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Pierre-Antoine Guihéneuf

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École doctorale 142
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Laboratoire de Mathématiques d'Orsay

THÈSE DE DOCTORAT

présentée pour obtenir le grade de
docteur en sciences de l'Université Paris-Sud

discipline : mathématiques

par

Pierre-Antoine Guihéneuf

Sujet :

Discrétisations spatiales de systèmes dynamiques génériques

Soutenue le 26 juin 2015 devant la commission d'examen :

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Résumé

Dans quelle mesure peut-on lire les propriétés dynamiques (quand le temps tend vers l'infini) d'un système sur des simulations numériques ? Pour tenter de répondre à cette question, on étudie dans cette thèse un modèle rendant compte de ce qui se passe lorsqu'on calcule numériquement les orbites d'un système à temps discret f (par exemple un homéomorphisme). L'ordinateur travaillant à précision numérique finie, il va remplacer f par une *discrétisation spatiale* de f, notée f_N (où l'ordre de la discrétisation N rend compte de la précision numérique). On s'intéresse en particulier au comportement dynamique des applications finies f_N pour un système f *générique* et pour l'ordre N tendant vers l'infini, où générique sera à prendre dans le sens de Baire (principalement parmi des ensembles d'homéomorphismes ou de C^1-difféomorphismes).

La première partie de cette thèse est consacrée à l'étude de la dynamique des discrétisations f_N lorsque f est un homéomorphisme conservatif/dissipatif générique d'une variété compacte. L'étude montre qu'il est illusoire de vouloir retrouver la dynamique du système de départ f à partir de celle d'une seule discrétisation f_N : la dynamique de f_N dépend fortement de l'ordre N. Pour détecter certaines dynamiques de f il faut considérer l'ensemble des discrétisations f_N, lorsque N parcourt $\mathbf{N}$.

La seconde partie traite du cas linéaire, qui joue un rôle important dans l'étude du cas des C^1-difféomorphismes génériques, abordée dans la troisième partie de cette thèse. Sous ces hypothèses, on obtient des résultats similaires à ceux établis dans la première partie, bien que plus faibles et de preuves plus difficiles.

Abstract

How is it possible to read the dynamical properties (ie when the time goes to infinity) of a system on numerical simulations ? To try to answer this question, we study in this thesis a model reflecting what happens when the orbits of a discrete time system f (for example an homeomorphism) are computed numerically . The computer working in finite numerical precision, it will replace f by a *spacial discretization* of f, denoted by f_N (where the order N of discretization stands for the numerical accuracy). In particular, we will be interested in the dynamical behaviour of the finite maps f_N for a *generic* system f and N going to infinity, where generic will be taken in the sense of Baire (mainly among sets of homeomorphisms or C^1-diffeomorphisms).

The first part of this manuscript is devoted to the study of the dynamics of the discretizations f_N, when f is a generic conservative/dissipative homeomorphism of a compact manifold. We show that it would be mistaken to try to recover the dynamics of f from that of a single discretization f_N : its dynamics strongly depends on the order N. To detect some dynamical features of f, we have to consider all the discretizations f_N when N goes through $\mathbf{N}$.

The second part deals with the linear case, which plays an important role in the study of C^1-generic diffeomorphisms, discussed in the third part of this manuscript. Under these assumptions, we obtain results similar to those established in the first part, though weaker and harder to prove.

Avant-propos

Voilà, c'est fini. Quatre ans de travail, résumés dans ces quelques 300 pages. C'est beaucoup, diront certains, mais cela ne fait qu'une page pour quatre jours de travail ; pour le profane, cette productivité pourrait passer pour indécente. Si les matheux écrivent si lentement, c'est parce qu'ils passent la plus grande partie de leur temps à apprendre, discuter, expliquer, lire, tâtonner, dire n'importe quoi, tout recommencer depuis le début, se désespérer, enseigner, oublier, réapprendre, recommencer, etc. Je n'ai malheureusement pas encore le talent de certains [1], et j'aurais bien aimé que dans ce manuscrit, il reste un peu plus les échafaudages qui ont aidé à sa construction : je vais essayer de réparer un peu ça ici.

Ce manuscrit ne raconte pas qu'en fait, je n'ai commencé à travailler sur mon sujet de thèse qu'en janvier 2012. Le début de ma thèse, je l'ai passé à reprendre mon rapport de stage de M2 pour en faire un survol sur les propriétés génériques des homéomorphismes conservatifs. Une très bonne idée de François : j'ai été ravi d'explorer un peu plus ce sujet passionnant ; ravi aussi de pouvoir, quelques mois plus tard, avoir entre les mains un exemplaire édité par la SBM. Une autre des excellentes idées de François fut de me proposer le sujet des discrétisations après que j'ai fini de travailler sur le sujet de M2 : je retrouvai le théorème I de Tomasz Miernowski en quelques jours seulement, d'autres résultats suivirent assez vite. Je n'aurais pas pu rêver mieux comme initiation à la recherche : François, tu as eu beaucoup de nez [2]. Ce sujet était essentiellement vierge il y a quatre ans, et après m'avoir guidé parmi les homéomorphismes conservatifs, tu m'as laissé explorer le cas différentiable et le cas linéaire, malgré presque deux ans sans résultat réellement satisfaisant. Tu n'as pas compté les heures passées à discuter sans trop savoir où ça nous mènerait ; durant cette thèse, j'ai appris que bien souvent, les idées viennent en discutant de tout et de rien. Merci enfin pour tes innombrables relectures de ce que j'ai pu produire, parfois à l'état de brouillon, truffé de fautes. Sans elles, ce manuscrit n'aurait certainement pas pu être bouclé à temps !

Durant cette thèse, j'ai aussi eu la chance d'avoir, dans le bureau d'en face, un des plus grands spécialistes de la théorie des difféomorphismes génériques. J'ai exploré avec bonheur une partie de ce sujet à la faune exotique : lemme de fermeture, classe homocline, cycle hétérodimensionel, décomposition dominée… Sylvain a toujours été très disponible pour répondre à mes questions — souvent naïves, parfois stupides — avec beaucoup de gentillesse et de pédagogie. S'il l'avait été, il n'aurait certainement pas usurpé le titre de directeur adjoint, et sans ses références précises, la rédaction de la partie 3 m'aurait pris un temps infini.

1. Voir par exemple [Ber09], qui m'a par ailleurs beaucoup inspiré lors de mes recherches sur les discrétisations d'applications linéaires.
2. En particulier concernant les références [Lax71] et [AA13].

Une autre rencontre décisive a été celle d'Yves Meyer. C'est lui qui m'a fait découvrir la théorie des ensembles modèle. Sans cet outil, je ne serais sans doute jamais venu à bout du cas linéaire. J'ai été frappé par son enthousiasme constant, qui pour certains s'effrite un peu avec les années de pratique : il est resté chez lui intact. Les discussions avec Yves sont toujours peuplées de souvenirs de lieux, de gens : les mathématiques, souvent vues comme froides et mécaniques, deviennent avec lui chaleureuses et humaines.

Justement, cette question du taux d'injectivité dans le cas linéaire m'a occupé une grande partie du temps de cette thèse. Parfois un peu désespéré, j'ai été glaner des idées un peu partout ; cela m'a donné l'occasion de faire la connaissance de matheux d'horizons très variés, comme Frédéric Le Roux (une des trois têtes du trio Béguin-Crovisier-Le Roux), Emmanuel Militon, Yves Benoist, Yuri Lima, Pierre Pansu, Sébastien Gouëzel, Nicolas Gourmelon, Nalini Anantharaman, Raphaël Cerf, Pierre Arnoux, Michele Triestino, Nicolas Curien… Merci à tous, le théorème Q doit un peu à chacun de vous.

Enfin, je dois l'idée de la preuve du théorème 7.29 à Émilien. Sans ses capacités calculatoires hors normes, je n'aurais jamais réussi à trouver comment démontrer ce fichu énoncé ! Merci à lui, à Lucie et à son chat.

Les maths auraient beaucoup moins d'intérêt si une fois écrites, elles n'étaient pas partagées. Étienne Ghys et Enrique Pujals m'ont fait l'honneur d'accepter de relire ce manuscrit "un peu enveloppé". J'espère que le premier aura pris autant de plaisir à lire que moi à travailler sur les questions qu'il avait soulevées il y a presque vingt ans. J'espère aussi que le second n'aura pas trop souffert de la lecture des parties en français. Un grand merci à eux, ainsi qu'à Valérie Berthé, Jérôme Buzzi et Sylvain Crovisier, d'avoir accepté de faire partie de mon jury de thèse.

J'ai aussi eu la chance, tout au long de cette thèse, de pouvoir assister à des conférences un peu partout dans le monde, à Gdynia, Rio, Bordeaux, Marseille, et Sao Paulo. Cela n'aurait pas été possible sans les financements de l'école doctorale et de l'équipe topologie et dynamique. Je tiens aussi à remercier le GDR Platon (en particulier Françoise Dal'Bo et Frédéric Paulin), pour l'organisation des superbes *paroles aux jeunes chercheurs*, ainsi que les participants à la conférence *surfaces in Sao Paulo*, pour les discussions sur l'ensemble de rotation qui ont abouti à la rédaction du chapitre 6 de ce manuscrit. J'y ai découvert de jeunes chercheurs passionnés par la dynamique, ainsi que les caïpirinhas. Merci à tous ceux que j'ai rencontré durant ces conférence, en particulier Samuel Roth, Martyna Kłos, Lenka Sivakova, Lenka Rucka, Maria Sorokina, Alejandro Passeggi, Juliana Xavier, Juliette Bavard, Gabriel Fuhrmann et Pablo Dávalos.

Dès la première année de mon contrat doctoral, l'équipe d'enseignants de l'IUT de Cachan m'a proposé la charge d'un cours/TD en autonomie. Même si j'ai passé un certain temps à la préparation des séances, j'ai beaucoup appris de cette expérience : loin des leçons un peu formatées de l'agreg', il a fallu adapter le cours et les exercices au fur et à mesure, pour donner envie à des élèves qui conçoivent les maths comme un outil plutôt que comme une fin en soi. J'ai aussi eu la chance d'avoir carte blanche lors des deux dernières années de mon contrat. L'équipe de l'IUT m'a fait confiance lorsque je leur ai proposé de remplacer l'habituelle préparation des concours par des projets de modélisation et simulation : un grand merci à eux, j'espère que le bilan de ce cours est positif. Enfin, j'ai passé une formidable semaine au ski avec une partie des enseignants et des élèves de Cachan, merci à tous pour cette super ambiance !

Le labo d'Orsay voit passer beaucoup de doctorants. J'ai eu la chance, pendant ces quatre ans, d'en côtoyer quelques uns. Tout d'abord, dans le bureau 108, avec Sébastien, Jérémie et Schweta, puis Caroline, Émilie, Céline A., Gabriele et Céline K. En particulier, je tire mon chapeau aux deux Mme Mout' qui, depuis bientôt trois ans, supportent mes ronchonnages quotidiens, intermèdes musicaux immondes et crapauseries diverses. Sans cette ambiance au bureau 108, j'aurais peut-être fini ma thèse un peu plus rapidement, mais je me serais certainement beaucoup moins amusé ! Dans le bureau d'à-côté, il y avait les connexes : Olivier (et Cyrielle), Vincent, Valérie Robert [3], Solenne, Arthur, Aurélien, Carolina, Ramon, Ramla, Adrien… Dans le reste du couloir, c'étaient Igor, Laure, Thierry, Lionel, Lucie, Cagri… Enfin, au rez-de-chaussée, Élodie, Pierre, Bernardo, Thomas (Morzy), Alba, Ilia [4], Lison, Joseph… Et ne l'oublions pas, il y a des doctorants à Orsay dans les bâtiments 425 et 430 : Loïc [5], Nina, Vincent, Mélina… J'ai passé beaucoup de bons moments en votre compagnie : les pauses café avaient tendance à manger un peu du temps de travail légal, occupées par les discussions à propos des tickets de CESFO et du principe du sas.

N'oublions pas ce petit capybara qui m'a bien aidé lors de la rédaction !

L'administration française n'est pas forcément connue pour sa simplicité. Valérie Lavigne guide avec beaucoup de gentillesse les doctorants d'Orsay dans le moments les plus stressants de la thèse ; elle n'est certainement pas étrangère à la bonne ambiance qui règne dans le couloir du premier étage du bâtiment 430. L'école doctorale doit aussi beaucoup à l'implication de ses deux directeurs successifs, David Harari et Frédéric Paulin, merci à eux.

Pendant ces quatre ans, j'ai pu partager la vie de l'équipe topologie et dynamique d'Orsay. Elle m'a accueilli les bras ouverts, en particulier Jérôme Buzzi et son successeur Frédéric Bourgeois, ainsi que sa secrétaire Fabienne Jacquemin, et Rémi Leclercq, Anne Vaugon, Sylvie Ruette, Hans-Henrik Rugh, Romain Tessera… J'ai énormément apprécié le principe du café culturel, qui m'a bien aidé à comprendre ce que font les non-dynamiciens de l'équipe (et il y en a beaucoup !). Merci aussi à tous les participants du groupe de travail "ergodique et dynamique", un vrai succès quand on pense qu'il a lieu le lundi matin !

Je tiens à remercier les numériciens d'Orsay qui m'ont autorisé à utiliser le cluster du labo. La plupart des simulations de ce manuscrit ont tourné de longues heures sur « Cinaps », utilisant une quantité affolante de mémoire vive.

Un grand bravo à toute l'équipe d'organisation du $\mathbb{TFJM}^2$, en premier lieu Matthieu, ainsi que David, Igor, Giancarlo, Martin… pour le travail incroyable qu'ils font en faveur des lycéens férus de mathématiques. J'ai pris un plaisir immense à encadrer les équipes de l'Essonne deux années de suite (Alain, Alexandre, Clara, Colin, Cyril, Florent, Hector, Nicolas) : quelques heures par semaine, j'ai un peu eu l'impression d'être à la place de mon directeur de thèse ! J'ai aussi eu la chance de pouvoir partir une semaine en Allemagne pour coacher une des deux équipes nationales qualifiées pour l'ITYM, en compagnie d'Ambroise, Thomas et Guillaume (avec en plus Cécile et Alexandre parmi les élèves). Je pense que j'ai autant appris à encadrer ces compétitions que les élèves à y participer. On a aussi pu constater que les biélorusses ne conçoivent pas du tout les mathématiques comme les français !

Et il n'y a pas que les maths dans la vie ! J'ai passé de nombreuses heures à "faire

3. Son rêve est qu'on cite son nom de famille.
4. Dans la même classe en sixième, soutenance la même année dans le même labo !
5. Qui a repris l'organisation du séminaire des doctorants avec Émilie et avec brio.

de la varappe" un peu partout, à Bleau [6], Céüse, Siurana [7], Bercy [8], Châteauvert, Massy, Terradets, St-Léger [9], Finale Ligure, en compétition aux quatre coins de la France, dans le Tarn, la Jonte, le Verdon, etc., en compagnie de grimpeurs aux surnoms évocateurs : Goret [10], Barnouze (le crêteux) et sa compagne Roca, Marca [11], Mme la présidente [12], Petite So, Rox [13], Bristov (le crêteux n°2), Coquille, Ludo [14], Yann et Vaness [15], Fuego, Simon, Tristoon, Mél, Le Contellec, Grand Chef, l'Anglais et son gîte, Gus, Shortwood, Yo, Nina, Céline, Marco, Alice, Tonton, Toinou, Bonito, Coco, Panpan, Poupou, Zaza, Pascal, La Caze, Jérôme, Pascal, Cécile... L'escalade est un sport hautement social : grimper, c'est aussi un prétexte pour passer du temps avec vous tous ; sans ça je serais sans doute devenu un peu autiste. En particulier, un grand merci à La coloc de Perthes-en-Gâtinais qui m'a hébergé tant de dimanches matins, et à tout le club de Massy (et ça fait du monde !), où j'ai passé un nombre incalculable de soirées à me faire exploser les avant-bras.

Et je n'oublie pas la bande de Rennais (ou plutôt d'ex-rennais et associés) : Rominou, toujours aussi enthousiaste à l'idée de grimper ou de se goinfrer de Nutella, Popoff, toujours aussi jovialement insupportable, Julie Connor qui est maintenant bien loin de sa Bretagne natale, Pierrot (le Guignolo), Marine et sa maison qui m'aura vu traîner la patte pendant une semaine, Christophe (le PU aux biceps gonflables), KRon (grand spécialiste de l'aménagement de piscines), Élise, Mama. Sans vous, je n'aurais jamais pu connaître Kraftwerk, TNGHT, Graou ou mon orthopédiste.

Je pense aussi à une partie de la bande de cachanais que je n'ai pas encore cité : Rémi, co-bureau par alliance, Kevin et Guillaume, toujours prêts à "coincher la chauffe", Bruno, qui m'a hébergé maintes fois à Bordeaux avec une hospitalité déconcertante, Antoine...

Un énorme merci à tous ceux qui m'ont dépanné d'une lessive de temps en temps, en particulier Ben et Maud ! Merci aussi au RER B, dans lequel j'ai passé beaucoup de temps, et où j'ai eu une partie non négligeable des idées présentées dans ce manuscrit.

Enfin, une petite pensée pour ceux qui suivront : je laisse ma place au Kinou, la petite jeune qui commence bientôt sa thèse dans le sud, et à Tom, qui me succède en tant qu'élève de François. Vous verrez, ça vaut le coup !

En quatre ans, il peut s'en passer des choses : un décès, deux déménagements, une maladie, des travaux, des pattes cassées... Ma famille — en particulier ma mère, ma sœur [16], mon père, ma tante, mes grands parents — a toujours été là pour me soutenir, malgré quelques moments difficiles. Elle a aussi compris quand j'ai eu besoin d'un peu plus de distance [17]. La fin de cette thèse signe aussi la fin du statut d'éternel étudiant : est-ce que ça veut dire que je vais devenir adulte ?

6. J'ai pu y perfectionner mon anglais : il y avait du boulot !
7. Minimal Techno, un grand souvenir !
8. Où j'ai été sacré double champion du monde de l'assurage !
9. Et la rencontre du "Prince du lactique".
10. Avec qui j'ai passé des nuits de folie dans son camion jaune. Merci pour cet accueil chaleureux !
11. Le roi des traits.
12. Mais il faut aussi que je cite Marion, sinon elle ne sera pas contente.
13. Bravo pour la coupe de cheveux, une vraie pro !
14. Merci pour tout ce que j'ai appris pendant ces stages d'ouverture ; gérer l'ouverture d'une compétition, c'est finalement un peu comme gérer la rédaction d'une thèse : on a l'impression que ça n'avance pas, qu'on fait n'importe quoi, on fait des modifications de dernière minute alors qu'on s'était promis de ne pas le faire, et finalement ça ne se passe pas si mal !
15. Et Maël !
16. Qui passe le bac cette année !
17. J'ai parfois été assez insupportable...

Je vais bientôt rejoindre le labo de maths de Niteroi, près de Rio. Une aventure se termine ici, j'ai la chance de pouvoir en commencer une nouvelle bientôt. Les maths présentées dans ce manuscrit seront peut-être oubliées dans dix ans, mais ce n'est finalement pas si important : je me souviendrai longtemps de ce que j'ai appris pendant cette thèse. Bonne lecture !

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Chapitre 1

Introduction

1.1 Présentation du problème

Le but de cette thèse est d'étudier la dynamique des discrétisations spatiales de systèmes à temps discret. Ce problème est motivé par des questions d'approximation numérique : considérons une dynamique (X, f), où X est un espaces des configurations « gentil » (à ce stade, on peut penser à un domaine borné de $\mathbf{R}^n$ ou au tore $\mathbf{T}^n$), et $f : X \to X$ une application continue, donnée par une formule explicite. On veut essayer de comprendre la dynamique de f, c'est-à-dire le comportement asymptotique des itérés f^k de f, à l'aide de simulations numériques. Pour ce faire, l'idée la plus naïve consiste à prendre un point $x \in X$, et à demander à l'ordinateur de calculer successivement ses images $f^t(x)$, pour un temps t « assez grand ».

Analysons ce qui se passe alors : l'ordinateur travaille à précision numérique fixée, disons qu'il considère 10 chiffres après la virgule [1]. Il ne calcule donc pas réellement dans l'espace des phases X, mais dans un espace discret E_{10}, qui est l'ensemble des points de X dont les coordonnées sont décimales avec au plus 10 chiffres après la virgule. Il remplace aussi l'application f par une *discrétisation* $f_{10} : E_{10} \to E_{10}$ de f, qui est une approximation de la dynamique f sur l'ensemble discret E_{10}. Cet espace fini est une assez bonne approximation [2] de l'espace continu X, mais il se peut que les petites erreurs d'arrondi faites à chaque itération de f_{10} s'accumulent au fil du temps, si bien qu'à partir d'un certain moment, les orbites de x par l'application f et par sa discrétisation f_{10} n'ont plus grand chose à voir. Si on ne sait rien de plus sur la dynamique f, on ne peut par conséquent rien dire du comportement asymptotique de l'orbite de x par f en regardant son orbite par la discrétisation f_{10}, et cela même si à première vue une précision de 10 chiffres après la virgule est relativement bonne.

Néanmoins, cette orbite discrète $\left(f_{10}^t(x)\right)_{t\in\mathbf{N}}$ est proche d'une vraie orbite de f sur tout intervalle de temps de longueur raisonnable. On peut donc espérer que le comportement *collectif* des orbites calculées numériquement — où par collectif on entend que l'on considère beaucoup de points de départ x — reflète la dynamique *globale* de l'application f.

L'objet principal de cette thèse n'est pas d'essayer de trouver le meilleur algorithme qui permette de retrouver la dynamique de f, ni de mener des études numériques sur

1. Ce n'est pas exactement ce qui se passe en général, étant donné que le format de nombre le plus répandu est le flottant, où ce n'est pas le nombre de chiffres après la virgule qui est fixé mais le nombre de chiffres significatifs. Au cours de cette thèse, on négligera cet aspect.

2. Par exemple au sens de la distance de Hausdorff.

des exemples particuliers, visant à constater si oui ou non on observe en pratique la dynamique du système initial f. Le but ici est d'analyser l'algorithme naïf évoqué plus haut, qui décrit ce qui se passe quand on fait des calculs numériques sans aucune précaution ; autrement dit, on veut comprendre les phénomènes qui apparaissent lorsqu'on discrétise en espace un système dynamique. En d'autres termes, on aimerait relier la dynamique asymptotique (quand le temps tend vers l'infini) de l'application de départ f avec celle de l'application discrète f_{10}.

Il faut bien noter que cette problématique est bien différente de celle de l'analyse numérique classique : en quelque sorte, étant donnés un temps t_0 et une précision δ, les numériciens déterminent une méthode, en particulier une grille, qui permet d'approcher la solution réelle pendant un temps t_0 et avec une précision d'au moins δ. Dans notre cas, on se fixe une grille assez fine, et on compare les comportements asymptotiques du système réel et du système discrétisé sur cette grille.

Dans cette thèse, on considérera des systèmes dynamiques à temps discret $f : X \to X$, où X est une variété compacte (avec ou sans bord) munie d'une métrique d, et f une application continue de X dans lui-même. L'opération de discrétisation qui transforme X en l'espace discret et f en sa discrétisation sera modélisée de la façon suivante. On se donne une suite de grilles $(E_N)_{N\in\mathbf{N}}$, qui sont donc des sous-ensembles finis de X, et dont la maille tend vers [3] 0. On définit une projection $P_N : X \to E_N$ qui envoie chaque point de X sur le point de E_N le plus proche [4]. La *discrétisation* d'un point $x \in X$ est alors définie comme étant le point $P_N(x)$, et la *discrétisation* de f comme étant l'application

$$\begin{array}{rcl} f_N : E_N & \longrightarrow & E_N \\ x & \longmapsto & P_N\big(f(x)\big) : \end{array}$$

pour calculer l'image $f_N(x)$ de $x \in E_N$, on calcule l'image de x par f puis on projette sur E_N *via* l'application P_N. Remarquons que cette modélisation de l'erreur d'arrondi est imparfaite : en pratique, l'ordinateur travaille aussi en précision numérique finie lorsqu'il fait des calculs intermédiaires nécessaires pour calculer $f(x)$, il fait donc plusieurs discrétisations successives. Nous ne tiendrons pas compte de cet aspect par la suite.

Le cadre de notre étude étant mieux fixé, on peut préciser un peu plus la question qui nous intéresse.

Question. *Quelles propriétés dynamiques de f peuvent être lues sur la dynamique (globale) des discrétisations $(f_N)_{N\in\mathbf{N}}$?*

Pour tenter de répondre à cette question, le premier réflexe est d'observer ce qui se passe pour une dynamique simple et bien connue. C'est ce que fait É. Ghys dans un article paru au journal des élèves de l'ENS de Lyon [Ghy94], où il étudie les discrétisations de l'automorphisme d'Anosov du tore, dit du « chat d'Arnold »,

$$\begin{array}{rcl} A : \mathbf{R}^2/\mathbf{Z}^2 & \longrightarrow & \mathbf{R}^2/\mathbf{Z}^2 \\ (x,y) & \longmapsto & (y, x+y), \end{array}$$

sur les grilles

$$E_N = \left\{ \left(\frac{i}{N}, \frac{j}{N} \right) \in \mathbf{R}^2/\mathbf{Z}^2 \;\middle|\; 1 \le i,j \le N \right\}. \tag{1.1}$$

3. Par « la maille de la grille tend vers 0 », on entend « pour tout $\varepsilon > 0$ et tout N assez grand, la grille E_N est ε-dense dans l'espace X ».

4. On fait une fois pour toutes un choix raisonnable lorsqu'il y a plusieurs points de E_N qui sont les plus proches.

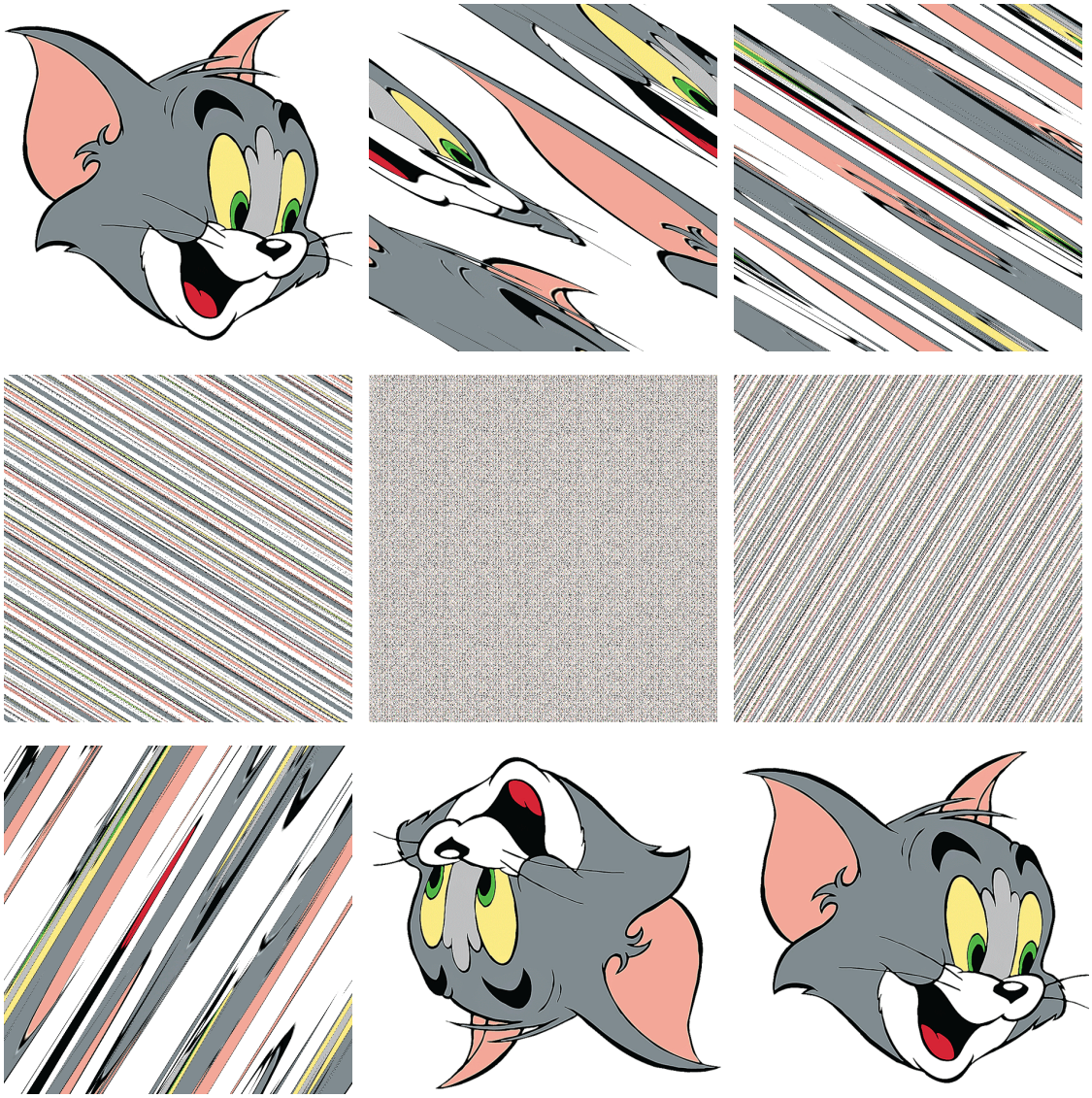

FIGURE 1.1: Images d'une image de chat par la discrétisation du chat d'Arnold, sur une grille de taille 610×610. De gauche à droite et de bas en haut, temps égaux à 0, 2, 4, 8, 14, 20, 26, 30 et 60.

La figure 1.1 montre ce qu'on obtient en prenant les itérés par A d'une image numérique : une telle image est constituée de pixels — ici l'image fait 610×610 pixels —, elle correspond donc formellement à une application $\phi : E_{610} \to [\![1, 256]\!]^3$, dont on considère les poussées en avant successifs par l'application A_{610} (rappelons que l'on note A_N la discrétisation de A suivant la grille E_N). On observe alors deux phénomènes très particuliers.

– Premièrement, lorsqu'on itère l'image, on obtient encore des images ; autrement dit la couleur de chaque pixel des itérés est bien définie. Ce n'est *a priori* pas évident : il n'y a aucune raison pour que les discrétisations d'un homéomorphisme [5] quelconque du tore dans lui-même soient bijectives. Ici, c'est la forme particulière de A, à savoir le fait que c'est un automorphisme linéaire à coefficients

5. Ou plus généralement d'une application continue.

entiers, qui fait que pour tout ordre $N \in \mathbf{N}$, l'application A_N est une permutation de la grille E_N. En fait, dans le cas de A, on n'a même pas besoin de projeter : pour tout point $x \in E_N$, on a automatiquement $Ax \in E_N$.

– Ensuite, on retrouve l'image de départ après un nombre extrêmement court d'itérations (60 dans l'exemple de la figure 1.1). Or, on sait que l'ordre global d'une permutation typique d'un ensemble à N^2 éléments est équivalente à $N^{2\ln N}$ (voir par exemple [Bol01]) ; appliquée à $N = 610$, cette asymptotique donne un ordre d'environ 5.10^{35} : si l'application A_{610} était une permutation quelconque, il faudrait itérer à peu près 10^{35} fois avant de revoir apparaître notre pauvre chat, ce qui est notoirement supérieur à 60 fois ! De fait, on a le résultat suivant, démontré par F. Dyson et H. Falk dans [DF92] : *pour tout* $N \in \mathbf{N}$, *l'ordre de la permutation* A_N *est inférieur ou égal à* $3N$. *De plus, il existe une suite d'entiers* $(N_k)_{k\in\mathbf{N}}$, *tendant vers l'infini à vitesse exponentielle, telle que l'ordre de la permutation* A_{N_k} *vaut* $2k$.

Ce comportement des discrétisations contraste fortement avec la dynamique réelle de l'automorphisme linéaire A : cette application est un paradigme de système exponentiellement mélangeant. Si la dynamique des discrétisations devait refléter celle de A, les images obtenues devraient rapidement devenir uniformément grises (ce qu'on observe au centre de la figure 1.1, à la $14^{\text{ème}}$ itération) et le rester à tout jamais (au contraire de ce qu'on observe sur les dernières images de la figure 1.1).

Il ressort de ces observations que l'exemple de l'automorphisme linéaire A est très particulier : en quelque sorte, les grilles E_N entrent en résonance avec la dynamique A ; ceci parce que les grilles E_N sont régulièrement espacées, et parce que A possède des propriétés arithmétiques particulières.

Pour éviter ce type de phénomènes « exceptionnels », É. Ghys propose (toujours dans [Ghy94]) d'étudier les discrétisations d'homéomorphismes *génériques* du tore préservant l'aire. C'est ce que nous ferons dans cette thèse : nous étudierons les discrétisations de systèmes dynamiques génériques.

On peut donner un sens formel très précis au mot « générique ». La plupart des espaces de fonctions raisonnables sont complets (ce sera toujours le cas ici) ; en particulier, on peut y appliquer le théorème de Baire : *toute intersection dénombrable d'ouverts denses, est elle-même dense.* On dit alors qu'une propriété (P) portant sur la classe de fonctions considérée est *générique* si elle est vérifiée sur au moins une intersection dénombrable d'ouverts denses de telles fonctions. Notons que cette notion de généricité possède deux propriétés agréables :

– une propriété générique est satisfaite sur un ensemble dense ;
– si (P) et (Q) sont génériques, alors la propriété « vérifier à la fois (P) et (Q) » est elle-même générique ; ceci reste même vrai pour une famille dénombrable de propriétés.

Ainsi, nous étudierons la dynamique des discrétisations d'homéomorphismes ou de C^1-difféomorphismes génériques, tant conservatifs (c'est-à-dire sous l'hypothèse de préservation d'une bonne mesure sur X, fixée une fois pour toutes) que dissipatifs (c'est-à-dire sans hypothèse de préservation de mesure).

Par abus de langage, on parlera souvent d'« homéomorphisme générique », et on listera ses propriétés. Il faudra avoir à l'esprit qu'un tel homéomorphisme générique n'existe pas : un peu comme en mécanique quantique, où le fait d'effectuer une mesure perturbe inévitablement le système étudié ; essayez de choisir un élément générique d'un espace, de ce fait il cessera de l'être. Cet abus de langage est donc très mauvais, et lorsqu'il la rencontrera, le lecteur devra remplacer mentalement la phrase « soit f un

élément générique de tel espace, alors il a la propriété (P) » par la phrase plus correcte « la propriété (P) est générique dans tel espace ».

Précisons maintenant ce qu'on entend par « la dynamique des discrétisations ». Toute discrétisation $f_N : E_N \to E_N$ est une application d'un ensemble fini dans lui-même. En particulier, chacune de ses orbites est pré-périodique, c'est-à-dire que pour tout point $x \in E_N$, l'orbite de x sous f_N est périodique à partir d'un certain rang : il existe deux entiers C et T tels que pour tout $k \geq C$, on ait $f_N^{k+T}(x) = f_N^k(x)$. Ainsi, on peut partitionner l'ensemble E_N en deux sous-ensembles : l'*ensemble récurrent* $\Omega(f_N)$ de f_N, qui est égal à l'union des orbites périodiques de f_N, et son complémentaire, appelé *ensemble errant* de f_N. Notons que l'ensemble récurrent $\Omega(f_N)$ est stable par f_N, et que la restriction de f_N à cet ensemble est bijective. De plus, chaque $x \in E_N$ « tombe » dans $\Omega(f_N)$, c'est-à-dire qu'il existe $t \in \mathbf{N}$ tel que $f_N^t(x) \in \Omega(f_N)$; $\Omega(f_N)$ est d'ailleurs le plus petit sous-ensemble de E_N vérifiant cette propriété.

Ainsi, d'un point de vue purement combinatoire, la dynamique de f_N est caractérisée par un petit nombre de quantités, comme par exemple le nombre d'orbites périodiques de f_N et la répartition de leurs longueurs, les tailles de leurs bassins d'attraction [6], le temps de stabilisation de f_N (*i.e* le plus petit $t \in \mathbf{N}$ tel que $f_N^t(E_N) = \Omega(f_N)$), etc. Nous nous intéresserons plus particulièrement au *degré de récurrence* $D(f_N)$ de f_N, qui est égal au rapport entre le cardinal de l'ensemble récurrent $\Omega(f_N)$ et celui de la grille E_N.

Ces propriétés dynamiques combinatoires de f_N ne tiennent pas compte de la géométrie des grilles E_N. Or, on a supposé que ces grilles sont de bonnes approximations de l'espace X au sens métrique, autrement dit que pour tout $\varepsilon > 0$ et tout N assez grand, la grille E_N est ε-dense dans l'espace X. On peut aussi munir X d'une bonne mesure λ, et supposer que les mesures uniformes sur les grilles E_N tendent vers λ. Ces hypothèses permettent d'étudier des propriétés dynamiques géométriques ou ergodiques des discrétisations : on peut par exemple se demander si les orbites périodiques de f_N tendent vers les orbites périodiques de f. Plus généralement, on peut se demander si les compacts invariants par les discrétisations tendent (pour la distance de Hausdorff) vers les compacts invariants par f. Le même type de questions peut être posé pour les mesures invariantes, munies de la topologie faible-*.

Il est assez étonnant que ce problème n'ait été que très peu étudié. À ma connaissance, si on met de côté des travaux analysant des exemples numériques bien choisis, ou expliquant des phénomènes par des arguments heuristiques (voir la section 2.2), il existe très peu de travaux théoriques sur ce problème. En fait,

En fait, seuls P.P. Flockermann et T. Miernowski ont vraiment entrepris une étude théorique systématique pour une large classe de systèmes dynamiques [7]. Dans sa thèse sous la direction de O. E. Lanford, P.P. Flockermann s'est intéressé aux discrétisations des applications dilatantes du cercle. Au cours de sa thèse [Mie05], effectuée sous la direction d'É. Ghys, T. Miernowski a quant à lui étudié le cas des homéomorphismes du cercle (voir aussi [Mie06]). Il montre essentiellement que les propriétés dynamiques d'un homéomorphisme/difféomorphisme du cercle typique (générique ou prévalent) peuvent être lues sur la dynamique de ses discrétisations. Les preuves de ces résultats utilisent de manière cruciale l'existence d'un ordre cyclique préservé par la dynamique ;

6. C'est-à-dire le cardinal du nombre de points dont l'orbite positive tombe dans cette orbite périodique.

7. Voir le chapitre 2 pour un historique complet.

en particulier, les comportement dynamiques des discrétisations dépendent fortement du nombre de rotation de l'homéomorphisme. Cette étude est donc assez spécifique et ne saurait être généralisée à d'autres types de systèmes.

Durant sa thèse [Mie05], T. Miernowski s'est aussi brièvement intéressé au cas des homéomorphismes conservatifs génériques du tore : on munit le tore $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ de la mesure de Lebesgue et des grilles de discrétisation canoniques (définies par l'équation (1.1)), et on note $\mathrm{Homeo}(\mathbf{T}^2, \mathrm{Leb})$ l'ensemble des homéomorphismes du tore qui préservent la mesure de Lebesgue. T. Miernowski démontre alors le résultat suivant.

Theorème I (Miernowski)**.** *Pour un homéomorphisme conservatif générique* $f \in \mathrm{Homeo}(\mathbf{T}^2, \mathrm{Leb})$*, il existe une sous-suite* f_{N_k} *de discrétisations de* f *dont les éléments sont des permutations de l'ensemble* $\mathrm{E}_{\mathrm{N}_k}$*.*

Ce théorème est en quelque sorte le point de départ de cette thèse : on aimerait montrer d'autres résultats de ce type, pour mieux comprendre la dynamique des discrétisations d'un homéomorphisme ou d'un difféomorphisme générique.

1.2 Quelques résultats que l'on pourra trouver dans cette thèse

Ce manuscrit contient un grand nombre d'énoncés, concernant les propriétés combinatoires, topologiques, ou ergodiques, des discrétisations de systèmes dynamiques différentiables ou seulement continus, conservatifs ou dissipatifs. Dans cette section, j'ai voulu en isoler quelques-uns, qui me semblent les plus représentatifs et intéressants. Pour une présentation plus systématique et exhaustive de nos résultats, on renvoie aux introductions des différentes parties du manuscrit. Enfin, dans le chapitre 13, nous discuterons et comparerons les résultats obtenus dans différents contextes.

Dans cette section, pour plus de simplicité, nous nous plaçons dans le cas où X est le tore $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$, muni s'il y a lieu de la mesure de Lebesgue notée Leb, et où les grilles sont les grilles uniformes sur le tore, définies par

$$\mathrm{E}_{\mathrm{N}} = \left\{ \left(\frac{i}{\mathrm{N}}, \frac{j}{\mathrm{N}} \right) \,\middle|\, 1 \leq i, j \leq \mathrm{N} \right\}. \tag{1.1}$$

Les résultats présentés sont en réalité vrais dans divers contextes plus généraux ; on se référera aux parties du manuscrit concernées pour les définitions précises de ces contextes.

1.2.1 Degré de récurrence

Commençons par deux théorèmes relatifs à la dynamique combinatoire des discrétisations, plus précisément une des quantités les plus simples à étudier, le degré de récurrence. Rappelons que le degré de récurrence $\mathrm{D}(f_{\mathrm{N}})$ d'une discrétisation f_{N} de f est défini comme étant le rapport entre le cardinal de l'ensemble récurrent de f_{N} (qui est l'union des orbites périodiques de f_{N}, voir page 19) et celui de la grille E_{N}. Étant donné que l'orbite positive de tout point de E_{N} par la discrétisation f_{N} tombe dans cet ensemble récurrent, le degré de récurrence est aussi égal au rapport $\mathrm{Card}\left(f_{\mathrm{N}}^t(\mathrm{E}_{\mathrm{N}})\right)/\mathrm{Card}(\mathrm{E}_{\mathrm{N}})$ pour tout t assez grand.

Le premier résultat concerne le comportement de $\mathrm{D}(f_\mathrm{N})$ pour un homéomorphisme conservatif[8] générique du tore : dans le chapitre 5, on démontre le résultat suivant (corollaire 5.24 page 83).

Theorème II. *Pour un homéomorphisme conservatif générique* $f \in \mathrm{Homeo}(\mathbf{T}^2, \mathrm{Leb})$, *la suite* $(\mathrm{D}(f_\mathrm{N}))_{\mathrm{N}\geq 0}$ *s'accumule sur tout le segment* $[0,1]$.

Le fait que ce degré de récurrence s'accumule sur 1 est un corollaire trivial du théorème I de T. Miernowski. Pris indépendamment, ce théorème I peut être vu comme très positif : il exprime qu'une infinité de discrétisations d'un homéomorphisme conservatif générique se comporte de la même manière que cet homéomorphisme, à savoir que c'est une bijection qui préserve la mesure. Ce phénomène n'est en fait qu'un cas particulier d'un comportement très irrégulier de la suite $\left(\mathrm{D}(f_\mathrm{N})\right)_{\mathrm{N}\geq 0}$: le théorème II affirme qu'elle s'accumule en fait sur le plus gros ensemble sur lequel elle a le droit de s'accumuler.

Nous étudions ensuite le cas des C^1-difféomorphismes conservatifs génériques. On démontre le théorème suivant dans le chapitre 11 (théorème 11.15 page 235).

Theorème III. *Pour un difféomorphisme de classe* C^1 *conservatif générique* $f \in \mathrm{Diff}^1(\mathbf{T}^2, \mathrm{Leb})$, *on a*

$$\lim_{\mathrm{N}\to+\infty} \mathrm{D}(f_\mathrm{N}) = 0.$$

Ce théorème est à mettre en parallèle avec le théorème II : il exprime que le comportement global des discrétisations évolue moins irrégulièrement avec l'ordre N que dans le cas des homéomorphismes. Néanmoins, le fait que le degré de récurrence tende vers 0 signifie qu'il y a une perte d'information arbitrairement grande lorsque N est assez grand.

Pour démontrer ce théorème, on fait le lien entre les comportements macroscopique et mésoscopique des discrétisations d'un difféomorphisme générique. Pour commencer, on définit les *taux d'injectivité* de f

$$\tau^t_\mathrm{N}(f) = \frac{\mathrm{Card}(f^t_\mathrm{N}(\mathrm{E}_\mathrm{N}))}{\mathrm{Card}(\mathrm{E}_\mathrm{N})};$$

par définition, leur limite pour t tendant vers $+\infty$ est le degré de récurrence $\mathrm{D}(f_\mathrm{N})$. On est donc ramené à l'étude de ces taux d'injectivité ; en particulier on démontre un résultat les reliant à des quantités similaires pour les différentielles de f (théorème 11.3 page 229, voir aussi le théorème 11.26 page 247).

Theorème IV. *Pour tout* $r \geq 1$, *et pour un difféomorphisme de classe* C^r *conservatif générique* $f \in \mathrm{Diff}^r(\mathbf{T}^2, \mathrm{Leb})$, *pour tout* $t \in \mathbf{N}^*$, *on a*

$$\lim_{\mathrm{N}\to+\infty} \tau^t_\mathrm{N}(f) = \int_{\mathbf{T}^2} \tau(\mathrm{D}f_{f^{t-1}(x)}, \cdots, \mathrm{D}f_x)\,\mathrm{d}\,x,$$

où le taux d'injectivité d'une suite de matrices est défini de manière similaire à celui d'un difféomorphisme (voir la définition 7.19 page 152).

Ce résultat permet de ramener l'étude du degré de récurrence d'un difféomorphisme générique à celle du taux d'injectivité d'une suite de matrices générique ; cela fait l'objet de la partie 2 du présent manuscrit.

8. Rappelons que par « conservatif » on entend « qui préserve la mesure de Lebesgue ».

1.2.2 Mesures physiques

Nous nous intéressons maintenant à la dynamique ergodique des discrétisations. Commençons par définir la notion de mesure physique. On pose $\mu^f_{x,\mathrm{T}}$ la mesure de probabilité uniforme sur le segment d'orbite $\left(x, f(x), \cdots, f^{\mathrm{T}-1}(x)\right)$. Cela permet de définir le *bassin d'attraction* d'une mesure de probabilité μ comme étant l'ensemble des points $x \in \mathbf{T}^2$ tels que les mesures $\mu^f_{x,\mathrm{T}}$ tendent vers μ (au sens de la topologie faible-*) lorsque T tend vers $+\infty$. On dit alors que la mesure μ est *physique* pour f si son bassin d'attraction est de mesure de Lebesgue strictement positive. Une mesure physique est une mesure que l'on devrait pouvoir détecter lors d'expérience physiques car elle est « vue » par un ensemble de points x de mesure de Lebesgue positive. La question est de savoir s'il est aussi possible de repérer de telles mesures sur des expériences numériques.

Signalons que dans cet esprit, il y a de nombreux résultats sur la stabilité *stochastique* des mesures physiques, pour tenir compte du fait que, dans une expérience physique, il y a toujours du bruit (voir par exemple les notes de cours [Via97] de M. Viana). Dans un grand nombre de cas, il s'avère que malgré le bruit, on retrouve les mesures physiques de la dynamique de départ. Ces travaux supposent toujours le bruit aléatoire, et surtout indépendant à chaque itération, ce qui est loin d'être le cas des troncatures numériques. La question est donc de savoir si ce type de résultats perdure pour les discrétisations. Comme les applications f_{N} sont finies, les mesures $\mu^{f_{\mathrm{N}}}_{x,\mathrm{T}}$ convergent vers la mesure uniforme sur l'orbite périodique dans laquelle tombe l'orbite positive de x_{N} par f_{N} ; on note $\mu^{f_{\mathrm{N}}}_x$ cette mesure. Le but est de caractériser le comportement de ces mesures pour « la plupart » des points $x \in \mathrm{E}_{\mathrm{N}}$. Dans cette optique, on démontre le théorème suivant (théorème 5.43 page 90).

Theorème V. *Pour un homéomorphisme conservatif générique $f \in \mathrm{Homeo}(\mathbf{T}^2, \mathrm{Leb})$, pour toute mesure de probabilité μ invariante par f, il existe une sous-suite de discrétisations $(f_{\mathrm{N}_k})_k$ telle que f_{N_k} possède une unique mesure invariante μ_k, qui tend vers μ. Autrement dit, pour toute mesure de probabilité f-invariante μ, il existe une suite d'entiers $(\mathrm{N}_k)_{k\geq 0}$ telle que, pour tout $x \in \mathbf{T}^2$, on a*

$$\mu^{f_{\mathrm{N}_k}}_x \underset{k\to+\infty}{\longrightarrow} \mu.$$

Précisons que le théorème d'Oxtoby-Ulam (un des tout premiers résultats de généricité, voir [OU41]) implique qu'un homéomorphisme conservatif générique $f \in \mathrm{Homeo}(\mathbf{T}^2, \mathrm{Leb})$ ne possède qu'une seule mesure physique, à savoir la mesure de Lebesgue. Le théorème V exprime donc que d'une certaine manière, toute mesure f-invariante est « physique du point de vue d'une infinité de discrétisations ». Ce résultat peut être considéré comme positif ou négatif selon le point de vue : si on est content de pouvoir retrouver toutes les mesures invariantes par f, rien ne nous permet à l'heure actuelle de détecter la mesure physique de f à partir des discrétisations, alors que la définition même de mesure physique est censée exprimer que ce sont les mesures observables en pratique.

Ce phénomène apparaît même sur des simulations numériques d'exemples d'homéomorphismes conservatifs : sur la figure 1.2, on a représenté les résultats des simulations numériques des mesures $\mu^{f_{\mathrm{N}}}_{\mathbf{T}^2}$, où $\mu^{f_{\mathrm{N}}}_{\mathbf{T}^2}$ est la limite au sens de Cesàro des poussées en avant par f_{N} de la mesure uniforme sur E_{N}. On observe que sur cet exemple, que non seulement les mesures $\mu^{f_{\mathrm{N}}}_{\mathbf{T}^2}$ ne convergent pas du tout vers la mesure de Lebesgue, mais qu'en plus elles n'ont rien à voir les unes avec les autres, et cela même si les ordres de discrétisation sont très proches.

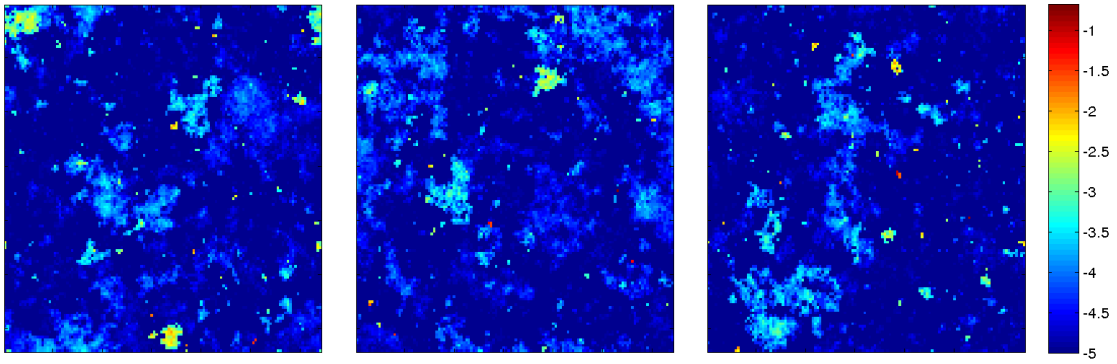

FIGURE 1.2: Densité de la mesure invariante canonique $\mu_{\mathbf{T}^2}^{f_N}$ de f_N, obtenue comme limite au sens de Cesàro des poussés en avant par f_N de la mesure uniforme sur E_N. La densité est représentée en échelle logarithmique : un pixel orange porte une mesure d'environ 10^{-2}. L'homéomorphisme f est une petite perturbation C^0 de l'identité (voir page 99). De gauche à droite, $N = 20\,000, 20\,001, 20\,002$.

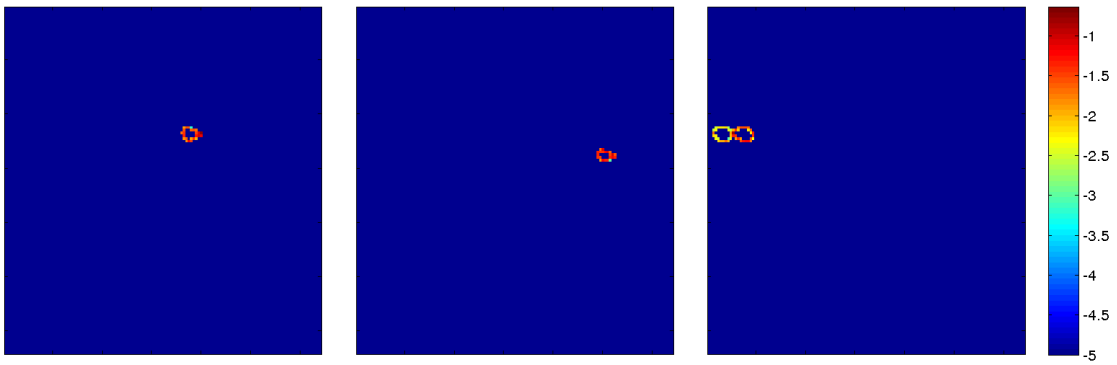

FIGURE 1.3: Densité de la mesure invariante canonique $\mu_x^{f_N}$ de f_N, obtenue comme limite au sens de Cesàro des poussés en avant par f_N de la mesure δ_x, avec $x = (1/2, 1/2)$. La densité est représentée en échelle logarithmique : un pixel orange porte une mesure d'environ 10^{-2}. L'homéomorphisme f est une petite perturbation C^1 de l'identité (voir la section 12.3). De gauche à droite, $N = 2^{20}+1, 2^{20}+2, 2^{20}+3$.

Dans le chapitre 12 de cette thèse, on obtient un théorème similaire pour les difféomorphismes de classe C^1. Ce résultat n'est malheureusement pas assez fort pour expliquer le comportement des mesures $\mu_x^{f_N}$ pour tous les points $x \in \mathbf{T}^2$; il se contente de le décrire pour un ensemble générique de points (théorème 12.1 page 251).

Theorème VI. *Pour un difféomorphisme de classe C^1 conservatif générique $f \in \mathrm{Diff}^1(\mathbf{T}^2, \mathrm{Leb})$, pour un point générique $x \in \mathbf{T}^2$, tout mesure μ invariante par f, il existe une sous-suite $(f_{N_k})_k$ de discrétisations telles que*

$$\mu_x^{f_{N_k}} \underset{k\to+\infty}{\longrightarrow} \mu.$$

Par rapport au Théorème V, on notera aussi que la suite $(N_k)_k$ dépend maintenant du point x choisi. Toujours est-il que si on choisit un difféomorphisme conservatif f, ainsi qu'un point $x \in \mathbf{T}^2$, et que l'on calcule numériquement la mesure uniforme sur l'orbite $x, f(x), \cdots, f^T(x)$, avec T très grand, alors le théorème VI exprime qu'il se peut que l'on

ne tombe pas du tout sur une mesure physique de f, mais plutôt sur une mesure invariante par f quelconque. Ce phénomène peut s'observer en pratique, et de manière très prononcée, comme le montrent les résultats de simulations numériques des mesures $\mu_x^{f_N}$ (voir la figure 1.3).

La preuve de ce résultat est assez longue et technique. Elle utilise de manière cruciale un énoncé linéaire (démontré à la fin de la partie 2) assez proche de celui utilisé pour démontrer le théorème III. Elle utilise aussi deux lemmes de fermeture, classiques en dynamique C^1 générique : le lemme de connexion de pseudo-orbites de C. Bonatti et S. Crovisier [BC04], et une amélioration du lemme de fermeture ergodique de F. Abdenur, C. Bonatti et S. Crovisier [ABC11].

1.2.3 Détection de dynamiques rares vis-à-vis de la mesure

Dans le chapitre 6 de cette thèse, on s'intéresse au calcul numérique de l'ensemble de rotation d'un homéomorphisme conservatif générique du tore $\mathbf{T}^2$. On y tire parti d'un phénomène illustré par le théorème V : les discrétisations d'un homéomorphisme conservatif générique permettent de retrouver toutes les mesures invariantes de l'homéomorphisme de départ, et pas seulement sa mesure physique. Ainsi, on montre que paradoxalement, les erreurs numériques sont utiles, voire nécessaires, pour calculer l'ensemble de rotation d'un homéomorphisme conservatif générique.

L'ensemble de rotation d'un homéomorphisme du tore est un sous-ensemble compact et convexe de $\mathbf{R}^2$, défini modulo $\mathbf{Z}^2$ (voir la définition page 6.1.1). C'est une généralisation à la dimension 2 de la notion de nombre de rotation d'un homéomorphisme du cercle : cet ensemble décrit à quelle vitesse les orbites de points s'enroulent autour du tore $\mathbf{T}^2$. Pour un homéomorphisme conservatif générique, ainsi que pour un difféomorphisme de classe C^1 conservatif générique, il est d'intérieur non vide (voir les propositions 6.2 et 10.13 pages 115 et 224).

Dans ce manuscrit, on définit l'*ensemble de rotation observable* : un vecteur v est dans l'ensemble de rotation observable si, pour tout $\varepsilon > 0$, il existe un ensemble de mesure de Lebesgue positive de points x dont l'orbite a un vecteur de rotation ε-proche de v (voir la définition 6.9 page 117). Cette définition est censée représenter l'ensemble de rotation que l'on obtiendrait en faisant des mesures exactes, mais en n'étant autorisé à faire qu'un nombre fini d'expériences. On démontre alors le théorème suivant (théorème 6.23 page 122).

Theorème VII. *Pour un homéomorphisme conservatif générique* $f \in \mathrm{Homeo}(\mathbf{T}^2, \mathrm{Leb})$,
- *l'ensemble de rotation observable est réduit à un point ;*
- *la limite supérieure des ensembles de rotation des discrétisations coïncide avec l'ensemble de rotation de* f.

La seconde assertion du théorème reste vraie pour un difféomorphisme conservatif générique de classe C^1 si on considère les enveloppes convexes des ensembles de rotation des discrétisations (voir le théorème 10.15 page 224). La première assertion est aussi vraie dès lors que la mesure de Lebesgue est ergodique pour f ; on conjecture que c'est le cas pour un difféomorphisme conservatif générique de classe C^1 (en tous cas, il existe des ouverts de tels difféomorphismes dans lesquels un difféomorphisme générique est ergodique, voir page 250).

Ce théorème nous donne un algorithme d'approximation de l'ensemble de rotation, qui consiste à calculer les ensembles de rotations des discretisations. Sur la Figure 1.4, on voit que l'algorithme « naturel » ne réussit à détecter qu'une toute petite partie de

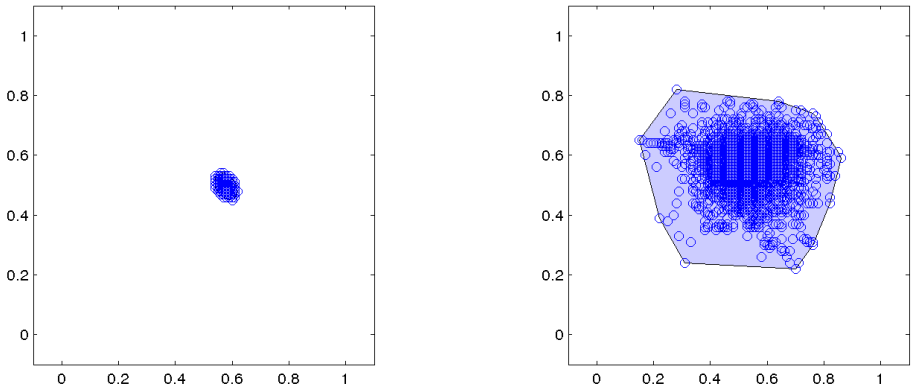

FIGURE 1.4: Comparaison des méthodes de calcul de l'ensemble de rotation sur un même homéomorphisme f « quelconque » (voir sa définitionsection 6.4). À gauche, ensemble obtenu à l'aide de l'algorithme naturel, qui consiste à calculer des vecteurs de rotation d'orbites très longues. À droite, ensemble obtenu en accumulant les ensembles de rotation de discrétisation sur différentes grilles. Les temps de calcul sont similaires (environ 4h).

l'ensemble rotation, contrairement à l'algorithme utilisant des discrétisations (comme prédit par le théorème VII). Dans le chapitre 6, on verra sur d'autre exemples que cet algorithme est assez efficace. C'est une application inattendue de la notion de discrétisation : la troncature, qui semble a priori être un phénomène gênant, peut en fait être mise à profit pour détecter certaines dynamiques ; paradoxalement, pour retrouver ces dynamiques, il peut être utile de faire volontairement des erreurs d'arrondi grossières.

Ce phénomène de stabilisation de chaque invariant dynamique de f par une infinité de discrétisations peut être expliqué de la manière suivante. Chaque invariant dynamique de f que l'on étudie peut être approché par une orbite périodique ω (ou une union finie de telles orbites). Génériquement, les points de ω sont de type Liouville par rapport à la suite de grilles, c'est-à-dire qu'il va exister une sous-suite d'ordres pour lesquels chaque point de ω est très proche d'un point de la grille [9], en tous cas assez proche pour qu'à certains ordres N, si $x \in \omega$, alors $f_N(x_N) = (f(x))_N$: on aura une orbite discrétisée proche de l'orbite ω. Pour un système pratique (qui ne vérifie donc pas forcément les conclusions des théorèmes, qui supposent que les systèmes sont génériques), on peut espérer qu'un phénomène similaire ait lieu : si l'orbite ω n'est pas pas trop longue, on a de grandes chances pour qu'en considérant un nombre raisonnable de discrétisations sur des grilles différentes, on retrouve l'orbite ω sur au moins une de ces grilles.

1.3 Guide de lecture

Le présent manuscrit est divisé en quatre parties. La partie 1 concerne les discrétisations d'homéomorphismes génériques. La partie 2 s'intéresse aux discrétisations d'ap-

9. Le « très proche » étant ajusté à la maille de la grille et à des quantités telles que le module de continuité de f.

plications linéaires. Dans la partie 3, on étudie les discrétisations de difféomorphismes de classe C^1 génériques. Enfin, la partie 4 fait office de conclusion.

Dans la partie 1, le chapitre 3 est un chapitre de notations et préliminaires, qui serviront tout au long de la partie 1. Le lecteur pressé trouvera un index des notations à la fin du mémoire. Les chapitres 4 et 5 sont largement indépendants, tandis que le chapitre 6 utilise des notions développées dans les chapitres 4 et 5.

La partie 2 est complètement indépendante des parties 1 et 3. Plus précisément, le chapitre 7 est en quelque sorte un chapitre introductif au chapitre 8, où on trouvera la plupart des résultats originaux de cette partie. Le chapitre 9, très court, utilise le formalisme mis en place dans les deux chapitres précédents.

Enfin, la partie 3 utilise les résultats de la partie 2. Dans le détail, le chapitre 10 est indépendant du reste du manuscrit, mais on y comparera régulièrement les résultats obtenus à ceux de la partie 1. Le chapitre 11 utilise de manière cruciale l'analyse du cas linéaire menée dans la partie 2. Pour finir, dans le chapitre 12, on combine des techniques développées dans la partie 2 avec des énoncés du chapitre 10.

Enfin, la partie 4 fait office de conclusion. Dans le premier chapitre (chapitre 13), on confronte suivant divers points de vue les résultats obtenus durant cette thèse. Le second chapitre (chapitre 14), quant à lui, donne des exemples de questions ouvertes qui me sembles pertinentes au sujet des discrétisations spatiales de systèmes dynamiques.

Les chapitres 4 et 5 reprennent largement le contenu de [Gui15a]. Par rapport à cet article, certains énoncés ont été améliorés, d'autres ont été ajoutés, quelques coquilles ont été corrigées, des simulations supplémentaires ont été insérées…

Le chapitre 6 est lui quasiment identique à l'article [Gui15b].

Chapitre 2

Historique du sujet

2.1 (Très) bref survol des résultats classiques de dynamique générique

Avant de s'intéresser à la dynamique des discrétisations d'applications génériques, il est bon de connaître la dynamique réelle de ces applications (c'est-à-dire sans discrétisation de l'espace). De plus, les techniques développées pour comprendre la dynamiques des homéomorphismes (ou difféomorphismes) génériques nous seront très utiles pour étudier leurs discrétisations.

L'un des tout premiers résultats de dynamique générique est le théorème d'Oxtoby-Ulam (datant de 1941, voir [OU41]), qui affirme qu'un homéomorphisme conservatif générique d'une variété compacte de dimension au moins deux est ergodique. Depuis lors, de nombreux travaux ont permis de bien comprendre la dynamique d'un homéomorphisme conservatif générique. On notera que cette dynamique n'est ni complètement triviale, ni extrêmement chaotique : par exemple, un homéomorphisme conservatif générique est topologiquement mélangeant, mais pas fortement mélangeant (au sens de la mesure). On notera aussi que les techniques mises en œuvre pour démontrer les énoncés sont variées et jolies : on a en particulier un théorème de transfert des propriétés ergodiques génériques de l'espace des automorphismes à celui des homéomorphismes conservatifs (dû à S. Alpern, voir [Alp79]), ou une loi du 0-1 pour la généricité des propriétés dynamiques ergodiques (due à E. Glasner et J. King, voir [GK98]). Le lecteur intéressé pourra consulter le survol [Gui12]. Donnons quelques éléments de la preuve de la généricité de la transitivité (sans doute l'une des plus simples) : considérant un homéomorphisme conservatif f,

- on commence par casser la dynamique f en discrétisant l'espace et en appliquant le théorème de Lax (voir [Lax71] et le théorème 5.5 page 76), qui affirme qu'un homéomorphisme conservatif est arbitrairement proche d'une permutation cyclique d'une grille de la variété ayant de bonnes propriétés ;
- on reconstruit ensuite un homéomorphisme conservatif à partir de cette permutation cyclique, à l'aide d'un lemme de perturbation C^0 (voir la proposition 3.3 page 50).

Cette technique de preuve peut être utilisée pour obtenir des résultats concernant les discrétisations d'homéomorphismes conservatifs génériques ; c'est ce que nous faisons dans le chapitre 5.

Le cas des homéomorphismes *dissipatifs* génériques n'a été étudié que beaucoup plus tard. En fait, l'étude systématique menée par E. Akin, M. Hurley et J. Kennedy dans leur survol [AHK03] s'avère relativement décevante, dans la mesure où les résultats vont tous dans la même direction et procèdent d'une même heuristique : la dynamique d'un homéomorphisme dissipatif générique « contient » simultanément tous les comportements sauvages que l'on peut imaginer.

Ce survol ne concerne que la dynamique *topologique* des homéomorphismes ; très récemment, F. Abdenur et M. Anderson se sont intéressés aux propriétés *ergodiques* des homéomorphismes dissipatifs génériques : il s'agit de considérer le comportement des sommes de Birkhoff le long de l'orbite d'un point typique pour la mesure de Lebesgue (voir [AA13]). À cette fin, ils établissent un résultat technique qu'ils appellent le *shredding lemma*, qui leur permet de montrer qu'un homéomorphisme générique est *weird* (voir la définition 4.1 page 58). C'est ce lemme technique que nous utiliserons pour établir la dynamique des discrétisations d'un homéomorphisme dissipatif générique.

L'étude des propriétés génériques des difféomorphismes de classe C^1 est un sujet très actif — aussi bien dans le cadre conservatif que dans le cadre dissipatif — ; il est devenu de ce fait extrêmement touffu. Dans le cas conservatif, l'absence d'analogue C^1 au lemme de reconstruction C^0 oblige à établir des lemmes de fermeture et de connexion d'orbites. On a notamment le lemme de fermeture de C. Pugh [Pug67], qui implique qu'un C^1-difféomorphisme conservatif générique possède un ensemble dense de points périodiques. On a aussi le lemme de connexion de pseudo-orbites de C. Bonatti et S. Crovisier [BC04], qui permet de montrer qu'un C^1-difféomorphisme conservatif générique est transitif. La question de savoir si l'ergodicité est générique ou non parmi les C^1-difféomorphismes conservatifs génériques reste toujours ouverte ; on notera l'avancée récente d'A. Avila, S. Crovisier et A. Wilkinson [ACW14], qui établit une dichotomie générique : pour un difféomorphisme conservatif générique f, soit tous les exposants de Lyapunov de f sont nuls, soit f est Anosov et ergodique. Il existe bien d'autres résultats sur ce sujet, on pourra notamment consulter le survol [Cro06b] de S. Crovisier, ou l'introduction de [ACW14].

Dans le cadre dissipatif, les travaux sont guidés par un ensemble de conjectures dues à J. Palis qui décrivent ce que pourrait être la dynamique — d'un point de vue topologique comme d'un point de vue ergodique — d'un difféomorphisme dissipatif générique de classe C^1. On pourra se faire une idée de la richesse de ce sujet en consultant le séminaire Bourbaki [Bon03] ou le compte-rendu de conférence de l'ICM [Bon02] de C. Bonatti, le mémoire d'habilitation [Cro13] ou le survol récent [Cro14] de S. Crovisier.

Si on la compare à la situation pour les difféomorphismes de classe C^1, l'étude des propriétés dynamiques génériques des difféomorphismes de classe C^r avec $r > 1$ est bloquée par l'absence d'un lemme de fermeture en topologie C^r pour $r > 1$ (concernant le lemme de fermeture en topologie C^1, voir le livre [Arn98] de M.-C. Arnaud). À ma connaissance, le seul cas dans lequel on ait des résultats autres qu'anecdotiques, outre le cas trivial de la dimension 1, est celui des difféomorphismes conservatifs des surfaces ; on consultera à ce sujet l'article [FLC03] de J. Franks et P. Le Calvez. Notons qu'en régularité assez grande ($r \geq 4$), le théorème KAM implique que, sur toute surface compacte, il existe des ouverts de difféomorphismes conservatifs de classe C^r (pour la topologie C^r) sur lesquels les éléments ne sont pas transitifs (voir par exemple la section 4 de [Her98]).

2.2 Survol général des travaux sur les discrétisations spatiales de systèmes dynamiques

Les premières simulations numériques de systèmes dynamiques apparaissent dès les années 1960, avec par exemple les célèbres travaux de E. Lorenz et M. Hénon. Ce dernier conduit entre autres une étude numérique poussée de ce qui est désormais appelé « application de Hénon conservative » (voir [Hén69]) :

$$f_\alpha(x,y) = \Big(x\cos\alpha - (y-x^2)\sin\alpha\ ,\ x\sin\alpha + (y-x^2)\cos\alpha\Big).$$

M. Hénon constate que les phénomènes que l'on voit apparaître numériquement (en particulier, « des îlots elliptiques dans une mer chaotique ») sont en accord avec ce qu'on sait déjà de ces applications d'un point de vue théorique ; il ne semble pas réellement se préoccuper d'éventuels effets indésirables induits par la troncature numérique.

Quelques années plus tard, P. Lax, dans [Lax71], fait remarquer que le fait de discrétiser ces applications de Hénon conservatives fait perdre le caractère bijectif de la dynamique. Pour pallier ce problème, P. Lax démontre qu'un homéomorphisme conservatif du tore est arbitrairement bien approché par des permutations des grilles « naturelles » sur le tore ; dans un certain sens, si la discrétisation d'un homéomorphisme n'est pas bijective, on sait qu'il existe au moins une application finie proche de cet homéomorphisme qui l'est. Ce théorème est à ma connaissance le premier résultat théorique concernant la dynamique des discrétisations ; sa jolie preuve, essentiellement combinatoire, repose sur le lemme des mariages (voir page 76). Une amélioration de ce résultat est démontrée par S. Alpern [Alp78] : dans l'énoncé du théorème de Lax (théorème 5.5 page 76), on peut remplacer le terme « permutation » par le terme « permutation cyclique ». Ce résultat indique que la dynamique de tout homéomorphisme conservatif est proche d'une application finie transitive. Dans les années 1990, dans [DKP93, KM97], le théorème de Lax est généralisé au cas des des applications préservant la mesure de Lebesgue (sans hypothèse de bijectivité, ni de continuité) ; les auteurs y donnent aussi des exemples d'obtention effective de la permutation approchante en dimension 1.

À partir de la fin des années 1970, des physiciens et des mathématiciens s'intéressent aux discrétisations de systèmes dynamiques particuliers, dont le comportement est bien connu. Certains effectuent des simulations numériques, et évaluent la cohérence entre les résultats de ces simulations et la vraie dynamique du système ; d'autres encore étudient cette cohérence d'un point de vue théorique. Par exemple, en 1978, G. Benettin *et al.* s'intéressent à l'obtention numérique des mesures physiques de certains difféomorphismes d'Anosov [BCG+78, BCG+79], dont l'application du chat d'Arnold et quelques unes de ses perturbations.

En 1983, dans [GT83], J.-M. Gambaudo et C. Tresser montrent, à l'aide de simulations numériques bien choisies, qu'il arrive que les puits d'un homéomorphisme soient difficiles à détecter sur les simulations, simplement parce que la taille de leurs bassins peut être minuscule, même pour des homéomorphismes ayant une définition assez simple.

Toujours dans les années 80, on voit apparaître des travaux théoriques sur la dynamique des discrétisations de systèmes dynamiques classiques, comme dans [HYG88], où les auteurs s'intéressent à la propriété de pistage pour certains paramètres de l'application de Hénon, dans [HYG87], où ils étudient cette même propriété pour l'application logistique, dans [NY88], où les auteurs s'intéressent à la propriété de pistage pour l'application tente et l'application quadratique, ou encore dans [SB86], où les auteurs étu-

dient les caractéristiques des orbites des discrétisations de l'application tente, à savoir leurs nombre, leurs longueurs et leurs répartition.

Aussi, dans [Bin92], l'auteur étudie numériquement le nombre de cycles limites et la longueur du plus grand de ces cycles pour les discrétisations de l'application logistique.

Dans [DKPV95, DSKP96, DKKP96, DP96, DKKP97], les auteurs remarquent que certaines quantités relatives aux discrétisations de systèmes dynamiques de dimension 1, comme la proportion de points des discrétisations dans le bassin d'attraction du point fixe 0, la distribution des longueurs des cycles, le temps de stabilisation, etc., sont similaires aux mêmes quantités que pour des applications aléatoires avec un centre attractif, notamment lorsqu'on fait la moyenne de ces quantités sur plusieurs discrétisations successives. Leur étude est principalement basée sur des hypothèses de convergence de la dynamique des discrétisations vers celle de l'application de départ, vérifiées expérimentalement sur des exemples bien choisis.

Enfin, dans l'un de ses tout derniers papiers [Lan98], O.E. Lanford étudie des simulations de discrétisations de systèmes dynamiques en dimension 1, en s'intéressant entres autres aux périodes et aux tailles des bassins des orbites périodiques des discrétisations. Il fait la remarque que pour retrouver la dynamique de l'application de départ, il faudrait dans l'idéal adapter le temps jusqu'auquel on observe les itérés des discrétisations aux tailles des grilles de ces discrétisations. Ce point de vue, bien que très intéressant, est très différent de celui adopté dans ce manuscrit.

Tous ces résultats concernent des exemples très particuliers de systèmes ; ils seraient difficilement généralisables à des classes plus larges de dynamiques.

L'idée de O.E. Lanford consistant à s'intéresser à la longueur des segments d'orbites avait en fait déjà été développée à la fin des années 1980 par A. Boyarsky. Dans [Boy86], celui-ci explique de manière heuristique pourquoi, en général, on retrouve sur les simulations les mesures absolument continues par rapport à la mesure de Lebesgue. Ses arguments sont basés sur le pistage de longs segments d'orbites ; le seul obstacle à l'obtention d'une preuve rigoureuse est l'absence d'uniformité dans le théorème ergodique de Birkhoff [1].

Dans [GB88], A. Boyarsky et P. Góra établissent le résultat suivant, qui concerne encore l'obtention de mesures absolument continues par rapport à la mesure de Lebesgue à partir des discrétisations.

Theorème VIII. *Si f admet une unique mesure invariante absolument continue par rapport à Lebesgue μ, et si $\alpha > 0$ est tel qu'il existe une sous-suite f_N admettant un segment d'orbite de longueur supérieure à αq_n (où q_n est le cardinal de la grille E_N), si on pose ν_N la mesure uniforme sur ce segment d'orbite, alors $\nu_N \rightharpoonup \mu$.*

Si le fait de posséder une mesure invariante absolument continue est établi pour certaines classes assez larges de systèmes (applications dilatantes du cercle de classe $C^{1+\alpha}$, difféomorphismes d'Anosov de classe $C^{1+\alpha}$...), l'existence d'un segment d'orbite de longueur proportionnelle à celle de la grille semble n'être presque jamais vérifiée (par exemple, elle n'est pas vraie pour une application dilatante générique du cercle, simplement parce que le degré de récurrence est nul). Malgré cela, si on met de côté le théorème de Lax, ce théorème constitue l'un des tout premiers résultats théoriques concernant les discrétisations de systèmes dynamiques.

1. De fait, l'intuition de A. Boyarski se révèle correcte pour un homéomorphisme uniquement ergodique (proposition IX de T. Miernowski), mais faux dans des situations plus générales (théorème V pour les homéorphismes conservatifs génériques, et théorème VI pour les difféomorphismes conservatifs C^1-génériques).

Parallèlement, des recherches sont effectuées par des chercheurs soviétiques, notamment M. Blank dans [Bla84, Bla86, Bla89, Bla94], qui s'intéresse entre autres à la question suivante : comment retrouver sur les discrétisations les orbites périodiques du système dynamique de départ, aussi bien attractives que répulsives ? Il présente le phénomène de multiplication de la période : sur les discrétisations, il peut apparaître une orbite dont la longueur est un multiple non trivial de l'orbite réelle qu'elle est censée représenter.

Dans les années 90 un groupe de chercheurs comprenant entre autres P. Diamond, P. Kloeden, V. Kozyakin, J. Mustard et A. Pokrovskii publie toute une série d'articles sur la détection de propriétés dynamiques à partir des discrétisations. Par exemple, en 1993, dans [DK93], les auteurs font remarquer que tout attracteur est détecté par les discrétisations. L'étude des discrétisations amène les auteurs à définir quelques variantes de propriétés dynamiques stables, adaptées à l'étude des discrétisations. Par exemple, dans [DKP94], ils définissent une notion de transitivité par chaînes adaptée aux applications finies, la propriété de « récurrence faible par chaînes ». Ils établissent ensuite bon nombre de propriétés de manipulation de cette notion ; ils vérifient notamment qu'elle est stable par passage aux discrétisations. Dans le même ordre d'idées, dans [DKKP95a, DKKP95b], les auteurs définissent ce qu'ils nomment les applications semi-hyperboliques (qui incluent les applications hyperboliques) ; ils établissent ensuite que applications semi-hyperboliques possèdent la propriété de pistage, ainsi que divers résultats tels que la stabilité structurelle de la semi-hyperbolicité ou bien la semi-continuité de l'entropie. Enfin, les auteurs définissent ce qu'est le *centre minimal d'attraction de Birkhoff*, un ensemble comprenant l'ensemble non errant [DKP96] ; de par sa définition, cet ensemble est stabilisé par passage aux discrétisations.

Ces travaux ont le mérite de démontrer formellement que certaines propriétés dynamiques peuvent être détectés sur des discrétisations spatiales. On notera cependant qu'ils ne concernent que des propriétés qui sont par définition robustes (existence d'orbites attractives, récurrence faible par chaînes, etc.), et donc passent naturellement aux discrétisations.

Un des exemples les plus simples de systèmes dynamiques chaotiques est celui des automorphismes d'Anosov linéaires du tore $\mathbf{T}^2$. En 1992, F. Dyson et H. Falk [DF92] font une étude théorique assez complète de la période globale des discrétisations de l'application du chat d'Arnold $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. En particulier, la dynamique des discrétisations est complètement différente de celle de l'automorphisme d'Anosov linéaire (voir le début de cette introduction, page 16). Ce travail est repris et généralisé à tous les automorphismes d'Anosov linaires de $\mathbf{T}^2$ par É. Ghys dans [Ghy94]. Ce dernier note que la période des discrétisations est très faible comparativement à l'ordre d'une application aléatoire sur un ensemble ayant le même nombre d'éléments que la grille ; il attribue ce phénomène aux très fortes propriétés arithmétiques des automorphismes linéaires, ce qui peut laisser penser qu'il est en un certain sens exceptionnel. Cela l'amène à proposer d'étudier le comportement des discrétisations d'homéomorphismes génériques.

Cette question a été résolue en partie par T. Miernowski dans le cas des homéomorphismes génériques du cercle (voir [Mie06] et [Mie05]). Nous survolerons ces résultats intéressants dans la section 2.3. Dans [Mie06], T. Miernowski établit aussi un théorème de convergence de la dynamique des discrétisations vers celle de l'application de départ d'un point de vue ergodique, sous l'hypothèse cependant assez restrictive que l'homéomorphisme considéré est uniquement ergodique :

Proposition IX. *Soit* $f : M \to M$ *un homéomorphisme ayant une unique mesure de probabilité* f*-invariante* μ*. Pour tout* $N \in \mathbf{N}$*, soit* γ_N *un cycle périodique de la* N*-ième discrétisation* f_N *de* f*. Soit* ν_N *la mesure de probabilité uniforme sur le cycle* γ_N*. Alors on a la convergence faible* $\nu_N \rightharpoonup \mu$ *quand* N *tend vers l'infini, indépendamment du choix de chaque cycle* γ_N*.*

Cette convergence de la dynamique discrète vers la dynamique continue est à nouveau mise en valeur dans le cas des homéomorphismes conservatifs génériques du tore par le théorème 2.2.2 de [Mie05] (que nous avons déjà rencontré, au début de cette introduction, voir le théorème I) : Pour un homéomorphisme conservatif générique, il existe une sous-suite de discrétisations de qui sont des permutations des grilles. L'auteur conjecture néanmoins que ce phénomène n'est plus typique dans le cas des difféomorphismes hyperboliques génériques du tore de classe C^r, $r \geq 1$: selon lui, le comportement des discrétisations de telles applications devrait se rapprocher de celui d'une application aléatoire d'un ensemble ayant le même nombre d'éléments que la grille de discrétisation.

La question du comportement des discrétisations d'un système générique a aussi été abordée par P.P. Flockermann dans sa thèse [Flo02] sous la direction de O.E. Lanford. P.P. Flockermann s'intéresse aux applications dilatantes du cercle de régularité au moins C^2 ; il montre principalement qu'en temps fini, les discrétisations de telles applications se comportent de la même manière que des perturbations aléatoires (voir la section 2.4). Ces résultats, bien qu'à mon avis intéressants et non triviaux, ne concernent que le comportement des discrétisations en temps fini et ne disent rien de la dynamique des discrétisations.

Il existe plusieurs autres point de vues sur la modélisation des simulations numériques. On peut par exemple se demander si, étant données une famille d'applications et une propriété dynamique, une machine peut décider en temps fini si un élément de cette famille vérifie cette propriété. Par exemple, dans [AM04], A. Arbieto et C. Matheus montrent que les propriétés « avoir des exposants de Lyapunov positifs » et « posséder une mesure SRB » sont indécidables parmi la famille quadratique et la famille d'applications de Hénon, et que la propriété « être d'entropie topologique strictement positive » est décidable parmi la famille quadratique.

Une étude concernant la modélisation d'une discrétisation d'un homéomorphisme par une application multivaluée a été faite dans [LP11] (voir aussi [Mro96] et [DKP96]). On y trouve une discussion très intéressante sur le passage du continu au discret : *a priori*, on ne peut retrouver les propriétés du système initial à partir d'un nombre fini de discrétisations que lorsque ces propriétés sont robustes. Cela amène les auteurs à définir ce qu'est une propriété à résolution finie : en particulier ce sont des propriétés dynamiques qui peuvent être vérifiables en temps fini par des méthodes calculatoires.

Pour un point de vue différent sur l'histoire des discrétisations, voir le court survol de J. Buzzi [Buz15], ou le survol plus complet [GHR12] de S. Galatolo, M. Hoyrup et C. Rojas.

Dans un tout autre registre, les discrétisations d'applications linéaires ont déjà été étudiées, en vue d'applications au traitement de l'image. Le but est par exemple de répondre à la question suivante : comment effectuer une rotation d'une image numérique sans trop diminuer sa qualité ? On veut en particulier estimer la perte d'information causée par la non bijectivité des discrétisations de rotations, ou bien essayer d'éviter des

phénomènes d'aliasing (l'apparition de résonances entre un motif régulier sur l'image et la grille de l'image elle-même). Les travaux existants se concentrent principalement sur l'étude des itérations de discrétisations d'applications linéaires en temps court ; en particulier ils s'intéressent au comportent local des images de $\mathbf{Z}^2$ par ces discrétisations. De plus, les applications linéaires considérées sont systématiquement supposées à coefficients rationnels. On pourra se faire une idée du sujet en consultant les thèses [Nou06] et [Thi10] (voir aussi page 140 pour une bibliographie plus détaillée).

2.3 Travaux de T. Miernowski sur les discrétisations d'homéomorphismes du cercle

Dans cette sous-section, nous donnons les principaux résultats concernant la dynamique des discrétisations uniformes d'homéomorphismes génériques du cercle $\mathbf{R}/\mathbf{Z}$, obtenus par de T. Miernowski dans [Mie06]. Par « uniforme », on entend que ces discrétisations sont prises selon les grilles $\mathrm{E_N} = \{i/\mathrm{N} \mid 0 \leq i \leq \mathrm{N}-1\}$.

La préservation de l'ordre cyclique sur le cercle implique que si f est un homéomorphisme croissant, alors les orbites périodiques d'une discrétisation f_N ont toutes la même longueur, que l'on note q_N ; c'est sur cette quantité que nous nous concentrons principalement. Il existe alors un entier p_N, qui est tel que si $\widetilde{f}$ est un relevé de f à $\mathbf{R}$, et $\widetilde{f_\mathrm{N}}$ une discrétisation de $\widetilde{f}$ sur la grille uniforme $\mathbf{Z}/\mathrm{N}$, alors pour tout point $\widetilde{x} \in \mathbf{Z}/\mathrm{N}$ dont le projeté sur $\mathbf{S}^1$ est dans une orbite périodique de f_N, on a $\widetilde{f_\mathrm{N}}^{q_\mathrm{N}}(\widetilde{x}) - \widetilde{x} = p_\mathrm{N}$. On montre alors très facilement que la fraction $p_\mathrm{N}/q_\mathrm{N}$ converge vers le nombre de rotation $\rho(f)$ de f.

L'exemple le plus simple d'homéomorphisme du cercle est celui de la rotation (on a même de célèbres énoncés de conjugaison à une rotation dans le cas irrationnel, voir par exemple [Her79]). T. Miernowski fait remarquer que la discrétisation d'une rotation est elle-même une rotation, en particulier qu'elle est bijective. Si on note r_N le nombre de cycles périodiques de la rotation d'angle α, on a bien sûr $\mathrm{N} = r_\mathrm{N} q_\mathrm{N}$: la dynamique de f_N est donc complètement déterminée par ce nombre r_N. Dans le cas où $\alpha \in \mathbf{R} \setminus \mathbf{Q}$, le comportement de r_N se déduit facilement d'un résultat connu de comptage de points rationnels dans une région du plan (voir par exemple [Wat53], voir aussi [DKKP96]) ; on obtient alors :

$$\lim_{\mathrm{M}\to+\infty} \frac{\mathrm{Card}\{\mathrm{N} \leq \mathrm{M} \mid r_\mathrm{N} = k\}}{\mathrm{M}} = \frac{6}{\pi^2 k^2}.$$

La proposition 3.2 de [Mie06] étend ce résultat au cas rationnel. Avec ces résultats, on peut en déduire que la longueur q_N des orbites périodiques converge au sens de Cesàro.

Proposition X. *Si $\alpha \in \mathbf{R} \setminus \mathbf{Q}$, alors*

$$\frac{1}{\mathrm{N}} \sum_{i=1}^{\mathrm{N}} q_i \sim \mathrm{N} \frac{3}{\pi^2} \zeta(3).$$

Si en revanche $\alpha = \frac{p}{q} \in \mathbf{Q}$, avec p et q premiers entre eux, on a deux cas :

$$\frac{1}{\mathrm{N}} \sum_{i=1}^{\mathrm{N}} q_i \sim \begin{cases} \mathrm{N} \dfrac{q-1}{q^2} \displaystyle\sum_{c=1}^{\frac{q-1}{2}} \frac{1}{c^2} \sum_{r=1}^{\frac{q-1}{2c}} \frac{\varphi(r)}{r} & \text{si } q \text{ impair,} \\ \mathrm{N} \dfrac{q-1}{q^2} \displaystyle\sum_{c=1}^{\frac{q}{2}} \frac{1}{c^2} \sum_{r=1}^{\frac{q}{2c}} \frac{\varphi(r)}{r} + \frac{\mathrm{N}}{2} \frac{(q-1)^2(q+1)}{q} \varphi(q/2) & \text{si } q \text{ pair.} \end{cases}$$

Par conséquent, en un certain sens, on a une convergence de la dynamique des discrétisations au sens de Cesàro. Ce phénomène est complètement opposé à celui apparaissant dans la cas des homéomorphismes conservatifs génériques, où même si on essaie de moyenner sur les discrétisations successives, la dynamique combinatoire des discrétisations continue à être fortement erratique (voir le théorème XXXII page 302).

T. Miernowski s'intéresse ensuite au cas des homéomorphismes génériques du cercle. Le résultat décrivant le comportement de la longueur des orbites périodiques des discrétisations est le suivant.

Proposition XI (Miernowski)**.** *Un homéomorphisme croissant générique f du cercle possède un point périodique stable. En particulier, son nombre de rotation est rationnel, noté $\rho(f) = p/q$. Dans ce cas, il existe un entier $N_0 \in \mathbf{N}$ tel que pour tout $N \geq N_0$, on a $p_N = p$ et $q_N = q$.*

Ce résultat facile est quelque peu décevant : le comportement dynamique des discrétisations est le même pour toutes les discrétisations assez fines, et découle en réalité du fait que la dynamique de l'homéomorphisme de départ est une dynamique de puits (ou de Morse-Smale). C'est pourquoi on cherche à éviter ce type de dynamiques : T. Miernowski étudie ensuite le cas plus difficile des dynamiques génériques parmi celles qui ont un nombre de rotation rationnel mais aucune orbite attractive, ou bien celles ayant un nombre de rotation irrationnel.

Dans le premier cas, T. Miernowski montre le théorème suivant.

Theorème XII (Miernowski)**.** *Dans l'ensemble des homéomorphismes semi-stables*[2] *de nombre de rotation p/q, il existe un ouvert dense d'homéomorphismes pour lesquels il existe une sous-suite $(N_k)_{k\in\mathbf{N}}$ de discrétisations vérifiant $q_{N_k}(f) = q$.*

Comme on peut l'imaginer, la preuve de ce résultat utilise de manière cruciale l'ordre naturel sur le cercle.

Dans le cas du nombre de rotation irrationnel, T. Miernowski propose deux approches. La première concerne les difféomorphismes *génériques*.

Theorème XIII (Miernowski)**.** *Soit $\Psi : \mathbf{N} \to \mathbf{R}_+$ une fonction croissante tendant vers $+\infty$ en $+\infty$. Soit $r \in \mathbf{N} \cup \{\infty, \omega\}$ (en particulier, on peut avoir $r = 0$). Alors pour un difféomorphisme irrationnel de classe C^r générique f, on a*

$$\varliminf_{N\to+\infty} \frac{q_N(f)}{\Psi(N)} = 0.$$

Autrement dit, la suite $q_N(f)$ tend vers $+\infty$, mais possède une sous-suite qui le fait arbitrairement lentement. La preuve de ce résultat est assez jolie, car elle repose essentiellement sur l'étude menée dans le cas semi-stable et sur l'approximation de tout difféomorphisme irrationnel par des difféomorphismes rationnels semi-stables.

Dans la seconde approche, T. Miernowski s'intéresse au cas des difféomorphismes *prévalents*. Le résultat obtenu est alors bien différent de ce qui se passe pour des difféomorphismes génériques : la suite $q_N(f)$ tend vers $+\infty$ presque aussi vite que $\sqrt{N}$.

Theorème XIV (Miernowski)**.** *Soit f un difféomorphisme prévalent parmi les difféomorphismes irrationnels de classe C^r, $3 \leq r \leq \omega$. Alors pour tout $\varepsilon > 0$, il existe $K > 0$ tel que $q_N(f) \geq KN^{1/(2+\varepsilon)}$.*

2. C'est-à-dire que le relevé de f^q à $\mathbf{R}$ est soit toujours supérieur à la translation de vecteur p, soit toujours inférieur à la translation de vecteur p.

La preuve de ce théorème est principalement basée sur un théorème classique de dynamique prévalente sur le cercle, à savoir le théorème de conjugaison différentiable à une rotation, dû à M. Herman (voir [Her77]).

2.4 Travaux de P.P. Flockermann et O.E. Lanford sur les discrétisations d'applications dilatantes génériques du cercle

Dans sa thèse [Flo02], P.P. Flockermann étudie le comportement des discrétisations [3] des applications dilatantes du cercle de classe (au moins) C^2. Outre des études numériques assez poussées, P.P. Flockermann y démontre deux résultats principaux ; ceux-ci expriment qu'en temps k fixé et sous des hypothèses d'indépendance linéaire des dérivées, l'opération de discrétisation d'une application dilatante du cercle typique agit essentiellement comme une perturbation aléatoire.

Tout d'abord, on peut faire des statistiques sur les erreurs d'arrondi faites à chaque itération : pour $x \in \mathrm{E_N}$, on note $\varepsilon_k(x,\mathrm{N}) = \mathrm{N}\Big(f_{\mathrm{N}}^k(x) - f(f_{\mathrm{N}}^{k-1}(x))\Big)$ l'erreur normalisée faite à la k-ième itération ; *a priori*, cette erreur est comprise entre $-1/2$ et $1/2$. Le théorème que démontre P.P. Flockermann est alors le suivant (théorème 9 de [Flo02]).

Theorème XV (Flockermann)**.** *Soit f une application dilatante de degré 2 de $\mathbf{S}^1$ telle que*
- *soit f est générique parmi les applications dilatantes de classe C^r, $r \geq 2$;*
- *soit f est analytique, mais différente de $x \mapsto 2x$.*

Soit aussi $\vartheta : \mathbf{N} \to \mathbf{R}$ une fonction telle que $1/\mathrm{N} \ll \vartheta(\mathrm{N}) \ll 1$. Alors il existe un ensemble dénombrable $\mathrm{D} \subset \mathbf{S}^1$ tel que pour tout $x_0 \in \mathbf{S}^1 \setminus \mathrm{D}$, et pour tout $k_0 \in \mathbf{N}$, les distributions des k_0-uplets $\Big(\varepsilon_k(x,\mathrm{N})\Big)_{k\leq k_0} \in [-1/2,1/2]^{k_0}$, pour $x \in \mathrm{E_N} \cap [x_0 - \vartheta(\mathrm{N}), x_0 + \vartheta(\mathrm{N})]$, convergent vers la distribution uniforme sur $[-1/2,1/2]^{k_0}$.

Autrement dit, si on se place à une échelle d'espace $\vartheta(\mathrm{N})$ intermédiaire, ou « mésoscopique », autour d'un point x_0 générique, alors les erreurs d'arrondi que l'on fait à chaque itération lorsqu'on discrétise sont indépendantes du temps et suivent une loi uniforme sur $[-1/2,1/2]$. La preuve de ce résultat se décompose en 2 parties indépendantes. La première est un résultat conduisant aux conclusions du théorème sous des hypothèses d'indépendance des dérivées successives des applications dilatantes. On montre ensuite que ces hypothèses sont vérifiées par une application dilatante C^r générique, ou analytique différente de $x \mapsto 2x$.

À l'aide du formalisme des ensembles modèle, nous donnerons une généralisation de ce théorème au cas de la dimension quelconque dans le chapitre 9 du présent manuscrit (voir la proposition 9.1).

Dans un second temps, P.P. Flockermann étudie la répartition de $\mathrm{Card}(f_{\mathrm{N}}^{-k}(x))$, avec k fixé et pour un grand nombre de points x dans une petite région de $\mathbf{S}^1$. Pour plus de simplicité, nous n'énonçons le théorème que dans le cas où le temps k vaut 1 ; on imaginera sans peine un énoncé en toute généralité (cet énoncé est néanmoins assez délicat à formuler, voir le théorème 12 et le corollaire 3 de [Flo02]).

Theorème XVI (Flockermann)**.** *Soit f une application dilatante de degré 2 de $\mathbf{S}^1$, générique parmi les applications dilatantes de classe C^r, $r \geq 2$. Soit aussi $\vartheta : \mathbf{N} \to \mathbf{R}$ une fonction telle*

3. Toujours par rapport aux grilles $\mathrm{E_N} = [0,1[\cap \mathbf{Z}/\mathrm{N}$.

que $1/N \ll \vartheta(N) \ll 1$. *Alors il existe un ensemble dénombrable* $D \subset \mathbf{S}^1$ *tel que pour tout* $x_0 \in \mathbf{S}^1 \setminus D$, *si on note* $\{y_1, y_2\} = f^{-1}(x_0)$, *lorsque* N *tend vers* $+\infty$,

- *la proportion de points* $x \in E_N \cap [x_0 - \vartheta(N), x_0 + \vartheta(N)]$ *qui n'ont pas d'image réciproque par* f_N *tend vers*
$$\left(1 - \frac{1}{f'(y_1)}\right)\left(1 - \frac{1}{f'(y_2)}\right);$$
- *la proportion de points* $x \in E_N \cap [x_0 - \vartheta(N), x_0 + \vartheta(N)]$ *qui ont exactement une image réciproque par* f_N *tend vers*
$$\left(1 - \frac{1}{f'(y_1)}\right)\frac{1}{f'(y_2)} + \frac{1}{f'(y_1)}\left(1 - \frac{1}{f'(y_2)}\right);$$
- *la proportion de points* $x \in E_N \cap [x_0 - \vartheta(N), x_0 + \vartheta(N)]$ *qui ont deux images réciproques par* f_N *tend vers*
$$\frac{1}{f'(y_1)}\frac{1}{f'(y_2)}.$$

Ce théorème exprime lui aussi qu'en temps court, l'opération de discrétisation se comporte comme une perturbation aléatoire : si on se donne une dilatation h_λ de $\mathbf{R}$ de rapport $\lambda > 1$, alors la proportion de points dans l'image de la discrétisation de h_λ vaut $1/\lambda$. Or, le graphe de f au voisinage de $f^{-1}(x_0)$ possède exactement deux branches ; la première est bien approchée par la dilatation $h_{f'(y_1)}$ et la seconde par la dilatation $h_{f'(y_2)}$. Si la famille $(1, f'(y_1), f'(y_2))$ est libre, alors les événements « posséder une image réciproque par f_N dans la première branche » et « posséder une image réciproque par f_N dans la seconde branche » deviennent indépendants, ce qui permet d'obtenir le théorème. Celui-ci implique le résultat dynamique suivant.

Theorème XVII (Flockermann, Lanford, non publié)**.** *Soit* f *une application dilatante de degré 2 de* $\mathbf{S}^1$, *générique parmi les applications dilatantes de classe* C^r, $r \geq 2$. *Alors le degré de récurrence de* f *est égal à 0.*

Nous démontrerons des résultats similaires dans la section 11.5 du présent manuscrit. On y énoncera notemment un résultat reliant le comportemnt local au comportement global d'une application dilatante générique (théorème 11.19).

Part 1

Discretizations of generic homeomorphisms

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Introduction

In this part, we consider the dynamics of discretizations of generic homeomorphisms, *i.e.* we tackle the following question:

Question. *Which dynamical properties of a generic homeomorphism f can be read on the dynamics of its discretizations $(f_N)_{N\geq 0}$?*

We will establish properties for both *dissipative*, *i.e.* arbitrary homeomorphisms of X, and *conservative* homeomorphisms, *i.e.* homeomorphisms of X that preserve a given good probability measure. In this part, our results concern generic homeomorphisms of a compact manifold (with boundary) of dimension $n \geq 2$.

We will prove many results, concerning various aspects of the dynamics of the discretizations, adapting some classical tools of study of the generic dynamics of homeomorphisms. Morally, our results express that in the dissipative generic case, the dynamics of the discretizations tends to the "physical" dynamics of the initial homeomorphism [4] whereas in the conservative generic setting, the dynamics of the discretizations accumulates on all the possible dynamics of the initial homeomorphism, and moreover the physical dynamics cannot be detected on discretizations. In the rest of this introduction, we try to organize our results according to some "lessons":

1) The dynamics of discretizations of a generic dissipative homeomorphism tends to the "physical dynamics" of the initial homeomorphism.

We first study properties of discretizations of generic *dissipative* homeomorphisms [5]. The ergodic behaviour of such a generic homeomorphism is deduced from the *shredding lemma* of F. Abdenur and M. Andersson [AA13] (Lemma 4.2), which implies that a generic homeomorphism has a "attractor dynamics" (see Corollaries 4.5 and 4.8):

Theorem A. *For a generic homeomorphism f, the closure $\overline{A_0}$ of the set of Lyapunov-stable periodic points is a Cantor set of dimension 0 which attracts almost every point of* X. *Moreover, the measure μ_X^f (see Definition 3.10 page 52) is well defined, atomless and is supported by $\overline{A_0}$.*

This behaviour easily transmits to discretizations, for example every attractor of the homeomorphism can be seen on all the fine enough discretizations (Proposition 4.13).

Theorem B. *For a generic homeomorphism f, the recurrent set of its discretization $\Omega(f_N)$ tends to $\overline{A_0}$ in the following weak sense: for all $\varepsilon > 0$, there exists $N_0 \in \mathbf{N}$ such that for*

4. That is, the dynamics that occur for almost every point with respect to Lebesgue measure.
5. Without assumption of preservation of a given measure.

all $N \geq N_0$, *there exists a subset* $\widetilde{E}_N$ *of* E_N, *stabilized by* f_N, *such that, noting* $\widetilde{\Omega}(f_N)$ *the corresponding recurrent set, we have* $\frac{\mathrm{Card}(\widetilde{E}_N)}{\mathrm{Card}(E_N)} > 1-\varepsilon$ *and* $d_H(A_0, \widetilde{\Omega}(f_N)) < \varepsilon$.

Moreover, the Cesàro limit of the pushforwards of the uniform measures on the grid by the discretizations tend to the Cesàro limit of the pushforwards of λ by the homeomorphism. Indeed, we will prove the following (Theorem 4.16).

Theorem C. *For a generic homeomorphism* f, *the measures* $\frac{1}{m}\sum_{i=0}^{m-1} f_*^i \lambda$ *converge to a measure that we denote by* μ_X^f *(see Definition 3.10).*

Concerning the discretizations, for a generic homeomorphism f, *the measures* $\mu_X^{f_N}$ *tend to the measure* μ_X^f *when* N *goes to infinity, where* $\mu_X^{f_N}$ *is the measure on the periodic orbits of* f_N *such that the global measure of each periodic orbit is proportional to the size of its basin of attraction (see Definition 3.10).*

Moreover, there is *shadowing* of the dynamics of f by that of its discretizations f_N (Corollary 4.12).

Theorem D. *For a generic homeomorphism* f, *for all* $\varepsilon > 0$ *and all* $\delta > 0$, *there is a full measure dense open subset* O *of* X *such that for all* $x \in O$, *all* $\delta > 0$ *and all* N *large enough, the orbit of* x_N *under* f_N δ*-shadows* [6] *the orbit of* x *under* f.

Thus, it is possible to detect on discretizations the "physical" dynamics of a generic dissipative homeomorphism, that is the dynamics that can be seen by almost every point of X. This dynamics is mainly characterized by the position of the attractors and of the corresponding basins of attraction.

2) The dynamics of a single discretization of a generic conservative homeomorphism cannot be inferred from the dynamics of the initial homeomorphism.

We then turn to the study of the conservative case. The starting point of our study is a question from É. Ghys (see [Ghy94, Section 6]): for a generic conservative homeomorphism of the torus, what is the asymptotical behaviour of the sequence of degrees of recurrence of f_N? A partial answer to this question was obtained by T. Miernowski in the second chapter of his thesis (see Corollary 5.9).

Theorem E (Miernowski)**.** *For a generic conservative homeomorphism* [7] f, *there are infinitely many integers* N *such that the discretization* f_N *is a cyclic permutation.*

To prove this theorem, T. Miernowski combines a genericity argument with a quite classical technique in generic dynamics: Lax's theorem [Lax71] (Theorem 5.5), which states that any conservative homeomorphism can be approximated by cyclic permutations of the discretization grids. In fact this proof can be generalized to obtain many results about discretizations. We will establish some variants of Lax theorem; each of them, combined with a genericity argument, leads to a result for discretizations of generic homeomorphisms. For instance, we will prove the following (Corollary 5.36).

Theorem F. *For a generic conservative homeomorphism* f, *there exists infinitely many integers* N *such that the cardinality of* $\Omega(f_N)$ *is equal to the smallest period of periodic points of* f.

6. That is, for all $k \in \mathbf{N}$, $d(f_N^k(x_N), f^k(x)) < \delta$.

7. That is, there is a G_δ dense subset of the set of conservative homeomorphisms of the torus on which the conclusion of the theorem holds.

Note that the combination of these two theorems answer É. Ghys's question: for a generic homeomorphism f, the sequence of the degrees of recurrence of f_N accumulates on both 0 and 1; we can even show that it accumulates on the whole segment $[0,1]$ (Corollary 5.24).

Another variation of Lax's theorem leads to a theorem that throws light on the behaviour of the discretizations on their recurrent set (Corollary 5.22).

Theorem G. *For a generic conservative homeomorphism f and for all $M \in \mathbf{N}$, there are infinitely many integers N such that f_N is a permutation of E_N having at least M periodic orbits.*

To summarize, generically, infinitely many discretizations are cyclic permutations, but also infinitely many discretizations are highly non-injective or else permutations with many cycles. In particular, it implies that for all $x \in X$, there exists infinitely many integers N such that the orbit of x_N under f_N does not shadow the orbit of x under f: in this sense, generically, the dynamics of discretizations does not reflect that of the homeomorphism. Note that this behaviour is in the opposite of the dissipative case, here the individual behaviour of discretizations does not indicate anything about the actual dynamics of the homeomorphism.

3) A dynamical property of a generic conservative homeomorphism cannot be deduced from the frequency it appears on discretizations either.

The previous theorems express that the dynamics of a single discretization does not reflect the actual dynamics of the homeomorphism. However, we might reasonably expect that the properties of the homeomorphism are transmitted to many discretizations. More precisely, we may hope that given a property (P) about discretizations, if there are many integers N such that the discretization f_N satisfies (P), then the homeomorphism satisfies a similar property. It is not so, for instance, we will prove the following result (Theorem 5.31).

Theorem H. *For a generic conservative homeomorphism f, when M goes to infinity, the proportion of integers N between 1 and M such that f_N is a cyclic permutation accumulates on both 0 and 1.*

In fact, for all the properties considered in the previous paragraph, the frequency with which they appear on discretizations of orders smaller than M accumulates on both 0 and 1 when M goes to infinity. Remark that these result imply that, even by looking at the frequency at which some properties occur, the discretizations of a generic conservative homeomorphism do not behave like typical random maps, as for a random map of a set with q elements, the average number of periodic orbits is asymptotically $\log q$ (see for example [Bol01, XIV.5]).

4) Many dynamical properties of a generic conservative homeomorphism can be detected by looking at the dynamics of *all* the discretizations.

We have observed that we cannot detect the dynamics of a generic homeomorphism when looking at the dynamics of its discretizations, or even at the frequency with which some dynamics appears on discretizations. Nevertheless, the dynamical properties of a generic conservative homeomorphism can be deduced from the analogous dynamical properties of *all* the discretizations. More precisely, we have a shadowing property of

the dynamical properties of the homeomorphism: for each dynamical property of the homeomorphism, its discrete analogue can be seen on an infinite number of discretizations. It is worthwhile to note the intriguing fact that this shadowing property occurs for all the dynamical properties of a generic conservative homeomorphism, independently of the measure λ, while for a generic dissipative homeomorphism the dynamics of the discretizations converges to the "physical" dynamics of the homeomorphism (that is, the dynamics depending of λ).

This idea of convergence of the dynamics when looking at arbitrary large precisions can be related to the work of P. Diamond *et al* (see page 31). For instance, we will prove that the periodic orbits of a homeomorphism can be detected by looking at the periodic orbits of its discretizations (Theorem 5.36).

Theorem I. *For a generic homeomorphism f, for every $\varepsilon > 0$ and every periodic orbit of f, this periodic orbit is ε-shadowed by an infinite number of periodic orbits of the same period of the discretizations*

We will also prove a theorem in the same vein for invariant measures (respectively invariant compact sets), which expresses that the set of invariant measures (respectively compact sets) of the homeomorphism can be deduced from the sets of invariant measures (respectively invariant sets) of its discretizations. More precisely, we will prove the following result (Theorems 5.45 and 5.51, see Theorems 5.49 and 5.55 for the compact versions).

Theorem J. *For a generic conservative homeomorphism f and for every convex closed set (for Hausdorff topology) $\mathscr{M}$ of f-invariant Borel probability measures there exists an increasing sequence of integers N_k such that the set of f_{N_k}-invariant probability measures tends to $\mathscr{M}$ (for Hausdorff topology).*

Moreover, if $\mathscr{M}$ is reduced to a single measure, then f_{N_k} can be supposed to bear a unique invariant measure.

In the third chapter of this part, we will present an application of the notion of discretization to the practical problem of computing numerically the rotation set of a torus homeomorphism. In particular, we will prove a theorem which expresses the shadowing property of the rotation set of a generic conservative torus homeomorphism by the rotation sets of its discretizations (Corollary 6.23).

Theorem K. *If f is a generic conservative homeomorphism, then there exists a subsequence f_{N_i} of discretizations such that $\rho_{\mathrm{N}_i}(\mathrm{F})$ tends to $\rho(\mathrm{F})$ for Hausdorff topology (where F is a lift of f to $\mathbf{R}^2$).*

This will give us a convenient way to compute the rotation set in practice, as we will prove that if we compute the rotation set corresponding to a starting point $x \in \mathbf{T}^2$ without roundoff error, then for almost every $x \in \mathbf{T}^2$, the obtained rotation set is reduced to a single vector, that is the mean rotation vector of f (Proposition 6.19).

Theorem L. *Let f be a generic conservative homeomorphism of the torus $\mathbf{T}^2$ and F a lift of f to $\mathbf{R}^2$. Then for almost every $\tilde{x} \in \mathbf{R}^2$, the corresponding rotation set*

$$\overline{\rho}(x) = \bigcap_{\mathrm{M} \in \mathbf{N}} \overline{\bigcup_{m \geq \mathrm{M}} \left\{ \frac{\mathrm{F}^m(\tilde{x}) - \tilde{x}}{m} \right\}}$$

is reduced to the mean rotation vector with respect to the measure λ.

This shows that if we try to compute the rotation set by calculating segments of orbits *without* making any roundoff error, we will only find the mean rotation vector of the homeomorphism. In Chapter 6, we will introduce the notion of *observable rotation set*, which expresses which rotation vectors can be found by looking at almost every periodic orbit. We will compute this set for some examples, in particular for both generic conservative (that is, Theorem L) and dissipative (Proposition 6.15) homeomorphisms.

5) The "physical dynamics" of a generic conservative homeomorphism plays no particular role for discretizations.

The heuristic idea underlying the concept of physical measure is that these measures are the invariant ones which can be detected "experimentally" (since many initial conditions lead to these measures). Indeed, some experimental results on specific examples of dynamical systems show that they are actually the measures that are detected in practice for these examples (see for example [BCG+78, BCG+79] or [Boy86, GB88]). Moreover, if the dynamical system is uniquely ergodic, then the invariant measure appears naturally on discretizations (see [Mie06, Proposition 8.1] and Proposition 5.42).

According to this heuristic and these results, we could expect from physical measures to be the only invariant measures that can be detected on discretizations of generic conservative homeomorphisms. This is not the case: for a generic conservative homeomorphism, there exists a unique physical measure, namely λ (that follows directly from the celebrated Oxtoby-Ulam's theorem [OU41]). According to the previous theorem, invariant measures of the discretizations accumulate on all the invariant measures of the homeomorphism and not only on Lebesgue measure.

However, we could still hope to distinguish the physical measure from other invariant measures. For this purpose, we define the canonical physical measure $\mu_X^{f_N}$ associated to a discretization f_N: it is the limit in the sense of Cesàro of the images of the uniform measure on E_N by the iterates of f_N: if λ_N is the uniform measure on E_N, then

$$\mu_X^{f_N} = \lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} (f_N^i)_* \lambda_N.$$

This measure is supported by the recurrent set $\Omega(f_N)$; it is uniform on every periodic orbit and the total weight of a periodic orbit is proportional to the size of its basin of attraction. The following theorem expresses that these measures accumulate on the whole set of f-invariant measures: physical measures cannot be distinguished from other invariant measures on discretizations, at least for generic homeomorphisms (see Theorems 5.51 and 5.53).

Theorem M. *For a generic conservative homeomorphism f, the set of limit points of the sequence $(\mu_X^{f_N})_{N\in\mathbf{N}}$ is the set of all f-invariant measures. Also, for every f-invariant measure μ, there exists a subsequence f_{N_k} of discretizations such that for every $x \in X$, the sequence of measures*[8] *$\mu_x^{f_{N_k}}$ tends to μ.*

The same phenomenon appears for compact invariant subsets: the the recurrent subsets of the discretizations f_N accumulate on the whole set of invariant compact subsets for f (Proposition 5.55).

8. Recall that by Definition 3.10, $\mu_x^{f_N}$ is the Cesàro limit of the pushforwards of the Dirac measure δ_{x_N} by the discretization f_N.

6) On the numerical experiments we performed, the dynamics of a dissipative homeomorphism can be detected on discretizations, and a lot of different dynamical behaviours can be observed on discretizations of a conservative homeomorphism.

We will compare our theoretical results with the reality of numerical simulations. Indeed, it is not clear that the behaviour predicted by our results can be observed on computable discretizations of a homeomorphism defined by a simple formula. On the one hand, all our results are valid "for *generic* homeomorphisms"; nothing indicates that these results apply to actual examples of homeomorphisms defined by simple formulas. On the other hand, results such as "there are infinitely many integers N such that the discretization of order N..." provide no control over the integers N involved; they may be so large that the associated discretizations are not computable in practice.

We first carried out simulations of dissipative homeomorphisms. The results of discretizations of a small perturbation of identity (in C^0 topology) may seem disappointing at first sight: the trapping regions of the initial homeomorphisms cannot be detected, and there is little difference with the conservative case. This behaviour is similar to that highlighted by J.-M. Gambaudo and C. Tresser in [GT83] (see page 2.2). That is why it seemed to us useful to test a homeomorphism which is C^0 close to the identity, but whose basins are large enough. In this case the simulations point to a behaviour that is very similar to that described by theoretical results, namely that the dynamics converges to the dynamics of the initial homeomorphism. In fact, we have actually observed behaviours as described by theorems only for examples of homeomorphisms with a very small number of attractors.

For conservative homeomorphisms, our numerical simulations produce mixed results. From a quantitative viewpoint, the behaviour predicted by our theoretical result cannot be observed on our numerical simulations. For example, we do not observe any discretization whose degree of recurrence is equal to 1 (i.e. which is a permutation). This is nothing but surprising: the events pointed out by the theorems are *a priori* very rare. For instance, there is a very little proportion of bijective maps among maps from a given finite set into itself. From a more qualitative viewpoint, the behaviour of the simulations is quite in accordance with the predictions of the theoretical results. For example, for a given conservative homeomorphism, the degree of recurrence of a discretization depends a lot on the size of the grid used for the discretization. Similarly, the canonical invariant measure associated with a discretization of a homeomorphism f does depend a lot on the size of the grid used for the discretization.

7) Discretizations can actually be very useful and efficient to compute some dynamical invariants like the rotation set of a torus homeomorphism.

We have also performed numerical simulations of rotation sets. To obtain numerically an approximation of the observable rotation set, we have calculated rotation vectors of long segments of orbits for a lot of starting points, these points being chosen randomly fore some simulations and being all the points of a grid on the torus for other simulations. For the numerical approximation of the asymptotic discretized rotation set we chosed a fine enough grid on the torus and have calculated the rotation vectors of periodic orbits of the discretization of the homeomorphism on this grid.

We made these simulations on an example where the rotation set is known to be the square $[0,1]^2$. It makes us sure of the shape of the rotation set we should obtain numerically, however it limits a bit the "genericity" of the examples we can produce.

In the dissipative case we made attractive the periodic points which realize the vertex of the rotation set $[0,1]^2$. It is obvious that these rotation vectors, which are realized by attractive periodic points with basin of attraction of reasonable size, will be detected by the simulations of both observable and asymptotic discretized rotation sets; that is we observe in practice: we can recover quickly the rotation set in both cases.

In the conservative setting we observe the surprising behaviour predicted by the theory: when we compute the rotation vectors of long segments of orbits we obtain mainly rotation vectors which are quite close to the mean rotation vector, in particular we do not recover the initial rotation set. More precisely, when we perform simulations with less than one hour of calculation we only obtain rotation vectors close to the mean rotation vector, and when we let three hours to the computer we only recover one vertex of the rotation set $[0,1]^2$. On the other hand, the rotation set is detected very quickly by the convex hulls of discretized rotation sets (less than one second of calculation). Moreover, when we calculate the union of the discretized rotation sets over several grids to obtain a simulation of the asymptotic discretized rotation set, we obtain a set which is quite close to $[0,1]^2$ for Hausdorff distance. As for theoretical results, this suggests the following lesson:

When we compute segments of orbits with very good precision it is very difficult to recover the rotation set. However, when we decrease the number of digits used in computations we can obtain quickly a very good approximation of the rotation set. In fact, we have to adapt the precision of the calculation to the number of orbits we can obtain numerically.

This phenomenon can be explained by the fact that each grid of the torus is stabilized by the corresponding discretization of the homeomorphism. Thus, there exists an infinite number of grids such that every periodic point of the homeomorphism is shadowed by some periodic orbits of the discretizations on these grids.

Chapter 3

Grids, discretizations, measures

3.1 The manifold X and the measure λ

The results stated in the introduction for the torus $\mathbf{T}^n$ and the Lebesgue measure Leb actually extend to any smooth connected manifold X with dimension $n \geq 2$, compact and possibly with boundary, endowed with a Riemannian metric d. *We fix once and for all such a manifold* X *endowed with the metric* d. In the general case, Lebesgue measure on $\mathbf{T}^n$ can be replaced by a *good measure* λ on X:

Definition 3.1. A Borel probability measure λ on X is called a *good measure*, or an *Oxtoby-Ulam* measure, if it is non-atomic, it has total support (it is positive on every non-empty open sets) and it is zero on the boundary of X.

These restrictions are supported by Oxtoby-Ulam theorem (Theorem 3.14). *We fix once and for all a good measure* λ *on* X.

Notation 3.2. We denote by Homeo(X) the set of homeomorphisms of X, endowed by the metric d defined by:

$$d(f,g) = \sup_{x\in X} d\Big(f(x), g(x)\Big).$$

We denote by $\mathrm{Homeo}(X,\lambda)$ the subset of Homeo(X) made of the homeomorphisms that preserve the measure λ (*i.e.* for every Borel set A, $\lambda(f^{-1}(A)) = \lambda(A)$), endowed with the same metric d. Elements of Homeo(X) will be called *dissipative* homeomorphisms, and elements of $\mathrm{Homeo}(X,\lambda)$ *conservative* homeomorphisms.

3.2 Generic properties

All the functional spaces we will use are Baire spaces, *i.e.* the intersection of every countable collection of dense open sets is dense. In particular, the topological spaces Homeo(X) and $\mathrm{Homeo}(X,\lambda)$ are Baire spaces. Indeed, we easily check that the map

$$\delta(f,g) = d(f,g) + d(f^{-1}, g^{-1})$$

defines a distance on Homeo(X) and $\mathrm{Homeo}(X,\lambda)$ which defines the same topology as d, and is complete in Homeo(X) and $\mathrm{Homeo}(X,\lambda)$ (see for example [Gui12, Appendice A.1]).

We will sometimes abusively use the phrase "for a generic element of $\mathscr{E}$, we have the property (P)". By that we will mean that "the property (P) is generic in $\mathscr{E}$", *i.e.* "there exists G_δ dense subset G of $\mathscr{E}$, such that every $f \in G$ satisfy the property (P)".

3.3 C^0 extension of finite maps

The basic tool we will use to perturb a homeomorphism in the C^0 topology is the finite map extension proposition. It asserts that if we have given a (conservative) homeomorphism and a pseudo-orbit of this homeomorphism, we can perturb this homeomorphism such that this pseudo-orbit becomes a real orbit.

Proposition 3.3 (Finite map extension)**.** *Let* $E \subset X \setminus \partial X$ *be a finite set and* $\sigma : E \to X \setminus \partial X$ *be an injective map. Then there exists an homeomorphism* $g \in \mathrm{Homeo}(X, \lambda)$ *such that* $g_{|E} = \sigma$. *Moreover, if* $d(\sigma, \mathrm{Id}_E) < \delta$, *then the homeomorphism* g *can be chosen such that* $d(g, \mathrm{Id}_X) < \delta$.

The last condition ensures that if the finite map σ is close to a given homeomorphism g — in other words σ is a good approximation of g —, then the extension of this finite map can be chosen close to the map g.

The idea of the proof is quite simple: it suffices to compose correctly homeomorphisms whose support's size is smaller than ε and which are central symmetries within this support. We denote $E = \{x_1, \cdots, x_q\}$ and $\sigma(x_i) = y_i$. For each i, we construct a sequence $(z_j)_{1 \leq j \leq k}$ such that $z_1 = x_i$, $y_i = z_k$ and $d(z_j, z_{j+1}) < \varepsilon/10$. Composing $k-1$ homeomorphisms as above, such that each one sends z_j on z_{j+1}, we obtain a conservative homeomorphism which sends x_i on y_i. Implementing these remarks is then essentially technical and boring. A detailed proof can be found in [Gui12, Section 2.2]. This proposition is valid in the neighbourhood of the identity; to have a result concerning the neighbourhood of any homeomorphism f it suffices to compose by the inverse of f. The following corollary also includes the case where the map σ is not injective (as a non injective map is arbitrarily close to an injective one).

Corollary 3.4. *Let* $E \subset X \setminus \partial X$ *be a finite set,* $\sigma : E \to X \setminus \partial X$, $f \in \mathrm{Homeo}(X)$ *(respectively* $f \in \mathrm{Homeo}(X, \lambda)$*) and* $\delta > 0$. *If* $d(f, \sigma) < \varepsilon$, *then there exists an homeomorphism* $g \in \mathrm{Homeo}(X)$ *(respectively* $\mathrm{Homeo}(X, \lambda)$*) such that* $d(g_{|E}, \sigma) < \delta$ *and* $d(f, g) < \varepsilon$. *Moreover, if* σ *is injective, we can suppose that* $g_{|E} = \sigma$.

3.4 Discretization grids, discretizations of a homeomorphism

We now define a more general notion of discretization grid than in the introduction. Depending on the case, some additional assumptions about these grids will be needed (see also the next section).

Definition 3.5 (Discretization grids)**.** A *sequence of discretization grids* on X is a sequence $(E_N)_{N \in \mathbf{N}}$ of discrete subsets of $X \setminus \partial X$, called *grids*, such that the mesh of these grids tends to 0: for all $\varepsilon > 0$, there exists $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$, the grid E_N is ε-dense [1]. We denote by q_N the cardinality of E_N.

We fix once and for all a sequence $(E_N)_{N \in \mathbf{N}}$ *of discretization grids on* X.
To each grid is associated a discretization map in the following way.

Definition 3.6 (Discretizations)**.** Let P_N be a projection of X on E_N (the projection of $x_0 \in X$ on E_N is some $y_0 \in E_N$ minimizing the distance $d(x_0, y)$ when y runs through E_N). Such a projection is uniquely defined out of the set E'_N consisting of the points $x \in X$ for which there exists at least two points minimizing the distance between x and E_N. On E'_N the map P_N is chosen arbitrarily (nevertheless measurably). For $x \in X$ we denote by

1. That is, for every point $x \in X$ there exists a point $y \in E_N$ such that $d(x, y) < \varepsilon$.

x_N the *discretization of order* N of x, defined by $x_N = P_N(x)$. For $f \in \mathrm{Homeo}(X)$ we denote by $f_N : E_N \to E_N$ the *discretization of order* N of f, defined by $f_N = P_N \circ f$. We denote by $\mathscr{D}_N$ be the set of homeomorphisms f such that $f(E_N) \cap E'_N = \emptyset$.

If $\sigma : E_N \to E_N$ and $f \in \mathrm{Homeo}(X)$, we denote by $d_N(f, \sigma)$ the distance between $f_{|E_N}$ and σ, considered as maps from E_N into X.

Remark 3.7. We might wonder why the points of the discretization grids are supposed to be *inside* X. The reason is simple: a homeomorphism f sends ∂X on ∂X. Putting points of some grids on the edge could perturb the dynamics of discretizations f_N. In particular, it would introduce at least one orbit with length smaller than $\mathrm{Card}(E_N \cap \partial X)$.

Remark 3.8. As the exponential map is a local diffeomorphism, the sets E'_N are closed and have empty interior for every N large enough. Subsequently, we will implicitly suppose that the union $\bigcup_{N \in \mathbf{N}} E'_N$ is an F_σ with empty interior. It is not a limiting assumption as we will focus only on the behaviour of the discretizations for N going to $+\infty$. It will allow us to restrict the study to the G_δ dense set $\bigcap_{N \in \mathbf{N}} \mathscr{D}_N$, which is the set of homeomorphisms whose N-th discretization is uniquely defined for all $N \in \mathbf{N}$.

3.5 Probability measures on X

In the sequel, we will study some *ergodic* properties of discretizations of f. Denote by $\mathscr{P}$ the set of Borel probability measures on X endowed with the weak-star topology: a sequence $(\nu_m)_{m \in \mathbf{N}}$ of $\mathscr{P}$ tends to $\nu \in \mathscr{P}$ (denoted by $\nu_m \rightharpoonup \nu$) if for every continuous function $\varphi : X \to \mathbf{R}$,

$$\lim_{m \to \infty} \int_X \varphi \, \mathrm{d}\nu_m = \int_X \varphi \, \mathrm{d}\nu.$$

It is well known that under these conditions, the space $\mathscr{P}$ is metrizable and compact, therefore separable (Prohorov, Banach-Alaoglu-Bourbaki theorem); we equip this space with a distance dist which is compatible with this topology.

Let $f \in \mathrm{Homeo}(X)$. For $x \in \mathbf{T}^2$, we denote by $p\omega(x)$ the set of limit points of the sequence

$$\left\{ \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)} \right\}_{n \in \mathbf{N}^*}.$$

It is a compact subset of the set $\mathscr{M}^f$ of f-invariant Borel probability measures.

We will pay a particular attention to physical measures

Definition 3.9. A Borel probability measure μ is called *physical* (sometimes called SRB, see [You02]) for the map f if its basin of attraction has positive λ-measure, where the *basin of attraction* of μ for f is the set

$$\left\{ x \in X \;\middle|\; \frac{1}{M} \sum_{m=0}^{M-1} \delta_{f^m(x)} \underset{M \to +\infty}{\longrightarrow} \mu \right\}$$

of points whose Birkhoff's limit coincides with μ.

Heuristically, the basin of a measure is the set of points that can see the measure, and physical measures are the ones that can be seen in practice.

To study ergodic properties of homeomorphisms and their discretizations, we define natural invariant probability measures associated with these maps:

Definition 3.10. For any non-empty open subset U of X, we denote by λ_U the normalized restriction of λ on U, i.e. $\lambda_U = \frac{1}{\lambda(U)}\lambda_{|U}$. We also denote by $\lambda_{N,U}$ the uniform probability measure on $E_N \cap U$.

For $x \in X$ we denote by $\mu^f_{x,m}$ the uniform measure on the segment of orbit $x, f(x), \cdots, f^{m-1}(x)$:

$$\mu^f_{x,m} = \frac{1}{m}\sum_{i=0}^{m-1} f_*^i \delta_x.$$

When the limit exists, we denote by μ^f_x the *Birkhoff limit* of x by:

$$\mu^f_x = \lim_{m\to\infty} \mu^f_{x,m}.$$

Similarly, for f_N and $x \in X$ (recall that x_N is the projection of x on the grid E_N):

$$\mu^{f_N}_{x,m} = \frac{1}{m}\sum_{i=0}^{m-1} (f_N)_*^i \delta_{x_N};$$

in this case the Birkhoff limit is always well defined: it is the uniform measure on the periodic orbit on which the positive orbit of x_N under f_N falls:

$$\mu^{f_N}_x = \lim_{m\to\infty} \mu^{f_N}_{x,m},$$

We also define a similar quantity for sets of points. We set

$$\mu^f_{U,m} = \frac{1}{m}\sum_{i=0}^{m-1} f_*^i \lambda_U,$$

and when it is well defined,

$$\mu^f_U = \lim_{m\to\infty} \mu^f_{U,m}.$$

Likewise,

$$\mu^{f_N}_{U,m} = \frac{1}{m}\sum_{i=0}^{m-1} (f_N)_*^i \lambda_{N,U},$$

and in this case the Birkhoff limit always exists: it is a measure supported by the periodic orbits of f_N, such that the measure of each periodic orbit is proportional to the number of points of $E_N \cap U$ whose orbit under f_N eventually belongs to this periodic orbit:

$$\mu^{f_N}_U = \lim_{m\to\infty} \mu^{f_N}_{U,m}.$$

We have defined two types of invariant measures: one from a point x, the other from the uniform measure λ. The link between them is done by the following remark which easily follows from the dominated convergence theorem.

Remark 3.11. When U is an open set almost every point (for λ_U) in which admits a Birkhoff limit, the measure μ^f_U is well defined and satisfies, for every continuous map $\varphi : X \to \mathbf{R}$,

$$\int_X \varphi \, \mathrm{d}\mu^f_U = \int_U \left(\int_X \varphi \, \mathrm{d}\mu^f_x \right) \mathrm{d}\lambda_U.$$

Similarly,

$$\int_X \varphi \, \mathrm{d}\mu^{f_N}_U = \int_U \left(\int_X \varphi \, \mathrm{d}\mu^{f_N}_x \right) \mathrm{d}\lambda_{N,U}.$$

3.6 Hypothesis on discretization grids

Previously, we have given a very general definition of the concept of sequence of discretization grids. In some cases, we will need additional technical assumptions about these sequences of grids; of course all of them will be satisfied by good sequences of uniform discretization grids on the torus (as defined in the introduction).

The first assumption is useful for proving Lax's theorem (Theorem 5.5), and therefore necessary only in the part concerning conservative homeomorphisms.

Definition 3.12 (Well distributed and well ordered grids)**.** We say that a sequence of discretization grids $(\mathrm{E_N})_{\mathrm{N}\in\mathbf{N}}$ is *well distributed* if we can associate to each $x \in \mathrm{E_N}$ a subset $\mathrm{C_{N,}}_x$ of X, which will be called a *cube of order* N, such that:

- for all N and all $x \in \mathrm{E_N}$, $x \in \mathrm{C_{N,}}_x$;
- for all N, $\{\mathrm{C_{N,}}_x\}_{x\in\mathrm{E_N}}$ is a measurable partition of X: $\bigcup_{x\in\mathrm{E_N}} \mathrm{C_{N,}}_x$ has full measure and for two distinct points x, y of $\mathrm{E_N}$, the intersection $\mathrm{C_{N,}}_x \cap \mathrm{C_{N,}}_y$ is null measure;
- for a fixed N, the cubes $\{\mathrm{C_{N,}}_x\}_{x\in\mathrm{E_N}}$ have all the same measure;
- the diameter of the cubes of order N tends to 0: $\max_{x\in\mathrm{E_N}} \mathrm{diam}(\mathrm{C_{N,}}_x) \underset{\mathrm{N}\to+\infty}{\longrightarrow} 0$.

If $(\mathrm{E_N})_{\mathrm{N}\in\mathbf{N}}$ is well distributed and if $\{\mathrm{C_{N,}}_x\}_{\mathrm{N}\in\mathbf{N},x\in\mathrm{E_N}}$ is a family of cubes as above, we will say that $(\mathrm{E_N})_{\mathrm{N}\in\mathbf{N}}$ is *well ordered* if, for a fixed N, the cubes $\{\mathrm{C_{N,}}_x\}_{x\in\mathrm{E_N}}$ can be indexed by $\mathbf{Z}/q_\mathrm{N}\mathbf{Z}$ (recall that $q_\mathrm{N} = \mathrm{Card}(\mathrm{E_N})$) such that two consecutive cubes (in $\mathbf{Z}/q_\mathrm{N}\mathbf{Z}$) are close to each other:

$$\max_{i\in\mathbf{Z}/q_\mathrm{N}\mathbf{Z}} \mathrm{diam}(\mathrm{C}_\mathrm{N}^i \cup \mathrm{C}_\mathrm{N}^{i+1}) \underset{\mathrm{N}\to+\infty}{\longrightarrow} 0$$

(in particular, it is true when the boundaries of two consecutive cubes overlap).

At first glance, it can seem surprising that there is no link between the cubes and the projections. In fact, the existence of such cubes expresses that the grids "fit" the measure λ.

The following definition describe assumptions on grids that will be useful for obtaining properties on average.

Definition 3.13 (self similar grids)**.** We say that a sequence of discretization grids $(\mathrm{E_N})_{\mathrm{N}\in\mathbf{N}}$ is *sometimes self similar* if for all $\varepsilon > 0$ and all $\mathrm{N}_0 \in \mathbf{N}$, there exists two integers N_1 and N_2 satisfying $\mathrm{N}_2 \geq \mathrm{N}_1 \geq \mathrm{N}_0$, such that the set $\mathrm{E_{N_2}}$ contains disjoint subsets $\widetilde{\mathrm{E}}^1_{\mathrm{N}_2}, \cdots, \widetilde{\mathrm{E}}^\alpha_{\mathrm{N}_2}$ whose union fills a proportion greater than $1-\varepsilon$ of $\mathrm{E_{N_2}}$:

$$\frac{\mathrm{Card}\left(\widetilde{\mathrm{E}}^1_{\mathrm{N}_2} \cup \cdots \cup \widetilde{\mathrm{E}}^\alpha_{\mathrm{N}_2}\right)}{\mathrm{Card}(\mathrm{E_{N_2}})} > 1-\varepsilon,$$

and such that for all i, the set $\widetilde{\mathrm{E}}^i_\mathrm{N}$ is the image of the grid $\mathrm{E_{N_1}}$ by a bijection h_i which is ε-close to identity.

We say that a sequence of discretization grids $(\mathrm{E_N})_{\mathrm{N}\in\mathbf{N}}$ is *sometimes strongly self similar* if it is sometimes self similar for the parameter $\varepsilon = 0$ (that is, the sets $\widetilde{\mathrm{E}}^i_{\mathrm{N}_2}$ form a partition of $\mathrm{E_{N_2}}$).

We say that a sequence of discretization grids $(\mathrm{E_N})_{\mathrm{N}\in\mathbf{N}}$ is *always self similar* if for all $\varepsilon > 0$ and all $\mathrm{N}_0 \in \mathbf{N}$, there exists $\mathrm{N}_1 \geq \mathrm{N}_0$ such that for all $\mathrm{N}_2 \geq \mathrm{N}_1$, the set $\mathrm{E_{N_2}}$ contains disjoint subsets $\widetilde{\mathrm{E}}^1_{\mathrm{N}_2}, \cdots, \widetilde{\mathrm{E}}^{\alpha_{\mathrm{N}_2}}_{\mathrm{N}_2}$ whose union fills a proportion greater than $1-\varepsilon$ of $\mathrm{E_{N_2}}$:

$$\frac{\mathrm{Card}\left(\widetilde{\mathrm{E}}^1_{\mathrm{N}_2} \cup \cdots \cup \widetilde{\mathrm{E}}^{\alpha_{\mathrm{N}_2}}_{\mathrm{N}_2}\right)}{\mathrm{Card}(\mathrm{E_{N_2}})} > 1-\varepsilon,$$

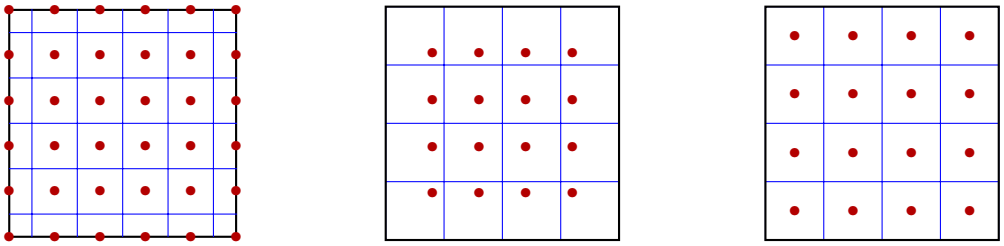

Figure 3.1: Uniform discretization grids of order 5 on the torus $\mathbf{T}^2$ (E_N, left) and on the cube I^2 (E^0_N, middle, and E^1_N, right) and their associated cubes

and such that for all i, $\widetilde{\mathrm{E}}^i_\mathrm{N}$ is the image of the grid $\mathrm{E}_{\mathrm{N}_1}$ by a bijection h_i which is ε-close to identity, and such that for all i, j and all $\mathrm{N}_2, \mathrm{N}_2' \geq \mathrm{N}_1$, either $\widetilde{\mathrm{E}}^i_{\mathrm{N}_2} \cap \widetilde{\mathrm{E}}^j_{\mathrm{N}_2'} = \emptyset$, or $\widetilde{\mathrm{E}}^i_{\mathrm{N}_2} = \widetilde{\mathrm{E}}^j_{\mathrm{N}_2'}$.

We say that a sequence of discretization grids $(\mathrm{E}_\mathrm{N})_{\mathrm{N}\in\mathbf{N}}$ is *always strongly self similar* if it is always self similar for the parameter $\varepsilon = 0$ (that is, the sets $\widetilde{\mathrm{E}}^i_{\mathrm{N}_2}$ form a partition of $\mathrm{E}_{\mathrm{N}_2}$).

3.7 Some examples of discretization grids

In the previous section we set properties about discretizations — being well distributed, being well ordered, being sometimes/always (strongly) self similar — for a later use. In this section, we study these properties for some natural examples of grids.

Uniform discretization grids on the torus

The simplest example of grid of discretization, which will be used for the simulations, is that of the torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ of dimension $n \geq 1$ endowed with discretizations called *uniform* discretizations, defined from the fundamental domain $\mathrm{I}^n = [0,1]^n$ of $\mathbf{T}^n$: take an increasing sequence of integers $(k_\mathrm{N})_{\mathrm{N}\in\mathbf{N}}$ and set (see Figure 3.1, left)

$$\mathrm{E}_\mathrm{N} = \left\{ \left(\frac{i_1}{k_\mathrm{N}}, \cdots, \frac{i_n}{k_\mathrm{N}} \right) \in \mathbf{T}^n \,\middle|\, \forall j,\, 0 \leq i_j \leq k_\mathrm{N} - 1 \right\},$$

$$\mathrm{C}_{\mathrm{N},(i_1/\mathrm{N},\cdots,i_n/\mathrm{N})} = \prod_{j=1}^{n} \left[\frac{i_j - 1/2}{k_\mathrm{N}}, \frac{i_j + 1/2}{k_\mathrm{N}} \right].$$

We easily verify that this sequence of grids is well distributed, well ordered and always self similar. If we assume that for for every $\mathrm{N}_0 \in \mathbf{N}$, there exists $\mathrm{N}_2 \geq \mathrm{N}_1 \geq \mathrm{N}_0$ such that k_{N_1} divides k_{N_2}, then the sequence is sometimes strongly self similar. If we further assume that for any $\mathrm{N} \in \mathbf{N}$, k_N divides $k_{\mathrm{N}+1}$ (which is true when $k_\mathrm{N} = p^\mathrm{N}$ with $p \geq 2$), then the sequence is always strongly self similar. When $k_\mathrm{N} = p^\mathrm{N}$ with $p = 2$ (respectively, $p = 10$) the discretization performs what we can expect from a numerical simulation: doing a binary (respectively, decimal) discretization at order N is the same as truncating each binary (respectively decimal) coordinate of the point $x \in \mathrm{I}^n$ to the N-th digit, *i.e.* working with a fixed digital precision [2].

2. In practice, the computer works in floating point format, so that the number of decimal places is not the same when the number is close to 0 or not.

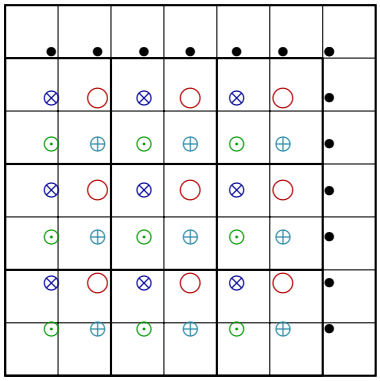

Figure 3.2: Self similarity of grids E_N^0

Uniform discretization grids on the cube

In the case of the cube $I^n = [0,1]^n$, there are two natural types of grids of discretization. In the first one the grids are obtained by taking the same definition as the grids E_N (but avoiding taking points on the boundary of the cube):

$$E_N^0 = \left\{ \left(\frac{i_1}{k_N}, \cdots, \frac{i_n}{k_N} \right) \in I^n \middle| \, \forall j,\, 1 \leq i_j \leq k_N - 1 \right\},$$

with an increasing sequence $(k_N)_{N \in \mathbf{N}}$ of integers. To these grids are associated the cubes [3] (see Figure 3.1, middle)

$$C_{N,(i_1/N,\cdots,i_n/N)} = \prod_{j=1}^{n} \left[\frac{i_j - 1}{k_N - 1}, \frac{i_j}{k_N - 1} \right].$$

As before, we easily verify that this sequence of grids is well distributed, well ordered and always self similar (see Figure 3.2). If we assume that for every $N_0 \in \mathbf{N}$, there exists $N_2 \geq N_1 \geq N_0$ such that $k_{N_1} - 1$ divides $k_{N_2} - 1$, then the sequence is sometimes strongly self similar. If we further assume that for any $N \in \mathbf{N}$, $k_N - 1$ divides $k_{N+1} - 1$ (that is, for every N, there exists an integer $\ell_N > 1$ such that $k_{N+1} = k_N + \ell_N k_N (k_N + 1)$), and either k_N always divides k_{N+1} or k_N never divides k_{N+1}, then the sequence is always strongly self similar.

We can also take discretizations according to the centres of the cubes:

$$E_N^1 = \left\{ \left(\frac{i_1 + 1/2}{k_N}, \cdots, \frac{i_n + 1/2}{k_N} \right) \in I^n \middle| \, \forall j,\, 0 \leq i_j \leq k_N - 1 \right\}.$$

This time, the cubes are much more natural (see Figure 3.1, right):

$$C_{N,(i_1/N,\cdots,i_n/N)} = \prod_{j=1}^{n} \left[\frac{i_j}{k_N}, \frac{i_j + 1}{k_N} \right].$$

Again, we easily verify that this sequence of grids is well distributed, well ordered and always self similar. Moreover, it is sometimes strongly self similar if for every $N_0 \in \mathbf{N}$, there exists $N_2 \geq N_1 \geq N_0$ such that k_{N_2}/k_{n_1} is an odd integer, and always strongly self similar if k_{N+1}/k_N is an odd integer for all $N \in \mathbf{N}$.

3. Be careful, these cubes have their vertices on the grid of order $k_N - 1$.

Discretization grids on an arbitrary manifold

In fact, discretizations on arbitrary manifolds can be easily obtained from discretization grids on the cube I^n (for example the grids E_N^0) by applying the Oxtoby-Ulam theorem (see [OU41] or [Gui12, Corollaire 1.4]). This theorem asserts that every smooth connected compact manifold of dimension n, possibly with boundary, equipped with a good measure λ, is obtained from the unit cube I^n by gluing together some parts of the boundary of the cube. Moreover, this gluing map can be chosen such that the image of Lebesgue measure on the cube is λ.

Theorem 3.14 (Oxtoby, Ulam)**.** *Under the assumptions that have been made on* X *and* λ*, there exists a map* $\phi : I^n \to X$ *such that:*

1. ϕ *is surjective;*
2. $\phi_{|\mathring{I}^n}$ *is a homeomorphism on its image;*
3. $\phi(\partial I^n)$ *is a closed subset of* X *with empty interior and disjoint from* $\phi(\mathring{I}^n)$*;*
4. $\lambda(\phi(\partial I^n)) = 0$*;*
5. $\phi^*(\lambda) = \mathrm{Leb}$*, where* Leb *is Lebesgue measure.*

This theorem allows to define discretization grids on X from discretizations on the cube; this is outlined by the following informal proposition.

Proposition 3.15. *If there is a sequence* $(E_N)_N$ *of grids on* I^n *whose elements are not on the edge of the cube, then its image by* ϕ *defines a sequence of grids on* X *which satisfy the same properties as the initial grid on the cube.*

Remark 3.16. The example also includes the case where $X = I^n$, $\lambda = \mathrm{Leb}$ and where the grids are the images of the grids E_N by a unique homeomorphism of X preserving Lebesgue measure.

Chapter 4

Discretizations of a generic dissipative homeomorphism

In this chapter, we study the dynamical behaviour of the discretizations of generic *dissipative* homeomorphisms, *i.e.* without the assumption of preservation of a given measure.

The dynamics of a generic dissipative homeomorphism is very stable: the *shredding lemma* of F. Abdenur and M. Andersson (Lemma 4.2) implies that for a generic homeomorphism, there exists a Cantor set of dimension 0 which attracts almost every point of X (see Corollary 4.5). From that we will deduce easily the dynamics of the discretizations of a generic homeomorphism from that of the homeomorphism itself.

This study of the dynamics of a generic homeomorphism suggests that in the generic dissipative setting, "the physical dynamics of the discretizations of a homeomorphism f converges to the physical dynamics of f", both from topological and measurable viewpoints. Indeed, we will prove that for a generic homeomorphism and for almost every point $x \in X$, the orbit of x_N by the discretization f_N shadows the orbit of x by f (Corollary 4.12). Moreover, we will prove that the recurrent set $\Omega(f_N)$ tends to the closure of the set of Lyapunov stable periodic points of f in a weak sense defined in Proposition 4.13; from a combinatorial viewpoint, the cardinality of this recurrent set is as small as possible (Corollary 4.14) and the length of the longest periodic orbit tends to $+\infty$ when N goes to $+\infty$ (Corollary 4.15). Finally, we will prove that the measures $\mu_X^{f_N}$ tend to the measure μ_X^f, which is the Cesàro limit of the sequence of pushforwards of λ by f (Theorem 4.16).

The end of this chapter will be dedicated to the results of the numerical simulations.

Throughout this chapter, we fix a compact manifold X *of dimension* $n \geq 2$. *Here, homeomorphisms are not supposed to be conservative; more formally, we recall that we denote by* Homeo(X) *the set of all* dissipative *homeomorphisms of* X *(without assumption of conservation of a given measure). The manifold* X *is equipped with a sequence* $(E_N)_{N\in\mathbf{N}}$ *of discretization grids and a good measure* λ *as defined in Section 3.1 (we will use this measure to look at the physical dynamics, that is the dynamics of almost every point), we xsuppose that the sequence of grids of discretization satisfy* $\sum_{x\in E_N} \delta_x \underset{N\to+\infty}{\longrightarrow} \lambda$. *We are interested in properties of discretizations of generic elements of* Homeo(X) *with respect to the sequence of grids* (E_N).

4.1 Dynamics of a generic dissipative homeomorphism

In their article [AA13], F. Abdenur and M. Andersson try to identify the generic ergodic properties of continuous maps and homeomorphisms of compact manifolds. More precisely, they study the behaviour of Birkhoff limits [1] μ_x^f for a generic homeomorphism $f \in \mathrm{Homeo}(X)$ and almost every point x for λ. To do this, they define some interesting behaviours of homeomorphisms related to Birkhoff limits, including one they call *weird*.

Definition 4.1. A homeomorphism f is said *weird* if almost every point $x \in X$ (for λ) has a Birkhoff's limit μ_x^f, and if f is *totally singular* (*i.e.* there exists a Borel set with total measure whose image by f is null measure) and does not admit any physical measure.

This definition is supported by their proof, based on the shredding lemma, that a generic homeomorphism is weird. We state an improvement of this lemma, whose main consequence is that a generic homeomorphism has many open attractive sets, all of small measure, and decomposable into a small number of small diameter open sets:

Lemma 4.2 (Shredding lemma, F. Abdenur, M. Andersson, [AA13])**.** *For every homeomorphism $f \in \mathrm{Homeo}(X)$, for all $\varepsilon, \delta > 0$, there exists integers ℓ and $\ell_1, \cdots, \ell_\ell$, bigger than $1/\varepsilon$, and a family of regular pairwise disjoints open sets* [2] *$U_1, \cdots, U_\ell$ such that for all $\varepsilon' > 0$, there exists $g \in \mathrm{Homeo}(X)$ such that $d(f,g) < \delta$ and:*

(i) $g(\overline{U_j}) \subset U_j$,

(ii) $\lambda(U_j) < \varepsilon$,

(iii) $\lambda\left(\bigcup_{j=1}^{\ell} U_j\right) > 1 - \varepsilon$,

(iv) $\lambda(g(U_j)) < \varepsilon' \lambda(U_j)$,

(v) for all $j \leq \ell$, there exists open sets $W_{j,1}, \cdots, W_{j,\ell_j}$ such that:

a) $\mathrm{diam}(W_{j,i}) < \varepsilon'$ *for all* $i \in \{1, \cdots, \ell_j\}$,

b) $g(\overline{W_{j,i}}) \subset W_{j,i+1}$, *for every* $i \in \{1, \cdots, \ell_j - 1\}$ *and* $g(\overline{W_{j,\ell_j}}) \subset W_{j,1}$,

c)

$$\overline{U_j} \subset \bigcup_{m \geq 0} g^{-m}\Big(\bigcup_{i=1}^{\ell_j} W_{j,i}\Big).$$

d) The sets $W_{j,i}$ have disjoints attractive sets, i.e. *for all $j \neq j'$, we have* [3]

$$\left(\bigcap_{m \geq 0} \bigcup_{m' \geq m} g^{m'}(W_{j,1})\right) \cap \left(\bigcap_{m \geq 0} \bigcup_{m' \geq m} g^{m'}(W_{j,i'})\right) = \emptyset.$$

e) We can further assume that each set $W_{j,i}$ contains a Lyapunov stable periodic point (see Lemma 4.4).

Moreover, these properties remain true on a neighbourhood of the homeomorphism g.

We outline the proof of this lemma: by using arguments such as the Oxtoby-Ulam theorem (Theorem 3.14) or the concept of uniform grid on the cube I^n, it is possible to shorten a little the arguments of [AA13].

1. We recall that the Birkhoff limit μ_x^f is the limit in the Cesàro sense of the pushforwards of the Dirac measure δ_x by the homeomorphism f.
2. An open set is said *regular* if it is equal to the interior of its closure.
3. These two sets are decreasing intersections of compact sets, thus compact and non-empty.

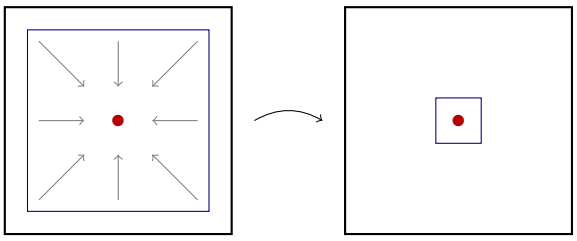

Figure 4.1: Local perturbation for Lemma 4.2

Proof of Lemma 4.2. Let $f \in \mathrm{Homeo}(X)$ and $\delta, \varepsilon > 0$. To begin with, we endow X with a collection of "cubes": for N large enough, the images of the cubes associated to the the grid E^1_N on the cube I^n (see page 55) by the map given by Oxtoby-Ulam theorem (Theorem 3.14) give us a collection of cubes (C_i) such that their union has full measure, their boundaries have null measure, they all have the same measure and their diameters are smaller than δ. To each cube C_i we associate its centre e_i. These properties assert that this collection of cubes behaves like the collection of cubes on I^n endowed with Lebesgue measure. Thus, in the sequel, we will only treat the case of I^n endowed with Leb.

We define a finite map $\sigma : \{e_i\} \to \{e_i\}$ such that $f(e_i) \in C_{\sigma(e_i)}$. Making a perturbation if necessary, we can suppose that σ does not contain any periodic orbit of length smaller than $1/\varepsilon$ (simply because if N is large enough, then for all e_i there exists at least $1/\varepsilon$ other points e_j such that $d(e_i, e_j) < \delta$). We then use Corollary 3.4 to move the points $f(e_i)$ closer to the points e_j, in other words we build a homeomorphism g_1 which is δ-close to f and such that $\max_i \left(\min_j d(g_1(e_i), e_j) \right) < \min_i \mathrm{diam}(C_i)/10$.

We then set $g_2 = g_1 \circ h$, where h is a homeomorphism close to the identity whose restriction to every cube is a huge contraction (see Figure 4.1): such a contraction can be easily expressed on the cube $[-1,1]^n$: let α be a "big" number (determined by $\min(\varepsilon, \varepsilon')$) and

$$\phi_\alpha(x) = \begin{cases} \|x\|_\infty^\alpha x & \text{if } \|x\|_\infty \leq 1 \\ x & \text{otherwise.} \end{cases}$$

We then build easily h by composing such contractions on each cube, each of them being obtained by conjugating by a translation and a homothety.

The "physical" dynamics of the map g_2 is then close to that of the finite map σ which maps e_i on the unique centre e_j such that $g(e_i) \in C_j$. In particular, the periodic sets $W_{j,1}, \cdots, W_{j,\ell_j}$ are obtained as neighbourhoods of a given periodic orbit of σ, and the corresponding basins of attraction U_j are unions of subsets of the cubes [4] whose centres have some iterates by σ which fall in this periodic orbit. We then easily check that these sets satisfy the conclusions of the lemma, apart from the point *(ii)*.

Thus, we have to prove that the basins of attraction can be supposed to have small measure. To do that, we consider the cubes of order $M = kN$ with $k > 1/\varepsilon$, and partition this set of cubes into k^n sets of cubes in the way defined by the fact that these grids are strongly self similar (see page 55). We then use this self similarity to perturb f such that σ stabilizes each one of these subgrids, and apply the same technique as previously. □

4. More precisely, for every cube C_j, the sets U_j contain $\{x \in C \mid d(x\partial C_j) > \mathrm{diam}(C)(1-\varepsilon)\}$.

Definition 4.3. For a homeomorphism $f \in \mathrm{Homeo}(X)$, we denote by A_0 the set of *Lyapunov stable* periodic points of f, *i.e.* the set of periodic points x such that for all $\delta > 0$, there exists $\eta > 0$ such that if $d(x,y) < \eta$, then $d(f^m(x), f^m(y)) < \delta$ for all $m \in \mathbf{N}$.

For two compact sets K and K′, by $K' \subset\subset K$ we mean that there exists an open set O such that $K' \subset O \subset K$. In the sequel the set K will be called *strictly periodic* if there exists an integer $i > 0$ such that $f^i(K) \subset\subset K$.

The following lemma ensures that for a generic homeomorphism, every set $W_{j,i}$ of the shredding lemma contains at least one (in fact an infinite number of) Lyapunov stable periodic point.

Lemma 4.4. *For a generic homeomorphism $f \in \mathrm{Homeo}(X)$, for every strictly periodic topological ball O (*i.e. *there exists $i > 0$ such that $f^i(O) \subset\subset O$), there exists a Lyapunov stable periodic point $x \in O$.*

Proof of Lemma 4.4. We begin by choosing a countable basis of closed sets of X: for example we can take $\mathscr{K}_N$ the set of unions of the closures of the cubes of order N. We also denote by $\mathscr{B}$ the set of all closed topological balls of X. We define $\mathscr{U}_{k,\varepsilon,N}$ as the set of homeomorphisms such that each large enough strictly periodic ball contains a smaller strictly periodic ball with the same period [5]:

$$\mathscr{U}_{k,\varepsilon,N} = \left\{ f \in \mathrm{Homeo}(X) \;\middle|\; \begin{array}{l} \forall K \in \mathscr{K}_N \cap \mathscr{B} \text{ s.t. } \exists i \le k \text{ s.t.} \\ f^i(K) \subset\subset K \text{ and } \mathrm{diam}_{int}(K) > \varepsilon, \\ \exists K' \subset K, K' \in \mathscr{B} \text{ s.t. } \mathrm{diam}(K') < \varepsilon/2 \\ \text{and } f^i(K') \subset\subset K' \end{array} \right\}.$$

Then for every k, ε, N, it is straightforward that the set $\mathscr{U}_{k,\varepsilon,N}$ is an open subset of Homeo(X). To show that it is dense it suffices to apply Brouwer's theorem to each K such that $f^i(K) \subset\subset K$ and to make the obtained periodic point attractive.

We now prove that every $f \in \bigcap_{k,\varepsilon,N} \mathscr{U}_{k,\varepsilon,N}$ satisfies the conclusions of the lemma. First of all, remark that for every topological ball K with non-empty interior which is strictly i-periodic, there exits $N \in \mathbf{N}$ and a smaller topological ball $\tilde{K} \subset K$ which is strictly i periodic such that $\tilde{K} \in \mathscr{K}_N$. It implies that if f belongs to the G_δ dense set $\bigcap_{k,\varepsilon,N} \mathscr{U}_{k,\varepsilon,N}$, then for every topological ball K with non-empty interior which is strictly i-periodic, there exits $N \in \mathbf{N}$ and a topological ball $K' \subset \tilde{K} \subset K$ which is strictly i periodic and at least twice smaller. Taking the intersection of such balls, we obtain a periodic point with period i which is Lyapunov stable by construction. □

The shredding lemma tells us a lot about the dynamics of a generic homeomorphism, which becomes quite clear: there are many attractors whose basins of attraction are small and attract almost all the manifold X. Moreover there is convergence of the attractive sets of the shredding lemma to the closure of the set of Lyapunov stable periodic points of f.

Corollary 4.5. *Let $f \in \mathrm{Homeo}(X)$ verifying the conclusions of the shredding lemma for all $\varepsilon = \varepsilon' > 0$, and $W_{j,i,\varepsilon}$ be the corresponding open sets. Such homeomorphisms form a G_δ dense subset of* Homeo(X). *Then the sets*

$$A_\varepsilon = \overline{\bigcup_{j,i} W_{j,i,\varepsilon}}$$

5. For a compact set K, $\mathrm{diam}_{int}(K)$ denotes the diameter of the biggest euclidean ball included in K.

converge for Hausdorff distance when ε *tends to* 0 *to a closed set which coincides generically with the set* $\overline{A_0}$ *(see Definition 4.3).*

Moreover, generically, the set A_0 *is a Cantor set (that is, it is compact, without any isolated point and totally disconnected) whose Hausdorff dimension* [6] *is 0.*

Remark 4.6. Thus, for a generic homeomorphism $f \in \mathrm{Homeo}(X)$, the ω-limit set of almost every point $x \in X$ is included in $\overline{A_0}$.

Proof of Corollary 4.5. Let f verifying the hypothesis of the corollary. We want to show that the sets A_ε tend to A_0 for Hausdorff distance when ε goes to 0. This is equivalent to show that for all $\delta > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, $A_0 \subset B(A_\varepsilon, \delta)$ and $A_\varepsilon \subset B(A_0, \delta)$ (where $B(A, \delta)$ denotes the set of points of X whose distance to A is smaller than δ). Subsequently we will denote by $U_{j,\varepsilon}$ and by $W_{j,i,\varepsilon}$ the open sets given by the shredding lemma for the parameters $\varepsilon = \varepsilon'$.

Let $\delta > 0$. We start by taking $x \in X$ whose orbit is periodic with period p and Lyapunov stable. Then there exists $\eta > 0$ such that if $d(x, y) < \eta$, then $d(f^m(x), f^m(y)) < \delta/2$ for all $m \in \mathbf{N}$; we note $O = B(x, \eta)$. As $\lambda(O) > 0$ there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, there exists $j \in \mathbf{N}$ such that the intersection between O and $U_{j,\varepsilon}$ is non-empty. Let y be an element of this intersection. By compactness, there exists a subsequence of $(f^{pm}(y))_{m\in\mathbf{N}}$ which tends to a point x_0; moreover $f^m(y) \in \bigcup_i W_{j,i}$ eventually and $d(f^{pm}(x), f^{pm}(y)) < \delta/2$, thus at the limit $m \to +\infty$, $d(x, x_0) < \delta/2$. We deduce that $x \in B\left(\overline{\bigcup_{j,i} W_{j,i,\varepsilon_0}}, \delta/2\right)$ for all ε_0 small enough. Since $\overline{A_0}$ is compact, it is covered by a finite number of balls of radius $\delta/2$ centred at some points x_i whose orbits attract non-empty open sets. Taking ε_0' the minimum of all the ε_0 associated to the x_i, the inclusion $A_0 \subset B(A_\varepsilon, \delta)$ occurs for all $\varepsilon < \varepsilon_0'$.

Conversely, let $\delta > 0$, $\varepsilon < \delta$ and focus on the set $W_{j,i,\varepsilon}$. By Lemma 4.4, we can suppose that there exists $x \in W_{j,i,\varepsilon}$ whose orbit is periodic and Lyapunov stable. Thus $x \in A_0$ and since the diameter of $W_{j,i,\varepsilon}$ is smaller than δ, $W_{j,i,\varepsilon} \subset B(A_0, \delta)$.

We now prove that the set $\overline{A_0}$ has no isolated point. Let $x \in \overline{A_0}$ and $\delta > 0$, we want to find another point $y \in \overline{A_0}$ such that $d(x, y) < \delta$. If $x \in \overline{A_0} \setminus A_0$ then it is trivial that x is an accumulation point of A_0; thus we suppose that $x \in A_0$. As x is a Lyapunov-stable periodic point, there exists a neighbourhood O of x whose diameter is smaller than δ and which is periodic (*i.e.* there exists $t > 0$ such that $f^t(0) \subset O$). For ε small enough, the open set O meets at least two sets $U_{j,\varepsilon}$ and $U_{j',\varepsilon}$ of the shredding lemma. Thus, applying the shredding lemma, we deduce that O contains at least two different strictly periodic sets $W_{j,i,\varepsilon}$ and $W_{j',i',\varepsilon}$; by Lemma 4.4 they both contain a Lyapunov-stable periodic point, and by construction their distance to x is smaller than δ.

Finally we prove that generically, the set $\overline{A_0}$ has 0 Hausdorff dimension. Let $s > 0$. We consider the set of homeomorphisms verifying the conclusions of the shredding lemma for ε' such that $\sum_j \ell_j \varepsilon'^s < 1$. This equality implies that for all $\varepsilon > 0$, the s-Hausdorff measure of the set $\bigcap_{\delta\in]0,\varepsilon[} A_\delta$ is smaller than 1. As we have

$$\overline{A_0} = \bigcup_{k\geq 0} \bigcap_{\delta\in]0,1/k[} A_\delta,$$

the set $\overline{A_0}$ is a countable union of sets of Hausdorff dimension smaller than s for every $s > 0$. Thus, $\overline{A_0}$ has zero Hausdorff dimension; this also proves that $\overline{A_0}$ is perfect. □

6. And better, if we are given a countable family $(\lambda_m)_{m\in\mathbf{N}}$ of good measures, generically the Hausdorff dimension of this set with respect to these measures is zero.

Remark 4.7. We can also prove that for every $\tau > 0$, the set of periodic points with period τ of a generic homeomorphism is either empty, either a Cantor set with zero Hausdorff dimension.

From the shredding lemma, we can easily deduce that the sequence of pushforwards of the uniform measure on an open set U is Cauchy. Thus, it converges to a measure μ_{U}^f, which is supported by $\overline{\mathrm{A}_0}$ and is atomless (because the μ_{U}^f-measure of each set $\bigcup_i \mathrm{W}_{j,i,\varepsilon}$ is between ε and 2ε).

Corollary 4.8. *For a generic homeomorphism $f \in \mathrm{Homeo}(\mathrm{X})$, for every open subset* U *of* X*, the measure μ_{U}^f is well defined, atomless and is supported by $\overline{\mathrm{A}_0}$.*

4.2 Dynamics of discretizations of a generic homeomorphism

We now establish a discrete counterpart of the shredding lemma. Since having basins of attraction is stable by perturbation, we have a similar statement for discretizations of a generic homeomorphism. In what follows these arguments are developed.

To each point $x_{\mathrm{N}} \in \mathrm{E}_{\mathrm{N}}$, we associate a closed set $\overline{\mathrm{P}_{\mathrm{N}}^{-1}(\{x_{\mathrm{N}}\})}$, made of the points in X one of whose projections on E_{N} is x_{N}. The closed sets $\overline{\mathrm{P}_{\mathrm{N}}^{-1}(\{x_{\mathrm{N}}\})}$ form a basis of the topology of X when N runs through $\mathbf{N}$ and x_{N} runs through E_{N}. Let $f \in \mathrm{Homeo}(\mathrm{X})$, $\mathrm{N} \in \mathbf{N}$ and $\vartheta : \mathbf{N} \to \mathbf{R}_+^*$ be a function that tends to $+\infty$ at $+\infty$. Let $\delta, \varepsilon > 0$ and U_j be the sets obtained by the shredding lemma for f, δ and ε.

For all $j \leq \ell$ we denote by $\widetilde{\mathrm{U}}_j^{\mathrm{N}}$ the union over $x_{\mathrm{N}} \in \mathrm{E}_{\mathrm{N}}$ of the closed sets $\overline{\mathrm{P}_{\mathrm{N}}^{-1}(\{x_{\mathrm{N}}\})}$ whose intersection with U_j are non-trivial. Then $\widetilde{\mathrm{U}}_j^{\mathrm{N}}$ tends to U_j for the metric $d(\mathrm{A},\mathrm{B}) = \lambda(\mathrm{A}\Delta\mathrm{B})$, and for the Hausdorff metric; the shredding lemma states that these convergences are independent from the choice of ε'. Thus, for all k big enough, properties (i) to (iii) of the shredding lemma remain true for the discretizations g_{N} (for arbitrary g satisfying the properties of the lemma). Taking ε' small enough and modifying a little g if necessary, there exists sets $\mathrm{W}_{j,i}$ and $g \in \mathrm{Homeo}(\mathrm{X})$ such that $d(f,g) < \delta$, $\mathrm{W}_{j,i} \subset \overline{\mathrm{P}_{\mathrm{N}}^{-1}(\{x_{\mathrm{N}}\})}$ and $\mathrm{Card}(\bigcup_{j,i} \mathrm{W}_{j,i} \cap \mathrm{E}_{\mathrm{N}}) \leq \vartheta(\mathrm{N})$. The others estimations over the sizes of the sets involved in the lemma are obtained similarly. Finally, we obtain the following lemma:

Lemma 4.9 (Discrete shredding lemma)**.** *For all $f \in \mathrm{Homeo}(\mathrm{X})$, for all $\varepsilon, \delta > 0$ and all function $\vartheta : \mathbf{N} \to \mathbf{R}_+^*$ that tends to $+\infty$ at $+\infty$, there exists integers N_0, ℓ and $\ell_1, \cdots, \ell_\ell$ bigger than $1/\varepsilon$, and a homeomorphism $g \in \mathrm{Homeo}(\mathrm{X})$ such that $d(f,g) < \delta$ and for all $\mathrm{N} \geq \mathrm{N}_0$, there exists a family of subsets $\mathrm{U}_1^{\mathrm{N}}, \cdots, \mathrm{U}_\ell^{\mathrm{N}}$ of E_{N} such that:*

(i) $g_{\mathrm{N}}(\mathrm{U}_j^{\mathrm{N}}) \subset \mathrm{U}_j^{\mathrm{N}}$,

(ii) $\mathrm{Card}(\mathrm{U}_j^{\mathrm{N}}) < \varepsilon q_{\mathrm{N}}$,

(iii) $\mathrm{Card}\left(\bigcup_{j=1}^{\ell} \mathrm{U}_j^{\mathrm{N}}\right) > (1-\varepsilon) q_{\mathrm{N}}$,

(iv) $\mathrm{Card}(g_{\mathrm{N}}(\mathrm{U}_j^{\mathrm{N}})) < \mathrm{Card}(\mathrm{U}_j^{\mathrm{N}})/(\ell\vartheta(\mathrm{N}_0))$,

(v) for all j, there exists subsets $\mathrm{W}_{j,1}^{\mathrm{N}}, \cdots, \mathrm{W}_{j,\ell_j}^{\mathrm{N}}$ of E_{N} such that

a) $\mathrm{Card}(\bigcup_{j,i} \mathrm{W}_{j,i}^{\mathrm{N}}) \leq q_{\mathrm{N}}/\vartheta(\mathrm{N}_0)$,

b) $g_{\mathrm{N}}(\mathrm{W}_{j,i}^{\mathrm{N}}) \subset \mathrm{W}_{j,i+1}^{\mathrm{N}}$, *for every $i \in \{1, \cdots, \ell_j - 1\}$ and* $g_{\mathrm{N}}(\mathrm{W}_{j,\ell_j}^{\mathrm{N}}) \subset \mathrm{W}_{j,1}^{\mathrm{N}}$,

c)

$$\mathrm{U}_j^{\mathrm{N}} \subset \bigcup_{m\geq 0} g_{\mathrm{N}}^{-m}\Big(\bigcup_{i=1}^{\ell_j} \mathrm{W}_{j,i}^{\mathrm{N}}\Big),$$

(vi) for all j and all i, there exists sets U_j and $\mathrm{W}_{j,i}$ satisfying properties (i) to (v) of the shredding lemma such that $\mathrm{U}_j \subset \mathrm{P}_{\mathrm{N}}^{-1}(\mathrm{U}_j^{\mathrm{N}})$ and $\mathrm{W}_{j,i} \subset \mathrm{P}_{\mathrm{N}}^{-1}(\mathrm{W}_{j,i}^{\mathrm{N}})$.

(vii) if $\mathrm{N} \geq \mathrm{N}_0$, then for all j and all i, we have

$$\sum_j d_{\mathrm{H}}(\overline{\mathrm{U}_j}, \overline{\mathrm{P}_{\mathrm{N}}^{-1}(\mathrm{U}_j^{\mathrm{N}})}) < \varepsilon \quad \textit{and} \quad \sum_{i,j} d_{\mathrm{H}}(\overline{\mathrm{W}_{j,i}}, \overline{\mathrm{P}_{\mathrm{N}}^{-1}(\mathrm{W}_{j,i}^{\mathrm{N}})}) < \varepsilon$$

(where d_{H} is the Hausdorff metric), and

$$\sum_j \lambda(\mathrm{U}_j \Delta \mathrm{P}_{\mathrm{N}}^{-1}(\mathrm{U}_j^{\mathrm{N}})) < \varepsilon \quad \textit{and} \quad \sum_{i,j} \lambda(\mathrm{W}_{j,i} \Delta \mathrm{P}_{\mathrm{N}}^{-1}(\mathrm{W}_{j,i}^{\mathrm{N}})) < \varepsilon.$$

Remark 4.10. Properties *(i)* to *(v)* are discrete counterparts of properties *(i)* to *(v)* of the continuous shredding lemma, but the properties *(vi)* and *(vii)* reflect the convergence of the dynamics of discretizations to that of the original homeomorphism.

This lemma implies that we can theoretically deduce the behaviour of a generic homeomorphism from the dynamics of its discretizations. The next section details this remark.

To begin with, we deduce from the shredding lemma that the dynamics of discretizations f_{N} tends to that of the homeomorphism f. More precisely almost all orbits of the homeomorphism are δ-*shadowed* by the orbits of the corresponding discretizations.

Definition 4.11. Let f and g be two maps from a metric space X into itself, $x, y \in \mathrm{X}$ and $\delta > 0$. We say that the orbit of x by f δ-*shadows* the orbit of y by g if for all $m \in \mathbf{N}$, $d(f^m(x), g^m(y)) < \delta$.

Corollary 4.12. *For a generic homeomorphism $f \in \mathrm{Homeo}(\mathrm{X})$, for all $\varepsilon > 0$ and all $\delta > 0$, there exists an open set A such that $\lambda(\mathrm{A}) > 1 - \varepsilon$ and $\mathrm{N}_0 \in \mathbf{N}$, such that for all $\mathrm{N} \geq \mathrm{N}_0$ and all $x \in \mathrm{A}$, the orbit of $x_{\mathrm{N}} = \mathrm{P}_{\mathrm{N}}(x)$ by f_{N} δ-shadows the orbit of x by f.*

Therefore, for a generic homeomorphism f, there exists a full measure dense open set O such that for all $x \in \mathrm{O}$, all $\delta > 0$ and all N large enough, the orbit of x_{N} by f_{N} δ-shadows that of x by f.

Proof of Corollary 4.12. This easily follows from the discrete shredding lemma, and especially from the fact that the sets $\mathrm{W}_{j,i}^{\mathrm{N}}$ tend to the sets $\mathrm{W}_{j,i}$ for Hausdorff metric, in particular $\mathrm{O} = \bigcup_{j,\varepsilon} \mathrm{U}_{j,\varepsilon}$. □

This statement is a bit different from the genericity of shadowing (see [PP99]): here the starting point is not a pseudo-orbit but a point $x \in \mathrm{X}$; Corollary 4.12 expresses that we can "see" the dynamics of f on that of f_{N}, with arbitrarily high precision, provided that N is large enough. Among other things, this allows us to observe the basins of attraction of the neighbourhoods of the Lyapunov stable periodic points of f on discretizations. Better yet, to each family of attractors $(\mathrm{W}_{j,i})_i$ of the basin U_j of the homeomorphism corresponds a unique family of sets $(\mathrm{W}_{j,i}^{\mathrm{N}})_i$ that are permuted cyclically by f_{N} and attract a neighbourhood of U_j. Thus, attractors are shadowed by cyclic orbits

of f_N and we can detect the "period" of the attractor (*i.e.* the integer ℓ_j) on discretizations: when $N = N_0$, the sets $W_{j,i}^N$ each contain only one point; a phenomenon of period multiplication might appear for N bigger [Bla89, Bla86, Bla84, Bla94, AHK03]. This behaviour is the opposite of what happens in the conservative case, where discretized orbits and true orbits are very different for most points.

Again, in order to show that the dynamics of discretizations converges to that of the initial homeomorphism, we establish the convergence of attractive sets of f_N to that of f. Recall that A_0 is the closure of the set of Lyapunov stable periodic points of f (see Definition 4.3).

Proposition 4.13. *For a generic homeomorphism $f \in \mathrm{Homeo}(X)$, the recurrent sets $\Omega(f_N)$ tend weakly to A_0 in the following sense: for all $\varepsilon > 0$, there exists $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$, there exists a subset $\widetilde{E}_N$ of E_N, stabilized by f_N, such that, noting $\widetilde{\Omega}(f_N)$ the corresponding recurrent set, we have $\frac{\mathrm{Card}(\widetilde{E}_N)}{\mathrm{Card}(E_N)} > 1 - \varepsilon$ and $d_H(A_0, \widetilde{\Omega}(f_N)) < \varepsilon$.*

Proof of Proposition 4.13. Let $\varepsilon > 0$. For all $N \in \mathbf{N}$, let $\widetilde{E}_N$ be the union of the sets U_j^N of Lemma 4.9 for the parameter ε. This lemma ensures that $\widetilde{E}_N$ is stable by f_N and fills a proportion greater than $1 - \varepsilon$ of E_N. We also denote by $\widetilde{\Omega}(f_N)$ the associated recurrent set:

$$\widetilde{\Omega}(f_N) = \bigcup_{x \in \bigcup_{j=1}^{\ell} \widetilde{U}_j^N} \omega_{f_N}(x).$$

Property *(viii)* of Lemma 4.9 ensures that

$$\overline{\lim_{N \to +\infty}} \, d_H(A_\varepsilon, \widetilde{\Omega}(f_N)) < \varepsilon.$$

To conclude, it suffices to apply Corollary 4.5 which asserts that $A_\varepsilon \to A_0$ for Hausdorff distance. □

We now set a final consequence of Lemma 4.9, which reflects that the ratio between the cardinality of the image of discretizations and which of the grid is smaller and smaller:

Corollary 4.14. *Let $\vartheta : \mathbf{R}_+ \to \mathbf{R}_+^*$ be a function that tends to $+\infty$ at $+\infty$. Then for a generic homeomorphism $f \in \mathrm{Homeo}(X)$,*

$$\overline{\lim_{N \to +\infty}} \, \vartheta(N) \frac{\mathrm{Card}(f_N(E_N))}{\mathrm{Card}(E_N)} = 0;$$

more precisely, for every $M \in \mathbf{N}$, there exists $N_0 \geq M$ such that for every $N \geq N_0$, we have

$$\frac{\mathrm{Card}(f_N(E_N))}{\mathrm{Card}(E_N)} \leq \frac{1}{\vartheta(N_0)}.$$

In particular, the degree of recurrence satisfies $\lim_{N \to +\infty} D(f_N) = 0$.

Remark that as $\Omega(f_N) \subset f_N(E_N)$, the same estimation holds for the recurrent set. This corollary can be seen as a discrete analogue of the fact that a generic homeomorphism is totally singular, *i.e.* that there exists a Borel set of full measure whose image under f is zero measure. Again, it reflects the regularity of the behaviour of the discretizations of a dissipative homeomorphism: generically, the behaviour of *all* (sufficiently fine) discretizations is the same as the physical behaviour of the initial homeomorphism.

This is very different from the conservative case, where sometimes $f_N(E_N) = E_N$ and sometimes $\mathrm{Card}(f_N(E_N)) \leq \vartheta(N)$ where $\vartheta : \mathbf{R}_+ \to \mathbf{R}_+^*$ is a given map that tends to $+\infty$ at $+\infty$.

The discrete shredding lemma also allows us to have an estimation about the combinatorial behaviour of $f_{N\Omega(f_N)}$: f_N has a lot of periodic orbits, and along these periodic orbits a lot have long lengths.

Corollary 4.15. *For a generic homeomorphism $f \in \mathrm{Homeo}(X)$ and for every $M \in \mathbf{N}$, there exists $N_0 \in \mathbf{N}$ such that for every $N \geq N_0$, there exists a subset $\widetilde{E}_N$ of E_N, stabilized by f_N, such that we have $\frac{\mathrm{Card}(\widetilde{E}_N)}{\mathrm{Card}(E_N)} > 1 - \varepsilon$ (as in Proposition 4.13), such that all the periodic orbits of $f_{N|\widetilde{E}_N}$ have length bigger than M, and such that the number of such orbits is bigger than M.*

It remains to study the behaviour of measures $\mu_U^{f_N}$ (see Definition 3.10). To do that, we have to suppose that the sequence of grids well behaves with respect to the measure λ. Again, the results are very different from the conservative case: for any open set U, the measures $\mu_U^{f_N}$ tend to a single measure, say μ_U^f.

Theorem 4.16. *For a generic homeomorphism $f \in \mathrm{Homeo}(X)$ and an open subset U of X, the measure μ_U^f is well defined*[7] *and is supported by the set $\overline{A_0}$. Moreover the measures $\mu_U^{f_N}$ tend weakly to μ_U^f.*

Sketch of proof of Theorem 4.16. The proof of this theorem is based on the shredding lemma: the set of homeomorphisms which satisfy the conclusions of the lemma is a G_δ dense, so it suffices to prove that such homeomorphisms f satisfy the conclusion of the proposition. Let U be an open subset of X and $\varphi : X \to \mathbf{R}$ be a continuous function.

We want to show that on the one hand the integral $\int_X \varphi \, \mathrm{d}\mu_U^f$ is well defined, *i.e.* that the Birkhoff limits for the function φ

$$\lim_{m\to+\infty} \frac{1}{m} \sum_{i=0}^{m-1} \varphi(f^i(x))$$

are well defined for almost every $x \in U$; and on the other hand we have the convergence

$$\int_X \varphi \, \mathrm{d}\mu_U^{f_N} \underset{N\to+\infty}{\longrightarrow} \int_X \varphi \, \mathrm{d}\mu_U^f,$$

For the first step, the idea of the proof is that most of the points (for λ) eventually belong to a set $W_{j,i}$. Since the iterates of the sets $W_{j,i}$ have small diameter, by uniform continuity, the function φ is almost constant on the sets $f^m(W_{j,i})$. Thus the measure μ_x^f is well defined and almost constant on the set of points whose iterates eventually belong to $W_{j,i}$. And by the same construction, since the dynamics of f_N converge to that of f, and in particular that the sets U_j^N and $\{w_{j,i}^N\}$ converge to the sets U_j and $W_{j,i}$, the measures $\mu_U^{f_N}$ tend to the measures μ_U^f. □

7. In other words, a generic homeomorphism is weird, see Definition 4.1, see also [AA13].

4.3 Numerical simulations

We now present some numerical simulations of dissipative homeomorphisms. Again, our aim is to compare the theoretical results with the reality of numerical simulations: for simple homeomorphisms and reasonable orders of discretization, do we have convergence of the dynamics of the discretizations to that of the homeomorphism, as suggested by the above theorems?

We simulate homeomorphisms of the form

$$f(x,y) = (\mathrm{R} \circ \mathrm{Q} \circ \mathrm{P})(x,y),$$

where P and Q are two homeomorphisms of the torus that modify only one coordinate:

$$\mathrm{P}(x,y) = \Big(x, y + p(x)\Big) \quad \text{and} \quad \mathrm{Q}(x,y) = \Big(x + q(y), y\Big),$$

so that the homeomorphism $\mathrm{Q} \circ \mathrm{P}$ preserves Lebesgue measure. We discretize these examples according to the uniform grids on the torus

$$\mathrm{E_N} = \left\{ \left(\frac{i_1}{\mathrm{N}}, \cdots, \frac{i_n}{\mathrm{N}} \right) \in \mathbf{T}^n \,\middle|\, \forall j,\ 0 \leq i_j \leq \mathrm{N}-1 \right\}.$$

We have tested two homeomorphisms:

– To begin with we studied $f_1 = \mathrm{R}_1 \circ \mathrm{Q} \circ \mathrm{P}$, with

$$p(x) = \frac{1}{209}\cos(2\pi \times 187x) + \frac{1}{271}\sin(2\pi \times 253x) - \frac{1}{703}\cos(2\pi \times 775x),$$

$$q(y) = \frac{1}{287}\cos(2\pi \times 241y) + \frac{1}{203}\sin(2\pi \times 197y) - \frac{1}{841}\sin(2\pi \times 811y)$$

and

$$\begin{aligned}\Big(\mathrm{R}_1(x,y)\Big)_x = x &- 0.00227\sin(2\pi \times 95(x+\alpha)) \\ &+ 0.000224\cos(2\pi \times 197(y+\alpha)) \\ &- 0.00111\sin(2\pi \times 343(x+\alpha))\end{aligned}$$

$$\begin{aligned}\Big(\mathrm{R}_1(x,y)\Big)_y = y &- 0.00376\sin(2\pi \times 107(y+\beta)) \\ &- 0.000231\cos(2\pi \times 211(x+\beta)) \\ &+ 0.00107\cos(2\pi \times 331(y+\beta)),\end{aligned}$$

with $\alpha = 0.00137$ and $\beta = 0.00159$. This dissipative homeomorphism is a small C^0 perturbation of the identity, whose derivative has many oscillations whose amplitudes are close to 1. That creates many fixed points which are attractors, sources or saddles.

– It has also seemed to us useful to simulate a homeomorphism close to the identity in C^0 topology, but with a small number of attractors. Indeed, as explained heuristically by J.-M. Gambaudo and C. Tresser in [GT83], a homeomorphism like f_1 can have a large number of attractors whose basins of attraction are small. It turns out that the dissipative behaviour of f_1 cannot be detected for reasonable orders discretization. We therefore defined another homeomorphism close to the

identity in C^0 topology, but with much less attractors, say $f_2 = R_2 \circ Q \circ P$, with P and Q identical to those used for f_1, and R_2 defined by

$$R_2\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x - 0.00227\sin(2\pi\times 14(x+\alpha)) + 0.000324\cos(2\pi\times 33(y+\alpha))\\ y - 0.00376\cos(2\pi\times 15(y+\beta)) - 0.000231\sin(2\pi\times 41(x+\beta))\end{pmatrix},$$

with $\alpha = 0.00137$ and $\beta = 0.00159$.

Remark that we have chosen to define the homeomorphisms we compute with lacunary trigonometric series, to "mimic" the action of Baire theorem.

4.3.1 Algorithm used for the calculus of invariant measures

The algorithm we used to conduct simulations is quite fast (in fact, it is linear in the number of points of the grid). It detects all the periodic orbits of the discretizations f_N in the following way. It takes a first point $x_1 \in E_N$ and iterates it until the orbit meets a point that has ever been visited by the orbit. The points belonging to the orbit of x_1 are labelled as falling into the periodic orbit number 1. The algorithm also notes the number of points that have been attracted by this orbit, and the coordinates of the points of the periodic orbit. It then takes another point $x_2 \in E_N$ which does not belong to the orbit of x_1. There are two cases: either an iterate of x_2 is equal to an iterate of x_1, and in this case it updates the number of points which fall into the periodic orbit number 1; or an iterate of x_2 meets another iterate of x_2, and in this case it creates a periodic orbit number 2, which attracts all the orbit of x_2. This procedure is iterated until there is no more points of E_N that have not been visited. Remark that this algorithm computes the image of a point at most twice.

This algorithm allows to compute quantities like the cardinality of the recurrent set $\Omega(f_N)$, the number of periodic orbits of f_N, the maximal size of a periodic orbit of f_N, etc. It also allows to represent the invariant measure $\mu_X^{f_N}$ of f_N. Recall that this measure is defined as the limit in the Cesàro sense of the push forward of the uniform measures on E_N by the discretizations f_N. It is supported by the union $\Omega(f_N)$ of the periodic orbits of f_N; the measure of each of these periodic orbits is proportional to the size of its basin of attraction.

We present images of sizes 128×128 pixels representing in logarithmic scale the density of the measures $\mu_X^{f_N}$: each pixel is coloured according to the measure carried by the set of points of E_N it covers. Blue corresponds to a pixel with very small measure and red to a pixel with very high measure. Scales on the right of each image corresponds to the measure of one pixel on the $\log 10$ scale: if green corresponds to -3, then a green pixel will have measure 10^{-3} for $\mu_X^{f_N}$. For information, when Lebesgue measure is represented, all the pixels have a value about -4.2.

We also compute the distance between the measure $\mu_X^{f_N}$ and Lebesgue measure. The distance we have chosen is given by the formula

$$d(\mu,\nu) = \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{i,j=0}^{2^k-1} \left|\mu(C_{i,j,k}) - \nu(C_{i,j,k})\right| \in [0,2],$$

where

$$C_{i,j,k} = \left[\frac{i}{2^k}, \frac{i+1}{2^k}\right] \times \left[\frac{j}{2^k}, \frac{j+1}{2^k}\right].$$

This distance spans the weak-* topology, which makes compact the set of probability measures on $\mathbf{T}^2$. In practice, we have computed an approximation of this quantity by summing only on the $k \in [\![0,7]\!]$.

From a practical point of view, we have restricted ourselves to grids of sizes smaller than $2^{15} \times 2^{15}$: the initial data become quickly very large, and the algorithm creates temporary variables that are of size of the order of five times the size of the initial data. For example, for a grid $2^{15} \times 2^{15}$, the algorithm needs between 25 and 30 Go of RAM, and takes about two days of calculus for a single order of discretization.

4.3.2 Combinatorial behaviour

We simulated some quantities related to the combinatorial behaviour of discretizations of homeomorphisms, namely:

- the cardinality of the recurrent set $\Omega(f_N)$,
- the number of periodic orbits of f_N,
- the maximal size of a periodic orbit of f_N.

We calculated these quantities for discretizations of orders $128k$ for k from 1 to 150, and represented it graphically (see Figure 4.2). For information, if $N = 128 \times 150$, then $q_N \simeq 3.6.10^8$.

Theoretically, the degree of recurrence, that is, the ratio between the cardinality of $\Omega(f_N)$ and q_N, should tend to 0 (see Corollary 4.14); this is what we observe on simulations. This is not really surprising: we will even see it for discretizations of conservative homeomorphisms (see Figure 5.7). In this context, it is interesting to compare the behaviour of $\Omega(f_N)$ in the conservative and the dissipative case. The result is a little disappointing: the graphic for f_1, the dissipative homeomorphism, is very similar to that of f_3, the corresponding conservative homeomorphism, while in theory they should be very different. This is quite different for f_2, where the cardinality of $\Omega((f_2)_N)$ is more or less linear in N. We have no explanation to the linear shape of this function; if the maps f_N were typical random maps, their degree of recurrence would be linear in N, with a value would close to $2.4.10^4$ for $N = 150 \times 128$ (here the value is about three times bigger).

The theoretical results assert that the number of periodic orbits of f_N should tend to $+\infty$ (as a generic dissipative homeomorphism has an infinite number of attractors). We can hope that this quantity reflects the fact that the dynamics converges to that of the initial homeomorphism: among others, we can test if it is of the same order as the number of attractors of the homeomorphism. In practice, this number of periodic orbits of f_N first increases rather quickly, to stabilize to around a value of $1.5.10^4$ for f_1 and 9.10^3 for f_2. We could be tempted to interpret this phenomenon by the fact that after a while, the discretization has detected all the attractors of f and thus, the number of periodic of the discretizations reflects the number of attractors of f. This idea may be reasonable for f_2 (as we will see in observing the invariant measures of $(f_2)_N$ in Figure 4.5), but if we compare these graphs in the dissipative case with that of the conservative case (Figure 5.7), we see that they are as alike as two peas in a pod. Thus, this is not clear at all that this behaviour is due to the dissipative character of the homeomorphism or not.

Since the dynamics of discretizations is assumed to converge to that of the initial homeomorphism, we could expect that the length of the longest periodic orbit of discretizations $(f_i)_N$ is almost always a multiple of that of an attractive periodic orbit of f_i. The graphic of this length for f_1 looks like the conservative case (see Figure 5.7), so we can say that the dissipative behaviour of this homeomorphism is not detected in practical by this quantity. For f_2, the length of the longest orbit is much smaller than

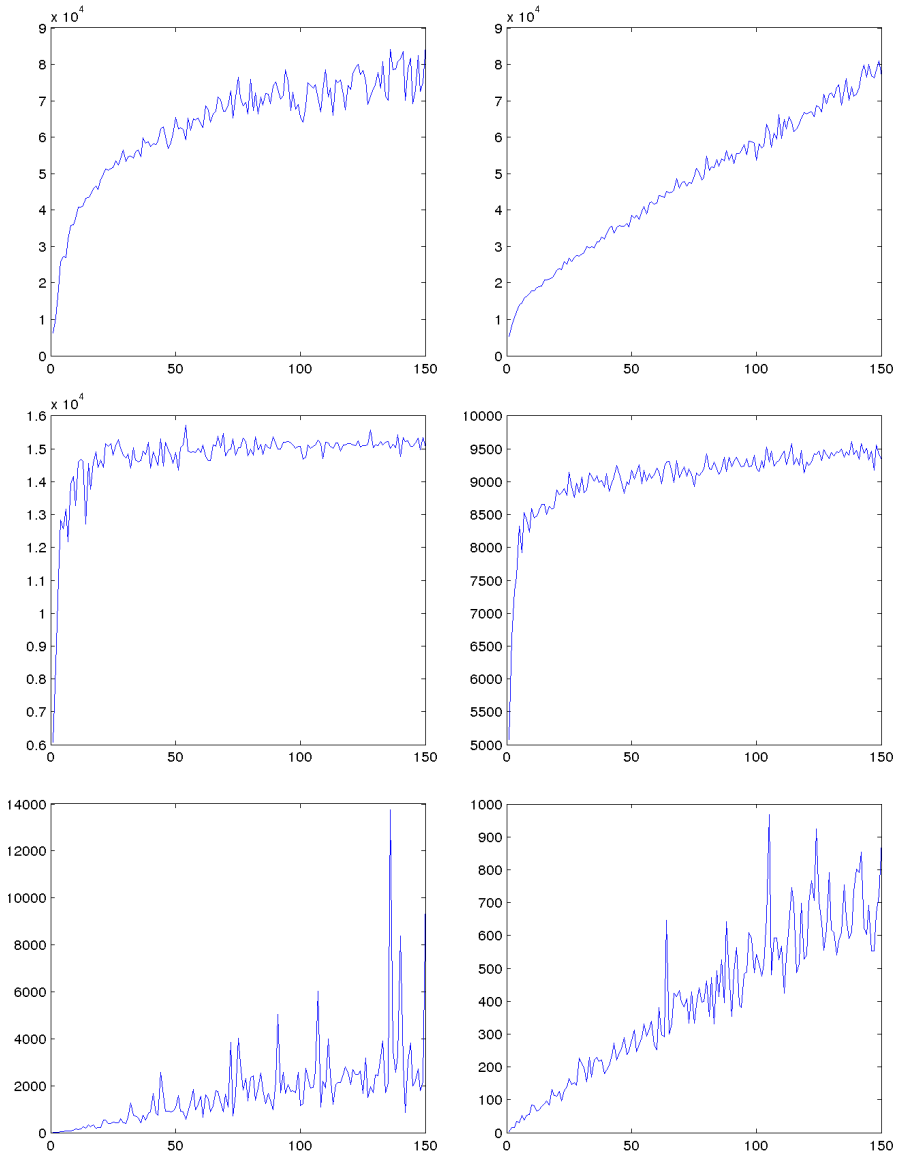

Figure 4.2: Size of the recurrent set $\Omega((f_i)_N)$ (top), number of periodic orbits of $(f_i)_N$ (middle) and length of the largest periodic orbit of $(f_i)_N$ (bottom) depending on N, for f_1 (left) and f_2 (right), on the grids E_N with $N = 128k$, $k = 1, \cdots, 150$.

for f_1 (up to a factor 10), and seem to increase linearly in N. This may be imputed to the fact that f_2 is "almost conservative" around its attractive periodic points; thus it has a conservative behaviour, but at a smaller scale than f_1.

4.3.3 Behaviour of invariant measures

We have computed the invariant measures $\mu_{\mathbf{T}^2}^{(f_i)_N}$ of dissipative homeomorphisms f_1 and f_2 as defined on page 66. Our aim is to test whether Theorem 4.16 applies in practice or if there are technical constraints such that this behaviour cannot be observed on these examples. For a presentation of the representations of the measures, see page 67.

On Figure 4.3, we have represented the distance between the measure $\mu_{\mathbf{T}^2}^{(f_i)_N}$ and

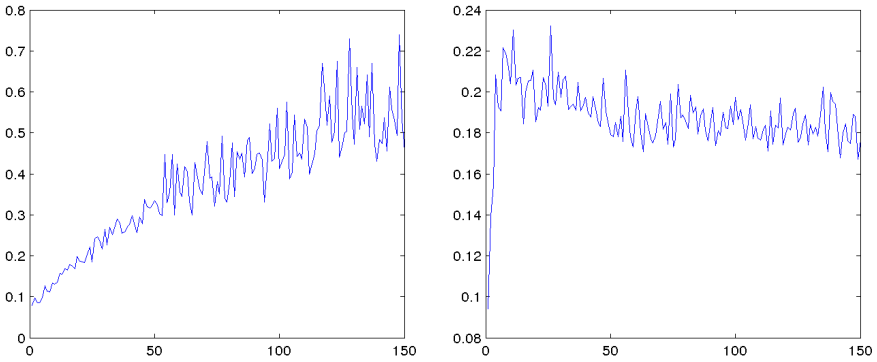

Figure 4.3: Distance between Lebesgue measure and the measure $\mu_{\mathbf{T}^2}^{f_N}$ depending on N for f_1 (left) and f_2 (right), on the grids E_N with $N = 128k$, $k = 1, \cdots, 150$.

Lebesgue measure. Theorem 4.16 says that for a generic dissipative homeomorphism, this quantity converges to the distance between $\mu_{\mathbf{T}^2}^{f}$ and Lebesgue measure. Clearly, this is not what happens in practice for f_1: the distance between both measures globally increases when N increases. Locally, the behaviour of the map $N \mapsto \mathrm{dist}(\mu_{\mathbf{T}^2}^{f_N}, \mathrm{Leb})$ is quite erratic: there is no sign of convergence to any measure. When we compare this with the conservative case (Figure 5.8), we see that both behaviours are very similar. In other words, we do not see the dissipative nature of f_1 on simulations. The behaviour of the distance between the measure $\mu_{\mathbf{T}^2}^{(f_2)_N}$ and Lebesgue measure is much more interesting. First of all, we observe that this distance is smaller than for f_1. Moreover, it is globally slightly decreasing in N, and it seem to converge to a value close to 0.2. So it suggests that the measures $\mu_{\mathbf{T}^2}^{(f_2)_N}$ converge to a given measure whose distance to Leb is close to 0.2, as predicted by the theory. In the view of these graphics, we can say that the behaviour of f_2 is much more close to that of a generic dissipative homeomorphism that that of f_1. This can be interpreted in the view of the article of J.-M. Gambaudo and C. Tresser [GT83]: for f_1, the attractors are much too small to be observed in practice for reasonable orders of discretization.

The behaviour of invariant measures for f_1 (see Figure 4.4), which is a small C^0 dissipative perturbation of identity, is relatively similar to that of invariant measures for f_3 *i.e.* the corresponding conservative case (see Figure 5.9): when the order discretization is large enough, there is a strong variation of the measure $\mu_{\mathbf{T}^2}^{(f_1)_N}$. Moreover, this measure has a significant absolutely continuous component with respect to Lebesgue measure. This is very different from what is expected from the theoretical results (in particular Theorem 4.16), which say that for a generic dissipative homeomorphism, the measures $\mu_{\mathbf{T}^2}^{f_N}$ must converge to the measure $\mu_{\mathbf{T}^2}^{f}$. Thus, we can say that it is impossible to detect the dissipative character of f_1 on these simulations. Again, as noted by J.-M. Gambaudo and C. Tresser in [GT83], the size of the attractors of f_1 can be very small compared to the numbers involved in the definition of f_1. So even in orders discretization such as 2^{15}, the dissipative nature of the homeomorphism is undetectable on discretizations. This is why the discretizations of f_1 are very similar to those of its conservative counterpart f_3.

Recall that what happens for the dissipative homeomorphism f_1 is rather close to what happens for the conservative homeomorphism f_3. For their part, simulations of

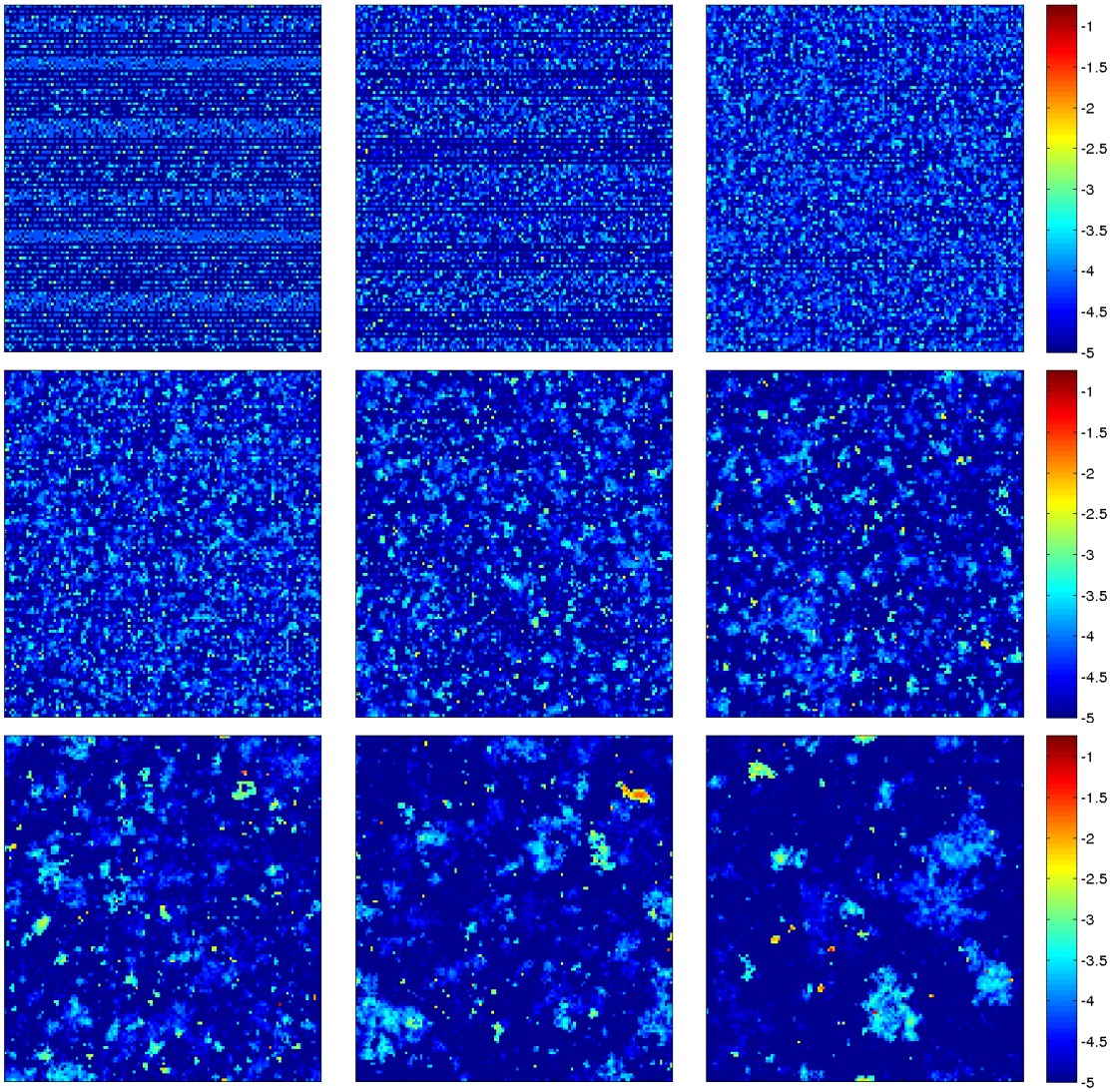

Figure 4.4: Simulations of $\mu_{\mathbf{T}^2}^{(f_1)_N}$ on grids E_N, with $N = 2^k$, $k = 7, \cdots, 15$ (from left to right and top to bottom).

invariant measures for f_2 on grids of size $2^k \times 2^k$ (see Figure 4.5) highlight that we expect from a generic dissipative homeomorphism: the measures $\mu_{\mathbf{T}^2}^{(f_2)_N}$ seem to tend to a single measure (that is also observed on a series of simulations), which is carried by the attractors of f_2. The fact that f_2 has few attractors allows the discretizations of reasonable orders (typically 2^{11}) to find the actual attractors of the initial homeomorphism, contrary to what we observed for f_1. We also present a zoom of the density of the measures $\mu_{\mathbf{T}^2}^{(f_2)_N}$ (Figure 4.6). On these simulations, we can see that the attracting regions that we observe on the simulations of Figure 4.5 are in fact crumbled, it particular they are not connected. This is what is predicted by the theory: the set of Lyapunov periodic points of a generic dissipative homeomorphism is a Cantor set. On these zoomed simulations, we also observe that the dynamics of the discretizations has not completely converged at the order $N = 2^{15}$: locally, the measures $\mu_{\mathbf{T}^2}^{(f_2)_N}$ are quite different for different orders N; locally, the homeomorphism still behaves like in the conservative one.

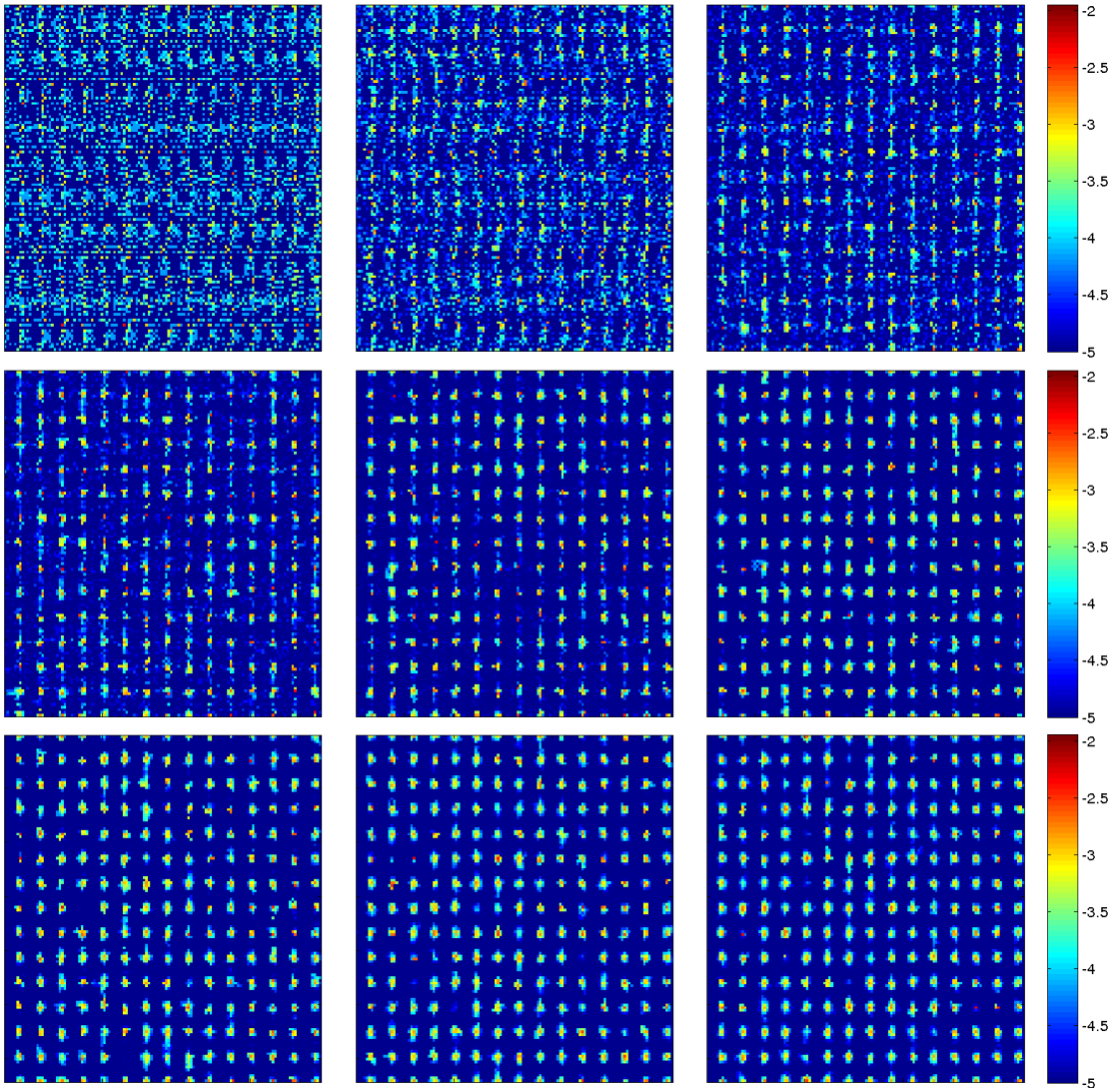

Figure 4.5: Simulations of $\mu_{\mathbf{T}^2}^{(f_2)_N}$ on grids E_N, with $N = 2^k$, $k = 7, \cdots, 15$ (from left to right and top to bottom).

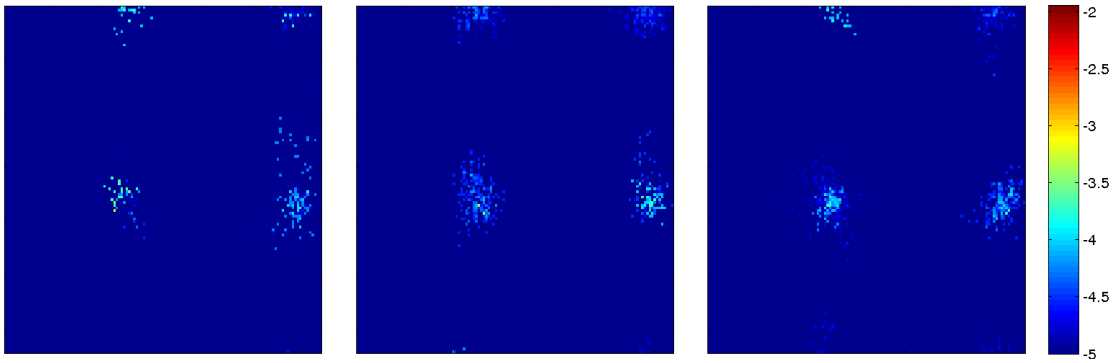

Figure 4.6: Zoom on the density of the measures $\mu_{\mathbf{T}^2}^{(f_2)_N}$ on grids E_N, with $N = 2^k$, $k = 13, 14, 15$ (from left to right); the zoom is made on the top right of the representation of Figure 4.5, on a square of size $1/8 \times 1/8$.

Chapter 5

Discretizations of a generic conservative homeomorphism

We now study the conservative case, *i.e.* we study the dynamical behaviour of discretizations of homeomorphisms of X which are generic among those which preserve a given good measure λ. In the previous chapter, we proved that the dynamics of the discretizations of a generic *dissipative* homeomorphism f converges to the dynamics of f. This was due to the fact that such a generic homeomorphism possesses attractors whose basins cover almost all the phase space X. Now, the hypothesis of preservation of a measure λ precludes this kind of behaviours, as conservative maps do not have attractors and thus are chain transitive.

We will begin by introducing the concept of *dense type of discrete approximation* (Definition 5.1), which expresses that a given set of finite maps on the grids can approach any conservative homeomorphism. Then, using the proposition of finite maps extension (Proposition 3.3), we prove that any dense type of approximation occurs infinitely many times in the discretizations of a generic homeomorphism (Theorem 5.2). Those dense types of approximations will be obtained by using Lax's theorem (Theorem 5.5). This strategy of proof will lead us to many results concerning the dynamical properties of discretizations of generic conservative homeomorphisms. In particular, for a generic conservative homeomorphism f, under the assumptions that the grids are well distributed and well ordered:

- there are infinitely many discretizations f_N such that f_N is a cyclic permutation (Corollary 5.9);
- for every $\varepsilon > 0$, there are infinitely many discretizations f_N such that f_N is ε-topologically weakly mixing (Corollary 5.13, see Definition 5.11);
- for every period p of a periodic orbit of f, there are infinitely many discretizations f_N such that f_N has a unique injective orbit whose corresponding periodic orbit has length p (Corollary 5.36);

– for every map $\vartheta : \mathbf{N} \to \mathbf{R}_+^*$ which tends to $+\infty$ at $+\infty$, there are infinitely many discretizations f_N such that $\mathrm{Card}(f_N(E_N)) \leq \vartheta(N)$ (Corollary 5.20);
– if we further assume that the grids are self similar, for every map $\vartheta : \mathbf{N} \to \mathbf{R}$ such that $\vartheta(N) = o(q_N)$, there are infinitely many discretizations f_N such that f_N has at least $\vartheta(N)$ cycles which are pairwise conjugated (Corollary 5.22).

Moreover, we prove that we can not hope to observe the actual dynamics of a generic homeomorphism by looking at the frequency with which some properties appear on discretizations. For example, if the sequence of grids is strongly self similar, we will prove that for a generic conservative homeomorphism, for every $\varepsilon > 0$, there are infinitely many $M \in \mathbf{N}$ such that the proportion of discretizations f_N, with $1 \leq N \leq M$, such that f_N is a cyclic permutation, is greater than $1 - \varepsilon$ (Theorem 5.31). We also prove that under the assumption that the sequence of grids is self similar, the same property holds when we replace "f_N is a cyclic permutation" by "$\mathrm{Card}(\Omega(f_N)) = o(q_N)$" (Corollary 5.27), "$f_N$ is ε-topologically weakly mixing" (Corollary 5.28), "$\frac{\mathrm{Card}(f_N(E_N))}{\mathrm{Card}(E_N)} < \varepsilon$" (Corollary 5.29) or "$f_N$ has at least M periodic orbits" (Corollary 5.30).

However, we will prove that it is possible to recover the dynamics of the initial homeomorphism by looking at the dynamics of *all* its discretizations. For example, it is possible to detect the periodic orbits of f and their periods (Theorem 5.36), to recover the set of invariant measures of f (Theorem 5.45), or to recover the set of compact subsets of X that are invariant under f (Theorem 5.49).

From this point of view of the shadowing property of the dynamics of the homeomorphism by that of all its discretizations, the "physical dynamics", that is the dynamical properties that depend on the good measure λ, plays non important role among all the dynamical properties of f. For instance, we will prove that if the sequence of grids is self similar, then for a generic homeomorphism f, the sequence of f_N-physical [1] measures $(\mu_X^{f_N})_{N\in\mathbf{N}}$ accumulates on the the whole set of f-invariant measures (Theorem 5.51). And with the same techniques of proof we will show a similar result for invariant compact sets (Proposition 5.55).

The end of this chapter is dedicated to the results of the numerical simulations.

Recall that we have fixed once and for all a manifold X, a good measure λ on X and a sequence of discretization grids $(E_N)_{N\in\mathbf{N}}$ (see Definition 3.5). We further assume that this sequence of grids is well distributed and well ordered (see Chapter 3). In this chapter, we will focus on discretizations of elements of $\mathrm{Homeo}(X, \lambda)$, *so homeomorphisms will always be supposed conservative.*

5.1 Dense types of approximation

To begin with, we define the notion of *type of approximation*.

Definition 5.1. A *type of approximation* $\mathscr{T} = (\mathscr{T}_N)_{N\in\mathbf{N}}$ is a sequence of subsets of the set $\mathscr{F}(E_N, E_N)$ of applications from E_N into itself.

Let $\mathscr{U}$ be an open subset of $\mathrm{Homeo}(X, \lambda)$ (therefore $\mathscr{U}$ is a set of homeomorphisms). A type of approximation $\mathscr{T}$ is said to be *dense* in $\mathscr{U}$ if for all $f \in \mathscr{U}$, all $\varepsilon > 0$ and all $N_0 \in \mathbf{N}$, there exists $N \geq N_0$ and $\sigma_N \in \mathscr{T}_N$ such that $d_N(f, \sigma_N) < \varepsilon$ (recall that d_N is the distance between $f_{|E_N}$ and σ_N considered as maps from E_N into X).

1. Recall that $\mu_X^{f_N}$ is the limit in the sense of Cesàro of the pushforwards of the uniform measure on E_N by the iterates of f_N

The goal of this paragraph is to obtain the following result: every dense type of approximation appears on infinitely many discretizations of a generic homeomorphism. This will lead us to a systematic study of dense types of approximation.

Theorem 5.2. *Let $\mathscr{T}$ be a type of approximation which is dense in* $\mathrm{Homeo}(X,\lambda)$. *Then for a generic homeomorphism $f \in \mathrm{Homeo}(X,\lambda)$ and for all $N_0 \in \mathbf{N}$, there exists $N \geq N_0$ such that $f_N \in \mathscr{T}_N$.*

Thus, we have decomposed the obtaining of generic properties of discretizations into two steps.

1. The first step consists in proving that a given type of approximation is dense. This part will be done in Section 5.3, the systematic use of Lax's theorem (see Section 5.2) will reduce the proof of density of types of approximation to combinatorial problems.
2. The second step is always the same, it simply consists in applying Theorem 5.2.

Remark 5.3. However, some properties can be proved in different ways: for instance Corollary 5.20 can be proved by inserting horseshoes in the dynamics of a given homeomorphism, using the local modification theorem (Theorem 5.40, for a presentation of the technique in another context see [Gui12, Section 3.3]); also a variation of Corollary 5.15 can be shown in perturbing any given homeomorphism such that it has a periodic orbit whose distance to the grid E_N is smaller than the modulus of continuity of f, so that the actual periodic orbit and the discrete orbit fit together.

To prove Theorem 5.2, we start from a dense type of approximation — in other words a sequence of discrete applications — and we want to get properties of homeomorphisms. The tool that allows us to restore a homeomorphism from a finite map $\sigma_N : E_N \to E_N$ is the finite map extension proposition (Proposition 3.3).

Lemma 5.4. *Let $\mathscr{U}$ be an open subset*[2] *of* $\mathrm{Homeo}(X,\lambda)$ *and $\mathscr{T} = (\mathscr{T}_N)_{N\in\mathbf{N}}$ be a type of approximation which is dense in $\mathscr{U}$. Then for all $N_0 \in \mathbf{N}$, there exists an open and dense subset of $\mathscr{U}$ on which every homeomorphism f satisfies: there exists $N \geq N_0$ such that the discretization f_N belongs to $\mathscr{T}_N$.*

Proof of Lemma 5.4. Let $f \in \mathscr{U}$, $N_0 \in \mathbf{N}$ and $\varepsilon > 0$. Since $\mathscr{T}$ is a type of approximation which is dense in $\mathscr{U}$, there exists $N \geq N_0$ and $\sigma_N \in \mathscr{T}_N$ such that $d_N(f,\sigma_N) < \varepsilon$. Let $x_1,\cdots,x_{q_N}$ be the elements of E_N. Then for all ℓ, $d(f(x_\ell),\sigma_N(x_\ell)) < \varepsilon$. Now, the problem is that σ_N is not injective. To solve this problem, we take advantage of the fact that P_N is highly non injective to modify σ_N into a injection $\sigma'_N : E_N \to X$ whose discretization on E_N equals to σ_N: set $\sigma'_N(x_1) = \sigma_N(x_1)$, σ'_N is defined by induction by choosing $\sigma'_N(x_\ell)$ such that $\sigma'_N(x_\ell)$ is different from $\sigma'_N(x_i)$ for $i \leq \ell$, such that $P_N(\sigma'_N(x_\ell)) = P_N(\sigma_N(x_\ell))$ and that $d(f(x_\ell),\sigma'_N(x_\ell)) < \varepsilon$.

Since σ'_N is a bijection, Proposition 3.3 can be applied to $f(x_\ell)$ and $\sigma'_N(x_\ell)$; this gives a measure-preserving homeomorphism φ such that $\varphi(f(x_\ell)) = \sigma'_N(x_\ell)$ for all ℓ and such that $d(\varphi,\mathrm{Id}) < \varepsilon$. Set $f' = \varphi \circ f$, we have $d(f,f') \leq \varepsilon$ and $f'(x_\ell) = \sigma'_N(x_\ell)$ for all ℓ, and therefore $f_N = \sigma_N$. □

Now, proof of Theorem 5.2 is straightforward.

2. In most cases we will take $\mathscr{U} = \mathrm{Homeo}(X,\lambda)$.

Proof of Theorem 5.2. Let $(x_{N,\ell})_{1\leq\ell\leq q_N}$ be the elements of E_N and consider the set (where ρ_N is the minimal distance between two distinct points of E_N)

$$\bigcap_{N_0\in\mathbf{N}} \bigcup_{\substack{N\geq N_0\\ \sigma_N\in\mathscr{S}_N}} \left\{ f\in \mathrm{Homeo}(X,\lambda) \mid \forall \ell,\, d_N\big(f(x_{N,\ell}),\sigma_N(x_{N,\ell})\big) < \frac{\rho_N}{2} \right\}.$$

This set is clearly a G_δ set and its density follows directly from Lemma 5.4. Moreover we can easily see that its elements satisfy the conclusions of the theorem. □

5.2 Lax's theorem

Now that we have shown Theorem 5.2, it remains to obtain dense types of approximation. Systematic use will be made of Lax's theorem [Lax71], more precisely its improvement as stated by S. Alpern [Alp76], which allows to approach any conservative homeomorphism by a cyclic permutation of the elements of a discretization grid.

Theorem 5.5 (Lax, Alpern)**.** *Let $f\in\mathrm{Homeo}(X,\lambda)$ and $\varepsilon>0$. Then there exists $N_0\in\mathbf{N}$ such that for all $N\geq N_0$, there exists a cyclic permutation σ_N of E_N such that $d_N(f,\sigma_N)<\varepsilon$.*

It is worthwhile to note that this theorem was initially stated in the view of numerical approximation of conservative homeomorphisms: P. Lax noticed that in some pioneering works showing numerical simulations of chaotic dynamical systems (e.g. [Hén69]), the discretizations of conservative maps were only approximately measure-preserving. His theorem states that among all possible approximations of a conservative homeomorphism, some of them are bijective, which is a discrete counterpart of measure preservation. This theorem was later improved by S. Alpern in [Alp76, Alp78] to establish generic properties of conservative homeomorphisms (see also [DKP93, KM97] for a generalisation and some simulations in dimension 1). This theorem is now one of the keystones of the theory of generic properties of homeomorphisms (see [Gui12]), together with other theorems of approximation by permutations [3].

As a compact metric set can be seen as a "finite set up to ε", Lax's theorem allows us to see a homeomorphism as a "cyclic permutation up to ε". Genericity of transitivity in $\mathrm{Homeo}(X,\lambda)$ follows easily from Lax's theorem together with the finite map extension proposition (see [AP00] or part 2.4 of [Gui12]). In our case, applying Theorem 5.2, we deduce that infinitely many discretizations of a generic homeomorphism are cyclic permutations. The purpose of Section 5.3 is to find variations of Lax's theorem (which are at the same time corollaries of it) concerning other properties of discretizations.

We give briefly the beautiful proof of Lax's theorem, which is essentially combinatorial and based on marriage lemma and on a lemma of approximation of permutations by cyclic permutations due to S. Alpern. Readers wishing to find proofs of these lemmas may consult [Gui12, Section 2.1]. It is here that we use the fact that discretizations grids E_N are well distributed and well ordered (see Section 3.6).

Lemma 5.6 (Marriage lemma)**.** *Let E and F be two finite sets and $\approx$ a relation between elements of E and F. Suppose that the number of any subset $E'\subset E$ is smaller than the number of elements in F that are associated with an element of E', i.e.:*

$$\forall E'\subset E,\quad \mathrm{Card}(E')\leq \mathrm{Card}\{f\in F \mid \exists e\in E' : e\approx f\},$$

3. See also [KS67, KS70] for approximations in weak topology or [Hal56, page 65] for approximations of conservative automorphisms.

then there exists an injection $\Phi : E \to F$ *such that for all* $e \in E$, $e \approx \Phi(e)$.

Lemma 5.7 (Cyclic approximations in $\mathfrak{S}_q$, [Alp76])**.** *Let* $q \in \mathbf{N}^*$ *and* $\sigma \in \mathfrak{S}_q$ *($\mathfrak{S}_q$ is seen as the permutation group of $\mathbf{Z}/q\mathbf{Z}$). Then there exists* $\tau \in \mathfrak{S}_q$ *such that* $|\tau(k) - k| \leq 2$ *for all* k *(where $|.|$ is the distance in $\mathbf{Z}/q\mathbf{Z}$) and such that the permutation $\tau\sigma$ is cyclic.*

Proof of Theorem 5.5. Let $f \in \mathrm{Homeo}(X, \lambda)$ and $\varepsilon > 0$. Consider $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$, the diameter of the cubes of order N (given by the hypothesis "E_N is well distributed") and their images by f is smaller than ε. We define a relation $\approx$ between cubes of order $N-1$: $C \approx C'$ if and only if $f(C) \cap C' \neq \emptyset$. Since f preserves λ, the image of the union of ℓ cubes intersects at least ℓ cubes, so the marriage lemma (Lemma 5.6) applies: there exists an injection Φ_N from the set of cubes of order N into itself (then a bijection) such that for all cube C, $f(C) \cap \Phi_N(C) \neq \emptyset$. Let σ_N be the application that maps the centre of any cube C to the centre of the cube $\Phi_N(C)$, we obtain:

$$d_N(f, \sigma_N) \leq \sup_C \Big(\mathrm{diam}(C) \Big) + \sup_C \Big(\mathrm{diam}(f(C)) \Big) \leq 2\varepsilon.$$

It remains to show that σ_N can be chosen as a cyclic permutation. Increasing N if necessary, and using the hypothesis that the grids are well ordered, we number the cubes such that the diameter of the union of two consecutive cubes is smaller than ε. Then we use Lemma 5.7 to obtain a cyclic permutation σ'_N whose distance to σ_N is smaller than ε. Thus we have found $N_0 \in \mathbf{N}$ and for all $N \geq N_0$ a cyclic permutation σ'_N of E_N whose distance to f is smaller than 3ε. □

Remark 5.8. In this proof, the marriage lemma can be replaced by Birkhoff's theorem about bistochastic matrices: we define the transition matrix M for the partition of X by the cubes by $m_{i,j} = \lambda\Big(C_i \cap f(C_j)\Big)/\lambda(C_i)$; the hypothesis of preservation of λ implies that M is bistochastic [4]. Then, Birkhoff's theorem ensures that bistochastic matrices form a convex set whose extremal points are permutation matrices; thus M can be written as a convex combination of permutation matrices and each one of the corresponding permutation provides an approximating permutation for f.

5.3 Individual behaviour of discretizations

Now we have set the technical Theorems 5.2 and 5.5, we can establish results concerning the behaviour of discretizations of a generic conservative homeomorphism. Here we study *individual* behaviour of discretizations, *i.e.* properties about only one order of discretization. As has already been said, applying Theorem 5.2, it suffices to find dense types of approximation to obtain properties about discretizations. In practice, these dense types of approximations are obtained from variations of Lax's theorem (Theorem 5.5).

Recall that the sequence $(E_N)_{n \in \mathbf{N}}$ of discretization grids is well distributed and well ordered (see Definition 3.12), we denote by f_N the discretization of a homeomorphism f and by $\Omega(f_N)$ the recurrent set of f_N (*i.e.* the union of periodic orbits of f_N).

We will show that for a lot of simple dynamical properties (P) about finite maps and for a generic conservative homeomorphism f, infinitely many discretizations f_N satisfy (P) as well as infinitely many discretizations satisfy its contrary. For instance, for

4. That is, its entries are nonnegative and the sum of the elements of each of its columns and rows is equal to 1.

a generic homeomorphism f, the recurrent set $\Omega(f_N)$ is sometimes as large as possible, *i.e.* $\Omega(f_N) = E_N$ (Corollary 5.9), sometimes very small (Corollary 5.15) and even better sometimes the number of elements of the image of E_N is small (Corollary 5.20). In the same way stabilization time [5] is sometimes zero (Corollary 5.9, for example), sometimes around $\mathrm{Card}(E_N)$ (Corollary 5.36). Finally, concerning the dynamics of $f_{N|\Omega(f_N)}$, sometimes it is a cyclic permutation (Corollary 5.9) or a bicyclic permutation (Corollary 5.13), sometimes it has many orbits (Corollary 5.22)...

Firstly, we deduce directly from Lax's theorem that cyclic permutations of the sets E_N form a dense type of approximation in $\mathrm{Homeo}(X, \lambda)$. Combining this with Theorem 5.2, we obtain directly:

Corollary 5.9 (Miernowski, [Mie05])**.** *For a generic homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$, for every $N_0 \in \mathbf{N}$, there exists $N \geq N_0$ such that f_N is a cyclic permutation* [6].

This theorem states that for every generic conservative homeomorphism, there exists a subsequence of discretizations which are "transitive". Recall that generic homeomorphism are transitive (see [Oxt37]). So, in a certain sense, transitivity can be detected on discretizations. Remark that this result implies that the discretizations of a generic conservative homeomorphism do not behave like typical random maps, as for a random map of a set with q elements the average number of periodic orbits is asymptotically $\log q$ (see for example [Bol01, XIV.5]).

There exists a variation of Lax's Theorem for bicyclic permutations, which are permutations having exactly two orbits whose lengths are relatively prime (see [Gui12, lemme 2.9 page 28]); this variation leads to a proof of genericity in $\mathrm{Homeo}(X, \lambda)$ of topological weak mixing (see for example [AP00] or [Gui12, Part 2.4]). In the following we define a discrete analogue to topological weak mixing and state that this property occurs infinitely often on discretizations of a generic homeomorphism.

Definition 5.10. A homeomorphism f is said to be *topologically weakly mixing* if for all non-empty open sets $(U_i)_{i \leq M}$ and $(U'_i)_{i \leq M}$, there exists $m \in \mathbf{N}$ such that $f^m(U_i) \cap U'_i$ is non-empty for all $i \leq M$.

The proof of the genericity of topological weak mixing starts by an approximation of every conservative homeomorphism by another having ε-dense periodic orbits whose lengths are relatively prime. The end of the proof primarily involves the use of Baire's theorem and Bézout's identity. In the discrete case, the notion of weak mixing is replaced by the following.

Definition 5.11. Let $\varepsilon > 0$. A finite map σ_N is said ε-*topologically weakly mixing* if for all $M \in \mathbf{N}$ and all balls $(B_i)_{i \leq M}$ and $(B'_i)_{i \leq M}$ with diameter ε, there exists $m \in \mathbf{N}$ such that for all i

$$\sigma_N^m(B_i \cap E_N) \cap (B'_i \cap E_N) \neq \emptyset.$$

The first step of the proof is replaced by the following variation of Lax's theorem:

5. That is, the smallest integer k such that $f_N^k(E_N) = \Omega(f_N)$.

6. In fact, T. Miernowski proves "permutation" but his arguments, combined with S. Alpern's improvement of the Lax's theorem, show "cyclic permutation".

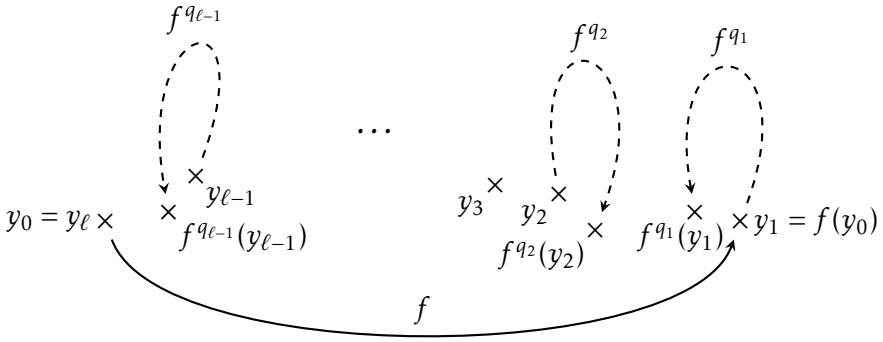

Figure 5.1: Construction of the sequence $(y_m)_{1\leq m\leq \ell}$

Proposition 5.12 (First variation of Lax's theorem)**.** *Let $f \in \mathrm{Homeo}(X, \lambda)$ be a homeomorphism such that all the iterates of f are topologically transitive. Then for all $\varepsilon > 0$ and all $M \in \mathbf{N}^*$, there exists $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$, there exists $\sigma_N : E_N \to E_N$ which has M ε-dense periodic orbits whose lengths are pairwise relatively prime, and such that $d_N(f, \sigma_N) < \varepsilon$.*

Proof of Proposition 5.12. We prove the proposition in the case where $M = 2$, the other cases being easily obtained by an induction. Let $\varepsilon > 0$ and f be a homeomorphism whose all iterates are topologically transitive. Then there exists $x_0 \in X$ and $p \in \mathbf{N}^*$ such that $\{x_0, \cdots, f^{p-1}(x_0)\}$ is ε-dense and $d(x_0, f^p(x_0)) < \varepsilon/2$. Since transitive points of f^p form a dense G_δ subset of X, while the orbit of x_0 form a F_σ set with empty interior, the set of points whose orbit under f^p is dense and disjoint from that of x_0 is dense. So we can pick such a transitive point y_0. Set $y_1 = f(y_0)$. Then there exists a multiple q_1 of p such that the orbit $\{y_1, \cdots, f^{q_1-1}(y_1)\}$ is ε-dense and $d(y_1, f^{q_1}(y_1)) < \varepsilon/2$. Again, by density, we can choose a transitive point y_2 whose orbit is disjoint from that of x_0 and y_1, with $d(y_1, y_2) < \varepsilon/2$ and $d(y_0, y_1) - d(y_0, y_2) > \varepsilon/4$. Then there exists a multiple q_2 of p such that $d(y_2, f^{q_2}(y_2)) < \varepsilon/2$. And so on, we construct a sequence $(y_m)_{1\leq m\leq \ell}$ such that (see figure 5.1):

(i) for all m, there exists $q_m > 0$ such that $p | q_m$ and $d(y_m, f^{q_m}(y_m)) < \varepsilon/2$,

(ii) the orbits $\{x_0, \cdots, f^{p-1}(x_0)\}$ and $\{y_m, \cdots, f^{q_m-1}(y_m)\}$ (m going from 0 to $\ell - 1$) are pairwise disjoints,

(iii) for all m, $d(y_m, y_{m+1}) < \varepsilon/2$ and $d(y_0, y_m) - d(y_0, y_{m+1}) > \varepsilon/4$,

(iv) $y_\ell = y_0$.

Let σ_N be a finite map given by Lax's theorem. For all N large enough, σ_N satisfies the same properties (i) to (iii) than f. Changing σ_N at the points $\sigma_N^{q_m-1}((y_m)_N)$ and $\sigma_N^{p-1}((x_0)_N)$, we obtain a finite map σ'_N such that $\sigma'^{q_N}_N((y_m)_N) = (y_{m+1})_N$ and $\sigma'^{p}_N((x_0)_N) = (x_0)_N$. Thus the orbit of $(x_0)_N$ under σ'_N is 2ε-dense and has period p and the orbit of $(y_0)_N$ under σ'_N is 2ε-dense, disjoint from which of $(x_0)_N$ and has period $1 + q_1 + \cdots + q_{\ell-1}$ relatively prime to p. □

Corollary 5.13. *For a generic homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$, for all $\varepsilon > 0$ and all $N_0 \in \mathbf{N}$, there exists $N \geq N_0$ such that f_N is ε-topologically weakly mixing.*

Proof of Corollary 5.13. Again, we prove the corollary in the case where $M = 2$, other cases being easily obtained by induction. Let $\varepsilon > 0$ and $N_0 \in \mathbf{N}$. All iterates of a generic homeomorphism f are topologically transitive: it is an easy consequence of the genericity of transitivity (see for example Corollary 5.9 or [Gui12, Theorem 2.11]); we pick such a homeomorphism. Combining Theorem 5.2 and Proposition 5.12, we obtain $N \geq N_0$ such that f_N has two $\varepsilon/3$-dense periodic orbits whose lengths p et q are coprime. We now have to prove that f_N is ε-topologically weakly mixing. Let B_1, B_2, B_1' and B_2' be balls with diameter ε. Since each one of these orbits is $\varepsilon/3$-dense, there exists $x_N \in X$ which is in the intersection of the orbit whose length is p and B_1, and $y_N \in X$ which is in the intersection of the orbit whose length is q and B_2. Similarly, there exists two integers a and b such that $f_N^a(x_N) \in B_1'$ and $f_N^b(y_N) \in B_2'$.

Recall that we want to find a power of f_N which sends both x_N in B_1' and y_N in B_2'. It suffices to pick $m \in \mathbf{N}$ such that $m = a + \alpha p = b + \beta q$. Bézout's identity states that there exists two integers α and β such $\alpha p - \beta q = b - a$. Set $m = a + \alpha p$, adding a multiple of pq if necessary, we can suppose that m is positive. Thus $f_N^m(x_N) \in B_1'$ and $f_N^m(y_N) \in B_2'$. □

For now, the two approximation types we studied concern analogues of properties that are generic among $\mathrm{Homeo}(X, \lambda)$. We now show that some discrete analogues of properties that are not generic among conservative homeomorphisms also occur infinitely often in the discretizations of generic homeomorphisms. The second variation of Lax's theorem concerns the approximation of applications whose recurrent set is small.

Proposition 5.14 (Second variation of Lax's theorem)**.** *Let $f \in \mathrm{Homeo}(X, \lambda)$. Then for all $\varepsilon, \varepsilon' > 0$, there exists $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$, there exists a map $\sigma_N : E_N \to E_N$ such that $d_N(f, \sigma_N) < \varepsilon$ and*

$$\frac{\mathrm{Card}(\Omega(\sigma_N))}{\mathrm{Card}(E_N)} = \frac{\mathrm{Card}(\Omega(\sigma_N))}{q_N} < \varepsilon',$$

and such that E_N is made of a unique (pre-periodic) orbit of σ_N.

Proof of Proposition 5.14. Let $f \in \mathrm{Homeo}(X, \lambda)$, $\varepsilon > 0$ and a recurrent point x of f. There exists $\tau \in \mathbf{N}^*$ such that $d(x, f^\tau(x)) < \frac{\varepsilon}{8}$; this inequality remains true for fine enough discretizations: there exists $N_1 \in \mathbf{N}$ such that if $N \geq N_1$, then

$$d(x, x_N) < \frac{\varepsilon}{8}, \quad d\Big(f^\tau(x), f^\tau(x_N)\Big) < \frac{\varepsilon}{8} \quad \text{and} \quad \frac{\tau}{q_N} < \varepsilon'.$$

Using the modulus of continuity of f^τ and Lax's theorem (Theorem 5.5), we obtain an integer $N_0 \geq N_1$ such that for all $N \geq N_0$, there exists a cyclic permutation σ_N of E_N such that $d_N(f, \sigma_N) < \frac{\varepsilon}{2}$ and $d_N(f^\tau, \sigma_N^\tau) < \frac{\varepsilon}{8}$. Then

$$\begin{aligned} d(x_N, \sigma_N^\tau(x_N)) \leq & d(x_N, x) + d\Big(x, f^\tau(x)\Big) + d\Big(f^\tau(x), f^\tau(x_N)\Big) \\ & + d\Big(f^\tau(x_N), \sigma_N^\tau(x_N)\Big) \\ < & \frac{\varepsilon}{2}. \end{aligned}$$

We compose σ_N by the (non bijective) application mapping $\sigma_N^{\tau-1}(x_N)$ on x_N and being identity anywhere else (see Figure 5.2), in other words we consider the application

$$\sigma_N'(x) = \begin{cases} x_N & \text{if } x = \sigma_N^{\tau-1}(x_N) \\ \sigma_N(x) & \text{otherwise.} \end{cases}$$

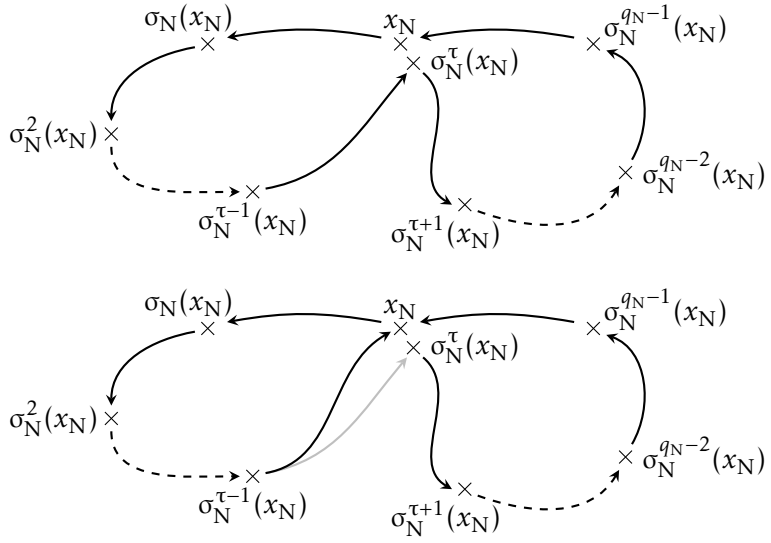

Figure 5.2: Modification of a cyclic permutation in the proof of Proposition 5.14

The map σ'_N has a unique injective orbit whose associated periodic orbit $\Omega(\sigma'_N)$ has length τ (it is $(x_N, \sigma_N(x_N), \cdots, \sigma_N^{\tau-1}(x_N))$). Since $d(f, \sigma'_N) < \varepsilon$, the map σ'_N verifies the conclusions of the proposition. □

A direct application of Theorem 5.2 leads to the following corollary.

Corollary 5.15. *For a generic homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$,*

$$\underline{\lim}_{N\to+\infty} \frac{\mathrm{Card}(\Omega(f_N))}{\mathrm{Card}(E_N)} = 0.$$

Specifically for all $\varepsilon > 0$ and all $N_0 \in \mathbf{N}$, there exists $N \geq N_0$ such that $\mathrm{Card}(\Omega(f_N))/\mathrm{Card}(E_N) < \varepsilon$ and such that E_N is made of a unique (pre-periodic) orbit of f_N.

We now improve this corollary in stating that for a generic homeomorphism, there exists $C > 0$ such that we have $\mathrm{Card}(\Omega(f_N)) \leq C$ for an infinite number of orders N; in particular these discretizations are highly non transitive, which is the opposite behaviour to the dynamics of a generic homeomorphism.

The same kind of idea than in the proof of Proposition 5.14 leads to the following variation of Lax's theorem.

Proposition 5.16. *Let $f \in \mathrm{Homeo}(X, \lambda)$ having at least one periodic point x of period p. Then for all $\varepsilon > 0$, there exists $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$, there exists an application $\sigma_N : E_N \to E_N$ with $d_N(f, \sigma_N) < \varepsilon$, such that E_N is made of a unique (pre-periodic) orbit of σ_N and such that the unique periodic orbit of f_N is of length p and ε-shadows the f-orbit of x.*

Proof of proposition 5.16. Simply replace the recurrent point by a periodic point of period p in the proof of Proposition 5.14. □

We will use this proposition in Section 5.5 to obtain Theorem 5.36. In particular, it will imply the following statement.

Corollary 5.17. *For a generic homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$, for every period p of a periodic point of f and for infinitely many integers N, f_N has a unique periodic orbit, whose length is p, and such that E_N is covered by a single (pre-periodic) orbit of f_N.*

Remark 5.18. In particular, if p_0 denotes the minimal period of the periodic points of f, then there are infinitely many discretizations such that the cardinality of their recurrent set is equal to p_0. Note that however, the shortest period of periodic points of generic homeomorphisms has no global upper bound in $\mathrm{Homeo}(X, \lambda)$: for example, for all $p \in \mathbf{N}$, there is an open set of homeomorphisms of the torus without periodic point of period less than p (*e.g.* the neighbourhood of an irrational rotation) and this property remains true for discretizations.

Corollary 5.17 states that for a generic homeomorphism, $\mathrm{Card}(\Omega(f_N))/\mathrm{Card}(E_N)$ is as small as possible for an infinite number of orders N. The next result states that this loss of injectivity can even occur from the first iteration of f_N.

Proposition 5.19 (Third variation of Lax's theorem)**.** *Let $f \in \mathrm{Homeo}(X, \lambda)$ and $\vartheta : \mathbf{N} \to \mathbf{R}_+^*$ a map which tends to $+\infty$ at $+\infty$. Then for all $\varepsilon > 0$, there exists $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$, there exists a map $\sigma_N : E_N \to E_N$ such that $\mathrm{Card}(\sigma_N(E_N)) < \vartheta(N)$ and $d_N(f, \sigma_N) < \varepsilon$.*

Proof of Proposition 5.19. Let $f \in \mathrm{Homeo}(X, \lambda)$, $\vartheta : \mathbf{N} \to \mathbf{R}_+^*$ a map which tends to $+\infty$ at $+\infty$ and $\varepsilon > 0$. By Lax's theorem (Theorem 5.5) there exists $N_1 \in \mathbf{N}$ such that for all $N \geq N_1$, there exists a cyclic permutation $\sigma_N : E_N \to E_N$ whose distance to f is smaller than $\varepsilon/2$. For $N \geq N_1$, set $\sigma'_N = P_{N_1} \circ \sigma_N$. Increasing N_1 if necessary we have $d(f, \sigma'_N) < \varepsilon$, regardless of N. Moreover $\mathrm{Card}(\sigma'_N(E_N)) \leq q_{N_1}$; if we choose N_0 large enough such that for all $N \geq N_0$ we have $q_{N_1} < \vartheta(N)$, then $\mathrm{Card}(\sigma'_N(E_N)) \leq \vartheta(N)$. We have shown that the map σ'_N satisfies the conclusions of proposition for all $N \geq N_0$. □

Corollary 5.20. *Let $\vartheta : \mathbf{N} \to \mathbf{R}_+^*$ a map which tends to $+\infty$ at $+\infty$. Then for a generic homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$,*

$$\lim_{N \to +\infty} \frac{\mathrm{Card}(f_N(E_N))}{\vartheta(N)} = 0.$$

In particular, generically, $\underline{\lim} \frac{\mathrm{Card}(f_N(E_N))}{\mathrm{Card}(E_N)} = 0$.

Proof of Corollary 5.20. Remark that if we replace $\vartheta(N)$ by $\sqrt{\vartheta(N)}$, it suffices to prove that for a generic homeomorphism, $\underline{\lim} \frac{\mathrm{Card}(f_N(E_N))}{\vartheta(N)} \leq 1$. This is easily obtained by combining Theorem 5.2 and Proposition 5.19. □

So far all variations of Lax's theorem have constructed finite maps with a small number of cycles. With the additional assumption that the sequence of grids is sometimes self similar, we show a final variation of Lax's theorem, approaching every homeomorphism by a finite map with a large number of orbits.

Proposition 5.21 (Fourth variation of Lax's theorem)**.** *Assume that the sequence of grids $(E_N)_{N \in \mathbf{N}}$ is sometimes self similar. Let $f \in \mathrm{Homeo}(X, \lambda)$ and $\vartheta : \mathbf{N} \to \mathbf{R}$ such that $\vartheta(N) = o(q_N)$. Then for all $\varepsilon > 0$, there exists $N_1 \in \mathbf{N}$ such that for all $N \geq N_1$, there exists a permutation σ_N of E_N such that $d_N(f, \sigma_N) < \varepsilon$ and that the number of cycles of σ_N is greater than $\vartheta(N)$. Moreover, $\vartheta(N)$ of these cycles of σ_N are conjugated to a cyclic permutation of E_{N_0} by bijections whose distance to identity is smaller than ε.*

Proof of Proposition 5.21. Let $\varepsilon > 0$, for all $N_0 \in \mathbf{N}$ large enough, Lax's theorem gives us a cyclic permutation σ'_{N_0} of E_{N_0} whose distance to f is smaller than ε. Since the grids are sometimes self similar, there exists $N_1 \in \mathbf{N}$ such that for all $N \geq N_1$, the set E_N contains $q_N/q_{N_0} \geq \vartheta(N)$ disjoint subsets $\widetilde{E}^j_N$, each one conjugated to a grid E_{N_0} by a bijection h_j whose distance to identity is smaller than ε. On each $\widetilde{E}^j_N$, we define σ_N as the conjugation of σ'_{N_0} by h_j; outside these sets we just pick σ_N such that $d_N(f, \sigma_N) < \varepsilon$. Since the distance between h_j and identity is smaller than ε, we have $d_N(f, \sigma_N) < 2\varepsilon$. Moreover, σ_N has at least $\vartheta(N)$ cycles which are conjugated to a cyclic permutation of E_{N_0} by the maps h_j; this completes the proof. □

The application of Theorem 5.2 leads to the following corollary, which ensures that an infinite number of discretizations f_N have a lot of periodic orbits.

Corollary 5.22. *We still assume that the sequence of grids $(E_N)_{N\in\mathbf{N}}$ is sometimes self similar. Let $\vartheta : \mathbf{N} \to \mathbf{R}$ such that $\vartheta(N) = o(q_N)$. Then for a generic homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$ and for infinitely many integers N, the discretization f_N of f has at least $\vartheta(N)$ periodic orbits (which all have the same period).*

If we compose the map obtained by Proposition 5.21 by an appropriate map of the finite set $\{h_i(x_0)\}_i$ into itself (x_0 being fixed), we can prove the following result.

Proposition 5.23. *Let $f \in \mathrm{Homeo}(X, \lambda)$, $\varepsilon > 0$ and $]a, b[\subset [0, 1]$. Then there exists $N_1 \in \mathbf{N}$ such that for all $N \geq N_1$, there exists a permutation σ_N of E_N such that $d_N(f, \sigma_N) < \varepsilon$ and that the degree of recurrence of σ_N (that is, the ratio between the cardinality of the recurrent set of σ_N and the cardinality of E_N) belongs to $]a, b[$.*

This proposition implies trivially the following corollary.

Corollary 5.24. *For a generic conservative homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$, the sequence $D(f_N)$ of the degrees of recurrence of the discretizations accumulates on the whole segment $[0, 1]$.*

5.4 Average behaviour of discretizations

We now want to study the average behaviour of discretizations of a generic homeomorphism. For example, we could imagine that even if for a generic homeomorphism f, the event "f_N is a cyclic permutation" appears for infinitely many orders N, it is statistically quite rare.

More precisely, we study the frequency of occurrence of properties related to the discretizations of generic homeomorphisms: given a property (P) concerning discretizations, we are interested in the behaviour of the proportion between 1 and M of discretizations satisfying the property (P), when M goes to infinity. For this study, we assume that the sequence of discretization grids is always self similar (which is true for example for the torus equipped with discretizations upon uniform grids of orders powers of an integer, see Section 3.7). This prevents us from having to deal with tricky arithmetic problems about overlay of grids.

Definition 5.25. Let $f \in \mathrm{Homeo}(X, \lambda)$. We say that a property (P) concerning discretizations is *satisfied in average* if for all $N_0 \in \mathbf{N}$ and all $\varepsilon > 0$, there exists $N \geq N_0$ such that

the proportion of integers $M \in \{0, \cdots, N\}$ such that f_M satisfies (P) is greater than $1-\varepsilon$, *i.e.*

$$\overline{\lim_{N\to+\infty}} \frac{1}{N+1} \operatorname{Card}\Big\{M \in \{0,\cdots,N\} \mid f_M \text{ satisfies (P)}\Big\} = 1.$$

We will show that most of the dynamical properties studied in the previous section are actually satisfied on average for generic homeomorphisms. To start with, we set out a technical lemma:

Lemma 5.26. *Let $\mathscr{T}$ be a dense type of approximation in* $\mathrm{Homeo}(X,\lambda)$. *Then for a generic homeomorphism $f \in \mathrm{Homeo}(X,\lambda)$, for all $\varepsilon > 0$ and all $\alpha > 0$, the property* (P)*: "E_N contains at least α disjoints subsets which fills a proportion greater than $1-\varepsilon$ of E_N, each of them stabilized by f_N and such that the restriction of f_N to each one is conjugated to a map of $\mathscr{T}$ by a bijection whose distance to identity is smaller than ε" is satisfied in average.*

In practice, this lemma provides many properties satisfied on average, for instance:
- quantitative properties on discretizations, such as possessing at least M periodic orbits;
- properties of existence of sub-dynamics on discretizations, such as possessing at least one dense periodic orbit.

Proof of Lemma 5.26. Let us consider the set

$$\mathscr{C} = \bigcap_{\substack{\varepsilon>0\\ N_0\in\mathbf{N}}} \bigcup_{N\geq N_0} \left\{ \begin{array}{c} f \in \mathrm{Homeo}(X,\lambda) \,\Big|\\ \frac{1}{N+1}\operatorname{Card}\Big\{M \in \{0,\cdots,N\} \mid f_M \text{ satisfies (P)}\Big\} > 1-\varepsilon \end{array} \right\}.$$

We want to show that $\mathscr{C}$ contains a dense G_δ of $\mathrm{Homeo}(X,\lambda)$. The set $\mathscr{C}$ is a G_δ of the generic set $\bigcap_{N\in\mathbf{N}} \mathscr{D}_N$, it suffices to prove that it is dense in $\bigcap_N \mathscr{D}_N$. Let $f \in \mathrm{Homeo}(X,\lambda)$, $N_0 \in \mathbf{N}$, $\delta > 0$ and $\varepsilon > 0$. To prove it we want to find a homeomorphism g whose distance to f is smaller than δ and an integer $N \geq N_0$ such that

$$\frac{1}{N+1}\operatorname{Card}\Big\{M \in \{0,\cdots,N\} \mid g_M \text{ satisfies (P)}\Big\} > 1-\varepsilon.$$

It is simply obtained in combining the density of the type of approximation $\mathscr{T}$ and the fact that the sequence of grids is always self similar. □

This lemma allows us to obtain some properties about the average behaviour of discretizations. For instance here is an improvement of Corollary 5.36.

Corollary 5.27. *For a generic homeomorphism $f \in \mathrm{Homeo}(X,\lambda)$, the property "$f_N$ has a ε-dense periodic orbit and the cardinality of $\Omega(f_N)$ satisfies $\operatorname{Card}(\Omega(f_N)) = o(q_N)$" is satisfied in average.*

Or an improvement of Corollary 5.13.

Corollary 5.28. *For a generic homeomorphism $f \in \mathrm{Homeo}(X,\lambda)$ and for all $\varepsilon > 0$, the property "f_N is ε-topologically weakly mixing" (see Definition 5.11) is satisfied in average.*

Or even an improvement of Corollary 5.20.

Corollary 5.29. *For a generic homeomorphism $f \in \mathrm{Homeo}(X,\lambda)$ and for all $\varepsilon > 0$, the property "$\frac{\operatorname{Card}(f_N(E_N))}{\operatorname{Card}(E_N)} < \varepsilon$" is satisfied in average.*

And an improvement of Corollary 5.22.

Corollary 5.30. *For a generic homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$ and for all $M \in \mathbf{N}$, property "f_N has at least M periodic orbits" is satisfied in average.*

However, note that the most simple property about discretizations, *i.e.* being a cyclic permutation, cannot be proved by using Lemma 5.26. To do this, we need a slightly more precise result, that requires the hypothesis that the grids are always strongly self similar.

Theorem 5.31. *For a generic homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$, the property "$f_N$ is a cyclic permutation" is satisfied in average.*

Note that this implies that most of the discretizations of f does not behave like a random map of a set of cardinality q_N, as a random map of a set with q elements has in average $\log q$ periodic orbits (see for example [Bol01, XIV.5]).

Lemma 5.32. *Let $f \in \mathrm{Homeo}(X, \lambda)$ and $\varepsilon > 0$. The sequence of grids is supposed to be always strongly self similar. Then there exists $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$, there exists a permutation σ_N of E_N, such that $d_N(f, \sigma_N) < \varepsilon$, $\sigma_{N|E_{N_0}}$ is a cyclic permutation of E_{N_0} and for all $M \in \{N_0, \cdots, N-1\}$, σ_N is a cyclic permutation of $E_{M+1} \setminus E_M$.*

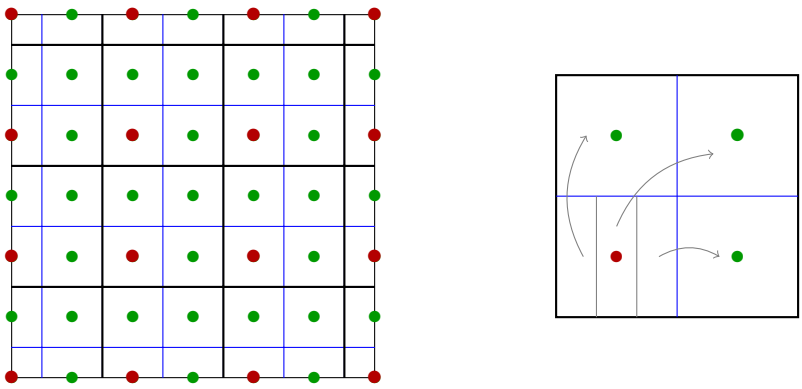

Figure 5.3: Construction[7] of Lemma 5.32 for two grids of orders 3 and 6 and zoom around a point of a grid of order 3

Proof of Lemma 5.32. Applying Lax's theorem (Theorem 5.5), we can find an integer N_0 and a cyclic permutation σ_{N_0} of E_{N_0} such that $d_{N_0}(f, \sigma_{N_0}) < \varepsilon$. As the sequence of grids is always strongly self similar, we can suppose that N_0 is big enough to verify the conclusions of Definition 3.13.

Let us observe what happens for the order $N_0 + 1$. We will define an application σ'_{N_0+1} on E_{N_0+1}, which will be close to the homeomorphism f. On E_{N_0}, we define σ'_{N_0+1} as being equal to σ_{N_0}. The idea is to repeat the proof of Lax's theorem for the elements of $E_{N_0+1} \setminus E_{N_0}$. To do that, we have to find a partition of X into sets with the same measure, such that every element A_x of this partition is one to one associated to a point x of $E_{N_0+1} \setminus E_{N_0}$, and has "small" diameter. So it suffices to split equitably the mass of the cubes E_{N_0} to the other cubes.

7. Do not forget that we identify some points of the boundary!

For this purpose, we cut each cube of E_{N_0+1} corresponding to a point x_{N_0} of the grid E_N into $\alpha_N - 1$ subsets of the same measure $1/(q_{N_0+1}(\alpha_N - 1))$ (see Figure 5.3). Each of these subsets is assigned to the cube corresponding to the point $h_i(x_{N_0})$. For each $x \in E_{N_0+1} \setminus E_{N_0}$, we define A_x as the union of the cube C_x with the subset that is associated to it. Then all the sets A_x have the same measure and have a diameter smaller than twice the diameter of a cube of E_{N_0+1} plus ε. We can now apply Lax's theorem to the homeomorphism f and the partition $\{A_x\}_{x\in E_N}$ of X, this partition being numbered the same way as the partition E_{N_0+1} (some numbers are skipped); this gives a cyclic permutation of $E_{N_0+1} \setminus E_{N_0}$ which is close to f. This cyclic permutation defines the permutation σ'_{N_0+1} where it was not yet. This σ'_{N_0+1} is close to f and permutes cyclically the elements of E_{N_0} and these of $E_{N_0+1} \setminus E_{N_0}$.

We finish the proof by an induction, iterating $N - N_0$ times. □

Corollary 5.33. *Let $f \in \mathrm{Homeo}(X, \lambda)$ and $\varepsilon > 0$ (we recall that the sequence of grids is supposed to be always strongly self similar). Then there exists $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$, there exists a homeomorphism g whose distance to f is smaller than ε such that for all $M \in \{N, \cdots, N_0\}$, the discretization g_M is a cyclic permutation of E_M. Moreover, this property can be supposed to be verified on a neighbourhood of g.*

Proof of Corollary 5.33. Let $f \in \mathrm{Homeo}(X, \lambda)$ and $\varepsilon > 0$. Lemma 5.32 gives us an integer $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$, there exists a permutation σ_N of E_N, whose distance to f is smaller than $\varepsilon/2$, such that σ_N is a cyclic permutation of E_{N_0} and for all $M \in \{N_0+1, \cdots, N\}$, σ_M is a cyclic permutation of $E_M \setminus E_{M-1}$. Set $N \geq N_0$, the idea is to modify slightly the orbits of σ_N such that σ_N becomes a cyclic permutation on all the sets E_M.

We begin by choosing two points $x_{N_0} \in E_{N_0}$ and $x'_{N_0+1} \in E_{N_0+1}$ such that x'_{N_0+1} belongs to the cube corresponding to the point x_{N_0}. Then, we modify σ_N in interchanging the points x_{N_0} et x'_{N_0+1}, in other words we set

$$\sigma_N^{N_0}(x) = \begin{cases} \sigma_N(x'_{N_0+1}) & \text{if } x = x_{N_0} \\ \sigma_N(x_{N_0}) & \text{if } x = x'_{N_0+1} \\ \sigma_N(x) & \text{otherwise.} \end{cases}$$

Thus, $\sigma_N^{N_0}$ is a cyclic permutation of E_{N_0+1} and the discretization of order N_0+1 of $\sigma_N^{N_0}$ is also cyclic. We build the same way some maps $\sigma_N^{N_0+1}, \cdots, \sigma_N^N$ in interchanging the images of two adjacent points lying in the grids E_M et E_{M+1}. Then, for all $M \in \{N_0, \cdots, N\}$, the discretization of σ_N^{M-1} on E_M is a cyclic permutation. Moreover, the distance between the map σ_N^N and σ_N is smaller than $\varepsilon/2$. Combined with Proposition 3.3, this allows us to interpolate σ_N^N to a homeomorphism g whose distance to f is smaller than ε such that for all $M \in \{N_0, \cdots, N\}$, the discretization of order M of g is a cyclic permutation of E_M. A careful reader would notice that it may happen that in this construction, the discretizations of g are not uniquely defined; depending on the choices made during the definition of P_N on E'_N, these discretizations may not be cyclic permutations. If we want to avoid this problem, it suffices to modify slightly the map σ_N^M; moreover this ensures that the conclusions of the corollary are verified on a whole neighbourhood of g. □

Proof of Theorem 5.31. Let the set

$$\mathscr{C} = \bigcap_{\substack{\varepsilon>0 \\ N_0\in\mathbf{N}}} \bigcup_{N\geq N_0} \left\{ \begin{array}{c} f \in \mathrm{Homeo}(\mathbf{T}^n, \mathrm{Leb}) \mid \\ \frac{1}{N+1}\,\mathrm{Card}\left\{M \in \{0, \cdots, N\} \mid f_M \text{ cyclic permutation}\right\} > 1 - \varepsilon \end{array} \right\},$$

we want to show that $\mathscr{C}$ contains a G_δ dense subset of $\mathrm{Homeo}(X,\lambda)$. Let $f \in \mathrm{Homeo}(X,\lambda)$, $N_0 \in \mathbf{N}$, $\delta > 0$ and $\varepsilon > 0$. This is equivalent to find a homeomorphism g whose distance to f is smaller than δ, and an integer $N \geq N_0$, such that for all homeomorphism g' close enough to g, we have

$$\frac{1}{N+1}\mathrm{Card}\left\{M \in \{0,\cdots,N\} \mid g'_M \in P\right\} > 1-\varepsilon.$$

Such a homeomorphism g is simply given by Corollary 5.33 for $N \geq N_0$ such that $\varepsilon k_N > k_{N_0}$. □

Remark 5.34. However, the property of approximation by bicyclic permutations in average cannot be proven directly with this technique.

Remark 5.35. A simple calculation shows that everything that has been done in this section also applies to the behaviour of discretizations in average of Cesàro average, in average of average of Cesàro average etc., *i.e.* when studying quantities

$$\frac{1}{N_2+1}\sum_{N_1=0}^{N_2}\frac{1}{N_1+1}\mathrm{Card}\left\{M \in \{0,\cdots,N_1\} \mid f_M \text{ satisfies (P)}\right\},$$

$$\frac{1}{N_3+1}\sum_{N_2=0}^{N_3}\frac{1}{N_2+1}\sum_{N_1=0}^{N_2}\frac{1}{N_1+1}\mathrm{Card}\left\{M \in \{0,\cdots,N_1\} \mid f_M \text{ satisfies (P)}\right\}\cdots$$

5.5 Behaviour of all the discretizations

In the previous sections, we showed that the dynamical behaviour of discretizations depends drastically on the order of discretization: even when looking at the frequency a property appears on discretizations, a lot of different behaviours can occur. In contrast, the dynamics of a generic homeomorphism is well known (see for example [Gui12]) and "independent from the homeomorphism". We even have a 0-1 law on $\mathrm{Homeo}(X,\lambda)$ (see [GK98] or the final chapter of [Gui12]) which states that either a given ergodic property is generic among conservative homeomorphisms, or its contrary is generic. Thus, the dynamics of a generic homeomorphism and that of its discretizations seem quite of independent. In this chapter, we will see that the variations of the dynamics of the discretizations are as large as possible. Indeed, in general, from the convergence $d_N(f,f_N) \to 0$, it can be deduced by a compactness argument that the accumulation set of the dynamical invariants (for example, invariant measures) of f_N is included in that of f. We will see that in the generic case, the inverse inclusion is also true: every dynamical invariant of f is an accumulation point of the corresponding dynamical invariants of f_N. Thus, in a certain sense, it is possible to deduce some dynamical features of a generic homeomorphism from the corresponding dynamical features of *all* its discretizations. In other words, there is a shadowing property of the dynamical features of the homeomorphism by that of its discretizations: for each dynamical feature of the homeomorphism, its discrete analogues appear on an infinite number of discretizations.

For example, we will show that every periodic orbit of f is shadowed by a periodic orbit of an infinite number of f_N (Theorem 5.36). We will also show that every f-invariant measure is the limit of a subsequence of f_N-invariant measures (Theorem 5.43), and every f-invariant chain transitive compact set is the limit of a subsequence of f_N-invariant sets (Theorem 5.47). Moreover, in Chapter 6, we will give an application

of this convergence of all the dynamics of the discretizations to that of the homeomorphism. This will give an algorithm to obtain numerically the rotation set of a generic conservative homeomorphism: the upper limit (for the inclusion) of the rotation sets of the discretizations of a generic conservative homeomorphism of the torus is equal to the rotation set of the homeomorphism itself (see Corollary 6.23).

5.5.1 Periodic orbits

Firstly, we deduce from Proposition 5.16 that every periodic orbit of a generic homeomorphism is shadowed by a periodic orbit with the same period of an infinite number of discretizations. We will see later that this property of shadowing is true for a larger class of compact sets (namely the chain transitive invariant compact sets, see Theorem 5.47), but the following theorem is also concerned with the period of the periodic orbits of the discretizations. Moreover, the case of periodic orbits seems natural enough to be handled separately.

Theorem 5.36. *Let $f \in \mathrm{Homeo}(X, \lambda)$ be a generic homeomorphism. Then, for every periodic point x of period p for f, for every $\delta > 0$ and for every $N_0 \in \mathbf{N}$, there exists $N \geq N_0$ such that f_N has a unique periodic orbit, whose length is p, which δ-shadows the f-orbit of x, and such that E_N is covered by a single (pre-periodic) orbit of f_N (in particular, this implies that this periodic orbit attracts the whole set E_N).*

In particular, this theorem implies that it is possible to detect the periods of the periodic orbits of a generic homeomorphism from that of its discretizations.

Corollary 5.37. *Let f be a generic homeomorphism of $\mathrm{Homeo}(X, \lambda)$ and p be an integer. Then f has a periodic orbit of period p if and only if there exists infinitely many integers N such that f_N has a periodic orbit with period p.*

Proof of Corollary 5.37. The first implication is Corollary 5.36, and the other follows easily from a compactness argument. □

F. Daalderop and R. Fokkink have proved in [DF00] (see also [Gui12, Part 3.2], the proof is similar to that of Theorem 5.36) that for a generic element of $f \in \mathrm{Homeo}(X, \lambda)$, the set of f-periodic points is dense in X.

To prove Theorem 5.36, the concept of *persistent point* will allow us to define open properties about periodic points.

Definition 5.38. Let $f \in \mathrm{Homeo}(X)$. A periodic point x of f with period p is said *persistent* if for all neighbourhood U of x, there exists a neighbourhood $\mathscr{V}$ of f in Homeo(X) such that every $\widetilde{f} \in \mathscr{V}$ has a periodic point $\widetilde{x} \in U$ with period p.

Example 5.39. The endomorphism $h = \mathrm{Diag}(\lambda_1, \cdots, \lambda_n)$ of $\mathbf{R}^n$, with $\prod \lambda_i = 1$ and $\lambda_i \neq 1$ for all i, is measure-preserving and has a persistent fixed point at the origin (see for example [KH95, page 319]). Let s be a reflection of $\mathbf{R}^n$, the application $h \circ s$ is also measure-preserving and has a persistent fixed point at the origin.

To create persistent periodic points, we use the theorem of local modification of conservative homeomorphisms, which allows to replace locally a homeomorphism by another. Although it seems "obvious" and has an elementary proof in dimension two, the proof in higher dimensions uses the (difficult) *annulus theorem*. For more details we may refer to [DF00] or [Gui12, Part 3.1].

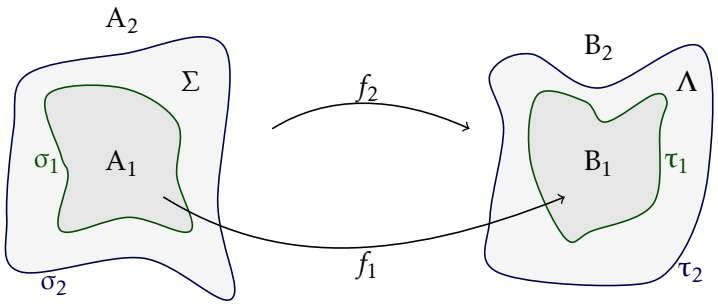

Figure 5.4: Technique of local modification

Theorem 5.40 (Local modification). *Let σ_1, σ_2, τ_1 and τ_2 be four bicollared embeddings*[8] *of $\mathbf{S}^{n-1}$ in $\mathbf{R}^n$, such that σ_1 is in the bounded connected component*[9] *of σ_2 and τ_1 is in the bounded connected component of τ_2. Let A_1 be the bounded connected component of $\mathbf{R}^n \setminus \sigma_1$ and B_1 the bounded connected component of $\mathbf{R}^n \setminus \tau_1$; Σ be the connected component of $\mathbf{R}^n \setminus (\sigma_1 \cup \sigma_2)$ having $\sigma_1 \cup \sigma_2$ as boundary and Λ the connected component of $\mathbf{R}^n \setminus (\tau_1 \cup \tau_2)$ having $\tau_1 \cup \tau_2$ as boundary; A_2 be the unbounded connected component of $\mathbf{R}^n \setminus \sigma_2$ and B_2 the unbounded connected component of $\mathbf{R}^n \setminus \tau_2$ (see Figure 5.4).*

Suppose that $\mathrm{Leb}(A_1) = \mathrm{Leb}(B_1)$ and $\mathrm{Leb}(\Sigma) = \mathrm{Leb}(\Lambda)$. Let $f_i : A_i \to B_i$ be two measure-preserving homeomorphisms such that either each one preserves the orientation, or each one reverses it. Then there exists a measure-preserving homeomorphism $f : \mathbf{R}^n \to \mathbf{R}^n$ whose restriction to A_1 equals f_1 and restriction to A_2 equals f_2.

Since this theorem is local, it can be applied to an open space O instead of $\mathbf{R}^n$ or, even better, together with the Oxtoby-Ulam theorem (Theorem 3.14), to any domain of chart of a manifold X instead of $\mathbf{R}^n$ and measure λ instead of Lebesgue measure.

By perturbing a homeomorphism, we can make every periodic point persistent, as stated by the following lemma.

Lemma 5.41. *Let $f \in \mathrm{Homeo}(X, \lambda)$, $\varepsilon > 0$, $\delta > 0$ and $P \in \mathbf{N}$. Then there exists $g \in \mathrm{Homeo}(X, \lambda)$ such that $d(f, g) < \delta$ and such that for every f-periodic point x of period smaller than P, there exists a persistent periodic point of g with the same period which ε-shadows the f-orbit of x.*

Proof of Lemma 5.41. Indeed, by compactness of the set of compact subsets of X endowed with Hausdorff topology, it suffices to prove it only for a finite number of f-periodic orbits. If for each of these periodic orbits, we perform the perturbation of f outside of a neighbourhood of the other periodic orbits, we reduce the proof to the case of a single periodic orbit. More formally, we choose a neighbourhood D of x such that the sets D, $f(D), \cdots, f^{p-1}(D)$ are pairwise disjoints (and disjoint from a neighbourhood of the others periodic orbits) and we locally replace f by the map $h \circ f^{-(p-1)}$ (where h is one of the two maps of Example 5.39, depending of whether $f^{-(p-1)}$ preserves orientation or not) in the neighbourhood of $f^{-1}(x)$, using the theorem of local modification (Theorem 5.40), such that f do not change outside the union of the sets $f^i(D)$. □

8. An embedding i of a manifold M in $\mathbf{R}^n$ is said *bicollared* if there exists an embedding $j : M \times [-1, 1] \to \mathbf{R}^n$ such that $j_{M \times \{0\}} = i$.

9. By the Jordan-Brouwer theorem the complement of a set which is homeomorphic to $\mathbf{S}^{n-1}$ has exactly two connected components: one bounded and one unbounded.

Proof of Theorem 5.36. We choose a basis $\{C_k\}$ of the topology of X. Now, for $p, N, k \in \mathbf{N}$, we define the set $\mathcal{U}_{p,N,k}$ as the set of $f \in \mathrm{Homeo}(X, \lambda)$ such that if f possesses a persistent periodic point of period p which intersects C_k, then f_N has a unique injective orbit whose periodic orbit intersects C_k and has length p. Note that the fact that we consider *persistent* periodic points ensures that these sets are open.

Then, the combination of Proposition 5.16, Lemma 5.4 and Lemma 5.41 implies that the set

$$\bigcap_{p\in\mathbf{N}^*} \bigcap_{M\in\mathbf{N}} \bigcup_{N\geq M} \mathcal{U}_{p,N}$$

is a dense G_δ of $\mathrm{Homeo}(X, \lambda)$, consisting of the homeomorphisms that verify the conclusions of the theorem. □

5.5.2 Invariant measures

We now try to obtain information about invariant measures of a generic homeomorphism from invariant measures of its discretizations. More precisely, given all the invariant measures of discretizations of a generic homeomorphism, what can be deduced about invariant measures of the initial homeomorphism? A first step in this study was taken in 2006 by T. Miernowski in [Mie06, Part 8]:

Proposition 5.42. *Let $f \in \mathrm{Homeo}(X)$. For all $N \in \mathbf{N}$ let $\gamma_N \subset E_N$ be a periodic cycle of f_N and ν_N the uniform probability measure on γ_N. Then, any limit point of the sequence of measures $(\nu_N)_N$ is f-invariant.*

In particular, if f is uniquely ergodic, whose unique invariant probability measure is denoted by μ, then the sequence of measures $(\nu_N)_N$ tends to μ.

The proof of this proposition essentially consists in an appropriate use of the compactness of the set of probability measures on X. We now state the same kind of result for *generic* homeomorphisms.

Theorem 5.43. *Let $f \in \mathrm{Homeo}(X, \lambda)$ be a generic homeomorphism and suppose that the sequence of grids $(E_N)_{N\in\mathbf{N}}$ is sometimes strongly self similar. Then for every f-invariant measure μ, there exists a subsequence ν_{N_k} of f_{N_k}-invariant periodic measures such that $(\nu_{N_k})_k$ tends to μ.*

This result is a particular case of the more general Theorem 5.45, for which we will need the following notation.

Notation 5.44. For $f \in \mathrm{Homeo}(X)$, we denote by $\mathcal{M}^f$ the set of Borel probability measures on X which are invariant under f, and by $\mathcal{M}_N^f$ the set of Borel probability measures on E_N which are invariant under f_N

Theorem 5.45. *Let $f \in \mathrm{Homeo}(X, \lambda)$ be a generic homeomorphism and suppose that the sequence of grids $(E_N)_{N\in\mathbf{N}}$ is sometimes strongly self similar. Let $\mathcal{M}_N$ be the set of probability measures on E_N that are invariant under f_N. Then for every closed and convex subset $\mathcal{M}$ of f-invariant measures, there exists a subsequence of $\{\mathcal{M}_N\}$ which tends to $\mathcal{M}$ for Hausdorff topology.*

Proposition 5.42 asserts that the upper limit of the sets of measures that are invariant under f_N is included in the set of f-invariant measures. As the set of f_N-invariant measures is always convex, Theorem 5.45 asserts that generically, the sets of f_N-invariant measures accumulate on all the possible sets of measures.

Before giving a detailed proof of Theorem 5.45, let us give its main arguments. An *ad hoc* application of Baire's theorem, together with compactness arguments, reduces the proof to that of the following variation of Lax's theorem, which asserts that for every finite collection $\mathscr{M}$ of measures that are invariant under f, there exists a homeomorphism g which is close to f and a big order of discretization N such that the set of periodic measures that are invariant under g_N is close to $\mathscr{M}$ for Hausdorff distance.

Lemma 5.46 (Ergodic variation of Lax's theorem)**.** *Suppose that the sequence of grids $(E_N)_{N\in\mathbf{N}}$ is sometimes strongly self similar. Then, for all $f \in \mathrm{Homeo}(X,\lambda)$, for all collection of f-invariant measures $\nu_1,\cdots,\nu_\ell$, for all $\varepsilon>0$ and all $k_0, N_0 \in \mathbf{N}$, there exists $g \in \mathrm{Homeo}(X,\lambda)$ and $N \geq N_0$ such that $d(f,g)<\varepsilon$, and that g_N possesses exactly ℓ periodic orbits, and the the invariant periodic measures $\{\nu^g_{1,N},\cdots,\nu^g_{\ell,N}\}$ supported by these periodic orbits satisfy $\mathrm{dist}(\nu_i,\nu^g_{i,N}) < \frac{1}{k_0}$ for all i. Moreover, we can suppose that these properties are still true on a whole neighbourhood of g.*

For the proof of Theorem 5.45, we will need a slightly weaker result; however this lemma will also be useful in the next part concerning the physical dynamics.

Before giving the formal proof of Lemma 5.46, we sketch its main arguments. Suppose first that $\ell=1$ and that ν_1 is ergodic. For this purpose we apply Birkhoff's theorem to f, ν_1 and a recurrent point x: for all M large enough,

$$\frac{1}{M}\sum_{k=0}^{M-1}\delta_{f^k(x)} \simeq \nu_1.$$

Since x is recurrent we can choose an integer M large enough such that $x \simeq f^M(x)$. First we approximate f by a cyclic permutation σ_N given by Lax's theorem, then we slightly modify σ_N into a map σ'_N by choosing $\sigma'_N(\sigma_N^M(x_N)) = x_N$, as in Proposition 5.14. The measure $\nu_N^{\sigma'_N}$ is the uniform measure on the orbit $x_N,\cdots,\sigma_N^{M-1}(x_N)$, so it is close to ν_1; we then just have to apply the proposition of extension of finite maps to finish the proof in the case where $\ell=1$ and ν_1 ergodic. The proof of the lemma when ν_1 is not ergodic but only invariant is obtained by approximating the invariant measure by a finite convex combination of ergodic measures. We use the hypothesis of self similarity of the grids to apply the previous process on each subgrid; we approximate each ergodic measure of the convex combination on a number of subgrids proportional to the coefficient of this measure in the convex combination. We then make a final perturbation to merge the periodic orbits of the different subgrids and apply the proposition of extension of finite maps; this proves the case $\ell=1$. Finally, the general case (when there are several measures to approximate) is obtained is using (again) the fact that the grids are sometimes strongly self similar.

Proof of Theorem 5.45. Recall that we denote by $\mathscr{P}$ the set of all Borel probability measures on X and that $\mathscr{P}$ is equipped with a distance dist compatible with the weak-* topology. The set of compact sets of Borel probability measures on X is equipped with the Hausdorff distance dist_H which makes this set compact; we consider a sequence $\{\mathscr{M}_\ell\}_\ell$ of compact sets of Borel probability measures which is dense for this topology. Thereafter homeomorphisms will be taken in the generic set

$$\bigcap_{N\in\mathbf{N}} \mathscr{D}_N,$$

made of homeomorphisms whose N-th discretization is uniquely defined for all N. Consider

$$\mathscr{A} = \bigcap_{(\ell,N_0,k_0)\in\mathbf{N}^3} \mathscr{O}_{\ell,N_0,k_0},$$

where $\mathscr{O}_{\ell,N_0,k_0}$ is the set of homeomorphisms $f \in \bigcap_{N\in\mathbf{N}} \mathscr{D}_N$ such that if there exists a closed convex set $\mathscr{M}^f$ of f-invariant measures such that $\mathrm{dist}_H(\mathscr{M}_\ell, \mathscr{M}^f) \leq \frac{1}{k_0}$, then there exists $N \geq N_0$ such that the set $\mathscr{M}_N^f$ of f_N-invariant measures satisfies $\mathrm{dist}_H(\mathscr{M}_\ell, \mathscr{M}_N^f) < \frac{2}{k_0}$.

We easily check that if $f \in \mathscr{A}$, then it satisfies the conclusions of the theorem.

The fact that the sets $\mathscr{O}_{\ell,N_0,k_0}$ are open follows from the upper semi-continuity of $f \mapsto \mathscr{M}^f$ and the fact that in the set $\bigcap_{N\in\mathbf{N}} \mathscr{D}_N$, the map $g \mapsto \mathscr{M}_N^g$ is locally constant.

It remains to show that the sets $\mathscr{O}_{\ell,N_0,k_0}$ are dense, it follows from Lemma 5.46 in the following way. Let $f \in \bigcap_{N\in\mathbf{N}} \mathscr{D}_N$ and ℓ, N_0 and k_0. If $\mathrm{dist}_H(\mathscr{M}_\ell, \mathscr{M}^f) > \frac{1}{k_0}$, there is nothing to prove. So we suppose that $\mathrm{dist}_H(\mathscr{M}_\ell, \mathscr{M}^f) \leq \frac{1}{k_0}$. By compactness, there exists a finite collection of f-invariant measures $\nu_1, \cdots, \nu_\ell$ such that $\mathscr{M}^f \subset \bigcup_i \mathrm{B}\left(\nu_i, \frac{1}{k_0}\right)$ (where $\mathrm{B}\left(\nu_i, \frac{1}{k_0}\right)$ denotes the set of measures whose distance to ν_i is smaller than $1/k_0$). So it suffices to find g close to f such that the set of g_N-invariant measures is included in $\bigcup_i \mathrm{B}\left(\nu_i, \frac{1}{k_0}\right)$, but this fact is implied by Lemma 5.46. □

Proof of Lemma 5.46. To begin with, we prove the lemma for only one measure $\nu = \nu_1$ which is ergodic. We want to show that there exists a homeomorphism g whose distance to f is smaller than ε, and an integer $N \geq N_0$ such that $\mathrm{dist}(\nu, \mu_X^{g_N}) < \frac{2}{k_0}$.

Since ν is ergodic, for all continuous map φ, by Birkhoff's theorem,

$$\frac{1}{M}\sum_{m=0}^{M-1} \delta_{f^m(x)} \underset{M\to+\infty}{\longrightarrow} \nu \tag{5.1}$$

for ν-almost every x. Let $x \in X$ be a recurrent point for f satisfying equation (5.1) (such points form ν-full measure set). There exists $M_0 \in \mathbf{N}$ such that for all $M \geq M_0$ we have

$$\mathrm{dist}\left(\frac{1}{M}\sum_{m=0}^{M-1} \delta_{f^m(x)}, \nu\right) \leq \frac{1}{2k_0}. \tag{5.2}$$

Since x is recurrent, there exists $\tau \geq M_0$ such that $d(x, f^\tau(x)) \leq \varepsilon/4$. Let σ_N be a map from E_N into itself given by Lax's theorem: it is a cyclic permutation and its distance to f is smaller than $\varepsilon/2$. For all N large enough the orbit $(\sigma_N^m(x_N))_{0\leq m\leq\tau}$ shadows the orbit $(f^m(x))_{0\leq m\leq\tau}$, thus $d(x_N, \sigma_N^\tau(x_N)) < \varepsilon/2$. Then, we "close" the orbit of x_N between the points x_N and $\sigma_N^\tau(x_N)$, *i.e.* we set (as in Proposition 5.14, see also Figure 5.2)

$$\sigma'_N(y_N) = \begin{cases} x_N & \text{if } y_N = \sigma_N^{\tau-1}(x_N) \\ \sigma_N(y_N) & \text{otherwise.} \end{cases}$$

Then $d_N(\sigma'_N, f) < \varepsilon$ and $(x_N, \cdots, \sigma_N'^{\tau-1}(x_N))$ is a periodic orbit for σ'_N whose basin of attraction is the whole set E_N. If ε is small enough, then

$$\mathrm{dist}\left(\frac{1}{\tau}\sum_{m=0}^{\tau-1} \delta_{f^m(x)}, \frac{1}{\tau}\sum_{m=0}^{\tau-1} \delta_{\sigma'^m_N(x_N)}\right) \leq \frac{1}{2k_0}. \tag{5.3}$$

Since the periodic orbit $(\sigma_N'^m(x))_{0\leq m\leq \tau}$ attracts E_N, for all M' large enough and $y_N \in E_N$ we have

$$\frac{1}{M'}\sum_{m=0}^{M'-1}\delta_{\sigma_N'^m(y_N)} \underset{M'\to\infty}{\longrightarrow} \frac{1}{\tau}\sum_{m=0}^{\tau-1}\delta_{\sigma_N'^m(x_N)}. \tag{5.4}$$

With Proposition 3.3, the same way as in Lemma 5.4, we construct a homeomorphism g' from the map σ_N' such that the discretization g_N' and σ_N' fit together, and such that g' and the restriction of g_N' to the orbit of x_N fit together. Since $d_N(f,\sigma_N') < \varepsilon$, we can furthermore assume that $d(f,g') < \varepsilon$. Thus, Equation (5.4) implies that

$$\mu_X^{g_N'} = \frac{1}{\tau}\sum_{m=0}^{\tau-1}\delta_{\sigma_N'^m(x_N)}. \tag{5.5}$$

We now have all the necessary estimations to compute the distance between $\mu_N^{g'}$ and ν: applying Equations (5.2), (5.3), (5.5) and triangle inequality, we obtain:

$$\mathrm{dist}(\mu_X^{g_N'},\nu) < \frac{1}{k_0}.$$

In the general case the measure ν is only invariant (and not ergodic). So in the second step of the proof we treat the case where there is only one measure that is not ergodic. It reduces to the ergodic case by the fact that the set of invariant measures is a compact convex set whose extremal points are exactly the ergodic measures: by the Krein-Milman theorem, for all $M \geq 1$ there exists an f-invariant measure ν' which is a finite convex combination of ergodic measures:

$$\nu' = \sum_{j=1}^{r}\lambda_j\nu_j^e,$$

and whose distance to ν is smaller than $\frac{1}{k_0}$. To simplify the proof, we treat the case where $r = 2$, the other cases being treated the same way. So $\nu' = \lambda_1\nu_1^e + \lambda_2\nu_2^e$. We use the hypothesis of self similarity of the grids to build a permutation of a grid E_N which is close to f and has a unique periodic orbit whose associated measure is close to ν' in the way described by Figure 5.5. More precisely, the self similarity of the grids imply that for all $\varepsilon > 0$, there exists $N_0 \in \mathbf{N}$ such that for all $N \geq N_0$, the set E_N is partitioned by disjoint subsets $\widetilde{E}_N^1,\cdots,\widetilde{E}_N^p$ such that for all i, $\widetilde{E}_N^i$ is the image of the grid E_{N_0} by a

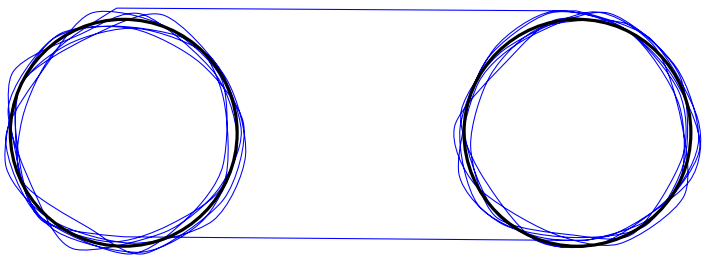

Figure 5.5: Construction of Lemma 5.46: the measure carried by the blue cycle is close to a convex combination of the measures carried by the black cycles

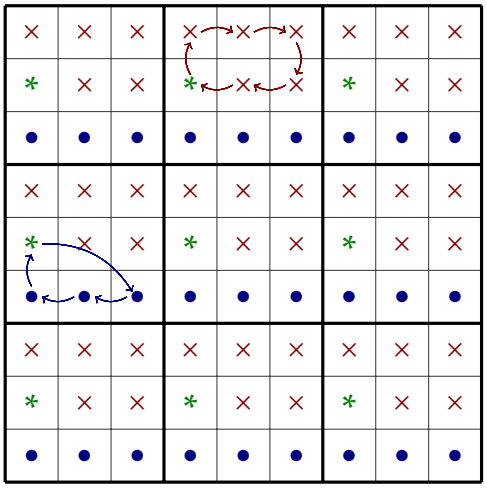

Figure 5.6: Construction of Lemma 5.46: before the perturbation, the red (crosses) subgrids carry a measure close to ν_1^e, the blue (points) subgrids carry a measure close to ν_1^e and the green (stars) subgrid carries a cyclic permutation; we compose σ'_N by two cyclic permutations (red and blue arrows) to create a long periodic orbit which attracts the whole grid E_N and carries a measure close to ν'

bijection h_i which is ε-close to identity; in particular we have $P = q_N/q_{N_0}$. Then, Lax's theorem (Theorem 5.5) imply that there exists a permutation σ_N whose distance to f is smaller than ε and which is a cyclic permutation on each set $\widetilde{E}_N^i$. We denote by P the number of such cycles and approximate the λ_i by multiples of 1/P:

$$\lambda_1' = \frac{\lfloor \lambda_1 P \rfloor}{P}$$

and

$$\lambda_2' = 1 - \lambda_1' - \frac{1}{P}.$$

Then we have $|\lambda_1 - \lambda_1'| \leq \frac{1}{P}$ and $|\lambda_2 - \lambda_2'| \leq \frac{3}{P}$. Increasing N_0 if necessary, we make two different modifications of the map σ_N on the sets $\widetilde{E}_N^i$ to obtain a map σ'_N:

1. on the sets $\widetilde{E}_N^i$ with $1 \leq i \leq P\lambda_1'$, we do the previous construction concerning ergodic measures: on each of these sets, $\sigma'_{N|\widetilde{E}_N^i}$ carries an unique invariant probability measure $\nu_i^{e\prime}$ satisfying $\mathrm{dist}(\nu_i^e, \nu_1^e) \leq \frac{1}{10k_0}$;
2. on the sets $\widetilde{E}_N^i$ with $2 + P\lambda_1' \leq i \leq P$, we do the previous construction concerning ergodic measures: on each of these sets, $\sigma'_{N|\widetilde{E}_N^i}$ carries an unique invariant probability measure $\nu_i^{e\prime}$ satisfying $\mathrm{dist}(\nu_i^e, \nu_2^e) \leq \frac{1}{10k_0}$.

It remains to merge the periodic orbits on the sets $\widetilde{E}_N^i$ to create a periodic orbit which carries a measure which is close to ν' (see Figure 5.6). We first consider a point $x_1 \in E_N$ such that x_1 belongs to the periodic orbit of $\sigma'_{N|\widetilde{E}_N^1}$, and compose σ'_N by a cyclic permutation of the set $\{h_i(h_1^{-1}(x_1))\}_{1 \leq i \leq P(\lambda_1'+1)}$. Then, we consider a point $x_2 \in E_N$ such that x_2 belongs to the periodic orbit of $\sigma'_{N|\widetilde{E}_N^P}$, and compose σ'_N by a cyclic permutation of the set $\{h_i(h_1^{-1}(x_2))\}_{P(\lambda_1'+1) \leq i \leq P}$.

Thus, by construction, the obtained map σ''_N carries a single periodic orbit (thus this orbit attracts the whole grid E_N); we denote by $\mu^{\sigma''_N}$. Moreover, if we consider a

point x_N belonging to this orbit, its orbit distributes like the measure ν_1 during a time proportional to λ'_1, like the measure ν_2 during a time proportional to λ'_2, whereas it covers the rest of the orbit during a (small) time proportional to 1/P. Thus, increasing N (therefore P) if necessary, it means that the distance between the measures $\mu^{\sigma''_N}$ and ν' is smaller than $1/k_0$. To finish the proof of the lemma when there is only one measure to approximate, it suffices to interpolate the map σ''_N by a homeomorphism close to f in applying the proposition of extension of finite maps (Proposition 3.3).

To treat the case where there are several measures to approximate, we use once more the fact that the grids are sometimes strongly self similar. On each of the subgrids, (and considering sub-subgrids if necessary), we apply the process of the previous part of the proof, and apply the proposition of extension of finite maps to find a homeomorphism which satisfies the conclusions of the lemma. □

5.5.3 Invariant compact sets

There is a similarity between invariant measures and invariant compact sets, thus the previous theorem is also true for invariant compact sets. As in the previous section, we begin by a particular case of the main theorem of this section (Theorem 5.49).

Theorem 5.47. *Let $f \in \mathrm{Homeo}(X,\lambda)$ be a generic homeomorphism and suppose that the sequence of grids $(E_N)_{N\in\mathbf{N}}$ is sometimes strongly self similar. Then for every f-invariant chain transitive compact set $K \subset X$, there exists a subsequence K_{N_k} of f_{N_k}-periodic orbits which tends to K for Hausdorff topology.*

In the ergodic case, the set of invariant measures of a given map had to be closed and convex; concerning compact sets, there are also conditions for a collection of compact sets to be the set of compact invariant sets of a map. This motivates the following definition.

Definition 5.48. Let $f \in \mathrm{Homeo}(X)$. We say that a set $\mathscr{K}$ of compact subsets of X that are invariant under f is *admissible for f* if there exists $\mathscr{K}_0 \subset \mathscr{K}$ such that:

(i) every $K_0 \in \mathscr{K}_0$ is chain transitive;

(ii) every $K \in \mathscr{K}$ is a finite union of elements of $\mathscr{K}_0$.

Theorem 5.49. *Let $f \in \mathrm{Homeo}(X,\lambda)$ be a generic homeomorphism and suppose that the sequence of grids $(E_N)_{N\in\mathbf{N}}$ is sometimes strongly self similar. Let $\mathscr{K}_N$ be the set of compact subsets of E_N that are invariant under f_N. Then for every collection $\mathscr{K}$ of compact subsets of X that is admissible for f, there exists a subsequence of $\mathscr{K}_N$ which tends to $\overline{\mathscr{K}}$ for Hausdorff topology* [10].

Like in the ergodic case, this theorem expresses that the sets of invariant compact sets for f_N accumulate on all the admissible sets of invariant sets for f.

The proof of Theorem 5.49 is very similar to that of Theorem 5.45. An appropriate application of Baire's theorem reduces the proof to that of the following variation of Lax's theorem.

Lemma 5.50 (Compact variation of Lax's theorem)**.** *Suppose that the sequence of grids $(E_N)_{N\in\mathbf{N}}$ is sometimes strongly self similar. For all $f \in \mathrm{Homeo}(X,\lambda)$, for all collection $\mathscr{K}$ of compact subsets of X that is admissible for f, for all $\varepsilon > 0$ and $k_0, N_0 \in \mathbf{N}$, there exists $g \in \mathrm{Homeo}(X,\lambda)$ and $N \geq N_0$ such that $d(f,g) < \varepsilon$, and that the set of g_N-invariant sets* [11] *on*

10. That is, Hausdorff topology on the sets of compacts subsets of X endowed with Hausdorff distance.
11. Which consists of unions of periodic orbits of g_N.

E_N is $1/k_0$-close to $\overline{\mathscr{K}}$ for Hausdorff distance. Moreover, we can suppose that these properties are satisfied on a whole neighbourhood of g.

Sketch of proof of Lemma 5.50. The proof is very similar to that of Lemma 5.46. We take a collection $\mathscr{K}$ of compact subsets of X which is admissible for f. By compactness of $\overline{\mathscr{K}}$, it is close for Hausdorff topology to a finite set $\{K_1\}\cup\cdots\cup\{K_p\}$. Then, every compact set K_i is close to a finite union $\tilde{K}_i^1\cup\cdots\cup\tilde{K}_i^{\ell_i}$ of chain transitive compact sets that are invariant under f. As each set $\tilde{K}_i^j$ is chain transitive, there exists a f-pseudo orbit $\tilde{\omega}_i^j$ which is close to $\tilde{K}_i^j$ for Hausdorff distance. We then apply the same proof strategy as for Lemma 5.46 to the pseudo orbits $\tilde{\omega}_i^j$ to obtain directly the conclusions of Lemma 5.50. □

5.6 Physical dynamics

In the previous section, we proved that generically, the upper limit of the sets of invariant measures of discretizations is the set of invariant measures of the initial homeomorphism. This expresses that the sets of invariant measures of discretizations accumulate on "the biggest possible set". However, we might expect that physical measures — that is, Borel measures which attracts a lot of points with respect to λ (see Definition 3.9 page 51) — play a specific part: their definition expresses that they are the measures that can be observed in practice on numerical experiments, because they governs the ergodic behaviour of λ almost every point [12]; in the case of a generic conservative homeomorphism, Oxtoby-Ulam theorem [OU41] implies that λ is the unique physical measure. Moreover, results of stochastic stability are known to be true in various contexts (for example, expanding maps [Kif86a],[Kif86b], [Kel82], uniformly hyperbolic attractors [Kif86b], [You86]...). These theorems suggest that the physical measures can always be observed in practice, even if the system is noisy. Given this background, we are tempted to think that the natural invariant measures $\mu_X^{f_N}$ of f_N, which can be seen as the physical measures of f_N, converge to the physical measures of f.

Recall that $\mu_X^{f_N}$ is the limit in the sense of Cesàro of the pushforwards by iterates of f_N of uniform measure λ_N on E_N (see Definition 3.10 page 52):

$$\mu_X^{f_N} = \lim_{m\to\infty} \frac{1}{m}\sum_{i=0}^{m-1} (f_N)_*^i \lambda_N.$$

The measure $\mu_X^{f_N}$ is supported by the recurrent set of f_N; it is uniform on every periodic orbit and the total weight of a periodic orbit is proportional to the size of its basin of attraction.

The expectation that the measure λ plays a specific part is supported by the following variation of Proposition 5.42, obtained by replacing ν_N by $\mu_X^{f_N}$: *if f is uniquely ergodic, then the measures $\mu_X^{f_N}$ converge weakly to the only measure μ that is invariant under f.*

Unfortunately, we show that this is not at all the case: the sequence of measures $(\mu_X^{f_N})$ accumulates of the whole set of f-invariant measures. More precisely we have the following theorem:

12. Recall that λ is a good measure, in particular it is positive on every non-empty open set.

Theorem 5.51. *If the sequence of grids* $(E_N)_{N\in\mathbf{N}}$ *is sometimes strongly self similar, then for a generic homeomorphism* $f \in \mathrm{Homeo}(X,\lambda)$*, the set of limit points of the sequence* $(\mu_X^{f_N})_{N\in\mathbf{N}}$ *is exactly the set of* f*-invariant measures.*

This theorem can be seen as a discretized version of the following conjecture.

Conjecture 5.52 (F. Abdenur, M. Andersson, [AA13])**.** *A homeomorphism* f *which is generic in the set of homeomorphisms of* X *(without measure preserving hypothesis) that do not have any open trapping set is* wicked*, i.e. it is not uniquely ergodic and the measures*

$$\frac{1}{m}\sum_{k=0}^{m-1} f_*^k(\mathrm{Leb})$$

accumulate on the whole set of invariant measures under f*.*

The behaviour described in this conjecture is the opposite of that consisting of possessing a physical measure.

Sketch of proof of theorem 5.51. The proof is similar to which of theorem 5.45: the set $\mathscr{A}$ is replaced by

$$\mathscr{A}' = \bigcap_{(\ell,N_0,k_0)\in\mathbf{N}^3} \mathscr{O}_{\ell,N_0,k_0},$$

where

$$\mathscr{O}_{\ell,N_0,k_0} = \left\{ f \in \bigcap_{N\in\mathbf{N}} \mathscr{D}_N \;\middle|\; \begin{array}{l} \left(\exists \nu\, f\text{-inv.} : \mathrm{dist}(\nu,\tilde{\nu}_\ell) \le \frac{1}{k_0}\right) \Longrightarrow \\ \left(\exists N \ge N_0 : \mathrm{dist}(\tilde{\nu}_\ell, \mu_X^{f_N}) < \frac{2}{k_0}\right) \end{array} \right\},$$

and $\{\nu_\ell\}$ is a countable set of Borel probability measures that is dense in the whole set of Borel probability measures. A direct application of Lemma 5.46 leads to the fact that every set $\mathscr{O}_{\ell,N_0,k_0}$ is open and dense. □

We have also another corollary of Lemma 5.46 about physical dynamics, which can be seen as a combination of Theorem 5.51 and of a discrete Birkhoff's theorem.

Theorem 5.53. *If the sequence of grids* $(E_N)_{N\in\mathbf{N}}$ *is sometimes strongly self similar, then for a generic homeomorphism* $f \in \mathrm{Homeo}(X,\lambda)$ *and for every* f*-invariant measure* μ*, there exists a subsequence* f_{N_k} *of discretizations such that for every* $x \in X$*, the sequence of measures*[13] $\mu_x^{f_{N_k}}$ *tends to* μ*.*

In particular, this theorem asserts that it is impossible to detect the physical measure of a generic conservative homeomorphism by looking at the physical measures of its discretizations:

Remark 5.54. This seems to contradict the empirical observations made by A. Boyarsky in 1986 (see [Boy86] or [GB88]): when a homeomorphism f has only one ergodic measure μ which is absolutely continuous with respect to Lebesgue measure, "most of" the measures $\mu_x^{f_N}$ tend to measure μ. However, the author does not specify in what sense he means "most of the points", or if his remark is based on a tacit assumption of regularity for f.

13. Recall that by Definition 3.10, $\mu_x^{f_N}$ is the Cesàro limit of the pushforwards of the Dirac measure δ_{x_N} by the discretization f_N.

Note that as in the previous section, we have a compact counterpart of Theorem 5.51:

Theorem 5.55. *If the sequence of grids $(E_N)_{N\in\mathbf{N}}$ is sometimes strongly self similar, then for a generic homeomorphism $f \in \mathrm{Homeo}(X,\lambda)$, the set of limit points of the sequence $(\Omega(f_N))_{N\in\mathbf{N}}$ is exactly the set of limit points of finite unions of f-invariant chain transitive compact sets.*

5.7 Addendum: generic conjugates and generic grids

In this addendum we want to adapt the proofs of this chapter to the case of discretizations of conjugated generic homeomorphisms. More precisely, we fix a conservative homeomorphism f and we look at the dynamics of the discretizations of hfh^{-1}, where h is a generic conservative homeomorphism. This is equivalent to fix a dynamics f and to wonder what is the dynamics of the discretizations of generic realizations of f. Another point of view is to see these discretizations as discretizations of a fixed homeomorphism on a generic sequence of grids, where by definition a generic sequence of grids is the image of a good sequence of grids by a generic homeomorphism.

The following property is a finite maps extension result for conjugations of a given homeomorphism. It implies that under some weak hypothesis, the results are the same for generic conjugations of a fixed homeomorphism and for generic conservative homeomorphisms.

Proposition 5.56. *Let f be a conservative homeomorphism whose set of fixed points has empty interior, and $\sigma : E_N \to E_N$ be such that $d_N(f,\sigma) < \varepsilon$. Then there exists a conservative homeomorphism h such that $d(h,\mathrm{Id}) < \varepsilon$ and such that the homeomorphism $g = hfh^{-1}$ satisfies $g_N = \sigma$ (and these properties remain true on a whole neighbourhood of h).*

Proof of Proposition 5.56. Let $\delta = \min\{1/(2N), \varepsilon - d(h,\mathrm{Id})\}$. Let $x_1,\cdots,x_{q_N}$ be the points of E_N. It may happen that some of these points are fixed points of f, thus we define an alternative sequence of points $(x'_1,y'_1,\cdots,x'_{q_N},y'_{q_N})$ by induction, which is close to the sequence $(x_1,f(x_1),\cdots,x_{q_N},f(x_{q_N}))$ and contains no fixed points. Suppose that we have constructed the sequence until $x'_{\ell-1}$ and, $y'_{\ell-1}$. Then there exists $x'_\ell \in X$ δ-close to x_ℓ, which is not a fixed point of f and which is different from all the points $x'_1, f(x'_1), f^{-1}(x'_1),\cdots,x'_{\ell-1}, f(x'_{\ell-1}), f^{-1}(x'_{\ell-1})$. Similarly, there exists a point y'_ℓ δ-close to $\sigma(x_\ell)$ and different from all the points $y_1,\cdots,y_{\ell-1}$ and $x_1,\cdots,x_{q_N}$. So on, we define a set $(x'_1,y'_1,\cdots,x'_{q_N},y'_{q_N})$.

We can now build the homeomorphism h with the proposition of extension of finite maps (Proposition 3.3): we choose a conservative homeomorphism h such that for all ℓ, we have $h(x'_\ell) = x_\ell$ and $h(f(x'_\ell)) = y'_\ell$. As for all ℓ, the point x'_ℓ is ε-close to x_ℓ, the point $\sigma(x_\ell)$ is $\varepsilon - d(h,\mathrm{Id})$-close to y'_ℓ and the point $f(x_\ell)$ is $d(h,\mathrm{Id})$-close to $\sigma(x_\ell)$, we can choose h such that $d(h,\mathrm{Id}) < \varepsilon$. Moreover, if we set $g = hfh^{-1}$, then by construction we have $g(x_\ell) = h(f(x'_\ell)) = y'_\ell$. As y'_ℓ is $1/(2N)$-close to $\sigma(x_\ell)$, we obtain $g_N(x_\ell) = \sigma(x_\ell)$ for all ℓ. □

As in this chapter, the only perturbation result we used was the proposition of finite maps extension (Proposition 3.3), this proposition implies the following corollary.

Corollary 5.57. *All the results stated in this chapter concerning the dynamical behaviour of the discretizations of generic conservative homeomorphisms remain true for the dynamical behaviour of the discretizations of conservative homeomorphisms which are generic among those conjugated to a given conservative homeomorphism whose set of fixed points has empty interior.*

5.8 Numerical simulations

We now present the results of numerical simulations of conservative homeomorphisms. We are interested in whether it is possible to observe the behaviours described by our theoretical results on actual simulations. It is not clear that the orders of discretization involved in these results can be reached in practice, or if simple examples of homeomorphisms behave the same way as generic homeomorphisms. For instance, can we observe a lot of discretizations that are cyclic permutations of the grid, or that have a small degree of recurrence? Given a homeomorphism f, is it possible to see the invariant measures of the discretizations accumulating on the set of invariant measures of f? Is it possible to recover the periodic points of the homeomorphism?

We simulate homeomorphisms $f(x,y) = (\mathrm{Q} \circ \mathrm{P})(x,y)$, where both P and Q are homeomorphisms of the torus that modify only one coordinate:

$$\mathrm{P}(x,y) = \Big(x, y + p(x)\Big) \quad \text{and} \quad \mathrm{Q}(x,y) = \Big(x + q(y), y\Big),$$

so that P and Q both preserve Lebesgue measure. We discretize these examples according to the uniform grids on the torus

$$\mathrm{E}_\mathrm{N} = \left\{ \left(\frac{i}{\mathrm{N}}, \frac{j}{\mathrm{N}} \right) \in \mathbf{T}^2 \,\middle|\, 0 \leq i, j \leq \mathrm{N} - 1 \right\}.$$

We perform simulations of two conservative homeomorphisms which are small perturbations of identity or of the standard Anosov automorphism $\mathrm{A} : (x,y) \mapsto (x+y, x+2y)$.

– To begin with we study $f_3 = \mathrm{Q} \circ \mathrm{P}$, with

$$p(x) = \frac{1}{209}\cos(2\pi \times 187x) + \frac{1}{271}\sin(2\pi \times 253x) - \frac{1}{703}\cos(2\pi \times 775x),$$

$$q(y) = \frac{1}{287}\cos(2\pi \times 241y) + \frac{1}{203}\sin(2\pi \times 197y) - \frac{1}{841}\sin(2\pi \times 811y).$$

This conservative homeomorphism is a small C^0 perturbation of identity. Experience shows that even dynamical systems with fairly simple definitions behave quite chaotically (see for example [GT83]). We can expect that a homeomorphism such as f_3 has a complex dynamical behaviour and even more, behaves essentially as a generic homeomorphism, at least for orders of discretization that can be reached in practice. Note that we choose coefficients that have (virtually) no common divisors to avoid arithmetical phenomena such as periodicity or resonance. Also, we have chosen to simulate a homeomorphism close to the identity to avoid phenomenons like "every orbit of the discretization f_N is ε-dense for all N large enough": in this case it would be difficult to see something interesting in the images of the invariant measures.

– We also simulate f_4 the composition of f_3 with the standard Anosov automorphism

$$\mathrm{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

say $f_4 = \mathrm{Q} \circ \mathrm{P} \circ \mathrm{A}$. This makes it a small C^0-perturbation of A. Thus f_4 is semi-conjugated to A but not conjugated: to each periodic orbit of A corresponds many periodic orbits of f_4. As for f_3, we define f_4 in the hope that the behaviour of its discretizations is fairly close to that of discretizations of a generic homeomorphism.

We have chosen to define the homeomorphisms we compute with lacunary trigonometric series, to "mimic" the action of Baire theorem.

Note that the homeomorphisms f_3 and f_4 have at least one fixed point (for f_3, simply make simultaneously $p(x)$ and $q(y)$ vanish; for f_4, note that 0 is a persistent fixed point of A). Theoretical results indicate that for a generic homeomorphism f which has a fixed point, infinitely many discretizations has a unique fixed point; moreover a subsequence of $(\mu_{\mathbf{T}^2}^{f_N})_{N\in\mathbf{N}}$ tends to an invariant measure under f supported by a fixed point; we can test if these properties are true on simulations or not.

We refer to the page 67 for a presentation of the algorithm we used for the simulations.

5.8.1 Combinatorial behaviour

As a first step, we are interested in some quantities related to the combinatorial behaviour of discretizations of homeomorphisms. These quantities are:

- the cardinality of the recurrent set $\Omega(f_N)$;
- the number of periodic orbits of the map f_N;
- the maximal size of a periodic orbit of f_N.

Recall that according to the theoretical results we obtained, for a generic homeomorphism, the degree of recurrence $\frac{\mathrm{Card}(\Omega(f_N))}{q_N}$ should accumulate on the whole segment $[0,1]$ (Corollary 5.24), the number of orbits of f_N should be 1 for some N and should be bigger than (for example) $\sqrt{q_N}$ for other orders N (Corollaries 5.9 and 5.22) and the maximal size of a periodic orbit of f_N should be sometimes 1, and sometimes equal to q_N (Corollaries 5.9 and 5.22).

We have calculated these quantities for discretizations of orders $128k$, for k from 1 to 150 and have represented them graphically (see Figure 5.7). For information, if $N = 128\times 150$, then $q_N \simeq 3.7\times 10^8$.

We begin with the cardinality of the recurrent set $\Omega(f_N)$ (Figure 5.7, top). Contrary to what the theoretical results provide for a generic homeomorphism, for all simulations, the degree of recurrence $\frac{\mathrm{Card}(\Omega(f_N))}{q_N}$ tends clearly to 0 as N increases. More specifically, the cardinality of $\Omega(f_N)$ evolves much more regularly for f_3 than for f_4: for f_3 the value of this cardinality seems to be the sum of a smooth increasing function and a random noise, but for f_4 this value seems to be the product of a smooth increasing function with a random noise. We have no explanation for the parabolic shape of the curve for f_3: it reflects the fact that the cardinality of $\Omega((f_3)_N)$ evolves in the same way as $\sqrt{N} = \sqrt[4]{q_N}$ (whereas for a random map of a finite set with q elements into itself, it evolves as $\sqrt{q}$). Finally, it is interesting to note that for f_4, the size of the recurrent set is distributed more or less around the size of the recurrent set of a random map of a set with q_N elements into itself, which depends (asymptotically) linearly of N (because it is of the order of $\sqrt{q_N}$) and is worth about 24 000 for $N = 128\times 150$ (see [Bol01, XIV.5] or the Theorem 2.3.1 of [Mie05]).

According to the results of the previous sections, for a generic conservative homeomorphism f, the number of periodic orbits of f_N should be sometimes equal to 1 and sometimes bigger than (for example) $\sqrt{q_N}$. It is clearly not the case for these simulations. In addition, its behaviour is clearly not the same for f_3 and for f_4 (Figure 5.7, middle): for f_3 the number of orbits reaches quickly a value around 1.5×10^4 to stagnate thereafter, while for f_4 it oscillates between 1 and 16, regardless the order of discretization. Note that contrary to what we have observed for the cardinality of the recurrent set, this seem to contradict the fact that the discretizations of f_4 behave like random

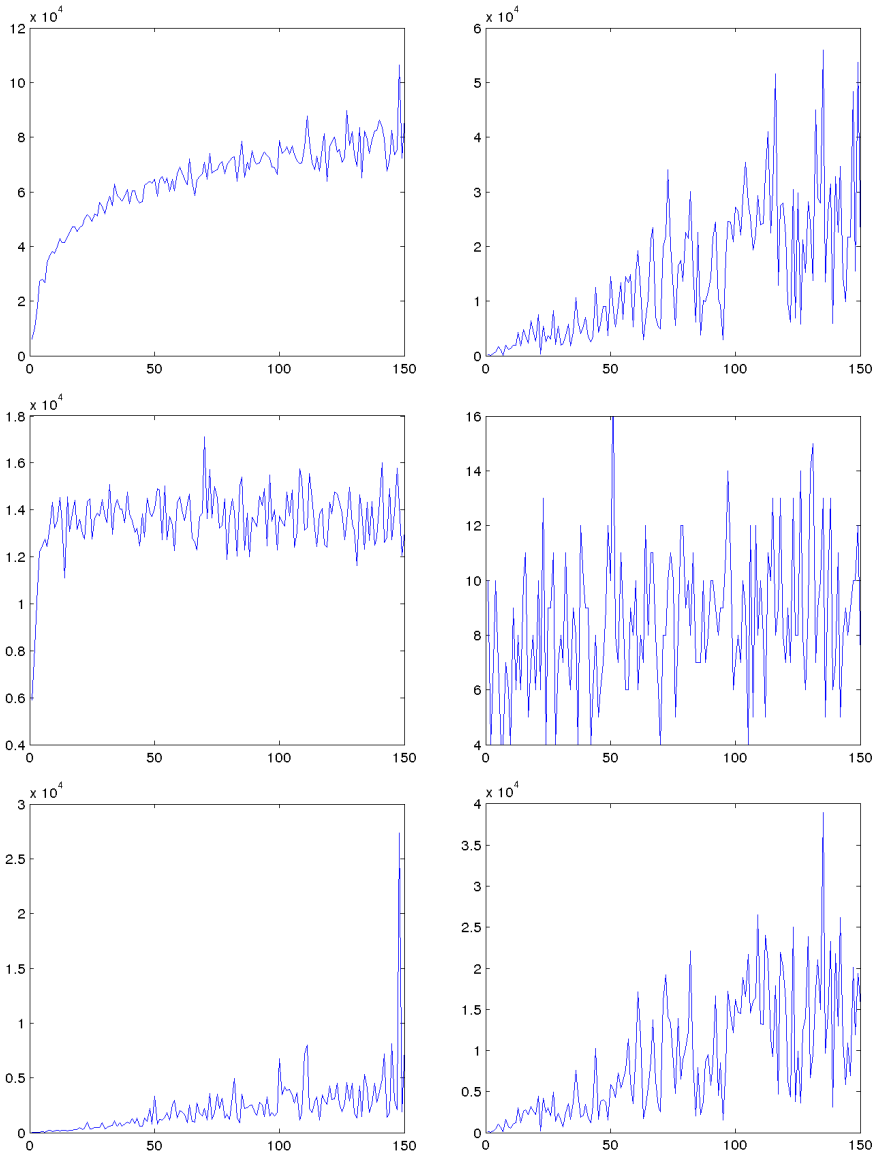

Figure 5.7: Size of the recurrent set $\Omega((f_i)_{\mathrm{N}})$ (top), number of periodic orbits of $(f_i)_{\mathrm{N}}$ (middle) and length of the largest periodic orbit of $(f_i)_{\mathrm{N}}$ (bottom) depending on N, for f_3 (left) and f_4 (right), on the grids E_{N} with $\mathrm{N} = 128k$, $k = 1, \cdots, 150$.

maps: if this were the case, the behaviour of this quantity should be proportional to $\log \mathrm{N}$, with a value close to 25 for $\mathrm{N} = 128 \times 150$. These graphics can be compared with those representing the size of the recurrent set $\Omega((f_i)_{\mathrm{N}})$: the number of periodic orbits and the size of the recurrent set are of the same order of magnitude for f_3 (up to a factor 5), which means that the average period of a periodic orbit is small (which is not surprising, since f_3 is a small perturbation of identity). They differ by a factor roughly equal to 10^3 for f_4, which means that this time the average period of a periodic orbit is very large. This can be explained partly by the fact that the standard Anosov automorphism tends to mix what happens in the neighbourhood of identity. The fact remains that these simulations (such as the size of the recurrent set $\Omega((f_i)_{\mathrm{N}})$) suggest that the behaviour in the neighbourhood of identity and of the standard Anosov automorphism

are quite different, at least for reasonable orders of discretization.

Regarding the maximum size of a periodic orbit of f_N (Figure 5.7, bottom), again its behaviour does not correspond to that of a generic homeomorphism: it should oscillate between 1 and q_N, and this is not the case. However, it varies widely depending on N, especially when N is large. The qualitative behaviours are quite similar for the three simulations, but there are some quantitative differences: the maximum of the maximal length of a periodic orbit is twice greater for f_4 than for f_3. For these simulations, there is no significant difference between the behaviours of the discretizations of f_3: if we remove the very big value that is attained by the maximal length of the periodic orbit of f_3 (for N close to 145), the graphics are very similar.

5.8.2 Behaviour of invariant measures

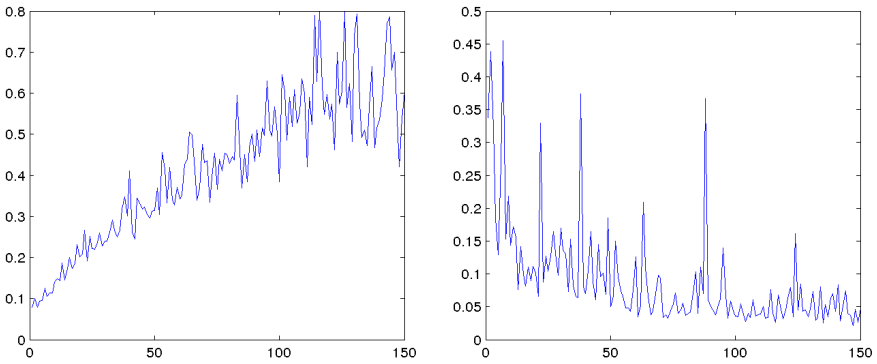

Figure 5.8: Distance between Lebesgue measure and the measure $\mu_{\mathbf{T}^2}^{(f_i)_N}$ depending on N for f_3 (left) and f_4 (right), on the grids E_N with $N = 128k$, $k = 1, \cdots, 150$.

We have simulated the measure $\mu_{\mathbf{T}^2}^{(f_i)_N}$ for the two examples of conservative homeomorphisms f_3 and f_4 as defined page 99. Our purpose is to test whether phenomena as described by Theorem 5.51 can be observed in practice or not. We obviously cannot expect to see the sequence $(\mu_{\mathbf{T}^2}^{(f_i)_N})_{N\in\mathbf{N}}$ accumulating on *all* the invariant probability measures of f, since these measures generally form an infinite-dimensional convex set [14], but we can still test if it seems to converge or not. In particular, we can try to detect whether Lebesgue measure (which is the unique physical measure by Oxtoby-Ulam theorem [OU41]) play a particular role for these invariant measures or not.

We present images of sizes 128×128 pixels representing in logarithmic scale the density of the measures $\mu_{\mathbf{T}^2}^{f_N}$: each pixel is coloured according to the measure carried by the set of points of E_N it covers. Blue corresponds to a pixel with very small measure and red to a pixel with very high measure. Scales on the right of each image corresponds to the measure of one pixel on the $\log 10$ scale: if green corresponds to -3, then a green pixel will have measure 10^{-3} for $\mu_{\mathbf{T}^2}^{f_N}$. For information, when Lebesgue measure is represented, all the pixels have a value about -4.2.

14. To see that, simply observe that the set of periodic points is a union of Cantor sets; the uniform measures on these periodic orbits form an uncountable free family.

Firstly, we have computed the distance between the measure $\mu_{\mathbf{T}^2}^{f_N}$ and Lebesgue measure. The distance we have chosen is given by the formula

$$d(\mu,\nu)=\sum_{k=0}^{\infty}\frac{1}{2^k}\sum_{i,j=0}^{2^k-1}\left|\mu(C_{i,j,k})-\nu(C_{i,j,k})\right|\in[0,2],$$

where

$$C_{i,j,k}=\left[\frac{i}{2^k},\frac{i+1}{2^k}\right]\times\left[\frac{j}{2^k},\frac{j+1}{2^k}\right].$$

In practice, we have computed an approximation of this quantity by summing only on the $k\in[\![0,7]\!]$. Theoretically, the distance between the measure $\mu_{\mathbf{T}^2}^{f_N}$ and Leb (Figure 5.8) should oscillate asymptotically between 0 and the supremum over the set of f-invariant measures μ of the distance between Lebesgue measure and μ. For f_3 (left of Figure 5.8), wee see that the distance between $\mu_{\mathbf{T}^2}^{f_N}$ and Leb seem to increase with N: the measure $\mu_X^{f_N}$ is close to Lebesgue measure for small values of N, and then is more and more far away from Lebesgue measure. It can be explained by the fact that for a small N, the discretization f_N does not see the irregularities of f and is more or less close to the discretization of identity. The behaviour for f_4 is very different: overall, the distance between $\mu_{\mathbf{T}^2}^{f_N}$ and Leb is much smaller and decreasing, but we observe peaks of this distance: there are a few values of N for which $\mu_{\mathbf{T}^2}^{f_N}$ is far away from Leb.

The results of simulations of invariant measures of discretizations of f_3 (which is a C^0 conservative perturbation of identity) are fairly positive: they agree with theoretical results about discretizations of generic conservative homeomorphisms, in particular with Theorem 5.51. Indeed, when we compute discretizations of the homeomorphism f_3 on the grids of size $2^k\times 2^k$ (Figure 5.9), we first observe that $\mu_{\mathbf{T}^2}^{f_N}$ is fairly well distributed, say for $2^k=128,256,512$. When the order of the discretization increases, we can observe places where the measure accumulates; moreover these places changes a lot when the order of discretization varies (see Figure 5.10): for the orders N we tested, the measures $\mu_{\mathbf{T}^2}^{f_N}$ and $\mu_{\mathbf{T}^2}^{f_{N+1}}$ are always completely different, and do not have anything in common with Lebesgue measure. This agrees fairly with what happens in the C^0 generic case, where we have proved that the measure $\mu_X^{f_N}$ depends very much on the order of discretization, rather than on the homeomorphism itself. There is also an other phenomenon: when the size of the grid is large enough (around $10^{12}\times 10^{12}$), some areas uniformly charged by the measure $\mu_{\mathbf{T}^2}^{f_N}$ appear; their sizes seem to be inversely proportional to the common mass of the pixels of the area; we do not know why this phenomenon appears...

For the discretizations of f_4, which is a C^0 conservative perturbation of the linear Anosov automorphism A, the simulations on grids of size $2^k\times 2^k$ might suggest that the measures $\mu_{\mathbf{T}^2}^{(f_4)_N}$ tend to Lebesgue measure (Figure 5.11). In fact, making a large number of simulations, we realize that there are also strong variations of the behaviour of the measures (Figure 5.12): the measure is often well distributed in the torus, and sometimes quite singular with respect to Lebesgue measure (as it can be seen in Figure 5.8). For example, when we discretize on the grid of size $22\,400\times 22\,400$ (middle right of Figure 5.12), we observe an orbit of length 369 which mass 84% of the total measure.In fact the behaviour of discretizations looks the same that in the neighbourhood of identity, modulo the fact that the linear Anosov automorphism A tends to spread the attractive periodic orbits of the discretizations on the entire torus: for many values of

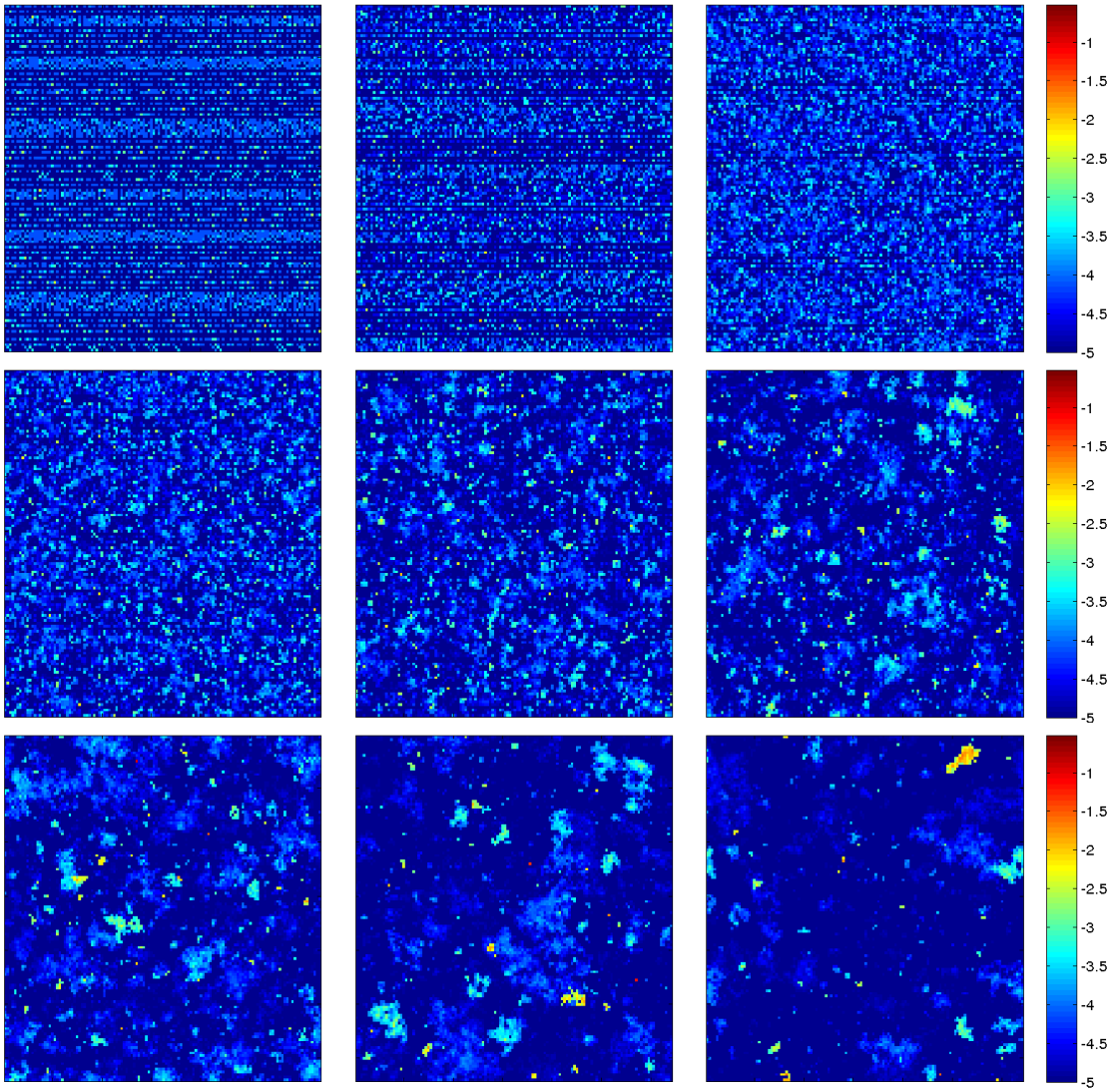

Figure 5.9: Simulations of invariant measures $\mu_{\mathbf{T}^2}^{(f_3)_N}$ on the grids E_N, with $N = 2^k$, $k = 7, \cdots, 15$ (from left to right and top to bottom).

N, composing by A spreads the behaviour of the measure $\mu_{\mathbf{T}^2}^{(f_4)_N}$, but sometimes (in fact, seldom) a fixed point of $(f_3)_N$ which attracts a large part of E_N is located around one of the few periodic points of small period for A. This then creates a periodic orbit for $(f_4)_N$ with a big measure for $\mu_{\mathbf{T}^2}^{(f_4)_N}$.

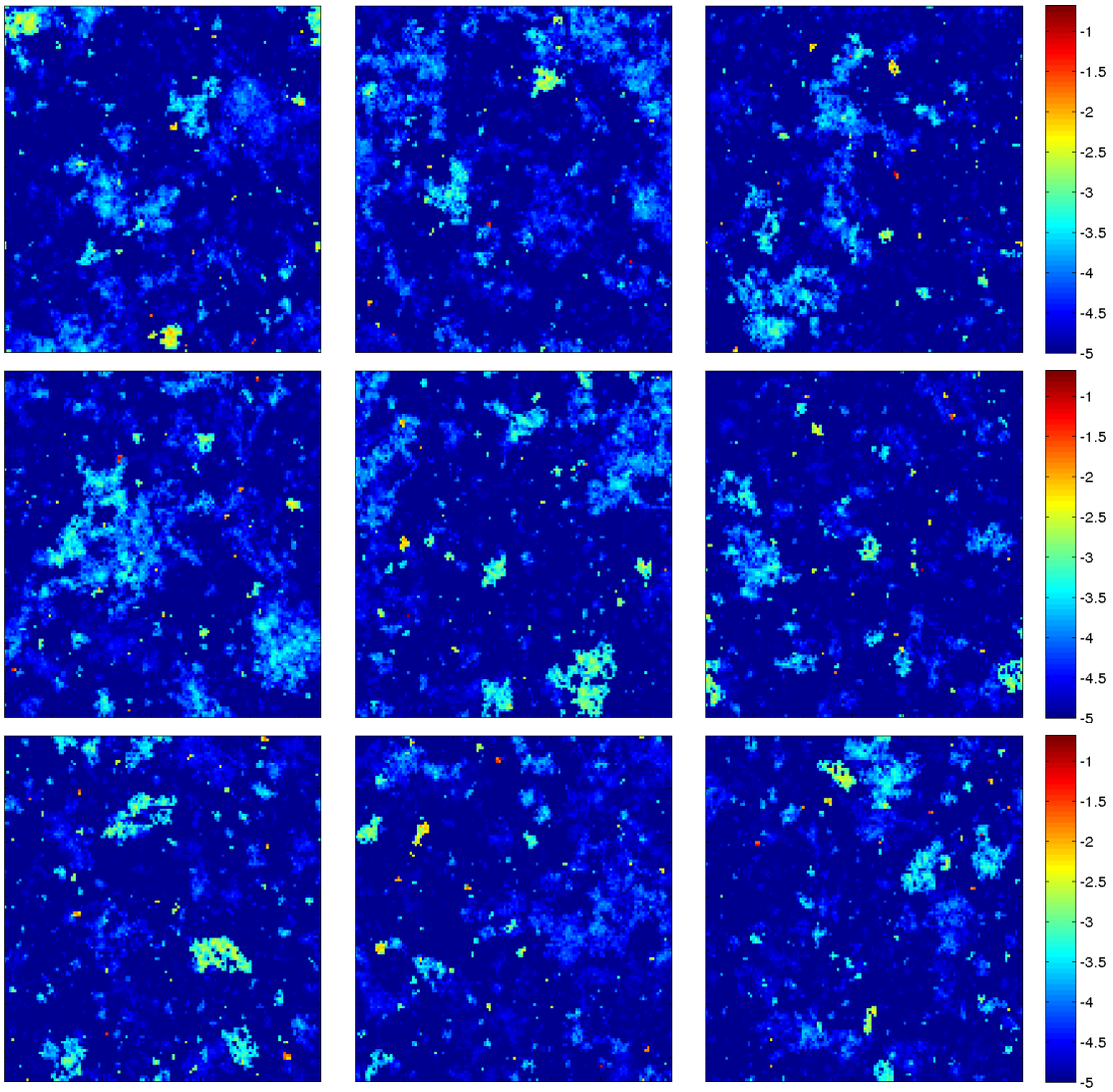

Figure 5.10: Simulations of invariant measures $\mu_{\mathbf{T}^2}^{(f_3)_N}$ on the grids E_N, with $N = 20\,000, \cdots, 20\,008$ (from left to right and top to bottom).

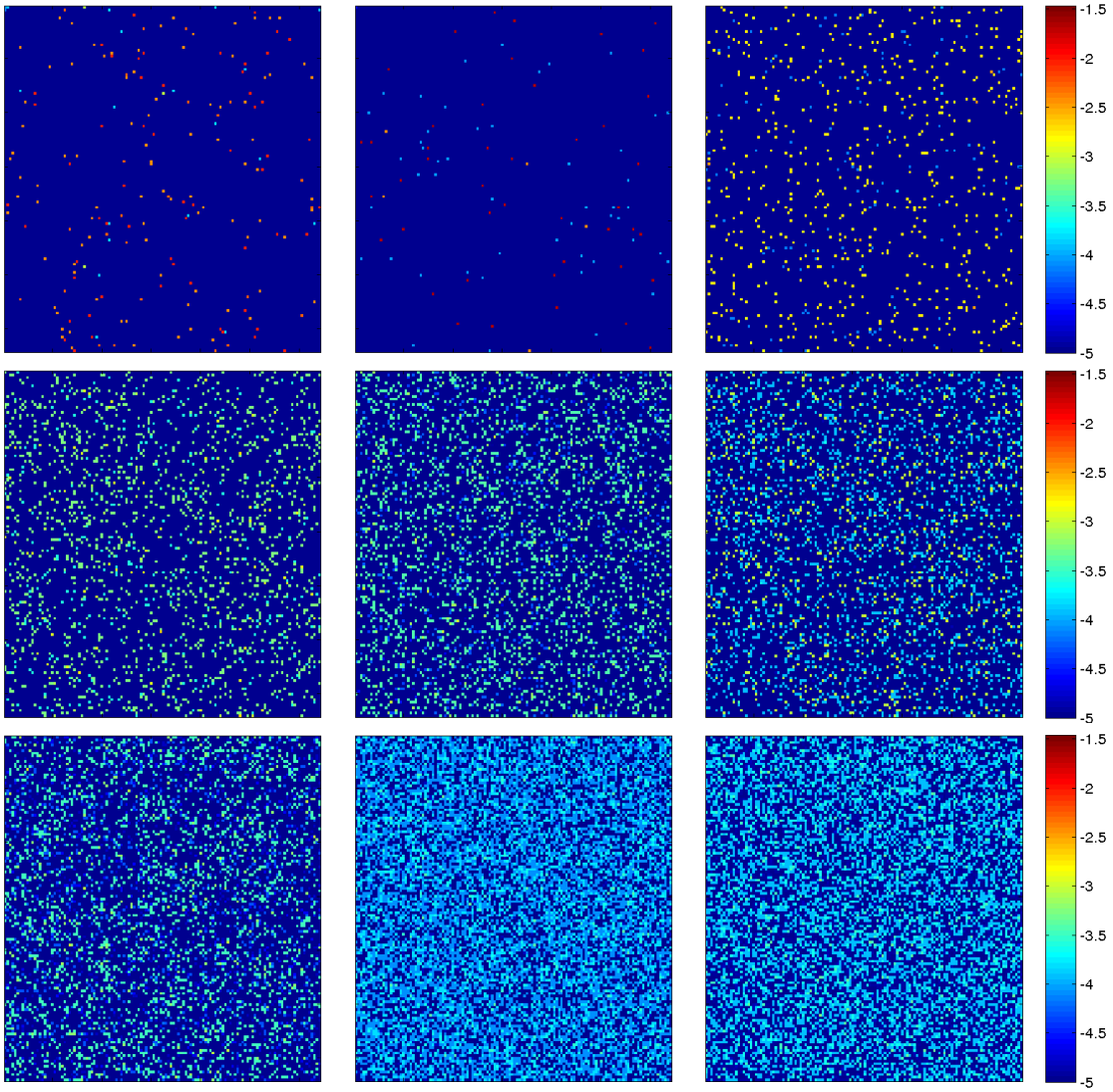

Figure 5.11: Simulations of invariant measures $\mu_{\mathbf{T}^2}^{(f_4)_N}$ on the grids E_N, with $N = 2^k$, $k = 7, \cdots, 15$ (from left to right and top to bottom).

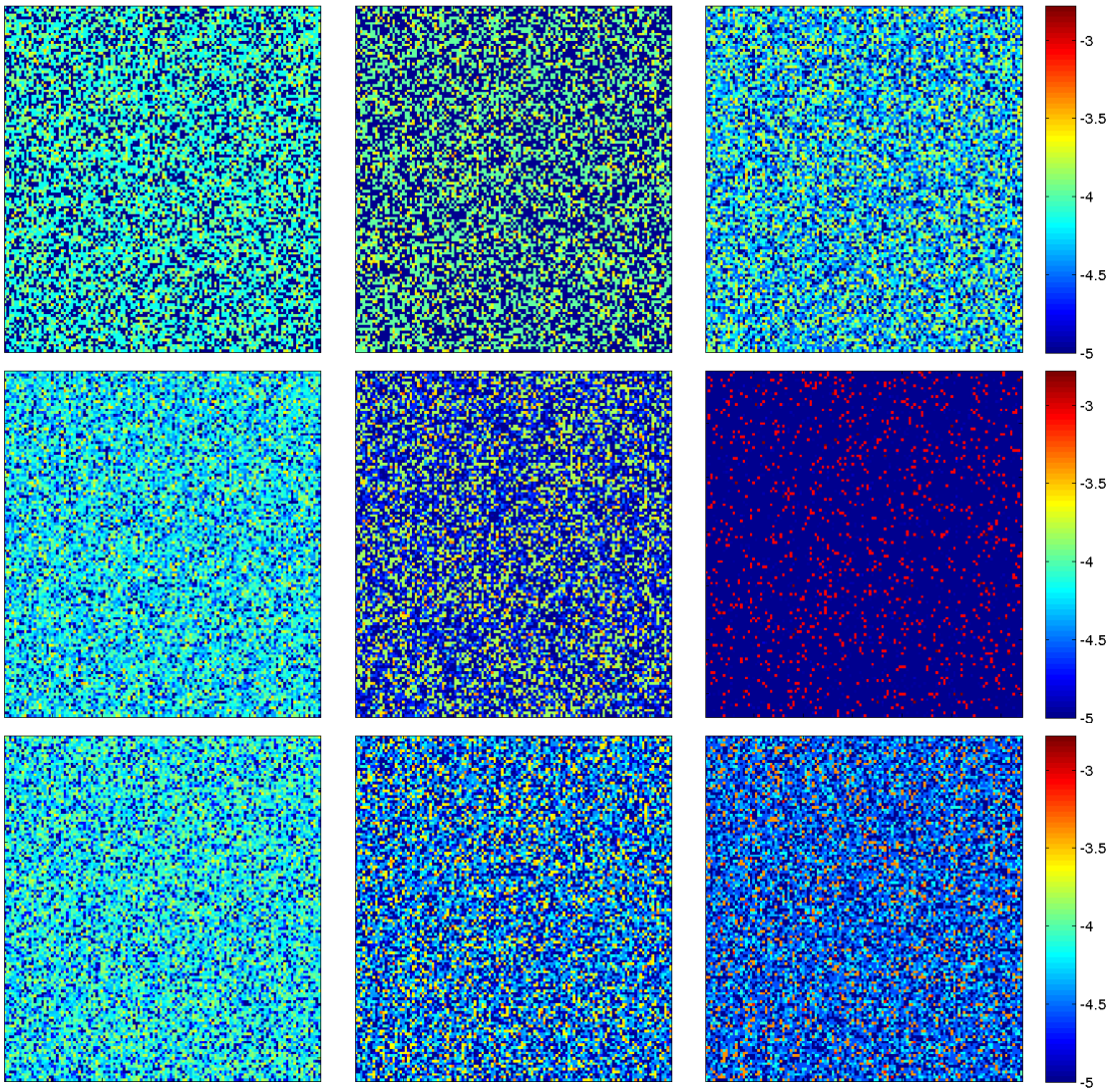

Figure 5.12: Simulations of invariant measures $\mu_{\mathbf{T}^2}^{(f_4)_N}$ on the grids E_N, with $N = 22\,395, \cdots, 22\,403$ (from left to right and top to bottom).

5.8.3 Periodic points

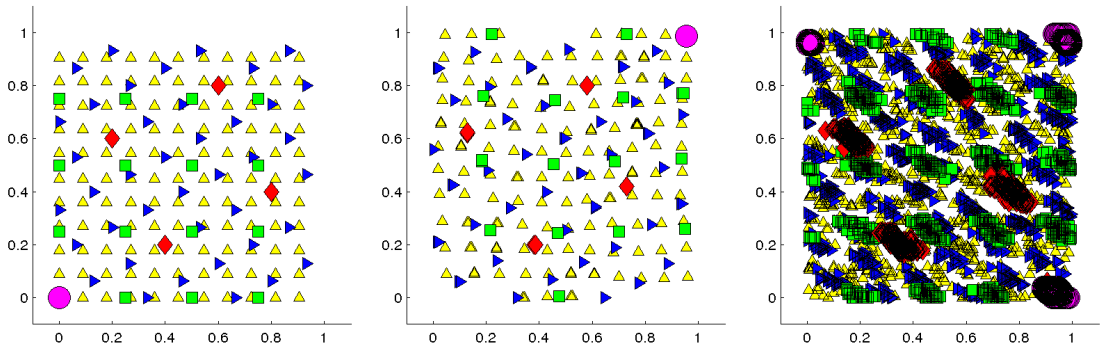

Figure 5.13: Unions of the sets of periodic points of period smaller than 5 of all the discretizations on grids E_N with $1\,000 \leq N \leq 2\,000$ for A (left), a small C^1 perturbation of A (middle) and a small C^0 perturbation of A (right); period 1: purple circle, period 2: red diamond, period 3: green square, period 4: blue triangle pointing right, period 5: yellow triangle pointing up.

We have also conducted simulations of the periodic orbits of period smaller than 5 of the discretizations for three conservative homeomorphisms:

– the linear map

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix};$$

– a small C^1 perturbation of A, say $g_1 = Q_1 \circ P_1 \circ A$, with

$$P_1(x,y) = \left(x - 0.01975\ ,\ y + \frac{1}{41}\cos(2\pi \times x) + \frac{1}{351}\sin(2\pi \times 7x) + 0.02478\right),$$

$$Q_1(x,y) = \left(x + \frac{1}{47}\cos(2\pi \times y) + \frac{1}{311}\sin(2\pi \times 5y) + 0.01237\ ,\ y + 0.00975\right);$$

– and a small C^0 perturbation of A, say $g_2 = Q_2 \circ P_2 \circ A$, with

$$P_2(x,y) = \left(x - 0.01975\ ,\ y + \frac{1}{31}\cos(2\pi \times 17x) + \frac{1}{41}\sin(2\pi \times 233x) + 0.02478\right),$$

$$Q_2(x,y) = \left(x + \frac{1}{27}\cos(2\pi \times 23y) + \frac{1}{37}\sin(2\pi \times 217y) + 0.01237\ ,\ y + 0.00975\right).$$

We represent the accumulation of the periodic points of period smaller than 5 of the discretizations f_N, for $1\,000 \leq N \leq 2\,000$.

It can be easily seen that the periodic points of A are rational points; it is also easy to compute by hand the coordinates of the periodic points of period smaller than 5. As A is Anosov and as Anosov maps are structurally stable (see [KH95], Theorem 2.6.3 for the linear maps of the torus and Corollary 18.2.2 for the general case), a small C^1 perturbation of A is (C^0) conjugated to A: if $d_{C^1}(A, g)$ is small enough, then there exists a homeomorphism $h \in \mathrm{Homeo}(\mathbf{T}^2)$ which maps bijectively each set of periodic points of a given period of A to the set of periodic points with the same period of g. If the perturbation of A is only C^0, then if $d_{C^0}(A, g)$ is small enough but $d_{C^1}(A, g)$ is big, then it can happen that the homeomorphism g is only semi-conjugated to A. But for a generic

homeomorphism, the set of periodic points of a given period is either empty, either a Cantor set with zero Hausdorff dimension. Thus, to any periodic point of the linear map A will correspond a Cantor set of periodic points of g (in other words, a generic small C^0 perturbation blows up each periodic point).

The theoretical results about discretizations assert that any periodic point of a generic homeomorphism is shadowed by a periodic orbit with the same period of an infinite number of discretizations (see Theorem 5.36); the same result holds for a generic C^1-diffeomorphism (see Lemma 10.2). Thus, on the simulations, we should recover all the periodic points of the initial map for both a small C^1 and C^0 generic perturbation.

On the left of Figure 5.13, we see that we recover all the periodic points of period ≤ 5 of the linear Anosov map A. It is logical: these periodic points have rational coordinates with small denominators, thus they are located on a lot of grids of order $N \in [\![1\,000, 2\,000]\!]$.

In the case of a small C^1 perturbation g_1 of A (middle of Figure 5.13), we also recover all the periodic points of period ≤ 5 of the map: we can see that we detect one (and almost always only one) periodic point of g_1 in the neighbourhood of each periodic point af A. In a certain sense, the simulation detects the conjugation between A and g_1.

For the small C^0 perturbation g_2 of A (right of Figure 5.13), we observe the phenomenon of blow up of the periodic points of A: to each periodic point of A corresponds a lot of periodic points of the discretizations of g_2. Thus, this simulation show two phenomenon. Firstly, the fact that the set of periodic points of a generic homeomorphism is a Cantor set is illustrated by some simple examples of homeomorphisms. Secondly, the behaviour predicted by Theorem 5.36 can be observed on practice: we recover (all) the periodic points of the homeomorphism with small period by looking at all the periodic points of the discretizations on grids of reasonable orders.

Chapter 6

How roundoff errors help to compute the rotation set of torus homeomorphisms

Introduction

We now present a practical application of the concept of discretization to the numerical computation of rotation set of torus homeomorphisms.

In this chapter, to be able to talk about the rotation set of the homeomorphisms, we will only consider homeomorphisms of the torus $\mathbf{T}^2$ which are homotopic to the identity. In other words, the space phase will always be $X = \mathbf{T}^2$. We fix once for all a good measure λ on $\mathbf{T}^2$ (see Definition 3.1), and a sequence of grids E_N on $\mathbf{T}^2$, which is sometimes strongly self-similar (see Definition 3.13). As an example of such grids, when $\lambda = \text{Leb}$, the reader can think about the canonical grids

$$E_N^0 = \left\{ \left(\frac{i}{N}, \frac{j}{N} \right) \in \mathbf{T}^2 \,\middle|\, 1 \leq i, j \leq N \right\}$$

(see Section 3.7 for other examples of grids).

We begin with the presentation of the concept of rotation set of a torus homeomorphism.

The concept of rotation number for circle homeomorphisms was introduced by H. Poincaré in 1885. In [Poi85], he states the theorem of classification of orientation-preserving circle homeomorphisms: if a homeomorphism has a rational rotation number, then it posses a periodic point and all its periodic points have the same period; moreover the ω-limit set of every point is a periodic orbit (the dynamics is asymptotically periodic). On the contrary, if a homeomorphism has an irrational rotation number α, then it is semi-conjugated to the rigid rotation of angle α (the dynamics contains that of the irrational rotation). Ever since, the rotation number has been the fundamental tool in the study of the dynamics of circle homeomorphisms (see for example [Her79]).

About a century after was introduced a generalisation to dimension 2 of this rotation number, the rotation set for homeomorphisms of the torus which are homotopic to the identity. Due to the loss of natural cyclic order on the phase space, there is no longer a single speed of rotation for orbits; informally the rotation set is defined as the set of all possible rotation speeds of all possible orbits. Like in dimension 1, this topological invariant gives precious informations about the dynamics of the homeomorphism; for

example, depending of the shape of this set, we can ensure the existence of periodic points of a given period ([Fra88], [Fra89]). Moreover, the size of this convex set gives lower bounds on the topological entropy of the homeomorphism ([LM91] and [Kwa93] for an explicit estimation)... See the course [Bég07] of F. Béguin for a quite complete survey of the results about the rotation sets.

The aim of this chapter is to tackle the question of numerical approximation of the rotation set: given a homeomorphism of the torus homotopic to the identity, is it possible to compute numerically its rotation set? In particular, is it possible to detect its dimension? Is it possible to approximate it in Hausdorff topology? And what algorithm shall we use to compute it?

First of all, we build a theoretical model of what happens when we try to calculate the rotation set of a homeomorphism with a computer. To do that, we first take into account the fact that the computer can calculate only a finite number of orbits; in particular it will detect only phenomenon that occur on λ-positive measure sets. This leads us to the notion of *observable rotation set*: a rotation vector is called observable if it is the rotation vector of an observable measure in the sense given by E. Catsigeras and H. Enrich in [CE11]; more precisely, a measure μ is observable if for every $\varepsilon > 0$, the set of points which have a Birkhoff limit whose distance to μ is smaller than ε has λ-positive measure (see Definition 6.4).

However, this notion of observable measure does not take into account the fact that the computer uses finite precision numbers and can calculate only finite length orbits; this observation leads to the definition leads to the definition of the *asymptotic discretized rotation set* in the following way. We fix a sequence of finite grids on the torus with precision going to 0; the discretized rotation set on one of these grids is the collection of rotation vectors of periodic orbits of the discretization of the homeomorphism on this grid (see section 6.1); the asymptotic discretized rotation set is then the upper limit of these discretized rotation sets on the grids.

We focus mainly on the generic behaviour of both observable and asymptotic discretized rotation sets. We recall that a result of A. Passeggi states that for a generic dissipative homeomorphism of the torus the rotation set is a polygon with rational vertices, possibly degenerated [1] [Pas14]. In this chapter we will prove the following result about generic dissipative homeomorphisms.

Theorem N. *For a generic dissipative homeomorphism,*

1. *the observable rotation set is the closure of the set of rotation vectors corresponding to Lyapunov stable periodic points (Lemma 6.16);*
2. *the convex hull of the observable rotation set, the convex hull of the asymptotic discretized rotation set and the rotation set are equal;*
3. *if the rotation set has non-empty interior, there is no need to take convex hulls,* i.e. *both observable and asymptotic discretized rotation sets coincide with the rotation set (Propositions 6.15 and 6.21).*

Thus, it is possible to obtain the rotation set of a generic dissipative homeomorphism from the observable or the asymptotic discretized rotation set. In other words, from the theoretical point of view, it is possible to recover numerically the rotation set of a generic homeomorphism. The generic conservative setting is quite different.

1. Namely it can be a segment or a singleton. However there are open sets of homeomorphisms where the rotation set has non-empty interior.

Theorem O. *For a generic conservative homeomorphism,*

1. *the rotation set has non-empty interior (Proposition 6.2);*
2. *the observable rotation set consists in a single point: the mean rotation vector (Proposition 6.19). On the other hand, the asymptotic discretized rotation set coincides with the rotation set (Corollary 6.23).*

These results suggest the quite surprising moral that to recover the rotation set of a conservative homeomorphisms, it is better to do coarse roundoff errors at each iteration. More precisely, if we compute a finite number of orbits with arbitrarily good precision and long length, we will find only the mean rotation vector of the homeomorphism; but if we make roundoff errors while computing, we will be able to retrieve the whole rotation set.

We have performed numerical simulations to see whether these behaviours can be observed in practice or not. To obtain numerically an approximation of the observable rotation set, we have calculated rotation vectors of long segments of orbits for a lot of starting points with high precision (these points being chosen randomly). For the numerical approximation of the asymptotic discretized rotation set we have chosen a fine enough grid on the torus and have calculated the rotation vectors of periodic orbits of the discretization of the homeomorphism on this grid.

We have chosen to make these simulations on some examples where the rotation set is known to be the square $[0,1]^2$. It makes us sure of the shape of the rotation set we should obtain numerically, however it limits a bit the "genericity" of the examples we can produce. We also produced simulations for a homeomorphism for which we do not know the shape of the rotation set.

In the dissipative case, we made attractive the periodic points which realize the vertex of the rotation set $[0,1]^2$. It is obvious that these rotation vectors, which are realized by attractive periodic points with basin of attraction of reasonable size, will be detected by the simulations of both observable and asymptotic discretized rotation sets; that is we observe in practice: we can recover quickly the rotation set in both cases (Figures 6.1 and 6.2).

In the conservative setting, we observe the surprising behaviour predicted by the theory: when we compute the rotation vectors of long segments of orbits we obtain mainly rotation vectors which are quite close to the mean rotation vector, in particular we do not recover the initial rotation set. More precisely, when we perform simulations with three hours of calculation we only obtain rotation vectors close to the mean rotation vector (Figure 6.7). On the other hand, when we calculate the union of the discretized rotation sets over several grids to obtain a simulation of the asymptotic discretized rotation set, the rotation set is detected very quickly by the convex hulls of discretized rotation sets (less than one minute of calculation) and when we compute more orders of discretizations, we obtain a set which is quite close to $[0,1]^2$ for Hausdorff distance (Figure 6.9). Moreover, when we compute the observable rotation set of a homeomorphism whose rotation set is unknown, we obtain a single rotation vector (Figure 6.10); but when we simulate the asymptotic discretized rotation set, then we obtain a sequence of "thick" sets whose convex hulls seem to converge (Figure 6.12). As for theoretical results, this suggests the following lesson:

When we compute segments of orbits with very good precision it is very difficult to recover the rotation set. However, when we decrease the number of digits used in computations we can obtain quickly a very good approximation of the rotation set.

This phenomenon can be explained by the fact that each grid of the torus is stabilized by the corresponding discretization of the homeomorphism. Thus, there exists an infinite number of grids such that every periodic point of the homeomorphism is shadowed by some periodic orbits of the discretizations on these grids.

6.1 Notations and preliminaries

6.1.1 Rotation sets

The definition of the rotation set is made to mimic the rotation number for homeomorphisms of the circle. At first sight the natural generalisation to dimension 2 of this notion is the point rotation set, defined as follows. For every homeomorphism f of the torus $\mathbf{T}^2$ homotopic to the identity [2] we take a lift $F : \mathbf{R}^2 \to \mathbf{R}^2$ of f to the universal cover $\mathbf{R}^2$ of $\mathbf{T}^2$. The difference with the one dimensional case is that as we lose the existence of a total order on our space, the sequence $\frac{F^n(\tilde{x})-\tilde{x}}{n}$ no longer need to converge. Thus, we have to consider all the possible limits of such sequences, called *rotation vectors*; the set of rotation vectors associated to $\tilde{x} \in \mathbf{R}^2$ will be denoted by $\overline{\rho}(\tilde{x})$:

$$\overline{\rho}(\tilde{x}) = \bigcap_{N_0 \in \mathbf{N}} \overline{\bigcup_{n \geq N_0} \left\{ \frac{F^n(\tilde{x}) - x}{n} \right\}}.$$

Then, the *point rotation set* is defined as $\rho_{pts}(F) = \bigcup_{\tilde{x} \in \mathbf{R}^2} \overline{\rho}(\tilde{x})$. Unfortunately this definition is not very convenient and it turns out that when we interchange the limits in the previous definition, we obtain the *rotation set*

$$\rho(F) = \bigcap_{M \in \mathbf{N}} \overline{\bigcup_{m \geq M} \left\{ \frac{F^m(\tilde{x}) - \tilde{x}}{m} \mid \tilde{x} \in \mathbf{R}^2 \right\}}$$

which has much better properties and is easier to manipulate. In particular, it is compact and convex (see [MZ89]), and it is the convex hull of $\rho_{pts}(F)$. Moreover, it coincides with the *measure rotation set*: if we denote by $D(F)$ the *displacement function*, defined on $\mathbf{T}^2$ by $D(F)(x) = F(\tilde{x}) - \tilde{x}$, where $\tilde{x}$ is a lift of x to $\mathbf{R}^2$ (we easily check that this quantity does not depend of the lift), then (recall that $\mathscr{M}^f$ is the set of f-invariant probability measures)

$$\rho(F) = \left\{ \int_{\mathbf{T}^2} D(F)(x) \ \mathrm{d}\mu \mid \mu \in \mathscr{M}^f \right\}.$$

Finally, for a homeomorphism f preserving λ, we denote by $\rho_{mean}(F)$ the *mean rotation vector* of F:

$$\rho_{mean}(F) = \int_{\mathbf{T}^2} D(F)(x) \ \mathrm{d}\lambda(x).$$

The geometry of the rotation set of a generic dissipative homeomorphism is given by a recent result published by A. Passeggi:

Theorem 6.1 (Passeggi, [Pas14]). *On an open and dense set of homeomorphisms $f \in \mathrm{Homeo}(\mathbf{T}^2)$, the rotation set is locally constant around f and is equal to a rational polygon.*

We end this paragraph by giving a proof that if f is a generic conservative homeomorphism of the torus, then $\rho(F)$ has non-empty interior.

2. In this chapter every homeomorphism will be supposed homotopic to the identity.

Proposition 6.2. *If f is generic* [3] *among* Homeo($\mathbf{T}^2, \lambda$), *then* $\rho(F)$ *has non-empty interior.*

Remark 6.3. We do not know the shape of the boundary of the rotation set of a generic conservative homeomorphism. In particular we do not know if it is a polygon or not.

Proof of Proposition 6.2. We use an argument due to S. Crovisier. If $\rho(F)$ consists in a single point, we use classical perturbation techniques for conservative homeomorphisms to create a persistent periodic point x for f. Then, by composing by a small rotation of the torus, we can move a little the mean rotation vector; in particular as the rotation set still contains the rotation vector of the periodic point x, it is not reduced to a single point. Now if the rotation set is a segment, by a C^0 ergodic closing lemma we can create a persistent periodic point whose rotation vector is close to the mean rotation vector in the following way. A small perturbation allows us to suppose that the homeomorphism we obtained, still denoted by f, is ergodic (it is the Oxtoby-Ulam theorem, see [OU41]). We then choose a recurrent point $y \in \mathbf{T}^2$ which verifies the conclusion of Birkhoff's theorem: for N large enough, the measure $\frac{1}{N}\sum_{k=0}^{N-1} \delta_{f^k(y)}$ is close to the measure λ. As this point is recurrent, by making a little perturbation, we can make it periodic (like in the proof of Lemma 5.46) and even persistent (see Definition 5.38 and Theorem 5.40 page 89); by construction $\overline{\rho}(y)$ is close to the mean rotation vector. We now have two persistent periodic points, say x and y, whose rotation vectors are different. It then suffices to compose by an appropriate rotation such that the mean rotation vector goes outside of the line generated by these two rotation vectors, and to repeat the construction to find a persistent periodic point whose rotation vector is close to this new mean rotation vector. Thus, we obtain a homeomorphism g which is arbitrarily close to f and possesses three periodic points x, y and z whose rotation vectors are non-aligned; therefore the rotation set of this homeomorphism has nonempty interior. Moreover, as the periodic points x, y and z are persistent, this property remains true on a neighbourhood of g, which concludes the proof for Homeo($\mathbf{T}^2$, Leb). □

6.1.2 Observable measures

From the ergodic point of view, we could be tempted to define the observable rotation set to be the set of rotation vectors associated to physical measures (see Definition 3.9), which are defined to express which measures can be observed in practice. However, such measures do not need to exist for every dynamical system, in this case the associated observable rotation set would be empty. To solve this problem of non existence of physical measures, E. Catsigeras and H. Enrich have defined in [CE11] the weaker notion of *observable measure*:

Definition 6.4. A probability measure μ is *observable* for f if, for every $\varepsilon > 0$, the set

$$A_\varepsilon(\mu) = \{x \in \mathbf{T}^2 \mid \exists \nu \in p\omega(x) : \mathrm{dist}(\nu, \mu) < \varepsilon\} \tag{6.1}$$

has λ-positive measure (recall that $p\omega(x)$ is the set of Birkhoff limits of x, see page 51). The set of observable measures is denoted by $\mathrm{Obs}(f)$.

The interesting property of these measures is that, unlike physical measures, they always exist. More precisely, the set $\mathrm{Obs}(f)$ is a non-empty compact subset of the set of invariant measures of f containing the set of physical measures (see [CE11]).

3. In fact on a open dense subset of Homeo($\mathbf{T}^2, \lambda$).

Remark 6.5. The behaviour of observable measures is compatible with topological conjugacy in the following sense: if μ is observable for f and h is a homeomorphism which preserves null sets, then $h^*\mu$ is observable for hfh^{-1}.

Example 6.6. 1. If $f = \mathrm{Id}$, then $\mathrm{Obs}(f) = \{\delta_x \mid x \in X\}$, but f has no physical measure.

2. If a dynamical system possesses a collection of physical measures whose basins of attraction cover almost all the phase space X (for example if it is ergodic with respect to a smooth measure), then the set of physical measures coincides with the set of observable measures.

Proposition 6.7. *If f is generic among* $\mathrm{Homeo}(\mathbf{T}^2)$, *then*

$$\mathrm{Obs}(f) = \mathrm{Cl}\{\delta_\omega \mid \omega \textit{ is a Lyapunov stable periodic orbit}\},$$

where Cl *denotes the closure.*

Thus, a generic homeomorphism f has a lot of observable measures [4], but no physical measure (it is a direct consequence of the shredding lemma 4.2, see [AA13]).

Proof of Proposition 6.7. The first inclusion is easy: it suffices to remark that every stable measure supported by a Lyapunov stable periodic orbit is observable.

For the other inclusion , let f be a generic dissipative homeomorphism, $\mu \in \mathrm{Obs}(f)$ and $\varepsilon > 0$. By hypothesis $\lambda(A_\varepsilon(\mu)) > 0$ (see Equation (6.1)), then $\varepsilon' = \frac{1}{2}\min(\varepsilon, \lambda(A_\varepsilon(\mu))) > 0$. As f is generic, it satisfies the conclusions of the shredding lemma (see Section 4.1) applied to f and ε', in particular there exists a Borel set $B \subset A_\varepsilon(\mu)$ and an open set $O \subset \mathbf{T}^2$ such that:

- $\lambda(B) > 0$;
- O is strictly periodic: $\exists i > 0 : f^i(O) \subset\subset O$;
- $\mathrm{diam}(O) < \varepsilon'$,
- every orbit of every point of B belongs to O eventually.

By Lemma 4.4 page 60, O contains a Lyapunov stable periodic point whose orbit is denoted by ω; thus for every $x \in B$ and every $\nu \in p\omega(x)$, we have $\mathrm{dist}(\nu, \delta_\omega) < \varepsilon'$. But by hypothesis $\mathrm{dist}(\nu, \mu) < \varepsilon$, then $\mathrm{dist}(\mu, \delta_\omega) < 2\varepsilon$, with ω a Lyapunov stable periodic orbit. □

Lemma 6.8. *If f is generic among* $\mathrm{Homeo}(\mathbf{T}^2, \lambda)$, *then* $\mathrm{Obs}(f) = \{\lambda\}$ *coincide with the set of physical measures.*

Proof of Lemma 6.8. A classical theorem of J. Oxtoby and S. Ulam [OU41] states that a generic conservative homeomorphism $f \in \mathrm{Homeo}(\mathbf{T}^2, \lambda)$ is ergodic with respect to the measure λ. But Remark 1.8 of [CE11] states that if the measure λ is ergodic, then $\mathrm{Obs}(f) = \{\lambda\}$. □

6.2 Observable rotation sets

6.2.1 Definitions

As said before, from the notion of observable measure, it is easy to define a notion of observable ergodic rotation set. Another definition, more topologic, seemed reasonable to us for observable rotation sets:

4. The set of Lyapunov stable periodic orbits is a Cantor set.

Definition 6.9.

$$\rho^{obs}(\mathrm{F}) = \Big\{ v \in \mathbf{R}^2 \mid \forall \varepsilon > 0,\ \lambda\Big\{ x \mid \exists u \in \overline{\rho}(x) : d(u,v) < \varepsilon \Big\} > 0 \Big\}.$$

$$\rho^{obs}_{mes}(\mathrm{F}) = \left\{ \int_{\mathbf{T}^2} \mathrm{D}(\mathrm{F})(x)\, \mathrm{d}\mu(x) \mid \mu \in \mathrm{Obs}(f) \right\}.$$

These two sets are non-empty compact subsets of the classical rotation set, and the first one is even a subset of $\rho_{pts}(\mathrm{F})$. The next lemma states that these two definitions coincide:

Lemma 6.10. $\rho^{obs}_{mes}(\mathrm{F}) = \rho^{obs}(\mathrm{F})$.

Proof of Lemma 6.10. We first prove that $\rho^{obs}_{mes}(\mathrm{F}) \subset \rho^{obs}(\mathrm{F})$. Let $v \in \rho^{obs}_{mes}(\mathrm{F})$ and $\varepsilon > 0$. Then there exists $\mu \in \mathrm{Obs}(f)$ such that $v = \int_{\mathbf{T}^2} \mathrm{D}(\mathrm{F})\, \mathrm{d}\mu$, in particular $\lambda(\mathrm{A}_{\varepsilon/2}(\mu)) > 0$. But if $x \in \mathrm{A}_{\varepsilon/2}(\mu)$, then there exists a strictly increasing sequence of integers $(n_i(x))_i$ such that for every $i \geq 0$,

$$\mathrm{dist}\left(\frac{1}{n_i(x)} \sum_{k=0}^{n_i(x)-1} \delta_{f^k(x)}, \mu \right) < \varepsilon.$$

Thus,

$$\left| \frac{1}{n_i(x)} \sum_{k=0}^{n_i(x)-1} \mathrm{D}(\mathrm{F})(f^k(x)) - \int_{\mathbf{T}^2} \mathrm{D}(\mathrm{F})\, \mathrm{d}\mu \right| < \varepsilon,$$

in other words the inequality

$$\left| \frac{\mathrm{F}^{n_i(x)}(x) - x}{n_i(x)} - v \right| < \varepsilon$$

holds for every i and on a λ-positive measure set of points x.

For the other inclusion, let $v \in \rho^{obs}(\mathrm{F})$ and set

$$\tilde{\mathrm{A}}_\varepsilon(v) = \{ x \in \mathbf{T}^2 \mid \exists u \in \overline{\rho}(x) : d(u,v) < \varepsilon \}.$$

By hypothesis, $\lambda(\tilde{\mathrm{A}}_\varepsilon(v)) > 0$ for every $\varepsilon > 0$. To each $x \in \tilde{\mathrm{A}}_\varepsilon(v)$ we associate the set $p\omega^v_\varepsilon(x)$ of limit points of the sequence of measures

$$\frac{1}{n_i(x)} \sum_{k=0}^{n_i(x)-1} \delta_{f^k(x)},$$

where $(n_i(x))_i$ is a strictly increasing sequence such that

$$\left| \frac{\mathrm{F}^{n_i(x)}(x) - x}{n_i(x)} - v \right| < \varepsilon.$$

By compactness of $\mathscr{P}$, the set $p\omega^v_\varepsilon(x)$ is non-empty and compact. In the sequel we will use the following easy remark: if $0 < \varepsilon < \varepsilon'$ and $x \in \tilde{\mathrm{A}}_\varepsilon$, then $p\omega^v_\varepsilon(x) \subset p\omega^v_{\varepsilon'}(x)$.

By contradiction, suppose that for every $\mu \in \mathscr{P}$, there exists $\varepsilon_\mu > 0$ such that

$$\lambda\Big\{ x \in \tilde{\mathrm{A}}_{\varepsilon_\mu}(v) \mid \exists \nu \in p\omega^v_{\varepsilon_\mu}(x) : \mathrm{dist}(\nu, \mu) < \varepsilon_\mu \Big\} = 0.$$

By compactness, $\mathscr{P}$ is covered by a finite number of balls $B(\mu_j, \varepsilon_{\mu_j})$. Taking $\varepsilon = \min \varepsilon_{\mu_j}$, for every j we have

$$\lambda\left\{x \in \tilde{A}_\varepsilon(v) \mid \exists \nu \in p\omega_\varepsilon^v(x) : \operatorname{dist}(\nu, \mu) < \varepsilon_{\mu_j}\right\} = 0,$$

thus, as balls $B(\mu_j, \varepsilon_{\mu_j})$ cover $\mathscr{P}$,

$$\lambda\left\{x \in \tilde{A}_\varepsilon(v) \mid p\omega_\varepsilon^v(x) \cap \mathscr{P} \neq \emptyset\right\} = 0,$$

which is a contradiction.

Therefore, there exists $\mu_0 \in \mathscr{P}$ such that for every $\varepsilon > 0$,

$$\lambda\left\{x \in \tilde{A}_\varepsilon(v) \mid \exists \nu \in p\omega_\varepsilon^v(x) : \operatorname{dist}(\nu, \mu_0) < \varepsilon\right\} > 0,$$

in particular $\mu_0 \in \mathrm{Obs}(f)$. Furthermore, for $\varepsilon > 0$, there exists $x \in \tilde{A}_\varepsilon(v)$ and $\mu_x \in p\omega_\varepsilon^v(x)$ such that $\operatorname{dist}(\mu_x, \mu_0) < \varepsilon$. As $\mu_x \in p\omega_\varepsilon^v(x)$, there exists a sequence $(n_i(x))_i$ such that

$$\operatorname{dist}\left(\mu_x, \frac{1}{n_i(x)} \sum_{k=0}^{n_i(x)-1} \delta_{f^k(x)}\right) < \varepsilon \qquad \text{and} \qquad \left|\frac{F^{n_i(x)}(x) - x}{n_i(x)} - v\right| < \varepsilon.$$

Thus,

$$\operatorname{dist}\left(\mu_0, \frac{1}{n_i(x)} \sum_{k=0}^{n_i(x)-1} \delta_{f^k(x)}\right) < 2\varepsilon.$$

Integrating this estimation according to the function $D(F)$, we obtain:

$$\left|\int_{\mathbf{T}^2} D(F)\, d\mu_0 - \frac{F^{n_i(x)}(x) - x}{n_i(x)}\right| < 2\varepsilon,$$

so

$$\left|\int_{\mathbf{T}^2} D(F)\, d\mu_0 - v\right| < 3\varepsilon,$$

for every $\varepsilon > 0$, in other words,

$$v = \int_{\mathbf{T}^2} D(F)\, d\mu_0.$$

□

6.2.2 Properties of the observable rotation set

We begin by giving two lemmas which state the dynamical behaviour of the observable rotation sets.

Lemma 6.11. *For every $q \in \mathbf{N}$, $\rho^{obs}(F^q) = q\rho^{obs}(F)$.*

Proof of Lemma 6.11. It suffices to remark that $\overline{\rho}_{F^q}(x) = q\overline{\rho}_F(x)$ (one inclusion is trivial and the other is easily obtained by Euclidean division). □

Remark 6.12. In general $\rho^{obs}(F^{-1}) \neq -\rho^{obs}(F)$: see for instance the point 3. of Example 6.14.

Lemma 6.13. *If* H *is a homeomorphism of* $\mathbf{R}^2$ *commuting with integral translations and preserving null sets, then* $\rho^{obs}(H \circ F \circ H^{-1}) = \rho^{obs}(F)$.

Proof of Lemma 6.13. It follows easily from the fact that the notion of observable measure is stable by conjugacy (see Remark 6.5). □

We now give a few simple examples of calculation of observable rotation sets.

Example 6.14. 1. If $f = \mathrm{Id}$, then $\rho^{obs}(F) = \{(0,0)\}$.

2. If
$$F(x,y) = (x + \cos(2\pi y), y),$$
then $\rho_{pts}(F) = \rho^{obs}(F) = [-1,1] \times \{0\}$.

3. If
$$F(x,y) = \left(x + \cos(2\pi y), y + \frac{1}{100}\sin(2\pi y)\right),$$
then $\rho_{pts}(F) = \{(0,-1),(0,1)\}$, but $\rho^{obs}(F) = \{(0,-1)\}$ and $\rho^{obs}(F^{-1}) = \{(0,1)\}$.

4. Let
$$P\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x + \frac{1}{2}\cos(2\pi y) + 1\\ y\end{pmatrix} \quad \text{and} \quad Q\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\ y + \frac{1}{2}\cos(2\pi x) + 1\end{pmatrix}.$$
Then the rotation set of the (conservative) homeomorphism $F = P \circ Q$ is equal to $[0,1]^2$. Moreover, we can perturb F into a (conservative) homeomorphism $\tilde{F}$ such that $\tilde{F}$ is the identity on the neighbourhoods of the points whose coordinates belong to $1/2\mathbf{Z}$ (applying for example Theorem 5.40). Then, the vertices of the square $[0,1]^2$ belong to the observable rotation set of $\tilde{F}$.

5. Let P be a convex polygon with rational vertices. In [Kwa92], J. Kwapisz has constructed an axiom A diffeomorphism f of $\mathbf{T}^2$ whose rotation set is the polygon P. It is possible to modify slightly Kwapisz's construction so that all the sinks of f are fixed points, and so that the union of the basins of these sinks have λ-full measure. Hence, the observable rotation set of f_P is reduced to $\{(0,0)\}$.

We now give the results about the link between the rotation set and the observable rotation set in the generic setting. We begin by the dissipative case.

Proposition 6.15. *If* f *is generic among* $\mathrm{Homeo}(\mathbf{T}^2)$, *then* $\rho(F) = \mathrm{conv}(\rho^{obs}(F))$. *If moreover* f *is generic with a non-empty interior rotation set, then* $\rho(F) = \rho^{obs}(F)$.

To prove this proposition, we will use the following lemma, which is a direct consequence of Proposition 6.7.

Lemma 6.16. *If* f *is generic among* $\mathrm{Homeo}(\mathbf{T}^2)$, *then*
$$\rho^{obs}(F) = \mathrm{Cl}\{\overline{\rho}(\tilde{x}) \mid x \text{ is a Lyapunov stable periodic point}\}.$$

We will also need a theorem of realization of rotation vectors by periodic points.

Theorem 6.17 (J. Franks, [Fra89])**.** *For every* $f \in \mathrm{Homeo}(\mathbf{T}^2)$, *every rational point of the interior of* $\rho(F)$ *is realized as the rotation vector of a periodic point of the homeomorphism* f.

Proof of Proposition 6.15. Theorem 6.1 states that for an open dense set of homeomorphisms, the rotation set is a rational polygon. Then, a theorem of realization of J. Franks [Fra88, Theorem 3.5] implies that every vertex of this polygon is realized as the rotation vector of a periodic point of the homeomorphism, which can be made attractive by a little perturbation of the homeomorphism. Then generically we can find a Lyapunov stable periodic point which shadows the previous periodic point (by Lemma 4.4), in particular it has the same rotation vector. Thus every vertex of $\rho(F)$ belongs to $\rho^{obs}(F)$ and $\rho(F) = \mathrm{conv}(\rho^{obs}(F))$.

For $\varepsilon > 0$, we can find a finite ε-dense subset R_ε of $\rho(F)$ made of rational points. Thus, Theorem 6.17 associates to each of these rational vectors a periodic point of the homeomorphism which realizes this rotation vector; we can even make these periodic points of the homeomorphism attractive. Thus, for every $\varepsilon > 0$, the set O_ε made of the homeomorphisms such that every vector of R_ε is realized by a strictly periodic open subset of $\mathbf{T}^2$ is open and dense in the set of homeomorphisms with non-empty interior rotation set. Applying Lemma 4.4 we find a G_δ dense subset of O_ε on which every strictly periodic open subset of $\mathbf{T}^2$ contains a Lyapunov stable periodic point; on this set the Hausdorff distance between $\rho(F) = \rho^{obs}(F)$ is smaller than ε. The conclusion of the proposition then easily follows from Baire theorem. □

Remark 6.18. It is not true that $\rho(F) = \rho^{obs}(F)$ holds for a generic homeomorphism: see for instance the point 3 of Example 6.14, where on a neighbourhood of f the set ρ^{obs} is contained in a neighbourhood of the points $(0,-1)$ and $(0,1)$.

For the conservative case, we recall the result of Proposition 6.2: the rotation set of a generic conservative homeomorphism has non-empty interior. The following result states that in this case the observable rotation set is much smaller, more precisely it consists in a single vector, namely the mean rotation vector.

Proposition 6.19. *If f is generic among* $\mathrm{Homeo}(\mathbf{T}^2, \lambda)$, *then* $\rho^{obs}(F) = \{\rho_\lambda(F)\}$, *where* $\rho_\lambda(F)$ *is the mean rotation vector with respect to the measure* λ.

Thus, for almost every $x \in \mathbf{T}^2$ (with respect to the measure λ), the set $\overline{\rho}(x)$ is reduced to a single point which is the mean rotation vector.

Proof of Proposition 6.19. It is easily implied by the fact that the measure λ is the only observable measure (Lemma 6.8, which easily follows from Oxtoby-Ulam theorem). □

6.3 Discretized rotation sets

We now take into account the fact that the computer has a finite digital precision. It will be the occasion to apply the techniques of proof explained in Chapter 5.

The *discretized rotation set* is defined as follows. Consider a lift $F : \mathbf{R}^2 \to \mathbf{R}^2$ of f and a lift $\tilde{E}_N$ of the grid E_N to $\mathbf{R}^2$. Then

$$\rho(F_N) = \bigcap_{M \in \mathbf{N}} \overline{\bigcup_{m \geq M} \left\{ \frac{F_N^m(\tilde{x}) - \tilde{x}}{m} \mid \tilde{x} \in \mathbf{R}^2 \right\}}.$$

Remark that this set coincides with the set of rotation vectors of the periodic orbits of f_N. Then the *asymptotic discretized rotation set* is the upper limit of the sets $\rho(F_N)$:

$$\rho^{discr}(F) = \bigcap_{M \in \mathbf{N}} \overline{\bigcup_{N \geq M} \rho(F_N)}.$$

The first result is that for every homeomorphism f, the discretized rotation set $\rho(F_N)$ is almost included in the rotation set $\rho(F)$ when N is large enough. This property follows easily with a compactness argument from the convergence of the sequence f_N to the homeomorphism f (for example for the Hausdorff distance on the graphs of these maps).

Proposition 6.20. *For every homeomorphism f and every $\varepsilon > 0$, it exists $N_0 \in \mathbf{N}$ such that for every $N \geq N_0$, we have $\rho(F_N) \subset B(\rho(F), \varepsilon)$, where $B(\rho(F), \varepsilon)$ denotes the set of points whose distance to $\rho(F)$ is smaller than ε. In particular $\rho^{discr}(F) \subset \rho(F)$.*

Proof of Proposition 6.20. By definition of the rotation set, for $\varepsilon > 0$ there exists $m \in \mathbf{N}$ such that

$$\left\{ \frac{F^m(\tilde{x}) - \tilde{x}}{m} \mid \tilde{x} \in \mathbf{R}^2 \right\} \subset B(\rho(F), \varepsilon).$$

Then there exists $N_0 \in \mathbf{N}$ such that for every $N \geq N_0$,

$$\left| \frac{F^m(\tilde{x}) - \tilde{x}}{m} - \frac{F_N^m(\tilde{x}_N) - \tilde{x}_N}{m} \right| \leq \varepsilon.$$

This allows us to handle the case of long periodic orbits of the discretizations: by euclidean division, each periodic orbit of f_N of length bigger than m/ε will be in the ε neighbourhood of the convex hull of the set

$$\frac{F_N^m(\tilde{x}_N) - \tilde{x}_N}{m},$$

so in the 3ε-neighbourhood of the rotation set $\rho(f)$.

For short orbits we argue by contradiction: suppose that there exists $\varepsilon > 0$ such that for every $N_0 \in \mathbf{N}$ there exists $N \geq N_0$ and $x_N \in E_N$ which is periodic under f_N with period smaller than m/ε and whose associated rotation vector is not in $B(\rho(F), \varepsilon)$. Then up to take subsequences these periodic points x_N have the same period and converge to a periodic point $x \in \mathbf{T}^2$ whose associated rotation vector (for F) is not in $B(\rho(F), \varepsilon)$, which is impossible. □

The other inclusion depends on the properties of the map f. We begin by the dissipative case.

Proposition 6.21. *If f is generic among* $\mathrm{Homeo}(\mathbf{T}^2)$, *then* $\mathrm{conv}\left(\rho(F_N)\right)$ *tends to* $\rho(F)$ *for the Hausdorff topology. In particular* $\mathrm{conv}\left(\rho^{discr}(F)\right) = \rho(F)$. *Moreover, if $\rho(F)$ has nonempty interior, then there is no need to take convex hulls.*

Proof of Proposition 6.21. The fact that the upper limit of $\rho(F_N)$ is included in $\rho(F)$ follows directly from Lemma 6.20.

It remains to prove that the lower limit of $\mathrm{conv}\left(\rho(F_N)\right)$ contains $\rho(F)$. To do that, we prove that $\rho(F_N)$ converges to $\rho^{obs}(F)$. First of all the rotation set is the closure of the convex hull of the rotation vectors of Lyapunov stable periodic points (Propositions 6.7 and 6.15). To each one of these points we can associate a periodic closed set K with non-empty interior and with period τ which has the same rotation vector. Then there exists an open set $O \subset K$ such that for N large enough and $x \in K$ we also have $f_N^\tau(x_N) \in O \subset K$. Thus there exists $i \in \mathbf{N}^*$ such that $f_N^{\tau i}(x_N) = f_N^{2\tau i}(x_N)$ and $f_N^{\tau i}(x_N)$ has the same rotation vector as K, thus the same rotation vector as the initial Lyapunov stable periodic point. □

For the conservative case, with the same techniques as in Chapter 5, we can prove the following result:

Lemma 6.22. *If f is generic among* $\mathrm{Homeo}(\mathbf{T}^2,\lambda)$, *then for every finite collection of rotation vectors $\{v_1,\cdots,v_n\}$, each one realized by a periodic orbit of f, there exists a subsequence f_{N_i} of discretizations such that for every i, $\rho_{N_i}(f)=\{v_1,\cdots,v_n\}$.*

Proof of Lemma 6.22. We denote by $\mathscr{D}_q$ the set of subsets of $\mathbf{Q}^2$ made of elements whose coordinates are of the type p'/q', with $0<q'<q$ and $-q^2<p'<q^2$. Consider the set

$$\bigcap_{q,N_0}\bigcap_{D\in\mathscr{D}_q}\bigcup_{N\geq N_0}\left\{\begin{array}{c} f\in\mathrm{Homeo}(\mathbf{T}^2,\lambda)\,|\,(\forall v\in D,\ v \text{ is realised by a}\\ \text{persistent periodic point of } f)\implies \rho(F_N)=D\end{array}\right\}.$$

To prove the lemma it suffices to prove that this set contains a G_δ dense. This is obtained with the same kind of proof as for Proposition 5.14.

Let $f\in\mathrm{Homeo}(\mathbf{T}^2,\lambda)$, $\varepsilon>0$, $q,N_0\in\mathbf{N}$ and $D\in\mathscr{D}_q$. We suppose that for all $v\in D$, v is realizable by a persistent periodic orbit ω_i of f. For all of these orbits $\omega_1,\cdots,\omega_\ell$, we denote by p_i the length of the orbit ω_i and choose a point x_i belonging to ω_i. We then apply Lax's theorem (Theorem 5.5): if N is large enough, then there exists a cyclic permutation σ_N of E_N such that $d_N(f,\sigma_N)<\varepsilon$. If N is large enough, then the families

$$\left\{(x_1)_N,\cdots,\sigma_N^{p_1-1}((x_1)_N)\right\},\cdots,\left\{(x_\ell)_N,\cdots,\sigma_N^{p_\ell-1}((x_\ell)_N)\right\}$$

are disjoint and satisfy $d\Big((x_i)_N,\sigma_N^{p_i-1}((x_i)_N)\Big)<\varepsilon$ for all i. We then use the same technique as in the proof of Proposition 5.14 (see also Figure 5.2) to close each orbit $\{(x_i)_N,\cdots,\sigma_N^{p_i-1}((x_i)_N)\}$. The discrete map σ'_N we obtain has then exactly ℓ periodic orbits, and each of them has the same rotation vector as the corresponding real periodic orbit of f. As in the proof of Lemma 5.4, we then use the proposition of finite maps extension (Proposition 3.3) to build a homeomorphism g which is ε-close to f and whose discretization g_N satisfies $\rho(G_N)=D$, and moreover we can suppose that this occurs on a whole neighbourhood of g. □

The combination of the realisation theorem of J. Franks [Fra89, Theorem 3.2] and the fact that for a generic conservative homeomorphism the rotation set has non-empty interior (Proposition 6.2) leads to the following corollary.

Corollary 6.23. *If f is generic among* $\mathrm{Homeo}(\mathbf{T}^2,\lambda)$, *then for every compact subset* K *of the rotation set of* F *there exists a subsequence f_{N_i} of discretizations such that $\rho_{N_i}(F)$ tends to* K *for the Hausdorff topology. In particular $\rho^{discr}(F)=\rho(F)$.*

6.4 Numerical simulations

We have conducted numerical simulations of the rotation sets associated to both dissipative and conservative homeomorphisms. For the first examples we treated, we have made the deliberate choice to choose homeomorphisms whose rotation set is known to be the square $[0,1]^2$. Of course these homeomorphisms are not the best candidates for "generic" homeomorphisms, but at least we are sure of what is the shape of the rotation set we want to obtain.

As an example of dissipative homeomorphism we have taken $f_1 = R_1 \circ Q_1 \circ P_1$, and for the conservative homeomorphism we have chosen the very similar expression $g_1 = Q_1 \circ P_1$, where

$$\begin{aligned} P_1(x,y) = &\Big(x\ ,\ y + \frac{1}{2}\Big(\cos(2\pi(x+\alpha)) + 1\Big) \\ &+ 0.0234\sin^2(4\pi(x+\alpha))\Big(\sin(6\pi(x+\alpha)) + 0.3754\cos(26\pi(x+\alpha))\Big)\Big), \end{aligned}$$

$$\begin{aligned} Q_1(x,y) = &\Big(x + \frac{1}{2}\Big(\cos(2\pi(y+\beta)) + 1\Big) \\ &+ 0.0213\sin^2(4\pi(y+\beta))\Big(\sin(6\pi(y+\beta)) + 0.4243\cos(22\pi(y+\beta))\Big)\ ,\ y\Big), \end{aligned}$$

$$\begin{aligned} R_1(x,y) = &\big(x - 0.0127\sin(8\pi(x+\alpha)) + 0.000324\sin(33\pi(x+\alpha))\ , \\ &y - 0.0176\sin(12\pi(y+\beta)) + 0.000231\sin(41\pi y)\big), \end{aligned}$$

with $\alpha = 0.00137$ and $\beta = 0.00159$.

The homeomorphisms P_1 and Q_1 are close to the homeomorphisms

$$\tilde{P}(x,y) = \Big(x\ ,\ y + \frac{1}{2}\Big(\cos(2\pi(x+\alpha)) + 1\Big)\Big)$$

and

$$\tilde{Q}(x,y) = \Big(x + \frac{1}{2}\Big(\cos(2\pi(y+\beta)) + 1\Big)\ ,\ y\Big);$$

it can easily be seen that the rotation set of the homeomorphism $\tilde{Q} \circ \tilde{P}$ is the square $[0,1]^2$, whose vertices are realized by the points $(0,0)$, $(0,1/2)$, $(1/2,0)$ and $(1/2,1/2)$. The perturbations P_1 and Q_1 of $\tilde{P}$ and $\tilde{Q}$ are small enough (in C^2 topology) to ensure that the rotation set remains the square $[0,1]^2$; these perturbations are made in order to make f_1 "more generic" (in particular, the periodic orbits whose rotation vectors realize the vertices of the square do not belong to the grids). The key property of the homeomorphism R_1 is that is has the fixed points of f_1 which realize the vertices of $[0,1]^2$ as fixed attractive points; this creates fixed attractive points which realize the vertices of the rotation set.

We have chosen R_1 to be very close to the identity in C^1-topology to ensure that the basins of the sinks and sources are large enough. Indeed, J.-M. Gambaudo and C. Tresser have shown in [GT83] that, even for dissipative diffeomorphisms defined by very simple formulas, sinks and sources are often undetectable in practice because the size of the their basins are too small.

We have conducted other series of simulations for two other examples of conservative homeomorphisms. The first one has an expression which is very similar to that of g_1, but the cosines are replaced by a piecewise affine map with the same following properties: s is 1-periodic, $s(0) = 1$, $s(1/2) = 0$ and s is affine between 0 and 1/2 and between 1/2 and 1. More precisely, we set $g_2 = Q_2 \circ P_2$, with

$$\begin{aligned} P_2(x,y) = &\Big(x\ ,\ y + 2s(x+\alpha) + 0.0234s(2(x+\alpha)) \\ &+ 0.0167s(10(x+\alpha)\Big); \end{aligned}$$

$$Q_2(x,y) = \Big(x + 2s(y+\beta) + 0.0213s(2(y+\beta)) \\ + 0.0101s(6(y+\beta))\ ,\ y\Big).$$

The properties of s imply that the rotation set of g_2 is also the square $[0,1]^2$; the difference with g_1 is that the vertices of this rotation set are no longer realized by elliptic periodic points, which makes them harder to detect.

For the last conservative homeomorphism we tested, we made "random" choices of the coefficients; we do not know *a priori* what is its rotation set. More precisely, we took $g_3 = Q_3 \circ P_3$, with

$$P_3(x,y) = \Big(x\ ,\ y + 0.3\sin(2\pi(x+0.34137)) \\ + 0.2\sin(3\pi(x+0.21346)) + 0.578675)\Big);$$

$$Q_3(x,y) = \Big(x + 0.25\sin(2\pi(y+0.9734)) \\ + 0.35\sin(3\pi(y-0.20159)) + 0.551256\ ,\ y\Big).$$

We will test on simulations whether the computed rotation sets seem to converge or not. If so, it could be a good indication that the rotation set we obtained is close to the actual rotation set.

We have made two kinds of simulations of the rotation set.

- In the first one we have computed the rotation vectors of segments of orbits of length 1 000 with good precision (52 binary digits); in other words for N random starting points $x \in \mathbf{T}^2$, we have computed $\frac{F^{1000}(x)-x}{1000}$. We have made these tests for N = 100, which takes around 1s of calculation, N = 10 000, which takes about 2min of calculation, and N = 1 000 000, which takes about 4h of calculation. This is maybe the most simple process that can be used to find numerically the rotation set. It should lead to a good approximation of the observable rotation set; in particular, Proposition 6.15 suggests that, for the dissipative homeomorphism f_1, we should obtain a set which is close (for Hausdorff distance) to the square $[0,1]^2$, and if not at least a set whose convex hull is this square. On the other hand, for the conservative homeomorphisms g_i, Proposition 6.19 suggests that we should only obtain the mean rotation vector, which is close to $(1/2, 1/2)$.
- In the second kind of simulations we have computed the rotation vectors of the periodic orbits of the discretization $(f_i)_N$ on a grid $N \times N$; these simulations calculate the discretized rotation sets. For each homeomorphism we have represented these sets for N = 499, N = 500 and N = 501, each calculation taking about 2s of calculation. We have also computed the union of the discretized rotation sets for $100 \le N \le M$, which represents the asymptotic discretized rotation set. We represent these sets for M = 100 ($\simeq$ 0.5s of calculation), M = 150 ($\simeq$ 15s of calculation), M = 200 ($\simeq$ 45s of calculation), M = 500 ($\simeq$ 13min of calculation), M = 1 000 ($\simeq$ 1h 45min of calculation), M = 2 000 ($\simeq$ 14h of calculation) and for g_3, M = 3950 ($\simeq$ 100h of calculation). The theory tells us that in both conservative and dissipative cases, for some N, the discretized rotation set should be close (for Hausdorff distance) to the square $[0,1]^2$; a weaker property would be that its convex hull should be close to this square. Moreover this should also be true for the asymptotic discretized rotation sets.

We shall notice that these two methods are formally the same: making simulations on a grid $N \times N$ is equivalent to calculate with $-\log_2 N$ binary digits (for example about 10 for $N = 1\,000$). The only difference is that for the second method we use deliberately a very bad numerical precision, which allows us to detect the actual dynamics of the discretizations.

Moreover, in practice, for a given calculation time, the calculation of the rotation set by discretization (*i.e.* by the second method) allows to compute much more orbits than the other method. More precisely, the algorithm we have used to compute the asymptotic discretized rotation set visits each point of the grid $N \times N$ once. Thus, for N^2 starting points we only have to compute N^2 images of the discretization of the homeomorphism on the grid; the number of rotation vectors we obtain is simply the number of periodic orbits of the discretization. So in a certain sense this second algorithm is much faster than the naive algorithm consisting in computing long segments of orbits. All the simulations have been performed on a computer equipped with a processor Intel Core I5 2.40GHz.

We shall notice that the calculated rotation sets we obtained for f_1, g_1 and g_2 are always contained inside of the square $[0,1]^2$ (see the figures below). Indeed, the very definition of the rotation set ensures that if T is large enough, then the rotation vector of every segment of orbit of length T belongs to a neighbourhood of the rotation set; therefore the computed observable rotation set should be included in a small neighbourhood of the rotation set provided we have chosen a large enough length of orbit. Concerning the discretized rotation set, Proposition 6.20 ensures that if the order of discretization is large enough, then the discretized rotation set is included in a neighbourhood of the actual rotation set. However, there is no global estimation of these integers, as can be seen in the following example. Set $f : \mathbf{T}^2 \to \mathbf{T}^2$ defined by

$$f(x,y) = \Big(x + \alpha\,,\, y + \sin(2\pi x)\Big),$$

with α an irrational number close to (say) $1/(2T)$. As $\alpha \in \mathbf{R} \setminus \mathbf{Q}$, we easily obtain that the rotation set of f is equal to $(\alpha, 0)$; but if we compute the rotation numbers of all the segment of orbit of length T, they will form a set which is close to the segment $\Big[(\alpha, -2/\pi), (\alpha, 2/\pi)\Big]$.

In the dissipative case, a lot of the obtained rotation vectors are close to one of the vertices of the real rotation set $[0,1]^2$ of f_1, the others being located around $(1/2, 1/2)$ (see Figure 6.1). This is what is predicted by the theory, in particular by Lemma 6.16: we detect rotation vectors realized by Lyapunov stable periodic points. The fact that the rotation vectors are not located exactly on the vertices of $[0,1]^2$ can be explained by the slow convergence of the orbits to the attractive points: it may take a while until the orbit become close to one of the Lyapunov stable periodic points. We will see that this behaviour is very different from the one in the conservative case, even if the homeomorphism f_1 is very close to g_1 (approximately 10^{-2} close).

For the discretized rotation set for f_1, the vertices of $[0,1]^2$ are also detected, and we only have a few points in the interior of the square (see Figure 6.2). However, when we compute the asymptotic discretized rotation set (see Figure 6.3), we observe that the computed rotation vectors fill a great proportion of the square $[0,1]^2$, as predicted by the theory.

In the conservative case, the rotation vectors of the observable rotation set are mainly quite close to the mean rotation vector of g_1, as predicted by Proposition 6.19.

In particular in Figure 6.4, left, all the 100 rotation vectors of the computed observable rotation set are in the neighbourhood of $(1/2, 1/2)$. Thus, the behaviour of these vectors is governed by Birkhoff's ergodic theorem with respect to the ergodic measure Leb; *a priori* this behaviour is quite chaotic and converges slowly: a typical orbit will visit every measurable subset with a frequency proportional to the measure of this set, so the rotation vectors will take time to converge. When the number of computed orbits increases (Figure 6.4, middle and rignt), we observe that a few rotation vectors are not close to the mean rotation vector; for 10^6 different orbits we even detect three of the vertices of the actual rotation set. Anyway, even after 4 hours of calculation, we are unable to recover completely the initial rotation set of the homeomorphism.

On the other hand, the convex hull of the discretized rotation set gives quickly a very good approximation of the rotation set. For example on a grid 500×500 (Figure 6.5), with 2s of calculation (and even on a grid 100×100 and $0.2s$ of calculation), we obtain a rotation set whose convex hull is already very close to $[0,1]^2$. However, for a single size of grid, we do not obtain exactly the conclusions of Corollary 6.23 which states that for some integers N the discretized rotation set should be close to the rotation set for Hausdorff distance; here for each N we only have a few points in the interior of $[0,1]^2$. That is why we represented the union of the discretized rotation sets on grids $N \times N$ with $100 \leq N \leq 1\,000$ (Figure 6.6). In this case we recover almost all the rotation set of g_1, except from the points which are close to one edge of the square but far from its vertices. The fact that we can obtain very easily the vertices of the rotation set can be due to the fact that in our example f_1 these vertices are realized by elliptic periodic points of the homeomorphism (in fact the derivative on this points is the identity). That is why we also conducted simulations of the homeomorphism g_2 which rotation set is also the square $[0,1]^2$ whose vertices are realized by non-elliptic periodic points.

In fact, when we compute the observable rotation set for g_2 (Figure 6.7), we only find rotation vectors which are close to the mean rotation vector, even after 4h of calculation. As the periodic points which realize the vertices of the rotation set are no longer elliptic, they are much more unstable and thus they are not detected by these simulations.

The sets detected by the discretized rotation sets of order 499, 500 and 501 (Figure 6.8) are quite bigger than those detected by the simulations of the observable rotation set, even if the time of calculation is much smaller. However, we do not recover the whole rotation set of the homeomorphism (we conducted simulations for higher orders around $N = 1\,000$ and $N = 2\,000$ and the behaviour is similar). By contrast, the simulations of the asymptotic discretized rotation set (Figure 6.9) allows us to see the actual rotation set of the homeomorphism: when we represent all the rotation vectors of the discretizations of order $100 \leq N \leq M$ with $M = 200$ (which takes about 45s of calculation) we obtain a set which is very close to the square $[0,1]^2$; for $M = 100$ ($\simeq$ 1h 45min of calculation) we recover almost exactly the initial rotation set.

Finally, the behaviour of the observable rotation set of g_3 is very similar to that of g_2 (see Figure 6.10): even when we compute $1\,000\,000$ different orbits with random starting points, we only obtain rotation vectors which are close to $(0.55, 0.5)$, which should be a good approximation of the mean rotation vector.

Like for g_2, the simulations of the discretized rotation sets for the grids E_N with $N \in \{499, 500, 501\}$ (Figure 6.11) are not very convincing: the sets do not seem to converge to anything. We have to compute the asymptotic discretized rotation sets (Figure 6.11) to see something that looks like a convergence for the Hausdorff topology of the computed rotation sets. However, this convergence in practical is just an indication that the set we

compute is close to the actual rotation set of g_3. To our knowledge, it is impossible to ensure that for a given order of discretization, the asymptotic rotation set computed to this order is close to the rotation set of g_3.

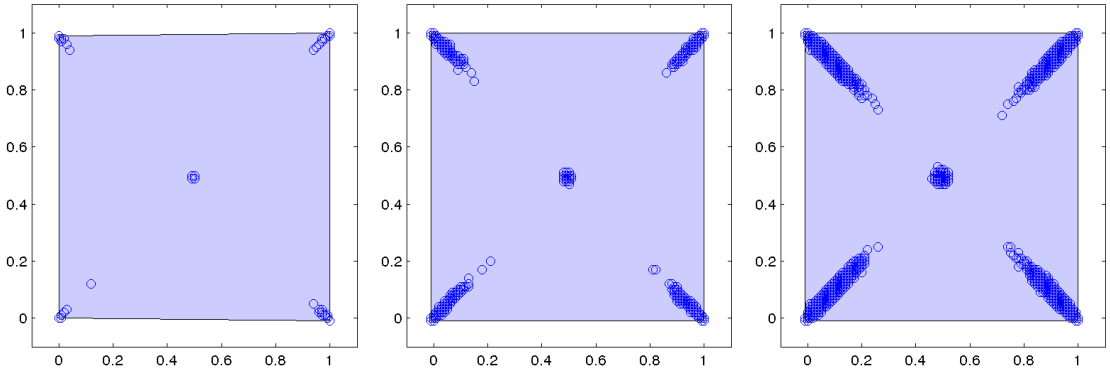

Figure 6.1: Observable rotation set of f_1, k orbits of length 1 000 with random starting points with $k = 100$ (left), 10 000 (middle) and 1 000 000 (right)

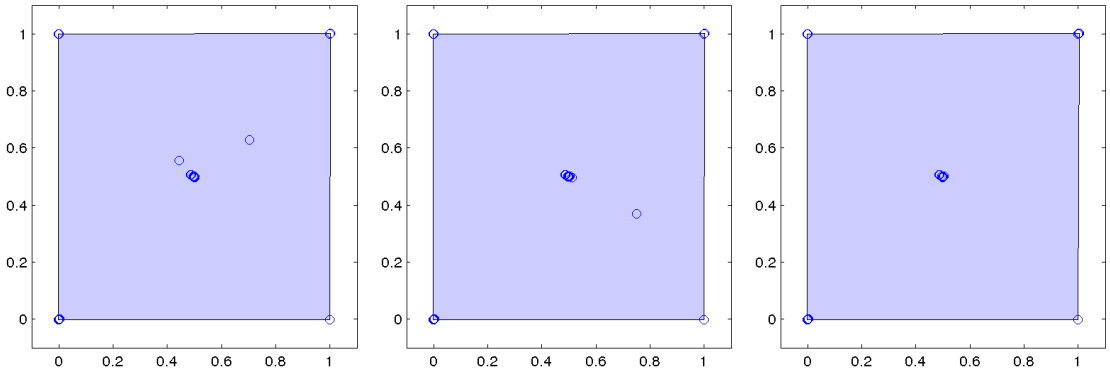

Figure 6.2: Discretized rotation set of f_1 on grids E_N, with N = 499 (left), N = 500 (middle) and N = 501 (right)

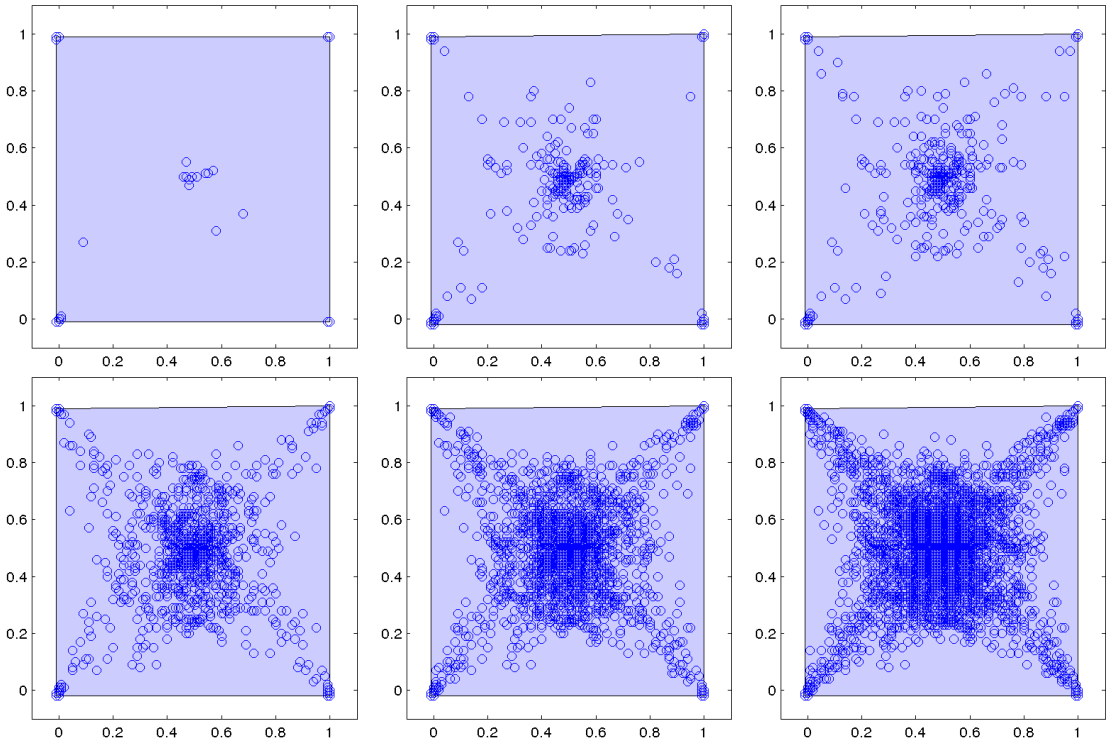

Figure 6.3: Asymptotic discretized rotation set of f_1 as the union of the discretized rotation sets on grids E_N with $100 \leq N \leq M$ with $M = 100$ (top left), $M = 150$ (top middle), $M = 200$ (top right), $M = 500$ (bottom left), $M = 1\,000$ (bottom middle) and $M = 2\,000$ (bottom right)

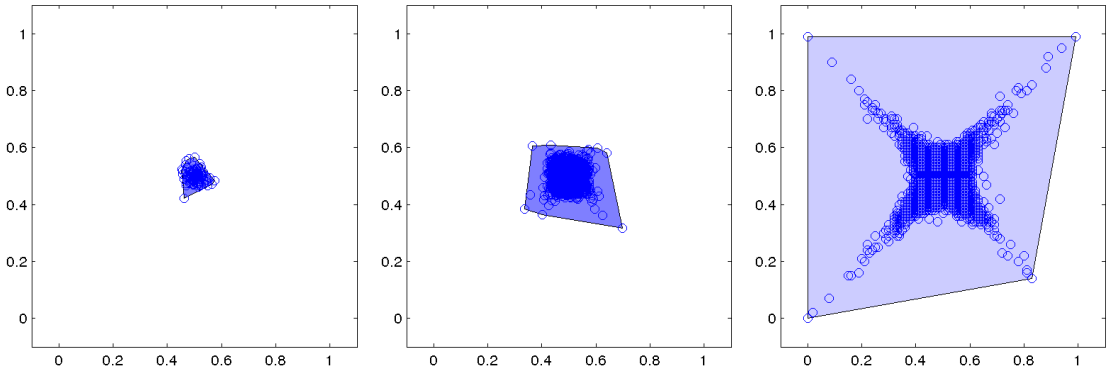

Figure 6.4: Observable rotation set of g_1, k orbits of length 1 000 with random starting points with $k = 100$ (left), 10 000 (middle) and 1 000 000 (right)

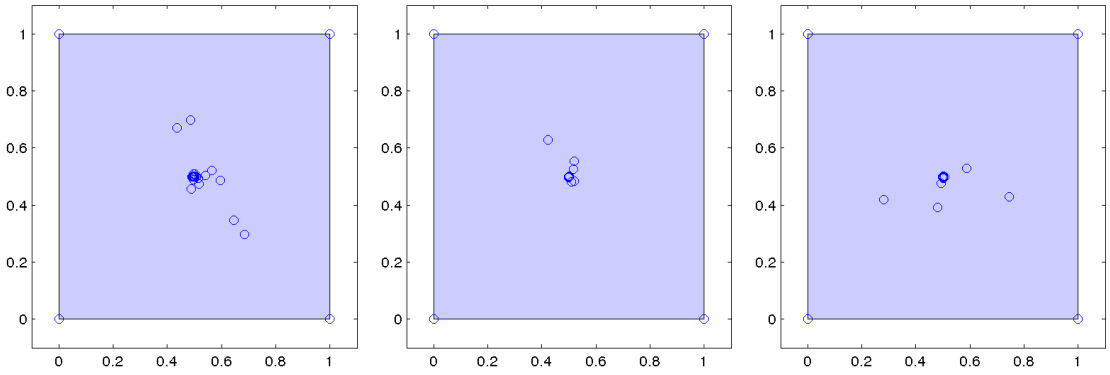

Figure 6.5: Discretized rotation set of g_1 on grids E_N, with N = 499 (left), N = 500 (middle) and N = 501 (right)

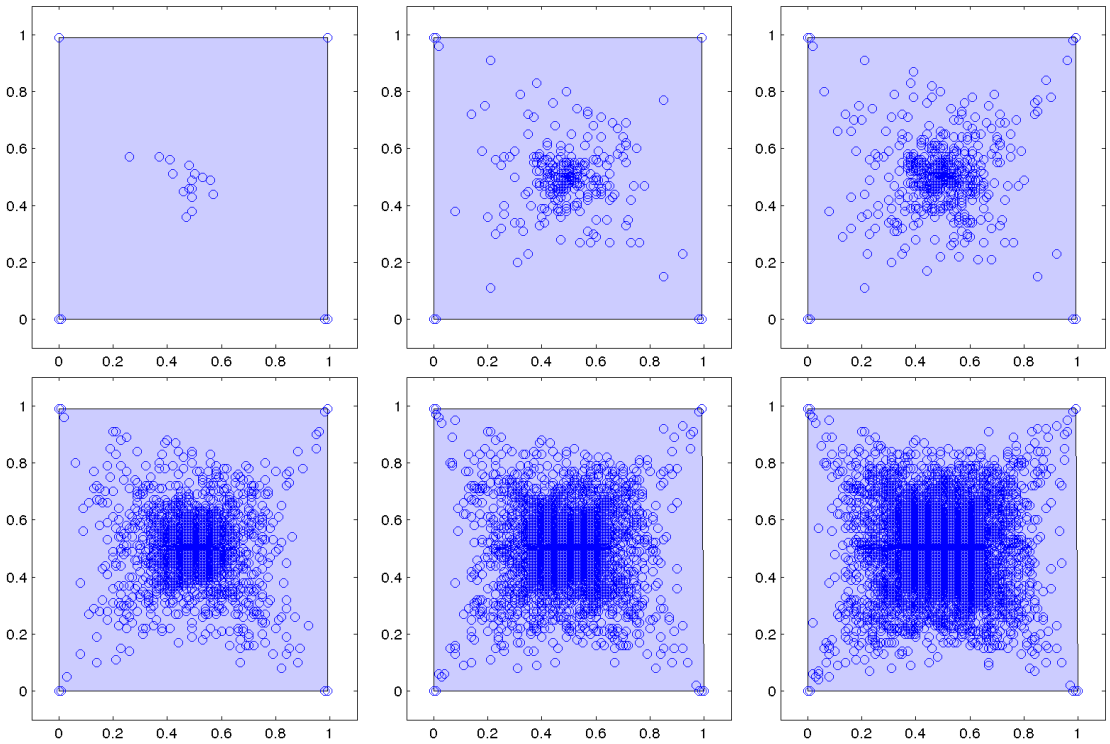

Figure 6.6: Asymptotic discretized rotation set of g_1 as the union of the discretized rotation sets on grids E_N with $100 \leq N \leq M$ with M = 100 (top left), M = 150 (top middle), M = 200 (top right), M = 500 (bottom left), M = 1 000 (bottom middle) and M = 2 000 (bottom right)

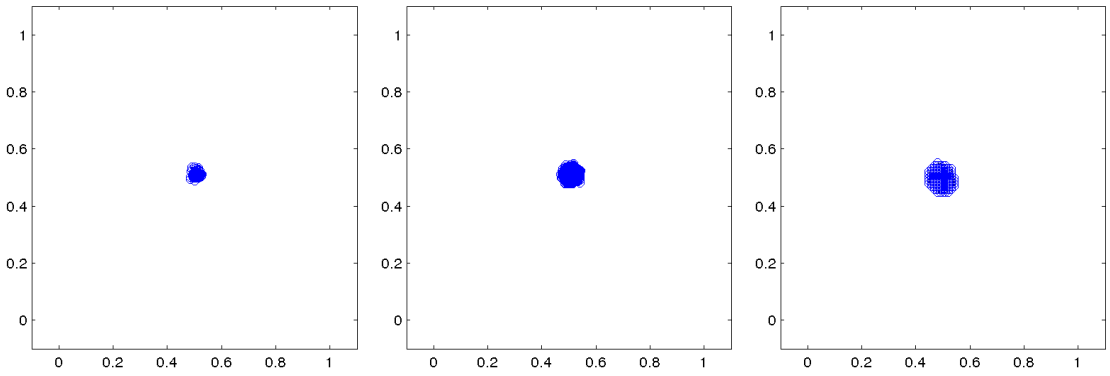

Figure 6.7: Observable rotation set of g_2, k orbits of length 1 000 with random starting points with $k = 100$ (left), 10 000 (middle) and 1 000 000 (right)

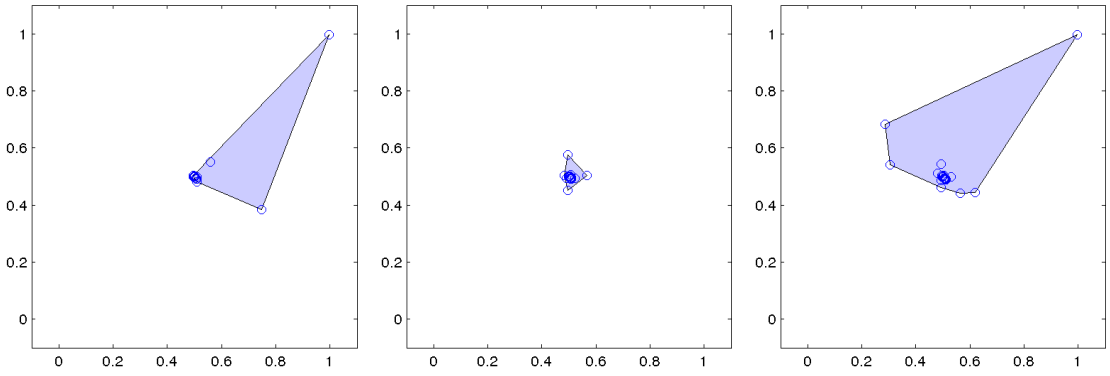

Figure 6.8: Discretized rotation set of g_2 on grids E_N, with N = 499 (left), N = 500 (middle) and N = 501 (right)

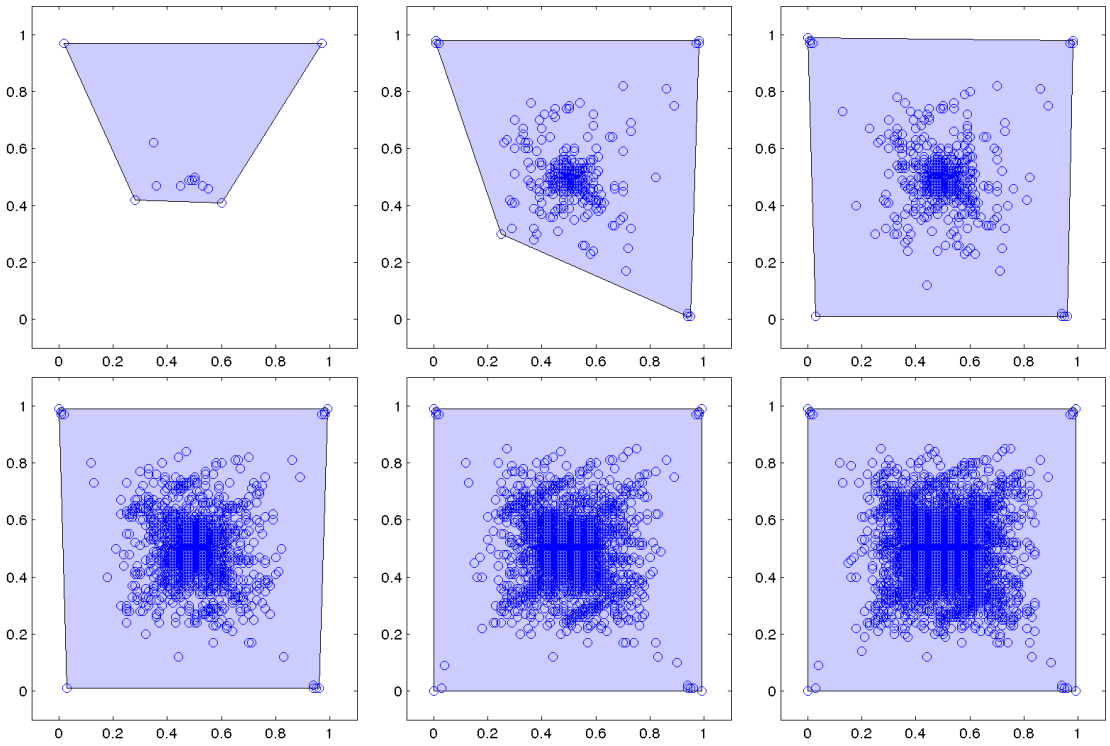

Figure 6.9: Asymptotic discretized rotation set of g_2 as the union of the discretized rotation sets on grids E_N with $100 \leq N \leq M$ with $M = 100$ (top left), $M = 150$ (top middle), $M = 200$ (top right), $M = 500$ (bottom left), $M = 1\,000$ (bottom middle) and $M = 2\,000$ (bottom right)

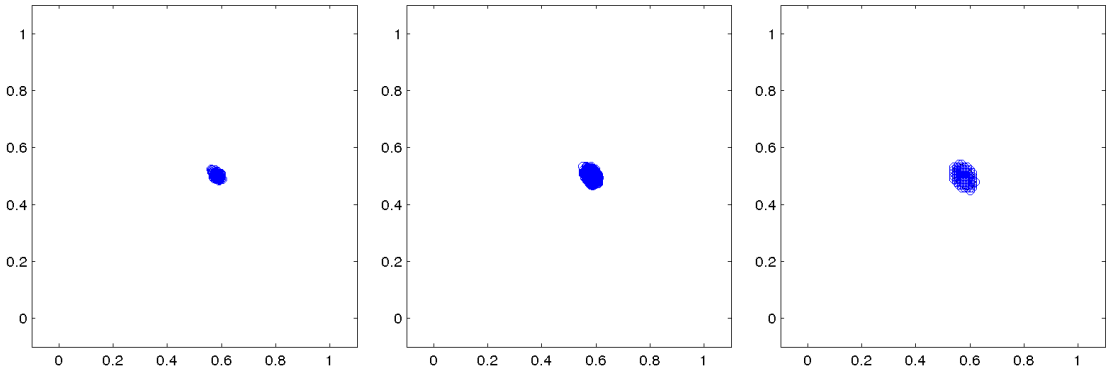

Figure 6.10: Observable rotation set of g_3, k orbits of length 1 000 with random starting points with $k = 100$ (left), 10 000 (middle) and 1 000 000 (right)

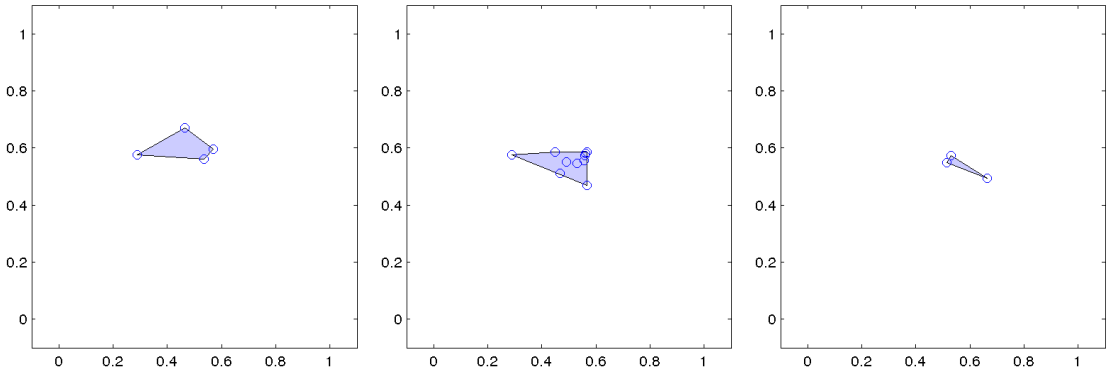

Figure 6.11: Discretized rotation set of g_3 on grids E_N, with N = 499 (left), N = 500 (middle) and N = 501 (right)

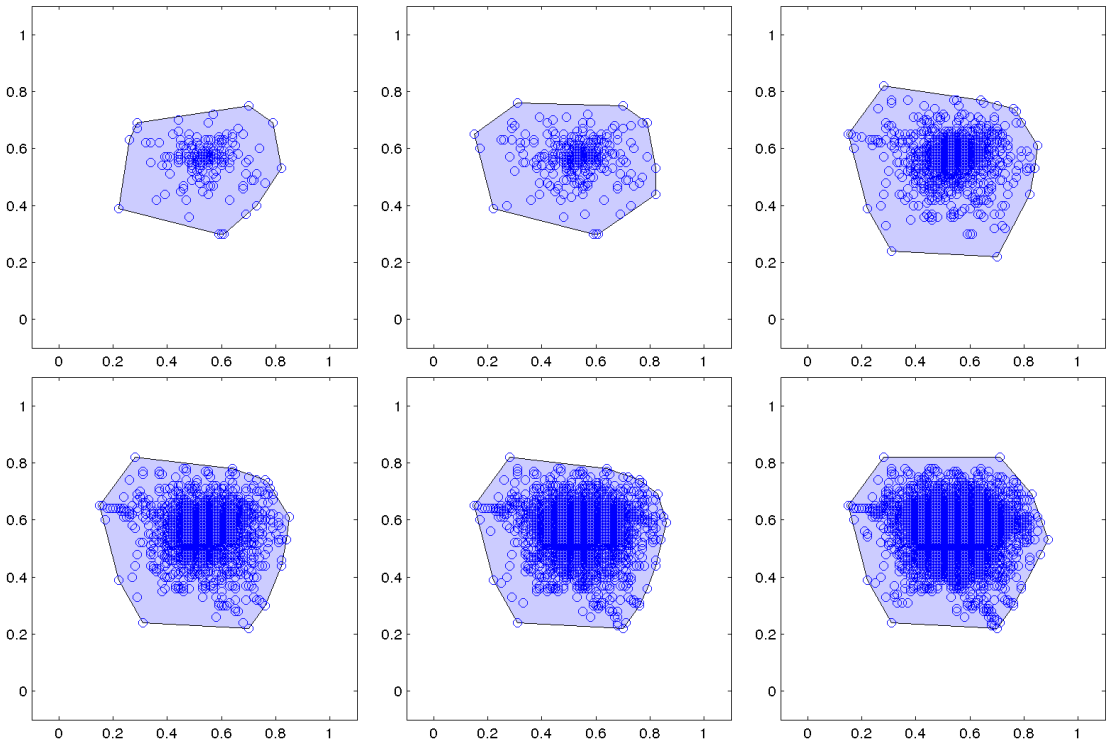

Figure 6.12: Asymptotic discretized rotation set of g_3 as the union of the discretized rotation sets on grids E_N with $100 \leq N \leq M$ with M = 150 (top left), M = 200 (top middle), M = 500 (top right), M = 1 000 (bottom left), M = 2 000 (bottom middle) and M = 3 950 (bottom right)

Part 2

Discretizations of linear maps

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Introduction

In the second part of this manuscript, we will consider the dynamical behaviour of the discretizations of *linear maps*. We were led to this question by the study of the dynamics of the discretizations of generic C^1-*diffeomorphisms*; for exmaple, we will see in the next part that the study of the degree of recurrence of a generic conservative C^1-diffeomorphism requires a good understanding of the dynamics of the discretizations of linear maps. However, it seemed to us that the study of the linear case could form a whole part. Indeed, it appeared that this subject is itself quite interesting and does not use the tools developed in the other parts of this manuscript. Moreover, the techniques involved in this second part of the thesis are very different from that used in the others: we will use tools like almost periodic patterns, model sets or lattice tilings of the Euclidean space by unit cubes.

The definition of the discretization of a linear map of $\mathbf{R}^n$ ($n \geq 2$) is made to mimic that of the discretization of a map of a compact space. The idea underlying this definition is to take the grid $\frac{1}{N}\mathbf{Z}^n \cap [-1,1]^n$ for a large N, and to consider the images of this grid by the discretization of linear maps on the grid $\frac{1}{N}\mathbf{Z}^n$. As the action of a linear map is invariant under homothety, it is equivalent to look at the grids $\mathbf{Z}^n \cap [-N,N]^n$ and to consider the images of these grids by the discretization of linear maps on the grid $\mathbf{Z}^n$. Thus, we define a projection π of $\mathbf{R}^n$ onto $\mathbf{Z}^n$, which maps any point of $\mathbf{R}^n$ onto (one of) the closest point of the lattice $\mathbf{Z}^n$ (see Definition 7.11); given $A \in GL_n(\mathbf{R})$, the *discretization* of A is the map

$$\widehat{A} = \pi \circ A : \mathbf{Z}^n \to \mathbf{Z}^n.$$

Given a sequence $(A_k)_{k\geq 1}$ of matrices of $GL_n(\mathbf{R})$, we want to study the dynamics of the sequence $(\widehat{A_k})_{k\geq 1}$, and in particular, the *density* of the set $\Gamma_k = (\widehat{A_k} \circ \cdots \circ \widehat{A_1})(\mathbf{Z}^n)$. This density is defined in the following way, where B_R denotes the infinite ball $B_\infty(0,R)$:

$$D^+(\Gamma_k) = \overline{\lim_{R\to+\infty}} \frac{\mathrm{Card}(\Gamma_k \cap B_R)}{\mathrm{Card}(\mathbf{Z}^n \cap B_R)}. \tag{6.2}$$

First of all, we study the structure of the image sets Γ_k. It appears that there is a kind of "regularity at infinity" of the behaviour of Γ_k. More precisely, this set is an *almost periodic pattern*: for every $\varepsilon > 0$ there exists a relatively dense [5] set of ε-translations of Γ_k, where a vector $v \in \mathbf{Z}^n$ is an ε-translation of Γ_k if $D^+\Big((\Gamma_k - v)\Delta\Gamma_k\Big) \leq \varepsilon$ (see Definition 7.3). Roughly speaking, for R large enough, the set $\Gamma_k \cap B_R$ determines the whole set Γ_k up to an error of density smaller than ε. We prove that the image of an almost periodic pattern by the discretization of a linear map is still an almost periodic pattern (Theorem 7.12);

5. A set $\mathcal{N}$ is *relatively dense* if there exists $R > 0$ such that every ball of radius R contains at least one point of $\mathcal{N}$.

thus, given a sequence $(A_k)_{k\geq 1}$ of invertible matrices, the sets $(\widehat{A_k}\circ\cdots\circ\widehat{A_1})(\mathbf{Z}^n)$ are almost periodic patterns. In particular, these sets possess a uniform density: the superior limit in Equation (6.2) is in fact a limit (Corollary 7.7). This allows us to define the rate of injectivity: given a sequence $(A_k)_{k\geq 1}$ of linear maps, the *rate of injectivity in time* k of this sequence is the quantity (see Definition 7.19)

$$\tau^k(A_1,\cdots,A_k)=\lim_{R\to+\infty}\frac{\mathrm{Card}\left((\widehat{A_k}\circ\cdots\circ\widehat{A_1})(B_R\cap\mathbf{Z}^n)\right)}{\mathrm{Card}(B_R\cap\mathbf{Z}^n)}\in]0,1],$$

where $B_R=B_\infty(0,R)$. As these quantities are decreasing in k, we can also define the *asymptotic rate of injectivity*

$$\tau^\infty\left((A_k)_{k\geq 1}\right)=\lim_{k\to+\infty}\tau^k(A_1,\cdots,A_k)\in[0,1].$$

These rates on injectivity can be seen as the quantity of information we lose when we apply a sequence of discretizations of matrices.

The link between the rate of injectivity and the density of the images sets is made by the following formula (Proposition 7.20):

$$\tau^k(A_1,\cdots,A_k)=\det(A_1)\cdots\det(A_k)D^+\left((\widehat{A_k}\circ\cdots\circ\widehat{A_1})(\mathbf{Z}^n)\right).$$

The goal of the second chapter of this part is to study the behaviour of the asymptotic rate of injectivity of a generic sequence of matrices of $\mathrm{SL}_n(\mathbf{R})$ (in fact, the same holds for matrices with determinant ± 1). It is given by the main theorem of this chapter (Theorem 8.24).

Theorem P. *For a generic sequence* $(A_k)_{k\geq 1}$ *of matrices of* $\mathrm{SL}_n(\mathbf{R})$, *we have* $\tau^\infty\left((A_k)_k\right)=0$.

If this result can seem quite natural, its proof is far from being trivial. First of all, the sets $\Gamma_k=(\widehat{A_k}\circ\cdots\widehat{A_1})(\mathbf{Z}^n)$ are "more and more complex" when k increases: *a priori*, the radius R_0 for which $\Gamma_k\cap B_{R_0}$ determines almost all Γ_k is more and more large when k increases; it is very difficult to have an idea of the local geometry of these sets, thus to decide which is the best possible perturbation of the matrices (especially since we have to get practical estimates on the loss of injectivity). Moreover, once the set Γ_k has a density smaller than 1/2, it may be impossible to make the rate decrease in one step of time. For example, if we set

$$A_1=\begin{pmatrix}4 & \\ & 1/4\end{pmatrix}\quad\text{and}\quad A_2=\begin{pmatrix}1/2 & \\ & 2\end{pmatrix},$$

then $\left(\widehat{A_2}\circ\widehat{A_1}\right)(\mathbf{Z}^2)=(2\mathbf{Z})^2$, and for every B_3 close to the identity, we have $\tau^3(A_1,A_2,B_3)=\tau^2(A_1,A_2)=1/4$. In such cases, it seems difficult to have a long term strategy to make the rate decrease...

The idea of the proof of Theorem P is to take advantage of the fact that for a generic sequence of matrices, the coefficients of all the matrices are rationally independent (in fact, we will only need a weaker assumption about he independence). For example, for a generic matrix $A\in\mathrm{SL}_n(\mathbf{R})$, the set $A(\mathbf{Z}^n)$ is uniformly distributed modulo $\mathbf{Z}^n$. We then remark that the local pattern of the image set $\widehat{A}(\mathbf{Z}^n)$ around $\widehat{A}(x)$ is only determined by

A and the the remainder of Ax modulo $\mathbf{Z}^n$: the global behaviour of $\widehat{\mathrm{A}}(Z^n)$ is coded by the quotient $\mathbf{R}^n/\mathbf{Z}^n$. This somehow reduces the study to a local problem.

As a first application of this remark, we state that the rate of injectivity in time 1 can be seen as the area of an intersection of cubes (Proposition 8.11). This observation, combined with considerations about the frequency of differences $\rho_{\Gamma_k}(v) = \mathrm{D}\Big((\Gamma_k - v)\cap\Gamma_k\Big)$, allows to prove a weak version of Theorem P: the asymptotic rate of injectivity of a generic sequence of matrices of $\mathrm{SL}_n(\mathbf{R})$ is smaller than 1/2 (Theorem 8.30). It also leads to a proof of the following result.

Theorem Q. *Let $(\mathrm{P}_k)_{k\geq 1}$ be a generic sequence of matrices of $\mathrm{O}_n(\mathbf{R})$. Then $\tau^\infty((\mathrm{P}_k)_k) = 0$.*

In particular, for a generic sequence of rotations of the plane, we have a total loss of information when we apply successively the discretizations of these rotations (see Figure 6.17).

At the end of the second chapter of this part, we make a full use of the equidistribution property to see the rate of a sequence $\mathrm{A}_1, \cdots, \mathrm{A}_k$ of matrices of $\mathrm{SL}_n(\mathbf{R})$ in terms of areas of intersections of cubes in $\mathbf{R}^{nk}$ (Proposition 8.40). The proof of this formula is based on the notion of model set [6] (Definition 7.32), which is a particular class of almost periodic patterns. Using this formula, we replace the iteration by a passage in high dimension. These considerations allow to prove Theorem P, without having to make "clever" perturbations of the sequence of matrices (that is, the perturbations made a each iteration are chosen independently from that made in the past or in the future).

Finally, the last chapter of this part is devoted to the study of the statistics of the roundoff errors induced by the discretization process. The main result is the following (Proposition 9.1).

Proposition R. *For a generic sequence $(\mathrm{A}_k)_{k\geq 1}$ of matrices of $\mathrm{GL}_n(\mathbf{R})$, or $\mathrm{SL}_n(\mathbf{R})$, or $\mathrm{O}_n(\mathbf{R})$, for every fixed integer k, the finite sequence of errors $\varepsilon_x = \left(\varepsilon_x^1, \cdots, \varepsilon_x^k\right)$ is equidistributed in $(\mathbf{R}^n/\mathbf{Z}^n)^k$ when x ranges over $\mathbf{Z}^n$.*

This chapter presents a work in progress; the goal of these considerations is to study in more depth a conjecture of O.E. Lanford (Conjecture 12.13) concerning the physical measures of expanding maps of the circle.

Throughout this part, we will use several notions of almost periodicity: Delone sets, almost periodic patterns, weakly almost periodic sets (which is, to our knowledge, an original concept), model sets, Bohr and Besicovitch almost periodic functions. We now give a quick overview of the history of these different concepts. The history of almost periodic sets is made of many independent rediscoveries, made in different areas of mathematics and physics. The first traces of studies of some notions of almost periodicity date back to 1893 with the master's thesis of P. Bohl "*Über die Darstellung von Funktionen einer Variabeln durch trigonometrische Reihen mit mehreren einer Variabeln proportionalen Argumenten*", where the author introduces quasi-periodic functions. This class of functions were rediscovered about ten years later by E. Esclangon, in the view of the study of celestial mechanic. Almost periodic functions have been studied in more depth by H. Bohr in 1924. In the paper [Boh24], he proves in particular the fundamental theorem stating that any uniform almost periodic function (now called Bohr almost

6. Sometimes called "cut-and-project" set.

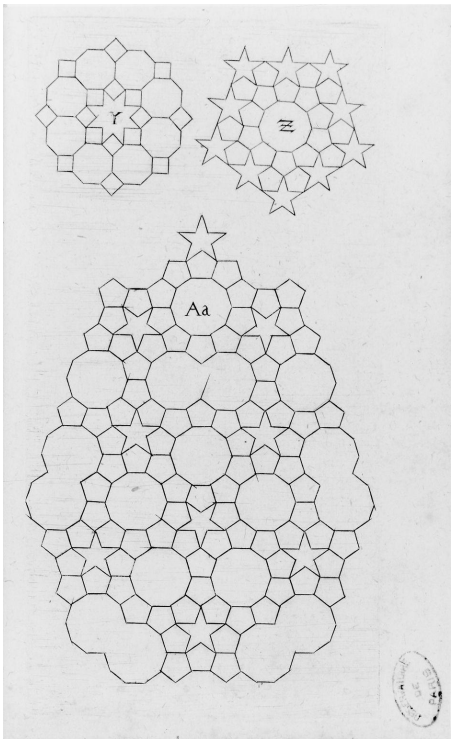

Figure 6.13: Some drawings of J. Kepler (1619).

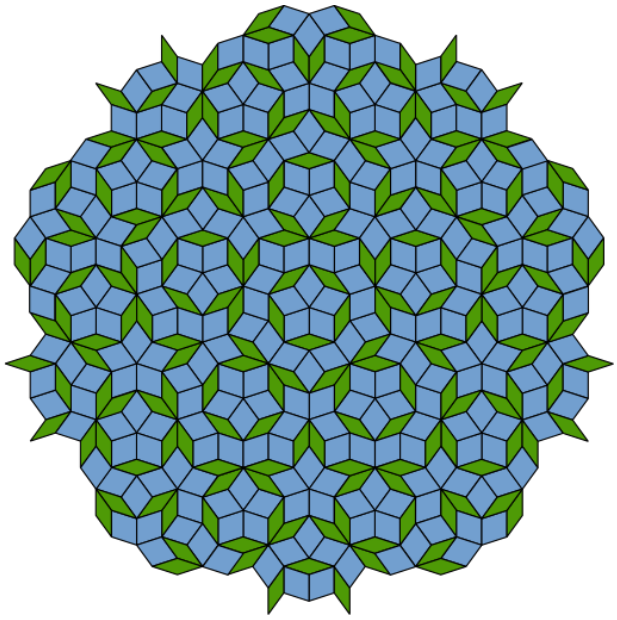

Figure 6.14: A Penrose tiling.

periodic function) is the limit of generalized trigonometric series (and reciprocally). Two years later, A. Besicovitch defined a weaker notion of almost periodicity [Bes26]; the results of these papers were summarized later in the books [Boh47] and [Bes55]. In the early seventies, Y. Meyer [Mey72] introduced *model sets* in the context of harmonic analysis (more precisely, he wanted to study Pisot and Salem numbers). These sets were rediscovered independently in 1981 by N. G. de Bruijn (see [dB81]) to study Penrose aperiodic tilings. These aperiodic tilings we introduced in 1974 by the physicist R. Penrose [Pen74] (see Figure 6.14), inspired by the work of J. Kepler (see Figure 6.13). In 1984 was made the fundamental discovery of quasicrystals [SBGC84] (see Figure 6.15), that is, solids that are ordered but not periodic. Quickly, the link was made with the previous works of H. Bohr and A. Besicovitch. Since then, the various notions of almost periodicity play an important role in the study of quasicrystals and aperiodic tilings, but also in many other parts of mathematics (see for example the survey [Moo00]). Various mathematical formulations of quasicrystals have been proposed, such as Meyer sets (see [Mey95]), and the harmonic study of these sets has been investigated (see [Mey12]). Quite recently, P. Lu, an American physicist, discovered that some very old Uzbek mosaics have the same structure than the Penrose tiling (see Figure 6.16). It is amusing to note that in our case, the study of almost periodic sets arises from a branch of mathematics which is still different: the discretizations of generic diffeomorphisms of the torus.

The particular problem of the discretization of linear maps has been only little studied. To our knowledge, what has been made in this direction has been initiated by image processing. The goal of these studies is to try to answer to the question: what is the best

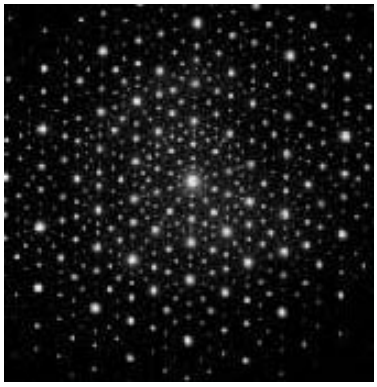

Figure 6.15: The figure of diffraction behind the discovery of quasicrystals; this figure possesses a symmetry of order 5, which is impossible for regular crystals.

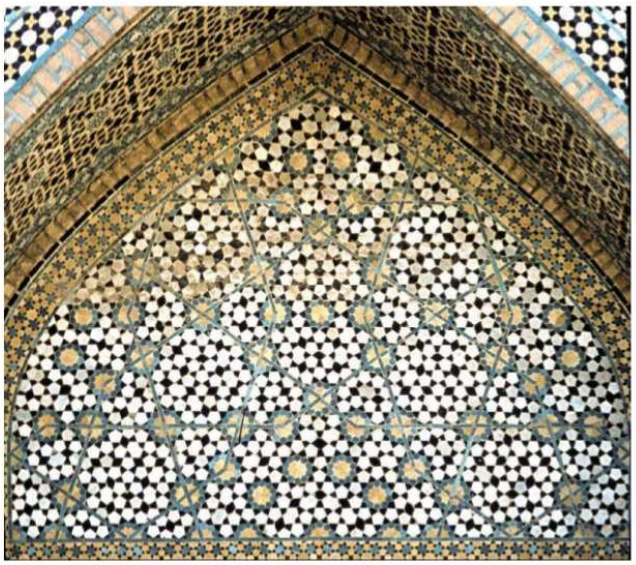

Figure 6.16: Mosaic in a madrassa of Boukhara (Ouzbekistan, 15th century). In 2007, P. Lu discovered that it has the same structure as a Penrose tiling.

way to define the action of a linear map on the lattice $\mathbf{Z}^2$? In particular, how can we compute the image of a numerical image by a linear map? More precisely, we want to avoid phenomenons like loss of information (due to the fact that discretizations of linear maps are not injective) or aliasing (the apparition of undesirable periodic patterns in the image, due for example to a resonance between a periodic pattern in the image and the discretized map). For example, in Figure 6.17, we have applied 40 successive random rotations to a 500×684 pixels picture, using a consumer software. These discretized rotations induce a very strong blur in the resulting image, thus a big loss of information. To our knowledge, the existing studies are mostly interested in the linear maps with *rational coefficients* (see for example [Jac95], [Neh96] or [JDC01]), and also in the specific case of *rotations* (see for example [And96], [Nou06], [Thi10], [BV00]). These works mainly focus on the local behaviour of the images of $\mathbf{Z}^2$ by discretizations of linear maps: given a radius R, what pattern can follow the intersection of this set with any ball of radius R? What is the number of such patterns, what are their frequencies? Are they complex (in a sense to define) or not? Are these maps bijections? In particular, the thesis [Nou06] of B. Nouvel gives a characterization of the angles for which the discrete rotation is a bijection (such angles are countable and accumulate only on 0).

Here, our point of view is quite different: we want to determine the dynamical behaviour of discretizations of *generic* linear maps; in particular generic matrices are totally irrational (by this, we mean that the image of the lattice $\mathbf{Z}^n$ by the matrix is equidistributed modulo $\mathbf{Z}^n$). We will see that the behaviour of discretizations of generic linear maps is in a certain sense smoother than that of rational linear maps: for example the rate of injectivity is continuous when restricted to totally irrational matrices (Proposition 8.4), while it is not on some rational matrices (Proposition 8.1). However, we study the rate of injectivity of a sequence of matrices, which measures the loss of information we have when we apply several discretized linear maps to an image; The-

Figure 6.17: Original image (left) and 40 successive random rotations of this image, obtained with a consumer software.

orem P expresses that in the general case, when we apply a lot of linear maps, then we lose most of the information contained in the image, and Theorem Q states that this phenomenon also appears for a generic sequence of rotations.

Chapter 7

Almost periodic sets

In this chapter, we introduce the basic notions that we will use during the study of discretizations of linear maps of $\mathbf{R}^n$, $n \geq 1$. First of all, we introduce the notion of almost periodic pattern: roughly speaking, a set $\Gamma \subset \mathbf{Z}^n$ is an almost periodic pattern if for R large enough, the set $\Gamma \cap [-\mathrm{R}, \mathrm{R}]^n$ determines the whole set Γ up to an error of density smaller than ε (see Definition 7.3). It can be easily seen that an almost periodic pattern Γ possesses a uniform density, that is, that the limit

$$\mathrm{D}(\Gamma) = \lim_{\mathrm{R} \to +\infty} \frac{\mathrm{Card}(\Gamma \cap [-\mathrm{R}, \mathrm{R}]^n)}{\mathrm{Card}(\mathbf{Z}^n \cap [-\mathrm{R}, \mathrm{R}]^n)}$$

is well defined (Corollary 7.7).

Recall that the *discretization* of a linear map $\mathrm{A} \in \mathrm{GL}_n(\mathbf{R})$ is the map $\widehat{\mathrm{A}} = \pi \circ \mathrm{A} : \mathbf{Z}^n \to \mathbf{Z}^n$, where π is a projection from $\mathbf{R}^n$ to the nearest element of $\mathbf{Z}^n$ (see Definition 7.11). The definition of an almost periodic pattern is supported by Theorem 7.12, which states that the image of an almost periodic pattern by the discretization of a linear map is still an almost periodic pattern. In particular, the successive images of the integer lattice $\mathbf{Z}^n$ by the discretizations of linear maps are almost periodic patterns, and possess a uniform density.

We then study the frequency of any difference v in the almost periodic pattern Γ, defined by $\rho_\Gamma(v) = \mathrm{D}\big((\Gamma - v) \cap \Gamma\big)$ (Definition 7.24); in particular we prove a Minkowski-like theorem for these differences (Theorem 7.29, obtained in collaboration with É. Joly): if S is a centrally symmetric convex body, then the sum of the frequency of the differences $v \in \mathrm{S}$ is bigger than a quantity which depends linearly on the measure of S.

Finally, we introduce the notion of model set (Definition 7.32): the model set modelled on a lattice Λ of $\mathbf{R}^{m+n}$ and on a "regular" set $\mathrm{W} \subset \mathbf{R}^m$ is the projection on the n last coordinates of the points of Λ whose projection on the m first coordinates belongs to W. We state that the images sets $(\widehat{\mathrm{A}_k} \circ \cdots \circ \widehat{\mathrm{A}_1})(\mathbf{Z}^n)$ are model sets, and prove that these sets are almost periodic patterns (Theorem 7.36, obtained in collaboration with Y. Meyer).

We fix once for all an integer $n \geq 1$. We will denote by $[\![a, b]\!]$ the integer segment $[a, b] \cap \mathbf{Z}$. In this part, every ball will be taken with respect to the infinite norm; in particular, for $x = (x_1, \cdots, x_n)$, we will have

$$\mathrm{B}(x, \mathrm{R}) = \mathrm{B}_\infty(x, \mathrm{R}) = \Big\{ y = (y_1, \cdots, y_n) \in \mathbf{R}^n \mid \forall i \in [\![1, n]\!], |x_i - y_i| < \mathrm{R} \Big\}.$$

We will also denote $B_R = B(0,R)$. Finally, we will denote by $\lfloor x \rfloor$ the biggest integer that is smaller than x and $\lceil x \rceil$ the smallest integer that is bigger than x. For a set $B \subset \mathbf{R}^n$, we will denote $[B] = B \cap \mathbf{Z}^n$.

7.1 Almost periodic patterns: definitions and first properties

In this section, we define the notion of almost periodic pattern and prove that these sets possess a uniform density.

Definition 7.1. Let Γ be a subset of $\mathbf{R}^n$.
- We say that Γ is *relatively dense* if there exists $R_\Gamma > 0$ such that each ball with radius at least R_Γ contains at least one point of Γ.
- We say that Γ is a *uniformly discrete* if there exists $r_\Gamma > 0$ such that each ball with radius at most r_Γ contains at most one point of Γ.

The set Γ is called a *Delone* set if it is both relatively dense and uniformly discrete.

Definition 7.2. For a discrete set $\Gamma \subset \mathbf{R}^n$ and $R \geq 1$, we define the uniform R-density:

$$D_R^+(\Gamma) = \sup_{x \in \mathbf{R}^n} \frac{\mathrm{Card}\Big(B(x,R) \cap \Gamma\Big)}{\mathrm{Card}\Big(B(x,R) \cap \mathbf{Z}^n\Big)},$$

and the uniform upper density:

$$D^+(\Gamma) = \overline{\lim_{R \to +\infty}} D_R^+(\Gamma).$$

Remark that if $\Gamma \subset \mathbf{R}^n$ is Delone for the parameters r_Γ and R_Γ, then its upper density satisfies:

$$\frac{1}{(2R_\Gamma + 1)^n} \leq D^+(\Gamma) \leq \frac{1}{(2r_\Gamma + 1)^n}.$$

We can now define the notion of almost periodic pattern that we will use throughout this chapter. Roughly speaking, an almost periodic pattern Γ is a set for which there exists a relatively dense set of translations of Γ, where a vector v is a translation of Γ if $\Gamma - v$ is equal to Γ up to a set of upper density smaller than ε. More precisely, we state the following definition.

Definition 7.3. A Delone set Γ is an *almost periodic pattern* if for every $\varepsilon > 0$, there exists $R_\varepsilon > 0$ and a relatively dense set $\mathcal{N}_\varepsilon$, called the *set of ε-translations* of Γ, such that

$$\forall R \geq R_\varepsilon,\ \forall v \in \mathcal{N}_\varepsilon,\ D_R^+\Big((\Gamma + v) \Delta \Gamma\Big) < \varepsilon. \tag{7.1}$$

Remark that if Γ is a subset of $\mathbf{Z}^n$ with positive upper density and which satisfies the condition of this definition, then it is a Delone set. In the sequel, we will only use this definition for subsets of the lattice $\mathbf{Z}^n$. Remark that by [Fav12, Theorem 3], an almost periodic pattern which is also a Meyer set [1] is "almost included" in a finite union of lattices.

Of course, every lattice, or every finite union of translates of a given lattice, is an almost periodic pattern. We will see in next section a large class of examples of almost periodic patterns: images of $\mathbf{Z}^n$ by discretizations of linear maps.

This definition is stronger than the following one, that we initially used for this study.

1. A set Γ is called a *Meyer set* if $\Gamma - \Gamma$ is a Delone set. It is equivalent to ask that there exists a finite set F such that $\Gamma - \Gamma \subset \Gamma + F$ (see [Lag96]).

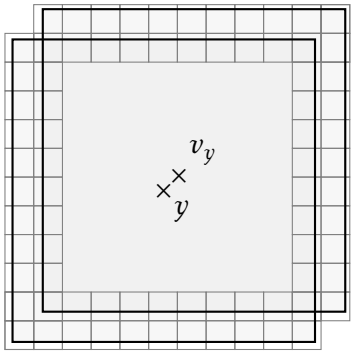

Figure 7.1: Covering the set $B(y,R)\Delta B(v_y,R)$ by cubes of radius r_Γ.

Definition 7.4. We say that a Delone set Γ is *weakly almost periodic* if for every $\varepsilon > 0$, there exists $R > 0$ such that for every $x, y \in \mathbf{R}^n$, there exists $v \in \mathbf{R}^n$ such that

$$\frac{\mathrm{Card}\Big(\big(B(x,R)\cap\Gamma\big)\Delta\big(\big(B(y,R)\cap\Gamma\big)-v\big)\Big)}{\mathrm{Card}(B_R\cap\mathbf{Z}^n)} \le \varepsilon. \tag{7.2}$$

Remark that *a priori*, the vector v is different from $y - x$.

We had defined this concept because it seemed to us that it was the weakest to imply the existence of a uniform density. Unfortunately, this notion is not very convenient to manipulate and we have not succeeded to prove that it is stable under the action of discretization of linear maps. Of course, we have the following result (see also the addendum [GM14] of [Mey12] for more details on the subject).

Proposition 7.5. *Every almost periodic pattern is weakly almost periodic.*

We do not know if the converse is true or not (that is, if there exists weakly almost periodic sets that are not almost periodic patterns).

Proof of Proposition 7.5. We prove that an almost periodic pattern satisfies Equation (7.2) for $x = 0$, the general case being obtained by applying this result twice.

Let Γ be an almost periodic pattern and $\varepsilon > 0$. Then by definition, there exists $R_\varepsilon > 0$ and a relatively dense set $\mathscr{N}_\varepsilon$ (for a parameter $R_{\mathscr{N}_\varepsilon} > 0$) such that

$$\forall R \ge R_\varepsilon,\ \forall v \in \mathscr{N}_\varepsilon,\ D_R^+\big((\Gamma+v)\Delta\Gamma\big) < \varepsilon. \tag{7.3}$$

Moreover, as Γ is Delone, there exists $r_\Gamma > 0$ such that each ball with radius smaller than r_Γ contains at most one point of Γ.

As $\mathscr{N}_\varepsilon$ is relatively dense, for every $y \in \mathbf{R}^n$, there exists $v_y \in -\mathscr{N}_\varepsilon$ such that $d_\infty(y, v_y) < R_{\mathscr{N}_\varepsilon}$. This v_y is the vector v we look for to have the property of Definition 7.4. Indeed, by triangle inequality, for every $R \ge R_\varepsilon$, we have

$$\begin{aligned}\mathrm{Card}\Big(\big(B(0,R)\cap\Gamma\big)\Delta\big(\big(B(y,R)\cap\Gamma\big)-v_y\big)\Big)&\\ \le \mathrm{Card}\Big(\big(B(0,R)\cap\Gamma\big)\Delta\big(\big(B(v_y,R)\cap\Gamma\big)-v_y\big)\Big)&\\ + \mathrm{Card}\Big(\big(B(v_y,R)\cap\Gamma\big)\Delta\big(B(y,R)\cap\Gamma\big)\Big).&\end{aligned} \tag{7.4}$$

By Equation (7.3), the first term of the right side of the inequality is smaller than $\varepsilon \mathrm{Card}\Big(\mathrm{B}(0,\mathrm{R}) \cap \mathbf{Z}^n\Big)$. It remains to bound the second one.

For every $y \in \mathbf{R}^n$, as $d_\infty(y, v_y) < \mathrm{R}_{\mathscr{N}_\varepsilon}$, the set $\mathrm{B}(y,\mathrm{R})\Delta\mathrm{B}(v_y,\mathrm{R})$ is covered by

$$\frac{2n(\mathrm{R}+r_\Gamma)^{n-1}(\mathrm{R}_{\mathscr{N}_\varepsilon}+r_\Gamma)}{r_\Gamma^n}$$

disjoint cubes of radius r_Γ (see Figure 7.1). Thus, as each one of these cubes contains at most one point of Γ, this implies that

$$\mathrm{Card}\Big(\Big(\mathrm{B}(y,\mathrm{R})\Delta\mathrm{B}(v_y,\mathrm{R})\Big)\cap\Gamma\Big) \leq 2n\frac{(\mathrm{R}+r_\Gamma)^{n-1}(\mathrm{R}_{\mathscr{N}_\varepsilon}+r_\Gamma)}{r_\Gamma^n}.$$

Increasing R_ε if necessary, for every $\mathrm{R} \geq \mathrm{R}_\varepsilon$, we have

$$2n\frac{(\mathrm{R}+r_\Gamma)^{n-1}(\mathrm{R}_{\mathscr{N}_\varepsilon}+r_\Gamma)}{r_\Gamma^n} \leq \varepsilon\mathrm{Card}\Big(\mathrm{B}(0,\mathrm{R})\cap\mathbf{Z}^n\Big),$$

so,

$$\mathrm{Card}\Big(\Big(\mathrm{B}(y,\mathrm{R})\Delta\mathrm{B}(v_y,\mathrm{R})\Big)\cap\Gamma\Big) \leq \varepsilon\mathrm{Card}\Big(\mathrm{B}(0,\mathrm{R})\cap\mathbf{Z}^n\Big).$$

This bounds the second term of Equation (7.4). We finally get

$$\mathrm{Card}\Big(\Big(\mathrm{B}(0,\mathrm{R})\cap\Gamma\Big)\Delta\Big(\mathrm{B}(y,\mathrm{R})\cap\Gamma - v_y\Big)\Big) \leq 2\varepsilon\mathrm{Card}\Big(\mathrm{B}(0,\mathrm{R})\cap\mathbf{Z}^n\Big),$$

which proves the proposition. □

Weakly almost periodic sets — and in particular almost-periodic patterns — have a regular enough behaviour at the infinity to possess a density.

Proposition 7.6. *Let Γ be a weakly almost periodic set. Then the uniform upper density of Γ is a limit, more precisely:*

$$\mathrm{D}^+(\Gamma) = \lim_{\mathrm{R}\to+\infty} \mathrm{D}^+_{\mathrm{R}}(\Gamma) = \mathrm{D}(\Gamma).$$

Moreover, we have uniformity of the convergence of the density with respect to the base point, that is: for every $x \in \mathbf{R}^n$, we have

$$\mathrm{D}(\Gamma) = \lim_{\mathrm{R}\to+\infty} \frac{\mathrm{Card}\Big(\mathrm{B}(x,\mathrm{R})\cap\Gamma\Big)}{\mathrm{Card}\Big(\mathrm{B}(x,\mathrm{R})\cap\mathbf{Z}^n\Big)}.$$

In this case, we say that $\mathrm{D}(\Gamma)$ *is the* uniform density *of* Γ.

Combined with Proposition 7.5, this proposition directly implies the following corollary.

Corollary 7.7. *Let Γ be an almost periodic pattern. Then the uniform upper density of Γ is a limit; in other words Γ possesses a uniform density.*

Remark 7.8. The same proof also shows that the same property holds if instead of considering the density D^+, we take a *Jordan-measurable* [2] set J and consider the density $D^+_J(\Gamma)$ of a set $\Gamma \subset \mathbf{Z}^n$ defined by

$$D^+_J(\Gamma) = \overline{\lim_{R\to+\infty}} \sup_{x\in\mathbf{R}^n} \frac{\mathrm{Card}\left(J_R \cap \Gamma\right)}{\mathrm{Card}\left(J_R \cap \mathbf{Z}^n\right)},$$

where J_R denotes the set of points $x \in \mathbf{R}^n$ such that $x/R \in J$.

Proof of Proposition 7.6. Let Γ be a weakly almost periodic set and $\varepsilon > 0$. Then, by definition, there exists $R > 0$ such that for all $x, y \in \mathbf{R}^n$, there exists $v \in \mathbf{R}^n$ such that Equation (7.2) holds. We take a "big" $M \in \mathbf{R}$, $x \in \mathbf{R}^n$ and $R' \geq MR$. We use the tiling of $\mathbf{R}^n$ by the collection of squares $\{B(Ru, R)\}_{u\in(2\mathbf{Z})^n}$ and the Equation (7.2) (applied to the radius R' and the points 0 and Ru) to find the number of points of Γ that belong to $B(x, R')$: as $B(x, R')$ contains at least $\lfloor M\rfloor^n$ disjoint cubes $B(Ru, R)$ and is covered by at most $\lceil M\rceil^n$ such cubes, we get (recall that $B_R = B(0, R)$)

$$\frac{\lfloor M\rfloor^n\Big(\mathrm{Card}(B_R \cap \Gamma) - 2\varepsilon\mathrm{Card}(B_R \cap \mathbf{Z}^n)\Big)}{\lceil M\rceil^n \mathrm{Card}(B_R \cap \mathbf{Z}^n)}$$

$$\leq \frac{\mathrm{Card}\left(B(x, R') \cap \Gamma\right)}{\mathrm{Card}\left(B(x, R') \cap \mathbf{Z}^n\right)} \leq$$

$$\frac{\lceil M\rceil^n\Big(\mathrm{Card}(B_R \cap \Gamma) + 2\varepsilon\mathrm{Card}(B_R \cap \mathbf{Z}^n)\Big)}{\lfloor M\rfloor^n \mathrm{Card}(B_R \cap \mathbf{Z}^n)},$$

thus

$$\frac{\lfloor M\rfloor^n}{\lceil M\rceil^n}\left(\frac{\mathrm{Card}(B_R \cap \Gamma)}{\mathrm{Card}(B_R \cap \mathbf{Z}^n)} - 2\varepsilon\right)$$

$$\leq \frac{\mathrm{Card}\left(B(x, R') \cap \Gamma\right)}{\mathrm{Card}\left(B(x, R') \cap \mathbf{Z}^n\right)} \leq$$

$$\frac{\lceil M\rceil^n}{\lfloor M\rfloor^n}\left(\frac{\mathrm{Card}(B_R \cap \Gamma)}{\mathrm{Card}(B_R \cap \mathbf{Z}^n)} + 2\varepsilon\right).$$

For M large enough, this ensures that for every $R' \geq MR$ and every $x \in \mathbf{R}^n$, the density

$$\frac{\mathrm{Card}\left(B(x, R') \cap \Gamma\right)}{\mathrm{Card}\left(B(x, R') \cap \mathbf{Z}^n\right)} \quad \text{is close to} \quad \frac{\mathrm{Card}(B_R \cap \Gamma)}{\mathrm{Card}(B_R \cap \mathbf{Z}^n)};$$

this finishes the proof of the proposition. □

We end this section by an easy lemma which asserts that for ε small enough, the set of translations $\mathscr{N}_\varepsilon$ is "stable under additions with a small number of terms".

2. We say taht a set J is Jordan-measurable if for every $\varepsilon > 0$, there exists $\eta > 0$ such that there exists two disjoint unions $\mathscr{C}$ and $\mathscr{C}'$ of cubes of radius η, such that $\mathscr{C} \subset J \subset \mathscr{C}'$, and that $\mathrm{Leb}(\mathscr{C}' \setminus \mathscr{C}) < \varepsilon$.

Lemma 7.9. *Let Γ be an almost periodic pattern, $\varepsilon > 0$ and $\ell \in \mathbf{N}$. Then if we set $\varepsilon' = \varepsilon/\ell$ and denote by $\mathscr{N}_{\varepsilon'}$ the set of translations of Γ and $R_{\varepsilon'} > 0$ the corresponding radius for the parameter ε', then for every $k \in [\![1, \ell]\!]$ and every $v_1, \cdots, v_\ell \in \mathscr{N}_{\varepsilon'}$, we have*

$$\forall R \geq R_{\varepsilon'},\ D_R^+\Big(\big(\Gamma + \sum_{i=1}^{\ell} v_i\big)\Delta\Gamma\Big) < \varepsilon.$$

Proof of Lemma 7.9. Let Γ be an almost periodic pattern, $\varepsilon > 0$, $\ell \in \mathbf{N}$, $R_0 > 0$ and $\varepsilon' = \varepsilon/\ell$. Then there exists $R_{\varepsilon'} > 0$ such that

$$\forall R \geq R_{\varepsilon'},\ \forall v \in \mathscr{N}_{\varepsilon'},\ D_R^+\big((\Gamma + v)\Delta\Gamma\big) < \varepsilon'.$$

We then take $1 \leq k \leq \ell$, $v_1, \cdots, v_k \in \mathscr{N}_{\varepsilon'}$ and compute

$$\begin{aligned} D_R^+\Big(\big(\Gamma + \sum_{i=1}^{k} v_i\big)\Delta\Gamma\Big) &\leq \sum_{m=1}^{k} D_R^+\Big(\big(\Gamma + \sum_{i=1}^{m} v_i\big)\Delta\big(\Gamma + \sum_{i=1}^{m-1} v_i\big)\Big) \\ &\leq \sum_{m=1}^{k} D_R^+\Big(\big((\Gamma + v_m)\Delta\Gamma\big) + \sum_{i=1}^{m-1} v_i\Big). \end{aligned}$$

By the invariance under translation of D_R^+, we deduce that

$$\begin{aligned} D_R^+\Big(\big(\Gamma + \sum_{i=1}^{k} v_i\big)\Delta\Gamma\Big) &\leq \sum_{m=1}^{k} D_R^+\big((\Gamma + v_m)\Delta\Gamma\big) \\ &\leq k\varepsilon'. \end{aligned}$$

As $k \leq \ell$, this ends the proof. □

Remark 7.10. In particular, this lemma implies that each set $\mathscr{N}_\varepsilon$ contains arbitrarily large arithmetical progressions. More precisely, for every almost periodic pattern Γ, $\varepsilon > 0$ and $\ell \in \mathbf{N}$, if we set $\varepsilon' = \varepsilon/\ell$, then for every $k \in [\![1, \ell]\!]$ and every $v \in \mathscr{N}_{\varepsilon'}$, we have

$$\forall R \geq R_{\varepsilon'},\ D_R^+\big((\Gamma + kv)\Delta\Gamma\big) < \varepsilon.$$

It also implies that the set $\mathscr{N}_\varepsilon$ contains arbitrarily large patches of lattices of $\mathbf{R}^n$: for every almost periodic pattern Γ, $\varepsilon > 0$ and $\ell \in \mathbf{N}$, there exists $\varepsilon' > 0$ such that for every $k_i \in [\![-\ell, \ell]\!]$ and every $v_1, \cdots, v_n \in \mathscr{N}_{\varepsilon'}$, we have

$$\forall R \geq R_{\varepsilon'},\ D_R^+\Big(\big(\Gamma + \sum_{i=1}^{n} k_i v_i\big)\Delta\Gamma\Big) < \varepsilon.$$

7.2 Almost periodic patterns and linear maps

In this section, we prove that the notion of almost periodic pattern is invariant under discretizations of linear maps: the image of an almost periodic pattern by the discretization of a linear map is still an almost periodic pattern. First of all, we define precisely what we mean by discretization of a linear map.

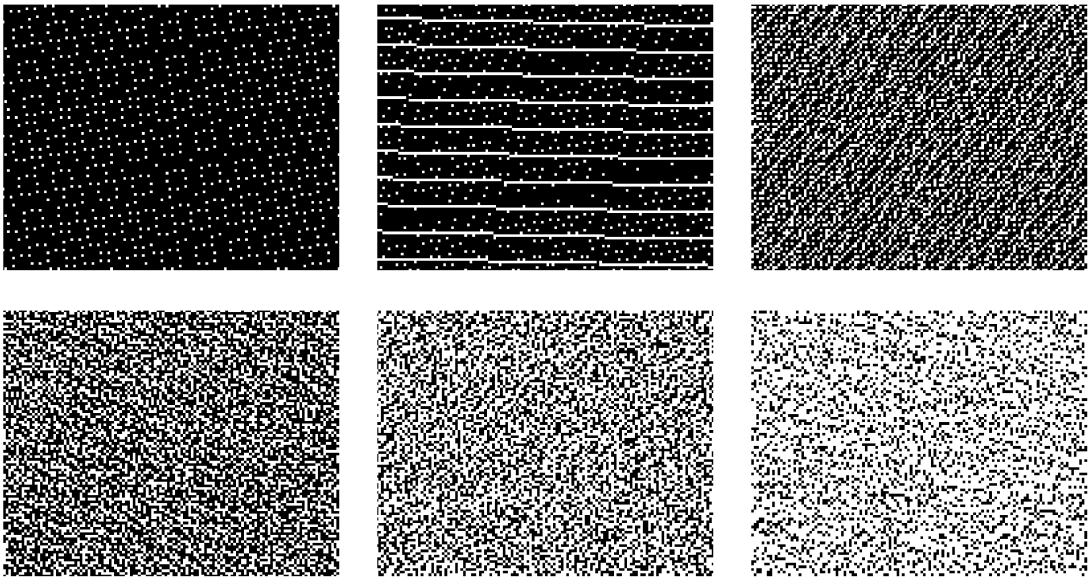

Figure 7.2: Successive images of $\mathbf{Z}^2$ by discretizations of random matrices in $\mathrm{SL}_2(\mathbf{R})$, a point is black if it belongs to $(\widehat{\mathrm{A}_k} \circ \cdots \circ \widehat{\mathrm{A}_1})(\mathbf{Z}^2)$. The A_i are chosen randomly, using the singular value decomposition: they are chosen among the matrices of the form $\mathrm{R}_\theta \mathrm{D}_t \mathrm{R}_{\theta'}$, with R_θ the rotation of angle θ and D_t the diagonal matrix $\mathrm{Diag}(e^t, e^{-t})$, the θ, θ' being chosen uniformly in $[0, 2\pi]$ and t uniformly in $[-1/2, 1/2]$. From left to right and top to bottom, $k = 1, 2, 3, 5, 10, 20$.

Definition 7.11. The map $\mathrm{P} : \mathbf{R} \to \mathbf{Z}$ is defined as a projection from $\mathbf{R}$ onto $\mathbf{Z}$. More precisely, for $x \in \mathbf{R}$, $\mathrm{P}(x)$ is the unique [3] integer $k \in \mathbf{Z}$ such that $k - 1/2 < x \leq k + 1/2$. This projection induces the map

$$\begin{array}{rccc} \pi : & \mathbf{R}^n & \longmapsto & \mathbf{Z}^n \\ & (x_i)_{1\leq i\leq n} & \longmapsto & \big(\mathrm{P}(x_i)\big)_{1\leq i\leq n} \end{array}$$

which is an Euclidean projection on the lattice $\mathbf{Z}^n$. Let $\mathrm{A} \in \mathrm{M}_n(\mathbf{R})$. We denote by $\widehat{\mathrm{A}}$ the *discretization* of the linear map A, defined by

$$\begin{array}{rccc} \widehat{\mathrm{A}} : & \mathbf{Z}^n & \longrightarrow & \mathbf{Z}^n \\ & x & \longmapsto & \pi(\mathrm{A}x). \end{array}$$

The main result of this section is the following theorem.

Theorem 7.12. *Let $\Gamma \subset \mathbf{Z}^n$ be an almost periodic pattern and $\mathrm{A} \in \mathrm{GL}_n(\mathbf{R})$. Then the set $\widehat{\mathrm{A}}(\Gamma)$ is an almost periodic pattern.*

In particular, for every lattice Λ of $\mathbf{R}^n$, the set $\pi(\Lambda)$ is an almost periodic pattern. More generally, given a sequence $(\mathrm{A}_k)_{k\geq 1}$ of invertible matrices of $\mathbf{R}^n$, the successive images $(\widehat{\mathrm{A}_k} \circ \cdots \circ \widehat{\mathrm{A}_1})(\mathbf{Z}^n)$ are almost periodic patterns. See Figure 7.2 for an example of the successive images of $\mathbf{Z}^2$ by a random sequence of bounded matrices of $\mathrm{SL}_2(\mathbf{R})$.

Notation 7.13. For $\mathrm{A} \in \mathrm{GL}_n(\mathbf{R})$, we denote $\mathrm{A} = (a_{i,j})_{i,j}$. We denote by $\mathrm{I}_{\mathbf{Q}}(\mathrm{A})$ the set of indices i such that $a_{i,j} \in \mathbf{Q}$ for every $j \in [\![1, n]\!]$

3. Remark that the choice of where the inequality is strict and where it is not is arbitrary.

The proof of Theorem 7.12 relies on the following remark:

Remark 7.14. If $a \in \mathbf{Q}$, then there exists $q \in \mathbf{N}^*$ such that $\{ax \mid x \in \mathbf{Z}\} \subset \frac{1}{q}\mathbf{Z}$. On the contrary, if $a \in \mathbf{R} \setminus \mathbf{Q}$, then the set $\{ax \mid x \in \mathbf{Z}\}$ is equidistributed in $\mathbf{R}/\mathbf{Z}$.

Thus, in the rational case, the proof will lie in an argument of periodicity. On the contrary, in the irrational case, the image $\mathrm{A}(\mathbf{Z}^n)$ is equidistributed modulo $\mathbf{Z}^n$: on every large enough domain, the density does not move a lot when we perturb the image set $\mathrm{A}(\mathbf{Z}^n)$ by small translations. This reasoning is formalized by Lemmas 7.15 and 7.16.

More precisely, for R large enough, we would like to find vectors w such that $\mathrm{D}_{\mathrm{R}}^{+}\Big((\pi(\mathrm{A}\Gamma) + w)\Delta\pi(\mathrm{A}\Gamma)\Big)$ is small. We know that there exists vectors v such that $\mathrm{D}_{\mathrm{R}}^{+}\Big((\Gamma+v)\Delta\Gamma\Big)$ is small; this implies that $\mathrm{D}_{\mathrm{R}}^{+}\Big((\mathrm{A}\Gamma+\mathrm{A}v)\Delta\mathrm{A}\Gamma\Big)$ is small, thus that $\mathrm{D}_{\mathrm{R}}^{+}\Big(\pi(\mathrm{A}\Gamma+\mathrm{A}v)\Delta\pi(\mathrm{A}\Gamma)\Big)$ is small. The problem is that in general, we do not have $\pi(\mathrm{A}\Gamma + \mathrm{A}v) = \pi(\mathrm{A}\Gamma)+\pi(\mathrm{A}v)$. However, this is true if we have $\mathrm{A}v \in \mathbf{Z}^n$. Lemma 7.15 shows that in fact, it is possible to suppose that $\mathrm{A}v$ "almost" belongs to $\mathbf{Z}^n$, and Lemma 7.16 asserts that this property is sufficient to conclude.

The first lemma is a consequence of the pigeonhole principle.

Lemma 7.15. *Let $\Gamma \subset \mathbf{Z}^n$ be an almost periodic pattern, $\varepsilon > 0$, $\delta > 0$ and $\mathrm{A} \in \mathrm{GL}_n(\mathbf{R})$. Then we can suppose that the elements of $\mathrm{A}(\mathscr{N}_\varepsilon)$ are δ-close to $\mathbf{Z}^n$. More precisely, there exists $\mathrm{R}_{\varepsilon,\delta} > 0$ and a relatively dense set $\widetilde{\mathscr{N}_{\varepsilon,\delta}}$ such that*

$$\forall \mathrm{R} \geq \mathrm{R}_{\varepsilon,\delta},\ \forall v \in \widetilde{\mathscr{N}_{\varepsilon,\delta}},\ \mathrm{D}_{\mathrm{R}}^{+}\Big((\Gamma + v)\Delta\Gamma\Big) < \varepsilon,$$

and that for every $v \in \widetilde{\mathscr{N}_{\varepsilon,\delta}}$, we have $d_\infty(\mathrm{A}v, \mathbf{Z}^n) < \delta$. Moreover, we can suppose that for every $i \in \mathrm{I}_{\mathbf{Q}}(\mathrm{A})$ and every $v \in \widetilde{\mathscr{N}_{\varepsilon,\delta}}$, we have $(\mathrm{A}v)_i \in \mathbf{Z}$.

The second lemma states that in the irrational case, we have continuity of the density under perturbations by translations.

Lemma 7.16. *Let $\varepsilon > 0$ and $\mathrm{A} \in \mathrm{GL}_n(\mathbf{R})$. Then there exists $\delta > 0$ and $\mathrm{R}_0 > 0$ such that for all $w \in \mathrm{B}_\infty(0, \delta)$ (such that for every $i \in \mathrm{I}_{\mathbf{Q}}(\mathrm{A})$, $w_i = 0$), and for all $\mathrm{R} \geq \mathrm{R}_0$, we have*

$$\mathrm{D}_{\mathrm{R}}^{+}\Big(\pi(\mathrm{A}\mathbf{Z}^n)\Delta\pi(\mathrm{A}\mathbf{Z}^n + w)\Big) \leq \varepsilon.$$

Remark 7.17. In Section 8.1 of Chapter 8, we will present an example which shows that the assumption "for every $i \in \mathrm{I}_{\mathbf{Q}}(\mathrm{A})$, $v_i = 0$" is necessary to obtain the conclusion of the lemma.

Remark 7.18. When $\mathrm{I}_{\mathbf{Q}}(\mathrm{A}) = \emptyset$, and in particular when A is totally irrational (see Definition 8.3), the map $v \mapsto \tau(\mathrm{A} + v)$ is continuous in 0; the same proof as that of this lemma implies that this function is globally continuous.

We begin by the proofs of both lemmas, and prove Theorem 7.12 thereafter.

Proof of Lemma 7.15. Let us begin by giving the main ideas of the proof of this lemma. For R_0 large enough, the set of remainders modulo $\mathbf{Z}^n$ of vectors $\mathrm{A}v$, where v is a ε-translation of Γ belonging to $\mathrm{B}_{\mathrm{R}_0}$, is close to the set of remainders modulo $\mathbf{Z}^n$ of vectors $\mathrm{A}v$, where v is any ε-translation of Γ. Moreover (by the pigeonhole principle), there exists an integer k_0 such that for each ε-translation $v \in \mathrm{B}_{\mathrm{R}_0}$, there exists $k \leq k_0$ such that $\mathrm{A}(kv)$ is close to $\mathbf{Z}^n$. Thus, for every ε-translation v of Γ, there exists a $(k_0 - 1)\varepsilon$-translation $v' = (k-1)v$, belonging to $\mathrm{B}_{k_0\mathrm{R}_0}$, such that $\mathrm{A}(v + v')$ is close to $\mathbf{Z}^n$. The vector

$v + v'$ is then a $k_0\varepsilon$-translation of Γ (by additivity of the translations) whose distance to v is smaller than $k_0 R_0$.

We now formalize these remarks. Let Γ be an almost periodic pattern, $\varepsilon > 0$ and $A \in GL_n(\mathbf{R})$. First of all, we apply the pigeonhole principle. We partition the torus $\mathbf{R}^n/\mathbf{Z}^n$ into squares whose sides are smaller than δ; we can suppose that there are at most $\lceil 1/\delta \rceil^n$ such squares. For $v \in \mathbf{R}^n$, we consider the family of vectors $\{A(kv)\}_{0 \leq k \leq \lceil 1/\delta \rceil^n}$ modulo $\mathbf{Z}^n$. By the pigeonhole principle, at least two of these vectors, say $A(k_1 v)$ and $A(k_2 v)$, with $k_1 < k_2$, lie in the same small square of $\mathbf{R}^n/\mathbf{Z}^n$. Thus, if we set $k_v = k_2 - k_1$ and $\ell = \lceil 1/\delta \rceil^n$, we have

$$1 \leq k_v \leq \ell \quad \text{and} \quad d_\infty\Big(A(k_v v), \mathbf{Z}^n\Big) \leq \delta. \tag{7.5}$$

To obtain the conclusion in the rational case, we suppose in addition that $v \in q\mathbf{Z}^n$, where $q \in \mathbf{N}^*$ is such that for every $i \in I_{\mathbf{Q}}(A)$ and every $1 \leq j \leq n$, we have $q\, a_{i,j} \in \mathbf{Z}$ (which is possible by Remark 7.10).

We set $\varepsilon' = \varepsilon/\ell$. By the definition of an almost periodic pattern, there exists $R_{\varepsilon'} > 0$ and a relatively dense set $\mathscr{N}_{\varepsilon'}$ such that Equation (7.1) holds for the parameter ε':

$$\forall R \geq R_{\varepsilon'},\ \forall v \in \mathscr{N}_{\varepsilon'},\ D_R^+\Big((\Gamma + v)\Delta\Gamma\Big) < \varepsilon', \tag{7.1'}$$

We now set

$$P = \Big\{Av \bmod \mathbf{Z}^n \mid v \in \mathscr{N}_{\varepsilon'}\Big\} \quad \text{and} \quad P_R = \Big\{Av \bmod \mathbf{Z}^n \mid v \in \mathscr{N}_{\varepsilon'} \cap B_R\Big\}.$$

We have $\bigcup_{R>0} P_R = P$, so there exists $R_0 > R_{\varepsilon'}$ such that $d_H(P, P_{R_0}) < \delta$ (where d_H denotes Hausdorff distance). Thus, for every $v \in \mathscr{N}_{\varepsilon'}$, there exists $v' \in \mathscr{N}_{\varepsilon'} \cap B_{R_0}$ such that

$$d_\infty(Av - Av', \mathbf{Z}^n) < \delta. \tag{7.6}$$

We then remark that for every $v' \in \mathscr{N}_{\varepsilon'} \cap B_{R_0}$, if we set $v'' = (k_{v'} - 1)v'$, then by Equation (7.5), we have

$$d_\infty(Av' + Av'', \mathbf{Z}^n) = d_\infty\Big(A(k_{v'} v'), \mathbf{Z}^n\Big) \leq \delta.$$

Combining this with Equation (7.6), we get

$$d_\infty(Av + Av'', \mathbf{Z}^n) \leq 2\delta,$$

with $v'' \in B_{\ell R_0}$.

On the other hand, $k_{v'} \leq \ell$ and Equation (7.1') holds, so Lemma 7.9 (more precisely, the first point of Remark 7.10) implies that $v'' \in \mathscr{N}_\varepsilon$, that is

$$\forall R \geq R_{\varepsilon'},\ D_R^+\Big((\Gamma + v'')\Delta\Gamma\Big) < \varepsilon.$$

In other words, for every $v \in \mathscr{N}_{\varepsilon'}$, there exists $v'' \in \mathscr{N}_\varepsilon \cap B_{\ell R_0}$ (with ℓ and R_0 independent from v) such that $d_\infty\Big(A(v + v''), \mathbf{Z}^n\Big) < 2\delta$. The set $\widetilde{\mathscr{N}_{2\varepsilon,2\delta}}$ we look for is then the set of such sums $v + v''$. □

Proof of Lemma 7.16. Under the hypothesis of the lemma, for every $i \notin I_{\mathbf{Q}}(A)$, the sets

$$\left\{\sum_{j=1}^n a_{i,j} x_j \mid (x_j) \in \mathbf{Z}^n\right\},$$

are equidistributed modulo $\mathbf{Z}$. Thus, for all $\varepsilon > 0$, there exists $R_0 > 0$ such that for every $R \geq R_0$,

$$D_R^+\Big\{v \in \mathbf{Z}^n \,\Big|\, \exists i \notin I_{\mathbf{Q}}(A) : d\Big((Av)_i, \mathbf{Z} + \frac{1}{2}\Big) \leq \varepsilon\Big\} \leq 2(n+1)\varepsilon.$$

As a consequence, for all $w \in \mathbf{R}^n$ such that $\|w\|_\infty \leq \varepsilon/(2(n+1))$ and that $w_i = 0$ for every $i \in I_{\mathbf{Q}}(A)$, we have

$$D_R^+\Big(\pi(A\mathbf{Z}^n)\Delta\pi(A(\mathbf{Z}^n + w))\Big) \leq \varepsilon.$$

Then, the lemma follows from the fact that there exists $\delta > 0$ such that $\|A(w)\|_\infty \leq \varepsilon/(2(n+1))$ as soon as $\|w\| \leq \delta$. □

Proof of Theorem 7.12. Let $\varepsilon > 0$. Lemma 7.16 gives us a corresponding $\delta > 0$, that we use to apply Lemma 7.15 and get a set of translations $\widetilde{\mathscr{N}_{\varepsilon,\delta}}$. Then, for every $v \in \widetilde{\mathscr{N}_{\varepsilon,\delta}}$, we write $\pi(Av) = Av + \big(\pi(Av) - Av\big) = Av + w$. The conclusions of Lemma 7.15 imply that $\|w\|_\infty < \delta$, and that $w_i = 0$ for every $i \in I_{\mathbf{Q}}(A)$.

We now explain why $\hat{A}v = \pi(Av)$ is a ε-translation for the set $\widehat{A}(\Gamma)$. Indeed, for every $R \geq \max(R_{\varepsilon,\delta}, MR_0)$, where M is the maximum of the greatest modulus of the eigenvalues of A and of the greatest modulus of the eigenvalues of A^{-1}, we have

$$\begin{aligned} D_R^+\Big(\pi(A\Gamma)\Delta\Big(\pi(A\Gamma) + \widehat{A}v\Big)\Big) \leq\ & D_R^+\Big(\pi(A\Gamma)\Delta\Big(\pi(A\Gamma) + w\Big)\Big) \\ & + D_R^+\Big(\Big(\pi(A\Gamma) + w\Big)\Delta\Big(\pi(A\Gamma) + \widehat{A}v\Big)\Big) \end{aligned}$$

(where $w = \pi(Av) - Av$). By Lemma 7.16, the first term is smaller than ε. For its part, the second term is smaller than

$$D_R^+\Big((A\Gamma + Av)\Delta A\Gamma\Big) \leq M^2 D_{RM}^+\Big((\Gamma + v)\Delta\Gamma\Big),$$

which is smaller than ε because $v \in \mathscr{N}_\varepsilon$. □

Theorem 7.12 motivates the following definition.

Definition 7.19. Let $A \in GL_n(\mathbf{R})$. The *rate of injectivity* of A is the quantity [4]

$$\tau(A) = \lim_{R\to+\infty} \frac{\mathrm{Card}(\widehat{A}[B_R])}{\mathrm{Card}[B_R]} \in]0,1].$$

More generally, for $A_1, \cdots, A_k \in GL_n(\mathbf{R})$, we set

$$\tau^k(A_1, \cdots, A_k) = \lim_{R\to+\infty} \frac{\mathrm{Card}\Big((\widehat{A_k} \circ \cdots \circ \widehat{A_1})[B_R]\Big)}{\mathrm{Card}[B_R]} \in]0,1],$$

and for an infinite sequence $(A_k)_{k\geq 1}$ of invertible matrices, as the previous quantity is decreasing, we can define the *asymptotic rate of injectivity*

$$\tau^\infty\big((A_k)_{k\geq 1}\big) = \lim_{k\to+\infty} \tau^k(A_1, \cdots, A_k) \in [0,1].$$

An easy calculation shows that the rate of injectivity can be deduced from the uniform density of the image $\widehat{A}(\mathbf{Z}^n)$ (in particular, it uses the fact that for every $A \in GL_n(\mathbf{R})$ and every $R > 0$, the set $A(B_R)$ is Jordan-measurable, see Remark 7.8).

4. For the definition of the discretization $\widehat{A}$, see Definition 7.11. By definition, $[B_r] = B_\infty(0,R) \cap \mathbf{Z}^n$.

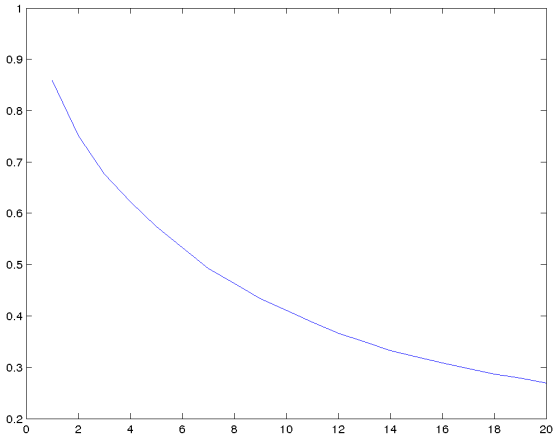

Figure 7.3: Expectation of the rate of injectivity of random sequences of matrices: the graphic represents the mean of the rate of injectivity $\tau^k(A_1, \cdots, A_k)$ depending on k, $1 \leq k \leq 20$, for 50 random draws of matrices A_i. The A_i are chosen randomly using the singular value decomposition: they are chosen among the matrices of the form $R_\theta D_t R_{\theta'}$, with R_θ the rotation of angle θ and D_t the diagonal matrix $\mathrm{Diag}(e^t, e^{-t})$, and θ, θ' chosen uniformly in $[0, 2\pi]$ and t chosen uniformly in $[-1/2, 1/2]$. Note that the behaviour is experimentally not exponential. Also note that contrary to what happens in the case of isometries (see Figure 8.14), some hyperbolic-like phenomenons can occur in a random sequences of matrices of $\mathrm{SL}_2(\mathbf{R})$; so that in practical, it becomes difficult to plot a graph up to a long time (20 iterations for $\mathrm{SL}_2(\mathbf{R})$ versus 200 iterations for $\mathrm{O}_2(\mathbf{R})$) for reasons of memory constraints.

Proposition 7.20. *For every matrix* $A \in \mathrm{GL}_n(\mathbf{R})$, *we have*

$$\tau(A) = |\det(A)| D\Big(\widehat{A}(\mathbf{Z}^n)\Big). \tag{7.7}$$

More generally, for every $A_1, \cdots, A_k \in \mathrm{GL}_n(\mathbf{R})$, *we have*

$$\tau^k(A_1, \cdots, A_k) = |\det(A_1)| \cdots |\det(A_k)| D\Big((\widehat{A_k} \circ \cdots \circ \widehat{A_1})(\mathbf{Z}^n)\Big).$$

The same convergence holds for every affine map, thus we will also use the notion of rate of injectivity (and the notation τ) in this more general context.

The quantity $\tau(A)$ does not change when we multiply A on his right by an element of $\mathrm{SL}_n(\mathbf{Z})$. Similarly, $\tau^k(A_1, \cdots, A_k)$ in invariant under the multiplication of A_1 on his right by an element of $\mathrm{SL}_2(\mathbf{Z})$. It is difficult to detect more simple invariants for the rate τ (or even τ^k), for example it has no obvious good behaviour with respect to the geodesic flow (or horocyclic flow) on the modular surface $\mathrm{SL}_2(\mathbf{R})/\mathrm{SL}_2(\mathbf{Z})$ (see Figure 8.4)

The goal of Chapter 8 will be to study in detail this quantity τ, and in particular to prove that the behaviour of the rate suggested by Figure 7.3 actually holds, namely that the asymptotic rate of injectivity of a generic sequence of $\mathrm{SL}_n(\mathbf{R})$ is zero (Theorem 8.24).

We end this section by a technical lemma that we will use a lot in next chapter. It expresses that given an almost periodic pattern Γ, a generic matrix $A \in \mathrm{GL}_n(\mathbf{R})$ is non resonant with respect to Γ.

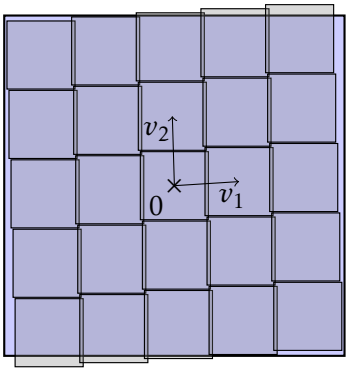

Figure 7.4: "Almost tiling" of $B_{\ell R_0}$ by cubes $B(\sum_{i=1}^n k_i v_i, R_0)$, with $-\ell \le k_i \le \ell$.

Lemma 7.21. *Let $\Gamma \subset \mathbf{Z}^n$ be an almost periodic pattern with positive uniform density. Then the set of $A \in GL_n(\mathbf{R})$ (respectively $SL_n(\mathbf{R})$, $O_n(\mathbf{R})$) such that $A(\Gamma)$ is equidistributed modulo $\mathbf{Z}^n$ is generic. More precisely, for every $\varepsilon > 0$, there exists an open and dense set of $A \in GL_n(\mathbf{R})$ (respectively $SL_n(\mathbf{R})$, $O_n(\mathbf{R})$) such that there exists $R_0 > 0$ such that for every $R > R_0$, the projection on $\mathbf{R}^n/\mathbf{Z}^n$ of the uniform measure on $A(\Gamma \cap B_R)$ is ε-close to Lebesgue measure on $\mathbf{R}^n/\mathbf{Z}^n$.*

Remark 7.22. The proof also allows to suppose that the radius R_0 is uniform in a whole neighbourhood of every matrix A of this open set of matrices.

Proof of Lemma 7.21. During this proof, we consider a distance dist on $\mathscr{P}(\mathbf{R}^n/\mathbf{Z}^n)$ which is invariant under translations. We also suppose that this distance satisfies the following convexity inequality: if $\mu, \nu_1, \cdots, \nu_d \in \mathscr{P}(\mathbf{R}^n/\mathbf{Z}^n)$, then

$$\operatorname{dist}\left(\mu, \frac{1}{d}\sum_{i=1}^d \nu_i\right) \le \frac{1}{d}\sum_{i=1}^d \operatorname{dist}(\mu, \nu_i).$$

For the simplicity of the notations, when μ and ν have not total mass 1, we will denote by $\operatorname{dist}(\mu, \nu)$ the distance between the normalizations of μ and ν.

We consider the set $\mathscr{U}_\varepsilon$ of matrices $A \in GL_n(\mathbf{R})$ satisfying: there exists $R_0 > 0$ such that for all $R \ge R_0$,

$$\operatorname{dist}\left(\operatorname{Leb}_{\mathbf{R}^n/\mathbf{Z}^n}, \sum_{x \in B_R \cap \Gamma} \bar{\delta}_{Ax}\right) < \varepsilon,$$

where $\bar{\delta}_x$ is the Dirac measure of the projection of x on $\mathbf{R}^n/\mathbf{Z}^n$. We show that for every $\varepsilon > 0$, $\mathscr{U}_\varepsilon$ contains an open dense set. Then, the set $\bigcap_{\varepsilon>0} \mathscr{U}_\varepsilon$ will be a G_δ dense set made of matrices $A \in GL_n(\mathbf{R})$ such that $A(\Gamma)$ is well distributed.

Let $\varepsilon > 0$, $\delta > 0$, $\ell > 0$ and $A \in GL_n(\mathbf{R})$. We apply the second part of Remark 7.10 to obtain a parameter $R_0 > 0$ and a family $v_1, \cdots, v_n$ of ε-translations of Γ such that the family of cubes $\left(B(\sum_{i=1}^n k_i v_i, R_0)\right)_{-\ell \le k_i \le \ell}$ is an "almost tiling" of $B_{\ell R_0}$ (in particular, each v_i is close to the vector having $2R_0$ in the i-th coordinate and 0 in the others, see Figure 7.4):

(1) this collection of cubes fills almost all $B_{\ell R_0}$:

$$\frac{\operatorname{Card}\left(\Gamma \cap \left(\bigcup_{-\ell \le k_i \le \ell} B(\sum_{i=1}^n k_i v_i, R_0) \Delta B_{\ell R_0}\right)\right)}{\operatorname{Card}(\Gamma \cap B_{\ell R_0})} \le \varepsilon;$$

(2) the overlaps of the cubes are not too big: for all collections (k_i) and (k_i') such that $-\ell \le k_i, k_i' \le \ell$,

$$\frac{\mathrm{Card}\left(\Gamma \cap \left(\mathrm{B}(\sum_{i=1}^n k_i v_i, \mathrm{R}_0) \Delta \mathrm{B}(\sum_{i=1}^n k_i' v_i, \mathrm{R}_0)\right)\right)}{\mathrm{Card}(\Gamma \cap \mathrm{B}_{\ell \mathrm{R}_0})} \le \varepsilon;$$

(3) the vectors $\sum_{i=1}^n k_i v_i$ are translations for Γ: for every collection (k_i) such that $-\ell \le k_i \le \ell$,

$$\frac{\mathrm{Card}\left(\left(\Gamma \Delta (\Gamma - \sum_{i=1}^n k_i v_i)\right) \cap \mathrm{B}_{\mathrm{R}_0}\right)}{\mathrm{Card}(\Gamma \cap \mathrm{B}_{\mathrm{R}_0})} \le \varepsilon.$$

Increasing R_0 and ℓ if necessary, there exists $\mathrm{A}' \in \mathrm{GL}_n(\mathbf{R})$ (respectively $\mathrm{SL}_n(\mathbf{R})$, $\mathrm{O}_n(\mathbf{R})$) such that $\|\mathrm{A} - \mathrm{A}'\| \le \delta$ and that we have

$$\mathrm{dist}\left(\mathrm{Leb}_{\mathbf{R}^n/\mathbf{Z}^n}, \sum_{-\ell \le k_i \le \ell} \bar{\delta}_{\mathrm{A}'(\sum_{i=1}^n k_i v_i)}\right) \le \varepsilon. \tag{7.8}$$

Indeed, if we denote by Λ the lattice spanned by the vectors $v_1, \cdots, v_n$, then the set of matrices A' such that $\mathrm{A}'\Lambda$ is equidistributed modulo $\mathbf{Z}^n$ is dense in $\mathrm{GL}_n(\mathbf{R})$ (respectively $\mathrm{SL}_n(\mathbf{R})$ and $\mathrm{O}_n(\mathbf{R})$).

Then, we have,

$$\begin{aligned}\mathrm{dist}\left(\mathrm{Leb}_{\mathbf{R}^n/\mathbf{Z}^n}, \sum_{\substack{-\ell \le k_i \le \ell \\ x \in \Gamma \cap \mathrm{B}(\sum_{i=1}^n k_i v_i, \mathrm{R}_0)}} \bar{\delta}_{\mathrm{A}'x}\right) \le \\ \mathrm{dist}\left(\mathrm{Leb}_{\mathbf{R}^n/\mathbf{Z}^n}, \sum_{\substack{-\ell \le k_i \le \ell \\ x \in \Gamma \cap \mathrm{B}(0, \mathrm{R}_0)}} \bar{\delta}_{\mathrm{A}'(\sum_{i=1}^n k_i v_i) + \mathrm{A}'x}\right) \\ + \mathrm{dist}\left(\sum_{\substack{-\ell \le k_i \le \ell \\ x \in \Gamma \cap \mathrm{B}(0, \mathrm{R}_0)}} \bar{\delta}_{\mathrm{A}'(\sum_{i=1}^n k_i v_i) + \mathrm{A}'x}, \sum_{\substack{-\ell \le k_i \le \ell \\ x \in \Gamma \cap \mathrm{B}(\sum_{i=1}^n k_i v_i, \mathrm{R}_0)}} \bar{\delta}_{\mathrm{A}'x}\right)\end{aligned}$$

By the property of convexity of dist, the first term is smaller than

$$\frac{1}{\mathrm{Card}\left(\Gamma \cap \mathrm{B}(0, \mathrm{R}_0)\right)} \sum_{x \in \Gamma \cap \mathrm{B}(0, \mathrm{R}_0)} \mathrm{dist}\left(\mathrm{Leb}_{\mathbf{R}^n/\mathbf{Z}^n}, \sum_{-\ell \le k_i \le \ell} \bar{\delta}_{\mathrm{A}'(\sum_{i=1}^n k_i v_i) + \mathrm{A}'x}\right);$$

by Equation (7.8) and the fact that dist is invariant under translation, this term is smaller than ε. As by hypothesis, the vectors $\sum_{i=1}^n k_i v_i$ are ε-translations of Γ (hypothesis (3)), the second term is also smaller than ε. Thus, we get

$$\mathrm{dist}\left(\mathrm{Leb}_{\mathbf{R}^n/\mathbf{Z}^n}, \sum_{\substack{-\ell \le k_i \le \ell \\ x \in \Gamma \cap \mathrm{B}(\sum_{i=1}^n k_i v_i, \mathrm{R}_0)}} \bar{\delta}_{\mathrm{A}'x}\right) \le 2\varepsilon$$

By the fact that the family of cubes $\left(\mathrm{B}(\sum_{i=1}^n k_i v_i, \mathrm{R}_0)\right)_{-\ell \le k_i \le \ell}$ is an almost tiling of $\mathrm{B}_{\ell \mathrm{R}_0}$ (hypotheses (1) and (2)), we get, for every $v \in \mathbf{R}^n$,

$$\mathrm{dist}\left(\mathrm{Leb}_{\mathbf{R}^n/\mathbf{Z}^n}, \sum_{x \in \Gamma \cap \mathrm{B}_{\ell \mathrm{R}_0}} \bar{\delta}_{\mathrm{A}'x}\right) < 4\varepsilon.$$

Remark that we can suppose that this remains true on a whole neighbourhood of A′. We use the fact that Γ is an almost periodic pattern to deduce that A′ belongs to the interior of $\mathscr{U}_\varepsilon$. □

7.3 Differences in almost periodic patterns

From now, we suppose that the almost periodic patterns we consider are subsets of $\mathbf{Z}^n$.

In the sequel, we will use the concept of difference in an almost periodic pattern. The following lemma, whose proof is straightforward, states that the occurrences of a given difference in an almost periodic pattern form an almost periodic pattern.

Lemma 7.23. *Let $v \in \mathbf{R}^n$ and Γ be an almost periodic pattern. Then the set*

$$\{x \in \Gamma \mid x + v \in \Gamma\} = \Gamma \cap (\Gamma - v)$$

is an almost periodic pattern.

Then Proposition 7.6 allows to associate to each translation of $\mathbf{Z}^n$ the frequency it appears in the almost periodic pattern.

Definition 7.24. For $v \in \mathbf{Z}^n$, we set

$$\rho_\Gamma(v) = \frac{\mathrm{D}\{x \in \Gamma \mid x + v \in \Gamma\}}{\mathrm{D}(\Gamma)} = \frac{\mathrm{D}\big(\Gamma \cap (\Gamma - v)\big)}{\mathrm{D}(\Gamma)} \in [0, 1]$$

the *frequency* of the difference v in the almost periodic pattern Γ.

The function ρ_Γ is itself almost periodic in the sense given by H. Bohr (see [Boh24]).

Definition 7.25. Let $f : \mathbf{Z}^n \to \mathbf{R}$. Denoting by T_v the translation of vector v, we say that f is *Bohr almost periodic* (also called *uniformly almost periodic*) if for every $\varepsilon > 0$, the set

$$\mathscr{N}_\varepsilon = \Big\{v \in \mathbf{Z}^n \mid \|f - f \circ \mathrm{T}_v\|_\infty < \varepsilon\Big\},$$

is relatively dense.

If $f : \mathbf{Z}^n \to \mathbf{R}$ is a Bohr almost periodic function, then it possesses a *mean* $\mathscr{M}(f)$ (see for example the historical paper of H. Bohr [Boh24, Satz VIII]), which satisfies: for every $\varepsilon > 0$, there exists $\mathrm{R}_0 > 0$ such that for every $\mathrm{R} \geq \mathrm{R}_0$ and every $x \in \mathbf{R}^n$, we have

$$\left| \mathscr{M}(f) - \frac{1}{\mathrm{Card}[\mathrm{B}(x, \mathrm{R})]} \sum_{v \in [\mathrm{B}(x,\mathrm{R})]} f(v) \right| < \varepsilon.$$

The fact that ρ_Γ is Bohr almost periodic is straightforward.

Lemma 7.26. *If Γ is an almost periodic pattern, then the function ρ_Γ is Bohr almost periodic.*

In fact, we can compute precisely the mean of $\rho(\Gamma)$.

Proposition 7.27. *If Γ is an almost periodic pattern, then we have*

$$\mathscr{M}(\rho_\Gamma) = \mathrm{D}(\Gamma).$$

Proof of Proposition 7.27. This proof lies primarily in an inversion of limits.

Let $\varepsilon > 0$. As Γ is an almost periodic pattern, there exists $R_0 > 0$ such that for every $R \geq R_0$ and every $x \in \mathbf{R}^n$, we have

$$\left| D(\Gamma) - \frac{\Gamma \cap [B(x,R)]}{\mathrm{Card}[B_R]} \right| \leq \varepsilon. \tag{7.9}$$

So, we choose $R \geq R_0$, $x \in \mathbf{Z}^n$ and compute

$$\begin{aligned}
\frac{1}{\mathrm{Card}[B_R]} \sum_{v \in [B(x,R)]} \rho_\Gamma(v) &= \frac{1}{\mathrm{Card}[B_R]} \sum_{v \in [B(x,R)]} \frac{D\big((\Gamma - v) \cap \Gamma\big)}{D(\Gamma)} \\
&= \frac{1}{\mathrm{Card}[B_R]} \sum_{v \in [B(x,R)]} \lim_{R' \to +\infty} \frac{1}{\mathrm{Card}[B_{R'}]} \sum_{y \in [B_{R'}]} \frac{\mathbf{1}_{y \in \Gamma - v} \mathbf{1}_{y \in \Gamma}}{D(\Gamma)} \\
&= \frac{1}{D(\Gamma)} \lim_{R' \to +\infty} \frac{1}{\mathrm{Card}[B_{R'}]} \sum_{y \in [B_{R'}]} \mathbf{1}_{y \in \Gamma} \frac{1}{\mathrm{Card}[B_R]} \sum_{v \in [B(x,R)]} \mathbf{1}_{y \in \Gamma - v} \\
&= \frac{1}{D(\Gamma)} \underbrace{\lim_{R' \to +\infty} \frac{1}{\mathrm{Card}[B_{R'}]} \sum_{y \in [B_{R'}]} \mathbf{1}_{y \in \Gamma}}_{\text{first term}} \underbrace{\frac{1}{\mathrm{Card}[B_R]} \sum_{v' \in [B(y+x,R)]} \mathbf{1}_{v' \in \Gamma}}_{\text{second term}}.
\end{aligned}$$

By Equation (7.9), the second term is ε-close to $D(\Gamma)$. Considered independently, the first term is equal to $D(\Gamma)$ (still by Equation (7.9)). Thus, we have

$$\left| \frac{1}{\mathrm{Card}[B(x,R)]} \sum_{v \in [B(x,R)]} \rho_\Gamma(v) - D(\Gamma) \right| \leq \varepsilon,$$

that we wanted to prove. □

We now state a Minkowski-type theorem for the map ρ_Γ. To begin with, we recall the classical Minkowski theorem (see for example [Ber09, IX.3] or the whole book [GL87]).

Theorem 7.28 (Minkowski). *Let Λ be a lattice of $\mathbf{R}^n$, $k \in \mathbf{N}$ and $S \subset \mathbf{R}^n$ be a centrally symmetric convex body. If $\mathrm{Leb}(S/2) > k\,\mathrm{covol}(\Lambda)$, then S contains at least $2k$ distinct points of $\Lambda \setminus \{0\}$.*

In particular, if $\mathrm{Leb}(S/2) > \mathrm{covol}(\Lambda)$, then S contains at least one point of $\Lambda \setminus \{0\}$. This theorem is optimal in the following sense: for every lattice Λ and for every $k \in \mathbf{N}$, there exists a centrally symmetric convex body S such that $\mathrm{Leb}(S/2) = k\,\mathrm{covol}(\Lambda)$ and that S contains less than $2k$ distinct points of $\Lambda \setminus \{0\}$. In Chapter 8, we will make use of Hajós theorem (Theorem 8.21), which specify what can happen in the case where S is an infinite ball satisfying $\mathrm{Leb}(S/2) = \mathrm{covol}(\Lambda)$.

Proof of Theorem 7.28. We consider the function

$$\varphi = \sum_{\lambda \in \Lambda} \mathbf{1}_{\lambda + S/2}.$$

The hypothesis about the covolume of Λ and the volume of $S/2$ imply that the mean of the periodic function φ satisfies $\mathscr{M}(\varphi) > k$. In particular, there exists $x_0 \in \mathbf{R}^n$ such that $\varphi(x_0) \geq k+1$ (note that this argument is similar to the pigeonhole principle). Then, there

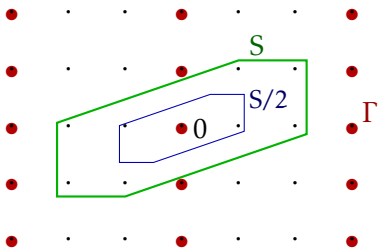

Figure 7.5: Example 7.31 of equality case in Theorem 7.29 for $k = 3$

exists $\lambda_0, \cdots, \lambda_k \in \Lambda$, with the λ_i sorted in lexicographical order (for a chosen basis), such that the points $x_0 - \lambda_i$ belong to S/2. As S/2 is centrally symmetric, $\lambda_i - x_0$ belongs to S/2 and as S/2 is convex, $\big((x_0 - \lambda_0) + (\lambda_i - x_0)\big)/2 = (\lambda_i - \lambda_0)/2$ also belongs to S/2. Then, $\lambda_i - \lambda_0 \in \Lambda \setminus \{0\} \cap S$ for every $i \in [\![1, k]\!]$. By hypothesis, these k vectors are all different. To obtain $2k$ different points of $S \cap \Lambda \setminus \{0\}$ (instead of k different points), it suffices to consider the points $\lambda_0 - \lambda_i$; this collection is disjoint from the collection of $\lambda_i - \lambda_0$ by the fact that the λ_i are sorted in lexicographical order. This proves the theorem. □

With the definitions we introduced, Minkowski theorem can be seen as a result about the function ρ_Γ: *Let Λ be a lattice of $\mathbf{R}^n$, and $S \subset \mathbf{R}^n$ be a centrally symmetric convex body, then*

$$\sum_{u \in S} \rho_\Lambda(u) \geq 2\lceil D(\Lambda)\,\mathrm{Leb}(S/2)\rceil - 1.$$

We now state a similar statement in the more general case of almost periodic patterns.

Theorem 7.29 (in collaboration with É. Joly)**.** *Let $\Gamma \subset \mathbf{Z}^n$ be an almost periodic pattern, and $S \subset \mathbf{R}^n$ be a centrally symmetric convex body. Then (recall that $[S] = S \cap \mathbf{Z}^n$)*

$$\sum_{u \in [S]} \rho_\Gamma(u) \geq D(\Gamma)\,\mathrm{Card}[S/2].$$

Remark that by Theorem 7.28, we have $\mathrm{Card}[S/2] \geq 2\lceil \mathrm{Leb}(S/4)\rceil - 1$. Thus,

$$\sum_{u \in [S]} \rho_\Gamma(u) \geq D(\Gamma)\Big(2\lceil \mathrm{Leb}(S/4)\rceil - 1\Big).$$

Remark 7.30. It is possible to prove in an easier way a weak version of Theorem 7.29 which is sufficient for the use we will make (Theorem 8.36). However, it seemed nicer to us to state a more optimal property.

The case of equality in the theorem is attained even in the non trivial case where $\mathrm{Card}[S/2] > 1$, as shown by the following example.

Example 7.31. If k is an odd number, if Γ is the lattice $k\mathbf{Z} \times \mathbf{Z}$, and if S is a centrally symmetric convex set such that (see Figure 7.5)

$$S \cap \mathbf{Z}^2 = \{(i, 0) \mid i \in [\![-(k-1), k-1]\!]\} \cup \{\pm(i, 1) \mid i \in [\![1, k-1]\!]\},$$

then $\sum_{u \in S} \rho(u) = 1$, $D(\Gamma) = 1/k$ and $\mathrm{Card}(S/2 \cap \mathbf{Z}^n) = k$.

The strategy of proof of this theorem is similar to that of the classical Minkowski theorem: we consider the set $\Gamma + S/2$ and define a suitable auxiliary function based on this set.

Proof of Theorem 7.29. We first define

$$\rho_a^R = \frac{1}{\mathrm{Card}[B_R]} \sum_{v\in[B_R]} \mathbf{1}_{v\in[S/2+a]} \sum_{u\in[S]} \frac{\mathbf{1}_{v\in\Gamma}\mathbf{1}_{u+v\in\Gamma}}{D(\Gamma)}.$$

We use this function to apply an argument of double counting. Firstly, we have

$$\begin{aligned}\sum_{a\in\mathbf{Z}^n} \rho_a^R &= \frac{1}{\mathrm{Card}[B_R]} \sum_{v\in[B_R]} \sum_{u\in[S]} \frac{\mathbf{1}_{v\in\Gamma}\mathbf{1}_{u+v\in\Gamma}}{D(\Gamma)} \sum_{a\in\mathbf{Z}^n} \mathbf{1}_{a\in[S/2+v]} \\ &= \frac{1}{\mathrm{Card}[B_R]} \sum_{v\in[B_R]} \sum_{u\in[S]} \frac{\mathbf{1}_{v\in\Gamma}\mathbf{1}_{u+v\in\Gamma}}{D(\Gamma)} \mathrm{Card}[S/2+v] \\ &= \mathrm{Card}[S/2] \sum_{u\in[S]} \frac{1}{\mathrm{Card}[B_R]} \sum_{v\in[B_R]} \frac{\mathbf{1}_{v\in\Gamma}\mathbf{1}_{u+v\in\Gamma}}{D(\Gamma)}.\end{aligned}$$

Thus, by the definition of ρ_Γ, we get

$$\lim_{R\to+\infty} \sum_{a\in\mathbf{Z}^n} \rho_a^R = \mathrm{Card}[S/2] \sum_{u\in[S]} \rho_\Gamma(u). \tag{7.10}$$

The conclusion of the theorem is an estimate on the right side of this equality; to prove it we compute the left side in another way.

First of all, we remark that as S is a centrally symmetric convex body, $v, w \in S/2$ implies that $w - v \in S$, so (applying this property to $w = u + v$)

$$\mathbf{1}_{v\in S/2+a}\mathbf{1}_{u\in S} \geq \mathbf{1}_{v\in S/2+a}\mathbf{1}_{u+v\in S/2+a},$$

and thus

$$\mathbf{1}_{v\in[S/2+a]}\mathbf{1}_{u\in[S]}\mathbf{1}_{v\in\Gamma}\mathbf{1}_{u+v\in\Gamma} \geq \mathbf{1}_{v\in\Gamma\cap[S/2+a]}\mathbf{1}_{u+v\in\Gamma\cap[S/2+a]}.$$

We now sum this inequality over $u \in \mathbf{Z}^n$:

$$\begin{aligned}\sum_{u\in[S]} \mathbf{1}_{v\in[S/2+a]}\mathbf{1}_{v\in\Gamma}\mathbf{1}_{u+v\in\Gamma} &\geq \mathbf{1}_{v\in\Gamma\cap[S/2+a]} \sum_{u\in\mathbf{Z}^n} \mathbf{1}_{u+v\in\Gamma\cap[S/2+a]} \\ &\geq \mathbf{1}_{v\in\Gamma\cap[S/2+a]} \sum_{u'\in\mathbf{Z}^n} \mathbf{1}_{u'\in\Gamma\cap[S/2+a]} \\ &\geq \mathbf{1}_{v\in\Gamma\cap[S/2+a]} \mathrm{Card}\Big(\Gamma\cap[S/2+a]\Big);\end{aligned}$$

so

$$\rho_a^R \geq \frac{1}{D(\Gamma)} \frac{1}{\mathrm{Card}[B_R]} \sum_{v\in[B_R]} \mathbf{1}_{v\in\Gamma\cap[S/2+a]} \mathrm{Card}(\Gamma\cap[S/2+a]).$$

We denote by B_R^S the S-interior of B_R, that is

$$B_R^S = \Big(B_R^C + S\Big)^C = \{x \in B_R \mid \forall s \in S, x + s \in B_R\}.$$

In particular, $a \in [\mathrm{B}_r^{\mathrm{S}}]$ implies that $[\mathrm{S}/2 + a] \subset [\mathrm{B}_r]$, then

$$\begin{aligned}\sum_{a\in\mathbf{Z}^n} \rho_a^{\mathrm{R}} &\geq \frac{1}{\mathrm{D}(\Gamma)} \sum_{a\in\mathbf{Z}^n} \frac{1}{\mathrm{Card}[\mathrm{B}_{\mathrm{R}}]} \sum_{v\in[\mathrm{B}_{\mathrm{R}}]} \mathbf{1}_{v\in\Gamma\cap[\mathrm{S}/2+a]} \mathrm{Card}\Big(\Gamma \cap [\mathrm{S}/2 + a]\Big) \\ &\geq \frac{1}{\mathrm{D}(\Gamma)} \frac{1}{\mathrm{Card}[\mathrm{B}_{\mathrm{R}}]} \sum_{a\in[\mathrm{B}_{\mathrm{R}}^{\mathrm{S}}]} \sum_{v\in[\mathrm{B}_{\mathrm{R}}]} \mathbf{1}_{v\in\Gamma\cap[\mathrm{S}/2+a]} \mathrm{Card}\Big(\Gamma \cap [\mathrm{S}/2 + a]\Big) \\ &\geq \frac{1}{\mathrm{D}(\Gamma)} \frac{1}{\mathrm{Card}[\mathrm{B}_{\mathrm{R}}]} \sum_{a\in[\mathrm{B}_{\mathrm{R}}^{\mathrm{S}}]} \mathrm{Card}\Big(\Gamma \cap [\mathrm{S}/2 + a]\Big)^2.\end{aligned}$$

We then use the fact that the family $\{\mathrm{B}_{\mathrm{R}}\}_{\mathrm{R}>0}$ is van Hove when R goes to infinity (see for example [Moo02, Equation 4]), that is

$$\lim_{\mathrm{R}\to+\infty} \frac{\mathrm{Card}[\mathrm{B}_{\mathrm{R}}] - \mathrm{Card}[\mathrm{B}_{\mathrm{R}}^{\mathrm{S}}]}{\mathrm{Card}[\mathrm{B}_{\mathrm{R}}]} = 0;$$

thus,

$$\lim_{\mathrm{R}\to+\infty} \sum_{a\in\mathbf{Z}^n} \rho_a^{\mathrm{R}} \geq \frac{1}{\mathrm{D}(\Gamma)} \overline{\lim_{\mathrm{R}\to+\infty}} \frac{1}{\mathrm{Card}[\mathrm{B}_{\mathrm{R}}]} \sum_{a\in[\mathrm{B}_{\mathrm{R}}]} \mathrm{Card}\Big(\Gamma \cap [\mathrm{S}/2 + a]\Big)^2.$$

Using the convexity of $x \mapsto x^2$, we deduce that

$$\lim_{\mathrm{R}\to+\infty} \sum_{a\in\mathbf{Z}^n} \rho_a^{\mathrm{R}} \geq \overline{\lim_{\mathrm{R}\to+\infty}} \frac{1}{\mathrm{D}(\Gamma)} \left(\frac{1}{\mathrm{Card}[\mathrm{B}_{\mathrm{R}}]} \sum_{a\in[\mathrm{B}_{\mathrm{R}}]} \mathrm{Card}\Big(\Gamma \cap [\mathrm{S}/2 + a]\Big) \right)^2. \tag{7.11}$$

Now, it remains to compute

$$\begin{aligned}\frac{1}{\mathrm{Card}[\mathrm{B}_{\mathrm{R}}]} \sum_{a\in[\mathrm{B}_{\mathrm{R}}]} \mathrm{Card}\Big(\Gamma \cap [\mathrm{S}/2 + a]\Big) &= \frac{1}{\mathrm{Card}[\mathrm{B}_{\mathrm{R}}]} \sum_{a\in[\mathrm{B}_{\mathrm{R}}]} \sum_{s\in[\mathrm{S}/2]} \mathbf{1}_{s+a\in\Gamma} \\ &= \sum_{s\in[\mathrm{S}/2]} \frac{1}{\mathrm{Card}[\mathrm{B}_{\mathrm{R}}]} \sum_{a'\in[\mathrm{B}_{\mathrm{R}}]-s} \mathbf{1}_{a'\in\Gamma}.\end{aligned}$$

But for every $s \in \mathbf{R}^n$, we have

$$\lim_{\mathrm{R}\to+\infty} \frac{1}{\mathrm{Card}[\mathrm{B}_{\mathrm{R}}]} \sum_{a'\in[\mathrm{B}_{\mathrm{R}}]-s} \mathbf{1}_{a'\in\Gamma} = \mathrm{D}(\Gamma),$$

thus

$$\frac{1}{\mathrm{Card}[\mathrm{B}_{\mathrm{R}}]} \sum_{a\in[\mathrm{B}_{\mathrm{R}}]} \mathrm{Card}\Big(\Gamma \cap [\mathrm{S}/2 + a]\Big) = \mathrm{Card}[\mathrm{S}/2]\mathrm{D}(\Gamma).$$

Applied to Equation (7.11), this gives

$$\lim_{\mathrm{R}\to+\infty} \sum_{a\in\mathbf{Z}^n} \rho_a^{\mathrm{R}} \geq \mathrm{Card}[\mathrm{S}/2]^2\mathrm{D}(\Gamma).$$

To finish the proof, we combine it with the first estimate in Equation (7.10) and get:

$$\mathrm{Card}[\mathrm{S}/2] \sum_{u\in\mathrm{S}} \rho_\Gamma(u) \geq \mathrm{Card}[\mathrm{S}/2]^2\mathrm{D}(\Gamma),$$

so

$$\sum_{u\in\mathrm{S}} \rho_\Gamma(u) \geq \mathrm{Card}[\mathrm{S}/2]\mathrm{D}(\Gamma),$$

□

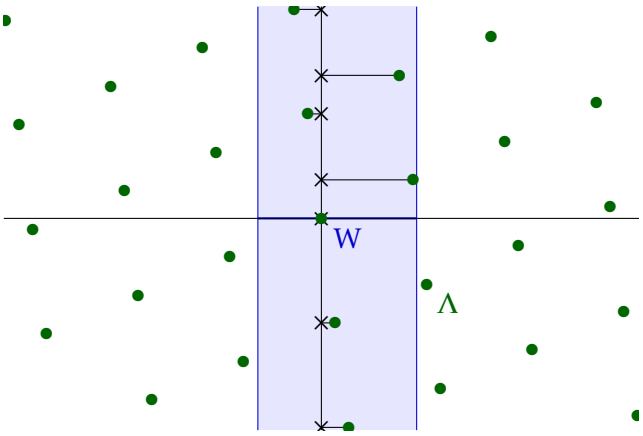

Figure 7.6: Construction of a model set.

7.4 Model sets

We now turn to a more precise notion about almost periodicity: model sets. We begin this section by motivating the introduction of these sets: we give an alternative construction of the sets $\widehat{\mathrm{A}}(\mathbf{Z}^n)$ in terms of model sets.

A point $x \in \mathbf{Z}^n$ belongs to $\widehat{\mathrm{A}}(\mathbf{Z}^n)$ if and only if there exists $y \in \mathbf{Z}^n$ such that $\|x - \mathrm{A}y\|_\infty < \frac{1}{2}$. In other words, if we set

$$\mathrm{M}_\mathrm{A} = \begin{pmatrix} \mathrm{A} & -\mathrm{Id} \\ & \mathrm{Id} \end{pmatrix} \in \mathrm{M}_{2n}(\mathbf{R}),$$

if we note p_1 and p_2 the projections of $\mathbf{R}^{2n}$ on respectively the n firsts and the n last coordinates, and if we set $\mathrm{W} =]-\frac{1}{2}, \frac{1}{2}]^n$, then

$$\widehat{\mathrm{A}}(\mathbf{Z}^n) = \left\{ p_2(\mathrm{M}_\mathrm{A} v) \mid v \in \mathbf{Z}^{2n},\, p_1(\mathrm{M}_\mathrm{A} v) \in \mathrm{W} \right\}.$$

This notion is close to that of model set introduced by Y. Meyer in the early seventies [Mey72], but in our case the projection p_2 is not injective. Model sets are sometimes called "cut and project" sets in the literature.

Definition 7.32. Let Λ be a lattice of $\mathbf{R}^{m+n}$, p_1 and p_2 the projections of $\mathbf{R}^{m+n}$ on respectively $\mathbf{R}^m \times \{0\}_{\mathbf{R}^n}$ and $\{0\}_{\mathbf{R}^m} \times \mathbf{R}^n$, and W a Riemann integrable subset of $\mathbf{R}^m$. The *model set* modelled on the lattice Λ and the *window* W is (see Figure 7.6)

$$\Gamma = \left\{ p_2(\lambda) \mid \lambda \in \Lambda,\, p_1(\lambda) \in \mathrm{W} \right\}.$$

Here, we will call *model set* every set of this type, even if the projection p_2 is not injective. Indeed, this phenomenon is what is interesting for us, if it did not occur there would be no loss of injectivity when applying discretizations of linear maps. Notice that this definition, which could seem very restrictive for the set Γ, is in fact quite general: as stated by Y. Meyer in [Mey72], every Meyer set [5] is a subset of a model set. Conversely, model sets are Meyer sets (see [Mey95]).

Returning to our problem of images of the lattice $\mathbf{Z}^n$ by discretizations of linear maps, we have the following (trivial) result.

5. A set Γ is a *Meyer set* if $\Gamma - \Gamma$ is a Delone set. It is equivalent to ask that there exists a finite set F such that $\Gamma - \Gamma \subset \Gamma + \mathrm{F}$ (see [Lag96]).

Proposition 7.33. *Let $A_1, \cdots, A_k \in \mathrm{GL}_n(\mathbf{R})$ be k invertible maps, then the k-th image $\Gamma_k = \widehat{A_k} \circ \cdots \circ \widehat{A_1}(\mathbf{Z}^n)$ of $\mathbf{Z}^n$ by the discretizations is the model set modelled on the window $W =]-\frac{1}{2}, \frac{1}{2}]^{nk}$ and the lattice $M_{A_1,\cdots,A_k}\mathbf{Z}^{n(k+1)}$, where*

$$M_{A_1,\cdots,A_k} = \begin{pmatrix} A_1 & -\mathrm{Id} & & & \\ & A_2 & -\mathrm{Id} & & \\ & & \ddots & \ddots & \\ & & & A_k & -\mathrm{Id} \\ & & & & \mathrm{Id} \end{pmatrix} \in \mathrm{M}_{n(k+1)}(\mathbf{R}).$$

Remark 7.34. This notion has the advantage that it builds the k-th image directly: the concept of time disappears, so we will be able "anticipate" the behaviour of successive images. The downside is the increasing of the dimension; thus it will be more difficult to have a geometric intuition...

In the sequel, we will only consider model sets whose window is regular.

Definition 7.35. Let W be a subset of $\mathbf{R}^n$. We say that W is *regular* if for every affine subspace $V \subset \mathbf{R}^n$, we have

$$\mathrm{Leb}_V\Big(B_V\big(\partial(V \cap W), \eta\big)\Big) \underset{\eta\to 0}{\longrightarrow} 0,$$

where Leb_V denotes the Lebesgue measure on V, and $B_V\big(\partial(V \cap W), \eta\big)$ the set of points of V whose distance to $\partial(V \cap W)$ is smaller than η (of course, the boundary is also take in restriction to V).

The link with the previous sections is made by the following theorem.

Theorem 7.36 (in collaboration with Y. Meyer)**.** *A model set modelled on a regular window is an almost periodic pattern.*

In other words, for every $\varepsilon > 0$, there exists $R_0 > 0$ and a relatively dense set $\mathcal{N}$ such that for every $v \in \mathcal{N}$ and every $R \geq R_0$, most of the points (*i.e.* a proportion greater than ε) of the model set Γ also belong to $v + \Gamma$ (see Definition 7.3).

We begin by proving a weak version of this theorem.

Lemma 7.37. *A model set modelled on a window with nonempty interior is relatively dense.*

Proof of Lemma 7.37. We prove this lemma in the specific case where the window is B_η (recall that B_η is the infinite ball of radius η centred at 0). We will use this lemma only in this case (and the general case can be treated the same way).

Let Γ be a model set modelled on a lattice Λ and a window B_η. We will use the fact that for any centrally symmetric convex set $S \subset \mathbf{R}^n$, if there exists a basis $e_1, \cdots, e_n$ of Λ such that for each i, $\lceil n/2 \rceil e_i \in S$, then S contains a fundamental domain of $\mathbf{R}^n/\Lambda$, that is to say, for every $v \in \mathbf{R}^n$, we have $(S + v) \cap \Lambda \neq \emptyset$. This is due to the fact that the parallelepiped spanned by the vectors e_i is included into the simplex spanned by the vectors $\lceil n/2 \rceil e_i$.

We set

$$V = \bigcap_{\eta'>0} \mathrm{span}\left(p_1^{-1}(B_{\eta'}) \cap \Lambda\right) = \bigcap_{\eta'>0} \mathrm{span}\left\{\lambda \in \Lambda \mid d_\infty(\lambda, \ker p_1) \leq \eta'\right\},$$

and remark that $\operatorname{im} p_2 = \ker p_1 \subset V$, simply because for every vectorial line $D \subset \mathbf{R}^n$ (and in particular for $D \subset \ker p_1$), there exists some points of $\Lambda \setminus \{0\}$ arbitrarily close to D. We also take $R > 0$ such that

$$V \subset V' \doteq \operatorname{span}\Big(p_1^{-1}(B_{\eta/\lceil n/2\rceil}) \cap \Lambda \cap p_2^{-1}(B_R)\Big).$$

We then use the remark made in the beginning of this proof and apply it to the linear space V', the set $S = \Big(p_2^{-1}(B_R) \times p_1^{-1}(B_\eta)\Big) \cap V'$, and the module $V' \cap \Lambda$. This leads to

$$\forall v \in V,\ \Big(\big(p_1^{-1}(B_\eta) \cap p_2^{-1}(B_R)\big) + v\Big) \cap \Lambda \neq \emptyset,$$

and as $\operatorname{im} p_2 \subset V$, we get

$$\forall v' \in \mathbf{R}^n,\ \big(p_1^{-1}(B_\eta) \cap p_2^{-1}(B_R + v')\big) \cap \Lambda \neq \emptyset;$$

this proves that the model set is relatively dense for the radius R. □

Proof of Theorem 7.36. Let Γ be a model set modelled on a lattice Λ and a window W.

First of all, we decompose Λ into three supplementary modules: $\Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3$, such that (see [Bou98, Chap. VII, §1, 2]):

1. $\Lambda_1 = \ker p_1 \cap \Lambda$;
2. $p_1(\Lambda_2)$ is discrete;
3. $p_1(\Lambda_3)$ is dense in the vector space V it spans (and such a vector space is unique), and $V \cap p_1(\Lambda_2) = \{0\}$.

As $\Lambda_1 = \ker p_1 \cap \Lambda = \operatorname{im} p_2 \cap \Lambda$, we have $\Lambda_1 = p_2(\Lambda_1)$. Thus, for every $\lambda_1 \in \Lambda_1$ and every $\gamma \in \Gamma$, we have $\lambda_1 + \gamma \in \Gamma$. So Λ_1 is a set of periods for Γ. Therefore, considering the quotients $\mathbf{R}^n/\operatorname{span}\Lambda_1$ and Λ/Λ_1 if necessary, we can suppose that $p_{1|\Lambda}$ is injective (in other words, $\Lambda_1 = \{0\}$).

Under this assumption, the set $p_2(\Lambda_3)$ spans the whole space $\operatorname{im} p_2$. Indeed, as $\ker p_1 \cap \Lambda = \{0\}$, we have the decomposition

$$\mathbf{R}^{m+n} = \underbrace{\ker p_1}_{=\operatorname{im} p_2} \oplus \underbrace{\operatorname{span}\big(p_1(\Lambda_2)\big) \oplus \operatorname{span}\big(p_1(\Lambda_3)\big)}_{=\operatorname{im} p_1}. \tag{7.12}$$

Remark that as $p_1(\Lambda_2)$ is discrete, we have $\dim \operatorname{span}\big(p_1(\Lambda_2)\big) = \dim \Lambda_2$; thus, considering the dimensions in the decomposition (7.12), we get

$$\dim \operatorname{span}(\Lambda_3) = \dim\Big(\ker p_1 \oplus \operatorname{span}\big(p_1(\Lambda_3)\big)\Big). \tag{7.13}$$

The following matrix represents a basis of $\Lambda = \Lambda_2 \oplus \Lambda_3$ in a basis adapted to the decomposition (7.12).

$$\begin{array}{l} \\ \ker p_1 = \operatorname{im} p_2 \\ \operatorname{span}\big(p_1(\Lambda_2)\big) \\ \operatorname{span}\big(p_1(\Lambda_3)\big) \end{array} \begin{array}{c} \overbrace{}^{\Lambda_2}\ \overbrace{}^{\Lambda_3} \\ \left(\begin{array}{c|c} * & * \\ \hline * & 0 \\ \hline 0 & * \end{array}\right) \end{array}$$

We can see that the projection of the basis of Λ_3 on $\operatorname{im} p_2 \oplus \operatorname{span}\big(p_1(\Lambda_3)\big)$ form a free family; by Equation (7.13), this is in fact a basis. Thus, $\operatorname{span}(\Lambda_3) \supset \ker p_1 = \operatorname{im} p_2$, so $\operatorname{span}\big(p_2(\Lambda_3)\big) = \operatorname{im}(p_2)$.

For $\eta > 0$, let $\mathscr{N}(\eta)$ be the model set modelled on Λ and $\mathrm{B}(0,\eta)$, that is

$$\mathscr{N}(\eta) = \{p_2(\lambda_3) \mid \lambda_3 \in \Lambda_3, \|p_1(\lambda_3)\|_\infty \leq \eta\}.$$

Lemma 7.37 asserts that $\mathscr{N}(\eta)$ is relatively dense in the space it spans, and the previous paragraph asserts that this space is $\operatorname{im} p_2$. The next lemma, which obviously implies Theorem 7.36, expresses that if η is small enough, then $\mathscr{N}(\eta)$ is the set of translations we look for.

Lemma 7.38. *For every $\varepsilon > 0$, there exists $\eta > 0$ and a regular model set $\mathrm{Q}(\eta)$ such that $\mathrm{D}^+(\mathrm{Q}(\eta)) \leq \varepsilon$ and*

$$v \in \mathscr{N}(\eta) \Rightarrow (\Gamma + v)\Delta\Gamma \subset \mathrm{Q}(\eta).$$

We have now reduced the proof of Theorem 7.36 to that of Lemma 7.38. □

Proof of Lemma 7.38. We begin by proving that $(\Gamma + v) \setminus \Gamma \subset \mathrm{Q}(\eta)$ when $v \in \mathscr{N}(\eta)$. As $v \in \mathscr{N}(\eta)$, there exists $\lambda_0 \in \Lambda_3$ such that $p_2(\lambda_0) = v$ and $\|p_1(\lambda_0)\|_\infty \leq \eta$.

If $x \in \Gamma + v$, then $x = p_2(\lambda_2 + \lambda_3) + p_2(\lambda_0) = p_2(\lambda_2 + \lambda_3 + \lambda_0)$ where $\lambda_2 \in \Lambda_2$, $\lambda_3 \in \Lambda_3$ and $p_1(\lambda_2 + \lambda_3) \in \mathrm{W}$. If moreover $x \notin \Gamma$, it implies that $p_1(\lambda_2 + \lambda_3 + \lambda_0) \notin \mathrm{W}$. Thus, $p_1(\lambda_2 + \lambda_3 + \lambda_0) \in \mathrm{W}_\eta$, where (recall that $\mathrm{V} = \operatorname{span}(p_1(\Lambda_3))$)

$$\mathrm{W}_\eta = \big\{k + w \mid k \in \partial\mathrm{W}, w \in \mathrm{V} \cap \mathrm{B}_\eta\big\}.$$

We have proved that $\Gamma + v \setminus \Gamma \subset \mathrm{Q}(\eta)$, where

$$\mathrm{Q}(\eta) = \big\{p_2(\lambda) \mid \lambda \in \Lambda, p_1(\lambda) \in \mathrm{W}_\eta\big\}.$$

Let us stress that the model set $\mathrm{Q}(\eta)$ does not depend on v. We now observe that as W is regular, we have

$$\sum_{\lambda_2 \in \Lambda_2} \mathrm{Leb}_{\mathrm{V}+p_1(\lambda_2)}\big(\mathrm{W}_\eta \cap (\mathrm{V} + p_1(\lambda_2))\big) \underset{\eta \to 0}{\longrightarrow} 0.$$

As $p_1(\Lambda_3)$ is dense in V (thus, it is equidistributed), the uniform upper density of the model set $\mathrm{Q}(\eta)$ defined by the window W_η can be made smaller than ε by taking η small enough.

The treatment of $\Gamma \setminus (\Gamma + v)$ is similar; this ends the proof of Lemma 7.38. □

Chapter 8

Rate of injectivity of linear maps

In this chapter, we focus in more detail on the rate of injectivity of a sequence of linear maps (see Definition 7.19).

To begin with, study the easier case of the rate of injectivity in time 1. First of all, we study the continuity of the map $\tau = \tau^1$. Unfortunately, this map is not continuous (Proposition 8.1). However, τ is continuous at every totally irrational matrix [1] (Proposition 8.4). More precisely, we define the *mean rate of injectivity* $\overline{\tau}(A)$ of a matrix A: it is the mean of the rates of injectivity of the affine maps $A + v$, for $v \in \mathbf{R}^n/\mathbf{Z}^n$ (see Definition 8.3). It turns out that this quantity is much more convenient to use: by an argument of equirepartition, it coincides with the rate of injectivity when A is totally irrational, and it is continuous on the whole set $\mathrm{GL}_n(\mathbf{R})$ (Proposition 8.4). We finish this study by estimating the lack of continuity of τ — that is, the difference between $\overline{\tau}$ and τ — at the matrices which are not totally irrational: this difference is arbitrarily small out of a topologically small set, that is, out of a locally finite union of hyperplanes (Proposition 8.7).

These considerations allow us to focus on the mean rate of injectivity. The equidistribution property we gain by using the mean rate of injectivity allows to have a formula to compute it directly: for $A \in \mathrm{GL}_n(\mathbf{R})$, we have (Proposition 8.11, see also Proposition 8.18 and Figures 8.2 and 8.9)

$$\overline{\tau}(A) = \det(A)\mathrm{D}\left(\bigcup_{\lambda \in A\mathbf{Z}^n} \mathrm{B}_\infty(\lambda, 1/2)\right).$$

As the set $A\mathbf{Z}^n$ is a lattice, this quantity is equal to the area of the intersection between a fundamental domain of $A\mathbf{Z}^n$ and the union of cubes $\bigcup_{\lambda \in A\mathbf{Z}^n} \mathrm{B}_\infty(\lambda, 1/2)$: this reduces the computation of the mean rate of injectivity to that of the area of a finite union of cubes. This formula proves that the mean rate of injectivity is continuous and piecewise polynomial (Corollary 8.13). It also allows to compute the mean rate of injectivity of some practical examples (see Applications 8.14 and 8.15), and gives a quick method to compute numerically the mean rate of injectivity (see Figure 8.3).

To characterize the matrices for which the mean rate of injectivity is equal to 1, we combine this formula with the classical Hajós theorem (Theorem 8.21) and get directly Corollary 8.22: a matrix $A \in \mathrm{SL}_n(\mathbf{R})$ has a mean rate of injectivity equal to 1 if and only if there exists $B \in \mathrm{SL}_n(\mathbf{Z})$ such that in a canonical basis of $\mathbf{R}^n$ (that is, permuting

1. We say that a matrix $A \in \mathrm{GL}_n(\mathbf{R})$ is *totally irrational* when the set $A(\mathbf{Z}^n)$ is equidistributed modulo $\mathbf{Z}^n$.

coordinates if necessary), the matrix AB is upper triangular with ones on the diagonal. Thus, the set of such matrices forms a locally finite union of manifolds of positive codimension.

We then study the behaviour of the asymptotic rate of injectivity of a generic sequence of matrices of $\mathrm{SL}_n(\mathbf{R})$. It is given by the main theorem of this chapter (Theorem P, see also Theorem 8.24): for a generic sequence $(A_k)_{k\geq 1}$ of matrices of $\mathrm{SL}_n(\mathbf{R})$, we have $\tau^\infty\Big((A_k)_k\Big) = 0$.

The end of the chapter is devoted to the study of this theorem and some variations on it. The first variant is given in the (easy) case where all the matrices are diagonal (Proposition 8.27): for a generic sequence $(A_k)_{k\geq 1}$ of diagonal matrices, we have $\tau^\infty((A_k)_k) = 0$.

In the general case, we give a proof of the following weak version of Theorem 8.24 (Theorem 8.30): if $(A_k)_{k\geq 1}$ is a generic sequence of matrices of $\mathrm{SL}_n(\mathbf{R})$, then $\tau^\infty((A_k)_k) \leq 1/2$. This proof involves the study of the differences in an almost periodic pattern: the frequency of the difference $v \in \mathbf{Z}^n$ in the almost periodic pattern Γ is the quantity $\rho_\Gamma(v) = \mathrm{D}\Big((\Gamma - v)\cap\Gamma\Big)$ (see Definition 7.24). To start the proof, we remark that if $\Gamma \subset \mathbf{Z}^n$ is an almost periodic pattern whose density is bigger than $1/2+\delta$, then for every $v \in \mathbf{Z}^n$, the frequency of the difference v in Γ satisfies $\rho_\Gamma(v) \geq 2\delta$ (Lemma 8.29). The theorem results from this remark and from the fact that matrices with a rate of injectivity equal to 1 are very rare (Corollary 8.22). The proof also uses crucially the geometric construction used to compute the mean rate of injectivity $\overline{\tau}$ (Proposition 8.11).

We then state a variation of Theorem P in the case of isometries (Theorem 8.36) : if $(P_k)_{k\geq 1}$ is a generic sequence of matrices of $\mathrm{O}_n(\mathbf{R})$, then $\tau^\infty((P_k)_k) = 0$.
To prove this result, we study in more detail the action of the discretizations of linear maps on the frequencies of the differences of an almost periodic pattern (Proposition 8.33). We show that the discretization $\widehat{A}$ of $A \in \mathrm{GL}_n(\mathbf{R})$ acts smoothly on the frequency of differences: when we want to compute $\widehat{A}(\Gamma)$, we apply A and then make a projection, whereas when we want to compute $\rho_{\widehat{A}(\Gamma)}$, we apply A and then make a *weighted* projection. To prove Theorem Q, we combine this computation with the Minkowski-like theorem for the differences in an almost periodic pattern we obtained in the previous chapter (Theorem 7.29).

Finally, we use the notion of model set [2] (Definition 7.32) to improve Theorem 8.30. It is defined as follows: given a lattice Λ of $\mathbf{R}^{m+n}$ and a "regular" set $W \subset \mathbf{R}^m$, we select the points of the lattice whose projection on the m first coordinates belongs to W; the projection on the n last coordinates of these points is called the *model set* modelled on Λ and W. Model sets form a subclass of almost periodic patterns (Theorem 7.36). The introduction of this notion is supported by the fact that for every sequence $(A_k)_{k\geq 1}$ of invertible linear maps, the set $(\widehat{A_k}\circ\cdots\circ\widehat{A_1})(\mathbf{Z}^n)$ is a model set (Proposition 7.33). This fact gives us a more precise information about the structure of the successive images of $\mathbf{Z}^n$ by discretizations of linear maps, as the definition of model set is stronger than that of almost periodic pattern. This viewpoint is fruitful to study the question raised in Theorem P: first of all, an argument of equidistribution leads to a generalization of the geometric construction to compute the mean rate of injectivity in time 1 to the rate of injectivity in arbitrary time k (Propositions 8.40 and 8.46). Moreover, these constructions allows us to give an alternative proof of Theorem 8.30, and to finally prove Theorem P.

2. Sometimes called "cut-and-project" set.

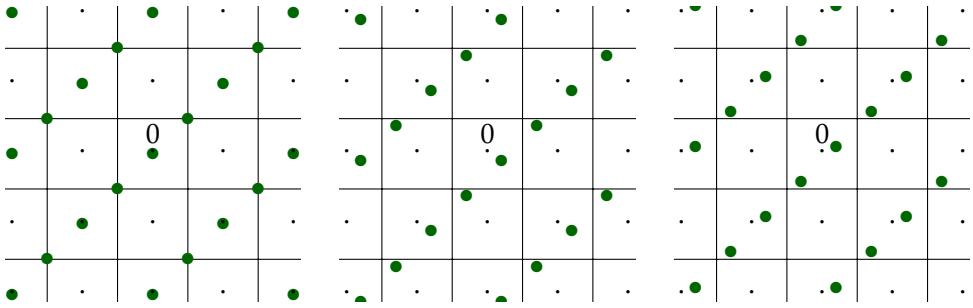

Figure 8.1: Sets $f_0(\mathbf{Z}^n)+v$ for $v=(0,0)$, $v=(0.2,-0.1)$ and $v=(0.2,0.1)$

8.1 Study of the continuity of the rate of injectivity

In this section, we study the properties of continuity of the function τ. We first show that there exists some matrices in which τ is not continuous (Proposition 8.1). However, we show in Proposition 8.4 that τ is continuous at every totally irrational matrix; more precisely, the map τ coincides with the *mean rate of injectivity* $\overline{\tau}$ (Definition 8.3) on the set of totally irrational matrices, and the map $\overline{\tau}$ is continuous on the whole set $\mathrm{GL}_n(\mathbf{R})$. We finish this section in estimating the lack of continuity of τ at the matrices which are not totally irrational (Proposition 8.7).

In the following example, we show that on some rational matrices, the rate of injectivity is not continuous. In particular, it will give us an example where Lemma 7.16 is not true when there is no restriction about the value of v on $\mathrm{I}_{\mathbf{Q}}(A)$ (the set of indices corresponding to rational coefficients, see Notation 7.13).

Our counterexample is given by irrational perturbations of the matrix

$$f_0=\begin{pmatrix}\frac{1}{2} & -1\\ \frac{1}{2} & 1\end{pmatrix}.$$

It can be easily seen that the rate of injectivity of f_0 is $1/2$ (see Figure 8.1). Remark that it depends on the choice made for the projection of $\mathbf{R}^2$ on $\mathbf{Z}^2$: if we make the same choice for both directions (that is what we have chosen in Definition 7.11) then the rate of injectivity is $1/2$, otherwise this rate of injectivity is 1. For ε "small", we consider the linear map

$$f_\varepsilon=\begin{pmatrix}1+\varepsilon & 0\\ 0 & \frac{1}{1+\varepsilon}\end{pmatrix}f_0,$$

which is close to f_0 when ε is small.

Proposition 8.1. *The rate of injectivity of f_ε tends $\frac{3}{4}$ when $\varepsilon\notin\mathbf{Q}$ tends to 0.*

In particular, the rate of injectivity τ is not continuous in f_0.

Lemma 8.2.
- *For all $v\in]0,\frac{1}{2}[^2\cup]\frac{1}{2},1[^2$ modulo $\mathbf{Z}^n$, the rate of injectivity of f_0+v is $\frac{1}{2}$.*
- *For all $v\in]0,\frac{1}{2}[\times]\frac{1}{2},1[\cup]\frac{1}{2},1[\times]0,\frac{1}{2}[$ modulo $\mathbf{Z}^n$, the rate of injectivity of f_0+v is 1.*

Proof of Lemma 8.2. We want to know when it is possible to find two different vectors $x,y\in\mathbf{Z}^n$ such that $\pi(f_0x+v)=\pi(f_0y+v)$. This implies that $d_\infty(f_0x,f_0y)<1$, a simple calculation (see Figure 8.1) shows that in this case, $x=y\pm(1,0)$. As $f_0(2\mathbf{Z}\times\mathbf{Z})\subset\mathbf{Z}^2$, we

can suppose that $x = 0$; we want to know if for $y = \pm(1,0)$, we have $\pi(f_0 x+v) = \pi(f_0 y+v)$, that is $\pi(v) = \pi(v \pm (1/2, 1/2))$. Again, a simple calculation shows that this occurs if and only if $v \in]0, \frac{1}{2}[^2 \cup]-\frac{1}{2}, 0[^2$. □

Thus, for half of the vectors v (for Lebesgue measure), the rate of injectivity of $f_0 + v$ is 1 and for the other half of the vectors v, the rate of injectivity of $f_0 + v$ is $\frac{1}{2}$. Proposition 8.1 then follows from an argument of equirepartition. To make it rigorous, we define the mean rate of injectivity.

Definition 8.3. For $A \in GL_n(\mathbf{R})$, the quantity

$$\overline{\tau}(A) = \int_{\mathbf{T}^n} \tau(A + v)\, d\, v$$

is called the *mean rate of injectivity* of A.

We say that a matrix $A \in GL_n(\mathbf{R})$ is *totally irrational* if the image $A(\mathbf{Z}^n)$ is equidistributed [3] modulo $\mathbf{Z}^n$; in particular, this is true when the coefficients of A form a $\mathbf{Q}$-free family.

The motivation of this definition is that the mean rate of injectivity is continuous and coincides with the rate of injectivity on totally irrational matrices.

Proposition 8.4. *The mean rate of injectivity $\overline{\tau}$ is continuous on $GL_n(\mathbf{R})$. Moreover, if A is totally irrational, then $\overline{\tau}(A) = \tau(A)$; and even more, $\tau(A) = \tau(A + v)$ for every $v \in \mathbf{T}^n$.*

Thus, the restriction of the rate of injectivity to totally irrational matrices is a continuous function, which extends to $GL_n(\mathbf{R})$ into a continuous function. In particular, the restriction of this function to any bounded subset of $SL_n(\mathbf{R})$ is uniformly continuous.

Proposition 8.4, combined with Lemma 8.2, obviously implies Proposition 8.1 (as for $\varepsilon \notin \mathbf{Q}$, the map f_ε is totally irrational).

Proof of Proposition 8.4. The continuity of $\overline{\tau}$ will be obtained later as a direct consequence of Corollary 8.13.

Let $A \in GL_n(\mathbf{R})$ be a totally irrational matrix. We want to prove that $\tau(A) = \overline{\tau}(A)$. To do that, we show that $\tau(A) = \tau(A+v)$ for every $v \in \mathbf{T}^n$. As $\{Ax \mid x \in \mathbf{Z}^n\}$ is equidistributed modulo $\mathbf{Z}^n$, for every $v \in \mathbf{T}^n$, there exists $x \in \mathbf{Z}^n$ such that $d_\infty(v - Ax, \mathbf{Z}^n) < \varepsilon$. But as the density is a limit independent from the choice of the centre of the ball (Corollary 7.7 and Theorem 7.12), we have

$$\lim_{R\to+\infty} \frac{\mathrm{Card}\left(\pi(A(\mathbf{Z}^n)) \cap B_R\right)}{\mathrm{Card}[B_R]} = \lim_{R\to+\infty} \frac{\mathrm{Card}\left(\pi(A(\mathbf{Z}^n) + Ax) \cap B(Ax, R)\right)}{\mathrm{Card}[B_R]}.$$

This proves that $\tau(A) = \tau(A + Ax)$. We then use Remark 7.18, which states that as A is totally irrational, the function $v \mapsto \tau(A+v)$ is continuous. As the vector Ax is arbitrarily close to the vector v, we get $\tau(A) = \tau(A + v)$. □

Remark 8.5. The same proof shows that if $\Gamma \subset \mathbf{Z}^n$ is an almost periodic pattern, and $A \in GL_n(\mathbf{R})$ is such that $A(\Gamma)$ is equidistributed modulo $\mathbf{Z}^n$ (which is true for a generic A, see Lemma 7.21), then $B \mapsto D\left(\widehat{B(\Gamma)}\right)$ is continuous in $B = A$.

3. It is equivalent to require that it is dense instead of equidistributed; it is also equivalent to ask that $\mathbf{Z}^n$ is equidistributed modulo $A(\mathbf{Z}^n)$.

We now study the behaviour of τ at the matrices where it is not continuous. In particular, we prove that the set of matrices on which the rate of injectivity makes "big jumps" is small. More precisely, we define the oscillation of a map.

Definition 8.6. For $A \in GL_n(\mathbf{R})$, the *oscillation* of τ at A is the quantity

$$\omega_\tau(A) = \overline{\lim_{r\to 0}} \sup\Big\{|\tau(B_1) - \tau(B_2)| \;\Big|\; \|A - B_i\| \le r\Big\}.$$

Proposition 8.7. *For every $\varepsilon > 0$, the set $\Big\{A \in GL_n(\mathbf{R}) \mid \omega_\tau(A) > \varepsilon\Big\}$ is locally contained in a finite union of hyperplanes.*

To prove this proposition, we will need a technical lemma which requires the following definition.

Definition 8.8. Let Λ be a lattice of $\mathbf{R}^n$. Then [Bou98, Chap. VII, §1, 2] implies that the $\mathbf{Z}$-module $\Lambda + \mathbf{Z}^n$ can be decomposed into two complementary modules

$$\Lambda + \mathbf{Z}^n = \Lambda_{\text{cont}} \oplus \Lambda_{\text{discr}},$$

such that Λ_{cont} is dense in the vector space it spans and Λ_{discr} is discrete.

During the proof of Proposition 8.7, we will make use of the following lemma, that we will not prove.

Lemma 8.9. *For every $\varepsilon > 0$, the set of matrices $A \in GL_n(\mathbf{R})$ such that $(A\mathbf{Z}^n)_{discr}$ has a fundamental domain of diameter smaller than ε contains the complement of a locally finite union of hyperplanes.*

Remark that if A is totally irrational, then $(A\mathbf{Z}^n)_{\text{discr}} = \{0\}$

Proof of Proposition 8.7. Let $\varepsilon > 0$. We consider a matrix $A \in GL_n(\mathbf{R})$ such that $(A\mathbf{Z}^n)_{\text{discr}}$ has a fundamental domain $\mathscr{D}$ of diameter smaller than ε; Lemma 8.9 asserts that such matrices contains the complement of a locally finite union of hyperplanes. We denote $\Lambda = A\mathbf{Z}^n$. We use Proposition 8.4, which states that $\overline{\tau}$ is continuous and coincides with τ on totally irrational matrices, to estimate the oscillation of τ in A : it is smaller than the maximal value of $\big|D(\pi(\Lambda+v)) - D(\pi(\Lambda+v'))\big|$, when the vectors v and v' run through $\mathbf{T}^n$.

First of all, we remark that the map

$$v \mapsto D(\pi(\Lambda + v))$$

is $\Lambda+\mathbf{Z}^n$-periodic. Thus, we only have to estimate $\big|D(\pi(\Lambda+v)) - D(\pi(\Lambda+v'))\big|$ for v and v' in $\mathscr{D}$. We denote by $(\mathbf{Z}^n)'$ the set of points in $\mathbf{R}^n$ whose discretization is not canonically defined, more precisely,

$$(\mathbf{Z}^n)' = \Big\{(x_i)_{1\le i\le n} \in \mathbf{R}^n \mid \exists i : x_i \in \mathbf{Z} + 1/2\Big\}.$$

Then, for every $v, v' \in \mathbf{R}^n$, we have

$$\big|D(\pi(\Lambda + v)) - D(\pi(\Lambda + v'))\big| \le D\{x \in \Lambda \mid \exists w \in \mathscr{D} : x + w \in (\mathbf{Z}^n)'\},$$

and as $\operatorname{diam}(\mathscr{D})$ is smaller than ε, we get

$$\big|D(\pi(\Lambda + v)) - D(\pi(\Lambda + v'))\big| \le \frac{\operatorname{Card}\Big\{x \in \Lambda_{\text{discr}} \cap [0,1]^n \mid d(x,(\mathbf{Z}^n)') \le \varepsilon\Big\}}{\operatorname{Card}(\Lambda_{\text{discr}} \cap [0,1]^n)}.$$

The proposition then follows from the fact that this last quantity is small (uniformly in ε). □

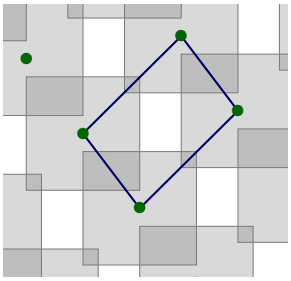

Figure 8.2: First geometric construction: the green points are the elements of Λ, the blue parallelogram is a fundamental domain of Λ and the grey squares are centred on the points of Λ and have radii 1/2; their union form the set U. A point $x \in \mathbf{Z}^n$ belongs to $\pi(\Lambda)$ if and only if it belongs to at least one grey square.

8.2 A geometric viewpoint on the mean rate

In this section, we present two geometric constructions to compute the rate of injectivity of a matrix.

First construction Let $A \in \mathrm{GL}_n(\mathbf{R})$ and $\Lambda = A(\mathbf{Z}^n)$. The density of $\pi(\Lambda)$ is the proportion of $x \in \mathbf{Z}^n$ belonging to $\pi(\Lambda)$; in other words the proportion of $x \in \mathbf{Z}^n$ such that there exists $\lambda \in \Lambda$ whose distance to x (for $\|\cdot\|_\infty$) is smaller than 1/2. This property only depends on the value of x modulo Λ. If we consider the union

$$U = \bigcup_{\lambda \in \Lambda} B(\lambda, 1/2)$$

of balls of radius 1/2 centred on the points of Λ (see Figure 8.2), then $x \in \pi(\Lambda)$ if and only if $x \in U \cap \mathbf{Z}^n$. So, if we set ν the measure of repartition of the $x \in \mathbf{Z}^n$ modulo Λ, that is

$$\nu = \lim_{R \to +\infty} \frac{1}{\mathrm{Card}(B_R \cap \mathbf{Z}^n)} \sum_{x \in B_R \cap \mathbf{Z}^n} \delta_{\mathrm{pr}_{\mathbf{R}^n/\Lambda}(x)},$$

then we obtain the following formula (using also Equation (7.7), which links $\tau(A)$ and $D(\Lambda)$).

Proposition 8.10. *For every $A \in \mathrm{GL}_n(\mathbf{R})$ (we identify U with its projection of $\mathbf{R}^n/\Lambda$),*

$$\tau(A) = |\det(A)| D\big(\pi(\Lambda)\big) = |\det(A)| \nu\big(\mathrm{pr}_{\mathbf{R}^n/\Lambda}(U)\big).$$

In particular, when the matrix A is totally irrational, the measure ν is the uniform measure; thus if $\mathscr{D}$ is a fundamental domain of $\mathbf{R}^n/\Lambda$, then $\tau(A)$ is the area of $\mathscr{D} \cap U$. The same holds for the mean rate of injectivity and any matrix (not necessarily totally irrational).

Proposition 8.11. *For every $A \in \mathrm{GL}_n(\mathbf{R})$,*

$$\overline{\tau}(A) = |\det(A)| \mathrm{Leb}\Big(\mathrm{pr}_{\mathbf{R}^n/\Lambda}(U)\Big) = |\det(A)| \mathrm{Leb}(U \cap \mathscr{D}).$$

Thus, the mean rate of injectivity can be seen as the area of an intersection of parallelepipeds (see Figure 8.2).

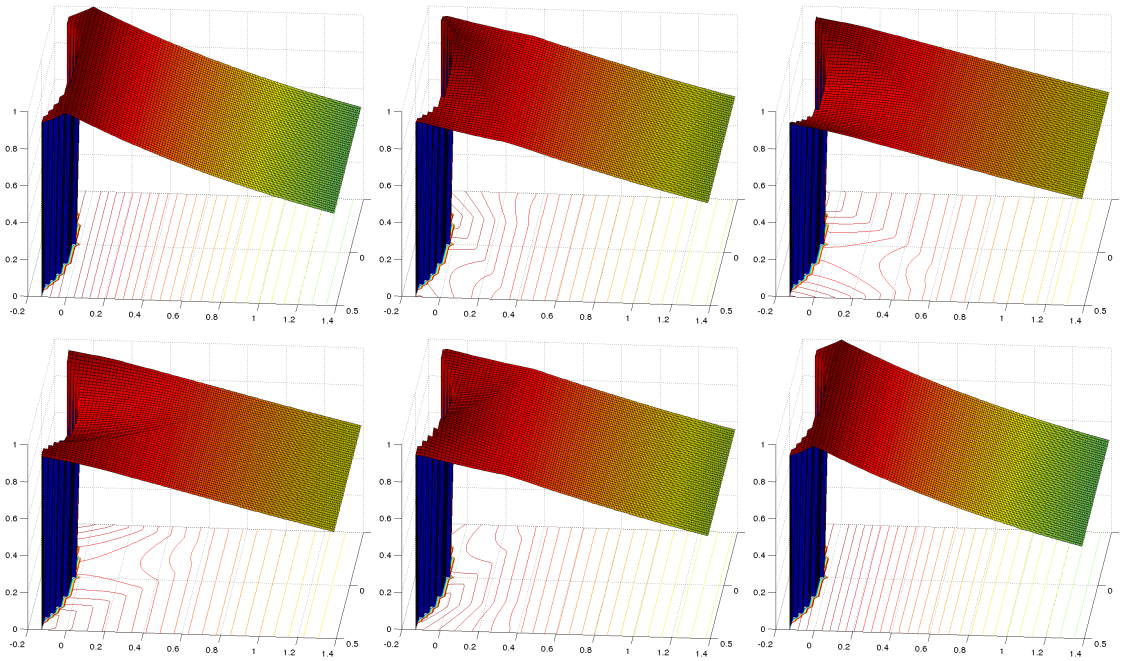

Figure 8.3: Mean rate of injectivity on $\mathrm{SL}_2(\mathbf{R})$ on the fundamental domain $\mathscr{D}$ (more precisely, $\mathscr{D} = \{z \in \mathbf{C} \mid |z| > 1, \operatorname{Im} z > 0, -1/2 < \operatorname{Re} z < 1/2\}$) of the modular surface, for various angles in $\mathrm{T}^1\mathscr{D}$ $(0, \pi/10, \pi/5, 3\pi/10, 2\pi/5, \pi/2)$.

With the same kind of arguments, we easily obtain a formula for $\rho_{\widehat{\mathrm{A}}(\mathbf{Z}^n)}(v)$ (the frequency of the difference v in $\widehat{\mathrm{A}}(\mathbf{Z}^n)$, see Definition 7.24).

Proposition 8.12. *If $\mathrm{A} \in \mathrm{GL}_n(\mathbf{R})$ is totally irrational, then for every $v \in \mathbf{Z}^n$,*

$$\rho_{\widehat{\mathrm{A}}(\mathbf{Z}^n)}(v) = \mathrm{Leb}\big(\mathrm{B}(v, 1/2) \cap \mathrm{U}\big).$$

Sketch of proof of Proposition 8.12. We want to know which proportion of points $x \in \Gamma = \widehat{\mathrm{A}}(\mathbf{Z}^n)$ are such that $x + v$ also belongs to Γ. But modulo $\Lambda = \mathrm{A}(\mathbf{Z}^n)$, x belongs to Γ if and only if $x \in \mathrm{B}(0, 1/2)$. Similarly, $x + v$ belongs to Γ if and only if $x \in \mathrm{B}(-v, 1/2)$. Thus, by equirepartition, $\rho_{\widehat{\mathrm{A}}(\mathbf{Z}^n)}(v)$ is equal to the area of $\mathrm{B}(v, 1/2) \cap \mathrm{U}$. □

From Proposition 8.11, we deduce the continuity of $\overline{\tau}$.

Corollary 8.13. *The mean rate of injectivity is a continuous function on $\mathrm{GL}_n(\mathbf{R})$, which is locally polynomial with degree smaller than n in the coefficients of the matrix.*

Moreover, this construction gives a quick algorithm to compute numerically the mean rate of injectivity of some matrices: this algorithm is much more efficient than the naive one consisting in computing the cardinality of the image of a large ball by the discretization, see Figure 8.3 and Figure 8.4.

It also allows to compute simply the mean rate of injectivity of some examples of matrices.

Application 8.14. For $\theta \in [0, \pi/2]$, the mean rate of injectivity of a rotation of $\mathbf{R}^2$ of angle θ is (see Figure 8.5).

$$\overline{\tau}(\mathrm{R}_\theta) = 1 - (\cos(\theta) + \sin(\theta) - 1)^2.$$

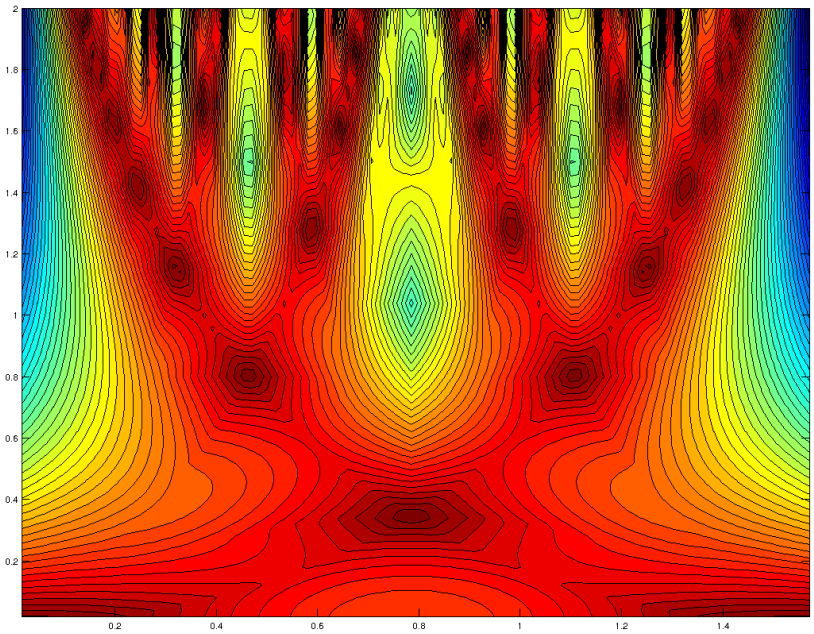

Figure 8.4: Action of the geodesic flow on the mean rate of injectivity: the red corresponds to a rate of injectivity close to 1 and the blue to a rate of injectivity close to 0. The figure represents the rate of injectivity on geodesics of the space $SL_2(\mathbf{R})/SL_2(\mathbf{Z})$, identified with the unitary tangent space T^1S of the modular surface S starting from the point i, depending on the time (on the vertical axis) and on the starting angle of the geodesic (on the horizontal axis). The mean rate of injectivity of each lattice has been computed in using Proposition 8.18

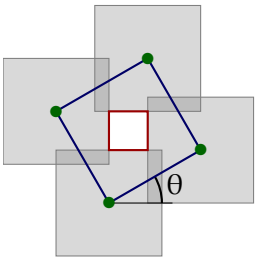

Figure 8.5: Computation of the mean rate of injectivity of a rotation of $\mathbf{R}^2$: it is equal to 1 minus the area of the interior of the red square.

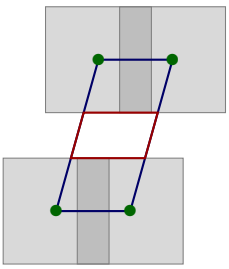

Figure 8.6: Computation of the mean rate of injectivity of a lattice having one vector parallel with the horizontal axis with length $\ell < 1$.

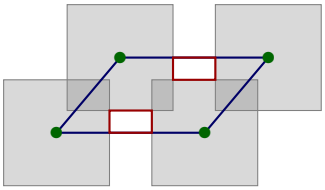

Figure 8.7: Computation of the mean rate of injectivity of a lattice having one vector parallel with the horizontal axis with length $\ell \in]1,2[$ for a parameter $x > \ell - 1$.

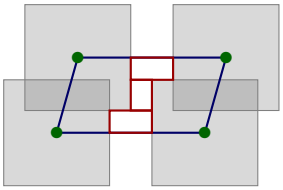

Figure 8.8: Computation of the mean rate of injectivity of a lattice having one vector parallel with the horizontal axis with length $\ell \in]1,2[$ for a parameter $x < \ell - 1$.

Application 8.15. We consider a lattice of $\mathbf{R}^2$ of covolume 1 with one basis vector parallel with the horizontal axis. We have several cases:

1. If the length ℓ of the vector parallel with the horizontal axis is smaller than 1, then the mean rate of injectivity is equal to ℓ, independently from the choice of the second basis vector (see Figure 8.6).
2. If the length ℓ is bigger than 1 but smaller than 2, we choose $v = (x, 1/\ell)$ another basis vector of the lattice. We can suppose that $-\ell/2 \leq x \leq \ell/2$; by symmetry we only treat the case $x \in [0, \ell/2]$. The mean rate of injectivity is then a piecewise affine map with respect to x:
   - if $\ell - 1 \leq x \leq \ell/2$, then the mean rate of injectivity is constant and equal to $1 - (\ell - 1)(2/\ell - 1) = \ell - 2 + 2/\ell$ (see Figure 8.7);
   - if $0 \leq x \leq \ell - 1$, then the mean rate of injectivity is equal to $1 - (\ell - 1)(2/\ell - 1) - (1 - 1/\ell)(\ell - 1 - x) = 1/\ell + x - x/\ell$ (see Figure 8.8).

We do not treat the other cases where $\ell > 2$.

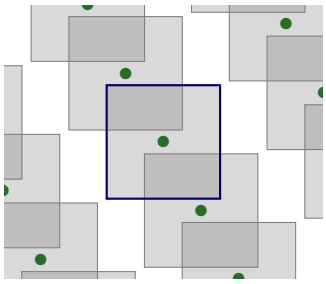

Figure 8.9: Second geometric construction: the green points are the elements of Λ, the blue square is a fundamental domain of $\mathbf{Z}^n$ and the grey squares are centred on the points of Λ and have radii 1/2. The rate of injectivity is equal to the integral on the blue square of the inverse of the number of different squares the point belongs to.

Second construction Here, we take another viewpoint to compute the rate of injectivity of A. To begin with, we remark that for every map f defined on a finite set E, the cardinality of $f(\mathrm{E})$ can be computed by the formula

$$\mathrm{Card}\big(f(\mathrm{E})\big) = \sum_{x \in \mathrm{E}} \frac{1}{\mathrm{Card}\big(f^{-1}(\{f(x)\})\big)};$$

in particular,

$$\tau(A) = \lim_{R\to+\infty} \frac{\mathrm{Card}\left(\widehat{A}([B_R])\right)}{\mathrm{Card}[B_R]} = \lim_{R\to+\infty} \frac{1}{\mathrm{Card}[B_R]} \sum_{x\in[B_R]} \frac{1}{\mathrm{Card}\left(\widehat{A}^{-1}(\{\hat{A}(x)\})\right)}. \tag{8.1}$$

Thus, we are reduced to compute $\mathrm{Card}\left(\widehat{A}^{-1}(\{\hat{A}(x)\})\right)$ for $x \in \mathbf{Z}^n$. This cardinality only depends on the class of Ax modulo $\mathbf{Z}^n$, thus we can suppose that $Ax \in B_{1/2}$ (in other words, $\widehat{A}x = 0$).

Lemma 8.16.

$$\mathrm{Card}\left(\widehat{A}^{-1}(\{\hat{A}(x)\})\right) = \psi(Ax),$$

where ψ is the function (different from the function $\mathbf{1}_U$ used previously; recall that $\Lambda = A\mathbf{Z}^n$)

$$\psi = \sum_{\lambda\in\Lambda} \mathbf{1}_{B(\lambda,1/2)}$$

which is the sum of the indicator functions of the balls of radius $1/2$ centred on the points of Λ (see Figure 8.9).

Proof of Lemma 8.16. We denote by $\tilde{x}$ the projection of Ax on the fundamental domain $B_{1/2}$ of $\mathbf{Z}^n$. We want to prove that there exists k different vectors $y \in \mathbf{Z}^n$ such that $\widehat{A}(x) = \widehat{A}(y)$ if and only if $\psi(\tilde{x}) = k$. Indeed, if $\lambda \in \Lambda$ is such that $\tilde{x} \in B(\lambda, 1/2)$, then $\tilde{x} - \lambda \in B_{1/2}$ and thus $Ax + \lambda$ projects on the same point of $\mathbf{Z}^n$ as Ax; the number of such vectors λ determines the number of points $y \in \mathbf{Z}^n$ such that $\widehat{A}x = \widehat{A}y$. □

If we set ν' the measure of repartition of the $\lambda \in \Lambda$ modulo $\mathbf{Z}^n$, that is

$$\nu' = \lim_{R\to+\infty} \frac{1}{\mathrm{Card}(B_R \cap \Lambda)} \sum_{\lambda\in\Lambda\cap B_R} \delta_{\mathrm{pr}_{\mathbf{R}^n/\mathbf{Z}^n}(\lambda)},$$

then, by combining Equation (8.1) with Lemma 8.16, we obtain

Proposition 8.17.

$$\tau(A) = \int_{B_{1/2}} \frac{1}{\psi(\lambda)}\, \mathrm{d}\,\nu'(\lambda).$$

And we have a similar statement for the mean rate of injectivity.

Proposition 8.18.

$$\overline{\tau}(A) = \int_{B_{1/2}} \frac{1}{\psi(\lambda)}\, \mathrm{d}\,\mathrm{Leb}(\lambda).$$

These properties can be also directly deduced from the first geometric construction by applying a double counting argument. We state it in more general context because we will need a more precise statement in the sequel (here, we only need the case $m = 0$, thus $B_1 = B$); this lemma allows to compute the area of the projection of a set on a quotient by a lattice.

Lemma 8.19. *Let Λ_1 be a subgroup of $\mathbf{R}^m$ which is a lattice in the vector space it spans, Λ_2 be such that $\Lambda_1 \oplus \Lambda_2$ is a lattice of covolume 1 of $\mathbf{R}^m$, and B be a compact subset of $\mathbf{R}^m$. Let*

B_1 *be the projection of* B *on the quotient* $\mathbf{R}^m/\Lambda_1$, *and* B_2 *the projection of* B *on the quotient* $\mathbf{R}^m/(\Lambda_1 \oplus \Lambda_2)$. *We denote by*

$$a_i = \mathrm{Leb}\Big\{x \in B_1 \mid \mathrm{Card}\{\lambda_2 \in \Lambda_2 \mid x \in B_1 + \lambda_2\} = i\Big\}.$$

Then,

$$\mathrm{Leb}(B_2) = \sum_{i \geq 1} \frac{a_i}{i}.$$

In particular, the area of B_2 (the projection on the quotient by the direct sum of both lattices) is smaller than (or equal to) that of B_1 (the projection on the quotient by the first lattice). The loss of area is given by the following corollary.

Corollary 8.20. *With the same notations as for Lemma 8.19, if we denote by*

$$D_1 = \mathrm{Leb}\Big\{x \in B_1 \mid \mathrm{Card}\{\lambda_2 \in \Lambda_2 \mid x \in B_1 + \lambda_2\} \geq 2\Big\},$$

then,

$$\mathrm{Leb}(B_2) \leq \mathrm{Leb}(B_1) - \frac{D_1}{2}.$$

Proof of Lemma 8.19. Consider the function

$$\begin{array}{rccl} \phi: & \mathbf{R}^n/\Lambda_1 & \longrightarrow & \mathbf{N}^* \\ & x & \longmapsto & \begin{cases} \Big(\mathrm{Card}\{\lambda_2 \in \Lambda_2 \mid x \in B_1 + \lambda_2\}\Big)^{-1} & \text{if } x \in B_1 \\ 0 & \text{if } x \notin B_1. \end{cases} \end{array}$$

Obviously, it satisfies $\int_{B_1} \phi = \sum_{i \geq 1} \frac{a_i}{i}$. Moreover, if we set

$$\begin{array}{rccl} \Phi: & \mathbf{R}^n/(\Lambda_1 \oplus \Lambda_2) & \longrightarrow & \mathbf{N}^* \\ & x & \longmapsto & \sum_{\lambda_2 \in \Lambda_2} \phi(x - \lambda_2), \end{array}$$

then we easily see that on the one hand, $\Phi = \mathbf{1}_{\Lambda_2 + B_1}$, thus $\int \Phi = \mathrm{Leb}(B_2)$, and on the other hand, $\int \Phi = \mathrm{covol}(\Lambda_2) \int_{B_2} \phi = \sum_{i \geq 1} \frac{a_i}{i}$. □

8.3 When is the rate close to 1?

Obviously, the rate of injectivity $\tau(A)$ of any matrix of $A \in \mathrm{SL}_n(\mathbf{Z})$ is equal to 1. In Section 8.1, we have found other examples of affine maps with determinant 1 whose rate of injectivity is also 1. Also, in Application 8.15, taking $\ell = 1$, this gives another class of examples where the rate of injectivity is one. In this section, we investigate more in detail this question: what are the matrices with determinant 1 whose mean rate of injectivity is 1? With Proposition 8.11, we can reformulate this question in terms of intersection of cubes: if $\det(A) = 1$, $\overline{\tau}(A) = 1$ if and only if the cubes $B(\lambda, 1/2)$, with $\lambda \in A\mathbf{Z}^n$, tile the space $\mathbf{R}^n$. This is a classical problem raised by H. Minkowski in 1896 (see [Min10]), and answered by G. Hajós in 1941.

Theorem 8.21 (Hajós, [Haj41])**.** *Let* Λ *be a lattice of* $\mathbf{R}^n$. *Then the collection of squares* $\{B(\lambda, 1/2)\}_{\lambda \in \Lambda}$ *tiles the plane if and only if in a canonical basis of* $\mathbf{R}^n$ *(that is, permuting coordinates if necessary),* Λ *admits a generating matrix which is upper triangular with ones on the diagonal.*

This theorem can be stated geometrically: if the collection of squares tiles the space $\mathbf{R}^n$, then at least two of these squares have a face in common. The conclusion of Theorem 8.21 can then be inferred by induction, quotienting in the direction which is orthogonal to this face and iterating the argument. The proof of this theorem involves fine results of group theory; we will not prove it here in the general case (see for example the excellent book [SS94] for a complete investigation on the subject). Remark that Hajós theorem studies a particular case of the case of equality in Minkowski's theorem (Theorem 7.28): it states a necessary and sufficient condition for a lattice with covolume 1 to possess a non-trivial point in the boundary of the unit square centred at 0, but no non-trivial point in its interior. We give an elementary proof of Hajós theorem in dimension 2 (for dimension 3, there is also an elementary proof, see for example the book of H. Minkowski [Min07]).

Proof of Theorem 8.21 in dimension 2. Let Λ be a lattice of $\mathbf{R}^2$ such that the family $\{\mathrm{B}(\lambda, 1/2)\}_{\lambda\in\Lambda}$ tiles the plane. We consider the point $x_1 = (1/2, 0)$. By the hypothesis of tiling, there exists $\lambda_1 \in \Lambda \setminus \{0\}$ such that $x_1 \in \overline{\mathrm{B}(\lambda_1, 1/2)}$. Remark that this implies that λ_1 has the form $\lambda_1 = (1, y)$. We have two cases.

1. If $\lambda_1 = (1, 0)$, then we consider the point $x_2 = (0, 1/2)$ and again, there exists $\lambda_2 \in \Lambda\setminus\{0\}$ such that $x \in \overline{\mathrm{B}(\lambda_2, 1/2)}$. The vector λ_2 has the form $\lambda_2 = (*, 1)$, thus the basis (λ_1, λ_2) of Λ has the desired form.
2. If $\lambda_1 = (1, y) \neq (1, 0)$, we suppose that $y > 0$ (doing a symmetry if necessary). Considering the point $x_2 = (1/2, 1/2)$, this implies that $\lambda_2 = (0, 1)$ belongs to Λ (in other words, $x_2 \in \mathrm{B}(\lambda_2, 1/2)$). The basis (λ_1, λ_2) of Λ has the desired form.

□

Combined with the results of the previous section (more precisely, Proposition 8.18), Hajós theorem (Theorem 8.21) leads to the following corollary.

Corollary 8.22. *A matrix $\mathrm{A} \in \mathrm{SL}_n(\mathbf{R})$ has a mean rate of injectivity equal to 1 if and only if there exists $\mathrm{B} \in \mathrm{SL}_n(\mathbf{Z})$ such that in a canonical basis of $\mathbf{R}^n$ (that is, permuting coordinates if necessary), the matrix AB is upper triangular with ones on the diagonal (equivalently, if and only if there exists a permutation matrix P and a matrix $\mathrm{B} \in \mathrm{SL}_n(\mathbf{Z})$ such that PAB is upper triangular with ones on the diagonal). Remark that the set of such matrices is a locally finite union of manifolds of positive codimension.*

8.4 Rate of injectivity of a sequence of matrices

The aim of the end of this chapter is to study the asymptotic rate of injectivity of sequences of matrices with determinant 1. More precisely, we want to estimate the rate of injectivity $\tau^\infty((\mathrm{A}_k)_k)$ (that is, the limit of the densities of the sets $(\widehat{\mathrm{A}_k} \circ \cdots \circ \widehat{\mathrm{A}_1})(\mathbf{Z}^n)$ when k goes to infinity, see Definition 7.19) when $(\mathrm{A}_k)_{k\geq 1}$ is a *generic* sequence of matrices of $\mathrm{SL}_n(\mathbf{R})$ in the following sense.

Definition 8.23. We fix once for all a norm $\|\cdot\|$ on $\mathrm{M}_n(\mathbf{R})$. For a bounded sequence $(\mathrm{A}_k)_{k\geq 1}$ of matrices of $\mathrm{SL}_n(\mathbf{R})$, we set

$$\|(\mathrm{A}_k)_k\| = \sup_{k\geq 1} \|\mathrm{A}_k\|.$$

In other words, we consider the space $\ell^\infty(\mathrm{SL}_n(\mathbf{R}))$ of uniformly bounded sequences of matrices of determinant 1.

This metric is complete, thus there is a good notion of genericity on the set of bounded sequences of matrices of determinant 1 (see page 18). We give the main theorem of this chapter, which states the behaviour of the asymptotic rate of injectivity of a sequence of matrices (stated in the introduction as Theorem P).

Theorem 8.24. *For a generic* [4] *sequence of matrices* $(A_k)_{k\geq 1}$ *of* $SL_n(\mathbf{R})$*, we have* [5]

$$\tau^{\infty}\Big((A_k)_{k\geq 1}\Big) = 0.$$

This statement was motivated by some numerical simulations (see Figure 7.3 page 153). In Part 3, we will deduce from this theorem a similar statement for generic diffeomorphisms of the torus (Theorem 11.15).

Remark 8.25. This statement remain true if we replace "for a generic sequence of matrices $(A_k)_{k\geq 1}$ of $SL_n(\mathbf{R})$" by "for a generic sequence of matrices $(A_k)_{k\geq 1}$ among matrices of determinant ± 1": the property we will need in the proofs is that the matrices preserve the volume of $\mathbf{R}^n$.

Remark 8.26. Remark 8.5 implies that for a generic sequence $(A_i)_{i\geq 1}$ of matrices of $SL_n(\mathbf{R})$ and for every $k \in \mathbf{N}^*$, the map $(B_i)_{i\geq 1} \mapsto \tau^k(B_1, \cdots, B_k)$ is continuous in $A_1, \cdots, A_k$. Thus, as the asymptotic rate of injectivity is the infimum of the rates of injectivity in times k for $k \in \mathbf{N}^*$, it is upper semi continuous on every generic sequence of matrices. In particular, for every generic sequence $(A_i)_{i\geq 1}$ of matrices of $SL_n(\mathbf{R})$, and every $\varepsilon > 0$ there exists a neighbourhood of this sequence on which the asymptotic rate of injectivity is smaller than ε.

We begin the study of Theorem 8.24 by handling the easiest case: the result is true when we restrict to diagonal matrices (in the canonical basis). More precisely, the set of diagonal matrices is a Baire space, so we can talk about generic diagonal matrices, and we have the following property.

Proposition 8.27. *Let* $(A_k)_{k\geq 1}$ *be a generic sequence of diagonal matrices of* $SL_n(\mathbf{R})$*. Then* $\tau^{\infty}((A_k)_k) = 0$.

Proof of Proposition 8.27. We denote $A_k = \mathrm{Diag}(\lambda_{k,1}, \cdots, \lambda_{k,n})$. Since $A_k \in SL_n(\mathbf{R})$ for every k, there exists $i_0 \in [\![1, n]\!]$ and an infinite number of integers ℓ such that $\prod_{k=0}^{\ell} \lambda_{k,i_0} \leq 1$. We then use the following lemma.

Lemma 8.28. *Let* $\delta_0 > 0$*, and* $(\lambda_1, \cdots, \lambda_\ell)$ *be a collection of real numbers such that* $\lambda_i > \delta_0$ *for every* i *and* $\prod_{k=1}^{\ell} \lambda_k \leq 1$*. Then, for every* $\delta \in]0, \delta_0[$*, we have*

$$\prod_{k=1}^{\ell} (\lambda_k - \delta) \leq \frac{1}{1 + \delta(\ell + 1)}.$$

4. Generic for the topology ℓ^{∞} on $(SL_n(\mathbf{R}))^{\mathbf{N}}$, see Definition 8.23.
5. For a definition of τ^{∞}, see Definition 7.19.

Proof of Lemma 8.28. We decompose the difference $\prod_{k=1}^{\ell}\lambda_k - \prod_{k=1}^{\ell}(\lambda_k-\delta)$ to have a telescopic sum:

$$\begin{aligned}\prod_{k=1}^{\ell}\lambda_k - \prod_{k=1}^{\ell}(\lambda_k-\delta) &= \sum_{i=1}^{\ell}\left(\prod_{k=1}^{i}\lambda_k \prod_{k=i+1}^{\ell}(\lambda_k-\delta) - \prod_{k=1}^{i-1}\lambda_k \prod_{k=i}^{\ell}(\lambda_k-\delta)\right)\\ &= \sum_{i=1}^{\ell}\left(\prod_{k=1}^{i-1}\lambda_k \prod_{k=i+1}^{\ell}(\lambda_k-\delta)\right)\Big(\lambda_i - (\lambda_i-\delta)\Big)\\ &= \sum_{i=1}^{\ell}\left(\prod_{k=1}^{i}\lambda_k \prod_{k=i+1}^{\ell}(\lambda_k-\delta)\right)\frac{\delta}{\lambda_i}\\ &\geq \delta\left(\prod_{k=1}^{\ell}(\lambda_k-\delta)\right)\sum_{i=1}^{\ell}\frac{1}{\lambda_i}.\end{aligned}$$

As a consequence, using the inequality of arithmetic and geometric means,

$$\prod_{k=1}^{\ell}\lambda_k - \prod_{k=1}^{\ell}(\lambda_k-\delta) \geq (\ell+1)\delta\left(\prod_{k=1}^{\ell}(\lambda_k-\delta)\right)\left(\prod_{i=1}^{\ell}\frac{1}{\lambda_i}\right)^{1/\ell},$$

and as $\prod_{i=1}^{\ell}\lambda_i \leq 1$, we get

$$\prod_{k=1}^{\ell}\lambda_k - \prod_{k=1}^{\ell}(\lambda_k-\delta) \geq (\ell+1)\delta\left(\prod_{k=1}^{\ell}(\lambda_k-\delta)\right).$$

This implies that

$$\prod_{k=1}^{\ell}(\lambda_k-\delta) \leq \frac{\prod_{k=1}^{\ell}\lambda_k}{(\ell+1)\delta+1} \leq \frac{1}{1+\delta(\ell+1)}.$$

□

Proposition 8.27 easily follows from Lemma 8.28 and the following property, which is specific to the case where the matrices are diagonal: the rate of injectivity of $A_1,\cdots,A_k$ is smaller than the smallest eigenvalue of $A_1\circ\cdots\circ A_k$. Indeed, the eigenspace associated to this eigenvalue λ is stable by discretization, and on this eigenspace the image of a big ball of radius R is included in a ball of radius $\lambda R + o(R)$. □

8.5 Frequency of differences and a first result: the asymptotic rate of injectivity is generically smaller than 1/2

In this section, we prove a weak version of Theorem 8.24, which states that the rate of injectivity of a generic sequence of matrices is smaller than one half. It is based on the following lemma, which asserts that if a difference appears rarely in Γ, then the rate of injectivity of Γ is small.

Lemma 8.29. *Let $\Gamma \subset \mathbf{Z}^n$ be an almost periodic pattern, and $\rho_0 \in]0,1[$. If $D(\Gamma) \geq \frac{1}{2-\rho_0}$, then for every $v \in \mathbf{Z}^n$, we have $\rho_\Gamma(v) \geq \rho_0$.*

Proof of Lemma 8.29. We argue by contraposition and suppose that there exists $v_0 \in \mathbf{Z}^n$ such that $\rho_\Gamma(v_0) \leq \rho_0$. We set $\Gamma_{v_0} = \{x \in \Gamma \mid x + v_0 \notin \Gamma\}$, thus we have $D(\Gamma_{v_0}) \geq (1-\rho_0)D(\Gamma)$. But $\Gamma \cap (\Gamma_{v_0} + v_0) = \emptyset$, so $D(\Gamma) + D(\Gamma_{v_0} + v_0) \leq 1$, which implies that $D(\Gamma)(1+1-\rho_0) \leq 1$. □

Theorem 8.30. *Let $(A_k)_{k\geq 1}$ be a generic sequence of matrices of $\mathrm{SL}_n(\mathbf{R})$. Then there exists a parameter $\lambda \in]0,1[$ such that for every $k \geq 1$, we have $\tau^k(A_1, \cdots, A_k) \leq (\lambda^k + 1)/2$. In particular, $\tau^\infty((A_k)_k) \leq 1/2$.*

The proof of this theorem is based on an argument of equirepartition, which translates the problem in terms of areas of intersections of cubes.

Lemma 8.31. *Let $\delta > 0$ and $M > 0$. Then there exists $V_0 > 0$ such that for all $A \in \mathrm{SL}_n(\mathbf{R})$ with $\|A\| \leq M$, there exists $B \in \mathrm{SL}_n(\mathbf{R})$ totally irrational, with $\|B - A\| \leq \delta$, there exist a polygon $P \subset \mathbf{T}^n$ whose volume is bigger than V_0, and some $v_0 \in \mathbf{Z}^n \setminus \{0\}$, such that if $Bx \in P$ mod $\mathbf{Z}^n$, then $\pi(Bx) = \pi(B(x + v_0))$.*

Proof of Lemma 8.31. Hajós theorem (Theorem 8.21, see also Corollary 8.22) proves that for $\varepsilon > 0$ small enough, the set of matrices $B \in \mathrm{SL}_n(\mathbf{R})$ such that for every $v \in \mathbf{Z}^n \setminus \{0\}$, $\|Bv\|_\infty > 1 - \varepsilon$ is a small neighbourhood of a locally finite union of manifolds. Thus, there exists $\varepsilon = \varepsilon(\delta, M) > 0$ such that for all $A \in \mathrm{SL}_n(\mathbf{R})$ with $\|A\| \leq M$, there exists $B \in \mathrm{SL}_n(\mathbf{R})$ and $v_0 \in \mathbf{Z}^n \setminus \{0\}$ such that $\|B - A\| \leq \delta$ and $\|Bv_0\|_\infty < 1 - \varepsilon$. As totally irrational matrices are dense among $\mathrm{SL}_n(\mathbf{R})$, we can moreover suppose that B is totally irrational. We now set $P = B(0, 1/2) \cap B(-v_0, 1/2) \subset \mathbf{R}^n/\mathbf{Z}^n$ (we identify $\mathbf{R}^n/\mathbf{Z}^n$ with its fundamental domain $B(0, 1/2)$). The volume of P is bigger than $V_0 = \varepsilon^n$. Reasoning as in the proof of Proposition 8.17, we get that if $Bx \in P \bmod \mathbf{Z}^n$, then $\pi(Bx) = \pi(B(x + v_0))$. □

Proof of Theorem 8.30. We prove the theorem by induction. Suppose that the theorem is proved for a rank $k \in \mathbf{N}$. We set

$$\Gamma_k = (\widehat{B_k} \circ \cdots \circ \widehat{B_1})(\mathbf{Z}^n) \qquad \text{and} \qquad \rho_0 = 2 - \frac{1}{D(\Gamma_k)}.$$

By Theorem 7.12, Γ_k is an almost periodic pattern.

Lemma 8.31 applied to $A = A_{k+1}$, δ and $M = \|(A_i)_i\| + \delta$ gives us a parameter $V_0 > 0$ (depending only on δ and M), a polygon $P \subset \mathbf{T}^n$ of volume greater than V_0, a matrix $B = B_{k+1}$ and a vector $v_0 \in \mathbf{Z}^n \setminus \{0\}$. Lemma 8.29 implies that $\rho_{\Gamma_k}(v_0) \geq \rho_0$. But Lemma 7.21 asserts that for a generic set of matrices B, the set $B(\Gamma_k)$ is equidistributed modulo $\mathbf{Z}^n$; perturbing a little B if necessary, we suppose that it is true. These facts imply that

$$D\{x \in \Gamma_k \mid Bx \in P \bmod \mathbf{Z}^n, x + v_0 \in \Gamma_k\} \geq V_0 \rho_0 D(\Gamma_k),$$

and thus, as $Bx \in P \bmod \mathbf{Z}^n$ implies $\widehat{B}(x) = \widehat{B}(x + v_0)$, we get

$$D(\widehat{B_{k+1}}(\Gamma_k)) \leq D(\Gamma_k) - V_0 \rho_0 D(\Gamma_k) = D(\Gamma_k)(1 - V_0 \rho_0).$$

But $\rho_0 = 2 - \frac{1}{D(\Gamma)}$, thus (we can suppose that $V_0 \leq 1/2$)

$$D(\Gamma_{k+1}) = D(\widehat{B_{k+1}}(\Gamma_k)) \leq D(\Gamma_k)\left(1 - V_0\left(2 - \frac{1}{D(\Gamma_k)}\right)\right) = V_0 + D(\Gamma_k)(1 - 2V_0).$$

Setting $\lambda = 1 - 2V_0$, we obtain the general term of an arithmetico-geometric sequence, this proves the theorem for the rank $k+1$. □

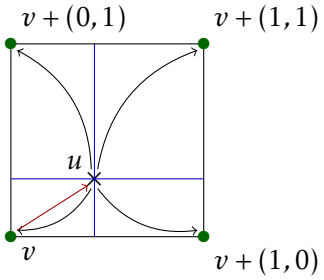

Figure 8.10: The function φ_u in dimension 2: its value on one vertex of the square is equal to the area of the opposite rectangle; in particular, $\varphi_u(v)$ is the area of the rectangle with the vertices u and $v+(1,1)$.

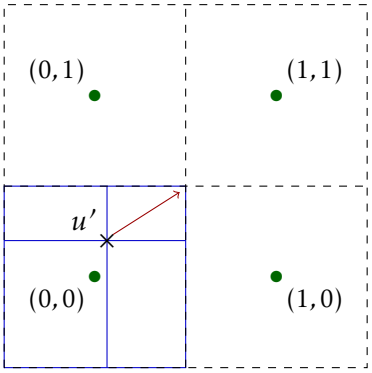

Figure 8.11: The red vector is equal to that of Figure 8.10 for $u = Av$. If Ax belongs to the bottom left rectangle, then $\pi(Ax + Av) = y \in \mathbf{Z}^2$; if Ax belongs to the top left rectangle, then $\pi(Ax + Av) = y + (0,1)$ etc.

8.6 Diffusion process and the case of isometries

Unfortunately, problems arise when we try to perturb a sequence of matrices to make its asymptotic rate smaller than 1/2. First of all, if the density of an almost periodic pattern Γ is smaller than 1/2, then the set of differences may not be the full set $\mathbf{Z}^n$. Even worse, it might happen that this set of differences has big holes, as shown by the following example. We take the almost periodic pattern

$$\Gamma = \bigcup_{i=0}^{39} 100\mathbf{Z} + i.$$

For every $x \in \Gamma$ and $v \in [\![40, 60]\!]$, $x + v \notin \Gamma$. In other words, for every $v \in [\![40, 60]\!]$, we have $\rho_\Gamma(v) = 0$, whereas $D(\Gamma) = 0.4$. However, the things are not too bad for the frequencies of differences when we are close to 0, as shown by the Minkowski-like theorem for almost periodic patterns (Theorem 7.29).

Diffusion process In this paragraph, we study the action of a discretization of a matrix on the set of differences of an almost periodic pattern Γ; more precisely, we study the link between the functions ρ_Γ and $\rho_{\widehat{A(\Gamma)}}$.

For $u \in \mathbf{R}^n$, we define the function φ_u, which is a "weighted projection" of u on $\mathbf{Z}^n$.

Definition 8.32. Given $u \in \mathbf{R}^n$, the function $\varphi_u = \mathbf{Z}^n \to [0,1]$ is defined by

$$\varphi_u(v) = \begin{cases} 0 & \text{if } d_\infty(u,v) \geq 1 \\ \prod_{i=1}^n (1 - |u_i + v_i|) & \text{if } d_\infty(u,v) < 1. \end{cases}$$

In particular, the function φ_u satisfies $\sum_{v \in \mathbf{Z}^n} \varphi_u(v) = 1$, and is supported by the vertices of the integral unit cube [6] that contains [7] u. Figure 8.10 gives a geometric interpretation of this function φ_u.

6. By definition, an integral cube has its faces parallel to the canonical hyperplanes.

7. More precisely, the support of φ_u is the smallest integral unit cube of dimension $n' \leq n$ which contains u.

The following property asserts that $\widehat{\mathrm{A}}$ acts "smoothly" on the frequency of differences. In particular, when $\mathrm{D}(\Gamma) = \mathrm{D}(\widehat{\mathrm{A}}\Gamma)$, the function $\rho_{\widehat{\mathrm{A}}\Gamma}$ is obtained from the function ρ_Γ by applying a linear operator $\mathscr{A}$, acting on each Dirac function δ_v in the following way: consider the Dirac function $\delta_{\mathrm{A}v}$ and spread it on the vertices of the integer cube in which $\mathrm{A}v$ belongs to. More precisely, $\mathscr{A}\delta_v$ is supported by these vertices and its integral is equal to 1, the value assigned to each vertex V is proportional to absolute value of the product of the coordinates of $\mathrm{V} - \mathrm{A}v$. Roughly speaking, to compute $\mathscr{A}\delta_v$, we take $\delta_{\mathrm{A}v}$ and apply a diffusion process. In the other case where $\mathrm{D}(\widehat{\mathrm{A}}\Gamma) < \mathrm{D}(\Gamma)$, we only have inequalities involving the operator $\mathscr{A}$ to compute the function $\rho_{\widehat{\mathrm{A}}\Gamma}$.

Proposition 8.33. *Let $\Gamma \subset \mathbf{Z}^n$ be an almost periodic pattern and $\mathrm{A} \in \mathrm{SL}_n(\mathbf{R})$ be a generic matrix.*

(i) If $\mathrm{D}(\widehat{\mathrm{A}}(\Gamma)) = \mathrm{D}(\Gamma)$, then for every $u \in \mathbf{Z}^n$,

$$\rho_{\widehat{\mathrm{A}}(\Gamma)}(u) = \sum_{v \in \mathbf{Z}^n} \varphi_{\mathrm{A}(v)}(u)\rho_\Gamma(v).$$

(ii) In the general case, for every $u \in \mathbf{Z}^n$, we have

$$\frac{\mathrm{D}(\Gamma)}{\mathrm{D}(\widehat{\mathrm{A}}(\Gamma))} \sup_{v \in \mathbf{Z}^n} \varphi_{\mathrm{A}(v)}(u)\rho_\Gamma(v) \le \rho_{\widehat{\mathrm{A}}(\Gamma)}(u) \le \frac{\mathrm{D}(\Gamma)}{\mathrm{D}(\widehat{\mathrm{A}}(\Gamma))} \sum_{v \in \mathbf{Z}^n} \varphi_{\mathrm{A}(v)}(u)\rho_\Gamma(v).$$

Remark 8.34. This proposition expresses that the action of the discretization of a linear map A on the differences is more or less that of a multivalued map.

Proof of Proposition 8.33. We begin by proving the first point of the proposition. We suppose that $\mathrm{A} \in \mathrm{SL}_n(\mathbf{R})$ is generic and that $\mathrm{D}(\widehat{\mathrm{A}}(\Gamma)) = \mathrm{D}(\Gamma)$. Let $x \in \Gamma \cap (\Gamma - v)$. We consider the projection x' of $\mathrm{A}x$, and the projection u' of $-u = -\mathrm{A}v$, on the fundamental domain $]-1/2, 1/2]^n$ of $\mathbf{R}^n/\mathbf{Z}^n$. If x' belongs to the parallelepiped whose vertices are $(-1/2, \cdots, -1/2)$ and u' (see Figure 8.11), then we obtain $\pi(\mathrm{A}(x+v)) = \pi(\mathrm{A}x + \lfloor \mathrm{A}v \rfloor) = \pi(\mathrm{A}x) + \lfloor \mathrm{A}v \rfloor$. The same kind of results holds for the other parallelepipeds whose vertices are u' and one vertex of $[-1/2, 1/2[^n$.

We set $\Gamma = \widehat{\mathrm{A}}(\mathbf{Z}^n)$. The genericity of A ensures that for every $v \in \mathbf{Z}^n$, the set $\Gamma \cap (\Gamma - v)$, which has density $\mathrm{D}(\Gamma)\rho_\Gamma(v)$ (by definition of ρ_Γ), is equidistributed modulo $\mathbf{Z}^n$ (by Lemma 7.21). Thus, the points x' are equidistributed modulo $\mathbf{Z}^n$. In particular, the difference v will spread into the differences which are the support of the function $\varphi_{\mathrm{A}v}$, and each of them will occur with a frequency given by $\varphi_{\mathrm{A}v}(x)\rho_\Gamma(v)$. The hypothesis about the fact that the density of the sets does not decrease imply that the contributions of each difference of Γ to the differences of $\widehat{\mathrm{A}}(\Gamma)$ add.

In the general case, the contributions to each difference of Γ may overlap. However, applying the argument of the previous case, we can easily prove the second part of the proposition. □

Remark 8.35. We also remark that:

(i) the density strictly decreases (that is, $\mathrm{D}(\widehat{\mathrm{A}}(\Gamma)) < \mathrm{D}(\Gamma)$) if and only if there exists $v_0 \in \mathbf{Z}^n$ such that $\rho_\Gamma(v_0) > 0$ and $\|\mathrm{A}v_0\|_\infty < 1$;

(ii) if there exists $v_0 \in \mathbf{Z}^n$ such that

$$\sum_{v \in \mathbf{Z}^n} \varphi_{\mathrm{A}(v)}(v_0)\rho_\Gamma(v_0) > 1,$$

then the density strictly decreases by at least $\sum_{v \in \mathbf{Z}^n} \varphi_{\mathrm{A}(v)}(v_0)\rho_\Gamma(v_0) - 1$;

(iii) we can compute which differences will go to the difference 0, that is, the differences $u \in \mathbf{Z}^n$ such that there exists $x \in \Gamma \cap (\Gamma - u)$ such that $\widehat{A}(x) = \widehat{A}(x+u)$ (that will make the rate of injectivity decrease). The set of such differences u is $A^{-1}(B(0,1)) \cap \mathbf{Z}^n$ (recall that $B(0,1)$ is an infinite ball). By Hajós theorem (Theorem 8.21, see also Corollary 8.22), this set contains a non zero vector when A is generic. More generally, the set of differences $u \in \Gamma \cap (\Gamma - u)$ such that $\widehat{A}(x) + v = \widehat{A}(x+u)$ is $A^{-1}(B(v,1)) \cap \mathbf{Z}^n$. Iterating this process, it is possible to compute which differences will go to 0 in time t.

Rate of injectivity of a generic sequence of isometries Proposition 8.33 allows to give an alternative proof of Theorem 8.24 for isometries.

Theorem 8.36. *Let $(P_k)_{k\geq 1}$ be a generic sequence of matrices of $O_n(\mathbf{R})$. Then $\tau^\infty((P_k)_k) = 0$.*

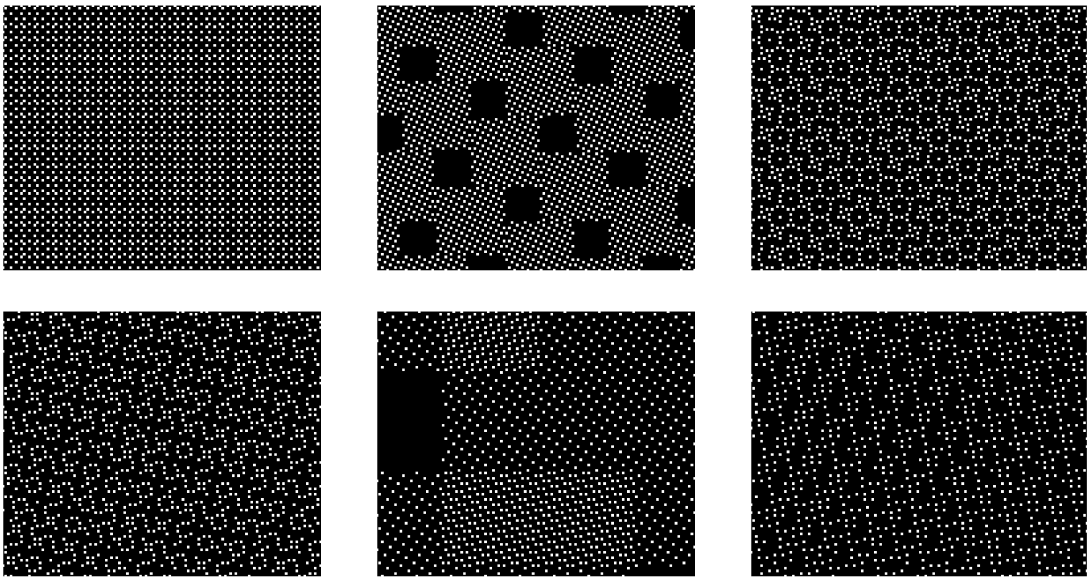

Figure 8.12: Images of $\mathbf{Z}^2$ by discretizations of rotations, a point is black if it belongs to the image of $\mathbf{Z}^2$ by the discretization of the rotation. From left to right and top to bottom, angles $\pi/4$, $\pi/5$, $\pi/6$, $\pi/7$, $\pi/8$ and $\pi/9$.

In the previous section, the starting point of the proof was Lemma 8.29, which ensures that when the rate of injectivity is bigger than 1/2, the frequency of *any* difference is bigger than a constant depending on the rate. Here, the starting point is the Minkowski theorem for almost periodic patterns (Theorem 7.29), which gives *one* nonzero difference whose frequency is positive. The rest of the proof of Theorem 8.36 consists in using again an argument of equidistribution. More precisely, we apply successively the following lemma, which asserts that given an almost periodic pattern Γ of density D_0, a sequence of isometries and $\delta > 0$, then, perturbing each isometry of at most δ if necessary, we can make the density of the k_0-th image of Γ smaller than $\lambda_0 D_0$, with k_0 and λ_0 depending only on D_0 and δ. The proof of this lemma involves the study of the action of the discretizations on differences made in Proposition 8.33

Lemma 8.37. *Let $(P_k)_{k\geq 1}$ be a sequence of matrices of $O_n(\mathbf{R})$ and $\Gamma \subset \mathbf{Z}^n$ an almost periodic pattern. Given $\delta > 0$ and $D > 0$ such that $D(\Gamma) \geq D$, there exists $k_0 = k_0(D)$ (decreasing in*

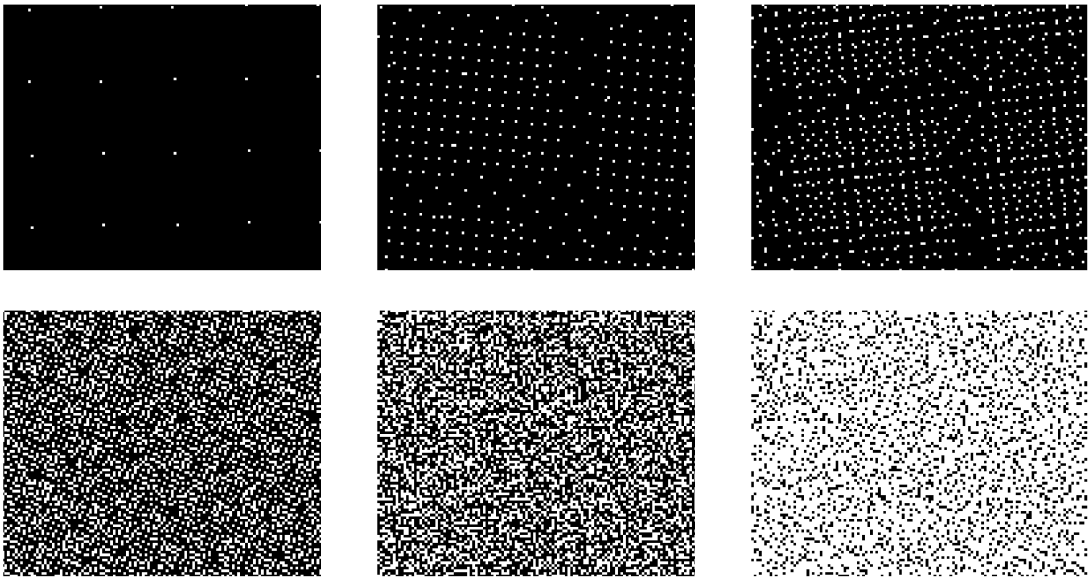

Figure 8.13: Successive images of $\mathbf{Z}^2$ by discretizations of random rotations, a point is black if it belongs to $(\widehat{\mathrm{R}_{\theta_k}} \circ \cdots \circ \widehat{\mathrm{R}_{\theta_1}})(\mathbf{Z}^2)$, where the θ_i are chosen uniformly randomly in $[0, 2\pi]$. From left to right and top to bottom, $k = 1, 2, 3, 5, 10, 50$.

D*)*, $\lambda_0 = \lambda_0(\mathrm{D}, \delta) < 1$ *(decreasing in* D *and in* δ*), and a sequence* $(\mathrm{Q}_k)_{k\geq 1}$ *of totally irrational matrices of* $\mathrm{O}_n(\mathbf{R})$*, such that* $\|\mathrm{P}_k - \mathrm{Q}_k\| \leq \delta$ *for every* $k \geq 1$ *and*

$$\mathrm{D}\Big((\widehat{\mathrm{Q}_{k_0}} \circ \cdots \circ \widehat{\mathrm{Q}_1})(\Gamma)\Big) < \lambda_0 \mathrm{D}.$$

We begin by proving that this lemma implies Theorem 8.36.

Proof of Theorem 8.36. Suppose that Lemma 8.37 is true. Let $\tau_0 \in]0,1[$ and $\delta > 0$. We want to prove that we can perturb the sequence $(\mathrm{P}_k)_k$ into a sequence $(\mathrm{Q}_k)_k$ of isometries, which is δ-close to $(\mathrm{P}_k)_k$ and is such that its asymptotic rate is smaller than τ_0 (and that this remains true on a whole neighbourhood of these matrices).

Thus, we can suppose that $\tau^\infty((\mathrm{P}_k)) > \tau_0$. We apply Lemma 8.37 to obtain the parameters $k_0 = k_0(\tau_0/2)$ (because $k_0(\mathrm{D})$ is decreasing in D) and $\lambda_0 = \lambda_0(\tau_0/2, \delta)$ (because $\lambda_0(\mathrm{D}, \delta)$ is decreasing in D). Applying the lemma ℓ times, this gives a sequence $(\mathrm{Q}_k)_k$ of isometries, which is δ-close to $(\mathrm{P}_k)_k$, such that, as long as $\tau^{\ell k_0}(\mathrm{Q}_0, \cdots, \mathrm{Q}_{\ell k_0}) > \tau_0/2$, we have $\tau^{\ell k_0}(\mathrm{Q}_1, \cdots, \mathrm{Q}_{\ell k_0}) < \lambda_0^\ell \mathrm{D}(\mathbf{Z}^n)$. But for ℓ large enough, $\lambda_0^\ell < \tau_0$, which proves the theorem. □

Proof of Lemma 8.37. The idea of the proof is the following. Firstly, we apply the Minkowski-type theorem for almost periodic patterns (Theorem 7.29) to find a uniform constant $\mathrm{C} > 0$ and a point $u_0 \in \mathbf{Z}^n \setminus \{0\}$ whose norm is "not too big", such that $\rho_\Gamma(u_0) > \mathrm{CD}(\Gamma)$. Then, we apply Proposition 8.33 to prove that the difference u_0 in Γ eventually goes to 0; that is, that there exists $k_0 \in \mathbf{N}^*$ and an almost periodic pattern $\widetilde{\Gamma}$ of positive density (that we can compute) such that there exists a sequence $(\mathrm{Q}_k)_k$ of isometries, with $\|\mathrm{Q}_i - \mathrm{P}_i\| \leq \delta$, such that for every $x \in \widetilde{\Gamma}$,

$$(\widehat{\mathrm{Q}_{k_0}} \circ \cdots \circ \widehat{\mathrm{Q}_1})(x) = (\widehat{\mathrm{Q}_{k_0}} \circ \cdots \circ \widehat{\mathrm{Q}_1})(x + u_0).$$

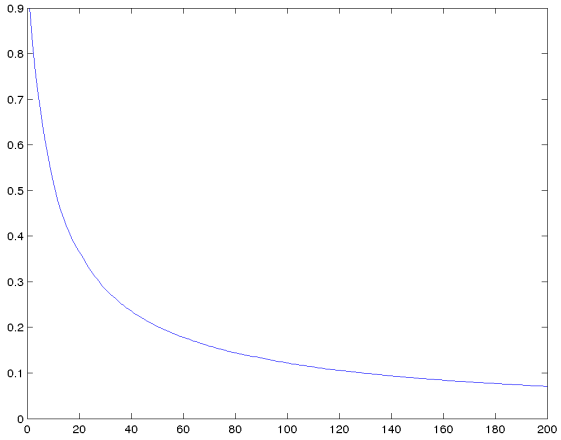

Figure 8.14: Expectation of the rate of injectivity of a random sequences of rotations: the graphic represents the mean of the rate of injectivity $\tau^k(R_{\theta_k}, \cdots, R_{\theta_1})$ depending on k, $1 \leq k \leq 200$, for 50 random draws of sequences of angles $(\theta_i)_i$, with each θ_i chosen independently and uniformly in $[0, 2\pi]$. Note that the behaviour is not exponential.

We begin by applying the Minkowsky-like theorem for almost periodic patterns (Theorem 7.29) to a *Euclidean* ball B'_R with radius R (recall that [B] denotes the set of integer points inside B):

$$\frac{1}{\mathrm{Card}[B'_R]} \sum_{u \in [B'_R]} \rho_\Gamma(u) \geq D(\Gamma) \frac{\mathrm{Card}[B'_{R/2}]}{\mathrm{Card}[B'_R]}. \tag{8.2}$$

An easy estimation of the number of integer points inside of the balls B'_R and $B'_{R/2}$ leads to [8]:

$$\frac{\mathrm{Card}[B'_{R/2}]}{\mathrm{Card}[B'_R]} \geq \left(\frac{\sqrt{R/2} - \sqrt{n}/2}{\sqrt{R} + \sqrt{n}/2} \right)^{2n},$$

in particular, if $R \geq n$, we obtain

$$\frac{\mathrm{Card}[B'_{R/2}]}{\mathrm{Card}[B'_R]} \geq \left(\frac{\sqrt{2} - 1}{3} \right)^{2n} \geq \frac{1}{53^n},$$

thus Equation (8.2) becomes

$$\frac{1}{\mathrm{Card}[B'_R]} \sum_{u \in [B'_R]} \rho_\Gamma(u) \geq D(\Gamma) \frac{1}{53^n}. \tag{8.3}$$

We now want that $\rho_\Gamma(0) = 1$ plays only a little role in Equation (8.3), that is

$$\frac{1}{\mathrm{Card}[B'_R]} \leq \frac{D(\Gamma)}{2 \times 53^n},$$

8. See for example [Fri82, page 5], this book also performs a complete investigation on the subject of finding the number of integer points in a Euclidean ball.

using the same estimate as previously for $\mathrm{Card}[\mathrm{B}'_{\mathrm{R}}]$, that is true if

$$\frac{1}{(\sqrt{\mathrm{R}}-\sqrt{n}/2)^n} \leq \frac{\mathrm{D}(\Gamma)}{2\times 53^n};$$

a direct calculation shows that it is true for example if

$$\mathrm{R} = \mathrm{R}_0 = n + \frac{212}{\sqrt[n]{\mathrm{D}(\Gamma)}}. \tag{8.4}$$

In this case, Equation (8.3) implies

$$\frac{1}{\mathrm{Card}[\mathrm{B}'_{\mathrm{R}}]} \sum_{u\in[\mathrm{B}'_{\mathrm{R}}]\setminus\{0\}} \rho_\Gamma(u) \geq \mathrm{D}(\Gamma)\frac{1}{53^n} - \frac{\rho_\Gamma(0)}{\mathrm{Card}[\mathrm{B}'_{\mathrm{R}}]} \geq \frac{\mathrm{D}(\Gamma)}{2\times 53^n},$$

thus there exists $u_0 \in [\mathrm{B}'_{\mathrm{R}}]\setminus\{0\}$ such that

$$\rho_\Gamma(u_0) \geq \frac{\mathrm{D}(\Gamma)}{2\times 53^n}. \tag{8.5}$$

We now take $\mathrm{R} \geq \mathrm{R}_0$, and perturb each matrix P_k into a totally irrational matrix Q_k such that for every point $x \in [\mathrm{B}'_{\mathrm{R}}]\setminus\{0\}$, the point $\mathrm{Q}_k(x)$ is far away from the lattice $\mathbf{Z}^n$. More precisely, as the set of matrices $\mathrm{Q} \in \mathrm{O}_n(\mathbf{R})$ such that $\mathrm{Q}([\mathrm{B}'_{\mathrm{R}}]) \cap \mathbf{Z}^n \neq \{0\}$ is finite, there exists a constant $d_0(\mathrm{R},\delta)$ such that for every $\mathrm{P} \in \mathrm{O}_n(\mathbf{R})$, there exists $\mathrm{Q} \in \mathrm{O}_n(\mathbf{R})$ such that $\|\mathrm{P}-\mathrm{Q}\| \leq \delta$ and for every $x \in [\mathrm{B}'_{\mathrm{R}}]\setminus\{0\}$, we have $d_\infty(\mathrm{Q}(x),\mathbf{Z}^n) > d_0(\mathrm{R},\delta)$. Applying Lemma 7.21 (which states that if the sequence $(\mathrm{Q}_k)_k$ is generic, then the matrices Q_k are "non resonant"), we build a sequence $(\mathrm{Q}_k)_{k\geq 1}$ of totally irrational matrices of $\mathrm{O}_n(\mathbf{R})$ such that for every $k \in \mathbf{N}^*$, we have:

- $\|\mathrm{P}_k - \mathrm{Q}_k\| \leq \delta$;
- for every $x \in [\mathrm{B}'_{\mathrm{R}}]\setminus\{0\}$, we have $d_\infty(\mathrm{Q}_k(x),\mathbf{Z}^n) > d_0(\mathrm{R},\delta)$;
- the set $\mathrm{Q}_k \circ \widehat{\mathrm{Q}_{k-1}} \circ \cdots \circ \widehat{\mathrm{Q}_1}(\Gamma)$ is equidistributed modulo $\mathbf{Z}^n$.

We then consider the difference u_0 (given by Equation (8.5)). We denote by $\lfloor \mathrm{P} \rfloor(u)$ the point of the smallest integer cube of dimension $n' \leq n$ that contains $\mathrm{P}(u)$ which has the smallest Euclidean norm (that is, the point of the support of $\varphi_{\mathrm{P}(u)}$ with the smallest Euclidean norm). In particular, if $\mathrm{P}(u) \notin \mathbf{Z}^n$, then $\|\lfloor \mathrm{P} \rfloor(u)\|_2 < \|\mathrm{P}(u)\|_2$ (where $\|\cdot\|_2$ is the Euclidean norm). Then, the point (ii) of Proposition 8.33 shows that

$$\begin{aligned}\rho_{\widehat{\mathrm{Q}_1}(\Gamma)}(\lfloor \mathrm{Q}_1 \rfloor(u_0)) &\geq \frac{\mathrm{D}(\Gamma)}{\mathrm{D}(\widehat{\mathrm{Q}_1}(\Gamma))}\varphi_{\mathrm{Q}_1(\lfloor \mathrm{Q}_1 \rfloor(u_0))}(u_0)\rho_\Gamma(u_0)\\ &\geq \frac{\big(d_0(\mathrm{R},\delta)\big)^n}{2\times 53^n}\mathrm{D}(\Gamma),\end{aligned}$$

and so on, for every $k \in \mathbf{N}^*$,

$$\rho_{(\widehat{\mathrm{Q}_k}\circ\cdots\circ\widehat{\mathrm{Q}_1})(\Gamma)}\Big(\big(\lfloor \mathrm{Q}_k \rfloor \circ \cdots \circ \lfloor \mathrm{Q}_1 \rfloor\big)(u_0)\Big) \geq \left(\frac{\big(d_0(\mathrm{R},\delta)\big)^n}{2\times 53^n}\right)^k \mathrm{D}(\Gamma).$$

We then remark that the Euclidean norm of $\lfloor \mathrm{Q}_1 \rfloor(u_0)$ can only take a finite number of values (it lies in $\sqrt{\mathbf{Z}}$). Then, there exists $k_0 \leq \mathrm{R}^2$ such that

$$\big(\lfloor \mathrm{Q}_{k_0} \rfloor \circ \cdots \circ \lfloor \mathrm{Q}_1 \rfloor\big)(u_0) = 0;$$

in particular, by Equation (8.4), we have

$$k_0 \leq 2n^2 + \frac{89\,888}{(\mathrm{D}(\Gamma))^{n/2}}.$$

Then, point (ii) of Remark 8.35 implies that the density of the image set satisfies

$$\mathrm{D}\Big((\widehat{\mathrm{Q}_k} \circ \cdots \circ \widehat{\mathrm{Q}_1})(\Gamma)\Big) \leq \left(1 - \left(\frac{\big(d_0(\mathrm{R},\delta)\big)^n}{2 \times 53^n}\right)^{k_0}\right) \mathrm{D}(\Gamma).$$

We obtain the conclusions of the lemma by setting $\lambda_0 = 1 - \left(\frac{(d_0(\mathrm{R},\delta))^n}{2\times 53^n}\right)^{k_0}$. □

Remark 8.38. We could be tempted to try to apply this strategy of proof to any sequence of matrices in $\mathrm{SL}_n(\mathbf{R})$. However, this does not work in the general case, because if we consider a non-Euclidean norm N, there could be some $x \in \mathbf{R}^n$ such that for every vertex y of the cube containing x, we have $\mathrm{N}(y) \geq \mathrm{N}(x)$.

We now set out a conjecture which states a generic dichotomy for the behaviour of the frequency of differences for discretizations of generic sequences of matrices, in the same vein as [Boc02] (for dimension 2) or [ACW14] (in the general case).

Conjecture 8.39. *Let $(\mathrm{A}_k)_{k\geq 1}$ be a generic sequence of matrices of $\mathrm{SL}_n(\mathbf{R})$. Then, if we note $\Gamma_k = (\widehat{\mathrm{A}_k} \circ \cdots \circ \widehat{\mathrm{A}_1})(\mathbf{Z}^n)$, the standard deviation of ρ_{Γ_k} is either very small, either as big as possible. More precisely, the standard deviation of*

$$\frac{1}{\sqrt{\mathrm{D}(\Gamma_k)\big(1-\mathrm{D}(\Gamma_k)\big)}}\rho_{\Gamma_k}$$

either tends to 0 as k goes to infinity (which corresponds to the case of zero Lyapunov exponent), or tends to 1 as k goes to infinity (which corresponds to the hyperbolic case).

The idea underlying the conjecture is that for for every $\varepsilon > 0$, for every generic sequence $(\mathrm{A}_k)_{k\geq 1}$ of matrices and every k big enough, we have two cases.

– Either

$$\mathrm{D}\left\{v \in \mathbf{Z}^n \;\middle|\; \left|\frac{1}{\mathrm{D}(\Gamma_k)}\rho_{\Gamma_k}(v) - 1\right| \geq \varepsilon\right\} < \varepsilon,$$

which means that $\rho_{\Gamma_k}(v)$ is very close to its mean on a set of density almost 1. This corresponds to the zero Lyapunov exponent case, and the idea is that diffusion process wins on the hyperbolicity of the sequence of matrices.

– Or

$$\mathrm{D}\left\{v \in \mathbf{Z}^n \;\middle|\; \rho_{\Gamma_k}(v) \in [\varepsilon, 1-\varepsilon]\right\} < \varepsilon,$$

which means that $\rho_{\Gamma_k}(v)$ is very close to 0 or 1 on a set of density almost 1. This corresponds to the hyperbolic case and the idea is that the hyperbolicity of the sequence of matrices wins on the diffusion process.

8.7 Applications of the notion of model set

This section is devoted to the application of the notion of model set to the computation of the rate of injectivity of a finite number of matrices.

We take advantage of the rational independence between the matrices of a generic sequence to generalize to arbitrary times the geometric formulas of Section 8.2. The end of the present section is devoted to the proof of the main theorem of this chapter (Theorem 8.24).

Let us summarize the different notations we will use throughout this section. We will denote by 0^k the origin of the space $\mathbf{R}^k$, and $W^k =]-1/2, 1/2]^{nk}$ (unless otherwise stated). In this section, we will denote $D_c(E)$ the density of a "continuous" set $E \subset \mathbf{R}^n$, defined as (when the limit exists)

$$D_c(E) = \lim_{R \to +\infty} \frac{\mathrm{Leb}(B_R \cap E)}{\mathrm{Leb}(B_R)},$$

while for a discrete set $E \subset \mathbf{R}^n$, the notation $D_d(E)$ will indicate the discrete density of E, defined as (when the limit exists)

$$D_d(E) = \lim_{R \to +\infty} \frac{\mathrm{Card}(B_R \cap E)}{\mathrm{Card}(B_R \cap \mathbf{Z}^n)},$$

We will consider $(A_k)_{k \geq 1}$ a sequence of matrices of $SL_n(\mathbf{R})$, and denote

$$\Gamma_k = (\widehat{A_k} \circ \cdots \circ \widehat{A_1})(\mathbf{Z}^n).$$

Also, Λ_k will be the lattice $M_{A_1,\cdots,A_k}\mathbf{Z}^{n(k+1)}$, with

$$M_{A_1,\cdots,A_k} = \begin{pmatrix} A_1 & -\mathrm{Id} & & & \\ & A_2 & -\mathrm{Id} & & \\ & & \ddots & \ddots & \\ & & & A_k & -\mathrm{Id} \\ & & & & \mathrm{Id} \end{pmatrix} \in M_{n(k+1)}(\mathbf{R}),$$

and $\widetilde{\Lambda}_k$ will be the lattice $\widetilde{M}_{A_1,\cdots,A_k}\mathbf{Z}^{nk}$, with

$$\widetilde{M}_{A_1,\cdots,A_k} = \begin{pmatrix} A_1 & -\mathrm{Id} & & & \\ & A_2 & -\mathrm{Id} & & \\ & & \ddots & \ddots & \\ & & & A_{k-1} & -\mathrm{Id} \\ & & & & A_k \end{pmatrix} \in M_{nk}(\mathbf{R}).$$

Finally, we will denote

$$\overline{\tau}^k(A_1, \cdots, A_k) = D_c\left(W^{k+1} + \Lambda_k\right).$$

These quantities will be related during this section.

8.7.1 A geometric viewpoint to compute the rate of injectivity in arbitrary times

We recall the (trivial) equality stated in Proposition 7.33: let $A_1, \cdots, A_k \in \mathrm{GL}_n(\mathbf{R})$, then

$$\begin{aligned}\Gamma_k &= (\widehat{A_k} \circ \cdots \circ \widehat{A_1})(\mathbf{Z}^n) \\ &= \left\{p_2(\lambda_k) \mid \lambda_k \in \Lambda_k,\, p_1(\lambda_k) \in W^k\right\} \\ &= p_2\Big(\Lambda_k \cap \big(p_1^{-1}(W^k)\big)\Big),\end{aligned}$$

with

$$M_{A_1,\cdots,A_k} = \begin{pmatrix} A_1 & -\mathrm{Id} & & & \\ & A_2 & -\mathrm{Id} & & \\ & & \ddots & \ddots & \\ & & & A_k & -\mathrm{Id} \\ & & & & \mathrm{Id} \end{pmatrix} \in \mathrm{M}_{n(k+1)}(\mathbf{R}), \tag{8.6}$$

$\Lambda_k = M_{A_1,\cdots,A_k}\mathbf{Z}^{n(k+1)}$, p_1 the projection on the nk first coordinates, p_2 the projection on the n last coordinates and $W^k =]-1/2, 1/2]^{nk}$.

Here, we suppose that the set $p_1(\Lambda_k)$ is dense (thus, equidistributed) in the image set $\mathrm{im}\, p_1$ (note that this condition is generic among the sequences of invertible linear maps). In particular, the set $\{p_2(\gamma) \mid \gamma \in \Lambda_k\}$ is equidistributed in the window W^k.

The following property makes the link between the density of Γ_k — that is, the rate of injectivity of $A_1, \cdots, A_k$ — and the density of the union of unit cubes centred on the points of the lattice Λ_k.

Proposition 8.40. *For a generic sequence of matrices $(A_k)_k$ of $\mathrm{SL}_2(\mathbf{R})$, we have*

$$\mathrm{D}_d(\Gamma_k) = \mathrm{D}_c\left(W^k + \widetilde{\Lambda}_k\right).$$

Equivalently, we have

$$\mathrm{D}_d(\Gamma_k) = \mathrm{D}_c\left(W^{k+1} + \Lambda_k\right).$$

Of course, this proposition generalizes Proposition 8.11 to an arbitrary number of matrices $A_1, \cdots, A_k$.

Remark 8.41. The density on the left of the equality is the density of a discrete set (that is, with respect to counting measure), whereas the density on the right of the equality is that of a continuous set (that is, with respect to Lebesgue measure). The two notions coincide when we consider discrete sets as sums of Dirac masses.

This proposition leads to the definition of the mean rate of injectivity in time k.

Definition 8.42. Let $(A_k)_{k\geq 1}$ be a sequence of matrices of $\mathrm{SL}_n(\mathbf{R})$. The *mean rate of injectivity in time k* of this sequence is defined by

$$\overline{\tau}^k(A_1, \cdots, A_k) = \mathrm{D}_c\left(W^{k+1} + \Lambda_k\right).$$

Remark 8.43. As in Section 8.2 for the rate of injectivity in time 1, Proposition 8.40 asserts that for a generic sequence of matrices, the rate of injectivity τ^k in time k coincides with the mean rate of injectivity $\overline{\tau}^k$, which is continuous and piecewise polynomial of degree $\leq nk$ in the coefficients of the matrix.

Remark 8.44. The formula of Proposition 8.40 could be used to compute numerically the mean rate of injectivity in time k of a sequence of matrices: it is much faster to compute the area of a finite number of intersections of cubes (in fact, a small number) than to compute the cardinalities of the images of a big set $[-\mathrm{R},\mathrm{R}]^n \times \mathbf{Z}^n$.

Proof of Proposition 8.40. We want to determine the proportion of integral points $x \in \mathbf{Z}^n$ such that there exists $\lambda \in \Lambda_k$ such that $p_2(\lambda) = x$ and $p_1(\lambda) \in \mathrm{W}^k$. This condition is equivalent to $p_1(\Lambda_k \cap p_2^{-1}(x)) \in \mathrm{W}^k$, which depends only on the position of the set $p_1(\Lambda_k \cap p_2^{-1}(x))$. More precisely, if we set

$$\widetilde{\mathrm{M}}_{\mathrm{A}_1,\cdots,\mathrm{A}_k} = \begin{pmatrix} \mathrm{A}_1 & -\mathrm{Id} & & & \\ & \mathrm{A}_2 & -\mathrm{Id} & & \\ & & \ddots & \ddots & \\ & & & \mathrm{A}_{k-1} & -\mathrm{Id} \\ & & & & \mathrm{A}_k \end{pmatrix} \in \mathrm{M}_{nk}(\mathbf{R}) \tag{8.7}$$

and $\widetilde{\Lambda}_k = \widetilde{\mathrm{M}}_{\mathrm{A}_1,\cdots,\mathrm{A}_k}\mathbf{Z}^{nk}$ (which is a lattice of covolume 1), then we have

$$p_1\Big(\Lambda_k \cap p_2^{-1}(x)\Big) = \widetilde{\Lambda}_k + (0^{(k-1)n}, -x) \subset \mathbf{R}^{nk}. \tag{8.8}$$

Thus, the two sets $p_1(p_2^{-1}(0) \cap \Lambda_k)$ and $p_1(p_2^{-1}(x) \cap \Lambda_k)$ only differ by a translation of the form $(0^{(k-1)n}, -x)$ (see Figure 8.15). Recall that our aim is to decide whether there exists a point in the set $\widetilde{\Lambda}_k + (0^{(k-1)n}, -x) \subset \mathbf{R}^{nk}$ (see Equation (8.8)) belonging to W^k or not. But $\Big(\widetilde{\Lambda}_k + (0^{(k-1)n}, -x)\Big) \cap \mathrm{W}^k \neq \emptyset$ if and only if $(0^{(k-1)n}, x) \in \widetilde{\Lambda}_k + \mathrm{W}^k$. The proposition follows directly from the fact that the points of the form $(0^{(k-1)n}, x)$, with $x \in \mathbf{Z}^n$, are equidistributed in $\mathbf{R}^{nk}/\widetilde{\Lambda}_k$.

To prove this equidistribution, we compute the inverse matrix of $\widetilde{\mathrm{M}}_{\mathrm{A}_1,\cdots,\mathrm{A}_k}$:

$$\widetilde{\mathrm{M}}^{-1}_{\mathrm{A}_1,\cdots,\mathrm{A}_k} = \begin{pmatrix} \mathrm{A}_1^{-1} & \mathrm{A}_1^{-1}\mathrm{A}_2^{-1} & \mathrm{A}_1^{-1}\mathrm{A}_2^{-1}\mathrm{A}_3^{-1} & \cdots & \mathrm{A}_1^{-1}\cdots\mathrm{A}_k^{-1} \\ & \mathrm{A}_2^{-1} & \mathrm{A}_2^{-1}\mathrm{A}_3^{-1} & \cdots & \mathrm{A}_2^{-1}\cdots\mathrm{A}_k^{-1} \\ & & \ddots & \ddots & \vdots \\ & & & \mathrm{A}_{k-1}^{-1} & \mathrm{A}_{k-1}^{-1}\mathrm{A}_k^{-1} \\ & & & & \mathrm{A}_k^{-1} \end{pmatrix}.$$

Thus, the set of points of the form $(0^{(k-1)n}, x)$ in $\mathbf{R}^{nk}/\widetilde{\Lambda}_k$ corresponds to the image of the action

$$\mathbf{Z}^n \ni x \longmapsto \begin{pmatrix} \mathrm{A}_1^{-1}\cdots\mathrm{A}_k^{-1} \\ \mathrm{A}_2^{-1}\cdots\mathrm{A}_k^{-1} \\ \vdots \\ \mathrm{A}_{k-1}^{-1}\mathrm{A}_k^{-1} \\ \mathrm{A}_k^{-1} \end{pmatrix} x$$

of $\mathbf{Z}^n$ on the canonical torus $\mathbf{R}^{nk}/\mathbf{Z}^{nk}$. But this action is trivially ergodic (even in restriction to the first coordinate) when the sequence of matrices is generic. □

As in the case of the time 1 (Proposition 8.12), there is also a geometric method to compute the frequency of a difference in Γ_k.

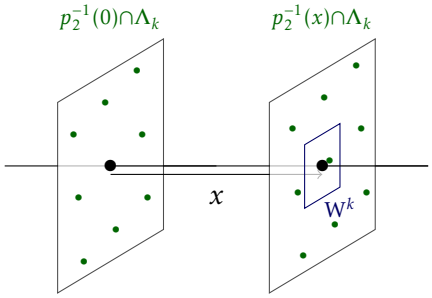

Figure 8.15: Construction of Proposition 8.40.

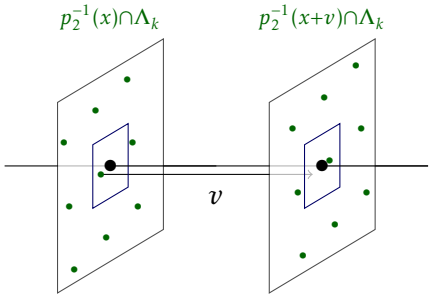

Figure 8.16: Construction of Proposition 8.45

Proposition 8.45. *For a generic sequence of matrices* $(A_k)_k$*, we have*

$$\rho_{\Gamma_k}(v) = \mathrm{Leb}\Big(\big(W^{k+1}+\Lambda_k\big)\cap\big(W^{k+1}+(0^{kn},v)\big)\Big).$$

Proof of Proposition 8.45. To compute $\rho_{\Gamma_k}(v)$, we want to know for which proportion of $x\in\mathbf{Z}^n$ such that $x\in\Gamma_k$, we have $x+v\in\Gamma_k$, that is, knowing that $p_1\big(p_2^{-1}(x)\cap\Lambda_k\big)\cap W^k\neq\emptyset$, we have $p_1\big(p_2^{-1}(x+v)\cap\Lambda_k\big)\cap W^k\neq\emptyset$. But

$$\begin{aligned} p_1\big(p_2^{-1}(x+v)\cap\Lambda_k\big)\cap W^k\neq\emptyset \quad &\Longleftrightarrow \quad \big(\widetilde{\Lambda}_k+(0^{(k-1)n},-x-v)\big)\cap W^k\neq\emptyset \\ &\Longleftrightarrow \quad (0^{(k-1)n},x+v)\in\widetilde{\Lambda}_k+W^k. \end{aligned}$$

As the sets $p_1(p_2^{-1}(x)\cap\Lambda_k)$ are equidistributed in W^k (see the proof of Proposition 8.40), the proportion of such points x is equal to the area of the intersection $(W^{k+1}+\Lambda_k)\cap(W^{k+1}+(0^{kn},v))$. □

As in the case of time 1 (see Section 8.2), an argument of equidistribution combined with Proposition 7.27 allows to deduce Proposition 8.40 from Proposition 8.45.

There is also a dual method to compute the rate of injectivity of a sequence of matrices: we define $\psi:\mathbf{R}^{nk}\to\mathbf{R}$ by

$$\psi=\sum_{\lambda\in\Lambda_k}\mathbf{1}_{W^k+\lambda},$$

and obtain the following formula (which is a generalization of Proposition 8.18).

Proposition 8.46. *For a generic sequence of matrices* $(A_k)_k$*, we have*

$$D_d(\Gamma_k)=\int_{B_{1/2}}\frac{1}{\psi(\lambda)}\,\mathrm{d}\,\mathrm{Leb}(\lambda).$$

The proof of this proposition is similar to that of Proposition 8.18 page 174 (see in particular Lemma 8.19).

Recall the problem raised by Theorem 8.24: we want to make τ^k as small as possible. By an argument of equidistribution, generically, it is equivalent to make the mean rate of injectivity $\overline{\tau}^k$ as small as possible when k goes to infinity, by perturbing every matrix in $\mathrm{SL}_n(\mathbf{R})$ of at most $\delta>0$ (fixed once for all). In the framework of model sets, Theorem 8.24 is motivated by the phenomenon of concentration of the measure on a neighbourhood of the boundary of the cubes in high dimension.

Proposition 8.47. *Let* W^k *be the infinite ball of radius* $1/2$ *in* $\mathbf{R}^k$ *and* v^k *the vector* $(1,\cdots,1) \in \mathbf{R}^k$. *Then, for every* $\varepsilon, \delta > 0$, *there exists* $k_0 \in \mathbf{N}^*$ *such that for every* $k \geq k_0$, *we have* $\mathrm{Leb}\Big(W^k \cap (W^k + \delta v^k)\Big) < \varepsilon$.

Applying Hajós theorem (Theorem 8.21), it is easy to see when the density of Γ_k is equal to 1: combining this theorem with Proposition 8.40, we obtain that this occurs if and only if the lattice given by the matrix $M_{A_1,\cdots,A_k}$ satisfies the conclusions of Hajós theorem [9]. The heuristic suggested by the phenomenon of concentration of the measure is that if we perturb "randomly" any sequence of matrices, we will go "far away" from the lattices satisfying Hajós theorem and then the rate of injectivity will be close to 0.

Remark 8.48. The kind of questions addressed by Hajós theorem are in general quite delicate. For example, we can wonder what happens if we do not suppose that the centres of the cubes form a lattice of $\mathbf{R}^n$. O. H. Keller conjectured in [Kel30] that the conclusion of Hajós theorem is still true under this weaker hypothesis. This conjecture was proven to be true for $n \leq 6$ by O. Perron in [Per40a, Per40b], but remained open in higher dimension until 1992, when J. C. Lagarias and P. W. Shor proved in [LS92] that Keller's conjecture is false for $n \geq 10$ (this result was later improved by [Mac02] which shows that it is false as soon as $n \geq 8$; the case $n = 7$ is still open).

8.7.2 New proof that the asymptotic rate of injectivity is generically smaller than $1/2$

As a first application of the concept of model set, we give a new proof of the fact that rate of injectivity of a generic sequence of $\mathrm{SL}_n(\mathbf{R})$ is smaller than $1/2$. To begin with, we give a lemma estimating the sizes of intersections of cubes when the rate is bigger than $1/2$ (which is the geometric counterpart of Lemma 8.29).

Lemma 8.49. *Let* $W^k =]-1/2, 1/2]^k$ *and* $\Lambda \subset \mathbf{R}^k$ *be a lattice with covolume 1 such that* $D_c(W^k + \Lambda) \geq 1/2$. *Then, for every* $v \in \mathbf{R}^k$, *we have*

$$D_c\Big((W^k + \Lambda + v) \cap (W^k + \Lambda)\Big) \geq 2D_c(W^k + \Lambda) - 1.$$

Proof of Lemma 8.49. We first remark that $D_c(\Lambda + W^k)$ is equal to the volume of the projection of W^k on the quotient space $\mathbf{R}^k/\Lambda$. For every $v \in \mathbf{R}^k$, the projection of $W^k + v$ on $\mathbf{R}^k/\Lambda$ has the same volume; as this volume is greater than $1/2$, and as the covolume of Λ is 1, the projections of W^k and $W^k + v$ overlap, and the volume of the intersection is bigger than $2D_c(W^k + \Lambda) - 1$. Returning to the whole space $\mathbf{R}^k$, we get the conclusion of the lemma. □

Remark 8.50. In the case where Λ is the lattice spanned by $M_{A_1,\cdots,A_k}$, where $A_1, \cdots, A_k$ is a generic family, Lemma 8.49 can be deduced directly from Lemma 8.29 (more precisely, improves its conclusion of a factor 2).

This lemma allows us to give another proof of Theorem 8.30 (page 179), which implies in particular that the asymptotic rate of injectivity of a generic sequence of matrices of $\mathrm{SL}_n(\mathbf{R})$ is smaller than $1/2$ (and even is smaller than a sequence converging exponentially fast to $1/2$).

9. Of course, this property can be obtained directly by saying that the density is equal to 1 if and only if the rate of injectivity of every matrix of the sequence is equal to 1.

How to read these figures : The top of the figure represents the set $W^k + \widetilde{\Lambda}_k$ by the 1-dimensional set $[-1/2, 1/2] + v\mathbf{Z}$ (in dark blue), for a number $v > 1$. The bottom of the figure represents the set $W^{k+1} + \widetilde{\Lambda}_{k+1}$ by the set $[-1/2, 1/2]^2 + \Lambda$, where Λ is the lattice spanned by the vectors $(0, v)$ and $(1, 1-\varepsilon)$ for a parameter $\varepsilon > 0$ close to 0. The dark blue cubes represent the "old" cubes, that is, the thickening in dimension 2 of the set $W^k + \widetilde{\Lambda}_k$, and the light blue cubes represent the "added" cubes, that is, the rest of the set $W^{k+1} + \widetilde{\Lambda}_{k+1}$.

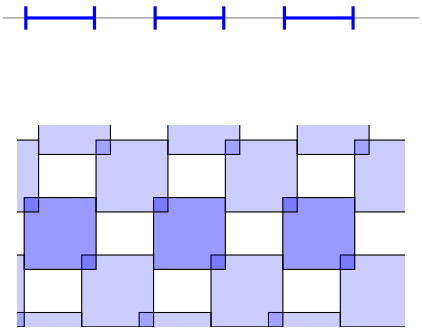

Figure 8.17: In the case where the rate is bigger than 1/2, some intersections of cubes appear automatically between times k and $k+1$.

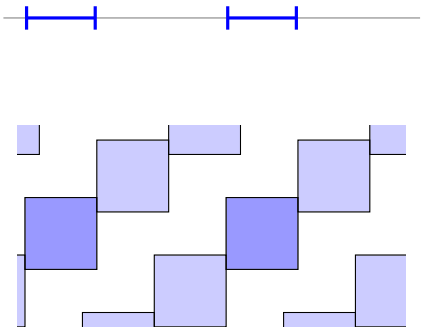

Figure 8.18: In the case where the rate is smaller than 1/2, there is not necessarily new intersections between times k and $k+1$.

Alternative proof of Theorem 8.30. Let $(A_k)_{k\geq 1}$ be a bounded sequence of matrices of $SL_n(\mathbf{R})$ and $\delta > 0$. As in the previous proof of Theorem 8.30, we proceed by induction on k and suppose that the theorem is proved for a rank $k \in \mathbf{N}^*$. Let $\widetilde{\Lambda}_k$ be the lattice spanned by the matrix $\widetilde{M}_{B_1,\cdots,B_k}$ (defined by Equation (8.7) page 189) and $W^k =]-1/2, 1/2]^{nk}$ be the window corresponding to the model set Γ_k modelled on Λ_k (the lattice spanned by the matrix $M_{B_1,\cdots,B_k}$, see Equation (8.6)). By Proposition 8.40, we have

$$\tau^k(B_1, \cdots, B_k) = D_c\left(W^k + \widetilde{\Lambda}_k\right).$$

We now choose a matrix B_{k+1} satisfying $\|A_{k+1} - B_{k+1}\| \leq \delta$, such that there exists $x_1 \in \mathbf{Z}^n \setminus \{0\}$ such that $\|B_{k+1}x_1\|_\infty \leq 1 - \varepsilon$, with $\varepsilon > 0$ depending only on δ and $\|(A_k)_k\|$ (and n): indeed, for every matrix $B \in SL_n(\mathbf{R})$, Minkowski theorem implies that there exists $x_1 \in \mathbf{Z}^n \setminus \{0\}$ such that $\|Bx_1\|_\infty \leq 1$; it then suffices to modify slightly B to decrease $\|Bx_1\|_\infty$. We can also suppose that the sequence of matrices $B_1, \cdots, B_{k+1}$ is generic. Again, Proposition 8.40 reduces the calculation of the rate on injectivity $\tau^{k+1}(B_1, \cdots, B_{k+1})$ to that of the density of $W^{k+1} + \widetilde{\Lambda}_{k+1}$. By the form of the matrix $M_{B_1,\cdots,B_k}$, this set can be decomposed into

$$W^{k+1} + \widetilde{\Lambda}_{k+1} = W^{k+1} + \begin{pmatrix} \widetilde{\Lambda}_k \\ 0^n \end{pmatrix} + \begin{pmatrix} 0^{n(k-1)} \\ -\mathrm{Id} \\ B_{k+1} \end{pmatrix} \mathbf{Z}^n.$$

In particular, as $|\det(B_{k+1})| = 1$, this easily implies that $D_c\left(W^{k+1} + \widetilde{\Lambda}_{k+1}\right) \leq D_c\left(W^k + \widetilde{\Lambda}_k\right)$.

What we need is a more precise bound. We apply Corollary 8.20 to

$$\Lambda_1 = \left(\widetilde{\Lambda}_k, 0^n\right), \qquad \Lambda_2 = \begin{pmatrix} 0^{n(k-1)} \\ -\mathrm{Id} \\ B_{k+1} \end{pmatrix} \mathbf{Z}^n \qquad \text{and} \qquad B = W^{k+1}.$$

Then, the decreasing of the rate of injectivity between times k and $k+1$ is bigger than the D_1 defined in Corollary 8.20: using Lemma 8.49, we have

$$D_c\Big(\Big(W^k+\widetilde{\Lambda}_k\Big)\cap\Big(W^k+\widetilde{\Lambda}_k+\big(0^{n(k-1)},-x_1\big)\Big)\Big)\geq 2D_d(\Gamma_k)-1;$$

thus, as $\|x_1\|_\infty<1-\varepsilon$

$$D_1\geq \varepsilon^n\big(2D_d(\Gamma_k)-1\big).$$

From Corollary 8.20 we deduce that

$$\begin{aligned}D(\Gamma_{k+1})&=D_c\Big(W^{k+1}+\widetilde{\Lambda}_{k+1}\Big)\\&\leq D_c\Big(W^k+\widetilde{\Lambda}_k\Big)-\frac{1}{2}D_1\\&\leq D_d(\Gamma_k)-\frac{1}{2}\varepsilon^n\big(2D_d(\Gamma_k)-1\big).\end{aligned}$$

This proves the theorem for the rank $k+1$. □

8.7.3 Proof of Theorem 8.24: generically, the asymptotic rate is zero

We now come to the proof of Theorem 8.24. The strategy of proof is identical to that we used in the previous section to state that generically, the asymptotic rate is smaller than 1/2: we will use an induction to decrease the rate step by step. Recall that $\overline{\tau}^k(A_1,\cdots,A_k)$ indicates the density of the set $W^{k+1}+\Lambda_k$ (see Definition 8.42).

Unfortunately, if the density of $W^k+\widetilde{\Lambda}_k$ — which is generically equal to the density of the k-th image $\big(\widehat{A}_k\circ\cdots\circ\widehat{A}_1\big)(\mathbf{Z}^n)$ — is smaller than 1/2, then we can not apply exactly the strategy of proof of the previous section (see Figure 8.18). For example, if we take

$$A_1=\begin{pmatrix}4&\\&1/4\end{pmatrix}\quad\text{and}\quad A_2=\begin{pmatrix}1/2&\\&2\end{pmatrix},$$

then $\big(\widehat{A_2}\circ\widehat{A_1}\big)(\mathbf{Z}^2)=(2\mathbf{Z})^2$, and for every B_3 close to the identity, we have $\tau^3(A_1,A_2,B_3)=\tau^2(A_1,A_2)=1/4$.

To overcome this difficulty, we prove that for a generic sequence $(A_k)_{k\geq 1}$, if $\overline{\tau}^k(A_1,\cdots,A_k)>1/\ell$, then $\overline{\tau}^{k+\ell-1}(A_1,\cdots,A_{k+\ell-1})$ is strictly smaller than $\overline{\tau}^k(A_1,\cdots,A_k)$. An argument of equirepartition (in fact, Propostion 8.40) allows to see this problem in terms of area of intersections of cubes. More precisely, we consider the maximal number of disjoint translates of $W^k+\widetilde{\Lambda}_k$ in $\mathbf{R}^{nk}$: we easily see that if the density of $W^k+\widetilde{\Lambda}_k$ is bigger than $1/\ell$, then there can not be more than ℓ disjoint translates of $W^k+\widetilde{\Lambda}_k$ in $\mathbf{R}^{nk}$(Lemma 8.51). Then, Lemma 8.52 states that if the sequence of matrices is generic, either the density of $W^{k+1}+\widetilde{\Lambda}_{k+1}$ is smaller than that of $W^k+\widetilde{\Lambda}_k$ (see Figure 8.19), or there can not be more than $\ell-1$ disjoint translates of $W^{k+1}+\widetilde{\Lambda}_{k+1}$ in $\mathbf{R}^{n(k+1)}$(see Figure 8.20). Applying this reasoning (at most) $\ell-1$ times, we obtain that the density of $W^{k+\ell-1}+\widetilde{\Lambda}_{k+\ell-1}$ is smaller than that of $W^k+\widetilde{\Lambda}_k$. For example if $D_c\big(W^k+\widetilde{\Lambda}_k\big)>1/3$, then $D_c\big(W^{k+2}+\widetilde{\Lambda}_{k+2}\big)<D\big(W^k+\widetilde{\Lambda}_k\big)$ (see Figure 8.21). To apply this strategy in practice, we have to obtain quantitative estimates about the loss of density we get between times k and $k+\ell-1$.

Remark that with this strategy, we do not need to make "clever" perturbations of the matrices: provided that the coefficients of the matrices are rationally independent, the perturbation of each matrix is made independently from that of the others. However,

this reasoning does not tell when exactly the rate of injectivity decreases (likely, in most of cases, the speed of decreasing of the rate of injectivity is much faster than the one obtained by this method), and does not say either where exactly the loss of injectivity occurs in the image sets.

Firstly, we give a more precise the statement of Theorem 8.24.

Theorem 8.24. *For a generic sequence of matrices $(A_k)_{k\geq 1}$ of $SL_n(\mathbf{R})$, for every $\ell \in \mathbf{N}$, there exists $\lambda_\ell \in]0,1[$ such that for every $k \in \mathbf{N}$,*

$$\tau^{\ell k}(A_1, \cdots, A_{\ell k}) \leq \lambda_\ell^k + \frac{1}{\ell}. \tag{8.9}$$

Also, for every $\nu < 1$, we have

$$\tau^k(A_1, \cdots, A_k) = o\Big(\ln(k)^{-\nu}\Big). \tag{8.10}$$

In particular, the asymptotic rate of injectivity $\tau^\infty\Big((A_k)_{k\geq 1}\Big)$ is equal to zero.

The following lemma is a generalization of Lemma 8.49. It expresses that if the density of $W^k + \widetilde{\Lambda}_k$ is bigger than $1/\ell$, then there can not be more than ℓ disjoint translates of $W^k + \widetilde{\Lambda}_k$, and gives an estimation on the size of these intersections.

Lemma 8.51. *Let $W^k =]-1/2, 1/2]^k$ and $\Lambda \subset \mathbf{R}^k$ be a lattice with covolume 1 such that $D_c(W^k + \Lambda) \geq 1/\ell$. Then, for every collection $v_1, \cdots, v_\ell \in \mathbf{R}^k$, there exists $i \neq i' \in [\![1, \ell]\!]$ such that*

$$D_c\Big((W^k + \Lambda + v_i) \cap (W^k + \Lambda + v_{i'})\Big) \geq 2\frac{\ell D_c(W^k + \Lambda) - 1}{\ell(\ell - 1)}.$$

Proof of Lemma 8.51. For every $v \in \mathbf{R}^k$, the density $D_c(W^k + \Lambda + v)$ is equal to the volume of the projection of W^k on the quotient space $\mathbf{R}^k/\Lambda$. As this volume is greater than $1/\ell$, and as the covolume of Λ is 1, the projections of the $W^k + v_i$ overlap, and the volume of the points belonging to at least two different sets is bigger than $\ell D_c(W^k + \Lambda) - 1$. As there are $\ell(\ell-1)/2$ possibilities of intersection, there exists $i \neq i'$ such that the volume of the intersection between the projections of $W^k + v_i$ and $W^k + v_{i'}$ is bigger than $2(\ell D_c(W^k + \Lambda) - 1)/(\ell(\ell-1))$. Returning to the whole space $\mathbf{R}^k$, we get the conclusion of the lemma. □

Recall that we denote $\widetilde{\Lambda}_k$ the lattice spanned by the matrix

$$\widetilde{M}_{A_1, \cdots, A_k} = \begin{pmatrix} A_1 & -\mathrm{Id} & & & \\ & A_2 & -\mathrm{Id} & & \\ & & \ddots & \ddots & \\ & & & A_{k-1} & -\mathrm{Id} \\ & & & & A_k \end{pmatrix} \in M_{nk}(\mathbf{R}),$$

and W^k the cube $]-1/2, 1/2]^{nk}$.

Lemma 8.52. *For every $\delta > 0$ and every $M > 0$, there exists $\varepsilon > 0$ and an open set of matrices $\mathcal{O} \subset SL_n(\mathbf{R})$ which is δ-dense in the set of matrices of norm $\leq M$, such that if $\ell \geq 2$ and $D_0 > 0$ are such that for every collection of vectors $v_1, \cdots, v_\ell \in \mathbf{R}^n$, there exists $j, j' \in [\![1, \ell]\!]$ such that*

$$D_c\Big(\Big(W^k + \widetilde{\Lambda}_k + (0^{(k-1)n}, v_j)\Big) \cap \Big(W^k + \widetilde{\Lambda}_k + (0^{(k-1)n}, v_{j'})\Big)\Big) \geq D_0,$$

then for every $B \in \mathcal{O}$, if we denote by $\widetilde{\Lambda}_{k+1}$ the lattice spanned by the matrix $\widetilde{M}_{A_1, \cdots, A_k, B}$,

(1) either $D_c(W^{k+1} + \widetilde{\Lambda}_{k+1}) \le D_c(W^k + \widetilde{\Lambda}_k) - \varepsilon D_0/(4\ell)$;

(2) or for every collection of vectors $w_1, \cdots, w_{\ell-1} \in \mathbf{R}^n$, *there exists* $i \neq i' \in [\![1, \ell-1]\!]$ *such that*

$$D_c\Big(\big(W^{k+1} + \widetilde{\Lambda}_{k+1} + (0^{kn}, w_i)\big) \cap \big(W^{k+1} + \widetilde{\Lambda}_{k+1} + (0^{kn}, w_{i'})\big)\Big) \ge \varepsilon D_0/\ell^2.$$

Remark 8.53. If $\ell = 2$, then we have automatically the conclusion (1) of the lemma.

In a certain sense, the conclusion (1) corresponds to the hyperbolic case, and the conclusion (2) expresses that there is a diffusion between times k and $k+1$.

Proof of Lemma 8.52. Let $\mathscr{O}_\varepsilon$ be the set of the matrices $B \in SL_n(\mathbf{R})$ satisfying: for any collection of vectors $w_1, \cdots, w_{\ell-1} \in \mathbf{R}^n$, there exists a set $U \subset \mathbf{R}^n/B\mathbf{Z}^n$ of measure $> \varepsilon$ such that every point of U belongs to at least ℓ different cubes of the collection $(Bv + w_i + W^1)_{v\in\mathbf{Z}^n, 1\le i\le \ell-1}$. In other words [10], every $x \in \mathbf{R}^n$ whose projection $\overline{x}$ on $\mathbf{R}^n/B\mathbf{Z}^n$ belongs to U satisfies

$$\sum_{i=1}^{\ell-1} \sum_{v\in\mathbf{Z}^n} \mathbf{1}_{x\in Bv+w_i+W^1} \ge \ell. \tag{8.11}$$

We easily see that the sets $\mathscr{O}_\varepsilon$ are open and that the union of these sets over $\varepsilon > 0$ is dense (it contains the set of matrices B whose entries are all irrational). Thus, if we are given $\delta > 0$ and $M > 0$, there exists $\varepsilon > 0$ such that $\mathscr{O} = \mathscr{O}_\varepsilon$ is δ-dense in the set of matrices of $SL_n(\mathbf{R})$ whose norm is smaller than M.

We then choose $B \in \mathscr{O}$ and a collection of vectors $w_1, \cdots, w_{\ell-1} \in \mathbf{R}^n$. Let $x \in \mathbf{R}^n$ be such that $\overline{x} \in U$. By hypothesis on the matrix B, x satisfies Equation (8.11), so there exists $\ell + 1$ integer vectors $v_1, \cdots, v_\ell$ and ℓ indices $i_1, \cdots, i_\ell$ such that the couples (v_j, i_j) are pairwise distinct and that

$$\forall j \in [\![1, \ell+1]\!], \quad x \in Bv_j + w_{i_j} + W^1. \tag{8.12}$$

The following formula makes the link between what happens in the n last and in the n penultimates coordinates of $\mathbf{R}^{n(k+1)}$:

$$W^{k+1} + \widetilde{\Lambda}_{k+1} + \big(0^{(k-1)n}, 0^n, w_{i_j}\big) = W^{k+1} + \widetilde{\Lambda}_{k+1} + \big(0^{(k-1)n}, -v_j, w_{i_j} + Bv_j\big), \tag{8.13}$$

(we add a vector belonging to $\widetilde{\Lambda}_{k+1}$).

We now apply the hypothesis of the lemma to the vectors $-v_1, \cdots, -v_{\ell+1}$: there exists $j \neq j' \in [\![1, \ell]\!]$ such that

$$D_c\Big(\big(W^k + \widetilde{\Lambda}_k + (0^{(k-1)n}, -v_j)\big) \cap \big(W^k + \widetilde{\Lambda}_k + (0^{(k-1)n}, -v_{j'})\big)\Big) \ge D_0. \tag{8.14}$$

Let y be a point belonging to this intersection. Applying Equations (8.12) and (8.14), we get that

$$(y, x) \in W^{k+1} + \big(\widetilde{\Lambda}_k, 0^n\big) + \big(0^{(k-1)n}, -v_j, w_{i_j} + Bv_j\big) \tag{8.15}$$

and the same for j'.

Two different cases can occur.

10. Matrices that does not possess this property are such that the union of cubes form a k-fold tiling. This was the subject of Furtwängler conjecture, see [Fur36], proved false by G. Hajós. R. Robinson gave a characterization of such k-fold tilings in some cases, see [Rob79] or [SS94, p. 29].

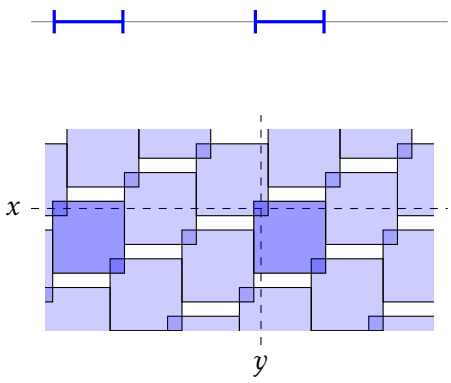

Figure 8.19: First case of Lemma 8.52, in the case $\ell = 3$: the set $W^{k+1} + \widetilde{\Lambda}_{k+1}$ auto-intersects.

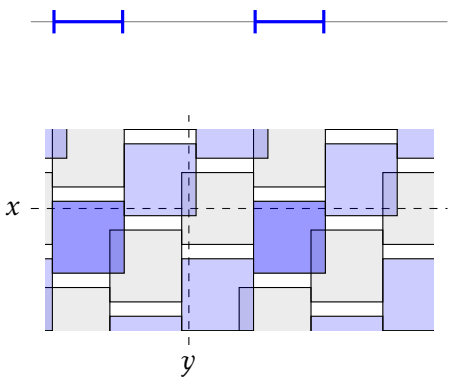

Figure 8.20: Second case of Lemma 8.52, in the case $\ell = 3$: two distinct vertical translates of $W^{k+1} + \widetilde{\Lambda}_{k+1}$ intersect (the first translate contains the dark blue thickening of $W^k + \widetilde{\Lambda}_k$, the second is represented in grey).

(i) Either $i_j = i_{j'}$ (that is, the translation vectors w_{i_j} and $w_{i_{j'}}$ are equal). As a consequence, applying Equation (8.15), we have

$$(y,x) + \left(0^{(k-1)n}, v_j, -Bv_j - w_{i_j}\right) \in \left(W^{k+1} + \left(\widetilde{\Lambda}_k, 0^n\right)\right) \cap \left(W^{k+1} + \left(\widetilde{\Lambda}_k, 0^n\right) + v'\right),$$

with

$$v' = \left(0^{(k-1)n}, -(v_{j'} - v_j), B(v_{j'} - v_j)\right) \in \widetilde{\Lambda}_{k+1} \setminus \widetilde{\Lambda}_k.$$

This implies that the set $W^{k+1} + \widetilde{\Lambda}_{k+1}$ auto-intersects (see Figure 8.19).

(ii) Or $i_j \neq i_{j'}$ (that is, $w_{i_j} \neq w_{i_{j'}}$). Combining Equations (8.15) and (8.13) (note that $\left(\widetilde{\Lambda}_k, 0^n\right) \subset \widetilde{\Lambda}_{k+1}$), we get

$$(y,x) \in \left(W^{k+1} + \widetilde{\Lambda}_{k+1} + \left(0^{kn}, w_{i_j}\right)\right) \cap \left(W^{k+1} + \widetilde{\Lambda}_{k+1} + \left(0^{kn}, w_{i_{j'}}\right)\right).$$

This implies that two distinct vertical translates of $W^{k+1} + \widetilde{\Lambda}_{k+1}$ intersect (see Figure 8.20).

We now look at the global behaviour of all the x such that $\overline{x} \in U$. Again, we have two cases.

(1) Either for more than the half of such x (for Lebesgue measure), we are in the case (i). To each of such x corresponds a translation vector w_i. We choose w_i such that the set of corresponding x has the biggest measure; this measure is bigger than $\varepsilon/\left(2(\ell-1)\right) \geq \varepsilon/(2\ell)$. Reasoning as in the second proof of Theorem 8.30 (page 192), and in particular applying Corollary 8.20, we get that the density D_1 of the auto-intersection of $W^{k+1} + \widetilde{\Lambda}_{k+1} + (0, w_i)$ is bigger than $D_0\varepsilon/(2\ell)$. This leads to

$$D_c(W^{k+1} + \widetilde{\Lambda}_{k+1}) < D_c(W^k + \widetilde{\Lambda}_k) - \frac{D_0\varepsilon}{4\ell}.$$

In this case, we get the conclusion (1) of the lemma.

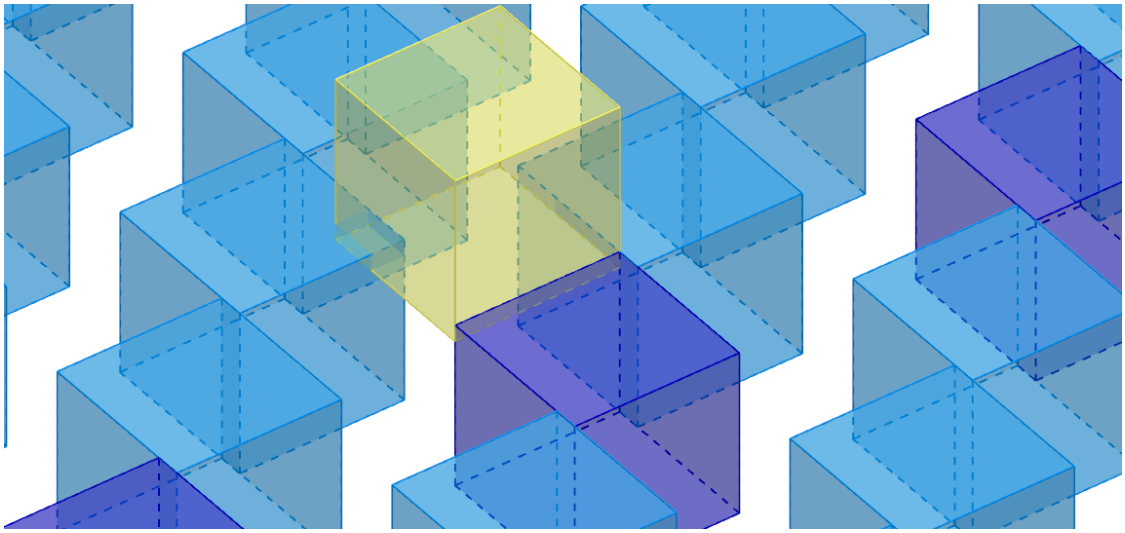

Figure 8.21: Intersection of cubes in the case where the rate is bigger than 1/3. The thickening of the cubes of $W^k + \widetilde{\Lambda}_k$ is represented in dark blue and the thickening of the rest of the cubes of $W^{k+1} + \widetilde{\Lambda}_{k+1}$ is represented in light blue; we have also represented another cube of $W^{k+2} + \widetilde{\Lambda}_{k+2}$ in yellow. We see that if the projection on the z-axis of the centre of the yellow cube is smaller than 1, then there is automatically an intersection between this cube and one of the blue cubes.

(2) Or for more than the half of such x, we are in the case (ii). Choosing the couple $(w_i, w_{i'})$ such that the measure of the set of corresponding x is the greatest, we get

$$D_c\Big(\Big(W^{k+1} + \widetilde{\Lambda}_{k+1} + (0^{kn}, w_i)\Big) \cap \Big(W^{k+1} + \widetilde{\Lambda}_{k+1} + (0^{kn}, w_{i'})\Big)\Big) \geq \frac{D_0 \varepsilon}{(\ell-1)(\ell-2)}.$$

In this case, we get the conclusion (2) of the lemma.

□

We can now prove Theorem 8.24.

Proof of Theorem 8.24. As in the previous proofs of such results, we proceed by induction on k. Suppose that $\widetilde{\Lambda}_k$ is such that $D_c(W^k + \widetilde{\Lambda}_k) > 1/\ell$. Then, Lemma 8.51 ensures that it is not possible to have ℓ disjoint translates of $W^k + \widetilde{\Lambda}_k$. Applying Lemma 8.52, we obtain that either $D_c(W^{k+1} + \widetilde{\Lambda}_{k+1}) < D_c(W^k + \widetilde{\Lambda}_k)$, or it is not possible to have $\ell - 1$ disjoint translates of $W^{k+1} + \widetilde{\Lambda}_{k+1}$. And so on, applying Lemma 8.52 at most $\ell - 1$ times, there exists $k' \in [\![k+1, k+\ell-1]\!]$ such that $W^{k'} + \widetilde{\Lambda}_{k'}$ has additional auto-intersections. Quantitatively, combining Lemmas 8.51 and 8.52, we get

$$D_c\Big(W^{k+\ell-1} + \widetilde{\Lambda}_{k+\ell-1}\Big) \leq D\Big(W^k + \widetilde{\Lambda}_k\Big) - \frac{\varepsilon}{4\ell}\left(\frac{\varepsilon}{\ell^2}\right)^{\ell-1} 2\frac{\ell D_c(W^k + \widetilde{\Lambda}_k) - 1}{\ell(\ell-1)},$$

thus

$$D_c\Big(W^{k+\ell-1} + \widetilde{\Lambda}_{k+\ell-1}\Big) - 1/\ell \leq \left(1 - \frac{1}{2}\left(\frac{\varepsilon}{\ell^2}\right)^{\ell}\right)\Big(D_c\Big(W^k + \widetilde{\Lambda}_k\Big) - 1/\ell\Big),$$

in other words, if we denote $\overline{\tau}^k = \overline{\tau}^k(B_1, \cdots, B_k)$ and $\lambda_\ell = 1 - \left(\frac{\varepsilon}{\ell^2}\right)^{\ell}$,

$$\overline{\tau}^{k+\ell-1} - 1/\ell \leq \lambda_\ell\Big(\overline{\tau}^k - 1/\ell\Big). \tag{8.16}$$

This implies that for every $\ell > 0$, the sequence of rates $\overline{\tau}^k$ is smaller than a sequence converging exponentially fast to $1/\ell$: we get Equation (8.9). In particular, the asymptotic rate of injectivity is generically equal to zero.

We now prove the estimation of Equation (8.10). Suppose that $\overline{\tau}^k \in [1/(\ell-2), 1/(\ell-1)]$, we compute how long it takes for the rate to be smaller than $1/(\ell-1)$. We apply Equation (8.16) j times to ℓ and get

$$\overline{\tau}^{k+j\ell} - 1/\ell \leq \lambda_\ell^j(\overline{\tau}^k - 1/\ell),$$

with $\lambda = 1 - \left(\frac{\varepsilon}{\ell^2}\right)^\ell$. In the worst case, we have $\overline{\tau}^k = 1/(\ell-2)$, thus if j' satisfies

$$\frac{1}{\ell-1} - \frac{1}{\ell} = \lambda_\ell^{j'}\left(\frac{1}{\ell-2} - \frac{1}{\ell}\right), \tag{8.17}$$

then $j = \lceil j' \rceil$ is such that $\overline{\tau}^{k+j\ell} \leq 1/(\ell-1)$. Equation (8.17) is equivalent to

$$j' = \frac{-1}{\log \lambda_\ell}\left(\log 2 - \log\left(1 - \frac{1}{\ell-1}\right)\right).$$

And for ℓ very big, we have the equivalent (recall that $\lambda_\ell = 1 - \frac{1}{2}\left(\frac{\varepsilon}{\ell^2}\right)^\ell$)

$$j' \sim \left(\frac{\ell^2}{\varepsilon}\right)^\ell 2\log 2.$$

Thus, when ℓ is large enough, the time it takes for the rate to decrease from $1/(\ell-2)$ to $1/(\ell-1)$ is smaller than $\ell^{2(\ell+1)} = e^{2(\ell+1)\log \ell}$.

On the other hand, if we set $f(k) = (\log k)^{-\nu}$, the time it takes for f to go from $1/(\ell-2)$ to $1/(\ell-1)$ is equal to

$$e^{(\ell-1)^{1/\nu}} - e^{(\ell-2)^{1/\nu}} = e^{(\ell-1)^{1/\nu}}\left(1 - e^{(\ell-2)^{1/\nu} - (\ell-1)^{1/\nu}}\right) \sim e^{(\ell-1)^{1/\nu}}$$

when l goes to infinity, thus smaller than $e^{(\ell-1)^{1/\nu}+1}$ when ℓ is large enough. But we have $2(\ell+1)\log \ell = o\left((\ell-1)^{1/\nu}+1\right)$. So, when ℓ is large enough, it takes arbitrarily much more time for $\overline{\tau}^k$ to decrease from $1/(\ell-2)$ to $1/(\ell-1)$ than for f to decrease from $1/(\ell-2)$ to $1/(\ell-1)$. As a consequence, $\overline{\tau}(k) = o(f(k))$. □

In the next part, we will need another technical statement, whose proof is similar to that of Theorem 8.24.

Lemma 8.54. *There exists an open dense set $\mathscr{O}$ of the set $\ell^\infty(\mathrm{SL}_n(\mathbf{R}))$ of infinite sequences of matrices of $\mathrm{SL}_n(\mathbf{R})$ such that for every $\mathrm{R}_0 > 0$, there exists $k_0 \in \mathbf{N}$ such that for every $(\mathrm{A}_k)_{k\geq 1} \in \mathscr{O}$, there exists a sequence $(w_k)_{k\geq 1}$ of translation vectors belonging to $[-1/2, 1/2]^n$, and a vector $\widetilde{y}_0 \in \mathbf{Z}^n$, with norm bigger than R_0, such that (where $\pi(\mathrm{A}+w)$ denotes the discretization of the affine map $\mathrm{A}+w$)*

$$\left(\pi(\mathrm{A}_{k_0}+w_{k_0}) \circ \cdots \circ \pi(\mathrm{A}_1+w_1)\right)(\widetilde{y}_0) = \left(\widehat{\mathrm{A}_{k_0}+w_{k_0}} \circ \cdots \circ \widehat{\mathrm{A}_1+w_1}\right)(0) = 0.$$

Moreover, the point $\widetilde{y}_0$ being fixed, this property can be supposed to remain true on a whole neighbourhood of the sequence $(\mathrm{A}_k)_{k\geq 1} \in \mathscr{O}$.

Proof of Lemma 8.54. In fact, we will prove the following statement.

There exists an open dense set $\mathscr{O}$ of the set $\ell^\infty(\mathrm{SL}_n(\mathbf{R}))$ of infinite sequences of matrices of $\mathrm{SL}_n(\mathbf{R})$ such that for every $m \in \mathbf{N}$, there exists $k_0 \in \mathbf{N}$ such that for every $(\mathrm{A}_k)_{k\geq 1} \in \mathscr{O}$, there exists a vector $y_0 \in \mathbf{Z}^n$ such that

$$\mathrm{Card}\left(\widehat{\mathrm{A}_{k_0}} \circ \cdots \circ \widehat{\mathrm{A}_1}\right)^{-1}(y_0) \geq m.$$

The lemma is deduced from this statement by remarking that on the one hand, for every $\mathrm{R}_0 > 0$, there exists $m \in \mathbf{N}$ such that every subset of $\mathbf{Z}^n$ with cardinality bigger than m contains at least one point with norm bigger than R_0, and that on the other hand, if we choose $w_k \in [-1/2, 1/2]^n$ such that

$$w_k = \mathrm{A}_k^{-1}\left(\left(\widehat{\mathrm{A}_{k_0}} \circ \cdots \circ \widehat{\mathrm{A}_{k-1}}\right)^{-1}(y_0)\right) \mod \mathbf{Z}^n,$$

then the properties of the cardinality of the inverse image of y_0 are transferred to the point 0.

To prove the statement of the beginning of the proof, we remark that if a loss of injectivity created by the proof of Theorem 8.24 occurs at a time k, then the intersections of cubes that are created are of the form $\left(\mathrm{W}^k + v_1\right) \cap \left(\mathrm{W}^k + v_1 + v_2\right)$, with the vector v_2 having nonzero values only in the $2n$ lasts coordinates. Thus, if the proof of the theorem creates m intersections of cubes (in times differing of at least 2), there is a point which belongs simultaneously to at least m cubes. To a point which belongs to this intersection will correspond a point y_0 as desired by the statement. And the perturbation we have made to do that is arbitrarily small, moreover the property remains true on a whole neighbourhood of the perturbed sequence.

The only thing it remains to prove is that the choice of the time k_0 is independent from the choice of the sequence of matrices. In fact, we can choose $k_0 = m^2$. We have two cases. Either in time k_0, the rate is smaller than $1/m$, thus automatically the conclusion of the statement will be satisfied by at least one point y_0. Or in time k_0, the rate is bigger than $1/m$, thus applying the reasoning above m times we get the conclusion of the statement. □

Chapter 9

Statistics of roundoff errors

In this (very) short chapter, we use the formalism of model sets to study the statistics of the errors made when we compute the images of a point of $\mathbf{Z}^n$ by the discretizations of a generic sequence of linear maps. Here, the main result is that when we consider all the points of $\mathbf{Z}^n$, the roundoff errors made at each iteration are equidistributed in $[-1/2, 1/2]^n$ (Proposition 9.1). This result had already been obtained by P.P. Flockermann in [Flo02, Theorem 10 page 44] in dimension 1 (see Proposition 9.2), with quite different techniques. From this result, we deduce the statistics of the cumulative errors made after k iterations. In particular, in dimension $n = 1$, it allows us to compute the *discrepancy* (see Definition 9.3) between the appropriate uniform measure on $\mathbf{R}$ and the image sets (Proposition 9.4).

Ultimately, we hope that these notions can be used to tackle a conjecture of O.E. Lanford (Conjecture 12.13) concerning the physical measures of expanding maps of the circle.

9.1 Roundoff errors

Given $x \in \mathbf{Z}^n$ and a sequence $(\mathrm{A}_m)_{m\geq 1}$ of invertible matrices of $\mathbf{R}^n$, we want to compute the sequence $(\varepsilon_x^m)_m$ of roundoff errors made at each iteration. They are defined by

$$\varepsilon_x^m = \left(\widehat{\mathrm{A}}_m - \mathrm{A}_m\right)\left((\widehat{\mathrm{A}}_{m-1} \circ \cdots \circ \widehat{\mathrm{A}}_1)(x)\right) \in [-1/2, 1/2]^n.$$

We also set

$$v_x^m = (\widehat{\mathrm{A}}_m \circ \widehat{\mathrm{A}}_{m-1} \circ \cdots \circ \widehat{\mathrm{A}}_1)(x) \in \mathbf{Z}^n.$$

We fix $k \geq 0$ and set $v_x = \left(v_x^1, \cdots, v_x^k\right)$, $\varepsilon_x = \left(\varepsilon_x^1, \cdots, \varepsilon_x^k\right)$, $u_x = (\mathrm{A}_1 x, 0^{(n-1)k}) \in \mathbf{R}^{nk}$ and

$$\mathrm{N}_{\mathrm{A}_1,\cdots,\mathrm{A}_k} = \begin{pmatrix} -1 & & & & \\ \mathrm{A}_2 & -1 & & & \\ & \mathrm{A}_3 & \ddots & & \\ & & \ddots & -1 & \\ & & & \mathrm{A}_k & -1 \end{pmatrix} \in \mathrm{M}_{nk}(\mathbf{R}).$$

As

$$\mathrm{N}_{\mathrm{A}_1,\cdots,\mathrm{A}_k} v_x = \begin{pmatrix} -v_x^1 \\ \mathrm{A}_2 v_x^1 - v_x^2 \\ \mathrm{A}_3 v_x^2 - v_x^3 \\ \vdots \\ \mathrm{A}_k v_x^{k-1} - v_x^k \end{pmatrix},$$

the vector u_x can be decomposed into

$$u_x = \mathrm{N}_{\mathrm{A}_1,\cdots,\mathrm{A}_k} v_x - \varepsilon_x,$$

with $v_x \in \mathbf{Z}^{nk}$ and $\varepsilon_x \in \mathrm{W}^k$ (recall that $\mathrm{W}^k =]-1/2, 1/2]^{nk}$). The vector u_x being fixed, this condition characterizes completely ε_x and v_x, as W^k is a fundamental domain of $\mathrm{N}_{\mathrm{A}_1,\cdots,\mathrm{A}_k}\mathbf{Z}^{nk}$ (remark that we are in the case of the conclusion of Hajós theorem, see Theorem 8.21). Thus, ε_x is equal to the projection of u_x on W^k modulo $\mathrm{N}_{\mathrm{A}_1,\cdots,\mathrm{A}_k}\mathbf{Z}^{nk}$; equivalently, $\mathrm{N}^{-1}_{\mathrm{A}_1,\cdots,\mathrm{A}_k}\varepsilon_x$ is equal to the projection of $\mathrm{N}^{-1}_{\mathrm{A}_1,\cdots,\mathrm{A}_k} u_x$ on $\mathrm{N}^{-1}_{\mathrm{A}_1,\cdots,\mathrm{A}_k}\mathrm{W}^k$ modulo $\mathbf{Z}^{nk}$.

This implies that the sequences of errors ε_x are equidistributed in $(\mathbf{R}^n/\mathbf{Z}^n)^k$ when x ranges over $\mathbf{Z}^n$ if and only if the vectors $\left(\mathrm{N}^{-1}_{\mathrm{A}_1,\cdots,\mathrm{A}_k} u_x\right)_{x\in\mathbf{Z}^n}$ are equidistributed modulo $\mathbf{Z}^{nk}$. For this purpose, the matrix $\mathrm{N}^{-1}_{\mathrm{A}_1,\cdots,\mathrm{A}_k}$ can be easily computed:

$$\mathrm{N}^{-1}_{\mathrm{A}_1,\cdots,\mathrm{A}_k} = \begin{pmatrix} -1 & & & & \\ -\mathrm{A}_2 & -1 & & & \\ -\mathrm{A}_3\mathrm{A}_2 & -\mathrm{A}_3 & \ddots & & \\ \vdots & \vdots & \ddots & -1 & \\ -\mathrm{A}_k\cdots\mathrm{A}_2 & -\mathrm{A}_k\cdots\mathrm{A}_3 & \cdots & -\mathrm{A}_k & -1 \end{pmatrix},$$

thus

$$\mathrm{N}^{-1}_{\mathrm{A}_1,\cdots,\mathrm{A}_k} u_x = -\begin{pmatrix} \mathrm{A}_1 \\ \mathrm{A}_2\mathrm{A}_1 \\ \mathrm{A}_3\mathrm{A}_2\mathrm{A}_1 \\ \vdots \\ \mathrm{A}_k\cdots\mathrm{A}_1 \end{pmatrix} x. \tag{9.1}$$

As a consequence, the sequences of errors ε_x are equidistributed in $(\mathbf{R}^n/\mathbf{Z}^n)^k$ if and only if the action of $\mathbf{Z}^n$ on $(\mathbf{R}^n/\mathbf{Z}^n)^k$ given by Equation (9.1) is ergodic. This leads to the following proposition.

Proposition 9.1. *For a generic sequence $(\mathrm{A}_k)_{k\geq 1}$ of matrices of $\mathrm{GL}_n(\mathbf{R})$, or $\mathrm{SL}_n(\mathbf{R})$, or $\mathrm{O}_n(\mathbf{R})$, for every fixed integer k, the finite sequence of errors $\varepsilon_x = \left(\varepsilon_x^1, \cdots, \varepsilon_x^k\right)$ is equidistributed in $(\mathbf{R}^n/\mathbf{Z}^n)^k$ when x ranges over $\mathbf{Z}^n$.*

In particular, this proposition implies that the errors ε^k are globally independent.

If we take $n = 1$, we obtain an alternative proof of the following statement, which was already observed in the thesis [Flo02] of P.P. Flockermann (Theorem 10 page 44).

Proposition 9.2. *If $n = 1$, we denote $\mathrm{A}_m = \lambda_m$ and for $m \leq k$, set $\overline{\lambda}^k_m = \lambda_k\lambda_{k-1}\cdots\lambda_m$, with the convention $\overline{\lambda}^k_{k+1} = 1$. If the coefficients $(\overline{\lambda}^k_m)_{1\leq m\leq k+1}$ are linearly independent over $\mathbf{Q}$, then for every fixed integer k, the sequence $\left(\varepsilon_x^1, \cdots, \varepsilon_x^k\right)$ is equidistributed in $\mathbf{R}^k/\mathbf{Z}^k$ when x ranges over $\mathbf{Z}$.*

9.2 Cumulative errors and discrepancy

From the previous study, it is possible to deduce the statistics of the global error

$$\mathscr{E}_x^k = \big(\widehat{\mathrm{A}}_k \circ \widehat{\mathrm{A}}_{k-1} \circ \cdots \circ \widehat{\mathrm{A}}_1\big)(x) - \big(\mathrm{A}_k \circ \mathrm{A}_{k-1} \circ \cdots \circ \mathrm{A}_1\big)(x)$$

made after k iterations. Indeed, we have

$$\mathscr{E}_x^{k+1} = \mathrm{A}_{k+1}\mathscr{E}_x^k + \varepsilon_x^{k+1}.$$

From this recurrence relation, we deduce that

$$\mathscr{E}_x^k = \sum_{m=1}^{k} \mathrm{B}_m \varepsilon_x^m,$$

where $\mathrm{B}_m = \mathrm{A}_k\mathrm{A}_{k-1}\cdots\mathrm{A}_{i+1}$. As the ε_x^m are independent and equidistributed, this gives the law of the global error $\mathscr{E}_k$. In particular, the covariance of $\mathscr{E}_k$ is equal to

$$\mathrm{Var}(\mathscr{E}_x^k) = \frac{1}{12}\sum_{m=1}^{k} \mathrm{B}_m\mathrm{B}_m^\top,$$

where $\mathrm{B}^\top$ denotes the transpose of the matrix B.

We assume $n = 1$, and that the linear maps A_i are expanding. In this case, if we denote $\mathrm{A}_i = \lambda_i \geq 1$, the variance of $\mathscr{E}_x^k$ is equal to

$$\mathrm{Var}(\mathscr{E}_x^k) = \frac{1}{12}\sum_{m=1}^{k} (\lambda_k\lambda_{k-1}\cdots\lambda_m)^2. \tag{9.2}$$

In particular, if there exists $\alpha > 1$ such that $\lambda_k \geq \alpha$ for every k, then

$$\mathrm{Var}(\mathscr{E}_x^k) \geq \frac{1}{12}\sum_{m=1}^{k} \alpha^{2m} = \frac{\alpha^2(\alpha^{2k}-1)}{12(\alpha-1)}.$$

These considerations allow us to compute the discrepancy of the image set $(\widehat{\mathrm{A}}_k \circ \cdots \circ \widehat{\mathrm{A}}_1)(x)$.

Definition 9.3. Let $\mathrm{E} \subset \mathbf{Z}$ and μ a Borel measure on $\mathbf{R}$. We call *discrepancy of* E with respect to the measure μ the quantity (when it is well defined)

$$\mathrm{Disc}(\mathrm{E}, \mu) = \lim_{\mathrm{R}\to+\infty} \left(\frac{1}{2\mathrm{R}} \int_{-\mathrm{R}}^{\mathrm{R}} \Big(\mathrm{Card}([-x, x] \cap \mathrm{E}) - \mu([-x, x]) \Big)^2 \, \mathrm{d}\, x \right)^{1/2}.$$

For a complete investigation of the subject of geometric discrepancy and an extensive bibliography, see the excellent book [Mat10] of J. Matoušek.

In dimension 1, the discrepancy of the set $(\widehat{\mathrm{A}}_k \circ \cdots \circ \widehat{\mathrm{A}}_1)(\mathbf{Z})$ can be easily computed: the following proposition is a direct consequence of Equation (9.2).

Proposition 9.4. *The discrepancy of the set* $(\widehat{\mathrm{A}}_k \circ \cdots \circ \widehat{\mathrm{A}}_1)(\mathbf{Z})$ *is equal to the standard deviation of* $(\mathscr{E}_x^k)_x$ *:*

$$\mathrm{Disc}\Big((\widehat{\mathrm{A}}_k \circ \cdots \circ \widehat{\mathrm{A}}_1)(\mathbf{Z}), \det(\mathrm{A}_k^{-1}\cdots\mathrm{A}_1^{-1})\,\mathrm{Leb}\Big) = \mathrm{Var}(\mathscr{E}_x^k)^{1/2} = \left(\frac{1}{12}\sum_{m=1}^{k} (\lambda_k\lambda_{k-1}\cdots\lambda_m)^2\right)^{1/2}.$$

In particular, if there exists $\alpha > 1$ such that $\lambda_k \geq \alpha$ for every k, then

$$\mathrm{Disc}\Big((\widehat{\mathrm{A}}_k \circ \cdots \circ \widehat{\mathrm{A}}_1)(\mathbf{Z}), \det(\mathrm{A}_k^{-1} \cdots \mathrm{A}_1^{-1})\,\mathrm{Leb}\Big) \geq \left(\frac{\alpha^2(\alpha^{2k}-1)}{12(\alpha-1)}\right)^{1/2} \underset{k\to+\infty}{\sim} \frac{\alpha^{k+1}}{\sqrt{12(\alpha-1)}}.$$

As has already been said, we hope that these kind of considerations about the statistics of the deviation of the uniform measure on the image set with respect to Lebesgue measure can help to understand the behaviour of the image measures $(f_{\mathrm{N}}^*)^k(\lambda_{\mathrm{N}})$ for $k \gg \log \mathrm{N}$ (but not too big either). In particular, we would like to use discrepancy to improve the results of Section 12.2 and possibly explain the behaviour observed in Figure 12.18.

Part 3

Discretization of maps in higher regularity

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Introduction

In this third part of the manuscript, we will study the dynamics of the discretizations of generic C^r-maps, namely both conservative and dissipative diffeomorphisms, and expanding maps. We conducted this study for several reasons.

First of all, it is commonly accepted that the generic C^1 dynamics represents some physical systems better than the C^0 generic dynamics: a lot of these concrete systems are smooth, and the generic C^0 dynamics contains "wild" behaviours (for example, for both generic conservative and dissipative homeomorphisms, whether the set of periodic points of a given period is empty, whether it forms a Cantor set, thus it is uncountable). Moreover, in the C^1 generic case, a wider class of behaviours can occur, as there are "generic dichotomy theorems" (see for example [ACW14]).

Furthermore, some aspects of the behaviour of the discretizations of generic conservative homeomorphisms are a bit disappointing. The combinatorics of the discretizations varies a lot depending on the order of discretization, and does not tell anything about the actual dynamics of the continuous system. In addition, nothing in the results we have proved in the first part gives a method to to detect the "physical" behaviour of the homeomorphism on the discretizations. We would like to know if in the C^1 case, it is possible to obtain a method to recover the physical measures from some dynamical features of the discretizations.

In this part, we will consider a compact boundaryless manifold X of dimension $n \geq 2$ (we will also study the specific case of the circle), equipped with a measure λ derived froma volume form. We will use the following notations.

Notation 9.5. We denote by $\mathrm{Diff}^1(\mathrm{X})$ the set of C^1-diffeomorphisms of X, endowed by the metric d_{C^1} defined by:

$$d_{C^1}(f,g) = \sup_{x\in \mathrm{X}} d\Big(f(x), g(x)\Big) + \sup_{x\in \mathrm{X}} \Big\|\mathrm{D}f_x - \mathrm{D}g_x\Big\|.$$

We denote by $\mathrm{Diff}^1(\mathrm{X},\lambda)$ the subset of $\mathrm{Diff}^1(\mathrm{X})$ made of the diffeomorphisms that preserve the measure λ, endowed with the same metric d_{C^1}.

The metric d_{C^1} on $\mathrm{Diff}^1(\mathrm{X})$ and $\mathrm{Diff}^1(\mathrm{X},\lambda)$ makes them Baire spaces. In this introduction, we will state all the results for the phase space $\mathrm{X} = \mathbf{T}^n$, the measure $\lambda = \mathrm{Leb}$ and the uniform grids

$$\mathrm{E_N} = \left\{ \left(\frac{i_1}{\mathrm{N}}, \cdots, \frac{i_n}{\mathrm{N}} \right) \in \mathbf{T}^n \,\middle|\, \forall j,\, 0 \leq i_j \leq \mathrm{N}-1 \right\},$$

However, these results are true in a more general setting (see the concerned chapters for precisions on these settings).

The first chapter of this part will be devoted to the direct application of classical perturbation lemmas for C^1-diffeomorphisms.

These perturbations lemmas are in general much more difficult to obtain than in the case of homeomorphisms (proposition of finite maps extension, see Proposition 3.3): when we perturb a diffeomorphism, we have to take care of the norm of the differentials of the perturbation. This is a crucial difference. Indeed, in the C^0 case, if we want to perturb the image of a point of ε, we only have to make a perturbation on a ball of size ε. If we want to do the same in topology C^1, the support of the perturbation has to contain a ball whose size is way bigger than ε. Thus, for homeomorphisms, and for N large enough, we are able to move independently each point of a grid E_N, while for C^1 diffeomorphisms we are only authorized to move a small proportion [1] of the points of E_N.

Thus, if the C^1 generic dynamics is mainly considered as more interesting than the C^0 dynamics, the price to pay is that our results are in general weaker: they only concern a small proportion of the points of the grids. Note that the case of the regularity C^1 is somehow the "limit case": to our knowledge, there is no general statement of interesting perturbation result in regularity C^r, with $r > 1$.

As a consequence, the results we obtain in this chapter — by applying directly classical perturbation lemmas — only concern a sub-dynamics of the discretizations. More precisely, in Part 1, we were able to determine the dynamics of *all* the points of the discretizations of generic conservative homeomorphisms. Here, we will only control the dynamics of a few points of the grid. For example, even if the dynamical property we are interested in concern a ε-dense subset of X, it may happen that the mesh of the grids which satisfies the conclusions of the theorems is very small compared to ε; in this case we only know the dynamics (for the discretizations) of a small proportion of points of these grids.

The results of this first chapter of Part 3 will be based on the following statement (Corollary 10.2).

Proposition S. *Let $f \in \mathrm{Diff}^1(\mathbf{T}^n)$ (or $f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$) be a generic diffeomorphism. Then, for every $\tau \in \mathbf{N}$, every $N_0 \in \mathbf{N}$ and every $\varepsilon > 0$, there exists $N \geq N_0$ such that every periodic orbit of f of period smaller than τ is ε-shadowed by a periodic orbit of f_N with the same period.*

This proposition states that every periodic orbit of a generic diffeomorphism is detected by an infinite number of discretizations. It will also allow us to deduce a lot of results about the dynamics of the discretizations of generic diffeomorphisms by applying classical perturbation lemmas. For example, applying the connecting lemma of C. Bonatti and S. Crovisier, [BC04], we get the following result (Corollary 10.4).

Proposition T. *Let f be a generic diffeomorphism of $\mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$. Then, for any $\varepsilon > 0$ and any $N_0 > 0$, there exists $N \geq N_0$ such that f_N has a periodic orbit which is ε-dense.*

The statement is still true in the dissipative generic case if we restrict to a maximal invariant chain-transitive set.

This statement can be seen as a weak version of Corollary 5.9, which states that for a generic conservative homeomorphism, there exists an infinite number of discretizations

1. This proportion is asymptotically independent from the order N of the discretization, and depends on the size of the C^1 perturbation.

which are cyclic permutations of the grids. We can also apply an ergodic closing lemma (due to R. Mañé [Mañ82] and F. Abdenur, C. Bonatti and S. Crovisier [ABC11]); we then get the following result for discretizations (Corollary 10.9).

Proposition U. *Let f be a generic diffeomorphism of* $\mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$. *Then, for any f-invariant measure μ, any $\varepsilon > 0$ and any $N_0 > 0$, there exists $N \geq N_0$ such that f_N supports a periodic measure which is ε-close to μ.*

The statement is still true in the dissipative generic case for every measure μ supported by a maximal invariant chain-transitive set.

Again, this result can be seen as a weak version of Theorem 5.51 for homeomorphisms.

Finally, we can say that all these results of Chapter 10 go in the direction of the following heuristic.

For a generic diffeomorphism $f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$, each "sub-dynamics" of f (periodic orbits, chain-transitive invariant compact sets, invariant measure, rotation set...) can be detected by some discretizations f_N. However, we have no control over the orders of discretization N for which it is true, and no control over the global dynamics of f_N. In the dissipative case, the same holds on each chain-recurrent class of f.

Remark that for diffeomorphisms, the behaviours in the dissipative case are quite close to those in the conservative case, which was not true for generic homeomorphisms. Indeed, for a generic dissipative homeomorphism, the chain recurrent classes are totally disconnected, while for generic dissipative diffeomorphisms they can contain nonempty open sets (for example, there are open sets of transitive Anosov diffeomorphisms). On each of these chain recurrent classes, the dynamics generically resembles to that of a conservative diffeomorphism.

Recall that the class of C^1-diffeomorphisms bears open sets of Anosov maps. One of the important properties of such systems is that they satisfy the shadowing lemma: if f is Anosov, then for every $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo-orbit of f is ε-shadowed by a true orbit of f. As orbits of discretizations are in particular pseudo-orbits, orbits of discretizations of Anosov maps are shadowed by real orbits of the initial map. However, we do not control the dynamics of the shadowing orbit: for example, if the Anosov map is ergodic, we can not impose to the shadowing orbit to be typical with respect to Lebesgue measure. This phenomenon could seem anecdotic, but its consequences are in fact quite bad. Recall, for example, the behaviour of the discretizations of the linear automorphisms of the torus on the canonical grids (see Figure 1.1): in this case, the discretizations are permutations with a very small global period; thus the dynamics of the discretizations does not reflect the mixing properties of the initial dynamics. Thus, we can say that the shadowing lemma does not imply that the dynamics of f_N looks like that of f.

In a certain sense, the statements we prove in this first chapter express that for a generic diffeomorphism f, any dynamical behaviour of f is shadowed by a similar dynamical behaviour of the discretizations of f. More precisely, by a sub-dynamics of the discretizations; the dynamics of the rest of the grid is not totally random, as by the shadowing lemma it is close to some dynamics of the initial system. The bad news is that in the theorems we prove, we have no explicit control on the orders of discretization for which we detect this or that dynamical feature of f, whereas in the shadowing lemma the dependence of the parameter δ to ε is explicit.

The second chapter of this part is devoted to the study of the degree of recurrence of a generic C^1-diffeomorphism, in both conservative and dissipative cases, and of a generic expanding map.

Indeed, in Chapter 10, we do not say anything about the global dynamics of the discretizations; we obtain results about sub-dynamics of the discretizations. In Chapter 11, we will study the simplest global combinatorial quantity for the discretizations: the degree of recurrence. We say that this invariant is the simplest to study because it is obtained as a decreasing limit in time. Thus, the study of the degree of recurrence reduces to that of finite time quantities. This study will also be the occasion to observe that the local behaviour of the discretizations in small time is governed by the behaviour of the differentials of the diffeomorphism; in particular we will use crucially the study of the discretizations of linear maps we have conducted in Part 2 of this manuscript.

Recall that the degree of recurrence $D(f_N)$ of a homeomorphism f is defined as the ratio between the cardinality of the recurrent set of the discretization f_N and that of the grid E_N (Definition 11.1). This degree of recurrence somehow represents the amount of information we lose when we iterate the discretization. The study of the degree of recurrence of a generic diffeomorphism easily reduces to that of the rates of injectivity (see Definition 11.2)

$$\tau^t(f) = \overline{\lim_{N\to+\infty}} \frac{\mathrm{Card}\left((f_N)^t(E_N)\right)}{\mathrm{Card}(E_N)}.$$

We first show a local-global formula for this rate of injectivity: the rate of injectivity of a generic diffeomorphism is linked with the rates of injectivity of its differentials. To do that, we define the discretization of a linear map $A : \mathbf{R}^n \to \mathbf{R}^n$ as the map $\widehat{A} = \pi \circ A : \mathbf{Z}^n \to \mathbf{Z}^n$, where $\pi : \mathbf{R}^n \to \mathbf{Z}^n$ is a projection on (one of) the nearest integer point for the euclidean distance (see Definition 7.11). Then, the rate of injectivity of a sequence $(A_k)_{k\geq 0}$ of matrices is defined as (see Definition 7.19)

$$\tau^k(A_1,\cdots,A_k) = \lim_{R\to+\infty} \frac{\mathrm{Card}\left((\widehat{A_k}\circ\cdots\circ\widehat{A_1})[B_R]\right)}{\mathrm{Card}[B_R]} \in]0,1].$$

The local-global formula is the following (Theorem 11.8, see also Theorem 11.26).

Theorem V. *Let $r \geq 1$ and $f \in \mathrm{Diff}^r(\mathbf{T}^n)$ (or $f \in \mathrm{Diff}^r(\mathbf{T}^n, \mathrm{Leb})$) be a generic diffeomorphism. Then $\tau^t(f)$ is well defined (that is, the limit exists) and satisfies:*

$$\tau^t(f) = \int_{\mathbf{T}^n} \tau^t\left(Df_x,\cdots,Df_{f^{t-1}(x)}\right) \mathrm{d}\,\mathrm{Leb}(x).$$

Moreover, the function τ^t is continuous in f.

Remark that the hypothesis of genericity is necessary to get this theorem (see Example 11.4). The proof of this result involves the local linearization of a diffeomorphism (Lemma 11.5), which is a difficult result in the conservative case (it uses the smoothing of a conservative C^1-diffeomorphism of A. Avila, see [Avi10]). It also uses crucially the study of the continuity of the rate we have conducted in Part 2 of this manuscript.

In the last section of Chapter 11, we give a variation of Theorem V for expanding maps. This statement links the rate of injectivity of a generic C^r-expanding map with the probability of percolation $\overline{D}$ of a random graph associated to the derivatives of this map (Theorem 11.19).

Theorem W. *For every $r \geq 1$, and every generic C^r-expanding map f on $\mathbf{S}^1$, the rate of injectivity $\tau^k(f)$ satisfies*

$$\tau^k(f) = \int_{\mathbf{T}^n} \overline{\mathrm{D}}\Big((\det \mathrm{D}f_x^{-1})_{\substack{1\leq m\leq k \\ x\in f^{-m}(y)}}\Big)\, \mathrm{d}\,\mathrm{Leb}(y)$$

(see Definition 11.18 for the definition of $\overline{\mathrm{D}}$).

The techniques developed to prove this statement also lead to the proof of Theorem V in the C^r regularity, for every $r \geq 1$. Notice that even if theorem V is true for generic C^r-diffeomorphisms for every $r \geq 1$, it is not sufficient to get that the degree of recurrence is generically equal to 0 in the C^r topology: to prove Theorem X, we need to perturb the derivative of a diffeomorphism on a big subset of $\mathbf{T}^n$. However, there is a hope that in the case of expanding maps, Theorem W leads to the fact that the degree of recurrence is generically equal to 0 in any topology of C^r convergence.

The behaviour described by Theorem V is very different from the C^0 case. For a generic dissipative homeomorphism f, we have $\tau^1(f) = 0$ (Corollary 4.14), while for a generic conservative homeomorphism f, for every $t > 0$, the quantity $\mathrm{Card}\Big((f_\mathrm{N})^t(\mathrm{E}_\mathrm{N})\Big)/\mathrm{Card}(\mathrm{E}_\mathrm{N})$ accumulates on both 0 and 1 when N goes to $+\infty$ (Corollaries 5.9 and 5.20). The behaviour for generic diffeomorphisms does not depend on the assumption of preservation of a measure or not, and is in a certain sense smoother.

In short, the rate of injectivity of a diffeomorphism can be computed by integrating the rates of injectivity of its differentials. This allows us to use the study of the rate of injectivity of linear maps we have conducted in the second part of this thesis, and in particular Theorem 8.24. Applying classical techniques of C^1 dynamics (in particular Rokhlin tower lemma), the application of this theorem is quite straightforward and leads to the following theorem.

Theorem X. *For a generic conservative diffeomorphism $f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$, we have*

$$\lim_{t\to\infty} \tau^t(f) = 0.$$

This implies in particular that $\lim_{\mathrm{N}\to+\infty} \mathrm{D}(f_\mathrm{N}) = 0$.

This theorem must be compared with the case of homeomorphisms (see Corollary 5.24): for a generic conservative homeomorphism f, the sequence $\mathrm{D}(f_\mathrm{N})$ accumulates on the whole segment $[0,1]$. Thus, in the case of diffeomorphisms, the behaviour of the rate of injectivity is less irregular than for homeomorphisms; despite this, the theorem shows that when we iterate the discretizations of a generic conservative diffeomorphism, we lose a great amount of information. Moreover, although f is conservative, its discretizations tend to behave like dissipative maps. This can be compared with the work of P. Lax [Lax71]: for any conservative homeomorphism f, there is a bijective finite map arbitrarily close to f (see Theorem 5.5). The previous theorem states that for a generic conservative C^1 diffeomorphism, the discretizations never possess this property.

To summarize, for a generic conservative C^1-diffeomorphism, the recurrent sets $\Omega(f_\mathrm{N})$ are such that their limit superior is $\mathbf{T}^n$ (see Proposition T), and their cardinality behaves as $o(\mathrm{Card}(\mathrm{E}_\mathrm{N}))$.

The same behaviour of the rate of injectivity holds in the generic dissipative case (Corollary 11.13): for a generic dissipative diffeomorphism $f \in \mathrm{Diff}^1(\mathbf{T}^n)$, we have

$\lim_{N\to+\infty} D(f_N) = 0$. This result is an easy consequence of a theorem of A. Avila and J. Bochi (Theorem 11.11, see also [AB06]).

Finally, in Chapter 12, we tackle the question of the physical behaviour of the discretizations. In section 12.1, we study the measures $\mu_x^{f_{N_k}}$ for "a lot" of points x. Recall that for every x, the orbit of x_N under f_N falls in a periodic orbit of f_N; we denote by $\mu_x^{f_{N_k}}$ the uniform measure on this periodic orbit. In the third chapter of this part, we will prove the following statement (Theorem 12.1).

Theorem Y. *For a generic diffeomorphism $f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$, for a generic point $x \in X$, and for any f-invariant probability measure μ, there exists a subsequence $(N_k)_k$ of discretizations such that*

$$\mu_x^{f_{N_k}} \underset{k\to+\infty}{\longrightarrow} \mu.$$

Again, the study of the linear case (more precisely, Lemma 8.54) plays a pivotal role in the proof of this theorem. It also uses an ergodic closing lemma adapted from [ABC11] (Lemma 12.6), and the connecting lemma for pseudo-orbits of [BC04] (see also Theorem 10.3).

Recall that in the case of homeomorphisms, we have proved that for a generic conservative homeomorphism f, and for any f-invariant measure μ, there exists a subsequence $(f_{N_k})_k$ of discretizations such that for every $x \in \mathbf{T}^n$, we have $\mu_x^{f_{N_k}} \underset{k\to+\infty}{\longrightarrow} \mu$. The result we have for diffeomorphisms — even if it is much more difficult to prove — is much weaker: it not only concerns the behaviour of a generic subset of the torus instead of all the points, but the orders of discretization for which the conclusions of the theorem are true strongly depend on the point x.

Despite this, this theorem says that in practice, we must be very careful when we want to find a physical measure. Suppose that we want to find the unique physical measure μ_0 of a generic C^1-diffeomorphism f. We could think that it is sufficient to compute the measures μ_x^f for a certain number of points x; since μ_0 is the unique physical measure, μ_x^f should be close to μ_0 for most of the points x. In practice, of course, we will consider a finite number of such points x, and we will compute with a given precision, that is we will iterate a discretization f_N. The result says that for some N, we observe a measure $\mu_x^{f_{N_k}}$ that has nothing to do with the physical measure μ_0.

Section 12.2 is devoted to the study of the measures $(f_N^*)^m \lambda_N$ for "small" times m (recall that λ_N is the uniform measure on the grid E_N), in the case where f is a $C^{1+\alpha}$ expanding map of the circle. In this setting, a classical result asserts that f has a single physical measure μ_0, which coincides with the SRB measure of f, and is also the unique invariant probability measure with a C^α density (see Theorem 12.17). Our theorem asserts that if N goes to infinity much faster than m, the measures $(f_N^*)^m \lambda_N$ converge to μ_0 (Theorem 12.15).

Theorem Z. *For every $0 < \alpha \leq 1$ and every $C^{1+\alpha}$ expanding map $f \in \mathscr{E}_d^{1+\alpha}(\mathbf{S}^1)$, there exists a constant $c_0 = c_0(f) > 0$ such that if $(N_m)_m$ is a sequence of integers going to infinity and satisfying $\log N_m > c_0 m$, then the convergence $(f_{N_m}^*)^m(\lambda_{N_m}) \to \mu_0$ holds.*

This result answers partially to a conjecture of O.E. Lanford (see Conjecture 12.13). Its proof consists in a calculus of the difference of action on measures between the Ruelle-Perron-Frobenius operator and the pushforwards by the discretization f_N^*.

In Section 12.3, we first present numerical experiments simulating the measures $\mu_x^{f_N}$ and $\mu_{T^2}^{f_N}$ for some examples of conservative C^1-diffeomorphisms of the torus. The results of these simulations are quite striking: for all the diffeomorphisms we have tested (even a diffeomorphism C^1-close to a linear Anosov map, thus C^0 conjugated to it), for some large orders of discretization N, the measures $\mu_x^{f_N}$ do not look like Lebesgue measure (see for instance Figures 12.7, 12.8 and 12.9). Thus, in practice, we have to be very careful when we compute numerically some measures $\mu_{x,T}^{f_N}$, it may happen that they do not reflect the physical behaviour of the initial diffeomorphism at all.

The end of this section is devoted to numerical simulations of invariant measures of expanding maps; in particular we observe on Figure 12.18 that as predicted by Theorem Z, the distance $\mathrm{dist}(\mu_0, (f_{N_m}^*)^m(\lambda_{N_m}))$ is very close to 0 for small values of m, but then increases with m to a significant value.

To summarize, we take stock about what we have proved to answer the question: is is possible to observe physical measures on discretizations? In practice, this question can be formalized in many ways [2].

- First, we can wonder if the measures $\mu_x^{f_N}$ tend to a physical measure of f for "many" points x. The answer is no, when by "many points x" we mean "generic points $x \in X$", for generic conservative and dissipative C^1-diffeomorphisms (Theorems 12.1 and 12.3) and generic expanding maps (Proposition 12.4). The behaviours described by these statements can even be observed in practice (see Section 12.3) However, we do not know what happens when we consider a set of points x which is typical for Lebesgue measure.
- Second, we can ask whether the measures [3] $\mu_X^{f_N}$ converge to a physical measure of f or not. We have no theoretical statement about this question, even for expanding maps of the circle. Moreover, the results of numerical simulations are not clear, as they suggest that the answer to this question could be "yes, if we consider averages among a wide range of orders N" (see Figures 12.21 and 12.22).
- As we do not understand the behaviours of the measures [4] $\mu_X^{f_N}$, as suggested by O.E. Lanford (see Conjecture 12.13), we can ask if the measures $(f_N^*)^m \lambda_N$ converge to a physical measure for both m and N going to infinity, with m not too big with respect to N. Theorem Z answers this question for some m satisfying $m = O(\log N)$ and for a C^α expanding map on the circle: in this case, the measures $(f_N^*)^m \lambda_N$ converge exponentially fast to the unique physical measure of the map. For now, we do not know what happens for bigger times m. The numerical experiments suggest a quite surprising evolution of the measures $(f_N^*)^m \lambda_N$: for a fixed order of discretization N, these measures converge very fast towards the SRB measure, and then moves away from it slowly. This phenomenon is maybe due to the fact that the numerical errors due to the discretization process are equidistributed (see Proposition 9.4; this phenomenon could be specific to the dimension 1).
- Finally, we can investigate whether the two first questions are connected or not. More precisely, we wonder if there exits a discrete Birkhoff's theorem, which

2. Here, we consider only the case of the regularity C^r, $r \geq 1$; the case of homeomorphisms has been treated in Part 1.

3. Recall that the measures $\mu_X^{f_N}$ are supported by the union of periodic orbits of f_N, such that the total measure of each periodic orbit is proportional to the cardinality of its basin of attraction.

4. Note that the measures $\mu_X^{f_N}$ are obtained by averaging the measures $(f_N^*)^m \lambda_N$ over m.

would state that if the measures $\mu_X^{f_N}$ converge to a measure μ, then the measures $\mu_x^{f_N}$ tend to μ for "a lot" of points x (in a sense to define).

Chapter 10

Applications of perturbation lemmas

In this chapter, we apply classical perturbation lemmas of C^1-diffeomorphisms to deduce statements about the dynamics of the discretizations of C^1 generic conservative and dissipative diffeomorphisms.

In this chapter, X will be a smooth compact boundaryless Riemannian manifold of dimension $n \geq 2$. It will be equipped with a measure λ, derived from a volume form (independent from the metric on X). We will be interested in the dynamics of the discretizations of generic C^1-diffeomorphisms, in both dissipative and conservative cases; *i.e.* generic elements of $\mathrm{Diff}^1(X)$ and $\mathrm{Diff}^1(X, \lambda)$.

We will also fix a sequence $(E_N)_{N\geq 0}$ of discretizations grids on X. Recall that the very definition of discretization grid (Definition 3.5) supposes that the mesh of these discrete sets tends to 0, that is: for every $\varepsilon > 0$, there exists N_0 such that for every $N \geq N_0$, the grid E_N is ε-dense in X. In this chapter, this is the only assumption we will make on the grids. Given a diffeomorphism $f \in \mathrm{Diff}(X, \lambda)$, we denote by f_N the discretization of f with respect to the grid E_N.

We will begin this chapter by stating that the periodic orbits of a generic diffeomorphism are shadowed by periodic orbits of its discretizations (Corollary 10.2). This will allow us to apply classical closing lemmas, which state that some dynamical invariants of a generic diffeomorphism are shadowed by periodic orbits.

We first apply the connecting lemma for pseudo-orbits of C. Bonatti and S. Crovisier [BC04], to prove that for a generic diffeomorphism $f \in \mathrm{Diff}^1(X, \lambda)$ and for every $\varepsilon > 0$, an infinite number of discretizations f_N have a periodic orbit which is ε-dense in X (Corollary 10.4). We also apply a closing lemma of F. Abdenur and S. Crovisier [AC12] to obtain that an infinite number of discretizations f_N are "ε-topologically mixing" (Corollary 10.7). The same results holds in the dissipative case for every chain-recurrent class.

We then apply an ergodic closing lemma of R. Mañé [Mañ82] and F. Abdenur, C. Bonatti and S. Crovisier [ABC11] to state that if $f \in \mathrm{Diff}^1(X, \lambda)$ is a generic conservative diffeomorphism, then every f-invariant measure is shadowed by periodic measures of the discretizations f_N (Corollary 10.9). The same kind of results holds for chain-transitive invariant compact sets (Corollary 10.11), by applying a connecting lemma of S. Crovisier [Cro06a]. Again, the same results holds in the dissipative case when restricted to a chain-recurrent class.

Finally, we will use a theorem of J. Franks [Fra89] of realization of rotation vectors by periodic points to prove that the rotation set of a generic conservative diffeomor-

phism $f \in \mathrm{Diff}^1(X, \lambda)$ is well approximated by rotation sets of discretizations (Proposition 10.15). The same holds for generic dissipative C^1-diffeomorphisms, provided that we take convex hulls of the rotation sets of the discretizations (Proposition 10.16).

10.1 Shadowing of periodic orbits

First of all, we give the statement of an elementary perturbation lemma that we will use all along this chapter. This is a C^1 counterpart of the proposition of extension of finite maps for the C^0 topology (Proposition 3.3).

Lemma 10.1 (Elementary perturbation lemma in C^1 topology)**.** *For every diffeomorphism $f \in \mathrm{Diff}^1(X)$ and every $\delta > 0$, there exists $\eta > 0$ and $r_0 > 0$ such that the following property holds: for every $x, y \in X$ such that $d(x,y) < r_0$, there exists a diffeomorphism $g \in \mathrm{Diff}^1(X)$ satisfying $d_{C^1}(f,g) < \delta$, such that $g(x) = f(y)$ and that f and g are equal out of the ball $B\left(\frac{x+y}{2}, \frac{1+\eta}{2} d(x,y)\right)$.*

Moreover, if $f \in \mathrm{Diff}^1(X, \lambda)$, then we can also suppose that $g \in \mathrm{Diff}^1(X, \lambda)$.

Before discussing the consequences of this lemma, let us underline the differences with the C^0 case. *A priori*, we will apply the lemma to a small neighbourhood of f, thus a small parameter $\delta > 0$. In facts, the number $\eta > 0$ tends to $+\infty$ when δ goes to 0; thus the ball $B\left(\frac{x+y}{2}, \frac{1+\eta}{2} d(x,y)\right)$ in which lies the support of the perturbation is large with respect to the distance between x and y. This phenomenon does not happen in the C^0 topology, where the proposition of extension of finite maps (Proposition 3.3) states that the size of the support of the perturbation is almost equal to the distance between x and y (in fact, it can be supposed to be included in the neighbourhood of a given path joining x to y). Remark that the situation would be even worse in the C^r topology, for $r > 1$: in this case, the C^r size of the perturbation $\delta > 0$ being fixed, the ratio between the size of the support of the perturbation and $d(x,y)$ would go to infinity when $d(x,y)$ goes to 0.

Thus, the difference between the statements we get for homeomorphisms and for C^1-diffeomorphisms lies in this difference between the elementary perturbation lemmas: in the C^1 case, perturbations are less local than in the C^0 case. In the view of obtaining generic properties of discretizations, the impossibility of making perturbations of all the points of the discretization grids has the effect that we can not obtain *global* properties of discretizations of generic diffeomorphisms with this strategy of proof.

Sketch of proof of Lemma 10.1. We sketch the proof in the conservative case; the dissipative case being obtained by the same arguments (and even, by simpler arguments).

The idea of the proof of this lemma is to apply "local rotations", as in the proof of the proposition of extension of finite maps in the C^0 topology.

More precisely, in dimension 2, g is obtained by composing f by the time 1 of a Hamiltonian flow whose orbits are circles with a common centre, such that one of these circles meets both x and y (see Figure 10.1). As the time 1 of this Hamiltonian maps x into y, its C^0 norm is bigger than $d(x,y)$. But as we also want that the norm of its differential is smaller than δ, its support should contain a ball or radius $(1 + 1/\delta)d(x,y)$. This explains why when δ is small, the size of the perturbation has to be big with respect to $d(x,y)$.

A precise proof can be found for example in [Arn98, Proposition 5.1.1]. □

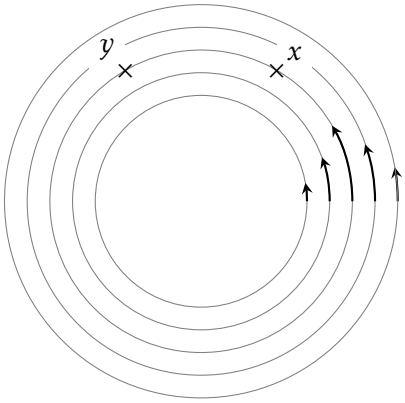

Figure 10.1: Flow of the Hamiltonian used to prove Lemma 10.1 ("local rotation").

From Lemma 10.1, we now deduce another perturbation result which will be at the basis of the rest of the chapter. It states that every periodic orbit of a generic conservative diffeomorphism is shadowed by periodic orbits of its discretizations.

Corollary 10.2. *Let $f \in \mathrm{Diff}^1(X)$ (or $f \in \mathrm{Diff}^1(X,\lambda)$) be a generic diffeomorphism. Then, for every $\tau \in \mathbf{N}$, every $N_0 \in \mathbf{N}$ and every $\varepsilon > 0$, there exists $N \geq N_0$ such that every periodic orbit of f of period smaller than τ is ε-shadowed by a periodic orbit of f_N with the same period.*

Proof of Corollary 10.2. We prove the corollary in the dissipative case, the conservative case being identical.

We set $\mathscr{S}_{\tau,\varepsilon,N}$ the set of $f \in \mathrm{Diff}^1(X)$ such that for every periodic orbit ω of f of period smaller than τ, there exists a periodic orbit of f_N with the same period which is ε-close to ω. Then, the set of diffeomorphisms satisfying the conclusion of the lemma is the set

$$\bigcap_{\tau,\varepsilon,N_0} \bigcup_{N \geq N_0} \mathscr{S}_{\tau,\varepsilon,N}.$$

Thus, it suffices to prove that for every τ, ε, N_0, the set $\bigcup_{N \geq N_0} \mathscr{S}_{\tau,\varepsilon,N}$ contains an open and dense subset of $\mathrm{Diff}^1(X)$.

Let $\tau > 0$, $\varepsilon > 0$, $N_0 \in \mathbf{N}$, $f \in \mathrm{Diff}^1(X,\lambda)$ and $\delta > 0$; we want to find $g \in \mathrm{Diff}^1(X)$ such that $d(f,g) < \delta$ and $g \in \bigcup_{N \geq N_0} \mathscr{S}_{\tau,\varepsilon,N}$.

We first use a classical result of C. Robinson, which asserts that that for a generic diffeomorphism $f \in \mathrm{Diff}^1(X)$, the set of periodic points of period smaller than τ is finite, and moreover is continuous in the diffeomorphism f: if $g \in \mathrm{Diff}^1(X)$ is C^1-close to f, then its set of periodic points of period smaller than τ is close to that of f for Hausdorff topology (see [Rob70a, Rob70b], [Rob73], see also [Cro06b]). Thus, perturbing a little f if necessary, we can suppose that this property actually holds; taking a smaller δ if necessary, we can also suppose that there is no creation of periodic orbit when we perturb f into the δ-neighbourhood of f.

So we can enumerate $\omega_1, \cdots, \omega_\ell$ the periodic orbits of f of period smaller than τ. For each i we denote $\omega_i = \{x_{i,1}, \cdots, x_{i,\tau_i}\}$, with $f(x_{i,j}) = x_{i,j+1}$ (j belonging to $\mathbf{Z}/\tau_i\mathbf{Z}$). Using Lemma 10.1, there exists a radius $r_1 \in]0,\varepsilon[$ (which depends on δ, on the C^1-norm of f and on the minimal distance between two distinct points of the union of these orbits) such that for every collection of points $y_{i,j}$ satisfying $d(y_{i,j}, x_{i,j}) < r_1$ for every $i \in [\![1,\ell]\!]$

and $j \in [\![1, \tau_i]\!]$, there exists a diffeomorphism $g \in \mathrm{Diff}^1(X)$ such that $d_{C^1}(f,g) < \delta$ and that $g(y_{i,j}) = y_{i,j+1}$.

By the hypothesis on the grids E_N, there exists $N_1 \geq N_0$ such that if $N \geq N_1$, then every ball of radius r_1 contains at least one point of E_N. We can apply the property stated in the last paragraph to the discretizations $(x_{i,j})_N$ of the points $x_{i,j}$ on the grid E_N, and get a diffeomorphism g which belongs to the interior of $\bigcup_{n \geq N_0} \mathscr{S}_{\tau,\varepsilon,N}$. This completes the proof of the lemma. □

Corollary 10.2 states that for a generic conservative diffeomorphism, any periodic orbit is shadowed by periodic orbits of the same period of an infinite number of discretizations. In particular, if we denote by $\mathrm{Ord}(f)$ the set of orders of periodic orbits of f, for a generic diffeomorphism $f \in \mathrm{Diff}^1(X,\lambda)$ and for any $M \in \mathbf{N}$, the supremum limit over N of $[0,M] \cap \mathrm{Ord}(f_N)$ is equal $[0,M] \cap \mathrm{Ord}(f)$. Moreover, Corollary 10.2 indicates that it is theoretically possible to recover the set of periodic points of a generic conservative C^1-diffeomorphism by looking at the corresponding periodic points of its discretizations (see Figure 5.13 for numerical simulations corresponding to this property).

10.2 Discrete counterparts of transitivity and topological mixing

As a first application of Corollary 10.2, we apply the connecting lemma of C. Bonatti and S. Crovisier [BC04] to get a C^1 counterpart of Corollary 5.9.

Theorem 10.3 (Bonatti-Crovisier)**.** *A generic diffeomorphism $f \in \mathrm{Diff}^1(X,\lambda)$ is transitive. More precisely, for any generic diffeomorphism $f \in \mathrm{Diff}^1(X,\lambda)$ and any $\varepsilon > 0$, there exists a periodic orbit of f which is ε-dense.*

In the dissipative case, for a generic diffeomorphism $f \in \mathrm{Diff}^1(X)$, every maximal invariant chain-transitive set is transitive. More precisely, for every maximal invariant chain-transitive set K and every $\varepsilon > 0$, there exists a periodic orbit of f which is ε-dense in K.

In the conservative case, this difficult theorem is somehow a weak answer to the question asking whether a generic conservative C^1-diffeomorphism is ergodic or not.

Combining this theorem with Corollary 10.2, we obtain directly the following corollary.

Corollary 10.4. *Let f be a generic diffeomorphism of $\mathrm{Diff}^1(X,\lambda)$. Then, for any $\varepsilon > 0$ and any $N_0 > 0$, there exists $N \geq N_0$ such that f_N has a periodic orbit which is ε-dense.*

Let f be a generic diffeomorphism of $\mathrm{Diff}^1(X)$. Then, for every maximal invariant chain-transitive set K, for every $\varepsilon > 0$ and every $N_0 > 0$, there exists $N \geq N_0$ such that f_N has a periodic orbit which is ε-dense in K.

In the conservative case, this corollary expresses that for every $\varepsilon > 0$, an infinite number of discretizations are "ε-topologically transitive", thus contain a sub-dynamics which is similar to that of the initial diffeomorphism.

In the dissipative case, it implies that it is possible to recover the chain transitive set $\mathscr{R}(f)$ of a generic dissipative diffeomorphism f, by using the recurrent sets of the discretizations: $\mathscr{R}(f) = \overline{\lim}_{N \to +\infty} \Omega(f_N)$.

More recently, F. Abdenur and S. Crovisier have obtained in [AC12] a more sophisticated closing lemma, which allows to prove that a generic diffeomorphism is topologically mixing [1].

Theorem 10.5 (Closing lemma with time control, Abdenur-Crovisier)**.** *Let $f \in \mathrm{Diff}^1(X)$, $\ell \geq 2$ be an integer, x be either a non-periodic point or a non-resonant periodic point [2]. Assume that each neighbourhood V of x intersects some iterate $f^m(V)$ such that m is not a multiple of ℓ. Then, there exists diffeomorphisms $g \in \mathrm{Diff}^1(X)$, arbitrarily C^1-close to f, such that x is periodic under g whose period is not a multiple of ℓ.*

Moreover, if $f \in \mathrm{Diff}^1(X, \lambda)$, then g can be supposed to be conservative too.

This perturbation lemma can be seen as a weak version of Proposition 5.12, which implies that under the same conditions, but in the C^1 topology, the period of x can be supposed to be coprime with ℓ. However, combining this lemma with arguments that are specific to diffeomorphisms, we get a stronger result of genericity, that is that a generic diffeomorphism is topologically strongly mixing.

From this theorem, it is possible to deduce the following statement.

Corollary 10.6. *For a generic $f \in \mathrm{Diff}^1(X, \lambda)$ and for any $\varepsilon > 0$, f has two ε-dense periodic points, whose periods are not multiples on to the other. The same property holds for a generic dissipative diffeomorphism on every maximal invariant chain-transitive set* K.

Proof of Corollary 10.6. Let $\varepsilon > 0$; we show that the set of $f \in \mathrm{Diff}^1(X, \lambda)$ satisfying the conclusions of the corollary for ε is open and dense. By the connecting lemma of C. Bonatti and S. Crovisier (Theorem 10.3), for a generic $f \in \mathrm{Diff}^1(X, \lambda)$ and for any $\varepsilon > 0$, f has a ε-dense periodic point, that we denote by p, whose period is ℓ; we can furthermore assume that the periodic point p is persistent. Then, by Proposition 2.3 of [AC12], the map f^ℓ is transitive on the pointwise homoclinic class [3] of p. Thus, for any neighbourhood V of p, there exists $k \in \mathbf{N}$ such that $f^{k\ell}\big(f(V)\big) \cap V \neq \emptyset$; this allows us to apply Theorem 10.5 to perturb the diffeomorphism f to a diffeomorphism g. Making both of the obtained periodic points of g persistent (if necessary), a whole neighbourhood of g satisfies the conclusions of the corollary.

The dissipative case is identical: it suffices to apply the dissipative version of Theorem 10.3. □

From Corollary 10.6, and using Corollary 10.2, we deduce directly a statement about discretizations, which is somehow a C^1 counterpart of Corollary 5.13.

Corollary 10.7. *For a generic $f \in \mathrm{Diff}^1(X, \lambda)$, for any $\varepsilon > 0$ and any $N_0 > 0$, there exists $N \geq N_0$ such that f_N has two ε-dense periodic points, whose periods are not multiples on to the other. The same property holds for a generic dissipative diffeomorphism on every maximal invariant chain-transitive set* K.

Thus, in a certain sense, for every $\varepsilon > 0$, an infinite number of discretizations are ε-topologically mixing on every maximal invariant chain-transitive set.

1. Recall that a continuous map $f : X \to X$ is *topologically mixing* if for any non-empty open sets $U, V \subset X$, there exists $M \in \mathbf{N}$ such that $f^m(U) \cap V \neq \emptyset$ for every $m \geq M$.

2. All the periodic points of a generic conservative diffeomorphism are non resonant, see Definition 3.1 of [AC12].

3. The *pointwise homoclinic class* is the closure of the set of transverse intersection points between the manifolds $W^s(p)$ and $W^u(p)$

10.3 Shadowing of invariant measures

We now come to the ergodic properties of the discretizations of a generic diffeomorphism. We make use of an ergodic closing lemma, which allows to approximate every invariant measure by periodic invariant measures

Theorem 10.8 (Mañé, Abdenur-Bonatti-Crovisier)**.** *Let $f \in \mathrm{Diff}^1(X, \lambda)$ be a generic diffeomorphism. Then, every f-invariant measure μ is the weak limit of periodic measures.*

Let $f \in \mathrm{Diff}^1(X)$ be a generic diffeomorphism. Then, every f-invariant measure μ which is supported by an invariant chain-transitive set is the weak limit of periodic measures.

The case where the invariant measure is ergodic has been obtained by R. Mañé in [Mañ82], the general case has been treated by F. Abdenur, C. Bonatti and S. Crovisier in [ABC11, Theorem 3.5] (the theorem also holds for generic conservative C^1-diffeomorphisms).

Applying Corollary 10.2, we get the following corollary, which is a C^1 counterpart of Theorems 5.51 and 5.45.

Corollary 10.9. *Let f be a generic conservative diffeomorphism of $\mathrm{Diff}^1(X, \lambda)$. Then, for any f-invariant measure μ, any $\varepsilon > 0$ and any $N_0 > 0$, there exists $N \geq N_0$ such that f_N supports a periodic measure which is ε-close to μ.*

More generally, for a generic diffeomorphism $f \in \mathrm{Diff}^1(X, \lambda)$, for any $\varepsilon > 0$ and $N_0 > 0$, there exists $N \geq N_0$ such that the set of f_N-invariant measures is ε-close to the set of f-invariant measures (for the Hausdorff distance on the space of compact sets of probability measures on X).

The statement is still true for a generic dissipative diffeomorphism $f \in \mathrm{Diff}^1(X)$, if the measure μ is supported by an invariant chain-transitive set.

Roughly speaking, every invariant measure supported by a maximal invariant chain-transitive set is "seen" by an infinite number of discretizations.

In Chapter 12, we will obtain an improvement of this corollary (Theorem 12.1), which will describe the basin of attraction of the periodic measure of the discretization.

10.4 Shadowing of chain-transitive invariant sets

We now treat the case of the shadowing of invariant sets of a generic diffeomorphism. In [Cro06a, Theorem 4], S. Crovisier stated the following result, which asserts that for a generic diffeomorphism, it is possible to recover the chain-transitive invariant sets by looking only at the periodic points.

Theorem 10.10 (Crovisier)**.** *Any chain-transitive compact invariant set of a generic diffeomorphism $f \in \mathrm{Diff}^1(X)$ of $f \in \mathrm{Diff}^1(X, \lambda)$ is approximated in the Hausdorff topology by periodic orbits.*

Obviously, the converse is always true: any point of accumulation of periodic orbits is a chain-transitive compact invariant set.

Combined with Corollary 10.2, Theorem 10.10 leads to the following corollary, which is a C^1 counterpart of Theorem 5.47, or Theorem 5.49.

Corollary 10.11. *Let f be a generic diffeomorphism of $\mathrm{Diff}^1(X)$ or $\mathrm{Diff}^1(X, \lambda)$. Then, for any f-invariant chain-transitive compact set K, any $\varepsilon > 0$ and any $N_0 > 0$, there exists $N \geq N_0$*

such that f_N has an invariant compact set which is ε-close to K for Hausdorff distance on the set of compact subsets of X.

More generally, for a generic diffeomorphism $f \in \mathrm{Diff}^1(X)$ (or $f \in \mathrm{Diff}^1(X,\lambda)$), for any $\varepsilon > 0$ and $N_0 > 0$, there exists $N \geq N_0$ such that the set of f_N-invariant sets is ε-close to the set of f-invariant chain-transitive compact sets (for Hausdorff distance on the set of compact sets of compact subsets of X).

10.5 Rotation sets

In this section, we are interested in the approximation of the rotation set of a generic diffeomorphism by the rotation sets of its discretizations (see Chapter 6). Thus, we will take $X = \mathbf{T}^n$ and consider only diffeomorphisms in the connected component of the identity. We also take the same notations as in Chapter 6.

We will see that in the C^1-case, applying the elementary perturbation lemma (Corollary 10.2), it is possible to obtain a weaker result about discretized rotation sets (in this case, we can not control what happens on the whole grid E_N, but only on a subgrid of E_N): for a generic conservative diffeomorphism, the upper limit of the discretized rotation sets is equal to the rotation set of the diffeomorphism itself.

To begin with, we state an approximation lemma we will use in this section; this lemma is a quite direct consequence of Lemma 10.1.

Lemma 10.12. *If f is generic among $\mathrm{Diff}^1(\mathbf{T}^n)$ or $\mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$, then for every finite collection of rotation vectors $\{v_1, \cdots, v_n\}$, each one realized by a periodic orbit of f, there exists a subsequence f_{N_i} of discretizations such that for every i, $\{v_1, \cdots, v_n\} \subset \rho_{N_i}(f)$.*

Proof of Lemma 10.12. The proof of this lemma is very similar to that of Lemma 6.22; we take the same notations (in particular, $D_q \subset \mathbf{Q}$ is the set of fractions whose numerator is smaller than q^2 and whose denominator is smaller than q). We prove this lemma in the conservative setting, the dissipative case being identical.

Consider the set

$$\bigcap_{q,N_0} \bigcap_{D \in \mathscr{D}_q} \bigcup_{N \geq N_0} \left\{ \begin{array}{c} f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb}) \mid (\forall v \in D,\ v \text{ is realised by a} \\ \text{persistent periodic point of } f) \implies D \subset \rho(F_N) \end{array} \right\}. \tag{10.1}$$

To prove the lemma, it suffices to prove that this set contains a G_δ dense subset of $\mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$.

Let $f \in \mathrm{Homeo}(\mathbf{T}^n, \mathrm{Leb})$, $\varepsilon > 0$, $q, N_0 \in \mathbf{N}$ and $D \in \mathscr{D}_q$. We suppose that for all $v \in D$, v is realizable by a persistent periodic orbit ω_i of f. Then, by the elementary perturbation lemma (Lemma 10.1), it is possible to perturb f into a diffeomorphism g such that $d_{C^1}(f,g) < \varepsilon$ and that there exists $N \geq N_0$ such that for every i, there exists a periodic orbit ω_i' of g which is close to ω_i (in particular, it has the same rotation vector) and such that $\omega_i' \subset E_N$. Moreover, perturbing a little g if necessary (as in the proof of Proposition 6.2), we can suppose that the periodic orbits ω_i' are persistent. This proves that the set of (10.1) contains a G_δ dense subset of $\mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$. □

To begin with, we treat the conservative case. We will combine Lemma 10.12 with a realization theorem of rotation vectors by periodic orbits; but to use this realization theorem we need to prove that the rotation set of a generic conservative diffeomorphism is nonempty. It is a C^1 counterpart of Proposition 6.2.

Proposition 10.13. *On a open dense*[4] *subset of* $\mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$, $\rho(F)$ *has non-empty interior.*

Remark 10.14. We do not know the shape of the boundary of the rotation set of a generic conservative C^1-diffeomorphism. In particular we do not know if it is a polygon or not.

Proof of Proposition 10.13. It suffices to resume the proof of Proposition 6.2, and to replace the C^0 ergodic closing lemma is replaced by the C^1-ergodic closing lemma which is stated in Theorem 10.8. To make the obtained periodic point persistent, we just have to apply Franks lemma [Fra71] to perturb the differential of g on the periodic orbit to avoid having the eigenvalue 1, so that the periodic point becomes persistent (see [KH95, page 319]). The rest of the proof is identical to the C^0 case. □

Combined with the realisation theorem of J. Franks (Theorem 6.17, see also [Fra89, Theorem 3.2]) and the approximation lemma (Lemma 10.12), this proposition directly implies the following result on the discretizations.

Proposition 10.15. *If f is generic among* $\mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$, *then there exists a subsequence f_{N_i} of discretizations such that $\rho_{N_i}(F)$ tends to $\rho(F)$ for the Hausdorff topology; in particular, the asymptotic discretized rotation set tends to the rotation set of f.*

In particular, when f is a generic conservative C^1-diffeomorphism, the asymptotic discretized rotation set

$$\rho^{discr}(F) = \bigcap_{M \in \mathbf{N}} \bigcup_{N \geq M} \rho(F_N)$$

coincides with the rotation set.

For a generic C^1-diffeomorphism, we do not know if the observable rotation set is reduced to a singleton or not. This is true when the diffeomorphism is ergodic, which is conjectured to be a generic property (see also page 250).

Remark that the behaviour described by Proposition 10.15 can actually be observed on actual examples of conservative C^∞-diffeomorphisms, see Section 6.4.

The dissipative case is less straightforward, as Proposition 10.13 is false for generic dissipative C^1-diffeomorphisms (there are open sets of diffeomorphisms whose rotation set has empty interior). However, the connected component of Id in $\mathrm{Diff}^1(\mathbf{T}^2)$, includes three disjoint open subsets, whose union is dense.

1. The first one is the open subset $\mathscr{O}_1$ of diffeomorphisms f such that $\rho(F)$ has a nonempty interior (this set is open by the continuity[5] of $F \mapsto \rho(F)$ on every homeomorphism whose rotation set has nonempty interior, see [LM91]). On $\mathscr{O}_1$, the same arguments than in the conservative case can be applied, thus if f is generic among $\mathscr{O}_1$, then there exists a subsequence f_{N_i} of discretizations such that $\rho_{N_i}(F)$ tends to $\rho(F)$.
2. The second one is the open subset $\mathscr{O}_2$ of diffeomorphisms f such that $\rho(F)$ is stably a segment. By the ergodic closing lemma of R. Mañé (Theorem 10.8), for a generic $f \in \mathscr{O}_2$, every rotation vector of an ergodic measure (in particular, the vertices of the segment) is arbitrarily approximated by a sequence of rotation vectors of (persistent) periodic points. In particular, on an open and dense subset of $\mathscr{O}_2$, the rotation set is a segment with rational slope, which coincides with the closure of the convex hull of

4. In particular, if f is generic.

5. With the C^0 topology on the domain of ρ and the Hausdorff distance on compact subsets of $\mathbf{R}^2$ on its image.

rotation vectors of (persistent) periodic points [6]. Thus, applying Lemma 10.12, we get that if f is generic among $\mathscr{O}_2$, then there exists a subsequence $\left(f_{\mathrm{N}_i}\right)_i$ of discretizations such that $\mathrm{conv}(\rho_{\mathrm{N}_i}(\mathrm{F}))$ tends to $\rho(\mathrm{F})$.

3. The third one is the open subset $\mathscr{O}_3$ of diffeomorphisms f such that $\rho(\mathrm{F})$ is stably a singleton. Then, as on an open dense subset of $\mathrm{Diff}^1(\mathbf{T}^2)$, every diffeomorphism possesses a periodic point, on an open and dense subset of $\mathscr{O}_3$, the rotation vector is realized by a periodic point. Then, trivially, we get that for f belonging to an open and dense subset of $\mathscr{O}_3$, the rotation sets of f_{N} converge globally to the rotation set of f.

To summarize, we have proved the following property.

Proposition 10.16. *If $f \in \mathrm{Diff}^1(\mathbf{T}^n)$ is a generic dissipative diffeomorphism, then there exists a subsequence $\left(f_{\mathrm{N}_i}\right)_i$ of discretizations such that $\mathrm{conv}\left(\rho_{\mathrm{N}_i}(\mathrm{F})\right)$ tends to $\rho(\mathrm{F})$ for the Hausdorff topology; in particular, the convex hull of the asymptotic discretized rotation set tends to the rotation set of f. Moreover, if $\rho(\mathrm{F})$ has nonempty interior, or is reduced to a singleton, then there is no need to take convex hulls.*

Remark that even if the techniques of proof are quite different, this behaviour is identical to what happens for generic dissipative homeomorphisms (Proposition 6.21).

6. Remark that in general, as the slope of the rotation set is rational, we can not hope to get a (for instance) dense subset of the rotation set on which each vector is realized by a periodic point, see for example [Bég07, Section 3.3]

Chapter 11

Degree of recurrence of a generic diffeomorphism

In this chapter, we begin the study of the the global dynamics of the discretizations of a generic C^1-diffeomorphism (both conservative and dissipative) by focusing on the degree of recurrence of the discretizations.

We will consider that the space phase is the torus $\mathbf{T}^n$, the measure is Lebesgue measure and the grids are the uniform grids

$$\mathrm{E_N} = \left\{ \left(\frac{i_1}{\mathrm{N}}, \cdots, \frac{i_n}{\mathrm{N}} \right) \in \mathbf{R}^n/\mathbf{Z}^n \,\middle|\, 1 \leq i_1, \cdots, i_n \leq \mathrm{N} \right\}.$$

We will see in Section 11.4 that this quite restrictive framework can be generalized to arbitrary manifolds, provided that the discretizations grids behave locally (and almost everywhere) like the canonical grids on the torus.

Let us recall the definition of the degree of recurrence.

Definition 11.1. Let E be a finite set and $\sigma : \mathrm{E} \to \mathrm{E}$ be a finite map on E. The *recurrent set* of σ is the union $\Omega(\sigma)$ of the periodic orbits of σ; it is also equal to the set $\sigma^t(\mathrm{E})$ for every t large enough.

The *degree of recurrence* of the finite map σ is the ratio $\mathrm{D}(\sigma)$ between the cardinality of the recurrent set and the cardinality of E, that is

$$\mathrm{D}(\sigma) = \frac{\mathrm{Card}\left(\Omega(\sigma)\right)}{\mathrm{Card}(\mathrm{E})}.$$

The goal of this chapter is to study the behaviour of the degree of recurrence $\mathrm{D}(f_\mathrm{N})$ as N goes to infinity and for a generic conservative/dissipative diffeomorphism. This degree of recurrence somehow represents the amount of information we lose when we iterate the discretization.

As it can be obtained as the decreasing limit of finite time quantities, the degree of recurrence is maybe the easiest combinatorial invariant to study: we will deduce the behaviour of the degree of recurrence from that of the rate of injectivity.

Definition 11.2. Let $f \in \mathrm{End}(\mathbf{T}^n)$ be an endomorphism of the torus and $t \in \mathbf{N}$. The *rate of injectivity* in time t and for the order N is the quantity

$$\tau^t_\mathrm{N}(f) = \frac{\mathrm{Card}\left((f_\mathrm{N})^t(\mathrm{E_N})\right)}{\mathrm{Card}(\mathrm{E_N})}.$$

Then, the *upper rate of injectivity* of f in time t is defined as

$$\tau^t(f) = \overline{\lim_{N\to+\infty}} \tau^t_N(f).$$

The link between the degree of recurrence and the rates of injectivity is made by the following formula:

$$D(f_N) = \lim_{t\to+\infty} \tau^t_N(f).$$

The study of the rates of injectivity will be the opportunity to understand the local behaviour of the discretizations of diffeomorphisms: Theorem 11.8 asserts that that these rates of injectivity are obtained by averaging the corresponding quantities for the differentials of the diffeomorphism. The proof of this result involves the local linearization of a diffeomorphism (Lemma 11.6), together with estimates of the lack of continuity of the rate of injectivity in the linear case (Proposition 8.7). We generalize later this result to the C^r topology, for generic diffeomorphisms (Theorem 11.26) and for generic expanding maps (Theorem 11.19).

In the conservative case, we will use the study of the rate of injectivity of matrices with determinant 1 in Part 2 of this manuscript (see Theorem 8.24). It will lead to the proof of the fact that the sequence $D(f_N)$ of degrees of recurrence of a generic conservative C^1-diffeomorphism tends to 0 (Theorem 11.15).

For a generic dissipative C^1-diffeomorphism, the sequence $D(f_N)$ also converges to 0 (Corollary 11.13); it is an easy consequence of a theorem of A. Avila and J. Bochi (Theorem 11.11, see also [AB06]).

Note that the fact that the local-global formula is true for C^r-generic diffeomorphisms does not help to conclude about the degree of recurrence of such maps: a priori, we need to perturs the derivative of such maps on a large subset of the torus. However, we can hope that

We now explain in more detail why the behaviour of $D(f_N)$ can be deduced from that of τ^t. When N is fixed, the sequence $(\tau^t_N(f))_t$ is decreasing in t, so $D(f_N) \le \tau^t_N(f)$ for every $t \in \mathbf{N}$. Taking the upper limit in N, we get

$$\overline{\lim_{N\to+\infty}} D(f_N) \le \tau^t(f)$$

for every $t \in \mathbf{N}$, so

$$\overline{\lim_{N\to+\infty}} D(f_N) \le \lim_{t\to+\infty} \tau^t(f) \tag{11.1}$$

(as the sequence $(\tau^t(f))_t$ is decreasing, the limit is well defined). In particular, if we have an upper bound on $\lim_{t\to+\infty} \tau^t(f)$, this will give a bound on $\overline{\lim_{N\to+\infty}} D(f_N)$. Thus, the proof of Theorem 11.15 is reduced to the study of the quantity $\lim_{t\to+\infty} \tau^t(f)$.

At the end of this chapter, we recall the results of the simulations we have conducted about the degree of recurrence of C^1-diffeomorphisms; it shows that in practice, the degree of recurrence tends to 0, at least for the examples of diffeomorphisms we have tested.

11.1 Local-global formula

In this section, we state a local-global formula which links the rate of injectivity of a generic diffeomorphism f and the rates of injectivity of its differentials. The fact that the map f is C^1 — then possesses differentials — introduces a mesoscopic scale for the study of the action of the discretizations in small time:

– at the macroscopic scale, the discretization of f acts as f;
– at the intermediate mesoscopic scale, the discretization of f acts as a linear map;
– at the microscopic scale, we are able to see that the discretization is a finite map and we see that the phase space is discrete.

Recall that the discretization of a linear map $A : \mathbf{R}^n \to \mathbf{R}^n$ is the map $\widehat{A} : \mathbf{Z}^n \to \mathbf{Z}^n$ defined by $\widehat{A}(x) = \pi(Ax)$, where $\pi : \mathbf{R}^n \to \mathbf{Z}^n$ is a projection on (one of) the nearest integer point for the euclidean distance (see Definition 7.11).

The rate of injectivity in time k of the matrices $A_1, \cdots, A_k \in \mathrm{GL}_n(\mathbf{R})$ is then defined as (see Definition 7.19; see also Corollary 7.7 and Theorem 7.12 for the fact that this limit is well defined)

$$\tau^k(A_1, \cdots, A_k) = \lim_{R \to +\infty} \frac{\mathrm{Card}\left((\widehat{A_k} \circ \cdots \circ \widehat{A_1})[B_R]\right)}{\mathrm{Card}[B_R]} \in]0, 1];$$

and for an infinite sequence $(A_k)_{k \geq 1}$ of invertible matrices, as the previous quantity is decreasing, we define the asymptotic rate of injectivity

$$\tau^\infty\left((A_k)_{k \geq 1}\right) = \lim_{k \to +\infty} \tau^k(A_1, \cdots, A_k) \in [0, 1].$$

So, the link between local and global behaviours of the rates of injectivity is given by the following theorem.

Theorem 11.3. *Let $f \in \mathrm{Diff}^1(\mathbf{T}^n)$ (or $f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$) be a generic diffeomorphism. Then $\tau^1(f)$ is well defined (that is, the superior limit is actually a limit) and satisfies:*

$$\tau^1(f) = \int_{\mathbf{T}^n} \tau^1(Df_x)\, \mathrm{d}x. \tag{L-G}$$

Moreover, the function τ is continuous in f.

Example 11.4. Theorem 11.3 becomes false if we do not suppose that the diffeomorphism is generic. For example, take a diffeomorphism $f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$, which is equal to

$$f_0 + v = \begin{pmatrix} \frac{1}{2} & -1 \\ \frac{1}{2} & 1 \end{pmatrix} + v.$$

in a ball of radius $r > 0$, with a vector v such that $v = (0.2/\mathrm{N}, -0.1/\mathrm{N})$ modulo $\mathbf{Z}^2/\mathrm{N}$, and "generic" where it does not coincide with this affine map. To construct this example more rigorously, we can for example apply Lemma 11.6 to have a C^1-diffeomorphism which is equal equal to $f_0 + v$ in a ball of radius $r > 0$, and apply the process of proof of Theorem 11.3 outside of this ball, such that outside of this ball, a formula similar to Equation L-G holds.

Then, by Lemma 8.2, the rate of injectivity of f_0 is equal to $1/2$ and that of $f_0 + v$ is equal to 1. Because f is generic anywhere else, both sides of Equation (L-G) are different. Moreover, the rate of injectivity τ^1 is not continuous in f.

To show Theorem 11.3, we have to show that Equation (L-G) holds on a dense G_δ set of diffeomorphisms. The "density" part is easily deduced from the following lemma.

Lemma 11.5. *Let $f_1 \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$ (respectively $f_1 \in \mathrm{Diff}^1(\mathbf{T}^n)$). Then there exists $f_2 \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$ (respectively $f_2 \in \mathrm{Diff}^1(\mathbf{T}^n)$) arbitrarily close to f_1, and a subset $\mathscr{C} \in \mathbf{T}^n$ (which is a finite union of cubes, see Figure 11.1) of measure arbitrarily close to 1, such that the differential of f_2 is piecewise constant and totally irrational [1] on $\mathscr{C}$.*

1. A matrix $A \in \mathrm{GL}_n(\mathbf{R})$ is *totally irrational* if $A\mathbf{Z}^n$ is equidistributed modulo $\mathbf{Z}^n$, see Definition 8.3.

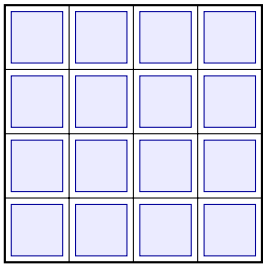

Figure 11.1: The set $\mathscr{C}$ (in blue) in the torus.

This lemma follows from the following result of A. Avila, S. Crovisier and A. Wilkinson.

Lemma 11.6 (Avila, Crovisier, Wilkinson)**.** *Let* C *be the unit ball of* $\mathbf{R}^n$ *for* $\|\cdot\|_\infty$ *and* $\varepsilon > 0$. *Then, there exists* $\delta > 0$ *such that for every* $g_1 \in \mathrm{Diff}^\infty(\mathbf{R}^n)$ *such that* $d_{C^1}(g_{1|C}, \mathrm{Id}_{|C}) < \delta$, *there exists* $g_2 \in \mathrm{Diff}^\infty(\mathbf{R}^n)$ *such that:*

(i) $d_{C^1}(g_{2|C}, g_{1|C}) < \varepsilon$;

(ii) $g_{2|(1-\varepsilon)C} = \mathrm{Id}_{|(1-\varepsilon)C}$;

(iii) $g_{2|C^\complement} = g_{1|C^\complement}$.

Moreover, if g_1 *preserves Lebesgue measure, then* g_2 *can be chosen to preserve it as well.*

The dissipative case of this lemma is easily obtained by interpolating the diffeomorphism with identity, using a smooth bump function. The proof in the conservative case is more difficult and involves a result of J. Moser [Mos65]. The reader may refer to [ACW14, Corollary 6.9] for a complete proof of this lemma [2].

Proof of Lemma 11.5. First of all, we regularize the diffeomorphism f_1 to get a C^∞ diffeomorphism f_3 which is close to f_1 in the C^1 topology. In the dissipative case, this is easily obtained (for example) by convolving it by an approximation of the identity. The conservative case is much more difficult; this result has been obtained recently by A. Avila [Avi10].

We then obtain the lemma by applying Lemma 11.6 to the restriction of f_3 on each cube of a fine enough cubulation of $\mathbf{T}^n$ (see Figure 11.1). □

The proof of Theorem 11.3 will primarily consist in applying Lemma 11.5 together with the following lemma.

Lemma 11.7. *Let* C *be a cube of* $\mathbf{T}^n$. *Then, for every totally irrational matrix* $A \in \mathrm{GL}_n(\mathbf{R})$, *every* $v \in \mathbf{T}^n$ *and every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for every* C^1 *map* $f : C \to \mathbf{T}^n$ *satisfying* $\sup_{x\in C} \|Df_x - A\| \leq \delta$, *there exists* $N_0 \in \mathbf{N}$ *such that for every* $N \geq N_0$,

$$\frac{\mathrm{Card}\Big((A+v)_N(E_N \cap C)\,\Delta\, f_N(E_N \cap C)\Big)}{\mathrm{Card}(E_N \cap C)} \leq \varepsilon.$$

Before proving Lemma 11.7, we explain how it implies Theorem 11.3.

2. Le 24/02/2015, cette version n'est pas encore en ligne…

Proof of Theorem 11.3. We perform the proof in the dissipative setting, the conservative case being identical. Let $f \in \mathrm{Diff}^1(\mathbf{T}^n)$. The idea is to cut the torus $\mathbf{T}^n$ into small pieces on which f is very close to its Taylor expansion at order 1.

Let $\mathscr{U}_\ell$ ($\ell \in \mathbf{N}^*$) be the set of diffeomorphisms f such that the set of accumulation points of the sequence $\left(\tau^1_N(f)\right)_N$ is included in the ball of radius $1/\ell$ and centre $\int_{\mathbf{T}^n} \tau^1(Df_x)\,dx$ (that is, the right side of Equation (L-G)). We want to show that $\mathscr{U}_\ell$ contains a dense open subset of $\mathrm{Diff}^1(\mathbf{T}^n)$. In other words, we pick a diffeomorphism f, an integer ℓ and $\delta > 0$, and we want to find another diffeomorphism g which is δ-close to f, and which belongs to the interior of $\mathscr{U}_\ell$.

To do that, we first use Lemma 11.5, which gives a diffeomorphism g which is δ-close to f and whose differential is piecewise constant and irrational on a finite union of cubes $\mathscr{C}$ whose measure is bigger than $1 - 1/(10\ell)$. Then by construction, $g \in \mathscr{U}_{2\ell}$. Indeed, as the differentials of g are irrational on C, the rates of injectivity of the translates of the differentials are all equal to that of the differential itself (see Proposition 8.4), thus

$$\frac{\mathrm{Card}\left(g_N(E_N \cap \mathscr{C})\right)}{\mathrm{Card}(E_N \cap \mathscr{C})} \underset{N\to+\infty}{\longrightarrow} \int_{\mathscr{C}} \tau^1(Df_x)\,dx.$$

Moreover, as the measure of $\mathscr{C}$ is bigger than $1 - 1/(10\ell)$, what happens on the complement of $\mathscr{C}$ can be neglected, more precisely,

$$\begin{aligned}
\left|\tau^1_N(g) - \frac{\mathrm{Card}\left(g_N(E_N \cap \mathscr{C})\right)}{\mathrm{Card}(E_N \cap \mathscr{C})}\right| &\le \left|\frac{\mathrm{Card}\left(g_N(E_N)\right)}{\mathrm{Card}(E_N)} - \frac{\mathrm{Card}\left(g_N(E_N)\right)}{\mathrm{Card}(E_N \cap \mathscr{C})}\right| \\
&\quad + \left|\frac{\mathrm{Card}\left(g_N(E_N)\right)}{\mathrm{Card}(E_N \cap \mathscr{C})} - \frac{\mathrm{Card}\left(g_N(E_N \cap \mathscr{C})\right)}{\mathrm{Card}(E_N \cap \mathscr{C})}\right| \\
&\le \left|1 - \frac{\mathrm{Card}(E_N)}{\mathrm{Card}(E_N \cap \mathscr{C})}\right| + \frac{\mathrm{Card}(E_N \cap \mathscr{C})}{\mathrm{Card}(E_N)} \\
&\underset{N\to+\infty}{\longrightarrow} \frac{1}{1 - 1/(10\ell)} - 1 + \frac{1}{10\ell} \le \frac{1}{4\ell},
\end{aligned}$$

and as for every x, $0 \le \tau^1(Df_x) \le 1$, we also have

$$\left|\int_{\mathbf{T}^n} \tau^1(Df_x)\,dx - \int_{\mathscr{C}} \tau^1(Df_x)\,dx\right| \le \mathrm{Leb}\left(\mathscr{C}^{\complement}\right) \le \frac{1}{10\ell}.$$

To show that a whole neighbourhood of g belongs to $\mathscr{U}_\ell$, it suffices to apply Lemma 11.7 to each cube of $\mathscr{C}$; as the measure of $\mathscr{C}$ is bigger than $1 - /(10\ell)$ (again, what happens on $\mathscr{C}^{\complement}$ does not count), we get the conclusion of the theorem. □

Thus, it remains to prove Lemma 11.7.

Proof of Lemma 11.7. Let $A \in \mathrm{GL}_n(\mathbf{R})$ be a totally irrational matrix. Then, there exists a parameter $\delta > 0$ and a radius $R_0 > 0$ such that every matrix $B \in Gl_n(\mathbf{R})$ such that $\|A - B\| \le \delta$ is "almost totally irrational", so that:

(H1) the rates of injectivity of the translates of B are close one to each other and can be seen on every ball of radius bigger than R_0: there exists $R_0 > 0$ such that for every $R > R_0$, every B satisfying $\|A - B\| \le \delta$ and every $v \in \mathbf{R}^n$, we have

$$\left|\tau^1(B) - \det(B) D^+_R\left(\pi(B(\mathbf{Z}^n) + v)\right)\right| \le 1/(10\ell);$$

(H2) the image sets of the differentials are well distributed on every ball of radius bigger than R_0: if we note $(\mathbf{Z}^n)'$ the set of points of $\mathbf{R}^n$ at least one coordinate of which belongs to $\mathbf{Z}+1/2$, then for every $R \geq R_0$ and every B satisfying $\|A-B\| \leq \delta$, we have

$$\det(B) D_R^+\Big\{y \in \mathbf{Z}^n \mid d(B(y),(\mathbf{Z}^n)') < 1/(40\ell n)\Big\} \leq 1/(10\ell).$$

In other words, everything about the rate of injectivity is uniform. The fact that Property (H1) is true on an open dense set of matrices is obtained by applying Proposition 8.4 and Proposition 8.7, and that Property (H2) is true on an open dense set is a direct consequence of Lemma 7.21. Remark that the uniformity of the R_0 comes from Remark 7.22.

Let $f : C \to \mathbf{T}^n$ be a C^1 map satisfying $\sup_{x\in C} \|Df_x - A\| \leq \delta$. We write the Taylor expansion of order 1 of f at the neighbourhood of $x \in C$; by compactness we obtain

$$\sup\left\{\frac{1}{\|h\|}\Big\|f(x+h)-f(x)-Df_x(h)\Big\| \;\Big|\; x \in C, h \in B(0,\varepsilon)\right\} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Thus, for every $\eta > 0$, there exists $\varepsilon > 0$ such that for all $x \in C$ and all $h \in B(0,\varepsilon)$, we have

$$\Big\|f(x+h)-\Big(f(x)+Df_x(h)\Big)\Big\| < \eta\|h\| \leq \eta\varepsilon. \tag{11.2}$$

We now take $R \geq R_0$ (given by hypotheses (H1) and (H2)). We want to find an order of discretization N such that the error made by linearizing f on $B(x,R/N)$ is small compared to N, that is, for every $h \in B(0,R/N)$, we have

$$\Big\|f(x+h)-\Big(f(x)+Df_x(h)\Big)\Big\| < \frac{1}{40\ell n N}.$$

To do that, we take $\eta = 1/(40\ell n R)$, then get a $\varepsilon > 0$ given by Equation (11.2) (we can take ε as small as we want), and set $N = \lceil R/\varepsilon \rceil$ (thus, we can take N as big as we want). By Equation 11.2, for every $y \in B(0,R)$, we obtain

$$\Big\|f(x+y/N)-\Big(f(x)+Df_x(y/N)\Big)\Big\| < \frac{1}{40\ell n N}.$$

Combining this result with Hypothesis (H2), we obtain that the proportion of points of $E_N \cap B(x,R/N)$ whose images by the discretizations of f and that of the linearization of f in x is smaller than $1/(10\ell)$. By Hypothesis (H1), the density of the discretization of the linearization of f in x is close to $\tau^1(Df_x)$, that is (recall that P_N is the projection of X on E_N)

$$\Big|\tau^1(Df_x) - \det(Df_x) D_R^+\Big(P_N(Df_x(\mathbf{Z}^n)+v)\Big)\Big| \leq 1/(10\ell).$$

These two facts lead to

$$\left|\frac{\mathrm{Card}\Big(P_N\big(f(x+B(0,R/N)\cap E_N)\big)\Big)}{\mathrm{Card}\Big(B(0,R/N)\cap E_N\Big)} - D_R^+(Df_x)\right| < \frac{1}{2\ell}. \tag{11.3}$$

We then tile C by smaller cubes of size of the order of $\varepsilon \simeq R/N$. On each of these squares, by Equation (11.3), the rate of injectivity of f and that of its differential are $1/2\ell$-close. More precisely, using $\|f\|_{C^1}$, we can find $\delta > 0$ arbitrarily small so that the images by f_N of the δ-interiors of the cubes of the tiling are disjoint; so that the rates of injectivity on the cubes add. This proves the lemma. □

With the same kind of proof, we get the same result for arbitrary times t.

Theorem 11.8. *Let $f \in \mathrm{Diff}^1(\mathbf{T}^n)$ (or $f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$) be a generic diffeomorphism. Then $\tau^k(f)$ is well defined (that is, the limit exists) and satisfies:*

$$\tau^k(f) = \int_{\mathbf{T}^n} \tau^k\left(Df_x, \cdots, Df_{f^{k-1}(x)}\right) \mathrm{d}\,\mathrm{Leb}(x).$$

Moreover, the function τ^k is continuous in f.

Corollary 11.9. *The mean rate of injectivity in time t (see also Definition 8.3)*

$$\overline{\tau}^k(f) = \int_{\mathbf{T}^n} \overline{\tau}^k\left(Df_x, \cdots, Df_{f^{k-1}(x)}\right) \mathrm{d}\,\mathrm{Leb}(x)$$

is continuous (and even locally Lipschitz) on $\mathrm{Diff}^1(\mathbf{T}^n)$, and coincides with the rate of injectivity when the diffeomorphism f is generic.

Corollary 11.10. *As the asymptotic rate of injectivity $\lim_{k\to+\infty} \tau^k(f)$ is upper semi continuous at every generic diffeomorphism f, that is, for every $\varepsilon > 0$ and every g close enough to f, we have*

$$\lim_{k\to+\infty} \tau^k(g) \leq \lim_{k\to+\infty} \tau^k(f) + \varepsilon.$$

11.2 Asymptotic rate of injectivity for a generic dissipative diffeomorphism

First of all, we tackle the issue of the asymptotic rate of injectivity of generic dissipative diffeomorphisms. We will deduce it from a theorem of A. Avila and J. Bochi. Again, we will consider the torus $\mathbf{T}^n$ equipped with Lebesgue measure Leb and the canonical measures E_N, see Section 11.4 for a more general setting where the result is still true. The study of the rate of injectivity for generic dissipative diffeomorphisms is based on the following theorem of A. Avila and J. Bochi.

Theorem 11.11 (Avila, Bochi)**.** *Let f be a generic C^1 maps of $\mathbf{T}^n$. Then for every $\varepsilon > 0$, there exists a compact set $\mathrm{K} \subset \mathbf{T}^n$ and an integer $m \in \mathbf{N}$ such that*

$$\mathrm{Leb}(\mathrm{K}) > 1 - \varepsilon \qquad \textit{and} \qquad \mathrm{Leb}(f^m(\mathrm{K})) < \varepsilon.$$

This statement is obtained by combining Lemma 1 and Theorem 1 of [AB06].

Remark 11.12. As C^1 expanding maps of $\mathbf{T}^n$ and C^1 diffeomorphisms of $\mathbf{T}^n$ are open subsets of the set of C^1 maps of $\mathbf{T}^n$, the same theorem holds for generic C^1 expanding maps and C^1 diffeomorphisms of $\mathbf{T}^n$ (this had already been proved in the case of C^1-expanding maps by A. Quas in [Qua99]).

This theorem can be used to compute the asymptotic rate of injectivity of a generic diffeomorphism.

Corollary 11.13. *The asymptotic rate of injectivity of a generic dissipative diffeomorphism $f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$ is equal to 0. In particular, the degree of recurrence $\mathrm{D}(f_\mathrm{N})$ of a generic dissipative diffeomorphism tends to 0 when N goes to infinity.*

Proof of Corollary 11.13. The proof of this corollary mainly consists in stating which good properties can be supposed to possess the compact set K of Theorem 11.11. Thus, for f a generic diffeomorphism and $\varepsilon > 0$, there exists $m > 0$ and a compact set K such that $\mathrm{Leb}(K) > 1 - \varepsilon$ and $\mathrm{Leb}(f^m(K)) < \varepsilon$.

First of all, it can be easily seen that Theorem 11.11 is still true when the compact set K is replaced by an open set O: simply consider an open set $O' \supset f^m(K)$ such that $\mathrm{Leb}(O') < \varepsilon$ (by regularity of the measure Leb) and set $O = f^{-m}(O') \supset K$. We then approach the set O by unions of dyadic cubes of $\mathbf{T}^n$: we define the cubes of order 2^M on $\mathbf{T}^n$

$$C_{M,i} = \prod_{j=1}^{n} \left[\frac{i_j}{2^M}, \frac{i_j + 1}{2^M} \right],$$

and set

$$U_M = \mathrm{Int}\left(\overline{\bigcup_{C_{M,i} \subset O} C_{M,i}} \right),$$

where Int denotes the interior. Then, the union $\bigcup_{M \in \mathbf{N}} U_M$ is increasing in M and we have $\bigcup_{M \in \mathbf{N}} U_M = O$. In particular, there exists $M_0 \in \mathbf{N}$ such that $\mathrm{Leb}(U_{M_0}) > 1 - \varepsilon$, and as $U_{M_0} \subset O$, we also have $\mathrm{Leb}(f^m(U_{M_0})) < \varepsilon$. We denote $U = U_{M_0}$. Finally, as U is a finite union of cubes, and as f is a diffeomorphism, there exists $\delta > 0$ such that the measure of the δ-neighbourhood of $f^m(U)$ is smaller than ε. We call V this δ-neighbourhood.

As U is a finite union of cubes, there exists $N_0 \in \mathbf{N}$ such that if $N \geq N_0$, then the proportion of points of E_N which belong to U is bigger than $1 - 2\varepsilon$, and the proportion of points of E_N which belong to V is smaller than 2ε. Moreover, if N_0 is large enough, then for every $N \geq N_0$, and for every $x_N \in E_N \cap U$, we have $f_N^m(x_N) \in V$. This implies that

$$\frac{\mathrm{Card}(f_N^m(E_N))}{\mathrm{Card}(E_N)} \leq 4\varepsilon,$$

which proves the corollary. □

11.3 Asymptotic rate of injectivity for a generic conservative diffeomorphism

The goal of this section is to prove that the degree of recurrence of a generic conservative C^1-diffeomorphism is equal to 0. It will be obtained by using the local-global formula (Theorem 11.8) and the result about the asymptotic rate of injectivity of a generic sequence of matrices (Theorem 8.24). As a warm-up, we begin by a weak and very easy version of this statement, which already shows that the discretizations of generic conservative C^1-diffeomorphism do not behave in the same fashion as discretization of a generic conservative homeomorphism.

Proposition 11.14. *For a generic conservative diffeomorphism $f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$, we have $\overline{\lim}_{N \to +\infty} \tau_N^1(f) < 1$, in particular $\overline{\lim}_{N \to +\infty} D(f_N) < 1$.*

Proof of Proposition 11.14. To prove this proposition, we prove that on an open and dense set of diffeomorphisms f, we have $\overline{\lim}_{N \to +\infty} \tau_N^1(f) < 1$. This is easily obtained by perturbing locally any diffeomorphism. Indeed, we take a diffeomorphism $f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$ and a point $x \in \mathbf{T}^n$. Then, we perturb the differential of f at x such that $\overline{\tau}^1(Df_x) < 1$ (the set of matrices satisfying this property is locally included in a finite

union of submanifolds of codimension at least 1, see Corollary 8.22). We then apply the local linearization theorem to get a diffeomorphism which is linear in a neighbourhood of x. Thus, $\overline{\tau}^1(f) < 1$ by the local-global formula Theorem 11.8, and this remains true on a whole neighbourhood of f by the continuity of $\overline{\tau}^1$ (Corollary 11.9). □

Theorem 11.15. *For a generic conservative diffeomorphism $f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$, we have*

$$\lim_{t\to\infty} \tau^k(f) = 0;$$

more precisely, for every $\varepsilon > 0$, the set of diffeomorphisms $f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$ such that $\lim_{t\to+\infty} \tau^k(f) < \varepsilon$ is open and dense.

In particular[3]*, we have $\lim_{N\to+\infty} D(f_N) = 0$.*

Proof of theorem 11.15. We show that for every $\ell \in \mathbf{N}$ and every $\varepsilon > 0$, the set of conservative diffeomorphisms such that $\overline{\lim}_{t\to\infty} \tau^t_\infty < 1/\ell + \varepsilon$ contains an open dense subset of $\mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$. To begin with, we fix $f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$ and $\delta > 0$ (which will be a size of perturbation of f). By Theorem 8.24, and in particular Equation (8.9), there exists a parameter $\lambda \in]0,1[$ (depending only on δ, ℓ and $\|f\|_{C^1}$), such that for every sequence $(A_k)_{k\geq 1}$ of linear maps in $\mathrm{SL}_n(\mathbf{R})$, there exists a sequence $(B_k)_{k\geq 1}$ of (generic) linear maps in $\mathrm{SL}_n(\mathbf{R})$ such that for each k, we have $\|A_k - B_k\| \leq \delta$ and $\tau^{\ell k}(B_1, \cdots, B_{\ell k}) \leq \lambda^k + 1/\ell$ (as the sequence is generic, this property remains true on a whole neighbourhood of $(B_k)_{k\geq 1}$, see Remark 8.43). From that parameter λ, we deduce a time $k_0 > 0$ such that

$$\frac{1}{k_0} \sum_{k=1}^{k_0} \lambda^k = \frac{\lambda}{k_0} \frac{1-\lambda^{k_0}}{1-\lambda} < \varepsilon/100.$$

Applying a classical technique in this context (see for example [Boc02]), we use a Rokhlin tower of height k_0 with an open basis U:

- The sets $U, f(U), \cdots, f^{k_0-1}(U)$ are pairwise disjoint;
- the measure of the union of the "floors" $U \cup f(U) \cup \cdots \cup f^{k_0-1}(U)$ is bigger than $1 - \varepsilon/100$

For the existence of such towers, see for example [Gui12, Lemme 6.8] or [Hal56, Chapter "Uniform topology"].

We then approach the basis U by a union of cubes of a dyadic subdivision of $\mathbf{T}^n$ (as in page 234, from now we suppose that U is a union of such cubes). If that dyadic subdivision is fine enough, on each cube C, it is possible to perturb f into a diffeomorphism g such that on each set $(1 - \varepsilon/100)C$, g is affine and irrational, using Lemma 11.6 and Franks lemma (see [Fra71] or [Cro06b]), which is valid only in the C^1 topology. We do the same thing on the $k_0 - 1$ first images of each cube and perturb g such that on each set $g^k((1 - \varepsilon/100)C)$, the perturbed diffeomorphism h is linear and equal to B_k. By what we have said at the beginning of the proof, we can moreover suppose that the sequence $B_1, \cdots, B_{k_0}$ of linear maps is generic, and satisfies $\tau^k(B_1, \cdots, B_k) \leq \lambda^k + 1/\ell$ for

3. Using Equation (11.1) page 228.

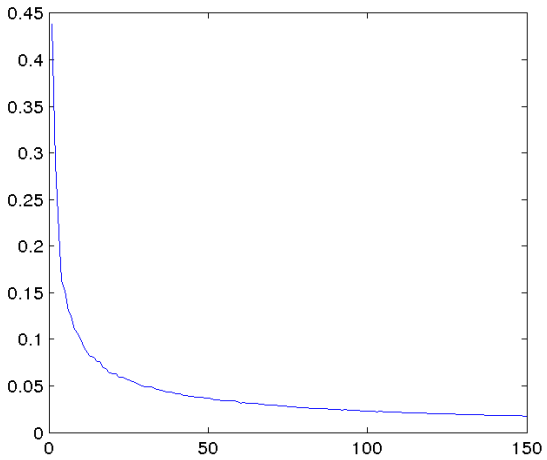

Figure 11.2: Simulation of the degree of recurrence $D((f_5)_N)$ depending on N, on the grids E_N with $N = 128k$, $k = 1, \cdots, 150$.

every $k \leq k_0 - 1$. By the choice of k_0 we have made, this implies that

$$\begin{aligned}
\tau^{k_0}(h) \leq & \sum_{k=0}^{k_0-1} \mathrm{Leb}(h^k(U))\tau^{k_0}\Big(h_{|h^k(U)}\Big) \\
& + \mathrm{Leb}\Big((U \cup \cdots \cup h^{k_0-1}(U))^{\complement}\Big)\tau^{k_0}\Big(h_{(U\cup\cdots\cup h^{k_0-1}(U))^{\complement}}\Big) \\
\leq & \sum_{k=0}^{k_0-1} \mathrm{Leb}(h^k(U))\tau^{k_0-k}\Big(h_{|h^k(U)}\Big) + \mathrm{Leb}\Big((U \cup \cdots \cup h^{k_0-1}(U))^{\complement}\Big) \\
\leq & \mathrm{Leb}(U) \sum_{k=0}^{k_0-1} \Big(\lambda^{k_0-k} + \frac{1}{\ell}\Big) + \varepsilon/100 \\
\leq & \frac{1}{k_0} \sum_{k=1}^{k_0} (\lambda^k + 1)/2 + \varepsilon/100 \\
\leq & 1/\ell + \varepsilon/2.
\end{aligned}$$

Moreover, the differentials of h form generic sequences on a set of measure at least $1-\varepsilon/10$. This implies that the rate of injectivity os continuous in h when restricted to this subset of $\mathbf{T}^n$. Thus, the inequality $\tau^{k_0}(h) \leq 1/\ell + \varepsilon$ still holds on a whole neighbourhood of h. This proves the theorem. □

Numerical simulation

We recall the results of numerical simulations we have presented in Part 1 (Figure 5.7). We have computed numerically the degree of recurrence of the diffeomorphism f_5, which is C^1-close to Id. Recall that $f_5 = Q \circ P$, with

$$P(x,y) = \Big(x, y + p(x)\Big) \quad \text{and} \quad Q(x,y) = \Big(x + q(y), y\Big),$$

$$p(x) = \frac{1}{209}\cos(2\pi \times 17x) + \frac{1}{271}\sin(2\pi \times 27x) - \frac{1}{703}\cos(2\pi \times 35x),$$

$$q(y) = \frac{1}{287}\cos(2\pi \times 15y) + \frac{1}{203}\sin(2\pi \times 27y) - \frac{1}{841}\sin(2\pi \times 38y).$$

On Figure 11.2, we have represented graphically the quantity $\mathrm{D}((f_5)_{128k})$ for k from 1 to 150. It appears that, as predicted by Theorem 11.15, this degree of recurrence goes to 0. In fact, it is even decreasing, and converges quite fast to 0: as soon as $\mathrm{N} = 128$, the degree of recurrence is smaller than 1/2, and if $\mathrm{N} \gtrsim 1000$, then $\mathrm{D}((f_5)_\mathrm{N}) \leq 1/10$.

11.4 A more general setting where the theorems are still true

Here, we give weaker assumptions under which the theorems of Sections 11.1, 11.2, 11.3 and 12.1.1 (that is, Theorems 11.3, 11.8, 11.15 and 12.1, and Corollaries 11.9, 11.13 and 11.10) are still true: the framework "torus $\mathbf{T}^n$ with grids E_N and Lebesgue measure" could be seen as a little too restrictive.

So, we take a compact smooth manifold M (possibly with boundary) and choose a partition $\mathrm{M}_1, \cdots, \mathrm{M}_k$ of M into closed sets [4] with smooth boundaries, such that for every i, there exists a chart $\varphi_i : \mathrm{M}_i \to \mathbf{R}^n$. We endow $\mathbf{R}^n$ with the euclidean distance, which defines a distance on M *via* the charts ϕ_i (this distance is not necessarily continuous). From now, we study what happens on a single chart, as what happens on the neighbourhoods of the boundaries of these charts "counts for nothing" from the Lebesgue measure viewpoint.

Finally, we suppose that the uniform measures on the grids $\mathrm{E}_\mathrm{N} = \bigcup_i \mathrm{E}_{\mathrm{N},i}$ converge to a smooth measure λ on M when N goes to infinity.

This can be easily seen that these conditions are sufficient for Corollary 11.13 to be still true.

For the rest of the statements of this chapter, we need that the grids behave locally as the canonical grids on the torus.

For every i, we choose a sequence $(\kappa_{\mathrm{N},i})_\mathrm{N}$ of positive real numbers such that $\kappa_{\mathrm{N},i} \underset{\mathrm{N}\to+\infty}{\longrightarrow} 0$. This defines a sequence $\mathrm{E}_{\mathrm{N},i}$ of grids on the set M_i by $\mathrm{E}_{\mathrm{N},i} = \varphi_i^{-1}(\kappa_{\mathrm{N},i}\mathbf{Z}^n)$. Also, the canonical projection $\pi : \mathbf{R}^n \to \mathbf{Z}^n$ (see Definition 7.11) allows to define the projection $\pi_{\mathrm{N},i}$, defined as the projection on $\kappa_{\mathrm{N},i}\mathbf{Z}^n$ in the coordinates given by φ_i:

$$\begin{array}{rccl} \pi_{\mathrm{N},i} : & \mathrm{M}_i & \longrightarrow & \mathrm{E}_{\mathrm{N},i} \\ & x & \longmapsto & \varphi_i^{-1}\Big(\kappa_{\mathrm{N},i}\,\pi\Big(\kappa_{\mathrm{N},i}^{-1}\varphi_i(x)\Big)\Big). \end{array}$$

We easily check that under these conditions, Theorems 11.3, 11.8 and 11.15 are still true, that is if we replace the torus $\mathbf{T}^n$ by M, the uniform grids by the grids E_N, the canonical projections by the projections $\pi_{\mathrm{N},i}$, and Lebesgue measure by the measure λ.

11.5 A local-global formula for C^r-generic expanding maps and C^r-generic diffeomorphisms

The goal of this section is to study the rate of injectivity of generic C^r-expanding maps of the torus $\mathbf{T}^n$. Here, the term expanding map is taken from the point of view of discretizations: we say that a linear map A is expanding if there does not exist two distinct integer points $x, y \in \mathbf{Z}^n$ such that $\widehat{\mathrm{A}}(x) = \widehat{\mathrm{A}}(y)$. The main result of this section

4. That is, $\bigcup_i \mathrm{M}_i = \mathrm{M}$, and for $i \neq j$, the intersection between the interiors of M_i and M_j are empty.

is that, as for generic C^1-diffeomorphisms of the torus (see Theorem 11.8), the rate of injectivity of a generic C^r-expanding map of the torus $\mathbf{T}^n$ is obtained from a local-global formula (Theorem 11.19).

Here, the definition of the linear analogue of the rate of injectivity of an expanding map in time k is more complicated than for diffeomorphisms: in this case, the set of preimages has a structure of d-ary tree. We define the rate of injectivity of a tree — with edges decorated by linear expanding maps — as the probability of percolation of a random graph associated to this decorated tree (see Definition 11.18). In particular, if all the expanding maps are equal, then the connected component of the root of this random graph is a Galton-Watson tree.

From the local-global formula for expanding maps, we can easily deduce that for every $d \geq 2$, there exists a map ψ_d tending to 0 in $1/d$, such that if f is a generic C^r-expanding map of the torus $\mathbf{T}^n$, then its asymptotic degree of recurrence satisfies $\tau^\infty(f) \leq \psi_d(1/\|f\|_\infty)$ (Corollary 11.20); in particular on the circle, if f is C^1-close to $x \mapsto dx$, then its asymptotic degree of recurrence is close to 0. In dimension 1, this result is a weak version of a theorem of P.P. Flockermann (Corollary 2 page 69 and Corollary 3 page 71 of [Flo02]), stating that for a generic $C^{1+\alpha}$ expanding map f of the circle, the asymptotic rate of injectivity $\tau^\infty(f)$ is equal to 0. We hope that more sophisticated techniques can be used to deduce from Theorem 11.19 a similar result for any dimension.

The proof of the local-global formula can be reduced to a linear lemma (Lemma 11.21) by a suitable application of Taylor's formula. To prove this linear lemma, we use crucially the formalism of model sets, and an improvement of Weyl's criterion.

The strategy of proof of Theorem 11.19 can be used to generalize the local-global formula for generic C^1-diffeomorphisms (Theorem 11.8) to generic C^r-diffeomorphisms, with $r \geq 1$, see Theorem 11.26.

We begin by the definition of the set of expanding maps.

Definition 11.16. For $r \geq 1$ and $d \geq 2$, we denote by $\mathscr{D}^r(\mathbf{T}^n)$ the set of C^r "$\mathbf{Z}^n$-expanding maps" of $\mathbf{T}^n$ for the infinite norm. More precisely, $\mathscr{D}^r(\mathbf{T}^n)$ is the set of maps $f : \mathbf{T}^n \to \mathbf{T}^n$, which are local diffeomorphisms, such that the derivative $f^{(\lfloor r \rfloor)}$ is well defined and belongs to $C^{r-\lfloor r \rfloor}(\mathbf{T}^n)$ and such that for every $x \in \mathbf{T}^n$ and every $v \in \mathbf{Z}^n \setminus \{0\}$, we have $\|\mathrm{D}f_x v\|_\infty \geq 1$.

In particular, for $f \in \mathscr{D}^r(\mathbf{T}^n)$, the number of preimages of any point of $\mathbf{T}^n$ is equal to a constant, that we denote by d.

Remark that in dimension $n = 1$, the set $\mathscr{D}^r(\mathbf{S}^1)$ coincides with the classical set of expanding maps: $f \in \mathscr{D}^r(\mathbf{S}^1)$ if and only if it belongs to $C^r(\mathbf{S}^1)$ and $f'(x) \geq 1$ for every $x \in \mathbf{S}^1$.

First of all, we define the linear setting corresponding to a map $f \in \mathscr{D}(\mathbf{T}^n)$.

Definition 11.17. We set (see also Figure 11.3)

$$\mathrm{I}_k = \bigsqcup_{m=1}^{k} [\![1, d]\!]^m$$

the set of m-tuples of integers of $[\![1, d]\!]$, for $1 \leq m \leq k$.

For $\boldsymbol{i} = (i_1, \cdots, i_m) \in [\![1, d]\!]^m$, we set $\text{length}(\boldsymbol{i}) = m$ and $\text{father}(\boldsymbol{i}) = (i_1, \cdots, i_{m-1}) \in [\![1, d]\!]^{m-1}$ (with the convention $\text{father}(i_1) = \emptyset$).

The set I_k is the linear counterpart of the set $\bigsqcup_{m=1}^{k} f^{-m}(y)$. Its cardinal is equal to $d(1-d^k)/(1-d)$.

Definition 11.18. Let $k \in \mathbf{N}$. The *complete tree of order k* is the rooted d-ary tree T_k whose vertices are the elements of I_k together with the root, denoted by $\emptyset$, and whose edges are of the form $(\text{father}(\boldsymbol{i}), \boldsymbol{i})_{\boldsymbol{i} \in \mathrm{I}_k}$ (see Figure 11.3).

Let $(p_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathrm{I}_k}$ be a family of numbers belonging to $[0, 1]$. These probabilities will be seen as decorations of the edges of the tree T_k. We will call *random graph associated to* $(p_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathrm{I}_k}$ the random subgraph $\mathrm{G}_{(p_i)_i}$ of T_k, such that the laws of appearance the edges $(\text{father}(\boldsymbol{i}), \boldsymbol{i})$ of $\mathrm{G}_{(p_i)_i}$ are independent Bernoulli laws of parameter $p_{\boldsymbol{i}}$. In other words, $\mathrm{G}_{(p_i)_i}$ is obtained from T_k by erasing independently each vertex of T_k with probability $1 - p_{\boldsymbol{i}}$.

We define the *mean density* $\overline{\mathrm{D}}((p_{\boldsymbol{i}})_{\boldsymbol{i}})$ of $(p_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathrm{I}_k}$ as the probability that in $\mathrm{G}_{(p_i)_i}$, there is at least one path linking the root to a leaf.

Remark that if the probabilities $p_{\boldsymbol{i}}$ are constant equal to p, the random graph $\mathrm{G}_{(p_i)_i}$ is a Galton-Watson tree, where the probability for a vertex to have i children is equal to $\binom{d}{i} p^i (1-p)^{d-i}$.

By the notation $\overline{\mathrm{D}}((\det \mathrm{D} f_x^{-1})_{x \in f^{-m}(y), 1 \le m \le k})$, we will mean that the mean density is taken with respect to the decorated tree whose edges are the $f^{-m}(y)$ for $0 \le m \le k$, and whose vertices are of the form $(f(x), x)$ for $x \in f^{-m}(y)$ with $1 \le m \le k$, each one being decorated by the number $\det \mathrm{D} f_x^{-1}$ (see Figure 11.4).

Recall that the rates of injectivity are defined by(see also Definition 11.2)

$$\tau_{\mathrm{N}}^k(f) = \frac{\mathrm{Card}\left((f_{\mathrm{N}})^k(\mathrm{E}_{\mathrm{N}})\right)}{\mathrm{Card}(\mathrm{E}_{\mathrm{N}})} \qquad \text{and} \qquad \tau^k(f) = \overline{\lim_{\mathrm{N} \to +\infty}} \tau_{\mathrm{N}}^k(f).$$

Theorem 11.19. *Let $r \ge 1$, f a generic element of $\mathscr{D}^r(\mathbf{T}^n)$ and $k \in \mathbf{N}$. Then, $\tau^k(f)$ is a limit (that is, the sequence $(\tau_{\mathrm{N}}^k(f))_{\mathrm{N}}$ converges), and we have*

$$\tau^k(f) = \int_{\mathbf{T}^n} \overline{\mathrm{D}}\Big((\det \mathrm{D} f_x^{-1})_{\substack{1 \le m \le k \\ x \in f^{-m}(y)}} \Big) \, \mathrm{d}\,\mathrm{Leb}(y). \tag{11.4}$$

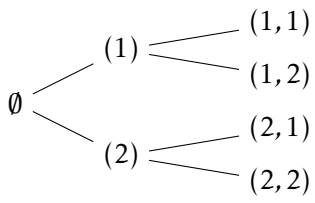

Figure 11.3: The tree T_2 for $d = 2$.

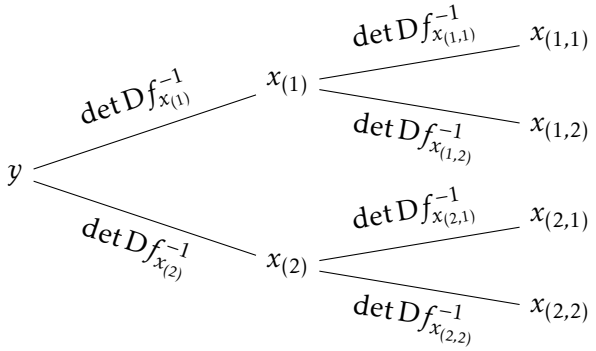

Figure 11.4: The probability tree associated to the preimages of y, for $k = 2$ and $d = 2$. We have $f(x_{(1,1)}) = f(x_{(1,2)}) = x_{(1)}$, etc.

Moreover, the map $f \mapsto \tau^k(f)$ is continuous in f.

Note that in dimension 1, P.P. Flockermann has proved that for a generic $C^{1+\alpha}$ expanding map f of the circle, the asymptotic rate of injectivity $\tau^\infty(f)$ is equal to 0 (Corollary 2 page 69 and Corollary 3 page 71 of [Flo02]). Note also that in C^1 regularity, the equality $\tau^\infty(f) = 0$ for a generic f is a consequence of Theorem 11.11 of A. Avila and J. Bochi (see also Corollary 11.13); the same theorem even proves that the asymptotic rate of injectivity of a generic C^1 endomorphism of the circle is equal to 0. We hope to recover these results as corollaries of Theorem 11.19. For now, we can only deduce the following result.

Corollary 11.20. *Let $r \geq 1$ and f a generic element of $\mathscr{D}^r(\mathbf{T}^n)$. If every point of $\mathbf{T}^n$ has d preimages by f, then $\tau^\infty(f) \leq \psi_d(1/\|f\|_\infty)$, where $\psi_d(p)$ denotes the unique fixed point of the map*

$$\begin{array}{rccc} Q_{d,p} :]0,1] & \longrightarrow &]0,1] \\ x & \longmapsto & 1-(1-px)^d. \end{array}$$

Moreover, the function ψ_d satisfies

$$2\frac{p-1/d}{p^2(d-1)} \leq \psi_d(p) \leq 2\frac{p-1/d}{p^2(d-1)(1-p)^{d-2}},$$

in particular $\psi_d(p) \underset{p\to 1/d}{\longrightarrow} 0$.

For example, if $d = 2$, then $\psi_d(p) = (2p-1)/p^2$. Also, if f is a generic C^r-expanding map of the circle C^1-close to $x \mapsto dx$, then Corollary 11.20 implies that its asymptotic rate of injectivity is close to 0.

Proof of Corollary 11.20. By the definition of $\overline{D}$, and the inclusion-exclusion principle, we see that

$$\tau^{k+1}(f) \leq \sum_{j=1}^{d} (-1)^{j+1} \binom{d}{j} p^j \big(\tau^k(f)\big)^j = Q_{d,p}\big(\tau^k(f)\big).$$

The existence and uniqueness of the fixed point of $Q_{d,p}$, and also the estimation on $\psi_d(p) = 0$, are easily deduced from Taylor's theorem, and the fact that $Q_{d,p}$ is increasing and concave on $]0,1]$. □

We now begin the proof of Theorem 11.19. It is mainly based on the following lemma, which treats the linear corresponding case. Its statement is divides into two parts, the second one being a quantitative version of the first.

Lemma 11.21. *Let $k \in \mathbf{N}$, and a family $(A_{\boldsymbol{i}})_{\boldsymbol{i}\in I_k}$ of invertible matrices, such that for any $\boldsymbol{i} \in I_k$ and any $v \in \mathbf{Z}^n \setminus \{0\}$, we have $\|A_{\boldsymbol{i}} v\|_\infty \geq 1$.*

If the image of the map

$$\mathbf{Z}^n \ni x \mapsto \bigoplus_{\boldsymbol{i}\in I_k} A_{\boldsymbol{i}}^{-1} A_{\mathrm{father}(\boldsymbol{i})}^{-1} \cdots A_{\mathrm{father}^{\mathrm{length}(\boldsymbol{i})}(\boldsymbol{i})}^{-1} x$$

projects on a dense subset of the torus $\mathbf{R}^{n\,\mathrm{Card}\,I_k}/\mathbf{Z}^{n\,\mathrm{Card}\,I_k}$, then we have

$$D\left(\bigcup_{\boldsymbol{i}\in[\![1,d]\!]^k} \big(\widehat{A}_{\mathrm{father}^{k-1}(\boldsymbol{i})} \circ \cdots \circ \widehat{A}_{\boldsymbol{i}}\big)(\mathbf{Z}^n)\right) = \overline{D}\big((\det A_{\boldsymbol{i}}^{-1})_{\boldsymbol{i}}\big).$$

More precisely, for every $\ell', c \in \mathbf{N}$, there exists a locally finite union of positive codimension submanifolds V_q of $(\mathrm{GL}_n(\mathbf{R}))^{\mathrm{Card}\, I_k}$ such that for every $\eta' > 0$, there exists a radius $R_0 > 0$ such that if $(A_{\boldsymbol{i}})_{\boldsymbol{i}\in I_k}$ satisfies $d((A_{\boldsymbol{i}})_{\boldsymbol{i}}, V_q) > \eta'$ for every q, then for every $R \geq R_0$, and every family $(v_{\boldsymbol{i}})_{\boldsymbol{i}\in I_k}$ of vectors of $\mathbf{R}^n$, we have [5]

$$\left| D_R^+ \left(\bigcup_{\boldsymbol{i}\in[\![1,d]\!]^k} \left(\pi(A_{\mathrm{father}^{k-1}(\boldsymbol{i})} + v_{\mathrm{father}^{k-1}(\boldsymbol{i})}) \circ \cdots \circ \pi(A_{\boldsymbol{i}} + v_{\boldsymbol{i}})\right)(\mathbf{Z}^n) \right) - \overline{D}\left((\det A_{\boldsymbol{i}}^{-1})_{\boldsymbol{i}}\right) \right| < \frac{1}{\ell'} \tag{11.5}$$

(the density of the image set is "almost invariant" under perturbations by translations), and for every $m \leq k$ and every $\boldsymbol{i} \in [\![1,d]\!]^k$, we have [6]

$$D_R^+ \left\{ x \in \left(A_{\mathrm{father}^m(\boldsymbol{i})} + v_{\mathrm{father}^m(\boldsymbol{i})}\right)(\mathbf{Z}^n) \;\middle|\; d\left(x, (\mathbf{Z}^n)'\right) < \frac{1}{c\ell'(2n+1)} \right\} < \frac{1}{c\ell'} \tag{11.6}$$

(there is only a small proportion of the points of the image sets which are obtained by discretizing points close to $(\mathbf{Z}^n)'$).

The local-global formula (11.4) will later follow from this lemma and an appropriate application of Taylor's theorem.

The next lemma uses the strategy of proof of Weyl's criterion to get a uniform convergence in Birkhoff's theorem for rotations in the torus $\mathbf{T}^n$ whose rotation vectors are outside of a neighbourhood of a finite union of hyperplanes.

Lemma 11.22 (Weyl)**.** *Let* dist *be a distance generating the weak-* topology on $\mathscr{P}$. Then, for every $\varepsilon > 0$, there exists a locally finite family of affine hyperplanes $H_i \subset \mathbf{R}^n$, such that for every $\eta > 0$, there exists $M_0 \in \mathbf{N}$, such that for every $\boldsymbol{\lambda} \in \mathbf{R}^n$ satisfying $d(\boldsymbol{\lambda}, H_q) > \eta$ for every q, and for every $M \geq M_0$, we have*

$$\mathrm{dist}\left(\frac{1}{M} \sum_{m=0}^{M-1} \bar{\delta}_{m\boldsymbol{\lambda}},\ \mathrm{Leb}_{\mathbf{R}^n/\mathbf{Z}^n} \right) < \varepsilon,$$

where $\bar{\delta}_x$ is the Dirac measure of the projection of x on $\mathbf{R}^n/\mathbf{Z}^n$.

Proof of Lemma 11.22. As dist generates the weak-* topology on $\mathscr{P}$, it can be replaced by any other distance also generating the weak-* topology on $\mathscr{P}$. So we consider the distance dist_W defined by:

$$\mathrm{dist}_W(\mu,\nu) = \sum_{\boldsymbol{k}\in\mathbf{N}^n} \frac{1}{2^{k_1+\cdots+k_n}} \left| \int_{\mathbf{R}^n/\mathbf{Z}^n} e^{i2\pi\boldsymbol{k}\cdot x}\, \mathrm{d}(\mu-\nu)(x) \right|;$$

there exists $K > 0$ and $\varepsilon' > 0$ such that if a measure $\mu \in \mathscr{P}$ satisfies

$$\forall \boldsymbol{k} \in \mathbf{N}^n : 0 < k_1 + \cdots + k_n \leq K, \quad \left| \int_{\mathbf{R}^n/\mathbf{Z}^n} e^{i2\pi\boldsymbol{k}\cdot x}\, \mathrm{d}\,\mu(x) \right| < \varepsilon', \tag{11.7}$$

then $\mathrm{dist}(\mu, \mathrm{Leb}) < \varepsilon$.

For every $\boldsymbol{k} \in \mathbf{N}^n \setminus \{0\}$ and $j \in \mathbf{Z}$, we set

$$H_{\boldsymbol{k}}^j = \{\boldsymbol{\lambda} \in \mathbf{R}^n \mid \boldsymbol{k}\cdot\boldsymbol{\lambda} = j\}.$$

5. The map $\pi(A+v)$ is the discretization of the affine map $A+v$.
6. Recall that $(\mathbf{Z}^n)'$ stands for the set of points of $\mathbf{R}^n$ at least one coordinate of which belongs to $\mathbf{Z}+1/2$.

Remark that the family $\{H_{\boldsymbol{k}}^j\}$, with $j \in \mathbf{Z}$ and $\boldsymbol{k}$ such that $0 < k_1 + \cdots + k_n \leq K$, is locally finite. We denote by $\{H_q\}_q$ this family, and choose $\boldsymbol{\lambda} \in \mathbf{R}^n$ such that $d(\boldsymbol{\lambda}, H_q) > \eta$ for every q. We also take

$$M_0 \geq \frac{2}{\varepsilon' \left|1 - e^{i2\pi\eta}\right|}. \tag{11.8}$$

Thus, for every $\boldsymbol{k} \in \mathbf{N}^n$ such that $k_1 + \cdots + k_n \leq K$, and every $M \geq M_0$, the measure

$$\mu = \frac{1}{M} \sum_{m=0}^{M-1} \bar{\delta}_{m\boldsymbol{\lambda}}.$$

satisfies

$$\left| \int_{\mathbf{R}^n/\mathbf{Z}^n} e^{i2\pi \boldsymbol{k}x} \, \mathrm{d}\mu(x) \right| = \frac{1}{M} \left| \frac{1 - e^{i2\pi M \boldsymbol{k}\cdot\boldsymbol{\lambda}}}{1 - e^{i2\pi \boldsymbol{k}\cdot\boldsymbol{\lambda}}} \right| \leq \frac{2}{M_0} \frac{1}{\left|1 - e^{i2\pi \boldsymbol{k}\cdot\boldsymbol{\lambda}}\right|}.$$

By (11.8) and the fact that $d(\boldsymbol{k}\cdot\boldsymbol{\lambda}, \mathbf{Z}) \geq \eta$, we deduce that

$$\left| \int_{\mathbf{R}^n/\mathbf{Z}^n} e^{i2\pi \boldsymbol{k}x} \, \mathrm{d}\mu(x) \right| \leq \varepsilon'.$$

Thus, the measure μ satisfies the criterion (11.7), which proves the lemma. □

Proof of Lemma 11.21. To begin with, let us treat the case $d = 1$. Let $A_1, \cdots, A_k$ be k invertible matrices. We want to compute the rate of injectivity of $\widehat{A}_k \circ \cdots \circ \widehat{A}_1$. Recall that we set

$$\widetilde{M}_{A_1,\cdots,A_k} = \begin{pmatrix} A_1 & -\mathrm{Id} & & & \\ & A_2 & -\mathrm{Id} & & \\ & & \ddots & \ddots & \\ & & & A_{k-1} & -\mathrm{Id} \\ & & & & A_k \end{pmatrix} \in M_{nk}(\mathbf{R}),$$

$\widetilde{\Lambda}_k = \widetilde{M}_{\lambda_1,\cdots,\lambda_k}\mathbf{Z}^{nk}$ and $W^k =]-1/2, 1/2]^{nk}$. Resuming the proof of Proposition 8.40, we see that $x \in \left(\widehat{A}_k \circ \cdots \circ \widehat{A}_1\right)(\mathbf{Z}^n)$ if and only if $(0^{n(k-1)}, x) \in W^k + \widetilde{\Lambda}_k$. This implies the following statement.

Lemma 11.23. *We have*

$$\det(A_k \cdots A_1) D\left(\widehat{A}_k \circ \cdots \circ \widehat{A}_1\right)(\mathbf{Z}^n) = \nu(\mathrm{pr}_{\mathbf{R}^{nk}/\widetilde{\Lambda}_k}(W^k)), \tag{11.9}$$

where ν is the uniform measure on the submodule $\mathrm{pr}_{\mathbf{R}^{nk}/\widetilde{\Lambda}_k}(0^{n(k-1)}, \mathbf{Z}^n)$ of $\mathbf{R}^{nk}/\widetilde{\Lambda}_k$.

In particular, if the image of the map

$$\mathbf{Z}^n \ni x \mapsto \bigoplus_{m=1}^{k} (A_m)^{-1} \cdots (A_k)^{-1} x$$

projects on a dense subset of the torus $\mathbf{R}^{nk}/\mathbf{Z}^{nk}$, then the quantity (11.9) is equal to the volume of the intersection between the projection of W^k on $\mathbf{R}^{nk}/\widetilde{\Lambda}_k$ and a fundamental domain of $\widetilde{\Lambda}_k$. By the hypothesis made on the matrices A_m, that is, for any $v \in \mathbf{Z}^n \setminus \{0\}$, $\|A_m v\|_\infty \geq 1$, this volume is equal to 1 (simply because the restriction to W^k of the projection $\mathbf{R}^{nk} \mapsto \mathbf{R}^{nk}/\widetilde{\Lambda}_k$ is injective). Thus, the density of the set $\left(\widehat{A}_k \circ \cdots \circ \widehat{A}_1\right)(\mathbf{Z}^n)$ is equal to $1/(\det(A_k \cdots A_1))$.

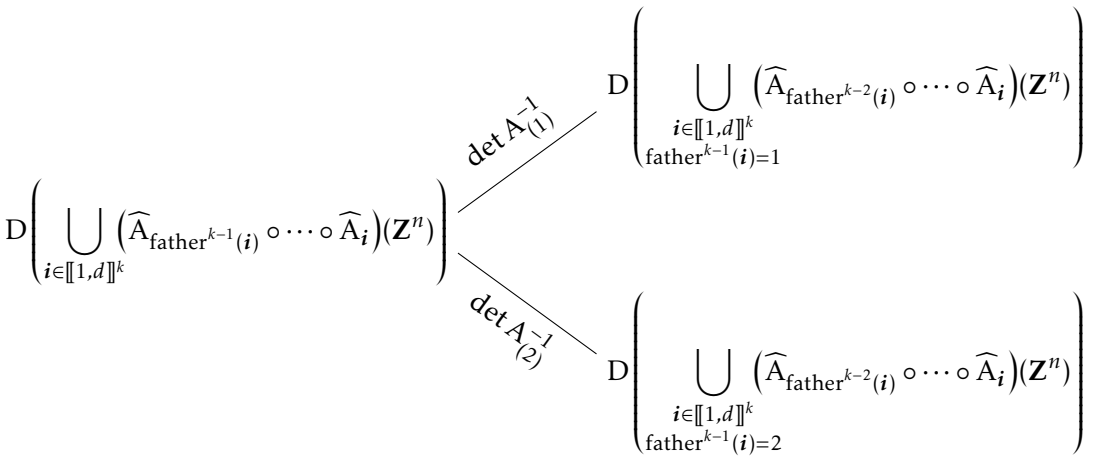

Figure 11.5: Calculus of the density of the image set at the level k according to the density of its sons.

We now consider a family $(\mathrm{A}_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathrm{I}_k}$ of invertible matrices, such that for any $\boldsymbol{i}\in\mathrm{I}_k$ and any $v\in\mathbf{Z}^n\setminus\{0\}$, we have $\|\mathrm{A}_{\boldsymbol{i}}v\|_\infty\geq 1$. A point $x\in\mathbf{Z}^n$ belongs to

$$\bigcup_{\boldsymbol{i}\in[\![1,d]\!]^k}\left(\widehat{\mathrm{A}}_{\mathrm{father}^{k-1}(\boldsymbol{i})}\circ\cdots\circ\widehat{\mathrm{A}}_{\boldsymbol{i}}\right)(\mathbf{Z}^n)$$

if and only if there exists $\boldsymbol{i}\in[\![1,d]\!]^k$ such that $(0^{m-1},x)\in\mathrm{W}^k+\widetilde{\Lambda}_{\boldsymbol{i}}$. Equivalently, a point $x\in\mathbf{Z}^n$ does not belong to the set

$$\widehat{\mathrm{A}}_{\mathrm{father}^{k-1}(\boldsymbol{i})}\circ\cdots\circ\widehat{\mathrm{A}}_{\boldsymbol{i}}$$

if and only if for every $\boldsymbol{i}\in[\![1,d]\!]^k$, we have $(0^{n(k-1)},x)\notin\mathrm{W}^k+\widetilde{\Lambda}_{\boldsymbol{i}}$. Thus, if the image of the map

$$\mathbf{Z}^n\ni x\mapsto\bigoplus_{\boldsymbol{i}\in\mathrm{I}_k}\mathrm{A}_{\boldsymbol{i}}^{-1}\mathrm{A}_{\mathrm{father}(\boldsymbol{i})}^{-1}\cdots\mathrm{A}_{\mathrm{father}^{\mathrm{length}(i)}(\boldsymbol{i})}^{-1}x$$

projects on a dense subset of the torus $\mathbf{R}^{n\operatorname{Card}\mathrm{I}_k}/\mathbf{Z}^{n\operatorname{Card}\mathrm{I}_k}$, then the events $x\in\mathrm{S}_i$, with

$$\mathrm{S}_i=\bigcup_{\substack{\boldsymbol{i}\in[\![1,d]\!]^k\\ \mathrm{father}^{k-1}(\boldsymbol{i})=i}}\left(\widehat{\mathrm{A}}_{\mathrm{father}^{k-1}(\boldsymbol{i})}\circ\cdots\circ\widehat{\mathrm{A}}_{\boldsymbol{i}}\right)(\mathbf{Z}^n)$$

are independent (see Figure 11.5), meaning that for every $\mathrm{F}\subset[\![1,d]\!]$, we have

$$\mathrm{D}\left(\bigcap_{i\in\mathrm{F}}\mathrm{S}_i\right)=\prod_{i\in\mathrm{F}}\mathrm{D}\big(\mathrm{S}_i\big). \tag{11.10}$$

Thus, by the inclusion-exclusion principle, we get

$$\mathrm{D}\left(\bigcup_{i\in[\![1,d]\!]}\mathrm{S}_i\right)=\sum_{\emptyset\neq\mathrm{F}\subset[\![1,d]\!]}(-1)^{\operatorname{Card}(\mathrm{F})+1}\prod_{i\in\mathrm{F}}\mathrm{D}\big(\mathrm{S}_i\big).$$

Moreover, the fact that for any $\boldsymbol{i} \in \mathrm{I}_k$ and any $v \in \mathbf{Z}^n \setminus \{0\}$, we have $\|\mathrm{A}_{\boldsymbol{i}} v\|_\infty \geq 1$ leads to

$$\mathrm{D}(\mathrm{S}_i) = \det \mathrm{A}^{-1}_{\mathrm{father}^{k-1}(\boldsymbol{i})}\, \mathrm{D}\left(\bigcup_{\substack{\boldsymbol{i} \in [\![1,d]\!]^k \\ \mathrm{father}^{k-1}(\boldsymbol{i}) = i}} \left(\widehat{\mathrm{A}}_{\mathrm{father}^{k-2}(\boldsymbol{i})} \circ \cdots \circ \widehat{\mathrm{A}}_{\boldsymbol{i}} \right)(\mathbf{Z}^n) \right).$$

These facts imply that the density we look for follows the same recurrence relation as $\overline{\mathrm{D}}\big((\det \mathrm{A}_{\boldsymbol{i}}^{-1})_{\boldsymbol{i}}\big)$, thus

$$\mathrm{D}\left(\bigcup_{\boldsymbol{i} \in [\![1,d]\!]^k} \left(\widehat{\mathrm{A}}_{\mathrm{father}^{k-1}(\boldsymbol{i})} \circ \cdots \circ \widehat{\mathrm{A}}_{\boldsymbol{i}} \right)(\mathbf{Z}^n) \right) = \overline{\mathrm{D}}\big((\det \mathrm{A}_{\boldsymbol{i}}^{-1})_{\boldsymbol{i}}\big).$$

The second part of the lemma is an effective improvement of the first one. To obtain the bound (11.5), we combine Lemma 11.22 with Lemma 11.23 to get that for every $\varepsilon > 0$, there exists a locally finite collection of submanifolds V_q of $(\mathrm{GL}_n(\mathbf{R}))^{\mathrm{Card}\,\mathrm{I}_k}$ with positive codimension, such that for every $\eta' > 0$, there exists $\mathrm{R}_0 > 0$ such that if $d((\mathrm{A}_{\boldsymbol{i}})_{\boldsymbol{i}}, \mathrm{V}_q) > \eta'$ for every q, then Equation (11.10) is true up to ε.

The other bound (11.6) is obtained independently from the rest of the proof by a direct application of Lemma 11.22 and of Lemma 11.23 applied to $k = 1$. □

Lemma 11.24 (Perturbations in C^r topology)**.** *Let $r \geq 1$ and f a generic element of $\mathscr{D}^r(\mathbf{T}^n)$. Then, for every $k \in \mathbf{N}$, every $\ell' \in \mathbf{N}$ and every finite collection (V_q) of submanifolds of positive codimension of $(\mathrm{GL}_n(\mathbf{R}))^{dm}$, there exists $\eta > 0$ such that the set*

$$\mathrm{T}_\eta = \left\{ y \in \mathbf{T}^n \;\middle|\; \forall q,\, d\Big(\big(\mathrm{D}f_x \big)_{\substack{1 \leq m \leq k \\ x \in f^{-m}(y)}}, \mathrm{V}_q \Big) > \eta \right\}$$

contains a finite union of cubes, whose union has measure bigger than $1 - 1/\ell'$.

Proof of Lemma 11.24. By Thom's transversality theorem, for a generic map $f \in \mathscr{D}^r(\mathbf{T}^n)$, the set

$$\left\{ y \in \mathbf{T}^n \;\middle|\; \forall q,\, \big(\mathrm{D}f_x \big)_{\substack{1 \leq m \leq k \\ x \in f^{-m}(y)}} \in \mathrm{V}_q \right\}$$

if finite. Thus, the sets $\mathrm{T}_\eta^{\mathrm{C}}$ are compact sets and their (decreasing) intersection over η is a finite set. So, there exists $\eta > 0$ such that $\mathrm{T}_\eta^{\mathrm{C}}$ is close enough to this finite set for Hausdorff topology to have the conclusions of the lemma. □

We can now begin the proof of Theorem 11.19.

Proof of Theorem 11.19. Let $f \in \mathscr{D}^r(\mathbf{T}^n)$. The idea is to cut the torus $\mathbf{T}^n$ into small pieces on which f is very close to its Taylor expansion at order 1.

Let $m \in \mathbf{N}$, and $\mathscr{U}_\ell$ ($\ell \in \mathbf{N}^*$) be the set of maps $f \in \mathscr{D}^r(\mathbf{T}^n)$ such that the set of accumulation points of the sequence $(\tau_{\mathrm{N}}^m(f))_{\mathrm{N}}$ is included in the ball of radius $1/\ell$ and centre

$$\int_{\mathbf{T}^n} \overline{\mathrm{D}}\Big((\mathrm{D}f_x)_{\substack{1 \leq m \leq k \\ x \in f^{-m}(y)}} \Big)\, \mathrm{d}\,\mathrm{Leb}(y).$$

(that is, the right side of Equation (11.4)). We want to show that $\mathscr{U}_\ell$ contains an open and dense subset of $\mathscr{D}^r(\mathbf{T}^n)$. In other words, we pick a map f, an integer ℓ and $\delta > 0$,

and we want to find another map $g \in \mathscr{D}^r(\mathbf{T}^n)$ which is δ-close to f for the C^r distance, and which belongs to the interior of $\mathscr{U}_\ell$.

To do that, we first set $\ell' = 3\ell$ and $c = d(1-d^k)/(1-d) = \mathrm{Card}(\mathrm{I}_k)$, and use Lemma 11.21 to get a locally finite union of positive codimension submanifolds V_q of $(\mathrm{GL}_n(\mathbf{R}))^{\mathrm{Card}(\mathrm{I}_k)}$. We then apply Lemma 11.24 to these submanifolds, to the δ we have fixed at the beginning of the proof and to $\ell' = 4\ell$; this gives us a parameter $\eta > 0$ and a map $g \in \mathscr{D}^r(\mathbf{T}^n)$ such that $d_{C^r}(f,g) < \delta$, and such that the set

$$\left\{ y \in \mathbf{T}^n \;\middle|\; \forall q,\, d\Big(\big(\mathrm{D}g_x\big)_{\substack{1\le m\le k \\ x\in g^{-m}(y)}}, \mathrm{V}_q\Big) < \eta \right\}$$

is contained in a finite union $\mathscr{C}$ of cubes, whose union has measure bigger than $1 - 1/(4\ell)$. Finally, we apply Lemma 11.21 to $\eta' = \eta/2$; this gives us a radius $\mathrm{R}_0 > 0$ such that if $(\mathrm{A}_{\boldsymbol{i}})_{\boldsymbol{i}\in \mathrm{I}_k}$ is a family of matrices of $\mathrm{GL}_n(\mathbf{R})$ satisfying $d((\mathrm{A}_{\boldsymbol{i}})_{\boldsymbol{i}}, \mathrm{V}_q) > \eta/2$ for every q, then for every $\mathrm{R} \ge \mathrm{R}_0$, and every family $(v_{\boldsymbol{i}})_{\boldsymbol{i}\in \mathrm{I}_k}$ of vectors of $\mathbf{R}^n$, we have

$$\left| \mathrm{D}^+_{\mathrm{R}} \left(\bigcup_{\boldsymbol{i}\in[\![1,d]\!]^k} \big(\pi(\mathrm{A}_{\mathrm{father}^{k-1}(\boldsymbol{i})} + v_{\mathrm{father}^{k-1}(\boldsymbol{i})}) \circ \cdots \circ \pi(\mathrm{A}_{\boldsymbol{i}} + v_{\boldsymbol{i}})\big)(\mathbf{Z}^n) \right) - \overline{\mathrm{D}}\big((\det \mathrm{A}_{\boldsymbol{i}}^{-1})_{\boldsymbol{i}}\big) \right| < \frac{1}{3\ell}, \tag{11.11}$$

and for every i, j,

$$\mathrm{D}^+_{\mathrm{R}} \left\{ x \in \big(\mathrm{A}_{j^m(\boldsymbol{i})} + v_{j^m(\boldsymbol{i})}\big)(\mathbf{Z}^n) \;\middle|\; d\big(x, (\mathbf{Z}^n)'\big) < \frac{1}{3\ell(2n+1)\,\mathrm{Card}\,\mathrm{I}_k} \right\} < \frac{1}{3\ell\,\mathrm{Card}\,\mathrm{I}_k}. \tag{11.12}$$

We now take a map $h \in \mathscr{D}^r(\mathbf{T}^n)$ such that $d_{C^1}(g,h) < \delta'$, and prove that if δ' is small enough, then h belongs to the interior of $\mathscr{U}_\ell$. First of all, we remark that if δ' is small enough, then the set

$$\left\{ y \in \mathbf{T}^n \;\middle|\; \forall q,\, d\Big(\big(\mathrm{D}h_x\big)_{\substack{1\le m\le k \\ x\in h^{-m}(y)}}, \mathrm{V}_q\Big) > \eta/2 \right\}$$

contains a set $\mathscr{C}'$, which is a finite union of cubes whose union has measure bigger than $1 - 1/(3\ell)$.

Let C be a cube of $\mathscr{C}'$, $y \in \mathrm{C}$ and $x \in f^{-m}(y)$, with $1 \le m \le k$. We write the Taylor expansion of order 1 of h at the neighbourhood of x; by compactness we obtain

$$\sup\left\{ \frac{1}{\|z\|} \big\| h(x+z) - h(x) - \mathrm{D}h_x(z) \big\| \;\middle|\; x \in \mathrm{C},\, z \in \mathrm{B}(0,\rho) \right\} \underset{\rho\to 0}{\longrightarrow} 0.$$

Thus, for every $\varepsilon > 0$, there exists $\rho > 0$ such that for all $x \in \mathrm{C}$ and all $z \in \mathrm{B}(0,\rho)$, we have

$$\big\| h(x+z) - \big(h(x) + \mathrm{D}h_x(z)\big) \big\| < \varepsilon \|z\| \le \varepsilon\rho. \tag{11.13}$$

We now take $\mathrm{R} \ge \mathrm{R}_0$. We want to find an order of discretization N such that the error made by linearizing h on $\mathrm{B}(x, \mathrm{R}/\mathrm{N})$ is small compared to N, that is, for every $z \in \mathrm{B}(0, \mathrm{R}/\mathrm{N})$, we have

$$\big\| h(x+z) - \big(h(x) + \mathrm{D}h_x(z)\big) \big\| < \frac{1}{3\ell(2n+1)\,\mathrm{Card}\,\mathrm{I}_k} \cdot \frac{1}{\mathrm{N}}.$$

To do that, we apply Equation (11.13) to

$$\varepsilon = \frac{1}{3\mathrm{R}\ell(2n+1)\operatorname{Card} \mathrm{I}_k},$$

to get a radius $\rho > 0$ (we can take ρ as small as we want), and we set $\mathrm{N} = \lceil \mathrm{R}/\rho \rceil$ (thus, we can take N as big as we want). By (11.13), for every $z \in \mathrm{B}(0,\mathrm{R})$, we obtain the desired bound:

$$\Big\| h(x+z/\mathrm{N}) - \big(h(x) + \mathrm{D}h_x(z/\mathrm{N})\big) \Big\| < \frac{1}{3\ell(2n+1)\operatorname{Card} \mathrm{I}_k} \cdot \frac{1}{\mathrm{N}}.$$

Combined with (11.12), this leads to

$$\frac{\operatorname{Card}\Big(h_{\mathrm{N}}\big(\mathrm{B}(x,\mathrm{R}/\mathrm{N})\big) \Delta \mathrm{P}_{\mathrm{N}}\big(h(x) + \mathrm{D}h_x(\mathrm{B}(0,\mathrm{R}/\mathrm{N}))\big)\Big)}{\operatorname{Card}\big(\mathrm{B}(x,\mathrm{R}/\mathrm{N}) \cap \mathrm{E}_{\mathrm{N}}\big)} \leq \frac{1}{3\ell \operatorname{Card} \mathrm{I}_k}; \tag{11.14}$$

in other words, on every ball of radius R/N, the image of E_{N} by h_{N} and the discretization of the linearization of h are almost the same (that is, up to a proportion $1/(3\ell \operatorname{Card} \mathrm{I}_k)$ of points).

We now set $\mathrm{R}_1 = \mathrm{R}_0 \|f'\|_\infty^m$, and choose $\mathrm{R} \geq \mathrm{R}_1$, to which is associated a number $\rho > 0$ and an order $\mathrm{N} = \lceil \mathrm{R}/\rho \rceil$, that we can choose large enough so that $2\mathrm{R}/\mathrm{N} \leq \|f'\|_\infty$. We also choose $y \in \mathrm{C}$. As

$$\operatorname{Card}\big(h_{\mathrm{N}}^m(\mathrm{E}_{\mathrm{N}}) \cap \mathrm{B}(y,\mathrm{R}/\mathrm{N})\big) = \operatorname{Card}\left(\bigcup_{x \in h^{-m}(y)} h_{\mathrm{N}}^m\big(\mathrm{B}(x,\mathrm{R}/\mathrm{N}) \cap \mathrm{E}_{\mathrm{N}}\big) \cap \mathrm{B}(y,\mathrm{R}/\mathrm{N}) \right),$$

and using the estimations (11.11) and (11.14), we get

$$\left| \frac{\operatorname{Card}\big(h_{\mathrm{N}}^m(\mathrm{E}_{\mathrm{N}}) \cap \mathrm{B}(y,\mathrm{R}/\mathrm{N})\big)}{\operatorname{Card}\big(\mathrm{B}(y,\mathrm{R}/\mathrm{N}) \cap \mathrm{E}_{\mathrm{N}}\big)} - \overline{\mathrm{D}}\Big((\det \mathrm{D}f_x^{-1})_{\substack{1 \leq m \leq k \\ x \in f^{-m}(y)}} \Big) \right| < \frac{2}{3\ell},$$

and as such an estimation holds on a subset of $\mathbf{T}^n$ of measure bigger than $1 - 1/(3\ell)$, we get the conclusion of the theorem. □

We can easily adapt the proof of Lemma 11.21 to the case of sequences of matrices, without the hypothesis of expansivity.

Lemma 11.25. *For every $k \in \mathbf{N}$ and every $\ell', c \in \mathbf{N}$, there exists a locally finite union of positive codimension submanifolds V_q of $(\mathrm{GL}_n(\mathbf{R}))^k$ (respectively $(\mathrm{SL}_n(\mathbf{R}))^k$) such that for every $\eta' > 0$, there exists a radius $\mathrm{R}_0 > 0$ such that if $(\mathrm{A}_m)_{1 \leq m \leq k}$ is a finite sequence of matrices of $(\mathrm{GL}_n(\mathbf{R}))^k$ (respectively $(\mathrm{SL}_n(\mathbf{R}))^k$) satisfying $d((\mathrm{A}_m)_m, \mathrm{V}_q) > \eta'$ for every q, then for every $\mathrm{R} \geq \mathrm{R}_0$, and every family $(v_m)_{1 \leq m \leq k}$ of vectors of $\mathbf{R}^n$, we have*

$$\Big| \mathrm{D}_{\mathrm{R}}^+\big((\widehat{\mathrm{A}}_k \circ \cdots \circ \widehat{\mathrm{A}}_1)(\mathbf{Z}^n) \big) - \det(\mathrm{A}_k^{-1} \cdots \mathrm{A}_1^{-1}) \overline{\tau}^k(\mathrm{A}_1, \cdots, \mathrm{A}_k) \Big| < \frac{1}{\ell'}$$

(the density of the image set is "almost invariant" under perturbations by translations), and for every $m \leq k$, we have [7]

$$\mathrm{D}_{\mathrm{R}}^+ \left\{ x \in \big(\mathrm{A}_m + v_m\big)(\mathbf{Z}^n) \;\middle|\; d\big(x, (\mathbf{Z}^n)'\big) < \frac{1}{c\ell'(2n+1)} \right\} < \frac{1}{c\ell'}$$

(there is only a small proportion of the points of the image sets which are obtained by discretizing points close to $(\mathbf{Z}^n)'$).

7. Recall that $(\mathbf{Z}^n)'$ stands for the set of points of $\mathbf{R}^n$ at least one coordinate of which belongs to $\mathbf{Z} + 1/2$.

With the same proof as Theorem 11.19, Lemma 11.25 leads to the following improvement of Theorem 11.8, for higher regularity.

Theorem 11.26. *Let $r \geq 1$, and $f \in \mathrm{Diff}^r(\mathbf{T}^n)$ (or $f \in \mathrm{Diff}^r(\mathbf{T}^n, \mathrm{Leb})$) be a generic diffeomorphism. Then $\tau^k(f)$ is well defined (that is, the limit exists) and satisfies:*

$$\tau^k(f) = \int_{\mathbf{T}^n} \tau^k\left(Df_x, \cdots, Df_{f^{k-1}(x)}\right) \mathrm{d}\,\mathrm{Leb}(x).$$

Moreover, the function τ^k is continuous in f.

Numerical simulation

We present the results of the numerical simulation we have conducted for the degree of recurrence of the expanding map of the circle f, defines dy

$$f(x) = 2x + \varepsilon_1 \cos(2\pi x) + \varepsilon_2 \sin(6\pi x),$$

with $\varepsilon_1 = 0.127\,943\,563\,72$ and $\varepsilon_2 = 0.008\,247\,359\,61$.

On Figure 11.6, we have represented the quantity $\mathrm{D}((f)_{128k})$ for k from 1 to 1 000. It appears that, as predicted by the theorem of P.P. Flockermann (Corollary 2 page 69 and Corollary 3 page 71 of [Flo02]), this degree of recurrence seems to tend to 0. In fact, it is even decreasing, and converges quite fast to 0: as soon as $\mathrm{N} = 128$, the degree of recurrence is smaller than 1/5, and if $\mathrm{N} \gtrsim 25\,000$, then $\mathrm{D}((f)_{\mathrm{N}}) \leq 1/50$.

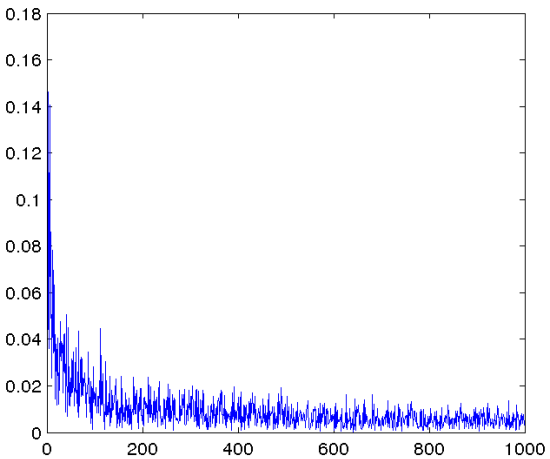

Figure 11.6: Simulation of the degree of recurrence $\mathrm{D}((f_5)_{\mathrm{N}})$ of the expanding map N, depending on N, on the grids E_{N} with $\mathrm{N} = 128k$, $k = 1, \cdots, 1\,000$.

Chapter 12

Physical measures of discretizations of generic diffeomorphisms and expanding maps

12.1 Physical measures of discretizations of generic diffeomorphisms

In this section, we will consider the torus $\mathbf{T}^n$, equipped with Lebesgue measure Leb and the uniform grids

$$\mathrm{E_N} = \left\{ \left(\frac{i_1}{\mathrm{N}}, \cdots, \frac{i_n}{\mathrm{N}} \right) \in \mathbf{R}^n/\mathbf{Z}^n \,\middle|\, 1 \leq i_1, \cdots, i_n \leq \mathrm{N} \right\}.$$

Fore a more general setting where the theorems of this section are still true, see Section 11.4.

This chapter is devoted to the study of the physical measures of the discretizations of a generic conservative diffeomorphism. Recall the classical definition of a physical measure for a map f: a Borel probability measure μ is called *physical* for the map f if its basin of attraction has positive Lebesgue measure, where the basin of attraction of μ is the set

$$\left\{ x \in \mathbf{T}^n \,\middle|\, \frac{1}{\mathrm{M}} \sum_{m=0}^{\mathrm{M}-1} \delta_{f^m(x)} \underset{\mathrm{M}\to+\infty}{\longrightarrow} \mu \right\}.$$

(see also Definition 3.9). Heuristically, the physical measures are the ones that can be observed in practical experiments, because they are "seen" by a set of points x of positive Lebesgue measure. Here, our aim is to study similar concepts in the view of discretizations: which measures can be seen by the discretizations of generic conservative C^1-diffeomorphisms?

Remember that we denote by $\mu_x^{f_\mathrm{N}}$ the limit of the Birkhoff sums

$$\frac{1}{\mathrm{M}} \sum_{m=0}^{\mathrm{M}-1} \delta_{f_\mathrm{N}^m(x_\mathrm{N})}.$$

More concretely, $\mu_x^{f_\mathrm{N}}$ is the f_N-invariant probability measure supported by the periodic orbit on which the positive orbit of $x_\mathrm{N} = \mathrm{P_N}(x)$ falls after a while (see also Definition 3.10

page 52). We would like to know the answer the following question: for a generic conservative C^1-diffeomorphism f, does the sequence of measure $\mu_x^{f_N}$ tend to a physical measure of f for most of the points x as N goes to infinity?

Recall that in the C^0 case, we have proved Theorem 5.51, which implies in particular that for a generic homeomorphism $f \in \mathrm{Homeo}(\mathbf{T}^n, \mathrm{Leb})$ and every $x \in \mathbf{T}^n$, the measures $\mu_x^{f_N}$ accumulate on the whole set of f-invariant measures when N goes to infinity (moreover, given an f-invariant measure μ, the sequence $(N_k)_{k\geq 0}$ such that $\mu_x^{f_{N_k}}$ tends to μ can be chosen independently of x). In a certain sense, this theorem in the case of homeomorphisms expresses that from the point of view of the discretizations, all the f-invariant measures are physical.

In the C^1-case, we have already proved Corollary 10.9 (in Chapter 10), which states that for a generic conservative C^1-diffeomorphism f, any f-invariant measure is the limit of a sequence of f_N-invariant measures. This corollary has been obtained as a simple consequence of an ergodic closing lemma of R. Mañé and F. Abdenur, C. Bonatti and S. Crovisier (Theorem 10.8); however it does not say anything about the basin of attraction of these discrete measures.

In the theorem we prove in this chapter (Theorem 12.1), we improve the previous statement for generic conservative C^1-diffeomorphisms, in order to describe the basin of attraction of the discrete measures. In particular, we prove that for points x belonging to a generic subset of points of the torus, the measures $\mu_x^{f_N}$ accumulate on the whole set of f-invariant measure (Theorem 12.1). Notice that given an f-invariant measure μ, the sequence $(N_k)_k$ such that $\mu_x^{f_{N_k}}$ converges to μ depends on the point x, contrary to what happens in the C^0 case.

Moreover, if we fix a countable subset $D \subset \mathbf{T}^n$, then for a generic conservative C^1-diffeomorphism f and for any $x \in D$, the measures $\mu_x^{f_N}$ accumulate on the whole set of f-invariant measures (Addendum 12.2). This is a process that is usually applied in practice to detect the f-invariant measures: fix a finite set $D \subset \mathbf{T}^n$ and compute the measure $\mu_x^{f_N}$ for $x \in D$ and for a large order of discretization N. The theoretical result expresses that it is possible that the measure that we observe on numerical experiments is very far away from the physical measure.

Note that in the space $\mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$, there are open sets where generic diffeomorphisms are ergodic: the set of Anosov diffeomorphisms is open in $\mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$, and a generic Anosov conservative C^1-diffeomorphism is ergodic (it is a consequence of the fact that any C^2 Anosov conservative diffeomorphism is ergodic, together with the theorem of regularization of conservative diffeomorphisms of A. Avila [Avi10]). More generally, A. Avila, S. Crovisier and A. Wilkinson have set recently in [ACW14] a generic dichotomy for a conservative diffeomorphism f: either f is ergodic, either all the Lyapunov exponents of f vanish. In short, there are open sets where generic conservative diffeomorphisms have only one physical measure; in this case, our result asserts that this physical measure is not detected on discretizations by computing the measures $\mu_x^{f_N}$.

Recall that results of stochastic stability are known to be true in various contexts (for example, expanding maps [Kif86a],[Kif86b], [Kel82], uniformly hyperbolic attractors [Kif86b], [You86], etc.). These theorems suggest that the physical measures can always be observed in practice, even if the system is noisy. Our Theorem 12.1 indicates that the effects of discretizations (i.e. numerical truncation) might be quite different from those of a random noise.

However, we shall remark that the same proof as for Theorem 12.1 implies that for a generic diffeomorphism $f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$ and a generic point $x \in \mathbf{T}^n$ (or equivalently, for any $x \in \mathbf{T}^n$ and for a generic $f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$), the measures

$$\mu^f_{x,m} = \frac{1}{m} \sum_{i=0}^{m-1} f_*^i \delta_x$$

accumulate on the whole set of f-invariant measures.

The proof of Theorem 12.1 uses crucially the results of Part 2 on the fact that the asymptotic rate is null (in particular Lemma 8.54); it also uses two connecting lemmas (Theorem 10.3 of [BC04] and an improvement of Theorem 10.8 of [ABC11]).

At the end of this chapter, we present numerical experiments simulating the measures $\mu_x^{f_\mathrm{N}}$ for some examples of conservative C^1-diffeomorphisms f of the torus. The results of these simulations are quite striking for an example of f C^1-close to Id (see Figure 12.7): even for very large orders N, the measures $\mu_x^{f_\mathrm{N}}$ do not converge to Lebesgue measure at all, and are very different ones from the others. This illustrates perfectly Theorem 12.1 (more precisely, Addendum 12.2), which states that if x is fixed, then for a generic $f \in \mathrm{Diff}^1(\mathbf{T}^2, \mathrm{Leb})$, the measures $\mu_x^{f_\mathrm{N}}$ accumulate on the whole set of f-invariant measures, but do not say anything about, for instance, the frequency of orders N such that $\mu_x^{f_\mathrm{N}}$ is not close to Lebesgue measure. Moreover, the same phenomenon (although less pronounced) occurs for diffeomorphisms close to a translation of $\mathbf{T}^2$ (Figure 12.8) or a linear Anosov automorphism (Figure 12.9).

12.1.1 Statement of the theorem

Theorem 12.1. *For a generic diffeomorphism $f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$, for a generic point $x \in \mathrm{X}$, for any f-invariant probability measure μ, there exists a subsequence $(\mathrm{N}_k)_k$ of discretizations such that*

$$\mu_x^{f_{\mathrm{N}_k}} \underset{k \to +\infty}{\longrightarrow} \mu.$$

Remark that the theorem in the C^0 case is almost the same, except that here, the starting point $x \in \mathbf{T}^n$ is no longer arbitrary but has to be chosen in a generic subset of the torus, and that the sequence $(\mathrm{N}_k)_k$ depends on the starting point x. The proof of this theorem will also lead to the two following statements.

Addendum 12.2. *For a generic diffeomorphism $f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$, for any $\varepsilon > 0$ there exists a ε-dense subset $(x_1, \cdots, x_m)$ such that for any f-invariant probability measure μ, there exists a subsequence $(\mathrm{N}_k)_k$ of discretizations such that for every j,*

$$\mu_{x_j}^{f_{\mathrm{N}_k}} \underset{k \to +\infty}{\longrightarrow} \mu.$$

Also, for any countable subset $\mathrm{D} \subset \mathbf{T}^n$, for a generic diffeomorphism $f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$, for any f-invariant probability measure μ, and for any finite subset $\mathrm{E} \subset \mathrm{D}$, there exists a subsequence $(\mathrm{N}_k)_k$ of discretizations such that for every $x \in \mathrm{E}$, we have

$$\mu_x^{f_{\mathrm{N}_k}} \underset{k \to +\infty}{\longrightarrow} \mu.$$

The first statement asserts that if f is a generic conservative C^1-diffeomorphism, then for any f-invariant measure μ, there exists an infinite number of discretizations f_N which possess an invariant measure which is close tu μ, and whose basin of attraction is ε-dense. Basically, for an infinite number of N any f-invariant will be seen from any region of the torus.

In the second statement, a countable set of starting points of the experiment is chosen "by the user". This is quite close to what happens in practice: we take a finite number of points $x_1, \cdots, x_m$ and compute the measures $\mu_{x_m,T}^{f_{N_k}}$ for all m, for a big $N \in \mathbf{N}$ and for "large" times T (we can expect that T is large enough to have $\mu_{x_m,T}^{f_{N_k}} \simeq \mu_{x_m}^{f_{N_k}}$). In this case, the result expresses that it may happen (in fact, for arbitrarily large N) that the measures $\mu_{x_m,T}^{f_{N_k}}$ are not close to the physical measure of f but are rather chosen "at random" among the set of f-invariant measures.

We also have a dissipative counterpart of Theorem 12.1, whose proof is easier.

Theorem 12.3. *For a generic dissipative diffeomorphism $f \in \mathrm{Diff}^1(\mathbf{T}^n)$, for any f-invariant probability measure μ such that the sum of the Lyapunov exponents of μ is negative (or equal to 0), for a generic point x belonging to the same chain recurrent class as μ, there exists a subsequence $(N_k)_k$ of discretizations such that*

$$\mu_x^{f_{N_k}} \underset{k\to+\infty}{\longrightarrow} \mu.$$

Remark that if we also consider the inverse f^{-1} of a generic diffeomorphism $f \in \mathrm{Diff}^1(\mathbf{T}^n)$, we can recover any invariant measure μ of f by looking at the measures $\mu_x^{f_{N_k}}$ for generic points x in the chain recurrent class of μ.

The proof of this result is obtained by applying Lemma 12.12 during the proof of Theorem 12.1.

We also have the same statement as Theorem 12.1 but for expanding maps of the circle. We denote $\mathscr{E}_d^1(\mathbf{S}^1)$ the set of C^1-expanding maps of the circle of degree d.

Proposition 12.4. *For a generic expanding map $f \in \mathscr{E}_d^1(\mathbf{S}^1)$, for any f-invariant probability measure μ, for a generic point $x \in \mathbf{S}^1$, there exists a subsequence $(N_k)_k$ of discretizations such that*

$$\mu_x^{f_{N_k}} \underset{k\to+\infty}{\longrightarrow} \mu.$$

The proof of this statement is far easier than that of Theorem 12.1 , as it can be obtained by coding any expanding map of class C^1 (that is, any $f \in \mathscr{E}_d^1(\mathbf{S}^1)$ is homeomorphic to a full shift on a set with d elements).

We will use the connecting lemma for pseudo-orbits (see [BC04], see also Theorem 10.3), together with an ergodic closing lemma (adapted from [ABC11]) and the results of Part 2 on the fact that the asymptotic rate is null (in particular Lemma 8.54), to prove that any invariant measure of the diffeomorphism can be observed by starting at any point of a generic subset of $\mathbf{T}^n$.

By Baire theorem and the fact that for a generic conservative diffeomorphism, a generic invariant measure is ergodic, non periodic and has no zero Lyapunov exponent (see Theorem 3.5 of [ABC11]), the proof of Theorem 12.1 can be reduced easily to that of the following approximation lemma.

Lemma 12.5. *For every $f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$, for every f-invariant measure μ which is ergodic, not periodic and has no zero Lyapunov exponent, for every open subset $U \subset \mathbf{T}^n$, for every C^1-neighbourhood $\mathscr{V}$ of f, for every $\varepsilon > 0$ and every $N_0 \in \mathbf{N}$, there exists $g \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$ such that $g \in \mathscr{V}$, there exists $y \in U$ and $N \geq N_0$ such that $\mathrm{dist}(\mu, \mu_y^{g_N}) < \varepsilon$. Moreover, we can suppose that this property remains true on a whole neighbourhood of g.*

First of all, we explain how to deduce Theorem 12.1 from Lemma 12.5.

Proof of Theorem 12.1. We consider a sequence $(\nu_\ell)_{\ell \geq 0}$ of Borel probability measures, which is dense in the whole set of probability measures. We also consider a sequence $(U_i)_{i \geq 0}$ of open subsets of $\mathbf{T}^n$ which spans the topology of $\mathbf{T}^n$. This allows us to set

$$\mathscr{S}_{N_0,k_0,\ell,i} = \left\{ f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb}) \;\middle|\; \begin{array}{l} \exists \mu \ f\text{-inv.} : \mathrm{dist}(\mu, \nu_\ell) \leq 1/k_0 \implies \\ \exists N \geq N_0, y \in U_i : d(\mu_y^{f_N}, \nu_\ell) < 2/k_0 \end{array} \right\}.$$

We easily see that the set

$$\bigcap_{N_0,k_0,\ell,i \geq 0} \mathscr{S}_{N_0,k_0,\ell,i}$$

in contained in the set of diffeomorphisms satisfying the conclusions of the theorem.

It remains to prove that each set $\mathscr{S}_{N_0,k_0,\ell,i}$ contains an open and dense subset of $\mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$. Actually the interior of each set $\mathscr{S}_{N_0,k_0,\ell,i}$ is dense. This follows from [1] the upper semi-continuity of the set of f-invariant measures with respect to f and from the combination of Lemma 12.5 with the fact that for a generic diffeomorphism, a generic invariant measure is ergodic, non periodic and has no zero Lyapunov exponent (see Theorem 3.5 of [ABC11]). □

The rest of this chapter is devoted to the proof of Lemma 12.5. We now outline the main arguments of this quite long and technical proof.

Sketch of proof of Lemma 12.5. First of all, we take a point $x \in \mathbf{T}^n$ which is typical for the measure μ. In particular, by an ergodic closing lemma derived from that of F. Abdenur, C. Bonatti and S. Crovisier [ABC11] (Lemma 12.6), there is a perturbation of f (still denoted by f) so that the orbit ω of x is periodic of period τ_1; moreover, ω can be supposed to bear an invariant measure close to μ, to have an arbitrary large length, and to have Lyapunov exponents and Lyapunov subspaces close to that of μ under f. Applying the (difficult) connecting lemma for pseudo-orbits of C. Bonatti and S. Crovisier [BC04], we get another perturbation of the diffeomorphism (still denoted by f), such that the stable manifold of x under f meets the open set U at a point that we denote by y.

So, we need to perturb the diffeomorphism f so that:

- the periodic orbit x is stabilized by f_N. This can be easily made by a small perturbation of f;
- the positive orbit of y under f_N falls on the periodic orbit of x under f_N. This is the difficult part of the proof: we can apply the previous strategy to put every point of the positive orbit of y on the grid only during a finite time. It becomes impossible to perform perturbations to put the orbit of y on the grid — without perturbing the orbit of x — as soon as this orbit comes into a C/N- neighbourhood of the orbit of x (where C is a constant depending on $\mathscr{V}$).

1. See the proof of Theorem 5.45 for more details on the same kind of arguments.

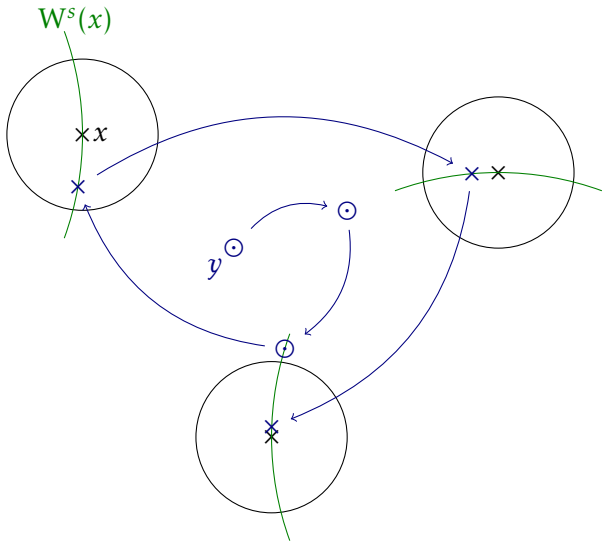

Figure 12.1: During the proof of Lemma 12.5, it is easy to perturb the first points of the orbit of y (small disks) until the orbit meets the neighbourhoods of the orbits of x where the diffeomorphism is linear (inside of the circles). The difficulty of the proof is to make appropriate perturbations in these small neighbourhoods.

To solve this problem, we need the results about the linear case we have proved in Chapter 8. They allow us to find a point z whose distance to the orbit of x is bigger than C/N, and such that for t large enough, $f_N^t(z)$ belongs to the orbit of x under f_N.

In more detail, we use Lemma 11.6 to linearize locally the diffeomorphism in the neighbourhood of the periodic orbit of ω. In particular, the positive orbit of y eventually belongs to this linearizing neighbourhood, from a time T_1. We denote $y' = f^{T_1}(y)$. To summarize, the periodic orbit ω bears a measure close to μ, its Lyapunov exponents are close to that of μ, its Lyapunov linear subspaces are close to that of μ (maybe not all along the periodic orbit, but at least for the first iterates of x). The diffeomorphism f is linear around each point of ω. Finally, the stable manifold of ω meets U at y, and the positive orbit of y is included in the neighbourhood of ω where f is linear from the point $y' = f^{T_1}(y)$.

We then choose an integer N large enough, and perturb the orbit of x such that it is stabilized by the discretization f_N. We want to make another perturbation of f such that the backward orbit of x by f_N also contains y' (recall that f_N is not necessarily one-to-one). This is done by a perturbation supported in the neighbourhood of ω where f is linear. First of all, during a time $t_4 \geq 0$, we apply Lemma 8.54 of Chapter 8 to find a point z in the neighbourhood of $f^{-t_4}(x)$ where f is linear, but far enough from $f^{-t_4}(x)$ compared to 1/N, such that the t_4-th image of z by the discretization f_N is equal to x. Next, we perturb the orbit of z under f^{-1} during a time $t_3 \geq 0$ such that $f^{-t_3}(z)$ belongs to the stable subspace of $f^{-t_4-t_3}(x)$. Note that the support of this perturbation must be disjoint from ω; this is the reason why z must be "far enough from x". Finally, we find another time t_2 such that the negative orbit $\{f^{-t}(z')\}_{t\geq 0}$ of $z' = f^{-t_3-t_2}(z)$ has an hyperbolic behaviour. We then perturb each point of the negative orbit of z' (within the stable manifold of ω), so that it contains an arbitrary point of the stable manifold of ω, far enough from ω. This allows us to meet the point y', provided that the order of

discretizations N is large enough.

To complete the proof, we we consider the segment of f-orbit joining y to z; we perturb each one of these points to put them on the grid E_N (with a perturbation whose supports size is proportional to 1/N).

Notice that we shall have chosen carefully the parameters of the first perturbations in order to make this final perturbation possible. Also, remark that the length of the periodic orbit ω must be very large compared to the times t_2, t_3 and t_4. This is why we will perform the proof in the opposite direction : we will begin by choosing the times t_i and make the perturbation of the dynamics afterwards.

Note that the Addendum 12.2 can be proved by using a small variation on Lemma 12.5, that we will explain at the end of Section 12.1.3.

12.1.2 An improved ergodic closing lemma

The proof of Theorem 12.1 begins by the approximation of any invariant measure μ of any conservative C^1-diffeomorphism by a periodic measure of a diffeomorphism g close to f. This is done by R. Mañé's ergodic closing lemma, but we will need the fact that the obtained periodic measure inherits some of the properties of the measure μ. More precisely, given a C^1-diffeomorphism f, we will have to approach any non periodic ergodic measure of f with nonzero Lyapunov exponent by a periodic measure of a diffeomorphism g close to f, such that the Lyapunov exponents and the Lyapunov subspaces of the measure are close to that of f by μ. We will obtain this result by modifying slightly the proof of a lemma obtained by F. Abdenur, C. Bonatti and S. Crovisier in [ABC11] (Proposition 6.1).

Lemma 12.6 (Ergodic closing lemma)**.** *Let $f \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$. We consider*

- *a number $\varepsilon > 0$;*
- *a C^1-neighbourhood $\mathscr{V}$ of f;*
- *a time $\tau_0 \in \mathbf{N}$;*
- *an ergodic measure μ without zero Lyapunov exponent;*
- *a point $x \in X$ which is typical for μ (see the beginning of the paragraph 6.1 of [ABC11]);*

moreover, we denote by λ the smallest absolute value of the Lyapunov exponents of μ, by F^f_x the stable subspace at x and by G^f_x the unstable subspace[2] at x. Then, there exists a diffeomorphism $g \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$ and a time $\widetilde{t_0} > 0$ (depending only in f, μ and x) such that:

1. *$g \in \mathscr{V}$;*
2. *the point x is periodic for g of period $\tau \geq \tau_0$;*
3. *for any $t \leq \tau$, we have $d\left(f^t(x), g^t(x)\right) < \varepsilon$;*
4. *x has no zero Lyapunov exponent for g and the smallest absolute value of the Lyapunov exponents of x is bigger than $\lambda/2$, we denote by F^g_x the stable subspace and G^g_x the unstable subspace;*
5. *the angles between F^f_x and F^g_x, and between G^f_x and G^g_x, are smaller than ε;*
6. *for any $t \geq \widetilde{t_0}$, for any vectors of unit norm $v_F \in F^g_x$ and $v_G \in G^g_x$, we have*

$$\frac{1}{t}\log\left(\|Dg_x^{-t}(v_F)\|\right) \geq \frac{\lambda}{4} \qquad \text{and} \qquad \frac{1}{t}\log\left(\|Dg_x^{t}(v_G)\|\right) \geq \frac{\lambda}{4}$$

2. Stable and unstable in the sense of Oseledets splitting.

Remark that the proof of Proposition 6.1 of [ABC11] yields a similar lemma but with the weaker conclusion

5. *"the angle between* G^f_x *and* G^g_x*, is smaller than* ε*".*

Indeed, the authors obtain the linear space G^g_x by a fixed point argument: Lemma 6.5 of [ABC11] states that the cone $C^s_{j,4C}$ is invariant by $Df_n^{-t_n}$, and thus contains both G^f_x and G^g_x. Taking C as big as desired, the cone $C^s_{j,4C}$ is as thin as desired and thus the angle between G^f_x and G^g_x, is as small as desired. Unfortunately, in the original proof of Proposition 6.1 of [ABC11], the linear space F^g_x is not defined in the same way ; it is an invariant subspace which belongs to $C^u_{j,4C}$, which is an arbitrarily thick cone. Thus, the angle between F^f_x and F^g_x, is not bounded by this method of proof. Our goal here is to modify the proof of Proposition 6.1 of [ABC11] to have simultaneously two thin cones $C'^u_{j,4C}$ and $C^s_{j,4C}$ which are invariant under respectively $Df_n^{t_n}$ and $Df_n^{-t_n}$

We begin by modifying the Lemma 6.2 of [ABC11]: we replace its forth point

– *a sequence of linear isometries* $P_n \in O_d(\mathbf{R})$ *such that* $\|P_n - \mathrm{Id}\| < \varepsilon$,

by the point

– *two sequences of linear isometries* $P_n, Q_n \in O_d(\mathbf{R})$ *such that* $\|P_n - \mathrm{Id}\| < \varepsilon$ *and* $\|Q_n - \mathrm{Id}\| < \varepsilon$,

and its forth conclusion

d) For every $i \le j \in \{1, \cdots, k\}$ *the inclination* [3] *of* $Df_n^{t_n}.E_{i,j}$ *with respect to* $E_{i,j}$ *is less than* C.

by the conclusion

d) For every $i \le j \in \{1, \cdots, k\}$ *the inclination of* $Df_n^{t_n}.E_{i,j}$ *with respect to* $E_{i,j}$ *is less than* C*, and the inclination of* $Df_n^{-t_n}.E_{i,j}$ *with respect to* $E_{i,j}$ *is less than* C.

These replacements in the lemma are directly obtained by replacing Claim 6.4 of [ABC11] by the following lemma.

Lemma 12.7. *For any* $\eta > 0$*, there exists a constant* $C > 0$ *such that for any matrix* $A \in GL_n(\mathbf{R})$ *and any linear subspace* $E \subset \mathbf{R}^n$*, there exists two orthogonal matrices* $P, Q \in O_n(\mathbf{R})$ *satisfying* $\|P - \mathrm{Id}\| < \eta$ *and* $\|Q - \mathrm{Id}\| < \eta$*, such that the inclinations of* $(PAQ)(E)$ *and* $(PAQ)^{-1}(E)$ *with respect to* E *are smaller than* C.

Proof of Lemma 12.7. Given $\eta > 0$, there exists a constant $C > 0$ and a matrix $P_0 \in O_n(\mathbf{R})$ such that $\|P_0 - \mathrm{Id}\| < \eta$, satisfying: for any linear subspace $E' \subset \mathbf{R}^n$, one of the two inclinations of E′ and of $P_0(E')$ with respect to E is smaller then C.

We then choose an orthogonal matrix $Q \in O_n(\mathbf{R})$ such that $\|Q - \mathrm{Id}\| < \eta$ and that (taking a bigger C if necessary) both inclinations of $Q^{-1}\big(A^{-1}(E)\big)$ and $Q^{-1}\big((A^{-1}P_0^{-1})(E)\big)$ with respect to E are smaller than C. There are two cases: either the inclination of $(AQ)(E)$ with respect to E is smaller than C, and in this case we choose $P = \mathrm{Id}$, or the inclination of $(AQ)(E)$ with respect to E is bigger than C, and in this case we can choose $P = P_0$. In both cases, the lemma is proved. □

The rest of the proof of Lemma 12.6 can be easily adapted from the proof of Proposition 6.1 of [ABC11].

3. The *inclination* of a linear subspace $E \subset \mathbf{R}^n$ with respect to another subspace $E' \subset \mathbf{R}^n$ with the same dimension is the minimal norm of the linear maps $f : E \to E^\perp$ whose graph are equal to E.

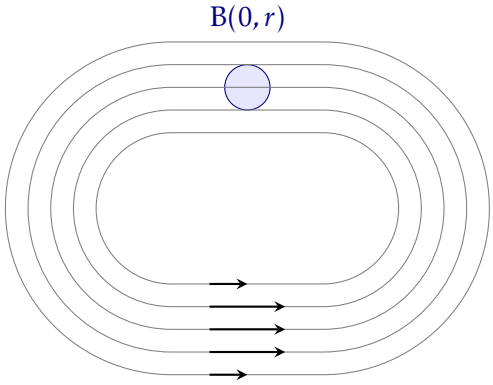

Figure 12.2: Flow of the Hamiltonian used to prove Lemma 12.9 ("staduim").

12.1.3 Proof of the perturbation lemma (Lemma 12.5)

We now come to the proof of Lemma 12.5. We first do this proof in dimension 2, to simplify some arguments and to be able to make pictures.

Proof of Lemma 12.5. Let f be a conservative C^1-diffeomorphism, $\mathscr{V}$ a C^1-neighbourhood of f, $\varepsilon > 0$ and $N_0 \in \mathbf{N}$. We denote $M = \max\left(\|Df\|_\infty, \|Df^{-1}\|_\infty\right)$. We also choose an f-invariant measure μ which is ergodic, not periodic and has no zero Lyapunov exponent, and an open set $U \subset \mathbf{T}^2$. We will make several successive approximations of f in $\mathscr{V}$; during the proof we will need to decompose this neighbourhood: we choose $\delta > 0$ such that the open δ-interior $\mathscr{V}'$ of $\mathscr{V}$ is non-empty.

Step 0: elementary perturbation lemmas. During the proof of Lemma 12.5, we will use three different elementary perturbation lemmas.

The first one is the elementary perturbation lemma in C^1 topology we stated in Chapter 10 (Lemma 10.1). It allows to perturb locally the orbit of a diffeomorphism.

The second one is an easy corollary of the first one. We will use it to perturb a segment of orbit such that for any N large enough, each point of this segment of orbit belongs to the grid E_N.

Lemma 12.8 (Perturbation of a point such that it belongs to the grid)**.** *For every open set $\mathscr{V}'$ of $\mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$, there exists $\eta' > 0$ such that for N large enough an every $x \in \mathbf{T}^n$, there exists $g \in \mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$ such that*

- *$g \in \mathscr{V}'$;*
- *$g(x_N) = \left(f(x)\right)_N$;*
- *$f = g$ outside of $B\left(x, (1+\eta')/N\right)$.*

Applying this lemma to several points $x_i \in \mathbf{T}^n$ which are far enough one from the others (for $i \neq j$, $d(x_i, x_j) \geq 2(1+\eta')/N$), it is possible to perturb f into a diffeomorphism g such that for every i, $g\left((x_i)_N\right) = \left(f(x_i)\right)_N$.

These two perturbations will be applied locally.

The third perturbation lemma is an improvement of Lemma 10.1; it states that the perturbation can be supposed to be a translation in a small neighbourhood of the perturbed point.

Lemma 12.9 (Elementary perturbation with local translation)**.** *For every open set $\mathscr{V}'$ of* $\mathrm{Diff}^1(\mathbf{T}^n, \mathrm{Leb})$*, and every $r > 0$, there exists $\mathrm{N}_1 > 0$ such that for every $\mathrm{N} \geq \mathrm{N}_1$ and every $\|v\|_\infty \leq 1/(2\mathrm{N})$, there exists $g \in \mathrm{Diff}^1(\mathbf{R}^n, \mathrm{Leb})$ such that:*

- *$g \in \mathscr{V}'$;*
- *$\mathrm{Supp}(g) \subset \mathrm{B}(0, 10\,r)$;*
- *for every $x \in \mathrm{B}(0, r)$, $g(x) = x + v$.*

Proof of Lemma 12.9. Take an appropriate Hamiltonian, see Figure 12.2. □

Step 1: choice of the starting point x of the orbit. Let λ be the smallest absolute value of the Lyapunov exponents of μ (in particular, $\lambda > 0$).

We choose a point x which is regular for the measure μ: we suppose that it satisfies the conclusions of Oseledets and Birkhoff theorems, and Mañé's ergodic closing lemma (see Paragraph 6.1 of [ABC11]). We denote by F^f_x the stable subspace and G^f_x the unstable subspace for the Oseledets splitting at the point x. By Oseledets theorem, the growth of the angles $\angle\big(\mathrm{F}^f_{f^i(x)}, \mathrm{G}^f_{f^i(x)}\big)$ between the stable and unstable subspaces is subexponential (in both positive and negative times).

Step 2: choice of the parameters we use to apply the ergodic closing lemma. In this second step, we determine the time during which we need an estimation of the angle between the stable and unstable subspaces of f and its perturbations, and the minimal length of the approximating periodic orbit.

We first use the "hyperbolic-like" behaviour of f near the orbit of x: for well chosen times t_1 and t_2, each vector which is not too close to $\mathrm{G}^f_{f^{t_1}(x)}$ is mapped by $\mathrm{D}f^{-t_2}$ into a vector which is close to $\mathrm{F}^f_{f^{t_1-t_2}(x)}$. Given a vector $v \in \mathrm{T}\mathbf{T}^n_{f^{t_1}(x)}$, it will allow us to perturb f into g such that an iterate of v under $\mathrm{D}g^{-1}$ belongs to $\mathrm{F}^f_{f^{t_1-t_2}(x)}$.

Lemma 12.10. *For every $\alpha > 0$, there exists two times t_1 and $t_2 \geq 0$ such that if $v \in \mathrm{T}\mathbf{T}^n_{f^{t_1}(x)}$ is such that the angle between v and $\mathrm{G}_{f^{t_1}(x)}$ is bigger than α, then the angle between $\mathrm{D}f^{-t_2}_{f^{t_1}(x)}v$ and $\mathrm{F}_{f^{t_1-t_2}(x)}$ is smaller than α (see Figure 12.4).*

Proof of Lemma 12.10. It easily follows from Oseledets theorem, and more precisely from the fact that the function $\exp(t\lambda)/\angle(\mathrm{F}_{f^t(x)}, \mathrm{G}_{f^t(x)})$ goes to $+\infty$ when t goes to $+\infty$. □

So, we fix two times t_1 and $t_2 \geq 0$, obtained by applying Lemma 12.10 to $\alpha = \arcsin\big(1/(1+\eta)\big)$, where η is the parameter obtained by applying the elementary perturbation lemma (Lemma 10.1) to $\delta/2$ (see Figure 12.3).

We also choose a time $t_3 \geq \widetilde{t_0}$ ($\widetilde{t_0}$ being given by Lemma 12.6) such that

$$e^{\lambda(t_3+t_2)/4} \geq \mathrm{M}^{t_2}.$$

This estimation will be applied to point 6. of Lemma 12.6. It will imply that for every $v \in \mathrm{F}^f_{f^{t_1}(x)}$ and for every $t \geq t_2 + t_3$, we have

$$\|\mathrm{D}f^{-t}_{f^{t_1}(x)}(v)\| \geq \|\mathrm{D}f^{-t_2}_{f^{t_1}(x)}(v)\| \geq \frac{1}{\mathrm{M}^{t_2}}\|v\|. \tag{12.1}$$

We then apply Lemma 8.54 of Chapter 8 to

$$\mathrm{R}_0 = \mathrm{M}^{t_2+t_3}(1+\eta'), \tag{12.2}$$

where η' is given by Lemma 12.8 applied to the parameter $\delta/2$. This gives us a parameter $k_0 = t_4$. Note that R_0 is chosen so that if $v \in T_{f^{t_1}(x)}\mathbf{T}^n$ is such that $\|v\| \geq R_0/N$, then for any $t \in [\![0, t_2 + t_3]\!]$, we have

$$\left\|Df^{-t}_{f^{t_1}(x)}(v)\right\| \geq (1+\eta')/N. \tag{12.3}$$

Thus, we will be able to apply Lemma 12.8 to the points $f^{-t}\left(f^{t_1}(x)+v\right)$, with $t \in [\![0, t_2 + t_3]\!]$, without perturbing the points of the orbit of x.

Step 3: global perturbation of the dynamics. We can now apply the ergodic closing lemma we have stated in the previous subsection (Lemma 12.6) to the neighbourhood $\mathscr{V}'$, the measure μ, the point $x_1 = f^{t_1-t_2-t_3}(x)$ and $\tau_0 \geq t_2 + t_3 + t_4$ large enough so that $\tau_0\lambda/4 \geq 3$. We also need that the expansion of vectors F^{g_1} along the segment of orbit $\left(x_2, g_2(x_2), \cdots, g_2^{\tau_0-t_2-t_3-t_4}(x_2)\right)$ is bigger than 3, but it can be supposed true by taking a bigger τ_0 if necessary. This gives us a first perturbation g_1 of the diffeomorphism f, such that the point x_1 is periodic under g_1 with period $\tau_1 \geq \tau_0$, and such that the Lyapunov exponents of x_1 for g_1 are close to that of x_1 under f, and the stable and unstable subspaces of g_1 at the point $g_1^t(x_1)$ are close to that of f at the point $g_1^t(x_1)$ for every $t \in [\![0, t_3 + t_2]\!]$.

Remark that by the hypothesis on τ_0, the Lyapunov exponent of $g_1^{\tau_1}$ at x_1 is bigger than 3, thus we will be able to apply Lemma 12.8 to every point of the orbit belonging to $F^{g_1}_{x_1}$, even when the orbit returns several times near x_1. Also note that these properties are stable under C^1 perturbation.

We then use the connecting lemma for pseudo-orbits of C. Bonatti and S. Crovisier (Theorem 10.3, see [BC04]), which implies that the stable manifolds of the periodic orbits of a generic conservative C^1-diffeomorphism are dense. This allows us to perturb the diffeomorphism g_1 into a diffeomorphism $g_2 \in \mathscr{V}'$ such that there exists a point x_2 close to x_1 such that:

(1) x_2 is periodic for g_2 with the same period than that of x_1 under g_1, and moreover the periodic orbit of x_2 under g_2 shadows that of x_1 under g_1;
(2) the Lyapunov exponents and the Lyapunov subspaces of x_2 for g_2 are very close to that of x_1 for g_1 (see the conclusions of Lemma 12.6, in particular the Lyapunov subspaces are close during a time $t_3 + t_2$);
(3) the stable manifold of x_2 under g_2 meets the set U, at a point denoted by y_2.

Step 4: linearization near the periodic orbit. We then use Franks lemma (see [Fra71]) to perturb slightly the differentials of g_2 at the points $g_2^{t_2+t_3}(x_2), \cdots, g_2^{t_2+t_3+t_4}(x_2)$, such that these differentials belong to the open set of matrices defined by Lemma 8.54 of Chapter 8. This gives us another diffeomorphism $g_3 \in \mathscr{V}'$ close to g_2, such that the point x_2 still satisfies the nice properties (1), (2) and (3).

By Lemma 11.6, there exists a parameter $r > 0$ such that it is possible to linearize g_3 in the r-neighbourhood of the periodic orbit of x_2, without changing the nice properties (1), (2) and (3) of the periodic orbit of x_2. We can choose r small enough so that the $10\,r$-neighbourhoods of the points of the periodic orbit of x_2 are pairwise disjoint. This gives us a diffeomorphism g_4, to which are associated two points x_4 and y_4, such that x_4 satisfies the properties (1), (2) and (3), and such that:

(4) the differentials of f at the points $g_4^{t_2+t_3}(x_4), \cdots, g_4^{t_2+t_3+t_4}(x_4)$ lie in the open dense set of matrices of Lemma 8.54 of Chapter 8;
(5) g_4 is linear in the r-neighbourhood of each point of the periodic orbit of x_4.

Step 5: choice of the order of discretization. We choose a neighbourhood $\mathscr{V}'' \subset \mathscr{V}'$ of g_4 such that properties (1) to (3) are still true for every diffeomorphism $g \in \mathscr{V}''$. We denote by ω_{x_4} the periodic orbit of x_4 under g_4, and by $\mathrm{B}(\omega_{x_4}, r)$ the r-neighbourhood of this periodic orbit. We also denote T_1 the smallest integer such that $g_4^t(y_4) \in \mathrm{B}(\omega_{x_4}, r/2)$ for every $t \geq \mathrm{T}_1$, and set $y_4' = g_4^{\mathrm{T}_1}(y_4)$. Thus, the positive orbit of y_4' will stay forever in the linearizing neighbourhood of ω_{x_4}. Taking T_1 bigger if necessary, we can suppose that y_4' belongs to the linearizing neighbourhood of the point x_4. We can also suppose that for every $t \in [\![0, \tau_1]\!]$,

$$3d\Big(g_4^{\mathrm{T}_1 - t}(y_4), g_4^{-t}(x_4)\Big) \leq \min_{\tau_1 \leq t' \leq \mathrm{T}_1} d\Big(g_4^{\mathrm{T}_1 - t'}(y_4), g_4^{-t}(x_4)\Big). \tag{12.4}$$

We can now choose the order N of the discretization, such that

(i) $\mathrm{N} \geq \mathrm{N}_0$ (N_0 has been chosen at the very beginning of the proof);

(ii) applying Lemma 12.9 to the parameter r and the neighbourhood $\mathscr{V}''$ to get an integer N_1, we have $\mathrm{N} \geq \mathrm{N}_1$, so that it is possible to choose the value of the points of ω_{x_4} modulo E_N without changing the properties (1) to (5);

(iii) the distance between two distinct points of the segment of orbit y_4, $g_4(y_4), \cdots, g_4^{\mathrm{T}_1}(y_4) = y_4'$ is bigger than $2(1+\eta')/\mathrm{N} + 2/\mathrm{N}$, so that it will be possible to apply Lemma 12.8 simultaneously to each of these points, even after the perturbation made during the point (ii), such that these points belong to E_N;

(iv) every $\sqrt{2}/\mathrm{N}$-pseudo-orbit [4] starting at a point of the periodic orbit ω_{x_4} stays during a time $\mathrm{T}'\tau_1$ in the $d(y_4', \omega_{x_4})$-neighbourhood of the periodic orbit, where T' the smallest integer such that

$$\left(1 + \frac{1}{3(1+\eta)}\right)^{\mathrm{T}'} \geq \nu, \tag{12.5}$$

and ν is the maximal modulus of the eigenvalues of $(\mathrm{D}g_4)_{x_4}^{\tau_1}$. A simple calculus shows that this condition is true if for example

$$\mathrm{N} \geq \frac{2\sqrt{n}(\mathrm{M}^{\mathrm{T}'\tau_1} - 1)}{r(\mathrm{M} - 1)}.$$

This condition will be used to apply the process described by Lemma 12.11.

Step 6: application of the linear theorem. By the hypothesis (ii) on N, we are able to use Lemma 12.9 (elementary perturbation with local translation) to perturb each point of the periodic orbit ω_{x_4} such that we obtain a diffeomorphism $g_5 \in \mathscr{V}''$ and points x_5, y_5 and y_5' satisfying properties (1) to (5) and moreover:

(6) for every $t \in [\![t_2 + t_3, t_2 + t_3 + t_4]\!]$, the value of $g_5^t(x_5)$ modulo E_N is equal to w_k/N, where w_k is given by Lemma 8.54 of Chapter 8;

(7) for any other t, $g_5^t(x_5)$ belongs to E_N.

In particular, the periodic orbit of x_5 under g_5 is stabilized by the discretization $(g_5)_\mathrm{N}$ (indeed, recall that $w_k \in [-1/2, 1/2]^k$).

By construction of the diffeomorphism g_5 (more precisely, the hypotheses (4), (5), (6) and (7)), it satisfies the conclusions of Lemma 8.54 of Chapter 8; thus there exists a point $z \in \mathrm{B}\Big(g_5^{t_2+t_3}(x_5), r\Big)$ such that $(g_5)_\mathrm{N}^{t_4}(z) = (g_5)_\mathrm{N}^{t_2+t_3+t_4}(x_5)$ and that $\|z - g_5^{t_2+t_3}(x_5)\| \geq \mathrm{R}_0/\mathrm{N}$ (where R_0 is defined by Equation (12.2)). Remark that hypothesis (iv) implies that $\|z - g_5^{t_2+t_3}(x_5)\| \ll r$.

4. The constant $\sqrt{n}/\mathrm{N}$ comes from the fact that an orbit of the discretization is a $\sqrt{2}/\mathrm{N}$-pseudo-orbit.

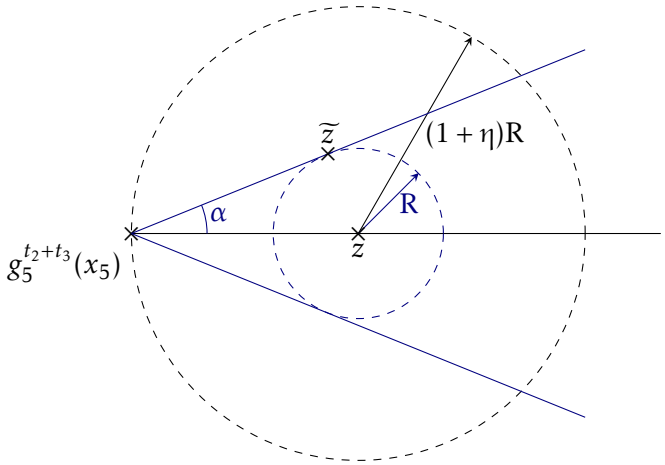

Figure 12.3: Perturbation we make to apply Lemma 12.10 (see also Figure 12.4): we make an elementary perturbation in a neighbourhood of z mapping z into $\widetilde{z}$, such that the angle between the lines $\left(g_5^{t_2+t_3}(x_5)\ z\right)$ and $\left(g_5^{t_2+t_3}(x_5)\ \widetilde{z}\right)$ is bigger than $\alpha = \arcsin\left(1/(1+\eta)\right)$, and such that the support of the perturbation does not contain $g_5^{t_2+t_3}(x_5)$.

Step 7: perturbations in the linear world. In this step, our aim is to perturb the negative orbit of z under g_5 such that it meets the point y_5'. Remark that by hypothesis (iv), every point of $z, g_5^{-1}(z), \cdots, g_5^{-t_2}(z)$ is in the linearizing neighbourhood of ω_{x_5}.

From now, all the perturbations we will make will be local, and we will only care of the positions of a finite number of points. Thus, it will not be a problem if these perturbations make hypotheses (3) and (5) become false, provided that they have a suitable behaviour on this finite set of points.

First, if necessary, we make a perturbation in the way of Figure 12.3, so that the angle between the lines $\left(g_5^{t_2+t_3}(x_5)\ z\right)$ and $\mathrm{G}^{g_5}_{g_5^{t_2+t_3}(x_5)}$ is bigger than α; this gives us a diffeomorphism g_6. More precisely, the support of the perturbation we apply is contained in a ball centred at z and with radius $d(z, x_6)$, so that this perturbation does not change the orbit of x_6. Under these conditions, we satisfy the hypotheses of Lemma 12.10, thus the angle between $\left(g_6^{t_3}(x_6)\ g_6^{-t_2}(z)\right)$ and $\mathrm{F}^{g_6}_{g_6{}^{t_3}(x_6)}$ is smaller than α. Another perturbation, described by Figure 12.3, allows us to suppose that $g_6^{-t_2}(z)$ belongs to $\mathrm{F}^{g_6}_{g_6{}^{t_3}(x_6)}$. This gives us a diffeomorphism that we still denote by g_6. Remark that it was possible to make these perturbations independently because the segment of negative orbit of the point z we considered does not enter twice in the neighbourhood of a point of ω_{x_6} where the diffeomorphism is linear.

Thus, the points $z' = g_6^{-t_2}(z)$ and $y_6' = y_5'$ both belong to the local stable manifold of the point $x_6 = x_5$ for g_6 (which coincides with the Oseledets linear subspace $\mathrm{F}^{g_6}_{x_6}$ since g_6 is linear near x_6).

The next perturbation takes place in the neighbourhood of the point x_6 (and not in all the linearizing neighbourhoods of the points of ω_{x_6}).

Lemma 12.11. *For every $y' \in \mathrm{F}^{g_6}_{x_6}$ such that $d(y', x_6) > d(z', x_6)\nu^{\mathrm{T}'\tau_1}$ (T' being defined by Equation (12.5)), there exists a diffeomorphism g_7 close to g_6 and $\mathrm{T}'' \in \mathbf{N}$ such that*

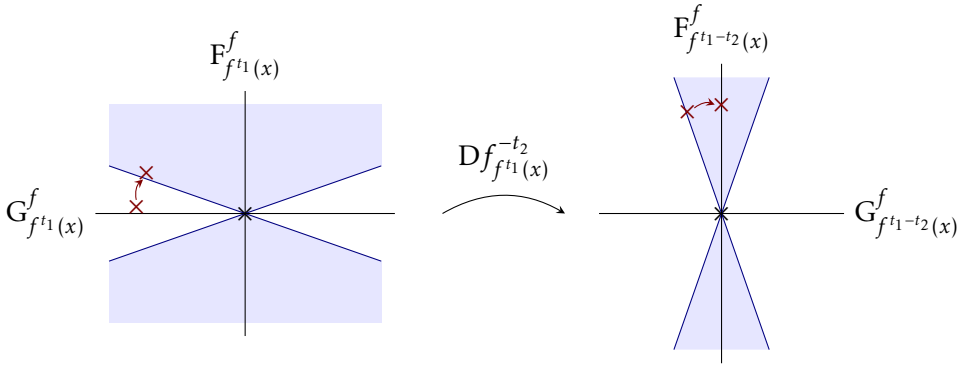

Figure 12.4: Proof of Lemma 12.10: make a small perturbation at times t_1 and $t_1 - t_2$ (in red), the hyperbolic-like behaviour of f does the rest of the work for you. In red: the perturbation that we will make during step 7.

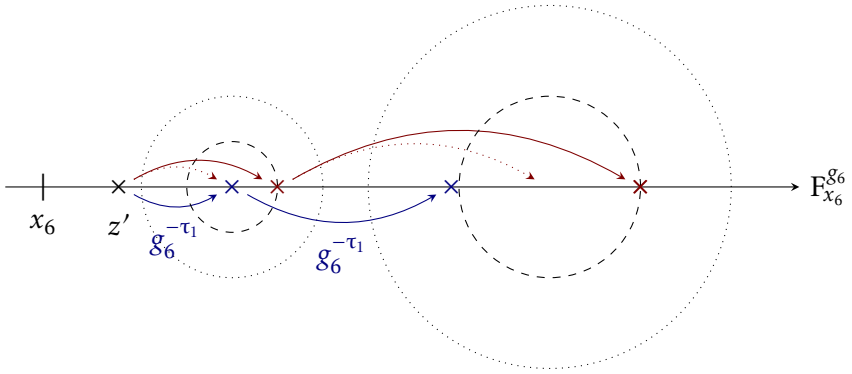

Figure 12.5: Perturbation such that the point y'_6 belongs to the negative orbit of z': the initial orbit is drawn in blue (below) and the perturbed orbit in red (above). From a certain time, the red orbit overtakes the blue orbit.

$g_7^{-\tau_1 T''}(z') = y'$. *Moreover, the perturbations made to obtain g_7 are contained in the linearizing neighbourhood of ω_{x_6}, do not modify the images of ω_{x_6}, nor these of the negative orbit of z' by the discretization or these of the positive orbit of y' in the linearizing neighbourhood of ω_{x_6}*

Proof of Lemma 12.11. During this proof, if r and s are two points of $W^s(x_6)$, we will denote by $[r,s]$ the segment of $W^s(x_6)$ between r and s. Remark that if r and s lie in the neighbourhood of x_6 where g_6 is linear, then $[r,s]$ is a real segment, included in $F^{g_6}_{x_6}$ moreover, we will denore $[r,+\infty[$ the connected component of $W^s(x_6) \setminus \{r\}$ which does not contain x_6.

Consider the point $z' \in F^{g_6}_{x_6}$, and choose a point

$$p \in \left[g_6^{-\tau_1}(z'), \left(1 + \frac{1}{2(1+\eta)}\right) g_6^{-\tau_1}(z') \right].$$

By applying an elementary perturbation (Lemma 10.1) whose support is contained into $B\left(g_6^{-\tau_1}(z'), d(x_6, g_6^{-\tau_1}(z'))/2\right)$, it is possible to perturb g_6 into a diffeomorphism g_7 such

that $g_6^{\tau_1}(z') = p$ (see Figure 12.5). Applying this process t times, for every

$$p \in \left[g_6^{-\tau_1 t}(z'), \left(1 + \frac{1}{2(1+\eta)}\right)^t g_6^{-\tau_1 t}(z') \right],$$

it is possible to perturb g_6 into a diffeomorphism g_7 such that $g_7^{-\tau_1 t}(z') = p$ (the supports of the perturbations are disjoint because the expansion of $g_{|\mathrm{F}_{x_6}^{g_6}}^{-\tau_1}$ is bigger than 3). But as T′ satisfies Equation (12.5), the union

$$\bigcup_{t\geq 0} \left[g_6^{-\tau_1 t}(z'), \left(1 + \frac{1}{2(1+\eta)}\right)^t g_6^{-\tau_1 t}(z') \right]$$

covers all the interval $[g_6^{-\tau_1 \mathrm{T}'}(z'), +\infty[$. By the hypothesis made on y', we also have $y' \in [g_6^{-\tau_1 \mathrm{T}'}(z'), +\infty[$; this proves the lemma. □

Thus, by hypothesis (iv), it is possible to apply Lemma 12.11 to our setting. This gives us a diffeomorphism g_7.

Step 8: final perturbation to put the segment of orbit on the grid. To summarize, we have a diffeomorphism $g_7 \in \mathscr{V}'$, and periodic orbit ω_{x_7} of g_7, stabilized by $(g_7)_\mathrm{N}$, which bears a measure close to μ. We also have a segment of real orbit of g_7 which links the points $y_7 \in \mathrm{U}$ and z, where z is such that $(g_7)_\mathrm{N}^{t_4}(z) \in (\omega_{x_7})_\mathrm{N}$. To finish the proof of the lemma, it remains to perturb g_7 so that the segment of orbit which links the points y_7 and z is stabilized by the discretization $(g_7)_\mathrm{N}$.

We now observe that by the construction we have made, the distance between two different points of the segment of orbit under g_7 between y_7 and z is bigger than $2(1+\eta')/\mathrm{N}$, and the distance between one point of this segment of orbit and a points of ω_{x_7} is bigger than $(1+\eta')/\mathrm{N}$.

Indeed, if we take one point of the segment of forward orbit $z, g_7^{-1}(z), \cdots, g_7^{-t_2-t_3}(z)$, and one point in the periodic orbit ω_{x_7}, this is due to the hypothesis $\|z - x_7\| \geq \mathrm{R}_0/\mathrm{N}$ (R_0 being defined by Equation (12.2)) combined with Equation (12.3). If we take one point in this segment $z, g_7^{-1}(z), \cdots, g_7^{-t_2-t_3}(z)$, and one among the rest of the points (that is, the segment of orbit between y_7 and z), this is due to the fact that the Lyapunov exponent of $g_7^{\tau_1}$ in x_7 is bigger than 3, and to Equation (12.4).

If we take one point of the form $g_7^{-t}(z)$, with $t > t_2 + t_3$, but belonging to the neighbourhood of ω_{x_7} where g_7 is linear, and one point of ω_{x_7}, this follows from the estimation given by Equation (12.1) applied to $\|v\| \geq \mathrm{R}_0/\mathrm{N}$. If for the second point, instead of considering a point of ω_{x_7}, we take an element of the segment of orbit between y_7 and z, this follows from the fact that the Lyapunov exponent of $g_7^{\tau_1}$ in x_7 is bigger than 3.

Finally, for the points of the orbit that are not in the neighbourhood of ω_{x_7} where g_7 is linear, the property arises from hypothesis (iii) made on N.

Thus, by Lemma 12.8, we are able to perturb each of the points of the segment of orbit under g_7 between y_7 and z, such that each of these points belongs to the grid. This gives us a diffeomorphism $g_8 \in \mathscr{V}$.

To conclude, we have a point $y_8 \in \mathrm{U}$ whose orbit under $(g_8)_\mathrm{N}$ falls on the periodic orbit $(\omega_{x_7})_\mathrm{N}$, which bears a measure ε-close to μ. The lemma is proved.

The proof in higher dimensions is almost identical. The perturbation lemmas are still true[5], and the arguments easily adapts by considering the "super-stable" manifold of the orbit ω_x, that is the set of points $y \in \mathbf{T}^n$ whose positive orbit is tangent to the Oseledets subspace corresponding to the maximal Lyapunov exponent. In particular, Lemma 12.6 is still true in this setting, and the connecting lemma (Theorem 10.3) implies that generically, this "super-stable" manifold is dense in $\mathbf{T}^n$. □

The proofs of the two statements of the addendum are almost identical.

For the first statement (the fact that for every $\varepsilon > 0$, the basin of attraction of the discrete measure can be supposed to contain a ε-dense subset of the torus), we apply exactly the same proof than that of Lemma 12.5: making smaller perturbations of the diffeomorphism if necessary, we can suppose that the stable manifold of y_8 is ε-dense. Thus, there exists a segment of backward orbit of y_8 which is ε-dense, and we apply the same strategy of proof consisting in putting this segment of orbit on the grid.

For the second statement, it suffices to apply the strategy of the first statement, and to conjugate the obtained diffeomorphism g_9 by an appropriate conservative diffeomorphism with small norm (this norm can be supposed to be as small as desired by taking ε small), so that the image of the ε-dense subset of $\mathbf{T}^2$ by the conjugation contains the set E.

To obtain Theorem 12.3 (dealing with the dissipative case) it suffices to replace the use of Lemmas 12.10 and 8.54 by the following easier statement.

Lemma 12.12. *For every $\alpha > 0$ and every $R_0 > 0$, there exists three times t_1, $t_2 \geq 0$ and $t_4 \geq 0$ such that:*

- *there exists $v \in TT^n_{f^{t_1}(x)} \cap \mathbf{Z}^n$ such that $\|v\| \geq R_0$ and*

$$\left(\widehat{Df}_{f^{t_1+t_4}(x)} \circ \cdots \circ \widehat{Df}_{f^{t_1}(x)}\right)(v) = 0;$$

- *if $v \in TT^n_{f^{t_1}(x)}$ is such that the angle between v and $G_{f^{t_1}(x)}$ is bigger than α, then the angle between $Df^{-t_2}_{f^{t_1}(x)} v$ and $F_{f^{t_1-t_2}(x)}$ is smaller than α (see Figure 12.4).*

Proof of Lemma 12.12. This comes from Oseledets theorem and the hypotheses made on the Lyapunov exponents of x, and in particular that their sum is strictly negative. □

12.2 Transfer operators and physical measures

The aim of this section is to study the behaviour of the measures $(f^*_N)^m \lambda_N$ for "small" times m (recall that λ_N is the uniform measure on the grid E_N). We will focus on the case where f is a $C^{1+\alpha}$ expanding map of the circle: a classical theorem asserts that in this case, the map f has a single physical measure (which is also the SRB measure), that we will denote by μ_0 (see Theorem 12.17). In this setting, O.E. Lanford has stated the following conjecture.

Conjecture 12.13 (Lanford). *Let $f : \mathbf{S}^1 \to \mathbf{S}^1$ be a generic C^2 expanding map. Then the convergence $(f^*_N)^m(\lambda_N) \to \mu_0$ holds for N, m both going to $+\infty$, with $\log N \ll m \ll \sqrt{N}$.*

5. In particular, Lemma 12.9 can be obtained by considering a plane (P) containing both x and y and taking a foliation of $\mathbf{R}^n$ by planes parallel to (P). The desired diffeomorphism is then defined on each leave by the time-$\psi(t)$ of the Hamiltonian given in the proof of the lemma, with ψ is a smooth compactly supported map on the space $\mathbf{R}^n/(P)$, equal to 1 in 0 and with small C^1 norm.

In this section, we will prove a kind of weak version of this conjecture (Theorem 12.15). We begin by some notations.

Definition 12.14. For $0 < \alpha \leq 1$ and $d \geq 2$, the set $\mathscr{E}_d^{1+\alpha}(\mathbf{S}^1)$ is the set of maps $f : \mathbf{S}^1 \to \mathbf{S}^1$ of degree d, whose derivative f' belongs to $\mathrm{C}^\alpha(\mathbf{S}^1)$ and satisfies $f'(x) > 1$ for every $x \in \mathbf{S}^1$.

Here, we will only consider parameters $\alpha \leq 1$. In particular, the set $\mathscr{E}_d^2$ will denote the set of maps $f : \mathbf{S}^1 \to \mathbf{S}^1$ of degree d, whose derivative f' is Lipschitz and satisfies $f'(x) > 1$ for every $x \in \mathbf{S}^1$.

The main result of this section is the following.

Theorem 12.15. *For every $0 < \alpha \leq 1$ and every $\mathrm{C}^{1+\alpha}$ expanding map $f \in \mathscr{E}_d^{1+\alpha}(\mathbf{S}^1)$, there exists a constant $c_0 = c_0(f) > 0$ such that if $(\mathrm{N}_m)_m$ is a sequence of integers going to infinity and satisfying $\log \mathrm{N}_m > c_0 m$, then the convergence $(f_{\mathrm{N}_m}^*)^m(\lambda_{\mathrm{N}_m}) \to \mu_0$ holds.*

This theorem will be obtained by combining the classical result of convergence of the pushforwards of any smooth measure towards the SRB measure μ_0 (Theorem 12.17), together with the convergence of the discrete operators f_{N}^* to the Ruelle-Perron-Frobenius operator (Theorem 12.19). Both of these theorems are effective: the constant c_0 of Theorem 12.15 can be computed in practical.

We fix once for all an expanding map $f \in \mathscr{E}_d^{1+\alpha}(\mathbf{S}^1)$, with $0 < \alpha \leq 1$. In the sequel, we will freely identify a measure with its density.

First of all, we recall an effective result about the convergence of the pushforwards of any smooth measure towards the SRB measure.

The *transfer operator* associated to the map f (usually called Ruelle-Perron-Frobenius operator), which acts on densities of probability measures, will be denoted by $\mathscr{L}_f$. It is defined by

$$\mathscr{L}_f \phi(y) = \sum_{x \in f^{-1}(y)} \frac{\phi(x)}{f'(x)}.$$

Then, for every map $\phi \in \mathrm{C}^\alpha(\mathbf{S}^1)$ which is the density of a probability measure on $\mathbf{S}^1$, the sequence $\mathscr{L}_f^m \phi$ converges exponentially fast towards a map $\phi_0 \in \mathrm{C}^\alpha(\mathbf{S}^1)$; this map is the density of the unique f-invariant probability measure μ_0 having a C^α density. This measure is also the unique SRB measure of f. More precisely, we have Theorem 12.17. To state it, we will need the following notation.

Definition 12.16. For $0 < \alpha \leq 1$ and $g \in \mathrm{C}^\alpha(\mathbf{S}^1)$, we denote by $[g]_\alpha$ the Hölder norm of g, i.e.

$$[g]_\alpha = \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x - y|^\alpha}$$

Theorem 12.17 ([FJ03], see also Theorem 2.5 of [Liv95] for the C^2 case)**.** *Let $\phi \in \mathrm{C}^\alpha(\mathbf{S}^1)$ be a positive function whose integral is equal to 1. Then, there exists an integer $\mathrm{M} = \mathrm{M}(\phi) > 0$ such that for every $m \geq \mathrm{M}$,*

$$\|\mathscr{L}_f^m(\phi) - \phi_0\|_\infty \leq \mathrm{C}_0 \Lambda^{m-\mathrm{M}} \|\phi\|_\infty,$$

where ϕ_0 denotes the density of the measure μ_0,

$$\mathrm{C}_0 = e^\Delta \|\phi_0\|_\infty \left(\|\phi_0\|_\infty + \frac{2}{\min_{x \in \mathbf{S}^1} \phi_0(x)} \right),$$

and $\Lambda = \tanh(\Delta/4)$, *with*

$$\Delta \leq 2\log\left(\frac{1+\tau}{1-\tau}\right) + 2\Big(3 - \tau + 2\|f'\|_\infty\Big)\frac{2[\log f']_\alpha}{\lambda^\alpha - 1}\|f'\|_\infty^{-\alpha},$$

where $\lambda = \min_{x\in \mathbf{S}^1} f'(x)$, *and* $\tau = (1+\lambda^{-\alpha})/2$.

Moreover, if ϕ *is the constant function equal to 1, then* $\mathrm{M}(\phi) = 0$.

This theorem is obtained by taking $\mathrm{K} = 2\mathrm{K}_0/(\lambda^\alpha - 1)$ in the Theorem 10.1 of [FJ03] (where K is defined at the beginning of section 10 of [FJ03]).

We now come to the original part of the proof of Theorem 12.15, and compare the actions of the transfer operator and of its discrete counterpart on smooth densities of measures. The discrete operator will be denoted by f_{N}^*; it acts on the probability measures supported by E_{N}:

$$(f_{\mathrm{N}}^*\mu)(y) = \sum_{x\in f_{\mathrm{N}}^{-1}(y)} \mu(x).$$

We will show that in a certain sense, we have

$$f_{\mathrm{N}}^* \underset{\mathrm{N}\to+\infty}{\longrightarrow} \mathscr{L}_f.$$

Thus, for a fixed time $m \in \mathbf{N}$, the measures $(f_{\mathrm{N}}^*)^m(\lambda_{\mathrm{N}})$ tend to the measure μ_0 when N goes to $+\infty$. The problem is to have precise estimates about the speed of this convergence. This speed will be measured by the following distance on measures.

Definition 12.18. For μ and ν two probability Borel measures, we define $d_{\mathrm{Lip}}(\mu,\nu)$ the distance

$$d_{\mathrm{Lip}}(\mu,\nu) = \sup\left\{\left|\int_{\mathbf{S}^1} \psi\ \mathrm{d}(\mu-\nu)\right| \mid \psi \in \mathrm{Lip}_1(\mathbf{S}^1)\right\},$$

where $\mathrm{Lip}_1(\mathbf{S}^1)$ is the space of Lipschitz functions whose Lipschitz constant is smaller than 1.

This distance d_{Lip} spans the weak-* topology on $\mathscr{P}$. Remark that by a theorem of L. Kantorovich and G. Rubinstein, this distance coincides with Wasserstein distance W_1 (see [KR58]). Note that the distance between Lebesgue measure and the uniform measure λ_{N} on E_{N} is $d_{\mathrm{Lip}}(\mathrm{Leb},\lambda_{\mathrm{N}}) = 1/(4\mathrm{N})$.

Theorem 12.19. *Let* $m \in \mathbf{N}$. *To every map* $\phi \in \mathrm{C}^\alpha(\mathbf{S}^1)$ *which is the density of a probability measure, we associate the measure* φ_{N} *supported by* E_{N}: $\varphi_{\mathrm{N}}(\mathrm{A}) = \int_{\mathrm{P}_{\mathrm{N}}^{-1}(\mathrm{A})}\phi$ *(where* P_{N} *is the projection on* E_{N}*). Then, for every map* $f \in \mathscr{E}_d^{1+\alpha}(\mathbf{S}^1)$,

$$f_{\mathrm{N}}^{*\,m}(\varphi_{\mathrm{N}}) \underset{\mathrm{N}\to+\infty}{\longrightarrow} \mathscr{L}_f^m(\phi).$$

More precisely, we have

$$d_{\mathrm{Lip}}\Big((f_{\mathrm{N}}^*)^m(\varphi_{\mathrm{N}}), \mathscr{L}_f^m\phi\Big) \leq \eta$$

as soon as

$$\mathrm{N} > (4/\eta)^{1+1/\alpha}\Big(2+\|f'\|_\infty\Big)^{m(1+1/\alpha)}\left(d\Big(1+[f']_\alpha\Big)\Big([\phi]_\alpha + \frac{1}{d}\Big)\right)^{m/\alpha}.$$

Corollary 12.20. *For example, we have (recall that λ_N is the uniform measure on E_N)*

$$d_{\mathrm{Lip}}\Big((f_N^*)^m(\lambda_N), \mathscr{L}_f^m \mathrm{Leb}\Big) \leq \eta$$

as soon as

$$N > \left(\frac{4}{\eta}\right)^{1+1/\alpha} \Big(\big(2+\|f'\|_\infty\big)^{1+1/\alpha}\big(1+[f']_\alpha\big)^{1/\alpha}\Big)^m.$$

In particular, if f is C^2, this gives

$$d_{\mathrm{Lip}}\Big((f_N^*)^m(\lambda_N), \mathscr{L}_f^m \mathrm{Leb}\Big) \leq \eta$$

as soon as

$$N > \frac{16}{\eta^2}\Big(\big(2+\|f'\|_\infty\big)^2\big(1+[f']_\alpha\big)\Big)^m.$$

Notice that this theorem is only a partial answer to the Conjecture 12.13 of O.E. Lanford. In particular, it does not answer to the following question.

Question. *Let $f : \mathbf{S}^1 \to \mathbf{S}^1$ be a generic $C^{1+\alpha}$ expanding map. What is the best function $m(N)$ such that the convergence $(f_N^*)^m(\lambda_N) \to \mu_0$ holds for N, m both going to $+\infty$, with $\log N \ll m \ll m(N)$?*

It is reasonable to think (see Conjecture 12.13) that $m(N) \leq \sqrt{N}$. Indeed, if $\sigma : E \to E$ is a typical random map of a set E with q elements, and x is a typical point of E, then the smallest integer m such that $\sigma^m(x) \in \Omega(\sigma)$ (recall that $\Omega(\sigma)$ is the union of the periodic orbits of σ) is of order $\sqrt{q}$ (see [Bol01, XIV.5] or the Theorem 2.3.1 of [Mie05]).

Notice that in Theorem 12.19, there is no hypothesis of genericity. We can hope that the uniform distribution of the roundoff errors of a generic map (see Proposition 9.2) allows to have finer results about the best function $m(N)$.

We do not know what happens in the C^1 generic case. It is possible that the behaviour is very different from the $C^{1+\alpha}$ case. Indeed, a theorem of J. Campbell and A. Quas (see [CQ01]) asserts that a generic C^1 expanding map of the circle has a single physical measure, but that this measure is singular with respect to Lebesgue measure.

We could also imagine similar studies in more general settings. For example, we can expect that the results on the circle generalize to $C^{1+\alpha}$ expanding maps of the torus $\mathbf{T}^n$, with $n \geq 1$. It could be less straightforward to have generalizations to Anosov diffeomorphisms of $\mathbf{T}^n$: in this case, the transfer operator is quite different from that used in the case of expanding maps.

We now begin the proof of Theorem 12.19 by the easier case where the time m is equal to 1.

Lemma 12.21. *For $\varepsilon < 1/5$ and*

$$N > \varepsilon^{-1-1/\alpha} \max\Big\{1,\ [\phi]_\alpha^{1/\alpha},\ [f']_\alpha^{1/\alpha}\Big\},$$

we have

$$d_{\mathrm{Lip}}\Big(f_N^*(\varphi_N),\ \mathscr{L}_f\phi\Big) \leq \varepsilon\Big(5 + 2\|f'\|_\infty\Big).$$

Proof of Lemma 12.21. Let ϕ be the $C^\alpha(\mathbf{S}^1)$ density of a probability measure on $\mathbf{S}^1$, i.e. $\phi \geq 0$ and $\int_{\mathbf{S}^1} \phi = 1$. Let $\varepsilon > 0$ and $f \in \mathscr{C}_d^{1+\alpha}$. As $f \in C^{1+\alpha}(\mathbf{S}^1)$, we can estimate its lack of linearity on an interval depending on its length. This will allow us to estimate the difference between $\mathscr{L}_f$ and f_N on every interval based on the length of this interval. The global difference between $\mathscr{L}_f$ and f_N will be obtained by a summation on a partition of $\mathbf{S}^1$ of intervals of appropriate length.

Lemma 12.22. *Let* I *be an interval of* $\mathbf{S}^1$ *with length smaller than*

$$\left(\frac{\varepsilon}{[f']_\alpha}\right)^{1/\alpha},$$

and $x_0 \in I$. *Then*

$$\left|\frac{\mathrm{Card}(I \cap E_N)}{\mathrm{Card}(E_N)} - \frac{\mathrm{Card}(f(I) \cap E_N)}{f'(x_0)\mathrm{Card}(E_N)}\right| \leq \mathrm{Leb}(I)\left(\varepsilon + \frac{2}{N}\frac{f'(x_0)}{\mathrm{Leb}(f(I))}\frac{1-\varepsilon}{1-2\varepsilon}\right).$$

Proof of Lemma 12.22. The hypothesis on the length of I implies that if $x, y \in I$, then $|f'(x) - f'(y)| < \varepsilon$. Using the mean value inequality, we get

$$\left|\mathrm{Leb}(I) - \frac{\mathrm{Leb}(f(I))}{f'(x_0)}\right| \leq \varepsilon;$$

we deduce the bound

$$\left|\mathrm{Leb}(I) - \frac{\mathrm{Leb}(f(I))}{f'(x_0)}\right| \leq \frac{\varepsilon\,\mathrm{Leb}(I)}{f'(x_0)}. \tag{12.6}$$

This implies that

$$\left|\mathrm{Leb}(I) - \frac{\mathrm{Leb}(f(I))}{f'(x_0)}\right| \leq \frac{\varepsilon/f'(x_0)}{1-\varepsilon/f'(x_0)}\mathrm{Leb}(f(I)),$$

thus (because $f'(x_0) \geq 1$)

$$\left|\mathrm{Leb}(I) - \frac{\mathrm{Leb}(f(I))}{f'(x_0)}\right| \leq \frac{\varepsilon}{f'(x_0)(1-\varepsilon)}\mathrm{Leb}(f(I));$$

in particular,

$$\mathrm{Leb}(I) \geq \frac{\mathrm{Leb}(f(I))}{f'(x_0)}\frac{1-2\varepsilon}{1-\varepsilon}. \tag{12.7}$$

Moreover, for every interval J,

$$\left|\mathrm{Leb}(J) - \frac{\mathrm{Card}(J \cap E_N)}{\mathrm{Card}(E_N)}\right| \leq \frac{1}{N},$$

as a result, using Equation (12.6),

$$\left|\frac{\mathrm{Card}(I \cap E_N)}{\mathrm{Card}(E_N)} - \frac{\mathrm{Card}(f(I) \cap E_N)}{f'(x_0)\mathrm{Card}(E_N)}\right| \leq \frac{1}{N} + \frac{\varepsilon\,\mathrm{Leb}(I)}{f'(x_0)} + \frac{1}{Nf'(x_0)}.$$

We deduce that (still because $f'(x_0) \geq 1$)

$$\left|\frac{\mathrm{Card}(I \cap E_N)}{\mathrm{Card}(E_N)} - \frac{\mathrm{Card}(f(I) \cap E_N)}{f'(x_0)\mathrm{Card}(E_N)}\right| \leq \mathrm{Leb}(I)\left(\varepsilon + \frac{2}{N\,\mathrm{Leb}(I)}\right),$$

using Equation (12.7), this leads to

$$\left|\frac{\mathrm{Card}(I \cap E_N)}{\mathrm{Card}(E_N)} - \frac{\mathrm{Card}(f(I) \cap E_N)}{f'(x_0)\mathrm{Card}(E_N)}\right| \leq \mathrm{Leb}(I)\left(\varepsilon + \frac{2}{N}\frac{f'(x_0)}{\mathrm{Leb}(f(I))}\frac{1-\varepsilon}{1-2\varepsilon}\right).$$

□

Let $\psi \in \mathrm{Lip}_1(\mathbf{S}^1)$ be a test function. We want to compute the difference

$$\left|\langle f_N^*(\varphi_N), \psi\rangle - \langle \mathscr{L}\phi^*, \psi\rangle\right|.$$

So, we compute:

$$\begin{aligned}
&\left|\langle f_N^*(\varphi_N), \psi\rangle - \langle \mathscr{L}_f \phi, \psi\rangle\right| \\
&\qquad = \left| \sum_{y_N \in E_N} \int_{P_N^{-1}(y_N)} \psi(y)\Big(f_N^*(\varphi_N) - \mathscr{L}_f\phi\Big)(y)\,\mathrm{d}y \right| \\
&\qquad = \left| \sum_{y_N \in E_N} \left(\psi(y_N) \sum_{x_N \in f_N^{-1}(y_N)} \varphi_N(x_N) - \int_{P_N^{-1}(y_N)} \psi(y) \sum_{x \in f^{-1}(y)} \frac{\phi(x)}{f'(x)}\,\mathrm{d}y \right) \right|.
\end{aligned}$$

We remark that by preservation of the total mass (the operators f_N^* and $\mathscr{L}$ map probability measures to probability measures), this expression is independent from the mean of ψ. We can therefore freely add or remove a constant to the function ψ.

We introduce an intermediate scale between that of the grid E_N and that of the circle $\mathbf{S}^1$: the grid E_M. When M is large enough, we have:

(C1) $|\psi(y) - \psi(y')| < \varepsilon$ for every points y, y' such that $|y - y'| < 1/M$,

(C2) $|f'(x) - f'(x')| < \varepsilon$ et $|\phi(x) - \phi(x')| < \varepsilon$ for every points x, x' in the same connected component of $f^{-1}\Big(P_M^{-1}(y_M)\Big)$,

(C3) $1/M \leq (\varepsilon/[f']_\alpha)^{1/\alpha}$.

(C4) $M/N \leq \varepsilon$.

In particular, for every points x, x' in the same connected component of the preimage $f^{-1}(P_M^{-1}(y_M))$, we have

$$\left|\varphi_N(x_N) - \frac{1}{N}\phi(x')\right| \leq \frac{\varepsilon}{N}, \tag{12.8}$$

and

$$\begin{aligned}
\left|\frac{\phi(x)}{f'(x)} - \frac{\phi(x')}{f'(x')}\right| &\leq \phi(x)\left|\frac{1}{f'(x)} - \frac{1}{f'(x')}\right| + \frac{1}{f'(x')}\left|\phi(x) - \phi(x')\right| \\
&\leq \frac{\phi(x)}{f'(x)}\left|\frac{f'(x') - f'(x)}{f'(x')}\right| + \frac{\varepsilon}{f'(x)}\frac{f'(x)}{f'(x')} \\
&\leq \frac{\phi(x)}{f'(x)}\frac{\varepsilon}{\min f'} + \frac{\varepsilon}{f'(x)}\left(1 + \frac{\varepsilon}{\min f'}\right) \\
&\leq \varepsilon\left(\frac{\phi(x)}{f'(x)} + \frac{1+\varepsilon}{f'(x)}\right)
\end{aligned} \tag{12.9}$$

(we want to have a $f'(x)$ at the denominator to be able to integrate properly and keep working with L^1 norms instead of L^∞ norms).

We begin by cutting out the distance we want to compute by using the intermediate scale E_M:

$$\begin{aligned}
&\left|\langle f_N^*(\varphi_N), \psi\rangle - \langle \mathscr{L}_f \phi, \psi\rangle\right| \\
&\qquad \leq \sum_{y'_M \in E_M} \left| \sum_{y_N \in E_N \cap P_M^{-1}(y'_M)} \left(\psi(y_N) \sum_{x_N \in f_N^{-1}(y_N)} \varphi_N(x_N) - \int_{P_N^{-1}(y_N)} \psi(y) \sum_{x \in f^{-1}(y)} \frac{\phi(x)}{f'(x)}\,\mathrm{d}y \right) \right|.
\end{aligned}$$

Using condition (C1) and the fact that $\langle f_N^*(\varphi_N), 1\rangle = \langle \mathscr{L}\phi^*, 1\rangle = 1$, we get:

$$\begin{aligned}\Big|\langle f_N^*(\varphi_N), \psi\rangle - \langle \mathscr{L}_f \phi, \psi\rangle\Big| &\leq 2\varepsilon \\ &\quad + \|\psi\|_\infty \sum_{y'_M \in E_M} \left| \sum_{y_N \in E_N \cap P_M^{-1}(y'_M)} \left(\sum_{x_N \in f_N^{-1}(y_N)} \varphi_N(x_N) - \int_{P_N^{-1}(y_N)} \sum_{x \in f^{-1}(y)} \frac{\phi(x)}{f'(x)}\, \mathrm{d}y \right) \right|.\end{aligned}$$

For $y'_M \in E_M$, we denote by $(I_{y'_M,k})_{1\leq k\leq d}$ the connected components of $f^{-1}\big(P_M^{-1}(y'_M)\big)$. For every k, we also denote by $x_{y'_M,k}$ the unique point of $I_{y'_M,k} \cap f^{-1}(y'_M)$. Using the bound (12.8), we get:

$$\begin{aligned}\Big|\langle f_N^*(\varphi_N), \psi\rangle - \langle \mathscr{L}_f \phi, \psi\rangle\Big| &\leq \varepsilon(2 + \|\psi\|_\infty) \\ &\quad + \|\psi\|_\infty \sum_{y'_M \in E_M} \left| \sum_{k=1}^{d} \mathrm{Card}(I_{y'_M,k} \cap E_N) \frac{\phi(x_{y'_M,k})}{N} - \sum_{y_N \in E_N \cap P_M^{-1}(y'_M)} \int_{P_N^{-1}(y_N)} \sum_{x \in f^{-1}(y)} \frac{\phi(x)}{f'(x)}\, \mathrm{d}y \right|.\end{aligned}$$

Combined with a change of variables, the bound (12.9) leads to:

$$\begin{aligned}\Big|\langle f_N^*(\varphi_N), \psi\rangle - \langle \mathscr{L}_f \phi, \psi\rangle\Big| &\leq \varepsilon\Big(2 + \|\psi\|_\infty\big(1 + 1 + (1+\varepsilon)\big)\Big) \\ &\quad + \|\psi\|_\infty \sum_{y'_M \in E_M} \left| \sum_{k=1}^{d} \mathrm{Card}(I_{y'_M,k} \cap E_N) \frac{\phi(x_{y'_M,k})}{N} - \sum_{y_N \in E_N \cap P_M^{-1}(y'_M)} \frac{1}{N} \sum_{k=1}^{d} \frac{\phi(x_{y'_M,k})}{f'(x_{y'_M,k})} \right| \\ &\leq \varepsilon\Big(2 + \|\psi\|_\infty(3+\varepsilon)\Big) \\ &\quad + \|\psi\|_\infty \sum_{y'_M \in E_M} \left| \sum_{k=1}^{d} \frac{\phi(x_{y'_M,k})}{N} \left(\mathrm{Card}(I_{y'_M,k} \cap E_N) - \frac{\mathrm{Card}(P_M^{-1}(y'_M))}{f'(x_{y'_M,k})} \right) \right| \\ &\leq \varepsilon\Big(2 + \|\psi\|_\infty(3+\varepsilon)\Big) \\ &\quad + \|\psi\|_\infty \sum_{y'_M \in E_M} \sum_{k=1}^{d} \phi(x_{y'_M,k}) \left| \frac{\mathrm{Card}(I_{y'_M,k} \cap E_N)}{N} - \frac{\mathrm{Card}(f(I_{y'_M,k}) \cap E_N)}{f'(x_{y'_M,k})N} \right|.\end{aligned}$$

Using Lemma 12.22 (which is valid by Condition (C3)), we deduce that:

$$\begin{aligned}\Big|\langle f_N^*(\varphi_N), \psi\rangle - \langle \mathscr{L}_f \phi, \psi\rangle\Big| &\leq \varepsilon\Big(2 + \|\psi\|_\infty(3+\varepsilon)\Big) \\ &\quad + \|\psi\|_\infty \sum_{y'_M \in E_M} \sum_{k=1}^{d} \phi(x_{y'_M,k})\, \mathrm{Leb}(I_{y'_M,k}) \left(\varepsilon + \frac{2}{N} \frac{f'(x_{y'_M,k})}{\mathrm{Leb}(f(I_{y'_M,k}))} \frac{1-\varepsilon}{1-2\varepsilon} \right).\end{aligned}$$

As a result, using Condition (C2) and the fact that $\|\phi\|_{L^1} = 1$, we get

$$\begin{aligned}\left|\langle f_N^*(\varphi_N), \psi\rangle - \langle \mathscr{L}_f \phi, \psi\rangle\right| &\leq \varepsilon\Big(2 + \|\psi\|_\infty(3+\varepsilon)\Big) \\ &\quad + \|\psi\|_\infty(1+\varepsilon)\left(\varepsilon + 2\frac{M}{N}\|f'\|_\infty \frac{1-\varepsilon}{1-2\varepsilon}\right)\end{aligned}$$

finally, condition (C4) gives

$$\begin{aligned}\left|\langle f_N^*(\varphi_N), \psi\rangle - \langle \mathscr{L}_f \phi, \psi\rangle\right| &\leq \varepsilon\left(2 + \|\psi\|_\infty\left(4 + 2\varepsilon + 2\|f'\|_\infty \frac{1+\varepsilon}{1-2\varepsilon}\right)\right).\end{aligned}$$

We now use the fact that $\psi \in \mathrm{Lip}_1(\mathbf{S}^1)$. In this case (recall that we are allowed to add any constant to ψ), we can suppose that $\|\psi\|_\infty \leq 1/2$. We have a bound on the minimal order M to have Conditions (C1) to (C3) :

$$\frac{1}{M} < \min\left\{\varepsilon, \left(\frac{\varepsilon}{[\phi]_\alpha}\right)^{1/\alpha}, \left(\frac{\varepsilon}{[f']_\alpha}\right)^{1/\alpha}\right\}.$$

So, for $\varepsilon \leq 1$, Condition (C4) gives

$$\frac{1}{N} < \varepsilon^{1+1/\alpha} \min\left\{1, \frac{1}{[\phi]_\alpha^{1/\alpha}}, \frac{1}{[f']_\alpha^{1/\alpha}}\right\},$$

and in this case,

$$d_{\mathrm{Lip}}\Big(f_N^*(\varphi_N), \mathscr{L}_f\phi\Big) \leq \varepsilon\left(4 + \varepsilon + \|f'\|_\infty \frac{1+\varepsilon}{1-2\varepsilon}\right).$$

If moreover, we suppose that $\varepsilon < 1/5$, we get the conclusion of the lemma:

$$d_{\mathrm{Lip}}\Big(f_N^*(\varphi_N), \mathscr{L}_f\phi\Big) \leq \varepsilon\Big(5 + 2\|f'\|_\infty\Big).$$

□

We now focus on the general case $m > 1$. It follows from two easy lemmas.

Lemma 12.23. *Let $\varphi_1, \varphi_2 : E_N \to \mathbf{R}_+$. Then, for every $j \in \mathbf{N}$, we have*

$$d_{\mathrm{Lip}}\Big((f_N^*)^j\varphi^1, (f_N^*)^j\varphi^2\Big) \leq \Big(2 + \|f'\|_\infty\Big)^j d_{\mathrm{Lip}}(\varphi^1, \varphi^2).$$

Proof of Lemma 12.23. For $\psi \in \mathrm{Lip}_1(\mathbf{S}^1)$, we compute

$$\begin{aligned}\Big\langle f_N^*\varphi^1 - f_N^*\varphi^2, \psi\Big\rangle &= \sum_{y_N \in E_N}\Big(f_N^*\varphi^1(y_N) - f_N^*\varphi^2(y_N)\Big)\psi(y_N) \\ &= \sum_{y_N \in E_N}\sum_{x_N \in f_N^{-1}(y)}\Big(\varphi^1(x_N) - \varphi^2(x_N)\Big)\psi(y_N).\end{aligned}$$

But, for $x_N, x_N' \in E_N$, we have (remark that either $x_N = x_N'$, or $|x_N - x_N'| \geq 1/N$)

$$\Big|f_N(x_N) - f_N(x_N')\Big| \leq \Big(2 + \|f'\|_\infty\Big)|x_N - x_N'|.$$

As a result, the map $x_N \mapsto \psi(f_N(x_N))$ is $\Big(2 + \|f'\|_\infty\Big)$-Lipschitz, so

$$d_{\mathrm{Lip}}\Big(f_N^*\varphi^1, f_N^*\varphi^2\Big) \leq \Big(2 + \|f'\|_\infty\Big)d_{\mathrm{Lip}}(\varphi^1, \varphi^2).$$

The lemma easily follows from an induction. □

Lemma 12.24.

$$\left[\mathscr{L}^j\phi\right]_\alpha \le d^j\left(1+[f']_\alpha\right)^j\left([\phi]_\alpha+\frac{1}{d}\right).$$

Proof of lemma 12.24. We have

$$\left|(\mathscr{L}\phi)(y)-(\mathscr{L}\phi)(y')\right| \le \left|\sum_{x\in f^{-1}(y)}\frac{\phi(x)}{f'(x)}-\sum_{x'\in f^{-1}(y')}\frac{\phi(x')}{f'(x')}\right|.$$

We write $f^{-1}(y)=\{x_1,\cdots,x_d\}$ and $f^{-1}(y')=\{x'_1,\cdots,x'_d\}$, so that for every k, we have $|x_k-x'_k|\le|y-y'|$. Thus

$$\begin{aligned}\left|(\mathscr{L}\phi)(y)-(\mathscr{L}\phi)(y')\right| &\le \sum_{k=1}^d\left|\frac{\phi(x_k)}{f'(x_k)}-\frac{\phi(x'_k)}{f'(x'_k)}\right| \\ &\le \sum_{k=1}^d|\phi(x_k)|\left|\frac{1}{f'(x_k)}-\frac{1}{f'(x'_k)}\right|+\frac{1}{f'(x'_k)}\left|\phi(x_k)-\phi(x'_k)\right|.\end{aligned}$$

Using the fact that $\min_{x\in\mathbf{S}^1}f'(x)\ge 1$, this leads to

$$\begin{aligned}\left|(\mathscr{L}\phi)(y)-(\mathscr{L}\phi)(y')\right| &\le \sum_{k=1}^d|\phi(x_k)|\left|f'(x_k)-f'(x'_k)\right|+\left|\phi(x_k)-\phi(x'_k)\right| \\ &\le \sum_{k=1}^d\|\phi\|_\infty[f']_\alpha|x_k-x'_k|^\alpha+[\phi]_\alpha|x_k-x'_k|^\alpha \\ &\le d\left(\|\phi\|_\infty[f']_\alpha+[\phi]_\alpha\right)|y-y'|^\alpha.\end{aligned}$$

Moreover, as $\phi\ge 0$ and as there exists $x\in\mathbf{S}^1$ such that $\phi(x)\le 1$, we have $\|\phi\|_\infty\le 1+[\phi]_\alpha$. So

$$\left[\mathscr{L}\phi\right]_\alpha \le d\left([\phi]_\alpha\left(1+[f']_\alpha\right)+[f']_\alpha\right).$$

By iterating, we get

$$\begin{aligned}\left[\mathscr{L}^j\phi\right]_\alpha &\le d^j\left(1+[f']_\alpha\right)^j\left([\phi]_\alpha+\frac{[f']_\alpha}{d\left(1+[f']_\alpha\right)-1}\right) \\ &\le d^j\left(1+[f']_\alpha\right)^j\left([\phi]_\alpha+\frac{1}{d}\right).\end{aligned}$$

□

Proof of Theorem 12.19. To begin with, we decompose the distance between $(f_N^*)^m(\varphi_N)$ and $\mathscr{L}_f^m\phi$:

$$d_{\mathrm{Lip}}\left((f_N^*)^m(\varphi_N),\mathscr{L}_f^m\phi\right) \le \sum_{j=1}^m d_{\mathrm{Lip}}\left((f_N^*)^j\left(\mathscr{L}_f^{m-j}(\phi)\right)_N,(f_N^*)^{j-1}\left(\mathscr{L}_f^{m+1-j}(\phi)\right)_N\right).$$

We then use Lemma 12.23:

$$\begin{aligned}&d_{\mathrm{Lip}}\left((f_N^*)^m(\varphi_N),\mathscr{L}_f^m\phi\right) \\ &\qquad\le \sum_{j=1}^m\left(2+\|f'\|_\infty\right)^{j-1}d_{\mathrm{Lip}}\left(f_N^*\left(\mathscr{L}_f^{m-j}(\phi)\right)_N,\mathscr{L}_f\left(\mathscr{L}_f^{m-j}(\phi)_N\right)\right).\end{aligned}$$

Applying Lemma 12.21 to $\varepsilon < 1/5$ and N such that for every j,

$$\mathrm{N} > \varepsilon^{-1-1/\alpha} \max\left\{1, \left[\mathscr{L}_f^{m-j}(\phi)\right]_\alpha^{1/\alpha}, [f']_\alpha^{1/\alpha}\right\}, \tag{12.10}$$

we deduce that

$$d_{\mathrm{Lip}}\left(f_{\mathrm{N}}^*(\varphi_{\mathrm{N}}), \mathscr{L}_f\phi\right) \le \varepsilon\left(5 + 2\|f'\|_\infty\right).$$

$$\begin{aligned} d_{\mathrm{Lip}}\left((f_{\mathrm{N}}^*)^m(\varphi_{\mathrm{N}}), \mathscr{L}_f^m\phi\right) &\le \sum_{j=1}^{m}\left(2 + \|f'\|_\infty\right)^{j-1}\left(5 + 2\|f'\|_\infty\right) \\ &\le \frac{\left(2 + \|f'\|_\infty\right)^m - 1}{\left(2 + \|f'\|_\infty\right) - 1}\left(5 + 2\|f'\|_\infty\right). \end{aligned}$$

As $\|f'\|_\infty \ge 2$, we get

$$d_{\mathrm{Lip}}\left((f_{\mathrm{N}}^*)^m(\varphi_{\mathrm{N}}), \mathscr{L}_f^m\phi^*\right) \le 3\varepsilon\left(2 + \|f'\|_\infty\right)^m.$$

By Lemma 12.24, (12.10) is satisfied if

$$\mathrm{N} > \varepsilon^{-1-1/\alpha} \max\left\{1, \left(d\left(1 + [f']_\alpha\right)\left([\phi]_\alpha + \frac{1}{d}\right)\right)^{m/\alpha}, [f']_\alpha^{1/\alpha}\right\},$$

i.e.

$$\mathrm{N} > \varepsilon^{-1-1/\alpha}\left(d\left(1 + [f']_\alpha\right)\left([\phi]_\alpha + \frac{1}{d}\right)\right)^{m/\alpha}.$$

Thus, for every $\eta \le 2$, we have

$$d_{\mathrm{Lip}}\left((f_{\mathrm{N}}^*)^m(\varphi_{\mathrm{N}}), \mathscr{L}_f^m\phi^*\right) \le \eta$$

as soon as

$$\mathrm{N} > \eta^{-1-1/\alpha} 4^{1+1/\alpha}\left(2 + \|f'\|_\infty\right)^{m(1+1/\alpha)}\left(d\left(1 + [f']_\alpha\right)\left([\phi]_\alpha + \frac{1}{d}\right)\right)^{m/\alpha}.$$

□

12.3 Numerical simulations

In this section, we present the results of the numerical simulations we have conducted in connection with Theorems 12.1 and 12.15.

12.3.1 Simulations of the measures $\mu_x^{f_{\mathrm{N}}}$ for conservative torus diffeomorphisms

We have computed numerically the measures $\mu_x^{f_{\mathrm{N}}}$ for conservative diffeomorphisms $f \in \mathrm{Diff}^1(\mathbf{T}^2, \mathrm{Leb})$, for the uniform grids

$$\mathrm{E}_{\mathrm{N}} = \left\{\left(\frac{i}{\mathrm{N}}, \frac{j}{\mathrm{N}}\right) \in \mathbf{T}^2 \,\middle|\, 0 \le i, j \le \mathrm{N} - 1\right\},$$

and for starting points x either equal to $(1/2, 1/2)$, or chosen at random. We present images of sizes 128×128 pixels representing in logarithmic scale the density of the measures $\mu_x^{f_N}$: each pixel is coloured according to the measure carried by the set of points of E_N it covers. Blue corresponds to a pixel with very small measure and red to a pixel with very high measure. Scales on the right of each image corresponds to the measure of one pixel on the $\log 10$ scale: if green corresponds to -3, then a green pixel will have measure 10^{-3} for $\mu_x^{f_N}$. For information, when Lebesgue measure is represented, all the pixels have a value about -4.2.

We have carried out the simulations on three different diffeomorphisms.

– The first conservative diffeomorphism h_1 is of the form $h_1 = Q \circ P$, where both P and Q are homeomorphisms of the torus that modify only one coordinate:

$$P(x,y) = \Big(x, y + p(x)\Big) \quad \text{and} \quad Q(x,y) = \Big(x + q(y), y\Big),$$

with

$$p(x) = \frac{1}{209}\cos(2\pi \times 17x) + \frac{1}{471}\sin(2\pi \times 29x) - \frac{1}{703}\cos(2\pi \times 39x),$$

$$q(y) = \frac{1}{287}\cos(2\pi \times 15y) + \frac{1}{403}\sin(2\pi \times 31y) - \frac{1}{841}\sin(2\pi \times 41y).$$

This C^∞-diffeomorphism is in fact C^1-close to the identity. This allows h_1 to admit periodic orbits with not too large periods. Note that h_1 is also chosen so that it is not C^2-close to the identity.

– The second conservative diffeomorphism h_2 is the composition $h_2 = h_1 \circ R$, with the translation of the torus

$$R(x,y) = \Big(x + 1/10, y + 1/15\Big).$$

Again, for R, we have chosen a translation with a relatively small order (here 30) to ensure that the discretizations can have periodic orbits with small periods.

– The third conservative diffeomorphism h_3 is the composition $h_2 = h_1 \circ A$, with A the linear Anosov map

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

As h_1 is C^1-close to Id, the diffeomorphism h_3 is C^0-conjugated to the linear automorphism A, which is in particular ergodic.

To compute these measures, we used Floyd's Algorithm (or the "tortoise and the hare algorithm"). It has appeared that on the examples of diffeomorphisms we have tested, we were able to test orders of discretization $N \simeq 2^{20}$. Thus, the first figures represent the measures $\mu_x^{f_N}$ for $N \in [\![2^{20}+1, 2^{20}+9]\!]$. We have also computed the distance between the measure $\mu_x^{f_N}$ and Lebesgue measure (see Figure 12.6). The distance we have chosen is given by the formula

$$d(\mu,\nu) = \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{i,j=0}^{2^k-1} \Big|\mu(C_{i,j,k}) - \nu(C_{i,j,k})\Big| \in [0,2],$$

where

$$C_{i,j,k} = \left[\frac{i}{2^k}, \frac{i+1}{2^k}\right] \times \left[\frac{j}{2^k}, \frac{j+1}{2^k}\right].$$

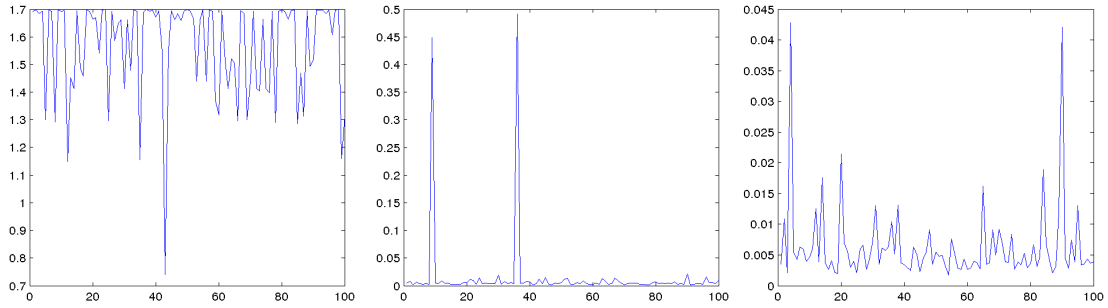

Figure 12.6: Distance between Lebesgue measure and the measure $\mu^{(h_i)_N}_{(1/2,1/2)}$ depending on N for h_1 (left), h_2 (middle) and h_3 (right), on the grids E_N with $N = 2^{20} + k$, $k = 1, \cdots, 100$.

In practice, we have computed an approximation of this quantity by summing only on the $k \in [\![0, 7]\!]$.

In the case of the diffeomorphism h_1, which is close to the identity, we observe a strong variation of the measure $\mu_x^{(h_1)_N}$ depending on N (left of Figure 12.6 and Figure 12.7). More precisely, for 7 on the 9 orders of discretization represented, these measures seems to be supported by a small curve; for $N = 2^{20} + 3$, this measure seems to be supported by a figure-8 curve, and for $N = 2^{20} + 5$, the support of the measure is quite complicated and looks like an interlaced curve. The fact that the measures $\mu_x^{(h_1)_N}$ strongly depend on N reflects the behaviour predicted by Theorem 12.1: in theory, for a generic C^1 diffeomorphism, the measures $\mu_x^{f_N}$ should accumulate on the whole set of f-invariant measures; here we see that these measures strongly depend on N (moreover, we can see on Figure 12.6 that on the orders of discretization we have tested, these measures are never close to Lebesgue measure). We have no satisfying explanation to the specific shape of the supports of the measures. When we fix the order of discretization and make vary the starting point x, the behaviour is very similar: the measures $\mu_x^{(h_1)_N}$ widely depend on the point x (see Figure 12.10). We also remark that increasing the order of discretizations does not make the measures $\mu_x^{(h_1)_N}$ evolve more smoothly.

The measures $\mu_x^{(h_2)_N}$ vary less than the measures $\mu_x^{(h_1)_N}$ (recall that h_2 is a small perturbation of a rotation of order 30). For the 8 first measures of the Figure 12.8, we obtain a measure which is very close to Lebesgue measure. The fact that we obtain measures closer to Lebesgue measures was predictable: the dynamics h_2 is close to the rotation R, whose orbits are better distributed in the torus than that of the identity. But for the order $N = 2^{20}+9$, the measure we obtain is very different from the previous one: its support seems quite close to a real orbit of R (of order 30), in particular this support covers a very small proportion of $\mathbf{T}^2$. This is what is predicted by Theorem 12.1: at least sometimes, the measures $\mu_x^{(h_2)_N}$ should not be close to Lebesgue measure. We observe exactly this behaviour when we make simulations for more different orders of discretization N (middle of Figure 12.6): for two orders N between $2^{20} + 1$ and $2^{20} + 100$, the measure is far away from Lebesgue measure. Remark that the same behaviour holds when we fix the order of discretization and make the point x vary (Figure 12.11); however, we observe that the frequency of occurrence of the event "the measure $\mu_x^{(h_2)_N}$ is close to a periodic measure with small period" is smaller in the case of Figure 12.11 than in the

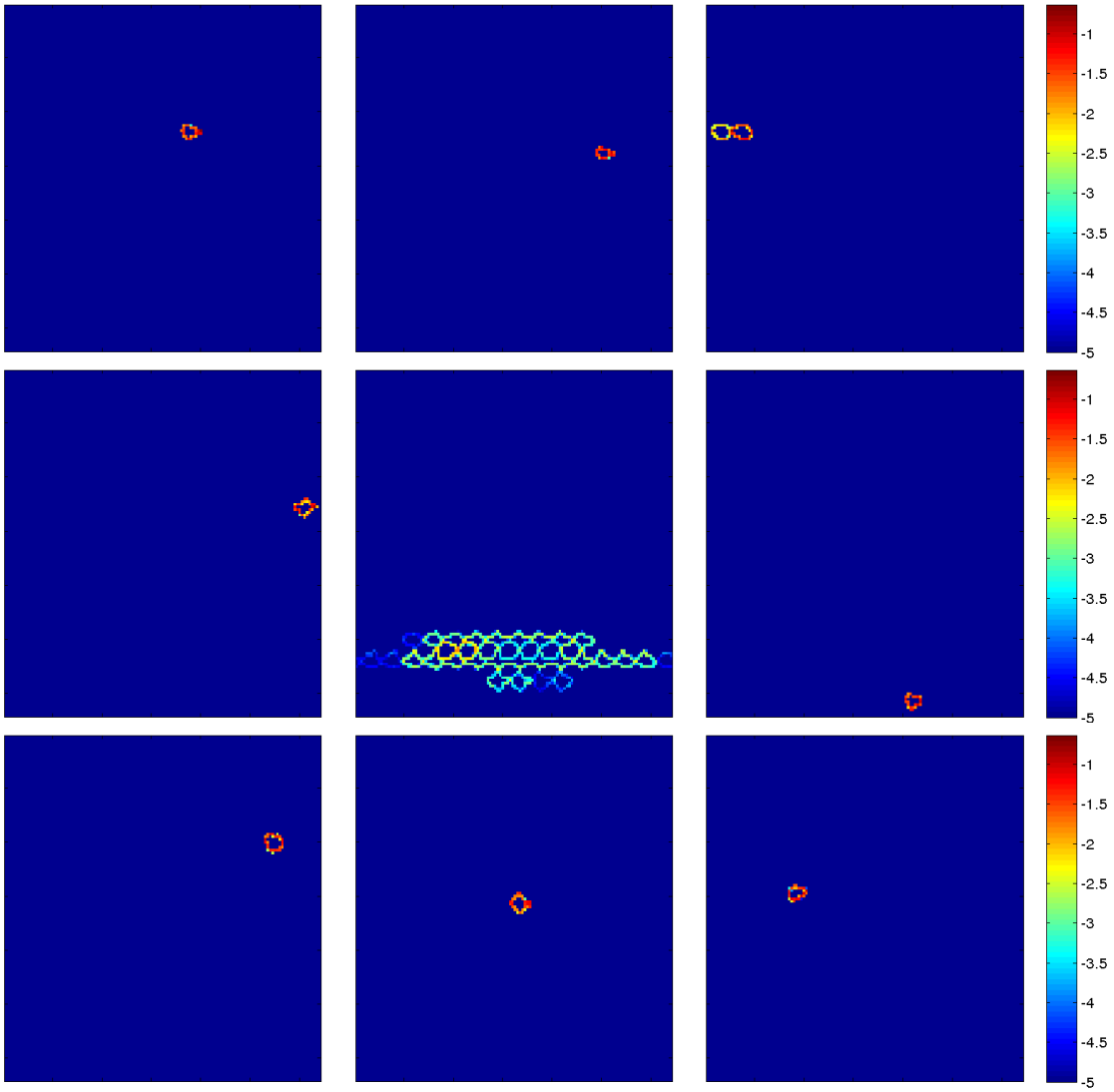

Figure 12.7: Simulations of invariant measures $\mu_x^{(h_1)_N}$ on the grids E_N, with $N = 2^{20} + i$, $i = 1, \cdots, 9$ and $x = (1/2, 1/2)$ (from left to right and top to bottom).

case of Figure 12.8. We think that it is more due to the fact that the order of discretization is bigger in the first case, than to a fundamental difference of the processes used to produce these simulations.

The behaviour of the measures $\mu_x^{(h_3)_N}$, where h_3 is a small C^1-perturbation of the linear Anosov map A, is quite close to that of the the measures $\mu_x^{(h_2)_N}$ (see Figure 12.9): most of the time, these measures are close to Lebesgue measure, but for one order of discretization N (here, $N = 2^{20} + 4$), the measure becomes very different from Lebesgue measure (we can see on the right of Figure 12.6 that this phenomenon appears twice when $N \in [\![2^{20} + 1, 2^{20} + 100]\!]$). The difference with the case of h_2 is that here, the "exceptional" measure is much better distributed than for h_2 (the maximal measure of a region of size $1/128 \times 1/128$ is close to $10^{-3.5}$ for h_3 and close to $10^{-1.5}$ for h_2). The same phenomenon holds when we fix the order of discretization but change the starting point x (see Figure 12.12), except that (as for f_2) the number of apparition of measures that

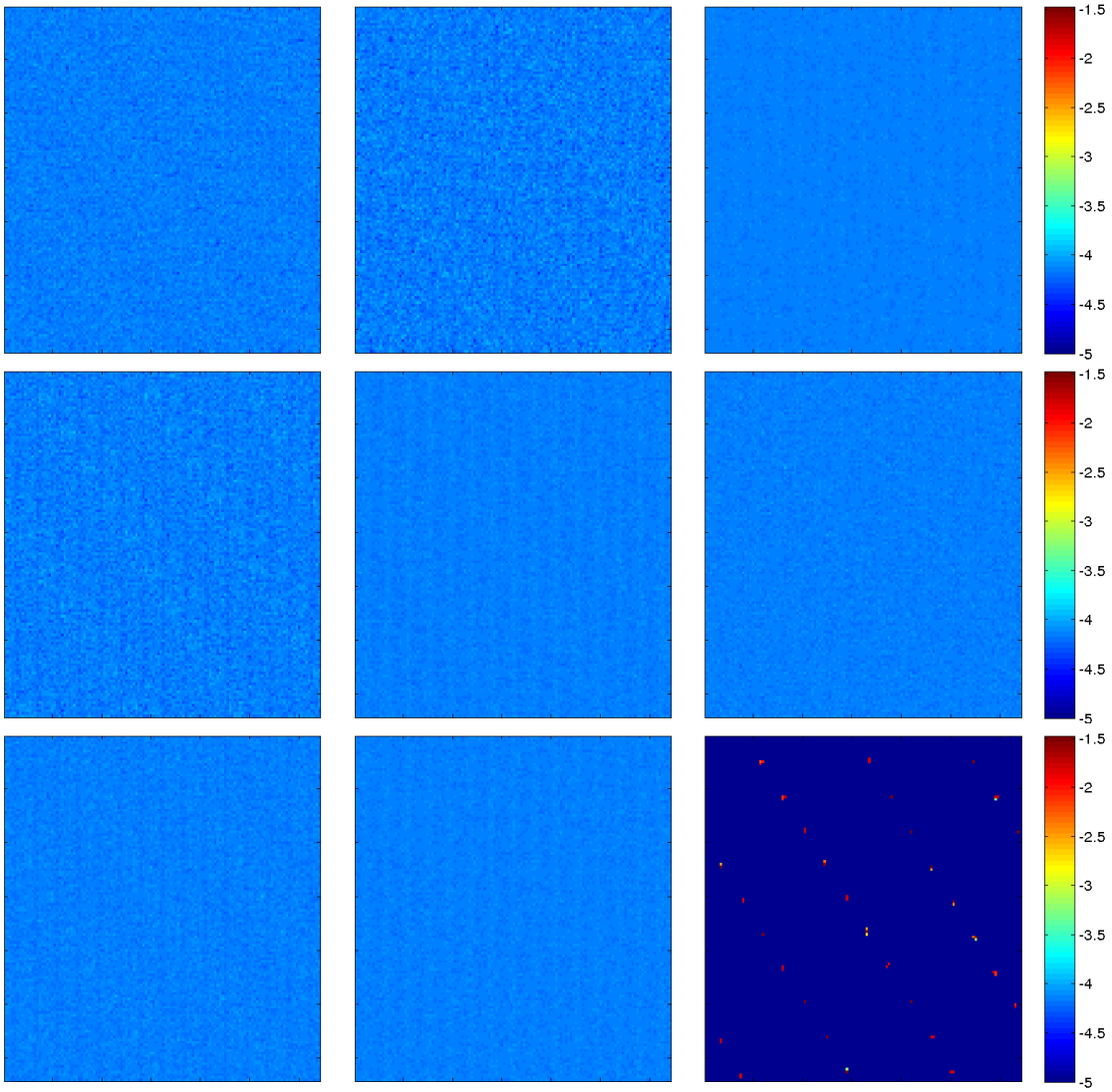

Figure 12.8: Simulations of invariant measures $\mu_x^{(h_2)_N}$ on the grids E_N, with $N = 2^{20} + i$, $i = 1, \cdots, 9$ and $x = (1/2, 1/2)$ (from left to right and top to bottom).

are singular with respect to Lebesgue measure is smaller than in Figure 12.9. Again, we think that this follows from the fact that the orders of discretizations tested are bigger. In this case, the simulations suggest the following behaviour: when the order of discretization N increases, the frequency of apparition of measures $\mu_x^{(h_3)_N}$ far away from Lebesgue measures tends to 0.

Recall that Addendum 12.2 states that if x is fixed, then for a generic $f \in \mathrm{Diff}^1(\mathbf{T}^2, \mathrm{Leb})$, the measures $\mu_x^{f_N}$ accumulate on the whole set of f-invariant measures, but do not say anything about, for instance, the frequency of orders N such that $\mu_x^{f_N}$ is not close to Lebesgue measure. It is natural to think that this frequency depends a lot on f; for example that such N are very rare close to an Anosov diffeomorphism and more frequent close to an "elliptic" dynamics like the identity. The results of numerical simulations seem to confirm this heuristic.

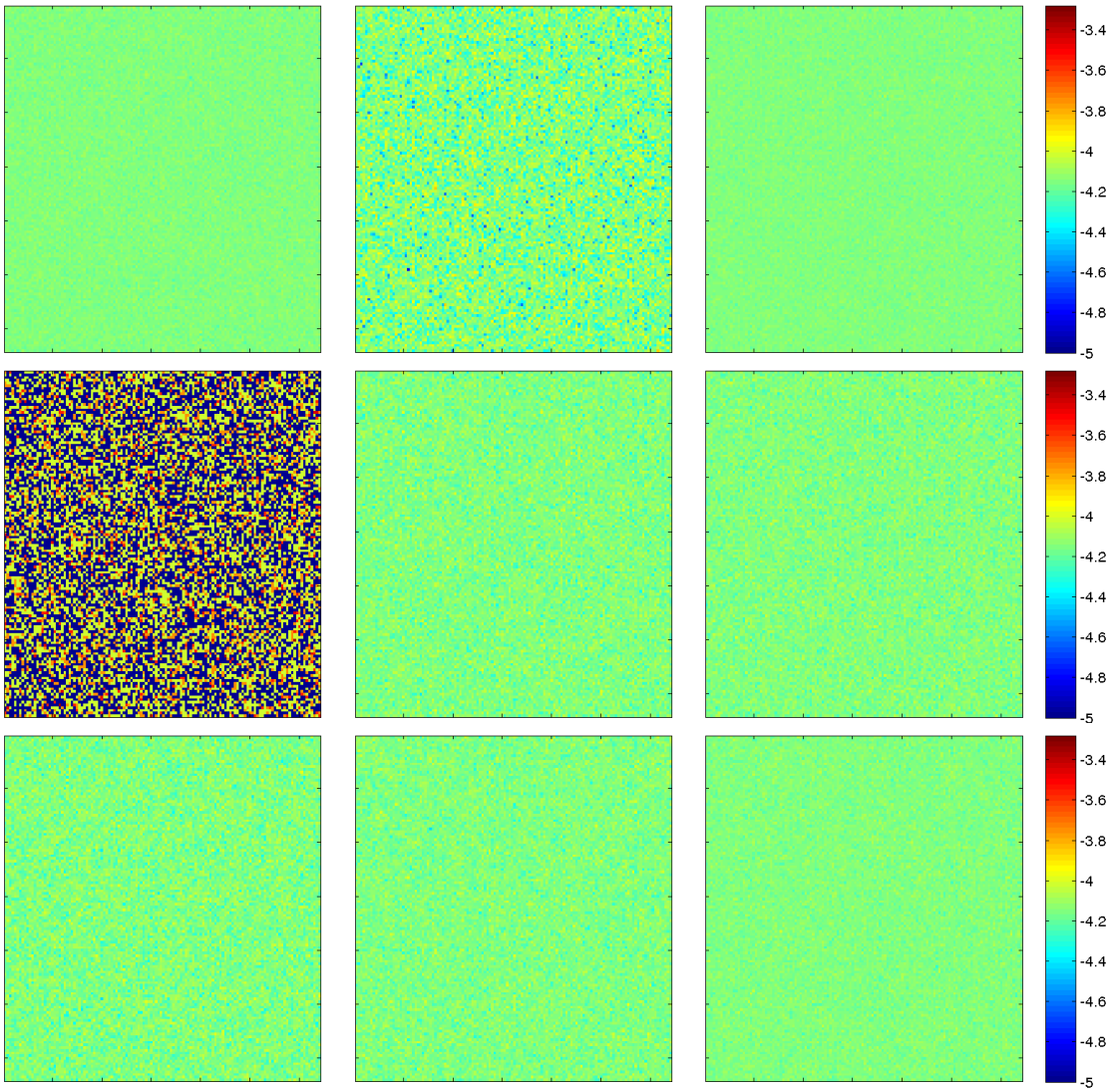

Figure 12.9: Simulations of invariant measures $\mu_x^{(h_3)_N}$ on the grids E_N, with $N = 2^{20} + i$, $i = 1, \cdots, 9$ and $x = (1/2, 1/2)$ (from left to right and top to bottom).

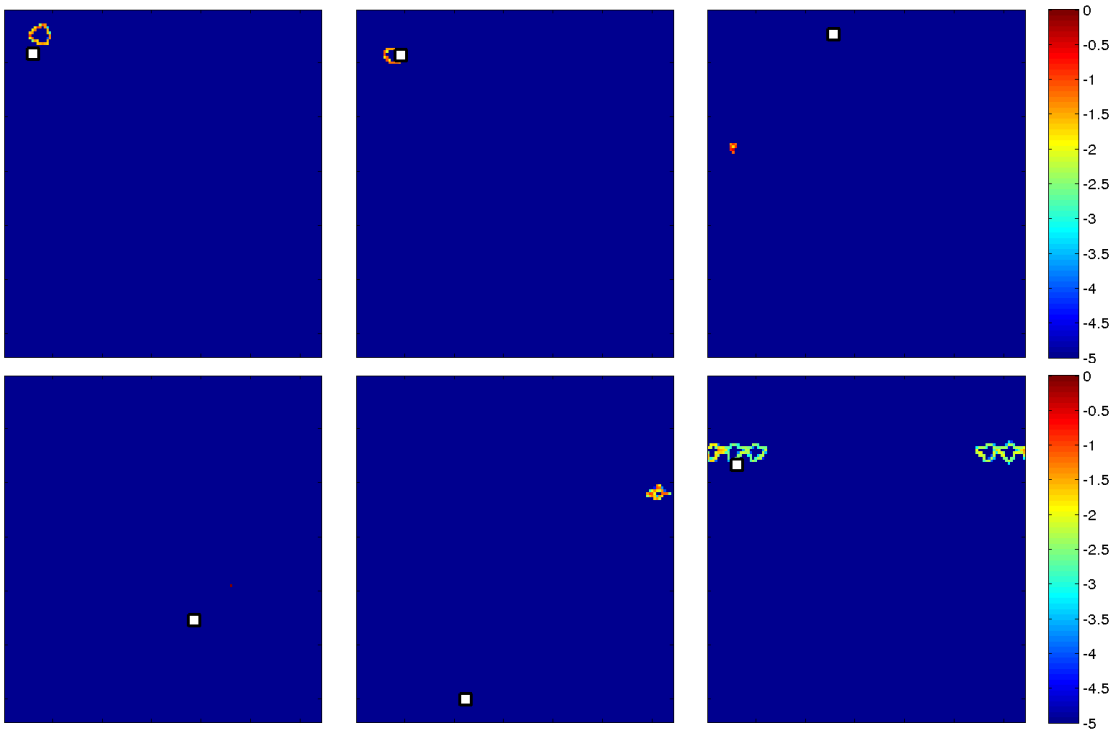

Figure 12.10: Simulations of invariant measures $\mu_x^{(h_1)_N}$ on the grid E_N, with $N = 2^{23}$, and x a random point of $\mathbf{T}^2$, represented by the black and white box. The behaviour observed on the top left picture is the most frequent, but we also observe other kind of measures: for example, the measures has a very small support like on the bottom left picture on about 10 of the 100 random draws we have made; we even see appearing the strange behaviour of the last picture once.

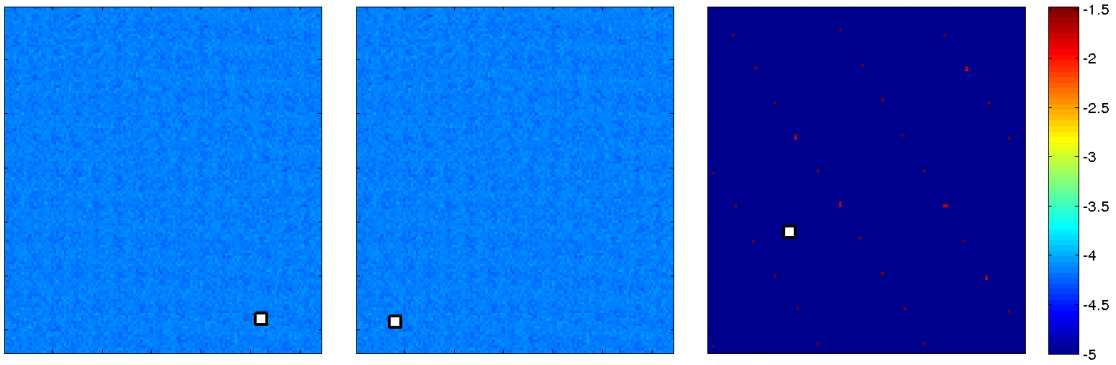

Figure 12.11: Simulations of invariant measures $\mu_x^{(h_2)_N}$ on the grid E_N, with $N = 2^{23} + 5$, and x a random point of $\mathbf{T}^2$, represented by the black and white box. The behaviour observed on the two firsts pictures is the most frequent, but sometimes (in fact, twice on 1 000 random draws), we also observe a measure whose support is very close to a periodic point with small period, like on the right picture.

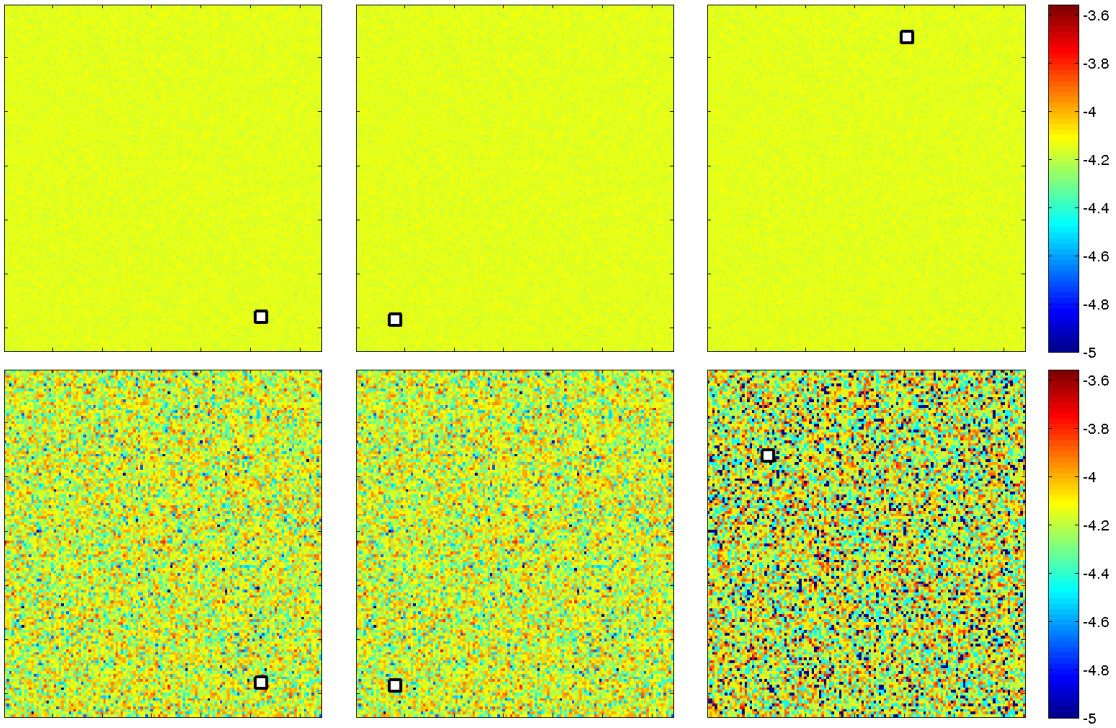

Figure 12.12: Simulations of invariant measures $\mu_x^{(h_3)_N}$ on the grid E_N, with $N = 2^{23} + 1$ (top) and $N = 2^{23} + 17$ (bottom), and x a random point of $\mathbf{T}^2$, represented by the black and white box. The behaviour observed on the three top pictures is the most frequent (for 17 over the 20 orders $N = 2^{23} + i$, $i = 0, \cdots, 19$, all the 100 random draws we have made gave a measure very close to Leb), but seldom we also observe measures further from Lebesgue measure, like what happens for $N = 2^{23} + 17$ (bottom), where 99 over the 100 random draws of x produce a measure identical to the two first pictures, and the other random draw gives a measure a bit more singular with respect to Lebesgue measure (bottom right).

12.3.2 Simulations of the measures $\mu_{\mathbf{T}^2}^{f_N}$ for conservative torus diffeomorphisms

We now present the results of numerical simulations of the measures $\mu_{\mathbf{T}^2}^{f_N}$. Recall that the measure $\mu_{\mathbf{T}^2}^{f_N}$ is supported by the union of periodic orbits of f_N, and is such that the total measure of each periodic orbit is equal to the cardinality of its basin of attraction.

First, we simulate a conservative diffeomorphism f_5 which is close to the identity in the C^1 topology. We have chosen $f_5 = Q \circ P$, with

$$p(x) = \frac{1}{209}\cos(2\pi \times 17x) + \frac{1}{271}\sin(2\pi \times 27x) - \frac{1}{703}\cos(2\pi \times 35x),$$

$$q(y) = \frac{1}{287}\cos(2\pi \times 15y) + \frac{1}{203}\sin(2\pi \times 27y) - \frac{1}{841}\sin(2\pi \times 38y).$$

We have also simulated the conservative diffeomorphism $f_6 = f_5 \circ A$, with A the standard Anosov automorphism

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

thus f_6 is a small C^1 perturbation of A; in particular the theory asserts that it is topologically conjugated to A. We can test whether this property can be observed on simulations or not.

First of all, we present the results of the simulations of the size of the recurrent set $\Omega((f_i)_N)$, the number of periodic orbits of $(f_i)_N$ (middle) and the length of the largest periodic orbit of $(f_i)_N$, for $N = 128k$ and k going from 1 to 150 (Figure 12.13).

For f_5, the cardinality of the recurrent set is clearly increasing; it looks as if it behaves like $N \ln N$. Fact remains that this behaviour is very different from the one we observe for the simulations of the homeomorphisms f_3 and f_4 (which are conservative homeomorphisms, which are respectively small C^0 perturbations of Id and A, see Section 5.8, and more precisely Figure 5.7). For its part, the evolution of the cardinality of the recurrent set of f_6 is much more erratic, and is quite similar to that observed in the corresponding C^0 case (top right of Figure 5.7). Anyway, the behaviours are completely different at the neighbourhood of the identity and of the linear automorphism A.

As for the cardinality of the recurrent set, the number of periodic orbits of $(f_5)_N$ is very smooth and seems to evolve linearly in N. Obviously, this behaviour is very different from the case of homeomorphisms; however, we have no explanation to this "smooth" behaviour. In the neighbourhood of A, the number of periodic orbits (middle right of Figure 12.13) seems to be uniformly bounded in N by 18. Again, this behaviour is very similar to what happens to f_4, which is a small C^0 perturbation of A (Figure 5.7). For now, we do not have explanation to this behaviour.

The maximal period of a periodic orbit of $(f_5)_N$ evolves less smoothly then the number of periodic orbits. It is even quite similar to the corresponding C^0 case (Figure 5.7). The same holds in the neighbourhood of A for the diffeomorphism f_6. Seemingly, the maximal period of a periodic orbit of f_N does not give a good criterion to test if a map behaves like a C^1 system or not.

For f_5, the distance $d(\mu_{\mathbf{T}^2}^{f_N}, \mathrm{Leb})$ is quite quickly smaller than 0.1, and oscillates between 0.05 and 0.1 from $N = 128 \times 30$. It is not clear if in this case, the sequence of

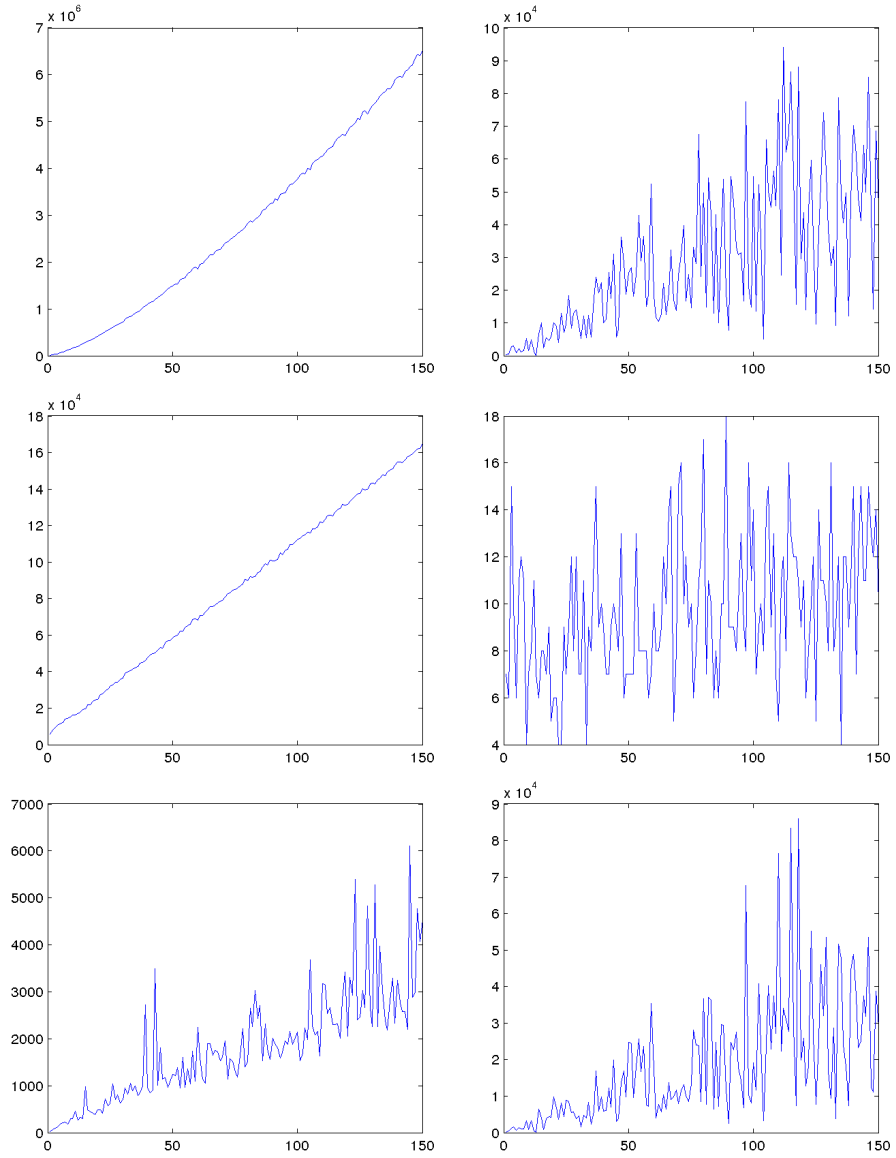

Figure 12.13: Size of the recurrent set $\Omega((f_i)_N)$ (top), number of periodic orbits of $(f_i)_N$ (middle) and length of the largest periodic orbit of $(f_i)_N$ (bottom) depending on N, for f_5 (left) and f_6 (right), on the grids E_N with $N = 128k$, $k = 1, \cdots, 150$.

measures $(\mu_{\mathbf{T}^2}^{f_N})_N$ converge towards Lebesgue measure or not (while for the C^0 perturbation of the identity we have tested, it is clear that these measures do not converge to anything, see Figure 5.8). The distance $d(\mu_{\mathbf{T}^2}^{f_N}, \text{Leb})$ even seem to increase slowly (in average — there are a lot of oscillations) from $N = 50 \times 128$. We have the same kind of conclusion for f_6: by looking at Figure 12.14, we can not decide if the sequence of measures $(\mu_{\mathbf{T}^2}^{f_N})$ seem to tend to Lebesgue measure or not.

The behaviour of the computed invariant measures $\mu_{\mathbf{T}^2}^{(f_5)_N}$, where f_5 is a small C^1 perturbation of the identity, is way smoother than in the C^0 case (compare Figure 12.15 with Figure 5.9). Indeed, the measure $\mu_{\mathbf{T}^2}^{(f_5)_N}$ has quickly a big component which is close

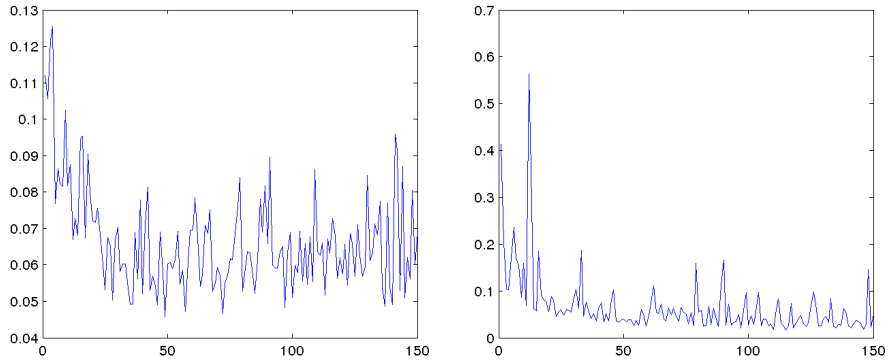

Figure 12.14: Distance between Lebesgue measure and the measure $\mu_{\mathbf{T}^2}^{(f_i)_\mathrm{N}}$ depending on N for f_5 (left) and f_6 (right), on the grids E_N with $\mathrm{N} = 128k$, $k = 1, \cdots, 150$.

to Lebesgue measure: the images contain a lot of light blue. Thus, we could be tempted to conclude that these measures converge to Lebesgue measure. However, there are still little regions that have a big measure: in the example of Figure 12.15, it seems that there are invariant curves that attract a lot of the points of the grid (as can also be observed on Figure 12.7). We have no explanation to this phenomenon, and we do not know if it still occurs when the order of discretization is very large.

For the discretizations of f_6, the simulations on grids of size $2^k \times 2^k$ might suggest that the measures $\mu_{\mathbf{T}^2}^{(f_6)_\mathrm{N}}$ tend to Lebesgue measure (Figure 12.16). Actually, when we perform a lot of simulations, we realize that there are also big variations of the behaviour of the measures (Figure 12.17): the measure is often well distributed in the torus, and sometimes quite singular with respect to Lebesgue measure (as it can be seen in Figure 12.14). This behaviour is almost identical to that observed in the C^0 case in the neighbourhood of A (see Figures 5.11 and 5.12, and also the corresponding discussion).

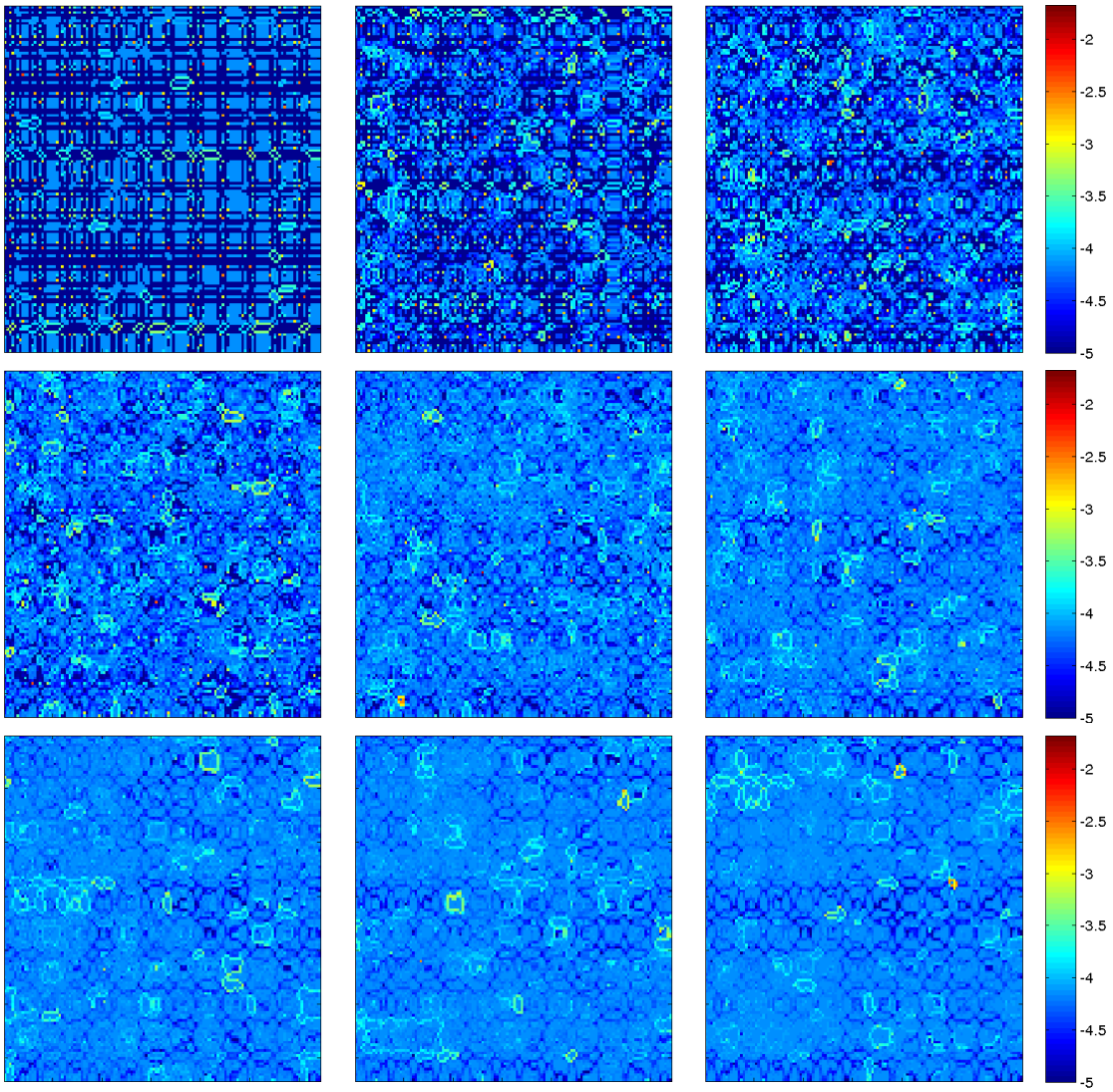

Figure 12.15: Simulations of invariant measures $\mu_{\mathbf{T}^2}^{(f_5)_N}$ on the grids E_N, with $N = 2^k$, $k = 7, \cdots, 15$ (from left to right and top to bottom).

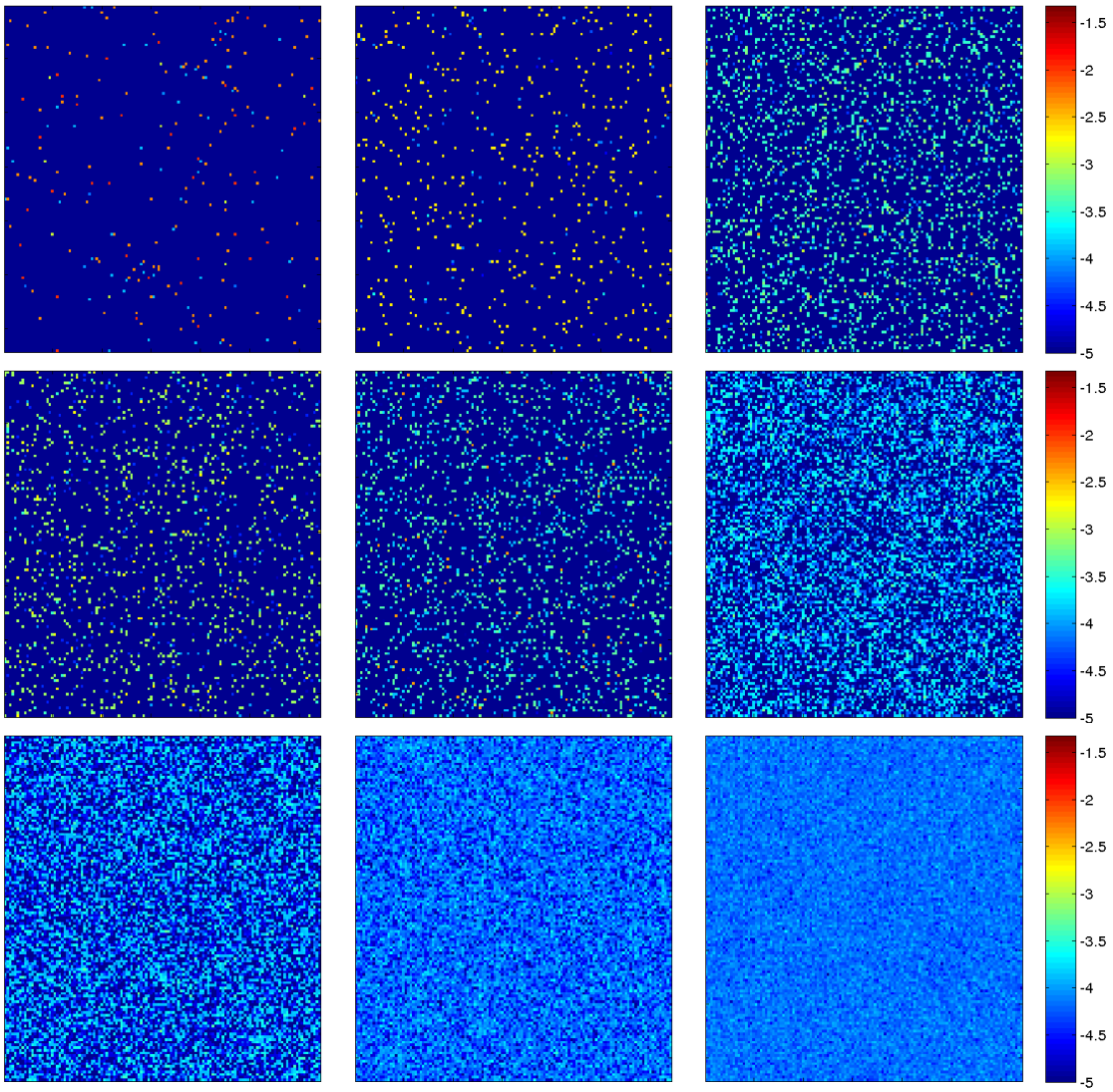

Figure 12.16: Simulations of invariant measures $\mu_{\mathbf{T}^2}^{(f_6)_N}$ on the grids E_N, with $N = 2^k$, $k = 7, \cdots, 15$ (from left to right and top to bottom).

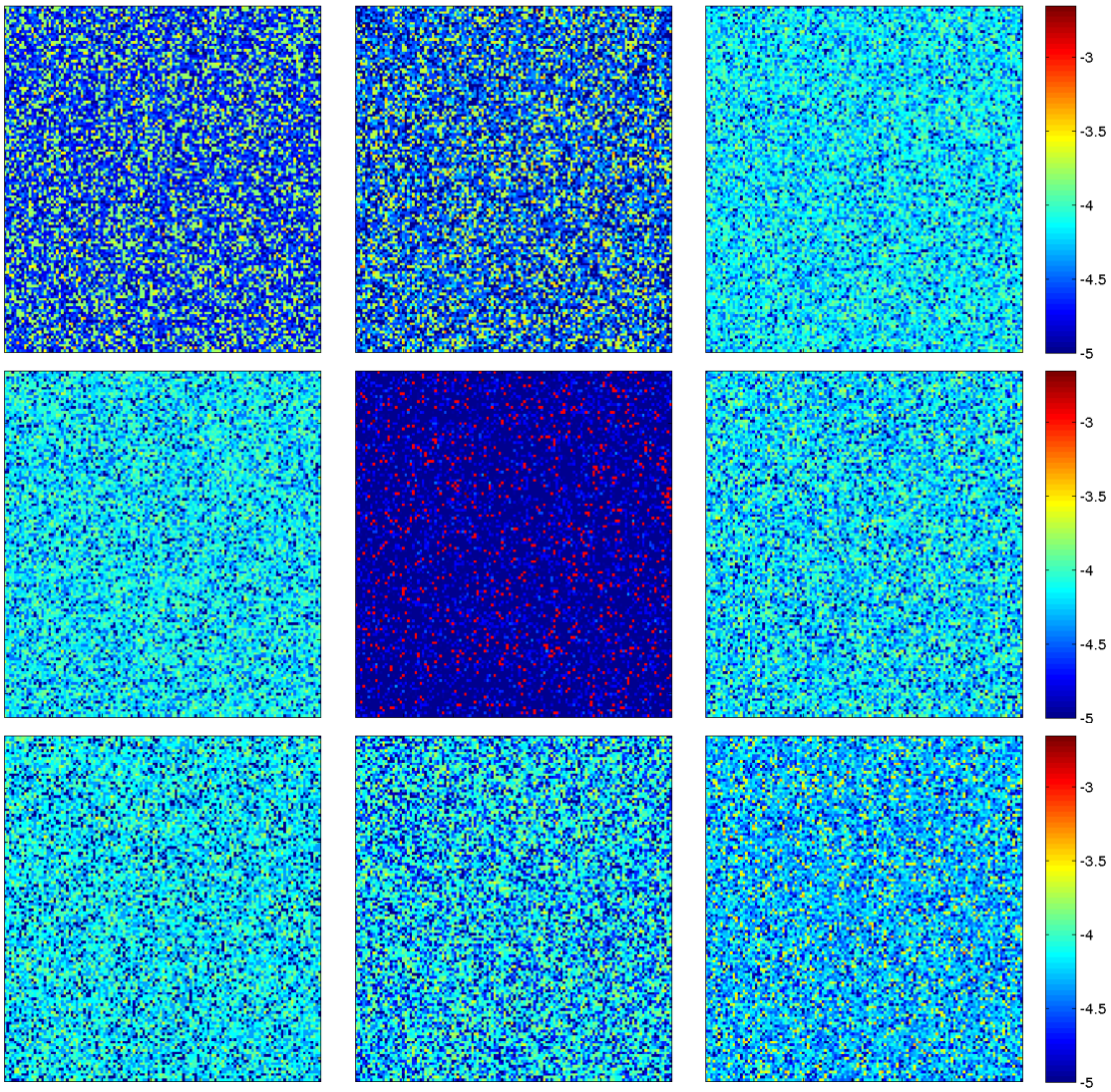

Figure 12.17: Simulations of invariant measures $\mu_{\mathbf{T}^2}^{(f_6)_N}$ on the grids E_N, with $N = 11\,516, \cdots, 11\,524$ (from left to right and top to bottom).

12.3.3 The case of expanding maps

We now present the results of the numerical simulations we have conducted for expanding maps of the circle. We have tested the following expanding map of degree 2:

$$f(x) = 2x + \varepsilon_1 \cos(2\pi x) + \varepsilon_2 \sin(6\pi x),$$

with $\varepsilon_1 = 0.127\,943\,563\,72$ and $\varepsilon_2 = 0.008\,247\,359\,61$.

We focus on the simulations of both measures $(f_N^*)^k \lambda_N$ and $\mu_{\mathbf{S}^1}^{f_N}$. Recall that the former is simply the push-forward of the uniform measure on E_N by the iterate f_N^k of the discretization. The latter is the measure supported by the union of periodic orbits of f_N, such that the total measure of a periodic orbit is equal to the size of its basin of attraction.

Recall that as f is expanding and belongs to $C^2(\mathbf{S}^1)$, it has a single SRB measure, that we denote by μ_0. This measure can be computed quite easily using the Ruelle-Perron-Frobenius transfer operator. In Figure 12.18, we have represented the distance $d(\mu_0, (f_N^*)^k \lambda_N)$ depending on the time k, for various orders of discretization N. The distance d is defined by

$$d(\mu, \nu) = \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{i=0}^{2^k-1} \left| \mu\left(\left[\frac{i}{2^k}, \frac{i+1}{2^k}\right]\right) - \nu\left(\left[\frac{i}{2^k}, \frac{i+1}{2^k}\right]\right) \right| \in [0, 2].$$

This distance spans the weak-* topology, which makes compact the set of probability measures on $\mathbf{T}^2$. In practice, we have computed an approximation of this quantity by summing only on the $k \in [\![0, 7]\!]$.

On Figure 12.18, we observe that for N being fixed, the distance between the SRB measure μ_0 and the image measure $(f_N^*)^k \lambda_N$ reaches quite quickly a value quite close to 0, to increase thereafter. We shall notice that for the smaller values of N ($N = 2^7$ and $N = 2^9$), the distance $d(\mu_0, (f_N^*)^k \lambda_N)$ looks eventually periodic: it seems that the stabilization time of the discretization is attained, or at least that for almost all the points $x \in E_N$, we have $f_N^{1\,024}(x) \in \Omega(f_N)$.

The fact that the image measure $(f_N^*)^k \lambda_N$ is very close to the SRB measure for short times k is not surprising, as it was predicted by Theorem 12.15. For example, in Figure 12.19, we can see that for $N = 2^{19}$, the distance between μ_0 and $(f_N^*)^8 \lambda_N$ is approximatively 10^{-4}

We think that growth of the distance $d(\mu_0, (f_N^*)^k \lambda_N)$ is increasing from a certain point is due to the fact that the roundoff errors arising from the discretization process imply that locally, the discrepancy between the image measure and the appropriate uniform measure grows exponentially with k (see Proposition 9.4). However, for now, we have neither made the precise study of the discrepancy of a union of "independent" image sets (what happens for maps of degree $d \geq 2$), nor the application of the discrepancy to the non-linear case.

Figures 12.20 and 12.21display the density of the measure $\mu_{\mathbf{S}^1}^{f_N}$ for various values of N. Observe that even when N is quite large (for $N = 2^{19}$ on Figure 12.20 and for $N \geq 2^{28}$ on Figure 12.21), the measure $\mu_{\mathbf{S}^1}^{f_N}$ is quite far away from the SRB measure μ_0. Thus, there is no evidence (at least on this example) that the measures $\mu_{\mathbf{S}^1}^{f_N}$ (or the

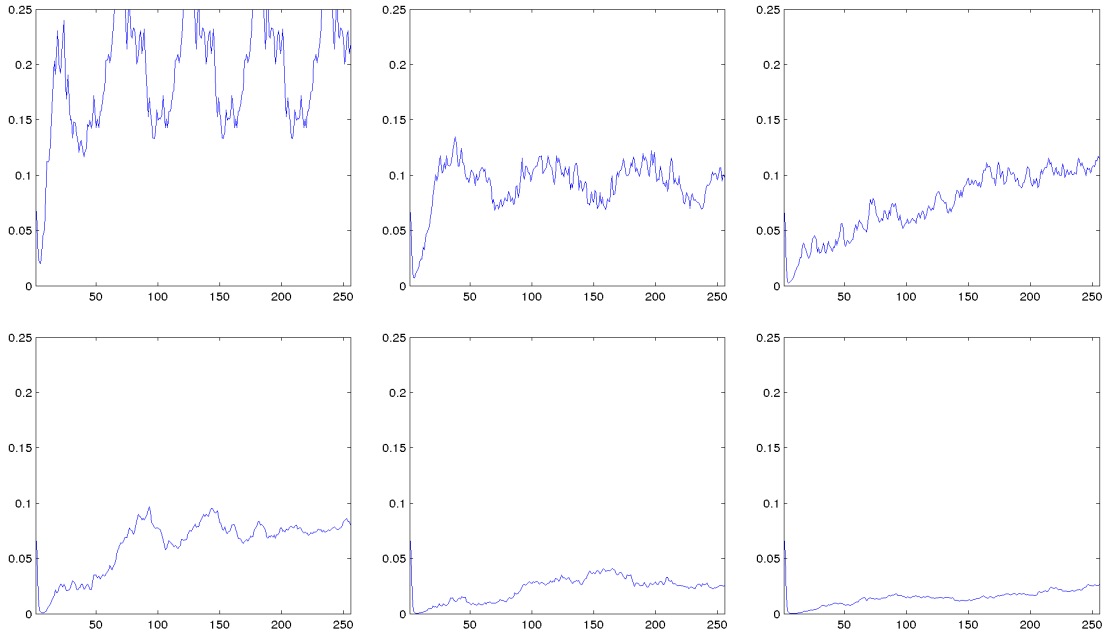

Figure 12.18: Distance between the SRB measure of f and the measure $(f_N^*)^k \lambda_N$ depending on the time k ($1 \leq k \leq 1\,024$), for $N = 2^j$, with (from left to right and top to bottom) $j = 9$, 11, 13, 15, 17 and 19.

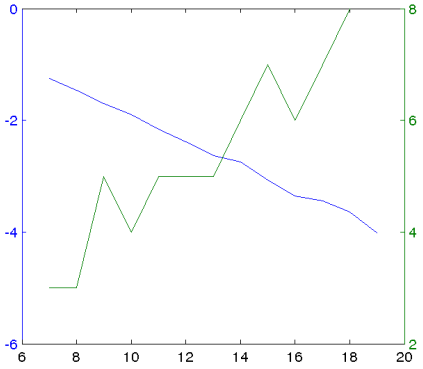

Figure 12.19: Logarithm in base 10 of the minimum over k of $d(\mu_0, (f_N^*)^k \lambda_N)$ depending on N (in blue, left axis), and time k realizing this minimal distance (in green, right axis), for $N = 2^j$, with $j \in [\![7, 19]\!]$.

measures $(f_N^*)^k \lambda_N$ for "large" times k) converge towards the SRB measure when N goes to infinity.

However, we can see on Figure 12.22 that when we take the average of these measures on a large number of different grids, the measure we obtain is very close to the SRB measure. This suggests that on this example, the averages

$$\frac{1}{M} \sum_{N=0}^{M} \mu_{S^1}^{f_N}$$

might converge towards the measure μ_0.

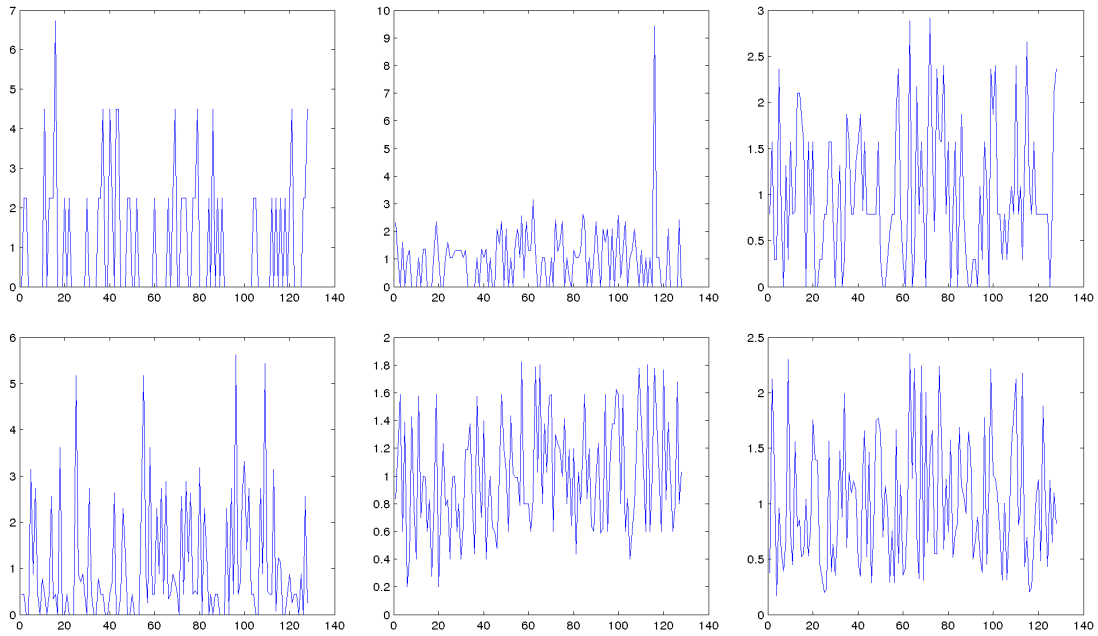

Figure 12.20: Density of the measure $\mu_{S^1}^{f_N}$, for $N = 2^j$, with (from left to right and top to bottom) $j = 7, 9, 11, 13, 15, 17$ and 19.

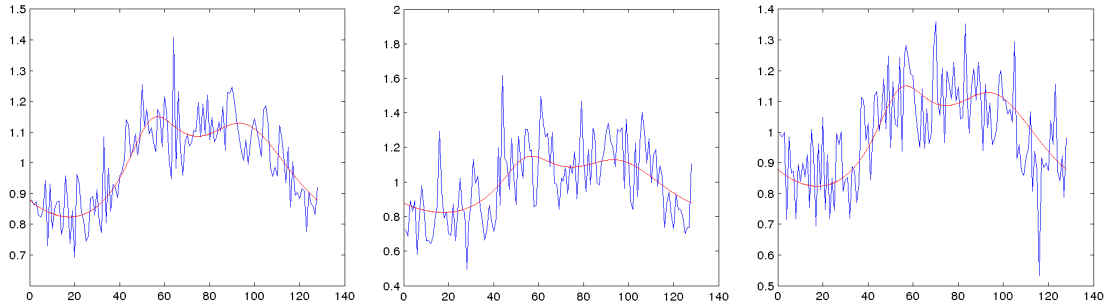

Figure 12.21: Density of the measure $\mu_{S^1}^{f_N}$, for $N = 5.10^8 + j$, $j = 1, 2, 3$. The red curves represent the density of the SRB measure μ_0.

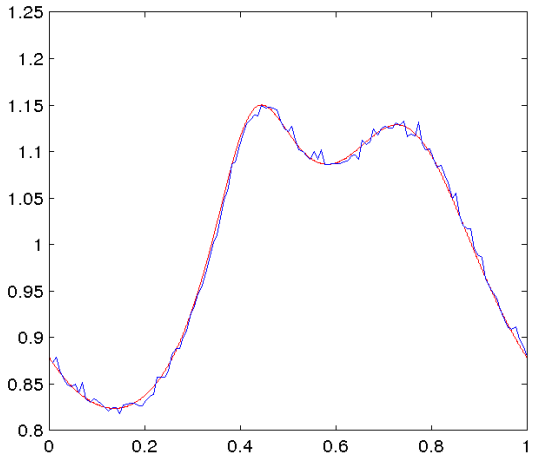

Figure 12.22: Density of the SRB measure μ_0 of f, and mean of the densities of the measures $\mu_{S^1}^{f_N}$, for $N = 5.10^8 + j$, $j \in [\![1, 1\,000]\!]$.

Quatrième partie

Discussion des résultats et pistes de recherche

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Chapitre 13

Différents facteurs dont dépendent les résultats

Cette thèse est consacrée à l'étude des discrétisations dans plusieurs contextes (dynamiques conservatives ou dissipatives, continues ou différentiables, etc.) et selon plusieurs point de vue (combinatoire, topologique, ergodique ; étude d'une seule discrétisation ou de toutes les discrétisations, etc.). Dans ce chapitre, on discute de l'influence de ces divers facteurs sur la dynamique des discrétisations. Cela semble d'autant plus nécessaire que certains énoncés concernant des questions similaires se retrouvent éloignés de presque 150 pages dans ce manuscrit, simplement parce qu'ils concernent des dynamiques de départ différentes (c'est le cas par exemple des corollaires 5.9 et 10.4).

Comme dans l'introduction, nous nous plaçons pour simplifier dans le cas où X est le tore $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$, muni s'il y a lieu de la mesure de Lebesgue notée Leb, et où les grilles sont les grilles uniformes sur le tore, définies par

$$\mathrm{E_N} = \left\{ \left(\frac{i}{\mathrm{N}}, \frac{j}{\mathrm{N}} \right) \in \mathbf{R}^2/\mathbf{Z}^2 \;\middle|\; 1 \leq i, j \leq \mathrm{N} \right\}.$$

Les résultats présentés sont en réalité vrais dans divers contextes plus généraux ; on se référera aux parties du manuscrit concernées pour les définitions précises de ces contextes.

13.1 Système particulier *versus* système générique

Au début de l'introduction (page 17), on a vu que la dynamique des discrétisations d'un système particulier — par exemple un difféomorphisme d'Anosov linéaire du tore — peut être assez singulière. Plus précisément, dans le cas de l'exemple donné (voir la figure 1.1 et [Ghy94]), on observe que d'une part les discrétisations sont toutes bijectives, et que d'autre part on a une récurrence très rapide. Nous avons interprété cela comme étant dû à un phénomène de résonance entre l'automorphisme linéaire, qui est une dynamique très particulière, et les grilles [1].

1. On a un phénomène similaire pour le doublement de l'angle sur le cercle, défini sur $\mathbf{R}/\mathbf{Z}$ par $x \mapsto 2x$; avec $\mathbf{R}/\mathbf{Z}$ muni des grilles $\mathrm{E_N} = \mathbf{Z}/\mathrm{N}$. Si N est une puissance de 2, alors la discrétisation f_N possède un unique point fixe, qui attire tous les points de la grille. Si, en revanche, N est premier avec 2, alors f_N est une permutation cyclique de la grille. Dans aucun de ces deux cas la dynamique de la discrétisation ne reflète la vraie dynamique de $x \mapsto 2x$.

Ce phénomène contraste avec les résultats concernant les discrétisations d'homéomorphismes conservatifs génériques obtenus au cours de cette thèse. Par exemple, rappelons le théorème II de l'introduction (corollaire 5.24 page 83).

Theorème XVIII. *Pour un homéomorphisme conservatif générique* $f \in \mathrm{Homeo}(\mathbf{T}^2, \mathrm{Leb})$, *la suite* $(\mathrm{D}(f_N))_{N\geq 0}$ *s'accumule sur tout le segment* $[0,1]$.

En particulier, il existe une sous-suite de discrétisations dont le cardinal de l'ensemble récurrent [2] est arbitrairement petit par rapport au cardinal de la grille : une infinité de discrétisations sont très peu injectives. Le comportement dynamique générique des discrétisations est donc bien différent de celui des discrétisations de certains systèmes particuliers.

On remarquera tout de même que le fait de considérer des homéomorphismes génériques une fois la suite de grilles fixée induit une certaine résonance, certes moins marquée que dans l'exemple ci-dessus, entre la dynamique et certaines grilles de la suite : les résultats que l'on obtient expriment que pour un homéomorphisme conservatif générique f, chaque partie de la dynamique de f entre en résonance avec une infinité de grilles. Par exemple, ceci est illustré par le théorème V de l'introduction (théorème 5.43 page 90).

Theorème XIX. *Pour un homéomorphisme conservatif générique* $f \in \mathrm{Homeo}(\mathbf{T}^2, \mathrm{Leb})$, *pour toute mesure de probabilité* μ *invariante par* f, *il existe une sous-suite de discrétisations* $(f_{N_k})_k$ *telle que* f_{N_k} *possède une unique mesure invariante* μ_k, *qui tend vers* μ. *Autrement dit, pour toute mesure de probabilité* f*-invariante* μ, *il existe une suite d'entiers* $(N_k)_{k\geq 0}$ *telle que, pour tout* $x \in \mathbf{T}^2$, *on a*

$$\mu_x^{f_{N_k}} \underset{k\to+\infty}{\longrightarrow} \mu.$$

Le même phénomène se produit — en un sens plus faible — pour les difféomorphismes de classe C^1 conservatifs génériques : voir par exemple le théorème VI de l'introduction (théorème 12.1 page 251).

Theorème XX. *Pour un difféomorphisme de classe* C^1 *conservatif générique* $f \in \mathrm{Diff}^1(\mathbf{T}^2, \mathrm{Leb})$, *pour un point générique* $x \in \mathbf{T}^2$ *(dépendant de* f*), toute mesure* μ *invariante par* f, *il existe une sous-suite* $(f_{N_k})_k$ *de discrétisations telles que*

$$\mu_x^{f_{N_k}} \underset{k\to+\infty}{\longrightarrow} \mu.$$

Signalons aussi que certains des résultats de généricité, qui concernent *a priori* des dynamiques abstraites, s'observent sur des exemples concrets (voir la section 13.8 pour une discussion plus détaillée).

13.2 Conservatif *versus* dissipatif

Lorsqu'on observe les discrétisations d'une dynamique dissipative f, on arrive bien évidemment à détecter les ouverts attractifs de f : c'est une propriété bien connue, vérifiée plus généralement par les pseudo-orbites.

2. Rappelons que l'ensemble récurrent est l'union des orbites périodiques.

Dans le cas d'un homéomorphisme dissipatif générique, cette convergence de la dynamique des discrétisations vers celle de l'homéomorphisme s'observe plus généralement sur bon nombre de propriétés. Par exemple, lorsqu'on s'intéresse à la dynamique combinatoire des discrétisations d'un homéomorphisme dissipatif générique, on observe un comportement assez régulier ; on démontre par exemple l'énoncé suivant (corollaire 4.14 page 64).

Proposition XXI. *Pour un homéomorphisme dissipatif générique* $f \in \mathrm{Homeo}(\mathbf{T}^2)$, *le taux d'injectivité* $\mathrm{Card}\big(f_N(E_N)\big)/\mathrm{Card}(E_N)$ *de* f_N *tend vers 0 lorsque* N *tend vers l'infini.*

A fortiori, le degré de récurrence $D(f_N) = \mathrm{Card}\big(\Omega(f_N)\big)/\mathrm{Card}(E_N)$ tend vers 0. Ce résultat se déduit directement d'une propriété similaire pour la dynamique initiale : un homéomorphisme dissipatif générique f est *totalement singulier*, c'est-à-dire que pour tout $\varepsilon > 0$, il existe un ouvert de mesure de Lebesgue supérieure à $1-\varepsilon$ dont l'image par f est de mesure de Lebesgue inférieure à ε (voir [AA13] et la définition 4.1 page 58 du présent manuscrit).

Du point de vue ergodique aussi, la dynamique des discrétisations converge vers celle de f. Plus précisément, nous démontrons le résultat suivant (théorème 4.16 page 65).

Theorème XXII. *Pour un homéomorphisme dissipatif générique* f, *les mesures* $\mu_{\mathbf{T}^2}^{f_N}$, *qui sont les limites au sens de Cesàro des poussés en avant de la mesure uniforme sur* E_N *par les itérés de* f_N, *tendent vers la mesure* $\mu_{\mathbf{T}^2}^{f}$, *qui est la limite au sens de Cesàro des poussés en avant de la mesure de Lebesgue par les itérés de* f.

Ainsi, pour un homéomorphisme dissipatif générique, la dynamique « physique » des discrétisations, *i.e.* la dynamique d'une proportion arbitrairement grande des points de la grille, converge vers la dynamique « physique » de l'homéomorphisme, *i.e.* la dynamique de presque tous les points relativement à la mesure de Lebesgue. C'est un phénomène qui était prévisible, vu qu'à une échelle donnée, la dynamique d'un homéomorphisme dissipatif générique est essentiellement une dynamique de « puits ».

Dans le cas conservatif, le comportement des discrétisations est bien moins régulier. Par exemple, le théorème XVIII affirme que, contrairement au cas dissipatif générique, le degré de récurrence des discrétisations d'un homéomorphisme conservatif générique s'accumule sur tout le segment $[0,1]$. D'un point de vue ergodique, le théorème XIX affirme que toute mesure invariante (et pas seulement la mesure physique) d'un homéomorphisme conservatif générique est « vue » par une infinité de discrétisations.

Concernant la dichotomie conservatif/dissipatif pour les homéomorphismes, la morale que l'on peut retenir est la suivante. Si on ne suppose pas que l'homéomorphisme préserve la mesure de Lebesgue, alors la dynamique des discrétisations converge vers la dynamique de l'homéomorphisme relative à la mesure de Lebesgue. Si, en revanche, on suppose que l'homéomorphisme préserve la mesure de Lebesgue, alors l'ensemble des dynamiques des discrétisations reflète toutes les dynamiques possibles de l'homéomorphisme (orbite périodique, compact invariant, mesure invariante, vecteur de rotation, etc.), et pas seulement celles de presque tout point pour la mesure de Lebesgue.

Cette grande différence entre les comportements des discrétisations dans le cas conservatif et le cas dissipatif semble assez spécifique aux homéomorphismes : le cas des

C^1-difféomorphismes génériques paraît moins contrasté. Cela s'explique par le fait que les classes de récurrence par chaînes d'un homéomorphisme dissipatif générique sont toutes totalement discontinues, alors que pour un C^1-difféomorphisme elles peuvent être stablement d'intérieur non vide. Globalement, les résultats concernant la dynamique des discrétisations qui sont valables pour les C^1-difféomorphismes conservatifs génériques le sont aussi pour les C^1-difféomorphismes dissipatifs génériques, pour peu que l'on se restreigne à une classe de récurrence par chaînes. On démontre par exemple le résultat suivant (Corollaire 10.4 page 220).

Proposition XXIII. *Si $f \in \mathrm{Diff}^1(\mathbf{T}^2, \mathrm{Leb})$ est un difféomorphisme conservatif générique, alors pour tout $\varepsilon > 0$ et tout $N_0 \in \mathbf{N}$, il existe $N \geq N_0$ tel que f_N ait une orbite périodique qui est ε-dense dans $\mathbf{T}^2$.*

Si $f \in \mathrm{Diff}^1(\mathbf{T}^2)$ est un difféomorphisme dissipatif générique, alors pour toute classe de récurrence par chaînes K, pour tout $\varepsilon > 0$ et tout $N_0 \in \mathbf{N}$, il existe $N \geq N_0$ tel que f_N ait une orbite périodique qui est ε-dense dans K.

Ainsi, le comportement des discrétisations sur une classe de récurrence par chaînes d'un difféomorphisme dissipatif générique semble très proche de celui observé sur les discrétisations d'un difféomorphisme conservatif générique ; cela est dû au fait que les dynamiques réelles sont elles-mêmes relativement proches dans ce cas.

13.3 Dimension 1 *versus* dimension ≥ 2

Comme on peut l'imaginer, les dynamiques des discrétisations d'homéomorphismes en dimension 1 d'une part, et en dimension supérieure à deux d'autre part, sont bien distinctes. Cela reflète le fait que les dynamiques elles-mêmes de tels systèmes sont très différentes.

En dimension 1, la longueur des cycles périodiques des discrétisations a un comportement uniforme en $+\infty$. Pour un homéomorphisme générique du cercle, donc de nombre de rotation rationnel $\rho(f) = p/q$, la longueur de ces cycles périodiques est constante égale à q à partir d'un certain ordre de discrétisation (proposition XI page 34, due à T. Miernowski). En revanche, en dimension supérieure à 2, une application du théorème de Baire donne le résultat suivant.

Proposition XXIV. *Pour un homéomorphisme dissipatif générique $f \in \mathrm{Homeo}(\mathbf{T}^2)$, pour tout entier m, il existe une sous-suite $(f_{N_k})_{k\geq 0}$ de discrétisations de f telle que pour tout k, f_{N_k} possède au moins m orbites périodiques, de périodes deux à deux distinctes.*

Dans le cas des C^r-difféomorphismes du cercle, génériques parmi ceux de nombre de rotation irrationnel, ce qui correspond plus ou moins au cas conservatif [3] en dimension supérieure, le théorème XIII (page 34) de T. Miernowski affirme que la longueur commune $q_N(f)$ des orbites périodiques de f_N tend vers $+\infty$ quand N tend vers $+\infty$, mais arbitrairement lentement. En dimension supérieure, on a le résultat suivant sur les longueurs possibles des orbites périodiques (obtenu en combinant les preuves des théorèmes 5.36 et 5.47 pages 88 et 95).

3. Dans le sens où en régularité $C^{1+\alpha}$, le théorème de Denjoy affirme que la dynamique est transitive.

Theorème XXV. *Soit* $f \in \mathrm{Homeo}(\mathbf{T}^2, \mathrm{Leb})$ *un homéomorphisme conservatif générique. Alors pour tout entier* m*, il existe des entiers* $p_1, \cdots, p_m$ *deux à deux distincts, et une sous-suite* $(f_{\mathrm{N}_k})_{k\geq 0}$ *de discrétisations de* f *tels que l'ensemble des périodes des orbites périodiques de* f_{N_k} *est exactement* $\{p_1, \cdots, p_m\}$.

En particulier, pour un homéomorphisme conservatif générique, on a une sous-suite des discrétisations dont le nombre d'orbites périodiques tend vers $+\infty$, ainsi qu'une sous-suite de discrétisations dont les orbites périodiques sont de longueurs uniformément bornées. Ces phénomènes s'opposent fortement à ce qui se passe sur le cercle.

Le comportement dynamique des discrétisations d'une application *dilatante* générique sur le cercle se rapproche assez de celui d'un homéomorphisme/difféomorphisme d'une variété de dimension supérieure. En effet, comme pour les difféomorphismes génériques (voir le théorème IV page 21 de l'introduction), on a dans ce cas un résultat reliant les comportements local et global des discrétisations, et plus précisément le taux d'injectivité des discrétisations à l'intégrale des taux d'injectivité des dérivées (théorème 11.19 page 239).

Theorème XXVI. *Pour tout* $r \geq 1$*, et toute application dilatante générique* f *de classe* C^r *sur le cercle, le taux d'injectivité* $\tau^k(f)$ *vérifie*

$$\tau^k(f) = \int_{\mathbf{T}^n} \overline{\mathrm{D}}\Big((\det \mathrm{D} f_x^{-1})_{\substack{1\leq m\leq k \\ x\in f^{-m}(y)}} \Big) \, \mathrm{d}\,\mathrm{Leb}(y)$$

(voir la définition 11.18 page 239 pour la définition de $\overline{\mathrm{D}}$*).*

De même, le comportement ergodique des applications C^1-génériques dilatantes sur le cercle ressemble fortement à celui d'un C^1-difféomorphisme conservatif générique du tore de dimension $n \geq 2$, comme on peut le voir en comparant le théorème 12.1 page 251 avec la proposition 12.4 page 252.

13.4 Régularité C^0 *versus* régularité C^r, $r \geq 1$

Le comportement des homéomorphismes génériques est bien souvent considéré comme n'étant pas très pertinent d'un point de vue physique : il est localement bien trop compliqué pour représenter une grande classe de systèmes concrets. Malgré cela, l'étude de telles dynamiques n'est pas sans intérêt.

- D'une part, on obtient des résultats concernant ce qui peut arriver lorsque le système considéré n'est pas très régulier en pratique (par exemple une dynamique C^1 dont les dérivées sont très grandes). C'est en partie ce qui se passe dans notre cas : on voit apparaître des phénomènes soulignés par la théorie sur des exemples relativement simples de difféomorphismes de classe C^∞ avec de grandes dérivées (voir la figure 1.2 page 23).
- D'autre part, les dynamiques C^0 génériques sont en quelque sorte des « modèles-jouets » ; leur étude constitue une première étape dans la compréhension de phénomènes apparaissant en plus grande régularité.

Cette thèse ne déroge pas à la règle : les résultats obtenus en régularité C^1 sont bien plus faibles que ceux obtenus en régularité C^0. En effet, pour démontrer des résultats de généricité, la difficulté est bien souvent d'obtenir des énoncés de densité. Or, le lemme de perturbation que l'on utilise en régularité C^0 (proposition 3.3 page 50), dont

la preuve est relativement simple, permet de perturber indépendamment un nombre fini arbitraire de points de notre espace. Un tel résultat n'existe pas en topologie C^r, pour $r \geq 1$; on a seulement des résultats partiels pour le cas $r = 1$, appelés « lemmes de fermeture » ou de « connexion ». Ces lemmes classiques, de preuves bien plus difficiles que dans le cas C^0, permettent de transférer gratuitement les résultats que l'on a concernant le comportement global des discrétisations d'homéomorphismes génériques, vers des résultats concernant le comportement d'une sous-dynamique des discrétisations de C^1- difféomorphismes génériques.

Par exemple, on a l'énoncé suivant (corollaire 5.9 page 78) pour les homéomorphismes conservatifs génériques, qui est une petite amélioration du théorème I page 20 de T. Miernowski.

Proposition XXVII. *Pour un homéomorphisme conservatif générique du tore, il existe une sous-suite de discrétisations qui sont des permutations cycliques des grilles.*

Ce théorème exprime que pour un homéomorphisme conservatif générique, pour tout $\varepsilon > 0$, une infinité de discrétisations est topologiquement transitive à ε près.

Dans le cas des difféomorphismes, en appliquant le lemme de connexion de C. Bonatti et S. Crovisier [BC04], ce théorème devient l'énoncé suivant (corollaire 10.4 page 220), qui ne concerne qu'une partie des points de la grille.

Proposition XXVIII. *Pour un C^1-difféomorphisme conservatif générique du tore et pour tout $\varepsilon > 0$, il existe une sous-suite de discrétisations ayant au moins une orbite ε-dense dans le tore.*

Remarquons que la différence entre les théorèmes XXVII et XXVIII est de même nature que la différence entre les théorèmes XIX et XX.

Le comportement global des difféomorphismes conservatifs génériques est bien plus difficile à capter. On obtient tout de même un résultat (non sans peine !) en considérant une quantité « semi-dynamique », puisque décroissante au cours du temps : le degré de récurrence. Si celui-ci s'accumule sur tout le segment $[0,1]$ pour un homéomorphisme conservatif générique (théorème XVIII), il tend vers 0 pour un difféomorphisme conservatif générique : rappelons le théorème III de l'introduction (théorème 11.15 page 235).

Theorème XXIX. *Pour un difféomorphisme de classe C^1 conservatif générique $f \in \mathrm{Diff}^1(\mathbf{T}^2, \mathrm{Leb})$, on a*

$$\lim_{N\to+\infty} D(f_N) = 0.$$

La preuve de ce résultat concernant les difféomorphismes utilise de manière cruciale le fait que ceux-ci possèdent des différentielles, puisqu'elle s'appuie sur une étude poussée du cas linéaire (partie 2) et la formule suivante, reliant les comportements local et global des discrétisations (théorème 11.3 page 229, déjà énoncé dans l'introduction en tant que théorème IV).

Theorème XXX. *Pour tout $r \geq 1$, et pour un difféomorphisme de classe C^r conservatif générique $f \in \mathrm{Diff}^r(\mathbf{T}^2, \mathrm{Leb})$, pour tout $t \in \mathbf{N}^*$, on a*

$$\lim_{N\to+\infty} \tau_N^t(f) = \int_{\mathbf{T}^2} \tau(Df_{f^{t-1}(x)}, \cdots, Df_x)\, \mathrm{d}x,$$

où le taux d'injectivité d'une suite de matrices est défini de manière similaire à celui d'un difféomorphisme (voir la définition 7.19 page 152).

Remarquons qu'il existe des ouverts de difféomorphismes de classe C^1 constitués de difféomorphismes d'Anosov, qui satisfont ainsi le lemme de pistage : si f est un tel difféomorphisme, alors pour tout $\varepsilon > 0$, il existe $\delta > 0$ tel que toute δ-pseudo-orbite est ε-pistée par une vraie orbite de f. En particulier, ce lemme est vérifié par les orbites de toute discrétisation sur une grille assez fine. On pourrait espérer que cette propriété très forte implique que la dynamique des discrétisations reflète bien celle du difféomorphisme de départ. La réalité est un peu plus complexe. Par exemple, on a vu que les discrétisations de l'application du chat d'Arnold, qui est un exemple de difféomorphisme d'Anosov bien connu, ne rendent pas du tout compte du caractère mélangeant de la dynamique : les discrétisations attrapent uniquement la dynamique des points périodiques de petites périodes (voir la figure 1.1 page 17). Cela est dû au fait qu'on ne sait *a priori* rien de l'orbite qui piste la pseudo-orbite ; il se peut qu'elle ne soit pas du tout représentative de la dynamique globale de l'application.

En quelque sorte, les énoncés que l'on démontre concernant la dynamique des discrétisations d'un difféomorphisme générique f de classe C^1 expriment que chaque comportement dynamique de f est pisté par un comportement similaire d'une sous-dynamique de certaines discrétisations f_{N_k} (voir par exemple la proposition XXVIII). Le souci est que ces résultats ne disent rien du reste de la dynamique de f_{N_k}. La bonne nouvelle est que si f vérifie le lemme de pistage, celui-ci assure que le reste de la dynamique n'est pas du tout arbitraire : cette dynamique est en fait pistée par de vraies orbites de f.

Enfin, on montre qu'en temps court, les poussées en avant de la mesure uniforme sur les grilles par les discrétisations d'une application dilatante du cercle suffisament régulière convergent vers une unique mesure, qui est la mesure SRB μ_0 de l'application dilatante (mesure SRB qui est par ailleurs l'unique mesure physique du système). Plus précisément, on a le théorème suivant (théorème 12.15).

Theorème XXXI. *Pour tout $\alpha \in]0,1[$ et pour toute fonction dilatante du cercle f de classe $C^{1+\alpha}$, il existe une constante $c_0 = c_0(f) > 0$ telle que si $(N_m)_m$ est une suite d'entiers tendant vers $+\infty$ mais telle que $\ln N_m > c_0 m$, alors on a la convergence $(f^*_{N_m})^m(\lambda_{N_m}) \to \mu_0$.*

Remarquons que ce résultat ne fait pas du tout intervenir de notion de généricité. Remarquons aussi que ce théorème ne dit rien sur ce qui se passe pour les applications dilatantes génériques de classe C^1 : dans ce cas, les techniques utilisées lors de la preuve (à savoir l'opérateur de transfert de Ruelle-Perron-Frobenius) ne fonctionnent plus du tout.

13.5 Une discrétisation *versus* la plupart des discrétisations *versus* toutes les discrétisations

Pour les homéomorphismes conservatifs génériques, la philosophie des résultats n'est pas du tout la même selon que l'on considère une seule discrétisation ou bien toutes les discrétisations [4]. Si on regarde la dynamique d'une seule discrétisation, le comportement observé dépend sauvagement de l'ordre de discrétisation ; c'est en tous cas ce qu'expriment les résultats de la section 5.3 du chapitre 5. Par exemple, pour un homéomorphisme conservatif générique, il existe une constante $P > 0$ telle que les propriétés suivantes sont toutes vérifiées par une infinité d'ordres de discrétisation N :

4. Dans ce paragraphe, nous ne traiterons pas le cas dissipatif, puisque la dynamique des discrétisations converge uniformément vers la dynamique de l'homéomorphisme de départ.

- f_N est une permutation cyclique (corollaire 5.9 page 78) ;
- f_N possède un seul cycle, qui est de longueur inférieure à P, et qui attire tous les points de la grille E_N (corollaire 5.36 page 88) ;
- f_N possède au moins $\sqrt{\mathrm{Card}(E_N)}$ cycles différents (corollaire 5.22 page 83).

Pire encore, cette variabilité des comportements des discrétisations perdure lorsqu'on tente de faire des statistiques sur les fréquences d'apparition des propriétés parmi les discrétisations ; (voir la section 5.4 du chapitre 5). Par exemple, on démontre le résultat suivant (théorème 5.31 page 85 et corollaire 5.27 page 84).

Theorème XXXII. *On pose* C *l'ensemble des* $N \in \mathbf{N}$ *tels que* f_N *soit une permutation cyclique. Pour un homéomorphisme conservatif générique* $f \in \mathrm{Homeo}(\mathbf{T}^2, \mathrm{Leb})$, *la suite* $\mathrm{Card}(C \cap [\![1, M]\!])/M$ *s'accumule à la fois sur 0 et 1.*

Néanmoins, les choses se passent bien mieux lorsqu'on considère la dynamique de *toutes* les discrétisations de l'homéomorphisme : dans ce cas, ce qui ressort des résultats que nous démontrons dans la section 5.5 est qu'il est possible de détecter sur les discrétisations un grand nombre de propriétés dynamiques de l'homéomorphisme. Par exemple, on peut détecter les orbites périodiques de f, ainsi que leurs périodes (théorème 5.36 page 88).

Theorème XXXIII. *Soit* $f \in \mathrm{Homeo}(\mathbf{T}^2, \mathrm{Leb})$ *un homéomorphisme conservatif générique. Alors, pour toute orbite périodique* ω *de période* p *pour* f, *et tout* $\delta > 0$, *il existe une sous-suite de discrétisations* f_{N_k} *telle que pour tout* k, f_{N_k} *possède une unique orbite périodique*[5], *de longueur* p, *qui de plus* δ*-piste l'orbite* ω.

De même, le théorème XIX sur les mesures invariantes affirme qu'il est possible de retrouver l'ensemble des mesures invariantes de l'homéomorphisme initial en regardant les mesures invariantes de toutes les discrétisations de cet homéomorphisme. En quelque sorte, ces énoncés expriment que la dynamique de l'homéomorphisme est pistée par celle de ses discrétisations.

Concernant les difféomorphismes conservatifs de classe C^1 génériques, les résultats sont pour l'instant trop partiels pour pouvoir conclure à une convergence ou non de la dynamique des discrétisations.

D'une part, on sait qu'un certain nombre d'invariants dynamiques du difféomorphisme peuvent être détectés sur une sous-dynamique d'une infinité de discrétisations. Par exemple, on peut récupérer les orbites périodiques à partir des discrétisations (voir le lemme 10.2 page 219).

Proposition XXXIV. *Soit* $f \in \mathrm{Diff}^1(\mathbf{T}^2, \mathrm{Leb})$ *un difféomorphisme de classe* C^1 *conservatif générique. Alors, pour toute orbite périodique* ω *de période* p *pour* f, *et tout* $\delta > 0$, *il existe une sous-suite de discrétisations* f_{N_k} *telle que pour tout* k, f_{N_k} *possède au moins une orbite périodique*[6] *de longueur* p, *qui de plus* δ*-piste l'orbite* ω.

De même, le théorème XX indique qu'il est possible de retrouver les mesures invariantes d'un difféomorphisme conservatif générique (voir aussi le corollaire 10.9 pour une preuve d'un énoncé plus simple).

5. Cette orbite périodique attire donc tous les points de la grille.
6. Remarquons que la différence avec le cas C^0 réside dans le fait que cette orbite n'est pas unique.

D'autre part on a tout de même un comportement global plus régulier de l'ensemble des discrétisations que dans le cas C^0, comme le souligne le fait que le degré de récurrence tend vers 0 pour un difféomorphisme conservatif générique (théorème XXIX). Malheureusement, à l'heure actuelle, on ne sait pas décrire mieux la nature de la dynamique globale des discrétisations. Par exemple, on n'a pour l'instant aucune idée du comportement asymptotique de la suite de mesures [7] $\left(\mu_{\mathbf{T}^2}^{f_N}\right)_{N\in\mathbf{N}}$.

13.6 Propriétés combinatoires *versus* propriétés topologiques et ergodiques

Comme on l'a déjà vu, certaines quantités combinatoires associées aux discrétisations d'un homéomorphisme conservatif générique évoluent de manière très erratique en fonction de l'ordre de discrétisation N. Par exemple, le degré de récurrence s'accumule sur tout le segment $[0,1]$ (théorème XVIII), la limite inférieure du nombre d'orbites périodiques vaut 1 (théorème XXVII) et sa limite supérieure vaut $+\infty$ (théorème XXV, voir aussi le corollaire 5.22 page 83), etc. On démontre même un théorème général dans ce sens : on dit qu'une propriété portant sur les applications finies des grilles E_N est *dense* si tout homéomorphisme conservatif est arbitrairement bien approché par des applications finies ayant cette propriété ; le théorème 5.2 page 75 affirme que si une propriété est dense, alors elle se retrouve sur une infinité de discrétisations d'un homéomorphisme générique.

On peut donc dire que ces quantités combinatoires sont très peu utiles pour détecter la dynamique de l'application de départ. De plus, ces résultats impliquent que les discrétisations ne se comportent pas du tout comme une application aléatoire typique d'un ensemble fini : par exemple, une telle application $\sigma : E \to E$, où E est un ensemble fini à q éléments, est telle que le cardinal de son ensemble récurrent est de l'ordre de $\sqrt{N}$ (voir [Bol01, XIV.5] ou le théorème 2.3.1 de [Mie05]). Pire encore, cela n'est toujours pas vrai lorsqu'on considère par exemple la moyenne des degrés de récurrence des discrétisations sur de longs segments d'ordres de discrétisation (voir le théorème XXXII).

En ce qui concerne les difféomorphismes, on ne sait que très peu de choses sur le comportement combinatoire global des discrétisations. Pour ce problème, le théorème principal que nous obtenons est le théorème XXIX, qui illustre le fait que les discrétisations d'un difféomorphisme de classe C^1 conservatif générique se comportent différemment de celles d'un homéomorphisme conservatif générique.

Pour retrouver la dynamique initiale, il faut utiliser la géométrie des grilles E_N, et s'intéresser aux propriétés topologiques ou ergodiques de l'ensemble des discrétisations. Comme on l'a déjà dit, aussi bien en topologie C^0 qu'en topologie C^1, on peut de cette manière retrouver par exemple l'ensemble des mesures invariantes par la dynamique de départ (théorèmes XIX et XX), ou bien l'ensemble des points périodiques (théorème XXXIII et proposition XXXIV), ou encore l'ensemble de rotation, comme le montre le théorème VII de l'introduction (théorème 6.23 page 122).

Theorème XXXV. *Pour un homéomorphisme conservatif générique* $f \in \mathrm{Homeo}(\mathbf{T}^2, \mathrm{Leb})$,

– *l'ensemble de rotation observable est réduit à un point ;*

7. Rappelons qu'étant donnée une discrétisation f_N, la mesure $\mu_{\mathbf{T}^2}^{f_N}$ est définie comme la limite au sens de Cesàro des poussées en avant par f_N de la mesure uniforme sur le grille E_N

– *la limite supérieure des ensembles de rotation des discrétisations coïncide avec l'ensemble de rotation de* f.

13.7 Microscopique *versus* mésoscopique *versus* macroscopique

Un homéomorphisme générique (tant conservatif que dissipatif) a un comportement local très chaotique. Par exemple, dès qu'il possède un point périodique de période p, alors l'ensemble de ses points périodiques de période p est un ensemble de Cantor (il est donc indénombrable) ; de plus pour tout multiple q de p, l'ensemble des points périodiques de période q est alors non vide, et tout point périodique de période p est dans l'adhérence des points périodiques de période q. Ce manque de régularité du comportement local de l'application de départ fait que le comportement local des discrétisations d'un homéomorphisme générique est lui-même très chaotique, et dépend fortement de l'ordre de la discrétisation. Par exemple, une discrétisation sera tantôt localement bijective (proposition XXVII), tantôt localement très peu injective, comme le montre le résultat suivant (corollaire 5.20 page 82).

Theorème XXXVI. *Soit* $\vartheta : \mathbf{N} \to \mathbf{R}_+^*$ *une fonction tendant vers* $+\infty$ *en* $+\infty$. *Alors, pour un homéomorphisme conservatif générique* $f \in \mathrm{Homeo}(\mathbf{T}^2, \mathrm{Leb})$, *on a* [8]

$$\varlimsup_{N\to+\infty} \frac{\mathrm{Card}\left(f_N(E_N)\right)}{\vartheta(N)} = 0.$$

Par conséquent, le comportement local des discrétisations d'un homéomorphisme conservatif générique est assez sauvage, et donc relativement peu pertinent. Ce phénomène se traduit au niveau temporel : par exemple, en comparant les théorèmes XXVII et XXXVI, on voit que des discrétisations de deux ordres distincts (arbitrairement grands) peuvent avoir des comportements opposés dès la première itération.

Pour les difféomorphismes, les choses sont bien différentes. L'existence de différentielles — qui plus est continues — pour un difféomorphisme f, dicte le comportement local des discrétisations de f. Cela est illustré par le théorème XXX, qui exprime que génériquement, pour un temps t fixé, et pour des ordres de discrétisation assez grands, le cardinal des images de la grille par les discrétisations en temps inférieur à t est déterminé par les différentielles de f. On a donc trois échelles d'espace différentes :

– tout d'abord, l'échelle du tore, l'échelle macroscopique, pour laquelle en temps fini, et pour des ordres de grilles assez grands, la dynamique des discrétisations ressemble à celle du difféomorphisme initial ;
– ensuite, l'échelle des différentielles de f, l'échelle mésoscopique ; c'est l'échelle à laquelle on perçoit l'action des dérivées de f ; autrement dit pour laquelle en temps fini, et pour des ordres de grilles assez grands, la dynamique des discrétisations ressemble à celle d'une suite de matrices ;
– enfin, l'échelle de la grille, l'échelle microscopique, où on voit de toute façon les points de la grille, les « atomes ».

Comme dans le cas des homéomorphismes, ces échelles spatiales se transposent au niveau temporel : le fait d'avoir une échelle mésoscopique induit un régime transitoire

8. Remarquons que ce résultat est une amélioration du fait que le degré de récurrence d'un homéomorphisme conservatif générique s'accumule sur 0.

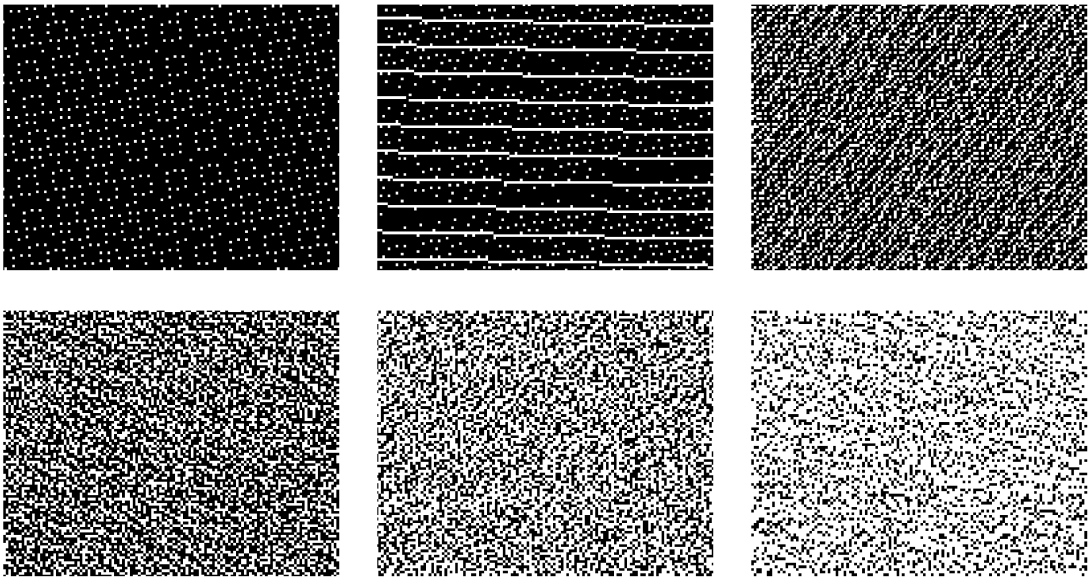


FIGURE 13.1: Images successives de $\mathbf{Z}^2$ par les discrétisations de k matrices de $\mathrm{SL}_2(\mathbf{R})$ choisies aléatoirement parmi un compact de $\mathrm{SL}_2(\mathbf{R})$ (voir la figure 7.2 pour une explication précise). Un point de $\mathbf{Z}^2$ est noir s'il appartient à l'image. De gauche à droite et de haut en bas, $k = 1, 2, 3, 5, 10, 20$.

pour le comportement local des discrétisations, c'est d'ailleurs ce qu'exprime le théorème XXX, qui n'est de ce fait pas vraiment un énoncé purement « dynamique », car il ne concerne qu'un nombre fini d'itérés. Pour en déduire un résultat asymptotique, on triche en quelque sorte, puisque la quantité étudiée, à savoir le degré de récurrence, décroît au cours du temps.

On est donc amenés à considérer, pour étudier la dynamique des difféomorphismes, la dynamique des discrétisations de suites d'applications linéaires. Plus précisément, la discrétisation d'une matrice $A \in \mathrm{GL}_2(\mathbf{R})$ est l'application $\widehat{A} = P \circ A_{|\mathbf{Z}^2}$, où P est une projection de $\mathbf{R}^2$ sur le point de $\mathbf{Z}^2$ le plus proche. On cherche alors à étudier les ensembles $(\widehat{A_k} \circ \cdots \circ \widehat{A_1})(\mathbf{Z}^n)$, pour une suite générique [9] $(A_k)_{k\geq 1}$ de matrices (éventuellement de déterminant 1). On trouvera dans la figure 13.1 des images typiques des ensembles images que l'on obtient. On observe notamment — et c'est sur la première image que cela est le plus flagrant — un phénomène de presque-périodicité de l'ensemble obtenu. On démontre en effet un résultat dans ce sens pour toutes les itérations [10] (théorème 7.12) : de manière informelle, un ensemble Γ est *presque périodique* si pour R assez grand, l'ensemble $B(0,R) \cap \Gamma$ détermine l'ensemble Γ à un ensemble de densité inférieure à ε près (voir la définition 7.3 page 144).

Theorème XXXVII. *Pour toute suite $(A_k)_{k\geq 1}$ de matrices inversibles, les ensembles $(\widehat{A_k} \circ \cdots \circ \widehat{A_1})(\mathbf{Z}^n)$ sont presque périodiques.*

En observant la figure 13.1, on remarque aussi que la densité des ensembles images semble décroître au cours du temps. Nous démontrons effectivement un résultat dans

9. Générique au sens de la topologie de la norme infinie de la suite, relativement à une norme sur $\mathrm{M}_2(\mathbf{R})$ fixée une fois pour toutes.

10. Même si cela ne se voit pas du premier coup d'œil sur les dernières images de la figure…

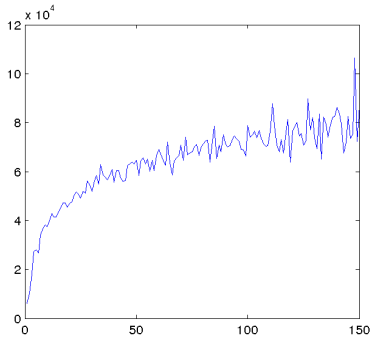


FIGURE 13.2: Cardinal de l'ensemble récurrent $\Omega(f_N)$ en fonction de N, pour $N = 128k$, $k = 1, \cdots, 150$, et f un exemple d'homéomorphisme conservatif proche de l'identité (voir page 99 pour la définition de cet homéomorphisme).

ce sens (théorème 8.24 page 177).

Theorème XXXVIII. *Pour une suite générique* $(A_k)_{k\geq 1}$ *de matrices de déterminant 1, la densité des ensembles* $(\widehat{A_k} \circ \cdots \circ \widehat{A_1})(\mathbf{Z}^n)$ *tend vers 0 lorsque k tend vers* $+\infty$.

Ce théorème est le résultat principal de la partie 2 de cette thèse. Sa preuve, assez longue et délicate, repose essentiellement sur des arguments d'équirépartition, qui permettent de ramener le calcul de la densité des ensembles images (qui n'est pas très pratique à manipuler) à celui de l'aire d'une intersection de cubes en grande dimension.

13.8 En théorie *versus* en pratique

Pour chaque contexte étudié dans ce manuscrit (homéomorphismes et C^1-difféomorphismes, tant dissipatifs que conservatifs), nous confrontons les énoncés obtenus aux résultats de simulations numériques menées sur des exemples censés représenter le mieux possible le cas « générique ».

Commençons par le cas des homéomorphismes conservatifs. À première vue, les simulations de quantités telles que le degré de récurrence peuvent passer pour décevantes : en pratique, le degré de récurrence ne s'accumule pas du tout sur tout le segment $[0,1]$, mais tend relativement vite vers 0 (voir la figure 13.2, voir aussi la section 5.8). Ceci contredit les conclusions du théorème XVIII : en pratique, on peut ne pas observer les comportements décrits par les énoncés concernant le cas générique, même lorsque la définition de la dynamique est faite pour mimer ce cas générique.

Pour observer effectivement les phénomènes prédits par résultats théoriques, il faut cesser de s'intéresser à la dynamique combinatoire des discrétisations. Par exemple, nous simulons les mesures $\mu_{\mathbf{T}^2}^{f_N}$, obtenues rappelons-le en poussant en avant par f_N la mesure uniforme sur E_N, et en prenant la limite au sens de Cesàro de ces mesures. En pratique, on ne peut pas vraiment s'attendre à voir converger les mesures $\mu_{\mathbf{T}^2}^{f_N}$ vers l'ensemble des mesures invariantes par f, puisque cet ensemble est un convexe de dimension infinie ; on peut néanmoins déterminer si cette suite de mesures semble converger ou non. Sur la figure 1.2 page 23, on observe que non seulement les mesures $\mu_{\mathbf{T}^2}^{f_N}$ ne convergent pas du tout vers la mesure de Lebesgue, mais qu'en plus elles n'ont rien

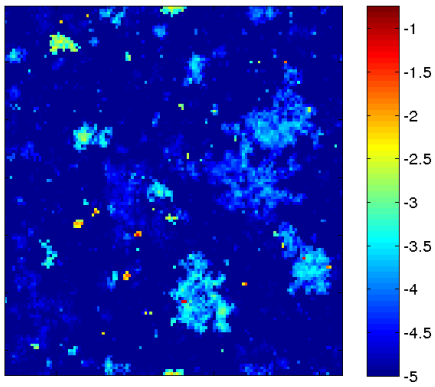


FIGURE 13.3: Densité de la mesure $\mu_{\mathbf{T}^2}^{f_N}$, pour f un exemple d'homéomorphisme dissipatif proche de l'identité avec de petits ensembles attractifs, et N = 32768 (voir page 66 pour la définition de cet homéomorphisme).

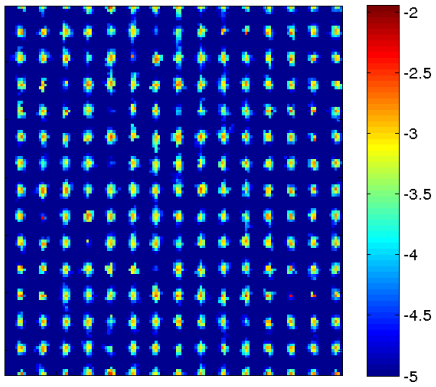


FIGURE 13.4: Densité de la mesure $\mu_{\mathbf{T}^2}^{f_N}$, pour f un exemple d'homéomorphisme dissipatif proche de l'identité avec de gros ensembles attractifs, et N = 32768 (voir page 66 pour la définition de cet homéomorphisme).

à voir les unes avec les autres, et cela même si les ordres de discrétisation sont très proches. Comme prévu par le théorème XIX, on ne retrouve pas la mesure physique de l'homéomorphisme de départ sur les discrétisations.

Concernant les homéomorphismes dissipatifs, nous avons aussi simulé les mesures $\mu_{\mathbf{T}^2}^{f_N}$, sur deux exemples différents. Le premier est un homéomorphisme dissipatif proche de l'identité, mais avec de petits puits (voir la figure 13.3, voir aussi la section 4.3). Il s'avère que le comportement de ces mesures est identique à celui observé dans le cas conservatif : on ne détecte pas du tout le caractère dissipatif de la dynamique. Ce phénomène avait déjà été souligné par J.-M. Gambaudo et C. Tresser dans [GT83] : si les puits sont trop petits — et cela se produit assez facilement, même si la définition de l'application est relativement simple —, alors ils ne sont pas observés sur des simulations.

Pour éviter ce phénomène, nous avons aussi testé un exemple d'homéomorphisme dissipatif proche de l'identité, mais ayant des puits volontairement beaucoup plus gros (voir la figure 13.4, voir aussi la section 4.3). Dans ce cas, comme prédit par le théorème XXII, les mesures $\mu_{\mathbf{T}^2}^{f_N}$ semblent converger assez rapidement vers une mesure portée par les ensembles attractifs de l'homéomorphisme.

Remarquons aussi que la fréquence d'apparition des phénomènes prédits par les théorèmes peut changer radicalement selon le type de dynamique considérée. Par exemple, la figure 13.5 illustre le théorème XX, qui exprime que dans certains cas (en fait, la plupart des cas au sens topologique), si on se fixe $x \in \mathbf{T}^2$ et $f \in \mathrm{Diff}^1(\mathbf{T}^2, \mathrm{Leb})$, alors les mesures [11] $\mu_x^{f_N}$ s'accumulent sur l'ensemble des mesures invariantes par f. La figure 13.5 représente la distance $\mathrm{dist}(\mu_x^{f_N}, \mathrm{Leb})$ en fonction de N pour trois difféomorphismes conservatifs différents (voir la section 12.3 pour des définitions précises). Pour

11. $\mu_x^{f_N}$ est rappelons-le la mesure uniforme sur l'orbite périodique dans laquelle tombe l'orbite positive de x_N sous f_N.

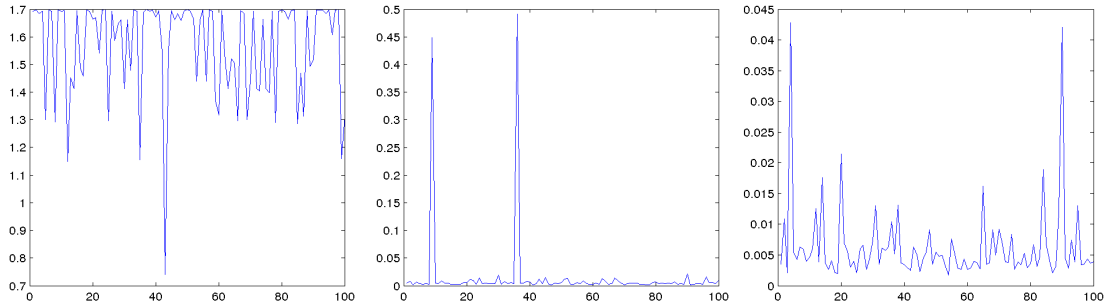


FIGURE 13.5: Distance entre la mesure de Lebesgue et les mesures $\mu_x^{(f_i)_N}$ en fonction de N, for f_1 une petite perturbation de Id en topologie C^1 (gauche), f_2 une petite perturbation d'une translation d'ordre 30 en topologie C^1 (milieu) et f_3 une petite perturbation d'un automorphisme d'Anosov linéaire en topologie C^1 (droite), sur les grilles E_N, avec $N = 2^{20} + k$, $k = 1, \cdots, 100$ (voir page 274 pour une définition précise de ces difféomorphismes).

le premier difféomorphisme, qui est proche d'une dynamique « elliptique », à savoir l'identité, cette distance est toujours assez élevée : la mesure obtenue est systématiquement loin de la mesure de Lebesgue (voir aussi la figure 1.3 page 23). Le deuxième difféomorphisme testé est une petite perturbation (en topologie C^1) d'une translation d'ordre 30 sur le tore. Dans ce cas, la mesure est presque toujours très proche de la mesure de Lebesgue, sauf pour de rares ordres N où elle en est assez loin. Enfin, le troisième difféomorphisme est une petite C^1-perturbation d'un automorphisme d'Anosov linéaire. Là encore, la mesure calculée est presque systématiquement très proche de la mesure de Lebesgue, sauf pour de rares ordres N (mais ce phénomène est moins marqué que pour le second difféomorphisme). Il semble donc (et cela paraît naturel) qu'en pratique, plus la dynamique initiale est chaotique, plus les ordres de discrétisation pour lesquels les conclusions des énoncés qu'on obtient sont vraies deviennent rares.

Chapitre 14

Pistes de recherche

14.1 Dynamique combinatoire des discrétisations de difféomorphismes génériques

À ce jour, on ne sait que très peu de choses sur la dynamique combinatoire des discrétisations de difféomorphismes conservatifs génériques. Comme on l'a déjà dit, on sait que le degré de récurrence tend vers 0 (théorème XXIX), mais c'est en fait le seul résultat qu'on ait sur le comportement global des discrétisations. On ne sait toujours pas répondre à des questions simples telles que la suivante.

Question. *Est-ce que le cardinal de l'ensemble récurrent des discrétisations f_N d'un difféomorphisme conservatif générique f tend vers l'infini lorsque* N *tend vers l'infini ? Si oui, est-ce que la longueur de la plus longue orbite périodique de f_N, ou bien le nombre d'orbites périodiques de f_N, tendent vers l'infini ?*

L'étape suivante serait de déterminer si les discrétisations de difféomorphismes génériques ont tendance à se comporter comme des applications aléatoires (voir la section 14.8). Ce type de questions me semble pour l'instant hors de portée.

14.2 Convergence des mesures invariantes canoniques

On ne connaît toujours pas le comportement des mesures invariantes « canoniques » $\mu_{\mathbf{T}^2}^{f_N}$ de f_N lorsque f est un C^1-difféomorphisme conservatif générique. Rappelons que chacune de ces mesures $\mu_{\mathbf{T}^2}^{f_N}$ est portée par l'ensemble récurrent de f_N ; la mesure totale de chaque orbite périodique est proportionnelle à la taille de son bassin d'attraction. Dans le cas des homéomorphismes conservatifs génériques, ces mesures s'accumulent sur l'ensemble des mesures invariantes par f (théorème XIX) ; au contraire, dans le cas des homéomorphismes dissipatifs génériques, celles-ci convergent vers une unique mesure (théorème XXII). Dans le cas des difféomorphismes conservatifs génériques, leur comportement n'est à ce jour pas connu.

Question. *Pour un difféomorphisme conservatif générique $f \in \mathrm{Diff}^1(\mathbf{T}^2, \mathrm{Leb})$, est-ce que les mesures $\mu_{\mathbf{T}^2}^{f_N}$ tendent vers la mesure de Lebesgue ? Est-ce qu'elles s'accumulent sur l'ensemble des mesures invariantes par f ?*

Les simulations numériques que nous présentons ne permettent pas clairement d'intuiter une réponse à cette question (voir la figure 12.15) : si ces mesures sont beaucoup plus proches de la mesure de Lebesgue que dans le cas C^0, on observe tout de même de petites régions du tore qui portent une mesure assez grande.

On peut aussi se demander, toujours dans le cas des difféomorphismes conservatifs génériques, ce qu'il advient des mesures $\mu_x^{f_N}$. Rappelons que la mesure $\mu_x^{f_N}$ est la mesure uniforme sur l'orbite périodique de f_N dans laquelle tombe l'orbite de x par f_N. Le théorème XX affirme que pour un ensemble générique de points x, ces mesures s'accumulent sur l'ensemble des mesures invariantes par f, mais les questions suivantes restent ouvertes.

Question. *Pour un difféomorphisme conservatif générique $f \in \mathrm{Diff}^1(\mathbf{T}^2, \mathrm{Leb})$, quel est le comportement des mesures $\mu_x^{f_N}$ pour un ensemble de points x de mesure totale ? Du point de vue des grilles, quel est le comportement des mesures $\mu_x^{f_N}$ pour « la plupart » des points $x_N \in E_N$?*

14.3 Détection de certaines propriétés dynamiques

Dans cette thèse, nous ne traiterons pas du tout de la détection d'un certain nombre d'invariants dynamiques. Par exemple, on ne donne aucun moyen de déterminer si une dynamique est transitive ou non. De fait, les systèmes étudiés ont par essence des dynamiques bien spécifiques : un homéomorphisme dissipatif générique n'est jamais transitif, alors qu'un homéomorphisme — ou un difféomorphisme de classe C^1 — conservatif générique l'est toujours. On pourrait aussi chercher à détecter la conservativité d'une dynamique. Nous n'avons malheureusement pas d'algorithme permettant de répondre à ce type de question. Nous ne savons pas non plus répondre à des questions comme la suivante.

Question. *Est-ce que si pour tout $\varepsilon > 0$, une infinité de discrétisations possède une orbite ε-dense, alors la dynamique de départ est transitive ?*

Une problématique classique en dynamique est aussi le calcul effectif de l'entropie ou des exposants de Lyapunov d'un système. Les résultats obtenus au cours de cette thèse ne permettent pas non plus de répondre à ce type de questions, qui ne sont d'ailleurs pertinentes que dans le cas des difféomorphismes génériques (pour un homéomorphisme conservatif générique, l'entropie métrique est nulle et l'entropie topologique est infinie). Tout juste peut-on avoir une minoration (éventuellement très mauvaise) de l'entropie topologique d'un difféomorphisme conservatif du tore à l'aide du calcul de son ensemble de rotation, et des bornes sur l'entropie en fonction de la taille de l'ensemble de rotation obtenues par J. Kwapisz dans [Kwa93].

Question. *Connaissant seulement la dynamique des discrétisations, est-il possible de calculer l'entropie de l'application de départ ?*

Pour traiter ce type de problème, la bonne méthode est sans doute de ne pas regarder la dynamique des discrétisations, c'est-à-dire d'attendre un temps « infini », mais plutôt de regarder le comportement des discrétisations pour un temps bien choisi, vraisemblablement logarithmique (voir la section suivante).

14.4 Estimations explicites de la convergence de la dynamique

Un des défauts des résultats qui sont présentés dans ce manuscrit est qu'ils ne sont pas du tout explicites (on utilise le théorème de Baire, donc l'axiome du choix dénombrable). Par exemple, lorsqu'on obtient des résultats de simulation d'ensembles de rotation à l'aide de discrétisations grossières (voir par exemple la figure 1.4 page 25), on n'a aucun moyen de savoir si les ensembles obtenus sont proches du vrai ensemble de rotation ou non. Il serait sans doute assez facile d'avoir des estimations pratiques telles que « *étant donné* $\delta > 0$, *alors il existe* $N_0 > 0$ *explicite tel que si l'ordre de la grille est supérieur à* N_0, *alors l'ensemble de rotation de la discrétisation est contenu dans le* δ*-voisinage du vrai ensemble de rotation* ». Par contre, on n'a pour l'instant aucun moyen de savoir à partir de quel moment l'ensemble de rotation de la discrétisation « remplit bien » le vrai ensemble de rotation. On peut seulement avoir une indication de la convergence lorsque la suite des ensembles obtenus semble se stabiliser, mais cela n'assure en aucun cas que l'ensemble de rotation simulé est effectivement proche de l'ensemble de rotation de l'homéomorphisme de départ. Il serait donc très intéressant de pouvoir apporter des éléments de réponse à la question suivante.

Question. *Peut-on trouver un algorithme raisonnablement rapide d'approximation de l'ensemble de rotation tel qu'on ait des estimations rigoureuses sur la distance entre l'ensemble de rotation calculé et le vrai ensemble de rotation, en fonction des paramètres du problème ?*

14.5 Différentes notions de généricité

Dans cette thèse, nous n'abordons les résultats de généricité que dans deux cas spécifiques, celui de la topologie C^0 et celui de la topologie C^1 ; on peut très bien imaginer des études dans des contextes différents. On pourrait bien sûr s'intéresser à ce qui se passe dans le cas intermédiaire des applications Hölder génériques. Il serait aussi naturel de regarder ce qui se passe pour des applications C^r génériques, avec $r > 1$, voire $r = \infty$ ou $r = \omega$ (applications analytiques réelles), mais les résultats obtenus sont déjà assez partiels en topologie C^1 ; il se peut qu'il soit très difficile d'avoir des énoncés satisfaisants dans ces cadres plus rigides.

Dans ce manuscrit, on fait la distinction entre les comportement des discrétisations de systèmes génériques conservatifs et dissipatifs. On pourrait très bien imaginer étudier ce qui se passe pour d'autres types de dynamiques, par exemple pour les homéomorphismes transitifs par chaînes génériques [1].

Il existe aussi d'autres notions de généricité que celle au sens de Baire. Citons par exemple celle de *prévalence*, qui est censée se rapprocher de la notion d'ensemble de mesure totale dans un espace mesuré. C'est d'ailleurs aux difféomorphismes prévalents que s'intéresse une partie de l'article [Mie06] de T. Miernowski sur les discrétisations des difféomorphismes du cercle. La notion la plus plus naturelle pour l'étude des discrétisations est peut-être celle de généricité au sens de Kolmogorov : étant donnée une fonction f donnée par une formule explicite, la complexité au sens de Kolmogorov décrit le nombre d'opérations élémentaires nécessaires pour calculer les images $f(x)$. En effet, pour faire des simulations numériques, on a forcément une expression explicite

1. On trouvera une étude de la dynamique des applications continues transitives par chaînes génériques (attention, ce ne sont pas nécessairement des homéomorphismes) dans le livre de E. Akin [Aki99]. Il est possible que les techniques présentées puissent être mises à profit pour l'étude des discrétisations de telles applications.

de la loi d'évolution ; on veut donc une notion de « généricité » pour les applications définissables par une formule. On peut alors se poser la question suivante.

Question. *Quelles sont les propriétés des discrétisations de la plupart des fonctions ayant une complexité inférieure à une borne donnée ?*

Ce problème semble pour le moment bien trop complexe pour qu'on puisse y apporter des réponses satisfaisantes.

Remarquons que certaines propriétés sont vraies sur des ouverts denses d'applications, par exemple, pour tout $\tau_0 > 0$, la propriété « $\lim_{t\to+\infty} \tau^t(f) \leq \tau_0$ » est vérifiée sur un ouvert dense de difféomorphismes conservatifs de classe C^1. Dans ce cas, les notions de généricité et de prévalence coïncident plus ou moins.

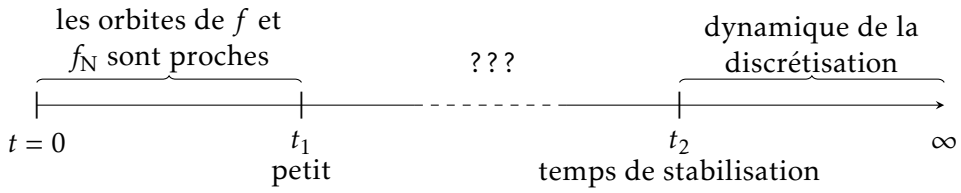
14.6 Comportements génériques parmi les conjugués à un homéomorphisme donné

Enfin, on peut se demander ce qui se passe lorsqu'on fixe une dynamique f, et que l'on considère ses discrétisations par rapport à une suite de grilles générique. Une partie de la réponse à cette question est donnée dans la section 5.7 du chapitre 5 : si on considère un homéomorphisme conservatif f, et si par « suite de grilles générique » on entend « image par un homéomorphisme conservatif générique d'une suite de grilles fixée », alors il se passe exactement la même chose que pour les homéomorphismes conservatifs génériques suivant une suite de grilles fixée, pour peu que l'ensemble des points fixes de f soit d'intérieur vide. Il serait intéressant de savoir ce qui se passe dans d'autres cas, en particulier d'avoir des réponses à la question suivante.

Question. *Si on fixe un homéomorphisme quelconque f et si on considère les discrétisations de f par rapport à l'image d'une suite de grilles par un homéomorphisme conservatif générique, peut-on retrouver les classes de récurrence par chaînes de f, voire mieux, le graphe de transition entre ces classes de récurrence ? Que dire d'un point de vue pratique ? Que se passe-t-il sur chaque classe de récurrence ?*

14.7 Temps caractéristique

En général, lorsqu'on demande à un ordinateur de calculer l'orbite d'un point x par une application f, celui-ci travaille en précision double, avec 52 chiffres binaires significatifs. Si on itère numériquement f un nombre raisonnable t de fois, il est donc très peu probable que l'orbite discrète en temps t de x tombe dans une orbite périodique de la discrétisation, c'est-à-dire qu'il existe un temps $t' < t$ tel que $f_N^{t'}(x_N) = f_N^t(x_N)$. En effet, pour une application aléatoire d'un ensemble à 2^{52} éléments, ce temps t' est typiquement de l'ordre de 84 millions (voir par exemple le théorème 2.3.1 de [Mie05]). Or, les résultats que l'on obtient dans cette thèse concernent précisément ce cas récurrent ; *a priori* ils ne peuvent donc pas expliquer ce qui se passe en pratique.



La ligne du temps ci-dessus concerne le comportement temporel d'une discrétisation f_N donnée. On a représenté un temps t_1 qui correspond au temps jusqu'auquel l'orbite de n'importe quel point de départ x sous f_N piste celle de x sous f. Les estimations théoriques qu'on a pour ce temps jusqu'auquel on a un « pistage fort » sont assez mauvaises. Par exemple, si l'application f est lipschitzienne, ce temps t_1 est logarithmique en la taille de la grille. En pratique, lorsqu'on effectue des simulations, on itère bien au-delà de ce temps généralement assez court. Le temps t_2, lui, est le temps de stabilisation de f_N, c'est-à-dire le temps à partir duquel l'orbite de tout point de la grille est tombée dans l'ensemble récurrent de f_N. À partir de ce temps-ci, la dynamique de f_N est strictement périodique ; on peut considérer que ce n'est qu'à partir de t_2 que l'on voit apparaître la dynamique de la discrétisation.

Il reste donc tout un intervalle de temps — à savoir $[t_1, t_2]$ — sur lequel on ne sait pas vraiment comment évoluent les discrétisations.

Dans cette thèse, on s'intéresse principalement à la *dynamique* des discrétisations de systèmes. Le comportement en temps fini des discrétisations a déjà été étudié par P.P. Flockermann (voir la section 2.4). Il serait extrêmement intéressant de déterminer un temps caractéristique « intermédiaire » $t_1 \ll t_1' \ll t_2$, jusqu'auquel on voit à coup sûr la dynamique de l'application de départ, sans pour autant subir les phénomènes de récurrence induits par le fait de discrétiser. À ce sujet, O.E. Lanford a émis une conjecture.

Conjecture (Lanford). *Soit $f : \mathbf{S}^1 \to \mathbf{S}^1$ une application dilatante de classe C^2 générique. Si on note Leb_N la mesure uniforme sur la grille E_N, et μ l'unique mesure physique de f, alors $f_N^k(\mathrm{Leb}_N) \rightharpoonup \mu$, lorsque N, k tendent vers $+\infty$, avec $\ln N \ll k \ll \sqrt{N}$.*

Dans ce manuscrit, nous démontrons un théorème allant dans le sens de la conjecture (voir le théorème XXXI), mais qui ne concerne malheureusement que des temps $\ln N = O(k)$. J'espère que les techniques présentées dans le chapitre 9 permettront d'avancer un peu plus dans le sens de cette conjecture.

14.8 Comparaison avec les applications aléatoires

Dans cette thèse, nous ne comparons (presque) pas la dynamique des discrétisations avec celle d'une application aléatoire typique d'un ensemble à q éléments. Par typique, on entend la chose suivante. On se fixe un entier q et on considère l'ensemble des applications d'un ensemble à q éléments dans lui-même ; on munit cet ensemble fini d'applications de la probabilité uniforme. On se demande alors quelles propriétés sont satisfaites par la plupart de ces applications. Ce type de questions a entre autres été étudié par P. Erdős et P. Turán (voir par exemple [ET65] et [ET67]), on trouvera un survol des résultats dans [Bol01]. Par exemple, le degré de récurrence de telles applications (qui est rappelons le le rapport entre le cardinal de l'ensemble récurrent et q) est de l'ordre de $q^{-1/2}$. Bien sûr, l'étude que nous menons dans le cas des homéomorphismes conservatifs génériques montre que les discrétisations de telles applications ne se comportent pas du tout comme des applications aléatoires, et cela même d'un point de vue expérimental. Par contre, ce qui se passe pour les discrétisations de difféomorphismes de classe C^r, avec $r \geq 1$, n'est pas clair du tout. Malheureusement, l'étude précise de la combinatoire des discrétisations de telles applications semble pour l'instant hors de portée.

14.9 Discrétisations spatiales *versus* perturbations stochastiques *versus* applications multivaluées

Dans cette thèse, nous n'abordons pas du tout les problèmes des perturbations stochastiques de dynamiques ou de discrétisations multivaluées.

Le premier problème est le suivant : on considère une dynamique f, et on s'intéresse non pas aux orbites de f, mais aux suites $(x_k)_{k\geq 0}$, où x_{k+1} est obtenu en prenant une petite perturbation aléatoire de l'image $f(x_k)$. On peut formaliser de plusieurs manières l'idée de « petite perturbation », on peut par exemple choisir de prendre l'image de x_k par une fonction choisie aléatoirement parmi celles qui sont proches de f, ou bien prendre un point aléatoirement dans un voisinage de $f(x_k)$, etc. On pourra consulter les notes de cours [Via97] de M. Viana pour se faire une idée de la richesse des résultats obtenus (par exemple, dans le cas des applications dilatantes [Kif86a],[Kif86b], [Kel82], ou des applications uniformément hyperboliques [Kif86b], [You86], etc.).

En règle générale, ces résultats supposent que la dynamique soit assez régulière (typiquement au moins $C^{1+\alpha}$) ; une méthode classique pour les obtenir est de passer par certains opérateurs de transfert liés à la dynamique. Ils suggèrent tous que la dynamique physique des perturbations stochastiques converge vers la dynamique physique de l'application de départ. Comme on l'a déjà dit, ce qu'on obtient pour les discrétisations de systèmes génériques va dans un sens bien différent : au bout d'un moment, l'orbite de chaque point de la grille tombe dans une orbite périodique. Ce phénomène de récurrence détruit tout espoir de voir les discrétisations se comporter comme des perturbations stochastiques sur des échelles de temps assez longues ; cela peut influer sur la dynamique ergodique des discrétisations, vue l'absence de convergence uniforme dans le théorème de Birkhoff. Cela n'empêche que les discrétisations de certaines dynamiques peuvent êtres proches de perturbations stochastiques pendant un certain temps, c'est par exemple ce que démontre P.P Flockermann dans sa thèse [Flo02] dans le cas des applications dilatantes du cercle. Toutefois, il n'obtient pas d'estimation effective sur le temps pendant lequel ce comportement reste vrai.

Question. *Pendant combien de temps les discrétisations d'une application dilatante sur le cercle générique, ou d'un difféomorphisme générique, se comportent comme des perturbations stochastiques ?*

Dans ce manuscrit, nous ne parlons pas non plus d'approximation de la dynamique par des applications multivaluées. Ce point de vue peut être assez intéressant pour déterminer quelles propriétés dynamiques d'une application peuvent être retrouvées en considérant des approximations discrètes de la dynamique, voir par exemple [LP11] (voir aussi [Mro96] et [DKP96]). En approchant une dynamique par une application multivaluée, on perd *a priori* beaucoup moins d'information qu'en l'approchant par une application finie. Malgré cela, dans le cas des homéomorphismes conservatifs génériques, on montre qu'en fait, on peut retrouver « tous » les invariant dynamiques de l'application de départ en considérant l'ensemble de ses discrétisations (voir par exemple le théorème XIX). Par conséquent, en théorie, on ne perd pas plus d'information en discrétisant qu'en considérant des approximations multivaluées. De fait, dans le chapitre 5, on utilise implicitement des applications multivaluées : la preuve du théorème de Lax consiste à extraire une vraie application d'une application multivaluée. La suite du chapitre 5 utilise le même type d'arguments : les candidats à être des discrétisations sont contenus dans une application multivaluée ; le théorème de Baire nous autorise alors

à choisir, parmi toutes les extractions possibles, celles qui ont des propriétés que l'on veut voir apparaître une infinité de fois sur la suite des discrétisations.

Néanmoins, dans le cas des applications multivaluées, il est peut-être possible d'obtenir des estimations effectives de l'ordre de la discrétisation jusqu'auquel il faut aller pour pouvoir retrouver les invariants dynamiques avec une précision donnée. Cela n'est clairement pas le cas pour les discrétisations de systèmes génériques, en tous cas avec les méthodes utilisées dans cette thèse. Il faut noter que le fait de considérer des applications multivaluées pose des contraintes pratiques supplémentaires : par exemple, il est beaucoup plus coûteux de détecter numériquement les cycles d'une application multivaluée que ceux d'une application classique.

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