Vortex dynamics for some non-linear transport models

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Dynamique des tourbillons pour quelques modèles de transport non linéaires.

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Je dédie cette thèse à mes parents
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Résumé en français

Les travaux effectués dans cette thèse portent sur l'étude théorique de quelques modèles d'évolution non-linéaires issus de la mécanique des fluides. Nous distinguons trois parties indépendantes.

La première partie de la thèse traite essentiellement de l'existence des V-states pour un modèle quasi-géostrophique non visqueux. Les V-states sont des poches de tourbillons animées d'un mouvement rigide et ne se déforment pas au cours de l'évolution. Notre étude est répartie sur deux chapitres où les poches présentent des structures topologiques différentes : dans le premier chapitre nous étudions le cas simplement connexe correspondant à une poche avec une seule interface. Nous validons l'existence de ces structures dans un voisinage des tourbillons de Rankine par des techniques de bifurcation. Plus précisément, nous démontrons l'émergence d'une famille dénombrable de courbes bifurquant à partir de la solution triviale à des vitesses angulaires explicitement calculées via la fonction gamma. Dans le deuxième chapitre nous abordons le cas doublement connexe où la poche admet un seul trou. La dynamique est décrite par des termes d'interaction supplémentaires et la recherche des V-states devient un peu plus délicate et gagne en complexité. En déployant les techniques de bifurcation conjuguées avec des outils d'analyse complexe et de fonctions spéciales, nous obtenons un résultat positif analogue à celui qui était démontré récemment pour les équations d'Euler incompressibles par Hmidi, de la Hoz, Mateu et Verdera [64]. Notre étude théorique a été complétée par des simulations numériques portant sur les V-states limites et un bon nombre de constatations numériques intéressantes ont été formulées ouvrant la porte à de nouvelles perspectives de recherche.

La seconde partie concerne l'étude du problème de Cauchy pour le système de Boussinesq non visqueux en 2D avec des données initiales de type Yudovich. Le problème est dans un certain sens critique à cause de quelques termes comportant la transformée de Riesz dans la formulation tourbillon-densité et qui, en général, se prête mal vis-à-vis de l'espace L^∞. Nous donnons une réponse positive pour une sous-classe comprenant les poches de tourbillon régulières et singulières. Notons que pour ce dernier cas, nous supposons en outre que la densité initiale est constante autour de la partie singulière du bord de la poche initiale.

Dans la dernière partie nous analysons le problème de la limite incompressible pour les équations d'Euler isentropiques 2D associées à des données initiales très mal préparées et pour lesquelles les tourbillons ne sont pas forcément bornés. En fait, les tourbillons appartiennent à des espaces de type BMO à poids et qui s'intercalent strictement entre L^∞ et BMO. On utilise principalement deux ingrédients : d'un côté les estimations de Strichartz pour contrôler la partie acoustique et démontrer qu'elle ne contribue pas en faible nombre de Mach. D'un autre côté, on se sert de la structure de transport compressible du tourbillon et on démontre une estimation de propagation linéaire dans l'esprit du travail récent de Bernicot et Keraani [12] mené dans le cas incompressible.

0.1 Généralités et motivations

Nous nous intéressons dans cette thèse à l'étude mathématique de quelques problèmes d'évolution issus de la mécanique des fluides dans des régimes faiblement compressibles et incompressibles. Pour le premier régime nous étudions le système d'Euler isentropique en faible nombre de Mach avec des tourbillons qui ne soient pas uniformément bornés dans L^∞. Par contre, pour les fluides incompressibles nous centrons notre discussion sur la dynamique des poches de tourbillon pour des modèles de transport comme les équations quasi-géostrophiques généralisées et le système d'Euler stratifié. Il est à noter que la dynamique des tourbillons, du moins pour les équations d'Euler incompressibles, est un sujet très ancien et toujours à la mode. Il est largement étudié dans tous ses aspects théoriques, numériques et expérimentaux. Il a débuté avec les travaux de Helmholtz [57], Kirchhoff [76], Kelvin [73] et bien d'autres et la littérature ne cesse de s'enrichir avec de nouvelles contributions. Il est alors hors de portée de faire une synthèse complète et l'on va se restreindre à quelques résultats significatifs.

Pour commencer, considérons un fluide parfait incompressible homogène et non visqueux occupant l'espace tout entier $\mathbb{R}^d$ et dont l'évolution est décrite par le système d'Euler :

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla p = 0, \\ \operatorname{div} v = 0, \\ v_{|t=0} = v_0. \end{cases} \tag{1}$$

Ici, les inconnues sont le champ des vitesses $v = (v^1, .., v^d)$ dépendant de la variable spatio-temporelle $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ et la pression p qui est un scalaire intervenant à travers la force conservative ∇p. La première équation de (1) décrit la conservation du moment tandis que la seconde traduit la conservation de la masse.

Il est bien connu depuis les travaux de Wölibner [126] que le système d'Euler (1) est localement bien posé pour des données initiales régulières. Ce résultat a été nettement amélioré au fil des années par de nombreux auteurs et pour plusieurs espaces fonctionnels. Le travail novateur dans ce cadre là est réalisé par Kato et Ponce dans [72] où ils utilisent la théorie des commutateurs pour démontrer que le système (1) est localement bien posé dans l'espace de Sobolev H^s pour $s > \frac{d}{2} + 1$. La limitation sur l'indice de régularité découle de la structure hyperbolique du système qui requiert dans le contexte des solutions classiques une vitesse lipschitzienne. Ce résultat a été depuis généralisé à d'autres espaces fonctionnels de type Hölder, Besov, Triebel,.. et pour plus de détails on renvoie aux références [20, 25, 101, 120, 131]. Cependant la question d'existence globale des solutions classiques reste jusqu'à présent un problème ouvert sauf pour quelques cas particuliers : la dimension deux et qui est résolu depuis longtemps dans [126] et la dimension trois mais avec une géométrie axisymétrique sans swirl [118]. Dans ces cas, l'existence globale découle de la structure spéciale du tourbillon qui fournit de fortes lois de conservation permettant d'établir la persistance de la régularité initiale. Signalons au passage que le tourbillon est une quantité physique très importante dans la dynamique des fluides et joue un rôle crucial dans le décryptage des mouvements de rotation locale à l'intérieur d'un fluide donné.

La mise en place des fondements de la théorie des tourbillons a démarré avec le travail pionnier de Helmholtz [57] en 1858. Dans ce document, Helmholtz souligne l'importance du tourbillon dans l'étude des fluides incompressibles en formulant quelques lois de base qui le régissent. Depuis, des axes de recherches ont été fondés autour de cet objet avec des approches différentes allant de l'aspect topologique comme par exemple pour les tubes de tourbillons en dimension trois vers un aspect plus analytique comme par exemple pour l'évolution des poches de tourbillon 2D ou aussi les points vortex.

Par la suite, nous allons nous restreindre à l'étude de la dynamique des tourbillons dans le cas bidimensionnel. La particularité de cette dimension peut-être illustrée à travers le tourbillon qui s'identifie à une fonction scalaire $\omega = \partial_1 v_2 - \partial_2 v_1$ et vérifie une simple équation de transport dite l'équation de Helmholtz,

$$\partial_t \omega + v \cdot \nabla \omega = 0. \tag{2}$$

Remarquons au passage qu'à partir de cette équation nous pouvons tirer une famille infinie de lois de conservation. Pour chaque $p \in [1, +\infty]$ nous avons

$$\forall t \geq 0, \quad \|\omega(t)\|_{L^p} = \|\omega_0\|_{L^p}.$$

Cette observation est fondamentale pour démontrer l'existence gloable des solutions classiques de type Kato ou aussi la construction des solutions faibles de type Yudovich ; un sujet que nous allons aborder juste après.

Pour compléter l'équation (2) et avoir un système fermé on se sert de la condition d'incompressibilité qui permet sous des hypothèses raisonnables de reconstituer la vitesse v à partir de son rotationnel ω. A cet effet, on introduit la fonction de courant ψ via la formule

$$\Psi(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x - y|\, \omega(y) dy$$

et il s'ensuit alors que l'on a la loi de Biot-Savart

$$v(t, x) = \nabla^\perp \Psi = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x - y)^\perp}{|x - y|^2} \omega(y) dy. \tag{3}$$

Ceci permet de réécrire le système (1) sous une forme équivalente donnée par $\big($(2),(3)$\big)$.

À partir de cette formulation tourbillon-vitesse et en utilisant les lois de conservation L^p, Yudovich a pu aller au-delà de la limitation fixée par la théorie classique des systèmes hyperboliques et construire des solutions faibles et globales en temps pour le système (1) en travaillant en dessous de la régularité lipschitzienne. Il a obtenu le résultat suivant.

Théorème 0.1 (Yudovich) *Soit $\omega_0 \in L^2 \cap L^\infty$. Alors le système* (1) *admet une unique solution globale de tourbillon $\omega \in L^\infty(\mathbb{R}, L^2 \cap L^\infty)$. De plus la vitesse v admet un flot ψ continu dans les deux variables et qui est l'unique solution de l'équation integrale :*

$$\psi(t, x) = x + \int_0^t v(\tau, \psi(\tau, x)) d\tau.$$

En outre, ψ est un homéomorphisme préservant la mesure de Lebesgue .

L'une des conséquences reamrquables de ce théorème est de pouvoir décrire le tourbillon $\omega(t)$ à partir de sa valeur initiale et du flot via la relation suivante

$$\omega(t, x) = \omega_0(\psi_t^{-1}(x))\cdot$$

On dit que le tourbillon est transporté le long des trajectoires. Cela entraîne en particulier la persistance globale des structures de poches de tourbillon. Rappelons qu'une poche est un tourbillon de la forme $\mathbf{1}_D$, avec D une partie bornée du plan. Dans ce cas, on a à chaque instant $\omega(t) = \mathbf{1}_{D_t}$ avec $D_t = \psi_t(D_0)$ et la dynamique est réduite à l'évolution du bord de la poche. Sans beaucoup rentrer dans les détails, si l'on désigne par $\gamma_t : \mathbb{T} \to \partial D_t$ la paramétrisation lagrangienne du bord alors elle satisfait une équation non linéaire et non locale de type,

$$\partial_t \gamma_t(s) = -\frac{1}{2\pi} \int_0^{2\pi} \log|\gamma_t(s) - \gamma_t(\tau)| \partial_\tau \gamma_t(\tau) d\tau. \tag{4}$$

L'évolution de la régularité de la frontière ne peut en aucun cas être résolue par le théorème de Yudovich qui ne donne pas suffisamment d'informations sur la régularité du flot dans le cas particulier des poches. D'ailleurs il est bien connu que le flot a une régularité höldérienne dégénerescente avec le temps de type $C^{\exp(-Ct)}$ et pour l'optimalité de ce résultat on renvoie à l'exemple de de Bahouri-Chemin [7].

La persistance globale de la régularité du bord a été résolue de manière profonde par Chemin [25, Chapitre 5] qui a mis au point un formalisme très robuste se prêtant bien pour des données initiales qui ne sont pas seulement des poches uniformes mais aussi des poches à densités régulières. Il a démontré en particulier que lorsque le bord ∂D_0 est de classe $C^{1+\varepsilon}$ pour $0 < \varepsilon < 1$, alors ∂D_t préserve la même régularité pour tout instant futur t sans aucune perte. Sa preuve s'appuie sur une estimation logarithmique qui relie la norme Lipschitz de la vitesse à la régularité co-normale du tourbillon et qui se transporte par le flot. Une preuve appropriée au cadre strict des poches de tourbillon a été donnée par Bertotzzi et Constantin dans [13].

Lorsque la frontière initiale contient des singularités il est bien connu, comme l'exemple d'un carré l'indique, que la solution n'est plus lipschitzienne en général. L'étude des poches singulières a été accompli par Danchin [30] qui démontre que la dynamique de la partie régulière n'est pas affectée par la partie singulière. De manière plus précise, un bord initial qui est de calsse $C^{1+\varepsilon}$ en dehors d'un ensemble singulier va garder cette régularité en dehors de l'image par le flot de l'ensemble singulier initial. De nombreuses études similaires ont été menées ensuite par de nombreux auteurs et dans différents contextes, voir [25, 31, 30, 36, 46, 58, 62, 113].

Dans cette thèse, nous avons étendu les résultats de Chemin et Danchin pour le modèle d'Euler stratifié qui couple le champ des vitesses avec la densité et qui satisfait une simple équation de transport.

En général, la dynamique du bord est très complexe et extrêmement difficile à suivre en raison des effets non linéaires de la vitesse induite. En littérature on ne connait que peu d'exemples de poches avec une dynamique explicite. Le premier exemple correspond au tourbillon de Rankine pour lequel D_0 est un disque et qui génère une poche stationnaire. Le second est dû à Kirchhoff [76] qui démontre que toute poche elliptique de demi-axes a et b tourne uniformément autour de son centre avec une vitesse angulaire $\Omega = \frac{ab}{(a+b)^2}$.

Une généralisation du tourbillon de Kirchhoff a été faite par Chaplygin dans [24]. Il a étudié le comportement des poches elliptiques sous l'effet d'un champ de vitesses extérieur avec un cisaillement pur. Il a démontré que le tourbillon conserve sa forme elliptique et la dynamique est dictée par trois scénarios dépendant de l'amplitude du champ : rotation, nutation et élongation. Ce résultat a été étendu quelques décennies après par Moore et Saffman [110], Kida [74] et Neu [98] à des champs de déformation uniformes. Plus de détails sur ce sujet peuvent être consultés dans [5].

L'exploration des poches animées d'un mouvement de rotation uniforme en dehors des exemples explicites précédents a démarré il y a maintenant quelques décennies avec les travaux numériques de Deem et Zabusky [34]. Ils ont mis en évidence l'existence de ces structures et qu'ils nomment par V-states. Ce sont des domaines simplement connexes ayant le même groupe de symétries qu'un polygone régulier à m côtés : on les appelle $m-$folds. Le cas $m = 2$ n'est autre que les ellipses de Kirchhoff.

Une démonstartion de ces observations numériques a été obtenue quelques années plus tard par Burbea dans [16]. Son approche consiste à formuler le problème, qui devient stationnaire dans le référentiel de la poche, avec les applications conformes et d'utiliser les techniques de la théorie de la bifurcation. Il a obtenu une famille dénombrable de branches unidimensionnels formées de V-states et qui bifurquent à partir du tourbillon de Rankine aux vitesses angulaires $\left\{\frac{m-1}{2m},\ m \geq 2\right\}$. Le paramètre m n'est autre que la symétrie $m-$ fold de la poche. La preuve de Burbea a été revisité récemment avec plus de détails par Hmidi, Mateu et Verdera [65]. Ils ont également montré que, proche du disque, les V-states sont convexes et de classe C^∞.

Les équations des V-states peuvent se faire de plusieurs manières exigeant des différents niveaux de régularité du bord. Nous allons en premier lieu discuter la formulation avec la fonction de courant Ψ que l'on rappelle ici,

$$\Psi(t,x) = \frac{1}{2\pi}\int_D \log|x-y|dy.$$

Dans le référentiel de la poche (qui tourne avec une vitesse angulaire constante Ω) le bord est fixe, ce qui veut dire qu'il est une ligne de niveau de la fonction de courant relative $x \mapsto \Psi(x) - \frac{1}{2}\Omega|x|^2$. Ceci se traduit tout simplement par l'équation suivante :

$$\frac{1}{2\pi}\int_D \log|x-y|dy - \frac{1}{2}\Omega|x|^2 - \mu = 0 \quad \forall x \in \partial D, \tag{5}$$

avec μ une constante. Compte tenu de cette équation, les V-states sont en fait définis à travers une forte interaction entre le potentiel newtonien et le potentiel quadratique qui doivent atteindre une forme d'équilibre sur le bord de la poche. L'existence de solutions à l'équation (5) ainsi que leur classification est loin d'être achevées et l'on dispose jusqu'ici que de résultats partiels dépendant étroitement du signe de Ω et de son amplitude. Par exemple, lorsque $\Omega = 0$, ce qui correspond à des poches stationnaires, Fraenkel [43] a démontré en utilisant la méthode des plans mobiles que le disque est la seule solution du problème des V-states. Cette méthode a été adaptée très récemment par Hmidi dans [59] pour le cas $\Omega < 0$ sous une contrainte de convexité et avec la même conclusion. Dans le cas attractif $\Omega > 0$, l'interaction entre les potentiels est plus fructueuse et fournit, d'après le résultat de Burbea, une infinité de solutions non triviales qui tournent avec une vitesse angulaire $\Omega \in]0, \frac{1}{2}[$. Par ailleurs, aucune solution non triviale avec une vitesse angulaire hors cette bande n'est connue et l'approche des plans mobiles ne s'adapte malheureusement pas pour le cas $\Omega \geq \frac{1}{2}$ à cause de la perte de monotonie dûe à la compétition active entre les deux potentiels. Cependant le point extrémal $\Omega = \frac{1}{2}$ est très particulier puisque le membre de gauche de l'équation (5) est harmonique à l'intérieur du domaine D et s'annule sur la frontière ∂D. Cela conduit à une formule explicite de type quadratique du potentiel newtonien dans D, ce qui permet de conclure que ∂D est forcément un cercle. Ce raisonnement est bien expliqué dans [59].

Un autre sujet aussi intéressant que le précédent est de mener une étude similaire pour des poches à plusieurs interfaces. L'analyse dans ce cas est beaucoup plus compliquée en raison de la forte interaction entre les interfaces. Le premier résultat dans cette direction remonte à Flierl et Polvani [44] qui ont ont révélé l'existence des ellipses de Kirchhoff généralisées tournant sans cesse autour de leurs centres. Dans un papier récent [66], les auteurs ont donné, en utilisant des outils d'analyse complexe, une caractérisation complète des poches en rotation à deux interfaces où l'une d'eux est prescrite dans la classe des ellipses. Ils ont prouvé en particulier que lorsque l'interface intérieure est une ellipse les tourbillons de Flierl et Polvani sont en fait les seules V-states.

L'étude des V-states doublement connexes (poches connexes avec un seul trou) a été menée tout récemment dans [64] en suivant l'approche de Burbea mais l'analyse dans ce cas est plus complexe à cause des termes d'interaction entre les interfaces. Dans ce papier les auteurs ont démontré que pour $b \in (0,1)$ et m un nombre entier satisfaisant l'inégalité

$$1 + b^m - \frac{1-b^2}{2}m < 0$$

alors il existe deux courbes formées de m-folds doublement connexes bifurquant de l'anneau

$\{z; b < |z| < 1\}$ en deux valeurs $\Omega_m^\pm$ données explicitement par la formule

$$\Omega_m^\pm = \frac{1-b^2}{4} \pm \frac{1}{2m}\sqrt{\Big[\frac{m}{2}(1-b^2)-1\Big]^2 - b^{2m}}.$$

Une bonne partie de ma thèse s'articule autour de l'existence des V-states pour un autre modèle de transport qui généralise l'équation du tourbillon en 2D appelé le modèle quasi-géostrophique généralisé. L'objectif de la prochaine section est de présenter mes résulats dans ce sujet.

0.2 Présentation des résultats

0.2.1 V-states simplement connexes pour le modèle QG généralisé

Dans ce chapitre nous proposons d'étudier l'existence des poches de tourbillon en rotation uniforme pour le modèle quasi-géostrophique généralisé non visqueux. Il décrit l'évolution d'une température potentielle θ via l'équation de transport suivante :

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ u = -\nabla^\perp(-\Delta)^{-1+\frac{\alpha}{2}}\theta, \\ \theta_{|t=0} = \theta_0. \end{cases} \tag{6}$$

où α est un paramètre dans $]0,2[$ et l'opérateur $(-\Delta)^{-1+\frac{\alpha}{2}}$ est de type convolution défini par

$$(-\Delta)^{-1+\frac{\alpha}{2}}\theta(x) = \frac{C_\alpha}{2\pi}\int_{\mathbb{R}^2}\frac{\theta(y)}{|x-y|^\alpha}dy, \tag{7}$$

avec $C_\alpha \triangleq \dfrac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(\frac{2-\alpha}{2})}$.

Ce modèle a été proposé dans [28] comme une interpolation entre l'équation d'Euler, qui coïncide avec le cas $\alpha = 0$, et le modèle SQG correspondant à $\alpha = 1$. Ce dernier modèle apparaît dans la description asymptotique des fluides startifiés en faible nombre de Froude et de Rossby, pour plus détails sur sa dérivation voir par exemple [102, 56, 71] .

L'existence locale en temps des solutions régulières du système (6) est faite dans la littérature de manière classique dans des espaces fonctionnels variés en utilisant la théorie des commutateurs. Cependant, l'existence globale pour $\alpha > 0$ est une question epineuse et sans réponse jusqu'à maintenant. Contrairement aux équations d'Euler, l'existence des solutions faibles à la Yudovich pour $\alpha > 0$, même pour une courte durée, n'est pas encore résolue sauf dans le cas des poches de tourbillon. En effet, lorsque $\theta_0(x) = \mathbf{1}_{D_0}$, où D_0 est un domaine borné régulier du plan, alors le système (6) admet une unique solution locale de la forme $\theta(t) = \chi_{D_t}$. Dans ce cas la dynamique du bord ∂D_t peut-être décrite par l'équation suivante :

$$\partial_t\gamma(t,\sigma) = \frac{C_\alpha}{2\pi}\int_0^{2\pi}\frac{\partial_s\gamma(t,s)}{|\gamma(t,\sigma)-\gamma(t,s)|^\alpha}ds. \tag{8}$$

où γ_t est une paramétrisation régulière du bord ∂D_t ; voir [47, 107] pour plus de détails sur l'existence de telles solutions. La persistence globale de la régularité du bord n'est connue que dans le cas $\alpha = 0$, d'après le résultat de Chemin. Par contre pour $\alpha > 0$ des simulations numériques montrent la formation, en temps fini, de singularités sur le bord, voir par exemple [28].

La principale contribution de ce chapitre consiste à démontrer l'existence des V-States pour le modèle (6) lorsque $\alpha \in]0,1[$. Notre résultat est formulé comme suit :

Théorème 0.2 [54] *Soient* $\alpha \in]0,1[$ *et* $m \in \mathbb{N}^\star \backslash \{1\}$. *Alors, il existe une courbe* V_m *formée de V-states de l'équation* (6) *avec la symétrie* $m-$*fold et qui bifurque de la solution triviale* $\theta_0 = \mathbf{1}_\mathbb{D}$ *à la vitesse angulaire*

$$\Omega_m^\alpha \triangleq \frac{\Gamma(1-\alpha)}{2^{1-\alpha}\Gamma^2(1-\frac{\alpha}{2})}\left(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})} - \frac{\Gamma(m+\frac{\alpha}{2})}{\Gamma(m+1-\frac{\alpha}{2})}\right), \tag{9}$$

où Γ *désigne la fonction gamma.*

Il convient de rappeler que l'existence de solutions globales non stationnaires pour (6) n'est pas connue pour $\alpha > 0$. Notre théorème permet de construire une famille très large de solutions globales non triviales avec une dynamique périodique. Ceci se résume comme suit :

Corollaire 0.1 *L'équation* (6) *admet une famille de solutions globales qui sont périodiques en temps.*

La preuve du théorème 0.2 se fait dans l'esprit de l'approche de Burbea mise au point pour les équations d'Euler incompressibles [16, 65]. Il s'agit de formuler soigneuesmet le problème des V-states dans un language exploitable analytiquement utilisant la structure complexe du plan et d'appliquer la théorie de la bifurcation. Comme le bord est transporté par le flot alors la composante normale de la vitesse de la particule et du point matériel occupant le même point du bord sont égales. Ainsi, l'équation obtenue peut se mettre sous la forme complexe,

$$\Omega\,\mathrm{Re}\big\{z\,\overline{z'}\big\} = \mathrm{Im}\big\{u_0(z)\,\overline{z'}\big\}, \quad \forall z \in \partial D_0. \tag{10}$$

où z' est un vecteur tangent au bord ∂D_0 au point z et la vitesse u_0 est donnée par la formule,

$$u_0(z) = \frac{C_\alpha}{2\pi}\int_{\partial D_0} \frac{1}{|z-\xi|^\alpha} d\xi. \tag{11}$$

avec $d\xi$ est l'intégration complexe sur le contour ∂D_0 orientée positivement. On choisit ensuite la pramétrisation conforme $\phi : \mathbb{D}^c \to D_0^c$ qui s'étend par exemple de manière injective et continue jusqu'au bord lorsque ce dernier est une courbe de Jordan. En procédant ainsi, l'équation (10) peut être transformée en une équation non linéaire sur le cercle unité :

$$F_\alpha\big(\Omega,\phi\big)(w) \triangleq \mathrm{Im}\bigg\{\bigg(\Omega\phi(w) - \frac{C_\alpha}{2i\pi}\int_\mathbb{T} \frac{\phi'(\tau)}{|\phi(w)-\phi(\tau)|^\alpha}d\tau\bigg)\overline{w}\,\overline{\phi'(w)}\bigg\} = 0, \quad \forall w \in \mathbb{T}, \tag{12}$$

Il n'est pas compliqué de vérifier que pour tout $\Omega \in \mathbb{R}$ nous avons

$$F(\Omega, \mathrm{Id})(w) = 0, \quad \forall w \in \mathbb{T}. \tag{13}$$

Cela signifie que le disque unité tourne à toute vitesse, ce qui est en concordance avec sa structure radiale et le fait que c'est une solution stationnaire évidente. Rappelons que les ellipses sont des V-states pour les équations d'Euler mais on peut vérifier à partir de (12) qu'elles ne le sont plus pour $\alpha > 0$. En effet, comme la la pramétrisation conforme de l'ellipse est explicite :

$$\phi : w \in \mathbb{T} \mapsto w + Q\overline{w}, \quad \text{with} \quad Q = \frac{a-b}{a+b} \in (0,1)$$

où a et b sont les demi-axes de l'ellipse, alors moyennant des calculs simples on démontre qu'il n'y a aucune valeur de Ω pour laquelle $F(\Omega,\phi)(w) = 0, \forall w \in \mathbb{T}$. Ce résultat était récemment démontré dans [17] et nous donnons dans ce chapitre une autre preuve plus fléxible. Maintenant, à partir de l'observation (13), il est légitime d'essayer de trouver des solutions non triviales en utilisant la théorie de la bifurcation. Dans le cadre de cette théorie il est nécessaire de comprendre la structure de l'opérateur linéarisé autour de la solution triviale Id et d'identifier les valeurs de Ω pour lesquelles cet opérateur n'est pas inversible. Plus précisément, nous devons déterminer l'ensemble des Ω où l'opérateur linéarisé appartient à la classe de Fredholm d'indice zéro et possède un noyau simple.

Les espaces fonctionnels dans lesquels nous avons validé la bifurcation sont de type Hölder et seront précisés plus tard dans le chapitre 1. Notons que le calcul du linéarisé est beaucoup plus coûteux que le cas eulerien, qui est assez spécifique vu que le noyau qui apparaît dans l'équation analogue à (12) est algébrique. Après un long calcul, on obtient l'expression suivante : pour $h(w) = \sum_{n\in\mathbb{N}^\star} b_n \overline{w}^n$,

$$\partial_\phi F_\alpha(\Omega, \mathrm{Id})h(w) = \frac{i}{2}\sum_{n\geq 2} n\left(\Omega - \Omega_n^\alpha\right)b_{n-1}\big(w^n - \overline{w}^n\big).$$

où Ω_n^α est donné par (9). Il en découle que le linéarisé agit comme un multiplicateur de Fourier dans l'espace des phases. Plus précisément on démontre qu'il agit comme un opérateur différentiel d'ordre 1 car $\sup_n \Omega_n^\alpha < \infty$. De plus, toutes les hypothèses du théorème de Crandall-Rabinowitz sont satisfaites y compris la condition de transversalité. Il est bon de noter que les espaces fonctionnels utilisés atteignent leur limitation dans le cas $\alpha = 1$ et malheureusement nous n'avons pas pu démontrer la bifurcation à cause de la croissance logarithmique du spectre non linéaire (Ω_n^1). Nous avons calculé le linéarisé qui prend cette forme, après avoir modifié l'équation fonctionnelle,

$$\partial_\phi F_1(\Omega, \mathrm{Id})h(w) = \frac{1}{2}b_0\Omega\, i\big(w - \overline{w}\big) + \frac{i}{2}\sum_{n\geq 2} n\left(\Omega - \Omega_n^1\right)b_{n-1}\Big(w^n - \overline{w}^n\Big),$$

avec

$$\Omega_n^1 = \frac{2}{\pi}\sum_{k=1}^{n-1}\frac{1}{2k+1}. \tag{14}$$

Trouver un espace fonctionnel décryptant cette perte logarithmique et se prêtant bien vis-à-vis des conditions de Crandall-Rabiowitz est résolu tout récemment par Córdoba, Castro et Gómez-Serrano dans [18] où ils démontrent l'existence des V-states pour $\alpha \in [1, 2[$. Ils ont également démontrer que, proche du disque, le bord des V-states est de calsse C^∞.

Dans [55], nous avons complété l'étude théorique avec des simulations numériques décrivant la structure des V-states. Nous avons comparé ces structures limites à celles pour les équations d'Euler et là nous avons mis en exergue beaucoup de différences et encore une fois les équations d'Euler retrouve sa forme singulière. Par exemple, on sait que les 2−folds pour les équations d'Euler sont simplement des ellipses et lorsque on suit cette branche on finit avec un segment. Par contre pour le modèle (6), la branche des 2−folds se termine avec une sorte de deux yeux attachés et ceci étant pour tout $\alpha \in]0, 1[$, voir le schéma ci-après.

0.2.2 Existence des V-states doublement connexes pour les équations QG generalisées

Dans ce chapitre nous continuons notre exploration des V-states pour les équations quasi-géostrophiques mais avec une topologie différente : poches à un seul trou dites aussi doublement connexes. Nous avons étendu les résultats obtenus dans [64] pour les équations d'Euler au modèle

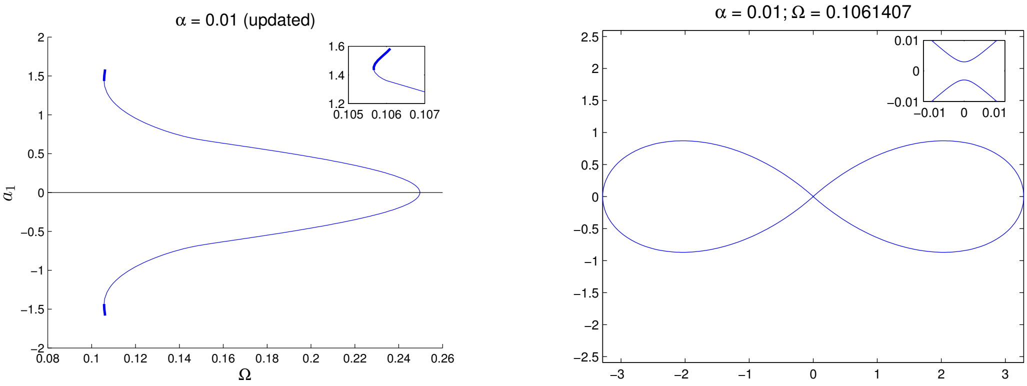

FIGURE 1 – Gauche : la courbe de bifurcation, pour $m = 2$, et $\alpha = 0,01$. Droite : 2-folds limite.

(6) avec $\alpha \in]0,1[$. On part d'une poche annulaire telle que le domaine D est donné par l'anneau $\mathcal{A}_b := \{z \in \mathbb{C}, b < |z| < 1\}$ avec $b \in]0,1[$. C'est une solution stationnaire de l'équation (6) et l'on souhaite chercher des V-states en déformant continûment ces anneaux. Ceci est fait avec les techniques de bifurcation qui repose sur l'étude du linéarisé qui est assez complexe comme nous allons le voir. Pour la clareté de l'énoncé de notre résultat nous aurons besoin d'introduire quelques notations : on pose

$$\Lambda_m(b) \triangleq \frac{1}{b}\int_0^{+\infty} J_m(bt)J_m(t)\frac{dt}{t^{1-\alpha}},$$

et

$$\Theta_m \triangleq \Lambda_1(1) - \Lambda_m(1),$$

où J_m désigne la fonction de Bessel de première espèce. Notre résultat est le suivant.

Théorème 0.3 *Soient $\alpha \in [0,1[$ et $b \in (0,1)$. Alors, il existe $N \in \mathbb{N}$ tel que pour tout $m > N$; il existe deux courbes formées de V-states doublement connexes qui bifurquent de l'anneau $\mathcal{A}_b$ aux vitesses angulaires*

$$\Omega_m^{\alpha,\pm} \triangleq \frac{1-b^2}{2}\Lambda_1(b) + \frac{1}{2}\big(1-b^{-\alpha}\big)\Theta_m \pm \sqrt{\Delta_m(\alpha,b)},$$

avec

$$\Delta_m(\alpha,b) \triangleq \Big[(b^{-\alpha}+1)\Theta_m - (1+\Lambda_1(b))\Big]^2 - 4b^2\Lambda_m^2(b). \tag{15}$$

Avant de donner les grandes lignes de la démonstration, nous allons formuler quelques remarques.

Remarque 0.1 *1. Pour tous $\alpha, b \in (0,1)$, on a*

$$\lim_{m\to\infty} \Omega_m^{\alpha,-} = -b^{-\alpha}\Lambda_1(1) + \Lambda_1(b) < 0.$$

Par conséquent, il existe des V-states tournant dans le sens des aiguilles d'une montre. Ce phénomène est complètement nouveau comparé à ce que l'on connait déjà dans les autres cas : simplement connexe ou le cas eulerien. Signalons que dans le cas eulerien le spectre non linéaire est logé dans la partie positive de l'axe réel avec zéro comme un point d'accumulation. Dès lors le paramètre α peut-être vu comme un paramètre de perturbation et il permet dans notre contexte de déplacer le spectre vers la partie négative. Des V-states non triviales et non stationnaires sont également observées numériquement.

2. *Le spectre non linéaire dépend continûmenet avec les paramètres α et b. En effet, quand α tend vers 0 alors on retrouve le spectre eulerien. Par contre quand b tend vers zéro, on retrouve le spectre du cas simplement connexe.*

La preuve de ce théorème suit le schéma général développé dans les papiers antérieurs [16, 54, 64, 65]. Le domaine du V-state recherché est de la forme $D = D_1 \backslash D_2$ avec D_1 et D_2 sont deux domaines simplement connexes emboités l'un dans l'autre $D_2 \subset D_1$. Soient $\phi_j : \mathbb{D}^c \to D_j^c$ les applications conformes associées. Alors en partant de l'équation complexe (10) nous obtenons un système de deux équations couplées non linéaires qui relient les paramétrisations conformes : pour tout $j \in \{1, 2\}$

$$F_j(\Omega, \phi_1, \phi_2)(w) \triangleq \ \operatorname{Im}\left\{\Big(\Omega\, \phi_j(w) + S(\phi_2, \phi_j)(w) - S(\phi_1, \phi_j)(w)\Big)\overline{w}\,\overline{\phi_j'(w)}\right\} = 0, \quad \forall w \in \mathbb{T},$$

avec $S(\phi_i, \phi_j)(w) = C_\alpha \displaystyle\fint_{\mathbb{T}} \frac{\phi_i'(\tau)}{|\phi_j(w) - \phi_i(\tau)|^\alpha} d\tau.$

Maintenant, pour pouvoir appliquer la théorie de la bifurcation nous devrions comprendre la structure de l'opérateur linéarisé autour de la solution triviale $(\mathrm{Id}, b\,\mathrm{Id})$, qui correspond à l'anneau de rayons 1 et $b \in (0, 1)$, et d'identifier les valeurs de Ω pour lesquelles cet opérateur admet un noyau unidimensionnel. Les calculs du linéarisé en termes des coefficients de Fourier sont assez longs et pénibles et reposent sur quelques structures algébriques des fonctions hypergéométriques. Alors nous obtenons que le linéarisé admet une structure de multplictateur de Fourier matriciel. Plus précisément, pour

$$h_1(w) = \sum_{n \geq 1} \frac{a_n}{w^n}, \quad h_2(w) = \sum_{n \geq 1} \frac{c_n}{w^n},$$

on a

$$DF(\Omega, \mathrm{Id}, b\,\mathrm{Id})\big(h_1, h_2\big)(w) = \frac{i}{2} \sum_{n \geq 2} n\, M_n^\alpha \begin{pmatrix} a_{n-1} \\ c_{n-1} \end{pmatrix} \Big(w^n - \overline{w}^n\Big),$$

où $F \triangleq (F_1, F_2)$ et la matrice M_n^α est donnée pour tout $n \geq 2$ par

$$M_n^\alpha \triangleq \begin{pmatrix} \Omega - \Theta_n + b^2 \Lambda_1(b) & -b^2 \Lambda_n(b) \\ \\ b\Lambda_n(b) & b\Omega + b^{1-\alpha}\Theta_n - b\Lambda_1(b) \end{pmatrix}.$$

Par conséquent les valeurs de Ω générant un noyau non trivial sont les solutions du polynôme du second degré,

$$\det M_n^\alpha = 0. \tag{16}$$

Cela est possible lorsque le discriminant $\Delta_n(\alpha, b)$ donné par (15) est positive. L'étape suivante consiste à identifier la dimension du noyau. Ceci est alors directement lié à un probème de dénombrement ; il s'agit de calculer le cardinal de l'ensemble discret suivant :

$$\big\{n \geq 2, \det M_n^\alpha = 0\big\}.$$

Dans le cas simplement connexe ou le cas eulerien analysés dans [54, 64] cet ensemble est formé d'un seul élément (et donc le noyau et de dimension 1). Ce constat découle de la stricte monotonie du spectre par rapport à la fréquence. Dans le cas traité ici, la monotonie est une question assez tordue vu la structure implicite et non linéaire par rapport aux paramètres n, α et b des coefficients qui apparaissent dans l'expression des $\Omega_n^{\alpha,\pm}$.

Pour parachever l'étude spectrale et appliquer le théorème de Crandall-Rabinowitz il reste à vérifier que l'image est de co-dimension 1 et que la condition de transversalité est satisfaite. Nous démontrons que cette dernière condition est satisfaite seulement lorsque le spectre n'est pas double, ce qui veut dire que le discriminant $\Delta_n(\alpha, b) > 0$.

Concernant le cas limite $\alpha = 1$ qui correspond à l'équation SQG, notre analyse échoue pour les mêmes raisons que le cas simplement connexe : il y a une perte logarithmique dans les multiplicateurs en Fourier et l'usage des espaces tarditionnels de type Hölder n'est pas trop bien approprié. Il est alors naturel d'essayer les espaces de Hilbert introduits dans le papier [18] et nous pensons que ce sont de bons candidats pour établir la bifurcation. Remarquons qu'en faisant un passage à la limite dans (16) quand α tend vers 1, on s'attend à ce que l'opérateur linéarisé prenne la forme suivante :

$$DF(\Omega, \mathrm{Id}, b\,\mathrm{Id})\big(h_1, h_2\big)(w) = \frac{i}{2}\sum_{n\geq 2} n\, M_n^1 \begin{pmatrix} a_{n-1} \\ c_{n-1} \end{pmatrix} \Big(w^n - \overline{w}^n\Big),$$

où la matrice M_n^1 est donnée, pour tout $n \geq 2$, par

$$M_n^1 \triangleq \begin{pmatrix} \Omega - \Theta_n^1 + b^2\Lambda_1^1(b) & -b^2\Lambda_n^1(b) \\ \\ b\Lambda_n^1(b) & b\Omega + \Theta_n^1 - b\Lambda_1^1(b) \end{pmatrix},$$

avec

$$\Theta_n^1 = \frac{2}{\pi}\sum_{k=1}^{n-1}\frac{1}{2k+1} \quad \text{and} \quad \Lambda_n^1(b) = \frac{1}{b}\int_0^{+\infty} J_n(bt)J_n(t)dt.$$

Nous avons également fait des simulations numériques et là encore une fois on découvre des comportements différents de ceux observés pour les équations d'Euler sur la structure des V-states limites ainsi que sur la durée de vie des courbes de bifurcation. Par exemple il y a des cas où il n' y a plus de gap spectral, ce qui entraîne en particulier qu'il y a des V-states avec des symétries différentes mais qui tournent à la même vitesse angulaire.

0.2.3 Fluides stratifiés non visqueux

Ce chapitre est consacré aux équations d'Euler starifiées dites aussi de Boussinesq non visqueux,

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla p = \rho\vec{e}_2, \quad t\geq 0, x\in\mathbb{R}^2 \\ \partial_t\rho + v\cdot\nabla\rho = 0, \\ \operatorname{div} v = 0, \\ v_{|t=0} = v_0, \quad \rho_{|t=0} = \rho_0. \end{cases} \tag{17}$$

Ici, $v = (v_1, v_2)$ désigne le champ de vitesses, p la pression et ρ joue le rôle de la densité ou la température. Le couplage s'effectue à travers la force gravitationnelle $\rho\vec{e}_2$ où $\vec{e}_2 = (0, 1)$.

Le système de Boussinesq sert à modéliser les effets de stratification dans certains phénomènes océanographiques et géophysiques. La justification rigoureuse dans le cas visqueux comme une limite incompressible des équations complètes de Navier-Stokes-Fourier a été discutée récemment dans[43].

Le système (17) apparaît comme une extension des équations d'Euler incompressibles. En effet si l'on prend la densité ρ identiquement constante, la force gravitationnelle est conservative et on peut l'insérer dans la pression et donc le système (17) coïncide avec le système d'Euler incompressible classique (1). Alors il est tout-à-fait légitime de vérifier si les résultats connus pour les équations d'Euler, discutés dans la première section, restent valable pour le système de Boussinesq.

La littérature traitant de ce système et ses variantes visqueuses est très abondante et beaucoup de recherches ont été menées dans les dernières décennies. A titre d'exemple on cite [1, 32, 33, 63, 67, 69, 70, 81, 96, 127]. Dans le cas complètement non visqueux qui nous intéresse ici on va se limiter à quelques résultats de base. Il bien connu que le système (17) peut être considéré comme un un système hyperbolique et donc la théorie des commutateurs de Kato et Ponce développée dans [72] peut être facilement appliquée. Cela a été fait par Chae et Nam dans [21] qui ont prouvé que le système (17) est localement bien posé lorsque les données initiales (v_0, ρ_0) appartiennent aux espaces de Sobolev sous-critiques ; H^s avec $s > 2$. Un résultat similaire a été établi plus tard par les mêmes auteurs [22] pour des données initiales se trouvant dans les espaces de type Hölder C^r avec $r > 1$. Le cas des espaces de Besov critiques $B^{2/p+1}_{p,1}$, avec $p \in]1, +\infty[$ a été traité dans [84]. Toutefois, le question d'existence globale des solutions régulières demeure jusqu'ici sans issue.

Pour des données moins régulières de type Yudovich, le problème d'existence locale des solutions faibles semble très difficile en raison du couplage fort dans la formulation tourbillon-densité. Ce couplage génère des opérateurs homogènes de type Riesz et qui ne sont pas compatibles avec l'espace L^∞. Pour être plus claire, rappelons que la formulation tourbillon-densité est donnée par,

$$\begin{cases} \partial_t \omega + v \cdot \nabla \omega = \partial_1 \rho, \\ \partial_t \rho + v \cdot \nabla \rho = 0, \end{cases} \tag{18}$$

En appliquant le principe du maximum à l'équation de la vorticité ci-dessus on obtient

$$\|\omega(t)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} + \int_0^t \|\nabla \rho(\tau)\|_{L^\infty} d\tau.$$

L'estimation de la dernière intégrale exige que la régularité des données initiales soient plus forte que ce qui est permis par la classe de Yudovich. En effet, la dérivé partielle $\partial_j \rho$ vérifie l'équation suivante :

$$(\partial_t + v \cdot \nabla)\partial_j \rho = \partial_j v \cdot \nabla \rho = \partial_j (\Delta)^{-1} \nabla^\perp \omega \cdot \nabla \rho. \tag{19}$$

Ainsi l'estimation de la norme de Lipschitz de la densité $\|\nabla \rho(t)\|_{L^\infty}$ nécessite une estimation de la transformée de Riesz de ω dans L^∞ et du coup on sort du cadre générique des solutions de type Yudovich.

Notre principale contribution dans le troisième chapitre est d'apporter une réponse positive à la théorie locale pour le système (17) pour une classe spéciale des données de type Yudovich. Nous démontrons dans la première partie de ce chapitre le résultat pour des données de type poches avec un bord régulier. Notre premier résultat est le suivant :

Théorème 0.4 *Soient $\varepsilon \in]0, 1[$, $a \in]1, 2[$ et Ω_0 un domaine borné du plan $\mathbb{R}^2$ dont le bord est de classe $C^{1+\varepsilon}$. Considérons v_0 un champ de vecteurs de divergence nulle et de tourbillon $\omega_0 = 1_{\Omega_0}$. Soit ρ_0 une fonction scalaire dans $L^2 \cap C^{1+\varepsilon}$ et etlle que $\nabla \rho_0 \in L^a$. Alors, il existe $T > 0$ tel que le système de Boussinesq* (17) *admet une unique solution locale $(v, \rho) \in L^\infty\big([0, T], W^{1,\infty}\big) \times L^\infty\big([0, T], W^{1,\infty}\big)$. De plus, pour tout $t \in [0, T]$ le bord du domaine transporté $\Omega_t = \psi(t, \Omega_0)$ est de classe $C^{1+\varepsilon}$.*

Remarque 0.2 *Le résultat du théorème 0.4 sera étendu à des structures de tourbillon plus générales. Nous obtenons également une borne inférieure pour la durée de vie, qui est infini pour* *des densités constantes, ce qui correspond au résultat global pour les équations d'Euler. Plus précisément nous obtenons*

$$T^\star \geq \frac{1}{C_0} \log\Big(1 + C_0 \log\big(1 + C_0/\|\nabla\rho_0\|_{L^\infty}\big)\Big).$$

où $C_0 \triangleq C_0(\omega_0, \rho_0)$ dépend continûment des normes concernées.

Avant de discuter la démonstration de ce résultat, il convient de souligner que le même problème surgit dans différentes EDP comme les équations SQG longuement discutées dans la section précédente, les équations de la MHD non visqueuses ou aussi les équations d'Euler incompressibles inhomogènes. Pour plus de détails on renvoie aux papiers [60, 42].

La preuve du théorème 0.4 est basée sur le formalisme général des poches de tourbillon développé par Chemin dans [25]. Il s'appuie fortement sur une estimation logarithmique statique qui relie la norme Lipschitz de la vitesse à la régularité co-normale du tourbillon mesurée via une famille convenable de champs de vecteurs (X_t) dans un espace de Hölder d'indice négatif. Ces champs de vecteurs sont choisis de sorte qu'ils soient non dégénérés, tangents à la poche initiale et transportés par le flot,

$$\partial_t X + v \cdot \nabla X = X \cdot \nabla v.$$

L'avantage principal de ce choix est la commutation de ces champs avec l'opérateur de transport $\partial_t + v \cdot \nabla$ qui conduit à son tour à

$$\begin{aligned}(\partial_t + v \cdot \nabla)\partial_X \omega &= \partial_X \partial_1 \rho \\ &= \partial_1(\partial_X \rho) + [\partial_X, \partial_1]\rho.\end{aligned}$$

En utilisant le calcul paradifférentiel nous pouvons démontrer que le commutateur qui apparaît dans la dernière équation se comporte bien et donc le problème se réduit à l'estimation de $\|\partial_X \rho\|_{C^\varepsilon}$. Pour ce dernier terme, nous utilisons à nouveau la commutation entre ∂_X et l'opérateur de transport et le fait que la densité est conservée le long des trajectoires des particules. C'est ainsi que nous obtenons l'équation

$$(\partial_t + v \cdot \nabla)\partial_X \rho = 0.$$

Cette structure est très importante dans notre analyse et permet de boucler les estimations à priori sans avoir aucune perte supplémentaire.

Passons maintenant à la deuxième contribution de ce chapitre. Nous supposons toujours que $\omega_0 = \mathbf{1}_{\Omega_0}$, mais cette fois-ci le bord peut contenir un sous-ensemble singulier. Cela apporte des complications techniques supplémentaires dûes au fait que le champ de vitesse n'est pas en général lipschitzien. Notre résultat dans un cadre restreint peut être formulé comme suit :

Théorème 0.5 *Soient Ω_0 un domaine borné et ε un réel appartenant à $]0,1[$. On suppose que le bord $\partial\Omega_0$ est une courbe de classe $C^{1+\varepsilon}$ en dehors d'un ensemble fermé Σ_0. Considérons un champ de vecteurs de divergence nulle v_0 de vorticité $\omega_0 = 1_{\Omega_0}$. Soit $\rho_0 \in L^2 \cap C^{\varepsilon+1}$ avec $\nabla\rho_0 \in L^a$ pour certain $1 < a < 2$. Supposons que ρ_0 est constante sur un petit voisinage de Σ_0. Alors le système* (17) *admet une unique solution locale (ω, ρ) telle que*

$$\omega, \rho \in L^\infty\big([0,T], L^2 \cap L^\infty\big), \quad \nabla\rho \in L^\infty\big([0,T], L^a \cap L^\infty\big).$$

De plus, on a

$$\sup_{h \in (0, e^{-1}]} \frac{\|\nabla v(t)\|_{L^\infty((\Sigma_t)_h^c)}}{-\log h} \in L^\infty([0,T]),$$

où l'ensemble $(\Sigma_t)_h^c$ *est défini par,*

$$(\Sigma_t)_h^c \triangleq \{x \in \mathbb{R}^2;\, d\big(x, \Sigma(t)\big) \geq h\}, \quad \Sigma_t \triangleq \psi(t, \Sigma_0).$$

En outre, le bord de $\psi(t, \Omega_0)$ *est localement dans* $C^{1+\varepsilon}$ *en dehors de l'ensemble* Σ_t.

Nous allons dans ce qui suit donner quelques éléments de la démonstration. D'une manière similaire au cas d'un bord régulier, nous devons contrôler en premier lieu la norme lipschitzienne de la densité $\|\nabla\rho(t)\|_{L^\infty}$. On a d'abord,

$$\|\partial_j \rho(t)\|_{L^\infty} \leq \|\nabla \rho_0\|_{L^\infty} + \int_0^t \|\partial_j v \cdot \nabla \rho(t)\|_{L^\infty}.$$

Parallèlement, on s'attend à ce que les singularités, initialement situées sur le bord, soient propagées le long des trajectoires des particules. Donc l'idée cruciale, pour traiter le terme intégral dans la dernière estimation, consiste à éliminer les effets des singularités de la vitesse par certaines hypothèses spécifiques sur la densité. Comme un choix possible, nous supposons que la densité initiale est constante autour de l'ensemble de la singularité. A partir de la structure de transport, la densité reste constante autour de l'image par le flot de l'ensemble singulier. Cela permet en un sens de masquer l'ensemble singulier et d'annuler ses effets indésirables via la densité.

0.2.4 Limite incompressible pour une vorticité non bornée

Il s'agit dans ce chapitre d'étudier le système d'Euler isentropique régissant l'évolution d'un fluide non visqueux et faiblement compressible. L'état du fluide est décrit par le champ des vitesses v_ε et la vitesse du son c_ε à travers le système hyperbolique suivant

$$\begin{cases} \partial_t v_\varepsilon + v_\varepsilon \cdot \nabla v_\varepsilon + \frac{1}{\varepsilon}\nabla c_\varepsilon + \overline{\gamma} c_\varepsilon \cdot \nabla c_\varepsilon = 0, \\ \partial_t c_\varepsilon + v_\varepsilon \cdot \nabla c_\varepsilon + \frac{1}{\varepsilon}\mathrm{div}\, v_\varepsilon + \overline{\gamma} c_\varepsilon \mathrm{div}\, v_\varepsilon = 0, \\ (v_\varepsilon, c_\varepsilon)_{|t=0} = (v_{0,\varepsilon}, c_{0,\varepsilon}). \end{cases} \tag{20}$$

où $\overline{\gamma}$ est un réel strictement positif et ε un paramètre désignant le nombre de Mach qui est censé tendre vers zéro dans notre contexte.

D'un point de vu mathématique, ce modèle a été largement étudié durant les dernières décennies et l'un des problématiques abordées est la justification rigoureuse de la convergence vers le système d'Euler incompressible (1) lorsque le nombre de Mach ε tends vers zéro. Plusieurs facteurs rentrent en jeu et peuvent affecter sérieusement l'étude comme la géométrie du domaine englobant le fluide, que nous n'allons pas en tenir compte dans cette discussion, ou l'état des données initiales ; si elles sont bien préparées ou non. Dans le cas bien préparé les données initiales sont supposées être légèrement compressibles, ce qui signifie que leurs parties compressibles div $v_{0,\varepsilon}$ et acoustiques $\nabla c_{0,\varepsilon}$ sont de taille ε. Sous ces hypothèses, on peut facilement vérifier que la dérivée en temps de la solution est uniformément bornée par rapport à ε et donc la justification de la limite incompressible découle par un argument de compacité en temps ; pour une discussion détaillée on renvoie le lecteur au papiers de Klainerman et Majda [77, 78]. Toutefois, dans le cas mal préparé la famille $(v_{0,\varepsilon}, c_{0,\varepsilon})_\varepsilon$ est supposée être bornée dans des espaces de Sobolev H^s avec $s > 2$ et $(v_{0,\varepsilon})_\varepsilon$ convergent fortement dans un espace adéquat, disons dans l'espace L^2, vers un certain champ de vecteurs incompressible v_0. Dans ce cadre là, la dérivée en temps $\partial_t v_\varepsilon$ se propage avec une vitesse ε^{-1}. Ainsi l'argument de compacité ci-dessus ne semble pas opérationnel. Pour surmonter cette difficulté Ukai [117] a utilisé les effets de dispersion générés par les ondes acoustiques afin de prouver que la partie compressible de la vitesse et le terme acoustique disparaissent quand ε tend vers zéro. Des études similaires, mais dans des situations plus complexes et pour différents modèles ont été réalisées après et nous citons ici une courte liste de références [2, 6, 58, 68, 82, 83, 95]. Un aperçu général et complet qui met en lumière l'état de l'art sur la question de la limite incompressible et les différentes approches évoquées dans la littérature est discuté par Gallagher dans [45].

Notons que la régularité des données initiales peut également être une source de difficultés supplémentaires. En effet, Hmidi et Sulaiman ont établi récemment dans [68] un résultat de limite incompressible en dimension deux lorsque la régularité des données initiales est critique, c'est à dire, qu'elles sont prises dans l'espace de Besov $B^2_{2,1}$. Ils ont démontré en particulier la convergence forte dans l'espace des données initiales et donner une borne inférieure du temps d'existence dépendant du profil de la donnée initiale. Le même problème a été également résolu en dimension trois dans [58] pour des données initiales axisymétriques. Le fait que la régularité est optimale pour le système incompressible engendre plus de difficultés techniques et la structure de la vorticité est d'une grande importance et permet de surmonter ces difficultés.

Remarquons que dans quasiment toutes les contributions citées auparavant, la vitesse initiale doit être uniformément lipschitzienne par rapport au paramètre ε. Cette contrainte a été légèrement relaxée dans [41] en travaillant avec des données initiales qui sont des régularisations lentes des données de Yudovich. Ceci correspond à un cas de données très mal préparées qui peuvent exploser en ε, mais lentement, dans les espaces de Sobolev H^s avec $s > 2$ alors que la partie incompressible converge vers une solution de type Yudovich pour le système incompressible donné ici par la formulation vitesse-tourbillon,

$$\partial_t \omega + v \cdot \nabla \omega = 0, \quad \Delta v = \nabla^\perp \omega. \tag{21}$$

L'objectif principal du chapitre 4 est de justifier la limite incompressible dans un cas très mal préparé où le tourbillon n'est pas nécessairement borné. Il appartient à une variété d'espaces de type BMO à poids qui s'intercalent strictement entre L^∞ et BMO. Ceci est inspireé par le travail récent de Bernicot et Keraani [12] où ils démontrent que le système incompressible est globalement bien posé lorsque la donnée initiale appartient à l'espace LBMO ; cet espace est strictement plus grand que L^∞ et est inclus strictement dans l'espace BMO usuel. On le définit comme l'ensemble des fonctions localement intégrables f telles que

$$\|f\|_{LBMO} \triangleq \|f\|_{BMO} + \sup_{2B_2 \subset B_1} \frac{\left| \frac{1}{|B_2|} \int_{B_2} f - \frac{1}{|B_1|} \int_{B_1} f \right|}{1 + \ln\left(\frac{1 - \ln r_2}{1 - \ln r_1}\right)} < +\infty,$$

où le supremum est pris sur toutes les boules $B_1 = B(x_1, r_1)$ et $B_2 = B(x_2, r_2)$ avec $0 < r_1 < 1$. Par un souci de clarté de l'énoncé nous allons nous restreindre à l'espace LBMO et une extension sera donnée dans le chapitre 4.

Théorème 0.6 [52] *Soient $s, \alpha \in]0, 1[$ et $p \in]1, 2[$. Considérons une famille de données initiales $(v_{0,\varepsilon}, c_{0,\varepsilon})_{0<\varepsilon<1}$ telle qu'il existe une constante $C > 0$ qui ne dépend pas de ε et vérifiant*

$$\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{H^{s+2}} \leq C(\log \varepsilon^{-1})^\alpha \quad \text{and} \quad \|\omega_{0,\varepsilon}\|_{L^p \cap LBMO} \leq C.$$

Alors le système $(E.C)$ admet une unique solution $(v_\varepsilon, c_\varepsilon) \in C([0, T_\varepsilon[; H^{s+2})$ avec les propriétés suivantes :

1. *Le temps de vie T_ε de la solution satisfait la minoration :*

$$T_\varepsilon \geqslant \log\log\log \varepsilon^{-1} \triangleq \tilde{T}_\varepsilon,$$

et pout tout $t \leq \tilde{T}_\varepsilon$ on a

$$\|\omega_\varepsilon(t)\|_{LBMO \cap L^p} \quad \leq \quad C_0 e^{C_0 t}. \tag{22}$$

De plus, les parties compressibles et acoustiques des solutions convergent en moyenne vers zéro :

$$\lim_{\varepsilon\to 0}\|(\mathrm{div}\, v_\varepsilon, \nabla c_\varepsilon)\|_{L^1_{T_\varepsilon}L^\infty} = 0.$$

2. *Supposons de plus que* $\lim_{\varepsilon\to 0}\|\omega_{0,\varepsilon}-\omega_0\|_{L^p}=0$, *pour certain tourbillon* $\omega_0\in LBMO\cap L^p$ *associé à une vitesse incompressible* v_0. *Alors les tourbillons* $(\omega_\varepsilon)_\varepsilon$ *convergent fortement vers la solution faible* ω *de* (21) *associé à la donnée initiale* ω_0 *:*
Pour tout $t\in\mathbb{R}_+$ *on a*

$$\lim_{\varepsilon\to 0}\|\omega_\varepsilon(t)-\omega(t)\|_{L^q}=0,\quad \forall q\in[p,+\infty[. \tag{23}$$

De plus,

$$\|\omega(t)\|_{LBMO\cap L^p} \quad\leq\quad C_0 e^{C_0 t}. \tag{24}$$

La constante C_0 *depend seulement de la taille de la donnée initiale et ne dépend pas de* ε.

Remarque 0.3 *A partir de l'inégalité* (4.4), *il est facile de voir que le théorème* 0.6 *permet de restituer le résultat de* [*12*].

La démonstration de ce théorème est fondée sur deux ingrédients principaux : le premier et qui est le plus pertinent et a un intérêt en soi concerne la persistance de la régularité $LBMO$ pour un modèle de transport compressible régissant le tourbillon,

$$\partial_t\omega_\varepsilon + v_\varepsilon\cdot\nabla\omega_\varepsilon + \omega_\varepsilon \mathrm{div}\, v_\varepsilon = 0. \tag{25}$$

Lorsque le champ est incompressible Bernicot et Keraani ont démontré dans [12] l'estimation logarithmique suivante :

$$\|\omega(t)\|_{LBMO\cap L^p} \quad\leq\quad C\|\omega_0\|_{LBMO\cap L^p}\Big(1+\int_0^t \|v(\tau)\|_{LL}d\tau\Big), \tag{26}$$

où LL désigne la norme associée à l'espace log-Lipshitz.

Pour étendre ce résultat au modèle de transport compressible (25) nous faisons usage de la méthode developée dans [12] et qui consiste à suivre dans l'espace physique la dynamique des oscillations et notamment à comprendre leurs interactions mutuelles. Cependant la compressibilité du champ des vitesses avec lequel on transporte le tourbillon et l'émergence de la structure quadratique $\omega_\varepsilon\,\mathrm{div}\, v_\varepsilon$ contribuent avec beaucoup plus de difficultés techniques que nous devons soigneusement analyser. De manière plus précise, nous établissons le résultat suivant.

Théorème 0.7 *Soit* ω_ε *une solution régulière de l'équation* (25). *Alors pour tout* $s\in]0,1[$, *il existe une constante* $C>0$ *telle que* $\forall\, 1\leq p\leq\infty$ *et* $\forall\, t\geq 0$,

$$\begin{aligned}\|\omega_\varepsilon(t)\|_{LBMO\cap L^p} &\leq C\|\omega_{0,\varepsilon}\|_{LBMO\cap L^p}\Big(1+\int_0^t\|v_\varepsilon(\tau)\|_{LL}d\tau\Big)\\ &\times\Big[1+\|\mathrm{div}\, v_\varepsilon\|_{L^1_t C^s}\int_0^t\|v_\varepsilon(\tau)\|_{LL}d\tau\Big]e^{C\|\mathrm{div}\, v_\varepsilon\|_{L^1_t L^\infty}}.\end{aligned}$$

On observe que l'on peut dériver facilement le résultat (4.5) à partir de cette dernière estimation en prenant div $v_\varepsilon = 0$. La démonstration de ce résultat se fait en quelques étapes. D'abord, on filtre la partie de transport compressible qui a l'avantage de transformer le problème à une estimation logarithmique pour la composition dans l'espace $LBMO$. Le défaut cette composition est que le flot impliqué ne conserve pas nécessairement la mesure de Lebesgue mais la méthode développée dans [12] est suffisamment robuste pour permettre de traiter ce cas. La dernière étape consiste à revenir à l'inconnue de départ avant la filtration et là nous étions obligées d'établir des lois de produit adaptées au espace $LBMO$ et d'identifier quelques espaces de multpilicateurs de type LMO.

Le deuxième ingrédient de la démonstration du théorème 0.6 est l'utilisation des estimations de Strichartz de manière similaire à [41] et qui représentent un outil éfficace pour traiter les données initiales très mal préparées. La convergence vers le système incompressible est faite directement avec le tourbillon.

Première partie

Existence of the V-states for the generalized quasi-geostrophic equations

Chapitre 1

On the V-states for the generalized quasi-geostrophic equations

This chapter is the subject of the following publication :
Hassainia, Zineb ; Hmidi, Taoufik, *On the V-states for the Generalized Quasi-Geostrophic Equations*.
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Abstract. We prove the existence of the V-states for the generalized inviscid SQG equations with $\alpha \in]0,1[$. These structures are special rotating simply connected patches with $m-$ fold symmetry bifurcating from the trivial solution at some explicit values of the angular velocity. This produces, inter alia, an infinite family of non stationary global solutions with uniqueness.

1.1 Introduction

In this chapter we shall investigate some special structures of the vortical motions for the generalized inviscid surface quasi-geostrophic equation arising in fluid dynamics. This model describes the evolution of the potential temperature θ by the transport equation,

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = 0, \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ u = -\nabla^\perp(-\Delta)^{-1+\frac{\alpha}{2}}\theta, \\ \theta_{|t=0} = \theta_0. \end{cases} \tag{1.1}$$

Here u refers to the velocity field, $\nabla^\perp = (-\partial_2, \partial_1)$ and α is a real parameter taken in the interval $[0,1[$. The operator $(-\Delta)^{-1+\frac{\alpha}{2}}$ is of convolution type and is defined by

$$(-\Delta)^{-1+\frac{\alpha}{2}}\theta(x) = \frac{C_\alpha}{2\pi}\int_{\mathbb{R}^2}\frac{\theta(y)}{|x-y|^\alpha}dy \tag{1.2}$$

with $C_\alpha = \frac{\Gamma(\alpha/2)}{2^{1-\alpha}\Gamma(\frac{2-\alpha}{2})}$. This model was proposed by Córdoba et al. in [28] as an interpolation between Euler equations and the surface quasi-geostrophic model, hereafter denoted by SQG, corresponding to $\alpha = 0$ and $\alpha = 1$, respectively. The SQG equation was used by Juckes [71] and Held et al. [56] as a concise model of the atmosphere circulation near the tropopause. It was also developed by Lapeyre and Klein [80] to describe the ocean dynamics in the upper layers. We note that there is a strong mathematical and physical analogy with the three-dimensional incompressible Euler equations, and it can be viewed as a simplified model for that system ; see [27] for details.

The local well-posedness of classical solutions can be performed in various function spaces. For instance, this was implemented in the framework of Sobolev space [23] by using the commutator theory. However, it is so delicate to extend the Yudovich theory of weak solutions known for the two-dimensional Euler equations [128] to the case $\alpha > 0$ because the velocity is in general below the Lipschitz class. Nonetheless, one can say more about this issue for some special class of concentrated vortices. More precisely, when the initial datum has a vortex patch structure, that is, $\theta_0(x) = \chi_D$ is the characteristic function of a bounded simply connected smooth domain D, then there is a unique local solution in the patch form $\theta(t) = \chi_{D_t}$. In this case, the boundary motion of the domain D_t is described by the contour dynamics formulation ; see the papers [47, 107]. The global persistence of the boundary regularity is only known for $\alpha = 0$ according to the result of Chemin [25]. For $\alpha > 0$ there are some numerical simulations showing the singularity formation in finite time, see for instance [28].

The technique of contour dynamics was originally devised by Zabusky et al. [130] and has found many applications in the study of two-dimensional flows. We shall use this technique to track the boundary motion of the patch for the generalized SQG equation. According to the Green formula one can recover the velocity from the boundary through the formula,

$$u(t,x) = \frac{C_\alpha}{2\pi} \int_{\partial D_t} \frac{1}{|x-\xi|^\alpha} d\xi, \tag{1.3}$$

where $d\xi$ denotes the complex integration over the positively oriented curve ∂D_t. To write down the equation of the boundary, one can use for instance the Lagrangian parametrization $\gamma_t : [0, 2\pi] \to \mathbb{C}$, given by the nonlinear ode,

$$\begin{cases} \partial_t \gamma(t,\sigma) = u(t, \gamma(t,\sigma)), \\ \gamma(0,\sigma) = \gamma_0(\sigma) \end{cases}$$

where γ_0 is a periodic smooth parametrization of the initial boundary and consequently the contour dynamics equation becomes

$$\partial_t \gamma(t,\sigma) = \frac{C_\alpha}{2\pi} \int_0^{2\pi} \frac{\partial_s \gamma(t,s)}{|\gamma(t,\sigma) - \gamma(t,s)|^\alpha} ds. \tag{1.4}$$

The main objective of this chapter is to focus on some special vortices, called V-states or rotating patches, whose dynamics is described by a rigid body transformation. The problem consists in finding some domains D subject to a uniform rotation around their centers of mass. In which case the support of the patch D_t does not change its shape and is given by $D_t = \mathbf{R}_{x_0,\Omega t} D$, where $\mathbf{R}_{x_0,\Omega t}$ stands for the planar rotation with center x_0 and angle Ωt. The parameter Ω is called the angular velocity of the rotating domain.

This problem was investigated first for the two-dimensional Euler equations ($\alpha = 0$) a long time ago and is still a subject of intensive research combining analytical and numerical studies. It is worthy noting that explicit non trivial rotating patches are known in the literature and goes back to Kirchhoff [76] who discovered that an ellipse of semi-axes a and b is subject to a perpetual rotation with uniform angular velocity $\Omega = ab/(a+b)^2$; see for instance [14, p. 304] and [79, p. 232]. In the seventies of the last century, Deem and Zabusky [34] wrote an equation for the V-states and gave partial numerical solutions. They put in evidence the existence of the V-states with m-fold symmetry for each integer $m \geq 2$ and in this countable cascade the case $m = 2$ corresponds to the known Kirchhoff's ellipses. Recall that a domain is said to be m-fold symmetric if it has the same group invariance of a regular polygon with m sides. This means that the domain is invariant by the action of the dihedral group D_m. At each frequency m these V-states can be seen as a continuous deformation of the disc with respect to a hidden bifurcation parameter corresponding to the angular velocity. An analytical proof was given by Burbea in [16] and his approach consists in writing the problem with the conformal mapping of the domain and to look at the non trivial solutions by using the technique of the bifurcation theory. Actually, Burbea's proof is not completely rigorous and one can find a complete one in [65]. In this latter paper Burbea's approach was revisited with more details and explanations. We also studied the boundary regularity of the V-states and showed that they are of class C^∞ and convex close to the disc.

The formulation of the rotating patches can be done in several ways requiring different levels of regularity for the solution. We shall give here a short glimpse with an emphasis on two different approaches. The first one uses the elliptic equation governing the stream function ψ associated to the domain D of the initial patch. As to the second approach, it uses the conformal parametrization of the boundary combined with the contour dynamics formulation. To be more precise, recall that the function ψ is defined by the Newtonian potential through the formula,

$$\psi(x) = \frac{1}{2\pi}\int_D \log|x-y|\,dy, \quad \Delta\psi = \chi_D.$$

Note that a patch with a smooth boundary rotates uniformly around its center, which can be taken equal to zero, means that in its own frame the boundary is stationary. In other words, the relative stream function $x \mapsto \psi(x) - \frac{1}{2}\Omega|x|^2$ should be constant on the boundary and therefore we get the equation

$$\frac{1}{2\pi}\int_D \log|x-y|\,dy - \frac{1}{2}\Omega|x|^2 = \mu, \quad \forall x \in \partial D, \tag{1.5}$$

with μ a constant. By virtue of this equation, the domains D are in fact defined through a strong interaction between the Newtonian and the quadratic potentials. The issue depends heavily on the sign of Ω. To fix the terminology, we say that the potential is repulsive when $\Omega \le 0$ and attractive when it has an opposite sign. It seems that the situation in the repulsive case $\Omega \le 0$ is trivial in the sense that only the discs are solutions of the rotating patch problem. This means that all the V-states must rotate counterclockwise. This result is the subject of a work in progress by the second author [59]. The proof relies on the moving plane method, which allows us to show that any solution of (1.5) must be radial with respect to some specific point, which is the center of mass of the domain D, and is strictly monotone. In the attractive case $\Omega > 0$, the interaction between the potentials is more fruitful and leads to infinite nontrivial solutions called the V-states as we have already mentioned. We point out that Burbea shows that for each frequency $m \ge 2$ the V-states V_m can be assimilated to a bifurcating curve from the disc at the angular velocity $\Omega_m = \frac{m-1}{2m}$. His idea is to use the conformal mapping parametrization $\phi : \mathbb{D}^c \to D^c$ which satisfies the nonlinear integral equation

$$\begin{aligned} F\big(\Omega, \phi(w)\big) &\triangleq \operatorname{Im}\bigg\{\Big((1-2\Omega)\overline{\phi(w)} - \frac{1}{2i\pi}\int_{\mathbb{T}} \frac{\overline{\phi(\tau)-\phi(w)}}{\phi(\tau)-\phi(w)}\phi'(\tau)d\tau\Big) w\,\phi'(w)\bigg\} \\ &= 0, \quad \forall w \in \mathbb{T}, \end{aligned} \tag{1.6}$$

where $\mathbb{D}$ denotes the open unit disc and $\mathbb{T}$ its boundary. Now we observe that $F(\Omega, \mathrm{Id}) = 0$ and thus we may try to find non trivial solutions by using the bifurcation theory. For this end Burbea computes the linearized operator of F around this solution and shows that it has a nontrivial kernel if and only if $\Omega \in \{\Omega_m, m \ge 2\}$. In this case $\partial_f F(\Omega, \mathrm{Id})$ is a Fredholm operator with one-dimensional kernel. Consequently, one may apply the bifurcation theory through, for instance, the Crandall-Rabinowitz theorem. This allows us to prove the existence of the non trivial branch of solutions emerging from the trivial one at each frequency level Ω_m.

One cannot escape mentioning that other explicit vortex solutions are discovered in the literature for the incompressible Euler equations in the presence of an external shear flow ; see for instance [24, 74, 98]. A general review about vortex dynamics can be found in the papers [5, 99]. Another closely related subject is to conduct a similar study for the patches with multiple interfaces, which is inherently complicated due to the strong interaction between the interfaces. In this context, Flierl and Polvani [44] proved that confocal ellipses with some compatibility relations rotate as a rigid body motion. Recently, we developed a complete characterization of rotating patches with two interfaces, provided one of them is prescribed in the ellipses class.

In this chapter, we shall address the same problem for the generalized SQG equations and look for the existence of the V-states. The question was raised by Diego Córdoba and was the initial motivation for this work. As we shall see later in Proposition 1.4, the equation (1.6) becomes

$$F_\alpha\big(\Omega,\phi(w)\big) \triangleq \operatorname{Im}\bigg\{\Big(\Omega\phi(w)-\frac{C_\alpha}{2i\pi}\int_{\mathbb{T}}\frac{\phi'(\tau)}{|\phi(w)-\phi(\tau)|^\alpha}d\tau\Big)\overline{w}\,\overline{\phi'}(w)\bigg\}=0,\quad \forall w\in\mathbb{T},$$

with $C_\alpha=\dfrac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(\frac{2-\alpha}{2})}$. Note that the structure of the singular nonlinear part is different from (1.6). Indeed, the singular kernel is not algebraic with respect to the conformal mapping which is holomorphic outside the unit disc. This property is profoundly important for Euler equations because it yields at different levels of the analysis, especially in the spectral study, to simple computations through Residue Theorem. Another disadvantage of the kernel structure concerns the computations of the regularity of the functional F_α which are heavy and more involved.

The main contribution of this chapter is to give a positive answer for the existence of the V-states when $\alpha\in]0,1[$. For the sake of clarity we shall now give an elementary statement and a complete one is postponed to Theorem 1.3.

Theorem 1.1 *Let $\alpha\in]0,1[$ and $m\in\mathbb{N}^\star\backslash\{1\}$. Then, there exists a family of m-fold symmetric V-states $(V_m)_{m\geq 2}$ for the equation (1.1). Moreover, for each $m\geq 2$ the curve V_m bifurcates from the trivial solution $\theta_0=\chi_{\mathbb{D}}$ at the angular velocity*

$$\Omega_m^\alpha \triangleq \frac{\Gamma(1-\alpha)}{2^{1-\alpha}\Gamma^2(1-\frac{\alpha}{2})}\bigg(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})}-\frac{\Gamma(m+\frac{\alpha}{2})}{\Gamma(m+1-\frac{\alpha}{2})}\bigg),$$

where Γ denotes the gamma function.

In addition, the boundary of the V-states belongs to the Hölder class $C^{2-\alpha}$.

The proof of this theorem will be done in the spirit of the incompressible Euler equations by using the bifurcation theory through the Crandall-Rabinowitz Theorem. In the framework of this theory one should understand the structure of the linearized operator around the trivial solution Id and identify the range of Ω where this operator is not invertible. More precisely, we should determine where this operator belongs to the Fredholm class with zero index and possesses a simple kernel. By using some tricky integral formulae summarized in Lemma 2.1 one finds : for $h(w)=\sum_{n\in\mathbb{N}}b_n\overline{w}^n$

$$\partial_\phi F_\alpha(\Omega,\mathrm{Id})h(w)=\frac{1}{2}b_0\Omega\, i\big(w-\overline{w}\big)+\frac{i}{2}\sum_{n\geq 1}(n+1)\big(\Omega-\Omega_{n+1}^\alpha\big)b_n\Big(w^{n+1}-\overline{w}^{n+1}\Big).$$

Consequently, the linearized operator acts as a Fourier multiplier in the phase space. It behaves as a differential operator of order one because $\sup_n\Omega_n^\alpha<\infty$. Afterwards, we prove that this operator sends $C^{2-\alpha}$ to $C^{1-\alpha}$ and fulfills the required assumptions of the Crandall-Rabinowitz Theorem :

it is of Fredholm type with zero index and satisfies the transversality assumption. This latter one means that when we look for the linearized operator with Ω close to Ω_m^α, then the eigenvalue that is close to zero (it depends on Ω) must cross the real axis with non zero velocity at the value $\Omega = \Omega_m^\alpha$.

Next, we shall make few comments about the statement of the main theorem.

Remarks 1.1 1) *For the incompressible Euler equations the dilation has no effects on the angular vorticity of the V-states. However, this property fails for the generalized SQG model because we change the homogeneity of the equation. As we shall see later in Proposition* 1.3 *the angular velocity depends on the inverse of the dilation parameter raised to the power α. This means that we can find small patches rotating quickly and also big ones rotating very slowly. In addition, the bifurcation set $\{\Omega_m^\alpha, m \geq 2\}$ introduced in Theorem* 1.1 *concerns only the bifurcation from the unit disc. However, to get a bifurcation from a disc of radius r we have to scale this set as follows $\{r^{-\alpha}\Omega_m^\alpha, m \geq 2\}$.*

2) *For the SQG equation corresponding to $\alpha = 1$ the situation is more delicate as we shall discuss later in the end of the chapter. Indeed, one can modify the function F_α in order to get a less singular kernel, but we note a regularity loss for the linearized operator. This appears more clearly when we compute the linearized operator, which is given by*

$$\partial_\phi F_1(\Omega, \mathrm{Id})h(w) = \frac{1}{2} b_0 \Omega\, i\big(w - \overline{w}\big) + \frac{i}{2} \sum_{n \geq 1} (n+1)\big(\Omega - \Omega_{n+1}^1\big) b_n \Big(w^{n+1} - \overline{w}^{n+1}\Big),$$

with

$$\Omega_n^1 = \frac{2}{\pi} \sum_{k=1}^{n-1} \frac{1}{2k+1}. \tag{1.7}$$

We see that this operator acts as a Fourier multiplier with an additional logarithmic growth compared to the case $\alpha \in [0, 1[$. As a consequence, this operator does not send $C^{1+\varepsilon}$ to C^ε and it seems complicated to find suitable function spaces X and Y such that the Crandall-Rabinowitz Theorem can be applied. More discussion will be brought forward the end of this chapter ; see Section 1.10. *We also mention that the preceding dispersion relation was computed formally in [4] by using Bessel functions. In Section* 1.10 *we shall give another proof of this relation.*

3) *The boundary of the rotating patches belongs to Hölder space $C^{2-\alpha}$. For $\alpha = 0$, we get better result as it was shown in [65] ; the boundary is C^∞ and convex when the V-states are close to the circle. The proof in this particular case uses in a deep way the algebraic structure of the kernel according to some recurrence formulae. It is not clear whether this approach can be implemented for the generalized SQG equation but we do believe that the boundary is also C^∞.*

4) *The global existence of non stationary solutions for* (1.1) *is not known for $\alpha > 0$. The V-states offer a suitable class of initial data with global existence because they generate periodic solutions in time.*

5) *As we shall see later in Lemma* 1.3, *there is continuity of the spectrum Ω_m^α with respect to α. This means that,*

$$\lim_{\alpha \to 0} \Omega_m^\alpha = \frac{m-1}{2m} \quad \textit{and} \quad \lim_{\alpha \to 1} \Omega_m^\alpha = \Omega_m^1,$$

where Ω_m^1 is defined in (1.7).

The chapter is organized as follows. In the next section we shall fix some notation. In Section 3, we discuss some general properties of the V-states. In Section 4, we shall introduce and review some background material on the bifurcation theory and singular integrals. In Section 5, we will study the elliptic patches and show that they never rotate. This was recently proved in [17] and we intend to give another proof by using complex analysis formulation. Section 6 is devoted to a general statement of Theorem 1.1. The proof of Theorem 1.3 will be discussed in Sections 7, 8 and 9. Last, in Section 10 we will pay a special attention to the SQG model corresponding to the limit case $\alpha = 1$. We shall reformulate the boundary equation in order to kill the violent singularity of the kernel. In this case we give a complete description of the linearized operator and the dispersion relation. However we are not able to give a complete proof of the bifurcation of the V-states, which should require a slightly different mathematical machinery than does the sub-critical case $\alpha \in [0, 1[$.

1.2 Notation

In this section we shall fix some notation that will be frequently used along this chapter.

- We denote by C any positive constant that may change from line to line.
- For any positive real numbers A and B, the notation $A \lesssim B$ means that there exists a positive constant C independent of A and B such that $A \le CB$.
- We denote by $\mathbb{D}$ the unit disc. Its boundary, the unit circle, is denoted by $\mathbb{T}$.
- Let $f : \mathbb{T} \to \mathbb{C}$ be a continuous function. We define its mean value by,
$$\fint_{\mathbb{T}} f(\tau)d\tau \triangleq \frac{1}{2i\pi}\int_{\mathbb{T}} f(\tau)d\tau,$$
where $d\tau$ stands for the complex integration.
- Let X and Y be two normed spaces. We denote by $\mathcal{L}(X, Y)$ the space of all continuous linear maps $T : X \to Y$ endowed with its usual strong topology.
- For a linear operator $T : X \to Y$, we denote by $N(T)$ and $R(T)$ the kernel and the range of T, respectively.
- If Y is a vector space and R is a subspace, then Y/R denotes the quotient space.

1.3 Preliminaries on the V-states

In this introductory section we will focus on some general results on the rotating patches called also V-states according to Deem and Zabusky terminology. These results were proved in [66] for Euler equations and will be likewise extended to the SQG model (1.1).

1.3.1 General facts

Now we intend to fix some vocabulary and prove in particular that the center of rotation of any V-state should coincide with its center of mass. We shall also deal with the effects of the dilation of the geometry on the angular velocity of the rotating patches.

Definition 1.1 *Let D_0 be a simply connected domain in the plane with smooth boundary. We say that $\theta_0 = \mathbf{1}_{D_0}$ is a rotating patch if the associated solution of* (1.1) *is given by*
$$\theta(t, x) = \mathbf{1}_{D_t} \quad \textit{with} \quad D_t = \mathbf{R}_{x_0,\varphi(t)}D_0.$$

Here we denote by $\mathbf{R}_{x_0,\varphi(t)}$ *the planar rotation of center* x_0 *and angle* $\varphi(t)$*. In addition, we assume that the function* $t \mapsto \varphi(t)$ *is smooth and non-constant.*

The velocity dynamics in the framework of rotating patches is described as follows.

Proposition 1.1 *Let* θ_0 *be a rotating patch as in Definition* 1.1*. Then the velocity* $u(t)$ *can be recovered from its initial value* u_0 *according to the formula*

$$u(t,x) = \mathbf{R}_{x_0,\varphi(t)} u_0(\mathbf{R}_{x_0,-\varphi(t)} x).$$

Proof : We shall use the formula

$$-(-\Delta)^{1-\frac{\alpha}{2}} u(t,x) = \nabla^\perp \theta(t,x).$$

Performing some algebraic computations we get

$$\begin{aligned}
\nabla^\perp \theta(t,x) &= \mathbf{R}_{x_0,\varphi(t)} \nabla^\perp \theta_0(\mathbf{R}_{x_0,-\varphi(t)} x) \\
&= -\mathbf{R}_{x_0,\varphi(t)} (-\Delta)^{1-\frac{\alpha}{2}} v_0(\mathbf{R}_{x_0,-\varphi(t)} x) \\
&= -(-\Delta)^{1-\frac{\alpha}{2}} \big(\mathbf{R}_{x_0,\varphi(t)} v_0(\mathbf{R}_{x_0,-\varphi(t)} x)\big).
\end{aligned}$$

Here we have used the commutation between the operator $(-\Delta)^{1-\frac{\alpha}{2}}$ and the rotation transformations which can be checked easily from the integral representation of the fractional Laplacian. Therefore, the result follows from a uniqueness argument. □

Now, we will discuss a special result concerning the evolution of the center of mass of the patch $\theta(t) = \mathbf{1}_{D_t}$, defined by

$$X(t) = \frac{1}{|D_t|} \int_{D_t} x\, dx = \frac{1}{|D_0|} \int_{D_t} x\, dx.$$

We have used the fact that the volume of a patch is an invariant of the motion since the velocity is divergence free. Next, we prove that the center of mass is stationary for any patch solution of (1.1). This is known for Euler equation and we shall give here a similar proof.

Proposition 1.2 *Let* $\theta(t) = \mathbf{1}_{D_t}$ *be a solution of* (1.1) *then the center of mass is fixed, that is*

$$X(t) = X(0).$$

Proof : The invariance of the center of mass follows from the constancy of the functions

$$f_j(t) \triangleq \int_{\mathbb{R}^2} x_j\, \theta(t,x)\, dx, \quad j = 1, 2.$$

Differentiating this function with respect to the time variable combined with the equation (1.1) and integration by parts yields

$$\begin{aligned}
f_j'(t) &= \int_{\mathbb{R}^2} x_j\, \partial_t \theta(t,x)\, dx \\
&= -\int_{\mathbb{R}^2} x_j\, (u \cdot \nabla \theta)(t,x)\, dx \\
&= \int_{\mathbb{R}^2} u^j(t,x)\, \theta(t,x)\, dx.
\end{aligned}$$

Using the relation between u and θ and integrating once again by parts we get

$$\begin{aligned} f_1'(t) &= -\int_{\mathbb{R}^2} \theta\, \nabla^{\perp}(-\Delta)^{-1+\frac{\alpha}{2}}\theta\, dx \\ &= -\int_{\mathbb{R}^2} \big\{(-\Delta)^{-\frac{1}{2}+\frac{\alpha}{4}}\theta\big\}\, \nabla^{\perp}\big\{(-\Delta)^{-\frac{1}{2}+\frac{\alpha}{4}}\theta\big\}\, dx \\ &= 0. \end{aligned}$$

This completes the proof of the desired result. □

In consequence, we obtain the following result.

Corollary 1.1 *Let $\theta_0 = \mathbf{1}_{D_0}$ be a rotating patch center around some point x_0. Then necescenter-sarily x_0 is the center of mass of the domain D_0.*

Proof : By a change of variables

$$\begin{aligned} X(t) &= \frac{1}{|D_0|}\int_{\mathbb{R}^2} x\,\theta_0(\mathbf{R}_{x_0,-\varphi(t)}x)dx \\ &= \frac{1}{|D_0|}\int_{\mathbb{R}^2} (\mathbf{R}_{x_0,\varphi(t)}x)\theta_0(x)dx \\ &= \frac{1}{|D_0|}\mathbf{R}_{x_0,\varphi(t)}\left(\int_{\mathbb{R}^2} x\theta_0(x)dx\right) \\ &= \mathbf{R}_{x_0,\varphi(t)}X(0). \end{aligned}$$

Since $X(t) = X(0)$ by Proposition (1.2), $X(0)$ is fixed by the rotation and thus $X(0) = x_0$, as claimed. □

Next we shall discuss how the dilation affects the angular velocity. We point out that the following notation that we shall use $D_\lambda = \lambda D$ means a dilation of the domain D with respect to its center of mass.

Proposition 1.3 *Let $\theta_0 = \mathbf{1}_D$ be a rotating patch with constant angular velocity Ω. Let $\lambda > 0$ and denote by $D_\lambda = \lambda D$. Then D_λ is also a rotating patch with angular velocity $\Omega_\lambda = \frac{\Omega}{\lambda^\alpha}\cdot$*

Proof : Without loss of generality we can assume that the center of rotation is the origin. Then according to the equation (1.13) we have

$$\Omega \mathrm{Re}\big\{z\overline{z'}\big\} = \mathrm{Im}\Big\{\frac{C_\alpha}{2\pi}\int_{\partial D}\frac{d\zeta}{|z-\zeta|^\alpha}\overline{z'}\Big\}, \quad \forall\, z\in\partial D.$$

Let $z \in D$ and $\tau = \lambda z$, then multiplying the preceding equation by λ^2 and using the change of variables $w = \lambda\zeta$ we get

$$\big(\lambda^{-\alpha}\Omega\big)\mathrm{Re}\big\{\tau\overline{\tau'}\big\} = \mathrm{Im}\Big\{\frac{C_\alpha}{2\pi}\int_{\partial D_\lambda}\frac{dw}{|\tau-w|^\alpha}\overline{\tau'}\Big\}, \quad \forall\, \tau\in\partial D_\lambda.$$

This shows that D_λ rotates with the angular velocity $\lambda^{-\alpha}\Omega$ as it is claimed. □

1.3.2 Boundary equation

Before proceeding further with the consideration of the V-states, we shall recall Riemann mapping theorem which is one of the most important results in complex analysis. To restate this result we shall recall the definition of *simply connected* domains. Let $\widehat{\mathbb{C}} \triangleq \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere. We say that a domain $\Omega \subset \widehat{\mathbb{C}}$ is *simply connected* if the set $\widehat{\mathbb{C}} \backslash \Omega$ is connected.

Riemann Mapping Theorem. Let $\mathbb{D}$ denote the unit open ball and $\Omega \subset \mathbb{C}$ be a simply connected bounded domain. Then there is a unique bi-holomorphic map called also conformal map, $\Phi : \mathbb{C}\backslash\overline{\mathbb{D}} \to \mathbb{C}\backslash\overline{\Omega}$ taking the form

$$\Phi(z) = az + \sum_{n\in\mathbb{N}} \frac{a_n}{z^n} \quad \text{with} \quad a > 0.$$

In this theorem the regularity of the boundary has no effect regarding the existence of the conformal mapping but it contributes in the boundary behavior of the conformal mapping, see for instance [104, 122]. Here, we shall recall the following result.

Kellogg-Warschawski's theorem. It can be found in [122] or in [104, Theorem 3.6]. It asserts that if the boundary $\Phi(\mathbb{T})$ is a Jordan curve of class $C^{n+1+\beta}$, with $n \in \mathbb{N}$ and $0 < \beta < 1$, then the conformal map $\Phi : \mathbb{C}\backslash\overline{\mathbb{D}} \to \mathbb{C}\backslash\overline{\Omega}$ has a continuous extension to $\mathbb{C}\backslash\mathbb{D}$ which is of class $C^{n+1+\beta}$.

Next, we shall write down the equation governing the boundary of the V-states ; it is highly nonlinear and non local as the next proposition shows.

Proposition 1.4 *Let $\alpha \in]0,1[$, D_0 be a smooth simply connected domain and $D_t = R_{x_0,\varphi(t)} D_0$ be a V-state of the model* (1.1). *Then, the following claims hold true.*

1. *The point x_0 is the center of mass of D_0 and $\dot{\varphi}(t) = \Omega$ is constant.*
2. *Assume that $x_0 = 0$ and let $\phi : \mathbb{D}^c \to D_0^c$ be the conformal mapping, then*

$$\mathrm{Im}\bigg\{ \Big(\Omega\, \phi(w) - C_\alpha \fint_{\mathbb{T}} \frac{\phi'(\tau) d\tau}{|\phi(w) - \phi(\tau)|^\alpha} \Big) \overline{w}\, \overline{\phi'(w)} \bigg\} = 0, \quad \forall w \in \mathbb{T}, \tag{1.8}$$

with $C_\alpha = \dfrac{\Gamma(\alpha/2)}{2^{1-\alpha}\Gamma(\frac{2-\alpha}{2})}$.

Proof :

($\mathbf{1}$) The first claim was proved in Proposition 1.1 and so it remains to check that the angular velocity is constant. For this aim we shall start with writing the boundary equation of a V-state. Loosely speaking, the boundary ∂D_t is a material surface and there is no flux matter across it. In other words, it is transpocenterrted by the flow $\psi(t)$ defined in the next few lines. For a smooth initial boundary, say of class C^1, there exists a function $\varphi_0 : \mathbb{R}^2 \to \mathbb{R}$ of class C^1 such that

$$\partial D_0 = \Big\{ x \in \mathbb{R}^2; \varphi_0(x) = 0 \Big\},$$

with the additional constraints : $\forall x \in \partial D_0, \ \ \nabla\varphi_0(x) \neq 0$,

$$\varphi_0 < 0 \ \text{ on } \ D_0 \quad \text{and} \quad \varphi_0 > 0 \ \text{ on } \ \mathbb{R}^2 \setminus \overline{D_0}.$$

One says in this case that φ_0 is a defining function for ∂D_0. Set

$$F(t,x) = \varphi_0(\psi^{-1}(t,x)),$$

where ψ is the flow associated to the velocity u and given by the integral equation

$$\psi(t,x) = x + \int_0^t u\big(\tau, \psi(\tau,x)\big)d\tau.$$

It follows that the maps $x \mapsto F(t,x)$ is a defining function for $\partial D_t = \psi(t, \partial D_0)$ and satisfies the transport equation

$$\partial_t F + u \cdot \nabla F = 0.$$

Now, let $\sigma \in [0, 2\pi] \mapsto \gamma_t(\sigma)$ be a parametrization of ∂D_t, continuously differentiable in t, and let $\vec{n}_t$ be the unit outward normal vector to ∂D_t. Differentiating the equation $F(t, \gamma_t(\sigma)) = 0$ with respect to t yields

$$\partial_t F + \partial_t \gamma_t \cdot \nabla F = 0.$$

Since for $x \in \partial D_t$ the vector $\nabla F(t,x)$ is colinear to the normal vector $\vec{n}_t$ then

$$(\partial_t \gamma_t - u(t, \gamma_t)) \cdot \vec{n}_t = 0. \tag{1.9}$$

The meaning of (1.9) is that the velocity of the boundary and the the velocity of the fluid particle occupying the same position have the same normal components. We observe that the equation (1.9) can be written in a complex form which seems to be more convenient in our case,

$$\text{Im}\Big\{(\partial_t \gamma_t - v(t, \gamma_t))\overline{\gamma_t'}\Big\} = 0, \tag{1.10}$$

where the "prime" denotes the derivative with respect to the σ variable.

We now take a closer look at the case of a rotating connected patch. Assume that the boundary rotates with the angular velocity $\dot{\theta}(t)$ around its center of mass which can be assumed to be the origin. According to the Proposition 1.1 the velocity $u(t)$ can be recovered from the initial velocity u_0 through to the formula

$$u(t,x) = e^{i\theta(t)} u_0(e^{-i\theta(t)}x). \tag{1.11}$$

Hence

$$\text{Im}\big\{u(t,\gamma_t)\overline{\gamma_t'}\big\} = \text{Im}\big\{u_0(\gamma_0)\overline{\gamma_0'}\big\}.$$

The rotating patch has a standard parametrization given by $\gamma_t = e^{i\theta(t)}\gamma_0$ which yields

$$\text{Im}\big\{\partial_t \gamma_t \overline{\gamma_t'}\big\} = \dot{\theta}(t)\text{Re}\big\{\gamma_0 \overline{\gamma_0'}\big\}.$$

Consequently the equation (1.10) becomes

$$\dot{\theta}(t)\text{Re}\big\{\gamma_0 \overline{\gamma_0'}\big\} = \text{Im}\big\{u_0(\gamma_0)\overline{\gamma_0'}\big\}$$

which is equivalent to

$$\frac{\dot{\theta}(t)}{2}\frac{d}{ds}|\gamma_0(s)|^2 = \text{Im}\Big\{u_0(\gamma_0)\,\overline{\gamma_0'}\Big\}.$$

If there exists some s with $\frac{d}{ds}|\gamma_0(s)|^2 \neq 0$ then, since the right-hand side does not depend on the time variable, we conclude that $\dot{\theta}(t) = \Omega$ is constant. Otherwise, $\frac{d}{ds}|\gamma_0(s)|^2$ vanishes everywhere, which tells us that the initial domain is a disc and therefore it rotates with any angular velocity. Finally we get the boundary equation

$$\Omega\,\text{Re}\big\{z\,\overline{z'}\big\} = \text{Im}\big\{u_0(z)\,\overline{z'}\big\}, \quad \forall z \in D_0. \tag{1.12}$$

Recall that z' is a tangent vector to the boundary ∂D_0 at the point z.

(**2**) Combining (1.12) with the velocity formula (1.3) we get

$$\Omega\,\mathrm{Re}\big\{z\overline{z'}\big\} = C_\alpha \mathrm{Im}\bigg\{\frac{1}{2\pi}\int_{\partial D_0}\frac{d\zeta}{|z-\zeta|^\alpha}\overline{z'}\bigg\},\quad \forall\, z\in\partial D_0. \tag{1.13}$$

We shall now parametrize the domain with the outside conformal mapping $\phi:\mathbb{D}^c\to D_0^c$.

$$\phi(w) = w + \sum_{n\geq 0}\frac{b_n}{w^n} \tag{1.14}$$

Setting $z=\phi(w)$ and $\zeta=\phi(\tau)$, then for $w\in\mathbb{T}$ a tangent vector is given by

$$\overline{z'} = -i\overline{w}\,\overline{\phi'(w)}.$$

Inserting this in the equation (1.13) gives

$$G(\Omega,\phi)(w) \triangleq \mathrm{Im}\bigg\{\bigg(\Omega\phi(w) - \frac{C_\alpha}{2i\pi}\int\limits_{\mathbb{T}}\frac{\phi'(\tau)d\tau}{|\phi(w)-\phi(\tau)|^\alpha}\bigg)\overline{w}\,\overline{\phi'(w)}\bigg\} = 0,\quad \forall\, w\in\mathbb{T}. \tag{1.15}$$

This achieves the proof of the proposition. □

1.4 Tools

The purpose of this introductory section is to review and collect some technical tools that will be used quite often in the remainder of this chapter. We will firstly recall some basic elements of the bifurcation theory. We will focus on the Crandall-Rabinowitz's theorem, hereafter referred by C-R Theorem, which is very crucial for the proof of our main result. Secondly, some simple facts about Hölder spaces $C^{n+\gamma}(\mathbb{T})$ will be recalled and we shall also explore some results on the action of singular integral operators on these spaces. Last, we end this section with some integral computations that will be frequently used in the study of the linearized operator.

1.4.1 Elements of the bifurcation theory

We intend now to give some formal explanations and general principles of the bifurcation theory. This discussion will be closed by stating C-R theorem. Roughly speaking, the main objective of this theory is to look for the solutions of the equation

$$F(\lambda, x) = 0$$

where $F:\mathbb{R}\times X\to Y$ is continuous function and satisfies some additional regularity assumptions. The vector spaces X and Y are Banach spaces. We assume in addition that $x=0$ is a trivial solution for any λ, that is, $F(\lambda,0)=0$. Whether close to the solution $(\lambda_0,0)$ one can find a branch of non trivial ones is the main problem discussed in this theory. If this is the case we say that there is a bifurcation at the point $(\lambda_0,0)$. As the Implicit Function Theorem tells us, the first test that should be carried out is to analyze the linear operator $\mathcal{L}_\lambda \triangleq \partial_x F(\lambda,0):X\to Y$. If this operator is an isomorphism then such non trivial solutions cannot exist. Thus a necessary condition for the bifurcation is to get a nontrivial kernel of $\mathcal{L}_\lambda$. In many instances, the involved Banach spaces are infinite-dimensional and thus the bifurcation analysis is in general very complex. However, if the linearized operator is of Fredholm type one can reduce the problem to finite-dimensional spaces by using the so-called Lyapunov-Schmidt reduction. Recall that a Fredholm operator means that it is continuous and whose kernel $N(\mathcal{L}_\lambda)$ and cokernel $Y/R(\mathcal{L}_\lambda)$ are finite-dimensional, where $R(\mathcal{L}_\lambda)$ denotes the range of $\mathcal{L}_\lambda$. If moreover the index of this operator is zero then the bifurcation may occur despite that some suitable conditions are satisfied. Here we shall only discuss the bifurcation with one dimensional kernel which is the most common one and appears in many dynamical systems as for our generalized SQG model. With the preceding assumptions on the linear operator a one-parameter curve bifurcates from the trivial solution provided a transversality assumption is satisfied. Roughly speaking, this latter assumption means that the linear operator $\mathcal{L}_\lambda$ possesses a one-parameter eigenvalues $\lambda \mapsto \mu(\lambda)$ that should cross the real axis at λ_0 with non zero velocity. This is the classical theorem proved by Crandall and Rabinowitz [29] which is a basic tool in the bifurcation theory and that will be used in this chapter. More general results are summarized in the book of Kielhöfer [75]. Now we recall Crandall-Rabinowitz Theorem.

Theorem 1.2 *Let X, Y be two Banach spaces, V a neighborhood of 0 in X and let $F : \mathbb{R} \times V \to Y$ with the following properties :*

1. *$F(\lambda, 0) = 0$ for any $\lambda \in \mathbb{R}$.*
2. *The partial derivatives F_λ, F_x and $F_{\lambda x}$ exist and are continuous.*
3. *$N(\mathcal{L}_0)$ and $Y/R(\mathcal{L}_0)$ are one-dimensional.*
4. *Transversality assumption : $F_{tx}(0,0)x_0 \notin R(\mathcal{L}_0)$, where*

$$N(\mathcal{L}_0) = span\{x_0\}, \quad \mathcal{L}_0 \triangleq \partial_x F(0,0).$$

If Z is any complement of $N(\mathcal{L}_0)$ in X, then there is a neighborhood U of $(0,0)$ in $\mathbb{R} \times X$, an interval $(-a,a)$, and continuous functions $\varphi : (-a,a) \to \mathbb{R}$, $\psi : (-a,a) \to Z$ such that $\varphi(0) = 0$, $\psi(0) = 0$ and

$$F^{-1}(0) \cap U = \Big\{ \big(\varphi(\xi), \xi x_0 + \xi\psi(\xi)\big) \, ; \, |\xi| < a \Big\} \cup \Big\{ (\lambda, 0) \, ; \, (\lambda, 0) \in U \Big\}.$$

1.4.2 Singular integrals

In this paragraph we shall briefly recall the classical Hölder spaces on the periodic case and state some classical facts on the continuity of singular integrals over these spaces. It is convenient to think of 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$ as a function of the complex variable $w = e^{i\eta}$ rather than a function of the real variable η. To be more precise, let $f : \mathbb{T} \to \mathbb{R}^2$, be a continuous function, then it can be assimilated to a $2\pi-$ periodic function $g : \mathbb{R} \to \mathbb{R}$ via the relation

$$f(w) = g(\eta), \quad w = e^{i\eta}.$$

Hence when f is smooth enough we get

$$f'(w) \triangleq \frac{df}{dw} = -ie^{-i\eta} g'(\eta).$$

Because d/dw and $d/d\eta$ differ only by a smooth factor with modulus one we shall in the sequel work with d/dw instead of $d/d\eta$ which appears to be more convenient in the computations. Moreover, if f has real Fourier coefficients and is of class C^1 then we have the identity

$$\{\overline{f}\}'(w) = -\frac{1}{w^2}\overline{f'(w)}. \tag{1.16}$$

Now we shall introduce Hölder spaces on the unit circle $\mathbb{T}$.

Definition 1.2 *Let $0 < \gamma < 1$. We denote by $C^\gamma(\mathbb{T})$ the space of continuous functions f such that*

$$\|f\|_{C^\gamma(\mathbb{T})} \triangleq \|f\|_{L^\infty(\mathbb{T})} + \sup_{x\neq y\in\mathbb{T}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty.$$

For any integer n the space $C^{n+\gamma}(\mathbb{T})$ stands for the set of functions f of class C^n whose $n-th$ order derivatives are Hölder continuous with exponent γ. It is equipped with the usual norm,

$$\|f\|_{C^{n+\gamma}(\mathbb{T})} \triangleq \|f\|_{L^\infty(\mathbb{T})} + \Big\|\frac{d^n f}{dw^n}\Big\|_{C^\gamma(\mathbb{T})}.$$

Recall that the Lipschitz (semi)-norm is defined as follows.

$$\|f\|_{\mathrm{Lip}(\mathbb{T})} = \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|}.$$

Now we list some classical properties that will be used later especially in Section 1.7.

1. For $n \in \mathbb{N}, \gamma \in]0,1[$ the space $C^{n+\gamma}(\mathbb{T})$ is an algebra.
2. For $K \in L^1(\mathbb{T})$ and $f \in C^{n+\gamma}(\mathbb{T})$ we have the convolution law,

$$\|K * f\|_{C^{n+\gamma}(\mathbb{T})} \le \|K\|_{L^1(\mathbb{T})}\|f\|_{C^{n+\gamma}(\mathbb{T})}.$$

The next result is used often. It deals with singular integrals of the following type,

$$\mathcal{T}(f)(w) = \int_{\mathbb{T}} K(w,\tau)\, f(\tau) d\tau, \tag{1.17}$$

with $K : \mathbb{T}\times\mathbb{T} \to \mathbb{C}$ a singular kernel satisfying some properties. This problem will appear naturally when we shall deal with the regularity of the nonlinear operator in the rotating patches formalism, see Section 1.7. The result that we shall discuss with respect to this subject is classical and for the self-containing of the chapter we shall provide a complete proof which is similar to [91].

Lemma 1.1 *Let $0 \le \alpha < 1$ and consider a function $K : \mathbb{T}\times\mathbb{T} \to \mathbb{C}$ with the following properties. There exits $C_0 > 0$ such that,*

1. *K is measurable on $\mathbb{T}\times\mathbb{T}\backslash\{(w,w),\ w\in\mathbb{T}\}$ and*

$$\big|K(w,\tau)\big| \le \frac{C_0}{|w-\tau|^\alpha}, \quad \forall\, w\neq\tau\in\mathbb{T}.$$

2. *For each $\tau\in\mathbb{T}$, $w\mapsto K(w,\tau)$ is differentiable in $\mathbb{T}\backslash\{\tau\}$ and*

$$\big|\partial_w K(w,\tau)\big| \le \frac{C_0}{|w-\tau|^{1+\alpha}}, \quad \forall\, w\neq\tau\in\mathbb{T}.$$

Then the operator $\mathcal{T}$ defined by (1.17) is continuous from $L^\infty(\mathbb{T})$ to $C^{1-\alpha}(\mathbb{T})$. More precisely, there exists a constant C_α depending only on α such that

$$\|\mathcal{T}(f)\|_{1-\alpha} \le C_\alpha C_0 \|f\|_{L^\infty}.$$

Proof :

We first prove that $\mathcal{T}(f)$ is bounded on $\mathbb{T}$. Let $w\in\mathbb{T}$, then by the condition (1),

$$\begin{aligned}|\mathcal{T}(f)(w)| &\leq C_0\|f\|_{L^\infty}\Big|\int_{\mathbb{T}}\frac{d\tau}{|w-\tau|^\alpha}\Big|\\ &\leq C_\alpha C_0\|f\|_{L^\infty}.\end{aligned}$$

Next, take $w_1,w_2\in\mathbb{T}$, set $r=|w_1-w_2|$ and define $B_r(w_1)=\big\{\tau\in\mathbb{T};\,|\tau-w_1|\leq r\big\}$. Then,

$$\begin{aligned}\Big|\mathcal{T}(f)(w_1)-\mathcal{T}(f)(w_2)\Big| &\leq \Big|\int_{B_{2r}(w_1)}|f(\tau)||K(w_1,\tau)|d\tau\Big|+\Big|\int_{B_{2r}(w_1)}|f(\tau)||K(w_2,\tau)|d\tau\Big|\\ &+\Big|\int_{B^c_{2r}(w_1)}|f(\tau)||K(w_1,\tau)-K(w_2,\tau)|d\tau\Big|\\ &\triangleq J_1+J_2+J_3.\end{aligned}$$

By using again the condition (1), J_1 and J_2 can be estimated by

$$\begin{aligned}J_1+J_2 &\leq C_0\|f\|_{L^\infty}\Big(\Big|\int_{B_{2r}(w_1)}\frac{d\tau}{|w_1-\tau|^\alpha}\Big|+\Big|\int_{B_{3r}(w_2)}\frac{d\tau}{|w_2-\tau|^\alpha}\Big|\Big)\\ &\leq C_\alpha C_0\|f\|_{L^\infty}|w_1-w_2|^{1-\alpha}.\end{aligned}$$

To estimate the third term J_3 we shall use the condition (2) combined with the mean value theorem,

$$|K(w_1,\tau)-K(w_2,\tau)|\leq CC_0\frac{|w_1-w_2|}{|w_1-\tau|^{1+\alpha}},\quad \forall\,\tau\in B^c_{2r}(w_1).$$

Consequently we get

$$\begin{aligned}J_3 &\leq CC_0\|f\|_{L^\infty}\Big|\int_{B^c_{2r}(w_1)}\frac{|w_1-w_2|}{|w_1-\tau|^{1+\alpha}}d\tau\\ &\leq C_\alpha C_0\|f\|_{L^\infty}|w_1-w_2|^{1-\alpha}.\end{aligned}$$

This concludes the result. □

As a by-product we obtain the result.

Corollary 1.2 *Let $0\leq\alpha<1$, $\phi:\mathbb{T}\to\phi(\mathbb{T})$ be a bi-Lipschitz function with real Fourier coefficients and define the operator*

$$\mathcal{T}_\phi: f\mapsto \fint_{\mathbb{T}}\frac{f(\tau)}{|\phi(w)-\phi(\tau)|^\alpha}d\tau,\quad w\in\mathbb{T}.$$

Then $\mathcal{T}_\phi: L^\infty\big(\mathbb{T}\big)\to C^{1-\alpha}\big(\mathbb{T}\big)$ is continuous with the estimation,

$$\|\mathcal{T}_\phi(f)\|_{C^{1-\alpha}(\mathbb{T})}\leq C\Big(\|\phi^{-1}\|^\alpha_{\text{Lip}(\mathbb{T})}+\|\phi\|^2_{\text{Lip}(\mathbb{T})}\|\phi^{-1}\|^{1+\alpha}_{\text{Lip}(\mathbb{T})}\Big)\|f\|_{L^\infty(\mathbb{T})},$$

where C is a positive constant depending only on α.

Proof : We set

$$K(w,\tau)=\frac{1}{|\phi(w)-\phi(\tau)|^\alpha},\quad \forall\,w\neq\tau\in\mathbb{T}.$$

Since ϕ is bi-Lipschitz then we deduce that

$$|K(w,\tau)| \leq \|\phi^{-1}\|^{\alpha}_{\text{Lip}(\mathbb{T})}\frac{1}{|w-\tau|^{\alpha}} \quad \forall w \neq \tau \in \mathbb{T}. \tag{1.18}$$

To get the second assumption (2) of Lemma 1.1 we shall compute $\partial_w K(w,\tau)$.

$$\begin{aligned}\partial_w K(w,\tau) &= \frac{-\alpha}{2}\Bigg(\phi'(w)\frac{\overline{\phi(w)}-\overline{\phi(\tau)}}{|\phi(w)-\phi(\tau)|^{\alpha+2}}+(\overline{\phi})'(w)\frac{\phi(w)-\phi(\tau)}{|\phi(w)-\phi(\tau)|^{\alpha+2}}\Bigg)\\ &= \frac{-\alpha}{2}\Bigg(\phi'(w)\frac{\overline{\phi(w)}-\overline{\phi(\tau)}}{|\phi(w)-\phi(\tau)|^{2}}-\frac{\overline{\phi'(w)}}{w^2}\frac{\phi(w)-\phi(\tau)}{|\phi(w)-\phi(\tau)|^{2}}\Bigg)K(w,\tau) \quad w\neq\tau \in \mathbb{T}. \end{aligned}\tag{1.19}$$

We have used the fact that the Fourier coefficients of ϕ are real and therefore we can apply the identity (1.16). It follows that,

$$\begin{aligned}|\partial_w K(w,\tau)| &\leq C\|\phi\|^2_{\text{Lip}(\mathbb{T})}\frac{1}{|\phi(w)-\phi(\tau)|^{\alpha+1}}\\ &\leq C\|\phi\|^2_{\text{Lip}(\mathbb{T})}\|\phi^{-1}\|^{1+\alpha}_{\text{Lip}(\mathbb{T})}\frac{1}{|w-\tau|^{\alpha+1}}\cdot\end{aligned}$$

We can conclude by Lemma 1.1 and get the desired result. □

1.4.3 Basic integrals

This section presents some basic computations of few integrals that will appear later in the study of the linearized operator. But before going further into the details we shall recall some facts on the gamma function which emerges in a natural way in our computations. The function $\Gamma:\mathbb{C}\backslash(-\mathbb{N})\to\mathbb{C}$ refers to the gamma function which is the analytic continuation to the negative half plane of the usual gamma function defined on the positive half-plane $\big\{\text{Re}z>0\big\}$ by the integral representation

$$\Gamma(z)=\int_0^{+\infty}t^{z-1}\,e^{-t}dt.$$

It satisfies the relation

$$\Gamma(z+1)=z\,\Gamma(z), \quad \forall z\in\mathbb{C}\backslash(-\mathbb{N}). \tag{1.20}$$

Note that this function does not vanish and its poles $\{-n, n\in\mathbb{N}\}$ are simple and so the reciprocal gamma function $\frac{1}{\Gamma}$ is an entire function. There are some particular values of the gamma function that will be used later,

$$\Gamma(n+1)=n!, \quad \Gamma(1/2)=\sqrt{\pi}. \tag{1.21}$$

Now we shall introduce another related function called the digamma function which is nothing but the logarithmic derivative of the function gamma and often denoted by $\digamma$. It is given by

$$\digamma(x)=\frac{\Gamma'(x)}{\Gamma(x)}\cdot$$

For a future use we need the following identity,

$$\forall n\in\mathbb{N}, \quad \digamma(n+\frac{1}{2})=-\gamma-2\ln 2+2\sum_{k=0}^{n-1}\frac{1}{2k+1}\cdot \tag{1.22}$$

Now for $x \in \mathbb{R}$ we denote by $(x)_n$ the Pokhhammer's symbol defined by

$$(x)_n = \begin{cases} x(x+1)...(x+n-1), & \text{if} \quad n \geq 1, \\ 1, & \text{if} \quad n = 0. \end{cases} \tag{1.23}$$

Note that in the literature the above notation is replaced by $(x)^n$ which can introduce in our context a lot of confusion with the power x^n and for this reason we prefer not to use it.

It is obvious that

$$(x)_n = x\,(1+x)_{n-1}, \quad (x)_{n+1} = (x+n)\,(x)_n. \tag{1.24}$$

From the identity (1.20) we deduce the relations

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, \quad (x)_n = (-1)^n \frac{\Gamma(1-x)}{\Gamma(1-x-n)}, \tag{1.25}$$

provided all the quantities in the right terms are well-defined.

In the sequel we shall prove the following lemma which is the main result of this section.

Lemma 1.2 *Let $n \in \mathbb{N}$ and $\alpha \in (0,1)$. Then for any $w \in \mathbb{T}$ we have the following formulae.*

$$\fint_{\mathbb{T}} \frac{\tau^n}{|\tau - w|^\alpha} d\tau = \frac{\Gamma(1-\alpha)}{\Gamma^2(1-\alpha/2)} \frac{\left(\frac{\alpha}{2}\right)_{n+1}}{(1-\frac{\alpha}{2})_{n+1}} w^{n+1}. \tag{1.26}$$

$$\fint_{\mathbb{T}} \frac{(w-\tau)(w^n - \tau^n)}{|w-\tau|^{\alpha+2}} d\tau = \frac{\left(1+\frac{\alpha}{2}\right)\Gamma(1-\alpha)}{(2-\alpha)\Gamma^2(1-\frac{\alpha}{2})} \left(1 - \frac{\left(2+\frac{\alpha}{2}\right)_n}{\left(2-\frac{\alpha}{2}\right)_n}\right) w^{n+2}. \tag{1.27}$$

$$\fint_{\mathbb{T}} \frac{(\overline{w}-\overline{\tau})(\overline{w}^n - \overline{\tau}^n)}{|1-\tau|^{\alpha+2}} d\tau = -\frac{\Gamma(1-\alpha)}{2\Gamma^2(1-\frac{\alpha}{2})} \left(1 - \frac{\left(\frac{\alpha}{2}\right)_n}{(-\frac{\alpha}{2})_n}\right) \overline{w}^n. \tag{1.28}$$

Proof : We start with the change of variables $\tau = w\zeta$,

$$\begin{aligned} \fint_{\mathbb{T}} \frac{\tau^n}{|\tau - w|^\alpha} d\tau &= w^{n+1} \fint_{\mathbb{T}} \frac{\zeta^n}{|\zeta - 1|^\alpha} d\tau \\ &= w^{n+1} \frac{1}{2^{\alpha+1}\pi} \int_0^{2\pi} \frac{e^{i(n+1)\eta}}{|\sin(\eta/2)|^\alpha} d\eta. \end{aligned}$$

Again by the change of variables $\eta/2 \mapsto \eta$ one gets

$$\fint_{\mathbb{T}} \frac{\tau^n}{|\tau - w|^\alpha} d\tau = w^{n+1} \frac{1}{2^\alpha \pi} \int_0^\pi \frac{e^{2i(n+1)\eta}}{\sin^\alpha \eta} d\eta.$$

We shall now recall the following identity, see for instance [87, p.8] and [123, p.449].

$$\int_0^\pi \sin^x(\eta) e^{iy\eta} d\eta = \frac{\pi e^{i\frac{\pi y}{2}} \Gamma(x+1)}{2^x \Gamma(1+\frac{x+y}{2})\Gamma(1+\frac{x-y}{2})}, \quad \forall\, x > -1, \quad \forall\, y \in \mathbb{R}. \tag{1.29}$$

As it was pointed before the gamma function has no real zeros but simple poles located at $-\mathbb{N}$ and therefore the function $\frac{1}{\Gamma}$ admits an analytic continuation on $\mathbb{C}$. Apply this formula with $x = -\alpha$ and $y = 2(n+1)$ yields,

$$\frac{1}{2^\alpha \pi} \int_0^\pi \frac{e^{2i(n+1)\eta}}{\sin^\alpha \eta} d\eta = \frac{(-1)^{n+1}\Gamma(1-\alpha)}{\Gamma(n+2-\frac{\alpha}{2})\Gamma(-n-\frac{\alpha}{2})}. \tag{1.30}$$

It is easy to see that from the relations (1.25) we may write for any $n \in \mathbb{N}$,

$$\begin{aligned}\Gamma(1+n-\alpha/2) &= \Gamma(1-\alpha/2)\Big(1-\frac{\alpha}{2}\Big)_n\\ \Gamma(1-n-\alpha/2) &= (-1)^n\frac{\Gamma(1-\alpha/2)}{\Big(\frac{\alpha}{2}\Big)_n}.\end{aligned}$$

It follows that

$$\Gamma(1-n-\alpha/2)\Gamma(1+n-\alpha/2) = (-1)^n\Gamma^2\big(1-\frac{\alpha}{2}\big)\frac{\Big(1-\frac{\alpha}{2}\Big)_n}{\Big(\frac{\alpha}{2}\Big)_n}\cdot$$

By replacing n with $n+1$ we get

$$\Gamma(-n-\alpha/2)\Gamma(2+n-\alpha/2) = (-1)^{n+1}\Gamma^2\big(1-\frac{\alpha}{2}\big)\frac{\Big(1-\frac{\alpha}{2}\Big)_{n+1}}{\Big(\frac{\alpha}{2}\Big)_{n+1}}\cdot$$

Inserting this identity into (2.3.4) gives

$$\frac{1}{2^\alpha\pi}\int_0^\pi\frac{e^{2i(n+1)\theta}}{\sin^\alpha\theta}d\theta = \frac{\Gamma(1-\alpha)}{\Gamma^2(1-\frac{\alpha}{2})}\frac{\Big(\frac{\alpha}{2}\Big)_{n+1}}{\Big(1-\frac{\alpha}{2}\Big)_{n+1}}\cdot \tag{1.31}$$

Consequently

$$\fint_{\mathbb{T}}\frac{\tau^n}{|\tau-w|^\alpha}d\tau = \frac{\Gamma(1-\alpha)}{\Gamma^2(1-\frac{\alpha}{2})}\frac{\Big(\frac{\alpha}{2}\Big)_{n+1}}{\Big(1-\frac{\alpha}{2}\Big)_{n+1}}\,w^{n+1}.$$

This completes the proof of (1.26).

We intend now to compute the second integral. To this end we use a change of variable as before,

$$J_n \triangleq \fint_{\mathbb{T}}\frac{(w-\zeta)(w^n-\zeta^n)}{|w-\zeta|^{\alpha+2}}d\zeta = w^{n+2}\fint_{\mathbb{T}}\frac{(1-\zeta)(1-\zeta^n)}{|1-\zeta|^{\alpha+2}}d\zeta.$$

Using once again the change of variables $\zeta \mapsto e^{i\eta}$ and $\eta \mapsto 2\eta$ one gets

$$\begin{aligned}J_n &= \frac{w^{n+2}}{2\pi}\int_0^{2\pi}\frac{(1-e^{i\eta})(1-e^{in\eta})e^{i\eta}}{2^{\alpha+2}|\sin(\eta/2)|^{\alpha+2}}d\eta\\ &= \frac{w^{n+2}}{2^{\alpha+2}\pi}\int_0^\pi\frac{(1-e^{i2\theta})(1-e^{i2n\theta})e^{i2\theta}}{\big(\sin\theta\big)^{\alpha+2}}d\theta.\end{aligned}$$

Observe that

$$J_n = \frac{w^{n+2}}{2^{\alpha+1}i\pi}\int_0^\pi\frac{(e^{2i\theta}-e^{i2(n+1)\theta})e^{i\theta}}{\sin^{\alpha+1}\theta}d\theta$$

and therefore

$$J_n = \frac{w^{n+2}}{2^{\alpha+1}i\pi}\bigg(\int_0^\pi\Big(e^{i2\theta}-e^{i2(n+1)\theta}\Big)\frac{\cos\theta}{\sin^{\alpha+1}\theta}d\theta + i\int_0^\pi\frac{e^{i2\theta}-e^{i2(n+1)\theta}}{\sin^\alpha\theta}d\theta\bigg).$$

Integrating by parts implies

$$\int_0^\pi \Big(e^{i2\theta}-e^{i2(n+1)\theta}\Big)\frac{\cos\theta}{\sin^{\alpha+1}\theta}d\theta=\frac{2i}{\alpha}\int_0^\pi\frac{\big(e^{i2\theta}-(n+1)e^{i2(n+1)\theta}\big)}{\sin^\alpha\theta}d\theta.$$

Note that in this formula the contribution coming from the boundary terms is zero for $\alpha\in[0,1[$. Hence we get

$$J_n=\frac{w^{n+2}}{2^{\alpha+1}\pi}\bigg(\frac{2+\alpha}{\alpha}\int_0^\pi\frac{e^{i2\theta}}{\sin^\alpha\theta}d\theta-\frac{2(n+1)+\alpha}{\alpha}\int_0^\pi\frac{e^{i2(n+1)\theta}}{\sin^\alpha\theta}d\theta\bigg).$$

Combining this formula with the identity (1.31) gives

$$\begin{aligned}J_n&=w^{n+2}\frac{(2+\alpha)\Gamma(1-\alpha)}{2(2-\alpha)\Gamma^2(1-\frac{\alpha}{2})}-\frac{\Gamma(1-\alpha)}{\Gamma^2(1-\frac{\alpha}{2})}\frac{2n+2+\alpha}{2\alpha}\frac{\big(\frac{\alpha}{2}\big)_{n+1}}{\big(1-\frac{\alpha}{2}\big)_{n+1}}\\&=w^{n+2}\frac{(1+\frac{\alpha}{2})\Gamma(1-\alpha)}{(2-\alpha)\Gamma^2(1-\frac{\alpha}{2})}\bigg(1-\frac{1-\frac{\alpha}{2}}{1+\frac{\alpha}{2}}\frac{n+1+\frac{\alpha}{2}}{\frac{\alpha}{2}}\frac{\big(\frac{\alpha}{2}\big)_{n+1}}{\big(1-\frac{\alpha}{2}\big)_{n+1}}\bigg).\end{aligned}$$

By (1.24) we may transform this formula into,

$$J_n=w^{n+2}\frac{(1+\frac{\alpha}{2})\Gamma(1-\alpha)}{(2-\alpha)\Gamma^2(1-\frac{\alpha}{2})}\bigg(1-\frac{\big(2+\frac{\alpha}{2}\big)_n}{\big(2-\frac{\alpha}{2}\big)_n}\bigg).$$

Next we shall now move to the computation of the last integral (1.28),

$$Z_n\triangleq\fint_{\mathbb{T}}\frac{(\overline{w}-\overline{\tau})(\overline{w}^n-\overline{\tau}^n)}{|1-\tau|^{\alpha+2}}d\tau=\overline{w}^n\fint_{\mathbb{T}}\frac{(1-\overline{\zeta})(1-\overline{\zeta}^n)}{|1-\zeta|^{\alpha+2}}d\zeta.$$

Making a standard change of variables as for the preceding integral we obtain

$$\begin{aligned}Z_n&=\frac{\overline{w}^n}{2\pi}\int_0^{2\pi}\frac{(1-e^{-i\eta})(1-e^{-in\eta})e^{i\eta}}{2^{\alpha+2}|\sin(\eta/2)|^{\alpha+2}}d\eta\\&=\frac{\overline{w}^n}{2^{\alpha+2}\pi}\int_0^\pi\frac{(1-e^{-i2\eta})(1-e^{-i2n\eta})e^{i2\eta}}{\sin^{\alpha+2}\eta}d\eta\\&=\frac{i\overline{w}^n}{2^{\alpha+1}\pi}\int_0^\pi\frac{(1-e^{-i2n\theta})e^{i\theta}}{\sin^{\alpha+1}\theta}d\theta\\&=\frac{i\overline{w}^n}{2^{\alpha+1}\pi}\bigg(\int_0^\pi\frac{\cos\theta(1-e^{-i2n\theta})}{\sin^{\alpha+1}\theta}d\theta+i\int_0^\pi\frac{(1-e^{-i2n\theta})}{\sin^\alpha\theta}d\theta\bigg).\end{aligned}$$

Integrating by parts gives

$$\int_0^\pi\frac{\cos\theta(1-e^{-i2n\theta})}{\sin^{\alpha+1}\theta}d\theta=\frac{2in}{\alpha}\int_0^\pi\frac{e^{-i2n\theta}}{\sin^\alpha\theta}d\theta.$$

This implies that

$$Z_n=-\frac{\overline{w}^n}{2^{\alpha+1}\pi}\bigg(\int_0^\pi\frac{1}{\sin^\alpha\theta}d\theta+\frac{2n-\alpha}{\alpha}\int_0^\pi\frac{e^{-i2n\theta}}{\sin^\alpha\theta}d\theta\bigg).$$

Using once again (1.31) and (1.24) we obtain

$$\begin{aligned}Z_n&=-\overline{w}^n\frac{\Gamma(1-\alpha)}{2\Gamma^2(1-\frac{\alpha}{2})}\bigg(1+\frac{n-\frac{\alpha}{2}}{\frac{\alpha}{2}}\frac{(\frac{\alpha}{2})_n)}{(1-\frac{\alpha}{2})_n}\bigg)\\&=-\overline{w}^n\frac{\Gamma(1-\alpha)}{2\Gamma^2(1-\frac{\alpha}{2})}\bigg(1-\frac{(\frac{\alpha}{2})_n}{(-\frac{\alpha}{2})_n}\bigg).\end{aligned}$$

Note that we have used the following fact which can be deduced easily from (1.31) by conjugation,

$$\int_0^\pi \frac{e^{-i2n\theta}}{\sin^\alpha\theta}d\theta = \int_0^\pi \frac{e^{i2n\theta}}{\sin^\alpha\theta}d\theta$$

and therefore the proof of the lemma is now completed. □

1.5 Elliptic patches

Given a simply connected domain, to check whether or not it is a rotating patch can be done through the equation of Proposition 1.4 provided that a parametric representation of the boundary is known (for example the one given by the conformal mapping) and the computations of the integral term are feasible. In what follows we shall concretize this program for some elementary domains. We shall prove that the ellipses never rotate except for the degenerate case where they coincide with discs. We point out this result was recently shown in [17] and we will give here a flexible proof with less computations.

Proposition 1.5 *The following holds true*

1. *The discs are rotating patches for any* $\Omega \in \mathbb{R}$.
2. *The ellipses are not rotating patches.*

Proof :

$(\mathbf{1})$ Recall from (1.15) that the conformal mapping of a rotating domain must satisfy the equation

$$G(\Omega, \phi(w)) = 0, \quad \forall w \in \mathbb{T}$$

To check whether or not the unit disc is a solution, it suffices to prove that

$$G(\Omega, \mathrm{Id}) = 0.$$

It is easy to see that,

$$\begin{aligned} G(\Omega, \mathrm{Id})(w) &= \mathrm{Im}\Big\{\Big(\Omega w - C_\alpha \fint_{\mathbb{T}} \frac{d\tau}{|w-\tau|^\alpha}\Big)\frac{1}{w}\Big\} \\ &= -C_\alpha \mathrm{Im}\Big\{\fint_{\mathbb{T}} \frac{d\tau}{w|w-\tau|^\alpha}\Big\}. \end{aligned}$$

Using the formula (1.26) with $n = 0$ we may conclude that for any $\Omega \in \mathbb{R}$,

$$G(\Omega, \mathrm{Id}) = 0.$$

We observe that this result is known and expected because the disc corresponds to a stationary solution for (1.1) and is invariant by rotation.

$(\mathbf{2})$ By translation, dilation and rotation we can assume that the ellipse $\mathcal{E}$ is parametrized by the conformal mapping

$$\phi_Q : w \in \mathbb{T} \mapsto w + Q\overline{w}, \quad \text{with} \quad Q = \frac{a-b}{a+b} \in (0,1)$$

where a and b denote the major and minor axes, respectively. This map sends conformally the exterior of the unit disc to the exterior of the ellipse. Performing straightforward computations leads in view of (1.15) to

$$G\big(\Omega,\phi_Q\big)(w) \quad = \quad -\mathrm{Im}\Big\{2Q\Omega w^2+C_\alpha\big(\overline{w}-Qw\big)\fint_{\mathbb{T}}\frac{\big(1-Q\overline{\tau}^2\big)d\tau}{|w-\tau+Q(\overline{w}-\overline{\tau})|^\alpha}\Big\}.$$

By using the identity

$$\big|z+Q\overline{z}\big|^2=(1+Q^2)|z|^2+2Q\,\mathrm{Re}(z^2),\quad \forall z\in\mathbb{C},$$

one gets

$$G\big(\Omega,\phi_Q\big)(w)=-\mathrm{Im}\Big\{2Q\Omega w^2+C_\alpha\big(\overline{w}-Qw\big)\fint_{\mathbb{T}}\frac{\big(1-Q\overline{\tau}^2\big)d\tau}{\Big[(1+Q^2)|w-\tau|^2+2Q\mathrm{Re}\big\{(w-\tau)^2\big\}\Big]^{\alpha/2}}\Big\}.$$

Making the change of variables $\tau=w\zeta$ and using the identity

$$\big(1-z\big)^2=-z|1-z|^2,\quad \forall\, z\in\mathbb{T}$$

we find

$$G\big(\Omega,\phi_Q\big)(w)=-\mathrm{Im}\Big\{2Q\Omega w^2+\frac{C_\alpha}{(1+Q^2)^{\frac{\alpha}{2}}}\big(1-Qw^2\big)\fint_{\mathbb{T}}\frac{\big(1-Q\overline{w}^2\overline{\zeta}^2\big)d\zeta}{|1-\zeta|^\alpha\Big[1-\frac{2Q}{1+Q^2}\mathrm{Re}\{w^2\zeta\}\Big]^{\alpha/2}}\Big\}.$$

We shall transform the last integral term as follows,

$$\fint_{\mathbb{T}}\frac{1-Q\overline{w}^2\overline{\zeta}^2}{|1-\zeta|^\alpha\Big[1-\frac{2Q}{1+Q^2}\mathrm{Re}\{w^2\zeta\}\Big]^{\alpha/2}}d\zeta=J(w)-Q\overline{w}^2\overline{J(w)},$$

with

$$J(w)\triangleq\fint_{\mathbb{T}}\frac{d\zeta}{|1-\zeta|^\alpha\big(1-\frac{2Q}{1+Q^2}\mathrm{Re}\{w^2\zeta\}\big)^{\alpha/2}}.$$

Therefore we get,

$$\begin{aligned}G\big(\Omega,\phi_Q\big)(w) \quad &= \quad -\mathrm{Im}\Big\{2Q\Omega w^2+\frac{C_\alpha}{(1+Q^2)^{\frac{\alpha}{2}}}\Big(J(w)+Q^2\overline{J(w)}-Q\Big[w^2J(w)+\overline{w}^2\overline{J(w)}\Big]\Big)\Big\}\\ &= \quad -\mathrm{Im}\Big\{2Q\Omega w^2+C_\alpha\frac{1-Q^2}{(1+Q^2)^{\frac{\alpha}{2}}}J(w)\Big\}.\end{aligned}\tag{1.32}$$

Since $\big|\frac{2Q}{1+Q^2}\mathrm{Re}\{w^2\zeta\}\big|<1$ then we can use the Taylor series

$$\Big(1-\frac{2Q}{1+Q^2}\mathrm{Re}\{w^2\zeta\}\Big)^{-\alpha/2}=\sum_{n=0}^{\infty}2^nA_n\big(\mathrm{Re}\{w^2\zeta\}\big)^n,$$

with

$$A_n=\frac{(\alpha/2)_n}{n!}\Big(\frac{Q}{1+Q^2}\Big)^n,\quad \forall\, n\in\mathbb{N}.$$

Consequently we get

$$\begin{aligned} J(w) &= \sum_{n=0}^{\infty} 2^n A_n \fint_{\mathbb{T}} \frac{\big(\mathrm{Re}\{w^2\zeta\}\big)^n}{|1-\zeta|^\alpha} d\zeta \\ &= a_\alpha + \sum_{n=1}^{\infty} A_n \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} w^{2(n-2k)} \fint_{\mathbb{T}} \frac{\zeta^{n-2k}}{|1-\zeta|^\alpha} d\zeta. \end{aligned}$$

By the Lemma 2.1 the coefficient a_α is real and therefore it does not contribute in $\mathrm{Im} J(w)$. Our goal now is to compute the coefficients of w^4 and $\overline{w}^4$ of the function between the bracket in (1.32), denoted by B_4 and B_{-4}, respectively. First we observe that the coefficient B_4 can be obtained by summing over the set

$$\Big\{n \geq 1, 0 \leq k \leq n \setminus n - 2k = 2\Big\} = \Big\{n \geq 1, k \geq 0 \setminus n = 2k + 2\Big\}.$$

This is equivalent to write

$$\begin{aligned} B_4 &= \sum_{k=0}^{\infty} A_{2k+2} \begin{pmatrix} 2k+2 \\ k \end{pmatrix} \fint_{\mathbb{T}} \frac{\zeta^2 d\zeta}{|1-\zeta|^\alpha} \\ &= \sum_{k=1}^{\infty} A_{2k} \begin{pmatrix} 2k \\ k-1 \end{pmatrix} a_2, \qquad\qquad a_2 \triangleq \fint_{\mathbb{T}} \frac{\zeta^2 d\zeta}{|1-\zeta|^\alpha}. \end{aligned}$$

Next we shall compute the coefficient of $\overline{w}^4$ denoted by B_{-4}. This may be done by summing over the set

$$\Big\{n \geq 1, 0 \leq k \leq n \setminus n - 2k = -2\Big\} = \Big\{n \geq 1, k \geq 2 \setminus n = 2k - 2\Big\}.$$

Hence by change of variables,

$$\begin{aligned} B_{-4} &= \sum_{k=2}^{\infty} A_{2k-2} \begin{pmatrix} 2k-2 \\ k \end{pmatrix} \fint_{\mathbb{T}} \frac{\xi^{-2} d\xi}{|1-\xi|^\alpha} \\ &= \sum_{k=1}^{\infty} A_{2k} \begin{pmatrix} 2k \\ k+1 \end{pmatrix} \fint_{\mathbb{T}} \frac{d\zeta}{|1-\zeta|^\alpha} \\ &\triangleq \sum_{k=1}^{\infty} A_{2k} \begin{pmatrix} 2k \\ k-1 \end{pmatrix} a_0. \end{aligned}$$

But, in view of Lemma 2.1, one has

$$\frac{a_2}{a_0} = \frac{(2+\alpha)(4+\alpha)}{(4-\alpha)(6-\alpha)} \neq 1 \quad \text{for } \alpha \neq 1.$$

Thus $B_4 \neq B_{-4}$ and therefore the coefficient of $w^4 - \overline{w}^4$ of $G(\Omega, \phi_Q)(w)$ does not vanish. It follows that the equation $G(\Omega, \phi_Q)(w) = 0, \forall w \in \mathbb{T}$ is not true for any Ω. This concludes the proof of the desired result. □

1.6 General statement

In this section we shall give a more precise statement of Theorem 1.1. In particular we shall give a description of the conformal mapping which parametrizes the rotating patches close to the unit disc.

Theorem 1.3 *Let $\alpha \in]0,1[$ and $m \in \mathbb{N}^*\backslash\{1\}$. Then there exists $a > 0$ and two continuous functions $\Omega : (-a,a) \to \mathbb{R}$, $\phi : (-a,a) \to C^{2-\alpha}(\mathbb{T})$ satisfying $\Omega(0) = \Omega_m^\alpha$, $\phi(0) = \mathrm{Id}$, such that $(\phi_s)_{-a<s<a}$ is a one-parameter non trivial solution of the equation* (1.8)*, where*

$$\Omega_m^\alpha \triangleq \frac{\Gamma(1-\alpha)}{2^{1-\alpha}\Gamma^2(1-\frac{\alpha}{2})}\bigg(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})} - \frac{\Gamma(m+\frac{\alpha}{2})}{\Gamma(m+1-\frac{\alpha}{2})}\bigg),$$

Moreover, ϕ_s admits the expansion

$$\phi_s(w) = w\Big(1 + s\frac{1}{w^m} + s\sum_{n\geq 2} a_{nm-1}(s)\frac{1}{w^{nm}}\Big), \quad \forall\, w \in \mathbb{T},$$

and it is conformal on $\mathbb{C}\backslash\mathbb{D}$ and the complement D_s of $\phi_s\big(\mathbb{C}\backslash\mathbb{D}\big)$ is an $m-$fold rotating patch with the angular velocity $\Omega(s)$. In addition, the boundary of this patch belongs to the class $C^{2-\alpha}$.

• **Outline of the proof**. The proof of this theorem will be divided into several steps. The main key is the Crandall Rabinowitz Theorem, sometimes denoted by C-R, which requires one to check many properties for the linear and the nonlinear functionals of the equation (1.8) defining the V-states. Firstly, we shall check the regularity assumptions that will be separated into weak and strong ones. Secondly, we will conduct a spectral study of the linearized operator around the trivial solution. In this context, we are able to describe the complete bifurcation set made of the values Ω such that the linearized operator is Fredholm with one-dimensional kernel. We shall also check in this section the transversality assumption of the C-R Theorem. In the last step, we give the complete proof for the existence of the V-states and check their m-fold structure.

1.7 Regularity of the Functional F

This section is devoted to the study of the regularity assumptions stated in the C-R Theorem. The object that we shall study is the nonlinear functional G introduced in (1.15) and given by

$$G(\Omega,\phi)(w) \triangleq \mathrm{Im}\bigg\{\bigg(\Omega\phi(w) - C_\alpha \fint_{\mathbb{T}} \frac{\phi'(\tau)}{|\phi(w)-\phi(\tau)|^\alpha}d\tau\bigg)\overline{w}\,\overline{\phi'(w)}\bigg\}, \quad \forall\, w \in \mathbb{T}.$$

Because we are interested in the bifurcation from the disc (corresponding to $\phi = \mathrm{Id}$), it is more convenient to make a translation and study the bifurcation from zero. To this end, we introduce the function F defined by

$$F(\Omega, f)(w) = G(\Omega, w + f(w)), \quad \forall w \in \mathbb{T}.$$

In order to apply C-R Theorem we need first to fix the function spaces and check the regularity of the functional F with respect to these spaces. We should look for Banach spaces X and Y such that $F : \mathbb{R} \times X \mapsto Y$ is well-defined and satisfies the assumptions of Theorem **??**. These spaces will be defined in the spirit of the work done for the incompressible Euler equations [65]. They are given by,

$$X = \Big\{f \in C^{2-\alpha}(\mathbb{T}),\ f(w) = \sum_{n\geq 0} b_n\overline{w}^n, b_n \in \mathbb{R},\ w \in \mathbb{T}\Big\}$$

and

$$Y = \Big\{g \in C^{1-\alpha}(\mathbb{T}),\ g(w) = i\sum_{n\geq 1} g_n\big(w^n - \overline{w}^n\big), g_n \in \mathbb{R},\ w \in \mathbb{T}\Big\}.$$

For $r \in (0,1)$ we denote by B_r the open ball of X with center 0 and radius r,

$$B_r = \Big\{ f \in X, \quad \|f\|_{C^{2-\alpha}} \leq r \Big\}.$$

It is straightforward that for any $f \in B_r$ the function $w \mapsto \phi(w) = w + f(w)$ is conformal on $\mathbb{C}\backslash\overline{\mathbb{D}}$. Moreover according to Kellog-Warshawski result [122], the boundary of $\phi(\mathbb{C}\backslash\overline{\mathbb{D}})$ is a Jordan curve of class $C^{2-\alpha}$. This gives the proof of the last result of Theorem 1.3 provided that the regularity of ϕ is shown. Note that we can prove the regularity of the boundary without making appeal to the result [122]. We just look for the conformal parametrization $\theta \mapsto \phi(e^{i\theta})$ which is regular and prove that it belongs to $C^{2-\alpha}$. This last fact is equivalent to $\phi \in C^{2-\alpha}(\mathbb{T})$.

1.7.1 Weak regularity

Our objective is to prove that the functional F is well-defined and admits Gâteaux derivatives for any given direction. More precisely, we shall prove the following result.

Proposition 1.6 *For any $r \in (0,1)$ the following holds true.*

1. *$F : \mathbb{R} \times B_r \to Y$ is well-defined.*
2. *For each point $(\Omega, f) \in \mathbb{R} \times B_r$, the Gâteaux derivative of F, $\partial_f F(\Omega, f) : X \to Y$ exists and belongs to $\mathcal{L}(X,Y)$*

Proof : $(\mathbf{1})$ First, because the space $C^{1-\alpha}(\mathbb{T})$ is an algebra, it is clear that the first part of the functional G given by, $w \mapsto \Omega\, \phi(w)\, \overline{w}\, \overline{\phi'(w)}$ belongs to $C^{1-\alpha}(\mathbb{T})$. To prove that the second term of G belongs to $C^{1-\alpha}(\mathbb{T})$ it suffices to check that

$$S(\phi) : w \mapsto \fint_{\mathbb{T}} \frac{\phi'(\tau)}{|\phi(w) - \phi(\tau)|^\alpha} d\tau \in C^{1-\alpha}(\mathbb{T}).$$

This follows immediately from Corollary 1.2. Therefore it remains to check that the Fourier coefficients of $G(\Omega, \phi)$ belong to $i\mathbb{R}$. By the assumption, the Fourier coefficients of $\phi = \mathrm{Id} + f$ are real and thus the coefficients of $\overline{\phi'}$ are real too. Now using the stability of this property under the multiplication and the conjugation we deduce that the Fourier coefficients of $w \mapsto \Omega\phi(w)\overline{\phi'(w)}\overline{w}$ are real. To complete the proof we shall check that the Fourier coefficients of $S(\phi)$ are also real for every $f \in B_r$. From the regularity of $\phi \in C^{1-\alpha}(\mathbb{T})$ we can pointwise expand this function into its Fourier series, that is,

$$S(\phi)(w) = \sum_{n \in \mathbb{Z}} a_n w^n, \quad a_n = \fint_{\mathbb{T}} \frac{S(\phi)(w)}{w^{n+1}} dw = \fint_{\mathbb{T}} \fint_{\mathbb{T}} \frac{\phi'(\tau)}{|\phi(\tau) - \phi(w)|^\alpha} d\tau \frac{dw}{w^{n+1}}.$$

This coefficient can also be written in the form

$$a_n = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{\phi'(e^{i\theta}) e^{i\theta} e^{-in\eta}}{|\phi(e^{i\theta}) - \phi(e^{i\eta})|^\alpha} d\theta\, d\eta.$$

By taking the conjugate of a_n and using the properties

$$\overline{\phi(e^{i\theta})} = \phi(e^{-i\theta}), \quad \overline{\phi'(e^{i\theta})} = \phi'(e^{-i\theta}) \quad \text{and} \quad |z| = |\overline{z}|$$

one may obtain by change of variables

$$\begin{aligned}\overline{a_n} &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{\phi'(e^{-i\theta})e^{-i\theta}e^{in\eta}}{|\phi(e^{-i\theta})-\phi(e^{-i\eta})|^\alpha}d\theta\, d\eta\\ &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{\phi'(e^{i\theta})e^{i\theta}e^{-in\eta}}{|\phi(e^{i\theta})-\phi(e^{i\eta})|^\alpha}d\theta\, d\eta\\ &= a_n.\end{aligned}$$

Consequently the Fourier coefficients of $S(\phi)$ are real and therefore $F(\Omega, f)$ belongs to Y.

(**2**) We shall compute the Gâteaux derivative of F at the point $f \in B_r$ in the direction $h \in X$. A refined analysis concerning its connection with Fréchet derivative will be developed in the next section. The Gâteaux derivative of $\partial_f F(\Omega, f)h$ is defined through the formula,

$$\begin{aligned}\partial_f F(\Omega, f)h(w) &= \lim_{t\to 0}\frac{F(\Omega, f(w)+th(w)) - F(\Omega, f(w))}{t}\\ &= \frac{d}{dt}\Big|_{t=0} F(\Omega, f+th)(w).\end{aligned}$$

This limit is taken in the strong topology of $C^{1-\alpha}(\mathbb{T})$. Thus we shall first prove the existence of this limit for every point $w \in \mathbb{T}$ and after check that this limit exists in $C^{1-\alpha}(\mathbb{T})$.
With the notation $\phi = \mathrm{Id} + f$,

$$\begin{aligned}\partial_f F(\Omega, f)h(w) &= \frac{d}{dt}\Big|_{t=0} F(\Omega, f+th)(w)\\ &= \Omega\,\mathrm{Im}\Big\{\phi(w)\,\overline{w}\,\overline{h'(w)} + h(w)\,\overline{w}\,\overline{\phi'(w)}\Big\}\\ &\quad - C_\alpha\,\mathrm{Im}\Big\{S(\phi(w))\overline{w}\overline{h'(w)} + \overline{w}\,\overline{\phi'(w)}\frac{d}{dt}\Big|_{t=0} S(\phi+th)(w)\Big\}\\ &\triangleq \mathcal{L}(f)(h(w)). \end{aligned} \tag{1.33}$$

We shall make use of the following identity : let $A \in \mathbb{C}^\star$, $B \in \mathbb{C}$, $\alpha \in \mathbb{R}$ and introduce the function $K : t \mapsto |A + Bt|^\alpha$ which is smooth close to zero, then we have

$$K'(0) = \alpha|A|^{\alpha-2}\mathrm{Re}(\overline{A}B). \tag{1.34}$$

Combining this formula with few easy computations one gets

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} S(\phi+th)(w) &= \fint_{\mathbb{T}}\frac{h'(\tau)}{|\phi(w)-\phi(\tau)|^\alpha}d\tau - \frac{\alpha}{2}\fint_{\mathbb{T}}\frac{\big(\phi(w)-\phi(\tau)\big)\big(\overline{h(w)}-\overline{h(\tau)}\big)}{|\phi(w)-\phi(\tau)|^{\alpha+2}}\phi'(\tau)d\tau\\ &\quad - \frac{\alpha}{2}\fint_{\mathbb{T}}\frac{\big(\overline{\phi(w)}-\overline{\phi(\tau)}\big)\big(h(w)-h(\tau)\big)}{|\phi(w)-\phi(\tau)|^{\alpha+2}}\phi'(\tau)d\tau\\ &\triangleq A(\phi,h)(w) - \frac{\alpha}{2}\Big(B(\phi,h)(w) + C(\phi,h)(w)\Big).\end{aligned} \tag{1.35}$$

Therefore we obtain from (1.33) the identity

$$\begin{aligned}\mathcal{L}(f)(h)(w) &= \mathrm{Im}\Big\{\Omega\left[\phi(w)\,\overline{w}\,\overline{h'(w)} + h(w)\,\overline{w}\,\overline{\phi'(w)}\right] - C_\alpha\, S(\phi(w))\,\overline{w}\,\overline{h'(w)}\Big\}\\ &\quad - C_\alpha\,\mathrm{Im}\Big\{\overline{w}\,\overline{\phi'(w)}\Big[A(\phi,h)(w) - \frac{\alpha}{2}\Big(B(\phi,h)(w)+C(\phi,h)(w)\Big)\Big]\Big\}.\end{aligned} \tag{1.36}$$

Set

$$\mathcal{L}_1(f)h(w) \triangleq \text{Im}\Big\{\Omega\left[\phi(w)\,\overline{w}\,\overline{h'(w)}+h(w)\,\overline{w}\,\overline{\phi'(w)}\right]-C_\alpha\,S(\phi(w))\,\overline{w}\,\overline{h'(w)}\Big\}$$

Since $C^{1-\alpha}(\mathbb{T})$ is an algebra and using some classical Hölder embeddings, we get

$$\|\mathcal{L}_1(f)h\|_{C^{1-\alpha}((T)}\lesssim\|\phi\|_{C^{2-\alpha}}\|h\|_{C^{2-\alpha}}+\|S(\phi)\|_{C^{1-\alpha}}\|h\|_{C^{2-\alpha}}.$$

To estimate $S(\phi)$ we use Corollary 1.2 combined with the estimate $\|\phi\|_{\text{Lip}}+\|\phi^{-1}\|_{\text{Lip}}\le C(r)$. Therefore

$$\|S(\phi)\|_{C^{1-\alpha}(\mathbb{T})}\le C.$$

It follows that

$$\|\mathcal{L}_1(f)h\|_{C^{1-\alpha}((T)}\le C\|h\|_{C^{2-\alpha}}. \tag{1.37}$$

Now using once again Corollary 1.2 we get that $A(\phi,h)\in C^{1-\alpha}$ and

$$\begin{aligned}\|A(\phi,h)\|_{C^{1-\alpha}(\mathbb{T})}&\le C\|h'\|_{L^\infty}\\&\le C\|h\|_{C^{2-\alpha}(\mathbb{T})}.\end{aligned} \tag{1.38}$$

So, it remains to show that $B(f,h)$ and $C(f,h)$ are of class $C^{1-\alpha}\big(\mathbb{T}\big)$. For this end we set

$$K_1(w,\tau)\triangleq\frac{\big(\phi(w)-\phi(\tau)\big)\big(\overline{h(w)}-\overline{h(\tau)}\big)}{|\phi(w)-\phi(\tau)|^{\alpha+2}}.$$

Clearly we have for $\tau\neq w\in\mathbb{T}$,

$$\begin{aligned}|K_1(w,\tau)|&\le\frac{\|\phi\|_{\text{Lip}}\|h\|_{\text{Lip}}}{|w-\tau|^\alpha}\\&\le C\frac{\|h\|_{C^{2-\alpha}(\mathbb{T})}}{|w-\tau|^\alpha}.\end{aligned} \tag{1.39}$$

Moreover, in view of the formula (1.16) we readily obtain

$$\begin{aligned}\partial_w K_1(w,\tau)=\phi'(w)&\frac{\overline{h(w)}-\overline{h(\tau)}}{|\phi(w)-\phi(\tau)|^{\alpha+2}}-\frac{\overline{h'(w)}}{w^2}\frac{\phi(w)-\phi(\tau)}{|\phi(w)-\phi(\tau)|^{\alpha+2}}\\&-\frac{\alpha+2}{2}\bigg(\frac{\phi'(w)}{|\phi(w)-\phi(\tau)|^{\alpha+2}}-\frac{\overline{\phi'(w)}}{w^2}\frac{\big(\phi(w)-\phi(\tau)\big)^2}{|\phi(w)-\phi(\tau)|^{\alpha+4}}\bigg)\Big(\overline{h(w)}-\overline{h(\tau)}\Big).\end{aligned}$$

Therefore one has

$$\begin{aligned}\big|\partial_w K_1(w,\tau)\big|&\le C\frac{\|\phi'\|_{L^\infty}\|h'\|_{L^\infty}}{|w-\tau|^{\alpha+1}}\\&\le C\frac{\|h\|_{C^{2-\alpha}(\mathbb{T})}}{|w-\tau|^{1+\alpha}}.\end{aligned} \tag{1.40}$$

Hence, combining the inequalities (1.39) and (1.40) with Lemma 1.1 we get

$$\begin{aligned}\|B(\phi,h)\|_{C^{1-\alpha}(\mathbb{T})}&\le C\|h\|_{C^{2-\alpha}(\mathbb{T})}\|\phi'\|_{L^\infty(\mathbb{T})}\\&\le C\|h\|_{C^{2-\alpha}(\mathbb{T})}.\end{aligned} \tag{1.41}$$

To estimate the last term $C(\phi,h)$ we observe that

$$C(\phi,h)(w)=\fint_{\mathbb{T}}\overline{K_1(w,\tau)}\phi'(\tau)d\tau$$

and consequently similar proof of the estimate (1.41) allows to get,

$$\|C(\phi,h)\|_{C^{1-\alpha}(\mathbb{T})}\le C\|h\|_{C^{2-\alpha}(\mathbb{T})}.$$

By putting together this estimate with (1.36), (1.37), (1.38) and (1.41) one concludes

$$\|\mathcal{L}(f)h\|_{C^{1-\alpha}(\mathbb{T})}\le C\|h\|_{C^{2-\alpha}(\mathbb{T})}.$$

This means that $\mathcal{L}(f)\in\mathcal{L}(X,Y)$. To achieve the proof it remains to check that the convergence in (1.33) towards $\mathcal{L}(f)(h)$ occurs in the strong topology of $C^{1-\alpha}(\mathbb{T})$. The convergence of the quadratic terms containing the parameter Ω can be easily obtained from the algebra structure of $C^{1-\alpha}(\mathbb{T})$. Therefore the problem reduces to verify only the convergence in the formula (1.35). We shall check only the convergence for the term involving $A(\phi,h)$ and the analysis for the other terms leading to $B(\phi,h)$ and $C(\phi,h)$ is quite similar and we omit here the details. We start with showing

$$\lim_{t\to0}\fint_{\mathbb{T}}\frac{h'(\tau)}{|\{\phi(w)-\phi(\tau)\}+t\{h(w)-h(\tau)\}|^\alpha}d\tau=\fint_{\mathbb{T}}\frac{h'(\tau)}{|\phi(w)-\phi(\tau)|^\alpha}d\tau\quad\text{in}\quad C^{1-\alpha}.$$

Set

$$\begin{aligned}K(t,w,\tau)&=\frac{1}{|\{\phi(w)-\phi(\tau)\}+t\{h(w)-h(\tau)\}|^\alpha}-\frac{1}{|\phi(w)-\phi(\tau)|^\alpha}\\&\triangleq g(t,w,\tau)-g(0,w,\tau).\end{aligned}$$

Then according to Lemma 1.1, the convergence happens provided that

$$\big|K(t,w,\tau)\big|\le C|t|\frac{1}{|w-\tau|^\alpha},\quad\big|\partial_wK(t,w,\tau)\big|\le C|t|\frac{1}{|w-\tau|^{1+\alpha}}.$$

Let $t>0$ such that $t||h||_{\text{Lip}(\mathbb{T})}\le\frac12$ then

$$\begin{aligned}\big|K(t,w,\tau)\big|&\le C\frac{\Big||\{\phi(w)-\phi(\tau)\}+t\{h(w)-h(\tau)\}|^\alpha-|\{\phi(w)-\phi(\tau)\}|^\alpha\Big|}{|\tau-w|^{2\alpha}}\\&\le C|t|\|h\|_{\text{Lip}(\mathbb{T})}|\tau-w|\frac{|\tau-w|^{\alpha-1}}{|\tau-w|^{2\alpha}}\\&\le C|t|\|h\|_{\text{Lip}(\mathbb{T})}\frac{1}{|\tau-w|^\alpha},\end{aligned}$$

where we have used the inequality : for $\alpha\in(0,1)$, there exists $C_\alpha>0$, such that

$$|a^\alpha-b^\alpha|\le C_\alpha\frac{|a-b|}{a^{1-\alpha}+b^{1-\alpha}},\quad\forall a,b\in\mathbb{R}_+^*.\tag{1.42}$$

To estimate $\partial_wK(t,w,\tau)$ we shall use the Mean value Theorem,

$$K(t,w,\tau)=\int_0^t\partial_sg(s,w,\tau)ds$$

and therefore

$$\big|\partial_w K(t,w,\tau)\big| \le \int_0^t \big|\partial_w \partial_s g(s,w,\tau)\big| ds,$$

with

$$g(t,w,\tau) = \frac{1}{|\{\phi(w)-\phi(\tau)\} + t\{h(w)-h(\tau)\}|^\alpha}.$$

Using (1.19) leads to

$$\begin{aligned}\partial_w g(t,w,\tau) = \frac{-\alpha}{2}&\frac{g(t,w,\tau)}{|\phi(w)-\phi(\tau)+t\big(h(w)-h(\tau)\big)|^2}\times \\ &\Big\{\big(\phi'(w)+th'(w)\big)\Big(\overline{\phi(w)-\phi(\tau)}+t\overline{h(w)-h(\tau)}\Big) \\ &-\frac{\overline{\phi'(w)+th'(w)}}{w^2}\Big(\phi(w)-\phi(\tau)+t\big(h(w)-h(\tau)\big)\Big)\Big\}.\end{aligned}$$

Using straightforward computations yield for any $s \in [0,t]$,

$$\Big|\partial_s \partial_w g(s,w,\tau)\Big| \le C\frac{1}{|w-\tau|^{1+\alpha}}.$$

Hence we get

$$\big|\partial_w K(t,w,\tau)\big| \le C|t|\frac{1}{|w-\tau|^{1+\alpha}}.$$

This completes the proof of the estimate of the kernel and the required statement follows immediately. □

1.7.2 Strong regularity

In this subsection we shall discuss the existence of Fréchet derivative of F and prove that F is continuously differentiable on the domain $\mathbb{R}\times B_r$. More precisely, we shell establish the following result.

Proposition 1.7 *For any $r \in (0,1)$ the following holds true.*

1. *$F:\mathbb{R}\times B_r \to Y$ is of class C^1.*
2. *The partial derivative $\partial_\Omega \partial_f F:\mathbb{R}\times B_r \to \mathcal{L}(X,Y)$ exists and is continuous.*

Proof : $(\mathbf{1})$ This amounts to showing that the partial derivatives $\partial_\Omega F$ and $\partial_f F$ in the Gâteaux sense exist and are continuous. For the first derivative, we observe the the linear dependence of F on Ω allows to get,

$$\partial_\Omega F(\Omega,f)(w) \quad = \quad \mathrm{Im}\Big\{\overline{w}\,\phi(w)\,\overline{\phi'(w)}\Big\}.$$

Obviously this is polynomial on ϕ and ϕ' and therefore it is continuous in the strong topology of X. The next step is to prove that for given Ω, $\partial_f F\big(\Omega,f\big)$ is continuous as a function of f taking values in the space of bounded linear operators from X to Y. In other words, we will show that, for a fixed $f,g \in B_r$,

$$\|\partial_f F(\Omega,f)(h) - \partial_g F(\Omega,g)(h)\|_{C^{1-\alpha}(\mathbb{T})} \le C\|f-g\|_{C^{2-\alpha}(\mathbb{T})}\|h\|_{C^{2-\alpha}(\mathbb{T})}. \qquad (1.43)$$

Now because $C^{1-\alpha}$ is an algebra and from (1.35) and (1.36) the problem reduces to show the required inequality for the quantities $\mathcal{L}_1(f)h$, $A(\phi,h)$, $B(\phi,h)$ and $C(\phi,h)$. The crucial tool for this task is Lemma 1.1 which will be frequently used here. We shall start with proving the estimate

$$\|\mathcal{L}_1(f)h-\mathcal{L}_1(g)h\|_{C^{1-\alpha}}\leq C\|f-g\|_{C^{2-\alpha}(\mathbb{T})}\|h\|_{C^{2-\alpha}(\mathbb{T})}.$$

For this end , it sufficient to establish that

$$\|S(\phi)-S(\psi)\|_{C^{1-\alpha}}\leq C\|f-g\|_{C^{2-\alpha}(\mathbb{T})},$$

with $\phi=\mathrm{Id}+f$ and $\psi=\mathrm{Id}+g$. Write

$$\begin{aligned}S(\phi)(w)-S(\psi)(w)&=\fint_{\mathbb{T}}\Big(\frac{\phi'(\tau)}{|\phi(w)-\phi(\tau)|^{\alpha}}-\frac{\psi'(\tau)}{|\psi(w)-\psi(\tau)|^{\alpha}}\Big)d\tau\\&=\fint_{\mathbb{T}}\Big(\frac{1}{|\phi(w)-\phi(\tau)|^{\alpha}}-\frac{1}{|\psi(w)-\psi(\tau)|^{\alpha}}\Big)\psi'(\tau)d\tau\\&\quad+\fint_{\mathbb{T}}\frac{\phi'(\tau)-\psi'(\tau)}{|\phi(w)-\phi(\tau)|^{\alpha}}d\tau.\end{aligned}$$

The estimate of the last term follows immediately from Corollary 1.2, that is,

$$\begin{aligned}\Big\|\fint_{\mathbb{T}}\frac{\phi'(\tau)-\psi'(\tau)}{|\phi(\cdot)-\phi(\tau)|^{\alpha}}d\tau\Big\|_{C^{1-\alpha}(\mathbb{T})}&\leq C\|f'-g'\|_{L^\infty}\\&\leq C\|f-g\|_{C^{2-\alpha}(\mathbb{T})}.\end{aligned}$$

As to the estimate of first term it can be deduced easily from the next general one : let T be the operator defined by

$$T\chi(w)=\fint_{\mathbb{T}}\Big(\frac{1}{|\phi(w)-\phi(\tau)|^{\alpha}}-\frac{1}{|\psi(w)-\psi(\tau)|^{\alpha}}\Big)\chi(\tau)d\tau$$

then

$$\big\|T\chi\big\|_{C^{1-\alpha}(\mathbb{T})}\leq C\|\psi-\phi\|_{\mathrm{Lip}(\mathbb{T})}\|\chi\|_{L^\infty(\mathbb{T})}. \tag{1.44}$$

To prove this control we shall introduce the kernel

$$K_2(w,\tau)\triangleq\frac{1}{|\phi(w)-\phi(\tau)|^{\alpha}}-\frac{1}{|\psi(w)-\psi(\tau)|^{\alpha}}$$

and prove that it satisfies the estimates,

$$|K_2(w,\tau)|\lesssim\frac{\|\psi'-\phi'\|_{L^\infty}}{|w-\tau|^{\alpha}}\quad\text{and}\quad|\partial_wK_2(w,\tau)|\lesssim\frac{\|\psi'-\phi'\|_{L^\infty}}{|w-\tau|^{\alpha+1}}.$$

Whence these estimates are proved we can then apply Lemma 1.1 and get the desired result. The first estimate is easy to obtain by using (1.42). On other hand, in view of (1.16) the derivative of $K_2(w,\tau)$ with respect to w is given by

$$\partial_wK_2(w,\tau)=-\frac{\alpha}{2}\Big(\overline{\mathcal{I}(w,\tau)}-\frac{\mathcal{I}(w,\tau)}{w^2}\Big)$$

where

$$\mathcal{I}(w,\tau) \triangleq \overline{\phi'(w)}\frac{\phi(w)-\phi(\tau)}{|\phi(w)-\phi(\tau)|^{\alpha+2}} - \overline{\psi'(w)}\frac{\psi(w)-\psi(\tau)}{|\psi(w)-\psi(\tau)|^{\alpha+2}}.$$

We shall transform this quantity into,

$$\mathcal{I}(w,\tau) = \mathcal{I}_1(w,\tau) + \mathcal{I}_2(w,\tau) + \mathcal{I}_3(w,\tau),$$

with

$$\mathcal{I}_1(w,\tau) \triangleq \overline{\phi'(w)}\,\frac{(\phi-\psi)(w)-(\phi-\psi)(\tau)}{|\phi(w)-\phi(\tau)|^{\alpha+2}},$$

$$\mathcal{I}_2(w,\tau) \triangleq \big(\overline{\phi'(w)}-\overline{\psi'(w)}\big)\frac{\psi(w)-\psi(\tau)}{|\psi(w)-\psi(\tau)|^{\alpha+2}},$$

and

$$\mathcal{I}_3(w,\tau) \triangleq \overline{\phi'(w)}\big(\psi(\tau)-\psi(w)\big)\frac{|\phi(w)-\phi(\tau)|^{\alpha+2}-|\psi(w)-\psi(\tau)|^{\alpha+2}}{|\phi(w)-\phi(\tau)|^{\alpha+2}|\psi(w)-\psi(\tau)|^{\alpha+2}}.$$

For the first and the second term one readily gets

$$|\mathcal{I}_1(w,\tau)| + |\mathcal{I}_2(w,\tau)| \lesssim \frac{\|\phi-\psi\|_{\text{Lip}(\mathbb{T})}}{|w-\tau|^{\alpha+1}}. \tag{1.45}$$

Concerning the last term we shall use the following inequality whose proof is classical.

$$|a^{k+1+\alpha}-b^{k+1+\alpha}| \le C(k,\alpha)|a-b|\big(a^{k+\alpha}+b^{k+\alpha}\big), \quad a,b\in\mathbb{R}_+, k\in\mathbb{N}^*,\ 0<\alpha<1. \tag{1.46}$$

Thus we find

$$\Big||\phi(w)-\phi(\tau)|^{\alpha+2} - |\psi(w)-\psi(\tau)|^{\alpha+2}\Big| \le C\|\phi-\psi\|_{\text{Lip}(\mathbb{T})}\,|\tau-w|^{\alpha+2}$$

and consequently,

$$|\mathcal{I}_3(w,\tau)| \le C\frac{\|\phi-\psi\|_{\text{Lip}(\mathbb{T})}}{|w-\tau|^{\alpha+1}}. \tag{1.47}$$

Putting together (1.45) and (1.47) we find,

$$|\mathcal{I}(w,\tau)| \le C\frac{\|\phi-\psi\|_{\text{Lip}(\mathbb{T})}}{|w-\tau|^{\alpha+1}}.$$

Therefore

$$|\partial_w K_2(w,\tau)| \le C\frac{\|\phi-\psi\|_{\text{Lip}(\mathbb{T})}}{|w-\tau|^{\alpha+1}}.$$

This achieves the suitable estimates for the kernel K_2. Let us now move to the continuity estimate of $A(\phi,h)$. We write from the definition,

$$A(\phi,h)(w) - A(\psi,h)(w) = \fint_{\mathbb{T}}\Big(\frac{1}{|\phi(w)-\phi(\tau)|^{\alpha}} - \frac{1}{|\psi(w)-\psi(\tau)|^{\alpha}}\Big)h'(\tau)d\tau \tag{1.48}$$

Using (2.38) we immediately obtain,

$$\begin{aligned}\big\|A(\phi,h)-A(\psi,h)\big\|_{C^{1-\alpha}} &\le C\|\phi-\psi\|_{\text{Lip}(\mathbb{T})}\|h\|_{\text{Lip}(\mathbb{T})}\\ &\le C\|f-g\|_{\text{Lip}(\mathbb{T})}\|h\|_{C^{2-\alpha}(\mathbb{T})}.\end{aligned} \tag{1.49}$$

This completes the proof of the estimate of the term $A(\phi, h)$ which fits with (1.43).

Now we shall investigate the continuity estimate of $B(\phi, h)$ defined in (1.35). According to this definition, one has

$$\big(B(\phi,h)-B(\psi,h)\big)(w)=\fint_{\mathbb{T}} K_3(w,\tau)d\tau,$$

with

$$K_3(w,\tau)\triangleq\left(\phi'(\tau)\frac{\phi(w)-\phi(\tau)}{|\phi(w)-\phi(\tau)|^{\alpha+2}}-\psi'(\tau)\frac{\psi(w)-\psi(\tau)}{|\psi(w)-\psi(\tau)|^{\alpha+2}}\right)\Big(\overline{h(w)}-\overline{h(\tau)}\Big).$$

We can rewrite this kernel in the form,

$$K_3(w,\tau)=\Big(K_3^1(w,\tau)+K_3^2(w,\tau)+K_3^3(w,\tau)\Big)\Big(\overline{h(w)}-\overline{h(\tau)}\Big),$$

with

$$K_3^1(w,\tau)\triangleq\phi'(\tau)\,\frac{(\phi-\psi)(w)-(\phi-\psi)(\tau)}{|\phi(w)-\phi(\tau)|^{\alpha+2}},$$
$$K_3^2(w,\tau)\triangleq\big(\phi'-\psi'\big)(\tau)\frac{\psi(w)-\psi(\tau)}{|\psi(w)-\psi(\tau)|^{\alpha+2}},$$

and

$$K_3^3(w,\tau)\triangleq\phi'(\tau)\big(\psi(\tau)-\psi(w)\big)\frac{|\phi(w)-\phi(\tau)|^{\alpha+2}-|\psi(w)-\psi(\tau)|^{\alpha+2}}{|\psi(w)-\psi(\tau)|^{\alpha+2}|\phi(w)-\phi(\tau)|^{\alpha+2}}.$$

In view of the inequality (1.46) we may conclude that

$$\begin{aligned}|K_3^1(w,\tau)|+|K_3^2(w,\tau)|+|K_3^3(w,\tau)|&\le C\frac{\|\phi-\psi\|_{\mathrm{Lip}(\mathbb{T})}}{|w-\tau|^{\alpha+1}}\\&\le C\frac{\|f-g\|_{\mathrm{Lip}(\mathbb{T})}}{|w-\tau|^{\alpha+1}}.\end{aligned}$$

Consequently we find for $w\neq\tau\in\mathbb{T}$

$$|K_3(w,\tau)|\le C\|h\|_{\mathrm{Lip}(\mathbb{T})}\frac{\|f-g\|_{\mathrm{Lip}(\mathbb{T})}}{|w-\tau|^{\alpha}}. \tag{1.50}$$

Now we intend to estimate $\partial_w K_3(w,\tau)$. Easy computations yield

$$\begin{aligned}\partial_w K_3(w,\tau)=&\Big(-\frac{\alpha}{2}\mathcal{N}_1(w,\tau)-\big(1+\frac{\alpha}{2}\big)\mathcal{N}_2(w,\tau)\Big)\big(\overline{h(w)}-\overline{h(\tau)}\big)\\&-\mathcal{N}_3(w,\tau)\frac{\overline{h'(w)}}{w^2},\end{aligned} \tag{1.51}$$

with

$$\mathcal{N}_1(w,\tau)\triangleq\frac{\phi'(\tau)\phi'(w)}{|\phi(w)-\phi(\tau)|^{\alpha+2}}-\frac{\psi'(\tau)\psi'(w)}{|\psi(w)-\psi(\tau)|^{\alpha+2}},$$
$$\mathcal{N}_2(w,\tau)\triangleq\frac{\overline{\phi'(w)}}{w^2}\frac{\phi'(\tau)\big(\phi(w)-\phi(\tau)\big)^2}{|\phi(w)-\phi(\tau)|^{\alpha+4}}-\frac{\overline{\psi'(w)}}{w^2}\frac{\psi'(\tau)\big(\psi(w)-\psi(\tau)\big)^2}{|\psi(w)-\psi(\tau)|^{\alpha+4}},$$

and

$$\mathcal{N}_3(w,\tau)\triangleq\phi'(\tau)\frac{\phi(w)-\phi(\tau)}{|\phi(w)-\phi(\tau)|^{\alpha+2}}-\psi'(\tau)\frac{\psi(w)-\psi(\tau)}{|\psi(w)-\psi(\tau)|^{\alpha+2}}.$$

The estimate of the last term $\mathcal{N}_3$ can be done exactly as for $\mathcal{I}$. Concerning $\mathcal{N}_1(w,\tau)$ we may write

$$\mathcal{N}_1(w,\tau) = (\phi'(w)-\psi'(w))\frac{\phi'(\tau)}{|\phi(w)-\phi(\tau)|^{\alpha+2}} + \big(\phi'(\tau)-\psi'(\tau)\big)\frac{\psi'(w)}{|\psi(w)-\psi(\tau)|^{\alpha+2}}$$
$$-\phi'(\tau)\psi'(w)\frac{|\phi(w)-\phi(\tau)|^{\alpha+2}-|\psi(w)-\psi(\tau)|^{\alpha+2}}{|\phi(w)-\phi(\tau)|^{\alpha+2}|\psi(w)-\psi(\tau)|^{\alpha+2}}.$$

Hence, using inequality (1.46) we immediately deduce that

$$|\mathcal{N}_1(w,\tau)| \le \frac{C\|\phi'-\psi'\|_{L^\infty}}{|w-\tau|^{\alpha+2}}.$$

Now we shall split the term $\mathcal{N}_2(w,\tau)$ as follows,

$$\mathcal{N}_2(w,\tau) = \sum_{k=1}^{4}\mathcal{N}_{2,k}(w,\tau)$$

with

$$\mathcal{N}_{2,1}(w,\tau) = \phi'(\tau)\big(\overline{\phi'(w)}-\overline{\psi'(w)}\big)\frac{\big(\phi(w)-\phi(\tau)\big)^2}{w^2|\phi(w)-\phi(\tau)|^{\alpha+4}},$$

$$\mathcal{N}_{2,2}(w,\tau) = \phi'(\tau)\overline{\psi'(w)}\,\frac{\big(\phi(w)-\phi(\tau)\big)^2-\big(\psi(w)-\psi(\tau)\big)^2}{w|\phi(w)-\phi(\tau)|^{\alpha+4}},$$

$$\mathcal{N}_{2,3}(w,\tau) = \big(\phi'(\tau)-\psi'(\tau)\big)\overline{\psi'(w)}\frac{\big(\psi(w)-\psi(\tau)\big)^2}{w^2|\phi(w)-\phi(\tau)|^{\alpha+4}},$$

and

$$\mathcal{N}_{2,4}(w,\tau) = \psi'(\tau)\overline{\psi'(w)}\big(\psi(w)-\psi(\tau)\big)^2\frac{|\psi(w)-\psi(\tau)|^{\alpha+4}-|\phi(w)-\phi(\tau)|^{\alpha+4}}{|\psi(w)-\psi(\tau)|^{\alpha+4}|\phi(w)-\phi(\tau)|^{\alpha+4}}.$$

Similar computations as before lead to,

$$|\mathcal{N}_2(w,\tau)| \lesssim \frac{\|\phi'-\psi'\|_{L^\infty}}{|w-\tau|^{\alpha+2}}.$$

Hence, in view of the identity (1.51) we obtain

$$|\partial_w K_3(w,\tau)| \lesssim \frac{\|h'\|_{L^\infty}\|\phi'-\psi'\|_{L^\infty}}{|w-\tau|^{\alpha+1}}. \tag{1.52}$$

At this stage we can use (1.50), (1.52) and Lemma 1.1,

$$\begin{aligned}\big\|B(\phi,h)-B(\psi,h)\big\|_{C^{1-\alpha}} &\lesssim \|f-g\|_{\mathrm{Lip}(\mathbb{T})}\|h\|_{\mathrm{Lip}(\mathbb{T})}\\ &\lesssim \|f-g\|_{C^{2-\alpha}(\mathbb{T})}\|h\|_{C^{2-\alpha}(\mathbb{T})}.\end{aligned} \tag{1.53}$$

Finally, we observe from (1.35) that

$$C(\phi,h)(w)-C(\psi,h)(w) = \fint_{\mathbb{T}}\overline{K_3(w,\tau)}\phi'(\tau)d\tau$$

and therefore we find similar estimate to (1.53). This ends the proof of (1.43).

(**2**) Now we shall compute $\partial_\Omega \partial_f F(\Omega, f)$ and prove the continuity of this function. Let $f \in B_r$ and $h \in C^{2-\alpha}(\mathbb{T})$ be a fixed direction, then in view of (1.33) one has

$$\partial_\Omega \partial_f F(\Omega, f)h(w) = \text{Im}\Big\{\phi(w)\,\overline{w}\,\overline{h'(w)} + h(w)\,\overline{w}\,\overline{\phi'(w)}\Big\}.$$

It follows that for $f, g \in B_r$,

$$\begin{aligned}\|\partial_\Omega \partial_f F(\Omega, f)h - \partial_\Omega \partial_f F(\Omega, f)h\|_{C^{1-\alpha}(\mathbb{T})} &\leq C\|f-g\|_{C^{1-\alpha}(\mathbb{T})}\|h\|_{C^{2-\alpha}(\mathbb{T})}\\ &\leq C\|f-g\|_{C^{2-\alpha}(\mathbb{T})}\|h\|_{C^{2-\alpha}(\mathbb{T})}.\end{aligned}$$

This proves the continuity of $\partial_\Omega \partial_f F(\Omega, f) : \mathbb{R} \times B_r \to \mathcal{L}(X, Y)$ and therefore the proof of the second point is now achieved. □

1.8 Spectral study

In this section we concentrate on the spectral study of the linearized operator of F around zero and denoted by $\partial_f F(\Omega, 0)$. We shall peculiarly look for the values of Ω where the kernel is non trivial. We will be seeing that the kernel is necessarily simple and all the required assumptions of the C-R Theorem are satisfied. According to the Proposition 1.7, the functional $F : \mathbb{R} \times B_r \to Y$ is C^1 and therefore Gâteaux and Fréchet derivatives with respect to f and in the direction $h \in X$ coincide. Now putting together the formulas (1.35) and (1.36) with $\phi = \text{Id}$, we find

$$\begin{aligned}\partial_f F(\Omega, 0)h(w) =& \text{Im}\Big\{\Omega\Big(\overline{h'(w)} + \frac{h(w)}{w}\Big) - C_\alpha \frac{\overline{h'(w)}}{w} \fint_{\mathbb{T}} \frac{d\tau}{|w-\tau|^\alpha} - \frac{C_\alpha}{w}\fint_{\mathbb{T}} \frac{h'(\tau)}{|w-\tau|^\alpha} d\tau\\ &+ \frac{\alpha C_\alpha}{2w}\fint_{\mathbb{T}} \frac{(w-\tau)\big(\overline{h(w)} - \overline{h(\tau)}\big)}{|w-\tau|^{\alpha+2}} d\tau + \frac{\alpha C_\alpha}{2w}\fint_{\mathbb{T}} \frac{(\overline{w}-\overline{\tau})\big(h(w) - h(\tau)\big)}{|w-\tau|^{\alpha+2}} d\tau\Big\}\\ \triangleq& \text{Im}\Big\{\text{I}_1(h(w)) + \text{I}_2(h(w)) + \text{I}_3(h(w)) + \text{I}_4(h(w)) + \text{I}_5(h(w))\Big\}. \end{aligned} \tag{1.54}$$

Recall that the spaces X and Y are successively given by,

$$X = \Big\{f \in C^{2-\alpha}(\mathbb{T}),\ f(w) = \sum_{n\geq 0} b_n \overline{w}^n, b_n \in \mathbb{R},\ w \in \mathbb{T}\Big\}$$

and

$$Y = \Big\{g \in C^{1-\alpha}(\mathbb{T}),\ g(w) = i\sum_{n\geq 1} g_n\big(w^n - \overline{w}^n\big), g_n \in \mathbb{R},\ w \in \mathbb{T}\Big\}.$$

To state our main result we shall introduce a special set $\mathcal{S}$ describing the dispersion relation which plays a central role in the bifurcation of non trivial solutions.

$$\mathcal{S} \triangleq \left\{\Omega \in \mathbb{R},\, \exists\, m \geq 2,\ \ \Omega = \Omega_m^\alpha \triangleq \frac{\Gamma(1-\alpha)}{2^{1-\alpha}\Gamma^2(1-\frac{\alpha}{2})}\left(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})} - \frac{\Gamma(m+\frac{\alpha}{2})}{\Gamma(m+1-\frac{\alpha}{2})}\right)\right\}. \tag{1.55}$$

We shall discuss soon some elementary properties of this set. Now we state our result.

Proposition 1.8 *The following assertions hold true.*

1. *The kernel of $\partial_f F(\Omega, 0)$ is non trivial if and only if $\Omega = \Omega_m^\alpha \in \mathcal{S}$ and, in this case, it is one-dimensional vector space generated by*

$$v_m(w) = \overline{w}^{m-1}.$$

2. *The range of $\partial_f F(\Omega_m^\alpha, 0)$ is closed in Y and is of co-dimension one. It is given by*

$$R\big(\partial_f F(\Omega_m^\alpha, 0)\big) = \Big\{g \in C^{1-\alpha}(\mathbb{T});\quad g(w) = i\sum_{\substack{n\geq 1\\ n\neq m}}^{\infty} g_n(w^n - \overline{w}^n), g_n \in \mathbb{R}\Big\}.$$

3. *Transversality assumption :*

$$\partial_\Omega\partial_f F(\Omega_m^\alpha, 0)(v_m) \notin R\big(\partial_f F(\Omega_m^\alpha, 0)\big).$$

Before proving this result we collect some properties on the asymptotic behavior of the sequence $\{\Omega_n^\alpha\}$ with respect to α and n. This is summarized in the next lemma.

Lemma 1.3 *We have the following results.*

1. *Let $n \geq 2$, then*

$$\lim_{\alpha\to 0} \Omega_n^\alpha = \frac{n-1}{2n}, \qquad \lim_{\alpha\to 1} \Omega_n^\alpha = \frac{2}{\pi}\sum_{k=1}^{n-1}\frac{1}{2k+1}\cdot$$

2. *For any $\alpha \in (0,1)$, we get $\Omega_n^\alpha > 0$ and $n \mapsto \Omega_n^\alpha$ is strictly increasing. Moreover,*

$$\mathcal{S} \subset \Theta_\alpha\Big[\frac{1-\alpha}{2-\frac{\alpha}{2}}, 1\Big[,$$

with

$$\Theta_\alpha \triangleq \frac{2^\alpha\Gamma(1+\frac{\alpha}{2})\Gamma(1-\alpha)}{(2-\alpha)\Gamma^3(1-\frac{\alpha}{2})}\cdot$$

3. *For $\alpha \in (0,1)$ fixed and n sufficiently large,*

$$\Omega_n^\alpha = \Theta_\alpha - \big(1-\alpha/2\big)\Theta_\alpha\frac{e^{\alpha\gamma+c_\alpha}}{n^{1-\alpha}} + O\Big(\frac{1}{n^{2-\alpha}}\Big), \tag{1.56}$$

where γ denotes Euler constant, c_α is the sum of the series

$$c_\alpha \triangleq \sum_{m=1}^{\infty}\frac{\alpha^{2m+1}}{2^{2m-1}(2m+1)}\zeta(2m+1).$$

and $s \mapsto \zeta(s)$ is the Riemann zeta function.

Proof : $(\mathbf{1})$ Recall first that for $n \geq 2$,

$$\Omega_n^\alpha \triangleq \frac{\Gamma(1-\alpha)}{2^{1-\alpha}\Gamma^2(1-\frac{\alpha}{2})}\bigg(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})} - \frac{\Gamma(n+\frac{\alpha}{2})}{\Gamma(n+1-\frac{\alpha}{2})}\bigg).$$

Passing to the limit in this formula when α goes to zero yields

$$\begin{aligned}\lim_{\alpha\to 0}\Omega_n^\alpha &= \frac{1}{2}\Big(\frac{\Gamma(1)}{\Gamma(2)} - \frac{\Gamma(n)}{\Gamma(n+1)}\Big)\\ &= \frac{1}{2}\Big(1 - \frac{(n-1)!}{n!}\Big)\\ &= \frac{n-1}{2n}\cdot\end{aligned}$$

As to the second limit, we shall introduce for a fixed n the function

$$\phi_n(\alpha)=\frac{\Gamma(n+\alpha/2)}{\Gamma(n+1-\alpha/2)}.$$

Therefore we obtain according to (1.20) and (1.21) and the relation $\phi_n(1)=1$,

$$\begin{aligned}\lim_{\alpha\to 1}\Omega_n^\alpha&=\frac{-1}{\Gamma^2(1/2)}\lim_{\alpha\to 1}\big\{(1-\alpha)\Gamma(1-\alpha)\big\}\lim_{\alpha\to 1}\Big\{\frac{\phi_1(\alpha)-\phi_1(1)}{\alpha-1}-\frac{\phi_n(\alpha)-\phi_n(1)}{\alpha-1}\Big\}\\&=\frac{-1}{\pi}\Big\{\phi_1'(1)-\phi_n'(1)\Big\}.\end{aligned}$$

By applying the logarithm function to ϕ_n and differentiating with respect to α one obtains the relation

$$2\frac{\phi_n'(\alpha)}{\phi_n(\alpha)}=\digamma(n+\alpha/2)+\digamma(n+1-\alpha/2).$$

Now using the fact that $\phi_n(1)=1$ combined with the preceding identity and (2.16), we find

$$\begin{aligned}\lim_{\alpha\to 1}\Omega_n^\alpha&=\frac{-1}{\pi}\Big\{\digamma(3/2)-\digamma(n+1/2)\Big\}\\&=\frac{2}{\pi}\sum_{k=1}^{n-1}\frac{1}{2k+1},\end{aligned}$$

which is the desired result.

(**2**) Using the identities (1.25) we find the alternative formula

$$\Omega_n^\alpha\triangleq\Theta_\alpha\bigg(1-\frac{\big(1+\frac{\alpha}{2}\big)_{n-1}}{\big(2-\frac{\alpha}{2}\big)_{n-1}}\bigg),\tag{1.57}$$

with

$$\Theta_\alpha\triangleq\frac{2^\alpha\Gamma(1+\frac{\alpha}{2})\Gamma(1-\alpha)}{(2-\alpha)\Gamma^3(1-\frac{\alpha}{2})}.$$

and $(x)_n$ denotes Pokhhammer's symbol introduced in (1.23). Now because $x\mapsto(x)_{n-1}$ is increasing in the set $\mathbb{R}_+$ provided that $\alpha<1$ we conclude easily that $\Omega_n^\alpha>0$.

To prove that $n\mapsto\Omega_n^\alpha$ is strictly increasing, it suffices according to (1.57) to check that the sequence $n\mapsto u_n=\frac{\left(1+\frac{\alpha}{2}\right)_{n-1}}{\left(2-\frac{\alpha}{2}\right)_{n-1}}$ is strictly decreasing. This follows from the obvious fact that for $\alpha\in(0,1)$, one has

$$\frac{u_{n+1}}{u_n}=\frac{n+\frac{\alpha}{2}}{n+1-\frac{\alpha}{2}}<1.$$

From this it is apparent that

$$\mathcal{S}\subset\big[\Omega_2^\alpha,\lim_{n\to\infty}\Omega_n^\alpha[\subset\Theta_\alpha\Big[\frac{1-\alpha}{2-\frac{\alpha}{2}},1\Big[.$$

Note that we have used in the last limit that $\lim_{n\to\infty}u_n=0$ which can be deduced for instance from the proof of the point (**3**) of this lemma.

$(\mathbf{3})$ First recall that Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, s > 1.$$

To get the required asymptotic behavior we shall first study the sequence,

$$U_n \triangleq \log\left(\frac{\left(1+\frac{\alpha}{2}\right)_n}{\left(1-\frac{\alpha}{2}\right)_n}\right).$$

Making use of the definition of $(x)_n$, we can rewrite this sequence in the manner

$$U_n = \sum_{k=1}^{n}\Big\{\log\Big(1+\frac{\alpha}{2k}\Big) - \log\Big(1-\frac{\alpha}{2k}\Big)\Big\}.$$

Using the Taylor expansion of $\log(1+x)$ around zero one gets

$$\begin{aligned}
U_n &= \sum_{k=1}^{n}\Big\{2\sum_{m=0}^{\infty}\frac{1}{2m+1}\Big(\frac{\alpha}{2k}\Big)^{2m+1}\Big\}\\
&= \sum_{k=1}^{n}\frac{\alpha}{k} + 2\sum_{m=1}^{\infty}\frac{\alpha^{2m+1}}{2^{2m}(2m+1)}\sum_{k=1}^{n}\frac{1}{k^{2m+1}}\\
&= \sum_{k=1}^{n}\frac{\alpha}{k} + 2\sum_{m=1}^{\infty}\Big(\frac{\alpha^{2m+1}}{2^{2m}(2m+1)}\zeta(2m+1) + O\Big(\frac{1}{n^{2m}}\Big)\Big)\\
&= \sum_{k=1}^{n}\frac{\alpha}{k} + c_\alpha + O\Big(\frac{1}{n^2}\Big).
\end{aligned}$$

Note that we have used the following estimate for the remainder term of the zeta function

$$\sum_{k=n+1}^{\infty}\frac{1}{k^{2m+1}} \le \frac{1}{n^{2m}}.$$

Now we use the classical expansion of the harmonic series

$$\sum_{k=1}^{n}\frac{1}{k} = \log n + \gamma + O\Big(\frac{1}{n}\Big)$$

with γ the Euler constant. Therefore we get

$$U_n = \alpha\log n + \alpha\gamma + c_\alpha + O\Big(\frac{1}{n}\Big)$$

and consequently by raising to the exponential we find

$$\begin{aligned}
e^{U_n} &= \frac{\left(1+\frac{\alpha}{2}\right)_n}{\left(1-\frac{\alpha}{2}\right)_n}\\
&= e^{\alpha\gamma+c_\alpha}n^{\alpha}e^{O(1/n)}\\
&= e^{\alpha\gamma+c_\alpha}n^{\alpha} + O\Big(\frac{1}{n^{1-\alpha}}\Big).
\end{aligned}$$

It is apparent that

$$\begin{aligned}\frac{\left(1+\frac{\alpha}{2}\right)_{n-1}}{\left(2-\frac{\alpha}{2}\right)_{n-1}} &= \frac{1-\frac{\alpha}{2}}{n-\frac{\alpha}{2}}\frac{\left(1+\frac{\alpha}{2}\right)_{n-1}}{\left(1-\frac{\alpha}{2}\right)_{n-1}}\\ &= \frac{1-\frac{\alpha}{2}}{n-\frac{\alpha}{2}}e^{U_{n-1}}\end{aligned}$$

and consequently by making appeal to the formula (1.57) we obtain

$$\begin{aligned}\Omega_n^\alpha &= \Theta_\alpha\Big(1-\frac{1-\frac{\alpha}{2}}{n-\frac{\alpha}{2}}e^{U_{n-1}}\Big)\\ &= \Theta_\alpha\Big(1-\big(1-\frac{\alpha}{2}\big)\frac{e^{\alpha\gamma+c_\alpha}}{n^{1-\alpha}}\Big)+O\Big(\frac{1}{n^{2-\alpha}}\Big).\end{aligned}$$

This concludes the proof of Lemma 1.3. □

In what follows we shall give the proof of the Proposition 1.8. *Proof* :

$(\mathbf{1})$ We begin by calculating $\mathrm{I}_1(h)$ in (1.54) which is easy compared to the other terms. Let $h\in X$ taking the form $h(w)=\sum_{n\geq 0}\frac{b_n}{w^n}$, then straightforward computations give

$$\mathrm{I}_1(h(w))=\Omega\sum_{n\geq 0}\Big(b_n\overline{w}^{n+1}-nb_nw^{n+1}\Big). \tag{1.58}$$

To compute the second term $I_2(h(w))$ we write

$$\begin{aligned}\mathrm{I}_2(h(w)) &\triangleq -C_\alpha\overline{w}\,\overline{h'(w)}\fint_{\mathbb{T}}\frac{d\tau}{|w-\tau|^\alpha}\\ &= C_\alpha\sum_{n\geq 1}nb_nw^n\fint_{\mathbb{T}}\frac{d\tau}{|w-\tau|^\alpha}.\end{aligned}$$

Applying the formula (1.26) with $n=0$ we get

$$\mathrm{I}_2(h(w))=\frac{\alpha C_\alpha\Gamma(1-\alpha)}{(2-\alpha)\Gamma^2(1-\alpha/2)}\sum_{n\geq 1}nb_nw^{n+1}. \tag{1.59}$$

Regarding the third term $\mathrm{I}_3(h(w))$ it may be rewritten in the manner

$$\begin{aligned}\mathrm{I}_3(h(w)) &\triangleq -C_\alpha\overline{w}\fint_{\mathbb{T}}\frac{h'(\tau)}{|w-\tau|^\alpha}d\tau\\ &= C_\alpha\sum_{n\geq 1}nb_n\overline{w}\fint_{\mathbb{T}}\frac{\overline{\tau}^{n+1}}{|w-\tau|^\alpha}d\tau.\end{aligned}$$

Using change of variables allows to get

$$\fint_{\mathbb{T}}\frac{\overline{\tau}^{n+1}}{|w-\tau|^\alpha}d\tau=\overline{w}^n\fint_{\mathbb{T}}\frac{\tau^{n-1}}{|1-\tau|^\alpha}d\tau$$

which yields in view of the formula (1.26) to the expression

$$\mathrm{I}_3(h(w))=\frac{C_\alpha\Gamma(1-\alpha)}{\Gamma^2(1-\alpha/2)}\sum_{n\geq 1}nb_n\frac{\left(\frac{\alpha}{2}\right)_n}{\left(1-\frac{\alpha}{2}\right)_n}\overline{w}^{n+1}. \tag{1.60}$$

Concerning the term $I_4(h(w))$ we start with the expansion,

$$\begin{aligned}\mathrm{I}_4(h(w)) &\triangleq \frac{\alpha C_\alpha}{2}\fint_{\mathbb{T}}\frac{(w-\tau)\big(\overline{h(w)}-\overline{h(\tau)}\big)}{w|w-\tau|^{\alpha+2}}d\tau\\ &= \frac{\alpha C_\alpha}{2}\sum_{n\geq 1}b_n\fint_{\mathbb{T}}\frac{(w-\tau)(w^n-\tau^n)}{w|w-\tau|^{\alpha+2}}d\tau. \end{aligned}\tag{1.61}$$

Hence, using the identity (1.27) one gets

$$\mathrm{I}_4(h(w)) \quad = \quad \frac{\alpha(1+\frac{\alpha}{2})}{2(2-\alpha)}\frac{C_\alpha\Gamma(1-\alpha)}{\Gamma^2(1-\alpha/2)}\sum_{n\geq 1}b_n\left(1-\frac{\big(2+\frac{\alpha}{2}\big)_n}{\big(2-\frac{\alpha}{2}\big)_n}\right)w^{n+1}. \tag{1.62}$$

It remains to compute the last term I_5 of (1.34) which can be written in the form

$$\begin{aligned}\mathrm{I}_5(h(w)) &= \frac{\alpha C_\alpha}{2}\fint_{\mathbb{T}}\frac{(\overline{w}-\overline{\tau})\big(h(w)-h(\tau)\big)}{w|w-\tau|^{\alpha+2}}d\tau\\ &= \frac{\alpha C_\alpha}{2}\sum_{n\geq 1}b_n\fint_{\mathbb{T}}\frac{(\overline{w}-\overline{\tau})(\overline{w}^n-\overline{\tau}^n)}{w|w-\tau|^{\alpha+2}}d\tau.\end{aligned}$$

Using the identity (1.28) gives

$$\mathrm{I}_5(h(w)) \quad = \quad -\frac{\alpha C_\alpha\Gamma(1-\alpha)}{4\Gamma^2(1-\alpha/2)}\sum_{n\geq 1}b_n\left(1-\frac{\big(\frac{\alpha}{2}\big)_n}{\big(-\frac{\alpha}{2}\big)_n}\right)\overline{w}^{n+1}. \tag{1.63}$$

Collecting the identities (1.60), (1.63) and using (1.24) we find

$$\begin{aligned}\mathrm{I}_3(h(w))+\mathrm{I}_5(h(w)) &= \frac{C_\alpha\Gamma(1-\alpha)}{2\Gamma^2(1-\frac{\alpha}{2})}\sum_{n\geq 1}b_n\left(\frac{2n\big(\frac{\alpha}{2}\big)_n}{\big(1-\frac{\alpha}{2}\big)_n}-\frac{\alpha}{2}-\frac{\big(-\frac{\alpha}{2}\big)\big(\frac{\alpha}{2}\big)_n}{\big(-\frac{\alpha}{2}\big)_n}\right)\overline{w}^{n+1}\\ &= \frac{C_\alpha\Gamma(1-\alpha)}{2\Gamma^2(1-\frac{\alpha}{2})}\sum_{n\geq 1}b_n\left(\frac{2n\big(\frac{\alpha}{2}\big)_n}{\big(1-\frac{\alpha}{2}\big)_n}-\frac{\alpha}{2}-\big(n-\frac{\alpha}{2}\big)\frac{\big(\frac{\alpha}{2}\big)_n}{\big(1-\frac{\alpha}{2}\big)_n}\right)\overline{w}^{n+1}\\ &= \frac{C_\alpha\Gamma(1-\alpha)}{2\Gamma^2(1-\frac{\alpha}{2})}\sum_{n\geq 1}b_n\left(-\frac{\alpha}{2}+\big(n+\frac{\alpha}{2}\big)\frac{\big(\frac{\alpha}{2}\big)_n}{\big(1-\frac{\alpha}{2}\big)_n}\right)\overline{w}^{n+1}\\ &= -\frac{C_\alpha\Gamma(1-\alpha)}{2\Gamma^2(1-\frac{\alpha}{2})}\sum_{n\geq 1}b_n\left(\frac{\alpha}{2}-\frac{\big(\frac{\alpha}{2}\big)_{n+1}}{\big(1-\frac{\alpha}{2}\big)_n}\right)\overline{w}^{n+1}\\ &\triangleq -\sum_{n\geq 1}b_n\beta_n\overline{w}^{n+1},\end{aligned}$$

with

$$\beta_n = \frac{C_\alpha\Gamma(1-\alpha)}{2\Gamma^2(1-\alpha/2)}\left(\frac{\alpha}{2}-\frac{\big(\frac{\alpha}{2}\big)_{n+1}}{\big(1-\frac{\alpha}{2}\big)_n}\right).$$

Now by summing up (1.59) and (1.62) we deduce that,

$$\begin{aligned}\mathrm{I}_2(h(w))+\mathrm{I}_4(h(w)) &= \frac{\alpha C_\alpha\Gamma(1-\alpha)}{2(2-\alpha)\Gamma^2(1-\frac{\alpha}{2})}\sum_{n\geq 1}b_n\left(2n+1+\frac{\alpha}{2}-\frac{(1+\frac{\alpha}{2})\big(2+\frac{\alpha}{2}\big)_n}{\big(2-\frac{\alpha}{2}\big)_n}\right)w^{n+1}\\ &= \frac{\alpha C_\alpha\Gamma(1-\alpha)}{2(2-\alpha)\Gamma^2(1-\frac{\alpha}{2})}\sum_{n\geq 1}b_n\left(2n+1+\frac{\alpha}{2}-\frac{\big(1+\frac{\alpha}{2}\big)_{n+1}}{\big(2-\frac{\alpha}{2}\big)_n}\right)w^{n+1}\\ &\triangleq \sum_{n\geq 1}b_n\alpha_n w^{n+1},\end{aligned}$$

with

$$\alpha_n \triangleq \frac{\alpha C_\alpha \Gamma(1-\alpha)}{2(2-\alpha)\Gamma^2(1-\alpha/2)}\bigg(2n+1+\frac{\alpha}{2}-\frac{\big(1+\frac{\alpha}{2}\big)_{n+1}}{\big(2-\frac{\alpha}{2}\big)_n}\bigg).$$

Then inserting (1.58) and the two preceding identities into (1.54) one can readily verify that

$$\begin{aligned}\partial_f F(\Omega,0)(h)(w) &= \mathrm{Im}\Big\{\Omega b_0\overline{w}-\sum_{n\geq 1} b_n\Big(n\Omega-\alpha_n\Big)w^{n+1}+\sum_{n\geq 1} b_n\Big(\Omega-\beta_n\Big)\overline{w}^{n+1}\Big\}\\ &= \frac{\Omega b_0}{2}i\big(w-\overline{w}\big)+i\sum_{n\geq 1}\frac{b_n}{2}\Big((n+1)\Omega-\big(\alpha_n+\beta_n\big)\Big)\Big(w^{n+1}-\overline{w}^{n+1}\Big).\end{aligned} \tag{1.64}$$

By using (1.24) combined with the foregoing expressions for α_n and β_n one may write,

$$\begin{aligned}\alpha_n+\beta_n &= \frac{\alpha C_\alpha \Gamma(1-\alpha)}{2(2-\alpha)\Gamma^2(1-\alpha/2)}\bigg(2n+2-\frac{\big(1+\frac{\alpha}{2}\big)_{n+1}}{\big(2-\frac{\alpha}{2}\big)_n}-\frac{\big(1-\frac{\alpha}{2}\big)\big(\frac{\alpha}{2}\big)_{n+1}}{\frac{\alpha}{2}\big(1-\frac{\alpha}{2}\big)_n}\bigg)\\ &= \frac{\alpha C_\alpha \Gamma(1-\alpha)}{2(2-\alpha)\Gamma^2(1-\alpha/2)}\bigg(2n+2-\frac{\big(1+\frac{\alpha}{2}\big)_{n+1}}{\big(2-\frac{\alpha}{2}\big)_n}-\frac{\big(1+\frac{\alpha}{2}\big)_{n}}{\big(2-\frac{\alpha}{2}\big)_{n-1}}\bigg)\\ &= \frac{\alpha C_\alpha \Gamma(1-\alpha)}{(2-\alpha)\Gamma^2(1-\alpha/2)}\big(n+1\big)\bigg(1-\frac{\big(1+\frac{\alpha}{2}\big)_{n}}{\big(2-\frac{\alpha}{2}\big)_n}\bigg).\end{aligned} \tag{1.65}$$

Coming back to the definition of C_α, see for instance Proposition 1.4, and setting

$$\Theta_\alpha \triangleq \frac{\alpha C_\alpha \Gamma(1-\alpha)}{(2-\alpha)\Gamma^2(1-\frac{\alpha}{2})}=\frac{\alpha\Gamma(\frac{\alpha}{2})\Gamma(1-\alpha)}{2^{1-\alpha}(2-\alpha)\Gamma^3(1-\frac{\alpha}{2})},$$

one finds that

$$\alpha_n+\beta_n=\Theta_\alpha\big(n+1\big)\bigg(1-\frac{\big(1+\frac{\alpha}{2}\big)_{n}}{\big(2-\frac{\alpha}{2}\big)_n}\bigg).$$

Making appeal to the definition (1.57), the linearized operator (1.64) takes the form,

$$\partial_f F(\Omega,0)(h)(w)=\frac{\Omega b_0}{2}i\big(w-\overline{w}\big)+\frac{1}{2}i\sum_{n\geq 1}(n+1)b_n\Big(\Omega-\Omega_{n+1}^\alpha\Big)\Big(w^{n+1}-\overline{w}^{n+1}\Big). \tag{1.66}$$

We should mention in passing that the linearized operator has a special structure : it acts as a Fourier multiplier and as we shall see this will be very useful in the explicit computations for the kernel and the range of this operator. Now let us look for the values of Ω corresponding to non trivial kernel. It is easy to see that this will be the case if and only if Ω belongs to the dispersion set $\mathcal{S}$ introduced in (1.55). This corresponds to the values of Ω such that there exists $m\geq 1$ with

$$\begin{aligned}\Omega &= \Omega_{m+1}^\alpha\\ &= \Theta_\alpha\bigg(1-\frac{\big(1+\frac{\alpha}{2}\big)_{m}}{\big(2-\frac{\alpha}{2}\big)_m}\bigg).\end{aligned}$$

From Lemma 1.3-(2) the sequence $n\mapsto\Omega_n^\alpha$ is strictly increasing and therefore for any $n\neq m$

$$(1+n)\big(\Omega_{m+1}^\alpha-\Omega_{n+1}^\alpha\big)\neq 0.$$

From these last facts it is apparent that the kernel of $\partial_f F(\Omega_{m+1}^\alpha,0)$ is one-dimensional vector space generated by the function $v_m(w)=\overline{w}^m$.The claim of Proposition 1.8 follows by shifting the index m.

($\mathbf{2}$) Now we are going to show that for any $m \geq 2$ the range $R(\partial_f F(\Omega_m^\alpha, 0))$ coincides with the subspace

$$Z_m \triangleq \Big\{ g \in C^{1-\alpha}(\mathbb{T}); \quad g(w) = \sum_{\substack{n\geq 1\\ n\neq m}}^{\infty} i g_n(w^n - \overline{w}^n), g_n \in \mathbb{R} \Big\}.$$

Note that this sub-space is closed and of co-dimension one in the ambient space Y. In addition, one may easily deduce from (1.64) the trivial inclusion $R(\partial_f F(\Omega_m^\alpha, 0)) \subset Z_m$ and therefore it remains to check just the converse. For this end, let $g \in Z_m$ we shall look for a pre-image $h(w) = \sum_{n\geq 0} b_n \overline{w}^n \in X$ satisfying $\partial_f F(\Omega_m^\alpha, 0)(h) = g$. From the relation (1.66) this is equivalent to

$$\frac{\Omega_m^\alpha}{2} b_0 = g_1 \quad \text{and} \quad \frac{n b_{n-1}}{2}\Big(\Omega_m^\alpha - \Omega_n^\alpha\Big) = g_n, \ \forall n \geq 2, \, n \neq m.$$

This determines uniquely the sequence $(b_n)_{n\neq m-1}$ and one has

$$b_0 = \frac{2g_1}{\Omega_m^\alpha} \quad \text{and} \quad b_n = \frac{2g_{n+1}}{(n+1)\big(\Omega_m^\alpha - \Omega_{n+1}^\alpha\big)}, \quad \forall n \neq m-1, n \geq 1.$$

However the value b_{m-1} is free and it can be taken zero. Then the proof of $h \in X$ reduces to show that $h \in C^{2-\alpha}\big(\mathbb{T}\big)$. For this end, it suffices to show that the function $H(w) = \sum_{n\geq m} b_n \overline{w}^n$ belongs to this latter Hölder space. First we shall transform H in the form

$$\begin{aligned} H(w) &= 2\sum_{n\geq m} \frac{g_{n+1}}{(n+1)\big(\Omega_m^\alpha - \Omega_{n+1}^\alpha\big)} \overline{w}^n \\ &= 2w \sum_{n\geq m+1} \frac{g_n}{n\big(\Omega_m^\alpha - \Omega_n^\alpha\big)} \overline{w}^n. \end{aligned}$$

Using (1.56) one may write down

$$H(w) = 2w \sum_{n\geq m+1} \frac{g_n}{n\Big(\Omega_m^\alpha - \Theta_\alpha - d_n\Big)} \overline{w}^n,$$

where

$$d_n = -\Theta_\alpha\big(1-\alpha/2\big)\frac{e^{\alpha\gamma + c_\alpha}}{n^{1-\alpha}} + O\big(\frac{1}{n^{2-\alpha}}\big)\cdot \tag{1.67}$$

Denote $A = \Omega_m^\alpha - \Theta_\alpha$ then one may use the general decomposition : for $k \in \mathbb{N}$,

$$\frac{1}{A - d_n} = \frac{A^{-k-1} d_n^{k+1}}{A - d_n} + \sum_{j=0}^{k} A^{-j-1} d_n^j.$$

This allows to rewrite $H(w)$ in the manner

$$\begin{aligned} H(w) &= 2A^{-k-1} w \sum_{n\geq m+1} \frac{g_n d_n^{k+1}}{n(A-d_n)} \overline{w}^n + 2w \sum_{j=0}^{k} A^{-j-1} \sum_{n\geq m+1} \frac{g_n d_n^j}{n} \overline{w}^n \\ &\triangleq 2A^{-k-1} w H_{k+1}(\overline{w}) + 2w \sum_{j=0}^{k} A^{-j-1} L_j(\overline{w}). \end{aligned}$$

Fix k such that $(1-\alpha)(k+1)>2$ then $H_{k+1}\in C^2(\mathbb{T})$. Indeed as the sequence $(g_n)_n$ is bounded then we get by (1.67)

$$\Big|\frac{g_n d_n^{k+1}}{n(A-d_n)}\Big|\lesssim\frac{|d_n|^{k+1}}{n}\lesssim\frac{1}{n^{1+(1-\alpha)(k+1)}}.$$

Therefore the regularity follows from the polynomial decay of the Fourier coefficients. Concerning the estimate of L_j we shall restrict the analysis to $j=0$ and $j=1$ and the higher terms can be treated in a similar way. We write

$$L_0(w)=\sum_{n\geq m+1}\frac{g_n}{n}w^n.$$

Using Cauchy-Schwarz we deduce that

$$\begin{aligned}\|L_0\|_{L^\infty}&\lesssim\sum_{n\geq m+1}\frac{|g_n|}{n}\\&\lesssim\Big(\sum_{n\geq 1}\frac{1}{n^2}\Big)^{1/2}\Big(\sum_{n\geq m+1}|g_n|^2\Big)^{1/2}\\&\lesssim\|g\|_{L^2}.\end{aligned}$$

Hence, by the embedding $C^{1-\alpha}\big(\mathbb{T}\big)\hookrightarrow L^\infty\big(\mathbb{T}\big)\hookrightarrow L^2\big(\mathbb{T}\big)$ we conclude that

$$\|L_0\|_{L^\infty}\lesssim\|g\|_{1-\alpha}.$$

It remains to prove that $L_0'\in C^{1-\alpha}(\mathbb{T})$. For this end one need first to check that one can differentiate the series term by term. Fix $N\geq m+1$ and define

$$L_0^N(w)\triangleq\sum_{n=m+1}^{N}\frac{g_n}{n}w^n.$$

Then it is obvious from Cauchy-Schwarz inequality that

$$\lim_{N\to\infty}\|L_0^N-L_0\|_{L^\infty(\mathbb{T})}=0. \tag{1.68}$$

Now differentiating L_0^N term by term one should get

$$\begin{aligned}(L_0^N)'(w)&=\overline{w}\sum_{n=m+1}^{N}g_nw^n\\&\triangleq\overline{w}\,G_N(w).\end{aligned}$$

Assume for a while that $w\mapsto G(w)=\displaystyle\sum_{n\geq m+1}g_nw^n$ belongs to $C^{1-\alpha}(\mathbb{T})$, then by virtue of a classical result on Fourier series one gets

$$\lim_{N\to\infty}\|G_N-G\|_{L^\infty(\mathbb{T})}=0$$

and consequently

$$\lim_{N\to\infty}\|(L_0^N)'-\overline{w}\,G\|_{L^\infty(\mathbb{T})}=0. \tag{1.69}$$

Putting together (1.68) and (1.69) we obtain that L_0 is differentiable and

$$L_0'(w)=\overline{w}\,G(w),\quad w\in\mathbb{T}.$$

This concludes that $L_0 \in C^{2-\alpha}$. Now to complete rigorously the reasoning it remains to prove the preceding claim asserting that $G \in C^{1-\alpha}(\mathbb{T})$. Actually, this is based on the continuity of Szegö projection

$$\Pi : \sum_{n\in\mathbb{Z}} a_n w^n \mapsto \sum_{n\in\mathbb{N}} a_n w^n$$

on Hölder spaces $C^\varepsilon, \varepsilon \in (0,1)$. To see this we write

$$G(w) = \Pi\big(-i\, g(w) - \sum_{n=0}^{m} g_n w^n\big).$$

From which we deduce that

$$\begin{aligned}\|G\|_{C^{1-\alpha}(\mathbb{T})} &\le C\Big(\|g\|_{C^{1-\alpha}} + \sum_{n=0}^{m} |g_n|\|w^n\|_{C^{1-\alpha}}\Big)\\ &\le C_m\big(\|g\|_{C^{1-\alpha}} + \|g\|_{L^2}\big)\\ &\le C_m\|g\|_{C^{1-\alpha}}\end{aligned} \tag{1.70}$$

and this concludes the proof of the claim.
As to the term L_1 we write down by the definition

$$L_1(w) = \sum_{n\ge m+1} \frac{g_n d_n}{n} w^n.$$

As before we can easily get $L_1 \in L^\infty$ and we shall check that $L_1' \in C^{1-\alpha}\big(\mathbb{T}\big)$. Arguing in a similar way to L_0 we can differentiate term by term the series defining L_1 leading to

$$w\, L_1'(w) = \sum_{n\ge m+1} g_n d_n w^n.$$

We shall write down this series in the convolution form. With the notation $w = e^{i\theta}$, we may write

$$\begin{aligned}w\, L_1'(w) &= (K * G)(w),\\ &= \frac{1}{2\pi}\int_0^{2\pi} K(e^{i\eta})G(e^{i(\theta-\eta)})d\eta\end{aligned}$$

where

$$K(w) \triangleq \sum_{n\ge m+1} d_n w^n.$$

Making use of the definition (1.67) we find the expansion

$$\begin{aligned}K(w) &= -\Theta_\alpha\big(1-\alpha/2\big)e^{\alpha\gamma + c_\alpha} \sum_{n\ge m+1} \frac{w^n}{n^{1-\alpha}} + \sum_{n\ge m+1} O\big(\frac{1}{n^{2-\alpha}}\big)w^n.\\ &\triangleq -\Theta_\alpha\big(1-\alpha/2\big)e^{\alpha\gamma+c_\alpha}K_1(w) + K_2(w).\end{aligned}$$

The second term is easy to analyze because we have an absolute series as follows,

$$\|K_2\|_{L^\infty} \lesssim \sum_{n\ge m+1} \frac{1}{n^{2-\alpha}} \le C$$

and therefore $K_2 \in L^1(\mathbb{T})$. It suffices now to combine this fact with the classical convolution law $L^1(\mathbb{T}) * C^{1-\alpha}(\mathbb{T}) \to C^{1-\alpha}(\mathbb{T})$ with (1.70). Next, we shall concentrate on the first term $K_1 * G$ and prove that it belongs to $C^{1-\alpha}(\mathbb{T})$. For this end it is enough to show that $K_1 \in L^1(\mathbb{T})$ for $\alpha \in [0,1[$ which is more tricky. This claim is an immediate consequence of a more precise estimate : for any $\beta \in (\alpha, 1)$

$$|K_1(e^{i\theta})| \lesssim \frac{1}{\sin^\beta(\frac{\theta}{2})}, \quad \forall \theta \in (0, 2\pi)\cdot \tag{1.71}$$

This estimate sounds classical and for the convenience of the reader we shall give here a complete proof. The basic tool is Abel transform. We set

$$K_1^n(w) \triangleq \sum_{k=m+1}^{n} \frac{w^k}{k^{1-\alpha}} \quad \text{and} \quad U_n(w) \triangleq \sum_{k=0}^{n} w^k.$$

Then it is apparent that

$$\begin{aligned} K_1^n(w) &\triangleq \sum_{k=m+1}^{n} \frac{U_k(w) - U_{k-1}(w)}{k^{1-\alpha}} \\ &= \sum_{k=m+1}^{n} \frac{U_k(w)}{k^{1-\alpha}} - \sum_{k=m}^{n-1} \frac{U_k(w)}{(1+k)^{1-\alpha}} \\ &= \sum_{k=m+1}^{n-1} U_k(w)\Big(\frac{1}{k^{1-\alpha}} - \frac{1}{(1+k)^{1-\alpha}}\Big) + \frac{U_n(w)}{n^{1-\alpha}} - \frac{U_m(w)}{(m+1)^{1-\alpha}} \\ &\triangleq K_{1,1}^n(w) + K_{1,2}^n(w) + K_{1,3}(w). \end{aligned}$$

The last term is bounded independently of n and w. For the second term, it converges to zero as n goes to infinity for any $w \in \mathbb{T}\backslash\{1\}$. This follows easily from the estimate,

$$\begin{aligned} |K_{1,2}^n(w)| &\le \frac{|1 - w^{n+1}|}{|1-w|} \frac{1}{n^{1-\alpha}} \\ &\le \frac{2}{|1-w|} \frac{1}{n^{1-\alpha}}. \end{aligned}$$

As regards the first term $K_{1,1}^n$, we shall use the mean value theorem through the simple fact

$$0 \le \frac{1}{k^{1-\alpha}} - \frac{1}{(1+k)^{1-\alpha}} \lesssim \frac{1}{k^{2-\alpha}}\cdot$$

Hence we get

$$|K_{1,1}^n(w)| \lesssim \sum_{k=m+1}^{n-1} \frac{|U_k(w)|}{k^{2-\alpha}}.$$

Now we use the classical estimates

$$|U_k(w)| \le k+1, \quad |U_k(w)| \le \frac{1}{|\sin\frac{\theta}{2}|}, \; w = e^{i\theta}.$$

By an obvious convexity inequality we get for any $\beta \in [0,1]$

$$|U_k(w)| \le \frac{(k+1)^{1-\beta}}{|\sin\frac{\theta}{2}|^\beta}$$

and therefore

$$|K_{1,1}^n(w)| \lesssim \frac{1}{|\sin\frac{\theta}{2}|^\beta} \sum_{k=m+1}^{n-1} \frac{1}{k^{1-\alpha+\beta}}.$$

The partial sum of the series converges provided that we choose $\beta \in (\alpha, 1)$. Collecting the preceding estimates and passing to the limit when n goes to infinity we may write,

$$|K_1(w)| \lesssim \frac{1}{|\sin \frac{\theta}{2}|^\beta}$$

and this completes the proof of the inequality (2.98).

$(\mathbf{3})$ Now, we intend to check the transversality assumption. According to the continuity property of the second derivative $\partial_\Omega \partial_f F$ seen in Proposition 1.7, this assumption reduces to

$$\big\{\partial_\Omega \partial_f F(\Omega, 0)(h)\big\}\Big|_{\Omega=\Omega_m^\alpha, h=v_m} \notin R\Big(\partial_f F(\Omega_m^\alpha, 0)\Big).$$

Differentiating (1.54) with respect to Ω one gets

$$\partial_\Omega \partial_f F(\Omega, 0)(h)(w) = \operatorname{Im}\Big\{\overline{h'}(w) + \overline{w}h(w)\Big\}.$$

Then obviously

$$\partial_\Omega \partial_f F(\Omega_m^\alpha, 0)(\overline{w}^{m-1}) = i\frac{m}{2}\big(w^m - \overline{w}^m\big).$$

which is not in the range of $\partial_f F(\Omega_m^\alpha, 0)$ as it was described in the part (2) of Proposition 1.8. □

1.9 $m-$fold symmetry

Now we are ready to complete the proof of Theorem 1.3 started and developed throughout the preceding sections. We have gathered all the required elements to apply Theorem 1.2 of Crandal-Rabinowitz. Combining Proposition 1.7 and Proposition 1.8 we deduce the existence of non trivial curves $\{\mathcal{C}_m, m \geq 2\}$ bifurcating at the points Ω_m^α of the dispersion set $\mathcal{S}$ introduced in (1.55). Each point of the branch $\mathcal{C}_m$ represents a V-state and we shall now see that it is an m-fold symmetric in a similar way to the case of the incompressible Euler equations. This will be done by showing the bifurcation in spaces including the m-fold symmetry. To be more precise, let $m \geq 2$ and define the spaces X_m and Y_m as follows : the space X_m is the set of those functions $f \in X$ with a Fourier expansion of the type

$$f(w) = \sum_{n=1}^{\infty} a_{nm-1}\overline{w}^{nm-1}, \quad w \in \mathbb{T}.$$

equipped with the usual strong topology of $C^{2-\alpha}(\mathbb{T})$. We define the ball of radius $r \in (0, 1)$ by

$$B_r^m = \Big\{f \in X_m, \, \|f\|_{C^{2-\alpha}(\mathbb{T})} \leq r\Big\}.$$

If $f \in B_r^m$ the expansion of the associated conformal mapping ϕ in $\{z : |z| \geq 1\}$ is given by

$$\phi(z) = z + f(z) = z\Big(1 + \sum_{n=1}^{\infty} \frac{a_{nm-1}}{z^{nm}}\Big).$$

This will provide the $m-$fold symmetry of the associated patch $\phi(\mathbb{T})$, via the relation

$$\phi\big(e^{i2\pi/m}z\big) = e^{i2\pi/m}\phi(z), \quad |z| \geq 1. \tag{1.72}$$

The space Y_m is the subspace of Y consisting of those $g \in Y$ whose Fourier expansion is of the type

$$g(w) = i\sum_{n=1}^{\infty} g_{nm}\big(w^{nm} - \overline{w}^{nm}\big), \quad w \in \mathbb{T}.$$

To apply Crandall-Rabinowitz's Theorem and get the symmetry property stated in Theorem 1.3 it suffices to show the following result.

Proposition 1.9 *The following assertions hold true. Let $m \geq 2$ and $r \in (0,1)$, then*

1. *$F : \mathbb{R} \times B_r^m \to Y_m$ is well-defined.*
2. *The kernel of $\partial_f F(\Omega_m^\alpha, 0)$ is one-dimensional and generated by $w \mapsto \overline{w}^{m-1}$.*
3. *The range of $\partial_f F(\Omega_m^\alpha, 0)$ is closed in Y_m and is of co-dimension one.*

Proof :

$(\mathbf{1})$ Let $f \in B_r^m$, we shall show that $F(\Omega, f) = G(\Omega, \phi) \in Y_m$. Recall that the functional G is defined by

$$G(\Omega,\phi)(w) = \operatorname{Im}\Big\{\Big(\Omega\, \overline{w}\, \phi(w) - \overline{w}\, S(\phi)(w)\Big)\, \overline{\phi'(w)}\Big\},$$

with

$$S(\phi)(w) = C_\alpha \fint_{\mathbb{T}} \frac{\phi'(\tau)}{|\phi(w) - \phi(\tau)|^\alpha} d\tau.$$

It is easy to verify from (1.72) that the functions ϕ' and $w \mapsto \frac{\phi(w)}{w}$ belong to the space $C^{1-\alpha}(\mathbb{T})$ and their Fourier coefficients vanish at frequencies which are not integer multiples of m. Since this latter space is an algebra and stable by conjugation then the map $w \mapsto \operatorname{Im}\big\{\overline{\phi'(w)}\frac{\phi(w)}{w}\big\}$ belongs to the space Y_m. Therefore, it remains to show that $w \mapsto \operatorname{Im}\big\{\overline{\phi'(w)}\big(\overline{w}S(\phi)(w)\big)\big\} \in Y_m$. This follows easily once we have proved that $w \mapsto \overline{w}S(\phi)(w)$ satisfies (1.72). For this end, set

$$\begin{aligned}\Phi(w) &\triangleq \overline{w}S(\phi)(w)\\ &= \overline{w}\fint_{\mathbb{T}} \frac{\phi'(\tau)}{|\phi(w) - \phi(\tau)|^\alpha} d\tau.\end{aligned}$$

Then

$$\Phi(e^{i2\pi/m}w) = e^{-i2\pi/m}\overline{w}\fint_{\mathbb{T}} \frac{\phi'(\tau)}{|\phi(e^{i2\pi/m}w) - \phi(\tau)|^\alpha} d\tau.$$

By the change of variables $\tau = e^{i2\pi/m}\zeta$ and according to (1.72) we get for any $w \in \mathbb{T}$

$$\begin{aligned}\Phi(e^{i2\pi/m}w) &= \frac{1}{w}\fint_{\mathbb{T}} \frac{\phi'(e^{i2\pi/m}\zeta)}{|\phi(e^{i2\pi/m}w) - \phi(e^{i2\pi/m}\zeta)|^\alpha} d\zeta\\ &= \frac{1}{w}\fint_{\mathbb{T}} \frac{\phi'(\zeta)}{|\phi(w) - \phi(\zeta)|^\alpha} d\zeta\\ &= \Phi(w).\end{aligned}$$

Hence, The Fourier coefficients of Φ vanish at frequencies which are not integer multiples of m and this concludes the proof of the result,

$$f \in B_r^m \Longrightarrow F(\Omega, f) = G(\Omega,\phi) \in Y_m.$$

$(\mathbf{2})$ Since the generator $w \mapsto \overline{w}^{m-1}$ of the kernel of $\partial_f F(\Omega_m^\alpha, 0)$ belongs to X_m, we still have that the dimension of the kernel is 1.

$(\mathbf{3})$ Using Proposition 1.8, we deduce that

$$R\Big(\partial_f F(\Omega_m^\alpha, 0)\Big) = \Big\{g \in C^{1-\alpha}(\mathbb{T});\quad g(w) = i\sum_{n\geq 1, n\neq m}^{\infty} g_n(w^n - \overline{w}^n), g_n \in \mathbb{R}\Big\} \cap Y_m$$

$$= \Big\{g \in C^{1-\alpha}(\mathbb{T});\quad g(w) = i\sum_{n\geq 2}^{\infty} g_{nm}(w^{nm} - \overline{w}^{nm}), g_{nm} \in \mathbb{R}\Big\}.$$

Obviously, the range of $\partial_f F(\Omega_m, 0)$ is of co-dimension 1 in the space Y_m.

Therefore we can apply Crandall-Rabinowitz's Theorem to X_m and Y_m and obtain the existence of the $m-$fold symmetric patches for each integer $m \geq 2$. This achieves the proof of Theorem 1.3.

□

1.10 Limiting case $\alpha = 1$

In this section we shall discuss the limiting case $\alpha = 1$ corresponding to the SQG model. This case was excluded from Theorem 1.1 at least because the rotating patch model seen in (1.15) does not work due to the higher singularity of the kernel. Thus we shall modify a little bit this model as in [28] and give an equation of the boundary of the V-states. Although the model seems to be coherent and satisfactory, it is completely different from the sub-critical one $\alpha \in [0, 1[$ and generates more technical difficulties in studying the rotating patches. As we shall see later when we compute formally the linearized operator $\mathcal{L}_\Omega$ around the trivial solution we find that it behaves as a Fourier multiplier with an extra loss compared to the case $\alpha \in [0, 1[$ which is of logarithmic type. Thus the property $\mathcal{L}_\Omega : C^{1+\varepsilon} \to C^\varepsilon$ fails and one should find other suitable spaces X and Y satisfying the assumptions of C-R Theorem. We do believe that such spaces must exist but certainly this would require more sophisticated analysis than what we shall do here. Among our objective is to describe in details the dispersion relation which tells us where the bifurcating curves emerge from the trivial one and also shed light on the main difficulties encountered in this case.

1.10.1 Rotating patch model

First recall from in the equation (1.12) one can change the velocity at the boundary by subtracting a tangential vector to the boundary without changing the full equation. Thus we shall work with the following modified velocity : let γ_0 be a 2π periodic parametrization of the boundary of D, and define

$$u_0(\gamma_0(\sigma)) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial_s \gamma_0(s) - \partial_\sigma \gamma_0(\sigma)}{|\gamma_0(s) - \gamma_0(\sigma)|} ds. \tag{1.73}$$

Then, by substituting the expression of the velocity in the equation of the boundary (1.12) one gets

$$\Omega \text{Re}\big\{z\ \overline{z'}\big\} = \text{Im}\Big\{\frac{1}{2\pi}\int_{\partial D} \frac{\big(\zeta' - z'\big)}{|z - \zeta|^\alpha} \frac{d\zeta}{\zeta'}\ \overline{z'}\Big\}, \quad \forall z \in \partial D. \tag{1.74}$$

Next, we parametrize the domain with the outside conformal mapping $\phi : \mathbb{D}^c \to D^c$,

$$\phi(z) = z + \sum_{n\geq 0} \frac{b_n}{z^n} \tag{1.75}$$

by setting $z = \phi(w)$ and $\zeta = \phi(\tau)$. Then, we obtain the equation

$$G(\Omega, \phi)(w) \triangleq \mathrm{Im}\left\{ \left(\Omega\phi(w) - \fint_{\mathbb{T}} \frac{\tau\phi'(\tau) - w\phi'(w)}{|\phi(w) - \phi(\tau)|} \frac{d\tau}{\tau} \right) \frac{\overline{\phi'(w)}}{w} \right\} = 0, \quad \forall w \in \mathbb{T}, \tag{1.76}$$

which is nothing but the boundary equation of the rotating patches. As for the sub-critical case we define,

$$F(\Omega, f)(w) \triangleq G(\Omega, \mathrm{Id} + f)(w), \quad f(w) = \sum_{n\geq 0} \frac{b_n}{w^n}, \; w \in \mathbb{T}, \; b_n \in \mathbb{R}.$$

We point out that the Rankine vortices correspond to the trivial solutions $F(\Omega, 0) = 0$. This can be checked as follows.

$$\begin{aligned} F(\Omega, 0)(w) &= \mathrm{Im}\left\{ \left(\Omega w - \fint_{\mathbb{T}} \frac{\tau - w}{|w - \tau|} \frac{d\tau}{\tau} \right) \frac{1}{w} \right\} \\ &= -\mathrm{Im}\left\{ \frac{1}{w} \fint_{\mathbb{T}} \frac{\tau - w}{|w - \tau|} \frac{d\tau}{\tau} \right\}. \end{aligned}$$

In view of the next identity (1.77) applied with $n = 1$ we conclude that

$$F(\Omega, 0)(w) = 0, \quad \forall \Omega \in \mathbb{R}, \forall w \in \mathbb{T}.$$

We should mention in passing that we can get a similar result to the Proposition 1.5 and prove that the ellipses never rotate.

1.10.2 Integral computations

We shall discuss some elementary integrals that will appear later in the computations of the linearized operator.

Lemma 1.4 *Let $n \geq 1$ and $w \in \mathbb{T}$, then we have*

$$\fint_{\mathbb{T}} \frac{\tau^n - w^n}{|w - \tau|} \frac{d\tau}{\tau} = -\frac{2w^n}{\pi} \sum_{k=0}^{n-1} \frac{1}{2k+1}. \tag{1.77}$$

$$\fint_{\mathbb{T}} \frac{(\tau - w)^2(\tau^n - w^n)}{|w - \tau|^3} \frac{d\tau}{\tau} = \frac{2w^{n+2}}{\pi} \sum_{k=1}^{n} \frac{1}{2k+1}. \tag{1.78}$$

Proof : To prove (1.77) we use successively the change of variables $\tau = w\zeta$ and $\zeta = e^{i\eta}$

$$\begin{aligned} \fint_{\mathbb{T}} \frac{\tau^n - w^n}{|w - \tau|} \frac{d\tau}{\tau} &= w^n \fint_{\mathbb{T}} \frac{\zeta^n - 1}{|1 - \zeta|} \frac{d\zeta}{\zeta} \\ &= -w^n \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - e^{in\eta}}{|1 - e^{i\eta}|} d\eta \\ &= -w^n \frac{1}{\pi} \int_0^{\pi} \frac{1 - e^{i2n\eta}}{|1 - e^{i2\eta}|} d\eta. \end{aligned}$$

Using the identity
$$|1-e^{i2\eta}| = i(1-e^{i2\eta})e^{-i\eta}, \quad \eta\in[0,\pi].$$
we find easily
$$\begin{aligned}\fint_{\mathbb{T}}\frac{\tau^n-w^n}{|w-\tau|}\frac{d\tau}{\tau} &= -w^n\frac{1}{i\pi}\int_0^\pi\frac{1-e^{i2n\eta}}{1-e^{i2\eta}}e^{i\eta}d\eta\\ &= -w^n\frac{1}{i\pi}\sum_{k=0}^{n-1}\int_0^\pi e^{i(2k+1)\eta}d\eta\\ &= -w^n\frac{2}{\pi}\sum_{k=0}^{n-1}\frac{1}{2k+1}\cdot\end{aligned}$$
To compute the second integral we argue as before by using suitable change of variables,
$$\begin{aligned}\fint_{\mathbb{T}}\frac{(\tau-w)^2(\tau^n-w^n)}{|w-\tau|^3}\frac{d\tau}{\tau} &= -w^{n+2}\fint_{\mathbb{T}}\frac{(\zeta-1)^2(1-\zeta^n)}{|1-\zeta|^3}\frac{d\zeta}{\zeta}\\ &= -w^{n+2}\fint_{\mathbb{T}}\frac{(1-\zeta)(1-\zeta^n)}{(1-\overline{\zeta})|1-\zeta|}\frac{d\zeta}{\zeta}\\ &= -w^{n+2}\frac{1}{2\pi}\int_0^{2\pi}\frac{(1-e^{i\eta})(1-e^{in\eta})}{(1-e^{-i\eta})|1-e^{i\eta}|}d\eta.\end{aligned}$$
Thus we get
$$\begin{aligned}\fint_{\mathbb{T}}\frac{(\tau-w)^2(\tau^n-w^n)}{|w-\tau|^3}\frac{d\tau}{\tau} &= -w^{n+2}\frac{i}{\pi}\int_0^\pi\frac{e^{i3\eta}(1-e^{i2n\eta})}{1-e^{i2\eta}}d\eta\\ &= -w^{n+2}\frac{i}{\pi}\sum_{k=0}^{n-1}\int_0^\pi e^{i(2k+3)\eta}d\eta\\ &= w^{n+2}\frac{2}{\pi}\sum_{k=0}^{n-1}\frac{1}{2k+3}\cdot\end{aligned}$$
This concludes the proof of the lemma. □

1.10.3 Dispersion relation

We shall now compute the Gâteaux derivative of F with respect to f in the direction h, denoted as before by $\partial_f F(\Omega,f)h$. Afterwards, we shall exhibit the dispersion set corresponding to the values of Ω where the kernel of the linearized operator around $f=0$ is non trivial. Using (1.76)
$$\begin{aligned}\partial_f F(\Omega,f)h(w) &= \frac{d}{dt}F(\Omega,f+th)(w)_{|t=0}\\ &= \mathrm{Im}\bigg\{\Omega\Big(\phi(w)\overline{h'(w)}+\frac{\overline{\phi'(w)}}{w}h(w)\Big)\\ &\quad -\frac{\overline{h'(w)}}{w}\fint_{\mathbb{T}}\frac{\tau\phi'(\tau)-w\phi'(w)}{|\phi(w)-\phi(\tau)|}\frac{d\tau}{\tau}-\frac{\overline{\phi'(w)}}{w}\fint_{\mathbb{T}}\frac{\tau h'(\tau)-wh'(w)}{|\phi(w)-\phi(\tau)|}\frac{d\tau}{\tau}\\ &\quad +\frac{\overline{\phi'(w)}}{w}\fint_{\mathbb{T}}\frac{\big(\tau\phi'(\tau)-w\phi'(w)\big)\mathrm{Re}\Big\{\big(\overline{h(\tau)}-\overline{h(w)}\big)\big(\phi(\tau)-\phi(w)\big)\Big\}}{|\phi(w)-\phi(\tau)|^3}\frac{d\tau}{\tau}\bigg\}\end{aligned}\tag{1.79}$$

with the notation $\phi = \text{Id} + f$.

In the particular case $f = 0$ one has

$$\begin{aligned}\partial_f F(\Omega,0)h(w) &= \frac{d}{dt}F(\Omega,th)(w)_{|t=0}\\ &= \text{Im}\bigg\{\Omega\big(\overline{h'(w)}+\overline{w}h(w)\big) - \fint_{\mathbb{T}}\frac{\tau h'(\tau)-wh'(w)}{w|w-\tau|}\frac{d\tau}{\tau} - \frac{\overline{h'(w)}}{w}\fint_{\mathbb{T}}\frac{\tau-w}{|w-\tau|}\frac{d\tau}{\tau}\\ &\quad + \frac{1}{2w}\fint_{\mathbb{T}}\frac{h(\tau)-h(w)}{|w-\tau|}\frac{d\tau}{\tau} + \frac{1}{2w}\fint_{\mathbb{T}}\frac{(\tau-w)^2\big(\overline{h(\tau)}-\overline{h(w)}\big)}{|w-\tau|^3}\frac{d\tau}{\tau}\bigg\}. \qquad (1.80)\end{aligned}$$

with $h(w) = \sum_{n\geq 0}\frac{b_n}{w^n}$ and $b_n \in \mathbb{R}$ for all $n \geq 1$.

Our next goal is to look for the values Ω where the linearized operator fails to be injective. We will be seeing that the function spaces that we shall use differs from the ones of the case $\alpha \in (0,1)$. We will abandon the use of Hölder spaces which generate more technical difficulties. We introduce the spaces,

$$B^s(\mathbb{T}) = \Big\{f(w) = \sum_{n\geq 0} b_n\overline{w}^n, b_n \in \mathbb{R},\ \|f\|_{B^s} < \infty\Big\},\ \|f\|_{B^s} = |b_0| + \sum_{n\geq 1} n^s|b_n|$$

and

$$B^s_{\text{Log}}(\mathbb{T}) = \Big\{g(w) = i\sum_{n\geq 1} g_n\big(w^n - \overline{w}^n\big), g_n \in \mathbb{R},\ \|g\|_{B^s_{\text{Log}}} < \infty\Big\},\ \|g\|_{B^s_{\text{Log}}} = \sum_{n\geq 1}\frac{n^s}{1+\ln n}|g_n|.$$

First we define the dispersion set

$$\mathcal{S}_1 = \Big\{\Omega = \frac{2}{\pi}\sum_{k=1}^{m-1}\frac{1}{2k+1} \quad m \geq 2\Big\}.$$

The main result of this section reads as follows.

Proposition 1.10 *Let $s \geq 1$, $m \geq 2$.*

1. *For any $\Omega \in \mathbb{R}$, $\partial_f F(\Omega,0) : B^s(\mathbb{T}) \to B^{s-1}_{\text{Log}}(\mathbb{T})$ is continuous.*
2. *The kernel of $\partial_f F(\Omega,0)$ is non trivial if and only if $\Omega \in \mathcal{S}$ and, in this case, it is a one-dimensional vector space generated by*

$$v_m(w) = \overline{w}^{m-1} \quad \textit{with} \quad \Omega = \Omega^1_m \triangleq \frac{2}{\pi}\sum_{k=1}^{m-1}\frac{1}{2k+1}.$$

3. *The range of $\partial_f F(\Omega^1_m,0)$ is closed in $B^{s-1}_{\text{Log}}(\mathbb{T})$ and is of co-dimension one. It is given by*

$$R\big(\partial_f F(\Omega^1_m,0)\big) = \Big\{g \in B^s_{\text{Log}}(\mathbb{T}),\quad g(w) = i\sum_{\substack{n\geq 1\\ n\neq m}}^{\infty} g_n(w^n - \overline{w}^n),\ g_n \in \mathbb{R}\Big\}.$$

4. *Transversality assumption :*

$$\partial_\Omega\partial_f F(\Omega^1_m,0)(v_m) \notin R\big(\partial_f F(\Omega^1_m,0)\big).$$

Before giving the proof of this result, we should make few comments.

Remark 1.1 *1. The dispersion relation was discovered in [4] by using another analytical approach based on Bessel functions. The proof that we shall present is different and is somehow elementary.*

2. The spaces $B^s(\mathbb{T})$ and $B^{s-1}_{\text{Log}}(\mathbb{T})$ introduced above are well-adapted to the study of the linear operator but it is not at all clear whether the nonlinear function F sends $B^s(\mathbb{T})$ into $B^{s-1}_{\text{Log}}(\mathbb{T})$ and satisfies the regularity properties required by C-R Theorem. If this is the case then one can show the existence of the V-states for the SQG equation.

Proof : $(\mathbf{1}) - (\mathbf{2})$. We shall prove in the same time the two points. We start with replacing h and $\overline{h'}$ in the identity (1.80) by their Fourier expansions,

$$h(w) = \sum_{n\geq 0} b_n \overline{w}^n \quad \text{and} \quad \overline{h'(w)} = -\sum_{n\geq 0} n b_n w^{n+1}.$$

Therefore we get

$$\begin{aligned}\partial_f F(\Omega,0)h(w) = \text{Im}\Big\{\Omega\sum_{n\geq 0} b_n\big(\overline{w}^{n+1} - n w^{n+1}\big) + \sum_{n\geq 1} n b_n \overline{w} \fint_{\mathbb{T}} \frac{\overline{\tau}^n - \overline{w}^n}{|w-\tau|}\frac{d\tau}{\tau} \\ + \sum_{n\geq 1} n b_n w^n \fint_{\mathbb{T}} \frac{\tau - w}{|w-\tau|}\frac{d\tau}{\tau} + \frac{1}{2}\sum_{n\geq 1} b_n \overline{w}\fint_{\mathbb{T}} \frac{\overline{\tau}^n - \overline{w}^n}{|w-\tau|}\frac{d\tau}{\tau} \\ + \frac{1}{2}\sum_{n\geq 1} b_n \overline{w}\fint_{\mathbb{T}} \frac{(w-\tau)^2(\tau^n - w^n)}{|w-\tau|}\frac{d\tau}{\tau}\Big\}.\end{aligned}$$

Note that

$$\fint_{\mathbb{T}} \frac{\overline{\tau}^n - \overline{w}^n}{|w-\tau|}\frac{d\tau}{\tau} = \overline{\fint_{\mathbb{T}} \frac{\tau^n - w^n}{|w-\tau|}\frac{d\tau}{\tau}}, \tag{1.81}$$

and consequently we can rewrite in view of Lemma 1.4 the linear operator as follows,

$$\begin{aligned}\partial_f F(\Omega,0)h(w) =& \text{Im}\Big\{\Omega\sum_{n\geq 0} b_n\big(\overline{w}^{n+1} - n w^{n+1}\big) - \frac{2}{\pi}\sum_{n\geq 1} n b_n \overline{w}^{n+1}\sum_{k=0}^{n-1}\frac{1}{2k+1} \\ & - \frac{2}{\pi}\sum_{n\geq 1} n b_n w^{n+1} - \frac{1}{\pi}\sum_{n\geq 0} b_n \overline{w}^{n+1}\sum_{k=0}^{n-1}\frac{1}{2k+1} + \frac{1}{\pi}\sum_{n\geq 0} b_n w^{n+1}\sum_{k=1}^{n}\frac{1}{2k+1}\Big\} \\ =& \frac{b_0\Omega}{2}i(w-\overline{w}) + \text{Im}\Big\{\sum_{n\geq 1} b_n\Big(\Omega - \alpha_n\Big)\overline{w}^{n+1} - \sum_{n\geq 1} b_n\Big(n\Omega - \beta_n\Big)w^{n+1}\Big\} \\ =& \frac{b_0\Omega}{2}i(w-\overline{w}) + \frac{1}{2i}\sum_{n\geq 1}(n+1)\Big(\Omega - \frac{\alpha_n+\beta_n}{n+1}\Big)b_n\big(\overline{w}^{n+1} - w^{n+1}\big),\end{aligned} \tag{1.82}$$

where

$$\alpha_n \triangleq \frac{2n+1}{\pi}\sum_{k=0}^{n-1}\frac{1}{2k+1}$$

and

$$\beta_n \triangleq -\frac{2n}{\pi} + \frac{1}{\pi}\sum_{k=1}^{n}\frac{1}{2k+1}.$$

It is plain to see that

$$\begin{aligned}\alpha_n+\beta_n &= \frac{2n+1}{\pi}\Big(1-\frac{1}{2n+1}+\sum_{k=1}^{n}\frac{1}{2k+1}\Big)-\frac{2n}{\pi}+\frac{1}{\pi}\sum_{k=1}^{n}\frac{1}{2k+1}\\ &= \frac{2(n+1)}{\pi}\sum_{k=1}^{n}\frac{1}{2k+1}\\ &\triangleq (n+1)\Omega^1_{n+1}.\end{aligned}$$

Inserting this formula into (1.82) we obtain

$$\partial_f F(\Omega,0)h(w)=\frac{b_0\Omega}{2}i(w-\overline{w})+\frac{i}{2}\sum_{n\geq 1}(n+1)\Big(\Omega-\Omega^1_{n+1}\Big)b_n\big(w^{n+1}-\overline{w}^{n+1}\big).\qquad(1.83)$$

To check that $\partial_f F(\Omega,0):B^s(\mathbb{T})\to B^{s-1}_{\text{Log}}(\mathbb{T})$ is continuous we write

$$\begin{aligned}\big\|\partial_f F(\Omega,0)h\big\|_{B^{s-1}_{\text{Log}}} &= \frac{1}{2}|b_0\Omega|+\frac{1}{2}\sum_{n\geq 1}\frac{(1+n)^s}{1+\ln(1+n)}\big|\Omega-\Omega^1_{n+1}\big|\,|b_n|\\ &\leq \frac{1}{2}|b_0\Omega|+C\sum_{n\geq 1}\frac{n^s}{1+\ln n}\big|\Omega-\Omega^1_{n+1}\big|\,|b_n|\\ &\leq C\Omega\|h\|_{B^s}+C\sum_{n\geq 1}\frac{n^s}{1+\ln n}\sum_{k=1}^{n}\frac{1}{2k+1}\,|b_n|.\end{aligned}$$

To estimate the last term we shall use the asymptotic behavior of the harmonic series

$$\sum_{k=1}^{n}\frac{1}{2k+1}=\frac{1}{2}\ln n+\frac{1}{2}\gamma+\ln 2-1+O(\frac{1}{n})\qquad(1.84)$$

which yields,

$$\big\|\partial_f F(\Omega,0)h\big\|_{B^{s-1}_{\text{Log}}}\leq C\|h\|_{B^s}.$$

This concludes the continuity of the linear operator $\partial_f F(\Omega,0):B^s(\mathbb{T})\to B^{s-1}_{\text{Log}}$.

Now we shall study the kernel of this operator. From the formulae (1.83) we immediately deduce that the kernel of $\partial_f F(\Omega,0)$ is non trivial if and only if there exists $m\geq 2$ such that

$$\Omega=\Omega^1_m=\frac{2}{\pi}\sum_{k=1}^{m-1}\frac{1}{2k+1}.$$

In which case the kernel contains the eigenfunction $w\mapsto\overline{w}^{m-1}$. Moreover, it is one-dimensional vector space because the sequence $n\mapsto\Omega_n$ is strictly increasing and therefore the Fourier coefficients in (1.83) satisfy

$$(1+n)\Big(\Omega^1_m-\Omega^1_{n+1}\Big)\neq 0,\quad\forall\, n\neq m-1.$$

This achieves the proof of a simple kernel.

(**3**) Denote by

$$Z_m\triangleq\Big\{g\in B^s_{\text{Log}}(\mathbb{T}),\quad g(w)=i\sum_{\substack{n\geq 1\\ n\neq m}}^{\infty}g_n(w^n-\overline{w}^n)\Big\}.$$

Clearly Z_m is a closed subspace of $B^s_{\text{Log}}(\mathbb{T})$ and from (1.83) we deduce the obvious embedding $R(\partial_f F(\Omega_m, 0)) \subset Z_m$. Thus it remains to check the converse, that is, for any $g \in Z_m$ there exists $w \mapsto h(w) = \sum_{n\geq 0} b_n \overline{w}^n \in B^s(\mathbb{T})$ such that $\partial_f F(\Omega_m, 0)h = g$. In terms of Fourier coefficients this is equivalent to

$$b_0 \Omega^1_m = 2g_0, \quad n\Big(\Omega^1_m - \Omega^1_n\Big) b_{n-1} = 2g_n, \quad \forall n \geq 2,$$

This defines only one sequence $(b_n)_{n\neq m-1}$ and the coefficient b_{m-1} is free. To check the regularity of h it suffices to prove that

$$w \mapsto H(w) = \sum_{n\geq m} b_n \overline{w}^n \in B^s(\mathbb{T}).$$

According to the definition of the norm of B^s and (1.84) one gets

$$\begin{aligned}
\|H\|_{B^s} &= \sum_{n\geq m} n^s |b_n| \\
&= 2\sum_{n\geq m} n^s \frac{|g_{n+1}|}{(1+n)(\Omega^1_{n+1} - \Omega^1_m)} \\
&\leq C\sum_{n\geq m} \frac{n^{s-1}}{\ln n} |g_{n+1}| \\
&\leq \|g\|_{B^s_{\text{Log}}}.
\end{aligned}$$

This completes the proof of $Z_m = R(\partial_f F(\Omega^1_m, 0))$.

(4) We shall now check the transversality assumption

$$\partial_\Omega \partial_f F(\Omega^1_m, 0)(v_m) \notin R\big(\partial_f F(\Omega^1_m, 0)\big).$$

Differentiating (1.83) one gets

$$\partial_\Omega \partial_f F(\Omega, 0)(h)(w) = \text{Im}\Big\{\overline{h'(w)} + \overline{w} h(w)\Big\}.$$

Then

$$\partial_\Omega \partial_f F(\Omega^1_m, 0)(v_m) = i\frac{m}{2}\big(w^m - \overline{w}^m\big),$$

which is not clearly in the subspace $Z_m = \partial_f F(\Omega_m, 0)$ as it was claimed. The proof of Proposition 1.10 is now completed. □

1.11 Numerical study of the simply-connected V-states

This part is done for us by F. de la Hoz and taken from the paper [55].

Even if there is a number of references on the numerical obtention of rotating V-states for the vortex patch problem (see for instance [34] and [38], and more recently [64]), up to our knowledge nothing similar has been done for the quasi-geostrophic problem. Therefore, for the sake of completeness, we will discuss in this section the numerical obtention of V-states for the quasi-geostrophic problem in the simply-connected case. Since the procedure is very similar to that in the vortex patch problem, we will omit some details, which can be consulted in [64].

We gather the main theoretical arguments from [54]. Given a simply-connected domain D with boundary $z(\theta)$, where $\theta \in [0, 2\pi)$ is the Lagrangian parameter, and z is counterclockwise parameterized, the contour dynamics equation for the quasi-geostrophic problem is

$$z_t(\theta, t) = \frac{C_\alpha}{2\pi} \int_0^{2\pi} \frac{z_\phi(\phi, t) d\phi}{|z(\phi, t) - z(\theta, t)|^\alpha}; \tag{1.85}$$

from now on, in order not to burden the notation, we will not indicate explicitly the dependence on t.

The simply-connected domain D is a V-state rotating with constant angular velocity Ω, if and only if its boundary satisfies the following equation :

$$\mathrm{Re}\left[\left(\Omega z(\theta) - \frac{C_\alpha}{2\pi i} \int_0^{2\pi} \frac{z_\phi(\phi) d\phi}{|z(\phi) - z(\theta)|^\alpha}\right)\overline{z_\theta(\theta)}\right] = 0. \tag{1.86}$$

However, it is convenient to rewrite (1.86) in the following equivalent form :

$$\mathrm{Re}\left[\left(\Omega z(\theta) - \frac{C_\alpha}{2\pi i} \int_0^{2\pi} \frac{(z_\phi(\phi) - z_\theta(\theta)) d\phi}{|z(\phi) - z(\theta)|^\alpha}\right)\overline{z_\theta(\theta)}\right] = 0. \tag{1.87}$$

We use a pseudo-spectral method to find m-fold V-states from (1.87). We discretize $\theta \in [0, 2\pi)$ in N equally spaced nodes $\theta_i = 2\pi i/N$, $i = 0, 1, \ldots, N-1$. Observe that, although (1.86) and (1.87) are trivially equivalent, the addition of $z_\theta(\theta)$ in the numerator cancels the singularity in the denominator ; indeed,

$$\lim_{\phi \to \theta} \frac{z_\phi(\phi) - z_\theta(\theta)}{|z(\phi) - z(\theta)|^\alpha} = 0. \tag{1.88}$$

Therefore, bearing in mind (1.88), we can evaluate numerically with spectral accuracy the integral in (1.87) at a node $\theta = \theta_i$, by means of the trapezoidal rule, provided that N is large enough :

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{(z_\phi(\phi) - z_\theta(\theta_i)) d\phi}{|z(\phi) - z(\theta_i)|^\alpha} \approx \frac{1}{N} \sum_{\substack{j=0 \\ j \neq i}}^{N-1} \frac{z_\phi(\phi_j) - z_\theta(\theta_i)}{|z(\phi_j) - z(\theta_i)|^\alpha}. \tag{1.89}$$

Now, in order to obtain m-fold V-states, we approximate the boundary z as

$$z(\theta) = e^{i\theta}\left[1 + \sum_{k=1}^{M} a_k \cos(m\,k\,\theta)\right], \quad \theta \in [0, 2\pi), \tag{1.90}$$

where the mean radius is 1 ; and we are imposing that $z(-\theta) = \overline{z(\theta)}$, i.e., we are looking for V-states symmetric with respect to the x-axis. For sampling purposes, N has to be chosen such that $N \geq 2mM + 1$; additionally, it is convenient to take N a multiple of m, in order to be able to reduce the N-element discrete Fourier transforms to N/m-element discrete Fourier transforms. If we write $N = m2^r$, then $M = \lfloor (m2^r - 1)/(2m) \rfloor = 2^{r-1} - 1$.

We introduce (1.90) into (1.87), and approximate the error in that equation by an M-term sine expansion :

$$\mathrm{Re}\left[\left(\Omega z(\theta) - \frac{C_\alpha}{2\pi i} \int_0^{2\pi} \frac{(z_\phi(\phi) - z_\theta(\theta)) d\phi}{|z(\phi) - z(\theta)|^\alpha}\right)\overline{z_\theta(\theta)}\right] = \sum_{k=1}^{M} b_k \sin(m\,k\,\theta). \tag{1.91}$$

This last expression can be represented in a very compact way as

$$\mathcal{F}_{\alpha,\Omega}(a_1,\ldots,a_M)=(b_1,\ldots,b_M), \tag{1.92}$$

for a certain $\mathcal{F}_{\alpha,\Omega}\,:\,\mathbb{R}^M\to\mathbb{R}^M$.

Remark that, for any value of the parameters α and Ω, we have trivially $\mathcal{F}_{\alpha,\Omega}(\mathbf{0})=\mathbf{0}$, i.e., the unit circumference is a solution of the problem. Therefore, the obtention of a simply-connected V-state is reduced to finding numerically a nontrivial root $(a_1,\ldots,a_M)$ of (1.92). To do so, we discretize the $(M\times M)$-dimensional Jacobian matrix $\mathcal{J}$ of $\mathcal{F}_{\alpha,\Omega}$ using first-order approximations. Fixed $|h|\ll 1$ (we have chosen $h=10^{-9}$), we have

$$\frac{\partial}{\partial a_1}\mathcal{F}_{\alpha,\Omega}(a_1,\ldots,a_M)\approx\frac{\mathcal{F}_{\alpha,\Omega}(a_1+h,\ldots,a_M)-\mathcal{F}_{\alpha,\Omega}(a_1,\ldots,a_M)}{h}. \tag{1.93}$$

Then, the sine expansion of (1.93) gives us the first row of $\mathcal{J}$, and so on. Hence, if the n-th iteration is denoted by $(a_1,\ldots,a_M)^{(n)}$, then the $(n+1)$-th iteration is given by

$$(a_1,\ldots,a_M)^{(n+1)}=(a_1,\ldots,a_M)^{(n)}-\mathcal{F}_{\alpha,\Omega}\left((a_1,\ldots,a_M)^{(n)}\right)\cdot[\mathcal{J}^{(n)}]^{-1}, \tag{1.94}$$

where $[\mathcal{J}^{(n)}]^{-1}$ denotes the inverse of the Jacobian matrix at $(a_1,\ldots,a_M)^{(n)}$. This iteration converges in a small number of steps to a nontrivial root for a large variety of initial data $(a_1,\ldots,a_M)^{(0)}$. In particular, it is usually enough to perturb the unit circumference by assigning a small value to ${a_1}^{(0)}$, and leave the other coefficients equal to zero. Our stopping criterion is

$$\max\left|\sum_{k=1}^{M}b_k\sin(m\,k\,\theta)\right|<tol, \tag{1.95}$$

where $tol=10^{-11}$. For the sake of coherence, we change eventually the sign of all the coefficients $\{a_k\}$, in order that, without loss of generality, $a_1>0$.

Before moving forward, let us mention that it is possible to work numerically with other parametrizations than (1.90), like for example

$$z(\theta)=e^{i\theta}\Big(1+\sum_{k=1}^{M}a_k e^{-imk\theta}\Big),\quad \theta\in[0,2\pi), \tag{1.96}$$

where we have used a conformal mapping. However, a caveat should be make here. Indeed, unlike in the vortex patch problem, given a V-state (z,Ω), and $\mu>0$, $(\mu z,\Omega)$ is no longer a V-state, but, from (1.87), $(\mu z,\mu^{-\alpha}\Omega)$ is. Therefore, since we bifurcate from the unit circumference at a certain angular velocity $\Omega=\Omega_m^\alpha$, we always obtain, by uniqueness, the same V-states up to a scaling that implies also a modification on Ω, irrespectively of the chosen numerical representation of z. An equivalent observation can be done for the doubly-connected case, etc.

Numerical experiments

According to Theorem 1 in [54], for fixed $\alpha\in(0,1)$ and $m\geq 2$, there is a family of m-fold V-states bifurcating from the unit circumference at the angular velocity

$$\Omega_m^\alpha\triangleq\frac{\Gamma(1-\alpha)}{2^{1-\alpha}\Gamma^2(1-\frac{\alpha}{2})}\left(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})}-\frac{\Gamma(m+\frac{\alpha}{2})}{\Gamma(m+1-\frac{\alpha}{2})}\right). \tag{1.97}$$

Given an Ω slightly smaller than Ω_m^α, it is straightforward to obtain the corresponding V-state with the technique described above. Then, we can use that V-state as a new initial datum to obtain another V-state with smaller Ω, and so on. However, it seems impossible to obtain numerically V-states for Ω strictly larger than Ω_m^α. This means that the bifurcation is pitchfork and this fact follows from a symmetry argument : if (Ω, z) is a solution of (1.86) then $(\Omega, -z)$ is a solution too.

Bearing in mind (1.97), we are able to obtain V-states for an arbitrary large number of symmetries m. For instance, in Figure 1.1, we have plotted simultaneously the V-states corresponding to $\alpha = 0.5$, $m = 10$, for $\Omega = 0.5592$ and $\Omega = 0.556, 0.552, \ldots, 0.528$. In all the numerical experiments in this section, we take $N = 256 \times m$ nodes. Since, according to (1.97), $\Omega_{10}^{0.5} = 0.559238\ldots$, the V-state corresponding to $\Omega = 0.5592$, in black, is practically a circumference, as expected. On the other hand, the V-state corresponding to $\Omega = 0.528$ is plotted in red. Observe that we have been unable to obtain the V-state corresponding to, say, $\Omega = 0.527$; this makes us wonder whether the V-state in red might be close from developing some kind of singularity.

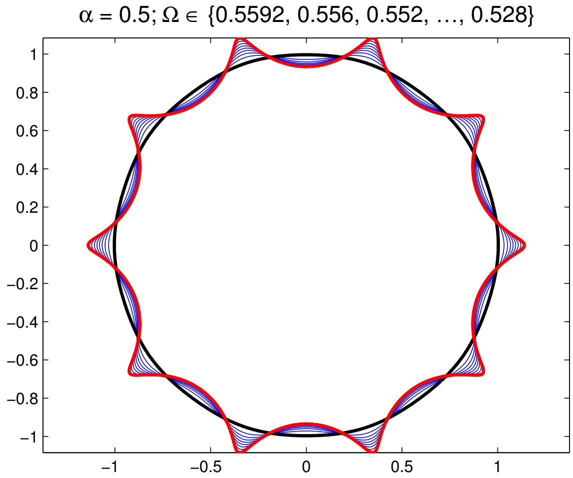

FIGURE 1.1 – 10-fold V-states corresponding to $\alpha = 0.5$, for different values of Ω.

It is an established fact that simply-connected limiting V-states do exist for $\alpha = 0$, which corresponds to the vortex patch problem. These V-states are obtained after bifurcating from the circumference at $\Omega_m \triangleq (m-1)/(2m)$, which corresponds to (1.97) evaluated at $\alpha = 0$, and decreasing Ω as much as possible, until corner-shaped singularities appear. Furthermore, it has been proved in [100, 125] that the angle at the corners is always $\pi/2$, irrespectively of the number m of symmetries. Therefore, we are interested in understanding what happens when $\alpha > 0$.

In Figure 1.2, we have plotted V-states corresponding to $m = 3, 4, 5, 6, 7$; for the vortex patch problem, and for $\alpha = 0.1$, $\alpha = 0.5$, and $\alpha = 0.9$ (i.e, a value of α rather close to zero, an intermediate value, and a value rather close to one). We have used the smallest possible (four-digit) values of Ω, which are offered in Table 1.1, in such a way that the experiments become numerically instable for smaller values of Ω. In order to facilitate the comparison between different m, we have plotted $z(\theta)/\max_\theta(|z(\theta)|)$, rather than $z(\theta)$.

Figure 1.2 confirms graphically that the angles developed by the five limiting V-states in the vortex patch problem are identical. On the other hand, when $\alpha = 0.1$, the V-states depicted are very similar to the limiting V-states in the vortex patch problem, whereas, for $\alpha = 0.5$, and

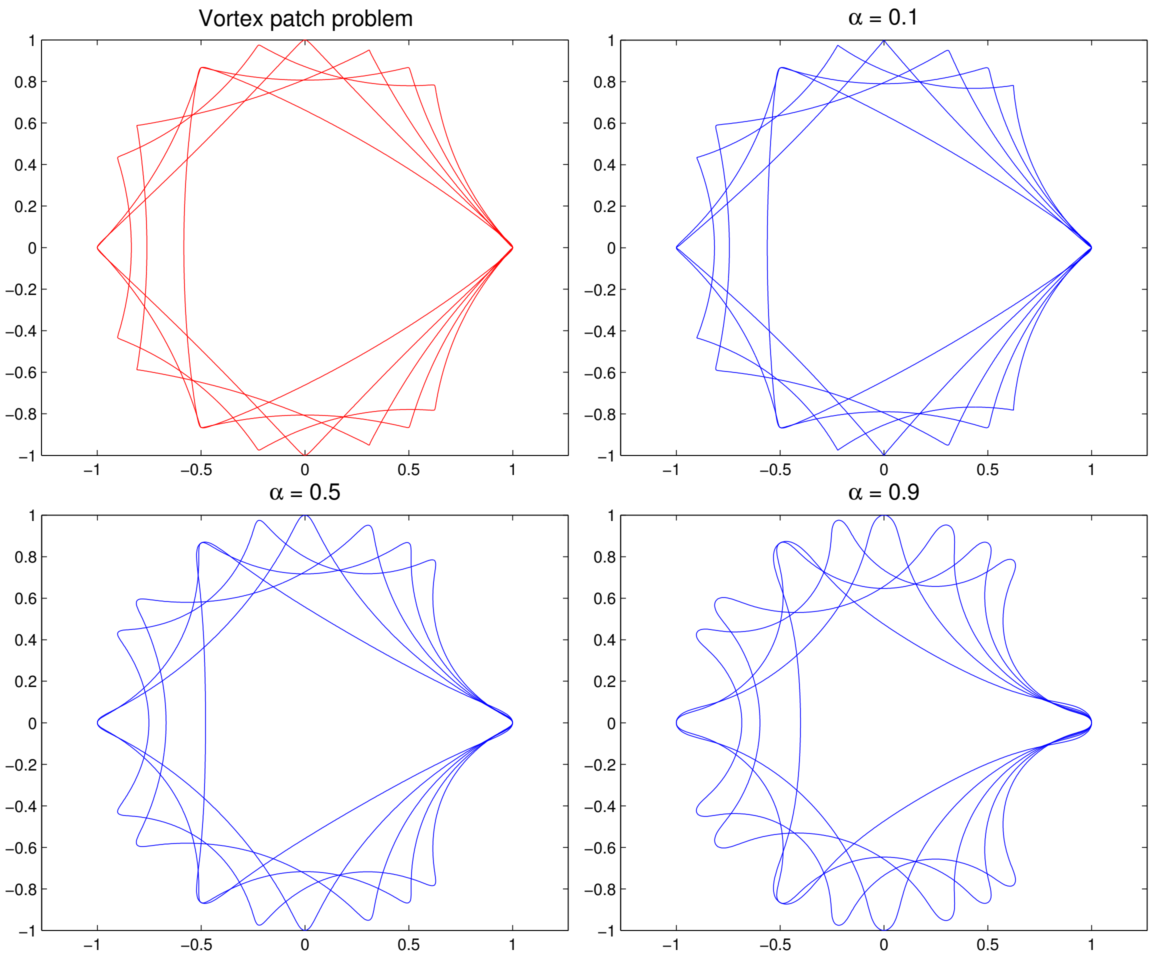

FIGURE 1.2 – V-states corresponding to the vortex patch problem (i.e., $\alpha = 0$), and to $\alpha = 0.1$, $\alpha = 0.5$, and $\alpha = 0.9$; with $m = 3, 4, 5, 6, 7$. To facilitate the comparison between different m, we have plotted $z(\theta)/\max_\theta(|z(\theta)|)$, instead of $z(\theta)$. We have chosen the smallest possible (four-digit) values of Ω, which are offered in Table 1.1.

α / m	3	4	5	6	7
0	0.3013	0.3540	0.3842	0.4040	0.4180
0.1	0.2965	0.3544	0.3884	0.4112	0.4275
0.5	0.2690	0.3498	0.4021	0.4399	0.4689
0.9	0.2191	0.3202	0.3918	0.4476	0.4933

TABLE 1.1 – Values of Ω for the V-states plotted in Figure 1.2. The case $\alpha = 0$ corresponds to the vortex patch problem.

especially for $\alpha = 0.9$, it is unclear whether any singularity has happened at all. In order to shed some light on this, we have plotted in Figure 1.3 the respective bifurcation diagrams of a_1 in (1.90) with respect to Ω. In the vortex patch problem, we have a family of monotonic curves already shown in [108]. When $\alpha = 0.1$, the curves are very similar to those in the vortex patch problem, but slightly bigger and more spaced. Then, as α grows, the curves become bigger and bigger, and more and more spaced. Furthermore, when $\alpha = 0.9$, the curves are partially superposed ; for example, there are 2-fold and 3-fold V-states with the same Ω. This phenomenon also happens in the last four curves, when $\alpha = 0.5$. However, the most striking fact from Figure 1.3 is that all the fifteen curves, corresponding to $\alpha = 0.1$, $\alpha = 0.5$ and $\alpha = 0.9$, lose monotonicity at their left ends. In fact, especially for $\alpha = 0.9$, incipient hooks are clearly visible. This seems to suggest the presence of saddle-node bifurcation points (see for instance [75]) at a certain $\Omega = \Omega_c$, which is indeed the case, as we will show in the following lines.

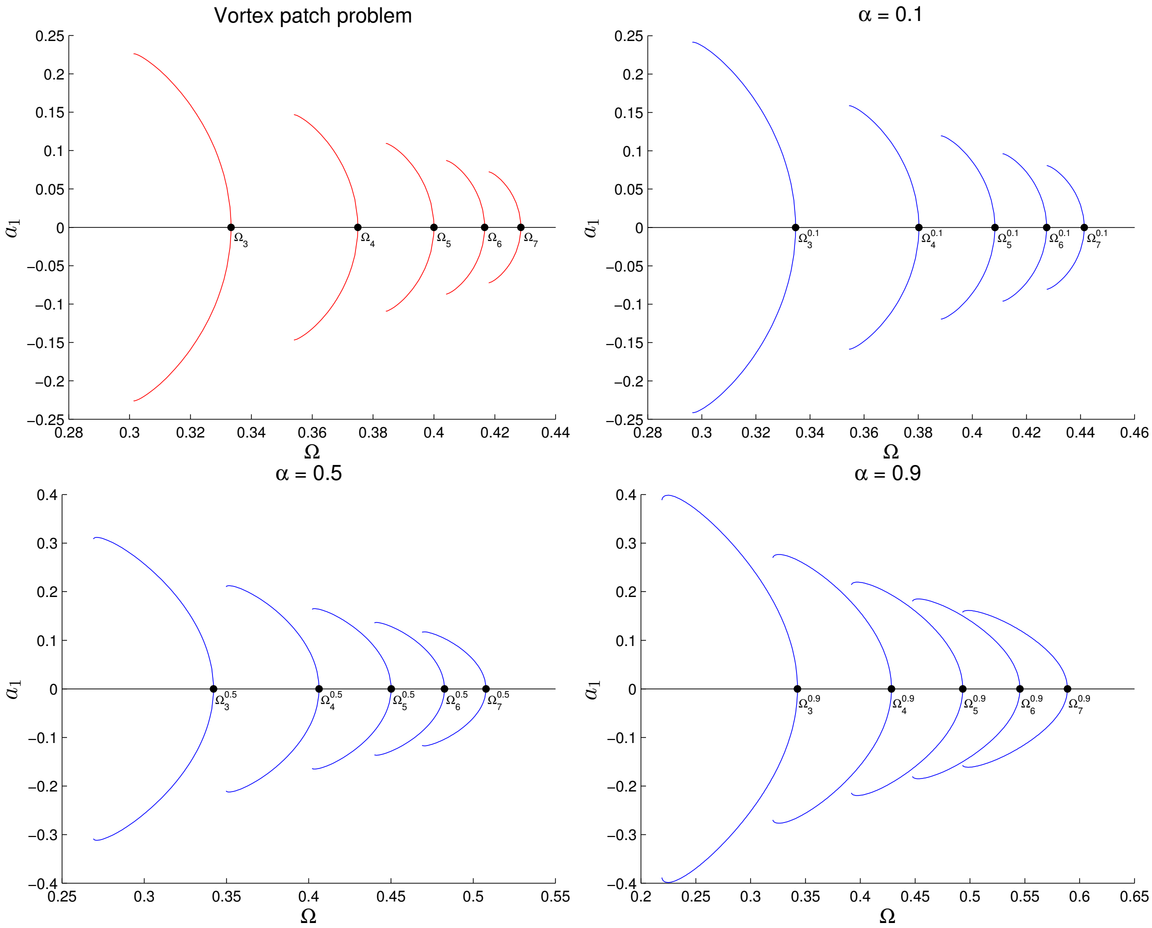

FIGURE 1.3 – Bifurcations diagrams, for the vortex patch problem (i.e., $\alpha = 0$), and for $\alpha = 0.1$, $\alpha = 0.5$, and $\alpha = 0.9$; with $m = 3, 4, 5, 6, 7$.

We work with the bifurcation curve corresponding to $\alpha = 0.9$ and $m = 3$ in Figure 1.3, because it has the most pronounced hook, but everything that follows is applicable to the other bifurcation curves as well. We need to estimate the corresponding Ω_c with enough accuracy. In our case, we have taken $\Omega_c = 0.21904$. Then, given $0 < \epsilon \ll 1$ (we have taken here $\epsilon = 10^{-4}$), we calculate the V-states corresponding to $\Omega^{(A)} \equiv \Omega_c + 4\epsilon$, $\Omega^{(B)} \equiv \Omega_c + 3\epsilon$, $\Omega^{(C)} \equiv \Omega_c + 2\epsilon$, $\Omega^{(D)} \equiv \Omega_c + \epsilon$, and $\Omega^{(E)} \equiv \Omega_c$, whose coefficients in (1.90) are respectively denoted as $\{a_k^{(A)}\}$, $\{a_k^{(B)}\}$, $\{a_k^{(C)}\}$, $\{a_k^{(D)}\}$, and $\{a_k^{(E)}\}$. The main idea is to introduce a new parameter λ, instead of Ω, in such a way that $\lambda = \lambda^{(A)}$ corresponds to $\Omega = \Omega^{(A)}$, and so on. We set $\lambda^{(A)} = 0$, $\lambda^{(B)} = \lambda^{(A)} + [(\Omega^{(B)} - \Omega^{(A)})^2 + (a_1^{(B)} - a_1^{(A)})^2]^{1/2}$, $\lambda^{(C)} = \lambda^{(B)} + [(\Omega^{(C)} - \Omega^{(B)})^2 + (a_1^{(C)} - a_1^{(B)})^2]^{1/2}$, $\lambda^{(D)} = \lambda^{(C)} + [(\Omega^{(D)} - \Omega^{(C)})^2 + (a_1^{(D)} - a_1^{(C)})^2]^{1/2}$, $\lambda^{(E)} = \lambda^{(D)} + [(\Omega^{(E)} - \Omega^{(D)})^2 + (a_1^{(E)} - a_1^{(D)})^2]^{1/2}$. Therefore, our problem is reduced to finding the V-state corresponding to some λ slightly larger than $\lambda^{(E)}$; and a fairly good initial guess for that V-state can be obtained by means of a four-degree Lagrange interpolation polynomial. More precisely, let us define

$$c^{(A)} = \frac{(\lambda - \lambda^{(B)})(\lambda - \lambda^{(C)})(\lambda - \lambda^{(D)})(\lambda - \lambda^{(E)})}{(\lambda^{(A)} - \lambda^{(B)})(\lambda^{(A)} - \lambda^{(C)})(\lambda^{(A)} - \lambda^{(D)})(\lambda^{(A)} - \lambda^{(E)})}, \tag{1.98}$$

and, in a similar way, $c^{(B)}$, $c^{(C)}$, $c^{(D)}$, and $c^{(E)}$. Then,

$$\Omega(\lambda) = c^{(A)}\Omega^{(A)} + c^{(B)}\Omega^{(B)} + c^{(C)}\Omega^{(C)} + c^{(D)}\Omega^{(D)} + c^{(E)}\Omega^{(E)}, \tag{1.99}$$

$$a_k(\lambda) = c^{(A)}a_k^{(A)} + c^{(B)}a_k^{(B)} + c^{(C)}a_k^{(C)} + a_k\Omega^{(D)} + a_k\Omega^{(E)}, \quad k = 1, \ldots, M. \tag{1.100}$$

Remark that a couple of trials may be needed until a good choice of $\Omega^{(A)}, \ldots$, and of λ is found, i.e., values that enable us to continue the bifurcation curve, and not to come back to some already known V-state. In our case, we have chosen λ equal to $\lambda^{(E)}$ plus the mean of the four previous increments of λ, i.e.,

$$\begin{aligned}\lambda &= \lambda^{(E)} + \frac{(\lambda^{(E)} - \lambda^{(D)}) + (\lambda^{(D)} - \lambda^{(C)}) + (\lambda^{(C)} - \lambda^{(B)}) + (\lambda^{(B)} - \lambda^{(A)})}{4} \\ &= \frac{5\lambda^{(E)} - \lambda^{(A)}}{4}.\end{aligned} \tag{1.101}$$

After applying this technique just once, we have successfully obtained a V-state corresponding to $\Omega = 0.219054\ldots$, i.e., a V-state beyond the critical point. It may be useful (and sometimes even convenient) to iterate several times the procedure, after updating $\lambda^{(A)} = \lambda^{(B)}$, $\lambda^{(B)} = \lambda^{(C)}$,$\lambda^{(C)} = \lambda^{(C)}$, $\lambda^{(C)} = \lambda^{(D)}$, and $\lambda^{(E)} = \lambda$. In fact, it can be even applied from the very beginning, to obtain all the bifurcation curves in 1.3 in their integrity, with an important spare of computational time. In Figure 1.4, we plot on the left-hand side the completed bifurcation curve until $\Omega = 0.236$; the piece of curve beyond the saddle-node bifurcation point, absent in Figure 1.3, is shown in thicker stroke.

It is possible to still continue the bifurcation curve, although the results are to be taken with prudence, because higher spatial resolution is needed. Further numerical experiments, which include the use of alternative parameterizations of z, would suggest the eventual formation of cusp-shaped singularities at the corners. They would also suggest the presence of additional saddle-node bifurcation points, in such a way that the bifurcation curve in 1.4 would show spiral-like structures at its ends. Nevertheless, since our results are still inconclusive, we postpone this challenging issue for the future.

On the other hand, based on the previous pages and on additional numerical experiments that we have carried on, we conjecture the existence of saddle-node bifurcation points for all $m \geq 3$ and for all $\alpha \in (0, 1)$. However, as α decreases, smaller and smaller structures are expected at the ends of the bifurcation curves, until $\alpha = 0$, when they disappear. Indeed, in the vortex patch problem, as mentioned above, the bifurcation curves are always monotonic.

We cannot finish this section, without saying something about the case $m = 2$, which has a pretty different behavior and is interesting per se. In Figure 1.5, we have plotted 2-fold V-states for $\alpha = 0.01$ and different values of Ω, starting from $\Omega = 0.2496$, which is close to $\Omega_2^{0.01} = 0.249667\ldots$, so the corresponding V-state, in black, is practically a unit circumference. Since α is small, we might expect to have a similar behavior to that in the vortex patch problem, where the V-states tend to degenerate to a segment as Ω decreases. However, although this is true for Ω close enough to $\Omega_2^{0.01}$, there is an instant when convexity is lost, and the V-states get a more and

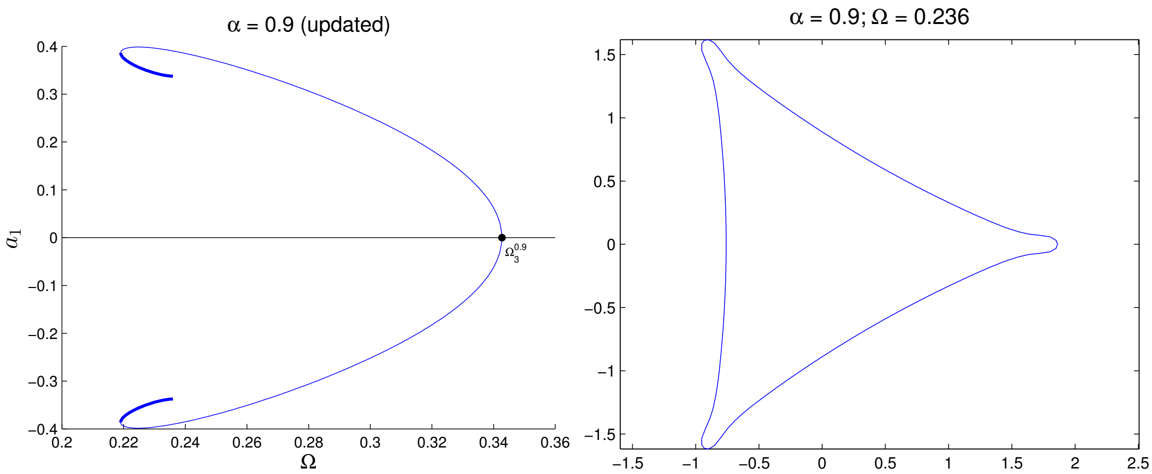

FIGURE 1.4 – Left : Extended bifurcation curve, for $m = 3$, and $\alpha = 0.9$. Right : V-state beyond the saddle-node bifurcation point.

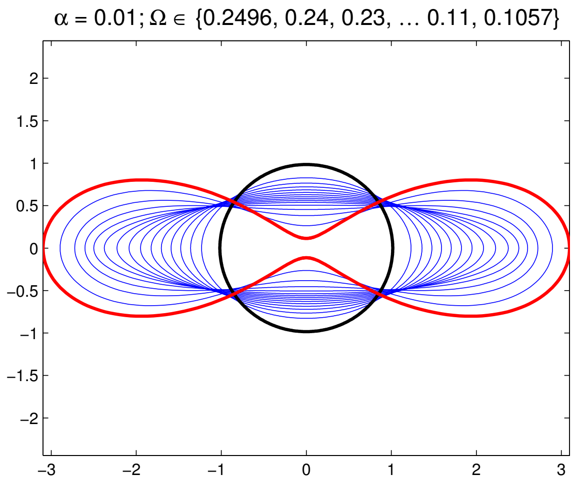

FIGURE 1.5 – 2-fold V-states corresponding to $\alpha = 0.2$, for different values of Ω.

more pronounced ∞-shape, as Ω decreases. Remark that Ω cannot be smaller than a certain value, which corresponds approximately to $\Omega = 0.1057$, and whose corresponding V-state is plotted in red. Let us mention that the situation is very similar for greater α, even for those close to 1.

In Figure 1.5, the V-state in red seems to have developed no singularity. Again, insight into what is happening is given by the bifurcation curve of a_1 in (1.90) with respect to Ω, for $\alpha = 0.01$, which is plotted in Figure 1.6. In that figure, we have also plotted the bifurcation curves for $\alpha = 0.1, 0.2, 0.3$, being the four curves very similar to each other.

As with $m \geq 3$, the bifurcation curves suggest the existence of saddle-node bifurcation points. To see whether this is indeed the case, we have used the previously described continuation method for $\alpha = 0.01$, taking $\Omega_c = 0.10567$, and $\epsilon = 10^{-4}$. Figure 1.7 confirms our suspicions. On the left-hand side, we plot the completed bifurcation curve until $\Omega = 0.1061$; the piece of curve beyond the saddle-node bifurcation point, absent in Figure 1.6, is shown in thicker stroke. Remark that, unlike Figure 1.4, a zoom is necessary in order for the hook to be appreciated. On the right-hand side, we plot the new V-state corresponding to a slightly larger Ω, i.e., $\Omega = 0.1061407$ (and such that $\Omega = 0.1061408$ is unstable), with twice as many nodes, i.e, $N = 512 \times 2$. Apparently, a

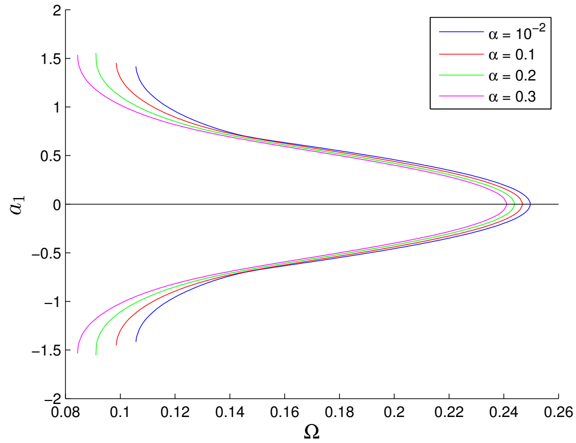

FIGURE 1.6 – Bifurcation curves, for $m = 2$, and $\alpha = 0.01, 0.1, 0.2, 0.3$. We plot the value of the first coefficient a_1 in (1.90) with respect to Ω. The right-most curve corresponds to the smallest α, and so on.

self-intersection has happened, although a powerful zoom shows that the distance between the two inner pieces of curve is approximately 5.9×10^{-3} ; moreover, there are apparently enough nodes in that region, so it seems that we could decrease that distance even further, by increasing the eight decimal of Ω, and so on. Even if we rather think that a self-intersection will eventually occur (see [19] and [85] for similar phenomena in the vortex patch problem), further study is required here.

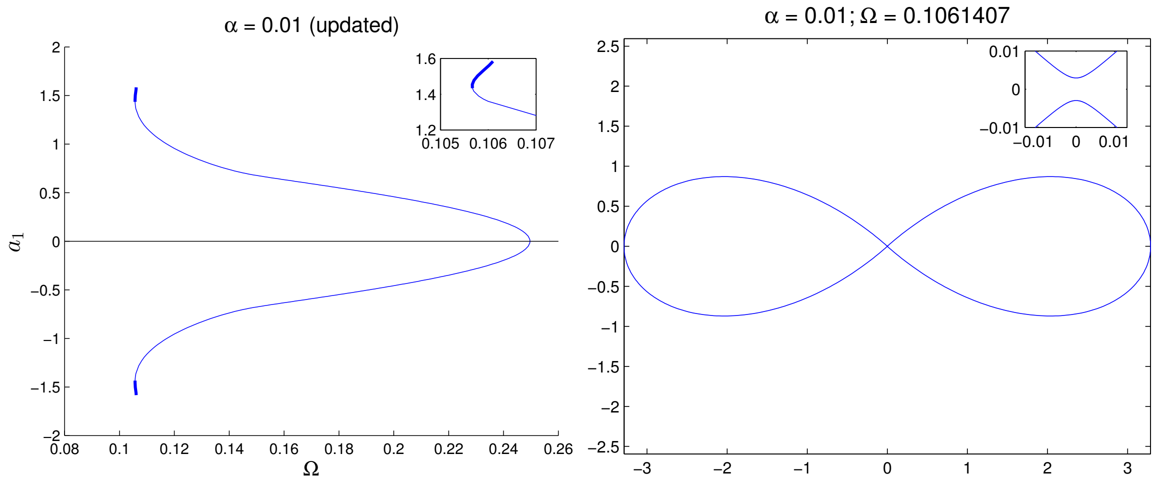

FIGURE 1.7 – Left : Extended bifurcation curve, for $m = 2$, and $\alpha = 0.01$. Right : V-state beyond the saddle-node bifurcation point. A self-intersection has almost occurred.

Chapitre 2

Doubly connected V-states for the generalized quasi-geostrophic equations

This chapter is the object of a submitted paper
in joint work with T. Hmidi and F. de la Hoz

Abstract. In this chapter, we prove the existence of doubly connected V-states for the generalized SQG equations with $\alpha \in]0,1[$. They can be described by countable branches bifurcating from the annulus at some explicit "eigenvalues" related to Bessel functions of the first kind. Contrary to Euler equations [64], we find V-states rotating with positive and negative angular velocities. At the end of the chapter we discuss some numerical experiments concerning the limiting V-states.

2.1 Introduction

The present work deals with the generalized surface quasi-geostrophic equation (gSQG) arising in fluid dynamics and which describes the evolution of the potential temperature θ by the transport equation :

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ u = -\nabla^\perp (-\Delta)^{-1+\frac{\alpha}{2}} \theta, \\ \theta_{|t=0} = \theta_0. \end{cases} \tag{2.1}$$

Here u refers to the velocity field, $\nabla^\perp = (-\partial_2, \partial_1)$ and α is a real parameter taken in $]0,2[$. The singular operator $(-\Delta)^{-1+\frac{\alpha}{2}}$ is of convolution type and defined by,

$$(-\Delta)^{-1+\frac{\alpha}{2}} \theta(x) = \frac{C_\alpha}{2\pi} \int_{\mathbb{R}^2} \frac{\theta(y)}{|x-y|^\alpha} dy, \tag{2.2}$$

with $C_\alpha = \frac{\Gamma(\alpha/2)}{2^{1-\alpha}\Gamma(\frac{2-\alpha}{2})}$ where Γ stands for the gamma function. This model was proposed by Córdoba et al. in [28] as an interpolation between Euler equations and the surface quasi-geostrophic model (SQG) corresponding to $\alpha = 0$ and $\alpha = 1$, respectively. The SQG equation was used by Juckes [71] and Held et al. [56] to describe the atmosphere circulation near the tropopause. It was also used by Lapeyre and Klein [80] to track the ocean dynamics in the upper layers. We note that there is a strong mathematical and physical analogy with the three-dimensional incompressible Euler equations ; see [27] for details.

In the last few years there has been a growing interest in the mathematical study of these active scalar equations. Special attention has been paid to the local well-posedness of classical solutions which can be performed in various functional spaces. For instance, this was implemented in the framework of Sobolev spaces [23] by using the commutator theory. Wether or not these solutions are global in time is an open problem except for Euler equations $\alpha = 0$. The second restriction with the gSQG equation concerns the construction of Yudovich solutions – known to exist globally in time for Euler equations [128] – which are not at all clear even locally in time. The main difficulty is due to the velocity which is in general singular and scales below the Lipschitz class. Nonetheless one can say more about this issue for some special class of concentrated vortices. More precisely, when the initial datum has a vortex patch structure, that is, $\theta_0(x) = \chi_D$ is the characteristic function of a bounded simply connected smooth domain D, then there is a unique local solution in the patch form $\theta(t) = \chi_{D_t}$. In this case, the boundary motion of the domain D_t is described by the contour dynamics formulation ; see the papers [47, 107]. The global persistence of the boundary regularity is only known for $\alpha = 0$ according to Chemin's result [25] ; for another proof see the paper of Bertozzi and Constantin [13]. Notice that for $\alpha > 0$ the numerical experiments carried out in [28] provide strong evidence for the singularity formation in finite time. Let us mention that the contour dynamics equation remains locally well-posed when the domain of the initial patch is assumed to be multi-connected meaning that the boundary is composed with finite number of disjoint smooth Jordan curves.

The main concern of this work is to explore analytically and numerically some special vortex patches called V-states ; they correspond to patches which do not change their shapes during the motion. The emphasis will be put on the V-states subject to uniform rotation around their center of mass, that is, $D_t = \mathbf{R}_{x_0,\Omega t}D$, where $\mathbf{R}_{x_0,\Omega t}$ stands for the planar rotation with center x_0 and angle Ωt. The parameter Ω is called the angular velocity of the rotating domain. Along the chapter we call these structures rotating patches or simply V-states. Their existence is of great interest for at least two reasons : first they provide non trivial initial data with global existence, and second this might explain the emergence of some ordered structures in the geophysical flows. This study has been conducted first for the two-dimensional Euler equations ($\alpha = 0$) a long time ago and a number of analytical and numerical studies are known in the literature. The first result in this setting goes back to Kirchhoff [76] who discovered that an ellipse of semi-axes a and b rotates uniformly with the angular velocity $\Omega = ab/(a+b)^2$; see for instance the references [14, p. 304] and [79, p. 232]. Till now this is the only known explicit V-states ; however the existence of implicit examples was established about one century later. In fact, Deem and Zabusky [34] gave numerical evidence of the existence of the V-states with m-fold symmetry for each integer $m \geq 2$; remark that the case $m = 2$ coincides with Kirchhoff's ellipses. To fix the terminology, a planar domain is said m-fold symmetric if it has the same group invariance of a regular polygon with m sides. Note that at each frequency m these V-states can be seen as a continuous deformation of the disc with respect to to the angular velocity. An analytical proof of this fact was given few years later by Burbea in [16]. His approach consists in writing a stationary problem in the frame of the patch with the conformal mapping of the domain and to look for the non trivial solutions by using the technique of the bifurcation theory. Quite recently, in [65] Burbea's approach was revisited with more details and explanations. The boundary regularity of the V-states was also studied and it was shown to be of class C^∞ and convex close to the disc.

We mention that explicit vortex solutions similar to the ellipses are discovered in the literature for the incompressible Euler equations in the presence of an external shear flow ; see for instance [24, 74, 98]. A general review about vortex dynamics can be found in the papers [5, 99].

With regard to the existence of the simply connected V-states for the (gSQG) it has been discussed very recently in the papers [18, 17, 54]. In [17], it was shown that the ellipses cannot rotate for any $\alpha \in (0,2)$ and to the authors' best knowledge no explicit example is known in the literature. Lately, in [54] the last two authors proved the analogous of Burbea's result and showed the existence of the m-folds rotating patches for $\alpha \in]0,1[$. In addition, the bifurcation from the unit disc occurs at the angular velocities,

$$\Omega_m^\alpha \triangleq \frac{\Gamma(1-\alpha)}{2^{1-\alpha}\Gamma^2(1-\frac{\alpha}{2})}\bigg(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})}-\frac{\Gamma(m+\frac{\alpha}{2})}{\Gamma(m+1-\frac{\alpha}{2})}\bigg), \quad m\geq 2,$$

where Γ denotes the usual gamma function.The remaining case $\alpha \in [1,2)$ has been explored and solved by Castro, Córdoba and Gómez-Serrano in [18]. They also show that the V-states are C^∞ and convex close to the discs. To complete these works we discussed in this work (which is forwarded in Section 1.11) some numerical experiments concerning these V-states and their limiting structures when we go to the end of each branch ; new behaviors will be observed compared to the numerical experiments achieved for Euler case [109].

We want in this chapter to learn more about the V-states but with different topological structure compared to the preceding discussion. More precisely, we propose to scrutinize rotating patches with only one hole, also called doubly connected V-states. Recall that a patch $\theta_0 = \chi_D$ is said to be doubly connected if the domain $D = D_1\backslash D_2$, with D_1 and D_2 being two simply connected bounded domains satisfying $\overline{D_2} \subset D_1$. This structure is preserved for Euler system globally in time but known to be for short time when $\alpha \in]0,1[$ see [23, 47, 107]. We notice that compared to the simply connected case the boundaries evolve through extra nonlinear terms coming from the interaction between the boundaries and therefore the existence of the V-states is relatively more complicate to analyze. This problem is not well studied from the analytical point of view and recent progress has been made for Euler equations in the papers [44, 66, 64]. In [66], the authers proved the existence of explicit V-states similar to Kirchhoff ellipses seems to be out of reach. Indeed, it was stated that if one of the boundaries of the V-state is a circle then necessarily the other one should be also a circle. Moreover, if the inner curve is an ellipse then there is no rotation at all. Another closely related subject is to deal with some vortex magnitude μ inside the domain D_2 and try to find explicit rotating patches. This was done by Flierl and Polvani [44] who proved that confocal ellipses rotate uniformly provided some compatibility relations are satisfied between the parameter μ and the semi-axes of the ellipses. We note that another approach based upon complex analysis tools with a complete discussion can be found in [66].

Now, from the equations (2.1) we may easily conclude that the annulus is a stationary doubly connected patch, and therefore it rotates with any angular velocity Ω. From this obvious fact, one can wonder wether or not the bifurcation to nontrivial V-states still happens as for the simply connected case. This has been recently investigated in [64] for Euler equations following basically Burbea's approach but with more involved calculations. It was shown that for $b \in (0,1)$ and m being an integer satisfying the inequality

$$1+b^m-\frac{1-b^2}{2}m<0 \tag{2.3}$$

then there exist two curves of non-annular m-fold doubly connected patches bifurcating from the annulus $\big\{z; b<|z|<1\big\}$ at different eigenvalues $\Omega_m^\pm$ given explicitly by the formula

$$\Omega_m^\pm = \frac{1-b^2}{4} \pm \frac{1}{2m}\sqrt{\Big[\frac{m}{2}(1-b^2)-1\Big]^2-b^{2m}}.$$

Now we come to the main contribution of the current work. We propose to study the doubly connected V-states for the gSQG model (2.1) when $\alpha \in]0,1[$. Before stating our result we need to make some notation. We define

$$\Lambda_n(b) \triangleq \frac{1}{b}\int_0^{+\infty} J_n(bt)J_n(t)\frac{dt}{t^{1-\alpha}},$$

and

$$\Theta_n \triangleq \Lambda_1(1) - \Lambda_n(1),$$

where J_n refers to Bessel function of the first kind. Our result reads as follows.

Theorem 2.1 *Let $\alpha \in [0,1[$ and $b \in]0,1[$; there exists $N \in \mathbb{N}$ with the following property :*
For each $m > N$ there exists two curves of m-fold doubly connected V-states that bifurcate from the annulus $\{z \in \mathbb{C}, b < |z| < 1\}$ at the angular velocities

$$\Omega_m^{\alpha,\pm} \triangleq \frac{1-b^2}{2}\Lambda_1(b) + \frac{1}{2}(1-b^{-\alpha})\Theta_m \pm \frac{1}{2}\sqrt{\Delta_m(\alpha,b)},$$

with

$$\Delta_m(\alpha,b) \triangleq \Big[(b^{-\alpha}+1)\Theta_m - (1+b^2)\Lambda_1(b)\Big]^2 - 4b^2\Lambda_m^2(b).$$

Remarks 2.1 *1. The number N is the smallest integer such that*

$$\Theta_N \geq \frac{1+b^2}{b^{-\alpha}+1}\Lambda_1(b) + \frac{2b}{b^{-\alpha}+1}\Lambda_N(b). \tag{2.4}$$

This restriction appear in the spectral study of the linearized operator and gives only sufficient condition for the existence of the V-states which is due in part to some technical reasons that inconvenience the proof of its necessity.

2. *As we shall see later in Lemma 3.3 , for $\alpha = 0$ we find the result of Euler equations established in [64] and the condition* (2.4) *is in accordance with that given by* (2.3).
3. *We can check by using the strict monotonicity of $b \mapsto \Lambda_1(b)$ that for any $b, \alpha \in (0,1)$,*

$$\lim_{m\to+\infty} \Omega_m^{\alpha,-} = -b^{-\alpha}\Lambda_1(1) + \Lambda_1(b) < 0.$$

Consequently the corresponding bifurcating curves generate close to the annulus non trivial clockwise doubly connected V-states. This fact is completely new compared to what we know for Euler equations or for the simply connected case where the bifurcation occurs at positive angular velocities. The numerical experiments discussed in Section 2.7 reveal the existence of non radial stationary patches for the generalized quasi-geostrophic equations and it would be very interesting to establish this fact analytically. In a connected subject, we point out that the last author has shown quite recently in [59] that for Euler equations clockwise convex V-states reduce to the discs.

Now we shall sketch the proof of Theorem 2.1 which is mainly based upon the bifurcation theory via Crandall-Rabinowitz's Theorem. The first step is to write down the analytical equations of the boundaries of the V-states. This can be done for example through the conformal parametrization of the domains D_1 and D_2 : we denote by $\phi_j : \mathbb{D}^c \to D_j^c$ the conformal mappings possessing the following structure,

$$\forall\, |w| \geq 1, \quad \phi_1(w) = w + \sum_{n\in\mathbb{N}} \frac{a_n}{w^n}, \quad \phi_2(w) = bw + \sum_{n\in\mathbb{N}} \frac{c_n}{w^n}.$$

We assume in addition that the Fourier coefficients are real which means that we look only for the V-states which are symmetric with respect to the real axis. Moreover using the subordination principle we deduce that $b \in]0,1[$; the parameter b coincides with the small radius of the annulus that we slightly perturb. As we shall see later in Section 2.2, the conformal mappings are subject to two coupled nonlinear equations defined as follows : for $j \in \{1,2\}$

$$\begin{aligned} F_j(\Omega,\phi_1,\phi_2)(w) &\triangleq \operatorname{Im}\left\{\Big(\Omega\,\phi_j(w)+S(\phi_2,\phi_j)(w)-S(\phi_1,\phi_j)(w)\Big)\overline{w}\,\overline{\phi_j'(w)}\right\}\\ &= 0, \quad \forall w\in\mathbb{T}, \end{aligned}$$

with

$$S(\phi_i,\phi_j)(w)=C_\alpha \fint_{\mathbb{T}} \frac{\phi_i'(\tau)}{|\phi_j(w)-\phi_i(\tau)|^\alpha}d\tau \quad \text{and} \quad C_\alpha \triangleq \frac{\Gamma(\alpha/2)}{2^{1-\alpha}\Gamma(\frac{2-\alpha}{2})}.$$

In order to apply the bifurcation theory we should understand the structure of the linearized operator around the trivial solution $(\phi_1,\phi_2)=(\mathrm{Id}, b\,\mathrm{Id})$, corresponding to the annulus with radii b and 1, and identify the range of Ω where this operator has a one-dimensional kernel. The computations of the linear operator $DF(\Omega,\mathrm{Id},b\,\mathrm{Id})$ with $F=(F_1,F_2)$ in terms of its Fourier coefficients are long and tricky. They are connected to the hypergeometric functions ${}_2F_1(a,b;c;z)$ simply denoted by $F(a,b;c;z)$ throughout this chapter. To find compact formula we use at several steps some algebraic identities described by the contiguous function relations (2.19)-(2.25). Similarly to the Euler equations [64] the linearized operator acts as a matrix Fourier multiplier. More precisely, for

$$h_1(w)=\sum_{n\geq 1}\frac{a_n}{w^n}, \quad h_2(w)=\sum_{n\geq 1}\frac{c_n}{w^n},$$

we obtain the formula,

$$DF(\Omega,\mathrm{Id},b\,\mathrm{Id})\big(h_1,h_2\big)(w)=\frac{i}{2}\sum_{n\geq 1}(n+1)M_{n+1}^\alpha\begin{pmatrix} a_n\\ c_n\end{pmatrix}\Big(w^{n+1}-\overline{w}^{n+1}\Big),$$

where the matrix M_n is given for $n\geq 2$ by

$$M_n^\alpha \triangleq \begin{pmatrix} \Omega-\Theta_n+b^2\Lambda_1(b) & -b^2\Lambda_n(b)\\ \\ b\Lambda_n(b) & b\Omega+b^{1-\alpha}\Theta_n-b\Lambda_1(b)\end{pmatrix}.$$

Therefore the values of Ω associated to non trivial kernels are the solutions of a second-degree polynomial,

$$\det M_n^\alpha = 0. \tag{2.5}$$

This can be solved when the discriminant $\Delta_n(\alpha,b)$ introduced in Theorem 2.1 is positive. The computations of the dimension of the kernel are more complicate than the cases raised before in the references [54, 64]. The matter reduces to count the following discrete set

$$\big\{n\geq 2, \det M_n^\alpha = 0\big\}.$$

Note that in [54, 64] this set has only one element and therefore the kernel is one-dimensional. This follows from the monotonicity of the "nonlinear eigenvalues" sequence $n\mapsto\Omega$ which is not very hard to get due to the explicit polynomial structure of the coefficients of the analogous polynomial to (2.5). Unfortunately, in the current situation this structure is broken because the matrix coefficients of M_n^α are related to Bessel functions. Therefore the monotonicity of the eigenvalues is more subtle and will require more refined analysis. This subject will be discussed later with ample details in the Subsection 2.5.3. To achieve the spectral study and check the complete assumptions of Crandall-Rabinowitz's Theorem it remains to prove the transversality assumption and check that the image is of co-dimension one. This will be done in Section 2.6 in a straightforward way and without serious difficulties. We also mention that the transversality assumption is obtained only when the discriminant $\Delta_n(\alpha, b) > 0$ meaning that there is no crossing roots for the equation (2.5). The proof of the bifurcation will be achieved in Section 2.6. Next, we shall make few comments about the statement of the main theorem.

Remarks 2.2 1) *For the SQG equation corresponding to $\alpha = 1$ the situation is more delicate due to some logarithmic loss. The simply connected case has been achieved recently in [18] by using Hilbert spaces where we take into account this loss. The same approach could lead to the existence of the doubly-connected V-states for the (SQG) equation. For the spectral study, the linearized operator can be obtained as a limit of* (2.5) *when α goes to* 1. *More precisely, we get*

$$DF(\Omega, Id, b\, Id)\big(h_1, h_2\big)(w) = \frac{i}{2}\sum_{n\geq 1}(n+1)M^1_{n+1}\begin{pmatrix} a_n \\ c_n \end{pmatrix}\Big(w^{n+1} - \overline{w}^{n+1}\Big),$$

where the matrix M_n is given for $n \geq 2$ by

$$M^1_n \triangleq \begin{pmatrix} \Omega - \Theta_n + b^2\Lambda_1(b) & -b^2\Lambda_n(b) \\ \\ b\Lambda_n(b) & b\Omega + \Theta_n - b\Lambda_1(b) \end{pmatrix},$$

with

$$\Theta_n = \frac{2}{\pi}\sum_{k=1}^{n-1}\frac{1}{2k+1} \quad \textit{and} \quad \Lambda_n(b) = \frac{1}{b}\int_0^{+\infty} J_n(bt)J_n(t)dt.$$

3) *The boundary of the V-states belongs to Hölder space $C^{2-\alpha}$. For Euler equations corresponding to $\alpha = 0$, we get better result in the simply connected geometry as it was shown in [65] ; the boundary is C^∞ and convex when the V-states are close to the circle. The proof in this particular case uses in a deep way the algebraic structure of the kernel according to some recurrence formulae. The extension of this result to $\alpha \in]0, 2[$ was done in [18]. We expect that the latter approach could be also adapted to the (gSQG) model for the doubly connected case.*

4) *In the setting of the vortex patches the global existence with smooth boundaries is not known for $\alpha \in]0, 2[$. The simply connected V-states discussed in [18, 54] offer a first class of global solutions which are periodic in time. We find here a second class of global solutions which are the doubly connected V-states.*

The remainder of the chapter is organized as follows. In the next section, we shall write down the boundary equations through the conformal parametrization. In Section 3, we shall introduce and review some background material on the bifurcation theory and Gauss hypergeometric functions. In Section 4, we will study the regularity of the nonlinear functionals involved in the boundary equations. In Section 5, we conduct the spectral study and formulate the suitable assumptions to get a Fredholm operator of zero index. In Section 6 we prove Theorem 2.1. Finally, the last section will be devoted to some numerical experiments dealing with the simply and doubly connected V-states.

Notation. We need to fix some notation that will be frequently used along this chapter.

- We denote by C any positive constant that may change from line to line.
- For any positive real numbers A and B, the notation $A \lesssim B$ means that there exists a positive constant C independent of A and B such that $A \leq CB$.
- We denote by $\mathbb{D}$ the unit disc. Its boundary, the unit circle, is denoted by $\mathbb{T}$.
- Let $f : \mathbb{T} \to \mathbb{C}$ be a continuous function. We define its mean value by,

$$\fint_{\mathbb{T}} f(\tau)d\tau \triangleq \frac{1}{2i\pi}\int_{\mathbb{T}} f(\tau)d\tau,$$

  where $d\tau$ stands for the complex integration.
- Let X and Y be two normed spaces. We denote by $\mathcal{L}(X, Y)$ the space of all continuous linear maps $T : X \to Y$ endowed with its usual strong topology.
- For a linear operator $T : X \to Y$, we denote by $N(T)$ and $R(T)$ the kernel and the range of T, respectively.
- If Y is a vector space and R is a subspace, then Y/R denotes the quotient space.

2.2 Boundary equations

Before proceeding further with the consideration of the V-states, we shall recall Riemann mapping theorem which is one of the most important results in complex analysis. To restate this result we need to recall the definition of *simply connected* domains. Let $\widehat{\mathbb{C}} \triangleq \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere. We say that a domain $\Omega \subset \widehat{\mathbb{C}}$ is *simply connected* if the set $\widehat{\mathbb{C}}\backslash\Omega$ is connected.

Riemann Mapping Theorem. Let $\mathbb{D}$ denote the unit open ball and $\Omega \subset \mathbb{C}$ be a simply connected bounded domain. Then there is a unique bi-holomorphic map called also conformal map, $\Phi : \mathbb{C}\backslash\overline{\mathbb{D}} \to \mathbb{C}\backslash\overline{\Omega}$ taking the form

$$\Phi(z) = az + \sum_{n\in\mathbb{N}} \frac{a_n}{z^n} \quad \text{with} \quad a > 0.$$

In this theorem the regularity of the boundary has no effect regarding the existence of the conformal mapping but it contributes in the boundary behavior of the conformal mapping, see for instance [104, 122]. Here, we shall recall the following result.

Kellogg-Warschawski's theorem. It can be found in [122] or in [104, Theorem 3.6]. It asserts that if the conformal map $\Phi : \mathbb{C}\backslash\overline{\mathbb{D}} \to \mathbb{C}\backslash\overline{\Omega}$ has a continuous extension to $\mathbb{C}\backslash\mathbb{D}$ which is of class $C^{n+1+\beta}$, with $n \in \mathbb{N}$ and $0 < \beta < 1$, then the boundary $\Phi(\mathbb{T})$ is a Jordan curve of class $C^{n+1+\beta}$.

Next, we shall write down the equation governing the boundary of the doubly connected V-states. Let $D = D_1\backslash D_2$ be a doubly connected domain, that is, D_1 and D_2 are two simply connected domains with $D_2 \subset D_1$. Denote by Γ_1 and Γ_2 their boundaries, respectively. Consider the parametrization by the conformal mapping : $\phi_j : \mathbb{D}^c \to D_j^c$ satisfying

$$\phi_1(z) = z + f_1(z) = z\Big(1 + \sum_{n=1}^{\infty} \frac{a_n}{z^n}\Big),$$

and

$$\phi_2(z) = bz + f_2(z) = z\Big(b + \sum_{n=1}^{\infty} \frac{b_n}{z^n}\Big), \quad 0 < b < 1.$$

Now assume that $\theta_0 = \chi_D$ is a rotating patch for the model (2.1) then according to [54] the boundary equations are given by

$$\begin{aligned}\Omega\,\mathrm{Re}\big\{z\overline{z'}\big\} &= C_\alpha\,\mathrm{Im}\left\{\frac{1}{2\pi}\int_{\partial D}\frac{d\zeta}{|z-\zeta|^\alpha}\overline{z'}\right\}, \qquad \forall\, z\in\partial D=\Gamma_1\cup\Gamma_2.\\ &= C_\alpha\,\mathrm{Im}\left\{\Big(\frac{1}{2\pi}\int_{\Gamma_1}\frac{d\zeta}{|z-\zeta|^\alpha}-\frac{1}{2\pi}\int_{\Gamma_2}\frac{d\zeta}{|z-\zeta|^\alpha}\Big)\overline{z'}\right\}, \qquad (2.6)\end{aligned}$$

where z' denotes a tangent vector to the boundary ∂D at the point z. We shall now rewrite the equations by using the conformal parametrizations ϕ_1 and ϕ_2. First remark that for $w\in\mathbb{T}$ a tangent vector on the boundary Γ_j at the point $\phi_j(w)$ is given by

$$\overline{z'}=-i\overline{w}\,\overline{\phi_j'(w)}.$$

Inserting this into the equation (2.6) and using the change of variables $\tau=\phi_j(w)$ give

$$\forall\, w\in\mathbb{T},\quad F_j\big(\Omega,\phi_1,\phi_2\big)(w)=0;\quad j=1,2,$$

with

$$\begin{aligned}&F_j\big(\Omega,\phi_1,\phi_2\big)(w)\triangleq\Omega\ \mathrm{Im}\Big\{\phi_j(w)\overline{w}\,\overline{\phi_j'(w)}\Big\}\\ &+\ C_\alpha\,\mathrm{Im}\left\{\Big(\fint_{\mathbb{T}}\frac{\phi_2'(\tau)d\tau}{|\phi_j(w)-\phi_2(\tau)|^\alpha}-\fint_{\mathbb{T}}\frac{\phi_1'(\tau)d\tau}{|\phi_j(w)-\phi_1(\tau)|^\alpha}\Big)\overline{w}\,\overline{\phi_j'(w)}\right\} \qquad (2.7)\end{aligned}$$

and $C_\alpha\triangleq\dfrac{\Gamma(\alpha/2)}{2^{1-\alpha}\Gamma(\frac{2-\alpha}{2})}$. We shall introduce the functionals

$$G_j(\Omega,f_1,f_2)\triangleq F_j\big(\Omega,\phi_1,\phi_2\big)\quad j=1,2.$$

Then equations of the V-states become,

$$\forall w\in\mathbb{T},\quad G_j\big(\Omega,f_1,f_2\big)(w)=0,\quad j=1,2. \qquad (2.8)$$

Now it is easy to ascertain that the annulus is a rotating patch for any $\Omega\in\mathbb{R}$. Indeed, replacing ϕ_1 and ϕ_2 in (2.7) by Id and $b\,$Id, respectively, we get

$$F_1\big(\Omega,\mathrm{Id},b\,\mathrm{Id}\big)(w) = C_\alpha\,\mathrm{Im}\left\{b\overline{w}\fint_{\mathbb{T}}\frac{d\tau}{|w-b\tau|^\alpha}-\overline{w}\fint_{\mathbb{T}}\frac{d\tau}{|w-\tau|^\alpha}\right\}.$$

Using the change of variables $\tau=w\zeta$ in the two preceding integrals we find

$$F_1\big(\Omega,\mathrm{Id},b\,\mathrm{Id}\big)(w) = C_\alpha\,\mathrm{Im}\left\{b\fint_{\mathbb{T}}\frac{d\zeta}{|1-b\zeta|^\alpha}-\fint_{\mathbb{T}}\frac{d\zeta}{|1-\zeta|^\alpha}\right\}.$$

Note that each integral in the right side is real since,

$$\begin{aligned}\forall a\in(0,1],\quad \overline{\fint_{\mathbb{T}}\frac{d\zeta}{|1-a\zeta|^\alpha}} &= -\fint_{\mathbb{T}}\frac{d\overline{\zeta}}{|1-a\overline{\zeta}|^\alpha}\\ &= \fint_{\mathbb{T}}\frac{d\xi}{|1-a\xi|^\alpha}.\end{aligned}$$

Therefore we obtain,

$$\forall\, w \in \mathbb{T}, \quad F_1\big(\Omega, \mathrm{Id}, b\,\mathrm{Id}\big)(w) \;=\; 0.$$

Arguing similarly for the second component F_2 we get for any $w \in \mathbb{T}$,

$$F_2\big(\Omega, \mathrm{Id}, b\,\mathrm{Id}\big)(w) = C_\alpha \,\mathrm{Im}\left\{ b^{1-\alpha} \fint_{\mathbb{T}} \frac{d\zeta}{|1-\zeta|^\alpha} - \fint_{\mathbb{T}} \frac{d\zeta}{|b-\zeta|^\alpha} \right\} = 0,$$

which is the desired result.

2.3 Tools

In this section we shall recall in the first part some simple facts about Hölder spaces on the unit circle $\mathbb{T}$. In the second part we state Crandall-Rabinowitz's Theorem which is a crucial tool in the proof of Theorem 2.1. We shall also recall some important properties of the hypergeometric functions which appear in a natural way in the spectral study of the linearized operator. The last part is devoted to the computations of some integrals used later in the spectral study.

2.3.1 Functional spaces

Throughout this chapter it is more convenient to think of 2π-periodic function $g : \mathbb{R} \to \mathbb{C}$ as a function of the complex variable $w = e^{i\eta}$. To be more precise, let $f : \mathbb{T} \to \mathbb{R}^2$, be a continuous function, then it can be assimilated to a $2\pi-$ periodic function $g : \mathbb{R} \to \mathbb{R}$ via the relation

$$f(w) = g(\eta), \quad w = e^{i\eta}.$$

Hence when f is smooth enough we get

$$f'(w) \triangleq \frac{df}{dw} = -ie^{-i\eta} g'(\eta).$$

Because d/dw and $d/d\eta$ differ only by a smooth factor with modulus one we shall in the sequel work with d/dw instead of $d/d\eta$ which appears more suitable in the computations.
Moreover, if f has real Fourier coefficients and is of class C^1 then we can easily check that

$$\{\overline{f}\}'(w) = -\frac{1}{w^2}\overline{f'(w)}. \tag{2.9}$$

Now we shall introduce Hölder spaces on the unit circle $\mathbb{T}$.

Definition 2.1 *Let $0 < \gamma < 1$. We denote by $C^\gamma(\mathbb{T})$ the space of continuous functions f such that*

$$\|f\|_{C^\gamma(\mathbb{T})} \triangleq \|f\|_{L^\infty(\mathbb{T})} + \sup_{\tau \neq w \in \mathbb{T}} \frac{|f(\tau) - f(w)|}{|\tau - w|^\alpha} < \infty.$$

For any integer n, the space $C^{n+\gamma}(\mathbb{T})$ stands for the set of functions f of class C^n whose $n-$th order derivatives are Hölder continuous with exponent γ. It is equipped with the usual norm,

$$\|f\|_{C^{n+\gamma}(\mathbb{T})} \triangleq \|f\|_{L^\infty(\mathbb{T})} + \Big\| \frac{d^n f}{dw^n} \Big\|_{C^\gamma(\mathbb{T})}.$$

Recall that the Lipschitz semi-norm is defined by,

$$\|f\|_{\mathrm{Lip}(\mathbb{T})} = \sup_{\tau \neq w \in \mathbb{T}} \frac{|f(\tau) - f(w)|}{|\tau - w|}.$$

Now we list some classical properties that will be used later at many places.

1. For $n \in \mathbb{N}, \gamma \in]0, 1[$ the space $C^{n+\gamma}(\mathbb{T})$ is an algebra.
2. For $K \in L^1(\mathbb{T})$ and $f \in C^{n+\gamma}(\mathbb{T})$ we have the convolution law,

$$\|K * f\|_{C^{n+\gamma}(\mathbb{T})} \leq \|K\|_{L^1(\mathbb{T})} \|f\|_{C^{n+\gamma}(\mathbb{T})}.$$

2.3.2 Elements of the bifurcation theory

We intend now to recall Crandall-Rabinowitz's Theorem which is a basic tool of the bifurcation theory and will be useful in the proof of Theorem 2.1. Let $F : \mathbb{R} \times X \to Y$ be a continuous function with X and Y being two Banach spaces. Assume that $F(\lambda, 0) = 0$ for any λ belonging in a non empty interval I. Whether close to a trivial solution $(\lambda_0, 0)$ we can find a branch of non trivial solutions of the equation $F(\lambda, x) = 0$ is the main concern of the bifurcation theory. If this happens we say that we have a bifurcation at the point $(\lambda_0, 0)$. We shall restrict ourselves here to the classical result of Crandall and Rabinowitz [29]. For more general results we refer the reader to the book of Kielhöfer [75].

Theorem 2.2 *Let X, Y be two Banach spaces, V a neighborhood of 0 in X and let $F : \mathbb{R} \times V \to Y$ with the following properties :*

1. *$F(\lambda, 0) = 0$ for any $\lambda \in \mathbb{R}$.*
2. *The partial derivatives F_λ, F_x and $F_{\lambda x}$ exist and are continuous.*
3. *$N(\mathcal{L}_0)$ and $Y/R(\mathcal{L}_0)$ are one-dimensional.*
4. *Transversality assumption : $\partial_\lambda \partial_x F(0,0) x_0 \notin R(\mathcal{L}_0)$, where*

$$N(\mathcal{L}_0) = span\{x_0\}, \quad \mathcal{L}_0 \triangleq \partial_x F(0,0).$$

If Z is any complement of $N(\mathcal{L}_0)$ in X, then there is a neighborhood U of $(0,0)$ in $\mathbb{R} \times X$, an interval $(-a, a)$, and continuous functions $\varphi : (-a, a) \to \mathbb{R}$, $\psi : (-a, a) \to Z$ such that $\varphi(0) = 0$, $\psi(0) = 0$ and

$$F^{-1}(0) \cap U = \Big\{ \big(\varphi(\xi), \xi x_0 + \xi \psi(\xi)\big) \, ; \, |\xi| < a \Big\} \cup \Big\{ (\lambda, 0) \, ; \, (\lambda, 0) \in U \Big\}.$$

2.3.3 Special functions

We shall give a short introduction on the Gauss hypergeometric functions and discuss some of their basic properties. The formulae listed below will be crucial in the computations of the linearized operator associated to the V-states equations. Recall that for any real numbers $a, b \in \mathbb{R}$, $c \in \mathbb{R} \backslash (-\mathbb{N})$ the hypergeometric function $z \mapsto F(a, b; c; z)$ is defined on the open unit disc $\mathbb{D}$ by the power series

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad \forall z \in \mathbb{D}.$$

Here, $(x)_n$ is the Pochhammer symbol defined by,

$$(x)_n = \begin{cases} 1 & n = 0 \\ x(x+1)\cdots(x+n-1) & n \geq 1. \end{cases}$$

It is obvious that

$$(x)_n = x\,(1+x)_{n-1}, \quad (x)_{n+1} = (x+n)\,(x)_n. \tag{2.10}$$

For a future use we recall an integral representation of the hypergeometric function, for instance see [105, p. 47]. Assume that Re$(c) >$ Re$(b) > 0$, then

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\,\mathrm{d}x, \quad |z| < 1. \tag{2.11}$$

The function $\Gamma : \mathbb{C}\backslash\{-\mathbb{N}\} \to \mathbb{C}$ refers to the gamma function which is the analytic continuation to the negative half plane of the usual gamma function defined on the positive half-plane $\{\mathrm{Re}\ z > 0\}$ by the integral representation :

$$\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}\mathrm{d}t.$$

It satisfies the relation

$$\Gamma(z+1) = z\,\Gamma(z), \quad \forall z \in \mathbb{C}\backslash(-\mathbb{N}). \tag{2.12}$$

From this we deduce the identities

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, \quad (x)_n = (-1)^n \frac{\Gamma(1-x)}{\Gamma(1-x-n)}, \tag{2.13}$$

provided all the quantities in the right terms are well-defined. Later we need the following values,

$$\Gamma(n+1) = n!, \quad \Gamma(1/2) = \sqrt{\pi}. \tag{2.14}$$

Another useful identity is the Euler's reflection formula,

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \quad \forall z \notin \mathbb{Z}. \tag{2.15}$$

Now we shall introduce the digamma function which is nothing but the logarithmic derivative of the gamma function and often denoted by $\digamma$. It is given by

$$\digamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \quad z \in \mathbb{C}\backslash(-\mathbb{N}).$$

The following identity is classical,

$$\forall n \in \mathbb{N}, \quad \digamma(n + \frac{1}{2}) = -\gamma - 2\ln 2 + 2\sum_{k=0}^{n-1} \frac{1}{2k+1}. \tag{2.16}$$

When $\mathrm{Re}(c-a-b) > 0$ then it can be shown that the hypergeometric series is absolutely convergent on the closed unit disc and one has the expression,

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{2.17}$$

the proof can be found in [105, p. 49],

Now recall the Kummer's quadratic transformation

$$F\Big(a,b;2b;\frac{4z}{(1+z)^2}\Big)=(1+z)^{2a}F\Big(a,a+\frac12-b;b+\frac12;z^2\Big),\quad \forall\, z\in[0,1[.\tag{2.18}$$

Next we recall some contiguous function relations of the hypergeometric series, see [105].

$$c(c+1)\,F(a,b;c;z)-c(c+1)\,F(a,b;c+1;z)-ab\,zF(a+1,b+1;c+2;z)=0,\tag{2.19}$$

$$c\,F(a,b;c;z)-c\,F(a+1,b;c;z)+b\,zF(a+1,b+1;c+1;z)=0,\tag{2.20}$$

$$c\,F(a,b;c;z)-c\,F(a,b+1;c;z)+a\,zF(a+1,b+1;c+1;z)=0,\tag{2.21}$$

$$c\,F(a,b;c;z)-(c-b)\,F(a,b;c+1;z)-b\,F(a,b+1;c+1;z)=0,\tag{2.22}$$

$$c\,F(a,b;c;z)-(c-a)\,F(a,b;c+1;z)-a\,F(a+1,b;c+1;z)=0,\tag{2.23}$$

$$b\,F(a,b+1;c;z)-a\,F(a+1,b;c;z)+(a-b)\,F(a,b;c;z)=0,\tag{2.24}$$

$$(b-a)(1-z)\,F(a,b;c;z)-(c-a)\,F(a-1,b;c;z)+(c-b)\,F(a,b-1;c;z)=0.\tag{2.25}$$

Now we close this discussion with recalling Bessel function J_n of the first kind of index $n\in\mathbb{N}$ and review some important identities. It is defined in the full space $\mathbb{C}$ by the power series

$$J_n(z)=\sum_{k\geq0}\frac{(-1)^k}{k!\,(n+k)!}\Big(\frac z2\Big)^{2k+n}.$$

The following identity called Sonine-Schafheitlin's formula will be very useful later.

$$\begin{aligned}\int_0^{+\infty}\frac{J_\mu(at)J_\nu(bt)}{t^\lambda}\mathrm{d}t &= \frac{a^{\lambda-\nu-1}b^\nu\,\Gamma\Big(\frac12\mu+\frac12\nu-\frac12\lambda+\frac12\Big)}{2^\lambda\Gamma(\nu+1)\Gamma\Big(\frac12\lambda+\frac12\mu-\frac12\nu+\frac12\Big)}\\ &\times\ F\Big(\frac{\mu+\nu-\lambda+1}{2},\frac{\nu-\lambda-\mu+1}{2};\nu+1;\frac{b^2}{a^2}\Big),\end{aligned}\tag{2.26}$$

provided that $0<b<a$ and that the integral is convergent. A detailed proof of this result can be found in [123, p. 401].

2.3.4 Basic integrals

The main goal of this paragraph is to compute explicitly some integrals that will appear later in the spectral study.

Lemma 2.1 *Let α, $b\in(0,1)$ and $n\in\mathbb{N}$. Then for any $w\in\mathbb{T}$ we have the following formulae :*

$$\fint_{\mathbb{T}}\frac{\tau^{n-1}}{|w-b\tau|^\alpha}d\tau\ =\ w^nb^n\frac{(\frac\alpha2)_n}{n!}F\Big(\frac\alpha2,n+\frac\alpha2;n+1;b^2\Big).\tag{2.27}$$

$$\begin{aligned}\fint_{\mathbb{T}}\frac{(w-b\tau)(aw^n-c\tau^n)}{|w-b\tau|^{\alpha+2}}d\tau=w^{n+2}b\Big[a\Big(1+\frac\alpha2\Big)F\Big(\frac\alpha2,2+\frac\alpha2;2;b^2\Big)\\ -\,c\,b^n\frac{(1+\frac\alpha2)_{n+1}}{(n+1)!}F\Big(\frac\alpha2,n+2+\frac\alpha2;n+2;b^2\Big)\Big].\end{aligned}\tag{2.28}$$

$$\fint_{\mathbb{T}} \frac{(\overline{w}-b\overline{\tau})(a\overline{w}^n - c\overline{\tau}^n)}{|w-b\tau|^{\alpha+2}} d\tau = \overline{w}^n \Big[ab\frac{\alpha}{2}\, F(\frac{\alpha}{2}+1,\frac{\alpha}{2}+1;2;b^2) - c\, b^{n-1}\frac{(1+\frac{\alpha}{2})_{n-1}}{(n-1)!} F\Big(\frac{\alpha}{2}, n+\frac{\alpha}{2}; n; b^2\Big)\Big]. \tag{2.29}$$

$$\fint_{\mathbb{T}} \frac{(bw-\tau)(aw^n - c\tau^n)}{|bw-\tau|^{\alpha+2}} d\tau = -w^{n+2}b^2 \Big[a\frac{\alpha}{4}\Big(\frac{\alpha}{2}+1\Big) F\Big(1+\frac{\alpha}{2}, 2+\frac{\alpha}{2}; 3; b^2\Big) - c\, b^{n}\frac{(\frac{\alpha}{2})_{n+2}}{(n+2)!} F\Big(1+\frac{\alpha}{2}, n+2+\frac{\alpha}{2}; n+3; b^2\Big)\Big]. \tag{2.30}$$

$$\fint_{\mathbb{T}} \frac{(b\overline{w}-\overline{\tau})(a\overline{w}^n - c\overline{\tau}^n)}{|bw-\tau|^{\alpha+2}} d\tau = -\overline{w}^n \Big[aF\Big(\frac{\alpha}{2}, \frac{\alpha}{2}+1; 1; b^2\Big) - c\, b^{n}\frac{(\frac{\alpha}{2})_{n}}{n!} F\Big(\frac{\alpha}{2}+1, n+\frac{\alpha}{2}; n+1; b^2\Big)\Big]. \tag{2.31}$$

Proof : To prove the first identity we use successively the change of variables $\tau = w\zeta$ and $\zeta = e^{i\eta}$,

$$\begin{aligned}\fint_{\mathbb{T}} \frac{\tau^{n-1}}{|w-b\tau|^{\alpha}} d\tau &= w^n \fint_{\mathbb{T}} \frac{\zeta^{n-1}}{|1-b\zeta|^{\alpha}} d\zeta \\ &= w^n \frac{1}{2(1+b)^{\alpha}\pi} \int_0^{2\pi} \frac{e^{in\eta}}{\big(1-\frac{4b}{(1+b)^2}\cos^2(\eta/2)\big)^{\frac{\alpha}{2}}} d\eta.\end{aligned}$$

Again by the change of variables $\eta/2 \mapsto \eta$ one gets

$$\fint_{\mathbb{T}} \frac{\tau^{n-1}}{|w-b\tau|^{\alpha}} d\tau = w^n \frac{1}{(1+b)^{\alpha}\pi} \int_0^{\pi} \frac{e^{i2n\eta}}{\big(1-\frac{4b}{(1+b)^2}\cos^2\eta\big)^{\frac{\alpha}{2}}} d\eta.$$

Since $\Big|\frac{4b}{(1+b)^2}\cos^2\eta\Big| < 1$ then we can use the Taylor series

$$\Big(1-\frac{4b}{(1+b)^2}\cos^2\eta\Big)^{-\alpha/2} = \sum_{m=0}^{\infty} \frac{\big(\alpha/2\big)_m}{m!} \frac{2^{2m}b^m}{(1+b)^{2m}} \cos^{2m}\eta.$$

Consequently, we get

$$\fint_{\mathbb{T}} \frac{\tau^{n-1}}{|w-b\tau|^{\alpha}} d\tau = w^n \frac{1}{\pi(1+b)^{\alpha}} \sum_{m=0}^{\infty} \frac{\big(\alpha/2\big)_m}{m!} \frac{2^{2m}b^m}{(1+b)^{2m}} \int_0^{\pi} \cos^{2m}\eta e^{i2n\eta} d\eta.$$

We shall now recall the following identity, see for instance [87, p. 8] and [123, p. 449],

$$\int_0^{\pi} \cos^x(\eta) e^{iy\eta} d\eta = \frac{\pi\Gamma(x+1)}{2^x\Gamma\Big(1+\frac{x+y}{2}\Big)\Gamma\Big(1+\frac{x-y}{2}\Big)}, \quad \forall\, x > -1, \quad \forall\, y \in \mathbb{R}.$$

As it was pointed before the gamma function has no real zeros but simple poles located at $-\mathbb{N}$ and therefore the function $\frac{1}{\Gamma}$ admits an analytic continuation on $\mathbb{C}$. Apply this formula with $x = 2m$ and $y = 2n$ yields,

$$\frac{1}{\pi}\int_0^{\pi} \cos^{2m}(\eta) e^{2in\eta} d\eta = \frac{\Gamma(2m+1)}{2^{2m}\Gamma(m+n+1)\Gamma(m-n+1)}.$$

Hence,

$$\begin{aligned}
\fint_{\mathbb{T}} \frac{\tau^{n-1}}{|w-b\tau|^\alpha}d\tau &= w^n\frac{1}{(1+b)^\alpha}\sum_{m=n}^{\infty}\frac{\big(\alpha/2\big)_m}{m!}\frac{\Gamma(2m+1)}{\Gamma(m+n+1)\Gamma(m-n+1)}\frac{b^m}{(1+b)^{2m}}\\
&= w^n\frac{1}{(1+b)^\alpha}\sum_{m=0}^{\infty}\frac{\big(\alpha/2\big)_{m+n}}{(m+n)!}\frac{\Gamma(2m+2n+1)}{\Gamma(m+2n+1)\Gamma(m+1)}\frac{b^{m+n}}{(1+b)^{2(m+n)}}.
\end{aligned}$$

We shall use Legendre's duplication formula,

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right)=2^{1-2z}\,\sqrt{\pi}\,\Gamma(2z),\quad z\in\mathbb{C}\backslash\{-\mathbb{N}\},$$

which gives

$$\begin{aligned}
\frac{\Gamma(2m+2n+1)}{(m+n)!} &= \frac{\Gamma(2m+2n+1)}{\Gamma(m+n+1)}\\
&= \frac{2^{2m+2n}}{\sqrt{\pi}}\Gamma(m+n+1/2).
\end{aligned}$$

Therefore using the identity (2.13) and $\Gamma(1/2)=\sqrt{\pi}$ we find

$$\frac{\Gamma(2m+2n+1)}{(m+n)!} = 2^{2m+2n}(1/2)_{m+n}.$$

From the elementary fact

$$(x)_{m+n}=(x)_n(n+x)_m\quad \forall x\in\mathbb{R}$$

one deuces

$$\fint_{\mathbb{T}} \frac{\tau^{n-1}}{|w-b\tau|^\alpha}d\tau = w^n\frac{(\frac{\alpha}{2})_n(\frac{1}{2})_n}{(1+b)^\alpha(2n)!}\Big(\frac{4b}{(1+b)^2}\Big)^n\sum_{m=0}^{\infty}\frac{(n+\frac{\alpha}{2})_m(n+1/2)_m}{(2n+1)_m m!}\frac{2^{2m}b^m}{(1+b)^{2m}}.$$

By definition of the hypergeometric series we conclude that

$$\fint_{\mathbb{T}} \frac{\tau^{n-1}}{|w-b\tau|^\alpha}d\tau = w^n\frac{(\frac{\alpha}{2})_n}{n!}\frac{2^{2n}b^n}{(1+b)^{2n+\alpha}}F\Big(n+\frac{\alpha}{2},n+\frac{1}{2};2n+1;\frac{4b}{(1+b)^2}\Big).$$

Using Kummer's quadratic transformation (2.18) the last identity becomes

$$\fint_{\mathbb{T}} \frac{\tau^{n-1}}{|w-b\tau|^\alpha}d\tau = w^n b^n\frac{(\frac{\alpha}{2})_n}{n!}F\Big(\frac{\alpha}{2},n+\frac{\alpha}{2};n+1;b^2\Big).$$

This completes the proof of (2.27).

We intend now to compute the second integral (2.28). To this end we use the change of variables as before,

$$\begin{aligned}
I_n &\triangleq \fint_{\mathbb{T}} \frac{(w-b\zeta)(aw^n-c\zeta^n)}{|w-b\zeta|^{\alpha+2}}d\zeta\\
&= w^{n+2}\fint_{\mathbb{T}} \frac{(1-b\zeta)(a-c\zeta^n)}{|1-b\zeta|^{\alpha+2}}d\zeta\\
&= w^{n+2}\Big(a\,A_0-c\,A_n\Big) \qquad (2.32)
\end{aligned}$$

where

$$A_n \triangleq \fint_{\mathbb{T}} \frac{\zeta^n d\zeta}{|1-b\zeta|^{\alpha+2}} - b\fint_{\mathbb{T}} \frac{\zeta^{n+1} d\zeta}{|1-b\zeta|^{\alpha+2}}.$$

It follows from the identity (2.27) that

$$A_n = b^{n+1}\frac{(1+\frac{\alpha}{2})_{n+1}}{(n+1)!}\Big[F\Big(\frac{\alpha}{2}+1, n+2+\frac{\alpha}{2}; n+2; b^2\Big) - \frac{(\frac{\alpha}{2}+n+2)}{n+2} b^2 F\Big(\frac{\alpha}{2}+1, n+3+\frac{\alpha}{2}; n+3; b^2\Big)\Big].$$

Then, in view of the formula (2.20) one gets

$$A_n = b^{n+1}\frac{(1+\frac{\alpha}{2})_{n+1}}{(n+1)!} F\Big(\frac{\alpha}{2}, n+2+\frac{\alpha}{2}; n+2; b^2\Big).$$

Replacing A_n by its expression in (2.32) we conclude that

$$I_n = w^{n+2}\Big[ab\big(1+\frac{\alpha}{2}\big)F\Big(\frac{\alpha}{2}, 2+\frac{\alpha}{2}; 2; b^2\Big) - cb^{n+1}\frac{(1+\frac{\alpha}{2})_{n+1}}{(n+1)!}F\Big(\frac{\alpha}{2}, n+2+\frac{\alpha}{2}; n+2; b^2\Big)\Big].$$

We shall now compute the integral (2.29). We write

$$\begin{aligned} K_n &\triangleq \fint_{\mathbb{T}} \frac{(\overline{w}-b\overline{\tau})(a\overline{w}^n - c\overline{\tau}^n)}{|1-b\tau|^{\alpha+2}} d\tau \\ &= \overline{w}^n \fint_{\mathbb{T}} \frac{(1-b\overline{\zeta})(a-c\overline{\zeta}^n)}{|1-b\zeta|^{\alpha+2}} d\zeta \\ &\triangleq \overline{w}^n\Big(aB_0 - cB_n\Big). \end{aligned}$$

Using the identity (2.27), B_0 can be rewritten as

$$\begin{aligned} B_0 &\triangleq \fint_{\mathbb{T}} \frac{d\zeta}{|1-b\zeta|^{\alpha+2}} - b\fint_{\mathbb{T}} \frac{\overline{\zeta} d\zeta}{|1-b\zeta|^{\alpha+2}} \\ &= \fint_{\mathbb{T}} \frac{d\zeta}{|1-b\zeta|^{\alpha+2}} - b\fint_{\mathbb{T}} \frac{\zeta^{-1} d\zeta}{|1-b\zeta|^{\alpha+2}} \\ &= b\big(1+\frac{\alpha}{2}\big)F\Big(\frac{\alpha}{2}+1, 2+\frac{\alpha}{2}; 2; b^2\Big) - bF\Big(\frac{\alpha}{2}+1, \frac{\alpha}{2}+1; 1; b^2\Big). \end{aligned}$$

Then, in view of the formula (2.22) we get

$$B_0 = \frac{\alpha}{2} bF\Big(\frac{\alpha}{2}+1, \frac{\alpha}{2}+1; 2; b^2\Big).$$

To compute B_n we first observe that by the change of variables $\overline{\zeta} \mapsto \zeta$ we find

$$\fint_{\mathbb{T}} \frac{\overline{\zeta}^n}{|1-b\zeta|^{\alpha+2}} d\zeta = \fint_{\mathbb{T}} \frac{\zeta^{n-2}}{|1-b\zeta|^{\alpha+2}} d\zeta$$

which yields in turn

$$\begin{aligned} B_n &= A_{n-2} \\ &= b^{n-1}\frac{(1+\frac{\alpha}{2})_{n-1}}{(n-1)!} F\Big(\frac{\alpha}{2}, n+\frac{\alpha}{2}; n; b^2\Big). \end{aligned}$$

Consequently,

$$K_n = \overline{w}^n\Big[a\,b\frac{\alpha}{2}F(\frac{\alpha}{2}+1,\frac{\alpha}{2}+1;2;b^2)-c\,b^{n-1}\frac{(1+\frac{\alpha}{2})_{n-1}}{(n-1)!}F\Big(\frac{\alpha}{2},n+\frac{\alpha}{2};n;b^2\Big)\Big].$$

Concerning the integral (2.30) we use a change of variable as before in order to get

$$\begin{aligned} P_n &\triangleq \fint_{\mathbb{T}}\frac{(bw-\zeta)(aw^n-c\zeta^n)}{|bw-\zeta|^{\alpha+2}}d\zeta\\ &= w^{n+2}\fint_{\mathbb{T}}\frac{(b-\zeta)(a-c\zeta^n)}{|b-\zeta|^{\alpha+2}}d\zeta\\ &= w^{n+2}\Big(a\,C_0-c\,C_n\Big), \end{aligned} \tag{2.33}$$

with

$$C_n \triangleq b\fint_{\mathbb{T}}\frac{\zeta^n d\zeta}{|b-\zeta|^{\alpha+2}}-\fint_{\mathbb{T}}\frac{\zeta^{n+1}d\zeta}{|b-\zeta|^{\alpha+2}}.$$

Observe that

$$|b-\zeta|=|1-b\zeta|\quad\forall\zeta\in\mathbb{T}.$$

Then, it follows from the formula (2.27) that

$$\begin{aligned} C_n = b^{n+2}\frac{(1+\frac{\alpha}{2})_{n+1}}{(n+2)!}\Big[&(n+2)F\Big(\frac{\alpha}{2}+1,n+2+\frac{\alpha}{2};n+2;b^2\Big)\\ &-\Big(\frac{\alpha}{2}+n+2\Big)F\Big(\frac{\alpha}{2}+1,n+3+\frac{\alpha}{2};n+3;b^2\Big)\Big]. \end{aligned}$$

Using once again the identity (2.22) implies

$$C_n = -b^{n+2}\frac{(\frac{\alpha}{2})_{n+2}}{(n+2)!}F\Big(1+\frac{\alpha}{2},n+2+\frac{\alpha}{2};n+3;b^2\Big). \tag{2.34}$$

Plugging the latter expression of C_n into (2.33) yields

$$\begin{aligned} P_n = -w^{n+2}\Big[&ab^2\frac{\alpha}{4}\Big(1+\frac{\alpha}{2}\Big)F\Big(1+\frac{\alpha}{2},2+\frac{\alpha}{2};3;b^2\Big)\\ &-cb^{n+2}\frac{(\frac{\alpha}{2})_{n+2}}{(n+2)!}F\Big(1+\frac{\alpha}{2},n+2+\frac{\alpha}{2};n+3;b^2\Big)\Big]. \end{aligned}$$

We shall now move to the computation of the last integral (2.31),

$$\begin{aligned} Q_n &\triangleq \fint_{\mathbb{T}}\frac{(b\overline{w}-\overline{\tau})(a\overline{w}^n-c\overline{\tau}^n)}{|bw-\tau|^{\alpha+2}}d\tau\\ &= \overline{w}^n\fint_{\mathbb{T}}\frac{(b-\overline{\zeta})(a-c\overline{\zeta}^n)}{|1-b\zeta|^{\alpha+2}}d\zeta\\ &\triangleq \overline{w}^n\Big(aD_0-c\,D_n\Big). \end{aligned}$$

From the identity (2.27) we may write

$$\begin{aligned} D_0 &\triangleq b\fint_{\mathbb{T}}\frac{d\zeta}{|1-b\zeta|^{\alpha+2}}-\fint_{\mathbb{T}}\frac{\overline{\zeta}d\zeta}{|1-b\zeta|^{\alpha+2}}\\ &= b^2\Big(1+\frac{\alpha}{2}\Big)F\Big(\frac{\alpha}{2}+1,2+\frac{\alpha}{2};2;b^2\Big)-F\Big(\frac{\alpha}{2}+1,\frac{\alpha}{2}+1;1;b^2\Big). \end{aligned}$$

Therefore by the formula (2.20) we obtain

$$D_0 \;=\; -F\Big(\frac{\alpha}{2},\frac{\alpha}{2}+1;1;b^2\Big).$$

To compute D_n we write through a change of variables,

$$\begin{aligned} D_n &\triangleq b\fint_{\mathbb{T}}\frac{\overline{\zeta}^n d\zeta}{|1-b\zeta|^{\alpha+2}}-\fint_{\mathbb{T}}\frac{\overline{\zeta}^{n+1} d\zeta}{|1-b\zeta|^{\alpha+2}}\\ &= b\fint_{\mathbb{T}}\frac{\zeta^{n-2} d\zeta}{|1-b\zeta|^{\alpha+2}}-\fint_{\mathbb{T}}\frac{\zeta^{n-1} d\zeta}{|1-b\zeta|^{\alpha+2}}\\ &= C_{n-2}, \end{aligned}$$

which implies in view of (2.34)

$$D_n=-b^n\frac{(\frac{\alpha}{2})_n}{n!}F\Big(1+\frac{\alpha}{2},n+\frac{\alpha}{2};n+1;b^2\Big).$$

Hence we find

$$Q_n \;=\; \overline{w}^n\Big[-aF\Big(\frac{\alpha}{2},\frac{\alpha}{2}+1;1;b^2\Big)+c\,b^n\frac{(\frac{\alpha}{2})_n}{n!}F\Big(\frac{\alpha}{2}+1,n+\frac{\alpha}{2};n+1;b^2\Big)\Big]$$

and therefore the proof of the lemma is now complete. □

2.4 Regularity of the nonlinear functional

This section is devoted to the regularity study of the nonlinear functional G introduced in (2.8) and which defines the V-states equations . We shall check the regularity assumptions required by Crandall-Rabinowitz's Theorem. The computations are vey heavy and can be done in a straightforward way without new difficulties compared to the simply connected case treated in the paper [54]. Many of the details may be found in that work and will not be reiterated here. Therefore for the sake of concise presentation we shall study the new terms involving the interaction between the boundaries. However, regarding the self-induced terms we only recall the results from the paper [54]. To begin with, we introduce the function spaces that we shall use. We set,

$$X=C^{2-\alpha}_{ar}(\mathbb{T})\times C^{2-\alpha}_{ar}(\mathbb{T}),\quad Y=H\times H, \tag{2.35}$$

with

$$C^{2-\alpha}_{ar}(\mathbb{T})=\Big\{f\in C^{2-\alpha}(\mathbb{T}); f(w)=\sum_{n\geq 1}a_n\overline{w}^n, w\in\mathbb{T}, a_n\in\mathbb{R}, n\in\mathbb{N}^\star\Big\}$$

and

$$H=\Big\{g\in C^{1-\alpha}(\mathbb{T}); g(w)=\frac{i}{2}\sum_{n\geq 1}a_n(w^n-\overline{w}^n), w\in\mathbb{T}, a_n\in\mathbb{R}, n\in\mathbb{N}\Big\}.$$

For $b\in(0,1)$, let V denote the product $B_r\times B_r$, where B_r is the open ball of X with center 0 and radius $r=(1/4)\min\{b,1-b\}$. We note that this choice is done in order to guarantee that $\phi_1=\mathrm{Id}+f_1$ and $\phi_2=b\,\mathrm{Id}+f_2$ are conformal for $f_1,f_2\in B_r$ and to prevent the intersection

between the curves $\phi_1(\mathbb{T})$ and $\phi_2(\mathbb{T})$ which represent the boundaries of the V-states.
Now recall from (2.8) the form of the functional $G = (G_1, G_2)$,

$$G_j\big(\Omega, f_1, f_2\big) \triangleq \text{Im}\Big\{\Big(\Omega\,\phi_j(w) + S(\phi_2, \phi_j)(w) - S(\phi_1, \phi_j)(w)\Big)\overline{w}\overline{\phi_j'(w)}\Big\}, \quad w \in \mathbb{T},\, j = 1, 2,$$

where S is defined by

$$S(\phi_i, \phi_j)(w) = C_\alpha \fint_{\mathbb{T}} \frac{\phi_i'(\tau)d\tau}{|\phi_j(w) - \phi_i(\tau)|^\alpha}, \quad i, j = 1, 2. \tag{2.36}$$

We shall rewrite G_j as follows,

$$G_j\big(\Omega, f_1, f_2\big) = L_j\big(\Omega, f_j\big) + N_j\big(f_1, f_2\big), \quad j = 1, 2,$$

with

$$L_j\big(\Omega, f_j\big) \triangleq \text{Im}\Big\{\Big(\Omega\,\phi_j(w) + (-1)^j S(\phi_j, \phi_j)(w)\Big)\overline{w}\,\overline{\phi_j'(w)}\Big\},$$

and

$$N_j\big(f_1, f_2\big) \triangleq (-1)^{j-1}\text{Im}\Big\{S(\phi_i, \phi_j)(w)\overline{w}\,\overline{\phi_j'(w)}\Big\}, \qquad i \neq j,$$

usually with the notation $\phi_1 = \text{Id} + f_1$, $\phi_2 = b\,\text{Id} + f_2$.

We propose to prove the following result concerning the regularity of G.

Proposition 2.1 *The following holds true.*

1. $G : \mathbb{R} \times V \to Y$ *is well-defined.*
2. $G : \mathbb{R} \times V \to Y$ *is of class* C^1.
3. *The partial derivative* $\partial_\Omega DG : \mathbb{R} \times V \to \mathcal{L}(X, Y)$ *exists and is continuous.*

Proof :

Notice that the terms $L_j, j = 1, 2$ appears modulo the sign of $(-1)^j$ in the simply connected case discussed in the paper [54] and all the computations were done there. Therefore we shall restrict ourselves to recalling just the results of those computations :

1. $L_j : \mathbb{R} \times B_r \to H$ is well-defined.
2. $L_j : \mathbb{R} \times B_r \to H$ is of class C^1.

Moreover the differential DL_j is given for $f_j \in B_r, h_j \in C^{2-\alpha}_{ar}(\mathbb{T})$ by

$$\begin{aligned} DL_j(\Omega, f_j)h_j(w) =& \text{Im}\Big\{\Omega\,\Big(\overline{w}\,\overline{h_j'(w)}\,\phi_j(w) + \overline{w}\,\overline{\phi_j'(w)}\,h_j(w)\Big) \\ &+ (-1)^j\overline{w}\,\overline{h_j'(w)}\,S(\phi_j, \phi_j)(w) + (-1)^j C_\alpha \overline{w}\overline{\phi_j'(w)} \fint_{\mathbb{T}} \frac{h_j'(\tau)}{|\phi_j(w) - \phi_j(\tau)|^\alpha}d\tau \\ &+ (-1)^{j+1}\alpha C_\alpha \overline{w}\overline{\phi_j'(w)} \fint_{\mathbb{T}} \frac{\text{Re}\big[\big(\phi_j(w) - \phi_j(\tau)\big)\big(\overline{h_j(w)} - \overline{h_j(\tau)}\big)\big]\phi_j'(\tau)}{|\phi_j(w) - \phi_j(\tau)|^{\alpha+2}}d\tau\Big\}. \end{aligned} \tag{2.37}$$

In addition, the partial derivative $\partial_\Omega DL_j : \mathbb{R}\times B_r \to \mathcal{L}(C^{2-\alpha}_{ar}(\mathbb{T}), H)$ exists and is continuous. It is given by the formula,

$$\partial_\Omega DL_j(\Omega, f_j)h_j(w) = \text{Im}\Big\{\overline{w}\,\overline{h_j'(w)}\,\phi_j(w) + \overline{w}\,\overline{\phi_j'(w)}\,h_j(w)\Big\}.$$

If we prove the regularity properties for the second part N_j then we can easily deduce that

$$\begin{aligned}\partial_\Omega DG_j(\Omega, f_1, f_2)(h_1, h_2)(w) &= \partial_\Omega DL_j(\Omega, f_j)h_j(w)\\ &= \text{Im}\Big\{\overline{w}\,\overline{h_j'(w)}\,\phi_j(w) + \overline{w}\,\overline{\phi_j'(w)}\,h_j(w)\Big\}.\end{aligned}$$

Therefore all the regularity assumptions are satisfied for the terms L_j and to complete the proof of the proposition we should check these assumptions for N_j. More precisely, we shall prove that $N_j : V \to H$ is well-defined and is of class C^1.

(**1**) First, we shall prove that for $(f_1, f_2)\in V$ we have $N_j(f_1, f_2)\in C^{1-\alpha}(\mathbb{T})$. Because the space $C^{1-\alpha}(\mathbb{T})$ is an algebra the problem reduces to show that for $i\neq j$, the function $S(\phi_i, \phi_j)$ belongs to $C^{1-\alpha}(\mathbb{T})$. This can be deduced easily from the next general result. Let $(f_1, f_2)\in V$ and $\phi_1 = \text{Id} + f_1$, $\phi_2 = b\,\text{Id} + f_2$ and define the operator

$$\mathcal{T}\chi(w) \triangleq \fint_{\mathbb{T}} \frac{\chi(\tau)}{|\phi_j(w)-\phi_i(\tau)|^\alpha}d\tau, \quad w\in\mathbb{T}.$$

Then

$$\big\|\mathcal{T}\chi\big\|_{C^{1-\alpha}(\mathbb{T})} \le C\|\phi_j\|_{C^{1-\alpha}(\mathbb{T})}\|\chi\|_{L^\infty(\mathbb{T})}. \tag{2.38}$$

The proof of this inequality will be done in a straightforward way since as we shall see the kernel is not singular. This is due to the fact that the inner and the outer boundaries do not intersect. Indeed, for all $w, \tau\in\mathbb{T}$ we can write

$$\begin{aligned}\big|\phi_1(w)-\phi_2(\tau)\big| &\ge |w-b\tau| - \big|f_1(w)\big| - \big|f_2(\tau)\big|\\ &\ge \big(1-b\big) - \|f_1\|_{L^\infty} - \|f_2\|_{L^\infty} > \frac{1-b}{2}\cdot\end{aligned}$$

The same result remains true if we change τ by w and therefore we get for $i\neq j$

$$\big|\phi_i(w)-\phi_j(\tau)\big| \;\ge\; \frac{1-b}{2}\cdot \tag{2.39}$$

It follows that

$$\big|\mathcal{T}\chi(w)\big| \;\lesssim\; \int_{\mathbb{T}} \frac{\big|\chi(\tau)\big|}{|\phi_j(w)-\phi_i(\tau)|^\alpha}|d\tau| \lesssim \|\chi\|_{L^\infty(\mathbb{T})},$$

which implies that

$$\|T\chi\|_{L^\infty(\mathbb{T})} \le C\|\chi\|_{L^\infty(\mathbb{T})}.$$

Next take $w_1\neq w_2\in\mathbb{T}$. Using the inequality (2.39) gives

$$\begin{aligned}\big|\mathcal{T}\chi(w_1)-\mathcal{T}\chi(w_2)\big| &\lesssim \int_{\mathbb{T}}\Bigg|\frac{1}{|\phi_j(w_1)-\phi_i(\tau)|^\alpha} - \frac{1}{|\phi_j(w_2)-\phi_i(\tau)|^\alpha}\Bigg|\big|\chi(\tau)\big||d\tau|\\ &\lesssim \|\chi\|_{L^\infty(\mathbb{T})}\int_{\mathbb{T}}\Big||\phi_j(w_1)-\phi_i(\tau)|^\alpha - |\phi_j(w_2)-\phi_i(\tau)|^\alpha\Big||d\tau|\\ &\lesssim \|\chi\|_{L^\infty(\mathbb{T})}\big|\phi_j(w_1)-\phi_j(w_2)\big|,\end{aligned}$$

where we have used in the last estimate the following inequality : for $\alpha \in (0,1)$, there exists a constant $C>0$, such that

$$\forall a,b\in\mathbb{R}_+^*,\quad |a^\alpha-b^\alpha|\le C\frac{|a-b|}{a^{1-\alpha}+b^{1-\alpha}}\cdot \tag{2.40}$$

Finally, using the fact that $\phi_i\in C^{2-\alpha}(\mathbb{T})\hookrightarrow C^{1-\alpha}(\mathbb{T})$ one can conclude that

$$\big|\mathcal{T}\chi(w_1)-\mathcal{T}\chi(w_2)\big|\lesssim \|\chi\|_{L^\infty(\mathbb{T})}\|\phi_j\|_{C^{1-\alpha}(\mathbb{T})}|w_1-w_2|^{1-\alpha},$$

which is the desired result. Now applying (2.38) to the operator S we get

$$\begin{aligned}\big\|S(\phi_i,\phi_j)\big\|_{C^{1-\alpha}(\mathbb{T})} &\le C\|\phi_j\|_{C^{1-\alpha}(\mathbb{T})}\|\phi_i'\|_{L^\infty(\mathbb{T})}\\ &\le C\|\phi_j\|_{C^{2-\alpha}(\mathbb{T})}\|\phi_i\|_{C^{2-\alpha}(\mathbb{T})}.\end{aligned}\tag{2.41}$$

To complete the proof of the first point we shall verify that the Fourier coefficients of $N_j(f_1,f_2)$ belong to $i\mathbb{R}$. From the definition of the space X in (2.35) the mapping ϕ_j has real Fourier coefficients and thus the Fourier coefficients of ϕ_j' are real too. Since this property is stable under the conjugation and the multiplication the problem reduces to prove that the Fourier coefficients of $S(\phi_i,\phi_j)$ are real. For this last purpose, we take the conjugate and make a change of variables,

$$\begin{aligned}\overline{S(\phi_i,\phi_j)(w)} &= -C_\alpha \fint_{\mathbb{T}}\frac{\phi_i'(\overline{\tau})d\overline{\tau}}{|\phi_j(\overline{w})-\phi_i(\overline{\tau})|^\alpha}\\ &= C_\alpha\fint_{\mathbb{T}}\frac{\phi_i'(\zeta)d\zeta}{|\phi_j(\overline{w})-\phi_i(\zeta)|^\alpha}\\ &= S(\phi_i,\phi_j)(\overline{w}).\end{aligned}$$

This proves that the Fourier coefficients of the functions $S(\phi_i,\phi_j)$ are real and the proof of the first part (1) is now achieved.

(2) The strategy to get that N_j is of class C^1 on V consists first in checking the existence of its Gâteaux derivative. Second we show that the Gâteaux derivative is strongly continuous. This will ensure in the same time the existence of Fréchet derivative and its continuity.
The Gâteaux derivative of the function N_j at (f_1,f_2) in the direction $h=(h_1,h_2)\in X$ is given by the formula

$$\begin{aligned}DN_j(f_1,f_2)h &= D_{f_1}N_j(f_1,f_2)h_1\;+\;D_{f_2}N_j(f_1,f_2)h_2\\ &\triangleq \lim_{t\to0}\frac1t\Big[N_j(f_1+th_1,f_2)-N_j(f_1,f_2)\Big]+\lim_{t\to0}\frac1t\Big[N_j(f_1,f_2+th_2)-N_j(f_1,f_2)\Big],\end{aligned}\tag{2.42}$$

where the limits are taken in the strong topology of Y. Thus we shall first prove the existence of these limit in the pointwise sense, that is for every point $w\in\mathbb{T}$, and after check that these limits exist in the strong topology of $C^{1-\alpha}(\mathbb{T})$.

Let us first check for each point $(f_1,f_2)\in V$ the existence of $D_{f_j}N_j(f_1,f_2)$ as a linear and bounded operator, that is, $D_{f_j}N_j(f_1,f_2)\in\mathcal{L}(C^{2-\alpha}_{ar}(\mathbb{T}),H)$. With the notation $\phi_1=\text{Id}+f_1$ and $\phi_2=b\,\text{Id}+f_2$, one has

$$\begin{aligned}D_{f_j}N_j(f_1,f_2)h_j(w) = (-1)^{j-1}\text{Im}\Big\{&\overline{w}\,\overline{h_j'(w)}S(\phi_i,\phi_j)(w)\\ &+\overline{w}\,\overline{\phi_j'(w)}\frac{d}{dt}\Big|_{t=0}S\big(\phi_i(w),\phi_j(w)+th_j(w)\big)\Big\}.\end{aligned}\tag{2.43}$$

We shall make use of the following identity : let $A \in \mathbb{C}^\star$, $B \in \mathbb{C}$, $\alpha \in \mathbb{R}$ and define the function $K : t \mapsto |A + Bt|^\alpha$ which is smooth close to zero, then we have

$$K'(0) = \alpha |A|^{\alpha-2} \text{Re}(\overline{A}B). \tag{2.44}$$

Combining this formula with few easy computations yield

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} S(\phi_i, \phi_j + th_j)(w) &= -\frac{\alpha}{2} C_\alpha \Big[\overline{h_j(w)} \fint_{\mathbb{T}} \frac{\big(\phi_j(w) - \phi_i(\tau)\big)\phi_i'(\tau)}{|\phi_j(w) - \phi_i(\tau)|^{\alpha+2}} d\tau \\ &\quad + h_j(w) \fint_{\mathbb{T}} \frac{\big(\overline{\phi_j(w)} - \overline{\phi_i(\tau)}\big)\phi_i'(\tau)}{|\phi_j(w) - \phi_i(\tau)|^{\alpha+2}} d\tau\Big] \\ &\triangleq -\frac{\alpha}{2} C_\alpha \Big[\overline{h_j(w)} A_i\big(\phi_j\big)(w) + h_j(w) B_i\big(\phi_j\big)(w)\Big].\end{aligned}$$

Therefore,

$$\begin{aligned}D_{f_j} N_j(\Omega, f_1, f_2) h_j(w) = (-1)^{j-1}\, \text{Im}\Big\{&\overline{w}\, \overline{h_j'(w)} S\big(\phi_i, \phi_j\big)(w) \\ &- \frac{\alpha}{2} C_\alpha \overline{w}\, \overline{\phi_j'(w)} \Big[\overline{h_j(w)} A_i\big(\phi_j\big)(w) + h_j(w) B_i\big(\phi_j\big)(w)\Big]\Big\}.\end{aligned} \tag{2.45}$$

Using the algebra structure of $C^{1-\alpha}(\mathbb{T})$ combined with the estimate (2.41), we get

$$\big\|D_{f_j} N_j(f_1, f_2) h_j\big\|_{C^{1-\alpha}(\mathbb{T})} \lesssim \Big(1 + \|A_i(\phi_j)\|_{C^{1-\alpha}(\mathbb{T})} + \|B_i\big(\phi_j\big)\|_{C^{1-\alpha}}\Big) \|h_j\|_{C^{2-\alpha}(\mathbb{T})}. \tag{2.46}$$

It remains to estimate the terms $A_i(\phi_j)$ and $B_i(\phi_j)$. For the first one, we get by virtue of (2.39),

$$\begin{aligned}\big|A_i\big(\phi_j\big)(w)\big| &\lesssim \int_{\mathbb{T}} \frac{\big|\phi_i'(\tau)\big|}{|\phi_j(w) - \phi_i(\tau)|^{\alpha+1}} |d\tau| \\ &\lesssim \|\phi_i'\|_{L^\infty(\mathbb{T})}.\end{aligned}$$

Now let $w_1 \neq w_2 \in \mathbb{T}$, then

$$\begin{aligned}\Big|A_i\big(\phi_j\big)(w_1) - A_i\big(\phi_j\big)(w_2)\Big| &\leq \int_{\mathbb{T}} \Big|\frac{\overline{\phi_j(w_1)} - \overline{\phi_i(\tau)}}{|\phi_j(w_1) - \phi_i(\tau)|^{\alpha+2}} - \frac{\overline{\phi_j(w_2)} - \overline{\phi_i(\tau)}}{|\phi_j(w_2) - \phi_i(\tau)|^{2+\alpha}}\Big| |\phi_i'(\tau)| |d\tau| \\ &\triangleq \int_{\mathbb{T}} \big|K(w_1, \tau) - K(w_2, \tau)\big| |\phi_i'(\tau)| |d\tau|.\end{aligned} \tag{2.47}$$

Few easy computations show that

$$\begin{aligned}\big|K(w_1, \tau) - K(w_2, \tau)\big| &\lesssim \frac{\big|\phi_j(w_2) - \phi_j(w_1)\big|}{|\phi_j(w_1) - \phi_i(\tau)|^{\alpha+2}} \\ &\quad + \frac{\Big||\phi_j(w_2) - \phi_i(\tau)|^{\alpha+2} - |\phi_j(w_1) - \phi_i(\tau)|^{\alpha+2}\Big|}{|\phi_j(w_1) - \phi_i(\tau)|^{\alpha+2} |\phi_j(w_2) - \phi_i(\tau)|^{\alpha+1}}.\end{aligned} \tag{2.48}$$

Concerning the last term we shall use the following inequality whose proof is classical.

$$|a^{k+1+\alpha} - b^{k+1+\alpha}| \leq C(k, \alpha) |a - b| \big(a^{k+\alpha} + b^{k+\alpha}\big), \quad a, b \in \mathbb{R}_+, k \in \mathbb{N}^*,\ 0 < \alpha. \tag{2.49}$$

Hence, we get

$$\frac{\Big||\phi_j(w_2)-\phi_i(\tau)|^{\alpha+2}-|\phi_j(w_1)-\phi_i(\tau)|^{\alpha+2}\Big|}{|\phi_j(w_1)-\phi_i(\tau)|^{\alpha+2}|\phi_j(w_2)-\phi_i(\tau)|^{\alpha+1}}\lesssim\frac{\big|\phi_j(w_2)-\phi_j(w_1)\big|}{|\phi_j(w_1)-\phi_i(\tau)|^{\alpha+2}}+\frac{\big|\phi_j(w_2)-\phi_j(w_1)\big|}{|\phi_j(w_1)-\phi_i(\tau)||\phi_j(w_2)-\phi_i(\tau)|^{\alpha+1}}.$$

Inserting this in the estimate (2.48) and using the inequality (2.39) we find

$$\begin{aligned}\big|K(w_1,\tau)-K(w_2,\tau)\big| &\lesssim \big|\phi_j(w_2)-\phi_j(w_1)\big|\\ &\lesssim \|\phi_j\|_{C^{1-\alpha}}|w_2-w_1|^{1-\alpha}.\end{aligned}$$

Now by plugging the latter estimate into (2.47) one gets

$$\Big|A_i\big(\phi_j\big)(w_1)-A_i\big(\phi_j\big)(w_2)\Big| \le C|w_2-w_1|^{1-\alpha}$$

which is the desired result. The estimate of the term $B_i\big(\phi_j\big)$ can be done in a similar way by observing that

$$B_i\big(\phi_j\big)(w)=\fint_{\mathbb{T}}\overline{K(w,\tau)}\phi_i'(\tau)d\tau.$$

Consequently, from (2.46) we deduce that

$$\|D_{f_j}N_j(f_1,f_2)h_j\|_{C^{1-\alpha}(\mathbb{T})}\le C\|h_1\|_{C^{2-\alpha}(\mathbb{T})}.$$

This means that $D_{f_j}N_j(f_1,f_2)\in\mathcal{L}(C^{1-\alpha}_{ar}(\mathbb{T}),H)$.

Let us now move to the computation of $D_{f_i}N_j(f_1,f_2)h_i$, for $i\neq j$ when $(f_1,f_2)\in V$ and $h_i\in C^{2-\alpha}_{ar}(\mathbb{T})$. From the definition, we obtain the formula

$$D_{f_i}N_j(f_1,f_2)h_i(w)=(-1)^{j-1}\mathrm{Im}\bigg\{\overline{w}\,\overline{\phi_j'(w)}\frac{d}{dt}\Big|_{t=0}S\big(\phi_i(w)+th_i(w),\phi_j(w)\big)\bigg\}.$$

Some easy computations combined with the relation (2.44) allow to get

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}S\big(\phi_i(w)+th_i(w),\phi_j(w)\big)&=C_\alpha\fint_{\mathbb{T}}\frac{h_i'(\tau)}{\big|\phi_j(w)-\phi_i(\tau)\big|^\alpha}d\tau\\ &+\frac{\alpha}{2}C_\alpha\fint_{\mathbb{T}}\frac{\big(\overline{\phi_j(w)}-\overline{\phi_i(\tau)}\big)h_i(\tau)\phi_i'(\tau)}{|\phi_j(w)-\phi_i(\tau)|^{\alpha+2}}d\tau\\ &+\frac{\alpha}{2}C_\alpha\fint_{\mathbb{T}}\frac{\big(\phi_j(w)-\phi_i(\tau)\big)\overline{h_i(\tau)}\phi_i'(\tau)}{|\phi_j(w)-\phi_i(\tau)|^{\alpha+2}}d\tau\\ &\triangleq C_i\big(\phi_j,h_i\big)(w)+D_i\big(\phi_j,h_i\big)(w)+E_i\big(\phi_j,h_i\big)(w)\cdot\end{aligned}$$

It follows that,

$$D_{f_i}N_j(f_1,f_2)h_i(w)=\mathrm{Im}\bigg\{\overline{w}\overline{\phi_j'(w)}\Big[C_i\big(\phi_j,h_i\big)(w)+D_i\big(\phi_j,h_i\big)(w)+E_i\big(\phi_j,h_i\big)(w)\Big]\bigg\}.\tag{2.50}$$

Since $C^{1-\alpha}(\mathbb{T})$ is an algebra one finds that

$$\big\|D_{f_j}N_j(\Omega,f_1,f_2)h_j\big\|_{C^{1-\alpha}(\mathbb{T})}\lesssim\|C_i(\phi_j,h_i)\|_{C^{1-\alpha}}+\|D_i(\phi_j,h_i)\|_{C^{1-\alpha}(\mathbb{T})}+\|E_i(\phi_j,h_i)\|_{C^{1-\alpha}(\mathbb{T})}.$$

The estimate of the term $C_i(\phi_j, h_i)$ follows immediately from (2.38) and we get,

$$\|C_i(\phi_j, h_i)\|_{C^{1-\alpha}(\mathbb{T})} \lesssim \|\phi_j\|_{C^{1-\alpha}(\mathbb{T})}\|h_i'\|_{L^\infty(\mathbb{T})}.$$

For the terms $D_i(\phi_j, h_i)$ and $E_i(\phi_j, h_i)$ we can proceed similarly as for $A_i(\phi_j, h_j)$ and we find

$$\|D_i(\phi_j, h_i)\|_{C^{1-\alpha}(\mathbb{T})} + \|E_i(\phi_j, h_i)\|_{C^{1-\alpha}(\mathbb{T})} \lesssim \|h_i\|_{L^\infty(\mathbb{T})}.$$

Putting together the preceding estimates yields,

$$\|D_{f_i} N_j(f_1, f_2)h_i\|_{C^{1-\alpha}(\mathbb{T})} \le C\|h_i\|_{C^{2-\alpha}(\mathbb{T})}.$$

This shows that $D_{f_i} N_j(f_1, f_2) \in \mathcal{L}(C_{ar}^{1-\alpha}(\mathbb{T}), H)$.

To achieve the existence proof of the Gâteaux derivatives it remains to check that the convergence in (2.42) occurs in the strong topology of $C^{1-\alpha}(\mathbb{T})$. There are many terms to analyze and they can be treated in a similar way. The computations are straightforward but slightly long and we prefer just to treat a significant term and the remaining ones are quite similar. For example in the first term of the right-hand side of (2.43) we need to check

$$\lim_{t\to 0} S\big(\phi_i, \phi_j + th_j\big) - S\big(\phi_i, \phi_j\big) = 0 \quad \text{in} \quad C^{1-\alpha}(\mathbb{T}).$$

To simplify the notation we set

$$T_{ij}(t, w) = S\big(\phi_i, \phi_j + th_j\big)(w) - S\big(\phi_i, \phi_j\big)(w).$$

Let $t > 0$ such that $t\|h_j\|_{L^\infty} < (1 - b)/2$. Then by (2.36) we have

$$\begin{aligned} T_{ij}(t, w) &= \fint_{\mathbb{T}} \Big(\frac{\phi_i'(\tau)}{|\phi_j(w) - \phi_i(\tau) + th_j(w)|^\alpha} - \frac{\phi_i'(\tau)}{|\phi_j(w) - \phi_i(\tau)|^\alpha}\Big) d\tau \\ &\triangleq \fint_{\mathbb{T}} K(t, w, \tau)\phi_i'(\tau) d\tau. \end{aligned}$$

It follows from the inequalities (2.39) and (2.40) that

$$\big|K(t, w, \tau)\big| \lesssim t\|h_j\|_{L^\infty(\mathbb{T})}$$

which implies in turn,

$$\big|T_{ij}(t, w)\big| \lesssim t\|h_j\|_{L^\infty(\mathbb{T})}.$$

Therefore we get

$$\lim_{t\to 0} \big\|T_{ij}(t, \cdot)\big\|_{L^\infty(\mathbb{T})} = 0.$$

Now for $w_1 \neq w_2 \in \mathbb{T}$, we write by the Mean Value Theorem

$$\begin{aligned} \big|T_{ij}(t, w_1) - T_{ij}(t, w_2)\big| &\lesssim \int_{\mathbb{T}} \Big|K(t, w_1, \tau) - K(t, w_2, \tau)\Big||d\tau| \\ &\lesssim \big|w_1 - w_2\big| \int_{\mathbb{T}} \sup_{w\in\mathbb{T}} \big|\partial_w K(t, w, \tau)\big||d\tau|. \end{aligned} \tag{2.51}$$

Observe that $K(t, w, \tau)$ can be rewritten in the integral form

$$K(t, w, \tau) = \int_0^t \partial_s g(s, w, \tau) ds, \quad \text{with} \quad g(t, w, \tau) \triangleq \frac{1}{|\phi_j(w) - \phi_i(\tau) + th_j(w)|^\alpha}.$$

Thus,

$$\big|\partial_w K(t,w,\tau)\big| \leq \int_0^t \big|\partial_w \partial_s g(s,w,\tau)\big| ds.$$

In view of the formula (2.9) we readily obtain

$$\partial_w g(t,w,\tau) = \frac{-\alpha}{2}\Bigg[\big(\phi_j'(w)+th_j'(w)\big)\frac{\overline{\phi_j(w)}-\overline{\phi_i(\tau)}+t\overline{h_j(w)}}{|\phi_j(w)-\phi_i(\tau)+th_j(w)\big)|^{2+\alpha}}$$

$$-\frac{\overline{\phi_j'(w)}+t\overline{h_j'(w)}}{w^2}\frac{\Big(\phi_j(w)-\phi_i(\tau)+th_j(w)\big)\Big)}{|\phi_j(w)-\phi_i(\tau)+th_j(w)\big)|^{2+\alpha}}\Bigg].$$

Using straightforward computations combined with the inequality (2.39) yield for any $s\in[0,t]$,

$$\Big|\partial_s\partial_w g(s,w,\tau)\Big| \leq C.$$

Hence we get,

$$\big|\partial_w K(t,w,\tau)\big| \leq C|t|.$$

This implies according to the estimate (2.51) that

$$\big|T_{ij}(t,w_1)-T_{ij}(t,w_2)\big| \;\;\leq\;\; C|t|\big|w_1-w_2\big|$$

and consequently,

$$\lim_{t\to 0}\|T_{ij}(t,\cdot)\|_{C^{1-\alpha}(\mathbb{T})}=0.$$

This concludes the desired result.

The next task is to show that the Gâteaux derivatives are continuous operators from the neighborhood V into the Banach space $\mathcal{L}(C^{1-\alpha}_{ar}(\mathbb{T}),H)$. From the identities (2.45) and (2.50) and since $C^{1-\alpha}(\mathbb{T})$ is an algebra the problem amounts to showing the continuity of the terms $S(\phi_i,\phi_j)$, $A_i(\phi_j)$, $B_i(\phi_j)$, $C_i(\phi_j,h_i)$, $D_i(\phi_j,h_i)$ and $E_i(\phi_j,h_i)$. We shall present here the complete details for the term $S(\phi_i,\phi_j)$, with $i\neq j$ and the other terms can be dealt via straightforward variations. Set

$$\phi_1=\mathrm{Id}+f_1,\quad \psi_1=\mathrm{Id}+g_1,\quad \phi_2=b\,\mathrm{Id}+f_2,\quad \psi_2=b\,\mathrm{Id}+g_2,$$

with (f_1,f_2) and $(g_1,g_2)\in V$. We shall prove the estimate

$$\|S(\phi_i,\phi_j)-S(\psi_i,\psi_j)\|_{C^{1-\alpha}} \leq C\Big(\|f_1-g_1\|_{C^{2-\alpha}(\mathbb{T})}+\|f_2-g_2\|_{C^{2-\alpha}(\mathbb{T})}\Big).$$

In view of (2.36) we may write

$$\begin{aligned} S(\phi_i,\phi_j)(w)-S(\psi_i,\psi_j)(w) &= \fint_{\mathbb{T}}\Big(\frac{\phi_i'(\tau)}{|\phi_j(w)-\phi_i(\tau)|^\alpha}-\frac{\psi_i'(\tau)}{|\psi_j(w)-\psi_i(\tau)|^\alpha}\Big)d\tau \\ &= \fint_{\mathbb{T}}\tilde K(w,\tau)\psi_i'(\tau)d\tau+\fint_{\mathbb{T}}\frac{\phi_i'(\tau)-\psi_i'(\tau)}{|\phi_j(w)-\phi_i(\tau)|^\alpha}d\tau, \end{aligned} \tag{2.52}$$

with

$$\tilde K(w,\tau)\triangleq\frac{1}{|\phi_j(w)-\phi_i(\tau)|^\alpha}-\frac{1}{|\psi_j(w)-\psi_i(\tau)|^\alpha}.$$

The estimate of the last term in (2.52) follows immediately from (2.38), that is,

$$\begin{aligned}\Big\|\fint_{\mathbb{T}}\frac{\phi_i'(\tau)-\psi_i'(\tau)}{|\phi_j(\cdot)-\phi_i(\tau)|^\alpha}d\tau\Big\|_{C^{1-\alpha}(\mathbb{T})} &\leq C\|f_i'-g_i'\|_{L^\infty} \\ &\leq C\|f_i-g_i\|_{C^{2-\alpha}(\mathbb{T})}. \end{aligned}\tag{2.53}$$

To control the remaining term we introduce the functional

$$L(w) \triangleq \fint_{\mathbb{T}} \tilde{K}(w,\tau)\psi'(\tau)d\tau.$$

Owing to the inequalities (2.39) and (2.40) one has

$$\begin{aligned} |L(w)| &\lesssim \|\phi_i\|_{\text{Lip}(\mathbb{T})} \int_{\mathbb{T}} \frac{\big||\psi_j(w)-\psi_i(\tau)|^\alpha - |\phi_j(w)-\phi_i(\tau)|^\alpha\big|}{|\phi_j(w)-\phi_i(\tau)|^\alpha|\psi_j(w)-\psi_i(\tau)|^\alpha}|d\tau| \\ &\lesssim \|\psi_j-\phi_j\|_{L^\infty(\mathbb{T})} \\ &\lesssim \|f_j-f_j\|_{L^\infty(\mathbb{T})}. \end{aligned} \tag{2.54}$$

Now let $w_1 \neq w_2 \in \mathbb{T}$, then we have

$$\begin{aligned} |L(w_1)-L(w_2)| &\lesssim \int_{\mathbb{T}} \big|\tilde{K}(w_1,\tau)-\tilde{K}(w_2,\tau)\big||d\tau| \\ &\lesssim |w_1-w_2| \int_{\mathbb{T}} \sup_{w\in\mathbb{T}} \big|\partial_w \tilde{K}(w,\tau)\big||d\tau|. \end{aligned} \tag{2.55}$$

In view of (2.9) the derivative of $K(w,\tau)$ with respect to w is given by

$$\partial_w \tilde{K}(w,\tau) = -\frac{\alpha}{2}\Big(\overline{\mathcal{I}(w,\tau)} - \frac{\mathcal{I}(w,\tau)}{w^2}\Big),$$

where

$$\mathcal{I}(w,\tau) \triangleq \overline{\phi_j'(w)}\frac{\phi_j(w)-\phi_i(\tau)}{|\phi_j(w)-\phi_i(\tau)|^{\alpha+2}} - \overline{\psi_j'(w)}\frac{\psi_j(w)-\psi_i(\tau)}{|\psi_j(w)-\psi_i(\tau)|^{\alpha+2}}.$$

We shall transform this quantity into,

$$\mathcal{I}(w,\tau) = \mathcal{I}_1(w,\tau) + \mathcal{I}_2(w,\tau) + \mathcal{I}_3(w,\tau),$$

with

$$\mathcal{I}_1(w,\tau) \triangleq \overline{\phi_j'(w)}\,\frac{(\phi_j-\psi_j)(w)-(\phi_i-\psi_i)(\tau)}{|\phi_j(w)-\phi_i(\tau)|^{\alpha+2}},$$

$$\mathcal{I}_2(w,\tau) \triangleq \big(\overline{\phi_j'(w)} - \overline{\psi_j'(w)}\big)\frac{\psi_j(w)-\psi_i(\tau)}{|\psi_j(w)-\psi_i(\tau)|^{\alpha+2}},$$

and

$$\mathcal{I}_3(w,\tau) \triangleq \overline{\phi_j'(w)}\big(\psi_i(\tau)-\psi_j(w)\big)\frac{|\phi_j(w)-\phi_i(\tau)|^{\alpha+2} - |\psi_j(w)-\psi_i(\tau)|^{\alpha+2}}{|\phi_j(w)-\phi_i(\tau)|^{\alpha+2}|\psi_j(w)-\psi_i(\tau)|^{\alpha+2}}.$$

For the first and the second terms one readily gets by Sobolev embeddings

$$|\mathcal{I}_1(w,\tau)| + |\mathcal{I}_2(w,\tau)| \lesssim \|\phi_i-\psi_i\|_{C^{2-\alpha}(\mathbb{T})} + \|\phi_j-\psi_j\|_{C^{2-\alpha}(\mathbb{T})}. \tag{2.56}$$

To estimate the last term we shall use the inequality (4.10) combined with the estimate (2.39),

$$\frac{\big||\phi_j(w)-\phi_i(\tau)|^{\alpha+2} - |\psi_j(w)-\psi_i(\tau)|^{\alpha+2}\big|}{|\phi_j(w)-\phi_i(\tau)|^{\alpha+2}|\psi_j(w)-\psi_i(\tau)|^{\alpha+2}} \lesssim \|\phi_j-\psi_j\|_{L^\infty(\mathbb{T})}$$

and consequently,

$$|\mathcal{I}_3(w,\tau)| \lesssim \|\phi_j-\psi_j\|_{C^{2-\alpha}(\mathbb{T})}. \tag{2.57}$$

Putting together (2.56) and (2.57) we find,

$$|\mathcal{I}(w,\tau)| \lesssim \|\phi_1-\psi_1\|_{C^{2-\alpha}(\mathbb{T})}+\|\phi_2-\psi_2\|_{C^{2-\alpha}(\mathbb{T})}.$$

Therefore

$$|\partial_w \tilde{K}(w,\tau)| \leq C\big(f_1-g_1\|_{C^{2-\alpha}(\mathbb{T})}+\|f_2-g_2\|_{C^{2-\alpha}(\mathbb{T})}\big).$$

Inserting this inequality into the estimate (2.55) we get

$$|L(w_1)-L(w_2)| \quad \leq \quad C\big(f_1-g_1\|_{C^{2-\alpha}(\mathbb{T})}+\|f_2-g_2\|_{C^{2-\alpha}(\mathbb{T})}\big)|w_1-w_2|^{1-\alpha}.$$

Putting together the last estimate with the estimate (2.53), we obtain

$$\|S(\phi_i,\phi_j)-S(\psi_i,\psi_j)\|_{C^{1-\alpha}(\mathbb{T})} \lesssim \|f_1-g_1\|_{C^{2-\alpha}(\mathbb{T})}+\|f_2-g_2\|_{C^{2-\alpha}(\mathbb{T})}.$$

This concludes the proof of the Proposition 2.1. □

2.5 Spectral study

The main goal of this section is to perform a spectral study of the linearized operator of G at the annular solution $(\Omega,0,0)$ and denoted by the differential $DG(\Omega,0,0)$. In particular, we shall identify the values Ω for which the kernel of $DG(\Omega,0,0)$ is not trivial leading to what we call the dispersion relation. Therefore the next step is to look among the "nonlinear eigenvalues" Ω those corresponding to one-dimensional kernels which is an important assumption in Crandall-Rabinowitz's Theorem. This task is very complicate compared to the previous cases discussed in [15, 54, 64]. This is due to the multiple parameters α, b and m in this problem and especially to the nonlinear and implicit structure of the coefficients appearing in the dispersion relation. We will be able to validate only a sufficient, but still a satisfactory result, with a restriction on the symmetry of the V-states. This will be deeply discussed in the Subsection 2.5.3 devoted to the monotonicity of the eigenvalues.

2.5.1 Linearized operator

We propose to compute explicitly the differential $DG(\Omega,0,0)$ and show that it acts as a Fourier multiplier. Since $G=(G_1,G_2)$ then for given $(h_1,h_2)\in X$, we have

$$DG(\Omega,0,0)(h_1,h_2) \quad = \quad \begin{pmatrix} D_{f_1}G_1(\Omega,0,0)h_1+D_{f_2}G_1(\Omega,0,0)h_2 \\ D_{f_1}G_2(\Omega,0,0)h_1+D_{f_2}G_2(\Omega,0,0)h_2 \end{pmatrix}.$$

where we recall the function spaces

$$X=C^{2-\alpha}_{ar}(\mathbb{T})\times C^{2-\alpha}_{ar}(\mathbb{T}),$$

and

$$C^{2-\alpha}_{ar}(\mathbb{T})=\Big\{f\in C^{2-\alpha}(\mathbb{T}); f(w)=\sum_{n\geq 1}a_n\overline{w}^n, w\in\mathbb{T}, a_n\in\mathbb{R}, n\in\mathbb{N}^\star\Big\}.$$

Putting together the formulas (2.37), (2.45) and (2.50) with $j=1$ and $j=2$, where we replace ϕ_1 by Id and ϕ_2 by $b\,$Id we get

$$DG_1(\Omega,0,0)h(w)=\Omega\,\mathcal{L}_0\big(h_1\big)(w)-C_\alpha\,\mathcal{L}_1\big(h_1\big)(w)+C_\alpha\,\mathcal{L}_2\big(h_1,h_2\big)(w), \tag{2.58}$$

$$DG_2(\Omega,0,0)h(w)=b\Big(\Omega\,\mathcal{L}_0\big(h_2\big)(w)+b^{-\alpha}C_\alpha\,\mathcal{L}_1\big(h_2\big)(w)-C_\alpha\,\mathcal{L}_3\big(h_1,h_2\big)(w)\Big),\tag{2.59}$$

with

$$\mathcal{L}_0\big(h_j\big)(w)\triangleq \mathrm{Im}\Big\{\overline{h_j'(w)}+\overline{w}h_j(w)\Big\},$$

$$\mathcal{L}_1\big(h_j\big)(w)\triangleq \mathrm{Im}\Big\{\overline{w}\overline{h_j'(w)}\fint_{\mathbb{T}}\frac{d\tau}{|w-\tau|^\alpha}+\overline{w}\fint_{\mathbb{T}}\frac{h_j'(\tau)}{|w-\tau|^\alpha}d\tau - \alpha\overline{w}\fint_{\mathbb{T}}\frac{\mathrm{Re}\,\big[(w-\tau)\big(\overline{h_j(w)}-\overline{h_j(\tau)}\big)\big]}{|w-\tau|^{\alpha+2}}d\tau\Big\},$$

$$\mathcal{L}_2\big(h_1,h_2\big)(w)\triangleq \mathrm{Im}\left\{b\overline{w}\overline{h_1'(w)}\fint_{\mathbb{T}}\frac{d\tau}{|w-b\tau|^\alpha}+\overline{w}\fint_{\mathbb{T}}\frac{h_2'(\tau)d\tau}{|w-b\tau|^\alpha} - \alpha b\overline{w}\fint_{\mathbb{T}}\frac{\mathrm{Re}\,\big[(w-b\tau)\big(\overline{h_1(w)}-\overline{h_2(\tau)}\big)\big]d\tau}{|w-b\tau|^{\alpha+2}}\right\},$$

and

$$\mathcal{L}_3\big(h_1,h_2\big)(w)\triangleq \mathrm{Im}\left\{b\overline{w}\,\overline{h_2'(w)}\fint_{\mathbb{T}}\frac{d\tau}{|bw-\tau|^\alpha}+\overline{w}\fint_{\mathbb{T}}\frac{h_1'(\tau)}{|bw-\tau|^\alpha}d\tau - \alpha\overline{w}\fint_{\mathbb{T}}\frac{\mathrm{Re}\,\big[(bw-\tau)\big(\overline{h_2(w)}-\overline{h_1(\tau)}\big)\big]}{|bw-\tau|^{\alpha+2}}d\tau\right\}.$$

We shall now compute the Fourier series of the mapping $w\mapsto DG(\Omega,0,0)(h_1,h_2)(w)$ with

$$h_1(w)=\sum_{n=1}^{\infty}a_n\overline{w}^n\quad\text{and}\quad h_2(w)=\sum_{n=1}^{\infty}c_n\overline{w}^n,\quad w\in\mathbb{T},$$

where a_n and c_n are real for all $n\in\mathbb{N}^\star$. This is summarized in the following lemma.

Lemma 2.2 *Let $\alpha\in(0,1)$ and $b\in(0,1)$. We set*

$$\Lambda_n(b)\triangleq\frac{1}{b}\int_0^{+\infty}\frac{J_n(bt)J_n(t)}{t^{1-\alpha}}dt,\tag{2.60}$$

and

$$\Theta_n \triangleq \Lambda_1(1)-\Lambda_n(1),$$

where J_n refers to the Bessel function of the first kind. Then, we have

$$DG(\Omega,0,0)\big(h_1,h_2\big)(w)=\frac{i}{2}\sum_{n\geq 1}\big(n+1\big)M_{n+1}^\alpha\begin{pmatrix}a_n\\ c_n\end{pmatrix}\Big(w^{n+1}-\overline{w}^{n+1}\Big).\tag{2.61}$$

where the matrix M_n is given for $n\geq 2$ by

$$M_n^\alpha\triangleq\begin{pmatrix}\Omega-\Theta_n+b^2\Lambda_1(b) & -b^2\Lambda_n(b)\\ b\Lambda_n(b) & b\Omega+b^{1-\alpha}\Theta_n-b\Lambda_1(b)\end{pmatrix}.\tag{2.62}$$

The determinant of this matrix is given by

$$\det\big(M_n^\alpha\big) = \Big(\Omega-\Theta_n+b^2\Lambda_1(b)\Big)\Big(b\Omega+b^{1-\alpha}\Theta_n-b\Lambda_1(b)\Big)+b^3\Lambda_n^2(b)\cdot\tag{2.63}$$

Proof : First we shall compute $DG_1(\Omega,0,0)(h_1,h_2)$. For this goal we start with calculating the term $\mathcal{L}_0\big(h_1(w)\big)$ of the right-hand side of (2.58) which is easy compared to the other terms. Thus by straightforward computations we obtain

$$\begin{aligned}\mathcal{L}_0\big(h_1\big)(w) &= \operatorname{Im}\Big\{\sum_{n\geq 1}\Big(a_n\overline{w}^{n+1}-na_nw^{n+1}\Big)\Big\} \\ &= \frac{i}{2}\sum_{n\geq 1}(n+1)a_n\Big(w^{n+1}-\overline{w}^{n+1}\Big).\end{aligned} \tag{2.64}$$

The computation of the second term $\mathcal{L}_1\big(h_1\big)(w)$ was done in the paper [54] dealing with the simply connected domain. It is given by

$$C_\alpha\mathcal{L}_1\big(h_1\big)(w)=\frac{i}{2}\sum_{n\geq 1}a_n(n+1)\frac{\Gamma(1-\alpha)}{2^{1-\alpha}\Gamma^2(1-\frac{\alpha}{2})}\bigg(\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})}-\frac{\Gamma(n+1+\frac{\alpha}{2})}{\Gamma(n+2-\frac{\alpha}{2})}\bigg)\Big(w^{n+1}-\overline{w}^{n+1}\Big).$$

We shall later establish the identity (2.83) which gives here

$$C_\alpha\,\mathcal{L}_1\big(h_1\big)(w) = \frac{i}{2}\sum_{n\geq 1}a_n(n+1)\Theta_{n+1}\Big(w^{n+1}-\overline{w}^{n+1}\Big). \tag{2.65}$$

To compute the term $\mathcal{L}_2\big(h_1,h_2\big)(w)$ we first split it into two parts as follows,

$$\mathcal{L}_2\big(h_1,h_2\big)(w) = \operatorname{Im}\Big\{\mathrm{I}_1(w)+\mathrm{I}_2(w)\Big\}, \tag{2.66}$$

with

$$\mathrm{I}_1(w)\triangleq\frac{b\overline{h_1'(w)}}{w}\fint_{\mathbb{T}}\frac{d\tau}{|w-b\tau|^\alpha}-\frac{\alpha b}{2w}\fint_{\mathbb{T}}\frac{(w-b\tau)\big(\overline{h_1(w)}-\overline{h_2(\tau)}\big)}{|w-b\tau|^{\alpha+2}}d\tau$$

and

$$\mathrm{I}_2(w)\triangleq\frac{1}{w}\fint_{\mathbb{T}}\frac{h_2'(\tau)}{|w-b\tau|^\alpha}d\tau-\frac{\alpha b}{2w}\fint_{\mathbb{T}}\frac{(\overline{w}-b\overline{\tau})\big(h_1(w)-h_2(\tau)\big)}{|w-b\tau|^{\alpha+2}}d\tau.$$

By using the Fourier expansions of h_1 and h_2 we get,

$$\mathrm{I}_1(w)=-b\sum_{n\geq 1}na_nw^n\fint_{\mathbb{T}}\frac{d\tau}{|w-b\tau|^\alpha}-\frac{\alpha}{2}b\overline{w}\sum_{n\geq 1}\fint_{\mathbb{T}}\frac{(w-b\tau)(a_nw^n-c_n\tau^n)}{|w-b\tau|^{\alpha+2}}d\tau.$$

Then by applying the formula (2.27) with $n=1$ to the first term and the formula (2.28) to the second term we find

$$\begin{aligned}\mathrm{I}_1(w)=&-\frac{\alpha}{2}b^2\sum_{n\geq 1}na_nw^{n+1}F\Big(\frac{\alpha}{2},1+\frac{\alpha}{2};2;b^2\Big)\\ &-\frac{\alpha}{2}b^2\sum_{n\geq 0}w^{n+1}\bigg[a_n\Big(1+\frac{\alpha}{2}\Big)F\Big(\frac{\alpha}{2},2+\frac{\alpha}{2};2;b^2\Big)\\ &\qquad\qquad -c_nb^n\frac{(1+\frac{\alpha}{2})_{n+1}}{(n+1)!}F\Big(\frac{\alpha}{2},n+2+\frac{\alpha}{2};n+2;b^2\Big)\bigg]\\ \triangleq&-\frac{\alpha}{2}b^2\sum_{n\geq 0}\big(a_n\gamma_n+c_n\delta_n\big)w^{n+1},\end{aligned} \tag{2.67}$$

where we have used in the last equality the notation,

$$\gamma_n\triangleq\Big(1+\frac{\alpha}{2}\Big)F\Big(\frac{\alpha}{2},2+\frac{\alpha}{2};2;b^2\Big)+nF\Big(\frac{\alpha}{2},1+\frac{\alpha}{2};2;b^2\Big)$$

and

$$\delta_n = -b^n \frac{(1+\frac{\alpha}{2})_{n+1}}{(n+1)!} F\Big(\frac{\alpha}{2}, n+2+\frac{\alpha}{2}; n+2; b^2\Big).$$

Similarly, the second term $\mathrm{I}_2(w)$ may be written in the form,

$$\mathrm{I}_2(w) = -\overline{w}\sum_{n\geq 0} nc_n \fint_{\mathbb{T}} \frac{\overline{\tau}^{n+1}d\tau}{|w-b\tau|^{\alpha}} - \frac{\alpha}{2}b\overline{w}\sum_{n\geq 0}\fint_{\mathbb{T}} \frac{(\overline{w}-b\overline{\tau})(a_n\overline{w}^n - c_n\overline{\tau}^n)}{|w-b\tau|^{\alpha+2}}d\tau.$$

Using the elementary fact

$$\fint_{\mathbb{T}} \frac{\overline{\tau}^{n+1}d\tau}{|w-b\tau|^{\alpha}} = \overline{\fint_{\mathbb{T}} \frac{\tau^{n-1}d\tau}{|w-b\tau|^{\alpha}}} \tag{2.68}$$

combined with the formulas (2.27) and (2.29) we obtain

$$\begin{aligned}
\mathrm{I}_2(w) =& -\sum_{n\geq 1} nc_n \frac{(\frac{\alpha}{2})_n}{n!}\overline{w}^{n+1}b^n F\Big(\frac{\alpha}{2}, n+\frac{\alpha}{2}; n+1; b^2\Big)\\
&-\frac{\alpha}{2}\sum_{n\geq 1}\overline{w}^{n+1}\Big[a_n b^2\frac{\alpha}{2}F\Big(\frac{\alpha}{2}+1,\frac{\alpha}{2}+1;2;b^2\Big) - c_n b^n\frac{(1+\frac{\alpha}{2})_{n-1}}{(n-1)!}F\Big(\frac{\alpha}{2},n+\frac{\alpha}{2};n;b^2\Big)\Big]\\
=& -\frac{\alpha}{2}\sum_{n\geq 1}\overline{w}^{n+1}\Big[a_n b^2\frac{\alpha}{2}F(\frac{\alpha}{2}+1,\frac{\alpha}{2}+1;2;b^2)\\
&+c_n\frac{(1+\frac{\alpha}{2})_{n-1}}{(n-1)!}b^n\Big(F\Big(\frac{\alpha}{2},n+\frac{\alpha}{2};n+1;b^2\Big) - F\Big(\frac{\alpha}{2},n+\frac{\alpha}{2};n;b^2\Big)\Big)\Big].
\end{aligned} \tag{2.69}$$

Thus owing to the formula (2.19) applied with $a=\frac{\alpha}{2}, b=\frac{\alpha}{2}+n$ and $c=n$ one gets

$$\mathrm{I}_2(w) = -\frac{\alpha}{2}b^2\sum_{n\geq 1}\Big(\alpha_n a_n + \beta_n c_n\Big)\overline{w}^{n+1}, \tag{2.70}$$

where α_n and β_n are defined by,

$$\alpha_n \triangleq \frac{\alpha}{2}F\Big(\frac{\alpha}{2}+1,\frac{\alpha}{2}+1;2;b^2\Big)$$

and

$$\beta_n \triangleq -\frac{(\frac{\alpha}{2})_{n+1}}{(n+1)!}b^n F\Big(\frac{\alpha}{2}+1, n+1+\frac{\alpha}{2}; n+2; b^2\Big).$$

Inserting the identities (2.70) and (2.67) into (2.66) we find

$$\begin{aligned}
\mathcal{L}_2\big(h_1,h_2\big)(w) &= -\frac{\alpha}{2}b^2\mathrm{Im}\Big\{\sum_{n\geq 1}\big(a_n\gamma_n + c_n\delta_n\big)w^{n+1} + \sum_{n\geq 1}\big(a_n\alpha_n + c_n\beta_n\big)\overline{w}^{n+1}\Big\}\\
&= i\frac{\alpha}{4}b^2\sum_{n\geq 1}\Big(w^{n+1}-\overline{w}^{n+1}\Big)\Big[a_n\big(\gamma_n-\alpha_n\big) + c_n\big(\delta_n-\beta_n\big)\Big].
\end{aligned}$$

To compute $\gamma_n - \alpha_n$ we shall use the formula (2.24) which gives

$$\begin{aligned}
\gamma_n - \alpha_n &= \Big(1+\frac{\alpha}{2}\Big)F\Big(\frac{\alpha}{2},2+\frac{\alpha}{2};2;b^2\Big) - \frac{\alpha}{2}F\Big(\frac{\alpha}{2}+1,\frac{\alpha}{2}+1;2;b^2\Big) + n\,F\Big(\frac{\alpha}{2},1+\frac{\alpha}{2};2;b^2\Big)\\
&= (n+1)\,F\Big(\frac{\alpha}{2},1+\frac{\alpha}{2};2;b^2\Big).
\end{aligned}$$

Similarly we have

$$\delta_n-\beta_n=-b^n\frac{(1+\frac{\alpha}{2})_n}{(n+1)!}\Big[\Big(n+1+\frac{\alpha}{2}\Big)F\Big(\frac{\alpha}{2},n+2+\frac{\alpha}{2};n+2;b^2\Big)$$
$$-\frac{\alpha}{2}F\Big(\frac{\alpha}{2}+1,n+1+\frac{\alpha}{2};n+2;b^2\Big)\Big]$$

and therefore using once again the identity (2.24) we find

$$\delta_n-\beta_n=-b^n\frac{(1+\frac{\alpha}{2})_n}{n!}F\Big(\frac{\alpha}{2},n+1+\frac{\alpha}{2},n+2,b^2\Big).$$

Consequently the Fourier expansion of $\mathcal{L}_2\big(h_1,h_2\big)$ is described by the formula

$$\mathcal{L}_2\big(h_1,h_2\big)(w)=\frac{i}{2}b^2\sum_{n\geq1}(n+1)\Big[a_n\frac{\alpha}{2}F\Big(\frac{\alpha}{2},1+\frac{\alpha}{2},2,b^2\Big)$$
$$-c_nb^n\frac{(\frac{\alpha}{2})_{n+1}}{(n+1)!}F\Big(\frac{\alpha}{2},n+1+\frac{\alpha}{2},n+2,b^2\Big)\Big]\Big(w^{n+1}-\overline{w}^{n+1}\Big).$$

By virtue of the identity (2.81) we get

$$C_\alpha\,\mathcal{L}_2\big(h_1,h_2\big)(w)=\frac{i}{2}\sum_{n\geq1}(n+1)\Big[a_nb^2\Lambda_1(b)-c_nb^2\Lambda_{n+1}(b)\Big]\Big(w^{n+1}-\overline{w}^{n+1}\Big).\tag{2.71}$$

Finally inserting (2.64), (2.65) and (2.71) into (2.58) we find

$$DG_1(\Omega,0,0)(h_1,h_2)(w)=\frac{i}{2}\sum_{n\geq1}(n+1)\Big[a_n\Big(\Omega-\Theta_{n+1}+b^2\Lambda_1(b)\Big)-c_nb^2\Lambda_{n+1}(b)\Big]$$
$$\times\Big(w^{n+1}-\overline{w}^{n+1}\Big).\tag{2.72}$$

Next, we shall move to the computations of $DG_2(\Omega,0,0)(h_1,h_2)$ defined in (2.59). The first two terms are done in the preceding step and therefore it remains just to compute the term $\mathcal{L}_3\big(h_1,h_2\big)$. It may be splitted into two terms,

$$\mathcal{L}_3\big(h_1,h_2\big)(w)=\mathrm{Im}\Big\{\tilde{\mathrm{I}}_1(w)+\tilde{\mathrm{I}}_2(w)\Big\},\tag{2.73}$$

with

$$\tilde{\mathrm{I}}_1(w)\triangleq\frac{\overline{h_2'(w)}}{bw}\fint_{\mathbb{T}}\frac{d\tau}{|bw-\tau|^\alpha}-\frac{\alpha}{2w}\fint_{\mathbb{T}}\frac{(bw-\tau)\big(\overline{h_2(w)}-\overline{h_1(\tau)}\big)}{|bw-\tau|^{\alpha+2}}d\tau$$

and

$$\tilde{\mathrm{I}}_2(w)\triangleq\frac{1}{w}\fint_{\mathbb{T}}\frac{h_1'(\tau)d\tau}{|bw-\tau|^\alpha}-\frac{\alpha}{2w}\fint_{\mathbb{T}}\frac{(b\overline{w}-\overline{\tau})\big(h_2(w)-h_1(\tau)\big)}{|bw-\tau|^{\alpha+2}}d\tau.$$

To compute the first term $\tilde{\mathrm{I}}_1(w)$ we write

$$\tilde{\mathrm{I}}_1(w)\;=\;-\sum_{n\geq1}n\frac{c_n}{b}w^n\fint_{\mathbb{T}}\frac{d\tau}{|bw-\tau|^\alpha}-\frac{\alpha}{2w}\sum_{n\geq1}\fint_{\mathbb{T}}\frac{(bw-\tau)\big(c_nw^n-a_n\tau^n\big)}{|bw-\tau|^{\alpha+2}}d\tau.$$

Thus applying successively the formula (2.27) to the first term with $n = 1$ and the formula (2.30) to the second one we get

$$\begin{aligned}\tilde{\text{I}}_1(w) =& -\frac{\alpha}{2}\sum_{n\geq 1} n\, c_n\; w^{n+1}F\Big(\frac{\alpha}{2},1+\frac{\alpha}{2};2;b^2\Big)\\ &+\frac{\alpha}{2}\sum_{n\geq 1} w^{n+1}\Big[c_n b^2\frac{\alpha}{4}(1+\frac{\alpha}{2})F\Big(1+\frac{\alpha}{2},2+\frac{\alpha}{2};3;b^2\Big)\\ &\qquad -a_n b^{n+2}\frac{(\frac{\alpha}{2})_{n+2}}{(n+2)!}F\Big(1+\frac{\alpha}{2},n+2+\frac{\alpha}{2};n+3;b^2\Big)\Big]\\ =& -\frac{\alpha}{2}\sum_{n\geq 1}\big(a_n\;\tilde{\gamma}_n+c_n\,\tilde{\delta}_n\big)w^{n+1} \end{aligned} \tag{2.74}$$

with

$$\tilde{\gamma}_n \triangleq b^{n+2}\frac{(\frac{\alpha}{2})_{n+2}}{(n+2)!}F\Big(1+\frac{\alpha}{2},n+2+\frac{\alpha}{2};n+3;b^2\Big)$$

and

$$\tilde{\delta}_n \triangleq nF\Big(\frac{\alpha}{2},1+\frac{\alpha}{2};2;b^2\Big)-b^2\frac{\alpha}{4}\Big(1+\frac{\alpha}{2}\Big)F\Big(1+\frac{\alpha}{2},2+\frac{\alpha}{2};3;b^2\Big).$$

As to the term $\tilde{\text{I}}_2(w)$ we write

$$\tilde{\text{I}}_2(w) \;=\; -\frac{1}{w}\sum_{n\geq 1} na_n \fint_{\mathbb{T}}\frac{\overline{\tau}^{n+1}d\tau}{|bw-\tau|^{\alpha}}-\frac{\alpha}{2w}\sum_{n\geq 1}\fint_{\mathbb{T}}\frac{(b\overline{w}-\overline{\tau})\big(c_n\overline{w}^n-a_n\overline{\tau}^n\big)d\tau}{|bw-\tau|^{\alpha+2}}.$$

Owing to (2.68) and using the formulae (2.27) and (2.31), one gets

$$\begin{aligned}\tilde{\text{I}}_2(w) =& -\sum_{n\geq 1} na_n\overline{w}^{n+1}b^n\frac{(\frac{\alpha}{2})_n}{n!}F\Big(\frac{\alpha}{2},n+\frac{\alpha}{2};n+1;b^2\Big)\\ &-\frac{\alpha}{2}\sum_{n\geq 1}\overline{w}^{n+1}\Big[-c_nF\Big(\frac{\alpha}{2},\frac{\alpha}{2}+1;1;b^2\Big)+a_nb^n\frac{(\frac{\alpha}{2})_n}{n!}F\Big(\frac{\alpha}{2}+1,n+\frac{\alpha}{2};n+1;b^2\Big)\Big]\\ =& -\frac{\alpha}{2}\sum_{n\geq 1}\big(a_n\tilde{\alpha}_n+c_n\tilde{\beta}_n\big)\overline{w}^{n+1}, \end{aligned} \tag{2.75}$$

with

$$\tilde{\alpha}_n \triangleq \frac{(1+\frac{\alpha}{2})_{n-1}}{n!}b^n\Big[n\,F\Big(\frac{\alpha}{2},n+\frac{\alpha}{2};n+1;b^2\Big)+\frac{\alpha}{2}F\Big(1+\frac{\alpha}{2},n+\frac{\alpha}{2};n+1;b^2\Big)\Big]$$

and

$$\tilde{\beta}_n \triangleq -F\Big(\frac{\alpha}{2},\frac{\alpha}{2}+1;1;b^2\Big).$$

Now inserting the identities (2.74) and (2.75) into (2.73) we find

$$\begin{aligned}\mathcal{L}_3\big(h_1,h_2\big)(w) \;&=\; -\frac{\alpha}{2}\text{Im}\Big\{\sum_{n\geq 1}\big(a_n\tilde{\gamma}_n+c_n\tilde{\delta}_n\big)w^{n+1}+\sum_{n\geq 1}\big(a_n\tilde{\alpha}_n+c_n\tilde{\beta}_n\big)\overline{w}^{n+1}\Big\}\\ &=\; i\frac{\alpha}{4}\sum_{n\geq 1}\Big[a_n\big(\tilde{\gamma}_n-\tilde{\alpha}_n\big)+c_n\big(\tilde{\delta}_n-\tilde{\beta}_n\big)\Big]\Big(w^{n+1}-\overline{w}^{n+1}\Big). \end{aligned} \tag{2.76}$$

From the foregoing expressions for $\tilde{\gamma}_n$ and $\tilde{\alpha}_n$ one may write,

$$\begin{aligned}
\tilde{\gamma}_n - \tilde{\alpha}_n = & \, b^{n+2}\frac{(\frac{\alpha}{2})_{n+2}}{(n+2)!} \quad F\Big(1+\frac{\alpha}{2}, n+2+\frac{\alpha}{2}; n+3; b^2\Big) \\
& - \frac{(1+\frac{\alpha}{2})_{n-1}}{n!} b^n \quad \Big[nF\Big(\frac{\alpha}{2}, n+\frac{\alpha}{2}; n+1; b^2\Big) + \frac{\alpha}{2} F\Big(1+\frac{\alpha}{2}, n+\frac{\alpha}{2}; n+1; b^2\Big)\Big] \\
= & \, \frac{(1+\frac{\alpha}{2})_{n-1}}{n!} b^n \Big[b^2 \frac{\frac{\alpha}{2}\big(\frac{\alpha}{2}+n\big)\big(\frac{\alpha}{2}+1+n\big)}{(n+2)(n+1)} F\Big(1+\frac{\alpha}{2}, n+2+\frac{\alpha}{2}; n+3; b^2\Big) \\
& - nF\Big(\frac{\alpha}{2}, n+\frac{\alpha}{2}; n+1; b^2\Big) - \frac{\alpha}{2} F\Big(1+\frac{\alpha}{2}, n+\frac{\alpha}{2}; n+1; b^2\Big)\Big].
\end{aligned}$$

Hence using the formula (2.19) with $a = \frac{\alpha}{2}, b = n+1+\frac{\alpha}{2}$ and $c = n+1$ yields

$$\begin{aligned}
\tilde{\gamma}_n - \tilde{\alpha}_n = & \, \frac{(1+\frac{\alpha}{2})_{n-1}}{n!} b^n \Big[\big(\frac{\alpha}{2}+n\big) F\Big(\frac{\alpha}{2}, n+1+\frac{\alpha}{2}; n+1; b^2\Big) \\
& - \big(\frac{\alpha}{2}+n\big) F\Big(\frac{\alpha}{2}, n+1+\frac{\alpha}{2}; n+2; b^2\Big) \\
& - \; nF\Big(\frac{\alpha}{2}, n+\frac{\alpha}{2}; n+1; b^2\Big) - \frac{\alpha}{2} F\Big(1+\frac{\alpha}{2}, n+\frac{\alpha}{2}; n+1; b^2\Big)\Big].
\end{aligned}$$

Applying the formula (2.24) with $a = \frac{\alpha}{2}, b = \frac{\alpha}{2}+n$ and $c = n+1$ we get

$$\big(\frac{\alpha}{2}+n\big) F\Big(\frac{\alpha}{2}, n+1+\frac{\alpha}{2}; n+1; b^2\Big) - \frac{\alpha}{2} F\Big(1+\frac{\alpha}{2}, n+\frac{\alpha}{2}; n+1; b^2\Big) = nF\Big(\frac{\alpha}{2}, n+\frac{\alpha}{2}; n+1; b^2\Big).$$

This implies,

$$\tilde{\gamma}_n - \tilde{\alpha}_n = -\frac{(1+\frac{\alpha}{2})_n}{n!} b^n F\Big(\frac{\alpha}{2}, n+1+\frac{\alpha}{2}; n+2; b^2\Big).$$

Using the expressions of $\tilde{\delta}_n$ and $\tilde{\beta}_n$ combined with the identity (2.19) applied with $a = \frac{\alpha}{2}, b = 1+\frac{\alpha}{2}$ and $c = 1$ we find the compact formula

$$\begin{aligned}
\tilde{\delta}_n - \tilde{\beta}_n = & \, nF\Big(\frac{\alpha}{2}, 1+\frac{\alpha}{2}; 2; b^2\Big) - b^2\frac{\alpha}{4}\big(1+\frac{\alpha}{2}\big) F\Big(1+\frac{\alpha}{2}, 2+\frac{\alpha}{2}; 3; b^2\Big) \\
& + F\Big(\frac{\alpha}{2}, \frac{\alpha}{2}+1; 1; b^2\Big) \\
= & \, (n+1) F\Big(\frac{\alpha}{2}, 1+\frac{\alpha}{2}; 2; b^2\Big).
\end{aligned}$$

Putting together the preceding identities allows to write

$$\begin{aligned}
\mathcal{L}_3\big(h_1, h_2(w)\big) = & \, i\frac{\alpha}{4}\sum_{n\geq 1}(n+1)\Big(w^{n+1} - \overline{w}^{n+1}\Big) \\
& \times \Big[c_n F\Big(\frac{\alpha}{2}, 1+\frac{\alpha}{2}; 2; b^2\Big) - a_n b^n \frac{(1+\frac{\alpha}{2})_n}{(n+1)!} F\Big(\frac{\alpha}{2}, n+1+\frac{\alpha}{2}, n+2, b^2\Big)\Big].
\end{aligned}$$

According to the identity (2.81) we get

$$C_\alpha \mathcal{L}_3\big(h_1, h_2(w)\big) = \frac{i}{2}\sum_{n\geq 1}(n+1)\Big(c_n\Lambda_1(b) - a_n\Lambda_{n+1}(b)\Big)\Big(w^{n+1} - \overline{w}^{n+1}\Big).$$

Finaly, inserting the preceding identity and the expressions (2.65) and (2.73) into (2.59) one can readily verify that

$$DG_2(\Omega,0,0)(h_1,h_2)(w)=\frac{i}{2}\sum_{n\geq 1}(n+1)\Big[a_nb\Lambda_{n+1}(b)+c_n\Big(b\Omega+b^{1-\alpha}\Theta_{n+1}-b\Lambda_1(b)\Big)\Big]\times\Big(w^{n+1}-\overline{w}^{n+1}\Big).\tag{2.77}$$

This concludes the proof of the Lemma 2.2. □

2.5.2 Asymptotic behavior

We shall collect some useful properties on the asymptotic behavior of the sequences $(\Theta_n)_n$ and $(\Lambda_n)_n$ introduced in Lemma 2.2. The study is done with respect to the parameters α and n. This is summarized in the next lemma.

Lemma 2.3 *Let $\alpha\in(0,1)$ and $b\in(0,1)$. Then the following results hold true.*

1. *For all $n\in\mathbb{N}^*$, $\Theta_n\geq 0$, $\Lambda_n\geq 0$. Moreover, $b\mapsto\Lambda_n(b)$ is strictly increasing, $n\mapsto\Theta_n$ is strictly increasing and $n\mapsto\Lambda_n(b)$ is strictly decreasing.*
2. *Let $n\geq 2$, then*

$$\lim_{\alpha\to 0}\Theta_n=\frac{n-1}{2n},\quad \lim_{\alpha\to 0}\Lambda_n(b)=\frac{b^{n-1}}{2n}$$

and

$$\lim_{\alpha\to 1}\Theta_n=\frac{2}{\pi}\sum_{k=1}^{n-1}\frac{1}{2k+1},\quad \lim_{\alpha\to 1}\Lambda_n(b)=\frac{1}{b}\int_0^{+\infty}J_n(bt)J_n(t)dt.$$

3. *For n sufficiently large,*

$$\Lambda_n(b)=O(b^{n-1}),\quad \lim_{n\to\infty}\Lambda_n(b)=0,\tag{2.78}$$

$$\Theta_n=\Lambda_1(1)-\big(1-\alpha/2\big)\Lambda_1(1)\frac{e^{\alpha\gamma+c_\alpha}}{n^{1-\alpha}}+O\Big(\frac{1}{n^{2-\alpha}}\Big),\quad \lim_{n\to\infty}\Theta_n=\Lambda_1(1).\tag{2.79}$$

4. *The determinant of the matrix M_n^α introduced in* (2.62) *satisfies*

$$\det(M_n^\alpha)\;=\;\mu+\frac{\nu}{n^{1-\alpha}}+O\Big(\frac{1}{n^{2-\alpha}}\Big)\quad\text{and}\quad\lim_{n\to+\infty}\det(M_n)=\mu,\tag{2.80}$$

with

$$\mu\triangleq\Big(\Omega-\Lambda_1(1)+b^2\Lambda_1(b)\Big)\Big(b\Omega+b^{1-\alpha}\Lambda_1(1)-b\Lambda_1(b)\Big)$$

and

$$\nu\triangleq\big(1-\alpha/2\big)\Lambda_1(1)\Big(2b^{1-\alpha}\Lambda_1(1)+(b-b^{1-\alpha})\Omega-b(1+b^{2-\alpha})\Lambda_1(b)\Big)e^{\alpha\gamma+c_\alpha},$$

with γ denotes Euler constant, c_α is the sum of the series

$$c_\alpha\triangleq\sum_{m=1}^{\infty}\frac{\alpha^{2m+1}}{2^{2m-1}(2m+1)}\zeta(2m+1).$$

and $s\mapsto\zeta(s)$ is the Riemann zeta function.

Remark 2.1 *The assertion* (2) *from the last lemma shows that the spectrum is continuous with respect to* α*. In other words, we have*

$$M_n^0 \triangleq \lim_{\alpha\to 0} M_n^\alpha = \begin{pmatrix} \Omega - \frac{n-1}{2n} + \frac{b^2}{2} & -\frac{b^{n+1}}{2n} \\ \frac{b^{n+1}}{2n} & b\big(\Omega + \frac{n-1}{2n} - \frac{1}{2}\big) \end{pmatrix}$$

Hence, by the change of variable $\lambda = 1 - 2\Omega$ *we can see that* M_n^0 *is exactly the same matrix obtained in [64]. However, for* $\alpha = 1$ *the dispersion relation established in [51] involves the following matrix*

$$\widetilde{M_n} = \begin{pmatrix} \Omega - \frac{2}{\pi}\sum_{k=1}^{n-1}\frac{1}{2k+1} + b^2\Lambda_1(b) & -b^3\Lambda_n(b) \\ \frac{1}{b}\Lambda_n(b) & \Omega + \frac{2}{b\pi}\sum_{k=1}^{n-1}\frac{1}{2k+1} - \Lambda_1(b) \end{pmatrix}.$$

This discrepancy with the matrix M_n^1 *is due to the parametrization used in [51] for the interior curve. Indeed, in that paper the perturbation of the interior curve is dilated by* b*. Thus with our parametrization we should multiply the second column of* $\widetilde{M_n}$ *by* b *and the matrix* $\widetilde{M_n}$ *becomes*

$$\begin{pmatrix} \Omega - \frac{2}{\pi}\sum_{k=1}^{n-1}\frac{1}{2k+1} + b^2\Lambda_1(b) & -b^4\Lambda_n(b) \\ \frac{1}{b}\Lambda_n(b) & b\Omega + \frac{2}{\pi}\sum_{k=1}^{n-1}\frac{1}{2k+1} - b\Lambda_1(b) \end{pmatrix}.$$

As we can easily see, this matrix has exactly the same determinant of the matrix M_n^1 *and therefore we find the same dispersion relations.*

Let us now prove Lemma 3.3. *Proof* : **(1)** To study the sign of Λ_n we shall make use of Sonine-Schafheitlin's formula (2.26) leading to the identity

$$\Lambda_n(b) = \frac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})}\frac{(\frac{\alpha}{2})_n}{n!}b^{n-1}F\Big(\frac{\alpha}{2}, n+\frac{\alpha}{2}, n+1, b^2\Big), \tag{2.81}$$

which is obviously positive for all $\alpha, b \in (0,1)$.
Let us now prove that the mapping $n \mapsto \Lambda_n(b)$ is decreasing. For this end, we rewrite the hypergeometric series F appearing in right-hand side of (2.81) according to the identity (2.11), which yields

$$F\Big(\frac{\alpha}{2}, n+\frac{\alpha}{2}, n+1, b^2\Big) = \frac{\Gamma(n+1)}{\Gamma(n+\frac{\alpha}{2})\Gamma(1-\frac{\alpha}{2})}\int_0^1 x^{n-1+\frac{\alpha}{2}}(1-x)^{-\frac{\alpha}{2}}(1-b^2x)^{-\frac{\alpha}{2}}\,\mathrm{d}x.$$

From the relation (2.13) we get

$$\Lambda_n(b) = \frac{b^{n-1}}{2^{1-\alpha}\Gamma^2(1-\frac{\alpha}{2})}\int_0^1 x^{n-1+\frac{\alpha}{2}}(1-x)^{-\frac{\alpha}{2}}(1-b^2x)^{-\frac{\alpha}{2}}\,\mathrm{d}x, \quad b\in(0,1). \tag{2.82}$$

Therefore it is easily seen that $b \in (0,1) \mapsto \Lambda_n(b)$ is increasing and $n \in \mathbb{N}^\star \mapsto \Lambda_n(b)$ is decreasing. This implies in turn that $n \in \mathbb{N}^\star \mapsto \Theta_n = \Lambda_1(1) - \Lambda_n(1)$ is increasing and thus it should be positive. Notice that these properties can be also proven from the series expansion (2.81).

(2) Passing to the limit in the formula defining $\Lambda_n(b)$ when α goes to one yields

$$\lim_{\alpha\to 1}\Lambda_n(b)=\frac{1}{b}\int_0^{+\infty}J_n(bt)J_n(t)dt.$$

As to the second limit, we have

$$\lim_{\alpha\to 0}\Lambda_n(b)=\frac{1}{b}\int_0^{+\infty}\frac{J_n(bt)J_n(t)}{t}dt.$$

Since $b\in(0,1)$ we can use the following identity,

$$\int_0^{+\infty}\frac{J_n(bt)J_n(t)}{t}dt=\frac{b^n}{2n},$$

whose proof can be found for example in [123, p. 405]. Consequently,

$$\lim_{\alpha\to 0}\Lambda_n(b)=\frac{b^{n-1}}{2n}.$$

Now to compute the limits of Θ_n when α goes to the values 0 and 1 we shall rewrite Θ_n by using the identity(2.81) in the form

$$\begin{aligned}\Theta_n &\triangleq \Lambda_1(1)-\Lambda_n(1)\\ &= \frac{(\frac{\alpha}{2})\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})}\Big[F\Big(\frac{\alpha}{2},1+\frac{\alpha}{2},2,1\Big)-\frac{(\frac{\alpha}{2}+1)_{n-1}}{n!}F\Big(\frac{\alpha}{2},n+\frac{\alpha}{2},n+1,1\Big)\Big].\end{aligned}$$

This gives in view of the formula (2.17) and (2.13),

$$\Theta_n=\frac{\Gamma(1-\alpha)}{2^{1-\alpha}\Gamma^2(1-\frac{\alpha}{2})}\bigg(\frac{\Gamma(n+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})}-\frac{\Gamma(n+\frac{\alpha}{2})}{\Gamma(n+1-\frac{\alpha}{2})}\bigg).\tag{2.83}$$

This expression coincides with the "eigenvalues" in the simply connected case, see [54]. It follows that,

$$\begin{aligned}\lim_{\alpha\to 0}\Theta_n &= \frac{1}{2}\Big(\frac{\Gamma(1)}{\Gamma(2)}-\frac{\Gamma(n)}{\Gamma(n+1)}\Big)\\ &= \frac{1}{2}\Big(1-\frac{(n-1)!}{n!}\Big)\\ &= \frac{n-1}{2n}.\end{aligned}$$

We note that these values coincide with the "eigenvalues" for Euler equations in the simply connected case. To compute the second limit, we shall introduce for a fixed n the function

$$\phi_n(\alpha)=\frac{\Gamma(n+\alpha/2)}{\Gamma(n+1-\alpha/2)}.$$

Therefore we obtain according to (2.12), (2.14) and the relation $\phi_n(1)=1$,

$$\begin{aligned}\lim_{\alpha\to 1}\Theta_n &= \frac{-1}{\Gamma^2(1/2)}\lim_{\alpha\to 1}\big\{(1-\alpha)\Gamma(1-\alpha)\big\}\lim_{\alpha\to 1}\Big\{\frac{\phi_1(\alpha)-\phi_1(1)}{\alpha-1}-\frac{\phi_n(\alpha)-\phi_n(1)}{\alpha-1}\Big\}\\ &= \frac{-1}{\pi}\Big\{\phi_1'(1)-\phi_n'(1)\Big\}.\end{aligned}$$

By applying the logarithm function to ϕ_n and differentiating with respect to α one obtains the relation

$$2\frac{\phi_n'(\alpha)}{\phi_n(\alpha)} = F(n+\alpha/2) + F(n+1-\alpha/2).$$

Now using the fact that $\phi_n(1)=1$ combined with the preceding identity and (2.16), we find

$$\begin{aligned}\lim_{\alpha\to 1}\Theta_n &= \frac{-1}{\pi}\Big\{F(3/2) - F(n+1/2)\Big\}\\ &= \frac{2}{\pi}\sum_{k=1}^{n-1}\frac{1}{2k+1},\end{aligned}$$

which is the desired result.

(3)-(4) The asymptotic behavior of Λ_n may be easily obtained from the integral formula (2.82). The proof of (2.79) was done in details in [54]. Finally, by combining (2.79), (2.78) and the expression of det(M_n^α) given by (2.63) one can deduce the identity (2.80). □

2.5.3 Monotonicity of the eigenvalues

In this section we shall discuss some important properties concerning the monotonicity of the eigenvalues associated to the matrix M_n^α already seen in Lemma 2.2. This will be crucial in the study of the kernel of the linearized operator $DG(\Omega,0,0)$. Recall that

$$M_n^\alpha = \begin{pmatrix}\Omega - \Theta_n + b^2\Lambda_1(b) & -b^2\Lambda_n(b)\\ b\Lambda_n(b) & b\Omega + b^{1-\alpha}\Theta_n - b\Lambda_1(b)\end{pmatrix}.$$

The determinant of this matrix given by (2.63) is a second order polynomial on the variable Ω and therefore it has two roots depending on all the parameters n, b and α. For our deal it is important to formulate sufficient conditions to avoid the eigenvalues crossing in order to guarantee a one-dimensional kernel which is an essential assumption in Crandall-Rabionwitz's Theorem. In what follows we shall use the variable $\lambda \triangleq 1-2\Omega$ instead of Ω in the spirit of the work of [64]. Thus easy computations show that the determinant (2.63) takes the form,

$$\det\big(M_n^\alpha\big) = b\Big(\lambda^2 - 2C_n\lambda + D_n\Big), \tag{2.84}$$

with

$$C_n \triangleq 1 + \big(b^{-\alpha}-1\big)\Theta_n - \big(1-b^2\big)\Lambda_1(b),$$

and

$$\begin{aligned}D_n \triangleq -4b^{-\alpha}\Theta_n^2 + 2\Big[b^{-\alpha} - 1 + 2\big(1+b^{2-\alpha}\big)\Lambda_1(b)\Big]\Theta_n - 4b^2\Big(\Lambda_1^2(b) - \Lambda_n^2(b)\Big)\\ -2\big(1-b^2\big)\Lambda_1(b) + 1.\end{aligned} \tag{2.85}$$

Note that the quantities $\Lambda_n(b)$ and Θ_n have been introduced in Lemma 2.2. It is easy to check through straightforward computations that the reduced discriminant of the second order polynomial appearing in (2.84) is given by

$$\Delta_n = \Big((b^{-\alpha}+1)\Theta_n - (1+b^2)\Lambda_1(b)\Big)^2 - 4b^2\Lambda_n^2(b). \tag{2.86}$$

Our result reads as follows.

Proposition 2.2 *There exists $N \geq 2$ such that the following holds true.*

1. *For all $n \geq N$ we get $\Delta_n > 0$ and the equation* (2.84) *admits two different real solutions given by*
$$\lambda_n^{\pm} \triangleq C_n \pm \sqrt{\Delta_n}.$$
2. *The sequences $(\Delta_n)_{n\geq N}$ and $(\lambda_n^+)_{n\geq N}$ are strictly increasing and $(\lambda_n^-)_{n\geq N}$ is strictly decreasing.*
3. *For all $m > n > N$ we have*
$$\lambda_m^- < \lambda_n^- < \lambda_n^+ < \lambda_m^+.$$

Remark 2.2 1. *The number N in (2) in the previous proposition is the smallest integer satisfying,*
$$\Theta_N > \frac{1+b^2}{b^{-\alpha}+1}\Lambda_1(b) + \frac{2b}{b^{-\alpha}+1}\Lambda_N(b). \tag{2.87}$$
2. *In the known cases like the simply connected case with $\alpha \in [0,2[$ or the doubly connected case with $\alpha = 0$ the analysis is more easier because the dispersion relation is a kind of fractional polynomial but in the present case it is highly nonlinear with respect to the frequencies and the parameters α and b. Therefore the program is achieved with only a sufficient condition on the existence of the eigenvalue and which is given by* (2.87)*. This condition coincides with that given in [64] for $\alpha = 0$.*

Proof : **(1)** We intend to discuss the conditions leading to the positivity of the discriminant defined in (2.86) which ensures in turn that the polynomial (2.84) has two real solutions. We can see that $(\Delta_n)_{n\geq 2}$ can be extended to a smooth function defined on $[1, +\infty[$ as follows
$$\Delta_x = \Big((b^{-\alpha}+1)\Theta_x - (1+b^2)\Lambda_1(b)\Big)^2 - 4b^2\Lambda_x^2(b).$$
It is strictly positive if and only if
$$E_b(x) \triangleq (b^{-\alpha}+1)\Theta_x - (1+b^2)\Lambda_1(b) - 2b\Lambda_x(b) > 0, \tag{2.88}$$
or
$$F_b(x) \triangleq (b^{-\alpha}+1)\Theta_x - (1+b^2)\Lambda_1(b) + 2b\Lambda_x(b) < 0.$$
Remark first that for x and b verifying the inequality (2.88) we have $F_b(x) > 0$. On other hand, from Lemma 3.3 one has : for all $b \in (0,1)$ the mapping $x \mapsto E_b(x)$ is strictly increasing, continuous and verifying
$$\begin{aligned}\lim_{x\to+\infty} E_b(x) &= (b^{-\alpha}+1)\Lambda_1(1) - (1+b^2)\Lambda_1(b)\\ &\geq \big(b^{-\alpha} - b^2)\Lambda_1(1) > 0,\end{aligned}$$
$$E_b(1) = -(1+b)^2\Lambda_1(b) < 0.$$
Consequently the set
$$\mathcal{I}^\alpha(b) \triangleq \big\{x > 1;\ E_b(x) > 0\big\}.$$
is connected and of the form $]\beta, +\infty[$, with
$$E_b(\beta) = 0.$$

Hence the integer N is chosen as

$$N = [\beta].$$

(2) To prove that $x \mapsto \Delta_x$ is decreasing we shall compute its derivative with respect to x. The plain computations give,

$$\partial_x \Delta_x = 2(b^{-\alpha}+1)\Big[(b^{-\alpha}+1)\Theta_x - \big(1+b^2\big)\Lambda_1(b)\Big]\partial_x\Theta_x - 8b^2\Lambda_x(b)\partial_x\Lambda_x(b).$$

Since $x \mapsto \Lambda_x(b)$ is decreasing, $x \mapsto \Theta_x$ is increasing and $\Lambda_x(b) \geq 0$ (see (1) from Lemma 3.3) then we deduce that

$$\begin{aligned}\partial_x \Delta_x &> 2(b^{-\alpha}+1)\Big[(b^{-\alpha}+1)\Theta_x - \big(1+b^2\big)\Lambda_1(b)\Big]\partial_x\Theta_x\\ &> 2(b^{-\alpha}+1)E_x(b)\partial_x\Theta_x.\end{aligned}$$

Hence, for all $x \in \mathcal{I}^\alpha(b)$ we have

$$\partial_x \Delta_x > 0.$$

This shows that $x \mapsto \Delta_x$ is strictly increasing.
Now, recall that

$$\lambda_x^+ \triangleq 1 + (b^{-\alpha}-1)\Theta_x - (1-b^2)\Lambda_1(b) + \sqrt{\Delta_x}.$$

Using the fact that $x \mapsto \Theta_x$ is increasing (according to Lemma 3.3) combined with the increasing property of $x \mapsto \Delta_x$ we get the desired result.

So it remains to establish that the mapping $x \mapsto \lambda_x^-$ is strictly decreasing. For this aim we calculate its derivative with respect to x,

$$\begin{aligned}\partial_x\lambda_x^- =&(b^{-\alpha}-1)\partial_x\Theta_x - \frac{\partial_x\Delta_x}{2\sqrt{\Delta_x}}\\ =&(b^{-\alpha}-1)\partial_x\Theta_x - (b^{-\alpha}+1)\Big[\frac{(b^{-\alpha}+1)\Theta_x - \big(1+b^2\big)\Lambda_1(b)}{\sqrt{\Delta_x}}\Big]\partial_x\Theta_x + 4b^2\partial_x\Lambda_x(b)\frac{\Lambda_x(b)}{\sqrt{\Delta_x}}\\ <&b^{-\alpha}\Big[1 - \frac{(b^{-\alpha}+1)\Theta_x - \big(1+b^2\big)\Lambda_1(b)}{\sqrt{\Delta_x}}\Big]\partial_x\Theta_x\\ &-\Big[1 + \frac{(b^{-\alpha}+1)\Theta_x - \big(1+b^2\big)\Lambda_1(b)}{\sqrt{\Delta_x}}\Big]\partial_x\Theta_x.\end{aligned}$$

where we have used in the last inequality the decreasing property of the mapping $x \mapsto \Lambda_x(b)$ and the fact that $\Lambda_x(b) \geq 0$. Now from the inequality

$$(b^{-\alpha}+1)\Theta_x - \big(1+b^2\big)\Lambda_1(b) > E_x(b) > 0,$$

and the expression of Δ_x we deduce that

$$\partial_x\lambda_x^- < 0.$$

This gives the desired result.

(3) This follows easily from (2) and the obvious fact

$$\lambda_n^- \leq \lambda_n^+.$$

□

2.6 Bifurcation at simple eigenvalues

In this section we shall prove Theorem 2.1 which is deeply related to the spectral study developed in the preceding section combined with Crandall-Rabinowitz's Theorem. To construct the function spaces where the bifurcation occurs we shall take into account the restriction to the high frequencies stated in Proposition 2.2 and include the m-fold symmetry of the V-states. To proceed, fix $b \in (0,1)$ and $m \geq N$, where N is defined in Proposition 2.2 and Remark 2.2. Set,

$$X_m = C_m^{2-\alpha}(\mathbb{T}) \times C_m^{2-\alpha}(\mathbb{T}),$$

where $C_m^{2-\alpha}(\mathbb{T})$ is the space of the $2\pi-$periodic functions $f \in C^{2-\alpha}(\mathbb{T})$ whose Fourier series is given by

$$f(w) = \sum_{n=1}^{\infty} a_n \overline{w}^{nm-1}, \quad w \in \mathbb{T}, \quad a_n \in \mathbb{R}.$$

This space is equipped with its usual norm. We define the ball of radius $r \in (0,1)$ by

$$B_r^m = \Big\{ f \in X_m,\, \|f\|_{C^{2-\alpha}(\mathbb{T})} \leq r \Big\}$$

and we introduce the neighborhood of zero,

$$V_{m,r} \triangleq B_r^m \times B_r^m.$$

The set $V_{m,r}$ is endowed with the induced topology of the product spaces.
Take $(f_1, f_2) \in V_{m,r}$ then the expansions of the associated conformal mappings ϕ_1, ϕ_2 outside the unit disc $\big\{ z \in \mathbb{C};\, |z| \geq 1 \big\}$ are given successively by

$$\phi_1(z) = z + f_1(z) = z\Big(1 + \sum_{n=1}^{\infty} \frac{a_n}{z^{nm}}\Big)$$

and

$$\phi_2(z) = bz + f_2(z) = z\Big(b + \sum_{n=1}^{\infty} \frac{b_n}{z^{nm}}\Big).$$

This structure provides the $m-$fold symmetry of the associated boundaries $\phi_1(\mathbb{T})$ and $\phi_2(\mathbb{T})$, via the relation

$$\phi_j\big(e^{2i\pi/m} z\big) = e^{2i\pi/m} \phi_j(z), \quad j = 1, 2 \quad \text{and} \quad |z| \geq 1. \tag{2.89}$$

For functions f_1 and f_2 with small size the boundaries can be seen as a small perturbation of the boundaries of the annulus $\big\{ z \in \mathbb{C};\, b \leq |z| \leq 1 \big\}$. Set

$$H_m = \left\{ g \in C^{1-\alpha}(\mathbb{T}),\, g(w) = i \sum_{n \geq 1} A_n \big(w^{mn} - \overline{w}^{mn} \big),\, A_n \in \mathbb{R},\, n \in \mathbb{N}^* \right\}$$

and define the product space Y_m by

$$Y_m = H_m \times H_m.$$

From Proposition 2.2 recall the definition of the eigenvalues $\lambda_m^{\pm}$ and the associated angular velocities are

$$\begin{aligned} \Omega_m^{\pm} &= \frac{1}{2} - \frac{1}{2}\lambda_m^{\pm} \\ &= \frac{1}{2}\widehat{C}_m \mp \frac{1}{2}\sqrt{\Delta_m} \end{aligned}$$

with
$$\Delta_m \triangleq \Big((b^{-\alpha}+1)\Theta_m-(1+b^2)\Lambda_1(b)\Big)^2-4b^2\Lambda_m^2(b)$$
and
$$\widehat{C}_m \triangleq (1-b^{-\alpha})\Theta_m+(1-b^2)\Lambda_1(b).$$
Note that Θ_m and $\Lambda_m(b)$ were introduced in Lemma 2.2. The V-states equations are described in (2.7) and (2.8) which we restate here : for $j\in\{1,2\}$,
$$F_j\big(\Omega,\phi_1,\phi_2\big)(w)\triangleq G_j\big(\Omega,f_1,f_2\big)(w)=0,\quad \forall\, w\in\mathbb{T};\quad \text{and}\quad G\triangleq(G_1,G_2).$$
with
$$\begin{aligned}F_j\big(\Omega,\phi_1,\phi_2\big)(w) &\triangleq \Omega\,\mathrm{Im}\Big\{\phi_j(w)\overline{w}\,\overline{\phi_j'(w)}\Big\}\\ &+ C_\alpha\,\mathrm{Im}\bigg\{\bigg(\fint_{\mathbb{T}}\frac{\phi_2'(\tau)d\tau}{|\phi_j(w)-\phi_2(\tau)|^\alpha}-\fint_{\mathbb{T}}\frac{\phi_1'(\tau)d\tau}{|\phi_j(w)-\phi_1(\tau)|^\alpha}\bigg)\overline{w}\,\overline{\phi_j'(w)}\bigg\}.\end{aligned}$$

Now, to apply Crandall-Rabinowitz's Theorem it suffices to show the following result.

Proposition 2.3 *Let N be as in the part (1) of Proposition 2.2 and $m\geq N$, and take $\Omega\in\big\{\Omega_m^\pm\big\}$. Then, the following assertions hold true.*

1. *There exists $r>0$ such that $G:\mathbb{R}\times V_{m,r}\to Y_m$ is well-defined and of class C^1.*
2. *The kernel of $DG(\Omega,0,0)$ is one-dimensional and generated by*
$$v_{0,m}:\,w\in\mathbb{T}\mapsto\begin{pmatrix}\Omega+b^{-\alpha}\Theta_m-\Lambda_1(b)\\ -\Lambda_m(b)\end{pmatrix}\overline{w}^{m-1}.$$
3. *The range of $DG(\Omega,0,0)$ is closed and is of co-dimension one in Y_m.*
4. *Transversality assumption : If Ω is a simple eigenvalue $(\Delta_m>0)$ then*
$$\partial_\Omega DG(\Omega_m^\pm,0,0)v_{0,m}\notin R\big(DG(\Omega_m^\pm,0,0)\big)\cdot$$

Proof :

(1) Compared to Proposition 2.1 we need just to check that $G=(G_1,G_2)$ preserves the $m-$fold symmetry and maps X_m into Y_m. For this end, it is sufficient to check that for given $(f_1,f_2)\in X_m$, the coefficients of the Fourier series of $F_j(\Omega,\phi_1,\phi_2)$ vanish at frequencies which are not integer multiple of m. This amounts to proving that,
$$F_j(\Omega,\phi_1,\phi_2)\Big(e^{i\frac{2\pi}{m}}w\Big)=F_j(\Omega,\phi_1,\phi_2)(w),\quad w\in\mathbb{T},\quad j=1,2\cdot$$
This property is obvious for the first term $\mathrm{Im}\Big\{\overline{w}\overline{\phi_j'(w)}\phi_j(w)\Big\}$. For the two last terms in the expression of F_j it is enough to check the identity,
$$\forall w\in\mathbb{T},\quad \Phi_j\big(e^{i\frac{2\pi}{m}}w\big)=e^{i\frac{2\pi}{m}}\Phi_j(w),$$
with
$$\Phi_j(w)\ \triangleq\ \fint_{\mathbb{T}}\frac{\phi_2'(\tau)}{|\phi_j(w)-\phi_2(\tau)|^\alpha}d\tau$$

This follows easily by making the change of variables $\tau = e^{i2\pi/m}\zeta$ and from (2.89). Indeed,

$$\begin{aligned}\Phi_j\big(e^{i\frac{2\pi}{m}}w\big) &= e^{i2\pi/m}\fint_{\mathbb{T}}\frac{\phi_2'(e^{i2\pi/m}\zeta)}{|\phi_j(e^{i\frac{2\pi}{m}}w)-\phi_2(e^{i2\pi/m}\zeta)|^\alpha}d\zeta\\ &= e^{i\frac{2\pi}{m}}\fint_{\mathbb{T}}\frac{\phi_2'(\zeta)}{|\phi_j(w)-\phi_2(\zeta)|^\alpha}d\zeta\\ &= e^{i\frac{2\pi}{m}}\Phi_j(w).\end{aligned}$$

This concludes the proof of the following statement,

$$(f_1,f_2)\in V_{m,r}\Longrightarrow G(\Omega,f_1,f_2)\in Y_m.$$

(2) We shall describe the kernel of the linear operator $DF(\Omega_m^\pm,0,0)$ and show that it is one-dimensional. Let h_1,h_2 be two functions in $C_m^{2-\alpha}\big(\mathbb{T}\big)$ such that

$$h_1(w)=\sum_{n=1}^\infty a_n\overline{w}^{nm-1}\quad\text{and}\quad h_2(w)=\sum_{n=1}^\infty c_n\overline{w}^{nm-1},\quad w\in\mathbb{T},\tag{2.90}$$

Recall from Lemma 2.2 the following expression,

$$DG(\Omega,0,0)\big(h_1,h_2\big)(w)=\frac{i}{2}\sum_{n\geq1}nmM_{nm}^\alpha\begin{pmatrix}a_n\\c_n\end{pmatrix}\Big(w^{nm}-\overline{w}^{nm}\Big).\tag{2.91}$$

where the matrix M_n is given for $n\geq2$ by

$$M_n^\alpha\triangleq\begin{pmatrix}\Omega-\Theta_n+b^2\Lambda_1(b) & -b^2\Lambda_n(b)\\ b\Lambda_n(b) & b\Omega+b^{1-\alpha}\Theta_n-b\Lambda_1(b)\end{pmatrix}.\tag{2.92}$$

Now, if $\Omega\in\big\{\Omega_m^\pm\big\}$ then

$$\det\big(M_m^\alpha\big)=0.$$

Thus, the kernel of $DG(\Omega,0,0)$ is non trivial and it is one-dimensional if and only if

$$\det\big(M_{nm}^\alpha\big)\neq0,\quad\forall\,n\geq2.\tag{2.93}$$

This condition is ensured by the part (1) of the Proposition 2.2. Then, (h_1,h_2) is in the kernel of $DG\big(\Omega,0,0\big)$ if and only the Fourier coefficients in the identity (2.91) vanish, namely,

$$a_n=c_n=0\quad\text{for all}\quad n\geq2\quad\text{and}\quad(a_1,c_1)\in\text{Ker}M_m^\alpha.$$

Hence, a generator of $\text{Ker}\Big(DG(\Omega,0,0)\Big)$ can be chosen as the pair of functions

$$w\in\mathbb{T}\mapsto\begin{pmatrix}\Omega+b^{-\alpha}\Theta_m-\Lambda_1(b)\\-\Lambda_m(b)\end{pmatrix}\overline{w}^{m-1},\quad w\in\mathbb{T}.\tag{2.94}$$

(3) We are going to show that for any $m\geq N$ the range $R\big(DG(\Omega,0,0)\big)$ coincides with the space of the functions $(g_1,g_2)\in C^{1-\alpha}(\mathbb{T})\times C^{1-\alpha}(\mathbb{T})$ such that

$$g_1(w)=\sum_{n\geq1}iA_n\big(w^{mn}-\overline{w}^{mn}\big),\quad g_1(w)=\sum_{n\geq1}iC_n\big(w^{mn}-\overline{w}^{mn}\big)\tag{2.95}$$

where $A_n, C_n \in \mathbb{R}$ for all $n \in \mathbb{N}^*$ and there exists $(a_1, c_1) \in \mathbb{R}^2$ such that

$$M_m \begin{pmatrix} a_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} A_1 \\ C_1 \end{pmatrix}. \tag{2.96}$$

For the sake of simple notation we remove in this part the parameter α from M_n^α. The range of operator $DG(\Omega, 0, 0)$ is obviously included in the space defined above which is closed and of co-dimension 1 in Y_m. Therefore it remains to check just the converse. Let g_1 and g_2 be two functions in $C^{1-\alpha}(\mathbb{T})$ with Fourier series expansions as in (2.95) and (2.96). We shall prove that the equation

$$\frac{1}{m} DG(\Omega, 0, 0)(h_1, h_2) = (g_1, g_2)$$

admits a solution (h_1, h_2) in the space X_m, where the Fourier series expansions of these functions are given in (2.90). Then according to (2.91), the preceding equation is equivalent to

$$n\, M_{mn} \begin{pmatrix} a_n \\ c_n \end{pmatrix} = \begin{pmatrix} A_n \\ C_n \end{pmatrix}, \quad \forall n \in \mathbb{N}^\star.$$

For $n = 1$, the existence follows from the condition (2.96) and therefore we shall only focus on $n \geq 2$. Owing to (2.93) the sequences $(a_n)_{n \geq 2}$ and $(c_n)_{n \geq 2}$ are uniquely determined by the formulae

$$\begin{pmatrix} a_n \\ c_n \end{pmatrix} = \frac{1}{n} M_{mn}^{-1} \begin{pmatrix} A_n \\ C_n \end{pmatrix}, \quad n \geq 2. \tag{2.97}$$

By computing the matrix M_{mn}^{-1} we deduce that for all $n \geq 2$,

$$a_n = \frac{b\big(\Omega + b^{-\alpha}\Theta_{nm} - \Lambda_1(b)\big)}{n \det\big(M_{nm}\big)} A_n + \frac{b^2 \Lambda_{nm}(b)}{n \det\big(M_{nm}\big)} C_n$$

and

$$c_n = -\frac{b\, \Lambda_{nm}(b)}{n \det\big(M_{nm}\big)} A_n + \frac{\big(\Omega - b\Theta_{nm} + b^2 \Lambda_1(b)\big)}{n \det\big(M_{nm}\big)} C_n.$$

Therefore the proof of $(h_1, h_2) \in X_m$ amounts to showing that

$$w \mapsto \begin{pmatrix} h_1(w) - a_1 \overline{w}^{m-1} \\ h_2(w) - c_1 \overline{w}^{m-1} \end{pmatrix} \in C^{2-\alpha}(\mathbb{T}) \times C^{2-\alpha}(\mathbb{T}).$$

We shall develop the computations only for the first component and the second one can be done in a similar way. We set $\widetilde{h}_1(w) = h_1(w) - a_1 \overline{w}^{m-1}$ and

$$H(w) \triangleq \sum_{n \geq 2} \frac{A_n}{n \det\big(M_{nm}\big)} w^n, \quad H_1(w) \triangleq \sum_{n \geq 2} \frac{C_n}{n} w^n.$$

Then in view of (2.79) the function $\widetilde{h}_1(w)$ can be rewritten as follows

$$\begin{aligned} \widetilde{h}_1(w) &= C_1\, w H(\overline{w}^m) + C_2\, w\big(H * K_1\big)(\overline{w}^m) \\ &+ w\,\big(H * K_2\big)(\overline{w}^m) + b^2 w\big(H_1 * K_3\big)(\overline{w}^m), \end{aligned}$$

with C_1 and C_2 two constants. The kernels are defined by

$$K_1(w) \triangleq \sum_{n \geq 2} \frac{w^n}{n^{1-\alpha}}, \quad K_2(w) \triangleq \sum_{n \geq 2} O\Big(\frac{1}{n^{2-\alpha}}\Big) w^n,$$

and

$$K_3(w) \triangleq \sum_{n\geq 2} \frac{\Lambda_{nm}(b)}{\det\big(M_{nm}\big)} w^n.$$

The convolution is understood in the usual one : for two continuous functions $f, g; \mathbb{T} \to \mathbb{C}$ we define

$$\forall w \in \mathbb{T}, \quad f * g(w) = \fint_{\mathbb{T}} f(\tau) g(\tau \overline{w}) \frac{d\tau}{\tau}\cdot$$

Assume for a while that $w \mapsto H(w)$ belongs to $C^{2-\alpha}(\mathbb{T})$. Then by virtue of the classical convolution law $L^1(\mathbb{T}) * C^{2-\alpha}(\mathbb{T}) \to C^{2-\alpha}(\mathbb{T})$, it suffices to show that the kernels K_1, K_2 and K_3 belong to $L^1(\mathbb{T})$. The second and the third kernels are easy to analyze because the series converge absolutely,

$$\|K_2\|_{L^\infty(\mathbb{T})} \lesssim \sum_{n\geq 1} \frac{1}{n^{2-\alpha}} \leq C\cdot$$

Similarly, owing to (2.78) one has

$$\|K_3\|_{L^\infty(\mathbb{T})} \lesssim \sum_{n\geq 0} b^n \leq C\cdot$$

and therefore $K_2, K_3 \in L^1(\mathbb{T})$. Note that to bound the series we have used the fact that the sequence $\big(\det M_{nm}\big)_{n\geq 2}$ does not vanish and converges to a strictly positive number D_∞ defined in (**??**). It remains to show that $K_1 \in L^1(\mathbb{T})$. For this end we shall use the following estimate : for any $\beta \in \big(\alpha, 1\big)$

$$|K_1(e^{i\theta})| \lesssim \frac{1}{\sin^\beta(\frac{\theta}{2})}, \quad \forall \theta \in (0, 2\pi)\cdot \tag{2.98}$$

which is true for all $\alpha \in [0, 1[$ and for a proof we can see [54].

Now to complete the reasoning it remains to prove the preceding claim asserting that the function H belongs to the space $C^{2-\alpha}(\mathbb{T})$. To prove this we write in view of (2.80),

$$\begin{aligned}
\det(M_{nm}) &= \mu + \frac{\nu}{n^{1-\alpha}} + O\Big(\frac{1}{n^{2-\alpha}}\Big) \\
&= \mu - \rho_n,
\end{aligned}$$

where μ and ν are two constants (depending on m) and

$$\rho_n \triangleq -\frac{\nu}{n^{1-\alpha}} + O\Big(\frac{1}{n^{2-\alpha}}\Big)\cdot \tag{2.99}$$

This allows to get

$$H(w) = \sum_{n\geq 2} \frac{A_n}{n\big(\mu - \rho_n\big)} w^n.$$

Then one may use the general decomposition : for $k \in \mathbb{N}$,

$$\frac{1}{\mu - \rho_n} = \frac{\mu^{-k-1}\rho_n^{k+1}}{\mu - \rho_n} + \sum_{j=0}^{k} \mu^{-j-1}\rho_n^j$$

which yields,

$$\begin{aligned}
H(w) &= \mu^{-k-1} \sum_{n\geq 2} \frac{A_n \rho_n^{k+1}}{n\big(\mu - \rho_n\big)} w^n + \sum_{j=0}^{k} \mu^{-j-1} \sum_{n\geq 2} \frac{A_n \rho_n^j}{n} w^n \\
&\triangleq \mu^{-k-1} H_{k+1}(w) + \sum_{j=0}^{k} \mu^{-j-1} L_j(w)\cdot
\end{aligned}$$

Since the sequence $(A_n)_{n\geq 2}$ is bounded then by (2.99) we obtain

$$\left|\frac{A_n\rho_n^{k+1}}{n\big(\mu-\rho_n\big)}\right|\lesssim\frac{\big|\rho_n\big|^{k+1}}{n}\lesssim\frac{1}{n^{1+(1-\alpha)(k+1)}}.$$

Thus for k large enough we get $H_{k+1}\in C^{2-\alpha}(\mathbb{T})$. Concerning the estimate of L_j we shall restrict the analysis to $j=0$ and $j=1$ and the higher terms can be treated in a similar way. We write

$$L_0(w)=\sum_{n\geq 2}\frac{A_n}{n}w^n.$$

Using the Cauchy-Schwarz inequality we get

$$\begin{aligned}\|L_0\|_{L^\infty} &\leq \sum_{n\geq 2}\frac{|A_n|}{n}\\ &\leq \Big(\sum_{n\geq 1}\frac{1}{n^2}\Big)^{1/2}\Big(\sum_{n\geq 2}|A_n|^2\Big)^{1/2}\\ &\lesssim \|g_1\|_{L^2}.\end{aligned}$$

By the embedding $C^{1-\alpha}\big(\mathbb{T}\big)\hookrightarrow L^\infty\big(\mathbb{T}\big)\hookrightarrow L^2\big(\mathbb{T}\big)$ we conclude that

$$\|L_0\|_{L^\infty}\lesssim\|g_1\|_{1-\alpha}.$$

It remains to prove that $L_0'\in C^{1-\alpha}(\mathbb{T})$. For this end, one needs first to check that we can differentiate the series term by term. Fix $N\geq 1$ and define

$$L_0^N(w)\triangleq\sum_{n=2}^{N}\frac{A_n}{n}w^n.$$

From Cauchy-Schwarz inequality we find

$$\begin{aligned}\|L_0^N-L_0\|_{L^\infty(\mathbb{T})} &\lesssim \Big(\sum_{n\geq N+1}\frac{1}{n^2}\Big)^{1/2}\|g_1\|_{1-\alpha}\\ &\lesssim \frac{\|g_1\|_{1-\alpha}}{N^{1/2}}.\end{aligned}$$

Hence,

$$\lim_{N\to+\infty}\|L_0^N-L_0\|_{L^\infty(\mathbb{T})}=0. \tag{2.100}$$

Differentiating L_0^N term by term one gets

$$\begin{aligned}(L_0^N)'(w) &= \overline{w}\sum_{n=2}^{N}A_nw^n\\ &\triangleq \overline{w}\,g_1^N(w).\end{aligned}$$

Put

$$g_1^+(w)=\sum_{n\geq 2}A_nw^n,$$

then using the continuity of Szegö protection :

$$\Pi : \sum_{n\in\mathbb{Z}} a_n w^n \mapsto \sum_{n\in\mathbb{N}} a_n w^n$$

on Hölder spaces $C^{1-\alpha}(\mathbb{T})$ for $\alpha \in (0,1)$ we may conclude that g_1^+ belongs to $C^{1-\alpha}(\mathbb{T})$, (for more details see for example [54]). By virtue of a classical result on Fourier series one gets

$$\lim_{N\to+\infty} \|g_1^N - g_1^+\|_{L^\infty(\mathbb{T})} = 0$$

and consequently

$$\lim_{N\to+\infty} \|(L_0^N)' - \overline{w}\, g_1^+\|_{L^\infty(\mathbb{T})} = 0. \tag{2.101}$$

Putting together (2.100) and (2.101) we deduce that L_0 is differentiable and

$$L_0'(w) = \overline{w}\, g_1^+(w), \quad w \in \mathbb{T}.$$

This concludes that $L_0 \in C^{2-\alpha}$. Now, as before, we can easily get $L_1 \in L^\infty(\mathbb{T})$ and we shall check that $L_1' \in C^{1-\alpha}(\mathbb{T})$. Arguing in a similar way to L_0 we can differentiate term by term the series defining L_j leading to

$$L_1'(w) = \overline{w} \sum_{n\geq 2} A_n \rho_n w^n.$$

Note that with the same kernels K_1 and K_2 as before one can write

$$wL'{}_1(w) = -\nu\big(K_1 * g_1^+\big)(w) + \big(K_2 * g_1^+\big)(w).$$

Using the fact that g_1^+ belongs to $C^{1-\alpha}(\mathbb{T})$ and $K_1, K_2 \in L^1(\mathbb{T})$ we obtain the desired result.

(4) *The transversality condition.* Let $\Omega \in \{\Omega_m^\pm\}$ be a simple eigenvalue associated to the frequencies m and $v_{0,m}$ be the generator of the kernel $DG(\Omega,0,0)$ defined in the part (2) of Proposition 2.3. We shall prove that

$$\partial_\Omega DG(\Omega,0,0)v_{0,m} \notin R\big(DG(\Omega,0,0)\big),$$

with

$$v_{0,m}(w) = \begin{pmatrix} \Omega + b^{-\alpha}\Theta_m - \Lambda_1(b) \\ -\Lambda_m(b) \end{pmatrix} \overline{w}^{m-1}, \quad w \in \mathbb{T}.$$

Differentiating (2.58) and (2.59) with respect to Ω we get

$$\partial_\Omega DG_1(\Omega,0,0)(h_1,h_2)(w) = \operatorname{Im}\Big\{\overline{h_1'(w)} + \overline{w}\, h_1(w)\Big\}$$

and

$$\partial_\Omega DG_2(\Omega,0,0)(h_1,h_2)(w) = b\operatorname{Im}\Big\{\overline{h_2'(w)} + \overline{w}\, h_2(w)\Big\}.$$

Hence,

$$\begin{aligned}\partial_\Omega DG(\Omega,0,0)(v_{0,m})(w) &= \frac{m}{2} i \begin{pmatrix} \Omega + b^{-\alpha}\Theta_m - \Lambda_1(b) \\ -b\Lambda_m(b) \end{pmatrix} \big(w^m - \overline{w}^m\big) \\ &\triangleq \frac{m}{2} i\, e_m\, \big(w^m - \overline{w}^m\big).\end{aligned}$$

This pair of functions is in the range of $DG(\Omega,0,0)$ if and only if the vector $e_m\in\mathbb{R}^2$ is a scalar multiple of one column of the matrix M_m^α seen in (2.92), which happens if and only if

$$\Big(\Omega+b^{-\alpha}\Theta_m-\Lambda_1(b)\Big)^2-b^2\Lambda_m^2(b)=0. \tag{2.102}$$

Combining this equation with det $M_m^\alpha=0$ we get

$$\Big(\Omega-\Theta_m+b^2\Lambda_1(b)\Big)\Big(\Omega+b^{-\alpha}\Theta_m-\Lambda_1(b)\Big)+\Big(\Omega+b^{-\alpha}\Theta_m-\Lambda_1(b)\Big)^2=0.$$

This yields

$$\Big(\Omega+b^{-\alpha}\Theta_m-\Lambda_1(b)\Big)\Big[2\Omega+(b^2-1)\Lambda_1(b)+(b^{-\alpha}-1)\Theta_m\Big]=0$$

which is equivalent to

$$\Omega+b^{-\alpha}\Theta_m-\Lambda_1(b)=0 \quad\text{or}\quad \Omega=\frac{1}{2}\Big(\big(1-b^2\big)\Lambda_1(b)+\big(1-b^{-\alpha}\big)\Theta_m\Big).$$

This first possibility is excluded by (2.102) because $\Lambda_m(b)\neq 0$ and the second one is also impossible because it corresponds to a multiple eigenvalue which is not the case here. This concludes the proof of Proposition 2.3.

□

2.7 Numerical study of the doubly-connected V-states

For the sake of completeness, we will discuss in this section the numerical obtention of V-states for the quasi-geostrophic problem in both the doubly-connected case. Since the procedure is very similar to that in the vortex patch problem, we will omit some details, which can be consulted in [64].

Given a doubly-connected domain D with outer boundary $z_1(\theta)$ and inner boundary $z_2(\theta)$, where $\theta\in[0,2\pi)$ is the Lagrangian parameter, and z_1 and z_2 are counterclockwise parameterized, the contour dynamics equations for the quasi-geostrophic problem are

$$\begin{aligned} z_{1,t}(\theta,t)&=\frac{C_\alpha}{2\pi}\int_0^{2\pi}\frac{z_{1,\phi}(\phi,t)d\phi}{|z_1(\phi,t)-z_1(\theta,t)|^\alpha}-\frac{C_\alpha}{2\pi}\int_0^{2\pi}\frac{z_{2,\phi}(\phi,t)d\phi}{|z_2(\phi,t)-z_1(\theta,t)|^\alpha},\\ z_{2,t}(\theta,t)&=\frac{C_\alpha}{2\pi}\int_0^{2\pi}\frac{z_{1,\phi}(\phi,t)d\phi}{|z_1(\phi,t)-z_2(\theta,t)|^\alpha}-\frac{C_\alpha}{2\pi}\int_0^{2\pi}\frac{z_{2,\phi}(\phi,t)d\phi}{|z_2(\phi,t)-z_2(\theta,t)|^\alpha}; \end{aligned} \tag{2.103}$$

where, as in the simply-connected case, we will omit the explicit dependence on t.

The doubly-connected domain D is a V-state if and only if its boundaries satisfy the following equations :

$$\operatorname{Re}\left[\left(\Omega z_1(\theta)-\frac{C_\alpha}{2\pi i}\int_0^{2\pi}\frac{z_{1,\phi}(\phi)d\phi}{|z_1(\phi)-z_1(\theta)|^\alpha}+\frac{C_\alpha}{2\pi i}\int_0^{2\pi}\frac{z_{2,\phi}(\phi)d\phi}{|z_2(\phi)-z_1(\theta)|^\alpha}\right)\overline{z_{1,\theta}(\theta)}\right]=0, \tag{2.104}$$

$$\operatorname{Re}\left[\left(\Omega z_2(\theta)-\frac{C_\alpha}{2\pi i}\int_0^{2\pi}\frac{z_{1,\phi}(\phi)d\phi}{|z_1(\phi)-z_2(\theta)|^\alpha}+\frac{C_\alpha}{2\pi i}\int_0^{2\pi}\frac{z_{2,\phi}(\phi)d\phi}{|z_2(\phi)-z_2(\theta)|^\alpha}\right)\overline{z_{2,\theta}(\theta)}\right]=0. \tag{2.105}$$

However, as we did in (1.87), it is convenient to rewrite them in the following equivalent form :

$$\begin{aligned}\operatorname{Re}\Bigg[\bigg(\Omega z_1(\theta) - \frac{C_\alpha}{2\pi i}\int_0^{2\pi}\frac{(z_{1,\phi}(\phi)-z_{1,\theta}(\theta))d\phi}{|z_1(\phi)-z_1(\theta)|^\alpha} \\ + \frac{C_\alpha}{2\pi i}\int_0^{2\pi}\frac{z_{2,\phi}(\phi)d\phi}{|z_2(\phi)-z_1(\theta)|^\alpha}\bigg)\overline{z_{1,\theta}(\theta)}\Bigg] = 0,\end{aligned} \tag{2.106}$$

$$\begin{aligned}\operatorname{Re}\Bigg[\bigg(\Omega z_2(\theta) - \frac{C_\alpha}{2\pi i}\int_0^{2\pi}\frac{z_{1,\phi}(\phi)d\phi}{|z_1(\phi)-z_2(\theta)|^\alpha} \\ + \frac{C_\alpha}{2\pi i}\int_0^{2\pi}\frac{(z_{2,\phi}(\phi)-z_{2,\theta}(\theta))d\phi}{|z_2(\phi)-z_2(\theta)|^\alpha}\bigg)\overline{z_{2,\theta}(\theta)}\Bigg] = 0.\end{aligned} \tag{2.107}$$

We use again a pseudo-spectral method to find V-states. We discretize $\theta \in [0, 2\pi)$ in N equally spaced nodes $\theta_i = 2\pi i/N$, $i = 0, 1, \ldots, N-1$. Since z_1 and z_2 never intersect, the second integral in (2.106) and the first integral in (2.107) can be evaluated numerically with spectral accuracy at a node $\theta = \theta_i$ by means of the trapezoidal rule ; e.g.,

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{z_{2,\phi}(\phi)d\phi}{|z_2(\phi)-z_1(\theta_i)|^\alpha} \approx \frac{1}{N}\sum_{j=0}^{N-1}\frac{z_{2,\phi}(\phi_j)}{|z_2(\phi_j)-z_1(\theta_i)|^\alpha}. \tag{2.108}$$

Likewise, the first integral in (2.106) and the second integral in (2.107) can also be evaluated with spectral accuracy, reasoning as in (1.88) ; e.g.,

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{(z_{1,\phi}(\phi)-z_{1,\theta}(\theta_i))d\phi}{|z_1(\phi)-z_1(\theta_i)|^\alpha} \approx \frac{1}{N}\sum_{\substack{j=0\\ j\neq i}}^{N-1}\frac{z_{1,\phi}(\phi_j)-z_{1,\theta}(\theta_i)}{|z_1(\phi_j)-z_1(\theta_i)|^\alpha}. \tag{2.109}$$

Therefore, for N large enough, we can evaluate with spectral accuracy all the terms in (2.106)-(2.107) at $\theta = \theta_i$.

In order to obtain doubly connected m-fold V-states, we approximate z_1 and z_2 as in (1.90) :

$$z_1(\theta) = e^{i\theta}\left[1 + \sum_{k=1}^{M} a_{1,k}\cos(m\,k\,\theta)\right], \qquad z_2(\theta) = e^{i\theta}\left[b + \sum_{k=1}^{M} a_{2,k}\cos(m\,k\,\theta)\right], \tag{2.110}$$

where $\theta \in [0, 2\pi)$, the mean outer and inner radii are respectively 1 and b ; and we are imposing that $z_1(-\theta) = \overline{z_1(\theta)}$ and $z_2(-\theta) = \overline{z_2(\theta)}$, i.e., we are looking for V-states symmetric with respect to the x-axis. Again, if we choose N of the form $N = m2^r$, then $M = \lfloor(m2^r-1)/(2m)\rfloor = 2^{r-1}-1$.

We introduce (2.110) into (2.106)-(2.107), and approximate the error in those equations by an M-term sine expansion :

$$\begin{aligned}\operatorname{Re}\Bigg[\bigg(\Omega z_1(\theta) - \frac{C_\alpha}{2\pi i}\int_0^{2\pi}\frac{(z_{1,\phi}(\phi)-z_{1,\theta}(\theta))d\phi}{|z_1(\phi)-z_1(\theta)|^\alpha} \\ + \frac{C_\alpha}{2\pi i}\int_0^{2\pi}\frac{z_{2,\phi}(\phi)d\phi}{|z_2(\phi)-z_1(\theta)|^\alpha}\bigg)\overline{z_{1,\theta}(\theta)}\Bigg] = \sum_{k=1}^{M} b_{1,k}\sin(m\,k\,\theta), \\ \operatorname{Re}\Bigg[\bigg(\Omega z_2(\theta) - \frac{C_\alpha}{2\pi i}\int_0^{2\pi}\frac{z_{1,\phi}(\phi)d\phi}{|z_1(\phi)-z_2(\theta)|^\alpha} \\ + \frac{C_\alpha}{2\pi i}\int_0^{2\pi}\frac{(z_{2,\phi}(\phi)-z_{2,\theta}(\theta))d\phi}{|z_2(\phi)-z_2(\theta)|^\alpha}\bigg)\overline{z_{2,\theta}(\theta)}\Bigg] = \sum_{k=1}^{M} b_{2,k}\sin(m\,k\,\theta).\end{aligned} \tag{2.111}$$

As in (1.92), this last system of equations can be represented in a very compact way as

$$\mathcal{F}_{b,\alpha,\Omega}(a_{1,1},\ldots,a_{1,M}\ ,\ a_{2,1},\ldots,a_{2,M}) = (b_{1,1},\ldots,b_{1,M}\ ,\ b_{2,1},\ldots,b_{2,M}), \tag{2.112}$$

for a certain $\mathcal{F}_{\alpha,\Omega}\ :\ \mathbb{R}^{2M}\to\mathbb{R}^{2M}$.

Remark that, for any value of the parameters b, α and Ω, we have trivially $\mathcal{F}_{b,\alpha,\Omega}(\mathbf{0}) = \mathbf{0}$, i.e., any circular annulus is a solution of the problem. Therefore, the obtention of a doubly-connected V-state is reduced to finding numerically $\{a_{1,k}\}$ and $\{a_{2,k}\}$, such that $(a_{1,1},\ldots,a_{1,M}\ ,\ a_{2,1},\ldots,a_{2,M})$ is a nontrivial root of (2.112). To do so, we discretize the $(2M\times 2M)$-dimensional Jacobian matrix $\mathcal{J}$ of $\mathcal{F}_{b,\alpha,\Omega}$ as in (1.93), taking $h = 10^{-9}$:

$$\begin{aligned}&\frac{\partial}{\partial a_{1,1}}\mathcal{F}_{b,\alpha,\Omega}(a_{1,1},\ldots,a_{1,M}\ ,\ a_{2,1},\ldots,a_{2,M})\\ &\qquad\approx\frac{\mathcal{F}_{b,\alpha,\Omega}(a_{1,1}+h,a_{1,2},\ldots,a_{1,M}\ ,\ a_{2,1},\ldots,a_{2,M})-\mathcal{F}_{b,\alpha,\Omega}(a_{1,1},\ldots,a_{1,M}\ ,\ a_{2,1},\ldots,a_{2,M})}{h},\end{aligned} \tag{2.113}$$

Then, the sine expansion of (2.113) gives us the first row of $\mathcal{J}$, and so on. Hence, if the n-th iteration is denoted by $(a_{1,1},\ldots,a_{1,M}\ ,\ a_{2,1},\ldots,a_{2,M})^{(n)}$, then the $(n+1)$-th iteration is given by

$$\begin{aligned}&(a_{1,1},\ldots,a_{1,M}\ ,\ a_{2,1},\ldots,a_{2,M})^{(n+1)}\\ &\quad=(a_{1,1},\ldots,a_{1,M}\ ,\ a_{2,1},\ldots,a_{2,M})^{(n)}-\mathcal{F}_{b,\alpha,\Omega}\left((a_{1,1},\ldots,a_{1,M}\ ,\ a_{2,1},\ldots,a_{2,M})^{(n)}\right)\cdot[\mathcal{J}^{(n)}]^{-1},\end{aligned} \tag{2.114}$$

where $[\mathcal{J}^{(n)}]^{-1}$ denotes the inverse of the Jacobian matrix at $(a_{1,1},\ldots,a_{1,M}\ ,\ a_{2,1},\ldots,a_{2,M})^{(n)}$. To make this iteration converge, it is usually enough to perturb the annulus by assigning a small value to ${a_{1,1}}^{(0)}$ or ${a_{2,1}}^{(0)}$, and leave the other coefficients equal to zero. Our stopping criterion is

$$\max\left|\sum_{k=1}^{M}b_{1,k}\sin(m\,k\,\theta)\right| < tol \quad\wedge\quad \max\left|\sum_{k=1}^{M}b_{2,k}\sin(m\,k\,\theta)\right| < tol, \tag{2.115}$$

where $tol = 10^{-11}$. As in the vortex patch problem, $a_{1,1}\cdot a_{2,1} < 0$, so, for the sake of coherence, we change eventually the sign of all the coefficients $\{a_{1,k}\}$ and $\{a_{2,k}\}$, in order that, without loss of generality, $a_{1,1} > 0$ and $a_{2,1} < 0$.

Numerical experiments

As we have seen, the procedure to find doubly-connected m-fold V-states is very similar in the vortex patch and in the quasi-geostrophic problems. However, as evidenced in the simply-connected case, the numerical study of the V-states for the quasi-geostrophic problem reveals itself as a much richer task. Indeed, unlike in the vortex patch problem, where we had just two parameters b and Ω, we have now a third parameter α. Furthermore, since (2.106) and (2.107) are not homogeneous, choosing the mean outer radius not to be equal to one, as we are doing, would introduce a fourth parameter. Therefore, we will limit ourselves here to exposing a few relevant facts.

Theorem 2.1 states that, for any $\alpha\in(0,1)$, the inner mean radius b must be greater than b_0, which is the unique solution of the equation

$$b^2\Lambda_1(b)-\Lambda_1(1)+\frac{1}{2}=0, \qquad \text{with } \Lambda_1(1)=\frac{\Gamma(1-\alpha)}{2^{1-\alpha}\Gamma^2(1-\frac{\alpha}{2})}\frac{\Gamma(1+\frac{\alpha}{2})}{\Gamma(2-\frac{\alpha}{2})}. \tag{2.116}$$

This is an important difference with the vortex patch problem, where no lower bound for b exists.

In order to obtain b_0 (and other relevant quantities), we need to compute $\Lambda_n(b)$ accurately :

$$\begin{aligned}\Lambda_n(b) &= \frac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})}\frac{(\frac{\alpha}{2})_n}{n!}b^{n-1}F\left(\frac{\alpha}{2}, n+\frac{\alpha}{2}, n+1, b^2\right) \\ &= \frac{b^{n-1}\Gamma(\frac{\alpha}{2}+n)}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})n!}F\left(\frac{\alpha}{2}, n+\frac{\alpha}{2}, n+1, b^2\right). \end{aligned} \quad (2.117)$$

The hypergeometric function F is commonly implemented in most scientific packages ; for instance, in MATLAB©, $F(a,b,c,z)$ can be evaluated by means of the command `hypergeom([a, b], c, z)`, so we can find the only value b_0 satisfying (2.116) efficiently and with the greatest possible accuracy, by means a simple bisection technique. In Figure 2.1, we have plotted b_0 against 200 different values of α, i.e., $\alpha = 10^{-4}, 10^{-3}$, and $\alpha = 0.005, 0.01, \ldots, 0.995$. Observe that b_0 tends very quickly to 1 ; for example, $b_0(0.76) > 0.99$. On the other hand, $b_0(0) = 0$, which is coherent with the vortex patch problem, where there is no lower bound for b.

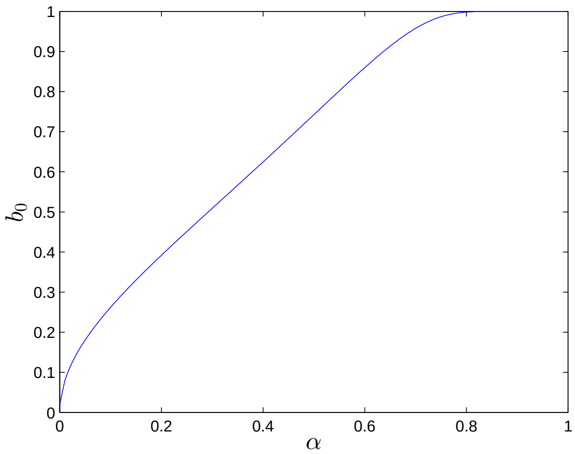

FIGURE 2.1 – Unique solution of (2.116), b_0, in function of α.

However, we have discovered in our numerical experiments that (2.116) is not sharp. Indeed, we have been able to find V-states with b much smaller than b_0. In all the numerical experiments in this section, we take $N = 64 \times m$ nodes. In Figure 2.2, we have plotted V-states corresponding to $\alpha = 0.9$ and $b = 0.2$, where $b_0(0.9) - 1 = \mathcal{O}(10^{-9})$. On the left-hand side, we have started to bifurcate from $\Omega_4^+(0.2) = 0.4077\ldots$. Observe that z_2 is practically a circumference, for all Ω. Moreover, the V-state corresponding to $\Omega = 0.4076$, in black, is practically a circular annulus, whereas the outer boundary of the V-state corresponding to $\Omega = 0.296$, in red, has a marked star shape. For Ω slightly smaller than $\Omega = 0.296$, the numerical experiments become instable. On the right-hand side, we have started to bifurcate from $\Omega_4^-(0.2) = -1.3055\ldots$. The most remarkable fact is that Ω is always negative, i.e, the V-states rotate clockwise, which is an important difference with respect to the vortex patch problem. On the other hand, z_1 is practically a circumference, for all Ω. The V-state corresponding to $\Omega = -1.305$, in black, is practically a circular annulus, whereas the inner boundary of the V-state corresponding to $\Omega = -0.849$, in red, has a marked star shape. For Ω slightly larger than $\Omega = -0.849$, the numerical experiments become instable.

Figure 2.2 shows the obvious parallelism with [64] in the vortex patch problem : as b becomes smaller, bifurcating at $\Omega_m^+(b)$ yields doubly-connected V-states closer and closer to simply-

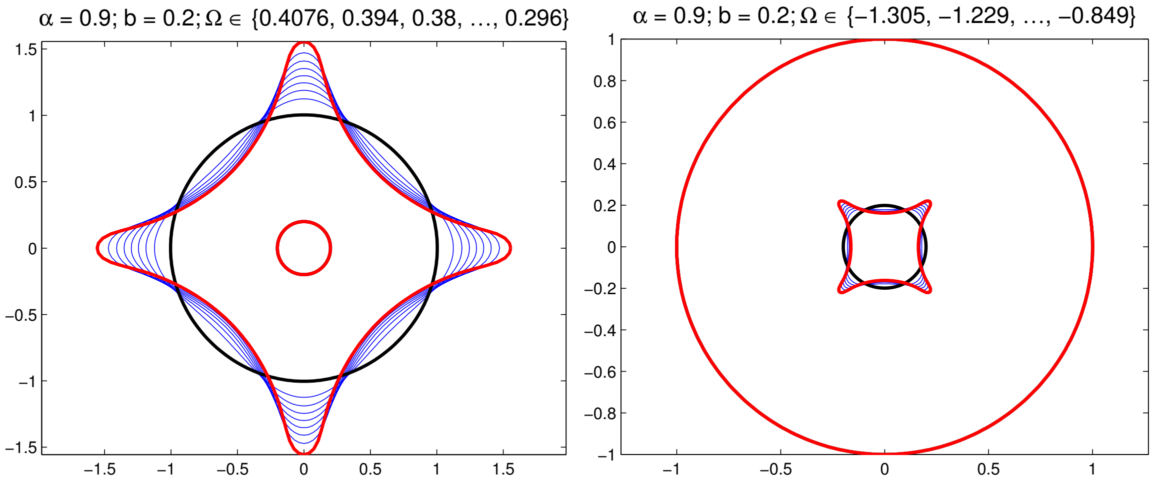

FIGURE 2.2 – 4-fold V-states corresponding to $\alpha = 0.9$, for different values of Ω. On the left-hand side, we have started to bifurcate from $\Omega_4^+(0.2) = 0.4077\ldots$; on the right-hand side, from $\Omega_4^-(0.2) = -1.3055\ldots$.

connected V-states ; whereas, bifurcating at $\Omega_m^-(b)$ yields double-connected V-states closer and closer to the unit circumference. This explains why there are two bifurcation values of Ω in the doubly-connected case, and just one single bifurcation value of Ω in the simply-connected case. Nonetheless, unlike what would have happened in the vortex patch problem, the V-states corresponding to $\Omega = 0.296$ and to $\Omega = -0.849$ seem to have developed no singularity. Again, as in the simply-connected case, the explanation is given by the loss of monotonicity in the bifurcation curves of $a_{1,1}$ and $a_{2,1}$ in (2.110) with respect to Ω, plotted in Figure 2.3, which predicts the existence of saddle-node bifurcation points. Observe also that, when we bifurcate from $\Omega_4^+(0.2)$, $a_{1,1}$ clearly dominates ; whereas, when we bifurcate from $\Omega_4^-(0.2)$, $a_{2,1}$ dominates and $a_{1,1}$ is of the order of $\mathcal{O}(10^{-6})$. This confirms our previous observations.

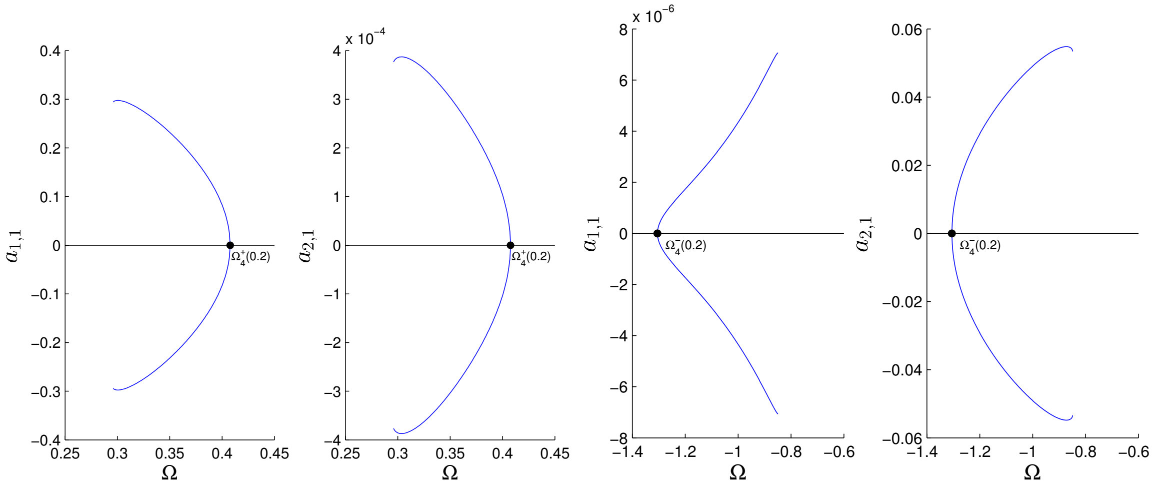

FIGURE 2.3 – Bifurcation curves, for $m = 4$, $\alpha = 0.9$, and $b = 0.2$. The leftmost two correspond to a bifurcation from $\Omega_4^+(0.2)$, whereas the rightmost two correspond to a bifurcation from $\Omega_4^-(0.2)$. Remember that $a_{1,1}$ and $a_{2,1}$ are such that $a_{1,1} \cdot a_{2,1} < 0$.

Following the procedure explained in the simply-connected case (where we set $\lambda^{(A)} = 0$, $\lambda^{(B)} = \lambda^{(A)} + [(\Omega^{(B)} - \Omega^{(A)})^2 + (a_{1,1}^{(B)} - a_{1,1}^{(A)})^2 + (a_{2,1}^{(B)} - a_{2,1}^{(A)})^2]^{1/2}$, and so on), we have continued the bifurcation curves at $\Omega = 0.29507$, and $\Omega = -0.84878$, until $\Omega = 0.3$ and $\Omega = -0.86$, respectively, as is shown in Figure 2.4. It is still possible to go a bit further, but, in order not to lose accuracy, a larger number of nodes is convenient. The pieces of curve beyond the saddle-node bifurcation points are shown in thicker stroke.

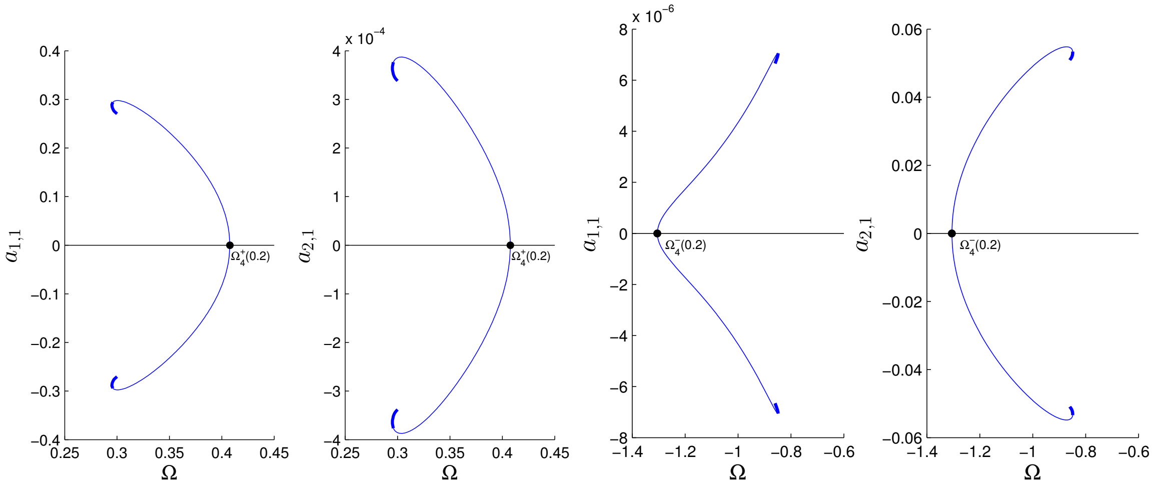

FIGURE 2.4 – Extended bifurcation curves, for $m = 4$, $\alpha = 0.9$, and $b = 0.2$. The leftmost two correspond to a bifurcation from $\Omega_4^+(0.2)$, whereas the rightmost two correspond to a bifurcation from $\Omega_4^-(0.2)$. Remember that $a_{1,1}$ and $a_{2,1}$ are such that $a_{1,1} \cdot a_{2,1} < 0$.

In Figure 2.5, we have plotted the V-states corresponding to $\Omega = 3$ and $\Omega = -0.86$, but beyond the saddle-node bifurcation points. The differences with respect to the V-states in red from Figure 2.2 are evident. It would be interested to calculate which kind of limiting V-states develops. Finally, whether the lower bound restriction for b can be ignored completely or not is another relevant question.

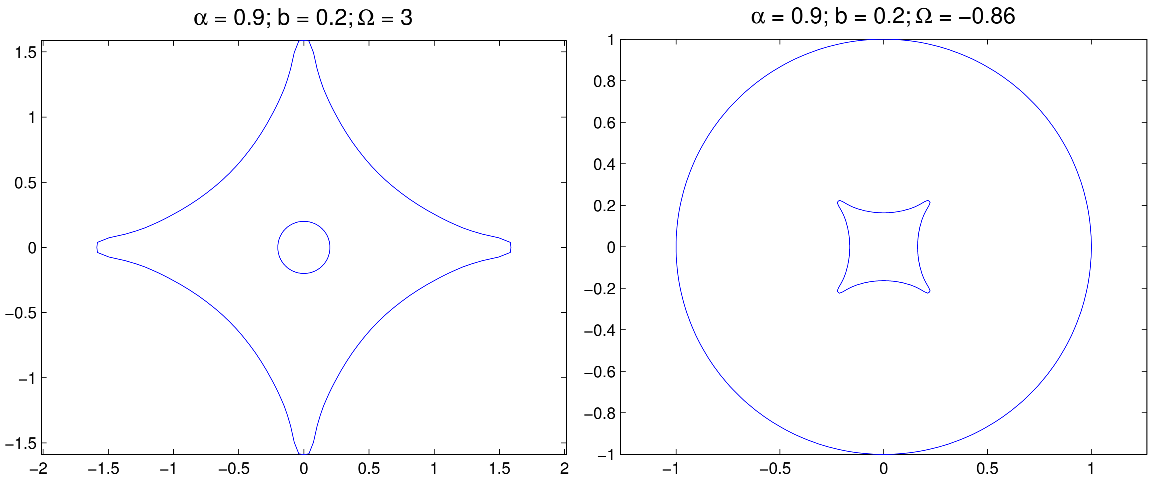

FIGURE 2.5 – V-states beyond the saddle-node bifurcation points.

Another relevant fact is the possibility to find examples where V-states exist for all $\Omega \in [\Omega_m^-, \Omega_m^+]$; such situation was also found in the vortex patch problem. We have considered, for instance, $m = 4$, $\alpha = 0.5$, and $b = 0.65$, with $\Omega_4^+(0.65) = 0.1480\ldots$ and $\Omega_4^-(0.65) = 0.08168\ldots$. Observe that $b_0(0.5) = 0.7424\ldots$, so we are violating the restriction on b_0 again. We have computed successfully all the V-states with $\Omega = 0.082, 0.083, \ldots, 0.148$. On the left-hand side of Figure

2.6, we plot the V-states corresponding to $\Omega = 0.082, 0.093, \ldots, 0.148$. The V-state corresponding to $\Omega = 0.148$ (in red), and to $\Omega = 0.082$ (in black), are practically circular annuli (we use no thicker stroke here, because all the V-states are very close to each other). On the right-hand side of Figure 2.6, we plot the bifurcation curves of $a_{1,1}$ and $a_{2,1}$ in (2.110), with respect to Ω. As expected, the curves are closed ; however, the curve corresponding to $a_{2,1}$ is much more symmetrical and reminds us of an ellipse.

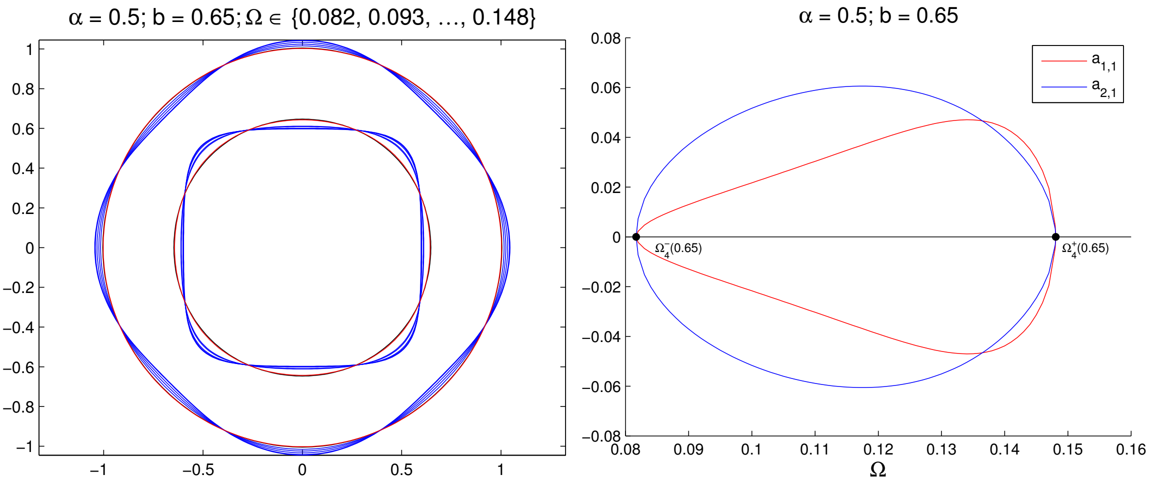

FIGURE 2.6 – Left : 4-fold V-states corresponding to $\alpha = 0.5$, $b = 0.65$, for different values of Ω ; there are V-states for all $\Omega \in [\Omega_m^-, \Omega_m^+]$. Right : bifurcation curves of $a_{1,1}$ and $a_{2,1}$ in (2.110) with respect to Ω. The curve corresponding to $a_{2,1}$ is much more symmetrical. Remember that $a_{1,1}$ and $a_{2,1}$ are such that $a_{1,1} \cdot a_{2,1} < 0$.

Let us finish this section by mentioning the existence of stationary doubly-connected V-states when $\alpha > 0$, i.e., V-states with $\Omega = 0$. Like the examples with $\Omega < 0$ shown above, they have no particularity from a numerical point of view, yet they are a completely new phenomenon with respect to the vortex patch problem. To obtain them, it is necessary to choose m, α and b, such that $\Omega_m^- < 0$, but $|\Omega_m^-| \ll 1$, since we bifurcate from the annulus at $\Omega = \Omega_m^-$. We have chosen $m = 4$ and $\alpha = 0.5$, as in the last experiment, but with a b even smaller, $b = 0.5$, in such a way that $\Omega_4^-(0.5) = -0.02760\ldots$. Figure 2.7 shows the corresponding stationary V-state.

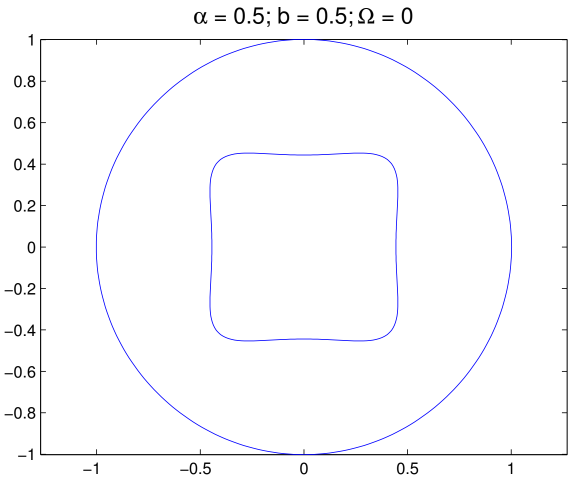

FIGURE 2.7 – Example of a stationary V-state.

Deuxième partie

Stratified inviscid fluids

Chapitre 3

On the inviscid Boussinesq system with rough initial data

This chapter is the subject of the following publication :
Hassainia, Zineb ; Hmidi, Taoufik ; *On the inviscid Boussinesq system with rough initial data*,
J. Math. Anal. Appl. 430 (2015), no. 2, 777-809.

Abstract. We deal with the local well-posedness theory for the two-dimensional inviscid Boussinesq system with rough initial data of Yudovich type. The problem is in some sense critical due to some terms involving Riesz transforms in the vorticity-density formulation. We give a positive answer for a special sub-class of Yudovich data including smooth and singular vortex patches. For the latter case we assume in addition that the initial density is constant around the singular part of the patch boundary.

3.1 Introduction

We consider the inviscid Boussinesq system describing the planar motion of a perfect incompressible fluid evolving under an external vertical force whose amplitude is proportional to the density which is in turn transported by the flow associated to the velocity field. The corresponding equations are given by,

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = \rho \vec{e_2}, \quad t \geq 0, x \in \mathbb{R}^2 \\ \partial_t \rho + v \cdot \nabla \rho = 0, \\ \operatorname{div} v = 0, \\ v_{|t=0} = v_0, \quad \rho_{|t=0} = \rho_0. \end{cases} \tag{3.1}$$

Here the vector field $v = (v_1, v_2)$ and the scalar function p denote the fluid velocity and the pressure respectively. The density ρ is a passive scalar quantity and the buoyancy force $\rho\vec{e_2}$ in the velocity equation models the gravity effect on the fluid motion, where $\vec{e_2}$ stands for the unit vertical vector $(0, 1)$.

This system serves as a simplified model for the fluid dynamics of the oceans and atmosphere. It takes into account the stratification which plays a dominant role for large scales. For more details about this subject see for instance [10] and [102]. The derivation of the above system can be formally done from the density dependent Euler equations through the Oberbeck-Boussinesq approximation where the density fluctuation is neglected everywhere in the momentum equation except in the buoyancy force. We point out that Feireisl and Novotný provide in [43] a rigorous justification of the viscous model by means of scale analysis and singular limit of the full compressible Navier-Stokes-Fourier system. Note that the system (3.1) coincides with the classical incompressible Euler equations when the initial density ρ_0 is identically constant. Recall that Euler system is given by,

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla p = 0, \\ \operatorname{div} v = 0, \\ v_{|t=0} = v_0. \end{cases} \tag{3.2}$$

Before discussing some theoretical results on the well-posedness problem for the inviscid Boussinesq equations we shall first start with the state of the art for the system (3.2). The local existence and uniqueness of very smooth solutions for (3.2) goes back to Wolibner [126] in the thirties of the last century. This result has been improved through the years by numerous authors and for several functional spaces. The pioneering work in this field is accomplished by Kato and Ponce in [72] who proved the local well-posedness in the framework of Sobolev spaces H^s, with $s > \frac{d}{2}+1$. This result was later generalized for other spaces, see for instance [20, 25, 101, 121, 131] and the references therein. Whether or not classical solutions develop singularities in finite time is still open except some special cases as the planar motion or the axisymmetric flows without swirl. Unlike the viscous models, the global theory for Euler equations has a geometric feature and relies crucially on the vorticity dynamics. Historically, the concept of the vorticity $\omega \triangleq \operatorname{rot} v$ and their basic laws were studied by Helmholtz in his seminal work on the vortex motion theory [57]. More recently, a blow-up vorticity criterion for Kato's solutions was given by Beale, Kato and Majda in [9] : the lifespan $T^\star$ is finite if and only if $\int_0^{T^\star} \|\omega(\tau)\|_{L^\infty} d\tau = +\infty$. In two dimensions the vorticity can be identified to the scalar function $\omega = \partial_1 v_2 - \partial_2 v_1$ and it is transported by the flow,

$$\partial_t \omega + v\cdot\nabla\omega = 0. \tag{3.3}$$

This leads to an infinite family of conservation laws. For example, we have $\|\omega(t)\|_{L^p} = \|\omega_0\|_{L^p}$ for any $p \in [1,\infty]$. Hence the global well-posedness of Kato's solution follows from the Beale-Kato-Majda criterion.

By using the formal L^p conservation laws it seems that we can relax the classical regularity and construct global weak solutions in L^p spaces for $p > 1$. This question has been originally addressed by Yudovich in [128], where he proved the existence and uniqueness of weak solution to 2D Euler system only with the assumption $\omega_0 \in L^p \cap L^\infty$. Under this pattern, the velocity is no longer in the Lipschitz class but belongs to the log-Lipschitz functions. With a velocity being in this latter class, it is proved that the flow map ψ defined below is uniquely defined in the class of continuous functions in both space and time variables,

$$\psi(t,x) = x + \int_0^t v(\tau,\psi(\tau))d\tau.$$

We can find more details about this subject in the book [25]. As a by-product we obtain the global persistence of the vortex patch structure. More precisely, if the initial vorticity $\omega_0 = \mathbf{1}_{\Omega_0}$ is a patch, that is, the characteristic function of a bounded domain Ω_0 then its evolution is given $\omega(t) = \mathbf{1}_{\Omega_t}$ with $\Omega_t \triangleq \psi(t,\Omega_0)$. The regularity persistence of the boundary is very subtle and was successfully accomplished by Chemin in [25] who showed in particular that when the boundary $\partial\Omega_0$ is better than C^1, say in $C^{1+\varepsilon}$ for $0<\varepsilon<1$, then $\partial\Omega_t$ will keep its initial regularity for all the time without any loss. The proof relies heavily on the estimate of the Lipschitz norm of the velocity with the co-normal regularity $\partial_X\omega$ of the vorticity. The vector fields (X_t) are transported by the flow, that i s,

$$\partial_t X + v\cdot\nabla X = X\cdot\nabla v. \tag{3.4}$$

The main advantage of this choice is the commutation of these vector fields with the transport operator $\partial_t + v \cdot \nabla$ which leads in turn to the master equation

$$(\partial_t + v \cdot \nabla)\partial_X \omega = 0. \tag{3.5}$$

This means that the tangential derivative of the vorticity is also transported by the flow and this is crucial in the framework of the vortex patches. The proof given by Chemin is not restrictive to the usual patches but covers more singular data called generalized vortex patches. We point out that there is another proof in the special case of the vortex patches that can be found in [13]. It is also important to mention that Chemin got in fact more accurate result for patches with singular boundary. In broad terms, he showed that the regular part of the initial boundary $\partial\Omega_0$ propagates with the same regularity without being affected by the singular part which by the reversibility of the problem cannot be smoothed out by the dynamics and becomes better than C^1 . Furthermore, the velocity v is Lipschitz far from the singular set and may undergo a blowup behavior near this set with a rate bounded by the logarithm of the distance from the singular set. Many similar studies have been subsequently implemented by numerous authors for bounded domains or viscous flows, see for instance [40, 30, 31, 36, 46, 61, 62, 113] and the references therein.

Now, bearing in mind that the system (3.1) is at a formal level a perturbation of the incompressible Euler equations, it is legitimate to see whether the known results for Euler equations work for the Boussinesq system as well. The earliest mathematical studies of the Boussinesq system and its dissipative counterpart are relatively recent and a great deal of attention has been paid to the local/global well-posedness problem, see for instance [1, 32, 33, 63, 67, 69, 70, 81, 96, 127]. Hereafter, we shall primarily restrict the discussion to the inviscid model described by (3.1) and recall some known facts on the classical solutions. We stress that this system can be seen as a hyperbolic one and therefore the commutator theory developed by Kato can be applied in a straightforward way. This was done by Chae and Nam in [21] who proved the local well-posedness when the initial data (v_0, ρ_0) belong to the sub-critical Sobolev space H^s with $s > 2$. A similar result was also established later by the same authors [22] for initial data lying in Hölderian spaces C^r with $r > 1$. Another local well-posedeness result is recently obtained in [84] for the critical Besov spaces $B^{2/p+1}_{p,1}$, with $p \in]1, +\infty[$. Furthermore, an analogous Beale-Kato-Majda criterion can be stated for the sub-critical cases. More precisely, it can be shown that Kato's solutions cease to exist in finite time $T^\star$ if and only if

$$\int_0^{T^\star} \|\nabla\rho(t)\|_{L^\infty} dt = +\infty.$$

For more details we refer the reader for instance to [84, 116]. Whether or not $T^\star$ is finite remains an outstanding open problem.

The main scope of this paper is to deal with the local well-posedness for (3.1) when the initial data are rough and belong to Yudovich class. Contrary to the incompressible Euler equations the problem sounds extremely hard to solve for generic Yudovich data due to the violent coupling between the vorticity and the density. The difficulties can be illustrated from the vorticity-density formulation,

$$\begin{cases} \partial_t\omega + v \cdot \nabla\omega = \partial_1\rho, \\ \partial_t\rho + v \cdot \nabla\rho = 0, \end{cases} \tag{3.6}$$

According to the first equation in the above system one gets

$$\|\omega(t)\|_{L^\infty} \le \|\omega_0\|_{L^\infty} + \int_0^t \|\nabla\rho(\tau)\|_{L^\infty} d\tau.$$

As we shall now see the estimate of the last integral requires the initial data to be more strong than what is allowed by Yudovich class. Indeed, the partial derivative $\partial_j\rho$ obeys to the following transport model,

$$(\partial_t + v\cdot\nabla)\partial_j\rho = \partial_j v\cdot\nabla\rho. \tag{3.7}$$

Consequently, the estimate of $\|\nabla\rho(t)\|_{L^\infty}$ requires the velocity field to be at least Lipschitz with respect to the space variable and unfortunately this is not necessary satisfied with a bounded vorticity. The main goal of this paper is to give a positive answer for the local well-posedness problem for a special class of Yudovich data. We shall in the first part prove the result for vortex patches with smooth boundary. In the second part we conduct the same study for patches with singular boundaries. Our first result reads as follows

Theorem 3.1 *Let $0<\varepsilon<1$ and Ω_0 be a bounded domain of the plane with a boundary $\partial\Omega_0$ in Hölder class $C^{1+\varepsilon}$. Let v_0 be a divergence-free vector field of vorticity $\omega_0 = 1_{\Omega_0}$ and consider $\rho_0\in L^2\cap C^{1+\varepsilon}$ a real-valued function with $\nabla\rho_0\in L^a$ and $1<a<2$. Then, there exists $T>0$ such that the Boussinesq system* (3.1) *admits a unique local solution $v,\rho\in L^\infty\big([0,T],W^{1,\infty}\big)$. Moreover, for all $t\in[0,T]$ the boundary of the advected domain $\Omega_t=\psi(t,\Omega_0)$ is of class $C^{1+\varepsilon}$.*

Before giving some details about the proof we shall discuss few remarks.

Remark 3.1 *The result of Theorem 3.1 will be extended in Theorem* 4.3 *to more general vortex structures. We shall get in particular a lower bound for the lifespan which is infinite for constant densities corresponding to the global result for Euler equations. More precisely we get*

$$T^\star \geq \frac{1}{C_0}\log\Big(1+C_0\log\Big(1+C_0/\|\nabla\rho_0\|_{L^\infty}\Big)\Big).$$

where $C_0 \triangleq C_0(\omega_0,\rho_0)$ depends continuously on the involved norms.

Remark 3.2 *For the sake of a clear presentation we have assumed in Theorem* 3.1 *that the density $\rho_0\in C^{1+\varepsilon}$. The persistence of such regularity is not clear and requires more than the Lipschitz norm for the velocity. However, as we shall see in Theorem* 4.3 *this condition can be relaxed to one that can be transported without loss : we replace this space by an anisotropic one.*

The proof of Theorem 3.1 is firmly based on the formalism of vortex patches developed by Chemin in [25, 26]. The key is to estimate the tangential regularity $\partial_X\omega$ in the Hölder space of negative index $C^{\varepsilon-1}$, with respect to a suitable family of vector fields. Since this family commutes with the transport operator $\partial_t+v\cdot\nabla$ one gets easily the equation

$$\begin{aligned}(\partial_t+v\cdot\nabla)\partial_X\omega &= \partial_X\partial_1\rho\\ &= \partial_1(\partial_X\rho)+[\partial_X,\partial_1]\rho.\end{aligned}$$

By using para-differential calculus we can show that the commutator term is well-behaved and therefore the problem reduces to the estimate $\|\partial_X\rho\|_{C^\varepsilon}$. For this latter term we use anew the commutation between ∂_X and the transport operator combined with the fact that the density is also conserved along the particle trajectories. Hence we find the equation

$$(\partial_t+v\cdot\nabla)\partial_X\rho = 0.$$

This structure is very important in our analysis in order to derive some crucial a priori estimates.

Let us move on to the second contribution of this paper which is concerned with the singular vortex patches. We shall assume that $\omega_0 = \mathbf{1}_{\Omega_0}$ but the boundary may now contain a singular subset. As the example of the square indicates, the velocity associated to a vortex patch is not in general Lipschitz and this will bring more technical difficulties. Similarly to the smooth boundary one needs to bound $\|\nabla\rho(t)\|_{L^\infty}$ and from the characteristic method we obtain

$$\|\partial_j\rho(t)\|_{L^\infty} \leq \|\nabla\rho_0\|_{L^\infty} + \int_0^t \|\partial_j v \cdot \nabla\rho(t)\|_{L^\infty}.$$

We expect the singularities initially located at the boundary to be frozen in the particle trajectories and the idea to treat the last integral term is to annihilate the effects of the velocity singularities by some specific assumptions on the density. As a possible choice we shall assume the initial density to be constant around the singularity set and from its transport structure the density will remain constant around the image by the flow of the singular set. This allows to track the singularities and kill their nasty effects by the density. Our result reads as follows,

Theorem 3.2 *Let $0 < \varepsilon < 1$ and Ω_0 be a bounded domain of the plane whose boundary $\partial\Omega_0$ is a curve of class $C^{1+\varepsilon}$ outside a closed set Σ_0. Let us consider a divergence-free vector field v_0 of vorticity $\omega_0 = 1_{\Omega_0}$ and take $\rho_0 \in L^2 \cap C^{\varepsilon+1}$ with $\nabla\rho_0 \in L^a$ for some $1 < a < 2$. Suppose that ρ_0 is constant in a small neighborhood of Σ_0. Then the system* (3.1) *admits a unique local solution (ω, ρ) such that*

$$\omega, \rho \in L^\infty\big([0,T], L^2 \cap L^\infty\big), \quad \nabla\rho \in L^\infty\big([0,T], L^a \cap L^\infty\big).$$

Furthermore, the velocity v is Lipschitz outside $\Sigma_t \triangleq \psi(t, \Sigma_0)$. More precisely, we have

$$\sup_{h\in(0,e^{-1}]} \frac{\|\nabla v(t)\|_{L^\infty((\Sigma_t)_h^c)}}{-\log h} \in L^\infty([0,T]),$$

where the set $(\Sigma_t)_h^c$ is defined by,

$$(\Sigma_t)_h^c \triangleq \big\{x \in \mathbb{R}^2;\ d\big(x, \Sigma(t)\big) \geq h\big\}.$$

In addition, the boundary of $\psi(t, \Omega_0)$ is locally in $C^{1+\varepsilon}$ outside the set Σ_t.

Remark 3.3 *Let us mention that the initial singular set is not arbitrary and should satisfy a weak condition of the following type : there exists two strictly positive real numbers $\tilde{\gamma}$ and C and a neighborhood V_0 of $\partial\Omega_0$ such that for any point $x \in V_0$ we have*

$$|\nabla f(x)| \geq C d(x, \Sigma_0)^{\tilde{\gamma}}.$$

Here the function $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth and satisfies

$$\Omega_0 = \big\{x, f(x) > 0\big\}, \partial\Omega_0 = \{x \in \mathbb{R}^2, f(x) = 0\}.$$

This means that the curves defining the boundary of Ω_0 are not tangent to one another at infinite order at the singular points.

The general outline of the paper is as follows. In the next section we recall some function spaces and give some of their useful properties, we also gather some preliminary estimates. Section 3 is devoted to the study of the regular vortex patches and the last section concerns the singular case. We close this paper with an appendix covering the proof of a technical lemma.

3.2 Tools and function spaces

Throughout this paper, C stands for some real positive constant which may be different in each occurrence and C_0 for a positive constant depending on the size of the initial data. We shall sometimes alternatively use the notation $X \lesssim Y$ for an inequality of the type $X \leq CY$.

Let us start with the dyadic partition of the unity whose proof can be found for instance in [25]. There exists a radially symmetric function φ in $\mathcal{D}(\mathbb{R}^2\backslash\{0\})$ such that

$$\forall \xi \in \mathbb{R}^2\backslash\{0\}, \quad \sum_{q\in\mathbf{Z}} \varphi(2^{-q}\xi) = 1.$$

We define the function $\chi \in \mathcal{D}(\mathbb{R}^2)$ by

$$\forall \xi \in \mathbb{R}^2, \quad \chi(\xi) = 1 - \sum_{q\geq 0} \varphi(2^{-q}\xi).$$

For every $u \in \mathcal{S}'(\mathbb{R}^2)$ one defines the non homogeneous Littlewood-Paley operators by,

$$\Delta_{-1} v = \mathcal{F}^{-1}\big(\chi \hat{v}\big), \quad \forall q \in \mathbb{N} \quad \Delta_q v = \mathcal{F}^{-1}\big(\varphi(2^{-q}\cdot)\hat{v}\big) \quad \text{and} \quad S_q v = \sum_{-1\leq j\leq q-1} \Delta_j v.$$

We notice that these operators map continuously L^p to itself uniformly with respect to q and p. Furthermore, one can easily check that for every tempered distribution v, we have

$$v = \sum_{q\geq -1} \Delta_q v.$$

By choosing in a suitable way the support of φ one can easily check the almost orthogonality properties : for any $u, v \in \mathcal{S}'(\mathbb{R}^2)$,

$$\Delta_p \Delta_q u = 0 \quad \text{if} \quad |p-q| \geq 2$$

$$\Delta_p(S_{q-1}u\Delta_q v) = 0 \quad \text{if} \quad |p-q| \geq 5.$$

We can now give a characterization of the Hölder spaces using the Littlewood-Paley decomposition.

Definition 3.1 *For all $s \in \mathbb{R}$, we denote by C^s the space of tempered distributions v such that*

$$\|v\|_s \triangleq \sup_{q\geq -1} 2^{qs}\|\Delta_q v\|_{L^\infty} < +\infty.$$

Remark 3.4 *We notice that for any strictly positive non integer real number s this definition coincides with the usual Hölder space C^s with equivalent norms. For example if $s \in]0,1[$,*

$$\|v\|_s \lesssim \|v\|_{C^s} \triangleq \|v\|_{L^\infty} + \sup_{x\neq y} \frac{|v(x)-v(y)|}{|x-y|^s} \lesssim \|v\|_s.$$

Next, we recall Bernstein inequalities, see for example [25].

Lemma 3.1 *There exists a constant $C > 0$ such that for all $q \in N, k \in \mathbb{N}, 1 \leq a \leq b \leq \infty$ and for every tempered distribution u we have*

$$\sup_{|\alpha|\leq k} \|\partial^\alpha S_q u\|_{L^b} \leq C^k 2^{q(k+2(\frac{1}{a}-\frac{1}{b}))}\|S_q u\|_{L^a},$$

$$C^{-k}2^{qk}\|\dot{\Delta}_q u\|_{L^b} \leq \sup_{|\alpha|=k} \|\partial^\alpha \dot{\Delta}_q u\|_{L^b} \leq C^k 2^{qk}\|\dot{\Delta}_q u\|_{L^b}.$$

Now, we introduce the Bony's decomposition [15] which is the basic tool of the para-differential calculus. Formally the product of two tempered distributions u and v is splitted into three parts as follows :

$$uv = T_u v + T_v u + R(u, v),$$

where

$$T_u v = \sum_q S_{q-1} u \Delta_q v \quad \text{and} \quad R(u, v) = \sum_q \Delta_q u \tilde{\Delta}_q v,$$

$$\text{with} \quad \tilde{\Delta}_q = \sum_{i=-1}^{1} \Delta_{q+i}.$$

The following lemma clarifies the behavior of the paraproduct operators in the Hölder spaces.

Lemma 3.2 *Let s be a real number. If $s < 0$ the bilinear operator T is continuous from $L^\infty \times C^s$ in C^s and from $C^s \times L^\infty$ in C^s. Moreover, we have*

$$\|T_u v\|_s + \|T_v u\|_s \le C \|u\|_{L^\infty} \|v\|_s.$$

If $s > 0$ the remainder operator R is continuous from $L^\infty \times C^s$ in C^s. Furthermore, we have

$$\|R(u, v)\|_s \le C \|u\|_{L^\infty} \|v\|_s.$$

Where C is a positive constant depending only on s.

As a result, we have the following corollary.

Corollary 3.1 *Let $\varepsilon \in]0, 1[$, X be a vector field belonging to C^ε as well as its divergence and f be a Lipschitz scalar function. Then for $j \in \{1, 2\}$ we have*

$$\|(\partial_j X) \cdot \nabla f\|_{\varepsilon-1} \le C \|\nabla f\|_{L^\infty} \big(\|\text{div} X\|_\varepsilon + \|X\|_\varepsilon\big).$$

Proof : In view of Bony's decomposition we write

$$\|(\partial_j X) \cdot \nabla f\|_{\varepsilon-1} \quad \le \quad \|T_{\partial_j X^i} \partial_i f\|_{\varepsilon-1} + \|T_{\partial_i f} \partial_j X^i\|_{\varepsilon-1} + \|R(\partial_j X^i, \partial_i f)\|_{\varepsilon-1},$$

where we have adopted in the right-hand side of the last inequality the Einstein summation convention for the index i. Since $\varepsilon - 1 < 0$ the previous lemma ensures that

$$\|T_{\partial_j X^i} \partial_i f\|_{\varepsilon-1} + \|T_{\partial_i f} \partial_j X^i\|_{\varepsilon-1} \le C \|\nabla f\|_{L^\infty} \|X\|_\varepsilon.$$

For the remainder term we write

$$R(\partial_j X^i, \partial_i f) = \partial_j R(X^i, \partial^i f) - \partial_i R(X^i, \partial_j f) + R(\text{div}\, X, \partial_j f).$$

Using once again Lemma 3.2 we get

$$\begin{aligned} \|R(\partial_j X^i, \partial_i f)\|_{\varepsilon-1} &\lesssim \|R(X^i, \partial^i f)\|_\varepsilon + \|R(X, \partial_j f)\|_\varepsilon + \|R(\text{div}\, X, \partial_j f)\|_\varepsilon \\ &\lesssim \|\nabla f\|_{L^\infty} \|X\|_\varepsilon + \|\nabla f\|_{L^\infty} \|\text{div}\, X\|_\varepsilon. \end{aligned}$$

This concludes the proof of the corollary. □

In the next section we will need the following result dealing with the Hölderian regularity persistence for the transport equations. Its proof is given in page 66 from [25].

Lemma 3.3 *Let v be a smooth divergence-free vector field and let $r \in]-1,1[$. Let us consider (f,g) a couple of functions belonging to $L^\infty_{loc}(\mathbb{R}, C^r) \times L^1_{loc}(\mathbb{R}, C^r)$ and such that*

$$\partial_t f + v \cdot \nabla f = g.$$

Then we have

$$\|f(t)\|_r \lesssim \|f(0)\|_r e^{C\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau} + \int_0^t \|g(\tau)\|_r e^{C\int_\tau^t \|\nabla v(\sigma)\|_{L^\infty} d\sigma} d\tau \tag{3.8}$$

The constant C depends only on r.

Next, we notice that if v is divergence-free and decaying at infinity then it can be recovered from its vorticity $\omega \triangleq \mathrm{rot} v$ by means of the Biot-Savart law

$$v(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega(y)dy. \tag{3.9}$$

Now we briefly recall the Calderón-Zygmund estimate that will be frequently used through this paper.

Proposition 3.1 *There exists a positive constant C satisfying the following property. For any smooth divergence-free vector field v with vorticity $\omega \in L^p$ and $p \in]1, \infty[$ one has*

$$\|\nabla v\|_{L^p} \le C\frac{p^2}{p-1}\|\omega\|_{L^p}. \tag{3.10}$$

In order to extend the results stated in the introduction to various geometries, we shall introduce some useful notations and definitions. Namely, we define the anisotropic Besov spaces with respect to slight smooth vector fields. This approach has been initially developed by J.-Y. Chemin in [25] in order to treat the vortex patch problem for the incompressible Euler system.

Definition 3.2 *Let Σ be a closed set of the plane and $\varepsilon \in (0,1)$. Let $X = (X_\lambda)_{\lambda\in\Lambda}$ be a family of vector fields. We say that this family is admissible of class C^ε outside Σ if and only if :*

1. *Regularity :* $X_\lambda, \mathrm{div}\, X_\lambda \in C^\varepsilon$.
2. *Non degenracy :*
$$I(\Sigma, X) \triangleq \inf_{x\notin\Sigma}\sup_{\lambda\in\Lambda}|X_\lambda(x)| > 0.$$

We set

$$\widetilde{\|}X_\lambda\|_\varepsilon \triangleq \|X_\lambda\|_\varepsilon + \|\mathrm{div} X_\lambda\|_{\varepsilon-1},$$

and

$$N_\varepsilon(\Sigma, X) \triangleq \sup_{\lambda\in\Lambda}\frac{\widetilde{\|}X_\lambda\|_\varepsilon}{I(\Sigma,X)}.$$

For each element X_λ of the preceding family we define its action on bounded real-valued functions u in the weak sense as follows :

$$\partial_{X_\lambda} u \triangleq \mathrm{div}(u\, X_\lambda) - u\,\mathrm{div} X_\lambda.$$

Definition 3.3 *Let $\varepsilon \in (0,1)$, $k \in \mathbb{N}$ and Σ be a closed set of the plane. Consider a family of vector fields $X = (X_\lambda)_\lambda$ as in the Definition 4.1. We denote by $C^{\varepsilon+k}(\Sigma, X)$ the space of functions $u \in W^{k,\infty}$ such that*

$$\sum_{|\alpha|\le k} \|\partial^\alpha u\|_{L^\infty} + \sup_{\lambda\in\Lambda} \|\partial_{X_\lambda} u\|_{\varepsilon+k-1} < +\infty,$$

and we set

$$\|u\|^{\varepsilon+k}_{\Sigma,X} \triangleq N_\varepsilon(\Sigma, X) \sum_{|\alpha|\le k} \|\partial^\alpha u\|_{L^\infty} + \sup_{\lambda\in\Lambda} \frac{\|\partial_{X_\lambda} u\|_{\varepsilon+k-1}}{I(\Sigma, X)}.$$

Remark 3.5 *When Σ is empty we will merely say that the set of vector fields $(X_\lambda)_{\lambda\in\Lambda}$ is admissible and to make the notation less cluttered we shall withdraw the symbol Σ from the previous definitions. For example, we use simply $I(X)$ instead of $I(X,\Sigma)$and $\|.\|^{\varepsilon+k}_X$ instead of $\|.\|^{\varepsilon+k}_{\Sigma,X}$.*

The next result deals with a logarithmic estimate established in [25] which is the main key in the study of the generalized vortex patches.

Theorem 3.3 *There exists an absolute constant C such that for any $a \in (1,\infty), \varepsilon \in (0,1)$ we have the following property. Let Σ be a closed set of the plane and X be a family of vector fields as in Definition 4.1. Consider a function $\omega \in C^\varepsilon(\Sigma, X) \cap L^a$. Let v be the divergence-free vector field with vorticity ω, then we get :*

$$\|\nabla v\|_{L^\infty(\Sigma)} \le Ca\|\omega\|_{L^a} + \frac{C}{\varepsilon}\|\omega\|_{L^\infty} \log\Big(e + \frac{\|\omega\|^\varepsilon_{\Sigma,X}}{\|\omega\|_{L^\infty}}\Big).$$

3.3 Smooth patches

In this section we shall state a local well-posedness result for the system (3.1) with general initial data covering the result of Theorem 3.1. The main result of this section is the following.

Theorem 3.4 *Let $0 < \varepsilon < 1, a \in (1,\infty)$ and $X_0 = (X_{0,\lambda})_{\lambda\in\Lambda}$ be an admissible family of vector fields of class C^ε. Let v_0 be a divergence-free vector field whose vorticity ω_0 belongs to $L^a \cap C^\varepsilon(X_0)$ and ρ_0 be a real-valued function belonging to $L^2 \cap C^{\varepsilon+1}(X_0)$ with $\nabla\rho_0 \in L^a$. Then there exists $T > 0$ such that the inviscid Boussinesq system (3.1) admits a unique solution $(v,\rho) \in L^\infty_{loc}\big([0,T], Lip(\mathbb{R}^2)\big) \times L^\infty_{loc}\big([0,T], Lip(\mathbb{R}^2)\cap L^2\big)$ such that $\omega \in L^\infty\big([0,T], L^a \cap L^\infty\big)$. Moreover, for all $t \in [0,T]$ the transported X_t of X_0 by the flow ψ, defined by*

$$X_{t,\lambda}(x) \triangleq \big(\partial_{X_{0,\lambda}}\psi(t)\big)(\psi^{-1}(t,x)), \tag{3.11}$$

is admissible of class C^ε and

$$\rho(t) \in C^{\varepsilon+1}(X_t) \quad \text{and} \quad \omega(t) \in C^\varepsilon(X_t).$$

In addition,

$$T \ge \frac{1}{C_0}\log\Big(1 + C_0\log\Big(1 + C_0/\|\nabla\rho_0\|_{L^\infty}\Big)\Big) \triangleq T_0$$

where $C_0 \triangleq C_0(\omega_0,\rho_0)$ depends continuously on the norms of the initial data

Remark 3.6 *The Theorem 4.3 can be applied to a larger class of initial data than the vortex patches class. For example, we may take $\omega_0 = \tilde\omega_0 1_{\Omega_0}$ with Ω_0 a bounded domain of class $C^{\varepsilon+1}$ and $\tilde\omega_0$ a function of class $C^\varepsilon(\mathbb{R}^2)$ for some $\varepsilon \in]0,1[$.*

We shall now make precise the boundary regularity used in the main theorems.

Definition 3.4 *Let $0 < \varepsilon < 1$ and Ω be a bounded domain in $\mathbb{R}^d$. We say that Ω is of class $C^{1+\varepsilon}$ if there exists a compactly supported function $f \in C^{1+\varepsilon}(\mathbb{R}^2)$ and a neighborhood V of $\partial\Omega$ such that*

$$\partial\Omega = f^{-1}(\{0\}) \cap V \quad \text{and} \quad \nabla f(x) \neq 0 \quad \forall x \in V.$$

Let us see how to deduce the results of Theorem 3.1 from the preceding one.

3.3.1 Proof of Theorem 3.1

To begin with, we shall construct an admissible family of vector fields X_0 for which the initial vorticity $\omega_0 = 1_{\Omega_0}$ satisfies the tangential regularity property. In view of the previous definition, there exists a real function $f_0 \in C^{1+\varepsilon}$ and a neighborhood V_0 such that $\partial\Omega_0 = V_0 \cap f^{-1}(\{0\})$ and $\nabla f_0 \not\equiv 0$ on V_0. Let $\tilde{\alpha}$ be a smooth function supported in V_0 and taking the value 1 in a small neighborhood of $V_1 \subset V_0$. We set

$$X_{0,0} = \nabla^{\perp} f_0, \quad X_{0,1} = (1 - \tilde{\alpha}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The first vector field is of class C^{ε} with zero divergence, the second is C^{∞} and a simple verification shows that the family of vector fields $(X_{0,i})_{i\in\{0,1\}}$ is admissible . Besides, since the derivative of ω_0 along the direction $\nabla^{\perp} f_0$ is zero and $1 - \tilde{\alpha}$ vanishes on V_1 then we have $\partial_{X_{0,i}}\omega_0 = 0$.

Also, the fact that $\rho_0 \in C^{\varepsilon+1}$ implies that $\rho_0 \in C^{\varepsilon}\big((X_{0,i})_{i\in\{1,2\}}\big)$. Therefore Theorem 3.1 provides a unique local solution $(v, \rho) \in L^{\infty}_{loc}\big([0, T_0], Lip(\mathbb{R}^2)\big)^2$ to (3.1). For the regularity of the transported initial domain $\Omega_t = \psi(t, \Omega_0)$, we consider $\gamma^0 \in C^{\varepsilon+1}(\mathbb{R}_+, \mathbb{R}^2)$ a parametrization of $\partial\Omega_0$ given by

$$\begin{cases} \partial_\sigma \gamma^0 = \nabla^{\perp} f_0(\gamma^0(\sigma)), \\ \gamma^0(0) = x_0 \in \partial\Omega_0. \end{cases}$$

Set $\gamma_t(\sigma) = \psi(t, \gamma^0(\sigma))$, then by differentiating with respect to the parameter σ we get

$$\begin{cases} \partial_\sigma \gamma_t(\sigma) = (\partial_{X_{0,0}}\psi)(t, \gamma^0(\sigma)), \\ \gamma_t(0) = \psi(t, x_0). \end{cases}$$

From Theorem 4.3, $\partial_{X_{0,0}}\psi$ belongs to $L^{\infty}_{loc}([0, T_0], C^{\varepsilon})$, then γ_t belongs to $L^{\infty}_{loc}(\mathbb{R}_+, C^{\varepsilon+1})$ for all $t \leq T_0$. Finally, as $X_{0,0}$ does not vanish on V_0, then it is the same for $\partial_{X_{0,0}}\psi$, therefore, $\partial_\sigma\gamma_t$ does not vanish on $\mathbb{R}$ as indicated by the estimate (3.15). Consequently, γ_t is a regular parameterization of $\partial\Omega_t$.

3.3.2 A priori estimates

This part is the core of the proof of Theorem 4.3. As a matter of fact, we aim here to propagate the regularity of the initial data, namely, to bound the norms $\|\omega(t)\|_{L^a\cap L^\infty}$ and $\|\nabla\rho(t)\|_{L^a\cap L^\infty}$. Even though these quantities seem to be less regular than $\|\nabla v(t)\|_{L^1_t L^\infty}$, it is not at all clear how to estimate them without involving the latter quantity. It comes then to show the two following propositions : The first deals with the L^p estimates and the second is related on the estimate of the Lipschitz norm for the solution of the system 3.1.

Proposition 3.2 *Let* (v,ρ) *be a smooth solution of the Boussinesq system* (3.1) *defined on the time interval* $[0,T]$. *Then, for all* $p\in[1,+\infty]$ *and* $t\le T$ *we have*

$$\|\omega(t)\|_{L^p} \quad\le\quad \|\omega_0\|_{L^p}+\|\nabla\rho_0\|_{L^p}e^{CV(t)}t. \tag{3.12}$$

and

$$\|\nabla\rho(t)\|_{L^p}\le\|\nabla\rho_0\|_{L^p}e^{CV(t)}. \tag{3.13}$$

with the notation :

$$V(t)\triangleq\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau.$$

Proof : Using the vorticity equation (3.6) we can easily see that for all $1\le p\le\infty$,

$$\|\omega(t)\|_{L^p} \quad\le\quad \|\omega_0\|_{L^p}+\int_0^t\|\nabla\rho(\tau)\|_{L^p}d\tau. \tag{3.14}$$

Next, applying the partial derivative operator ∂_j to the second equation of the system (3.1), we get

$$\partial_t\partial_j\rho+v\cdot\nabla(\partial_j\rho)=\partial_j v\cdot\nabla\rho.$$

Hence, for all $1\le p\le\infty$, we obtain

$$\|\partial_j\rho(t)\|_{L^p} \quad\le\quad \|\nabla\rho_0\|_{L^p}+\int_0^t\|\nabla\rho(\tau)\|_{L^p}\|\nabla v(\tau)\|_{L^\infty}d\tau.$$

According to the Gronwall lemma we conclude that

$$\|\nabla\rho(t)\|_{L^p}\le\|\nabla\rho_0\|_{L^p}e^{CV(t)}.$$

Plugging this estimate into (3.14) gives,

$$\|\omega(t)\|_{L^p} \quad\le\quad \|\omega_0\|_{L^p}+\|\nabla\rho_0\|_{L^p}e^{CV(t)}t;\quad \forall 1\le p\le\infty.$$

□

Next we shall discuss the Lipschitz norm of the velocity. This parts uses the formalism of the vortex patches. Our result reads as follows.

Proposition 3.3 *Let* $0<\varepsilon<1$, $a>1$ *and* X_0 *be an admissible family of vector fields of class* C^ε. *Let* (v,ρ) *be a smooth solution of the Boussinesq system* (3.1) *defined on the time interval* $[0,T^\star[$. *Then there exists* $0<T_0\le T^\star$ *such that for all time* $t\le T_0$ *we have*

$$\|\nabla v(t)\|_{L^\infty}\le C_0.$$

The proof of this proposition is firmly based on the following lemma.

Lemma 3.4 *There exists a constant* C *such that for any smooth solution* (v,ρ) *of* (3.1) *on* $[0,T]$, *and any time dependent family of vector field* X_t *transported by the flow of* v, *we have for all* $t\in[0,T]$,

$$I(X_t)\ge I(X_0)e^{-V(t)}. \tag{3.15}$$

$$\|\mathrm{div}X_{t,\lambda}\|_\varepsilon\le\|\mathrm{div}X_{0,\lambda}\|_\varepsilon e^{CV(t)}. \tag{3.16}$$

$$\|X_t\|_\varepsilon + \|\partial_{X_{t,\lambda}}\omega\|_{\varepsilon-1} \le C\Big(\tilde{\|}X_{0,\lambda}\|_\varepsilon + \|\partial_{X_{0,\lambda}}\omega_0\|_{\varepsilon-1} + \|\partial_{X_{0,\lambda}}\rho_0\|_\varepsilon\Big) \times e^{C(t+V(t))}\exp\big(t\|\nabla\rho_0\|_{L^\infty}e^{CV(t)}\big). \quad (3.17)$$

$$\|\partial_{X_{t,\lambda}}\rho\|_\varepsilon \le \|\partial_{X_{0,\lambda}}\rho_0\|_\varepsilon e^{CV(t)}.$$

where

$$V(t) \triangleq \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau.$$

Proof : Taking the derivative of term $\partial_{X_{0,\lambda}}\psi(t,x)$ with respect to the time t we get

$$\begin{cases} \partial_t\partial_{X_{0,\lambda}}\psi(t,x) = \nabla v(t,\psi(t,x))\partial_{X_{0,\lambda}}\psi(t,x). \\ \partial_{X_{0,\lambda}}\psi(0,x) = X_{0,\lambda}. \end{cases} \quad (3.18)$$

Using the time reversibility of this equation combined with Gronwall's lemma we find

$$|X_{0,\lambda}(x)| \le |\partial_{X_{0,\lambda}}\psi(t,x)|e^{V(t)}.$$

From the Definition 4.1 and the relation (4.4) we obtain the desired estimate (3.15).
It is easy to check from the relation (3.18) that

$$\partial_t X_{t,\lambda} + v\cdot\nabla X_{t,\lambda} = \partial_{X_{t,\lambda}} v. \quad (3.19)$$

Applying the divergence operator to the equation (3.19) we obtain,

$$\big(\partial_t + v\cdot\nabla\big)\mathrm{div}X_{t,\lambda} = 0,$$

and therefore we may use Lemma 3.3 leading to the estimate (3.16).
Next, we intend to establish (3.17). For this goal we start with the following result whose proof is given in Lemma 3.3.2 of [25],

$$\|\partial_{X_{t,\lambda}} v(t)\|_\varepsilon \lesssim \|\nabla v(t)\|_{L^\infty}\tilde{\|}X_{t,\lambda}\|_\varepsilon + \|\partial_{X_{t,\lambda}}\omega(t)\|_{\varepsilon-1}.$$

Applying Lemma 3.3 to equation (3.19) we get

$$\|X_{t,\lambda}\|_\varepsilon \le e^{CV(t)}\Big(\|X_{0,\lambda}\|_\varepsilon + C\int_0^t \big(\|\nabla v(\tau)\|_{L^\infty}\tilde{\|}X_{\tau,\lambda}\|_\varepsilon + \|\partial_{X_{\tau,\lambda}}\omega(\tau)\|_{\varepsilon-1}\big)e^{-CV(\tau)}d\tau\Big).$$

Putting this estimate with (3.16) yields

$$\tilde{\|}X_{t,\lambda}\|_\varepsilon \le e^{CV(t)}\Big(\tilde{\|}X_{0,\lambda}\|_\varepsilon + C\int_0^t \big(\|\nabla v(\tau)\|_{L^\infty}\tilde{\|}X_{\tau,\lambda}\|_\varepsilon + \|\partial_{X_{\tau,\lambda}}\omega(\tau)\|_{\varepsilon-1}\big)e^{-CV(\tau)}d\tau\Big). \quad (3.20)$$

Since $\partial_{X_{t,\lambda}}$ commutes with the transport operator $\partial_t + v\cdot\nabla$, then

$$\big(\partial_t + v\cdot\nabla\big)\partial_{X_{t,\lambda}}\omega = \partial_{X_{t,\lambda}}\partial_1\rho$$

and consequently we get in view of Lemma 3.3

$$\|\partial_{X_{t,\lambda}}\omega(t)\|_{\varepsilon-1} \le e^{CV(t)}\Big(\|\partial_{X_{0,\lambda}}\omega_0\|_{\varepsilon-1} + C\int_0^t \|\partial_{X_{\tau,\lambda}}\partial_1\rho(\tau)\|_{\varepsilon-1}e^{-CV(\tau)}d\tau\Big). \quad (3.21)$$

Observe that

$$\partial_{X_{\tau,\lambda}}\partial_1\rho = \partial_1(\partial_{X_{\tau,\lambda}}\rho) - \partial_{\partial_1 X_{\tau,\lambda}}\rho, \quad (3.22)$$

and thus

$$\begin{aligned}\|\partial_{X_{\tau,\lambda}}\partial_1\rho(\tau)\|_{\varepsilon-1} &\lesssim \|\partial_{X_{\tau,\lambda}}\rho(\tau)\|_{\varepsilon} + \|(\partial_1 X_{\tau,\lambda})\cdot\nabla\rho(\tau)\|_{\varepsilon-1}\\ &\lesssim \|\partial_{X_{\tau,\lambda}}\rho(\tau)\|_{\varepsilon} + \|\nabla\rho(\tau)\|_{L^\infty}\tilde{\|}X_{\tau,\lambda}\|_{\varepsilon},\end{aligned} \tag{3.23}$$

where we have used in the last inequality Corollary 3.1.
To estimate the term $\|\partial_{X_{\tau,\lambda}}\rho(\tau)\|_{\varepsilon}$ we use once again the commutation between $\partial_{X_{t,\lambda}}$ and the transport operator leading to,

$$\big(\partial_t + v\cdot\nabla\big)\partial_{X_{t,\lambda}}\rho = 0. \tag{3.24}$$

Applying Lemma 3.3 gives

$$\|\partial_{X_{t,\lambda}}\rho(t)\|_{\varepsilon} \lesssim \|\partial_{X_{0,\lambda}}\rho_0\|_{\varepsilon} e^{CV(\tau)}$$

which yields according to (3.23)

$$\|\partial_{X_{\tau,\lambda}}\partial_1\rho(\tau)\|_{\varepsilon-1} \lesssim \|\partial_{X_{0,\lambda}}\rho_0\|_{\varepsilon} e^{CV(\tau)} + \|\nabla\rho(\tau)\|_{L^\infty}\tilde{\|}X_{\tau,\lambda}\|_{\varepsilon}.$$

Plugging this estimate into (3.21) implies

$$\|\partial_{X_{t,\lambda}}\omega(t)\|_{\varepsilon-1} \lesssim e^{CV(t)}\Big(\|\partial_{X_{0,\lambda}}\omega_0\|_{\varepsilon-1} + \|\partial_{X_{0,\lambda}}\rho_0\|_{\varepsilon} t + \int_0^t \|\nabla\rho(\tau)\|_{L^\infty}\tilde{\|}X_{\tau,\lambda}\|_{\varepsilon} e^{-CV(\tau)}d\tau\Big).$$

Hence, putting together the foregoing estimate and (3.20) we get

$$\Gamma(t) \lesssim \Gamma(0) + \|\partial_{X_{0,\lambda}}\rho_0\|_{\varepsilon} t + \int_0^t \big(\|\nabla\rho\|_{L^\infty} + \|\nabla v\|_{L^\infty} + 1\big)\Gamma(\tau)d\tau.$$

with $\Gamma(t) \triangleq \big(\|\partial_{X_{t,\lambda}}\omega(t)\|_{\varepsilon-1} + \tilde{\|}X_{t,\lambda}\|_{\varepsilon}\big)e^{-CV(t)}$. Then Gronwall's lemma implies that

$$\Gamma(t) \lesssim \Big(\Gamma(0) + \|\partial_{X_{0,\lambda}}\rho_0\|_{\varepsilon}\Big) e^{C\int_0^t \big(\|\nabla\rho\|_{L^\infty} + \|\nabla v\|_{L^\infty} + 1\big)d\tau}.$$

Finally, Proposition 3.2 gives the desired result. □

Proof of the Proposition 3.3. Combining the inequality (3.12) with the estimate (3.17) we find

$$\begin{aligned}\|\partial_{X_{t,\lambda}}\omega(t)\|_{\varepsilon-1} + \|\omega(t)\|_{L^\infty}\tilde{\|}X_{t,\lambda}\|_{\varepsilon} &\le \big(1 + \|\omega_0\|_{L^\infty}\big)\Big(\Gamma(0) + \|\partial_{X_{0,\lambda}}\rho_0\|_{\varepsilon}\Big)\\ &\quad\times e^{C(t+V(t))}\exp\big(Ct\|\nabla\rho_0\|_{L^\infty}e^{CV(t)}\big).\end{aligned}$$

Putting together the last estimate and the inequality (3.15) then we get according to the Definition 3.3

$$\|\omega(t)\|^{\varepsilon}_{X_t} \le C_0 e^{C(t+V(t))}\exp\big(Ct\|\nabla\rho_0\|_{L^\infty}e^{CV(t)}\big), \tag{3.25}$$

According to the Proposition 3.3 and the monotonicity of the map $x \mapsto x\log\big(e + \frac{a}{x}\big)$ we find

$$\|\nabla v(t)\|_{L^\infty} \le C\Big(\|\omega_0\|_{L^a\cap L^\infty} + t\|\nabla\rho_0\|_{L^a\cap L^\infty}e^{CV(t)}\Big)\log\bigg(e + \frac{\|\omega(t)\|^{\varepsilon}_{X_t}}{\|\omega_0\|_{L^\infty}}\bigg).$$

It follows from the estimate (4.17) that

$$\begin{aligned}\|\nabla v(t)\|_{L^\infty} &\le C\Big(\|\omega_0\|_{L^a\cap L^\infty} + t\|\nabla\rho_0\|_{L^a\cap L^\infty}e^{CV(t)}\Big)\\ &\quad\times\Big(C_0 + t + t\|\nabla\rho_0\|_{L^a\cap L^\infty}e^{CV(t)} + V(t)\Big),\end{aligned} \tag{3.26}$$

We shall take $T > 0$ such that

$$T\|\nabla\rho_0\|_{L^a\cap L^\infty}e^{CV(T)} \leq \min\big(1, \|\omega_0\|_{L^1\cap L^\infty}\big). \tag{3.27}$$

Then we deuce from (3.26)

$$\|\nabla v(t)\|_{L^\infty} \leq C\|\omega_0\|_{L^a\cap L^\infty}\Big(C_0 + t + \int_0^t \|\nabla v(\tau)\|_{L^\infty}d\tau\Big), \quad \forall t \in [0, T].$$

which yields in view of Gronwall lemma

$$\|\nabla v(t)\|_{L^\infty} \quad \leq \quad C\|\omega_0\|_{L^a\cap L^\infty}(C_0 + t)e^{C\|\omega_0\|_{L^a\cap L^\infty}t} \quad \text{for all} \quad t \in [0, T],$$

Therefore in order to satisfy the assumption (3.27), it suffices that

$$T\|\nabla\rho_0\|_{L^a\cap L^\infty} \exp\Big((C_0 + T)\big(e^{C\|\omega_0\|_{L^a\cap L^\infty}T} - 1\big)\Big) \leq \min\big(1, \|\omega_0\|_{L^1\cap L^\infty}\big).$$

Hence, a possible choice for T is given by the formula

$$T \triangleq \frac{C}{\|\omega_0\|_{L^a\cap L^\infty}} \log\Bigg(1 + \frac{\|\omega_0\|_{L^a\cap L^\infty}}{C_0\|\omega_0\|_{L^a\cap L^\infty} + 1} \log\Big(1 + \frac{C\min\big\{\|\omega_0\|_{L^a\cap L^\infty}, \|\omega_0\|^2_{L^a\cap L^\infty}\big\}}{\|\nabla\rho_0\|_{L^\infty}}\Big)\Bigg). \tag{3.28}$$

3.3.3 Existence

The main goal of this paragraph is to answer to the local existence part mentioned in Theorem 3.1. For this aim we shall we consider the following system

$$\begin{cases} \partial_t v_n + v_n \cdot \nabla v_n + \nabla p_n = \rho_n \vec{e}_2, \\ \partial_t \rho_n + v_n \cdot \nabla \rho_n = 0, \\ \operatorname{div} v_n = 0, \\ v_{0,n} = S_n v_0, \quad \rho_{0,n} = S_n \rho_0. \end{cases}$$

where S_n is the usual cut-off in frequency defined in Section 2. Since the initial data $v_{0,n}, \rho_{0,n}$ are smooth and belong to $C^s, s > 1$ then we can apply Chae's result [22] and get for each n a unique local solution $v_n, \rho_n \in C\big([0, T_n^*[, C^s\big)$. The maximal time existence T_n^* obeys to the following blow-up criterion.

$$T_n^\star < \infty \Longrightarrow \int_0^{T_n^*} \|\nabla v_n(\tau)\|_{L^\infty}d\tau = +\infty. \tag{3.29}$$

To get a uniform time existence, that is, $\liminf_{n\to\infty} T_n^\star > 0$ it suffices to check that the time existence $\liminf_{n\to\infty} T_n \geq T$, where T is given by (3.28) and T_n is defined by (3.28) with the smooth data. To do so, it suffices first to check the uniformness of the constant depending on the size of the initial data and we shall see second how to achieve the argument. First, we should bound uniformly the quantities

$$\|\omega_{0,n}\|_{L^a\cap L^\infty}, \|\nabla\rho_{0,n}\|_{L^a\cap L^\infty}, \|\omega_{0,n}\|^\varepsilon_{X_0}, \|\rho_{0,n}\|^{\varepsilon+1}_{X_0}.$$

This follows from the uniform continuity of the operator $S_n : L^p \to L^p$ and by the following estimates stated in pages 62, 63 from [25]

$$\|\partial_{X_{0,\lambda}}\omega_{0,n}\|_{\varepsilon-1} \leq C\big(\|\partial_{X_{0,\lambda}}\omega_0\|_{\varepsilon-1} + \tilde{\|}X_{0,\lambda}\|_\varepsilon\|\omega_0\|_{L^\infty}\big).$$

By the same way we may prove that

$$\|\partial_{X_{0,\lambda}}\rho_{0,n}\|_\varepsilon \leq C\big(\|\partial_{X_{0,\lambda}}\rho_0\|_\varepsilon + \tilde{\|}X_{0,\lambda}\|_\varepsilon\|\nabla\rho_0\|_{L^\infty}\big).$$

To complete the proof of the claim, we assume that for some n we have $T_n^\star \leq T_0$ where T_0 is given by (3.28) , then all the a priori estimates done in the preceding section are justified and therefore we obtain according to the Proposition 3.3

$$\|\nabla v_n(t)\|_{L^\infty} \quad \leq \quad C_0,$$

$$\|\omega_n(t)\|_{L^a\cap L^\infty} + \|\nabla\rho_n(t)\|_{L^a\cap L^\infty} \leq C_0$$

and

$$\|\rho_n(t)\|^{\varepsilon+1}_{X_{t,n}} + \|\omega_n(t)\|^{\varepsilon}_{X_{t,n}} + + \sup_{\lambda\in\Lambda}\|\partial_{X_{0,\lambda}}\psi_n(t)\|_\varepsilon \leq C_0.$$

Where ψ_n is the flow associated to the vector field v_n. This contradicts the blow-up criterion (3.29) and consequently $T_n^* > T_0$. By standard compactness arguments we can show that this family $(v_n, \rho_n)_{n\in\mathbb{N}}$ converges to (v, ρ) which satisfies our initial value problem. We omit here the details and we will next focus on the uniqueness part.

3.3.4 Uniqueness

We shall now focus on the uniqueness part which will be performed in the functions space $\mathcal{X}_{T_0} = L^\infty([0, T_0], L^q \cap W^{1,\infty})$for some $2 < q < \infty$. We point out that this space is larger than the space of the existence part and the restriction to $q > 2$ comes from the fact that the velocity associated to a vortex patch is not in L^2, due to its slow decay at infinity, but belongs to the spaces $L^q, \forall q > 2$. Let (v_1, p_1, ρ_1) and (v_2, p_2, ρ_2) be two solutions of the system (3.1) belonging to the space $\mathcal{X}_{T_0}$ and let us denote by

$$v = v_1 - v_2, \quad p = p_1 - p_2 \quad \text{and} \quad \rho = \rho_1 - \rho_2.$$

Then we have the system

$$\begin{cases} \partial_t v + v_2\cdot\nabla v = -v\cdot\nabla v_1 - \nabla p + \rho\vec{e}_2, \\ \partial_t\rho + v_2\cdot\nabla\rho = -v\cdot\nabla\rho_1, \\ v_{|t=0} = v_0, \quad \rho_{|t=0} = \rho_0. \end{cases}$$

The L^q estimate of the density is given by

$$\|\rho(t)\|_{L^q} \leq \|\rho_0\|_{L^q} + \int_0^t \|v(\tau)\|_{L^q}\|\nabla\rho_1\|_{L^\infty}d\tau. \tag{3.30}$$

Similarly we estimate the velocity as follows,

$$\|v(t)\|_{L^q} \leq \|v_0\|_{L^q} + \int_0^t \big(\|v(\tau)\|_{L^q}\|\nabla v_1\|_{L^\infty} + \|\nabla p(\tau)\|_{L^q} + \|\rho(\tau)\|_{L^q}\big)d\tau. \tag{3.31}$$

But using the incompressibility condition we get

$$\begin{aligned} \nabla p &= \nabla\Delta^{-1}\text{div}\big(-v\cdot\nabla v_1 + \rho\vec{e_2}\big) - \nabla\Delta^{-1}\text{div}(v_2\cdot\nabla v) \\ &= \nabla\Delta^{-1}\text{div}\big(-v\cdot\nabla(v_1+v_2) + \rho e_2\big). \end{aligned}$$

where we have used in the last equality the fact that $\mathrm{div}(v_2\cdot\nabla v)=\mathrm{div}(v\cdot\nabla v_2)$. By the continuity of Riesz transform on L^q we obtain

$$\|\nabla p\|_{L^q}\le C\Big(\|v\|_{L^q}\big(\|\nabla v_1\|_{L^\infty}+\|\nabla v_2\|_{L^\infty}\big)+\|\rho\|_{L^q}\Big).$$

Inserting the last estimate into (3.31) and using the continuity of Riesz transforms one gets

$$\|v(t)\|_{L^q}\le\|v_0\|_{L^q}+C\int_0^t\Big(\|v(\tau)\|_{L^q}\big(\|\nabla v_1(\tau)\|_{L^\infty}+\|\nabla v_2(\tau)\|_{L^\infty}\big)+\|\rho\|_{L^q}\Big)d\tau.$$

Combining the last estimate with (3.30) and using Gronwall inequality we find that for all $t\le T_0$ we have

$$\|(v(t),\rho(t))\|_{L^q}\quad\le\quad\|(v_0,\rho_0)\|_{L^q}e^{Ct}\exp\Big(\int_0^t\big(\|\nabla v_1\|_{L^\infty}+\|\nabla v_2\|_{L^\infty}+\|\nabla\rho_1\|_{L^\infty}\big)d\tau\Big).$$

This achieves the proof of the uniqueness part.

3.4 Singular patches

In this section, we move on to some results concerning singular vortex patches. Our main goal is to prove Theorem 3.2 and enlarge its statement for more general initial data belonging to Yudovich class. To the best of our knowledge, even for the simple case of patches with singular boundary no results on the local well-posedness are known in the literature. In this special case and as it was previously stressed in Theorem 3.2 we must take a density with constant magnitude around the singularity. By this assumption we wish to kill the singularity effects and reduce their violent interaction with the density which is the main obstacle of this problem.

The generalization of Theorem 3.2 will require some specific material that were developed by Chemin in [25]. In this new pattern we assume that the initial boundary contains a singular subset and therefore the vector fields which encode the regularity should vanish close to it. This forces us to work with degenerate vector fields and a cut-off procedure near the singular set becomes necessary. Therefore we shall deal with infinite family of vector fields parametrized by the distance to the singular set and the control of the blowup with respect to this parameter is mostly the main difficulty in this problem.

3.4.1 Preliminaries

We shall introduce and recall some basic definitions and results in connection with singular vortex patches. These tools are mostly introduced in [25] with sufficient details and for the completeness of the manuscript we shall recall them here without any proof.

Definition 3.5 *Let Σ be a closed set of the plane.We denote by $L(\Sigma)$ the set of the functions v such that*

$$\|v\|_{L(\Sigma)}\triangleq\sup_{0<h\le e^{-1}}\frac{\|v\|_{L^\infty(\Sigma_h^c)}}{-\log h}<\infty.$$

In this definition and for the remaing of the paper we shall adopt the following notation : For $h>0$

$$\Sigma_h=\big\{x\in\mathbb{R}^2;\quad\mathrm{dist}(x,\Sigma)\le h\big\}\quad and\quad\Sigma_h^c=\big\{x\in\mathbb{R}^2;\quad\mathrm{dist}(x,\Sigma)\ge h\big\}.$$

Next we introduce log-Lipschitz space which is frequently used in the framework of Yudovich solutions. This space appears in a natural way thanks to the fact the velocity associated to a bounded and integrable vorticity is not in general Lipschitz but belongs to a slight bigger one called log-Lipschitz class.

Definition 3.6 *We denote by LL the space of log-Lipschitz functions, that is the set of bounded functions v in $\mathbb{R}^2 \to \mathbb{R}$ satisfying*

$$\|v\|_{LL} \triangleq \|v\|_{L^\infty} + \sup_{0<|x-y|<1} \frac{|v(x)-v(y)|}{|x-y|\log\frac{e}{|x-y|}} < +\infty.$$

We have the following classical estimate which is a simple consequence of the embedding $B^1_{\infty,\infty} \subset LL$ combined with Bernstein inequality and Biot-Savart law (4.7).

Lemma 3.5 *For any finite $a > 1$ we have*

$$\|v\|_{LL} \leq C\|\omega\|_{L^a\cap L^\infty},$$

with C depending only on a.

It is well-known, thanks to Osgood lemma, that a vector field v belonging to the space LL has a unique global flow map ψ in the class of continuous functions on the space and time variables. This map is defined by the nonlinear integral equation,

$$\psi(t,x) = x + \int_0^t v(\tau, \psi(\tau,x))d\tau \quad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2.$$

For more details about this issue we refer the reader to Section 3.3 in [8].

The next result deals with some general aspect of the dynamics of a given set through the flow associated to a vector field in the LL space. Such result was proved in [25].

Lemma 3.6 *Let A_0 be a subset of $\mathbb{R}^2$ and v be a vector field belonging to $L^1_{loc}(\mathbb{R}_+; LL)$. We denote by $\psi(t)$ the flow associated to this vector field. Then setting $A(t) \triangleq \psi(t, A_0)$ we get,*

$$\psi\big(t, (A_0)_h^c\big) \subset \big(A(t)\big)^c_{\delta_t(h)}, \quad \text{with} \quad \delta_t(h) \triangleq h^{\exp\int_0^t \|v(\tau)\|_{LL}d\tau}.$$

For all $0 \leq \tau \leq t$,

$$\psi\big(\tau, \psi^{-1}\big(t, (A_t)_h^c\big)\big) \subset \big(A(\tau)\big)^c_{\delta_{\tau,t}(h)}, \quad \text{with} \quad \delta_{\tau,t}(h) \triangleq h^{\exp\int_\tau^t \|v(\sigma)\|_{LL}d\sigma}.$$

Next, we discuss the regularity persistence for a transport model and the proof can be found in [25].

Proposition 3.4 *Let $\varepsilon \in (-1,1)$, $a \in (1,+\infty)$ and v be a smooth divergence-free vector field. Set*

$$W(t) \triangleq \Big(\|\nabla v(t)\|_{L(\Sigma_t)} + \|\omega(t)\|_{L^a\cap L^\infty}\Big)\exp\Big(\int_0^t \|v(\tau)\|_{LL}d\tau\Big), \quad \Sigma_t = \psi(t,\Sigma_0).$$

Let $f \in L^\infty_{loc}([0,T], C^\varepsilon)$ be a solution of transport model,

$$\begin{cases} \partial_t f + v\cdot\nabla f = g, \\ f_{|t=0} = f_0, \end{cases}$$

where $g = g_1 + g_2$ is given and belongs to $L^1([0,T];C^\varepsilon)$. We assume that supp $f_0 \subset (\Sigma_0)^c_h$ and supp $g(t) \subset (\Sigma_t)^c_{\delta(t,h)}$ for any $t \in [0,T]$, and for some small h

$$\|g_2(t)\|_\varepsilon \leq -CW(t)\|f(t)\|_\varepsilon \log h.$$

Then the following inequality holds true

$$\|f(t)\|_\varepsilon \leq \|f_0\|_\varepsilon h^{-C\int_0^t W(\tau)d\tau} + \int_0^t h^{-C\int_\tau^t W(\tau')d\tau'}\|g_1(\tau)\|_\varepsilon d\tau.$$

Here the constant C is universal and does not depend on h.

Next, we recall the following definition introduced in [25].

Definition 3.7 *Let Σ be a closed subset of $\mathbb{R}^d$ and $\Xi = (\alpha,\beta,\gamma)$ be a triplet of real numbers. We consider a family $\mathcal{X} = (X_{\lambda,h})_{(\lambda,h)\in\Lambda\times]0,e^{-1}]}$ of vector fields belonging to C^ε as well as their divergences, with $\varepsilon \in]0,1[$ and we denote by $\mathcal{X}_h = (X_{\lambda,h})_{\lambda\in\Lambda}$.*

The familly $\mathcal{X}$ will be said $\Sigma-$admissible of order Ξ if and only if the following properties are satisfied :

$$\forall(\lambda,h) \in \Lambda\times]0,e^{-1}],\ \mathrm{supp}X_{\lambda,h} \subset \Sigma^c_{h^\alpha},$$

$$\inf_{h\in]0,e^{-1}]} h^\gamma I(\Sigma_h,\mathcal{X}_h) > 0,$$

$$\sup_{h\in]0,e^{-1}]} h^{-\beta} N_\varepsilon(\Sigma_h,\mathcal{X}_h) < \infty,$$

where we adopt the following notation : for $\eta \geq h^\alpha$,

$$I(\Sigma_\eta,\mathcal{X}_h) \triangleq \inf_{x\in\Sigma^c_\eta}\sup_{\lambda\in\Lambda}|X_{\lambda,h}(x)| \quad \text{and} \quad N_\varepsilon(\Sigma_\eta,\mathcal{X}_h) \triangleq \sup_{\lambda\in\Lambda}\frac{\widetilde{\|X_{\lambda,h}\|}_\varepsilon}{I(\Sigma_\eta,\mathcal{X}_h)}.$$

Remark 3.7 *Concretely, the family of vector fields $\mathcal{X}$ that we shall work with vanishes near the singular set and therefore we should get $\gamma,\beta < 0$. Moreover the parameter $\alpha > 1$.*

Similarly to the smooth patches we shall introduce for $\eta \geq h^\alpha$,

$$\|u\|^{\varepsilon+k}_{\Sigma_\eta,\mathcal{X}_h} \triangleq N_\varepsilon(\Sigma_\eta,\mathcal{X}_h)\sum_{|\alpha|\leq k}\|\partial^\alpha u\|_{L^\infty} + \sup_{\lambda\in\Lambda}\frac{\|\partial_{X_\lambda}u\|_{\varepsilon+k-1}}{I(\Sigma_\eta,\mathcal{X}_h)}. \tag{3.32}$$

3.4.2 General statement

We intend now to extend the result of Theorem 3.2 and see in turn how to deduce the result of this theorem. The proof of the general statement will be carried out in multiple steps and will be postponed in the next subsections.

Theorem 3.5 *Let $0 < \varepsilon < 1$, $0 < r < e^{-1}$, $1 < a < 2$ and Σ_0 be a closed subset of the plane. Let v_0 be a divergence-free vector field with vorticity ω_0 belonging to $L^a \cap L^\infty$ and ρ_0 be a real-valued function in $W^{1,a} \cap W^{1,\infty}$ and taking constant value on $(\Sigma_0)_r$. Consider $\mathcal{X}_0 = (X_{0,\lambda,h})_{(\lambda,h)\in\Lambda\times]0,e^{-1}]}$ a family of vector fields of class C^ε as well as their divergences and suppose that this family is Σ_0-admissible of order $\Xi_0 = (\alpha_0,\beta_0,\gamma_0)$ such that*

$$\sup_{h\in]0,e^{-1}]} h^{-\beta_0}\|\rho_0\|^{\varepsilon+1}_{(\Sigma_0)_h,(\mathcal{X}_0)_h} + \sup_{h,\in]0,e^{-1}]} h^{-\beta_0}\|\omega_0\|^{\varepsilon}_{(\Sigma_0)_h,(\mathcal{X}_0)_h} < \infty.$$

Then, there exists $T > 0$ such that the Boussinesq system (3.6) *has a unique solution*

$$(\omega, \rho) \in L^\infty\big([0,T], L^a \cap L^\infty\big) \times L^\infty\big([0,T], W^{1,a} \cap W^{1,\infty}\big).$$

In addition, we have

$$\sup_{h \in (0,e^{-1}]} \frac{\|\nabla v(t)\|_{L^\infty((\Sigma(t))^c_h)}}{-\log h} \in L^\infty([0,T]),$$

where $\Sigma(t) = \psi(t, \Sigma_0)$.

• *Proof of Theorem* 3.2. Let us briefly show how this result leads to Theorem 3.2 stated in the Introduction. Let Ω_0 be a bounded open set whose boundary belongs to $C^{\varepsilon+1}$ outside the closed singular set Σ_0. In view of the Definition 3.4 we may show the existence of a neighborhood V_0 of $\partial\Omega_0$ and a real function $f_0 \in C^{\varepsilon+1}$ such that $\partial\Omega_0 = f_0^{-1}(0) \cap V_0$ and whose gradient does not vanish on $V_0 \backslash \Sigma_0$. We also assume that there exists a positive number $\tilde{\gamma}_0 > 0$ such that for all $x \in V_0$,

$$|\nabla f_0(x)| \geq d(x, \Sigma_0)^{\tilde{\gamma}_0}. \tag{3.33}$$

This means that the curves defining the boundary of Ω_0 are not tangent to one another at infinite order at the singular points. Consider $(\theta_h)_{h\in(0,e^{-1}]}$ a family of infinitely differentiable functions, supported in $(\Sigma_0)^c_{h/2}$ and taking the value 1 on the set $(\Sigma_0)^c_h$ and satisfying for all $h \in]0, e^{-1}]$ and any positive real number r,

$$\|\theta_h\|_r \leq C_r h^{-r}.$$

The existence of such functions can be proved by dilation. Consider $\tilde{\alpha}$ a function of class C^∞ supported in V_0 and taking the value 1 on V_1, where V_1 is a neighborhood of $\partial\Omega_0$ such that $V_1 \subset\subset V_0$. We define the family $\mathcal{X}_0 = (X_{0,\lambda,h})_{\lambda\in\{0,1\}, h\in]0,e^{-1}]}$ of vector fields by :

$$X_{0,0,h} = \nabla^\perp(\theta_h f_0), \quad X_{0,1,h} = (1-\tilde{\alpha})\begin{pmatrix}1\\0\end{pmatrix}.$$

It comes to see if the family $\mathcal{X}_0$ is Σ_0-admissible of certain order $\Xi_0 = (\alpha_0, \beta_0, \gamma_0)$. The first vector field is of class C^ε with zero divergence and the second is C^∞. On other hand, by construction, supp $X_{0,i,h} \subset (\Sigma_0)^c_{h/2} \subset (\Sigma_0)^c_{h_0^{\alpha}}$ with $\alpha_0 > 1$. Moreover, thanks to the hypothesis (3.33) we may choose $\gamma_0 = -\tilde{\gamma}_0$. Finally, we easily show that

$$\tilde{\|} X_{0,\lambda,h}\|_\varepsilon \leq C h^{-\varepsilon-1}.$$

Hence, it suffices to take $\beta_0 = \gamma_0 - \varepsilon - 1$ and $\Xi_0 = (\alpha_0, \beta_0, \gamma_0)$. Besides, we have

$$\partial_{X_{0,0,h}}\omega_0 = \theta_h \partial_{\nabla^\perp f_0}\omega_0 + f_0 \partial_{\nabla^\perp \theta_h}\omega_0.$$

First we observe that the derivative of ω_0 in the direction $\nabla^\perp f_0$ is zero and second $\partial_{\nabla^\perp\theta_h}\omega_0$ is a distribution of order zero supported on the boundary $\partial\Omega_0$. As the function f_0 vanishes on $\partial\Omega_0$ then

$$f_0 \partial_{\nabla^\perp \theta_h} 1_{\Omega_0} = 0$$

Thus we deduce that $\partial_{X_{0,0,h}}\omega_0 = 0$. For the second vector field we use that $1 - \tilde{\alpha}$ vanishes on a small neighborhood of $\partial\Omega_0$ and therefore $\partial_{X_{0,1,h}}\omega_0 = 0$. It remains to check the regularity assumption on the density ρ_0. This function is constant in a neighborhood of Σ_0 and thus $\nabla\rho_0 \nabla^\perp\theta_h = 0$ and moreover $\partial_{\nabla^\perp f_0}\rho_0 \in C^\varepsilon$. It is then immediate that $\partial_{X_{0,0,h}}\rho_0 \in C^\varepsilon$. Hence the hypothesis of Theorem 4.2 are satisfied and the local well-posedness result of Theorem 3.2 is now established. To infer that the boundary Ω_t is a curve of class $C^{1+\varepsilon}$ outside $\Sigma(t)$ we argue as in the case of regular vortex patches seen in the previous section.

3.4.3 A priori estimates

This section is devoted to some a priori estimates of L^p type for both the density and the vorticity functions. As the velocity may loose regularity and becomes rough close to the singular set, our assumption to work with constant density near this set seems to be crucial and unavoidable in our analysis. Without this assumption the problem remains open and may be one should expect to propagate the regularity with some loss.

Proposition 3.5 *Let Σ_0 be a closed set of $\mathbb{R}^2$ and (v,ρ) be a smooth solution of the system* (3.1) *defined on the time interval $[0,T]$. We suppose that ρ_0 is constant in the set $(\Sigma_0)_r = \big\{x\in\mathbb{R}^2;\, d(x,\Sigma_0)\leq r\big\}$ for some $r\in(0,e^{-1})$. Then for all $p\in[1,+\infty]$ and for any $t\in[0,T]$ we have*

$$\|\nabla\rho(t)\|_{L^p}\leq\|\nabla\rho_0\|_{L^p}r^{-C\int_0^t W(\tau)d\tau}$$

and

$$\|\omega(t)\|_{L^p}\quad\leq\quad\|\omega_0\|_{L^p}+t\,\|\nabla\rho_0\|_{L^p}r^{-C\int_0^t W(\tau)d\tau},$$

with C an absolute constant.

Proof : In order to prove the first estimate, we apply the partial derivative ∂_j to the second equation of the system (3.1),

$$\partial_t\partial_j\rho+v\cdot\nabla(\partial_j\rho)=\partial_j v\cdot\nabla\rho. \tag{3.34}$$

Hence, for all $1\leq p\leq\infty$ one has

$$\|\partial_j\rho(t)\|_{L^p}\leq\|\partial_j\rho_0\|_{L^p}+\int_0^t\|\partial_j v\cdot\nabla\rho(\tau)\|_{L^p}d\tau. \tag{3.35}$$

Since ρ_0 is transported by the flow ψ,

$$\rho(\tau,x)=\rho_0(\psi^{-1}(t,x)),$$

then $\rho(\tau)$ is constant in $\psi\big(\tau,(\Sigma_0)_r\big)$ and therefore,

$$\text{supp}\;\nabla\rho(\tau)\subset\psi\big(\tau,(\Sigma_0)_r\big)^c=\psi\big(\tau,(\Sigma_0)_r^c\big).$$

Using Lemma 3.6 we get easily

$$\text{supp}\;\nabla\rho(\tau)\subset\big(\Sigma_\tau\big)^c_{\delta_\tau(r)},\qquad\delta_\tau(r)\triangleq r^{\exp\big(\int_0^\tau\|v(\sigma)\|_{LL}d\sigma\big)}.$$

Accordingly we obtain

$$\|\partial_j v\cdot\nabla\rho(\tau)\|_{L^p}\leq\|\nabla v(\tau)\|_{L^\infty\big((\Sigma_\tau)^c_{\delta_\tau(r)}\big)}\|\nabla\rho(\tau)\|_{L^p}.$$

Thus coming back to the Definition 3.5 we may write

$$\begin{aligned}\|\nabla v(\tau)\|_{L^\infty\big((\Sigma_\tau)^c_{\delta_\tau(r)}\big)}&\leq-\|\nabla v(\tau)\|_{L(\Sigma_\tau)}\log\delta_\tau(r)\\&\leq-(\log r)\|\nabla v(\tau)\|_{L(\Sigma_\tau)}\exp\Big(\int_0^\tau\|v(\sigma)\|_{LL}d\sigma\Big)\\&\leq-W(\tau)\log r.\end{aligned}$$

Recall that the function $W(t)$ was introduced in Proposition 3.4. It follows that

$$\|\nabla\rho(t)\|_{L^p} \leq \|\nabla\rho_0\|_{L^p} - C\log r \int_0^t \|\nabla\rho(\tau)\|_{L^p} W(\tau)\, d\tau.$$

By Gronwall lemma we conclude that

$$\|\nabla\rho(t)\|_{L^p} \leq \|\nabla\rho_0\|_{L^p} r^{-C\int_0^t W(\tau)d\tau}.$$

Therefore, in view of the vorticity equation (3.6) we obviously have for all $1 \leq p \leq \infty$,

$$\begin{aligned}\|\omega(t)\|_{L^p} &\leq \|\omega_0\|_{L^p} + \int_0^t \|\nabla\rho(\tau)\|_{L^p} d\tau\\ &\leq \|\omega_0\|_{L^p} + \|\nabla\rho_0\|_{L^p} r^{-C\int_0^t W(\tau)d\tau} t.\end{aligned}$$

This completes the proof of the proposition. □

Now we shall discuss the a priori estimates which are the key of the proof of Theorem 4.2.

Proposition 3.6 *Let $0 < \varepsilon < 1$, $a > 1$ and Σ_0 a closed set of the plane and X_0 be a vector field of class C^ε as well as its divergence and whose support is embedded in $(\Sigma_0)^c_h$. Let (v, ρ) be a smooth solution of the system (3.1) defined on a time interval $[0, T]$ and with the initial data (v_0, ρ_0) such that ρ_0 is constant in the set $(\Sigma_0)_r$. Let X_t be the solution of :*

$$\begin{cases} \big(\partial_t + v\cdot\nabla\big)X_t = \partial_{X_t} v.\\ X_{|t=0} = X_0. \end{cases} \tag{3.36}$$

then we have the estimates,

$$\text{supp}\, X_t \subset \big(\Sigma_t\big)^c_{\delta_t(h)} \quad \text{with} \quad \delta_t(h) \triangleq h^{\exp\left(\int_0^t \|v(\tau)\|_{LL} d\tau\right)}.$$

$$\|\text{div} X_t\|_\varepsilon \leq \|\text{div} X_0\|_{C^\varepsilon} h^{-C\int_0^t W(\tau)d\tau},$$

$$\begin{aligned}\tilde{\|} X_t\|_\varepsilon + \|\partial_{X_t}\omega(t)\|_{\varepsilon-1} &\leq Ce^{Ct}\Big(\tilde{\|} X_0\|_\varepsilon + \|\partial_{X_0}\omega_0\|_{\varepsilon-1} + \|\partial_{X_0}\rho_0\|_\varepsilon\Big) h^{-C\int_0^t W(\tau)d\tau}\\ &\times\ \exp\Big(Ct\|\nabla\rho_0\|_{L^\infty} r^{-C\int_0^t W(\tau)d\tau}\Big).\end{aligned}$$

$$\|\partial_{X_t}\rho(t)\|_\varepsilon \leq \|\partial_{X_t}\rho_0\|_{C^\varepsilon} r^{-C\int_0^t W(\tau)d\tau},$$

where

$$W(t) \triangleq \Big(\|\nabla v(t)\|_{L(\Sigma(t))} + \|\omega(t)\|_{L^a\cap L^\infty}\Big)\exp\Big(\int_0^t \|v(\tau)\|_{LL} d\tau\Big).$$

Proof : The embedding result on the support of X_t can be deduced from Lemma 3.6 and the complete proof can be found in [25]. The second result concerning the estimate of div X_t follows from the equation

$$\big(\partial_t + v\cdot\nabla\big)\text{div}\, X_t = 0,$$

and Proposition 3.4. As to the estimate of $\|X_t\|_\varepsilon$ we shall admit the following assertion and more details see for instance Chapter 9 from [25].

$$\partial_{X_t} v(t) = g_1(t) + g_2(t),$$

with

$$g_1(t) \quad\leq\quad C\|\partial_{X_t}\omega(t)\|_{\varepsilon-1}+C\|\mathrm{div}X_t\|_\varepsilon\|\omega(t)\|_{L^\infty},$$

and

$$g_2(t)\quad\leq\quad -C\|X_t\|_\varepsilon W(t)\log h.$$

Then applying once again Proposition 3.4 to the equation (3.36) we get

$$\begin{aligned}\|X_t\|_{C^\varepsilon}\lesssim \|X_0\|_\varepsilon h^{-C\int_0^t W(\tau)d\tau} &+ \int_0^t \|\mathrm{div}X_\tau\|_\varepsilon\|\omega(\tau)\|_{L^\infty}h^{-C\int_\tau^t W(\tau')d\tau'}d\tau\\ &+ \int_0^t\|\partial_{X_\tau}\omega(\tau)\|_{\varepsilon-1}h^{-C\int_\tau^t W(\tau')d\tau'}d\tau.\end{aligned}$$

According to the definition of the function W we may write

$$\int_0^t\|\omega(\tau)\|_{L^\infty}d\tau\leq h^{-C\int_0^t W(\tau)d\tau}.$$

This together with the estimate of $\|\mathrm{div}X_t\|_{C^\varepsilon}$ yields

$$\tilde{\|}X_t\|_\varepsilon\lesssim\tilde{\|}X_0\|_\varepsilon h^{-C\int_0^t W(\tau)d\tau}+\int_0^t\|\partial_{X_\tau}\omega(\tau)\|_{\varepsilon-1}h^{-C\int_\tau^t W(\tau')d\tau'}d\tau. \tag{3.37}$$

Now applying the operator ∂_{X_t} to the vorticity equation (3.6) gives

$$\big(\partial_t+v\cdot\nabla\big)\partial_{X_t}\omega=\partial_{X_t}\partial_1\rho,$$

then from Proposition 3.4 and taking advantage of the inequality (3.23) we get

$$\begin{aligned}\|\partial_{X_t}\omega(t)\|_{\varepsilon-1}&\lesssim\|\partial_{X_0}\omega_0\|_{\varepsilon-1}h^{-C\int_0^t W(\tau)d\tau}+\int_0^t\|\partial_{X_\tau}\partial_1\rho(\tau)\|_{\varepsilon-1}h^{-C\int_\tau^t W(\tau')d\tau'}d\tau\\ &\lesssim\|\partial_{X_0}\omega_0\|_{\varepsilon-1}h^{-C\int_0^t W(\tau)d\tau}+\int_0^t\|\partial_{X_\tau}\rho(\tau)\|_\varepsilon h^{-C\int_\tau^t W(\tau')d\tau'}d\tau\\ &+\int_0^t\|\nabla\rho(\tau)\|_{L^\infty}\tilde{\|}X_\tau\|_\varepsilon h^{-C\int_\tau^t W(\tau')d\tau'}d\tau.\end{aligned}$$

As it has been shown for smooth patches the scalar function $\partial_{X_t}\rho(t)$ is transported by the flow,

$$(\partial_t+v\cdot\nabla)\partial_{X_t}\rho(t)=0.$$

It is easy to check that $\partial_{X_0}\rho_0$ is supported in $(\Sigma_0)_r^c$ and thus Proposition 3.4 gives

$$\|\partial_{X_\tau}\rho(\tau)\|_\varepsilon\leq\|\partial_{X_0}\rho_0\|_\varepsilon\, r^{-C\int_0^\tau W(\tau')d\tau'}.$$

Hence, we obtain

$$\begin{aligned}\|\partial_{X_t}\omega(t)\|_{\varepsilon-1}&\lesssim\big(\|\partial_{X_0}\omega_0\|_{\varepsilon-1}+\|\partial_{X_0}\rho_0\|_\varepsilon t\big)h^{-C\int_0^t W(\tau)d\tau}\\ &+\int_0^t\|\nabla\rho(\tau)\|_{L^\infty}\tilde{\|}X_\tau\|_\varepsilon h^{-C\int_\tau^t W(\tau')d\tau'}d\tau.\end{aligned}$$

Putting together the preceding estimate and (3.37) we find

$$\Gamma(t)\lesssim\Gamma(0)+\|\partial_{X_0}\rho_0\|_\varepsilon t+\int_0^t\big(\|\nabla\rho(\tau)\|_{L^\infty}+1)\Gamma(\tau)d\tau,$$

where

$$\Gamma(t) \triangleq \big(\|\partial_{X_t}\omega(t)\|_{\varepsilon-1} + \tilde{\|}X_t\|_\varepsilon\big)h^{C\int_0^t W(\tau)d\tau}.$$

So Gronwall lemma ensures that

$$\Gamma(t) \le \big(\Gamma(0) + \|\partial_{X_0}\rho_0\|_\varepsilon\big)e^{C\int_0^t \|\nabla\rho(\tau)\|_{L^\infty}d\tau}e^{Ct}. \tag{3.38}$$

Then Proposition 3.5 completes the proof. □

Now, we have to control the Lipschitz norm of the velocity outside the transported of the singular set Σ_0 by the flow.

Proposition 3.7 *Let $0 < \varepsilon < 1$, $0 < r < e^{-1}$ and $a > 1$ and Σ_0 be a closed set of the plane. Let $\mathcal{X}_0 = \big(X_{0,\lambda,h}\big)_{\lambda\in\Lambda, h\in(0,e^{-1}]}$ be a family vector field which is Σ_0-admissible of order $\Xi_0 = (\alpha,\beta_0,\gamma_0)$ and let (v,ρ) be a smooth solution of the system (3.1) defined on a time interval $[0,T^\star[$. We assume that the initial data satisfy ω_0, $\nabla\rho_0 \in L^a$, ρ_0 is constant in the set $(\Sigma_0)_r$ and*

$$\sup_{h\in]0,e^{-1}]} h^{-\beta_0}\|\rho_0\|^{\varepsilon+1}_{(\Sigma_0)_h,(\mathcal{X}_0)_h} + \sup_{h,\in]0,e^{-1}]} h^{-\beta_0}\|\omega_0\|^{\varepsilon}_{(\Sigma_0)_h,(\mathcal{X}_0)_h} < \infty.$$

Then there exists $0 < T < T^\star$ such that

$$\|\nabla v(t)\|_{L(\Sigma(t))} \in L^\infty\big([0,T]\big).$$

Proof : The dynamical vector fields $\{X_{t,\lambda,h}\}$ are nothing but the transported of the initial family by the flow. They are given by the identity

$$Y_{t,\lambda,h}(x) \triangleq X_{t,\lambda,h}\big(\psi(t,x)\big) = \partial_{X_{0,\lambda,h}}\psi(t,x)$$

and clearly they satisfy

$$\partial_t Y_{t,\lambda,h}(x) = \{\nabla v(t,\psi(t,x))\}\cdot Y_{t,\lambda,h}(x).$$

Fix $t > 0$ and set for $\tau \in [0,t]$, $Z(\tau,x) = Y_{t-\tau,\lambda,h}(x)$, then

$$\partial_\tau Z(\tau,x) = -\{\nabla v(t-\tau,\psi(t-\tau,x))\}\cdot Z(\tau,x).$$

Hence using Gronwall lemma we get

$$\begin{aligned} |Z(t,x)| &\le |Z(0,x)|\, e^{\int_0^t |\nabla v(t-\tau,\psi(t-\tau,x))|d\tau} \\ &\le |Z(0,x)|\, e^{\int_0^t |\nabla v(\tau,\psi(\tau,x))|d\tau}, \end{aligned}$$

which is equivalent to

$$|Y_{0,\lambda,h}(x)| \le |Y_{t,\lambda,h}(x)|e^{\int_0^t |\nabla v(\tau,\psi(\tau,x))|d\tau}.$$

This gives in turn,

$$|X_{0,\lambda,h}(\psi^{-1}(t,x))| \le |X_{t,\lambda,h}(x)|\, e^{\int_0^t |\nabla v(\tau,\psi(\tau,\psi^{-1}(t,x)))|d\tau}.$$

Denoting by δ_t^{-1} the inverse function of δ_t given by the formula,

$$\delta_t^{-1}(h) \triangleq h^{\exp(-\int_0^t \|v(\tau)\|_{LL}d\tau)},$$

the last estimate yields

$$\inf_{x\in(\Sigma_t)^c_{\delta_t^{-1}(h)}}\sup_{\lambda\in\Lambda}|X_{0,\lambda,h}(\psi^{-1}(t,x))| \leq \inf_{x\in(\Sigma_t)^c_{\delta_t^{-1}(h)}}\sup_{\lambda\in\Lambda}|X_{t,\lambda,h}(x)| \times \exp\Big(\int_0^t \big\|\nabla v\big(\tau,\psi(\tau,\psi^{-1}(t,\cdot))\big)\big\|_{L^\infty((\Sigma_t)^c_{\delta_t^{-1}(h)})}d\tau\Big), \tag{3.39}$$

where we recall that $\Sigma_t = \psi(t,\Sigma_0)$ and $(\Sigma_t)^c_\eta = \big\{x\in\mathbb{R}^2;\, d\big(x,\Sigma_t\big)\geq\eta\big\}$.
According to Lemma 3.6 we have

$$\psi^{-1}\big(t,(\Sigma_t)^c_{\delta_t^{-1}(h)}\big)\subset\big(\Sigma_0\big)^c_{\delta_t(\delta_t^{-1}(h))} = (\Sigma_0)^c_h.$$

Then we immediately deduce that

$$\begin{aligned}\inf_{x\in(\Sigma_t)^c_{\delta_t^{-1}(h)}}\sup_{\lambda\in\Lambda}|X_{0,\lambda,h}(\psi^{-1}(t,x))| &= \inf_{y\in\psi^{-1}(t,(\Sigma_t)^c_{\delta_t^{-1}(h)})}\sup_{\lambda\in\Lambda}|X_{0,\lambda,h}(y)|\\ &\geq \inf_{y\in(\Sigma_0)^c_h}\sup_{\lambda\in\Lambda}|X_{0,\lambda,h}(y)|\\ &\geq I\big((\Sigma_0)_h,(\mathcal{X}_0)_h\big).\end{aligned} \tag{3.40}$$

Moreover, in view of the same lemma we have

$$\psi\Big(\tau,\psi^{-1}\big(t,(\Sigma_t)^c_{\delta_t^{-1}(h)}\big)\Big)\subset\big(\Sigma_\tau\big)^c_{\delta_{\tau,t}(\delta_t^{-1}(h))} = \big(\Sigma_\tau\big)^c_{\delta_\tau^{-1}(h)}\subset\big(\Sigma_\tau\big)^c_h.$$

Consequently, we may write

$$\begin{aligned}\big\|\nabla v\big(\tau,\psi(\tau,\psi^{-1}(t,\cdot))\big)\big\|_{L^\infty\big((\Sigma_t)^c_{\delta_t^{-1}(h)}\big)} &\leq \|\nabla v(\tau)\|_{L^\infty((\Sigma_\tau)^c_h)}\\ &\leq -(\log h)\|\nabla v(\tau)\|_{L(\Sigma_\tau)}\\ &\leq -(\log h)W(\tau).\end{aligned}$$

Combining the last estimate with (3.39) and(3.40) we get

$$I\big((\Sigma_t)_{\delta_t^{-1}(h)},(\mathcal{X}(t))_h\big)\geq I\big((\Sigma_0)_h,(\mathcal{X}_0)_h\big)h^{\int_0^t W(\tau)d\tau} \tag{3.41}$$

We introduce

$$\Upsilon(t)\triangleq\|\omega(t)\|_{L^\infty}\tilde{\|}X_{t,\lambda,h}\|_\varepsilon + \|\partial_{X_{t,\lambda,h}}\omega(t)\|_{\varepsilon-1}.$$

Then combining Proposition 3.6 and Proposition 3.5 we get

$$\begin{aligned}\Upsilon(t) &\lesssim e^{Ct}\big(1+\|\omega_0\|_{L^\infty}\big)\Big(\tilde{\|}X_{0,\lambda,h}\|_\varepsilon + \|\partial_{X_{0,\lambda,h}}\omega_0\|_{\varepsilon-1} + \|\partial_{X_{0,\lambda,h}}\rho_0\|_{\varepsilon-1}\Big)\\ &\times \exp\Big(Ct\|\nabla\rho_0\|_{L^\infty}r^{-C\int_0^t W(\tau)d\tau}\Big)h^{-C\alpha_0(\int_0^t W(\tau)d\tau)}.\end{aligned}$$

Hence in view of the Definition 3.7 and (3.41) we immediately deduce that

$$\begin{aligned}\Upsilon(t) &\leq Ce^{Ct}\Big(N_\varepsilon\big((\Sigma_0)_h,(\mathcal{X}_0)_h\big) + \big(1+\|\omega_0\|_{L^\infty}\big)\big(\|\omega_0\|^\varepsilon_{(\Sigma_0)_h,(\mathcal{X}_0)_h} + \|\rho_0\|^{\varepsilon+1}_{(\Sigma_0)_h,(\mathcal{X}_0)_h}\big)\Big)\\ &\quad\times I\big((\Sigma_0)_h,(\mathcal{X}_0)_h\big)h^{-C\alpha_0\int_0^t W(\tau)d\tau}e^{Ct\|\nabla\rho_0\|_{L^\infty}r^{-C\int_0^t W(\tau)d\tau}}\\ &\lesssim \sup_{0<h\leq e^{-1}}h^{-\beta_0}\Big(N_\varepsilon\big((\Sigma_0)_h,(\mathcal{X}_0)_h\big) + \big(1+\|\omega_0\|_{L^\infty}\big)\big(\|\omega_0\|^\varepsilon_{(\Sigma_0)_h,(\mathcal{X}_0)_h} + \|\rho_0\|^{\varepsilon+1}_{(\Sigma_0)_h,(\mathcal{X}_0)_h}\big)\Big)\\ &\quad\times I\big((\Sigma_t)_{\delta_t^{-1}(h)},(\mathcal{X}(t))_h\big)h^{\beta_0-C\alpha_0\int_0^t W(\tau)d\tau}e^{\{Ct\|\nabla\rho_0\|_{L^\infty}r^{-C\int_0^t W(\tau)d\tau}\}}e^{Ct}.\end{aligned}$$

From this estimate and the definition (3.32), one has

$$\|\omega(t)\|^{\varepsilon}_{(\Sigma(t))_{\delta_t^{-1}(h)},(\mathcal{X}(t))_h} \leq C_0 e^{Ct} h^{\beta_0 - C\int_0^t W(\tau)d\tau} e^{\{Ct\|\nabla\rho_0\|_{L^\infty} r^{-C\int_0^t W(\tau)d\tau}\}}. \tag{3.42}$$

Now, we shall combine Theorem 3.3 with the Proposition 3.5 and the monotonicity of the map $x \longmapsto x\log\left(e+\frac{a}{x}\right)$ to get

$$\|\nabla v(t)\|_{L^\infty((\Sigma_t)^c_{\delta_t^{-1}(h)})} \leq C\Big(\|\omega_0\|_{L^a\cap L^\infty} + t\|\nabla\rho_0\|_{L^a\cap L^\infty} r^{-C\int_0^t W(\tau)d\tau}\Big) \times \log\left(e + \frac{\|\omega\|^{\varepsilon}_{(\Sigma_t)_{\delta_t^{-1}(h)},(\mathcal{X}(t))_h}}{\|\omega_0\|_{L^\infty}}\right).$$

So according to the estimate (3.42), we find

$$\|\nabla v(t)\|_{L^\infty((\Sigma_t)^c_{\delta_t^{-1}(h)})} \leq C\Big(\|\omega_0\|_{L^1\cap L^\infty} + t\|\nabla\rho_0\|_{L^a\cap L^\infty} r^{-C\int_0^t W(\tau)d\tau}\Big) \times \Big(C_0 + t + t\|\nabla\rho_0\|_{L^a\cap L^\infty} r^{-C\int_0^t W(\tau)d\tau} + \Big(\beta_0 - C\int_0^t W(\tau)d\tau\Big)\log h\Big).$$

It follows that

$$\frac{\|\nabla v(t)\|_{L^\infty((\Sigma_t)^c_{\delta_t^{-1}(h)})}}{-\log\delta_t^{-1}(h)} \leq C\Big(\|\omega_0\|_{L^1\cap L^\infty} + t\|\nabla\rho_0\|_{L^a\cap L^\infty} r^{-C\int_0^t W(\tau)d\tau}\Big) \times \Big(C_0 + t + t\|\nabla\rho_0\|_{L^\infty} r^{-C\int_0^t W(\tau)d\tau} + \int_0^t W(\tau)d\tau\Big) e^{\int_0^t \|v(\tau)\|_{LL}d\tau}.$$

By the definition of $W(t)$ introduced in the Proposition 3.4 we have

$$\begin{aligned} W(t) &\leq C\Big(\|\omega_0\|_{L^a\cap L^\infty} + \|\nabla\rho_0\|_{L^a\cap L^\infty} r^{-C\int_0^t W(\tau)d\tau} t\Big) \\ &\times \Big(C_0 + t + \|\nabla\rho_0\|_{L^\infty} r^{-C\int_0^t W(\tau)d\tau} t + \int_0^t W(\tau)d\tau\Big) e^{2\int_0^t \|v(\tau)\|_{LL}d\tau}. \end{aligned}$$

We choose T such that

$$T\|\nabla\rho_0\|_{L^a\cap L^\infty} r^{-C\int_0^T W(\tau)d\tau} \leq \min\big(1, \|\omega_0\|_{L^1\cap L^\infty}\big). \tag{3.43}$$

From Lemma 3.5 and Proposition 3.5 we get for $t \in [0,T]$

$$\begin{aligned} \|v(t)\|_{LL} &\leq \|\omega(t)\|_{L^a\cap L^\infty} \\ &\leq \|\omega_0\|_{L^a\cap L^\infty} + \|\nabla\rho_0\|_{L^a\cap L^\infty} r^{-C\int_0^t W(\tau)d\tau} t \\ &\leq 2\|\omega_0\|_{L^a\cap L^\infty}. \end{aligned} \tag{3.44}$$

Then,

$$W(t) \leq C\|\omega_0\|_{L^1\cap L^\infty}\Big(C_0 + t + \int_0^t W(\tau)d\tau\Big) e^{C\|\omega_0\|_{L^a\cap L^\infty} t}.$$

Therefore by using to Gronwall lemma we conclude that for $t \in [0,T]$

$$W(t) \leq C\|\omega_0\|_{L^a\cap L^\infty}(C_0 + t) e^{C\|\omega_0\|_{L^a\cap L^\infty} t} \exp\Big(e^{C\|\omega_0\|_{L^1\cap L^\infty} t}\Big).$$

It follows that

$$\int_0^t W(\tau)d\tau \leq (C_0 + t)\exp\Big(e^{C\|\omega_0\|_{L^a\cap L^\infty} t}\Big).$$

Then,

$$r^{-\int_0^t W(\tau)d\tau} \leq r^{-(C_0+t)\exp\left(e^{Ct\|\omega_0\|_{L^a\cap L^\infty}}\right)}.$$

Hence in order to ensure the assumption (3.43) it suffices to impose,

$$T\|\nabla\rho_0\|_{L^\infty} r^{-(C_0+T)\exp\left(e^{CT\|\omega_0\|_{L^a\cap L^\infty}}\right)} = \min\bigl(1, \|\omega_0\|_{L^1\cap L^\infty}\bigr).$$

The existence of such $T > 0$ can be justified by a continuity argument and this ends the proof of the proposition. □

3.4.4 Existence

The existence part can be done in a similar way to the case of smooth patches by smoothing out the initial data. However we should be careful about this procedure which must preserve the imposed geometric structure. Especially we have seen in the a priori estimates that a constant density close to the singular set is a crucial fact and thereby this must be satisfied for the smooth approximation of the density. Hence we smooth the initial velocity as before by setting $v_{0,n} = S_n v_0$ but for the density we have to choose a compactly supported mollifiers. More precisely, we take $\phi \in C_0^\infty(\mathbb{R}^2)$ a positive function supported in the ball of center 0 and radius 1 and of integral 1 over $\mathbb{R}^2$. We denote by $(\phi_n)_{n\in\mathbb{N}}$ the usual mollifiers :

$$\phi_n(x) = n^2\phi(nx)$$

and we set

$$\rho_{0,n} = \phi_n * \rho_0.$$

Then the following uniform bounds hold true,

$$\|\rho_{0,n}\|_{L^2} \leq \|\nabla\rho_0\|_{L^2}, \quad \|\nabla\rho_{0,n}\|_{L^1\cap L^\infty} \leq \|\nabla\rho_0\|_{L^1\cap L^\infty}$$

and

$$\|\partial_{X_{0,\lambda}}\rho_{0,n}\|_\varepsilon \leq \|\partial_{X_{0,\lambda}}\rho_0\|_\varepsilon + \tilde{\|}X_{0,\lambda}\|_\varepsilon\|\nabla\rho_0\|_{L^\infty}.$$

The first two estimates are easy to get by using the classical properties of the convolution laws. The proof of the last estimate is obtained by writing the identity

$$\partial_{X_{0,\lambda}}\rho_{0,n} = \phi_n * (\partial_{X_{0,\lambda}}\rho_0) + [\partial_{X_{0,\lambda}}, \phi_n *]\rho_0,$$

where we use the notation $[A, B*]f = A(B * f) - B * (Af)$. We can easily check that the first term is uniformly bounded in C^ε, as to the second one we use Proposition 3.8 which gives

$$\|[\partial_{X_{0,\lambda}}, \phi_n *]\rho_0\|_{C^\varepsilon} \leq C\|X_{0,\lambda}\|_{C^\varepsilon}\|\nabla\rho_0\|_{L^\infty}.$$

It remains to show that $\rho_{0,n}$ is constant in small neighborhood of Σ_0. For this aim we consider the set

$$(\Sigma_0)_{r-\frac{1}{n}} \triangleq \Bigl\{x \in \mathbb{R}^2;\, d(x, \Sigma_0) \leq r - \frac{1}{n}\Bigr\}.$$

We can easily check according to the support property of the convolution that $\rho_{0,n}$ is constant in the set $\Sigma^0_{r-\frac{1}{n}}$ which contains $(\Sigma_0)_{r/2}$ for n big enough. The remaining of the proof is similar to the one of the Theorem 3.1 and we omit here the details.

3.4.5 Uniqueness

It seems that the uniqueness argument performed in the smooth patches cannot be easily extended to singular patches because the velocity is not Lipschitz. To avoid this difficulty we shall use the original argument of Yudovich [128]. First let us observe according to [25] that the velocity does not necessary belong to L^2 but to an affine space of type $\sigma + L^2$. The vector field σ is a stationary solution for Euler equations and can be constructed as follows : let g be a radial function in C_0^∞ supported away from the origin and set,

$$\sigma(x) = \frac{x^\perp}{|x|^2}\int_0^{|x|} rg(r)dr. \tag{3.45}$$

Such σ is a smooth stationary solutions of the incompressible Euler system,

$$\partial_t\sigma = \mathbb{P}(\sigma\cdot\nabla\sigma) = 0$$

where $\mathbb{P} \triangleq \Delta^{-1}$div is Leray's projector onto divergence-free vector fields. It behaves like $1/|x|$ at infinity and $\nabla\sigma$ belongs to $H^s(\mathbb{R}^2)$ for all $r\in\mathbb{R}$. For a vortex patch we can show that its velocity given by Biot-Savart law belongs to some $\sigma + L^2$.
Let us state the following lemma

Lemma 3.7 *Let σ be a stationary vector field satisfying* (3.45) *and (v_0,ρ_0) be a smooth initial data belonging to $(\sigma+L^2)\times L^2$. Then any local solution $(v(t),\rho(t))$ of the system* (3.1) *associated to the initial data (v_0,ρ_0) belongs to $(\sigma+L^2)\times L^2$*

Proof : Setting $v = u+\sigma$ then the system(3.1) can be written in the form,

$$\begin{cases} \partial_t u + (u+\sigma)\cdot\nabla u = -u\cdot\nabla\sigma - \nabla p + \rho\vec{e}_2, \\ \partial_t\rho + (u+\sigma)\cdot\nabla\rho = 0, \\ u_{|t=0} = v_0+\sigma, \quad \rho_{|t=0} = \rho_0. \end{cases} \tag{3.46}$$

Since div$u =$ div$\sigma = 0$, then we have the following L^2 estimates

$$\|u(t)\|_{L^2} \le \|u_0\|_{L^2} + \|\rho_0\|_{L^2}t + \|\nabla\sigma\|_{L^\infty}\int_0^t \|u(\tau)\|_{L^2}d\tau.$$

By Gronwall inequality we conclude that

$$\|u(t)\|_{L^2} \le \big(\|u_0\|_{L^2} + \|\rho_0\|_{L^2}t\big)e^{\|\nabla\sigma\|_{L^\infty}t}.$$

This conclude the proof of the lemma. □

Now we shall prove the uniqueness part. As we have already seen the velocity belongs to $\sigma + L^2$ and the uniqueness in the space $L^\infty_T L^2$ should be done by using the formulation (3.46). However for the clarity of the proof we shall assume that $\sigma = 0$ and the proof works for non trivial σ as well. Let (v_1,ρ_1,p_1) and (v_2,ρ_2,p_2) two solutions of the system (3.1) with the same initial data. We notice that $(v,\rho,p) \triangleq (v_2-v_1,\rho_2-\rho_1,p_2-p_1)$ satisfies

$$\begin{cases} \partial_t v + v_2\cdot\nabla v = -v\cdot\nabla v_1 - \nabla p + \rho\vec{e}_2, \\ \partial_t\rho + v_2\cdot\nabla\rho = -v\cdot\nabla\rho_1, \\ v_{|t=0} = v_0, \quad \rho_{|t=0} = \rho_0. \end{cases} \tag{3.47}$$

A standard energy method with Hölder inequality yield for all $q \in [a, +\infty[$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 &\leq \|\nabla v_1(t)\|_{L^q}\|v(t)\|_{L^{2q'}}^2 + \|\rho(t)\|_{L^2}\|v(t)\|_{L^2}\\ &\lesssim q\|\omega_1(t)\|_{L^a\cap L^\infty}\|v(t)\|_{L^\infty}^{\frac{2}{q}}\|v(t)\|_{L^2}^{\frac{2}{q'}} + \|\rho(t)\|_{L^2}\|v\|_{L^2}.\end{aligned}$$

with $q' = \frac{q}{q-1}$ and

$$\frac{1}{2}\frac{d}{dt}\|\rho(t)\|_{L^2}^2 \leq \|\nabla\rho_1(t)\|_{L^\infty}\|v(t)\|_{L^2}\|\rho(t)\|_{L^2}.$$

Let η be a small parameter and set,

$$\Gamma_\eta(t) \triangleq \sqrt{\|\rho(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 + \eta}.$$

Then,

$$\frac{d}{dt}\Gamma_\eta(t) \leq Cq\|\omega_1(t)\|_{L^a\cap L^\infty}\|v(t)\|_{L^\infty}^{\frac{2}{q}}\Gamma_\eta(t)^{1-\frac{2}{q}} + (1 + \|\nabla\rho_1(t)\|_{L^\infty})\Gamma_\eta(t).$$

Setting

$$\Upsilon_\eta(t) \triangleq e^{-\int_0^t(1+\|\nabla\rho_1(\tau)\|_{L^\infty})d\tau}\Gamma_\eta(t),$$

we obtain

$$\frac{d}{dt}\Upsilon_\eta(t) \leq Cq\|\omega_1(t)\|_{L^a\cap L^\infty}\|v(t)\|_{L^\infty}^{\frac{2}{q}}\Upsilon_\eta(t)^{1-\frac{2}{q}}e^{-\frac{2}{q}\int_0^t(1+\|\nabla\rho_1(\tau)\|_{L^\infty})d\tau}$$

which gives

$$\frac{2}{q}\Upsilon_\eta(t)^{\frac{2}{q}-1}\frac{d}{dt}\Upsilon_\eta(t) \leq C\|\omega_1(t)\|_{L^a\cap L^\infty}\|v(t)\|_{L^\infty}^{\frac{2}{q}}.$$

Integrating in time we get

$$\Upsilon_\eta(t) \leq \Big(\eta^{\frac{1}{q}} + C\int_0^t \|\omega_1(\tau)\|_{L^a\cap L^\infty}\|v(\tau)\|_{L^\infty}^{\frac{2}{q}}d\tau\Big)^{\frac{q}{2}}.$$

Letting η go to 0 leads

$$\|\rho(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 \leq \|v(t)\|_{L^\infty_{T_0}L^\infty}^2\Big(C\int_0^t \|\omega_1(\tau)\|_{L^a\cap L^\infty}d\tau\Big)^q.$$

Then, from the Biot-Savart law we have for $a \in (1, 2)$

$$\begin{aligned}\|\rho(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 &\leq \|\omega(t)\|_{L^\infty_{T_0}(L^a\cap L^\infty}^2\Big(C\int_0^t \|\omega_1(\tau)\|_{L^a\cap L^\infty}d\tau\Big)^q\\ &\leq C_0\Big(C\int_0^t \|\omega_1(\tau)\|_{L^a\cap L^\infty}d\tau\Big)^q.\end{aligned}$$

Therefore, We may find T such that $\int_0^T \|\omega_1(\tau)\|_{L^a\cap L^\infty}d\tau < \frac{1}{C}$. Letting first q tend to $+\infty$ and using bootstrap arguments we can conclude that $(v, \rho) \equiv 0$ on $[0, T_0]$

3.5 Appendix

Proposition 3.8 *Given* $0 < \varepsilon < 1$*,* X *a vector field belonging to* C^ε *and* $f \in Lip(\mathbb{R}^2)$*. Let* $\phi \in C_0^\infty(\mathbb{R}^2)$ *be a positive function supported in the ball of center* 0 *and radius* 1 *and such that* $\int_{\mathbb{R}^2} \phi(x)dx = 1$*. For all* $n \in \mathbb{N}$ *we set*

$$\phi_n(x) = n^2\phi(nx),$$

and we denote by R_n *the convolution operator with the function* ϕ_n*. Then we have the following estimate*

$$\|[\partial_X, R_n]f\|_{C^\varepsilon} \leq C\|X\|_\varepsilon \|\nabla f\|_{L^\infty}.$$

Proof : By definition we write

$$[\partial_X, R_n]f(x) = \int_{\mathbb{R}^2} \phi_n(x-y)\big(X(x) - X(y)\big)\nabla f(y)dy.$$

Then for all $x_1, x_2 \in \mathbb{R}^2$ such that $|x_1 - x_2| < 1$ one has

$$[\partial_X, R_n]f(x_1) - [\partial_X, R_n]f(x_2) = \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} = \int_{\mathbb{R}^2} \phi_n(x_1 - y)\big(X(x_1) - X(x_2)\big)\nabla f(y)dy$$

$$\mathrm{II} = \int_{\mathbb{R}^2} \big(\phi_n(x_1 - y) - \phi_n(x_2 - y)\big)\big(X(x_2) - X(y)\big)\nabla f(y)dy.$$

By straightforward computations we get,

$$|\mathrm{I}| \leq \|\nabla f\|_{L^\infty}\|\phi\|_{L^1}|x_1 - x_2|^\varepsilon \|X\|_{C^\varepsilon}.$$

As ϕ is supported in the ball of center 0 and radius $\frac{1}{n}$ then we may write

$$\begin{aligned} |\mathrm{II}| &\leq \|\nabla f\|_{L^\infty} \int_{B(x_1,\frac{1}{n})\cup B(x_2,\frac{1}{n})} |\phi_n(x_1 - y) - \phi_n(x_2 - y)||X(x_2) - X(y)|dy \\ &\leq \|\nabla f\|_{L^\infty}\big(\mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3\big). \end{aligned}$$

with

$$\mathrm{II}_1 = \int_{B(x_1,\frac{1}{n})} |\phi_n(x_1 - y) - \phi_n(x_2 - y)||X(x_2) - X(x_1)|dy$$

$$\mathrm{II}_2 = \int_{B(x_1,\frac{1}{n})} |\phi_n(x_1 - y) - \phi_n(x_2 - y)||X(x_1) - X(y)|dy$$

and

$$\mathrm{II}_3 = \int_{B(x_2,\frac{1}{n})} |\phi_n(x_1 - y) - \phi_n(x_2 - y)||X(x_2) - X(y)|dy.$$

The term II_1 can be treated by the same way as I. For term II_2 we have

$$\begin{aligned} \mathrm{II}_2 &\leq n^{2+\varepsilon}\|\phi\|_\varepsilon \|X\|_\varepsilon \int_{B(x_1,\frac{1}{n})} |x_1 - y|^\varepsilon dy \\ &\leq \|\phi\|_\varepsilon \|X\|_\varepsilon. \end{aligned}$$

The bound of and II_3 is done similarly. □

Troisième partie

Incompressible limit

Chapitre 4

On the 2D isentropic Euler system with unbounded initial vorticity

This chapter is the subject of the following publication :
Z. Hassainia, *On the 2D isentropic Euler system with unbounded initial vorticity*. J. DifferentialEquations 259 (2015) 264–317.

Abstract. This chapter is devoted to the study of the low Mach number limit for the 2D isentropic Euler system associated to ill-prepared initial data with slow blow up rate on $\log \varepsilon^{-1}$. We prove in particular the strong convergence to the solution of the incompressible Euler system when the vorticity belongs to some weighted BMO spaces allowing unbounded functions. The proof is based on the extension of the result of [12] to a compressible transport model.

4.1 Introduction

The equations of motion governing a perfect compressible fluid evolving in the whole space $\mathbb{R}^2$ are given by Euler system :

$$\begin{cases} \rho(\partial_t v + v \cdot \nabla v) + \nabla p = 0, \, t \geq 0, \, x \in \mathbb{R}^2 \\ \partial_t \rho + \text{div}\,(\rho v) = 0, \\ (v, \rho)_{|t=0} = (v_0, \rho_0). \end{cases}$$

Here, the vector field $v = (v_1, v_2)$ describes the velocity of the fluid particles and the scalar functions p and $\rho > 0$ stand for the pressure and the density, respectively. From now onwards, we shall be concerned only with the isentropic case corresponding to the law

$$p = \rho^\gamma,$$

where the parameter $\gamma > 1$ is the adiabatic exponent.

Following the idea of Kawashima, Makino and Ukai [90], this system can be symmetrized by using the sound speed c defined by

$$c = 2\frac{\sqrt{\gamma}}{\gamma - 1}\rho^{\frac{\gamma-1}{2}}.$$

The main scope of this chapter is to deal with the weakly compressible fluid and particularly we intend to get a lower bound for the lifespan and justify the convergence towards the incompressible system. But before reviewing the state of the art and giving a precise statement of our main result we shall briefly describe the way how to get formally the weakly compressible fluid. In broad terms, the basic idea consists in writing the foregoing system around the equilibrium state $(0, c_0)$: let $\varepsilon > 0$ be a small parameter called the Mach number and set

$$v(t,x) = \bar{\gamma} c_0 \varepsilon v_\varepsilon(\varepsilon \bar{\gamma} c_0 t, x) \quad \text{and} \quad c(t,x) = c_0 + \bar{\gamma} c_0 \varepsilon c_\varepsilon(\varepsilon \bar{\gamma} c_0 t, x) \quad \text{with} \quad \bar{\gamma} = \frac{\gamma - 1}{2}.$$

Then the resulting system will be the following

$$\begin{cases} \partial_t v_\varepsilon + v_\varepsilon \cdot \nabla v_\varepsilon + \frac{1}{\varepsilon} \nabla c_\varepsilon + \bar{\gamma} c_\varepsilon \nabla c_\varepsilon = 0, \\ \partial_t c_\varepsilon + v_\varepsilon \cdot \nabla c_\varepsilon + \frac{1}{\varepsilon} \mathrm{div}\, v_\varepsilon + \bar{\gamma} c_\varepsilon \mathrm{div}\, v_\varepsilon = 0, \\ (v_\varepsilon, c_\varepsilon)_{|t=0} = (v_{0,\varepsilon}, c_{0,\varepsilon}). \end{cases} \tag{E.C}$$

As we can easily observe this system contains singular terms in ε that might affect dramatically the dynamics when the Mach number is close to zero. For more details about the derivation of the above model we invite the interested reader to consult the papers [68, 78, 41] and the references therein.

From mathematical point of view this system has been intensively investigated in the few last decades. One of the basic problems is the construction of the solutions $(v_\varepsilon, c_\varepsilon)$ in suitable function spaces with a non-degenerate time existence and most importantly the asymptotic behavior for small Mach number. Formally, one expects the velocity v_ε to converge to v the solution of the incompressible Euler system given by

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = 0, \\ \mathrm{div}\, v = 0, \\ v_{|t=0} = v_0. \end{cases} \tag{E.I}$$

As a matter of fact, the singular parts are antisymmetric and do not contribute in the energy estimates built over Sobolev spaces H^s. Accordingly, a uniform time existence can be shown by using just the theory of hyperbolic systems, see [77]. However it is by no means obvious that the constructed solutions will converge to the expected incompressible Euler solution and the problem can be highly non-trivial when it is coupled with the geometry of the domain, a fact that we ignore here. In most of the papers dealing with this recurrent subject there are essentially two kinds of hypothesis on the initial data : the first one concerns the well-prepared case where the initial data are assumed to be slightly compressible meaning that $(\mathrm{div}\, v_{0,\varepsilon}, \nabla c_{0,\varepsilon}) = O(\varepsilon)$ for ε close to zero. In this context it can be proved that the time derivative of the solutions is uniformly bounded and therefore the justification of the incompressible limit follows from Aubin-Lions compactness lemma. For a complete discussion we refer the reader to the papers of Klainerman and Majda [77, 78]. The second class of initial data is the ill-prepared case where the family $(v_{0,\varepsilon}, c_{0,\varepsilon})_\varepsilon$ is assumed to be bounded in Sobolev spaces H^s with $s > 2$ and the incompressible parts of $(v_{0,\varepsilon})_\varepsilon$ converge strongly to some divergence-free vector field v_0 in L^2. In this framework, the main difficulty that one has to face, as regards the incompressible limit, is the propagation of the time derivative $\partial_t v_\varepsilon$ with the speed ε^{-1}, a phenomenon which does not occur in the case of the well-prepared data. To deal with this trouble, Ukai used in [117] the dispersive effects generated by the acoustic waves in order to prove that the compressible part of the velocity and the acoustic term vanish when ε goes to zero. Similar studies but in more complex situations and for various models were accomplished later in different works and for the convenient of the reader we quote here a short list of references [2, 6, 45, 58, 68, 82, 83, 95].

Regarding the lifespan of these solutions, it is well-known that in contrast to the incompressible case where the classical solutions are global in dimension two, the compressible Euler system (E.C) may develop singularities in finite time for some smooth initial data. This was shown in space dimension two by Rammaha [106], and by Sideris [113] for dimension three. It seems that in dimension two we can generically get a lower bound for the lifespan T_ε that goes to infinity for small ε. More precisely, when the initial data are bounded in H^s with $s > 2$ then by taking benefit of the vorticity structure coupled with Strichartz estimates we get,

$$T_\varepsilon \geq C \log\log \varepsilon^{-1}.$$

Besides, we can get precise information on the lifespans when the initial data enjoy some specific structures. In fact, Alinhac [3] showed that in two-dimensional space and for axisymmetric data the lifespan is equivalent to ε^{-1}. Also, for the three-dimensional system, Sideris [114] proved the almost global existence of the solution for potential flows. In other words, it was shown that the lifespan T_ε is bounded below by $\exp(c/\varepsilon)$. To end this short discussion we mention that global existence results were obtained in [48, 111] for some restrictive initial data.

Recently the incompressible limit to (E.C) for ill-prepared initial data lying to the critical Besov space $B^2_{2,1}$ was carried out in [68]. It was also shown that the strong convergence occurs in the space of the initial data. The same program was equally accomplished in dimension three in [58] for the axisymmetric initial data. The fact that the regularity is optimal for the incompressible system will contribute with much more technical difficulties and unfortunately the perturbation theory cannot be easily adapted. In these studies, the geometry of the vorticity is of crucial importance.

In the contributions cited before, the velocity should be in the Lipschitz class uniformly with respect to ε. This constraint was slightly relaxed in [41] by allowing the initial data to be so ill-prepared in order to permit Yudovich solutions for the incompressible system. Recall that these latter solutions are constructed globally in time for (E.I) when the initial vorticity ω_0 belongs to $L^1 \cap L^\infty$, see [128]. In the incompressible framework the vorticity ω, defined for a vector field $v = (v_1, v_2)$ by $\omega = \partial_1 v_2 - \partial_2 v_1$, is advected by the flow,

$$\partial_t \omega + v \cdot \nabla \omega = 0, \quad \Delta v = \nabla^\perp \omega. \tag{4.1}$$

Working in larger spaces than the Yudovich's one for the system (4.1) and peculiarly with unbounded vorticity, possibly without uniqueness, is not in general an easy task and often leads to more technical complications. Nevertheless, in the last decade slight progress were done and we shall here comment only some of them which fit with the scope of this paper. For a complete list of references we invite the reader to check the papers [35, 37, 129]. One of the basic result in this subject is due to Vishik in [120] who gave various results when the vorticity belongs to the class B_Γ : a kind of functional space characterized by the slow growth of the partial sum built over the dyadic Fourier blocks. The results of Vishik which cover global and local existence with or without uniqueness depending on some analytic properties of Γ suffer from one inconvenient : the persistence regularity is not proved and an instantaneous loss of regularity may happen.

Recently, Bernicot and Keraani proved in [12] the global existence and uniqueness without any loss of regularity for the incompressible Euler system when the initial vorticity is taken in a weighted BMO space called $LBMO$ and denotes the set of functions with $\log$-*bounded mean oscillations*. This space is strictly larger than L^∞ and smaller than the usual BMO space.

The main task of this paper is to conduct the incompressible limit study for (E.C) when the limiting system (E.I) is posed for initial data lying in the $LBMO$ space. As we shall discuss later we will be also able to generalize the result of [12] for more general spaces. To give a clear statement we need to introduce the $LBMO$ space and a precise discussion will be found in the next section. First, take $f:\mathbb{R}^2\to\mathbb{R}$ be a locally integrable function. We say that f belongs to BMO space if

$$\|f\|_{BMO}\triangleq\sup_{B\,ball}\fint_B\Big|f-\fint_B f\Big|.$$

Second, we say that f belongs to the space $LBMO$ if

$$\|f\|_{BMO_F}\triangleq\|f\|_{BMO}+\sup_{2B_2\subset B_1}\frac{\Big|\fint_{B_2}f-\fint_{B_1}f\Big|}{1+\ln\Big|\frac{\ln r_2}{\ln r_1}\Big|}<+\infty,$$

where the supremum is taken over all the pairs of balls $B_2=B(x_2,r_2)$ and $B_1=B(x_1,r_1)$ in $\mathbb{R}^2$ with $0<r_1\le\frac12$. We have used the notation $\fint_B f$ to refer to the average $\frac{1}{|B|}\int_B f(x)\,dx$.

Next we shall state our main result in the special case of $LBMO$ space and whose extension will be given in Theorem 4.4.

Theorem 4.1 *Let $s,\alpha\in]0,1[$ and $p\in]1,2[$. Consider a family of initial data $(v_{0,\varepsilon},c_{0,\varepsilon})_{0<\varepsilon<1}$ such that there exists a constant $C>0$ which does not depend on ε and verifying*

$$\|(v_{0,\varepsilon},c_{0,\varepsilon})\|_{H^{s+2}}\le C(\log\varepsilon^{-1})^{\alpha},$$

$$\|\omega_{0,\varepsilon}\|_{L^p\cap LBMO}\le C.$$

Then, the system (E.C) admits a unique solution $(v_\varepsilon,c_\varepsilon)\in C([0,T_\varepsilon[;H^{s+2})$ with the following properties :

1. *The lifespan T_ε of the solution satisfies the lower bound :*

$$T_\varepsilon\geqslant\log\log\log\varepsilon^{-1}\triangleq\tilde T_\varepsilon,$$

and for all $t\le\tilde T_\varepsilon$ we have

$$\|\omega_\varepsilon(t)\|_{LBMO\cap L^p}\quad\le\quad C_0e^{C_0t}.\tag{4.2}$$

Moreover, the compressible and acoustic parts of the solutions converge to zero :

$$\lim_{\varepsilon\to0}\|(\operatorname{div}v_\varepsilon,\nabla c_\varepsilon)\|_{L^1_{\tilde T_\varepsilon}L^\infty}=0.$$

2. *Assume in addition that $\lim_{\varepsilon\to0}\|\omega_{0,\varepsilon}-\omega_0\|_{L^p}=0$, for some vorticity $\omega_0\in LBMO\cap L^p$. associated to a divergence-free vector field v_0. Then the vortices $(\omega_\varepsilon)_\varepsilon$ converge strongly to the weak solution ω of (4.1) associated to the initial data ω_0 : for all $t\in\mathbb{R}_+$ we have*

$$\lim_{\varepsilon\to0}\|\omega_\varepsilon(t)-\omega(t)\|_{L^q}=0,\quad\forall q\in[p,+\infty[,\tag{4.3}$$

and

$$\|\omega(t)\|_{LBMO\cap L^p}\quad\le\quad C_0e^{C_0t}.\tag{4.4}$$

The constant C_0 depends only on the size of the initial data and does not depend on ε.

Before giving a brief account of the proof, we shall summarize some comments in order to clarify some points in the theorem.

Remark 4.1 *1. The estimate* (4.3) *can be translated to the velocity according to the Biot-Savart law* (4.7) *as follows*

$$\lim_{\varepsilon\to 0}\|\mathbb{P}v_\varepsilon - v\|_{L^\infty_t W^{1,r}} = 0 \quad \forall r\in\Big[\frac{2p}{2-p},+\infty\Big[.$$

where $\mathbb{P}v_\varepsilon = v_\varepsilon - \nabla\Delta^{-1}\mathrm{div}\, v_\varepsilon$ *denotes the Leray's projector over solenoidal vector fields.*

2. *We can generically construct a family* $(v_{0,\varepsilon})$ *satisfying the assumptions of Theorem* 4.1. *In fact, let* v_0 *be a divergence-free vector field with* $\omega_0 = \mathrm{curl}\, v_0 \in L^p\cap LBMO$. *Take two functions* $\chi,\rho\in C_0^\infty(\mathbb{R}^2)$ *with*

$$\rho\geq 0 \quad \textit{and} \quad \int_{\mathbb{R}^2}\rho(x)dx = 1.$$

Denote by $(\rho_k)_{k\in\mathbb{N}^*}$ *the usual mollifiers :*

$$\rho_k(x)\triangleq k^2\rho(kx).$$

For $R>0$, *we set*

$$v_{0,\varepsilon} = \rho_k * \Big(\chi(\frac{\cdot}{R})v_0\Big).$$

By the convolution laws we obtain

$$\|v_{0,\varepsilon}\|_{H^{s+2}} \quad\leq\quad Ck^{s+2}R\|v_0\|_{L^\infty}.$$

We choose carefully k *and* R *with slight growth with respect to* ε *in order to get*

$$k^{s+2}R\leq C_0(\ln\varepsilon^{-1})^\alpha.$$

The uniform boundedness of $(\omega_{0,\varepsilon})$ *in the space* $LBMO$ *is more subtle and will be the object of Proposition* 4.1 *and Proposition* 4.2 *in the next section.*

Let us now outline the basic ideas for the proof of Theorem 4.1. It is founded on two main ingredients : the first one which is the most relevant and has an interest in itself concerns the persistence of the regularity $LBMO$ for a compressible transport model governing the vorticity,

$$\partial_t\omega_\varepsilon + v_\varepsilon\cdot\nabla\omega_\varepsilon + \omega_\varepsilon\mathrm{div}\, v_\varepsilon = 0.$$

In the incompressible case (4.1), Bernicot and Keraani have shown recently in [12] the following estimate

$$\|\omega(t)\|_{LBMO\cap L^p} \quad\leq\quad C\|\omega_0\|_{LBMO\cap L^p}\Big(1+\int_0^t\|v(\tau)\|_{LL}d\tau\Big). \tag{4.5}$$

Where LL refers to the norm associated to the log-Lipshitz space.

Our goal consists in extending this result to the compressible model cited before. To do so, we shall proceed in the spirit of the work [12] by following the dynamics of the oscillations and especially understand the interaction between them and how the global mass is distributed. However, the lack of the incompressibility of the velocity and the quadratic structure of the nonlinearity $\omega_\varepsilon\mathrm{div}\, v_\varepsilon$ will bring more technical difficulties that we should carefully analyze. Our result whose extension will be given later in Theorem 4.2 reads as follows,

$$\begin{aligned}\|\omega_\varepsilon(t)\|_{LBMO\cap L^p} \quad&\leq\quad C\|\omega_{0,\varepsilon}\|_{LBMO\cap L^p}\Big(1+\int_0^t\|v_\varepsilon(\tau)\|_{LL}d\tau\Big)\\ &\times\quad \Big(1+\|\mathrm{div}\, v_\varepsilon\|_{L^1_tC^s}\int_0^t\|v_\varepsilon(\tau)\|_{LL}d\tau\Big)e^{C\|\mathrm{div}\, v_\varepsilon\|_{L^1_tL^\infty}}.\end{aligned}$$

From this estimate the result (4.5) follows easily by taking $\operatorname{div} v_\varepsilon = 0$. Now to prove this result we shall first filtrate the compressible part and then reduce the problem to the establishment of a logarithmic estimate for the composition in the space $LBMO$ but with a flow which does not necessarily preserve the Lebesgue measure. For this part we follow the ideas of [12]. Once this logarithmic estimate is proven we should come back to the real solution and thus we are led to establish some law products invoking some weighted ***LMO*** spaces acting as multipliers of $LBMO$ space.

The second ingredient of the proof of Theorem 4.1 is the use of the Strichartz estimates which are an efficient tool to deal with the so ill-prepared initial data. As it has already been mentioned, this fact was used in [41] for Yudovich solutions and here we follow the same strategy but with slight modifications for the strong convergence. This is done directly by manipulating the vorticity equation.

The remainder of this paper is organized as follows. In the next section we recall basic results about Littlewood-Paley operators, Besov spaces and gather some preliminary estimates. We shall also introduce some functional spaces and prove some of their basic properties. In Section 3 we shall examine the regularity of the flow map and establish a logarithmic estimate for the compressible transport model. Section 4 is devoted to some classical energy estimate for the system (E.C) and the corresponding Strichartz estimates. In the last section, we generalize the result of Theorem 4.1 and give the proofs. We close this paper with an appendix covering the proof of some technical lemmas.

4.2 Functional tool box

In this section, we shall recall the definition of the frequency localization operators, some of their elementary properties and the Besov spaces. We will also introduce some function spaces and discuss few basic results that will be used later.
First of all, we fix some notations that will be intensively used in this paper.

- In what follows, C stands for some real positive constant which may be different in each occurrence and C_0 a constant which depends on the initial data.
- For any X and Y, the notation $X \lesssim Y$ means that there exists a positive universal constant C such that $X \leq CY$.
- For a ball B and $\lambda > 0$, λB denotes the ball that is concentric with B and whose radius is λ times the radius of B.
- We will denote the mean value of f over the ball B by

$$\fint_B f \triangleq \frac{1}{|B|}\int_B f(x)\,dx.$$

- For $p \in [1,\infty]$, the notation $L^p_T X$ stands for the set of measurable functions $f : [0,T] \to X$ such that $t \longmapsto \|f(t)\|_X$ belongs to $L^p([0,T])$.

4.2.1 Littlewood-Paley theory

Let us recall briefly the classical dyadic partition of the unity, for a proof see for instance [25] : there exists two positive radial functions $\chi \in \mathcal{D}(\mathbb{R}^2)$ and $\varphi \in \mathcal{D}(\mathbb{R}^2\backslash\{0\})$ such that

$$\forall \xi \in \mathbb{R}^2, \quad \chi(\xi) + \sum_{q\geq 0} \varphi(2^{-q}\xi) = 1;$$

$$\forall \xi \in \mathbb{R}^2 \backslash \{0\}, \quad \sum_{q\in \mathbf{Z}} \varphi(2^{-q}\xi) = 1;$$

$$|j-q| \geq 2 \Rightarrow \quad \text{Supp}\ \varphi(2^{-j}\cdot) \cap \text{Supp}\ \varphi(2^{-q}\cdot) = \varnothing;$$

$$q \geq 1 \Rightarrow \quad \text{Supp}\ \chi \cap \text{Supp}\ \varphi(2^{-q}\cdot) = \varnothing.$$

For every $u \in \mathcal{S}'(\mathbb{R}^2)$ one defines the non-homogeneous Littlewood-Paley operators by,

$$\Delta_{-1}v = \mathcal{F}^{-1}\big(\chi\hat{v}\big), \quad \forall q \in \mathbb{N} \quad \Delta_q v = \mathcal{F}^{-1}\big(\varphi(2^{-q}\cdot)\hat{v}\big) \quad \text{and} \quad S_q v = \sum_{-1\leq j\leq q-1} \Delta_j v.$$

Similarly, we define the homogeneous operators by

$$\forall q \in \mathbb{Z} \quad \dot{\Delta}_q v = \mathcal{F}^{-1}\big(\varphi(2^{-q}\cdot)\hat{v}\big) \quad \text{and} \quad \dot{S}_q v = \sum_{-\infty\leq j\leq q-1} \dot{\Delta}_j v.$$

We notice that these operators map continuously L^p to itself uniformly with respect to q and p. Furthermore, one can easily check that for every tempered distribution v, we have

$$v = \sum_{q\geq -1} \Delta_q v,$$

and for all $v \in \mathcal{S}'(\mathbb{R}^2)/\{\mathcal{P}[\mathbb{R}^2]\}$

$$v = \sum_{q\in\mathbb{Z}} \dot{\Delta}_q v,$$

where $\mathcal{P}[\mathbb{R}^2]$ is the space of polynomials.
The following lemma (referred in what follows as Bernstein inequalities) describes how the derivatives act on spectrally localized functions.

Lemma 4.1 *There exists a constant $C > 0$ such that for all $q \in N, k \in \mathbb{N}, 1 \leq a \leq b \leq \infty$ and for every tempered distribution u we have*

$$\sup_{|\alpha|\leq k} \|\partial^\alpha S_q u\|_{L^b} \leq C^k 2^{q(k+2(\frac{1}{a}-\frac{1}{b}))} \|S_q u\|_{L^a},$$

$$C^{-k} 2^{qk} \|\dot{\Delta}_q u\|_{L^b} \leq \sup_{|\alpha|=k} \|\partial^\alpha \dot{\Delta}_q u\|_{L^b} \leq C^k 2^{qk} \|\dot{\Delta}_q u\|_{L^b}.$$

Based on Littlewood-Paley operators, we can define Besov spaces as follows. Let $(p,r) \in [1,+\infty]^2$ and $s \in \mathbb{R}$. The non-homogeneous Besov space $B^s_{p,r}$ is the set of tempered distributions v such that

$$\|v\|_{B^s_{p,r}} \triangleq \Big\| \big(2^{qs}\|\Delta_q v\|_{L^p}\big)_{q\in\mathbb{Z}} \Big\|_{\ell^r(\mathbb{Z})} < +\infty.$$

The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined as the set of $\mathcal{S}'(\mathbb{R}^2)/\{\mathcal{P}[\mathbb{R}^2]\}$ such that

$$\|v\|_{\dot{B}^s_{p,r}} \triangleq \Big\| \big(2^{qs}\|\dot{\Delta}_q v\|_{L^p}\big)_{q\in\mathbb{Z}} \Big\|_{\ell^r(\mathbb{Z})} < +\infty.$$

We point out that, for a strictly positive non-integer real number s the Besov space $B^s_{\infty,\infty}$ coincides with the usual Hölder space C^s. For $s \in]0,1[$, this means that

$$\|v\|_{B^s_{\infty,\infty}} \lesssim \|v\|_{L^\infty} + \sup_{x\neq y} \frac{|v(x)-v(y)|}{|x-y|^s} \lesssim \|v\|_{B^s_{\infty,\infty}}.$$

Also we can identify $B^s_{2,2}$ with the Sobolev space H^s for all $s \in \mathbb{R}$.
The following embeddings are an easy consequence of Bernstein inequalities,

$$B^s_{p_1,r_1} \hookrightarrow B^{s+2(\frac{1}{p_2}-\frac{1}{p_1})}_{p_2,r_2} \quad p_1 \le p_2 \quad \text{and} \quad r_1 \le r_2.$$

Next, we recall the log-Lipschitz space, denoted by LL. It is the set of bounded functions v such that

$$\|v\|_{LL} \triangleq \sup_{0<|x-y|<1} \frac{|v(x)-v(y)|}{|x-y|\log\frac{e}{|x-y|}} < +\infty.$$

Note that the space $B^1_{\infty,\infty}$ is a subspace of LL. More precisely, we have the following inequality, see [8] for instance.

$$\|v\|_{LL} \lesssim \|\nabla v\|_{B^0_{\infty,\infty}}. \tag{4.6}$$

If in addition v is divergence-free and under sufficient conditions of integrability, the velocity v is determined by the vorticity $\omega \triangleq \text{rot}v$ by means of the Biot-Savart law

$$v(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega(y)dy. \tag{4.7}$$

The following result is a deep estimate of harmonic analysis and related to the singular integrals of Calderón-Zygmund type,

$$\|\nabla v\|_{L^p} \le C\frac{p^2}{p-1}\|\omega\|_{L^p}. \tag{4.8}$$

where C is a universal constant and $p \in]1,\infty[$.

Lemma 4.2 *Let v be a smooth vector field and ω its vorticity. Define the compressible part of v by*

$$\mathbb{Q}v \triangleq \nabla\Delta^{-1}\text{div}\, v.$$

Then,

$$\|v\|_{LL} \lesssim \|\mathbb{Q}v\|_{L^\infty} + \|\omega\|_{L^p \cap B^0_{\infty,\infty}} + \|\text{div}\, v\|_{B^0_{\infty,\infty}}. \tag{4.9}$$

Proof : The identity $v = \mathbb{P}v + \mathbb{Q}v$ and the estimate (4.6) ensure that

$$\|v\|_{LL} \lesssim \|\nabla\mathbb{P}v\|_{B^0_{\infty,\infty}} + \|\nabla\mathbb{Q}v\|_{B^0_{\infty,\infty}}.$$

Bernstein inequality, the continuity of $\dot{\Delta}_q\mathbb{P} : L^p \longrightarrow L^p, \forall p \in [1,\infty]$ uniformly in q and the classical fact $\|\dot{\Delta}_q v\|_{L^\infty} \sim 2^{-q}\|\dot{\Delta}_q\omega\|_{L^\infty}$ give

$$\begin{aligned}
\|\nabla\mathbb{P}v\|_{B^0_{\infty,\infty}} &\le \|\Delta_{-1}\nabla\mathbb{P}v\|_{L^\infty} + \sup_{q\in\mathbb{N}}\|\dot{\Delta}_q\nabla\mathbb{P}v\|_{L^\infty} \\
&\lesssim \|\Delta_{-1}\nabla\mathbb{P}v\|_{L^p} + \sup_{q\in\mathbb{N}} 2^q\|\dot{\Delta}_q\mathbb{P}v\|_{L^\infty} \\
&\lesssim \|\nabla\mathbb{P}v\|_{L^p} + \sup_{q\in\mathbb{N}}\|\dot{\Delta}_q\omega\|_{L^\infty}.
\end{aligned}$$

Since the incompressible part $\mathbb{P}v$ has the same vorticity ω of the total velocity :

$$\text{curl}\;\mathbb{P}v = \omega,$$

then by the inequality (4.8) we get

$$\|\nabla\mathbb{P}v\|_{B^0_{\infty,\infty}} \lesssim \|\omega\|_{L^p} + \|\omega\|_{B^0_{\infty,\infty}}.$$

On other hand,

$$\begin{aligned}\|\nabla\mathbb{Q}v\|_{B^0_{\infty,\infty}} &\leq \|\Delta_{-1}\nabla\mathbb{Q}v\|_{L^\infty} + \sup_{q\in\mathbb{N}}\|\dot{\Delta}_q\nabla^2\Delta^{-1}\mathrm{div}\, v\|_{L^\infty}\\ &\lesssim \|\mathbb{Q}v\|_{L^\infty} + \|\mathrm{div}\, v\|_{B^0_{\infty,\infty}}.\end{aligned}$$

This concludes the proof of the lemma. □

The following result generalizes the classical Gronwall inequality, it will be very useful in the proof of the lifespan part of Theorem 4.3. For its proof see Lemma 5.2.1 in [25] .

Lemma 4.3 *[Osgood Lemma] Let* $a, C > 0$, $\gamma : [t_0, T] \to \mathbb{R}_+$ *be a locally integrable function and* $\mu : [a, +\infty[\longrightarrow \mathbb{R}_+$ *be a continuous non-decreasing function. Let* $\rho : [t_0, T] \to [a, +\infty[$ *be a measurable, positive function satisfying*

$$\rho(t) \leq C + \int_{t_0}^t \gamma(s)\mu(\rho(s))ds.$$

Set $\mathcal{M}(y) = \displaystyle\int_a^y \frac{dx}{\mu(x)}$ *and assume that* $\lim_{y\to+\infty}\mathcal{M}(y) = +\infty$. *Then*

$$\forall t \in [t_0, T], \quad \rho(t) \leq \mathcal{M}^{-1}\Big(\mathcal{M}(C) + \int_{t_0}^t \gamma(s)ds\Big).$$

Next, we will introduce a new space which play a crucial role in the study of Euler equations as we will see later.

4.2.2 The BMO_F space

The main goal of this section is to introduce the weighted BMO spaces denoted by BMO_F where the function F measures the rate between two oscillations. Thereafter, we shall focus on the analysis of some of their useful topological properties.

To start with, let us recall the classical BMO space (bounded mean oscillation), which is nothing but the set of locally integrable function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\|f\|_{BMO} \triangleq \sup_B \fint_B \Big|f - \fint_B f\Big| < +\infty,$$

where B runs over all the balls in $\mathbb{R}^2$.

It is well-known that the quotient space BMO space by the constants is a Banach space. Moreover, the BMO space enjoys with the following classical properties :

1. BMO is imbricated between the space of bounded functions and the $B^0_{\infty,\infty}$ space, that is,

$$L^\infty \hookrightarrow BMO \hookrightarrow B^0_{\infty,\infty}.$$

2. The unit ball of the BMO space is a weakly compact set.

Another interesting property of the BMO space concerns the rate between two oscillations which is estimated as follows : take two balls $B_1 = B(x_1, r_1)$ and $B_2 = B(x_2, r_2)$ such that $2B_2 \subset B_1$, then

$$\Big|\fint_{B_2} f - \fint_{B_1} f\Big| \lesssim \ln(1 + \frac{r_1}{r_2})\|f\|_{BMO}. \tag{4.10}$$

It is worthy pointing out that the local well-posedness theory for the incompressible Euler equations is still an open problem when the initial vorticity belongs to the BMO space. However, it was proved recently in [12] that the global well-posedness can be achieved for the $LBMO$ space, which is larger than the bounded functions and smaller than the BMO space. To be precise this space is defined by

$$\|f\|_{LBMO} \triangleq \|f\|_{BMO} + \sup_{2B_2 \subset B_1} \frac{\left| \fint_{B_2} f - \fint_{B_1} f \right|}{1 + \ln \left| \dfrac{\ln r_2}{\ln r_1} \right|} < +\infty,$$

where the radius r_1 of B_1 is smaller than 1. It seems that we can perform the same result for more general spaces by replacing the "outside" logarithm in the second part of the norm by a general function F which must satisfy some special assumptions listed below. We mention that a similar extension was done in the paper [11]. Before stating the definition of these spaces we need the following notions.

Definition 4.1 *Let $F : [1, +\infty[\longrightarrow [1, +\infty[$ be a nondecreasing function.*

- *We say that F belongs to the class $\mathcal{F}$ if there exits a constant $C > 0$ such that the following properties hold true :*
  - *(a) Blow up at infinity : $\lim_{x \to 0} F(x) = +\infty$.*
  - *(b) Asymptotic behavior : For any $\lambda \in [1, +\infty[$ and $x \in [\lambda, +\infty[$, we have*

    $$\int_x^{+\infty} e^{-\frac{y}{\lambda}} F(y)dy \le C\,\lambda\, e^{-\frac{x}{\lambda}} F(x).$$

  - *(c) Polynomial growth : For all $(x, y) \in ([1, +\infty[)^2$*

    $$F(xy) \le CF(x)F(y).$$

- *We say that F belongs to the class $\mathcal{F}'$ if it belongs to $\mathcal{F}$ and satisfies the Osgood condition :*

  $$\int_1^{\infty} \frac{dx}{xF(x)} = +\infty.$$

Now we shall give some elementary facts listed in the following remark.

Remark 4.2 *1. Note that the condition* (c) *in the previous definition implies that the function F has at most a polynomial growth. More precisely, there exists $\alpha > 0$ such that*

$$F(x) \le Cx^{\alpha} \quad \forall x \in [1, +\infty[. \tag{4.11}$$

2. It turns out from the monotony of F that

$$F(n)e^{-\frac{n+1}{\lambda}} \le \int_n^{n+1} F(x)e^{-\frac{x}{\lambda}} dx \le F(n+1)e^{-\frac{n}{\lambda}}.$$

Thus, we get the equivalence

$$e^{-\frac{1}{\lambda}} \sum_{n \ge N} e^{-\frac{n}{\lambda}} F(n) \le \int_N^{+\infty} e^{-\frac{x}{\lambda}} F(x)dx \le e^{\frac{1}{\lambda}} \sum_{n \ge N} e^{-\frac{n}{\lambda}} F(n).$$

Consequently, using the asymptotic behavior of F described by the point (b) and making a change of variable we obtain for all $a, b \in \mathbb{R}_+^*$,

$$\frac{1}{C}\sum_{n>N} e^{-\frac{n}{\lambda}} F\Big(\frac{n+a}{b}\Big) \leq \int_N^{+\infty} e^{-\frac{x}{\lambda}} F(\frac{x+a}{b})dx \leq C\lambda e^{-\frac{N}{\lambda}} F\Big(\frac{N+a}{b}\Big). \qquad (4.12)$$

3. *Several examples of functions belonging to the class $\mathcal{F}'$ can be found, for instance, we mention : $x \mapsto 1 + \ln^\alpha(x)$ with $0 < \alpha \leq 1$; $x \mapsto 1 + \ln\ln(e + x)\ln(x)$.*
4. *$\mathcal{F}$ is strictly embedded in $\mathcal{F}'$: For any $\beta > 0$, the function $x \mapsto x^\beta$ belongs to the class $\mathcal{F}\backslash\mathcal{F}'$.*

The BMO_F space is given by the following definition.

Definition 4.2 *Let F in $\mathcal{F}$, we denote by BMO_F the space of the locally integrable functions $f : \mathbb{R}^2 \to \mathbb{R}$ such that*

$$\|f\|_{BMO_F} = \|f\|_{BMO} + \sup_{B_1,B_2} \frac{\Big|⨍_{B_2} f - ⨍_{B_1} f\Big|}{F\Big(\frac{1-\ln r_2}{1-\ln r_1}\Big)} < +\infty,$$

where the supremum is taken over all the pairs of balls $B_2 = B(x_2, r_2)$ and $B_1 = B(x_1, r_1)$ in $\mathbb{R}^2$ with $0 < r_1 \leq 1$ and $2B_2 \subset B_1$.

In the next proposition we shall deal with some topological properties for the BMO_F spaces.

Proposition 4.1 *The following properties hold true.*

1. *The space BMO_F is complete, included in BMO and containing $L^\infty(\mathbb{R}^2)$.*
2. *If $F, G \in \mathcal{F}$ with $F \lesssim G$ then $BMO_F \hookrightarrow BMO_G$.*
3. *If $\ln(1 + x) \lesssim F(x), \forall x \geq 1$, then L^∞ is strictly embedded in BMO_F.*
4. *Let $p \in]1, \infty]$, then $BMO_F \cap L^p$ is weakly compact.*
5. *For $g \in L^1$ and $f \in BMO_F$ one has*

$$\|g * f\|_{BMO_F} \leq \|g\|_{L^1}\|f\|_{BMO_F}.$$

Proof :

(1) The two embeddings are straightforward. For the completeness of the space we consider a Cauchy sequence $(u_n)_n$ in BMO_F. Since BMO_F is contained in BMO which is complete, then this sequence converges in BMO and then in L^1_{loc} (see [50] for instance). Using the definition of the second term of the BMO_F norm and the convergence in L^1_{loc}, we get the convergence in BMO_F.

(2) The embedding is obvious from the definition of the second term of the BMO_F norm.

(3) According to the assertion (ii) we have the embedding $LBMO = BMO_{1+\ln} \hookrightarrow BMO_F$. Now, we conclude by the result of [12] where it is proved that the unbounded function defined by

$$x \longmapsto \begin{cases} \ln(1 - \ln|x|) & \text{if} \quad |x| \leq 1 \\ 0 \quad \text{if} \quad |x| \geq 1. \end{cases}$$

belongs to $BMO_{1+\ln}$.
(4) Let (w_n) be a bounded sequence of $BMO_F \cap L^p$, that is

$$\sup_n \|w_n\|_{BMO_F \cap L^p} = M < \infty.$$

We shall prove that up to an extraction we can find a subsequence denoted also by (w_n) which converges weakly to some $w \in BMO_F \cap L^p$. The bound of $(w_n)_n$ in L^p implies the existence of a subsequence, denoted also by $(w_n)_n$, and a function $w \in L^p$ such that (w_n) converges weakly in L^p and consequently for all $B = B(x,r)$ we have

$$\lim_{n\to+\infty} \fint_B w_n dx = \fint_B w dx. \tag{4.13}$$

In addition we have

$$\|w\|_{L^p} \leq \lim_{n\to\infty} \inf \|w_n\|_{L^p} \leq M.$$

Let B_1 and B_2 be two balls in $\mathbb{R}^2$ such that $2B_2 \subset B_1$ and $0 < r_1 \leq 1$. As F is larger than 1 we may write

$$\begin{aligned}
\frac{\left|\fint_{B_1} w - \fint_{B_2} w\right|}{F\left(\frac{1-\ln r_2}{1-\ln r_1}\right)} &\leq \left|\fint_{B_1}(w-w_n) - \fint_{B_2}(w-w_n)\right| + \frac{\left|\fint_{B_1} w_n - \fint_{B_2} w_n\right|}{F\left(\frac{1-\ln r_2}{1-\ln r_1}\right)} \\
&\leq \left|\fint_{B_1}(w-w_n)\right| + \left|\fint_{B_2}(w-w_n)\right| + M.
\end{aligned}$$

Hence, we get from (4.13) that

$$\frac{\left|\fint_{B_1} w - \fint_{B_2} w\right|}{F\left(\frac{1-\ln r_2}{1-\ln r_1}\right)} \leq M.$$

Moreover, from the weak compactness in the BMO space we have

$$\|w\|_{BMO} \leq \lim_{n\to\infty} \inf \|w_n\|_{BMO} \leq M.$$

(5) The result follows immediately from the identity

$$x \longmapsto \fint_{B(x,r)} (g * f) = \Big(g * \fint_{B(.,r)} f\Big)(x) \quad \forall r > 0.$$

□

Remark 4.3 *By using the Hölder inequality we observe that for any ball B of radius r we have*

$$\fint_B |f - \fint_B f| \leq C r^{-\frac{2}{p}} \|f\|_{L^p}.$$

In this respect we will only need to deal with balls whose radius is smaller than a universal constant, say 1.

The following proposition gives a rigorous justification for the choice of the initial data given by Remark 4.1.

Proposition 4.2 *Let $1 < p < 2$, $R > 1$ and χ a be smooth compactly supported function equal to 1 in the neighborhood of the unit ball. Let $\omega \in BMO_F \cap L^p$ be the vorticity of a divergence-free vector filed v. Then*

$$\Big\|\mathrm{rot}\Big(\chi(\frac{\cdot}{R})v\Big)\Big\|_{BMO_F\cap L^p} \lesssim \big(\|\chi\|_{L^\infty} + \|\nabla\chi\|_{L^\infty}\big)\|\omega\|_{BMO_F\cap L^p},$$

and

$$\lim_{R\to+\infty}\Big\|\mathrm{rot}\Big(\chi(\frac{\cdot}{R})v\Big) - \omega\Big\|_{L^p} = 0.$$

Proof : It is obvious that the term $\mathrm{rot}\big(\chi(\frac{\cdot}{R})v\big)$ can be splitted into

$$\mathrm{rot}\Big(\chi(\frac{x}{R})v\Big)(x) = \chi(\frac{x}{R})\omega(x) + \frac{1}{R}\nabla^\perp\chi(\frac{x}{R})v(x). \tag{4.14}$$

By Hölder inequality, it follows that

$$\Big\|\mathrm{rot}\Big(\chi(\frac{\cdot}{R})v\Big)\Big\|_{L^p} \leq \|\chi\|_{L^\infty}\|\omega\|_{L^p} + \frac{1}{R}\big\|\nabla\chi(\frac{\cdot}{R})\big\|_{L^2}\|v\|_{L^{\frac{2p}{2-p}}}. \tag{4.15}$$

In view of the Biot-Savart law and the classical Hardy-Littlewood- Sobolev inequality we have

$$\|v\|_{L^{\frac{2p}{2-p}}} \lesssim \|\omega\|_{L^p}.$$

Inserting this in (4.15) we conclude that

$$\Big\|\mathrm{rot}\Big(\chi(\frac{\cdot}{R})v\Big)\Big\|_{L^p} \leq \big(\|\chi\|_{L^\infty} + \|\nabla\chi\|_{L^\infty}\big)\|\omega\|_{L^p}.$$

For the BMO part of the norm we use (4.14) combined with the embedding $L^\infty \hookrightarrow BMO$,

$$\begin{aligned}\Big\|\mathrm{rot}\Big(\chi(\frac{\cdot}{R})v\Big)\Big\|_{BMO} &\leq \|\chi(\frac{\cdot}{R})\omega\|_{BMO} + \frac{1}{R}\|\nabla^\perp\chi(\frac{\cdot}{R})\cdot v\|_{BMO}\\ &\lesssim \|\chi(\frac{\cdot}{R})\omega\|_{BMO} + \frac{1}{R}\|\nabla\chi\|_{L^\infty}\|v\|_{L^\infty}.\end{aligned} \tag{4.16}$$

As $1 < p < 2$, the Biot-Savart law ensures that

$$\|v\|_{L^\infty} \lesssim \|\omega\|_{L^p\cap L^{2p}}.$$

Recall the classical result of interpolation, see [50] for instance :

$$\|\omega\|_{L^q} \lesssim \|\omega\|_{L^p}^{\frac{p}{q}}\|\omega\|_{BMO}^{1-\frac{p}{q}} \quad \forall q \in [p, +\infty[. \tag{4.17}$$

Combining the preceding two inequalities we get

$$\begin{aligned}\|v\|_{L^\infty} &\lesssim \|\omega\|_{L^p}^{\frac12}\|\omega\|_{BMO}^{\frac12} + \|\omega\|_{L^p}\\ &\lesssim \|\omega\|_{BMO\cap L^p}.\end{aligned} \tag{4.18}$$

To estimate the first term of the right-hand side term of inequality (4.16) we write

$$\begin{aligned}\chi(\frac{x}{R})\omega(x) - \fint_B \chi(\frac{y}{R})\omega(y)dy &= \omega(x)\Big(\chi(\frac{x}{R}) - \fint_B \chi(\frac{y}{R})dy\Big)\\ &+ \Big(\fint_B \chi(\frac{y}{R})dy\Big)\Big(\omega(x) - \fint_B \omega(y)dy\Big)\\ &+ \fint_B \omega(y)\Big(\Big(\fint_B \chi(\frac{z}{R})dz\Big) - \chi(\frac{y}{R})\Big)dy.\end{aligned}$$

Hence,

$$\begin{aligned}\fint_B\Big|\chi(\frac{x}{R})\omega(x)dx-\fint_B\chi(\frac{y}{R})\omega(y)dy\Big| \leq& 2\fint_B|\omega(x)|\Big|\chi(\frac{x}{R})-\fint_B\chi(\frac{y}{R})dy\Big|dx\\&+\Big|\fint_B\chi(\frac{x}{R})dx\Big|\fint_B\Big|\omega-\fint_B\omega\Big|\\ \lesssim& \fint_B\fint_B|\omega(x)|\Big|\chi(\frac{x}{R})-\chi(\frac{y}{R})\Big|dydx\\&+\|\chi\|_{L^\infty}\|\omega\|_{BMO}.\end{aligned}$$

By the mean value theorem, we get

$$\begin{aligned}\fint_B\fint_B|\omega(x)|\Big|\chi(\frac{x}{R})-\chi(\frac{y}{R})\Big|dydx &\leq \|\nabla\chi\|_{L^\infty}\fint_B\fint_B|\omega(x)|\Big|\frac{x-y}{R}\Big|dydx\\&\lesssim r\|\nabla\chi\|_{L^\infty}\fint_B|\omega(x)|dx.\end{aligned}$$

Using Hölder inequality, the facts $p>1$, $r<1$, and inequality (4.17) we find

$$\begin{aligned}\fint_B\fint_B|\omega(x)|\Big|\chi(\frac{x}{R})-\chi(\frac{y}{R})\Big|dydx &\lesssim r^{1-1/p}\|\nabla\chi\|_{L^\infty}\|\omega\|_{L^{2p}}\\&\lesssim \|\nabla\chi\|_{L^\infty}\|\omega\|_{BMO\cap L^p}.\end{aligned}$$

Concerning the second term of the norm in BMO_F we start with the identity,

$$\begin{aligned}\fint_{B_2}\chi(\frac{x}{R})\omega(x)dx-\fint_{B_1}\chi(\frac{y}{R})\omega(y)dy =& \fint_{B_2}\chi(\frac{x}{R})\Big(\omega(x)-\fint_{B_2}\omega(y)dy\Big)dx\\&+\Big(\fint_{B_2}\chi(\frac{x}{R})dx\Big)\Big(\fint_{B_2}\omega(y)dy-\fint_{B_1}\omega(y)dy\Big)\\&+\Big(\fint_{B_1}\omega(y)dy\Big)\Big(\fint_{B_2}\chi(\frac{x}{R})dx-\fint_{B_1}\chi(\frac{x}{R})dx\Big)\\&+\fint_{B_1}\chi(\frac{x}{R})\Big(\Big(\fint_{B_1}\omega(y)dy\Big)-\omega(x)\Big)dx\\&\triangleq \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4.\end{aligned}$$

Using the definition of BMO_F space we get

$$\frac{|\mathrm{I}_1|+|\mathrm{I}_2|+|\mathrm{I}_4|}{F\Big(\dfrac{1-\ln r_2}{1-\ln r_1}\Big)}\leq \|\chi\|_{L^\infty}\|\omega\|_{BMO_F}.$$

For the remainder term I_3 we proceed as follows : we take $x_0\in B_1\cap B_2$ and we use the main value theorem with $F\geq 1$

$$\begin{aligned}\frac{|\mathrm{I}_3|}{F\Big(\dfrac{1-\ln r_2}{1-\ln r_1}\Big)} &\lesssim r_1^{-\frac{1}{p}}\|\omega\|_{L^{2p}}\Big(\fint_{B_2}\Big|\chi(\frac{x}{R})-\chi(\frac{x_0}{R})\Big|dx+\fint_{B_1}\Big|\chi(\frac{x}{R})-\chi(\frac{x_0}{R})\Big|dx\Big)\\&\lesssim r_1^{-\frac{1}{p}}\|\omega\|_{L^{2p}}\|\nabla\chi\|_{L^\infty}(r_2/R+r_1/R)\\&\lesssim \|\omega\|_{BMO\cap L^p}\|\nabla\chi\|_{L^\infty}.\end{aligned}$$

We have now to check the strong convergence in L^p space. By considering the identity (4.14), Hölder inequality and the fact that $\chi \equiv 1$ in the neighborhood of the unit ball,

$$\begin{aligned}\Big\|\mathrm{rot}\Big(\chi(\frac{\cdot}{R})v\Big)-\omega\Big\|_{L^p} &\le \|\chi(\frac{\cdot}{R})-1\|_{L^\infty}\|\omega\|_{L^p(B^c(0,R))}+\frac{1}{R}\|\nabla^\perp\chi(\frac{\cdot}{R})\|_{L^2}\|v\|_{L^{\frac{2p}{2-p}}(B^c(0,R))}\\ &\lesssim \|\omega\|_{L^p(B^c(0,R))}+\|\nabla\chi\|_{L^\infty}\|v\|_{L^{\frac{2p}{2-p}}(B^c(0,R))}.\end{aligned}$$

Passing to the limit completes the proof of the desired result. □

4.2.3 LMO_F spaces and law products

Here we endeavor to define a functional space whose elements can be served as pointwise multipliers for BMO_F space. In this context, we point out that BMO is stable by multiplication by $LMO \cap L^\infty$ functions, we refer to [97] for the proof. Where LMO is a subspace of BMO, not comparable to L^∞ and equipped with the semi-norm

$$\|f\|_{LMO} \triangleq \sup_B |\ln r| \fint_B \Big|f-\fint_B f\Big|.$$

In this definition B runs over the balls of radius lesser than 1. In order to validate a similar result for the BMO_F space, we have to define the following function space.

Definition 4.3 *Let F be in the class $\mathcal{F}$ and $f \in L^1_{loc}(\mathbb{R}^2,\mathbb{R})$, we say that f belongs to the LMO_F space if*

$$\|f\|_{LMO_F} = \sup_{\substack{B\\ r\le 1}} F(1-\ln r)\fint_B\Big|f-\fint_B f\Big| + \sup_{\substack{2B_2\subset B_1\\ r_1\le 1}} F(1-\ln r_1)\Big|\fint_{B_2} f-\fint_{B_1} f\Big| < +\infty,$$

We shall prove the following law product.

Proposition 4.3 *Let $g \in BMO_F \cap L^p$, with $1 \le p < \infty$, and $f \in LMO_F \cap L^\infty$. Then $fg \in BMO_F \cap L^p$ and*

$$\|fg\|_{BMO_F} \le C\|f\|_{LMO_F\cap L^\infty}\|g\|_{BMO_F\cap L^p},$$

where C is independent of f and g.

Proof : In view Remark 4.3 we will consider throughout the proof B a ball of radius $r < \frac{1}{4}$. We start with writing the following plain identity

$$fg-\fint_B fg = f\Big(g-\fint_B g\Big)+\Big(\fint_B g\Big)\Big(f-\fint_B f\Big)+\fint_B\Big\{f\Big((\fint_B g)-g\Big)\Big\}, \tag{4.19}$$

which gives in turn

$$\begin{aligned}\fint_B\Big|fg-\fint_B fg\Big| &\le 2\fint_B|f|\Big|g-\fint_B g\Big|+\Big|\fint_B g\Big|\fint_B\Big|f-\fint_B f\Big|\\ &\lesssim \|f\|_{L^\infty}\|g\|_{BMO}+\Big|\fint_B g\Big|\fint_B\Big|f-\fint_B f\Big|.\end{aligned} \tag{4.20}$$

We denote by $\hat{B}$ the ball which is concentric to B and whose radius is equal to 1. According to the definition of the second part of the BMO_F norm we get

$$\begin{aligned}\Big|\fint_B g\Big| &\le \Big|\fint_B g - \fint_{\hat{B}} g\Big| + \Big|\fint_{\hat{B}} g\Big| \\ &\lesssim F(1-\ln r)\|g\|_{BMO_F} + \|g\|_{L^p}. \end{aligned} \tag{4.21}$$

It follows that

$$\Big|\fint_B g\Big| \fint_B \Big|f - \fint_B f\Big| \lesssim \|g\|_{BMO_F\cap L^p}\|f\|_{LMO_F}.$$

Inserting this in (4.20) we find

$$\|fg\|_{BMO} \le C\|g\|_{BMO_F\cap L^p}\|f\|_{LMO_F\cap L^\infty}.$$

For the second term of the BMO_F-norm we will make use of the following identity

$$\begin{aligned}\fint_{B_2} fg - \fint_{B_1} fg &= \fint_{B_2} f\Big(g - \fint_{B_2} g\Big) + \Big(\fint_{B_2} f\Big)\Big(\fint_{B_2} g - \fint_{B_1} g\Big) \\ &+ \Big(\fint_{B_1} g\Big)\Big(\fint_{B_2} f - \fint_{B_1} f\Big) + \fint_{B_1} f\Big(\Big(\fint_{B_1} g\Big) - g\Big). \end{aligned} \tag{4.22}$$

Since F is larger than 1 we may write

$$\frac{\Big|\fint_{B_2} fg - \fint_{B_1} fg\Big|}{F\Big(\frac{1-\ln r_2}{1-\ln r_1}\Big)} \le \text{I+II+III+IV},$$

where

$$\text{I} \triangleq \Big|\fint_{B_2} f(g - \fint_{B_2} g)\Big|,$$

$$\text{II} \triangleq \frac{\Big|\fint_{B_2} f\Big|\Big|\fint_{B_2} g - \fint_{B_1} g\Big|}{F\Big(\frac{1-\ln r_2}{1-\ln r_1}\Big)},$$

$$\text{III} \triangleq \Big|\fint_{B_1} g\Big|\Big|\fint_{B_2} f - \fint_{B_1} f\Big|,$$

$$\text{IV} \triangleq \Big|\fint_{B_1} f(g - \fint_{B_1} g)\Big|.$$

It is clear that I and IV are bounded by $\|f\|_{L^\infty}\|g\|_{BMO}$ and II is bounded by $\|f\|_{L^\infty}\|g\|_{BMO_F}$. It remains to estimate III. Reproducing the same argument used in (4.21) we get

$$\begin{aligned}\text{III} &\le \|g\|_{BMO_F\cap L^p} F(1-\ln r_1)\Big|\fint_{B_2} f - \fint_{B_1} f\Big| \\ &\le \|g\|_{BMO_F\cap L^p}\|f\|_{LMO_F}. \end{aligned}$$

This completes the proof of the proposition. □

The next proposition deals with some useful properties of the LMO_F space.

Proposition 4.4 *1. The $LMO_F \cap L^\infty$ space is an algebra. More precisely, there exists an absolute constant C such that for any $f, g \in LMO_F \cap L^\infty$ one has*

$$\|fg\|_{LMO_F \cap L^\infty} \leq C\big(\|f\|_{L^\infty}\|g\|_{LMO_F} + \|g\|_{L^\infty}\|f\|_{LMO_F}\big).$$

2. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an entire real-function, which vanishes at 0. For any real-valued function u in $LMO_F \cap L^\infty$, the function $f \circ u$ belongs to the same space. Moreover, there exists a positive constant C and an entire real-function g such that we have*

$$\|f \circ u\|_{LMO_F \cap L^\infty} \leq C\|u\|_{LMO_F}\, g\big(\|u\|_{L^\infty}\big).$$

3. *For any $s > 0$ we have the embedding $C^s \hookrightarrow LMO_F$, that is, there exits $C > 0$ such that for any $f \in C^s$,*

$$\|f\|_{LMO_F} \leq C\|f\|_{C^s}.$$

Proof $(\mathbf{1})$ Making appeal to inequality (4.20) we get

$$\begin{aligned} F(1-\ln r)\fint_B \Big|fg - \fint_B fg\Big| \leq{}& 2F(1-\ln r)\fint_B |f|\Big|g - \fint_B g\Big| \\ &+ F(1-\ln r)\Big|\fint_B g\Big|\fint_B \Big|f - \fint_B f\Big| \\ \leq{}& 2\|f\|_{L^\infty}\|g\|_{LMO_F} + \|g\|_{L^\infty}\|f\|_{LMO_F}. \end{aligned}$$

Likewise, as $r_2 \leq r_1$ and F is a nondecreasing function, we immediately deduce from identity (4.22) that

$$\begin{aligned} F(1-\ln r_1)\Big|\fint_{B_2} fg - \fint_{B_1} fg\Big| \leq{}& F(1-\ln r_2)\fint_{B_2} |f|\Big|g - \fint_{B_2} g\Big| \\ &+ F(1-\ln r_1)\fint_{B_2} |f|\Big|\fint_{B_2} g - \fint_{B_1} g\Big| \\ &+ F(1-\ln r_1)\Big(\fint_{B_1} |g|\Big|\fint_{B_2} f - \fint_{B_1} f\Big| \\ &+ \fint_{B_1} |f|\Big|g - \fint_{B_1} g\Big|\Big) \\ \leq{}& 2\|f\|_{L^\infty}\|g\|_{LMO_F} + 2\|g\|_{L^\infty}\|f\|_{LMO_F}. \end{aligned}$$

This completes the proof of the assertion (i).
$(\mathbf{2})$ By definition, there exists a sequence $(a_n)_n \subset \mathbb{R}$ such that for all $x \in \mathbb{R}$,

$$f(x) = \sum_{n \geq 1} a_n x^n$$

and thus

$$f \circ u(x) = \sum_{n=1}^{\infty} a_n u^n(x).$$

Consequently

$$\|f \circ u\|_{LMO_F \cap L^\infty} \leq \sum_{n=1}^{\infty} |a_n|\|u^n\|_{LMO_F \cap L^\infty}.$$

According to $(\mathbf{1})$ and using the induction principle, we infer that for all $n \geq 2$ one has,

$$\|u^n\|_{LMO_F \cap L^\infty} \leq C^{n-1}\|u\|_{LMO_F}\|u\|_{L^\infty}^{n-1}.$$

Therefore,

$$\begin{aligned}\|f\circ u\|_{LMO_F\cap L^\infty} &\leq \|u\|_{LMO_F}\sum_{n=1}^{\infty}|a_n|C^{n-1}\|u\|_{L^\infty}^{n-1}\\ &\leq \|u\|_{LMO_F}\sum_{n=0}^{\infty}|a_{n+1}|C^{n}\|u\|_{L^\infty}^{n}\\ &\triangleq \|u\|_{LMO_F}\,g\big(\|u\|_{L^\infty}\big).\end{aligned}$$

($\mathbf{3}$) Using the definition of the Hölder space, we can write

$$\begin{aligned}F(1-\ln r)\fint_B\Big|f-\fint_B f\Big| &\leq F(1-\ln r)\fint_B\fint_B\big|f(x)-f(y)\big|dxdy\\ &\lesssim F(1-\ln r)r^s\|f\|_{C^s}\\ &\lesssim \|f\|_{C^s}.\end{aligned}$$

The last inequality follows from (4.11) which implies that

$$\lim_{r\to0}F(1-\ln r)r^s=0.$$

The second term of the norm can be handled exactly as the first one. For $x_0\in B_1\cap B_2$ we have

$$\begin{aligned}F(1-\ln r_1)\Big|\fint_{B_2}f-\fint_{B_1}f\Big| &\leq F(1-\ln r_2)\fint_{B_2}\big|f(x)-f(x_0)\big|dx\\ &+ F(1-\ln r_1)\fint_{B_1}\big|f(x)-f(x_0)\big|dx\\ &\leq \big(F(1-\ln r_2)r_2^s+F(1-\ln r_1)r_1^s\big)\|f\|_{C^s},\end{aligned}$$

which is bounded for the same reason as before.

4.3 Compressible transport model

We focus in this section on the study of the persistence regularity of the initial data measured in BMO_F space for the following compressible transport model :

$$\begin{cases}\partial_t f+v\cdot\nabla f+f\operatorname{div}v=0;\\ f_{|t=0}=f_0.\end{cases}\tag{4.23}$$

This model describes the vorticity dynamics for the system (E.C) and the advection is governed by a compressible velocity which not necessary in the Lipschitz class. According to the inequality (4.6) the velocity belongs to the log-Lipschitz class and as it was revealed in [8] the solution in the incompressible case may exhibit a loss of regularity in the classical spaces like Sobolev and Hölder spaces. This possible loss cannot occur in the $LBMO$ space as it was recently proved in [12]. Here we shall extend this latter result to the model (4.23) and we will see later an application for the incompressible limit problem. Before stating our main result we will recall the log-Lipschitz norm :

$$\|v\|_{LL}=\sup_{0<|x-y|<1}\frac{|v(x)-v(y)|}{|x-y|\log\frac{e}{|x-y|}}<\infty$$

Our result reads as follows.

Theorem 4.2 *Let v be a smooth vector field and f be a smooth solution of the system* (4.23). *Then, there exits an absolute constant $C > 0$ such that for every $1 \leq p \leq \infty$,*

$$\|f(t)\|_{BMO_F\cap L^p} \leq C\|f_0\|_{BMO_F\cap L^p} e^{C\|\mathrm{div}\, v\|_{L^1_t L^\infty}} F\big(e^{CV(t)}\big)\Big(1 + F\big(e^{CV(t)}\big)\|\mathrm{div}\, v\|_{L^1_t(LMO_F\cap L^\infty)}\Big).$$

with

$$V(t) \triangleq \int_0^t \|v(\tau)\|_{LL} d\tau.$$

Remark 4.4 *When the velocity is divergence-free the estimate of the preceding theorem becomes*

$$\|f(t)\|_{BMO_F\cap L^p} \leq C\|f_0\|_{BMO_F\cap L^p} F\big(e^{CV(t)}\big). \tag{4.24}$$

Therefore, Theorem 4.2 *recovers the result stated in [12] for $F(x) = 1 + \ln(x)$*

$$\|f(t)\|_{LBMO\cap L^p} \quad \leq \quad C\|f_0\|_{LBMO\cap L^p}\Big(1 + \int_0^t \|v(\tau)\|_{LL} d\tau\Big).$$

As well, we mention that a similar estimate to (4.24) *has been established in [11] in the incompressible framework for some $L^\alpha mo_F$ space.*

To prove this result, we shall use the same approach of [11] and [12]. However the lack of the incompressibility brings more technical difficulties. Before going further into the details we should point out that the solution of the system (4.23) has, as we will see later, the following structure

$$f(t,x) = f_0(\psi^{-1}(t,x)) \exp\Big(- \int_0^t (\mathrm{div}\, v)(\tau, \psi(\tau, \psi^{-1}(t,x))) d\tau\Big).$$

Where ψ is the flow associated to the vector field v, that is, the solution of the differential equation,

$$\partial_t \psi(t,x) = v(t, \psi(t,x)), \quad \psi(0,x) = x.$$

It turns out that the study of the propagation in the BMO_F space returns to the study of the composition by the flow. Based on that, we shall firstly examine the regularity of the flow map, discuss its left composition with the elements of BMO_F and finally give the proof of Theorem 4.2.

4.3.1 The regularity of the flow map

Although the vector field v is not Lipschitz in our context, we still have existence and uniqueness of the flow but a loss of regularity may occur. In fact ψ is not necessary Lipschitz but belongs to the class C^{s_t} with $s_t < 1$, as indicated in the lemma below. For more details see for instance [25] and [88].

Lemma 4.4 *Let v be a smooth vector field on $\mathbb{R}^2$ and ψ its flow . Then for all $t \in \mathbb{R}_+$, we have*

$$|x_1 - x_2| < e^{-\beta(t)} \Longrightarrow |\psi^{\pm 1}(t,x_1) - \psi^{\pm 1}(t,x_2)| \leq e|x_1 - x_2|^{\frac{1}{\beta(t)}}.$$

where $\psi^1 = \psi$ and ψ^{-1} is the inverse of ψ. The function $\beta(t)$ is given by

$$\beta(t) = \exp\Big(\int_0^t \|v\|_{LL} d\tau\Big).$$

As a consequence, we obtain the following lemma which is with an extreme importance in the proof of the composition result. Its proof can be found in [12].

Lemma 4.5 *Under the assumptions of Lemma* 4.4 *and for* $r \leq \exp(-\beta(t))$ *we have*

$$4\psi(B(x_0, r)) \subset B(\psi(x_0), g_\psi(r)),$$

where

$$g_\psi(r) \triangleq 4er^{\frac{1}{\beta(t)}}. \tag{4.25}$$

In particular

$$\sup\left\{\frac{1-\ln g_\psi(r)}{1-\ln r}, \frac{1-\ln r}{1-\ln g_\psi(r)}\right\} \lesssim 1+\beta(t). \tag{4.26}$$

The following inequality will be frequently used in the rest of this paper, see for instance [25].

Lemma 4.6 *Let* ψ *be the flow associated to a smooth vector field* v*. Then for all* $t \in \mathbb{R}_+$

$$e^{-\|\text{div}\, v\|_{L^1_t L^\infty}} \leq |J_{\psi_t^{\pm 1}}(x)| \leq e^{\|\text{div}\, v\|_{L^t L^\infty}} \quad \forall x \in \mathbb{R}^2.$$

Where $J_{\psi_t^{\pm}}(t, x)$ *is the Jacobian of* $\psi^{\pm}(t, x)$*.*

4.3.2 Composition in the BMO_F space

The problem of the composition in the BMO space can be easily solved when ψ is a bi-Lipschitz map which is unfortunately not necessarily verified in our case. Such difficulty could in general induce a losing regularity but as we will see we can face up this loss by working in a suitable space and replace BMO space with the BMO_F spaces. We will be also led to deal with another technical difficulty linked to the fact that ψ is no longer measure-preserving map. Our result is the following,

Theorem 4.3 *There exists a positive constant* C *such that, for any function* f *taken in* $BMO_F \cap L^p$*, with* $1 \leq p \leq \infty$ *and for* ψ *the flow associated to a smooth vector field* v*, we have*

$$\|f \circ \psi\|_{BMO_F \cap L^p} \leq C\|f\|_{BMO_F \cap L^p} F\left(e^{C\|v\|_{L^1_t LL}}\right) e^{C\|\text{div}\, v\|_{L^1_t L^\infty}}.$$

Proof : We know that, by a change of variable, the composition in the L^p space gives

$$\|f \circ \psi\|_{L^p} \leq \|J_{\psi^{-1}}\|_{L^\infty}^{\frac{1}{p}} \|f\|_{L^p}.$$

The composition in the BMO_F space is more subtle and we shall use the idea of [12]. In fact, the proof is divided into two steps : in the first one we deal with the BMO term of the norm and in the second we consider the other term.

• **Step 1 : Persistence of the BMO regularity.** We shall start with the persistence of the first part of the norm BMO_F. For this purpose we distinguish two cases depending whether the radius r is small or not.

<u>**Case 1**</u> : $r < (4e)^{-\beta(t)} \min\left\{1, \frac{1}{\|J_\psi\|_{L^\infty}}\right\}$. According to the definition (4.25) this condition implies

$$g_\psi(r) < 1.$$

We denote by $\tilde{B}$ the ball of center $\psi(x_0)$ and radius $g_\psi(r)$. It is easily seen that

$$\fint_B \Big|f\circ\psi - \fint_B f\circ\psi\Big| \;\leq\; 2\fint_B \Big|f\circ\psi - \fint_{\tilde{B}} f\Big|.$$

Then by a change of variable one has

$$\fint_B \Big|f\circ\psi - \fint_B f\circ\psi\Big| \;\leq\; \frac{2\|J_{\psi^{-1}}\|_{L^\infty}}{|B|}\int_{\psi(B)} \Big|f - \fint_{\tilde{B}} f\Big|dx.$$

At this stage the strategy consists in the partition of the open set $\psi(B)$ into countable balls with variable sizes and to try to measure their interactions with the biggest ball $\tilde{B}$. For this goal we shall use Whitney covering lemma [115] which asserts in our case the existence of a collection of countable open balls $(O_k)_k$ such that :

- The collection of double balls is a bounded covering :

$$\psi(B) \subset \bigcup_k 2O_k.$$

- The collection is disjoint and for all k,

$$O_k \subset \psi(B).$$

- The Whitney property is verified : the radius r_k of O_k satisfies

$$r_k \approx d(O_k, \psi(B)^c).$$

So by the first property we may write

$$\begin{aligned}\fint_B \Big|f\circ\psi - \fint_B f\circ\psi\Big|dx &\lesssim \frac{\|J_{\psi^{-1}}\|_{L^\infty}}{|B|}\sum_j |O_j|\fint_{2O_j}\Big|f - \fint_{\tilde{B}} f\Big| \\ &\lesssim \|J_{\psi^{-1}}\|_{L^\infty}\big(I_1 + I_2\big),\end{aligned}$$

where

$$\mathrm{I}_1 \triangleq \frac{1}{|B|}\sum_j |O_j|\fint_{2O_j}\Big|f - \fint_{2O_j} f\Big|$$

and

$$\mathrm{I}_2 \triangleq \frac{1}{|B|}\sum_j |O_j|\Big|\fint_{2O_j} f - \fint_{\tilde{B}} f\Big|.$$

Using the fact

$$\sum_j |O_j| \leq |\psi(B)| \leq \|J_\psi\|_{L^\infty}|B|, \tag{4.27}$$

we immediately deduce that

$$\mathrm{I}_1 \lesssim \|J_\psi\|_{L^\infty}\|f\|_{BMO}.$$

According to Lemma 4.5 we have $4O_j \subset \tilde{B}$. In addition, as $g_\psi(r) < 1$ and in view of the definition of the BMO_F-norm, we infer that

$$\begin{aligned}\mathrm{I}_2 &\lesssim \frac{1}{|B|}\sum_j |O_j| F\Big(\frac{1-\ln 2r_j}{1-\ln g_\psi(r)}\Big)\|f\|_{BMO_F} \\ &\lesssim \frac{1}{|B|}\sum_j |O_j| F\Big(\frac{1-\ln r_j}{1-\ln g_\psi(r)}\Big)\|f\|_{BMO_F}.\end{aligned}$$

From (4.27) we get $r_j \leq \|J_\psi\|_{L^\infty}^{1/2} r$ for all j. Set

$$h(r) \triangleq r \max\{1, \|J_\psi\|_{L^\infty}\}, \quad U_k \triangleq \sum_{e^{-k-1}h(r) < r_j \leq e^{-k}h(r)} |O_j|, \quad k \in \mathbb{N}.$$

Hence as F is non-decreasing we may write

$$\begin{aligned} \mathrm{I}_2 &\lesssim \frac{1}{|B|}\sum_{k\geq 0} U_k F\Big(\frac{k+2-\ln h(r)}{1-\ln g_\psi(r)}\Big)\|f\|_{BMO_F} \\ &\lesssim \frac{1}{|B|}\sum_{k\geq 0} U_k F\Big(\frac{k+2-\ln r}{1-\ln g_\psi(r)}\Big)\|f\|_{BMO_F}. \end{aligned}$$

Let N be a fixed positive integer that will be carefully chosen later. We split the preceding sum into two parts

$$\begin{aligned} \mathrm{I}_2 &\lesssim \Big(\frac{1}{|B|}\sum_{k\leq N} U_k F\Big(\frac{k+2-\ln r}{1-\ln g_\psi(r)}\Big) + \frac{1}{|B|}\sum_{k> N} U_k F\Big(\frac{k+2-\ln r}{1-\ln g_\psi(r)}\Big)\Big)\|f\|_{BMO_F} \\ &\triangleq \big(\mathrm{I}_{2,1} + \mathrm{I}_{2,2}\big)\|f\|_{BMO_F}. \end{aligned}$$

Since $\sum U_k \leq \|J_\psi\|_{L^\infty}|B|$ and F is non-decreasing then

$$\mathrm{I}_{2,1} \leq \|J_\psi\|_{L^\infty} F\Big(\frac{N+2-\ln r}{1-\ln g_\psi(r)}\Big)\|f\|_{BMO_F}. \tag{4.28}$$

To estimate $\mathrm{I}_{2,2}$ we need the following bound of U_k whose proof will be given in Lemma 4.8 of the Appendix.

$$U_k \lesssim \big(1+\|J_\psi\|_{L^\infty}\big)^2 e^{\frac{-k}{\beta(t)}} r^{1+\frac{1}{\beta(t)}} \quad \forall k \geq \beta(t).$$

Therefore for N taken larger then $\beta(t)$ we have

$$\mathrm{I}_{2,2} \lesssim \big(1+\|J_\psi\|_{L^\infty}\big)^2 \sum_{k>N} e^{-\frac{k}{\beta(t)}} r^{\frac{1}{\beta(t)}-1} F\Big(\frac{k+2-\ln r}{1-\ln g_\psi(r)}\Big)\|f\|_{BMO_F}.$$

Inequality (4.12) from Remark 4.2 gives

$$\mathrm{I}_{2,2} \lesssim \big(1+\|J_\psi\|_{L^\infty}\big)^2 e^{-\frac{N}{\beta(t)}} \beta(t) r^{\frac{1}{\beta(t)}-1} F\Big(\frac{N+2-\ln r}{1-\ln g_\psi(r)}\Big)\|f\|_{BMO_F}. \tag{4.29}$$

Combining this estimate with (4.28) we obtain

$$\mathrm{I}_2 \lesssim \big(1+\|J_\psi\|_{L^\infty}\big)^2 \Big(1+e^{\frac{-N}{\beta(t)}}\beta(t) r^{\frac{1}{\beta(t)}-1}\Big) F\Big(\frac{N+2-\ln r}{1-\ln g_\psi(r)}\Big)\|f\|_{BMO_F}.$$

Taking $N = \big[\beta(t)(\beta(t)-\ln r)\big]+1$ we get

$$\begin{aligned} \mathrm{I}_2 &\lesssim \big(1+\|J_\psi\|_{L^\infty}\big)^2 \big(1+e^{-\beta(t)} e^{-\frac{1}{\beta(t)}(1-\ln r)}\beta(t)\big) F\Big(\frac{\beta(t)(\beta(t)-\ln r)+3-\ln r}{1-\ln g_\psi(r)}\Big)\|f\|_{BMO_F} \\ &\lesssim \big(1+\|J_\psi\|_{L^\infty}\big)^2 F\Big(\frac{\beta(t)^2+3-(1+\beta(t))\ln r}{1-\ln g_\psi(r)}\Big)\|f\|_{BMO_F}. \end{aligned}$$

Where we have used in the last inequality the fact that $\sup_{\beta>1}\beta e^{-\beta} < 1$. Furthermore, from estimate (4.26) we have

$$\begin{aligned} F\Big(\frac{\beta(t)^2+2-(1+\beta(t))\ln r}{1-\ln g_\psi(r)}\Big) &\lesssim F\Big((1+\beta(t))\frac{\beta(t)^2+3-(1+\beta(t))\ln r}{1-\ln r}\Big)\\ &\lesssim F\big((1+\beta(t))^3\big)\\ &\lesssim F(\beta^3(t)). \end{aligned}$$

Hence,

$$\mathrm{I}_2 \lesssim \big(1+\|J_\psi\|_{L^\infty}^2\big)F\big(\beta^3(t)\big)\|f\|_{BMO_F}.$$

Putting together the previous estimates gives

$$⨍_B \Big| f\circ\psi - ⨍_B f\circ\psi\Big| \;\leq\; \|J_{\psi^{-1}}\|_{L^\infty}\big(1+\|J_\psi\|_{L^\infty}^2\big)F\big(\beta^3(t)\big)\|f\|_{BMO_F}.$$

According to Lemma 4.6 we find

$$⨍_B \Big| f\circ\psi - ⨍_B f\circ\psi\Big| \leq Ce^{C\|\mathrm{div}\, v\|_{L^1_t L^\infty}} F\big(\beta^3(t)\big)\|f\|_{BMO_F}. \tag{4.30}$$

Let us now move to the second case.

<u>**Case 2**</u> : $1 \geqslant r \geq (4e)^{-\beta(t)}\min\big\{1, \frac{1}{\|J_\psi\|_{L^\infty}}\big\}$. Under this assumption we can easily check that

$$|\ln r| \lesssim \beta(t) + \big|\ln\|J_\psi\|_{L^\infty}\big|. \tag{4.31}$$

By a change of variable we can write

$$\begin{aligned} ⨍_B \Big| f\circ\psi - ⨍_B f\circ\psi\Big| &\leq 2⨍_B |f\circ\psi|\\ &\lesssim \frac{1}{|B|}\int_{\psi(B)} |f(x)||J_{\psi^{-1}}(x)|dx\\ &\lesssim \frac{1}{|B|}\sum_j |O_j|\|J_{\psi^{-1}}\|_{L^\infty} ⨍_{2O_j} |f|\\ &\lesssim \|J_{\psi^{-1}}\|_{L^\infty}\Big(\frac{1}{|B|}\sum_j |O_j| ⨍_{2O_j}\Big|f - ⨍_{2O_j} f\Big| + \frac{1}{|B|}\sum_j \Big|\int_{2O_j} f\Big|\Big)\\ &\lesssim \|J_{\psi^{-1}}\|_{L^\infty}\Big(\|J_\psi\|_{L^\infty}\|f\|_{BMO} + \mathrm{I}_1 + \mathrm{I}_2\Big), \end{aligned}$$

where

$$\mathrm{I}_1 \triangleq \frac{1}{|B|}\sum_{j\backslash r_j>\frac14} \Big|\int_{2O_j} f\Big|,$$

and

$$\mathrm{I}_2 \triangleq \frac{1}{|B|}\sum_{j\backslash r_j\leq\frac14} |O_j|\Big| ⨍_{2O_j} f\Big|.$$

Hölder inequality implies that

$$\begin{aligned} \mathrm{I}_1 &\leq \frac{1}{|B|}\sum_{j\backslash r_j>\frac14} |O_j|^{1-\frac1p}\|f\|_{L^p}\\ &\lesssim \frac{1}{|B|}\sum_j |O_j|\|f\|_{L^p}\\ &\lesssim \|J_\psi\|_{L^\infty}\|f\|_{L^p}. \end{aligned}$$

In order to estimate the term I_2, we consider a collection of open balls $(\tilde{O}_j)_j$ such that, for all j in $\mathbb{N}$, $\tilde{O}_j$ is concentric to O_j and of radius equal to 1. Then, as $r_j \leq \frac{1}{4}$ we have $4O_j \subset \tilde{O}_j$ and thus using the definition of the BMO_F-norm gives

$$\begin{aligned}
\mathrm{I}_2 &\lesssim \frac{1}{|B|}\sum_{j\backslash r_j\leq\frac{1}{4}}|O_j|\Big|\fint_{2O_j} f - \fint_{\tilde{O}_j} f\Big| + \frac{1}{|B|}\sum_{j\backslash r_j\leq\frac{1}{4}}|O_j|\Big|\fint_{\tilde{O}_j} f\Big| \\
&\lesssim \frac{1}{|B|}\sum_{j\backslash r_j\leq\frac{1}{4}}|O_j|F(1-\ln 2r_j)\|f\|_{BMO_F} + \|J_\psi\|_{L^\infty}\|f\|_{L^p}.
\end{aligned}$$

We set

$$V_k \triangleq \sum_{e^{-k-1}\leq 4r_j\leq e^{-k}} |O_j|.$$

Then,

$$\frac{1}{|B|}\sum_{r_j\leq\frac{1}{4}}|O_j|F(1-\ln 2r_j) \leq \frac{1}{|B|}\sum_{k\geq 0} V_k F(k+4).$$

Fix $N \in \mathbb{N}^\star$ and split the last sum into two parts according to $k \geq N$ and $k > N$ gives

$$\frac{1}{|B|}\sum_{r_j\leq\frac{1}{4}}|O_j|F(1-\ln 2r_j) \leq \frac{1}{|B|}\sum_{k\leq N} V_k F(N+4) + \frac{1}{|B|}\sum_{k>N} V_k F(k+4).$$

For $N \geq \beta(t)$, we may use Lemma 4.8 and inequality (4.12) leading to

$$\begin{aligned}
\frac{1}{|B|}\sum_{j}|O_j|F(1-\ln 2r_j) &\lesssim \|J_\psi\|_{L^\infty}\Big(F(N+4) + \sum_{k>N} e^{-\frac{k}{\beta(t)}} r^{-1} F(k+4)\Big) \\
&\lesssim \|J_\psi\|_{L^\infty}\big(1 + e^{-\frac{N}{\beta(t)}}\beta(t) r^{-1}\big)F(N+4).
\end{aligned}$$

We choose $N = \big[\beta(t)(\beta(t) - \ln r)\big]$ and by using (4.31) we obtain

$$\frac{1}{|B|}\sum_{j}|O_j|F(1-\ln 2r_j) \lesssim \|J_\psi\|_{L^\infty} F\big(1 + \beta(t)^2 + \beta(t)\big|\ln\|J_\psi\|_{L^\infty}\big|\big).$$

Putting together the previous estimates gives

$$\begin{aligned}
\fint_B \Big|f\circ\psi - \fint_B f\circ\psi\Big| &\leq \|J_{\psi^{-1}}\|_{L^\infty}\|J_\psi\|_{L^\infty}\|f\|_{BMO_F\cap L^p} F\big(\beta(t)^2 + \beta(t)\big|\ln\|J_\psi\|_{L^\infty}\big|\big) \\
&\leq C e^{C\|\mathrm{div}\, v\|_{L^1_t L^\infty}}\|f\|_{BMO_F\cap L^p} F\Big(e^{C\|v\|_{L^1_t LL}}\|\mathrm{div}\, v\|_{L^1_t L^\infty}\Big).
\end{aligned}$$

Combining this estimate with (4.30) we obtain

$$\|f\circ\psi\|_{BMO} \lesssim C\|f\|_{BMO_F\cap L^p} e^{C\|\mathrm{div}\, v\|_{L^1_t L^\infty}} F\Big(e^{C\|v\|_{L^1_t LL}}\|\mathrm{div}\, v\|_{L^1_t L^\infty}\Big).$$

In view of the polynomial growth condition of F seen in (4.11) we get

$$\|f\circ\psi\|_{BMO} \leq C\|f\|_{BMO_F\cap L^p} e^{C\|\mathrm{div}\, v\|_{L^1_t L^\infty}} F\big(e^{C\|v\|_{L^1_t LL}}\big).$$

Now we shall move to the treatment of the second part of the BMO_F norm.

• **Step 2 : Estimate of the second part of the** BMO_F **norm**. This will be done in a similar way to the first part. Denote $B_i = B(x_i, r_i)$ and $\tilde{B}_i = B(x_i, g_\psi(r_i))$ for $i \in \{1,2\}$, with $2B_2 \subset B_1$ and $r_1 < 1$. Set

$$J \triangleq \frac{\Big|\fint_{B_2} f\circ\psi - \fint_{B_1} f\circ\psi\Big|}{F\Big(\dfrac{1-\ln r_2}{1-\ln r_1}\Big)},$$

We have three cases to discuss :
<u>**Case 1**</u> : $r_1 \le (4e)^{-\beta(t)}\min\big\{1, \frac{1}{\|J_\psi\|_{L^\infty}}\big\}$. Since the denominator of the quantity J is larger than one, we may write

$$J \le J_1 + J_2 + J_3.$$

Where

$$\begin{aligned} J_1 &\triangleq \Big|\fint_{B_2} f\circ\psi - \fint_{\tilde{B}_2} f\Big| + \Big|\fint_{B_1} f\circ\psi - \fint_{\tilde{B}_1} f\Big|,\\ J_2 &\triangleq \frac{\Big|\fint_{\tilde{B}_2} f - \fint_{2\tilde{B}_1} f\Big|}{F\Big(\dfrac{1-\ln r_2}{1-\ln r_1}\Big)},\\ J_3 &\triangleq \Big|\fint_{\tilde{B}_1} f - \fint_{2\tilde{B}_1} f\Big|. \end{aligned}$$

The treatment of J_1 will be exactly the same as for the case 1 from step 1. For J_2, by definition of the second part of the BMO_F norm, we have

$$J_2 \le \frac{F\Big(\frac{1-\ln g_\psi(r_2)}{1-\ln g_\psi(r_1)}\Big)}{F\Big(\dfrac{1-\ln r_2}{1-\ln r_1}\Big)}\|f\|_{BMO_F}.$$

According to the inequality (4.26), we get

$$\begin{aligned} F\Big(\frac{1-\ln g_\psi(r_2)}{1-\ln g_\psi(r_1)}\Big) &= F\Big(\frac{1-\ln r_1}{1-\ln g_\psi(r_1)}\;\frac{1-\ln r_2}{1-\ln r_1}\;\frac{1-\ln g_\psi(r_2)}{1-\ln r_2}\Big)\\ &\lesssim F\Big((1+\beta(t))^2\frac{1-\ln r_2}{1-\ln r_1}\Big). \end{aligned}$$

This gives in view of Definition 4.1,

$$J_2 \lesssim F\big(\beta^2(t)\big)\|f\|_{BMO_F}.$$

Concerning J_3 we use the inequality (4.10) to get

$$J_3 \lesssim \|f\|_{BMO}.$$

<u>**Case 2**</u> : $r_2 \geqslant (4e)^{-\beta(t)}\min\big\{1, \frac{1}{\|J_\psi\|_{L^\infty}}\big\}$. As F is larger than 1, we write

$$J \le \fint_{B_2}|f\circ\psi| + \fint_{B_1}|f\circ\psi|.$$

Both terms can be handled as in the analysis of the case 2 from step 1.

Case 3 : $r_1 \geqslant (4e)^{-\beta(t)} \min\{1, \frac{1}{\|J_\psi\|_{L^\infty}}\}$ and $r_2 \leq (4e)^{-\beta(t)} \min\{1, \frac{1}{\|J_\psi\|_{L^\infty}}\}$. We decompose J as follows :

$$\begin{aligned} J &\leq \fint_{B_2} \Big|f\circ\psi - \fint_{\tilde{B}_2} f\Big| + \frac{\Big|\fint_{\tilde{B}_2} f\Big|}{F\Big(\frac{1-\ln r_2}{1-\ln r_1}\Big)} + \fint_{B_1} |f\circ\psi| \\ &\triangleq J_1 + J_2 + J_3. \end{aligned}$$

Let $\tilde{B}_2'$ the ball of center $\psi(x_2)$ and radius 1. Then

$$\begin{aligned} J_2 &\leq \frac{\Big|\fint_{\tilde{B}_2} f - \fint_{\tilde{B}_2'} f\Big| + \Big|\fint_{\tilde{B}_2'} f\Big|}{F\Big(\frac{1-\ln r_2}{1-\ln r_1}\Big)} \\ &\leq \frac{F(1-\ln g_\psi(r_2))}{F\Big(\frac{1-\ln r_2}{1-\ln r_1}\Big)} \|f\|_{BMO_F} + \|f\|_{L^p}. \end{aligned}$$

From Definition 4.1 combined with the inequality (4.26), we obtain

$$\begin{aligned} \frac{F(1-\ln g_\psi(r_2))}{F\Big(\frac{1-\ln r_2}{1-\ln r_1}\Big)} &= \frac{F\Big(\frac{1-\ln g_\psi(r_2)}{1-\ln r_2} \quad \frac{1-\ln r_2}{1-\ln r_1} \quad 1-\ln r_1\Big)}{F\Big(\frac{1-\ln r_2}{1-\ln r_1}\Big)} \\ &\lesssim \frac{F\Big(\frac{1-\ln r_2}{1-\ln r_1} \quad (1+\beta(t))(1-\ln r_1)\Big)}{F\Big(\frac{1-\ln r_2}{1-\ln r_1}\Big)} \\ &\lesssim F\big((1+\beta(t)(\beta + |\ln\|J_\psi\|_{L^\infty}|)\big) \\ &\lesssim F(e^{C\|v\|_{L^1_t LL}}) F\big(\|\mathrm{div} v\|_{L^1_t L^\infty}\big). \end{aligned}$$

The terms J_1 and J_3 can be handled in the same way as the cases 1 and 2 from step 1.
The proof is now achieved.
□

Our next task is to study the composition in the space $LMO_F \cap L^\infty$. This will more easier than the BMO_F space since we shall use in a crucial way the L^∞ norm.

Proposition 4.5 *There exists a positive constant C such that, for any function $f \in LMO_F \cap L^\infty$ and for ψ a flow associated to a smooth vector field v, we have*

$$\|(f\circ\psi)(t)\|_{LMO_F} \leq CF\big(e^{C\|v\|_{L^1_t LL}}\big) e^{C\|\mathrm{div}\, v\|_{L^1_t L^\infty}} \|f\|_{LMO_F \cap L^\infty}.$$

Proof : Identically to the proof of the Theorem 4.5, we will proceed in two steps ; the first one concerns the first term of the norm and the second one is devoted to the other term.

• **Step1 : Estimate of the first part of the norm.** One distinguishes two cases :

Case 1 : $r \leq (4e)^{-\beta(t)}$. We may write

$$F(1-\ln r)\fint_B \Big|f\circ\psi - \fint_B f\circ\psi\Big|dx \;\;\leq\;\; 2F(1-\ln r)\fint_B \Big|f\circ\psi - \fint_{\tilde{B}} f\Big|dx.$$

Recall that $\tilde{B}$ is the ball of center $\psi(x_0)$ and radius $g_\psi(r)$. A change of variable gives

$$F(1-\ln r)\fint_B \Big|f\circ\psi - \fint_B f\circ\psi\Big|dx \;\;\lesssim\;\; \|J_{\psi^{-1}}\|_{L^\infty}\frac{F(1-\ln r)}{|B|}\int_{\psi(B)}\Big|f-\fint_{\tilde{B}} f\Big|dx.$$

Using the Whitney covering lemma used in the proof of Theorem 4.5 we get

$$\begin{aligned}F(1-\ln r)\fint_B \Big|f\circ\psi - \fint_B f\circ\psi\Big|dx &\lesssim \|J_{\psi^{-1}}\|_{L^\infty}\frac{F(1-\ln r)}{|B|}\sum_j |O_j|\fint_{2O_j}\Big|f-\fint_{\tilde{B}} f\Big|\\ &\lesssim \|J_{\psi^{-1}}\|_{L^\infty}\big(\mathrm{I}_1+\mathrm{I}_2\big),\end{aligned}$$

where

$$\mathrm{I}_1 \triangleq \frac{F(1-\ln r)}{|B|}\sum_j |O_j|\fint_{2O_j}\Big|f-\fint_{2O_j} f\Big|,$$

and

$$\mathrm{I}_2 \triangleq \frac{F(1-\ln r)}{|B|}\sum_j |O_j|\Big|\fint_{2O_j} f-\fint_{\tilde{B}} f\Big|.$$

In view of the polynomial growth property of F, we have

$$\begin{aligned}\mathrm{I}_1 &= \frac{1}{|B|}\sum_j |O_j| F\Big(\frac{1-\ln r}{1-\ln 2r_j}\big(1-\ln(2r_j)\big)\Big)\fint_{2O_j}\Big|f-\fint_{2O_j} f\Big|\\ &\lesssim \frac{1}{|B|}\sum_j |O_j| F\Big(\frac{1-\ln r}{1-\ln(2r_j)}\Big)F(1-\ln(2r_j))\fint_{2O_j}\Big|f-\fint_{2O_j} f\Big|\\ &\lesssim \frac{1}{|B|}\sum_j |O_j| F\Big(\frac{1-\ln r}{1-\ln(2r_j)}\Big)\|f\|_{LMO_F}\end{aligned}$$

As $r_j \leq g_\psi(r)$ and by (4.26) one has

$$\begin{aligned}\frac{1-\ln r}{1-\ln 2r_j} &\lesssim \frac{1-\ln r}{1-\ln 2g_\psi(r)}\\ &\lesssim 1+\beta(t).\end{aligned}$$

Consequently we get in view of (4.27),

$$\mathrm{I}_1 \lesssim \|J_\psi\|_{L^\infty}F\big(\beta(t)\big)\|f\|_{LMO_F}.$$

Since $4O_j \subset \tilde{B}$, $g_\psi(r)\leq 1$ and by the definition of the second part of the BMO_F-norm, I_2 can be estimated as follows

$$\begin{aligned}\mathrm{I}_2 &\lesssim \frac{1}{|B|}\sum_j |O_j| F\Big(\frac{1-\ln r}{1-\ln g_\psi(r)}\big(1-\ln g_\psi(r)\big)\Big)\Big|\fint_{2O_j} f-\fint_{\tilde{B}} f\Big|\\ &\lesssim \frac{1}{|B|}\sum_j |O_j| F\big(1+\beta(t)\big)F(1-\ln g_\psi(r))\Big|\fint_{2O_j} f-\fint_{\tilde{B}} f\Big|\\ &\lesssim \|J_\psi\|_{L^\infty}F\big(\beta(t)\big)\|f\|_{LMO_F}.\end{aligned}$$

Case 2 : $1 \geqslant r \geq (4e)^{-\beta(t)}$. Under this assumption we have $|\ln r| \lesssim \beta(t)$ and then we can immediately deduce that

$$\begin{aligned} F(1-\ln r) \fint_B \Big| f\circ\psi - \fint_B f\circ\psi \Big| &\lesssim F\big(1+\beta(t)\big) \fint_B \big| f\circ\psi \big| \\ &\lesssim F\big(\beta(t)\big)\|f\|_{L^\infty}. \end{aligned}$$

Step 2 : Estimate of the second part of the norm. Denote $B_i = B(x_i, r_i)$ and $\tilde{B}_i = (x_i, g_\psi(r_i))$ for $i \in \{1,2\}$ with $2B_2 \subset B_1$ and $r_1 < 1$. We shall estimate J defined by,

$$J \triangleq F(1-\ln r_1)\Big| \fint_{B_2} f\circ\psi - \fint_{B_1} f\circ\psi \Big|.$$

There are two cases to discuss depending on the size of r_1.
Case 1 : $r_1 \leq (4e)^{-\beta(t)}$. We write

$$J \leq J_1 + J_2 + J_3,$$

where

$$\begin{aligned} J_1 &\triangleq F(1-\ln r_1)\Big(\Big| \fint_{B_2} f\circ\psi - \fint_{\tilde{B}_2} f \Big| + \Big| \fint_{B_1} f\circ\psi - \fint_{\tilde{B}_1} f \Big|\Big) \\ J_2 &\triangleq F(1-\ln r_1)\Big| \fint_{\tilde{B}_2} f - \fint_{2\tilde{B}_1} f \Big| \\ J_3 &\triangleq F(1-\ln r_1)\Big| \fint_{\tilde{B}_1} f - \fint_{2\tilde{B}_1} f \Big|. \end{aligned}$$

Reproducing the same arguments as for the case 1 from step 1 we can estimate J_1. For J_2, we use the polynomial growth property of F with the inequality (4.26) to get

$$\begin{aligned} J_2 &\leq F\Big(\frac{1-\ln r_1}{1-\ln 2g_\psi(r_1)}\big(1-\ln 2g_\psi(r_1)\big)\Big)\Big| \fint_{\tilde{B}_2} f - \fint_{2\tilde{B}_1} f \Big| \\ &\leq F\Big(\frac{1-\ln r_1}{1-\ln 2g_\psi(r_1)}\Big) F\big(1-\ln 2g_\psi(r_1)\big)\Big| \fint_{\tilde{B}_2} f - \fint_{2\tilde{B}_1} f \Big| \\ &\leq F\big(1+\beta(t)\big)\|f\|_{LMO_F}. \end{aligned}$$

Similarly,

$$\begin{aligned} J_3 &\leq F\Big(\frac{1-\ln r_1}{1-\ln 2g_\psi(r_1)}\Big) F\big(1-\ln 2g_\psi(r_1)\big)\Big| \fint_{\tilde{B}_1} f - \fint_{2\tilde{B}_1} f \Big| \\ &\leq F\big(1+\beta(t)\big)\|f\|_{LMO_F}. \end{aligned}$$

Case 2 : $r_1 \geqslant (4e)^{-\beta(t)}$. Since F is non-decreasing then,

$$\begin{aligned} J &\leq F(1-\ln r_1)\Big(\fint_{B_2} |f\circ\psi| + \fint_{B_1} |f\circ\psi| \Big) \\ &\lesssim F(\beta(t))\|f\|_{L^\infty}. \end{aligned}$$

This completes the proof of Proposition 4.5. □

Now we have all the necessary ingredients for the proof of Theorem 4.2

4.3.3 Proof of Theorem 2

We set $g(t, x) = f(t, \psi(t, x))$. Then, in view of (4.23), we see that g satisfies the following equation

$$\partial_t g(t, x) + (\text{div}\, v)(t, \psi(t, x))g(t, x) = 0, \quad g(0, x) = f_0(x).$$

It follows that

$$\begin{aligned} g(t, x) &= f_0(x)e^{-\int_0^t (\text{div}\, v)(\tau,\psi(\tau))d\tau} \\ &= f_0(x) + f_0(x)\Big(e^{-\int_0^t (\text{div}\, v)(\tau,\psi(\tau))d\tau} - 1\Big). \end{aligned}$$

According to the law product stated in Proposition 4.3, we have

$$\|g(t)\|_{BMO_F\cap L^p} \leq C\|f_0\|_{BMO_F\cap L^p}\Big(1 + \big\|e^{-\int_0^t (\text{div}\, v)(\tau,\psi(\tau))d\tau} - 1\big\|_{LMO_F\cap L^\infty}\Big).$$

Therefore by applying the assertion (ii) of Proposition 4.4 to the function $x \longmapsto e^x - 1$, we get

$$\|g(t)\|_{BMO_F\cap L^p} \leq C\|f_0\|_{BMO_F\cap L^p}\Big(1 + e^{C\|\text{div}\, v\|_{L^1_t L^\infty}} \int_0^t \|(\text{div}\, v)(\tau, \psi(\tau))\|_{LMO_F\cap L^\infty} d\tau\Big).$$

Furthermore, according to Proposition 4.5 we infer that

$$\|g(t)\|_{BMO_F\cap L^p} \leq C\|f_0\|_{BMO_F\cap L^p}\Big(1 + e^{C\|\text{div}\, v\|_{L^1_t L^\infty}} F\Big(e^{C\|v\|_{L^1_t LL}}\Big)\|\text{div}\, v\|_{L^1_t(LMO_F\cap L^\infty)}\Big).$$

Finally, Theorem 4.3 gives

$$\begin{aligned} \|f(t)\|_{BMO_F\cap L^p} \leq\ & C \quad \|f_0\|_{BMO_F\cap L^p} e^{C\|\text{div}\, v\|_{L^1_t L^\infty}} F\Big(e^{C\|v\|_{L^1_t LL}}\Big) \\ & \times \quad \Big(1 + F\big(e^{C\|v\|_{L^1_t LL}}\big)\|\text{div}\, v\|_{L^1_t(LMO_F\cap L^\infty)}\Big). \end{aligned}$$

This completes the proof of the theorem.

4.4 Some classical estimates

The aim of this section is to highlight two useful estimates for the system (E.C), those will be of great importance for obtaining a lower bound of the lifespan of the solution v_ε. In the first instance we shall recall classical energy estimates for the full system, afterwards, we lead a short discussion on Strichartz estimates for the wave operator.

4.4.1 Energy estimates

The following energy estimates are classical and for the proof we refer the reader for instance to [41, 58, 77].

Proposition 4.6 *Let* $(v_\varepsilon, c_\varepsilon)$ *be a smooth solution of* (E.C). *For* $s > 0$ *there exists a constant* $C > 0$ *such that*

$$\|(v_\varepsilon, c_\varepsilon)(t)\|_{H^s} \leq C\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{H^s} e^{CV_\varepsilon(t)}, \quad \forall t \geq 0$$

with

$$V_\varepsilon(t) \triangleq \|\nabla v_\varepsilon\|_{L^1_t L^\infty} + \|\nabla c_\varepsilon\|_{L^1_t L^\infty}.$$

4.4.2 Strichartz estimates

The main interest of using Strichartz estimates is to deal with the ill-prepared data in the presence of the singular terms in $\frac{1}{\varepsilon}$. Actually, it has been shown that the average in time of the compressible and the acoustic parts, which are governed by a coupling non-linear wave equations, disappear when the Mach number approaches zero. The details of this assumption has been discussed for instance in [58] and [68] for initial data in Besov spaces, but for the convenience of the reader we briefly outline the main arguments of the proof. The system (E.C) can be rewritten under the form

$$\begin{cases} \partial_t v_\varepsilon + \frac{1}{\varepsilon}\nabla c_\varepsilon = -v_\varepsilon.\nabla v_\varepsilon - \bar{\gamma} c_\varepsilon \nabla c_\varepsilon \triangleq f_\varepsilon \\ \partial_t c_\varepsilon + \frac{1}{\varepsilon}\text{div}\, v_\varepsilon = -v_\varepsilon.\nabla c_\varepsilon - \bar{\gamma} c_\varepsilon \text{div}\, v_\varepsilon \triangleq g_\varepsilon \\ (v_\varepsilon, c_\varepsilon)_{|t=0} = (v_{\varepsilon,0}, c_{\varepsilon,0}). \end{cases}$$

We denote by $\mathbb{Q}v_\varepsilon \triangleq \nabla\Delta^{-1}\text{div}\, v_\varepsilon$ the compressible part of the velocity v_ε. Then the complex-valued functions

$$\Gamma_\varepsilon \triangleq \mathbb{Q}v_\varepsilon - i\nabla|\mathrm{D}|^{-1}c_\varepsilon \quad \text{and} \quad \Upsilon_\varepsilon \triangleq |\mathrm{D}|^{-1}\text{div}\, v_\varepsilon + ic_\varepsilon$$

satisfy the following wave equations

$$(\partial_t + \frac{i}{\varepsilon}|\mathrm{D}|)\Gamma_\varepsilon = \mathbb{Q}f_\varepsilon - i\nabla|\mathrm{D}|^{-1}g_\varepsilon \tag{4.32}$$

and

$$(\partial_t + \frac{i}{\varepsilon}|\mathrm{D}|)\Upsilon_\varepsilon = |\mathrm{D}|^{-1}\text{div}\, f_\varepsilon - ig_\varepsilon \tag{4.33}$$

with $|\mathrm{D}| = (-\Delta)^{\frac{1}{2}}$.
Now we can following Strichartz estimates whose proof can be found for instance in [8, 41, 49].

Lemma 4.7 *Let φ be a solution of the wave equation*

$$(\partial_t + \frac{i}{\varepsilon}|\mathrm{D}|)\varphi = G, \quad \varphi_{|t=0} = \varphi_0.$$

Then, there exists a constant $C > 0$ such that for all $T > 0$ and $2 < p \leq +\infty$,

$$\|\varphi\|_{L^r_T L^p} \leq C\varepsilon^{\frac{1}{4}-\frac{1}{2p}}\Big(\|\varphi_0\|_{\dot{B}^{\frac{3}{4}-\frac{3}{2p}}_{2,1}} + \int_0^T \|G(\tau)\|_{\dot{B}^{\frac{3}{4}-\frac{3}{2p}}_{2,1}} d\tau\Big),$$

with $r = 4 + \frac{8}{p-2}$.

As an application we get the following result.

Corollary 4.1 *Let $s > 0$ and $(v_{0,\varepsilon}, c_{0,\varepsilon})$ be a family in H^{2+s}. Then any solution of (E.C) defined in the time interval $[0, T]$ satisfies*

$$\|(\mathbb{Q}v_\varepsilon, c_\varepsilon)\|_{L^4_T L^\infty} \leq C_0^\varepsilon \varepsilon^{\frac{1}{4}}(1+T)e^{CV_\varepsilon(T)}. \tag{4.34}$$

Moreover, there exists a positive real number η which depends only on s such that

$$\|(\text{div}\, v_\varepsilon, \nabla c_\varepsilon)\|_{L^1_T B^{s/3}_{\infty,\infty}} \leq C_0^\varepsilon(1 + T^{\frac{7}{4}})\varepsilon^\eta e^{CV_\varepsilon(T)}. \tag{4.35}$$

Where

$$V_\varepsilon(T) \triangleq \|\nabla v_\varepsilon\|_{L^1_T L^\infty} + \|\nabla c_\varepsilon\|_{L^1_T L^\infty},$$

and C_0^ε depends only on the quantity $\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{H^{2+s}}$ and with polynomial growth.

Proof : Applying Lemma 4.7 to Eq. (4.32), we get

$$\|\Gamma_\varepsilon\|_{L^4_T L^\infty} \lesssim \varepsilon^{\frac{1}{4}}\Big(\|\Gamma^0_\varepsilon\|_{\dot{B}^{\frac{3}{4}}_{2,1}} + \int_0^T \|(\mathbb{Q}f_\varepsilon - i\nabla|\mathrm{D}|^{-1}g_\varepsilon)(\tau)\|_{\dot{B}^{\frac{3}{4}}_{2,1}}\,d\tau\Big).$$

Since $\mathbb{Q}$ and $\nabla|\mathrm{D}|^{-1}$ act continuously on the homogeneous Besov spaces, we get

$$\begin{aligned}\|\Gamma_\varepsilon\|_{L^4_T L^\infty} &\lesssim \varepsilon^{\frac{1}{4}}\Big(\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{\dot{B}^{\frac{3}{4}}_{2,1}} + \int_0^T \|(f_\varepsilon, g_\varepsilon)(\tau)\|_{\dot{B}^{\frac{3}{4}}_{2,1}}\,d\tau\Big)\\ &\lesssim \varepsilon^{\frac{1}{4}}\Big(\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{H^1} + T\|(f_\varepsilon, g_\varepsilon)\|_{L^\infty_T H^1}\Big). \end{aligned} \tag{4.36}$$

Where we have used in the last inequality the fact that $H^1 \hookrightarrow B^{\frac{3}{4}}_{2,1} \hookrightarrow \dot{B}^{\frac{3}{4}}_{2,1}$.
To estimate $\|(f_\varepsilon, g_\varepsilon)\|_{H^1}$ we use the following law product

$$\|u\cdot\nabla w\|_{H^1} \lesssim \|u\|_{L^\infty}\|w\|_{H^2} + \|w\|_{L^\infty}\|u\|_{H^2}.$$

Then, by definition of $(f_\varepsilon, g_\varepsilon)$ we have

$$\|(f_\varepsilon, g_\varepsilon)\|_{H^1} \lesssim \|(v_\varepsilon, c_\varepsilon)\|_{L^\infty}\|(v_\varepsilon, c_\varepsilon)\|_{H^2}.$$

Using the embedding $H^2 \hookrightarrow L^\infty$ combined with the energy estimates, we get

$$\begin{aligned}\|(f_\varepsilon, g_\varepsilon)\|_{L^\infty_T H^1} &\lesssim \|(v_\varepsilon, c_\varepsilon)\|^2_{L^\infty_T H^2}\\ &\lesssim \|(v_{0,\varepsilon}, c_{0,\varepsilon})\|^2_{H^2} e^{CV_\varepsilon(T)}.\end{aligned}$$

Inserting this into the estimate (4.36) we find

$$\begin{aligned}\|\Gamma_\varepsilon\|_{L^4_T L^\infty} &\lesssim \varepsilon^{\frac{1}{4}}\Big(\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{H^2} + T\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|^2_{H^2} e^{CV_\varepsilon(T)}\Big)\\ &\lesssim C_0^\varepsilon \varepsilon^{\frac{1}{4}}(1+T)e^{CV_\varepsilon(T)}.\end{aligned}$$

As the compressible part of v_ε is the imaginary part of Γ_ε, then

$$\|\mathbb{Q}v_\varepsilon\|_{L^4_T L^\infty} \lesssim C_0^\varepsilon \varepsilon^{\frac{1}{4}}(1+T)e^{CV_\varepsilon(T)}.$$

By the same manner, we use (4.33) to prove

$$\|c_\varepsilon\|_{L^4_T L^\infty} \lesssim C_0^\varepsilon \varepsilon^{\frac{1}{4}}(1+T)e^{CV_\varepsilon(T)}.$$

To prove the second estimate we use an interpolation procedure between the Strichartz estimates for lower frequencies and the energy estimates for higher frequencies. More precisely, consider N an integer that will be judiciously fixed later. Then, using the embedding $B^{s/3}_{\infty,1} \hookrightarrow B^{s/3}_{\infty,\infty}$, Bernstein inequality and the continuity of the operator $\mathbb{Q}$ on the Lebesgue space L^2, we find

$$\begin{aligned}\|\mathrm{div}\, v_\varepsilon\|_{B^{s/3}_{\infty,\infty}} &= \|\mathrm{div}\,\mathbb{Q}v_\varepsilon\|_{B^{s/3}_{\infty,\infty}}\\ &\leq \sum_{q<N} 2^{q\frac{s}{3}}\|\Delta_q \mathrm{div}\,\mathbb{Q}v_\varepsilon\|_{L^\infty} + \sum_{q\geqslant N} 2^{q\frac{s}{3}}\|\Delta_q \mathrm{div}\,\mathbb{Q}v_\varepsilon\|_{L^\infty}\\ &\lesssim \sum_{q<N} 2^{q(\frac{s}{3}+1)}\|\Delta_q\mathbb{Q}v_\varepsilon\|_{L^\infty} + 2^{-N\frac{s}{3}}\sum_{q\geqslant N} 2^{q(\frac{2s}{3}+2)}\|\Delta_q\mathbb{Q}v_\varepsilon\|_{L^2}\\ &\lesssim 2^{N(\frac{s}{3}+1)}\|\mathbb{Q}v_\varepsilon\|_{L^\infty} + 2^{-N\frac{s}{3}}\|v_\varepsilon\|_{B^{\frac{2s}{3}+2}_{2,1}}\\ &\lesssim 2^{N(\frac{s}{3}+1)}\|\mathbb{Q}v_\varepsilon\|_{L^\infty} + 2^{-N\frac{s}{3}}\|v_\varepsilon\|_{H^{2+s}}.\end{aligned}$$

Where we have used in last inequality the embedding $H^{2+s} \hookrightarrow B_{2,1}^{\frac{2s}{3}+2}$. Integrating in time and combining (4.34) with the energy estimates we get

$$\begin{aligned}\|\mathrm{div}\, v_\varepsilon\|_{L^1_T B^{s/3}_{\infty,\infty}} &\lesssim C_0^\varepsilon 2^{N(\frac{s}{3}+1)}\varepsilon^{\frac14}T^{\frac34}(1+T)e^{CV_\varepsilon(t)} + C_0^\varepsilon 2^{-\frac{s}{3}N}Te^{CV_\varepsilon(T)}\\ &\lesssim C_0^\varepsilon(1+T^{\frac74})\big(2^{N(\frac{s}{3}+1)}\varepsilon^{\frac14}+2^{-\frac{s}{3}N}\big)e^{CV_\varepsilon(T)}.\end{aligned}$$

By similar computations, we get

$$\|\nabla c_\varepsilon\|_{L^1_T B^{s/3}_{\infty,\infty}} \lesssim C_0^\varepsilon(1+T^{\frac74})\big(2^{N(\frac{s}{3}+1)}\varepsilon^{\frac14}+2^{-\frac{s}{3}N}\big)e^{CV_\varepsilon(T)}.$$

We choose N such that $2^{N(\frac{2s}{3}+1)} \approx \varepsilon^{\frac{-1}{4}}$. This is equivalent to take

$$N \approx \frac{1}{4(\frac{2s}{3}+1)}\log_2 \varepsilon^{-1}.$$

Consequently

$$\|(\mathrm{div}\, v_\varepsilon, \nabla c_\varepsilon)\|_{L^1_t B^{s/3}_{\infty,\infty}} \lesssim C_0^\varepsilon(1+t^{\frac74})\varepsilon^{\frac{s}{4(2s+3)}}e^{CV_\varepsilon(t)}.$$

This ends the proof of the corollary.
□

We are now in position to prove our main theorem.

4.5 Main result

In this section, we shall state more general result than Theorem 1, afterwards, the rest of this section will be devoted to the discussion of the proof. Our result reads as follows.

Theorem 4.4 *Let $s, \alpha \in]0,1[$, $p \in]1,2[$ and $F \in \mathcal{F}$. Consider a family of initial data $\{(v_{0,\varepsilon}, c_{0,\varepsilon})_{0<\varepsilon<1}\}$ such that there exists a positive constant C which does not depend on ε and verifying*

$$\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{H^{2+s}} \le C(\log \varepsilon^{-1})^\alpha,$$

$$\|\omega_{0,\varepsilon}\|_{L^p\cap BMO_F} \le C.$$

Then, the system (E.C) *admits a unique solution $(v_\varepsilon, c_\varepsilon) \in C\big([0,T_\varepsilon], H^{2+s}\big)$ with the alternative :*

(1) If F belongs to the class $\mathcal{F}'$ then the lifespan T_ε of the solution satisfies the lower bound :

$$T \geqslant \frac{1}{C_0}\mathcal{M}\Big((1-\alpha)\ln\ln\varepsilon^{-1}\Big) \triangleq \tilde{T}_\varepsilon.$$

and the vorticity ω_ε satisfies

$$\forall t \in [0,\tilde{T}_\varepsilon], \quad \|\omega_\varepsilon(t)\|_{BMO_F\cap L^p} \le C_0(\mathcal{M}^{-1})'\big(C_0(1+t)\big). \tag{4.37}$$

Where $\mathcal{M} : [0,+\infty[\longrightarrow [0,+\infty[$ is defined by

$$\mathcal{M}(x) \triangleq \int_0^x \frac{dy}{F(e^{Cy})}$$

and $(\mathcal{M}^{-1})'$ denotes the derivative of $\mathcal{M}^{-1}$.

(2) *If F belongs to the class $\mathcal{F}\backslash\mathcal{F}'$ then there exists $T_0 > 0$ independent of ε such that for all $t \leq T_0$ we have*

$$\|\omega_\varepsilon(t)\|_{BMO_F\cap L^p} \quad \leq \quad C_0. \tag{4.38}$$

Moreover, in both cases the compressible and acoustic parts of the solutions tend to zero : there exists $\eta > 0$ such that for small ε and for all $0 \leq T \leq \tilde{T}_\varepsilon$ (respectively $0 \leq T \leq T_0$)

$$\|(\operatorname{div} v_\varepsilon, \nabla c_\varepsilon)\|_{L^1_T B^{s/3}_{\infty,\infty}} \lesssim \varepsilon^{\eta/2}. \tag{4.39}$$

Assume in addition that $\lim_{\varepsilon\to 0}\|\omega_{0,\varepsilon}-\omega_0\|_{L^p} = 0$, for some vorticity $\omega_0 \in BMO_F \cap L^p$ associated to a divergence-free vector field v_0. Then, the vortices $(\omega_\varepsilon)_\varepsilon$ converge strongly to the solution ω of (4.1) *associated to the initial data ω_0. More precisely, for all $t \in \mathbb{R}_+$ $(0 \leq t \leq T_0$ respectively$)$*

$$\lim_{\varepsilon\to 0}\|\big(\omega_\varepsilon - \omega\big)(t)\|_{L^q} = 0 \quad \forall q \in [p, +\infty[.$$

Furthermore,

1. *if $F \in \mathcal{F}'$ then for all $t \in \mathbb{R}_+$*

$$\|\omega(t)\|_{BMO_F\cap L^p} \quad \leq \quad C_0(\mathcal{M}^{-1})'\big(C_0(1+t)\big). \tag{4.40}$$

2. *if $F \in \mathcal{F}\backslash\mathcal{F}'$ then for all $t < T_0$*

$$\|\omega(t)\|_{L^p\cap BMO_F} \leq C_0. \tag{4.41}$$

Remark 4.5 *Theorem* 4.4 *recovers the local and global well-posedness theory of the incompressible Euler system according to the inequalities* (4.40) *and* (4.41).

The proof of Theorem 4.4 will be divided into two parts : in the first one we estimate the lifespan of the solutions, thereafter, we discuss in the second part the incompressible limit problem.

4.5.1 Lower bound of the lifespan

We will give an a priori bound of T_ε and show that the acoustic parts vanish when the Mach number goes to zero. We denote by

$$W_\varepsilon(t) \triangleq \|v_\varepsilon\|_{L^1_t LL} \quad \text{and} \quad V_\varepsilon(t) \triangleq \|\nabla v_\varepsilon\|_{L^1_t L^\infty} + \|\nabla c_\varepsilon\|_{L^1_t L^\infty}.$$

In view of Theorem 4.2 and using the embedding (iii) from Proposition 4.4, we have

$$\|\omega_\varepsilon(t)\|_{BMO_F\cap L^p} \leq C_0 F\Big(e^{CW_\varepsilon(t)}\Big)\bigg(1 + F\Big(e^{CW_\varepsilon(t)}\Big)\|\operatorname{div} v_\varepsilon\|_{L^1_t B^{\frac{s}{3}}_{\infty,\infty}}\bigg)e^{C\|\operatorname{div} v_\varepsilon\|_{L^1_t L^\infty}}.$$

According to Remark 4.2, the function F has at most a polynomial growth : $F(x) \lesssim 1 + x^\beta$. Also, since $\|v_\varepsilon\|_{LL} \lesssim \|\nabla v_\varepsilon\|_{L^\infty}$ we may write

$$\|\omega_\varepsilon(t)\|_{BMO_F\cap L^p} \quad \leq \quad C_0 F\Big(e^{CW_\varepsilon(t)}\Big)\bigg(1 + e^{CV_\varepsilon(t)}\|\operatorname{div} v_\varepsilon\|_{L^1_t B^{\frac{s}{3}}_{\infty,\infty}}\bigg)e^{C\|\operatorname{div} v_\varepsilon\|_{L^1_t L^\infty}}.$$

It follows from Corollary 4.1 that

$$\begin{aligned}\|\omega_\varepsilon(t)\|_{BMO_F\cap L^p} \quad &\leq \quad C_0 F\Big(e^{C\|v_\varepsilon\|_{L^1_t LL}}\Big)\Big(1 + C_0^\varepsilon(1+t^{\frac{7}{4}})\varepsilon^\eta e^{CV_\varepsilon(t)}\Big)e^{C_0^\varepsilon(1+t^{\frac{7}{4}})\varepsilon^\eta e^{CV_\varepsilon(t)}}\\ &\leq \quad C_0 F\Big(e^{CW_\varepsilon(t)}\Big)e^{C_0^\varepsilon(1+t^{\frac{7}{4}})\varepsilon^\eta e^{CV_\varepsilon(t)}},\end{aligned} \tag{4.42}$$

we recall that C_0^ε is a positive constant depending polynomially on $\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{H^{2+s}}$.
Now Lemma 4.2 and the embeddings $B^s_{\infty,\infty} \hookrightarrow BMO \hookrightarrow B^0_{\infty,\infty}$ ensure that

$$\|v_\varepsilon\|_{LL} \quad \lesssim \quad \|\omega_\varepsilon\|_{BMO_F\cap L^p} + \|\mathbb{Q}v_\varepsilon\|_{L^\infty} + \|\text{div}\, v_\varepsilon\|_{B^s_{\infty,\infty}}. \tag{4.43}$$

Integrating in time and using Corollary 4.1 we get

$$W_\varepsilon(t) \quad \leq \quad C\int_0^t \|\omega_\varepsilon(\tau)\|_{BMO_F\cap L^p} d\tau + C_0^\varepsilon(1+t^{\frac{7}{4}})\varepsilon^\eta e^{CV_\varepsilon(t)}. \tag{4.44}$$

Setting

$$\rho_\varepsilon(t) \triangleq C_0^\varepsilon(1+t^{\frac{7}{4}})\varepsilon^\eta e^{CV_\varepsilon(t)}$$

and inserting the estimate (4.42) into (4.44) give

$$W_\varepsilon(t) \leq C_0\int_0^t F\big(e^{CW_\varepsilon(\tau)}\big)e^{\rho_\varepsilon(\tau)}d\tau + \rho_\varepsilon(t). \tag{4.45}$$

At this stage we distinguish two cases depending whether $F \in \mathcal{F}'$ or not.
(1) If $F \in \mathcal{F}'$, we fix $T > 0$ an arbitrary real number. So the inequality (4.45) becomes

$$\forall t \in [0,T] \quad W_\varepsilon(t) \quad \leq \quad C_0\int_0^t F\big(e^{CW_\varepsilon(\tau)}\big)e^{\rho_\varepsilon(\tau)}d\tau + \rho_\varepsilon(T).$$

We introduce the function $\mathcal{M} : [0,+\infty[\longrightarrow [0,+\infty[$ defined by

$$\mathcal{M}(x) \triangleq \int_0^x \frac{dy}{F(e^{Cy})}.$$

Since $\mathcal{M}$ is a nondecreasing function and $\lim_{x\to\infty} \mathcal{M}(x) = +\infty$ then $\mathcal{M}$ is one-to one and Lemma 4.3 implies that

$$\forall t \in [0,T] \quad W_\varepsilon(t) \quad \leq \quad \mathcal{M}^{-1}\Big(\mathcal{M}\big(\rho_\varepsilon(T)\big) + C_0 e^{\rho_\varepsilon(t)}t\Big).$$

Then,

$$W_\varepsilon(T) \quad \leq \quad \mathcal{M}^{-1}\Big(\mathcal{M}\big(\rho_\varepsilon(T)\big) + C_0 e^{\rho_\varepsilon(T)}T\Big).$$

Inserting this estimate into (4.42) leads to

$$\|\omega_\varepsilon(T)\|_{BMO_F\cap L^p} \quad \leq \quad C_0 F\Big(e^{C\mathcal{M}^{-1}\big(\mathcal{M}(\rho_\varepsilon(T)) + C_0 e^{\rho_\varepsilon(T)}T\big)}\Big)e^{\rho_\varepsilon(T)}. \tag{4.46}$$

Now we need the following estimate whose proof is given in the Appendix :

$$\|\nabla v_\varepsilon(T)\|_{L^\infty} \lesssim \|(v_{0,\varepsilon}, c_{0\varepsilon})\|_{H^{s+2}} + \|\omega_\varepsilon(T)\|_{BMO\cap L^p} V_\varepsilon(T).$$

This, combined with Corollary 4.1 yield

$$V_\varepsilon(T) \leq \rho_\varepsilon(T) + C_0^\varepsilon T + C\int_0^T \|\omega_\varepsilon(t)\|_{BMO\cap L^p} V_\varepsilon(t)dt.$$

Hence Gronwall's inequality implies that

$$V_\varepsilon(T) \leq \big(C_0^\varepsilon T + \rho_\varepsilon(T)\big)\exp\Big(C\int_0^T \|\omega_\varepsilon(t)\|_{BMO\cap L^p}dt\Big). \tag{4.47}$$

Putting (4.46) in the last estimate we get

$$V_\varepsilon(T) \leq \big(C_0^\varepsilon T + \rho_\varepsilon(T)\big)\exp\Big(C_0 e^{\rho_\varepsilon(T)}\int_0^T F\big(e^{C\mathcal{M}^{-1}(\mathcal{M}(\rho_\varepsilon(t))+C_0 e^{\rho_\varepsilon(t)}t)}\big)dt\Big).$$

Assuming $\rho_\varepsilon(T) \leq 1$, which is true at least for small T, and using the fact that $\mathcal{M}^{-1}$ is non-decreasing we find

$$V_\varepsilon(T) \lesssim C_0^\varepsilon(1+T)\exp\Big(C_0\int_0^T F\big(e^{C\mathcal{M}^{-1}(C_0(1+t))}\big)dt\Big).$$

Moreover the inequality (4.46) becomes

$$\|\omega_\varepsilon(T)\|_{BMO_F\cap L^p} \leq C_0 F\Big(e^{C\mathcal{M}^{-1}\big(C_0(1+T)\big)}\Big).$$

A straightforward computation gives

$$F\big(e^{C\mathcal{M}^{-1}(x)}\big) = (\mathcal{M}^{-1})'(x).$$

Then we immediately deduce that

$$\|\omega_\varepsilon(T)\|_{BMO_F\cap L^p} \leq C_0(\mathcal{M}^{-1})'\big(C_0(1+T)\big),$$

and

$$V_\varepsilon(T) \lesssim C_0^\varepsilon(1+T)\exp\Big(\mathcal{M}^{-1}\big(C_0(1+T)\big)\Big)$$

So from the condition $\rho_\varepsilon(T) \leq 1$ we can conclude a lower bound of the lifespans of the solution. Indeed, we have

$$\begin{aligned}\rho_\varepsilon(T) &= C_0^\varepsilon \varepsilon^\eta(1+T^{\frac{7}{4}})e^{CV_\varepsilon(T)}\\ &\leq \varepsilon^\eta\exp\Big(C_0^\varepsilon(1+T)e^{\mathcal{M}^{-1}\big(C_0(1+T)\big)}\Big).\end{aligned}$$

Now let $\alpha(\varepsilon)$ be a function going to ∞ as ε approaches zero and choosing T such that

$$C_0(1+T) = \mathcal{M}(\alpha(\varepsilon))$$

Then in order to get $\rho_\varepsilon(T) \leq \varepsilon^{\frac{\eta}{2}}$ we should impose the constraint

$$\exp\Big(C_0^\varepsilon\mathcal{M}(\alpha(\varepsilon))e^{\alpha(\varepsilon)}\Big) \leq \varepsilon^{-\frac{\eta}{2}}.$$

Since $F(x) \geq 1$, for $x \geq 1$ then from the definition of $\mathcal{M}$ we infer that

$$\mathcal{M}(x) \leq x$$

and thus to get the preceding inequality it suffices to assume

$$\exp\Big(C_0^\varepsilon e^{\alpha(\varepsilon)}\Big) \leq \varepsilon^{-\frac{\eta}{2}}.$$

At this stage we see that one can impose the following conditions,

$$C_0^\varepsilon \leq C(\ln\varepsilon^{-1})^\alpha \quad \text{and} \quad \alpha(\varepsilon) \approx (1-\alpha)\ln\ln\frac{1}{\varepsilon}$$

for some $\alpha \in (0,1)$. In particular, from Corollary 4.1, we conclude that

$$\|(\text{div}v_\varepsilon, \nabla c_\varepsilon)\|_{(L^1_T\cap L^4_T)B^{\frac{s}{3}}_{\infty,\infty}} + \|(\mathbb{Q}v_\varepsilon, c_\varepsilon)\|_{(L^1_T\cap L^4_T)L^\infty} \lesssim \varepsilon^{\eta/2} \tag{4.48}$$

(2) Let $F \in \mathcal{F}\backslash\mathcal{F}'$ then we return to the estimate (4.45) and we assume that $\rho_\varepsilon(t) \leq 1$. Using again the fact that F has at most a polynomial growth, we get

$$W_\varepsilon(t) \leq C_0 e^{CW_\varepsilon(t)}t + 1.$$

Consequently we can find $T_0 \in (0,1)$ independent of ε such that

$$\forall t \in [0, T_0] \quad W_\varepsilon(t) \leq 2.$$

Plugging this estimate into (4.42) leads to

$$\|\omega_\varepsilon(t)\|_{BMO_F\cap L^p} \leq C_0.$$

This combined with (4.47) and the constraint on ρ_ε give

$$\begin{aligned} V_\varepsilon(T) &\leq C_0^\varepsilon(T+1)e^{C_0 T} \\ &\lesssim C_0^\varepsilon. \end{aligned}$$

Hence,

$$\begin{aligned} \rho_\varepsilon(T) &= C_0^\varepsilon \varepsilon^\eta(1+T^{\frac{7}{4}})e^{CV_\varepsilon(T)} \\ &\lesssim \varepsilon^\eta e^{2C_0^\varepsilon}. \end{aligned}$$

Choosing $C_0^\varepsilon \leq C(\ln \varepsilon^{-1})^\alpha$, the last expression will be bounded by $\varepsilon^{\frac{\eta}{2}}$ and thus we find $\rho_\varepsilon(t) \leq 1$.

This concludes the proof of the first part of Theorem 4.4.

4.5.2 Incompressible limit

Proof : As it has already pointed out, the vorticity ω_ε has the following structure

$$\omega_\varepsilon(t) = \omega_{0,\varepsilon}(\psi_\varepsilon^{-1}(t))e^{-\int_0^t \text{div}v_\varepsilon(\tau,\psi(\tau,\psi_\varepsilon^{-1}(t)))d\tau}. \tag{4.49}$$

So the question of the convergence of the vortices $\{\omega_\varepsilon\}$ can be examined through the convergence of the flow maps $\{\psi_\varepsilon\}$. In other words, we shall establish the existence of the particle trajectories ψ as a uniform limit of a subsequence of $\{\psi_\varepsilon\}$. Once this flow is constructed, we can propose a candidate for the solution of the incompressible Euler system given by

$$\omega(t,x) = \omega_0(\psi^{-1}(t,x)) \quad \text{and} \quad v(t) = K \star \omega(t), \quad K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}. \tag{4.50}$$

At this stage and in order to get a solution for (E.I) we need to show that ψ is the flow map associated to the velocity v. For this goal we develop strong convergence results of the velocities $\{v_\varepsilon\}$.

To begin with, let T and R be two positive real numbers such that $T \leq \tilde{T}_\varepsilon$. Then, for all $t \in [0,T]$ and $x \in \bar{B}(0,R)$ we use the integral equation of the flow ψ_ε to get

$$\begin{aligned} |\psi_\varepsilon^{-1}(t,x) - x| &= \left|\int_0^t v_\varepsilon(\tau, \psi_\varepsilon(\tau, \psi_\varepsilon^{-1}(t,x)))d\tau\right| \\ &\leq \int_0^t \|v_\varepsilon(\tau)\|_{L^\infty} d\tau \\ &\leq \|\mathbb{P}v_\varepsilon\|_{L^1_T L^\infty} + \|\mathbb{Q}v_\varepsilon(\tau)\|_{L^1_T L^\infty}. \end{aligned} \tag{4.51}$$

Since the incompressible part $\mathbb{P}v_\varepsilon$ has the same vorticity as the total velocity

$$\text{curl}\,\mathbb{P}v_\varepsilon = \text{curl}\,v_\varepsilon,$$

and $1 < p < 2$, the Biot-Savart law implies that

$$\|\mathbb{P}v_\varepsilon(\tau)\|_{L^\infty} \lesssim \|\omega_\varepsilon(\tau)\|_{L^p\cap L^{2p}}. \tag{4.52}$$

Hence, according to the identity (4.49) we find

$$\|\mathbb{P}v_\varepsilon(\tau)\|_{L^\infty} \quad \lesssim \quad \|\omega_{0,\varepsilon}(\psi_\varepsilon^{-1}(\tau))\|_{L^p\cap L^{2p}} e^{\|\text{div}\,v_\varepsilon\|_{L^1_T L^\infty}}.$$

Using a change of variable combined with Lemma 4.6 and the estimate (4.48) we get

$$\begin{aligned}\|\mathbb{P}v_\varepsilon(\tau)\|_{L^\infty} &\lesssim \|\omega_{0,\varepsilon}\|_{L^p\cap L^{2p}} e^{C\|\text{div}\,v_\varepsilon\|_{L^1_T L^\infty}}\\ &\lesssim \|\omega_{0,\varepsilon}\|_{L^p\cap BMO} e^{C\varepsilon^{\eta/2}}\\ &\leq C_0,\end{aligned} \tag{4.53}$$

where we have used the classical interpolation inequality (4.17). Plugging the estimates (4.48) and (4.53) into (4.51) we find

$$|\psi_\varepsilon^{-1}(t,x)| \quad \leq \quad C_0 T + R + 1. \tag{4.54}$$

So the family $\{\psi_\varepsilon^{-1}\}$ is uniformly bounded on every compact $[0,T]\times\bar{B}(0,R)$ and it remains to study its equicontinuity. According to Lemma 4.4 we have

$$\forall (x,y)\in\bar{B}(0,R)^2,\quad |x-y|\leq e^{-\exp\left(\|v_\varepsilon\|_{L^1_T LL}\right)} \Rightarrow |\psi_\epsilon^{-1}(t,x)-\psi_\epsilon^{-1}(t,y)|\leq e|x-y|^{\exp(-\|v_\varepsilon\|_{L^1_T LL})}.$$

But estimate (4.43) combined with (4.37) and (4.48) ensures the existence of an explicit time continuous function $\alpha(t) > 0$ such that

$$\|v_\varepsilon\|_{L^1_T LL} \leq C_0\alpha(T).$$

Hence, for all $(x,y)\in\bar{B}(0,R)^2$ with $|x-y|\leq e^{-\exp(C_0\alpha(T))}$ we have

$$\left|\psi_\epsilon^{-1}(t,x)-\psi_\epsilon^{-1}(t,y)\right| \leq e|x-y|^{\exp(-C_0\alpha(T))}. \tag{4.55}$$

Consider the backward particle trajectories that we denote by $(\phi_\varepsilon(s,t,x))_\varepsilon$ and which satisfies,

$$\phi_\varepsilon(s,t,x) = x - \int_s^t v_\varepsilon(\tau,\phi_\varepsilon(\tau,t,x))d\tau.$$

Then it is a well-known fact that

$$\phi_\varepsilon(0,t,x) = \psi^{-1}(t,x) \quad \text{and} \quad \phi_\varepsilon(0,t_2,x) = \phi_\varepsilon\big(0,t_1,\phi_\varepsilon(t_1,t_2,x)\big) \quad \forall (t_1,t_2)\in[0,T]^2.$$

Consequently we get in view of (4.55)

$$\begin{aligned}\left|\psi_\varepsilon^{-1}(t_1,x)-\psi_\varepsilon^{-1}(t_2,x)\right| &= \left|\psi_\varepsilon^{-1}(t_1,x)-\psi_\varepsilon^{-1}(t_1,\phi_\varepsilon(t_1,t_2,x))\right|\\ &\leq e\big|x-\phi_\varepsilon(t_1,t_2,x)\big|^{\exp(-C_0\alpha(T))}\\ &= e\Big|\int_{t_1}^{t_2} v_\varepsilon(\tau,\phi_\varepsilon(\tau,t_2,x))d\tau\Big|^{\exp(-C_0\alpha(T))}\\ &\leq e\Big|\int_{t_1}^{t_2}\big(\|\mathbb{P}v_\varepsilon(\tau)\|_{L^\infty}+\|\mathbb{Q}v_\varepsilon(\tau)\|_{L^\infty}\big)d\tau\Big|^{\exp(-C_0\alpha(T))}.\end{aligned}$$

despite that

$$\big|x - \phi_\varepsilon(t_1, t_2, x)\big| \le e^{-\exp(C_0\alpha(T))}. \tag{4.56}$$

It follows from (4.48) and (4.53) that

$$\big|\psi_\varepsilon^{-1}(t_1, x) - \psi_\varepsilon^{-1}(t_2, x)\big| \le e\Big|C_0|t_1 - t_2| + C\varepsilon^{\eta/2}|t_1 - t_2|^{3/4}\Big|^{\exp(-C_0\alpha(T))}.$$

By taking $|t_2 - t_2| << 1$ we see that the condition (4.56) is satisfied and the preceding estimate is justified. Thus with (4.55) we find that the family $\{\psi_\epsilon^{-1}\}$ is equicontinuous on every compact $[0, T] \times \bar{B}(0, R)$. Consequently, the Arzela-Ascoli theorem ensures the existence of a subsequence, denoted also by $\{\psi_\epsilon^{-1}\}$ and a particle trajectories ψ^{-1}, such that $\{\psi_\varepsilon^{-1}\}$ converges uniformly to ψ^{-1} on every compact $[0, T] \times \bar{B}(0, R)$. Observe that the subsequence may in principle depend on this compact but we can suppress this dependence by using Cantor's diagonal argument.

Performing the same analysis as previously to the integral equation of the flow ϕ_ε we can readily obtain that up to an extraction $\{\phi_\varepsilon\}$ converges uniformly in any compact to some continuous function ϕ. Moreover for any $t, s \in [0, T]$, $\phi(t, s)$ is a homeomorphism with

$$\phi^{-1}(t, s, x) = \phi(s, t, x), \quad \psi^{-1}(t, x) = \phi(0, t, x), \quad \psi(t, x) = \phi(t, 0, x).$$

In addition, for all $t \in [0, T]$, $\psi(t)$ is a Lebesgue measure preserving map. More precisely, for $q \in [1, \infty[$ and $f \in L^q(\mathbb{R}^2)$ we have

$$\|f \circ \psi_t^{\pm 1}\|_{L^q} = \|f(t)\|_{L^q}. \tag{4.57}$$

Indeed, for all continuous, compactly supported function f, $\{f \circ \psi_{t,\varepsilon}^{\pm 1}\}$ converges pointwisely to $f \circ \psi_t^{\pm 1}$. By the uniform boundedness of $\{\psi_{t,\varepsilon}^{\pm 1}\}$ with respect to ε, we get from the integral equation,

$$|x| \le |\psi_\varepsilon^{\pm 1}(t, x)| + C_0 T \quad \forall t \in [0, T].$$

Since f is compactly supported, we conclude the existence of $M > 0$ such that

$$\text{supp}(f \circ \psi_{t,\varepsilon}) \subset B(0, M + C_0 T).$$

Therefore, by Lebesgue dominated convergence theorem, we get

$$\lim_{\varepsilon \to 0} \|f \circ \psi_{t,\varepsilon}^{\pm 1} - f \circ \psi_t^{\pm 1}\|_{L^q} = 0. \tag{4.58}$$

In particular,

$$\lim_{\varepsilon \to 0} \|f \circ \psi_\varepsilon^{\pm 1}\|_{L^q} = \|f \circ \psi^{\pm 1}\|_{L^q}.$$

On other hand, a change of variable combined with Lemma 4.6 lead to

$$\|f\|_{L^q} e^{-C\int_0^t \|\text{div} v_\varepsilon(\tau)\|_{L^\infty} d\tau} \lesssim \|f \circ \psi_\varepsilon^{\pm 1}\|_{L^q} \lesssim \|f\|_{L^q} e^{C\int_0^t \|\text{div} v_\varepsilon(\tau)\|_{L^\infty} d\tau}.$$

Taking into consideration the estimate (4.48), the passage to the limit in the last inequality gives the identity (4.57). To finish the proof we use a density argument.

With this flow ψ in hand we construct (v, ω) via (4.50) and we shall prove some strong convergence results which give in turn that (v, ω) is a solution of the incompressible Euler equations. Recall that ω_0 and $(\omega_\varepsilon)_\varepsilon$ belong to L^q for all $q \in [p, +\infty[$ according to the classical interpolation result between Lebesgue and BMO spaces, see (4.17). Then, we shall prove following convergence result,

$$\lim_{\varepsilon \to 0} \|(\omega_\varepsilon - \omega)(t)\|_{L^q} \quad \forall q \in [p, +\infty[, \quad \forall t \in [0, T].$$

For this aim we write

$$\begin{aligned}\|(\omega-\omega_\varepsilon)(t)\|_{L^q} &= \|\omega_0\circ\psi^{-1}(t)-\omega_{0,\varepsilon}\circ\psi_\varepsilon^{-1}(t)e^{-\int_0^t \mathrm{div} v_\varepsilon(\tau,\psi(\tau,\psi_\varepsilon^{-1}(t)))d\tau}\|_{L^q}\\ &\le \mathrm{I}_\varepsilon+\mathrm{II}_\varepsilon.\end{aligned}$$

Where

$$\mathrm{I}_\varepsilon \triangleq \|\omega_0\circ\psi^{-1}(t)-\omega_0\circ\psi_\varepsilon^{-1}(t)\|_{L^q},$$

and

$$\mathrm{II}_\varepsilon \triangleq \|\omega_0\circ\psi_\epsilon^{-1}(t)-\omega_0^\epsilon\circ\psi_\varepsilon^{-1}(t)e^{-\int_0^t \mathrm{div} v_\varepsilon(\tau,\psi(\tau,\psi_\varepsilon^{-1}(t)))d\tau}\|_{L^q}.$$

In view of the equality (4.58), we can confirm that

$$\lim_{\varepsilon\to 0}\mathrm{I}_\varepsilon=0.$$

To estimate II_ε we make a change of variable and we use Lemma 4.6 to get

$$\begin{aligned}\mathrm{II}_\varepsilon &\le e^{C\|\mathrm{div} v_\varepsilon\|_{L^1_tL^\infty}}\|\omega_0-\omega_{0,\varepsilon}e^{-\int_0^t \mathrm{div} v_\varepsilon(\tau,\psi(\tau))d\tau}\|_{L^q}\\ &\le e^{C\|\mathrm{div} v_\varepsilon\|_{L^1_tL^\infty}}\Big(\|\omega_0\big(1-e^{-\int_0^t \mathrm{div} v_\varepsilon(\tau,\psi(\tau))d\tau}\big)\|_{L^q}+\|\big(\omega_0-\omega_{0,\varepsilon}\big)e^{-\int_0^t \mathrm{div} v_\varepsilon(\tau,\psi(\tau))d\tau}\|_{L^q}\Big)\\ &\le e^{C\|\mathrm{div} v_\varepsilon\|_{L^1_tL^\infty}}\Big(\|\omega_0\|_{L^q}\|1-e^{-\int_0^t \mathrm{div} v_\varepsilon(\tau,\psi(\tau))d\tau}\|_{L^\infty}+\|\omega_0-\omega_{0,\varepsilon}\|_{L^q}\Big)\\ &\le e^{C\|\mathrm{div} v_\varepsilon\|_{L^1_tL^\infty}}\Big(\|\omega_0\|_{L^q}\int_0^t\|\mathrm{div} v_\varepsilon(\tau)\|_{L^\infty}d\tau+\|\omega_0-\omega_{0,\varepsilon}\|_{L^q}\Big).\end{aligned}$$

Where we have used in the last inequality the estimate

$$\|e^u-1\|_{L^\infty}\le\|u\|_{L^\infty}e^{\|u\|_{L^\infty}}.$$

Then, from (4.39) and (4.17) we find

$$\begin{aligned}\mathrm{II}_\varepsilon &\lesssim C_0\varepsilon^\eta+\|\omega_0-\omega_{0,\varepsilon}\|_{L^q}\\ &\lesssim C_0\varepsilon^\eta+\|\omega_0-\omega_{0,\varepsilon}\|_{L^p}^{\frac{p}{q}}\|\omega_0-\omega_{0,\varepsilon}\|_{BMO}^{1-\frac{p}{q}}\\ &\lesssim C_0\varepsilon^\eta+C_0\|\omega_0-\omega_{0,\varepsilon}\|_{L^p}^{\frac{p}{q}}.\end{aligned}$$

Passing to the limit in the last estimate gives the desired result. Now we shall translate these results to the velocities via Biot-Savart law : we get since $1<p<2$,

$$\|\big(\mathbb{P}v_\varepsilon-v\big)(t)\|_{L^\infty}\lesssim\|\big(\omega_\varepsilon-\omega\big)(t)\|_{L^p\cap L^{2p}}. \tag{4.59}$$

Furthermore, by the classical Hardy-Littlewood-Sobolev inequality, one has

$$\|\big(\mathbb{P}v_\varepsilon-v\big)(t)\|_{L^r}\lesssim\|\big(\omega_\varepsilon-\omega\big)(t)\|_{L^q}. \tag{4.60}$$

where $r\in[\frac{2p}{2-p},+\infty[$ and $q\in[p,+\infty[$. Moreover, in view of the Calderón-Zygmund inequality (4.8) we have

$$\|\nabla\big(\mathbb{P}v_\varepsilon-v\big)(t)\|_{L^q}\lesssim\|\big(\omega_\varepsilon-\omega\big)(t)\|_{L^q}\quad\forall q\in[p,+\infty[.$$

Then, the convergence of $(\mathbb{P}v_\varepsilon)$ to v holds true in $W^{1,r}$ for all $r\in[\frac{2p}{2-p},+\infty[$.
It remains to show that ω is a solution of (4.1) associated to the initial vorticity ω_0. But before doing it, we have to verify that ψ is the flow associated to v. Using the preceding convergence and

the uniform convergence of $\{\psi_\varepsilon\}$ and according to the estimate (4.48), the passage to the limit in the integral equation of the flow,

$$\psi_\varepsilon(t,x) = x + \int_0^t \mathbb{P}v_\varepsilon(\tau,\psi_\varepsilon(\tau,x))d\tau + \int_0^t \mathbb{Q}v_\varepsilon(\tau,\psi_\varepsilon(\tau,x))d\tau,$$

yields

$$\psi(t,x) = x + \int_0^t v(\tau,\psi(\tau,x))d\tau.$$

As $v \in LL$ then by the uniqueness of the flow associated to v we can confirm the assumption. Next, let ϕ be an element of $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^2)$. By definition of ω and using a change of variable we have

$$\int_0^\infty \int_{\mathbb{R}^2} \omega(t,x)\partial_t\phi(t,x)dxdt \quad = \quad \int_0^\infty \int_{\mathbb{R}^2} \omega_0(x)(\partial_t\phi)(t,\psi(t,x))dxdt.$$

But

$$\begin{aligned}(\partial_t\phi)(t,\psi(t,x)) &= \partial_t\big(\phi(t,\psi(t,x))\big) - \partial_t\psi(t,x)\cdot\nabla\phi(t,\psi(t,x))\\ &= \partial_t\big(\phi(t,\psi(t,x))\big) - (v\cdot\nabla\phi)(t,\psi(t,x)).\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^2} \omega(t,x)\partial_t\phi(t,x)dxdt &= -\int_{\mathbb{R}^2} \omega_0(x)\phi(0,x)dx - \int_0^\infty \int_{\mathbb{R}^2} \omega_0(x)(v\cdot\nabla\phi)(t,\psi(t,x))dxdt\\ &= -\int_{\mathbb{R}^2} \omega_0(x)\phi(0,x)dx - \int_0^\infty \int_{\mathbb{R}^2} \omega(t,x)(v\cdot\nabla\phi)(t,x)dxdt.\end{aligned}$$

Hence, ω verifies the velocity-vorticity weak formulation :

$$\int_0^\infty \int_{\mathbb{R}^2} \omega(t,x)(\partial_t\phi + v\cdot\nabla\phi)(t,x)dxdt + \int_{\mathbb{R}^2} \omega_0(x)\phi(0,x)dx = 0.$$

Moreover, from (iv) of Proposition 4.1 and the estimates (4.37), (4.38) we immediately deduce (4.40) and (4.41). Finally, the uniqueness of the limit can be concluded by uniqueness of the solution of the incompressible Euler system since the velocity v belongs to $L^1_T LL$. □

4.6 Appendix

Lemma 4.8 *Let B be a ball of center 0 and radius $r > 0$ and ψ be the flow associated to a given smooth vector field v. Consider a Whitney covering of the open connected set $\psi(t,B)$ that is a collection of countable open balls $(O_k)_k$ introduced in the proof of Theorem 4.3.*
For all $k \in \mathbb{N}$ we set

$$U_k \triangleq \sum_{e^{-k-1}h(r) < r_j \le e^{-k}h(r)} |O_j|,$$

and

$$V_k \triangleq \sum_{e^{-k-1} < 4r_j \le e^{-k}} |O_j|,$$

with $h(r) \triangleq r\max\{1, \|J_\psi\|_{L^\infty}\}$ *and* J_ψ *is the Jacobian of* ψ.
Then, there exists an absolute constants C *such that for all* $k \geq \beta(t)$, *we have*
If $r \lesssim e^{-\beta(t)}\min\left\{1, \frac{1}{\|J_\psi\|_{L^\infty}}\right\}$, *then*

$$U_k \leq C\big(1+\|J_\psi\|_{L^\infty}\big)^2 e^{-\frac{k}{\beta(t)}} r^{1+\frac{1}{\beta(t)}}, \tag{4.61}$$

For all $r \in \mathbb{R}_+^*$,

$$V_k \leq C\|J_\psi\|_{L^\infty} e^{-\frac{k}{\beta(t)}} r. \tag{4.62}$$

Where $\beta(t) = \exp\Big(\int_0^t \|v(\tau)\|_{LL} d\tau\Big)$.

Proof By the definition of U_k, we have

$$U_k \leq \Big|\Big\{y \in \psi(B) : d(y, \psi(B)^c) \leq Ce^{-k} r\max\{1, \|J_\psi\|_{L^\infty}\}\Big\}\Big|$$

Since $|\psi(A)| \leq |A|\|J_\psi\|_{L^\infty}$ for any measurable set $A \subset \mathbb{R}^2$, we can deduce that

$$U_k \leq \Big|\Big\{x \in B : d(\psi(x), \psi(B^c)) \leq Ce^{-k} r\max\{1, \|J_\psi\|_{L^\infty}\}\Big\}\Big|\|J_\psi\|_{L^\infty}. \tag{4.63}$$

We set

$$D_k = \Big\{x \in B : d(\psi(x), \psi(B^c)) \leq Ce^{-k} r\max\{1, \|J_\psi\|_{L^\infty}\}\Big\}.$$

According to the fact $d(\psi(x), \psi(B^c)) = d(\psi(x), \partial\psi(B))$ and $\partial\psi(B) = \psi(\partial B)$, we can write

$$D_k \subset \Big\{x \in B : \exists y \in \partial B : |\psi(x) - \psi(y)| \leq Ce^{-k} r\max\{1, \|J_\psi\|_{L^\infty}\}\Big\}.$$

As $r \lesssim e^{-\beta(t)}\min\left\{1, \frac{1}{\|J_\psi\|_{L^\infty}}\right\}$, then

$$e^{-k} r\max\{1, \|J_\psi\|_{L^\infty}\} \lesssim e^{-\beta(t)}.$$

Then, Lemma 4.4 applied with ψ^{-1} gives

$$D_k \quad\subset\quad \Big\{x \in B : \exists y \in \partial B : |x-y| \leq Ce^{1-\frac{k}{\beta(t)}} r^{\frac{1}{\beta(t)}}(1+\|J_\psi\|_{L^\infty})\Big\}.$$

Therefore,

$$D_k \subset A = \Big\{x \in B : d(x, \partial B) : |x-y| \leq Ce^{1-\frac{k}{\beta(t)}} r^{\frac{1}{\beta(t)}}(1+\|J_\psi\|_{L^\infty})\Big\}.$$

Inserting this into (4.63) gives

$$U_k \leq \|J_\psi\|_{L^\infty}|D_k| \lesssim \big(1+\|J_\psi\|_{L^\infty}\big)^2 e^{\frac{-k}{\beta(t)}} r^{1+\frac{1}{\beta(t)}}$$

as claimed.

Reproducing the same procedure as previously with replacing $Ce^{-k} r\max\{1, \|J_\psi\|_{L^\infty}\}$ by Ce^{-k} in (4.63) and considering the fact that $c_0 e^{-k} \lesssim e^{-\beta(t)}$, we get the estimation of V_k.

Lemma 4.9 *Let* $(v_\varepsilon, c_\varepsilon)$ *be a smooth solution of the compressible Euler system* (E.C) *and* ω_ε *be the vorticity of* v_ε. *Then there exists a positive constant* C *such that*

$$\|\nabla v_\varepsilon(t)\|_{L^\infty} \leq C\Big(\|\omega_\varepsilon(t)\|_{BMO\cap L^p} V_\varepsilon(t) + \|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{H^{s+2}}\Big),$$

with

$$V_\varepsilon(t) = \int_0^t \big(\|\nabla v_\varepsilon(\tau)\|_{L^\infty} + \|\nabla c_\varepsilon(\tau)\|_{L^\infty}\big) d\tau$$

Proof : According to Bernstein inequality and the fact that $\|\dot{\Delta v}_{\varepsilon q}\|_{L^\infty} \sim 2^{-q}\|\dot{\Delta}_q\omega\|_{L^\infty}$, we have

$$\begin{aligned}
\|\nabla v_\varepsilon\|_{L^\infty} &\leq \|\Delta_{-1}\nabla v_\varepsilon\|_{L^\infty} + \sum_{0\leq q\leq N}\|\Delta_q\nabla v_\varepsilon\|_{L^\infty} + \sum_{q\geq N}\|\Delta_q\nabla v_\varepsilon\|_{L^\infty} \\
&\lesssim \|\Delta_{-1}\nabla v_\varepsilon\|_{L^p} + \sum_{0\leq q\leq N}\|\Delta_q\omega_\varepsilon\|_{L^\infty} + \sum_{q\geq N}2^q\|\Delta_q v_\varepsilon\|_{L^\infty} \\
&\lesssim \|\omega_\varepsilon\|_{L^p} + N\|\omega_\varepsilon\|_{B^0_{\infty,\infty}} + \|v_\varepsilon\|_{B^{s+1}_{\infty,\infty}}\sum_{q\geq N}2^{-qs} \\
&\lesssim N\|\omega_\varepsilon\|_{BMO\cap L^p} + 2^{-Ns}\|v_\varepsilon\|_{H^{s+2}}.
\end{aligned}$$

where we have used in the last inequality the fact that $H^{s+2}\hookrightarrow B^{s+1}_{\infty,\infty}$. Then from the energy estimates we deduce that

$$\|\nabla v_\varepsilon(t)\|_{L^\infty} \lesssim N\|\omega_\varepsilon(t)\|_{BMO\cap L^p} + 2^{-Ns}\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{H^{s+2}}e^{CV_\varepsilon(t)}.$$

Choosing N such that $2^{Ns}\simeq e^{CV_\varepsilon(t)}$, gives the desired result.
□

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Résumé

Cette thèse est consacrée à l'étude théorique de quelques modèles d'évolution non linéaires issus de la mécanique des fluides. Nous distinguons trois parties indépendantes.

La première partie de la thèse traite essentiellement de l'existence des poches de tourbillon en rotation uniforme (appelées aussi V-states) pour un modèle quasi-géostrophique non visqueux. Notre étude est répartie sur deux chapitres où les poches présentent des structures topologiques différentes. Dans le premier chapitre nous étudions le cas simplement connexe et nous validons l'existence de ces structures dans un voisinage du tourbillon de Rankine en utilisant des techniques de bifurcation. Dans le deuxième chapitre nous abordons le cas doublement connexe où la poche admet un seul trou. Plus précisément, proche d'un anneau donné, nous décrivons cette famille par des branches dénombrables bifurquant de cet anneau à certaines valeurs explicites des vitesses angulaires liées aux fonctions de Bessel. Notre étude théorique a été complétée par des simulations numériques portant sur les V-states limites et un bon nombre de constatations ont été formulées ouvrant la porte à de nouvelles perspectives de recherche.

La seconde partie concerne l'étude du problème de Cauchy pour le système de Boussinesq non visqueux 2D avec des données initiales de type Yudovich. Le problème est critique à cause de quelques termes comportant la transformée de Riesz dans la formulation tourbillon-densité. Nous donnons une réponse positive pour une sous-classe comprenant les poches de tourbillon régulières et singulières.

Dans la dernière partie nous analysons le problème de la limite incompressible pour les équations d'Euler isentropiques 2D associées à des données initiales très mal préparées et pour lesquelles les tourbillons ne sont pas forcément bornés mais appartiennent plutôt à des espaces de type BMO à poids. On utilise principalement deux ingrédients: d'un côté les estimations de Strichartz pour contrôler la partie acoustique. D'un autre côté, on se sert de la structure de transport compressible du tourbillon et on démontre une estimation de propagation linéaire dans des espaces BMO à poids.

Abstract

In this dissertation, we are concerned with the study of some non-linear evolution models arising in fluid mechanics. We distinguish three independent parts.

The first part deals with the existence of the rotating vortex patches (called also V-states) for an inviscid quasi-geostrophic model. Our study is divided into two chapters dealing with different topological structures of the V-states. In the first chapter we study the simply connected case and we prove the existence of such structures in a neighborhood of the Rankine vortices by using the bifurcation theory. In the second chapter we discuss the doubly connected case where the patches admit only one hole. More precisely, close to a given annulus we describe this family by countable branches bifurcating from this annulus at some explicit angular velocities related to Bessel functions. Our theoretical study is completed by numerical simulations on the limiting V-states and a number of numerical observations was formulated opening new research perspectives.

The second part of the thesis concerns the local well-posedness theory for the inviscid Boussinesq system with rough initial data. The problem is critical due to some terms involving Riesz transforms in the vorticity-density formulation. We give a positive answer for a special sub-class of Yudovich data including smooth and singular vortex patches.

In the last part we address the problem of the incompressible limit for the 2D isentropic fluids associated to ill-prepared initial data and for which the vortices are not necessarily bounded and belong to some weighted BMO spaces. We mainly use two ingredients: On one hand, the Strichartz estimates to control the acoustic part and prove that it does not contribute for low Mach number. On the other hand, we use the transport compressible structure of the vorticity and we establish a linear propagation estimate in the weighted BMO spaces.