Estimation and control of descriptor systems

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Doctoral Thesis

ESTIMATION AND CONTROL OF DESCRIPTOR SYSTEMS

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Keywords : Descriptor systems, Takagi-Sugeno models, control design, observer design, linear matrix inequalities.

Mots-clés: Systèmes descripteurs, modèles Takagi-Sugeno, commande, observateur, inégalités matricielles linéaires.

"No hay límites para el deterioro"

Vargas-Llosa

"I came here to do some business, not shoot the breeze.

You want to expound your personal philosophy, write another book.

[...] I don't have to like you. You're a client, and you pay well"

The Ninth Gate, dir. Polansky

Abstract

This thesis addresses the estimation and control for nonlinear descriptor systems. The developments are focused on a family of nonlinear descriptor models with a full-rank descriptor matrix. The proposed approaches are based on a Takagi-Sugeno (TS) descriptor representation of a given nonlinear descriptor model. This type of TS models is a generalization of the standard TS ones. One of the mains goals is to obtain conditions in terms of linear matrix inequalities (LMIs). In the existing literature, the observer design for TS descriptor models has led to bilinear matrix inequality (BMI) conditions. In addition, to the best of our knowledge, there are no results in the literature on controller/observer design for discrete-time TS descriptor models (with a non-constant and invertible descriptor matrix).

Three problems have been addressed: state feedback controller design, observer design, and static output feedback controller design. LMI conditions have been obtained for both continuous and discrete-time TS descriptor models. In the continuous-time case, relaxed LMI conditions for the state feedback controller design have been achieved via parameter-dependent LMI conditions. For the observer design, pure LMI conditions have been developed by using a different extended estimation error. For the static output feedback controller, LMI constraints can be obtained once an auxiliary matrix is fixed. In the discrete-time case, results in the LMI form are provided for state/output feedback controller design and observer design; thus filling the gap in the literature. Several examples have been included to illustrate the applicability of the obtained results and the importance of keeping the original descriptor structure instead of computing a standard state-space.

Keywords – Descriptor systems, Takagi-Sugeno models, controller design, observer design, linear matrix inequalities.

Résumé

Cette thèse est consacrée au développement des techniques d'estimation et de commande pour systèmes descripteurs non linéaires. Les développements sont centrés sur une famille particulière de systèmes descripteurs non linéaires avec une matrice descripteur de rang plein. Toutes les approches présentées utilisent un formalisme de modélisation du type Takagi-Sugeno (TS) pour représenter les modèles descripteurs non linéaires. Un objectif très important est de développer des conditions sous la forme d'inégalités matricielles linéaires (LMI, en anglais). Dans la littérature, les conditions pour l'estimation des modèles TS descripteurs s'écrivent sous forme d'inégalités matricielles bilinéaires (BMI, en anglais). En plus, à notre connaissance, il n'y pas de résultats dans la littérature concernant la commande/estimation pour les modèles TS descripteurs en temps discret (avec une matrice descripteur régulière non linéaire).

Trois problèmes ont été examinés : commande par retour d'état, estimation de l'état et commande statique par retour de la sortie. Dans le cas continu, des conditions moins conservatives ont été développées pour la commande par retour d'état. Pour l'estimation d'état, des conditions LMI ont été obtenues (au lieu des usuelles BMI) en utilisant un différent vecteur d'erreur augmenté. Pour la commande statique par retour de la sortie, des conditions LMI sont proposées si une matrice auxiliaire est fixée. Pour le temps discret, des nouveaux résultats sous la forme LMI ont été développées pour la commande/estimation, comblant ainsi certains manques de la littérature. Des exemples ont été inclus pour montrer l'applicabilité de tous les résultats que nous avons obtenus et ainsi l'importance de garder la structure originale des descripteurs.

Mots clés – Systèmes descripteurs, modèles Takagi-Sugeno, commande, observateur, inégalités matricielles linéaires.

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Valenciennes, October 2015.

Víctor Estrada

Index

Chapter 1. Introduction

1.1. Context of the thesis

Fuzzy models are based on IF-THEN rules originally considered to represent operator experience and thus avoid the necessity of the mathematical representation of the system. Therefore, Takagi-Sugeno (TS) models were considered an approach that emulate human operators (Takagi and Sugeno, 1985) and were regarded as a heuristic technique. Breaking with this initial way, model-based approaches have been introduced by Tanaka and Sugeno, (1992). These approaches keep a framework similar to the initial fuzzy modelling one but the analysis and synthesis methods used have resulted in losing the heuristic point of view: classical tools such as Lyapunov analysis and synthesis have been introduced.

In the past decades, TS models have been widely used to represent nonlinear systems. Two frequently used model-based methodologies to obtain a TS representation are 1) linearization around several points (Johansen et al., 2000) and 2) the sector nonlinearity approach (Ohtake et al., 2001). The former provides a TS model which is an approximation, while the latter gives a TS model that exactly represents the given nonlinear model in a compact set corresponding to the sectors considered. Therefore, the sector nonlinearity approach has been widely adopted. Nevertheless, the sector nonlinearity approach has an important shortcoming: the number of local linear models (rules) is exponentially related to the number of non-linear terms in the original nonlinear model.

A TS model is a collection of linear models blended together with membership functions (MFs), which are nonlinear and hold the convex sum property (Lendek et al., 2010; Tanaka and Sugeno, 1992; Tanaka and Wang, 2001; Wang et al., 1996). The direct Lyapunov method is employed for the analysis and controller/observer design for such systems; usually, the conditions are cast as linear matrix inequalities (LMIs). The interest of obtaining conditions in LMI form comes from the fact that they can be efficiently solved via convex optimization techniques (Boyd et al., 1994; Scherer and Weiland, 2005).

The conditions developed within the TS-LMI framework, similarly to LPV, quasi-LPV, and piecewise models are only sufficient, that is, if the LMI problem is unfeasible, no conclusion can be drawn. Unfeasible conditions for stable / controllable / observable systems may be obtained due to several reasons: the type of Lyapunov function chosen, the way MFs (the nonlinear parts of the TS model) are removed to obtain an LMI formulation, the non-uniqueness of a TS representation from a given nonlinear model, etc.

Quadratic Lyapunov functions have been extensively employed for the stability analysis or controller / observer design for TS models. They led to several results within the TS-LMI framework (Bergsten and Driankov, 2002; Ichalal et al., 2008; Lendek et al., 2010; Tanaka et al., 1998; Tanaka and Sugeno, 1992; Tanaka and Wang, 2001; Wang et al., 1996). However, since a common Lyapunov matrix is used for all the linear local models of the TS model, this type of Lyapunov function is in some cases highly conservative.

To alleviate the conservativeness, in (Blanco et al., 2001) a non-quadratic (fuzzy) Lyapunov function has been introduced; this Lyapunov function uses the same MFs as the TS model under study. For the continuous-time case, the use of such Lyapunov functions leads to dealing with the time-derivatives of the MFs. Several methods have been proposed to tackle this problem: by bounding a priori the time-derivatives of the MFs and checking a posteriori such bounds (Bernal et al., 2006; Blanco et al., 2001; Mozelli et al., 2009; Tanaka et al., 2003); via piecewise Lyapunov functions (Campos et al., 2013; Johansson et al., 1999); via line-integral Lyapunov functions (Mozelli et al., 2009; Rhee and Won, 2006); or by bounding the partial derivatives of the MFs, leading to local conditions (Bernal and Guerra, 2010; Guerra et al., 2012a; Guerra and Bernal, 2009; Lee and Kim, 2014; Pan et al., 2012). For the discrete-time case, the time derivative is replaced by a one sample delay that appears to have fewer drawbacks. The use of non-quadratic (NQ) Lyapunov functions has led to important improvements (Ding et al., 2006; Guerra et al., 2012b, 2009; Guerra and Vermeiren, 2004; Kruszewski et al., 2008; Lee et al., 2010; Lendek et al., 2015).

If the full information of the state is not available, one alternative is the use of state observers (Luenberger, 1971). Usually two cases are considered for the observer design: 1) the MFs depend on measured (available) variables and, 2) the MFs depend on unmeasurable variables (Bergsten and Driankov, 2002; Ichalal et al., 2008; Tanaka et al., 1998). The first case can be seen sometimes (for example when quadratic Lyapunov functions are used) as the dual of the controller design, while the second one requires extra conditions, e.g., Lipchitz conditions (see Bergsten et al. 2002), to guarantee the convergence of the estimation error. Another alternative when only partial information of the state is available is the design of output feedback controllers (Cao et al., 1998; Chadli and Guerra, 2012; Kau et al., 2007; Syrmos et al., 1997). However, the existing conditions for output feedback are not always "pure" LMIs.

Based on nonlinear descriptor models (Luenberger, 1977) – that naturally appear in mechanical systems (Dai, 1989; Lewis, 1986; Lewis et al., 2004; Luenberger, 1977) –, TS descriptor models have been introduced in (Taniguchi et al., 1999). TS descriptor models use two families of MFs: one for the nonlinearities in the left-hand side (descriptor matrix) and another one for the nonlinear terms in the right-hand side. Tools developed for descriptor models have also been used for models which do not appear in a natural descriptor form. For example, the so-called descriptor redundancy approach (Tanaka and Sugie, 1997) has been adopted in order to relax existing conditions (Cao and Lin, 2004; Chen, 2004; Guelton et al., 2009; Tanaka and Sugie, 1997; Tanaka et al., 2007).

Since the descriptor matrix may be singular, descriptor models are also called singular systems, differential-algebraic equation (DAE) systems, partial state space representation, etc. (Dai, 1989). For linear singular systems, generally, it is not sufficient to study their stability, but their admissibility has to be investigated. Therefore, concepts such as regular and impulse-free systems have been introduced. A descriptor system is admissible if it is regular, impulse-free, and stable (Dai, 1989). The concepts of controllability, observability, and duality have been stated in (Cobb, 1984). Controller design has been carried out in (Mukunda and Dayawansa, 1983). Observer design conditions have been developed in (Dai, 1988; Darouach and Boutayeb, 1995), but these conditions are not in LMI form. Later, LMI conditions have been given in (Chadli and Darouach, 2012; Feng and Yagoubi, 2013; Fridman and Shaked, 2002; Garcia et al., 2002, 1998; Masubuchi and Ohta, 2013; Rehm and Allgöwer, 2002; Zhang et al., 2008).

Recently, in the linear-parameter-varying (LPV) field, several works concerning controller/observer design have appeared (Chadli et al., 2008a, 2008b; Hamdi et al., 2009; López-Estrada et al., 2014). For nonlinear systems with a constant rank-deficient descriptor matrix, few results that involve LMI constraints (Wang et al., 2012; Yang et al., 2013) exist.

All the above results on descriptor models consider a constant rank-deficient matrix. This thesis focuses on nonlinear systems with a non-constant full-rank descriptor matrix. In such case a standard state space model can be computed; however, it is important to keep the original descriptor structure (Taniguchi et al., 1999). Stability conditions based on quadratic Lyapunov functions for continuous-time TS descriptors models have been established in (Taniguchi et al., 1999). Controller design conditions also based on quadratic Lyapunov functions for TS descriptor models have been given in (Taniguchi et al., 2000). These conditions have been improved in (Guerra et al., 2007) and extended to robust control in (Vermeiren et al., 2012). Observer design for continuous-time TS descriptors has been addressed in (Guerra et al., 2004); the procedure leads to a set of bilinear matrix inequalities (BMIs). Sufficient LMI conditions can be derived by fixing beforehand one of the decision variables. To the best of our knowledge, discrete-time TS descriptor models with nonsingular descriptor matrix have not been considered in the literature.

1.2. Scope and objectives

This work is concerned with developing conditions for nonlinear descriptor models in order to improve the conditions found in the literature both for controller and observer designs. In addition, since there are no results concerning the discrete-time case when the descriptor matrix is invertible, LMI conditions for controller/observer design for discrete-time TS descriptor models have been developed. The problems considered are:

- State feedback controller design.
- Observer design.
- Output feedback controller design.

The methods developed in this thesis are based on TS descriptor representations of a given nonlinear descriptor model (in both continuous and discrete time), using Lyapunov's direct method, and with the objective of developing "pure" LMI conditions.

1.3. Outline

The thesis is organized as follows:

Chapter 2 gives the necessary background on the TS-LMI framework, the descriptor form, and motivates the use of TS descriptor models.

Chapters 3-5 develop design conditions for TS descriptor models.

Chapter 3 considers conditions for state feedback controller design. With respect to previous LMI conditions, a larger solution set for the continuous-time case is achieved. In the discrete-time case, LMI conditions are given with different NQ Lyapunov functions, thus filling the gap in the literature.

Chapter 4 deals with observer design for TS descriptor models. For the continuous-time case, LMI conditions are obtained by using a different extended estimation error. For the discrete-time case, results in LMI form are provided via several types of Lyapunov functions.

Chapter 5 considers static output feedback controller design. For both continuous and discrete-time, the developed conditions are still BMI and become LMI only if a slack variable is fixed.

Chapter 6 concludes this work with final remarks and some future research directions.

Additionally, a brief introduction to LMIs and some properties used throughout this thesis are given in *Appendix A*. *Appendix B* is devote to give some sum relaxations.

1.4. Publications

The results presented in this thesis have been disseminated in the following publications:

International journal publications:

1. V. Estrada-Manzo, Zs. Lendek, T. M. Guerra, and P. Pudlo. (2015). *Controller design for discrete-time descriptor models: a systematic LMI approach*. IEEE Transactions on Fuzzy Systems, vol. 23 (5), pp. 1608-1621.
2. T. M. Guerra, V. Estrada-Manzo, and Zs. Lendek. (2015). *Observer design for nonlinear descriptor systems: an LMI approach*. Automatica (52), pp. 154-159.

Book chapters:

1. V. Estrada-Manzo, Zs. Lendek, and T. M. Guerra. (2015). *Observer design for robotic systems via Takagi-Sugeno models and linear matrix inequalities*. In Handling Uncertainty and Networked Structure in Robot Control. Ed. Springer.

Conference publications:

1. V. Estrada-Manzo, Zs. Lendek, and T.M. Guerra. (2015). *Unknown input estimation for nonlinear descriptor systems via LMIs and Takagi-Sugeno models.* In Proceedings of the 54th IEEE Conference on Decision and Control (CDC). Osaka, Japan. pp. 1-6.
2. V. Estrada-Manzo, T.M. Guerra, and Zs. Lendek. (2015). *Static output feedback control for continuous-time TS descriptor models: decoupling the Lyapunov function.* In Proceedings of the 2015 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE). Istanbul, Turkey. pp. 1-5.
3. V. Estrada-Manzo, Zs. Lendek, and T. M. Guerra. (2015). *Improving observer design for discrete-time TS descriptor models under the quadratic framework.* In Proceedings of 2nd IFAC Conference on Embedded Systems, Computational Intelligence and Telematics in Control (CESCIT). Maribor, Slovenia, pp. 276-281.
4. V. Estrada-Manzo, Zs. Lendek, and T. M. Guerra. (2014). *Output feedback control for T-S discrete-time nonlinear descriptor models*. In Proceedings of the 53rd IEEE Conference on Decision and Control (CDC), Los Angeles, USA, pp. 860-865.
5. V. Estrada-Manzo, Zs. Lendek, and T. M. Guerra. (2014). *H_∞ control for discrete-time Takagi-Sugeno descriptor models: a delayed approach*. In Proceedings of the 23rd Rencontres francophones sur la logique floue et ses applications (LFA), Ajaccio, France. pp 175-182.
6. V. Estrada-Manzo, T. M. Guerra, and Zs. Lendek. (2014). *An LMI approach for observer design for Takagi-Sugeno descriptor models*. In Proceedings of the 2014 IEEE International Conference on Automation, Quality and Testing, Robotics (AQTR), Cluj-Napoca, Romania, pp. 1-5.
7. V. Estrada-Manzo, Zs. Lendek, and T. M. Guerra. (2014). *Discrete-time Takagi-Sugeno descriptor models: observer design.* In Proceedings of the 19th IFAC World Congress, Cape Town, South Africa, pp. 7965-7969.

8. V. Estrada-Manzo, T. M. Guerra, Zs. Lendek, and P. Pudlo. (2014) *Discrete-time Takagi-Sugeno descriptor models: controller design*. In Proceedings of the 2014 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE). Beijing, China, pp. 2277-2281.
9. V. Estrada-Manzo, T. M. Guerra, Zs. Lendek, and M. Bernal. (2013). *Improvements on non-quadratic stabilization of continuous-time Takagi-Sugeno descriptor models*. In Proceedings of the 2013 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), Hyderabad, India, pp. 1-6.

Chapter 2. Takagi-Sugeno models

This chapter will provide the reader with the basic knowledge on Takagi-Sugeno (TS) models as well as an introduction to the existing results in this framework. It is not intended to be an exhaustive survey but rather the necessary background to follow the developments in next chapters. In addition, it motivates the use of TS descriptor models instead of standard TS ones when a nonlinear models in the descriptor form. The final remarks in the chapter enumerate the problems to be faced in the present thesis.

2.1. Standard TS models

A TS model is a collection of linear systems and nonlinear membership functions (MFs) of the form (Takagi and Sugeno, 1985):

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\left(z(t)\right)\left(A_i x(t) + B_i u(t)\right), \tag{2.1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $z(t) \in \mathbb{R}^{p}$ is the premise vector, and r is the number of rules (local models). Matrices (A_i, B_i), $i \in \{1,2,\ldots,r\}$ represent the i-th linear model of the TS one (2.1). The scheduling vector $z(t)$ may depend on the state, input, exogenous parameters, or time, on measurable and/or unmeasurable variables. The MFs $h_i\left(z(t)\right)$, $i \in \{1,2,\ldots,r\}$ are nonlinear functions and hold the convex sum property in a compact set of the state space Ω:

$$\sum_{i=1}^{r} h_i\left(z(t)\right)=1, \qquad 0 \le h_i\left(z(t)\right) \le 1. \tag{2.2}$$

There are several model-based procedures to obtain a TS representation from a given nonlinear model. Two of them are frequently used. The first one is a method based on linearization in several operating points (Johansen et al., 2000). The second method is called the sector nonlinearity approach (Ohtake et al., 2001) and consists of an algebraic rewriting of the original nonlinear model based on the known bounds of the nonlinearities. The former provides a TS model which approximates the nonlinear one, while the latter gives a TS model that exactly represents the nonlinear one in a compact set (Lendek et al., 2010; Tanaka and Wang, 2001).

2.1.1. The sector nonlinearity approach

This thesis focuses on TS models derived by using the sector nonlinearity approach, although LMI conditions can be applied regardless of the origin of the TS model. The idea of this approach is to rewrite a nonlinear expression as a convex combination of nonlinear membership functions (MFs). This is summarized in the following steps (Ohtake et al., 2001; Tanaka and Wang, 2001).

Consider the following nonlinear model:

$$\dot{x}(t)=f\left(x(t),u(t)\right), \tag{2.3}$$

where $f(\bullet)$ is a nonlinear function whose elements are smooth and bounded in a compact set of the state space Ω. In what follows, arguments will be omitted when their meaning is straightforward.

Step 1. Assume that the nonlinear system (2.3) can be expressed as the affine-in control model:

$$\dot{x}=A(x)x+B(x)u, \tag{2.4}$$

where $A(x)\in\mathbb{R}^{n_x\times n_x}$ and $B(x)\in\mathbb{R}^{n_x\times n_u}$ are matrices whose entries may be non-constant terms, which are assumed to be bounded in Ω. Thus consider the p non-constant terms that appear in $\left(A(x),B(x)\right)$, i.e., $nl_j(\bullet)\in\left[\underline{nl}_j \quad \overline{nl}_j\right]$, $\underline{nl}_j=\inf\left(nl_j(\bullet)\right)$, $\overline{nl}_j=\sup\left(nl_j(\bullet)\right)$, $j\in\{1,2,\ldots,p\}$; these non-constant terms constitute the premise vector $z\in\mathbb{R}^p$.

Step 2. Construct, for each $nl_j(\bullet)$, $j \in \{1,2,\ldots,p\}$, a pair of weighting functions (WFs) as follows:

$$\omega_0^j(\bullet) = \frac{nl_j(\bullet) - \underline{nl}_j}{\overline{nl}_j - \underline{nl}_j}, \qquad \omega_1^j(\bullet) = 1 - \omega_0^j(\bullet), \qquad j \in \{1,2,\ldots,p\}. \tag{2.5}$$

By construction, each pair of WFs holds the convex sum property in the compact set Ω.

Step 3. Define the $r = 2^p$ membership functions (MFS) using WFs in (2.5):

$$h_i(z) = \prod_{j=1}^{p} \omega_{i_j}^j(z_j), \qquad i \in \{1,2,\ldots,r\}, \qquad i_j \in \{0,1\}. \tag{2.6}$$

These MFs hold the convex sum property (2.2) in Ω.

Step 4. Compute the linear local models (A_i, B_i), $i \in \{1,2,\ldots,r\}$ of (2.1). To this end, it is necessary to substitute into $(A(z), B(z))$ the values of the bounds $\underline{nl}_j$, $\overline{nl}_j$, $j \in \{1,2,\ldots,p\}$ that activate each rule, i.e., when $h_i(z) = 1$, $i \in \{1,2,\ldots,r\}$.

Based on the above definitions, the nonlinear model (2.3) is exactly represented by the TS model (2.1) in the considered set Ω.

Remark 2.1: Formally, it is possible to consider also systems that are not affine-in-control, i.e., $\dot{x} = A(x,u)x + B(x,u)u$. However the fuzzy control laws will generally include the MFs $h_i(z(x,u))$. This means that implicit equations, i.e., $u = f(h(x,u),x)$ have to be solved, which are nonlinear and difficult to work with. In the context of this work only affine in the control models are considered.

Remark 2.2. The total number of rules r depends on the number of nonlinear terms p, that is, $r = 2^p$. Since the relation is exponential, this can be a problem when modeling complex nonlinear systems as it can lead to computationally intractable problems.

Remark 2.3. From a given nonlinear model, several TS representations can be obtained, since different algebraic manipulations may lead to different premise vectors. Since the resulting vertices (A_i, B_i), $i \in \{1,2,\ldots,r\}$ may have different properties, e.g., they may be stable/unstable or uncontrollable/unobservable; thus different TS representations may lead to different results. This non-uniqueness is considered one of the shortcomings of TS models.

Example 2.1. Consider the following nonlinear model in the compact set $\Omega = \{x : |x_i| \le 1, i = 1,2\}$:

$$\dot{x}_1 = -x_1 + x_1 x_2, \qquad \dot{x}_2 = x_1 - 3x_2. \tag{2.7}$$

One possible way of rewriting the system in the form (2.4) is by defining:

$$\dot{x} = A(x)x, \quad \text{with} \quad A(x) = \begin{bmatrix} -1 & x_1 \\ 1 & -3 \end{bmatrix}. \tag{2.8}$$

Following the procedure above one can identify $nl_1 = x_1$ as the non-constant term in $A(x)$, and it is bounded inside Ω as $nl_1 \in [-1 \quad 1]$; the premise vector is $z = nl_1 = x_1$. Thus (2.8) writes:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & nl_1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2.9}$$

Then, the following WFs can be constructed:

$$\omega_0^1(x_1) = \frac{x_1 - (-1)}{1 - (-1)} = \frac{x_1 + 1}{2}, \quad \omega_1^1(x_1) = 1 - \omega_0^1(x_1) = \frac{1 - x_1}{2}. \tag{2.10}$$

Using the WFs in (2.10), the MFs are $h_1(z) = \omega_0^1(x_1)$ and $h_2(z) = \omega_1^1(x_1)$. The local matrices are computed as follows:

$$A_1 = \begin{bmatrix} -1 & \overline{nl}_1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -3 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & \underline{nl}_1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}. \tag{2.11}$$

Finally, the TS model is

$$\dot{x} = \sum_{i=1}^{2} h_i(z) A_i x = \left(h_1(z) A_1 + h_2(z) A_2\right) x, \tag{2.12}$$

which exactly represents the nonlinear model (2.7) in the compact set Ω.

Recall that for a given nonlinear model there are many TS representations (Remark 2.3). For instance, by choosing $A(x)$ as $A(x) = \begin{bmatrix} -1 + x_2 & 0 \\ 1 & -3 \end{bmatrix}$, we obtain $nl_1 = x_2 \in [-1 \quad 1]$, the WFs are $\omega_0^1 = 0.5(x_2 + 1)$ and $\omega_1^1 = 1 - \omega_0^1$; thus the MFs are $h_1 = \omega_0^1$ and $h_2 = \omega_1^1$. The local matrices are calculated as:

$$A_1 = \begin{bmatrix} -1+\overline{nl}_1 & 0 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1+\underline{nl}_1 & 0 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix}. \tag{2.13}$$

Note that A_1 is not Hurwitz. ♦

2.1.2. Notation

Throughout this thesis, the following shorthand notation is adopted to represent convex sums of matrix expressions:

$$\Upsilon_h = \sum_{i=1}^{r} h_i(z)\Upsilon_i, \quad \Upsilon_h^{-1} = \left(\sum_{i=1}^{r} h_i(z)\Upsilon_i\right)^{-1}, \quad \text{and} \quad \Upsilon_{hhv} = \sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{j=1}^{r} h_{i_1}(z)h_{i_2}(z)v_j(z)\Upsilon_{i_1 i_2}^{j}.$$

In the discrete-time case, MFs with delays may appear:

$$\Upsilon_h = \sum_{i=1}^{r} h_i(z(k))\Upsilon_i, \quad \Upsilon_{h^+} = \sum_{i_x=1}^{r} h_{i_x}(z(k+1))\Upsilon_{i_x}, \quad \text{and} \quad \Upsilon_{h^-} = \sum_{i_x=1}^{r} h_{i_x}(z(k-1))\Upsilon_{i_x}.$$

Subscripts will change to v if the respective MFs are v_j, e.g., $\Upsilon_v = \sum_{j=1}^{r} v_j(z)\Upsilon_j$. Using the aforementioned notation, the TS model (2.1) is written as $\dot{x} = A_h x + B_h u$.

An asterisk $(*)$ will be used in matrix expressions to denote the transpose of the symmetric element; for in-line expressions it will denote the transpose of the terms on its left side, for example:

$$\begin{bmatrix} A & B^T \\ B & C \end{bmatrix} = \begin{bmatrix} A & (*) \\ B & C \end{bmatrix}, \qquad A + B + A^T + B^T + C = A + B + (*) + C.$$

In addition, in matrix expressions, the symbols ">" and "<" will stand for positive and negative-definiteness, respectively. Arguments will be omitted when their meaning is clear.

2.1.3. Overview of existing results

The main advantage of expressing a nonlinear model as a TS one is that the direct Lyapunov method can be systematized. The main objective is to express the conditions in terms of LMIs, which can be efficiently solved via convex optimization techniques (Boyd et al., 1994; Scherer and Weiland, 2005).

The continuous-time case

This section will briefly present established results on the analysis and design for standard continuous-time TS models. Recall the TS model:

$$\dot{x} = \sum_{i=1}^{r} h_i(z)(A_i x + B_i u), \qquad y = \sum_{i=1}^{r} h_i(z) C_i x, \tag{2.14}$$

When $u = 0$, system (2.14) has an equilibrium point in $x = 0$. Sufficient conditions for the stability of (2.14) with $u = 0$ are given in the sense of Lyapunov. Effectively, the stability of the equilibrium point of the autonomous TS model (2.14) is analyzed using the quadratic Lyapunov function

$$V(x) = x^T P x, \qquad P = P^T > 0. \tag{2.15}$$

The equilibrium point is asymptotically stable if there exists a matrix $P = P^T$ such that (Tanaka and Wang, 2001):

$$P > 0, \qquad A_i^T P + P A_i < 0, \qquad \forall i \in \{1, 2, \ldots, r\}. \tag{2.16}$$

The LMI conditions (2.16) are directly obtained when the time-derivative of (2.15) is taken:

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = x^T \left(A_h^T P + P A_h \right) x = x^T \left(\sum_{i=1}^{r} h_i(z) \left(A_i^T P + P A_i \right) \right) x. \tag{2.17}$$

Since $\sum_{i=1}^{r} h_i(z) = 1$ and $0 \le h_i(z) \le 1$, a sufficient condition for $\dot{V}(x) < 0$ is given by the LMI conditions (2.16). Note that (2.16) is reduced to the Lyapunov stability theorem for linear systems, i.e., when $r = 1$.

Remark 2.4. Conditions (2.16) do not take into account the information of the MFs; in addition, the Lyapunov function candidate is restricted to a quadratic one. Hence, the given LMI conditions are only sufficient, i.e., if the LMI problem is unfeasible, no conclusion can be drawn. Moreover, notice that conditions (2.16) are valid for a family of TS models with the same vertex matrices. Therefore, it is also equivalent to LPV quadratic stability.

Example 2.2. Recall the nonlinear model in Example 2.1:

$$\dot{x}_1 = -x_1 + x_1 x_2, \qquad \dot{x}_2 = x_1 - 3x_2. \tag{2.18}$$

Consider two different TS representations for (2.18):

$$\mathrm{TS}_1 : \dot{x} = A_h x, \quad \text{with} \quad A_1 = \begin{bmatrix} 0 & 0 \\ 1 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix}, \quad \begin{cases} h_1 = 0.5(x_2 + 1) \\ h_2 = 1 - h_1 \end{cases} \tag{2.19}$$

$$\mathrm{TS}_2 : \dot{x} = A_h x, \quad \text{with} \quad A_1 = \begin{bmatrix} -1 & 1 \\ 1 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}, \quad \begin{cases} h_1 = 0.5(x_1 + 1) \\ h_2 = 1 - h_1 \end{cases} \tag{2.20}$$

The stability analysis using LMIs (2.16) for TS_1 (2.19) yields unfeasible LMIs (A_1 is not Hurwitz), but since conditions (2.16) are only sufficient, no conclusion can be drawn from this result (see Remark 2.4). Indeed, for TS_2 (2.20), the LMI conditions are feasible and provide the following Lyapunov matrix:

$$P = \begin{bmatrix} 0.5510 & 0.0443 \\ 0.0443 & 0.2635 \end{bmatrix}. \tag{2.21}$$

Since the LMI conditions do not consider the information on the MFs beside their convex structure, any TS model with vertex matrices (2.20) will be stable regardless of the original nonlinear model. To see this, consider:

$$\dot{x}_1 = -x_1 + \cos(x_2) x_2, \qquad \dot{x}_2 = x_1 - 3x_2. \tag{2.22}$$

The nonlinear system (2.22) can be exactly represented in $\Omega \in \{x : x \in \mathbb{R}^2\}$ by a TS model (2.14) with local models:

$$A_1 = \begin{bmatrix} -1 & 1 \\ 1 & -3 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}, \tag{2.23}$$

with $h_1 = 0.5(\cos(x_2) + 1)$ and $h_2 = 1 - h_1$. Since the vertex matrices are the same as (2.20), and the LMI conditions are feasible, the stability of (2.22) is also established. ◆

Remark 2.5. If the set (2.16) of LMI conditions is feasible, it establishes global stability of the TS model. However, since the TS model is usually valid locally in a compact set of the state space Ω, this does not mean that the original nonlinear model is globally stable. Thus, it is necessary to study the outermost Lyapunov level in the modelling region Ω.

For the controller design, (Wang et al., 1995) have proposed the following parallel distributed compensator (PDC):

$$u = \sum_{i=1}^{r} h_i(z) K_i x = K_h x, \quad \text{with} \quad K_i \in \mathbb{R}^{n_u \times n_x}, \quad i \in \{1, 2, \ldots, r\}. \tag{2.24}$$

This type of controllers incorporates the system's MFs in the control law; therefore, it relaxes the linear controller $u = Kx$ by introducing r control gains. The TS model (2.14) together with PDC control law (2.24) produces

$$\dot{x} = \sum_{i_1=1}^{r}\sum_{i_2=1}^{r} h_{i_1}(z) h_{i_2}(z)\left(A_{i_1} + B_{i_1} K_{i_2}\right)x. \tag{2.25}$$

Stabilization conditions can be obtained via the quadratic Lyapunov function (2.15) as follows:

$$\dot{V}(x) = x^T \left(\left(A_h + B_h K_h\right)^T P + P\left(A_h + B_h K_h\right)\right)x < 0,$$

which after using the congruence property with the matrix $X = P^{-1}$ and a change of variables $M_h = K_h X$ gives

$$\dot{V}(x) < 0 \Leftrightarrow A_h X + B_h M_h + (*) < 0. \tag{2.26}$$

This is a co-negativity problem. To get more relaxed conditions than the trivial $A_{i_1} X + B_{i_1} M_{i_2} + (*) < 0$, $\forall i_1, i_2 \in \{1, 2, \ldots, r\}$, *sum relaxations* are used and the MFs are dropped. In Appendix B, several sum relaxations are given.

The quadratic framework presents an important drawback: a single matrix $P = P^T > 0$ must satisfy the conditions for each linear local model of a given TS model. In the continuous-time case, piecewise quadratic Lyapunov functions have been investigated as a natural option for those TS models which do not have all linear models activated at once (Johansson et al., 1999). This approach cannot be applied to TS models constructed by the sector nonlinearity approach because all the rules are active at the same time. On the other hand, non-quadratic Lyapunov functions (or fuzzy Lyapunov functions) have first been used by (Blanco et al., 2001; Tanaka et al., 2003):

$$V(x) = x^T \left(\sum_{i=1}^{r} h_i(z) P_i\right)x, \qquad P_i = P_i^T > 0, \qquad i \in \{1, 2, \ldots, r\}. \tag{2.27}$$

The analysis of (Tanaka et al., 2003) is based on the existence of scalars ϕ_i such that $\left|\dot{h}_i(z)\right| \le \phi_i$ $i \in \{1, 2, \ldots, r\}$, these bounds must be checked a *posteriori*. A way to avoid this problem has been presented in (Rhee and Won, 2006): a path-independent Lyapunov function has been proposed. This type of Lyapunov function avoids the time-derivative of the MFs and provides global results; however, it is restricted to a specific family of TS models and the controller design leads to conditions in bilinear matrix inequality (BMI) form. In (Bernal et al., 2006) another controller structure has been proposed:

$$u=\left(\sum_{i=1}^{r}h_i(z)K_i\right)\left(\sum_{i=1}^{r}h_i(z)P_i\right)^{-1}x=K_hP_h^{-1}x. \tag{2.28}$$

The controller (2.28) is known as a non-PDC control law – it first appeared in the discrete-time framework (Guerra and Vermeiren, 2004) – ; the stabilization conditions are derived via a non-quadratic Lyapunov function of the form $V(x)=x^TP_h^{-1}x$; this approach allows the inclusion of $|u(t)|<\mu$ into the MFs, however $|\dot{h}_i(z)|\le\phi_i$ must still be given a *priori*.

Example 2.3 Consider the following nonlinear system (Pan et al., 2012; Tanaka et al., 2007):

$$\dot{x}=ax+\left(x^3+b\right)u. \tag{2.29}$$

By employing the sector nonlinearity approach, the following TS model is obtained:

$$\dot{x}=\sum_{i=1}^{2}h_i(z)\left(A_ix+B_iu\right). \tag{2.30}$$

where $A_1=A_2=a$, $B_1=d^3+b$, $B_2=-d^3+b$; the MFs are $h_1(z)=w_0^1=\left(x^3+d^3\right)/2d^3$ and $h_2(z)=w_1^1=\left(d^3-x^3\right)/2d^3$; they hold the convex sum property in the compact set $\Omega=\{x:|x|\le d\}$. By computing $\dot{h}_1(z)$, it gives

$$\left|\dot{h}_1(z)\right|=\left|\frac{3}{2d^3}x^2\dot{x}\right|=\left|\frac{3}{2d^3}x^2\left(ax+\left(x^3+b\right)u\right)\right|\le\phi_1. \tag{2.31}$$

Via this simple example, it can be seen that since $\dot{h}_1(z)$ depends on the control law to be designed, the assumption on an *a priori* bound of the time-derivatives of the MFs is an important drawback (Tanaka et al., 2007, 2003). The validity of these conditions must be checked *a posteriori*, which makes their usefulness questionable. ◆

Recently, another alternative has been stated in (Bernal and Guerra, 2010; Guerra and Bernal, 2009); the main idea is to develop local stability conditions instead of global ones by bounding the partial derivatives of the MFs; these bounds can be calculated a *priori* and incorporated in the LMI conditions: therefore they no longer need to be verified. This idea has been extended to the controller design in (Guerra et al., 2012a; Pan et al., 2012).

When the full state is not available for control tasks, an observer can be implemented. In case of the state observer for the TS model (2.25), whose output is given by $y = C_h x$, two cases can be considered (Lendek et al., 2010): 1) the MFs depend only on measured premise variables; 2) the MFs depend on some of the unmeasured variables. This thesis considers the former case, i.e., the nonlinear terms must depend on available (measurable) variables. State observers usually have the form:

$$\dot{\hat{x}} = A_h \hat{x} + B_h u + L_h \left(y - \hat{y} \right), \qquad \hat{y} = C_h \hat{x}, \tag{2.32}$$

where $L_h = \sum_{i=1}^{r} h_i(z) L_i$ is the observer gain. By defining the estimation error $e = x - \hat{x}$, its dynamics yield $\dot{e} = \left(A_h - L_h C_h \right) e$. Thus, via a Lyapunov function $V(e) = e^T P e$, $P = P^T > 0$ the following conditions are obtained:

$$\dot{V}(e) < 0 \Leftrightarrow PA_h - N_h C_h + (*) < 0, \tag{2.33}$$

where $N_h = PL_h$. In order to achieve LMI conditions, sufficient conditions for (2.33) to hold are obtained via sum relaxations (see Appendix B). For the case of unmeasurable premise variables the interested reader is referred to (Bergsten et al., 2002; Ichalal et al., 2008).

The discrete-time case

Consider a discrete-time TS model of the form

$$x(k+1) = A_h x(k) + B_h u(k), \tag{2.34}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state vector, $u(k) \in \mathbb{R}^{n_u}$ is the input, k stands for the current sample. Recall the short hand notation $A_h = \sum_{i=1}^{r} h_i(z(k)) A_i$ and $B_h = \sum_{i=1}^{r} h_i(z(k)) B_i$. In addition, in the sequel $x(k+1)$ and $x(k)$ will be denoted by x_{k+1} and x_k, respectively.

Via a quadratic Lyapunov function $V(x_k) = x_k^T P x_k > 0$, the stability of (2.34) when $u(k) = 0$ is ensured if the following LMI problem is feasible (Tanaka and Sugeno, 1992):

$$P > 0, \qquad A_i^T P A_i - P < 0, \qquad \forall i \in \{1, 2, \ldots, r\}. \tag{2.35}$$

In (Wang et al., 1996), a PDC control law of the form (2.24) is proposed. A stabilizing PDC controller of the form (2.24) can be designed. Via the Schur complement the resulting inequality is:

$$\begin{bmatrix} -X & (*) \\ A_h X + B_h M_h & -X \end{bmatrix} < 0, \qquad \text{with} \quad M_h = K_h X, \quad X = P^{-1} > 0. \tag{2.36}$$

Sufficient LMI conditions for (2.36) to hold can be obtained via sum relaxations (see Appendix B).

Contrary to the continuous-time case, since the appearance of the non-quadratic Lyapunov function (Guerra and Vermeiren, 2004) analysis and design conditions for the discrete-time case has witnessed several improvements. This is thanks to the fact that the derivatives of the MFs do not appear. Thus, a non-PDC controller has been proposed (Guerra and Vermeiren, 2004):

$$u_k = K_h P_h^{-1} x_k. \tag{2.37}$$

Consider the following non-quadratic Lyapunov function (Guerra and Vermeiren, 2004):

$$V(x_k) = x_k^T \left(\sum_{i=1}^{r} h_i(z(k)) P_i \right)^{-1} x_k, \qquad P_i = P_i^T > 0, \qquad i \in \{1,2,\ldots,r\}. \tag{2.38}$$

The controller design conditions are:

$$\begin{bmatrix} -P_h & (*) \\ A_h P_h + B_h K_h & -P_{h^+} \end{bmatrix} < 0. \tag{2.39}$$

Another controller proposed in (Guerra and Vermeiren, 2004) is

$$u_k = \left(\sum_{i=1}^{r} h_i(z(k)) K_i \right) \left(\sum_{i=1}^{r} h_i(z(k)) G_i \right)^{-1} x_k = K_h G_h^{-1} x_k, \tag{2.40}$$

which is a generalization of (2.37) in the sense that in the worst case $G_h = P_h$. The Lyapunov function used to synthesize this controller is

$$V(x_k) = x_k^T \left(\sum_{i=1}^{r} h_i(z(k)) G_i \right)^{-T} \left(\sum_{i=1}^{r} h_i(z(k)) P_i \right) \left(\sum_{i=1}^{r} h_i(z(k)) G_i \right)^{-1} x_k. \tag{2.41}$$

The stabilization conditions are:

$$\begin{bmatrix} -P_h & (*) \\ A_h G_h + B_h K_h & -G_{h^+} - G_{h^+}^T + P_{h^+} \end{bmatrix} < 0. \tag{2.42}$$

Within the discrete-time framework, the α-sample variation has been developed in (Kruszewski et al., 2008). The approach is based on the idea to avoid the requirement for the difference of the Lyapunov function $\left(V\left(x(k+1)\right)-V\left(x(k)\right)<0\right)$ to decrease at each consecutive sample. Instead it is required that $\Delta V\left(x(k)\right)$ decreases at every α-samples $\left(V\left(x(k+\alpha)\right)-V\left(x(k)\right)<0\right)$.

Recently, a novel Lyapunov function has been proposed in (Guerra et al., 2012b) for the observer design, that is:

$$V\left(e_k\right)=e_k^T\left(\sum_{i=1}^{r} h_i\left(z(k-1)\right)P_i\right)e_k, \qquad P_i=P_i^T>0, \qquad i\in\{1,2,\ldots,r\}. \tag{2.43}$$

The idea is to use past samples in the MFs of the observer gains as well as in the Lyapunov function, thus the proposed observer reads:

$$\hat{x}_{k+1}=A_h\hat{x}_k+B_h u_k+G_{hh^-}^{-1}L_{hh^-}\left(y_k-\hat{y}_k\right), \qquad \hat{y}_k=C_h\hat{x}_k. \tag{2.44}$$

This small change allows adding extra degrees of freedom to the LMI conditions without altering the number of conditions and thus achieving relaxed results. The delayed approach has been generalized in (Lendek et al., 2015) for controller design.

Remark 2.6. One of the main advantages of the TS-LMI framework (both the continuous and discrete time case) is that one can easily include specifications and/or constraints such as decay rate, H_∞ disturbance attenuation, constraint on the input, constraint on the output, etc. (Lendek et al., 2010, Chapter 3; Tanaka and Wang, 2001, Chapter 3).

2.2. TS descriptor models

This section presents a more general state space representation. So-called descriptor models naturally appear when dealing with mechanical systems (Lewis, 1986; Luenberger, 1977). Consider the following descriptor model:

$$g(x)\dot{x}=f(x,u), \tag{2.45}$$

where $g(x) \in \mathbb{R}^{n_x \times n_x}$ may be a rank deficient matrix, i.e., $rank(g(x)) \le n_x$. This is the reason for the names for (2.45): Differential-algebraic equations (DAE) systems, partial state space representation, singular systems, etc. (Dai, 1989). Nevertheless, in this work the matrix $g(x)$ is considered full-rank at least in a compact set of the state space Ω. For instance, in mechanical systems, the matrix $g(x)$ contains the inertia matrix and is positive definitive (Guelton et al., 2008; Lewis et al., 2004; Spong et al., 2005; Vermeiren et al., 2011). Moreover, a nonsingular matrix $g(x)$ allows using classical ODE solvers.

The class of nonlinear descriptors treated in this thesis can be expressed as the affine-in control model (see Remark 2.1):

$$E(x)\dot{x} = A(x)x + B(x)u, \tag{2.46}$$

where $x \in \mathbb{R}^{n_x}$ is the state vector and $u \in \mathbb{R}^{n_u}$ is the control input; $A(x)$, $B(x)$, and $E(x)$ are matrices of appropriate sizes, whose entries may be non-constant.

The sector nonlinearity methodology has been extended to descriptor models in (Taniguchi et al., 1999); hence, the p_a nonlinearities in the right-hand side of (2.46) – those in $A(x)$ and $B(x)$ – are captured via MFs $h_i(z)$, $i \in \{1, 2, \ldots, 2^{p_a}\}$. Proceeding similarly, the p_e nonlinear terms in the left-hand side of (2.46) – those in $E(x)$ – give the MFs $v_j(z)$, $j \in \{1, 2, \ldots, 2^{p_e}\}$. These MFs have the convex sum property in the compact set Ω, i.e., $\sum_{i=1}^{r_a} h_i(z) = 1$, $h_i(z) \ge 0$, $\sum_{j=1}^{r_e} v_j(z) = 1$, $v_j(z) \ge 0$, with $r_a = 2^{p_a}$, $r_e = 2^{p_e}$. Recall that this work considers that the premise vector $z \in \mathbb{R}^p$, $p = p_a + p_e$, depends on measured variables.

Therefore, the nonlinear descriptor model (2.46) can be exactly rewritten in the considered compact set as the following TS descriptor model (Taniguchi et al., 1999):

$$\sum_{j=1}^{r_e} v_j(z) E_j \dot{x} = \sum_{i=1}^{r_a} h_i(z)(A_i x + B_i u), \tag{2.47}$$

or in shorthand notation $E_v \dot{x} = A_h x + B_h u$; where matrices A_i and B_i, represent the *i*-th right-hand side local model of (2.47), while E_j is the *j*-th left-hand side local model of the TS descriptor model.

2.2.1. General definitions and properties

In order to correctly place the reader in the context of the current research a short summary of results for linear singular systems follows. Consider the linear descriptor system (Luenberger, 1977):

$$E\dot{x} = Ax + Bu, \qquad y = Cx \tag{2.48}$$

where $x \in \mathbb{R}^{n_x}$ is the state vector, $u \in \mathbb{R}^{n_u}$ is the control input, and $y \in \mathbb{R}^{n_y}$ is the output; A, B, C, and E are real matrices of adequate sizes. Matrix E is not full rank, i.e., $rank(E) < n_x$. In the case of autonomous singular systems $u = 0$, consider the following definitions (Dai, 1989):

Definition 2.1.

- The pair (E, A) is said to be *regular* if $\det(sE - A) \neq 0$.
- The pair (E, A) is said to be *impulse-free* if $\deg(\det(sE - A)) = rank(E)$.
- The pair (E, A) is said to be *stable* if $(sE - A)$ is Hurwitz.
- The pair (E, A) is said to be *admissible* if it is regular, impulse-free, and stable.

For a given pair (E, A), there always exist nonsingular matrices M and N such that

$$E = M\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}N \quad \text{and} \quad A = M\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}N. \tag{2.49}$$

The matrices M and N can be computed via the singular value decomposition of the matrix E followed by scaling of the bases. Thus the singular system (2.48) is casual (impulse-free) if and only if $\det(A_4) \neq 0$. Hence, the stability of (2.48) is determined (Dai, 1989) by the stability of $A_1 - A_2A_4^{-1}A_3$. A similar discussion applies for discrete-time singular systems.

Now, let us recall the TS descriptor model with $E_v = \sum_{j=1}^{r_e} v_j(z)E_j$ regular in Ω:

$$E_v\dot{x} = A_hx + B_hu, \qquad y = C_hx, \tag{2.50}$$

with $y \in \mathbb{R}^{n_y}$ being the output of the system.

In (Taniguchi et al., 1999), the open-loop system (2.50) with $u = 0$ is expressed as

$$\bar{E}\dot{\bar{x}} = \bar{A}_{hv}\bar{x}, \quad \text{with} \quad \bar{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \quad \bar{E} = \begin{bmatrix} I_{n_x} & 0 \\ 0 & 0_{n_x} \end{bmatrix}, \quad \bar{A}_{hv} = \begin{bmatrix} 0_{n_x} & I_{n_x} \\ A_h & -E_v \end{bmatrix}. \tag{2.51}$$

This procedure is the so-called descriptor redundancy approach in (Tanaka and Sugie, 1997). The TS descriptor system (2.51) is quadratically stable if

$$\frac{dV\left(\bar{x}(t)\right)}{dt} \leq -\alpha \left\| \left(\bar{x}(t)\right) \right\|_2, \tag{2.52}$$

where $V\left(\bar{x}(t)\right) = \bar{x}^T(t)\bar{E}^T P \bar{x}(t)$ and the following conditions are satisfied

1) $\det\left(s\bar{E} - \bar{A}_{hv}\right) \neq 0$.
2) The open-loop is impulse free. Note that the representation (2.51) is impulse free due to $\det\left(E_v\right) \neq 0$.
3) There exists a common matrix P and $\alpha > 0$ such that: $P \in \mathbb{R}^{2n_x \times 2n_x}$, $\bar{E}^T P = P^T \bar{E} \geq 0$, $\det\left(P\right) \neq 0$.

2.2.2. Regular $E(x)$: motivation (part I)

A large part of the thesis focuses on the case when the descriptor matrix $E(x)$ is invertible. A motivation for this lies in models based on mechanical fundamentals. Generally, when studying the dynamics of robotic systems, a nonlinear descriptor model is obtained (Guelton et al., 2008; Lewis et al., 2004; Luenberger, 1977). Since the matrix $E(x)$ is the inertia matrix and is therefore nonsingular and positive definite in Ω, the descriptor model (2.46) can be written in the standard state-space form (2.4):

$$\begin{aligned} \dot{x} &= E^{-1}(x)A(x)x + E^{-1}(x)B(x)u \\ &= \tilde{A}(x)x + \tilde{B}(x)u, \end{aligned} \tag{2.53}$$

thus standard tools can be applied. For example, in our case, a standard TS model can be constructed from this nonlinear model. However, even if the nonlinear models (2.46) and (2.53) are equivalent in the considered state space, (2.53) may have the following shortcomings:

1. The total number of rules is generally higher because $E^{-1}(x)$ has in most cases a more 'complicated' structure than $E(x)$.
2. If the input matrix $B(x)$ is state-independent in (2.46), i.e., $B(x)=B$, the controller design complexity is significantly reduced. Inverting $E(x)$ produces a state-dependent input matrix $\bar{B}(x)=E^{-1}(x)B$ in (2.53), thus leading to more complexity in the controller design by introducing double sums and increasing the number of LMI constraints. For the observer design, this fact does not apply since the output matrix $C(x)$ is not multiplied by $E^{-1}(x)$.
3. The closer the TS model is to the nonlinear model structure the 'more natural' it is.

To summarize, keeping the descriptor structure may significantly reduce the number of local models as well as the number of LMIs; thus, it may increase the feasibility set (Tanaka and Wang, 2001; Taniguchi et al., 2000). In order to clarify these points, the following example is chosen.

Example 2.4. Consider the "*Futura pendulum*" system in descriptor form (Fantoni and Lozano, 2013):

$$E(x)\dot{x}=A(x)x+Bu, \tag{2.54}$$

with:

$$E(x)=\begin{bmatrix} I & 0 \\ 0 & M_a(x) \end{bmatrix}=\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & a+b\sin^2(x_2) & c\cos(x_2) \\ 0 & 0 & c\cos(x_2) & d \end{array}\right],\quad B=\left[\begin{array}{c} 0 \\ 0 \\ \hline 1 \\ 0 \end{array}\right],$$

$$A(x)=\begin{bmatrix} 0 & I \\ -G_r(x) & -C_f(x) \end{bmatrix}=\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & -e\sin(2x_2)x_4 & c\sin(x_2)x_4-e\sin(2x_2)x_3 \\ 0 & \dfrac{m_1 l_1 g\sin(x_2)}{x_2} & e\sin(2x_2)x_3 & 0 \end{array}\right];$$

with parameters $a=I_0+m_1L_0^2$, $b=m_1l_1^2$, $c=m_1l_1L_0$, $d=J_1+m_1l_1^2$, and $e=0.5m_1l_1^2$; where g is the gravitational acceleration, m_1 is the mass of the pendulum, I_0 is the inertia of the arm, L_0 is the total length of the arm, l_1 is the distance to the center of gravity of the pendulum,

J_1 is the inertia of the pendulum around its center of gravity. Note that the input matrix B is constant. A TS descriptor representation of (2.54) gives $r_e = 2^2 = 4$ due to the terms $\cos(x_2)$ and $\sin^2(x_2)$ in $E(x)$; and $r_a = 2^3 = 8$ due to $\sin(x_2)/x_2$, $\sin(2x_2)x_3$, and $\sin(2x_2)x_4$ in $A(x)$. To write (2.54) in the standard state space representation (2.3), it is necessary to invert the matrix $E(x)$, which gives

$$E^{-1}(x) = \begin{bmatrix} I & 0 \\ 0 & M_a^{-1}(x_2) \end{bmatrix}, \tag{2.55}$$

where

$$M_a^{-1}(x_2) = \frac{1}{\eta}\begin{bmatrix} d & -c\cos(x_2) \\ -c\cos(x_2) & a + b\sin^2(x_2) \end{bmatrix}, \quad \eta = ad - c^2 + \left(db + c^2\right)\sin^2(x_2).$$

The standard state space model is

$$\dot{x} = E^{-1}(x)A(x)x + E^{-1}(x)Bu. \tag{2.56}$$

The nonlinearities in (2.56) are: $\cos(x_2)$, $\sin^2(x_2)$, $\sin(x_2)/x_2$, $\sin(2x_2)x_3$, $\sin(2x_2)x_4$, $\sin(x_2)x_4$, and $1/\eta$; therefore a standard TS representation has $r = 2^7 = 128$ rules (vertices). Moreover, the new input matrix $E^{-1}(x)B$ is no longer constant. ◆

2.2.3. Regular $E(x)$: overview of existing results

Stability analysis as well as controller design for TS descriptor models has been introduced in (Taniguchi et al., 1999). In order to decouple the matrix E_v, the system (2.50) is rewritten by using the so-called descriptor-redundancy approach (or augmented system) used in (Tanaka and Sugie, 1997). The procedure is as follows: consider

$$\dot{x} = \dot{x} \quad \text{and}$$
$$0 \times \ddot{x} = A_h x + B_h u - E_v \dot{x}, \quad y = C_h x.$$

Using an augmented state vector $\bar{x} = \begin{bmatrix} x^T & \dot{x}^T \end{bmatrix}^T$, we have

$$\bar{E}\dot{\bar{x}} = \bar{A}_{hv}\bar{x} + \bar{B}_h u, \qquad y = \bar{C}_h \bar{x}, \tag{2.57}$$

with $\bar{E}=\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{A}_{hv}=\begin{bmatrix} 0 & I \\ A_h & -E_v \end{bmatrix}$, $\bar{B}_h=\begin{bmatrix} 0 \\ B_h \end{bmatrix}$, and $\bar{C}_h=\begin{bmatrix} C_h & 0 \end{bmatrix}$. Sufficient conditions for the stability of (2.57) when $u=0$ are obtained via the following Lyapunov function (Taniguchi et al., 2000, 1999):

$$V(\bar{x})=\bar{x}^T\bar{E}^T P\bar{x}, \qquad \bar{E}^T P=P^T\bar{E}\geq 0. \tag{2.58}$$

The time-derivative of (2.58) is

$$\dot{V}(\bar{x})=\dot{\bar{x}}^T\bar{E}^T P\bar{x}+\bar{x}^T P^T\bar{E}\dot{\bar{x}}=\bar{x}^T\left(\bar{A}_{hv}^T P+P^T\bar{A}_{hv}\right)\bar{x}. \tag{2.59}$$

Hence, $\dot{V}(\bar{x})<0$ implies $\bar{A}_{hv}^T P+P^T\bar{A}_{hv}<0$. Thus, the system (2.57) is stable if there exists a matrix P such that (once the MFs are removed):

$$\bar{E}^T P=P^T\bar{E}\geq 0, \qquad \bar{A}_{ij}^T P+P^T\bar{A}_{ij}<0, \qquad \forall i\in\{1,2,\ldots,r_a\},\ j\in\{1,2,\ldots,r_e\}. \tag{2.60}$$

Note that in conditions (2.60) the decision variable is $P\in\mathbb{R}^{2n_x\times 2n_x}$. This matrix should be regular, i.e., $\det(P)\neq 0$; and $\bar{E}^T P=P^T\bar{E}\geq 0$ must hold. One possible structure could be $P=\begin{bmatrix} P_1 & 0 \\ P_3 & P_1 \end{bmatrix}$ with $P_1=P_1^T>0$, which guarantees the regularity of $P\in\mathbb{R}^{2n_x\times 2n_x}$. Another possible structure is $P=\begin{bmatrix} P_1 & 0 \\ P_3 & P_4 \end{bmatrix}$ with $P_1=P_1^T>0$ and P_4 being a regular matrix. The latter structure provides more degrees of freedom; this choice turns (2.60) into

$$P_1>0, \quad \begin{bmatrix} A_i^T P_3+P_3^T A_i & (*) \\ P_4^T A_i-E_j^T P_3+P_1 & -P_4^T E_j-E_j^T P_4 \end{bmatrix}<0, \quad \forall i\in\{1,2,\ldots,r_a\},\ j\in\{1,2,\ldots,r_e\}. \tag{2.61}$$

The regularity of P_4 is guaranteed by the block (2,2). In effect, if (2.61) holds, then $-P_4^T E_v-E_v^T P_4<0$ also holds. Since E_v is nonsingular $\left(E_v x_0\neq 0, \quad \forall x_0\neq 0\right)$, let us assume P_4 is singular; then, there exists $x_0\neq 0$ such that $P_4 x_0=0$; consequently for that $x_0\neq 0$ it yields $x_0^T\left(-P_4^T E_v-E_v^T P_4\right)x_0=0$, which contradicts the condition $-P_4^T E_v-E_v^T P_4<0$. Thus if $\bar{A}_{hv}^T P+P^T\bar{A}_{hv}<0$ is true, then P_4 is nonsingular.

Regarding the controller design, a PDC controller

$$u=\sum_{i=1}^{r_a}\sum_{j=1}^{r_e}h_i(z)v_j(z)K_{ij}x \tag{2.62}$$

has been proposed in (Taniguchi et al., 2000), where $K_{ij} \in \mathbb{R}^{n_u \times n_x}$, $i \in \{1,2,\ldots,r_a\}$, $j \in \{1,2,\ldots,r_e\}$ are the controller gains to be designed. This control law incorporates nonlinearities from both sides of the TS descriptor model (2.50) via the MFs $h_i(z)$, $i \in \{1,2,\ldots,r_a\}$ and $v_j(z)$, $j \in \{1,2,\ldots,r_e\}$. Similarly to stability analysis, stabilization of (2.50) is done via the augmented system (2.57); thus the control law (2.62) should be rewritten using the extended vector $\bar{x}$, i.e., $u = \bar{K}_{hv}\bar{x}$ with $\bar{K}_{hv} = \begin{bmatrix} K_{hv} & 0_{n_u \times n_x} \end{bmatrix}$. The closed-loop model is:

$$\bar{E}\dot{\bar{x}} = \left(\bar{A}_{hv} + \bar{B}_h \bar{K}_{hv}\right)\bar{x}. \tag{2.63}$$

The time-derivative of the quadratic Lyapunov function (2.58) is:

$$\dot{V}(\bar{x}) = \dot{\bar{x}}^T \bar{E}^T P \bar{x} + \bar{x}^T P^T \bar{E}\dot{\bar{x}} = \bar{x}^T \left(\bar{A}_{hv} + \bar{B}_h \bar{K}_{hv}\right)^T P\bar{x} + \bar{x}^T P^T \left(\bar{A}_{hv} + \bar{B}_h \bar{K}_{hv}\right)\bar{x}. \tag{2.64}$$

Thus $\dot{V}(\bar{x}) < 0 \Leftrightarrow \left(\bar{A}_{hv} + \bar{B}_h \bar{K}_{hv}\right)^T P + (*) < 0$. LMI conditions are achieved by using the congruence property with $P^{-1} = X = \begin{bmatrix} X_1 & 0 \\ X_3 & X_4 \end{bmatrix}$, $X_1 = P_1^{-1}$, $X_3 = -P_4^{-1}P_3P_1^{-1}$, and $X_4 = P_4^{-1}$. With these choices (2.64) yields $\bar{A}_{hv}X + \bar{B}_h \bar{K}_{hv} X + (*) < 0$. Finally, via a change of variables $\bar{N}_{hv} = \bar{K}_{hv}X = \begin{bmatrix} K_{hv}P_1^{-1} & 0 \end{bmatrix} = \begin{bmatrix} N_{hv} & 0 \end{bmatrix}$, (2.64) produces

$$X_1 > 0, \quad \begin{bmatrix} X_3 + X_3^T & (*) \\ A_h X_1 + B_h N_h - E_v X_3 + X_4^T & -E_v X_4 - X_4^T E_v^T \end{bmatrix} < 0. \tag{2.65}$$

Sufficient LMI conditions can easily be obtained via sum relaxations.

Extensions to the previous results have been proposed by Guerra et al., (2007) via the following quadratic Lyapunov function:

$$V(\bar{x}) = \bar{x}^T \bar{E}^T P_{hh}^{-1}\bar{x} = x^T P_1^{-1} x, \qquad \bar{E}^T P_{hh}^{-1} = P_{hh}^{-T}\bar{E} \geq 0, \tag{2.66}$$

with $P_{hh} = \begin{bmatrix} P_1 & 0 \\ P_{3hh} & P_{4hh} \end{bmatrix}$, $P_1 = P_1^T > 0$. Since P_1 is constant, this structure avoids the appearance of the MFs' time-derivative. The control law to be designed is

$$u = K_{hv}\left(P_1\right)^{-1} x = \begin{bmatrix} K_{hv}P_1^{-1} & 0 \end{bmatrix}\bar{x}. \tag{2.67}$$

The conditions for designing (2.67) are:

$$P_1 > 0, \quad \begin{bmatrix} P_{3h} + P_{3h}^T & (*) \\ A_h P_1 + B_h K_h - E_v P_{3h} + P_{4h}^T & -E_v P_{4h} - P_{4h}^T E_v^T \end{bmatrix} < 0. \tag{2.68}$$

For the case when the full state is not available, the following observer has been designed in (Guerra et al., 2004):

$$\bar{E}\dot{\hat{\bar{x}}} = \bar{A}_{hv}\hat{\bar{x}} + \bar{B}_h u + \bar{L}_{hv}\left(y - \hat{y}\right), \qquad \hat{y} = \bar{C}_h \hat{\bar{x}}, \tag{2.69}$$

where $\hat{\bar{x}} = \begin{bmatrix} \hat{x}^T & \dot{\hat{x}}^T \end{bmatrix}^T$ is the augmented estimated state vector. The observer gain is $\bar{L}_{hv} = \begin{bmatrix} 0 & L_{hv}^T \end{bmatrix}^T$. The dynamics of the extended estimation error vector: $\bar{e} = \bar{x} - \hat{\bar{x}} = \begin{bmatrix} x - \hat{x} \\ \dot{x} - \dot{\hat{x}} \end{bmatrix}$, are given by $\bar{E}\dot{\bar{e}} = \left(\bar{A}_{hv} - \bar{L}_{hv}\bar{C}_h\right)\bar{e}$. The synthesis of the augmented observer (2.69) is done via the quadratic Lyapunov function candidate (Guerra et al., 2004):

$$V\left(\bar{e}\right) = \bar{e}^T \bar{E}^T P \bar{e}, \quad \bar{E}^T P = P^T \bar{E} \geq 0, \tag{2.70}$$

with $P = \begin{bmatrix} P_1 & 0 \\ P_3 & P_4 \end{bmatrix}$, $P_1 = P_1^T > 0$, P_4 being a regular matrix. The time-derivative of (2.70) yields $\dot{V}\left(\bar{e}\right) = \bar{e}^T \left(\left(\bar{A}_{hv} - \bar{L}_{hv}\bar{C}_h\right)^T P + P^T \left(\bar{A}_{hv} - \bar{L}_{hv}\bar{C}_h\right)\right)\bar{e}$. Thus, the estimation error is asymptotically stable if

$$P_1 > 0, \qquad \begin{bmatrix} P_3^T A_h - P_3^T L_{hv} C_h + (*) & (*) \\ P_4^T A_h - P_4^T L_{hv} C_h + P_1 - E_v^T P_3 & -E_v^T P_4 - P_4^T E_v \end{bmatrix} < 0. \tag{2.71}$$

Although the MFs can be removed via sum relaxations from (2.71), because of the terms $P_3^T L_{hv} C_h$ and $P_4^T L_{hv} C_h$ it is not possible to obtain LMI conditions. Conservative solutions such as fixing $P_3 = P_4$ or by designing the gains L_{ij}, $i \in \{1, 2, \ldots, r_a\}$, $j \in \{1, 2, \ldots, r_e\}$ via any technique (pole-placement, linear quadratic regulator, etc.) and using (2.71) to verify the convergence of the estimation error are given in (Guelton et al., 2008; Guerra et al., 2004):.

In what follows a preliminary technical result is stated on the equivalence between approaches involving descriptor-redundancy and Finsler's lemma in the case of continuous-time descriptors.

2.2.4. Relation between descriptor-redundancy and Finsler's lemma

For the analysis and design of controllers/observers for the TS descriptor model (2.50), the descriptor-redundancy approach has been used. This approach allows separating the matrix E_v from the derivative of the state vector. Briefly, the descriptor-redundancy approach consists in adding a virtual state variable to the original expression, and rewriting the model (2.50) as a singular system (Tanaka and Sugie, 1997; Taniguchi et al., 1999).

An alternative to descriptor-redundancy is the Finsler's lemma (see Appendix A, Lemma A.1), which avoids the explicit substitution of the close-loop dynamics of the considered problem (de Oliveira et al., 1999). Using this approach the closed-loop dynamics are rewritten as an equality constraint, the time-derivative of the Lyapunov function being an inequality constraint dependent on the state and its derivative.

In this section, we show that with the *proper* algebraic manipulations, the results (from the previous sections) obtained via the descriptor-redundancy approach can also be obtained using Finsler's lemma. To this end, recall the control law (2.67), i.e., $u = K_{hv}P_1^{-1}x$. The TS descriptor model (2.50) with the control law (2.67) gives

$$E_v\dot{x} = \left(A_h + B_hK_{hv}P_1^{-1}\right)x \quad \Leftrightarrow \quad \begin{bmatrix} A_h + B_hK_{hv}P_1^{-1} & -E_v \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = 0. \tag{2.72}$$

The following Lyapunov function is employed

$$V(x) = x^TP_1^{-1}x > 0, \quad P_1 = P_1^T > 0, \tag{2.73}$$

its time-derivative gives

$$\begin{aligned} \dot{V}(x) &= \dot{x}^TP_1^{-1}x + x^TP_1^{-1}\dot{x} \\ &= \begin{bmatrix} x \\ \dot{x} \end{bmatrix}^T \begin{bmatrix} 0 & P_1^{-1} \\ P_1^{-1} & 0 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} < 0. \end{aligned} \tag{2.74}$$

By selecting $\mathcal{Q} = \begin{bmatrix} 0 & P_1^{-1} \\ P_1^{-1} & 0 \end{bmatrix}$, $\mathcal{W} = \begin{bmatrix} A_h + B_hK_{hv}P_1^{-1} & -E_v \end{bmatrix}$, $\chi = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$ and using Finsler's lemma (see Appendix A, Lemma A.1), inequality (2.74) together with the equality constraint (2.72) gives

$$\mathcal{M}\begin{bmatrix} A_h + B_hK_{hv}P_1^{-1} & -E_v \end{bmatrix} + (*) + \begin{bmatrix} 0 & P_1^{-1} \\ P_1^{-1} & 0 \end{bmatrix} < 0, \tag{2.75}$$

where $\mathcal{M} \in \mathbb{R}^{2n_x \times n_x}$ is a free matrix. Congruence with the full-rank matrix $\begin{bmatrix} P_1 & P_3^T \\ 0 & P_4^T \end{bmatrix}$ yields

$$\begin{bmatrix} P_1 & P_3^T \\ 0 & P_4^T \end{bmatrix} \mathcal{M} \begin{bmatrix} A_h P_1 + B_h K_{hv} - E_v P_3 & -E_v P_4 \end{bmatrix} + (*) + \begin{bmatrix} P_3 + P_3^T & P_4 \\ P_4^T & 0 \end{bmatrix} < 0. \tag{2.76}$$

Since $\mathcal{M}$ is free a matrix, by choosing $\mathcal{M} = \begin{bmatrix} -P_1^{-1} P_3^T P_4^{-T} \\ P_4^{-T} \end{bmatrix}$, (2.76) produces exactly the conditions given in (2.68).

In the observer design case, recall the final form of the augmented observer (2.69):

$$E_v \dot{\hat{x}} = A_h \hat{x} + B_h u + L_{hv} \left(y - \hat{y} \right), \quad \hat{y} = C_h \hat{x}. \tag{2.77}$$

The matrix inequality (2.71) can be achieved by the use of Finsler's lemma. To this end, consider the estimation error $e = x - \hat{x}$ and its dynamics:

$$E_v \dot{e} = \left(A_h - L_{hv} C_h \right) e \quad \Leftrightarrow \quad \begin{bmatrix} A_h - L_{hv} C_h & -E_v \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \end{bmatrix} = 0. \tag{2.78}$$

The Lyapunov function under consideration is:

$$V(e) = e^T P_1 e > 0, \quad P_1 = P_1^T > 0. \tag{2.79}$$

The time-derivative is

$$\begin{aligned} \dot{V}(e) &= \dot{e}^T P_1 e + e^T P_1 \dot{e} \\ &= \begin{bmatrix} e \\ \dot{e} \end{bmatrix}^T \begin{bmatrix} 0 & P_1 \\ P_1 & 0 \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \end{bmatrix} < 0. \end{aligned} \tag{2.80}$$

By defining $\mathcal{Q} = \begin{bmatrix} 0 & P_1 \\ P_1 & 0 \end{bmatrix}$, $\mathcal{W} = \begin{bmatrix} A_h - L_{hv} C_h & -E_v \end{bmatrix}$, $\chi = \begin{bmatrix} e \\ \dot{e} \end{bmatrix}$, and using Finsler's lemma, (2.78) together with inequality (2.80) gives:

$$\mathcal{M} \begin{bmatrix} A_h - L_{hv} C_h & -E_v \end{bmatrix} + (*) + \begin{bmatrix} 0 & P_1 \\ P_1 & 0 \end{bmatrix} < 0, \tag{2.81}$$

where $\mathcal{M} \in \mathbb{R}^{2n_x \times n_x}$ is a free matrix. By choosing $\mathcal{M} = \begin{bmatrix} P_3^T \\ P_4^T \end{bmatrix}$, (2.81) yields (2.71). Generally, in this work we will prefer writing the problems via descriptor redundancy for the continuous-time case and using Finsler's lemma for the discrete-time case.

2.2.5. Regular $E(x)$: motivation (part II)

To conclude this chapter, considering regular $E(x)$, we summarize the results for regular TS descriptors for the continuous case. LMI conditions exist only for the controller design, and the observer design remains a BMI problem. For the discrete-time case, to the best of our knowledge there are no results in the literature. Therefore there is room for improvements as will be shown in the following example.

Example 2.5. Consider a discrete time nonlinear descriptor model

$$E(x_k)x_{k+1} = A(x_k)x_k + Bu_k, \qquad y_k = C(x_k)x_k, \tag{2.82}$$

where $E(x_k) = \begin{bmatrix} 2 & -\eta \\ \eta & 1 \end{bmatrix}$, $A(x_k) = \begin{bmatrix} \cos(x_1) & -1 \\ 0.7 & -1.1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $C(x_k) = \begin{bmatrix} \sin(x_1)/x_1 \\ 0.2 \end{bmatrix}^T$; with $\eta = 1/(1+x_1^2)$. Since $E(x)$ is regular for all $x \in \mathbb{R}^2$ $\det(E(x_k)) = (3+4x_1^2+2x_1^4)(1+x_1^2)^{-2} \neq 0$, a standard state-space model can be computed. The inverse of the descriptor matrix gives $E^{-1}(x_k) = \frac{1}{\det(E(x_k))}\begin{bmatrix} 1 & \eta \\ -\eta & 2 \end{bmatrix}$; this means that four different nonlinearities have to be considered, which results in $r = 16$. Consider the observer design problem. Using the Lyapunov function $V(e_k) = e_k^T \mathcal{P} e_k$, where e_k is the estimation error, and considering the compact set $\Omega = \{x \in \mathbb{R}^2\}$ no solution was obtained either for classical non-quadratic (NQ) approach in (Guerra and Vermeiren, 2004), i.e., $\mathcal{P} = \sum_{i_2=1}^{r} h_{i_2}(z(k)) P_{i_2}$ and

$$\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{i_x=1}^{r} h_{i_1}(z(k)) h_{i_1}(z(k)) h_{i_x}(z(k+1)) \begin{bmatrix} -P_{i_2} & (*) \\ G_{i_2}A_{i_1} - L_{i_2}C_{i_1} & -G_{i_2} - G_{i_2}^T + P_{i_x} \end{bmatrix} < 0,$$

or for the delayed non-quadratic (DNQ) approach in (Guerra et al., 2012b), i.e., $\mathcal{P} = \sum_{i_x=1}^{r} h_{i_x}(z(k-1)) P_{i_x}$ and

$$\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{i_x=1}^{r} h_{i_1}(z(k)) h_{i_2}(z(k)) h_{l_x}(z(k-1)) \begin{bmatrix} -P_{i_x} & (*) \\ G_{i_2 i_x}A_{i_1} - L_{i_2 i_x}C_{i_1} & -G_{i_2 i_x} - G_{i_2 i_x}^T + P_{i_2} \end{bmatrix} < 0.$$

Thus, what we can infer from this example is that even using recent results, there is no solution via a standard TS description. Whereas, using the descriptor formulation and associated LMI constraints a solution is available with a non quadratic Lyapunov function, as it will be shown in Chapter 4, Section 4.2.

2.3. Concluding remarks

This chapter briefly summarized the main results in the literature for TS models. Motivated by mechanical systems, the TS descriptor model is introduced. Since this thesis considers the case when the descriptor matrix is invertible, it is always possible to obtain a standard state-space form; however, within the TS-LMI framework this may increase the computational cost. The advantages of keeping the descriptor structure have been illustrated on examples.

Sections 2.2.3 and 2.2.5 showed that there are still many open problems, among them:

- Enlarging the feasible solution set of the existing results for controller and observer design.
- In the observer design case existing conditions are BMIs.
- To the best of our knowledge, there are no results in the literature for the discrete-time case.

Solutions for the problems above are presented in the following chapters.

Chapter 3. State feedback controller design

This chapter presents improvements of the state feedback controller design for both continuous and discrete time TS descriptor models. In continuous-time, the use the Finsler's lemma leads to the enlargement of the solution set of previous results (Guerra et al., 2007). For discrete-time TS descriptor models, results when the descriptor matrix is non-singular $\left(E^{-1}(x) \text{ exists } \forall x \in \Omega\right)$ are presented. In this case, relaxations can be achieved by using past samples in the MFs of the Lyapunov function and the control law. A systematic procedure is also given that generalizes the past samples approach.

3.1. Continuous-time TS descriptor models

This section presents a relaxed approach for stabilization and H_∞ disturbance rejection of continuous-time TS descriptor models. It has been shown in (Jaadari et al., 2012; Oliveira et al., 2011) that it is possible to generalize results even under the quadratic framework by applying the well-known Finsler's lemma.

Thus, by exploiting the fact that Finsler's lemma allows decoupling the control law from the Lyapunov function, a new structure of the control law is used. The derived conditions are LMIs up to fixing a scalar parameter.

3.1.1. Problem statement

Consider the following TS descriptor model:

$$\begin{aligned} E_v \dot{x} &= A_h x + B_h u + D_h w \\ y &= C_h x + J_h w. \end{aligned} \tag{3.1}$$

The analysis and design for (3.1) have been performed by rewriting the TS descriptor as follows (Taniguchi et al., 1999):

$$0 \times \ddot{x} = A_h x + B_h u + D_h w - E_v \dot{x}, \qquad y = C_h x + J_h w. \tag{3.2}$$

Then, by defining an extended vector $\bar{x} = \begin{bmatrix} x^T & \dot{x}^T \end{bmatrix}^T$, (3.2) can be written as

$$\begin{aligned} \bar{E}\dot{\bar{x}} &= \bar{A}_{hv}\bar{x} + \bar{B}_h u + \bar{D}_h w \\ y &= \bar{C}_h \bar{x} + J_h w, \end{aligned} \tag{3.3}$$

with $\bar{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{A}_{hv} = \begin{bmatrix} 0 & I \\ A_h & -E_v \end{bmatrix}$, $\bar{B}_h = \begin{bmatrix} 0 \\ B_h \end{bmatrix}$, $\bar{D}_h = \begin{bmatrix} 0 \\ D_h \end{bmatrix}$, and $\bar{C}_h = \begin{bmatrix} C_h & 0 \end{bmatrix}$. In (Taniguchi et al., 2000), the stabilization of TS descriptor models has been studied via the PDC control $u = K_{hv} x$. In (Guerra et al., 2007), relaxed conditions have been given using the following control law:

$$u = K_{hv} \left(P_1\right)^{-1} x = \begin{bmatrix} K_{hv} \left(P_1\right)^{-1} & 0 \end{bmatrix} \bar{x}. \tag{3.4}$$

Guerra et al., (2007) consider the Lyapunov function $V\left(\bar{x}\right) = \bar{x}^T \bar{E}^T P_{hh}^{-1} \bar{x}$, with $\bar{E}^T P_{hh}^{-1} = P_{hh}^{-T} \bar{E} \geq 0$ and $P_{hh} = \begin{bmatrix} P_1 & 0 \\ P_{3hh} & P_{4hh} \end{bmatrix}$, $P_1 = P_1^T > 0$. Thus, when $w = 0$, the conditions for designing the stabilizing control law (3.4) are: $P_1 = P_1^T > 0$ and

$$\Upsilon_{hh}^{v} := \begin{bmatrix} P_{3hh} + (*) & (*) \\ A_h P_1 + B_h K_{hv} - E_v P_{3hh} + P_{4hh}^T & -E_v P_{4hh} + (*) \end{bmatrix} < 0. \tag{3.5}$$

3.1.2. Results

3.1.2.1. Stabilization

The aim is to stabilize (3.1) via the augmented system (3.3) with the following non-PDC control law:

$$u = \begin{bmatrix} K_{1hv} & K_{2hv} \end{bmatrix} \begin{bmatrix} G_{1hv} & 0 \\ G_{3hh} & G_{4hh} \end{bmatrix}^{-1} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \bar{K}_{hv} \bar{G}_{hhv}^{-1} \bar{x}, \tag{3.6}$$

where $\bar{K}_{hv} \in \mathbb{R}^{n_u \times 2n_x}$ and $\bar{G}_{hhv} \in \mathbb{R}^{2n_x \times 2n_x}$ are matrices to be designed. These matrices depend on MFs $h_i(z)$, $i \in \{1, 2, \ldots, r_a\}$ and $v_j(z)$, $j \in \{1, 2, \ldots, r_e\}$.

Remark 3.1. The control law (3.6) corresponds to a new control structure since classically the inverted matrix is the one used for the Lyapunov function (see (3.4)). The regularity of $\bar{G}_{hhv}$ will be discussed later on.

First, consider the stabilization problem without disturbances $(w = 0)$. Substituting the control law (3.6) into the augmented TS descriptor (3.3) yields:

$$\bar{E}\dot{\bar{x}} = \left(\bar{A}_{hv} + \bar{B}_h \bar{K}_{hv} \bar{G}_{hhv}^{-1}\right)\bar{x} \quad \Leftrightarrow \quad \begin{bmatrix} \bar{A}_{hv} + \bar{B}_h \bar{K}_{hv} \bar{G}_{hhv}^{-1} & -I \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{E}\dot{\bar{x}} \end{bmatrix} = 0. \tag{3.7}$$

Consider the following Lyapunov function candidate:

$$V(\bar{x}) = \bar{x}^T \bar{E}^T P_{hhv}^{-1} \bar{x}, \qquad \bar{E}^T P_{hhv}^{-1} = P_{hhv}^{-T} \bar{E} \geq 0, \tag{3.8}$$

with $P_{hhv} = \begin{bmatrix} P_1 & 0 \\ P_{3hhv} & P_{4hhv} \end{bmatrix}$, $P_{hhv}^{-1} = \begin{bmatrix} P_1^{-1} & 0 \\ -P_{4hhv}^{-1} P_{3hhv} P_1^{-1} & P_{4hhv}^{-1} \end{bmatrix}$, with $P_1 = P_1^T$. P_1 is chosen as a constant matrix to prevent the time-derivatives of the MFs emerging in the following developments (Guerra et al., 2007), i.e., $\bar{E}^T \dot{P}_{hhv}^{-1} = 0$. Then, the time-derivative of (3.8) reads:

$$\dot{V}(\bar{x}) = \dot{\bar{x}}^T \bar{E}^T P_{hhv}^{-1} \bar{x} + \bar{x}^T P_{hhv}^{-T} \bar{E}\dot{\bar{x}} < 0. \tag{3.9}$$

Condition (3.9) can be expressed as

$$\begin{bmatrix} \bar{x} \\ \bar{E}\dot{\bar{x}} \end{bmatrix}^T \begin{bmatrix} 0 & P_{hhv}^{-T} \\ P_{hhv}^{-1} & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{E}\dot{\bar{x}} \end{bmatrix} < 0. \tag{3.10}$$

Via Finsler's lemma, the equality constraint (3.7) and the inequality (3.10) yields:

$$\mathcal{M}\mathcal{W}+\mathcal{W}^T\mathcal{M}^T+\mathcal{Q}<0, \tag{3.11}$$

where $\mathcal{Q}=\begin{bmatrix} 0 & P_{hhv}^{-T} \\ P_{hhv}^{-1} & 0 \end{bmatrix}$, $\mathcal{W}=\begin{bmatrix} \bar{A}_{hv}+\bar{B}_h\bar{K}_{hv}\bar{G}_{hhv}^{-1} & -I \end{bmatrix}$, and $\mathcal{M}\in\mathbb{R}^{2n_x\times 2n_x}$ is a free matrix. By selecting $\mathcal{M}=\begin{bmatrix} \bar{G}_{hhv}^{-T} \\ \varepsilon P_{hhv}^{-1} \end{bmatrix}$, $\varepsilon>0$, and multiplying by the full-rank matrix $diag\begin{bmatrix} \bar{G}_{hhv}^T, & P_{hhv} \end{bmatrix}$ the left-hand side and by its transpose $diag\begin{bmatrix} \bar{G}_{hhv}, & P_{hhv}^T \end{bmatrix}$ the right-hand side of (3.11) gives

$$\begin{bmatrix} I \\ \varepsilon I \end{bmatrix}\begin{bmatrix} \bar{A}_{hv}\bar{G}_{hhv}+\bar{B}_h\bar{K}_{hv} & -P_{hhv}^T \end{bmatrix}+(*)+\begin{bmatrix} 0 & \bar{G}_{hhv}^T \\ \bar{G}_{hhv} & 0 \end{bmatrix}<0. \tag{3.12}$$

The following theorem summarizes this result.

Theorem 3.1. The TS descriptor model (3.1) with $w=0$ under control law (3.6) is asymptotically stable if, for a given $\varepsilon>0$, there exist matrices $P_{i_1i_2j_1}$, $\bar{G}_{i_1i_2j_1}$, $\bar{K}_{i_2j_1}$, $i_1,i_2\in\{1,2,\ldots,r_a\}$, $j_1\in\{1,2,\ldots,r_e\}$ as defined in (3.6) and (3.8), such that:

$$\Upsilon_{i_1i_1}^{j_1}<0,\ \forall i_1,j_1; \qquad \frac{2}{r_a-1}\Upsilon_{i_1i_1}^{j_1}+\Upsilon_{i_1i_2}^{j_1}+\Upsilon_{i_2i_1}^{j_1}<0,\quad \forall j_1,\ i_i\neq i_2, \tag{3.13}$$

hold with

$$\Upsilon_{i_1i_2}^{j_1}=\begin{bmatrix} \bar{A}_{i_1j_1}\bar{G}_{i_1i_2j_1}+\bar{B}_{i_i}\bar{K}_{i_2j_1}+(*) & (*) \\ \bar{G}_{i_1i_2j_1}+\varepsilon\left(\bar{A}_{i_1j}\bar{G}_{i_1i_2j_1}+\bar{B}_{i_i}\bar{K}_{i_2j_1}\right)-P_{i_1i_2j_1} & -\varepsilon\left(P_{i_1i_2j_1}+P_{i_1i_2j_1}^T\right) \end{bmatrix}. \tag{3.14}$$

Proof. Developing (3.12), we obtain:

$$\Upsilon_{hh}^{v}:=\begin{bmatrix} \bar{A}_{hv}\bar{G}_{hhv}+\bar{B}_h\bar{K}_{hv}+(*) & (*) \\ \bar{G}_{hhv}+\varepsilon\left(\bar{A}_{hv}\bar{G}_{hhv}+\bar{B}_h\bar{K}_{hv}\right)-P_{hhv} & -\varepsilon\left(P_{hhv}+P_{hhv}^T\right) \end{bmatrix}<0, \tag{3.15}$$

which by applying the relaxation Lemma B.3 (Appendix B) yields conditions (3.13). The regularity of $\bar{G}_{hhv}$ is given as follows: if conditions in Theorem 3.1 hold, then (3.15) also holds. By means of the Schur complement (3.15) is equivalent to

$$\bar{A}_{hv}\bar{G}_{hhv}+\bar{B}_h\bar{K}_{hv}+(*)+(*)\frac{1}{\varepsilon}\left(P_{hhv}+P_{hhv}^T\right)^{-1}\left(\bar{G}_{hhv}+\varepsilon\left(\bar{A}_{hv}\bar{G}_{hhv}+\bar{B}_h\bar{K}_{hv}\right)-P_{hhv}\right)<0. \tag{3.16}$$

Suppose that $\bar{G}_{hhv}$ is singular. Therefore there exists $x_0\neq 0$ such that $\bar{G}_{hhv}x_0=0$, hence (3.16) yields:

$$x_0^T\left\{\bar{B}_h\bar{K}_{hv}+(*)+\left(\varepsilon\bar{B}_h\bar{K}_{hv}-P_{hhv}\right)^T\frac{1}{\varepsilon}\left(P_{hhv}+P_{hhv}^T\right)^{-1}\left(\varepsilon\bar{B}_h\bar{K}_{hv}-P_{hhv}\right)\right\}x_0<0, \qquad (3.17)$$

which is equivalent to

$$x_0^T\left\{\bar{B}_h\bar{K}_{hv}+(*)+(*)\frac{1}{\varepsilon}\left(P_{hhv}+P_{hhv}^T\right)^{-1}\left(\varepsilon\bar{B}_h\bar{K}_{hv}-\left(P_{hhv}+P_{hhv}^T\right)+P_{hhv}^T\right)\right\}x_0<0. \qquad (3.18)$$

After some algebraic manipulations (3.18) gives

$$\begin{aligned}x_0^T\Big\{&\varepsilon\bar{K}_{hv}^T\bar{B}_h^T\left(P_{hhv}+P_{hhv}^T\right)^{-1}\bar{B}_h\bar{K}_{hv}+\bar{K}_{hv}^T\bar{B}_h^T\left(P_{hhv}+P_{hhv}^T\right)^{-1}P_{hhv}^T\\&+P_{hhv}\left(P_{hhv}+P_{hhv}^T\right)^{-1}\bar{B}_h\bar{K}_{hv}+\frac{1}{\varepsilon}P_{hhv}\left(P_{hhv}+P_{hhv}^T\right)^{-1}P_{hhv}^T\Big\}x_0<0.\end{aligned} \qquad (3.19)$$

Multiplying by $\varepsilon>0$ and grouping terms results:

$$x_0^T\left\{\left(\varepsilon\bar{B}_h\bar{K}_{hv}+P_{hhv}^T\right)^T\left(P_{hhv}+P_{hhv}^T\right)^{-1}\left(\varepsilon\bar{B}_h\bar{K}_{hv}+P_{hhv}^T\right)+P_{hhv}\left(P_{hhv}+P_{hhv}^T\right)^{-1}P_{hhv}^T\right\}x_0<0, \qquad (3.20)$$

which contradicts (3.15), since $\left(P_{hhv}+P_{hhv}^T\right)^{-1}>0$ and therefore (3.20) cannot be true; as consequence if $\Upsilon_{hh}^v<0$ holds, then $\bar{G}_{hhv}$ is not singular. ■

Remark 3.2. The conditions in (3.13) are LMIs up to the selection of ε. Prefixing this sort of parameter has been a common practice in the LPV community in recent years (de Oliveira and Skelton, 2001; Jaadari et al., 2012; Oliveira et al., 2011; Shaked, 2001) since it allows searching for a feasible solution in a logarithmically spaced family of values $\varepsilon\in\left\{10^{-6},10^{-5},\ldots,10^{6}\right\}$, which avoids an exhaustive linear search.

Remark 3.3. The control law (3.6) could be implemented as follows:

$$u=\begin{bmatrix}K_{1hv} & K_{2hv}\end{bmatrix}\begin{bmatrix}G_{1hv}^{-1} & 0\\ -G_{4hh}^{-1}G_{3hh}G_{1hv}^{-1} & G_{4hh}^{-1}\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\end{bmatrix}=\mathcal{F}_1x+\mathcal{F}_2\dot{x}, \qquad (3.21)$$

with $\mathcal{F}_1=K_{1hv}G_{1hv}^{-1}+K_{2hv}\left(-G_{4hh}^{-1}G_{3hh}G_{1hv}^{-1}\right)$ and $\mathcal{F}_2=K_{2hv}G_{4hh}^{-1}$. Knowing that $\dot{x}=E_v^{-1}\left(A_hx+B_hu\right)$, (3.21) yields

$$\begin{aligned}u&=\mathcal{F}_1x+\mathcal{F}_2E_v^{-1}A_hx+\mathcal{F}_2E_v^{-1}B_hu\\&\Leftrightarrow\left(I-\mathcal{F}_2E_v^{-1}B_h\right)u=\left(\mathcal{F}_1+\mathcal{F}_2E_v^{-1}A_h\right)x\\&\Leftrightarrow u=\left(I-\mathcal{F}_2E_v^{-1}B_h\right)^{-1}\left(\mathcal{F}_1+\mathcal{F}_2E_v^{-1}A_h\right)x.\end{aligned} \qquad (3.22)$$

The existence of the inverse of $I-\mathcal{F}_2E_v^{-1}B_h$ can be deduced from the matrix inversion lemma: $\left(I-\mathcal{F}_2E_v^{-1}B_h\right)^{-1}=I-B_h\left(E_v-B_h\mathcal{F}_2\right)^{-1}\mathcal{F}_2$, which means that the regularity of $I-\mathcal{F}_2E_v^{-1}B_h$ is equivalent to $E_v-B_h\mathcal{F}_2$ being regular. In addition, if the LMI conditions (3.13) are satisfied, then (3.15) is also satisfied; by congruence with $diag\left[\bar{G}_{hhv}^{-T},\ \ P_{hhv}^{-1}\right]$, (3.15) gives:

$$\begin{bmatrix} \bar{G}_{hhv}^{-T}\bar{A}_{hv}+\bar{G}_{hhv}^{-T}\bar{B}_h\bar{K}_{hv}\bar{G}_{hhv}^{-1}+(*) & (*) \\ P_{hhv}^{-1}+\varepsilon P_{hhv}^{-1}\left(\bar{A}_{hv}+\bar{B}_h\bar{K}_{hv}\bar{G}_{hhv}^{-1}\right)-\bar{G}_{hhv}^{-1} & -\varepsilon\left(P_{hhv}^{-1}+P_{hhv}^{-T}\right) \end{bmatrix}<0. \tag{3.23}$$

Define $\bar{G}_{hhv}^{-1}=\begin{bmatrix} G_{1hv}^{-1} & 0 \\ -G_{4hh}^{-1}G_{3hh}G_{1hv}^{-1} & G_{4hh}^{-1}\end{bmatrix}=X=\begin{bmatrix} X_1 & 0 \\ X_3 & X_4\end{bmatrix}$; the (1,1) block of (3.23) writes $X^T\bar{A}_{hv}+X^T\bar{B}_h\bar{K}_{hv}X+(*)<0$ or:

$$\begin{bmatrix} X_3^T\left(A_h+B_h\mathcal{F}_1\right)+(*) & (*) \\ X_4^T\left(A_h+B_h\mathcal{F}_1\right)+X_1-\left(E_v-B_h\mathcal{F}_2\right)^T X_3 & -X_4^T\left(E_v-B_h\mathcal{F}_2\right)+(*) \end{bmatrix}<0. \tag{3.24}$$

Since (3.24) holds, then $-X_4^T\left(E_v-B_h\mathcal{F}_2\right)+(*)<0$ also holds. Suppose $E_v-B_h\mathcal{F}_2$ is singular; therefore there exist $x_0\neq 0$ such that $\left(E_v-B_h\mathcal{F}_2\right)x_0=0$ which contradicts (3.24). Thus if the LMI conditions (3.13) hold, $E_v-B_h\mathcal{F}_2$ is regular and the control law (3.22) can be computed.

Example 3.1. Consider the TS descriptor model (3.1) with $w=0$, $r_a=r_e=2$, and matrices $E_1=\begin{bmatrix}0.8 & 0\\ 0.2 & 0.5\end{bmatrix}$, $E_2=\begin{bmatrix}4.7 & 0\\ 0.4 & 0.7\end{bmatrix}$, $A_1=\begin{bmatrix}-4.3 & 4.8\\ -1.7 & 1\end{bmatrix}$, $A_2=\begin{bmatrix}4.4+0.5\delta & -4.6\\ 3.9 & -1.9\end{bmatrix}$, $B_1=\begin{bmatrix}5.6\\ 0.9\end{bmatrix}$, $B_2=\begin{bmatrix}8.1\\ -0.5\delta\end{bmatrix}$, and the parameter $\delta>0$. The maximum value of δ for which conditions in (Guerra et al., 2007) were found feasible is $\delta=0.17$; the conditions in Theorem 3.1 were feasible up to the value $\delta=0.47$.

Note that taking $\delta=0.40$ there is no solution for Theorem 1 in (Guerra et al., 2007), while employing the conditions of Theorem 3.1 with $\varepsilon=0.1$ the following values were found:

$$P_1 = \begin{bmatrix} 0.09 & 0.14 \\ 0.14 & 0.41 \end{bmatrix}, G_{11}^{(1)} = \begin{bmatrix} 0.08 & 0.12 \\ 0.14 & 0.40 \end{bmatrix}, \; G_{21}^{(3)} = \begin{bmatrix} -0.22 & -0.37 \\ -0.16 & -0.32 \end{bmatrix}, \; G_{22}^{(4)} = \begin{bmatrix} 0.51 & -0.12 \\ 0.30 & 0.19 \end{bmatrix},$$
$$\bar{K}_{11} = \begin{bmatrix} -0.21 & -0.32 & -0.26 & 0.08 \end{bmatrix}, \; \bar{K}_{21} = \begin{bmatrix} -0.25 & 0.08 & 0.01 & 0.02 \end{bmatrix},$$
$$\bar{K}_{12} = \begin{bmatrix} -0.19 & -0.30 & 0.30 & 0.12 \end{bmatrix}, \; \bar{K}_{22} = \begin{bmatrix} -0.10 & 0.03 & 0.28 & -0.01 \end{bmatrix}.$$

For simulation proposes, the MFs are chosen as $h_1 = 1/\left(1 + x_1^2\right)$, $h_2 = 1 - h_1$, $v_1 = \left(x_2 + 1\right)/2$, $v_2 = 1 - v_1$. Figure 3.1 shows the open-loop $u(t) = 0$ and the close-loop trajectories for the initial conditions $x(0) = \begin{bmatrix} 1 & -0.5 \end{bmatrix}^T$.

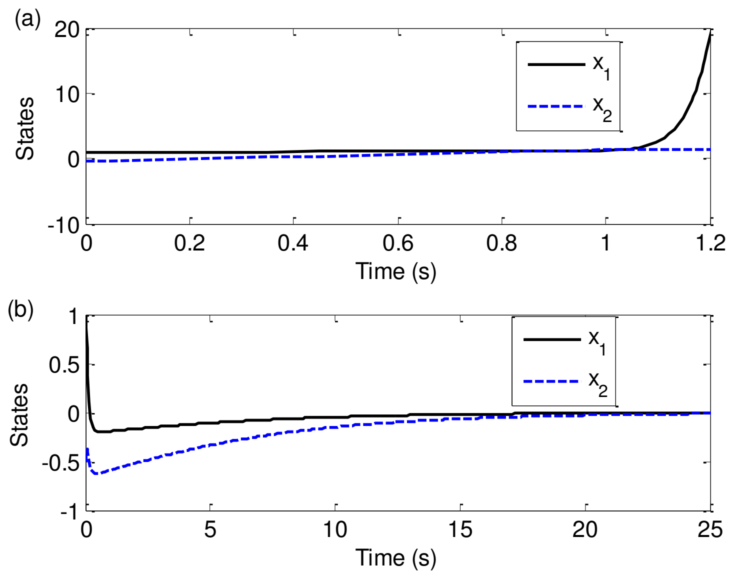

Figure 3.1. (a) State trajectories of the open-loop system. (b) State trajectories of the closed-loop system. ♦

3.1.2.2. H$_\infty$ control

Consider now the disturbance rejection problem $(w \neq 0)$. Substituting the control law (3.6) into the augmented TS descriptor (3.3) yields:

$$\bar{E}\dot{\bar{x}} = \left(\bar{A}_{hv} + \bar{B}_h \bar{K}_{hv} \bar{G}_{hhv}^{-1}\right)\bar{x} + \bar{D}_h w \quad \Leftrightarrow \quad \begin{bmatrix} \bar{A}_{hv} + \bar{B}_h \bar{K}_{hv} \bar{G}_{hhv}^{-1} & -I & \bar{D}_h \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{E}\dot{\bar{x}} \\ w \end{bmatrix} = 0. \qquad (3.25)$$

Recall that the output is $y = \bar{C}_h \bar{x} + J_h w$. The disturbance rejection can be realized by minimizing $\gamma > 0$ subject to

$$\sup_{\|w(t)\|_2 \neq 0} \frac{\|y(t)\|_2}{\|w(t)\|_2} \leq \gamma, \tag{3.26}$$

where $\|\bullet\|_2$ stands for l_2 norm. The following well-known condition (Tanaka and Wang, 2001):

$$\dot{V}(\bar{x}) + y^T y - \gamma^2 w^T w < 0, \qquad \forall \bar{x} \in \mathbb{R}^{2n_x}, \tag{3.27}$$

implies (3.26). Thus, condition (3.27) gives:

$$\begin{bmatrix} \bar{x} \\ \bar{E}\dot{\bar{x}} \\ w \end{bmatrix}^T \begin{bmatrix} \bar{C}_h^T \bar{C}_h & P_{hhv}^{-T} & \bar{C}_h^T J_h \\ P_{hhv}^{-1} & 0 & 0 \\ J_h^T \bar{C}_h & 0 & J_h^T J_h - \gamma^2 I \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{E}\dot{\bar{x}} \\ w \end{bmatrix} < 0. \tag{3.28}$$

Taking

$$\mathcal{Q} = \begin{bmatrix} \bar{C}_h^T \bar{C}_h & P_{hhv}^{-T} & \bar{C}_h^T J_h \\ P_{hhv}^{-1} & 0 & 0 \\ J_h^T \bar{C}_h & 0 & J_h^T J_h - \gamma^2 I \end{bmatrix} \quad \text{and} \quad \mathcal{W} = \begin{bmatrix} \bar{A}_{hv} + \bar{B}_h \bar{K}_{hv} \bar{G}_{hhv}^{-1} & -I & \bar{D}_h \end{bmatrix},$$

via Finsler's lemma we have

$$\mathcal{M} \begin{bmatrix} \bar{A}_{hv} + \bar{B}_h \bar{K}_{hv} \bar{G}_{hhv}^{-1} & -I & \bar{D}_h \end{bmatrix} + (*) + \begin{bmatrix} \bar{C}_h^T \bar{C}_h & P_{hhv}^{-T} & \bar{C}_h^T J_h \\ P_{hhv}^{-1} & 0 & 0 \\ J_h^T \bar{C}_h & 0 & J_h^T J_h - \gamma^2 I \end{bmatrix} < 0, \tag{3.29}$$

where $\mathcal{M} \in \mathbb{R}^{(2n_x + n_w) \times n_x}$. Condition (3.29) guarantees (3.28) under restriction (3.25). Multiplying (3.29) by $diag\begin{bmatrix} \bar{G}_{hhv}^T, & P_{hhv}, & I \end{bmatrix}$ on the left-hand side and by its transpose on the right-hand side and choosing $\mathcal{M} = \begin{bmatrix} \bar{G}_{hhv}^{-T} \\ \varepsilon P_{hhv}^{-1} \\ 0 \end{bmatrix}$, $\varepsilon > 0$ renders

$$\begin{bmatrix} I \\ \varepsilon I \\ 0 \end{bmatrix} \begin{bmatrix} \bar{A}_{hv} \bar{G}_{hhv} + \bar{B}_h \bar{K}_{hv} & -P_{hhv}^T & \bar{D}_h \end{bmatrix} + (*) + \begin{bmatrix} \bar{G}_{hhv}^T \bar{C}_h^T \bar{C}_h \bar{G}_{hhv} & \bar{G}_{hhv}^T & \bar{G}_{hhv}^T \bar{C}_h^T J_h \\ \bar{G}_{hhv} & 0 & 0 \\ J_h^T \bar{C}_h \bar{G}_{hhv} & 0 & J_h^T J_h - \gamma^2 I \end{bmatrix} < 0,$$

which can be expressed as

$$\begin{bmatrix} \bar{A}_{hv}\bar{G}_{hhv} + \bar{B}_h\bar{K}_{hv} + (*) & (*) & (*) \\ \bar{G}_{hhv} + \varepsilon\left(\bar{A}_{hv}\bar{G}_{hhv} + \bar{B}_h\bar{K}_{hv}\right) - P_{hhv} & -\varepsilon\left(P_{hhv} + P_{hhv}^T\right) & (*) \\ \bar{D}_h^T & \varepsilon\bar{D}_h^T & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} \bar{G}_{hhv}^T\bar{C}_h^T \\ 0 \\ J_h^T \end{bmatrix} \begin{bmatrix} \bar{C}_h\bar{G}_{hhv} & 0 & J_h \end{bmatrix} < 0.$$

Finally, applying the Schur complement yields

$$\Upsilon_{hh}^v := \begin{bmatrix} \bar{A}_{hv}\bar{G}_{hhv} + \bar{B}_h\bar{K}_{hv} + (*) & (*) & (*) & (*) \\ \bar{G}_{hhv} + \varepsilon\left(\bar{A}_{hv}\bar{G}_{hhv} + \bar{B}_h\bar{K}_{hv}\right) - P_{hhv} & -\varepsilon\left(P_{hhv} + P_{hhv}^T\right) & (*) & (*) \\ \bar{D}_h^T & \varepsilon\bar{D}_h^T & -\gamma^2 I & (*) \\ \bar{C}_h\bar{G}_{hhv} & 0 & J_h & -I \end{bmatrix} < 0. \quad (3.30)$$

Based on the developments above, the following theorem can be stated:

Theorem 3.2. The TS descriptor model (3.1) under control law (3.6) is asymptotically stable and ensures disturbance attenuation $\gamma > 0$ if, for a given $\varepsilon > 0$, there exist matrices $P_{i_1i_2j_1}$, $\bar{G}_{i_1i_2j_1}$, $\bar{K}_{i_2j_1}$, $i_1, i_2 \in \{1,2,\ldots,r_a\}$, $j_1 \in \{1,2,\ldots,r_e\}$ as defined in (3.6) and (3.8), such that

$$\Upsilon_{i_1i_1}^{j_1} < 0,\ \forall i_1, j_1; \qquad \frac{2}{r_a - 1}\Upsilon_{i_1i_1}^{j_1} + \Upsilon_{i_1i_2}^{j_1} + \Upsilon_{i_2i_1}^{j_1} < 0, \quad \forall j_1,\ i_i \neq i_2, \quad (3.31)$$

hold with

$$\Upsilon_{i_1i_2}^{j_1} = \begin{bmatrix} \bar{A}_{i_1j_1}\bar{G}_{i_1i_2j_1} + \bar{B}_{i_1}\bar{K}_{i_2j_1} + (*) & (*) & (*) & (*) \\ \bar{G}_{i_1i_2j_1} + \varepsilon\left(\bar{A}_{i_1j_1}\bar{G}_{i_1i_2j_1} + \bar{B}_{i_1}\bar{K}_{i_2j_1}\right) - P_{i_1i_2j_1} & -\varepsilon\left(P_{i_1i_2j_1} + P_{i_1i_2j_1}^T\right) & (*) & (*) \\ \bar{D}_{i_1}^T & \varepsilon\bar{D}_{i_1}^T & -\gamma^2 I & (*) \\ \bar{C}_{i_1}\bar{G}_{i_1i_2j_1} & 0 & J_{i_1} & -I \end{bmatrix}. \quad (3.32)$$

Proof. Applying the relaxation Lemma B.3 (Appendix B) to (3.30) ends the proof. ■

Example 3.2. Consider the TS descriptor model (3.1) with $r_a = r_e = 2$, $E_1 = \begin{bmatrix} 0.8 & 0 \\ 0.2 & 0.5 \end{bmatrix}$, $E_2 = \begin{bmatrix} 4.7 & 0 \\ 0.4 & 0.7 \end{bmatrix}$, $A_1 = \begin{bmatrix} -4.3 & 4.8 \\ -1.7 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 3.9 & -4.6 \\ 3.9 & -1.9 \end{bmatrix}$, $B_1 = \begin{bmatrix} 5.6 \\ 0.9 \end{bmatrix}$, $B_2 = \begin{bmatrix} 8.1 \\ 0.5 \end{bmatrix}$, $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $D_1 = \begin{bmatrix} -0.5\delta \\ 0 \end{bmatrix}$, $D_2 = \begin{bmatrix} 0 \\ -0.5\delta \end{bmatrix}$, $J_1 = -\delta$, $J_2 = \delta$, and the parameter $\delta \in [-2 \quad 0]$. Figure 3.2 shows the results when the optimal values for γ (min γ) are computed via Theorem 2 in (Guerra et al., 2007) (represented by black-O) and Theorem 3.2 (represented by blue-X). As can be seen, Theorem 3.2 obtains betters results.

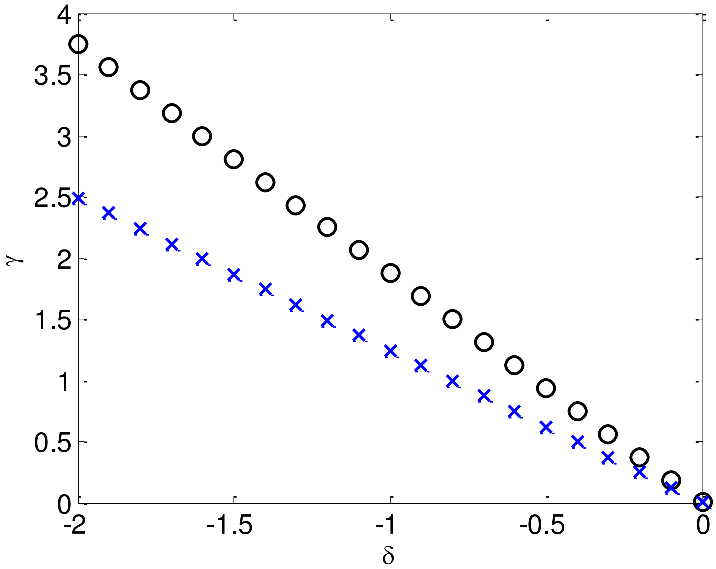

Figure 3.2. Optimal values for γ in Example 3.2. ♦

3.2. Discrete-time TS descriptor models

This section provides LMI conditions for the stabilization of discrete-time TS descriptor models by following recent advances: 1) using past samples in the MFs of the Lyapunov function and the observer gains as in (Guerra et al., 2012b), 2) a generalization via two different non-quadratic Lyapunov functions as in (Lendek et al., 2015).

3.2.1. Problem statement

Consider the following discrete-time TS model in the descriptor form:

$$\begin{aligned} E_v x_{k+1} &= A_h x_k + B_h u_k + D_h w_k \\ y_k &= C_h x_k + J_h w_k. \end{aligned} \tag{3.33}$$

For the controller design purpose, the following *nonlinear* control law is used:

$$u_k = \mathcal{K}\mathcal{G}^{-1} x_k, \tag{3.34}$$

where $\mathcal{K} \in \mathbb{R}^{n_u \times n_x}$ and $\mathcal{G} \in \mathbb{R}^{n_x \times n_x}$ are the controller gains to be designed. Their structure will be defined later on. The TS descriptor model (3.33) together with the control law (3.34) yields:

$$\begin{aligned} E_v x_{k+1} &= \left(A_h + B_h \mathcal{K}\mathcal{G}^{-1}\right) x_k + D_h w_k \\ y_k &= C_h x_k + J_h w_k. \end{aligned} \tag{3.35}$$

In order to design the state feedback controller (3.34), a generic Lyapunov function is considered:

$$V\left(x_k\right)=x_k^T \mathcal{P} x_k, \qquad \mathcal{P}=\mathcal{P}^T>0. \tag{3.36}$$

The structure of $\mathcal{P}$ depends on the case treated. The variation of the Lyapunov function (3.36) reads:

$$\Delta V\left(x_k\right)=x_{k+1}^T \mathcal{P}_+ x_{k+1}-x_k^T \mathcal{P} x_k<0. \tag{3.37}$$

3.2.2. Results

3.2.2.1. Stabilization

The closed-loop system (3.35) with $w_k=0$ can be written as the following equality constraint:

$$\begin{bmatrix} A_h+B_h \mathcal{K} \mathcal{G}^{-1} & -E_v \end{bmatrix}\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}=0. \tag{3.38}$$

The variation of the Lyapunov function, i.e., (3.37) can be expressed as:

$$\Delta V\left(x_k\right)=\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}^T\begin{bmatrix} -\mathcal{P} & 0 \\ 0 & \mathcal{P}_+ \end{bmatrix}\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}<0. \tag{3.39}$$

Denote $\mathcal{X}=\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$, $\mathcal{W}=\begin{bmatrix} A_h+B_h \mathcal{K} \mathcal{G}^{-1} & -E_v \end{bmatrix}$, and $\mathcal{Q}=\begin{bmatrix} -\mathcal{P} & 0 \\ 0 & \mathcal{P}_+ \end{bmatrix}$; via Finsler's lemma the inequality (3.39) and the equality constraint (3.38) can be combined in the following inequality:

$$\mathcal{M}\begin{bmatrix} A_h+B_h \mathcal{K} \mathcal{G}^{-1} & -E_v \end{bmatrix}+(*)+\begin{bmatrix} -\mathcal{P} & 0 \\ 0 & \mathcal{P}_+ \end{bmatrix}<0, \tag{3.40}$$

where $\mathcal{M} \in \mathbb{R}^{2n_x \times n_x}$ is a free matrix. Depending on the selection of $\mathcal{P} \in \mathbb{R}^{n_x \times n_x}$, the controller gains $\mathcal{K} \in \mathbb{R}^{n_u \times n_u}$ and $\mathcal{G} \in \mathbb{R}^{n_x \times n_x}$, and the matrix $\mathcal{M}$ several results can be obtained from (3.40). Two classes of Lyapunov functions are employed:

1) The non-quadratic (NQ) Lyapunov function (Guerra and Vermeiren, 2004).
2) The delayed non-quadratic (DNQ) Lyapunov function (Guerra et al., 2012b).

Non-quadratic approach

The following result uses the Lyapunov function (3.36) with $\mathcal{P} = G_h^{-T} P_h G_h^{-1}$, i.e.,

$$V(x_k) = x_k^T \left(\sum_{i=1}^{r_a} h_i(z(k)) G_i \right)^{-T} \left(\sum_{i=1}^{r_a} h_i(z(k)) P_i \right) \left(\sum_{i=1}^{r_a} h_i(z(k)) G_i \right)^{-1} x_k. \tag{3.41}$$

Theorem 3.3. The TS descriptor model (3.33) with $w_k = 0$ is asymptotically stabilized by the controller $u_k = K_{hv} G_h^{-1} x_k$ if there exist matrices $P_{i_2} = P_{i_2}^T > 0$, $K_{i_2 j_1}$, and G_{i_2}, $i_1, i_2, i_x \in \{1, 2, \ldots, r_a\}$, $j_1 \in \{1, 2, \ldots, r_e\}$ such that:

$$\Upsilon_{i_1 i_1 i_x}^{j_1} < 0, \ \forall i_1, i_x, j_1; \qquad \frac{2}{r_a - 1} \Upsilon_{i_1 i_1 i_x}^{j_1} + \Upsilon_{i_1 i_2 i_x}^{j_1} + \Upsilon_{i_2 i_1 i_x}^{j_1} < 0, \ \forall i_x, j_1, \ i_i \neq i_2, \tag{3.42}$$

are satisfied with

$$\Upsilon_{i_1 i_2 i_x}^{j_1} = \begin{bmatrix} -P_{i_2} & (*) \\ A_{i_1} G_{i_2} + B_{i_1} K_{i_2 j_1} & -E_{j_1} G_{i_x} - G_{i_x}^T E_{j_1}^T + P_{i_x} \end{bmatrix}. \tag{3.43}$$

Proof. Recall (3.40) and consider the controller gains $\mathcal{K} = K_{hv}$ and $\mathcal{G} = G_h$. The Lyapunov function is selected as $V(x_k) = x_k^T G_h^{-T} P_h G_h^{-1} x_k$. By using the congruence property with the full-rank matrix $diag\begin{bmatrix} G_h^T, & G_{h^+}^T \end{bmatrix}$ and selecting the free matrix as $\mathcal{M} = \begin{bmatrix} 0 & G_{h^+}^{-1} \end{bmatrix}^T$, (3.40) gives

$$\Upsilon_{hhh^+}^{v} := \begin{bmatrix} -P_h & (*) \\ A_h G_h + B_h K_{hv} & -E_v G_{h^+} - G_{h^+}^T E_v^T + P_{h^+} \end{bmatrix} < 0. \tag{3.44}$$

Finally, using the Lemma B.3 yields (3.42), thus concluding the proof. ■

Different conditions can be obtained when the structure of the Lyapunov function changes. The following theorem uses another structure of the Lyapunov function (3.36), $\mathcal{P} = P_h^{-1}$, i.e.,

$$V(x_k) = x_k^T \left(\sum_{i=1}^{r_a} h_i(z(k)) P_i \right)^{-1} x_k = x_k^T P_h^{-1} x_k, \tag{3.45}$$

and the controller gains are defined as $\mathcal{K} = K_{hv}$ and $\mathcal{G} = G_{hv}$. Then, (3.40) gives

$$\mathcal{M} \begin{bmatrix} A_h + B_h K_{hv} G_{hv}^{-1} & -E_v \end{bmatrix} + (*) + \begin{bmatrix} -P_h^{-1} & 0 \\ 0 & P_{h^+}^{-1} \end{bmatrix} < 0. \tag{3.46}$$

From (3.46), two results can be stated depending on the matrix used when the congruence property is applied. Theorems 3.4 and 3.5 summarize these results.

Theorem 3.4. The TS descriptor model (3.33) (when $w=0$) is asymptotically stabilized by the control law $u_k = K_{hv}G_{hv}^{-1}x_k$ if there exist matrices $P_{i_2} = P_{i_2}^T > 0$, $K_{i_2 j_1}$, and $G_{i_2 j_1}$, $i_1, i_2, i_x \in \{1, 2, \ldots, r_a\}$, $j_1 \in \{1, 2, \ldots, r_e\}$ such that:

$$\Upsilon_{i_1 i_1 i_x}^{j_1} < 0,\ \forall i_1, i_x, j_1; \qquad \frac{2}{r_a - 1}\Upsilon_{i_1 i_1 i_x}^{j_1} + \Upsilon_{i_1 i_2 i_x}^{j_1} + \Upsilon_{i_2 i_1 i_x}^{j_1} < 0,\ \forall i_x, j_1,\ i_i \neq i_2, \tag{3.47}$$

are satisfied with

$$\Upsilon_{i_1 i_2 i_x}^{j_1} = \begin{bmatrix} -G_{i_2 j_1} - G_{i_2 j_1}^T + P_{i_2} & (*) \\ A_{i_1} G_{i_2 j_1} + B_{i_1} K_{i_2 j_1} & -E_{j_1} P_{i_x} - P_{i_x} E_{j_1}^T + P_{i_x} \end{bmatrix}. \tag{3.48}$$

Proof. Recall (3.46). By congruence with the matrix $diag\begin{bmatrix} G_{hv}^T, & P_{h^+} \end{bmatrix}$ and setting the free matrix as $\mathcal{M} = \begin{bmatrix} 0 & P_{h^+}^{-1} \end{bmatrix}^T$, (3.46) gives

$$\Upsilon_{hhh^+}^{v} := \begin{bmatrix} -G_{hv}^T P_h^{-1} G_{hv} & (*) \\ A_h G_{hv} + B_h K_{hv} & -E_v P_{h^+} - P_{h^+} E_v^T + P_{h^+} \end{bmatrix} < 0. \tag{3.49}$$

Applying Property A.3 and the Lemma B.3 yields (3.47), thus concluding the proof. ■

Remark 3.4. Neither equivalence nor inclusion relation can be established between the LMI constraints in Theorems 3.3 and 3.4, since they have been derived from different Lyapunov structures (Lendek et al., 2012). This means that for one control problem the conditions in Theorem 3.3 could be feasible while those of Theorem 3.4 could be unfeasible; or vice-versa.

Example 3.3. Consider the TS descriptor model (3.33) with $r_a = r_e = 2$ and

$$E_1 = \begin{bmatrix} 0.9 & 0.2+a \\ -0.4-b & 1.3 \end{bmatrix},\quad E_2 = \begin{bmatrix} 0.9 & 0.2-a \\ -0.4+b & 1.3 \end{bmatrix},\quad A_1 = \begin{bmatrix} -0.9 & 1+a \\ -1.5 & -0.5 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -0.9 & 1-a \\ -1.5 & -0.5 \end{bmatrix},\quad B_1 = \begin{bmatrix} 1-b \\ 0 \end{bmatrix},\quad B_2 = \begin{bmatrix} 1+b \\ 0 \end{bmatrix}.$$

The real valued parameters are defined as $-3 \le a \le 3$ and $-1.2 \le b \le 1.2$. The LMI conditions in Theorem 3.3 and Theorem 3.4 have been tested in order to illustrate Remark 3.4. Figure 3.3 shows the feasibility sets for Theorem 3.3 $(\times)$ and Theorem 3.4 $(\square)$.

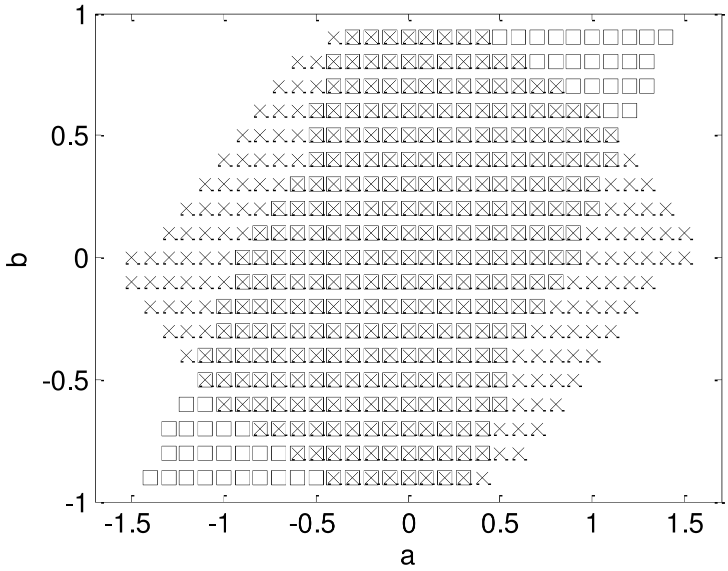

Figure 3.3. Feasibility set for Theorem 3.3 (×) and Theorem 3.4 (□) in Example 3.3. ♦

A refined result of Theorem 3.4 can be obtained using different matrices for congruence and for $\mathcal{M}$ in (3.46).

Theorem 3.5. The TS descriptor model (3.33) with $w_k = 0$ is asymptotically stabilized by the controller $u_k = K_{hv} G_{hv}^{-1} x_k$ if there exist matrices $P_{i_2} = P_{i_2}^T > 0$, $K_{i_2 j_1}$, $G_{i_2 j_1}$, and $F_{i_1 i_2 i_x}$, $i_1, i_2, i_x \in \{1, 2, \ldots, r_a\}$, $j_1 \in \{1, 2, \ldots, r_e\}$ such that:

$$\Upsilon^{j_1}_{i_1 i_1 i_x} < 0,\ \forall i_1, i_x, j_1; \qquad \frac{2}{r_a - 1}\Upsilon^{j_1}_{i_1 i_1 i_x} + \Upsilon^{j_1}_{i_1 i_2 i_x} + \Upsilon^{j_1}_{i_2 i_1 i_x} < 0,\ \forall i_x, j_1\ \ i_i \neq i_2, \tag{3.50}$$

are satisfied with

$$\Upsilon^{j_1}_{i_1 i_2 i_x} = \begin{bmatrix} -G_{i_2 j_1} - G_{i_2 j_1}^T + P_{i_2} & (*) & (*) \\ A_{i_1} G_{i_2 j_1} + B_{i_1} K_{i_2 j_1} & -E_{j_1} F_{i_1 i_2 i_x} - F^T_{\ 1 i_2 i_x} E^T_{j_1} & (*) \\ 0 & F_{i_1 i_2 i_x} & -P_{i_x} \end{bmatrix}. \tag{3.51}$$

Proof. Recall (3.46). By using the congruence property with the full-rank matrix $diag\left[G_{hv}^T,\ \ F_{hhh^+}^T\right]$ and setting the free matrix as $\mathcal{M} = \begin{bmatrix} 0 & F_{hhh^+}^{-1}\end{bmatrix}^T$, (3.46) gives:

$$\begin{bmatrix} -G_{hv}^T P_h^{-1} G_{hv} & (*) \\ A_h G_{hv} + B_h K_{hv} & -E_v F_{hhh^+} - F_{hhh^+}^T E_v^T + F_{hhh^+}^T P_{h^+}^{-1} F_{hhh^+} \end{bmatrix} < 0. \tag{3.52}$$

Applying the Schur complement on the entry (2,2) gives:

$$\Upsilon_{hhh^+}^{v} := \begin{bmatrix} -G_{hv}^{T} P_h^{-1} G_{hv} & (*) & (*) \\ A_h G_{hv} + B_h K_{hv} & -E_v F_{hhh^+} - F_{hhh^+}^{T} E_v^{T} & (*) \\ 0 & F_{hhh^+} & -P_{h^+} \end{bmatrix} < 0, \tag{3.53}$$

which by means of Property A.3 on entry (1,1) and via Lemma B.3 yields (3.50), thus concluding the proof. ■

Remark 3.5. The conditions given by Theorems 3.3, 3.4, and 3.5 hold if the matrix E_v is nonsingular; this fact can be seen, for instance, from conditions in Theorem 3.3: if Theorem 3.3 holds then (3.44) holds too, which means $-E_v G_{h^+} - G_{h^+}^T E_v^T + P_{h^+} < 0$. Assume that E_v is singular; then, there exist $x_0 \neq 0$ such that $E_v x_0 = 0$, thus the block (2,2) of (3.44) becomes $P_{h^+} < 0$ which cannot be true since $P_h > 0$. Thus, if conditions in Theorem 3.3 hold, then E_v is not singular.

Remark 3.6. Note that in Theorem 3.5 a new matrix F_{hhh^+} is introduced, which adds extra degrees of freedom without increasing the number of LMI constraints. Moreover, F_{hhh^+} is not used in the control law, therefore the use of the next sample MFs $h_{i_x}\left(z\left(k+1\right)\right)$, $i_x \in \{1, 2, \ldots, r_a\}$ is valid. Hence, the results in the Theorem 3.5 are more general than those in Theorem 3.4. To see this, let $F_{hhh^+} = P_{h^+}$. Applying the Schur complement on (3.53) gives (3.49).

Figure 3.4 illustrates Remarks 3.4 and 3.6.

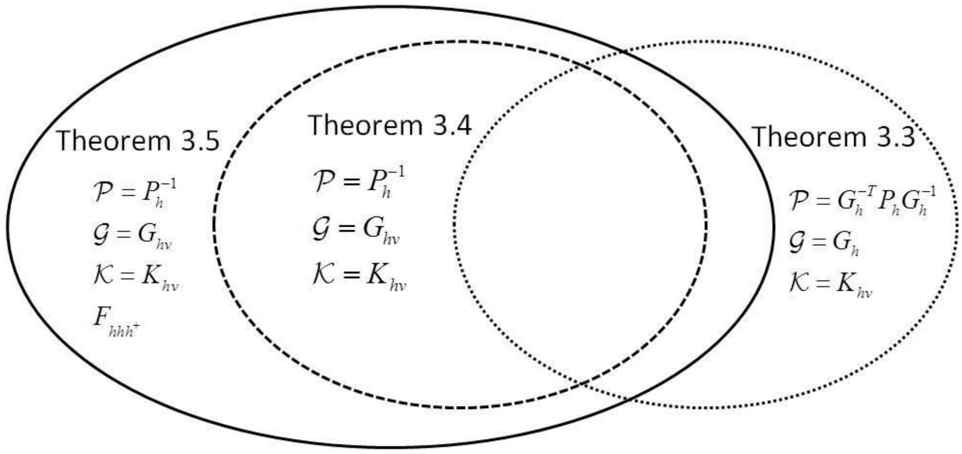

Figure 3.4. Illustration of Remark 3.4 and Remark 3.6.

Example 3.4. Recall Example 3.3. Figure 3.5 illustrates Remark 3.6 when the conditions of Theorems 3.4 $(\square)$ and 3.5 $(\times)$ are tested for parameter values $-3 \le a \le 3$ and $-1.2 \le b \le 1.2$.

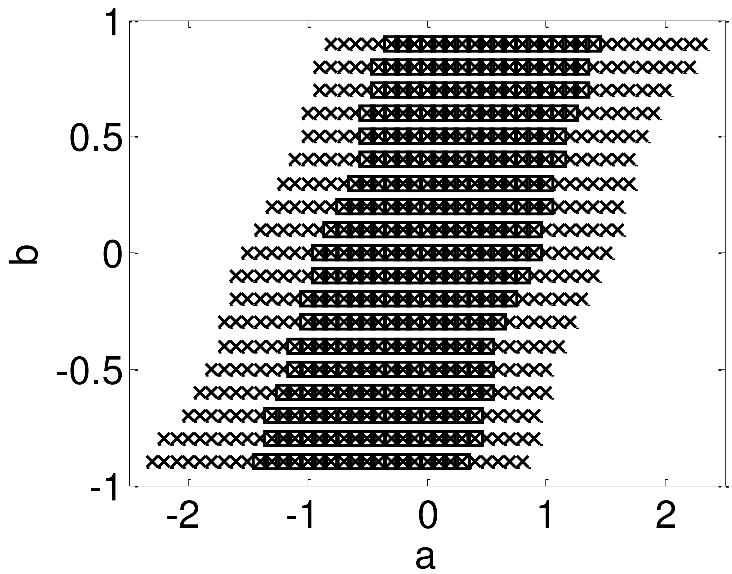

Figure 3.5. Solution set for Theorem 3.4 (□) and Theorem 3.5 (×) in Example 3.4. ♦

Delayed non-quadratic approach

This section introduces a way to improve the results obtained in the previous one. The main idea is to use delays in the MFs of the Lyapunov matrix, thus changing the structure of the controller matrices. This idea has been introduced in (Guerra et al., 2012b).

Recall the non-PDC control law:

$$u_k = \sum_{i=1}^{r_a} \sum_{j=1}^{r_e} h_i\left(z(k)\right) v_j\left(z(k)\right) K_{ij} \left(\sum_{i=1}^{r_a} h_i\left(z(k)\right) G_i \right)^{-1} x_k. \tag{3.54}$$

In order to introduce a delay in the MFs of (3.54), the simplest options for the Lyapunov function are:

1) $V\left(x_k\right) = x_k^T \left(\sum_{i=1}^{r_a} h_i\left(z(k)\right) P_i \right)^{-1} x_k$ or 2) $V\left(x_k\right) = x_k^T \left(\sum_{i_x=1}^{r_a} h_{i_x}\left(z(k-1)\right) P_{i_x} \right)^{-1} x_k$.

Since the controller must be causal, i.e., no future information can be implemented, it should not contain positive delays. On the one hand, Option 1 implies that the variation of the Lyapunov function $\Delta V\left(x(k)\right) = V\left(x(k+1)\right) - V\left(x(k)\right)$ depends on the advanced MF

$h_{i_x}\left(z(k+1)\right)$ $i_x \in \{1,2,\ldots,r_a\}$, which cannot be introduced in the control, i.e., $u_k = K_{hh^+v} G_{hh^+}^{-1} x_k$ cannot be implemented. On the other hand, Option 2 does not introduce future MFs but the delayed one $h_{i_x}\left(z(k-1)\right)$, $i_x \in \{1,2,\ldots,r_a\}$ which can be introduced in the controller (3.54), i.e., $u_k = K_{hh^-v} G_{hh^-v}^{-1} x_k$. For consistency, it is assumed that $z(-1) = z(0)$.

Based on the discussion above, the following results were obtained; since they are a 'direct' extension of Theorems 3.3, 3.4, and 3.5, they are summarized in the following corollary.

Corollary 3.1. Consider the delayed Lyapunov function $V(x_k) = x_k^T G_{h^-}^{-T} P_{h^-} G_{h^-}^{-1} x_k$ and the control law $u_k = K_{hh^-v} G_{h^-}^{-1} x_k$. We obtain (3.42) with

$$\Upsilon_{i_1 i_2 i_x}^{j_1} = \begin{bmatrix} P_{i_x} & (*) \\ A_{i_1} G_{i_x} + B_{i_1} K_{i_2 i_x j_1} & -E_{j_1} G_{i_2} - G_{i_2}^T E_{j_1}^T + P_{i_2} \end{bmatrix}, \tag{3.55}$$

$i_1, i_2, i_x \in \{1,2,\ldots,r_a\}$, $j_1 \in \{1,2,\ldots,r_e\}$. For the delayed Lyapunov function $V(x_k) = x_k^T P_{h^-}^{-1} x_k$ and the control law $u_k = K_{hh^-v} G_{hh^-v}^{-1} x_k$ we obtain (3.47) with

$$\Upsilon_{i_1 i_2 i_x}^{j_1} = \begin{bmatrix} -G_{i_2 i_x j_1} - G_{i_2 i_x j_1}^T + P_{i_x} & (*) \\ A_{i_1} G_{i_2 i_x j_1} + B_{i_1} K_{i_2 i_x j_1} & -E_{j_1} P_{i_2} - P_{i_2} E_{j_1}^T + P_{i_2} \end{bmatrix}, \tag{3.56}$$

$i_1, i_2, i_x \in \{1,2,\ldots,r_a\}$, $j_1 \in \{1,2,\ldots,r_e\}$. An improvement leads to (3.50) with

$$\Upsilon_{i_1 i_2 i_x}^{j_1} = \begin{bmatrix} -G_{i_2 i_x j_1} - G_{i_2 i_x j_1}^T + P_{i_x} & (*) & (*) \\ A_{i_1} G_{i_2 i_x j_1} + B_{i_1} K_{i_2 i_x j_1} & -E_{j_1} F_{i_1 i_2 i_x} - F_{i_1 i_2 i_x}^T E_{j_1}^T & (*) \\ 0 & F_{i_1 i_2 i_x} & -P_{i_2} \end{bmatrix}, \tag{3.57}$$

$i_1, i_2, i_x \in \{1,2,\ldots,r_a\}$, $j_1 \in \{1,2,\ldots,r_e\}$.

Proof. The results follow direct from inequality (3.40) using the same lines of proofs as for Theorem 3.3, Theorem 3.4, and Theorem 3.5, respectively. Table 3.1 provides a sketch of the proof. ■

Since all the approaches involve three convex sums in $h(\bullet)$ and one convex sum in $v(\bullet)$ the number of LMI constraints is $r_a \times r_a \times r_a \times r_e$ and is the same for Theorems 3.3, 3.4, and

3.5, and their respective delayed approaches;. Table 3.2 summarizes the obtained results in terms of number of decision variables, where n_x is the number of states, n_u is the number of inputs, r_a and r_e are the number of linear models in the right-hand side and in the left-hand side, respectively.

Remark 3.7. Note that when using past samples in the MFs to achieve relaxations, double sum relaxations should be taken into account, i.e., cross products sharing the same sample index should appear between the decision variables and the system matrices.

Table 3.1. Sketch of the proof for Corollary 3.1 (Delayed approaches).

Approach	Step 1	Step 2	Step 3	Result
Theorem 3.3 Delayed	$\mathcal{P}=G_{h^-}^{-T}P_{h^-}G_{h^-}^{-1}$ $\mathcal{K}=K_{hh^-v}$ $\mathcal{G}=G_{h^-}$	Congruence with $diag\left[G_{h^-}^T,\quad G_h^T\right]$ Set $\mathcal{M}=\left[0\quad G_h^{-1}\right]^T$	Lemma B.3	LMIs (3.42) with (3.55)
Theorem 3.4 Delayed	$\mathcal{P}=P_{h^-}^{-1}$ $\mathcal{K}=K_{hh^-v}$ $\mathcal{G}=G_{hh^-v}$	Congruence with $diag\left[G_{hh^-v}^T,\quad P_h\right]$ Set $\mathcal{M}=\left[0\quad P_h^{-1}\right]^T$	Property A.3 and Lemma B.3	LMIs (3.47) with (3.56)
Theorem 3.5 Delayed		Congruence with $diag\left[G_{hh^-v}^T,\quad F_{hhh^-}^T\right]$ Set $\mathcal{M}=\left[0\quad F_{hhh^-}^{-1}\right]^T$	Property A.3, Schur complement, and Lemma B.3	LMIs (3.50) with (3.57)

Example 3.5. Recall Example 3.3. Figure 3.7 illustrates Remark 3.7 when the conditions in Theorem 3.3 and its delayed approach $\left(\mathcal{P}=G_{h^-}^{-T}P_{h^-}G_{h^-}^{-1},\ \mathcal{K}=K_{hh^-v},\ \mathcal{G}=G_{h^-}\right)$ are implemented for parameter values $-3\le a\le 3$ and $-1.2\le b\le 1.2$. As can be seen from Figure 3.6, in this case, the delayed approach performs worse than Theorem 3.3. This is explained by the fact that there is no cross product at the same sample between the system matrix and the designed gain, i.e., no double sum relaxation scheme can be implemented on $\sum_{i_1=1}^{r_a}\sum_{i_x=1}^{r_a}h_{i_1}\left(z(k)\right)h_{i_x}\left(z(k-1)\right)A_{i_1}G_{i_x}$.

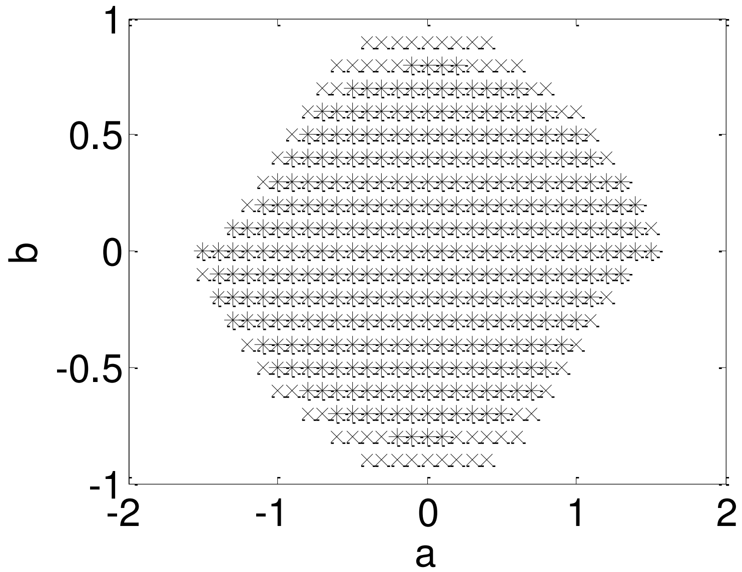

Figure 3.6. Solution set for conditions in Theorem 3.3 (x) and its delayed approach (+) in Example 3.5. ♦

Table 3.2. Comparison of Theorems 3.3, 3.4, and 3.5; and their delayed approaches (Corollary 3.1).

Approach	Lyapunov function (3.36)	Control law (3.34)	Number of decision variables
Theorem 3.3	$\mathcal{P}=G_h^{-T}P_hG_h^{-1}$	$\mathcal{K}=K_{hv}$ $\mathcal{G}=G_h$	$0.5n_x\times(n_x+1)\times r_a+(n_xn_ur_e+n_x^2)\times r_a$
Theorem 3.4	$\mathcal{P}=P_h^{-1}$	$\mathcal{K}=K_{hv}$ $\mathcal{G}=G_{hv}$	$0.5n_x\times(n_x+1)\times r_a+(n_xn_u+n_x^2)\times r_ar_e$
Theorem 3.5			$0.5n_x\times(n_x+1)\times r_a+(n_xn_u+n_x^2)\times r_ar_e+n_x^2r_a^3$
Theorem 3.3 Delayed	$\mathcal{P}=G_{h^-}^{-T}P_{h^-}G_{h^-}^{-1}$	$\mathcal{K}=K_{hh^-v}$ $\mathcal{G}=G_{h^-}$	$0.5n_x\times(n_x+1)\times r_a+(n_xn_ur_ar_e+n_x^2)\times r_a$
Theorem 3.4 Delayed	$\mathcal{P}=P_{h^-}^{-1}$	$\mathcal{K}=K_{hh^-v}$ $\mathcal{G}=G_{hh^-v}$	$0.5n_x\times(n_x+1)\times r_a+(n_xn_u+n_x^2)\times r_ar_e$
Theorem 3.5 Delayed			$0.5n_x\times(n_x+1)\times r_a+(n_xn_u+n_x^2)\times r_a^2r_e+n_x^2r_a^3$

3.2.2.2. H$_\infty$ control

In this subsection, the problem of disturbance attenuation is addressed. To end, rewrite the closed-loop system (3.35) as

$$\begin{bmatrix} A_h + B_h\mathcal{K}\mathcal{G}^{-1} & -E_v & D_h \end{bmatrix}\begin{bmatrix} x_k \\ x_{k+1} \\ w_k \end{bmatrix} = 0, \qquad y_k = C_h x_k + J_h w_k. \tag{3.58}$$

Then, consider Lyapunov function $V(x_k) = x_k^T \mathcal{P} x_k$, then $\Delta V(x_k) + y_k^T y_k - \gamma^2 w_k^T w_k < 0$ together with (3.37) writes

$$\begin{bmatrix} x_k \\ x_{k+1} \\ w_k \end{bmatrix}^T \begin{bmatrix} C_h^T C_h - \mathcal{P} & 0 & C_h^T J_h \\ 0 & \mathcal{P}_+ & 0 \\ J_h^T C_h & 0 & J_h^T J_h - \gamma^2 I \end{bmatrix}\begin{bmatrix} x_k \\ x_{k+1} \\ w_k \end{bmatrix} < 0. \tag{3.59}$$

Take $\mathcal{X} = \begin{bmatrix} x_k \\ x_{k+1} \\ w_k \end{bmatrix}$, $\mathcal{W} = \begin{bmatrix} A_h + B_h\mathcal{K}\mathcal{G}^{-1} & -E_v & D_h \end{bmatrix}$, $\mathcal{Q} = \begin{bmatrix} C_h^T C_h - \mathcal{P} & 0 & C_h^T J_h \\ 0 & \mathcal{P}_+ & 0 \\ J_h^T C_h & 0 & J_h^T J_h - \gamma^2 I \end{bmatrix}$.

Using Finsler's lemma, the equality constraint (3.58) together with the inequality (3.59) gives

$$\mathcal{M}\begin{bmatrix} A_h + B_h\mathcal{K}\mathcal{G}^{-1} & -E_v & D_h \end{bmatrix} + (*) + \begin{bmatrix} C_h^T C_h - \mathcal{P} & 0 & C_h^T J_h \\ 0 & \mathcal{P}_+ & 0 \\ J_h^T C_h & 0 & J_h^T J_h - \gamma^2 I \end{bmatrix} < 0, \tag{3.60}$$

where $\mathcal{M} \in \mathbb{R}^{(2n_x+n_w)\times n_x}$ is a free matrix. The following results are based on inequality (3.60). As in the previous section, the resulting LMI constraints depend on the selection of the Lyapunov matrix $\mathcal{P}$, controller gains $\mathcal{K}$, $\mathcal{G}$, and the slack matrix $\mathcal{M}$. For the sake of simplicity, only the proof of the first result is given. The others can be easily inferred from the previous developments.

Theorem 3.6. The TS descriptor model (3.33) under the control law $u_k = K_{hv} G_h^{-1} x_k$ is asymptotically stable and guarantees disturbance attenuation $\gamma > 0$ if there exist matrices $P_{i_2} = P_{i_2}^T > 0$, G_{i_2}, and $K_{i_2 j_1}$, $i_1, i_2, i_x \in \{1, 2, \ldots, r_a\}$, $j_1 \in \{1, 2, \ldots, r_e\}$ such that

$$\Upsilon_{i_1 i_1 i_x}^{j_1} < 0,\ \forall i_1, i_x, j_1; \qquad \frac{2}{r_a - 1}\Upsilon_{i_1 i_1 i_x}^{j_1} + \Upsilon_{i_1 i_2 i_x}^{j_1} + \Upsilon_{i_2 i_1 i_x}^{j_1} < 0,\ \forall i_x, j_1,\ i_i \neq i_2, \tag{3.61}$$

are satisfied with

$$\Upsilon_{i_1 i_2 i_x}^{j_1} = \begin{bmatrix} -P_{i_2} & (*) & (*) & (*) \\ A_{i_1} G_{i_2} + B_{i_1} K_{i_2 j_1} & -E_{j_1} G_{i_x} - G_{i_x}^T E_{j_1}^T + P_{i_x} & (*) & (*) \\ 0 & D_{i_1}^T & -\gamma^2 I & (*) \\ C_{i_1} G_{i_2} & 0 & J_{i_1} & -I \end{bmatrix}. \tag{3.62}$$

Proof. Recall (3.60). The Lyapunov matrix is selected as $\mathcal{P} = G_h^{-T} P_h G_h^{-1}$; the controller gains are selected as $\mathcal{K} = K_{hv}$ and $\mathcal{G} = G_h$. By congruence with the full-rank matrix $diag\left[G_h^T, \quad G_{h^+}^T, \quad I \right]$ and selecting $\mathcal{M} = \begin{bmatrix} 0 & G_{h^+}^{-1} & 0 \end{bmatrix}^T$, (3.60) yields

$$\begin{bmatrix} G_h^T C_h^T C_h G_h - P_h & (*) & (*) \\ A_h G_h + B_h K_{hv} & -E_v G_{h^+} - G_{h^+}^T E_v^T + P_{h^+} & (*) \\ J_h^T C_h G_h & D_h^T & J_h^T J_h - \gamma^2 I \end{bmatrix} < 0, \tag{3.63}$$

which can be expressed as

$$\Upsilon_{hhh^+}^v := \begin{bmatrix} -P_h & (*) & (*) \\ A_h G_h + B_h K_{hv} & -E_v G_{h^+} - G_{h^+}^T E_v^T + P_{h^+} & (*) \\ 0 & D_h^T & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} G_h^T C_h^T \\ 0 \\ J_h^T \end{bmatrix} \begin{bmatrix} C_h G_h & 0 & J_h \end{bmatrix} < 0.$$

Finally, applying the Schur complement and Lemma B.3 yields (3.61) . ■

The following results are based on the Lyapunov function $V(x_k) = x_k^T P_h^{-1} x_k$.

Theorem 3.7. The TS descriptor model (3.33) under the control law $u_k = K_{hv} G_{hv}^{-1} x_k$ is asymptotically stable and guarantees disturbance attenuation $\gamma > 0$ if there exist matrices $P_{i_2} = P_{i_2}^T > 0$, $G_{i_2 j_1}$, $K_{i_2 j_1}$, and $F_{i_1 i_2 i_x}$, $i_1, i_2, i_x \in \{1, 2, \ldots, r_a\}$, $j_1 \in \{1, 2, \ldots, r_e\}$ such that:

$$\Upsilon_{i_1 i_1 i_x}^{j_1} < 0, \ \forall i_1, i_x, j_1; \qquad \frac{2}{r_a - 1} \Upsilon_{i_1 i_1 i_x}^{j_1} + \Upsilon_{i_1 i_2 i_x}^{j_1} + \Upsilon_{i_2 i_1 i_x}^{j_1} < 0, \ \forall i_x, j_1, \ i_i \neq i_2, \tag{3.64}$$

are satisfied with

$$\Upsilon_{i_1 i_2 i_x}^{j_1} = \begin{bmatrix} -G_{i_2 j_1} - G_{i_2 j_1}^T + P_{i_2} & (*) & (*) & (*) & (*) \\ A_{i_1} G_{i_2 j_1} + B_{i_1} K_{i_2 j_1} & -E_{j_1} F_{i_1 i_2 i_x} - F_{i_1 i_2 i_x}^T E_{j_1}^T & (*) & (*) & (*) \\ 0 & F_{i_1 i_2 i_x} & -P_{i_x} & (*) & (*) \\ 0 & D_{i_1}^T & 0 & -\gamma^2 I & (*) \\ C_{i_1} G_{i_2 j_1} & 0 & 0 & J_{i_1} & -I \end{bmatrix}. \tag{3.65}$$

Remark 3.8. Since One can easily extend the results of Theorems 3.6 and 3.7 using past samples in the MFs. For instance, consider (3.60) and set the Lyapunov matrix as $\mathcal{P} = G_{h^-}^{-T} P_{h^-} G_{h^-}^{-1}$ and the control law as $u_k = K_{hh^- v} G_{h^-}^{-1} x_k$. Based on Theorem 3.6, the delayed approach gives (3.61) with:

$$\Upsilon_{i_1 i_2 i_x}^{j_1} = \begin{bmatrix} -P_{i_x} & (*) & (*) & (*) \\ A_{i_1} G_{i_x} + B_{i_1} K_{i_2 i_x j_1} & -E_{j_1} G_{i_2} - G_{i_2}^T E_{j_1}^T + P_{i_2} & (*) & (*) \\ 0 & D_{i_1}^T & -\gamma^2 I & (*) \\ C_{i_1} G_{i_x} & 0 & J_{i_1} & -I \end{bmatrix}. \tag{3.66}$$

Example 3.6. Consider the following nonlinear descriptor model:

$$E(x_k) x_{k+1} = A(x_k) x_k + B u_k + D(x_k) w_k, \qquad y_k = C(x_k) x_k + J w_k, \tag{3.67}$$

with

$$E(x_k) = \begin{bmatrix} 1.1 & -0.7\cos(x_1) \\ 0.7\cos(x_1) & 1.3 \end{bmatrix}, \quad A(x_k) = \begin{bmatrix} -0.9 & 0.6 \\ -1 & \cos(x_2) + 2.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix},$$
$$C(x_k) = \begin{bmatrix} 0.4 & 0.5 + x_2 \end{bmatrix}, \quad D(x_k) = \begin{bmatrix} 0 & 0.3 + 0.5\delta x_2 \end{bmatrix}^T, \quad \text{and} \quad J = 0.2\delta,$$

where δ is a real-valued parameter. Notice that since $|E(x_k)| = 1.43 + 0.49\cos^2(x_1) > 0$, $E(x_k)$ is regular $\forall x_1 \in \mathbb{R}$. Via the sector nonlinearity approach, a TS descriptor model results with $r_e = 2$ and $r_a = 4$ due to the number of nonlinearities in the left-hand side and in the right-hand side. Considering the compact set $\Omega = \{x : x_1 \in \mathbb{R}, |x_2| \le 1\}$, the MFs are defined as follows: $v_1 = (\cos(x_1) + 1)/2$, $v_2 = 1 - v_1$, $h_1 = \omega_0^1 \omega_0^2$, $h_2 = \omega_0^1 \omega_1^2$, $h_3 = \omega_1^1 \omega_0^2$, and $h_4 = \omega_1^1 \omega_1^2$; their corresponding weighting functions are: $\omega_0^1 = (\cos(x_2) + 1)/2$, $\omega_0^2 = (x_2 + 1)/2$, $\omega_1^1 = 1 - \omega_0^1$, and $\omega_1^2 = 1 - \omega_0^2$. The MFs hold the convex sum property in the compact set Ω. Hence, an exact TS representation is:

$$\sum_{j=1}^{r_e} v_j\left(z(k)\right)E_j x_{k+1} = \sum_{i=1}^{4} h_i\left(z(k)\right)\left(A_i x_k + B_i u_k + D_i w_k\right)$$
$$y_k = \sum_{i=1}^{4} h_i\left(z(k)\right)\left(C_i x_k + J_i w_k\right), \tag{3.68}$$

with local matrices as follows:

$$E_1 = \begin{bmatrix} 1.1 & -0.7 \\ 0.7 & 1.3 \end{bmatrix},\ E_2 = \begin{bmatrix} 1.1 & 0.7 \\ -0.7 & 1.3 \end{bmatrix},\ A_1 = A_2 = \begin{bmatrix} -0.9 & 0.6 \\ -1 & 3.5 \end{bmatrix},\ A_3 = A_4 = \begin{bmatrix} -0.9 & 0.6 \\ -1 & 3.05 \end{bmatrix},$$
$$B_i = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix},\ C_2 = C_4 = \begin{bmatrix} 0.4 \\ -0.5 \end{bmatrix}^T,\ C_1 = C_3 = \begin{bmatrix} 0.4 \\ 1.5 \end{bmatrix}^T,$$
$$D_1 = D_3 = \begin{bmatrix} 0 \\ 0.3+0.5\delta \end{bmatrix},\ D_2 = D_4 = \begin{bmatrix} 0 \\ 0.3-0.5\delta \end{bmatrix},\ \text{and}\ \ J_i = 0.2\delta,\ i \in \{1,2,\ldots,r_a\}.$$

Figure 3.7 shows the minimal value for γ is computed for $\delta \in [-2,\ \ 0]$ when employing the conditions in Theorems 3.6 (O) and its delayed approach $(\times)$ (see Remark 3.8). It can be seen that the delayed approach provides better attenuation.

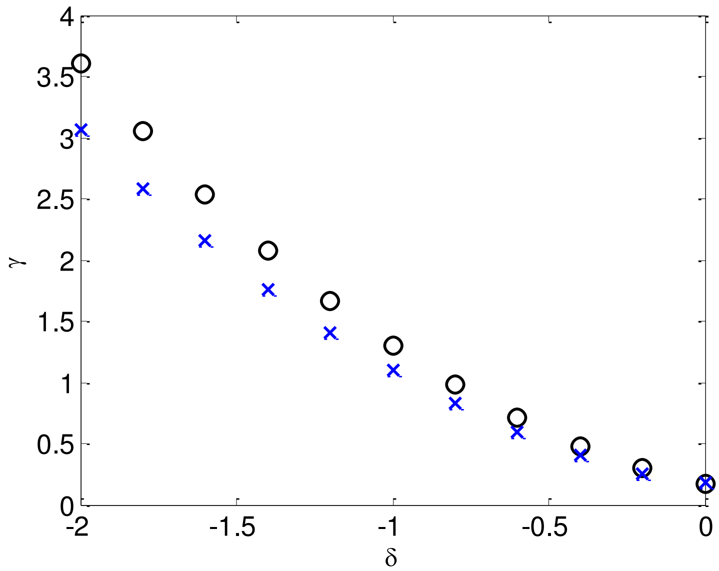

Figure 3.7. Minimal values for γ: Theorem 3.6 (O) and Theorem 3.6 delayed (×) in Example 3.6. ♦

3.2.3. Generalization

As it can be inferred from the previous subsections, extending the Lyapunov function and/or the control laws can significantly improve the results. Therefore, a natural question that arises is the generalization of this approach. The question is: if we add some more past samples, does it contribute to the effort of reducing conservativeness? Moreover, is there a "good" choice for the structure (in the sense of which and how many past samples) of the Lyapunov function and of the control law? The section hereafter answers these questions. To that end, the following notation is adopted from (Lendek et al., 2015).

Definition 3.1. (Multiple sum) A multiple sum with n_{Υ_h} terms and delays evaluated at sample k is of the form:

$$\Upsilon_{H_0^{\Upsilon}} = \sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\cdots\sum_{i_{n_{\Upsilon_h}}=1}^{r} h_{i_1}\left(z\left(k+d_1\right)\right)h_{i_2}\left(z\left(k+d_2\right)\right)\times\cdots\times h_{i_{n_{\Upsilon_h}}}\left(z\left(k+d_{n_{\Upsilon_h}}\right)\right)\Upsilon_{i_1 i_2\cdots i_{n_{\Upsilon_h}}},$$

where H_0^{Υ} is the multiset of delays $H_0^{\Upsilon} = \left\{d_1, d_2, \ldots, \mathrm{d}_{n_{\Upsilon_h}}\right\}$, $d_{(\cdot)} \in \mathbb{Z}$. The definition of $\mathcal{V}_0^{\Upsilon}$ is similar, i.e., $\Upsilon_{\mathcal{V}_0^{\Upsilon}} = \sum_{j_1=1}^{r}\cdots\sum_{j_{n_{\Upsilon_h}}=1}^{r} v_{j_1}\left(z\left(k+d_1\right)\right)\times\cdots\times v_{j_{n_{\Upsilon_v}}}\left(z\left(k+d_{n_{\Upsilon_h}}\right)\right)\Upsilon_{j_1\cdots j_{n_{\Upsilon_v}}}$.

Definition 3.2. (Multiset of delays) H_0^{Υ} denotes the multiset containing the delays in the multiple sum involving Υ at sample k. H_α^{Υ} denotes the multiset containing the delays in the sum Υ at sample $k+\alpha$.

Definition 3.3. (Cardinality) The cardinality of a multiset H, $|H| = n_H$, is defined as the number of elements in H.

Definition 3.4. (Index set) The index set of a multiple sum Υ_H is $\mathbb{I}_H = \left\{i_j : i_j = 1, 2, \ldots, r, j = 1, 2, \ldots, |H|\right\}$, the set of all indices that appear in the sum. An element $\boldsymbol{i}$ is a multiindex.

Definition 3.5. (Multiplicity) The multiplicity of an element x in a multiset H, $\mathbf{1}_H(x)$ denotes the number of times this element appears in the multiset H.

Definition 3.6. (Union) The union of two multisets H_A and H_B, denoted $H_C = H_A \cup H_B$, is such that: $\forall x \in H_C : \mathbf{1}_{H_C}(x) = \max\left\{\mathbf{1}_{H_A}(x), \mathbf{1}_{H_B}(x)\right\}$.

Definition 3.7. (Intersection) The intersection of two multisets H_A and H_A, denoted $H_C = H_A \cap H_B$, is such that $\forall x \in H_C$: $\mathbf{1}_{H_C}(x) = \min\left\{\mathbf{1}_{H_A}(x), \mathbf{1}_{H_B}(x)\right\}$.

Definition 3.8. (Sum) The sum of two multisets H_A and H_B, denoted $H_C = H_A \oplus H_B$, is such that $\forall x \in H_C : \mathbf{1}_{H_C}(x) = \mathbf{1}_{H_A}(x) + \mathbf{1}_{H_B}(x)$.

Definition 3.9. (Projection of an index) The projection of the index $\boldsymbol{i} \in \mathbb{I}_{H_A}$ to the multiset of delays H_B, $pr^{\boldsymbol{i}}_{H_B}$, is the part of the index that corresponds to the delays in $H_A \cap H_B$.

The following example illustrates the previous definitions.

Example 3.7. Consider the multiple sum:

$$\Upsilon_{H_0^\Upsilon} = \sum_{i_1=1}^{r_a}\sum_{i_2=1}^{r_a}\sum_{i_3=1}^{r_a}\sum_{i_4=1}^{r_a}\sum_{i_5=1}^{r_a} h_{i_1}\left(z(k)\right) h_{i_2}\left(z(k-1)\right) h_{i_3}\left(z(k-2)\right) h_{i_4}\left(z(k-3)\right) h_{i_5}\left(z(k-3)\right) \Upsilon_{i_1 i_2 i_3 i_4 i_5}.$$

Then, H_0^Υ is given by $H_0^\Upsilon = \{0,-1,-2,-3,-3\}$, or $H_\alpha^\Upsilon = \{\alpha, \alpha-1, \alpha-2, \alpha-3, \alpha-3\}$. The cardinality of H_0^Υ is $\left|H_0^\Upsilon\right| = n_{\Upsilon_h} = 5$. The index set of the multiple sum $\Upsilon_{H_0^\Upsilon}$ is $\mathbb{I}_{H_0^\Upsilon} = \{i_j : i_j = 1,2,\ldots,r_a, j = 1,2,\ldots,5\}$. The multiplicity of the elements in H_0^Υ is $\mathbf{1}_{H_0^\Upsilon}(0) = 1$, $\mathbf{1}_{H_0^\Upsilon}(-1) = 1$, $\mathbf{1}_{H_0^\Upsilon}(-2) = 1$, and $\mathbf{1}_{H_0^\Upsilon}(-3) = 2$. Now, let H_A and H_B be two multisets defined as $H_A = \{0,0,-1,-2,-3\}$ and $H_B = \{0,-3,-4\}$. The union of these multisets is $H_A \cup H_B = \{0,0,-1,-2,-3,-4\}$, the intersection is $H_A \cap H_B = \{0,-3\}$, and their sum is $H_A \oplus H_B = \{0,0,0,-1,-2,-3,-3,-4\}$. ♦

Considering the previous definitions, the discrete-time TS descriptor model (3.33) can be written as

$$\begin{aligned} E_{\mathcal{V}_0^E} x_{k+1} &= A_{H_0^A} x_k + B_{H_0^B} u_k + D_{H_0^D} w_k \\ y_k &= C_{H_0^C} x_k + J_{H_0^J} w_k, \end{aligned} \tag{3.69}$$

with $\mathcal{V}_0^E = H_0^A = H_0^B = H_0^C = H_0^D = H_0^J = \{0\}$, i.e., the system matrices are without delays.

In what follows, for design purposes, consider the following non-PDC control law:

$$u_k = K_{H_0^K \mathcal{V}_0^K} G^{-1}_{H_0^G \mathcal{V}_0^G} x_k, \tag{3.70}$$

where $K_{H_0^K \mathcal{V}_0^K}$ and $G_{H_0^G \mathcal{V}_0^G}$ are matrices to be determined of appropriate dimensions. The regularity of $G_{H_0^G \mathcal{V}_0^G}$ will be discussed further on. Obviously, for causality these matrices cannot contain positive delays, otherwise they incorporate future samples (Guerra et al., 2012b; Lendek et al., 2015). The delays are given by the multisets H_0^K, H_0^G, $\mathcal{V}_0^K$, and $\mathcal{V}_0^G$. Thus, when $w_k = 0$, the model (3.69) under the control law (3.70) gives the closed-loop dynamics

$$E_{\mathcal{V}_0^E} x_{k+1} = \left(A_{H_0^A} + B_{H_0^B} K_{H_0^K \mathcal{V}_0^K} G_{H_0^G \mathcal{V}_0^G}^{-1} \right) x_k. \tag{3.71}$$

Example 3.8. Recall that the multisets for the system matrices are $\mathcal{V}_0^E = H_0^A = H_0^B = \{0\}$ and by choosing $H_0^K = H_0^G = \mathcal{V}_0^K = \mathcal{V}_0^G = \{0\}$ for the controller gains, the closed-loop TS descriptor (3.71) renders:

$$\sum_{j_1=1}^{r_e} v_{j_1}\left(z(k)\right) E_{j_1} x_{k+1} = \sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{j_1=1}^{r_e} h_{i_1}\left(z(k)\right) h_{i_2}\left(z(k)\right) v_{j_1}\left(z(k)\right)$$
$$\times \left(A_{i_1} + B_{i_2} K_{i_2 j_1} \times \left(\sum_{i_2=1}^{r}\sum_{j_1=1}^{r_e} h_{i_2}\left(z(k)\right) v_{j_1}\left(z(k)\right) G_{i_2 j_1} \right)^{-1} \right) x_k,$$

which is exactly the same as Theorem 3.4 and Corollary 3.1. ♦

Following the same procedure as in the previous sections, i.e., using the generic Lyapunov function (3.36) and its variation (3.37), the closed-loop model (3.71) is

$$\begin{bmatrix} A_{H_0^A} + B_{H_0^B} K_{H_0^K \mathcal{V}_0^K} G_{H_0^G \mathcal{V}_0^G}^{-1} & -E_{\mathcal{V}_0^E} \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = 0; \tag{3.72}$$

while the variation of the Lyapunov function (3.37) is:

$$\Delta V\left(x_k\right) = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}^T \begin{bmatrix} -\mathcal{P} & 0 \\ 0 & \mathcal{P}_+ \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} < 0. \tag{3.73}$$

By taking $\mathcal{X} = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$, $\mathcal{W} = \begin{bmatrix} A_{H_0^A} + B_{H_0^B} K_{H_0^K \mathcal{V}_0^K} G_{H_0^G \mathcal{V}_0^G}^{-1} & -E_{\mathcal{V}_0^E} \end{bmatrix}$, $\mathcal{Q} = \begin{bmatrix} -\mathcal{P} & 0 \\ 0 & \mathcal{P}_+ \end{bmatrix}$, and using Finsler's lemma, inequality (3.73) under constraint (3.72) yields:

$$\mathcal{M}\begin{bmatrix} A_{H_0^A} + B_{H_0^B} K_{H_0^K \mathcal{V}_0^K} G_{H_0^G \mathcal{V}_0^G}^{-1} & -E_{\mathcal{V}_0^E} \end{bmatrix} + (*) + \begin{bmatrix} -\mathcal{P} & 0 \\ 0 & \mathcal{P}_+ \end{bmatrix} < 0. \tag{3.74}$$

From here, as in Section 3.2.2, two main configurations of $\mathcal{P}$ are considered:

Case 1: $\mathcal{P} = G_{H_0^G \mathcal{V}_0^G}^{-T} P_{H_0^P \mathcal{V}_0^P} G_{H_0^G \mathcal{V}_0^G}^{-1}$, thus $V\left(x_k\right) = x_k^T G_{H_0^G \mathcal{V}_0^G}^{-T} P_{H_0^P \mathcal{V}_0^P} G_{H_0^G \mathcal{V}_0^G}^{-1} x_k$, Case 2: $\mathcal{P} = P^{-1}_{H_0^P \mathcal{V}_0^P}$, therefore $V(x_k) = x_k^T P^{-1}_{H_0^P \mathcal{V}_0^P} x_k$,

where $P_{\boldsymbol{i}_0^P, \boldsymbol{j}_0^P} = P^T_{\boldsymbol{i}_0^P, \boldsymbol{j}_0^P} > 0$, $\boldsymbol{i} \in \mathbb{I}_{H_0^P}$, $\boldsymbol{j} \in \mathbb{I}_{\mathcal{V}_0^P}$. Considering *Case 1*, the conditions in Theorem 3.3 can be generalized as follows:

Theorem 3.9. The closed-loop TS descriptor model (3.71) is asymptotically stable if there exist $P_{\boldsymbol{i}_k^P, \boldsymbol{j}_k^P} = P^T_{\boldsymbol{i}_k^P, \boldsymbol{j}_k^P} > 0$, $\boldsymbol{i}_k^P = pr^{\boldsymbol{i}}_{H_k^P}$, $\boldsymbol{j}_k^P = pr^{\boldsymbol{j}}_{\mathcal{V}_k^P}$, $G_{\boldsymbol{i}_k^G, \boldsymbol{j}_k^G}$, $\boldsymbol{i}_k^G = pr^{\boldsymbol{i}}_{H_k^G}$, $\boldsymbol{j}_k^G = pr^{\boldsymbol{j}}_{\mathcal{V}_k^G}$, $k = 0,1$, and $K_{\boldsymbol{i}_0^K, \boldsymbol{j}_0^K}$, $\boldsymbol{i}_0^K = pr^{\boldsymbol{i}}_{H_0^K}$, $\boldsymbol{j}_0^K = pr^{\boldsymbol{j}}_{\mathcal{V}_0^K}$, $\boldsymbol{i} \in \mathbb{I}_{H_\Gamma}$, $\boldsymbol{j} \in \mathbb{I}_{\mathcal{V}_\Gamma}$, with $\mathcal{V}_\Gamma = \mathcal{V}_0^P \cup \mathcal{V}_1^P \cup \mathcal{V}_0^K \cup \mathcal{V}_0^G \cup \left(\mathcal{V}_0^E \oplus \mathcal{V}_1^G\right)$ $H_\Gamma = H_0^P \cup H_1^P \cup \left(H_0^B \oplus H_0^K\right) \cup \left(H_0^A \oplus H_0^G\right) \cup H_1^G$, such that

$$\begin{bmatrix} -P_{H_0^P \mathcal{V}_0^P} & (*) \\ A_{H_0^A} G_{H_0^G \mathcal{V}_0^G} + B_{H_0^B} K_{H_0^K \mathcal{V}_0^K} & -E_{\mathcal{V}_0^E} G_{H_1^G \mathcal{V}_1^G} - G^T_{H_1^G \mathcal{V}_1^G} E^T_{\mathcal{V}_0^E} + P_{H_1^P \mathcal{V}_1^P} \end{bmatrix} < 0. \tag{3.75}$$

Proof. Recall (3.74). Choosing $V(x_k) = x_k^T G^{-T}_{H_0^G \mathcal{V}_0^G} P_{H_0^P \mathcal{V}_0^P} G^{-1}_{H_0^G \mathcal{V}_0^G} x_k$, congruence with matrix $diag\left[G^T_{H_0^G \mathcal{V}_0^G}, \quad G^T_{H_1^G \mathcal{V}_1^G}\right]$ and selecting $\mathcal{M} = \begin{bmatrix} 0 & G^{-1}_{H_1^G \mathcal{V}_1^G} \end{bmatrix}^T$ gives directly (3.75). ■

Employing the Lyapunov function in *Case 2*, the following can be stated.

Theorem 3.10. The closed-loop TS descriptor model (3.71) is asymptotically stable if there exist $P_{\boldsymbol{i}_k^P, \boldsymbol{j}_k^P} = P^T_{\boldsymbol{i}_k^P, \boldsymbol{j}_k^P} > 0$, $\boldsymbol{i}_k^P = pr^{\boldsymbol{i}}_{H_k^P}$, $\boldsymbol{j}_k^P = pr^{\boldsymbol{j}}_{\mathcal{V}_k^P}$, $k = 0,1$, $K_{\boldsymbol{i}_0^K, \boldsymbol{j}_0^K}$, $\boldsymbol{i}_0^K = pr^{\boldsymbol{i}}_{H_0^K}$, $\boldsymbol{j}_0^K = pr^{\boldsymbol{j}}_{\mathcal{V}_0^K}$, and $G_{\boldsymbol{i}_0^G, \boldsymbol{j}_0^G}$, $\boldsymbol{i}_0^G = pr^{\boldsymbol{i}}_{H_0^G}$, $\boldsymbol{j}_0^G = pr^{\boldsymbol{j}}_{\mathcal{V}_0^G}$, $\boldsymbol{i} \in \mathbb{I}_{H_\Gamma}$, $\boldsymbol{j} \in \mathbb{I}_{\mathcal{V}_\Gamma}$, where $H_\Gamma = H_0^P \cup H_1^P \cup \left(H_0^B \oplus H_0^K\right) \cup \left(H_0^A \oplus H_0^G\right)$, $\mathcal{V}_\Gamma = \mathcal{V}_0^P \cup \mathcal{V}_1^P \cup \mathcal{V}_0^K \cup \mathcal{V}_0^G \cup \mathcal{V}_0^E$ such that

$$\begin{bmatrix} -G_{H_0^G \mathcal{V}_0^G} - G^T_{H_0^G \mathcal{V}_0^G} + P_{H_0^P \mathcal{V}_0^P} & (*) \\ A_{H_0^A} G_{H_0^G \mathcal{V}_0^G} + B_{H_0^B} K_{H_0^K \mathcal{V}_0^K} & -E_{\mathcal{V}_0^E} P_{H_1^P \mathcal{V}_1^P} - P_{H_1^P \mathcal{V}_1^P} E^T_{\mathcal{V}_0^E} + P_{H_1^P \mathcal{V}_1^P} \end{bmatrix} < 0. \tag{3.76}$$

Proof. Consider (3.74) with the Lyapunov function $V(x_k) = x_k^T P^{-1}_{H_0^P \mathcal{V}_0^P} x_k$. Applying the congruence property with the full-rank matrix $diag\left[G^T_{H_0^G \mathcal{V}_0^G}, \quad P_{H_1^P \mathcal{V}_1^P}\right]$ and choosing $\mathcal{M} = \begin{bmatrix} 0 & P^{-1}_{H_1^P \mathcal{V}_1^P} \end{bmatrix}^T$, (3.74) gives:

$$\begin{bmatrix} -G^T_{H_0^G \mathcal{V}_0^G} P^{-1}_{H_0^P \mathcal{V}_0^P} G_{H_0^G \mathcal{V}_0^G} & (*) \\ A_{H_0^A} G_{H_0^G \mathcal{V}_0^G} + B_{H_0^B} K_{H_0^K \mathcal{V}_0^K} & -E_{\mathcal{V}_0^E} P_{H_1^P \mathcal{V}_1^P} - P_{H_1^P \mathcal{V}_1^P} E^T_{\mathcal{V}_0^E} + P_{H_1^P \mathcal{V}_1^P} \end{bmatrix} < 0. \tag{3.77}$$

At last, by means of Property A.3 on the first block of (3.77) gives (3.76). ■

The next result provides more relaxed conditions than Theorem 3.10.

Theorem 3.11. The closed-loop TS descriptor model (3.71) is asymptotically stable if there exist $P_{\boldsymbol{i}_k^P,\boldsymbol{j}_k^P} = P_{\boldsymbol{i}_k^P,\boldsymbol{j}_k^P}^T > 0$, $\boldsymbol{i}_k^P = pr_{H_k^P}^{\boldsymbol{i}}$, $\boldsymbol{j}_k^P = pr_{\mathcal{V}_k^P}^{\boldsymbol{j}}$, $k = 0,1$, $K_{\boldsymbol{i}_0^K,\boldsymbol{j}_0^K}$, $\boldsymbol{i}_0^K = pr_{H_0^K}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^K = pr_{\mathcal{V}_0^K}^{\boldsymbol{j}}$, $G_{\boldsymbol{i}_0^G,\boldsymbol{j}_0^G}$, $\boldsymbol{i}_0^G = pr_{H_0^G}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^G = pr_{\mathcal{V}_0^G}^{\boldsymbol{j}}$, and $F_{\boldsymbol{i}_0^F,\boldsymbol{j}_0^F}$, $\boldsymbol{i}_0^F = pr_{H_0^F}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^F = pr_{\mathcal{V}_0^F}^{\boldsymbol{j}}$, $\boldsymbol{i} \in \mathbb{I}_{H_\Gamma}$, $\boldsymbol{j} \in \mathbb{I}_{\mathcal{V}_\Gamma}$, where: $H_\Gamma = H_0^P \cup H_1^P \cup \left(H_0^B \oplus H_0^K\right) \cup \left(H_0^A \oplus H_0^G\right) \cup H_0^F$, $\mathcal{V}_\Gamma = \mathcal{V}_0^P \cup \mathcal{V}_1^P \cup \mathcal{V}_0^K \cup \mathcal{V}_0^G \cup \left(\mathcal{V}_0^E \oplus \mathcal{V}_0^F\right)$ such that

$$\begin{bmatrix} -G_{H_0^G\mathcal{V}_0^G} - G_{H_0^G\mathcal{V}_0^G}^T + P_{H_0^P\mathcal{V}_0^P} & (*) & (*) \\ A_{H_0^A} G_{H_0^G\mathcal{V}_0^G} + B_{H_0^B} K_{H_0^K\mathcal{V}_0^K} & -E_{\mathcal{V}_0^E} F_{H_0^F\mathcal{V}_0^F} - F_{H_0^F\mathcal{V}_0^F}^T E_{\mathcal{V}_0^E}^T & (*) \\ 0 & F_{H_0^F\mathcal{V}_0^F} & -P_{H_1^P\mathcal{V}_1^P} \end{bmatrix} < 0. \tag{3.78}$$

Proof. Consider (3.74) with the Lyapunov function $V(x_k) = x_k^T P_{H_0^P\mathcal{V}_0^P}^{-1} x_k$. Applying the congruence property with the full-rank matrix $diag\left[G_{H_0^G\mathcal{V}_0^G}^T,\ F_{H_1^F\mathcal{V}_1^F}^T\right]$ and selecting $\mathcal{M} = \left[0 \quad F_{H_1^F\mathcal{V}_1^F}^{-1}\right]^T$, (3.74) gives

$$\begin{bmatrix} -G_{H_0^G\mathcal{V}_0^G}^T P_{H_0^P\mathcal{V}_0^P}^{-1} G_{H_0^G\mathcal{V}_0^G} & (*) \\ A_{H_0^A} G_{H_0^G\mathcal{V}_0^G} + B_{H_0^B} K_{H_0^K\mathcal{V}_0^K} & -E_{\mathcal{V}_0^E} F_{H_0^F\mathcal{V}_0^F} - F_{H_0^F\mathcal{V}_0^F}^T E_{\mathcal{V}_0^E}^T + F_{H_0^F\mathcal{V}_0^F}^T P_{H_1^P\mathcal{V}_1^P}^{-1} F_{H_0^F\mathcal{V}_0^F} \end{bmatrix} < 0. \tag{3.79}$$

Applying Property A.3 on the first block of (3.79) yields:

$$\begin{bmatrix} -G_{H_0^G\mathcal{V}_0^G} - G_{H_0^G\mathcal{V}_0^G}^T + P_{H_0^P\mathcal{V}_0^P} & (*) \\ A_{H_0^A} G_{H_0^G\mathcal{V}_0^G} + B_{H_0^B} K_{H_0^K\mathcal{V}_0^K} & -E_{\mathcal{V}_0^E} F_{H_0^F\mathcal{V}_0^F} - \mathrm{F}_{H_0^F\mathcal{V}_0^F}^T E_{\mathcal{V}_0^E}^T + \mathrm{F}_{H_0^F\mathcal{V}_0^F}^T P_{H_1^P\mathcal{V}_1^P}^{-1} F_{H_0^F\mathcal{V}_0^F} \end{bmatrix} < 0. \tag{3.80}$$

Finally, the Schur complement applied on (3.80) gives (3.78), thus ending the proof. ■

Note that the total number of sums – for MFs $h(\bullet)$ and $v(\bullet)$ – involved in Theorem 3.9, Theorem 3.10, and Theorem 3.11 is given by $n_{H\mathcal{V}} = |H_\Gamma| + |\mathcal{V}_\Gamma|$.

Remark 3.9. Note that the standard TS model is a special case of the TS descriptor one when $E_{\mathcal{V}_0^E} = I$, $\mathcal{V}_\Gamma = \varnothing$, where $\varnothing$ stands for the empty set; therefore Theorem 3.9 and 3.10 recover their respective theorems in (Lendek et al., 2015).

Example 3.9. Consider the closed-loop system (3.71) with $\mathcal{V}_0^E = H_0^A = H_0^B = \{0\}$ and the multisets $H_0^G = H_0^K = H_0^F = \{0,-1\}$, $\mathcal{V}_0^G = \mathcal{V}_0^F = \mathcal{V}_0^K = \{0,-1\}$, and $H_0^P = \mathcal{V}_0^P = \{-1\}$, i.e.,

$$P_{H_0^P\mathcal{V}_0^P} = P_{\{-1\},\{-1\}} = \sum_{i_x=1}^{r_a}\sum_{j_x=1}^{r_e} h_{i_x}\left(z(k-1)\right) v_{j_x}\left(z(k-1)\right) P_{i_x j_x},$$

$$K_{H_0^K\mathcal{V}_0^K} = K_{\{0,-1\},\{0,-1\}} = \sum_{i_1=1}^{r_a}\sum_{i_x=1}^{r_a}\sum_{j_1=1}^{r_e}\sum_{j_x=1}^{r_e} h_{i_1}\left(z(k)\right) h_{i_x}\left(z(k-1)\right) v_{j_1}\left(z(k)\right) v_{j_x}\left(z(k-1)\right) K_{i_1 i_x j_1 j_x},$$

$$G_{H_0^G\mathcal{V}_0^G} = G_{\{0,-1\},\{0,-1\}} = \sum_{i_1=1}^{r_a}\sum_{i_x=1}^{r_a}\sum_{j_1=1}^{r_e}\sum_{j_x=1}^{r_e} h_{i_1}\left(z(k)\right) h_{i_x}\left(z(k-1)\right)_{j_1}\left(z(k)\right) v_{j_x}\left(z(k-1)\right) G_{i_1 i_x j_1 j_x},$$

$$F_{H_0^G\mathcal{V}_0^G} = F_{\{0,-1\},\{0,-1\}} = \sum_{i_1=1}^{r_a}\sum_{i_x=1}^{r_a}\sum_{j_1=1}^{r_e}\sum_{j_x=1}^{r_e} h_{i_1}\left(z(k)\right) h_{i_x}\left(z(k-1)\right) v_{j_1}\left(z(k)\right) v_{j_x}\left(z(k-1)\right) F_{i_1 i_x j_1 j_x}.$$

Thus, conditions for Theorem 3.9 yield:

$$\begin{aligned}&\sum_{i_1=1}^{r_a}\sum_{i_2=1}^{r_a}\sum_{i_3=1}^{r_a}\sum_{i_4=1}^{r_a}\sum_{j_1=1}^{r_e}\sum_{j_2=1}^{r_e}\sum_{j_3=1}^{r_e}\sum_{j_4=1}^{r_e} h_{i_1}\left(z(k)\right) h_{i_2}\left(z(k)\right) h_{i_3}\left(z(k+1)\right) h_{i_4}\left(z(k-1)\right)\\ &\qquad\times v_{j_1}\left(z(k)\right) v_{j_2}\left(z(k)\right) v_{j_3}\left(z(k+1)\right) v_{j_4}\left(z(k-1)\right)\\ &\qquad\times\begin{bmatrix} -P_{i_3 j_3} & (*)\\ A_{i_1} G_{i_2 i_4 j_2 j_4} + B_{i_1} K_{i_2 i_3 j_2 j_3} & -E_{j_1} G_{i_2 i_3 j_2 j_3} - G_{i_2 i_3 j_2 j_3}^T E_{j_1}^T + P_{i_2 j_2}\end{bmatrix} < 0.\end{aligned} \tag{3.81}$$

Conditions in Theorem 3.10 write:

$$\begin{aligned}&\sum_{i_1=1}^{r_a}\sum_{i_2=1}^{r_a}\sum_{i_x=1}^{r_a}\sum_{j_1=1}^{r_e}\sum_{j_2=1}^{r_e}\sum_{j_x=1}^{r_e} h_{i_1}\left(z(k)\right) h_{i_2}\left(z(k)\right) h_{i_x}\left(z(k-1)\right)\\ &\qquad\times v_{j_1}\left(z(k)\right) v_{j_2}\left(z(k)\right) v_{j_x}\left(z(k-1)\right)\\ &\qquad\times\begin{bmatrix} -G_{i_2 i_x j_2 j_x} - G_{i_2 i_x j_2 j_x}^T + P_{i_x j_x} & (*)\\ A_{i_1} G_{i_2 i_x j_2 j_x} + B_{i_1} K_{i_2 i_x j_2 j_x} & -E_{j_1} P_{i_2 j_2} - P_{i_2 j_2} E_{j_1}^T + P_{i_2 j_2}\end{bmatrix} < 0.\end{aligned} \tag{3.82}$$

Finally conditions in Theorem 3.11 are:

$$\begin{aligned}&\sum_{i_1=1}^{r_a}\sum_{i_2=1}^{r_a}\sum_{i_x=1}^{r_a}\sum_{j_1=1}^{r_e}\sum_{j_2=1}^{r_e}\sum_{j_x=1}^{r_e} h_{i_1}\left(z(k)\right) h_{i_2}\left(z(k)\right) h_{i_x}\left(z(k-1)\right)\\ &\qquad\times v_{j_1}\left(z(k)\right) v_{j_2}\left(z(k)\right) v_{j_x}\left(z(k-1)\right)\\ &\qquad\times\begin{bmatrix} -G_{i_2 i_x j_2 j_x} - G_{i_2 i_x j_2 j_x}^T + P_{i_x j_x} & (*) & (*)\\ A_{i_1} G_{i_2 i_x j_2 j_x} + B_{i_1} K_{i_2 i_x j_2 j_x} & -E_{j_1} F_{i_2 i_3 j_2 j_x} - F_{i_2 i_x j_2 j_x}^T E_{j_1}^T & (*)\\ 0 & F_{i_2 i_x j_2 j_x} & -P_{i_2 j_2}\end{bmatrix} < 0.\end{aligned} \tag{3.83}$$

The number of sums in Theorem 3.9 is $n_{H\mathcal{V}} = |H_\Gamma| + |\mathcal{V}_\Gamma| = 8$, while for both Theorems 3.10 and 3.11 is $n_{H\mathcal{V}} = 6$. These differences are due to the fact that H_Γ and $\mathcal{V}_\Gamma$ depend on the chosen multisets for each theorem. ♦

Selecting multisets

At this point, it is important to clarify how to select the multisets involved in the control law and in the Lyapunov functions. The main idea is that multisets in $K_{H_0^K \mathcal{V}_0^K}$, $G_{H_0^G \mathcal{V}_0^G}$, $F_{H_0^F \mathcal{V}_0^F}$, and $P_{H_0^P \mathcal{V}_0^P}$ should be chosen such that sum relaxations can be used and the number of sums and the computational complexity of the resulting LMI should be reduced. Therefore, without considering solver limitations, the following reasoning applies:

Step 1: Since the system under study does not have delays in its matrices, i.e., $\mathcal{V}_0^E = H_0^A = H_0^B = \{0\}$, multisets H_0^K, H_0^G, H_0^F, $\mathcal{V}_0^K$, $\mathcal{V}_0^G$, and $\mathcal{V}_0^F$ should contain $\{0\}$. Double sum relaxations and the maximum number of variables should be used, but without increasing the number of sums. To illustrate the considerations above, consider conditions in Theorem 3.11 with $H_0^K = H_0^G = H_0^F = \mathcal{V}_0^K = \mathcal{V}_0^G = \mathcal{V}_0^F = \{0\}$:

$$\begin{bmatrix} -G_{\{0\},\{0\}} - G_{\{0\},\{0\}}^T + P_{H_0^P \mathcal{V}_0^P} & (*) & (*) \\ A_{\{0\}} G_{\{0\},\{0\}} + B_{\{0\}} K_{\{0\},\{0\}} & -E_{\{0\}} F_{\{0\},\{0\}} - F_{\{0\},\{0\}}^T E_{\{0\}}^T & (*) \\ 0 & F_{\{0\},\{0\}} & -P_{H_1^P \mathcal{V}_1^P} \end{bmatrix} < 0, \tag{3.84}$$

which after selecting $H_0^P = \mathcal{V}_0^P = \{-1\}$ gives

$$\begin{bmatrix} -G_{\{0\},\{0\}} - G_{\{0\},\{0\}}^T + P_{\{-1\},\{-1\}} & (*) & (*) \\ A_{\{0\}} G_{\{0\},\{0\}} + B_{\{0\}} K_{\{0\},\{0\}} & -E_{\{0\}} F_{\{0\},\{0\}} - F_{\{0\},\{0\}}^T E_{\{0\}}^T & (*) \\ 0 & F_{\{0\},\{0\}} & -P_{\{0\},\{0\}} \end{bmatrix} < 0, \tag{3.85}$$

it consists of three sums involving $h(\bullet): \sum_{i_1=1}^{r_a} \sum_{i_2=1}^{r_a} \sum_{i_x=1}^{r_a} h_{i_1}(z(k)) h_{i_2}(z(k)) h_{i_x}(z(k-1))$ and three sums of $v(\bullet): \sum_{j_1=1}^{r_e} \sum_{j_2=1}^{r_e} \sum_{j_x=1}^{r_e} v_{j_1}(z(k)) v_{j_2}(z(k)) v_{j_x}(z(k-1))$.

Step 2: Due to the structure of (3.85), it is possible to add the delay $\{-1\}$ in each multiple sum $K_{H_0^K \mathcal{V}_0^K}$, $G_{H_0^G \mathcal{V}_0^G}$, $F_{H_0^F \mathcal{V}_0^F}$ without increasing the number of sums:

$$\begin{bmatrix} -G_{\{0,-1\},\{0,-1\}} - G^T_{\{0,-1\},\{0,-1\}} + P_{\{-1\},\{-1\}} & (*) & (*) \\ A_{\{0\}} G_{\{0,-1\},\{0,-1\}} + B_{\{0\}} K_{\{0,-1\},\{0,-1\}} & -E_{\{0\}} F_{\{0,-1\},\{0,-1\}} - F^T_{\{0,-1\},\{0,-1\}} E^T_{\{0\}} & (*) \\ 0 & F_{\{0,-1\},\{0,-1\}} & -P_{\{0\},\{0\}} \end{bmatrix} < 0.$$

Step 3: Since the multiple sum $F_{H_0^F \mathcal{V}_0^F}$ does not multiply $A_{H_0^A}$ and $B_{H_0^B}$, one can add $\{0\}$ in H_0^F; similarly for the multiple sums $K_{H_0^K \mathcal{V}_0^K}$ and $G_{H_0^G \mathcal{V}_0^G}$: one can add $\{0\}$ in $\mathcal{V}_0^K$ and $\mathcal{V}_0^G$, respectively. Thus the "good" — more decision variables with less number of convex sums — multisets for this problem are:

$$\begin{bmatrix} -G_{\{0,-1\},\{0,0,-1\}} - G^T_{\{0,-1\},\{0,0,-1\}} + P_{\{-1\},\{-1\}} & (*) & (*) \\ A_{\{0\}} G_{\{0,-1\},\{0,0,-1\}} + B_{\{0\}} K_{\{0,-1\},\{0,0,-1\}} & -E_{\{0\}} F_{\{0,0,-1\},\{0,-1\}} - F^T_{\{0,0,-1\},\{0,-1\}} E^T_{\{0\}} & (*) \\ 0 & F_{\{0,0,-1\},\{0,-1\}} & -P_{\{0\},\{0\}} \end{bmatrix} < 0.$$

Table 3.3 shows how the number of decision variables changes at each step.

Table 3.3. Number of decision variables at each step for Theorem 3.11.

Step	Number of decision variables	Number of sums
Step 1	$0.5n_x \times (n_x + 1) \times (r_a r_e) + 2(n_x^2) \times (r_a r_e) + (n_u n_x) \times (r_a r_e)$	3 sums in $h(\bullet)$ 3 sums in $v(\bullet)$
Step 2	$0.5n_x \times (n_x + 1) \times (r_a r_e) + 2(n^2) \times (r_a^2 r_e^2) + (m_u n_x) \times (r_a^2 r_e^2)$	3 sums in $h(\bullet)$ 3 sums in $v(\bullet)$
Step 3	$0.5n_x \times (n_x + 1) \times (r_a r_e) + (n^2) \times (r_a^2 r_e^3) + (n^2) \times (r_a^3 r_e^2) + (n_u n_x) \times (r_a^2 r_e^3)$	3 sums in $h(\bullet)$ 3 sums in $v(\bullet)$

Remark 3.10. For a fixed combination of multisets, independent of the structure chosen, adding $\{0\}$ or $\{-1\}$ in every possible place will reduce the conservatism. Thus, the more delays $\{0\}, \{-1\}$ are used, the more relaxed the conditions are. Following the procedure given above, the maximum number of sums is given by $n_{H\mathcal{V}} = 2n_{P_h} + 2n_{P_v} + 2$.

The method for Theorem 3.9 is as follows:

Step 1: Select multisets $H_0^K = H_0^G = \mathcal{V}_0^K = \mathcal{V}_0^G = \{0\}$, thus conditions in Theorem 3.9 give:

$$\begin{bmatrix} -P_{H_0^P \mathcal{V}_0^P} & (*) \\ A_{\{0\}} G_{\{0\},\{0\}} + B_{\{0\}} K_{\{0\},\{0\}} & -E_{\{0\}} G_{\{1\},\{1\}} - G_{\{1\},\{1\}}^T E_{\{0\}}^T + P_{H_1^P \mathcal{V}_1^P} \end{bmatrix} < 0. \tag{3.86}$$

Since there are no double sums in $v(\bullet)$ at the current sample k, it is possible to add $\{0\}$ in $\mathcal{V}_0^K$, i.e., $\mathcal{V}_0^K = \left(\mathcal{V}_0^E \oplus \mathcal{V}_0^G\right)$. Then (3.86) yields

$$\begin{bmatrix} -P_{H_0^P \mathcal{V}_0^P} & (*) \\ A_{\{0\}} G_{\{0\},\{0\}} + B_{\{0\}} K_{\{0\},\{0,0\}} & -E_{\{0\}} G_{\{1\},\{1\}} - G_{\{1\},\{1\}}^T E_{\{0\}}^T + P_{H_1^P \mathcal{V}_1^P} \end{bmatrix} < 0, \tag{3.87}$$

which ends in three sums for $h(\bullet): \sum_{i_1=1}^{r_a} \sum_{i_2=1}^{r_a} \sum_{i_x=1}^{r_a} h_{i_1}(z(k)) h_{i_2}(z(k)) h_{i_x}(z(k+1))$ and three for $v(\bullet): \sum_{j_1=1}^{r_e} \sum_{j_2=1}^{r_e} \sum_{j_x=1}^{r_e} v_{j_1}(z(k)) v_{j_2}(z(k)) v_{j_x}(z(k+1))$.

Step 2: To keep the same number for sums as for Theorem 3.11, the best solution for the Lyapunov multiple sums is $H_0^P = \mathcal{V}_0^P = \{0\}$. Finally, (3.87) renders:

$$\begin{bmatrix} -P_{\{0\},\{0\}} & (*) \\ A_{\{0\}} G_{\{0\},\{0\}} + B_{\{0\}} K_{\{0\},\{0,0\}} & -E_{\{0\}} G_{\{1\},\{1\}} - G_{\{1\},\{1\}}^T E_{\{0\}}^T + P_{\{1\},\{1\}} \end{bmatrix} < 0. \tag{3.88}$$

Table 3.4 summarizes these results for an arbitrary cardinality of the multisets.

Example 3.10. Consider the TS descriptor (3.69) when $w_k = 0$, with $r_a = r_e = 2$, $E_1 = \begin{bmatrix} 1.1 & 0 \\ 0 & 0.36 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0.95 & 0 \\ 0 & 1 \end{bmatrix}$, $A_1 = \begin{bmatrix} 1.18 - 0.2\delta & -1.31 \\ -0.33 & 0.23 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.69 & 1.41 \\ -1.17 & 1.43 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \\ -1.05 \end{bmatrix}$, and $B_2 = \begin{bmatrix} 1 - 0.1\delta \\ 0 \end{bmatrix}$, where $\delta > 0$ is a real-valued parameter. Applying Theorem 3.9 with multisets:

- $H_0^P = H_0^K = H_0^G = \mathcal{V}_0^K = \{0\}$, $\mathcal{V}_0^P = \mathcal{V}_0^G = \varnothing$ (four sums are involved). The maximum value of β for which conditions were found feasible is $\delta = 0.86$. Using the same number of sums the conditions of Theorems 3.10 and 3.11 are not feasible for any δ.

- $H_0^P = H_0^G = H_0^K = \{0,-1\}$, $\mathcal{V}_0^K = \{0\}$, $\mathcal{V}_0^P = \mathcal{V}_0^G = \varnothing$ (five sums are involved). The maximum value of β for which conditions were found feasible is $\delta = 0.90$.
- $H_0^P = H_0^K = H_0^G = \mathcal{V}_0^P = \mathcal{V}_0^G = \{0\}$ and $\mathcal{V}_0^K = \{0,0\}$: three sums in $h(\bullet)$ and three sums in $v(\bullet)$, the maximum value was $\delta = 1.86$. ♦

Table 3.4. How to select multisets for Theorem 3.9 and Theorem 3.11.

Matrix	Multisets in Theorem 3.9	Multisets in Theorem 3.11
$P_{H_0^P \mathcal{V}_0^P}$	$H_0^P = \{0,0,\ldots,0\}$, $\lvert H_0^P \rvert = n_{P_h}$	$H_0^P = \{-1,-1,\ldots,-1\}$, $\lvert H_0^P \rvert = n_{P_h}$
	$\mathcal{V}_0^P = \{0,0,\ldots,0\}$, $\lvert \mathcal{V}_0^P \rvert = n_{P_v}$	$\mathcal{V}_0^P = \{-1,-1,\ldots,-1\}$, $\lvert \mathcal{V}_0^P \rvert = n_{P_v}$
$K_{H_0^K \mathcal{V}_0^K}$	$H_0^K = \{\underbrace{0,0,\ldots,0,}_{n_{P_h}}\}$, $\lvert H_0^K \rvert = n_{P_h}$	$H_0^K = \{\underbrace{0,0,\ldots,0,}_{n_{P_h}}\underbrace{-1,-1,\ldots,-1}_{n_{P_h}}\}$, $\lvert H_0^K \rvert = 2n_{P_h}$
	$\mathcal{V}_0^K = \{0,\underbrace{0,0,\ldots,0}_{n_{P_v}}\}$, $\lvert \mathcal{V}_0^K \rvert = 1 + n_{P_v}$	$\mathcal{V}_0^K = \{0,\underbrace{0,0,\ldots,0,}_{n_{P_v}}\underbrace{-1,-1,\ldots,-1}_{n_{P_v}}\}$, $\lvert \mathcal{V}_0^K \rvert = 1 + 2n_{P_v}$
$G_{H_0^G \mathcal{V}_0^G}$	$H_0^G = \{\underbrace{0,0,\ldots,0,}_{n_{P_h}}\}$, $\lvert H_0^G \rvert = n_{P_h}$	$H_0^G = \{\underbrace{0,0,\ldots,0,}_{n_{P_h}}\underbrace{-1,-1,\ldots,-1}_{n_{P_h}}\}$, $\lvert H_0^G \rvert = 2n_{P_h}$
	$\mathcal{V}_0^G = \{\underbrace{0,0,\ldots,0,}_{n_{P_v}}\}$, $\lvert \mathcal{V}_0^G \rvert = n_{P_v}$	$\mathcal{V}_0^G = \{0,\underbrace{0,0,\ldots,0,}_{n_{P_v}}\underbrace{-1,-1,\ldots,-1}_{n_{P_v}}\}$, $\lvert \mathcal{V}_0^G \rvert = 1 + 2n_{P_v}$
$F_{H_0^F \mathcal{V}_0^F}$	----------------	$H_0^F = \{0,\underbrace{0,0,\ldots,0,}_{n_{P_h}}\underbrace{-1,-1,\ldots,-1}_{n_{P_h}}\}$, $\lvert H_0^F \rvert = 1 + 2n_{P_h}$
	----------------	$\mathcal{V}_0^F = \{\underbrace{0,0,\ldots,0,}_{n_{P_v}}\underbrace{-1,-1,\ldots,-1}_{n_{P_v}}\}$, $\lvert \mathcal{V}_0^F \rvert = 2n_{P_v}$

H$_\infty$ attenuation

In this part, we consider disturbance attenuation. Recall the TS descriptor model (3.69). Using the control law (3.70) gives:

$$\begin{aligned} E_{\mathcal{V}_0^E} x_{k+1} &= \left(A_{H_0^A} + B_{H_0^B} K_{H_0^K \mathcal{V}_0^K} G^{-1}_{H_0^G \mathcal{V}_0^G} \right) x_k + D_{H_0^D} w_k \\ y_k &= C_{H_0^C} x_k + J_{H_0^J} w_k . \end{aligned} \tag{3.89}$$

Since the proofs follow the same lines as for the previous results, they are not stated here. For *Case 1* the following result is obtained.

Theorem 3.12. The closed-loop system (3.89) is asymptotically stable and the attenuation is γ if there exist $\gamma>0$, $P_{\boldsymbol{i}_k^P,\boldsymbol{j}_k^P}=P_{\boldsymbol{i}_k^P,\boldsymbol{j}_k^P}^T>0$, $\boldsymbol{i}_k^P=pr_{H_k^P}^{\boldsymbol{i}}$, $\boldsymbol{j}_k^P=pr_{\mathcal{V}_k^P}^{\boldsymbol{j}}$, $G_{\boldsymbol{i}_k^G,\boldsymbol{j}_k^G}$, $\boldsymbol{i}_k^G=pr_{H_k^G}^{\boldsymbol{i}}$, $\boldsymbol{j}_k^G=pr_{\mathcal{V}_k^G}^{\boldsymbol{j}}$, $k=0,1$, and $K_{\boldsymbol{i}_0^K,\boldsymbol{j}_0^K}$, $\boldsymbol{i}_0^K=pr_{H_0^K}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^K=pr_{\mathcal{V}_0^K}^{\boldsymbol{j}}$, $\boldsymbol{i}\in\mathbb{I}_{H_\Gamma}$, $\boldsymbol{j}\in\mathbb{I}_{\mathcal{V}_\Gamma}$, with $H_\Gamma=H_0^P\cup H_1^P\cup\left(H_0^B\oplus H_0^K\right)\cup\left(H_0^G\oplus\left(H_0^A\cup H_0^C\right)\right)\cup H_1^G$, $\mathcal{V}_\Gamma=\mathcal{V}_0^P\cup\mathcal{V}_1^P\cup\mathcal{V}_0^K\cup\mathcal{V}_0^G\cup\left(\mathcal{V}_0^E\oplus\mathcal{V}_1^G\right)$

$$\begin{bmatrix} -P_{H_0^P\mathcal{V}_0^P} & (*) & (*) & (*) \\ A_{H_0^A}G_{H_0^G\mathcal{V}_0^G}+B_{H_0^B}K_{H_0^K\mathcal{V}_0^K} & -E_{\mathcal{V}_0^E}G_{H_1^G\mathcal{V}_1^G}-G_{H_1^G\mathcal{V}_1^G}^TE_{\mathcal{V}_0^E}^T+P_{H_1^P\mathcal{V}_1^P} & (*) & (*) \\ 0 & D_{H_0^D}^T & -\gamma^2 I & (*) \\ C_{H_0^C}G_{H_0^G\mathcal{V}_0^G} & 0 & J_{H_0^J} & -I \end{bmatrix}<0. \quad (3.90)$$

For *Case 2*, the following result can be stated.

Theorem 3.13. The closed-loop system (3.89) is asymptotically stable and the attenuation is γ if there exist $\gamma>0$, $P_{\boldsymbol{i}_k^P,\boldsymbol{j}_k^P}=P_{\boldsymbol{i}_k^P,\boldsymbol{j}_k^P}^T$, $\boldsymbol{i}_k^P=pr_{H_k^P}^{\boldsymbol{i}}$, $\boldsymbol{j}_k^P=pr_{\mathcal{V}_k^P}^{\boldsymbol{j}}$, $k=0,1$, $K_{\boldsymbol{i}_0^K,\boldsymbol{j}_0^K}$, $\boldsymbol{i}_0^K=pr_{H_0^K}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^K=pr_{\mathcal{V}_0^K}^{\boldsymbol{j}}$, $G_{\boldsymbol{i}_0^G,\boldsymbol{j}_0^G}$, $\boldsymbol{i}_0^G=pr_{H_0^G}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^G=pr_{\mathcal{V}_0^G}^{\boldsymbol{j}}$, and $F_{\boldsymbol{i}_0^F,\boldsymbol{j}_0^F}$, $\boldsymbol{i}_0^F=pr_{H_0^F}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^F=pr_{\mathcal{V}_0^F}^{\boldsymbol{j}}$, $\boldsymbol{i}\in\mathbb{I}_{H_\Gamma}$, $\boldsymbol{j}\in\mathbb{I}_{\mathcal{V}_\Gamma}$, $H_\Gamma=H_0^P\cup H_1^P\cup\left(H_0^B\oplus H_0^K\right)\cup\left(H_0^G\oplus\left(H_0^A\cup H_0^C\right)\right)\cup H_0^F$, $\mathcal{V}_\Gamma=\mathcal{V}_0^P\cup\mathcal{V}_1^P\cup\mathcal{V}_0^K\cup\mathcal{V}_0^G\cup\left(\mathcal{V}_0^E\oplus\mathcal{V}_0^F\right)$ such that

$$\begin{bmatrix} -G_{H_0^G\mathcal{V}_0^G}-G_{H_0^G\mathcal{V}_0^G}^T+P_{H_0^P\mathcal{V}_0^P} & (*) & (*) & (*) & (*) \\ A_{H_0^A}G_{H_0^G\mathcal{V}_0^G}+B_{H_0^B}K_{H_0^K\mathcal{V}_0^K} & -E_{\mathcal{V}_0^E}\mathrm{F}_{H_0^F\mathcal{V}_0^F}-F_{H_0^F\mathcal{V}_0^F}^TE_{\mathcal{V}_0^E}^T & (*) & (*) & (*) \\ 0 & \mathrm{F}_{H_0^F\mathcal{V}_0^F} & -P_{H_1^P\mathcal{V}_1^P} & (*) & (*) \\ 0 & D_{H_0^D}^T & 0 & -\gamma^2 I & (*) \\ C_{H_0^C}G_{H_0^G\mathcal{V}_0^G} & 0 & 0 & J_{H_0^J} & -I \end{bmatrix}<0. \quad (3.91)$$

The following numerical example illustrates the performances of Theorems 3.12 and 3.13 for the options for multisets given in Table 3.5.

Example 3.11. Consider the TS descriptor model (3.69) with $r=r_e=2$, $E_1=\begin{bmatrix}0.9 & 0.1+\alpha\\ -0.4 & 1.1\end{bmatrix}$, $E_2=\begin{bmatrix}0.9 & 1.1\\ -0.4 & 1.1\end{bmatrix}$, $A_1=\begin{bmatrix}0 & -1.5\\ 0 & 0.5\end{bmatrix}$, $A_2=\begin{bmatrix}-1 & -1.5\\ 2 & 0.5\end{bmatrix}$, $B_1=B_2=\begin{bmatrix}0\\ 1\end{bmatrix}$, $C_1=\begin{bmatrix}0 & 1.28\end{bmatrix}$, $C_2=\begin{bmatrix}0 & 0.43\end{bmatrix}$, $D_1=\begin{bmatrix}0.23 & 0\end{bmatrix}^T$, $D_2=\begin{bmatrix}0 & 0.12\end{bmatrix}^T$, $J_1=0.12$, and

$J_2 = 0.09 + \alpha$, where α is a real-valued parameter. Table 3.6 shows the results for several parameter values when using the options in Table 3.5.

Table 3.5. Selection of multisets for Example 3.11

Approach	Option	Multisets	Number of convex sums
Theorem 3.12	1	$H_0^P = H_0^G = H_0^K = \mathcal{V}_0^K = \{0\}$ $\mathcal{V}_0^P = \mathcal{V}_0^G = \varnothing$	3 convex sums in $h(\bullet)$ 1 convex sum in $v(\bullet)$
	2	$H_0^P = H_0^K = H_0^G = \mathcal{V}_0^P = \mathcal{V}_0^G = \{0\}$ $\mathcal{V}_0^K = \{0,0\}$	3 convex sums in $h(\bullet)$ 3 convex sums in $v(\bullet)$
Theorem 3.13	3	$H_0^P = H_0^K = H_0^G = \mathcal{V}_0^K = \mathcal{V}_0^G = \{0\}$ $H_0^F = \{0,0,1\}$, $\mathcal{V}_0^P = \mathcal{V}_0^F = \varnothing$	3 convex sums in $h(\bullet)$ 1 convex sum in $v(\bullet)$
	4	$H_0^K = H_0^G = \mathcal{V}_0^F = \{0,-1\}$ $H_0^P = \mathcal{V}_0^P = \{-1\}$ $\mathcal{V}_0^K = \mathcal{V}_0^G = H_0^F = \{0,0,-1\}$	3 convex sums in $h(\bullet)$ 3 convex sums in $v(\bullet)$

Table 3.6. Minimum γ values in Example 3.11

Parameter α	Option 1	Option 2	Option 3	Option 4
$\alpha = -1.5$	$\gamma = 2.46$	$\gamma = 2.18$	$\gamma = 2.76$	$\gamma = 1.79$
$\alpha = -1$	$\gamma = 1.27$	$\gamma = 1.21$	$\gamma = 1.23$	$\gamma = 1.12$
$\alpha = -0.5$	$\gamma = 0.78$	$\gamma = 0.72$	$\gamma = 0.69$	$\gamma = 0.64$
$\alpha = 0$	$\gamma = 0.56$	$\gamma = 0.56$	$\gamma = 0.53$	$\gamma = 0.50$
$\alpha = 0.5$	$\gamma = 0.77$	$\gamma = 0.77$	$\gamma = 0.77$	$\gamma = 0.77$

The obtained results illustrate Remark 3.4, for instance, when $\alpha = -1.5$, Option 1 has provided better attenuation than Option 3; while for $\alpha = -0.5$ Option 3 has given better result than Option 1. ♦

Robust control

Consider a TS descriptor model with uncertainties:

$$\left(E_{\mathcal{V}_0^E}+\Delta E\right)x_{k+1}=\left(A_{H_0^A}+\Delta A\right)x_k+\left(B_{H_0^B}+\Delta B\right)u_k, \tag{3.92}$$

with the uncertainties defined as $\Delta E=D_{\mathcal{V}_{0,\mathrm{e}}^D}\Delta_e L_{\mathcal{V}_{0,\mathrm{e}}^L}$, $\Delta A=D_{H_{0,\mathrm{a}}^D}\Delta_a L_{H_{0,\mathrm{a}}^L}$, $\Delta B=D_{H_{0,\mathrm{b}}^D}\Delta_b L_{H_{0,\mathrm{b}}^L}$, and classical norm bounds $\Delta_e^T\Delta_e<I$, $\Delta_a^T\Delta_a<I$, and $\Delta_b^T\Delta_b<I$. The uncertain model (3.92) under the control law (3.70) gives

$$\left(E_{\mathcal{V}_0^E}+\Delta E\right)x_{k+1}=\left(A_{H_0^A}+B_{H_0^B}K_{H_0^K\mathcal{V}_0^K}G^{-1}_{H_0^G\mathcal{V}_0^G}+\Delta A+\Delta BK_{H_0^K\mathcal{V}_0^K}G^{-1}_{H_0^G\mathcal{V}_0^G}\right)x_k. \tag{3.93}$$

For *Case 1*, the following result can be stated:

Theorem 3.14. The closed-loop system (3.93) is asymptotically stable if there exist $P_{\boldsymbol{i}_k^P,\boldsymbol{j}_k^P}=P^T_{\boldsymbol{i}_k^P,\boldsymbol{j}_k^P}$, $\boldsymbol{i}_k^P=pr^{\boldsymbol{i}}_{H_k^P}$, $\boldsymbol{j}_k^P=pr^{\boldsymbol{j}}_{\mathcal{V}_k^P}$, $G_{\boldsymbol{i}_k^G,\boldsymbol{j}_k^G}$, $\boldsymbol{i}_k^G=pr^{\boldsymbol{i}}_{H_k^G}$, $\boldsymbol{j}_k^G=pr^{\boldsymbol{j}}_{\mathcal{V}_k^G}$, $k=0,1$, $K_{\boldsymbol{i}_0^K,\boldsymbol{j}_0^K}$, $\boldsymbol{i}_0^K=pr^{\boldsymbol{i}}_{H_0^K}$, $\boldsymbol{j}_0^K=pr^{\boldsymbol{j}}_{\mathcal{V}_0^K}$, $\tau_{\boldsymbol{i}^\tau_{0,a},\boldsymbol{j}^\tau_{0,a}}$, $\boldsymbol{i}_0^{\tau_a}=pr^{\boldsymbol{i}}_{H^\tau_{0,a}}$, $\boldsymbol{j}_0^{\tau_a}=pr^{\boldsymbol{j}}_{\mathcal{V}^\tau_{0,a}}$, $\tau_{\boldsymbol{i}^\tau_{0,b},\boldsymbol{j}^\tau_{0,b}}$, $\boldsymbol{i}_0^{\tau_b}=pr^{\boldsymbol{i}}_{H^\tau_{0,b}}$, $\boldsymbol{j}_0^{\tau_b}=pr^{\boldsymbol{j}}_{\mathcal{V}^\tau_{0,b}}$, and $\tau_{\boldsymbol{i}^\tau_{0,e},\boldsymbol{j}^\tau_{0,e}}$, $\boldsymbol{i}_0^{\tau_e}=pr^{\boldsymbol{i}}_{H^\tau_{0,e}}$, $\boldsymbol{j}_0^{\tau_e}=pr^{\boldsymbol{j}}_{\mathcal{V}^\tau_{0,e}}$, $\boldsymbol{i}\in\mathbb{I}_{H_\Gamma}$, $\boldsymbol{j}\in\mathbb{I}_{\mathcal{V}_\Gamma}$, where $H_\Gamma=H_0^P\cup H_1^P\cup H_1^G\cup$ $\left(H_0^K\oplus\left(H_0^B\cup H_{0,b}^L\right)\right)\cup\left(H_0^G\oplus\left(H_0^A\cup H_{0,\mathrm{a}}^L\right)\right)\cup\left(H_{0,a}^\tau\oplus H_{0,a}^D\right)\cup\left(H_{0,\mathrm{b}}^\tau\oplus H_{0,\mathrm{b}}^D\right)\cup H_{0,\mathrm{e}}^\tau$, $\mathcal{V}_\Gamma=\mathcal{V}_0^P\cup\mathcal{V}_1^P\cup\mathcal{V}_0^K\cup\mathcal{V}_0^G\cup\mathcal{V}_{0,\mathrm{a}}^\tau\cup\mathcal{V}_{0,\mathrm{b}}^\tau\cup\left(\mathcal{V}_1^G\oplus\left(\mathcal{V}_0^E\cup\mathcal{V}_{0,e}^L\right)\right)\cup\left(\mathcal{V}_{0,\mathrm{e}}^\tau\oplus\mathcal{V}_{0,\mathrm{e}}^D\right)$ such that

$$\left[\begin{array}{cc|c|c} -P_{H_0^P\mathcal{V}_0^P} & (*) & (*) & (*) \\ A_{H_0^A}G_{H_0^G\mathcal{V}_0^G}+B_{H_0^B}K_{H_0^K\mathcal{V}_0^K} & -E_{\mathcal{V}_0^E}G_{H_1^G\mathcal{V}_1^G}-G^T_{H_1^G\mathcal{V}_1^G}E^T_{\mathcal{V}_0^E}+P_{H_1^P\mathcal{V}_1^P} & & \\ \hline \multicolumn{2}{c|}{\tilde{\mathcal{T}}\tilde{\mathcal{D}}^T} & -\tilde{\mathcal{T}} & (*) \\ \hline \multicolumn{2}{c|}{\tilde{\mathcal{G}}} & 0 & -\tilde{\mathcal{T}} \end{array}\right]<0, \tag{3.94}$$

with $\tilde{\mathcal{D}}^T=\begin{bmatrix}0 & D^T_{H_{0,\mathrm{a}}^D}\\ 0 & D^T_{H_{0,\mathrm{b}}^D}\\ 0 & D^T_{\mathcal{V}_{0,\mathrm{e}}^D}\end{bmatrix}$, $\tilde{\mathcal{G}}=\begin{bmatrix}L_{H_{0,\mathrm{a}}^L}G_{H_0^G\mathcal{V}_0^G} & 0\\ L_{H_{0,\mathrm{b}}^L}K_{H_0^K\mathcal{V}_0^K} & 0\\ 0 & -L_{\mathcal{V}_{0,\mathrm{e}}^L}G_{H_1^G\mathcal{V}_1^G}\end{bmatrix}$, $\tilde{\mathcal{T}}=\begin{bmatrix}\tau_{H^\tau_{0,a}\mathcal{V}^\tau_{0,a}}I & 0 & 0\\ 0 & \tau_{H^\tau_{0,\mathrm{b}}\mathcal{V}^\tau_{0,\mathrm{b}}}I & 0\\ 0 & 0 & \tau_{H^\tau_{0,\mathrm{e}}\mathcal{V}^\tau_{0,\mathrm{e}}}I\end{bmatrix}$.

Proof. Using the results in Theorem 3.7 for the uncertain closed-loop model (3.93) gives:

$$\begin{bmatrix} -P_{H_0^P\mathcal{V}_0^P} & (*) \\ A_{H_0^A}G_{H_0^G\mathcal{V}_0^G}+B_{H_0^B}K_{H_0^K\mathcal{V}_0^K} & -E_{\mathcal{V}_0^E}G_{H_1^G\mathcal{V}_1^G}+(*)+P_{H_1^P\mathcal{V}_1^P}\end{bmatrix}+\Delta\Gamma<0, \tag{3.95}$$

where $\Delta\Gamma = \begin{bmatrix} 0 & (*) \\ \Delta A G_{H_0^G \mathcal{V}_0^G} + \Delta B K_{H_0^K \mathcal{V}_0^K} & -\Delta E G_{H_1^G \mathcal{V}_1^G} + (*) \end{bmatrix}$, which can be represented as

$$\Delta\Gamma = \tilde{\mathcal{D}}\bar{\Delta}\tilde{\mathcal{G}} + (*) = \begin{bmatrix} 0 & 0 & 0 \\ D_{H_{0,\mathrm{a}}^D} & D_{H_{0,\mathrm{b}}^D} & D_{\mathcal{V}_{0,\mathrm{e}}^D} \end{bmatrix} \begin{bmatrix} \Delta_a & 0 & 0 \\ 0 & \Delta_b & 0 \\ 0 & 0 & \Delta_e \end{bmatrix} \begin{bmatrix} L_{H_{0,\mathrm{a}}^L} G_{H_0^G \mathcal{V}_0^G} & 0 \\ L_{H_{0,\mathrm{b}}^L} K_{H_0^K \mathcal{V}_0^K} & 0 \\ 0 & -L_{\mathcal{V}_{0,\mathrm{e}}^L} G_{H_1^G \mathcal{V}_1^G} \end{bmatrix} + (*).$$

Employ Property A.4 (Appendix A) with $\mathcal{N} = \tilde{\mathcal{G}}$, $\mathcal{R}^T = \tilde{\mathcal{D}}\bar{\Delta}$, and $\mathcal{Q} = \tilde{\mathcal{T}} = diag\left[\tau_{H_{0,a}^\tau \mathcal{V}_{0,a}^\tau} I,\ \ \tau_{H_{0,b}^\tau \mathcal{V}_{0,b}^\tau} I,\ \ \tau_{H_{0,e}^\tau \mathcal{V}_{0,e}^\tau} I\right]$, $\tilde{\mathcal{T}} = \tilde{\mathcal{T}}^T > 0$. The uncertain terms can be expressed as $\Delta\Gamma \le \tilde{\mathcal{D}}\bar{\Delta}\left(\tilde{\mathcal{T}}\right)^{-1}\bar{\Delta}^T\tilde{\mathcal{D}}^T + \tilde{\mathcal{G}}^T\left(\tilde{\mathcal{T}}\right)^{-1}\tilde{\mathcal{G}}$. Consider $\bar{\Delta}\bar{\Delta}^T \le I$, thus

$$\Delta\Gamma \le \tilde{\mathcal{D}}\tilde{\mathcal{T}}\left(\tilde{\mathcal{T}}\right)^{-1}\tilde{\mathcal{T}}\tilde{\mathcal{D}}^T + \tilde{\mathcal{G}}^T\left(\tilde{\mathcal{T}}\right)^{-1}\tilde{\mathcal{G}}. \tag{3.96}$$

Substituting (3.96) in (3.95) writes:

$$\begin{bmatrix} -P_{H_0^P \mathcal{V}_0^P} & (*) \\ A_{H_0^A} G_{H_0^G \mathcal{V}_0^G} + B_{H_0^B} K_{H_0^K \mathcal{V}_0^K} & -E_{\mathcal{V}_0^E} G_{H_1^G \mathcal{V}_1^G} + (*) + P_{H_1^P \mathcal{V}_1^P} \end{bmatrix} + \begin{bmatrix} \tilde{\mathcal{D}}\tilde{\mathcal{T}} & \tilde{\mathcal{G}}^T \end{bmatrix} \begin{bmatrix} \tilde{\mathcal{T}} & 0 \\ 0 & \tilde{\mathcal{T}} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\mathcal{T}}\tilde{\mathcal{D}}^T \\ \tilde{\mathcal{G}} \end{bmatrix} < 0,$$

which by means of the Schur complement yields (3.94), thus concluding the proof. ■

For Case 2, the following result can be established:

Theorem 3.15. The closed-loop system (3.93) is asymptotically stable if there exist $P_{\boldsymbol{i}_k^P, \boldsymbol{j}_k^P} = P_{\boldsymbol{i}_k^P, \boldsymbol{j}_k^P}^T$, $\boldsymbol{i}_k^P = pr_{H_k^P}^{\boldsymbol{i}}$, $\boldsymbol{j}_k^P = pr_{\mathcal{V}_k^P}^{\boldsymbol{j}}$, $k = 0,1$, $K_{\boldsymbol{i}_0^K, \boldsymbol{j}_0^K}$, $\boldsymbol{i}_0^K = pr_{H_0^K}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^K = pr_{\mathcal{V}_0^K}^{\boldsymbol{j}}$, $G_{\boldsymbol{i}_0^G, \boldsymbol{j}_0^G}$, $\boldsymbol{i}_0^G = pr_{H_0^G}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^G = pr_{\mathcal{V}_0^G}^{\boldsymbol{j}}$, $F_{\boldsymbol{i}_0^F, \boldsymbol{j}_0^F}$, $\boldsymbol{i}_0^F = pr_{H_0^F}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^F = pr_{\mathcal{V}_0^F}^{\boldsymbol{j}}$, $\tau_{\boldsymbol{i}_{0,a}^\tau, \boldsymbol{j}_{0,a}^\tau}$, $\boldsymbol{i}_0^{\tau_a} = pr_{H_{0,a}^\tau}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^{\tau_a} = pr_{\mathcal{V}_{0,a}^\tau}^{\boldsymbol{j}}$, $\tau_{\boldsymbol{i}_{0,b}^\tau, \boldsymbol{j}_{0,b}^\tau}$, $\boldsymbol{i}_0^{\tau_b} = pr_{H_{0,b}^\tau}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^{\tau_b} = pr_{\mathcal{V}_{0,b}^\tau}^{\boldsymbol{j}}$, and $\tau_{\boldsymbol{i}_{0.e}^\tau, \boldsymbol{j}_{0,e}^\tau}$, $\boldsymbol{i}_0^{\tau_e} = pr_{H_{0,e}^\tau}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^{\tau_e} = pr_{\mathcal{V}_{0,e}^\tau}^{\boldsymbol{j}}$, $\boldsymbol{i} \in \mathbb{I}_{H_\Gamma}$, $\boldsymbol{j} \in \mathbb{I}_{\mathcal{V}_\Gamma}$, where $H_\Gamma = H_0^P \cup H_1^P \cup \left(H_0^K \oplus \left(H_0^B \cup H_{0,b}^L\right)\right) \cup \left(H_0^G \oplus \left(H_0^A \cup H_{0,\mathrm{a}}^L\right)\right) \cup H_0^F \cup \left(H_{0,a}^\tau \oplus H_{0,a}^D\right) \cup H_{0,\mathrm{e}}^\tau \cup \left(H_{0,\mathrm{b}}^\tau \oplus H_{0,\mathrm{b}}^D\right)$, $\mathcal{V}_\Gamma = \mathcal{V}_0^P \cup \mathcal{V}_1^P \cup \mathcal{V}_0^K \cup \mathcal{V}_0^G \cup \left(\mathcal{V}_0^F \oplus \left(\mathcal{V}_0^E \cup \mathcal{V}_{0,e}^L\right)\right) \cup \left(\mathcal{V}_{0,\mathrm{e}}^\tau \oplus \mathcal{V}_{0,\mathrm{e}}^D\right) \cup \mathcal{V}_{0,\mathrm{a}}^\tau \cup \mathcal{V}_{0,\mathrm{b}}^\tau$ such that

$$\begin{bmatrix} \begin{matrix} -G_{H_0^G\mathcal{V}_0^G} - G_{H_0^G\mathcal{V}_0^G}^T + P_{H_0^P\mathcal{V}_0^P} & (*) \\ A_{H_0^A} G_{H_0^G\mathcal{V}_0^G} + B_{H_0^B} K_{H_0^K\mathcal{V}_0^K} & -E_{\mathcal{V}_0^E} F_{H_0^F\mathcal{V}_0^F} - F_{H_0^F\mathcal{V}_0^F}^T E_{\mathcal{V}_0^E}^T \end{matrix} & (*) & (*) & (*) \\ \tilde{\mathcal{T}}\tilde{\mathcal{D}}^T & -\tilde{\mathcal{T}} & (*) & (*) \\ \tilde{\mathcal{G}} & 0 & -\tilde{\mathcal{T}} & (*) \\ \tilde{\mathcal{F}} & 0 & 0 & -P_{H_1^P\mathcal{V}_1^P} \end{bmatrix} < 0, \quad (3.97)$$

where $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{T}}$ are defined in Theorem 3.11, $\tilde{\mathcal{G}} = \begin{bmatrix} L_{H_{0,a}^L} G_{H_0^G\mathcal{V}_0^G} & 0 \\ L_{H_{0,b}^L} K_{H_0^K\mathcal{V}_0^K} & 0 \\ 0 & -L_{\mathcal{V}_{0,e}^L} F_{H_0^F\mathcal{V}_0^F} \end{bmatrix}$ and $\tilde{\mathcal{F}} = \begin{bmatrix} 0 & F_{H_0^F\mathcal{V}_0^F} \end{bmatrix}$.

Proof: The proof follows the same lines as the proof of Theorem 3.14 but using the Lyapunov function in *Case 2*. ■

3.3. Summary and concluding remarks

In this chapter, state feedback control design methods for TS descriptor models have been presented. The improvements are based on the well-known Finsler's lemma; this lemma allows handling the descriptor matrix as well as "cutting" the link between the Lyapunov matrix and the controller matrices. Nevertheless, when dealing with continuous-time TS descriptors the conditions are not "pure" LMIs since a scalar parameter must be fixed a priori. Therefore a logarithmically spaced search is performed. This increases the computational cost, but since all the computations are done offline, they are still realizable.

Chapter 4. Observer design

This chapter presents observer design for both continuous and discrete time nonlinear descriptor systems using an exact TS representation. In the case of the continuous-time TS descriptor model, strict LMI conditions are obtained by changing the extended estimated state vector and using a full observer gain. For discrete-time TS descriptors several LMI conditions are stated. These conditions depend on the selection of the Lyapunov function: quadratic, non-quadratic, or delayed non-quadratic. All the presented cases consider that the descriptor matrix is nonsingular in the considered compact set of the state space. Numerical examples are given in order to illustrate the performances of the provided improvements.

4.1. Continuous-time TS descriptor models

This section presents a novel observer design for continuous-time nonlinear descriptor systems using their Takagi-Sugeno representation, which overcomes BMI conditions existing in the literature. The main idea is to change the estimated state vector by using an auxiliary variable. This allows changing the structure of the observer and using a full observer gain LMI constraints are stated that improve results in the literature. In addition, some relaxations are achieved when a non-PDC-like observer is used. Finally, Finsler's lemma is used to enlarge the solution set by adding slack variables and decoupling the Lyapunov function from the observer gains.

4.1.1. Problem statement

Conditions in previous works (Guelton et al., 2008; Guerra et al., 2004) are given in BMI terms. Sufficient LMI conditions are obtained by fixing some of the decision variables, as will be shown in what follows.

Consider the following TS descriptor model

$$E_v\dot{x} = A_h x + B_h u, \qquad y = C_h x. \tag{4.1}$$

The following extended state vector is commonly used $\bar{x} = \begin{bmatrix} x^T & \dot{x}^T \end{bmatrix}^T$. Then (4.1) can be written as (Taniguchi et al., 1999):

$$\bar{E}\dot{\bar{x}} = \bar{A}_{hv}\bar{x} + \bar{B}_h u, \qquad y = \bar{C}_h\bar{x}, \tag{4.2}$$

with $\bar{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{A}_{hv} = \begin{bmatrix} 0 & I \\ A_h & -E_v \end{bmatrix}$, $\bar{B}_h = \begin{bmatrix} 0 \\ B_h \end{bmatrix}$, and $\bar{C}_h = \begin{bmatrix} C_h & 0 \end{bmatrix}$.

For the observer design the main task is to make the estimation error $e = x - \hat{x}$ converge to zero as $t \to \infty$. To this end, in (Guerra et al., 2004) the following estimated state for the extended model (4.2) was proposed:

$$\hat{\bar{x}}^* = \begin{bmatrix} \hat{x} \\ \dot{\hat{x}} \end{bmatrix}. \tag{4.3}$$

The corresponding observer is:

$$\begin{aligned} \bar{E}\dot{\hat{\bar{x}}}^* &= \bar{A}_{hv}\hat{\bar{x}}^* + \bar{B}_h u + \bar{L}_{hv}\left(y - \hat{y}\right) \\ \hat{y} &= \bar{C}_h\hat{\bar{x}}^*, \end{aligned} \tag{4.4}$$

where the observer gain is defined as $\bar{L}_{hv} = \begin{bmatrix} 0 & L_{2hv}^T \end{bmatrix}^T$. Defining an extended estimation error vector:

$$e^* = \bar{x} - \hat{\bar{x}}^* = \begin{bmatrix} x - \hat{x} \\ \dot{x} - \dot{\hat{x}} \end{bmatrix}, \tag{4.5}$$

its dynamic is given as

$$\bar{E}\dot{e}^* = \left(\bar{A}_{hv} - \bar{L}_{hv}\bar{C}_h\right)e^*. \tag{4.6}$$

The synthesis of the augmented observer (4.4) is done via the quadratic Lyapunov function candidate (Guerra et al., 2004):

$$V\left(e^*\right)=e^{*T}\bar{E}^T P e^*, \quad \bar{E}^T P = P^T \bar{E} \geq 0, \tag{4.7}$$

with $P=\begin{bmatrix} P_1 & 0 \\ P_3 & P_4 \end{bmatrix}$, $P_1 = P_1^T > 0$. Taking the time-derivative of (4.7) gives $\dot{V}\left(e^*\right)=\dot{e}^{*T}\bar{E}^T P e^* + e^{*T} P^T \bar{E}\dot{e}^*$, which by substituting (4.6) renders

$$\dot{V}\left(e^*\right)=e^{*T}\left(\left(\bar{A}_{hv}-\bar{L}_{hv}\bar{C}_h\right)^T P + P^T\left(\bar{A}_{hv}-\bar{L}_{hv}\bar{C}_h\right)\right)e^*.$$

Thus $\dot{V}\left(e^*\right)<0 \Leftrightarrow P^T\bar{A}_{hv}-P^T\bar{L}_{hv}\bar{C}_h+\left(*\right)<0$ or:

$$\begin{bmatrix} P_3^T A_h - P_3^T L_{hv} C_h + \left(*\right) & \left(*\right) \\ P_4^T A_h - P_4^T L_{hv} C_h + P_1 - E_v^T P_3 & -E_v^T P_4 - P_4^T E_v \end{bmatrix} < 0. \tag{4.8}$$

Remark 4.1. From inequality (4.8) is not possible to obtain LMI conditions because of the terms $P_3^T L_{hv} C_h$ and $P_4^T L_{hv} C_h$, thus (4.8) is a BMI problem. In (Guerra et al., 2004), a way to obtain LMIs is by fixing P_4 as $P_4 = P_3$. In (Guelton et al., 2008), the authors suggest a two-step algorithm: 1) design the gains L_{ij}, $i \in \{1,2,\ldots,r_a\}$, $j \in \{1,2,\ldots,r_e\}$ via the pole-placement technique; and 2) use (4.8) to verify the convergence of the estimation error.

The next subsection presents a way to overcome the BMI problem in (4.8).

4.1.2. Results

The first attempt to overcome the BMI problem in (4.8) is to consider a full observer gain, i.e., $\bar{L}_{hv}=\begin{bmatrix} L_{1hv}^T & L_{2hv}^T \end{bmatrix}^T$. Thus $P^T\bar{A}_{hv}-P^T\bar{L}_{hv}\bar{C}_h+\left(*\right)<0$ gives:

$$\begin{bmatrix} P_3^T A_h - \left(P_1 L_{1hv} + P_3^T L_{2hv}\right) C_h + \left(*\right) & \left(*\right) \\ P_4^T A_h - P_4^T L_{2hv} C_h + P_1 - E_v^T P_3 & -E_v^T P_4 - P_4^T E_v \end{bmatrix} < 0. \tag{4.9}$$

From here, a change of variables $N_{1hv} = P_1 L_{1hv} + P_3{}^T L_{2hv}$, $N_{2hv} = P_4^T L_{2hv}$ is possible and LMI constraints can be obtained. Nevertheless, the observer (4.4) with a full gain $\bar{L}_{hv}=\begin{bmatrix} L_{1hv}^T & L_{2hv}^T \end{bmatrix}^T$ reads:

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\hat{x}} \\ \ddot{\hat{x}} \end{bmatrix} = \begin{bmatrix} 0 & I \\ A_h & -E_v \end{bmatrix}\begin{bmatrix} \hat{x} \\ \dot{\hat{x}} \end{bmatrix} + \begin{bmatrix} 0 \\ B_h \end{bmatrix} u + \begin{bmatrix} L_{1hv} \\ L_{2hv} \end{bmatrix}\begin{bmatrix} C_h & 0 \end{bmatrix}\begin{bmatrix} x-\hat{x} \\ \dot{x}-\dot{\hat{x}} \end{bmatrix}. \quad (4.10)$$

The first row in (4.10) implies

$$\dot{\hat{x}} = \dot{\hat{x}} + L_{1hv} C_h \left(x - \hat{x} \right),$$

which is consistent only if $x - \hat{x} = 0$ or if $L_{1hv} C_h = 0$. When setting $L_{1hv} = 0$, the observer (4.4) is recovered. Hence, when using a full observer gain $\bar{L}_{hv} = \begin{bmatrix} L_{1hv}^T & L_{2hv}^T \end{bmatrix}^T$, the estimated state vector must be changed. Therefore, consider the following new estimated state vector:

$$\hat{\bar{x}} = \begin{bmatrix} \hat{x} \\ \beta \end{bmatrix}. \quad (4.11)$$

The definition of β depends on the observer under study and will be discussed later on. The main idea is that $\beta \to \dot{x}$ as $t \to \infty$. Based on the previous discussion, the following observer is proposed:

$$\bar{E}\dot{\hat{\bar{x}}} = \bar{A}_{hv}\hat{\bar{x}} + \bar{B}_h u + \bar{L}_{hv}\left(y - \hat{y} \right), \qquad \hat{y} = \bar{C}_h \hat{\bar{x}}, \quad (4.12)$$

where $\hat{\bar{x}} = \begin{bmatrix} \hat{x}^T & \beta^T \end{bmatrix}^T$ and $\bar{L}_{hv} = \begin{bmatrix} L_{1hv}^T & L_{2hv}^T \end{bmatrix}^T$. The extended estimation error is defined as:

$$\bar{e} = \bar{x} - \hat{\bar{x}} = \begin{bmatrix} x - \hat{x} \\ \dot{x} - \beta \end{bmatrix}, \quad (4.13)$$

and its dynamics are

$$\bar{E}\dot{\bar{e}} = \left(\bar{A}_{hv} - \bar{L}_{hv}\bar{C}_h \right)\bar{e}.$$

Consider the following Lyapunov function candidate:

$$V\left(\bar{e}\right) = \bar{e}^T \bar{E}^T \mathcal{P} \bar{e}, \qquad \bar{E}^T \mathcal{P} = \mathcal{P}^T \bar{E} \geq 0. \quad (4.14)$$

By considering that $\bar{E}^T \mathcal{P}$ is constant, the time-derivative of the Lyapunov function (4.14) is:

$$\dot{V}\left(\bar{e}\right) = \dot{\bar{e}}^T \bar{E}^T \mathcal{P} \bar{e} + \bar{e}^T \mathcal{P}^T \bar{E} \dot{\bar{e}}. \quad (4.15)$$

The following result can be stated.

Theorem 4.1. Consider the system (4.2) together with the observer (4.12). If there exist matrices $P_1 = P_1^T > 0$, P_3, P_4, $N_{1i_2j_1}$ and $N_{2i_2j_1}$, $i_1, i_2 \in \{1,2,\ldots,r_a\}$, $j_1 \in \{1,2,\ldots,r_e\}$ such that

$$\Upsilon_{i_1i_1}^{j_1} < 0, \quad \forall i_1, j_1; \qquad \frac{2}{r_a - 1}\Upsilon_{i_1i_1}^{j_1} + \Upsilon_{i_1i_2}^{j_1} + \Upsilon_{i_2i_1}^{j_1} < 0, \quad \forall j_1, i_1 \neq i_2, \tag{4.16}$$

holds with

$$\Upsilon_{i_1i_2}^{j_1} = \begin{bmatrix} P_3^T A_{i_1} - N_{1i_1j_1} C_{i_1} + (*) & (*) \\ P_4^T A_{i_1} - N_{2i_1j_1} C_{i_1} + P_1 - E_{j_1}^T P_3 & -P_4^T E_{j_1} - E_{j_1}^T P_4 \end{bmatrix},$$

then the estimation error e is asymptotically stable. The observer gains are recovered by $\bar{L}_{ij} = P^{-T}\bar{N}_{ij}$, $i \in \{1,2,\ldots,r_a\}$, $j \in \{1,2,\ldots,r_e\}$. Moreover, the final observer structure is

$$\begin{aligned} E_v \dot{\hat{x}} &= A_h \hat{x} + B_h u + \begin{bmatrix} E_v & I \end{bmatrix} \begin{bmatrix} L_{1hv} \\ L_{2hv} \end{bmatrix} (y - \hat{y}) \\ \hat{y} &= C_h \hat{x}. \end{aligned} \tag{4.17}$$

Proof. By taking $\mathcal{P} = P = \begin{bmatrix} P_1 & 0 \\ P_3 & P_4 \end{bmatrix}$, the condition $\dot{V}(\bar{e}) < 0 \Leftrightarrow P^T \bar{A}_{hv} - P^T \bar{L}_{hv} \bar{C}_h + (*) < 0$, together with the change of variables $\bar{N}_{hv} = P^T \bar{L}_{hv}$ gives:

$$\Upsilon_{hh}^{v} := \begin{bmatrix} P_3^T A_h - N_{1hv} C_h + (*) & (*) \\ P_4^T A_h - N_{2hv} C_h + P_1 - E_v^T P_3 & -P_4^T E_v - E_v^T P_4 \end{bmatrix} < 0. \tag{4.18}$$

By means of Lemma B.3 (4.18) gives (4.16). The proof of the regularity of P_4 is as follows: if the LMIs (4.16) hold, then inequality (4.18) also holds, which ensures $-P_4^T E_v - E_v^T P_4 < 0$. Since E_v is nonsingular $(E_v x_0 \neq 0, \ \forall x_0 \neq 0)$, let us assume P_4 is singular; then there exists $x_0 \neq 0$ such that $P_4 x_0 = 0$. Consequently for that $x_0 \neq 0$ it yields $x_0^T \left(-P_4^T E_v - E_v^T P_4\right) x_0 = 0$, which contradicts the condition $-P_4^T E_v - E_v^T P_4 < 0$. Thus if $\Upsilon_{hh}^{v} < 0$ is true, then P_4 is nonsingular.

The final observer form is obtained as follows: recall (4.12), i.e.,

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\hat{x}} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} 0 & I \\ A_h & -E_v \end{bmatrix} \begin{bmatrix} \hat{x} \\ \beta \end{bmatrix} + \begin{bmatrix} 0 \\ B_h \end{bmatrix} u + \begin{bmatrix} L_{1hv} C_h \\ L_{2hv} C_h \end{bmatrix} (x - \hat{x}),$$

or equivalently

$$\begin{aligned} \dot{\hat{x}} &= \beta + L_{1hv} C_h \left(x - \hat{x} \right) \\ E_v \beta &= A_h \hat{x} + B_h u + L_{2hv} C_h \left(x - \hat{x} \right). \end{aligned} \tag{4.19}$$

From (4.19), the definition of β arises:

$$\beta = \dot{\hat{x}} - L_{1hv} C_h \left(x - \hat{x} \right). \tag{4.20}$$

Finally, substituting the intermediate variable β into (4.19) gives

$$E_v \left(\dot{\hat{x}} - L_{1hv} C_h \left(x - \hat{x} \right) \right) = A_h \hat{x} + B_h u + L_{2hv} C_h \left(x - \hat{x} \right),$$

or

$$E_v \dot{\hat{x}} = A_h \hat{x} + B_h u + L_{2hv} C_h \left(x - \hat{x} \right) + E_v L_{1hv} C_h \left(x - \hat{x} \right), \tag{4.21}$$

which yields (4.17), thus concluding the proof. ■

Remark 4.2. If the LMI problem is feasible, it means that $\left(x - \hat{x} \right) \rightarrow 0$ and $\left(\dot{x} - \beta \right) \rightarrow 0$ as time goes to infinity.

Remark 4.3. Once the BMI problem in (4.8) is overcome, a more general observer structure can be achieved, thus relaxing the conditions given in Theorem 4.1. This will be shown in what follows.

Consider a non-PDC like observer of the form:

$$\begin{aligned} \bar{E} \dot{\hat{\bar{x}}} &= \bar{A}_{hv} \hat{\bar{x}} + \bar{B}_h u + P_h^{-T} \bar{L}_{hv} \left(y - \hat{y} \right) \\ \hat{y} &= \bar{C}_h \hat{\bar{x}}, \end{aligned} \tag{4.22}$$

where $\hat{\bar{x}} = \begin{bmatrix} \hat{x}^T & \beta^T \end{bmatrix}^T$ and $\bar{L}_{hv} = \begin{bmatrix} L_{1hv}^T & L_{2hv}^T \end{bmatrix}^T$. The structure of P_h is $P_h = \begin{bmatrix} P_1 & 0 \\ P_{3h} & P_{4h} \end{bmatrix}$, $P_1 = P_1^T > 0$, P_{4h} being a regular matrix, note that $P_h^{-1} = \begin{bmatrix} P_1^{-1} & 0 \\ -P_{4h}^{-1} P_{3h} P_1^{-1} & P_{4h}^{-1} \end{bmatrix}$. Recall the extended estimation error (4.13):

$$\bar{E} \dot{\bar{e}} = \left(\bar{A}_{hv} - P_h^{-T} \bar{L}_{hv} \bar{C}_h \right) \bar{e},$$

and the following result can be established.

Theorem 4.2. Consider the system (4.2) together with the observer (4.22). If there exist matrices $P_1 = P_1^T > 0$, P_{3i_2}, P_{4i_2}, $L_{1i_2 j_1}$ and $L_{2i_2 j_1}$, $i_1, i_2 \in \{1, 2, \ldots, r_a\}$, $j_1 \in \{1, 2, \ldots, r_e\}$ such that

$$\Upsilon_{i_1 i_1}^{j_1} < 0, \quad \forall i_1, j_1; \qquad \frac{2}{r_a - 1}\Upsilon_{i_1 i_1}^{j_1} + \Upsilon_{i_1 i_2}^{j_1} + \Upsilon_{i_2 i_1}^{j_1} < 0, \quad \forall j_1, i_1 \neq i_2, \tag{4.23}$$

hold with

$$\Upsilon_{i_1 i_2}^{j_1} = \begin{bmatrix} P_{3i_2}^T A_{i_1} - L_{1i_1 j_1} C_{i_1} + (*) & (*) \\ P_{4i_2}^T A_{i_1} - L_{2i_1 j_1} C_{i_1} + P_1 - E_{j_1}^T P_{3i_2} & -P_{4i_2}^T E_{j_1} - E_{j_1}^T P_{4i_2} \end{bmatrix},$$

then the estimation error e is asymptotically stable. Moreover, the final observer structure is

$$\begin{aligned} E_v \dot{\hat{x}} &= A_h \hat{x} + B_h u + \begin{bmatrix} E_v & I \end{bmatrix} P_h^{-T} \begin{bmatrix} L_{1hv} \\ L_{2hv} \end{bmatrix} (y - \hat{y}) \\ \hat{y} &= C_h \hat{x}. \end{aligned} \tag{4.24}$$

Proof. By setting $\mathcal{P} = P_h = \begin{bmatrix} P_1 & 0 \\ P_{3h} & P_{4h} \end{bmatrix}$, the time-derivative of the Lyapunov function (4.15) is $\dot{V}(\bar{e}) = \bar{e}^T P_h^T \left(\bar{A}_{hv} - P_h^{-T} \bar{L}_{hv} \bar{C}_h\right)\bar{e} + (*)$. Thus $\dot{V}(\bar{e}) < 0 \Leftrightarrow P^T \bar{A}_{hv} - \bar{L}_{hv} \bar{C}_h + (*) < 0$ or:

$$\Upsilon_{hh}^{v} := \begin{bmatrix} P_{3h}^T A_h - L_{1hv} C_h + (*) & (*) \\ P_{4h}^T A_h - L_{2hv} C_h + P_1 - E_v^T P_{3h} & -P_{4h}^T E_v - E_v^T P_{4h}^T \end{bmatrix} < 0. \tag{4.25}$$

By means of Lemma B.3, (4.25) gives (4.23). The proof of regularity of P_{4h} follows a procedure similar the one in Theorem 4.1. The final observer form is obtained as follows: recall (4.22), i.e.,

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\hat{x}} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} 0 & I \\ A_h & -E_v \end{bmatrix} \begin{bmatrix} \hat{x} \\ \beta \end{bmatrix} + \begin{bmatrix} 0 \\ B_h \end{bmatrix} u + P_h^{-T} \begin{bmatrix} L_{1hv} \\ L_{2hv} \end{bmatrix} (y - \hat{y}). \tag{4.26}$$

Define

$$\begin{bmatrix} N_{1hhv}(h) \\ N_{2hv}(h) \end{bmatrix} = \begin{bmatrix} P_1^{-1} & -P_1^{-1} P_{3h}^T P_{4h}^{-T} \\ 0 & P_{4h}^{-T} \end{bmatrix} \begin{bmatrix} L_{1hv} \\ L_{2hv} \end{bmatrix} = \begin{bmatrix} P_1^{-1}\left(L_{1hv} - P_{3h}^T P_{4h}^{-T} L_{2hv}\right) \\ P_{4h}^{-T} L_{2hv} \end{bmatrix}. \tag{4.27}$$

The subscripts h and v stand for the dependence on convex structures, while (h) means dependence on non-convex ones, for instance, $N_{2hv}(h)$ stands for $N_{2hv}(h) = \sum_{i_1=1}^{r_a} \sum_{j_1=1}^{r_e} h_{i_1}(z) v_{j_1}(z) \left(\sum_{i_2=1}^{r_a} h_{i_2}(z) P_{4i_2}\right)^{-T} L_{2i_1 j_1}$. Hence, (4.26) can be written as

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\hat{x}} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} 0 & I \\ A_h & -E_v \end{bmatrix}\begin{bmatrix} \hat{x} \\ \beta \end{bmatrix} + \begin{bmatrix} 0 \\ B_h \end{bmatrix} u + \begin{bmatrix} N_{1hhv}(h)C_h \\ N_{2hv}(h)C_h \end{bmatrix}(x-\hat{x}),$$

or equivalently

$$\begin{aligned} \dot{\hat{x}} &= \beta + N_{1hhv}(h)C_h(x-\hat{x}) \\ E_v\beta &= A_h\hat{x} + B_h u + N_{2hv}(h)C_h(x-\hat{x}). \end{aligned} \tag{4.28}$$

From (4.28), β is obtained as:

$$\beta = \dot{\hat{x}} - N_{1hhv}(h)C_h(x-\hat{x}). \tag{4.29}$$

Equation (4.28) by eliminating β gives

$$E_v\left(\dot{\hat{x}} - N_{1hhv}(h)C_h(x-\hat{x})\right) = A_h\hat{x} + B_h u + N_{2hv}(h)C_h(x-\hat{x}). \tag{4.30}$$

Substituting (4.27) and after some algebraic manipulations, (4.30) gives the final descriptor observer (4.24), thus concluding the proof. ■

Example 4.1. Consider a TS descriptor model (4.1) when $u=0$, $r_a = r_e = 2$ and matrices: $E_1 = \begin{bmatrix} 1.1 & -0.1 \\ -0.2+b & 1.5 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0.9 & -0.1 \\ 0.2 & 0.2 \end{bmatrix}$, $A_1 = \begin{bmatrix} -0.2 & -1 \\ -0.1 & -1.9 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1+a & 0.6 \\ 1.7 & -0.3 \end{bmatrix}$, $C_1 = \begin{bmatrix} 0 & -1 \end{bmatrix}$, and $C_2 = \begin{bmatrix} 0 & 0.6 \end{bmatrix}$. The real-valued parameters are defined as $a \in [-0.5 \quad 2.5]$ and $b \in [-1.5 \quad 1.5]$. Figure 4.1 shows the feasible regions for conditions (4.9) when $P_4 = P_3$ (see Remark 4.1) (O), for the conditions in Theorem 4.1 $(+)$, and therein Theorem 4.2 $(\times)$. As expected, the results obtained from Theorems 4.1 and 4.2 significantly outperform the ones obtained when fixing one of the decision variables.

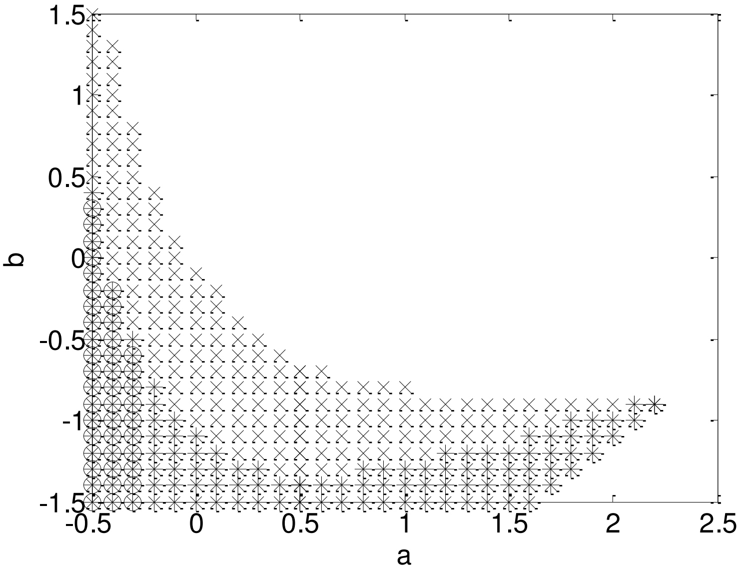

Figure 4.1. Feasible sets in Example 4.1. ◆

As it has been shown in Chapter 3, Section 3.1, Finsler's lemma allows decoupling the gains from the Lyapunov function. Therefore, a direct extension of the proposed observer in descriptor form is given by:

$$\bar{E}\dot{\hat{\bar{x}}} = \bar{A}_{hv}\hat{\bar{x}} + \bar{B}_h u + \bar{G}_{hv}^{-T}\bar{L}_{hhv}\left(y - \hat{y}\right), \qquad \hat{y} = \bar{C}_h\hat{\bar{x}}, \tag{4.31}$$

with $\bar{G}_{hv} = \begin{bmatrix} P_1 & 0 \\ G_{3hv} & G_{4hv} \end{bmatrix}$ and $\bar{L}_{hvv} = \begin{bmatrix} L_{1hvv} \\ L_{2hvv} \end{bmatrix}$. In this case, the dynamics of the error (4.13) are given by:

$$\bar{E}\dot{\bar{e}} = \left(\bar{A}_{hv} - \bar{G}_{hv}^{-T}\bar{L}_{hvv}\bar{C}_h\right)\bar{e} \quad \Leftrightarrow \quad \begin{bmatrix} \bar{A}_{hv} - \bar{G}_{hv}^{-T}\bar{L}_{hvv}\bar{C}_h & -I \end{bmatrix} \begin{bmatrix} \bar{e} \\ \bar{E}\dot{\bar{e}} \end{bmatrix} = 0. \tag{4.32}$$

Consider the Lyapunov function (4.14) with $\mathcal{P} = P_{hhv} = \begin{bmatrix} P_1 & 0 \\ P_{3hhv} & P_{4hhv} \end{bmatrix}$. The time-derivative of the Lyapunov function gives $\dot{V}(\bar{e}) = \dot{\bar{e}}^T\bar{E}^T P_{hhv}\bar{e} + \bar{e}^T P_{hhv}^T\bar{E}\dot{\bar{e}} < 0,$ which can be expressed as:

$$\dot{V}(\bar{e}) = \begin{bmatrix} \bar{e} \\ \bar{E}\dot{\bar{e}} \end{bmatrix}^T \begin{bmatrix} 0 & P_{hhv}^T \\ P_{hhv} & 0 \end{bmatrix} \begin{bmatrix} \bar{e} \\ \bar{E}\dot{\bar{e}} \end{bmatrix} < 0. \tag{4.33}$$

Taking $\mathcal{Q}=\begin{bmatrix} 0 & P_{hhv}^T \\ P_{hhv} & 0 \end{bmatrix}$ and $\mathcal{W}=\begin{bmatrix} \bar{A}_{hv}-\bar{G}_{hv}^{-T}\bar{L}_{hvv}\bar{C}_h & -I \end{bmatrix}$, via Finsler's lemma, the inequality constraint (4.33) together with the equality constraint (4.32) gives

$$\mathcal{M}\begin{bmatrix} \bar{A}_{hv}-\bar{G}_{hv}^{-T}\bar{L}_{hvv}\bar{C}_h & -I \end{bmatrix}+(*)+\begin{bmatrix} 0 & P_{hhv}^T \\ P_{hhv} & 0 \end{bmatrix}<0, \tag{4.34}$$

where $\mathcal{M}\in\mathbb{R}^{2n_x\times 2n_x}$ is a free matrix. Then, the following result can be stated.

Theorem 4.3. Consider the system (4.2) together with the observer (4.31). The estimation error e is asymptotically stable if there exist matrices $P_1=P_1^T>0$, $P_{3i_1i_2j_2}$, $P_{4i_1i_2j_2}$, $G_{3i_2j_2}$, $G_{4i_2j_2}$, $L_{1i_2j_1j_2}$ and $L_{2i_2j_1j_2}$, $i_1,i_2\in\{1,2,\ldots,r_a\}$, $j_1,j_2\in\{1,2,\ldots,r_e\}$ such that

$$\begin{aligned} &\Upsilon_{i_1i_1}^{j_1}<0, \quad \forall i_1,j_1; \\ &\frac{2}{r_a-1}\Upsilon_{i_1i_1}^{j_1j_1}+\Upsilon_{i_1i_2}^{j_1j_1}+\Upsilon_{i_2i_1}^{j_1j_1}<0, \quad \forall j_1, i_1\neq i_2, \\ &\frac{2}{r_e-1}\Upsilon_{i_1i_1}^{j_1j_1}+\Upsilon_{i_1i_1}^{j_1j_2}+\Upsilon_{i_1i_1}^{j_2j_1}<0, \quad \forall i_1, j_1\neq j_2, \\ &\frac{4}{(r_e-1)(r_a-1)}\Upsilon_{i_1i_1}^{j_1j_1}+\frac{2}{r_e-1}\left(\Upsilon_{i_1i_2}^{j_1j_1}+\Upsilon_{i_2i_1}^{j_1j_1}\right)+\frac{2}{r_a-1}\left(\Upsilon_{i_1i_1}^{j_1j_2}+\Upsilon_{i_1i_1}^{j_2j_1}\right) \\ &\qquad +\Upsilon_{i_1i_2}^{j_1j_2}+\Upsilon_{i_2i_1}^{j_1j_2}+\Upsilon_{i_1i_2}^{j_2j_1}+\Upsilon_{i_2i_1}^{j_2j_1}<0, \quad i_1\neq i_2, j_1\neq j_2, \end{aligned} \tag{4.35}$$

hold with

$$\Upsilon_{i_1i_2}^{j_1j_2}=\begin{bmatrix} G_{3i_1j_2}^TA_{i_1}-L_{1i_1j_1j_2}C_{i_1}+(*) & (*) & (*) & (*) \\ G_{4i_1j_2}^TA_{i_1}-L_{2i_1j_1j_2}C_{i_1}+P_1-E_{j_1}^TG_{3i_2j_2} & -G_{4i_1j_2}^TE_{j_1}+(*) & (*) & (*) \\ \varepsilon\left(G_{3i_1j_2}^TA_{i_1}-L_{1i_1j_1j_2}C_{i_1}\right) & \varepsilon\left(P_1-G_{3i_1j_2}^TE_{j_1}\right) & -2\varepsilon P_1 & (*) \\ \Gamma^{(4,1)} & \Gamma^{(4,2)} & 0 & \Gamma^{(4,4)} \end{bmatrix};$$

where $\Gamma^{(4,1)}=\varepsilon\left(G_{4i_1j_2}^TA_{i_1}-L_{2i_1j_1j_2}C_{i_1}\right)+E_{j_1}^T\left(P_{3i_1i_2j_2}-G_{3i_2j_2}\right)$, $\Gamma^{(4,2)}=-\varepsilon G_{4i_1j_2}^T+E_{j_1}^T\left(P_{4i_1i_2j_2}-G_{4i_2j_2}\right)$, and $\Gamma^{(4,4)}=-\varepsilon\left(G_{4i_1j_2}^TE_{j_1}+E_{j_1}^TG_{4i_1j_2}\right)$. Moreover, the final observer structure is

$$E_v\dot{\hat{x}}=A_h\hat{x}+B_hu+\begin{bmatrix} E_v & I \end{bmatrix}\bar{G}_{hv}^{-T}\begin{bmatrix} L_{1hhv} \\ L_{2hhv} \end{bmatrix}(y-\hat{y}), \qquad \hat{y}=C_h\hat{x}. \tag{4.36}$$

Proof. Recall (4.34). By selecting the free matrix as $\mathcal{M}=\begin{bmatrix} \bar{G}_{hv}^T \\ \varepsilon\mathcal{J}^{-T}\bar{G}_{hv}^T \end{bmatrix}$, congruence of (4.34) with a full-rank matrix $diag\begin{bmatrix} I, & \mathcal{J}^T \end{bmatrix}$ yields

$$\Upsilon_{hh}^{vv} := \begin{bmatrix} \bar{G}_{hv}^T \bar{A}_{hv} - \bar{L}_{hvv}\bar{C}_h + (*) & (*) \\ \varepsilon\left(\bar{G}_{hv}^T \bar{A}_{hv} - \bar{L}_{hvv}\bar{C}_h\right) + \mathcal{J}^T\left(P_{hhv} - \bar{G}_{hv}\right) & -\varepsilon\bar{G}_{hv}^T\mathcal{J} + (*) \end{bmatrix} < 0, \tag{4.37}$$

which by setting $\mathcal{J} = \begin{bmatrix} I & -P_1^{-1}G_{3hv}^T E_v \\ 0 & E_v \end{bmatrix}$ and applying the Lemma B.4 gives (4.35). The regularity of matrix $\bar{G}_{hv}$ is guaranteed as follows: recall that $\bar{G}_{hv} = \begin{bmatrix} P_1 & 0 \\ G_{3hv} & G_{4hv} \end{bmatrix}$ with $P_1 > 0$. If the LMI conditions (4.35) hold, it implies $-G_{4hv}^T E_v - E_v^T G_{4hv} < 0$. Since E_v is nonsingular $\left(E_v x_0 \neq 0, \quad \forall x_0 \neq 0\right)$, let us assume that G_{4hv} is singular; then it exists $x_0 \neq 0$ such that $G_{4hv}x_0 = 0$; and for that $x_0 \neq 0$ it yields $x_0^T\left(-G_{4hv}^T E_v - E_v^T G_{4hv}\right)x_0 = 0$, which contradicts the condition $-G_{4hv}^T E_v - E_v^T G_{4hv} < 0$. Thus, if $\Upsilon_{hh}^{vv} < 0$ is true, then $\bar{G}_{hv}$ is nonsingular.

The final form of the observer (4.36) can be obtained via manipulations similar to those in Theorem 4.2. ■

Remark 4.4. The conditions in Theorem 4.3 are LMIs when the scalar parameter $\varepsilon > 0$ is fixed. A logarithmically spaced family of values $\varepsilon \in \left\{10^{-6}, 10^{-5}, \ldots, 10^{6}\right\}$ (Jaadari et al., 2012; Oliveira et al., 2011; Shaked, 2001) can be used, see Remark 3.2.

Corollary 4.1. The results given by Theorem 4.2 are always included in those of Theorem 4.3 under the same relaxation scheme.

Proof. Suppose conditions of Theorem 4.2 hold; thus

$$\Upsilon_{hhv}^{Th4.2} := \begin{bmatrix} P_{3h}^T A_h - L_{1hv}C_h + (*) & (*) \\ P_{4h}^T A_h - L_{2hv}C_h + P_1 - E_v^T P_{3h} & -P_{4h}^T E_v - E_v^T P_{4h}^T \end{bmatrix}.$$

Choose for Theorem 4.3: $P_{hhv} = G_{hv} = P_h^{Th4.2}$ and $\bar{L}_{hvv} = \bar{L}_{hv}^{Th4.2}$. The conditions in Theorem 4.3 become

$$\left[\begin{array}{c:c} \Upsilon_{hhv}^{Th4.2} & (*) \\ \hdashline \varepsilon\Phi_{hhv} & -\varepsilon\begin{bmatrix} 2P_1 & 0 \\ 0 & P_{4h}^T E_v + E_v^T P_{4h} \end{bmatrix} \end{array}\right] < 0, \tag{4.38}$$

with $\Phi_{hhv} = \begin{bmatrix} P_{3h}^T A_h - L_{1hv} C_h & P_1 - P_{3h}^T E_v \\ P_{4h}^T A_h - L_{2hv} C_h & -P_{4h}^T E_v \end{bmatrix}$. Since the conditions in Theorem 4.2 hold, $\begin{bmatrix} 2P_1 & 0 \\ 0 & P_{4h}^T E_v + E_v^T P_{4h} \end{bmatrix} > 0$ also holds, thus via Schur complement (4.38) is equivalent to

$$\Upsilon_{hhv}^{Th4.2} + \varepsilon \Phi_{hhv}^T \begin{bmatrix} 2P_1 & 0 \\ 0 & P_{4h}^T E_v + E_v^T P_{4h} \end{bmatrix}^{-1} \Phi_{hhv} < 0. \tag{4.39}$$

If Theorem 4.2 holds, it always exists a sufficiently small $\varepsilon > 0$ such that (4.39) is true, (4.38) is also true and Theorem 4.3 holds. ■

Example 4.2. Recall Example 4.1. Corollary 4.1 is illustrated when the LMI conditions in Theorem 4.2 (O) and Theorem 4.3 $(\times)$ are implemented. From Figure 4.2 it can be seen that conditions in Theorem 4.3 are more relaxed (a larger solution set is obtained) than those in Theorem 4.2.

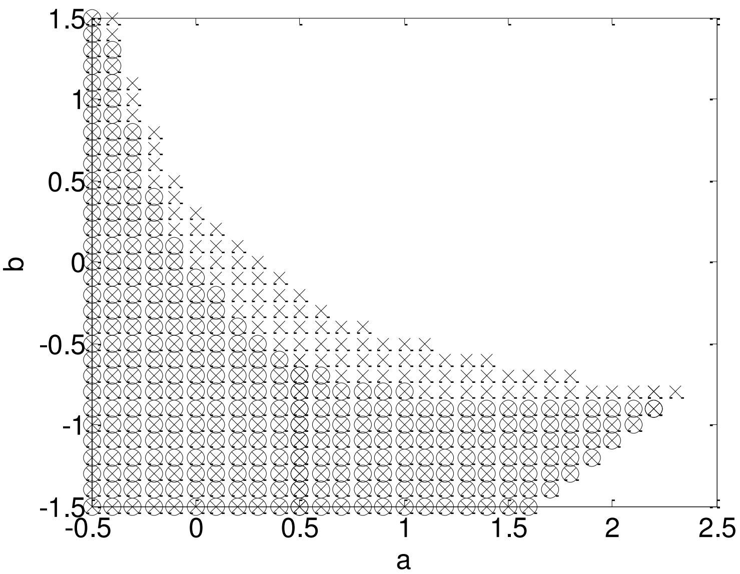

Figure 4.2. Feasible sets for Theorem 4.2 (O) and Theorem 4.3 (×). ◆

4.1.3. Unknown input observers

In (Guelton et al., 2008), the observer (4.4) has been extended to estimate unknown inputs; however it provides BMI conditions (see Remark 4.1). This section proposes to give LMI conditions via a simple extension of the previous work.

Consider the TS descriptor model:

$$E_v\dot{x} = A_h x + B_h x + M_h d$$
$$y = C_h x + G_h d, \tag{4.40}$$

where $d(t) \in \mathbb{R}^{n_d}$ stands for the unknown input vector and $M_h = \sum_{i=1}^{r_a} h_i(z) M_i \in \mathbb{R}^{n_x \times n_d}$. The goal is to design an observer capable to estimate both the state $x(t)$ and the unknown input $d(t)$. To this end, assume that the unknown inputs are given by an exo-system $\dot{d} = Sd$, where $S \in \mathbb{R}^{n_d \times n_d}$ is a known matrix. Using an extended vector $x^e = \begin{bmatrix} x^T & d^T \end{bmatrix}^T \in \mathbb{R}^{n_x + n_d}$, the TS descriptor model is expressed as:

$$E_v^e \dot{x}^e = A_h^e x^e + B_h^e u, \qquad y = C_h^e x^e, \tag{4.41}$$

with $E_v^e = \begin{bmatrix} E_v & 0 \\ 0 & I_{n_d} \end{bmatrix} \in \mathbb{R}^{(n_x+n_d)\times(n_x+n_d)}$, $B_h^e = \begin{bmatrix} B_h \\ 0 \end{bmatrix} \in \mathbb{R}^{(n_x+n_d)\times n_u}$, $A_h^e = \begin{bmatrix} A_h & M_h \\ 0 & S \end{bmatrix} \in \mathbb{R}^{(n_x+n_d)\times(n_x+n_d)}$, and $C_h^e = \begin{bmatrix} C_h & G_h \end{bmatrix} \in \mathbb{R}^{n_y \times (n_x+n_d)}$. A classical approach is to use an extended descriptor redundancy on (4.41) by defining $\bar{x} = \begin{bmatrix} x^e \\ \dot{x}^e \end{bmatrix} \in \mathbb{R}^{(2n_x+2n_d)\times(2n_x+2n_d)}$ the model writes directly:

$$\bar{E}\dot{\bar{x}} = \bar{A}_{hv}\bar{x} + \bar{B}_h u, \qquad y = \bar{C}_h \bar{x}, \tag{4.42}$$

where $\bar{E} = \begin{bmatrix} I_{n_x+n_d} & 0 \\ 0 & 0_{n_x+n_d} \end{bmatrix}$, $\bar{A}_{hv} = \begin{bmatrix} 0 & I_{n_x+n_d} \\ A_h^e & -E_v^e \end{bmatrix}$, $\bar{B}_h = \begin{bmatrix} 0 \\ B_h^e \end{bmatrix}$, and $\bar{C}_h = \begin{bmatrix} C_h^e & 0 \end{bmatrix}$. Then, consider the following nonlinear observer in TS descriptor form:

$$\bar{E}\dot{\hat{\bar{x}}} = \bar{A}_{hv}\hat{\bar{x}} + \bar{B}_h u + P^{-T}\bar{L}_{hv}\left(y - \hat{y}\right), \qquad \hat{y} = \bar{C}_h \hat{\bar{x}}, \tag{4.43}$$

with a full observer gain $\bar{L}_{hv} = \begin{bmatrix} L_{1hv}^T & L_{2hv}^T \end{bmatrix}^T$, L_{1hv}, $L_{2hv} \in \mathbb{R}^{(n_x+n_d)\times n_y}$. The matrix P will be defined later on. Define the estimated state as:

$$\hat{\bar{x}} = \begin{bmatrix} \hat{x}^e \\ \beta^e \end{bmatrix} \in \mathbb{R}^{(2n_x+2n_d)\times(2n_x+2n_d)}. \tag{4.44}$$

and the estimation error $e = x^e - \hat{x}^e$. As previously, β^e plays a role equivalent to $\dot{\hat{x}}^e$, therefore:

$$\bar{e} = \bar{x} - \hat{\bar{x}} = \begin{bmatrix} x^e - \hat{x}^e \\ \dot{x}^e - \beta^e \end{bmatrix} = \begin{bmatrix} x - \hat{x} \\ d - \hat{d} \\ \hdashline \dot{x} - \beta_1^e \\ \dot{d} - \beta_2^e \end{bmatrix}, \tag{4.45}$$

whose dynamics are given by

$$\bar{E}\dot{\bar{e}} = \left(\bar{A}_{hv} - P^{-T}\bar{L}_{hv}\bar{C}_h\right)\bar{e}.$$

Consider the following Lyapunov function:

$$V\left(\bar{e}\right) = \bar{e}^T \bar{E}^T P \bar{e}, \qquad \bar{E}^T P = P^T \bar{E} \geq 0, \tag{4.46}$$

with $P = \begin{bmatrix} P_1 & 0 \\ P_3 & P_4 \end{bmatrix} \in \mathbb{R}^{(2n_x+2n_d)\times(2n_x+2n_d)}$, $P_1 = P_1^T > 0$, P_4 being a full-rank matrix. The following result can be stated:

Theorem 4.4. Consider the model (4.42) together with the observer (4.43). If there exist matrices $P_1 = P_1^T > 0$, P_3, P_4, $L_{1i_2j_1}$, and $L_{2i_2j_1}$, $i_2 \in \{1,2,\ldots,r_a\}$, $j_1 \in \{1,2,\ldots,r_e\}$ such that

$$\Upsilon_{i_1i_1}^{j_1} < 0, \quad \forall i_1, j_1; \qquad \frac{2}{r_a - 1}\Upsilon_{i_1i_1}^{j_1} + \Upsilon_{i_1i_2}^{j_1} + \Upsilon_{i_2i_1}^{j_1} < 0, \quad \forall j_1, i_1 \neq i_2, \tag{4.47}$$

hold with

$$\Upsilon_{i_1i_2}^{j_1} = \begin{bmatrix} P_3^T A_{i_1}^e - L_{1i_2j_1} C_{i_1}^e + (*) & (*) \\ P_4^T A_{i_1}^e - L_{2i_2j_1} C_{i_1}^e + P_1 - \left(E_{j_1}^e\right)^T P_3 & -P_4^T E_{j_1}^e + (*) \end{bmatrix},$$

then the estimation error e is asymptotically stable and, the observer structure is

$$\begin{aligned} E_v^e \dot{\hat{x}}^e &= A_h^e \hat{x}^e + B_h^e u + \begin{bmatrix} E_v^e & I \end{bmatrix} P^{-T} \begin{bmatrix} L_{1hv} \\ L_{2hv} \end{bmatrix} \left(y - \hat{y}\right) \\ \hat{y} &= C_h^e \hat{x}^e. \end{aligned} \tag{4.48}$$

Proof. The time-derivative of the Lyapunov function (4.46) produces $\dot{V}\left(\bar{e}\right) = \bar{e}^T P^T \left(\bar{A}_{hv} - P^{-T}\bar{L}_{hv}\bar{C}_h\right)\bar{e} + (*)$; hence $\dot{V}\left(\bar{e}\right) < 0 \Leftrightarrow P^T\bar{A}_{hv} - \bar{L}_{hv}\bar{C}_h + (*) < 0$ or

$$\Upsilon_{hh}^v := \begin{bmatrix} P_3^T A_h^e - L_{1hv} C_h^e + (*) & (*) \\ P_4^T A_h^e - L_{2hv} C_h^e + P_1 - \left(E_v^e\right)^T P_3 & -P_4^T E_v^e + (*) \end{bmatrix} < 0, \tag{4.49}$$

which leads to conditions (4.47) via Lemma B.3. The final observer form (4.48) is computed as follows: define

$$\begin{bmatrix} N_{1hv} \\ N_{2hv} \end{bmatrix} = P^{-T}\bar{L}_{hv} = \begin{bmatrix} P_1^{-1}\left(L_{1hv} - P_3^T P_4^{-T} L_{2hv}\right) \\ P_4^{-T} L_{2hv} \end{bmatrix}. \tag{4.50}$$

The augmented observer (4.43) produces

$$\begin{bmatrix} I_{n_x+n_d} & 0 \\ 0 & 0_{n_x+n_d} \end{bmatrix}\begin{bmatrix} \dot{\hat{x}}^e \\ \dot{\beta}^e \end{bmatrix} = \begin{bmatrix} 0 & I_{n_x+n_d} \\ A_h^e & -E_v^e \end{bmatrix}\begin{bmatrix} \hat{x}^e \\ \beta^e \end{bmatrix} + \begin{bmatrix} 0 \\ B_h^e \end{bmatrix} u + \begin{bmatrix} N_{1hv} C_h^e \\ N_{2hv} C_h^e \end{bmatrix}\left(x^e - \hat{x}^e\right),$$

which is equivalent to:

$$\begin{aligned} \dot{\hat{x}}^e &= \beta^e + N_{1hv} C_h^e \left(x^e - \hat{x}^e\right) \\ E_v^e \beta^e &= A_h^e \hat{x}^e + B_h^e u + N_{2hv} C_h^e \left(x^e - \hat{x}^e\right). \end{aligned} \tag{4.51}$$

Using $\beta^e = \dot{\hat{x}}^e - N_{1h} C_h^e \left(x^e - \hat{x}^e\right)$, (4.51) gives:

$$E_v^e\left(\dot{\hat{x}}^e - N_{1hv}\left(y - \hat{y}\right)\right) = A_h^e \hat{x}^e + B_h^e u + N_{2hv}\left(y - \hat{y}\right). \tag{4.52}$$

Rearranging the terms, we have:

$$\begin{aligned} E_v^e \dot{\hat{x}}^e &= A_h^e \hat{x}^e + B_h^e u + \left(E_v^e N_{1hv} + N_{2hv}\right)\left(y - \hat{y}\right) \\ &= A_h^e \hat{x}^e + B_h^e u + \begin{bmatrix} E_v^e & I \end{bmatrix}\begin{bmatrix} N_{1hv} \\ N_{2hv} \end{bmatrix}\left(y - \hat{y}\right), \end{aligned} \tag{4.53}$$

substituting (4.50) into (4.53) leads to (4.48), thus and concludes the proof. ■

As it can be inferred, more relaxed results can be obtained by introducing MFs in the matrices P_3 and P_4; therefore $P_{3h} = \sum_{i=1}^{r_a} h_i\left(z\right) P_{3i}$ and $P_{4h} = \sum_{i=1}^{r_a} h_i\left(z\right) P_{4i}$ can be introduced without altering the number of LMIs to be verified. Effectively, the PDC-like observer (4.43) becomes:

$$\bar{E}\dot{\hat{\bar{x}}} = \bar{A}_{hv}\hat{\bar{x}} + \bar{B}_h u + P_h^{-T}\bar{L}_{hv}\left(y - \hat{y}\right), \qquad \hat{y} = \bar{C}_h \hat{\bar{x}}, \tag{4.54}$$

with $P_h = \begin{bmatrix} P_1 & 0 \\ P_{3h} & P_{4h} \end{bmatrix} \in \mathbb{R}^{(2n_x+2n_d)\times(2n_x+2n_d)}$. Observer (4.54) is a non-PDC-like observer.

Corollary 4.2. Consider the model (4.42) together with the observer (4.54). If there exists matrices $P_1 = P_1^T > 0$, P_{3i_2}, P_{4i_2}, $L_{1i_2j_1}$, and $L_{2i_2j_1}$, $i_2 \in \{1,2,\ldots,r_a\}$, $j_1 \in \{1,2,\ldots,r_e\}$ such that

$$\Upsilon_{i_1 i_1}^{j_1} < 0, \quad \forall i_1, j_1; \qquad \frac{2}{r_a - 1}\Upsilon_{i_1 i_1}^{j_1} + \Upsilon_{i_1 i_2}^{j_1} + \Upsilon_{i_2 i_1}^{j_1} < 0, \quad \forall j_1, i_1 \neq i_2, \tag{4.55}$$

hold with

$$\Upsilon_{i_1 i_2}^{j_1} = \begin{bmatrix} P_{3i_2}^T A_{i_1}^e - L_{1i_2 j_1} C_{i_1}^e + (*) & (*) \\ P_{4i_2}^T A_{i_1}^e - L_{2i_2 j_1} C_{i_1}^e + P_1 - \left(E_{j_1}^e\right)^T P_{3i_2} & -P_{4i_2}^T E_{j_1}^e + (*) \end{bmatrix},$$

then the estimation error e is asymptotically stable.

Proof. It follows a similar procedure as in Theorem 4.4. ■

Remark 4.5. The results given in Theorem 4.4 and Corollary 4.2 can be extended directly to the Proportional Integral (PI) or to Proportional Multi-Integral (PMI) observer. For a PI observer, set $S = 0$; while for the PMI observer consider $d^{(m)} = 0$, where $d^{(m)}$ is the mth-derivative of the unknown input.

Example 4.3. Consider the following nonlinear descriptor model:

$$E(x)\dot{x} = A(x)x + Bu + M(x)d, \qquad y = C(x)x + Gd, \tag{4.56}$$

with $E(x) = \begin{bmatrix} 0.87 & 0.33 + 0.5(1-2\eta) \\ 0.53 - \delta(1-2\eta) & 0.95 \end{bmatrix}$, $A(x) = \begin{bmatrix} -0.81 & 0.83 + \delta\cos(x_1) \\ -0.74 & 0.57 \end{bmatrix}$,

$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $M(x) = \begin{bmatrix} 2 & 1 + \delta\cos(x_1) \\ 1 & -0.5 \end{bmatrix}$, $C(x) = \begin{bmatrix} 1.5 + 0.5\cos(x_1) & 0 \\ 0 & 0.1 \end{bmatrix}$, and

$G(x) = \begin{bmatrix} 1 & -0.5 \\ 0.2 + 0.5\cos(x_1) & -0.4 \end{bmatrix}$, $\eta = 1/(1 + x_1^2)$, $\delta > 0$ a real-valued parameter known a priori. Note that $\det(E(x)) \neq 0$ for all x_1. Using the sector nonlinearity a TS descriptor model can be constructed with $r_a = 2$ due to the terms $\cos(x_1)$; $r_e = 2$ due to $\eta = 1/(1 + x_1^2)$. In total 4 vertices are needed to exactly represent the original nonlinear system in $\Omega_x = \{x \in \mathbb{R}^2\}$:

$$\sum_{k=1}^{2} v_k(z) E_k \dot{x} = \sum_{i=1}^{2} h_i(z)(A_i x + Bu + M_i d), \qquad y = \sum_{i=1}^{2} h_i(z)(C_i x + G_i d), \tag{4.57}$$

with $E_1 = \begin{bmatrix} 0.87 & -0.17 \\ 0.53 + \delta & 0.95 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0.87 & 0.83 \\ 0.53 - \delta & 0.95 \end{bmatrix}$, $A_1 = \begin{bmatrix} -0.81 & 0.83 + \delta \\ -0.74 & 0.57 \end{bmatrix}$,

$A_2 = \begin{bmatrix} -0.81 & 0.83 - \delta \\ -0.74 & 0.57 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $M_1 = \begin{bmatrix} 2 & 1 + \delta \\ 1 & -0.5 \end{bmatrix}$, $M_2 = \begin{bmatrix} 2 & 1 - \delta \\ 1 & -0.5 \end{bmatrix}$, $C_1 = \begin{bmatrix} 2 & 0 \\ 0 & -0.1 \end{bmatrix}$,

$C_2 = \begin{bmatrix} 1 & 0 \\ 0 & -0.1 \end{bmatrix}$, $G_1 = \begin{bmatrix} 1 & -0.5 \\ 0.7 & -0.4 \end{bmatrix}$, and $G_2 = \begin{bmatrix} 1 & -0.5 \\ -0.3 & -0.4 \end{bmatrix}$. Consider that x_1 is available. The MFs are $v_1 = 1/\left(1+x_1^2\right)$, $v_2 = 1 - v_1$, $h_1 = 0.5\left(\cos\left(x_1\right)+1\right)$, and $h_2 = 1 - h_1$. The dynamics of the unknown input are given by the exo-system $\dot{d} = \begin{bmatrix} 0 & 0.5 \\ -0.5 & 0 \end{bmatrix} d$. Note that in (Guelton et al., 2008; Ichalal et al., 2009), $\dot{d} = 0$ or $d^{(m)} = 0$ is considered. The exo-system in this example generates sinusoidal signals. In order to show the effectiveness of Theorem 4.4 and Corollary 4.2, two comparisons are done:

1. Comparing the conditions in Theorem 4.4 and Corollary 4.2 to those in (Ichalal et al., 2009): In order to use the methodology given in (Ichalal et al., 2009), a standard TS representation is needed. After the inversion of the matrix $E(x)$, a TS model with 8 vertices is obtained. By choosing common matrices as follows: $C = C_1$, $G = G_1$, and $S = 0$ ($\dot{d} = 0$), the following results were obtained: Theorem 4.4 and Corollary 4.2 were feasible up to the value $\delta = 0.91$; conditions in Theorem 1 in (Ichalal et al., 2009) were feasible until $\delta = 0.53$ (the larger δ is the more relaxed the approach is).
2. Comparing the conditions given in Theorem 2 (Guelton et al., 2008), Theorem 4.4 and Corollary 4.2. The aim is to design an UI observer for the TS descriptor model (4.57), considering the exo-system. For the conditions of Theorem 2 (Guelton et al., 2008), the maximum value of δ for which feasible solutions were found is $\delta = 1.16$. In case of Theorem 4.4, the maximum value of δ for which the conditions were found feasible is $\delta = 1.66$; while for Corollary 4.2 the maximum is $\delta = 1.84$.

When setting $\delta = 1.55$ and $\dot{d} = Sd$, conditions in Theorem 4.4 were found feasible. Figures 4.3 and 4.4 illustrate the trajectories with initial conditions $x^e(0) = \begin{bmatrix} -0.4 & 0.5 & 0.1 & -0.1 \end{bmatrix}$. Some matrices of the solution are:

$$P_1 = \begin{bmatrix} 0.14 & 0.01 & -1.54 & 0.33 \\ 0.01 & 0.02 & -0.21 & 0.29 \\ -1.54 & -0.21 & 20.96 & -2.79 \\ 0.33 & 0.29 & -2.79 & 22.44 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 0.15 & 0.02 & 0.07 & 0.06 \\ 0.04 & 0.09 & -0.02 & -0.02 \\ -0.08 & -0.05 & 0.78 & 0.00 \\ -0.01 & 0.01 & -0.05 & 0.62 \end{bmatrix},$$

$$\bar{L}_{11}=\begin{bmatrix}0.39 & -6.49\\ -0.62 & -4.04\\ 2.17 & -4.17\\ -1.92 & -2.21\\ 0.08 & -1.18\\ -0.11 & -0.81\\ -0.09 & 0.26\\ -0.33 & -0.93\end{bmatrix},\bar{L}_{12}=\begin{bmatrix}0.43 & -5.99\\ -0.64 & -3.89\\ 1.93 & -3.74\\ -1.88 & -2.06\\ 0.17 & 0.18\\ 0.16 & 0.34\\ -0.06 & 0.13\\ -0.40 & -0.59\end{bmatrix},\bar{L}_{21}=\begin{bmatrix}0.92 & -1.71\\ 0.25 & -2.36\\ -1.62 & -2.66\\ -0.79 & -0.01\\ -0.32 & -0.55\\ -0.45 & -0.46\\ -0.01 & 0.74\\ -0.81 & -0.59\end{bmatrix},\bar{L}_{22}=\begin{bmatrix}0.90 & -2.33\\ 0.26 & -2.40\\ -1.56 & -2.77\\ -0.85 & 0.00\\ 0.27 & 0.25\\ 0.31 & -0.16\\ 0.03 & 0.55\\ -0.81 & -0.45\end{bmatrix}.$$

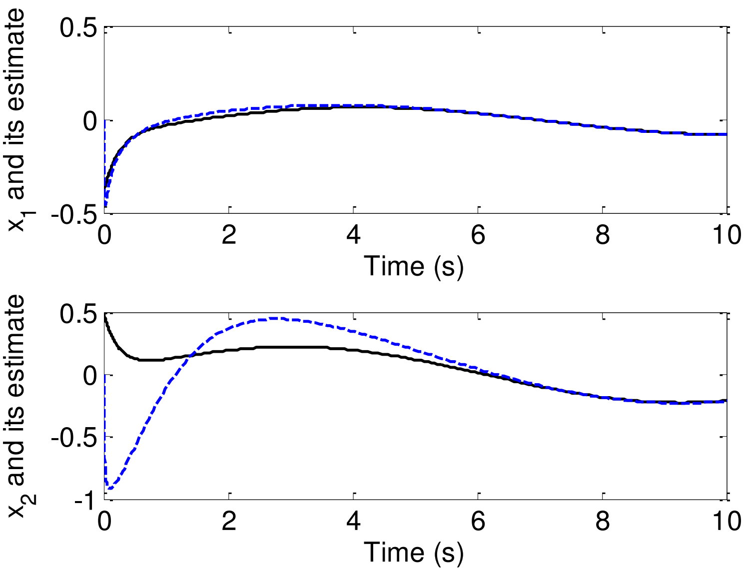

Figure 4.3. States in (black lines) and their estimates (blue-dashed lines) for Example 4.3 for $\delta=1.55$.

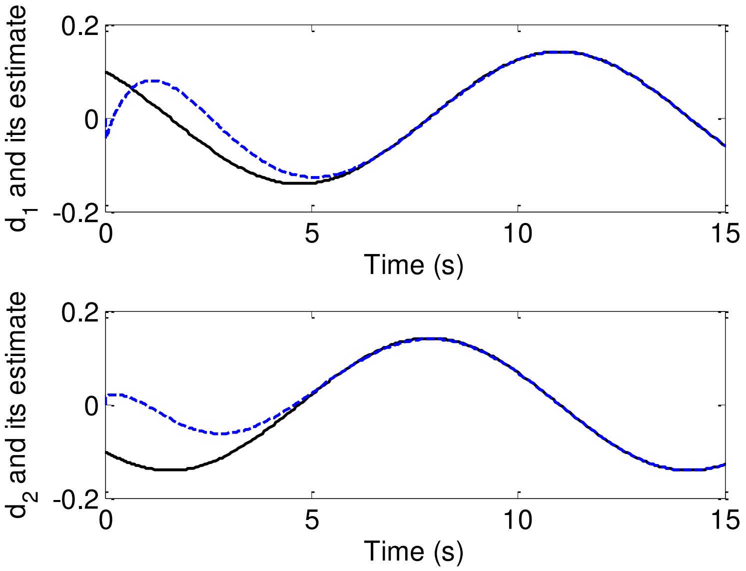

Figure 4.4. Unknown inputs (black lines) and their estimates (blue-dashed lines) for Example 4.3 for $\delta=1.55$. ◆

4.2. Discrete-time TS descriptor models

This section is dedicated to observer design for discrete-time nonlinear descriptor models using Lyapunov functions: quadratic, non-quadratic, and delayed-non-quadratic.. Throughout the section, the achieved improvements are illustrated on numerical examples.

4.2.1. Problem statement

Consider the following discrete-time TS descriptor model:

$$E_v x_{k+1} = A_h x_k + B_h u_k, \qquad y_k = C_h x_k. \tag{4.58}$$

Recall that $E_v = \sum_{j=1}^{r_e} v_j\left(z(k)\right)E_j$ is regular in the considered compact set of the state space Ω. Recall that, to the best of our knowledge, there are no results in the literature for systems of the form (4.58). The aim is to make the estimation error $e = x - \hat{x}$ converge to zero as $t \to \infty$. To this end the following generic observer is proposed:

$$\begin{aligned} E_v \hat{x}_{k+1} &= A_h \hat{x}_k + B_h u_k + \mathcal{G}\left(y - \hat{y}\right) \\ y_k &= C_h \hat{x}_k, \end{aligned} \tag{4.59}$$

where the observer gain $\mathcal{G} \in \mathbb{R}^{n_x \times n_y}$ may change according to the approach under study. Consider the error dynamics as follows:

$$E_v e_{k+1} = \left(A_h - \mathcal{G}C_h\right)e_k \quad \Leftrightarrow \quad \begin{bmatrix} A_h - \mathcal{G}C_h & -E_v \end{bmatrix}\begin{bmatrix} e_k \\ e_{k+1} \end{bmatrix} = 0. \tag{4.60}$$

4.2.2. Results

In order to investigate the stability of the estimation error (4.60)consider the following Lyapunov function

$$V\left(e_k\right) = e_k^T \mathcal{P} e_k > 0, \qquad \mathcal{P} = \mathcal{P}^T > 0, \tag{4.61}$$

where $\mathcal{P}$ may be constant (quadratic approach) or depend on MFs (non-quadratic approach). The variation of the Lyapunov function (4.61) gives

$$\begin{aligned} \Delta V\left(e_k\right) &= e_{k+1}^T \mathcal{P}_+ e_{k+1} - e_k^T \mathcal{P} e_k \\ &= \begin{bmatrix} e_k \\ e_{k+1} \end{bmatrix}^T \begin{bmatrix} -\mathcal{P} & 0 \\ 0 & \mathcal{P}_+ \end{bmatrix}\begin{bmatrix} e_k \\ e_{k+1} \end{bmatrix} < 0. \end{aligned} \tag{4.62}$$

Denote $\mathcal{X}=\begin{bmatrix} e_k \\ e_{k+1} \end{bmatrix}$, $\mathcal{Q}=\begin{bmatrix} -\mathcal{P} & 0 \\ 0 & \mathcal{P}_+ \end{bmatrix}$, and $\mathcal{W}=\begin{bmatrix} A_h-\mathcal{G}C_h & -E_v \end{bmatrix}$; via Finsler's lemma the inequality (4.62) together with the equality (4.60) yields

$$\mathcal{M}\begin{bmatrix} A_h-\mathcal{G}C_h & -E_v \end{bmatrix}+(*)+\begin{bmatrix} -\mathcal{P} & 0 \\ 0 & \mathcal{P}_+ \end{bmatrix}<0, \tag{4.63}$$

where $\mathcal{M}\in\mathbb{R}^{2n_x\times n_x}$ is a free matrix to be defined later on. From (4.63) many results can be derived, e.g., selecting $\mathcal{G}=L_{hv}$ and $\mathcal{P}=P$ gives a PDC-like observer designed via a quadratic Lyapunov function; setting $\mathcal{G}=G_h^{-1}L_{hv}$ and $\mathcal{P}=P_h$ yields a non-PDC-like observer designed under a non-quadratic Lyapunov function. In what follows three approaches are considered: the quadratic (Q) approach, the non-quadratic (NQ) approach, and the delayed non-quadratic (DNQ) approach.

4.2.2.1. Quadratic Lyapunov function

If a common quadratic Lyapunov function is used, the following result can be stated.

Theorem 4.5. The estimation error dynamics in (4.60) with $\mathcal{G}=L_{hv}$ is asymptotically stable if there exist matrices $P=P^T>0$ and $N_{i_2j_1}$, for $i_1,i_2\in\{1,2,\ldots,r_a\}$, $j_1\in\{1,2,\ldots,r_e\}$ such that:

$$\Upsilon_{i_1i_1}^{j_1}<0,\ \forall i_1,j_1; \qquad \frac{2}{r_a-1}\Upsilon_{i_1i_1}^{j_1}+\Upsilon_{i_1i_2}^{j_1}+\Upsilon_{i_2i_1}^{j_1}<0,\ \forall j_1,\ i_i\neq i_2, \tag{4.64}$$

hold with

$$\Upsilon_{i_1i_2}^{j_1}=\begin{bmatrix} -P & (*) \\ PA_{i_1}-N_{i_2j_1}C_{i_1} & -PE_{j_1}-E_{j_1}^TP+P \end{bmatrix}. \tag{4.65}$$

The observer gains are recovered with $L_{ij}=P^{-1}N_{ij}$, $i\in\{1,2,\ldots,r_a\}$, $j\in\{1,2,\ldots,r_e\}$. The final observer structure is

$$\begin{aligned} E_v\hat{x}_{k+}&=A_h\hat{x}_k+B_hu_k+L_{hv}\left(y_k-\hat{y}_k\right) \\ \hat{y}_k&=C_h\hat{x}_k. \end{aligned} \tag{4.66}$$

Proof. Recall (4.63). Selecting the Lyapunov function as $V(e_k)=e_k^TPe_k$, the observer gain as $\mathcal{G}=L_{hv}$, $\mathcal{M}=\begin{bmatrix} 0 & P \end{bmatrix}^T$, and defining $N_{hv}=PL_{hv}$ gives:

$$\Upsilon_{hh}^{v} := \begin{bmatrix} -P & (*) \\ PA_h - N_{hv}C_h & -PE_v - E_v^T P + P \end{bmatrix} < 0, \tag{4.67}$$

which by applying Lemma B.3 yields (4.64), thus concluding the proof. ■

By using a different but still quadratic Lyapunov function $V(e_k) = e_k^T F^T P^{-1} F e_k$, a more relaxed result can be obtained; this is summarized in the following theorem.

Theorem 4.6. The estimation error dynamics in (4.60) with $\mathcal{G} = L_{hv}$ are asymptotically stable if there exist matrices $P = P^T > 0$, F, and $N_{i_2 j_1}$ for $i_1, i_2 \in \{1, 2, \ldots, r_a\}$, $j_1 \in \{1, 2, \ldots, r_e\}$ such that:

$$\Upsilon_{i_1 i_1}^{j_1} < 0, \ \forall i_1, j_1; \qquad \frac{2}{r_a - 1}\Upsilon_{i_1 i_1}^{j_1} + \Upsilon_{i_1 i_2}^{j_1} + \Upsilon_{i_2 i_1}^{j_1} < 0, \ \forall j_1, \ i_i \neq i_2, \tag{4.68}$$

hold with

$$\Upsilon_{i_1 i_2}^{j_1} = \begin{bmatrix} -F - F^T + P & (*) & (*) \\ PA_{i_1} - N_{i_2 j_1} C_{i_1} & -PE_{j_1} - E_{j_1}^T P & (*) \\ 0 & F & -P \end{bmatrix}. \tag{4.69}$$

The observer gains are recovered as $L_{ij} = P^{-1} N_{ij}$, $i \in \{1, 2, \ldots, r_a\}$, $j \in \{1, 2, \ldots, r_e\}$. The final observer structure is (4.66).

Proof. Recall (4.63). By choosing the Lyapunov function as $V(e_k) = e_k^T F^T P^{-1} F e_k$, the observer gain as $\mathcal{G} = L_{hv}$, $\mathcal{M} = \begin{bmatrix} 0 & P \end{bmatrix}^T$, Finsler's lemma gives:

$$\begin{bmatrix} -F^T P^{-1} F & (*) \\ PA_h - N_{hv}C_h & -PE_v - E_v^T P + F^T P^{-1} F \end{bmatrix} < 0. \tag{4.70}$$

By applying Property A.3 and the Schur complement, (4.70) renders

$$\Upsilon_{hh}^{v} := \begin{bmatrix} -F - F^T + P & (*) & (*) \\ PA_h - N_{hv}C_h & -PE_v - E_v^T P & (*) \\ 0 & F & -P \end{bmatrix} < 0, \tag{4.71}$$

which by Lemma B.3 gives (4.68), thus concluding the proof. ■

Proposition 4.1. Under the same relaxation scheme, the conditions of Theorem 4.6 include those of Theorem 4.5. The reverse does not hold.

Proof: Consider (4.71), using $F = P$ gives:

$$\Upsilon_{hh}^{v} := \begin{bmatrix} -P & (*) & (*) \\ PA_h - N_{hv}C_h & -PE_v - E_v^T P & (*) \\ 0 & P & -P \end{bmatrix} < 0,$$

which by means of the Schur complement it is equivalent to (4.67). For the reverse, even if any positive matrix $P = P^T > 0$ can be decomposed in $F^T XF$ with $X = X^T > 0$ and F full-rank, (4.67) produces:

$$\begin{bmatrix} -F^T XF & (*) \\ F^T XFA_h - \left(F^T XF\right)L_{hv}C_h & -F^T XFE_v - E_v^T \left(F^T XF\right)^T + F^T XF \end{bmatrix} < 0,$$

which does not lead to the conditions in Theorem 4.6.■

In order to illustrate this proposition, the following example is given.

Example 4.4. Consider the TS descriptor (4.58) with $u_k = 0$, $r_a = r_e = 2$, and matrices: $E_1 = \begin{bmatrix} 0.9 & 0.1+a \\ -0.4-b & 1.1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0.9 & 0.1-a \\ -0.4+b & 1.1 \end{bmatrix}$, $A_1 = \begin{bmatrix} -1 & 1+a \\ -1.5 & 0.5 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 1-a \\ -1.5 & 0.5 \end{bmatrix}$, $C_1 = \begin{bmatrix} 0 \\ 1-b \end{bmatrix}^T$, and $C_2 = \begin{bmatrix} 0 \\ 1+b \end{bmatrix}^T$. The real-valued parameters are defined as $a \in [-0.7 \quad 0.7]$ and $b \in [-1 \quad 1]$. Figure 4.5 shows the results when conditions in Theorem 4.5 (O) are compared to those in Theorem 4.6 $(\times)$. Theorem 4.6 performs better than Theorem 4.5.

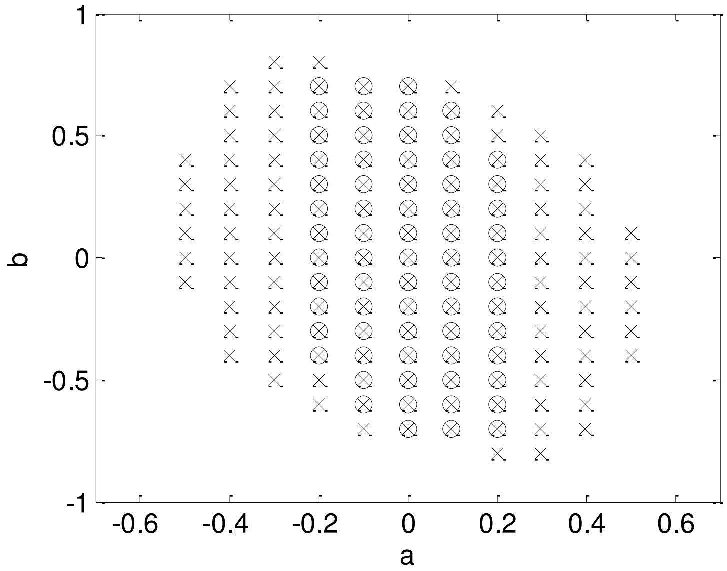

Figure 4.5. Feasible sets for theorems 4.5 and 4.6 in Example 4.4. ◆

The following developments use a non-PDC-like observer with $\mathcal{G}=G_h^{-1}L_{hv}$ under the quadratic framework.

Theorem 4.7. The estimation error dynamics in (4.60) with $\mathcal{G}=G_h^{-1}L_{hv}$ is asymptotically stable if there exist matrices $P=P^T>0$, $L_{i_2 j_1}$, and G_{i_2} for $i_2\in\{1,2,\ldots,r_a\}$, $j_1\in\{1,2,\ldots,r_e\}$ such that:

$$\Upsilon_{i_1 i_1}^{j_1}<0,\ \forall i_1,j_1; \qquad \frac{2}{r_a-1}\Upsilon_{i_1 i_1}^{j_1}+\Upsilon_{i_1 i_2}^{j_1}+\Upsilon_{i_2 i_1}^{j_1}<0,\ \forall j_1\ i_i\neq i_2, \tag{4.72}$$

hold with

$$\Upsilon_{i_1 i_2}^{j_1}=\begin{bmatrix} -P & (*) \\ G_{i_2}A_{i_1}-L_{i_2 j_1}C_{i_1} & -G_{i_2}E_{j_1}-E_{j_1}^T G_{i_2}^T+P \end{bmatrix}. \tag{4.73}$$

The final observer structure is

$$E_v\hat{x}_{k+1}=A_h\hat{x}_k+B_h u_k+G_h^{-1}L_{hv}\left(y_k-\hat{y}_k\right), \qquad \hat{y}_k=C_h\hat{x}_k. \tag{4.74}$$

Proof. Recall (4.63). Selecting the Lyapunov function as $V\left(e_k\right)=e_k^T P e_k$, the observer gain as $\mathcal{G}=G_h^{-1}L_{hv}$, and $\mathcal{M}=\begin{bmatrix}0 & G_h^T\end{bmatrix}^T$ we have:

$$\Upsilon_{hh}^{v}:=\begin{bmatrix} -P & (*) \\ G_h A_h-L_{hv}C_h & -G_h E_v-E_v^T G_h^T+P \end{bmatrix}<0, \tag{4.75}$$

which by Lemma B.3 yields (4.72). ■

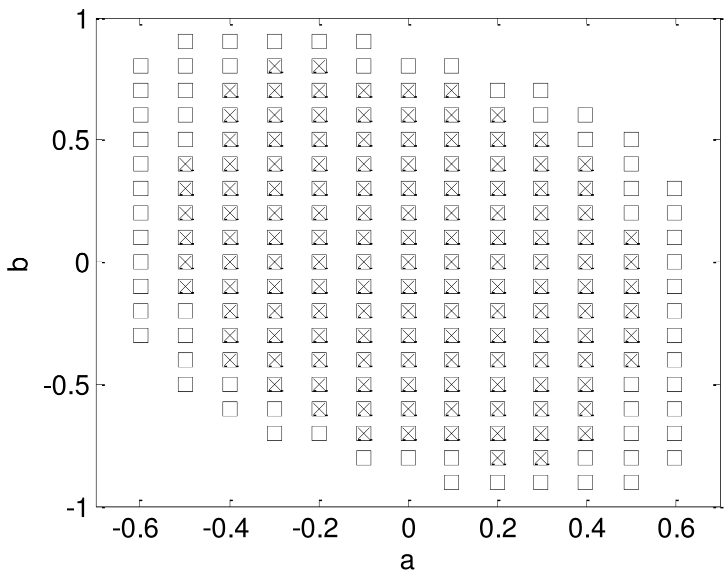

Figure 4.6. Feasible regions for Theorems 4.6 (×) and 4.7 (□) in Example 4.4. ♦

Example 4.5. Consider the local matrices of Example 4.4. Figure 4.6 shows the improvements brought via the non-PDC-like observer designed using Theorem 4.7.

Theorem 4.6 can be directly extended using the Lyapunov function $V(e_k) = e_k^T F^T P^{-1} F e_k$. However, the extension is equivalent to conditions in Theorem 4.7. Effectively, following the same path as Theorem 4.6 gives directly:

$$\Upsilon_{hh}^{v} := \begin{bmatrix} -F - F^T + P & (*) & (*) \\ G_h A_h - L_{hv} C_h & -G_h E_v - E_v^T G_h^T & (*) \\ 0 & F & -P \end{bmatrix} < 0. \tag{4.76}$$

By selecting $F = P$ as previously, (4.75) implies (4.76). Now, recall Proposition 4.1 where $P = F^T P^{-1} F$; substituting it in (4.75) produces:

$$\begin{bmatrix} -F^T P^{-1} F & (*) \\ G_h A_h - L_{hv} C_h & -G_h E_v - E_v^T G_h^T + F^T P^{-1} F \end{bmatrix} < 0. \tag{4.77}$$

Therefore if (4.76) holds, by means of Property A.3 and using Schur complement, it ends in (4.76) implying (4.75). Therefore there is no improvement in using $V(e_k) = e_k^T F^T P^{-1} F e_k$ over $V(e_k) = e_k^T P e_k$ when non-PDC-like observers are being designed.

4.2.2.2. Non-quadratic Lyapunov functions

The use of non-quadratic Lyapunov functions has been introduced in (Guerra and Vermeiren, 2004), where the benefits of this approach over its quadratic counterpart have been shown. In this part, results for the state estimation problem via a non-quadratic Lyapunov function and a non-PDC-like observer are presented. These results are summarized in the following theorem.

Theorem 4.8. The estimation error dynamics in (4.60) with $\mathcal{G} = G_h^{-1} L_{hv}$ is asymptotically stable if there exist matrices $P_{i_2} = P_{i_2}^T > 0$, $L_{i_2 j_1}$, and G_{i_2} for $i_1, i_2, i_x \in \{1, 2, \ldots, r_a\}$, $j_1 \in \{1, 2, \ldots, r_e\}$ such that:

$$\Upsilon_{i_1 i_1 i_x}^{j_1} < 0,\ \forall i_1, i_x, j_1; \qquad \frac{2}{r_a - 1} \Upsilon_{i_1 i_1 i_x}^{j_1} + \Upsilon_{i_1 i_2 i_x}^{j_1} + \Upsilon_{i_2 i_1 i_x}^{j_1} < 0,\ \forall i_x, j_1,\ i_i \neq i_2, \tag{4.78}$$

hold with

$$\Upsilon_{i_1 i_2 i_x}^{j_1} = \begin{bmatrix} -P_{i_2} & (*) \\ G_{i_2} A_{i_1} - L_{i_2 j_1} C_{i_1} & -G_{i_2} E_{j_1} - E_{j_1}^T G_{i_2}^T + P_{i_x} \end{bmatrix}.$$

The final observer structure is (4.74).

Proof. Recall inequality (4.63). Choose the observer gain as $\mathcal{G} = G_h^{-1} L_{hv}$, and $\mathcal{M} = \begin{bmatrix} 0 & G_h^T \end{bmatrix}^T$. Using Lemma B.3 on (4.63) gives (4.78). ■

Example 4.6. Recall the nonlinear descriptor model in Example 2.5 (Chapter 2, Section 2.2.5), i.e.,

$$E(x_k)x_{k+1} = A(x_k)x_k + Bu_k, \qquad y_k = C(x_k)x_k, \tag{4.79}$$

where $E(x_k) = \begin{bmatrix} 2 & -\eta \\ \eta & 1 \end{bmatrix}$, $A(x_k) = \begin{bmatrix} \cos(x_1) & -1 \\ 0.7 & -1.1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $C(x_k) = \begin{bmatrix} \sin(x_1)/x_1 \\ 0.2 \end{bmatrix}^T$; with $\eta = 1/(1+x_1^2)$. Employing the sector nonlinearity approach, an exact TS descriptor representation (4.58) in $\Omega = \{x \in \mathbb{R}^2\}$, with $r_e = 2$, $r_a = 4$, matrices as follows $E_1 = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $A_1 = A_2 = \begin{bmatrix} 1 & -1 \\ 0.7 & -1.1 \end{bmatrix}$, $A_3 = A_4 = \begin{bmatrix} -1 & -1 \\ 0.7 & -1.1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C_1 = C_3 = \begin{bmatrix} 1 & 0.2 \end{bmatrix}$, and $C_2 = C_4 = \begin{bmatrix} -0.2167 & 0.2 \end{bmatrix}$. The MFs are defined as $v_1 = 1/(1+x_2^2)$, $v_2 = 1 - v_1$, $h_1 = \omega_0^1 \omega_0^2$, $h_2 = \omega_0^1 \omega_1^2$, $h_3 = \omega_1^1 \omega_0^2$, $h_4 = \omega_1^1 \omega_1^2$, with $\omega_0^1 = 0.5(\cos(x_1)+1)$, $\omega_1^1 = 1 - \omega_0^1$, $\omega_0^2 = (\sin(x_1)/x_1 + 0.2167)/1.2167$, and $\omega_1^2 = 1 - \omega_0^2$. Applying conditions in Theorem 4.8 the following values have been obtained:

$$P_1 = \begin{bmatrix} 0.60 & -0.36 \\ -0.36 & 0.46 \end{bmatrix},\ P_2 = \begin{bmatrix} 0.66 & -0.32 \\ -0.32 & 0.45 \end{bmatrix},\ P_4 = \begin{bmatrix} 0.75 & -0.18 \\ -0.18 & 0.41 \end{bmatrix},\ L_{11} = \begin{bmatrix} 0.06 \\ -0.12 \end{bmatrix},$$

$$L_{32} = \begin{bmatrix} -0.44 \\ -0.04 \end{bmatrix},\ L_{41} = \begin{bmatrix} -0.57 \\ -0.16 \end{bmatrix},\ G_1 = \begin{bmatrix} 0.38 & -0.13 \\ -0.08 & 0.38 \end{bmatrix},\ \text{and}\ G_2 = \begin{bmatrix} 0.32 & -0.07 \\ -0.13 & 0.37 \end{bmatrix}.$$

Simulation results are shown in Figure 4.7 for initial conditions $x(0) = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$ and $\hat{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$; the input is $u(t) = 0.5\sin(t)$.

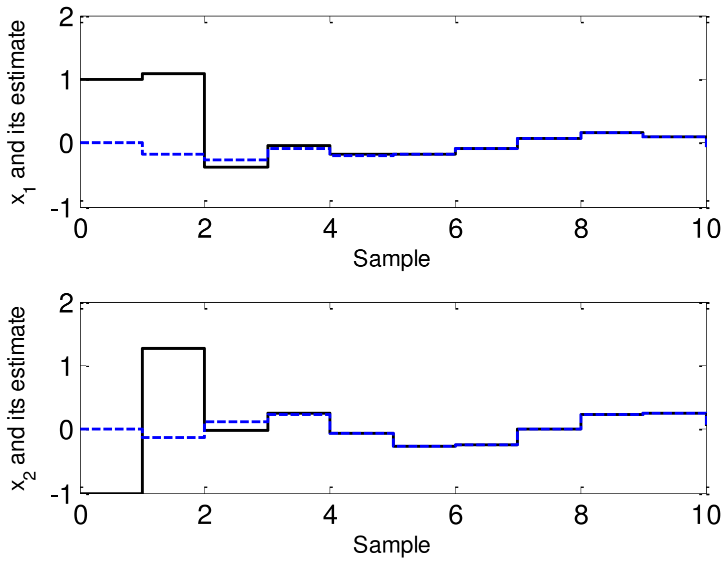

Figure 4.7. Simulation results for Example 4.6: States (black-solid-lie) and their estimates (blue-dashed-line).

Recall that by means of the approaches given in (Guerra and Vermeiren, 2004; Guerra et al., 2012b), no solution was found for a standard TS representation of (4.79). In addition via Theorem 4.8 only 132 LMI constraints are needed instead of 4112 for (Guerra and Vermeiren, 2004; Guerra et al., 2012b). ♦

Example 4.7. Consider a TS descriptor with $u=0$, $r_a=4$, $r_e=2$, and matrices:

$$E_1=\begin{bmatrix}0.9 & 1.8\\ -0.1 & 1.1\end{bmatrix},\qquad E_2=\begin{bmatrix}0.9 & 0.3\\ -1.4 & 0.8\end{bmatrix},\qquad A_1=\begin{bmatrix}-1 & 1.8\\ -1.5 & 0.5\end{bmatrix},\qquad A_2=\begin{bmatrix}-1.8 & 0.86\\ -1.5 & 0.5\end{bmatrix},$$

$$A_3=\begin{bmatrix}-1 & 0.2\\ -1.5 & 0.5\end{bmatrix},\ A_4=\begin{bmatrix}-0.2 & 1.14\\ -1.5 & 0.5\end{bmatrix},\ C_1=C_3=\begin{bmatrix}0\\ 1-\delta\end{bmatrix}^T,\ C_2=\begin{bmatrix}0\\ 2\end{bmatrix}^T,\text{ and } C_4=\begin{bmatrix}0\\ 1\end{bmatrix}^T,\text{ where}$$

$\delta>0$ is real-valued parameter. Conditions in Theorems 4.5 and 4.6 are not feasible for any $\delta\geq 0$; conditions in Theorem 4.7 provide solutions up to $\delta=0.16$; conditions in Theorem 4.8 are feasible up to $\delta=0.40$. ♦

4.2.2.3. Delayed non-quadratic Lyapunov function

As it has been stated in Chapter 3, Section 3.2, the use of past samples in the MFs of the Lyapunov function allows adding extra degrees of freedom while keeping the same number of convex sums, thus relaxing the results from the NQ approach. In this section the observer to be designed is:

$$E_v\hat{x}_{k+1}=A_h\hat{x}_k+B_hu_k+G_{hh^-}^{-1}L_{hh^-v}\left(y_k-\hat{y}_k\right),\qquad \hat{y}_k=C_h\hat{x}_k. \tag{4.80}$$

The following result is a delayed version of Theorem 4.8.

Theorem 4.9. The estimation error dynamics in (4.60) with $\mathcal{G}=G_{hh^-}^{-1}L_{hh^-v}$ is asymptotically stable if there exist matrices $P_{i_2}=P_{i_2}^T>0$, $L_{i_2i_xj_1}$, and $G_{i_2i_x}$ for $i_1,i_2,i_x\in\{1,2,\ldots,r_a\}$, $j_1\in\{1,2,\ldots,r_e\}$ such that:

$$\Upsilon_{i_1i_1i_x}^{j_1}<0,\ \forall i_1,i_x,j_1;\qquad \frac{2}{r_a-1}\Upsilon_{i_1i_1i_x}^{j_1}+\Upsilon_{i_1i_2i_x}^{j_1}+\Upsilon_{i_2i_1i_x}^{j_1}<0,\ \forall i_x,j_1,\ i_i\neq i_2, \tag{4.81}$$

hold with

$$\Upsilon_{i_1i_2i_x}^{j_1}=\begin{bmatrix}-P_{i_x} & (*)\\ G_{i_2i_x}A_{i_1}-L_{i_2i_xj_1}C_{i_1} & -G_{i_2i_x}E_{j_1}-E_{j_1}^TG_{i_2i_x}^T+P_{i_2}\end{bmatrix}. \tag{4.82}$$

Proof. Recall inequality (4.63). Choose the observer gain as $\mathcal{G}=G_{hh^-}^{-1}L_{hh^-v}$, and $\mathcal{M}=\begin{bmatrix}0 & G_{hh^-}^T\end{bmatrix}^T$, by means of Lemma B.3, (4.63) gives (4.81). ■

Example 4.7 (continued). Employing conditions in Theorem 4.9 on Example 4.7 increases the feasible solution set from $\delta=0.40$ up to $\delta=0.76$. Hence, the delayed approach provides a larger solution set than the classical approaches. ◆

4.2.3. Generalization

It has been shown in the previous subsection that more relaxed results can be achieved by incorporating delayed samples both in the Lyapunov function and in the observer gains. In order to generalize this approach, recall Definitions 3.1~3.9. Hence, the TS descriptor model (4.58) is rewritten as

$$E_{\mathcal{V}_0^E}x_{k+1}=A_{H_0^A}x_k+B_{H_0^B}u_k,\qquad y_k=C_{H_0^C}x_k, \tag{4.83}$$

with $\mathcal{V}_0^E=H_0^A=H_0^B=H_0^C=\{0\}$, i.e., without delays in the system matrices. The observer to be designed is:

$$\begin{aligned}E_{\mathcal{V}_0^E}\hat{x}_{k+1}&=A_{H_0^A}\hat{x}_k+B_{H_0^B}u_k+G_{H_0^G\mathcal{V}_0^G}^{-1}L_{H_0^L\mathcal{V}_0^L}\left(y_k-\hat{y}_k\right)\\ \hat{y}_k&=C_{H_0^C}\hat{x}_k,\end{aligned} \tag{4.84}$$

where $G_{H_0^G \mathcal{V}_0^G}$ and $L_{H_0^L \mathcal{V}_0^L}$ are the observer gains to be determined. They include delays given by multisets H_0^G, H_0^L, $\mathcal{V}_0^G$, and $\mathcal{V}_0^L$; these multisets must not contain positive delays, since a positive delay refers to future state variables, which are not available for computation.

The estimation error is $e_k = x_k - \hat{x}_k$ and its dynamics are

$$E_{\mathcal{V}_0^E} e_{k+1} = \left(A_{H_0^A} - G_{H_0^G \mathcal{V}_0^G}^{-1} L_{H_0^L \mathcal{V}_0^L} C_{H_0^C} \right) e_k,$$

which can be expressed as

$$\begin{bmatrix} A_{H_0^A} - G_{H_0^G \mathcal{V}_0^G}^{-1} L_{H_0^L \mathcal{V}_0^L} C_{H_0^C} & -E_{\mathcal{V}_0^E} \end{bmatrix} \begin{bmatrix} e_k \\ e_{k+1} \end{bmatrix} = 0. \tag{4.85}$$

Employing the Lyapunov function candidate

$$V(e_k) = e_k^T P_{H_0^P \mathcal{V}_0^P} e_k, \quad P_{i_0^P, j_0^P} = P_{i_0^P, j_0^P}^T > 0, \qquad \boldsymbol{i} \in \mathbb{I}_{H_0^P}, \quad \boldsymbol{j} \in \mathbb{I}_{\mathcal{V}_0^P}, \tag{4.86}$$

its variation is

$$\Delta V(e_k) = \begin{bmatrix} e_k \\ e_{k+1} \end{bmatrix}^T \begin{bmatrix} -P_{H_0^P \mathcal{V}_0^P} & 0 \\ 0 & P_{H_1^P \mathcal{V}_1^P} \end{bmatrix} \begin{bmatrix} e_k \\ e_{k+1} \end{bmatrix} < 0. \tag{4.87}$$

By defining $\mathcal{X} = \begin{bmatrix} e_k \\ e_{k+1} \end{bmatrix}$, $\mathcal{W} = \begin{bmatrix} A_{H_0^A} - G_{H_0^G \mathcal{V}_0^G}^{-1} L_{H_0^L \mathcal{V}_0^L} C_{H_0^C} & -E_{\mathcal{V}_0^E} \end{bmatrix}$, $\mathcal{Q} = \begin{bmatrix} -P_{H_0^P \mathcal{V}_0^P} & 0 \\ 0 & P_{H_1^P \mathcal{V}_1^P} \end{bmatrix}$, and employing Finsler's lemma, inequality (4.87) under the equality constraint (4.85) results in:

$$\mathcal{M} \begin{bmatrix} A_{H_0^A} - G_{H_0^G \mathcal{V}_0^G}^{-1} L_{H_0^L \mathcal{V}_0^L} C_{H_0^C} & -E_{\mathcal{V}_0^E} \end{bmatrix} + (*) + \begin{bmatrix} -P_{H_0^P \mathcal{V}_0^P} & 0 \\ 0 & P_{H_1^P \mathcal{V}_1^P} \end{bmatrix} < 0. \tag{4.88}$$

Therefore, the following result can be stated:

Theorem 4.10. The estimation error dynamics in (4.85) is asymptotically stable if there exist $P_{i_k^P, j_k^P} = P_{i_k^P, j_k^P}^T > 0$, $\boldsymbol{i}_k^P = pr_{H_k^P}^{\boldsymbol{i}}$, $\boldsymbol{j}_k^P = pr_{\mathcal{V}_k^P}^{\boldsymbol{j}}$, $L_{i_0^L, j_0^L}$, $\boldsymbol{i}_0^L = pr_{H_0^L}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^L = pr_{\mathcal{V}_0^L}^{\boldsymbol{j}}$, and $G_{i_0^G, j_0^G}$, $\boldsymbol{i}_0^G = pr_{H_0^G}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^G = pr_{\mathcal{V}_0^G}^{\boldsymbol{j}}$, $\boldsymbol{i} \in \mathbb{I}_{H_\Gamma}$, $\boldsymbol{j} \in \mathbb{I}_{\mathcal{V}_\Gamma}$, $k = 0,1$, where $\mathcal{V}_\Gamma = \mathcal{V}_0^P \cup \mathcal{V}_1^P \cup \mathcal{V}_0^L \cup \left(\mathcal{V}_0^G \oplus \mathcal{V}_0^E \right)$, $H_\Gamma = H_0^P \cup H_1^P \cup \left(H_0^L \oplus H_0^C \right) \cup \left(H_0^G \oplus H_0^A \right)$ such that

$$\begin{bmatrix} -P_{H_0^P \mathcal{V}_0^P} & (*) \\ G_{H_0^G \mathcal{V}_0^G} A_{H_0^A} - L_{H_0^L \mathcal{V}_0^L} C_{H_0^C} & -G_{H_0^G \mathcal{V}_0^G} E_{\mathcal{V}_0^E} - E_{\mathcal{V}_0^E}^T G_{H_0^G \mathcal{V}_0^G}^T + P_{H_1^P \mathcal{V}_1^P} \end{bmatrix} < 0. \tag{4.89}$$

The total number of sums involved in (4.89) is $n_{H\mathcal{V}} = |H_\Gamma| + |\mathcal{V}_\Gamma|$.

Proof. Consider (4.88). Selecting $\mathcal{M} = \begin{bmatrix} 0 & G^T_{H_0^G \mathcal{V}_0^G} \end{bmatrix}^T$ gives directly (4.89). ■

Remark 4.6. Considering that the system matrices do not contain delays, the maximum number of sums involved in (4.89) is given by $n_{H\mathcal{V}} \le 2n_{P_h} + 2n_{P_v} + n_{L_h} + n_{L_v} + n_{G_h} + n_{G_v} + 2$.

Corollary 4.3. The result for standard TS models stated in (Guerra et al., 2012b) is a special case of Theorem 4.10.

Proof. For (4.89), consider $E_{\mathcal{V}_0^E} = I$. Since $\mathcal{V}_0^E = \mathcal{V}_0^G = \mathcal{V}_0^L = \varnothing$, we have:

$$\begin{bmatrix} -P_{H_0^P} & (*) \\ G_{H_0^G} A_{H_0^A} - L_{H_0^L} C_{H_0^C} & -G_{H_0^G} - G^T_{H_0^G} + P_{H_1^P} \end{bmatrix} < 0. \tag{4.90}$$

By defining the multisets as $H_0^G = H_0^L = \{0, -1\}$ and $H_0^P = \{-1\}$ expression (4.90) directly yields the conditions in (Guerra et al., 2012b). ■

Selecting multisets

This part formalizes the delayed approach for the observer design. The main idea is that the multisets used in the design should be selected such that sums relaxations can be employed (double sums at the same instant). To this end, constructive steps are given.

Step 1: Recall that the system (4.83) does not contain delays in its matrices, i.e., $\mathcal{V}_0^E = H_0^A = H_0^B = H_0^C = \{0\}$; thus the multisets H_0^G, H_0^L, $\mathcal{V}_0^G$, and $\mathcal{V}_0^L$ should, at least, contain $\{0\}$. Thanks to the terms $G_{H_0^G \mathcal{V}_0^G} A_{H_0^A}$ and $L_{H_0^L \mathcal{V}_0^L} C_{H_0^C}$, a smart selection is $H_0^G = H_0^L = \mathcal{V}_0^G$. Hence (4.89) gives

$$\begin{bmatrix} -P_{H_0^P \mathcal{V}_0^P} & (*) \\ G_{\{0\},\{0\}} A_{\{0\}} - L_{\{0\},\{0\}} C_{\{0\}} & -G_{\{0\},\{0\}} E_{\{0\}} - E^T_{\{0\}} G^T_{\{0\},\{0\}} + P_{H_1^P \mathcal{V}_1^P} \end{bmatrix} < 0,$$

which by selecting the multisets for the Lyapunov matrix as $H_0^P = \mathcal{V}_0^P = \{-1\}$ (Guerra et al., 2012b) writes:

$$\begin{bmatrix} -P_{\{-1\},\{-1\}} & (*) \\ G_{\{0\},\{0\}} A_{\{0\}} - L_{\{0\},\{0\}} C_{\{0\}} & -G_{\{0\},\{0\}} E_{\{0\}} - E^T_{\{0\}} G^T_{\{0\},\{0\}} + P_{\{0\},\{0\}} \end{bmatrix} < 0. \tag{4.91}$$

Step 2: Note that the convex inequality (4.91) contains three sums of $h(\bullet)$: $\sum_{i_1=1}^{r_a}\sum_{i_2=1}^{r_a}\sum_{i_x=1}^{r_a} h_{i_1}\left(z(k)\right)h_{i_2}\left(z(k)\right)h_{i_x}\left(z(k-1)\right)$ and three sums of the form $v(\bullet)$: $\sum_{j_1=1}^{r_e}\sum_{j_2=1}^{r_e}\sum_{j_x=1}^{r_e} v_{j_1}\left(z(k)\right)v_{j_2}\left(z(k)\right)v_{j_x}\left(z(k-1)\right)$. Thus, it is possible to include the delay $\{-1\}$ in each multiple sum of $G_{H_0^G \mathcal{V}_0^G}$ and $L_{H_0^L \mathcal{V}_0^L}$ without altering the total number of sums:

$$\begin{bmatrix} -P_{\{-1\},\{-1\}} & (*) \\ G_{\{0,-1\},\{0,-1\}}A_{\{0\}} - L_{\{0,-1\},\{0,-1\}}C_{\{0\}} & -G_{\{0,-1\},\{0,-1\}}E_{\{0\}} - E_{\{0\}}^T G_{\{0,-1\},\{0,-1\}}^T + P_{\{0\},\{0\}} \end{bmatrix} < 0. \quad (4.92)$$

Step 3: Since there is no product involving $L_{H_0^L \mathcal{V}_0^L}$ and $E_{\mathcal{V}_0^E}$, the MFs $v(\bullet)$ of $L_{H_0^L \mathcal{V}_0^L}$ should be chosen as $\mathcal{V}_0^L = \mathcal{V}_0^G \oplus \mathcal{V}_0^E$, thus (4.92) gives

$$\begin{bmatrix} -P_{\{-1\},\{-1\}} & (*) \\ G_{\{0,-1\},\{0,-1\}}A_{\{0\}} - L_{\{0,-1\},\{0,0,-1\}}C_{\{0\}} & -G_{\{0,-1\},\{0,-1\}}E_{\{0\}} - E_{\{0\}}^T G_{\{0,-1\},\{0,-1\}}^T + P_{\{0\},\{0\}} \end{bmatrix} < 0.$$

Table 4.1 provides the generalization based on the previous steps (similar to Chapter 3, Section 3.2.3).

Table 4.1 How to select multisets for Theorem 4.10.

Matrix	Multisets in Theorem 4.10
$P_{H_0^P \mathcal{V}_0^P}$	$H_0^P = \{-1,-1,\ldots,-1\}$, $\lvert H_0^P \rvert = n_{P_h}$
	$\mathcal{V}_0^P = \{-1,-1,\ldots,-1\}$, $\lvert \mathcal{V}_0^P \rvert = n_{P_v}$,
$L_{H_0^L \mathcal{V}_0^L}$	$H_0^L = \{\underbrace{0,0,\ldots,0}_{n_{P_h}},\underbrace{-1,-1,\ldots,-1}_{n_{P_h}}\}$, $\lvert H_0^L \rvert = 2n_{P_h}$
	$\mathcal{V}_0^L = \{0,\underbrace{0,0,\ldots,0}_{n_{P_v}},\underbrace{-1,-1,\ldots,-1}_{n_{P_v}}\}$, $\lvert \mathcal{V}_0^L \rvert = 1+2n_{P_v}$
$G_{H_0^G \mathcal{V}_0^G}$	$H_0^G = \{\underbrace{0,0,\ldots,0}_{n_{P_h}},\underbrace{-1,-1,\ldots,-1}_{n_{P_h}}\}$, $\lvert H_0^G \rvert = 2n_{P_h}$
	$\mathcal{V}_0^G = \{\underbrace{0,0,\ldots,0}_{n_{P_v}},\underbrace{-1,-1,\ldots,-1}_{n_{P_v}}\}$, $\lvert \mathcal{V}_0^G \rvert = 2n_{P_v}$

Example 4.8. We turn back to the TS descriptor model in Example 4.4. Figure 4.8 shows the feasible regions for the proposed approach when $a \in [-1.5, 1.5]$ and $b \in [-1.5, 1.5]$. Two configurations for Theorem 4.10 are tested:

- Configuration 1: Multisets $H_0^P = H_0^G = H_0^L = \mathcal{V}_0^L = \{0\}$ and $\mathcal{V}_0^P = \mathcal{V}_0^G = \varnothing$, leading to the conditions in Theorem 4.8 The results are represented by $(\square)$ in Figure 4.8.
- Configuration 2: Multisets $H_0^G = H_0^L = \mathcal{V}_0^G = \{0,-1\}$, $\mathcal{V}_0^L = \{0,0,-1\}$, and $H_0^P = \mathcal{V}_0^P = \{-1\}$. The results are represented by $(\times)$ in Figure 4.8.

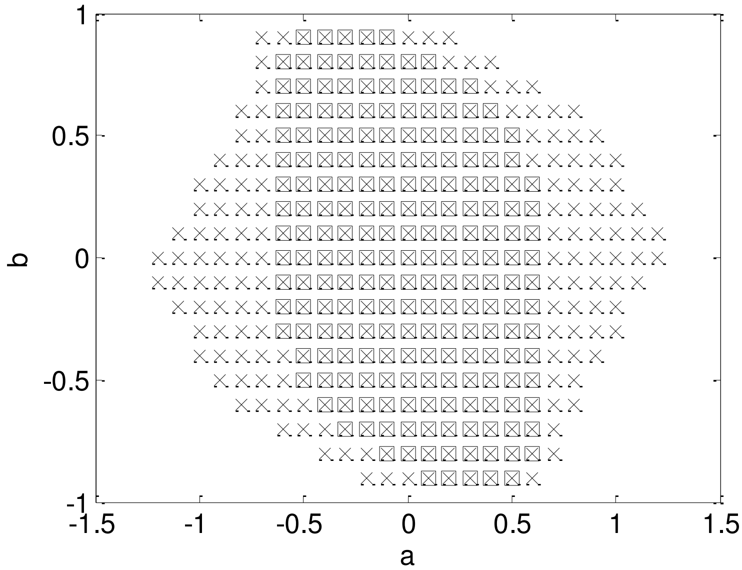

Figure 4.8. Feasible sets for different configurations for Theorem 4.10 in Example 4.8 . ◆

By selecting $(a,b) = (1, 0.3)$ Configuration 1 does not provide a solution, while Configuration 2 is feasible; some of the obtained gains are:

$$P_{11} = \begin{bmatrix} 0.16 & -0.05 \\ -0.05 & 0.03 \end{bmatrix},\ P_{22} = \begin{bmatrix} 0.15 & -0.17 \\ -0.17 & 0.20 \end{bmatrix},\ L_{11222} = \begin{bmatrix} 0.16 \\ -0.20 \end{bmatrix},\ L_{11211} = \begin{bmatrix} 0.34 \\ -0.18 \end{bmatrix},$$

$$L_{11111} = \begin{bmatrix} 0.16 \\ -0.06 \end{bmatrix},\ L_{12212} = \begin{bmatrix} 0.33 \\ -0.35 \end{bmatrix},\ G_{1111} = \begin{bmatrix} 0.07 & -0.06 \\ -0.03 & 0.08 \end{bmatrix},\ \text{and}\ G_{2222} = \begin{bmatrix} 0.12 & -0.06 \\ -0.08 & -0.09 \end{bmatrix}.$$

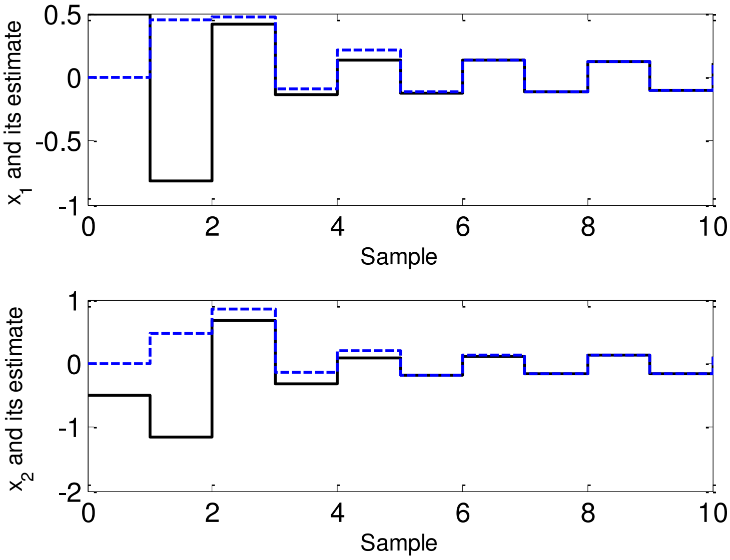

Figure 4.9. Simulation results for Example 4.8: States (black-solid-line) and their estimates (blue-dashed-line).

Simulation results are shown in Figure 4.9 for initial conditions $x(0)=\begin{bmatrix}0.5 & -0.5\end{bmatrix}^T$ and $\hat{x}(0)=\begin{bmatrix}0 & 0\end{bmatrix}^T$. The MFs used for simulation are $h_1 = x_2^2/4$, $h_2 = 1-h_1$, $v_1 = (x_2+2)/4$, and $v_2 = 1-v_1$. ♦

The α-sample variation

More relaxed conditions using the α-sample variation presented in (Kruszewski et al., 2008) are given in what follows. The main idea is to replace the classical one-sample variation of the Lyapunov function by its variation overall several samples, thus allowing the Lyapunov function to decrease at each α sample and not at each sample. This can be summarized in the following theorem.

Theorem 4.11. The estimation error dynamics in (4.85) is asymptotically stable if there exist $P_{\boldsymbol{i}_k^P,\boldsymbol{j}_k^P} = P_{\boldsymbol{i}_k^P,\boldsymbol{j}_k^P}^T > 0$, $\boldsymbol{i}_k^P = pr_{H_k^P}^{\boldsymbol{i}}$, $\boldsymbol{j}_k^P = pr_{\mathcal{V}_k^P}^{\boldsymbol{j}}$, $L_{\boldsymbol{i}_l^L,\boldsymbol{j}_l^L}$, $\boldsymbol{i}_l^L = pr_{H_l^L}^{\boldsymbol{i}}$, $\boldsymbol{j}_l^L = pr_{\mathcal{V}_l^L}^{\boldsymbol{j}}$, and $G_{\boldsymbol{i}_l^G,\boldsymbol{j}_l^G}$, $\boldsymbol{i}_l^G = pr_{H_l^G}^{\boldsymbol{i}}$, $\boldsymbol{j}_l^G = pr_{\mathcal{V}_l^G}^{\boldsymbol{j}}$, $\boldsymbol{i} \in \mathbb{I}_{H_\Gamma}$, $\boldsymbol{j} \in \mathbb{I}_{\mathcal{V}_\Gamma}$, $k = 0, \alpha$, $l \in \{0,1,\ldots,\alpha-1\}$ with $\mathcal{V}_\Gamma = \mathcal{V}_0^P \cup \mathcal{V}_\alpha^P \cup \bigcup_{l=0}^{\alpha-1} \mathcal{V}_l^L \cup \bigcup_{l=0}^{\alpha-1} \left(\mathcal{V}_l^G \oplus \mathcal{V}_l^E\right), H_\Gamma = H_0^P \cup H_\alpha^P \cup \bigcup_{l=0}^{\alpha-1} \left(H_l^L \oplus H_l^C\right) \cup \bigcup_{l=0}^{\alpha-1} \left(H_l^G \oplus H_l^A\right)$ such that:

$$\begin{bmatrix} -P_{H_0^P \mathcal{V}_0^P} & (*) & 0 & \cdots & 0 \\ \begin{pmatrix} G_{H_0^G \mathcal{V}_0^G} A_{H_0^A} \\ -L_{H_0^L \mathcal{V}_0^L} C_{H_0^C} \end{pmatrix} & -G_{H_0^G \mathcal{V}_0^G} E_{\mathcal{V}_0^E} + (*) & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & -G_{H_{\alpha-2}^G \mathcal{V}_{\alpha-2}^G} E_{\mathcal{V}_{\alpha-2}^E} + (*) & (*) \\ 0 & \cdots & 0 & \begin{pmatrix} G_{H_{\alpha-1}^G \mathcal{V}_{\alpha-1}^G} A_{H_{\alpha-1}^A} \\ -L_{H_{\alpha-1}^L \mathcal{V}_{\alpha-1}^L} C_{H_{\alpha-1}^C} \end{pmatrix} & \begin{pmatrix} -G_{H_{\alpha-1}^G \mathcal{V}_{\alpha-1}^G} E_{\mathcal{V}_{\alpha-1}^E} + (*) \\ + P_{H_\alpha^P \mathcal{V}_\alpha^P} \end{pmatrix} \end{bmatrix} < 0 \quad (4.93)$$

Proof. Consider the Lyapunov function (4.86) and its α-sample variation as follows (Guerra et al., 2012b; Kruszewski et al., 2008; Lendek et al., 2015):

$$\begin{aligned} \Delta V_\alpha &= V(e(k+\alpha)) - V(e(k)) \\ &= e^T(k+\alpha) P_{H_\alpha^P \mathcal{V}_\alpha^P} e(k+\alpha) - e^T(k) P_{H_0^P \mathcal{V}_0^P} e(k) \qquad (4.94) \\ &= \begin{bmatrix} e(k) \\ e(k+1) \\ \vdots \\ e(k+\alpha) \end{bmatrix}^T \begin{bmatrix} -P_{H_0^P \mathcal{V}_0^P} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_{H_\alpha^P \mathcal{V}_\alpha^P} \end{bmatrix} \begin{bmatrix} e(k) \\ e(k+1) \\ \vdots \\ e(k+\alpha) \end{bmatrix} < 0. \end{aligned}$$

The error dynamics (4.85) during α samples can be summarized as the following equality constraint:

$$\begin{bmatrix} S_0 & -E_{\mathcal{V}_0^E} & 0 & \cdots & 0 \\ 0 & S_1 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & -E_{\mathcal{V}_{\alpha-2}^E} & 0 \\ 0 & \cdots & 0 & S_{\alpha-1} & -E_{\mathcal{V}_{\alpha-1}^E} \end{bmatrix} \begin{bmatrix} e(k) \\ e(k+1) \\ \vdots \\ e(k+\alpha) \end{bmatrix} = 0, \qquad (4.95)$$

with $S_l = A_{H_l^A} - G_{H_l^G \mathcal{V}_l^G}^{-1} L_{H_l^L \mathcal{V}_l^L} C_{H_l^C}$, $l \in \{0, 1, \ldots, \alpha-1\}$. Applying Finsler's lemma, inequality (4.94) under constraint (4.95) is equivalent to

$$\mathcal{M} \begin{bmatrix} S_0 & -E_{\mathcal{V}_0^E} & 0 & \cdots & 0 \\ 0 & S_1 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & -E_{\mathcal{V}_{\alpha-2}^E} & 0 \\ 0 & \cdots & 0 & S_{\alpha-1} & -E_{\mathcal{V}_{\alpha-1}^E} \end{bmatrix} + (*) + \begin{bmatrix} -P_{H_0^P \mathcal{V}_0^P} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_{H_\alpha^P \mathcal{V}_\alpha^P} \end{bmatrix} < 0,$$

where $\mathcal{M} \in \mathbb{R}^{n_x(\alpha+1) \times n_x \alpha}$. In order to obtain strict LMI conditions a natural choice (Kruszewski et al., 2008) of matrix $\mathcal{M}$ is:

$$M_{H_0^G\mathcal{V}_0^G\ldots H_{\alpha-1}^G\mathcal{V}_{\alpha-1}^G} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ G_{H_0^G\mathcal{V}_0^G} & 0 & \cdots & 0 \\ 0 & G_{H_1^G\mathcal{V}_1^G} & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_{H_{\alpha-1}^G\mathcal{V}_{\alpha-1}^G} \end{bmatrix},$$

leading to (4.93) and concluding the proof. ■

Remark 4.7. Using $\alpha = 1$ in Theorem 4.11, Theorem 4.10 is recovered.

Example 4.9. Consider a discrete-time TS descriptor model as (4.58) with $u_k = 0$, $r_a = r_e = 2$, $A_1 = \begin{bmatrix} -1+a & -0.2 \\ -1.5 & 0.5 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 2.2 \\ -1.5 & 0.5 \end{bmatrix}$, $C_1 = \begin{bmatrix} 0 \\ 1.2-b \end{bmatrix}^T$, $C_2 = \begin{bmatrix} 0 \\ 1.2+b \end{bmatrix}^T$, $E_1 = \begin{bmatrix} 0.9 & -1.1 \\ -0.2 & 1.1 \end{bmatrix}$, and $E_2 = \begin{bmatrix} 0.9 & 1.3 \\ -0.6 & 1.1 \end{bmatrix}$. The parameters are defined as $a \in [-0.5, 1]$ and $b \in [-0.5, 0.5]$. Choose the multisets of Lyapunov matrix and of the observer gains as $H_0^G = H_0^L = \mathcal{V}_0^G = \{0, -1\}$, $\mathcal{V}_0^L = \{0, 0, -1\}$, and $H_0^P = \mathcal{V}_0^P = \{-1\}$. Two sets of conditions have been tested:

- Conditions in Theorem 4.11 for $\alpha = 1$, i.e., the conditions in Theorem 4.10. The resulting feasible solutions are represented by (□) in Figure 4.10.
- Conditions in Theorem 4.11 for $\alpha = 2$, the resulting feasible solutions are represented by (×) in Figure 4.10.

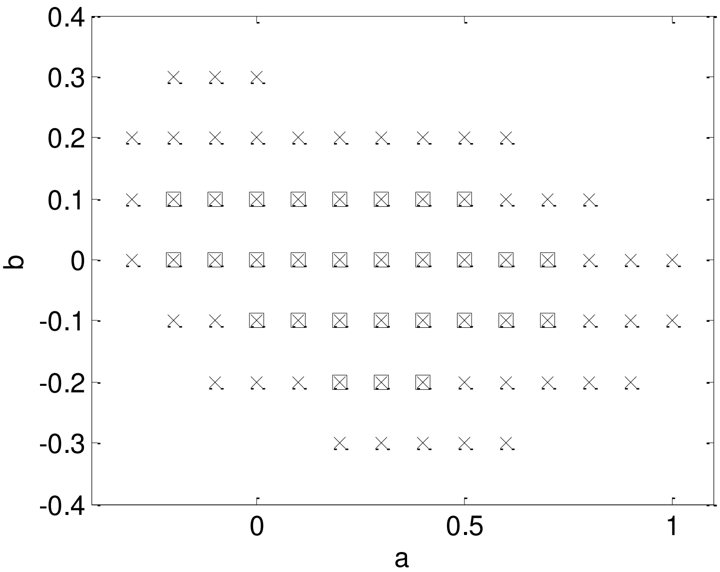

Figure 4.10. Feasible solution set for Theorem 4.11 for α=1 (□) and for α=2 (×) in Example 4.9. ◆

4.3. Summary and concluding remarks

In this chapter, recent results for observer design for TS descriptor models have been presented. In the continuous time case, the observer design is carried out efficiently by avoiding BMI conditions in the literature. The improvement is obtained by changing the estimated state vector. More, relaxed results can be achieved with different observer configurations. In the discrete-time case, several approaches have been provided; relaxed results are achieved via the delayed approach. An arbitrary number of past samples can be added into the MFs of both the Lyapunov function and the observer gains, and a systematic procedure is given to do this, thus providing the generalization of the delayed approach. Several numerical examples have been used to illustrate the performances of the given conditions.

Chapter 5. Static output feedback controller design

This chapter presents conditions for output feedback control design for both continuous and discrete time TS descriptor models under the assumption that the descriptor matrix is non-singular. In the continuous-time case, a quadratic Lyapunov function is used together with slack variables. In the discrete-time case, a delayed Lyapunov function is proposed together with delayed non-PDC controllers. In both cases, the design conditions are given in terms of LMIs up to the selection of a slack variable; naturally, different choices of this variable may lead to different degrees of conservatism, as illustrated via numerical examples.

5.1. Continuous-time TS descriptor models

This section deals with static output feedback controller design for continuous-time Takagi-Sugeno descriptor models. Via the well-known Finsler's lemma and the descriptor-redundancy approach a set of linear matrix inequalities are derived to solve this design problem.

5.1.1. Problem statement

Consider the following TS descriptor model (Taniguchi et al., 1999):

$$E_v \dot{x} = A_h x + B_h u, \qquad y = C_h x. \tag{5.1}$$

In the case of static output feedback control (SOFC) design for standard TS models, an iterated LMI (ILMI) approach has been presented in (Huang and Nguang, 2007), while sufficient LMI conditions have been developed in (Kau et al., 2007). In particular, (Kau et al., 2007) designed a SOFC PDC-like control law of the form:

$$u = K_h y = K_h C_h x. \tag{5.2}$$

Their analysis relied on the closed-loop system $\dot{x} = \left(A_h + B_h K_h C_h\right) x$. Stabilization conditions have been given in terms of LMIs together with equalities:

$$\begin{gathered} A_h P + B_h N_h C_h + (*) < 0, \quad M C_h - C_h P = 0, \\ N_h = K_h M, \quad P = P^T > 0. \end{gathered} \tag{5.3}$$

Conditions (5.3) are similar to those in (Crusius and Trofino, 1999). Effectively, in case of standard linear systems

$$\dot{x} = Ax + Bu, \qquad y = Cx.$$

Crusius and Trofino, (1999) gave sufficient conditions for the output feedback control problem. Two different approaches have been stated:

The *W-Problem*: given matrices A, B, and C with C full row rank, a controller $u = NM^{-1} y$ can be designed if there exist W, N, M so that the following conditions hold:

$$AW + BNC + (*) < 0, \quad MC - CW = 0, \quad W = W^T > 0. \tag{5.4}$$

The *P-Problem*: given matrices A, B, and C with B full column rank, a controller $u = M^{-1} N y$ can be designed if the exist P, N, M so that following conditions hold:

$$PA + BNC + (*) < 0, \quad BM - PB = 0, \quad P = P^T > 0. \tag{5.5}$$

Thus the conditions in (Kau et al., 2007) are a “direct” extension of the *W-Problem*. The result in (Bouarar et al., 2009) has established LMI conditions for stabilization of a standard TS model via the descriptor-redundancy approach together with a non-PDC control law $u = K_h \left(P_{1h}\right)^{-1} y$. Matrix P_{1h} is in the Lyapunov matrix, thus the conditions involve the time-derivative of the MFs.

Our aim is to control the TS descriptor model (5.1) via SOFC of the form

$$u = \left(G_{hhv}\right)^{-1} K_{hv} y = \left(G_{hhv}\right)^{-1} K_{hv} C_h x, \tag{5.6}$$

where $G_{hhv} \in \mathbb{R}^{n_u \times n_u}$ and $K_{hv} \in \mathbb{R}^{n_u \times n_y}$. Substituting the control law (5.6) in the system dynamics (5.1), it produces the closed-loop

$$E_v \dot{x} = \left(A_h + B_h \left(G_{hhv} \right)^{-1} K_{hv} C_h \right) x, \tag{5.7}$$

which is difficult to deal with.

5.1.2. Results

Using Finsler's lemma, it is possible to avoid the equality conditions in (Kau et al., 2007) and the ILMI conditions in (Huang and Nguang, 2007). The TS descriptor model together with the control law are written as the following equality constraint:

$$\begin{bmatrix} A_h & -E_v & B_h \\ \left(G_{hhv} \right)^{-1} K_{hv} C_h & 0_{n_u \times n_x} & -I_{n_u} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ u \end{bmatrix} = 0. \tag{5.8}$$

The expression (5.8) avoids the explicit appearance of the classical closed-loop $E_v \dot{x} = \left(A_h + B_h \left(G_{hhv} \right)^{-1} K_{hv} C_h \right) x$, and it also decouples the nonlinear matrix E_v.

Consider the following quadratic Lyapunov function candidate:

$$V(x) = x^T P x, \qquad P = P^T > 0. \tag{5.9}$$

Its time-derivate, adding the null-term $u^T 0_{n_u} u$ produces:

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} + u^T 0 u \\ &= \begin{bmatrix} x \\ \dot{x} \\ u \end{bmatrix}^T \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & 0_{n_u} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ u \end{bmatrix} < 0. \end{aligned} \tag{5.10}$$

Taking $\mathcal{X} = \begin{bmatrix} x \\ \dot{x} \\ u \end{bmatrix}$, $\mathcal{W} = \begin{bmatrix} A_h & -E_v & B_h \\ \left(G_{hhv} \right)^{-1} K_{hv} C_h & 0 & -I \end{bmatrix}$, and $\mathcal{Q} = \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Via Finsler's lemma the inequality constraint (5.10) together the equality constraint (5.8) yields

$$\mathcal{M} \begin{bmatrix} A_h & -E_v & B_h \\ \left(G_{hhv} \right)^{-1} K_{hv} C_h & 0 & -I \end{bmatrix} + (*) + \begin{bmatrix} 0 & P & 0 \\ P & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} < 0, \tag{5.11}$$

where $\mathcal{M} \in \mathbb{R}^{(2n_x+n_u)\times(n_x+n_u)}$ is a free matrix. Therefore, the following result can be stated.

Theorem 5.1. Consider $\eta \in \mathbb{R}^{n_x \times n_u}$ a constant matrix, the TS descriptor model (5.1) under the control law (5.6) is asymptotically stable if there exist matrices $P = P^T > 0$, M_{1i_2}, M_{3i_2}, M_{5i_2}, $G_{i_1 i_2 j_2}$, $K_{i_2 j_1}$, $i_1, i_2 \in \{1, 2, \ldots, r\}$, $j_1 \in \{1, 2, \ldots, r_e\}$ such that:

$$\Upsilon_{i_1 i_1}^{j_1} < 0, \quad \forall i_1, j_1; \qquad \frac{2}{r_a - 1}\Upsilon_{i_1 i_1}^{j_1} + \Upsilon_{i_1 i_2}^{j_1} + \Upsilon_{i_2 i_1}^{j_1} < 0, \quad \forall j_1, i_1 \neq i_2, \tag{5.12}$$

hold with

$$\Upsilon_{i_1 i_2}^{j_1} = \begin{bmatrix} M_{1i_2} A_{i_1} + \eta K_{i_2 j_1} C_{i_1} + (*) & (*) & (*) \\ M_{3i_2} A_{i_1} + \eta K_{i_2 j_1} C_{i_1} + \left(P - M_{1i_2} E_{j_1}\right)^T & -M_{3i_2} E_{j_1} - E_{j_1}^T M_{3i_2}^T & (*) \\ M_{5i_2} A_{i_1} + K_{i_2 j_1} C_{i_1} + \left(M_{1i_2} B_{i_1} - \eta G_{i_1 i_2 j_1}\right)^T & \begin{pmatrix} -M_{5i_2} E_{j_1} \\ +\left(M_{3i_2} B_{i_1} - \eta G_{i_1 i_2 j_1}\right)^T \end{pmatrix} & M_{5i_2} B_{i_1} - G_{i_1 i_2 j_1} + (*) \end{bmatrix}.$$

Proof. Going on from (5.11) we choose:

$$\mathcal{M} = \begin{bmatrix} M_{1h} & \eta G_{hhv} \\ M_{3h} & \eta G_{hhv} \\ M_{5h} & G_{hhv} \end{bmatrix}. \tag{5.13}$$

Then, inequality (5.11) renders

$$\Upsilon_{hh}^{v} := \begin{bmatrix} M_{1h} A_h + \eta K_{hv} C_h + (*) & (*) & (*) \\ \Gamma^{(2,1)} & -M_{3h} E_v - E_v^T M_{3h}^T & (*) \\ \Gamma^{(3,1)} & \Gamma^{(3,2)} & M_{5h} B_h - G_{hhv} + (*) \end{bmatrix} < 0, \tag{5.14}$$

where $\Gamma^{(2,1)} = M_{3h} A_h + \eta K_{hv} C_h + \left(P - M_{1h} E_v\right)^T$, $\Gamma^{(3,1)} = M_{5h} A_h + K_{hv} C_h + \left(M_{1h} B_h - \eta G_{hhv}\right)^T$, and $\Gamma^{(3,2)} = -M_{5h} E_v + \left(M_{3h} B_h - \eta G_{hhv}\right)^T$. Finally, applying Lemma B.3, (5.14) yields (5.12), thus concluding the proof. ■

Remark 5.1. The goal is to obtain an LMI problem. Since the slack matrices in $\mathcal{M}$ can be chosen, several options are available. The structure in (5.13) has been chosen following the idea from (Chadli and Guerra, 2012).

Remark 5.2. Several results can be obtained from Theorem 5.1, for instance, setting $\eta = 0_{n_x \times n_u}$ or $\eta = B_h$ or $M_5 = 0_{n_u \times n_x}$. Different configurations may lead to different results (Chadli and Guerra, 2012).

Remark 5.3. In this particular case (SOFC design), when using the extended vector $\begin{bmatrix} x^T & \dot{x}^T & u^T \end{bmatrix}^T$, equivalent conditions are obtained employing Finsler's lemma and descriptor-redundancy, as follows.

Consider $\bar{x} = \begin{bmatrix} x^T & \dot{x}^T & u^T \end{bmatrix}^T$; thus the system (5.1) together with the control law (5.6) writes:

$$\left.\begin{array}{l} 0 \times \ddot{x} = A_h x + B_h u - E_v \dot{x} \\ 0 \times \dot{u} = \left(G_{hhv}\right)^{-1} K_{hv} C_h x - u \end{array}\right\} \Leftrightarrow \bar{\mathcal{E}}\dot{\bar{x}} = \bar{\mathcal{A}}\bar{x}, \tag{5.15}$$

with $\bar{\mathcal{E}} = \begin{bmatrix} I_{n_x} & 0 & 0 \\ 0 & 0_{n_x} & 0 \\ 0 & 0 & 0_{n_u} \end{bmatrix}$ and $\bar{\mathcal{A}} = \begin{bmatrix} 0_{n_x} & I_{n_x} & 0_{n_u} \\ A_h & -E_v & B_h \\ \left(G_{hhv}\right)^{-1} K_{hv} C_h & 0 & -I_{n_u} \end{bmatrix}$. The Lyapunov function under consideration is $V(\bar{x}) = \bar{x}^T \bar{\mathcal{E}}^T \bar{P} \bar{x}$, where $\bar{\mathcal{E}}^T \bar{P} = \bar{P}^T \bar{\mathcal{E}} \geq 0$ and $\bar{P} = \begin{bmatrix} P_1 & 0 & 0 \\ P_2 & P_3 & P_4 \\ P_5 & P_6 & P_7 \end{bmatrix}$ with $P_1 = P_1^T > 0$. Therefore $\dot{V}(\bar{x}) < 0$ is equivalent to $\bar{\mathcal{A}}^T \bar{P} + \bar{P}^T \bar{\mathcal{A}} < 0$, or extending

$$\begin{bmatrix} P_2^T A_h + P_5^T G_{hhv}^{-1} K_{hv} C_h + (*) & (*) & (*) \\ P_3^T A_h + P_6^T G_{hhv}^{-1} K_{hv} C_h + \left(P_1 - P_2^T E_v\right)^T & -P_3^T E_v - E_v^T P_3 & (*) \\ P_4^T A_h + P_7^T G_{hhv}^{-1} K_{hv} C_h + \left(P_2^T B_h - P_5^T\right)^T & -P_4^T E_v + \left(P_3^T B_h - P_6\right)^T & P_4^T B_h - P_7^T + (*) \end{bmatrix} < 0,$$

which by setting $P_1 = P$, $P_2^T = M_{1h}$, $P_3^T = M_{3h}$, $P_4^T = M_{5h}$, $P_5^T = P_6^T = \eta G_{hhv}$, and $P_7^T = G_{hhv}$ gives exactly (5.14).

Corollary 5.1. The *P-Problem* conditions (5.5) given in (Crusius and Trofino, 1999) are included in those of Theorem 5.1.

Proof. By choosing $E_v = I$, $M_1 = P$, $M_2 = B_h G_{hh}$, $M_3 = \varepsilon P$, $M_4 = \varepsilon B_h G_{hh}$, $M_5 = 0$, and $M_6 = \varepsilon G_{hh}$, (5.11) gives:

$$\begin{bmatrix} PA_h + B_h K_h C_h + (*) & (*) & (*) \\ \varepsilon\left(PA_h + K_h C_h\right) & -2\varepsilon P & 0 \\ \varepsilon\left(K_h C_h + \frac{1}{\varepsilon}\left(PB_h - B_h G_{hh}\right)^T\right) & \varepsilon\left(PB_h - B_h G_{hh}\right)^T & -\varepsilon\left(G_{hh} + G_{hh}^T\right) \end{bmatrix} < 0. \quad (5.16)$$

Set $G_{hh} = G$. If the equality $PB_h - B_h G = 0$ holds, then

$$\begin{bmatrix} PA_h + B_h K_h C_h + (*) & (*) \\ \varepsilon\begin{bmatrix} PA_h + K_h C_h \\ K_h C_h \end{bmatrix} & -\varepsilon\begin{bmatrix} 2P & 0 \\ 0 & G + G^T \end{bmatrix} \end{bmatrix} < 0, \quad (5.17)$$

which by means of the Schur complement produces

$$PA_h + B_h K_h C_h + (*) + \varepsilon(*)\begin{bmatrix} 2P & 0 \\ 0 & G + G^T \end{bmatrix}^{-1}\begin{bmatrix} PA_h + K_h C_h \\ K_h C_h \end{bmatrix} < 0. \quad (5.18)$$

Considering a sufficiently small $\varepsilon > 0$, conditions similar to those of the *P-Problem* appear:

$$PA_h + B_h K_h C_h + (*) < 0, \quad PB_h - B_h G = 0, \quad P = P^T > 0. \quad (5.19)$$

Conditions (5.19) can be seen as the *P-Problem* for nonlinear systems in standard TS form. Moreover, when a standard linear system is under study, conditions (5.19) yield exactly the *P-Problem* conditions in (5.5). Note that in this case, η has been set as $\eta = B_h$. ■

The following example illustrates the performance of Theorem 5.1 when different options for η are tested.

Example 5.1. Consider a TS descriptor model of the form (5.1) with $r_a = r_e = 2$ and

$$E_1 = \begin{bmatrix} 1.05 & 0.7 & 0.7 \\ -0.1 & 1.1 & -0.2 \\ 0.1 & 0.5 & 0.9 - a \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.8 + b & 0.8 & 0.7 \\ -0.9 & 1.1 & -0.2 \\ 0.4 & 0.5 & 0.6 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1.15 & 0.1 & 1.8 + b \\ 0.3 & -1.3 & -0.5 \\ -0.1 & 0.8 & -0.8 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1.2 & -0.3 & -0.1 \\ 0.4 & -0.6 & 0.3 \\ -0.2 & -0.2 & -0.2 - a \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.6 & 1.2 \\ 0.3 & 1.5 - a \\ -0.6 & 1.3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1.3 & 2.1 \\ -2.7 & 0.5 \\ 1.5 & 1.6 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.4 \\ 1 \\ 0 \end{bmatrix}^T, \text{ and}$$

$C_2 = \begin{bmatrix} 0.8 & 1 & 0 \end{bmatrix}$, where $a \in [0 \;\; 1]$, $b \in [-0.5, 1]$ are real-valued parameters. Three configurations for Theorem 5.1 have been tested:

$$\text{Conf}_1\colon \eta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T, \quad \text{Conf}_2\colon \eta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^T, \quad \text{and} \quad \text{Conf}_3\colon \eta = B_h.$$

Figure 5.1 shows the feasible solution set for each of the configurations: Configuration 1 (O), Configuration 2 $(\times)$, and Configuration 3 (∇). In addition, Figure 5.1 illustrates Remark 5.2, since different solution sets have been obtained for different selections of the free matrix $\eta \in \mathbb{R}^{n_x \times n_u}$, i.e., they do not include each other.

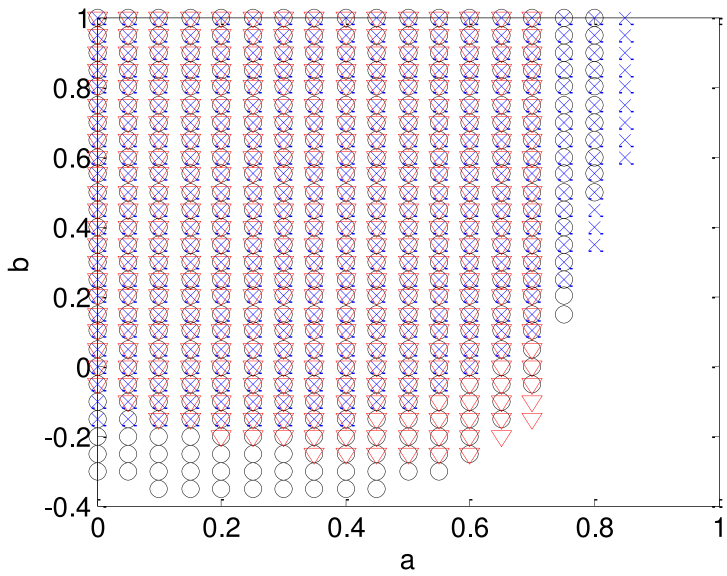

Figure 5.1. Feasible solution set in Example 5.1.

Observe that when Configuration 3 is implemented, another sum must be taken into account, turning the problem from three sums – $\sum_{i_1=1}^{r_a}\sum_{i_2=1}^{r_a}\sum_{j_1=1}^{r_e} h_{i_1}(\bullet)h_{i_2}(\bullet)v_{j_1}(\bullet)$ – to four sums – $\sum_{i_1=1}^{r_a}\sum_{i_2=1}^{r_a}\sum_{i_3=1}^{r_a}\sum_{j_1=1}^{r_e} h_{i_1}(\bullet)h_{i_2}(\bullet)h_{i_3}(\bullet)v_{j_1}(\bullet)$.♦

5.2. Discrete-time TS descriptor models

This section presents a static output feedback controller design for discrete-time TS descriptor models. The proposed method exploits the discrete-time nature of the TS model by the use of delayed Lyapunov functions, similarly to the previous chapters.

5.2.1. Problem statement

Consider the following discrete-time TS descriptor model (Taniguchi et al., 1999):

$$E_v x_{k+1} = A_h x_k + B_h u_k, \qquad y_k = C_h x_k. \tag{5.20}$$

Recall that E_v is full rank, thus a standard TS model can be constructed. For standard TS models, in (Chadli and Guerra, 2012; Kau et al., 2007; Lo and Lin, 2003), the following PDC control law is used

$$u_k = K_h y_k = K_h C_h x_k. \tag{5.21}$$

In case of the TS descriptor model (5.20), the following control law is used:

$$u_k = \left(G_{hhh^-v}\right)^{-1} K_{hh^-v} y_k. \tag{5.22}$$

The control law contains past samples incorporated via the MFs similar to Chapter 3.

5.2.2. Results

Controller design

The TS descriptor model (5.20) and the control law (5.22) can be expressed as:

$$\begin{bmatrix} A_h & -E_v & B_h \\ \left(G_{hhh^-v}\right)^{-1} K_{hh^-v} C_h & 0_{n_u \times n_x} & -I_{n_u} \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \\ u_k \end{bmatrix} = 0. \tag{5.23}$$

Consider the following delayed Lyapunov function (Guerra et al., 2012b):

$$V(x_k) = x_k^T \left(\sum_{i_1=1}^{r_a} h_{i_1}\left(z(k-1) P_{i_1}\right) \right) x_k = x_k^T P_{h^-} x_k > 0, \tag{5.24}$$

where $P_{i_1} = P_{i_1}^T > 0$, $i_1 \in \{1,2,\ldots,r_a\}$. The variation of the Lyapunov function (5.24) gives $\Delta V(x_k) = x_{k+1}^T P_h x_{k+1} - x_k^T P_{h^-} x_k$, which by adding the null-term $u_k^T 0 u_k$ can be rewritten as

$$\Delta V(x_k) = \begin{bmatrix} x_k \\ x_{k+1} \\ u_k \end{bmatrix}^T \begin{bmatrix} -P_{h^-} & 0 & 0 \\ 0 & P_h & 0 \\ 0 & 0 & 0_{n_u} \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \\ u_k \end{bmatrix} < 0. \tag{5.25}$$

Taking $\mathcal{X} = \begin{bmatrix} x_k \\ x_{k+1} \\ u_k \end{bmatrix}$, $\mathcal{W} = \begin{bmatrix} A_h & -E_v & B_h \\ \left(G_{hhh^-v}\right)^{-1} K_{hh^-v} C_h & 0 & -I \end{bmatrix}$, and $\mathcal{Q} = \begin{bmatrix} -P_{h^-} & 0 & 0 \\ 0 & P_h & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Via Finsler's lemma equality (5.23) together with inequality (5.25) yield

$$\mathcal{M} \begin{bmatrix} A_h & -E_v & B_h \\ \left(G_{hhh^-v}\right)^{-1} K_{hh^-v} C_h & 0 & -I \end{bmatrix} + (*) + \begin{bmatrix} -P_{h^-} & 0 & 0 \\ 0 & P_h & 0 \\ 0 & 0 & 0 \end{bmatrix} < 0, \tag{5.26}$$

where $\mathcal{M} \in \mathbb{R}^{(2n_x+n_u)\times(n_x+n_u)}$ is a free matrix. Therefore, the following result can be stated.

Theorem 5.2. Consider $\eta \in \mathbb{R}^{n_x \times n_u}$ a constant matrix. The TS descriptor model (5.20) under the control law (5.22) is asymptotically stable if there exist matrices $P_{i_2} = P_{i_2}^T > 0$, $M_{3i_2i_x}$ $G_{i_1i_2i_xj_1}$, and $K_{i_2i_xj_1}$, $i_1, i_2, i_x \in \{1, 2, \ldots, r_a\}$, $j_1 \in \{1, 2, \ldots, r_e\}$ such that

$$\Upsilon_{i_1i_1i_x}^{j_1} < 0, \ \forall i_1, i_x, j_1; \qquad \frac{2}{r_a - 1} \Upsilon_{i_1i_1i_x}^{j_1} + \Upsilon_{i_1i_2i_x}^{j_1} + \Upsilon_{i_2i_1i_x}^{j_1} < 0, \ \forall i_x, j_1, \ i_i \neq i_2, \tag{5.27}$$

hold with

$$\Upsilon_{i_1i_2i_x}^{j_1} = \begin{bmatrix} -P_{i_x} & (*) & (*) \\ M_{3i_2i_x} A_{i_1} + \eta K_{i_2i_xj_1} C_{i_1} & -M_{3i_2i_x} E_{j_1} - E_{j_1}^T M_{3i_2i_x}^T + P_{i_2} & (*) \\ K_{i_2i_xj_1} C_{i_1} & \left(M_{3i_2i_x} B_{i_1} - \eta G_{i_1i_2i_xj_1}\right)^T & -G_{i_1i_2i_xj_1} - G_{i_1i_2i_xj_1}^T \end{bmatrix}. \tag{5.28}$$

Proof. Recall (5.26), and select the free matrix as:

$$\mathcal{M} = \begin{bmatrix} 0_{n_x} & 0_{n_x \times n_u} \\ M_{3hh^-} & \eta G_{hhh^-v} \\ 0_{n_u \times n_x} & G_{hhh^-v} \end{bmatrix}, \tag{5.29}$$

where $M_{3hh^-} \in \mathbb{R}^{n_x \times n_x}$ and $G_{hhh^-v} \in \mathbb{R}^{n_u \times n_u}$ are decision variables. As previously, $\eta \in \mathbb{R}^{n_x \times n_u}$ is not a decision variable (Remark 5.2). Hence, (5.26) produces

$$\Upsilon_{hhh^-}^{v} := \begin{bmatrix} -P_{h^-} & (*) & (*) \\ M_{3hh^-} A_h + \eta K_{hh^-v} C_h & -M_{3hh^-} E_v - E_v^T M_{3hh^-}^T + P_h & (*) \\ K_{hh^-v} C_h & \left(M_{3hh^-} B_h - \eta G_{hhh^-v}\right)^T & -G_{hhh^-v} - G_{hhh^-v}^T \end{bmatrix} < 0, \tag{5.30}$$

which by means of Lemma B.3 yields (5.27). ■

Example 5.2. Consider the following nonlinear TS descriptor:

$$E(x_k)x_{k+1} = A(x_k)x_k + Bu_k, \qquad y_k = C(x_k)x_k, \tag{5.31}$$

with $E(x_k) = \begin{bmatrix} 0.9 & 0.1-0.1x_2 \\ -0.4-0.15x_2 & 1.1 \end{bmatrix}$, $A(x_k) = \begin{bmatrix} -0.5 & 0.8-0.1x_2^2 \\ -1.2 & 0.5 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $C(x_k) = \begin{bmatrix} 0 & 1.3-0.15x_2^2 \end{bmatrix}$. Consider the compact set $\Omega \in \{x : x_1 \in \mathbb{R}, |x_2| \le 2\}$: inside this compact set $E(x)$ is nonsingular: $\det(E(x)) \ne 0$, $\forall x \in \Omega$.

The TS descriptor model for (5.31) has $r_e = 2$ due to x_2 (left-hand side) and $r_a = 2$ due to x_2^2 (right-hand side). The matrices are $E_1 = \begin{bmatrix} 0.9 & 0.3 \\ -0.7 & 1.1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0.9 & -0.1 \\ -0.1 & 1.1 \end{bmatrix}$, $A_1 = \begin{bmatrix} -0.5 & 1.2 \\ -1.2 & 0.5 \end{bmatrix}$, $A_2 = \begin{bmatrix} -0.5 & 0.8 \\ -1.2 & 0.5 \end{bmatrix}$, $B_i = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $i = 1,2$, $C_1 = \begin{bmatrix} 0 & -0.7 \end{bmatrix}$, and $C_2 = \begin{bmatrix} 0 & 1.3 \end{bmatrix}$. On the right-hand side, the MFs are $h_1 = x_2^2/4$ and $h_2 = 1 - h_1$. On the left-hand side the MFs are $v_1 = (x_2 + 2)/4$ and $v_2 = 1 - v_1$. These sets of MFs hold the convex sum property in Ω.

Three configurations have been tested using the conditions in Theorem 5.2:

$$\text{Conf}_1 : \eta = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{Conf}_2 : \eta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{Conf}_3 : \eta = B_h.$$

Configuration 1 provides no feasible solution. In this case, Configurations 2 and 3 are exactly the same since $B_h = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. Theorem 5.2 with Configuration 2 provides the following matrices:

$$P_1 = \begin{bmatrix} 0.48 & -0.08 \\ -0.08 & 0.25 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.60 & -0.14 \\ -0.14 & 0.28 \end{bmatrix},$$
$$K_{111} = -0.25, \quad K_{222} = -0.10, \quad K_{121} = -0.23, \quad K_{122} = -0.27,$$
$$H_{1111} = 0.27, \quad H_{2222} = 0.30, \quad H_{1122} = 0.28, \quad H_{1221} = 0.45.$$

Figure 5.2 shows simulation results for initial conditions $x(0) = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$.

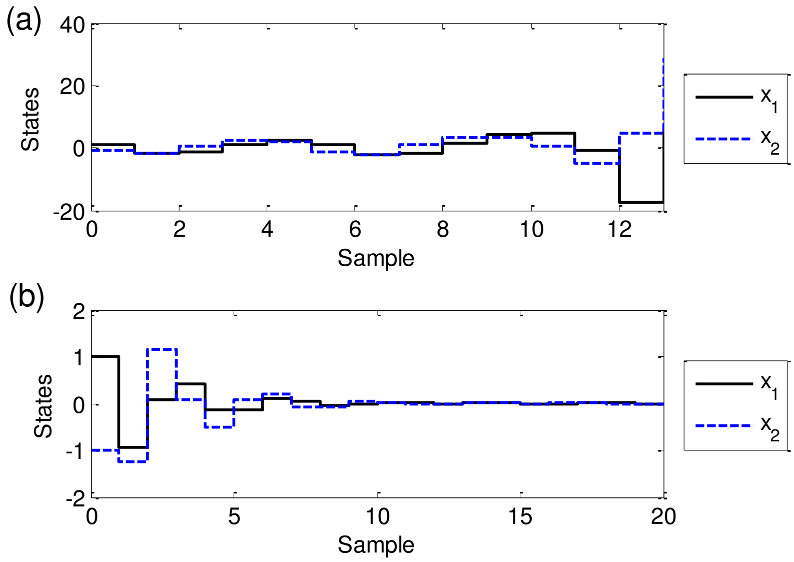

Figure 5.2. (a) State trajectories of the open-loop systems, (b) State trajectories of the closed-loop system in Example 5.2.

In order to apply the SOFC conditions in (Chadli and Guerra, 2012), it is necessary to write (5.31) in the standard form: $x_{k+1}=E^{-1}(x_k)(A(x_k)x_k+Bu_k)$ with $E^{-1}(x_k)=\frac{1}{\det(E(x_k))}\begin{bmatrix} 1.1 & -0.1+0.1x_2 \\ 0.4+0.15x_2 & 0.9 \end{bmatrix}$; this leads to $r=2^3=8$ linear models. Note that the input matrix B is no longer constant. Now, by employing conditions in (Chadli and Guerra, 2012), Configurations 1 and 2 yield $r^3+r=520$ LMIs, while Configuration 3 leads $r^4+r=4104$. Configurations 1 and 2 were not feasible; Configuration 3 gave numerical problems. ♦

Robust control

Consider the following uncertain TS descriptor model:

$$\begin{aligned}(E_v+\Delta E_v)x_{k+1}&=(A_h+\Delta A_h)x_k+(B_h+\Delta B_h)u_k\\ y_k&=(C_h+\Delta C_h)x_k,\end{aligned} \tag{5.32}$$

where $\Delta A_h=D_h^a\Delta_a(t)F_h^a$, $\Delta B_h=D_h^b\Delta_b(t)F_h^b$, $\Delta C_h=D_h^c\Delta_c(t)F_h^c$, and $\Delta E_v=D_v^e\Delta_e(t)F_v^e$ with $\Delta_a^T(t)\Delta_a(t)\le I$, $\Delta_b^T(t)\Delta_b(t)\le I$, $\Delta_c^T(t)\Delta_c(t)\le I$, and $\Delta_e^T(t)\Delta_e(t)\le I$. The TS descriptor model (5.32) together with the control law (5.6) can be expressed as

$$\begin{bmatrix} A_h + \Delta A_h & -E_v - \Delta E_v & B_h + \Delta B_h \\ \left(G_{hhh^-v}\right)^{-1} K_{hh^-v}\left(C_h + \Delta C_h\right) & 0_{n_u \times n_x} & -I_{n_u} \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \\ u_k \end{bmatrix} = 0. \tag{5.33}$$

Consider the delayed Lyapunov function (5.24). Through Finsler's lemma, its variation (5.25) under constraint (5.33) produces

$$\Upsilon^v_{hhh^-} + \mathcal{M}\Delta\bar{A} + \Delta\bar{A}^T \mathcal{M}^T < 0, \tag{5.34}$$

with $\Upsilon^v_{hhh^-}$ as in (5.30), $\mathcal{M}$ as in (5.29), and $\Delta\bar{A} = \begin{bmatrix} \Delta A_h & -\Delta E_v & \Delta B_h \\ \left(G_{hhh^-}\right)^{-1} K_{hh^-}\Delta C_h & 0_{n_u \times n_x} & 0_{n_u} \end{bmatrix}$. $\Delta\bar{A}$ can be written as $\Delta\bar{A} = \mathcal{D}\bar{\Delta}\mathcal{F}$ where

$$\mathcal{D} = \begin{bmatrix} D^a_h & D^b_h & 0 & -D^e_v \\ 0 & 0 & \left(G_{hhh^-v}\right)^{-1} K_{hh^-v} D^c_h & 0 \end{bmatrix}, \; \bar{\Delta} = \begin{bmatrix} \Delta_a & 0 & 0 & 0 \\ 0 & \Delta_b & 0 & 0 \\ 0 & 0 & \Delta_c & 0 \\ 0 & 0 & 0 & \Delta_e \end{bmatrix}, \; \mathcal{F} = \begin{bmatrix} F^a_h & 0 & 0 \\ 0 & 0 & F^b_h \\ F^c_h & 0 & 0 \\ 0 & F^e_v & 0 \end{bmatrix}.$$

Then, expression (5.34) produces

$$\Upsilon^v_{hhh^-} + \mathcal{M}\mathcal{D}\bar{\Delta}\mathcal{F} + \mathcal{F}^T \bar{\Delta}^T \mathcal{D}^T \mathcal{M}^T < 0. \tag{5.35}$$

By employing Property A.4 with $\mathcal{N} = \mathcal{M}\mathcal{D}\bar{\Delta}$, $\mathcal{R} = \mathcal{F}$, and $\mathcal{Q} = \mathcal{T} = diag\left[\tau^a_{hh^-v} I, \; \tau^b_{hh^-v} I, \; \tau^c_{hh^-v} I, \; \tau^e_{hhh^-} I\right]$, $\mathcal{T} = \mathcal{T}^T > 0$, (5.35) can be written as $\Upsilon^v_{hhh^-} + \mathcal{M}\mathcal{D}\bar{\Delta}\mathcal{T}^{-1}\bar{\Delta}^T \mathcal{D}^T \mathcal{M}^T + \mathcal{F}^T \mathcal{T}\mathcal{F} < 0$. Recall that $\bar{\Delta}^T\bar{\Delta} \le I$, and thus:

$$\Upsilon^v_{hhh^-} + \mathcal{M}\mathcal{D}\mathcal{T}^{-1}\mathcal{D}^T \mathcal{M}^T + \mathcal{F}^T \mathcal{T}\mathcal{F} < 0. \tag{5.36}$$

Therefore, the following result can be stated.

Theorem 5.3. Consider $\eta \in \mathbb{R}^{n_x \times n_u}$ a constant matrix. The uncertain TS descriptor system (5.32) together with the control law (5.22) is robustly asymptotically stable if there exist matrices $P_{i_2} = P^T_{i_2} > 0$, $M_{3i_2i_x}$ $G_{i_1i_2i_xj_1}$, $K_{i_2i_xj_1}$, and scalars $\tau^a_{i_2i_xj_1} > 0$, $\tau^b_{i_2i_xj_1} > 0$, $\tau^c_{i_2i_xj_1} > 0$, $\tau^e_{i_1i_2i_x} > 0$, $i_1, i_2, i_x \in \{1, 2, \ldots, r_a\}$, $j_1 \in \{1, 2, \ldots, r_e\}$, such that

$$\Gamma^{j_1}_{i_1i_1i_x} < 0, \; \forall i_1, i_x, j_1; \qquad \frac{2}{r_a - 1}\Gamma^{j_1}_{i_1i_1i_x} + \Gamma^{j_1}_{i_1i_2i_x} + \Gamma^{j_1}_{i_2i_1i_x} < 0, \; \forall i_x, j_1, \; i_i \ne i_2, \tag{5.37}$$

hold with

$$
\Gamma^{j_1}_{i_1 i_1 i_x} = \left[\begin{array}{c|c|c}
\Upsilon^{j_1}_{i_1 i_2 i_x} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ M_{3 i_2 i_x} D^a_{i_1} & M_{3 i_2 i_x} D^b_{i_1} & \eta K_{i_2 i_x j_1} D^c_{i_1} & -M_{3 i_2 i_x} D^e_{j_1} \\ 0 & 0 & K_{i_2 i_x j_1} D^c_{i_1} & 0 \end{bmatrix} & \begin{bmatrix} F^a_{i_1} & 0 & 0 \\ 0 & 0 & F^b_{i_1} \\ F^c_{i_1} & 0 & 0 \\ 0 & F^e_{j_1} & 0 \end{bmatrix}^T T_{i_1 i_2 i_x j_1} \\
\hline
(*) & -T_{i_1 i_2 i_x j_1} & 0 \\
\hline
(*) & (*) & -T_{i_1 i_2 i_x j_1}
\end{array}\right],
$$

where $\Upsilon^{j_1}_{i_1 i_2 i_x}$ has been defined in (5.28) and $\mathrm{T}_{i_1 i_2 i_x j_1} = diag\left(\tau^a_{i_2 i_x j_1} I, \tau^b_{i_2 i_x j_1} I, \tau^c_{i_2 i_x j_1} I, \tau^e_{i_1 i_2 i_x} I\right)$.

Proof. Note that inequality (5.36) can be rewritten as

$$
\Upsilon^v_{hhh^-} + \begin{bmatrix} \mathcal{M}\mathcal{D} & \mathcal{F}^T \mathcal{T} \end{bmatrix} \begin{bmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{T} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{D}^T \mathcal{M}^T \\ \mathcal{T}\mathcal{F} \end{bmatrix} < 0, \tag{5.38}
$$

which by means of the Schur complement gives

$$
\Gamma^v_{hhh^-} := \left[\begin{array}{c|c|c} \Upsilon^v_{hhh^-} & \mathcal{M}\mathcal{D} & \mathcal{F}^T \mathcal{T} \\ \hline (*) & -\mathcal{T} & 0 \\ \hline (*) & (*) & -\mathcal{T} \end{array}\right] < 0. \tag{5.39}
$$

By employing Lemma B.3 the proof is concluded. ■

To show the potential of the proposed approach and in order to compare it with other works, we propose the following corollary that applies the methodology to standard TS models, i.e. with, $E_v = I_{n_x}$, $x_{k+1} = A_h x_k + B_h u_k$.

Corollary 5.2. Consider the SOFC $u_k = G^{-1}_{hhh^-} K_{hh^-} y_k$ and $\eta \in \mathbb{R}^{n_x \times n_u}$ a constant matrix. The standard TS model is globally asymptotically stable if there exist matrices $P_{i_2} = P^T_{i_2} > 0$, $M_{3 i_2 i_x}$, $G_{i_1 i_2 i_x}$, and $K_{i_2 i_x}$, $i_1, i_2, i_x \in \{1, 2, \ldots, r_a\}$, such that:

$$\Upsilon_{i_1 i_1 i_x} < 0,\ \forall i_1, i_x; \qquad \frac{2}{r_a - 1}\Upsilon_{i_1 i_1 i_x} + \Upsilon_{i_1 i_2 i_x} + \Upsilon_{i_2 i_1 i_x} < 0\ \forall i_x,\ i_i \neq i_2, \tag{5.40}$$

hold with

$$\Upsilon_{i_1 i_2 i_x} = \begin{bmatrix} -P_{i_x} & (*) & (*) \\ M_{3 i_2 i_x} A_{i_1} + \eta K_{i_2 i_x} C_{i_1} & -M_{3 i_2 i_x} - M_{3 i_2 i_x}^T + P_{i_2} & (*) \\ K_{i_2 i_x} C_{i_1} & \left(M_{3 i_2 i_x} B_{i_1} - \eta G_{i_1 i_2 i_x} \right)^T & -G_{i_1 i_2 i_x} - G_{i_1 i_2 i_x}^T \end{bmatrix}.$$

Similar reasoning applies for the uncertain model (5.32): if $E_v = I_{n_x}$ and $\Delta E_v = 0$ then:

$$x_{k+1} = \left(A_h + \Delta A_h \right) x_k + \left(B_h + \Delta B_h \right) u_k, \qquad y_k = \left(C_h + \Delta C_h \right) x_k. \tag{5.41}$$

Thus (5.41) together the control law $u_k = G_{hhh^-}^{-1} K_{hh^-} y_k$ is robustly asymptotically stable if there exist matrices $P_{i_2} = P_{i_2}^T > 0$, $M_{3 i_2 i_x}$ $G_{i_1 i_2 i_x}$, $K_{i_2 i_x}$, and scalars $\tau_{i_2 i_x}^a > 0$, $\tau_{i_2 i_x}^b > 0$, $\tau_{i_2 i_x}^c > 0$, $i_1, i_2, i_x \in \{1, 2, \ldots, r_a\}$, such that:

$$\Gamma_{i_1 i_1 i_x} < 0,\ \forall i_1, i_x; \qquad \frac{2}{r_a - 1}\Gamma_{i_1 i_1 i_x} + \Gamma_{i_1 i_2 i_x} + \Gamma_{i_2 i_1 i_x} < 0\ \forall i_x,\ i_i \neq i_2, \tag{5.42}$$

hold with

$$\Gamma_{i_1 i_1 i_x} = \left[\begin{array}{c:c:c} \Upsilon_{i_1 i_2 i_x} & \begin{bmatrix} 0 & 0 & 0 \\ M_{3 i_2 i_x} D_{i_1}^a & M_{3 i_2 i_x} D_{i_1}^b & \eta K_{i_2 i_x} D_{i_1}^c \\ 0 & 0 & K_{i_2 i_x} D_{i_1}^c \end{bmatrix} & \begin{bmatrix} F_{i_1}^a & 0 & 0 \\ 0 & 0 & F_{i_1}^b \\ F_{i_1}^c & 0 & 0 \end{bmatrix}^T T_{i_1 i_2 i_x} \\ \hdashline (*) & -T_{i_1 i_2 i_x} & 0 \\ \hdashline (*) & (*) & -T_{i_1 i_2 i_x} \end{array} \right].$$

Proof. The proofs are straightforward from Theorems 5.2 and 5.3. ■

When $E_v = I$, the following numerical example compares the performance of Corollary 5.2 and Theorem 2 in (Chadli and Guerra, 2012). The example is adapted from (Chadli and

Guerra, 2012), by including a real-valued parameter in the uncertain terms. Also for this example different values for the arbitrary matrix η are tested (see Remark 5.2).

Example 5.3. Consider a TS model as in (5.41) with $r_a = 2$ and local matrices as follows (Chadli and Guerra, 2012):

$$A_1 = \begin{bmatrix} 0.55 & 0.12 & 0.27 & 0.23 \\ 0.37 & 0.51 & -0.39 & 0.36 \\ -0.14 & -0.25 & 0.65 & 0.47 \\ -0.53 & -0.15 & 0.22 & 0.46 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0.62 & -0.29 & -0.31 & 0.28 \\ 0.24 & 0.59 & -0.23 & 0.19 \\ 0.19 & -0.37 & 0.43 & 0.15 \\ 0.16 & 0.31 & 0.22 & 0.55 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.4 \\ -0.4 \\ 1.5 \\ 1.2 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.25 \\ 0.20 \\ -0.35 \\ 0.20 \end{bmatrix}, C_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0 \\ 0.2 & 1 \\ 0 & 0 \end{bmatrix}^T, C_2 = \begin{bmatrix} 0.41 & 0.5 \\ 0 & 0 \\ 0 & 0.7 \\ 0 & 0 \end{bmatrix}^T, D_i^a = \begin{bmatrix} 0.1+\delta \\ 0.2 \\ 0 \\ 0 \end{bmatrix}, F_i^a = \begin{bmatrix} 0.1 \\ 0 \\ 0.1 \\ 0 \end{bmatrix}^T, D_i^c = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},$$

$$F_i^c = \begin{bmatrix} 0.1 & 0.1-\delta & 0 & 0.01 \end{bmatrix}, \; D_i^b = \begin{bmatrix} 0.1 & 0.1+\delta & 0 & 0.12 \end{bmatrix}^T, \; F_i^b = 0.3, \quad i = 1, 2.$$

The parameter introduced is $\delta > 0$. The goal is to design a SOFC for as large values of δ as possible. Table 5.1 summarizes the obtained results.

Table 5.1. Results for Example 5.3.

Approach	Variable	Maximum parameter value
Theorem 2 in (Chadli and Guerra, 2012)	$\eta = B_h$	$\delta = 0.1$
	$\eta = 0_{n_x \times n_u}$	$\delta = 0.3$
Corollary 5.2	$\eta = B_h$	$\delta = 1.0$
	$\eta = 0_{n_x \times n_u}$	$\delta = 1.5$

Table 5.1 shows that a larger value of δ is obtained when Corollary 5.2 is applied, i.e., the new approach allows stabilizing the system for a larger size of the uncertainty than the one in (Chadli and Guerra, 2012). Furthermore, since there are different output matrices, the conditions in (Kau et al., 2007) are difficult to fulfill. Moreover, in both examples the output is nonlinear. ♦

5.3. Summary and concluding remarks

An LMI approach has been presented to deal with the static output feedback controller design for both continuous and discrete time TS descriptor models. These conditions circumvent those in the literature in the sense that no equality and/or rank constraints, which for TS models are considered an important drawback, are needed in the design procedure. The obtained conditions are LMIs up to fixing the matrix $\eta \in \mathbb{R}^{n_x \times n_u}$.

Chapter 6. Concluding remarks and future research directions

This section summarizes the results presented in the thesis and discusses future research directions within the TS-LMI framework.

Throughout the thesis the following family of nonlinear descriptor models has been considered:

$$\begin{aligned} &\text{Continuous-time:} \quad E(x)\dot{x} = A(x)x + B(x)u, \quad y = C(x)x. \\ &\text{Discrete-time:} \quad E(x_k)x_{k+1} = A(x_k)x_k + B(x_k)u_k, \quad y_k = C(x_k)x_k. \end{aligned} \tag{6.1}$$

The developed results are based on the assumption that the matrix descriptor matrix $E(x)$ is regular at least in a compact set of the state-space Ω including the origin $\left(\exists(E(x))^{-1}\ \forall x \in \Omega\right)$. Several examples have shown the importance of keeping the original descriptor structure instead of computing a standard state-space model – this is possible since $E(x)$ is regular.

Three problems have been addressed: 1) State feedback control design, 2) Observer design, and 3) Output feedback control design.

To develop the conditions, both for the continuous and discrete-time case, a TS representation of the nonlinear models (6.1) has been used. The conditions are given in LMI terms.

For the continuous-time case, the *state feedback control design* has been carried out by means of the descriptor redundancy approach together with Finsler's lemma. By enlarging the set of feasible solutions, we have improved previous results in the literature. For the discrete-time case, a systematic methodology has been presented, which allows including past samples in the MFs used in the Lyapunov function as well as in the controller gains.

For the *observer design*, since no "pure" LMI conditions were available in the literature, we proposed a new observer structure in order to solve the problem. This new structure does not fix any decision variable a priori and the feasibility sets in comparison to previous methods are significantly enlarged. In the discrete-time case, LMI conditions have been developed for the design of state estimators, thus filling this gap in the literature.

The *output feedback controller design* has led to LMI conditions up to the selection of an auxiliary matrix. Depending on the selection of this slack matrix, different results may be obtained. Table 6.1 summarizes the contributions of this thesis.

Table 13.1. Contributions of the thesis, where CT stands for continuous-time and DT means discrete-time.

	Contributions	Tools	Publications
State Feedback	CT: Enlarge the solution set via parameter-dependent LMI conditions.	CT: Finsler's Lemma / Descriptor redundancy.	CT: FUZZ-IEEE 2013
	DT: Provide strict LMI constraints / Generalization to an arbitrarily delayed MFs.	DT: Finsler's Lemma / Delayed Lyapunov functions.	DT: FUZZ-IEEE 2014 / LFA 2014 / IEEE Trans. on Fuzzy Systems 2015.
Observer Design	CT: Overcome a BMI problem by providing strict LMI constraints / Application to unknown input observers.	CT: Auxiliary variable in the extended estimation error.	CT: AQTR 2014 / CDC 2015 / book chapter 2016 / Automatica 2015.
	DT: Provide strict LMI constraints / Generalization to an arbitrarily delayed MFs.	DT: Finsler's Lemma / Delayed Lyapunov functions.	DT: IFAC World Congress 2014 / CESCIT 2014.
Output Feedback	CT/DT: Provide LMI constraints up to fixing a variable.	CT/DT: Finsler's Lemma / Extended vector with the input.	CT: FUZZ-IEEE 2015 DT: CDC 2014

Within the TS-LMI framework for descriptor models, beside direct extensions such as including more performance criteria in the conditions, reducing the complexity of the LMI problems and so on, we can enumerate some future research directions.

6.1. Use of NQ Lyapunov functions

For the continuous-time case, only quadratic Lyapunov functions have been employed in this work. A future research direction could be the investigation of recent NQ Lyapunov functions used for standard TS models. Several possibilities can come at hand:

- The most interesting would be to extend the line-integral Lyapunov function (Rhee and Won, 2006) from TS to TS descriptors, therefore obtaining to global conditions. The Lyapunov function $V(x)=2\int_{\Gamma(0,x)} f(\psi)d\psi$, with $\Gamma(0,x)$ being any path from the origin to the current state $x\in\mathbb{R}^{n_x}$, $\psi\in\mathbb{R}^{n_x}$ is a dummy vector for the integral, and $d\psi\in\mathbb{R}^{n_x}$ an infinitesimal displacement vector. Nevertheless, the condition for line-integral, i.e., $\frac{\partial f_i(x)}{\partial x_j}=\frac{\partial f_j(x)}{\partial x_i}$, $i,j\in\{1,2,\dots,n_x\}$ seems a huge problem. This problem has been solved in an efficient manner only for *second-order* TS descriptor systems with certain specific structure in (Marquez et al., 2014).
- A second possible NQ approach is to extend the local approach given in (Bernal and Guerra, 2010). By introducing Lyapunov functions such as:

$$V(x)=x^T\left(\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\cdots\sum_{i_q=1}^{r}h_{i_1}h_{i_2}\cdots h_{i_q}(z)P_{i_1i_2\cdots i_q}\right)x$$

together with given *a priori* bounds $\left|\frac{\partial w_0^k}{\partial z_k}\dot{z}_k\right|\le\beta_k$, its ensured that the future trajectories do not to escape from the prescribed region (Pan et al., 2012).

- A third approach can be the extension of TS models to sum-of-squares (SOS) tools (Prajna et al., 2004). Without entering into details, the idea of the SOS approach is that with an even integer d, any polynomial $p(x)$ can be written as: $p(x)=\gamma^T(x)\Pi\gamma(x)$ where $\gamma(x)$ is a vector of monomials and Π is a matrix directly obtained from the coefficients of

$p(x)$. It represents a "natural" extension of the LMI tools; moreover, it has already been used in the TS framework (Bernal et al., 2011; Sala, 2009; Tanaka et al., 2009).

Nevertheless, for the moment, the two last ideas present the drawback of huge increase of the computational cost. Only reduced order models could be considered.

6.2. Unmeasurable premise variables

In general MFs may depend on unmeasurable variables. Considering that the MFs depend on state variables that are not measurable is a more challenging problem for the observer design. Effectively, in this case, within the quadratic framework, a continuous-time TS descriptor observer will write:

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\hat{x}} \\ \dot{\beta} \end{bmatrix} = \sum_{i=1}^{r_a}\sum_{j=1}^{r_e} h_i(\hat{z}) v_j(\hat{z}) \begin{bmatrix} 0 & I \\ A_i & -E_j \end{bmatrix}\begin{bmatrix} \hat{x} \\ \beta \end{bmatrix} + \begin{bmatrix} 0 \\ B_i \end{bmatrix} u + \left(\sum_{i=1}^{r_a} h_i(\hat{z}) P_i \right)^{-T} \begin{bmatrix} L_{1ij} \\ L_{2ij} \end{bmatrix} (y-\hat{y}).$$

or in shorthand: $\bar{E}\dot{\hat{\bar{x}}} = \bar{A}_{\hat{h}\hat{v}}\hat{\bar{x}} + \bar{B}_{\hat{h}} u + P_{\hat{h}}^{-T}\bar{L}_{\hat{h}\hat{v}}(y-\hat{y})$. Therefore, similar procedure as in Chapter 4, the extended estimation error writes

$$\bar{E}\dot{\bar{e}} = \left(\bar{A}_{\hat{h}\hat{v}} - P_{\hat{h}\hat{v}}^{-T}\bar{L}_{\hat{h}\hat{v}}\bar{C}_{\hat{h}}\right)\bar{e} + \left(\bar{A}_{hv} - \bar{A}_{\hat{h}\hat{v}}\right)\bar{x} + \left(\bar{B}_h - \bar{B}_{\hat{h}}\right)u - P_{\hat{h}\hat{v}}^{-T}\bar{L}_{\hat{h}\hat{v}}\left(\bar{C}_h - \bar{C}_{\hat{h}}\right)\bar{x}.$$

Thus, it introduces terms as $E_v - E_{\hat{v}}$, $A_h - A_{\hat{h}}$, $B_h - B_{\hat{h}}$, etc. First ideas that can be exploited are the use of Lipchitz conditions as in (Bergsten et al., 2002; Bergsten, P. and Driankov, D., 2001; Lendek et al., 2010); but a more promising way is the use of the differential mean value theorem as for standard TS models in (Ichalal et al., 2011). For example, consider the case where the premise vector is the state, i.e., $z(t) \equiv x(t)$, then $A_h - A_{\hat{h}} = \sum_{i=1}^{r_a}\sum_{j=1}^{r_a}\left(h_i(x) - h_i(\hat{x})\right)A_i$; it exists a constant $c \in \left]x, \hat{x}\right[$ such that: $h_i(x) - h_i(\hat{x}) = \frac{\partial h_i}{\partial x}(c)(x-\hat{x})$. In this case it is possible to turn back to the estimation error $e = x - \hat{x}$, and the computation of the bounds $\left\| \partial h_i(x)/\partial x \right\| \le \lambda_i$ can be done a priori as they only depend on the shape of the MFs $h_i(x)$ and not on their time-derivative. Of course, the convergence of the estimation error will only be ensured in a ball around the origin depending of the bounds λ_i.

6.3. Extending the results to singular systems: singular $E(x)$

Naturally, one of the next steps is to extend the results to nonlinear descriptor models with rank deficient descriptor matrix $E(x)$. This could be addressed by exploring a reduced state-space representation as in (Feng and Yagoubi, 2013). For example, in the discrete case, consider the following decomposition with $rank\left(E(x_k)\right) = q < n_x$ and $E_I(x_k) \in \mathbb{R}^{n_x \times q}$, $E_D(x_k) \in \mathbb{R}^{n_x \times (n_x - q)}$:

$$\begin{bmatrix} E_I(x_k) & E_D(x_k) \end{bmatrix} \begin{bmatrix} x_{k+1}^1 \\ x_{k+1}^2 \end{bmatrix} = \begin{bmatrix} A_1(x_k) & A_2(x_k) \\ A_3(x_k) & A_4(x_k) \end{bmatrix} \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix} + \begin{bmatrix} B_1(x_k) \\ B_2(x_k) \end{bmatrix} u_k. \tag{6.2}$$

If it is possible to find a change of variable $\begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix} = \begin{bmatrix} I_q & -\Lambda \\ 0 & I_{n_x - q} \end{bmatrix} \begin{bmatrix} \xi_k^1 \\ x_k^2 \end{bmatrix}$, the descriptor (6.2) can be rewritten as:

$$\begin{bmatrix} E_I(x_k) & 0 \end{bmatrix} \begin{bmatrix} \xi_{k+1}^1 \\ x_{k+1}^2 \end{bmatrix} = \begin{bmatrix} A_1(x_k) & A_2(x_k) - A_1(x_k)\Lambda \\ A_3(x_k) & A_4(x_k) - A_3(x_k)\Lambda \end{bmatrix} \begin{bmatrix} \xi_k^1 \\ x_k^2 \end{bmatrix} + \begin{bmatrix} B_1(x_k) \\ B_2(x_k) \end{bmatrix} u_k.$$

This is a first track in the sense that for the change of variables Λ is not state dependent. Then in a sense for the vector $\xi_k^1 \in \mathbb{R}^q$, we are turning back to a descriptor with $rank\left(E_I(x_k)\right) = q$. Naturally, the control has to be re-designed in order to avoid the feedback of future states. Moreover, a TS form should be expressed only after the transformation in order to keep a lower number of vertices.

6.4. Fault diagnosis

Extension to fault diagnosis and/or fault tolerant control seems also natural. For instance, in (Koenig, 2005; Marx et al., 2003), an asymptotic estimation of both states and failures are obtained via proportional-integral (PI) observers. This approach has been developed for linear singular systems. In (Marx et al., 2007), the fault diagnosis of TS descriptor models (with a constant and singular descriptor matrix) has been addressed via an observer whose structure is not in the descriptor form:

$$\dot{\xi} = \sum_{i=1}^{r_a} h_i(z)\left(N_i \xi + M_i u + L_i y\right), \qquad \hat{x} = \xi + T_2 y, \tag{6.3}$$

where $\xi \in \mathbb{R}^{n_x}$ is an auxiliary variable, $\hat{x} \in \mathbb{R}^{n_x}$ is the estimated vector, N_i, M_i, L_i, and T_2, $i \in \{1, 2, \ldots, r_a\}$ are matrices of appropriate dimensions to be designed. Since this procedure has been stated for singular TS models, by using descriptor-redundancy (see Chapter 2), we can induce a singular TS system and therefore investigate the design conditions for an observer with a structure similar to (6.3).

6.5. Real-time applications

The work presented is also a preliminary theoretical study to cope with real-time problems. Effectively, applications at LAMIH UMR CNRS 8201 include the use of parallel robot manipulators and other mechanical systems that could be subject to the descriptor TS modeling. For parallel robots, preliminary results have been obtained on 2-DOF planar parallel robot so-called *biglide* (Vermeiren et al., 2012). Now, a very challenging problem to be faced concerns the step-crossing feasibility of a two-wheeled transporter (Allouche et al., 2014).

Another challenging problem concerns disabled people in wheelchair. The problem is the estimation of the forces in the shoulder during the push with electrical assistance; this problem is decomposed into two phases. The first one consists in an observer that from the measured speed of the wheels is able to compute the torque applied on the wheel by the person (Mohammad and Guerra, 2015). Once this estimated torque and the model of an arm are available, the goal is to find the efforts in the shoulder. The final aim is to produce an electrical assistance system that adapts to the estimated efforts as well as detects possible dissymmetry between the sides and compensates in real time. This activity is currently ongoing as a nonlinear descriptor has been designed in (Dequidt, 2015).

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Appendix A. Matrix inequalities

A.1. Some matrix properties

Lemma A.1. **(Finsler's lemma)** (de Oliveira and Skelton, 2001). Let $\chi \in \mathbb{R}^n$, $\mathcal{Q} = \mathcal{Q}^T \in \mathbb{R}^{n\times n}$, and $\mathcal{W} \in \mathbb{R}^{m\times n}$ such that $rank(\mathcal{W}) < n$; the following expressions are equivalent:

a) $\chi^T \mathcal{Q} \chi < 0, \quad \forall \chi \in \{\chi \in \mathbb{R}^n : \chi \neq 0, \mathcal{W}\chi = 0\}.$

b) $\exists \mathcal{M} \in \mathbb{R}^{n\times m} : \mathcal{M}\mathcal{W} + \mathcal{W}^T \mathcal{M}^T + \mathcal{Q} < 0.$

Property A.1 (Schur complement). Let $\mathcal{M} = \mathcal{M}^T = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}$, with M_{11} and M_{22} square matrices of appropriate dimensions. Then:

$$\mathcal{M} < 0 \iff \begin{cases} M_{11} < 0 \\ M_{22} - M_{12}^T M_{11}^{-1} M_{12} < 0 \end{cases} \iff \begin{cases} M_{22} < 0 \\ M_{11} - M_{12} M_{22}^{-1} M_{12}^T < 0 \end{cases}$$

Property A.2 (Congruence transformation). Consider $\mathcal{Q} = \mathcal{Q}^T \in \mathbb{R}^{n\times n}$ and a nonsingular matrix $\mathcal{M} \in \mathbb{R}^{n\times n}$. The following expression holds:

$$\left.\begin{matrix} \mathcal{Q} < 0 \\ \mathcal{M} \end{matrix}\right\} \iff \mathcal{M}\mathcal{Q}\mathcal{M}^T < 0; \qquad \left.\begin{matrix} \mathcal{Q} > 0 \\ \mathcal{M} \end{matrix}\right\} \iff \mathcal{M}\mathcal{Q}\mathcal{M}^T > 0. \tag{A.1}$$

Property A.3. Let $\mathcal{Q} = \mathcal{Q}^T > 0$ and $\mathcal{W}$ be matrices of appropriate sizes. The following expression holds:

$$(\mathcal{W}-\mathcal{Q})^T \mathcal{Q}^{-1}(\mathcal{W}-\mathcal{Q}) \geq 0 \Leftrightarrow \mathcal{W}^T\mathcal{Q}^{-1}\mathcal{W} \geq \mathcal{W}+\mathcal{W}^T-\mathcal{Q}. \tag{A.2}$$

Property A.4. Let $\mathcal{Q}=\mathcal{Q}^T>0$, $\mathcal{R}$ and $\mathcal{N}$ be matrices of appropriate sizes. The following expression holds:

$$\mathcal{R}^T\mathcal{N}+\mathcal{N}^T\mathcal{R} \leq \mathcal{R}^T\mathcal{Q}\mathcal{R}+\mathcal{N}^T\mathcal{Q}^{-1}\mathcal{N} \tag{A.3}$$

A.2. Linear matrix inequalities

A short introduction to linear matrix inequalities (LMIs) is given in this section. An LMI is a set of mathematical expressions whose variables are linearly-related matrices. A formal definition is (Boyd et al., 1994; Duan and Yu, 2013; Scherer and Weiland, 2005):

$$\begin{aligned} F(x) &= F_0 + x_1F_1 + X_2F_2 + \cdots + x_dF_d \\ &= F_0 + \sum_{i=1}^{d} x_iF_i < 0, \end{aligned} \tag{A.4}$$

where $x \in \mathbb{R}^d$ is the vector of decision variables and $F_j = F_j^T$, $j \in \{0,1,\ldots,d\}$ are known constant matrices. The symbol $<$ stands for negative definitiveness, while $>$ means positive definitiveness. In addition, non-strict LMIs can appear as $F(x) \leq 0$ (negative semi-definitiveness).

The feasibility set or the set of solutions of the LMI (A.4), denoted by $S = \{x \in \mathbb{R}^d : F(x) < 0\}$, is a convex subset of $\mathbb{R}^d$. Finding a solution of the LMI (A.4) is a convex optimization problem (Boyd et al., 1994). Basically, there are three well-known problems that often appear in control problems:

- The *Feasibility Problem* (FP) consist of determining if there exist elements $x \in X$ such that $F(x) < 0$. The LMI $F(x) < 0$ is called feasible if such an x exists, otherwise it is said to be infeasible.
- The *Eigenvalue Problem* (EVP) is the minimization of a linear combination of the decision variables $c^T x$ under some LMI constraints:

$$\begin{array}{ll} \text{min} & c^T x \\ \text{subjet to} & F(x) < 0, \end{array} \tag{A.5}$$

where c is a vector of appropriate dimensions.

- The *Generalized Eigenvalue Problem* (GEVP) consists of minimizing the eigenvalues of a pair of matrices which depend affinely on a variable, subject to a set of LMI constraints:

$$\begin{array}{ll} \min & \lambda \in \mathbb{R} \\ \text{subject to} & \lambda B(x) - A(x) < 0, \quad B(x) < 0, \quad C(x) < 0, \end{array} \tag{A.6}$$

where λ is scalar, the matrices $A(x)$, $B(x)$, and $C(x)$ are symmetric and affine in x.

Often, matrices appear as decision variables. For instance, consider the Lyapunov inequality:

$$\mathrm{A}^T P + PA < 0, \tag{A.7}$$

where matrix $A \in \mathbb{R}^{n\times n}$ is known and $P = P^T \in \mathbb{R}^{n\times n}$ is the Lyapunov matrix to be found, i.e., P is a decision variable. Inequality (A.7) can be written in the form (A.4), as shown in Example A.1.

Example A.1. For sake of clarity, let us consider $A \in \mathbb{R}^{2\times 2}$, then $n = 2$. The decision variable P with $d = 3$ unknown entries can be rewritten as

$$P = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} = x_1 \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_1} + x_2 \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{E_2} + x_3 \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_3}, \tag{A.8}$$

thus, the inequality $\mathrm{A}^T P + PA < 0$ can be represented as

$$F(x) = \sum_{i=1}^{3} x_i F_i < 0, \tag{A.9}$$

with $F_i = A^T E_i + E_i A$, $i \in \{1,2,3\}$. ♦

A bilinear matrix inequality (BMI) has the general form (Van Antwerp and Braatz, 2000):

$$F(x, y) = F_0 + \sum_{i=1}^{d_1} x_i F_i + \sum_{j=1}^{d_2} y_j G_j + \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} x_i y_j H_{ij} > 0, \tag{A.10}$$

where $F_0 = (F_0)^T$, $F_i = (F_i)^T$, $G_j = (G_j)^T$, and $H_{ij} = (H_{ij})^T$, $i \in \{1,2,\ldots,d_1\}$, $j \in \{1,2,\ldots,d_2\}$ are given constant matrices of appropriate dimensions; $x \in \mathbb{R}^{d_1}$ and $y \in \mathbb{R}^{d_2}$ are vectors of decision variables. Inequality (A.10) is not convex in x and y. A way to obtain an LMI from (A.4) is by fixing beforehand one of the decision variables: 1) the BMI (A.10) is an LMI in x for a fixed y, or 2) the BMI (A.10) is an LMI in y for a fixed x.

Appendix B. Sum relaxations

In the TS-LMI framework, it is natural to obtain inequality conditions involving convex sums; the weights in the convex combinations are nonlinear functions called membership functions (MFs). In order to obtain LMI conditions, the MFs must be removed. To this end, sum relaxations are employed. In what follows, some relaxation schemes that are employed throughout the thesis are presented.

First, consider the following problem with one convex sum

$$\Upsilon_h := \sum_{i=1}^{r_a} h_i\left(z(t)\right)\Upsilon_i < 0, \tag{B.1}$$

where $\Upsilon_i = \Upsilon_i^T$, $i \in \{1, 2, \dots, r_a\}$ are symmetric matrices of appropriate dimensions. The following lemma gives sufficient conditions for (B.1) to hold:

Lemma B.**1**. (Wang et al., 1996). The convex-sum (B.1) is negative if the following set of LMIs holds

$$\Upsilon_i < 0, \quad \forall i \in \{1, 2, \dots, r_a\}. \tag{B.2}$$

When dealing with controller/observer design within the quadratic framework, a double sum problem may appear:

$$\Upsilon_{hh} := \sum_{i_1=1}^{r_a}\sum_{i_2=1}^{r_a} h_{i_1}\left(z(t)\right)h_{i_2}\left(z(t)\right)\Upsilon_{i_1 i_2} < 0, \tag{B.3}$$

where $\Upsilon_{i_1 i_2} = \Upsilon_{i_1 i_2}^T$, $i_1, i_2 \in \{1, 2, \ldots, r_a\}$ are matrices of appropriate dimensions. The following lemmas give sufficient conditions for (B.3) to hold.

Lemma B.2. (Wang et al., 1996). The double convex-sum (B.3) is negative if the following set of LMIs holds

$$\begin{aligned} &\Upsilon_{i_1 i_1} < 0, \quad \forall i_1 \in \{1, 2, \ldots, r_a\}, \\ &\Upsilon_{i_1 i_2} + \Upsilon_{i_2 i_1} < 0, \quad i_1, i_2 \in \{1, 2, \ldots, r_a\}, \ i_1 < i_2. \end{aligned} \tag{B.4}$$

Lemma B.3. (Tuan et al., 2001). The double convex-sum (B.3) is negative if the following set of LMIs holds

$$\begin{aligned} &\Upsilon_{i_1 i_1} < 0, \quad \forall i_1 \in \{1, 2, \ldots, r_a\}, \\ &\frac{2}{r_a - 1} \Upsilon_{i_1 i_1} + \Upsilon_{i_1 i_2} + \Upsilon_{i_2 i_1} < 0, \quad i_1, i_2 \in \{1, 2, \ldots, r_a\}, \ i_1 \neq i_2. \end{aligned} \tag{B.5}$$

In this thesis unless otherwise specified, the sum relaxation scheme given by Lemma B.3 is adopted, especially since it does not involve extra slack matrices and therefore has a "reasonable" complexity. Other sum relaxations that include slack variables exist in the literature but they are beyond the scope of this thesis (Liu and Zhang, 2003; Sala and Ariño, 2007).

Usually, when dealing with stability/design of discrete-time TS or TS descriptor models, the co-negativity problem may involve more than 2 convex sums of matrices, for example:

$$\Upsilon_{hhh^+} := \sum_{i_1=1}^{r_a} \sum_{i_2=1}^{r_a} \sum_{j_1=1}^{r_a} h_{i_1}(z(k)) h_{i_2}(z(k)) h_{j_1}(z(k+1)) \Upsilon_{i_1 i_2}^{j_1} < 0, \tag{B.6}$$

where $\Upsilon_{i_1 i_2}^{j_1} = \left(\Upsilon_{i_1 i_2}^{j_1}\right)^T$, $i_1, i_2, j_1 \in \{1, 2, \ldots, r_a\}$, or

$$\Upsilon_{hh}^{v} := \sum_{i_1=1}^{r_a} \sum_{i_2=1}^{r_a} \sum_{j_1=1}^{r_e} h_{i_1}(z(t)) h_{i_2}(z(t)) v_{j_1}(z(t)) \Upsilon_{i_1 i_2}^{j_1} < 0. \tag{B.7}$$

where $\Upsilon_{i_1 i_2}^{j_1} = \left(\Upsilon_{i_1 i_2}^{j_1}\right)^T$, $i_1, i_2 \in \{1, 2, \ldots, r_a\}$, $j_1 \in \{1, 2, \ldots, r_e\}$ are matrices of appropriate dimensions. The following co-negativity problem also commonly appears:

$$\Upsilon_{hhh^+}^{v} := \sum_{i_1=1}^{r_a} \sum_{i_2=1}^{r_a} \sum_{i_x=1}^{r_a} \sum_{j_1=1}^{r_e} h_{i_1}(z(k)) h_{i_2}(z(k)) h_{i_x}(z(k+1)) v_{j_1}(z(k)) \Upsilon_{i_1 i_2 i_x}^{j_1} < 0, \tag{B.8}$$

where $\Upsilon_{i_1 i_2 i_x}^{j_1} = \left(\Upsilon_{i_1 i_2 i_x}^{j_1}\right)^T$, $i_1, i_2, i_x \in \{1, 2, \ldots, r_a\}$, $j_1 \in \{1, 2, \ldots, r_e\}$ are matrices of appropriate dimensions. In addition, inequalities involving two different pairs of convex sums may appear:

$$\Upsilon_{hh}^{vv} := \sum_{i_1=1}^{r_a}\sum_{i_2=1}^{r_a}\sum_{j_1=1}^{r_e}\sum_{j_2=1}^{r_e} h_{i_1}\left(z(t)\right) h_{i_2}\left(z(t)\right) v_{j_1}\left(z(t)\right) v_{j_2}\left(z(t)\right) \Upsilon_{i_1 i_2 j_1 j_2} < 0, \tag{B.9}$$

where $\Upsilon_{i_1 i_2}^{j_1 j_2} = \left(\Upsilon_{i_1 i_2}^{j_1 j_2}\right)^T$, $i_1, i_2 \in \{1, 2, \ldots, r_a\}$, $j_1, j_2 \in \{1, 2, \ldots, r_e\}$.

Remark B.1. Note that the co-negativity problems (B.6), (B.7), (B.8), and (B.9) share two sums of the same nature. In these cases, Lemma B.3 can be applied on the two common convex sums. For instance, sufficient LMI conditions for (B.8) to hold are

$$\begin{aligned} &\Upsilon_{i_1 i_1 i_x}^{j_1} < 0, \quad \forall i_1, i_x \in \{1, 2, \ldots, r_a\}, j_1 \in \{1, 2, \ldots, r_e\}; \\ &\frac{2}{r_a - 1}\Upsilon_{i_1 i_1 i_x}^{j_1} + \Upsilon_{i_1 i_2 i_x}^{j_1} + \Upsilon_{i_2 i_1 i_x}^{j_1} < 0, \quad \forall i_x \in \{1, 2, \ldots, r_a\}, j_1 \in \{1, 2, \ldots, r_e\},\ i_1 \neq i_2. \end{aligned} \tag{B.10}$$

Since (B.9) has two different pairs of convex sums, Lemma B.3 can be extended as follows:

Lemma B.4. Sufficient conditions for (B.9) to hold are:

$$\begin{cases} \Upsilon_{i_1 i_1}^{j_1 j_1} < 0, \quad \forall i_1 \in \{1, 2, \ldots, r_a\}, j_1 \in \{1, 2, \ldots, r_e\}, \\ \dfrac{2}{r_e - 1}\Upsilon_{i_1 i_1}^{j_1 j_1} + \Upsilon_{i_1 i_1}^{j_1 j_2} + \Upsilon_{i_1 i_1}^{j_2 j_1} < 0, \quad \forall i_1 \in \{1, 2, \ldots, r_a\}, \quad j_1 \neq j_2, \\ \dfrac{2}{r_a - 1}\Upsilon_{i_1 i_1}^{j_1 j_1} + \Upsilon_{i_1 i_2}^{j_1 j_1} + \Upsilon_{i_2 i_1}^{j_1 j_1} < 0, \quad \forall j_1 \in \{1, 2, \ldots, r_3\}, \quad i_1 \neq i_2, \\ \dfrac{4}{(r_e - 1)(r_a - 1)}\Upsilon_{i_1 i_1}^{j_1 j_1} + \dfrac{2}{r_e - 1}\left(\Upsilon_{i_1 i_2}^{j_1 j_1} + \Upsilon_{i_2 i_1}^{j_1 j_1}\right) + \dfrac{2}{r_a - 1}\left(\Upsilon_{i_1 i_1}^{j_1 j_2} + \Upsilon_{i_1 i_1}^{j_2 j_1}\right) \\ \quad + \Upsilon_{i_1 i_2}^{j_1 j_2} + \Upsilon_{i_2 i_1}^{j_1 j_2} + \Upsilon_{i_1 i_2}^{j_2 j_1} + \Upsilon_{i_2 i_1}^{j_2 j_1} < 0, \quad \forall i_1, i_2 \in \{1, 2, \ldots, r_a\}, j_1, j_2 \in \{1, 2, \ldots, r_e\},\ i_1 \neq i_2, \quad j_1 \neq j_2. \end{cases}$$

Proof. Applying Lemma B.3 on the double convex sum $h(z)h(z)$ in (B.9) yields

$$\Upsilon_{i_1 i_1}^{vv} < 0, \quad \forall i_1, \quad \frac{2}{r_a - 1}\Upsilon_{i_1 i_1}^{vv} + \Upsilon_{i_1 i_2}^{vv} + \Upsilon_{i_2 i_1}^{vv} < 0, \quad \forall i_1, i_2 \in \{1, 2, \ldots, r_a\},\ i_1 \neq i_2. \tag{B.11}$$

Using Lemma B.3 for the first inequality in (B.11) on the double sum of $v(z)v(z)$ it renders

$$\Upsilon_{i_1 i_1}^{vv} < 0 \Leftarrow \begin{cases} \Upsilon_{i_1 i_1}^{j_1 j_1} < 0, \quad \forall i_1 \in \{1,2,\ldots,r_a\}, j_1 \in \{1,2,\ldots,r_e\}, \\ \dfrac{2}{r_e - 1} \Upsilon_{i_1 i_1}^{j_1 j_1} + \Upsilon_{i_1 i_1}^{j_1 j_2} + \Upsilon_{i_1 i_1}^{j_2 j_1} < 0, \quad \forall i_1 \in \{1,2,\ldots,r_a\}, j_1, j_2 \in \{1,2,\ldots,r_e\}, \quad j_1 \neq j_2. \end{cases}$$

Following a similar procedure with the second inequality in (B.11), we obtain:

$$\frac{2}{r_a - 1} \Upsilon_{i_1 i_1}^{vv} + \Upsilon_{i_1 i_2}^{vv} + \Upsilon_{i_2 i_1}^{vv} < 0$$

$$\Leftarrow \begin{cases} \dfrac{2}{r_a - 1} \Upsilon_{i_1 i_1}^{j_1 j_1} + \Upsilon_{i_1 i_2}^{j_1 j_1} + \Upsilon_{i_2 i_1}^{j_1 j_1} < 0, \quad \forall i_1, i_2 \in \{1,2,\ldots,r_a\}, j_1 \in \{1,2,\ldots,r_e\}, \quad i_1 \neq i_2, \\ \dfrac{4}{(r_e - 1)(r_a - 1)} \Upsilon_{i_1 i_1}^{j_1 j_1} + \dfrac{2}{r_e - 1} \left(\Upsilon_{i_1 i_2}^{j_1 j_1} + \Upsilon_{i_2 i_1}^{j_1 j_1} \right) + \dfrac{2}{r_a - 1} \left(\Upsilon_{i_1 i_1}^{j_1 j_2} + \Upsilon_{i_1 i_1}^{j_2 j_1} \right) + \Upsilon_{i_1 i_2}^{j_1 j_2} + \Upsilon_{i_2 i_1}^{j_1 j_2} \\ \quad + \Upsilon_{i_1 i_2}^{j_2 j_1} + \Upsilon_{i_2 i_1}^{j_2 j_1} < 0, \quad \forall i_1, i_2 \in \{1,2,\ldots,r_a\}, j_2, j_1 \in \{1,2,\ldots,r_e\}, \ i_1 \neq i_2, \quad j_1 \neq j_2. \end{cases}$$

This concludes the proof. ■