Expansions géométriques et ampleur

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GEOMETRIC EXPANSIONS AND AMPLENESS

EXPANSIONS GÉOMÉTRIQUES ET AMPLEUR

by

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GEOMETRIC EXPANSIONS AND AMPLENESS

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I, JUAN FELIPE CARMONA, declare that this thesis titled, 'GEOMETRIC EXPANSIONS AND AMPLENESS' and the work presented in it are my own. I confirm that:

Signed: ________________________________

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Reason with its use of language has set up a satisfactory architecture, like the delightful, rhythmical composition in Renaissance painting.

Julio Cortázar.

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Résumé / Abstract

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Le résultat principal de cette thèse est l'étude de l'ampleur dans des expansion des structures géométriques et de SU-rang omega par un prédicat dense/codense indépendant. De plus, nous étudions le rapport entre l'ampleur et l'équationalité, donnant une preuve directe de l'équationalité de certaines théories CM-triviales. Enfin, nous considérons la topologie indiscernable et son lien avec l'équationalité et calculons la complexité indiscernable du pseudoplan libre.

The main result of this thesis is the study of how ampleness grows in geometric and SU-rank omega structures when adding a new independent dense/codense subset. In another direction, we explore relations of ampleness with equational theories; there, we give a direct proof of the equationality of certain CM-trivial theories. Finally, we study indiscernible closed sets—which are closely related with equations—and measure their complexity in the free pseudoplane.

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Contents

Introduction en français

La théorie des modèles a ses origines dans l'étude des structures mathématiques de point de vu de la logique du premier ordre, c'est à dire, dans un langage finitaire tel que l'on quantifie sur des éléments mais pas sur des sous-parties arbitraires de la structure (Ces restrictions, qui étaient imposées originalement pour des considérations fondationnelles et ensemble-théoriques, ont des conséquences rélévantes). Une question primordiale était si une structure donnée M pouvait être décrite complètement en premier ordre. Ceci s'avère faux lorsque M est infinie : en effet, deux résultats fondamentaux de la logique de premier ordre, la *compacité* et le *théorème de Löwenheim-Skolem* entraînent que, si M est infinie, alors pour tout cardinal λ infini, il existe une structure M_λ qui satisfait exactement tous les énoncés de premier ordre valables pour M. La structure M_λ est *élémentairement équivalente* à M. Elles ont la même théorie.

En revanche, certaines structures sont complètement caractérisées par leurs théories, à isomorphisme près. Par exemple, la structure $(\mathbb{Q}, <)$ est le seul ordre linéaire dense sans extrémes, à isomorphisme près. De même, le corps $(\mathbb{C}, +, \cdot, 0, 1)$ est le seul corps algébriquement clos en caractéristique 0 de cardinalité continue, à isomorphisme près. Une théorie est λ-*catégorique* si elle a un seul modèle, à isomorphisme près, de cardinalité λ.

Morley fut un des premiers à constater la rélévance de ces théories avec son célébre résultat suivant :

Théorème (Théorème de catégoricité de Morley)**.** Toute théorie dénombrable qui est catégorique pour un cardinal non-dénombrable est alors catégorique pour tout cardinal non-dénombrable.

Les techniques développées pour sa démonstration sont à la base de la *théorie des modèles géométrique* et ouvrent deux lignes fondamentales de recherche : l'une est le travail de classification, entamé par Shelah, qui cherche à décrire les théories complètes de premier ordre à partir de certains configurations de nature combinatoire, quoique les travaux de Shelah comprennent aussi le comptage de nombre des modèles d'une théorie. Le programme de Shelah a influencé de façon décisive la théorie de modèles, et certaine classes sortantes de sa classification, comme les théories stables, simples, NIP ou NTP_2, englobent beaucoup d'exemples de structures rélévantes en mathématiques.

L'autre direction de recherche, entamée par Cherlin, Harrington, Lachlan et Zilber, consiste à étudier des théories catégoriques. Une structure est catégorique en cardinalité non-dénombrable si sa théorie l'est. Une telle structure est construite à partir de certaines sous-parties définissables unidimensionelles irréductibles, dites *fortement minimales*. La clôture algébrique sur un ensemble fortement minimal induit une *prégéométrie*, c'est à dire, un matroïde infini. Étant donnée une structure catégorique en cardinalité non-dénombrable, les prégéométries de ses ensembles fortement minimaux sont toutes localement isomorphes. Deux structures catégoriques en cardinalité non-dénombrable sont *géométriquement équivalentes* si leurs prégéométries associés sont localement isomorphes.

Zilber [47] conjectura que toute structure catégorique en cardinalité non-dénombrable est géométriquement équivalente à un de types suivants :

Type 1 : Trivial, c'est à dire, le treilli des ensembles algébriquement clos (dans $\mathfrak{C}^{eq}$) est distributif.

Type 2 : Localement modulaire, si le treilli des ensembles algébriquement clos (dans $\mathfrak{C}^{eq}$) est modulaire.

Type 3 : La structure interprète un corps algébriquement clos.

La conjecture, qui est vraie pour des structures complètement catégoriques, est motivée par la citation suivante [48] :

> L'espoir originel de l'auteur dans [...] que toute structure catégorique en cardinalité non-dénombrable peut être récupérée à partir d'une structure classique (nommé le *principe de la trichotomie*), a ses motivations dans la croyance globale que les structures logiquement parfaites sont à la base du développement mathématique, ce qui induit à croire en une forte prédétermination logique dans les structures mathématiques primordiales.

Le principe de la trichotomie s'avère valable pour une large classe de structures, notamment celle des *Géométries de Zariski*, des structures munies d'un comportement topologique sur les ensembles définissables qui ressemble la topologique de Zariski dans un corps algébriquement clos. Les géométries de Zariski sont fondamentales pour la démonstration de Hrushovski de Mordell-Lang fonctionnel. Remarquons que les théories o-minimales satisfont aussi le principe de la trichotomie.

À la fin des années 80, Hrushovski montra que le principe de la trichotomie est faux en général [21]. Il construit, grâce à la méthode d'amalgamation de Fraïssé avec un contrôle précis de la dimension, une nouvelle classe d'ensembles fortement minimaux. Il propose aussi des nouvelles propriétés qui pourrait donner une analyse plus fine de la classification des structures catégoriques en cardinalité non-dénombrable. Nous allons principalement étudier une de ces propriétés suggérées, une généralisation de la modularité locale qu'il nome *CM-trivialité*. Nous ne donnerons pas l'explication du nom choisi (car nous ignorons la raison derrière).

Une réformulation de la modularité locale pour les ensembles fortement minimaux, qui permet de le généraliser à toute théorie stable, est la *monobasitude* :

Définition. Une théorie stable T est *monobasée* si pour tout uple réel c et tout modèle M de T, la base canonique $\mathrm{cb}(c/M)$ est algébrique sur c.

La base canonique $\mathrm{cb}(c/M)$ est le plus petit ensemble définisablement clos $D \subset M^{eq}$ tel que le type $tp(c/M)$ est définissable sur D. L'explication du nom *monobasitude* est que la base canonique d'un type stationnaire est algébrique sur une (toute) réalisation du type.

Une théorie stable est monobasée si et seulement si pour tout uple réel c et toutes sous-parties algébriquement closes $A \subset B$ dans T^{eq}, la base canonique $\mathrm{cb}(c/A)$ est algébrique sur $\mathrm{cb}(c/B)$. Cette caractérisation nous permet de voir que la CM-trivialité généralise la monobasitude :

Définition. Une théorie stable est *CM-triviale* si pour tout uple c et toutes sous-parties algébriquement closes $A \subset B$ dans T^{eq} avec $\mathrm{acl}^{\mathrm{eq}}(Ac) \cap B = A$, la base canonique $\mathrm{cb}(c/A)$ est algébrique sur $\mathrm{cb}(c/B)$.

Une théorie stable est monobasée si elle n'admet pas de pseudoplan type-définissable, c'est à dire, une configuration type-définissable d'incidence entre points et droites données par un type complet $tp(a, b)$ avec :

- $a \notin \mathrm{acl}(b)$ et $b \notin \mathrm{acl}(a)$,
- Si $a \neq a'$ et $ab \equiv a'b$, alors $b \in \mathrm{acl}(aa')$. De même, si $b \neq b'$ et $ab \equiv ab'$, alors $a \in \mathrm{acl}(bb')$.

L'intuition est que les théories CM-triviales correspondent à ces théories stables qui interdissent une certaine configuration type-définissable d'incidence entre points, droites et plans. Ceci reste encore à démontrer, quoique la thèse de Nübling [27] exhibe un résultat partiel dans cette direction. Il définit un *pseudoespace* 3*-dimensionnel indépendant type-définissable* comme la donnée d'un type complet $tp(a, b, c)$ tel que :

- $tp(a, b)$ et $tp(b, c)$ sont des pseudoplans type-définissables.
- a est indépendant de c sur b.

Nübling montre qu'aucun pseudoespace 3-dimensionnel indépendant type-définissable peut être défini dans une théorie supersimple CM-triviale de rang fini.

Dans le chapitre 2, des caractérisations équivalentes de la monobasitude et de la CM-trivialité seront exhibées, en termes du comportement de la relation d'indépendance, sans mention explicite des bases canoniques.

L'étude de la CM-trivialité apparait déjà dans [34] et [36]. Pillay montre qu'aucun groupe mauvais[1] peut être interprété dans une théorie CM-triviale. De plus, il montre que tout groupe CM-triviale de rang de Morley fini est nilpotent–par-fini.

Pillay [36] étend les notions de monobasitude et CM-trivialité, en introduisant la *hiérarchie ample*, selon laquelle ces deux notions correspondent aux premier et deuxième niveaux de la hiérarchie. L'amplitude est un effort de mieux classifier les ensembles fortement minimaux. Aucun ensemble fortement minimal n'interprétant pas de corps infinis n'est connu au delà de deux premiers niveaux.

Une de caractéristiques fondamentales des ensembles fortement minimaux est l'existence d'une notion de dimension, car la clôture algébrique satisfait le principe d'échange de Steinitz. Une théorie est *géométrique* si elle élimine le quanteurs $\exists^\infty$ et la clôture algébrique satisfait le principe de Steinitz. En particulier, il existe une borné uniforme aux instances algébriques d'une formule donnée, donc la dimension est définissable. Berenstein et Vassiliev [8, 10] ont étudié des expansions de théories géométriques par un prédicat dense et codense, que l'on interprète par un ensemble indépendant d'éléments. Ils démontrent que certaines notions, comme la stabilité ou la simplicité, sont préservées en ajoutant ce prédicat indépendant. Motivés par ces résultats, Berenstein et Kim [12] montrent que la NTP2 est aussi préservée.

Cependant, la monobasitude ne l'est pas :

Exemple. Soit $T = Th(V, +, \{\lambda_q\}_{q\in\mathbb{Q}})$ la théorie d'un espace vectoriel sur $\mathbb{Q}$, où V a dimension infinie. Soit B une base de V. Alors T est monobasée mais $T^{ind} = Th(V, +, \{\lambda_q\}_{q\in\mathbb{Q}}, B)$ ne l'est pas.

[1] Un *groupe mauvais* est un groupe connexe infini de rang de Morley non-résoluble tel que tous ses sous-groupes définissables connexes sont nilpotents. L'existence d'un tel groupe contredirait la conjecture de l'algébricité, qui affirme que tout groupe simple de rang de Morley fini interprète un corps algébriquement corps, tel que le groupe a une structure de groupe algébrique là-dessus.

Berenstein et Vassiliev posent comme question si la théorie T^{ind} de l'exemple précédent était CM-triviale. De plus, ils démandent si l'on pouvait obtenir des théories non CM-triviales à partir d'une théorie CM-triviale en ajoutant un prédicat indépendant. Nous [15] donnons des réponses complètes à ces deux questions : the theorem counter

Théorème 1. (cf. Théorème 3.2.7) Soit T^{ind} l'expansion d'une théorie T de rang SU 1 (par exemple, une théorie fortement minimale), par un prédicat indépendant dense. Pour $n \geq 2$, la théorie T est n-ample si et seulement si T^{ind} l'est.

Dans le chapitre 3, nous donnerons une preuve de ce résultat, ainsi que des conditions necéssaires et suffisantes pour que T^{ind} soit monobasée.

Une structure de rang de Lascar ω, quoique pas géométrique, est munie d'une prégéométrie naturelle par rapport à l'opérateur de clôture du type régulier de rang infini : l'élément est dans la *clôture* de l'ensemble A si $SU(a/A) < \omega$. Dans le cas de corps différentiels, ce correspond à que l'élément a soit différentiellement algébrique sur le corps différentiel engendré par A. En collaboration avec Berenstein et Vassiliev [11], nous étudions d'expansion de théories de rang ω par un prédicat dense indépendant, inspirés du cas geométrique. Cependant, nous ne pouvons pas démontrer le théorème A en toute généralité :

Théorème 2. (cf. Theorem 4.2.7) Si T a rang de Lascar ω, alors T est CM-triviale si et seulement si T^{ind} l'est.

Rappelons qu'une prégéométrie (M, cl) est *triviale* si $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(a)$ pour tout sous-partie $A \subset M$.

Théorème 3. (cf. Théorème 4.2.10) Soit T une théorie de rang de Lascar ω avec prégéométrie sous-jacente triviale. Alors T est n-ample si et seulement si T^{ind} l'est.

Ce théorème est démontré dans le chapitre 4, où nous considérons des structures de rang de Lascar ω et leurs expansions par un prédicat dense indépendant. De plus, nous donnons de conditions suffisantes pour que T^{ind} soit monobasée.

Comme mentionné auparavant, le principe de la trichotomie est valable pour les géométries de Zariski. Une géométrie de Zariski consiste en une structure M muni d'une collection compatible de topologies sur chaque produit cartesian qui engendrent tous les ensembles définissables. Une des propriétés fondamentales des géométries de Zariski est qu'ils ont une sous-classe distinguée d'ensembles définissables, dits *clos*, avec une noetherianité globale. Ce nous a motivé à considérer des théories équationnelles, introduites par Srour [40], pour adapter cette noetherianité localement. Une formule $\varphi(\bar{x}, \bar{y})$ est une *équation* si, pour tout ensemble des paramètres $\{\bar{b}_i\}_{i\in I}$, il existe un sous-ensemble fini $I_0 \subset I$ tel que $\bigcap_{i\in I} \varphi(\mathfrak{C}, \bar{b}_i) = \bigcap_{i\in I_0} \varphi(\mathfrak{C}, \bar{b}_i)$. Une théorie est *équationnelle* si toute formule est combinaison booléenne d'instances d'équations. Notons que la plus part des exemples de nature algébrique sont équationnels : les espaces vectoriels, les corps algébriquement clos et différentiellement clos, ainsi que les corps separablement clos de degré d'imperfection fini.

Toute théorie monobasée est équationnelle. Hrushovski même montre que son nouveau ensemble fortement minimal est équationnel, quoiqu'il n'a pas une structure de géométrie de Zariski.

Liée fortement à la notion d'équationalité est celle d'ensemble indiscernablement clos. Si X est un ensemble type-définissable, sa *clôture indiscernable* consiste en ces uples $\bar{a}$ du modèle ambient tels qu'il existe une suite indiscernable $I = (\bar{a}_0, \bar{a}_1, \bar{a}_2...)$ sur $\emptyset$ qui commence par $\bar{a}$ telle que $\bar{a}_i$ appartient à X pour tout $i \geq 1$. Un ensemble type-définissable X est *indiscernablement clos* si $X = \mathrm{icl}(X)$.

Junker et Lascar [26] traitent systématiquement l'équationalité. Ils montrent que la *topologie équationnelle*, où les ensembles fermés sont ceux définis par des équations, est fortement liée à la *topologie indiscernable*. Cette connexion leur permet de donner des conditions suffisantes pour qu'une théorie CM-triviale soit équationnelle. En outre, ils proposent une fonction i_T à valeurs ordinales qui mesure la complexité de la clôture indiscernable. Ils montrent que $i_T \leq 2$ si T est monobasée.

Dans le chapitre 5, nous étudierons les rapports possibles entre la CM-trivialité et l'équationalité, sous certaines conditions, et suggérons des généralisations possibles au cas ample. Nous calculons explicitement la valeur i_T pour la théorie du pseudoplan libre, qui est CM-triviale. Quoique nous soupçonnons que i_T est toujours borné par ω si T est CM-triviale, nous n'avons pas pu le démontrer en toute généralité.

Introduction

Model Theory has its origins in the study of mathematical structures under the light of first order languages, that is, finitary languages that can quantify over elements but not over subsets of the structure (these restrictions, originally imposed for the sake of set theory and foundations of mathematics, have strong consequences). An initial and natural question was whether a certain structure M could be completely described using a first order language. This turns out to be false whenever M is infinite: using two strong features of first order logic, *Compactness* and *Löwenheim-Skolem theorem*, it can be proved that for every infinite cardinal λ, there is a structure M_λ such that M_λ and M satisfy exactly the same properties expressible in first order. In this case we say that M is *elementary equivalent to* M_λ or that they have the same theory T.

However, there are some structures which are completely characterized in their own cardinality up to isomorphism. For example, the structure $(\mathbb{Q}, <)$ is the only countable "dense linear order without endpoints" and $(\mathbb{C}, +, \cdot, 0, 1)$ is the only "algebraically closed field of characteristic 0" of size $2^{\aleph_0}$.

A theory with exactly one model (up to isomorphism) of size λ is called λ-*categorical*. Morley drew his attention to these theories proving the following theorem:

Theorem (Morley's categoricity theorem)**.** Let T be a theory in a countable vocabulary. If T is λ-categorical for some uncountable cardinal λ, then T is κ-categorical for every uncountable cardinal κ.

The techniques developed for the proof of this theorem are the cornerstone of what has been called *geometric model theory*. After this theorem, one can distinguish two main lines of investigation. One of them is the impressive work of Shelah in *classification model theory*. Shelah's program can be seen, very roughly, as the classification of complete first-order theories by means of "encoding" certain combinatorial configurations (though the problem of counting models in each cardinal appears deeply at the core of Shelah's program). This program has permeated model theory everywhere, and several classes of theories which arose from this program: stable, simple, NIP, NTP2, have been widely studied and cover many examples of mathematical interest.

The other line, initiated by Cherlin, Harrington, Lachlan and Zilber, is the study of categorical theories: by a $\aleph_1$-categorical structure we mean a structure whose theory is $\aleph_1$-categorical. Every $\aleph_1$-categorical structure has some irreducible one-dimensional sets called *strongly minimal*; over these, algebraic closed subsets form a *pregeometry* (or an *finitary matroid*). For a fixed $\aleph_1$-categorical structure, all its pregeometries associated to strongly minimal sets are "alike", in the sense that they are isomorphic by localization in some finite set. Two $\aleph_1$-categorical structures are geometrically equivalent if their corresponding pregeometries are locally isomorphic.

Zilber conjectured that there were only three types of $\aleph_1$-categorical structures modulo geometric-equivalence (see [47]). These three types are characterized as follows:

Type 1: Trivial, meaning that the lattice of algebraically closed subsets (in $\mathfrak{C}^{eq}$) is distributive.

Type 2: Locally modular, meaning the lattice of algebraically closed subsets (in $\mathfrak{C}^{eq}$) is modular.

Type 3: The structure interprets an infinite field.

There were two motivations behind the conjecture. The first one is that the conjecture is true in totally categorical structures, the other one is more philosophical and can be elucidated from the following cite:

> The initial hope of the present author in [...], that any uncountably categorical structure comes from the classical context (the trichotomy conjecture), was based on the general belief that logically perfect structures could not be overlooked in the natural progression of mathematics. Allowing some philosophical licence here, this was also a belief in a strong logical predetermination of basic mathematical structures. (Zilber, [48]).

The trichotomy principle above holds in a variety of important contexts, namely Zariski geometries (topological structures on the definable sets which resemble the Zariski topology). Zariski geometries are crucial in Hrushovski's proof of the Mordell-Lang Conjecture. Recall that the trichotomy principle also holds in o-minimal structures.

In the late 80's, Hrushovski showed that the trichotomy principle is false in general (see [21]). He constructed, using a Fraïssé limit with a precise control of the dimension, a class of new strongly minimal sets. In that paper he defined some properties, pointing out directions for a more precise classification of $\aleph_1$-categorical structures. We will focus mainly on one of those properties, which he named *CM-triviality*. We will not attempt to explain the reason behind the name of CM-triviality (because we do not understand it); however, CM-triviality can be seen as a generalization of local modularity as we will later on explain.

The appropriate analogue of local modularity for stable theories is *1-basedness*

Definition. A stable theory T is *1-based* if for every real tuple c and every model M of T, we have that $\mathrm{cb}(c/M)$ is algebraic over c.

Here, by $\mathrm{cb}(c/A)$ we mean the *canonical base of* c *over* M, namely, the minimal definable-closed set $D \subset M^{eq}$ over which the type $tp(c/M)$ is definable. The name *1-based* comes from the fact that the canonical base of a stationary type is in the algebraic closure of *any* of its realizations. A strongly minimal theory is locally modular if and only if it is 1-based.

Equivalently, a stable theory is *1-based* if for any real tuple c and $A \subset B$ algebraically closed sets in T^{eq}, we have that $\mathrm{cb}(c/A)$ is algebraic over $\mathrm{cb}(c/B)$.

Using the previous definition, one may see (formally) that the concept of CM-triviality is a generalization of 1-basedness:

Definition. A stable theory is *CM-trivial* if for any c and $A \subset B$ algebraically closed sets in T^{eq}, whenever $\mathrm{acl}^{\mathrm{eq}}(cA) \cap B = A$ we have that $\mathrm{cb}(c/A)$ is algebraic over $\mathrm{cb}(c/B)$.

It has been proved that a stable theory is 1-based if and only if it forbids certain point-line type-definable configuration named a *type-definable pseudoplane*:

A *type-definable pseudoplane* is a complete type $tp(a, b)$ such that

- $a \notin \mathrm{acl}(b)$ and $b \notin \mathrm{acl}(a)$,
- If $a \neq a'$ and $ab \equiv a'b$ then $b \in \mathrm{acl}(aa')$. If $b \neq b'$ and $ab \equiv ab'$ then $a \in \mathrm{acl}(bb')$.

The intuition is that CM-trivial theories are exactly those that forbid a point-line-plane configuration; the latter has not yet been proved. As far as we now, the only work in this problem has been done by Nübling in his Ph.D thesis [27]:

A *type-definable independent 3-pseudospace* is a complete type $tp(a, b, c)$ such that:

- $tp(a, b)$ and $tp(b, c)$ are type-definable pseudoplanes.

- a is independent from c over b.

Nübling proved that simple CM-trivial theories of finite rank cannot have a type-definable independent 3-pseudospace. It is not known whether the converse is true.

We will see in Chapter 2 that 1-basedness and CM-triviality admit more general characterizations in terms of an independence relation, without mentioning canonical bases.

The notion of CM-triviality has been studied by Pillay in [34] and [36]. He proved that a bad group[2] cannot be interpreted in CM-trivial theory. Moreover, he proved that every stable CM-trivial group of finite Morley Rank is virtually nilpotent, i.e. it has a nilpotent subgroup of finite index.

In [36] Pillay defined the *ample hierarchy*, where 1-basedness corresponds to the first level and CM-triviality corresponds to the second level of the hierarchy. The definition of ampleness may be seen as an attempt to classify strongly minimal sets. Until now there is no known examples of strongly minimal sets in higher levels which do not interpret a field.

One of the crucial traits of strongly minimal structures is that there is a tame dimension, since the Steinitz exchange property holds for the algebraic closure. A theory is said to be *geometric* if it eliminates $\exists^{\infty}$ and algebraic closure has the Steinitz property (the first property establishes a uniform bound for the number of realizations of algebraic formulas; this is important because it allows to define the dimension of a formula in a first-order way). In [10] and [8] Berenstein and Vassiliev studied expansions of geometric theories by a dense/codense predicate interpreted by independent elements. In particular, they proved that properties such as stability and simplicity are preserved when adding a new predicate. In further work, Berenstein and Kim [12] showed that NTP2 is also preserved.

However, being 1-based is not transferred to the expansion:

Example. Let $T = Th(V, +, \{\lambda_q\}_{q\in\mathbb{Q}})$ be the theory of a vector space over $\mathbb{Q}$, and let B be a basis of V. Then T is 1-based but $T^{ind} = Th(V, +, \{\lambda_q\}_{q\in\mathbb{Q}}, B)$ is not 1-based.

Berenstein and Vassiliev asked if, in the last example, the theory T^{ind} was CM-trivial. Another question was whether a similar construction on a CM-trivial theory would give rise to a new non-CM-trivial theory. We managed to answer both questions (see [15]):

[2] A *bad group* is a non-solvable connected group of finite Morley rank, all whose proper connected definable subgroups are nilpotent. The existence of a bad group would contradict the algebraicity conjecture, which states that every simple group of finite Morley rank is an algebraic group over an algebraic closed field, which is interpretable in the group structure.

Theorem A. (cf. Theorem 3.2.7) Let T be a theory of SU-rank 1 (for example, a strongly minimal theory) with geometric elimination of imaginaries, and let T^{ind} be its expansion by a dense independent predicate. Then, for $n \geq 2$, we have that T is n-ample if and only if T^{ind} is.

In Chapter 3, we give a proof for this theorem and give a necessary and sufficient condition for T^{ind} to be 1-based.

Structures of SU-rank ω are not geometric but a natural pregeometry arises with respect to a different closure: we say that the element a is in the *closure* of a set A if $SU(a/A) < \omega$ (for example, in differentially closed fields, we have that $SU(a/A) < \omega$ holds if and only if a satisfies a differential equation over A). In joint work with Berenstein and Vassiliev [11], we study expansions of structures of SU-rank ω by a dense independent subset. A similar analysis can be done in these structures as in the geometric case. However we could not adapt Theorem A to its full generality. Nevertheless, we obtained the following results:

Theorem B. (cf. Theorem 4.2.7) Let T be a theory of SU-rank ω with geometric elimination of imaginaries. Then T is CM-trivial if and only if T^{ind} is.

Recall that a pregeometry (M, cl) is *trivial* if, for every $A \subset M$, we have $\mathrm{cl}(A) = \bigcup_{a\in A} \mathrm{cl}(a)$.

Theorem C. (cf. Theorem 4.2.10) Let T be of SU-rank ω and assume its associated pregeometry is trivial. Then T is n-ample if and only if T^{ind} is.

In Chapter 4. we study expansions of SU-rank ω structures and give the proof of this theorem. Also we provide conditions under which T^{ind} is 1-based.

As mentioned before, the trichotomy conjecture (even when false in general) is true in the context of Zariski Geometries. A Zariski geometry in a model M is a class of tame compatible topological structures on each M^n, which generate all definable sets by boolean combinations. A crucial feature of Zariski geometries is that they distinguish certain class of definable sets having a global Noetherian property. We call them *closed* sets. In this spirit we considered *equational theories*. These theories, defined by Srour in [40], may be thought of as generalization of Zariski geometries but having a local Noetherian property: a formula $\varphi(\bar{x}, \bar{y})$ is said to be an *equation* if for every set of parameters $\{\bar{b}_i\}_{i\in I}$ there is a finite set $I_0 \subset I$ such that $\bigcap_{i\in I} \varphi(\mathfrak{C}, \bar{b}_i) = \bigcap_{i\in I_0} \varphi(\mathfrak{C}, \bar{b}_i)$. A theory is *equational* if every formula is a Boolean combination of equations. It is worth to mention that many of the interesting examples of algebraic theories are equational:

vector-spaces, algebraic closed fields, differentially closed fields and separably closed fields of finite degree of imperfection.

Every 1-based stable theory is equational. Also, Hrushovski proved that the new strongly minimal sets he constructed are equational (however, it is not a Zariski structure).

A related concept to equationality is the one of *indiscernible closed set*. Given a set X, a tuple $\bar{a}$ is in $\mathrm{icl}(X)$ if there exists an indiscernible sequence $I = (\bar{a}_0, \bar{a}_1, \bar{a}_2 ...)$ over $\emptyset$, such that $\bar{a} = \bar{a}_0$ and $\bar{a}_i \in X$ for $i \geq 1$. We say that X is *indiscernible closed* if $X = \mathrm{icl}(X)$.

In [26] Junker and Lascar made a systematic study of equationality. They showed that the *equational topology* (this is, the topology whose closed sets are those defined by equations) and the *indiscernible topology* (this is, the topology whose closed sets are indiscernibly closed) are related in a strong way. They used this relation to point out sufficient conditions for equationality to hold in CM-trivial theories. In the same paper, they proposed an ordinal function i_T to measure the complexity of the indiscernible closure. They proved that stable one based theories have a good behaviour for this rank function:

Theorem 0.1 (Junker, Lascar [26])**.** *Let T be a stable 1-based theory, then $i_T \leq 2$.*

In Chapter 5. we study the relation of CM-triviality and equationality under suitable conditions, pointing out how this relation may be generalized to non-ample theories. Also we calculate i_T for the theory of the free pseudospace (which is a CM-trivial theory). We prove that $i_T \leq 3$ in this case. Our guess is that $i_T \leq \omega$ for CM-trivial theories and we prove it in certain cases. However, we did not manage to prove this result in general.

Chapter 1

Preliminaries

1.1 Conventions

- We will always consider complete first order theories, denoted by T. Also, the letter $\mathfrak{C}$ stands for κ-saturated and κ-strongly homogeneous model of T for κ sufficiently big.

- All sets are assumed to have cardinality less than κ.

- Unless otherwise stated, we will reserve the first letters of the alphabet a, b, c... for single elements in $\mathfrak{C}$. Finite tuples will be denoted by $\bar{a}, \bar{b}, \bar{c}$...

  In the same manner, by x, y, z... and $\bar{x}, \bar{y}, \bar{z}$... we denote single variables and tuples of variables respectively.

- For practical reasons, we shall always distinguish imaginary elements in $\mathfrak{C}^{eq}$ from elements of the real sort $\mathfrak{C}$.

- By $Aut(\mathfrak{C}/A)$, we mean the class of all the automorphisms of $\mathfrak{C}$ fixing A pointwise.

- The operators *definable closure* and *algebraic closure* are defined as follows:

  $\mathrm{dcl}^{\mathrm{eq}}(A) = \{a \in \mathfrak{C}^{eq} : f(a) = a \text{ for every } f \in Aut(\mathfrak{C}^{eq}/A)\}$,

  $\mathrm{dcl}(A) = \mathrm{dcl}^{\mathrm{eq}}(A) \cap \mathfrak{C}$,

  $\mathrm{acl}^{\mathrm{eq}}(A) = \{a \in \mathfrak{C}^{eq} : \{f(a) : f \in Aut(\mathfrak{C}^{eq}/A)\} \text{ is finite}\}$,

  $\mathrm{acl}(A) = \mathrm{acl}^{\mathrm{eq}}(A) \cap \mathfrak{C}$.

  Equivalently, an element $c \in \mathfrak{C}^{eq}$ is in $\mathrm{dcl}^{\mathrm{eq}}(A)$ (resp. $\mathrm{acl}^{\mathrm{eq}}(A)$), if there exists an L^{eq}-formula $\varphi(x, \bar{y})$ and a tuple $\bar{a} \in A$ such that $\varphi(c, \bar{a})$ and c is the unique element that satisfies $\varphi(x, \bar{a})$ (resp. $\varphi(x, \bar{a})$ has finitely many realizations).

- If A, B and C are sets, then $A \equiv_C B$ means that there is an automorphism $f \in Aut(\mathfrak{C}/C)$ such that f maps a certain enumeration of A onto a corresponding enumeration of B. Also, if A and B are tuples, then f respects the enumeration. By $A \equiv B$ we mean $A \equiv_\emptyset B$.

  Equivalently, we say that $A \equiv_C B$ if $\mathrm{tp}(A/C) = \mathrm{tp}(B/C)$, for some enumerations of A and B.

1.2 Imaginaries

In this section we define briefly what imaginaries are and describe different ways of "elimininating" them.

One of the purposes of imaginaries is to have canonical representatives of definable sets, and also to treat definable sets as elements. Let X be a definable set given by some formula $\varphi(\bar{x}, \bar{a})$. We may define an equivalence relation R such that $R(\bar{b}, \bar{b}')$ if and only if $\varphi(\mathfrak{C}, \bar{b}) = \varphi(\mathfrak{C}, \bar{b}')$. Notice that X is fixed setwise by all the automorphisms fixing the equivalence class of $\bar{a}$ and viceversa. Passing to the quotient by the equivalence relation R, the set of automorphisms fixing X setwise is exactly the set of automorphisms fixing $\bar{a}/R$. Hence, the class $\bar{a}/R$ is a good candidate to encode the definable set X as an element.

Definition 1.2.1. Let $R(\bar{x}, \bar{y})$ be a $\emptyset$-definable equivalence relation and consider the quotient $\mathfrak{C}/R$. Every element of this quotient is called an *imaginary*. By adding to $\mathfrak{C}$ a new sort $\mathfrak{C}/R$, for every quotient by a $\emptyset$-definable equivalence relation R, together with a projection function $\pi_R : \mathfrak{C} \to \mathfrak{C}/R$, one forms a new multi-sorted structure that is called $\mathfrak{C}^{eq}$. The theory T^{eq} stands for $Th(\mathfrak{C}^{eq})$.

Definition 1.2.2. Let $X = \varphi(\bar{x}, \bar{a})$ be a definable set and let R be the equivalence relation given by $R(\bar{b}, \bar{b}')$ if and only if $\varphi(\mathfrak{C}, \bar{b}) = \varphi(\mathfrak{C}, \bar{b}')$. A *canonical parameter* of X is the imaginary element $e = \bar{a}/R$.

Canonical parameters are unique up to interdefinability.

Definition 1.2.3. The theory T *eliminates imaginaries* if for every $\emptyset$-definable equivalence relation $R(\bar{x}, \bar{y})$ and for every $\bar{a}$, there exists a finite tuple $\bar{b}$ such that an automorphism $f \in Aut(\mathfrak{C})$ fixes $R(\mathfrak{C}, \bar{a})$ setwise if and only if f fixes $\bar{b}$ pointwise.

Clearly, this is equivalent to the following:

Definition 1.2.4. A theory T *eliminates imaginaries* if for every $e \in \mathfrak{C}^{eq}$ there exists a tuple $\bar{a}$ such that $e \in \mathrm{dcl}^{eq}(\bar{a})$ and $\bar{a} \in \mathrm{dcl}(e)$.

The theory T^{eq} eliminates imaginaries, therefore, it is convenient to work in $\mathfrak{C}^{eq}$ instead of $\mathfrak{C}$. It is similar to working with the skolemization of a theory in order to assume quantifier elimination.

Finally, we recall another useful encoding of imaginaries:

Definition 1.2.5.
- A theory T has *weak elimination of imaginaries* if for every $e \in \mathfrak{C}^{eq}$ there exists a tuple $\bar{a}$ such that $e \in \mathrm{acl}^{eq}(\bar{a})$ and $\bar{a} \in \mathrm{dcl}(e)$.
- We say that T has *geometric elimination of imaginaries* if for every $e \in \mathfrak{C}^{eq}$ there exists a tuple $\bar{a}$ such that $e \in \mathrm{acl}^{eq}(\bar{a})$ and $\bar{a} \in \mathrm{acl}(e)$.

The following easy fact will be useful in the present work:

Fact 1.2.6. Assume T has geometric elimination of imaginaries. Then, for every A and B subsets of $\mathfrak{C}$, we have $\mathrm{acl}^{eq}(\mathrm{acl}(A) \cap \mathrm{acl}(B)) = \mathrm{acl}^{eq}(A) \cap \mathrm{acl}^{eq}(B)$.

Proof. Clearly $\mathrm{acl}^{eq}(\mathrm{acl}(A) \cap \mathrm{acl}(B)) \subset \mathrm{acl}^{eq}(A) \cap \mathrm{acl}^{eq}(B)$. Assume $e \in \mathrm{acl}^{eq}(A) \cap \mathrm{acl}^{eq}(B)$. By hypothesis, there is a tuple $\bar{a} \in \mathfrak{C}$ such that $e \in \mathrm{acl}^{eq}(\bar{a})$ and $\bar{a} \in \mathrm{acl}(e)$. Hence $\bar{a} \in \mathrm{acl}(A) \cap \mathrm{acl}(B)$ and $e \in \mathrm{acl}^{eq}(\mathrm{acl}(A) \cap \mathrm{acl}(B))$. ■

1.3 Independence on strongly minimal theories

Strongly minimal sets are the "building blocks" of $\aleph_1$-categorical structures. First of all, every $\aleph_1$-categorical structure is prime over a strongly minimal set. Moreover, the geometric behaviour of $\aleph_1$-categorical structures is determined by the geometry of its strongly minimal sets. In this section we will describe the main ideas behind the geometry of strongly minimal sets, in order to motivate further definitions of geometrical complexity.

Definition 1.3.1. A definable set D of a structure M is *strongly minimal* if for every formula $\varphi(x, \bar{y})$, where x is a single variable, there exists a natural number k such that, for every $\bar{a} \in M^n$, either $|\varphi(D, \bar{a})| < k$ or $|\neg\varphi(D, \bar{a})| < k$.

A structure M is *strongly minimal* if it is strongly minimal as a set and a theory T is *strongly minimal* if it has a strongly minimal model.

If T is strongly minimal then all its models are.

Concerning imaginaries in strongly minimal theories, we have the following result:

Fact 1.3.2. Let T be a strongly minimal theory. If $\text{acl}(\emptyset)$ is infinite, then T has weak elimination of imaginaries.

Definition 1.3.3. A *Pregeometry* (or a *finitary Matroid*) is a pair (M, cl) where M is a set and cl is a closure operator $\text{cl} : P(M) \to P(M)$ such that, for every $A, B \subset M$:

- $A \subset \text{cl}(A)$.
- $\text{cl}(A) = \text{cl}(\text{cl}(A))$.
- $\text{cl}(A) = \bigcup_{A_0 \subset_{fin} A} \text{cl}(A_0)$.
- (Steinitz/Exchange property) If $a \in \text{cl}(bA) \setminus \text{cl}(A)$ then $b \in \text{cl}(aA)$.

Algebraic closure always satisfies the first three conditions. Moreover, in strongly minimal structure, algebraic closure also satisfies the exchange property, hence it induces a pregeometry.

We will now describe some general features of pregeometries and illustrate them in the case of strongly minimal structures.

Definition 1.3.4. Let (M, cl) be a pregeometry and $X \subset M$. We may construct a new pregeometry (M_X, cl_X) called the *localization at* X as follows: set $M_X = M$ and $\text{cl}_X(A) = \text{cl}(A \cup X)$ for every $A \subset M$.

Definition 1.3.5. Let (M, cl) be a pregeometry and $A \subset M$. We say that A is *independent* if $a \notin \text{cl}(A \setminus \{a\})$ for every $a \in A$.

The *dimension* of a set $A \subset M$ over $A_0 \subset A$, denoted by $dim(A/A_0)$, is the size of a maximal independent subset B contained in $A \setminus \text{cl}(A_0)$. We define $dim(A)$ as $dim(A/\emptyset)$.

It is easy to see that this dimension is well defined due to the exchange property.

The following definition distinguishes certain types of special pregeometries.

Definition 1.3.6. A pregeometry (M, cl) is:

- *Trivial* if, for every $A \subset M$, we have
$$\text{cl}(A) = \bigcup_{x \in A} \text{cl}(x).$$
- *Modular* if, for every $A, B \subset M$ of finite dimension, we have
$$dim(A \cup B) + dim(A \cap B) = dim(A) + dim(B).$$

- *Locally modular* if $(M_{\{x\}}, \mathrm{cl}_{\{x\}})$ is modular for some $x \in M$.

Assume that T is a strongly minimal theory and $M \models T$. We say that T is *trivial* (resp. *modular*, *locally modular*) if the algebraic closure acl in M is. This notion is well defined since does not depend on the model of T.

In strongly minimal structures, local modularity can be characterized in terms of an independence relation, which is useful when no good notion of dimension can be defined, yet a notion of independence exists.

Definition 1.3.7. Let M be strongly minimal and A, B, C subsets of M^{eq} with $C \subset B$. We say that A is *independent* from B over C, written $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$, if for every $a \in A$ such that $a \in \mathrm{acl}^{eq}((A \setminus \{a\})B)$ we have that $a \in \mathrm{acl}^{eq}((A \setminus \{a\})C)$.

Definition 1.3.8. A strongly minimal theory T is *1-based* if for every $A, B \subset M$, we have

$$A \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{acl}^{eq}(A) \cap \mathrm{acl}^{eq}(B)} B.$$

Fact 1.3.9 (see [35], Proposition 2.5.8)**.** A strongly minimal theory T is locally modular if and only if it is 1-based.

The following are the paradigmatic examples of strongly minimal structures:

Example 1.3.10.
- (**Trivial**) The theory of infinite sets without structure.
- (**Modular**) The theory $T = Th(V, +, \{\lambda_k\}_{k \in F})$ of a vector space V over a field F. In this theory, the algebraic closure coincides with the linear span. Therefore, dimension coincides with linear dimension, so this theory is modular.
- (**Locally modular**) Let V a vector space over $\mathbb{K}$. We define the *affine space* as follows: for every $k \in \mathbb{K}$ define $\lambda_k : (u, v) = ku + (1-k)v$. Also define $G(u, v, w) = u - v + w$. Then, the theory of the affine space $T = Th(V, \{\lambda_k\}_{k \in F}, G)$ is strongly minimal and locally modular, but not modular.

By Fact 1.3.9. these three examples are 1-based.

The following example provides a good insight on non-1-based theories.

- (**Non-Locally modular**)

  The theory AFC_p of algebraic closed fields in a fixed characteristic p (where p is a prime number or 0) is not 1-based (hence it is not locally modular): first of all, notice that algebraic closure in the model theoretic sense coincides with the algebraic closure in the field theoretic sense. To see that AFC_p is not 1-based, consider a, b, c trascendental independent elements and take $d = ac + b$. Due to quantifier elimination and elimination of imaginaries in AFC_p, it is not hard to show that $\mathrm{acl}^{eq}\{a, b\} \cap \mathrm{acl}^{eq}\{c, d\} = \overline{\mathbb{F}}_p$, where $\overline{\mathbb{F}}_p$ is the algebraic closure of the prime field $\mathbb{F}_p$. Nevertheless we have $a, b \not\mathop{\smile}\limits^{\mid}_{\overline{\mathbb{F}}_p} c, d$. Therefore, the theory ACF_p is not 1-based.

The point-line configuration we described in ACF_p is archetypical of every non-1-based structures:

Definition 1.3.11. A complete type $tp(\bar{a}, \bar{b})$ (possible in L^{eq}) is a *type-definable pseudo-plane* if:

1. $\bar{a} \notin \mathrm{acl}(\bar{b})$ and $\bar{b} \notin \mathrm{acl}(\bar{a})$.
2. If $\bar{a} \neq \bar{a}'$ and $\bar{a}\bar{b} \equiv \bar{a}'\bar{b}$ then $\bar{b} \in \mathrm{acl}(\bar{a}\bar{a}')$. If $\bar{b} \neq \bar{b}'$ and $\bar{a}\bar{b} \equiv \bar{a}\bar{b}'$ then $\bar{a} \in \mathrm{acl}(\bar{b}\bar{b}')$.

We may reformulate the axioms of the pseudoplane as:

- Every line has infinitely many points and every point is infinitely many lines.
- Given two points, there are finitely many lines passing through them. Given two lines, there are finitely many points contained in them.

Notice that, in the example of ACF_p, the type $tp(\overline{ab}, \overline{cd})$ is a type-definable pseudoplane.

1.4 Simple and stable theories

Simple theories where introduced originally by Shelah (see [38]) in an attempt to understand the "function" SP:

$SP(T) = \{(\lambda, \kappa)$: every model of T of size λ has a κ-saturated elementary extension of cardinality $\lambda\}$. The class $SP(T)$ behaves "well" when T is stable and it behaves "bad" when T is not simple. Hrushovski proved that it is consistent with ZFC that a simple theory T has bad behavior. (See [20] and [24] for definitions and explanations of "well" and "bad").

The original definition of simplicity is the following:

Definition 1.4.1.
- A formula $\varphi(\bar{x}, \bar{y})$ has the *tree property* if there exists a natural number $k \geq 2$ and a tree of parameters $\{\bar{a}_s | s \in \omega^{<\omega}\}$ such that:

1. For every $f \in \omega^\omega$, the set $\{\varphi(\bar{x}, \bar{a}_{f|n}) | n < \omega\}$ is consistent.
2. For each $s \in \omega^{<\omega}$, the set $\{\varphi(\bar{x}, \bar{a}_{s^i}) | i < \omega\}$ is k-inconsistent.

- We say that T is *simple* if no formula has the tree property.

Despite that these combinatorial properties are of great interest, we will take a different approach to this class of structures, mainly in terms of their independence relation. In 1997, Kim and Pillay proved, among other important results, that simplicity has a good characterization in terms of an abstract independence relation (Theorem 1.4.3), which is actually uniquely determined.

Definition 1.4.2. A ternary independence relation $R(A, B, C)$ over sets $A, B, C \subset \mathfrak{C}^{eq}$ is called an *abstract independence relation*, noted as $A \mathop{\smile\!\!\!\!\!\!|}_C B$, if it satisfies the following axioms:

1. (**Invariance**) If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $ABC \equiv A'B'C'$ then $A' \mathop{\smile\!\!\!\!\!\!|}_{C'} B'$.
2. (**Symmetry**) If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ then $B \mathop{\smile\!\!\!\!\!\!|}_C A$.
3. (**Transitivity**) Assume $D \subset C \subset B$. If $B \mathop{\smile\!\!\!\!\!\!|}_C A$ and $C \mathop{\smile\!\!\!\!\!\!|}_D A$, then $B \mathop{\smile\!\!\!\!\!\!|}_D A$.
4. (**Monotonicity**) Assume $A \mathop{\smile\!\!\!\!\!\!|}_C B$. If $A' \subset A$ and $B' \subset B$, then $A' \mathop{\smile\!\!\!\!\!\!|}_C B'$.
5. (**Base monotonicity**) If $D \subset C \subset B$ and $A \mathop{\smile\!\!\!\!\!\!|}_D B$, then $A \mathop{\smile\!\!\!\!\!\!|}_C B$.
6. (**Local character**) For every A, there exists a cardinal κ_A such that, for every B there is a subset $C \subset B$ with $|C| < \kappa_A$, where $A \mathop{\smile\!\!\!\!\!\!|}_C B$.
7. (**Finite character**) If $A_0 \mathop{\smile\!\!\!\!\!\!|}_C B$ for all finite $A_0 \subset A$, then $A \mathop{\smile\!\!\!\!\!\!|}_C B$.
8. (**Antireflexivity**) If $A \mathop{\smile\!\!\!\!\!\!|}_B A$ then $A \subset \mathrm{acl}^{eq}(B)$.
9. (**Existence**) For any A, B and C there is $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_C B$.

This list is not minimal at all; for example, Adler [1] proved that **Symmetry** can be deduced from the other axioms. However, we list all the relevant ones for the sake of completeness.

Theorem 1.4.3 (Kim-Pillay)**.** A complete theory is *simple* if and only if it has an independence relation that also satisfies:

1. (**Local character**) There exists a cardinal κ such that $\kappa_A = \kappa$ for every finite A.

2. (**Independence theorem over models**) Assume M is a model. If $A' \equiv_M B'$ with
$$A' \underset{M}{\downarrow} A, A \underset{M}{\downarrow} B \text{ and } B \underset{M}{\downarrow} B',$$
then there is some C such that $C \equiv_{MA} A'$, with $C \equiv_{MB} B'$ and $C \downarrow_M AB$.

As in the case of simple theories, stable theories may be defined in a combinatorial way as follows:

Definition 1.4.4.
- A formula $\varphi(\bar{x}; \bar{y})$ has the *order property* if there exist $(\bar{a}_i)_{i<\omega}$ and $(\bar{b}_j)_{j<\omega}$ such that $\mathfrak{C} \models \varphi(\bar{a}_i; \bar{b}_j)$ if and only if $i < j$.
- A theory T is *stable* if no formula has the order property.

Definition 1.4.5. Assume T is simple. Let $A \subset B$, $p \in S_n(A)$ and $q \in S_n(B)$ be an extension of p. We say that q is a *free* extension of p if for some (every) $c \models q$, we have $c \downarrow_A B$.

Theorem 1.4.6. A complete theory is *stable* if and only if is simple and the independence relation satisfies:

- (**Boundeness**) For every $p \in S_n(A)$ there is a cardinal μ such that, for every $A \subset B$, there are at most μ free extensions of p in $S(B)$.

We remark that every strongly minimal theory is stable and the independence relation defined in Definition 1.3.7. coincides with the independence relation of stable theories.

Definition 1.4.7. A theory is called *supersimple* (resp. *superstable*) if it is simple (resp. stable) and, for the local character of the independence, we have $\kappa_A = \omega$ for every finite A.

Finally, let us recall the meaning of indiscernible and Morley sequences:

Definition 1.4.8. A sequence $I = (\bar{a}_i)$ is an *indiscernible sequence over a set A* if for every $i_1 < ... < i_k$ and $j_1 < ... < j_k$ we have that $\bar{a}_{i_1}...\bar{a}_{i_k} \equiv_A \bar{a}_{j_1}...\bar{a}_{j_k}$.

We say that I is *totally indiscernible over A* if $\bar{a}_{i_1}...\bar{a}_{i_k} \equiv_A \bar{a}_{j_1}...\bar{a}_{j_k}$ for every subcollections $i_1, ..., i_k$ and $j_1...j_k$ of pairwise distinct indices.

Fact 1.4.9. A theory T is stable if and only if every indiscernible sequence is totally indiscernible.

Definition 1.4.10. Let T be a theory with an abstract independent relation. A sequence $I = (\bar{a}_i)$ is *Morley over A* if it is indiscernible over A and $\bar{a}_i \downarrow_A \{a_j | j < i\}$ for every i.

1.5 Forking and canonical bases

In this section we will describe the meaning of the independence relation on simple theories. For the sake of completeness we also define *canonical bases* in the more general frame.

Definition 1.5.1. • The formula $\varphi(\bar{x}, \bar{a})$ *divides over* A if there exists a natural number k and a sequence $\{\bar{a}_i\}_{i<\omega}$ such that:

1. For every $i < \omega$, we have $\bar{a}_i \equiv_A \bar{a}$.
2. The set $\{\varphi(\bar{x}, \bar{a}_i)\}_{i<\omega}$ is k-inconsistent.

- We say that $\varphi(\bar{x}, \bar{a})$ *forks over* A if there are formulas $\phi_1(\bar{x}, \bar{a}_1), ..., \phi_n(\bar{x}, \bar{a}_n)$, such that:

  1. $\varphi(\bar{x}, \bar{a}) \vdash \bigvee_{i \leq n} \phi_i(\bar{x}, \bar{a}_i)$,
  2. For every i, the formula $\phi_i(\bar{x}, \bar{a}_i)$ divides over A.

The definition of forking may be extended to types:

Definition 1.5.2. • A partial type p *divides* (resp. *forks*) *over* A if p implies a formula $\varphi(x, a)$ that divides (resp. forks) over A.

- Assume $A \subset B$, let $p \in S_n(A)$ and $q \in S_n(B)$ be complete types such that $q \supset p$. We say that q is a *non-forking (non-dividing) extension of* p if q does not fork over A.

- Let c be a realization of a type $p \in S(B)$ and $A \subset B$. We say that c is *forking-independent from* B *over* A if p does not fork over A.

Theorem 1.5.3 (Kim-Pillay)**.** Assume T is a simple theory. Then:

- A partial type $\pi(x)$ divides over A if and only if it forks over A.

- The independence relation of T is unique and coincides with forking-independence.

Now we will describe the concept of canonical bases. Basically, a canonical basis is the minimal set over which a type does not fork. For the purpose of this work, from now on all theories we consider will be either supersimple or stable.

Definition 1.5.4. We say that $p = tp(B'/A)$ is an *amalgamation base* if for every B, B', C and C' such that $B' \equiv_A C'$ and

$$B' \underset{A}{\downarrow} B, B \underset{A}{\downarrow} C \text{ and } C \underset{A}{\downarrow} C',$$

there exists D such that $D \equiv_{AC'} C$, $D \equiv_{AB'} B$ and $D \mathop{\smile\!\!\!\!\!\!|}_A BC$.

Notice that, by the *independence property* of the independence Relation (Definition 1.4.3), types over models are amalgamation bases.

It can also be proved that types over algebraic closed sets are amalgamation bases:

Remark 1.5.5 (see [16], Corollary 20.5)**.** Assume that T is supersimple (or stable). Let $a \in \mathfrak{C}^{eq}$ and $A = \mathrm{acl}^{\mathrm{eq}}(A) \subset \mathfrak{C}^{eq}$. Then $tp(a/A)$ is an amalgamation base.

Definition 1.5.6. Let $p \in S(B)$ be an amalgamation base and let $A \subset B$. We say that A is a *canonical basis of* p if $p|A$ is an amalgamation base and:

- The type p does not fork over A.
- If p does not fork over $A' \subset B$ then $A \subset \mathrm{dcl}(A')$.

Corollary 1.5.7. Let T be a supersimple theory. Then, for every $a \in \mathfrak{C}^{eq}$ and $A \subset \mathfrak{C}^{eq}$ with $A = \mathrm{acl}^{eq}(A)$, the canonical basis $\mathrm{cb}(a/A)$ exists in $\mathfrak{C}^{eq}$. (In fact, it can be proved that $\mathrm{cb}(a/A)$ is a single imaginary).

So, in supersimple theories, the algebraic closure of $\mathrm{cb}(A/B)$ may be seen as the minimal algebraically closed subset of $\mathrm{acl}^{eq}(B)$ over which A is independent from B. In practice we will only use the *minimality property* of canonical bases.

It is clear that if a supersimple/stable theory T has geometric elimination of imaginaries, then canonical bases are interalgebraic with real tuples (possibly infinite). Moreover, the converse is also true:

Fact 1.5.8. A supersimple/stable theory T has geometric elimination of imaginaries if and only if canonical bases of real tuples are interalgebraic with real tuples.

In the realm of stability, canonical bases admit a more useful characterization as we will see:

Definition 1.5.9. A type $p \in S_n(A)$ is *stationary* if it has a unique non-forking extension to any $B \supset A$.

Fact 1.5.10 (see [35], Remark 1.2.26)**.** Assume that T is stable. Let $a \in \mathfrak{C}^{eq}$, $B \subset \mathfrak{C}^{eq}$, $B = \mathrm{acl}^{eq}(B)$ and $c = \mathrm{cb}(a/B)$. Then:

1. The type $p = tp(a/c)$ is stationary.

2. For any automorphism $f \in Aut(\mathfrak{C})$, the automorphism f fixes p if and only if it fixes c.

From this we have:

Corollary 1.5.11. Assume that T is stable and let $c = \mathrm{cb}(a/B)$. If $c' \neq c$ and $c' \equiv_a c$ then $a \not\perp_c c'$.

Finally, we describe the *Morley Rank*, which says how many times a definable set can be "splitted", and *Lascar Rank*, which measures "how many" times a type may fork:

Definition 1.5.12. Let $\varphi(\bar{x}, \bar{a})$ be any formula. We define inductively the *Morley rank* as follows:

1. $MR(\varphi(\bar{x}, \bar{a})) \geq 0$ if $\varphi(\bar{x}, \bar{a})$ is consistent.
2. $MR(\varphi(\bar{x}, \bar{a})) \geq \alpha + 1$ if there are formulas $\{\varphi_i(\bar{x}, \bar{a}_i)\}_{i<\omega}$ of Morley rank $\geq \alpha$ such that $\varphi_i(\bar{x}, \bar{a}_i) \vdash \varphi(\bar{x}, \bar{a})$ and $\varphi_i(\bar{x}, \bar{a}_i) \wedge \varphi_j(\bar{x}, \bar{a}_j)$ is inconsistent for $i \neq j$.
3. $MR(\varphi(\bar{x}, \bar{a})) \geq \lambda$ for λ a limit ordinal if $MR(\varphi(\bar{x}, \bar{a})) \geq \beta$ for every $\beta < \lambda$.

Definition 1.5.13. For a partial type $\pi(\bar{x})$ we define

$$MR(\pi(\bar{x})) = inf\{MR(\varphi) | \varphi \in \pi(\bar{x})\}.$$

Notice that Morley-rank is a continuous function $MR : S_n(B) \to Ord \cup \{\infty\}$.

Definition 1.5.14. Assume that T is a simple theory. We define the *Lascar rank*, (usually named *SU-rank* in the context of simple theories and *U-rank* in the stable ones), inductively:

1. $SU(p) \geq 0$ for every type p.
2. $SU(p) \geq \alpha + 1$ if there exists a forking extension p' of p, such that $SU(p') \geq \alpha$.
3. For λ an ordinal limit, we say that $SU(p) \geq \lambda$ if $SU(p) \geq \beta$ for all $\beta < \lambda$.
4. $SU(p) = \infty$ if $SU(p) \geq \lambda$ for every ordinal λ.

By $SU(a/B)$, we mean $SU(tp(a/B))$. Notice that $a \perp_B C$ if and only if $SU(a/B) = SU(a/BC)$.

Definition 1.5.15. We say that a type p is *minimal* if $SU(p) = 1$.

Remark 1.5.16. Inside a minimal type p, algebraic closure induces a pregeometry.

Theorem 1.5.17 (Lascar inequality)**.** Let T be a simple theory. Then, for all tuples a, b and for every set C, we have:

$$SU(a/bC) + SU(b/C) \leq SU(ab/C) \leq SU(a/bC) \oplus SU(b/C).$$

Here, the symbol "+" denotes the usual sum for ordinals, while "$\oplus$" stands for the sum of ordinals in their Cantor-normal form, i.e. if $\alpha = \omega^{\beta_1}a_1 + \cdots + \omega^{\beta_k}a_k$ and $\beta = \omega^{\beta_1}b_1 + \cdots + \omega^{\beta_k}b_k$, then $\alpha \oplus \beta = \omega^{\beta_1}(a_1 + b_1) + \cdots + \omega^{\beta_k}(a_k + b_k)$.

Fact 1.5.18 (See [16], Proposition 13.13)**.** A theory T is supersimple if and only if every type p has U-rank $< \infty$.

Definition 1.5.19. Let T be a simple theory. By $SU(T)$ we mean $sup\{SU(c)|c \in \mathfrak{C}^{eq}\}$. In particular, a theory T has *finite rank* if there is a bound n such that $SU(c) \leq n$ for every element $c \in \mathfrak{C}$.

The U-rank is also a function $U : S_n(B) \to Ord \cup \{\infty\}$ not necessarily continuous. However, in certain cases of interest, we can assure continuity of U-rank due to the fact that Morley rank is continuous.

Fact 1.5.20. In the following cases, Morley rank coincides with U-rank:

- Strongly minimal theories.
- Groups of finite Morley rank.

In particular, the U-rank is continuous in these theories.

1.6 One-basedness

The concept of 1-basedness coincides with local modularity in strongly minimal theories. In general, the notion of modularity does not apply to theories without a good theory of dimension. However, the definition of 1-basedness (Definition 1.3.9) still makes sense in several classes with a tame notion of independence.

Let us recall the definition.

Definition 1.6.1. A theory with an abstract independent relation is 1-based if

$$A \underset{\mathrm{acl^{eq}}(A)\cap \mathrm{acl^{eq}}(B)}{\perp} B$$

for every $A, B \subset \mathfrak{C}^{eq}$.

Here are some examples of 1-based theories:

- (Strongly minimal) The theory of an infinite vector space V over a field $\mathbb{F}$.
- ($\aleph_1$-categorical) The theory of the group $(\mathbb{Z}/4\mathbb{Z}, +)$.
- (Superstable) The theory of a set endowed with infinitely many equivalence relations $\{E_i\}_{i<\omega}$, such that E_0 has 2-classes and E_{i+1} refines every class of E_i in two classes.
- (Stable) The theory of a set endowed with infinitely many equivalence relations $\{E_i\}_{i<\omega}$, each one with infinitely many infinite-classes and such that E_{i+1} refines every class of E_i in infinitely many classes.
- (Supersimple) The theory of the Random Graph.
- (O-minimal) The theory of $(\mathbb{R}, +, <, \pi()|(-1, 1))$. Where $\pi(r) = \pi \cdot r$ (the multiplication of the number r with the number π). Strictly speaking, this example is not 1-based, but, "philosophically", it belongs to this list. (See [9]).

The concept of 1-basedness has been widely studied in the context of stable theories. We list some of the most important results in order to have an idea on their relevance. This results and their proofs can be found in [35].

Theorem 1.6.2. Totally categorical theories are 1-based.

(This theorem plays an important role in the proof of the non-finite axiomatizability of totally categorical theories).

Theorem 1.6.3. Let T be a stable theory of finite U-rank. Then T is 1-based if and only if all its minimal types are locally modular.

Theorem 1.6.4. Assume T is stable and 1-based. If T does not interpret a group, then all its minimal types are trivial.

Definition 1.6.5. A group is virtually abelian if it has an abelian subgroup of finite index.

Theorem 1.6.6. If G is a 1-based stable group, then

1. The group G is virtually abelian.

2. Every definable set is a Boolean combination of cosets of $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$-definable subgroups.

Definition 1.6.7. A definable set X is *weakly normal* if, for every infinite family $\{X_i\}_{i\in I}$ of different conjugates of X under $Aut(\mathfrak{C}^{eq})$, we have $\bigcap_{i\in I} X_i = \emptyset$.

A theory is *weakly normal* if every definable set X is a Boolean combination of weakly normal definable sets.

Fact 1.6.8 (see [35], Proposition 4.1.5)**.** A theory T is weakly normal if and only if it is stable and 1-based.

Recall from Example 1.3.10 that ACF_p is not 1-based. As we indicated, this theory is not 1-based because one may construct a type-definable pseudoplane in it.

Let us summarize several characterizations of 1-basedness in the context of stable theories.

Theorem 1.6.9 (see [35], Chapter 4)**.** Let T be a stable theory, then each one of the following statements is equivalent to 1-basedness:

1. There is no type-definable pseudoplane.
2. T is weakly normal.
3. Every indiscernible sequence $(a_i)_{i<\omega}$ is Morley over a_0.
4. For any a and B, where B is the canonical basis of $stp(a/B)$, the type $tp(B/a)$ is algebraic.

Chapter 2

Ampleness

In this chapter we develop both the notions of CM-triviality, introduced by Hrushovski in [22], and ampleness, as introduced by Pillay in [36]. From now on, all theories are supersimple or stable.

2.1 CM-triviality

Definition 2.1.1. A theory T is *CM-trivial* if for every $c \in \mathfrak{C}^{eq}$ and $A, B \subset \mathfrak{C}^{eq}$ such that $\mathrm{acl}^{eq}(cA) \cap \mathrm{acl}^{eq}(B) = \mathrm{acl}^{eq}(A)$, we have $\mathrm{cb}(c/A) \subset \mathrm{acl}^{eq}(\mathrm{cb}(c/B))$.

Proposition 2.1.2. [Hrushovski [21]] The following definitions are equivalent to CM-triviality.

(CM1) For every $a, b, c \in \mathfrak{C}^{eq}$ such that $a \mathop{\smile\!\!\!\!\!\!|}_c b$ we have that $a \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}^{eq}(ab)\cap\mathrm{acl}^{eq}(c)} b$.

(CM2) For every $a, b, c \in \mathfrak{C}^{eq}$, if $a \mathop{\smile\!\!\!\!\!\!|}_b c$ and $\mathrm{acl}^{eq}(ac) \cap \mathrm{acl}^{eq}(ab) = \mathrm{acl}^{eq}(a)$, then $a \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}^{eq}(a)\cap\mathrm{acl}^{eq}(b)} c$.

The following equivalence was stated by Evans:

Proposition 2.1.3 (Evans, [18])**.** A theory T is CM-trivial if and only if, for every A, B, C algebraically closed sets (in $\mathfrak{C}^{eq}$), if $A \mathop{\smile\!\!\!\!\!\!|}_{A\cap B} B$, then $A \cap C \mathop{\smile\!\!\!\!\!\!|}_{A\cap B\cap C} B \cap C$.

It can be shown that every one-based theory is CM-trivial. In the context of stable theories, CM-triviality has been widely studied by Pillay in [34] and [36], where he shows the following:

Theorem 2.1.4 (Pillay [34])**.** Assume T stable and CM-trivial. Then T does not interpret an infinite field.

Definition 2.1.5. A *bad group* is a non-solvable connected group of finite Morley rank, all whose proper connected definable groups are nilpotent.

The existence of a bad group would contradict Cherlin-Zilber conjecture, which states that every simple group of finite Morley rank is an algebraic group over an algebraic closed field, which is interpretable in the group structure.

Theorem 2.1.6 (Pillay [34])**.** If G is a bad group, then G is not CM-trivial. Moreover, every CM-trivial group of finite Morley rank is virtually nilpotent, that is, it has a definable nilpotent subgroup of finite index.

Nowadays, several examples of CM-trivial structures are known: in finite Morley rank, Hrushovski's construction *ab-initio* [22] and Baudisch group [2], are both CM-trivial but not 1-based. Also, Baldwin [6] constructed a non-desarguesian projective plane or Morley rank 2, which is CM-trivial but not 1-based.

Concerning the case of stable theories of infinite rank, the canonical example is the theory of the *free pseudoplane*. This is, the theory of an infinitely many branching graph without cycles. We will explain this theory in detail in Chapter 5.

In the context of simple unstable theories we have the following result by Nübling:

Theorem 2.1.7 (Nübling [27])**.** If T is a CM-trivial supersimple theory, then T does not interpret an infinite field.

Every superstable ω-categorical theory is 1-based. In [24], Hrushovski constructed a non 1-based, supersimple and ω-categorical theory. This theory is CM-trivial, which leads to the following question:

Question 2.1.8. If T is supersimple and ω-categorical, is it then CM-trivial?

(In his Ph.D thesis, Palacín [32] obtained some partial results concerning this question).

Finally we remark that CM-triviality may be defined in theories with an independence notion, without mentioning canonical bases (CM1). Yoneda [45] studied CM-triviality in Rosy theories of thorn-rank 1, proving also the non-interpretability of infinite fields.

2.2 Ampleness

The hierarchy of ampleness, defined by Pillay in [36], and calibrated by Nübling in [27] and Evans in [17], has its origin in the study of CM-triviality. In this hierarchy, the notion of 1-basedness corresponds to the first level, while CM-triviality corresponds to the second level. Moreover, stable theories interpreting a field are n-ample for all n. This was proved by Pillay in [36], using a definition of ampleness slightly weaker than Evans definition (which is the one we will be working with), however Pillay's proof works as well for Evans definition. The original idea of ampleness is to classify forking complexity of strongly minimal structures. At the moment of writing, it is not known if there are strongly minimal structures in higher levels of ampleness which do not interpret an infinite field.

Definition 2.2.1. A theory T is *n-ample* if there exist $a_0, ..., a_n \in \mathfrak{C}^{eq}$ such that:

1. $\operatorname{acl}^{eq}(a_0, ..., a_i) \cap \operatorname{acl}^{eq}(a_0, ..., a_{i-1}a_{i+1}) = \operatorname{acl}^{eq}(a_0, ..., a_{i-1})$ for all $0 < i < n$.
2. $a_{i+1} \mathop{\smile\!\!\!\!\!\!|}_{a_i} a_0...a_{i-1}$ for all $i < n$.
3. $a_n \not\mathop{\smile\!\!\!\!\!\!|}_{\operatorname{acl}^{eq}(a_0)\cap\operatorname{acl}^{eq}(a_1)} a_0$.

A tuple $a_0, ..., a_n$ is *n-ample* if it satisfies the above conditions.

The original definition, given by Pillay in [36] is the following:

Definition 2.2.2 (Pillay)**.** A theory T is *n-ample* if, possibly after naming parameters, there exist $a_0, ..., a_n \in \mathfrak{C}^{eq}$ such that:

1. $\operatorname{acl}^{eq}(a_0, ..., a_i) \cap \operatorname{acl}^{eq}(a_0, ..., a_{i-1}a_{i+1}) = \operatorname{acl}^{eq}(a_0, ..., a_{i-1})$ for all $0 < i < n$.
2. $\operatorname{acl}^{eq}(a_0) \cap \operatorname{acl}^{eq}(a_1) = \operatorname{acl}^{eq}(\emptyset)$.
3. $a_n \mathop{\smile\!\!\!\!\!\!|}_{a_i} a_0...a_{i-1}$ for all $i < n$.
4. $a_n \not\mathop{\smile\!\!\!\!\!\!|} a_0$.

However, as it was remarked by Nübling in [27] Section 1.11, this definition has the following problem:

If $a_0, ..., a_n$ is ample in the sense of Pillay, then, taking $b_i = a_0...a_{i-2}a_i$, the tuple $b_0 = a_0$, $b_1 = a_1$, b_2,..., b_{n-1}, $b_n = a_n$ would be also ample (in Pillay's sense). But, for all $i < j-1 < j < n$, we have that $b_i \in \operatorname{acl}^{eq}(b_j)$. Which does not seem to agree with the notion of "ampleness".

On the other hand, the definition given by Nübling in [27] and Evans in [17], which replaces the third condition by the following one:

$$a_{i+1} \underset{a_i}{\downarrow} a_0 ... a_{i-1}$$

does not have that problem, (note that ampleness in the sense of Evans implies ampleness in the sense of Pillay).

Finally, we remark that our definition agrees with the last one:

If T is n-ample in our sense, then, by naming the parameters $\text{acl}^{\text{eq}}(a_0) \cap \text{acl}^{\text{eq}}(a_1)$, we get that T is n-ample in the sense of Evans.

On the other hand, if T is n-ample in the sense of Evans, meaning that there exists $a_0, ..., a_n$ and a set of parameters A satisfying the definition. Then the tuple $a'_0, ..., a'_n$, where $a'_i = a_i A$, satisfies our definition of ampleness.

Remark 2.2.3. If $a_0, ..., a_n$ satisfy condition (2) from Definition 2.2.1, then, by transitivity, we have $a_n, ...a_i \downarrow_{a_i} a_i, ..., a_0$.

The definition of ampleness may be extended to simple theories in general, or even to more general closure operators. The reader is referred to [31] for a detailed exposition of ampleness in these more general contexts.

Until now, the only known examples of ω-stable theories, in levels of the ample hierarchy above the second one (that do not interpret a field), are:

- The *n-dimensional free pseudospaces* (FP_n), constructed by Baudisch, Martin-Pizarro and Ziegler [4], and independently by Tent [44].
- The *right angled buildings*, studied by Baudisch, Martin-Pizarro and Ziegler in [5].

We will not explain the theory of the n-dimensional free pseudospace, but it is worth to state a few facts about the proof of the "hierarchy fitting" in [2]. Let us state the theorem first:

In order to prove that FP_n is not n+1-ample, Baudisch, Martin-Pizarro, and Ziegler [4] used the following result:

Proposition 2.2.4 (Baudisch, Martin-Pizarro, Ziegler)**.** Let T be a n-ample theory. Then there exist $a_0, ..., a_n$, such that

1. $a_n \downarrow_{a_{i+1}} a_i$
2. $\text{acl}^{eq}(a_n a_i) \cap \text{acl}^{\text{eq}}(a_{i+1} a_i) = \text{acl}^{\text{eq}}(a_i)$ for every $i < n - 1$.

3. $a_n \not\perp_{\mathrm{acl}^{eq}(a_i)\cap \mathrm{acl}^{eq}(a_{i+1})} a_i a_{i+1}$ for every $i < n$.

Their strategy to prove that FP_n is not $n+1$-ample consisted in proving that there were no $a_0, ..., a_{n+1}$ in the n-dimensional free pseudospace satisfying the conditions from the previous proposition. Hence, the full strength of ampleness was not necessary. This motivated us to look for weaker notions of ampleness.

2.3 Weak ampleness

In this section we develop "weaker" notions of ampleness. Besides the aforementioned reason, our motivation is the following:

Assume $a_0...a_n$ is an arbitrary tuple such that

$$a_0...a_{i-1} \mathop{\perp}_{a_i} a_{i+1} \text{ for all } i < n.$$

According to the definition, if this tuple is not n-ample, it may be due to the behaviour of their algebraic closures (condition (2) of Definition 2.2.1). However, for $n \leq 2$ we may define ampleness of the tuple in a different way:

If $n = 1$ we say that a_0a_1 is *1-ample* if

$$a_0 \mathop{\not\perp}_{\mathrm{acl}^{\mathrm{eq}}(a_0)\cap\mathrm{acl}^{\mathrm{eq}}(a_1)} a_1.$$

(This clearly coincides with the usual definition).

If $n = 2$ we say that $a_0a_1a_2$ is *2-ample* if

$$a_0 \mathop{\not\perp}_{\mathrm{acl}^{\mathrm{eq}}(a_0a_2)\cap\mathrm{acl}^{\mathrm{eq}}(a_1)} a_2.$$

(If we define a_0' such that $\mathrm{acl}^{\mathrm{eq}}(a_0') = \mathrm{acl}^{\mathrm{eq}}(a_0a_1) \cap \mathrm{acl}^{\mathrm{eq}}(a_0a_2)$, then this is equivalent to say that $a_0'a_1a_2$ is ample according to Definition 2.2.1).

However, for $n \geq 3$ it is unclear how to define n-ampleness of a tuple $a_0, ..., a_n$ without knowing the interaction of their algebraic closures (condition (2) of Definition 2.2.1).

Beyond the intuition about what ampleness should mean, the usual definition presents some technical difficulties that can be avoided by weakening the notion of ampleness. Moreover, we show in this section that all the examples known fit also in the weak ample hierarchy.

We introduce two new ample notions and prove that they agree for $n = 1, 2$. Moreover, they are equivalent for all n assuming SU-rank 1 and geometric elimination of imaginaries, which is a strong condition, though one should recall that the ample hierarchy was first introduced in an attempt to classify strongly minimal theories.

Definition 2.3.1. A theory T is *almost n-ample* if there exists a collection $A = \{a_0, ..., a_n\} \subset \mathfrak{C}^{eq}$ such that, possibly working over parameters, the following conditions hold:

1. $\operatorname{acl}^{eq}(S) \cap \operatorname{acl}^{eq}(A \setminus S) = \operatorname{acl}^{eq}(\emptyset)$ for every $S \subset A$.
2. $a_{i+1} \mathop{\smile\!\!\!\!\!\!|}_{a_i} a_0...a_{i-1}$ for all $i < n$.
3. $a_n \not\mathop{\smile\!\!\!\!\!\!|} a_0$.

Definition 2.3.2. A theory T is *weakly n-ample* if there exist $a_0, ..., a_n \in \mathfrak{C}^{eq}$ such that:

1. $\operatorname{acl}^{eq}(a_i a_{i+1}) \cap \operatorname{acl}^{eq}(a_i a_{i+2}) = \operatorname{acl}^{eq}(a_i)$ for all $i < n-1$.
2. $a_{i+1} \mathop{\smile\!\!\!\!\!\!|}_{a_i} a_0...a_{i-1}$ for all $i < n$.
3. $a_n \not\mathop{\smile\!\!\!\!\!\!|}_{\operatorname{acl}^{eq}(a_0) \cap \operatorname{acl}^{eq}(a_1)} a_0$.

Proposition 2.3.3. If T is n-ample then it is almost n-ample. Moreover, if a tuple is n-ample then it is almost n-ample, by naming the parameters $\operatorname{acl}^{\mathrm{eq}}(a_0) \cap \operatorname{acl}^{\mathrm{eq}}(a_1)$.

Proof. Let $A = \{a_0, ..., a_n\}$ be an n-ample tuple and work over $\operatorname{acl}^{eq}(a_1) \cap \operatorname{acl}^{eq}(a_0) = \operatorname{acl}^{\mathrm{eq}}(\emptyset)$ as parameters. In order to prove that A is an almost n-ample tuple, we need to show that $\operatorname{acl}^{eq}(S) \cap \operatorname{acl}^{eq}(A \setminus S) \subset \operatorname{acl}^{eq}(a_1) \cap \operatorname{acl}^{eq}(a_0)$, for every $S \subset A$.

Let S be any non-empty subset of A. Without loss of generality, we may assume that $a_n \in S$. Let $a_k \in A \setminus S$ be the last element in $A \setminus S$, i.e. such that $a_i \in S$ for every $i > k$. Then we have

$$\operatorname{acl}^{eq}(S) \cap \operatorname{acl}^{eq}(A \setminus S) \subseteq \operatorname{acl}^{eq}(a_n ... a_{k+1} a_{k-1}, ..., a_0) \cap \operatorname{acl}^{eq}(a_k, ..., a_0).$$

On the other hand, by Remark 2.2.3, we have

$$a_n, .., a_{k+1} \mathop{\smile\!\!\!\!\!\!|}_{a_{k+1}} a_k ... a_0,$$

hence

$$\begin{aligned} \operatorname{acl}^{eq}(a_n ... a_{k+1} a_{k-1}, ..., a_0) \cap \operatorname{acl}^{eq}(a_k, ..., a_0) &\subset \operatorname{acl}^{eq}(a_{k+1} a_{k-1}, ..., a_0) \cap \operatorname{acl}^{eq}(a_k, ..., a_0) \\ &= \operatorname{acl}^{eq}(a_{k-1}, ..., a_0). \end{aligned}$$

Now, notice that for every $X \subset A$ such that $a_{k-1} \notin X$, we have

$$Xa_{k-2}...a_0 \underset{a_k a_{k-2}...a_0}{\perp} a_{k-1}...a_0.$$

Thus, if $a_{k-1} \notin S$, then

$$\begin{aligned} \mathrm{acl}^{eq}(S) \cap \mathrm{acl}^{eq}(A \setminus S) &\subset \mathrm{acl}^{eq}(Sa_{k-2}...a_0) \cap \mathrm{acl}^{eq}(a_{k-1}, ..., a_0) \\ &\subset \mathrm{acl}^{eq}(a_k a_{k-2}...a_0) \cap \mathrm{acl}^{eq}(a_{k-1}, ..., a_0) \\ &= \mathrm{acl}^{eq}(a_{k-2}, ..., a_0). \end{aligned}$$

If not, take $a_{k-1} \in A \setminus S$ and we get the same conclusion changing S for $A \setminus S$.

Proceeding in the same way we get

$$\mathrm{acl}^{eq}(S) \cap \mathrm{acl}^{eq}(A \setminus S) \subset \mathrm{acl}^{eq}(a_0).$$

Finally, if $a_0 \notin S$ then $S \perp_{a_1} a_0$ (if not, then $A \setminus S \perp_{a_1} a_0$). Either way, we conclude that

$$\begin{aligned} \mathrm{acl}^{eq}(S) \cap \mathrm{acl}^{eq}(A \setminus S) &= \mathrm{acl}^{eq}(S) \cap \mathrm{acl}^{eq}(A \setminus S) \cap \mathrm{acl}^{eq}(a_0) \\ &\subset \mathrm{acl}^{eq}(a_1) \cap \mathrm{acl}^{eq}(a_0) \\ &= \mathrm{acl}^{\mathrm{eq}}(\emptyset) \end{aligned}$$

■

Proposition 2.3.4. If T is almost n-ample, then it is weakly n-ample.

Proof. Let $A = \{a_0, ..., a_n\} \subset \mathfrak{C}^{eq}$ such that, possibly working over parameters:

1. $\mathrm{acl}^{eq}(S) \cap \mathrm{acl}^{eq}(A \setminus S) = \mathrm{acl}^{eq}(\emptyset)$ for every $S \subset A$.
2. $a_{i+1} \perp_{a_i} a_0...a_{i-1}$ for all $i < n$.
3. $a_n \not\perp a_0$.

We will define a new tuple $a'_0, ..., a'_n$ satisfying the conditions of weakly n-ampleness:

- $a'_n = a_n$
- $a'_{n-1} = a_{n-1}$
- For $i < n-1$ define a'_i as an element such that

$$\operatorname{acl}^{eq}(a'_i) = \operatorname{acl}^{eq}(a_i a'_{i+1}) \cap \operatorname{acl}^{eq}(a_i a'_{i+2}).$$

Let us check conditions 1, 2 and 3 for weakly n-ampleness:

Condition 1: We need to prove that $\operatorname{acl}^{eq}(a'_i) = \operatorname{acl}^{eq}(a'_i a'_{i+1}) \cap \operatorname{acl}^{eq}(a'_i a'_{i+2})$. As $a'_i \subset \operatorname{acl}^{eq}(a_i a'_{i+1})$ then $\operatorname{acl}^{eq}(a'_i a'_{i+1}) \subset \operatorname{acl}^{eq}(a_i a'_{i+1})$. In the same way $\operatorname{acl}^{eq}(a'_i a'_{i+2}) \subset \operatorname{acl}^{eq}(a_i a'_{i+2})$. Hence,

$$\operatorname{acl}^{eq}(a'_i a'_{i+1}) \cap \operatorname{acl}^{eq}(a'_i a'_{i+2}) \subset \operatorname{acl}^{eq}(a_i a'_{i+1}) \cap \operatorname{acl}^{eq}(a_i a'_{i+2}) = \operatorname{acl}^{eq}(a'_i).$$

The other inclusion is clear.

Condition 2: First, we will prove that $a'_k \mathop{\smile\!\!\!\!\!\!\mid}_{a_{k-1}} a_0 ... a_{k-2}$ by induction on k:

If $k = n$ there is nothing to prove. Assume that the independence is valid for $k \geq i+1$. Let us check for $k = i$. From

$$a'_{i+1} \mathop{\smile\!\!\!\!\!\!\mid}_{a_i} a_0 ... a_{i-1} \text{ and } a_i \mathop{\smile\!\!\!\!\!\!\mid}_{a_{i-1}} a_0 ... a_{i-2},$$

we get

$$a'_{i+1} a_i \mathop{\smile\!\!\!\!\!\!\mid}_{a_{i-1}} a_0 ... a_{i-2},$$

by transitivity.

On the other hand, the inclusion $a'_i \subset \operatorname{acl}^{eq}(a_i a'_{i+1})$ implies $a'_i \mathop{\smile\!\!\!\!\!\!\mid}_{a_{i-1}} a_0 ... a_{i-2}$.

Finally, from $a'_{i+1} \mathop{\smile\!\!\!\!\!\!\mid}_{a_i} a_0 ... a_{i-1}$ it is easy to see that $a'_0 ... a'_{i-1} \mathop{\smile\!\!\!\!\!\!\mid}_{a'_i} a'_{i+1}$ because $a'_j \subset \operatorname{acl}^{eq}(a_j a'_{j+1})$ for every $j < n$.

Condition 3. Notice that, for every i, we have $a'_i \subset \operatorname{acl}^{eq}(a_k : k = i + 2l)$. In particular $a'_0 \subset \operatorname{acl}^{eq}(a_k : k \text{ even})$ and $a'_1 \subset \operatorname{acl}^{eq}(a_k : k \text{ odd})$. Using these inclusions and condition 1. of almost n-ampleness, it follows that $\operatorname{acl}^{eq}(a'_1) \cap \operatorname{acl}^{eq}(a'_0) = \operatorname{acl}^{eq}(\emptyset)$. Moreover, from condition 3. of almost n-ampleness we know that $a_0 \not\mathop{\smile\!\!\!\!\!\!\mid} a_n$. Therefore $a'_0 \not\mathop{\smile\!\!\!\!\!\!\mid} a'_n$. ∎

We do not know whether ampleness, almost ampleness and weak ampleness are different or not. However, on the first steps of the hierarchy, namely 1-basedness and CM-triviality, they coincide.

Lemma 2.3.5. (a) T is 1-based if and only if it is not weakly 1-ample.

(b) T is CM-trivial if and only if it is not weakly 2-ample.

Proof. (a) The definition of 1-ampleness and weakly 1-ampleness are exactly the same.

(b) The definition of weakly 2-ampleness coincide with the characterization (CM1) of CM-triviality (Proposition 2.1.2).

∎

Question 2.3.6. Do the notions of ampleness, almost-ampleness and weak-ampleness coincide?

Pillay proved that a stable theory interpreting a field is n-ample for all n [36]. This was generalized by Nübling to supersimple theories [27] and by Yoneda to Rosy theories of monomial thorn rank [45]. In [29], Ould Houcine and Tent proved that the theory of non abelian free groups is n-ample for all n. Clearly, all these examples are weakly n-ample for all n.

We will show that the n-dimensional free pseudospace is not weakly $(n+1)$-ample; for this, we just prove that the condition they isolated (Proposition 2.2.4) are also satisfied in weakly ample theories.

Proposition 2.3.7. Let T be a weakly n-ample theory. Then there exist $a_0, ..., a_n$, such that

1. $a_n \mathop{\smile\!\!\!\!\!\mid}_{a_{i+1}} a_i$
2. $\mathrm{acl}^{eq}(a_i a_n) \cap \mathrm{acl}^{eq}(a_i a_{i+1}) = \mathrm{acl}(a_i)$ for every $i < n-1$.
3. $a_n \not\mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}^{eq}(a_i) \cap \mathrm{acl}^{eq}(a_{i+1})} a_i a_{i+1}$ for every $i < n$.

Proof. Let $a_0 ... a_n$ be a weakly n-ample tuple. We will prove that this tuple satisfies the three conditions.

Condition 1. Follows by transitivity.

Condition 2. Notice that $a_n \mathop{\smile\!\!\!\!\!\mid}_{a_{i+2}} a_{i+1} a_i$, therefore $a_n a_i \mathop{\smile\!\!\!\!\!\mid}_{a_{i+2} a_i} a_{i+1} a_i$ and

$$\mathrm{acl}^{\mathrm{eq}}(a_i a_n) \cap \mathrm{acl}^{\mathrm{eq}}(a_i a_{i+1}) \subset \mathrm{acl}^{\mathrm{eq}}(a_i a_{i+2}) \cap \mathrm{acl}^{\mathrm{eq}}(a_i a_{i+1}).$$

On the other hand, by condition (1) of weakly n-ampleness, we have

$$\mathrm{acl}^{\mathrm{eq}}(a_i a_{i+2}) \cap \mathrm{acl}^{\mathrm{eq}}(a_i a_{i+1}) = \mathrm{acl}^{\mathrm{eq}}(a_i),$$

hence

$$\text{acl}^{\text{eq}}(a_i a_n) \cap \text{acl}^{\text{eq}}(a_i a_{i+1}) \subset \text{acl}^{\text{eq}}(a_i).$$

The other inclusion is obvious.

Condition 3. Assume the tuple does not satisfy the condition. Then, by transitivity

$$a_n \underset{acl^{eq}(a_i) \cap acl^{eq}(a_{i+1})}{\not\perp} a_0.$$

To get a contradiction it is enough to check that

$$acl^{eq}(a_i) \cap acl^{eq}(a_{i+1}) \subset acl^{eq}(a_0) \cap acl^{eq}(a_1).$$

On one hand we know

$$\text{acl}^{\text{eq}}(a_i) \cap \text{acl}^{\text{eq}}(a_{i+1}) \subset \text{acl}^{\text{eq}}(a_{i-1}a_i) \cap \text{acl}^{\text{eq}}(a_{i-1}a_{i+1}).$$

On the other hand, condition 1. in the definition of n-ample gives

$$\text{acl}^{\text{eq}}(a_{i-1}a_i) \cap \text{acl}^{\text{eq}}(a_{i-1}a_{i+1}) = \text{acl}^{\text{eq}}(a_{i-1}).$$

Hence

$$\text{acl}^{\text{eq}}(a_i) \cap \text{acl}^{\text{eq}}(a_{i+1}) \subset \text{acl}^{\text{eq}}(a_{i-1}) \cap \text{acl}^{\text{eq}}(a_i).$$

Continuing with the same procedure we get

$$\text{acl}^{\text{eq}}(a_i) \cap \text{acl}^{\text{eq}}(a_{i+1}) \subset \text{acl}^{\text{eq}}(a_0) \cap \text{acl}^{\text{eq}}(a_1),$$

which is the desired contradiction. ■

To end this section, we point out an easy generalization of a result by Yoneda ([45]) concerning geometric elimination of imaginaries.

Definition 2.3.8. T is n-ample *in the real sort* if in Definition 2.2.1, algebraic closure is taken in the real sort.

In [45] Yoneda proved that, if T is CM-trivial in the real sort (with respect to an abstract independence relation) then it has geometric elimination of imaginaries by using that a property called *Independence Intersection Property* is true for CM-trivial theories. We generalize this result to non n-ampleness directly. However, we do not know if non-n-ample theories have IIP.

Proposition 2.3.9. If T is not n-ample in the real sort, then T has geometric elimination of imaginaries.

Proof. Let $e \in \mathfrak{C}^{eq}$ and let a be a real tuple such that $a/E = e$, where E is a -definable equivalence relation. Let $A = \{a_0, ..., a_n\}$ be a set of realizations of $tp(a/e)$ independent over e. Note that for every i we have $e \in dcl^{eq}(a_i)$, hence

$$a_0...a_{i-1} \underset{a_i}{\downarrow} a_{i+1}.$$

Similarly $a_{i+1} \downarrow_{a_0...a_i} a_{i+2}$, therefore

$$\mathrm{acl}(a_i a_{i+1}) \cap \mathrm{acl}(a_i a_{i+2}) = \mathrm{acl}(a_i).$$

As T is not n-ample in the real sort, we have

$$a_n \underset{\mathrm{acl}(a_1)\cap\mathrm{acl}(a_0)}{\downarrow} a_0$$

Hence, as $a_1 \downarrow_e a_0$, we have

$$\mathrm{acl}(a_1) \cap \mathrm{acl}(a_0) \subset \mathrm{acl}(e).$$

Furthemore, as $e \in \mathrm{dcl}^{eq}(a_n) \cap \mathrm{dcl}^{\mathrm{eq}}(a_0)$, the independence $a_n \downarrow_{\mathrm{acl}(a_1)\cap\mathrm{acl}(a_0)} a_0$ gives that

$$e \in \mathrm{acl}^{\mathrm{eq}}(\mathrm{acl}(a_1) \cap \mathrm{acl}(a_0))$$

■

Chapter 3

Geometric structures with a dense independent subset

This chapter consists on two sections. In the first one we describe the work of Berenstein and Vassiliev [10] on expansions of geometric theories by a dense/codense predicate of independent elements, listing only the results which are relevant for our study of expansions of simple theories of SU-rank 1, and algebraic closure and canonical bases in such expansions. In the second section, we give a detailed exposition of our results in [15], in which we give a complete characterization of the forking geometry of the expansion, when the underlying theory is simple of SU-rank 1.

In this chapter, all theories are supersimple.

3.1 Independent predicates in geometric theories

Definition 3.1.1. A complete theory T is *geometric* if it eliminates $\exists^\infty$ and algebraic closure satisfies the exchange property in every model of T.

The exchange property gives rise to a tame notion of dimension (hence a tame notion of independence) with *generic elements*, i.e. in each type-definable set X, that is, an element x of the same dimension of X. Elimination of $\exists^\infty$ ensures that the dimension is definable. That is, for every formula $\varphi(x_1, ..., x_n, \bar{y})$ there exists a formula $\psi(\bar{y})$ such that $\models \psi(\bar{a})$ if and only if the formula $\varphi(x_1, ..., x_n, \bar{a})$ has dimension n.

All simple theories of SU-rank 1 are geometric, in particular all strongly minimal theories are geometric. Also, all Rosy theories of thorn-rank 1 are geometric, so all o-minimal structures are geometric.

Geometric theories are ubiquitous. In [23] Hrushovski and Pillay defined geometric theories and use their properties to study definable groups in local and pseudofinite fields.

Let T be a complete geometric theory in a language L and let $L_H = L \cup \{H\}$ where H is a new unary predicate. The theory T^{ind} is the L_H-theory extending T together with the axioms:

1. The set H is L-algebraically independent.
2. (**Density property**) For all L-formulas $\varphi(x, \bar{y})$,

$$\forall \bar{y}(\varphi(x, \bar{y})\text{non-algebraic} \rightarrow \exists x \in H \varphi(x, \bar{y}))$$

3. (**Extension property**) for all L-formulas $\varphi(x, \bar{y})$, for all $n \in \omega$ and for all $\psi(x, \bar{y}, \bar{z})$,

$$\forall \bar{y} \bar{z} \exists^{\leq n} x \psi(x, \bar{y}, \bar{z}) \rightarrow$$
$$\forall \bar{y}(\varphi(x, \bar{y})\text{non-algebraic} \rightarrow \exists x(\varphi(x, \bar{y}) \wedge \forall \bar{z} \in H \neg \psi(x, \bar{y}, \bar{z})))$$

In these axioms, *algebraic independence* and *non-algebraicity* are elementary way due to the elimination of $\exists^\infty$.

From now on, by acl() and $\downarrow$ we mean algebraic closure and algebraic independence in the sense of T. Also, given a model $M \models T$, we will usually denote $H(M)$ just by H.

Definition 3.1.2. An *H-structure* is a model (M, H) of T^{ind} such that:

1. (**Generalized density/coheir property**) If $A \subset M$ is finite dimensional and $q \in S_n(A)$ has dimension n, then there is $\bar{a} \in H^n$ such that $\bar{a} \models q$.
2. (**Generalized extension property**) If $A \subset M$ is finite dimensional and $q \in S_n(A)$ then there is $\bar{a} \models q$ such that $\bar{a} \downarrow_A H$.

Fact 3.1.3. If (M_1, H_1) and (M_2, H_2) are H-structures, then $(M_1, H_1) \equiv (M_2, H_2)$ and T^{ind} is the common complete theory. Moreover, any $|T|^+$-saturated model of T^{ind} is an H-structure.

If T is a simple theory of SU-rank 1, then T^{ind} is supersimple, so canonical bases exists as imaginaries. We will now briefly explain how canonical bases in T^{ind} can be described in terms of canonical bases in T, whenever T has SU-rank 1.

Definition 3.1.4. • Let (M, H) be an H-structure and $\bar{c}$ a tuple in M. We denote by $\mathrm{HB}(\bar{c})$, the *H-basis* of $\bar{c}$, the smallest tuple $\bar{h} \subset H$ such that $\bar{c} \underset{\bar{h}}{\downarrow} H$. (We will prove that H-bases always exist).

Also, for $A \subset M$, the *H-basis of* $\bar{c}$ *relative to* A, denoted by $\mathrm{HB}(\bar{c}/A)$, stands for the smallest tuple $\bar{h}_A \in H$ such that $\bar{c} \underset{\bar{h}_A A}{\downarrow} H$. (We will prove that such bases exist whenever $A = \mathrm{acl}(A)$ and $\mathrm{HB}(A) \subset A$).

We now prove of the existence of H-bases and relative H-bases, the proofs presented here are more detailed that the ones in [10].

Existence of H-bases. Let $\bar{c}$ be any tuple. We will prove that $\mathrm{HB}(\bar{c})$ exists. Let $\bar{h}$ and $\bar{h}'$ be tuples of H such that $\bar{c} \underset{\bar{h}}{\downarrow} H$ and $\bar{c} \underset{\bar{h}'}{\downarrow} H$. It suffices to prove that, if $\bar{h}'' = \bar{h} \cap \bar{h}'$, then $\bar{c} \underset{\bar{h}''}{\downarrow} H$.

We can write $\bar{c}$ as $\bar{c}_1\bar{c}_2$ where $\bar{c}_1$ is independent over H and $\bar{c}_2 \subseteq \mathrm{acl}(\bar{c}_1 H)$. By definition of $\bar{h}$ and $\bar{h}'$ we know that $\bar{c}_2 \subseteq \mathrm{acl}(\bar{c}_1\bar{h})$ and $\bar{c}_2 \subseteq \mathrm{acl}(\bar{c}_1\bar{h}')$. If $c_2 \nsubseteq \mathrm{acl}(c_1\bar{h}'')$ then, by exchange property, there is an element g in $\bar{h} \setminus \bar{h}'$ (or in $\bar{h}' \setminus \bar{h}$), such that $g \in \mathrm{acl}(\bar{c}_1\bar{h}')$. On the other hand, the tuple $\bar{c}_1$ was chosen to be independent from H so $g \in \mathrm{acl}(\bar{h}')$, which yields a contradiction, as H is an independent subset.

■

Existence of relative H-bases. Let $\bar{c}$ be any tuple and let A be an algebraic closed set such that $HB(A) \subset A$. We are going to prove that $HB(\bar{c}/A)$ exists. Again, let $\bar{h}$ and $\bar{h}'$ be minimal such that $\bar{c} \underset{\bar{h}A}{\downarrow} H$ and $\bar{c} \underset{\bar{h}'A}{\downarrow} H$. In particular we have that $\bar{h}\bar{h}' \cap A = \emptyset$.

Write $\bar{c}$ as $\bar{c}_1\bar{c}_2$ where $\bar{c}_1$ is independent over AH and $\bar{c}_2 \subseteq \mathrm{acl}(\bar{c}_1 AH)$. Then $\bar{c}_2 \subseteq \mathrm{acl}(\bar{c}_1 A\bar{h})$ and $\bar{c}_2 \subseteq \mathrm{acl}(\bar{c}_1 A\bar{h}')$. Let $\bar{h}'' = \bar{h} \cap \bar{h}'$. If $\bar{c}_2 \nsubseteq \mathrm{acl}(\bar{c}_1 A\bar{h}'')$, then, by the exchange property, there is an element $g \in \bar{h} \setminus \bar{h}'$ (or viceversa) such that $g \in \mathrm{acl}(\bar{c}_1 A\bar{h}')$.

Claim: we have that $g \notin \mathrm{acl}(A\bar{h}')$.

Proof of the claim. If not, as $g \notin \bar{h}'$, then by exchange there is an element a' and a subset A' of A such that $a' \notin \mathrm{acl}(A')$ and $a' \in \mathrm{acl}(A'g\bar{h}')$, then some (non empty) subset of $g\bar{h}'$ must be contained in $\mathrm{HB}(A)$, $\mathrm{HB}(A) \subset A$ and $\bar{h}'g \cap A \subset \bar{h}'\bar{h} \cap A = \emptyset$. Contradiction.

■

Therefore, as $g \in \operatorname{acl}(\bar{c}_1 A\bar{h}') \setminus \operatorname{acl}(A\bar{h}')$, the tuple $\bar{c}_1$ is not independent over AH. ■

Fact 3.1.5. If (M, H) is an H-structure and A is a subset of M, then

$$\operatorname{acl}(A, \operatorname{HB}(A)) = \operatorname{acl}_{\mathrm{H}}(A),$$

(where $\operatorname{acl}_{\mathrm{H}}(A)$ stands for the algebraic closure of A in the sense of T^{ind}).

Notice that Fact 3.1.5 and Existence of relative H-bases imply that H-bases always exist over H-algebraically closed sets. From now on, by $\operatorname{HB}(A/B)$ we mean $\operatorname{HB}(A/\operatorname{acl}_{\mathrm{H}}(B))$.

The next fact gives a characterization of canonical bases in T^{ind} in terms of H-bases.

Theorem 3.1.6. Assume that T is a simple theory of SU-rank 1, let (M, H) be an H-structure (sufficiently saturated) and let $\bar{a}$ be a tuple of M and $B \subset M$ $\operatorname{acl}_{\mathrm{H}}$-closed. Then the canonical base $\operatorname{cb}_H(\bar{a}/B)$ is interalgebraic (in the sense of L_H) with $\operatorname{cb}(\bar{a}\operatorname{HB}(a/B)/B)$.

In particular, if $A \subset B$ are $\operatorname{acl}_{\mathrm{H}}$-closed and $\bar{h} = \operatorname{HB}(\bar{c}/B)$, then

$$c \mathop{\smile\!\!\!\!\!\!\;|}_{A}^{H} B \text{ if and only if } \bar{c}\bar{h} \mathop{\smile\!\!\!\!\!\!\;|}_{A} B.$$

Example 3.1.7. Let V a vector space over $\mathbb{Q}$ such that $|V| > \aleph_0$ and let $H = \{h_0, h_1, ...\}$ be a countable independent subset of V. Then it is easy to check that (V, H) is an H-structure. Moreover, if t is a vector independent of H and $t_0 = t + h_0$ then $\operatorname{cb}_H(t/t_0)$ is interalgebraic with $\operatorname{cb}(th_0/t_0) = t_0$. So we have that $t \not\mathop{\smile\!\!\!\!\!\!\;|}^{H} t_0$, but $\operatorname{acl}_{\mathrm{H}}(t) \cap \operatorname{acl}_{\mathrm{H}}(t_0) = \emptyset$ hence $\operatorname{Th}(V, H)$ is not 1-based.

Frank Wagner noticed this theory is not 1-based since (V, H) is a stable group and H is a definable set, which is not a boolean combination of cosets of $\operatorname{acl}(\emptyset)$-definable subgroups (see Theorem 1.6.6). However, the above proof illustrates the failure of 1-basedness. In fact, we will see in the next section that, if T is a simple theory of SU-rank 1, then T^{ind} is 1-based if and only if T is trivial (the main ideas of the proof are a generalization of the above example).

3.2 Ampleness

From now on we will assume that T is a simple theory of SU-rank 1 with geometric elimination of imaginaries. By Theorem 3.1.6, canonical bases are interalgebraic with a tuple of elements of the home sort, hence T^{ind} has geometric elimination of imaginaries (Fact 1.5.8).

Proposition 3.2.1. The H-bases are transitive, in the sense that:

$$\mathrm{HB}(c/B) \cup \mathrm{HB}(B) = \mathrm{HB}(cB).$$

Proof. From $cB \downarrow_{\mathrm{HB}(cB)} H$ we have that $c \downarrow_{B\,\mathrm{HB}(cB)} H$ and $B \downarrow_{\mathrm{HB}(cB)} H$. Therefore $\mathrm{HB}(c/B) \subset \mathrm{HB}(cB)$ and $\mathrm{HB}(B) \subset \mathrm{HB}(cB)$. Hence

$$\mathrm{HB}(c/B) \cup \mathrm{HB}(B) \subset \mathrm{HB}(cB).$$

On the other hand, by definition, we have $B \underset{\mathrm{HB}(B)}{\downarrow} H$, which implies that

$$B\,\mathrm{HB}(c/B) \underset{\mathrm{HB}(c/B)\,\mathrm{HB}(B)}{\downarrow} H, \tag{3.1}$$

because $\mathrm{HB}(c/B) \subset H$.

Also, by definition

$$c \underset{B\,\mathrm{HB}(c/B)}{\downarrow} H. \tag{3.2}$$

Finally, by transitivity, conditions (3.1) and (3.2) imply

$$cB \underset{\mathrm{HB}(c/B)\,\mathrm{HB}(B)}{\downarrow} H,$$

which proves the inclusion $\mathrm{HB}(cB) \subset \mathrm{HB}(c/B) \cup \mathrm{HB}(B)$. ■

We will characterize ampleness of T^{ind} in terms of the geometry of T. We have seen that non 1-ampleness (1-basedness) is not preserved in T^{ind} (Example 3.1.7). Now, we exhibit a necessary and sufficient condition under which 1-basedness is preserved.

Definition 3.2.2. A geometric theory T is *trivial* if $\mathrm{acl}(A) = \bigcup_{a\in A} \mathrm{acl}(a)$ for every $A \subset \mathfrak{C}$.

Clearly, if T is trivial of SU-rank 1, then it is 1-based.

Lemma 3.2.3. If T is trivial, then $\mathrm{acl}(A) = \mathrm{acl}_{\mathrm{H}}(A)$ for every A.

Proof. By Theorem 3.1.5, we need only prove that $\mathrm{HB}(A) \subset \mathrm{acl}(A)$.

Consider $\bar{h} = \text{acl}(A) \cap H$, it suffices to show that $\text{HB}(A) \subset \bar{h}$. Recall that

$$A \underset{\text{acl}(A)\cap\text{acl}(H)}{\downarrow} H$$

by the previous remark. If $x \in (\text{acl}(A) \cap \text{acl}(H)) \setminus \text{acl}(\emptyset)$, then by triviality $x \in \text{acl}(h')$ for some $h' \in H \cap \text{acl}(A) = \bar{h}$. Hence $\text{acl}(A) \cap \text{acl}(H) \subset \text{acl}(\bar{h})$ and $A \downarrow_{\bar{h}} H$. This shows that $\text{HB}(A) \subset \bar{h}$. ∎

Lemma 3.2.4. Assume T is trivial of SU-rank 1. Then, for any sets A and $B = \text{acl}(B)$, we have

$$\text{HB}(A/B) \subset \text{acl}(A).$$

Proof. By the previous lemma and Proposition 3.2.1, we have

$$\text{HB}(A/B) \subset \text{acl}_H(AB) \setminus B = \text{acl}(AB) \setminus B \subset \text{acl}(A).$$

∎

Proposition 3.2.5. A simple theory T of SU-rank 1 is trivial if and only if T^{ind} is 1-based.

Proof. Assume T is trivial. Consider $\bar{a}$ and B with $B = \text{acl}_\text{H}(B)$. We need to show that $cb_H(\bar{a}/B) \subset \text{acl}_\text{H}(\bar{a})$. First, let us note that

$$\bar{h} = \text{HB}(\bar{a}/B) \subset \text{acl}(\bar{a}),$$

hence

$$\begin{aligned}
\text{acl}_H(\text{cb}_H(\bar{a}/B)) &= \text{acl}_H(\text{cb}(\bar{a}\bar{h}/B)) \text{ (By Fact 3.1.6)} \\
&\subset \text{acl}_H(\text{acl}(\bar{a}\bar{h})) \text{ (By 1-basedness of } T) \\
&= \text{acl}_H(\bar{a}) \text{ (Because } \bar{h} \subset \text{acl}(\bar{a})).
\end{aligned}$$

Suppose now that T^{ind} is 1-based and assume that T is not trivial, then there are a tuple $\bar{a}$ and elements b and h such that $b \in \text{acl}(\bar{a}h)$ and $b \notin \text{acl}(\bar{a}) \cup \text{acl}(h)$. Take $\bar{a}$ a tuple which is minimal with this property, so the elements in $\bar{a}$ are algebraically independent. By the *generalized extension property*, we may assume $\bar{a} \downarrow H$. Moreover, as $tp(h/\bar{a})$ is not algebraic, we may assume that h belongs to H, by the *generalized density property*.

It is clear that $h = \mathrm{HB}(b/\bar{a})$ (since $b \mathop{\smile\!\!\!\!\!\mid}_{h\bar{a}} H$, we have that $b \not\mathop{\smile\!\!\!\!\!\mid}_{\bar{a}} H$ and h is a single element), hence, by Theorem 3.1.6, we have that $\mathrm{cb}_H(b/\bar{a})$ is interalgebraic (in T^{ind}) with $\mathrm{cb}(bh/\bar{a})$. Now, the theory T^{ind} is 1-based, hence $\mathrm{acl_H}(\mathrm{cb}_H(b/\bar{a})) = \mathrm{acl_H}(b) \cap \mathrm{acl_H}(\bar{a})$.

As $\bar{a} \mathop{\smile\!\!\!\!\!\mid} H$ we have $\mathrm{HB}(\bar{a}) = \emptyset$. Hence $\mathrm{acl_H}(\bar{a}) = \mathrm{acl}(\bar{a})$.

Moreover $\bar{a} \mathop{\smile\!\!\!\!\!\mid} H$ implies $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_h H$. As $b \in \mathrm{acl}(\bar{a}h)$ we have that $b \mathop{\smile\!\!\!\!\!\mid}_h H$; but $b \notin \mathrm{acl}(h)$, hence $b \mathop{\smile\!\!\!\!\!\mid} h$ (recall that b is a single element) and by transitivity $b \mathop{\smile\!\!\!\!\!\mid} H$. Therefore $\mathrm{HB}(b) = \emptyset$ and $\mathrm{acl_H}(b) = \mathrm{acl}(b)$.

However, minimality of $\bar{a}$ yields $\mathrm{acl}(\mathrm{cb}(bh/\bar{a})) = \mathrm{acl}(\bar{a})$, hence $\mathrm{acl}(\bar{a}) \subset \mathrm{acl}(b)$ and $h \in \mathrm{acl}(\bar{a}b) \subset \mathrm{acl}(b)$. This is a contradiction. ■

We have characterized non 1-ampleness of T^{ind} in terms of the underlying geometry of T. We will now characterize non-n-ampleness for $n \geq 2$. First we need the following lemma.

Lemma 3.2.6. If $A \subset B$ and $\mathrm{acl_H}(cA) \cap \mathrm{acl_H}(B) = \mathrm{acl_H}(A)$ then

$$\mathrm{HB}(c/A) \subset \mathrm{HB}(c/B).$$

Proof. It is clear that $\mathrm{HB}(cA) \subseteq \mathrm{HB}(cB)$. By Proposition 3.2.1, this implies

$$HB(c/A) \cup HB(A) \subseteq HB(c/B) \cup HB(B).$$

In particular

$$HB(c/A) \subset HB(c/B) \cup HB(B).$$

It remains to show that $HB(c/A) \cap H(B) = \emptyset$. This follows from:

$$\begin{aligned} \mathrm{HB}(c/A) \cap \mathrm{HB}(B) &\subset \mathrm{acl_H}(cA) \cap \mathrm{acl_H}(B) \\ &= \mathrm{acl}_H(A) \end{aligned}$$

and $HB(c/A) \cap \mathrm{acl_H}(A) = \emptyset$. ■

Theorem 3.2.7. For every $n \geq 2$. A simple theory T of SU-rank 1 is n-ample if and only if T^{ind} is n-ample.

Proof. ($\Rightarrow$) Assume T is n-ample, then there are tuples $a_0, ..., a_n$ such that:

1. $\mathrm{acl}^{eq}(a_0...a_i) \cap \mathrm{acl}^{eq}(a_0...a_{i-1}a_{i+1}) = \mathrm{acl}^{eq}(a_0...a_{i-1})$ for all $i < n$.

2. $a_{i+1} \mathop{\smile\!\!\!\!\!\!|}_{a_i} a_0...a_{i-1}$ for all $i < n$.

3. $a_n \not\mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}^{eq}(a_0)\cap\mathrm{acl}^{eq}(a_1)} a_0$.

By the *generalized extension property* we may assume that $a_0...a_n \mathop{\smile\!\!\!\!\!\!|} H$.

As the H-bases of any subset of $\{a_0, ..., a_n\}$ are empty, the algebraic closure in T^{ind} of any of these sets is the same as in T. So condition (2) holds in T^{ind}.

Since all the corresponding H-bases are empty, by the characterization of algebraic closure in T^{ind} and geometric elimination of imaginaries [Theorem 3.1.6] condition (1) of ampleness also holds also in T^{ind}.

Note also that, in particular, $\mathrm{acl}(a_1) = \mathrm{acl_H}(a_1)$ and $\mathrm{acl}(a_0) = \mathrm{acl_H}(a_0)$. Therefore, if

$$a_n \mathop{\smile\!\!\!\!\!\!|}^{H}_{\mathrm{acl}_H(a_1)\cap\mathrm{acl}_H(a_0)} a_0,$$

then

$$a_n \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}(a_1)\cap\mathrm{acl}(a_0)} a_0,$$

which is a contradiction.

($\Leftarrow$)Assume T is not n-ample, and let $a_0...a_n$ be such that for all $1 \leq i \leq n-1$,

1. $a_{i+1} \mathop{\smile\!\!\!\!\!\!|}^{H}_{a_i} a_0...a_{i-1}$,

2. $\mathrm{acl_H}(a_0...a_{i-1}a_{i+1}) \cap \mathrm{acl_H}(a_0...a_{i-1}a_i) = \mathrm{acl_H}(a_0...a_{i-1})$.

We may assume that $\mathrm{acl}(a_i) = \mathrm{acl_H}(a_i)$ for every $i \leq n$.

Claim 1. In these conditions we have the following chain:

$$\mathrm{HB}(a_n/a_0) \subset \mathrm{HB}(a_n/a_0a_1) \subset ... \subset \mathrm{HB}(a_n/a_0...a_{n-1}).$$

Proof of Claim 1. By (1) and transitivity, we have

$$a_n \mathop{\smile\!\!\!\!\!\!|}^{H}_{a_{i+1}} a_i...a_0,$$

and thus

$$a_n \mathop{\smile\!\!\!\!\!\!|}^{H}_{a_0...a_{i-1}a_{i+1}} a_i...a_0,$$

therefore

$$\mathrm{acl}_H(a_0...a_{i-1}a_n) \cap \mathrm{acl}_H(a_0...a_i) \subset \mathrm{acl}_H(a_0...a_{i-1}a_{i+1}),$$

hence, by (2),

$$\mathrm{acl}_H(a_0...a_{i-1}a_n) \cap \mathrm{acl}_H(a_0...a_{i-1}a_i) = \mathrm{acl}_H(a_0...a_{i-1}).$$

The conclusion follows from Lemma 3.2.6 by making $A = a_0...a_{i-1}$, $B = a_i...a_0$ and $c = a_{i+1}$. Note that this makes sense only if $n \geq 2$. ■

In order to conclude $a_0 \mathop{\downarrow}^{H}_{\mathrm{acl}(a_0)\cap\mathrm{acl}(a_1)} a_n$, we cannot use that T is non n-ample because intersections may not satisfy condition (2) (more precisely, the set $\mathrm{acl}(a_0...a_{i-1}a_{i+1}) \cap \mathrm{acl}(a_i...a_0)$ is probably larger than $\mathrm{acl}(a_0...a_{i-1})$), thus we need first to enlarge the tuples in order to fullfill the condition of intersections while preserving condition 2. of the independences.

Set $h = \mathrm{HB}(a_n/a_0)$ and $h' = \mathrm{HB}(a_n/a_0...a_{n-1})$. Hence $h \subset h'$ by the previous claim. As the canonical base $\mathrm{cb}_H(a_n/\mathrm{acl}_\mathrm{H}(a_0...a_{n-1}))$ is interalgebraic (in T^{ind}) with $\mathrm{cb}(a_n h'/\mathrm{acl}_\mathrm{H}(a_0...a_{n-1}))$, we have

$$a_n h \mathop{\downarrow}_{a_{n-1}} \mathrm{acl}_H(a_{n-1}a_{n-2}...a_0).$$

Define recursively tuples a'_i, b_i for $0 \leq i \leq n-1$ in the following way:

For the case $i = 0$ let $a'_0 = \emptyset$ and $b_0 = a_0$.

For $i > 0$ let $a'_i \subset \mathrm{acl}_\mathrm{H}(a_i, b_{i-1}...b_0)$ be a maximal tuple independent of $\mathrm{acl}(a_i b_{i-1}...b_0)$ (in the sense of T), and $b_i = \mathrm{acl}(a_i a'_i)$.

Claim 2 We have that $\mathrm{acl}(b_0...b_k) = \mathrm{acl}_H(a_0...a_k)$.

Proof of Claim 2. By induction on k:

It is clear for $k = 0$.

Assume that the equality holds for $k = i$, i.e.

$$\mathrm{acl}(b_0...b_i) = \mathrm{acl}_\mathrm{H}(a_0...a_i).$$

Now, by definition we have

$$b_{i+1} \subset \mathrm{acl}_\mathrm{H}(a_{i+1}, b_i, ...b_0) = \mathrm{acl}_\mathrm{H}(a_{i+1}...a_0).$$

So $\operatorname{acl}(b_{i+1}...b_0) \subset \operatorname{acl}_H(a_{i+1}...a_0)$. On the other hand, the tuple a'_{i+1} is maximal independent of $\operatorname{acl}(a_i b_{i-1}...b_0)$, so, if $c \in \operatorname{acl_H}(a_{i+1}...a_0)$ then $c \in \operatorname{acl}(a'_{i+1} \operatorname{acl}(a_{i+1} b_i...b_0)) = \operatorname{acl}(b_{i+1}...b_0)$. ∎

Finally, we define b_n as $a_n h$.

Claim 3. For $i \leq n-1$ we have $b_i \underset{b_{i-1}}{\perp} b_0...b_{i-2}$.

Proof of Claim 3. By definition $a'_i \perp b_0...b_{i-1} a_i$, hence

$$a'_i \underset{a_i}{\perp} b_0...b_{i-1}.$$

On the other hand, by non n-ampleness of the tuple $(a_i)_{i \leq n}$ in T^{ind}, we have

$$a_i \underset{a_{i-1}}{\perp^H} a_0...a_{i-1},$$

therefore, as $\operatorname{acl}(b_0...b_{i-1}) = \operatorname{acl}_H(a_0...a_{i-1})$, and

$$a_{i-1} \subset b_{i-1} \subset \operatorname{acl_H}(a_0...a_{i-1}),$$

we have

$$a_i \underset{b_{i-1}}{\perp^H} b_0...b_{i-1}.$$

The last independence also holds in T and recall that $b_i = \operatorname{acl}(a_i a'_i)$. Hence, by transitivity,

$$b_i \underset{b_{i-1}}{\perp} b_{i-2}...b_0 \text{ for } i \leq n-1.$$

Note also that

$$b_n \underset{b_{n-1}}{\perp} b_0...b_{n-2}$$

by definition of h' and the characterization of canonical bases in T^{ind}. ∎

Claim 4. For $i \leq n-1$ we have

$$\operatorname{acl}(b_0...b_{i-1} b_{i+1}) \cap \operatorname{acl}(b_0...b_{i-1} b_i) = \operatorname{acl}(b_0...b_{i-1}).$$

Proof of Claim 4. For every $i \leq n$,

$$\operatorname{acl}(b_0...b_i) = \operatorname{acl}_H(a_0...a_i),$$

then by 2. in the definition of $a_0...a_n$, we have

$$\operatorname{acl}(b_0...b_{i-1}a_{i+1}) \cap \operatorname{acl}(b_0...b_{i-1}b_i) = \operatorname{acl}(b_0...b_{i-1}).$$

On the other hand, by definition of a_i' we have

$$b_0...b_i \underset{a_{i+1}}{\perp} a_{i+1}',$$

then

$$b_0...b_i \underset{b_0...b_{i-1}a_{i+1}}{\perp} a_{i+1}',$$

and

$$b_0...b_i \underset{b_0...b_{i-1}a_{i+1}}{\perp} b_{i+1},$$

since $b_{i+1} \subset \operatorname{acl}(a_{i+1}a_{i+1}')$. This implies that

$$\operatorname{acl}(b_0...b_{i-1}a_{i+1}) \cap \operatorname{acl}(b_0...b_i) = \operatorname{acl}(b_0...b_{i-1}b_{i+1}) \cap \operatorname{acl}(b_0...b_{i-1}b_i).$$

Finally, for $i = n-1$, notice that $b_n = a_n h \subset \operatorname{acl}_H(a_n a_0)$. Therefore

$$\begin{aligned} \operatorname{acl}(b_0...b_{n-1}) \cap \operatorname{acl}(b_0...b_{n-2}b_n) &\subset \operatorname{acl}_{\mathrm{H}}(a_0...a_{n-1}) \cap \operatorname{acl}_{\mathrm{H}}(a_0...a_{n-2}a_n) \\ &= \operatorname{acl}_{\mathrm{H}}(a_0...a_{n-2}) \\ &= \operatorname{acl}(b_0...b_{n-2}) \end{aligned}$$

■

Recall that $a_0a_1 \perp a_1'$, then $a_0 \perp_{a_1} a_1'$ and $a_0 \perp_{a_1} b_1$. In particular $\operatorname{acl}(a_0) \cap \operatorname{acl}(b_1) \subset \operatorname{acl}(a_1)$. Since $\operatorname{acl}(a_0) = \operatorname{acl}(b_0)$ and $\operatorname{acl}(a_1) \subset \operatorname{acl}(b_1)$ we have the following equality:

$$\operatorname{acl}(b_0) \cap \operatorname{acl}(b_1) = \operatorname{acl}(a_0) \cap \operatorname{acl}(a_1).$$

Claims 3 and 4 together with non n-ampleness of T imply that

$$b_0 \underset{\operatorname{acl}(b_0)\cap\operatorname{acl}(b_1)}{\downarrow} b_n,$$

thus

$$a_n h \underset{\operatorname{acl}(a_0)\cap\operatorname{acl}(a_1)}{\downarrow} a_0$$

.

Hence, again by definition of h and characterization of canonical bases we conclude that

$$a_0 \underset{\operatorname{acl}(a_0)\cap\operatorname{acl}(a_1)}{\downarrow^H} a_n,$$

which is the desired conclusion. ■

Question 3.2.8. Is non-ampleness preserved without assuming geometric elimination of imaginaries?

Chapter 4

Structure of SU-rank ω with a dense independent subset of generics

This chapter contains joint work with Berenstein and Vassiliev [11] on structures of SU-rank ω with a new dense independent predicate. In the first section, we exhibit the basic properties of these constructions. In the second section, we study ampleness in the expansion.

A word on attributions: the results from the first section were developed by Berenstein and Vassiliev, so we will refer to them without proofs, while the second section is the author's contribution to [11], which consists of a characterization of 1-basedness and the preservation of CM-triviality.

Finally, the last results when the closure is trivial, as well as preservation of weak-ampleness, were developed independently by the author and do not appear in [11].

4.1 Structure of SU-rank ω with a dense independent subset of generics

We aim to find anallogues of extensions of geometric theories to theories of SU-rank ω, in order to understand dense/codense expansions by an independent subset. For this, one needs a natural pregeometry as well as an analogue of elimination of $\exists^\infty$.

From now on, the theory T will denote a simple theory of SU-rank ω.

First of all, consider the closure operator cl where $\text{cl}(A) = \{a \in \mathfrak{C} : SU(a/A) < \omega\}$. We need to check that cl induces a pregeometry. The only non-trivial part is to see that cl is transitive, i.e. $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ for every A: assume $S(a/\bar{b}) < \omega$ where $SU(b_i/A) < \omega$ for every $b_i \in \bar{b}$. We need to check that $SU(a/A) < \omega$.

If not, we have $SU(a/A) = \omega$, hence, by Lascar's inequalities we have

$$\omega \leq SU(\bar{b}/aA) + SU(a/A) \leq SU(a\bar{b}/A) \leq SU(a/\bar{b}A) \oplus SU(\bar{b}/A) < \omega.$$

This is a contradiction.

We say that T eliminates $\exists^{large}$ if *dimension* is definable. This means, for every formula

$$\varphi(x_1 ... x_n; \bar{y})$$

there is a formula $\psi(\bar{y})$ such that there is a type of SU-rank ωn containing the formula $\varphi(x_1 ... x_n; \bar{a})$ if and only if $\psi(\bar{a})$ holds. This is a clear analogue for elimination of $\exists^{\infty}$.

Let us remark that one can do the same analogy to theories of monomial SU-rank and all results will work in this general context. Moreover, one can also study pregeometries associated to other regular types. However, for the exposition, we will focus on the case of SU-rank ω.

Let H be a new unary predicate and $L_H = L \cup \{H\}$. Let T' be the L_H-theory of all structures (M, H), where $M \models T$ and H is an independent subset of generic elements of M.

Notation 1. Let $(M, H) \models T'$ and let $A \subset M$. We write $H(A)$ for $H \cap A$.

Notation 2. As in the previous section, *independence* means independence in the sense of T and we use the symbol $\downarrow$. We write $\mathrm{tp}(\bar{a})$ for the L-type of a and dcl, acl for the definable closure and the algebraic closure in the language L. Similarly we write $\mathrm{dcl}_H, \mathrm{acl}_H, \mathrm{tp}_H$ for the definable closure, the algebraic closure and the type in the language L_H.

Definition 4.1.1. We say that (M, H) is an H-*structure* if

1. $(M, H) \models T'$.
2. (**Generalized density/coheir property**) If $A \subset M$ is finite dimensional and $q \in S_n(A)$ has SU-rank ωn, then there is $\bar{a} \in H^n$ such that $\bar{a} \models q$.
3. (**Generalized extension property**) If $A \subset M$ is finite dimensional and $q \in S_n(A)$, then there is $\bar{a} \in M^n$ realizing q such that $\mathrm{tp}(\bar{a}/A \cup H)$ does not fork over A.

Definition 4.1.2. Let A be a subset of an H-structure (M, H). We say that A is *H-independent* if A is independent from H over $H(A)$.

Fact 4.1.3. All H-structures are elementarily equivalent. We write T^{ind} for their common complete theory.

Definition 4.1.4. We say that T^{ind} is *axiomatizable* or *first order* if the $|T|^+$-saturated models of T^{ind} are again H-structures.

With elimination of $\exists^{large}$, we can axiomatize properties 1. and 2. of H-structures. For condition 3. we need, in addition, another technical condition.

Definition 4.1.5. Given $\psi(\bar{y}, \bar{z})$ and $\varphi(\bar{x}, \bar{y})$ be L-formulas, the predicate $Q_{\varphi,\psi}$ holds for a tuple $\bar{c}$ (in M) if for all $\bar{b}$ satisfying $\psi(\bar{y}, \bar{c})$, the formula $\varphi(\bar{x}, \bar{b})$ does not divide over $\bar{c}$.

Proposition 4.1.6 ([11])**.** Let T be a theory of SU-rank ω which eliminates $\exists^{large}$, then T^{ind} is axiomatizable if and only if the predicates $Q_{\varphi,\psi}$ are L-type-definable for all L-formulas $\varphi(\bar{x}, \bar{y})$ and $\psi(\bar{y}, \bar{z})$.

Example 4.1.7. Now we give a list of examples of simple theories T of SU-rank ω such that T^{ind} is first order. (see [11] for detailed explanations of these examples).

1. **Differentially closed fields of characteristic 0**.
2. **Vector spaces with a generic automorphism**.
3. **Theories of Morley rank omega with definable Morley rank**.
4. ***H*-structures**: Let T be a supersimple theory of SU-rank 1. If its pregeometry is not trivial, then T^{ind} (in the sense of the previous section) has SU-rank ω and $(T^{ind})^{ind}$ is first order (see [10]).

We will use the following example in the next section, so it requires a little explanation.

5. **Free pseudoplane or infinite branching tree**: Let T be the theory of the free pseudoplane, that is, a graph without cycles such that every vertex has infinitely many edges. The theory of the free pseudoplane is stable of U-rank ω. For every A we have $\mathrm{acl}(A) = \mathrm{dcl}(A) = A \cup \{x|$ there are points $a, b \in A$ and a path connecting them passing through $x\}$. For A an algebraically closed subset and a a single element, we have that $U(a/A) = d(a, A)$ where $d(a, A)$ is the minimum length of a path from a to an element of A or ω if there is no path; in this last case we say that a is at *infinite distance to* A or that a is *not connected to* A.

Note that there is a unique generic type over A, namely the type of an element which is not connected to A. The generic type is definable over $\emptyset$ and thus, by definability of types, we have that T eliminates $\exists^{large}$.

An H-structure (M, H) associated to T is an infinite forest with an infinite collection H of selected points lying at infinite distance one from the other, with infinite many trees not connected to them. If $(N, H) \models Th(M, H)$, then N has infinitely many selected points $H(N)$ at infinite distance one from the other.

If (N, H) is $\aleph_0$-saturated, then, by saturation, it has infinitely many trees which are not connected to the points $H(N)$. In this case, the model (N, H) is an H-structure. The density property is clear. Let $A \subset N$ be finite and assume that $A = \mathrm{dcl}(A)$ and let $c \in N$. If $U(c/A) = 0$, there is nothing to prove. If $U(c/A) = n > 0$, let a be the nearest point from A to c. Since there is at most one point of H connected to a and the trees are infinitely branching, we can choose a point b with $d(b, a) = n$ and such that $d(b, A \cup H) = n$. If $U(c/A) = \omega$, choose a point b in a tree not connected to $A \cup H$, then $\mathrm{tp}(c/A) = \mathrm{tp}(b/A)$ and $b \mathop{\smile\!\!\!\!|}_A H$; then $\mathrm{tp}(c/A) = \mathrm{tp}(b/A)$ and $b \mathop{\smile\!\!\!\!|}_A H$. This proves that (N, H) is an H-structure, so T^{ind} is first order.

As in the expansions of geometric theories, we have existence of H-bases and a good characterization of canonical bases in T^{ind} in terms of canonical bases of the original theory T.

Definition 4.1.8. Let (M, H) be an H-structure and $\bar{c}$ a tuple in M. We denote by $\mathrm{HB}(\bar{c})$, the *H-base* of $\bar{c}$, the smallest tuple $\bar{h} \subset H$ such that $\bar{c} \mathop{\smile\!\!\!\!|}_{\bar{h}} H$.

Also, for $A \subset M$ such that $A \mathop{\smile\!\!\!\!|}_{A \cap H} H$, the *H-basis of* $\bar{c}$ *relative to* A, denoted by $\mathrm{HB}(\bar{c}/A)$, stands for the smallest tuple $\overline{h_A} \in H$ such that $\bar{c} \mathop{\smile\!\!\!\!|}_{\overline{h_A} A} H$.

Fact 4.1.9. Let (M, H) an H-structure, then for every $\bar{c}$, the basis $\mathrm{HB}(\bar{c})$ exists. Moreover, if A is a subset of M such that $A = \mathrm{acl}(A)$ and $\mathrm{HB}(A) \subset A$, then $\mathrm{HB}(\bar{c}/A)$ exists.

Fact 4.1.10. If (M, H) is an H-structure and A is a subset of M then $\mathrm{acl}(A, \mathrm{HB}(A)) = \mathrm{acl_H}(A)$ (where $\mathrm{acl_H}(A)$ stands for the algebraic closure of A in the sense of T^{ind}).

The two previous facts imply that H-bases always exist over H-algebraically closed sets. From now on, by $\mathrm{HB}(A/B)$ we mean $\mathrm{HB}(A/\mathrm{acl_H}(B))$.

As in the case of geometric theories, there is a characterization of canonical bases.

Fact 4.1.11. Let T be a simple theory of *SU*-rank ω, and let (M, H) a sufficiently saturated H-structure. Given a a tuple of M and $B \subset M$ $\mathrm{acl_H}$-closed, the canonical base $\mathrm{cb}_H(a/B)$ is interalgebraic (in the sense of L_H) with $\mathrm{cb}(a\, \mathrm{HB}(a/B)/B)$.

4.2 Ampleness

In this section we examine the relation between the ampleness of T and T^{ind}. Using the ideas of Chapter 3 we show that CM-triviality is preserved and that non-n-ampleness is also preserved for $n > 2$ assuming cl is trivial. Finally, we prove that non-weak-n-ampleness is always preserved for every $n \geq 2$.

Remark 4.2.1. If T has geometric elimination of imaginaries, by Fact 4.1.11, canonical bases in T^{ind} are interalgebraic with real tuples. Thus T^{ind} has geometric elimination of imaginaries.

From now on, we assume that T has geometric elimination of imaginaries.

Example 4.1. *Let G be an 1-based stable group of U-rank ω and $T = Th(G)$. Notice that T^{ind} is again a stable theory, so (M, H) is a stable group. However, the set H is not a Boolean combination of cosets of subgroups, therefore T^{ind} is not 1-based.*

Notice that if G is a group of SU-rank ω, then cl is not trivial (take $a \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}} b$ both of rank ω and $c = a + b$, then $c \in \mathrm{cl}(a, b) \setminus \mathrm{cl}(a) \cup \mathrm{cl}(b)$).

Remark 4.2.2. In the theory of the free pseudoplane (see item 5 of Example 4.1.7.) the pregeometry generated by cl is trivial: if A is algebraically closed and b is a single element, then $U(b/A) = d(b, A)$, where $d(b, A)$ is the minimum length of a path from b to an element of A (or ω if there is no path). If $b \in \mathrm{cl}(A)$ then that there is a path to some element $a \in A$, hence $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(a)$.

We will now prove that 1-basedness is only preserved in T^{ind} when the pregeometry cl is trivial. It is worth noticing that, unlike the U-rank 1 case, the triviality of cl does not imply that T is 1-based, as the free pseudoplane shows.

Lemma 4.2.3. If cl is trivial in T, then for every $\bar{a}$ and for every $B = \mathrm{acl}_{\mathrm{H}}(B)$, we have

$$\mathrm{HB}(\bar{a}/B) \subset \mathrm{HB}(\bar{a}).$$

Proof. Let $h = \mathrm{HB}(\bar{a}/B) = \{h_i | i \in I\}$. By minimality of H-bases, for every $i \in I$, we have that $\bar{a} \not\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_{Bh \setminus h_i} h_i$, then $h_i \in \mathrm{cl}(\bar{a}Bh \setminus h_i)$.

As B is H-independent and $h_i \notin B$, we have that $h_i \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}} Bh \setminus h_i$, hence $h_i \notin \mathrm{cl}(Bh \setminus h_i)$.

On the other hand, triviality implies that $h_i \in \mathrm{cl}(a_i)$ for some $a_i \in \bar{a}$. By the *exchange property* we have $a_i \in \mathrm{cl}(h_i)$. This implies that $a_i \not\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}} h_i$ and $a_i \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_{h_i} H$ (because H is orthogonal to all the types of rank strictly less than ω). Therefore, we conclude that $h_i = \mathrm{HB}(a_i)$ and

$$\mathrm{HB}(\bar{a}/B) = \{h_i | i \in I\} = \bigcup_{i \in I} \mathrm{HB}(a_i) \subset \mathrm{HB}(\bar{a}).$$

■

Proposition 4.2.4. If T is 1-based, then T^{ind} is 1-based if and only if cl is trivial in T.

Proof. (⇐) Assume cl is trivial. Let $\bar{a}$ be a tuple and B an algebraic closed set in (M, H). Take $\bar{h} = \mathrm{HB}(\bar{a}/B)$. By the characterization of canonical bases, we have $\mathrm{acl_H}(cb_H(\bar{a}/B)) = \mathrm{acl_H}(\mathrm{cb}(\bar{a}\bar{h}/B))$. As T is 1-based, we have that $\mathrm{cb}(\bar{a}\bar{h}/B) \subset \mathrm{acl}(\bar{a}\bar{h})$. By the previous lemma we have $\bar{h} \subset \mathrm{HB}(\bar{a})$, then $\mathrm{cb}_H(\bar{a}/B) \subset \mathrm{acl_H}(\bar{a}\,\mathrm{HB}(\bar{a})) = \mathrm{acl_H}(\bar{a})$, i.e. the theory T^{ind} is 1-based.

(⇒) Assume T^{ind} is 1-based and cl is not trivial, then there are elements b and h and a tuple $\bar{a}$, such that $b \in \mathrm{cl}(\bar{a}h)$ but $b \notin \mathrm{cl}(\bar{a}) \cup \mathrm{cl}(h)$. We can take $\bar{a}$ cl-independent and minimal with this property. By the *Generalized Extension Property*, we may assume that $\bar{a} \mathop{\smile\!\!\!\!\!\mid}\, H$. Moreover, as $h \notin \mathrm{cl}(\bar{a})$, we may also assume that h belongs to H by the *Generalized Density Property.*

As $b \in \mathrm{cl}(\bar{a}h)$ and $\bar{a}h$ is H-independent, we have that $tp(b/\bar{a}h)$ is orthogonal to H, then $b \mathop{\smile\!\!\!\!\!\mid}_{h\bar{a}} H$. Now, recall that $b \not\mathop{\smile\!\!\!\!\!\mid}_{\bar{a}} h$ and h is a single element, then $h = \mathrm{HB}(b/\bar{a})$.

By hypothesis, the theory T^{ind} is 1-based, then $\mathrm{acl_H}(\mathrm{cb}_H(b/\bar{a})) = \mathrm{acl_H}(b) \cap \mathrm{acl_H}(\bar{a})$.

Also, we have $\mathrm{acl_H}(\bar{a}) = \mathrm{acl}(\bar{a})$, because $\bar{a} \mathop{\smile\!\!\!\!\!\mid}\, H$. On the other hand, from $\bar{a} \mathop{\smile\!\!\!\!\!\mid}\, H$ we get $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_h H$ and, from $b \mathop{\smile\!\!\!\!\!\mid}_{h\bar{a}} H$ we conclude that $b \mathop{\smile\!\!\!\!\!\mid}_h H$.

By hypothesis $b \notin \mathrm{cl}(h)$, so $b \mathop{\smile\!\!\!\!\!\mid}\, h$ (recall that b is a single element). Transitivity yields that $b \mathop{\smile\!\!\!\!\!\mid}\, H$. Thus, the H-basis $\mathrm{HB}(b) = \emptyset$ and $\mathrm{acl_H}(b) = \mathrm{acl}(b)$. This means $\mathrm{acl_H}(\mathrm{cb}_H(b/\bar{a})) = \mathrm{acl}(b) \cap \mathrm{acl}(\bar{a})$.

Recall that $\mathrm{acl_H}(\mathrm{cb}_H(b/\bar{a})) = \mathrm{acl_H}(\mathrm{cb}(bh/\bar{a}))$. Hence, a maximal cl-independent subset $\bar{d}$ of $\mathrm{cb}(bh/\bar{a})$ satisfies that $b \in \mathrm{cl}(\bar{d}h)$ and $b \notin \mathrm{cl}(\bar{d}) \cup \mathrm{cl}(h)$.

The minimality of the length of $\bar{a}$ yields $\mathrm{cl}(\mathrm{cb}(bh/\bar{a})) = \mathrm{cl}(\bar{a})$, therefore

$$\mathrm{cl}(\bar{a}) = \mathrm{cl}(\mathrm{acl}(a) \cap \mathrm{acl}(b)) \subset \mathrm{cl}(\bar{a}) \cap \mathrm{cl}(b),$$

thus $\bar{a} \in \mathrm{cl}(b)$ and $h \in \mathrm{cl}(\bar{a}b) \subset \mathrm{cl}(b)$, which is a contradiction. ■

Following the ideas in Chapter 3, we prove that CM-triviality is preserved in T^{ind}. First we need the following lemmas.

Lemma 4.2.5. The H-bases are transitive:

$$\mathrm{HB}(c/B) \cup \mathrm{HB}(B) = \mathrm{HB}(cB).$$

Lemma 4.2.6. If $A \subset B$ and $\mathrm{acl_H}(cA) \cap \mathrm{acl_H}(B) = \mathrm{acl_H}(A)$, then

$$\mathrm{HB}(c/A) \subset \mathrm{HB}(c/B).$$

The proofs of these two lemmas are exactly the same as in Proposition 3.2.1 and Lemma 3.2.6, respectively.

Theorem 4.2.7. The theory T is CM-trivial if and only if T^{ind} is.

Proof. Assume T is 2-ample. Let a_0, a_1, a_2 be tuples such that:

1. $a_2 \mathop{\smile\!\!\!\!\!\!|}\limits_{a_1} a_0$,
2. $\mathrm{acl}(a_0 a_2) \cap \mathrm{acl}(a_0 a_1) = \mathrm{acl}(a_0)$,
3. $a_2 \mathop{\not\smile\!\!\!\!\!\!|}\limits_{\mathrm{acl}(a_0)\cap\mathrm{acl}(a_1)} a_0$.

By the *generalized extension property*, we may assume that $a_0 a_1 a_2 \mathop{\smile\!\!\!\!\!\!|} H$.

As the H-bases of any subset of $\{a_0 a_1 a_2\}$ are empty, the algebraic closure in T^{ind} of any of these sets is the same as in T. So condition (2) holds in T^{ind}.

By the characterization of canonical bases, since H-bases are empty, condition (1) holds also in T^{ind}. If

$$a_2 \mathop{\smile\!\!\!\!\!\!|}\limits_{\mathrm{acl}_H(a_1)\cap\mathrm{acl}_H(a_0)}^{H} a_0,$$

then

$$a_2 \mathop{\smile\!\!\!\!\!\!|}\limits_{\mathrm{acl}(a_1)\cap\mathrm{acl}(a_0)} a_0,$$

which is a contradiction.

Assume now that T is CM-trivial. Let us see that T^{ind} is CM-trivial.

Let c be a tuple and $A \subset B$ be algebraically closed sets (in T^{ind}) such that $\mathrm{acl_H}(cA) \cap B = A$. Define $h = \mathrm{HB}(c/A)$, $h' = \mathrm{HB}(c/B)$ and $c' = ch$. By Proposition

4.1.11, we have that $\mathrm{acl_H}(\mathrm{cb_H}(c/A)) = \mathrm{acl_H}(\mathrm{cb}(ch/A))$. On the other hand, by Lemma 4.2.6, we have that $h \subset h'$.

Note that $\mathrm{acl}(c'A) \cap \mathrm{acl}(B) = \mathrm{acl}(A)$, because $\mathrm{acl}(c'A) \subset \mathrm{acl_H}(cA)$, $A = \mathrm{acl}(A)$ and $B = \mathrm{acl}(B)$. Thus, by CM-triviality of T, we have that $\mathrm{cb}(c'/A) \subset \mathrm{acl}(\mathrm{cb}(c'/B))$.

Recall that $c' = ch$, hence,

$$\begin{aligned}\mathrm{acl_H}(\mathrm{cb_H}(c/A)) &= \mathrm{acl_H}(\mathrm{cb}(ch/A))\\ &\subset \mathrm{acl_H}(\mathrm{cb}(ch/B))\\ &\subset \mathrm{acl_H}(\mathrm{cb}(ch'/B))\\ &= \mathrm{acl_H}(\mathrm{cb_H}(c/B)).\end{aligned}$$

Therefore, the theory T^{ind} is CM-trivial.

■

We can now modify the previous proof in order to show that, if T^{ind} is n-ample, then T is n-ample for every $n \geq 2$. The converse holds if cl is trivial.

Lemma 4.2.8. Assume cl is trivial. Let A and $B = \mathrm{acl_H}(B)$ subsets of M, then $\mathrm{HB}(AB) = \mathrm{HB}(A) \cup \mathrm{HB}(B)$.

Proof. By Proposition 4.2.5, we know that $\mathrm{HB}(AB) = \mathrm{HB}(A/B) \cup \mathrm{HB}(B)$. On the other hand, as cl is trivial, Lemma 4.2.3 implies that $\mathrm{HB}(A/B) \subset \mathrm{HB}(A)$. Therefore, we have $\mathrm{HB}(AB) \subset \mathrm{HB}(A) \cup \mathrm{HB}(B)$. The other inclusion is evident. ■

Corollary 4.2.9. Assume cl is trivial. If $A = \mathrm{acl_H}(A)$ and $B = \mathrm{acl_H}(B)$, then $\mathrm{acl}(AB) = \mathrm{acl_H}(AB)$.

Theorem 4.2.10. Assume cl is trivial. Then T is n-ample if and only if T^{ind} is.

Proof. For $n = 1$, this is Proposition 4.2.4. Also, as we just mentioned, for $n \geq 2$ we can adapt the proof of Theorem 4.2.7 in order to show that, if T is n-ample, then T^{ind} is n-ample. Hence, it remains to show that, if T is not n-ample, then T^{ind} is not n-ample.

Let $a_0, ..., a_n$ be tuples in $(\mathfrak{C}, H)$ such that:

1. $a_{i+1} \mathop{\smile\!\!\!\!\!\!|}\limits_{a_i}^{H} a_{i-1}...a_0$,

2. $\mathrm{acl_H}(a_0...a_{i-1}a_{i+1}) \cap \mathrm{acl_H}(a_0...a_{i-1}a_i) = \mathrm{acl_H}(a_0...a_{i-1})$.

We may assume that $\mathrm{acl}(a_i) = \mathrm{acl_H}(a_i)$ for every $i \leq n$. In order to show that T^{ind} is not n-ample, we need to show that $a_n \mathop{\smile\!\!\!\!\!\!\,|}^H_{\mathrm{acl_H}(a_1)\cap\mathrm{acl_H}(a_0)} a_0$. By Fact 4.1.11, this is equivalent to proving that $a_n h \mathop{\smile\!\!\!\!\!\!\,|}_{\mathrm{acl}(a_1)\cap\mathrm{acl}(a_0)} a_0$, where $h = \mathrm{HB}(a_n/a_0)$. Moreover, by Lemma 4.2.3 we have that $h \subset \mathrm{HB}(a_n) \subset \mathrm{acl}(a_n)$, so we only need to prove that

$$a_n \mathop{\smile\!\!\!\!\!\!\,|}_{\mathrm{acl}(a_1)\cap\mathrm{acl}(a_0)} a_0.$$

If the tuples $a_0, ..., a_n$ satisfy

1. $a_{i+1} \mathop{\smile\!\!\!\!\!\!\,|}_{a_i} a_{i-1}...a_0$,

2. $\mathrm{acl}(a_0...a_{i-1}a_{i+1}) \cap \mathrm{acl}(a_0...a_{i-1}a_i) = \mathrm{acl}(a_0...a_{i-1})$,

then, as T^{ind} is not n-ample, we may conclude that $a_n \mathop{\smile\!\!\!\!\!\!\,|}_{\mathrm{acl}(a_1)\cap\mathrm{acl}(a_0)} a_0$, which gaves the desired result.

Condition $a_{i+1} \mathop{\smile\!\!\!\!\!\!\,|}_{a_i} a_{i-1}...a_0$ follows directly from $a_{i+1} \mathop{\smile\!\!\!\!\!\!\,|}^H_{a_i} a_{i-1}...a_0$.

Now, from $\mathrm{acl_H}(a_i) = \mathrm{acl}(a_i)$ and Corollary 4.2.9, we can deduce that, for any $a_{i_1}...a_{i_k}$, where $i_k \leq n$, we have

$$\mathrm{acl}(a_{i_1}...a_{i_k}) = \mathrm{acl_H}(a_{i_1}...a_{i_k}).$$

In particular,

$$\begin{aligned} \mathrm{acl}(a_0...a_{i-1}a_{i+1}) \cap \mathrm{acl}(a_0...a_{i-1}a_i) &= \mathrm{acl_H}(a_0...a_{i-1}a_{i+1}) \cap \mathrm{acl_H}(a_0...a_{i-1}a_i) \\ &= \mathrm{acl_H}(a_0...a_{i-1}) \\ &= \mathrm{acl}(a_0...a_{i-1}), \end{aligned}$$

which finishes the proof. ∎

Finally, we prove that weak n-ampleness is preserved in this context:

Theorem 4.2.11. For $n \geq 2$, the theory T is weakly n-ample if and only if T^{ind} is.

Proof. Assume that T is weakly n-ample. Let $a_0, ..., a_n$ such that

1. $\mathrm{acl}(a_i a_{i+1}) \cap \mathrm{acl}(a_i a_{i+2}) = \mathrm{acl}(a_i)$ for all $i < n$,

2. $a_{i+1} \mathop{\smile\!\!\!\!\!\!\mid}_{a_i} a_0 ... a_{i-1}$ for all $i < n$,

3. $a_n \not\mathop{\smile\!\!\!\!\!\!\mid}_{\operatorname{acl}(a_1) \cap \operatorname{acl}(a_0)} a_0$.

By *the extension property*, we may assume that $a_0 ... a_n \mathop{\smile\!\!\!\!\!\!\mid} H$, hence $\operatorname{HB}(X) = \emptyset$ for every $X \subset \{a_0, ..., a_n\}$, which implies that $\operatorname{acl}(X) = \operatorname{acl}_H(X)$ for $X \subset \{a_0, ..., a_n\}$. Therefore

$$\begin{aligned} \operatorname{acl}_{\mathrm{H}}(a_i a_{i+1}) \cap \operatorname{acl}_{\mathrm{H}}(a_i a_{i+2}) &= \operatorname{acl}(a_i a_{i+1}) \cap \operatorname{acl}(a_i a_{i+2}) \\ &= \operatorname{acl}(a_i) \\ &= \operatorname{acl}_{\mathrm{H}}(a_i). \end{aligned}$$

On the other hand, we have

$$a_{i+1} \mathop{\smile\!\!\!\!\!\!\mid}^{H}_{a_i} a_0 ... a_{i-1},$$

because

$$a_{i+1} \mathop{\smile\!\!\!\!\!\!\mid}_{a_i} a_0 ... a_{i-1}$$

and $\operatorname{HB}(a_{i+1}/a_0 ... a_i) = \emptyset$.

Finally, since $a_n \not\mathop{\smile\!\!\!\!\!\!\mid}_{\operatorname{acl}(a_1) \cap \operatorname{acl}(a_0)} a_0$, we have that $a_n \not\mathop{\smile\!\!\!\!\!\!\mid}^{H}_{\operatorname{acl}(a_1) \cap \operatorname{acl}(a_0)} a_0$. Thus, the tuple $a_0, ... a_n$ is weakly n-ample in the sense of T^{ind}.

Assume now that T is not weakly n-ample, let $a_0, ..., a_n \in \mathfrak{C}$ be tuples such that:

1. $\operatorname{acl}_{\mathrm{H}}(a_i a_{i+1}) \cap \operatorname{acl}_{\mathrm{H}}(a_i a_{i+2}) = \operatorname{acl}_{\mathrm{H}}(a_i)$ for all $i < n$.

2. $a_{i+1} \mathop{\smile\!\!\!\!\!\!\mid}^{H}_{a_i} a_0 ... a_{i-1}$ for all $i < n$.

We may assume also that $\operatorname{acl}_{\mathrm{H}}(a_i) = \operatorname{acl}(a_i)$ for every i.

Consider $h_i = \operatorname{HB}(a_n/a_i)$. We claim that $h_0 \subset h_1 \subset \cdots \subset h_{n-1}$.

Proof of the claim. Since $a_n \mathop{\smile\!\!\!\!\!\!\mid}^{H}_{a_{i+2}} a_i a_{i+1}$, we have that $a_n a_i \mathop{\smile\!\!\!\!\!\!\mid}^{H}_{a_i a_{i+2}} a_i a_{i+1}$. Hence

$$\begin{aligned} \operatorname{acl}_{\mathrm{H}}(a_n a_i) \cap \operatorname{acl}_{\mathrm{H}}(a_i a_{i+1}) &\subset \operatorname{acl}_{\mathrm{H}}(a_i a_{i+2}) \cap \operatorname{acl}_{\mathrm{H}}(a_i a_{i+1}) \\ &\subset \operatorname{acl}_{\mathrm{H}}(a_i) \end{aligned}$$

Using Lemma 4.2.6 (by making $A = a_i$, $B = a_i a_{i+1}$ and $c = a_n$) we conclude that $h_i \subset h_{i+1}$ for every i. ■

In particular $h_0 \subset h_{n-1}$.

Notice that

$$h_0 \subset h_{n-2} \subset \mathrm{acl_H}(a_{n-2}a_n),$$

and that

$$a_n h_{n-1} \underset{a_{n-1}}{\perp} a_0 \ldots a_{n-1},$$

then, setting $a'_n = a_n h_0$ and $a'_i = a_i$ for $i < n$, we have that:

1. $\mathrm{acl}(a'_i a'_{i+1}) \cap \mathrm{acl}(a'_i a'_{i+2}) = \mathrm{acl}(a'_i)$ for all $i < n$.
2. $a'_{i+1} \perp_{a'_i} a'_0 \ldots a'_{i-1}$ for all $i < n$.

Since T is not weakly n-ample, it follows that

$$a'_n \underset{\mathrm{acl}(a'_0)\cap\mathrm{acl}(a'_1)}{\perp} a'_0$$

i.e.

$$a_n h_0 \underset{\mathrm{acl}(a_0)\cap\mathrm{acl}(a_1)}{\perp} a_0.$$

By the characterization of canonical bases, this is equivalent to

$$a_n \overset{H}{\underset{\mathrm{acl_H}(a_0)\cap\mathrm{acl_H}(a_1)}{\perp}} a_0.$$

Thus, T^{ind} is not weakly n-ample.

■

Question 4.2.12. Let $n \geq 3$. If T^{ind} almost n-ample, then is T?

Chapter 5

Equationality

In this chapter we introduce equational theories together with their main properties and describe the relation between CM-triviality and equationality. (For a more detailed exposition to equational theories, we refer the reader to [28]).

Equational theories, which form a subclass of stable theories, were defined by Srour in [40] in an attempt to capture the algebraic behaviour of certain categories of structures, such as algebraic closed fields or differential closed fields. For example, in ACF_0, given tuples $\bar{a}$, $\bar{b}$ and $\bar{c}$, it happens that $\bar{a} \not\perp_{\bar{c}} \bar{b}$ when there is some $a_i \in \bar{a}$, such that a_i satisfies a polynomial equation over $(\bar{a} \setminus a_i)\bar{b}\bar{c}$, but it does not satisfies any polynomial equation over $(\bar{a} \setminus a_i)\bar{c}$. Since polynomial equations satisfy a Noetherianity principle, then, forking in ACF_0 is witnessed by Noetherian formulas.

In [22], Hrushovski proved that the new strongly minimal set is equational using CM-triviality. However, the proof of this fact uses an auxiliary result from an unpublished work [23]. Junker and Lascar [26] reproved Hrushosvki's result by studying the relation between equational sets and indiscernible-closed sets.

Our interest in the notion of equationality was motivated by its relation with CM-triviality, our goal was to show that, in the context of strongly minimal theories, non-n-ampleness implies equationality. Although we were not able to prove it, we managed to merge the results from Lascar and Junker together with the result from Hrushovski, to exhibit a direct proof of the equationality of CM-trivial theories of finite, continuous SU-rank. We hope that, at least, this points possible connections between ampleness and equationality.

5.1 Equational theories and equations

Definition 5.1.1. Fix an L-theory T. An L-formula $\varphi(\bar{x};\bar{y})$ is an *equation* in $\bar{x}$, if, for every sequence $\{\bar{b}_i\}_{i<I}$ of tuples in $\mathfrak{C}$, of the same length of $\bar{y}$, there is a finite subset $I_0 \subset I$ such that $\bigcap_{i\in I}\varphi(\mathfrak{C},\bar{b}_i) = \bigcap_{i\in I_0}\varphi(\mathfrak{C},\bar{b}_i)$.

Remark 5.1.2. 1. Equations are closed under positive Boolean combinations.

2. If $\varphi(\bar{x};\bar{y})$ is an equation on $\bar{x}$, then it is an equation on $\bar{y}$.

3. Equations do not have the order property.

Proof. 1. It is easy to check that equations are closed under intersections. Let us see that they are closed under unions: let $\varphi_1(\bar{x}_1,\bar{y}_1)$ and $\varphi_2(\bar{x}_2,\bar{y}_2)$ be equations on x. If $\varphi_1 \vee \varphi_2$ is not an equation, then there are tuples $\bar{b}^i_1$ and $\bar{b}^i_2$ such that the sets $X^i_1 = \varphi_1(\mathfrak{C},\bar{b}^i_1)$ and $X^i_2 = \varphi_2(\mathfrak{C},\bar{b}^i_2)$ satisfy the following:

$$X^1_1 \cup X^1_2 \supsetneq \bigcup_{j,k\in\{1,2\}} (X^1_j \cap X^2_k) \supsetneq \bigcup_{j,k,l\in\{1,2\}} (X^1_j \cap X^2_k \cap X^3_l) \supsetneq ...$$

Then, using König's lemma, we obtain a chain of the form:

$$X^1_j \supsetneq X^1_j \cap X^2_k \supsetneq X^1_j \cap X^2_k \cap X^3_l \supsetneq ...$$

Moreover, as j,k,l, etc. have values over the set $\{1,2\}$, there must be a chain of the form:

$$X^{i_1}_m \supsetneq X^{i_1}_m \cap X^{i_2}_m \supsetneq X^{i_1}_m \cap X^{i_2}_m \cap X^{i_3}_m \supsetneq ...$$

for some $m \in \{1,2\}$, contradicting the equationality of φ_m.

2. Assume $\varphi(\bar{x},\bar{y})$ is not an equation on $\bar{y}$. Then there exists $(\bar{a}_i)_{i<\omega}$ and $(\bar{b}_i)_{i<\omega}$ such that $\varphi(\bar{a}_i,\bar{b}_j)$ if $i \leq j$ but $\neg\varphi(\bar{a}_{i+1},\bar{b}_i)$. Therefore, for every j, we have the following descending chain of length $j+1$

$$\varphi(\mathfrak{C},\bar{b}_j) \supsetneq \varphi(\mathfrak{C},\bar{b}_j) \cap \varphi(\mathfrak{C},\bar{b}_{j-1}) \supsetneq ... \supsetneq \varphi(\mathfrak{C},\bar{b}_j) \cap \varphi(\mathfrak{C},\bar{b}_{j-1}) \cap ... \cap \varphi(\mathfrak{C},\bar{b}_0).$$

By compactness, there is an infinite descending chain of the form

$$\varphi(\mathfrak{C},\bar{c}_0) \supsetneq \varphi(\mathfrak{C},\bar{c}_0) \cap \varphi(\mathfrak{C},\bar{c}_1) \supsetneq ... \supsetneq \varphi(\mathfrak{C},\bar{c}_0) \cap \varphi(\mathfrak{C},\bar{c}_1) \cap ... \cap \varphi(\mathfrak{C},\bar{c}_n) \supsetneq ....$$

Hence, the formula $\varphi(\bar{x},\bar{y})$ is not an equation in $\bar{x}$.

3. If $\varphi(\bar{x}, \bar{y})$ has the order property, then there exist tuples $\bar{a}_i$ and $\bar{b}_j$ such that $\models \varphi(\bar{a}_i, \bar{b}_j)$ if and only if $i \leq j$. Thus, we have the following chain

$$\varphi(\bar{a}_1, \mathfrak{C}) \supsetneq \varphi(\bar{a}_1, \mathfrak{C}) \cap \varphi(\bar{a}_2, \mathfrak{C}) \supsetneq \varphi(\bar{a}_1, \mathfrak{C}) \cap \varphi(\bar{a}_2, \mathfrak{C}) \cap \varphi(\bar{a}_3, \mathfrak{C})...$$

   Therefore, the formula $\varphi(\bar{x}, \bar{y})$ is not an equation on $\bar{y}$ and, by the previous item, is not an equation on x.

∎

Definition 5.1.3. A definable set X is *Srour-closed* if for every family $\{X_i\}_{i\in I}$ of conjugates of X under $Aut(\mathfrak{C})$, there is $I_0 \subset_{fin} I$ such that $\bigcap_{i\in I} X_i = \bigcap_{i\in I_0} X_i$.

Clearly, every set definable by an equation is Srour-closed. Moreover, every Srour-closed set X, definable over A, is of the form $\varphi(\mathfrak{C}, \bar{a})$, where $\varphi(\bar{x}, \bar{y})$ is an equation and $\bar{a}$ is a tuple of A: assume $X = \psi(\mathfrak{C}, \bar{a})$ and let $p(\bar{y}) = \mathrm{tp}(\bar{a})$. As X is Srour-closed, there exists $n \in \mathbb{N}$ such that there are no chains of the form

$$\psi(\mathfrak{C}, \bar{a}_1) \supsetneq \psi(\mathfrak{C}, \bar{a}_1) \cap \psi(\mathfrak{C}, \bar{a}_2) \supsetneq ... \supsetneq \psi(\mathfrak{C}, \bar{a}_1) \cap \psi(\mathfrak{C}, \bar{a}_2) \cap ... \cap \psi(\mathfrak{C}, \bar{a}_n),$$

where $\bar{a}_i \models p$. Therefore the type $p(\bar{y}_1) \cup ... \cup p(\bar{y}_n) \cup \{\bigwedge_{j<n}(\neg\forall\bar{x}(\bigwedge_{i\leq j}(\psi(\bar{x}, \bar{y}_i)) \rightarrow \bigwedge_{i\leq j+1} \psi(\bar{x}, \bar{y}_i))\}$ is inconsistent. By compactness there is a formula $\phi(\bar{y}) \in p$ such that

$$\phi(\bar{y_1}) \wedge ... \wedge \phi(\bar{y}_n) \rightarrow \bigvee_{j<n} \forall\bar{x}(\bigwedge_{i\leq j}(\psi(\bar{x}, \bar{y}_i)) \rightarrow \bigwedge_{i\leq j+1}(\psi(\bar{x}, \bar{y}_i)))$$

This implies that $\psi(\bar{x}, \bar{y}) \wedge \phi(\bar{y})$ is an equation. Since $X = \psi(\bar{x}, \bar{a}) \wedge \phi(\bar{a})$, the conclusion follows.

Definition 5.1.4. A complete theory T is *equational* if every formula $\psi(\bar{x}, \bar{y})$ is equivalent in T to a Boolean combination of equations.

Remark 5.1.5. Equational theories are stable.

This last remark comes from Remark 5.1.2, and Definition 1.4.4, since formulas without the order property are closed under Boolean combinations.

Junker proved the following "uniformity" result.

Fact 5.1.6 (Junker [25]). A theory T is equational if and only if every definable set is a Boolean combination of Srour-closed sets.

Definition 5.1.7. A definable set X is *weakly normal* if, for every infinite family $\{X_i\}_{i\in I}$ of different conjugates of X under $Aut(\mathfrak{C}^{eq})$, we have $\bigcap_{i\in I} X_i = \emptyset$.

Notice that, by compactness, we have $\bigcap_{i\in I_0} X_i = \emptyset$ for some $I_0 \subset_{fin} I$. Thus, every weakly normal definable set is Srour-closed.

Definition 5.1.8. A theory is *weakly normal* if every definable set X is a Boolean combination of weakly normal definable sets.

Theorem 5.1.9 (see [35], Proposition 4.1.5)**.** A theory T is weakly normal if and only if T is stable and 1-based.

In particular, we have the following result:

Corollary 5.1.10. Every stable 1-based theory is equational.

Therefore, in the context of stable theories, equationality is a generalization of 1-basedness.

Most of the natural examples in stability theory are equational. Namely:

- **One-based stable theories**.
- **Algebraic closed fields**. This is Hilbert's basis theorem.
- **Modules**. Let M be an R-module and $T = Th(M)$. Every formula is equivalent in T to a Boolean combination of *positive primitive formulas*, these are, formulas $\phi(\bar{z})$ of the form $\exists \bar{w}(\wedge_{j\leq n}\psi_j(\bar{w}, \bar{z}))$; where $\psi_j(\bar{w}, \bar{z})$ are atomic formulas. Ziegler [46] proved that, for every partition of the variable $\bar{z} = \bar{x}; \bar{y}$, we have that $\phi(\mathfrak{C}; \bar{a})$ is either empty or a coset of the subgroup $\phi(\mathfrak{C}, \bar{0})$. Hence, the formula $\phi(\bar{x}; \bar{y})$ is an equation and T is equational.
- **Differentiable closed fields**. The theory of differentially closed fields in characteristic 0 is equational for the same reason as ACF_0. Moreover, the set of all differential equations is an equational set: any intersection of solution-sets of differential equations is the intersection of a finite subfamily of solutions-sets (see [40]).
- **Separably closed fields of finite degree of imperfection** (see [41]).
- **The n-dimensional free pseudospace** (see [4]).

There are, until now, very few examples of stable non-equational theories:

- **The colored free pseudospace**. An unpublished construction due to Hrushovski [23], whose proof can be found in [13].
- **Free groups and torsion-free hyperbolic groups** (see [37]).

The terminology "equations" and "equational" reflects the behaviour of polynomial equations. Since equational theories in universal algebra have another precise meaning, which differs from the model theoretic one, we suggest *locally Noetherian theory* or *weakly Noetherian theory*.

5.2 CM-triviality and equationality

In [22] Hrushovski proved that every strongly minimal CM-trivial theory is equational. His proof, which uses a detour through a notion he calls *strongly equational*, does not work for a general stable theory of finite U-rank, since the U-rank need not be continuous. We will exhibit a direct proof of this result.

Definition 5.2.1. Let $A \subset \mathfrak{C}^n$ be any set (not necessary definable). Define $\mathrm{icl}(A)$ as the set of tuples $\bar{a}$ such that there is an $\emptyset$-indiscernible sequence $\{\bar{a}_i\}_{i<\omega}$ with $\bar{a} = \bar{a}_0$ and $\bar{a}_i \in A$ for $i \geq 1$.

We say that A is *indiscernible-closed* if $A = \mathrm{icl}(A)$.

Notice that for every A and B we have $\mathrm{icl}(A \cup B) = \mathrm{icl}(A) \cup \mathrm{icl}(B)$.

Fact 5.2.2. (Junker, Lascar [26]) Let X be a definable set. Then X is indiscernible-closed if and only if it is Srour-closed.

Lemma 5.2.3 (Junker, Lascar [26])**.** Assume T is a CM-trivial superstable theory of finite U-rank. Let X be a definable set over $\bar{c}$ and $\bar{a} \subset icl(X) \setminus X$. Then, there is $\bar{b} \in X$ such that $U(\bar{a}/\bar{c}) < U(\bar{b}/\bar{c})$.

Proof. Let $I = \{\bar{a}_i\}_{i<\omega}$ be an $\emptyset$-indiscernible sequence such that $\bar{a} = \bar{a}_0$ and $\bar{a}_i \in X$ for $i > 0$. As T is stable, the set I is $\emptyset$-indiscernible as a set. By superstability there exists $n \in \mathbb{N}$ such that $I' = I \setminus I_0$ is Morley over I_0 for every $I_0 \subset I$ of size n. In particular we can take I_0 not containing $\bar{a}_0$. We rename the elements of I' as $\bar{a}_0, \bar{a}'_1, \bar{a}'_2 ...$ and so on. By superstability, there exists $\bar{a}'_k \in I'$ such that $\bar{a}'_k \mathop{\smile\!\!\!\!\!\!|}_{I_0} \bar{c}$.

Let $D = \mathrm{acl}^{\mathrm{eq}}(I_0) \cap \mathrm{acl}^{\mathrm{eq}}(\bar{a}'_k\bar{c})$. By CM-triviality of T, we have $\bar{a}'_k \mathop{\smile\!\!\!\!\!\!|}_{D} \bar{c}$. Hence

$$\begin{aligned} U(\bar{a}'_k\bar{c}) &= U(\bar{a}'_k\bar{c}/D) + U(D) \\ &= U(\bar{a}'_k/D) + U(\bar{c}/D) + U(D) \\ &= U(\bar{a}_0/D) + U(\bar{c}/D) + U(D) \\ &\geq U(\bar{a}_0\bar{c}/D) + U(D) \\ &\geq U(\bar{a}_0\bar{c}). \end{aligned}$$

If equality holds, then $\bar{a}_0 \mathop{\smile\!\!\!\!\!\!\mid}_D \bar{c}$. Since the type $\mathrm{tp}(\bar{a}_k/D) = \mathrm{tp}(\bar{a}_0/D)$ is stationary, we would have $\bar{a}_k \equiv_{\bar{c}D} a_0$, which is a contradiction. Therefore $\bar{a}_0 \not\mathop{\smile\!\!\!\!\!\!\mid}_D \bar{c}$. Set $\bar{b} = a'_k$ to get the desired result. ∎

We previously defined U-rank for a complete type p. If X is an A-type-definable set, we define the U-rank of X as $sup\{U(tp(b/A)) \,|\, b \in X\}$.

Notice that if X is an A-definable set, then $\mathrm{icl}(X)$ is A-type-definable. Therefore $U(\mathrm{icl}(X) \setminus X)$ is defined.

We now prove the main result:

Theorem 5.2.4 (Hrushovski [21], Junker [25])**.** A CM-trivial stable theory of finite and continuous U-rank is equational.

Proof. Let X be a $\bar{c}$-definable set of U-rank n. By the previous lemma we have that $U(icl(X) \setminus X) < n$. Now, using the continuity of U-rank, there exists a $\bar{c}$-definable set $X_1 \supset \mathrm{icl}(X) \setminus X$ with $U(X_1) = U(\mathrm{icl}(X) \setminus X)$. Proceeding in the same way, using X_i instead of X and repeating the argument several times, we obtain a descending chain which must become stationary after at most n-steps:

$$U(X) > U(X_1) > ... > U(X_k) = U(X_{k+1}) = ...$$

Hence $X_k = \emptyset$ and X_{k-1} is Srour-Closed.

Moreover, notice that $X^* = X \cup X_1 \cup \cdots \cup X_{k-1}$ must be Srour-closed as well, because

$$\begin{aligned} \mathrm{icl}(X \cup X_1 \cup \cdots \cup X_{k-1}) \setminus (X \cup X_1 \cup \cdots \cup X_{k-1}) &\subset \mathrm{icl}(X_{k-1}) \setminus X_{k-1} \\ &\subset X_k \\ &= \emptyset \end{aligned}$$

Finally, using the inequality $U(X^* \setminus X) < U(X)$, we get that $X = X^* \setminus (X^* \setminus X)$ is a Boolean combination of Srour-closed sets of lower rank. Proceeding by induction, we conclude that every definable set is a Boolean combination of Srour-closed sets. Thus, the theory T is equational. ∎

Question 5.2.5. If T is any stable CM-trivial theory, then is it equational?

5.3 The indiscernible closure

In the proof of Theorem 5.2.4, applying repeatedly the operator icl over a definable set X eventually gets stationary after a finite number of steps (the number of iterations is bounded by the U-rank of X). This phenomenon was considered in general in [25] and applied to 1-based theories and algebraic closed fields. For general CM-trivial theories, this study seems to be out of reach with the methods we have, though we obtained a positive result for the free-pseudoplane.

Recall that if X be a C-type-definable set, then $\mathrm{icl}(X)$ is C-type-definable. In particular, if X is type-definable, then $\mathrm{icl}^n(X) = \underbrace{\mathrm{icl}\circ\ldots\circ\mathrm{icl}}_{n-times}(X)$, is type-definable. However, it is not known whether $\mathrm{icl}^\omega(X) = \bigcup_{n<\omega}\mathrm{icl}^n(X)$ is still type-definable. This suggests the following definition.

Definition 5.3.1. Let X be a type definable set. We define $\overline{\mathrm{icl}}^\lambda$ as follows:

1. If $\lambda = \alpha + 1$, then $\overline{\mathrm{icl}}^\lambda(X) = \mathrm{icl}(\overline{\mathrm{icl}}^\alpha)(X)$
2. If λ is a limit ordinal, then $\overline{\mathrm{icl}}^\lambda$ is the smallest type-definable set containing $\bigcup_{\alpha<\lambda}(\overline{\mathrm{icl}}^\alpha(X))$.

By $\overline{\mathrm{icl}}^\infty$ we mean $\bigcup_{\lambda\in Ord}\overline{\mathrm{icl}}^\lambda(X)$. This is the smallest indiscernible-closed, type-definable set containing X.

Observe that icl^n and $\overline{\mathrm{icl}}^n$ coincide for $n \in \mathbb{N}$.

Definition 5.3.2. Let T be any complete theory. Define the ordinal i_T as

$$i_T = \sup\{\alpha + 1 \mid \overline{\mathrm{icl}}^{\,\infty}(X) \neq \overline{\mathrm{icl}}^{\,\alpha}(X) \text{ for } X \text{ type-definable}\}$$

The ordinal i_T measures how far type-definable sets are from being indiscernible-closed. Notice that if T has infinite models, then $i_T > 1$.

Let I_α the class of theories T with $i_T \leq \alpha$.

Question 5.3.3. Is the class I_α closed under bi-interpretability?

In order to measure i_T, it suffices to study canonical types, that is types over their canonical base:

Theorem 5.3.4 (Junker, Lascar [26])**.** Let T be a stable theory.

1. If X is type-definable over A, then

$$\operatorname{icl}^{\alpha}(X)=\bigcup\{\operatorname{icl}^{\alpha}(p)\,|\,p\in S(\mathfrak{C}^{eq}),p\subset X \text{ and } p \text{ does not fork over } A\}.$$

2. For a stationary type $p\in S(A)$, the following condition holds:

$$p\subset p|_{\operatorname{cb}(p)}\subset icl(p).$$

Definition 5.3.5. A type p is a *canonical type* if $p=p|_{\operatorname{cb}(p)}$.

Corollary 5.3.6. If there exists a natural number n such that $icl^n(p)=\operatorname{icl}^{n+1}(p)$ for every canonical type p, then T is I_{n+1}.

Proof. If $\operatorname{icl}^{n+1}(p|_{\operatorname{cb}(p)})=\operatorname{icl}^{n}(p|_{\operatorname{cb}(p)})$, then, using $p\subset p|_{\operatorname{cb}(p)}\subset icl(p)$, we get that $\operatorname{icl}^{n+2}(p)=\operatorname{icl}^{n+1}(p)$.

Let X be a type definable set, then:

$$\begin{aligned}\operatorname{icl}^{n+2}(X) &= \bigcup\{\operatorname{icl}^{n+2}(p)\,|\,p\in S(\mathfrak{C}^{eq}),p\subset X \text{ and } p \text{ does not fork over } A\}\\ &= \bigcup\{\operatorname{icl}^{n+1}(p)\,|\,p\in S(\mathfrak{C}^{eq}),p\subset X \text{ and } p \text{ does not fork over } A\}\\ &= \operatorname{icl}^{n+1}(X).\end{aligned}$$

■

Corollary 5.3.7. (Junker, Lascar [26]) Every 1-based stable theory is in I_2.

Proof. It suffices to show that $\operatorname{icl}(p)=p$ for every canonical type p. Take p a canonical type and $\bar{a}\in\operatorname{icl}(p)$. Then there exists an indiscernible sequence $I=(\bar{a}=\bar{a}_0,\bar{a}_1,\bar{a}_2...)$ such that $\bar{a}_i\models p$ for $i\geq 1$. Since T is 1-based, then $\operatorname{cb}(p)\in\operatorname{acl}(\bar{a}_i)$ for every $i\geq 1$, in particular $\operatorname{cb}(p)\in\operatorname{acl}(\bar{a}_2)$.

As I is indiscernible, we have $\bar{a}_0\equiv_{\operatorname{acl}(\bar{a}_2)}\bar{a}_1$ (indiscernible sequences are also indiscernible over algebraic closures). Hence $\bar{a}_0\equiv_{\operatorname{cb}(p)}\bar{a}_1$ and $\bar{a}\models p$. ■

5.4 The free pseudoplane

Recall that the *free pseudoplane* is an infinite graph without cycles, such that every vertex has infinite valency. These two schemes of axioms form a complete theory, which we will call FP_1 because it is bi-interpretable with the 1-dimensional free pseudospace.

As we mentioned before in Example 4.1.7, the definable closure and the algebraic closure coincide: for every A, we have that

$$\mathrm{dcl}(A) = \mathrm{acl}(A) = \{c \mid \text{ there is a path between two elements of } A \text{ passing through } c\}.$$

In particular, the algebraic closure of any finite set is finite.

Definition 5.4.1. Let a and b two different vertices, if there is a path between a and b, it is unique (because there are no cycles), therefore, we may define the *distance* between two vertices a and b, noted by $d(a, b)$, as the length of the path connecting them.

If $a = b$ then $d(a, b) = 0$ and if $a \neq b$ and there is no path between a and b, we say that $d(a, b) = \infty$.

For sets B and C we define $d(B, C)$ as the minimum of $\{d(b, c) \mid b \in B, c \in C\}$.

Definition 5.4.2.

- We say that A *linked to* B if $d(A, B) < \infty$.
- A set A is *connected* if $d(a, a')$ is finite for every a and a' in A, and the path linking a and a' is contained in A. In particular, every connected set is definably closed.

Lemma 5.4.3. If c is a vertex linked to some set $A = \mathrm{dcl}(A)$, then there exists a unique element $a \in A$ such that $d(c, a) < d(c, a')$ for all $a' \in A$, with $a' \neq a$.

We call the element a the *projection* of c over A, denoted as $proj(c/A)$.

Proof. Let d be the minimal distance between c and A witnessed by a. Assume there is a different element a' such that $d(c, a) = d(c, a') = d$. Then, the paths $\gamma_1(c, a)$ and $\gamma_2(c, a')$ must be different. Take c' the last element where the paths $\gamma(c, a)$ and $\gamma_2(c, a')$ coincide, then there is a path between a and a' passing through c', so $c' \in A$ and $d(c, c') < d$, which is a contradiction. ∎

The theory FP_1 is stable of U-rank ω and has geometric elimination of imaginaries. Also, it has quantifier elimination by adding predicates $d_n(x, y)$ that say: "the distance between x and y is n".

Remark 5.4.4. Since FP_1 has quantifier elimination after adding predicates for the distances, we have that the type $tp(a_0...a_n)$ is totally determined by $(d(a_i, a_j) \,|\, i, j \leq n)$.

Forking-independence is characterized as follows:

Fact 5.4.5. For every A, B and C, we have that $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if and only if every path from A to B passes through C. In particular, if $\bar{c}$ is a tuple and $A = \mathrm{acl}(A)$, then

$$\mathrm{cb}(\bar{c}/A) = \{a \in A \,|\, a \text{ is the first vertex in } A \text{ of a path } \gamma \text{ from } c \text{ to } A\}.$$

From this we may conclude the following fact:

Fact 5.4.6. Forking is trivial, this is, if $\bar{a}, \bar{b}$ and $\bar{c} \in \mathfrak{C}$ are pairwise independent over a set D, then $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_D \bar{b}\bar{c}$.

The following facts about the free-pseudoplane are well known and have been generalized to the n-dimensional free pseudospaces. We include the proofs for the sake of completeness.

Proposition 5.4.7. The theory FP_1 is CM-trivial and not 1-based.

Proof. To check that FP_1 is not 1-based, take a_1 and a_2 two points linked by a path. By the characterization of independence we have $a_1 \not\mathop{\smile\!\!\!\!\!\!|}_\emptyset a_2$. On the other hand, notice that $\mathrm{acl}(\{a\}) = \{a\}$ for every point a, in particular $\mathrm{acl}(a_1) \cap \mathrm{acl}(a_2) = \{a_1\} \cap \{a_2\} = \emptyset$. Therefore $a_1 \not\mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}(a_1) \cap \mathrm{acl}(a_2)} a_2$ and FP_1 is not 1-based.

On the other hand, take $\bar{c}$ a tuple and $A \subset B$ such that $\mathrm{acl}(\bar{c}A) \cap \mathrm{acl}(B) = \mathrm{acl}(A)$. If a_i is any element in $\mathrm{cb}(\bar{c}/A)$, then there is a path γ_i from $\bar{c}$ to A, where a_i is the first element of γ_i in A. If b_i is the first element of γ_i in B, then $b_i \in \mathrm{acl}(\bar{c}A) \cap \mathrm{acl}(B) = \mathrm{acl}(A)$ and $b_i = a_i$. Hence $a_i \in \mathrm{cb}(\bar{c}/B)$ and $\mathrm{cb}(\bar{c}/A) \subset \mathrm{cb}(\bar{c}/B)$. Thus FP_1 is CM-trivial. ∎

Definition 5.4.8. A stable theory is *2-based* if for every type $p = tp(\bar{b}/A)$ and for all $\bar{b}_1$ and $\bar{b}_2$ realizations of p independent over A, we have that $\mathrm{cb}(\bar{b}/A) \subset \mathrm{acl}(\bar{b}_1\bar{b}_2)$.

Proposition 5.4.9. The theory FP_1 is 2-based.

Proof. Consider $p = \mathrm{tp}(\bar{b}/A)$ such that $C = \mathrm{cb}(\bar{b}/A)$ and let $\bar{b}_1$ and $\bar{b}_2$ be realizations of p independent over A. If $A = \emptyset$ then there is nothing to prove. If not, take any element $c \in C$. Since $C = \mathrm{cb}(\bar{b}_i/A)$ for $i = 1, 2$ there exists a point $b_i \in \bar{b}_i$ such that $c = proj(b_i/\,\mathrm{acl}(A))$. On the other hand b_1 and b_2 are linked because both are linked to c, hence the path connecting them must pass through C, then, it must pass through c

because c is the projection of b_1 and b_2 over C. Therefore $c \in \operatorname{acl}(b_1 b_2) \subset \operatorname{acl}(\bar{b}_1 \bar{b}_2)$. This implies that $C \subset \operatorname{acl}(\bar{b}_1 \bar{b}_2)$. ∎

Proposition 5.4.10. The theory FP_1 is equational.

Proof. The theory FP_1 has quantifier elimination by adding the predicates $d_n(x, y)$ for naming the distances. Consider the predicates $d_{\leq n}(x, y)$ saying that the distance between x and y is less or equal than n. Clearly the predicates $d_{\leq n}(x, y)$ are Boolean combination of the predicates $d_n(x, y)$ and viceversa. Then every formula $\varphi(\bar{x}, \bar{y})$ is a Boolean combination of formulas of the form $d_{\leq n}(x_i, y_i)$ where $x_i \in \bar{x}$ and $y_i \in \bar{y}$. Therefore, to see that FP_1 is equational it suffices to prove that $d_{\leq n}(x, y)$ is an equation. Moreover, using Fact 5.2.2, it suffices to check that $d_{\leq n}(\mathfrak{C}, b)$ is indiscernible-closed for any point b.

Take $(a_i)_{i \leq \omega}$ an $\emptyset$-indiscernible sequence such that $d_{\leq n}(a_i, b)$ for every $i \geq 1$. Since the elements a_i are linked between each other for $i \geq 1$ (because they are all linked to b) and the distance $d(a_i, a_j)$ is a constant (by indiscernibility), then they are linked through the same point b', which is between a_i and b for every $i \geq 1$.

Notice that $b' \in \operatorname{dcl}(b_1 b_2)$, then by indiscernibility we have that $d(a_0, b') = d(a_i, b')$ for every $i \geq 1$. Using this we conclude that

$$\begin{aligned} d(a_0, b) &\leq d(a_0, b') + d(b', b) \\ &= d(a_1, b') + d(b', b) \\ &= d(a_1, b) \text{ (Since } b' \text{ is between } a_1 \text{ and } b) \\ &= n \end{aligned}$$

Thus, the element a_0 is in $d_{\leq n}(\mathfrak{C}, b)$ and $d_{\leq n}(\mathfrak{C}, b)$ is indiscernible-closed.

∎

We are going to show that FP_1 is in I_3, for this we need to understand $\emptyset$-indiscernible sequences. The behaviour of those sequences motivates the following definition.

Definition 5.4.11. Assume that A and C are finite sets. A *star* of A over C is an infinite set S of realizations of $\operatorname{tp}(A/C)$ such that, for every A_i and A_j different elements in S, and γ_i, γ_j are paths from A_i to C and from A_j to C respectively, we have that $\gamma_i \cap \gamma_j \subset C$.

Remark 5.4.12. Assume S is a star over C and take A_i and A_j in S. If A_i is linked to C then so is A_j, as the have the same type over C. Hence A_i and A_j are linked. Moreover, any path from A_i to A_j must go through C. Then $A_i \perp_C A_j$. By triviality of forking, we have that S is an independent subset over C.

Our next goal is to characterize indiscernible sequences over $\emptyset$. Let us study first sequences of the form $I = \{\bar{a}_i\}_{i<\omega}$, where $\bar{a}_i$ is connected for some (all) $i < \omega$.

Lemma 5.4.13. Assume that $\bar{a}_0$ is connected. A infinite sequence $I = (\bar{a}_i)_{i<\omega}$ is indiscernible over $\emptyset$ if and only if $\bar{a}_i \equiv \bar{a}_0$ for every i, and it has one of following mutually exclusive forms:

1. For every $i \neq j$, we have that $\bar{a}_i$ is not linked with $\bar{a}_j$. In this case, the sequence I is a star of $\bar{a}_0$ over c, where c is any element which is not linked to any $\bar{a}_i$.
2. For every $i \neq j$, we have that $\bar{a}_i$ is linked with $\bar{a}_j$ and $\bar{a}_i \cap \bar{a}_j = \emptyset$. In this case I is a star of $\bar{a}_0$ over c, where c is a point in $\mathrm{dcl}(\bar{a}_0\bar{a}_1)$ such that $d(c, \bar{a}_i) = n$ for a fixed natural number n.
3. For every $i \neq j$, we have that $\bar{a}_i$ is linked with $\bar{a}_j$ and $\bar{a}_i \cap \bar{a}_j = C \neq \emptyset$. In this case, the sequence I is a star of $\bar{a}_0$ over C.

Proof. ($\Rightarrow$) Let I be an $\emptyset$-indiscernible sequence.

Case 1. Assume $\bar{a}_i$ is not linked to $\bar{a}_j$. Take any c which is not linked to any $\bar{a}_i$, then $\bar{a}_{i_1}...\bar{a}_{i_n}c \equiv \bar{a}_{j_1}...\bar{a}_{j_n}c$, for any $i_1, ..., i_n$ and $j_1, ..., j_n$ pairwise distinct. Thus I is a star over c.

Case 2. Assume that $\bar{a}_i$ is linked with $\bar{a}_j$ and $\bar{a}_i \cap \bar{a}_j = \emptyset$ for every $i \neq j$. Then there is a unique minimal path $\gamma_{i,j}$ connecting $\bar{a}_i$ and $\bar{a}_j$. Moreover the length of the paths $\gamma_{i,j}$ is a constant $n > 0$. Now, take $\bar{a}_i$, $\bar{a}_j$ and $\bar{a}_k$ three different tuples of I and consider the paths $\gamma_{i,j}$, $\gamma_{j,k}$ and $\gamma_{i,k}$. As there are no cycles, the paths $\gamma_{i,j}$ and $\gamma_{i,k}$ must intersect the path $\gamma_{j,k}$ in the same point $c_{i,j,k}$,i.e. $\gamma_{i,j} \cap \gamma_{i,k} \cap \gamma_{j,k} = \{c_{i,j,k}\}$. By indiscernibility of I, we have that $c_{i,j,k} = c_{i',j',k'} = c$ for every i', j', k'. Moreover, again by indiscernibility, the distance $d(c, \bar{a}_i)$ is constant for every i. This implies that I is a star of $\bar{a}_0$ over c.

Case 3. Assume that $\bar{a}_i$ is linked with $\bar{a}_j$ and $\bar{a}_i \cap \bar{a}_j = C \neq \emptyset$ for every $i \neq j$. By indiscernibility, we have that I is a set of realizations of $\bar{a}_0$ over C. Moreover, any path from $\bar{a}_i$ to C is contained in $\bar{a}_i$, hence, if γ_i and γ_j go from $\bar{a}_i$ to C and $\bar{a}_j$ to C respectively, then $\gamma_i \cap \gamma_j \subset \bar{a}_i \cap \bar{a}_j = C$. Thus, the sequence I is a star of $\bar{a}_0$ over C.

($\Leftarrow$) Assume that I has one of these forms, then, by, Remark 5.4.4 it is $\emptyset$-indiscernible. ∎

Lemma 5.4.14. Assume $\bar{a}$ is a connected tuple and let c be a single element connected to $\bar{a}$. Let us consider the sets:

$$D = \{(\bar{a}', c') \,|\, \bar{a}' \models \mathrm{tp}(\bar{a}/c) \text{ and } c' \text{ is in the path between } c \text{ and } proj(c/\bar{a})\}.$$

$$\mathcal{E} = \{S \,|\, S \text{ is a star of } \bar{a}' \text{ over } c', \text{ where } (\bar{a}', c') \in D\}$$

Then we have that $\mathrm{icl}(\mathrm{tp}(\bar{a}/c)) = \bigcup_{S \in \mathcal{E}} S$.

Proof. Take $p = \mathrm{tp}(\bar{a}/c)$. Notice that, for every $S \in \mathcal{E}$, the set $S \cap p$ is infinite: take $S \in \mathcal{E}$ a star of $\bar{a}'$ over c', where $(\bar{a}', c') \in D$. Since $\bar{a} \mathop{\smile\kern-0.65em\mid}_{c'} c$ and $\mathrm{tp}(\bar{a}/c')$ is stationary, then, for every $\bar{a}'' \in S$ such that $\bar{a}'' \mathop{\smile\kern-0.65em\mid}_{c'} c$, we have that $\bar{a} \equiv_{cc'} \bar{a}''$. In particular $\bar{a}'' \models p$, whenever $\bar{a}'' \in S$ with $\bar{a}'' \mathop{\smile\kern-0.65em\mid}_{c'} c$. Notice also that there are infinitely many such $\bar{a}''$ in S, therefore, if $\bar{b} \in S$ for some $S \in \mathcal{E}$, then, by Lemma 5.4.13, we have that $\bar{b}$ is in an indiscernible sequence, which has infinitely many elements in p. Thus $\bar{b}$ is in $\mathrm{icl}(p)$ and $\mathrm{icl}(p) \supset \bigcup_{S \in \mathcal{E}} S$.

To check the other inclusion, let $I = (\bar{a}_i)_{i<\omega}$ be an indiscernible sequence such that $\bar{a}_i \in p$ for every $i \geq 1$. Then, according to the Lemma 5.4.13, we have that I is a star of $\bar{a}_0$ over some set K. Note that $proj(c/K)$ must be between $\bar{a}_1$ and c. Let us call this element k and construct a new star S' of a_0 over k, such that the elements of $S' \setminus \{\bar{a}_0\}$ do not intersect $\mathrm{dcl}(S \cup c)$, outside $\{k\}$. Take any element $\bar{a}' \in S' \setminus \{\bar{a}_0\}$, since $\bar{a}_0 \equiv_K \bar{a}_1$ and $\bar{a}_0 \equiv_k \bar{a}'$, we conclude that $\bar{a}_1 \equiv_k \bar{a}'$. Moreover $c \mathop{\smile\kern-0.65em\mid}_k \bar{a'}$. Therefore, by stationarity $\bar{a'} \equiv_{ck} \bar{a}_1$. In particular $\bar{a'}$ is in p, this implies that S' is the star of $\bar{a}'$ over k, where $(\bar{a}', k) \in D$. Therefore $\mathrm{icl}(p) \subset \bigcup_{S \in \mathcal{E}} S$. ∎

This description of $\mathrm{icl}(\mathrm{tp}(\bar{a}/c))$ is useful as it helps to visualize why icl is idempotent: since icl covers all the possible stars with infinitely many elements in p, then, by applying icl twice, we do not get new stars. However, to make this intuition more precise, we need to describe $\mathrm{icl}(p)$ in another way:

Definition 5.4.15. Let $p = \mathrm{tp}(\bar{a}/c)$ and f be the function on $q = tp(\bar{a}/\emptyset)$ defined as follows:

If $\bar{b} \in p$, then $f(\bar{b}) = proj(c/\bar{b})$.

If $\bar{b} \notin p$, then $f(\bar{b})$ is the element that satisfies $\bar{b}f(\bar{b}) \equiv \bar{a}f(\bar{a})$. (i.e. the position of $f(\bar{b})$ in $\bar{b}$ is the same of $f(\bar{a})$ in $\bar{a}$).

Lemma 5.4.16. Let n be the distance $d(\bar{a}, c)$ and consider the set

$$J = \{\bar{a}' \equiv \bar{a} \,|\, d(f(\bar{a}'), c) = n - 2k, \text{ for some natural } k\}.$$

Then we have that $\text{icl}(p) = J$.

Proof. Using the previous lemma it suffices to show that

$$\bigcup_{S \in \mathcal{E}} S = J$$

Let $\bar{b} \in \bigcup_{S \in \mathcal{E}} S$, then $\bar{b}$ is in the star of some element $\bar{a}' \in p$ over some element c' such that c' is between c and $\bar{a}'$. Hence, either $d(f(\bar{b}), c) = n$ (and $\bar{b} \in J$) , or $f(\bar{b})$ is in the path between c and c'. Therefore

$$\begin{aligned} d(f(\bar{b}), c) &= d(c', c) - d(f(\bar{b}), c') \\ &= n - d(f(\bar{b}), c') - d(f(\bar{b}), c'). \end{aligned}$$

Naming $k = d(f(\bar{b}), c')$ we get that $\bar{b} \in J$. Thus, we have $\bigcup_{S \in \mathcal{E}} S \subset J$.

Assume now that $\bar{b} \equiv \bar{a}$ and $d(f(\bar{b}), c) = n - 2k$. Let $\gamma(f(\bar{b}), c')$ be a path of length k from $f(\bar{b})$ to a new vertex c' and such that $\gamma \cap \text{dcl}(\bar{b}, c) = f(\bar{b})$. Now consider a star S of $\bar{b}$ over c'. For all $\bar{a}_i \in S$ with $\bar{a}_i \neq \bar{b}$ we have that $d(f(\bar{a}_i), c) = d(f(\bar{b}, c)) + 2k = n$. Hence $\bar{a}_i$ is in p and the element c' is between $\bar{a}_i$ and c. Therefore we conclude that $\bar{b} \in \bigcup_{S \in \mathcal{E}} S$. Hence $\bigcup_{S \in \mathcal{E}} S \supset J$. ∎

With the last characterization we are now ready to prove the following:

Proposition 5.4.17. Let $\bar{a}$ be connected and $C = \text{dcl}(\text{cb}(\bar{a}/C))$. Then we have $\text{icl}^2(tp(\bar{a}/C)) = \text{icl}(tp(\bar{a}/C))$.

Proof. Let $p = \text{tp}(\bar{a}/C)$. We want to show that $\text{icl}^2(p) = \text{icl}(p)$.

Case 1: Assume $C = \emptyset$, then, if $\bar{a} \in \text{icl}(p)$, then there exists a sequence $(\bar{a}_i)_{i \geq 1}$ of realizations of p such that $(\bar{a}_i)_{0 \leq i}$ is $\emptyset$-indiscernible. In particular $\text{tp}(\bar{a}_0/C) = \text{tp}(\bar{a}_0) = \text{tp}(\bar{a}_1) = \text{tp}(\bar{a}_1/C)$. Therefore $\bar{a}_0 \models p$ and $\text{icl}(p) = p$.

Case 2: Assume C has more than one element. Take any two elements c and c' of C. Then there are a and a' in $\bar{a}$ such that $c = proj(a/C)$, and $c' = proj(a'/C)$. As $\bar{a}$ is connected and C is definable closed, then that the path from a to a' passes through c and c, in which case c and c' are both in $\bar{a}$. Since this is true for every c and c' in C, we conclude that $C \subset \bar{a}$. On the other hand, any indiscernible sequence I with infinitely many elements in p must be indiscernible over C. Therefore $\operatorname{icl}(p) = p$.

Case 3: Assume that $C = \{c\}$.

By Lemma 5.4.16 we know that

$$\operatorname{icl}(tp(\bar{a}/c)) = \bigcup_{\bar{a}' \in J} \operatorname{tp}(\bar{a}'/c)$$

Therefore

$$\operatorname{icl}^2(\operatorname{tp}(\bar{a}/c)) = \bigcup_{\bar{a}' \in J} \operatorname{icl}(\operatorname{tp}(\bar{a}'/c)).$$

It remains to check that, if $\bar{a}' \equiv \bar{a}$ and $d(f(\bar{a}'), c) = n - 2k$, then

$$\operatorname{icl}(\operatorname{tp}(\bar{a}'/c)) \subset \operatorname{icl}(\operatorname{tp}(\bar{a}/c)).$$

If $\bar{b} \in \operatorname{icl}(\operatorname{tp}(\bar{a}'/c))$, then there exist a tuple $\bar{a}'' \in \operatorname{tp}(\bar{a}'/c)$ and an element c'' between $\bar{a}''$ and c such that $\bar{b}$ is in a star of $\bar{a}''$ over c''. Hence

$$\begin{aligned} d(f(\bar{b}), c)) &= d(c'', c) - d(f(\bar{b}), c'') \\ &= d(\bar{a}'', c) - d(f(\bar{b}), c'') - d(f(\bar{b}), c'') \\ &= d(\bar{a}'', c) - 2d(f(\bar{b}), c'') \end{aligned}$$

Since $\bar{a}'' \in \operatorname{icl}(p)$, we have $d(\bar{a}'', c) = n - 2k$ for some k. Naming $l = d(f(\bar{b}, c''))$, we get $d(f(\bar{b}), c)) = n - 2(k + l)$. Therefore $\bar{b} \in \operatorname{icl}(tp(\bar{a}/c))$ and the conclusion follows. ∎

Finally, we need to consider canonical types $tp(\bar{a}/C)$ where $\bar{a}$ is not connected.

Proposition 5.4.18. Let $I = (\bar{a}_i \bar{b}_i)_{i<\omega}$ be an indiscernible sequence where $\bar{a}_0$ is not linked with $\bar{b}_0$. Then no $\bar{a}_i$ is linked to $\bar{b}_j$ with $i \neq j$.

Proof. Recall that FP_1 is stable, hence I is an indiscernible set. Assume there exists $i \neq j$ such that $\bar{a}_i$ is linked with $\bar{b}_j$. Then, by indiscernibility, the tuple $\bar{a}_i$ is linked with $\bar{b}_j$ for every $i \neq j$. In particular, we have the following path: $\bar{a}_0 - \bar{b}_1 - \bar{a}_2 - \bar{b}_0$, which is a contradiction. ∎

Proposition 5.4.19. Let $p = \mathrm{tp}(\bar{a}/C)$ be a canonical type, then $\mathrm{icl}(p) = \mathrm{icl}^2(p)$.

Proof. We may assume that $\bar{a} = \mathrm{dcl}(\bar{a})$, let $\bar{a}_1 ... \bar{a}_n$ be its connected components. We may also assume that $C = C_1 \cup ... \cup C_n$ where $C_i = \mathrm{cb}(\bar{a}_i/C)$. Set $p_i = \mathrm{tp}(\bar{a}_i/C_i)$. We have that $p = \{\bar{b}_1 ... \bar{b}_n \mid \bar{b}_i \models p_i$ and $\bar{b}_i$ is not linked to $\bar{b}_j$ for $i \neq j\}$. By the previous proposition, it follows that

$$\mathrm{icl}^2(p) = \{\bar{b}_1 ... \bar{b}_n \mid \bar{b}_i \in \mathrm{icl}^2(p_i) \text{ and } \bar{b}_i \text{ is not linked to } \bar{b}_j \text{ for } i \neq j\}.$$

Finally, as p_i is the canonical type of a connected component we have $\mathrm{icl}(p_i) = \mathrm{icl}^2(p_i)$. Hence $\mathrm{icl}(p) = \mathrm{icl}^2(p)$, which is the desired conclusion. ∎

From the previous lemma we may conclude the main result.

Corollary 5.4.20. The theory FP_1 is in I_3.

Question 5.4.21. Let FP_n be the theory of the n-dimensional free pseudospace. Is FP_n in I_{n+2}?

Question 5.4.22. If T is CM-trivial, then is it in I_3?

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