}(\cdot) = \int_0^{+\infty}\mathbb{P}_x^{\tau_l^x}(\cdot)e^{-\frac{l}{G(x,x)}}dl$$

Applying (2.3.14) to the above disintegration, we get (2.3.15). □

COROLLARY 2.3.10. *Let V be a positive continuous function on I. We consider a time change with speed V: $ds = V(x)dt$. Then*

$$\mu^*_{\frac{1}{V}L} = Speed_{V*}\mu^*_L \tag{2.3.18}$$

PROOF. By definition 2.2 and property 2.3.2 (vi):

$$\mu_{\frac{1}{V}L}(d\gamma) = \frac{1}{T(\gamma)}\int_0^{T(\gamma)} \frac{V(\gamma(0))}{V(\gamma(s))} ds \ Speed_{V*}(\mu_L(d\gamma))$$

Applying corollary 2.3.8 we obtain:

$$\frac{dSpeed_{V*}\mu_L}{d\mu_{\frac{1}{V}L}}_{|\pi^{-1}(\mathcal{B}_{\mathfrak{L}*})} = \frac{\int_0^1 V^{-1}(\gamma(vT(\gamma)))dv}{\frac{1}{T(\gamma)}\int_0^{T(\gamma)} V^{-1}(\gamma(s))ds} = 1$$

This concludes. □

In dimension two, the time change covariance of the measure μ^* on loops plays a key role for the construction of the Conformal Loop Ensembles (CLE) using loop soups as in [**SW12**]: Let D be an open domain of the complex plane, $(B_t)_{0\leqslant t<\zeta}$ the two-dimensional standard Brownian motion in D killed when hitting ∂D and μ^* the corresponding measure on loops. If $f : D \rightarrow D$ is a conformal map, then $(f(B_t))_{0\leqslant t<\zeta}$ is a time changed Brownian motion. If we consider μ^* not as a measure on loops parametrized by time but a measure on the geometrical drawings of loops, then μ^* is invariant by the transformation $(\gamma(t))_{0\leqslant t\leqslant T(\gamma)} \mapsto (f(\gamma(t)))_{0\leqslant t\leqslant T(\gamma)}$. This is proved in [**LW04**].

Given that μ^* is invariant through conjugations and covariant with the change of scale and change of time, if X is a recurrent diffusion, then up to a change of scale and time, μ^* is the same as for the Brownian motion on $\mathbb{R}$, and if X is a transient diffusion, even if the killing measure κ is non-zero, then up to a change of scale and time, μ^* is the same as for the Brownian motion on a bounded interval, killed when it hits the boundary.

2.3.4. Multiple local times. In this subsection we define the multiple local time functional on loops. Corollary 2.3.9 gives a link between the measure μ^* and the measures $(\mu^{x,x})_{x\in I}$. Using multiple local times we will get a further relation between μ^* and $(\mu^{x,y})_{x,y\in I}$. This will allow us to prove a converse to the property 2.3.6 (v): two diffusions that have the same measure on unrooted loops are related trough conjugation.

DEFINITION 2.3. *If $(\gamma(t))_{0\leqslant t\leqslant T(\gamma)}$ is a continuous path in I having a family of local times $(\ell_t^x(\gamma))_{x\in I,0\leqslant t\leqslant T(\gamma)}$ relative to the measure $m(x)dx$, we introduce multiple local times $\ell^{x_1,x_2,\dots,x_n}(\gamma)$ for $x_1, x_2, \dots, x_n \in I$:*

$$\ell^{x_1,x_2,\dots,x_n}(\gamma) := \int_{0\leqslant t_1\leqslant t_2\leqslant\dots\leqslant t_n\leqslant T(\gamma)} d_{t_1}\ell_{t_1}^{x_1}(\gamma)d_{t_2}\ell_{t_2}^{x_2}(\gamma)\dots d_{t_n}\ell_{t_n}^{x_n}(\gamma)$$

If $\gamma \in \mathfrak{L}$ and has local times, we introduce circular local times for γ:

$$\ell^{*x_1,x_2,\dots,x_n}(\gamma) := \sum_{\substack{c \text{ circular} \\ \text{permutation} \\ \text{of } \{1,2,\dots,n\}}} \ell^{x_{c(1)},x_{c(2)},\dots,x_{c(n)}}(\gamma)$$

$\ell^{*x_1,x_2,\dots,x_n}$ being invariant under the transformations $(shift_v)_{v\in[0,1]}$, we see it as a functional defined on $\mathfrak{L}^*$.

Multiple local times of the form $\ell^{x,x,\dots,x}(\gamma)$, called self intersection local times, were studied by Dynkin in [**Dyn84c**]. Circular local times were introduced by Le Jan in [**Jan11**].

Let $n \in \mathbb{N}^*$ and $p \in \{1, \dots, n\}$. Let $Shuffle_{p,n}$ be the set of permutations σ of $\{1, \dots, n\}$ such that for all $i \leqslant j \in \{1, \dots, p\}$, $\sigma(i) \leqslant \sigma(j)$ and for all $i \leqslant j \in \{p+1, \dots, n\}$, $\sigma(i) \leqslant \sigma(j)$. Permutations in $Shuffle_{p,n}$ are obtained by shuffling two card decks $\{1, \dots, p\}$ and $\{p+1, \dots, n\}$. Let $Shuffle'_{p,n}$ be the permutations of $\{1, \dots, n\}$ of the form $\sigma \circ c$ where c is a circular permutation of $\{p+1, \dots, n\}$ and $\sigma \in Shuffle_{p,n}$ satisfies $\sigma(1) = 1$. One can check that

PROPERTY 2.3.11. *For all $x_1, \dots, x_p, x_{p+1}, \dots, x_n \in I$:*

(i)
$$\ell^{x_1,\ldots,x_p}(\gamma)\ell^{x_{p+1},\ldots,x_n}(\gamma) = \sum_{\sigma\in Shuffle_{p,n}} \ell^{x_{\sigma(1)},\ldots,x_{\sigma(p)},x_{\sigma(p+1)},\ldots,x_{\sigma(n)}}(\gamma)$$

(ii)
$$\ell^{*x_1,\ldots,x_p}(\gamma)\ell^{*x_{p+1},\ldots,x_n}(\gamma) = \sum_{\sigma'\in Shuffle'_{p,n}} \ell^{x_{\sigma'(1)},\ldots,x_{\sigma'(p)},x_{\sigma'(p+1)},\ldots,x_{\sigma'(n)}}(\gamma)$$

The equality 2.3.11 (ii) appears in [**Jan11**]. It is also shown in [**Jan11**] that for transient Markov jump processes:

$$\int \ell^{*x_1,x_2,\ldots,x_n}(\gamma)\mu(d\gamma) = G(x_1,x_2)\times\cdots\times G(x_{n-1},x_n)\times G(x_n,x_1) \tag{2.3.19}$$

It turns out that we have more: We consider L a generator of a diffusion on I of form (2.2.11). If γ_i for $i\in\{1,2,\ldots,n-1\}$ is a continuous path from x_i to x_{i+1}, then we can concatenate $\gamma_1,\gamma_2,\ldots,\gamma_{n-1}$ to obtain a continuous path $\gamma_1\lhd\gamma_2\lhd\cdots\lhd\gamma_{n-1}$ from x_1 to x_n. Let $\mu^{x_1,x_2}\lhd\cdots\lhd\mu^{x_{n-1},x_n}$ be the image measure $\mu^{x_1,x_2}\otimes\cdots\otimes\mu^{x_{n-1},x_n}$ by this concatenation procedure.

Proposition 2.3.12. *The following absolute continuity relations hold:*

(i) $(\mu^{x_1,x_2}\lhd\cdots\lhd\mu^{x_{n-1},x_n})(d\gamma) = \ell^{x_2,\ldots,x_{n-1}}(\gamma)\mu^{x_1,x_n}(d\gamma)$

(ii) $\pi_*(\mu^{x_1,x_2}\lhd\cdots\lhd\mu^{x_{n-1},x_n}\lhd\mu^{x_n,x_1})(d\gamma) = \ell^{*x_1,x_2,\ldots,x_n}(\gamma)\mu^*(d\gamma)$

Proof. (i): Let $((X^{(j)}_t)_{0\leqslant t<\zeta_j})_{0\leqslant j\leqslant n-1}$ be $n-1$ independent diffusions of generator L, with $X^{(j)}_0 = x_j$. For $l\geqslant 0$, let

$$\tau^{x_{j+1}}_{j,l} := \inf\left\{t_j\geqslant 0|\ell^{x_{j+1}}_{t_j}(X^{(j)})>l\right\}$$

According to proposition 2.3.1, $(\mu^{x_1,x_2}\lhd\cdots\lhd\mu^{x_{n-1},x_n})(F(\gamma))$ equals

$$\mathbb{E}\Big[\int_{l_j<\ell^{x_{j+1}}_{\zeta_j}(X^{(j)}),1\leqslant j\leqslant n-1} F\big((X^{(1)}_t)_{0\leqslant t\leqslant\tau^{x_2}_{1,l_1}}\lhd\cdots\lhd(X^{(n-1)}_t)_{0\leqslant t\leqslant\tau^{x_n}_{n-1,l_{n-1}}}\big)dl_1\ldots dl_{n-1}\Big] \tag{2.3.20}$$

Let $(X_t)_{0\leqslant t<\zeta}$ be an other diffusion of generator L. Let

$$\tau_{l_1} := \inf\{t\geqslant 0|l^{x_2}_t(X)>l_1\}$$

and recursively defined

$$\tau_{l_1,\ldots l_{j-1},l_j} := \inf\{t\geqslant\tau_{l_1,\ldots l_{j-1}}|\ell^{x_{j+1}}_t(X)>l_j\}$$

Then by strong Markov property, (2.3.20) equals

$$\mathbb{E}\left[\int 1_{\tau_{l_1,\ldots,l_{n-1}}\leqslant\zeta}F\left((X_t)_{0\leqslant t\leqslant\tau_{l_1,\ldots,l_{n-1}}}\right)dl_1\ldots dl_{n-1}\right]$$

which in turn equals

$$\mathbb{E}\left[\int 1_{\forall j,t_j<\zeta}F\left((X_t)_{0\leqslant t\leqslant t_{n-1}}\right)d_{t_1}\ell^{x_2}_{t_1}(X)\ldots d_{t_{n-1}}\ell^{x_n}_{t_{n-1}}(X)\right] \tag{2.3.21}$$

By proposition 2.3.1, (2.3.21) equals $\int\ell^{x_1,\ldots,x_{n-1}}(\gamma)F(\gamma)\mu^{x_1,x_n}(d\gamma)$.

(ii): According to the identity (i) and corollary 2.3.8, we have

$$\pi_*(\mu^{x_1,x_2}\lhd\cdots\lhd\mu^{x_{n-1},x_n}\lhd\mu^{x_n,x_1})(d\gamma) = \int_0^1 \ell^{x_2,\ldots,x_n}(shift_v(\gamma))dv\,\pi_*\mu^{x_1,x_1}(d\gamma)$$

According to corollary 2.3.9

$$\int_0^1 \ell^{x_2,\ldots,x_n}(shift_v(\gamma))dv\pi_*\mu^{x_1,x_1}(d\gamma) = \ell^{x_1}(\gamma)\int_0^1 \ell^{x_2,\ldots,x_n}(shift_v(\gamma))dv\mu^*(d\gamma)$$

But

$$\ell^{x_1}(\gamma)\int_0^1 \ell^{x_2,\ldots,x_n}(shift_v(\gamma))dv = \ell^{*x_1,x_2,\ldots,x_n}(\gamma)$$

which ends the proof. □

The proposition 2.3.12 (ii) implies (2.3.19).

PROPOSITION 2.3.13. *If L and $\widetilde{L}$ are two generators of diffusions on I of the form (2.2.11) such that $\mu_L^* = \mu_{\widetilde{L}}^*$, then there is a positive continuous function h on I such that $\frac{d^2h}{dx^2}$ is a signed measure, Lh a negative measure and $\widetilde{L} = Conj(h, L)$. If the diffusion of generator L is recurrent then $\widetilde{L} = L$.*

PROOF. Let $m(x)dx$ be a speed measure for L and $\tilde{m}(x)dx$ be a speed measure for $\widetilde{L}$. First let's assume that both L and $\widetilde{L}$ are generators of transient diffusions. Applying the identity (2.3.19) to $\int_{\mathfrak{L}^*} \ell^{*x,y}(\gamma)\mu^*(d\gamma)$ we get that for all $x, y \in I$:

$$G_{\widetilde{L}}(x,y)G_{\widetilde{L}}(y,x)\tilde{m}(x)\tilde{m}(y) = G_L(x,y)G_L(y,x)m(x)m(y) \tag{2.3.22}$$

and for all $x, y, z \in I$:

(2.3.23)

$$G_{\widetilde{L}}(x,y)G_{\widetilde{L}}(y,z)G_{\widetilde{L}}(z,x)\tilde{m}(x)\tilde{m}(y)\tilde{m}(z) = G_L(x,y)G_L(y,z)G_L(z,x)m(x)m(y)m(z)$$

Fix $x_0 \in I$. Let h be

$$h(x) := \frac{G_{\widetilde{L}}(x_0,x)\tilde{m}(x)}{G_L(x_0,x)m(x)}$$

h is positive and continuous. $\frac{1}{h(x)}G_L(x,y)h(y)m(y)$ equals:

$$\frac{G_L(x_0,x)G_L(x,y)G_L(y,x_0)m(x_0)m(x)m(y)}{G_{\widetilde{L}}(x_0,x)G_{\widetilde{L}}(x,y)G_{\widetilde{L}}(y,x_0)\tilde{m}(x_0)\tilde{m}(x)\tilde{m}(y)} \times \frac{G_{\widetilde{L}}(x_0,y)G_{\widetilde{L}}(y,x_0)\tilde{m}(x_0)\tilde{m}(y)}{G_L(x_0,y)G_L(y,x_0)m(x_0)m(y)} \times G_{\widetilde{L}}(x,y)\tilde{m}(y) \tag{2.3.24}$$

Applying (2.3.22) and (2.3.23) to (2.3.24) we get that

$$\frac{1}{h(x)}G(x,y)h(y)m(y) = G_{\widetilde{L}}(x,y)\tilde{m}(y) \tag{2.3.25}$$

Applying (2.3.25) once to (x, y) and once do (x, x) we get that

$$h(y) = h(x)\frac{G_{\widetilde{L}}(x,y)}{G(x,y)}\frac{G(y,y)}{G_{\widetilde{L}}(y,y)} \tag{2.3.26}$$

From (2.3.26) we deduce that $\frac{d^2h}{dx^2}$ is a signed measure. From (2.3.25) we deduce that $\widetilde{L} = Conj(h, L)$. $-Lh$ is the killing measure of $\widetilde{L}$ and is positive.

If we no longer assume that L and $\widetilde{L}$ generate transient diffusions then consider $\lambda > 0$. Then $\mu_{L-\lambda}^* = \mu_{\widetilde{L}-\lambda}^*$. According to the above, there is h positive continuous function on I such that $\frac{d^2}{dx^2}$ is a signed measure and

$$\widetilde{L} - \lambda = Conj(h, L - \lambda) = Conj(h, L) - \lambda$$

Then $\widetilde{L} = Conj(h, L)$ and necessarily Lh is a negative measure.

The class of recurrent diffusions is preserved by conjugations. So if L is the generator of a recurrent diffusion then so is $\widetilde{L}$, and thus h is bound to satisfy $Lh = 0$. But since the diffusion of L is recurrent, the only solutions to $Lh = 0$ are constant functions. Thus $\widetilde{L} = L$. $\square$

2.3.5. A disintegration of μ^* induced by the Vervaat's transformation. By conditioning the measure μ by the life-time of loops we get a sum of bridge measures. In this section we will disintegrate the measure μ^* as a measure on the minimal value of the loop and its behaviour above this value. By doing this way we will obtain a sum of excursion measures $\eta_{exc}^{>x}$. In case of Brownian loops on $\mathbb{R}$ this disintegration will follow from the Vervaat's bridge to excursion transformation. The case of general diffusion will be obtained using covariance of the measure on loops by time and scale change, restriction to a subinterval, killing, as well as invariance by conjugations.

THEOREM(VERVAAT). *(**[Ver79]**,**[Bia86]**) Let $(\gamma(s))_{0\leqslant s\leqslant t}$ be a random path following the Brownian bridge probability measure $\mathbb{P}^t_{BM,0,0}(\cdot)$. Let $s_{min} := argmin\gamma$. Then the path*

$$s \mapsto -\min\gamma + (shift_{\frac{s_{min}}{t}}\gamma)(s)$$

has the law of a positive Brownian excursion of life-time t.

In the sequel if η is a measure on paths and $x \in \mathbb{R}$, we will write $(x + \eta)$ for the image of η by $\gamma \mapsto x + \gamma$. $\eta_{BM}^{>0}$ will be the Levy-Itô measure on positive Brownian excursions and $\eta_{t,BM}^{>0}$ the probability measure on positive Brownian excursions of duration t. Given a continuous loop $(\gamma_t)_{0\leqslant t\leqslant T(\gamma)}$ and t_{min} the first time γ hits $\min\gamma$, let $\mathcal{V}(\gamma)$ be the transformation $shift_{\frac{t_{min}}{T(\gamma)}}$. $\mathcal{V}$ is $\mathcal{B}_{\mathfrak{L}}$-measurable.

PROPOSITION 2.3.14. *Let μ^*_{BM} be the measure on loops associated to the Brownian motion on $\mathbb{R}$. Then:*

$$\mu^*_{BM}(d\gamma) = 2\int_{a\in\mathbb{R}} \pi_*(a + \eta_{BM}^{>0})(d\gamma)\, da \tag{2.3.27}$$

The measure on $(\min\gamma, \max\gamma)$ induced by μ^*_{BM} is $1_{a<b}(b-a)^{-2}dadb$. Let $a < b \in \mathbb{R}$ and $\rho, \tilde{\rho}$ two independent Bessel 3 processes starting from 0. Let T_{b-a} and $\widetilde{T}_{b-a}$ be the first times ρ respectively $\tilde{\rho}$ hits $b-a$. Let $(\beta_t)_{0\leqslant t\leqslant T_{b-a}+\widetilde{T}_{b-a}}$ be the path

$$\beta_t := \begin{cases} a + \rho_t & \text{if } t \leqslant T_{b-a} \\ a + \tilde{\rho}_{T_{b-a}+\widetilde{T}_{b-a}-t} & \text{if } t \geqslant T_{b-a} \end{cases}$$

Then the law of $(\beta_t)_{0\leqslant t\leqslant T_{b-a}+\widetilde{T}_{b-a}}$ is the probability measure obtained by conditioning the measure μ^*_{BM} by $(\min\gamma, \max\gamma) = (a, b)$.

PROOF. For the Brownian motion on $\mathbb{R}$, μ_{BM} writes

$$\mu_{BM}(\cdot) = \int_{x\in\mathbb{R}}\int_{t>0} (x + \mathbb{P}^t_{BM,0,0})(\cdot)\frac{dt}{\sqrt{2\pi t^3}}dx$$

Let $\chi(a)da$ be the law of the minimum of the bridge under $\mathbb{P}^t_{BM,0,0}$. Applying the Vervaat's transformation, we get that

$$\mathcal{V}_*\mu_{BM}(\cdot) = \int_{a\in\mathbb{R}}\int_{t>0}\left(\int_{x>a}\chi(x-a)dx\right)(a + \eta_{t,BM}^{>0})(\cdot)\frac{dt}{\sqrt{2\pi t^3}}da$$

Since $\int_{x>a}\chi(x-a)\,dx = 1$, the right-hand side above equals

$$\int_{a\in\mathbb{R}}\int_{t>0}(a + \eta_{t,BM}^{>0})(\cdot)\frac{dt}{\sqrt{2\pi t^3}}da$$

But

$$\int_{t>0}(a + \eta_{t,BM}^{>0})(\cdot)\frac{dt}{\sqrt{2\pi t^3}} = 2(a + \eta_{BM}^{>0})(\cdot)$$

The equality (2.3.27) follows. The rest of the proposition is a consequence of the William's representation of Brownian excursions. □

COROLLARY 2.3.15. *Let I be an open interval of $\mathbb{R}$ and $\lambda \geqslant 0$. Let μ^* be the measure on loops in I associated to the generator $\frac{1}{2}\frac{d^2}{dx^2} - \lambda$. Given a loop $(\gamma(t))_{0\leqslant t\leqslant T(\gamma)}$, let $R(\gamma)$ be the loop*

$$R(\gamma) := (\max\gamma + \min\gamma - \gamma(t))_{0\leqslant t\leqslant T(\gamma)}$$

that is the image of γ through reflection relative to $\frac{\max\gamma+\min\gamma}{2}$. Then

$$R_*\mu^* = \mu^*$$

PROOF. It is enough to prove this in case $\lambda = 0$ and $I = \mathbb{R}$. Otherwise we multiply the measure μ^*_{BM} by a density function that is left invariant by R. Then we use the description of the measure μ^*_{BM} conditioned by the value of $(\min\gamma, \max\gamma)$ and the fact that if $a > 0$, $(\rho_t)_{t\geqslant}$ is a Bessel 3 process starting from 0 and T_b is the first time it hits b, then $(y - \rho_{T_b - t})_{0\leqslant t\leqslant T_b}$ has the same law as $(\rho_t)_{0\leqslant t\leqslant T_b}$ (see **[RY99]**, chapter VII, §4). □

Now we consider that L is a generator of a diffusion on I of form (2.2.11). Given a point $x_0 \in I$, u^{+,x_0} and u^{-,x_0} will be the L-harmonic functions satisfying the initial conditions $u^{+,x_0}(x_0) = u^{-,x_0}(x_0) = 0$, $\frac{du^{+,x_0}}{dx}(x_0^+) = 1$ and $\frac{du^{-,x_0}}{dx}(x_0^-) = -1$. If $x \leqslant y \in I$ then

$$w(y)u^{-,y}(x) = w(x)u^{+,x}(y) \qquad (2.3.28)$$

Indeed, the Wronskian $W(u^{-,y}, u^{+,x})$ takes in x the value $u^{-,y}(x)$ and in y the value $u^{+,x}(y)$, and the ratio $\frac{1}{w(z)}W(u^{-,y}, u^{+,x})(z)$ is constant. If $\kappa = 0$, then the both sides of (2.3.28) equal $\int_x^y w(z)dz$. u^{+,x_0} is positive on $I \cap (x_0, +\infty)$ and u^{-,x_0} is positive on $I \cap (-\infty, x_0)$. Let L^{+,x_0} be $Conj(u^{+,x_0}, L)$ restricted to $I \cap (x_0, +\infty)$ and L^{-,x_0} be $Conj(u^{-,x_0}, L)$ restricted to $I \cap (-\infty, x_0)$. L^{+,x_0} and L^{-,x_0} are generators of transient diffusions without killing measures. If L is the generator of the Brownian motion on $\mathbb{R}$, then $L^{+,0}$ is just the generator of a Bessel 3 process. In general case, x_0 is an entrance boundary for L^{+,x_0} and L^{-,x_0}, that is to say a diffusion started from $x \neq x_0$ will never reach the boundary at x_0, and we can also start this diffusions at the boundary point x_0, in which case it will be immediately repelled away from x_0. Let $x \in I$ and $(\rho_t^{+,x})_{0\leqslant t<\zeta^{+,x}}$ be a diffusion of generator $L^{+,x}$ starting from x. Let $y \in I$, $y > x$. Let $T_y^{+,x}$ be the first time $\rho^{+,x}$ hits y and $\hat{T}_y^{+,x}$ the last time it visits y. Then $(\rho^{+,x}_{\hat{T}_y^{+,x}+t})_{0\leqslant t<\zeta^{+,x}-\hat{T}_y^{+,x}}$ is a diffusion of generator $L^{+,y}$ starting from y. Let $(\rho_t^{-,y})_{0\leqslant t<\zeta^{-,y}}$ be a diffusion of generator $L^{-,y}$ starting from y and $T_x^{-,y}$ the first time it hits x. Then $(\rho_t^{+,x})_{0\leqslant t\leqslant T_y^{+,x}}$ and $(\rho^{-,y}_{T_x^{-,y}-t})_{0\leqslant t\leqslant T_x^{-,y}}$ are equal in law: Indeed let C be the constant

$$C = \frac{w(z)}{W(u^{-,y}, u^{+,x})(z)}$$

The Green's operator of $\rho^{+,x}$ killed in y is

$$((-L^{+,x}_{|(x,y)})^{-1}f)(x') = C\int_x^y u^{+,x}(x' \wedge y')u^{-,y}(x' \vee y')\frac{u^{+,x}(y')}{u^{+,x}(x')}m(y')dy'$$

and the Green's operator of $\rho^{-,y}$ killed in x is

$$((-L^{-,y}_{|(x,y)})^{-1}f)(x') = C\int_x^y u^{+,x}(x' \wedge y')u^{-,y}(x' \vee y')\frac{u^{-,y}(y')}{u^{-,y}(x')}m(y')dy'$$

The potential measure of $(\rho_t^{+,x})_{0\leqslant t\leqslant T_y^{+,x}}$ starting from x is

$$U(x')dx' = Cu^{+,x}(x')u^{-,y}(x')m(x')dx'$$

and for any f, g bounded functions on (x, y)

$$\int_x^y ((-L^{+,x}_{|(x,y)})^{-1} f)(x') g(x') U(x') dx' = \int_x^y f(x') ((-L^{-,y}_{|(x,y)})^{-1} g)(x') U(x') dx' \tag{2.3.29}$$

The time reversal property for $(\rho^{+,x}_t)_{0 \leqslant t \leqslant T^{+,x}_y}$ follows from the duality relation (2.3.29). See [**RY99**], chapter VII, §4 for details on time reversal.

COROLLARY 2.3.16. *If L is a generator of a diffusion on I of form* (2.2.11)*, then*

$$\mu^*(\cdot) = \int_{a \in I} \pi_* \eta^{>a}(\cdot) w(a) da \tag{2.3.30}$$

The measure on $(\min \gamma, \max \gamma)$ induced by μ^* is $1_{a<b\in I} \frac{dadb}{u^{+,a}(b)u^{-,b}(a)}$. Let $a < b \in I$. Let $(\rho^{+,a}_t)_{0 \leqslant t < \zeta^{+,a}}$ and $(\rho^{-,b}_t)_{0 \leqslant t < \zeta^{-,b}}$ be two independent diffusion, the first of generator $L^{+,a}$ starting from a and the second of generator $L^{-,b}$ starting from b. Let $T^{+,a}_b$ be the first time $\rho^{+,a}$ hits b and $T^{-,b}_a$ the first time $\rho^{-,b}$ hits a. Let $(\beta_t)_{0 \leqslant t \leqslant T^{+,a}_b + T^{-,b}_a}$ be the path

$$\beta_t := \begin{cases} \rho^{+,a}_t & \text{if } t \leqslant T^{+,a}_b \\ \rho^{-,b}_{t - T^{+,a}_b} & \text{if } t \geqslant T^{+,a}_b \end{cases}$$

Then the law of $(\beta_t)_{0 \leqslant t \leqslant T^{+,a}_b + T^{-,b}_a}$ is the probability measure obtained by conditioning the measure μ^* by $(\min \gamma, \max \gamma) = (a, b)$.

PROOF. Both sides of (2.3.30) are covariant by scale and time change. Moreover both sides satisfy the property 2.3.6 (ii) for the restriction to a subinterval and the property 2.3.6 (iii) when adding a killing measure. Thus the general case (2.3.30) follows from the Brownian case (2.3.27) by this covariance properties.

If L is a generator without killing measure ($\kappa = 0$) then the description of the measure on $(\min \gamma, \max \gamma)$ and the probabilities obtained after conditioning by the value of $(\min \gamma, \max \gamma)$ follow through a change of scale and time from the analogous description in proposition 2.3.14. If $\kappa \neq 0$, then we can take u a positive L-harmonic function and deduce the result for L from the result for $Conj(u, L)$ using the fact that $\mu^*_L = \mu^*_{Conj(u,L)}$. □

The relation between the measure on loops and the excursions measures in dimension 1 (identity (2.3.30)) is analogous to the relation between the measure on Brownian loops and the so called bubble measures observed by Lawler and Werner in dimension 2. See propositions 7 and 8 in [**LW04**].

2.3.6. A generalization of the Vervaat's transformation. In this subsection we will show a conditioned version of the Vervaat's transformation that holds for any one-dimensional diffusion of form (2.2.11) and not just for the Brownian motion. L will be a generator of a diffusion on I of form (2.2.11). From corollary 2.3.9 and identity (2.3.30) follows that for every $x \in I$:

$$\int_{t>0} \mathcal{V}_* \mathbb{P}^t_{x,x}(d\gamma) p_t(x,x) dt = \int_{a \in I, a<x} \ell^x(\gamma) \eta^{>a}(d\gamma) w(a) da \tag{2.3.31}$$

Let $\mathbb{P}^t_{x,x}(d\gamma | \min \gamma = a)$ be the bridge probability measure condition by the value of the minimum to equal a. Further we will show that there is a version that depends continuously on (a, t). Let $\eta^{>a}_t$ the probability measure obtained from $\eta^{>a}$ by conditioning the excursion to have a life-time t. The identity (2.3.31) suggests the following:

PROPOSITION 2.3.17. *For every $a < x \in I$ and $t > 0$*

$$\mathcal{V}_* \mathbb{P}^t_{x,x}(d\gamma | \min \gamma = a) = \frac{\ell^x_t(\gamma) \eta^{>a}_t(d\gamma)}{\eta^{>a}_t(\ell^x_t(\gamma))} \tag{2.3.32}$$

The distribution of $\min\gamma$ *under* $\mathbb{P}^t_{x,x}$ *equals*

$$w(a)\eta^{>a}_t(\ell^x_t(\gamma))\frac{1}{p_t(x,x)}\frac{\eta^{>a}(T(\gamma)\in(t,t+dt))}{dt}da \tag{2.3.33}$$

where $\frac{\eta^{>a}(T(\gamma)\in(t,t+dt))}{dt}$ *is the density of the measure on the life-time of the excursion induced by* $\eta^{>a}$*. Given an excursion* γ *following the law* $\frac{\ell^x_t(\gamma)\eta^{>a}_t(d\gamma)}{\eta^{>a}_t(\ell^x_t(\gamma))}$*, the local time in* x *is a measure on* $\{s\in[0,t]|\gamma(s)=x\}$*. The transformation* $\mathcal{V}$ *sends the starting point of the bridge to a point* $s\in[0,t]$ *distributed conditional on the excursion* γ *according the measure* $\frac{d_s\ell^x_s(\gamma)}{\ell^x_t(\gamma)}$*.*

Identities (2.3.32) and (2.3.33) can be viewed as a conditioned analogue of the Vervaat's relation between the Brownian bridge and the Brownian excursion. The latter can be deduced from (2.3.32) and (2.3.33) using the translation invariance of the Brownian motion. From (2.3.32) we can only deduce that (2.3.32) and (2.3.33) hold for Lebesgue almost all t and a. We need to show the weak continuity in (a,t) of conditioned bridge probabilities and biased conditioned excursion probabilities to conclude. It is enough to prove the proposition 2.3.17 for L not containing any killing measure and such that for all $a<x\in I$, a diffusion starting from x reaches a almost surely. Indeed, for a general generator, $Conj(u_\downarrow,L)$ does satisfy the above constraints and if the proposition 2.3.17 is true for $Conj(u_\downarrow,L)$ then it is also true for L. From now on we assume that L satisfies the above constraints. Next we give a more constructive description of the conditioned bridges and biased conditioned excursions. We start with bridges.

Property 2.3.2 (viii) shows that the measure $\mathbb{P}^{T_a}_x\lhd\tilde{\mathbb{P}}^{\tilde{T}^a\wedge}_x$ conditioned on $T_a+\tilde{T}_a=t$ is a version of $\mathbb{P}^t_{x,x}(d\gamma|\min\gamma=a)$. Let $p^{(a\times)}_t(x,y)$ be the transition density on $I\cap(a,+\infty)$ relative to $m(y)dy$ of the semi-group generated by $L_{|I\cap(a,+\infty)}$. Then $p^{(a\times)}_t(x,a^+)=0$. According to [**McK56**], for all $t>0$, $y\mapsto p^{(a\times)}_t(x,y)$ is $\mathcal{C}^1$. Let $\partial_2 p^{(a\times)}_t(x,y)$ be the derivative relative to y. It has a positive limit $\partial_2 p^{(a\times)}_t(x,a^+)$ as $y\to a^+$. Extended in this way, the map $(t,x,y)\mapsto\partial_2 p^{(a\times)}_t(x,y)$ is continuous on $(0+\infty)\times I\cap(a,+\infty)\times I\cap[a,+\infty)$. The distribution of T_a under $\mathbb{P}_x$ is (see [**IM74**], page 154):

$$\frac{1}{w(a)}\partial_2 p^{(a\times)}_t(x,a^+)dt$$

Let $\mathbb{P}^{(a\times),t}_{x,y}$ be the bridge probability measures of $L_{|I\cap(a,+\infty)}$. It has a weak limit $\mathbb{P}^{(a\times),t}_{x,a^+}$ as $y\to a^+$. Let $\mathcal{F}_s$ be the sigma-algebra generated by the restriction of a continuous path to the time interval $[0,s]$. Let $\mathbb{P}^{+,a}_a$ be the law of $\rho^{+,a}$ starting from a. For all $s\in(0,t)$ we have the following absolute continuity relations:

$$\frac{d\mathbb{P}^{(a\times),t}_{x,a^+}}{d\mathbb{P}_x}_{|\mathcal{F}_s}=1_{s<T_a}\frac{\partial_2 p^{(a\times)}_{t-s}(X_s,a^+)}{\partial_2 p^{(a\times)}_t(x,a^+)} \tag{2.3.34}$$

and for the time reversed bridge

$$\frac{d\mathbb{P}^{(a\times),t\wedge}_{x,a^+}}{d\mathbb{P}^{+,a}_a}_{|\mathcal{F}_s}=\frac{p^{(a\times)}_{t-s}(\rho^{+,a}_s,x)}{\partial_2 p^{(a\times)}_t(x,a^+)} \tag{2.3.35}$$

Using the absolute continuity relation (2.3.34) and (2.3.35) one can prove in a similar way as in proposition 2.2.5 that the map $(t,y)\mapsto\mathbb{P}^{(a\times),t}_{x,a^+}$ is continuous for the weak topology. The first passage bridge $\mathbb{P}^{T_a}_x$ disintegrates as follows

$$\mathbb{P}^{T_a}_x(\cdot)=\frac{1}{w(a)}\int_{t>0}\mathbb{P}^{(a\times),t}_{x,a^+}(\cdot)\partial_2 p^{(a\times)}_t(x,a^+)dt \tag{2.3.36}$$

From the property 2.3.2 (viii) and (2.3.36) we get that

PROPERTY 2.3.18. *The distribution of* $\min\gamma$ *under* $P^t_{x,x}$ *is*

$$\frac{da}{w(a)p_t(x,x)}\int_0^t \partial_2 p_s^{(a\times)}(x,a^+)\partial_2 p_{t-s}^{(a)}(x,a^+)ds \tag{2.3.37}$$

There is a version of $\mathbb{P}^t_{x,x}(d\gamma|\min\gamma=a)$ *that disintegrates as*

$$\frac{\int_0^t \left(\mathbb{P}^{(a\times),s}_{x,a^+}\lhd\mathbb{P}^{(a\times),t-s\wedge}_{x,a^+}\right)(d\gamma)\partial_2 p_s^{(a\times)}(x,a^+)\partial_2 p_{t-s}^{(a\times)}(x,a^+)ds}{\int_0^t \partial_2 p_s^{(a\times)}(x,a^+)\partial_2 p_{t-s}^{(a\times)}(x,a^+)ds} \tag{2.3.38}$$

Next we show that the probability measure given by (2.3.38) depends continuously on (a,t).

LEMMA 2.3.19. *The functions* $(x,a,t)\mapsto p_t^{(a\times)}(x,a^+)$ *and* $(x,a,t)\mapsto \partial_2 p_t^{(a\times)}(x,a^+)$ *are continuous on* $\{(x,a)|x>a\in I\}\times(0,+\infty)$.

PROOF. As in [**McK56**], we can use the eigendifferential expansion of L to express $p_t^{(a\times)}(x,a^+)$ and $\partial_2 p_t^{(a\times)}(x,a^+)$. Let x_0. For $\lambda\in\mathbb{R}$ consider $e_1(\cdot,\lambda)$ and $e_2(\cdot,\lambda)$ two solutions to $Lu+\lambda u=0$ with initial conditions

$$e_1(x_0,\lambda)=1\quad \frac{\partial e_1}{\partial x}(x_0,\lambda)=0\quad e_2(x_0,\lambda)=0\quad \frac{\partial e_2}{\partial x}(x_0,\lambda)=1$$

Let $\mathfrak{e}(x,\lambda)$ be the 2-vector whose entries are $e_1(x,\lambda)$ and $e_2(x,\lambda)$. According to theorems 3.2 and 4.3 in [**McK56**], for all $a\in I$ there is a Radon measure $\mathfrak{f}^{(a)}$ on $(-\infty,0]$ with values in the space of 2×2 symmetric positive semi-definite matrices such that for all $x\in I\cap(a,+\infty)$

$$p_t^{(a\times)}(x,a^+)=\int_{-\infty}^0 e^{t\lambda\intercal}\mathfrak{e}(x,\lambda)\mathfrak{f}^{(a\times)}(d\lambda)\mathfrak{e}(a,\lambda)$$

$$\partial_2 p_t^{(a\times)}(x,a^+)=\int_{-\infty}^0 e^{t\lambda\intercal}\mathfrak{e}(x,\lambda)\mathfrak{f}^{(a\times)}(d\lambda)\frac{\partial\mathfrak{e}}{\partial x}(a,\lambda)$$

Let $x>a\in I$. Consider a two sequences $(x_n)_{n\geqslant 0}$ and $(a_n)_{n\geqslant 0}$ in $I\cap(-\infty,x)$ converging to x respectively a such that for all $n\geqslant 0$, $x_n>a_n$. Let $(b_j)_{j\geqslant 0}$ be an increasing sequence in $I\cap(x,\sup I)$ converging to $\sup I$. Let $\mathfrak{f}_{n,j}$ be the 2×2-matrix valued measure on $(-\infty,0]$ corresponding to the eigendifferential expansion of L restricted to (a_n,b_j). $\mathfrak{f}_{n,j}$ charges only a discrete set of atoms. As shown in the proof of theorem 3.2 in [**McK56**], the total mass of the measures $1\wedge|\lambda|^{-2}\|\mathfrak{f}_{n,j}\|(d\lambda)$, $1\wedge|\lambda|^{-2}\|\mathfrak{f}^{(a_n\times)}\|(d\lambda)$ and $1\wedge|\lambda|^{-2}\|\mathfrak{f}^{(a\times)}\|(d\lambda)$ is uniformly bounded. Moreover for a fixed n, as $j\to+\infty$, $1\wedge|\lambda|^{-2}\mathfrak{f}_{n,j}(d\lambda)$ converges vaguely, that is against continuous functions vanishing at infinity, to the measure $1\wedge|\lambda|^{-2}\mathfrak{f}^{(a_n\times)}(d\lambda)$. Moreover, for any increasing integer-valued sequence $(j_n)_{n\geqslant 0}$ converging to $+\infty$, $1\wedge|\lambda|^{-2}\mathfrak{f}_{n,j_n}(d\lambda)$ converges vaguely as $n\to+\infty$ to $1\wedge|\lambda|^{-2}\mathfrak{f}^{(a\times)}(d\lambda)$. Since the sequence $(j_n)_{n\geqslant 0}$ is arbitrary, this implies that $1\wedge|\lambda|^{-2}\mathfrak{f}^{(a_n\times)}(d\lambda)$ converges vaguely as $n\to+\infty$ to $1\wedge|\lambda|^{-2}\mathfrak{f}^{(a\times)}(d\lambda)$.

There are constants $C,c'>0$ such that for all $\lambda\leqslant 0$ and $n\geqslant 0$

$$\|\mathfrak{e}(x_n,\lambda)\|\leqslant Ce^{c'\sqrt{|\lambda|}}\quad \|\mathfrak{e}(a_n,\lambda)\|\leqslant Ce^{c'\sqrt{|\lambda|}}\quad \|\frac{\partial\mathfrak{e}}{\partial x}(a_n,\lambda)\|\leqslant Ce^{c'\sqrt{|\lambda|}} \tag{2.3.39}$$

Let $t>0$ and $(t_n)_{n\geqslant 0}$ a sequence of times converging to t. From (2.3.39) follows that

$$\lim_{\lambda\to-\infty}\sup_{n\geqslant 0}|\lambda|^2 e^{t_n\lambda}\|\mathfrak{e}(x_n,\lambda)\|\times\|\mathfrak{e}(a_n,\lambda)\|=0$$

$\lambda\mapsto 1\vee|\lambda|^2e^{t_n\lambda}\left(\mathfrak{e}(x_n,\lambda),\partial\mathfrak{e}(a_n,\lambda)\right)$ vanishes at infinity an converges uniformly on $(-\infty,0]$ to $\lambda\mapsto 1\vee|\lambda|^2e^{t\lambda}\left(\mathfrak{e}(x,\lambda),\mathfrak{e}(a,\lambda)\right)$. The vague convergence of measures implies that

$$\lim_{n\to+\infty}\int_{-\infty}^0 e^{t_n\lambda\intercal}\mathfrak{e}(x_n,\lambda)\mathfrak{f}^{(a_n\times)}(d\lambda)\mathfrak{e}(a_n,\lambda)=\int_{-\infty}^0 e^{t\lambda\intercal}\mathfrak{e}(x,\lambda)\mathfrak{f}^{(a\times)}(d\lambda)\mathfrak{e}(a,\lambda)$$

Similarly $\partial_2 p_{t_n}^{(a_n\times)}(x_n, a_n^+)$ converges to $\partial_2 p_t^{(a\times)}(x, a^+)$. □

LEMMA 2.3.20. *The map* $a \mapsto \mathbb{P}_a^{+,a}$ *is weakly continuous.*

PROOF. Let $a_0 \in I$. Consider the process $(\rho_t^{+,a_0})_{t\geqslant 0}$ following the law $\mathbb{P}_{a_0}^{+,a_0}$. For $a \in I \cap (a_0, +\infty)$, let $\hat{T}_a$ be the last time ρ^{+,a_0} visits a. Then $(\rho_{\hat{T}_a+t}^{+,a_0})_{t\geqslant 0}$ follows the law $\mathbb{P}_a^{+,a}$. The process valued map $a \mapsto (\rho_{\hat{T}_a+t}^{+,a_0})_{t\geqslant 0}$ is almost surely continuous on $I \cap (a_0, +\infty)$ and thus the laws depend weakly continuously on a. □

PROPOSITION 2.3.21. *The version of* $\mathbb{P}_{x,x}^t(d\gamma|\min\gamma = a)$ *given by* (2.3.38) *is weakly continuous in* (a, t).

PROOF. From the absolute continuity relations (2.3.34) for the bridge $\mathbb{P}_{x,a^+}^{(a\times),t}$ and (2.3.35) for its time reversal, together with the continuity of the densities which follows from lemma 2.3.19, and the weak continuity of $a \mapsto \mathbb{P}_a^{+,a}$, we can deduce in a very similar way as in proposition 2.2.5 that the map $(a,t) \mapsto \mathbb{P}_{x,a^+}^{(a\times),t}$ is weakly continuous on $(0,+\infty) \times I \cap (-\infty, x)$ and hence $(a, s, t) \mapsto \mathbb{P}_{x,a^+}^{(a\times),s} \lhd \mathbb{P}_{x,a^+}^{(a\times),t-s\wedge}$ is weakly continuous. Finally the densities that appear in expression (2.3.38) are continuous with respect to (a, s, t). □

Next we will give a decomposition of the measure $\eta^{>a}$ which is similar to the Bismut's decomposition of Brownian excursions (see [**RY99**], chapter XII, §4, theorem 4.7). Biane used this Bismut's decomposition to give an alternative proof for the Brownian Vervaat's transformation ([**Bia86**]). $\partial_2 p_t^{(a\times)}(x, a^+)$ is $\mathcal{C}^1$ relative to x and the derivative $\partial_{1,2} p_t^{(a\times)}(x, a^+)$ has a positive limit $\partial_{1,2} p_t^{(a\times)}(a^+, a^+)$ as $y \to a^+$. Moreover $t \mapsto \partial_{1,2} p_t^{(a\times)}(a^+, a^+)$ is continuous. The measure on the life-time of the excursion induced by $\eta^{>a}$ is (see [**SVY07**]):

$$\frac{1}{w(a)^2}\partial_{1,2} p_t^{(a\times)}(a^+, a^+)dt$$

Let $s \in [0, t]$. The measure $\eta_t^{>a}(\cdot)$ disintegrates as (see [**SVY07**]):

$$\int_{x\in I, x>a} \left(\mathbb{P}_{x,a^+}^{(a\times),s\wedge} \lhd \mathbb{P}_{x,a^+}^{(a\times),t-s}\right)(\cdot)\frac{\partial_2 p_s^{(a\times)}(x,a^+)\partial_2 p_{t-s}^{(a\times)}(x,a^+)m(y)}{\partial_{1,2} p_t^{(a\times)}(a^+,a^+)}dy \tag{2.3.40}$$

For every $s_1 < s_2 \in [0, s]$, under the bridge measure $\mathbb{P}_{y,z}^{(a\times),s}$

$$\mathbb{P}_{y,z}^{(a\times),t}(\ell_{s_2}^x(\gamma) - \ell_{s_1}^x(\gamma)) = \int_{s_1}^{s_2} \frac{p_r^{(a\times)}(y,x)p_{s-r}^{(a\times)}(x,z)}{p_s^{(a\times)}(y,z)}dr \tag{2.3.41}$$

and under the bridge measure $\mathbb{P}_{y,a^+}^{(a\times),s}$

$$\mathbb{P}_{y,a^+}^{(a\times),t}(\ell_{s_2}^x(\gamma) - \ell_{s_1}^x(\gamma)) = \int_{s_1}^{s_2} \frac{p_r^{(a\times)}(y,x)\partial_2 p_{s-r}^{(a\times)}(x,a^+)}{\partial_2 p_s^{(a\times)}(y,a^+)}dr \tag{2.3.42}$$

Combining (2.3.40) and (2.3.42) we get that for every $s_1 < s_2 \in [0, s]$:

$$\eta_t^{>a}(\ell_{s_2}^x(\gamma) - \ell_{s_1}^x(\gamma)) = \int_{s_1}^{s_2} \frac{\partial_2 p_s^{(a\times)}(x,a^+)\partial_2 p_{t-s}^{(a\times)}(x,a^+)}{\partial_{1,2} p_t^{(a\times)}(a^+,a^+)}ds \tag{2.3.43}$$

PROPOSITION 2.3.22. *Let F_1 and F_2 be two non-negative measurable functional on the paths with variable life-time. Then*

$$(2.3.44)\quad \eta_t^{>a}\left(\int_0^t F_1((\gamma(r))_{0\leqslant r\leqslant s})F_2((\gamma(s+r))_{0\leqslant r\leqslant t-s})d_s\ell_s^x(\gamma)\right) = \int_0^t \mathbb{P}^{(a\times),s\wedge}_{x,a^+}(F_1)\mathbb{P}^{(a\times),t-s}_{x,a^+}(F_2)\frac{\partial_2 p_s^{(a\times)}(x,a^+)\partial_2 p_{t-s}^{(a\times)}(x,a^+)}{\partial_{1,2}p_t^{(a)}(a^+,a^+)}\,ds$$

In particular

$$(2.3.45)\quad \ell_t^x(\gamma)\eta_t^{>a}(d\gamma) = \int_0^t \left(\mathbb{P}^{(a\times),s\wedge}_{x,a^+}\lhd\mathbb{P}^{(a\times),t-s}_{x,a^+}\right)(d\gamma)\frac{\partial_2 p_s^{(a\times)}(x,a^+)\partial_2 p_{t-s}^{(a\times)}(x,a^+)}{\partial_{1,2}p_t^{(a\times)}(a^+,a^+)}ds$$

PROOF. It is enough to prove the result in case F_1 and F_2 are non-negative, continuous and bounded. On top of that we may assume that there are $s_{min} < s_{max} \in (0,t)$ such that F_1 respectively F_2 takes value 0 if the life-time of a path is smaller than s_{min} respectively $t - s_{max}$, and that there is $C \in I$, $C > a$, such that F_1 and F_2 take value 0 if $\max\gamma > C$. For $j \leqslant n \in \mathbb{N}$ set $\Delta s_n := \frac{1}{n}(s_{max} - s_{min})$ and $s_{j,n} := s_{min} + j\Delta s_n$. Then almost surely

$$(2.3.46)\quad \int_0^t F_1((\gamma(r))_{0\leqslant r\leqslant s})F_2((\gamma(s+r))_{0\leqslant r\leqslant t-s})d_s\ell_s^x(\gamma) = \lim_{n\to+\infty}\sum_{j=0}^{n-1}F_1((\gamma(r))_{0\leqslant r\leqslant s_{j,n}})(\ell^x_{s_{j+1,n}}(\gamma)-\ell^x_{s_{j,n}}(\gamma))F_2((\gamma(s_{j+1,n}+r))_{0\leqslant r\leqslant t-s_{j+1,n}})$$

Moreover the right-hand side of (2.3.46) is dominated by $\ell_t^x(\gamma)\|F_1\|_\infty\|F_2\|_\infty$. Thus the $\eta_t^{>a}$-expectation converges too. Applying (2.3.40) and (2.3.41) we get

$$\eta_t^{>a}\big(F_1((\gamma(r))_{0\leqslant r\leqslant s_{j,n}})(\ell^x_{s_{j+1,n}}(\gamma)-\ell^x_{s_{j,n}}(\gamma))F_2((\gamma(s_{j+1,n}+r))_{0\leqslant r\leqslant t-s_{j+1,n}})\big) = \int_0^{\Delta s_n}\int_{(a,C)^2}\mathbb{P}^{(a\times),s_{j,n}\wedge}_{y,a^+}(F_1)\mathbb{P}^{(a\times),t-s_{j+1,n}}_{z,a^+}(F_2)q_n(r,y,z)m(y)dym(z)dzdr$$

where

$$q_n(r,y,z) = \frac{\partial_2 p^{(a\times)}_{s_{j,n}}(y,a^+)\partial_2 p^{(a\times)}_{t-s_{j+1,n}}(z,a^+)}{\partial_{1,2}p_t^{(a\times)}(a^+,a^+)}p_r^{(a\times)}(y,x)p^{(a\times)}_{\Delta s_n - r}(x,z)$$

The measure $1_{y,z>a\in I}\dfrac{1}{\Delta s_n}\int_0^{\Delta s_n} q_n(r,y,z)drdydz$ converges weakly as $n \to +\infty$ to $\delta_{(x,x)}$. The maps $(s,y) \mapsto \partial_2 p_s^{(a\times)}(x,a^+)$ and $(s,y) \mapsto \mathbb{P}_s^{(a\times),y,a^+}(\cdot)$ are continuous. Moreover $\partial_2 p^{(a\times)}_{s_{j,n}}(y,a^+)\partial_2 p^{(a\times)}_{t-s_{j+1,n}}(z,a^+)$ is uniformly bounded for $j \leqslant n \in \mathbb{N}$ and $y,z \in (a,C]$. All this ensures that the $\eta_t^{>a}$-expectation of the right-hand side of (2.3.46) converges as $n \to +\infty$ to the right-hand side of (2.3.44). □

Now we need only to match the preceding descriptions to prove proposition 2.3.17. (2.3.38) and (2.3.45) imply (2.3.32). (2.3.37) and (2.3.43) imply (2.3.33). The fact that the point where the excursion is split is distributed according to $\frac{d_s\ell_s^x(\gamma)}{\ell_t^x(\gamma)}$ follows from (2.3.44).

2.3.7. Restricting loops to a discrete subset. Let L be the generator of a diffusion on I of form (2.2.11) and $(X_t)_{0\leqslant t<\zeta}$ be the corresponding diffusion. Let $\mathbb{J}$ be a countable discrete subset of I. A Markov jump process to the nearest neighbours on $\mathbb{J}$ is naturally embedded in the diffusion X. In this section we will show that, given any $x,y \in \mathbb{J}$, the image of the measure $\mu_L^{x,y}$ through the restriction application that sends a sample paths of the diffusion $(X_t)_{0\leqslant t<\zeta}$ to a sample path of a Markov jump process on $\mathbb{J}$ is a measure on $\mathbb{J}$-valued paths that follows the pattern (2.3.2). From this we will deduce that the image of the measure μ_L^* through the restriction to $\mathbb{J}$ is a measure on $\mathbb{J}$-valued loops following the pattern (2.3.1) and which was studied in [**Jan11**]. This property will be used in section 2.4.2 to express the law of finite-dimensional marginals of the occupation field of a Poisson ensemble of intensity $\alpha\mu_L^*$.

For a continuous path $(\gamma(t))_{0\leqslant t\leqslant T(\gamma)}$ in I, endowed with continuous local times, let

$$\mathcal{I}_t^{\mathbb{J}}(\gamma) := \sum_{x\in\mathbb{J}} \ell_t^x(\gamma)m(x)$$

For $s \geqslant 0$, we introduce the stopping time

$$\tau_s^{\mathbb{J}}(\gamma) := \inf\{t \geqslant 0 | \mathcal{I}_t^{\mathbb{J}}(\gamma) \geqslant s\}$$

We write $\gamma^{\mathbb{J}}$ for the path $(\gamma(\tau_s^{\mathbb{J}}))_{0\leqslant s\leqslant \mathcal{I}_{T(\gamma)}^{\mathbb{J}}(\gamma)}$ on $\mathbb{J}$. Let $m_{\mathbb{J}}$ be the measure

$$m_{\mathbb{J}} := \sum_{x\in\mathbb{J}} m(x)\delta_x$$

The occupation measure of $\gamma^{\mathbb{J}}$ is

$$\sum_{x\in\mathbb{J}} \ell^x(\gamma)m(x)\delta_x$$

and $(\ell^x(\gamma))_{x\in\mathbb{J}}$ are also occupation densities of the restricted path $\gamma^{\mathbb{J}}$ with respect to $m_{\mathbb{J}}$.

The restricted diffusion $X^{\mathbb{J}}$ is a Markov jump process to nearest neighbours on $\mathbb{J}$, potentially with killing. If $x_0 < x_1$ are two consecutive points in $\mathbb{J}$, the jump rate from x_0 to x_1 is $\frac{1}{m(x_0)w(x_0)}\frac{1}{u^{+,x_0}(x_1)}$ and the jump rate from x_1 to x_0 is $\frac{1}{m(x_1)w(x_1)}\frac{1}{u^{-,x_1}(x_0)}$. If $x_0 < x_1 < x_2$ are three consecutive points in $\mathbb{J}$, then the rate of killing while in x_1 is

$$\frac{1}{m(x_1)w(x_1)}\left(\frac{W(u^{-,x_2},u^{+,x_0})(x_1)}{u^{-,x_2}(x_1)u^{+,x_0}(x_1)} - \frac{1}{u^{-,x_1}(x_0)} - \frac{1}{u^{+,x_1}(x_2)}\right)$$

If $\mathbb{J}$ has a minimum x_0 and x_1 is the second lowest point in $\mathbb{J}$, then the killing rate while in x_0 is

$$\frac{1}{m(x_0)w(x_0)}\left(\frac{W(u^{-,x_1},u_{\uparrow})(x_0)}{u^{-,x_1}(x_0)u_{\uparrow}(x_0)} - \frac{1}{u^{+,x_0}(x_1)}\right)$$

An analogous expression holds for the killing rate while in a possible maximum of $\mathbb{J}$. $X^{\mathbb{J}}$ is transient if and only if X is. Let $L_{\mathbb{J}}$ be the generator of $X^{\mathbb{J}}$. $L_{\mathbb{J}}$ is symmetric relative to $m_{\mathbb{J}}$. Its Green's function relative to $m_{\mathbb{J}}$ is $(G(x,y))_{x,y\in I}$, that is the restriction of the Green's function of L to $\mathbb{J}\times\mathbb{J}$. $X^{\mathbb{J}}$ may not be conservative even if the diffusion X is. In case if $\mathbb{J}$ is not finite, $X^{\mathbb{J}}$ may blow up performing an infinite number of jumps in finite time. Measures $(\mu_L^{x,y})_{x,y\in I}$, μ_L and μ_L^* have discrete space analogues $(\mu_{L_{\mathbb{J}}}^{x,y})_{x,y\in\mathbb{J}}$, $\mu_{L_{\mathbb{J}}}$ and $\mu_{L_{\mathbb{J}}}^*$ as defined in [**Jan11**], that follow the patterns (2.3.2) and (2.3.1).

PROPOSITION 2.3.23. *Let $x,y\in\mathbb{J}$. Then $\gamma\mapsto\gamma^{\mathbb{J}}$ transforms $\mu_L^{x,y}$ in $\mu_{L_{\mathbb{J}}}^{x,y}$ and μ_L^* in $\mu_{L_{\mathbb{J}}}^*$.*

PROOF. The representation (2.3.3) also holds for $\mu_{L_{\mathbb{J}}}^{x,y}$. For $l>0$, let

$$\tau_l^y := \inf\{t\geqslant 0 | \ell_t^y(X) > l\}$$

and

$$\tau_l^{y,\mathbb{J}} := \inf\{s\geqslant 0 | \ell_s^y(X^{\mathbb{J}}) > l\}$$

Then for any non-negative measurable functional F

$$\mu_{L_{\mathbb{J}}}^{x,y}(F(\gamma)) = \int_0^{+\infty} dl\mathbb{E}_x\left[1_{\tau_l^{y,\mathbb{J}}<\mathcal{I}_\zeta^{\mathbb{J}}}F((X_s^{\mathbb{J}})_{0\leqslant s\leqslant\tau_l^{y,\mathbb{J}}})\right]$$

But $(X^{\mathbb{J}}_s)_{0\leqslant s\leqslant \tau^{y,\mathbb{J}}_\lambda}$ is the image of $(X_t)_{0\leqslant t\leqslant \tau^y_\lambda}$ by the map $\gamma \mapsto \gamma^{\mathbb{J}}$ and $\tau^{y,\mathbb{J}}_l < \mathcal{I}^{\mathbb{J}}_\zeta$ if and only if $\tau^y_l < \zeta$. Thus $\mu^{x,y}_{L_{\mathbb{J}}}$ is the image of $\mu^{x,y}_L$ through the restriction on path to $\mathbb{J}$. The second part of the proposition can be deduced from that for any $x \in \mathbb{J}$

$$\ell^x(\gamma)\mu^*_L(d\gamma) = \pi_*\mu^{x,x}_L(d\gamma)$$

and as noticed in [**Jan11**]

$$\ell^x(\gamma)\mu^*_{L_{\mathbb{J}}}(d\gamma^{\mathbb{J}}) = \pi_*\mu^{x,x}_{L_{\mathbb{J}}}(d\gamma^{\mathbb{J}})$$

□

Previous restriction property and the time-change covariance of μ^* (corollary 2.3.10) can be treated in a unified framework of the time change by the inverse of a continuous additive functional. This is done in [**FR14**], section 7.

2.3.8. Measure on loops in case of creation of mass. We can further extend the definition of the measures $\mu^{x,y}$ on paths and μ and μ^* on loops to the case of L being a "generator" on I containing a creation of mass term as in (2.2.19). Doing so will enable us to emphasize further the conjugation invariance of the measure on loops and will be useful in section 2.4.2 to compute the exponential moments of the occupation field of Poisson ensembles of Markov loops. Let ν be signed measure on I. Let $L^{(0)} := \frac{1}{m(x)}\frac{d}{dx}\left(\frac{1}{w(x)}\frac{d}{dx}\right)$ and $L := L^{(0)} + \nu$.

DEFINITION 2.4.
- $\mu^{x,y}_L(d\gamma) := \exp\left(\int_I l^x(\gamma)m(x)\nu(dx)\right)\mu^{x,y}_{L^{(0)}}(d\gamma)$
- $\mu_L(d\gamma) := \exp\left(\int_I l^x(\gamma)m(x)\nu(dx)\right)\mu_{L^{(0)}}(d\gamma)$
- $\mu^*_L := \pi_*\mu_L$

Definition 2.4 is consistent with properties 2.3.2 (iv) and 2.3.6 (iii). If $\tilde{\nu}$ is any other signed measure on I, then

$$\mu^{x,y}_{L+\tilde{\nu}}(d\gamma) := \exp\left(\int_I \ell^x(\gamma)m(x)\tilde{\nu}(dx)\right)\mu^{x,y}_L(d\gamma) \tag{2.3.47}$$

Same holds for μ and μ^*. Under the extended definition, the measures $\mu^{x,y}$ still satisfy properties 2.3.2 (ii), (iii), (v) and (vi). Proposition 2.3.5 remains true. μ still satisfies properties 2.3.6 (i), (ii) and (iv). Proposition 2.3.7 and corollary 2.3.8 still hold. The identities (2.3.14) and (2.3.18) remain true for μ^*. Concerning the conjugations, we have:

PROPOSITION 2.3.24. *Let h be a continuous positive function on I such that $\frac{d^2h}{dx^2}$ is a signed measure. $h(x)^2m(x)dx$ is a speed measure for $Conj(h,L)$. Then for all $x,y \in I$, $\mu^{x,y}_{Conj(h,L)} = \frac{1}{h(x)h(y)}\mu^{x,y}_L$, and $\mu_{Conj(h,L)} = \mu_L$. Conversely, if L and $\widetilde{L}$ are two "generators" with or without creation of mass such that $\mu_L = \mu_{\widetilde{L}}$ then there is a positive continuous function h on I such that $\frac{d^2h}{dx^2}$ is a signed measure and $\widetilde{L} = Conj(h,L)$.*

PROOF. There is a positive Radon measure $\tilde{\kappa}$ on I such that both $L-\tilde{\kappa}$ and $Conj(h,L)-\tilde{\kappa}$ are generators of (killed) diffusions. But

$$Conj(h,L) - \tilde{\kappa} = Conj(h, L - \tilde{\kappa})$$

It follows that $\mu^{x,y}_{Conj(h,L)-\tilde{\kappa}} = \frac{1}{h(x)h(y)}\mu^{x,y}_{L-\tilde{\kappa}}$ and $\mu_{Conj(h,L)-\tilde{\kappa}} = \mu_{L-\tilde{\kappa}}$. Applying (2.3.47) we get the result.

If $\mu_L = \mu_{\widetilde{L}}$, we can again consider $\tilde{\kappa}$ a positive Radon measure on I such that both $L-\tilde{\kappa}$ and $\widetilde{L}-\tilde{\kappa}$ are generators of (killed) diffusions. Then according to proposition 2.3.13, there is a positive continuous function h on I such that $\frac{d^2h}{dx^2}$ is a signed measure and $\widetilde{L}-\tilde{k} = Conj(h, L-\tilde{k})$. Then $\widetilde{L} = Conj(h,L)$. □

Similarly to the case of generators of diffusions (section 2.3.5), one can consider L-harmonic functions $u^{-,x}$ and $u^{+,x}$ in case of L containing creation of mass. If $L \in \mathfrak{D}^{+}$, then $u^{-,x}$ respectively $u^{+,x}$ is not necessarily positive on $I \cap (-\infty, x)$ respectively $I \cap (x, +\infty)$. Let

$$M(x) := \sup\{y \in I, y \geqslant x | \forall z \in (x, y), u^{+,x}(z) > 0\} \in I \cup \{\sup I\}$$

If $L \in \mathfrak{D}^{0,-}$ then for all $x \in I$, $M(x) = \sup I$. Let $y \in I$, $y > x$. If $y < M(x)$, then $L_{|(x,y)} \in \mathfrak{D}^{-}$. If $y = M(x)$, then $L_{|(x,y)} \in \mathfrak{D}^{0}$. If $y > M(x)$, then $L_{|(x,y)} \in \mathfrak{D}^{+}$. The diffusion $\rho^{+,x}$ of generator $L^{+,x} = Conj(u^{+,x}, L^{+,x}_{|(x,M(x))})$ is defined on $(x, M(x))$. Similarly for $\rho^{-,y}$. Moreover if If $M(x) \in I$, then $L^{+,x}_{|(x,M(x))} = L^{-,M(x)}_{|(x,M(x))}$.

If $L \in \mathfrak{D}^{0,-}$, the description of the measure on $(\min\gamma, \max\gamma)$ induced by μ^{*} as well as of the probability measures obtained by conditioning μ^{*} by the value of $(\min\gamma, \max\gamma)$ is the same as given by corollary 2.3.16, with the same formal expressions. Next we state what happens if $L \in \mathfrak{D}^{+}$:

PROPOSITION 2.3.25. *Let $L \in \mathfrak{D}^{+}$. The measure on $(\min\gamma, \max\gamma)$ induced by μ^{*} and restricted to the set $\{a \in I, b \in (a, M(a))\}$ is $1_{a\in I, b\in(a,M(a))}\frac{dadb}{u^{+,a}(b)u^{-,b}(a)}$. If $a < b < M(a)$, then the probability measure obtained through conditioning by $(\min\gamma, \max\gamma) = (a, b)$ has the same description as in corollary 2.3.16. Outside the set $\{a \in I, b \in (a, M(a))\}$, the measure on $(\min\gamma, \max\gamma)$ is not locally finite. That is to say that, if $a < b \in I$ and $b \geqslant M(a)$, then for all $\varepsilon > 0$.*

$$\mu^{*}(\{\min\gamma \in (a, a+\varepsilon), \max\gamma \in (b-\varepsilon, b)\}) = +\infty \tag{2.3.48}$$

PROOF. For the behaviour on $\{a \in I, b \in (a, M(a))\}$: There is a countable collection $(I_j)_{j\geqslant 0}$ of open subintervals of I such that

$$\{a \in I, b \in (a, M(a))\} = \bigcup_{j\geqslant 0}\{x < y \in I_j\}$$

Since for all j, $L_{|I_j} \in \mathfrak{D}^{0,-}$, corollary 2.3.16 applies to $L_{|I_j}$. Combining the descriptions on different $\{a < b \in I_j\}$, we get the description on $\{a \in I, b \in (a, M(a))\}$.

For the behaviour outside $\{a \in I, b \in (a, M(a))\}$: Let $A < B \in \mathbb{R}$. Then

$$\mu^{*}_{BM}(\{\min\gamma < A, \max\gamma > B\}) = \int_{B}^{+\infty}\int_{-\infty}^{A}\frac{dadb}{(b-a)^2} = +\infty \tag{2.3.49}$$

If $a < b \in I$ and $M(a) = b$, then $1_{a<\gamma<b}\mu^{*}$ is the image of μ^{*}_{BM} through a change of scale and time. In this case (2.3.48) follows from (2.3.49). If $b > M(a)$, then $L_{|(a,b)} \in \mathfrak{D}^{+}$. According to proposition 2.2.9 (iv), there is a positive measure Radon measure κ on (a, b) such that $L_{|(a,b)} - \kappa \in \mathfrak{D}^{0}$. From what precedes, (2.3.48) holds for $\mu^{*}_{L_{|(a,b)}-\kappa}$. Moreover, $\mu^{*}_{L_{|(a,b)}} \geqslant \mu^{*}_{L_{|(a,b)}-\kappa}$. So (2.3.48) holds for $\mu^{*}_{L_{|(a,b)}}$. □

2.4. Occupation fields of the Poisson ensembles of Markov loops

2.4.1. Inhomogeneous continuous state branching processes with immigration. We will identify the occupation fields of the Poisson ensembles of Markov loops as inhomogeneous continuous state branching processes with immigration. This will be done in section 2.4.2. In the section 2.4.1 we will give the basic properties of such processes. In section 2.4.3 we will deal with the particular case of the intensity being $\frac{1}{2}\mu^{*}$, in relation with Dynkin's isomorphism.

Let I be an open interval of $\mathbb{R}$. We will consider stochastic processes where $x \in I$ is the evolution variable. We do not call it time because in the sequel it will rather represent a space variable. Let $(\mathbb{B}_x)_{x\in\mathbb{R}}$ be a standard Brownian motion. Consider the following SDE:

$$d\widetilde{Z}_x = \sigma(x)\sqrt{\widetilde{Z}_x}d\mathbb{B}_x + b(x)\widetilde{Z}_x dx \tag{2.4.1}$$

$$dZ_x = \sigma(x)\sqrt{Z_x}d\mathbb{B}_x + b(x)Z_x dx + c(x)dx \tag{2.4.2}$$

For our needs we will assume that σ is positive and continuous on I, that b and c are only locally bounded and that c is non negative. In this case existence and pathwise uniqueness holds for (2.4.1) and (2.4.2) (see [**RY99**], chapter IX, §3), and $\widetilde{Z}$ and Z take values in $\mathbb{R}_+$. 0 is an absorbing state for $\widetilde{Z}$.

(2.4.1) satisfies the branching property: if $\widetilde{Z}^{(1)}$ and $\widetilde{Z}^{(2)}$ are two independent processes solutions in law to (2.4.1), defined on $I \cap [x_0, +\infty)$, then $\widetilde{Z}^{(1)} + \widetilde{Z}^{(2)}$ is a solution in law to (2.4.1). If $\widetilde{Z}$ and Z are two independent processes, $\widetilde{Z}$ solution in law to (2.4.1) and Z solution in law to (2.4.2), defined on $I \cap [x_0, +\infty)$, then $Z + \widetilde{Z}$ is a solution in law to (2.4.2). Solutions to (2.4.2) are (inhomogeneous) continuous state branching processes with immigration. The branching mechanism is given by (2.4.1) and the immigration measure is $c(x)dx$. The homogeneous case (σ, b and c constant) was extensively studied. See [**KW71**].

The case of inhomogeneous branching without immigration reduces to the homogeneous case as follows: Let $x_0 \in I$ and let

$$C(x) := \exp\left(-\int_{x_0}^{x} b(y)\,dy\right) \qquad A(x) := \int_{x_0}^{x} \sigma(y)^2 C(y)^2 dy$$

If $(\widetilde{Z}_x)_{x\in I}$ is a solution to (2.4.1), then $(C(A^{-1}(a))\widetilde{Z}_{A^{-1}(a)})_{a\in A(I)}$ is a solution in law to

$$d\widetilde{\mathcal{Z}}_a = 2\sqrt{\widetilde{\mathcal{Z}}_a}d\mathbb{B}_a$$

Let $\widetilde{Z}$ be a solution to (2.4.1) defined on $I \cap [x_0, +\infty)$, starting at x_0 with the initial condition $\widetilde{Z}_{x_0} = z_0 \geqslant 0$. Then, for $\lambda \geqslant 0$ and $x \in I$, $x \geqslant x_0$:

$$\mathbb{E}_{\widetilde{Z}_{x_0}=z_0}\left[e^{-\lambda \widetilde{Z}_x}\right] = e^{-z_0\psi(x_0,x,\lambda)}$$

$\psi(x_0, x, \lambda)$ depends continuously on (x_0, x, λ). If $x = x_0$ then

$$\psi(x_0, x_0, \lambda) = \lambda \tag{2.4.3}$$

If $x_0 \leqslant x_1 \leqslant x_2 \in I$ then

$$\psi(x_0, x_2, \lambda) = \psi(x_0, x_1, \psi(x_1, x_2, \lambda))$$

ψ satisfies the differential equation

$$\frac{\partial\psi}{\partial x_0}(x_0, x, \lambda) = \frac{\sigma(x_0)^2}{2}\psi(x_0, x, \lambda)^2 - b(x_0)\psi(x_0, x, \lambda) \tag{2.4.4}$$

If b is not continuous, equation (2.4.4) should be understand in the weak sense. If be is continuous, then (2.4.4) satisfies the Cauchy-Lipschitz conditions, and ψ is uniquely determined by (2.4.4) and the initial condition (2.4.3). This is also the case even if b is not continuous. Indeed, by considering $C(x)\widetilde{Z}_x$ rather than $\widetilde{Z}_x$, that is to say considering $\frac{C(x)}{C(x_0)}\psi(x_0, x, \lambda)$ rather than $\psi(x_0, x, \lambda)$, we get rid of b.

Inhomogeneous branching processes are related to the local times of one-dimensional diffusions:

PROPOSITION 2.4.1. *Let $x_0 \in I$ and let $(X_t)_{0\leqslant t<\zeta}$ be a diffusion on I of generator L of form (2.2.11) starting from x_0. Let $z_0 > 0$ and*

$$\tau_{z_0}^{x_0} := \inf\{t \geqslant 0 | \ell_t^{x_0}(X) > z_0\}$$

Then conditional on $\tau_{z_0}^{x_0} < \zeta$, $(\ell^x_{\tau^{x_0}_{z_0}}(X))_{x\in I, x\geqslant x_0}$ *is a solution in law to the SDE:*

$$d\widetilde{Z}_x = \sqrt{2w(x)}\sqrt{\widetilde{Z}_x}d\mathbb{B}_x + 2\frac{d\log u_\downarrow}{dx}(x)\widetilde{Z}_x dx \tag{2.4.5}$$

PROOF. If X is the Brownian motion on $\mathbb{R}$, then $w \equiv 2$ and $u_\downarrow$ is constant. In this case the assertion is the second Ray-Knight theorem. See [**RY99**], chapter XI, §2. The equation (2.4.5) is then the equation of a square of Bessel 0 process. If $x_{min} < x_0$ and X is the Brownian motion on $(x_{min}, +\infty)$ killed in x_{min} then the law of $(\ell^x_{\tau^{x_0}_{z_0}}(X))_{x\in I, x\geqslant x_0}$ conditional on $\tau_{z_0}^{x_0} < \zeta$ does not depend on x_{min} and is the same as in case of the Brownian motion on $\mathbb{R}$. Equation (2.4.5) is still satisfied.

If X is a diffusion on I that satisfies that for all $x > a \in I$, starting from x, X reaches almost surly a, which is equivalent to $u_\downarrow$ being constant, then through a change of scale and time X is the Brownian motion on some $(x_{min}, +\infty)$ where $x_{min} \in [-\infty, +\infty)$. Time change does not change the local times because we defined them relative to the speed measure. Only the change of scale matters. If S is a primitive of w, then conditional on $\tau_{z_0}^{x_0} < \zeta$, $(\ell^{S^{-1}(2y)}_{\tau^{x_0}_{z_0}}(X))_{y\geqslant \frac{1}{2}S(x_0)}$ is a square of Bessel 0 process. The equation (2.4.5) follows from the equation of the square of Bessel 0 process by deterministic change of variable $dy := \frac{1}{2}w(x)dx$.

Now the general case: let $(\widetilde{X}_t)_{0\leqslant t<\tilde{\zeta}}$ be the diffusion of generator $Conj(u_\downarrow, L)$. $\frac{w(x)}{u_\downarrow(x)^2}dx$ is the natural scale measure of $\widetilde{X}$ and $u_\downarrow(x)^2 m(x)dx$ is its speed measure. We assume that both X and $\widetilde{X}$ start from x_0. The law of $\widetilde{X}$ up to the last time it visits x_0 is the same as for X. Let

$$\tilde{\tau} := \inf\left\{t \geqslant 0 | \ell_t^{x_0}(\widetilde{X}) > \frac{1}{u_\downarrow(x_0)^2}z_0\right\}$$

Then the law of $(\ell^x_{\tau^{x_0}_{z_0}}(X))_{x\in I, x\geqslant x_0}$ conditional on $\tau_{z_0}^{x_0} < \zeta$ is the same as the law of $(u_\downarrow(x)^2\ell^x_{\tilde{\tau}}(\widetilde{X}))_{x\in I, x\geqslant x_0}$ conditional on $\tilde{\tau} < \tilde{\zeta}$. The factor $u_\downarrow(x)^2$ comes from the fact that performing a conjugation we change the measure relative to which the local times are defined. For any $a < x_0 \in I$, $\widetilde{X}$ reaches a a.s. Thus $(\ell^x_{\tilde{\tau}}(\widetilde{X}))_{x\in I, x\geqslant x_0}$ satisfies the SDE

$$d\widetilde{Z}_x = \frac{\sqrt{2w(x)}}{u_\downarrow(x)}\sqrt{\widetilde{Z}_x}d\mathbb{B}_x$$

and $(u_\downarrow(x)^2\ell^x_{\tilde{\tau}^{x_0}_{z_0}}(\widetilde{X}))_{x\in I, x\geqslant x_0}$ satisfies (2.4.5). □

If there is immigration: Let Z be a solution to (2.4.2) defined on $I \cap [x_0, +\infty)$, starting at x_0 with the initial condition $Z_{x_0} = z_0 \geqslant 0$. Then, for $\lambda \geqslant 0$ and $x \in I$, $x \geqslant x_0$:

$$\mathbb{E}_{Z_{x_0}=z_0}\left[e^{-\lambda Z_x}\right] = \exp\left(-z_0\psi(x_0, x, \lambda) - \int_{x_0}^x \psi(y, x, \lambda)c(y)dy\right) \tag{2.4.6}$$

2.4.2. Occupation field. Let L be the generator of a diffusion on I of form (2.2.11). Let $\mathcal{L}_{\alpha,L}$ be a Poisson ensemble of intensity $\alpha\mu^*_L$. $\mathcal{L}_{\alpha,L}$ is a random infinite countable collection of unrooted loops supported in I. It is sometimes called "loop soup".

DEFINITION 2.5. *The occupation field of* $\mathcal{L}_{\alpha,L}$ *is* $(\widehat{\mathcal{L}}^x_{\alpha,L})_{x\in I}$ *where*

$$\widehat{\mathcal{L}}^x_{\alpha,L} := \sum_{\gamma\in\mathcal{L}_{\alpha,L}} \ell^x(\gamma)$$

We will drop out the subscript L whenever there is no ambiguity on L. In this subsection we will identify the law of $(\widehat{\mathcal{L}}^x_\alpha)_{x\in I}$ as an inhomogeneous continuous state branching process with immigration. If $\mathbb{J}$ is a discrete subset of I, then applying proposition 2.3.23 we deduce that $(\widehat{\mathcal{L}}^x_\alpha)_{x\in\mathbb{J}}$ is the occupation field of the Poisson ensemble of discrete loops of intensity

$\alpha\mu^*_{L_\mathbb{J}}$ as defined in [**Jan11**], chapter 4. This fact allows us to apply the results of [**Jan11**] in order to describe the finite-dimensional marginals of the occupation field. If the diffusion is recurrent, then for all $x \in I$, $\widehat{\mathcal{L}}^x_\alpha = +\infty$ a.s. If the diffusion is transient, then for all $x \in I$, $\widehat{\mathcal{L}}^x_\alpha < +\infty$ a.s. Next we state how does the occupation field behave if we apply various transformations on L.

PROPERTY 2.4.2. *Let L be the generator of a transient diffusion.*

(i) *If A is a change of scale function, then*

$$\widehat{\mathcal{L}}^{A(x)}_{\alpha, Scale^{gen}_A L} = \widehat{\mathcal{L}}^x_{\alpha,L}$$

(ii) *If V is a positive continuous function on I, then*

$$\widehat{\mathcal{L}}^x_{\alpha,\frac{1}{V}L} = \widehat{\mathcal{L}}^x_{\alpha,L}$$

(iii) *If h is a positive continuous function on I such that Lh is a negative measure, then*

$$\widehat{\mathcal{L}}^x_{\alpha,Conj(h,L)} = \frac{1}{h(x)^2}\widehat{\mathcal{L}}^x_{\alpha,L}$$

Previous equalities depend on a particular choice of the speed measure for the modification of L. For (i) we choose $\left(\frac{dA}{dx}\circ A^{-1}\right)^{-1} m\circ A^{-1}da$. For (ii) we choose $\frac{1}{V(x)}m(x)dx$. For (iii) we choose $h(x)^2 m(x)dx$. The fact that $\widehat{\mathcal{L}}^x_{\alpha,Conj(h,L)} \neq \widehat{\mathcal{L}}^x_{\alpha,L}$ despite $\mathcal{L}_{\alpha,Conj(h,L)} = \mathcal{L}_{\alpha,L}$ comes from a change of speed measure.

Next we characterize the finite-dimensional marginals of the occupation field by stating the results that appear in [**Jan11**], chapter 4.

PROPERTY 2.4.3. *The distribution of $\widehat{\mathcal{L}}^x_\alpha$ is*

$$\frac{(G_L(x,x))^\alpha}{\Gamma(\alpha)} l^{\alpha-1}\exp\left(-\frac{l}{G_L(x,x)}\right) 1_{l>0}dl$$

Let $x_1, x_2, \dots, x_n \in I$ and $\lambda_1, \lambda_2, \dots, \lambda_n \geqslant 0$. Then

$$\mathbb{E}\left[\exp\left(-\sum_{i=1}^n \lambda_i \widehat{\mathcal{L}}^{x_i}_\alpha\right)\right] = \left(\frac{\det(G_{L-\sum_{i=1}^n \lambda_i\delta_{x_i}}(x_i,x_j))_{1\leqslant i,j\leqslant n}}{\det(G_L(x_i,x_j))_{1\leqslant i,j\leqslant n}}\right)^\alpha \tag{2.4.7}$$

The moment $\mathbb{E}\left[\widehat{\mathcal{L}}^{x_1}_\alpha \widehat{\mathcal{L}}^{x_2}_\alpha \dots \widehat{\mathcal{L}}^{x_n}_\alpha\right]$ is an α-permanent:

$$\mathbb{E}\left[\widehat{\mathcal{L}}^{x_1}_\alpha \widehat{\mathcal{L}}^{x_2}_\alpha \dots \widehat{\mathcal{L}}^{x_n}_\alpha\right] = \sum_{\sigma\in\mathfrak{S}_n} \alpha^{\sharp\text{ cycles of }\sigma}\prod_{i=1}^n G(x_i, x_{\sigma(i)})$$

If $\mathbb{J}$ is a discrete subset of I, then $(\widehat{\mathcal{L}}^x_\alpha)_{x\in\mathbb{J}}$, viewed as a stochastic process that evolves when x increases, is an inhomogeneous continuous state branching process with immigration defined on the discrete set $\mathbb{J}$. In particular, for any $x_1 \leqslant x_2 \leqslant \dots \leqslant x_n \in I$ and $p \in \{1, 2, \dots, n\}$, $\left(\widehat{\mathcal{L}}^{x_1}_\alpha, \widehat{\mathcal{L}}^{x_2}_\alpha, \dots, \widehat{\mathcal{L}}^{x_p}_\alpha\right)$ and $\left(\widehat{\mathcal{L}}^{x_p}_\alpha, \widehat{\mathcal{L}}^{x_{p+1}}_\alpha, \dots, \widehat{\mathcal{L}}^{x_n}_\alpha\right)$ are independent conditional on $\widehat{\mathcal{L}}^{x_p}_\alpha$.

Next we show that the processes $x \mapsto \widehat{\mathcal{L}}^x_\alpha$ parametrized by $x \in I$, where x is assumed to increase, is an inhomogeneous branching process with immigration of form (2.4.2). In particular, it has a continuous version and is inhomogeneous Markov.

PROPOSITION 2.4.4. *$(\widehat{\mathcal{L}}^x_\alpha)_{x\in I}$ has the same finite-dimensional marginals as a solution to the stochastic differential equation*

$$dZ_x = \sqrt{2w(x)}\sqrt{Z_x}d\mathbb{B}_x + 2\frac{d\log u_\downarrow}{dx}(x)Z_x dx + \alpha w(x)dx \tag{2.4.8}$$

If L is the generator of a Brownian motion on $(0,+\infty)$ killed when it hits 0, then $(\widehat{\mathcal{L}}^x_\alpha)_{x>0}$ has the same law as the square of a Bessel process of dimension 2α starting from 0 at $x=0$. If L is the generator of a Brownian motion on $(0,x_{max})$, killed when hitting the boundary, then $(\widehat{\mathcal{L}}^x_\alpha)_{0<x<x_{max}}$ has the same law as the square of a Bessel bridge of dimension 2α from 0 at $x=0$ to 0 at $x=x_{max}$.

PROOF. Let $x_0<x\in I$ and $\lambda_0,\lambda\geqslant 0$. Applying the identity (2.4.7) to the case of two points, we get that
(2.4.9)

$$\mathbb{E}\left[\exp\left(-\lambda_0\widehat{\mathcal{L}}^{x_0}_\alpha-\lambda\widehat{\mathcal{L}}^{x}_\alpha\right)\right]=\left((1+\lambda_0 G(x_0,x_0))(1+\lambda G(x,x))-\lambda_0\lambda(G(x_0,x))^2\right)^{-\alpha}$$

Let

$$\Lambda(x_0,\lambda_0):=\mathbb{E}\left[e^{-\lambda_0\widehat{\mathcal{L}}^{x_0}_\alpha}\right]=\left(\frac{G(x_0,x_0)}{G(x_0,x_0)+\lambda_0}\right)^\alpha$$

For $y\leqslant x$, let

$$\psi(y,x,\lambda):=\frac{G(x,y)G(y,x)\lambda}{G(y,y)(G(y,y)+\lambda\det_{y,x}G)}$$

$$\varphi(y,x,\lambda):=-\log\left(\frac{G(y,y)}{G(y,y)+\lambda\det_{y,x}G}\right)$$

One can check that the right-hand side of (2.4.9) equals

$$\Lambda(x_0,\lambda_0+\psi(x_0,x,\lambda))\exp(-\alpha\varphi(x_0,x,\lambda))$$

In particular for the conditional Laplace transform:

$$\mathbb{E}\left[\exp\left(-\lambda\widehat{\mathcal{L}}^{x}_\alpha\right)|\widehat{\mathcal{L}}^{x_0}_\alpha\right]=\exp\left(-\widehat{\mathcal{L}}^{x_0}_\alpha\psi(x_0,x,\lambda)\right)\exp(-\alpha\varphi(x_0,x,\lambda))\text{ a.s.}\tag{2.4.10}$$

Moreover

$$\begin{aligned}\frac{\partial\psi}{\partial y}(y,x,\lambda)=&W(u_\downarrow,u_\uparrow)(y)\psi(y,x,\lambda)^2-\frac{2}{u_\downarrow(y)}\frac{du_\downarrow}{dy}(y)\psi(y,x,\lambda)\\=&w(y)\psi(y,x,\lambda)^2-2\frac{d\log u_\downarrow}{dy}(y)\psi(y,x,\lambda)\end{aligned}$$

and

$$\frac{\partial\varphi}{\partial y}(y,x,\lambda)=-W(u_\downarrow,u_\uparrow)(y)\psi(y,x,\lambda)=-w(y)\psi(y,x,\lambda)$$

and we have the initial conditions $\psi(x,x,\lambda)=\lambda$ and $\varphi(x,x,\lambda)=0$. Thus (2.4.10) has the same form as (2.4.6) where $c(y)=\alpha w(y)$. Let $(Z_y)_{y\in I,y\geqslant x_0}$ be a solution to (2.4.8) with the initial condition Z_{x_0} being a gamma random variable of parameter α with mean $\alpha G(x_0,x_0)$. It follows from what precedes that $(\widehat{\mathcal{L}}^{x_0}_\alpha,\widehat{\mathcal{L}}^{x}_\alpha)$ has the same law as (Z_{x_0},Z_x). Using the conditional independence satisfied by the occupation field, we deduce that $(\widehat{\mathcal{L}}^{y}_\alpha)_{y\in I,y\geqslant x_0}$ has the same finite-dimensional marginals as $(Z_y)_{y\in I,y\geqslant x_0}$. Making x_0 converge to $\inf I$ along a countable subset, we get a consistent family of continuous stochastic processes, which induces a continuous stochastic process $(Z_y)_{y\in I}$ defined on whole I. It satisfies (2.4.8) and has the same finite-dimensional marginals as $(\widehat{\mathcal{L}}^{y}_\alpha)_{y\in I}$.

In case of a Brownian motion in $(0,+\infty)$ killed in 0, the equation (2.4.8) becomes

$$dZ_x=2\sqrt{Z_x}d\mathbb{B}_x+2\alpha\,dx$$

which is the SDE satisfied by the square of a Bessel process of dimension 2α. Moreover $(\widehat{\mathcal{L}}^x_\alpha)_{x>0}$ has the same one-dimensional marginals as the latter, more precisely $\widehat{\mathcal{L}}^x_\alpha$ is a gamma r.v. of parameter α with mean $2\alpha x$. This shows the equality in law.

In case of a Brownian motion in $(0, x_{max})$ killed in 0 and x_{max} the equation (2.4.8) becomes

$$dZ_x = 2\sqrt{Z_x}d\mathbb{B}_x + \frac{1}{x_{max} - x}Z_x dx + 2\alpha dx$$

which is the SDE satisfied by the square of a Bessel bridge of dimension 2α from 0 at $x = 0$ to 0 at $x = x_{max}$. Moreover the latter process and $(\hat{\mathcal{L}}_\alpha^x)_{0<x<x_{max}}$ have the same one-dimensional marginals, more precisely gamma r.v. of parameter α with mean $2\alpha(x_{max} - x)\frac{x}{x_{max}}$. Thus the two have the same law. $\square$

We showed that $(\hat{\mathcal{L}}_\alpha^x)_{x\in I}$ has the same finite-dimensional marginals as a continuous stochastic process. We will assume in the sequel and prove in section 2.5.2 that one can couple the Poisson ensemble $\mathcal{L}_\alpha$ and a continuous version of its occupation field $(\hat{\mathcal{L}}_\alpha^x)_{x\in I}$ on the same probability space. This does not follow trivially from the fact that the process $(\hat{\mathcal{L}}_\alpha^x)_{x\in I}$ has a continuous version. Consider the following counterexample: Let U be an uniform r.v. on $(0,1)$. Let $\mathcal{E}$ be a countable random set of Brownian excursions defined as follows: conditional on U $\mathcal{E}$ is a Poisson ensemble with intensity $\eta_{BM}^{>U} + \eta_{BM}^{<U}$. Let $(\hat{\mathcal{E}}_x)_{x\in\mathbb{R}}$ be the occupation field of $\mathcal{E}$. Then $\hat{\mathcal{E}}$ is continuous on $(-\infty, U)$ and $(U, +\infty)$ but not at U. Indeed $\hat{\mathcal{E}}_U = 0$ and

$$\lim_{x\rightarrow U^-} \hat{\mathcal{E}}_x = \lim_{x\rightarrow U^-} \hat{\mathcal{E}}_x = 1$$

Let $(\hat{\mathcal{E}}'_x)_{x\in\mathbb{R}}$ be the field defined by: $\hat{\mathcal{E}}'_x = \hat{\mathcal{E}}_x$ if $x \neq U$ and $\hat{\mathcal{E}}'_U = 1$. $(\hat{\mathcal{E}}'_x)_{x\in\mathbb{R}}$ is continuous and for any fixed $x \in \mathbb{R}$ $\hat{\mathcal{E}}'_x = \hat{\mathcal{E}}_x$ a.s. Thus $(\hat{\mathcal{E}}'_x)_{x\in\mathbb{R}}$ is a continuous version of the process $(\hat{\mathcal{E}}_x)_{x\in\mathbb{R}}$ but it can not be implemented as a sum of local time across the excursions in $\mathcal{E}$. As we will show in section 2.5.2, such a difficulty does not arise in case of $\mathcal{L}_\alpha$.

$(\hat{\mathcal{L}}_\alpha^x)_{x\in I}$ is an inhomogeneous continuous state branching with immigration. The branching mechanism is the same as for the local times of the diffusion X, given by (2.4.1). The immigration measure is $\alpha w(x)dx$. The interpretation is the following: given a loop in $\mathcal{L}_\alpha$, its family of local times performs a branching according to the mechanism (2.4.1), independently from the other loops. The immigration between x and $x + \Delta x$ comes from the loops whose minima belong to $(x, x + \Delta x)$. It is remarkable that although the immigration measure is absolutely continuous with respect to Lebesgue measure, there is only a countable number of moments at which immigration occurs. These are the positions of the minima of loops in $\mathcal{L}_\alpha$. Moreover the local time of each loop at its minimum is zero. For $x > a \in I$, let

$$\hat{\mathcal{L}}_\alpha^{(a),x} := \sum_{\substack{\gamma\in\mathcal{L}_\alpha \\ \min\gamma>a}} \ell^x(\gamma)$$

Let $a < b \in I$. For $j \leqslant n \in \mathbb{N}$, let $\Delta x_n := \frac{1}{n}(b-a)$ and let $x_{j,n} := a + j\Delta x_n$. Then $\left(\hat{\mathcal{L}}_\alpha^{(x_{j-1}),x_j}\right)_{1\leqslant j\leqslant n}$ is a sequence of independent gamma r.v. of parameter α and the mean of $\hat{\mathcal{L}}_\alpha^{(x_{j-1}),x_j}$ is $\alpha\left(G(x_j,x_j) - \frac{G(x_{j-1},x_j)G(x_j,x_{j-1})}{G(x_{j-1},x_{j-1})}\right)$. For n large

$$G(x_j,x_j) - \frac{G(x_{j-1},x_j)G(x_j,x_{j-1})}{G(x_{j-1},x_{j-1})} = w(x_{j-1})\Delta x_n + o(\Delta x_n)$$

and $o(\Delta x_n)$ is uniform in j. Thus

$$\lim_{n\rightarrow+\infty}\mathbb{E}\Big[\sum_{j=1}^{n}\hat{\mathcal{L}}_{\alpha}^{(x_{j-1}),x_j}\Big]=$$

$$\lim_{n\rightarrow+\infty}\alpha\sum_{j=1}^{n}\Big(G(x_j,x_j)-\frac{G(x_{j-1},x_j)G(x_j,x_{j-1})}{G(x_{j-1},x_{j-1})}\Big)=\alpha\int_a^b w(x)dx$$

and

$$\lim_{n\rightarrow+\infty}Var\Big(\sum_{j=1}^{n}\hat{\mathcal{L}}_{\alpha}^{(x_{j-1}),x_j}\Big)=$$

$$\lim_{n\rightarrow+\infty}\alpha\sum_{j=1}^{n}\Big(G(x_j,x_j)-\frac{G(x_{j-1},x_j)G(x_j,x_{j-1})}{G(x_{j-1},x_{j-1})}\Big)^2=0$$

It follows that $\sum_{j=1}^{n}\hat{\mathcal{L}}_{\alpha}^{(x_{j-1}),x_j}$ converges in probability to $\alpha\int_a^b w(x)\,dx$. This is consistent with our interpretation of immigration.

Next proposition deals with the zeroes of the occupation field.

PROPOSITION 2.4.5. *Let $x_0\in I$. If $\int_{\inf I}^{x_0}w(x)dx<+\infty$ then*

$$\lim_{x\rightarrow\inf I}\hat{\mathcal{L}}_{\alpha}^{x}=0$$

Analogous result holds if $\int_{x_0}^{\sup I}w(x)\,dx<+\infty$.

If $\alpha\geqslant 1$, then the continuous process $(\hat{\mathcal{L}}_{\alpha}^{x})_{x\in I}$ stays almost surely positive on I. If $\alpha<1$ then $(\hat{\mathcal{L}}_{\alpha}^{x})_{x\in I}$ hits 0 infinitely many times on I.

PROOF. If $\int_{\inf I}^{x_0}w(x)dx<+\infty$, then $L+\kappa$, where κ is the killing measure of L, is also the generator of a transient diffusion. We can couple $(\hat{\mathcal{L}}_{\alpha,L}^{x})_{x\in I}$ and $(\hat{\mathcal{L}}_{\alpha,L+\kappa}^{x})_{x\in I}$ on the same probability space such that a.s. for all $x\in I$, $\hat{\mathcal{L}}_{\alpha,L}^{x}\leqslant\hat{\mathcal{L}}_{\alpha,L+\kappa}^{x}$. But according to property 2.4.2 (i), $(\hat{\mathcal{L}}_{\alpha,L+\kappa}^{x})_{x\in I}$ is just a scale changed square of Bessel process starting from 0 or square of a Bessel bridge from 0 to 0. Thus

$$\lim_{x\rightarrow\inf I}\hat{\mathcal{L}}_{\alpha,L}^{x}\leqslant\lim_{x\rightarrow\inf I}\hat{\mathcal{L}}_{\alpha,L+\kappa}^{x}=0$$

Regarding the number of zeros of $(\hat{\mathcal{L}}_{\alpha}^{x})_{x\in I}$ on I, property 2.4.2 ensures that it remains unchanged if we apply scale, time changes and conjugations to L. Since any generator of a transient diffusion is equivalent through latter transformation to the generator of a Brownian motion on $(0,+\infty)$ killed in 0, the result on the number of zeros of $(\hat{\mathcal{L}}_{\alpha}^{x})_{x\in I}$ follows from standard properties of Bessel processes. □

In [**SW12**] respectively [**JL13**] are studied the clusters of loops induced by a Poisson ensemble of loops in the setting of planar Brownian motion respectively Markovian jump processes on graphs. In our setting of one dimensional diffusions the description of such clusters is simple and is related to the zeros of the occupation field. We introduce an equivalence relation on the loops of $\mathcal{L}_\alpha$: γ is in the same class as $\tilde{\gamma}$ if there is a chain of loops $\gamma_0,\gamma_1,\dots,\gamma_n$ in $\mathcal{L}_\alpha$ such that $\gamma_0=\gamma$, $\gamma_n=\tilde{\gamma}$ and for all $i\in\{0,1,\dots,n-1\}$, $\gamma_i([0,T(\gamma_i)])\cap\gamma_{i+1}([0,T(\gamma_{i+1})])\neq\varnothing$. A cluster is the union of all $\gamma([0,T(\gamma)])$ where the loops γ belong to the same equivalence class. It is a subinterval of I. By definition clusters corresponding to different equivalence classes are disjoint.

PROPOSITION 2.4.6. *Let L be the generator of a transient diffusion on I. If $\alpha\geqslant 1$, the loops in $\mathcal{L}_\alpha$ form a single cluster: I. If $\alpha\in(0,1)$, there are infinitely many clusters.*

These are the maximal open intervals on which $(\hat{\mathcal{L}}_\alpha^x)_{x\in I}$ *is positive. In case of the Brownian motion on* $(0,+\infty)$ *killed at* 0*, the clusters correspond to the jumps of a stable subordinator with index* $1-\alpha$*. In case of a general diffusion, by performing a change of scale of derivative* $\frac{1}{2}\frac{w}{u_\downarrow^2}$*, we reduce the problem to the previous case. In case of the Brownian motion on* $(0,+\infty)$ *killed at* 0 *and with uniform killing* κ*, the clusters correspond to the jumps of a subordinator with Levy measure* $1_{x>0}\frac{e^{2\sqrt{2\kappa}x}dx}{(e^{2\sqrt{2\kappa}x}-1)^{2-\alpha}}$.

Proof. Assume that $\mathcal{L}_\alpha$ and a continuous version of $(\hat{\mathcal{L}}_\alpha^x)_{x\in I}$ are defined on the same probability space. Almost surely the following holds

- Given $\gamma\neq\gamma'\in\mathcal{L}_\alpha$, $\min\gamma\neq\max\gamma'$ and $\max\gamma\neq\min\gamma'$.
- For all $\gamma\in\mathcal{L}_\alpha$, $\ell^{\min\gamma}(\gamma)=\ell^{\max\gamma}(\gamma)=0$ and $\ell^x(\gamma)$ is positive for $x\in(\min\gamma,\max\gamma)$.

Whenever the above two conditions hold it follows deterministically that the clusters are the intervals on which $(\hat{\mathcal{L}}_\alpha^x)_{x\in I}$ stays positive. We deduce then the number of clusters from proposition 2.4.5.

If L is the generator of the Brownian motion on $(0,+\infty)$ killed at 0, then $(\hat{\mathcal{L}}_\alpha^x)_{x\in I}$ is the square of a Bessel process of dimension 2α and its excursions correspond to the jumps of a stable subordinator with index $1-\alpha$.

In general a generator L has the same measure on loops as $Conj(u_\downarrow,L)$. A diffusion of generator $Conj(u_\downarrow,L)$ transforms through a change of time and a change of scale of density $\frac{1}{2}\frac{w}{u_\downarrow^2}$ into a Brownian motion on $(0,+\infty)$ killed at 0. For the clusters, the change of time does not matter.

In case of a Brownian motion on $(0,+\infty)$ killed at 0 and with uniform killing κ, we can take $u_\downarrow(x)=e^{-\sqrt{2\kappa}x}$. The scale function is then

$$S(x)=\int_0^x\frac{dy}{u_\downarrow(y)^2}=\int_0^x e^{2\sqrt{2\kappa}y}dy=\frac{1}{2\sqrt{2\kappa}}(e^{2\sqrt{2\kappa}x}-1)$$

Let $(Y_t)_{t\geqslant 0}$ be an $1-\alpha$ stable subordinator with Levy measure $1_{y>0}y^{-(2-\alpha)}dy$. The clusters of $\mathcal{L}_{\alpha,\frac{1}{2}\frac{d^2}{dx^2}-\kappa}$ correspond to the jumps of the process $(S^{-1}(Y_t))_{t\geqslant 0}$, which is not a subordinator. We will that nevertheless the latter process the same set of jumps as a subordinator with Levy measure $1_{x>0}\frac{e^{2\sqrt{2\kappa}x}dx}{(e^{2\sqrt{2\kappa}x}-1)^{2-\alpha}}$. Let $\varepsilon>0$ and $(Y_{\varepsilon,t})_{t\geqslant 0}$ be the process obtained from $(Y_t)_{t\geqslant 0}$ by removing all the jumps of height less then ε. By construction $Y_{\varepsilon,t}\leqslant Y_t$. $(S^{-1}(Y_{\varepsilon,t}))_{t\geqslant 0}$ is a Markov process: given the position of $S^{-1}(Y_{\varepsilon,t})$ at time t, the process waits an exponential holding time with inverse of the mean equal to

$$\int_\varepsilon^{+\infty}\frac{dy}{y^{2-\alpha}}=\frac{1}{(1-\alpha)\varepsilon^{1-\alpha}}$$

Once a jump occurs, the jump of Y_ε is distributed according the probability

$$1_{y>\varepsilon}(1-\alpha)\varepsilon^{1-\alpha}\frac{dy}{y^{2-\alpha}}$$

The distribution of the corresponding jump of $S^{-1}(Y_{\varepsilon,t})$ is obtained by pushing forward the above probability by the map $y \mapsto S^{-1}(y+Y_{\varepsilon,t}) - S^{-1}(Y_{\varepsilon,t})$ which gives

$$1_{x>S^{-1}(\varepsilon+Y_{\varepsilon,t})-S^{-1}(Y_{\varepsilon,t})}(1-\alpha)\varepsilon^{1-\alpha}\frac{(2\sqrt{2\kappa})^{2-\alpha}e^{2\sqrt{2\kappa}(x+S^{-1}(Y_{\varepsilon,t}))}dx}{\left(e^{2\sqrt{2\kappa}(x+S^{-1}(Y_{\varepsilon,t}))}-e^{2\sqrt{2\kappa}S^{-1}(Y_{\varepsilon,t})}\right)^{2-\alpha}}$$
$$= 1_{x>S^{-1}(\varepsilon+Y_{\varepsilon,t})-S^{-1}(Y_{\varepsilon,t})}(1-\alpha)\varepsilon^{1-\alpha}(2\sqrt{2\kappa})^{2-\alpha}e^{-(1-\alpha)2\sqrt{2\kappa}S^{-1}(Y_{\varepsilon,t})}\frac{e^{2\sqrt{2\kappa}x}dx}{(e^{2\sqrt{2\kappa}x}-1)^{2-\alpha}}$$
$$= 1_{x>S^{-1}(\varepsilon+Y_{\varepsilon,t})-S^{-1}(Y_{\varepsilon,t})}(1-\alpha)\varepsilon^{1-\alpha}\frac{(2\sqrt{2\kappa})^{2-\alpha}}{(1+2\sqrt{2\kappa}Y_{\varepsilon,t})^{1-\alpha}}\frac{e^{2\sqrt{2\kappa}x}dx}{(e^{2\sqrt{2\kappa}x}-1)^{2-\alpha}}$$

Consider now the random time change

$$\tau_\varepsilon(v) := \inf\left\{t \geqslant 0 \Big| \int_0^t \frac{(2\sqrt{2\kappa})^{2-\alpha}}{(1+2\sqrt{2\kappa}Y_{\varepsilon,s})^{1-\alpha}}ds \geqslant v\right\}$$

and at the limit as $\varepsilon \to 0$

$$\tau(v) := \inf\left\{t \geqslant 0 \Big| \int_0^t \frac{(2\sqrt{2\kappa})^{2-\alpha}}{(1+2\sqrt{2\kappa}Y_{\varepsilon,s})^{1-\alpha}}ds \geqslant v\right\}$$

For the time-changed process $(S^{-1}(Y_{\varepsilon,\tau_\varepsilon(v)}))_{v\geqslant 0}$, the rate of jumps of height belonging to $[x, x+dx]$ is

$$\begin{cases} \dfrac{e^{2\sqrt{2\kappa}x}dx}{(e^{2\sqrt{2\kappa}x}-1)^{2-\alpha}} & \text{if } x > S^{-1}(\varepsilon+Y_{\varepsilon,\tau_\varepsilon(v)}) - S^{-1}(Y_{\varepsilon,\tau_\varepsilon(v)}) \\ 0 & \textit{otherwise} \end{cases}$$

Thus, as ε goes to 0, on one hand the process $(S^{-1}(Y_{\varepsilon,\tau_\varepsilon(v)}))_{v\geqslant 0}$ converges in law to $(S^{-1}(Y_{\tau(v)}))_{v\geqslant 0}$ and on the other hand it converges in law to a subordinator with Levy measure $1_{x>0}\frac{e^{2\sqrt{2\kappa}x}dx}{(e^{2\sqrt{2\kappa}x}-1)^{2-\alpha}}$. □

The clusters coalesce when α increases and fragment when α decreases. Some information on the coalescence of clusters delimited by the zeroes of Bessel processes is given in [**BP99**], section 3. This clusters can be obtained as a limit of clusters of discrete loops on discrete subsets. In case of a symmetric jump process to the nearest neighbours on $\varepsilon\mathbb{N}$, if $\alpha > 1$, there are finitely many clusters, and if $\alpha \in (0,1)$, there are infinitely many clusters and these clusters are given by the holding times of a renewal process, which suitable normalized converges in law as $\varepsilon \to 0^+$ to the inverse of a stable subordinator with index $1-\alpha$. See remark 3.3 in [**JL13**].

We can consider the occupation field $(\hat{\mathcal{L}}^x_{\alpha,L})_{x\in I}$ if L is not the generator of a diffusion but contains creation of mass as in (2.2.19). In this setting, if h is a positive continuous function on I such that $\frac{d^2h}{dx^2}$ is a signed measure, then for all $x \in I$

$$\hat{\mathcal{L}}^x_{\alpha,Conj(h,L)} = \frac{1}{h(x)^2}\hat{\mathcal{L}}^x_{\alpha,L}$$

It follows that if $L \in \mathfrak{D}^-$ then for all $x \in I$, $\hat{\mathcal{L}}^x_{\alpha,L} < +\infty$ a.s. and if $L \in \mathfrak{D}^0$ then for all $x \in I$, $\hat{\mathcal{L}}^x_{\alpha,L} = +\infty$ a.s. If $L \in \mathfrak{D}^+$, then according to proposition 2.2.9 (iv), there is a positive Radon measure $\tilde{\kappa}$ such that $L - \tilde{\kappa} \in \mathfrak{D}^0$. Then for all $x \in I$, $\hat{\mathcal{L}}^x_{\alpha,L} \geqslant \hat{\mathcal{L}}^x_{\alpha,L-\tilde{\kappa}} = +\infty$. If $L \in \mathfrak{D}^-$, then properties 2.4.2 (i) and (ii) still hold. The description given by the property 2.4.3 of the finite-dimensional marginals of $(\hat{\mathcal{L}}^x_\alpha)_{x\in I}$ is still true, although the case of creation of mass wasn't considered in [**Jan11**]. $(\hat{\mathcal{L}}^x_\alpha)_{x\in I}$ still satisfies the SDE (2.4.8).

PROPOSITION 2.4.7. *Let $L \in \mathfrak{D}^-$ and $\tilde{\nu}$ a finite signed measure with compact support in I. Then there is equivalence between*

(i) $\mathbb{E}\left[\exp\left(\int_I \hat{\mathcal{L}}^x_{\alpha,L}\tilde{\nu}(dx)\right)\right] < +\infty$

(ii) $L + \tilde{\nu} \in \mathfrak{D}^-$

If $L + \tilde{\nu} \in \mathfrak{D}^-$ then for $s \in [0,1]$

$$\mathbb{E}\left[\exp\left(\int_I \hat{\mathcal{L}}^x_{\alpha,L}\tilde{\nu}(dx)\right)\right] = \exp\left(\alpha\int_0^1\int_I G_{L+s\tilde{\nu}}(x,x)\tilde{\nu}(dx)ds\right) \tag{2.4.11}$$

PROOF. First observe that $\int_I \hat{\mathcal{L}}^x_{\alpha,L}|\tilde{\nu}|(dx)$ is almost surely finite because $|\tilde{\nu}|$ is finite and has compact support and $(\hat{\mathcal{L}}^x_{\alpha,L})_{x\in I}$ is continuous. Also observe that $\mathfrak{D}^-$ is convex. So if $L + \tilde{\nu} \in \mathfrak{D}^-$, then for all $s \in [0,1]$, $L + s\tilde{\nu} \in \mathfrak{D}^-$.

(i) implies (ii): Let $\mathbb{P}_{\mathcal{L}_{\alpha,L}}$ be the law of $\mathcal{L}_{\alpha,L}$ and $\mathbb{P}_{\mathcal{L}_{\alpha,L+\tilde{\nu}}}$ be the law of $\mathcal{L}_{\alpha,L+\tilde{\nu}}$. There is an absolute continuity relation between the intensity measures:

$$\mu_{L+\tilde{\nu}}(d\gamma) = \exp\left(\int_I \ell^x(\gamma)\right)\mu_L(d\gamma)$$

In case (i) is true $\mathbb{P}_{\mathcal{L}_{\alpha,L+\tilde{\nu}}}$ is absolutely continuous with respect to $\mathbb{P}_{\mathcal{L}_{\alpha,L}}$ and

$$d\mathbb{P}_{\mathcal{L}_{\alpha,L+\tilde{\nu}}} = \frac{\exp\left(\int_I \hat{\mathcal{L}}^x_{\alpha,L}\tilde{\nu}(dx)\right)}{\mathbb{E}\left[\exp\left(\int_I \hat{\mathcal{L}}^x_{\alpha,L}\tilde{\nu}(dx)\right)\right]}d\mathbb{P}_{\mathcal{L}_{\alpha,L}} \tag{2.4.12}$$

But this can not be if $L+\tilde{\nu} \notin \mathfrak{D}^-$ because then for any $x \in I$, $\hat{\mathcal{L}}^x_{\alpha,L} < +\infty$ and $\hat{\mathcal{L}}^x_{\alpha,L+\tilde{\nu}} = +\infty$. Thus necessarily $L + \tilde{\nu} \in \mathfrak{D}^-$.

(ii) implies (i): We first assume that $\tilde{\nu}$ is a positive measure and $L + \tilde{\nu} \in \mathfrak{D}^-$. Then $\mathbb{P}_{\mathcal{L}_{\alpha,L}}$ is absolutely continuous with respect to $\mathbb{P}_{\mathcal{L}_{\alpha,L+\tilde{\nu}}}$ and

$$d\mathbb{P}_{\mathcal{L}_{\alpha,L}} = \frac{\exp\left(-\int_I \hat{\mathcal{L}}^x_{\alpha,L+\tilde{\nu}}\tilde{\nu}(dx)\right)}{\mathbb{E}\left[\exp\left(-\int_I \hat{\mathcal{L}}^x_{\alpha,L+\tilde{\nu}}\tilde{\nu}(dx)\right)\right]}d\mathbb{P}_{\mathcal{L}_{\alpha,L+\tilde{\nu}}}$$

Inverting the above absolute continuity relation, we get that

$$\mathbb{E}\left[\exp\left(\int_I \hat{\mathcal{L}}^x_{\alpha,L}\tilde{\nu}(dx)\right)\right] = \mathbb{E}\left[\exp\left(-\int_I \hat{\mathcal{L}}^x_{\alpha,L+\tilde{\nu}}\tilde{\nu}(dx)\right)\right]^{-1} < +\infty$$

If $\tilde{\nu}$ is not positive, let $\tilde{\nu}^+$ and $-\tilde{\nu}^-$ be its positive respectively negative part. Then

$$\begin{aligned}\mathbb{E}\Bigg[\exp\left(\int_I \hat{\mathcal{L}}^x_{\alpha,L}\tilde{\nu}(dx)\right)\Bigg]& \\ &= \mathbb{E}\left[\exp\left(\int_I \hat{\mathcal{L}}^x_{\alpha,L-\tilde{\nu}^-}\tilde{\nu}^+(dx)\right)\right]\mathbb{E}\left[\exp\left(-\int_I \hat{\mathcal{L}}^x_{\alpha,L}\tilde{\nu}^-(dx)\right)\right] \\ &= \frac{\mathbb{E}\left[\exp\left(-\int_I \hat{\mathcal{L}}^x_{\alpha,L}\tilde{\nu}^-(dx)\right)\right]}{\mathbb{E}\left[\exp\left(-\int_I \hat{\mathcal{L}}^x_{\alpha,L+\tilde{\nu}}\tilde{\nu}^+(dx)\right)\right]} < +\infty\end{aligned}$$

For the expression (2.4.11) of exponential moments:

$$\frac{d}{ds}\mathbb{E}\left[\exp\left(s\int_I \hat{\mathcal{L}}^x_{\alpha,L}\tilde{\nu}(dx)\right)\right] = \mathbb{E}\left[\int_I \hat{\mathcal{L}}^x_{\alpha,L}\tilde{\nu}(dx)\exp\left(s\int_I \hat{\mathcal{L}}^x_{\alpha,L}\tilde{\nu}(dx)\right)\right] \tag{2.4.13}$$

From the absolute continuity relation (2.4.12) follows that the right-hand side of (2.4.13) equals

$$\alpha\int_I G_{L+s\tilde{\nu}}(x,x)\tilde{\nu}(dx)\mathbb{E}\left[\exp\left(s\int_I \hat{\mathcal{L}}^x_{\alpha,L}\tilde{\nu}(dx)\right)\right]$$

This implies (2.4.11). □

As in discrete space case, the above exponential moments can be expressed using determinants. On the complex Hilbert space $\mathbb{L}^2(d|\tilde{\nu}|)$ define for $s\in[0,1]$ the operators

$$(\mathfrak{G}_{s\tilde{\nu}}f)(x) := \int_I G_{L+s\tilde{\nu}}(x,y)f(y)\tilde{\nu}(dy)$$

$$(|\mathfrak{G}^*_{s\tilde{\nu}}|f)(x) := \int_I G_{L+s\tilde{\nu}}(x,y)f(y)|\tilde{\nu}|(dy)$$

The operator $|\mathfrak{G}^*_{s\tilde{\nu}}|$ is self-adjoint, positive semi-definite with continuous kernel function, and according to [**Sim05**], theorem 2.12, it is trace class. Since trace class operators form a two-sided ideal in the algebra of bounded operators, $\mathfrak{G}_{s\tilde{\nu}}$ is also trace class. Moreover

$$Tr(\mathfrak{G}_{s\tilde{\nu}}) = \int_I G_{L+s\tilde{\nu}}(x,x)\tilde{\nu}(dx) \tag{2.4.14}$$

The determinant $\det(Id+\mathfrak{G}_{s\tilde{\nu}})$ is well defined as a converging product of its eigenvalues (see [**Sim05**], chapter 3).

PROPOSITION 2.4.8.

$$\exp\left(\alpha\int_0^1\int_I G_{L+s\tilde{\nu}}(x,x)\tilde{\nu}(dx)ds\right) = (\det(Id+\mathfrak{G}_{\tilde{\nu}}))^\alpha$$

PROOF. $\mathfrak{G}_{\tilde{\nu}}$ has only real eigenvalues. Indeed, let λ be such an eigenvalue and f a non zero eigenfunction for λ. The sign of $\tilde{\nu}$, $\mathrm{sign}(\tilde{\nu})$, is a $\{-1,+1\}$-valued function defined $d|\tilde{\nu}|$ almost everywhere.

$$\int_I(\mathrm{sign}(\tilde{\nu})\bar{f})(x)|\mathfrak{G}_{\tilde{\nu}}|(\mathrm{sign}(\tilde{\nu})f)(x)|\tilde{\nu}|(dx) = \lambda\int_I|f|^2(x)\tilde{\nu}(dx) \tag{2.4.15}$$

The left-hand side of (2.4.15) is non-negative. If the right-hand side of (2.4.15) is non-zero, then λ is real. If it is zero, consider $f_\varepsilon := f+\varepsilon\,\mathrm{sign}(\tilde{\nu})f$. Then

$$\lambda = \lim_{\varepsilon\to0^+}\frac{1}{2\varepsilon}\left(\int_I(\mathrm{sign}(\tilde{\nu})\bar{f}_\varepsilon)(x)|\mathfrak{G}_{\tilde{\nu}}|(\mathrm{sign}(\tilde{\nu})f_\varepsilon)(x)|\tilde{\nu}|(dx)\right)\left(\int_I|f|^2(x)|\tilde{\nu}|(dx)\right)^{-1}$$

and thus λ is real.

The operators $\mathfrak{G}_{s\tilde{\nu}}$ are compact and the characteristic space corresponding to each of their non-zero eigenvalue is of finite dimension. Let $(\lambda_i)_{i\geqslant0}$ be the non-increasing sequence of positive eigenvalues of $\mathfrak{G}_{\tilde{\nu}}$. Each eigenvalue λ_i appears as many times as the dimension of its characteristic space $\ker(\mathfrak{G}_{\tilde{\nu}}-\lambda_i Id)^n$ (n large enough). Similarly let $(-\tilde{\lambda}_j)_{j\geqslant0}$ be the non-decreasing sequence of the negative eigenvalues of $\mathfrak{G}_{\tilde{\nu}}$. Let $s\in[0,1]$. According to the resolvent identity (lemma 2.2.8), the operators $\mathfrak{G}_{\tilde{\nu}}$ and $\mathfrak{G}_{s\tilde{\nu}}$ commute and satisfy the relation

$$\mathfrak{G}_{\tilde{\nu}}\mathfrak{G}_{s\tilde{\nu}} = \mathfrak{G}_{s\tilde{\nu}}\mathfrak{G}_{\tilde{\nu}} = \frac{1}{1-s}(\mathfrak{G}_{\tilde{\nu}}-\mathfrak{G}_{s\tilde{\nu}}) \tag{2.4.16}$$

Since $\mathfrak{G}_{\tilde{\nu}}$ and $\mathfrak{G}_{s\tilde{\nu}}$ commute, these operators have common characteristic spaces. From (2.4.16) follows that $\left(\frac{\lambda_i}{1+(1-s)\lambda_i}\right)_{i\geqslant0}$ is a non-increasing sequence of positive eigenvalues of $\mathfrak{G}_{s\tilde{\nu}}$. If $\frac{-1}{1-s}$ is not an eigenvalue of $\mathfrak{G}_{\tilde{\nu}}$, then $\left(\frac{-\tilde{\lambda}_j}{1-(1-s)\tilde{\lambda}_j}\right)_{j\geqslant0}$ is also a sequence of eigenvalues of $\mathfrak{G}_{s\tilde{\nu}}$. But the family of operators $(\mathfrak{G}_{s\tilde{\nu}})_{s\in[0,1]}$ is bounded. Thus none of $\frac{-\tilde{\lambda}_j}{1-(1-s)\tilde{\lambda}_j}$ can blow up when s varies. So it turns out that $\mathfrak{G}_{\tilde{\nu}}$ has no eigenvalues in $(-\infty,-1]$. From (2.4.14) we get

$$\int_I G_{L+s\tilde{\nu}}(x,x)\tilde{\nu}(dx) = \sum_{i\geqslant 0}\frac{\lambda_i}{1+(1-s)\lambda_i} - \sum_{j\geqslant 0}\frac{\tilde{\lambda}_j}{1-(1-s)\tilde{\lambda}_j}$$

The above sum is absolutely convergent, uniformly for $s\in[0,1]$. Integrating over $[0,1]$ yields

$$\int_0^1\int_I G_{L+s\tilde{\nu}}(x,x)\tilde{\nu}(dx)ds = \sum_{i\geqslant 0}\log(1+\lambda_i) + \sum_{j\geqslant 0}\log(1-\tilde{\lambda}_j)$$

This concludes the proof. □

2.4.3. Dynkin's isomorphism. In this subsection we recall the equality in law observed in [**Jan11**] between the occupation field $(\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{x\in I}$ and the square of a Gaussian Free Field and show how to derive from this particular versions of Dynkin's isomorphism.

Let L be a generator of a transient diffusion on I of form (2.2.11). Let $(\phi_x)_{x\in I}$ be a centred Gaussian process with variance-covariance function:

$$\mathbb{E}[\phi_x\phi_y] = G(x,y)$$

$(\phi_x)_{x\in I}$ is the Gaussian Free Field associated to L. Let $\widetilde{S}$ be a primitive of $\frac{w}{u_\downarrow^2}$. Then $\widetilde{S}(\sup I) = +\infty$. Moreover $\widetilde{S}(\inf I) > -\infty$ because L is the generator of a transient diffusion. $\left(\frac{1}{u_\downarrow(\widetilde{S}^{-1}(a))}\phi_{\widetilde{S}^{-1}(a)}\right)_{a\in\widetilde{S}(I)}$ is a standard Brownian motion starting from 0 at $\widetilde{S}(\inf I)$. In particular $(\phi_x)_{x\in I}$ is inhomogeneous Markov and has continuous sample paths.

It was shown in [**Jan11**], chapter 5, that when $\alpha=\frac{1}{2}$ $(\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{x\in I}$ has the same law as $(\frac{1}{2}\phi_x^2)_{x\in I}$. In case of a Brownian motion on $(0,+\infty)$ killed in 0, $(\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{x>0}$ is the square of a standard Brownian motion starting from 0. In case of a Brownian motion on $(0,x_{max})$ killed in 0 and x_{max}, $(\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{0<x<x_{max}}$ is the square of a standard Brownian bridge on $[0,x_{max}]$ from 0 to 0. In case of a Brownian motion on $\mathbb{R}$ with constant killing rate κ, $(\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{x\in\mathbb{R}}$ is the square of a stationary Ornstein–Uhlenbeck process.

The relation between the occupation field of a Poisson ensemble of Markov loops and the square of a Gaussian Fee Field extends the Dynkin's isomorphism which we state below (see [**Dyn84a**] and [**Dyn84c**]):

THEOREM(DYNKIN'S ISOMORPHISM). *Let $x_1,x_2,\dots,x_{2n}\in I$. Then for any non-negative measurable functional F on continuous paths on I,*

$$\text{(2.4.17)}\quad \mathbb{E}_\phi\left[\prod_{i=1}^{2n}\phi_{x_i}F((\frac{1}{2}\phi_x^2)_{x\in I})\right] = \sum_{pairings}\int\mathbb{E}_\phi\Big[F((\frac{1}{2}\phi_x^2+\sum_{j=1}^{n}\ell^x(\gamma_j))_{x\in I})\Big]\prod_{pairs}\mu^{y_j,z_j}(d\gamma_j)$$

where $\sum_{pairings}$ means that the n pairs $\{y_j,z_j\}$ are formed with all $2n$ points x_i in all $\frac{(2n)!}{2^n n!}$ possible ways.

Next we will show that in case $x_i = x_{i+n}$, for $i\in\{1,\dots,n\}$, i.e. $\prod_{i=1}^{2n}\phi_{x_i}$ being a product of squares $\prod_{i=1}^{n}\phi_{x_i}^2$, one can deduce the Dynkin's isomorphism from the relation between the square of the Gaussian Free Field and the occupation field. In [**LMR15**] and [**FR14**] this is only done in case $n=1$ and $x_1=x_2$ using the Palm's identity for Poissonian ensembles and the analogue of the relation (2.3.14). To generalize for any n we will use an extended version of Palm's identity and the absolute continuity relation given by proposition 2.3.19 (ii).

LEMMA 2.4.9. *Let $\mathcal{E}$ be an abstract Polish space. Let $\mathfrak{M}(\mathcal{E})$ be the space of locally finite measures on $\mathcal{E}$ and let $\mathcal{M} \in \mathfrak{M}(\mathcal{E})$. Let Φ be a Poisson random measure of intensity $\mathcal{M}$. Let H be a positive measurable function on $\mathfrak{M}(\mathcal{E}) \times \mathcal{E}^n$. Let $\mathfrak{P}_n$ be the set of partitions of $\{1, \dots, n\}$. If $\mathcal{P} \in \mathfrak{P}_n$ and $i \in \{1, \dots, n\}$, then $\mathcal{P}(i)$ will be the equivalence class of i under $\mathcal{P}$. The following identity holds:*

$$\mathbb{E}\Big[\int_{\mathcal{E}^n} H(\Phi, q_1, \dots, q_n) \prod_{i=1}^{n} \Phi(dq_i)\Big] = \sum_{\mathcal{P} \in \mathfrak{P}_n} \int_{\mathcal{E}^{\sharp\mathcal{P}}} \mathbb{E}\Big[H(\Phi + \sum_{c \in \mathcal{P}} \delta_{q_c}, q_{\mathcal{P}(1)}, \dots, q_{\mathcal{P}(n)})\Big] \prod_{c \in \mathcal{P}} \mathcal{M}(dq_c) \tag{2.4.18}$$

PROOF. We will make a recurrence over n. If $n = 1$, (2.4.18) is the Palm's identity for Poisson random measures. Assume that $n \geqslant 2$ and that (2.4.18) holds for $n - 1$. We set

$$\widetilde{H}(\Phi, q_1, \dots, q_{n-1}) := \int_{\mathcal{E}} H(\Phi, q_1, \dots, q_{n-1}, q_n) \Phi(dq_n)$$

Then

$$\begin{aligned} \mathbb{E}\Big[\int_{\mathcal{E}^n} H(\Phi, q_1, \dots, q_{n-1}, q_n) \prod_{i=1}^{n} \Phi(dq_i)\Big] &= \mathbb{E}\Big[\int_{\mathcal{E}^{n-1}} \widetilde{H}(\Phi, q_1, \dots, q_{n-1}) \prod_{i=1}^{n-1} \Phi(dq_i)\Big] \\ &= \sum_{\mathcal{P}' \in \mathfrak{P}_{n-1}} \int_{\mathcal{E}^{\sharp\mathcal{P}'}} \mathbb{E}\Big[\int_{\mathcal{E}} H(\Phi + \sum_{c' \in \mathcal{P}'} \delta_{q_{c'}}, q_{\mathcal{P}'(1)}, \dots, q_{\mathcal{P}'(n-1)}, q_n) \\ &\qquad \times (\Phi(dq_n) + \sum_{c' \in \mathcal{P}'} \delta_{q_{c'}}(dq_n))\Big] \prod_{c' \in \mathcal{P}'} \mathcal{M}(dq_{c'}) \end{aligned} \tag{2.4.19}$$

Given a partition $\mathcal{P}' \in \mathfrak{P}_{n-1}$, one can extend it to a partition of $\{1, \dots, n-1, n\}$ either by deciding that n is single in its equivalence class or by choosing an equivalence class $c' \in \mathcal{P}'$ and adjoining n to it. In the identity (2.4.19) the first case corresponds to the integration with respect to $\Phi(dq_n)$, and according to Palm's identity

$$\mathbb{E}\left[\int_{\mathcal{E}} H(\Phi + \sum_{c' \in \mathcal{P}'} \delta_{q_{c'}}, q_{\mathcal{P}'(1)}, \dots, q_{\mathcal{P}'(n-1)}, q_n) \Phi(dq_n)\right] = \int_{\mathcal{E}} \mathbb{E}\left[H(\Phi + \sum_{c' \in \mathcal{P}'} \delta_{q_{c'}}, q_{\mathcal{P}'(1)}, \dots, q_{\mathcal{P}'(n-1)}, q_n)\right] \mathcal{M}(dq_n)$$

The second case corresponds to the integration with respect to $\delta_{q_{c'}}(dq_n)$. Thus the right-hand side of (2.4.19) equals the right-hand side of (2.4.18). □

Next we show how derive a particular case of Dynkin's isomorphism using the above extended Palm's formula. Since $(\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{x \in I}$ and $(\frac{1}{2}\phi_x^2)_{x \in I}$ are equal in law:

$$\mathbb{E}_\phi\left[\prod_{i=1}^{n} \phi_{x_i}^2 F((\frac{1}{2}\phi_x^2)_{x \in I})\right] = 2^n \mathbb{E}_{\mathcal{L}_{\frac{1}{2}}}\left[\prod_{i=1}^{n} \widehat{\mathcal{L}}^{x_i}_{\frac{1}{2}} F((\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{x \in I})\right]$$

Applying lemma 2.4.9 we get that

$$\mathbb{E}_{\mathcal{L}_{\frac{1}{2}}}\left[\prod_{i=1}^{n}\hat{\mathcal{L}}_{\frac{1}{2}}^{x_i}F((\hat{\mathcal{L}}_{\frac{1}{2}}^{x})_{x\in I})\right]=$$
$$\sum_{\mathcal{P}\in\mathfrak{P}_n}\int\prod_{i=1}^{n}\ell^{x_i}(\gamma_{\mathcal{P}(i)})\mathbb{E}\left[F((\hat{\mathcal{L}}_{\frac{1}{2}}^{x}+\sum_{c\in\mathcal{P}}\ell^{x}(\gamma_c))_{x\in I})\right]\prod_{c\in\mathcal{P}}\frac{1}{2}\mu^*(d\gamma_c)$$

Let $\mathfrak{S}_n(\mathcal{P})$ be all the permutations σ of $\{1,\dots,n\}$ such that the classes of the partition $\mathcal{P}$ are the supports of the disjoint cycles of σ. Given a class $c\in\mathcal{P}$, let j_c be its smallest element. From property 2.3.11 (ii) follows that

$$\prod_{i=1}^{n}\ell^{x_i}(\gamma_{\mathcal{P}(i)})=\sum_{\sigma\in\mathfrak{S}_n(\mathcal{P})}\prod_{c\in\mathcal{P}}\ell^{*x_{j_c},x_{\sigma(j_c)},\dots,x_{\sigma^{|c|}(j_c)}}(\gamma_c)$$

Proposition 2.3.12 (ii) states that

$$\ell^{*x_{j_c},x_{\sigma(j_c)},\dots,x_{\sigma^{|c|}(j_c)}}(\gamma_c)\mu^*(d\gamma_c)=$$
$$\pi_*(\mu^{j_c,\sigma(j_c)}(d\tilde{\gamma}_{j_c})\lhd\cdots\lhd\mu^{\sigma^{|c|-1}(j_c),\sigma^{|c|}(j_c)}(d\tilde{\gamma}_{\sigma^{|c|-1}(j_c)})\lhd\mu^{\sigma^{|c|}(j_c),j_c}(d\tilde{\gamma}_{\sigma^{|c|}(j_c)}))$$

and if the loop γ_c is a concatenation of paths $\tilde{\gamma}_{j_c},\dots,\tilde{\gamma}_{\sigma^{|c|-1}(j_c)},\tilde{\gamma}_{\sigma^{|c|}(j_c)}$ then

$$\ell^{x}(\gamma_c)=\ell^{x}(\tilde{\gamma}_{j_c})+\cdots+\ell^{x}(\tilde{\gamma}_{\sigma^{|c|-1}(j_c)})+\ell^{x}(\tilde{\gamma}_{\sigma^{|c|}(j_c)})$$

It follows that

$$(2.4.20)\quad 2^n\mathbb{E}_{\mathcal{L}_{\frac{1}{2}}}\left[\prod_{i=1}^{n}\hat{\mathcal{L}}_{\frac{1}{2}}^{x_i}F((\hat{\mathcal{L}}_{\frac{1}{2}}^{x})_{x\in I})\right]=$$
$$\sum_{\sigma\in\mathfrak{S}_n}2^{n-\sharp\text{ cycles of }\sigma}\int\mathbb{E}_{\mathcal{L}_{\frac{1}{2}}}\left[F((\hat{\mathcal{L}}_{\frac{1}{2}}^{x}+\sum_{i=1}^{n}\ell^{x}(\tilde{\gamma}_i))_{x\in I})\right]\prod_{i=1}^{n}\mu^{i,\sigma(i)}(d\tilde{\gamma}_i)$$

But the right-hand side of (2.4.20) is just the same as the right-hand side of (2.4.17) in the specific case when for all $i\in\{1,\dots,n\}$, $x_{i+n}=x_i$. This finishes the derivation of the special case of Dynkin's isomorphism.

2.5. Decomposing paths into Poisson ensembles of loops

2.5.1. Glueing together excursions ordered by their minima. Let L be the generator of a diffusion on I of form (2.2.11). A loop of $\mathcal{L}_{\alpha,L}$ rooted at its minimal point is a positive excursion. For a given $x_0\in I$, we will consider the loops $\gamma\in\mathcal{L}_{\alpha,L}$ such that $\min\gamma\in(\inf I,x_0]$. We will root these loops at their minima and then order the obtained excursions in the decreasing sense of their minima. Then we will glue all these excursions together and obtain a continuous paths $\xi_{\alpha,L}^{(x_0)}$. The law of this path can be described as a one-dimensional projection of a two-dimensional Markov process. Moreover this path contains all the information on the ensemble of loops $\mathcal{L}_{\alpha,L}\cap\{\gamma\in\mathfrak{L}^*|\min\gamma<x_0\}$. So this is a way to sample the latter ensemble of loops. In the particular case of $\alpha=1$, $\xi_{1,L}^{(x_0)}$ is the sample paths of a one-dimensional diffusion. This is analogue of the link between $\mathcal{L}_1$ and the loop-erasure procedure already observed in [**LW04**] and in [**Jan11**], chapter 8 and will de described in detail in section 2.5.3 In the section 2.5.1 we will consider generalities about glueing together excursions ordered by their minima and probability laws won't be involved. In the section 2.5.2 we will deal with $\xi_{\alpha,L}^{(x_0)}$ and identify its law. In the section 2.5.3 we will focus on the case $\alpha=1$ and describe other ways of slicing sample paths of diffusions into Poisson ensembles of loops.

Let $x_0 \in \mathbb{R}$ and let $\mathcal{Q}$ be a countable everywhere dense subset of $(-\infty, x_0)$. We consider a deterministic collection of excursions $(\mathsf{e}_q)_{q\in\mathcal{Q}}$ where $(\mathsf{e}_q(t))_{0\leqslant t\leqslant T(\mathsf{e}_q)}$ is a continuous excursion above 0, $T(\mathsf{e}_q) > 0$ and

$$\mathsf{e}_q(0) = \mathsf{e}_q(T(\mathsf{e}_q)) = 0$$
$$\forall t \in (0, T(\mathsf{e}_q)),\ \mathsf{e}_q(t) > 0$$

We also assume that for all $C > 0$ and $a < x_0$, there are only finitely many $q \in \mathcal{Q} \cap (a, x_0)$ such that $\max \mathsf{e}_q > C$ and that for all $a < x_0$

$$\sum_{q\in\mathcal{Q}\cap(a,x_0)} T(\mathsf{e}_q) < +\infty \tag{2.5.1}$$

Let $\mathcal{T}(y)$ be the function defined on $[0, +\infty)$ by

$$\mathcal{T}(y) := \sum_{q\in\mathcal{Q}\cap(x_0-y,x_0)} T(\mathsf{e}_q)$$

$\mathcal{T}$ is a non-decreasing function. Since $\mathcal{Q}$ is everywhere dense, $\mathcal{T}$ is increasing. $\mathcal{T}$ is right-continuous and jumps when $x_0 - y \in \mathcal{Q}$. The height of the jump is then $T(\mathsf{e}_{-y})$.

Let $T_{max} := \mathcal{T}(+\infty) \in (0, +\infty]$. For $t \in [0, T_{max})$ we define

$$\theta(t) := x_0 - \sup\{y \in [0, +\infty) | \mathcal{T}(y) > t\}$$

θ is a non-increasing function from $[0, T_{max})$ to $(-\infty, x_0]$. Since $\mathcal{T}$ is increasing, θ is continuous. We define

$$b^-(t) = \inf\{s \in [0, T_{max}) | \theta(s) = \theta(t)\}$$
$$b^+(t) = \sup\{s \in [0, T_{max}) | \theta(s) = \theta(t)\}$$

$b^-(t) < b^+(t)$ if and only if $\theta(t) \in \mathcal{Q}$ and then $b^+(t) - b^-(t) = T(\mathsf{e}_{\theta(t)})$. We introduce the set

$$\mathfrak{b}^- := \{t \in [0, T_{max}) | \theta(t) \in \mathcal{Q},\, b^-(t) = \theta(t)\}$$

$\mathfrak{b}^-$ is in one to one correspondence with $\mathcal{Q}$ by $t \mapsto \theta(t)$.

Finally we define on $[0, T_{max})$ the function ξ:

$$\xi(t) := \begin{cases} \theta(t) & \text{if } \theta(t) \notin \mathcal{Q} \\ \theta(t) + \mathsf{e}_{\theta(t)}(t - b^-(t)) & \text{if } \theta(t) \in \mathcal{Q} \end{cases}$$

Intuitively ξ is the function obtained by gluing together the excursions $(q + \mathsf{e}_q)_{q\in\mathcal{Q}}$ ordered in decreasing sense of their minima. See figure 2.1 for an example of ξ and θ.

PROPOSITION 2.5.1. *ξ is continuous. For all $t \in [0, T_{max})$*

$$\theta(t) = \inf_{[0,t]} \xi \tag{2.5.2}$$

The set $\mathfrak{b}^-$ can be recovered from ξ as follows:

$$\mathfrak{b}^- = \{t \in [0, T_{max}) | \xi(t) = \inf_{[0,t]} \xi \text{ and } \exists \varepsilon > 0, \forall s \in (0, \varepsilon), \xi(t+s) > \xi(t)\} \tag{2.5.3}$$

If $t_0 \in \mathfrak{b}^-$ then

$$b^+(t_0) = \inf\{t \in [t_0, T_{max}] | \xi(t) < \xi(t_0)\} \tag{2.5.4}$$

PROOF. Let $t \in [0, T_{max})$. To prove the continuity of ξ at t we distinguish three case: the first case is when $\theta(t) \in \mathcal{Q}$ and $b^-(t) < t < b^+(t)$, the second case is when $\theta(t) \notin \mathcal{Q}$ and the third case is when $\theta(t) \in \mathcal{Q}$ and either $b^-(t) = t$ or $b^+(t) = t$.

In the first case, for all $s \in (b^-(t), b^+(t))$,

$$\xi(s) = \theta(t) + \mathsf{e}_{\theta(t)}(s - b^-(t))$$

$\mathsf{e}_{\theta(t)}$ being continuous, we get the continuity of ξ at t.

In the second case we consider a sequence $(t_n)_{n\geqslant 0}$ in $[0, T_{max})$ converging to t. Let $C > 0$. There are only finitely many $q \in \mathcal{Q}$ such that there is $n \geqslant 0$ such that $\theta(t_n) = q$ and $\max \mathsf{e}_q > C$. Moreover for any $q \in \mathcal{Q}$, there are only finitely many $n \geqslant 0$ such that $\theta(t_n) = q$. Thus there are only finitely many $n \geqslant 0$ such that $\theta(t_n) \in \mathcal{Q}$ and $\max \mathsf{e}_{\theta(t_n)} > C$. So for n large enough

$$\theta(t_n) \leqslant \xi(t_n) \leqslant \theta(t_n) + C \tag{2.5.5}$$

But $\xi(t) = \theta(t)$ and $\theta(t_n)$ converges to $\theta(t)$. Since we may take C arbitrarily small, (2.5.5) implies that $\xi(t_n)$ converges to $\theta(t)$.

Regarding the third case, assume for instance that $\theta(t) \in \mathcal{Q}$ and $t = b^-(t)$. The right-continuity of ξ at t follows from the same argument as in the first case and left-continuity from the same argument as in the second case.

By definition, for all $t \in [0, T_{max})$, $\theta(t) \leqslant \xi(t)$. θ being non-increasing, for all $t \in [0, T_{max})$

$$\theta(t) \leqslant \inf_{[0,t]} \xi$$

For the converse inequality, we have

$$\theta(t) = \xi(b^-(t)) \geqslant \inf_{[0,t]} \xi$$

Regarding (2.5.3) and (2.5.4) we have the following disjunction: if $\theta(t) \in \mathcal{Q}$ and $b^-(t) < t < b^+(t)$ then $\xi(t) > \theta(t)$. If $\theta(t) \in \mathcal{Q}$ and $t = b^-(t)$ then for all $s \in (0, b^+(t) - b^-(t))$, $\xi(t+s) > \xi(t)$. If either $\theta(t) \in \mathcal{Q}$ and $t = b^+(t)$ or $\theta(t) \notin \mathcal{Q}$ then $\xi(t) = \theta(t)$ and there is a positive sequence $(s_n)_{n\geqslant 0}$ decreasing to 0 such that $\theta(t+s_n) \notin \mathcal{Q}$ and $\xi(t+s_n) = \theta(t+s_n) < \theta(t)$. □

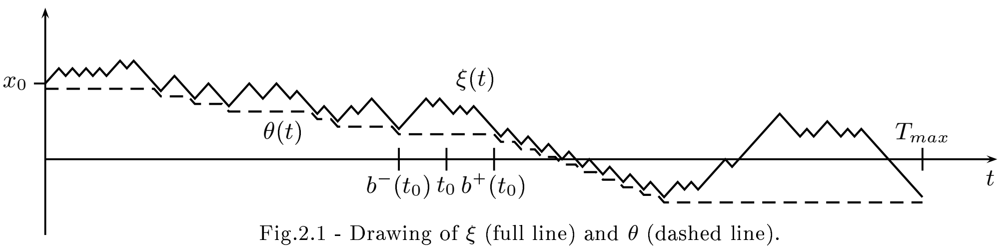

Fig.2.1 - Drawing of ξ (full line) and θ (dashed line).

Previous proposition shows that one can reconstruct $\mathcal{Q}$ and the family of excursions $(\mathsf{e}_q)_{q\in\mathcal{Q}}$ only knowing ξ. (2.5.2) shows how to recover θ from ξ. (2.5.3) and (2.5.4) show how to recover the left and the right time boundaries of the excursions of ξ above θ. Also observe that the set defined by the right-hand side of (2.5.3) is countable whatever the continuous function ξ is, even if it is not obtained by glueing together excursions.

2.5.2. Loops represented as excursions and glued together. Let $\alpha > 0$ and $\mathcal{L}_{\alpha,BM}$ the Poisson ensemble of loops of intensity $\alpha\mu^*_{BM}$ where μ^*_{BM} is the measure on loops associated to the Brownian motion on $\mathbb{R}$. Let $x_0 \in \mathbb{R}$. We consider the random countable set $\mathcal{Q}$:

$$\mathcal{Q} := \{\min \gamma | \gamma \in \mathcal{L}_{\alpha,BM}\} \cap (-\infty, x_0)$$

Almost surely $\mathcal{Q}$ is everywhere dense in $(-\infty, x_0)$ and for every $q \in \mathcal{Q}$ there is only one $\gamma \in \mathcal{L}_{\alpha,BM}$ such that $\min \gamma = q$. Almost surely $\gamma \in \mathcal{L}_{\alpha,BM}$ reaches its minimum at one single moment. Given $q \in \mathcal{Q}$ and $\gamma \in \mathcal{L}_{\alpha,BM}$ such that $\min \gamma = q$ we consider e_q to be the excursion above 0 equal to $\gamma - q$ where we root the unrooted loop γ at $argmin\gamma$. Then the random set of excursions $(\mathsf{e}_q)_{q\in\mathcal{Q}}$ almost surely satisfies the assumptions of the section 2.5.1. In particular the condition (2.5.1) follows from the fact that, according to (2.3.27)

$$\int_{\mathfrak{L}^*} 1 \wedge T(\gamma) 1_{\min \gamma \in (a,x_0)} \mu^*_{BM}(d\gamma) = (x_0 - a) \int_0^{+\infty} \frac{t \wedge 1}{\sqrt{2\pi t^3}} dt < +\infty$$

Thus we can consider the random continuous function $(\xi_{\alpha,BM}(t)^{(x_0)})_{t\geqslant 0}$ constructed by glueing together the excursions $(q+\mathsf{e}_q)_{q\in\mathcal{Q}}$ in the way described in section 2.5.1. Let

$$\theta^{(x_0)}_{\alpha,BM}(t)=\inf_{[0,t]}\xi^{(x_0)}_{\alpha,BM}$$

$$\Xi^{(x_0)}_{\alpha,BM}(t):=\big(\xi^{(x_0)}_{\alpha,BM}(t),\theta^{(x_0)}_{\alpha,BM}(t)\big)$$

Next we will describe the law of the two-dimensional process $\big(\Xi^{(x_0)}_{\alpha,BM}(t)\big)_{t\geqslant 0}$.

PROPOSITION 2.5.2. *Let $(\widetilde{B}_t)_{t\geqslant 0}$ be a standard Brownian motion on $\mathbb{R}$ starting from 0. $\big(\Xi^{(x_0)}_{\alpha,BM}(t)\big)_{t\geqslant 0}$ has the same law as*

$$\left(x_0+|\widetilde{B}_t|-\frac{1}{\alpha}\ell^0_t(\widetilde{B}),x_0-\frac{1}{\alpha}\ell^0_t(\widetilde{B})\right)_{t\geqslant 0}$$

In particular for $\alpha=1$, $(\xi^{(x_0)}_{1,BM}(t))_{t\leqslant 0}$ has the same law as a Brownian motion starting from x_0.

PROOF. For $a<x_0$ let T_a be the first time $\theta^{(x_0)}_{\alpha,BM}$ hits a. For $l>0$ let

$$\tilde{\tau}^0_l:=\inf\{t>0|\ell^0_t(\widetilde{B})>l\}$$

According to the disintegration (2.3.27) of the measure μ^*_{BM} in the proposition 2.3.14, for all $a<x_0$ the family $(\mathsf{e}_q)_{q\in\mathcal{Q}\cap(a,x_0)}$ of excursions above 0 is a Poisson point process of intensity $2\alpha\eta^{>0}_{BM}$. This implies the following equality in law

$$\big(\xi^{(x_0)}_{\alpha,BM}(t)-\theta^{(x_0)}_{\alpha,BM}(t)\big)_{0\leqslant t\leqslant T_a}\overset{(law)}{=}(|\widetilde{B}_t|)_{0\leqslant t\leqslant\tilde{\tau}^0_{\alpha(x_0-a)}}$$

Since the above holds for all $a<x_0$, we have the following equality in law

$$(\xi^{(x_0)}_{\alpha,BM}(t)-\theta^{(x_0)}_{\alpha,BM}(t),\alpha(x_0-\theta_{\alpha,BM}(t)))_{t\geqslant 0}\overset{(law)}{=}(|\widetilde{B}_t|,\ell^0_t(\widetilde{B}))_{t\geqslant 0}$$

which is exactly the equality in law we needed. Finally for $\alpha=1$, $(x_0+|\widetilde{B}_t|-\ell^0_t(\widetilde{B}))_{t\geqslant 0}$ has the law of a Brownian motion starting from x_0. See [**RY99**], chapter VI, §2. □

According to proposition 2.5.2 a Brownian sample path can be decomposed into a Poisson process of positive excursion with decreasing minima.This decomposition id for instance described in [**KS10**], section 6.2.D. In case $\alpha=1$, proposition 2.4.4 states that the occupation field of a the Poisson ensemble of loops associated to the Brownian motion on $(0,+\infty)$ killed at 0 is the square of a Bessel process of dimension 2 starting from 0 at 0. This result can also be obtained using the fact that $(\xi^{(x_0)}_{1,BM}(t))_{t\leqslant 0}$ is a Brownian sample path and applying the first Ray-Knight theorem which gives the law of the occupation field of a Brownian path stopped upon hitting 0.

From proposition 2.5.2 follows in particular that $\big(\Xi^{(x_0)}_{\alpha,BM}(t)\big)_{t\geqslant 0}$ is a sample path of a two-dimensional Feller process. Let

$$T^+(\mathbb{R}^2):=\{(x,a)\in\mathbb{R}^2|x\geqslant a\}\qquad Diag(\mathbb{R}^2):=\{(x,x)|x\in\mathbb{R}\}$$

For $(x_0,a_0)\in T^+(\mathbb{R}^2)$ we define the process

$$\begin{aligned}(2.5.6)\quad \big(\Xi^{(x_0,a_0)}_{\alpha,BM}(t)\big)_{t\geqslant 0}&=\big(\xi^{(x_0,a_0)}_{\alpha,BM}(t),\theta^{(x_0,a_0)}_{\alpha,BM}(t)\big)_{t\geqslant 0}\\&:=\left(a_0+|x_0-a_0+\widetilde{B}_t|-\frac{1}{\alpha}\ell^{a_0-x_0}_t(\widetilde{B}),a_0-\frac{1}{\alpha}\ell^{a_0-x_0}_t(\widetilde{B})\right)_{t\geqslant 0}\end{aligned}$$

where $(\widetilde{B}_t)_{t\geqslant 0}$ is a Brownian motion starting from 0. $\Xi^{(x_0,x_0)}_{\alpha,BM}$ has the same law as $\Xi^{(x_0)}_{\alpha,BM}$. The family of paths $\big(\Xi^{(x_0,a_0)}_{\alpha,BM}\big)_{x_0\geqslant a_0}$ are the sample paths of the same Feller semi-group on

$T^+(\mathbb{R}^2)$ starting from all possible positions. Next we describe this semi-group in terms of generator and domain. Let f be a continuous function on $T^+(\mathbb{R}^2)$, $\mathcal{C}^2$ on the interior of $T^+(\mathbb{R}^2)$, such that all its second order derivatives extend continuously to $Diag(\mathbb{R}^2)$. This implies in particular that the first order derivatives also extend continuously to $Diag(\mathbb{R}^2)$. We write $\partial_1 f$, $\partial_2 f$ and $\partial_{1,1} f$ for the first order derivative relative to the first variable, the second variable and the second order derivative relatively the first variable. Applying Itô-Tanaka's formula we get

$$f\big(\Xi^{(x_0,a_0)}_{\alpha,BM}(t)\big) = f(x_0,a_0) + \int_0^t \partial_1 f\big(\Xi^{(x_0,a_0)}_{\alpha,BM}(s)\big)\,\mathrm{sign}(x_0 - a_0 + \tilde{B}_s)d\tilde{B}_s +$$
$$\int_0^t \left(\left(1-\frac{1}{\alpha}\right)\partial_1 - \frac{1}{\alpha}\partial_2\right) f\big(\Xi^{(x_0,a_0)}_{\alpha,BM}(s)\big)d_s\ell_s^{a_0-x_0}(\tilde{B}) + \frac{1}{2}\int_0^t \partial_{1,1} f\big(\Xi^{(x_0,a_0)}_{\alpha,BM}(s)\big)ds$$

Let $\mathcal{D}_{\alpha,BM}$ be the set of continuous functions f on $D_{\mathbb{R}}$, $\mathcal{C}^2$ on the interior of $T^+(\mathbb{R}^2)$, such that all the second order derivatives extend continuously to $Diag(\mathbb{R}^2)$ and that moreover satisfy the following constraints: f and $\partial_{1,1} f$ are uniformly continuous and bounded (which also implies that $\partial_1 f$ is bounded by the inequality $\|\partial_1 f\|_\infty \leqslant 2\sqrt{\|f\|_\infty \|\partial_{1,1} f\|_\infty}$) and on $Diag(\mathbb{R}^2)$ the following equality holds:

$$\left(\left(1-\frac{1}{\alpha}\right)\partial_1 - \frac{1}{\alpha}\partial_2\right) f(x,x) = 0$$

If $f \in \mathcal{D}_{\alpha,BM}$ then $\frac{1}{t}\big(\mathbb{E}\big[f(\Xi^{x_0,a_0}_{\alpha,BM}(t))\big] - f(x_0,a_0)\big)$ converges as $t \to 0^+$, uniformly for $(x_0,a_0) \in T^+(\mathbb{R}^2)$, to $\frac{1}{2}\partial_{1,1} f(x_0,a_0)$. Moreover $\mathcal{D}_{\alpha,BM}$ is a core for $\frac{1}{2}\partial_{1,1}$ in the space of continuous bounded function on $T^+(\mathbb{R}^2)$.

Next we describe what we obtain if we glue together the loops, seen as excursion, ordered in the decreasing sense of their minima, where instead of $\mathcal{L}_{\alpha,BM}$ we use the Poisson ensemble of Markov loops associated to a general diffusion. Let I be an open interval of $\mathbb{R}$ and $\tilde{L}$ a generator on I of form

$$\tilde{L} = \frac{1}{\tilde{m}(x)}\frac{d}{dx}\left(\frac{1}{\tilde{w}(x)}\frac{d}{dx}\right)$$

with zero Dirichlet boundary conditions. Let $\tilde{S}$ be a primitive of $\tilde{w}(x)$. We assume that $\tilde{S}(\sup I) = +\infty$. Let

$$T^+(I^2) := \{(x,a) \in I^2 | x \geqslant a\} \qquad Diag(I^2) := \{(x,x) | x \in I\}$$

Let $\widehat{T^+(I^2)}$ be the closure of $T^+(I^2)$ in $(\inf I, \sup I]^2$.

Given any $x_0' \geqslant a_0' > \frac{1}{2}\tilde{S}(\inf I)$ let $\tilde{\zeta}_\alpha$ be the first time $\Xi^{(x_0',a_0')}_{\alpha,BM}$ hits $\frac{1}{2}\tilde{S}(\inf I)$. Let

$$\tilde{I}_t := \int_0^t \frac{1}{\tilde{m}}\big(\tilde{S}^{-1}\big(2\xi^{(x_0',a_0')}_{\alpha,BM}(s)\big)\big)ds$$

Let $(\tilde{I}_t^{-1})_{0\leqslant t<\tilde{I}_{\tilde{\zeta}_\alpha}}$ be the inverse function of $(\tilde{I}_t)_{0\leqslant t<\tilde{\zeta}_\alpha}$. It is a family of stopping times for $\Xi^{(x_0',a_0')}_{\alpha,BM}$. For $x_0 \geqslant a_0 \in I$ and $t < \tilde{I}_{\tilde{\zeta}_\alpha}$ let

$$\Xi^{(x_0,a_0)}_{\alpha,\tilde{L}}(t) = \big(\xi^{(x_0,a_0)}_{\alpha,\tilde{L}}(t), \theta^{(x_0,a_0)}_{\alpha,\tilde{L}}(t)\big) := \Xi^{(\tilde{S}(2x_0),\tilde{S}(2a_0))}_{\alpha,BM}(\tilde{I}_t^{-1})$$

If $\alpha = 1$ then $\xi^{(x_0,a_0)}_{\alpha,\tilde{L}}$ is just the sample paths starting x_0 of a diffusion of generator $\tilde{L}$. Let $\hat{\mathcal{D}}_{\alpha,\tilde{L}}$ be the space of continuous functions f on $T^+(I^2)$ satisfying

- $f \circ \tilde{S}^{-1}$ is $\mathcal{C}^2$ on the interior of $T^+(I^2)$ and all the second order derivatives extend continuously to $Diag(I^2)$.

- $f(x,a)$ and $\frac{1}{\tilde{m}(x)}\partial_1\left(\frac{1}{\tilde{w}(x)}\partial_1 f(x,a)\right)$ are bounded on $T^+(I^2)$ and extend continuously to $\widehat{T^+(I^2)}$.
- $f(x,a)$ and $\frac{1}{\tilde{m}(x)}\partial_1\left(\frac{1}{\tilde{w}(x)}\partial_1 f(x,a)\right)$ converge to 0 as a converges to $\inf I$ uniformly in x.
- On $Diag(I^2)$ the following equality holds:

$$\left(\left(1-\frac{1}{\alpha}\right)\partial_1 - \frac{1}{\alpha}\partial_2\right) f(x,x) = 0 \tag{2.5.7}$$

LEMMA 2.5.3. *$\left(\Xi_{\alpha,\tilde{L}}^{(x_0,a_0)}\right)_{x_0\geqslant a_0\in I}$ is a family of sample path starting from all possible positions of the same Markovian or sub-Markovian semi-group on $T^+(I^2)$. The law of the path $\Xi_{\alpha,\tilde{L}}^{(x_0,a_0)}$ depends weakly continuously on the starting point (x_0,a_0). The domain of the generator of this semi-group contains $\widehat{\mathcal{D}}_{\alpha,\tilde{L}}$, and on this space the generator equals*

$$\frac{1}{\tilde{m}(x)}\partial_1\left(\frac{1}{\tilde{w}(x)}\partial_1\right)$$

Moreover there is only one Markovian or sub-Markovian semi-group with such generator on $\widehat{\mathcal{D}}_{\alpha,\tilde{L}}$.

PROOF. Since a change of scale does not alter the validity of the above statement, we can assume that $\tilde{w}\equiv 2$. Then $\sup I = +\infty$. $\left(\Xi_{\alpha,\tilde{L}}^{(x_0,a_0)}(t)\right)_{0\leqslant t\leqslant \tilde{I}_{\tilde{\zeta}_\alpha}}$ is then obtained from $\left(\Xi_{\alpha,BM}^{(x_0,a_0)}(t)\right)_{0\leqslant t<\tilde{\zeta}_\alpha}$ by a random time change. The Markov property and the continuous dependence on the starting point for $\Xi_{\alpha,\tilde{L}}^{(x_0,a_0)}$ follows from analogous properties for $\Xi_{\alpha,BM}^{(x_0,a_0)}$. If $f\in\widehat{\mathcal{D}}_{\alpha,\tilde{L}}$ then

$$\left(f\left(\Xi_{\alpha,BM}^{(x_0,a_0)}(\tilde{I}_t^{-1}\wedge\tilde{\zeta}_\alpha)\right) - \frac{1}{2}\int_0^{\tilde{I}_t^{-1}\wedge\tilde{\zeta}_\alpha}\partial_{1,1}f\left(\Xi_{\alpha,BM}^{(x_0,a_0)}(s)\right)ds\right)_{t\geqslant 0}$$

is a local martingale. We can rewrite it as

$$\left(f\left(\Xi_{\alpha,\tilde{L}}^{(x_0,a_0)}(t\wedge\tilde{I}_{\tilde{\zeta}_\alpha})\right) - \int_0^t\frac{1}{2\tilde{m}\left(\xi_{\alpha,\tilde{L}}^{(x_0,a_0)}(s)\right)}\partial_{1,1}f\left(\Xi_{\alpha,\tilde{L}}^{(x_0,a_0)}(s)\right)1_{s<\tilde{I}_{\tilde{\zeta}_\alpha}}ds\right)_{t\geqslant 0}$$

The above local martingale is bounded on all finite time intervals and thus is a true martingale. Since $\frac{1}{2\tilde{m}(x)}\partial_{1,1}f(x,a)$ converges to 0 as a converges to $\inf I$, uniformly in x, it follows that

$$f\left(\Xi_{\alpha,\tilde{L}}^{(x_0,a_0)}(t\wedge\tilde{I}_{\tilde{\zeta}_\alpha})\right) = 1_{t<\tilde{I}_{\tilde{\zeta}_\alpha}}f\left(\Xi_{\alpha,\tilde{L}}^{(x_0,a_0)}(t)\right)$$

Thus

$$\lim_{t\to 0^+}\frac{1}{t}\left(\mathbb{E}\left[1_{t<\tilde{I}_{\tilde{\zeta}_\alpha}}f\left(\Xi_{\alpha,\tilde{L}}^{(x_0,a_0)}(t)\right)\right] - f(x_0,a_0)\right) = \frac{1}{2\tilde{m}(x_0)}\partial_{1,1}f(x_0,a_0)$$

Moreover the above convergence is uniform in (x_0,a_0) because $\frac{1}{2\tilde{m}(x)}\partial_{1,1}f(x,a)$ extends continuously to $\widehat{T^+(I^2)}$.

To prove the uniqueness of the semi-group we need to show that there is $\lambda>0$ such that

$$\left(\frac{1}{2\tilde{m}(x)}\partial_{1,1}-\lambda\right)(\widehat{\mathcal{D}}_{\alpha,\tilde{L}})$$

is sufficiently large, for instance that it contains all functions with compact support in $T^+(I^2)$. Let g be such a function and $\lambda > 0$. Consider the equation

$$\frac{1}{2\tilde{m}(x)}\partial_{1,1}f(x,a) - \lambda f(x,a) = g(x,a) \tag{2.5.8}$$

Let $\tilde{u}_{\lambda,\downarrow}$ be a positive decreasing solution to

$$\frac{1}{2\tilde{m}(x)}\frac{d^2u}{dx^2}(x) - \lambda u(x) = 0$$

Let

$$f_0(x,a) := \tilde{u}_{\lambda,\downarrow}(x)\int_x^{+\infty}\int_y^{+\infty} 2\tilde{m}(z)g(z,a)\tilde{u}_{\lambda,\downarrow}(z)\,dz\,\frac{dy}{\tilde{u}_{\lambda,\downarrow}(y)^2}$$

Then f_0 is a solution to (2.5.8) and it is compactly supported in $T^+(I^2)$. We look for the solutions to (2.5.8) of form

$$f(x,a) = f_0(x,a) + C(a)\tilde{u}_{\lambda,\downarrow}(x)$$

f satisfies the constraint (2.5.7) if and only if C satisfies

$$-\frac{1}{\alpha}\tilde{u}_{\lambda,\downarrow}(a)\frac{dC}{da}(a) + \left(1-\frac{1}{\alpha}\right)\frac{d\tilde{u}_{\lambda,\downarrow}}{dx}(a)C(a) + h(a) = 0$$

where

$$h(a) = \left(\left(1-\frac{1}{\alpha}\right)\partial_1 - \frac{1}{\alpha}\partial_2\right)f_0(a,a)$$

h is compactly supported in I. We can set

$$C(a) = \tilde{u}_{\lambda,\downarrow}(a)^{\alpha-1}\int_{\inf I}^x \frac{h(y)}{\tilde{u}_{\lambda,\downarrow}(y)^\alpha}dy$$

C is zero in the neighbourhood of $\inf I$. Moreover $\tilde{u}_{\lambda,\downarrow}$ has a limit at $+\infty$. It follows that $f \in \widehat{\mathcal{D}}_{\alpha,\tilde{L}}$. □

Let L be the generator of a diffusion on I of form (2.2.11). Let $x_0 \in I$. Consider the loops γ in $\mathcal{L}_{\alpha,L}$ such that $\min\gamma < x_0$, rooted at $argmin\gamma$, seen as excursions. Let $(\xi^{(x_0)}_{\alpha,L}(t))_{0\leqslant t<\zeta_\alpha}$ be the path on I obtained by glueing together this excursions ordered in the decreasing sense of their minima. Let

$$\theta^{(x_0)}_{\alpha,L}(t) := \min_{[0,t]}\xi^{(x_0)}_{\alpha,L}$$

$$\Xi^{(x_0)}_{\alpha,L} := \left(\xi^{(x_0)}_{\alpha,L}, \theta^{(x_0)}_{\alpha,L}\right)$$

PROPOSITION 2.5.4. *Let* $\widetilde{L} := Conj(u_\downarrow, L)$. *Then* $\left(\Xi^{(x_0)}_{\alpha,L}(t)\right)_{0\leqslant t<\zeta_\alpha}$ *has the same law as* $\left(\Xi^{(x_0,x_0)}_{\alpha,\widetilde{L}}(t)\right)_{0\leqslant t<\tilde{\zeta}_\alpha}$. *So it is a sample path of a two-dimensional Feller process. In particular for* $\alpha = 1$, $\xi^{(x_0)}_{1,L}$ *is the sample path of a diffusion of generator* $\widetilde{L}$. *For all* $\alpha > 0$

$$\liminf_{t\to\zeta_\alpha}\xi^{(x_0)}_{\alpha,L}(t) = \inf I$$

If L *is the generator of a recurrent diffusion then*

$$\limsup_{t\to\zeta_\alpha}\xi^{(x_0)}_{\alpha,L}(t) = \sup I$$

Otherwise

$$\limsup_{t\to\zeta_\alpha}\xi^{(x_0)}_{\alpha,L}(t) = \inf I$$

PROOF. First notice that if L is the generator of a recurrent diffusion then $\widetilde{L} = L$. Otherwise a diffusion of generator $\widetilde{L} = L$ is, put informally, a diffusion of generator L conditioned to converge to $\inf I$ (which may occur with zero probability). From conjugation invariance of the measure on loops follows that $\mathcal{L}_{\alpha,L} = \mathcal{L}_{\alpha,\widetilde{L}}$. From property 2.3.6 (iv) and corollary 2.3.10 follows that $\Xi_{\alpha,L}^{(x_0)}$ is obtained from $\Xi_{\alpha,BM}$ by scale and time change in the same way as $\Xi_{\alpha,\widetilde{L}}^{(x_0,x_0)}$ and thus $\Xi_{\alpha,L}^{(x_0)}$ and $\Xi_{\alpha,\widetilde{L}}^{(x_0,x_0)}$ have the same law. Regarding the limits of $\xi_{\alpha,L}^{(x_0)}$ at ζ_α, we need just to observe that they hold if L is the generator of the Brownian motion on an interval of form $(a,+\infty)$, $a \in [-\infty,+\infty)$, and by time and scale change they hold in general. □

As explained in the proposition 2.5.1, the knowledge of the path $\big(\xi_{\alpha,L}^{(x_0)}(t)\big)_{0\leqslant t<\zeta_\alpha}$ alone is enough to reconstruct $\mathcal{L}_{\alpha,L} \cap \{\gamma \in \mathfrak{L}^* | \min\gamma < x_0\}$. From this we deduce the following

COROLLARY 2.5.5. *If L is the generator of a transient diffusion, it is possible to construct on the same probability space $\mathcal{L}_{\alpha,L}$ and a continuous version of the occupation field $(\widehat{\mathcal{L}}_{\alpha,L}^x)_{x\in I}$.*

PROOF. By scale and time change covariance and conjugation invariance of the Poisson ensembles of loops, it is enough to prove the proposition in case of a Brownian motion on $(0,+\infty)$ killed at 0. Let $(x_n)_{n\geqslant 0}$ be an increasing sequence in $(0,+\infty)$ converging to $+\infty$. We consider a sequence of independent paths $\big(\xi_{\alpha,BM}^{(x_n,x_n)}\big)_{n\geqslant 0}$ defined by (2.5.6). Let

$$T_{n,x_{n-1}} := \inf\big\{t \geqslant 0 | \xi_{\alpha,BM}^{(x_n,x_n)}(t) = x_{n-1}\big\}$$

where conventionally we set $x_{-1} := 0$. By decomposing on $[0, T_{n,x_{n-1}}]$ the restricted path $\big(\xi_{\alpha,BM}^{(x_n,x_n)}(t)\big)_{0\leqslant t<T_{n,x_{n-1}}}$ one can reconstruct a family of loops γ such that $\min\gamma \in (x_{n-1},x_n)$: there is a random countable set $\mathscr{B}_n$ of disjoint compact subintervals $[b^-,b^+]$ of $[0,T_{n,x_{n-1}}]$ such that

$$\big\{\big(\xi_{\alpha,BM}^{(x_n,x_n)}(b^-+t)\big)_{0\leqslant t\leqslant b^+-b^-} | [b^-,b^+] \in \mathscr{B}_n\big\} = \mathcal{L}_{\alpha,BM} \cap \{\gamma \in \mathfrak{L}^* | \min\gamma \in (x_{n-1},x_n)\}$$

(see (2.5.3)). The union of all previous families of loops for $n \geqslant 0$ is a Poisson ensemble of loops $\mathcal{L}_{\alpha,BM} \cap \{\gamma \in \mathfrak{L}^* | \min\gamma > 0\}$.

Each of $\xi_{\alpha,BM}^{(x_n,x_n)}$ is a semi-martingale and its quadratic variation is

$$\big\langle \xi_{\alpha,BM}^{(x_n,x_n)}, \xi_{\alpha,BM}^{(x_n,x_n)} \big\rangle_t = t$$

Moreover for all $x \in \mathbb{R}$

$$\int_0^t 1_{\xi_{\alpha,BM}^{x_n,x_n}=x} d\xi_{\alpha,BM}^{(x_n,x_n)}(s) = \left(1-\frac{1}{\alpha}\right)\int_0^t 1_{\ell_s^0(\widetilde{B})=\alpha x} d_s\ell_s^0(\widetilde{B}) = 0$$

From theorems 1.1 and 1.7 in [**RY99**], chapter VI, §1, follows that we can construct on the same probability space $\xi_{\alpha,BM}^{(x_n,x_n)}$ and a space-time continuous version of local times $\big(\ell_t^x\big(\xi_{\alpha,BM}^{(x_n,x_n)}\big)\big)_{x\in\mathbb{R},t\geqslant 0}$ of $\xi_{\alpha,BM}^{(x_n,x_n)}$ relative to the Lebesgue measure. In particular $x \mapsto \ell_{T_{n,x_{n-1}}}^x\big(\xi_{\alpha,BM}^{(x_n,x_n)}\big)$ is continuous. If $[b^-,b^+] \in \mathcal{J}_n$, then

$$\big(\ell_{b^+}^x\big(\xi_{\alpha,BM}^{(x_n,x_n)}\big)\big) - \ell_{b^-}^x\big(\xi_{\alpha,BM}^{(x_n,x_n)}\big)\big)_{x>0}$$

is the occupation field of the loop corresponding to the time interval $[b^-,b^+]$. We need to check that a.s

$$\forall x > 0, \ell_{T_{n,x_{n-1}}}^x\big(\xi_{\alpha,BM}^{(x_n,x_n)}\big) = \sum_{[b^-,b^+]\in\mathscr{B}_n} \ell_{b^+}^x\big(\xi_{\alpha,BM}^{(x_n,x_n)}\big) - \ell_{b^-}^x\big(\xi_{\alpha,BM}^{(x_n,x_n)}\big) \tag{2.5.9}$$

For $x > 0$, consider the random set of times

$$\{t \in [0, T_{n,x_{n-1}}] | \xi^{(x_n,x_n)}_{\alpha,BM}(t) = x\} \backslash \bigcup_{[b^-,b^+]\in\mathscr{B}_n} [b^-, b^+] \tag{2.5.10}$$

If x is a minimum of a loop embedded in $\big(\xi^{(x_n,x_n)}_{\alpha,BM}(t)\big)_{0\leqslant t<T_{n,x_{n-1}}}$ or if $x \notin (x^{n-1}, x_n)$ then the set (2.5.10) is empty. Otherwise it is reduced to one point: the first hitting time of the level x. Almost surely, for all $x > 0$, the measure $d_t \ell^x_t\big(\xi^{(x_n,x_n)}_{\alpha,BM}\big)$ is supported in $\{t \geqslant 0 | \xi^{(x_n,x_n)}_{\alpha,BM}(t) = x\}$ and has no atoms, and thus does not charge the set (2.5.10). This implies (2.5.9). Finally we can conclude that $\big(\ell^x_{T_{n,x_{n-1}}}\big(\xi^{(x_n,x_n)}_{\alpha,BM}\big)\big)_{x>0}$ is the occupation field of $\mathcal{L}_{\alpha,BM} \cap \{\gamma \in \mathfrak{L}^* | \min\gamma \in (x_{n-1}, x_n)\}$.

The occupation field of $\mathcal{L}_{\alpha,BM} \cap \{\gamma \in \mathfrak{L}^* | \min\gamma > 0\}$ is

$$\Bigg(\sum_{n\geqslant 0} \ell^x_{T_{n,x_{n-1}}}\big(\xi^{(x_n,x_n)}_{\alpha,BM}\big)\Bigg)_{x>0}$$

The above sum is locally finite and thus varies continuously with x. □

2.5.3. The case $\alpha = 1$. According to proposition 2.5.4 in case $\alpha = 1$ the Poisson ensemble of loops $\mathcal{L}_{1,L}$ can be recovered from sample paths of one-dimensional diffusions. A similar property was observed for loops of the two-dimensional Brownian Motion and of Markov jump processes on graphs. In [**Jan11**], chapter 8, it is shown that by launching consecutively symmetric Markov jump processes from different vertices of a finite graph and applying the Wilson's algorithm ([**Wil96**]), one can simultaneously construct a uniform spanning tree of the graph with prescribed weights on the edges and an independent Poisson ensemble of Markov loops of parameter $\alpha = 1$. If $\mathbb{D}$ is a simply-connected open domain of $\mathbb{C}$ other than $\mathbb{C}$, it was shown in [**Zha12**] that one can couple a Brownian motion on $\mathbb{D}$, killed at hitting $\partial\mathbb{D}$, and a simple curve (SLE_2) with same extremal points such that the latter appears as the loop-erasure of the first. It is conjectured that given this loop-erased Brownian motion and an independent Poisson ensemble of Brownian loops of parameter 1, by attaching to the simple curve the loops that cross it one reconstructs a Brownian sample path. See [**LW04**], conjecture 1, and [**LSW03**], theorem 7.3.

in case of one-dimensional diffusions one can partially recover $\mathcal{L}_{1,L}$ from Markovian sample paths otherwise than slicing $\xi^{(x_0)}_{1,L}$ in excursions. The next result has an analogue for loops of Markov jump processes on graphs. See [**Jan11**], remark 21.

Proposition 2.5.6. *Assume that L is the generator of a transient diffusion. Let $x \in I$. Let $(X_t)_{0\leqslant t<\zeta}$ be the sample path of a diffusion of generator L started from x. Let $\widehat{T}_x$ the last time X visits x. For $l \geqslant 0$ let*

$$\tau^x_l := \{t \geqslant 0 | \ell^x_t(X) > l\}$$

Let $(q_j)_{j\in\mathbb{N}}$ be a Poisson-Dirichlet partition $PD(0,1)$ of $[0,1]$, independent from X, ordered in an arbitrary way. Let

$$l_j := \ell^x_\zeta(X) \sum_{i=0}^{j} q_i$$

The family of bridges $((X_t)_{\tau^x_{l_{j-1}}\leqslant t\leqslant \tau^x_{l_j}})_{j\geqslant 0}$ has, up to unrooting, the same law as the loops in

$$\mathcal{L}_{1,L} \cap \{\gamma \in \mathfrak{L}^* | x \in \gamma([0, T(\gamma)])\}$$

In particular $(X_t)_{0\leqslant t\leqslant \widehat{T}_x}$ can be obtained through sticking together all the loops in $\mathcal{L}_{\alpha,L}$ that visit x.

PROOF. According to corollary 2.3.9, $(\ell^x(\gamma))_{\gamma\in\mathcal{L}_{\alpha,L},\gamma \text{ visits } x}$ is a Poisson ensemble of intensity $e^{-\frac{l}{G(x,x)}}\frac{dl}{l}$. Thus $\widehat{\mathcal{L}}^x_{\alpha,L}$ is an exponential r.v. with mean $G(x,x)$ and has the same law as $\ell^x_\zeta(X)$. Moreover the Poisson ensemble $(\ell^x(\gamma))_{\gamma\in\mathcal{L}_{\alpha,L},\gamma \text{ visits } x}$ has up to reordering the same law as $(l_j-l_{j-1})_{j\geqslant 0}$. Almost surely $l\mapsto \tau^x_l$ does not jump at any l_j. Conditional on $(l_j)_{j\geqslant 0}$, $((X_t)_{\tau^x_{l_{j-1}}\leqslant t\leqslant \tau^x_{l_j}})_{j\geqslant 0}$ is an independent family of bridges and $(X_t)_{\tau^x_{l_{j-1}}\leqslant t\leqslant \tau^x_{l_j}}$ has the same law as $(X_t)_{0\leqslant t\leqslant \tau^x_{l_j-l_{j-1}}}$. We conclude using identity (2.3.15) and the theory of marked Poisson ensembles. □

Assume that L is the generator of a transient diffusion. Let $x\in I$ and let $(X_t)_{0\leqslant t<\zeta}$ be a sample path starting from x of the diffusion corresponding to L. We will describe two different ways to slice $(X_t)_{0\leqslant t<\zeta}$ so as to obtain the loops

$$\mathcal{L}_{1,L}\cap\{\gamma\in\mathfrak{L}^*|\gamma([0,T(\gamma)])\cap[X(0),X(\zeta^-)](\text{or }[X(\zeta^-),X(0)])\neq\varnothing\}$$

The first method corresponds to the "loop-erasure procedure" applied to $(X_t)_{0\leqslant t<\zeta}$ and the second to the "loop-erasure procedure" applied to the time-reversed path $(X_{\zeta-t})_{0<t\leqslant\zeta}$. Let $\widehat{T}_x$ be the last time $(X_t)_{0\leqslant t<\zeta}$ visits x. Let $\widetilde{T}$ be the first time X hits X_{ζ^-}. If $X_{\zeta^-}\in\partial I$ then $\widetilde{T}=\zeta$. Let $(q_j)_{j\in\mathbb{N}}$ be a Poisson-Dirichlet partition $PD(0,1)$ of $[0,1]$, independent from X. The first method of decomposition is the following:

- The path $(X_t)_{0\leqslant t\leqslant\widehat{T}_x}$ is decomposed in bridges $((X_t)_{\tau^x_{l_{j-1}}\leqslant t\leqslant\tau^x_{l_j}})_{j\geqslant 0}$ from x to x by applying the Poisson-Dirichlet partition $(q_j)_{j\in\mathbb{N}}$ to $\ell^x_\zeta(X)$, as described in proposition 2.5.6.
- Given the path $(X_{\widehat{T}_x+t})_{0\leqslant t<\zeta-\widehat{T}_x}$, if $X_{\zeta^-}<x$ we define

$$\mathfrak{b}^+ := \Big\{t\in[0,\zeta-\widehat{T}_x)|X_{\widehat{T}_x+t}=\sup_{s\in[t,\zeta-\widehat{T}_x)}X_{\widehat{T}_x+s} \text{ and } \exists\varepsilon\in(0,t)\text{ s.t. }\forall s\in(t-\varepsilon,t), X_{\widehat{T}_x+s}<X_{\widehat{T}_x+t}\Big\}$$

  $\mathfrak{b}^+$ is countable and we define on $\mathfrak{b}^+$ the map b^-:

$$b^-(t):=\sup\{s\in[0,t)|X_{\widehat{T}_x+s}=X_{\widehat{T}_x+t}\}$$

  $((X_{\widehat{T}_x+b^-(t)+s})_{0\leqslant s\leqslant t-b^-(t)})_{t\in\mathfrak{b}^+}$ is the family of negative excursions of the path $(X_{\widehat{T}_x+t})_{0\leqslant t<\zeta-\widehat{T}_x}$ below $(\sup_{[\widehat{T}_x+t,\zeta)}X)_{0\leqslant t<\zeta-\widehat{T}_x}$. If $X_{\zeta^-}>x$ then

$$\mathfrak{b}^+ := \Big\{t\in[0,\zeta-\widehat{T}_x)|X_{\widehat{T}_x+t}=\inf_{s\in[t,\zeta-\widehat{T}_x)}X_{\widehat{T}_x+s} \text{ and } \exists\varepsilon\in(0,t)\text{ s.t. }\forall s\in(t-\varepsilon,t), X_{\widehat{T}_x+s}>X_{\widehat{T}_x+t}\Big\}$$

  We define on $\mathfrak{b}^+$ the map b^-:

$$b^-(t):=\sup\{s\in[0,t)|X_{\widehat{T}_x+s}=X_{\widehat{T}_x+t}\}$$

  $((X_{\widehat{T}_x+b^-(t)+s})_{0\leqslant s\leqslant t-b^-(t)})_{t\in\mathfrak{b}^+}$ are the positive excursions of $(X_{\widehat{T}_x+t})_{0\leqslant t<\zeta-\widehat{T}_x}$ above $(\inf_{[\widehat{T}_x+t,\zeta)}X)_{0\leqslant t<\zeta-\widehat{T}_x}$.
- We denote $\mathscr{L}^1((X_t)_{0\leqslant t<\zeta})$ the set of loops

$$\{(X_{\tau^x_{l_{j-1}}+s})_{0\leqslant s\leqslant\tau^x_{l_j}-\tau^x_{l_{j-1}}}|j\geqslant 0\}\cup\{(X_{\widehat{T}_x+b^-(t)+s})_{0\leqslant s\leqslant t-b^-(t)}|t\in\mathfrak{b}^+\}$$

  where the loops are considered to be unrooted.

The second method of decomposition is the following:

- If $X_{\zeta^-} < x$ we define
$$\mathfrak{b}^- := \left\{t \in [0, \tilde{T}) | X_t = \inf_{[0,t]} X \text{ and } \exists \varepsilon > 0 \text{ s.t. } \forall s \in (t, t+\varepsilon), X_s > X_t\right\}$$
On $\mathfrak{b}^-$ we define the map b^+:
$$b^+(t) := \inf\{s \in (t, \tilde{T}) | X_s = X_t\}$$
$((X_{t+s})_{0\leqslant s\leqslant b^+(t)-t})_{t\in\mathfrak{b}^-}$ are the positive excursions of the path $(X_t)_{0\leqslant t<\tilde{T}}$ above $(\inf_{[0,t]} X)_{0\leqslant t\leqslant \tilde{T}}$. This is exactly the decomposition described in the previous section 2.5.2. If $X_{\zeta^-} > x$ then
$$\mathfrak{b}^- := \left\{t \in [0, \tilde{T}) | X_t = \sup_{[0,t]} X \text{ and } \exists \varepsilon > 0 \text{ s.t. } \forall s \in (t, t+\varepsilon), X_s < X_t\right\}$$
The map b^+ defined on $\mathfrak{b}^-$ is
$$b^+(t) := \inf\{s \in (t, \tilde{T}) | X_s = X_t\}$$
$((X_{t+s})_{0\leqslant s\leqslant b^+(t)-t})_{t\in\mathfrak{b}^-}$ are the negative excursions of the path $(X_t)_{0\leqslant t<\tilde{T}}$ below $(\sup_{[0,t]} X)_{0\leqslant t\leqslant \tilde{T}}$.
- If $\tilde{T} < \zeta$ we introduce:
$$\tilde{l}_j := \ell_{\zeta}^{X_{\zeta^-}}(X) \sum_{i=0}^{j} q_i$$
and
$$\tau_{\tilde{l}_j} := \inf\{t \in [\tilde{T}, \zeta) | \ell_t^{X_{\zeta^-}}(X) > \tilde{l}_j\}$$
We decompose the path $(X_t)_{\tilde{T}\leqslant t<\zeta}$ in bridges $((X_t)_{\tau_{\tilde{l}_{j-1}}\leqslant t\leqslant \tau_{\tilde{l}_j}})_{j\geqslant 0}$ from X_{ζ^-} to X_{ζ^-}.
- We denote $\mathscr{L}^2((X_t)_{0\leqslant t<\zeta})$ the set of loops
$$\left\{(X_{t+s})_{0\leqslant s\leqslant b^+(t)-t} | t \in \mathfrak{b}^-\right\} \cup \left\{(X_{\tau_{\tilde{l}_{j-1}}+s})_{0\leqslant s\leqslant \tau_{\tilde{l}_j}-\tau_{\tilde{l}_{j-1}}} | j \geqslant 0\right\}$$
where the loops are considered to be unrooted.

The loops in $\mathscr{L}^1((X_t)_{0\leqslant t<\zeta})$ and $\mathscr{L}^2((X_t)_{0\leqslant t<\zeta})$ are not the same but follow the same law.

PROPOSITION 2.5.7. *$\mathscr{L}^1((X_t)_{0\leqslant t<\zeta})$ and $\mathscr{L}^2((X_t)_{0\leqslant t<\zeta})$, considered as collections of unrooted loops, have the same law. Let $\mathcal{L}_{1,L}$ be a Poisson ensemble of loops independent from X_{ζ^-}. Then $\mathscr{L}^1((X_t)_{0\leqslant t<\zeta})$ and $\mathscr{L}^2((X_t)_{0\leqslant t<\zeta})$ have the same law as*

$$\mathcal{L}_{1,L} \cap \{\gamma \in \mathfrak{L}^* | \gamma([0, T(\gamma)]) \cap [X(0), X(\zeta^-)] \ (or\ [X(\zeta^-), X(0)]) \neq \varnothing\} \tag{2.5.11}$$

PROOF. First we will prove that $\mathscr{L}^2((X_t)_{0\leqslant t<\zeta})$ has the same law as (2.5.11). If $\mathbb{P}(X_{\zeta^-} = \inf I) > 0$, then conditional on $X_{\zeta^-} = \inf I$, $(X_t)_{0\leqslant t<\zeta}$ has the law of a sample path corresponding to the generator $Conj(u_\downarrow, L)$. If $y \in I \cap (-\infty, x]$ and y is in the support of κ (the killing measure in L) then conditional on $X_{\zeta^-} = y$, $(X_t)_{0\leqslant t<\zeta}$ is distributed according the measure $\frac{1}{G(x,y)}\mu_L^{x,y}$ (property 2.3.2 (i)). According to the lemma 2.3.3, $(X_t)_{0\leqslant t\leqslant \tilde{T}}$ and $(X_{\tilde{T}+t})_{0\leqslant t\leqslant \zeta-\tilde{T}}$ are independent conditionally $X_{\zeta^-} = y$, $(X_t)_{0\leqslant t\leqslant \tilde{T}}$ having the law of a sample path corresponding to the generator $Conj(u_\downarrow, L)$, run until hitting y, and $(X_{\tilde{T}+t})_{0\leqslant t\leqslant \zeta-\tilde{T}}$ following the law $\frac{1}{G(y,y)}\mu_L^{y,y}$. From proposition 2.5.4 and 2.5.6 follows that $\mathscr{L}^2((X_t)_{0\leqslant t<\zeta})$ and (2.5.11) have the same law on the event $X_{\zeta^-} \leqslant x$. Symmetrically this also true on the event $X_{\zeta^-} \geqslant x$.

The decomposition $\mathscr{L}^1((X_t)_{0\leqslant t<\zeta})$ is obtained by first applying the decomposition $\mathscr{L}^2$ to the time-reversed path $(X_{\zeta-t})_{0<t\leqslant\zeta}$ and then applying again the time-reversal to the obtained loops. The law of the loops in (2.5.11) is invariant by time-reversal. Let $y \in I$,

y in the support of κ. Conditional on $X_{\zeta-} = y$, the law of $(X_{\zeta-t})_{0<t\leqslant\zeta}$ is $\frac{1}{G(x,y)}\mu^{y,x}$. So applying the decomposition $\mathscr{L}^2$ to the path $(X_{\zeta-t})_{0<t\leqslant\zeta}$ conditioned by $X_{\zeta-} = y$ gives

$$\mathcal{L}_{1,L} \cap \{\gamma \in \mathfrak{L}^* | \gamma([0, T(\gamma)]) \cap [y, x] \text{ (or } [x, y]) \neq \varnothing\}$$

If $\mathbb{P}(X_{\zeta-} = \inf I) > 0$ then conditional on $X_{\zeta-} = \inf I$, the path $(X_t)_{0\leqslant t<\zeta}$ is a limit as $y \to \inf I$ of paths following the law $\frac{1}{G(x,y)}\mu^{x,y}$ (i.e. the latter are restrictions of the former). Thus conditional on $X_{\zeta-} = \inf I$ $\mathscr{L}^1((X_t)_{0\leqslant t<\zeta})$ is an increasing limit as $y \to \inf I$ of

$$\mathcal{L}_{1,L} \cap \{\gamma \in \mathfrak{L}^* | \gamma([0, T(\gamma)]) \cap [y, x] \neq \varnothing\}$$

which is

$$\mathcal{L}_{1,L} \cap \{\gamma \in \mathfrak{L}^* | \gamma([0, T(\gamma)]) \cap [\inf I, x] \neq \varnothing\}$$

Similar is true conditional on $X_{\zeta-} = \sup I$. $\square$

CHAPTER 3

The analogue of the Wilson's loop erasure algorithm for one-dimensional Brownian motion with killing

3.1. The algorithm and its output

3.1.1. Description of the algorithm. Given a finite undirected connected graph $\mathcal{G} = (V, E)$ and C a positive weight function on its edges, a Uniform Spanning Tree of the weighted graph $\mathcal{G}$ is a random spanning tree with the occurrence probability of a spanning tree $\mathcal{T}$ proportional to

$$\prod_{e \text{ edge of } \mathcal{T}} C(e)$$

The edges belonging to the Uniform Spanning Tree are a determinantal point process (transfer current theorem). In [**Wil96**] Wilson showed how to sample a Uniform Spanning Tree using successive random walks to nearest neighbours, with transition probabilities proportional to C, starting from different vertices, and erasing the loops created by these random walks. The edges left after loop-erasure form a Uniform Spanning Tree. This is known as Wilson's algorithm. See [**BLPS01**] for a review. In [**Jan11**], chapter 8, Le Jan shows that the loops erased during the execution of Wilson's algorithm are related to the Poisson ensemble of Markov loops of parameter 1.

In [**Jan11**], chapter 10, Le Jan suggests that Wilson's algorithm can be adapted to the situation where the random walk on a graph is replaced by a transient diffusion on a subinterval I of $\mathbb{R}$. In this section we will describe the algorithm in the latter setting. The algorithm returns on one hand a sequence of one-dimensional paths which can be decomposed into a Poisson ensemble of Markov loops of parameter 1 (section 3.1.2), and on the other hand a pair of interwoven determinantal point processes on I, which may be interpreted as some kind of Uniform Spanning Tree. In section 3.1.3 we will derive the law of this pair of determinantal point processes in the setting where the underlying is a Brownian motion on $\mathbb{R}$ with a killing measure. In section 3.1.4 we will give without proof the law in general case as it follows directly from the Brownian case.

Let I be a subinterval of $\mathbb{R}$ and L a generator of a transient diffusion on I of form 2.2.11. Let κ be the killing measure in L, which may be zero. Let $(x_n)_{n\geqslant 1}$ be a sequence of pairwise distinct points in I which is dense in I. Let $\left((X_t^{(x_n)})_{0\leqslant t<\zeta_n}\right)_{n\geqslant 1}$ be a sequence of independent sample paths of the diffusion of generator L, with starting points $X_0^{(x_n)} = x_n$. In the first step of Wilson's algorithm we will recursively define sequences $(T_n)_{n\geqslant 1}$, $(\mathcal{Y}_n)_{n\geqslant 1}$ and $(\mathcal{J})_{n\geqslant 1}$ where T_n is a killing time for $X^{(x_n)}$, $\mathcal{Y}_n$ is a finite subset of $Supp(\kappa) \cup \partial I$ and $\mathcal{J}_n$ is a finite set of disjoint compact subintervals of $\bar{I}$, some of which may be reduced to one point:

- $T_1 := \zeta_1$, $\mathcal{Y}_1 := \{X_{T_1^-}^{(x_1)}\}$, $\mathcal{J}_1 := \{[x_1, X_{T_1^-}^{(x_1)}]\}$ (or $\{[B_{T_1^-}^{(x_1)}, x_1]\}$).

- Assume that $\mathcal{Y}_n$ and $\mathcal{J}_n$ are constructed. If $x_{n+1} \in \bigcup_{J\subseteq\mathcal{J}_n} J$ then we set $T_{n+1} := 0$, $\mathcal{Y}_{n+1} := \mathcal{Y}_n$ and $\mathcal{J}_{n+1} := \mathcal{J}_n$. If $x_{n+1} \notin \bigcup_{J\subseteq\mathcal{J}_n} J$ then we define

$$T_{n+1} := \min\Big(\zeta_n, \inf\Big\{t \geqslant 0 | X_t^{(x_{n+1})} \in \bigcup_{J\subseteq\mathcal{J}_n} J\Big\}\Big)$$

If $X_{T_{n+1}^-}^{(x_{n+1})} \in \bigcup_{J\subseteq\mathcal{J}_n} J$ then there is a unique $J \in \mathcal{J}_n$ such that $X_{T_{n+1}^-}^{(x_{n+1})} \in J$. In this case we set $\mathcal{Y}_{n+1} := \mathcal{Y}_n$ and

$$\mathcal{J}_{n+1} := (\mathcal{J}_n\backslash\{J\}) \cup \big\{J \cup \big[x_{n+1}, X_{T_{n+1}^-}^{(x_{n+1})}\big]\big\}$$

$$\Big(\text{or } (\mathcal{J}_n\backslash\{J\}) \cup \big\{J \cup \big[X_{T_{n+1}^-}^{(x_{n+1})}, x_{n+1}\big]\big\}\Big)$$

If $X_{T_{n+1}^-}^{(x_{n+1})} \notin \bigcup_{J\subseteq\mathcal{J}_n} J$ then we set $\mathcal{Y}_{n+1} := \mathcal{Y}_n \cup \big\{X_{T_{n+1}^-}^{(x_{n+1})}\big\}$ and

$$\mathcal{J}_{n+1} := \mathcal{J}_n \cup \big\{\big[x_{n+1}, X_{T_{n+1}^-}^{(x_{n+1})}\big]\big\} \ \Big(\text{or } \mathcal{J}_n \cup \big\{\big[X_{T_{n+1}^-}^{(x_{n+1})}, x_{n+1}\big]\big\}\Big)$$

It is immediate to check by induction the following facts:

- $\mathcal{Y}_n \subseteq Supp(\kappa) \cup \partial I$. More precisely $\mathcal{Y}_n \subseteq Supp(\kappa) \cup \big\{y \in \partial I | \mathbb{P}\big(X_{\zeta_n^-}^{(x_n)} = y\big) > 0\big\}$.
- The intervals in $\mathcal{J}_n$ are pairwise disjoint.
- $\sharp\mathcal{Y}_n = \sharp\mathcal{J}_n \leqslant n$
- For every $y \in \mathcal{Y}_n$ there is one single $J \in \mathcal{J}_n$ such that $y \in J$.
- $\mathcal{Y}_n \subseteq \mathcal{Y}_{n+1}$
- If $n \leqslant n'$, then for every $J \in \mathcal{J}_n$ there is one single $J' \in \mathcal{J}_{n'}$ such that $J \subseteq J'$. We denote $\imath_{n,n'}$ the corresponding application from $\mathcal{J}_n$ to $\mathcal{J}_{n'}$. The application $\imath_{n,n'}$ is injective. Trivially for $n \leqslant n' \leqslant n''$, $\imath_{n,n''} = \imath_{n',n''} \circ \imath_{n,n'}$
- For any $J \in \mathcal{J}_n$, $\partial J \subseteq \mathcal{Y}_n \cup \{x_1, \dots, x_n\}$.

In the second step of Wilson's algorithm we will take the limit of $((\mathcal{Y}_n, \mathcal{J}_n))_{n\geqslant 1}$ and define $(\mathcal{Y}_\infty, \mathcal{J}_\infty)$ as follows:

$$\mathcal{Y}_\infty := \bigcup_{n\geqslant 1} \mathcal{Y}_n \qquad \mathcal{J}_\infty := \bigcup_{n\geqslant 1} \bigcup_{J\in\mathcal{J}_n} \Big\{ \bigcup_{n'\geqslant n} \imath_{n,n'}(J)\Big\}$$

$\mathcal{Y}_\infty$ is a finite or countable subset of $Supp(\kappa) \cup \partial I$. $\mathcal{J}_\infty$ is a finite of countable set of disjoint subintervals of $\bar{I}$, but these subintervals are not necessarily closed or bounded. For any $y \in \mathcal{Y}_\infty$, there is a single $J \in \mathcal{J}_\infty$ such that $y \in J$, and this induces a bijection between $\mathcal{Y}_\infty$ and $\mathcal{J}_\infty$. For any $J \in \mathcal{J}_n$, there is a single $J' \in \mathcal{J}_\infty$ such that $J \subseteq J'$. We define $\imath_{n,\infty}(J) = J'$. $\imath_{n,\infty}$ is injective. Trivially, for $n \leqslant n'$, $\imath_{n,\infty} = \imath_{n',\infty} \circ \imath_{n,n'}$. We will sometimes write $\mathcal{Y}_n(x_1, \dots, x_n)$, $\mathcal{J}_n(x_1, \dots, x_n)$, $\mathcal{Y}_\infty((x_n)_{n\geqslant 1})$ and $\mathcal{J}_\infty((x_n)_{n\geqslant 1})$ in order to emphasize the dependence on the starting points $(x_n)_{n\geqslant 1}$. In the sections 3.1.3 and 3.1.4 we will see that

- The set $\mathcal{Y}_\infty$ is a.s. discrete.
- A.s. for any intervals $J \in \mathcal{J}_\infty$, $J\backslash\partial I$ is open
- The subset $I\backslash\bigcup_{J\in\mathcal{J}_\infty} J$ is a.s. discrete.
- The law of $(\mathcal{Y}_\infty, \mathcal{J}_\infty)$ does not depend on the choice of starting points $(x_n)_{n\geqslant 1}$.

We introduce $\mathcal{Z}_\infty := I\backslash\Big(\bigcup_{J\in\mathcal{J}_\infty} J\Big)$. We will further see that $\mathcal{Y}_\infty$ and $\mathcal{Z}_\infty$ are determinantal point processes.

The couple $(\mathcal{Y}_\infty, \mathcal{J}_\infty)$ may be interpreted as a spanning tree. Consider the following undirected "graph": Its set of "vertices" is $\bar{I} \cup \{\dagger\}$ where $\dagger$ is a cemetery point outside of $\bar{I}$. Ever point $x \in I$ is connected by an "edge" to its two infinitesimal neighbours $x - dx$ and $x + dx$. Every point in $Supp(\kappa)$ is connected by an "edge" to $\dagger$. Finally any point in $y \in \partial I$ such that $\mathbb{P}\big(X^{(x_n)}_{\zeta_n^-} = y\big) > 0$ is connected by an "edge" to $\dagger$. On this "graph" $(\mathcal{Y}_\infty, \mathcal{J}_\infty)$ induces the following "spanning tree": Each point in $\bigcup_{J\in\mathcal{J}_\infty} J$ is connected to its infinitesimal neighbours in I and $\mathcal{Z}_\infty$ represents "edges" on I that are missing. Moreover every point in $\mathcal{Y}_\infty$ is connected to $\dagger$.

There are two trivial cases in which $(\mathcal{Y}_\infty, \mathcal{J}_\infty)$ is deterministic. In the first one $\kappa = 0$ and I has one single regular or exit boundary point y characterized by $\mathbb{P}\big(X^{(x_n)}_{\zeta_n^-} = y\big) > 0$ (see [**Bre92**], chapter 16, for the characterization of boundaries). Then $\mathcal{Y}_\infty$ is made of this boundary point and $\mathcal{J}_\infty$ contains one single interval $I \cup \mathcal{Y}_\infty$. $\mathcal{Z}_\infty$ is empty. In the second case I does not have regular or exit boundaries and κ is proportional to a Dirac measure $c\delta_{y_0}$. Then $\mathcal{Y}_\infty = \{y_0\}$ and $\mathcal{J}_\infty = \{I\}$. $\mathcal{Z}_\infty$ is again empty. In all other situation $\mathcal{Z}_\infty$ is non-empty and random. See figure 3.1.a for an illustration of $(\mathcal{Y}_n, \mathcal{J}_n)$ for $1 \leqslant n \leqslant 5$ and figure 3.1.b for an illustration of $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$.

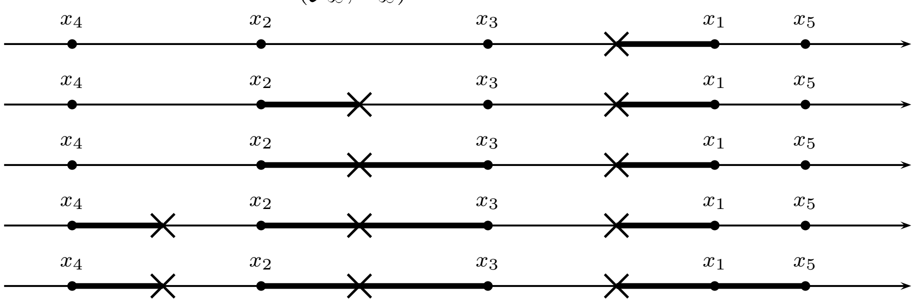

Fig.3.1.a - Illustration of $((\mathcal{Y}_n, \mathcal{J}_n))_{1\leqslant n\leqslant 5}$: x-dots represent the points of $\mathcal{Y}_n$ and thick lines the intervals in $\mathcal{J}_n$.

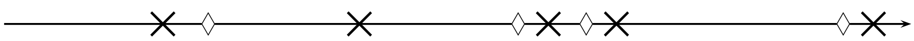

Fig.3.1.b - Illustration of $(\mathcal{Y}_\infty, \mathcal{J}_\infty)$: x-dots represent the points of $\mathcal{Y}_\infty$ and diamonds the points of $\mathcal{Z}_\infty$.

3.1.2. The erased paths. During the execution of Wilson's algorithm we used the paths $\Big(\big(X^{(x_n)}_t\big)_{0\leqslant t<T_n}\Big)_{n\geqslant 1}$. These paths can be further decomposed using the procedure described in the section 2.5.3.

PROPOSITION 3.1.1. *The family of unrooted loops*

$$\bigcup_{n\geqslant 1} \mathscr{L}^1\left(\big(X^{(x_n)}_t\big)_{0\leqslant t<T_n}\right)$$

has the same law as the Poisson ensemble $\mathcal{L}_{1,L}$. Moreover it is independent from $(\mathcal{Y}_\infty, \mathcal{J}_\infty)$.

PROOF. Let $\mathcal{L}_{1,L}$ be a Poisson ensemble of loops independent from the family of paths $\Big(\big(X^{(x_n)}_t\big)_{0\leqslant t<\zeta_n}\Big)_{n\geqslant 1}$. Using proposition 2.5.7 and induction is it immediate to show that the triple

$$\left(\mathcal{Y}_n, \mathcal{J}_n, \bigcup_{j=1}^{n} \mathscr{L}^1\left(\big(X^{(x_j)}_t\big)_{0\leqslant t<T_j}\right)\right)$$

has the same law as

$$\left(\mathcal{Y}_n, \mathcal{J}_n, \left\{(\gamma(t))_{0\leqslant t\leqslant T(\gamma)} \in \mathcal{L}_{1,L} \middle| \gamma([0,T(\gamma)]) \cap \bigcup_{J\in\mathcal{J}_n} J \neq \varnothing\right\}\right)$$

Since $(\mathcal{Y}_\infty, \mathcal{J}_\infty)$ is by construction independent from $\left(\left(X_t^{(x_j)}\right)_{0\leqslant t<T_j}\right)_{1\leqslant j\leqslant n}$ conditionally on $(\mathcal{Y}_n, \mathcal{J}_n)$, we further get that the triple

$$\left(\mathcal{Y}_\infty, \mathcal{J}_\infty, \bigcup_{j=1}^{n} \mathscr{L}^1\left(\left(X_t^{(x_j)}\right)_{0\leqslant t<T_j}\right)\right)$$

has the same law as

$$\left(\mathcal{Y}_\infty, \mathcal{J}_\infty, \left\{(\gamma(t))_{0\leqslant t\leqslant T(\gamma)} \in \mathcal{L}_{1,L} | \gamma([0,T(\gamma)]) \cap \bigcup_{J\in\mathcal{J}_n} J \neq \varnothing\right\}\right)$$

Taking the limit of the third component as n tends to infinity we get that

$$\left(\mathcal{Y}_\infty, \mathcal{J}_\infty, \bigcup_{j\geqslant 1} \mathscr{L}^1\left(\left(X_t^{(x_j)}\right)_{0\leqslant t<T_j}\right)\right)$$

has the same law as

$$\left(\mathcal{Y}_\infty, \mathcal{J}_\infty, \left\{(\gamma(t))_{0\leqslant t\leqslant T(\gamma)} \in \mathcal{L}_{1,L} | \gamma([0,T(\gamma)]) \cap \bigcup_{J\in\mathcal{J}_\infty} J \neq \varnothing\right\}\right)$$

To conclude we need only to show that almost surely

$$\left\{(\gamma(t))_{0\leqslant t\leqslant T(\gamma)} \in \mathcal{L}_{1,L} | \gamma([0,T(\gamma)]) \cap \bigcup_{J\in\mathcal{J}_\infty} J \neq \varnothing\right\} = \mathcal{L}_{1,L}$$

The latter is equivalent to $\bigcup_{J\in\mathcal{J}_\infty} J$ being dense in I, which will be proved in the next section. ☐

3.1.3. Determinantal point processes $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$: Brownian case. In this section we will describe $(\mathcal{Y}_\infty, \mathcal{J}_\infty)$ in the Brownian case by giving the joint law of the point processes $\mathcal{Y}_\infty$ and $\mathcal{Z}_\infty$. First we will study the case of a Brownian motion on a bounded interval (a,b), killed upon hitting a or b, and without killing measure. Then we will study the case of the Brownian motion on $\mathbb{R}$ with a non-zero Radon killing measure κ. We will write $\left(B_t^{(x_n)}\right)_{0\leqslant t<\zeta_n}$ instead of $\left(X_t^{(x_n)}\right)_{0\leqslant t<\zeta_n}$.

PROPOSITION 3.1.2. *In the case of a Brownian motion on a bounded interval (a,b), killed upon hitting a or b, and without killing measure, $\mathcal{Y}_\infty$ is deterministic and equals $\{a,b\}$ and $\mathcal{Z}_\infty$ is made of a single point distributed uniformly on (a,b).*

PROOF. For $n\geqslant 1$ we define $\tilde{x}_{n,0} < \tilde{x}_{n,1} < \cdots < \tilde{x}_{n,n+1}$ as the family $x_1,\dots,x_n,a,b$ ordered increasingly. According to this definition $\tilde{x}_{n,0} = a$ and $\tilde{x}_{n,n+1} = b$. As a convention we denote $\tilde{x}_{0,0} := a$ and $\tilde{x}_{0,1} := b$. For $n\geqslant 2$, one of the following situations may occur:

- $\mathcal{Y}_n = \{b\}$ and $\mathcal{J}_n = \{[\tilde{x}_{n,1}, b]\}$
- $\mathcal{Y}_n = \{a\}$ and $\mathcal{J}_n = \{[a, \tilde{x}_{n,n}]\}$
- $\mathcal{Y}_n = \{a,b\}$ and for some $j\in\{2,\dots,n\}$, $\mathcal{J}_n = \{[a,\tilde{x}_{n,j-1}], [\tilde{x}_{n,j}, b]\}$

In any case $(a,b)\backslash\left(\bigcup_{J\in\mathcal{J}_n} J\right)$ is an interval of form $(\tilde{x}_{n,j-1}, \tilde{x}_{n,j})$.

We set $\{J\}_0 = \varnothing$. Let $n\geqslant 1$. There is a $j\in\{1,\dots,n\}$ such that $x_n \in (\tilde{x}_{n-1,j-1}, \tilde{x}_{n-1,j})$. Conditional on $(a,b)\backslash\left(\bigcup_{J\in\mathcal{J}_{n-1}} J\right) = (\tilde{x}_{n-1,j-1}, \tilde{x}_{n-1,j})$, the point $B_{T_n^-}^{(x_n)}$ equals $\tilde{x}_{n-1,j-1}$ with probability $\frac{\tilde{x}_{n-1,j}-x_n}{\tilde{x}_{n-1,j})-\tilde{x}_{n-1,j-1}}$ and $\tilde{x}_{n-1,j}$ with probability $\frac{x_n-\tilde{x}_{n-1,j-1}}{\tilde{x}_{n-1,j})-\tilde{x}_{n-1,j-1}}$. By induction we get that

$$\mathbb{P}\left((a,b)\backslash\left(\bigcup_{J\in\mathcal{J}_n} J\right) = (\tilde{x}_{n,j-1}, \tilde{x}_{n,j})\right) = \frac{\tilde{x}_{n,j} - \tilde{x}_{n,j-1}}{b-a}$$

Hence

$$\mathbb{P}(\mathcal{Y}_\infty = \{a\}) \leqslant \lim_{n\to+\infty} \mathbb{P}\left((a,b)\backslash\Big(\bigcup_{J\in\mathcal{J}_n} J\Big) = (\tilde{x}_{n,0},\tilde{x}_{n,1})\right) = \lim_{n\to+\infty} \frac{\tilde{x}_{n,1}-\tilde{x}_{n,0}}{b-a} = 0$$

and similarly $\mathbb{P}(\mathcal{Y}_\infty = \{b\}) = 0$. Thus $\mathcal{Y}_\infty = \{a,b\}$. Almost surely for n large enough $\mathcal{J}_n$ will be of form $\{[a,\tilde{x}_{n,j-1}],[\tilde{x}_{n,j},b]\}$ for a random $j \in \{2,\dots,n\}$. We denote by $p^+_{n,1}$ respectively $p^-_{n,2}$ the random values of $\tilde{x}_{n,j-1}$ respectively $\tilde{x}_{n,j}$. Almost surely, neither of the non-decreasing sequence $(p^+_{n,1})_n$ or non-increasing sequence of $(p^-_{n,2})_n$ is stationary. This fact follows from the same argument according to which $\mathcal{Y}_\infty$ is not reduced to one point. Moreover $p^-_{n,2} - p^+_{n,1}$, bounded by $\sup_{2\leqslant j\leqslant n}(\tilde{x}_{n,j}-\tilde{x}_{n,j-1})$, converges to 0. It follows that a.s. $\mathcal{Z}_\infty$ is reduced to one point, the common limit of $p^+_{n,1}$ and $p^-_{n,2}$. Finally if $\tilde{a} < \tilde{b}$ are two values taken by the sequence $(x_n)_{n\geqslant 1}$ then

$$\mathbb{P}(\mathcal{Z}_\infty \subseteq (\tilde{a},\tilde{b})) = \frac{\tilde{b}-\tilde{a}}{b-a}$$

It follows that the unique point in $\mathcal{Z}_\infty$ is distributed uniformly on (a,b). □

We consider now the case of the Brownian motion on $\mathbb{R}$ with a non-zero Radon killing measure κ. $G(x,y) = u_\uparrow(x\wedge y)u_\downarrow(x\vee y)$ will be the Green's function of $\frac{1}{2}\frac{d^2}{dx^2} - \kappa$. The law of $(\mathcal{Y}_n,\mathcal{J}_n)$ may be expressed explicitly. Let Q_n be the cardinal of $\mathcal{Y}_n$. Let $Y_{n,1},Y_{n,2},\dots,Y_{n,Q(n)}$ be the points in $\mathcal{Y}_n$ ordered in the increasing sense. Denote by $[p^-_{n,1},p^+_{n,1}],[p^-_{n,2},p^+_{n,2}],\dots,$ $[p^-_{n,Q_n},p^+_{n,Q_n}]$ the intervals in $\mathcal{J}_n$ ordered in the increasing sense. For all $q \in \{1,\dots,Q_n\}$, $Y_{n,q} \in [p^-_{n,q},p^+_{n,q}]$. It happens with positive probability that for some q, $p^-_{n,q} = p^+_{n,q}$ if one of the starting points $x_1,\dots,x_n$ is an atom of κ. To compute recursively the joint law of above random variables we use the following facts: Given a killed Brownian path $\big(B^{(x)}_t\big)_{0\leqslant t<\zeta}$ starting from x, the distribution of $B^{(x)}_{\zeta^-}$ is $G(x,y)\kappa(dy)$ (see section 2.2.2). Given $a < x$, let T_a be the first time $B^{(x)}$ hits a. Then

$$\mathbb{P}_x(T_a \leqslant \zeta) = \frac{u_\downarrow(x)}{u_\downarrow(a)} = \frac{G(x,a)}{G(a,a)}$$

On the event $T_a > \zeta$, the distribution of $B^{(x)}_{\zeta^-}$ is:

$$(G(x,y) - \mathbb{P}_x(T_a \leqslant \zeta)G(a,y))1_{y>a}\kappa(dy) = \left(G(x,y) - \frac{G(x,a)G(a,y)}{G(a,a)}\right)1_{y>a}\kappa(dy)$$

More generally, if $a < x < b$ and $\tilde{\zeta}$ is the first time $B^{(x)}$ gets either killed by the killing measure κ or hits a or b then

- The probability that $B^{(x)}_{\tilde{\zeta}^-} = a$ is:

$$\frac{u_\downarrow(a)u_\uparrow(x) - u_\downarrow(x)u_\uparrow(a)}{u_\downarrow(a)u_\uparrow(b) - u_\downarrow(b)u_\uparrow(a)} = \frac{\det\begin{pmatrix} G(x,b) & G(a,b) \\ G(a,x) & G(a,a)\end{pmatrix}}{\det\begin{pmatrix} G(b,b) & G(a,b) \\ G(a,b) & G(a,a)\end{pmatrix}}$$

- The probability that $B^{(x)}_{\tilde{\zeta}^-} = b$ is:

$$\frac{u_\downarrow(x)u_\uparrow(b) - u_\downarrow(b)u_\uparrow(x)}{u_\downarrow(a)u_\uparrow(b) - u_\downarrow(b)u_\uparrow(a)} = \frac{\det\begin{pmatrix} G(a,x) & G(a,b) \\ G(x,b) & G(b,b)\end{pmatrix}}{\det\begin{pmatrix} G(a,a) & G(a,b) \\ G(a,b) & G(b,b)\end{pmatrix}}$$

- The distribution of $B^{(x)}_{\tilde{\zeta}^-}$ on (a,b) is:

$$\frac{\det\begin{pmatrix} G(x,y) & G(a,y) & G(y,b) \\ G(a,x) & G(a,a) & G(a,b) \\ G(x,b) & G(a,b) & G(b,b) \end{pmatrix}}{\det\begin{pmatrix} G(a,a) & G(a,b) \\ G(a,b) & G(b,b) \end{pmatrix}} 1_{a<y<b}\kappa(dy)$$

Above expressions give the law of $(\mathcal{Y}_1, \mathcal{J}_1)$ and the law of $(\mathcal{Y}_{n+1}, \mathcal{J}_{n+1})$ conditional on $(\mathcal{Y}_n, \mathcal{J}_n)$. By induction one can derive the law of $(\mathcal{Y}_n, \mathcal{J}_n)$. We will express it using a single identity involving a determinant. However this single identity may correspond to different configurations: We will divide the set of indices $\{1,\dots,Q_n\}$ in three categories E^-_n, E^+_n and $E^{-,+}_n$ where for $q \in E^-_n$, $Y_{n,q} = p^-_{n,q}$, for $q \in E^+_n$, $Y_{n,q} = p^+_{n,q}$ and for $q \in E^{-,+}_n$, $p^-_{n,q} < Y_{n,q} < p^+_{n,q}$. For instance on the figure 3.1.a, $Q_5 = 3$, $E^-_5 = \{3\}$, $E^+_5 = \{1\}$ and $E^{-,+}_5 = \{2\}$.

PROPOSITION 3.1.3. *Let $q \in \{1,\dots,n\}$. Let $(E^-_n, E^+_n, E^{-,+}_n)$ be a partition of $\{1,\dots,q\}$:*

$$\{1,\dots,q\} = E^-_n \amalg E^+_n \amalg E^{-,+}_n$$

Let x^- be an increasing function from $E^-_n \amalg E^{-,+}_n$ to $\{x_1,\dots,x_n\}$ and x^+ an increasing function from $E^+_n \amalg E^{-,+}_n$ to $\{x_1,\dots,x_n\}$. We assume that the sets $x^-(E^-_n \amalg E^{-,+}_n)$ and $x^+(E^+_n \amalg E^{-,+}_n)$ are disjoint, that for every $i \in E^{-,+}_n$ $x^-(i) < x^+(i)$ and that for every $i \in E^-_n \amalg E^{-,+}_n$ and $j \in E^+_q \amalg E^{-,+}_q$ such that $i \neq j$, $(x^+(j) - x^-(i))$ has the same sign as $(j-i)$. Let $(\Delta_i)_{1\leqslant i\leqslant n}$ be a family of disjoint bounded intervals each of which may be open, closed or semi-open such that for every $i<j$, $\max\Delta_i < \min\Delta_j$, that for every i, $\min\Delta_i \geqslant x^-(i)$ if $i \in E^-_n \amalg E^{-,+}_n$, $\max\Delta_i \leqslant x^+(i)$ if $i \in E^+_n \amalg E^{-,+}_n$, and that for all i

$$x^-(i-1), x^+(i-1) < \min\Delta_i,\ \max\Delta_i < x^-(i+1), x^+(i+1)$$

where in the previous inequalities one should only consider the terms that are defined. Let $p^-_i(y_i)$ and $p^+_i(y_i)$ be the functions defined by: $p^-_i(y_i) = x^-(i)$ if $i \in E^-_n \amalg E^{-,+}_n$ and y_i otherwise. $p^+_i(y_i) = x^+(i)$ if $i \in E^+_n \amalg E^{-,+}_n$ and y_i otherwise. Then

$$(3.1.1)\quad \mathbb{P}\big(Q_n = q, \forall i \in E^-_n, p^-_{n,i} = x^-(i), p^+_{n,i} = Y_{n,i} \forall i \in E^+_n, p^+_{n,i} = x^+(i), p^-_{n,i} = Y_{n,i}, \\ \forall i \in E^{-,+}_n, p^-_{n,i} = x^-(i), p^+_{n,i} = x^+(i), \forall r \in \{1,\dots,q\}, Y_{n,r} \in \Delta_r\big) = \\ \int_{y_1\in\Delta_1}\cdots\int_{y_q\in\Delta_q} \det\left(G(p^-_i(y_i), p^+_j(y_j))\right)_{1\leqslant i,j\leqslant q} \prod_{1\leqslant r\leqslant q}\kappa(dy_i)$$

$\det\left(G(p^-_i(y_i), p^+_j(y_j))\right)_{1\leqslant i,j\leqslant q}$ *may be rewritten as a simpler product:*

$$(3.1.2)\quad G(p^-_1(y_1), p^+_1(y_1)) \prod_{1\leqslant r\leqslant q-1}\Big(G(p^-_{r+1}(y_{r+1}), p^+_{r+1}(y_{r+1})) \\ - \frac{G(p^-_r(y_r), p^+_{r+1}(y_{r+1}))G(p^+_r(y_r), p^-_{r+1}(y_{r+1}))}{G(p^-_r(y_r), p^+_r(y_r))}\Big)$$

If σ is a permutation of $\{1,\dots,n\}$, then $(\mathcal{Y}_n(x_{\sigma(1)},\dots,x_{\sigma(n)}), \mathcal{J}_n(x_{\sigma(1)},\dots,x_{\sigma(n)}))$ has the same law as $(\mathcal{Y}_n(x_1,\dots,x_n), \mathcal{J}_n(x_1,\dots,x_n))$ Moreover, for any $n' > n$ and any permutation σ of $\{n+1,\dots,n'\}$, the law of $(\mathcal{Y}_{n'}(x_1,\dots,x_n,x_{\sigma(n+1)},\dots,x_{\sigma(n')}),$ $\mathcal{J}_{n'}(x_1,\dots,x_n,x_{\sigma(n+1)},\dots,x_{\sigma(n')}))$ conditional on $(\mathcal{Y}_n(x_1,\dots,x_n), \mathcal{J}_n(x_1,\dots,x_n))$ is the same as the law of $(\mathcal{Y}_{n'}(x_1,\dots,x_n,x_{n+1},\dots,x_{n'}), \mathcal{J}_{n'}(x_1,\dots,x_n,x_{n+1},\dots,x_{n'}))$ conditional on $(\mathcal{Y}_n(x_1,\dots,x_n), \mathcal{J}_n(x_1,\dots,x_n))$.

PROOF. We will only give the sketch of a short proof. First let's check that the determinant $\det\left(G(p_i^-(y_i),p_j^+(y_j))\right)_{1\leqslant i,j\leqslant q}$ may be indeed expressed as a product (3.1.2). We use the fact that for any $a<b<\tilde{a}<\tilde{b}\in\mathbb{R}$:

$$G(a,\tilde{b})G(b,\tilde{a})=G(a,\tilde{a})G(b,\tilde{b})=u_\uparrow(a)u_\uparrow(b)u_\downarrow(\tilde{a})u_\downarrow(\tilde{b})$$

By subtracting from the last line in the matrix $\left(G(p_i^-(y_i),p_j^+(y_j))\right)_{1\leqslant i,j\leqslant q}$, that is to say from $(G(p_q^-(y_q),p_j^+(y_j)))_{1\leqslant j\leqslant q}$, the second to last line $(G(p_{q-1}^-(y_{q-1}),p_j^+(y_j)))_{1\leqslant j\leqslant q}$ multiplied by $\dfrac{G(p_{q-1}^-(y_{q-1}),p_q^+(y_q))}{G(p_{q-1}^-(y_{q-1}),p_{q-1}^+(y_{q-1}))}$ we get zero for all coefficient on the last line, except the diagonal one. Thus $\det\left(G(p_i^-(y_i),p_j^+(y_j))\right)_{1\leqslant i,j\leqslant q}$ equals

$$\det\left(G(p_i^-(y_i),p_j^+(y_j))\right)_{1\leqslant i,j\leqslant q-1}\times\\ \left(G(p_q^-(y_q),p_q^+(y_q))-\frac{G(p_{q-1}^-(y_{q-1}),p_q^+(y_q))G(p_{q-1}^+(y_{q-1}),p_q^-(y_q))}{G(p_{q-1}^-(y_{q-1}),p_{q-1}^+(y_{q-1}))}\right)$$

By induction we get (3.1.2).

Next step is to check that $(\mathcal{Y}_n(x_1,\dots,x_{n-2},x_{n-1},x_n),\ \mathcal{J}_n(x_1,\dots,x_{n-2},x_{n-1},x_n))$ and $(\mathcal{Y}_n(x_1,\dots,x_{n-2},x_n,x_{n-1}),\mathcal{J}_n(x_1,\dots,x_{n-2},x_n,x_{n-1}))$ have the same law conditional on $(\mathcal{Y}_{n-2}(x_1,\dots,x_{n-2}),\mathcal{J}_{n-2}(x_1,\dots,x_{n-2}))$. This can be done using the explicit expressions for the conditional destitution of $B^{(x_{n-1})}_{T^-_{n-1}}$, $B^{(x_n)}_{T^-_n}$, $B^{(x_n)}_{T^-_{n-1}}$ and $B^{(x_{n-1})}_{T^-_n}$. This invariance by transposition of the two last starting points implies in turn all the invariances by permutation stated in the proposition.

From the invariance by permutation follows that one only needs to prove (3.1.1) in case $x_1<x_2<\dots<x_n$. In this case one can prove (3.1.1) by induction on n using the expression (3.1.2) for $\det\left(G(p_i^-(y_i),p_j^+(y_j))\right)_{1\leqslant i,j\leqslant q}$. □

The fact that the law of the tree obtained after n steps of Wilson's algorithm is invariant under permutations of the starting points $(x_1,\dots,x_n)$ is something that is also satisfied in case of random walks on a true finite graph. The product (3.1.2) can be further rewritten as

(3.1.3)

$$u_\uparrow(p_1^-(y_1))u_\downarrow(p_q^+(y_q))\prod_{1\leqslant r\leqslant q-1}(u_\downarrow(p_r^+(y_r))u_\uparrow(p_{r+1}^-(y_{r+1}))-u_\uparrow(p_r^+(y_r))u_\downarrow(p_{r+1}^-(y_{r+1})))$$

Next we will show that $\mathcal{Y}_\infty$ and $\mathcal{Z}_\infty$ are a.s. discrete.

LEMMA 3.1.4. *For all $n\geqslant 2$ and $q\in\{2,\dots,n\}$:*

$$\mathbb{P}\left(\mathcal{Y}_\infty\cap(p_{n,q-1}^-,p_{n,q}^+)=\varnothing|p_{n,q-1}^-,p_{n,q}^+,Q_n\geqslant q\right)=\\ \frac{2(p_{n,q}^+-p_{n,q-1}^-)}{u_\downarrow(p_{n,q-1}^-)u_\uparrow(p_{n,q}^+)-u_\uparrow(p_{n,q-1}^-)u_\downarrow(p_{n,q}^+)}$$

PROOF. Let n and q be fixed. For $n'>n$, let

$$N(n'):=\sharp(\{x_{n+1},\dots,x_{n'}\}\cap(p_{n,q-1}^-,p_{n,q}^+)$$

and $\tilde{x}_{n',1}<\tilde{x}_{n',2}<\dots<\tilde{x}_{n',N(n')}$ the points of $\{x_{n+1},\dots,x_{n'}\}\cap(p_{n,q-1}^-,p_{n,q}^+)$ ordered increasingly. Conventionally we define $\tilde{x}_{n',0}:=p_{n,q-1}^-$ and $\tilde{x}_{n',N(n')+1}:=p_{n,q}^+$. The condition $\mathcal{Y}_{n'}\cap(p_{n,q-1}^-,p_{n,q}^+)=\varnothing$ is satisfied if and only if for some $i\in\{1,2,\dots,N(n')+1\}$, necessarily unique, the following holds:

$$[p_{n,q-1}^-,\tilde{x}_{n',i-1}]\subseteq\bigcup_{J\in\mathcal{J}_m}J\quad\text{and}\quad[\tilde{x}_{n',i},p_{n,q}^+]\subseteq\bigcup_{J\in\mathcal{J}_{n'}}J$$

Thus

$$\mathbb{P}\left(\mathcal{Y}_{n'} \cap (p^-_{n,q-1}, p^+_{n,q}) = \varnothing | p^-_{n,q-1}, p^+_{n,q}, Q_n \geqslant q\right) = \sum_{i=1}^{N(n')+1} \mathbb{P}\Big([p^-_{n,q-1}, \tilde{x}_{n',i-1}] \subseteq \bigcup_{J\in\mathcal{J}_m} J, [\tilde{x}_{n',i}, p^+_{n,q}] \subseteq \bigcup_{J\in\mathcal{J}_{n'}} J \Big| p^-_{n,q-1}, p^+_{n,q}, Q_n \geqslant q\Big)$$

Let $T_{n',i}$ be the first time $B^{(\tilde{x}_{n',i})}$ hits either $p^-_{n,q-1}$ or $p^+_{n,q}$ or gets killed by the killing measure κ. For $i \in \{1, 2, \dots, N(n')+1\}$ let $T_{n',i,\tilde{x}_{n',i-1}}$ be the first time $B^{(\tilde{x}_{n',i})}$ hits $\tilde{x}_{n',i-1}$. Since the law of $(\mathcal{Y}_{n'}, \mathcal{J}_{n'})$ conditional on $(\mathcal{Y}_n, \mathcal{J}_n)$ is invariant by permutation of points in $(x_{n+1}, \dots, x_{n'})$, we get that

$$\mathbb{P}\Big([p^-_{n,q-1}, \tilde{x}_{n',i-1}] \subseteq \bigcup_{J\in\mathcal{J}_m} J, [\tilde{x}_{n',i}, p^+_{n,q}] \subseteq \bigcup_{J\in\mathcal{J}_{n'}} J \Big| p^-_{n,q-1}, p^+_{n,q}, Q_n \geqslant q\Big) = \mathbb{P}\left(B^{(\tilde{x}_{n',i-1})}_{T^-_{n',i-1}} = p^-_{n,q-1}, B^{(\tilde{x}_{n',i})}_{T^-_{n',i}} = p^+_{n,q}, T_{n',i} < T_{n',i,\tilde{x}_{n',i-1}} \Big| p^-_{n,q-1}, p^+_{n,q}, Q_n \geqslant q\right) = \frac{u_\downarrow(\tilde{x}_{n',i-1})u_\uparrow(\tilde{x}_{n',i}) - u_\uparrow(\tilde{x}_{n',i-1})u_\downarrow(\tilde{x}_{n',i})}{u_\downarrow(p^-_{n,q-1})u_\uparrow(p^+_{n,q}) - u_\uparrow(p^-_{n,q-1})u_\downarrow(p^+_{n,q})}$$

It follows that

$$\mathbb{P}\left(\mathcal{Y}_{n'} \cap (p^-_{n,q-1}, p^+_{n,q}) = \varnothing | p^-_{n,q-1}, p^+_{n,q}, Q_n \geqslant q\right) = \sum_{i=1}^{N(n')+1} \frac{u_\downarrow(\tilde{x}_{n',i-1})u_\uparrow(\tilde{x}_{n',i}) - u_\uparrow(\tilde{x}_{n',i-1})u_\downarrow(\tilde{x}_{n',i})}{u_\downarrow(p^-_{n,q-1})u_\uparrow(p^+_{n,q}) - u_\uparrow(p^-_{n,q-1})u_\downarrow(p^+_{n,q})}$$

If $\tilde{x}_{n',i-1}$ is close to $\tilde{x}_{n',i}$ then

$$\begin{aligned} u_\downarrow(\tilde{x}_{n',i-1})u_\uparrow(\tilde{x}_{n',i}) - u_\uparrow(\tilde{x}_{n',i-1})u_\downarrow(\tilde{x}_{n',i}) &= W(u_\downarrow, u_\uparrow)(\tilde{x}_{n',i-1})(\tilde{x}_{n',i} - \tilde{x}_{n',i-1}) + o(\tilde{x}_{n',i} - \tilde{x}_{n',i-1}) \\ &= 2(\tilde{x}_{n',i} - \tilde{x}_{n',i-1}) + o(\tilde{x}_{n',i} - \tilde{x}_{n',i-1}) \end{aligned}$$

The sequence $(x_{n'})_{n'\geqslant n+1}$ is dense in $(p^-_{n,q-1}, p^+_{n,q})$. Thus

$$\lim_{n'\to+\infty} \mathbb{P}\left(\mathcal{Y}_{n'} \cap (p^-_{n,q-1}, p^+_{n,q}) = \varnothing | p^-_{n,q-1}, p^+_{n,q}, Q_n \geqslant q\right) = \frac{2(p^+_{n,q} - p^-_{n,q-1})}{u_\downarrow(p^-_{n,q-1})u_\uparrow(p^+_{n,q}) - u_\uparrow(p^-_{n,q-1})u_\downarrow(p^+_{n,q})}$$

□

PROPOSITION 3.1.5. *Let $a < b \in \mathbb{R}$. Then for all $n \geqslant 1$*

$$\mathbb{E}\left[\sharp(\mathcal{Y}_n \cap [a,b))\right] \leqslant \int_{[a,b)} G(x,x)\kappa(dx) \tag{3.1.4}$$

It follows that a.s. for all $a < b \in \mathbb{R}$, $\mathcal{Y}_\infty \cap [a,b)$ is finite.

PROOF. Let $\tilde{a} < \tilde{b} \in [a,b]$ where $\tilde{a}$ is close to $\tilde{b}$. We will first show that for all $n \geqslant 1$

$$\mathbb{P}\left(\mathcal{Y}_n \cap [\tilde{a}, \tilde{b}) \neq \varnothing\right) \leqslant \int_{[\tilde{a},\tilde{b})} G(x,x)\kappa(dx) + o(\tilde{b} - \tilde{a}) \tag{3.1.5}$$

where $o(\tilde{b} - \tilde{a})$ is uniform over $\tilde{a}$ and $\tilde{b}$ close to each other in $[a,b]$. Then we will deduce (3.1.4) by partitioning the interval $[a,b)$ in small subintervals $[\tilde{a}, \tilde{b})$ and approximating the expected number of points in $[\tilde{a},\tilde{b})$ by the probability of presence of one point. Let $n \geqslant 1$. Then

$$\mathbb{P}\left(\mathcal{Y}_n(x_1,\dots,x_n)\cap[\tilde{a},\tilde{b})\neq\varnothing\right)\leqslant\mathbb{P}\left(\mathcal{Y}_{n+2}(x_1,\dots,x_n,\tilde{a},\tilde{b})\cap[\tilde{a},\tilde{b})\neq\varnothing\right)$$

Since the law of $\mathcal{Y}_{n+2}$ is invariant by permutation of the starting points:

$$\mathbb{P}\left(\mathcal{Y}_{n+2}(x_1,\dots,x_n,\tilde{a},\tilde{b})\cap[\tilde{a},\tilde{b})\neq\varnothing\right)=\mathbb{P}\left(\mathcal{Y}_{n+2}(\tilde{a},\tilde{b},x_1,\dots,x_n)\cap[\tilde{a},\tilde{b})\neq\varnothing\right)$$

But

$$\begin{aligned}(3.1.6)\quad \mathbb{P}\left(\mathcal{Y}_{n+2}(\tilde{a},\tilde{b},x_1,\dots,x_n)\cap[\tilde{a},\tilde{b})\neq\varnothing\right)&=\mathbb{P}\left(\mathcal{Y}_2(\tilde{a},\tilde{b})\cap[\tilde{a},\tilde{b})\neq\varnothing\right)\\&+\mathbb{P}\left(\mathcal{Y}_2(\tilde{a},\tilde{b})\cap[\tilde{a},\tilde{b})=\varnothing,\mathcal{Y}_{n+2}(\tilde{a},\tilde{b},x_1,\dots,x_n)\cap[\tilde{a},\tilde{b})\neq\varnothing\right)\end{aligned}$$

We start Wilson's algorithm by launching first $B^{(\tilde{a})}$ starting from $\tilde{a}$ followed by $B^{(\tilde{b})}$ starting $\tilde{b}$. Then

$$\mathbb{P}\left(\mathcal{Y}_2(\tilde{a},\tilde{b})\cap[\tilde{a},\tilde{b})\neq\varnothing\right)=\mathbb{P}\left(B^{(\tilde{a})}_{T_1^-}\in[\tilde{a},\tilde{b})\right)+\mathbb{P}\left(B^{(\tilde{a})}_{T_1^-}\notin[\tilde{a},\tilde{b}),B^{(\tilde{a})}_{T_1^-}\leqslant\tilde{a},B^{(\tilde{b})}_{T_2^-}\in[\tilde{a},\tilde{b})\right)$$

Applying proposition 3.1.3 we get that

$$\begin{aligned}&\mathbb{P}\left(\mathcal{Y}_2(\tilde{a},\tilde{b})\cap[\tilde{a},\tilde{b})\neq\varnothing\right)=\\&\qquad\int_{x\in[\tilde{a},\tilde{b})}\left(G(\tilde{a},x)+\int_{y\leqslant\tilde{a}}(G(y,\tilde{a})G(x,\tilde{b})-G(y,\tilde{b})G(\tilde{a},x))\kappa(dy)\right)\kappa(dx)\end{aligned}$$

For $x\in\mathbb{R}$, let $T_{1,x}$ be the first time $B^{(\tilde{a})}$ hits x. Then

$$\begin{aligned}&G(\tilde{a},x)+\int_{y\leqslant\tilde{a}}(G(y,\tilde{a})G(x,\tilde{b})-G(y,\tilde{b})G(\tilde{a},x))\kappa(dy)=\\&\qquad G(x,x)\left(\mathbb{P}(T_1\geqslant T_{1,x})+\frac{G(x,b)}{G(x,x)}\mathbb{P}(T_1<T_{1,x},B^{(\tilde{a})}_{T_1^-}\leqslant\tilde{a})\right)\leqslant G(x,x)\end{aligned}$$

Thus

$$(3.1.7)\qquad \mathbb{P}\left(\mathcal{Y}_2(\tilde{a},\tilde{b})\cap[\tilde{a},\tilde{b})\neq\varnothing\right)\leqslant\int_{[\tilde{a},\tilde{b})}G(x,x)\kappa(dx)$$

Further

$$\begin{aligned}&\mathbb{P}\left(\mathcal{Y}_2(\tilde{a},\tilde{b})\cap[\tilde{a},\tilde{b})=\varnothing,\mathcal{Y}_{n+2}(\tilde{a},\tilde{b},x_1,\dots,x_n)\cap[\tilde{a},\tilde{b})\neq\varnothing\right)\leqslant\\&\qquad\mathbb{P}\left(\mathcal{Y}_2(\tilde{a},\tilde{b})\cap[\tilde{a},\tilde{b})=\varnothing,\mathcal{Y}_\infty(\tilde{a},\tilde{b},(x_j)_{j\geqslant1})\cap[\tilde{a},\tilde{b})\neq\varnothing\right)\end{aligned}$$

Applying lemma 3.1.4 and proposition 3.1.3 we get that

$$\begin{aligned}&\mathbb{P}\Big(\mathcal{Y}_2(\tilde{a},\tilde{b})\cap[\tilde{a},\tilde{b})=\varnothing,\mathcal{Y}_\infty(\tilde{a},\tilde{b},(x_j)_{j\geqslant1})\cap[\tilde{a},\tilde{b})\neq\varnothing\Big)=\\&\mathbb{P}\Big(\mathcal{Y}_\infty(\tilde{a},\tilde{b},(x_j)_{j\geqslant1})\cap[\tilde{a},\tilde{b})\neq\varnothing|\mathcal{Y}_2(\tilde{a},\tilde{b})\cap[\tilde{a},\tilde{b})=\varnothing\Big)\times\mathbb{P}\left(\mathcal{Y}_2(\tilde{a},\tilde{b})\cap[\tilde{a},\tilde{b})=\varnothing\right)\\&=\left(1-\frac{2(\tilde{b}-\tilde{a})}{u_\downarrow(\tilde{a})u_\uparrow(\tilde{b})-u_\uparrow(\tilde{a})u\downarrow(\tilde{b})}\right)\\&\qquad\times\int_{y\leqslant a}\int_{z\geqslant b}1_{y,z\notin[\tilde{a},\tilde{b})}\det\begin{pmatrix}G(y,a)&G(y,z)\\G(a,b)&G(b,z)\end{pmatrix}\kappa(dy)\kappa(dz)\\&\leqslant(u_\downarrow(\tilde{a})u_\uparrow(\tilde{b})-u_\uparrow(\tilde{a})u\downarrow(\tilde{b})-2(\tilde{b}-\tilde{a}))\int_{y\leqslant b}u_\uparrow(y)\kappa(dy)\int_{z\geqslant a}u_\downarrow(z)\kappa(dz)\end{aligned}$$

But

$$u_{\downarrow}(\tilde{a})u_{\uparrow}(\tilde{b}) - u_{\uparrow}(\tilde{a})u\downarrow(\tilde{b}) - 2(\tilde{b}-\tilde{a}) = o(\tilde{b}-\tilde{a})$$

Thus

$$\mathbb{P}\left(\mathcal{Y}_2(\tilde{a},\tilde{b}) \cap [\tilde{a},\tilde{b}) = \emptyset, \mathcal{Y}_{n+2}(\tilde{a},\tilde{b},x_1,\dots,x_n) \cap [\tilde{a},\tilde{b}) \neq \emptyset\right) = o(\tilde{b}-\tilde{a}) \tag{3.1.8}$$

Combining (3.1.6), (3.1.7) and (3.1.8) we get (3.1.5).

Now for $j \in \mathbb{N}^*$ and $i \in \{1,\dots,2^j\}$ consider the intervals $\Delta_{i,j}$ defined by

$$\Delta_{i,j} = \begin{cases} [a+(i-1)2^{-j}(b-a), a+i2^{-j}(b-a)) & \text{if } i \leqslant 2^j-1 \\ [a+(1-2^{-j})(b-a), b] & \text{if } i = 2^j \end{cases}$$

Then $\mathbb{E}[\sharp(\mathcal{Y}_n \cap [a,b))]$ is the increasing limit of $\sum_{i=1}^{2^j}\mathbb{P}(\mathcal{Y}_n \cap \Delta_{i,j} \neq \emptyset)$. But

$$\sum_{i=1}^{2^j}\mathbb{P}(\mathcal{Y}_n \cap \Delta_{i,j} \neq \emptyset) \leqslant \sum_{i=1}^{2^j}\int_{\Delta_{i,j}} G(x,x)\kappa(dx) + 2^j o(2^{-j})$$

(3.1.4) follows. Since (3.1.4) holds for all n, it also holds at the limit when n tends to $+\infty$. This implies that $\mathcal{Y}_\infty \cap [a,b)$ is a.s. finite. □

Proposition 3.1.6. *Almost surely all the intervals in $\mathcal{J}_\infty$ are open.*

Proof. We need only to show that for any $n \geqslant 1$ and $q \in \{1,\dots,n\}$

$$\mathbb{P}\left(Q_n \geqslant q, \forall n' \geqslant n, \min(\imath_{n,n'}([p^-_{n,q}, p^+_{n,q}])) = p^-_{n,q}\right) = 0 \tag{3.1.9}$$

and

$$\mathbb{P}\left(Q_n \geqslant q, \forall n' \geqslant n, \max(\imath_{n,n'}([p^-_{n,q}, p^+_{n,q}])) = p^+_{q,n}\right) = 0$$

Let n and q be fixed. We will show (3.1.9). We will also assume that $q \geqslant 2$. The proof is similar if $q=1$. We need to show that a.s. the following conditional probability converges to 0:

$$\lim_{n'\to+\infty}\mathbb{P}\left(\min(\imath_{n,n'}([p^-_{n,q}, p^+_{n,q}])) = p^-_{n,q} | (\mathcal{Y}_n,\mathcal{J}_n), Q_n \geqslant q\right) = 0$$

We recall that for $n'' \geqslant n+1$, $B^{(x_{n''})}$ is a Brownian motion starting from $x_{n''}$ and it is independent from $(\mathcal{Y}_n,\mathcal{J}_n)$. Let $T_{n'',p^-_{q,n}}$ be the first time it hits $p^-_{q,n}$ and $\tilde{T}_{n''}$ the first time it either hits $\bigcup_{J\in\mathcal{J}_n} J$ or gets killed by the killing measure κ. Since the law of $(\mathcal{Y}_{n'},\mathcal{J}_{n'})$ conditional on $(\mathcal{Y}_n,\mathcal{J}_n)$ is invariant by permutation of points in $(x_{n+1},\dots,x_{n'})$, we get that

$$\begin{aligned}&\mathbb{P}\left(\min(\imath_{n,n'}([p^-_{n,q}, p^+_{n,q}])) = p^-_{n,q} | (\mathcal{Y}_n,\mathcal{J}_n), Q_n \geqslant q\right) \\ &\qquad \leqslant \inf_{n+1\leqslant n''\leqslant n'} 1 - 1_{p^+_{n,q-1}<x_{n''}<p^-_{n,q}}\mathbb{P}\left(\tilde{T}_{n''} = T_{n'',p^-_{q,n}} | p^+_{n,q-1}, p^-_{n,q}, Q_n \geqslant q\right)\end{aligned}$$

But $\mathbb{P}\left(\tilde{T}_{n''} = T_{n'',p^-_{q,n}} | p^+_{n,q-1}, p^-_{n,q}\right)$ is close to 1 if $x_{n''}$ is close enough to $p^-_{n,q}$. There is always a subsequence of $(x_{n''})_{n''\geqslant n+1}$ made of points in $(p^+_{n,q-1}, p^-_{n,q})$ which converges to $p^-_{n,q}$. It follows that

$$\inf_{n''\geqslant n+1} 1 - 1_{p^+_{n,q-1}<x_{n''}<p^-_{n,q}}\mathbb{P}\left(\tilde{T}_{n''} = T_{n'',p^-_{q,n}} | p^+_{n,q-1}, p^-_{n,q}, Q_n \geqslant q\right) = 0$$

which concludes the proof. □

From proposition 3.1.6 follows that $\mathcal{Z}_\infty$ is closed. Moreover it does not contain any of the points of the sequence $(x_n)_{n\geqslant 1}$. Since the sequence $(x_n)_{n\geqslant 1}$ is everywhere dense, the connected components of $\mathcal{Z}_\infty$ are single points. One can see that

- If $y < \tilde{y}$ are two consecutive points in $\mathcal{Y}_\infty$ then $\sharp(\mathcal{Z}_\infty \cap (y,\tilde{y})) = 1$.
- If $\mathcal{Y}_\infty$ is bounded from below and $y = \min\mathcal{Y}_\infty$ then $\mathcal{Z}_\infty \cap (-\infty, y] = \emptyset$.
- If $\mathcal{Y}_\infty$ is bounded from above and $y = \max\mathcal{Y}_\infty$ then $\mathcal{Z}_\infty \cap [y, +\infty) = \emptyset$.

See figure 3.1.b. The set $\mathcal{Z}_\infty$ may be empty, which for instance happens almost surely if κ is a Dirac measure. For $n \geqslant 1$ we define

$$\mathcal{Z}_n := \left\{ \frac{p_{n,q-1}^- + p_{n,q}^+}{2} \middle| 2 \leqslant q \leqslant Q_n \right\}$$

We will write $\mathcal{Z}_n(x_1, \ldots, x_n)$ and $\mathcal{Z}_\infty((x_n)_{n\geqslant 1})$ whenever we need to emphasize the dependence on the starting points.

PROPOSITION 3.1.7. *The law of* $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ *does not depend on the starting points* $(x_n)_{n\geqslant 1}$.

PROOF. Let $(\tilde{x}_n)_{n\geqslant 1}$ be another sequence of pairwise disjoint points in $\mathbb{R}$. We will show that the sequence $(\mathcal{Y}_{2n}(x_1, \ldots, x_n, \tilde{x}_1, \ldots, \tilde{x}_n), \mathcal{Z}_{2n}(x_1, \ldots, x_n, \tilde{x}_1, \ldots, \tilde{x}_n))$ converges in law to $(\mathcal{Y}_\infty((x_n)_{n\geqslant 1}), \mathcal{Z}_\infty((x_n)_{n\geqslant 1}))$ and that $(\mathcal{Y}_{2n}(\tilde{x}_1, \ldots, \tilde{x}_n, x_1, \ldots, x_n)$, $\mathcal{Z}_{2n}(\tilde{x}_1, \ldots, \tilde{x}_n, x_1, \ldots, x_n))$ converges to $(\mathcal{Y}_\infty((\tilde{x}_n)_{n\geqslant 1}), \mathcal{Z}_\infty((\tilde{x}_n)_{n\geqslant 1}))$. Since the two couples of point processes $(\mathcal{Y}_{2n}(x_1, \ldots, x_n, \tilde{x}_1, \ldots, \tilde{x}_n), \mathcal{Z}_{2n}(x_1, \ldots, x_n, \tilde{x}_1, \ldots, \tilde{x}_n))$ and $(\mathcal{Y}_{2n}(\tilde{x}_1, \ldots, \tilde{x}_n, x_1, \ldots, x_n)$, $\mathcal{Z}_{2n}(\tilde{x}_1, \ldots, \tilde{x}_n, x_1, \ldots, x_n))$ have the same law, this will finish the proof.

For the convergence in law we will use the topology of uniform convergence on compact sets of collections of points in $\mathbb{R}$. It can be defined using the following metric: Let d_H be the Hausdorff metric on compact subsets of $\mathbb{R}$. One may use the metric d_{PP} on point processes:

$$d_{PP}(\mathcal{X}, \widetilde{\mathcal{X}}) := d_H(\tan^{-1}(\mathcal{X}) \cup \{-1, 1\}, \tan^{-1}(\widetilde{\mathcal{X}}) \cup \{-1, 1\})$$

In order to simplify the notations we will write:

$$(\mathcal{Y}_n, \mathcal{Z}_n) := (\mathcal{Y}_n(x_1, \ldots, x_n), \mathcal{Z}_n(x_1, \ldots, x_n))$$

$$(\mathcal{Y}_\infty, \mathcal{Z}_\infty) := (\mathcal{Y}_\infty((x_n)_{n\geqslant 1}), \mathcal{Z}_\infty((x_n)_{n\geqslant 1}))$$

$$(\widetilde{\mathcal{Y}}_{2n}, \widetilde{\mathcal{Z}}_{2n}) := (\mathcal{Y}_{2n}(x_1, \ldots, x_n, \tilde{x}_1, \ldots, \tilde{x}_n), \mathcal{Z}_{2n}(x_1, \ldots, x_n, \tilde{x}_1, \ldots, \tilde{x}_n))$$

We can construct $((\mathcal{Y}_n, \mathcal{Z}_n))_{n\geqslant 1}$, $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ and $((\widetilde{\mathcal{Y}}_{2n}, \widetilde{\mathcal{Z}}_{2n}))_{n\geqslant 1}$ on the same probability space using independent Brownian motions starting from the points in $(x_n)_{n\geqslant 1}$ and $(\tilde{x}_n)_{n\geqslant 1}$ and killed by the measure κ. We construct the sequence $((\mathcal{Y}_n, \mathcal{Z}_n))_{n\geqslant 1}$ using the Wilson's algorithm described in introduction. This way $\mathcal{Y}_n \subseteq \mathcal{Y}_{n+1}$ and $\mathcal{Y}_\infty = \bigcup_{n\geqslant 1} \mathcal{Y}_n$. In order to construct $\widetilde{\mathcal{Y}}_{2n}$, we first construct $\mathcal{Y}_n$ and then continue the Wilson's algorithm using the Brownian motions starting from $\tilde{x}_1, \ldots, \tilde{x}_n$. This way $\mathcal{Y}_n \subseteq \widetilde{\mathcal{Y}}_{2n}$ but not necessarily $\widetilde{\mathcal{Y}}_{2n} \subseteq \widetilde{\mathcal{Y}}_{2(n+1)}$.

Let $C > 0$ and $\varepsilon \in (0, \frac{C}{2})$. Let $\delta \in (0, 1)$, δ small. There is $N \in \mathbb{N}^*$ such that

$$\mathbb{P}\left(\mathcal{Y}_N \cap [-C, C] = \mathcal{Y}_\infty \cap [-C, C]\right) \geqslant 1 - \delta$$

There is $\varepsilon' \in (0, \varepsilon)$ such that for all $a < b \in [-C, C]$ satisfying $b - a \leqslant \varepsilon'$ the following holds:

$$1 - \frac{2(b-a)}{u_\downarrow(a)u_\uparrow(b) - u_\uparrow(a)u_\downarrow(b)} \leqslant \frac{\delta}{N}$$

There is $N' \geqslant N$ such that with probability $1 - 2\delta$ the following two conditions hold:

$$\mathcal{Y}_N \cap [-C, C] = \mathcal{Y}_\infty \cap [-C, C] \tag{3.1.10}$$

$$Leb([-C, C] \backslash \bigcup_{J \in \mathcal{J}_{N'}} J) \leqslant \varepsilon' \tag{3.1.11}$$

We define the following two random variables:

$$K^- := \min_{J \in \mathcal{J}_{N'}, J \subseteq [-C, C]} (\min J) \qquad K^+ := \max_{J \in \mathcal{J}_{N'}, J \subseteq [-C, C]} (\max J)$$

If (3.1.11) holds, then $[-\frac{C}{2}, \frac{C}{2}] \subseteq [K^-, K^+]$. If (3.1.10) and (3.1.11) hold than for $n \geqslant N'$, $[K^-, K^+] \backslash \bigcup_{J \in \mathcal{J}_n} J$ is made of at most N intervals, each of length at most ε'. Consider the following condition on $\widetilde{\mathcal{Y}}_{2n}$:

$$\widetilde{\mathcal{Y}}_{2n} \cap [K^-, K^+] = \mathcal{Y}_n \cap [K^-, K^+] \tag{3.1.12}$$

Applying lemma 3.1.4 we get that for all $n \geqslant N'$

$$\mathbb{P}\left(\widetilde{\mathcal{Y}}_{2n} \text{ satisfies (3.1.12) } | \text{ (3.1.10) and (3.1.11) hold}\right) \geqslant 1 - \delta$$

This implies that for all $n \geqslant N'$

$$\mathbb{P}\left(\widetilde{\mathcal{Y}}_{2n} \text{ satisfies (3.1.12), and (3.1.10) and (3.1.11) hold.}\right) \geqslant 1 - 3\delta$$

Let $n \geqslant N'$. On the event when (3.1.10) and (3.1.11) hold and $\widetilde{\mathcal{Y}}_{2n}$ satisfies (3.1.12), which happens with probability at least $1 - 3\delta$, the following is true:

- $\widetilde{\mathcal{Y}}_{2n} \cap [K^-, K^+] = \mathcal{Y}_\infty \cap [K^-, K^+]$
- $d_H(\widetilde{\mathcal{Z}}_{2n} \cap [K^-, K^+], \mathcal{Z}_\infty \cap [K^-, K^+]) \leqslant \varepsilon$

In particular with probability at least $1 - 3\delta$

- $d_{PP}(\widetilde{\mathcal{Y}}_{2n}, \mathcal{Y}_\infty) \leqslant 1 - \tan^{-1}(\frac{C}{2})$
- $d_H(\widetilde{\mathcal{Z}}_{2n}, \mathcal{Z}_\infty) \leqslant \varepsilon + (1 - \tan^{-1}(\frac{C}{2}))$

Since C is arbitrary large and ε and δ are arbitrary small, this implies that $(\widetilde{\mathcal{Y}}_{2n}, \widetilde{\mathcal{Z}}_{2n})$ converges in law as $n \to +\infty$ to $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$. □

Next we identify the law of $\mathcal{Y}_\infty$ as a determinantal fermionic point process. For generalities on this processes see [**HKPV09**], chapter 4, and [**Sos00**].

PROPOSITION 3.1.8. *Let $n \geqslant 1$ and $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n \in \mathbb{R}$. Then*

$$\mathbb{E}\left[\prod_{r=1}^{n} \sharp(\mathcal{Y}_\infty \cap [a_r, b_r))\right] = \int_{[a_1,b_1)} \cdots \int_{[a_n,b_n)} \det\left(G(y_i, y_j)\right)_{1 \leqslant i,j \leqslant n} \prod_{r=1}^{n} \kappa(dy_r) \tag{3.1.13}$$

In other words $\mathcal{Y}_\infty$ is a determinantal point process on $\mathbb{R}$ with reference measure κ and determinantal kernel G.

PROOF. Consider points $\tilde{a}_r < \tilde{b}_r \in [a_r, b_r]$ for $r \in \{1, \ldots, n\}$. We will show that

$$\begin{aligned} \mathbb{P}\left(\forall r \in \{1, \ldots, n\}, \mathcal{Y}_\infty \cap [\tilde{a}_r, \tilde{b}_r) \neq \varnothing\right) = \\ \int_{[\tilde{a}_1,\tilde{b}_1)} \cdots \int_{[\tilde{a}_n,\tilde{b}_n)} \det\left(G(y_i, y_j)\right)_{1 \leqslant i,j \leqslant n} \prod_{r=1}^{n} \kappa(dy_r) \\ + \left(\sum_{r=1}^{n} O(\tilde{b}_r - \tilde{a}_r)\right) \times \prod_{r=1}^{n} \kappa([\tilde{a}_r, \tilde{b}_r)) + \sum_{\substack{E \subseteq \{1,\ldots,n\} \\ E \neq \varnothing}} \prod_{r \in E} o(\tilde{b}_r - \tilde{a}_r) \prod_{r \notin E} \kappa([\tilde{a}_r, \tilde{b}_r)) \end{aligned} \tag{3.1.14}$$

where the quantities $O(\tilde{b}_r - \tilde{a}_r)$ and $o(\tilde{b}_r - \tilde{a}_r)$ are uniform over $\tilde{a}_r < \tilde{b}_r \in [a_r, b_r]$, $\tilde{a}_r$ close to $\tilde{b}_r$. From (3.1.14) one deduces (3.1.13) by splitting the intervals $[a_r, b_r]$ in small subintervals and approximating the number of points in $\mathcal{Y}_\infty \cap [a_r, b_r)$ by the number of subintervals of $[a_r, b_r)$ that contain a point in $\mathcal{Y}_\infty$.

As the law of $\mathcal{Y}_\infty$ does not depend on the choice of everywhere dense sequence of starting points, we will assume that the first $2n$ starting points in Wilson's algorithm are in order

$\tilde{a}_1, \tilde{b}_1, \dots, \tilde{a}_n, \tilde{b}_n$. We will show that for all non-empty subsets E of $\{1, \dots, n\}$

$$\mathbb{P}\left(\forall r \in E, \mathcal{Y}_{2n} \cap [\tilde{a}_r, \tilde{b}_r) = \varnothing, \mathcal{Y}_\infty \cap [\tilde{a}_r, \tilde{b}_r) \neq \varnothing, \forall r \notin E, \mathcal{Y}_{2n} \cap [\tilde{a}_r, \tilde{b}_r) \neq \varnothing\right) = \prod_{r \in E} o(\tilde{b}_r - \tilde{a}_r) \prod_{r \notin E} \kappa([\tilde{a}_r, \tilde{b}_r)) \tag{3.1.15}$$

Further we will show that for any $r_0 \in \{1, \dots, n\}$

$$\mathbb{P}\big(\forall r \in \{1, \dots, n\}, \mathcal{Y}_{2n} \cap [\tilde{a}_r, \tilde{b}_r) \neq \varnothing, [\tilde{a}_{r_0}, \tilde{b}_{r_0}] \nsubseteq \bigcup_{J \in \mathcal{J}_{2n}} J\big) = O(\tilde{b}_{r_0} - \tilde{a}_{r_0}) \prod_{r=1}^{n} \kappa([\tilde{a}_r, \tilde{b}_r)) \tag{3.1.16}$$

If for all $r \in \{1, \dots, n\}$, $\mathcal{Y}_{2n} \cap [\tilde{a}_r, \tilde{b}_r) \neq \varnothing$ and $[\tilde{a}_r, \tilde{b}_r] \subseteq \bigcup_{J \in \mathcal{J}_{2n}} J$ then necessarily $Q_{2n} = n$ and $\mathcal{J}_{2n} = \{[\tilde{a}_r, \tilde{b}_r] | 1 \leqslant r \leqslant n\}$. We will use the fact that according to (3.1.1)

$$\begin{aligned} \mathbb{P}\left(Q_{2n} = n, \mathcal{J}_{2n} = \{[\tilde{a}_r, \tilde{b}_r] | 1 \leqslant r \leqslant n\}\right) &= \int_{[\tilde{a}_1, \tilde{b}_1)} \dots \int_{[\tilde{a}_n, \tilde{b}_n)} \det\left(G^{\tilde{a}_i, \tilde{b}_j}\right)_{1 \leqslant i,j \leqslant n} \prod_{r=1}^{n} \kappa(dy_r) \\ = \int_{[\tilde{a}_1, \tilde{b}_1)} \dots \int_{[\tilde{a}_n, \tilde{b}_n)} \det\left(G(y_i, y_j)\right)_{1 \leqslant i,j \leqslant n} &\prod_{r=1}^{n} \kappa(dy_r) + \left(\sum_{r=1}^{n} O(\tilde{b}_r - \tilde{a}_r)\right) \times \prod_{r=1}^{n} \kappa([\tilde{a}_r, \tilde{b}_r)) \end{aligned} \tag{3.1.17}$$

Let's show (3.1.15). A closed expression of the probability in (3.1.15) can be computed using (3.1.1) and lemma 3.1.4. Since many different configurations (different values of Q_{2n} and configurations of J_{2n}) contribute to the probability in (3.1.15), we won't give the closed expression and only give the estimates. Let E be a non-empty subset of $\{1, \dots, n\}$. If $r \notin E$, then the condition $\mathcal{Y}_{2n} \cap [\tilde{a}_r, \tilde{b}r) \neq \varnothing$ contributes by a factor $O(\kappa([\tilde{a}_r, \tilde{b}_r)))$ to the probability in (3.1.15). If $r \in E$, then the two conditions $\mathcal{Y}_{2n} \cap [\tilde{a}_r, \tilde{b}r) = \varnothing$ and $\mathcal{Y}_\infty \cap [\tilde{a}_r, \tilde{b}r) \neq \varnothing$ imply that $(\tilde{a}_r, \tilde{b}r) \cap \bigcup_{J \in \mathcal{J}_{2n}} J = \varnothing$. According to the identity (3.1.3), the condition $(\tilde{a}_r, \tilde{b}r) \cap \bigcup_{J \in \mathcal{J}_{2n}} J = \varnothing$ contributes to the probability in (3.1.15) by a factor

$$O(u_\downarrow(\tilde{a}_r) u_\uparrow(\tilde{b}_r) - u_\uparrow(\tilde{a}_r) u_\downarrow(\tilde{b}_r)) = O(\tilde{b}_r - \tilde{a}_r)$$

According to the lemma 3.1.4, the additional condition $\mathcal{Y}_\infty \cap [\tilde{a}_r, \tilde{b}r) \neq \varnothing$ contributes to the probability in (3.1.15) by a factor

$$1 - \frac{2(\tilde{b}_r - \tilde{a}_r)}{u_\downarrow(\tilde{a}_r) u_\uparrow(\tilde{b}_r) - u_\uparrow(\tilde{a}_r) u_\downarrow(\tilde{b}_r)} = o(1)$$

(3.1.15) follows.

We deal now with (3.1.16). As in the previous case, the condition that for all $r \in \{1, \dots, n\}$, $\mathcal{Y}_{2n} \cap [\tilde{a}_r, \tilde{b}_r) \neq \varnothing$ contributes by a factor $O\left(\prod_{r=1}^{n} \kappa([\tilde{a}_r, \tilde{b}r))\right)$ to the probability in (3.1.16). The condition $[\tilde{a}_{r_0}, \tilde{b}_{r_0}] \nsubseteq \bigcup_{J \in \mathcal{J}_{2n}} J$ implies that there is $i \in \{2, \dots, Q_{2n}\}$ such that $\tilde{a}_{r_0} < p^{+}_{2n,i-1} < p^{-}_{2n,i} < \tilde{b}_{r_0}$. As previously, this contributes by a factor $O(\tilde{b}_{r_0} - \tilde{a}_{r_0})$ to the probability. Combining (3.1.15), (3.1.16) and (3.1.17) yields (3.1.14). ☐

Let $\mathfrak{G}_\kappa$ be the following operator defined for functions in $\mathbb{L}^2(dk)$ with compact support:

$$(\mathfrak{G}_k f)(x) := \int_{\mathbb{R}} G(x, y) f(y) \kappa(dy)$$

A standard condition for a determinantal point process with kernel G relative to the measure κ to be well defined is $\mathfrak{G}_\kappa$ to be positive semi-definite, contracting and locally trace class.

We explain why this is true. Let f be a compactly supported $\mathbb{L}^2(d\kappa)$ function. Then the weak second derivative of $\mathfrak{G}_\kappa f$ is

$$d\Big(\frac{d(\mathfrak{G}_\kappa f)}{dx}\Big) = 2(\mathfrak{G}_\kappa f - f)d\kappa$$

$\mathfrak{G}_\kappa f$ and $\dfrac{d(\mathfrak{G}_\kappa f)}{dx}$ are square-integrable and

$$\begin{aligned}\int_{\mathbb{R}}(\mathfrak{G}_\kappa f)f d\kappa &= \int_{\mathbb{R}}(\mathfrak{G}_\kappa f)^2 d\kappa + \frac{1}{2}\int_{\mathbb{R}}(\mathfrak{G}_\kappa f)d\Big(\frac{d(\mathfrak{G}_\kappa f)}{dx}\Big)\\ &= \int_{\mathbb{R}}(\mathfrak{G}_\kappa f)^2 d\kappa + \frac{1}{2}\int_{\mathbb{R}}\Big(\frac{d(\mathfrak{G}_\kappa f)}{dx}\Big)^2 dx\end{aligned} \tag{3.1.18}$$

Identity (3.1.18) shows that $\mathfrak{G}_\kappa$ is positive semi-definite. It also shows that $\int_{\mathbb{R}}(\mathfrak{G}_\kappa f)^2 d\kappa \leqslant \int_{\mathbb{R}}(\mathfrak{G}_\kappa f)f d\kappa$, which implies that $\mathfrak{G}_\kappa$ is contracting and hence can be continuously extended to a contraction of the whole space $\mathbb{L}^2(d\kappa)$. $\mathfrak{G}_\kappa$ is locally trace class because it is positive semi-definite and its functional kernel is continuous (see theorem 2.12 in [**Sim05**], chapter 2).

Next we give a criterion for $\mathcal{Y}_\infty$ to be finite or just to be finite in the neighbourhood of either $+\infty$ or $-\infty$.

Proposition 3.1.9. *If $\int_{(0,+\infty)} x\kappa(dx) < +\infty$ then almost surely $\sharp(\mathcal{Y}_\infty \cap (0,+\infty))$ is finite. Moreover*

$$\mathbb{E}\left[\sharp(\mathcal{Y}_\infty \cap (0,+\infty))\right] = \int_{(0,+\infty)} G(x,x)\kappa(dx) < +\infty \tag{3.1.19}$$

If $\int_{(0,+\infty)} x\kappa(dx) = +\infty$ then almost surely $\sharp(\mathcal{Y}_\infty \cap (0,+\infty)) = +\infty$. In general, for all $a \in \mathbb{R}$

$$\mathbb{P}(\mathcal{Y}_\infty \cap (a,+\infty) = \varnothing) = u_\downarrow(+\infty)\int_{(-\infty,a]} u_\uparrow(x)\kappa(dx) \tag{3.1.20}$$

Similarly, if $\int_{\mathbb{R}} |x|\kappa(dx) < +\infty$ then a.s. $\sharp\mathcal{Y}_\infty$ is finite and

$$\mathbb{E}\left[\sharp\mathcal{Y}_\infty\right] = \int_{\mathbb{R}} G(x,x)\kappa(dx) < +\infty$$

If $\int_{\mathbb{R}} |x|\kappa(dx) = +\infty$ then a.s. $\sharp\mathcal{Y}_\infty = +\infty$.

Proof. We need only to deal with the finiteness of $\sharp(\mathcal{Y}_\infty \cap (0,+\infty))$. If $\int_{(0,+\infty)} x\,k(dx) < +\infty$ then (3.1.19) holds according to 2.2.3 and hence $\sharp(\mathcal{Y}_\infty \cap (0,+\infty))$ is finite is finite a.s.

We will prove (3.1.20). If $\int_{(0,+\infty)} x\kappa(dx) = +\infty$ then according 2.2.3 $u_\downarrow(+\infty) > 0$ and thus $\sharp(\mathcal{Y}_\infty \cap (0,+\infty)) = +\infty$ a.s. Let $a < b \in \mathbb{R}$. We assume that the two first starting points in Wilson's algorithm are a and b. Then

$$\begin{aligned}\mathbb{P}(\mathcal{Y}_\infty \cap (a,b] = \varnothing) &= \mathbb{P}\big(B^{(a)}_{T_1^-} > b\big) + \mathbb{P}\big(B^{(a)}_{T_1^-} \leqslant a, B^{(b)}_{T_2^-} = a\big)\\ &= \mathbb{P}\big(B^{(a)}_{\zeta_1^-} > b\big) + \mathbb{P}\big(B^{(a)}_{\zeta_1^-} \leqslant a\big) \times \mathbb{P}\big(B^{(b)} \text{ hits } a \text{ before time } \zeta_2\big)\\ &= \int_{(b,+\infty)} G(a,x)\kappa(dx) + \Big(\int_{(-\infty,a]} G(a,x)\kappa(dx)\Big) \times \frac{u_\downarrow(b)}{u_\downarrow(a)}\\ &= \int_{(b,+\infty)} G(a,x)\kappa(dx) + u_\downarrow(b)\int_{(-\infty,a]} u_\uparrow(x)\kappa(dx)\end{aligned} \tag{3.1.21}$$

Letting b go to $+\infty$ in (3.1.21) gives (3.1.20). $\square$

Next we will show that $\mathcal{Z}_\infty$ is a determinantal point process with kernel $\mathcal{K}$ relative to the Lebesgue measure where

$$\begin{aligned}\mathcal{K}(y,z) :=& -\frac{1}{2}\frac{du_\uparrow}{dx}((y\wedge z)^+)\frac{u_\downarrow}{dz}((y\vee z)^-)\\ =&2\int_{(-\infty,y\wedge z]} u_\uparrow(x)\kappa(dx)\times\int_{[y\vee z,+\infty)} u_\downarrow(x)\kappa(dx)\end{aligned}$$

Proposition 3.1.10. *Let $n\geqslant 1$ and $a_1<b_1<a_2<b_2<\dots<a_n<b_n\in\mathbb{R}$. Then*

$$\mathbb{E}\left[\prod_{r=1}^n \sharp(\mathcal{Z}_\infty\cap(a_r,b_r))\right]=\int_{(a_1,b_1)}\dots\int_{(a_n,b_n)}\det(\mathcal{K}(z_i,z_j))_{1\leqslant i,j\leqslant n}\prod_{r=1}^n dz_r \tag{3.1.22}$$

If for $r\in\{1,2,\dots,n\}$, $\kappa(\{a_r\})=\kappa(\{b_r\})=0$ then

$$\mathbb{P}\left(\forall r\in\{1,2,\dots,n\},\sharp(\mathcal{Z}_\infty\cap(a_r,b_r))=1\right)=\det(\mathcal{K}(a_i,b_j))_{1\leqslant i,j\leqslant n}\times\prod_{r=1}^n(b_r-a_r) \tag{3.1.23}$$

Proof. We will only prove (3.1.23). (3.1.22) can be deduced from (3.1.23) by diving the intervals (a_r,b_r) in small subintervals and approximating the expected number of points in these subintervals by the probability to have one single point per subinterval. Observe that if the measure κ has atoms then $\mathcal{K}$ is not continuous. Yet $z\mapsto\frac{du_\uparrow}{dx}(z^+)$ is right-continuous and $z\mapsto\frac{du_\downarrow}{dx}(z^-)$ is left-continuous. So the approximation can still be done.

Consider the Wilson's algorithm where the $2n$ first starting points are in order $a_1,b_1,a_2,b_2,\dots,a_n,b_n$. Then

$$\begin{aligned}&\mathbb{P}\left(\forall r\in\{1,2,\dots,n\},\sharp(\mathcal{Z}_\infty\cap(a_r,b_r))=1\right)=\\ &\qquad\mathbb{P}\Bigg(\forall r\in\{1,2,\dots,n\},(a_r,b_r)\subseteq\mathbb{R}\backslash\bigcup_{J\in\mathcal{J}_{2n}}J,(a_r,b_r)\cap\mathcal{Y}_\infty=\varnothing\Bigg)\end{aligned} \tag{3.1.24}$$

Applying lemma 3.1.4 we get that (3.1.24) equals

$$\mathbb{P}\Bigg(\forall r\in\{1,2,\dots,n\},(a_r,b_r)\subseteq\mathbb{R}\backslash\bigcup_{J\in\mathcal{J}_{2n}}J\Bigg)\times\prod_{r=1}^n\frac{2(b_r-a_r)}{u_\downarrow(a_r)u_\uparrow(b_r)-u_\uparrow(a_r)u_\downarrow(b_r)} \tag{3.1.25}$$

Further

$$\begin{aligned}&\mathbb{P}\Bigg(\forall r\in\{1,2,\dots,n\},(a_r,b_r)\subseteq\mathbb{R}\backslash\bigcup_{J\in\mathcal{J}_{2n}}J\Bigg)=\\ &\qquad\mathbb{P}\left(B^{(a_1)}_{T_1^-}\leqslant a_1,B^{(b_n)}_{T_{2n}^-}\geqslant b_n,\forall r\in\{1,\dots,n-1\},b_r\leqslant B^{(b_r)}_{T_{2r}^-}\leqslant B^{(a_{r+1})}_{T_{2r+1}^-}\leqslant a_{r+1}\right)\end{aligned} \tag{3.1.26}$$

Applying (3.1.1) and (3.1.3) we get that (3.1.26) equals

(3.1.27)
$$\begin{aligned}&\prod_{r=1}^n(u_\downarrow(a_r)u_\uparrow(b_r)-u_\uparrow(a_r)u_\downarrow(b_r))\times\int_{(-\infty,a_1]}u_\uparrow(y_1)\kappa(dy_1)\times\int_{[b_n,+\infty)}u_\downarrow(z_n)\kappa(dy_n)\\ &\quad\times\prod_{r=1}^{n-1}\left(\kappa([b_r,a_{r+1}])+\int_{b_r\leqslant y_r<\tilde{y}_r\leqslant a_{r+1}}(u_\downarrow(y_r)u_\uparrow(\tilde{y}_r)-u_\uparrow(y_r)u_\downarrow(\tilde{y}_r))\kappa(dy_r)\kappa(d\tilde{y}_r)\right)\end{aligned}$$

But

$$(3.1.28)\quad \int_{b_r\leqslant y_r<\tilde{y}_r\leqslant a_{r+1}} u_\downarrow(y_r)u_\uparrow(\tilde{y}_r)\kappa(dy_r)\kappa(d\tilde{y}_r)$$
$$= \frac{1}{2}\int_{b_r\leqslant y_r\leqslant a_{r+1}} u_\downarrow(y_r)\Big(\frac{du_\uparrow}{dx}(a_{r+1}) - \frac{du_\uparrow}{dx}(y_r^+)\Big)\kappa(dy_r)$$
$$= \frac{1}{4}\Big(\frac{du_\downarrow}{dx}(a_{r+1}) - \frac{du_\downarrow}{dx}(b_r)\Big)\frac{du_\uparrow}{dx}(a_{r+1}) - \frac{1}{2}\int_{b_r\leqslant y_r\leqslant a_{r+1}} u_\downarrow(y_r)\frac{du_\uparrow}{dx}(y_r^+)\kappa(dy_r)$$

and

$$(3.1.29)\quad -\int_{b_r\leqslant y_r<\tilde{y}_r\leqslant a_{r+1}} u_\uparrow(y_r)u_\downarrow(\tilde{y}_r)\kappa(dy_r)\kappa(d\tilde{y}_r)$$
$$= -\frac{1}{2}\int_{b_r\leqslant y_r\leqslant a_{r+1}} u_\uparrow(y_r)\Big(\frac{du_\downarrow}{dx}(a_{r+1}) - \frac{du_\downarrow}{dx}(y_r^+)\Big)\kappa(dy_r)$$
$$= -\frac{1}{4}\Big(\frac{du_\uparrow}{dx}(a_{r+1}) - \frac{du_\uparrow}{dx}(b_r)\Big)\frac{du_\downarrow}{dx}(a_{r+1}) + \frac{1}{2}\int_{b_r\leqslant y_r\leqslant a_{r+1}} u_\uparrow(y_r)\frac{du_\downarrow}{dx}(y_r)\kappa(dy_r)$$

Combining (3.1.28) and (3.1.29) we get that

$$\int_{b_r\leqslant y_r<\tilde{y}_r\leqslant a_{r+1}} (u_\downarrow(y_r)u_\uparrow(\tilde{y}_r) - u_\uparrow(y_r)u_\downarrow(\tilde{y}_r))\kappa(dy_r)\kappa(d\tilde{y}_r)$$
$$= \frac{1}{4}\Big(\frac{du_\uparrow}{dx}(b_r)\frac{du_\downarrow}{dx}(a_{r+1}) - \frac{du_\downarrow}{dx}(b_r)\frac{du_\uparrow}{dx}(a_{r+1})\Big)$$
$$-\frac{1}{2}\int_{b_r\leqslant y_r\leqslant a_{r+1}}\Big(u_\downarrow(y_r)\frac{du_\uparrow}{dx}(y_r^+) - u_\uparrow(y_r)\frac{du_\downarrow}{dx}(y_r^+)\Big)\kappa(dy_r)$$
$$= \frac{1}{4}\Big(\frac{du_\uparrow}{dx}(b_r)\frac{du_\downarrow}{dx}(a_{r+1}) - \frac{du_\downarrow}{dx}(b_r)\frac{du_\uparrow}{dx}(a_{r+1})\Big) - \kappa([b_r, a_{r+1}])$$

It follows that (3.1.27) equals

$$(3.1.30)\quad \prod_{r=1}^{n}(u_\downarrow(a_r)u_\uparrow(b_r) - u_\uparrow(a_r)u_\downarrow(b_r)) \times \Big(-\frac{1}{4}\frac{du_\uparrow}{dx}(a_1)\frac{du_\downarrow}{dx}(b_n)\Big)$$
$$\times\prod_{r=1}^{n-1}\Big(\frac{1}{4}\Big(\frac{du_\uparrow}{dx}(b_r)\frac{du_\downarrow}{dx}(a_{r+1}) - \frac{du_\downarrow}{dx}(b_r)\frac{du_\uparrow}{dx}(a_{r+1})\Big)\Big)$$
$$= \frac{1}{2^n}\prod_{r=1}^{n}(u_\downarrow(a_r)u_\uparrow(b_r) - u_\uparrow(a_r)u_\downarrow(b_r)) \times \det(\mathcal{K}(a_i, b_j))_{1\leqslant i,j\leqslant n}$$

(3.1.25) together with (3.1.30) gives (3.1.23). $\square$

To see that the operator induced by the kernel $\mathcal{K}$ on $\mathbb{L}^2(Leb)$ is positive semi-definite, one can check that for any $\mathbb{L}^2$ function f with compact support

$$\int_{\mathbb{R}^2} f(y)\mathcal{K}(y,z)f(z)dydz = \int_{\mathbb{R}^2} G(\tilde{y},\tilde{z})\left(\int_{\tilde{y}}^{\tilde{z}} f(x)dx\right)^2\kappa(d\tilde{y})\kappa(d\tilde{z})$$

Too see that $\mathcal{K}$ induces a contraction one can check that for any $\mathcal{C}^1$ function f with compact support

$$\int_{\mathbb{R}^2} f(y)\mathcal{K}(y,z)f(z)dydz = \int_{\mathbb{R}} f(x)^2dx - \frac{1}{2}\int_{\mathbb{R}^2}\frac{df}{dx}(\tilde{y})G(\tilde{y},\tilde{z})\frac{df}{dx}(\tilde{z})d\tilde{y}d\tilde{z}$$

and that $\int_{\mathbb{R}^2}\frac{df}{dx}(\tilde{y})G(\tilde{y},\tilde{z})\frac{df}{dx}(\tilde{z})d\tilde{y}d\tilde{z}\geqslant 0$.

The determinantal kernels G and $\mathcal{K}$ both satisfy the following relation: for any $x \leqslant y \leqslant z \in \mathbb{R}$

$$G(x,y)G(y,z) = G(x,z)G(y,y) \qquad \mathcal{K}(x,y)\mathcal{K}(y,z) = \mathcal{K}(x,z)\mathcal{K}(y,y) \tag{3.1.31}$$

For $x \in \mathbb{R}$ and $y, z > x$, we define

$$G^{(x\times)}(y,z) := G(y,z) - \frac{G(x,y)G(x,z)}{G(x,x)} \qquad \mathcal{K}^{(x\rhd)}(y,z) := \mathcal{K}(y,z) - \frac{\mathcal{K}(x,y)\mathcal{K}(x,z)}{\mathcal{K}(x,x)} \tag{3.1.32}$$

Relation (3.1.31) ensures that $\det(G(y_i,y_j))_{1\leqslant i,j\leqslant n}$ and $\det(\mathcal{K}(z_i,z_j))_{1\leqslant i,j\leqslant n}$ can be factorised as follows: If $y_1 < y_2 < \cdots < y_n$ then

$$\det(G(y_i,y_j))_{1\leqslant i,j\leqslant n} = G(y_1,y_1)\prod_{r=2}^{n} G^{(y_{r-1}\times)}(y_r,y_r) \tag{3.1.33}$$

If $z_1 < z_2 < \cdots < z_n$ then

$$\det(\mathcal{K}(z_i,z_j))_{1\leqslant i,j\leqslant n} = \mathcal{K}(z_1,z_1)\prod_{r=2}^{n} \mathcal{K}^{(z_{r-1}\rhd)}(z_r,z_r) \tag{3.1.34}$$

The relations (3.1.31) or equivalently the factorisations (3.1.33) and (3.1.34) imply that the spacings between consecutive points of $\mathcal{Y}_\infty$ respectively $\mathcal{Z}_\infty$ are independent, that is to say conditional on $\mathcal{Y}_\infty$ having a point at y_0, the position of the next higher point y is independent on $\mathcal{Y}_\infty \cap (-\infty, y_0)$, and similarly for $\mathcal{Z}_\infty$ ([**Sos00**], section 2.4). Conditional on $y_0 \in \mathcal{Y}_\infty$ the distribution of its higher neighbour in $\mathcal{Y}_\infty$ is of the form $f_G(y_0,y)\kappa(dy)$. Similarly denote $f_{\mathcal{K}}(z_0,z)dz$ the distribution between two consecutive points in $\mathcal{Z}_\infty$ conditional on z_0 be the lowest one. Following relations relate $G^{(y_0\times)}(y,y)$ respectively $\mathcal{K}^{(z_0\rhd)}(z,z)$ to f_G respectively $f_{\mathcal{K}}$:

$$\begin{aligned} G^{(y_0\times)}(y,y) &= f_G(y_0,y) \\ &\quad + \sum_{j\geqslant 2}\int_{y_0<y_1<\cdots<y_{j-1}<y} f_G(y_0,y_1)f_G(y_1,y_2)\dots f_G(y_{j-1},y)\kappa(dy_1)\dots\kappa(dy_{j-1}) \end{aligned}$$

$$\begin{aligned} \mathcal{K}^{(z_0\rhd)}(z,z) &= f_{\mathcal{K}}(z_0,z) \\ &\quad + \sum_{j\geqslant 2}\int_{z_0<z_1<\cdots<z_{j-1}<z} f_{\mathcal{K}}(z_0,z_1)f_{\mathcal{K}}(z_1,z_2)\dots f_{\mathcal{K}}(z_{j-1},z)dz_1\dots dz_{j-1} \end{aligned}$$

If $\int_{(0,+\infty)} x\, k(dx) < +\infty$, i.e. $\mathcal{Y}_\infty \cap (0,+\infty)$ a.s. finite, then $\int_{(y_0,+\infty)} f_G(y_0,y)\kappa(dy) < 1$ and $\int_{z_0}^{+\infty} f_{\mathcal{K}}(z_0,z)dz < 1$.

Given a couple of interwoven point processes $(\mathcal{Y},\mathcal{Z})$ on $\mathbb{R}$ such that between any two consecutive point in $\mathcal{Y}$ lies one single point of $\mathcal{Z}$ and such that for any J bounded subinterval of $\mathbb{R}$ $\mathcal{Y}$ satisfies the constraint

$$\mathbb{E}\big[\sharp(\mathcal{Y}\cap J)\big] < +\infty$$

the joint distribution of $(\mathcal{Y},\mathcal{Z})$ can be fully described by a family of measures $(M_n(\mathcal{Y},\mathcal{Z}))_{n\geqslant 0}$ defined by

$$\int_{\mathbb{R}} f(y_0)M_0(\mathcal{Y},\mathcal{Z})(dy_0) = \mathbb{E}\Big[\sum_{y_0\in\mathcal{Y}} f(y_0)\Big]$$

$$\int_{y_0<z_1<y_1<\dots z_n<y_n} f(y_0,z_1,y_1,\dots z_n,y_n) M_n(\mathcal{Y},\mathcal{Z})(dy_0,dz_1,dy_1,\dots dz_n,dy_n)$$
$$= \mathbb{E}\Bigg[\sum_{\substack{y_0,\dots,y_n \\ n+1 \text{ consecutive points in } \mathcal{Y} \\ z_1,\dots,z_n\in\mathcal{Z} \\ y_0<z_1<y_1<\dots z_n<y_n}} f(y_0,z_1,y_1,\dots z_n,y_n)\Bigg]$$

$M_n(\mathcal{Y},\mathcal{Z})(dy_0,dz_1,dy_1,\dots dz_n,dy_n)$ is the infinitesimal probability for $y_0,y_1,\dots y_n$ being $n+1$ consecutive points in $\mathcal{Y}$ and $z_1,\dots z_n$ being the n points in $\mathcal{Z}$ separating them. In case of $(\mathcal{Y}_\infty,\mathcal{Z}_\infty)$, $M_0(\mathcal{Y}_\infty,\mathcal{Z}_\infty)(dy_0) = G(y_0,y_0)\kappa(dy_0)$.

PROPOSITION 3.1.11. *For* $n \geqslant 1$

$$\begin{aligned} M_n(\mathcal{Y}_\infty,\mathcal{Z}_\infty)(dy_0,dz_1,\dots dz_n,dy_n) &= 2^n u_\uparrow(y_0)u_\downarrow(y_n)\kappa(dy_0)dz_1\dots dz_n\kappa(dy_n) \\ &= 2^n G(y_0,y_n)\kappa(dy_0)dz_1\dots dz_n\kappa(dy_n) \end{aligned} \tag{3.1.35}$$

Moreover

$$f_G(y_0,y) = 2(y-y_0)\frac{u_\downarrow(y)}{u_\downarrow(y_0)} \qquad \kappa(dy) - \textit{almost everywhere}$$
$$f_{\mathcal{K}}(z_0,z) = 2\kappa((z_0,z))\Big(\frac{du_\downarrow}{dx}(z_0)\Big)^{-1}\frac{du_\downarrow}{dx}(z) \qquad dz - \textit{almost everywhere}$$

The distribution on $\mathcal{Z}_\infty$ *conditional on* $\mathcal{Y}_\infty$ *is the following: given two consecutive points* $y_1 < y_2$ *in* $\mathcal{Y}_\infty$*, then the point of* $\mathcal{Z}_\infty$ *lying between them is distributed uniformly on* (y_1,y_2) *and independently on the behaviour of* $\mathcal{Z}_\infty$ *on* $(-\infty,y_1)\cup(y_2,+\infty)$*. The distribution on* $\mathcal{Y}_\infty$ *conditional on* $\mathcal{Z}_\infty$ *is the following: given two consecutive points* $z_1 < z_2$ *in* $\mathcal{Z}_\infty$*, then the point of* $\mathcal{Y}_\infty$ *lying between them is distributed on* (z_1,z_2) *according the measure* $1_{z_1<y<z_2}\frac{\kappa(dy)}{\kappa((z_1,z_2))}$ *and independently on the behaviour of* $\mathcal{Y}_\infty$ *on* $(-\infty,z_1)\cup(z_2,+\infty)$*. If* $\int_{(-\infty,0)}|x|\kappa(dx) < +\infty$*, then* $\min\mathcal{Y}_\infty$ *is distributed conditional on* $\mathcal{Z}_\infty$ *according to the measure* $1_{y<\min\mathcal{Z}_\infty}\frac{\kappa(dy)}{k((-\infty,\min\mathcal{Z}_\infty))}$ *and it is independent on the behaviour of* $\mathcal{Y}_\infty$ *on* $(-\infty,\min\mathcal{Z}_\infty)$*. Similarly for the distribution of* $\max\mathcal{Y}_\infty$ *conditional on* $\max\mathcal{Z}_\infty$ *if* $\int_{(0,+\infty)}x\kappa(dx) < +\infty$*.*

PROOF. Let $a_0 < b_0 < \tilde{a}_1 < \tilde{b}_1 < a_1 < b_1 < \dots < \tilde{a}_n < \tilde{b}_n < a_n < b_n \in \mathbb{R}$. Let $\mathscr{C}_n(a_0,b_0,\tilde{a}_1,\tilde{b}_1,a_1,b_1,\dots,\tilde{a}_n,\tilde{b}_n,a_n,b_n)$ corresponding to the following conditions:

- $\mathcal{Y}_\infty\cap[a_0,b_0]\neq\emptyset$, $\mathcal{Y}_\infty\cap[a_n,b_n]\neq\emptyset$
- $\forall r\in\{1,\dots,n\}, \sharp(\mathcal{Y}_\infty\cap[a_r,b_r]) = 1$
- $\forall r\in\{1,\dots,n\}, \sharp(\mathcal{Z}_\infty\cap(\tilde{a}_r,\tilde{b}_r)) = 1$
- $\forall r\in\{0,\dots,n-1\}, (\mathcal{Y}_\infty\cup\mathcal{Z}_\infty)\cap(b_r,\tilde{a}_r] = \emptyset, (\mathcal{Y}_\infty\cup\mathcal{Z}_\infty)\cap[\tilde{b}_r,a_{r+1}) = \emptyset$

We will compute the probability of $\mathscr{C}_n(a_0,b_0,\tilde{a}_1,\tilde{b}_1,a_1,b_1,\dots,\tilde{a}_n,\tilde{b}_n,a_n,b_n)$. Consider that we execute the Wilson's algorithm where the $2n$ first starting points are $\tilde{a}_1,\tilde{b}_1,\dots,\tilde{a}_n,\tilde{b}_n$. The only configurations that contribute to the studied event are those where $B^{(\tilde{a}_1)}_{T_1^-}\in[a_0,b_0]$, $B^{(\tilde{b}_n)}_{T_{2n}^-}\in[a_n,b_n]$ and for $r\in\{1,\dots,n-1\}$, $B^{(\tilde{b}_r)}_{T_{2r}^-} = B^{(\tilde{a}_{r+1})}_{T_{2r+1}^-}\in[a_{r+1},b_{r+1}]$. We further need that for $r\in\{1,\dots,n\}$, $\mathcal{Y}_\infty\cap(\tilde{a}_r,\tilde{b}_r)=\emptyset$. Thus applying (3.1.1), (3.1.3) and lemma 3.1.4 we get the probability of the event $\mathscr{C}_n(a_0,b_0,\tilde{a}_1,\tilde{b}_1,a_1,b_1,\dots,\tilde{a}_n,\tilde{b}_n,a_n,b_n)$ equals

$$\int_{[a_0,b_0]} u_\uparrow(y_0)\kappa(dy_0)\times\int_{[a_n,b_n]} u_\downarrow(y_n)\kappa(dy_n)\times\prod_{r=1}^{n-1}\kappa([a_r,b_r])\times\prod_{r=1}^{n}2(\tilde{b}_r-\tilde{a}_r)$$

The above probability also equals $M_n(\mathcal{Y}_\infty,\mathcal{Z}_\infty)([a_0,b_0]\times[\tilde{a}_1,\tilde{b}_1]\times[a_1,b_1]\times\dots\times[\tilde{a}_n,\tilde{b}_n]\times[a_n,b_n])$ and gives the expression of (3.1.35). To get the expressions of f_G and $f_{\mathcal{K}}$ just observe that

$$G(y_0,y_0)f_G(y_0,y)\kappa(dy_0)\kappa(dy) = M_1([y_0,y_0+dy_0]\times(y_0,y)\times[y,y+dy])$$

$$\mathcal{K}(z_0,z_0)f_{\mathcal{K}}(z_0,z)dz_0dz = M_3((-\infty,z_0)\times[z_0,z_0+dz_0]\times(z_0,z)\times[z,z+dz]\times(z,+\infty))$$

Expression (3.1.35) gives also the law of $\mathcal{Z}_\infty$ conditional on $\mathcal{Y}_\infty$ and the law of $\mathcal{Y}_\infty$ conditional on $\mathcal{Z}_\infty$, except for the possible extremal points of $\mathcal{Y}_\infty$. Let's deal with the distribution of $\max\mathcal{Y}_\infty$ conditional on $\max\mathcal{Z}_\infty$ in case $\int_{(0,+\infty)} x\kappa(dx) < +\infty$. Again according to (3.1.35), conditional on $z_0\in\mathcal{Z}_\infty$, the distribution of $\min\mathcal{Y}_\infty\cap(z_0,+\infty)$ is proportional to $1_{y>z_0}u_\downarrow(y)\kappa(dy)$. To obtain the distribution $\max\mathcal{Y}_\infty$ conditional on $\max\mathcal{Z}_\infty$, one must weight $u_\downarrow(y)$ by $1-\int_{\tilde{y}>y} f_G(y,\tilde{y})\kappa(d\tilde{y})$, i.e. the probability of not having any point in $\mathcal{Y}_\infty$ consecutive to y. But

$$\begin{aligned}\int_{\tilde{y}>y} f_G(y,\tilde{y})\kappa(d\tilde{y}) =& 2\int_{\tilde{y}>y}(\tilde{y}-y)\frac{u_\downarrow(\tilde{y})}{u_\downarrow(y)}\kappa(d\tilde{y})\\ =& \lim_{\tilde{y}\to+\infty}\frac{\tilde{y}-y}{u_\downarrow(y)}\frac{du_\downarrow}{dx}(\tilde{y}^+) - \frac{1}{u_\downarrow(y)}\int_{\tilde{y}>y}\frac{du_\downarrow}{dx}(\tilde{y}^+)d\tilde{y}\end{aligned}$$

But

$$(\tilde{y}-y)\frac{du_\downarrow}{dx}(\tilde{y}^+) = (\tilde{y}-y)\int_{(\tilde{y},+\infty)} 2u_\downarrow(x)\kappa(dx) \leqslant 2\int_{(\tilde{y},+\infty)}(x-y)u_\downarrow(x)\kappa(dx)\to 0$$

It follows that:

$$\int_{\tilde{y}>y} f_G(y,\tilde{y})\kappa(d\tilde{y}) = -\frac{1}{u_\downarrow(y)}\int_{\tilde{y}>y}\frac{du_\downarrow}{dx}(\tilde{y}^+)d\tilde{y} = 1-\frac{u_\downarrow(+\infty)}{u_\downarrow(y)}$$

Thus $1_{y>z_0}u_\downarrow(y)\big(1-\int_{\tilde{y}>y} f_G(y,\tilde{y})\kappa(d\tilde{y})\big)\kappa(dy)$ is simply proportional to $1_{y>z_0}\kappa(dy)$. □

PROPOSITION 3.1.12. *In case* $\int_{\mathbb{R}}|x|\kappa(dx) < +\infty$

$$\mathbb{P}(\sharp\mathcal{Y}_\infty = 1) = u_\uparrow(-\infty)u_\downarrow(+\infty)\kappa(\mathbb{R})$$

Conditional on $\sharp\mathcal{Y}_\infty = 1$ *the unique point in* $\mathcal{Y}_\infty$ *is distributed according* $\frac{\kappa(dy)}{\kappa(\mathbb{R})}$.

PROOF. The distribution of the unique point y_0 of $\mathcal{Y}_\infty$ on the event $\sharp\mathcal{Y}_\infty = 1$ is given by the following sieve identity:

$$\begin{aligned}\Big(G(y_0,y_0) &- \int_{y_{-1}<y_0} G(y_{-1},y_{-1})f_G(y_{-1},y_0)\kappa(dy_{-1})\\ &-\int_{y_1>y_0} G(y_0,y_0)f_G(y_0,y_1)\kappa(dy_1)\\ &+\int_{y_{-1}<y_0}\int_{y_1>y_0} G(y_{-1},y_{-1})f_G(y_{-1},y_0)f_G(y_0,y_1)\kappa(dy_{-1})\kappa(dy_1)\Big)\kappa(dy_0)\end{aligned}$$

It is the infinitesimal probability of $\mathcal{Y}_\infty$ having a point at y_0 minus the probability of having a point at y_0 and an other lower, minus the probability of having a point at y_0 and an other higher, plus the probability of having a point at y_0 surrounded by two neighbours on both sides. The identity can be further factorized as

$$\begin{aligned}&\Big(u_\uparrow(y_0) - 2\int_{y_{-1}<y_0}(y_0-y_{-1})u_\uparrow(y_{-1})\kappa(dy_{-1})\Big)\\ &\qquad\times\Big(u_\downarrow(y_0) - 2\int_{y_1>y_0}(y_1-y_0)u_\downarrow(y_1)\kappa(dy_1)\Big)\times\kappa(dy_0)\end{aligned}$$

According to the calculation done in the proof of proposition 3.1.11 this the above equals $u_\uparrow(-\infty)u_\downarrow(+\infty)\kappa(dy_0)$. □

Now let's describe $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ in two particular cases. If the killing rate is uniform, that is $\kappa(dy) = cdy$ where c is constant, then

$$cf_G(x_0, x) = f_{\mathcal{K}}(x_0, x) = 2c(x - x_0)e^{-\sqrt{2c}(x-x_0)}$$

Both the spacings of $\mathcal{Y}_\infty$ and $\mathcal{Z}_\infty$ are i.i.d. gamma-2 variables with mean $\sqrt{\frac{2}{c}}$. Actually the union $\mathcal{Y}_\infty \cup \mathcal{Z}_\infty$ is a Poisson point process with intensity $\sqrt{2c}dx$. If the killing measure is of form $\kappa = c\sum_{j\in\mathbb{Z}} \delta_j$ where c is constant, then again the spacings between consecutive points in $\mathcal{Y}_\infty$ are i.i.d random variables, this time integer valued. Let N_2 be a random variable with same distribution as this spacings. For any $j \in \mathbb{N}$

$$\mathbb{P}(N_2 = j) = 2cj(1 + \sqrt{2c})^{-j}$$

N_2 can be written as $N_2 = N_1 + \widetilde{N}_1 - 1$ where N_1 and $\widetilde{N}_1$ are two independent geometric variables of parameter $(1 + \sqrt{2c})^{-1}$. Actually, if $y_0 < y$ are two consecutive points in $\mathcal{Y}_\infty$ and z the point of $\mathcal{Z}_\infty$ lying between them, then conditional on y_0, $(\lfloor z\rfloor - y_0, y - \lfloor z\rfloor)$ has the same law as $(N_1 - 1, \widetilde{N})$. Moreover $\{\lfloor z\rfloor | z \in \mathcal{Z}_\infty\}$ has the same law as $\mathcal{Y}_\infty$.

3.1.4. Determinantal point processes $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$: general case. Let I be an open subinterval of $\mathbb{R}$ and L be the generator of a transient diffusion on I of form

$$L = \frac{1}{m(x)}\frac{d}{dx}\left(\frac{1}{w(x)}\frac{d}{dx}\right) - \kappa$$

with zero Dirichlet boundary conditions on ∂I with sample path denoted $(X_t)_{0\leqslant t<\zeta}$ We will describe, without proof, the law of $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ in this generic case. It can be derived in the same way as it was done in the previous section. Let G be the Green's function of L relative to the measure $m(y)dy$, factorisable as $G(x, y) = u_\uparrow(x \wedge y)u_\downarrow(x \vee y)$.

PROPOSITION 3.1.13. *$\mathcal{Y}_\infty$ and $\mathcal{Z}_\infty$ are a.s. discrete point processes. Let ∂I be the boundary of I in $\mathbb{R} \cup \{-\infty, +\infty\}$. Almost surely*

$$\mathcal{Y}_\infty \cap \partial I = \left\{y \in \partial I | \mathbb{P}(X_{\zeta^-} = y) > 0\right\}$$

If $\kappa \neq 0$, the points in $\mathcal{Y}_\infty \cap I$ are a determinantal point process with determinantal kernel $G(x, y)$ relatively the reference measure $m(y)\kappa(dy)$. $\mathcal{Z}_\infty$ is a determinantal point process on I with determinantal kernel

$$\frac{du_\uparrow}{dx}((y \wedge z)^+)\frac{du_\downarrow}{dx}((y \vee z)^-)$$

relative to the reference measure $\frac{dz}{w(z)}$. Given two consecutive points $y_1 < y_2$ in $\mathcal{Y}_\infty$, then the point of $\mathcal{Z}_\infty$ lying between them is distributed according to the measure $1_{y_1<z<y_2}\frac{w(z)dz}{\int_{(y_1,y_2)} w(a)da}$ and independently on the behaviour of $\mathcal{Z}_\infty$ on $(-\infty, y_1) \cup (y_2, +\infty)$. Given two consecutive points $z_1 < z_2$ in $\mathcal{Z}_\infty$, then the point of $\mathcal{Y}_\infty$ lying between them is distributed on (z_1, z_2) according the measure $1_{z_1<y<z_2}\frac{m(y)\kappa(dy)}{\int_{(z_1,z_2)} m(q)\kappa(dq)}$ and independently on the behaviour of $\mathcal{Y}_\infty$ on $(-\infty, z_1) \cup (z_2, +\infty)$.

3.2. Monotone couplings for the point processes $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$

3.2.1. Conditioning. In this chapter we will deal with monotone coupling for the determinantal point processes $\mathcal{Y}_\infty$ and $\mathcal{Z}_\infty$ intruded in chapter 3.1. We will restrict to the Brownian case. Consider two different killing measures κ and $\tilde{\kappa}$ on $\mathbb{R}$, with $\kappa \leqslant \tilde{\kappa}$, and the couples of determinantal point processes $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ respectively $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ corresponding to the Brownian motion on $\mathbb{R}$ with killing measure κ respectively $\tilde{\kappa}$. We will show that one can couple $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ and $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ on the same probability space such that $\mathcal{Z}_\infty \subseteq \widetilde{\mathcal{Z}}_\infty$ and

$\widetilde{\mathcal{Y}}_\infty \subseteq \mathcal{Y}_\infty \cup Supp(\tilde{\kappa} - \kappa)$. Moreover if κ and $\tilde{\kappa}$ are proportional we may also have $\mathcal{Y}_\infty \subseteq \widetilde{\mathcal{Y}}_\infty$. We will provide an explicit construction for the this couplings in the section 3.2.2.

In the section 3.2.1 we will prove conditionning results for $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$: what is obtained if $\mathcal{Y}_\infty$ or $\mathcal{Z}_\infty$ is conditioned by either containing a point at a given location or not containing any points in a given interval. These results will be used in the next section. The conditional law we will obtain are analogous to those of the Uniform Spanning Tree on a finite undirected connected graph: Let $\mathcal{G}$ be such a graph, E the set of its edges, C a weight function on E and Υ the corresponding Uniform Spanning Tree on $\mathcal{G}$. Let E_1 and E_2 be two disjoint subsets of E such that E_1 contains no cycles and such that erasing the edges in E_2 does not disconnect $\mathcal{G}$. The law of Υ conditioned by $E_1 \subseteq \Upsilon$ and $\Upsilon \cap E_2 = \varnothing$ can be described as follows: Let $\mathcal{G}'$ be the graph obtained from $\mathcal{G}$ trough erasing the edges in E_2 and contracting (i.e. identifying the two end vertices) the edges in E_1. The edges of $\mathcal{G}'$ are in one to one correspondence with $E\backslash E_2$. If we keep the same weight function C on these edges and take Υ' an Uniform Spanning Tree on $\mathcal{G}'$, then $\Upsilon' \cup E_1$ has the same law as Υ conditioned by $E_1 \subseteq \Upsilon$ and $\Upsilon \cap E_2 = \varnothing$ (see proposition 4.2 in [**BLPS01**]).

Let κ be a Radon measure on $\mathbb{R}$ and $G(x,y) = u_\uparrow(x \wedge y)u_\downarrow(x \vee y)$ the Green's function of $\frac{1}{2}\frac{d^2}{dx^2} - \kappa$. First we will restrict the Brownian motion with killing measure κ to a half-line by adding either a killing or a reflecting boundary point and describe what is obtained if we apply the Wilson's algorithm to it. This is related to some of the conditional laws we are interested in. Diffusions with reflection were not discussed so far.

For $x_0 < y$ let

$$u_\uparrow^{(x_0\times)}(y) := u_\uparrow(y) - \frac{u_\uparrow(x_0)}{u_\downarrow(x_0)}u_\uparrow(y)$$

and for $x_0 < y, z$ let

$$G^{(x_0\times)}(y,z) := u_\uparrow^{(x_0\times)}(y \wedge z)u_\downarrow(y \vee z)$$

$$\mathcal{K}^{(x_0\times)}(y,z) := -\frac{1}{2}\frac{du_\uparrow^{(x_0\times)}}{dx}((y \wedge z)^+)\frac{du_\downarrow}{dx}((y \vee z)^-)$$

$G^{(x_0\times)}$ was already introduced in (3.1.32). For $y < x_0$ let

$$u_\downarrow^{(\times x_0)}(y) := u_\downarrow(y) - \frac{u_\downarrow(x_0)}{u_\uparrow(x_0)}u_\uparrow(y)$$

and for $y, z < x_0$ let

$$G^{(\times x_0)}(y,z) := u_\uparrow(y \wedge z)u_\downarrow^{(\times x_0)}(y \vee z)$$

$$\mathcal{K}^{(\times x_0)}(y,z) := -\frac{1}{2}\frac{du_\uparrow}{dx}((y \wedge z)^+)\frac{du_\downarrow^{(\times x_0)}}{dx}((y \vee z)^-)$$

$G^{(x_0\times)}$ respectively $G^{(\times x_0)}$ is the Green's function of $\frac{1}{2}\frac{d^2}{dx^2} - \kappa$ restricted to the interval $(x_0, +\infty)$ respectively $(-\infty, x_0)$ with zero Dirichlet boundary condition at x_0.

Let $x_0 \in \mathbb{R}$ such that $\kappa(\{x_0\}) = 0$. For $x_0 < y$ let

$$u_\uparrow^{(x_0\triangleright)}(y) := u_\uparrow(y) + \Big(\frac{du_\downarrow}{dx}(x_0)\Big)^{-1}\frac{du_\uparrow}{dx}(x_0)u_\downarrow(y)$$

and for $y, z < x_0$ let

$$G^{(x_0\triangleright)}(y,z) := u_\uparrow^{(x_0\triangleright)}(y \wedge z)u_\downarrow(y \vee z)$$

$$\mathcal{K}^{(x_0\triangleright)}(y,z) := -\frac{1}{2}\frac{du_\uparrow^{(x_0\triangleright)}}{dx}((y \wedge z)^+)\frac{du_\downarrow}{dx}((y \vee z)^-)$$

$\mathcal{K}^{(x_0\triangleright)}$ was already introduced in (3.1.32). For $y < x_0$ let

$$u_\downarrow^{(\triangleleft x_0)}(y) := u_\downarrow(y) + \Big(\frac{du_\uparrow}{dx}(x_0)\Big)^{-1}\frac{du_\downarrow}{dx}(x_0)u_\uparrow(y)$$

and for $y, z < x_0$ let

$$G^{(\lhd x_0)}(y,z) := u_\uparrow(y \wedge z) u_\downarrow^{(\lhd x_0)}(y \vee z)$$

$$\mathcal{K}^{(\lhd x_0)}(y,z) := -\frac{1}{2}\frac{du_\uparrow}{dx}((y\wedge z)^+)\frac{du_\downarrow^{(\lhd x_0)}}{dx}((y\vee z)^-)$$

$G^{(x_0 \rhd)}$ respectively $G^{(\lhd x_0)}$ is the Green's function of $\frac{1}{2}\frac{d^2}{dx^2} - \kappa$ restricted to the interval $[x_0, +\infty)$ respectively $(-\infty, x_0]$ with zero Neumann boundary condition at x_0. Equivalently $G^{(x_0 \rhd)}$ respectively $G^{(\lhd x_0)}$ is the restriction to $[x_0, +\infty)$ respectively $(-\infty, x_0]$ of the Green's function on $\mathbb{R}$ of $\frac{1}{2}\frac{d^2}{dx^2} - 1_{[x_0,+\infty)}\kappa$ respectively $\frac{1}{2}\frac{d^2}{dx^2} - 1_{(-\infty,x_0]}\kappa$.

Consider now $x_0 \in \mathbb{R}$ and $(x_n)_{n\geqslant 1}$ a dense sequence of pairwise disjoint points in $(x_0, +\infty)$. We consider the Wilson's algorithm applied to the Brownian motion on $(x_0, +\infty)$ with killing measure κ and killing boundary x_0, where $(x_n)_{n\geqslant 0}$ is the sequence of starting points. Let $\mathcal{Y}_\infty^{(x_0 \times)}$ and $\mathcal{Z}_\infty^{(x_0\times)}$ be the interwoven point processes in $[x_0, +\infty)$ obtained as result. See figure 3.2.a for an illustration of the first four steps of Wilson's algorithm and of $(\mathcal{Y}_\infty^{(x_0 \times)}, \mathcal{Z}_\infty^{(x_0\times)})$. According to proposition 3.1.13, $x_0 \in \mathcal{Y}_\infty^{(x_0 \times)}$ a.s., $\mathcal{Y}_\infty^{(x_0 \times)} \cap (x_0, +\infty)$ is a determinantal point process with determinantal kernel $G^{(x_0\times)}$ relative to the measure $1_{(x_0,+\infty)}\kappa$ and $\mathcal{Z}_\infty^{(x_0\times)}$ is a determinantal point process with kernel $\mathcal{K}^{(x_0\times)}$ relative to the measure $1_{z>x_0}dz$. The distribution of the $2n$ closest to x_0 points in $(\mathcal{Y}_\infty^{(x_0 \times)} \cap (x_0,+\infty)) \cup \mathcal{Z}_\infty^{(x_0\times)}$, the odd-numbered belonging to $\mathcal{Y}_\infty^{(x_0 \times)} \cap (x_0, +\infty)$ and the even-numbered to $\mathcal{Z}_\infty^{(x_0\times)}$, is given by the measure

$$M_n^{(x_0\times)}(\mathcal{Y}_\infty^{(x_0 \times)}, \mathcal{Z}_\infty^{(x_0\times)})(dz_1, dy_1, \dots, dz_n, dy_n) := 2^n \frac{u_\downarrow(y_n)}{u_\downarrow(x_0)} dz_1 \kappa(dy_1)\dots dz_n\kappa(dy_n)$$

Its total mass equals $\mathbb{P}(\sharp\mathcal{Y}_\infty^{(x_0 \times)} \geqslant n+1)$. If the Wilson's algorithm is applied to the Brownian motion on $(-\infty, x_0)$, killed at x_0 and with killing measure κ, and $(\mathcal{Y}_\infty^{(\times x_0)}, \mathcal{Z}_\infty^{(\times x_0)})$ are the point processes returned by the algorithm, then the distribution of the $2n$ closest to x_0 points in $(\mathcal{Y}_\infty^{(\times x_0)} \cap (-\infty, x_0)) \cup \mathcal{Z}_\infty^{(\times x_0)}$ is given by the measure

$$M_n^{(\times x_0)}(\mathcal{Y}_\infty^{(\times x_0)}, \mathcal{Z}_\infty^{(\times x_0)})(dz_{-1}, dy_{-1}, \dots, dz_{-n}, dy_{-n}) := \\ 2^n\frac{u_\uparrow(y_{-n})}{u_\uparrow(x_0)} dz_{-1}\kappa(dy_{-1})\dots dz_{-n}\kappa(dy_{-n})$$

Let now $x_0 \in \mathbb{R}$ such that $\kappa(\{x_0\}) = 0$. If we replace the Brownian motion on $(x_0, +\infty)$ killed in x_0 by a Brownian motion on $[x_0, +\infty)$ reflected in x_0, and keep the killing measure κ, we get another pair $(\mathcal{Y}_\infty^{(x_0 \rhd)}, \mathcal{Z}_\infty^{(x_0 \rhd)})$ of interwoven point processes on $[x_0, +\infty)$. The pair $(\mathcal{Y}_\infty^{(x_0 \rhd)}, \mathcal{Z}_\infty^{(x_0 \rhd)})$ can be also obtain through applying Wilson's algorithm to a Brownian motion on $\mathbb{R}$ with the killing measure $1_{(x_0,+\infty)}\kappa$. See figure 3.2.b for an illustration of $(\mathcal{Y}_\infty^{(x_0 \rhd)}, \mathcal{Z}_\infty^{(x_0 \rhd)})$. Observe the difference with figure 3.2.a at the third step of Wilson's algorithm. $\mathcal{Y}_\infty^{(x_0 \rhd)}$ is a determinantal point process with determinantal kernel $G^{(x_0 \rhd)}$ relative to the measure $1_{(x_0,+\infty)}\kappa$. $\mathcal{Z}_\infty^{(x_0 \rhd)}$ is a determinantal point process with kernel $\mathcal{K}^{(x_0 \rhd)}$ relative to the measure $1_{z>x_0}dz$. The distribution of the $2n-1$ closest to x_0 points in $\mathcal{Y}_\infty^{(x_0 \rhd)} \cup \mathcal{Z}_\infty^{(x_0 \rhd)}$, the odd-numbered belonging to $\mathcal{Z}_\infty^{(x_0 \rhd)}$ and the even-numbered to $\mathcal{Y}_\infty^{(x_0 \rhd)}$, is given by the measure

$$M_n^{(x_0 \rhd)}(\mathcal{Y}_\infty^{(x_0 \rhd)}, \mathcal{Z}_\infty^{(x_0 \rhd)})(dy_1, dz_1, \dots dz_{n-1}, dy_n) := \\ -2^n\Big(\frac{du_\downarrow}{dx}(x_0)\Big)^{-1} u_\downarrow(y_n)\kappa(dy_1)dz_1\dots dz_{n-1}\kappa(dy_n)$$

If the Wilson's algorithm is applied to the Brownian motion on $(-\infty, x_0]$, reflected at x_0 and with killing measure κ, and $(\mathcal{Y}_\infty^{(\lhd x_0)}, \mathcal{Z}_\infty^{(\lhd x_0)})$ are the point processes returned by the algorithm, then the distribution of the $2n-1$ closest to x_0 points in $\mathcal{Y}_\infty^{(\lhd x_0)} \cup \mathcal{Z}_\infty^{(\lhd x_0)}$ is given by the measure

$$M_n^{(\lhd x_0)}(\mathcal{Y}_\infty^{(\lhd x_0)}, \mathcal{Z}_\infty^{(\lhd x_0)})(dy_{-1}, dz_{-1}, \dots dz_{-n+1}, dy_{-n}) := 2^n \Big(\frac{du_\uparrow}{dx}(x_0)\Big)^{-1} u_\uparrow(y_{-n})\kappa(dy_{-1})dz_{-1}\dots dz_{-n+1}\kappa(dy_{-n})$$

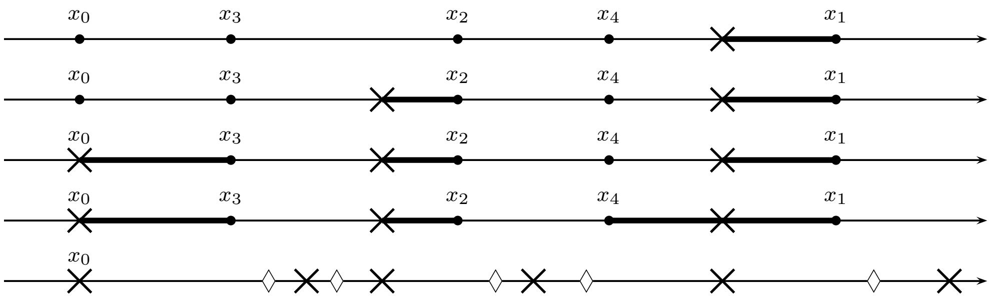

Fig.3.2.a - Illustration of the first four steps of Wilson's algorithm in case of killing at x_0 and of $(\mathcal{Y}_\infty^{(x_0\times)}, \mathcal{Z}_\infty^{(x_0\times)})$: x-dots represent the points of $\mathcal{Y}_n^{(x_0\times)}$, diamonds the points of $\mathcal{Z}_n^{(x_0\times)}$ and thick lines the intervals in $\mathcal{J}_n^{(x_0\times)}$.

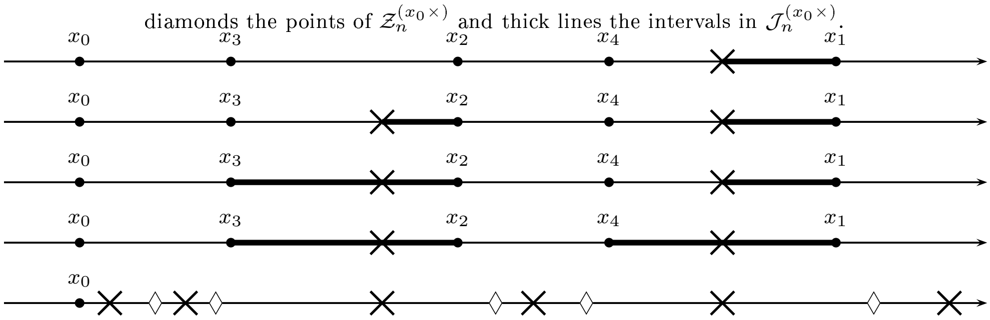

Fig.3.2.b - Illustration of the first four steps of Wilson's algorithm in case of reflection at x_0 and of $(\mathcal{Y}_\infty^{(x_0\rhd)}, \mathcal{Z}_\infty^{(x_0\rhd)})$: x-dots represent the points of $\mathcal{Y}_n^{(x_0\rhd)}$, diamonds the points of $\mathcal{Z}_n^{(x_0\rhd)}$ and thick lines the intervals in $\mathcal{J}_n^{(x_0\rhd)}$.

Let $\mathcal{Y}_\infty$ and $\mathcal{Z}_\infty$ be the determinantal point processes associated to the Brownian motion on $\mathbb{R}$ with killing measure κ. Let $n, n' \in \mathbb{N}^*$. The following two factorizations hold:

$$\begin{aligned} M_{n+n'}(\mathcal{Y}_\infty, \mathcal{Z}_\infty)(dy_{-n'}, dz_{-n'}, \dots dy_{-1}, dz_{-1}, dy_0, dz_1, dy_1, \dots, dz_n, dy_n) = \\ M_{n'}^{(\times y_0)}(\mathcal{Y}_\infty^{(\times y_0)}, \mathcal{Z}_\infty^{(\times y_0)})(dz_{-1}, dy_{-1}, \dots, dz_{-n'}, dy_{-n'}) \times G(y_0, y_0)\kappa(dy_0) \\ \times M_n^{(y_0\times)}(\mathcal{Y}_\infty^{(y_0\times)}, \mathcal{Z}_\infty^{(y_0\times)})(dz_1, dy_1, \dots, dz_n, dy_n) \end{aligned}$$

$$\begin{aligned} M_{n+n'-1}(\mathcal{Y}_\infty, \mathcal{Z}_\infty)(dy_{-n'}, dz_{-n'+1}, \dots dz_{-1}, dy_{-1}, dz_0, dy_1, dz_1, \dots, dz_{n-1}, dy_n) = \\ M_{n'}^{(\lhd z_0)}(\mathcal{Y}_\infty^{(\lhd z_0)}, \mathcal{Z}_\infty^{(\lhd z_0)})(dy_{-1}, dz_{-1}, \dots dz_{-n'+1}, dy_{-n'}) \times \mathcal{K}(z_0, z_0)dz_0 \\ \times M_n^{(z_0\rhd)}(\mathcal{Y}_\infty^{(z_0\rhd)}, \mathcal{Z}_\infty^{(z_0\rhd)})(dy_1, dz_1, \dots dz_{n-1}, dy_n) \end{aligned}$$

The above factorisations imply the following:

PROPERTY 3.2.1. *Let $\varepsilon > 0$ and let F_1 and F_2 be two measurable non-negative functionals on couples of point processes on $\mathbb{R}$ and f a measurable non-negative function on $\mathbb{R}$. Then*

$$\mathbb{E}\Big[\sum_{y_0\in\mathcal{Y}_\infty} f(y_0)F_1(\mathcal{Y}_\infty\cap(-\infty,y_0],\mathcal{Z}_\infty\cap(-\infty,y_0])F_2(\mathcal{Y}_\infty\cap[y_0,+\infty),\mathcal{Z}_\infty\cap[y_0,+\infty))\Big]$$
$$= \int_{\mathbb{R}} f(y_0)G(y_0,y_0)\mathbb{E}[F_1(\mathcal{Y}_\infty^{(\times y_0)},\mathcal{Z}_\infty^{(\times y_0)})]\mathbb{E}[F_2(\mathcal{Y}_\infty^{(y_0\times)},\mathcal{Z}_\infty^{(y_0\times)})]\kappa(dy_0)$$

and

$$\mathbb{E}\Big[\sum_{z_0\in\mathcal{Z}_\infty} f(z_0)F_1(\mathcal{Y}_\infty\cap(-\infty,z_0],\mathcal{Z}_\infty\cap(-\infty,z_0])F_2(\mathcal{Y}_\infty\cap[z_0,+\infty),\mathcal{Z}_\infty\cap[z_0,+\infty))\Big]$$
$$= \int_{\mathbb{R}} f(z_0)\mathcal{K}(z_0,z_0)\mathbb{E}[F_1(\mathcal{Y}_\infty^{(\lhd z_0)},\mathcal{Z}_\infty^{(\lhd z_0)})]\mathbb{E}[F_2(\mathcal{Y}_\infty^{(z_0\rhd)},\mathcal{Z}_\infty^{(z_0\rhd)})]dz_0$$

If $y_0 \in Supp(\kappa)$, then conditional on $y_0 \in \mathcal{Y}_\infty$, $(\mathcal{Y}_\infty\cap(-\infty,y_0],\mathcal{Z}_\infty\cap(-\infty,y_0])$ and $(\mathcal{Y}_\infty\cap[y_0,+\infty),\mathcal{Z}_\infty\cap[y_0,+\infty))$ are independent, $(\mathcal{Y}_\infty\cap(-\infty,y_0],\mathcal{Z}_\infty\cap(-\infty,y_0])$ has the same law as $(\mathcal{Y}_\infty^{(\times y_0)},\mathcal{Z}_\infty^{(\times y_0)})$ and $(\mathcal{Y}_\infty\cap[y_0,+\infty),\mathcal{Z}_\infty\cap[y_0,+\infty))$ has the same law as $(\mathcal{Y}_\infty^{(y_0\times)},\mathcal{Z}_\infty^{(y_0\times)})$.

If $\kappa((-\infty,z_0)) > 0$, $\kappa((z_0,+\infty)) > 0$ and $\kappa(\{z_0\}) = 0$, then conditional on $z_0 \in \mathcal{Z}_\infty$, $(\mathcal{Y}_\infty\cap(-\infty,z_0],\mathcal{Z}_\infty\cap(-\infty,z_0])$ and $(\mathcal{Y}_\infty\cap[z_0,+\infty),\mathcal{Z}_\infty\cap[z_0,+\infty))$ are independent, $(\mathcal{Y}_\infty\cap(-\infty,z_0],\mathcal{Z}_\infty\cap(-\infty,z_0])$ has the same law as $(\mathcal{Y}_\infty^{(\lhd z_0)},\mathcal{Z}_\infty^{(\lhd z_0)})$ and $(\mathcal{Y}_\infty\cap[z_0,+\infty)\mathcal{Z}_\infty\cap[z_0,+\infty))$ has the same law as $(\mathcal{Y}_\infty^{(z_0\rhd)},\mathcal{Z}_\infty^{(z_0\rhd)})$.

Let $y_0 \in \mathbb{R}$ and $c > 0$. We will denote by $(\mathcal{Y}_\infty^{(y_0)},\mathcal{Z}_\infty^{(y_0)})$ the pair of interwoven determinantal point processes corresponding to the killing measure $\kappa + c\delta_{y_0}$, conditioned on $\mathcal{Y}_\infty^{(y_0)}$ containing y_0. The law of $(\mathcal{Y}_\infty^{(y_0)},\mathcal{Z}_\infty^{(y_0)})$ does not depend on the value of c according to the property 3.2.1. $(\mathcal{Y}_\infty^{(y_0)}\cap(y_0,+\infty),\mathcal{Z}_\infty^{(y_0)}\cap(y_0,+\infty))$ and $(\mathcal{Y}_\infty^{(y_0)}\cap(-\infty,y_0),\mathcal{Z}_\infty^{(y_0)}\cap(-\infty,y_0))$ are independent. The distribution of the $2n$ closest to y_0 points in $(\mathcal{Y}_\infty^{(y_0)}\cup\mathcal{Z}_\infty^{(y_0)})\cap(y_0,+\infty)$, on the event $\sharp(\mathcal{Y}_\infty^{(y_0)}\cap(y_0,+\infty)) \geqslant n$, is

$$1_{y_0<z_1<y_1<\dots<z_n<y_n}2^n\frac{u_\downarrow(y_n)}{u_\downarrow(y_0)}dz_1\kappa(dy_1)\dots dz_n\kappa(dy_n) \tag{3.2.1}$$

The distribution of the $2n$ closest to y_0 points in $(\mathcal{Y}_\infty^{(y_0)}\cup\mathcal{Z}_\infty^{(y_0)})\cap(-\infty,y_0)$ is

$$1_{y_0>z_{-1}>y_{-1}>\dots>z_{-n}>y_{-n}}2^n\frac{u_\uparrow(y_{-n})}{u_\uparrow(y_0)}dz_{-1}\kappa(dy_{-1})\dots dz_{-n}\kappa(dy_{-n}) \tag{3.2.2}$$

Let $a < b \in \mathbb{R}$. Next we will describe what happens if we condition by $\mathcal{Z}_\infty\cap[a,b] = \varnothing$. This condition implies in particular that $\sharp(\mathcal{Y}_\infty\cap[a,b]) \leqslant 1$. Let $\widehat{\mathbb{R}}$ be the quotient space where in $\mathbb{R}$ we identify to one point all the points lying in $[a,b]$. $\widehat{\mathbb{R}}$ is homeomorphic to $\mathbb{R}$. Let $\hat{\pi}$ be the projection from $\mathbb{R}$ to $\widehat{\mathbb{R}}$. Let θ be the class of $[a,b]$ in $\widehat{\mathbb{R}}$. We define on $\widehat{\mathbb{R}}$ the metric $d_{\widehat{\mathbb{R}}}$:

- If $x < y < a$ or $b < x < y$ then $d_{\widehat{\mathbb{R}}}(\hat{\pi}(x),\hat{\pi}(y)) = y - x$.
- If $x < a$ and $y > b$ then $d_{\widehat{\mathbb{R}}}(\hat{\pi}(x),\hat{\pi}(y)) = (y-x)-(b-a)$.
- If $x < a$ then $d_{\widehat{\mathbb{R}}}(\hat{\pi}(x),\theta) = a - x$.
- If $x > b$ then $d_{\widehat{\mathbb{R}}}(\hat{\pi}(x),\theta) = x - b$.

$\widehat{\mathbb{R}}$ endowed with $d_{\widehat{\mathbb{R}}}$ is isometric to $\mathbb{R}$. So we can define a standard Brownian motion on $\widehat{\mathbb{R}}$. Let $\hat{\kappa}$ be the measure κ pushed forward by $\hat{\pi}$ on $\widehat{\mathbb{R}}$. In particular $\hat{\kappa}(\{\theta\}) = \kappa([a,b])$. Let

$(\widehat{\mathcal{Y}}_\infty, \widehat{\mathcal{Z}}_\infty)$ be the pair of interwoven determinantal point processes on $\widehat{\mathbb{R}}$ obtained by applying the Wilson's algorithm to the Brownian motion on $\widehat{\mathbb{R}}$ with killing measure $\hat{\kappa}$.

PROPOSITION 3.2.2. *Conditional on $\mathcal{Z}_\infty \cap [a,b] = \varnothing$, $(\hat{\pi}(\mathcal{Y}_\infty), \hat{\pi}(\mathcal{Z}_\infty))$ has the same distribution as $(\widehat{\mathcal{Y}}_\infty, \widehat{\mathcal{Z}}_\infty)$. Moreover on the event $\mathcal{Y}_\infty \cap [a,b] \neq \varnothing$, the unique point in $\mathcal{Y}_\infty \cap [a,b]$ is distributed according the probability measure $\frac{1_{a\leqslant y\leqslant b}\kappa(dy)}{\kappa([a,b])}$.*

PROOF. First we compute $\mathbb{P}(\mathcal{Z}_\infty \cap [a,b] = \varnothing)$. We consider that a and b are the first two starting points in the Wilson's algorithm. Then

$$\begin{aligned}\mathbb{P}(\mathcal{Z}_\infty \cap [a,b] = \varnothing) =& \mathbb{P}\Big(B^{(a)}_{T_1^-} > b\Big) + \mathbb{P}\Big(B^{(a)}_{T_1^-} < a, B^{(b)}_{T_2^-} = a\Big) + \mathbb{P}\Big(B^{(a)}_{T_1^-} = B^{(b)}_{T_2^-} \in [a,b]\Big)\\ =& \frac{1}{2}\frac{du_\uparrow}{dx}(a^-)u_\downarrow(b) - \frac{1}{2}u_\uparrow(a)\frac{du_\downarrow}{dx}(b^+) + u_\uparrow(a)u_\downarrow(b)\kappa([a,b])\end{aligned}$$

Next we determine the Green's function $\widehat{G}$ of $\frac{1}{2}\frac{d^2}{d\tilde{x}^2} - \hat{\kappa}$ on $\widehat{\mathbb{R}}$. Let $\hat{u}_\uparrow$ and $\hat{u}_\downarrow$ be two solutions on $\widehat{\mathbb{R}}$ to

$$\frac{1}{2}\frac{d\hat{u}}{dx} - \hat{u}\hat{\kappa} = 0$$

with the initial conditions $\hat{u}_\uparrow(\theta) = u_\uparrow(a)$, $\frac{d\hat{u}_\uparrow}{dx}(\theta^-) = \frac{du_\uparrow}{dx}(a^-)$, $\hat{u}_\downarrow(\theta) = u_\downarrow(b)$ and $\frac{d\hat{u}_\downarrow}{dx}(\theta^+) = \frac{du_\downarrow}{dx}(b^+)$. Then for $x \leqslant a$, $\hat{u}_\uparrow(\hat{\pi}(x)) = u_\uparrow(x)$ and for $x \geqslant b$, $\hat{u}_\downarrow(\hat{\pi}(x)) = u_\downarrow(x)$. $\hat{u}_\uparrow$ and $\hat{u}_\downarrow$ are positive, $\hat{u}_\uparrow$ is non-decreasing and $\hat{u}_\downarrow$ non-increasing. Moreover:

$$\frac{d\hat{u}_\uparrow}{dx}(\theta^+) = \frac{d\hat{u}_\uparrow}{dx}(\theta^-) + 2\hat{u}_\uparrow(\theta)\hat{\kappa}(\{\theta\}) = \frac{du_\uparrow}{dx}(a^-) + 2u_\uparrow(a)\kappa([a,b])$$

The Wronskian of $\hat{u}_\downarrow$ and $\hat{u}_\uparrow$ equals

$$\begin{aligned}W(\hat{u}_\downarrow, \hat{u}_\uparrow) =& \hat{u}_\downarrow(\theta)\frac{d\hat{u}_\uparrow}{dx}(\theta^+) - \hat{u}_\uparrow(\theta)\frac{d\hat{u}_\downarrow}{dx}(\theta^+)\\ =& \frac{du_\uparrow}{dx}(a^-)u_\downarrow(b) - u_\uparrow(a)\frac{du_\downarrow}{dx}(b^+) + 2u_\uparrow(a)u_\downarrow(b)\kappa([a,b])\\ =& 2\mathbb{P}(\mathcal{Z}_\infty \cap [a,b] = \varnothing)\end{aligned}$$

Thus $\widehat{G}$ equals

$$\widehat{G}(\tilde{x}, \tilde{y}) = \frac{\hat{u}_\uparrow(\tilde{x} \wedge \tilde{y})\hat{u}_\downarrow(\tilde{x} \vee \tilde{y})}{\mathbb{P}(\mathcal{Z}_\infty \cap [a,b] = \varnothing)}$$

In particular if $x \leqslant a$ and $y \geqslant b$ then

$$\widehat{G}(\hat{\pi}(x), \hat{\pi}(y)) = \frac{u_\uparrow(x)u_\downarrow(y)}{\mathbb{P}(\mathcal{Z}_\infty \cap [a,b] = \varnothing)} = \frac{G(x,y)}{\mathbb{P}(\mathcal{Z}_\infty \cap [a,b] = \varnothing)} \tag{3.2.3}$$

To prove the equality in law, we need to consider the probabilities of all the events $\mathscr{C}_n(a_0, b_0, \tilde{a}_1, \tilde{b}_1, a_1, b_1, \dots, \tilde{a}_n, \tilde{b}_n, a_n, b_n)$ where $n \geqslant 1$ and $a_0 < b_0 < \tilde{a}_1 < \tilde{b}_1 < a_1 < b_1 < \dots < \tilde{a}_n < \tilde{b}_n < a_n < b_n \in \mathbb{R}$, corresponding to following conditions:

- $\mathcal{Y}_\infty \cap [a_0, b_0] \neq \varnothing$, $\mathcal{Y}_\infty \cap [a_n, b_n] \neq \varnothing$
- $\forall r \in \{1, \dots, n\}, \sharp(\mathcal{Y}_\infty \cap [a_r, b_r]) = 1$
- $\forall r \in \{1, \dots, n\}, \sharp(\mathcal{Z}_\infty \cap (\tilde{a}_r, \tilde{b}_r)) = 1$
- $\forall r \in \{0, \dots, n-1\}, (\mathcal{Y}_\infty \cup \mathcal{Z}_\infty) \cap (b_r, \tilde{a}_r] = \varnothing, (\mathcal{Y}_\infty \cup \mathcal{Z}_\infty) \cap [\tilde{b}_r, a_{r+1}) = \varnothing$

We will also assume that either all of the $[a_r, b_r]$ do not intersect $[a,b]$ or one of the $[a_r, b_r]$ is contained in $[a,b]$ and the other do not intersect $[a,b]$. The probabilities of such events determine the joint law of $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ on the event $\sharp\mathcal{Y}_\infty \geqslant 2, \mathcal{Z}_\infty \cap [a,b] = \varnothing$. We will denote $\widehat{\mathscr{C}}_n(\cdot)$ the analogously defined events where we replace $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ by $(\widehat{\mathcal{Y}}_\infty, \widehat{\mathcal{Z}}_\infty)$. We do not need to deal with the event $\sharp\mathcal{Y}_\infty = 1$ because then $\mathcal{Z}_\infty = \varnothing$.

We first consider the case of $[a,b] \cap \left(\bigcup_{r=0}^n [a_r, b_r]\right) = \varnothing$. If there is $r_0 \in \{0, n-1\}$ such that $b_{r_0} < a$ and $b < a_{r_0+1}$ then

$$\begin{aligned}
&\mathbb{P}\left(\mathscr{C}_n(a_0, b_0, \tilde{a}_1, \tilde{b}_1, a_1, b_1, \ldots, \tilde{a}_n, \tilde{b}_n, a_n, b_n), \mathcal{Z}_\infty \cap [a,b] = \varnothing\right) \\
&\qquad = \int_{[a_0,b_0]} u_\uparrow(y_0)\kappa(dy_0) \times \int_{[a_n,b_n]} u_\downarrow(y_n)\kappa(dy_n) \times \prod_{r=1}^{n-1} \kappa([a_r, b_r]) \\
&\qquad\qquad \times \prod_{r \neq r_0} 2(\tilde{b}_r - \tilde{a}_r) \times 2Leb([\tilde{a}_{r_0}, \tilde{b}_{r_0}] \backslash [a,b])
\end{aligned}$$

Using (3.2.3) we get that the above equals

$$\mathbb{P}(\mathcal{Z}_\infty \cap [a,b] = \varnothing) \times \mathbb{P}\left(\widehat{\mathscr{C}}_n(\hat{\pi}(a_0), \hat{\pi}(b_0), \hat{\pi}(\tilde{a}_1), \hat{\pi}(\tilde{b}_1), \ldots, \hat{\pi}(a_n), \hat{\pi}(b_n))\right)$$

If $b < a_0$, then we consider a Wilson's algorithm where the $2(n+1)$ first starting points are $\tilde{a}_1, \tilde{b}_1, \ldots, \tilde{a}_n, \tilde{b}_n, a, b$. The conditions $\mathscr{C}_n(a_0, b_0, \tilde{a}_1, \tilde{b}_1, a_1, b_1, \ldots, \tilde{a}_n, \tilde{b}_n, a_n, b_n)$ and $\mathcal{Z}_\infty \cap [a,b] = \varnothing$ are satisfied if and only if the following is true:

- $B^{(\tilde{a}_1)}_{T_1^-} \in [a_0, b_0]$, $B^{(\tilde{b}_n)}_{T_{2n}^-} \in [a_n, b_n]$, for all $r \in \{1, \ldots, n-1\}$, $B^{(\tilde{b}_r)}_{T_{2r}^-} = B^{(\tilde{a}_{r+1})}_{T_{2r+1}^-} \in [a_r, b_r]$ and for all $r \in \{1, \ldots, n\}$, $\mathcal{Y}_\infty \cap (\tilde{a}_r, \tilde{b}_r) = \varnothing$.
- Either $B^{(a)}_{T_{2n+1}^-} \in (b, B^{(\tilde{a}_1)}_{T_1^-}]$ or $B^{(b)}_{T_{2n+2}^-} < a$ or $B^{(a)}_{T_{2n+1}^-} = B^{(b)}_{T_{2n+2}^-} \in [a,b]$.

Then

$$\begin{aligned}
&\mathbb{P}\Big(\mathscr{C}_n(a_0, b_0, \tilde{a}_1, \tilde{b}_1, a_1, b_1, \ldots, \tilde{a}_n, \tilde{b}_n, a_n, b_n), \mathcal{Z}_\infty \cap [a,b] = \varnothing\Big) \\
&= \int_{[a_n,b_n]} u_\downarrow(y_n)\kappa(dy_n) \times \prod_{r=1}^{n-1} \kappa([a_r, b_r]) \times \prod_{r=1}^{n} 2(\tilde{b}_r - \tilde{a}_r) \\
&\times \left(u_\uparrow(a) \int_{b<y<y_0, y_0 \in [a_0,b_0]} (u_\downarrow(y)u_\uparrow(y_0) - u_\uparrow(y)u_\downarrow(y_0))\kappa(dy)\kappa(dy_0) + u_\uparrow(a)\kappa([a_0, b_0])\right. \\
&+ \left(\int_{y_{-1}<a} u_\uparrow(y_{-1})\kappa(dy_{-1}) + u_\uparrow(a)\kappa([a_0, b_0])\right) \\
&\times \left.\int_{[a_0,b_0]} (u_\downarrow(b)u_\uparrow(y_0) - u_\uparrow(b)u_\downarrow(y_0))\kappa(dy_0)\right) \\
&= \int_{[a_n,b_n]} u_\downarrow(y_n)\kappa(dy_n) \times \prod_{r=1}^{n-1} \kappa([a_r, b_r]) \times \prod_{r=1}^{n} 2(\tilde{b}_r - \tilde{a}_r) \\
&\times \left(\frac{1}{2}u_\uparrow(a) \int_{[a_0,b_0]} \left(\frac{du_\downarrow}{dx}(b^+)u_\uparrow(y_0) - \frac{du_\uparrow}{dx}(b^+)u_\downarrow(y_0)\right)\kappa(dy_0)\right. \\
&+ \left(\frac{1}{2}\frac{du_\uparrow}{dx}(a^-) + u_\uparrow(a)\kappa([a_0, b_{01}])\right) \left.\int_{[a_0,b_0]} (u_\downarrow(b)u_\uparrow(y_0) - u_\uparrow(b)u_\downarrow(y_0))\kappa(dy_0)\right)
\end{aligned}$$

But for $y_0 \geqslant b$

$$\begin{aligned}
\hat{u}_\uparrow(\hat{\pi}(y_0)) =& \frac{1}{2}u_\uparrow(a)\left(\frac{du_\downarrow}{dx}(b^+)u_\uparrow(y_0) - \frac{du_\uparrow}{dx}(b^+)u_\downarrow(y_0)\right) \\
&+ \left(\frac{1}{2}\frac{du_\uparrow}{dx}(a^-) + u_\uparrow(a)\kappa([a_0, b_0])\right)(u_\downarrow(b)u_\uparrow(y_0) - u_\uparrow(b)u_\downarrow(y_0))
\end{aligned}$$

Indeed one can check the initial conditions $\hat{u}_\uparrow(\hat{\pi}(b)) = u_\uparrow(a)$ and $\frac{d\hat{u}_\uparrow}{dx}(\hat{\pi}(b)^+) = \frac{du_\uparrow}{dx}(a^-) + 2u_\uparrow(a)\kappa([a_0,b_0])$. It follows that

$$\begin{aligned}&\mathbb{P}\Big(\mathscr{C}_n(a_0,b_0,\tilde{a}_1,\tilde{b}_1,\dots,a_n,b_n), \mathcal{Z}_\infty \cap [a,b] = \varnothing\Big)\\ &= \int_{[\hat{\pi}(a_0),\hat{\pi}(b_0)]} \hat{u}_\uparrow(\tilde{y}_0)\kappa(d\tilde{y}_0) \times \int_{[\hat{\pi}(a_n),\hat{\pi}(b_n)]} \hat{u}_\downarrow(\tilde{y}_n)\kappa(d\tilde{y}_n) \times \prod_{r=1}^{n-1}\kappa([a_r,b_r]) \times \prod_{r=1}^{n} 2(\tilde{b}_r - \tilde{a}_r)\\ &= \mathbb{P}(\mathcal{Z}_\infty \cap [a,b] = \varnothing) \times \mathbb{P}\left(\widehat{\mathscr{C}}_n(\hat{\pi}(a_0),\hat{\pi}(b_0),\hat{\pi}(\tilde{a}_1),\hat{\pi}(\tilde{b}_1),\dots,\hat{\pi}(a_n),\hat{\pi}(b_n))\right)\end{aligned}$$

Similar holds if $b_n < a$.

Now we consider the case when there is $r_0 \in \{0,\dots,n\}$ such that $[a_{r_0}, b_{r_0}] \subseteq [a,b]$ and $[a,b] \cap \left(\bigcup_{r\neq r_0}[a_r,b_r]\right) = \varnothing$. If $1 \leqslant r_0 \leqslant n-1$ then

$$\begin{aligned}&\mathbb{P}\left(\mathscr{C}_n(a_0,b_0,\tilde{a}_1,\tilde{b}_1,\dots,a_n,b_n), \mathcal{Z}_\infty \cap [a,b] = \varnothing\right)\\ &= \int_{[a_0,b_0]} u_\uparrow(y_0)\kappa(dy_0) \times \int_{[a_n,b_n]} u_\downarrow(y_n)\kappa(dy_n) \times \prod_{r=1}^{n-1}\kappa([a_r,b_r])\\ &\quad\times \prod_{r\neq r_0,r_0+1} 2(\tilde{b}_r - \tilde{a}_r) \times 2Leb([\tilde{a}_{r_0},\tilde{b}_{r_0}]\backslash[a,b]) \times 2Leb([\tilde{a}_{r_0+1},\tilde{b}_{r_0+1}]\backslash[a,b])\\ &= \frac{\kappa([a_{r_0},b_{r_0}])}{\kappa([a,b])} \times \mathbb{P}(\mathcal{Z}_\infty \cap [a,b] = \varnothing) \times \mathbb{P}\left(\widehat{\mathscr{C}}_n(\hat{\pi}(a_0),\hat{\pi}(b_0),\hat{\pi}(\tilde{a}_1),\hat{\pi}(\tilde{b}_1),\dots,\hat{\pi}(a_n),\hat{\pi}(b_n))\right)\end{aligned}$$

Moreover $\hat{\pi}(a_{r_0}) = \hat{\pi}(b_{r_0}) = \theta$. If $r_0 = 0$ then

$$\begin{aligned}&\mathbb{P}\left(\mathscr{C}_n(a_0,b_0,\tilde{a}_1,\tilde{b}_1,\dots,a_n,b_n), \mathcal{Z}_\infty \cap [a,b] = \varnothing\right)\\ &= u_\uparrow(a)\kappa([a_0,b_0]) \times \int_{[a_n,b_n]} u_\downarrow(y_n)\kappa(dy_n)\\ &\quad\times \prod_{r=1}^{n-1}\kappa([a_r,b_r]) \times \prod_{r=2}^{n} 2(\tilde{b}_r - \tilde{a}_r) \times 2Leb([\tilde{a}_1,\tilde{b}_1]\backslash[a,b])\\ &= \frac{\kappa([a_0,b_0])}{\kappa([a,b])} \times \mathbb{P}(\mathcal{Z}_\infty \cap [a,b] = \varnothing) \times \mathbb{P}\left(\widehat{\mathscr{C}}_n(\hat{\pi}(a_0),\hat{\pi}(b_0),\hat{\pi}(\tilde{a}_1),\hat{\pi}(\tilde{b}_1),\dots,\hat{\pi}(a_n),\hat{\pi}(b_n))\right)\end{aligned}$$

and $\hat{\pi}(a_0) = \hat{\pi}(b_0) = \theta$. We have a similar expression if $r_0 = n$. □

Next we deal with the condition of the determinantal point process $\mathcal{Y}_\infty$ not charging a given subinterval of $\mathbb{R}$. We will consider the following more general situation: Let κ and $\tilde{\kappa}$ be two different killing measures on $\mathbb{R}$, with $\kappa \leqslant \tilde{\kappa}$, and the couples of determinantal point processes $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ respectively $(\tilde{\mathcal{Y}}_\infty, \tilde{\mathcal{Z}}_\infty)$ corresponding to the Brownian motion on $\mathbb{R}$ with killing measure κ respectively $\tilde{\kappa}$. Let $\widetilde{G}$ be the Green's function of $\frac{1}{2}\frac{d^2}{dx^2} - \tilde{\kappa}$, factorized as

$$\widetilde{G}(x,y) = \tilde{u}_\uparrow(x \wedge y)\tilde{u}_\downarrow(x \vee y)$$

Let

$$\widetilde{\mathcal{K}}(y,z) := -\frac{1}{2}\frac{d\tilde{u}_\uparrow}{dx}((y\wedge z)^+)\frac{d\tilde{u}_\downarrow}{dx}((y\vee z)^-)$$

We will assume that $\tilde{\kappa} - \kappa$ has a first moment, that is to say

$$\int_{\mathbb{R}} |x|(\tilde{\kappa}(dx) - \kappa(dx)) < +\infty$$

Let χ be the Radon-Nikodym derivative

$$\chi := \frac{d\kappa}{d\tilde{\kappa}}$$

By definition $0 \leqslant \chi \leqslant 1$. Let $\Delta\widetilde{\mathcal{Y}}$ be the point process obtained from $\widetilde{\mathcal{Y}}_\infty$ as follows: Given a point y in $\widetilde{\mathcal{Y}}_\infty$ we chose to erase it with probability $\chi(y)$ and keep it with probability $1-\chi(y)$, each choice being independent from the other choices and the position of other points. It is immediate to check that $\Delta\widetilde{\mathcal{Y}}$ is a determinantal point process with determinantal kernel $(\widetilde{G}(x,y))_{x,y\in\mathbb{R}}$ relative to the measure $(1-\chi)\tilde{\kappa}$, that is to say the measure $\tilde{\kappa}-\kappa$. We will show that conditional on $\Delta\widetilde{\mathcal{Y}} = \varnothing$, $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ has the same law as $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$. In case $1-\chi$ being the indicator function of a bounded subinterval of $\mathbb{R}$, this gives the law of $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ conditioned on $\widetilde{\mathcal{Y}}_\infty$ not charging this subinterval.

LEMMA 3.2.3. *$\Delta\widetilde{\mathcal{Y}}$ is a.s. finite. Let*

$$v_{\kappa,\tilde{\kappa}}(y) := \Big(\tilde{u}_\uparrow(y) - \int_{y_{-1}<y} \tilde{u}_\uparrow(y_{-1})(u_\downarrow(y_{-1})u_\uparrow(y) - u_\uparrow(y_{-1})u_\downarrow(y))(\tilde{\kappa}-\kappa)(dy_{-1})\Big) \times \Big(\tilde{u}_\downarrow(y) - \int_{y_1>y} \tilde{u}_\downarrow(y_1)(u_\uparrow(y_1)u_\downarrow(y) - u_\downarrow(y_1)u_\uparrow(y))(\tilde{\kappa}-\kappa)(dy_1)\Big)$$

Then

$$\mathbb{P}\big(\sharp\Delta\widetilde{\mathcal{Y}} = 1\big) = \int_{\mathbb{R}} v_{\kappa,\tilde{\kappa}}(y)(\tilde{\kappa}-\kappa)(dy)$$

The distribution of the unique point in $\Delta\widetilde{\mathcal{Y}}$ conditional on $\sharp\Delta\widetilde{\mathcal{Y}} = 1$ is

$$\frac{v_{\kappa,\tilde{\kappa}}(y)(\tilde{\kappa}-\kappa)(dy)}{\mathbb{P}\big(\sharp\Delta\widetilde{\mathcal{Y}} = 1\big)}$$

Furthermore

$$\mathbb{P}\big(\sharp\Delta\widetilde{\mathcal{Y}} \geqslant 2\big) \leqslant \frac{1}{2}\Big(\int_{\mathbb{R}} \widetilde{G}(y,y)(\tilde{\kappa}(dy)-\kappa(dy))\Big)^2$$

and $\mathbb{P}(\Delta\widetilde{\mathcal{Y}} = \varnothing) > 0$.

PROOF. First let us check that $\int_{\mathbb{R}} \widetilde{G}(y,y)(\tilde{\kappa}(dy)-\kappa(dy)) < +\infty$. Since $\tilde{\kappa}-\kappa$ has a first moment, we need only to show that $\widetilde{G}(y,y)$ grows sub-linearly in the neighbourhood of $-\infty$ and $+\infty$. Let $a < b \in \mathbb{R}$ such that $\tilde{\kappa}((a,b)) > 0$. Let $\widetilde{G}_{a,b}$ be the Green's function of $\frac{1}{2}\frac{d^2}{dx^2} - 1_{(a,b)}\tilde{\kappa}$. Then $\widetilde{G}_{a,b}(y,y)$ is affine on $(-\infty,a)$ and on $(b,+\infty)$. Moreover $\widetilde{G}(y,y) \leqslant \widetilde{G}_{a,b}(y,y)$. Thus we get

$$\mathbb{E}\big[\sharp\Delta\widetilde{\mathcal{Y}}\big] = \int_{\mathbb{R}} \widetilde{G}(y,y)(\tilde{\kappa}(dy)-\kappa(dy)) < +\infty$$

In particular $\Delta\widetilde{\mathcal{Y}}$ is a.s. finite.

To bound $\mathbb{P}\big(\sharp\Delta\widetilde{\mathcal{Y}} \geqslant 2\big)$ we use the following:

$$\begin{aligned}\mathbb{P}\big(\sharp\Delta\widetilde{\mathcal{Y}} \geqslant 2\big) &\leqslant \frac{1}{2}\mathbb{E}\big[\sharp\Delta\widetilde{\mathcal{Y}}(\sharp\Delta\widetilde{\mathcal{Y}}-1)\big] \\ &= \frac{1}{2}\int_{\mathbb{R}^2} (\widetilde{G}(x,x)\widetilde{G}(y,y) - \widetilde{G}(x,y)^2)(\tilde{\kappa}(dx)-\kappa(dx))(\tilde{\kappa}(dy)-\kappa(dy)) \\ &\leqslant \frac{1}{2}\Big(\int_{\mathbb{R}} \widetilde{G}(y,y)(\tilde{\kappa}(dy)-\kappa(dy))\Big)^2\end{aligned}$$

The expression of $\mathbb{E}\big[\sharp\Delta\widetilde{\mathcal{Y}}(\sharp\Delta\widetilde{\mathcal{Y}}-1)\big]$ that we used is general for determinantal point processes.

Let's prove now that $\mathbb{P}(\Delta\widetilde{\mathcal{Y}} = \varnothing) > 0$. $\Delta\widetilde{\mathcal{Y}}$ is determinantal point process associated to a trace-class self-adjoint positive semi-definite contraction operator on $\mathbb{L}^2(d\tilde{\kappa}-d\kappa)$. $\mathbb{P}(\Delta\widetilde{\mathcal{Y}} =$

$\varnothing) > 0$ if and only if all the eigenvalues of the operator are strictly less then 1 (see theorem 4.5.3 in [**HKPV09**]). Let $f \in \mathbb{L}^2(\tilde{\kappa} - \kappa)$. Let

$$F(x) := \int_{\mathbb{R}} \widetilde{G}(x,y) f(y)(\tilde{\kappa}(dy) - \kappa(dy))$$

F is continuous, dominated by

$$\widetilde{G}(x,x)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} \widetilde{G}(y,y)(\tilde{\kappa}(dy) - \kappa(dy)) \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}} f(y)^2 (\tilde{\kappa}(dy) - \kappa(dy)) \Big)^{\frac{1}{2}}$$

and has left-side and right-side derivatives at every point. F satisfies the equation

$$-\frac{1}{2}\frac{d^2 F}{dx^2} + F\tilde{\kappa} = f(\tilde{\kappa} - \kappa)$$

Assume by absurd that $f = F$ $(\tilde{\kappa} - \kappa)$-almost everywhere. Then

$$\begin{aligned} \int_{\mathbb{R}} F(x)^2 (\tilde{\kappa}(dx) - \kappa(dx)) &= \int_{\mathbb{R}} f(x) F(x) (\tilde{\kappa}(dx) - \kappa(dx)) \\ &= \int_{\mathbb{R}} F(x)^2 \tilde{\kappa}(dx) + \frac{1}{2} \int_{\mathbb{R}} \frac{dF}{dx}(x)^2 dx \end{aligned}$$

Thus F is necessarily constant. But then this means that $(\tilde{\kappa} - \kappa)(\mathbb{R}) = \tilde{\kappa}(\mathbb{R})$, which is impossible because κ is non zero. Thus 1 is not an eigenvalue of the operator defining the determinantal process $\Delta\widetilde{\mathcal{Y}}$ and thus $\mathbb{P}(\Delta\widetilde{\mathcal{Y}} = \varnothing) > 0$.

As for $\widetilde{\mathcal{Y}}_\infty$, the spacing between consecutive points of $\Delta\widetilde{\mathcal{Y}}$ are independent. By construction $\Delta\widetilde{\mathcal{Y}} \subseteq Supp(\tilde{\kappa} - \kappa)$. Given $y_0 \in Supp(\tilde{\kappa} - \kappa)$, let

$$1_{y > y_0} f_{\Delta\widetilde{\mathcal{Y}}}(y_0, y)(\tilde{\kappa}(dy) - \kappa(dy))$$

be the distribution of the lowest point in $\Delta\widetilde{\mathcal{Y}} \cap (y_0, +\infty)$ conditional on $y_0 \in \Delta\widetilde{\mathcal{Y}}$. Since y_0 may be the maximum of $\Delta\widetilde{\mathcal{Y}}$, $f_{\Delta\widetilde{\mathcal{Y}}}(y_0, y)(\tilde{\kappa}(dy) - \kappa(dy)) < 1$. For y to be $\min \Delta\widetilde{\mathcal{Y}} \cap (y_0, +\infty)$, y must belong to $\widetilde{\mathcal{Y}}_\infty$, all points in $y' \in \widetilde{\mathcal{Y}}_\infty \cap (y_0, y)$ must be erased (probability $\chi(y')$ for each), and y must be kept (probability $1 - \chi(y)$). For $y' > y_0$, let $f_{\widetilde{G}}(y_0, y')$ be

$$f_{\widetilde{G}}(y_0, y') = 2(y' - y_0) \frac{\tilde{u}_\downarrow(y')}{\tilde{u}_\downarrow(y_0)}$$

$1_{y' > y_0} f_{\widetilde{G}}(y_0, y') \tilde{\kappa}(dy')$ is the distribution of $\min \widetilde{\mathcal{Y}}_\infty \cap (y_0, +\infty)$ conditional on $y_0 \in \widetilde{\mathcal{Y}}_\infty$ (proposition 3.1.11). $f_{\Delta\widetilde{\mathcal{Y}}}$ and $f_{\widetilde{G}}$ are related as follows:

$$\begin{aligned} & f_{\Delta\widetilde{\mathcal{Y}}}(y_0, y) \\ &= f_{\widetilde{G}}(y_0, y) + \sum_{j \geqslant 2} \int_{y_0 < \cdots < y_{j-1} < y} f_{\widetilde{G}}(y_0, y_1) \dots f_{\widetilde{G}}(y_{j-1}, y) \prod_{i=1}^{j-1} \chi(y_i) \kappa(dy_i) \\ &= \frac{\tilde{u}_\downarrow(y)}{\tilde{u}_\downarrow(y_0)} \Big(2(y - y_0) + \sum_{j \geqslant 2} 2^j \int_{y_0 < \cdots < y_{j-1} < y} (y_1 - y_0) \dots (y - y_{j-1}) \prod_{i=1}^{j-1} \kappa(dy_i) \Big) \end{aligned}$$

But

$$\begin{aligned}
&2(y-y_0) + \sum_{j\geqslant 2} 2^j \int_{y_0<\dots<y_{j-1}<y} (y_1-y_0)\dots(y-y_{j-1})\prod_{i=1}^{j-1}\kappa(dy_i)\\
&= \frac{u_\downarrow(y_0)}{u_\downarrow(y)}\Big(f_G(y_0,y) + \sum_{j\geqslant 2}\int_{y_0<\dots<y_{j-1}<y} f_G(y_0,y_1)\dots f_G(y_{j-1},y)\prod_{i=1}^{j-1}\kappa(dy_i)\Big)\\
&= \frac{u_\downarrow(y_0)}{u_\downarrow(y)}\Big(G(y,y) - \frac{G(y_0,y)^2}{G(y_0,y_0)}\Big) = u_\downarrow(y_0)u_\uparrow(y) - u_\uparrow(y_0)u_\downarrow(y)
\end{aligned}$$

(see section 3.1.3). It follows that

$$f_{\Delta\widetilde{\mathcal{Y}}}(y_0,y) = \frac{\tilde{u}_\downarrow(y)}{\tilde{u}_\downarrow(y_0)}(u_\downarrow(y_0)u_\uparrow(y) - u_\uparrow(y_0)u_\downarrow(y))$$

In particular, if $y_0 < y_1 < \dots < y_n \in \mathbb{R}$, the infinitesimal probability that $\Delta\widetilde{\mathcal{Y}}$ has a point at each of the locations y_i and no points in-between is

$$\begin{aligned}
&\widetilde{G}(y_0,y_0)f_{\Delta\widetilde{\mathcal{Y}}}(y_0,y_1)\dots f_{\Delta\widetilde{\mathcal{Y}}}(y_{n-1},y_n)\prod_{i=0}^{n}(\tilde{\kappa}(dy_i)-\kappa(dy_i))\\
&\qquad = \tilde{u}_\uparrow(y_0)\tilde{u}_\downarrow(y_n)\prod_{i=1}^{n}(u_\downarrow(y_i)u_\uparrow(y_{i-1}) - u_\uparrow(y_i)u_\downarrow(y_{i-1}))\prod_{i=0}^{n}(\tilde{k}(dy_i)-k(dy_i))
\end{aligned}$$

Thus the expression of $v_{\kappa,\tilde{\kappa}}(y)$ is a sieve identity obtained as follows: $v_{\kappa,\tilde{\kappa}}(y)(\tilde{\kappa}-\kappa)(dy)$ is the infinitesimal probability that $\Delta\widetilde{\mathcal{Y}}$ contains a point at y, from which we subtract the infinitesimal probabilities to have a point at y at another below respectively above, and to which we add the infinitesimal probability to have a point at y and points both below and above y. □

Next we deal with the law of $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ conditional on $\Delta\widetilde{\mathcal{Y}} = 0$. Let $y_0 \in Supp(\tilde{\kappa}-\kappa)$. First we will compute the probability that $\Delta\widetilde{\mathcal{Y}} \cap (y_0,+\infty) \neq \varnothing$ conditional on $y_0 \in \widetilde{\mathcal{Y}}_\infty$.

LEMMA 3.2.4. *There are positive constants c_1 and c_2 such that for all $x \in \mathbb{R}$*

$$\int_{y<x}(u_\downarrow(y)u_\uparrow(x) - u_\uparrow(y)u_\downarrow(x))\tilde{u}_\uparrow(y)(\tilde{\kappa}(dy)-\kappa(dy)) = \tilde{u}_\uparrow(x) - c_1u_\uparrow(x) \tag{3.2.4}$$

$$\int_{y>x}(u_\uparrow(y)u_\downarrow(x) - u_\downarrow(y)u_\uparrow(x))\tilde{u}_\downarrow(y)(\tilde{\kappa}(dy)-\kappa(dy)) = \tilde{u}_\downarrow(x) - c_2u_\downarrow(x) \tag{3.2.5}$$

In particular

$$v_{\kappa,\tilde{\kappa}}(y) = c_1c_2u_\uparrow(y)u_\downarrow(y)$$

PROOF. We will prove (3.2.5). The proof of (3.2.4) is similar. Let f be the function

$$f(x) := \tilde{u}_\downarrow(x) - \int_{y>x}(u_\uparrow(y)u_\downarrow(x) - u_\downarrow(y)u_\uparrow(x))\tilde{u}_\downarrow(y)(\tilde{\kappa}(dy)-\kappa(dy))$$

The derivative of f, defined everywhere except at most countably many points, is

$$\frac{df}{dx}(x) = \frac{d\tilde{u}_\downarrow}{dx}(x) - \int_{y>x}\Big(u_\uparrow(y)\frac{du_\downarrow}{dx}(x) - u_\downarrow(y)\frac{du_\uparrow}{dx}(x)\Big)\tilde{u}_\downarrow(y)(\tilde{\kappa}(dy)-\kappa(dy))$$

The weak second derivative of f is:

$$\begin{aligned}\frac{d^2 f}{dx^2}(x) =& \frac{d^2\tilde{u}_\downarrow}{dx^2}(x) - \int_{y>x} \Big(u_\uparrow(y)\frac{d^2 u_\downarrow}{dx^2}(x) - u_\downarrow(y)\frac{d^2 u_\uparrow}{dx^2}(x)\Big)\tilde{u}_\downarrow(y)(\tilde{\kappa}(dy) - \kappa(dy))\\ &+ \Big(u_\uparrow(x)\frac{du_\downarrow}{dx}(x) - u_\downarrow(x)\frac{du_\uparrow}{dx}(x)\Big)\tilde{u}_\downarrow(x)(\tilde{\kappa}(dx) - \kappa(dx))\\ =&2\tilde{u}_\downarrow(x)\tilde{\kappa}(dx)\\ &- \int_{y>x}(u_\uparrow(y)u_\downarrow(x) - u_\downarrow(y)u_\uparrow(x))\tilde{u}_\downarrow(y)(\tilde{\kappa}(dy) - \kappa(dy)) \times \kappa(dx)\\ &+ 2\tilde{u}_\downarrow(x)(\tilde{k}(dx) - k(dx))\\ =&2\tilde{u}_\downarrow(x)\kappa(dx)\\ &- \int_{y>x}(u_\uparrow(y)u_\downarrow(x) - u_\downarrow(y)u_\uparrow(x))\tilde{u}_\downarrow(y)(\tilde{\kappa}(dy) - \kappa(dy)) \times \kappa(dx)\\ =&2f(x)\kappa(dx)\end{aligned}$$

Thus f satisfies the same differential equation as $u_\downarrow$. Moreover $|f|$ is dominated by

$$\tilde{u}_\downarrow(x) + u_\downarrow(x)\int_{y>x} G(y,y)(\tilde{\kappa}(dy) - \kappa(dy))$$

Thus f is bounded on the intervals of the type $(a, +\infty)$. It follows that there is a constant $c_2 \in \mathbb{R}$ such that $f \equiv c_2 u_\downarrow$. Thus we get the identity (3.2.5). Let's show that $c_2 > 0$. Let $x \in Supp(\tilde{\kappa})$. Then

$$\begin{aligned}1 - \frac{1}{\tilde{u}_\downarrow(x)}&\int_{y>x}(u_\uparrow(y)u_\downarrow(x) - u_\downarrow(y)u_\uparrow(x))\tilde{u}_\downarrow(y)(\tilde{\kappa}(dy) - \kappa(dy))\\ &= 1 - \int_{y>x} f_{\Delta\widetilde{\mathcal{Y}}}(x,y)(\tilde{\kappa}(dy) - \kappa(dy)) = \mathbb{P}\big(\Delta\widetilde{Y} \cap (x, +\infty) = \varnothing | x \in \widetilde{Y}_\infty\big)\end{aligned}$$

The above conditional probability is positive because according to the lemma 3.2.3, $\mathbb{P}\big(\Delta\widetilde{Y} = \varnothing\big) > 0$. Thus f is positive and $c_2 > 0$. ☐

LEMMA 3.2.5. *Conditional on the event $\Delta\widetilde{\mathcal{Y}} = \varnothing$, $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ has the same law as $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$.*

PROOF. It is enough to show that conditional on $\Delta\widetilde{\mathcal{Y}} = \varnothing$, $\widetilde{\mathcal{Y}}_\infty$ has the same law as $\mathcal{Y}_\infty$. Indeed in both cases the points of $\widetilde{\mathcal{Z}}_\infty$ respectively $\mathcal{Z}_\infty$ are distributed independently and uniformly between any two consecutive points of $\widetilde{\mathcal{Y}}_\infty$ respectively $\mathcal{Y}_\infty$. For $n \geqslant 1$ and $y_1 < \cdots < y_n$, let $\rho_n(dy_1, \dots dy_n)$ be the infinitesimal probability for $\widetilde{\mathcal{Y}}_\infty$ having a point at each of the locations y_i and none in-between, conditional on $\Delta\widetilde{\mathcal{Y}} = \varnothing$. We need only to show that

$$\rho_n(dy_1, \dots dy_n) = 2^{n-1}u_\uparrow(y_1)u_\downarrow(y_n)\prod_{i=2}^{n}(y_i - y_{i-1})\prod_{i=1}^{n}\kappa(dy_i) \tag{3.2.6}$$

For $y_1 < \cdots < y_n$ to be n consecutive points in $\widetilde{\mathcal{Y}}_\infty$ and for $\Delta\widetilde{\mathcal{Y}} = \varnothing$, we need $y_1 < \cdots < y_n$ to be n consecutive points in $\widetilde{\mathcal{Y}}_\infty$, to choose not to erase any of y_i (probability $\chi(y_i)$) and finally we need that $\Delta\widetilde{\mathcal{Y}} \cap (-\infty, y_1) = \varnothing$ and $\Delta\widetilde{\mathcal{Y}} \cap (y_n, +\infty) = \varnothing$. Thus

$$\begin{aligned}\rho_n(dy_1,\ldots,dy_n) = {} & \frac{1}{\mathbb{P}(\Delta\widetilde{\mathcal{Y}}=\varnothing)} 2^{n-1}\tilde{u}_\uparrow(y_1)\tilde{u}_\downarrow(y_n)\\ & \times \Big(1 - \frac{1}{\tilde{u}_\uparrow(y_1)}\int_{y<y_1}(u_\downarrow(y)u_\uparrow(y_1) - u_\uparrow(y)u_\downarrow(y_1))\tilde{u}_\uparrow(y)(\tilde{\kappa}(dy)-\kappa(dy))\Big)\\ & \times \Big(1 - \frac{1}{\tilde{u}_\downarrow(y_n)}\int_{y>y_n}(u_\uparrow(y)u_\downarrow(y_n) - u_\downarrow(y)u_\uparrow(y_n))\tilde{u}_\downarrow(y)(\tilde{\kappa}(dy)-\kappa(dy))\Big)\\ & \times \prod_{i=2}^{n}(y_i - y_{i-1})\prod_{i=1}^{n}\chi(y_i)\tilde{\kappa}(dy_i)\end{aligned}$$

Applying lemma 3.2.4 we get that

$$\rho_n(dy_1,\ldots,dy_n) = \frac{c_1c_2}{\mathbb{P}(\Delta\widetilde{\mathcal{Y}}=\varnothing)}2^{n-1}u_\uparrow(y_1)u_\downarrow(y_n)\prod_{i=2}^{n}(y_i-y_{i-1})\prod_{i=1}^{n}\kappa(dy_i)$$

Since the constant $\frac{c_1c_2}{\mathbb{P}(\Delta\widetilde{\mathcal{Y}}=\varnothing)}$ does not depend on n, the previous equations implies that

$$\mathbb{P}(\widetilde{\mathcal{Y}}_\infty \neq \varnothing | \Delta\widetilde{\mathcal{Y}} = \varnothing) = \frac{c_1c_2}{\mathbb{P}(\Delta\widetilde{\mathcal{Y}}=\varnothing)}\mathbb{P}(\mathcal{Y}_\infty \neq \varnothing)$$

But $\mathbb{P}(\widetilde{\mathcal{Y}}_\infty \neq \varnothing | \Delta\widetilde{\mathcal{Y}} = \varnothing) = \mathbb{P}(\mathcal{Y}_\infty \neq \varnothing) = 1$. Thus

$$\frac{c_1c_2}{\mathbb{P}(\Delta\widetilde{\mathcal{Y}}=\varnothing)} = 1$$

and 3.2.6 holds. □

COROLLARY 3.2.6. *Let $a < b \in \mathbb{R}$ such that $\tilde{\kappa}(\mathbb{R}\backslash[a,b]) > 0$. Conditional on $\widetilde{\mathcal{Y}}_\infty \cap [a,b] = \varnothing$, $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ has the same law as the pair of interwoven determinantal point processes obtained from the Wilson's algorithm applied to the Brownian motion with killing measure $1_{\mathbb{R}\backslash[a,b]}\kappa$.*

LEMMA 3.2.7. *Conditional on $\sharp\Delta\widetilde{\mathcal{Y}} = 1$ and on the position of the unique point Y in $\Delta\widetilde{\mathcal{Y}}$, $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ has the same law as $(\mathcal{Y}_\infty^{(Y)}, \mathcal{Z}_\infty^{(Y)})$.*

PROOF. It is enough to show that conditional on $\sharp\Delta\widetilde{\mathcal{Y}} = 1$ and on the position of the unique point Y in $\Delta\widetilde{\mathcal{Y}}$, $\widetilde{\mathcal{Y}}_\infty$ has the same law as $\mathcal{Y}_\infty^{(Y)}$. Indeed the points of $\widetilde{\mathcal{Z}}_\infty$ respectively $\mathcal{Z}_\infty^{(Y)}$ are independently and uniformly distributed between any two consecutive points in $\widetilde{\mathcal{Y}}_\infty$ respectively $\mathcal{Z}_\infty^{(Y)}$.

Let $n \geqslant 1$ and $i_0 \in \{1,\ldots,n\}$. Let $y_1 < \cdots < y_n \in \mathbb{R}$. The infinitesimal probability for $y_1,\ldots,y_n$ being n consecutive points in $\widetilde{\mathcal{Y}}_\infty$ and $\Delta\widetilde{\mathcal{Y}} = \{y_{i_0}\}$ is

$$\begin{aligned}& 2^{n-1}\tilde{u}_\uparrow(y_1)\tilde{u}_\downarrow(y_n)\\ & \qquad \times \Big(1 - \frac{1}{\tilde{u}_\uparrow(y_1)}\int_{y<y_1}(u_\downarrow(y)u_\uparrow(y_1) - u_\uparrow(y)u_\downarrow(y_1))\tilde{u}_\uparrow(y)(\tilde{\kappa}(dy)-\kappa(dy))\Big)\\ & \qquad \times \Big(1 - \frac{1}{\tilde{u}_\downarrow(y_n)}\int_{y>y_n}(u_\uparrow(y)u_\downarrow(y_n) - u_\downarrow(y)u_\uparrow(y_n))\tilde{u}_\downarrow(y)(\tilde{\kappa}(dy)-\kappa(dy))\Big)\\ & \qquad \times \prod_{i=2}^{n}(y_i - y_{i-1})\prod_{i\neq i_0}\kappa(dy_i) \times (\tilde{\kappa}-\kappa)(dy_{i_0})\\ & = c_1c_2 2^{n-1}u_\uparrow(y_1)u_\downarrow(y_n)\prod_{i=2}^{n}(y_i-y_{i-1})\prod_{i\neq i_0}\kappa(dy_i)\times(\tilde{\kappa}-\kappa)(dy_{i_0})\end{aligned}$$

$$\begin{aligned}&= v_{\kappa,\tilde{\kappa}}(y_{i_0})(\tilde{\kappa}-\kappa)(dy_{i_0}) \times 2^{i_0-1}\frac{u_\uparrow(y_1)}{u_\uparrow(y_{i_0})}\prod_{i=1}^{i_0-1}(y_{i+1}-y_i)\kappa(dy_i)\\ &\qquad \times 2^{n-i_0}\frac{u_\downarrow(y_n)}{u_\downarrow(y_{i_0})}\prod_{i=i_0+1}^{n}(y_i-y_{i-1})\kappa(dy_i)\end{aligned} \tag{3.2.7}$$

In 3.2.7 appears the infinitesimal probability for $\Delta\widetilde{\mathcal{Y}} = \{y_{i_0}\}$ times the infinitesimal probability for $y_1, \dots, y_n$ being n consecutive points in $\mathcal{Y}_\infty^{(y_0)}$ (compare with expressions 3.2.1 and 3.2.2). □

3.2.2. Couplings. In this section we will prove the monotone coupling results for $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ stated at the begining of section 3.2.1. The construction of the coupling will be explicit. However it will not appeal to Wilson's algorithm used to define $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$. First we will describe analogous monotone coupling results for Uniform Spanning Trees on finite graphs. In this case no explicit construction is known in general and the proof relies on Strassen's theorem and the conditions for stochastic domination between determinantal processes shown in [**Lyo03**].

PROPOSITION 3.2.8. *Let $\mathcal{G}$ be a finite connected undirected graph with E its set of edges, and $(C(e))_{e\in E}$ a positive weight function on E. Let F be a subset of E. Let $(\widetilde{C}(e))_{e\in E}$ be an other weight function such $\widetilde{C} \geqslant C$ and $\widetilde{C} = C$ on $E\backslash F$. Let Υ be the Uniform Spanning Tree of $\mathcal{G}$ corresponding to the weights C and $\widetilde{\Upsilon}$ the Uniform Spanning Tree of $\mathcal{G}$ corresponding to the weights $\widetilde{C}$. There is a coupling of Υ and $\widetilde{\Upsilon}$ such that*

$$\widetilde{\Upsilon} \cap (E\backslash F) \subseteq \Upsilon \cap (E\backslash F) \tag{3.2.8}$$

In case F is made of all edges adjacent to a particular vertex x_0, and $\widetilde{C}$ is proportional to C on F, then there is a coupling satisfying the additional condition

$$\Upsilon \cap F \subseteq \widetilde{\Upsilon} \cap F \tag{3.2.9}$$

PROOF. It is enough to prove the first coupling ((3.2.8)) in case F is a single edge ($F = \{e\}$). Then by induction on $\sharp F$ the general result will follow. From definition of Uniform Spanning Trees is clear that $\mathbb{P}(e \in \Upsilon) \leqslant \mathbb{P}(e \in \widetilde{\Upsilon})$. Moreover, Υ conditional on $e \in \Upsilon$ respectively $e \notin \Upsilon$ has the same law as $\widetilde{\Upsilon}$ conditional on $e \in \widetilde{\Upsilon}$ respectively $e \notin \widetilde{\Upsilon}$. A possible coupling is the following: first we couple $1_{e\in\Upsilon}$ with $1_{e\in\widetilde{\Upsilon}}$ in a way such that $1_{e\in\Upsilon} \leqslant 1_{e\in\widetilde{\Upsilon}}$. In case $1_{e\in\Upsilon} = 1_{e\in\widetilde{\Upsilon}} = 0$ respectively $1_{e\in\Upsilon} = 1_{e\in\widetilde{\Upsilon}} = 1$ we sample for both Υ and $\widetilde{\Upsilon}$ the same tree having the law of Υ conditioned by $e \notin \Upsilon$ respectively $e \in \Upsilon$. In case $1_{e\in\Upsilon} = 0$ and $1_{e\in\widetilde{\Upsilon}} = 1$, we use the fact that on the edges in $E\backslash\{e\}$, the law of Υ conditioned by $e \in \Upsilon$ is stochastically dominated by the law of Υ conditioned by $e \notin \Upsilon$, which implies the existence of a monotone coupling by Strassen's theorem. See theorems 5.2, 5.3 and 5.5 in [**Lyo03**].

Now we consider the case of F made of all edges adjacent to a particular vertex x_0, and $\widetilde{C}$ is proportional to C on F. Let $(\Upsilon, \widetilde{\Upsilon})$ be a coupling satisfying (3.2.8). In general it does not satisfy (3.2.9). To deal with this issue we will re-sample the edges of Υ and $\widetilde{\Upsilon}$ contained in F, that is to say sample Υ' having the same law as Υ, $\widetilde{\Upsilon}'$ having the same law as $\widetilde{\Upsilon}$, such that $\Upsilon' \cap (E\backslash F) = \Upsilon \cap (E\backslash F)$, $\widetilde{\Upsilon}' \cap (E\backslash F) = \widetilde{\Upsilon} \cap (E\backslash F)$ and such that $\Upsilon' \cap F \subseteq \widetilde{\Upsilon}' \cap F$. Let $\mathcal{T}_1, \dots, \mathcal{T}_N$ be the connected components of $\Upsilon \cap (E\backslash F)$. (3.2.8) ensures that each connected component of $\widetilde{\Upsilon}' \cap (E\backslash F)$ is contained in one of the $\mathcal{T}_i$. Let $\mathcal{T}_{1,1}, \dots, \mathcal{T}_{1,q_1}, \dots, \mathcal{T}_{N,1}, \dots, \mathcal{T}_{N,q_N}$ be the connected components of $\widetilde{\Upsilon}' \cap (E\backslash F)$, where $\mathcal{T}_{i,j} \subseteq \mathcal{T}_i$. Conditional on $\mathcal{T}_1, \dots, \mathcal{T}_N$, $\Upsilon \cap F$ has the following law: for each $\mathcal{T}_i$ one chooses an edge connecting x_0 to $\mathcal{T}_i$ with probability proportional to C, and independently from the edges of Υ that will connect x_0 to other $(\mathcal{T}_{i'})_{i'\neq i}$. Similarly for the law of $\widetilde{\Upsilon}$ conditional on $\mathcal{T}_{1,1}, \dots, \mathcal{T}_{1,q_1}, \dots, \mathcal{T}_{N,1}, \dots, \mathcal{T}_{N,q_N}$. To construct Υ' and $\widetilde{\Upsilon}'$ we use the fact that $\widetilde{C}$ is proportional to C on F:

- We start with Υ and $\widetilde{\Upsilon}$ satisfying (3.2.8).
- Then we remove from Υ and $\widetilde{\Upsilon}$ the edges contained in F.
- For each $\mathcal{T}_{i,j}$, we add to $\widetilde{\Upsilon}'$ an edge connecting x_0 to $\mathcal{T}_{i,j}$, chosen proportionally to its weight under C, each choice being independent from the others.
- For each $i \in \{1, \dots, N\}$, there are q_i edges in $\widetilde{\Upsilon}'$ connecting x_0 to $\mathcal{T}_i$, one for each $(\mathcal{T}_{i,j})_{1\leqslant j\leqslant q_i}$. In order to construct Υ', we need to chose one out of q_i to keep and remove the others. We chose to keep the edge corresponding to $\mathcal{T}_{i,j}$ with probability proportional to:
$$\sum_{\substack{e \text{ connecting} \\ x_0 \text{ to } \mathcal{T}_{i,j}}} C(e)$$
The choice is done independently for each $i \in \{1, \dots, N\}$.

By construction $\Upsilon' \cap F \subseteq \widetilde{\Upsilon}' \cap F$. □

Consider now two different killing measures κ and $\tilde{\kappa}$ on $\mathbb{R}$, with $\kappa \leqslant \tilde{\kappa}$, and the couples of determinantal point processes $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ respectively $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ corresponding to the Brownian motion on $\mathbb{R}$ with killing measure κ respectively $\tilde{\kappa}$. We want to show that one can couple $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ and $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ on the same probability space such that $\mathcal{Z}_\infty \subseteq \widetilde{\mathcal{Z}}_\infty$ and $\widetilde{\mathcal{Y}}_\infty \subseteq \mathcal{Y}_\infty \cup Supp(\tilde{\kappa} - \kappa)$, and if κ and $\tilde{\kappa}$ are proportional also have $\mathcal{Y}_\infty \subseteq \widetilde{\mathcal{Y}}_\infty$. The condition $\mathcal{Z}_\infty \subseteq \widetilde{\mathcal{Z}}_\infty$ and $\widetilde{\mathcal{Y}}_\infty \subseteq \mathcal{Y}_\infty \cup Supp(\tilde{\kappa} - \kappa)$ is analogous to (3.2.8). The condition $\mathcal{Y}_\infty \subseteq \widetilde{\mathcal{Y}}_\infty$ is analogous to (3.2.9), where the cemetery † plays the role of the distinguished vertex x_0. We used the stochastic domination principle ([**Lyo03**]) for determinantal point process with determinantal kernel a projection operator. It ensures the existence of a monotone coupling but does not give one explicitly (see open questions [**Lyo03**]). However for $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ and $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ we will construct a whole family of rather explicit monotone couplings.

Let $\widetilde{G}$ be the Green's function of $\frac{1}{2}\frac{d^2}{dx^2} - \tilde{\kappa}$, factorized as
$$\widetilde{G}(x, y) = \tilde{u}_\uparrow(x \wedge y)\tilde{u}_\downarrow(x \vee y)$$
Let
$$\widetilde{\mathcal{K}}(y, z) := -\frac{1}{2}\frac{d\tilde{u}_\uparrow}{dx}((y \wedge z)^+)\frac{d\tilde{u}_\downarrow}{dx}((y \vee z)^-)$$
Let $\mathfrak{G}_{\tilde{\kappa}}$ be the operator on $\mathbb{L}^2(d\tilde{\kappa})$ defined on functions with compact support as follows:
$$(\mathfrak{G}_{\tilde{\kappa}} f)(x) := \int_{\mathbb{R}} \widetilde{G}(x, y) f(y) \tilde{\kappa}(dy)$$
In case $\tilde{\kappa} = c\kappa$ where c is a constant, $c > 1$, we have the following resolvent identity, which follows from lemma 2.2.8:
$$\frac{1}{c}\mathfrak{G}_{ck}\mathfrak{G}_\kappa = \frac{1}{c}\mathfrak{G}_\kappa\mathfrak{G}_{c\kappa} = \frac{1}{c-1}\Big(\mathfrak{G}_\kappa - \frac{1}{c}\mathfrak{G}_{c\kappa}\Big) \tag{3.2.10}$$

Next we prove that a simple necessary but not sufficient condition for monotone couplings to exist is satisfied. It won't be used in the sequel but we prefer to give a direct proof for it.

PROPOSITION 3.2.9. *For any $z_1, \dots, z_n \in \mathbb{R}$ such that $\tilde{\kappa}(\{z_i\}) = 0$*
$$\det(\widetilde{\mathcal{K}}(z_i, z_j))_{1\leqslant i,j\leqslant n} \geqslant \det(\mathcal{K}(z_i, z_j))_{1\leqslant i,j\leqslant n} \tag{3.2.11}$$
If $\tilde{\kappa} = c\kappa$, $c > 1$, then for any $y_1, \dots, y_n \in Supp(\kappa)$
$$c^n \det(\widetilde{G}(y_i, y_j))_{1\leqslant i,j\leqslant n} \geqslant \det(G(y_i, y_j))_{1\leqslant i,j\leqslant n} \tag{3.2.12}$$

PROOF. We will first show (3.2.11). To begin with we will show that for any $z_1 \in \mathbb{R}$, $\widetilde{\mathcal{K}}(z_1, z_1) \geqslant \mathcal{K}(z_1, z_1)$. The Wronskian

$$W(u_\uparrow, \tilde{u}_\uparrow)(z) := u_\uparrow(z)\frac{d\tilde{u}_\uparrow}{dx}(z^+) - \tilde{u}_\uparrow(z)\frac{du_\uparrow}{dx}(z^+)$$

is non-negative. Indeed $W(u_\uparrow, \tilde{u}_\uparrow)(-\infty) = 0$ and

$$dW(u_\uparrow, \tilde{u}_\uparrow) = 2u_\uparrow\tilde{u}_\uparrow(d\tilde{\kappa} - d\kappa) \geqslant 0$$

Similarly the Wronskian

$$W(u_\downarrow, \tilde{u}_\downarrow)(z) := u_\downarrow(z)\frac{d\tilde{u}_\downarrow}{dx}(z^+) - \tilde{u}_\downarrow(z)\frac{du_\downarrow}{dx}(z^+)$$

is non-positive. Using the fact that

$$W(u_\downarrow, u_\uparrow) = W(\tilde{u}_\downarrow, \tilde{u}_\uparrow) \equiv 2$$

we get

$$\begin{aligned}\widetilde{\mathcal{K}}(z_1, z_1) - \mathcal{K}(z_1, z_1) =& \frac{1}{2}\Big(\frac{du_\uparrow}{dx}(z_1^+)\frac{du_\downarrow}{dx}(z_1^+) - \frac{d\tilde{u}_\uparrow}{dx}(z_1^+)\frac{d\tilde{u}_\downarrow}{dx}(z_1^+)\Big)\\ =&\frac{1}{4}\Big(\frac{du_\uparrow}{dx}(z_1^+)\frac{du_\downarrow}{dx}(z_1^+)W(\tilde{u}_\downarrow, \tilde{u}_\uparrow) - \frac{d\tilde{u}_\uparrow}{dx}(z_1^+)\frac{d\tilde{u}_\downarrow}{dx}(z_1^+)W(u_\downarrow, u_\uparrow)\Big)\\ =&\frac{1}{4}\Big(\frac{du_\downarrow}{dx}(z_1^+)\frac{d\tilde{u}_\downarrow}{dx}(z_1^+)W(u_\uparrow, \tilde{u}_\uparrow)(z_1)\\ &- \frac{du_\uparrow}{dx}(z_1^+)\frac{d\tilde{u}_\uparrow}{dx}(z_1^+)W(u_\downarrow, \tilde{u}_\downarrow)(z_1)\Big) \geqslant 0\end{aligned}$$

To prove (3.2.11) in general, we will use the factorization (3.1.34). For $x_0 < z$, let

$$\tilde{u}_\uparrow^{(x_0\rhd)}(z) := \tilde{u}_\uparrow(z) + \Big(\frac{d\tilde{u}_\downarrow}{dx}(x_0^-)\Big)^{-1}\frac{d\tilde{u}_\uparrow}{dx}(x_0^-)\tilde{u}_\downarrow(z)$$

Factorization (3.1.34) ensures that we only need to prove that for $x_0 < z$ with $\kappa(\{x_0\}) = 0$:

$$-\frac{d\tilde{u}_\uparrow^{(x_0\rhd)}}{dx}(z^+)\frac{d\tilde{u}_\downarrow}{dx}(z) \geqslant -\frac{du_\uparrow^{(x_0\rhd)}}{dx}(z^+)\frac{du_\downarrow}{dx}(z)$$

First observe that the Wronskian

$$W(u_\uparrow^{(x_0\rhd)}, \tilde{u}_\uparrow^{(x_0\rhd)})(z) := u_\uparrow^{(x_0\rhd)}(z)\frac{d\tilde{u}_\uparrow^{(x_0\rhd)}}{dx}(z^+) - \tilde{u}_\uparrow^{(x_0\rhd)}(z)\frac{du_\uparrow^{(x_0\rhd)}}{dx}(z^+)$$

is non-negative on $[x_0, +\infty)$. Indeed $W(u_\uparrow^{(x_0\rhd)}, \tilde{u}_\uparrow^{(x_0\rhd)})(x) = 0$ and

$$dW(u_\uparrow^{(x_0\rhd)}, \tilde{u}_\uparrow^{(x_0\rhd)}) = 2u_\uparrow^{(x_0\rhd)}(z)\tilde{u}_\uparrow^{(x_0\rhd)}(z)(d\tilde{\kappa} - d\kappa) \geqslant 0$$

The sequel of the proof works as in the previous case.

Let's prove now (3.2.12). First we consider the case $n = 1$. From the resolvent identity (3.2.10) follows that

$$\mathfrak{G}_{c\kappa} - \mathfrak{G}_\kappa = (c-1)(\mathfrak{G}_\kappa - \mathfrak{G}_{c\kappa}\mathfrak{G}_\kappa)$$

Since $\mathfrak{G}_{c\kappa}$ is contracting, this implies that $\mathfrak{G}_\kappa \leqslant \mathfrak{G}_{c\kappa}$, where the inequality stands for positive semi-definite operators on $\mathbb{L}^2(d\kappa)$. Let $y_1 \in Supp(\kappa)$. Then for any $\varepsilon > 0$

$$c\int_{(y_1-\varepsilon, y_1+\varepsilon)^2}\widetilde{G}(x,y)\kappa(dx)\kappa(dy) \geqslant \int_{(y_1-\varepsilon, y_1+\varepsilon)^2}G(x,y)\kappa(dx)\kappa(dy) \tag{3.2.13}$$

Since $y_1 \in Supp(\kappa)$, both sides of (3.2.13) are positive. The continuity of G and $\widetilde{G}$ ensures that $c\widetilde{G}(y_1, y_1) \geqslant G(y_1, y_1)$. In case of general n, we use the factorization (3.1.33). It is enough to prove that for any $x_0 < y$, $y \in Supp(\kappa)$

$$c\widetilde{G}^{(x_0\times)}(y,y) \geqslant G^{(x_0\times)}(y,y) \tag{3.2.14}$$

where

$$\widetilde{G}^{(x_0\times)}(y,y) := \widetilde{G}(y,y) - \frac{\widetilde{G}(x_0,y)^2}{\widetilde{G}(x_0,x_0)}$$

$\widetilde{G}$ is the restriction to $(x_0,+\infty)^2$ of the Green's function of $\frac{1}{2}\frac{d^2}{dx^2} - 1_{(x_0,+\infty)}\tilde{\kappa}$. Let $\mathfrak{G}_\kappa^{(x_0\times)}$ and $\mathfrak{G}_{c\kappa}^{(x_0\times)}$ be the operators on $\mathbb{L}^2(1_{(x_0,+\infty)}d\kappa)$ defined for functions f with compact support as

$$(\mathfrak{G}_\kappa^{(x_0\times)} f)(x) := \int_{(x_0,+\infty)} G^{(x_0\times)}(x,y) f(y)\kappa(dy)$$

$$(\mathfrak{G}_{c\kappa}^{(x_0\times)} f)(x) := c\int_{(x_0,+\infty)} \widetilde{G}^{(x_0\times)}(x,y) f(y)\kappa(dy)$$

$\mathfrak{G}_\kappa^{(x_0\times)}$ and $\mathfrak{G}_{c\kappa}^{(x_0\times)}$ are contractions and satisfy a resolvent identity similar to (3.2.10), which similarly implies (3.2.14). □

The resolvent identity (3.2.10) implies that $\mathfrak{G}_\kappa$ and $\mathfrak{G}_{c\kappa}$ commute and that $\mathfrak{G}_\kappa \leqslant \mathfrak{G}_{c\kappa}$. It was shown in case of determianatal point processes on discrete space that this a sufficient condition for a monotone coupling to exist. See theorem 7.1 in [**Lyo03**].

To construct the couplings we will give several procedures that take deterministic arguments, among which pairs of interwoven sets of points, and return pairs of interwoven random point processes. The first procedure we describe will be used as sub-procedure in subsequent procedures.

PROCEDURE 3.2.10. *Arguments:*

- *a pair $(\mathcal{Y}, \mathcal{Z})$ of disjoint discrete sets of points in $\mathbb{R}$ such that between any two points in $\mathcal{Y}$ lies a single point in $\mathcal{Z}$ and vice-versa, and such that* $\inf \mathcal{Y} \cup \mathcal{Z} \in \mathcal{Y} \cup \{-\infty\}$, $\sup \mathcal{Y} \cup \mathcal{Z} \in \mathcal{Y} \cup \{+\infty\}$
- *a positive Radon measure κ*
- *a point $y_0 \in \mathbb{R}$ such that $y_0 \notin \mathcal{Z}$*

Procedure:

(i) *If $y_0 \notin \mathcal{Y}$, we define a random variable Z distributed as follows:*

(i a) *If there are $y' \in \mathcal{Y}$, $z' \in \mathcal{Z} \cup \{+\infty\}$, such that $y' < z'$, $y_0 \in (y', z')$ and $\mathcal{Y} \cap (y', z') = \mathcal{Z}_\infty \cap (y', z') = \varnothing$ then Z is distributed according to*

$$\frac{1_{z\in(y',y_0)}}{u_\uparrow(y_0) - u_\uparrow(y')}\frac{du_\uparrow}{dx}(z)dz$$

(i b) *If there are $y' \in \mathcal{Y}$, $z' \in \mathcal{Z} \cup \{-\infty\}$, such that $z' < y'$, $y_0 \in (z', y')$ and $\mathcal{Y} \cap (z', y') = \mathcal{Z} \cap (z', y') = \varnothing$ then Z is distributed according to*

$$\frac{-1_{z\in(y_0,y')}}{u_\downarrow(y') - u_\downarrow(y_0)}\frac{du_\downarrow}{dx}(z)dz$$

(ii) *If there are $y' \in \mathcal{Y}$, $z' \in \mathcal{Z} \cup \{+\infty\}$, such that $y' < z'$, $y_0 \in (y', z')$ and $\mathcal{Y} \cap (y', z') = \mathcal{Z} \cap (y', z') = \varnothing$, then*

(ii a) *with probability $\dfrac{u_\uparrow(y')}{u_\uparrow(y_0)}$ we set*

$$(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}}) = (\mathcal{Y} \cup \{y_0\}\backslash\{y'\}, \mathcal{Z})$$

(ii b) *and with probability $1 - \dfrac{u_\uparrow(y')}{u_\uparrow(y_0)}$ we set*

$$(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}}) = (\mathcal{Y} \cup \{y_0\}, \mathcal{Z} \cup \{Z\})$$

(iii) *If there are $y' \in \mathcal{Y}$, $z' \in \mathcal{Z} \cup \{-\infty\}$, such that $z' < y'$, $y_0 \in (z', y')$ and $\mathcal{Y} \cap (z', y') = \mathcal{Z} \cap (z', y') = \varnothing$, then* (iii a) *with probability* $\dfrac{u_\downarrow(y')}{u_\downarrow(y_0)}$ *we set*

$$(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}}) = (\mathcal{Y} \cup \{y_0\} \backslash \{y'\}, \mathcal{Z})$$

(iii b) *and with probability* $1 - \dfrac{u_\downarrow(y')}{u_\downarrow(y_0)}$ *we set*

$$(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}}) = (\mathcal{Y} \cup \{y_0\}, \mathcal{Z} \cup \{Z\})$$

(iv) *If* $y_0 \in \mathcal{Y}$, *we set* $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}}) = (\mathcal{Y}, \mathcal{Z})$.

Return: $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}})$.

LEMMA 3.2.11. *If procedure 3.2.10 is applied to the pair of interwoven determinantal point processes* $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ *corresponding to the killing measure* κ, *then its result* $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}})$ *has the same law as* $(\mathcal{Y}_\infty^{(y_0)}, \mathcal{Z}_\infty^{(y_0)})$.

PROOF. By construction $y_0 \in \widetilde{\mathcal{Y}}$. Let $\widetilde{Z}_1 < \widetilde{Y}_1 < \cdots < \widetilde{Z}_n < \widetilde{Y}_n$ be the $2n$ closest points to y_0 in $(\widetilde{\mathcal{Y}} \cup \widetilde{\mathcal{Z}}) \cap (y_0, +\infty)$. On the event $\min(\mathcal{Y}_\infty \cup \mathcal{Z}_\infty) \cap (y_0, +\infty) \in \mathcal{Z}_\infty$ (point (ii) in procedure 3.2.10) their distribution is given by

$$1_{y_0 < z_1 < y_1 < \cdots < z_n < y_n} 2^n \Big(\int_{(-\infty, y_0)} u_\uparrow(y') \kappa(dy') \Big) u_\downarrow(y_n) dz_1 \kappa(dy_1) \dots dz_n \kappa(dy_n) \tag{3.2.15}$$

On the event $\min(\mathcal{Y}_\infty \cup \mathcal{Z}_\infty) \cap (y_0, +\infty) \in \mathcal{Y}_\infty$ (point (iii) in procedure 3.2.10), the distribution of $\min(\mathcal{Y}_\infty \cup \mathcal{Z}_\infty) \cap (y_0, +\infty)$ is (see proposition 3.1.11)

$$\begin{aligned}
1_{y' > y_0} 2 \Big(\int_{(-\infty, y_0)} & u_\uparrow(y_{-1})(y_0 - y_{-1}) \kappa(dy_{-1}) \Big) u_\downarrow(y') \kappa(dy') \\
&+ 1_{y' > y_0} \frac{u_\uparrow(+\infty)}{u_\uparrow(y')} G(y_0, y_0) \kappa(dy') \\
&= 1_{y' > y_0} (u_\uparrow(y_0) - u_\uparrow(+\infty)) u_\downarrow(y') \kappa(dy') + 1_{y' > y_0} u_\uparrow(+\infty) u_\downarrow(y') \kappa(dy') \\
&= 1_{y' > y_0} u_\uparrow(y_0) u_\downarrow(y') \kappa(dy')
\end{aligned}$$

Thus on the event $\min(\mathcal{Y}_\infty \cup \mathcal{Z}_\infty) \cap (y_0, +\infty) \in \mathcal{Y}_\infty$ (point (iii) in procedure 3.2.10), the distribution of $(Z_1, Y_1, \dots, Z_n, Y_n)$ is

$$1_{y_0 < z_1 < \cdots < y_n} \Big(\int_{y_0 < y' < z_1} \frac{u_\downarrow(y')}{u_\downarrow(y_0)} u_\uparrow(y_0) u_\downarrow(y') 2^n \frac{u_\downarrow(y_n)}{u_\downarrow(y')} \kappa(dy') \Big) dz_1 \kappa(dy_1) \dots dz_n \kappa(dy_n) \tag{3.2.16}$$

$$+ 1_{y_0 < z_1 < \cdots < y_n} \frac{-1}{u_\downarrow(y_0)} \frac{du_\downarrow}{dx}(z_1) u_\uparrow(y_0) u_\downarrow(y_1) 2^{n-1} \frac{u_\downarrow(y_n)}{u_\downarrow(y_1)} dz_1 \kappa(dy_1) \dots dz_n \kappa(dy_n) \tag{3.2.17}$$

The term (3.2.16) corresponds to the case when a point is removed from $\mathcal{Y}_\infty$ (case (iii a) in procedure 3.2.10) and (3.2.17) to the case when Z is added to $\mathcal{Z}_\infty$ (case (iii b) in procedure

3.2.10). The sum of the densities that appear in (3.2.15), (3.2.16) and (3.2.17) is

$$\begin{aligned}
&2^n\Big(\int_{(-\infty,y_0)} u_\uparrow(y')\kappa(dy')\Big)u_\downarrow(y_n) + \Big(\int_{y_0<y'<z_1} \frac{u_\downarrow(y')}{u_\downarrow(y_0)}u_\uparrow(y_0)u_\downarrow(y')2^n\frac{u_\downarrow(y_n)}{u_\downarrow(y')}\kappa(dy')\Big)\\
&\qquad + \frac{-1}{u_\downarrow(y_0)}\frac{du_\downarrow}{dx}(z_1)u_\uparrow(y_0)u_\downarrow(y_1)2^{n-1}\frac{u_\downarrow(y_n)}{u_\downarrow(y_1)}\\
&\quad =2^{n-1}\frac{du_\uparrow}{dx}(y_0)u_\downarrow(y_n) + \frac{2^{n-1}}{u_\downarrow(y_0)}\Big(\frac{du_\downarrow}{dx}(z_1) - \frac{du_\downarrow}{dx}(y_0^+)\Big)u_\uparrow(y_0)u_\downarrow(y_n)\\
&\qquad + \frac{-2^{n-1}}{u_\downarrow(y_0)}\frac{du_\downarrow}{dx}(z_1)u_\uparrow(y_0)u_\downarrow(y_n)\\
&\quad =2^{n-1}u_\downarrow(y_n)\Big(\frac{du_\uparrow}{dx}(y_0) + \frac{-1}{u_\downarrow(y_0)}\frac{du_\downarrow}{dx}(y_0^+)u_\uparrow(y_0)\Big) = 2^n\frac{u_\downarrow(y_n)}{u_\downarrow(y_0)}
\end{aligned}$$

So we obtain the density which appears in (3.2.1).

It remains to prove that $(\tilde{\mathcal{Y}}\cap(y_0,+\infty), \tilde{\mathcal{Z}}\cap(y_0,+\infty))$ and $(\tilde{\mathcal{Y}}\cap(-\infty,y_0), \tilde{\mathcal{Z}}\cap(-\infty,y_0))$ are independent. Let $Z_{-1} > Y_{-1} > \cdots > Z_{-n'} > Y_{-n'}$ be the n' closest points to y_0 in $(\tilde{\mathcal{Y}}\cup\tilde{\mathcal{Z}})\cap(-\infty,y_0)$. The distribution of the family of points $(Z_{-1}, Y_{-1}\dots, Z_{-n'}, Y_{-n'}, Z_1, Y_1,\dots, Z_n, Y_n)$ on the event $\sharp(\tilde{\mathcal{Y}}\cap(-\infty,y_0)) \geqslant n, \sharp(\tilde{\mathcal{Y}}\cap(y_0,+\infty)) \geqslant n'$ is

$$\Big(\int_{y_0<y'<z_1} 2^{n+n'}u_\uparrow(y_{-n'})u_\downarrow(y_n)\frac{u_\downarrow(y')}{u_\downarrow(y_0)}\kappa(dy') - \frac{2^{n+n'-1}u_\uparrow(y_{-n'})u_\downarrow(y_n)}{u_\downarrow(y_0)}\frac{du_\downarrow}{dx}(z_1) \tag{3.2.18}$$

$$+\int_{z_{-1}<y'<y_0} 2^{n+n'}u_\uparrow(y_{-n'})u_\downarrow(y_n)\frac{u_\uparrow(y')}{u_\uparrow(y_0)}\kappa(dy') + \frac{2^{n+n'-1}u_\uparrow(y_{-n'})u_\downarrow(y_n)}{u_\uparrow(y_0)}\frac{du_\uparrow}{dx}(z_{-1})\Big) \tag{3.2.19}$$

$$\times 1_{y_{-n'}<z_{-n'}<\cdots<z_{-1}<y_0<z_1<\cdots<z_n<y_n}\kappa(dy_{-n'})dz_{-n'}\dots dz_{-1}dz_1\dots dz_n\kappa(dy_n)$$

The term (3.2.18) corresponds to point (iii) in procedure 3.2.10 and (3.2.19) to point (ii) in procedure 3.2.10. One can check that the sum of the densities equals

$$2^{n+n'}\frac{u_\uparrow(y_{-n'})}{u_\uparrow(y_0)}\frac{u_\downarrow(y_n)}{u_\downarrow(y_0)}$$

Thus $(\tilde{\mathcal{Y}}\cap(y_0,+\infty), \tilde{\mathcal{Z}}\cap(y_0,+\infty))$ and $(\tilde{\mathcal{Y}}\cap(-\infty,y_0), \tilde{\mathcal{Z}}\cap(-\infty,y_0))$ are independent. □

LEMMA 3.2.12. *We consider the subspace of triples $((\mathcal{Y},\mathcal{Z}),\kappa,y_0)$ consisting of a pair of discrete sets of points $(\mathcal{Y},\mathcal{Z})$, a Radon measure κ and a point $y_0\in\mathbb{R}$, and which satisfies the restrictions on the arguments of procedure 3.2.10. We assume this subspace endowed with the product topology obtained from the topology of uniform convergence on compact subsets for the pairs $(\mathcal{Y},\mathcal{Z})$, the vague topology for the measures κ and standard order topology on $\mathbb{R}$. If $(\tilde{\mathcal{Y}},\tilde{\mathcal{Z}})$ is the pair of point processes obtained by applying procedure 3.2.10 to the arguments $((\mathcal{Y},\mathcal{Z}),\kappa,y_0)$, then its law depends continuously on $((\mathcal{Y},\mathcal{Z}),\kappa,y_0)$.*

PROOF. From lemma 2.2.4 it follows that the cumulative distribution function of Z (point (i) in procedure 3.2.10) depends uniformly continuously on $((\mathcal{Y},\mathcal{Z}),\kappa,y_0)$ in the neighbourhood of triples where $y_0\notin\mathcal{Y}$. Moreover the probabilities to make either the choice (ii a) or the choice (ii b), as well as to make either the choice (iii a) or the choice (iii b), depend continuously on $((\mathcal{Y},\mathcal{Z}),\kappa,y_0)$. Thus the law of $(\tilde{\mathcal{Y}},\tilde{\mathcal{Z}})$ depends continuously on $((\mathcal{Y},\mathcal{Z}),\kappa,y_0)$ in the neighbourhood of triples where $y_0\notin\mathcal{Y}$. Moreover in the neighbourhood of triples where $y_0\in\mathcal{Y}$, with high probability, converging to 1, $(\tilde{\mathcal{Y}},\tilde{\mathcal{Z}}) = (\mathcal{Y},\mathcal{Z})$. Thus the law of $(\tilde{\mathcal{Y}},\tilde{\mathcal{Z}})$ is continuous also at these triples. □

First we will describe a coupling in case when $\tilde{\kappa}$ and κ differ by an atom: $\tilde{\kappa} = \kappa + c\delta_{y_0}$. We construct the coupling as follows:

PROCEDURE 3.2.13. *Arguments:*

- *a pair* $(\mathcal{Y}, \mathcal{Z})$ *of disjoint discrete sets of points in* $\mathbb{R}$ *such that between any two points in* $\mathcal{Y}$ *lies a single point in* $\mathcal{Z}$ *and vice-versa, and such that* $\inf \mathcal{Y} \cup \mathcal{Z} \in \mathcal{Y} \cup \{-\infty\}$, $\sup \mathcal{Y} \cup \mathcal{Z} \in \mathcal{Y} \cup \{+\infty\}$
- *two positive Radon measures* κ *and* $\tilde{\kappa}$ *where* $\tilde{\kappa}$ *is of form* $\tilde{\kappa} = \kappa + c\delta_{y_0}$ *and* $y_0 \notin \mathcal{Z}$.

Procedure:

(i) *Let* β *be a Bernoulli r.v. of parameter* $c\widetilde{G}(y_0, y_0)$.
(ii) *If* $\beta = 0$ *we set* $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}}) = (\mathcal{Y}, \mathcal{Z})$.
(iii) *If* $\beta = 1$, *we apply the procedure 3.2.10 to the arguments* $(\mathcal{Y}, \mathcal{Z})$, κ *and* y_0 *and set* $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}})$ *to be its result.*

Return: $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}})$.

$(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}})$ constructed this way satisfies the following: between any two consecutive points in $\widetilde{\mathcal{Y}}$ lies a single point in $\widetilde{\mathcal{Z}}$ and between any two consecutive points in $\widetilde{\mathcal{Z}}$ lies a point in $\widetilde{\mathcal{Y}}$. By construction $\mathcal{Z} \subseteq \widetilde{\mathcal{Z}}$ and $\widetilde{\mathcal{Y}} \subseteq \mathcal{Y} \cup \{y_0\}$.

PROPOSITION 3.2.14. *If procedure 3.2.13 is applied to to the pair of interwoven determinantal point processes* $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ *corresponding to the measure* κ, *then the returned pair of point processes* $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}})$ *has the law of the interwoven determinantal point processes* $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ *corresponding to* $\tilde{\kappa} = \kappa + c\delta_{y_0}$.

PROOF. Observe that a.s. $y_0 \notin \mathcal{Z}_\infty$. First we deal with the case $\kappa(\{y_0\}) = 0$. Then almost surely $y_0 \notin \mathcal{Y}_\infty$ and $y_0 \in \widetilde{\mathcal{Y}}$ if and only if $\beta = 1$. But

$$\mathbb{P}(\beta = 1) = \mathbb{P}(y_0 \in \widetilde{\mathcal{Y}}) = c\widetilde{G}(y_0, y_0)$$

According to corollary 3.2.6, conditional on $y_0 \notin \widetilde{\mathcal{Y}}$, $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ has the same law as $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$, that is to say the same law as $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}})$ conditional on $\beta = 0$. According to lemma 3.2.11, conditional on $\beta = 1$, $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}})$ follows the same law as $(\widetilde{\mathcal{Y}} \cap (-\infty, y_0), \widetilde{\mathcal{Z}} \cap (-\infty, y_0))$, which is also the law of $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ conditioned on $y_0 \in \widetilde{\mathcal{Y}}_\infty$.

We deal now with the case $\kappa(\{y_0\}) > 0$.

$$\mathbb{P}(y_0 \in \widetilde{\mathcal{Y}}_\infty) = \tilde{\kappa}(\{y_0\})\widetilde{G}(y_0, y_0)$$

$$\begin{aligned}\mathbb{P}(y_0 \in \widetilde{\mathcal{Y}}) &= \mathbb{P}(\beta = 1) + \mathbb{P}(\beta = 0, y_0 \in \mathcal{Y}_\infty) \\ &= c\widetilde{G}(y_0, y_0) + (1 - c\widetilde{G}(y_0, y_0))\kappa(\{y_0\})G(y_0, y_0)\end{aligned}$$

But G and $\widetilde{G}$ satisfy the resolvent identity (see lemma 2.2.8):

$$\widetilde{G}(y_0, y_0)\kappa(\{y_0\})G(y_0, y_0) = \frac{\kappa(\{y_0\})}{\tilde{\kappa}(\{y_0\}) - \kappa(\{y_0\})}(G(y_0, y_0) - \widetilde{G}(y_0, y_0))$$

It follows that $\mathbb{P}(y_0 \in \widetilde{\mathcal{Y}}) = \mathbb{P}(y_0 \in \widetilde{\mathcal{Y}}_\infty)$. Let $\check{\kappa} := \kappa - \kappa(\{y_0\})\delta_{y_0}$ and $(\check{\mathcal{Y}}_\infty, \check{\mathcal{Z}}_\infty)$ be the interwoven determinantal point processes corresponding to $\check{\kappa}$.et $\tilde{\kappa}' := \tilde{\kappa} - \kappa(\{y_0\})\delta_{y_0}$ and $(\widetilde{\mathcal{Y}}'_\infty, \widetilde{\mathcal{Z}}'_\infty)$ be the interwoven determinantal processes corresponding to $\tilde{\kappa}'$. According to corollary 3.2.6, $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}})$ conditioned by $y_0 \notin \widetilde{\mathcal{Y}}$ has the same law as $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ conditioned by $y_0 \notin \mathcal{Y}_\infty$, which is the same law as $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ conditioned by $y_0 \notin \widetilde{\mathcal{Y}}_\infty$, and it is the law of $(\check{\mathcal{Y}}_\infty, \check{\mathcal{Z}}_\infty)$. For $y_0 \in \widetilde{\mathcal{Y}}$ there are two possibilities: either $y_0 \in \mathcal{Y}_\infty$ or $y_0 \notin \mathcal{Y}_\infty$ and $\beta = 1$. In the first case, it follows from proposition 3.2.1 that $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ conditioned on $y_0 \in \mathcal{Y}_\infty$ has the same law as $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ conditioned on $y_0 \in \widetilde{\mathcal{Y}}_\infty$. In the second case $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ conditioned on $y_0 \notin \mathcal{Y}_\infty$ has the same law as $(\check{\mathcal{Y}}_\infty, \check{\mathcal{Z}}_\infty)$. This bring us back to the situation $\kappa(\{y_0\}) = 0$. According to what was proved earlier, conditional on $y_0 \notin \mathcal{Y}_\infty$ and $\beta = 1$, $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}})$ has the same law as $(\widetilde{\mathcal{Y}}'_\infty, \widetilde{\mathcal{Z}}'_\infty)$ conditioned on $y_0 \in \widetilde{\mathcal{Y}}'_\infty$. But this is the same law as for $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ conditioned on $y_0 \in \widetilde{\mathcal{Y}}_\infty$. So again, $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}})$ has the same law as $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$. □

Next we consider the more general case where the measure $\tilde{\kappa} - \kappa$ has a first moment:

$$\int_{\mathbb{R}} |x|(\tilde{\kappa}(dx) - \kappa(dx)) < +\infty$$

First we describe a procedure that does not give a coupling between $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ and $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ but allows to approach it.

PROCEDURE 3.2.15. *Arguments:*

- *a pair $(\mathcal{Y}, \mathcal{Z})$ of disjoint discrete sets of points in $\mathbb{R}$ such that between any two points in $\mathcal{Y}$ lies a single point in $\mathcal{Z}$ and vice-versa, and such that* $\inf \mathcal{Y} \cup \mathcal{Z} \in \mathcal{Y} \cup \{-\infty\}$, $\sup \mathcal{Y} \cup \mathcal{Z} \in \mathcal{Y} \cup \{+\infty\}$
- *two positive Radon measures κ, $\tilde{\kappa}$ such that $\kappa \leqslant \tilde{\kappa}$ and $\int_{\mathbb{R}} |x| \,(\tilde{\kappa}(dx) - \kappa(dx)) < +\infty$ and $(\tilde{\kappa} - \kappa)(\mathcal{Z}) = 0$.*

Procedure:

(i) *Let β be a Bernoulli r.v. of parameter*

$$\int_{\mathbb{R}} v_{\kappa,\tilde{\kappa}}(y)(\tilde{\kappa} - \kappa)(dy)$$

(see notations of proposition 3.2.3)

(ii) *Let Y be a real r.v. independent from β distributed according to*

$$\frac{v_{\kappa,\tilde{\kappa}}(y)(\tilde{\kappa} - \kappa)(dy)}{\mathbb{P}(\beta = 1)}$$

(iii) *If $\beta = 0$ we set $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}}) = (\mathcal{Y}, \mathcal{Z})$.*

(iv) *If $\beta = 1$, we apply the procedure 3.2.10 to the arguments $(\mathcal{Y}, \mathcal{Z})$, κ and Y and set $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}})$ to be its result.*

Return: $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}})$.

Observe that in case $\tilde{\kappa}$ and κ differ only by an atom, procedure 3.2.15 is the same as procedure 3.2.13.

LEMMA 3.2.16. *Let $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ respectively $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ be the pair of interwoven determinantal point processes corresponding to the killing measure κ respectively $\tilde{\kappa}$. We assume that the procedure 3.2.15 is applied to $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ and that $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}})$ is the returned pair of point processes. Then the total variation distance between the law of $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}})$ and the law of $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ is less or equal to* $\left(\int_{\mathbb{R}} \widetilde{G}(y, y)(\tilde{\kappa}(dy) - \kappa(dy))\right)^2$.

PROOF. Let $\Delta\widetilde{\mathcal{Y}}$ be the determinantal point process defined in section 3.2.1 (see lemma 3.2.3). According to lemma 3.2.5, the law of $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}})$ conditional on $\beta = 0$ is the same as the law of $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ conditional on $\Delta\widetilde{\mathcal{Y}} = \varnothing$. From lemmas 3.2.11 and 3.2.7 follows that the law of $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}})$ conditional on $\beta = 1$ is the same as the law of $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ conditional on $\sharp\Delta\widetilde{\mathcal{Y}} = 1$. Moreover $\mathbb{P}(\beta = 1) = \mathbb{P}(\sharp\Delta\widetilde{\mathcal{Y}} = 1)$. However

$$\mathbb{P}(\beta = 0) = \mathbb{P}(\Delta\widetilde{\mathcal{Y}} = \varnothing) + \mathbb{P}(\sharp\Delta\widetilde{\mathcal{Y}} \geqslant 2) \geqslant \mathbb{P}(\Delta\widetilde{\mathcal{Y}} = \varnothing)$$

It follows that the total variation distance between the law of $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}})$ and the law of $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ is less or equal to $2\mathbb{P}(\sharp\Delta\widetilde{\mathcal{Y}} \geqslant 2)$, which according lemma 3.2.3 is less or equal to the square $\left(\int_{\mathbb{R}} \widetilde{G}(y, y)(\tilde{\kappa}(dy) - \kappa(dy))\right)^2$. □

COROLLARY 3.2.17. *Let $\kappa_0 \leqslant \kappa_1 \leqslant \cdots \leqslant \kappa_j$ be positive Radon measures such that $\int_{\mathbb{R}} |x|(\kappa_j(dx) - \kappa_0(dx)) < +\infty$. Let G_i be the Green's function of $\frac{1}{2}\frac{d^2}{dx^2} - \kappa_i$ and $(\mathcal{Y}_\infty^{(i)}, \mathcal{Z}_\infty^{(i)})$ the pair of interwoven determinantal point processes corresponding to κ_i. Let $((\mathcal{Y}^{(i)}, \mathcal{Z}^{(i)}))_{0 \leqslant i \leqslant j}$ be the sequence of pairs of interwoven point processes defined as follows: $(\mathcal{Y}^{(0)}, \mathcal{Z}^{(0)}) := (\mathcal{Y}_\infty^{(0)}, \mathcal{Z}_\infty^{(0)})$; given $(\mathcal{Y}^{(i-1)}, \mathcal{Z}^{(i-1)})$, $(\mathcal{Y}^{(i)}, \mathcal{Z}^{(i)})$ is obtained by applying procedure 3.2.15 to the arguments $(\mathcal{Y}^{(i-1)}, \mathcal{Z}^{(i-1)})$, κ_{i-1} and κ_i. Then the total variation distance between the law of $(\mathcal{Y}^{(j)}, \mathcal{Z}^{(j)})$ and the law of $(\mathcal{Y}_\infty^{(j)}, \mathcal{Z}_\infty^{(j)})$ is less or equal to*

$$\sum_{i=1}^{j} \Big(\int_{\mathbb{R}} G_{i-1}(y,y)(\kappa_i(dy) - \kappa_{i-1}(dy)) \Big)^2$$

PROOF. Let $(\mathcal{Y}'^{(i)}, \mathcal{Z}'^{(i)})$ be the pair of point processes obtained by applying procedure 3.2.15 to the arguments $(\mathcal{Y}_\infty^{(i-1)}, \mathcal{Z}_\infty^{(i-1)})$, κ_{i-1} and κ_i. According to lemma 3.2.16, the total variation distance between the law of $(\mathcal{Y}'^{(i)}, \mathcal{Z}'^{(i)})$ and the law of $(\mathcal{Y}_\infty^{(i)}, \mathcal{Z}_\infty^{(i)})$ is less or equal to $\Big(\int_{\mathbb{R}} G_{i-1}(y,y)(\kappa_i(dy) - \kappa_{i-1}(dy)) \Big)^2$. We denote by d_q the total variation distance between the law of $(\mathcal{Y}^{(q)}, \mathcal{Z}^{(q)})$ and the law of $(\mathcal{Y}_\infty^{(q)}, \mathcal{Z}_\infty^{(q)})$. The total variation distance between the law of $(\mathcal{Y}'^{(i)}, \mathcal{Z}'^{(i)})$ and the law of $(\mathcal{Y}^{(i)}, \mathcal{Z}^{(i)})$ is less or equal to d_{i-1}. It follows that

$$d_i \leqslant d_{i-1} + \Big(\int_{\mathbb{R}} G_{i-1}(y,y)(\kappa_i(dy) - \kappa_{i-1}(dy)) \Big)^2$$

and thus

$$d_j \leqslant \sum_{i=1}^{j} \Big(\int_{\mathbb{R}} G_{i-1}(y,y)(\kappa_i(dy) - \kappa_{i-1}(dy)) \Big)^2$$

□

Next we give a true monotone coupling between $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ and $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$. We still consider that $\kappa \leqslant \tilde{\kappa}$ and that $\int_{\mathbb{R}} |x|(\tilde{\kappa}(dx) - \kappa(dx)) < +\infty$. To construct the coupling we will use a continuous monotonic increasing path in the space of measures, $(\kappa_q)_{0 \leqslant q \leqslant 1}$, joining κ to $\tilde{\kappa}$ ($\kappa_0 = \kappa$, $\kappa_1 = \tilde{\kappa}$). Such a path is defined as follows: Let Λ be a positive Radon measure on $\mathbb{R} \times [0,1]$ satisfying the following constraints:

- For any $q \in [0,1]$, $\Lambda(\mathbb{R} \times \{q\}) = 0$
- For any A Borel subset of $\mathbb{R}, \Lambda(A \times [0,1]) = \tilde{\kappa}(A)$

For $q \in [0,1]$, we define κ_q as the measure on $\mathbb{R}$ satisfying, for any A Borel subset of $\mathbb{R}$

$$\kappa_q(A) = \kappa_0(A) + \Lambda(A \times [0,q])$$

For any $q \leqslant q' \in [0,1]$, $\kappa_q \leqslant \kappa_{q'}$. Moreover the map $q \mapsto \kappa_q$ is continuous for the vague topology. In the sequel we will denote G_q the Green's function of $\frac{1}{2}\frac{d^2}{dx^2} - \kappa_q$ (for $x \leqslant y$, $G_q(x,y) = u_{q,\uparrow}(x)u_{q,\downarrow}(y)$) and use the measure $G_q(y,y)\Lambda(dy,dq)$, which is finite.

PROCEDURE 3.2.18. *Arguments:*

- *a pair $(\mathcal{Y}, \mathcal{Z})$ of disjoint discrete sets of points in $\mathbb{R}$ such that between any two points in $\mathcal{Y}$ lies a single point in $\mathcal{Z}$ and vice-versa, and such that* $\inf \mathcal{Y} \cup \mathcal{Z} \in \mathcal{Y} \cup \{-\infty\}$, $\sup \mathcal{Y} \cup \mathcal{Z} \in \mathcal{Y} \cup \{+\infty\}$
- *two positive Radon measures κ, $\tilde{\kappa}$ such that $\kappa \leqslant \tilde{\kappa}$ and $\int_{\mathbb{R}} |x|(\tilde{\kappa}(dx) - \kappa(dx)) < +\infty$ and $(\tilde{\kappa} - \kappa)(\mathcal{Z}) = 0$.*
- *a continuous monotonic increasing path in the space of measures, $(\kappa_q)_{0 \leqslant q \leqslant 1}$, joining κ to $\tilde{\kappa}$, obtained by integrating the Radon measure Λ on $\mathbb{R} \times [0,1]$.*

Procedure:

(i) *First sample a Poisson point process of intensity* $G_q(y,y)\Lambda(dy,dq)$ *on* $\mathbb{R} \times [0,1]$*:* $((Y_j, q_j))_{1 \leqslant j \leqslant N}$*, the points being ordered in the increasing sense of* q_j.

(ii) *Then construct recursively the sequence* $((\mathcal{Y}^{(j)}, \mathcal{Z}^{(j)}))_{0 \leqslant j \leqslant N}$ *of pairs of interwoven point processes as follows:* $(\mathcal{Y}^{(0)}, \mathcal{Z}^{(0)})$ *is set to be* $(\mathcal{Y}, \mathcal{Z})$*.* $(\mathcal{Y}^{(j)}, \mathcal{Z}^{(j)})$ *is obtained by applying procedure 3.2.10 to the arguments* $(\mathcal{Y}^{(j-1)}, \mathcal{Z}^{(j-1)})$*,* κ_{q_j} *and* Y_j.

(iii) $(\tilde{\mathcal{Y}}, \tilde{\mathcal{Z}})$ *is set to be* $(\mathcal{Y}^{(N)}, \mathcal{Z}^{(N)})$

Return: $(\tilde{\mathcal{Y}}, \tilde{\mathcal{Z}})$.

The condition $(\tilde{\kappa} - \kappa)(\mathcal{Z}) = 0$ ensures that a.s., none of $\mathcal{Y}^{(j)}$ lies in $\mathcal{Z}$. By construction $\mathcal{Z} \subseteq \tilde{\mathcal{Z}}$ and $\tilde{\mathcal{Y}} \subseteq \mathcal{Y} \cup Supp(\tilde{\kappa} - \kappa)$. $(\tilde{\mathcal{Y}}, \tilde{\mathcal{Z}})$ differs from $(\mathcal{Y}, \mathcal{Z})$ only by a finite number of points. The law of $(\tilde{\mathcal{Y}}, \tilde{\mathcal{Z}})$ depends only on the "geometrical path" $(\kappa_q)_{0 \leqslant q \leqslant 1}$ and not on its parametrization: if θ is an increasing homomorphism from $[0,1]$ to itself, then procedure 3.2.18 applied the path $(\kappa_{\theta(q)})_{0 \leqslant q \leqslant 1}$ returns the same result (in law). Below an illustration of procedure 3.2.18:

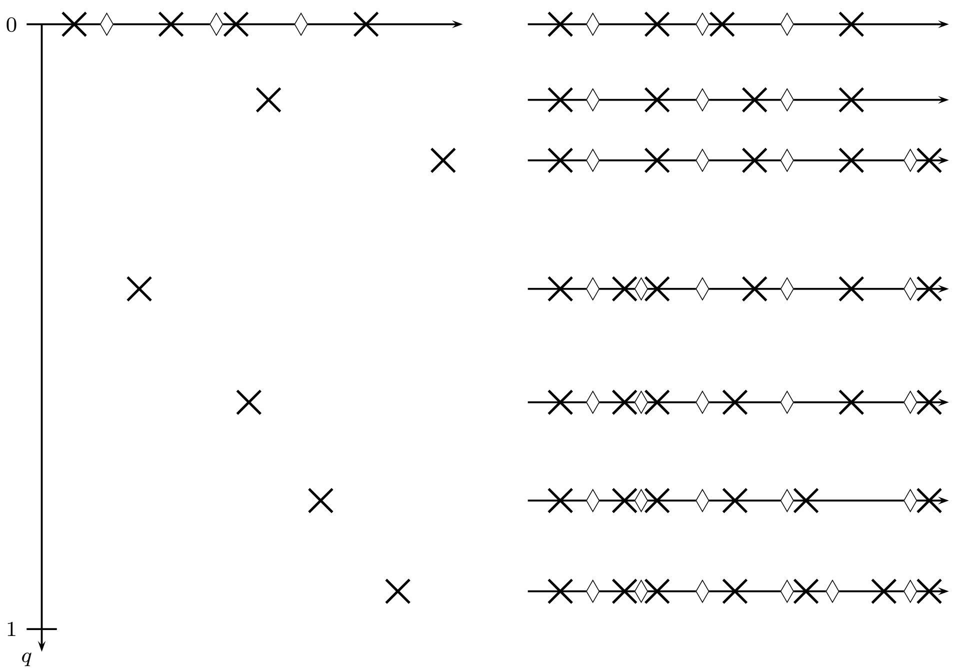

Fig.3.3 - Illustration of procedure 3.2.18: On the left are represented $(\mathcal{Y}, \mathcal{Z})$ and the Poisson process $((Y_j, q_j))_{1 \leqslant j \leqslant N}$. On the right are represented the successive $((\mathcal{Y}^{(j)}, \mathcal{Z}^{(j)}))_{0 \leqslant j \leqslant N}$. x-dots represent the points of $\mathcal{Y}^{(j)}$ and diamonds the points of $\mathcal{Z}^{(j)}$.

PROPOSITION 3.2.19. *Let* $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ *respectively* $(\tilde{\mathcal{Y}}_\infty, \tilde{\mathcal{Z}}_\infty)$ *be the couple of interwoven determinantal point processes corresponding to the killing measure* κ *respectively* $\tilde{\kappa}$*. We assume that the procedure 3.2.18 is applied to* $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ *and that* $(\tilde{\mathcal{Y}}, \tilde{\mathcal{Z}})$ *is the returned couple of point processes. Then* $(\tilde{\mathcal{Y}}, \tilde{\mathcal{Z}})$ *has the same law as* $(\tilde{\mathcal{Y}}_\infty, \tilde{\mathcal{Z}}_\infty)$.

PROOF. Observe that a.s. $(\tilde{\kappa} - \kappa)(\mathcal{Z}_\infty) = 0$. Let $n \in \mathbb{N}^*$. We define the family $((\mathcal{Y}^{(j,n)}, \mathcal{Z}^{(j,n)}))_{0 \leqslant j \leqslant n}$ of interwoven point processes as follows: $(\mathcal{Y}^{(0,n)}, \mathcal{Z}^{(0,n)})$ equals $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$. Given $(\mathcal{Y}^{(j-1,n)}, \mathcal{Z}^{(j-1,n)})$, $(\mathcal{Y}^{(j,n)}, \mathcal{Z}^{(j,n)})$ is obtained by applying procedure 3.2.15 to the arguments $(\mathcal{Y}^{(j-1,n)}, \mathcal{Z}^{(j-1,n)})$, $\kappa_{\frac{j-1}{n}}$ and $\kappa_{\frac{j}{n}}$. We will show that as n tends to infinity, the law of $(\mathcal{Y}^{(n,n)}, \mathcal{Z}^{(n,n)})$ converges in total variation to the law of $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ and converges weakly to the law of $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}})$, which will imply that $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}})$ and $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ have the same law.

Applying corollary 3.2.17, we get that the total variation distance between the law of $(\mathcal{Y}^{(n,n)}, \mathcal{Z}^{(n,n)})$ and the law of $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ is bounded by

$$
\begin{aligned}
\sum_{j=1}^{n} \Big(\int_{\mathbb{R}} & G_{\frac{j-1}{n}}(y,y)(\kappa_{\frac{j}{n}}(dy) - \kappa_{\frac{j-1}{n}}(dy)) \Big)^2 \\
& \leqslant \sup_{x\in\mathbb{R}} \left(\frac{G_0(x,x)}{1+|x|} \right)^2 \sum_{j=1}^{n} \Big(\int_{\mathbb{R}} (1+|y|)(\kappa_{\frac{j}{n}}(dy) - \kappa_{\frac{j-1}{n}}(dy)) \Big)^2 \\
& \leqslant \sup_{x\in\mathbb{R}} \left(\frac{G_0(x,x)}{1+|x|} \right)^2 \int_{\mathbb{R}} (1+|y|)(\tilde{\kappa}(dy) - \kappa(dy)) \\
& \quad \times \sup_{1\leqslant j\leqslant n} \int_{\mathbb{R}} (1+|y|)(\kappa_{\frac{j}{n}}(dy) - \kappa_{\frac{j-1}{n}}(dy))
\end{aligned}
$$

The continuity of the path $(\kappa_q)_{0\leqslant q\leqslant 1}$ ensures that

$$
\lim_{n\to+\infty} \sup_{1\leqslant j\leqslant n} \int_{\mathbb{R}} (1+|y|)(\kappa_{\frac{j}{n}}(dy) - \kappa_{\frac{j-1}{n}}(dy)) = 0
$$

and hence the total variation distance between the law of $(\mathcal{Y}^{(n,n)}, \mathcal{Z}^{(n,n)})$ and the law of $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ converges to 0 as n tends to infinity.

We define a random finite set E_n of points in $\mathbb{R} \times \left\{\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\right\}$ as follows: Take a family $(\beta_{1,n}, \beta_{2,n}, \dots, \beta_{n,n})$ of independent Bernoulli variables, $\beta_{i,n}$ being of parameter

$$
\int_{\mathbb{R}} v_{\kappa_{\frac{i-1}{n}}, \kappa_{\frac{i}{n}}}(y)(\kappa_{\frac{i}{n}} - \kappa_{\frac{i-1}{n}})(dy)
$$

Whenever $\beta_{i,n} = 1$, we add to E_n a point $(Y_{i,n}, \frac{i-1}{n})$ to E_n where $Y_{i,n}$ is a r.v. distributed according the measure

$$
\frac{1}{\mathbb{P}(\beta_{i,n} = 1)} v_{\kappa_{\frac{i-1}{n}}, \kappa_{\frac{i}{n}}}(y)(\kappa_{\frac{i}{n}} - \kappa_{\frac{i-1}{n}})(dy)
$$

The $(Y_{i,n}, \frac{i-1}{n})$ are assumed to be independent and independent from $(\beta_{1,n}, \beta_{2,n}, \dots, \beta_{n,n})$. The pair $(\mathcal{Y}^{(n,n)}, \mathcal{Z}^{(n,n)})$ is sampled as follows: starting from $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$, independent from E_n, we apply successively, for i ranging from 1 to n, the procedure 3.2.10 with the arguments $\kappa_{\frac{i-1}{n}}$ and $Y_{i,n}$ whenever $\beta_{i,n} = 1$. At the end we get $(\mathcal{Y}^{(n,n)}, \mathcal{Z}^{(n,n)})$. According to lemma 2.2.4, the law of the pair of point processes returned by procedure 3.2.10 depends continuously on the arguments. So to prove that $(\mathcal{Y}^{(n,n)}, \mathcal{Z}^{(n,n)})$ converges in law to $(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{Z}})$, we only need to show that the random set of point E_n converges in law to the Poisson point process $((Y_j, q_j))_{1\leqslant j\leqslant N}$ used in procedure 3.2.18. All of the functions $v_{\kappa_{\frac{i-1}{n}}, \kappa_{\frac{i}{n}}}(y)$ are dominated by $G_0(y,y)$. Moreover

$$
\begin{aligned}
&\left|v_{\kappa_{\frac{i-1}{n}},\kappa_{\frac{i}{n}}}(y) - G_{\frac{i}{n}}(y,y)\right| \\
&\quad\leqslant u_{\frac{i}{n},\downarrow}(y)\int_{y_{-1}<y} u_{\frac{i}{n},\uparrow}(y_{-1})(u_{\frac{i-1}{n},\downarrow}(y_{-1})u_{\frac{i-1}{n},\uparrow}(y) - u_{\frac{i-1}{n},\uparrow}(y_{-1})u_{\frac{i-1}{n},\downarrow}(y)) \\
&\qquad\times(\kappa_{\frac{i}{n}} - \kappa_{\frac{i-1}{n}})(dy_{-1}) \\
&\quad+u_{\frac{i}{n},\uparrow}(y)\int_{y_1>y} u_{\frac{i}{n},\downarrow}(y_1)(u_{\frac{i-1}{n},\uparrow}(y_1)u_{\frac{i-1}{n},\downarrow}(y) - u_{\frac{i-1}{n},\downarrow}(y_1)u_{\frac{i-1}{n},\uparrow}(y)) \\
&\qquad\times(\kappa_{\frac{i}{n}} - \kappa_{\frac{i-1}{n}})(dy_1) \\
&\quad+\int_{y_{-1}<y} u_{\frac{i}{n},\uparrow}(y_{-1})(u_{\frac{i-1}{n},\downarrow}(y_{-1})u_{\frac{i-1}{n},\uparrow}(y) - u_{\frac{i-1}{n},\uparrow}(y_{-1})u_{\frac{i-1}{n},\downarrow}(y))(\kappa_{\frac{i}{n}} - \kappa_{\frac{i-1}{n}})(dy_{-1}) \\
&\quad\times\int_{y_1>y} u_{\frac{i}{n},\downarrow}(y_1)(u_{\frac{i-1}{n},\uparrow}(y_1)u_{\frac{i-1}{n},\downarrow}(y) - u_{\frac{i-1}{n},\downarrow}(y_1)u_{\frac{i-1}{n},\uparrow}(y))(\kappa_{\frac{i}{n}} - \kappa_{\frac{i-1}{n}})(dy_1) \\
&\quad\leqslant G_0(y,y)\int_{y_{-1}<y} G_0(y_{-1},y_{-1})(\kappa_{\frac{i}{n}} - \kappa_{\frac{i-1}{n}})(dy_{-1}) \\
&\quad+G_0(y,y)\int_{y_1>y} G_0(y_1,y_1)(\kappa_{\frac{i}{n}} - \kappa_{\frac{i-1}{n}})(dy_1) \\
&\quad+G_0(y,y)\int_{y_{-1}<y} G_0(y_{-1},y_{-1})(\kappa_{\frac{i}{n}} - \kappa_{\frac{i-1}{n}})(dy_{-1})\int_{y_1>y} G_0(y_1,y_1)(\kappa_{\frac{i}{n}} - \kappa_{\frac{i-1}{n}})(dy_1)
\end{aligned}
$$

Thus given any bounded interval J

$$\lim_{n\to+\infty}\sup_{1\leqslant i\leqslant n}\sup_{y\in J}\left|v_{\kappa_{\frac{i-1}{n}},\kappa_{\frac{i}{n}}}(y) - G_{\frac{i}{n}}(y,y)\right| = 0$$

It follows that

$$\lim_{n\to+\infty}\sup_{1\leqslant i\leqslant n}\mathbb{P}(\beta_{i,n}=1)=0$$

and the measure

$$\sum_{i=1}^{n} v_{\kappa_{\frac{i-1}{n}},\kappa_{\frac{i}{n}}}(y)(\kappa_{\frac{i}{n}} - \kappa_{\frac{i-1}{n}})(dy)\otimes\delta_{\frac{i}{n}}(dq)$$

converges weekly to $G_q(y,y)\Lambda(dy,dq)$, which is the intensity of the Poisson point process $((Y_j,q_j))_{1\leqslant j\leqslant N}$. Thus the random sets E_n are compound Bernoulli approximations of the Poisson point process $((Y_j,q_j))_{1\leqslant j\leqslant N}$ and converge in law to the latter. □

Given a continuous monotonic increasing path $(\kappa_q)_{0\leqslant q\leqslant 1}$ in the space of Radon measures and a pair of interwoven determinantal point processes $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ corresponding to κ_0, used as argument, procedure 3.2.18 yields non-homogeneous Markov q-parametrized process in the space of interwoven pairs of discrete sets of points whose one-dimensional marginal at any value q_0 of the parameter is the pair of interwoven determinantal point processes corresponding to the killing measure κ_{q_0}. This corresponds to sampling only the partial Poisson point process of intensity $1_{0\leqslant q\leqslant q_0}G_q(y,y)\Lambda(dy,dq)$ and successively applying procedure 3.2.10 for each of its points. In general, multidimensional marginals corresponding to $q_1<\cdots<q_n$ depend not only on $\kappa_{q_1},\dots,\kappa_{q_n}$ but on the whole path $(\kappa_q)_{q_1\leqslant q\leqslant q_n}$. For instance consider two different paths $(\kappa_q)_{0\leqslant q\leqslant 1}$ and $(\hat{\kappa}_q)_{0\leqslant q\leqslant 1}$ where

- $\kappa_0 = \hat{\kappa}_0 = \delta_{-\frac{1}{2}} + \delta_{\frac{1}{2}}$
- $\kappa_1 = \hat{\kappa}_1 = \delta_{-\frac{3}{2}} + \delta_{-\frac{1}{2}} + \delta_{\frac{1}{2}} + \delta_{\frac{3}{2}}$
- $\kappa_q = 2q\delta_{-\frac{1}{2}} + \delta_{-\frac{1}{2}} + \delta_{\frac{1}{2}}$ for $q\in\left[0,\frac{1}{2}\right]$ and $\kappa_q = \delta_{-\frac{1}{2}} + \delta_{-\frac{1}{2}} + \delta_{\frac{1}{2}} + (2q-1)\delta_{\frac{3}{2}}$ for $q\in\left[\frac{1}{2},1\right]$

- $\hat{\kappa}_q = \delta_{-\frac{1}{2}} + \delta_{\frac{1}{2}} + 2q\delta_{\frac{1}{2}}$ for $q \in \left[0, \frac{1}{2}\right]$ and $\hat{\kappa}_q = (2q-1)\delta_{-\frac{1}{2}} + \delta_{-\frac{1}{2}} + \delta_{\frac{1}{2}} + \delta_{\frac{3}{2}}$ for $q \in \left[\frac{1}{2}, 1\right]$

Let $G_q(x,y) = u_{q,\uparrow}(x \wedge y)u_{q,\downarrow}(x \vee y)$ be the Green's function of $\frac{1}{2}\frac{d^2}{dx^2} - \kappa_q$ and $\widehat{G}_q(x,y) = \hat{u}_{q,\uparrow}(x \wedge y)\hat{u}_{q,\downarrow}(x \vee y)$ the Green's function of $\frac{1}{2}\frac{d^2}{dx^2} - \hat{\kappa}_q$. Let $((\mathcal{Y}_\infty, \mathcal{Z}_\infty), (\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty))$ be the coupling between the point process corresponding to κ_0 respectively κ_1 induced by the path $(\kappa_q)_{0\leqslant q\leqslant 1}$ and $((\mathcal{Y}_\infty, \mathcal{Z}_\infty), (\widehat{\mathcal{Y}}_\infty, \widehat{\mathcal{Z}}_\infty))$ the coupling induced by the path $(\hat{\kappa}_q)_{0\leqslant q\leqslant 1}$. Then

$$\begin{aligned}\mathbb{P}\Big(\mathcal{Y}_\infty = \Big\{-\frac{1}{2}, \frac{1}{2}\Big\}, \widetilde{\mathcal{Y}}_\infty = \Big\{-\frac{3}{2}, \frac{1}{2}, \frac{3}{2}\Big\}\Big)\\ = \mathbb{P}\Big(\mathcal{Y}_\infty = \Big\{-\frac{1}{2}, \frac{1}{2}\Big\}\Big) \times G_{\frac{1}{2}}\Big(-\frac{3}{2}\Big)\frac{u_{\frac{1}{2},\downarrow}(-\frac{1}{2})}{u_{\frac{1}{2},\downarrow}(-\frac{3}{2})}G_1\Big(\frac{3}{2}\Big)\Big(1 - \frac{u_{1,\uparrow}(\frac{1}{2})}{u_{1,\uparrow}(\frac{3}{2})}\Big)\end{aligned}$$

$$\begin{aligned}\mathbb{P}\Big(\mathcal{Y}_\infty = \Big\{-\frac{1}{2}, \frac{1}{2}\Big\}, \widehat{\mathcal{Y}}_\infty = \Big\{-\frac{3}{2}, \frac{1}{2}, \frac{3}{2}\Big\}\Big)\\ = \mathbb{P}\Big(\mathcal{Y}_\infty = \Big\{-\frac{1}{2}, \frac{1}{2}\Big\}\Big) \times \widehat{G}_{\frac{1}{2}}\Big(\frac{3}{2}\Big)\Big(1 - \frac{\hat{u}_{\frac{1}{2},\uparrow}(\frac{1}{2})}{\hat{u}_{\frac{1}{2},\uparrow}(\frac{3}{2})}\Big)\widehat{G}_1\Big(-\frac{3}{2}\Big)\frac{\hat{u}_{1,\downarrow}(-\frac{1}{2})}{\hat{u}_{1,\downarrow}(-\frac{3}{2})}\end{aligned}$$

But

$$\widehat{G}_{\frac{1}{2}}\Big(\frac{3}{2}\Big) = G_{\frac{1}{2}}\Big(-\frac{3}{2}\Big) \qquad \widehat{G}_1\Big(-\frac{3}{2}\Big) = G_1\Big(\frac{3}{2}\Big)$$

and

$$\frac{\hat{u}_{\frac{1}{2},\uparrow}(\frac{1}{2})}{\hat{u}_{\frac{1}{2},\uparrow}(\frac{3}{2})} = \frac{u_{\frac{1}{2},\downarrow}(-\frac{1}{2})}{u_{\frac{1}{2},\downarrow}(-\frac{3}{2})} \qquad \frac{\hat{u}_{1,\downarrow}(-\frac{1}{2})}{\hat{u}_{1,\downarrow}(-\frac{3}{2})} = \frac{u_{1,\uparrow}(\frac{1}{2})}{u_{1,\uparrow}(\frac{3}{2})}$$

Thus

$$\frac{\mathbb{P}\Big(\mathcal{Y}_\infty = \Big\{-\frac{1}{2}, \frac{1}{2}\Big\}, \widetilde{\mathcal{Y}}_\infty = \Big\{-\frac{3}{2}, \frac{1}{2}, \frac{3}{2}\Big\}\Big)}{\mathbb{P}\Big(\mathcal{Y}_\infty = \Big\{-\frac{1}{2}, \frac{1}{2}\Big\}, \widehat{\mathcal{Y}}_\infty = \Big\{-\frac{3}{2}, \frac{1}{2}, \frac{3}{2}\Big\}\Big)} = \frac{\dfrac{u_{\frac{1}{2},\downarrow}(-\frac{1}{2})}{u_{\frac{1}{2},\downarrow}(-\frac{3}{2})}}{1 - \dfrac{u_{\frac{1}{2},\downarrow}(-\frac{1}{2})}{u_{\frac{1}{2},\downarrow}(-\frac{3}{2})}} \times \frac{1 - \dfrac{u_{1,\uparrow}(\frac{1}{2})}{u_{1,\uparrow}(\frac{3}{2})}}{\dfrac{u_{1,\uparrow}(\frac{1}{2})}{u_{1,\uparrow}(\frac{3}{2})}}$$

But

$$\frac{u_{\frac{1}{2},\downarrow}(-\frac{1}{2})}{u_{\frac{1}{2},\downarrow}(-\frac{3}{2})} = \frac{3}{11} \qquad \frac{u_{1,\uparrow}(\frac{1}{2})}{u_{1,\uparrow}(\frac{3}{2})} = \frac{11}{41}$$

Thus

$$\frac{\mathbb{P}\Big(\mathcal{Y}_\infty = \Big\{-\frac{1}{2}, \frac{1}{2}\Big\}, \widetilde{\mathcal{Y}}_\infty = \Big\{-\frac{3}{2}, \frac{1}{2}, \frac{3}{2}\Big\}\Big)}{\mathbb{P}\Big(\mathcal{Y}_\infty = \Big\{-\frac{1}{2}, \frac{1}{2}\Big\}, \widehat{\mathcal{Y}}_\infty = \Big\{-\frac{3}{2}, \frac{1}{2}, \frac{3}{2}\Big\}\Big)} = \frac{45}{44} \neq 1$$

The two couplings are different.

If $\tilde{\kappa} - \kappa$ does not have a first moment we can still construct a coupling between $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ and $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ as follows: Consider a continuous monotonic increasing path $(\kappa_q)_{0\leqslant q\leqslant 1}$ joining κ to $\tilde{\kappa}$ satisfying the constraint

$$\forall q \in [0,1), \int_{\mathbb{R}} |x|(\kappa_q(dx) - \kappa_0(dx)) < +\infty$$

Given $q_0 \in (0,1)$, one can apply procedure 3.2.18 to the arguments $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$, κ, κ_{q_0} and the partial path $(\kappa_q)_{0\leqslant q\leqslant q_0}$. As result we get a two interwoven determinantal point processes corresponding to the killing measure κ_{q_0}. At the limit as q_0 tends to 1 we get something that has the same law as $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$.

Next we prove the existence of stronger couplings in case $\tilde{\kappa} = c\kappa$ where $c > 1$ is a constant.

PROPOSITION 3.2.20. *If $\tilde{\kappa} = c\kappa$ with $c > 1$ then there is a coupling between $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ and $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ such that $\mathcal{Z}_\infty \subseteq \widetilde{\mathcal{Z}}_\infty$ and $\mathcal{Y}_\infty \subseteq \widetilde{\mathcal{Y}}_\infty$.*

PROOF. Consider a coupling between $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$ and $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$ given by procedure 3.2.18, possible extended to the case where κ does not have a first moment. Then $\mathcal{Z}_\infty \subseteq \widetilde{\mathcal{Z}}_\infty$ but in general $\mathcal{Y}_\infty \nsubseteq \widetilde{\mathcal{Y}}_\infty$. So we will sample other point processes $\mathcal{Y}'_\infty$ and $\widetilde{\mathcal{Y}}'_\infty$ that conditional on $\mathcal{Z}_\infty$ respectively $\widetilde{\mathcal{Z}}_\infty$ have the same law as $\mathcal{Y}_\infty$ respectively $\widetilde{\mathcal{Y}}_\infty$, and such that $\mathcal{Y}'_\infty \subseteq \widetilde{\mathcal{Y}}'_\infty$. For each connected component $\widetilde{J}$ of $\mathbb{R}\backslash\widetilde{\mathcal{Z}}_\infty$ we sample a point $\widetilde{Y}_{\tilde{J}}$ according the measure $\frac{1_{y\in\tilde{J}}\tilde{\kappa}(dy)}{\tilde{\kappa}(\tilde{J})}$. We assume that conditional on $\widetilde{\mathcal{Z}}_\infty$, all the $\widetilde{Y}_{\tilde{J}}$ are independent from $\mathcal{Z}_\infty$ and independent one from another. We set

$$\widetilde{\mathcal{Y}}'_\infty := \{\widetilde{Y}_{\tilde{J}} | \widetilde{J} \text{ connected component of } \mathbb{R}\backslash\widetilde{\mathcal{Z}}_\infty\}$$

Then $(\widetilde{\mathcal{Y}}'_\infty, \widetilde{\mathcal{Z}}_\infty)$ has the same law as $(\widetilde{\mathcal{Y}}_\infty, \widetilde{\mathcal{Z}}_\infty)$. Let be J a connected component of $\mathbb{R}\backslash\mathcal{Z}_\infty$ and $\widetilde{J}_1, \dots, \widetilde{J}_{N_J}$ the connected components of $J\backslash\widetilde{\mathcal{Z}}_\infty$. On J we define the r.v. Y_J as follows: Y_J takes value in $\widetilde{Y}_{\tilde{J}_n}$ and

$$\mathbb{P}\big(Y_J = \widetilde{Y}_{\tilde{J}_n} | J, \widetilde{J}_1, \dots, \widetilde{J}_{N_J}\big) = \frac{\kappa(\widetilde{J}_n)}{\kappa(J)}$$

We set

$$\mathcal{Y}'_\infty := \{Y_J | J \text{ connected component of } \mathbb{R}\backslash\mathcal{Z}_\infty\}$$

By construction $\mathcal{Y}'_\infty \subseteq \widetilde{\mathcal{Y}}'_\infty$. Moreover the proportionality of κ and $\tilde{\kappa}$ ensures that $(\mathcal{Y}'_\infty, \mathcal{Z}_\infty)$ has the same law as $(\mathcal{Y}_\infty, \mathcal{Z}_\infty)$. □

CHAPTER 4

From loop clusters and random interlacements to the Gaussian free field

4.1. Introduction

Here we introduce our framework, some notations, state our main results and outline the layout of the paper.

We consider a connected undirected graph $\mathcal{G} = (V, E)$ where the set of vertices V is at most countable and every vertex has finite degree. We do not allow multiple edges nor loops from a vertex to itself. The edges are endowed with positive conductances $(C(e))_{e\in E}$ and vertices endowed with a non-negative killing measure $(\kappa(x))_{x\in V}$. κ may be uniformly zero. $(X_t)_{0\leqslant t<\zeta}$ is a continuous-time sub-Markovian jump process on V. Given two neighbouring vertices x and y, the transition rate from x to y equals the conductance $C(x, y)$. Moreover there is a transition rate $\kappa(x)$ from $x \in V$ to a cemetery point outside V. Once such a transition occurs, the process X is considered to be killed. Moreover we allow X to blow up in finite time, i.e. leave all finite sets. ζ is either $+\infty$ or the first time X gets killed or blows up. We assume that X is transient, which is a condition on C and κ. In particular if κ is not uniformly zero X is transient. $(G(x, y))_{x,y\in V}$ denotes the Green's function of X:

$$G(x, y) = \mathbb{E}_x\left[\int_0^{\zeta} 1_{X=y}dt\right]$$

G is symmetric.

Let $\left(\mathbb{P}^t_{x,y}(\cdot)\right)_{x,y\in V,t>0}$ be the bridge probability measures of X, conditioned on $\zeta > t$ and let $(p_t(x, y))_{x,y\in V,t\geqslant 0}$ be the transition probabilities of X. The measure μ on time-parametrized loops associated to X is, as defined in [**Jan11**],

$$\mu(\cdot) = \sum_{x\in V}\int_0^{+\infty}\mathbb{P}^t_{x,x}(\cdot)p_t(x, x)\frac{dt}{t} \tag{4.1.1}$$

Let $\alpha > 0$. $\mathcal{L}_\alpha$ is defined to be the Poisson point process in the space of loops on $\mathcal{G}$ with intensity $\alpha\mu$. It is sometimes called loop-soup of parameter α. The occupation field $(\hat{\mathcal{L}}^x_\alpha)_{x\in V}$ of $\mathcal{L}_\alpha$ is

$$\hat{\mathcal{L}}^x_\alpha = \sum_{\gamma\in\mathcal{L}_\alpha}\int_0^{t_\gamma} 1_{\gamma(t)=x}dt$$

where t_γ is the duration of the loop γ. The loops of $\mathcal{L}_\alpha$ may be partitioned into clusters: if $\gamma, \gamma' \in \mathcal{L}_\alpha$ belong to the same cluster if there is a chain $\gamma_0, \dots, \gamma_n$ of loops in $\mathcal{L}_\alpha$ such that $\gamma_0 = \gamma$, $\gamma_n = \gamma'$ and for all $i \in \{1, \dots, n\}$ γ_{i-1} and γ_i visit a common vertex ([**JL13**]). A cluster $\mathcal{C}$ is a set of loops, but it also induces a sub-graph of $\mathcal{G}$. Its vertices are the vertices of $\mathcal{G}$ visited by at least one loop in $\mathcal{C}$ and its edges are those that join two consecutive points of a loop in $\mathcal{C}$. Therefore we will also consider $\mathcal{C}$ as a subset of vertices and a subset of edges and use the notations $\gamma \in \mathcal{C}$, $x \in \mathcal{C}$ and $e \in \mathcal{C}$ where γ is a loop, x is a vertex and e is an edge. $\mathfrak{C}_\alpha$ will be the random set of all clusters of $\mathcal{L}_\alpha$. It induces a partition of V.

Let $(\phi_x)_{x\in V}$ be the Gaussian free field on $\mathcal{G}$, i.e. the mean-zero Gaussian field with $\mathbb{E}[\phi_x\phi_y] = G(x,y)$. In [**Jan11**], section 5, Le Jan showed that at intensity parameter $\alpha = \frac{1}{2}$ the occupation field $(\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{x\in V}$ has the same law as $(\frac{1}{2}\phi_x^2)_{x\in V}$. This equality in law may be seen as an extension of Dynkin's isomorphism ([**Dyn84a**], [**Dyn84b**]) and in turn enables an alternative derivation of some version of Dynkin's isomorphism through the use of Palm's identity for Poisson point processes ([**LMR15**],[**FR14**] and [**Lup13**], section 4.3). However the question of relating the sign of ϕ to $\mathcal{L}_{\frac{1}{2}}$ remained open. In this paper we show the following:

THEOREM 4.1. *There is a coupling between the Poisson ensemble of loops $\mathcal{L}_{\frac{1}{2}}$ and the Gaussian free field ϕ such that the two constraints hold:*

- *For all $x \in V$, $\widehat{\mathcal{L}}^x_{\frac{1}{2}} = \frac{1}{2}\phi_x^2$*
- *For all $\mathcal{C} \in \mathfrak{C}_{\frac{1}{2}}$ the sign of ϕ is constant on the vertices of $\mathcal{C}$.*

In section 4.2 we will construct the coupling that satisfies the constraints of theorem 4.1. To this end we will introduce the metric graph $\widetilde{\mathcal{G}}$ associated to the graph $\mathcal{G}$ and interpolate the loops in $\mathcal{L}_{\frac{1}{2}}$ by continuous loops on $\widetilde{\mathcal{G}}$. In section 4.3 we will give an alternative description of the same coupling that does not make use of the metric graph $\widetilde{\mathcal{G}}$ and the interpolation of loops. In section 4.4 we will give an alternative, direct, proof that the coupling holds using its description given in section 4.3.

In section 4.5 we will apply theorem 4.1 to the loop percolation problem. The loops of $\mathcal{L}_\alpha$ are said to percolate if there is an unbounded cluster of loops. This question of percolation was studied in [**JL13**] and [**CS14**]. Obviously from theorem 4.1 follows that the loops do not percolate if the sign clusters of ϕ are all bounded. But we will show that even in some situations where ϕ is known to have some (two) infinite sign clusters, the loops of $\mathcal{L}_{\frac{1}{2}}$ still do not percolate:

THEOREM 4.2. *Consider the following networks:*

- *$\mathbb{Z}^2$ with uniform conductances and a non-zero uniform killing measure*
- *the discrete half-plane $\mathbb{Z}\times\mathbb{N}$ with instantaneous killing on the boundary $\mathbb{Z}\times\{0\}$ and no killing elsewhere*
- *$\mathbb{Z}^d$, $d \geqslant 3$, with uniform conductances and no killing measure*

On all above networks $\mathcal{L}_{\frac{1}{2}}$ does not percolate.

We will also give a bound for the probability that two vertices belong to the same cluster of loops.

In section 4.6 we will show that on the discrete half-plane $\mathbb{Z}\times\mathbb{N}$, with instantaneous killing on the boundary $\mathbb{Z}\times\{0\}$, $\frac{1}{2}$ is actually the critical parameter for the loop percolation. In this section we will denote by $\mathtt{H} := \mathbb{Z}\times\mathbb{N}$, by $\mu^{\mathtt{H}}$ the measure on loops on the network $\mathtt{H}$ endowed with instantaneous killing on the boundary $\mathbb{Z}\times\{0\}$, by $\mathcal{L}^{\mathtt{H}}_\alpha$ the Poisson point process of intensity $\alpha\mu^{\mathtt{H}}$ and $\alpha^{\mathtt{H}}_*$ the critical value of α for the percolation by loops of $\mathcal{L}^{\mathtt{H}}_\alpha$. We will prove that

THEOREM 4.3. *For all $\alpha > \frac{1}{2}$ $\mathcal{L}^{\mathtt{H}}_\alpha$ has an infinite cluster of loops. Consequently $\alpha^{\mathtt{H}}_* = \frac{1}{2}$.*

For the proof of the inequality $\alpha^{\mathtt{H}}_* \leqslant \frac{1}{2}$ we will use completely different arguments than previously. We will use the fact that large discrete loops approximate two-dimensional Brownian loops on the continuum half plane $\mathbb{H} = \{z \in \mathbb{C}|\Im(z) > 0\}$. The measure $\mu^{\mathbb{H}}$ on the Brownian loops on $\mathbb{H}$ is defined as follows: Let $p_t(z,z')$ be the transition density of the Brownian motion on $\mathbb{H}$ killed on $\mathbb{R}$, let $P^{t,\mathbb{H}}_{z,z'}$ be the Brownian bridge probability measures conditioned on not hitting $\mathbb{R}$. Then

$$\mu^{\mathbb{H}}(\cdot) = \int_{\mathbb{H}} \int_0^{+\infty} P_{z,z}^{t,\mathbb{H}}(\cdot) p_t(z,z) \frac{dt}{t} \frac{d\bar{z} \wedge dz}{2i}$$

where $\frac{d\bar{z} \wedge dz}{2i}$ is the standard area form on $\mathbb{C}$. The measure $\mu^{\mathbb{H}}$ is invariant under Brownian scaling

$$(\gamma(t))_{0 \leqslant t \leqslant t_\gamma} \longmapsto \lambda^{-\frac{1}{2}}(\gamma(\lambda t))_{0 \leqslant t \leqslant \lambda^{-1} t_\gamma}$$

We will denote by $\mathcal{L}_\alpha^{\mathbb{H}}$ the Poisson point process of intensity $\alpha\mu^{\mathbb{H}}$. In [**SW12**] Werner and Sheffield consider the clusters of $\mathcal{L}_\alpha^{\mathbb{H}}$. They use the notion of *central charge* (denoted c) which comes from Conformal Field Theory. In [**SW12**] the authors use the same normalisation of the measure on loops as we, but contrary to what they claim, the central charge c is not the intensity parameter of the loop soup. Actually (see [**Law09**])

$$\alpha = \frac{c}{2}$$

The critical value of the central charge is $c_* = 1$. For $c > 1$, $\mathcal{L}_\alpha^{\mathbb{H}}$ has only one cluster and for $c \leqslant 1$ it has infinitely many clusters all bounded. This means that $\frac{1}{2}$ is the critical intensity parameter for $\mathcal{L}_\alpha^{\mathbb{H}}$. To conclude that $\alpha_*^{\mathbb{H}} \leqslant \frac{1}{2}$ we will use the result from [**LF07**] on approximation of Brownian loops by large discrete loops and block percolation arguments.

In section 4.7 we consider random interlacements on $\mathbb{Z}^d$ introduced by Sznitman ([**Szn10**]). We consider that the edges of $\mathbb{Z}^d$ have conductances equal to 1 and that $(G(x,y))_{x,y \in \mathbb{Z}^d}$ and $(\phi_x)_{x \in \mathbb{Z}^d}$ are the corresponding Green's function and Gaussian free field. Given K a finite subset of $\mathbb{Z}^d$, let e_K be the equilibrium measure of K (supported on K):

$$\forall x \in K, e_K(\{x\}) = \mathbb{P}_x(\forall j \geqslant 1, Y_j \notin K)$$

The capacity of K is

$$cap(K) = e_K(K)$$

Let Q_K be the measure on doubly infinite trajectories on $\mathbb{Z}^d$, $(x_j)_{j \in \mathbb{Z}}$ parametrized by discrete time $j \in \mathbb{Z}$, of total mass $cap(K)$, such that

- the measure on x_0 induced by Q_K is e_K
- conditional on x_0, $(x_j)_{j \geqslant 0}$ and $(y_j)_{j \leqslant 0}$ are independent
- conditional on x_0, $(x_j)_{j \geqslant 0}$ is a nearest neighbour random walk on $\mathbb{Z}^d$ starting from x_0
- conditional on x_0, $(x_{-j})_{j \geqslant 0}$ is a nearest neighbour random walk on $\mathbb{Z}^d$ starting from x_0 conditioned not to return in K for $j \geqslant 1$.

There is an (infinite) measure μ_{il} on right continuous doubly infinite trajectories $(w(t))_{t \in \mathbb{R}}$ on $\mathbb{Z}^d$, parametrized by continuous time, considered up to a translation of parametrization ($(w(t))_{t \in \mathbb{R}}$ same as $(w(t))_{t+t_0 \in \mathbb{R}}$) such that

- $\lim_{t \to +\infty} |w(t)| = \lim_{t \to -\infty} |w(t)| = +\infty$ μ_{il}-almost everywhere
- for any finite subset K of $\mathbb{Z}^d$, by restricting μ_{il} to trajectories visiting K, choosing the initial time $t = 0$ to be the first entrance time in K and taking the skeleton (the doubly infinite sequence of successively visited vertices) we get the measure Q_K
- under μ_{il}, conditional on the skeleton, the doubly infinite sequence of holding times of the trajectory (times spent at vertices before jumping to neighbours) is i.i.d with exponential distribution of mean $(2d)^{-1}$.

See [**Szn10**] and [**Szn12a**].

The random interlacement $\mathcal{I}^u$ of level $u > 0$ is the Poisson point process of intensity $u\mu_{il}$. The vacant set $\mathcal{V}^u$ of $\mathcal{I}^u$ is the set of vertices not visited by any of trajectories in $\mathcal{I}^u$.

There is $u_* \in (0,+\infty)$ such that for $u < u_*$ $\mathcal{V}^u$ has a.s. infinite connected components and for $u > u_*$ $\mathcal{V}_u$ has a.s. only finite connected components ([**Szn10**], [**SS09**]).

The occupation field $(L^x(\mathcal{I}^u))_{x\in\mathbb{Z}^d}$ of the interlacement $\mathcal{I}^u$ is defined as

$$L^x(\mathcal{I}^u) := \sum_{w\in\mathcal{I}_u}\int_{-\infty}^{+\infty} 1_{w(t)=x}dt$$

In [**Szn12a**] Sznitman showed the following isomorphism between $(L^x(\mathcal{I}^u))_{x\in\mathbb{Z}^d}$ and the Gaussian free field: Let $(\phi'_x)_{x\in\mathbb{Z}^d}$ be a copy of the free field independent of $(L^x(\mathcal{I}^u))_{x\in\mathbb{Z}^d}$. Then

$$\Big(L^x(\mathcal{I}^u) + \frac{1}{2}\phi'^2_x\Big)_{x\in\mathbb{Z}^d} \overset{(d)}{=} \Big(\frac{1}{2}(\phi_x - \sqrt{2u})^2\Big)_{x\in\mathbb{Z}^d} \tag{4.1.2}$$

This isomorphism can be used to relate the random interlacement to the level sets of the Gaussian free field. There is $h_* \in [0,+\infty)$ such that for $h < h_*$, the set $\{x\in\mathbb{Z}^d|\phi_x > h\}$ has an infinite connected components and for $h > h_*$ only finite connected components ([**RS13**], [**BLM87**]). h_* is positive if the dimension d high enough ([**RS13**]). In section 4.7 we will prove:

THEOREM 4.4. *For all $u > 0$, there is a coupling between $\mathcal{I}^u$ and ϕ such that a.s.*

$$\{x\in\mathbb{Z}^d|\phi_x > \sqrt{2u}\} \subseteq \mathcal{V}^u$$

In particular

$$h_* \leqslant \sqrt{2u_*}$$

This theorem is again obtained by replacing the discrete graph $\mathbb{Z}^d$ by a metric graph.

We would like to explain the interdependence of different sections. The sections 4.2, 4.3, 4.4 and 4.5 are closely related both for the results and for the notations. The section 4.7 is more independent but uses the main ideas and notation of the above mentioned sections. The section 4.6 is mostly independent of the rest, except for the most common notations in this article.

4.2. Coupling through interpolation by a metric graph

One can associate a measure on loops following the formal pattern of (4.1.1) to a wide range of Markovian or sub-Markovian processes. In the articles [**LMR15**] and [**FR14**] the authors give quite general definitions for a wide range of cases. The setting of [**FR14**] will cover our needs. In that article the measure on loops is defines for transient Borel right processes on a locally compact state space with with a countable base, that have 0-potential densities with respect some sigma-finite measure, the 0-potential densities being assumed to be finite everywhere (in particular on the diagonal) and continuous. In [**Lup13**] were specifically studied the measures on loops associated to one-dimensional diffusions and the corresponding loop ensembles. This case is of particular interest for the proof of Theorem 4.1. Indeed in the setting of one-dimensional diffusions the occupation fields are continuous space-parametrized processes with non-negative values and the clusters of loops correspond exactly to the excursions of the occupation field above zero (proposition 4.7 in [**Lup13**]). In particular for the loop ensemble of parameter $\frac{1}{2}$, the clusters of loops are exactly the sign clusters of the one-dimensional Gaussian free field.

The nice identity between the clusters of loops and the sign clusters of GFF in case of one-dimensional diffusions leads us to consider the metric graph or cable system $\widetilde{\mathcal{G}}$ associated to the graph $\mathcal{G}$ ([**BC84**], [**EK01**], [**Fol14**]). Topologically $\widetilde{\mathcal{G}}$ is constructed as follows: to each edge e of $\mathcal{G}$ corresponds a different compact interval, each endpoint of this interval being identified to one of the two vertices adjacent to e in $\mathcal{G}$; for every vertex $x\in V$ the intervals corresponding to the edges adjacent to x are glued together at the endpoints identified to the vertex x. We will consider V to be a subset of $\widetilde{\mathcal{G}}$. Given any $e \in E$, I_e will denote the subset of $\widetilde{\mathcal{G}}$ made of the interval corresponding to e minus its two endpoints. Topologically I_e is an open interval. $\widetilde{\mathcal{G}}$ is a disjoint union

$$\widetilde{\mathcal{G}} = V \cup \bigcup_{e \in E} I_e$$

We further endow $\widetilde{\mathcal{G}}$ with a metric structure by assigning a finite length to each of the $(I_e)_{e \in I}$. The length of I_e is set to be

$$\rho(e) := \frac{1}{2C(e)}$$

which makes I_e isometric to $(0, \rho(e))$. This particular choice of the lengths will be explained farther. Let m be the Borel measure on $\widetilde{\mathcal{G}}$ assigning a zero mass to V, a mass $\rho(e)$ to each of the I_e and to a subinterval of I_e a mass equal to its length. m is σ-finite.

On $\widetilde{\mathcal{G}}$ one can define a standard Brownian motion $B^{\widetilde{\mathcal{G}}}$. Here we give a description through chaining stopped Markovian paths on $\widetilde{\mathcal{G}}$ (see [**BC84**], [**EK01**] and [**Fol14**]). If $B^{\widetilde{\mathcal{G}}}$ starts in the interior I_e of an edge, it behaves as the standard Brownian motion on I_e until it reaches a vertex. To describe the behaviour of $B^{\widetilde{\mathcal{G}}}$ starting from a vertex we use the excursions. Let $x_0 \in V$, $\{x_1, \dots, x_{\deg(x_0)}\}$ the vertices adjacent to x_0 and $\{\{x_0, x_1\}, \dots, \{x_0, x_{\deg(x_0)}\}\}$ the edges joining x_0 to one of its neighbours. Let $(B_t)_{t \geqslant 0}$ be a standard Brownian motion on $\mathbb{R}$ starting from 0. To each excursion e of $(B_t)_{t \geqslant 0}$ away from 0 we associate a random variable $x(\mathsf{e})$ uniformly distributed in $\{x_1, \dots, x_{\deg(x_0)}\}$. We chose the different r.v.'s $x(\mathsf{e})$ to be independent conditional on the family of excursions of $(B_t)_{t \geqslant 0}$. e_t the excursion straddling the time t. Let be

$$T_{\{x_1, \dots, x_{\deg(x_0)}\}} := \inf\{t \geqslant 0 | |B_t| \geqslant \rho(\{x_0, x(\mathsf{e}_t)\})\}$$

To the path $(B_t)_{0 \leqslant t \leqslant T_{\{x_1, \dots, x_{\deg(x_0)}\}}}$ we associate a path in $\widetilde{\mathcal{G}}$: it starts at x_0 and each excursion e of $(B_t)_{0 \leqslant t \leqslant T_{\{x_1, \dots, x_{\deg(x_0)}\}}}$ is performed in $I_{\{x_0, x(\mathsf{e})\}}$ instead of $\mathbb{R}$. The obtained path has the law of $B^{\widetilde{\mathcal{G}}}$ starting at x_0 and stopped at reaching $\{x_1, \dots, x_{\deg(x_0)}\}$. Let $(L_t^y(B))_{t \geqslant 0, y \in \mathbb{R}}$ be the continuous family of local times of B and $(L_t^y(B^{\widetilde{\mathcal{G}}}))_{t \geqslant 0, y \in \widetilde{\mathcal{G}}}$ the family of local times of $B^{\widetilde{\mathcal{G}}}$ started at x_0, relative to the measure m. Let $y \in I_{\{x_0, x_i\}}$ and δ be the length of the subinterval (x_0, y) of $I_{\{x_0, x_i\}}$. Then

$$\begin{aligned} &(L_t^y(B^{\widetilde{\mathcal{G}}}))_{0 \leqslant t \leqslant T_{\{x_1, \dots, x_{\deg(x_0)}\}}} \\ &\qquad = \left(\int_0^t 1_{x(\mathsf{e}_s) = x_i} (dL_s^{\delta}(B) + dL_s^{-\delta}(B)) \right)_{0 \leqslant t \leqslant T_{\{x_1, \dots, x_{\deg(x_0)}\}}} \end{aligned}$$

and the limit, uniform in time, of the above process as y converges to x_0 is

$$\left(\frac{2}{\deg(x_0)} L_t^0(B) \right)_{0 \leqslant t \leqslant T_{\{x_1, \dots, x_{\deg(x_0)}\}}}$$

whatever the value of i. Let $\mathcal{B}(x_0, \delta)$ be the ball in $\widetilde{\mathcal{G}}$ around x_0 of radius δ. If $\delta \leqslant \min_{1 \leqslant i \leqslant \deg(x_0)} \rho(\{x_0, x_i\})$ then $m(\mathcal{B}(x_0, \delta)) = \deg(x_0)\delta$. Consequently for any time $t \in [0, T_{\{x_1, \dots, x_{\deg(x_0)}\}}]$

$$\begin{aligned} \lim_{\delta \to 0} \frac{1}{m(\mathcal{B}(x_0, \delta))} \int_0^t 1_{B_s^{\widetilde{\mathcal{G}}} \in \mathcal{B}(x_0, \delta)} ds &= \lim_{\delta \to 0} \frac{1}{\deg(x_0)\delta} \int_0^t 1_{|B_s| < \delta} ds \\ &= \frac{2}{\deg(x_0)} L_t^0(B) \end{aligned}$$

It follows that the process $(B_t^{\tilde{\mathcal{G}}})_{0\leqslant t\leqslant T_{\{x_1,\dots,x_{\deg(x_0)}\}}}$ has a space-time continuous family of local times. By concatenating different stopped paths we get that the whole process $B^{\tilde{\mathcal{G}}}$ has space-time continuous local times relative to the measure m. The measure on the height of excursions (in absolute value) induced by the measure on Brownian excursions is (see [**RY99**], chapter XII, §4)

$$1_{a>0}\frac{da}{a^2}$$

It follows that $L^0_{T_{\{x_1,\dots,x_{\deg(x_0)}\}}}(B)$ is an exponential random variable with mean

$$\frac{\deg(x_0)}{\sum_{i=1}^{\deg x_0}\rho(\{x_0,x_i\})^{-1}}=\frac{\deg(x_0)}{2\sum_{i=1}^{\deg x_0}C(x_0,x_i)}$$

$L^{x_0}_{T_{\{x_1,\dots,x_{\deg(x_0)}\}}}(B^{\tilde{\mathcal{G}}})$ is an exponential random variable with mean

$$\frac{1}{\sum_{i=1}^{\deg x_0}C(x_0,x_i)}$$

and

$$\mathbb{P}_{x_0}\left(B^{\tilde{\mathcal{G}}}_{T_{\{x_1,\dots,x_{\deg(x_0)}\}}}=x_j\right)=\frac{C(x_0,x_j)}{\sum_{i=1}^{\deg x_0}C(x_0,x_i)}$$

(see also theorem 2.1 in [**Fol14**]). This explains our particular choice of the lengths $(\rho(e))_{e\in E}$.

From now on the Brownian motion $B^{\tilde{\mathcal{G}}}$ on $\tilde{\mathcal{G}}$ is considered to be constructed and the starting point to be arbitrary. It is not excluded that $B^{\tilde{\mathcal{G}}}$ blows up in finite time. A necessary but not sufficient condition of this is the existence of a path of finite length that visits infinitely many vertices. Let $\tilde{\kappa}$ be the following measure on $\tilde{\mathcal{G}}$:

$$\tilde{\kappa}:=\sum_{x\in V}\kappa(x)\delta_x$$

Let $\tilde{\zeta}$ be the first time either $B^{\tilde{\mathcal{G}}}$ blows up or the additive functional

$$\int_{y\in\tilde{\mathcal{G}}}L^y_t(B^{\tilde{\mathcal{G}}})\tilde{\kappa}(dy)=\sum_{x\in V}L^x_t(B^{\tilde{\mathcal{G}}})\kappa(x)$$

hits an independent exponential time with mean 1. $\tilde{\zeta}=+\infty$ a.s. if $\kappa\equiv 0$ and $B^{\tilde{\mathcal{G}}}$ is conservative. For $l\geqslant 0$ let τ_l be the stopping time

$$\tau_l:=\inf\left\{T\geqslant 0\middle|\sum_{x\in V}L^x_t(B^{\tilde{\mathcal{G}}})\geqslant l\right\}$$

If the starting point of $B^{\tilde{\mathcal{G}}}$ is a vertex then the process $(B^{\tilde{\mathcal{G}}}_{\tau_l})_{0\leqslant l<\sum_{x\in V}L^x_{\tilde{\zeta}}(B^{\tilde{\mathcal{G}}})}$ has the same law as the Markov jump process X on V. In particular it follows that the process $(B^{\tilde{\mathcal{G}}}_t)_{0\leqslant t<\tilde{\zeta}}$ is transient.

The 0-potential of the process $(B^{\tilde{\mathcal{G}}}_t)_{0\leqslant t<\tilde{\zeta}}$ has a density relative to the measure m, the Green's function $(G(y,z))_{y,z\in\tilde{\mathcal{G}}}$. We use the same notation as for the Green's function of X because the latter is the restriction to V of the first. The value of $(G(y,z))_{y,z\in\tilde{\mathcal{G}}}$ on the interior of the edges is obtained from its value on the vertices by linear interpolation. Let (x_1,y_1) and (x_2,y_2) be two pairs of adjacent vertices in $\mathcal{G}$. Let z_1 respectively z_2 be a point in the interval $[x_1,y_1]$ respectively $[x_2,y_2]$ and r_1 respectively r_2 be the length of $[x_1,z_1]$

respectively $[x_2, z_2]$. Then

$$G(z_1,z_2)=\frac{1}{\rho(\{x_1,y_1\})\rho(\{x_2,y_2\})}\Big((\rho(\{x_1,y_1\})-r_1)(\rho(\{x_2,y_2\})-r_2)G(x_1,x_2) \\ +r_1r_2G(y_1,y_2)+r_1(\rho(\{x_2,y_2\})-r_2)G(y_1,x_2)+(\rho(\{x_1,y_1\})-r_1)r_2G(x_1,y_2)\Big) \tag{4.2.1}$$

Let $(\phi_y)_{y\in\widetilde{\mathcal{G}}}$ be the Gaussian free field on $\widetilde{\mathcal{G}}$ with variance-covariance function G. It's restriction to V is the Gaussian free field on the graph $\mathcal{G}$, hence the same notation. Conditional on $(\phi_x)_{x\in V}$, $(\phi_y)_{y\in\widetilde{\mathcal{G}}}$ is obtained by joining on every edge e the two values of ϕ on its endpoints by an independent bridge of length $\rho(e)$ of a Brownian motion with variance 2 at time 1 (not a standard Brownian bridge). In particular $(\phi_y)_{y\in\widetilde{\mathcal{G}}}$ has a continuous version.

The process $(B_t^{\widetilde{\mathcal{G}}})_{0\leqslant t<\tilde{\zeta}}$ fits into the framework of [**FR14**] and one can associate to it a measure on time-parametrized continuous loops $\tilde{\mu}$. Let $\widetilde{\mathcal{L}}_\alpha$ be the Poisson point process of loops of intensity $\alpha\tilde{\mu}$. We would like to stress that by loop we only mean a continuous paths with the same starting and endpoint without assumptions on its homotopy class and actually most loops in $\widetilde{\mathcal{L}}_\alpha$ are topologically trivial. Just as the process $(B_t^{\widetilde{\mathcal{G}}})_{0\leqslant t<\tilde{\zeta}}$ itself, the loop $\tilde{\gamma}\in\widetilde{\mathcal{L}}_\alpha$ can be endowed with space-time continuous local times $(L_t^y(\tilde{\gamma}))_{0\leqslant t\leqslant t_{\tilde{\gamma}},y\in\widetilde{\mathcal{G}}}$ relative to the measure m. The occupation field $(\widehat{\mathcal{L}}_\alpha^y)_{y\in\widetilde{\mathcal{G}}}$ is defined as

$$\widehat{\mathcal{L}}_\alpha^y=\sum_{\tilde{\gamma}\in\widetilde{\mathcal{L}}_\alpha}L_{t_{\tilde{\gamma}}}^y(\tilde{\gamma})$$

The restriction of $(\widehat{\mathcal{L}}_\alpha^y)_{y\in\widetilde{\mathcal{G}}}$ to the set of vertices V has the same law as the occupation field of the discrete loops $\mathcal{L}_\alpha$, hence the same notation. As in the discrete case, at $\alpha=\frac{1}{2}$, $(\widehat{\mathcal{L}}_{\frac{1}{2}}^y)_{y\in\widetilde{\mathcal{G}}}$ has the same law as $(\frac{1}{2}\phi_y^2)_{y\in\widetilde{\mathcal{G}}}$ (see Theorem 3.1 in [**FR14**]).

The discrete-space loops of $\mathcal{L}_\alpha$ can be obtained from the continuous loops $\widetilde{\mathcal{L}}_\alpha$ by taking the print of the latter on V. This is described in [**FR14**], section 7.3, or in a less general situation of the restriction of the loops of one-dimensional diffusions to a discrete subset in [**Lup13**], section 3.7. We explain how the restriction from $\widetilde{\mathcal{G}}$ to V works. First of all we consider only the subset $\{\tilde{\gamma}\in\widetilde{\mathcal{L}}_\alpha|\tilde{\gamma}\text{ visits }V\}$ because the print of other loops on V is empty. Next we re-root the loops so as to have the starting point in V: to each loop $\tilde{\gamma}$ visiting V we associate a uniform r.v. on $(0,1)$ $U_{\tilde{\gamma}}$, these different r.v.'s being independent conditional on the loops. We introduce the time

$$\tau^V(\tilde{\gamma}):=\inf\left\{t\in[0,t_{\tilde{\gamma}}]\Big|\sum_{x\in V}L_t^x(\tilde{\gamma})\geqslant U_{\tilde{\gamma}}\sum_{x\in V}L_{t_{\tilde{\gamma}}}^x(\tilde{\gamma})\right\}$$

For each loop $\tilde{\gamma}$ visiting V we make a rotation of parametrization so as to have the starting and end-time at $\tau^V(\tilde{\gamma})$ instead of 0. Let $\widetilde{\mathcal{L}}'$ be the set of the new re-parametrized loops. For each $\tilde{\gamma}'\in\widetilde{\mathcal{L}}'$ and $l\in\left[0,\sum_{x\in V}L_{t_{\tilde{\gamma}'}}^x(\tilde{\gamma}')\right]$ we define

$$\tau_l^V(\tilde{\gamma}'):=\inf\left\{t\in[0,t_{\tilde{\gamma}'}]\Big|\sum_{x\in V}L_t^x(\tilde{\gamma}')\geqslant l\right\}$$

The set of V-valued loops

$$\left\{(\tilde{\gamma}'_{\tau_l^V(\tilde{\gamma}')})_{0\leqslant l\leqslant\sum_{x\in V}L_{t_{\tilde{\gamma}'}}^x(\tilde{\gamma}')}|\tilde{\gamma}'\in\widetilde{\mathcal{L}}'\right\}$$

has the same law as $\mathcal{L}_\alpha$.

Next we explain how to reconstruct $\{\tilde{\gamma} \in \tilde{\mathcal{L}}_\alpha | \tilde{\gamma} \text{ visits } V\}$ from $\mathcal{L}_\alpha$ by adding random excursions to the discrete-space loops. We won't give the proof of this. For elements supporting what we explain see [**Jan11**], chapter 7, and [**Lup13**], corollary 3.11. Let $x_0 \in V$ and $\{x_1, \dots, x_{\deg(x_0)}\}$ the vertices adjacent to x_0. Let η_+ be the intensity measure of positive Brownian excursions. To every loop $\gamma \in \mathcal{L}_\alpha$ spending a time l in x_0 before jumping to one of its neighbours or before stopping one has to add excursions from x_0 to x_0 in $\bigcup_{i=1}^{\deg(x_0)} I_{\{x_0,x_i\}}$ according to a Poisson point process, the intensity of excursions that take place inside the edge $I_{\{x_0,x_i\}}$ being

$$l \times 1_{\text{height excursion } < \rho(\{x_0,x_i\})} \eta_+ \tag{4.2.2}$$

Let x, y be two adjacent vertices. Whenever a loop $\gamma \in \mathcal{L}_\alpha$ jumps from x to y one has to add a Brownian excursion from x to y inside $I_{\{x,y\}}$ (a Brownian excursion from 0 to $a > 0$ is a Bessel-3 process started from 0 run until hitting a). All the added excursions have to be independent conditional on $\mathcal{L}_\alpha$. At this stage we get a Poisson point process of continuous loops in $\tilde{\mathcal{G}}$, but all have a starting point lying in V. The final step is to choose for each a new random starting point distributed uniformly on their duration. What we get has the law of $\{\tilde{\gamma} \in \tilde{\mathcal{L}}_\alpha | \tilde{\gamma} \text{ visits } V\}$.

From now on we assume that $\mathcal{L}_\alpha$ and $\tilde{\mathcal{L}}_\alpha$ are naturally coupled on the same probability space through restriction. $\tilde{\mathcal{L}}_\alpha$ has loop clusters and we will denote by $\tilde{\mathfrak{C}}_\alpha$ the set of these clusters. Obviously each cluster of $\mathcal{L}_\alpha$ is contained in a cluster of $\tilde{\mathcal{L}}_\alpha$, but with positive probability a cluster of $\tilde{\mathcal{L}}_\alpha$ may contain several clusters of $\mathcal{L}_\alpha$. We will prove the following:

PROPOSITION 4.2.1. *There is a coupling between the Poisson ensemble of loops $\tilde{\mathcal{L}}_{\frac{1}{2}}$ and a continuous version of the Gaussian free field $(\phi_y)_{y \in \tilde{\mathcal{G}}}$ such that the two constraints hold:*

- *For all $y \in \tilde{\mathcal{G}}$, $\hat{\mathcal{L}}^y_{\frac{1}{2}} = \frac{1}{2}\phi_y^2$*
- *The clusters of loops of $\tilde{\mathcal{L}}_{\frac{1}{2}}$ are exactly the sign clusters of $(\phi_y)_{y \in \tilde{\mathcal{G}}}$*

Theorem 4.1 follows from the above proposition because the restriction of $(\hat{\mathcal{L}}^y_{\frac{1}{2}})_{y \in \tilde{\mathcal{G}}}$ to V is $(\hat{\mathcal{L}}^x_{\frac{1}{2}})_{x \in V}$, the restriction of $(\phi_y)_{y \in \tilde{\mathcal{G}}}$ to V is the Gaussian free field on $\mathcal{G}$ and the sign of ϕ is constant on the clusters of $\tilde{\mathcal{L}}_{\frac{1}{2}}$, hence also constant on the clusters of $\mathcal{L}_{\frac{1}{2}}$.

The first step in proving proposition 4.2.1 is to show that there is a realisation of $\tilde{\mathcal{L}}_\alpha$ such that its occupation field $(\hat{\mathcal{L}}^y_\alpha)_{y \in \tilde{\mathcal{G}}}$ is continuous. We know already that each individual loop in $\tilde{\mathcal{L}}_\alpha$ has space-time continuous local times and that the process $(\hat{\mathcal{L}}^y_\alpha)_{y \in \tilde{\mathcal{G}}}$ considered for itself, regardless of the loops, has a continuous version (see [**Lup13**], section 4.2). However this does not automatically imply that a realisation of $(\hat{\mathcal{L}}^y_\alpha)_{y \in \tilde{\mathcal{G}}}$ as the occupation field of $\tilde{\mathcal{L}}_\alpha$ can be made continuous (there are infinitely many loops above each point in $\tilde{\mathcal{G}}$ and the occupation field is an infinite sum of continuous functions). A counterexample is given in [**Lup13**], section 4.2, the remark after proposition 4.5.

LEMMA 4.2.2. *There is a realisation of $\tilde{\mathcal{L}}_\alpha$ such that its occupation field $(\hat{\mathcal{L}}^y_\alpha)_{y \in \tilde{\mathcal{G}}}$ is continuous.*

PROOF. We divide the loops of $\tilde{\mathcal{L}}_\alpha$ into three classes:

(i) The loops that visit at least two vertices in V
(ii) The loops that visit only one vertex in V
(iii) The loops that do not visit any vertex and are contained in the interior of an edge

Above any vertex $x \in V$ are only finitely many loops of type (i) (see [**Jan11**] chapter 2 for the exact expression of their intensity). Each individual loop of type (i) has a continuous occupation field and the sum of this occupation fields is locally finite and therefore continuous.

Let $x_0 \in V$ and $\{x_1, \dots, x_{\deg(x_0)}\}$ the vertices adjacent to x_0. We consider now the loops of type (ii) such that x_0 is the only vertex they visit, which we denote $(\tilde{\gamma}_j)_{j\geqslant 0}$. Conditional on $L^{x_0}_{t_{\tilde{\gamma}_j}}(\tilde{\gamma}_j)$, $\tilde{\gamma}_j$ is obtained by launching excursion from x_0 to x_0 in $\bigcup_{i=1}^{\deg(x_0)} I_{\{x_0,x_i\}}$ according to a Poisson point process, the intensity of excursions that take place inside the edge $I_{\{x_0,x_i\}}$ being (see (4.2.2))

$$L^{x_0}_{t_{\tilde{\gamma}_j}}(\tilde{\gamma}_j) \times 1_{\text{height excursion } <\rho(\{x_0,x_i\})}\eta_+$$

If we consider all the loops $(\tilde{\gamma}_j)_{j\geqslant 0}$ we obtain an intensity

$$\Big(\sum_{j\geqslant 0} L^{x_0}_{t_{\tilde{\gamma}_j}}(\tilde{\gamma}_j)\Big) \times 1_{\text{height excursion } <\rho(\{x_0,x_i\})}\eta_+$$

The continuity of the occupation field of $(\tilde{\gamma}_j)_{j\geqslant 0}$ follows from the continuity of Brownian local times.

Let e be an edge. We consider the loops of type (iii) that are contained in I_e. They have the same law as a Poisson ensemble of loops of parameter α associated to the standard Brownian motion on the bounded interval I_e killed upon reaching either of its boundary points. This situation was entirely covered in [**Lup13**]. According to corollary 5.5 in [**Lup13**] it is possible to construct these loops and a continuous version on their occupation field on the same probability space. All the subtlety of our lemma lies in this point. Moreover according to proposition 4.6 in [**Lup13**] the occupation field of these loops converges to 0 at the end-vertices of I_e. □

From now on we consider only the continuous realization of the occupation field $(\hat{\mathcal{L}}^y_\alpha)_{y\in\tilde{\mathcal{G}}}$. We call a positive component of $(\hat{\mathcal{L}}^y_\alpha)_{y\in\tilde{\mathcal{G}}}$ a maximal connected subset of $\tilde{\mathcal{G}}$ on which the occupation field is positive. It is open and by continuity the occupation field is zero on the boundary of a positive component. Given a continuous loop $\tilde{\gamma}$, $Range(\tilde{\gamma})$ will denote its range.

LEMMA 4.2.3. *Let $\tilde{\mathcal{C}} \in \tilde{\mathfrak{C}}_\alpha$ be a cluster of $\tilde{\mathcal{L}}_\alpha$. Then*

$$\bigcup_{\tilde{\gamma}\in\tilde{\mathcal{C}}} Range(\tilde{\gamma})$$

is a positive component of $(\hat{\mathcal{L}}^y_\alpha)_{y\in\tilde{\mathcal{G}}}$. Conversely every positive component of $(\hat{\mathcal{L}}^y_\alpha)_{y\in\tilde{\mathcal{G}}}$ is of this form.

PROOF. The following almost sure properties hold:

(i) For every $\tilde{\gamma} \in \tilde{\mathcal{L}}_\alpha$ the occupation field of $\tilde{\gamma}$ is positive in the interior of $Range(\tilde{\gamma})$ and zero on the boundary $\partial Range(\tilde{\gamma})$
(ii) For every $\tilde{\gamma} \in \tilde{\mathcal{L}}_\alpha$ and $y \in \partial Range(\tilde{\gamma})$, there is another loop $\tilde{\gamma}' \in \tilde{\mathcal{L}}_\alpha$ such that y is contained in the interior of $Range(\tilde{\gamma}')$

The property (i) comes from an analogous property of a finite duration one-dimensional Brownian path: its occupation field is positive on its range, except at the maximum and the minimum where it is zero.

We briefly explain why the property (ii) is true. First of all the boundary $\partial Range(\tilde{\gamma})$ is finite because it can intersect an edge in at most two points and a loop visits finitely many edges. Moreover any deterministic point in $\tilde{\mathcal{G}}$ is almost surely covered by the interior of the range of a loop. Applying Palm's identity one gets (ii):

$$\begin{aligned}&\mathbb{E}\left[\sharp\{\tilde{\gamma}\in\widetilde{\mathcal{L}}_{\alpha}|\partial Range(\tilde{\gamma}) \text{ not covered by interiors of the ranges of loops in } \widetilde{\mathcal{L}}_{\alpha}\}\right]\\ =&\alpha\int\mathbb{P}(\partial Range(\tilde{\gamma}) \text{ not covered by interiors of the ranges of loops in } \widetilde{\mathcal{L}}_{\alpha})\tilde{\mu}(d\tilde{\gamma})\\ =&0\end{aligned}$$

Properties (i) and (ii) imply on one hand that the zero set of $(\hat{\mathcal{L}}^{y}_{\alpha})_{y\in\widetilde{\mathcal{G}}}$ is exactly the set of all point in $\widetilde{\mathcal{G}}$ that are not visited by any loop in $\widetilde{\mathcal{L}}_{\alpha}$ and on the other hand that any point visited by a loop cannot belong to the boundary of a cluster of loops. This in turn implies the lemma. □

Proof of proposition 4.2.1. First sample $\widetilde{\mathcal{L}}_{\frac{1}{2}}$ with a continuous version of its occupation field. Consider $\left(\sqrt{2\hat{\mathcal{L}}^{y}_{\frac{1}{2}}}\right)_{y\in\widetilde{\mathcal{G}}}$ as a realization of $(|\phi_y|)_{y\in\widetilde{\mathcal{G}}}$ and sample the sign of the Gaussian free field ϕ independently from $\widetilde{\mathcal{L}}_{\frac{1}{2}}$ conditional on $\left(\sqrt{2\hat{\mathcal{L}}^{y}_{\frac{1}{2}}}\right)_{y\in\widetilde{\mathcal{G}}}$. Then according to lemma 4.2.3 the clusters of $\widetilde{\mathcal{L}}_{\frac{1}{2}}$ are exactly the positive components of $(|\phi_y|)_{y\in\widetilde{\mathcal{G}}}$ which are the sign clusters of $(\phi_y)_{y\in\widetilde{\mathcal{G}}}$. □

4.3. Alternative description of the coupling

In this section we give en alternative description on the coupling between $\mathcal{L}_{\frac{1}{2}}$ and $(\phi_x)_{x\in V}$ constructed in section 4.2 but that does not use $\widetilde{\mathcal{L}}_{\frac{1}{2}}$ as intermediate. First we deal with the law of the sign of ϕ conditional on $(|\phi_y|)_{y\in\widetilde{\mathcal{G}}}$. We will show that one has to chose the sign independently for each positive component of $(|\phi_y|)_{y\in\widetilde{\mathcal{G}}}$ and uniformly distributed in $\{-1,+1\}$. Then we will deal with the probability of a cluster of continuous loops occupying entirely an edge e conditional on discrete-space loops $\mathcal{L}_{\frac{1}{2}}$ and on the event that none of these loops occupies e.

Let K be a non-empty compact connected subset of $\widetilde{\mathcal{G}}$. ∂K is finite, $\widetilde{\mathcal{G}}\backslash K$ has finitely many connected components and the closure of each of these connected components is itself a metric graph associated to some discrete graph. Let T_K be the first time the Brownian motion $B^{\widetilde{\mathcal{G}}}$, started outside K, hits K. Let $(G^{\widetilde{\mathcal{G}}\backslash K}(y,z))_{y,z\in\widetilde{\mathcal{G}}\backslash K}$ be the Green's function relative to the measure m of the killed process $(B^{\widetilde{\mathcal{G}}}_t)_{0\leqslant t<\tilde{\zeta}\wedge T_K}$. $G^{\widetilde{\mathcal{G}}\backslash K}$ is symmetric, continuous and extends continuously to $\overline{\widetilde{\mathcal{G}}\backslash K}$ by taking value 0 on the boundary. Actually $G^{\widetilde{\mathcal{G}}\backslash K}$ is obtained by linear interpolation from its values on the vertices and ∂K as in (4.2.1). Let $(\phi^{\widetilde{\mathcal{G}}\backslash K}_y)_{y\in\widetilde{\mathcal{G}}\backslash K}$ be the Gaussian free field on $\widetilde{\mathcal{G}}\backslash K$ with variance-covariance function $G^{\widetilde{\mathcal{G}}\backslash K}$. Let f be a function on ∂K and $u_{f,K}$ be the following function on $\widetilde{\mathcal{G}}\backslash K$:

$$u_{f,K}(y):=\mathbb{E}_y\left[f(B^{\widetilde{\mathcal{G}}}_{T_K})1_{T_K<\tilde{\zeta}}\right]$$

By the Markov property of $(\phi_y)_{y\in\widetilde{\mathcal{G}}}$, conditional on $(\phi_y)_{y\in K}$, $(\phi_y)_{y\in\widetilde{\mathcal{G}}\backslash K}$ has the same law as $(u_{\phi,K}(y)+\phi^{\widetilde{\mathcal{G}}\backslash K}_y)_{y\in\widetilde{\mathcal{G}}\backslash K}$. We consider now a random connected compact subset $\mathcal{K}$. We use the equivalent σ-algebras on the connected compact subsets:

- the σ-algebra induced by the events $(\{\mathcal{K}\subseteq U\})_{U \text{ open subset of } \widetilde{\mathcal{G}}}$
- the σ-algebra induced by the events $(\{F\cap\mathcal{K}\neq\varnothing\})_{F \text{ closed subset of } \widetilde{\mathcal{G}}}$

Below we state a strong Markov property for the Gaussian free field $(\phi_y)_{y\in\widetilde{\mathcal{G}}}$. It can be derived from the simple Markov property (see [**Roz82**], chapter 2, §2.4, theorem 4).

STRONG MARKOV PROPERTY. *Let $\mathcal{K}$ be a random compact connected subset of $\widetilde{\mathcal{G}}$ such that for every deterministic open subset U of $\widetilde{\mathcal{G}}$ the event $\{\mathcal{K} \subseteq U\}$ is measurable with respect to $(\phi_y)_{y\in U}$. Then conditional on $\mathcal{K}$ and $(\phi_y)_{y\in\mathcal{K}}$, $(\phi_y)_{y\in\widetilde{\mathcal{G}}\backslash\mathcal{K}}$ has the same law as $(u_{\phi,\mathcal{K}}(y) + \phi_y^{\widetilde{\mathcal{G}}\backslash\mathcal{K}})_{y\in\widetilde{\mathcal{G}}\backslash\mathcal{K}}$.*

LEMMA 4.3.1. *Given $y_0 \in \widetilde{\mathcal{G}}$ we denote by F_{y_0} the closure of the positive component of $(|\phi_y|)_{y\in\widetilde{\mathcal{G}}}$ containing y_0 (a.s. $\phi_{y_0} \neq 0$). Then the field $(-1_{y\in F_{y_0}}\phi_y + 1_{y\notin F_{y_0}}\phi_y)_{y\in\widetilde{\mathcal{G}}}$ has the same law as the Gaussian free field $(\phi_y)_{y\in\widetilde{\mathcal{G}}}$.*

PROOF. By construction F_{y_0} is closed and connected, but not necessarily compact if V is not finite. ϕ is zero on ∂F_{y_0}.

We first consider the case of V being finite. Then F_{y_0} is compact. According to the strong Markov property, conditional on F_{y_0} and $(\phi_y)_{y\in F_{y_0}}$, $(\phi_y)_{y\in\widetilde{\mathcal{G}}\backslash F_{y_0}}$ has the same law as $(\phi_y^{\widetilde{\mathcal{G}}\backslash F_{y_0}})_{y\in\widetilde{\mathcal{G}}\backslash F_{y_0}}$. But $\phi^{\widetilde{\mathcal{G}}\backslash F_{y_0}}$ and $-\phi^{\widetilde{\mathcal{G}}\backslash F_{y_0}}$ have the same law. Thus $(1_{y\in F_{y_0}}\phi_y - 1_{y\notin F_{y_0}}\phi_y)_{y\in\widetilde{\mathcal{G}}}$ has the same law as ϕ. Since ϕ and $-\phi$ have the same law, $(-1_{y\in F_{y_0}}\phi_y + 1_{y\notin F_{y_0}}\phi_y)_{y\in\widetilde{\mathcal{G}}}$ has the same law as ϕ too.

If V is infinite, let $x_0 \in V$. Let V_n be the set of vertices separated from x_0 by at most n edges. V_n is finite. $V_0 = \{x_0\}$ and V_1 is made of x_0 and all its neighbours. For $n \geqslant 1$ let E_n be the set of edges either connecting two vertices in V_{n-1} or a vertex in $V_n\backslash V_{n-1}$ to a vertex in V_{n-1}. $\mathcal{G}_n := (V_n, E_n)$ is a connected sub-graph of $\mathcal{G}$. Let $\widetilde{\mathcal{G}}_n$ be the metric graph associated to the graph $\mathcal{G}_n$, viewed as a compact subset of $\widetilde{\mathcal{G}}$. For n large enough such that $y_0 \in \widetilde{\mathcal{G}}_n$, let $F_{y_0,n}$ be the positive component of $(|\phi_y^{\widetilde{\mathcal{G}}\backslash(V_n\backslash V_{n-1})}|)_{y\in\widetilde{\mathcal{G}}}$ containing y_0, which is compact. As in the previous case, $(-1_{y\in F_{y_0,n}}\phi_y^{\widetilde{\mathcal{G}}\backslash(V_n\backslash V_{n-1})} + 1_{y\notin F_{y_0,n}}\phi_y^{\widetilde{\mathcal{G}}\backslash(V_n\backslash V_{n-1})})_{y\in\widetilde{\mathcal{G}}}$ has the same law as $\phi^{\widetilde{\mathcal{G}}\backslash(V_n\backslash V_{n-1})}$. As n converges to $+\infty$, the first field converges in law to $(-1_{y\in F_{y_0}}\phi_y + 1_{y\notin F_{y_0}}\phi_y)_{y\in\widetilde{\mathcal{G}}}$ and the second field converges in law to ϕ, which proves the lemma. ☐

LEMMA 4.3.2. *Conditional on $(|\phi_y|)_{y\in\widetilde{\mathcal{G}}}$, the sign of ϕ on each of its connected components is distributed independently and uniformly in $\{-1, +1\}$.*

PROOF. Let $(y_n)_{n\geqslant 0}$ be a dense sequence in $\widetilde{\mathcal{G}}$. Let $(\sigma_n)_{n\geqslant 0}$ be an i.i.d. sequence of uniformly distributed variables in $\{-1, +1\}$ independent of ϕ. According to lemma 4.3.1, the field

$$\left(\prod_{n=0}^{N}(\sigma_n 1_{y\in F_{y_n}} + 1_{y\notin F_{y_n}}) \times \phi_y\right)_{y\in\widetilde{\mathcal{G}}}$$

has the same law as ϕ whatever the value of N. Moreover as N converges to $+\infty$, this field converges in law to the field obtained by choosing uniformly and independently a sign for each positive component of $(|\phi_y|)_{y\in\widetilde{\mathcal{G}}}$. ☐

Next we consider the discrete-space loops $\mathcal{L}_{\frac{1}{2}}$ and continuous loops $\widetilde{\mathcal{L}}_{\frac{1}{2}}$ coupled in the natural way though the restriction of the latter to V. We deal with the probability of a cluster of continuous loops occupying entirely an edge e conditional on $\mathcal{L}_{\frac{1}{2}}$ and on the event that none of discrete-space loops occupies e. This event is the same as the occupation field $\widehat{\mathcal{L}}_{\frac{1}{2}}$ staying positive on I_e and not having zeros there. Let $e = \{x, y\}$ be an edge joining vertices x and y. In case e is not occupied by a loop of $\mathcal{L}_{\frac{1}{2}}$, there are three kind of paths visiting I_e:

- the loops of entirely $\widetilde{\mathcal{L}}_{\frac{1}{2}}$ contained in I_e. These are independent $\mathcal{L}_{\frac{1}{2}}$ as they have no print on V. The occupation field of these loops is the square of a standard Brownian bridge of length $\rho(e)$ from 0 at x to 0 at y ([**Lup13**], proposition 4.5).

- the Poisson point process of excursions from x to x inside I_e of the loops in $\widetilde{\mathcal{L}}_{\frac{1}{2}}$ visiting x. The intensity of excursions is

$$\widehat{\mathcal{L}}^x_{\frac{1}{2}} \times 1_{\text{height excursion } <\rho(e)} \eta_+$$

Conditional on $\widehat{\mathcal{L}}^x_{\frac{1}{2}}$, this Poisson point process of excursions is independent from $\mathcal{L}_{\frac{1}{2}}$. Its occupation field is according to the second Ray-Knight theorem the square of a Bessel-0 process with initial value $\widehat{\mathcal{L}}^x_{\frac{1}{2}}$ at x conditioned to hit 0 before time $\rho(e)$.
- the Poisson point process of excursions from y to y inside I_e of the loops in $\widetilde{\mathcal{L}}_{\frac{1}{2}}$ visiting y. The picture is the same as above.

We will denote by $(b_t^{(T)})_{0\leqslant t\leqslant T}$ a standard Brownian bridge from 0 to 0 of length T and $(\beta_t^{(T,l)})_{t\geqslant 0}$ a square of a Bessel 0 process starting from l at $t=0$ and conditioned to hit 0 before time T. We have the following picture:

PROPERTY 4.3.3. *Conditional on the discrete-space loops $\mathcal{L}_{\frac{1}{2}}$, the events of the family $\Big(\{\widehat{\mathcal{L}}_{\frac{1}{2}}$ has a zero on $I_e\}\Big)_{e\in E\setminus\bigcup_{\mathcal{C}\in\mathfrak{C}_{\frac{1}{2}}}\mathcal{C}}$ are independent. Let $e=\{x,y\}$ be an edge. The probability*

$$\mathbb{P}\Big(\widehat{\mathcal{L}}_{\frac{1}{2}} \text{ has a zero on } I_e \Big| \mathcal{L}_{\frac{1}{2}}, e\in E\setminus \bigcup_{\mathcal{C}\in\mathfrak{C}_{\frac{1}{2}}} \mathcal{C}\Big)$$

is the same as for the sum of three independent processes

$$\Big(b_t^{(\rho(e))2} + \beta_t^{(\rho(e),\widehat{\mathcal{L}}^x_{\frac{1}{2}})} + \beta_{\rho(e)-t}^{(\rho(e),\widehat{\mathcal{L}}^y_{\frac{1}{2}})}\Big)_{0\leqslant t\leqslant \rho(e)}$$

having a zero on $(0,\rho(e))$.

LEMMA 4.3.4. *Let $T, l_1, l_2 > 0$. The probability that the sum of three independent processes*

$$\Big(b_t^{(T)2} + \beta_t^{(T,l_1)} + \beta_{T-t}^{(T,l_2)}\Big)_{0\leqslant t\leqslant T} \tag{4.3.1}$$

has a zero on $(0,T)$ is

$$\frac{1}{\sqrt{\pi}}\int_0^{+\infty} \exp\Big(-\frac{l_1 l_2}{(2T)^2 s} - s\Big)\frac{ds}{\sqrt{s}} \tag{4.3.2}$$

PROOF. We will break the symmetry of the expression (4.3.1) and use the fact that the process $\Big(b_t^{(T)2}+\beta_{T-t}^{(T,l_2)}\Big)_{0\leqslant t\leqslant T}$ has the same law as the square of a standard Brownian bridge of length T from 0 to $\sqrt{l_2}$ (see [**RY99**], chapter XI, §3). For the process (4.3.1) to have a zero on $(0,T)$, the process $\beta^{(T,l_1)}$ has to hit 0 before the last zero of $\Big(b_t^{(T)2} + \beta_{T-t}^{(T,l_2)}\Big)_{0\leqslant t\leqslant T}$.

According to Ray-Knight's theorem, the time when the square Bessel 0 started from l_1 hits 0 has the same law as the maximum of a standard Brownian motion started from 0 and stopped at its local time at 0 reaching the level l_1. The distribution of this maximum is

$$1_{a>0}\frac{l_1}{2a^2}\exp\Big(-\frac{l_1}{2a}\Big)\,da$$

In $\beta^{(T,l_1)}$ we condition on hitting zero before time T. So the distribution of the first zero is

$$1_{0<t_2<T}\frac{l_1}{2t_1^2}\exp\Big(\frac{l_1}{2T}-\frac{l_1}{2t_1}\Big)\,dt_1 \tag{4.3.3}$$

Let $(B_t)_{t\geqslant 0}$ be a standard Brownian motion on $\mathbb{R}$ started from 0 and

$$g_T := \sup\{t \in [0,T] | B_t = 0\}$$

The joint distribution of (g_T, B_T) is (see **[RY99]**, chapter XII, §3)

$$1_{0<a<T}\frac{|x|\exp\left(-\dfrac{x^2}{2(T-a)}\right)}{2\pi\sqrt{a(T-a)^3}}dadx$$

If we condition by $B_T = \sqrt{l_2}$ we get the distribution of the last zero of $\left(b_t^{(T)2} + \beta_{T-t}^{(T,l_2)}\right)_{0\leqslant t\leqslant T}$ which is

$$1_{0<t_1<T}\frac{\sqrt{l_2 T}\exp\left(\dfrac{l_2}{2T} - \dfrac{l_2}{2(T-t_2)}\right)}{\sqrt{2\pi t_2(T-t_2)^3}}dt_2 \tag{4.3.4}$$

Gathering (4.3.3) and (4.3.4) we get that the probability that we are interested in is

$$\sqrt{\frac{l_2T}{2\pi}}\exp\left(\frac{l_1+l_2}{2T}\right)\int_{0<t_1<t_2<T}\frac{l_1}{2t_1^2}\exp\left(-\frac{l_1}{2t_1}\right)\frac{\exp\left(-\dfrac{l_2}{2(T-t_2)}\right)}{\sqrt{t_2(T-t_2)^3}}dt_1dt_2$$
$$= \sqrt{\frac{l_2T}{2\pi}}\exp\left(\frac{l_1+l_2}{2T}\right)\int_{0<t_2<T}\exp\left(-\frac{l_1}{2t_2} - \frac{l_2}{2(T-t_2)}\right)\frac{dt_2}{\sqrt{t_2(T-t_2)^3}}$$

By performing the change of variables

$$s := \frac{l_2}{2T}\frac{t_2}{T-t_2}$$

we get the integral (4.3.2). □

LEMMA 4.3.5. *For all $\lambda \geqslant 0$*

$$\int_0^{+\infty}\exp\Big(-\frac{\lambda}{s} - s\Big)\frac{ds}{\sqrt{s}} = \sqrt{\pi}e^{-2\sqrt{\lambda}}$$

PROOF. Let

$$f(\lambda) := \int_0^{+\infty}\exp\Big(-\frac{\lambda}{s} - s\Big)\frac{ds}{\sqrt{s}}$$

Then $f(0) = \Gamma(\frac{1}{2}) = \sqrt{\pi}$ and

$$f'(\lambda) = -\int_0^{+\infty}\exp\Big(-\frac{\lambda}{s} - s\Big)\frac{ds}{\sqrt{s^3}}$$

By doing the change of variables $z = \frac{\lambda}{s}$ we get

$$f'(\lambda) = -\frac{1}{\sqrt{\lambda}}\int_0^{+\infty}\exp\Big(-z - \frac{\lambda}{z}\Big)\frac{dz}{\sqrt{z}}$$

f satisfies the ODE

$$f'(\lambda) = -\frac{1}{\sqrt{\lambda}}f(\lambda)$$

with initial condition $f(0) = \sqrt{\pi}$, thus $f(\lambda) = \sqrt{\pi}e^{-2\sqrt{\lambda}}$. □

COROLLARY 4.3.6. *Conditional on the discrete-space loops $\mathcal{L}_{\frac{1}{2}}$, the events of the family $\Big(\{\widehat{\mathcal{L}}_{\frac{1}{2}}$ has a zero on $I_e\}\Big)_{e\in E\setminus\bigcup_{\mathcal{C}\in\mathfrak{C}_{\frac{1}{2}}}\mathcal{C}}$ are independent and the corresponding probabilities are given by*

$$\mathbb{P}\Big(\widehat{\mathcal{L}}_{\frac{1}{2}} \text{ has a zero on } I_{\{x,y\}}\Big|\mathcal{L}_{\frac{1}{2}}, \{x,y\}\in E\setminus\bigcup_{\mathcal{C}\in\mathfrak{C}_{\frac{1}{2}}}\mathcal{C}\Big) = \exp\Big(-\frac{1}{\rho(\{x,y\})}\sqrt{\widehat{\mathcal{L}}^x_{\frac{1}{2}}\widehat{\mathcal{L}}^y_{\frac{1}{2}}}\Big)$$
$$= \exp\Big(-2C(x,y)\sqrt{\widehat{\mathcal{L}}^x_{\frac{1}{2}}\widehat{\mathcal{L}}^y_{\frac{1}{2}}}\Big)$$

From lemma 4.3.2 and corollary 4.3.6 we get the following alternative description of the coupling between $\mathcal{L}_{\frac{1}{2}}$ and $(\phi_x)_{x\in V}$ (see figure 4.1):

THEOREM 4.1. bis. *Consider the following construction:*

- *First sample the Poisson ensemble of loops $\mathcal{L}_{\frac{1}{2}}$ with $(\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{x\in V}$ being its occupation field and $\mathfrak{C}_{\frac{1}{2}}$ the set of its clusters.*
- *For any edge $\{x,y\}$ not visited by any loop in $\mathcal{L}_{\frac{1}{2}}$, choose to open it with probability $1-\exp\Big(-2C(x,y)\sqrt{\widehat{\mathcal{L}}^x_{\frac{1}{2}}\widehat{\mathcal{L}}^y_{\frac{1}{2}}}\Big)$. By doing so some cluster of $\mathfrak{C}_{\frac{1}{2}}$ may merge and this induces a partition $\mathfrak{C}'$ of V in larger clusters.*
- *For all clusters $\mathcal{C}'\in\mathfrak{C}'$ sample independent uniformly distributed in $\{-1,+1\}$ r.v. $\sigma(\mathcal{C}')$.*
- *Set $\phi_x := \sigma(\mathcal{C}'(x))\sqrt{2\widehat{\mathcal{L}}^x_{\frac{1}{2}}}$ where $\mathcal{C}'(x)$ is the cluster in $\mathfrak{C}'$ containing the vertex x.*

$(\phi_x)_{x\in V}$ is then a Gaussian free field on $\mathcal{G}$. Moreover the obtained coupling between $\mathcal{L}_{\frac{1}{2}}$ and ϕ is the same, in law, as the one constructed in section 4.2.

Observe that a posteriori the quantity $1-\exp\Big(-2C(x,y)\sqrt{\widehat{\mathcal{L}}^x_{\frac{1}{2}}\widehat{\mathcal{L}}^y_{\frac{1}{2}}}\Big)$ equals

$$1-e^{-C(x,y)|\phi_x\phi_y|}$$

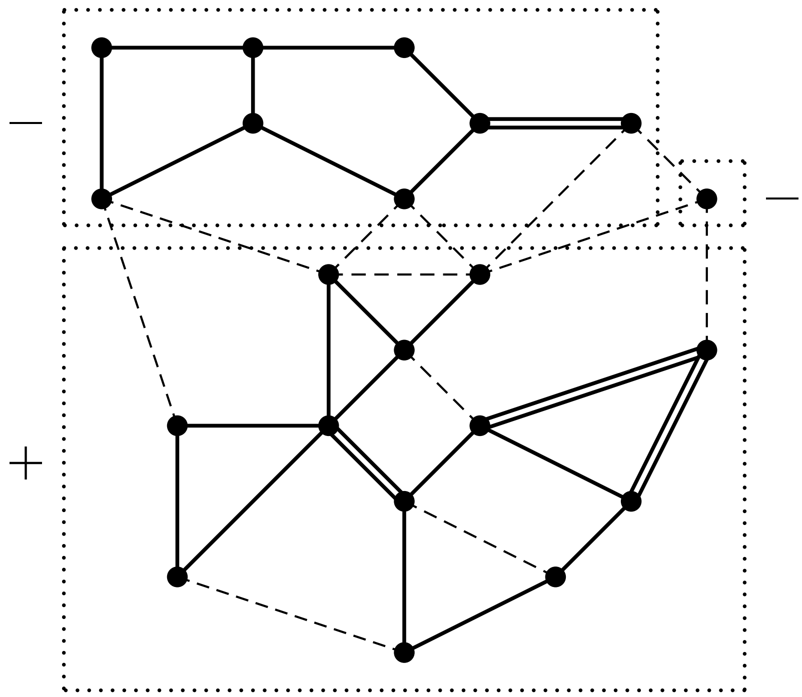

Fig.4.1: Full lines are the edges visited by discrete loops. Double lines are additionaly opened edges.

Dashed lines are edges left closed. Dotted contours surround clusters in $\mathfrak{C}'$.

4.4. Alternative proof of the coupling

In this section we prove directly, without using metric graphs, that the procedure described in theorem 4.1 bis provides a coupling between $\mathcal{L}_{\frac{1}{2}}$ and the Gaussian free field. We will denote by ϕ the field constructed by this procedure and ψ a generic Gaussian free field on $\mathcal{G}$, so as to avoid confusion.

Let $e_1 = \{x_1, y_1\}, \dots, e_n = \{x_n, y_n\}$ be n different edges of $\mathcal{G}$. Let $\mathcal{G}^{(e_1,\dots,e_n)}$ be the graph obtained by removing the edges $e_1, \dots, e_n$. $\mathcal{G}^{(e_1,\dots,e_n)}$ may not be connected. Let $\kappa^{(e_1,\dots,e_n)}$ be the killing measure on V defined as

$$\kappa^{(e_1,\dots,e_n)}(x) := \kappa(x) + \sum_{i=1}^{n} C(e_i)(1_{x=x_i} + 1_{x=y_i})$$

Let $(G^{(e_1,\dots,e_n)}(x,y))_{x,y\in V}$ be the Green's function of the Markov jump process on $\mathcal{G}^{(e_1,\dots,e_n)}$ with jump rates equal to conductances and killing rates given by $\kappa^{(e_1,\dots,e_n)}$. Let $(\psi_x^{(e_1,\dots,e_n)})_{x\in V}$ be the corresponding Gaussian free field on $\mathcal{G}^{(e_1,\dots,e_n)}$. Let H be the energy functional

$$H(f) := \frac{1}{2}\Big(\sum_{x\in V}\kappa(x)f_x^2 + \sum_{x,y\in V,\{x,y\}\in E} C(x,y)(f_x - f_y)^2\Big)$$

and let

$$H^{(e_1,\dots,e_n)}(f) := H(f) + \sum_{i=1}^{n} C(e_i) f_{x_i} f_{y_i}$$

If V is finite the distribution of ψ is

$$\frac{1}{(2\pi)^{\frac{|V|}{2}}\det(G)^{\frac{1}{2}}} e^{-H(f)} \prod_{x\in V} df_x$$

and the distribution of $\psi^{(e_1,\dots,e_n)}$ is

$$\frac{1}{(2\pi)^{\frac{|V|}{2}}\det(G^{(e_1,\dots,e_n)})^{\frac{1}{2}}}e^{-H^{(e_1,\dots,e_n)}(f)}\prod_{x\in V}df_x$$

Conditional on $e_i\notin\bigcup_{\mathcal{C}\in\mathfrak{C}_{\frac{1}{2}}}\mathcal{C}$ for every $i\in\{1,\dots,n\}$, $(\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{x\in V}$ has the same law as $\frac{1}{2}\psi^{(e_1,\dots,e_n)2}$. If V is finite then

$$\mathbb{P}\Big(\forall i\in\{1,\dots,n\},e_i\notin\bigcup_{\mathcal{C}\in\mathfrak{C}_{\frac{1}{2}}}\mathcal{C}\Big)=\frac{\det(G^{(e_1,\dots,e_n)})^{\frac{1}{2}}}{\det(G)^{\frac{1}{2}}}\tag{4.4.1}$$

See [**JL13**].

LEMMA 4.4.1. *Assume that V is finite. Let $e_1=\{x_1,y_1\},\dots,e_n=\{x_n,y_n\}$ be n different edges of $\mathcal{G}$. For any bounded functional on fields F*

$$\mathbb{E}\left[F(\widehat{\mathcal{L}}_{\frac{1}{2}});\forall i\in\{1,\dots,n\},e_i\notin\bigcup_{\mathcal{C}'\in\mathfrak{C}'}\mathcal{C}'\right]=\mathbb{E}\left[\prod_{i=1}^{n}e^{-C(e_i)(|\psi_{x_i}\psi_{y_i}|+\psi_{x_i}\psi_{y_i})}F\Big(\frac{1}{2}\psi^2\Big)\right]\tag{4.4.2}$$

$$\mathbb{E}\left[F(\widehat{\mathcal{L}}_{\frac{1}{2}});E\backslash\bigcup_{\mathcal{C}'\in\mathfrak{C}'}\mathcal{C}'=\{e_1,\dots,e_n\}\right]=\tag{4.4.3}$$
$$\mathbb{E}\left[\prod_{i=1}^{n}e^{-C(e_i)(|\psi_{x_i}\psi_{y_i}|+\psi_{x_i}\psi_{y_i})}\prod_{\substack{\{x,y\}\in\\ E\backslash\{e_1,\dots,e_n\}}}1_{\psi_x\psi_y>0}(1-e^{-2C(x,y)|\psi_x\psi_y|})F\Big(\frac{1}{2}\psi^2\Big)\right]$$

PROOF. We begin with the proof of (4.4.2). Conditional on $e_i\notin\bigcup_{\mathcal{C}\in\mathfrak{C}_{\frac{1}{2}}}\mathcal{C}$ for every $i\in\{1,\dots,n\}$, $(\widehat{\mathcal{L}}^x_{\frac{1}{2}})_{x\in V}$ has the same law as $\frac{1}{2}\psi^{(e_1,\dots,e_n)2}$, that is to say

$$\mathbb{E}\left[F(\widehat{\mathcal{L}}_{\frac{1}{2}})\Big|\forall i\in\{1,\dots,n\},e_i\notin\bigcup_{\mathcal{C}\in\mathfrak{C}_{\frac{1}{2}}}\mathcal{C}\right]=\mathbb{E}\left[F\Big(\frac{1}{2}\psi^{(e_1,\dots,e_n)2}\Big)\right]$$

Applying (4.4.1) we get that

$$\mathbb{E}\left[F(\widehat{\mathcal{L}}_{\frac{1}{2}});\forall i\in\{1,\dots,n\},e_i\notin\bigcup_{\mathcal{C}\in\mathfrak{C}_{\frac{1}{2}}}\mathcal{C}\right]=\frac{\det(G^{(e_1,\dots,e_n)})^{\frac{1}{2}}}{\det(G)^{\frac{1}{2}}}\mathbb{E}\left[F\Big(\frac{1}{2}\psi^{(e_1,\dots,e_n)2}\Big)\right]$$

But

$$\begin{aligned}&\frac{\det(G^{(e_1,\dots,e_n)})^{\frac{1}{2}}}{\det(G)^{\frac{1}{2}}}\mathbb{E}\left[F\Big(\frac{1}{2}\psi^{(e_1,\dots,e_n)2}\Big)\right]\\&=\frac{\det(G^{(e_1,\dots,e_n)})^{\frac{1}{2}}}{\det(G)^{\frac{1}{2}}}\frac{1}{(2\pi)^{\frac{|V|}{2}}\det(G^{(e_1,\dots,e_n)})^{\frac{1}{2}}}\int e^{-H^{(e_1,\dots,e_n)}(f)}F\Big(\frac{1}{2}f^2\Big)\prod_{x\in V}df_x\\&=\frac{1}{(2\pi)^{\frac{|V|}{2}}\det(G)^{\frac{1}{2}}}\int e^{-H(f)}\prod_{i=1}^{n}e^{-C(e_i)f_{x_i}f_{y_i}}F\Big(\frac{1}{2}f^2\Big)\prod_{x\in V}df_x\end{aligned}$$

It follows that

$$\mathbb{E}\left[F(\widehat{\mathcal{L}}_{\frac{1}{2}});\forall i\in\{1,\dots,n\},e_i\notin\bigcup_{\mathcal{C}\in\mathfrak{C}_{\frac{1}{2}}}\mathcal{C}\right]=\mathbb{E}\left[\prod_{i=1}^{n}e^{-C(e_i)\psi_{x_i}\psi_{y_i}}F\Big(\frac{1}{2}\psi^2\Big)\right]$$

Then

$$\mathbb{E}\left[F(\widehat{\mathcal{L}}_{\frac{1}{2}}); \forall i \in \{1,\dots,n\}, e_i \notin \bigcup_{\mathcal{C}'\in\mathfrak{C}'} \mathcal{C}'\right]$$
$$= \mathbb{E}\left[\prod_{i=1}^{n} \exp\Big(-2C(e_i)\sqrt{\widehat{\mathcal{L}}_{\frac{1}{2}}^{x_i}\widehat{\mathcal{L}}_{\frac{1}{2}}^{y_i}}\Big) F(\widehat{\mathcal{L}}_{\frac{1}{2}}); \forall i \in \{1,\dots,n\}, e_i \notin \bigcup_{\mathcal{C}\in\mathfrak{C}_{\frac{1}{2}}} \mathcal{C}\right]$$
$$= \mathbb{E}\left[\prod_{i=1}^{n} e^{-C(e_i)(|\psi_{x_i}\psi_{y_i}|+\psi_{x_i}\psi_{y_i})} F\Big(\frac{1}{2}\psi^2\Big)\right]$$

For the proof of (4.4.3) we will use the inclusion-exclusion principle.

$$\mathbb{E}\left[F(\widehat{\mathcal{L}}_{\frac{1}{2}}); E\backslash \bigcup_{\mathcal{C}'\in\mathfrak{C}'} \mathcal{C}' = \{e_1,\dots,e_n\}\right]$$
$$= \sum_{\substack{A\subseteq E\\ \{e_1,\dots,e_n\}\subseteq A}} (-1)^{|A|-n}\mathbb{E}\left[F(\widehat{\mathcal{L}}_{\frac{1}{2}}); \forall e \in A, e \notin \bigcup_{\mathcal{C}'\in\mathfrak{C}'} \mathcal{C}'\right]$$
$$= \sum_{\substack{A\subseteq E\\ \{e_1,\dots,e_n\}\subseteq A}} (-1)^{|A|-n}\mathbb{E}\left[\prod_{\{x,y\}\in A} e^{-C(x,y)(|\psi_x\psi_y|+\psi_x\psi_y)} F\Big(\frac{1}{2}\psi^2\Big)\right]$$
$$=\mathbb{E}\left[\prod_{i=1}^{n} e^{-C(e_i)(|\psi_{x_i}\psi_{y_i}|+\psi_{x_i}\psi_{y_i})} \prod_{\{x,y\}\in E\backslash\{e_1,\dots,e_n\}} \Big(1-e^{-C(x,y)(|\psi_x\psi_y|+\psi_x\psi_y)}\Big) F\Big(\frac{1}{2}\psi^2\Big)\right]$$

But

$$1-e^{-C(x,y)(|\psi_x\psi_y|+\psi_x\psi_y)} = 1_{\psi_x\psi_y>0}(1-e^{-2C(x,y)|\psi_x\psi_y|})$$

Thus we get (4.4.3). □

PROPOSITION 4.4.2. *The field $(\phi_x)_{x\in V}$ constructed in theorem 4.1 bis has the law of a Gaussian free field on $\mathcal{G}$.*

PROOF. First we consider the case of V being finite and use the identity (4.4.3). Let F be a bounded functional on fields. Given a subset of edges $A \subseteq E$, we will denote by $\mathfrak{C}(A)$ the partition of V obtained by removing from $\mathcal{G}$ the edges in A and taking the connected components. Let $\mathcal{S}_A(F)$ be the functional on non-negative fields defined as

$$\mathcal{S}_A(F)(f) := \frac{1}{2^{|\mathfrak{C}(A)|}} \sum_{\sigma\in\{-1,+1\}^{\mathfrak{C}(A)}} F(\sigma\sqrt{2f})$$

where $F(\sigma\sqrt{2f})$ means that we have made a choice of a sign which is the same on each equivalence class of the partition $\mathfrak{C}(A)$.

Let $e_1 = \{x_1, y_1\}, \dots, e_n = \{x_n, y_n\}$ be n different edges of $\mathcal{G}$. By construction

$$\mathbb{E}\left[F(\phi); E\backslash \bigcup_{\mathcal{C}'\in\mathfrak{C}'} \mathcal{C}' = \{e_1,\dots,e_n\}\right] \tag{4.4.4}$$
$$= \mathbb{E}\left[\mathcal{S}_{\{e_1,\dots,e_n\}}(F)(\widehat{\mathcal{L}}_{\frac{1}{2}}); E\backslash \bigcup_{\mathcal{C}'\in\mathfrak{C}'} \mathcal{C}' = \{e_1,\dots,e_n\}\right]$$

From (4.4.3) follows that this in turn equals

$$\mathbb{E}\Big[\prod_{i=1}^{n} e^{-C(e_i)(|\psi_{x_i}\psi_{y_i}|+\psi_{x_i}\psi_{y_i})} \prod_{\substack{\{x,y\}\in \\ E\backslash\{e_1,\dots,e_n\}}} 1_{\psi_x\psi_y>0}(1-e^{-2C(x,y)|\psi_x\psi_y|}) \times \mathcal{S}_{\{e_1,\dots,e_n\}}(F)\Big(\frac{1}{2}\psi^2\Big)\Big] \tag{4.4.5}$$

In (4.4.5) the factor

$$\prod_{i=1}^{n} e^{-C(e_i)|\psi_{x_i}\psi_{y_i}|} \prod_{\substack{\{x,y\}\in \\ E\backslash\{e_1,\dots,e_n\}}} (1-e^{-2C(x,y)|\psi_x\psi_y|})\mathcal{S}_{\{e_1,\dots,e_n\}}(F)\Big(\frac{1}{2}\psi^2\Big)$$

depends only on the absolute value $|\psi|$. Two other factors take in account the sign of ψ:

$$\prod_{i=1}^{n} e^{-C(e_i)\psi_{x_i}\psi_{y_i}} \tag{4.4.6}$$

and

$$\prod_{\substack{\{x,y\}\in \\ E\backslash\{e_1,\dots,e_n\}}} 1_{\psi_x\psi_y>0} \tag{4.4.7}$$

The factor (4.4.6) multiplied by the non normalized density $e^{-H(f)}$ of ψ gives the non-normalized density $e^{-H^{(e_1,\dots,e_n)}(f)}$ of $\psi^{(e_1,\dots,e_n)}$, the Gaussian free field on $\mathcal{G}^{(e_1,\dots,e_n)}$. $\psi^{(e_1,\dots,e_n)}$ is independent on each connected component of $\mathcal{G}^{(e_1,\dots,e_n)}$. The factor (4.4.7) means that we restrict to the event on which the field has constant sign on each connected component of $\mathcal{G}^{(e_1,\dots,e_n)}$. But conditional on $\psi^{(e_1,\dots,e_n)}$ having constant sign on each connected component of $\mathcal{G}^{(e_1,\dots,e_n)}$, these signs are independent on each connected component and $-$ and $+$ have equal probability $\frac{1}{2}$. This implies that (4.4.5) equals

$$\mathbb{E}\Big[\prod_{i=1}^{n} e^{-C(e_i)(|\psi_{x_i}\psi_{y_i}|+\psi_{x_i}\psi_{y_i})} \prod_{\substack{\{x,y\}\in \\ E\backslash\{e_1,\dots,e_n\}}} 1_{\psi_x\psi_y>0}(1-e^{-2C(x,y)|\psi_x\psi_y|})F(\psi)\Big]$$

Then summing on all possible values of $E\backslash\bigcup_{\mathcal{C}'\in\mathfrak{C}'}\mathcal{C}'$ we get $\mathbb{E}[F(\phi)]=\mathbb{E}[F(\psi)]$ and deduce that ϕ and ψ are equidistributed.

For the case of infinite V we approximate the graph $\mathcal{G}$ by an increasing sequence of finite connected sub-graphs. Let $x_0\in V$. Let V_n be the set of vertices separated from x_0 by at most n edges. For $n\geqslant 1$ let E_n be the set of edges either connecting two vertices in V_{n-1} or a vertex in $V_n\backslash V_{n-1}$ to a vertex in V_{n-1}. $\mathcal{G}_n:=(V_n,E_n)$ is a connected sub-graph of $\mathcal{G}$. We consider the Markov jump process on $\mathcal{G}_n$ with transition rates given by the conductances restricted to E_n, the killing measure κ restricted to V_{n-1} and an additional instant killing at reaching $V_n\backslash V_{n-1}$. Let $(G^{V_{n-1}}(x,y))_{x,y\in V_{n-1}}$ be the corresponding Green's function and $(\psi_x^{V_{n-1}})_{x\in V_{n-1}}$ the corresponding Gaussian free field. The associated Poisson ensemble of loops of parameter $\frac{1}{2}$ is $\{\gamma\in\mathcal{L}_{\frac{1}{2}}|\gamma \text{ stays in } V_{n-1}\}$. Let $(\phi_x^{V_{n-1}})_{x\in V_{n-1}}$ be the field obtained by applying the procedure described in theorem 4.1 bis to $\{\gamma\in\mathcal{L}_{\frac{1}{2}}|\gamma \text{ stays in } V_{n-1}\}$. As shown previously $\phi^{V_{n-1}}$ has same law as $\psi^{V_{n-1}}$. Moreover $\phi^{V_{n-1}}$ converges in law to ϕ and $\psi^{V_{n-1}}$ to ψ. Thus ϕ and ψ have same law. □

4.5. Application to percolation by loops

In this section we consider the lattices

- $\mathbb{Z}^2$ with uniform conductances and a non-zero uniform killing measure
- the discrete half-plane $\mathbb{Z} \times \mathbb{N}$ with instantaneous killing on the boundary $\mathbb{Z} \times \{0\}$ and no killing elsewhere
- $\mathbb{Z}^d$, $d \geqslant 3$, with uniform conductances and no killing measure

and show that there is no infinite loop cluster in $\mathcal{L}_{\frac{1}{2}}$. Obviously there cannot be such an infinite cluster if the Gaussian free field only has bounded sign clusters, which is the case for $\mathbb{Z}^2$ with uniform conductances and a non-zero uniform killing measure (see Theorem 14.3 in [**HJ06**]). However on $\mathbb{Z}^d$ for d sufficiently large the Gaussian free field has infinite sign clusters, one of each sign, at is it believed that is the case for all $d \geqslant 3$ ([**RS13**]). But at the level of the metric graph there are no unbounded sign clusters of the free field.

The uniqueness of an infinite cluster of loops on $\mathbb{Z}^d$, $d \geqslant 3$ and on $\mathbb{Z}^2$ with uniform killing measure was shown applying Burton-Keane's argument in [**CS14**]. Next we adapt this argument to the case of loops on the discrete half-plane.

PROPOSITION 4.5.1. *On the discrete half-plane $\mathbb{Z} \times \mathbb{N}$ with instantaneous killing on the boundary $\mathbb{Z} \times \{0\}$, a.s. $\mathcal{L}_{\frac{1}{2}}$ has at most one infinite cluster.*

PROOF. The general layout of the proof is the same as for the i.i.d. Bernoulli percolation. See section 8.2 in [**Gri99**]. The law of $\mathcal{L}_{\frac{1}{2}}$ is ergodic for the horizontal translations and hence the number of infinite clusters in $\mathfrak{C}_{\frac{1}{2}}$ is a.s. constant. The next step is to show that this constant can only be 0, 1 or $+\infty$. This can be proved similarly to the i.i.d. Bernoulli percolation case and we omit it. Then one has to rule out the case of infinitely many infinite clusters.

For $a \in \mathbb{N}$ let

$$\mathcal{L}_{\frac{1}{2}}^{>a} := \{\gamma \in \mathcal{L}_{\frac{1}{2}} | Range(\gamma) \subseteq \mathbb{Z} \times [a+1, +\infty)\}$$

$\mathcal{L}_{\frac{1}{2}}^{>0} = \mathcal{L}_{\frac{1}{2}}$ and all the $\mathcal{L}_{\frac{1}{2}}^{>a}$ have the same law up to a vertical translation. A vertex $(x_1, a+1) \in \mathbb{Z} \times \mathbb{N}^*$ will be an *upper trifurcation* if it is contained in an infinite cluster of $\mathcal{L}_{\frac{1}{2}}^{>a}$ and if this vertex and adjacent edges are removed the cluster splits in at least three infinite clusters. Every vertex of $\mathbb{Z} \times \mathbb{N}^*$ has equal probability to be an upper trifurcation. Let it be p_3. If with positive probability $\mathcal{L}_{\frac{1}{2}}$ has at least three infinite clusters then a vertex in $\mathbb{Z} \times \{1\}$ has a positive probability to be an upper trifurcation. This can be proved in the similar way as in i.i.d. Bernoulli case. Consequently $p_3 > 0$.

Let $\mathcal{T}_n$ be the set of upper trifurcations in $[-n, n] \times [1, n]$. Let $(z_i)_{1 \leqslant i \leqslant N_n}$ be an enumeration of $\mathcal{T}_n$ such that the sequence of second coordinates of z_i, $(a_i + 1)_{1 \leqslant i \leqslant N_n}$, is non-increasing. Given z_i, there are three simple paths $c_1(z_i)$, $c_2(z_i)$ and $c_3(z_i)$ that connect z_i to three different vertices on

$$\partial([-n-1, n+1] \times [1, n+1]) = \\ \{-n-1\} \times [1, n+1] \cup \{n+1\} \times [1, n+1] \cup [-n-1, n+1] \times \{n+1\}$$

that do not intersect outside z_i and such that $c_1(z_i) \backslash \{z_i\}$, $c_2(z_i) \backslash \{z_i\}$ and $c_3(z_i) \backslash \{z_i\}$ are contained in three different clusters induced by the clusters of $\mathcal{L}_{\frac{1}{2}}^{>a_i}$ after deleting the vertex z_i. For $i \geqslant 2$, two different paths $c_j(z_i) \backslash \{z_i\}$ and $c_{j'}(z_i) \backslash \{z_i\}$ cannot intersect the same connected component of

$$\bigcup_{1 \leqslant i' \leqslant i-1} (c_1(z_{i'}) \cup c_2(z_{i'}) \cup c_3(z_{i'}))$$

because the set above is covered by the loops in $\mathcal{L}_{\frac{1}{2}}^{>a_i}$. Then as in Burton-Keane's proof one sets $\tilde{c}_j(z_1) = c_j(z_1)$ and iteratively constructs the family of simple paths $(\tilde{c}_j(z_i))_{1\leqslant j\leqslant 3, 2\leqslant i\leqslant N_n}$ where the path $\tilde{c}_j(z_i)$ starts from z_i as $c_j(z_i)$ and as soon as it meets a path $\tilde{c}$ from the family $(\tilde{c}_{j'}(z_{i'}))_{1\leqslant j'\leqslant 3, 1\leqslant i'\leqslant i-1}$ it continues as $\tilde{c}$. The graph formed by the paths $(\tilde{c}_j(z_i))_{1\leqslant j\leqslant 3, 1\leqslant i\leqslant N_n}$ has no cycles, its leaves (vertices of degree 1) are contained in $\partial([-n-1, n+1]\times[1, n+1])$ and the vertices z_i have degree 3 at least. Thus

$$\sharp\mathcal{T}_n \leqslant \sharp\partial([-n-1, n+1]\times[1, n+1]) = 4n+3$$

The expectation of $\sharp\mathcal{T}_n$ cannot grow as fast as n^2 hence $p_3 = 0$. □

Next we give a simple upper bound for the probability of two vertices belonging to the same cluster of $\mathcal{L}_{\frac{1}{2}}$. This is an inequality that holds on all graphs and not specifically on periodic ones as considered previously in this section.

PROPOSITION 4.5.2. *Let $x, y \in V$. Let*

$$g(x,y) := \frac{G(x,y)}{\sqrt{G(x,x)G(y,y)}}$$

$$\text{(4.5.1)}\quad \mathbb{P}\left(x \text{ and } y \text{ belong to the same cluster of } \mathcal{L}_{\frac{1}{2}}\right) \leqslant$$
$$\mathbb{P}\left(x \text{ and } y \text{ belong to the same cluster of } \widetilde{\mathcal{L}}_{\frac{1}{2}}\right) = \frac{2}{\pi}\arcsin(g(x,y))$$

PROOF. Consider the set of extended clusters $\mathfrak{C}'$. The probability that x and y belong to the same cluster in $\mathfrak{C}'$ is exactly

$$\mathbb{E}\left[\operatorname{sign}(\phi_x)\operatorname{sign}(\phi_y)\right]$$

In our coupling if x and y belong to the same cluster in $\mathfrak{C}'$ then the product $sign(\phi_x)sign(\phi_y)$ equals 1, and if this is not the case $sign(\phi_x)sign(\phi_y)$ equals either 1 or -1 each with probability $\frac{1}{2}$

One must to check that

$$\mathbb{E}\left[\operatorname{sign}(\phi_x)\operatorname{sign}(\phi_y)\right] = \frac{2}{\pi}\arcsin(g(x,y))$$

Let Z_1 and Z_2 be two independent standard centred Gaussian r.v.'s. We have the equalities in law

$$(\phi_x, \phi_y) \overset{(law)}{=} (\sqrt{G(x,x)}Z_1, \sqrt{G(y,y)}(g(x,y)Z_1 + \sqrt{1-g(x,y)^2}Z_2))$$
$$(\operatorname{sign}(\phi_x), \operatorname{sign}(\phi_y)) \overset{(law)}{=} (\operatorname{sign}(Z_1), \operatorname{sign}(g(x,y)Z_1 + \sqrt{1-g(x,y)^2}Z_2))$$

Then

$$\mathbb{E}\left[\operatorname{sign}(\phi_x)\operatorname{sign}(\phi_y)\right] = \mathbb{P}\left(\frac{|Z_2|}{|Z_1|} \leqslant \frac{g(x,y)}{\sqrt{1-g(x,y)^2}}\right)$$

Z_2/Z_1 follows the Cauchy distribution

$$\frac{1}{\pi}\frac{dz}{1+z^2}$$

Thus

$$\mathbb{P}\left(\frac{|Z_2|}{|Z_1|} \leqslant \frac{g(x,y)}{\sqrt{1-g(x,y)^2}}\right) = \frac{2}{\pi}\arctan\left(\frac{g(x,y)}{\sqrt{1-g(x,y)^2}}\right) = \frac{2}{\pi}\arcsin(g(x,y))$$

□

In the case of a graph $\mathbb{Z}^d$ ($d \geqslant 2$) with positive constant killing measure, inequality (4.5.1) ensures an exponential decay of cluster size distribution. For $\mathbb{Z}^d$ ($d \geqslant 3$) with no killing inequality (4.5.1) implies

$$\mathbb{P}\left(x \text{ and } y \text{ belong to the same cluster of } \mathcal{L}_{\frac{1}{2}}\right) = O\left(\frac{1}{|y-x|^{d-2}}\right)$$

However this bound is certainly not sharp and one expects that for $d \geqslant 5$

$$\mathbb{P}\left(x \text{ and } y \text{ belong to the same cluster of } \mathcal{L}_{\frac{1}{2}}\right) = O\left(\frac{1}{|y-x|^{2(d-2)}}\right)$$

(see proposition 5.3 in [**CS14**]). This also means that the percolation by discrete loops on periodic lattices and the percolation by continuous loops on the corresponding metric graphs behave differently.

Proof of theorem 4.2. Assume that $\mathcal{L}_{\frac{1}{2}}$ has an infinite cluster. Let $\mathcal{C}_\infty$ be this infinite cluster. Let x be a vertex and

$$\theta(x) := \mathbb{P}(x \in \mathcal{C}_\infty)$$

Let u_1 be the unit vector corresponding to the first coordinate, $u_1 = (1, 0, \dots, 0)$. Let $x_n := x + nu_1$. From the invariance under translation by u_1 it follows that $\theta(x_n) = \theta(x)$.

$$\mathbb{P}\left(x \text{ and } y \text{ belong to the same cluster of } \mathcal{L}_{\frac{1}{2}}\right) \leqslant \mathbb{P}(x \in \mathcal{C}_\infty, x_n \in \mathcal{C}_\infty)$$

$\mathcal{L}_{\frac{1}{2}}$ satisfies Harris-FKG inequality ([**JL13**]). Thus

$$\mathbb{P}(x \in \mathcal{C}_\infty, x_n \in \mathcal{C}_\infty) \geqslant \theta(x)\theta(x_n) = \theta(x)^2$$

It follows that

$$\theta(x)^2 \leqslant \frac{2}{\pi}\arcsin(g(x, x_n))$$

Letting n go to $+\infty$ we get that $\theta(x) = 0$. □

Let $d \geqslant 3$. Let $\widetilde{\mathbb{Z}}^d$ be the metric graph associated to the graph $\mathbb{Z}^d$. All edges have length $\frac{1}{2}$. We consider the Gaussian free field $(\phi_z)_{z\in\widetilde{\mathbb{Z}}^d}$ on $\widetilde{\mathbb{Z}}^d$ and the following dependent percolation model on the edges of $\mathbb{Z}^d$: Let ω be the random configuration on the edges of $\mathbb{Z}^d$ with $\omega_e = 1$ (e is open) if $|\phi|$ has no zeros on I_e and $\omega_e = 0$ (e is closed) otherwise. The set of clusters of ω is exactly $\mathfrak{C}'$ which appears in the coupling of theorem 4.1 bis. The free field on the metric graph has an unbounded sign cluster if an only if there is an infinite cluster in $\mathfrak{C}'$, as the sign clusters of ϕ that are contained inside the intervals I_e corresponding to the edges are all bounded. We will show that this cannot happen. We will follow the same pattern as for the proof of theorem 4.2: first show that $\mathfrak{C}'$ can contain at most one infinite cluster, the show that ω satisfies the Harris-FKG inequality and conclude using inequality (4.5.1).

Lemma 4.5.3. *With probability one $\mathfrak{C}'$ has at most one infinite cluster.*

Proof. According to theorem 1 in [**GKN92**], the uniqueness of the infinite clusters is implied by translation invariance and positive finite energy property. We need only show the finite energy property:

$$\mathbb{P}(\omega_e = 1 | (\omega_f,\ f \text{ is an edge of } \mathbb{Z}^d \text{ and } f \neq e)) > 0 \text{ a.s.} \tag{4.5.2}$$

Let $e = \{x, y\}$ be an edge. We see $(\frac{1}{2}\phi_z^2)_{z\in\widetilde{\mathbb{Z}}^d}$ as the occupation field of continuous loops $\widetilde{\mathcal{L}}_{\frac{1}{2}}$. The loops inside I_e and the excursions inside I_e from x to x and y to y that do not cross entirely I_e are independent of $(\frac{1}{2}\phi_z^2)_{z\in\widetilde{\mathbb{Z}}^d\backslash I_e}$ conditional on $|\phi_x|$ and $|\phi_y|$. Thus, according to the computations made in section 4.3

$$\mathbb{P}\Big(\omega_e = 1 \Big| \Big(\frac{1}{2}\phi_z^2\Big)_{z\in\widetilde{\mathbb{Z}}^d\backslash I_e}\Big) \geqslant 1 - e^{-|\phi_x\phi_y|}$$

Hence

$$(4.5.3)\quad \mathbb{P}(\omega_e = 1 | (\omega_f,\ f \text{ is an edge of } \mathbb{Z}^d \text{ and } f \neq e)) \geqslant \\ \mathbb{E}\left[1 - e^{-|\phi_x\phi_y|} | (\omega_f,\ f \text{ is an edge of } \mathbb{Z}^d \text{ and } f \neq e)\right]$$

Since $|\phi_x\phi_y| > 0$ a.s., the right-hand side in (4.5.3) is a.s. non-zero and the condition (4.5.2) is satisfied. □

LEMMA 4.5.4. *ω satisfies the Harris-FKG inequality: given $A_1(\omega)$ and $A_2(\omega)$ two increasing events*

$$\mathbb{P}(A_1(\omega), A_2(\omega)) \geqslant \mathbb{P}(A_1(\omega))\mathbb{P}(A_2(\omega)) \tag{4.5.4}$$

PROOF. We see the field $(\frac{1}{2}\phi_y^2)_{y\in\widetilde{\mathbb{Z}}^d}$ as the occupation field of the metric graph loops $\widetilde{\mathcal{L}}_{\frac{1}{2}}$. If the events $A_1(\omega)$ and $A_2(\omega)$ are increasing in the sense that opening more edges in ω only helps their occurrence, then these events are also increasing in the sense that they are stable by adding more loops to $\widetilde{\mathcal{L}}_{\frac{1}{2}}$. The inequality (4.5.4) follows from the FKG inequality for Poisson point processes (lemma 2.1 in [**Jan84**]). □

PROPOSITION 4.5.5. *With probability one $\mathfrak{C}'$ has only finite clusters.*

4.6. Critical intensity parameter on the discrete half-plane

Let $\alpha, \delta > 0$. Given U an open subset of $\mathbb{H}$, we will denote by $\mathcal{L}_\alpha^{U,\geqslant\delta}$ respectively $\mathcal{L}_\alpha^{U\cap\mathtt{H},\geqslant\delta}$ the subset of $\mathcal{L}_\alpha^{\mathbb{H}}$ respectively $\mathcal{L}_\alpha^{\mathtt{H}}$ made of loops contained in U and with diameter greater or equal to δ. We will use the notations $\mathcal{L}_\alpha^{U}$ and $\mathcal{L}_\alpha^{U\cap\mathtt{H}}$ when there is a condition on the range but not on the diameter.

Let Q_{ext} and Q_{int} be the following rectangles:

$$Q_{ext} := (0,6)\times(0,3) \quad Q_{int} := (1,5)\times(1,2)$$

We consider the subset of Brownian loops $\mathcal{L}_\alpha^{Q_{ext},\geqslant\delta}$ which is a.s. finite. We introduce events $C_1(\mathcal{L}_\alpha^{Q_{ext},\geqslant\delta})$, $C_2(\mathcal{L}_\alpha^{Q_{ext},\geqslant\delta})$ and $C_3(\mathcal{L}_\alpha^{Q_{ext},\geqslant\delta})$ depending on the loops in $\mathcal{L}_\alpha^{Q_{ext},\geqslant\delta}$. The event $C_1(\mathcal{L}_\alpha^{Q_{ext},\geqslant\delta})$ will be satisfied if there is a cluster K_1 of loops in $\mathcal{L}_\alpha^{Q_{int},\geqslant\delta}$ such that in $\mathcal{L}_\alpha^{(0,6)\times(1,2),\geqslant\delta}$ there is a loop that intersects K_1 and $\{1\}\times(1,2)$ and a loop that intersects K_1 and $\{5\}\times(1,2)$. The two loops may be the same. $C_2(\mathcal{L}_\alpha^{Q_{ext},\geqslant\delta})$ will be satisfied if there a cluster K_2 in $\mathcal{L}_\alpha^{(1,2)^2,\geqslant\delta}$ such that in $\mathcal{L}_\alpha^{(1,2)\times(0,3),\geqslant\delta}$ there is a loop that intersects K_2 and $(1,2)\times\{1\}$ and a loop that intersects K_2 and $(1,2)\times\{2\}$. The event $C_3(\mathcal{L}_\alpha^{Q_{ext},\geqslant\delta})$ is similar to the event $C_2(\mathcal{L}_\alpha^{Q_{ext},\geqslant\delta})$ where the square $(1,2)^2$ is replaced by the square $(4,5)\times(1,2)$ and the rectangle $(1,2)\times(0,3)$ by the rectangle $(4,5)\times(0,3)$. Next figure illustrates the event $\bigcap_{i=1}^3 C_i(\mathcal{L}_\alpha^{Q_{ext},\geqslant\delta})$.

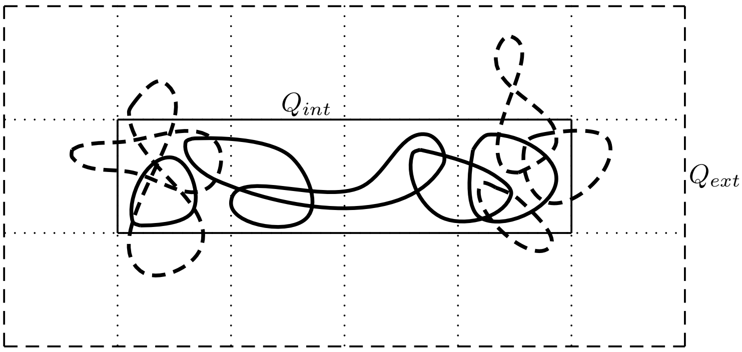

Fig.4.2: Illustration of the event $\bigcap_{i=1}^{3} C_i(\mathcal{L}_\alpha^{Q_{ext},\geqslant\delta})$.
One should imagine that the smooth loops are actually Brownian.
Only a set of loops that is sufficient for the event is represented.
Full line loops stay inside Q_{int}. Dashed loops cross the boundary of Q_{int}.

We will call the event $\bigcap_{i=1}^{3} C_i(\mathcal{L}_\alpha^{Q_{ext},\geqslant\delta})$ *special crossing event with exterior rectangle* Q_{ext} *and interior rectangle* Q_{int}. We will also consider translations, rotations and rescaling of Q_{ext} and Q_{int} and deal with *special crossing events* corresponding to the new rectangles. We are interested in the event $\bigcap_{i=1}^{3} C_i(\mathcal{L}_\alpha^{Q_{ext},\geqslant\delta})$ because then the loops in $\mathcal{L}_\alpha^{Q_{ext},\geqslant\delta}$ achieve the three crossings drawn on the figure 4.3:

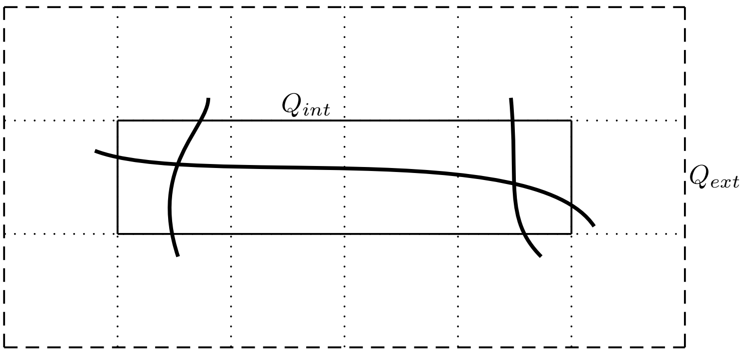

Fig.4.3: The three crossings we are interested in.

Next we show that if $\alpha > \frac{1}{2}$ and δ is small enough then the probability of the event $\bigcap_{i=1}^{3} C_i(\mathcal{L}_\alpha^{Q_{ext},\geqslant\delta})$ is close to 1.

LEMMA 4.6.1. *Let Q be a rectangle $Q := (-a, a) \times (0, b)$. Let $\alpha > 0$. Let $(B_t)_{t\geqslant 0}$ be the standard Brownian motion on $\mathbb{C}$ started from 0 and let $\mathcal{L}_\alpha^{Q}$ be a Poisson ensemble of loops independent from B. Then for all $\varepsilon > 0$ there is $t \in (0, \varepsilon)$ such that B at time t intersects a loop in $\mathcal{L}_\alpha^{Q}$.*

PROOF. First we consider a loops soup in $\mathbb{H}$, $\mathcal{L}_\alpha^{\mathbb{H}}$ independent of B. Let

$$T := \inf\{t > 0 | B_t \text{ is in the range of a loop in } \mathcal{L}_\alpha^{\mathbb{H}}\}$$

T is a.s. finite. Indeed a loop in $\mathcal{L}_\alpha^{\mathbb{H}}$ delimits a domain with non-empty interior. Since the Brownian motion on $\mathbb{C}$ is recurrent, B will visit this domain and thus intersect the loop. Let $\lambda > 0$. The Poisson ensemble of loops $\mathcal{L}_\alpha^{\mathbb{H}}$ is invariant in law under the Brownian scaling

$$(\gamma(t))_{0\leqslant t\leqslant t_\gamma} \longmapsto \lambda^{-\frac{1}{2}}(\gamma(\lambda t))_{0\leqslant t\leqslant \lambda^{-1}t_\gamma}$$

So does the Brownian motion B. Thus λT has the same law as T. It follows that $T = 0$ a.s.

The set of loops $\mathcal{L}_\alpha^{\mathbb{H}} \backslash \mathcal{L}_\alpha^Q$ is at positive distance from 0 thus B cannot intersect it immediately. It follows that B intersects immediately $\mathcal{L}_\alpha^Q$. □

LEMMA 4.6.2. *Let $a, \alpha > 0$. There is a.s. a loop in $\mathcal{L}_\alpha^{(-a,a)^2}$ that intersects the real line $\mathbb{R}$.*

PROOF. Let $\mathcal{L}_\alpha^{(n)}$ be the subset of $\mathcal{L}_\alpha^{(-a,a)^2}$ made of loops γ of duration t_γ comprised between 2^{-n-1} and 2^{-n}. The family $(\mathcal{L}_\alpha^{(n)})_{n \geqslant 0}$ is independent. By Brownian scaling, the probability that the loop in $\mathcal{L}_\alpha^{(n)}$ intersects $\mathbb{R}$ is the same as a loop in $\mathcal{L}_\alpha^{(-a2^{n/2}, a2^{n/2})^2}$ of duration comprised between $\frac{1}{2}$ and 1 intersects $\mathbb{R}$. This is at least as big as the similar probability for $\mathcal{L}_\alpha^{(0)}$. Since the latter probability is non-zero, the intersection events occurs a.s. for infinitely many of $\mathcal{L}_\alpha^{(n)}$. □

LEMMA 4.6.3. *Let $a, \alpha > 0$. There is a.s. a loop in $\mathcal{L}_\alpha^{(-a,a)^2}$ that intersects the real line $\mathbb{R}$ and a loop in $\mathcal{L}_\alpha^{(-a,a)\times(0,a)}$.*

PROOF. Consider the subset of $\mathcal{L}_\alpha^{(-a,a)^2}$ made of loops intersecting $\mathbb{R}$. It is non empty according the lemma 4.6.2. Moreover it is independent of $\mathcal{L}_\alpha^{(-a,a)\times(0,a)}$. The law of a Brownian loop that intersects $\mathbb{R}$ is locally, near the point of intersection, absolutely continuous with respect to the law of a Brownian motion started from there. Applying lemma 4.6.1, we get that it intersects a.s. a loop in $\mathcal{L}_\alpha^{(-a,a)\times(0,a)}$. □

LEMMA 4.6.4. *Let $\alpha > \frac{1}{2}$. Then*

$$\lim_{\delta \to 0^+} \mathbb{P}\Big(\bigcap_{i=1}^{3} C_i(\mathcal{L}_\alpha^{Q_{ext}, \geqslant \delta})\Big) = 1$$

PROOF. It is enough to show that the probability of each of the $C_i(\mathcal{L}_\alpha^{Q_{ext}, \geqslant \delta})$ converges to 1 as δ tends to 0. Since the three cases are very similar, we will do the proof only for $C_1(\mathcal{L}_\alpha^{Q_{ext}, \geqslant \delta})$. According to lemma 4.6.3 there is a loop γ in $\mathcal{L}_\alpha^{(0,6)\times(1,2)}$ that intersects $\{1\} \times (1,2)$ and a loop γ' in $\mathcal{L}_\alpha^{Q_{int}}$. Similarly there is a loop $\tilde{\gamma}$ in $\mathcal{L}_\alpha^{(0,6)\times(1,2)}$ that intersects $\{5\} \times (1,2)$ and a loop $\tilde{\gamma}'$ in $\mathcal{L}_\alpha^{Q_{int}}$. Since $\alpha > \frac{1}{2}$, γ' and $\tilde{\gamma}'$ belong to the same cluster in $\mathcal{L}_\alpha^{Q_{int}}$ ([**SW12**]). Thus there is a chain of loops $(\gamma_0, \dots, \gamma_n)$ in $\mathcal{L}_\alpha^{Q_{int}}$, with $\gamma_0 = \gamma'$ and $\gamma_n = \tilde{\gamma}'$, joining γ' and $\tilde{\gamma}'$. If δ is the minimum of diameters of $(\gamma_0, \dots, \gamma_n)$ and γ and $\tilde{\gamma}$ then $C_1(\mathcal{L}_\alpha^{Q_{ext}, \geqslant \delta})$ is satisfied. Let $\bar{\delta}$ be maximal value of δ such that $C_1(\mathcal{L}_\alpha^{Q_{ext}, \geqslant \delta})$ is satisfied. $\bar{\delta}$ is a well defined random variable with values in $(0, +\infty)$. Then

$$\lim_{\delta \to 0^+} \mathbb{P}(C_1(\mathcal{L}_\alpha^{Q_{ext}, \geqslant \delta})) = \lim_{\delta \to 0^+} \mathbb{P}(\delta \leqslant \bar{\delta}) = 1$$

□

Next we recall the result on approximation of Brownian loops by random walk loops from [**LF07**]. Let $N \in \mathbb{N}^*$. We consider the discrete loops γ on $\mathbb{Z} \times \mathbb{N}^*$. We define on these loops a map Φ_N to continuous loops on $\mathbb{H}$. Given γ a discrete loop and $(z_0, \dots, z_{n-1}, z_0)$ the sequence of the vertices it visits, the continuous loop $\Phi_N \gamma$ satisfies:

- The duration of $\Phi_N \gamma$ is $\frac{n}{2N^2}$.
- For $j \in \{0, \dots, n-1\}$, $\Phi_N \gamma(\frac{j}{2N^2}) = \frac{z_j}{N}$.
- $\Phi_N \gamma(\frac{n}{2N^2}) = \Phi_N \gamma(0) = \frac{z_0}{N}$.
- Between the times $\frac{j}{2N^2}$, $j \in \{0, \dots, n\}$, $\Phi_N \gamma$ interpolates linearly.

The number of jumps n of o discrete loop γ will be denoted s_γ. Let $\theta \in (\frac{2}{3}, 2)$ and $r \geqslant 1$. There is a coupling between $\mathcal{L}_\alpha^{\mathtt{H}}$ and $\mathcal{L}_\alpha^{\mathbb{H}}$ such that except on an event of probability at most $cste \cdot (\alpha + 1) r^2 N^{2-3\theta}$ there is a one to one correspondence between the two sets

- $\{\gamma \in \mathcal{L}_{\alpha}^{\mathtt{H}} | s_{\gamma} > 2N^{\theta}, |\gamma(0)| < Nr\}$
- $\{\tilde{\gamma} \in \mathcal{L}_{\alpha}^{\mathbb{H}} | t_{\tilde{\gamma}} > N^{\theta-2}, |\tilde{\gamma}(0)| < r\}$

such that given a discrete loop γ and the continuous loop $\tilde{\gamma}$ corresponding to it:

$$\left|\frac{s_{\gamma}}{2N^2} - t_{\tilde{\gamma}}\right| \leqslant \frac{5}{8}N^{-2} \qquad \sup_{0\leqslant u\leqslant 1}\left|\Phi_N\gamma\Big(u\frac{s_{\gamma}}{2N^2}\Big) - \tilde{\gamma}(ut_{\tilde{\gamma}})\right| \leqslant cste\cdot N^{-1}\log(N)$$

Next we state as without proof a lemma that follows immediately from this approximation.

LEMMA 4.6.5. *Let $\alpha > 0$ and $\delta > 0$. As N tends to $+\infty$ the random set of interpolating continuous loops*

$$\left\{\Phi_N\gamma | \gamma \in \mathcal{L}_{\alpha}^{NQ_{ext}\cap\mathtt{H},\geqslant N\delta}\right\}$$

converges in law to the set of Brownian loops $\mathcal{L}_{\alpha}^{Q_{ext},\geqslant\delta}$.

We need to show that the above convergence for the uniform norm also implies a convergence of the intersection relations, that is to say that

$$\{(\gamma,\gamma')|\gamma,\gamma' \in \mathcal{L}_{\alpha}^{NQ_{ext}\cap\mathtt{H},\geqslant N\delta}, \gamma \text{ intersects } \gamma'\}$$

converges in law to

$$\{(\tilde{\gamma},\tilde{\gamma}')|\tilde{\gamma},\tilde{\gamma}' \in \mathcal{L}_{\alpha}^{Q_{ext},\geqslant\delta}, \tilde{\gamma} \text{ intersects } \tilde{\gamma}'\}$$

Let $j\in\mathbb{N}$. Let γ be a continuous path on $\mathbb{C}$ (not necessarily a loop) of lifetime t_{γ}. For $r > 0$ let

$$T_r(\gamma) := \inf\{s > 0 || \gamma(s)| \geqslant r\} \in (0, +\infty]\}$$

If $T_r(\gamma) < +\infty$ let

$$e^{i\omega_r} := \frac{\gamma(T_r(\gamma))}{r}$$

Let I_j be the real interval

$$I_j := \Big(\frac{7}{12}2^{-j}, \frac{9}{12}2^{-j}\Big)$$

For $0 < r_1 < r_2$ let $\mathcal{A}(r_1,r_2)$ be the annulus

$$\mathcal{A}(r_1,r_2) := \{z\in\mathbb{C} | r_1 < |z| < r_2\}$$

For $r > 0$ let $HD(r)$ be the half-disc

$$HD(r) := B(0,r)\cap\{z\in\mathbb{C}|\Re(z) > 0\}$$

We will say that the path γ satisfies the condition $\mathcal{C}_j$ if

- $T_{\frac{11}{12}2^{-j}}(\gamma) < +\infty$
- After time $T_{\frac{11}{12}2^{-j}}(\gamma) < +\infty$, γ hits $e^{i(\omega_{2^{-j-1}}+\frac{\pi}{2})}I_j$ at a time $\tilde{t}_j$ before hitting the circle $S(0,2^{-j})$
- On the time interval $(T_{2^{-j-1}}(\gamma),\tilde{t}_j)$ γ stays in the half-disc $e^{i\omega_{2^{-j-1}}}HD(2^{-j})$
- From time $\tilde{t}_j$ the path γ stays in the annulus $\mathcal{A}(\frac{7}{12}2^{-j},\frac{9}{12}2^{-j})$ until surrounding once clockwise the disc $B(0,\frac{7}{12}2^{-j})$ once clockwise and hitting $e^{i(\omega_{2^{-j-1}}+\pi)}I_j$.

Figure 4.4 illustrates a path satisfying the condition $\mathcal{C}_j$. If this condition is satisfied than γ disconnects the disc $B(0,\frac{7}{12}2^{-j})$ from infinity. Moreover if one perturbs γ by any continuous function $f:[0,t_{\gamma}]\to\mathbb{C}$ such that $\|f\|_{\infty}\leqslant\frac{1}{12}2^{-j}$ then the path $(\gamma(s)+f(s))_{0\leqslant s\leqslant t_{\gamma}}$ disconnects the disc $B(0,2^{-j-1})$ from infinity. Moreover the disconnection is made inside the annulus $\mathcal{A}(2^{-j-1},2^{-j})$.

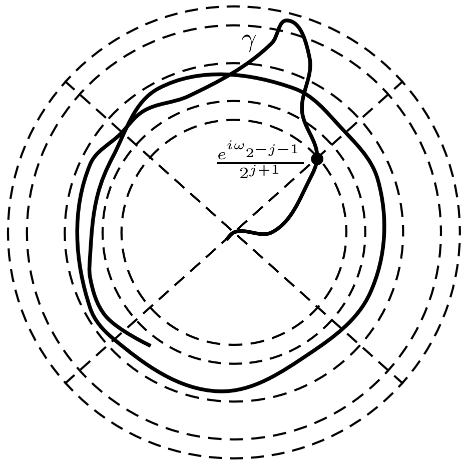

Fig.4.4: Representation of a path γ satisfying the condition $\mathcal{C}_j$

LEMMA 4.6.6. *Let $(B_t)_{0\leqslant t\leqslant T}$ be a standard Brownian path on $\mathbb{C}$ starting from 0. Then almost surely it satisfies the condition $\mathcal{C}_j$ for infinitely many values of $j\in\mathbb{N}$.*

PROOF. Let $\widetilde{B}$ be the Brownian path B continued on $t\in(0,+\infty)$. The events "$\widetilde{B}$ satisfies the condition $\mathcal{C}_j$" are i.i.d. Indeed such an event is rotation invariant and depends only on $\widetilde{B}$ on the time interval $(T_{2^{-j-1}}(\widetilde{B}),T_{2^{-j}}(\widetilde{B}))$. Moreover the probability of such an event is non-zero. Thus $\widetilde{B}$ satisfies the condition $\mathcal{C}_j$ for infinitely many values of $j\in\mathbb{N}$. Since

$$\lim_{j\to+\infty} T_{2^{-j}}(\widetilde{B})=0$$

so does B. □

LEMMA 4.6.7. *Let $z_1,z_2\in\mathbb{C}$ and $t_1,t_2>0$. Let $(b^{(1)}_s)_{0\leqslant s\leqslant t_1}$ and $(b^{(2)}_s)_{0\leqslant s\leqslant t_2}$ be two independent standard Brownian bridges from z_1 to z_1 and z_2 to z_2 respectively. On the event that $b^{(1)}$ intersects $b^{(2)}$ there is a.s. $\varepsilon>0$ such that for all continuous functions $f_1:[0,t_1]\to\mathbb{C}$ and $f_2:[0,t_2]\to\mathbb{C}$ of infinity norm $\|f_i\|_\infty\leqslant\varepsilon$, $(b^{(1)}_s+f_1(s))_{0\leqslant s\leqslant t_1}$ intersects $(b^{(2)}_s+f_2(s))_{0\leqslant s\leqslant t_2}$.*

PROOF. Let $T^{(1)}_2$ be the first time $b^{(1)}$ hits the range of $b^{(2)}$. If the two path do not intersect each other $T^{(1)}_2=+\infty$. On the event $T^{(1)}_2<+\infty$ the conditional law of $(b^{(1)}_{T^{(1)}_2+s}-b^{(1)}_{T^{(1)}_2})_{0\leqslant s\leqslant t_1-T^{(1)}_2-\varepsilon}$ ($\varepsilon>0$ a small constant) given the value $T^{(1)}_2$ is absolutely continuous with respect the law of a Brownian path starting from 0. From lemma 4.6.6 follows that the path $(b^{(1)}_{T^{(1)}_2+s}-b^{(1)}_{T^{(1)}_2})_{0\leqslant s\leqslant t_1-T^{(1)}_2}$ satisfies the condition $\mathcal{C}_j$ for infinitely many values of $j\in\mathbb{N}$. Let

$$\tilde{\jmath}:=\max\Big\{j\in\mathbb{N}|(b^{(1)}_{T^{(1)}_2+s}-b^{(1)}_{T^{(1)}_2})_{0\leqslant s\leqslant t_1-T^{(1)}_2}\text{ satisfies the condition }\mathcal{C}_j$$
$$\text{and }\exists s\in[0,t_2],|b^{(2)}_s-b^{(2)}_{T^{(1)}_2}|\geqslant\frac{13}{12}2^{-j}\Big\}$$

$\tilde{\jmath}$ is a r.v. defined on the event where $b^{(1)}$ and $b^{(2)}$ intersect. If f_1 and f_2 are such that $\|f_i\|\leqslant\frac{1}{12}2^{-\tilde{\jmath}}$ then the path $b^{(1)}+f_1$ disconnects the disc $B(b^{(1)}_{T^{(1)}_2},2^{-\tilde{\jmath}-1})$ from infinity inside the annulus $b^{(1)}_{T^{(1)}_2}+\mathcal{A}(2^{-\tilde{\jmath}-1},2^{-\tilde{\jmath}})$ and the path $b^{(2)}+f_2$ crosses from the circle $S(b^{(1)}_{T^{(1)}_2},2^{-\tilde{\jmath}-1})$ to the circle $S(b^{(1)}_{T^{(1)}_2},2^{-\tilde{\jmath}})$, so the two must intersect. □

Observe that two discrete loops γ and γ' intersect each other if and only if the continuous loops $\Phi_N\gamma$ and $\Phi_N\gamma'$ do. From lemmas 4.6.5 and 4.6.7 follows:

COROLLARY 4.6.8. *Let $\alpha > 0$ and $\delta > 0$. As N tends to $+\infty$ the random set of interpolating continuous loops*

$$\left\{\Phi_N\gamma | \gamma \in \mathcal{L}_\alpha^{NQ_{ext}\cap \mathtt{H}, \geqslant N\delta}\right\}$$

jointly with the intersection relations

$$\{(\gamma,\gamma')|\gamma,\gamma' \in \mathcal{L}_\alpha^{NQ_{ext}\cap \mathtt{H}, \geqslant N\delta}, \gamma \text{ intersects } \gamma'\}$$

converges in law to the set of Brownian loops $\mathcal{L}_\alpha^{Q_{ext},\geqslant\delta}$ jointly with the intersection relations

$$\{(\tilde\gamma,\tilde\gamma')|\tilde\gamma,\tilde\gamma' \in \mathcal{L}_\alpha^{Q_{ext},\geqslant\delta}, \tilde\gamma \text{ intersects } \tilde\gamma'\}$$

We consider the scaled up rectangle NQ_{ext} and NQ_{int}. The next lemma deals with the probability that the discrete loops $\mathcal{L}_\alpha^{NQ_{ext}\cap\mathtt{H}}$ realise *special crossing event with exterior rectangle NQ_{ext} and interior rectangle NQ_{int}*. See figures 4.2 and 4.3 and consider that Q_{ext} is replaced by NQ_{ext}, Q_{int} by NQ_{int} and $\mathcal{L}_\alpha^{Q_{ext},\geqslant\delta}$ by $\mathcal{L}_\alpha^{NQ_{ext}\cap\mathtt{H}}$.

LEMMA 4.6.9. *Let $\alpha > \frac{1}{2}$. As N tends to $+\infty$, the probability that the loops $\mathcal{L}_\alpha^{NQ_{ext}\cap\mathtt{H}}$ realise special crossing event with exterior rectangle NQ_{ext} and interior rectangle NQ_{int} converges to* 1.

PROOF. Let $\delta > 0$. The probability that that the loops $\mathcal{L}_\alpha^{NQ_{ext}\cap\mathtt{H}}$ realise *special crossing event with exterior rectangle NQ_{ext} and interior rectangle NQ_{int}* is at least as large as the probability that the loops $\mathcal{L}_\alpha^{NQ_{ext}\cap\mathtt{H},\geqslant N\delta}$ realise *special crossing event* with the same interior and exterior rectangle. From corollary 4.6.8 follows that the latter probability converges as $N \to +\infty$ to

$$\mathbb{P}\left(\bigcap_{i=1}^{3} C_i(\mathcal{L}_\alpha^{Q_{ext},\geqslant\delta})\right)$$

We conclude by applying the lemma 4.6.4. □

To conclude that for $\alpha > \frac{1}{2}$, $\mathcal{L}_\alpha^{\mathtt{H}}$ has an infinite cluster we will use a block percolation construction that will combine *special crossing events.* We will need the result of Liggett, Schonmann and Stacey in [**LSS97**] on locally dependent percolation models. Consider 1-dependent edge percolations on $\mathtt{H}$, $(\omega(e))_{e \text{ edge of } \mathtt{H}}$. By 1-dependent percolation we mean that if two disjoint subsets of edges E_1 and E_2 are at graph distance at least 1 then $(\omega(e))_{e\in E_1}$ and $(\omega(e))_{e\in E_2}$ are independent. For all such 1-dependent edge percolations, with a uniform probability p of an edge to be open, there is a universal $\tilde p(p) \in [0,1)$ such that the 1-dependent edge percolation contains an i.i.d. Bernoulli percolation with probability $\tilde p(p)$ of an edge to be open. Moreover the following constrain holds:

$$\lim_{p\to 1^-} \tilde p(p) = 1$$

Proof of theorem 4.3. From the Theorem 4.2 we know already that $\alpha_*^{\mathtt{H}} \leqslant \frac{1}{2}$. We need to show that For $\alpha > \frac{1}{2}$, $\mathcal{L}_\alpha^{\mathtt{H}}$ has an infinite cluster.

Let $\alpha > \frac{1}{2}$ and $N \geqslant 1$. We consider a depend edge percolation $(\omega^N(e))_{e \text{ edge of } \mathtt{H}}$ on the discrete half plane $\mathtt{H}$. If e is an edge of form $\{(j,k),(j+1,k)\}$ then $\omega^N(e) = 1$ (open edge) if $\mathcal{L}_\alpha^{(NQ_{int}+3Nj+i3Nk)\cap\mathtt{H}}$ achieves a *special crossing event with exterior rectangle $NQ_{ext}+3Nj+i3Nk$ and interior rectangle $NQ_{int}+3Nj+i3Nk$*. If e is an edge of form $\{(j,k),(j,k+1)\}$ then $\omega^N(e) = 1$ if $\mathcal{L}_\alpha^{(iNQ_{int}+3Nj+i3Nk)\cap\mathtt{H}}$ achieves a *special crossing event with exterior rectangle $iNQ_{ext}+3Nj+i3Nk$ and interior rectangle $iNQ_{int}+3Nj+i3Nk$*, where the multiplication by i means rotation by $+\frac{\pi}{2}$. ω^N is a 1-dependent edge percolation: if two disjoint subsets of edges E_1 and E_2 are such that no edge is adjacent to both E_1 and

E_2, then $(\omega^N(e))_{e\in E_1}$ and $(\omega^N(e))_{e\in E_2}$ are independent. This is due to the fact that the subsets of loops involved in the definition of *special crossing events* for edges in E_1 and and edges in E_2 are disjoint. To an open path in ω^N corresponds a cluster of $\mathcal{L}_\alpha^{\mathtt{H}}$ whose loops form crossings of related interior rectangles. Thus if ω^N has an unbounded cluster, then so does $\mathcal{L}_\alpha^{\mathtt{H}}$. See next picture.

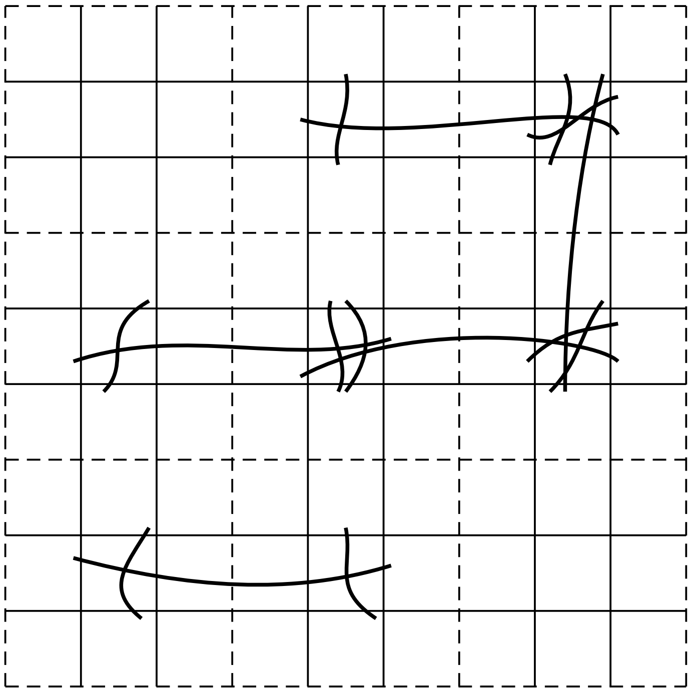

Fig.4.5: Crossings achieved by subsets of loops in $\mathcal{L}_\alpha^{\mathtt{H}}$, corresponding to five open edges in ω^N.

The probability $\mathbb{P}(\omega^N(e) = 1)$ is uniform and we will denote it p_N. According to the lemma 4.6.9

$$\lim_{N\to+\infty} p_N = 1$$

Thus for N large enough $\tilde{p}(p_N) > \frac{1}{2}$. $\frac{1}{2}$ is the critical probability for the i.i.d. Bernoulli edge percolation on $\mathtt{H}$. So for N large enough ω^N contains a supercritical i.i.d. Bernoulli edge percolation and percolates itself. Thus $\mathcal{L}_\alpha^{\mathtt{H}}$ percolates too. □

4.7. Random interlacements and level sets of the Gaussian free field

Let $d \geqslant 3$. As in section 4.2 we consider the metric graph $\widetilde{\mathbb{Z}}^d$ associated to the graph $\mathbb{Z}^d$. All edges have length $\frac{1}{2}$. We construct a continuous version $\widetilde{\mathcal{I}}^u$ of the random interlacement of level u on the metric graph $\widetilde{\mathbb{Z}}^d$. First we sample $\mathcal{I}^u$. Given a path w in $\mathcal{I}^u$ we replace each jump from a vertex to its neighbour by a Brownian excursion inside the linking edge and we add Brownian excursions from a vertex visited by w to itself inside adjacent edges such that the local time on the vertex equals the time w spends in it (as in (4.2.2) for loops). By construction $\mathcal{I}^u$ is the restriction of $\widetilde{\mathcal{I}}^u$ to the vertices. $\widetilde{\mathcal{I}}^u$ has an occupation field $\left(L^y(\widetilde{\mathcal{I}}^u)\right)_{y\in\widetilde{\mathbb{Z}}^d}$ which is continuous (because the occupation field of the Brownian excursions is) and its restriction to the vertices is $(L^x(\mathcal{I}^u))_{x\in\mathbb{Z}^d}$. We will show that the isomorphism (4.1.2) also holds in the continuous setting on $\widetilde{\mathbb{Z}}^d$. To this end we will use the approximation scheme of random interlacement by excursions that appeared in [**Szn12a**].

Let K be a finite subset of $\mathbb{Z}^d$. Let $\mathcal{I}_K^u$ be the set of trajectories in $\mathcal{I}^u$ that visit K. Given such a trajectory w we will denote by $(w^K(t))_{t\geqslant 0}$ the trajectory obtained by setting the origin of times at the entrance time of w in K and running w onward from this time. Conditional on w_0^K, $(w^K(t))_{t\geqslant 0}$ is a Markov jump process on $\mathbb{Z}^d$.

Let $\mathcal{G}_n$ be the (discrete) graph obtained from the subgraph $[-n,n]^d$ of $\mathbb{Z}^d$ by identifying to one vertex x_* the boundary of $[-n,n]^d$, that is the vertices

$$\{(x_1,x_2,\dots,x_d)\in[-n,n]^d|\exists! i\in\{1,\dots,d\},|x_i|=n\}$$

Between any two distinct adjacent vertices in $\mathcal{G}_n$ the conductance is 1. Let $(X_t^n)_{t\geqslant 0}$ be the recurrent Markov jump process on $\mathcal{G}_n$ starting from x_*. X^n jumps away from x_* with rate $2d(2n-1)^{d-1}$. Let

$$\tau_u^n := \inf\left\{t\geqslant 0\middle|\int_0^t 1_{X_s^n=x_*}ds=u\right\}$$

There are two sequences $(D_j^n)_{j\geqslant 1}$ and $(R_j^n)_{j\geqslant 1}$ with

$$0<D_1^n<R_1^n<D_2^n<R_2^n<\dots<D_j^n<R_j^n<\cdots$$

of successive departure and return times of X^n from and to x_*. By convention we set $R_0^n=0$. X^n is outside x_* on time intervals $[D_j^n,R_j^n)$ and in x_* on intervals $[R_{j-1}^n,D_j^n)$. Let

$$j_u^n := \max\{j\geqslant 0|R_j^n<\tau_u^n\}$$

Let K be a subset of $[-(n-1),n-1]^d$. Let

$$J_{u,K}^n := \{j\in\{1,\dots,j_u^n\}|X^n \text{ visits } K \text{ on } [D_j^n,R_j^n)\}$$

For $j\in J_{u,K}^n$ we define the stopping time $T_{K,j}^n$:

$$T_{K,j}^n := \inf\{t\in[D_j^n,R_j^n)|X_t^n\in K\}$$

Conditional on $j\in J_{u,K}^n$ and on the value of $X^n_{T^n_{K,j}}$, the trajectory

$$(X^n_{T^n_{K,j}+t})_{0\leqslant t\leqslant R_j^n-T^n_{K,j}}$$

is a Markov jump process on $\mathcal{G}_n$ run until hitting x_* or equivalently a Markov jump process on $\mathbb{Z}^d$ run until hitting the boundary of $[-n,n]^d$.

The next approximation result was shown in the first proof of theorem 2.1 in [**Szn12a**]:

LEMMA 4.7.1. *Let K be a finite subset of $\mathbb{Z}^d$. The set of points*

$$\{X^n_{T^n_{K,j}}|j\in J^n_{u,K}\}$$

converges in law as $n\to+\infty$ to

$$\{w^K(0)|w\in\mathcal{I}_K^u\}$$

The set of trajectories

$$\left\{(X^n_{T^n_{K,j}+t})_{0\leqslant t\leqslant R_j^n-T^n_{K,j}}|j\in J^n_{u,K}\right\}$$

converges in law to the set of trajectories

$$\{(w^K(t))_{t\geqslant 0}|w\in\mathcal{I}_K^u\}$$

Let $\widetilde{\mathcal{G}}_n$ be the metric graph associated to the graph $\mathcal{G}_n$. Let $(B_t^{\widetilde{\mathcal{G}}_n})_{t\geqslant 0}$ be a Brownian motion on $\widetilde{\mathcal{G}}_n$ starting from x_* and $(L_t^y(B^{\widetilde{\mathcal{G}}_n}))_{t\geqslant 0,y\in\widetilde{\mathcal{G}}_n}$ its family of local times. Let

$$\tilde{\tau}_u^n := \inf\{t\geqslant 0|L_t^{x_*}(B^{\widetilde{\mathcal{G}}_n})>u\}$$

For $r\in\mathbb{N}^*$ we denote by $\widetilde{\Lambda}_r$ the metric graph associated to the subgraph $[-r,r]^d$ of $\mathbb{Z}^d$ (without identification of boundary points).

LEMMA 4.7.2. *For all $r\in\mathbb{N}^*$ the occupation field $\left(L^y_{\tilde{\tau}_u^n}(B^{\widetilde{\mathcal{G}}_n})\right)_{y\in\widetilde{\Lambda}_r}$ converges in law as $n\to+\infty$ to $\left(L^y(\widetilde{\mathcal{I}}^u)\right)_{y\in\widetilde{\Lambda}_r}$.*

PROOF. The Markov jump process X^n on $\mathcal{G}_n$ is obtained from the Brownian motion $B^{\widetilde{\mathcal{G}}_n}$ through a time change by the inverse of the continuous additive functional

$$\sum_{x\in\mathcal{G}_n} L_t^x(B^{\widetilde{\mathcal{G}}_n})$$

Let $n \geqslant r+1$. The occupation field $\left(L^y_{\tilde{\tau}^n_u}(B^{\widetilde{\mathcal{G}}_n})\right)_{y\in\widetilde{\Lambda}_r}$ is obtained from the set of discrete-space trajectories

$$\left\{(X^n_{T^n_{[-r,r]^d,j}+t})_{0\leqslant t\leqslant R^n_j - T^n_{[-r,r]^d,j}} | j\in J^n_{u,[-r,r]^d}\right\} \tag{4.7.1}$$

in the same way as the occupation field $\left(L^y(\widetilde{\mathcal{I}}^u)\right)_{y\in\widetilde{\Lambda}_r}$ is obtained from the trajectories

$$\{(w^{[-r,r]^d}(t))_{t\geqslant 0} | w\in\mathcal{I}^u_{[-r,r]^d}\} \tag{4.7.2}$$

In both cases one adds Brownian excursions and takes the local times. According to lemma 4.7.1 applied to the set $[-r,r]^d$, (4.7.1) converges in law to (4.7.2). This implies the convergence in law of $\left(L^y_{\tilde{\tau}^n_u}(B^{\widetilde{\mathcal{G}}_n})\right)_{y\in\widetilde{\Lambda}_r}$ to $\left(L^y(\widetilde{\mathcal{I}}^u)\right)_{y\in\widetilde{\Lambda}_r}$. □

PROPOSITION 4.7.3. *Let $(\phi_y)_{y\in\widetilde{\mathbb{Z}}^d}$ be the Gaussian free field on the metric graph $\widetilde{\mathbb{Z}}^d$ and $(\phi'_y)_{y\in\widetilde{\mathbb{Z}}^d}$ a copy of ϕ independent of $\widetilde{\mathcal{I}}_u$. The following equality in law holds:*

$$\left(L^y(\widetilde{\mathcal{I}}^u)+\frac{1}{2}\phi'^2_y\right)_{y\in\widetilde{\mathbb{Z}}^d} \overset{(d)}{=} \left(\frac{1}{2}(\phi_y-\sqrt{2u})^2\right)_{y\in\widetilde{\mathbb{Z}}^d} \tag{4.7.3}$$

PROOF. Let ϕ^n be the Gaussian free field on the metric graph $\widetilde{\mathcal{G}}_n$ associated to the Brownian motion with instantaneous killing at x_* ($\phi^n_{x_*}=0$). Let ϕ'^n be a copy of ϕ^n independent of the Brownian motion $B^{\widetilde{\mathcal{G}}_n}$ starting from x_*. The second generalized Ray-Knight theorem (see theorem 8.2.2 in [**MR06**]) holds in this setting:

$$\left(L^y_{\tilde{\tau}^n_u}(B^{\widetilde{\mathcal{G}}_n})+\frac{1}{2}(\phi'^n_y)^2\right)_{y\in\widetilde{\mathcal{G}}_n} \overset{(d)}{=} \left(\frac{1}{2}(\phi^n_y-\sqrt{2u})^2\right)_{y\in\widetilde{\mathcal{G}}_n}$$

Since the ϕ^n converges in law to ϕ and according to lemma 4.7.2 $\left(L^y_{\tilde{\tau}^n_u}(B^{\widetilde{\mathcal{G}}_n})\right)_{y\in\widetilde{\mathcal{G}}_n}$ converges in law to $\left(L^y(\widetilde{\mathcal{I}}^u)\right)_{y\in\widetilde{\mathbb{Z}}^d}$ we get the isomorphism (4.7.3). □

Proof of theorem 4.4. The coupling is the following: take a discrete-space random interlacement $\mathcal{I}^u$ and extend it to a continuous interlacement $\widetilde{\mathcal{I}}^u$ of the metric graph $\widetilde{\mathbb{Z}}^d$. Take a Gaussian free field ϕ' on $\widetilde{\mathbb{Z}}^d$ independent on $\widetilde{\mathcal{I}}^u$. Using isomorphism (4.7.3) we see $\left(L^y(\widetilde{\mathcal{I}}^u)+\frac{1}{2}\phi'^2_y\right)_{y\in\widetilde{\mathbb{Z}}^d}$ as $\left(\frac{1}{2}(\phi_y-\sqrt{2u})^2\right)_{y\in\widetilde{\mathbb{Z}}^d}$ where ϕ is a Gaussian free field on $\widetilde{\mathbb{Z}}^d$ and sample the sign of $\phi-\sqrt{2u}$ using its conditional law given $|\phi-\sqrt{2u}|$.

The continuous occupation field $\left(L^y(\widetilde{\mathcal{I}}^u)\right)_{y\in\widetilde{\mathbb{Z}}^d}$ is strictly positive on all the vertices and inside the edges visited by the discrete random interlacement $\mathcal{I}^u$. In the isomorphism (4.7.3), $(|\phi_y-\sqrt{2u}|)_{y\in\widetilde{\mathbb{Z}}^d}$ is strictly positive on these vertices and inside these edges. This means that each trajectory in $\mathcal{I}^u$ is contained in a sign cluster of $\phi-\sqrt{2u}$, which is necessarily unbounded. But according proposition 4.5.5, ϕ has only bounded sign clusters on the metric graph and *a fortiori* the connected components of $\{y\in\widetilde{\mathbb{Z}}^d|\phi_y>\sqrt{2u}\}$ are all bounded. Thus in our coupling all the vertices visited by $\mathcal{I}^u$ are contained in $\{y\in\widetilde{\mathbb{Z}}^d|\phi_y<\sqrt{2u}\}$ and since these are vertices, they are contained in $\{x\in\mathbb{Z}^d|\phi_x<\sqrt{2u}\}$. □

The fact that for all $h > 0$, $\{x \in \mathbb{Z}^d | \phi_x < h\}$, seen as a dependent site percolation on $\mathbb{Z}^d$, has an infinite cluster was proved in [**BLM87**]. However theorem 4.4 may be used as an alternative proof of this fact.

CHAPTER 5

Convergence of the two-dimensional random walk loop soup clusters to CLE

5.1. Introduction

One can naturally associate to a wide class of Markov processes an infinite measure on time-parametrized loops. Roughly speaking, given a locally compact second-countable space S, a Markov process $(X_t)_{0\leqslant t<\zeta}$ on S, defined up to a killing time $\zeta \in (0, +\infty]$, with transition densities $p_t(x, y)$ with respect some σ-finite measure $m(dy)$ and with bridge probability measure $\mathbb{P}^t_{x,y}(\cdot)$, where the bridges are conditioned on $\zeta > t$, the loop measure associated to X is

$$\mu(\cdot) = \int_{x\in S}\int_{t>0} \mathbb{P}^t_{x,x}(\cdot)p_t(x, x)\frac{dt}{t}m(dx) \tag{5.1.1}$$

See [**LMR15**] for the precise setting and definition. A Poisson ensemble of Markov loops or loop soup of intensity parameter $\alpha > 0$ is a Poisson point process of loops of intensity $\alpha\mu$. It is a random collection of loops. We will deal with the clusters of loops. Two loops γ and γ' in a loop soup belong to the same cluster if there is a chain of loops $\gamma_0, \dots, \gamma_j$ such that $\gamma_0 = \gamma$, $\gamma_j = \gamma'$ and γ_i and γ_{i-1} visit a common point in S. These loop soups satisfy some universal properties, one of which being the relation to the Gaussian free field at intensity parameter $\alpha = \frac{1}{2}$ ([**Jan11**], [**Lup14**]).

We will consider loop soups in the following settings:

- On the continuum half-plane $\mathbb{H} = \{\Im(z) > 0\} \subset \mathbb{C}$ we will consider the loop soups associated to the Brownian motion on $\mathbb{H}$ killed at hitting the boundary $\mathbb{R}$ and denote them $\mathcal{L}^{\mathbb{H}}_{\alpha}$. These two-dimensional Brownian loop soups were introduced by Lawler and Werner in [**LW04**] and used by Sheffield and Werner in [**SW12**] to give a construction of Conformal Loop Ensembles (*CLE*). In (5.1.1) we use the same normalisation of the loop measure as in [**LW04**], [**SW12**], [**Jan11**] or [**LF07**]. However, contrary to what is claimed in [**SW12**], the intensity parameter α does not equal the so called *central charge c*. Actually
$$\alpha = \frac{c}{2}$$
The $\frac{1}{2}$ factor was pointed out by Werner in a private communication. It also appears in the Lawler's work [**Law09**].
- On the discrete rescaled half-plane
$$\mathtt{H}_n := \left(\frac{1}{n}\mathbb{Z}\right) \times \left(\frac{1}{n}\mathbb{N}\right)$$
we will consider the loop soups associated to the nearest neighbours Markov jump process with uniform transition rates and killed at hitting the boundary $\frac{1}{n}\mathbb{Z} \times \{0\}$. We will denote these loop soups $\mathcal{L}^{\mathtt{H}_n}_{\alpha}$. The loop soups associated to Markov jump processes on more general electrical networks were studied by Le Jan in [**Jan11**]. If one forgets the parametrisation by continuous time and the "loops" that visit

only one vertex, these are exactly the random walk loop soups studied by Lawler and Trujillo Ferreras in [**LF07**].

- We will use the metric graphs $\widetilde{\mathtt{H}}_n$ associated to $\mathtt{H}_n$: each edge $\{(\frac{i}{n},\frac{j}{n}),(\frac{i+1}{n},\frac{j}{n})\}$ or $\{(\frac{i}{n},\frac{j}{n}),(\frac{i}{n},\frac{j+1}{n})\}$ is replaced by a continuous line of length $\frac{1}{n}$. Let $(B_t^{\widetilde{\mathtt{H}}_n})_{0\leqslant t<\zeta_n}$ be the Brownian motion on $\widetilde{\mathtt{H}}_n$ killed at reaching the boundary, that is to say the vertices $\frac{1}{n}\mathbb{Z}\times\{0\}$ and all the lines joining $(\frac{i}{n},0)$ to $(\frac{i+1}{n},0)$. $(B_{2t}^{\widetilde{\mathtt{H}}_n})_{0\leqslant t<\frac{1}{2}\zeta_n}$ converges in law to the Brownian motion on the half-plane $\mathbb{H}$ killed at reaching $\mathbb{R}$. We will denote by $\mathcal{L}_\alpha^{\widetilde{\mathtt{H}}_n}$ the loop soups associated to $(B_t^{\widetilde{\mathtt{H}}_n})_{0\leqslant t<\zeta_n}$. The loop soups on metric graphs were first considered in [**Lup14**]. We will use the metric graphs because at intensity parameter $\alpha=\frac{1}{2}$ the probability that two points belong to the same cluster of loops can be explicitly expressed using a metric graph Gaussian free field. Indeed the clusters of loops are then exactly the sign clusters of the Gaussian free field ([**Lup14**]).

The discrete loops $\mathcal{L}_\alpha^{\mathtt{H}_n}$ can be deterministically recovered from the metric graph loops $\mathcal{L}_\alpha^{\widetilde{\mathtt{H}}_n}$. The first are the trace on the vertices of the latter. In particular each cluster of $\mathcal{L}_\alpha^{\mathtt{H}_n}$ is contained in a cluster of $\mathcal{L}_\alpha^{\widetilde{\mathtt{H}}_n}$, but the clusters of $\mathcal{L}_\alpha^{\widetilde{\mathtt{H}}_n}$ may be strictly larger ([**Lup14**]).

$c=1$ is known to be the critical central charge for the Brownian loop percolation on $\mathbb{H}$ (or any other simply connected proper subset of $\mathbb{C}$). This means that the critical intensity parameter is $\alpha=\frac{1}{2}$. For $\alpha>\frac{1}{2}$ $\mathcal{L}_\alpha^{\mathbb{H}}$ has only one cluster everywhere dense in $\mathbb{H}$. If $\alpha\in(0,\frac{1}{2}]$ there are infinitely many clusters and each is bounded ([**SW12**]). It was shown in [**Lup14**] that for discrete or metric graph Brownian loop soups on $\mathtt{H}_n$ respectively $\widetilde{\mathtt{H}}_n$ there are no unbounded clusters of loops if $\alpha\in(0,\frac{1}{2}]$. In all three settings, for $\alpha\in(0,\frac{1}{2}]$, we will consider the collection of outer boundaries of outermost clusters (not surrounded by any other cluster) and denote it $\mathcal{F}_{ext}(\mathcal{L}_\alpha^S)$, where S is $\mathbb{H}$, $\mathtt{H}_n$ or $\widetilde{\mathtt{H}}_n$. Next we give the formal definition of $\mathcal{F}_{ext}(\mathcal{L}_\alpha^S)$. We consider the set of all points in $\mathbb{H}$ visited by a loop in $\mathcal{L}_\alpha^S$ and take its complementary in $\mathbb{H}$. This complementary has only one unbounded connected component. We take the boundary in $\mathbb{H}$ of this connected component (by definition it does not intersect $\mathbb{R}$). The element of $\mathcal{F}_{ext}(\mathcal{L}_\alpha^S)$ are the connected components of this boundary. We will call the elements of $\mathcal{F}_{ext}(\mathcal{L}_\alpha^S)$ *contours*. The contours are two by two disjoint and non nested. See next picture for a representation of $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\widetilde{\mathtt{H}}_n})$.

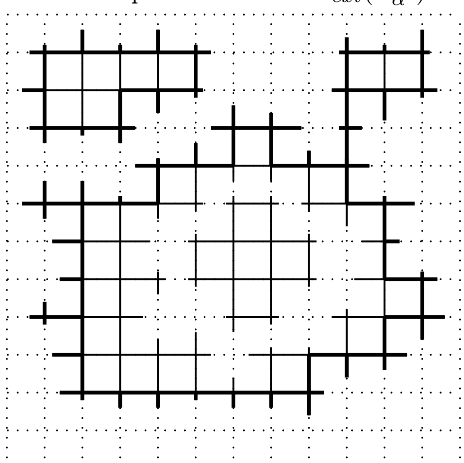

Fig.5.1: Illustration of three clusters (thin full lines) of $\mathcal{L}_\alpha^{\widetilde{\mathtt{H}}_n}$,
two of them being external and one being surrounded.
The thick lines represent the elements of $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\widetilde{\mathtt{H}}_n})$.

The contours in $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathbb{H}})$, $\alpha\in(0,\frac{1}{2}]$, are non self-intersecting loops, and are equal in law to a Conformal Loop Ensemble CLE_κ, $\kappa\in(\frac{8}{3},4]$ ([**SW12**]). The relation between α

and κ is given by

$$2\alpha = c = \frac{(3\kappa - 8)(6-\kappa)}{2\kappa} \tag{5.1.2}$$

We will denote by $\kappa(\alpha)$ the value of κ corresponding to a particular intensity parameter α.

We will show that both $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathtt{H}_n})$ and $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\widetilde{\mathtt{H}}_n})$ converge in law to $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathbb{H}}) \overset{(d)}{=} CLE_{\kappa(\alpha)}$ for $\alpha \in (0, \frac{1}{2}]$. Observe that $\kappa(\frac{1}{2}) = 4$ and $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\widetilde{\mathtt{H}}_n})$ and CLE_4 are both related to the Gaussian free field. $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\widetilde{\mathtt{H}}_n})$ is the collection of outer boundaries of outermost sign clusters of a GFF on the metric graph $\widetilde{\mathtt{H}}_n$ ([**Lup14**]) and the CLE_4 loops are in some sense zero level lines of the continuum GFF on $\mathbb{H}$ with zero boundary conditions on $\mathbb{R}$ ([**WW14**]).

Next we define the notion of convergence we will use. d_H will be Hausdorff distance on the compact subsets of $\mathbb{H}$. We introduce the distance d_H^* on finite sets of compact subsets of $\mathbb{H}$:

$$d_H^*(\mathcal{K}, \mathcal{K}') = \begin{cases} +\infty & \text{if } |\mathcal{K}| \neq |\mathcal{K}'| \\ \min_{\sigma \in Bij(\mathcal{K},\mathcal{K}')} \max_{K\in\mathcal{K}} d_H(K, \sigma(K)) & \text{otherwise} \end{cases}$$

Given $z \in \mathbb{H}$ we will denote by

$$\mathcal{F}_{ext}(\mathcal{L}_\alpha^S)(z)$$

the contour of $\mathcal{F}_{ext}(\mathcal{L}_\alpha^S)$ that contains or surrounds z, whenever it exists. It exists a.s. in the case $S = \mathbb{H}$. Given $z_1, \dots, z_j \in \mathbb{H}$ we will denote

$$\mathcal{F}_{ext}(\mathcal{L}_\alpha^S)[z_1, \dots, z_j] := \{\mathcal{F}_{ext}(\mathcal{L}_\alpha^S)(z_i) | 1 \leqslant i \leqslant j\}$$

By the convergence in law of $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathtt{H}_n})$ and $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\widetilde{\mathtt{H}}_n})$ to $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathbb{H}})$ we mean that for any $z_1, \dots, z_j \in \mathbb{H}$, the random sets $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathtt{H}_n})[z_1, \dots, z_j]$ and $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\widetilde{\mathtt{H}}_n})[z_1, \dots, z_j]$ converge in law to $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathbb{H}})[z_1, \dots, z_j]$ for the distance d_H^*.

In the article [**dBCL14**] Van de Brug, Camia and Lis consider clusters of rescaled two-dimensional random walk loops that are not too small. Given $T > 0$ let $\mathcal{L}_\alpha^{\mathtt{H}_n, T}$ be the subset of $\mathcal{L}_\alpha^{\mathtt{H}_n}$ consisting of random walk loops that do at least T jumps. In [**dBCL14**] it is almost shown that for $\theta \in (\frac{16}{9}, 2)$ and $\alpha \in (0, \frac{1}{2}]$, $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathtt{H}_n, n^\theta})$ converge in law to a $CLE_{\kappa(\alpha)}$ process in the sense described previously. The result uses the approximation of "not too small" Brownian loops by "not too small" random walk loops obtained by Lawler and Trujillo Ferreras in [**LF07**]. However the authors in [**dBCL14**] consider the loop soups only on bounded domains and their lattice approximations. We will fill the small gap to extend their result to the half-plane. Observe that in [**dBCL14**] the authors use the same normalisation of the measure on loops as we but do the widespread mistake to consider that the intensity parameter of the loop soups equals the central charge.

From above considerations one deduce that the limiting (in law) loops of $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathtt{H}_n})$ and *a fortiori* of $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\widetilde{\mathtt{H}}_n})$ are at least as big as $CLE_{\kappa(\alpha)}$ loops. We have a "lower bound". To conclude the convergence we need an "upper bound". We will construct such an "upper bound" for $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\widetilde{\mathtt{H}}_n})$, deduce the convergence to CLE_4 of $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\widetilde{\mathtt{H}}_n})$ and $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\mathtt{H}_n})$. Then from this we will deduce the desired convergences for $\alpha \in (0, \frac{1}{2})$. Next we explain how the "upper bound" will be constructed.

We will concentrate on the case $\alpha = \frac{1}{2}$. We will additionally introduce two Poisson point processes of excursions on $\widetilde{\mathtt{H}}_n$ and on $\mathbb{H}$. First we consider $\widetilde{\mathtt{H}}_n$. Let $x \in \frac{1}{n}\mathbb{Z}_- \times \{0\}$. Let $\nu_{exc}^{\widetilde{\mathtt{H}}_n}(x \to (-\infty, 0])$ be the measure on excursions of the metric graph Brownian motion $B^{\widetilde{\mathtt{H}}_n}$ from x to a point in $\frac{1}{n}\mathbb{Z}_- \times \{0\}$. It is defined as follows: Let $\mathbb{P}_{x+i\varepsilon}^{\widetilde{\mathtt{H}}_n}(\cdot, B_{\zeta_n^-}^{\widetilde{\mathtt{H}}_n} \in \frac{1}{n}\mathbb{Z}_- \times \{0\})$ be the law of a sample path of $B^{\widetilde{\mathtt{H}}_n}$, started at $x + i\varepsilon$, restricted to the event $B_{\zeta_n^-}^{\widetilde{\mathtt{H}}_n} \in \frac{1}{n}\mathbb{Z}_- \times \{0\}$

(we do not condition and the total mass is < 1). Then

$$\nu_{exc}^{\tilde{\mathbb{H}}_n}(x \to (-\infty, 0]) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{P}_{x+i\varepsilon}^{\tilde{\mathbb{H}}_n} \left(\cdot, B_{\zeta_n^-}^{\tilde{\mathbb{H}}_n} \in \frac{1}{n}\mathbb{Z}_- \times \{0\} \right)$$

Let $q \in (1, +\infty)$ and $x \in ((\frac{1}{n}\mathbb{Z}) \cap [1, q]) \times \{0\}$. We will similarly denote by $\nu_{exc}^{\tilde{\mathbb{H}}_n}(x \to [1, q])$ the measure on excursions from x to $((\frac{1}{n}\mathbb{Z}) \cap [1, q]) \times \{0\}$. Let

$$\nu_{exc}^{\tilde{\mathbb{H}}_n}((-\infty, 0]) := \frac{8\pi}{n} \sum_{x \in \frac{1}{n}\mathbb{Z}_- \times \{0\}} \nu_{exc}^{\tilde{\mathbb{H}}_n}(x \to (-\infty, 0]) \tag{5.1.3}$$

$$\nu_{exc}^{\tilde{\mathbb{H}}_n}([1, q]) := \frac{8\pi}{n} \sum_{x \in ((\frac{1}{n}\mathbb{Z}) \cap [1, q]) \times \{0\}} \nu_{exc}^{\tilde{\mathbb{H}}_n}(x \to [1, q]) \tag{5.1.4}$$

$\nu_{exc}^{\tilde{\mathbb{H}}_n}((-\infty, 0])$ is a measure on excursions from and to $\frac{1}{n}\mathbb{Z}_- \times \{0\}$. $\nu_{exc}^{\tilde{\mathbb{H}}_n}([1, q])$ is a measure on excursion from and to $((\frac{1}{n}\mathbb{Z}) \cap [1, q]) \times \{0\}$. Both measures are invariant under time reversal.

As n tends to infinity, $\nu_{exc}^{\tilde{\mathbb{H}}_n}((-\infty, 0])$ and $\nu_{exc}^{\tilde{\mathbb{H}}_n}([1, q])$ have limits which are measures on Brownian excursions in $\mathbb{H}$, from and to $(-\infty, 0] \times \{0\}$ respectively $[1, q] \times \{0\}$. We will denote them by $\nu_{exc}^{\mathbb{H}}((-\infty, 0])$ respectively $\nu_{exc}^{\mathbb{H}}([1, q])$. For $x, y \in \mathbb{R}$, let $\mathbb{P}_{x,y}^{\mathbb{H}}(\cdot)$ be the probability measure on Brownian excursions from x to y in $\mathbb{H}$. Then

$$\nu_{exc}^{\mathbb{H}}((-\infty, 0]) = 2 \int_{-\infty}^{0} \int_{-\infty}^{0} \mathbb{P}_{x,y}^{\mathbb{H}} \frac{dxdy}{(y-x)^2}$$

$$\nu_{exc}^{\mathbb{H}}([1, q]) = 2 \int_{1}^{q} \mathbb{P}_{x,y}^{\mathbb{H}} \frac{dxdy}{(y-x)^2}$$

In general, given $a < b \in \mathbb{R}$, we will use the notation

$$\nu_{exc}^{\mathbb{H}}([a, b]) := 2 \int_{a}^{b} \int_{a}^{b} \mathbb{P}_{x,y}^{\mathbb{H}} \frac{dxdy}{(y-x)^2}$$

We will consider on $\tilde{\mathbb{H}}_n$ three independent Poisson point processes:

- a loop soups $\mathcal{L}_{\frac{1}{2}}^{\tilde{\mathbb{H}}_n}$
- a Poisson point process of excursions of intensity $u\nu_{exc}^{\tilde{\mathbb{H}}_n}((-\infty, 0])$, $u > 0$, denoted by $\mathcal{E}_u^{\tilde{\mathbb{H}}_n}((-\infty, 0])$
- a Poisson point process of excursions of intensity $v\nu_{exc}^{\tilde{\mathbb{H}}_n}([1, q])$, $v > 0$, denoted by $\mathcal{E}_v^{\tilde{\mathbb{H}}_n}([1, q])$

We will consider the following event: either an excursion from $\mathcal{E}_u^{\tilde{\mathbb{H}}_n}((-\infty, 0])$ intersects an excursion from $\mathcal{E}_v^{\tilde{\mathbb{H}}_n}([1, q])$ or an excursion from $\mathcal{E}_u^{\tilde{\mathbb{H}}_n}((-\infty, 0])$ and from $\mathcal{E}_v^{\tilde{\mathbb{H}}_n}([1, q])$ intersect a common cluster of $\mathcal{L}_{\frac{1}{2}}^{\tilde{\mathbb{H}}_n}$. We will denote by $p_{\frac{1}{2},u,v}^{\tilde{\mathbb{H}}_n}(q)$ the probability of this event. The second condition of intersecting a common cluster is equivalent to intersecting a common contour in $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\tilde{\mathbb{H}}_n})$.

Similarly we will consider on $\mathbb{H}$ three independent Poisson point processes:

- a loop soups $\mathcal{L}_{\alpha}^{\mathbb{H}}$, $\alpha \in (0, \frac{1}{2}]$
- a Poisson point process of excursions of intensity $u\nu_{exc}^{\mathbb{H}}((-\infty, 0])$, $u > 0$, denoted by $\mathcal{E}_u^{\mathbb{H}}((-\infty, 0])$
- a Poisson point process of excursions of intensity $v\nu_{exc}^{\mathbb{H}}([1, q])$, $v > 0$, denoted by $\mathcal{E}_v^{\mathbb{H}}([1, q])$

Then we will consider the event when either an excursion from $\mathcal{E}_u^{\mathbb{H}}((-\infty,0])$ intersects an excursions from $\mathcal{E}_v^{\mathbb{H}}([1,q])$ or an excursion from $\mathcal{E}_u^{\mathbb{H}}((-\infty,0])$ and one from $\mathcal{E}_v^{\mathbb{H}}([1,q])$ intersect a common cluster of $\mathcal{L}_\alpha^{\mathbb{H}}$. This event is schematically represented on the figure 5.2. We denote by $p_{\alpha,u,v}^{\mathbb{H}}(q)$ its probability.

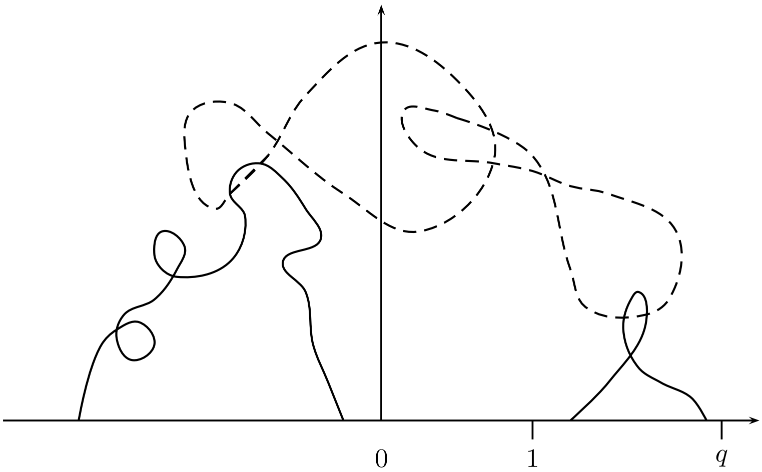

Fig.5.2: Two excursions (full lines) connected by a chain of two loops (doted lines).

In the section 5.2 we will compute $p_{\frac{1}{2},u,v}^{\tilde{\mathbb{H}}_n}(q)$ using the duality with the Gaussian free field, and compute its limit as n tends to 0. In section 5.3, for an arbitrary value of v and a particular value $u_0(\alpha)$ of u (depending on α) we will establish a differential equation in q for $1-p_{\alpha,u,v}^{\mathbb{H}}(q)$. Using this we will show that

$$\lim_{n\to+\infty} p_{\frac{1}{2},u_0(\frac{1}{2}),v}^{\tilde{\mathbb{H}}_n}(q) = p_{\frac{1}{2},u_0(\frac{1}{2}),v}^{\mathbb{H}}(q)$$

This will be our "upper bound". In the section 5.4 we will prove the convergences to CLE.

5.2. Computation of connexion probability on metric graph half-plane

Let $\mathcal{G}=(V,E)$ be a connected undirected graph. V is countable and each vertex is of finite degree. Each edge $\{x,y\}$ is endowed with a positive conductance $C(x,y)>0$. We also consider a metric graph $\tilde{\mathcal{G}}$ associated to $\mathcal{G}$ where each edge $\{x,y\}$ is replaced by a continuous line of length

$$r(x,y)=\frac{1}{2}C(x,y)^{-1} \tag{5.2.1}$$

Let $B^{\tilde{\mathcal{G}}}$ be the Brownian motion on the metric graph $\tilde{\mathcal{G}}$. Let F be a subset of V. Let ζ_F be the first time $B^{\tilde{\mathcal{G}}}$ hits F. Let measure $\mu^{\tilde{\mathcal{G}},\times F}$ be the measure on loops associated to $(B_t^{\tilde{\mathcal{G}}})_{0\leqslant t<\zeta_F}$ the Brownian motion killed at reaching F. It is defined according to (5.1.1). See [**Lup14**] for details. Let $\mathcal{L}_\alpha^{\tilde{\mathcal{G}},\times F}$ be the Poisson point process of intensity $\alpha\mu^{\tilde{\mathcal{G}},\times F}$.

$B^{\tilde{\mathcal{G}}}$ has a time-space continuous family of local times $L_t^z(B^{\tilde{\mathcal{G}}})$. The Green's function of the killed Brownian motion $(B_t^{\tilde{\mathcal{G}}})_{0\leqslant t<\zeta_F}$ satisfies

$$G^{\tilde{\mathcal{G}},\times F}(z,z')=G^{\tilde{\mathcal{G}},\times F}(z',z)=\mathbb{E}_z\left[\int_0^{\zeta_F} L_t^{z'}(B^{\tilde{\mathcal{G}}})\right]$$

Just as $B^{\widetilde{\mathcal{G}}}$, a loop $\gamma \in \mathcal{L}_{c\alpha}^{\widetilde{\mathcal{G}},\times F}$ has a family of continuous local times $L_t^z(\gamma)$. We will denote by t_γ the total life-time of the loop γ. The occupation field $(\widehat{\mathcal{L}}_\alpha^z)_{z\in\widetilde{\mathcal{G}}\backslash F}$ is defined as

$$\widehat{\mathcal{L}}_\alpha^z = \sum_{\gamma\in\mathcal{L}_\alpha^{\widetilde{\mathcal{G}},\times F}} L_{t_\gamma}^z(\gamma)$$

It is a continuous field. The clusters of $\mathcal{L}_\alpha^{\widetilde{\mathcal{G}},\times F}$ are delimited by the zero set of the occupation field.

At intensity parameter $\alpha = \frac{1}{2}$ the occupation field $(\widehat{\mathcal{L}}_\alpha^z)_{z\in\widetilde{\mathcal{G}}\backslash F}$ is related to the Gaussian free field $(\phi_z)_{z\in\widetilde{\mathcal{G}}\backslash F}$ with zero mean and covariance function $G^{\widetilde{\mathcal{G}},\times F}$. Given $z \in \widetilde{\mathcal{G}}\backslash F$ such that $\widehat{\mathcal{L}}_\alpha^z > 0$ we denote by $\mathcal{C}_{\frac{1}{2}}(z)$ the cluster of $\mathcal{L}_{\frac{1}{2}}^{\widetilde{\mathcal{G}},\times F}$ that contains z. We introduce a countable family $(\sigma(\mathcal{C}_{\frac{1}{2}}(z)))_{z\in\widetilde{\mathcal{G}}\backslash F}$ of i.i.d. random variables, independent of $\mathcal{L}_{\frac{1}{2}}^{\widetilde{\mathcal{G}},\times F}$ conditional on the clusters, which equal -1 or 1 with equal probability. There is an equality in law (see [**Lup14**]):

$$(\phi_z)_{z\in\widetilde{\mathcal{G}}\backslash F} \overset{(d)}{=} \left(\sigma(\mathcal{C}_{\frac{1}{2}}(z))\sqrt{2\widehat{\mathcal{L}}_{\frac{1}{2}}^z}\right)_{z\in\widetilde{\mathcal{G}}\backslash F} \tag{5.2.2}$$

Let $x, y \in V\backslash F$. Let $C^{eq}(x,y), \chi_{(x,y)}^{eq}(x), \chi_{(x,y)}^{eq}(y)$ be the quantities defined by

$$\begin{pmatrix} G^{\widetilde{\mathcal{G}},\times F}(x,x) & G^{\widetilde{\mathcal{G}},\times F}(x,y) \\ G^{\widetilde{\mathcal{G}},\times F}(x,y) & G^{\widetilde{\mathcal{G}},\times F}(y,y) \end{pmatrix}^{-1} = \begin{pmatrix} \chi_{(x,y)}^{eq}(x) + C^{eq}(x,y) & -C^{eq}(x,y) \\ -C^{eq}(x,y) & \chi_{(x,y)}^{eq}(y) + C^{eq}(x,y) \end{pmatrix}$$

Then $C^{eq}(x,y) > 0$, $\chi_{(x,y)}^{eq}(x), \chi_{(x,y)}^{eq}(y) \geqslant 0$, $(\chi_{(x,y)}^{eq}(x), \chi_{(x,y)}^{eq}(y)) \neq (0,0)$. $C^{eq}(x,y)$ is the equivalent conductance between x and y given that all points in F have the same (electrical) potential.

Let $\mathcal{N}_{\frac{1}{2}}(x,y)$ the number of loops in $\mathcal{L}_{\frac{1}{2}}^{\widetilde{\mathcal{G}},\times F}$ that visit both x and y.

LEMMA 5.2.1. *Let $u, v > 0$ and $x, y \in V\backslash F$.*

$$\mathbb{P}\left(\mathcal{C}_{\frac{1}{2}}(x) \neq \mathcal{C}_{\frac{1}{2}}(y) \middle| \widehat{\mathcal{L}}_{\frac{1}{2}}^x = u, \widehat{\mathcal{L}}_{\frac{1}{2}}^y = v, \mathcal{N}_{\frac{1}{2}}(x,y) = 0\right) = e^{-2C^{eq}(x,y)\sqrt{uv}} \tag{5.2.3}$$

PROOF. If $\mathcal{N}_{\frac{1}{2}}(x,y) > 0$ then $\mathcal{C}_{\frac{1}{2}}(x) = \mathcal{C}_{\frac{1}{2}}(y)$. Thus

$$\mathbb{P}\left(\mathcal{C}_{\frac{1}{2}}(x) \neq \mathcal{C}_{\frac{1}{2}}(y) \middle| \widehat{\mathcal{L}}_{\frac{1}{2}}^x = u, \widehat{\mathcal{L}}_{\frac{1}{2}}^y = v, \mathcal{N}_{\frac{1}{2}}(x,y) = 0\right) = \frac{\mathbb{P}\left(\mathcal{C}_{\frac{1}{2}}(x) \neq \mathcal{C}_{\frac{1}{2}}(y) \middle| \widehat{\mathcal{L}}_{\frac{1}{2}}^x = u, \widehat{\mathcal{L}}_{\frac{1}{2}}^y = v\right)}{\mathbb{P}\left(\mathcal{N}_{\frac{1}{2}}(x,y) = 0 \middle| \widehat{\mathcal{L}}_{\frac{1}{2}}^x = u, \widehat{\mathcal{L}}_{\frac{1}{2}}^y = v\right)} \tag{5.2.4}$$

The value of the denominator

$$\mathbb{P}\left(\mathcal{N}_{\frac{1}{2}}(x,y) = 0 \middle| \widehat{\mathcal{L}}_{\frac{1}{2}}^x = u, \widehat{\mathcal{L}}_{\frac{1}{2}}^y = v\right)$$

depends only on u, v and on $G^{\widetilde{\mathcal{G}},\times F}(x,x), G^{\widetilde{\mathcal{G}},\times F}(y,y), G^{\widetilde{\mathcal{G}},\times F}(x,y)$ (or equivalently on $C^{eq}(x,y), \chi_{(x,y)}^{eq}(x), \chi_{(x,y)}^{eq}(y)$). This a general property of the loop soups (see [**Jan11**], especially chapter 7).

As for the numerator, it can be computed using the duality with the Gaussian free field (5.2.2):

$$\begin{aligned}\mathbb{P}\Big(\mathcal{C}_{\frac{1}{2}}(x) \neq \mathcal{C}_{\frac{1}{2}}(y)\Big|\widehat{\mathcal{L}}^x_{\frac{1}{2}} = u, \widehat{\mathcal{L}}^y_{\frac{1}{2}} = v\Big) \\ &= 1 - \mathbb{E}\Big[\operatorname{sign}(\phi_x)\operatorname{sign}(\phi_y)\big||\phi_x| = \sqrt{2u}, |\phi_y| = \sqrt{2v}\Big] \\ &= \frac{e^{-2C^{eq}(x,y)\sqrt{uv}}}{\cosh(2C^{eq}(x,y)\sqrt{uv})}\end{aligned}$$

It follows that the probability (5.2.3) that we want to compute only depends on u, v and on $C^{eq}(x,y), \chi^{eq}_{(x,y)}(x), \chi^{eq}_{(x,y)}(y)$. Thus it is the same if we replace $\widetilde{\mathcal{G}}$ by the interval

$$I = \Big(-\frac{1}{2}\chi^{eq}_{(x,y)}(x)^{-1}, \frac{1}{2}C^{eq}(x,y)^{-1} + \frac{1}{2}\chi^{eq}_{(x,y)}(y)^{-1}\Big)$$

the Brownian motion on $\widetilde{\mathcal{G}}$ by the Brownian motion on I killed at endpoints, and the points x and y by 0 and $\frac{1}{2}C^{eq}(x,y)^{-1}$ respectively. According to the computation made in [**Lup14**], we get (5.2.3).

By the way we also get that

$$\mathbb{P}\left(\mathcal{N}_{\frac{1}{2}}(x,y) = 0\Big|\widehat{\mathcal{L}}^x_{\frac{1}{2}} = u, \widehat{\mathcal{L}}^y_{\frac{1}{2}} = v\right) = \cosh(2C^{eq}(x,y)\sqrt{uv})^{-1}$$

□

In [**Jan11**], chapter 7, there is a combinatorial representation of $C^{eq}(x,y)$. Given $z \in V$, we will denote

$$\lambda(z) := \sum_{z' \in V, z' \sim z} C(z,z')$$

where the sum is over the neighbours of z in the (discrete) graph $\mathcal{G}$. Then

$$C^{eq}(x,y) = \lambda(x) \sum_{j \geqslant 1} \sum_{\substack{(z_0,\dots,z_j) \in (V\backslash F)^{j+1} \\ z_0 = x, z_j = y, z_i \sim z_{i-1} \\ z_i \neq x,y \text{ for } 1 \leqslant i \leqslant j-1}} \prod_{i=1}^{j} \frac{C(z_{i-1},z_i)}{\lambda(z_{i-1})}$$

The sum is over all the discrete nearest neighbour paths joining x to y, that avoid F and only visit x and y at endpoints. The above equality can be rewritten as

$$C^{eq}(x,y) = \sum_{z \in V, z \sim x} C(x,z)\mathbb{P}_z(B^{\widetilde{\mathcal{G}}} \text{ hits } y \text{ before } F \text{ or } x) \tag{5.2.5}$$

Next we return to the metric graph half-plane $\widetilde{\mathbb{H}}_n$. Let $a > 0$. Let $\widetilde{\mathcal{G}}_{n,a}(q)$ be the metric graph obtained from $\widetilde{\mathbb{H}}_n$ by identifying the following vertices:

- All the vertices in $((\frac{1}{n}\mathbb{Z}) \cap [-a, 0]) \times \{0\}$ are identified into a single vertex $\lhd_n(a)$.
- All the vertices in $((\frac{1}{n}\mathbb{Z}) \cap [1, q]) \times \{0\}$ are identified into a single vertex $\rhd_n(q)$.

See the following picture.

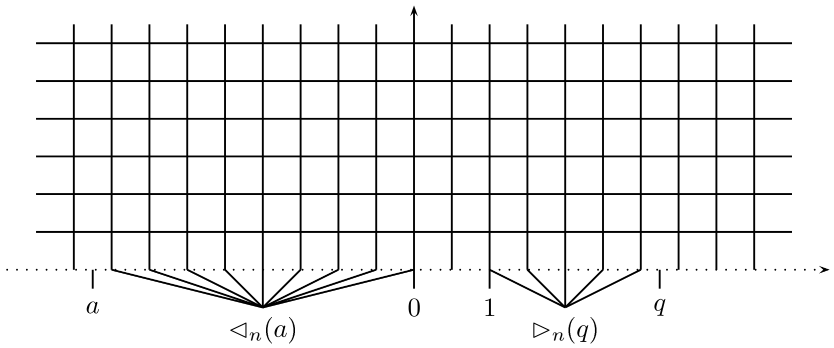

Fig.5.3: Illustration of points identified into $\lhd_n(a)$ and $\rhd_n(q)$.

As the length of the line joining $(\frac{i}{n},\frac{j}{n})$ to $(\frac{i+1}{n},\frac{j}{n})$ or $(\frac{i}{n},\frac{j}{n})$ to $(\frac{i}{n},\frac{j+1}{n})$ is $\frac{1}{n}$, the corresponding conductance is according (5.2.1) equal to $\frac{n}{2}$. Let $C^{eq}_{n,a}(q)$ be the equivalent conductance between $\lhd_n(a)$ and $\rhd_n(q)$ when all the points in $(\frac{1}{n})\mathbb{Z}\times\{0\}$ other than those identified to $\lhd_n(a)$ or $\rhd_n(q)$ have the same electrical potential. According to (5.2.5):

$$C^{eq}_{n,a}(q) = \frac{n}{2}\sum_{i=n}^{\lfloor nq\rfloor}\mathbb{P}_{(\frac{i}{n},\frac{1}{n})}\left(B^{\tilde{\mathtt{H}}_n} \text{ hits } \left(\frac{1}{n}\mathbb{Z}\right)\times\{0\} \text{ in } [-a,0]\times\{0\}\right)$$

As a tends to infinity, $C^{eq}_{n,a}(q)$ increases and converges to

$$C^{eq}_{n}(q) = \frac{n}{2}\sum_{i=n}^{\lfloor nq\rfloor}\mathbb{P}_{(\frac{i}{n},\frac{1}{n})}\left(B^{\tilde{\mathtt{H}}_n} \text{ hits } \left(\frac{1}{n}\mathbb{Z}\right)\times\{0\} \text{ in } (-\infty,0]\times\{0\}\right) \tag{5.2.6}$$

LEMMA 5.2.2. *For all $n\in\mathbb{N}^*$ and $x_0>0$, $C^{eq}_n(q)<+\infty$. Moreover*

$$\lim_{n\to+\infty}\frac{1}{n}C^{eq}_n(q) = \frac{1}{8\pi}\log(q)$$

PROOF. Let $G^{\mathtt{H}}(\cdot,\cdot)$ be the Green's function of the simple random walk on $\mathtt{H}=\mathbb{Z}\times\mathbb{N}$ killed at hitting $\mathbb{Z}\times\{0\}$. Let $i,j\in\mathbb{Z}$. Then

$$\mathbb{P}_{(\frac{i}{n},\frac{1}{n})}\left(B^{\tilde{\mathtt{H}}_n} \text{ hits } \left(\frac{1}{n}\mathbb{Z}\right)\times\{0\} \text{ in } \left(\frac{j}{n},0\right)\right) = \frac{1}{4}G^{\mathtt{H}}((i,1),(j,1)) = \frac{1}{4}G^{\mathtt{H}}((0,1),(j-i,1))$$

Indeed to go from $(\frac{i}{n},\frac{1}{n})$ to $(\frac{j}{n},0)$ the moving particle needs to reach $(\frac{j}{n},\frac{1}{n})$, possibly make excursions from and to this point without hitting $\left(\frac{1}{n}\mathbb{Z}\right)\times\{0\}$, and then with probability $\frac{1}{4}$ transition to $(\frac{j}{n},0)$. Replacing in (5.2.6) we get that

$$C^{eq}_n(q) = \frac{n}{8}\sum_{i=n}^{\lfloor nq\rfloor}\sum_{j=0}^{+\infty}G^{\mathtt{H}}((0,1),(i+j,1))$$

According to the asymptotic expansion given in [**LL10**], section 8.1.1,

$$G^{\mathtt{H}}((0,1),(j,1)) = \frac{1}{\pi j^2} + O\left(\frac{1}{j^3}\right)$$

This means that $C_n^{eq}(q) < +\infty$ and that

$$\begin{aligned}\frac{1}{n}C_n^{eq}(q) =& \frac{1}{8\pi}\sum_{i=n}^{\lfloor nq\rfloor}\sum_{j=0}^{+\infty}\frac{1}{(i+j)^2} + O\left(\sum_{i=n}^{\lfloor nq\rfloor}\sum_{j=0}^{+\infty}\frac{1}{(i+j)^3}\right)\\ =& \frac{1}{8\pi}\sum_{i=n}^{\lfloor nq\rfloor}\frac{1}{i} + O\left(\sum_{i=n}^{\lfloor nq\rfloor}\frac{1}{i^2}\right)\\ =& \frac{1}{8\pi}\log(q) + O\left(\frac{1}{n}\right)\end{aligned}$$

□

Let $\nu_{exc}^{\tilde{\mathbb{H}}_n}([-a,0])$ be the measure on excursions $\nu_{exc}^{\tilde{\mathbb{H}}_n}((-\infty,0])$ restricted to the excursions from and to $[-a,0]\times\{0\}$. Let $\mathcal{L}_\alpha^{\tilde{\mathcal{G}}_{n,a}(q)}$ be the loop soup associated to the Brownian motion on the metric graph $\tilde{\mathcal{G}}_{n,a}(q)$, killed at hitting $(\frac{1}{n})\mathbb{Z}\times\{0\}$ outside the points identified to $\lhd_n(a)$ or $\rhd_n(q)$. Let $(\hat{\mathcal{L}}^z_{n,a,q,\alpha})_{z\in\tilde{\mathcal{G}}_{n,a}(q)}$ be the occupation field of $\mathcal{L}_\alpha^{\tilde{\mathcal{G}}_{n,a}(q)}$. Let $\mathcal{N}_\alpha(\lhd_n(a),\rhd_n(q))$ be the number of loops in $\mathcal{L}_\alpha^{\tilde{\mathcal{G}}_{n,a}(q)}$ joining $\lhd_n(a)$ to $\rhd_n(q)$.

LEMMA 5.2.3. *Let $a,\alpha,u,v>0$. We consider $\mathcal{L}_\alpha^{\tilde{\mathcal{G}}_{n,a}(q)}$ conditioned on $\hat{\mathcal{L}}^{\lhd_n(a)}_{n,a,q,\alpha}=u$, $\hat{\mathcal{L}}^{\rhd_n(q)}_{n,a,q,\alpha}=v$ and $\mathcal{N}_\alpha(\lhd_n(a),\rhd_n(q))=0$. Then $\mathcal{L}_\alpha^{\tilde{\mathcal{G}}_{n,a}(q)}$ consists of three independent families of loops:*

- *The loops that visit neither $\lhd_n(a)$ nor $\rhd_n(q)$. These are the same as the loops in $\mathcal{L}_\alpha^{\tilde{\mathbb{H}}_n}$.*
- *The loops that visit $\lhd_n(a)$. The excursions these loops make outside $\lhd_n(a)$ form a Poisson point process of intensity $\frac{n}{8\pi}u\nu_{exc}^{\tilde{\mathbb{H}}_n}([-a,0])$.*
- *The loops that visit $\rhd_n(q)$. The excursions these loops make outside $\rhd_n(q)$ form a Poisson point process of intensity $\frac{n}{8\pi}v\nu_{exc}^{\tilde{\mathbb{H}}_n}([1,q])$.*

PROOF. This follows from universal properties of loop soups. See for instance [**Jan11**]. The factor $\frac{n}{8\pi}$ in $\frac{n}{8\pi}u\nu_{exc}^{\tilde{\mathbb{H}}_n}([-a,0])$ and $\frac{n}{8\pi}v\nu_{exc}^{\tilde{\mathbb{H}}_n}([1,q])$ comes from the normalisation factor $\frac{8\pi}{n}$ in the definition of $\nu_{exc}^{\tilde{\mathbb{H}}_n}([-a,0])$ ((5.1.3)) and $\nu_{exc}^{\tilde{\mathbb{H}}_n}([1,q])$ ((5.1.4)). □

PROPOSITION 5.2.4. *Let $u,v>0$, $q>1$ and $n\geqslant 1$.*

$$p^{\tilde{\mathbb{H}}_n}_{\frac{1}{2},u,v}(q) = 1-e^{-2C_n^{eq}(q)\frac{8\pi\sqrt{uv}}{n}} \tag{5.2.7}$$

$$\lim_{n\to+\infty} p^{\tilde{\mathbb{H}}_n}_{\frac{1}{2},u,v}(q) = 1-q^{-2\sqrt{uv}} \tag{5.2.8}$$

PROOF. Let $a>0$. Consider three independent Poisson point processes:

- a loop soup $\mathcal{L}_{\frac{1}{2}}^{\tilde{\mathbb{H}}_n}$
- a P.p.p of excursions of intensity $u\nu_{exc}^{\tilde{\mathbb{H}}_n}([-a,0])$
- a P.p.p of excursions of intensity $v\nu_{exc}^{\tilde{\mathbb{H}}_n}([1,q])$

The probability for the two P.p.p. of excursions to be connected either directly or through a cluster of $\mathcal{L}_{\frac{1}{2}}^{\tilde{\mathbb{H}}_n}$ equals, according lemma 5.2.3, the probability for $\lhd_n(a)$ and $\rhd_n(q)$ to be in the same cluster of $\mathcal{L}_{\frac{1}{2}}^{\tilde{\mathcal{G}}_{n,a}(q)}$ conditional on $\hat{\mathcal{L}}^{\lhd_n(a)}_{n,a,q,\frac{1}{2}}=\frac{8\pi}{n}u$, $\hat{\mathcal{L}}^{\rhd_n(q)}_{n,a,q,\frac{1}{2}}=\frac{8\pi}{n}v$ and $\mathcal{N}_{\frac{1}{2}}(\lhd_n(a),\rhd_n(q))=0$. According to lemma5.2.1 this probability equals

$$1-e^{-2C_{n,a}^{eq}(q)\sqrt{uv}}$$

Taking the limit as a tends to infinity we get (5.2.7). Using lemma 5.2.2 we get the limit (5.2.8). □

5.3. Computation of connexion probability on continuum half-plane

On the continuum upper half plane $\mathbb{H}$ we consider two independent Poisson point processes:

- a Brownian loop soup $\mathcal{L}_\alpha^{\mathbb{H}}$, $0 < \alpha \leqslant \frac{1}{2}$
- a P.p.p. of Brownian excursions from and to $(-\infty, 0] \times \{0\}$, $\mathcal{E}_u^{\mathbb{H}}((-\infty, 0])$, $u > 0$.

We will consider the clusters made out of loops in $\mathcal{L}_\alpha^{\mathbb{H}}$ and excursions in $\mathcal{E}_u^{\mathbb{H}}((-\infty, 0])$. Among these clusters we only take the clusters that contain at least one excursion and consider the right boundary of the rightmost cluster. This boundary is a non self-intersecting curve joining $\mathbb{R}$ to infinity. It can be formally defined as follows. Take the clusters that contain at least one excursion. The curve minus its starting point on $\mathbb{R}$ is the boundary in $\mathbb{H}$ of the closure in $\mathbb{H}$ of the set of points visited by the above clusters.

All the excursions $\mathcal{E}_u^{\mathbb{H}}((-\infty, 0])$ are located left to the curve and there are only clusters made of loops right to it. According to [**Wer03**] and [**WW13**] this boundary curve is an $SLE(\kappa, \rho)$ starting from 0, where κ is given by (5.1.2) and ρ by

$$u = \frac{(\rho + 2)(\rho + 6 - \kappa)}{4\kappa}$$

We will define

$$u_0(\alpha) := \frac{6 - \kappa(\alpha)}{2\kappa(\alpha)} \tag{5.3.1}$$

See next picture.

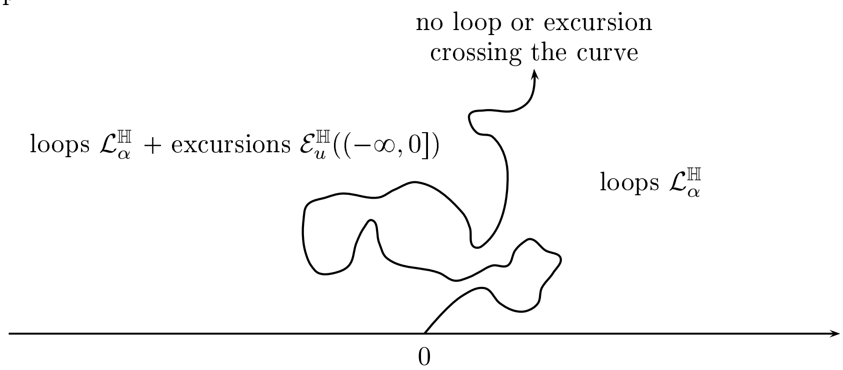

Fig.5.4: Illustration of the curve separting clusters with loops and excursions on the left from the clusters with only loops on the right.

For $u = u_0(\alpha)$, $\rho = 0$ and $SLE(\kappa, \rho)$ is a chordal SLE_κ curve starting from 0. For a description of SLE processes see [**Wer04**]. We will denote by $(\xi_t)_{t\geqslant 0}$ this curve. $\xi_0 = 0$. It does not return to $\mathbb{R}$ at positive times. There is only one conformal map g_t that sends $\mathbb{H}\backslash\xi([0, t])$ (half-plane minus the curve up to time t) onto $\mathbb{H}$ and that is normalized at infinity $z \to \infty$ as

$$g_t(z) = z + \frac{a_t}{z} + o(z^{-1})$$

The Loewner flow $(g_t)_{t\geqslant 0}$ satisfies the differential equation

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \sqrt{\kappa} W_t}$$

where $(W_t)_{t\geqslant 0}$ is a standard Brownian motion on $\mathbb{R}$.

$p^{\mathbb{H}}_{\alpha,u_0(\alpha),v}(q)$ is the probability that an excursion from $\mathcal{E}^{\mathbb{H}}_v([1,q])$ intersects an independent $SLE_{\kappa(\alpha)}$ curve. The excursions $\mathcal{E}^{\mathbb{H}}_v([1,q])$ satisfy a one-sided conformal restriction property (**[Wer05]**): if K is a compact subset of $\mathbb{C}$ that does not intersect $[1,q]\times\{0\}$ and such that $\mathbb{H}\backslash K$ is simply connected, if f is a conformal map from $\mathbb{H}\backslash K$ onto $\mathbb{H}$ such that $f(1) < f(q) \in \mathbb{R}$, then the probability that $\mathcal{E}^{\mathbb{H}}_v([1,q])$ does not intersect K equals

$$\left(\frac{f'(1)f'(q)(q-1)^2}{(f(q)-f(1))^2}\right)^v$$

Moreover conditional on this event the law of $f(\mathcal{E}^{\mathbb{H}}_v([1,q]))$ is $\mathcal{E}^{\mathbb{H}}_v([f(1),f(q)])$ up to a change of parametrization of the excursions. From this conformal restriction property immediately follows:

LEMMA 5.3.1. *Let $\kappa \in (0,4]$. Let $(\xi_t)_{t\geqslant 0}$ be an SLE_κ with the driving Brownian motion $(\sqrt{\kappa}W_t)_{t\geqslant 0}$ and the Loewner flow $(g_t)_{t\geqslant 0}$. Denote by g'_t the derivative of g_t with respect the complex variable:*

$$g'_t(z) = \frac{\partial g_t(z)}{\partial z}$$

Denote by $\bar{p}_{\kappa,v}(q)$ the probability that and independent family of excursions $\mathcal{E}^{\mathbb{H}}_v([1,q])$ does not intersect ξ. Then the conditional probability of the event that $\mathcal{E}^{\mathbb{H}}_v([1,q])$ does not intersect ξ conditional on $(\xi_s)_{0\leqslant s\leqslant t}$ (or equivalently conditional on $(W_s)_{0\leqslant s\leqslant t}$) and on not intersecting $(\xi_s)_{0\leqslant s\leqslant t}$ equals

$$\bar{p}_{\kappa,v}\left(\frac{g_t(q)-\sqrt{\kappa}W_t}{g_t(1)-\sqrt{\kappa}W_t}\right) \tag{5.3.2}$$

The conditional probability of the event that $\mathcal{E}^{\mathbb{H}}_v([1,q])$ does not intersect ξ conditional on $(\xi_s)_{0\leqslant s\leqslant t}$ is

$$\left(\frac{g'_t(1)g'_t(q)(q-1)^2}{(g_t(q)-g_t(1))^2}\right)^v \bar{p}_{\kappa,v}\left(\frac{g_t(q)-\sqrt{\kappa}W_t}{g_t(1)-\sqrt{\kappa}W_t}\right) \tag{5.3.3}$$

In particular for all $t \geqslant 0$

$$\bar{p}_{\kappa,v}(q) = \mathbb{E}\left[\left(\frac{g'_t(1)g'_t(q)(q-1)^2}{(g_t(q)-g_t(1))^2}\right)^v \bar{p}_{\kappa,v}\left(\frac{g_t(q)-\sqrt{\kappa}W_t}{g_t(1)-\sqrt{\kappa}W_t}\right)\right] \tag{5.3.4}$$

PROOF. (5.3.2) is the conditional probability that $g_t(\mathcal{E}^{\mathbb{H}}_v([1,q]))$ does not intersect the curve $(g_t(\xi_{t+s}))_{s\geqslant 0}$. To express it we used the fact that $g_t(\mathcal{E}^{\mathbb{H}}_v([1,q]))$ has same law as $\mathcal{E}^{\mathbb{H}}_v([g_t(1),g_t(q)])$ and that $(g_t(\xi_{t+s}))_{s\geqslant 0}$ is a chordal SLE_κ starting from $\sqrt{\kappa}W_t$. In (5.3.3) we multiplied the conditional probability that $\mathcal{E}^{\mathbb{H}}_v([1,q])$ does not intersect $(\xi_s)_{0\leqslant s\leqslant t}$ and the conditional probability that $g_t(\mathcal{E}^{\mathbb{H}}_v([1,q]))$ does not intersect $(g_t(\xi_{t+s}))_{s\geqslant 0}$. □

Next we derive a differential equation in q satisfied by $\bar{p}_{\kappa,v}(q)$ on $(1,+\infty)$:

LEMMA 5.3.2. *Let $\kappa \in (0,4]$, $v > 0$ and $q > 1$. Let f be a bounded, $\mathcal{C}^2$ function on $(1,+\infty)$. Then*

$$\left(\frac{g'_t(1)g'_t(q)(q-1)^2}{(g_t(q)-g_t(1))^2}\right)^v f\left(\frac{g_t(q)-\sqrt{\kappa}W_t}{g_t(1)-\sqrt{\kappa}W_t}\right)$$

is a martingale if and only if f satisfies the differential equation

$$f'' + \frac{1}{(q-1)q}\left(\left(2-\frac{4}{\kappa}\right)q - \frac{4}{\kappa}\right)f' - \frac{4v}{\kappa q^2}f = 0 \tag{5.3.5}$$

PROOF. Let

$$R_t := \frac{g_t'(1)g_t'(q)(q-1)^2}{(g_t(q)-g_t(1))^2} \qquad q_t := \frac{g_t(q)-\sqrt{\kappa}W_t}{g_t(1)-\sqrt{\kappa}W_t} \tag{5.3.6}$$

R_t has bounded variation (in t). Let

$$M_t := R_t^v f(q_t)$$

We apply the Itô's formula to $(M_t)_{t\geqslant 0}$.

$$dM_t = R_t^v\left(vf(q_t)\frac{dR_t}{R_t} + f'(q_t)dq_t + \frac{1}{2}f''(q_t)d\langle q\rangle_t\right)$$

Denote $\overline{\mathbb{H}} := \{\Im(z) \geqslant 0\}$. For $z \in \overline{\mathbb{H}}\backslash\xi([0,t])$:

$$\frac{\partial g_t'(z)}{\partial t} = \frac{\partial}{\partial z}\left(\frac{\partial g_t(z)}{\partial t}\right) = \frac{\partial}{\partial z}\left(\frac{2}{g_t(z)-\sqrt{\kappa}W_t}\right) = \frac{-2g_t'(z)}{(g_t(z)-\sqrt{\kappa}W_t)^2}$$

Thus

$$\begin{aligned}
dR_t =&\left(\frac{-2g_t'(1)g_t'(q)(q-1)^2}{(g_t(1)-\sqrt{\kappa}W_t)^2(g_t(q)-g_t(1))^2} + \frac{-2g_t'(1)g_t'(q)(q-1)^2}{(g_t(q)-\sqrt{\kappa}W_t)^2(g_t(q)-g_t(1))^2}\right.\\
&\left. + \frac{4g_t'(1)g_t'(q)(q-1)^2}{(g_t(1)-\sqrt{\kappa}W_t)(g_t(q)-g_t(1))^3} + \frac{-4g_t'(1)g_t'(q)(q-1)^2}{(g_t(q)-\sqrt{\kappa}W_t)(g_t(q)-g_t(1))^3}\right)dt\\
=&-2R_t\left(\frac{1}{(g_t(1)-\sqrt{\kappa}W_t)^2} + \frac{1}{(g_t(q)-\sqrt{\kappa}W_t)^2}\right.\\
&\left. - \frac{2}{(g_t(1)-\sqrt{\kappa}W_t)(g_t(q)-\sqrt{\kappa}W_t)}\right)dt\\
=&-2R_t\left(\frac{1}{g_t(1)-\sqrt{\kappa}W_t} - \frac{1}{g_t(q)-\sqrt{\kappa}W_t}\right)^2 dt\\
=&-2R_t\frac{(q_t-1)^2}{(g_t(q)-\sqrt{\kappa}W_t)^2}dt
\end{aligned}$$

Further

$$\begin{aligned}
dq_t =&\sqrt{\kappa}\left(\frac{-1}{g_t(1)-\sqrt{\kappa}W_t} + \frac{g_t(q)-\sqrt{\kappa}W_t}{(g_t(1)-\sqrt{\kappa}W_t)^2}\right)dW_t\\
&+\left(\frac{2}{(g_t(q)-\sqrt{\kappa}W_t)(g_t(1)-\sqrt{\kappa}W_t)} - 2\frac{g_t(q)-\sqrt{\kappa}W_t}{(g_t(1)-\sqrt{\kappa}W_t)^3}\right.\\
&\left.+\kappa\frac{g_t(q)-g_t(1)}{(g_t(1)-\sqrt{\kappa}W_t)^3}\right)dt\\
=&\frac{\sqrt{\kappa}(q_t-1)q_t}{g_t(q)-\sqrt{\kappa}W_t}dW_t + \frac{(q_t-1)q_t}{(g_t(q)-\sqrt{\kappa}W_t)^2}((\kappa-2)q_t-2)dt
\end{aligned}$$

$$d\langle q\rangle_t = \frac{\kappa(q_t-1)^2q_t^2}{(g_t(q)-\sqrt{\kappa}W_t)^2}dt$$

Finally

$$\begin{aligned}
dM_t =&R_t^v f'(q_t)\frac{\sqrt{\kappa}(q_t-1)q_t}{g_t(q)-\sqrt{\kappa}W_t}dW_t + \frac{R_t^v(q_t-1)}{(g_t(q)-\sqrt{\kappa}W_t)^2}\times\\
&\times\left(\frac{\kappa}{2}(q_t-1)q_t^2f''(q_t) + q_t((\kappa-2)q_t-2)f'(q_t) - 2v(q_t-1)f(q_t)\right)dt
\end{aligned}$$

It follows that $(M_t)_{t\geqslant 0}$ is a local martingale (hence a true one, f being bounded) if and only if

$$\frac{\kappa}{2}(q_t-1)q_t^2f''(q_t) + q_t((\kappa-2)q_t-2)f'(q_t) - 2v(q_t-1)f(q_t) \equiv 0$$

which gives the equation (5.3.5). □

(5.3.2) is the differential equation for $\bar{p}_{\kappa,v}$. However we do not know *a priori* that $\bar{p}_{\kappa,v}$ is $\mathcal{C}^2$-regular.

PROPOSITION 5.3.3. *Let $q > 1$, $v > 0$.*

$$\lim_{n\to+\infty} p^{\tilde{\mathbb{H}}_n}_{\frac{1}{2},u_0(\frac{1}{2}),v}(q) = p^{\mathbb{H}}_{\frac{1}{2},u_0(\frac{1}{2}),v}(q) = 1 - q^{-\sqrt{v}}$$

PROOF. By definition

$$p^{\mathbb{H}}_{\frac{1}{2},u_0(\frac{1}{2}),v}(q) = 1 - \bar{p}_{4,v}(q)$$

According to proposition 5.2.4

$$\lim_{n\to+\infty} p^{\tilde{\mathbb{H}}_n}_{\frac{1}{2},u_0(\frac{1}{2}),v}(q) = 1 - q^{-2\sqrt{u_0(\frac{1}{2})v}} = 1 - q^{-\sqrt{v}}$$

Let $f_v(q) := q^{-\sqrt{v}}$. With $\kappa = 4$ the ODE (5.3.5) becomes

$$f'' + \frac{1}{q}f' - \frac{v}{q^2}f = 0$$

and it is satisfied by f_v. According to the lemma 5.3.2, $(R_t^v f_0(q_t))_{t\geqslant 0}$ is a martingale (we use the notations (5.3.6) and $\kappa = 4$) for any initial value of q_0. In particular for any $t > 0$

$$f_v(q_0) = \mathbb{E}[R_t^v f_v(q_t)]$$

The same is true if we replace f_v by $\bar{p}_{4,v}$ ((5.3.4)). Thus

$$f_v(q_0) - \bar{p}_{4,v}(q_0) = \mathbb{E}[R_t^v(f_v(q_t) - \bar{p}_{4,v}(q_t))] \tag{5.3.7}$$

for any starting value of $q_0 \in (1, +\infty)$ and $t > 0$.

$\bar{p}_{4,v}$ is non-increasing on $(1, +\infty)$ with boundary limits

$$\bar{p}_{4,v}(1) = 1 \qquad \bar{p}_{4,v}(+\infty) = 0$$

Moreover $\bar{p}_{4,v}$ is continuous. Indeed, let $q \in (1, +\infty)$. A.s. there is no excursion in $\mathcal{E}_v^{\mathbb{H}}([1, q])$ with endpoint $(q, 0)$. This means that $\bar{p}_{4,v}$ is left-continuous at q. Moreover a.s. there is $\varepsilon > 0$ such that there is no excursion in $\mathcal{E}_v^{\mathbb{H}}([1, q + \varepsilon)$ with an endpoint in $[q, q + \varepsilon) \times \{0\}$ that intersects an independent SLE_4 curve. This implies that $\bar{p}_{4,v}$ is right-continuous at q. From the continuity of $\bar{p}_{4,v}$ follows that there is $\hat{q} \in (1, +\infty)$ such that

$$|f_v(\hat{q}) - \bar{p}_{4,v}(\hat{q})| = \max_{q\in(1,+\infty)} |f_v(q) - \bar{p}_{4,v}(q)|$$

Let $t > 0$ and let $\hat{q}$ be the initial value q_0 of $(q_s)_{s\geqslant 0}$. From (5.3.7) we get that

$$|f_v(\hat{q}) - \bar{p}_{4,v}(\hat{q})| \leqslant \mathbb{E}[R_t^v]|f_v(\hat{q}) - \bar{p}_{4,v}(\hat{q})|$$

But a.s. $R_t < 1$ and $\mathbb{E}[R_t^v] < 1$. This implies that

$$|f_v(\hat{q}) - \bar{p}_{4,v}(\hat{q})| = \max_{q\in(1,+\infty)} |f_v(q) - \bar{p}_{4,v}(q)| = 0$$

and that

$$\bar{p}_{4,v}(q) \equiv q^{-\sqrt{v}}$$

□

5.4. Convergence to *CLE*

In this section we prove the convergence results. Let $Q_l := (-l, l) \times (0, l)$. Let $\mathcal{L}_\alpha^{\mathbb{H}_n \cap Q_l, T}$ be the loops in $\mathcal{L}_\alpha^{\mathbb{H}_n}$ that are contained in Q_l and do at least T jumps. Let $\mathcal{L}_\alpha^{Q_l}$ be the Brownian loops in $\mathcal{L}_\alpha^{\mathbb{H}}$ that are contained in Q_l. From **[dBCL14]** follows that for $\alpha \in (0, \frac{1}{2}]$, $l > 0$ and $\theta \in (\frac{16}{9}, 2)$, $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathbb{H}_n \cap Q_l, n^\theta})$ converges in law to $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{Q_l})$.

LEMMA 5.4.1. *Let $\alpha \in (0, \frac{1}{2}]$ and $\theta \in (\frac{16}{9}, 2)$. $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathbb{H}_n, n^\theta})$ converges in law to $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathbb{H}})$.*

PROOF. Let $z_1, \dots, z_j \in \mathbb{H}$. To deduce that $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathbb{H}_n, n^\theta})[z_1, \dots, z_j]$ converges in law to $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathbb{H}})[z_1, \dots, z_j]$ from the result of **[dBCL14]** we need only to show that

$$\lim_{l \to +\infty} \liminf_{n \to +\infty} \mathbb{P}(\text{Contours of } \mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathbb{H}_n, n^\theta})[z_1, \dots, z_j] \text{ contained in } Q_l) = 1$$

Let $\varepsilon \in (0, \frac{1}{2})$. There is $l_0 > 0$ such that

$$\mathbb{P}\left(\text{Contours of } \mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathbb{H}})[z_1, \dots, z_j] \text{ contained in } Q_{l_0}\right) \geqslant 1 - \varepsilon$$

Denote

$$\partial_{\mathbb{H}} Q_l := (\{-l\} \times (0, l]) \cup (\{l\} \times (0, l]) \cup ([-l, l] \times \{l\})$$

There is $l_1 > l_0$ such that

$$\mathbb{P}(\exists \gamma \in \mathcal{L}_\alpha^{\mathbb{H}}, \gamma \cap Q_{l_0} \neq \varnothing, \gamma \cap \partial_{\mathbb{H}} Q_{l_1} \neq \varnothing) \leqslant \varepsilon$$

Then

$$\begin{aligned} &\lim_{n \to +\infty} \mathbb{P}(\text{Contours of } \mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathbb{H}_n \cap Q_{l_1}, n^\theta})[z_1, \dots, z_j] \text{ contained in } Q_{l_0}) \\ &\qquad = \mathbb{P}(\text{Contours of } \mathcal{F}_{ext}(\mathcal{L}_\alpha^{Q_{l_1}})[z_1, \dots, z_j] \text{ contained in } Q_{l_0}) \\ &\qquad \geqslant \mathbb{P}\left(\text{Contours of } \mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathbb{H}})[z_1, \dots, z_j] \text{ contained in } Q_{l_0}\right) \geqslant 1 - \varepsilon \end{aligned}$$

According the approximation of **[LF07]**

$$\begin{aligned} &\lim_{n \to +\infty} \mathbb{P}(\exists \gamma \in \mathcal{L}_\alpha^{\mathbb{H}_n, n^\theta}, \gamma \cap Q_{l_0} \neq \varnothing, \gamma \cap \partial_{\mathbb{H}} Q_{l_1} \neq \varnothing) \\ &\qquad\qquad = \mathbb{P}(\exists \gamma \in \mathcal{L}_\alpha^{\mathbb{H}}, \gamma \cap Q_{l_0} \neq \varnothing, \gamma \cap \partial_{\mathbb{H}} Q_{l_1} \neq \varnothing) \leqslant \varepsilon \end{aligned}$$

But

$$\begin{aligned} &\mathbb{P}(\text{Contours of } \mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathbb{H}_n, n^\theta})[z_1, \dots, z_j] \text{ contained in } Q_{l_0}) \geqslant \\ &\qquad \mathbb{P}(\text{Contours of } \mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathbb{H}_n \cap Q_{l_1}, n^\theta})[z_1, \dots, z_j] \text{ contained in } Q_{l_0}) \\ &\qquad - \mathbb{P}(\exists \gamma \in \mathcal{L}_\alpha^{\mathbb{H}_n, n^\theta}, \gamma \cap Q_{l_0} \neq \varnothing, \gamma \cap \partial_{\mathbb{H}} Q_{l_1} \neq \varnothing) \end{aligned}$$

Thus

$$\liminf_{n \to +\infty} \mathbb{P}(\text{Contours of } \mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathbb{H}_n, n^\theta})[z_1, \dots, z_j] \text{ contained in } Q_{l_0}) \geqslant 1 - 2\varepsilon$$

□

From now on $\theta \in (\frac{16}{9}, 2)$ will be fixed. α will belong to $(0, \frac{1}{2}]$. For $z_0 \in \mathbb{H}$, we define

$$\delta_{\alpha,n}(z_0) := \max\{d(z, \mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathbb{H}_n, n^\theta})(z_0)) | z \in \mathcal{F}_{ext}(\mathcal{L}_\alpha^{\tilde{\mathbb{H}}_n})(z_0)\}$$

By $z \in \mathcal{F}_{ext}(\mathcal{L}_\alpha^{\tilde{\mathbb{H}}_n})(z_0)$ we mean that z is a point on the contour $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\tilde{\mathbb{H}}_n})(z_0)$. The random variable $\delta_{\alpha,n}(z_0)$ is defined only when $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathbb{H}_n, n^\theta})(z_0)$ is defined, which happens with probability converging to 1.

LEMMA 5.4.2. *Assume that $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\tilde{\mathbb{H}}_n})$ does not converge in law to $\mathcal{F}_{ext}(\mathcal{L}_\alpha^{\mathbb{H}})$. Then there is $z_{\alpha,0} \in \mathbb{H}$ such that $\delta_{\alpha,n}(z_{\alpha,0})$ does not converge in law to 0.*

PROOF. If $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\tilde{\mathtt{H}}_n})$ does not converge in law to $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})$ then by definition there are $z_1, \dots, z_j \in \mathbb{H}$ such that $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\tilde{\mathtt{H}}_n})[z_1, \dots, z_j]$ does not converge in law to $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})[z_1, \dots, z_j]$. To the contrary $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathtt{H}_n, n^{\theta}})[z_1, \dots, z_j]$ does converge in law to $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})[z_1, \dots, z_j]$. Since each contour of $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathtt{H}_n, n^{\theta}})[z_1, \dots, z_j]$ is surrounded by a contour of $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\tilde{\mathtt{H}}_n})[z_1, \dots, z_j]$, one of $\delta_{\alpha,n}(z_i)$ must not converge in law to 0. □

Let $z_{\alpha,0}$ be defined by the previous lemma under the non convergence assumption. The set

$$\{z \in \mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\tilde{\mathtt{H}}_n})(z_{\alpha,0}) | d(z, \mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathtt{H}_n, n^{\theta}})(z_{\alpha,0})) = \delta_{\alpha,n}(z_{\alpha,0}) \wedge 1\}$$

is non-empty (when $\delta_{\alpha,n}(z_{\alpha,0})$ defined) because $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\tilde{\mathtt{H}}_n})(z_{\alpha,0})$ is connected and compact. Let $Z_{\alpha,n}$ be a random point in the above set, for instance the maximum for the lexicographical order.

LEMMA 5.4.3. *Assume that* $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\tilde{\mathtt{H}}_n})$ *does not converge in law to* $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})$. *Then there is a sub-sequence of indices* $n_{\alpha,0}$ *such that the joint law of*

$$(\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathtt{H}_{n_{\alpha,0}}, n_{\alpha,0}^{\theta}})(z_{\alpha,0}), Z_{\alpha, n_{\alpha,0}})$$

has a limit when $n_{\alpha,0} \to +\infty$, *the law of*

$$(\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})(z_{\alpha,0}), Z_{\alpha})$$

satisfying the property that with positive probability Z_{α} *is not contained or surrounded by* $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})(z_{\alpha,0})$.

PROOF. $\delta_{\alpha,n}(z_{\alpha,0})$ does not converge in law to 0. This means that there is $\varepsilon > 0$ and a sub-sequence of indices n' such that

$$\forall n', \mathbb{P}(d(Z_{\alpha,n'}, \mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathtt{H}_{n'}, n'^{\theta}})(z_{\alpha,0})) \geqslant \varepsilon) \geqslant \varepsilon \tag{5.4.1}$$

The sub-sequence of random variables

$$(\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathtt{H}_{n'}, n'^{\theta}})(z_{\alpha,0}), Z_{\alpha,n'})$$

is tight. Indeed the first component of the couple converges in law and the second is by definition at distance at most 1 from the first. Thus there is a sub-sequence of indices $n_{\alpha,0}$ out of n' such that there is a convergence in law. $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathtt{H}_{n_{\alpha,0}}, n_{\alpha,0}^{\theta}})(z_{\alpha,0})$ converges in law $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})(z_{\alpha,0})$. Let Z_{α} the limit in law $Z_{\alpha,n_{\alpha,0}}$. (5.4.1) implies that

$$\mathbb{P}(d(Z_{\alpha}, \mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})(z_{\alpha,0})) \geqslant \varepsilon) \geqslant \varepsilon$$

Moreover a.s. Z_{α} cannot be in the interior surrounded by $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})(z_{\alpha,0})$ because $Z_{\alpha,n}$ is non surrounded by $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathtt{H}_n, n^{\theta}})(z_{\alpha,0})$. □

From now on $(z_j)_{j\geqslant 1}$ will be a fixed everywhere dense sequence in $\mathbb{H}$.

LEMMA 5.4.4. *Assume that* $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\tilde{\mathtt{H}}_n})$ *does not converge in law to* $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})$. *Then there is a family of sub-sequences of indices* $n_{\alpha,j}$ *such that*

- $n_{\alpha,0}$ *is given by lemma 5.4.3.*
- $n_{\alpha,j+1}$ *is a sub-sequence of* $n_{\alpha,j}$.
- *The random variable*

$$(\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathtt{H}_{n_{\alpha,j}}, n_{\alpha,j}^{\theta}})[z_{\alpha,0}, z_1, \dots, z_j], Z_{\alpha,n_{\alpha,j}})$$

converges in law as $n_{\alpha,j} \to +\infty$ *and the limit defines the joint law of*

$$(\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})[z_{\alpha,0}, z_1, \dots, z_j], Z_{\alpha})$$

- *The family of joint laws* $(\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})[z_{\alpha,0}, z_1, \dots, z_j], Z_{\alpha})_{j\geqslant 1}$ *is consistent in the sense that the law on* $(\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})[z_{\alpha,0}, z_1, \dots, z_j], Z_{\alpha})$ *induced by the law of* $(\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})[z_{\alpha,0}, z_1, \dots, z_{j+1}], Z_{\alpha})$ *is the same as the one given by the convergence. In particular the law on* $(\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})(z_{\alpha,0}), Z_{\alpha})$ *is the one given by lemma 5.4.3.*
- *The family of laws of* $(\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})[z_{\alpha,0}, z_1, \dots, z_j], Z_{\alpha})_{j\geqslant 1}$ *uniquely defines a law on* $(\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}}), Z_{\alpha})$.

PROOF. The consistency of law follows from the fact that $n_{\alpha,j+1}$ is a sub-sequence of $n_{\alpha,j}$. A contour loop in $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})$ almost surely surrounds one of the z_j points. Thus the fact that a consistent family of laws on $(\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})[z_{\alpha,0}, z_1, \dots, z_j], Z_{\alpha})_{j\geqslant 1}$ uniquely defines a law on $(\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}}), Z_{\alpha})$ follows from the Kolmogorov extension theorem.

Next we explain how we extract $n_{\alpha,j+1}$ out of $n_{\alpha,j}$. By construction the sub-sequence $(\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}_{n_{\alpha,j}}, n_{\alpha,j}^{\theta}})[z_{\alpha,0}, z_1, \dots, z_j], Z_{\alpha,n_{\alpha,j}})$ converges in law as $n_{\alpha,j} \to +\infty$ and defines a joint law on $(\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})[z_{\alpha,0}, z_1, \dots, z_j], Z_{\alpha})$. Moreover we have the convergence in law of $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}_{n_{\alpha,j}}, n_{\alpha,j}^{\theta}})(z_{j+1})$ to $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})(z_{j+1})$. Thus the sub-sequence $(\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}_{n_{\alpha,j}}, n_{\alpha,j}^{\theta}})[z_{\alpha,0}, z_1, \dots, z_{j+1}], Z_{\alpha,n_{\alpha,j}})$ is tight and one can extract a subset of indices $n_{\alpha,j+1}$ such that it converges in law. The limit law is a law on $(\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})[z_{\alpha,0}, z_1, \dots, z_{j+1}], Z_{\alpha})$. □

THEOREM 5.1. *$\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\mathbb{H}_n})$ and $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\tilde{\mathbb{H}}_n})$ converge in law as $n \to +\infty$ to $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\mathbb{H}})$, that is to say to a CLE_4 on $\mathbb{H}$.*

PROOF. It is enough to prove the convergence of $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\tilde{\mathbb{H}}_n})$. Indeed we already have the convergence for $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\mathbb{H}_n, n^{\theta}})$ and each contour $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\mathbb{H}_n})(z)$ is comprised between the contour $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\mathbb{H}_n, n^{\theta}})(z)$ and the contour $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\tilde{\mathbb{H}}_n})(z)$.

Assume that $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\tilde{\mathbb{H}}_n})$ does not converge in law to $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\mathbb{H}})$. Let $z_{\frac{1}{2},0}$ be the point defined by lemma 5.4.2 and $n_{\frac{1}{2},j}$ the sub-sequences defined by lemma 5.4.4. We also consider the joint law of $(\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\mathbb{H}}), Z_{\frac{1}{2}})$ defined by 5.4.4.

For $u, v > 0$ and $q > 1$ we consider additional independent Poisson point processes of excursions $\mathcal{E}_u^{\mathbb{H}}((-\infty, 0])$ and $\mathcal{E}_v^{\mathbb{H}}([1, q])$. Let $A_{\frac{1}{2},u,v}(q)$ be the event that is satisfied if either an excursion from $\mathcal{E}_u^{\mathbb{H}}((-\infty, 0])$ and one from $\mathcal{E}_v^{\mathbb{H}}([1, q])$ intersect each other or both intersect a common contour from $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\mathbb{H}})$. By definition

$$\mathbb{P}(A_{\frac{1}{2},u,v}(q)) = p_{\frac{1}{2},u,v}^{\mathbb{H}}(q)$$

Let $A_{\frac{1}{2},u,v}^{+}(q)$ be the event that is satisfied if one of the following conditions holds:

- An excursion from $\mathcal{E}_u^{\mathbb{H}}((-\infty, 0])$ and one from $\mathcal{E}_v^{\mathbb{H}}([1, q])$ intersect each other.
- An excursion from $\mathcal{E}_u^{\mathbb{H}}((-\infty, 0])$ and one from $\mathcal{E}_v^{\mathbb{H}}([1, q])$ intersect a common contour from $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\mathbb{H}})$.
- An excursion from $\mathcal{E}_u^{\mathbb{H}}((-\infty, 0])$ intersects $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\mathbb{H}})(z_{\frac{1}{2},0})$ and an excursion from $\mathcal{E}_v^{\mathbb{H}}([1, q])$ hits or surrounds $Z_{\frac{1}{2}}$.
- An excursion from $\mathcal{E}_v^{\mathbb{H}}([1, q])$ intersects $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\mathbb{H}})(z_{\frac{1}{2},0})$ and an excursion from $\mathcal{E}_u^{\mathbb{H}}((-\infty, 0])$ hits or surrounds $Z_{\frac{1}{2}}$.

Since with positive probability $Z_{\frac{1}{2}}$ is not contained or surrounded by $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\mathbb{H}})(z_{\frac{1}{2},0})$

$$\mathbb{P}(A_{\frac{1}{2},u,v}^{+}(q)) > \mathbb{P}(A_{\frac{1}{2},u,v}(q)) = p_{\frac{1}{2},u,v}^{\mathbb{H}}(q)$$

See next picture for the illustration of $A^{+}_{\frac{1}{2},u,v}(q)\backslash A_{\frac{1}{2},u,v}(q)$.

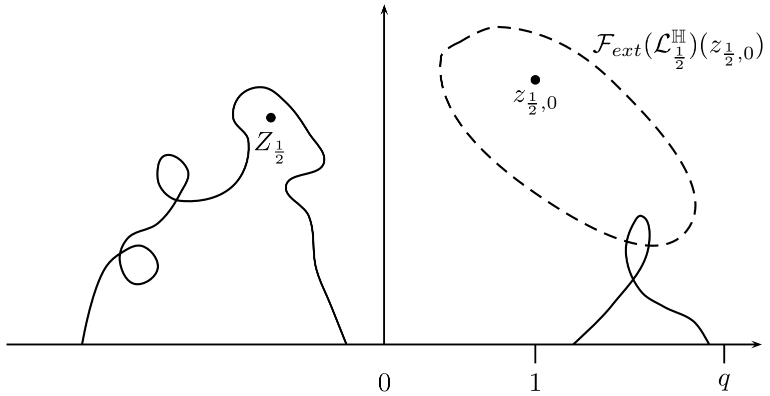

Fig.5.5: Illustration of $A^{+}_{\frac{1}{2},u,v}(q)$ where an excursion from $\mathcal{E}^{\mathbb{H}}_{u}((-\infty,0])$ surrounds $Z_{\frac{1}{2}}$ and an excursion from $\mathcal{E}^{\mathbb{H}}_{v}([1,q])$ intersects $\mathcal{F}_{ext}(\mathcal{L}^{\mathbb{H}}_{\frac{1}{2}})(z_{\frac{1}{2},0})$.

Let $j \geqslant 1$. The events $A_{\frac{1}{2},u,v}(q,j)$ respectively $A^{+}_{\frac{1}{2},u,v}(q,j)$ are defined similarly to $A_{\frac{1}{2},u,v}(q)$ respectively $A^{+}_{\frac{1}{2},u,v}(q)$ where the condition of $\mathcal{E}^{\mathbb{H}}_{u}((-\infty,0])$ and $\mathcal{E}^{\mathbb{H}}_{v}([1,q])$ intersecting a common contour of $\mathcal{F}_{ext}(\mathcal{L}^{\mathbb{H}}_{\frac{1}{2}})$ is replaced by the condition of intersecting a common contour of $\mathcal{F}_{ext}(\mathcal{L}^{\mathbb{H}}_{\frac{1}{2}})[z_{\frac{1}{2},0}, z_1, \dots, z_j]$. Then

$$\lim_{j\rightarrow+\infty}\mathbb{P}(A_{\frac{1}{2},u,v}(q,j)) = \mathbb{P}(A_{\frac{1}{2},u,v}(q)) \qquad \lim_{j\rightarrow+\infty}\mathbb{P}(A^{+}_{\frac{1}{2},u,v}(q,j)) = \mathbb{P}(A^{+}_{\frac{1}{2},u,v}(q))$$

We will denote by $A^{n}_{\frac{1}{2},u,v}(q,j)$ and $A^{n,+}_{\frac{1}{2},u,v}(q,j)$ the events defined similarly to $A_{\frac{1}{2},u,v}(q,j)$ and $A^{+}_{\frac{1}{2},u,v}(q,j)$ by doing the following replacements:

- $\mathcal{E}^{\mathbb{H}}_{u}((-\infty,0])$ replaced by $\mathcal{E}^{\widetilde{\mathbb{H}}_n}_{u}((-\infty,0])$ and $\mathcal{E}^{\mathbb{H}}_{v}([1,q])$ replaced by $\mathcal{E}^{\widetilde{\mathbb{H}}_n}_{v}([1,q])$
- $Z_{\frac{1}{2}}$ replaced by $Z_{\frac{1}{2},n}$
- $\mathcal{F}_{ext}(\mathcal{L}^{\mathbb{H}}_{\frac{1}{2}})$ replaced by $\mathcal{F}_{ext}(\mathcal{L}^{\mathbb{H}_n,n^{\theta}}_{\frac{1}{2}})$ and $\mathcal{F}_{ext}(\mathcal{L}^{\mathbb{H}}_{\frac{1}{2}})[z_{\frac{1}{2},0}, z_1, \dots, z_j]$ replaced by $\mathcal{F}_{ext}(\mathcal{L}^{\mathbb{H}_n,n^{\theta}}_{\frac{1}{2}})[z_{\frac{1}{2},0}, z_1, \dots, z_j]$

$\mathcal{F}_{ext}(\mathcal{L}^{\mathbb{H}_n,n^{\theta}}_{\frac{1}{2}})[z_{\frac{1}{2},0}, z_1, \dots, z_n]$ converges in law to $\mathcal{F}_{ext}(\mathcal{L}^{\mathbb{H}}_{\frac{1}{2}})[z_{\frac{1}{2},0}, z_1, \dots, z_j]$, the Poisson point process $\mathcal{E}^{\widetilde{\mathbb{H}}_n}_{u}((-\infty,0])$ to $\mathcal{E}^{\mathbb{H}}_{u}((-\infty,0])$ and $\mathcal{E}^{\widetilde{\mathbb{H}}_n}_{v}([1,q])$ to $\mathcal{E}^{\mathbb{H}}_{v}([1,q])$. Moreover in the limit, if an excursion intersects a contour loop in $\mathcal{F}_{ext}(\mathcal{L}^{\mathbb{H}}_{\frac{1}{2}})[z_{\frac{1}{2},0}, z_1, \dots, z_j]$ then a.s. it goes in the interior surrounded by the loop. Thus the intersection still holds for small deformations of the excursion and of the contour. Thus for all $j \geqslant 1$ we have the convergence

$$\lim_{n\rightarrow+\infty}\mathbb{P}(A^{n}_{\frac{1}{2},u,v}(q,j)) = \mathbb{P}(A_{\frac{1}{2},u,v}(q,j))$$

From lemma 5.4.4 follows that

$$\lim_{n_{\frac{1}{2},j}\rightarrow+\infty}\mathbb{P}(A^{n_{\frac{1}{2},j},+}_{\frac{1}{2},u,v}(q,j)) = \mathbb{P}(A^{+}_{\frac{1}{2},u,v}(q,j))$$

Each contour of $\mathcal{F}_{ext}(\mathcal{L}^{\mathbb{H}_n,n^{\theta}}_{\frac{1}{2}})$ is surrounded by a contour of $\mathcal{F}_{ext}(\mathcal{L}^{\widetilde{\mathbb{H}}_n}_{\frac{1}{2}})$ and $Z_{\frac{1}{2},n}$ belongs to $\mathcal{F}_{ext}(\mathcal{L}^{\widetilde{\mathbb{H}}_n}_{\frac{1}{2}})(z_{\frac{1}{2}},0)$. Thus on the event $A^{n,+}_{\frac{1}{2},u,v}(q,j)$, an excursion from $\mathcal{E}^{\widetilde{\mathbb{H}}_n}_{u}((-\infty,0])$ and one from $\mathcal{E}^{\widetilde{\mathbb{H}}_n}_{v}([1,q])$ either intersect each other or intersect a common contour from

$\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\tilde{\mathbb{H}}_n})[z_{\frac{1}{2},0}, z_1, \dots, z_j]$. Thus

$$p_{\frac{1}{2},u,v}^{\tilde{\mathbb{H}}_n}(q) \geqslant \mathbb{P}(A_{\frac{1}{2},u,v}^{n,+}(q,j))$$

Let u be equal to $u_0(\frac{1}{2})$. Then

$$p_{\frac{1}{2},u_0(\frac{1}{2}),v}^{\mathbb{H}}(q) = \lim_{n_{\frac{1}{2},j}\to+\infty} p_{\frac{1}{2},u_0(\frac{1}{2}),v}^{\tilde{\mathbb{H}}_{n_{\frac{1}{2},j}}}(q) \geqslant \lim_{n_{\frac{1}{2},j}\to+\infty} \mathbb{P}(A_{\frac{1}{2},u_0(\frac{1}{2}),v}^{n_{\frac{1}{2},j},+}(q,j)) = \mathbb{P}(A_{\frac{1}{2},u_0(\frac{1}{2}),v}^{+}(q,j))$$

Taking the limit as $j \to +\infty$ we get

$$p_{\frac{1}{2},u_0(\frac{1}{2}),v}^{\mathbb{H}}(q) \geqslant \lim_{j\to+\infty} \mathbb{P}(A_{\frac{1}{2},u_0(\frac{1}{2}),v}^{+}(q,j)) = \mathbb{P}(A_{\frac{1}{2},u_0(\frac{1}{2}),v}^{+}(q)) > \mathbb{P}(A_{\frac{1}{2},u_0(\frac{1}{2}),v}(q)) = p_{\frac{1}{2},u_0(\frac{1}{2}),v}^{\mathbb{H}}(q)$$

which is a contradiction. It follows that $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\tilde{\mathbb{H}}_n})$ converges in law to $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\mathbb{H}})$. □

THEOREM 5.2. *Let $\alpha \in (0, \frac{1}{2})$. $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}_n})$ and $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\tilde{\mathbb{H}}_n})$ converge in law as $n \to +\infty$ to $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})$, that is to say to a $CLE_{\kappa(\alpha)}$ on $\mathbb{H}$.*

PROOF. As for theorem 5.1 it is enough to prove that $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\tilde{\mathbb{H}}_n})$ converges in law to $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})$. Let's assume that this is not the case. Let $z_{\alpha,0}$ be the point and $n_{\alpha,0}$ the subsequence defined by lemma 5.4.2. We also consider the joint law of $(\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}}), Z_\alpha)$ defined by 5.4.4. Let $\tilde{z} \in \mathbb{H}$, $\tilde{z} \neq z_{\alpha,0}$.

Let $\bar{\alpha} := \frac{1}{2} - \alpha$. We take $\mathcal{L}_{\bar{\alpha}}^{\mathbb{H}}$ independent from $(\mathcal{L}_{\alpha}^{\mathbb{H}}, Z_\alpha)$ and $\mathcal{L}_{\bar{\alpha}}^{\tilde{\mathbb{H}}_n}$ independent from $(\mathcal{L}_{\alpha}^{\tilde{\mathbb{H}}_n}, Z_{\alpha,n})$. We define $\mathcal{L}_{\frac{1}{2}}^{\mathbb{H}}$ and $\mathcal{L}_{\frac{1}{2}}^{\tilde{\mathbb{H}}_n}$ as unions of two independent Poisson point processes:

$$\mathcal{L}_{\frac{1}{2}}^{\mathbb{H}} = \mathcal{L}_{\alpha}^{\mathbb{H}} \cup \mathcal{L}_{\bar{\alpha}}^{\mathbb{H}} \qquad \mathcal{L}_{\frac{1}{2}}^{\tilde{\mathbb{H}}_n} = \mathcal{L}_{\alpha}^{\tilde{\mathbb{H}}_n} \cup \mathcal{L}_{\bar{\alpha}}^{\tilde{\mathbb{H}}_n}$$

Let A_α be the event defined by $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\mathbb{H}})(z_{\alpha,0}) = \mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\mathbb{H}})(\tilde{z})$. Let A_α^+ be the event which holds if one of the below conditions is satisfied:

- $\mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\mathbb{H}})(z_{\alpha,0}) = \mathcal{F}_{ext}(\mathcal{L}_{\frac{1}{2}}^{\mathbb{H}})(\tilde{z})$.
- $\mathcal{F}_{ext}(\mathcal{L}_{\bar{\alpha}}^{\mathbb{H}})(\tilde{z})$ surrounds Z_α.

Since $\mathcal{L}_{\bar{\alpha}}^{\mathbb{H}}$ is independent from $(\mathcal{L}_{\alpha}^{\mathbb{H}}, Z_\alpha)$ and with positive probability Z_α is in the exterior of $\mathcal{F}_{ext}(\mathcal{L}_{\alpha}^{\mathbb{H}})(z_{\alpha,0})$

$$\mathbb{P}(A_\alpha^+) > \mathbb{P}(A_\alpha)$$

Next is an illustration of $A_\alpha^+ \backslash A_\alpha$.

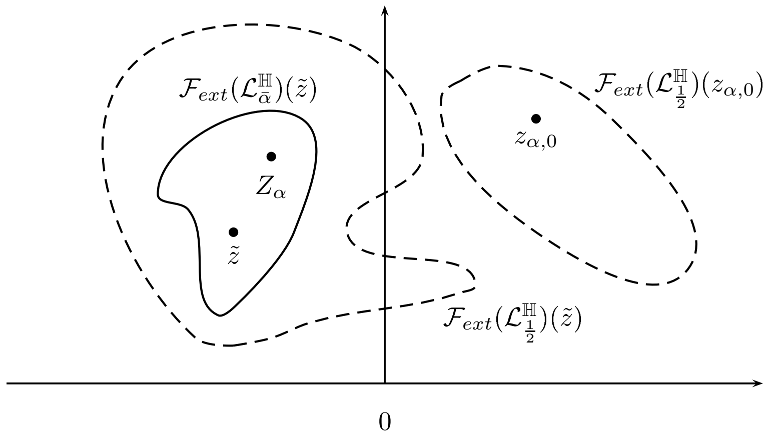

Fig.5.6: Illustration of $A^+_\alpha \backslash A_\alpha$.

Let A^n_α and $A^{n,+}_\alpha$ be the events defined similarly to A_α and A^+_α where the contours $\mathcal{F}_{ext}(\mathcal{L}^{\mathbb{H}}_{\frac{1}{2}})(z_{\alpha,0})$, $\mathcal{F}_{ext}(\mathcal{L}^{\mathbb{H}}_{\frac{1}{2}})(\tilde{z})$ and $\mathcal{F}_{ext}(\mathcal{L}^{\mathbb{H}}_{\tilde{\alpha}})(\tilde{z})$ are replaced by $\mathcal{F}_{ext}(\mathcal{L}^{\tilde{\mathtt{H}}_n}_{\frac{1}{2}})(z_{\alpha,0})$, $\mathcal{F}_{ext}(\mathcal{L}^{\tilde{\mathtt{H}}_n}_{\frac{1}{2}})(\tilde{z})$ and $\mathcal{F}_{ext}(\mathcal{L}^{\tilde{\mathtt{H}}_n}_{\tilde{\alpha}})(\tilde{z})$ respectively and Z_α is replaced by $Z_{\alpha,n}$. Since $Z_{\alpha,n}$ is on the contour $\mathcal{F}_{ext}(\mathcal{L}^{\tilde{\mathtt{H}}_n}_{\alpha})(z_{\alpha,0})$ we have the equality $A^{n,+}_\alpha = A^n_\alpha$. From theorem 5.1 follows that

$$\lim_{n \to +\infty} \mathbb{P}(A^n_\alpha) = \mathbb{P}(A_\alpha)$$

On the other hand

$$\lim_{n_{\alpha,0} \to +\infty} \mathbb{P}(A^{n_{\alpha,0},+}_\alpha) = \mathbb{P}(A^+_\alpha) > \mathbb{P}(A_\alpha)$$

which is a contradiction. It follows that $\mathcal{F}_{ext}(\mathcal{L}^{\tilde{\mathtt{H}}_n}_{\alpha})$ converges in law to $\mathcal{F}_{ext}(\mathcal{L}^{\mathbb{H}}_{\alpha})$. $\square$

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