MULTISYMPLECTIC GRAVITY

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Multisymplectic Geometry and Classical Field Theory

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Contents

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INTRODUCTION

The development of *Quantum Gravity* (QG) is related to the construction of an adequate alphabet which would permit the building of a bridge between the language of differential geometry (and its subdomain Riemanniann geometry) which forms the framework of Einstein's theory of *General Relativity* (GR), and the algebraic symbolism forming the framework of *Quantum Field Theory* (QFT). In pursuing this aim, we are led to consider the status of the following fundamental notions: symmetry, observable, space-time, matter and Relativity principle. We open the discussion with remarks about the first two of these: symmetries and observables.

1 Symmetry, Invariance and Observables

1.1 Symmetry and Invariance

The concept of *symmetry* and its role as a cornerstone in the modern formulation of *mathematical physics* was discussed by H. Weyl [237] and E.P. Wigner [242], who were amongst the earliest writers to stress its fundamental importance. Weyl recorded his conviction of that importance when he remarked [237] *"As far as I see, all a priori statements in physics have their origin in symmetry"*. However a crucial and intricate issue is posed by the recognition of two fundamentally distinct types of symmetry: the space-time symmetry (or external symmetry) *vs* the gauge symmetry (or internal symmetry). Furthermore, we must also note a further classification: some symmetries are termed *global symmetries* of the theory, while others are termed *local symmetries*. We concur with Weyl that symmetry principles are the cornerstone of modern physical theory and underline that at the heart of symmetry principles lies the question: *what is the geometrical object under consideration and how are we led to characterise its related invariants* ?

In physics, whenever a system is considered, we speak of a *symmetry* - and then about *invariance* with respect to this symmetry - by specifying *transformations* that leave some related quantities unchanged. Actually we define a symmetry as a change of coordinates or variables that leaves either the *action* invariant, the equations of motion or the field equations. Thereby, the first step, from a mathematical standpoint, arose with the theoretical definition of a *symmetry* as an invariance under a specific group of transformations, which brings with it a consideration of the theory of transformation groups. This recognition culminated in the *Erlangen program* of F. Klein. Klein's insight [146] was to show that geometry could be conceived as the study of *structures* on spaces defined by their *transformation groups*. This provided a principle of unification yielding a conceptual re-ordering of the subject-matter of geometry as traditionally conceived. The key point is that the space of any geometry is defined via a transitive group action revealing its invariance under group transformations. This development of group theory and its role in the generalization - and classification of geometries - henceforth spoken of in the plural, and the recognition of its central role in the physical interpretation of geometry - and its conceptual inverse, the geometrization of physics - played a key role in crystalising the outlook of the Göttingen school on both mathematics and physics in the early 20th Century. Klein's seminal work, in conjunction with that of S. Lie, opened the way to a conception of physical theories centered on their transformation properties. Lie groups and Lie algebras - the infinitesimal generators of vector fields - are al pervasive in mathematical physics and revealed the central importance of the theory of continuous canonical transformations, already anticipated in the insights of Lagrange and Hamilton. It was also in Göttingen, in the later work of D. Hilbert, Weyl and E. Noether, that the foundations and framework for the modern geometrical approach to the study of dynamical systems were put in place, forming the basis for our current and still developing understanding of the relationship between physical invariants and their mathematical counterparts. The Poincaré-Cartan integral invariants remain one of the most beautiful early expressions of that development.

1.2 Dirac constraints and Dirac observables

We will not enter into details here of the so-call Dirac constraints and the related Dirac-Bergman canonical quantization program, - see Dirac [60, 62, 63], M. Henneaux and C. Teitelboim, [119]. However we give an indication of its connection with the notion of an observable. Gauge invariance leads to a degenerate Legendre transform (canonical variables are related though *Legendre constraints*). The treatment of these constraints in the context of Hamiltonian dynamics leads - in particular through the notion of first class and second class constraints - to the notion of the constraint submanifold $\Sigma \subset T^{\star}\mathcal{M}$ of the phase space of the theory. Σ is defined by the data of m constraints $\{\boldsymbol{\chi}_i(q,p)\}_{1\leq i\leq m} = 0$. In the context of constrained Hamiltonian theory with first class constraint algebra, a Dirac observable $\mathcal{O}_{\text{Dirac}}$ is a function over $T^{\star}\mathcal{M}$ such that $\left\{\mathcal{O}_{\text{Dirac}}(q,p), \boldsymbol{\chi}_i\right\} \approx 0$. A Dirac observable $\mathcal{O}_{\text{Dirac}}$ - on the constraints surface Σ - has weakly vanishing Poison brackets with all of the first class constraints. Hence, a Dirac observable $\mathcal{O}_{\text{Dirac}}$ is defined as a physical, gauge-invariant quantity. Then dynamics with respect to first class constraints $\boldsymbol{\chi}_i$ is perceived as a gauge. We picture evolution of a function $\varphi(q,p)$ over phase space (1)(i).

$$\text{(i)} \quad \frac{d}{dt}\big(\varphi(q,p)\big) = \big\{\mathcal{H}, \varphi(q,p)\big\} \qquad\qquad \text{(ii)} \quad \big\{\mathcal{H}, \mathcal{O}_{\text{Dirac}}\big\} \approx 0 \tag{1}$$

In *generally covariant systems* like GR, the covariant Hamiltonian $\mathcal{H}$ (which generates dynamics and thus *time* evolution) is a constraint which vanishes identically (as a sum of first class constraints). A Dirac observable $\mathcal{O}_{\text{Dirac}}$ is given by property (1)(ii). It is this feature which gives rise to the problem of time in QG. This is also a reflection of the interplay between reparametrization invariance and the schizophrenic status of time: seen as a dimension or parametrization variable. In the context of a universal Hamiltonian formalism it is always possible to work in a more general Lepagean equivalent theory. In this case, thanks to the introduction of (huge) unphysical variables one can subsequently make the first class Dirac constraints set disappear. This point should underline the necessity of studying gauge theory on the basis of Lepage-Dedecker (LD) theory.

Indeed, since we understand *General Covariance* as the abandonment of any preferred coordinate system for field equations for space-time we are led to consider the idea of *background independence* as fundamental. For detailed discussion about the meaning of background independence and general covariance we follow the point of view of J. Stachel [217]. This insight leads to the problem of observables in GR: for pure gravity *no* observables are given. This implies the rejection as meaningless of the notion of any *a priori* given *space-time* structure. In such a picture the conclusion naturally follows that the Relativity principle[1] is intrinsically rooted in the relativity of *observable*. On this view of the philosophy of GR, the theory claims that observable quantities are not detected directly, but are only compared to one another. The key idea of the multisymplectic approach is to give a precise definition of the notion of an observable and a method to compare two observables without specify any volume form (which means for us without making any reference to a preferred space-time background). In this approach, we recover a crucial insight: *dynamics* just tells us how to compare two observations. To emphasize this fundamental point, we cite C. Rovelli [198]: *"What has physical meaning is only the relative localization of the dynamical objects of the theory (the gravitational field among them) with respect to one another."*

[1]We discuss Relativity Principles in the broad context of their development from Galilean Relativity to Special and General Relativiy.

In canonical quantum gravity, one formulates the concept of *observable* on the basis put forward by Dirac [62]. The *Loop Quantum Gravity* (LQG) program [10, 197, 221, 222] makes intensive use of Dirac observables. Dynamics is perceived as a gauge generated by first class constraints and an observable defined on the phase space commutes with all the constraints. Drawing inspiration from and studying the intersection of the formalisms of Yang-Mills theory (dynamical connections) and QFT Rovelli and L. Smolin [201] made use of Wilson loop[2] in the QG context (since the functional on the space of connections is invariant under gauge transformations) and studied the *loop representation* for Ashtekar variables [8, 9]. This brings to light the basis of Quantum Riemanniann Geometry. LQG is built upon two main kinematical operators, the volume and the area operators. However neither of these is a Dirac observable, and neither commutes with all the canonical constraints. Addressing the question of the *physical* meaning of such operators even within kinematics, - *i.e.* before confronting the issues in dynamics - concerning the status and meaning of an observable. As this example indicates, the obstructions we face in establishing a good notion of observable may be coming from several distinct, through related, conceptual and technical sources. One obviously concerns the issue of Dirac first class constraints and the subsequent notion of Dirac observable over phase space $\mathcal{O}_{\text{Dirac}}$. Secondly, it appears that suitable matter degrees of freedom need to be included to obtain a complete picture. Lastly, we have to face the more general issue of principle as to what constitutes a satisfactory notion of *observable* in the setting of a fully covariant field theory. The first two issues underline the necessity of Lepage-Dedecker (LD) theory for covariant Hamiltonian field theory. There, the Dirac constraint set can always be taken to be empty, this is possible by observing a total even-handedness as between space-time *and* matter fields. The last question is more subtle and general, but forces us towards an appreciation of the principle that any observable quantity should *emerge from* intrinsic properties, namely from *dynamics*. It is in this interplay between observables and dynamical considerations that we are led to recognize the astonishing beauty of the *universal Hamiltonian formalism*. One key underlying aim of this Thesis is to illustrate the subtlety of the concept of *observable* in the setting of multisymplectic geometry. Along the way we form an appreciation for the connection and interplay between the issues involved in our choice of a suitable notion of observable and our understanding of the Relativity principle.

2 The road ahead

2.1 Symbolic vision and diagram

The underlying aim of this discussion is to provide a fuller articulation of new insights into our notions of space, time, matter, observable and the relativity concept. To this end, we have chosen to introduce some new notation. We have made the choice to use symbolic drawing and new signs. Here we briefly introduce the nine unusual symbols which will be encountered below. Before describing them, we first motivate their introduction. We believe that diagrammatic notation can benefit mathematical physics, and not for extraneous or incidental reasons. Two recent developments in contemporary mathematical physics appear to us to illustrate a striking epistemological shift - namely that diagrams have ceased to be merely representational devices, but have themselves become objects of theoretical study. The modern development of pure mathematics already strikingly illustrates this development with the rise of Category Theory. As is well known, this had its origins in the work of S. MacLane and S. Eilenberg [68], in algebraic topology and homology. Later it was enormously extended and developed to become a framework for the organisation of almost

[2]The canonical variables of quantum geometry as developed in LQG are Wilson loops (given by the trace of the holonomy around the loop γ, $W_\gamma[\mathcal{A}] = \text{tr}[h_\gamma[\mathcal{A}]]$ of an SU(2)-connection $\mathcal{A}$) and fluxes of the conjugate momenta. Hence, $W[\mathcal{A}]$ is a functional of the connection that provides a rule for the parallel transport of the SU(2) connection.

all major fields of mathematics. Many names are associated with this, but one which cannot go unmentioned is that of A. Grothendieck.[3] [101] On the side of mathematical physics we cite three examples. The first - perhaps more anecdotal in flavour by comparison with the second - is the tensorial diagram representation developed by R. Penrose [190]. The second is the fundamental role of diagrams in the work of Feynman [76], which today still pervades the formalism of the modern canonical (quantum) gravity program. Indeed in LQG the analogue of a Feynman diagram is called a *spin foam* and constitute the core of recent developments of the theory. This is linked to general knot theory and its use in modern QG. Here once again, the role of diagrammatic representation is central - see the book edited by J.C. Baez [13] and references therein for exposition and discussion of examples. These are general remarks on the rise of diagrams in mathematical physics. The more particular suggestion informing the introduction of the new symbols below is that, in the quest for a Quantum Theory of Gravity, whatever the favoured approach, a new sort of *dynamical figure* will prove to play a central role in illustrating, and permitting a grasp of, the novel primordial concepts which will be involved. We now introduce our novel symbols in three groups.

The first is the set of three symbols: the *space-time* entity $\boxed{\text{S}}$, the *matter* entity $\boxed{\text{M}}$ and finally the *hybrid* entity $\boxed{\text{H}}$.[4]

We need to distinguish between the ontologic and dynamical aspects of variational problems but also more generally the ontologic and dynamical aspects of the structure of the theories. The first is chiefly concerned with kinematics - *phase space structure* -, in connection with which our interest is principally with the ontologic aspect of symmetry principles. The dynamical aspect comes to the fore when we consider dynamics. The Hamiltonian of a system, as much as it is to be understood as standing for some kind of potential - an dynamical potential -, brings with it a focus on the dynamical meaning of the symmetries of the system, as concerned with information about the system. As further illustration of the intended contrast, we may cite the perennial issue of the interpretation of the Einstein field equation - in particular the relationship of the purely geometrical part of the formalism (which may be seen as having a predominantly ontologic aspect) and the stress-energy tensor (regarded as having more of an dynamical flavour) This tension within the heuristics of GR as a classical theory also finds expression in the tension between the conceptual and structural ingredients of GR and QFT, the better understanding of which provides vital clues in analysing - and perhaps eventually clearing away - the obstacles - both conceptual and technical - blocking progress towards a Theory of Quantum Gravity. We denote:

$\triangleright$	Ontologic Space	$\dot{\triangleright}$	Dynamical Space
$\triangleleft$	Ontologic Form	$\dot{\triangleleft}$	Dynamical Form

A clearer understanding of the intended meaning of these symbols may be obtained when we study their use, together with that of two further symbols, in connection with the classification of different

[3] who developed many parts of the machinery of general category theory, and many key concepts - in particular the theoretical classification of the properties of limits and co-limits - in the course of his fundamental and path-breaking contributions, especially to algebraic geometry, although also in his earlier work in functional analysis.

[4] Note that these are not essentially diagrammatic but merely symbolic. They will be employed in two distinct contexts. The first concerns the setting of a variational problem. This involves the first two symbols, namely $\boxed{\text{S}}$ and $\boxed{\text{M}}$. They are to be understood more precisely as parametrization space and parametrized space[5]. This is connected with the distinction between what we shall term the ontological source or aspect - the *ontologic motif* $\boxed{\text{S}}\boxed{\text{M}}$ and the dynamical source or aspect - the *dynamical motif* $\boxed{\text{S}}$,$\boxed{\text{S}} \times \boxed{\text{M}}$ of the variational problem. The distinction or tension between these two aspects has its roots in the Grassmanian picture for variational calculus for field theory. We hope to show in what follows that these ideas can play a fruitfully role in connection with understanding the dual nature of the structures encountered in multisymplectic geometry. On the other side, those symbol would be understood in the last part as an heuristic clarification for ontologic mode $\boxed{\text{S}}\boxed{\text{M}}\boxed{\text{H}}$ of the multisymplectic space, in the final discussion of the dual nature of the multisymplectic form.

types of observable in the setting of both symplectic - see subSections (3.7) - (3.10) - and then multisymplectic geometry, see section (6).
In the chapter of the thesis devoted specifically to the discussion of observables - and also in the last part of the text (20.1), devoted to broader philosophical issues, we introduce two further symbols:

Ontologic Observer   Dynamical Observer

These use of these two symbols is intended to capture the distinction between the ontologic and dynamical aspects of the notion of observable. The use of the two symbols is connected not only with specific mathematical characteristics of the notion of an observable stemming from the ontologic and dynamical aspects of that notion but, more radically, with the two aspects of a fundamental duality - more precisely a double duality - which brings the notion of observable together with our understanding of symmetry principles and relativity to form what I shall label Eye-mirror monad. These two symbols for the Observers represent the movement in the search for observables within the *ontologic* domain (the ontologic observer) and the *dynamical* domain (the dynamical observer) respectively. We may say, picturesquely, that the notion of *physical observable* can be thought of analogous to the *place* where these two observers meet, having arrived at this double duality from the ontologic and dynamical directions respectively. In more mathematical language, we may say that the concepts of *pataplectic* manifold and of *dynamical observable functionals* are the mathematical keys to the understanding of the *double duality*.

2.2 Ontologic vs Dynamical

In this section we offer further exploration of the duality between the *ontologic* and *dynamical* aspects of the notion of an observable and of symmetry principles, stressing the distinction between invariance and covariance principles. To make the discussion more precise, we analyse two examples - the relation between the Einstein tensor and the energy-momentum tensor in Classical GR and the broader opposition between the deep structure and associated conceptual presuppositions of GR and QFT as the two pillars of 20th century physical theory.

Since our principal focus is the issue of space-time-matter organization, we shall open with a discussion of the Einstein equations. Einstein's equations connect the distribution of matter energy - given by the energy-momentum tensor $T_{\mu\nu}$ - and geometry - crystalized by the Einstein tensor $G_{\mu\nu}$. We write them $G_{\mu\nu} \propto T_{\mu\nu}$. These simplify in the vacuum case to $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\mathsf{g}_{\mu\nu}R = 0$. In classical GR, the Einstein tensor is usually understood ontologically, whereas the stress energy tensor is considered as having more an dynamical status.[6]

$G_{\mu\nu} \leftrightarrows$ Ontologic object $T_{\mu\nu} \leftrightarrows$ Dynamical object

Recall that the stress energy tensor depends on the representation, and is usually treated via a choice of cosmological model by imposing constraints on the *matter* distribution *in* the Universe. This arbitrary distinction is precisely the wrong approach to take from the standpoint of an intrinsically dynamical geometry. It will be a central claim of this Thesis that the framework of multisymplectic geometry provides a new vision of the ontology of space-time-matter. We shall set out the heuristic arguments for this conclusion below and most fully in the section (20). This claim regarding the

[6]The symbols and represent the algebraic duality and the dynamical duality, respectively, e.g. $(G_{\mu\nu})\leftrightarrows$ is a tensor object that belongs to ontologic considerations (i.e. relative to the structure of the geometry only), whereas the stress–energy tensor is an object constructed on the dynamical duality (which is translated as $T_{\mu\nu}\leftrightarrows$). Note also that the bold symbol $\leftrightarrows$ is symbolic, meaning that the object considered is related to ontologic or dynamical aspect, respectively.

consequences of the multisymplectic formalism for our understanding of these ontological issues is related to two further developments. The first lies in differential geometry and concerns the possible generalization of Cartan Geometry and the notion of Riemannn-Cartan space to the setting of multisymplectic geometry. The second is in cosmology and concerns a possible explanation of the origins of so-called dark matter and dark energy. We shall argue that issues concerning the ontological status of space-time in GR vis-a-vis its treatment in QFT - the clarification of which is indispensable for conceptual understanding of the proposed approaches to a quantum theory of gravity - has hitherto been hobbled by a mistaken identification between what we shall term distinct ontologic modes. This notion will be more fully defined in what follows: but briefly, we underline here that an ontologic mode $(\mathcal{M},\boldsymbol{\omega}) = \boxed{\text{S}}\boxed{\text{M}}\boxed{\text{H}}$ is to be thought of as given by a manifold structure together with a choice of multisymplectic form, which is to say to the choice of a Lepage-Dedecker (LD) theory.

• We have touched on the conceptual opposition between GR and QFT, but what is of particular interest from the viewpoint of what follows, is the way the gauge formalism points to geometrical structures beyond QFT. Here we immediately sense the main distinction between the two great pillars of modern physic. On one hand in GR, the *ontologic* space is space-time itself, described by the ontology $\boxed{\text{S}}\boxed{\text{M}}^{\text{GR}}$ - see section (11.2) - whereas the gauge picture as more of the general flavour of an epistemological representation. In the latter indeed, we describe the ontology of matter fields $\boxed{\text{S}}\boxed{\text{M}}^{\text{Gauge}}$ as sections of principal or associated bundle **over** a point of the manifold, which is taken to be space-time. We intuitively feel the lack of a unified conceptual framework in such a picture. Many attempts have been made to overcome this difficulty. The point we emphasis in the Thesis, within the enlarged multisymplectic framework for dynamical geometry is to make use of the concepts of *enlarged pseudofiber* $\mathbf{P}_q(z)$, *pseudofiber* $\mathbf{P}_q^h(z)$ and the *generalized pseudofiber direction* $\mathbf{L}_m^{\mathcal{H}}$. This is closely connected to the fact that the dynamical structure (namely the Hamiltonian and Hamiltonian equations of motion) for a multisymplectic manifold $(\mathcal{M},\boldsymbol{\omega})$ is invariant by deformations along pseudofiber $\mathbf{P}_q^h(z)$.

Gauge theory: Fiber bundles	Multisymplectic theory: Pseudofibers
Fiber bundle $(\mathcal{P},\mathcal{X},\pi)$	Multisymplectic space $\Lambda^n T^\star \mathfrak{Z}^\circ$
Fiber $\mathcal{P}_x$ over a point $x \in \mathcal{X}$	Pseudofibers $\mathbf{P}_q^h(z)$, $\mathbf{P}_q(z)$
Fibers do not intersects	Pseudofibers may intersect singularly
Gauge transformations	Deformations along pseudofibers

Once again, in the last part, see section (20), we shall develop heuristic considerations on the crucial importance of this insight. We believe it opens the way to a generalized *non-local* field theory, and provides the conceptual resources for reconciling the *local* features of gauge theory with the *non-local* features of gravity crystalized in the diffeomorphism group.

2.3 A glimpse of the journey

The Thesis falls into four main parts:

MULTISYMPLECTIC GEOMETRY comprises Sections (3) to (7) and concerns the general setting of multisymplectic geometry MG. First we offer a survey of symplectic geometry (3) with a special emphasis on the treatment of extended phase space. Then we pass to the multisymplectic setting developed by F. Hélein and J. Kouneiher (4). After a short discussion of the traditional presentation of the theory in terms of jet bundles and contact structure with the related graded Poisson structure (5) we pass to the main subject of this part: section (6) which is observable theory. Here we emphasize conceptual issues and make intensive use of diagrams. We conclude with some

remarks on the notion of pre-multisymplectic geometry and the notion of n-phase space - see (7).
MULTISYMPLECTIC MAXWELL THEORY is the study of some aspects of multisymplectic Maxwell theory. Here calculations are given in detail. We first treat the multisymplectic De-Donder-Weyl-Maxwell theory - see (8). In section (9) we set out some computations involved in the search for observable forms. Section (10) gives a simple example of Lepage-Dedecker higher theories. In this part we also develop the symbolic notation for various bracket constructions for forms of arbitrary degree.
TOWARDS MULTISYMPLECTIC GRAVITY comprises Sections (11) to (19) and offers an overview of multisymplectic gravity. This section is itself composed of three parts. Section (11) discusses the basic geometrical tools for GR, while (12) concentrates on the picture of gravity as a gauge theory which clearly underlies the modern LQG program. The aim of these two sections is purely introductory and descriptive. They offer a synthesis of basic tools and ideas. Sections (13) to (14) concern equations of movement for Palatini gravity and some remarks on the canonical variables that appear in LQG. One aim here is to highlight the way in which the work of Hélein and Kouneiher allows a natural geometrical treatment of *loop variables*. Section (15) offers a brief treatment of the Chern-Simon equations of movement with the aid of techniques drawn from the multisymplectic formalism. Section (16) concern the study of multisymplectic gravity chiefly through the example of Palatini-Hamilton equations in the simple DW multisymplectic setting while (17) treats the pre-multisymplectic case. Sections (18) to (19) concern observables, topological terms and the Lepage-Dedecker formalism.
SPACE–TIME–MATTER consists of section (20) and offers an overall perspective on the topics treated in the Thesis and the directions in which they may be developed in the future. These considerations rest on heuristic rather than mathematically rigorous arguments. They present predominantly conceptual claims. There are two main ideas. The first draws an analogy with Einstein's considerations about the dual nature of the metric field by reference to the dual nature of the multisymplectic form. Here we emphasize a heuristic reinterpretation of the main conceptual principle of GR within the extended multisymplectic setting. We replace the old ontological view on space-time with a new one resting on the basis of n-forms in the context of multisymplectic geometry. This opens the way to a further and more complex development concerning the possible extension of the notion of curvature and torsion to multisymplectic space. This brings with it a new conceptual setting for understanding the problem of dark matter and dark energy in Cosmology. Finally we introduce the notion of *monad* (the reason for this Leibnizian terminology will become apparent) which arises when we try to unify the notions of *observable* and *symmetry* on the basis of a single intrinsically dynamical principle.

To sum up, the Thesis contain:

- A presentation of MG, especially a synthesis of the principal motives and results presented in the work of Hélein, [111, 113, 114] and Hélein and Kouneiher [115, 116, 117, 118] (some material is directly taken from their work) together with the development of new symbolism based on conceptual reflection on new *principles* which are intended to make the connection between our understanding of relativity and symmetry principle and the notion of observable.

- A presentation of basic tools for the study of first order gravity. Here we shall be concerned only with the classical level and not with quantum developments and *spin foam* and a proposal for merging tools from MG and LQG in which multisymplectic theory provides a natural generalization for the loop variables.

- A more fundamental reflection on the nature of observables, relativity and space-time-matter within which our proposal for new symbolism finds a natural justification and which gives rise to the concept of Eye-mirror monad. We shall boldly claim to have given mathematical substance to the metaphysical thesis of the union of *oneness* with *duality*. We claim also to have provided a deeper understanding - one resting on the notions of algebraic observable forms and observable forms - into the distinction between covariance and invariance principles.

MULTISYMPLECTIC GEOMETRY

Within the context of covariant canonical quantization Multisymplectic Geometry MG is a generalization of symplectic geometry for field theory. It allows us to construct a general framework for the calculus of variations with several variables. Historically MG was developed in three distinct steps. Its origins are connected with the names of C. Carathéodory [36] (1929), T. De Donder [57, 56] (1935) on one hand and Weyl [236] (1935) on the other. We make this distinction since the motivations involved were different: Carathéodory and later Weyl, were involved with the generalization of the Hamilton-Jacobi equation to several variables and the line of development stemming from their work is concerned with the solution of variational problems in the setting of the action functional. On the other hand, E. Cartan [39] recognized the crucial importance of developing an *invariant language* for differential geometry not dependent on local coordinates. De Donder carried this development further. The two approaches merged in the so-called De Donder-Weyl theory based on the multisymplectic manifold $\mathcal{M}_{\mathsf{DW}}$. The second step arose with the work of T. Lepage and P. Dedecker. As was first noticed by Lepage [159, 160, 161], the De Donder-Weyl setting is a special case of the more general multisymplectic theory. The geometrical tools permitting a fully general treatment were provided by Dedecker [52, 53, 54, 55]. Indeed, for field theory, we are led to think of variational problems as n-dimensional submanifolds Σ embedded in a $(n+k)$-dimensional manifold $\mathfrak{Z}^{\circ}$. One observes the key role of the *Grassmannian bundle* as the analogue of the tangent bundle for variational problems for field theory.

The final step was taken by the Polish school in the seventies which further developed the geometric setting. W. Tulczyjew [224, 225], J. Kijowski [141, 142], K. Gawedski [93] and W. Szczyrba [144, 145] all formulated important steps. We find already in their work the notion of *algebraic form*, and in the work of Kijowski [141] a corresponding formulation of the notion of *dynamical observable* emerges. We emphasize, for the full geometrical multisymplectic approach, two fundamental points that will be treated later: the *generalized Legendre correspondence* - introduced by Lepage and Dedecker - and the issue of *observable* and *Poisson bracket*, two cornerstones within the *universal* Hamiltonian formalism developed by Hélein, [111, 113, 114] and Hélein and Kouneiher [115, 116, 117, 118]. However, in order to understand the difficult issues surrounding the choice of a *good* Poisson bracket structure for field theory in the multisymplectic setting, we will also describe some basic ideas about traditional multisymplectic theory in term of jet bundles and contact structures.

The first part of the Thesis is organized as follow. First we recall some basics of symplectic geometry in the Hamiltonian setting for classical mechanics and the transition to the relativistic case by means of the so-called Hamiltonian constraint. Later we give a basic exposition of MG for the calculus of variations with several variables, the tool for field theory. Finally we concentrate on the main ideas and motives concerning the treatment of observables found in [111, 113, 114, 115, 116, 117, 118]. Also we emphasize some points related to n-phase space theory.

3 Symplectic geometry and Hamiltonian dynamics

We refer to the classical textbooks of R. Abraham and J.E. Marsden [1], J.E. Marsden and T. Ratiu [169] for complete geometrical introductions to classical mechanics and symmetry. A great part of our present introduction is inspired by various works and papers of Hélein [113] - published and unpublished notes, in particular in Sections (3.1) and (3.2). Here we give a short treatment of his work, and present a synthesis on the symplectic roots of Hamiltonian dynamics.

3.1 Lagrangian systems and variational principles

First we focus on the description of non-relativistic dynamical systems. In this context, *time* is the evolution parameter, hence thought of as an *external parametrization.* The configuration space is denoted $\mathfrak{Z}$ - described as a k-dimensional manifold - and we describe the dynamical variables on $\mathfrak{Z}$ with coordinates $\{y^i\}_{1\leq i\leq k}$. The evolution of such a classical system is described by a curve $\gamma : I \to \mathfrak{Z} : t \mapsto \gamma(t)$. The tangent bundle $T\mathfrak{Z}$ is the *velocity configuration space* described by local coordinates $\{(y^i, v^i)\}_{1\leq i\leq k}$. A path γ is lifted to the tangent bundle $T\mathfrak{Z}$ in the following way: $(\gamma,\dot\gamma) : I \to T\mathfrak{Z} : t \mapsto (\gamma(t),\dot\gamma(t))$ and projects down to the configuration space for the same *motion.* In the case where $L = L(y, v)$ does *not* explicitly depend on *time*, we speak of an *autonomous* dynamical system. In the present development, we are also interested in the so-called *non-autonomous* case where the setting exhibits a Lagrangian defined on $I \times T\mathfrak{Z}$. In this case the Lagrangian depends explicitly on time so that $L = L(t, y, v)$. This consideration leads a definition of the *evolution space* as $I \times T\mathfrak{Z}$ which is the data (t, y^i, v^i). In such a context, the *Lagrangian* is defined on the evolution space $L : I \times T\mathfrak{Z} \to \mathbb{R}$ associates $L(t, y, v)$ to any $(t, y, v) \in I \times T\mathfrak{Z}$. The Lagrangian is defined as the functional:

$$\mathcal{L}[\gamma] = \int_I L(t,\gamma(t),\dot\gamma(t))dt = \int_{t_1}^{t_2} L(t,\gamma(t),\dot\gamma(t))dt.$$

We may compute the first variation of $\mathcal{L}[\gamma]$ along a trajectory $\gamma \in \mathcal{C}^2([t_1,t_2],\mathfrak{Z})$ induced by an infinitesimal variation $\delta\gamma \in \mathcal{C}^2([t_1,t_2],\gamma^\star T\mathfrak{Z})$ of the path. We write an infinitesimal deformation of a path γ as $\gamma+\sigma\delta\gamma$ where σ is an infinitesimal parameter $\mathcal{L}[\gamma+\sigma\delta\gamma] = \int_{t_1}^{t_2} L(t,\gamma+\sigma\delta\gamma,\dot\gamma+\sigma\dot{\delta\gamma})dt$. A critical point of $\mathcal{L}[\gamma]$ is any path or *trajectory* $\gamma \in \mathcal{C}^2(I,\mathfrak{Z})$ such that the infinitesimal deformation - with fixed endpoints $\gamma(t_1), \gamma(t_2)$ - for the functional action $\mathcal{L}[\gamma]$ is invariant up to first order. Since,

$$\mathcal{L}[\gamma+\sigma\delta\gamma] = \int_{t_1}^{t_2} L(t,\gamma,\dot\gamma)dt + \sigma\int_{t_1}^{t_2}\Big(\frac{\partial L}{\partial y^i}(t,\gamma,\dot\gamma)\delta\gamma^i + \frac{\partial L}{\partial v^i}(t,\gamma,\dot\gamma)\delta\dot\gamma^i\Big)dt + \mathcal{O}(\sigma^2),$$

we obtain the first variation as:

$$\begin{aligned}\delta\mathcal{L}_\gamma[\delta\gamma] &= \int_{t_1}^{t_2}\Big(\frac{\partial L}{\partial y^i}(t,\gamma,\dot\gamma)\delta\gamma^i + \frac{\partial L}{\partial v^i}(t,\gamma,\dot\gamma)\delta\dot\gamma^i\Big)dt\\ &= \Big[\frac{\partial L}{\partial v^i}(t,\gamma,\dot\gamma)\delta\gamma^i\Big]_{t_1}^{t_2} + \int_{t_1}^{t_2}\Big(\frac{\partial L}{\partial y^i}(t,\gamma,\dot\gamma) - \frac{d}{dt}\big(\frac{\partial L}{\partial v^i}(t,\gamma,\dot\gamma)\big)\Big)\delta\gamma^i dt.\end{aligned}$$

Following the hypothesis of fixed endpoints for the infinitesimal deformation $\gamma+\sigma\delta\gamma$ we conclude that γ is a critical point of $\mathcal{L}[\gamma]$ if and only if:

$$\frac{d}{dt}\big(\frac{\partial L}{\partial v^i}(t,\gamma,\dot\gamma)\big) - \frac{\partial L}{\partial y^i}(t,\gamma,\dot\gamma) = 0, \tag{2}$$

which are the Euler-Lagrange equations. Following Hélein [113], we shall focus on the variational principle for the Hamiltonian formulation. We want to describe a functional $\mathcal{L}^\circ[\gamma,\zeta]$ on the set of applications $(\gamma,\zeta) : I \to T\mathfrak{Z}$ such that the critical points would correspond to those of $\mathcal{L}[\gamma]$. A simple way to achieve this is to consider the restriction on the map (γ,ζ) for the functional

$\mathcal{L}^\circ[\gamma,\zeta]$ so that when we perform the first order variational calculation, we need to set restrictions on $\delta\mathcal{L}^\circ_{(\gamma,\zeta)}[\delta\gamma,\delta\zeta]$ for infinitesimal deformations $\delta\gamma$ and $\delta\zeta$. We summarize those conditions as:

$$\mathcal{L}^\circ[\gamma,\zeta] \;/\; \forall(\gamma,\zeta),\zeta=\dot\gamma \qquad \text{and} \qquad \delta\mathcal{L}^\circ_{(\gamma,\zeta)}[\delta\gamma,\delta\zeta] \;/\; \delta\zeta=\delta\dot\gamma.$$

In order to grasp the constraint $\zeta=\dot\gamma$ - and its counterpart on infinitesimal deformations $(\delta\gamma,\delta\zeta)$ namely $\delta\zeta=\delta\dot\gamma$ - we apply the *Lagrange multiplier* method on the functional space. We add to the unknown variables γ and ζ more unknown variables denoted λ and consider the following action functional:

$$\mathcal{L}^{\circ\circ}[\gamma,\zeta,\lambda] = \int_I \big(L(t,\gamma,\zeta)+\lambda_i(\dot\gamma^i-\zeta^i)\big)dt. \tag{3}$$

The functions λ_i are seen as the components of a section λ of $\gamma^\star T^\star\mathfrak{Z}$. We observe that $\forall t\in I, \lambda(t) = \lambda_i(t)dy^i$ is contained in $T^\star_{\gamma(t)}\mathfrak{Z}$ so that, in order to emphasize the source and target space involved, we describe the map $(\gamma,\zeta,\lambda): I\mapsto T\mathfrak{Z}\times T^\star\mathfrak{Z}$. Therefore for any (γ,ζ,λ) such that $\zeta=\dot\gamma$ we have $\mathcal{L}^{\circ\circ}[\gamma,\zeta,\lambda]=\mathcal{L}[\gamma]$. This is the key feature of the so-called Lagrange multiplier method. The first variation of $\mathcal{L}^{\circ\circ}_{(\gamma,\zeta,\lambda)}[\delta\gamma,\delta\zeta,\delta\lambda]$ is given by[7]:

$$\begin{aligned}\delta\mathcal{L}^{\circ\circ}_{(\gamma,\zeta,\lambda)}[\delta\gamma,\delta\zeta,\delta\lambda] &= \int_I \Big(\frac{\partial L}{\partial y^i}(t,\gamma,\zeta)\delta\gamma^i+\frac{\partial L}{\partial v^i}(t,\gamma,\zeta)\delta\zeta^i\Big)dt+\int_I\big(\delta\lambda_i(\dot\gamma^i-\zeta^i)+\lambda_i\delta\dot\gamma^i-\lambda_i\delta\zeta^i\big)dt\\ &= \int_I\Big(\frac{\partial L}{\partial y^i}(t,\gamma,\zeta)-\dot\lambda_i\Big)\delta\gamma^i dt+\int_I\Big(\frac{\partial L}{\partial v^i}(t,\gamma,\zeta)-\lambda^i\Big)\delta\zeta^i dt+\int_I\big(\dot\gamma^i-\zeta^i\big)\delta\lambda_i dt.\end{aligned}$$

Therefore, (γ,ζ,λ) is a critical point of $\delta\mathcal{L}^{\circ\circ}_{(\gamma,\zeta,\lambda)}[\delta\gamma,\delta\zeta,\delta\lambda]$ if and only if:

$$\begin{array}{lcl}\dfrac{\partial L}{\partial y^i}(t,\gamma,\zeta)-\dot\lambda_i &=& 0\\ \dfrac{\partial L}{\partial v^i}(t,\gamma,\zeta)-\lambda^i &=& 0\\ \dot\gamma^i-\zeta^i &=& 0\end{array} \quad\Longrightarrow\quad \begin{array}{llcl}\text{(i)} & \dfrac{d\lambda_i}{dt} &=& \dfrac{\partial L}{\partial y^i}(t,\gamma,\zeta)\\ \text{(ii)} & \lambda_i &=& \dfrac{\partial L}{\partial v^i}(t,\gamma,\zeta)\\ \text{(iii)} & \dfrac{d\gamma^i}{dt} &=& \zeta^i.\end{array} \tag{4}$$

The Lagrange multiplier method exhibits the interplay between the variables γ,ζ,λ and their dual equations (4). We find the desired relation $d\gamma^i/dt=\dot\gamma^i=\zeta^i$ and if we eliminate the variable λ_i we recover the Euler-Lagrange equations via the restriction of the action functional (3) $\mathcal{L}^{\circ\circ}[\gamma,\zeta,\lambda]$ on the path (γ,ζ,λ) such that (4)(iii) is satisfied.

When we turn to the Hamiltonian picture - and to the related variational formulation - we understand these Lagrange multipliers as the canonical momenta, but in the Hamiltonian setting we do not eliminate λ_i - the dual variables of equation (4)(iii) - but keep them and instead eliminate the unknown variables ζ^i - dual variables of equations (4)(ii) $\lambda_i(t)=\frac{\partial L}{\partial v^i}(t,\gamma,\zeta)$. This is the core of the so-called Legendre transform, on which the Hamiltonian picture is built. Before we give a more precise treatment of the Legendre transform and the Hamilton equations, we first offer an overview of variational principles in the Hamiltonian setting.

3.2 The Hamiltonian setting and variational principles

This part continues to develop the synthesis found in Hélein [113]. We now relabel the Lagrange multipliers by the variable π as is conventional in Hamilton mechanics. So that the previously

[7]Where we have cancelled one term due to the fact that $\delta\gamma$ is supposed to be compactly supported.

introduced action functional (3) becomes:

$$\mathcal{L}^{\circ\circ}[\gamma,\zeta,\pi] = \int_I \big(L(t,\gamma,\zeta) + \pi_i(\dot\gamma^i - \zeta^i)\big)dt.$$

With this formal change of notation the system under consideration (4) is now denoted by:

$$\begin{array}{lrcl} \text{(i)} & \frac{d\pi_i}{dt}(t) & = & \dfrac{\partial L}{\partial x^i}(t,\gamma,\zeta) \\ \text{(ii)} & \pi_i(t) & = & \dfrac{\partial L}{\partial v^i}(t,\gamma,\zeta) \\ \text{(iii)} & \frac{d\gamma^i}{dt}(t) & = & \zeta^i(t). \end{array} \tag{5}$$

The idea is to eliminate ζ while keeping $\pi_i(t) = \frac{\partial L}{\partial v^i}(t,\gamma,\zeta)$. The Lagrange multipliers are taken to be the canonical momenta $\pi_i(t)$ defined via the Legendre transform. We substitute $\pi_i(t)$ in place of ζ^i in equations (5)(i) and (5)(ii). The Hamiltonian picture[8] replaces the Euler-Lagrange system of equations involving $(\gamma(t),\dot\gamma(t))$ by a new system of equations involving $(\gamma(t),\pi(t))$. This process is equivalent to passing from the system (6)(i) to the system (6)(ii)

$$\text{(i)}\left|\begin{array}{rcl} \dfrac{d}{dt}\Big(\dfrac{\partial L}{\partial v^i}(t,\gamma(t),\zeta(t))\Big) & = & \dfrac{\partial L}{\partial y^i}(t,\gamma(t),\zeta(t)) \\ \dfrac{d\gamma}{dt}(t) & = & \zeta(t) \end{array}\right. \qquad \text{(ii)}\left|\begin{array}{rcl} \dfrac{d\pi_i}{dt}(t) & = & -\dfrac{\partial H}{\partial y^i}(t,\gamma(t),\pi(t)) \\ \dfrac{d\gamma^i}{dt}(t) & = & \dfrac{\partial H}{\partial p_i}(t,\gamma(t),\pi(t)). \end{array}\right. \tag{6}$$

where we defined $\pi_i(t)$ as $\pi_i(t) = \partial L/\partial v^i(\gamma(t),\zeta(t))$. Now we consider the following functional on the space of paths (γ,ζ,π) but with the restriction given by (5)(ii) provided by the Legendre hypothesis. Namely we construct a diffeomorphism $I\times T\mathfrak{Z} \to I\times T^\star\mathfrak{Z}$ so that the relation (4)(ii) is inverted: $\zeta^i = \mathcal{V}^i(t,\gamma,\pi)$. We consider the object $\mathcal{I}[\gamma,\pi] = \mathcal{L}^{\circ\circ}[\gamma,\zeta,\pi]$ such that ζ is a solution of $\frac{\partial L}{\partial v^i}(t,\gamma,\zeta) = \pi_i$. We obtain the functional:

$$\mathcal{I}[\gamma,\pi] = \int_I \big(L(t,\gamma,\mathcal{V}(t,\gamma,\pi)) + \pi_i(\dot\gamma^i - \mathcal{V}^i(t,\gamma,\pi))\big)dt = \int_I \big(\pi_i\dot\gamma^i - H(t,\gamma,\pi)\big)dt.$$

This is the main functional in the calculus of variations, discovered by H. Poincaré. We perform the first order variation of $\mathcal{I}[\gamma,\pi]$ denoted $\delta\mathcal{I}_{(\gamma,\pi)}[\delta\gamma,\delta\pi]$:

$$\begin{aligned} \delta\mathcal{I}_{(\gamma,\pi)}[\delta\gamma,\delta\pi] &= \int_I \Big(\dot\gamma^i\delta\pi_i + \pi_i\delta\dot\gamma^i - \frac{\partial H}{\partial y^i}(t,\gamma,\pi)\delta\gamma^i - \frac{\partial H}{\partial p_i}(t,\gamma,\pi)\delta\pi_i\Big)dt \\ &= -\int_I \Big(\dot\pi^i + \frac{\partial H}{\partial x_i}(t,\gamma,\pi)\Big)\delta\gamma^i dt + \int_I \frac{d}{dt}(\pi_i\delta\gamma^i)dt + \int_I \Big(\dot\gamma^i - \frac{\partial H}{\partial p_i}(t,\gamma,\pi)\Big)\delta\pi_i dt. \end{aligned}$$

If $\delta\gamma$ is taken with compact support, the first order variation $\delta\mathcal{I}_{(\gamma,\pi)}[\delta\gamma,\delta\pi] = 0$ gives Hamilton's equations. Now we proceed to the geometrization of these ideas, in the spirit of the work of Hélein [118]. In order to prepare the ground for later considerations related to the *Grassmanian standpoint*[9], we finally say a few words about geometrization. We introduce the following graphs $G[\gamma,\zeta]$, $G[\gamma,\pi]$ and $G[\gamma,\zeta,\pi]$ as the representations of the maps $(\gamma,\zeta): I\to T\mathfrak{Z}$, $(\gamma,\pi): I\to T^\star\mathfrak{Z}$ and $(\gamma,\zeta,\pi): I\to T\mathfrak{Z}\times_{\mathfrak{Z}} T^\star\mathfrak{Z}$. Notice that $G[\gamma,\zeta]\subset I\times T\mathfrak{Z}$, $G[\gamma,\zeta,\pi]\subset I\times T\mathfrak{Z}\times_{\mathfrak{Z}} T^\star\mathfrak{Z}$ and finally, $G[\gamma,\pi]\subset I\times T^\star\mathfrak{Z}$. The interplay of these three graphs is related to the geometrical description of

[8] thanks to Hamiltonian function $H: I\times T^\star\mathfrak{Z}\to\mathbb{R}: (t,y,p)\mapsto H(t,y,p) = p_i\mathcal{V}^i(t,y,p) - L(t,y,\mathcal{V}(t,y,p))$ see below

[9] which basically emphasize the central use of graph as dynamical objects.

each of the three previously described functionals: $\mathcal{L}^\circ[\gamma,\zeta]$, $\mathcal{L}^{\circ\circ}[\gamma,\zeta,\pi]$ and $\mathcal{I}[\gamma,\pi]$. We summarize this idea:

$$\begin{array}{lcl} G[\gamma,\zeta] & = & \{(t,\gamma(t),\zeta(z)), t\in I\} \\ G[\gamma,\zeta,\pi] & = & \{(t,\gamma(t),\zeta(z),\pi(t)), t\in I\} \\ G[\gamma,\pi] & = & \{(t,\gamma(t),\pi(t)), t\in I\} \end{array} \quad \text{and} \quad \begin{array}{lcl} \mathcal{L}^\circ[\gamma,\zeta] & = & \int_I L(t,\gamma,\zeta)dt = \int_{G(\gamma,\pi)} l \\ \mathcal{L}^{\circ\circ}[\gamma,\zeta,\pi] & = & \int_{G(\gamma,\zeta,\pi)} l - \pi_i\vartheta^i \\ \mathcal{I}[\gamma,\pi] & = & \int_{G(\gamma,\pi)} p_i dy^i - H(t,q,p)dt. \end{array}$$

Later we examine the condition $dt|_G \neq 0$, which mathematically allows us to express an oriented curve as a graph. Notice that $l = L(t,y,v)dt : I\times T\mathfrak{Z}$ is a 1-form defined on the evolution space.

3.3 Autonomous Hamiltonian setting and geometrization

In this section, we discuss the Legendre transform for autonomous Hamiltonian case in more detail. A similar construction is possible for the non-autonomous case. However, we will return to this point later. The Hamiltonian picture deals with structure on the cotangent bundle $T^\star\mathfrak{Z}$ by means of a non-degenerate Legendre transform: the map described by (7)(i) is a diffeomorphism. Then we are able to define its inverse $\mathfrak{J}^{-1}$ - (7)(ii). Notice that in this section, and in the following ones, we often abuse notation when specifying coordinates $(q^i,p_i) = (\gamma^i(t),\pi_i(t))$.

$$\text{(i)}\quad \mathfrak{J}: \begin{array}{ccc} T\mathfrak{Z} & \to & T^\star\mathfrak{Z} \\ (x,v) & \mapsto & \left(x, \frac{\partial L}{\partial v}(x,v)\right) \end{array} \qquad \text{(ii)}\quad \mathfrak{J}^{-1}: \begin{array}{ccc} T^\star\mathfrak{Z} & \to & T\mathfrak{Z} \\ (q,p) & \mapsto & (x,v) = (q,\mathcal{V}(q,p)) \end{array} \tag{7}$$

The *Legendre transform hypothesis* gives us a a characterization of $\mathcal{V}(q,p)$

$$p_i(t) = \frac{\partial L}{\partial v^i}(q,\mathcal{V}(q,p)) \qquad \text{and} \qquad \mathcal{V}^i(q,\frac{\partial L}{\partial v}(q,v)) = v^i \tag{8}$$

Hamiltonian dynamics gives the time evolution of coordinates (q,p) on $T^\star\mathfrak{Z}$, when (q,v) satisfy the Euler-Lagrange equations. From the path perspective, $\gamma:\mathbb{R}\to\mathfrak{Z}$ is a solution of the Euler-Lagrange equations if and only if the map $\mathfrak{z}^{\boxdot}: I\to T^\star\mathfrak{Z}/t\mapsto \mathfrak{z}^{\boxdot}(t) = (\gamma(t),\pi(t))$ is a solution of the Hamilton equations. This is possible thanks to the Hamiltonian function, defined in the autonomous case: $H: T^\star\mathfrak{Z}\to\mathbb{R}$. We observe the following definition:

Definition 3.3.1. *The Hamiltonian function is defined $H : T^\star\mathfrak{Z}\to\mathbb{R}$ such that*[10] $\forall(q,p)\in T^\star\mathfrak{Z}$, *we have* $H(q,p) = p_i\mathcal{V}^i(q,p) - L\circ\mathfrak{J}^{-1}(q,p)$

First we focus on the first equation of the system (6) - recall that here we are dealing with the autonomous case.

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial v^i}(\gamma(t),\zeta(t))\Big) = \frac{\partial L}{\partial x^i}(\gamma(t),\zeta(t)) \tag{9}$$

Since we want to express the problem with the variables $\pi_i(t) = \frac{\partial L}{\partial v^i}(\gamma(t),\zeta(t))$ rather than $\zeta^i(t)$, the left part of (9) is written $d\pi_i/dt(t)$. Now we come to the left part of equation (9). We first compute:

$$\frac{\partial H}{\partial q^i}(q,p) = \frac{\partial}{\partial q^i}p_j\mathcal{V}^j(q,p) - \frac{\partial}{\partial q^i}L\circ\mathfrak{J}^{-1}(q,p).$$

[10]For which the object $L\circ\mathfrak{J}^{-1}$ is seen as the map $L\circ\mathfrak{J}^{-1} : (q,p)\mapsto(q,\mathcal{V}(q,p))\longmapsto L\,(q,\mathcal{V}(q,p))$.

Since we work under the *Legendre hypothesis* (8) $\frac{\partial L}{\partial v^j}\left(q,\mathcal{V}(q,p)\right)=p_j$, we obtain:

$$\begin{aligned}\frac{\partial}{\partial q^i}\Big(L\circ\mathfrak{J}^{-1}\Big)(q,p) &= \frac{\partial L}{\partial x^i}\left(q,\mathcal{V}(q,p)\right)+\frac{\partial L}{\partial v^j}\left(q,\mathcal{V}(q,p)\right)\frac{\partial \mathcal{V}^j}{\partial q^i}(q,p)\\ &= \frac{\partial L}{\partial x^i}\left(q,\mathcal{V}(q,p)\right)+p_j\frac{\partial \mathcal{V}^j}{\partial q^i}(q,p)\\ &= \frac{\partial L}{\partial q^i}\left(q,\mathcal{V}(q,p)\right)=\Big(\frac{\partial}{\partial q^i}(L\circ\mathfrak{J}^{-1})(q,p)-p_j\frac{\partial \mathcal{V}^j}{\partial q^i}(q,p)\Big)=-\frac{\partial H}{\partial q^i}(q,p).\end{aligned}$$

In making this change of variables by means of the Legendre hypothesis, we consider $\zeta(t)=\mathcal{V}(\gamma(t),\pi(t))$ so that:

$$\frac{\partial H}{\partial q^i}(q,p)=-\frac{\partial L}{\partial x^i}(\gamma(t),\zeta(t))=-\frac{\partial}{\partial q^i}L\left(q,\mathcal{V}(q,p)\right)=-\frac{d}{dt}\Big(\frac{\partial L}{\partial v^i}(q,\mathcal{V}(q,p))\Big)=-\frac{d\pi_i}{dt}.$$

We find the first set of Hamilton equations (10):

$$\frac{d\pi_i}{dt}(t)=-\frac{\partial H}{\partial q^i}(q(t),p(t))\Big|_{1\leq i\leq k}. \tag{10}$$

Secondly we consider the condition: $\zeta(t)=\mathcal{V}(q(t),p(t))$, and thence obtain the condition: $d/dt[\gamma(t)]=\mathcal{V}(\gamma(t),\pi(t))$. Since

$$\begin{aligned}\frac{\partial H}{\partial p_i}(q,p) &= \frac{\partial}{\partial p_i}\Big[p_j\mathcal{V}^j(q,p)\Big]-\frac{\partial}{\partial p_i}\Big[L(q,\mathcal{V}(q,p))\Big]\\ &= \mathcal{V}^i(q,p)+p_j\frac{\partial \mathcal{V}^j}{\partial p_i}(q,p)-\frac{\partial L}{\partial z^j}(q,\mathcal{V}(q,p))\frac{\partial \mathcal{V}^j}{\partial p_i}(q,p)=\mathcal{V}^i(q,p),\end{aligned}$$

the second set of Hamilton equations (11) naturally arises:

$$\frac{d\gamma^i}{dt}(t)=\frac{\partial H}{\partial p_i}(\gamma(t),\pi(t))\ \Big|_{1\leq i\leq k}. \tag{11}$$

We picture the geometrization in the autonomous case by means of the Cartan-Poincaré form. Indeed, symplectic geometry is the natural arena to describe Hamiltonian dynamics. A symplectic structure ω defined on a manifold $\mathcal{M}$ is a closed ($d\omega=0$) and non-degenerate[11] 2-form. The geometrization of Hamiltonian dynamics is achieved via the use of the canonical Poincaré-Cartan form (12)(i) and the symplectic 2-form (12)(ii) given in coordinates:

$$\text{(i)}\quad \theta=\sum_{1\leq i\leq n}p_i dq^i \qquad\qquad \text{(ii)}\quad \omega=d\theta=\sum_{1\leq i\leq n}dp_i\wedge dq^i. \tag{12}$$

This process allows us to write the geometrical expression (13) for the Hamiltonian equation. Given a Hamiltonian function $H:T^\star\mathfrak{Z}\to\mathbb{R}$ and a Hamiltonian vector field ξ_H, we have:

$$\xi_H\ \lrcorner\ \omega=-dH. \tag{13}$$

Notice that a vector field $\xi\in T(T^\star\mathfrak{Z})$ is generally written as:

$$\xi=\frac{d}{dt}\Big((q(t),p(t))\Big)=\frac{dq^i}{dt}\frac{\partial}{\partial q^i}+\frac{dp_i}{dt}\frac{\partial}{\partial p_i},$$

[11]The non degeneracy condition means that we can construct an isomorphism between the vector fields ξ on $\mathcal{M}$ and the space of 1-forms: $T\mathcal{M}\to T^\star\mathcal{M}:\xi\to\xi\ \lrcorner\ \omega$. Then ω is non degenerate means $\forall\xi\in T\mathcal{M},\xi\ \lrcorner\ \omega=0\Rightarrow\xi=0$

whereas the Hamiltonian vector field ξ_H, is defined by the Hamilton equations:

$$\xi_H = (\frac{\partial H}{\partial p_i})\frac{\partial}{\partial q^i} - (\frac{\partial H}{\partial q^i})\frac{\partial}{\partial p_i} = \Big(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q^i}\Big) = \begin{pmatrix} 0 & \mathbf{Id} \\ -\mathbf{Id} & 0 \end{pmatrix} \cdot dH.$$

Integral curves of ξ_H are defined as maps $\mathfrak{z}^{\boxdot}_\gamma : \begin{cases} \mathbb{R} & \to & T^\star\mathfrak{Z} \\ t & \mapsto & (q(t), p(t)) \end{cases}$ that are solutions of Hamilton's equations - they are the dynamical trajectories of the Hamiltonian system $(\mathcal{M}, \omega, H)$. We want to characterize the flow on $T^\star\mathfrak{Z}$ that is encoded in the Hamilton equations. Equivalently, we picture the dynamical evolution for a point $(q(t), p(t)) \in T^\star\mathfrak{Z}$ via $\xi_H(q(t), p(t)) \in T(T^\star\mathfrak{Z})$:

$$\frac{d}{dt}(q(t), p(t)) = \frac{d\mathfrak{z}^{\boxdot}_\gamma}{dt}(t) = \xi_H(\mathfrak{z}_\gamma(t)) = \xi_H(q(t), p(t)).$$

The map $\mathfrak{z}^{\boxdot}(t)$ parametrizes an integral curve of ξ_H. In order to better understand later developments, we write more explicitly:

$$\mathfrak{z}^{\boxdot}_\gamma : \begin{cases} I & \to & T^\star\mathfrak{Z} \\ t & \mapsto & \mathfrak{z}^{\boxdot}_\gamma(t) = (q^1(t), \cdots q^k(t), p_1(t), \cdots, p_k(t)) \end{cases}$$

3.4 The Hamiltonian constraint and the presymplectic structure

In the previous treatment of Lagrangian and Hamiltonian classical mechanics, note that whatever the time dependence of the Lagrangian (autonomous or non-autonomous), we are faced with the notion of *time* as an exterior parameter which flows independently: it corresponds broadly to the Newtonian absolute time as an *evolution parameter* of the system, much as time does in QM. This is the *non-relativistic* case. This picture is altered when one treats time as a canonical variable. The key point is that in the previous development, *time* plays a twofold role. First as a *formal integration variable*, and secondly as the *external evolution parameter* in the Hamiltonian $H = H(t, q(t), p(t))$.

However, with the advent of relativistic dynamics comes an understanding of *space-time* itself as the fundamental dynamical entity of the theory. There is no *a priori* splitting of space-time into space and time. The fundamental covariance of the theory is related to the lack of such splitting. Then we need the notion of *extended phase space*, usually introduced in the context of classical mechanics. The subsequent development is twofold. On the one hand, we work on the proper covariant configuration space $\mathfrak{Z}^{\text{cov}}$ with local coordinates $\{q^{\boldsymbol{\mu}}\}_{0\leq\boldsymbol{\mu}\leq k}$. On the other, we choose a specific *time foliation* and work with $\mathfrak{Z}^{\text{ext}}$, the extended configuration space equivalently denoted $\mathfrak{Z}^\circ = \mathbb{R} \times \mathfrak{Z}$. In this case, coordinates are denoted $\{q^{\boldsymbol{\mu}}\}_{0\leq\boldsymbol{\mu}\leq k} = \{(q^\circ, q^i)\}_{1\leq i\leq k}$. Note that if we perform such a preferred *time foliation* we depart from the spirit of a purely covariant theory. Our idea is that in the search for a possible QG it is a fundamental necessity to express the field equations in a fully covariant way. This is the reason why we should distinguish between the *relativistic case* - where we work on $\mathfrak{Z}^\circ = \mathbb{R} \times \mathfrak{Z}$ - and the *covariant case* - where we work with $\mathfrak{Z}^{\text{cov}}$, without a preferred time direction.

Extended phase space. — The extended phase space - equivalently called extended phase space *with a preferred topology* - is built on the extended configuration space: the set of points $(q^\circ(\tau), q^i(\tau)) \in \mathfrak{Z} \times \mathbb{R} = \mathfrak{Z}^\circ$. The notation is chosen to emphasize that we add one dimension to the classical configuration space. Hence, the configuration space is no longer spanned by k variables - the position-coordinates q^i - but we consider instead the *trivial extended configuration space* as the data of $(k+1)$ variables treated on an equal footing. It appears that the role of the variable τ is that of a *parametrization* variable, whereas the *time* variable[12] $t(\tau) = q^\circ(\tau)$ is a variable parametrized

[12]as well as any coordinates on the extended phase space $T^\star\mathfrak{Z}^\circ = T^\star(\mathbb{R} \times \mathfrak{Z})$ denoted $q^\circ(\tau), q^i(\tau), p_\circ(\tau), p_i(\tau)$

by τ. In this connection we observe that $q^\circ(\tau) = t(\tau)$ plays the same role as the other variables. It is no longer viewed in the ambiguous role of *integration variable* vs *time* variable. Before further considering this dual aspect of time as parameter vs dynamical variable, we summarize the previous three cases in the following table:

Configuration Spaces	Phase Spaces
Configuration space $\mathfrak{Z}$ $\{q^i\}_{1\leq i\leq k}$	Phase space (PS) $T^\star\mathfrak{Z}$ $\{(q^i, p_i)\}_{1\leq i\leq k}$
Extended configuration space $\mathfrak{Z}^\circ$ $\{(q^\circ, q^i)\}_{1\leq i\leq k}$	Ext. PS $T^\star(\mathfrak{Z}\times\mathbb{R}) = T^\star\mathfrak{Z}^\circ$ $\{(q^\circ, q^i, p_\circ, p_i)\}_{1\leq i\leq k}$
Covariant configuration space $\mathfrak{Z}^{\mathsf{cov}}$ $\{q^{\boldsymbol{\mu}}\}_{0\leq\boldsymbol{\mu}\leq k}$	Cov. PS $T^\star\mathfrak{Z}^{\mathsf{cov}}$ $\{(q^{\boldsymbol{\mu}}, p_{\boldsymbol{\mu}})\}_{1\leq i\leq k}$

Notice that we generally describe the set of canonical variables by (q, p). Now we consider the *ontological* issues raised by the choice of dynamical variables. This leads to a consideration of the conceptual resources which can be provided by a graph theoretic treatment of dynamics. That treatment is deeply connected to a vision of the calculus of variations and to proposals for the correct mathematical treatment of Hamiltonian curves and dynamics which takes Grassmanian ideas as their point of departure. This leads us to emphasize the *obvious* and *natural* distinction between the *parametrized* space and the *parametrization* space. This is the first fundamental *duality* we discuss and - as explained below - its treatment is part of the ontological aspect of variational problems. Notice that the objects studied in such a variational formulation, are given by the paths γ , γ_{ext} and γ_{cov} respectively described by:

$$\left\{\begin{array}{ccc} I & \rightarrow & \mathfrak{Z} \\ t & \mapsto & \gamma(t) \end{array}\right. \qquad \left\{\begin{array}{ccc} I & \rightarrow & \mathfrak{Z}^\circ = \mathbb{R}\times\mathfrak{Z} \\ \tau & \mapsto & \gamma_{\mathsf{ext}}(\tau) = (\gamma^\circ_{\mathsf{ext}}(\tau), \gamma^i_{\mathsf{ext}}(\tau)) \end{array}\right. \qquad \left\{\begin{array}{ccc} I & \rightarrow & \mathfrak{Z} \\ t & \mapsto & \gamma_{\mathsf{cov}}(\tau) = (\gamma^{\boldsymbol{\mu}}_{\mathsf{cov}}(\tau)) \end{array}\right.$$

Therefore the *parametrized* space is given by the set of $(q^i(t))$, $(q^\circ(\tau), q^i(\tau))$ and $(q^{\boldsymbol{\mu}}(\tau))$ which are coordinates respectively on $\mathfrak{Z}$, $\mathfrak{Z}^{\mathsf{ext}}$ and $\mathfrak{Z}^{\mathsf{cov}}$, whereas the *parametrization* space is given in each case by means of an open subset $I \subset \mathbb{R}$ coordinated by t and τ[13] . Now we concentrate on the *extended* configuration space $\mathfrak{Z}^\circ$ so that paths: $\gamma_{\mathsf{ext}} : \tau \mapsto \gamma_{\mathsf{ext}}(\tau) = (\gamma^\circ_{\mathsf{ext}}(\tau), \gamma^i_{\mathsf{ext}}(\tau))$ are lifted respectively to $T\mathfrak{Z}^\circ$ and $T^\star\mathfrak{Z}^\circ$:

$$\gamma_{\mathsf{ext}}^{[T\mathfrak{Z}^\circ]} : \left\{\begin{array}{ccc} I & \rightarrow & T\mathfrak{Z}^\circ = T(\mathbb{R}\times\mathfrak{Z}) \\ \tau & \mapsto & (\gamma^\circ(\tau), \gamma^i(\tau), \dot\gamma^\circ(\tau), \dot\gamma^i(\tau)) \end{array}\right. \qquad \gamma_{\mathsf{ext}}^{[T^\star\mathfrak{Z}^\circ]} : \left\{\begin{array}{ccc} I & \rightarrow & T^\star\mathfrak{Z}^\circ = T^\star(\mathbb{R}\times\mathfrak{Z}) \\ \tau & \mapsto & (\gamma^\circ(\tau), \gamma^i(\tau), \pi_\circ(\tau), \pi_i(\tau)). \end{array}\right.$$

We insist on the fact that here *time* is encoded in the canonical variable $q^\circ(\tau) = \tau$. We exhibit the specific trivial case where time and parametrization are *identified*. In the case where $\gamma^\circ(\tau) = \tau$ we have paths $\gamma : \tau \mapsto (\tau, q^i(\tau))$ in $\mathfrak{Z}^\circ$ and the lifts are then described:

$$\gamma^{[T\mathfrak{Z}^\circ]} : \left\{\begin{array}{ccc} \mathbb{R} & \rightarrow & T\mathfrak{Z}^\circ = T(\mathbb{R}\times\mathfrak{Z}) \\ \tau & \mapsto & \gamma^\circ(\tau) = (\tau, \gamma^i(\tau), 1, \dot\gamma^i(\tau)) \end{array}\right. \qquad \gamma^{[T^\star\mathfrak{Z}^\circ]} : \left\{\begin{array}{ccc} \mathbb{R} & \rightarrow & T^\star\mathfrak{Z}^\circ \\ \tau & \mapsto & (\tau, \gamma^i(\tau), \pi_\circ(\tau), \pi_i(\tau)). \end{array}\right.$$

The extended phase space is the crucial step to the relativistic treatment of time and space on the same footing. By $q(\tau)$ we denote, as in the previous case of classical mechanics, the whole set of canonical positions. In this case, $q(\tau)$ represents both $\gamma^\circ(\tau)$ and $\gamma^i(\tau)$ or τ and $\gamma^i(\tau)$. We describe $T\mathfrak{Z}^\circ$ with coordinates $(q^\circ, q^i, z^\circ, z^i)$ and $T^\star\mathfrak{Z}^\circ$ with coordinates $(q^\circ, q^i, p_\circ, p_i)$. Notice we are now dealing with the core issue concerning the treatment of *time* as it will developed later in the Thesis. This picture corresponds to the parametrization of the time variable q° by another

[13] for classical phase space and for trivial extended phase space or covariant phase space.

additional variable $q^\circ(\tau)$. Then we write $L = L(\tau, q^\circ(\tau), q^i(\tau), z^\circ(\tau), z^i(\tau))$. Note that in the special case where $q^\circ(\tau) = \tau$, it is no longer possible to define $\mathfrak{J}^\circ$, the straightforward analogous Legendre map on $T\mathfrak{Z}^\circ$, for that would yield a meaningless expression:

$$\mathfrak{J}^\circ(\tau, \gamma^i(\tau), 1, \dot\gamma^i(\tau)) = \big(\tau, q^i(\tau), \frac{\partial L}{\partial z^\circ}(\tau, q^i(\tau), z^i(\tau)), \frac{\partial L}{\partial z^i}(\tau, q^i(\tau), z^i(\tau)\big).$$

It is not possible to specify $\partial L/\partial z^\circ(\tau, q^i(\tau), z^i(\tau))$. However $\left(\mathfrak{J}^\circ\right)^{-1}$ is well defined and gives rise to the Hamiltonian $\mathcal{H} : T^\star\mathfrak{Z}^\circ \to \mathbb{R}$. Since we want to shed light on the geometric construction that allows us to observe a time-invariant reparametrization feature which opens the way to the treatment of generally covariant systems. We imagine two Hamiltonian functions, those with and without an additional parameter τ. Then the general construction exhibits a Hamiltonian function as:

$$\mathcal{H}(q^\circ, q^i, p_\circ, p_i) = p_\circ \mathcal{Z}^\circ(q^\circ, q^i, p_\circ, p_i) + p_i \mathcal{Z}^i(q^\circ, q^i, p_\circ, p_i) - L\big(q^\circ, q^i, \mathcal{Z}^\circ(q^\circ, q^i, p_\circ, p_i), \mathcal{Z}^i(q^\circ, q^i, p_\circ, p_i)\big).$$

Here, all canonical coordinates are functions of τ. In the specific case where $q^\circ(\tau) = \tau$, we define a Hamiltonian function:

$$\mathcal{H}(\tau, q^i, p_\circ, p_i) = p_\circ + p_i \mathcal{Z}^i(\tau, q^i, p_i) - L(\tau, q^i, 1, \mathcal{Z}^i(\tau, q^i, p_i)).$$

This is always possible because in this case $\left(\mathfrak{J}^\circ\right)^{-1}(\tau, q^i, p_\circ, p_i) = (\tau, z, 1, \mathcal{Z}^i(\tau, z^i, p^i))$ is well defined. Relativistic mechanics is described with the *covariant Hamiltonian* $\mathcal{H} : T^\star(\mathfrak{Z} \times \mathbb{R}) \to \mathbb{R}$:

$$\mathcal{H}(\tau, q^i, p_\circ, p_i) = p_\circ + H(\tau, q^i, p_i). \tag{14}$$

Notice that $H = H(\tau, q^i, p_i)$ is identified with the Hamiltonian on the phase space $I \times T^\star\mathfrak{Z}$. In the relativistic formulation based on the *extended* configuration space $\mathfrak{Z}^\circ$ and the *extended* phase space $T^\star\mathfrak{Z}^\circ$, we build a geometrical picture analogous to the one developed for independent time mechanics - *i.e* we find a relation similar to (13) - but with objects - the Hamiltonian function and the symplectic 2-form - defined on $T^\star\mathfrak{Z}^\circ$. This leads to the relation (15):

$$\xi_{\mathcal{H}} \lrcorner\, \omega = -d\mathcal{H}. \tag{15}$$

However (15) makes use of the covariant Hamiltonian $\mathcal{H}(\tau, q^i, p_\circ, p_i) = p_\circ + H(\tau, q^i, p_i)$ on $T^\star\mathfrak{Z}^\circ$ Since the space $T^\star\mathfrak{Z}^\circ$ is a cotangent bundle it carries a canonical one-form $\theta = p_i dq^i + p_\circ dq^\circ$ and symplectic form $\omega = d\theta$. Notice that if $dim(\mathfrak{Z}) = k$, then $dim(T^\star\mathfrak{Z}^\circ) = 2k + 2$. We exhibit pre-symplectic dynamics by introducing the constraint hypersurface (16) $\Sigma_\circ$ as a $2k + 1$ dimensional submanifold of $T^\star\mathfrak{Z}^\circ$.

$$\Sigma_\circ \subset T^\star(\mathfrak{Z} \times \mathbb{R}) = \{(q^\circ, q^i, p_\circ, p_i) \in T^\star(\mathfrak{Z} \times \mathbb{R}) / p_\circ = -H(q, p)\}. \tag{16}$$

Let $\mathfrak{i} : \Sigma_\circ \to T^\star(\mathfrak{Z} \times \mathbb{R})$ be the inclusion map. The restrictions on the hypersurface $\Sigma_\circ$: $\theta_{|\Sigma_\circ} = \mathfrak{i}^\star\theta$ and $\omega_{|\Sigma_\circ} = \mathfrak{i}^\star\omega$ indicate a degenerate feature. This is why $\mathfrak{i}^\star\theta$ is called a *pre-symplectic* 2-form. Relativistic dynamics is given by the data of pre-symplectic space $(\Sigma_\circ, \omega_{|\Sigma_\circ})$ (with the additional condition that $(dq^\circ)_{|\Sigma_\circ} \neq 0$) whereas the pair $(T^\star\mathfrak{Z}^\circ, \omega)$ is symplectic. The analogue of (15) - namely the dynamical equations - is written in the pre-symplectic setting as follows:

$$\forall \xi \in \Gamma(\mathfrak{Z}^\circ, T\mathfrak{Z}^\circ), \qquad (\xi \lrcorner\, \omega)|_{\Sigma_\circ} = 0 \qquad \text{and} \qquad (dq^\circ)|_{\Sigma_\circ} \neq 0. \tag{17}$$

We return later to the general case of n-phase space, where pre-symplectic dynamics is fully described. The constraint $p_\circ = -H(q, p)$ leads to the picture of the problem on the constraint surface defined by the *Hamiltonian constraint* $\mathcal{H} = 0$. Here we have defined $\mathcal{H}$ in a natural way as in (14). This this usual approach, the theory takes on the form of a 1-phase space structure on the geometrical level. We denote by $\Sigma_\circ = \mathcal{H}^{-1}(0)$ the submanifold of $T^\star\mathfrak{Z}^\circ$ such that:

$$\Sigma_\circ = \mathcal{H}^{-1}(0) = \{(q,p) \in T^\star\mathfrak{Z}^\circ \ / \ \mathcal{H}(q,p) = 0\}.$$

This is equivalent to working with the Hamiltonian function $H(q^i(\tau), p_i(\tau)) = H(\tau, q^i, p_i)$ without explicitly making reference to $\mathcal{H}$, but by imposing the *Hamiltonian constraint.* In the most general context, using an additional parametrization variable τ, we consider the following paths:

$$\gamma : \begin{cases} [\tau_0, \tau_1] & \to & \Sigma_\circ \subset T^\star(\mathfrak{Z} \times \mathbb{R}) \\ \tau & \mapsto & (q^\circ(\tau), q^i(\tau), -H(q(\tau), p(\tau)), p_i(\tau)). \end{cases}$$

Proposition 3.1. *A path $\gamma : [\tau_1, \tau_2] \subset \mathbb{R} \to \Sigma_\circ$ is a solution of Hamilton equation if and only if its tangent vector $\gamma' = d\gamma(\tau)/d\tau$ satisfy:*

$$\big(\gamma' \lrcorner\, \omega\big)\big|_{\Sigma_\circ} = 0 \qquad \text{and} \qquad \frac{dq^\circ}{d\tau}(\tau) \neq 0.$$

We consider $X \in T\Sigma_\circ \subset T(T^\star\mathfrak{Z}^\circ)$:

$$X = \frac{d}{d\tau}\Big((q(\tau), p(\tau))\Big) = \frac{dq^\circ}{d\tau}\frac{\partial}{\partial q^\circ} + \frac{dq^i}{d\tau}\frac{\partial}{\partial q^i} + \frac{dp_\circ}{d\tau}\frac{\partial}{\partial p_\circ} + \frac{dp_i}{d\tau}\frac{\partial}{\partial p_i} = \Theta^\circ\frac{\partial}{\partial q^\circ} + \Theta^i\frac{\partial}{\partial q^i} + \Upsilon_\circ\frac{\partial}{\partial p_\circ} + \Upsilon_i\frac{\partial}{\partial p_i},$$

with $\Theta^\circ = dq^\circ/d\tau(\tau)$, $\Theta^i = dq^i/d\tau(\tau)$, $\Upsilon_\circ = dp_\circ/d\tau(\tau)$ and finally $\Upsilon_i = dp_i/d\tau(\tau)$. Since we work on $\Sigma_\circ$ we have the relation $p_\circ(\tau) = -H(q(\tau), p(\tau))$ and the term $\Upsilon_\circ \partial/\partial p_\circ$ gives:

$$\Upsilon_\circ\frac{\partial}{\partial p_\circ} = \frac{dp_\circ}{d\tau}(\tau)\frac{\partial}{\partial p_\circ} = -\frac{dH}{d\tau}(q(\tau), p(\tau))\frac{\partial}{\partial p_\circ} = -\Big(\frac{\partial H}{\partial q^i}\frac{dq^i}{d\tau} + \frac{\partial H}{\partial p_i}\frac{dp_i}{d\tau}\Big)\frac{\partial}{\partial p_\circ}.$$

Therefore we rewrite X as:

$$X = \Theta^\circ\frac{\partial}{\partial q^\circ} + \Theta^i\frac{\partial}{\partial q^i} - \Big(\frac{\partial H}{\partial q^i}\frac{dq^i}{d\tau} + \frac{\partial H}{\partial p_i}\frac{dp_i}{d\tau}\Big)\frac{\partial}{\partial p_\circ} + \Upsilon_i\frac{\partial}{\partial p_i}.$$

Thanks to $\omega = d\theta$ we expand:

$$\begin{aligned} (X \lrcorner\, \omega)\big|_{\Sigma_\circ} &= \big(X \lrcorner\, (dp_i \wedge dq^i + dp_\circ \wedge dq^\circ)\big)\big|_{\Sigma_\circ} \\ &= dp_i(\Upsilon_i\frac{\partial}{\partial p_i})dq^i - dq^i(\Theta^i\frac{\partial}{\partial q^i})dp_i - dp_\circ\Big((\frac{\partial H}{\partial q^i}\frac{dq^i}{d\tau} + \frac{\partial H}{\partial p_i}\frac{dp_i}{d\tau})\frac{\partial}{\partial p_0}\Big)dq^\circ \\ &\quad -dq^\circ(\Theta^\circ\frac{\partial}{\partial q^\circ})dp_\circ \\ &= \Upsilon_i dq^i - \Theta^i dp_i - \Big(\frac{\partial H}{\partial q^i}\frac{dq^i}{d\tau} + \frac{\partial H}{\partial p_i}\frac{dp_i}{d\tau}\Big)dq^\circ + \Theta^\circ\Big(\frac{\partial H}{\partial q^i}dq^i + \frac{\partial H}{\partial p_i}dp_i\Big). \end{aligned}$$

Therefore, $(X \lrcorner\, \omega)\big|_{\Sigma_\circ} = 0$ is equivalent to:

$$\Big(\Upsilon_i + \Theta^\circ(\frac{\partial H}{\partial q^i})\Big)dq^i - \Big(\Theta^i - \Theta^\circ(\frac{\partial H}{\partial p_i})\Big)dp_i - \Big(\Theta^i(\frac{\partial H}{\partial q^i}) + \Upsilon_i(\frac{\partial H}{\partial p_i})\Big)dq^\circ = 0$$

$$\left|\begin{array}{lcl} \Upsilon_i &=& -\Theta^\circ(\frac{\partial H}{\partial q^i}) \\ \Theta^i &=& \Theta^\circ(\frac{\partial H}{\partial p_i}) \\ 0 &=& \Theta^i(\frac{\partial H}{\partial q^i}) + \Upsilon_i(\frac{\partial H}{\partial p_i}) \end{array}\right. \qquad \text{so that} \qquad \left|\begin{array}{lcl} \frac{dp_i}{d\tau}(\tau) &=& -\Theta^\circ(\frac{\partial H}{\partial q^i}) \\ \frac{dq^i}{d\tau}(\tau) &=& \Theta^\circ(\frac{\partial H}{\partial q^i}) \\ 0 &=& \Theta^i(\frac{\partial H}{\partial q^i}) + \Upsilon_i(\frac{\partial H}{\partial p_i}). \end{array}\right.$$

The condition $\Theta^\circ \neq 0$ means that $q^\circ(\tau)$ is an invertible map and we can define the map $\big(\Theta^\circ\big)^{-1}$. This allows us to drop the variable τ and instead use q°. Let us call $q^\circ = t$ to recover our initial description. In the (pre)-relativistic formulation we obtain Hamilton's equations *and* conservation of energy:

$$\frac{dq^i}{dt}(t) = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt}(t) = -\frac{\partial H}{\partial q^i} \qquad \frac{dH}{dt} = 0.$$

The question that naturally arises here concerns the treatment of the parametrization Θ°. Then, in the case where $\Theta^\circ = dq^\circ/d\tau = 1$, we get $q^\circ(\tau) = \tau$, $z^\circ = 1$ so that the extended Hamiltonian $\mathcal{H} : T^\star(\mathfrak{Z} \times \mathbb{R}) \to \mathbb{R}$ is given by:

$$\mathcal{H}(q^\circ, q^i, p_\circ, p_i) = p_\circ(\frac{dq^\circ}{d\tau}) + p_i(\frac{dq^i}{d\tau}) - L(q^\circ, q^i, z^\circ, z^i) = p_\circ + H(q^\circ, q^i, p_i). \tag{18}$$

Classical mechanics falls under the case $\mathcal{H} = 0$. In the next section, we exhibit a way of treating time-dependent mechanics that allows us to treat the time issue directly from the specific construction itself.

3.5 Geometrical construction for relativistic dynamics

We exhibit the *Legendre correspondence hypothesis* developed in [115], applied to the case of classical mechanics. More precisely, we find that the specific construction on paths $\gamma : I \to \mathfrak{Z}$ allows us to recover dynamical evolution, where *time* and *energy* are seen as canonically conjugate variables in a natural way. Let us emphasize that the construction found in [115] gives a straightforward interpretation of *time* as a dimension and thus eliminates the schizophrenic opposition between its interpretation *either* as a parameter *or* as a variable in the extended configuration space. The *universal Hamiltonian formalism* was designed to address this point. Time and space are there treated on an equal footing in accordance with the principle of relativity and therefore, we need to separate the issue of parametrization of paths from the question of the number of dimensions within the theory. The question of how such apparently distinct fundamental notions as (*time, space* or *fields*) appear so different from each other is indeed an even more delicate one but finds its resolution in the *dynamics* itself. For it is the dynamical behavior of *observable forms* that allows us to address this question. The right and appropriate treatment of time dependent mechanics that led to the setting of a fully covariant field theory can be viewed as follows: we consider a path $\gamma : I \to \mathfrak{Z}$, and we denote by $\{\tau\}$ local coordinates on I as well as $\{y^1, \cdots, y^k\}$ local coordinates on $\mathfrak{Z}$. We denote $\{q^{\boldsymbol{\mu}}\}_{1\leq \boldsymbol{\mu} \leq (k+1)} = \{q^1, \cdots, q^{k+1}\} = \{\tau, y^1, \cdots, y^k\}$ coordinates on $I \times \mathfrak{Z}$. Then, we consider the following map:

$$\mathfrak{z}_\gamma : \begin{cases} I & \to & I \times \mathfrak{Z} \\ \tau & \mapsto & \mathfrak{z}(\tau) = (\tau, \gamma(\tau)). \end{cases}$$

The map $\mathfrak{z}_\gamma$ associates to any element $\tau \in I \subset \mathbb{R}$ the graph of γ, $G[\gamma] = \big\{(\tau, \gamma(\tau))/\tau \in I\big\}$. Therefore we picture the graph of γ, $G[\gamma]$ as the image of the map $\mathfrak{z}_\gamma(\tau) = G[\gamma]$. We associate to γ, $\gamma^\star T\mathfrak{Z} \otimes T^\star I = \gamma^\star T\mathfrak{Z} \otimes I \subset T\mathfrak{Z} \otimes I$, as a bundle over I. Notice that if we consider the tangent bundle $T\mathfrak{Z} \to \mathfrak{Z}$, and the smooth map $\gamma : I \to \mathfrak{Z}$, then by definition the bundle $\gamma^\star T\mathfrak{Z}$ is a fiber bundle over I, whose fiber over $\tau \in I$ is given by $(\gamma^\star T\mathfrak{Z})_\tau = (T\mathfrak{Z})_{\gamma(\tau)}$. In this picture we have the following two bundles, the tangent bundle and its associated pullback bundle by the map γ: respectively $T\mathfrak{Z} \longrightarrow \mathfrak{Z}$ and $\gamma^\star T\mathfrak{Z} \longrightarrow I$. We construct the bundle:

$$\begin{array}{c} \gamma^\star T\mathfrak{Z} \otimes T^\star I \\ \big\downarrow \pi \\ I \end{array} \tag{19}$$

The total space of this bundle is made with fibers over $\tau \in I$ which are taken to be objects in $\gamma^\star T\mathfrak{Z} \otimes T^\star I$. These are maps from the space TI to the total space of the previous pullback bundle, namely maps to the space $\gamma^\star T\mathfrak{Z}$. Therefore, we denote coordinates τ on I and $v^i = v^i_\tau$, with $1 \leq i \leq k = dim(\mathfrak{Z})$ such that a point in $\gamma^\star T\mathfrak{Z} \otimes T^\star I$ is described with coordinates (τ, v). Indeed, the object v is described by the map: $TI \cong I \to \gamma^\star T\mathfrak{Z}$. Notice also that to the map $\gamma : I \to \mathfrak{Z}$, we naturally associate the differential $d\gamma : \tau \mapsto d\gamma_\tau$, a 1-form that associate to any point $\tau \in I$ the differential of γ at τ[14]: $d\gamma : \tau \mapsto d\gamma_\tau : \begin{cases} T_\tau I & \to & T_{\gamma(\tau)}\mathfrak{Z} \\ \tau & \mapsto & d\gamma/d\tau(\tau) \end{cases}$.

This basic construction justifies the introduction of the bundle (19). The differential $d\gamma$ is a section of the bundle (19). Now we introduce another bundle: $T\mathfrak{Z} \otimes_{I\times\mathfrak{Z}} T^\star I \cong T\mathfrak{Z} \otimes_{I\times\mathfrak{Z}} I$

$$\begin{array}{c} T\mathfrak{Z} \otimes_{I\times\mathfrak{Z}} T^\star I \\ \big\downarrow \boldsymbol{\pi} \\ I \times \mathfrak{Z} \end{array} \tag{20}$$

We can think of $\gamma^\star T\mathfrak{Z} \otimes T^\star I$ as a sub-bundle $\gamma^\star T\mathfrak{Z} \otimes T^\star I \subset T\mathfrak{Z} \otimes_{I\times\mathfrak{Z}} I$. In doing so coordinates on the former are denoted by (τ, v) whereas coordinates on the latter are denoted (τ, y, v). Then, coordinates on $\gamma^\star T\mathfrak{Z} \otimes T^\star I$, namely (x, v), are naturally represented by:

$$v = \sum_{1\leq i\leq k} v^i \frac{\partial}{\partial y^i} \otimes d\tau.$$

The interest of the bundle (20) appears clearly since the base manifold in this case is the product $I \times \mathfrak{Z}$. Therefore, either parametrization space I - in classical mechanics, parametrization space is *time* - or the target space, namely the configuration space $\mathfrak{Z}$, are treated on equal footing as *graph*. This further stresses the underlying philosophy, which is to treat on the *same* footing time *and* space. Later, we apply the same principle to space-time *and* fields. To emphasize this we denote $\mathfrak{Z}^{\mathbf{o}} = I \times \mathfrak{Z}$. Notice the distinction with the notation for extended phase space $\mathfrak{Z}^\circ = \mathbb{R} \times \mathfrak{Z}$. Even if mathematically these spaces serve the same purpose (since we work in the symplectic setting) still conceptually they are obtained from different roots. The former is due to the graph idea whereas the latter is concerned with a topological choice. Now we identify the following two bundles, - the fiber over $\mathfrak{Z}^{\mathbf{o}} = I \times \mathfrak{Z}$ denoted ${}^{\mathfrak{s}}T_{(\tau,v)}\mathfrak{Z}^{\mathbf{o}}$ is described via:

$$\begin{array}{ccccc} T\mathfrak{Z} \otimes_{I\times\mathfrak{Z}} T^\star I & \cong & {}^{\mathfrak{s}}T_{(\tau,v)}\mathfrak{Z}^{\mathbf{o}} & = & {}^{\mathfrak{s}}T_{(\tau,v)}(I \times \mathfrak{Z}) \\ \big\downarrow \boldsymbol{\pi} & & \big\downarrow \boldsymbol{\pi}^{\mathbf{o}} & & \big\downarrow \boldsymbol{\pi}^{\mathbf{o}} \\ I \times \mathfrak{Z} & & I \times \mathfrak{Z} & & I \times \mathfrak{Z} \end{array} \tag{21}$$

We consider ${}^{\mathfrak{s}}T_{(\tau,v)}\mathfrak{Z}^{\mathbf{o}} = {}^{\mathfrak{s}}T_{(\tau,v)}(I \times \mathfrak{Z}) = \big\{(q, z) \in T(I \times \mathfrak{Z})/z \in T_q\mathfrak{Z}^{\mathbf{o}}, d\tau(z) = 1\big\}$. For any point $(\tau, v) = (\tau, \gamma(\tau)) \in I \times \mathfrak{Z}$, we identify the fiber ${}^{\mathfrak{s}}T_{(\tau,v)}\mathfrak{Z}^{\mathbf{o}}$ with $T_y\mathfrak{Z} \otimes T^\star_\tau I$ though the diffeomorphism:

$$\left| \begin{array}{ccc} T_y\mathfrak{Z} \otimes T^\star_\tau I & \to & {}^{\mathfrak{s}}T_{(\tau,v)}\mathfrak{Z}^{\mathbf{o}} \\ v = \displaystyle\sum_{1\leq i\leq k} v^i \frac{\partial}{\partial y^i} \otimes d\tau & \mapsto & z = \displaystyle\sum_{1\leq \boldsymbol{\mu}\leq k+1} z^{\boldsymbol{\mu}} \frac{\partial}{\partial q^{\boldsymbol{\mu}}} = \frac{\partial}{\partial \tau} + \sum_{1\leq i\leq k} v^i \frac{\partial}{\partial y^i}. \end{array} \right. \tag{22}$$

Indeed, the bold index $1 \leq \boldsymbol{\mu} \leq k+1$ is a multi-index meaning that $z^{\boldsymbol{\mu}} = 1$ for $\boldsymbol{\mu} = 1$ and $z^{\boldsymbol{\mu}} = v^i$ for $2 \leq \boldsymbol{\mu} \leq k+1$. The identification ${}^{\mathfrak{s}}T_{(\tau,z)}\mathfrak{Z}^{\mathbf{o}} \simeq T_z\mathfrak{Z} \otimes T^\star_\tau I$ gives alternatively the coordinates (τ, y^i, z^i)

[14] notice the canonical identification $TI \cong I$

or (τ, y^i, v^i). A time dependent Lagrangian density is described by $L : T\mathfrak{Z}\otimes T^\star I = T\mathfrak{Z}\otimes_{I\times\mathfrak{Z}} T^\star I \mapsto \mathbb{R}$. By considering the identification $T^\star I \simeq I$, we recover the usual description of such a Lagrangian - namely: $L = L(\tau, \gamma(\tau), d\gamma(\tau))$ - and its related functional: $\mathcal{L}[\gamma] = \int_I L(\tau, \gamma(\tau), d\gamma(\tau))d\tau$. Now we proceed to *Legendre correspondence* geometrization. This exhibits the following Legendre correspondence:

$$^{\mathfrak{s}}T(\mathfrak{Z}^{\mathbf{o}}) =^{\mathfrak{s}} T(I\times\mathfrak{Z}) \leftrightarrow T^\star(I\times\mathfrak{Z}) = T^\star(\mathfrak{Z}^{\mathbf{o}}).$$

This correspondence involves the analogue of cotangent space $T^\star(I\times\mathfrak{Z})$ and the analogue of the tangent space $T(I\times\mathfrak{Z})$. We emphasize that conceptually, $T^\star(\mathfrak{Z}^{\mathbf{o}}) = T^\star(I\times\mathfrak{Z})$ differs from $T^\star\mathfrak{Z}^\circ = T^\star(\mathbb{R}\times\mathfrak{Z})$ even if they formally agree. Then, on the bundle $T^\star(I\times\mathfrak{Z})$ - the analogue of the cotangent space - every point $(q,p)\in\mathcal{M} = T^\star(I\times\mathfrak{Z})$ has coordinates $q^{\boldsymbol{\mu}} = \{\tau, y^i\}$ and $p_{\boldsymbol{\mu}} = \{\mathfrak{e}, p_i\}$ so that $p = \mathfrak{e}d\tau + p_i dy^i$ represents any element of the bundle whereas every point $(q,v)\simeq(q,z)\in {}^{\mathfrak{s}}T(I\times\mathfrak{Z})$ has coordinates $q^{\boldsymbol{\mu}} = \{\tau, y^i\}$ and $v^i \simeq z^i$. We emphasize the dimension of the involved spaces:

$$\left|\begin{array}{l} \dim[\mathfrak{Z}] = k \\ \dim[\mathfrak{Z}^{\mathbf{o}}] = \dim[I\times\mathfrak{Z}] = k+1 \\ \dim[T(\mathfrak{Z}^{\mathbf{o}})] = 2k+2 \\ \dim[{}^{\mathfrak{s}}T(\mathfrak{Z}^{\mathbf{o}})] = 2k+1 \\ \dim[T\mathfrak{Z}\otimes_{I\times\mathfrak{Z}} T^\star I] = 2k+1 \dim[T^\star(\mathfrak{Z}^{\mathbf{o}})] = 2k+2 \end{array}\right.$$

Notice that the usual Lagrangian density $L(t,\gamma(t), d\gamma(t))$ for non-autonomous classical mechanics, is defined on a $2k+1$-dimensional space. The Legendre transform - in the classical presentation of non-autonomous dynamics - is described thus: $I\times T\mathfrak{Z}\to I\times T^\star\mathfrak{Z}$ and is generated by the function:

$$W^{\text{non-autonomous}} : \left\{\begin{array}{lcl} (I\times T\mathfrak{Z})\times(I\times T^\star\mathfrak{Z}) & \to & \mathbb{R} \\ (q,v,p) = (t, y^i, v^i, p_i) & \mapsto & p_i v^i - L(t,y,v). \end{array}\right.$$

Notice that we equivalently denote $I\times T\mathfrak{Z}$ by means of $\mathbf{o}$-notation. We obtain: $(T\mathfrak{Z})^{\mathbf{o}} = I\times T\mathfrak{Z}$ and $(T^\star\mathfrak{Z})^{\mathbf{o}} = I\times T^\star\mathfrak{Z}$. Following the method proposed in [115], we describe the Legendre correspondence. On the Lagrangian side we work with ${}^{\mathfrak{s}}T(\mathfrak{Z}^{\mathbf{o}})$. Notice that we can identify this space with the classical non-autonomous case ${}^{\mathfrak{s}}T(\mathfrak{Z}^{\mathbf{o}}) \simeq (T\mathfrak{Z})^{\mathbf{o}} = I\times T\mathfrak{Z}$ through a diffeomorphism. The difference comes into the play on the Hamiltonian side, where directly, in the construction, we add a further variable $\mathfrak{e}$. Here we work with $T^\star(\mathfrak{Z}^{\mathbf{o}})$. In this case we introduce the Legendre correspondence $W^{\mathbf{o}}(q,z,p)$:

$$W^{\mathbf{o}} : \left\{\begin{array}{ccc} {}^{\mathfrak{s}}T(\mathfrak{Z}^{\mathbf{o}})\times T^\star(\mathfrak{Z}^{\mathbf{o}}) & \to & \mathbb{R} \\ (q,z,p) = (\tau, y^i, z^i, \mathfrak{e}, p_i) & \mapsto & \langle p, z\rangle - L(q,z), \end{array}\right.$$

$$\text{with } \langle p,z\rangle = \langle \mathfrak{e}d\tau + p_i dy^i, \sum_{1\le\boldsymbol{\mu}\le k+1} z^{\boldsymbol{\mu}}\frac{\partial}{\partial q^{\boldsymbol{\mu}}}\rangle = \langle \mathfrak{e}d\tau + p_i dy^i, \frac{\partial}{\partial\tau} + \sum_{1\le i\le k} v^i\frac{\partial}{\partial y^i}\rangle = \mathfrak{e} + p_i z^i.$$

We define the Legendre transform hypothesis and we denote $(q,z,\varpi)\leftrightarrow(q,p)$ if and only if $\forall(q,z,\varpi) = (\tau, q^i, z^i, \varpi)\in {}^{\mathfrak{s}}T(\mathfrak{Z}^{\mathbf{o}})\times\mathbb{R}$ and $\forall(q,p) = (\tau, q^i, \mathfrak{e}, p_i)\in T^\star(\mathfrak{Z}^{\mathbf{o}})$:

$$(q,z,\varpi)\leftrightarrow(q,p) \quad\Longleftrightarrow\quad \left|\begin{array}{rcl} \langle p,z\rangle - L(q,z) = W^{\mathbf{o}}(q,z,p) & = & \varpi \\ \dfrac{\partial W}{\partial z}(q,z,p) & = & 0. \end{array}\right. \tag{23}$$

The Legendre Hypothesis states that $\forall(q,p) \in T^\star(\mathfrak{Z}^\circ)$ there exists a unique $z \in {}^{\mathfrak{s}}T(\mathfrak{Z}^\circ)$, denoted $\mathcal{Z}(q,p)$, a critical point of $z \mapsto W^\circ(q,z,p)$. The Hamiltonian function $\mathcal{H} : T^\star(\mathfrak{Z}^\circ) \to \mathbb{R}$ is constructed:

$$\mathcal{H} : (\tau, q^i, \mathfrak{e}, p_i) \mapsto \mathcal{H}(\tau, q^i, \mathfrak{e}, p_i) = \langle p, \mathcal{Z}(q,p)\rangle - L(q, \mathcal{Z}(q,p)) = W^\circ(q, \mathcal{Z}(q,p), p).$$

By construction, $\mathcal{H}(\tau, q^i, \mathfrak{e}, p_i) = \mathfrak{e} + p_i \mathcal{Z}^i(\tau, q^i, \mathfrak{e}, p_i) - L(\tau, q^i, \mathcal{Z}^i(q,p))$. The Hamilton equations are pictured as necessary and sufficient conditions on the map $\mathfrak{z}_\gamma^{\boxdot} : \tau \mapsto (q(\tau), p(\tau))$ such that there exists fields $\tau \mapsto \gamma(\tau)$ that verify two conditions. The first is that the Legendre condition is realized, $\forall \tau\ (\tau, \gamma(\tau), d\gamma(\tau)) \leftrightarrow (q(\tau), p(\tau))$, whereas the second is that fields $x \mapsto \gamma(\tau)$ are solutions of Euler Lagrange equations. In this geometrical construction for the time-dependent Lagrangian the map is:

$$\mathfrak{z}_\gamma^{\boxdot} : \begin{cases} I & \to \quad T^\star \mathfrak{Z}^\circ = T^\star(I \times \mathfrak{Z}) \\ \tau & \mapsto \quad \big(q(\tau), p(\tau)\big) = \big(\tau, q^i(\tau), \mathfrak{e}(\tau), p_i(\tau)\big). \end{cases}$$

The conditions for a map $\gamma : I \to \mathfrak{Z} : \tau \mapsto \gamma(\tau)$ such that $(\tau(t), \gamma(\tau), d\gamma(\tau)) \leftrightarrow (q(\tau), p(\tau))$ are:

- [1] First we must have: $q(\tau) = (\tau, \gamma(\tau)) = \mathfrak{z}(\tau)$. Let $\mathfrak{z} : \begin{cases} I & \to \quad I \times \mathfrak{Z} = \mathfrak{Z}^\circ \\ \tau & \mapsto \quad \mathfrak{z}(\tau) = (\tau, q^i(\tau)) \end{cases}$ be the map under consideration with $q(\tau) = (\tau, \gamma^i(\tau))$. The image $\mathfrak{z}(\tau)$ of the map is the graph $G[\gamma]$.
- [2] In the bundle $T\mathfrak{Z} \otimes_{I\times\mathfrak{Z}} T^\star I$, the coordinates $(\tau, y^i, v^i) = (\tau, y^i, \frac{dy^i}{d\tau})$ are identified with $\big(q(\tau), \mathcal{V}^i(q(\tau), p(\tau))\big)$. Hence, in ${}^{\mathfrak{s}}T(\mathfrak{Z}^\circ)$, we obtain $\frac{dq}{d\tau} = \frac{d\mathfrak{z}}{d\tau} = \mathcal{Z}(q(\tau), p(\tau)))$ and thus the following k equations:

$$\frac{dq^i}{d\tau}(\tau) = \mathcal{Z}^i(q(\tau), p(\tau)) \tag{24}$$

Since, by construction, $\mathcal{H}(q,p) = W^\circ(q, \mathcal{Z}(q,p), p)$ and recognizing the central role of the condition $\partial W^\circ / \partial z(q, \mathcal{Z}(q,p), p) = 0$, we realize that:

$$d\mathcal{H}(q,p) = \frac{\partial W^\circ}{\partial q^{\boldsymbol{\mu}}}(q, \mathcal{Z}(q,p), p) dq^{\boldsymbol{\mu}} + \frac{\partial W^\circ}{\partial p_{\boldsymbol{\mu}}}(q, \mathcal{Z}(q,p), p) dp_{\boldsymbol{\mu}}. \tag{25}$$

Also, by construction $\langle p, z\rangle - L(q,z) = W^\circ(q,z,p) = 0.$[15] Then we have $\forall 1 \leq \boldsymbol{\mu} \leq (k+1)$:

$$\frac{\partial W^\circ}{\partial q^{\boldsymbol{\mu}}}(q,z,p) = -\frac{\partial L}{\partial q^{\boldsymbol{\mu}}}(q,z,p).$$

We write the expression (25) for $d\mathcal{H}$ in the following way:

$$d\mathcal{H} = -\sum_{1\leq \boldsymbol{\mu} \leq k+1} \frac{\partial L}{\partial q^{\boldsymbol{\mu}}}(q, \mathcal{Z}(q,p), p) dq^{\boldsymbol{\mu}} + \sum_{1\leq \boldsymbol{\mu} \leq k+1} \mathcal{Z}^{\boldsymbol{\mu}}(q,p) dp_{\boldsymbol{\mu}}, \tag{26}$$

with $\mathcal{Z}^{\boldsymbol{\mu}}(q,p)$ components of the vector $\mathcal{Z}(q,p)$ in the basis $\{\frac{\partial}{\partial q^{\boldsymbol{\mu}}}\}_{1\leq\boldsymbol{\mu}\leq k+1}$.

$$\mathcal{Z}(q,p) = \sum_{\boldsymbol{\mu}} \mathcal{Z}^{\boldsymbol{\mu}}(q,p) \frac{\partial}{\partial q^{\boldsymbol{\mu}}} = \mathcal{Z}^\tau(q,p) \frac{\partial}{\partial \tau} + \sum_{1\leq i\leq k} \mathcal{Z}^i(q,p) \frac{\partial}{\partial q^i} = \mathcal{Z}^\tau \frac{\partial}{\partial \tau} + \mathcal{Z}^i \frac{\partial}{\partial q^i}.$$

But $\mathcal{Z}^\tau(q,p) = 1$ due to the condition expressed intrinsically by the construction of the space ${}^{\mathfrak{s}}T_{(\tau,v)}\mathfrak{Z}^\circ$. Indeed, if $\mathcal{Z}(q,p) \in {}^{\mathfrak{s}}T_{(\tau,v)}\mathfrak{Z}^\circ$, then $d\tau(\mathcal{Z}) = d\tau(\mathcal{Z}^\tau \frac{\partial}{\partial\tau} + \mathcal{Z}^i \frac{\partial}{\partial q^i}) = 1$. We found:

$$d\mathcal{H} = -\sum_{1\leq \boldsymbol{\mu} \leq k+1} \frac{\partial L}{\partial q^{\boldsymbol{\mu}}}(q, \mathcal{Z}(q,p), p) dq^{\boldsymbol{\mu}} + \sum_{1\leq \boldsymbol{\mu} \leq k+1} \mathcal{Z}^{\boldsymbol{\mu}}(q,p) dp_{\boldsymbol{\mu}}. \tag{27}$$

[15] we fix the value of the parameter ϖ to be zero

We make the following computation:

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial \mathfrak{e}}(q(\tau),p(\tau)) &= \frac{\partial W^{\mathbf{o}}}{\partial \mathfrak{e}}(q,\mathcal{Z}(q,p),p) = \frac{\partial}{\partial \mathfrak{e}}\Big[\mathfrak{e}\Big] + \frac{\partial}{\partial \mathfrak{e}}\Big[p_i\mathcal{Z}^i(q,p)\Big] - \frac{\partial}{\partial \mathfrak{e}}\Big[L(q,\mathcal{Z}(q,p))\Big] \\ &= 1 + p_i\frac{\partial \mathcal{Z}^i}{\partial \mathfrak{e}}(q,p) - \frac{\partial L}{\partial z^\mu}(q,\mathcal{Z}(q,p))\frac{\partial \mathcal{Z}^i}{\partial \mathfrak{e}}(q,p) = 1 + p_i\frac{\partial \mathcal{Z}^i}{\partial \mathfrak{e}}(q,p) - p_i\frac{\partial \mathcal{Z}^i}{\partial \mathfrak{e}}(q,p) \\ &= 1.\end{aligned}$$

This does not give any dynamical information. We exhibit the analogous relations for $2 \leq \boldsymbol{\mu} \leq (k+1)$ that corresponds to $1 \leq i \leq k$:

$$\frac{\partial \mathcal{H}}{\partial p_i}(q,p) = \mathcal{Z}^i(q,p).$$

Now following (24), we exploit the condition [$\mathfrak{z}$] in order to obtain $k+1$ equations:

$$\frac{\partial \mathcal{H}}{\partial \mathfrak{e}}(q,p) = \frac{d\tau}{d\tau}(\tau) \qquad \text{and} \qquad \frac{\partial \mathcal{H}}{\partial p_i}(q,p) = \frac{d\gamma^i}{d\tau}(\tau), \tag{28}$$

and obtain the desired result. Once again, the first is trivial, by construction. In such a context, we obtain the first set of Hamilton equations:

$$\frac{\partial q^{\boldsymbol{\mu}}}{\partial \tau}(\tau) = \frac{\partial \mathcal{H}}{\partial p_{\boldsymbol{\mu}}}(q(\tau),p(\tau)) = \frac{\partial \mathcal{H}}{\partial p_{\boldsymbol{\mu}}}(\tau, q^i(\tau), \mathfrak{e}(\tau), p_i(\tau)).$$

We have determined conditions on $\mathfrak{z}^{\boxdot}$ for the existence of a map γ such that $(\tau,\gamma(\tau),d\gamma(\tau))\leftrightarrow(q(\tau),p(\tau))$. The second part of the Hamilton equations is still needed. This second step brings the elimination of γ in the Euler Lagrange equations. We are looking for a relation on $\mathfrak{z}^{\boxdot}$ such that γ is a solution of the Euler-Lagrange equations. We compute:

$$\frac{d}{d\tau}\Big[\frac{\partial L}{\partial v^i}(\tau,\gamma(\tau),d\gamma(\tau))\Big] = \frac{d}{d\tau}\Big[\frac{\partial}{\partial v^i}\langle p,v\rangle\Big] = \frac{d}{d\tau}\Big[\frac{\partial}{\partial z^i}\langle p,z\rangle\Big] = \frac{dp_i}{d\tau}.$$

Due to the Euler Lagrange equations, we have $\frac{dp_i}{d\tau} = \frac{\partial L}{\partial y^i}(\tau,\gamma(\tau),d\gamma(\tau))$, so that we obtain the second set of Hamilton equations, the following k equations:

$$\frac{dp_i}{d\tau} = -\frac{\partial \mathcal{H}}{\partial q^i}(q,p).$$

The Hamilton equations can be completed with the following:

$$\frac{d\mathfrak{e}}{d\tau}(\tau) - \frac{d}{d\tau}\Big[\mathcal{H}(\tau,q^i,\mathfrak{e},p_i)\Big] = -\frac{\partial \mathcal{H}}{\partial \tau}(\tau,q^i,\mathfrak{e},p_i).$$

This relation is strictly a consequence of the relations described above. If we consider the Hamiltonian constraint, and write for $(\tau,q^i,\mathfrak{e},p_i) \in T^\star(\mathfrak{Z}^{\mathbf{o}})$ the old notation $(q^\circ,q^i,p_\circ,p_i) \in T^\star(\mathfrak{Z}\times\mathbb{R})$, we recover the description of the extended phase space as in the previous section (3.4). In this case $\mathfrak{e}(\tau) + H(\tau,q^i(\tau),p_i(\tau))$ is a constant which is set to zero.

3.6 Covariant Hamiltonian dynamics, preferred topology

Now we consider paths $\gamma^{\text{ext}} : I \to \mathfrak{Z}^\circ = \mathbb{R}\times\mathfrak{Z}$ - such that the target space itself is the extended phase space with preferred topology. We denote local coordinates on I by $\{\tau\}$ and bu $\{y^\circ, y^1, \cdots, y^k\}$ local coordinates on $\mathfrak{Z}^\circ = \mathbb{R}\times\mathfrak{Z}$. Then we denote $\{q^{\boldsymbol{\mu}}\}_{1\leq\boldsymbol{\mu}\leq(k+2)} = \{q^1, \cdots, q^{k+1}, q^{k+2}\} = \{\tau, y^\circ, y^1, \cdots, y^k\}$ coordinates on $\mathfrak{Z}^{\circ\mathbf{o}} = (\mathfrak{Z}^\circ)^{\mathbf{o}} = I\times\mathfrak{Z}^\circ = I\times\underbrace{\left[\mathbb{R}\times\mathfrak{Z}\right]}_{\mathfrak{Z}^{\mathrm{ext}}}$.

Now the map of interest appears to be:

$$\mathfrak{z}_{\Gamma^{ext}} = \mathfrak{z}_\Gamma : \begin{cases} I & \rightarrow \quad \mathfrak{Z}^{\circ\mathbf{o}} = I\times\mathfrak{Z}^\circ = I\times[\mathbb{R}\times\mathfrak{Z}] \\ \tau & \mapsto \quad \mathfrak{z}(\tau) = (\tau, \Gamma(\tau)) = (\tau, \gamma^\circ(\tau), \gamma^i(\tau)). \end{cases}$$

The map $\mathfrak{z}_\Gamma$ associates $\forall\tau\in I$ the graph of Γ, $G^\circ[\Gamma] = \big\{(\tau,\Gamma(\tau))/\tau\in I\big\}$. We picture the graph of Γ, $G^\circ[\Gamma]$ as the image of the map $\mathfrak{z}_\Gamma(\tau) = G^\circ[\Gamma]$. We associate to Γ, $\Gamma^\star T\mathfrak{Z}^\circ\otimes T^\star I = \Gamma^\star T\mathfrak{Z}^\circ\otimes I\subset T\mathfrak{Z}^\circ\otimes I$, as a bundle over I. Notice that, if we consider the tangent bundle $T\mathfrak{Z}^\circ\rightarrow\mathfrak{Z}^\circ$ and the smooth map $\Gamma : I\rightarrow\mathfrak{Z}^\circ$, then by definition the bundle $\Gamma^\star T\mathfrak{Z}^\circ$ is a fiber bundle over I, whose fiber over $\tau\in I$ is given by $(\Gamma^\star T\mathfrak{Z}^\circ)_\tau = (T\mathfrak{Z}^\circ)_{\Gamma(\tau)}$. In this picture we have the following two bundles, the tangent bundle and its associated pullback bundle by the map Γ:

$$\begin{array}{ccc} T\mathfrak{Z}^\circ = T(\mathbb{R}\times\mathfrak{Z}) & \qquad & \Gamma^\star T\mathfrak{Z}^\circ \\ \big\downarrow{\scriptstyle \pi_{\mathfrak{Z}^\circ}} & & \big\downarrow{\scriptstyle \pi_I^\gamma} \\ \mathfrak{Z}^\circ = \mathbb{R}\times\mathfrak{Z} & & I \end{array}$$

Hence, having constructed the bundle $\Gamma^\star T\mathfrak{Z}^\circ \xrightarrow{\pi_I^\gamma} I$, we construct the bundle $\Gamma^\star T\mathfrak{Z}^\circ\otimes T^\star I \xrightarrow{\pi} I$. We see that the total space of this bundle is made with fibers over $\tau\in I$ which are taken to be objects in $\gamma^\star T\mathfrak{Z}^\circ\otimes T^\star I$. These are maps from the space TI to the space $\Gamma^\star T\mathfrak{Z}^\circ$. Therefore we denote coordinates τ on I and $v_\tau^\circ = v^\circ$, $v_\tau^i = v^i$, with $1\leq i\leq k = dim(\mathfrak{Z})$ such that a point in $\Gamma^\star T\mathfrak{Z}^\circ\otimes T^\star I$ is denoted with coordinates (τ, v). The object v is described by the map: $TI\cong I\rightarrow\Gamma^\star T\mathfrak{Z}^\circ$. Again in this case, the differential $d\Gamma$ is a section of the bundle $\Gamma^\star T\mathfrak{Z}^\circ\otimes T^\star I$. Alternatively, the following bundle is constructed with $T\mathfrak{Z}^\circ\otimes_{I\times\mathfrak{Z}^\circ} T^\star I\cong T\mathfrak{Z}^\circ\otimes_{I\times\mathfrak{Z}^\circ} I$:

$$\begin{array}{ccc} T\mathfrak{Z}^\circ\otimes_{I\times\mathfrak{Z}^\circ} T^\star I & = & T\mathfrak{Z}^\circ\otimes_{\mathfrak{Z}^{\circ\mathbf{o}}} T^\star I \\ \big\downarrow{\scriptstyle \boldsymbol{\pi}} & & \big\downarrow{\scriptstyle \boldsymbol{\pi}} \\ I\times\mathfrak{Z}^\circ & & \mathfrak{Z}^{\circ\mathbf{o}} \end{array} \tag{29}$$

Hence $\Gamma^\star T\mathfrak{Z}^\circ\otimes T^\star I$ is a sub-bundle $\Gamma^\star T\mathfrak{Z}^\circ\otimes T^\star I\subset T\mathfrak{Z}^\circ\otimes_{I\times\mathfrak{Z}^\circ} I = T\mathfrak{Z}^\circ\otimes_{\mathfrak{Z}^{\circ\mathbf{o}}} I$. In doing this coordinates on the former are denoted (τ, v°, v^i) whereas coordinates on the latter are denoted $(\tau, y^\circ, y^i, v^\circ, v^i)$ Finally, coordinates (x, v) on $\Gamma^\star T\mathfrak{Z}^\circ\otimes T^\star I$ are represented:

$$v = \sum_{1\leq i\leq k+1} v^i\frac{\partial}{\partial y^i}\otimes d\tau = \sum_{1\leq i\leq k} v^i\frac{\partial}{\partial y^i}\otimes d\tau + v^\circ\frac{\partial}{\partial y^\circ}\otimes d\tau.$$

The interest of this choice of the bundle (29) appears clearly since the base manifold in this case is the product $I\times\mathfrak{Z}^\circ = \mathfrak{Z}^{\circ\mathbf{o}}$. Conceptually the manifold $\mathfrak{Z}^{\circ\mathbf{o}}$ is the right object, within this picture, to bring the parametrization *and* the parametrized space in the same geometrical construction. In this case, the parametrization space is denoted as usual by I whereas the target space, is the extended configuration space $\mathfrak{Z}^\circ$. Now we identify the following two bundles, - the fiber over $\mathfrak{Z}^{\circ\mathbf{o}} = I\times\mathfrak{Z}^\circ$ denoted $T_{(\tau,v)=(\tau,v^\circ,v^i)}\mathfrak{Z}^{\circ\mathbf{o}}$ is described below:

$$\begin{array}{ccccc} T\mathfrak{Z}^\circ\otimes_{I\times\mathfrak{Z}^\circ} T^\star I & \cong & {}^{\mathfrak{s}}T_{(\tau,v)}\mathfrak{Z}^{\circ\mathbf{o}} & = & {}^{\mathfrak{s}}T_{(\tau,v)}(I\times\mathfrak{Z}^\circ) \\ \big\downarrow{\scriptstyle \boldsymbol{\pi}^\circ} & & \big\downarrow{\scriptstyle \boldsymbol{\pi}^{\circ\mathbf{o}}} & & \big\downarrow{\scriptstyle \boldsymbol{\pi}^{\circ\mathbf{o}}} \\ I\times\mathfrak{Z}^\circ & & \mathfrak{Z}^{\circ\mathbf{o}} & & I\times\mathfrak{Z}^\circ \end{array} \tag{30}$$

We consider, ${}^{\mathfrak{s}}T_{(\tau,v)}\mathfrak{Z}^{\circ\mathbf{o}} = {}^{\mathfrak{s}}T_{(\tau,v)}(I\times\mathfrak{Z}^\circ) = \big\{(q,z)\in T(I\times\mathfrak{Z}^\circ)\ /\ z\in T_q\mathfrak{Z}^{\circ\mathbf{o}}\ ,\ d\tau(z)=1\big\}$. For any point $(\tau,v)=(\tau,\gamma^\circ(\tau),\gamma^i(\tau))\in\mathfrak{Z}^{\circ\mathbf{o}}$, we identify the fiber ${}^{\mathfrak{s}}T_{(\tau,v)}\mathfrak{Z}^{\circ\mathbf{o}}$ with $T_y\mathfrak{Z}^\circ\otimes T^\star_\tau I$ through the diffeomorphism (31):

$$\left\{\begin{array}{ccc} T_y\mathfrak{Z}^\circ\otimes T^\star_\tau I & \to & {}^{\mathfrak{s}}T_{(\tau,v)}\mathfrak{Z}^{\circ\mathbf{o}} \\ v=\displaystyle\sum_{1\le i\le k} v^i\frac{\partial}{\partial y^i}\otimes d\tau+\frac{\partial}{\partial y^\circ}\otimes d\tau & \mapsto & z=\displaystyle\sum_{1\le\boldsymbol{\mu}\le k+2} z^{\boldsymbol{\mu}}\frac{\partial}{\partial q^{\boldsymbol{\mu}}}=\frac{\partial}{\partial\tau}+v^\circ\frac{\partial}{\partial y^\circ}+\sum_{1\le i\le k}v^i\frac{\partial}{\partial y^i}\end{array}\right. \tag{31}$$

The bold index $1\le\boldsymbol{\mu}\le k+2$ is a multi-index meaning that $z^{\boldsymbol{\mu}}=1$ for $\boldsymbol{\mu}=1$, also $z^{\boldsymbol{\mu}}=v^\circ$ for $\boldsymbol{\mu}=2$ and $z^{\boldsymbol{\mu}}=v^i$ for $3\le\boldsymbol{\mu}\le k+2$. The identification ${}^{\mathfrak{s}}T_{(\tau,z)}\mathfrak{Z}^{\circ\mathbf{o}}\simeq T_z\mathfrak{Z}^\circ\otimes T^\star_\tau I$ gives either the coordinates $(\tau,y^\circ,y^i,z^\circ,z^i)$ or $(\tau,y^\circ,y^i,v^\circ,v^i)$. Note that a time-dependent Lagrangian density is described by:

$$L: T\mathfrak{Z}^\circ\otimes_{I\times\mathfrak{Z}^\circ}T^\star I = T\mathfrak{Z}^\circ\otimes_{\mathfrak{Z}^{\mathbf{o}}}T^\star I\mapsto\mathbb{R}.$$

Due to the identification $T^\star I\simeq I$, we recover the usual description $L=L(\tau,\gamma^{\text{ext}}(\tau),d\gamma^{\text{ext}}(\tau))$. Now we proceed to the *Legendre correspondence* geometrization.

$${}^{\mathfrak{s}}T\mathfrak{Z}^{\circ\mathbf{o}} ={}^{\mathfrak{s}} T(I\times\mathfrak{Z}^\circ)\leftrightarrow T^\star(I\times\mathfrak{Z}^\circ)=T^\star\mathfrak{Z}^{\circ\mathbf{o}}.$$

The Legendre correspondence is generated by the function $W^{\circ\mathbf{o}}(q,z,p)$ such that:

$$W^{\circ\mathbf{o}}:\left\{\begin{array}{ccc} {}^{\mathfrak{s}}T\mathfrak{Z}^{\circ\mathbf{o}}\times T^\star\mathfrak{Z}^{\circ\mathbf{o}} & \to & \mathbb{R} \\ (q,v,p)=(\tau,y^\circ,y^i,v^\circ,v^i,\mathfrak{e},p_\circ,p_i) & \mapsto & \langle p,z\rangle-L(q,z)\end{array}\right. \tag{32}$$

with $\langle p,v\rangle=\langle p,z\rangle=\mathfrak{e}+p_\circ z^\circ+p_iz^i$. We denote $(q,z,\varpi)\leftrightarrow(q,p)$ if and only if $\forall(q,z,\varpi)=(\tau,q^\circ,q^i,z^\circ,z^i,\varpi)\in{}^{\mathfrak{s}}T\mathfrak{Z}^{\circ\mathbf{o}}\times\mathbb{R}$ and $\forall(q,p)=(\tau,q^\circ,q^i,\mathfrak{e},p_\circ,p_i)\in T^\star\mathfrak{Z}^{\circ\mathbf{o}}$:

$$(q,z,\varpi)\leftrightarrow(q,p)\quad\Longleftrightarrow\quad\left|\begin{array}{rcl}\langle p,z\rangle-L(q,z)=W^{\circ\mathbf{o}}(q,z,p) & = & \varpi\\ \dfrac{\partial W^{\circ\mathbf{o}}}{\partial z}(q,z,p) & = & 0.\end{array}\right. \tag{33}$$

Then, the *Legendre Hypothesis* means that $\forall(q,p)\in T^\star\mathfrak{Z}^\circ$ there exists a unique $z\in{}^{\mathfrak{s}}T\mathfrak{Z}^\circ$ (denoted $\mathcal{Z}(q,p)$) such that $\mathcal{Z}(q,p)$ is a critical point of $z\mapsto W^{\circ\mathbf{o}}(q,z,p)$. We define a Hamiltonian function: $\mathcal{H}:T^\star(I\times\mathfrak{Z}^\circ)=T^\star(\mathfrak{Z}^{\circ\mathbf{o}})\to\mathbb{R}$

$$\mathcal{H}:(\tau,q^\circ,q^i,\mathfrak{e},p_\circ,p_i)\mapsto\mathcal{H}(\tau,q^\circ,q^i,\mathfrak{e},p_\circ,p_i)=\langle p,\mathcal{Z}(q,p)\rangle-L(q,\mathcal{Z}(q,p))=W^{\circ\mathbf{o}}(q,\mathcal{Z}(q,p),p).$$

In this case, we will not enter into details, however the conditions for a map $\mathfrak{z}^{\boxdot}_{\Gamma^{ext}}:I\to T^\star(\mathfrak{Z}^{\circ\mathbf{o}})$ such that: $(\tau,\gamma^{\text{ext}}(\tau),d\gamma^{\text{ext}}(\tau))\leftrightarrow(q(\tau),p(\tau))$ and that $\gamma^{\text{ext}}(\tau)$ is solution a of the Euler-Lagrange equation yields:

$$\frac{\partial\mathcal{H}}{\partial\mathfrak{e}}(q,p)=1,\qquad\frac{\partial\mathcal{H}}{\partial p_\circ}(q,p)=\frac{dq^\circ}{d\tau}(\tau)\qquad\text{and}\qquad\frac{\partial\mathcal{H}}{\partial p_i}(q,p)=\frac{dq^i}{d\tau}(\tau).$$

Therefore we obtain the second set of Hamilton equations:

$$\frac{\partial\mathcal{H}}{\partial q^\circ}(q,p)=-\frac{dp_\circ}{d\tau}\qquad\text{and}\qquad\frac{\partial\mathcal{H}}{\partial q^i}(q,p)=-\frac{dp_i}{d\tau},$$

with an additional relation. We remark that in order to regain the usual description for extended phase space, as set out in section (3.4) we do exhibit the relation between the two formalisms.

Formally we pass from the Hamiltonian $\mathcal{H} = \mathfrak{e} + p_\circ v^\circ + p_i v^i$ to the previously one, $\mathcal{H} = p_\circ + H(\tau, q^i(\tau), p_i(\tau))$ described in section (3.4). This is always possible where our time variable $q^\circ(\tau)$ is disconnected from parametrization considerations and treated as a proper canonical variable. Therefore, in the present general construction, taking into account either the parametrization *or* the full extended space - so that the Hamiltonian and the Legendre correspondence lie on $T^\star(\mathfrak{Z}^{\circ\mathbf{o}}) = T^\star[I \times \mathfrak{Z}^\circ]$. Time and energy appear as canonical variables in extended Hamiltonian equations. If further we *identify* time and parametrization so that $dq^\circ/d\tau(\tau) = 1$ we recover a result analogous to that in section (3.4). The principal feature here is that the variable $\mathfrak{e}$ does not play a role in the traditional Euler-Lagrange equations.

3.7 Symplectomorphisms and infinitesimal symplectomorphisms

In this section, we recall some basic facts concerning symmetry for Hamiltonian dynamics. The two central notions are: symplectomorphism and infinitesimal symplectomorphism, so that we have the following definitions.

Definition 3.7.1. *A* symplectomorphism *of* $(\mathcal{M}, \boldsymbol{\omega})$ *is a smooth diffeomorphism* $\varkappa : \mathcal{M} \to \mathcal{M}$ *such that* $\varkappa^\star \boldsymbol{\omega} = \boldsymbol{\omega}$

Definition 3.7.2. *An* infinitesimal symplectomorphism *of* $(\mathcal{M}, \boldsymbol{\omega})$ *is a vector field* $\xi \in \Gamma(\mathcal{M}, T\mathcal{M})$ *such that* $\mathcal{L}_\xi \boldsymbol{\omega} = 0$.

The fundamental symmetry of classical mechanics is contained in this statement $\mathcal{L}_\xi \omega = 0$. Since the symplectic 2-form is closed, this condition is equivalent to $d(\xi \lrcorner \omega) = 0$. The vector field $\xi \in \mathfrak{X}(\mathcal{M})$ that leaves the symplectic form invariant is called *symplectic* and is a generator of infinitesimal canonical transformations. We have the equivalent definition:

Definition 3.7.3. *An infinitesimal symplectomorphism of* $(\mathcal{M}, \boldsymbol{\omega})$ *is a vector field* $\xi \in \Gamma(\mathcal{M}, T\mathcal{M})$ *such that* $\xi \lrcorner \omega$ *is closed, namely such that* $d(\xi \lrcorner \omega) = 0$.

We denote by $\mathfrak{X}_{\mathsf{sym}}(\mathcal{M})$ the set of symplectic vector fields[16]. Consider the *flow* $\phi_t^\xi : \mathcal{M} \to \mathcal{M}$ induced by a vector field ξ as a one parameter family of diffeomorphisms. Then, a vector field is $\xi \in \mathfrak{X}_{\mathsf{sym}}(\mathcal{M})$ if its flow is defined by a symplectomorphism $(\phi_t^\xi)^\star \omega = \omega$. Such a symmetry of the symplectic manifold $(\mathcal{M}, \omega)$ is connected to what we shall refer to as the *ontologic* standpoint. Equivalently we describe the set of symplectic vector fields in $\mathfrak{X}_{\mathsf{sym}}(\mathcal{M})$ as a purely *ontologic* object. This emphasis on what we term *ontologic considerations* is related to the *kinematical* structure of the theory which is basically expressed by the data $(\mathcal{M}, \omega)$. In developing this idea we use the *four* symbols introduced in (2.1) - in order to indicate whether we are focussed on the *ontologic* or *dynamical aspect.* These symbols take on more interest in the multidimensional case, where the motivation for their introduction will become fully apparent. What we have termed the *ontologic* symmetry is to be thought of as captured by the preservation of the symplectic form $\mathcal{L}_\xi \boldsymbol{\omega} = 0$ and is what motivate the introduction of the first symbol **[▷]**. We draw attention to the ontologic flavour of symplectomorphisms by means of the following symbolic expression: $\mathfrak{X}_{\mathsf{sym}}(\mathcal{M})$⇆**[▷]**.

3.8 Locally and Globally Hamiltonian vector fields

In the next section (3.9) we will be concerned with the dual point of view *i.e.* with dynamical considerations. We will observe that the natural setting for dynamics is a Hamiltonian system

[16]Notice that the analogue of such an object in the multidimensional case is the set of infinitesimal symplectomorphisms $\mathfrak{sp}_\circ(\mathcal{M})$. We develop this point later.

$(\mathcal{M},\omega,\mathcal{H})$. Before taking this step, we need first to introduce and define another duality[17]. This is the duality of forms vis-a-vis symplectomorphisms: this duality is still tied to what we have termed *ontologic* considerations. These forms are usually called Hamiltonian forms.[18] We introduce the spaces $\mathfrak{X}_{\mathsf{ham}}(\mathcal{M})$ and $\mathfrak{X}_{\mathsf{loc}}(\mathcal{M})$, as the set of Hamiltonian vector fields and the set of *locally* Hamiltonian vector fields respectively. Notice that in the present section vector fields are described in connection with the function[19] $\varphi \in C^{\infty}(\mathcal{M})$ over the symplectic manifold. For the symplectic case ($n=1$) any function φ plays the same role in the classification of observables as does the Hamiltonian function $\mathcal{H} \in C^{\infty}(\mathcal{M})$. This is why we speak of the *triviality* of this duality in the symplectic case. We give further explanations later. We now introduce some basic definitions.

Definition 3.8.1. *A* globally *Hamiltonian vector field ξ_φ (or simply* Hamiltonian vector field*) is a vector field such that $\xi_\varphi \mathbin{\lrcorner} \omega$ is exact. In this case, there exists a* 0*-form $\varphi \in \Lambda^0 T^\star\mathcal{M} = C^{\infty}(\mathcal{M})$ such that $\xi \mathbin{\lrcorner} \omega + d\varphi = 0$.*

We denote by $\mathfrak{X}_{\mathsf{ham}}(\mathcal{M})$ the set of Hamiltonian vector fields. Notice that the set of Hamiltonian vector fields is given in connection with the so-called Hamiltonian 0-forms. This Hamiltonian function is given by $\varphi \in C^{\infty}(\mathcal{M})$ and we describe the set of Hamiltonian functions, denoted $\Omega^0_{\mathsf{ham}}(\mathcal{M})$ as the set: $\Omega^0_{\mathsf{ham}}(\mathcal{M}) = \big\{\varphi \in \Lambda^0 T^\star \mathfrak{Z} = C^{\infty}(\mathcal{M}),\ /\ \xi_\varphi \mathbin{\lrcorner} \omega = -d\varphi\big\}$.

Definition 3.8.2. *A locally Hamiltonian vector field ξ is a vector field such that $\xi \mathbin{\lrcorner} \omega$ is closed so that we write $\xi = \xi_\varphi$. Locally there exists $\varphi \in C^{\infty}(\mathcal{M})$ such that $\xi_\varphi \mathbin{\lrcorner} \omega = -d\varphi$.*

We denote by $\mathfrak{X}_{\mathsf{loc}}(\mathcal{M})$ the set of *locally* Hamiltonian vector fields. Any *locally* Hamiltonian vector field $\xi \in \mathfrak{X}_{\mathsf{loc}}(\mathcal{M})$ is an infinitesimal generator of symplectomorphisms, namely $\xi \in \mathfrak{X}_{\mathsf{sym}}(\mathcal{M})$. Notice the relation between $\mathfrak{X}_{\mathsf{ham}}(\mathcal{M})$ and $\mathfrak{X}_{\mathsf{loc}}(\mathcal{M})$. Each globally Hamiltonian vector field $\xi \in \mathfrak{X}_{\mathsf{ham}}(\mathcal{M})$ is a locally Hamiltonian vector field: $\forall \xi \in \mathfrak{X}_{\mathsf{ham}}(\mathcal{M}) \Rightarrow \xi \in \mathfrak{X}_{\mathsf{loc}}(\mathcal{M})$. The converse is false. Generally, a locally Hamiltonian vector field $\xi \in \mathfrak{X}_{\mathsf{loc}}(\mathcal{M})$ will not necessarily be a *globally* Hamiltonian vector field. Indeed, if $\xi \in \mathfrak{X}_{\mathsf{loc}}(\mathcal{M})$, then we have $\mathcal{L}_\xi \omega = 0$ which implies $d(\xi \mathbin{\lrcorner} \omega) = 0$ so that $\xi \mathbin{\lrcorner} \omega$ is a closed 1-form. Thanks to the Poincaré lemma, *locally* there exists a function φ such that $\xi \mathbin{\lrcorner} \omega = -d\varphi$. This justifies the introduction of Hamiltonian vector fields ξ_φ on $\mathcal{M}$. We next consider a vector fields $\xi_\varphi \in \mathfrak{X}_{\mathsf{ham}}(\mathcal{M})$ such that $\xi_\varphi \mathbin{\lrcorner} \omega + d\varphi = 0$. Therefore, we find that $\mathcal{L}_{\xi_\varphi}\omega = 0$ and hence $\xi_\varphi \in \mathfrak{X}_{\mathsf{loc}}(\mathcal{M})$. We picture the following inclusion of spaces:

$$\mathfrak{X}_{\mathsf{ham}}(\mathcal{M}) \subset \mathfrak{X}_{\mathsf{loc}}(\mathcal{M}) = \mathfrak{X}_{\mathsf{sym}}(\mathcal{M}) \subset \mathfrak{X}(\mathcal{M}).$$

Let us summarize the previous remarks in the following comparison between *vector field* $\xi \in \Gamma(\mathcal{M}, T\mathcal{M})$ and 1-*form* $(\xi \mathbin{\lrcorner} \omega) \in \Omega^1(\mathcal{M})$:

Locally Hamiltonian vector fields	Hamiltonian vector fields
$\xi \in \mathfrak{X}_{\mathsf{loc}}(\mathcal{M})$	$\xi \in \mathfrak{X}_{\mathsf{ham}}(\mathcal{M})$
$\xi \mathbin{\lrcorner} \boldsymbol{\omega}$ is closed	$\xi \mathbin{\lrcorner} \boldsymbol{\omega}$ is exact

Equivalently, we formulate the interplay of spaces $\mathfrak{X}_{\mathsf{sym}}(\mathcal{M})$ and $\mathfrak{X}_{\mathsf{ham}}(\mathcal{M})$ from the cohomological viewpoint. Indeed, if $\xi \in \mathfrak{X}_{\mathsf{sym}}(\mathcal{M})$ then $\xi \mathbin{\lrcorner} \omega$ is closed *e.g* $d(\xi \mathbin{\lrcorner} \omega) = 0$ whereas if $\xi \in \mathfrak{X}_{\mathsf{loc}}(\mathcal{M})$ then $\xi \mathbin{\lrcorner} \omega$ is exact *i.e* there exists a function $\varphi : \mathcal{M} \to \mathbb{R}$ such that $\xi_\varphi \mathbin{\lrcorner} \omega = d\varphi$. We observe the following exact sequence:

$$0 \longrightarrow \mathfrak{X}_{\mathsf{ham}}(\mathcal{M}) \longrightarrow \mathfrak{X}_{\mathsf{loc}}(\mathcal{M}) \longrightarrow \mathbf{H}^1_{\mathsf{deRham}}(\mathcal{M},\mathbb{R}) \longrightarrow 0,$$

[17] in addition to the ontologic-dynamical a.k.a kinematical-dynamical duality

[18] So that we are faced with an ambiguity in classification. Later we will introduce the term *algebraic observables*.

[19] this we will call an *algebraic observable* 0-form in the general multisymplectic setting. (See later developments).

where the right hand arrow is the map described by $\xi \mapsto (\xi \lrcorner \omega)$. Also, $\mathbf{H}^1_{\mathsf{deRham}}(\mathcal{M}, \mathbb{R})$ is the first de Rham cohomology: it measures the obstruction for a symplectic vector field to be Hamiltonian. If $\mathbf{H}^1_{\mathsf{deRham}}(\mathcal{M}, \mathbb{R})$ is trivial, then any symplectic vector field is Hamiltonian, globally. The sequence is an exact sequence of Lie algebras. The key point is that we introduce an additional symbol $[\lhd]$ to further specify the movement within the *ontologic* aspect (itself subordinate to the further opposition ontologic-dynamical) which will be discussed further in section (6). We emphasize the *relation* between vector fields and 1-forms given by the map: $\xi \mapsto (\xi \lrcorner \omega)$. In a manner analogous to the symbolic picture for classifying the *ontologic* space $\mathfrak{X}_{\mathsf{sym}}(\mathcal{M})$ (also to be further discussed in section (6)), we now introduce the symbolic representation for the space of related forms - so called Hamiltonian forms - $\Omega^0_{\mathsf{ham}}(\mathcal{M}) \leftrightarrows [\lhd]$. Notice that we are here still within the *ontologic* standpoint, since we have not yet introduced any dynamical considerations. In the symplectic case, we explain the relation with *Hamiltonian* 0-forms. We motivate the introduction of these symbols $[\rhd]$ and $[\lhd]$ by starting from a consideration of the duality between vector fields $\xi_\varphi \in \mathfrak{X}(\mathcal{M})$ and 1-forms $d\varphi$. Hence, we are concerned with a *topological duality* as opposed to a *dynamical duality* (a concept shortly to be defined). For the moment, we just keep in mind that this notion of *topological duality* is to be thought of as related to the *ontologic* aspect of the general ontologic-dynamical opposition. More precisely, we observe that the *ontologic symmetry* is described as the pairing via the symplectic form ω of the tangent space $T\mathcal{M}$ and the cotangent space $T^\star\mathcal{M}$. In the symplectic case, *topological* and *dynamical* duality are closely intertwined. The key point is that the symbolic expression $\Omega^0_{\mathsf{ham}}(\mathcal{M}) \leftrightarrows [\lhd]$ is tied to the feature that some 0-forms can be thought as manifesting the *ontologic* standpoint. In the general discussion the symbol $[\rhd]$ is connected with what we call the *ontologic space*, whereas $[\lhd]$ is tied to the *ontologic representation*. We summarize:

Ontologic space

$\mathfrak{X}_{\mathsf{sym}}(\mathcal{M})$ set of inf. symplectomorphisms
Symbolic vision: $\mathfrak{X}_{\mathsf{sym}}(\mathcal{M}) \leftrightarrows [\rhd]$

Ontologic representation

$\Omega^0_{\mathsf{ham}}(\mathcal{M})$ the set of Hamiltonian forms $(n-1)-$ forms
Symbolic vision: $\Omega^0_{\mathsf{ham}}(\mathcal{M}) \leftrightarrows [\lhd]$

3.9 Hamiltonian systems and Hamiltonian symmetry

Now we turn to dynamical considerations. Once again, let us begin with very basic definitions:

Definition 3.9.1. *A Hamiltonian system is described as a triple* $(\mathcal{M}, \omega, \mathcal{H})$ *where* $(\mathcal{M}, \omega)$ *is a symplectic manifold equipped with a Hamiltonian function* $\mathcal{H} : \mathcal{M} \to \mathbb{R}$.

Definition 3.9.2. *A continuous symmetry for a Hamiltonian system* $(\mathcal{M}, \omega, \mathcal{H})$ *is described by a vector field which preserves both the Hamiltonian function (*e.g $\mathcal{L}_\xi \mathcal{H} = 0$*),* and *the symplectic form.*

In the table below, the left side deals with the search for *ontologic* symmetry, namely symmetries that respect the *symplectic* or *phase space* structure. It is connected to *kinematics* and should be seen as concerned with the ontology of our mathematical framework: the symplectic structure given by the phase space together with the symplectic 2-form. On the other hand, the right side concerns *dynamics*: the search for *dynamical* symmetry or *dynamical* symmetry. This is connected to the study of variational problems, where the Hamiltonian function plays the role of a *potential* which selects a Hamiltonian symmetry. For that purpose we introduce the Hamiltonian curve, which crystalizes the expression of dynamical evolution. The key point is that these two aspects, reflecting the primitive duality *ontologic* vs *dynamical*, will reappear in our investigation of the notion of observables.

Ontologic

> The symplectic space $(\mathcal{M},\omega)$
> Infinitesimal symplectomorphism $\xi\in\Gamma(\mathcal{M},T\mathcal{M})$ / $\mathcal{L}_\xi\boldsymbol{\omega}=d(\xi\,\lrcorner\,\omega)=0$

Dynamical

> Hamiltonian data $(\Gamma,\mathcal{H})$
> Hamiltonian symmetry $\xi\in\Gamma(\mathcal{M},T\mathcal{M})$ / $\mathcal{L}_\xi\mathcal{H}=d\mathcal{H}(\xi)=0$

From this standpoint, the symmetry of a *Hamiltonian system* $(\mathcal{M},\omega,\mathcal{H})$ is described by a blend of the two aspects. We propose to call a vector field $\xi\in\Gamma(\mathcal{M},T\mathcal{M})$ such that $\mathcal{L}_\xi\mathcal{H}=0$ a *dynamical* vector field or an infinitesimal *dynamomorphism*. We shall see that the definition of this notion is connected with that of dynamical observable functionals - see later. The overarching claim is that the physical concepts are to be thought of as emerging from the intersection of the dynamical and ontologic side. Here, to the end of section (3.9), we are concerned with the *dynamical* aspect of the theory, the intuitive meaning of which is related to the symmetry of the Hamiltonian. We define the *vector field* $\xi\in\mathfrak{X}(\mathcal{M})$ from invariance of the Hamiltonian. Recall that in the previous section (3.8) we briefly introduced, via the concept of *topological duality*, the relation between vector fields and 1-forms - see the relation $\xi_\varphi\,\lrcorner\,\omega=-d\varphi$. From these considerations there emerged the concepts of *ontologic space* and *ontologic representation* crystalized through the related search for such vector fields and observable 0-forms. We now describe the counterpart for this move for the (purely) dynamical side. Here we are led to recognize the centrality of the notion of a Hamiltonian vector field. We describe the set of Hamiltonian vector fields - which form an aspect of the *dynamical space* - via the notation: $X_\mathcal{H}(m)\in[X]^\mathcal{H}_m\leftrightarrows[\,\triangleright\,]$. Let $\mathcal{H}:\mathcal{M}\to\mathbb{R}$ be a Hamiltonian function and let $X_\mathcal{H}(m)$ be a Hamiltonian vector field, $X_\mathcal{H}(m)\in[X]^\mathcal{H}_m$ is such that $\forall m\in\mathcal{M}$ we have $X_\mathcal{H}\,\lrcorner\,\omega_m=-d\mathcal{H}_m$. In the case of symplectic geometry these notions are trivial ; they were introduced in anticipation of the general multisymplectic case in which they become fully accurate. We are interested in the set of 0-forms φ such that $\langle X_\mathcal{H}(m),d\varphi_m\rangle$ does not depend on the choice of $X(m)$ but only on $d\mathcal{H}_m$. Equivalently, we can describe the evolution of an observable 0-form (a function over space phase) via $d\varphi(X_\mathcal{H})$. Here $X_\mathcal{H}$ is a vector field tangent to the Hamiltonian curve. This implies that $d\varphi(X_\mathcal{H})$ depends only on $d\mathcal{H}_m$. We describe the set of observable 0-forms as: $\Omega^0_\bullet(\mathcal{M})=\{\varphi\in\Lambda^0T^\star\mathfrak{Z},/\langle X(m),d\varphi_m\rangle$ depends only on $d\mathcal{H}_m,\forall X(m)\in[X]^\mathcal{H}_m\}$. The corresponding *dynamical representation* is written as $\Omega^0_\bullet(\mathcal{M})\leftrightarrows[\,\triangleleft\,]$. We have the analogous table:

Dynamical space

> $X_\mathcal{H}(m)\in[X]^\mathcal{H}_m$ Hamiltonian vector fields, Hamiltonian symmetry
> Symbolic vision: $[X]^\mathcal{H}_m\leftrightarrows[\,\triangleright\,]$

Dynamical representation

> $\Omega^0_\bullet(\mathcal{M})$ 0-forms such that the evaluation $d\varphi(\xi_m)$ only depends on $d\mathcal{H}_m$
> Symbolic vision: $\Omega^0_\bullet(\mathcal{M})\leftrightarrows[\,\triangleleft\,]$

Remark: In the symplectic case $n=1$ the distinction between the two side is trivial. We already introduced these concepts to prepare the ground for investigation of the multisymplectic case where the two movements cannot *a-priori* be identified. However we will see that the *ontologic* and the *dynamical* representation can be identified only for *pataplectic* manifolds.

3.10 Observables, dynamics and Poisson structure

Thanks to Hamiltonian dynamics, we have a good geometrical picture for the evolution of an observable on the phase space $\varphi : \mathcal{M} \to \mathbb{R}$. Then, we recover the duality discussed above from the standpoint of the notion of observable. We try to picture the evolution of an observable from the *ontologic* standpoint: we are led to exploit the relation $\xi \in \Gamma(\mathcal{M}, T\mathcal{M}) / \mathcal{L}_\xi \boldsymbol{\omega} = d(\xi \lrcorner \omega) = 0$. Therefore, when $\xi \lrcorner \omega$ is exact, there exists a 0-form φ on $\mathcal{M}$, such that:

$$\xi_\varphi \lrcorner \omega = -d\varphi. \tag{34}$$

Relation (34) emphasizes the *symmetry* based point of view. The evolution of an observable $d\varphi$ is given by means of the symplectic form and the related vector field ξ_φ. In the symplectic case $n = 1$, ω is a $n + 1 = 2$-form. Any function on $\mathcal{M}$ is an observable 0-form. We are naturally led to consider the *Poisson Bracket* between observables (35). We consider, $\varphi, \varrho \in \Omega^0_{\mathsf{ham}}(\mathcal{M})$. There is a natural Poisson bracket $\{,\}$: $\Omega^0_{\mathsf{ham}}(\mathcal{M}) \times \Omega^0_{\mathsf{ham}}(\mathcal{M}) \mapsto \Omega^0_{\mathsf{ham}}(\mathcal{M})$ defined by (35)(i):

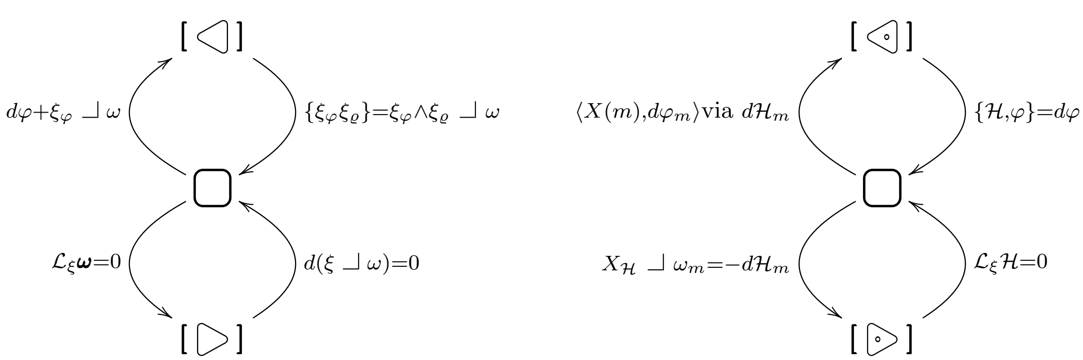
$$\text{(i)} \quad \{\varphi, \varrho\} = \omega(\xi_\varphi, \xi_\varrho) = \xi_\varphi \wedge \xi_\varrho \lrcorner \omega \qquad \text{(ii)} \quad \{\varphi, \varrho\} = \sum_i \Big(\frac{\partial \varphi}{\partial p_i}\frac{\partial \varrho}{\partial q^i} - \frac{\partial \varphi}{\partial q^i}\frac{\partial \varrho}{\partial p_i}\Big). \tag{35}$$

We have $\{\varphi, \varrho\} = \xi_\varphi(\varrho) = d\varrho(\xi_\varphi) = \mathcal{L}_{\xi_\varphi}\varrho = -\mathcal{L}_{\xi_\varrho}\varphi$ and $\omega(\xi_\varphi, \xi_\varrho) = \xi_\varrho \lrcorner (\xi_\varphi \lrcorner \omega) = -\xi_\varrho \lrcorner d\varphi = -d\varphi(\xi_\varrho) = -\{\varrho, \varphi\} = \{\varphi, \varrho\}$. We observe that the coordinate expression for the Poisson bracket is given by (35)(ii) following the notation found in [248]. We stress the *dynamical* insight provided by the relation (36)(i). Here the evolution of an observable $d\varphi/dt$ is given by the Poisson bracket with the Hamiltonian.

$$\text{(i)} \quad \{H, \varphi\} = \frac{d\varphi}{dt} \qquad\qquad \text{(ii)} \quad \big[\widehat{H}; \widehat{F}\big] = i\hbar\, \frac{d\widehat{F}}{dt}. \tag{36}$$

In the spirit of the canonical approach, the formulation of Hamiltonian dynamics is a good preliminary for development of the quantum theory. We replace the functions in $T^\star\mathfrak{Z}$ by Hermitian self-adjoint operators and the Poisson bracket by the commutator $[;]$. Roughly speaking we are led to the *Heisenberg picture* - see (36)(ii) - the quantum evolution equation. We conclude this part by recalling the symbolic representation for the two standpoints: ontologic *vs* dynamical. We observe for each one that the lower part - respectively [▷] and [⊳·] - is related to vector fields. Therefore symmetry incorporates two facets: the ontologic symmetry - the invariance of the symplectic form $\mathcal{L}_\xi\omega = 0$ - and dynamical symmetry[20] - the covariance or the data of Hamilton equations given by $X_\mathcal{H} \lrcorner \omega_m = -d\mathcal{H}_m$. The upper parts in each case - [◁] and [◁·] - are connected to the description of observables and their evolution:

Symbolic picture ontologic *vs* dynamical



[20] Notice that we also describe the invariance of the Hamiltonian function via $\mathcal{L}_\xi\mathcal{H} = 0$.

Notice, that in the context of symplectic geometry, $n = 1$ the figure above is trivial. In the diagram above, the Noether theorem falls naturally into place. The Noether theorem allows us to derive either a relation between global symmetries and conserved charges or a relation between local symmetries and gauge identities. They make a connection between continuous families of symmetries of Hamiltonian systems (or Lagrangian systems) and their first integrals or *conserved quantities*. They state that any differentiable smooth symmetry of a physical system has a corresponding conservation law. From the Lagrangian standpoint they relate symmetries that leave the action invariant and conservation laws of Euler-Lagrange equations. The Hamiltonian counterpart exhibits a homomorphism between between the Lie algebra of Noether symmetries and the algebra of conservation laws. The former is a Lie algebra under the Poisson bracket structure. The theorems[21] are concerned with continuous symmetries of the Hamiltonian system. They contain, as already emphasized above, two ingredients. The symmetry is depicted by a vector field $\xi \in \Gamma(\mathcal{M}, T\mathcal{M})$. The first ingredient of this picture is given by the fact that ξ is canonical: $d(\xi \lrcorner \omega) = 0$. Here we are concerned with what we have termed the *ontologic* part. This condition is independent of the *representation* of the phenomena, - the so-called *potential* landscape, determined by the Hamiltonian function. From this viewpoint, the set of infinitesimal symplectomorphisms is only a matter of intrinsic geometry of the vector field ξ. The other ingredient of a continuous symmetry is the condition that the vector field preserves the Hamiltonian function. Therefore, a symmetry of the Hamiltonian system $(\mathcal{M}, \omega, \mathcal{H})$ involves the two aspects and is obviously Hamiltonian dependent.

4 Multisymplectic geometry

4.1 Covariant Finite Hamiltonian Field Theories

In a finite dimensional system, the cotangent bundle is chosen as the mathematical model for the phase space. Elements of $T^\star\mathcal{M}$ are identified with initial data for the dynamical evolution. This determines the state space of the theory. Following the same line of thought, in field theory, we would like to describe the state space via the specification of Cauchy data for the system. In the canonical approach to field theory, canonical variables are defined on spacelike hypersurfaces. We thus have a *frozen time* picture. All the points on spacelike hypersurface are considered at the *same time*. Consequently, dynamical equations are understood as the evolution of canonical variables from one to another hypersurface of a foliation. This setting is the foundation of standard field theory and of modern approaches to the canonical quantization of gravity. In this approach the treatment of dynamics suffers from being postponed until the foliation of the manifold has been imposed. The underlying geometrical setting is that of the *instantaneous Hamiltonian formalism* - built on an infinite dimensional phase space. In such a viewpoint, *space* and *time* are not treated *symmetrically*, thus destroying the possibility of a fully *covariant* picture. This infinite dimensional phase space is a drawback. It presents several difficulties. Even at the classical level, the specific constraints of a given theory already reflect these difficulties - see the examples of non-linear Yang-Mills theory or the classical GR phase space. Further crucial difficulties appear when we set out to quantize the theory.[22] Therefore, we should recognize the potential gain of working in a *finite* dimensional formulation of canonical field theory. Such a formulation also allows the treatment of *space* and *time* on an equal footing. And this is precisely the insight of MG.

The project begun by Hélein and Kouneiher [111, 113, 114, 115, 116, 117, 118] has greatly developed ideas that already appeared one century ago. Their project concerns the construction of a *universal*

[21] See, in the broader context of covariant field theory the work found in [75, 81].

[22] The Stone-Von Neuman theorem does not apply to the infinite dimensional case and leads to the unitarily inequivalent representations of the canonical commutations relations.

Hamiltonian formalism. The key point turns on the investigation of Lepage-Dedecker theory. The aim is to generalize the calculus of variations in the setting of covariant finite dimensional field theory - see [26, 35, 38, 58, 77, 78, 91, 92, 96, 97, 98, 102, 132, 133, 134, 205, 206, 207, 208] and references therein.[23] At the core of this geometrical program, lies the construction of the generalized *Legendre correspondence*. The motivation for such a universal Hamiltonian formalism lies in the realm of integrable systems, the canonical structure of classical field theory and their quantization. Notice that a strong underlying motive is to find a setting in which the *principles* of GR naturally hold. In this approach the *intrinsically dynamical* nature of the geometrical background becomes apparent. Indeed, gravitational theory forces us to consider *space-time* structure as no longer given *a priori* ; rather it should emerge *from* dynamics. Following this line of thought we further pursue the consequences of the generalized Hamiltonian covariant formalism. We want not only a formalism where no *space-time* splitting is observed - following the principle of special relativity - but further than this, a setting where no splitting of *space-time-field* is given *a priori*. This specific aspect of the relativistic picture is linked in a very deep manner to our understanding of the notion of observable. In this approach, space-time coordinates - as well as those of matter fields - emerge from two more basic and closely interconnected notions: *observable* and *dynamics*.
The covariant Hamiltonian setting for the calculus of variations with several variables is concerned with the question of the Hamiltonian theory for fields $\sigma : \mathcal{X} \to \mathfrak{Z}$ which are critical points[24] of: $\mathcal{L}[\sigma] := \int_{\mathcal{X}} L(x, \sigma(x), d\sigma(x))\beta$. The aim to provide a setting for the treatment of more general variational problems. General variational problems are described geometrically as the study of n-dimensional submanifolds Σ chosen $\Sigma \subset \mathfrak{Z}^{\circ}$ - we consider $\dim(\mathfrak{Z}^{\circ}) = n + k$ - which are critical points of $\mathcal{L}(\mathbf{\Sigma}) := \int_{\Sigma} L(q, T_q\Sigma)\beta$ - where β is a n-form on $\mathfrak{Z}^{\circ}$. The works [111, 117, 118, 114] have treated the particular cases, where Σ is the graph in $\mathfrak{Z}^{\circ} = \mathcal{X} \times \mathfrak{Z}$ of some map $\sigma : \mathcal{X} \to \mathfrak{Z}$, or the section of a fiber bundle. In this context, the analogue of the *tangent bundle* in mechanics becomes the *Grassmannian bundle* $\mathbf{Gr}^n\mathfrak{Z}^{\circ}$ of oriented n-dimensional subspaces of tangent spaces to $\mathfrak{Z}^{\circ}$. The analogue of the cotangent bundle in mechanics is $\Lambda^n T^\star \mathfrak{Z}^{\circ}$. The geometrization of Lepage-Dedecker theory [52, 80, 117] leads us to extend this idea of treating all variables on the *same* footing. The unified picture asks for a *complete democracy* between *time, space* **and** *internal variables*. This viewpoint is the conceptual basis for Kaluza-Klein theory and supergravity theory and is an intrinsic consequence of the view of dynamical equations that seeks a common framework for the treatment of these variables.

The final aim of the development contained in the following sections concerns the notion of *observable forms*. We refer explicitly to the work of Hélein and Kouneiher [114, 117, 118] to emphasize the *dual* point of view concerning the notion of observable forms - namely *symmetry* vs *dynamics*. Our exposition leads up to a discussion of the *Poisson Bracket* on observable forms. In the MG landscape, the Poisson structure is defined via forms on $\Lambda^{n-1}T^\star\mathfrak{Z}^{\circ}$, insofar as we are concerned with the appropriate bracket on $(n-1)$-forms. In the case of an $(n-1)$-form, the generalization is rather straightforward (80) if we follow the *symmetry* point of view and leads to the notion of algebraic observable form AOF. Along the way we note the distinction between with AOF and OF with respect with their dynamical properties. This is why we distinguish between the Poisson bracket and the notion of *pseudobracket* which is connected to *dynamical* considerations (6.3.2). We also treat $(p-1)$-form (with $1 \leq p \leq n-1$). The symmetry viewpoint leads to inequivalent structures. We find a good definition of the Poisson bracket and consequently of canonical variables by means of a more careful use of the *Relativity Principle*. This open the way to a collective treatment of

[23] Note that we give particular attention to the presentations given by H.A. Kastrup [139], H. Rund [203], H. Goldschmidt and S. Sternberg [100]

[24] Here, related to this Lagrangian functional of maps σ we have $\mathcal{X}$ a n-dimensional manifold (space-time) and $\beta = dx^0 \wedge dx^1 \wedge ... \wedge dx^{n-1}$ a volume form on $\mathcal{X}$ and finally $\mathfrak{Z}$ is a k-dimensional manifold (fields).

forms via the notion of *copolarisation*, which is a key feature of an intrinsic dynamical geometry. This treatment provides insight for the appropriate generalization of the *Poisson bracket* on forms of arbitrary degree.

4.2 Multisymplectic zoology

We observe in the literature a wide use of the so-called multisymplectic formalism. The most direct and embracing presentation starts from the *multisymplectic setting* based on the multimomentum phase space $\mathcal{M}_{\mathsf{DW}} = \Lambda_2^n T^\star \mathfrak{Z}$. The bundle $\Lambda_2^n T^\star \mathfrak{Z} \to \mathfrak{Z}$ carries a canonical structure $\theta = \mathfrak{e}\beta + p_i^\mu dz^i \wedge \beta_\mu$ and leads to the multisymplectic structure $\boldsymbol{\omega} = d\mathfrak{e} \wedge \beta + dp_i^\mu \wedge dz^i \wedge \beta_\mu$. This is described by the DeDonder affine multisymplectic form (50). Most of the literature on the subject - within the traditional DW approach - focuses on the contact structure and jet bundles, [26, 35, 38, 58, 77, 78, 91, 92, 96, 97, 98, 102, 132, 133, 134, 205, 206, 207, 208], in the spirit of M.J. Gotay and *al.* We will say more about it in section (5). We next list some specific approaches within the development of the subject.

The first is that of I. Kanatchikov where the multisymplectic setting is based on the (poly)–multimomentum phase space described by the quotient bundle[25] $\mathbf{P}(\mathfrak{Z}) = \Lambda_2^n T^\star \mathfrak{Z} / \Lambda_1^n T^\star \mathfrak{Z}$. This object he called the *polymomentum phase space* [132, 133, 134, 135, 136, 137]. The polysymplectic structure on $\mathbf{P}(\mathfrak{Z})$ is described as an equivalence class of canonical forms. Within Kanatchikov polysymplectic structure, the main object is $\boldsymbol{\omega}^{\mathbf{V}}$, the *vertical part* of the multisymplectic form $\boldsymbol{\omega}^{\mathsf{DW}}$.

The word *polysymplectic* is used in a different but related meaning in other works - see for example G. Giachetta, L. Mangiarotti, and G. Sardanashvily [91, 92, 206, 207, 208]. The multimomentum phase space in these works is also called the *polymomentum phase space* but is here defined as $\mathbf{P}^{\mathsf{Poly}} = \pi^\star T\mathcal{X} \otimes \mathbf{V}^\star(\mathfrak{Z}) \otimes \pi^\star \Lambda^n T^\star \mathcal{X}$ where $\pi : \mathfrak{Z} \to \mathcal{X}$. We shall also denote this bundle as $\mathbf{P}^{\mathsf{Poly}} = T\mathcal{X} \otimes_{\mathfrak{Z}} \mathbf{V}^\star(\mathfrak{Z}) \otimes_{\mathfrak{Z}} \Lambda^n T^\star \mathcal{X}$ and notice that it is canonically described as $\mathbf{P}^{\mathsf{Poly}} \cong \mathbf{V}^\star(\mathfrak{Z}) \wedge \Lambda^{n-1} T^\star \mathcal{X}$ where the canonical *polysymplectic form* is given by $\boldsymbol{\omega}^{\mathsf{Poly}} = dp_i^\mu \wedge dz^i \wedge \beta \otimes \partial_\mu$.

Finally, a third line of research focussing on Lepagean equivalents was developed by D. Krupka, O. Krupkova and D. Smetanova [154, 155, 156, 157].

However the geometrization we are chiefly interested in rests on the seminal work of Lepage and T. Dedecker. We particularly notice how the work of Hélein and Kouneiher [111, 114, 117, 118] continues this development. Here the main focus is on the *generalized Legendre correspondence* which, by means of the *Grassmanian bundle*, makes the *universal Hamiltonian formalism* the basis of a unified geometrical picture.

4.3 Generalized Hamilton equations

Definition 4.3.1. *A multisymplectic manifold* $(\mathcal{M}, \boldsymbol{\omega})$ *is given by a manifold* $\mathcal{M}$ *together with a smooth* $(n+1)$*-form* $\boldsymbol{\omega}$ *such that:*

- $\boldsymbol{\omega}$ is non degenerate, $\forall m \in \mathcal{M}$, $\forall \boldsymbol{\zeta} \in T_m\mathcal{M}$, if $\boldsymbol{\zeta} \in \boldsymbol{\omega} = 0$, then $\boldsymbol{\zeta} = 0$.
- $\boldsymbol{\omega}$ is closed, $d\boldsymbol{\omega} = 0$.

We introduce the notion of a Hamiltonian n-curve which, in the multisymplectic context, is the analogue of the 1-dimensional Hamiltonian curve.

Definition 4.3.2. *Let* $\mathcal{H} : \mathcal{M} \to \mathbb{R}$ *be a smooth Hamiltonian function (such that* $d\mathcal{H} \neq 0$*). A* Hamiltonian n-curve *is a n-dimensional oriented submanifold* $\Gamma \subset \mathcal{M}$ *such that:*

$$\forall m \in \Gamma, \qquad \exists X \in \Lambda^n T_m \Gamma \qquad X \lrcorner \boldsymbol{\omega}_m = (-1)^n d\mathcal{H}_m. \tag{37}$$

[25] Recall that by we denote $\Lambda_p^n T^\star \mathfrak{Z}$ denotes the space of n-forms on $\mathfrak{Z}$ which are annihilated by p arbitrary vertical vectors of $\mathfrak{Z}$.

We denote by $\mathcal{E}^{\mathcal{H}}$ the set of Hamiltonian n-curves. As will be emphasized below, the role of the Hamiltonian n-curve is fundamental. It contains the geometrical description of the *generalized Hamilton equations*. It is the geometrical object in which we find a clear manifestation of dynamical properties. For example observable functionals are defined over the set $\mathcal{E}^{\mathcal{H}}$. We are led to consider the set of *decomposable* n-vectors - or decomposable n-multivectors - for any point $m \in \mathcal{M}$:

$$\mathbf{D}^n_m(\mathcal{M}) = \mathbf{D}^n_m \Lambda^n T_m\mathcal{M} = \{X_1 \wedge \cdots \wedge X_n \in \Lambda^n T_m\mathcal{M} \ / \ X_1, \cdots X_n \in T_m\mathcal{M}\}. \tag{38}$$

We also denote by $[X]^{\mathcal{H}}_m = \{X \in \mathbf{D}^n_m(\mathcal{M}) \ / \ X \lrcorner \boldsymbol{\omega} = (-1)^n d\mathcal{H}_m\}$ the class of Hamiltonian vector fields. Notice that the concept of decomposable n-vector[26] plays a fundamental in the development below, especially when we perfom the *classification* of observable forms.[27] The general setting leads us to view the Hamiltonian formalism in three ways. The first is based on the so-called generalized Hamilton equations. The second rests on the geometrization of the first approach by means of the Poincaré-Cartan form and the multisymplectic form. Finally, we can obtain the equations from a purely variational principle since our concern is with the calculus of variations in several variables. In the subsequent section we offer a brief view of these constructions, referring to [111, 113, 114] [115, 116, 117, 118] for a detailed description.

The generalized Hamilton equations are described step by step [115] and we have already presented the geometrical construction in the previous section (3.5) for relativistic dynamics. That construction dealt with the simple case where we are concerned with paths $\gamma : I \to \mathfrak{Z}$[28]. For field theory, we consider maps from $\mathcal{X}$ to $\mathfrak{Z}$. We replace the parametrizer space I by the space-time manifold. Thus we consider the map $\sigma : \mathcal{X} \to \mathfrak{Z}$, with $\{x^1, \cdots, x^n\}$ as local coordinates on $\mathcal{X}$ and also $\{y^1, \cdots, y^k\}$ as local coordinates on $\mathfrak{Z}$. We denote by $\{q^{\boldsymbol{\mu}}\}_{1 \leq \boldsymbol{\mu} \leq (n+k)} = \{q^1, \cdots, q^{n+k}\}$ the coordinates on $\mathfrak{Z}^{\circ} = \mathcal{X} \times \mathfrak{Z}$. We consider the map:

$$\mathfrak{z}_\sigma : \begin{cases} \mathcal{X} & \to \quad \mathfrak{Z}^{\circ} = (\mathcal{X} \times \mathfrak{Z}) \\ x & \mapsto \quad \mathfrak{z}_\sigma(x) = (x, \sigma(x)). \end{cases}$$

The map $\mathfrak{z}$ associates to all $x \in \mathcal{X}$ the graph of σ, $G[\sigma] = \{(x, \sigma(x))/x \in \mathcal{X}\}$. In this field generalization the role of the tangent bundle is played by the object $T\mathfrak{Z} \otimes_{\mathcal{X}\times\mathfrak{Z}} T^{\star}\mathcal{X}$. Therefore we can study the dynamics of the map σ[29] , $\sigma : \mathcal{X} \to \mathfrak{Z}$ by means of a Lagrangian function

[26] ⌈ Let $\mathcal{V}$ be a $(n+k)$-dimensional manifold. We consider $\mathbf{e}_{\boldsymbol{\mu}}$ a basis of $\mathcal{V}$ as well as $X \in \Lambda^n\mathcal{V}$ an element of the n-antisymmetric tensor product of $\mathcal{V}$. X is decomposable, namely $X = X_1 \wedge \cdots \wedge X_n$, if and only if $\forall 1 \leq \nu \leq n$, X_ν are n independent vector that satisfy $X_\nu \wedge X = 0$. We take the example with $n = 2, k = 4$. In this case we consider $X_\circ, X_\bullet \in T\mathcal{V}$, and a basis of $T\mathcal{V}$ as $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ so that we decompose $X_\circ, X_\bullet$ as $X_\circ = X_\circ^{\boldsymbol{\mu}}\mathbf{e}_{\boldsymbol{\mu}} = X_\circ^1\mathbf{e}_1 + X_\circ^2\mathbf{e}_2 + X_\circ^3\mathbf{e}_3 + X_\circ^4\mathbf{e}_4$ and $X_\bullet = X_\bullet^{\boldsymbol{\mu}}\mathbf{e}_{\boldsymbol{\mu}} = X_\bullet^1\mathbf{e}_1 + X_\bullet^2\mathbf{e}_2 + X_\bullet^3\mathbf{e}_3 + X_\bullet^4\mathbf{e}_4$. Therefore we write $X = X_\circ \wedge X_\bullet$

$$X = (X_\circ^1\mathbf{e}_1 + X_\circ^2\mathbf{e}_2 + X_\circ^3\mathbf{e}_3 + X_\circ^4\mathbf{e}_4) \wedge (X_\bullet^1\mathbf{e}_1 + X_\bullet^2\mathbf{e}_2 + X_\bullet^3\mathbf{e}_3 + X_\bullet^4\mathbf{e}_4)$$

$$\begin{aligned} X = {} & (X_\circ^1X_\bullet^2 - X_\circ^2X_\bullet^1)\mathbf{e}_1 \wedge \mathbf{e}_2 + (X_\circ^1X_\bullet^3 - X_\circ^3X_\bullet^1)\mathbf{e}_1 \wedge \mathbf{e}_3 + (X_\circ^1X_\bullet^4 - X_\circ^4X_\bullet^1)\mathbf{e}_1 \wedge \mathbf{e}_4 \\ & + (X_\circ^2X_\bullet^3 - X_\circ^3X_\bullet^2)\mathbf{e}_2 \wedge \mathbf{e}_3 + (X_\circ^2X_\bullet^4 - X_\circ^4X_\bullet^2)\mathbf{e}_2 \wedge \mathbf{e}_4 + (X_\circ^3X_\bullet^4 - X_\circ^4X_\bullet^3)\mathbf{e}_3 \wedge \mathbf{e}_4 \end{aligned}$$

Finally we compute $X_\circ \wedge X$, and obtain:

$$\begin{aligned} X_\circ \wedge X = {} & (X_\circ^3(X_\circ^1X_\bullet^2 - X_\circ^2X_\bullet^1) - X_\circ^2(X_\circ^1X_\bullet^3 - X_\circ^3X_\bullet^1) + X_\circ^1(X_\circ^2X_\bullet^3 - X_\circ^3X_\bullet^2))\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \\ & + (X_\circ^4(X_\circ^1X_\bullet^2 - X_\circ^2X_\bullet^1) - X_\circ^2(X_\circ^1X_\bullet^4 - X_\circ^4X_\bullet^1) + X_\circ^1(X_\circ^2X_\bullet^4 - X_\circ^4X_\bullet^2))\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4 \\ & + (X_\circ^4(X_\circ^2X_\bullet^3 - X_\circ^3X_\bullet^2) - X_\circ^3(X_\circ^2X_\bullet^4 - X_\circ^4X_\bullet^2) + X_\circ^2(X_\circ^3X_\bullet^4 - X_\circ^4X_\bullet^3))\mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4 \end{aligned}$$

We see that each coefficient associated with $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$, $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4$ and $\mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$ naturally vanishes. ⌋

[27] We refer to [33] for more general considerations on decomposable vector fields and forms within the context of exterior differential systems.

[28] with $\{\tau\}$ as local coordinate on $\mathbb{R}$ and also $\{y^1, \cdots, y^k\}$ as local coordinates on $\mathfrak{Z}$.

[29] The map σ *i.e.* the field

$L : T\mathfrak{Z}\otimes_{\mathcal{X}\times\mathfrak{Z}}T^\star\mathcal{X}\mapsto\mathbb{R}$. Notice that the bundle of interest, for the variational problem $\sigma:\mathcal{X}\to\mathfrak{Z}$ is given by $\sigma^\star T\mathfrak{Z}\otimes T^\star\mathcal{X}$ with natural coordinates x^μ for the space-time manifold and v^i_μ as coordinates on the fibers. A point (x,v) is given by:

$$v=\sum_{1\leq\mu\leq n}\sum_{1\leq i\leq k}v^i_\mu\frac{\partial}{\partial y^i}\otimes dx^\mu.$$

Then, we recover the Euler-Lagrange equations (39):

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial v^i_\mu}(x,\sigma(x),d\sigma_x)\right)=\frac{\partial L}{\partial y^i}(x,\sigma(x),d\sigma_x). \tag{39}$$

σ satisfy (39) if and only if it is a critical point of the functional $\mathcal{L}[\sigma]=\int_{\mathcal{X}}L(x,\sigma(x),d\sigma_x)d\mathfrak{y}$. As in the one dimensional case, we construct the space ${}^{\mathfrak{s}}\Lambda^nT\mathfrak{Z}^{\mathbf{o}}={}^{\mathfrak{s}}\Lambda^nT(\mathcal{X}\times\mathfrak{Z})$ given by:

$${}^{\mathfrak{s}}\Lambda^nT\mathfrak{Z}^{\mathbf{o}}=\{(q,z)\in\Lambda^nT\mathfrak{Z}^{\mathbf{o}}\ /\ z=z_1\wedge\cdots\wedge z_n/\forall 1\leq\mu\leq n z_\mu\in T_q\mathfrak{Z}^{\mathbf{o}}\ ,\ \beta(z_1,\cdots z_n)=1\}.$$

For any $(x,y)=(x,\sigma(x))\in\mathcal{X}\times\mathfrak{Z}$, we make the identification ${}^{\mathfrak{s}}\Lambda^nT_{(\tau,z)}\mathfrak{Z}^{\mathbf{o}}\simeq T\mathfrak{Z}\otimes T^\star\mathcal{X}$. Now an element $z=z_1\wedge\cdots\wedge z_n\in{}^{\mathfrak{s}}T_{(\tau,z)}\mathfrak{Z}^{\mathbf{o}}$ is given by considering for each z_ν with $1\leq\nu\leq n$,

$$z_\nu=\frac{\partial}{\partial x^\nu}+\sum_{1\leq i\leq k}v^i_\mu\frac{\partial}{\partial y^i}=\sum_{1\leq\boldsymbol{\mu}\leq n+k}z^{\boldsymbol{\mu}}_\nu\frac{\partial}{\partial q^{\boldsymbol{\mu}}}.$$

The bold index $1\leq\boldsymbol{\mu}\leq n+k$ is a multi-index meaning that $z^{\boldsymbol{\mu}}_\nu=\delta^{\boldsymbol{\mu}}_\nu$ for $1\leq\boldsymbol{\mu}\leq n$ and $z^{\boldsymbol{\mu}}_\nu=v^i_\nu$ for $n+1\leq\boldsymbol{\mu}\leq n+k$. The *Legendre correspondence* geometrization, is generated by the function $W(q,z,p)$ such that:

$$W:\left\{\begin{array}{ccc}{}^{\mathfrak{s}}\Lambda^nT\mathfrak{Z}^{\mathbf{o}}\times\Lambda^nT^\star\mathfrak{Z}^{\mathbf{o}} & \longrightarrow & \mathbb{R}\\ (q,z,p) & \longmapsto & \langle p,z\rangle-L(q,z)\end{array}\right.\quad\text{with }\langle p,z\rangle\cong\langle p,v\rangle=\sum_{\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n}p_{\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_n}z^{\boldsymbol{\mu}_1}_1\cdots z^{\boldsymbol{\mu}_n}_n$$

We describe the Legendre transform hypothesis $(q,z,\boldsymbol{\varpi})\leftrightarrow(q,p)$ if and only if $\forall(q,z,\boldsymbol{\varpi})\in{}^{\mathfrak{s}}\Lambda^nT\mathfrak{Z}^{\mathbf{o}}\times\mathbb{R}$ and $\forall(q,p)\in\Lambda^nT^\star\mathfrak{Z}^{\mathbf{o}}$:

$$(q,z,\varpi)\leftrightarrow(q,p)\qquad\Longleftrightarrow\qquad\left\{\begin{array}{rcl}\langle p,z\rangle-L(q,z)=W(q,z,p)&=&\varpi\\ \dfrac{\partial W}{\partial z}(q,z,p)&=&0.\end{array}\right. \tag{40}$$

The Legendre hypothesis states that $\forall(q,p)\in\mathcal{O}\subset\Lambda^nT^\star\mathfrak{Z}^{\mathbf{o}}$ there exists a unique $z\in{}^{\mathfrak{s}}\Lambda^nT\mathfrak{Z}^{\mathbf{o}}$, denoted $\mathcal{Z}(q,p)$, such that $\mathcal{Z}(q,p)$ is a critical point of $z\mapsto W(q,z,p)$. We define a Hamiltonian function: $\mathcal{H}:\mathcal{M}=\Lambda^nT^\star\mathfrak{Z}^{\mathbf{o}}\to\mathbb{R}$:

$$\mathcal{H}:(q,p)=(q^{\boldsymbol{\mu}},p_{\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_n})\mapsto\mathcal{H}(q,p)=\langle p,\mathcal{Z}(q,p)\rangle-L(q,\mathcal{Z}(q,p))=\mathcal{W}(q,\mathcal{Z}(q,p),p).$$

The Legendre hypothesis allows us to write the exterior differential $d\mathcal{H}$ as:

$$d\mathcal{H}=-\sum_{1\leq\boldsymbol{\mu}\leq n+k}\frac{\partial L}{\partial q^{\boldsymbol{\mu}}}dq^{\boldsymbol{\mu}}+\sum_{\boldsymbol{\mu}_1\leq\cdots\leq\boldsymbol{\mu}_n}\mathcal{Z}^{\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_n}_{1\cdots n}(q,p)dp_{\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_n} \tag{41}$$

with $\mathcal{Z}^{\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_n}_{1\cdots n}(q,p)$ the components of the n-multivector:

$$\mathcal{Z}=\mathcal{Z}_1(q,p)\wedge\cdots\wedge\mathcal{Z}_n(q,p)=\sum_{\boldsymbol{\mu}_1<\cdots<\boldsymbol{\mu}_n}\boldsymbol{\mathcal{Z}}^{\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_n}_{1\cdots n}\frac{\partial}{\partial q^{\boldsymbol{\mu}_1}}\wedge\cdots\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_n}}$$

$$\mathcal{Z} = \mathcal{Z}_1(q,p) \wedge \cdots \wedge \mathcal{Z}_n(q,p) = \sum_{\boldsymbol{\mu}_1 < \cdots < \boldsymbol{\mu}_n} \begin{vmatrix} \mathcal{Z}_1^{\boldsymbol{\mu}_1} & \cdots & \mathcal{Z}_n^{\boldsymbol{\mu}_1} \\ \vdots & & \vdots \\ \mathcal{Z}_1^{\boldsymbol{\mu}_n} & \ldots & \mathcal{Z}_n^{\boldsymbol{\mu}_n} \end{vmatrix} \frac{\partial}{\partial q^{\boldsymbol{\mu}_1}} \wedge \cdots \wedge \frac{\partial}{\partial q^{\boldsymbol{\mu}_n}}$$

We notice that this expression is the straightforward analogue of the (27). If we want to express a variational problem, in the multidimensional case, the question to ask is: which are the maps $\mathfrak{z} : x \to (q(x), p(x)) : \mathcal{X} \to \mathcal{M} = \Lambda^n T^\star \mathfrak{Z}^{\mathbf{o}}$ related to the critical point $\sigma : \mathcal{X} \to \mathfrak{Z}$? The detailed calculus is found in [115]. Therefore generalized Hamilton equations are to be thought of as necessary and sufficient conditions on the map $\mathfrak{z} : x \to (q(x), p(x))$ such that there exist fields $x \mapsto \sigma(x)$ that verify:

[1]. The generalized Legendre condition is realized. $\forall x \ (x, \sigma(x), d\sigma(x)) \leftrightarrow (q(x), p(x))$.

[2] Fields $x \mapsto \sigma(x)$ are solutions of Euler Lagrange equations for the Lagrangian.

We refer to [115] for details. The generalized Hamilton equations write:

$$\begin{aligned} \frac{\partial(q^{\boldsymbol{\mu}_1} \cdots q^{\boldsymbol{\mu}_n})}{\partial(x^1 \cdots x^n)} &= \frac{\partial \mathcal{H}}{\partial p_{\boldsymbol{\mu}_1 \cdots \boldsymbol{\mu}_n}}(q,p) \\ \sum_{\alpha} \sum_{\boldsymbol{\mu}_1 < \cdots < \boldsymbol{\mu}_n \ / \ \boldsymbol{\mu}_\alpha = n+i} \frac{\partial(q^{\boldsymbol{\mu}_1} \cdots q^{\boldsymbol{\mu}_{\alpha-1}} p_{\boldsymbol{\mu}_1 \cdots \boldsymbol{\mu}_n} q^{\boldsymbol{\mu}_{\alpha+1}} \cdots q^{\boldsymbol{\mu}_n})}{\partial(x^1 \cdots x^n)} &= -\frac{\partial \mathcal{H}}{\partial y^i}(q,p) \end{aligned} \tag{42}$$

4.4 Geometrization and Variational Principles

The geometrization of generalized Hamilton equations (42) is obtained by means of the multisymplectic $(n+1)$-form $\boldsymbol{\omega}$. The conditions on the map $(q(x), p(x))$ are expressed by $\mathcal{M} = \Lambda^n T^\star \mathfrak{Z}^{\mathbf{o}} = \Lambda^n T^\star (\mathcal{X} \times \mathfrak{Z})$

$$X \lrcorner \boldsymbol{\omega} = (-1)^n d\mathcal{H} \mod \boldsymbol{\mathcal{I}}, \tag{43}$$

where $\boldsymbol{\mathcal{I}} \subset \Lambda^n T^\star \mathcal{M}$ is the subspace generated by the dx^α and the n-vector $X \in \Lambda^n T_{(q,p)} \mathcal{M}$. It can be written $X = X_1 \wedge ... \wedge X_n$, where $\forall \mu = 1...n$ we have $X_\mu = \partial(q(x), p(x))/\partial(x^1, \ldots, x^n)$. The equations (43) are the *generalized Hamilton equations*. It is proven in [115, 117] that thanks to the variable $\mathfrak{e} = p_{1\cdots n}$ we can always deform a solution $x \mapsto (q_\circ(x), p_\circ(x))$ which is solution of (43) and obtain another solution $x \mapsto (q_\bullet(x), p_\bullet(x))$ which is again a solution of (43) and satisfies the Legendre hypothesis $(x, \sigma(x), d\sigma(x)) \leftrightarrow (q_\bullet(x), p_\bullet(x))$ where $\mathcal{H}(q_\bullet(x), p_\bullet(x))$ is constant. In this case, we describe the generalized multisymplectic Hamilton equations simply as:

$$X \lrcorner \boldsymbol{\omega}_m = (-1)^n d\mathcal{H}_m. \tag{44}$$

The *Poincaré-Cartan canonical* n-form θ (45) on $\Lambda^n T^\star \mathfrak{Z}^{\mathbf{o}}$ is defined as follow:

$$\forall z \in \mathfrak{Z}^{\mathbf{o}} \quad \forall p \in \Lambda^n T_z^\star \mathfrak{Z}^{\mathbf{o}} \qquad \theta_{(z,p)}(X_1, \cdots, X_n) := p(d\pi_{\mathcal{X}}(X_1), \cdots, d\pi_{\mathcal{X}}(X_n)) \tag{45}$$

Then, the *multisymplectic* $(n+1)$-form $\boldsymbol{\omega}$ is defined as the exterior derivative of the Poincaré-Cartan form. $\boldsymbol{\omega} = d\theta$. The case of classical mechanics case, we give the expression of θ and $\boldsymbol{\omega}$ in local coordinates. Let $(q^{\boldsymbol{\mu}})_{1 \leq \boldsymbol{\mu} \leq n+k}$ be coordinates on $\mathfrak{Z}^{\mathbf{o}}$ - a basis of $\Lambda^n T^\star \mathfrak{Z}^{\mathbf{o}}$ is the family $(dq^{\boldsymbol{\mu}_1} \wedge ... \wedge dq^{\boldsymbol{\mu}_n})_{1 \leq \boldsymbol{\mu}_1 < ... \boldsymbol{\mu}_n < n+k}$ - and $p_{\boldsymbol{\mu}_1 \cdots \boldsymbol{\mu}_n}$ coordinates on $\Lambda^n T^\star \mathfrak{Z}^{\mathbf{o}}$. We obtain the expression for the canonical n-form (46) and the pataplectic $(n+1)$-form (47) which is a straightforward generalization of the symplectic form (12)(ii).

$$\theta = \sum_{1 \leq \boldsymbol{\mu}_1 < \ldots \boldsymbol{\mu}_n < n+k} p_{\boldsymbol{\mu}_1 \cdots \boldsymbol{\mu}_n} dq^{\boldsymbol{\mu}_1} \wedge \ldots \wedge dq^{\boldsymbol{\mu}_n} \tag{46}$$

$$\boldsymbol{\omega} = \sum_{1 \leq \boldsymbol{\mu}_1 < \ldots \boldsymbol{\mu}_n < n+k} dp_{\boldsymbol{\mu}_1 \cdots \boldsymbol{\mu}_n} \wedge dq^{\boldsymbol{\mu}_1} \wedge \ldots \wedge dq^{\boldsymbol{\mu}_n} \tag{47}$$

Variational principle The idea here is to exhibit the relation between a variational formulation and Euler-Lagrange equations or *generalized* Hamilton equations. We consider the functional $\mathcal{I}[\Gamma] := \int_\Gamma = \theta - \mathcal{H}\beta$.

Proposition 4.1. *$\Gamma \subset \Lambda^n T^\star(\mathcal{X} \times \mathcal{Y})$ is the graph of a solution of the generalized Hamilton equation if and only if Γ is a critical point of the functional $\mathcal{I}[\Gamma] := \int_\Gamma = \theta - \mathcal{H}\beta$.*

Given the constraint $\mathcal{H}(q,p) = 0$[30], equivalence with the variational principle is given: Γ is the graph of the generalized Hamilton equations Γ if and only if it is a critical point of the functional $\mathcal{I}^{\mathbf{o}}[\Gamma] := \int_\Gamma = \theta$.

4.5 De Donder-Weyl multisymplectic theory

De Donder-Weyl multisymplectic manifold $\mathcal{M}_{\mathsf{DW}}$ The universal multisymplectic manifold $(\mathcal{M} = \Lambda^n T^\star \mathfrak{Z}, \boldsymbol{\omega})$ is very large. Usually multisymplectic theory is seen from the DW standpoint. In this context, one restricts the theory to the affine multisymplectic submanifold $\mathcal{M}_{\mathsf{DW}} \subset \Lambda^n T^\star \mathfrak{Z}$. We work in the framework of a fiber bundle $\mathfrak{Z} \to \mathcal{X}$ as configuration space. Coordinates on $\mathcal{X}$ are denoted by $\{x^\mu\}_{1 \leq \mu \leq n}$, and we choose a volume n-form $\beta = dx^1 \wedge \cdots \wedge dx^n$ on $\mathcal{X}$. Coordinates on $\mathfrak{Z}$ are denoted $\{z^i\}_{1 \leq i \leq k}$. We denote $\mathfrak{e} = p_{1 \cdots n}$, $p_i^\mu = p_{1 \cdots (\mu-1) i (\mu+1) \cdots n}$, $p_{i_1 i_2}^{\mu_1 \mu_2} = p_{1 \cdots (\mu_1 - 1) i_1 (\mu_1 + 1) \cdots (\mu_2 - 1) i_2 (\mu_2 + 1) \cdots n}$ etc... Also we use the notation $\beta_{\mu_1 \cdots \mu_p}^{i_1 \cdots i_p} = dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge \left(\partial_{\mu_1} \wedge \cdots \wedge \partial_{\mu_p} \lrcorner \beta\right)$ as well as $\beta_\mu = \partial_\mu \lrcorner \beta$. The De Donder-Weyl multisymplectic submanifold $\mathcal{M}_{\mathsf{DW}} \subset \Lambda^n T^\star \mathfrak{Z}$ is given as a constrained set of coordinates $(z,p) \in \Lambda^n T^\star \mathfrak{Z}$ obtained via the interior product of two vertical[31] vector fields $\xi, \chi \in T^{\mathbf{V}} \mathfrak{Z}$.

$$\mathcal{M}_{\mathsf{DW}} \quad = \quad \{(z,p) \in \Lambda^n T^\star \mathfrak{Z} \quad / \quad \forall \xi, \chi \in T^{\mathbf{V}} \mathfrak{Z} \quad \xi \wedge \chi \lrcorner p = 0\}. \tag{48}$$

In such a context, set $\theta^{\mathsf{DW}} := \theta|_{\mathcal{M}_{\mathsf{DW}}}$ the restriction of θ to $\mathcal{M}_{\mathsf{DW}}$. Working on $\mathcal{M}_{\mathsf{DW}}$ is equivalent to taking into account all the constraints $p_{i_1 \cdots i_2}^{\mu_1 \cdots \mu_2} = 0$ for all $j > 1$ in the expression of the full multisymplectic form (49).

$$\boldsymbol{\omega} = d\mathfrak{e} \wedge \beta + \sum_{j=1}^{n} \sum_{\mu_1 < \cdots < \mu_j} \sum_{i_1 < \cdots < i_j} dp_{i_1 \cdots i_j}^{\mu_1 \cdots \mu_j} \wedge \beta_{\mu_1 \cdots \mu_j}^{i_1 \cdots i_j}. \tag{49}$$

We obtain the corresponding De Donder-Weyl multisymplectic $(n+1)$-form $\boldsymbol{\omega}^{\mathsf{DW}}$ (50):

$$\boldsymbol{\omega}^{\mathsf{DW}} = d\mathfrak{e} \wedge \beta + \sum_\mu \sum_i dp_i^\mu \wedge dz^i \wedge \beta_\mu. \tag{50}$$

Classical De Donder-Weyl system. Starting from the Lagrangian side of the picture, the principal object of study are the fields, - seen as maps $\sigma : \mathcal{X} \longrightarrow \mathfrak{Z}$ with $dim(\mathcal{X}) = n$ and $dim(\mathfrak{Z}) = k$. We

[30]or any constant Hamiltonian function

[31]We call a vertical vector field any $\xi \in T_z \mathfrak{Z}$ such that $d\pi_{\mathcal{X}}(\xi) = 0$, then we denote $T^{\mathbf{V}} \mathfrak{Z}$ the set of vertical vector field.

recall that from the Lagrangian action $\mathcal{L}[\sigma] = \int_{\mathcal{X}} L(x, \sigma(x), d\sigma_x) d\mathfrak{y}$, we obtain the *Euler-Lagrange system* of equations (51).

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial v^i_\mu}(x, \sigma(x), d\sigma_x)\right) = \frac{\partial L}{\partial y^i}(x, \sigma(x), d\sigma_x) \tag{51}$$

We pass to the Hamiltonian side thanks to the Hamiltonian function: $H(x^\mu, y^i, p^\mu_i) = p^\mu_i v^i_\mu - L(x^\mu, y^i, v^i_\mu)$. Hamiltonian dynamics is obtained by a non degenerate Legendre transform. If $(x^\mu, y^i, v^i_\mu) \mapsto (x^\mu, y^i, p^\mu_i)$ is a good change of variables, with $p^\mu_i := \frac{\partial L}{\partial v^i_\mu}(x^\mu, y^i, v^i_\mu)$, Euler-Lagrange equations (51) are equivalent to the *Classical De Donder-Weyl* (DW) system of equations:

$$\frac{\partial \sigma^i}{\partial x^\mu}(x) = \frac{\partial H}{\partial p^\mu_i}(x^\mu, \sigma^i(x), \mathbf{p}^\mu_i(x)) \qquad \sum_\mu \frac{\partial \mathbf{p}^\mu_i}{\partial x^\mu}(x) = -\frac{\partial H}{\partial y^i}(x^\mu, \sigma^i(x), \mathbf{p}^\mu_i(x)) \tag{52}$$

The classical DW system of equations (52), when written under in this form, can be understood geometrically. This involves two aspects. The first is that we work in the DW multisymplectic manifold $(\mathcal{M}_{\mathsf{DW}}, \boldsymbol{\omega}^{\mathsf{DW}})$. This is the natural arena of system (52). Hence, we are forced to use the multisymplectic form (50). The second aspect concerns the *pre-multisymplectic* scenario where we impose *Hamiltonian constraint.* In keeping with these two aspects, we understand the geometrization of the classical DW system as the expression of a particular choice of LD theory. Working on the manifold $\mathcal{M}_{\mathsf{DW}}$ with a Hamiltonian function $\mathcal{H} : \mathcal{M}_{\mathsf{DW}} \subset \Lambda^n T^\star \mathfrak{Z}^\circ \to \mathbb{R}$, we restrict ourselves to the submanifold described by $\mathcal{H} = 0$. In this case, the multisymplectic $(n+1)$-form $\boldsymbol{\omega}^{\mathsf{DW}}$ (50) is:

$$\left(\boldsymbol{\omega}^{\mathsf{DW}}\right)_{\mathcal{H}=0} = \boldsymbol{\omega}^{\mathbf{o}} = dp^\mu_i \wedge dy^i \wedge \beta_\mu - dH \wedge \beta. \tag{53}$$

This interplay is possible thanks to the *canonical variable* $\mathfrak{e} = p_{1\cdots n}$ - seen as a canonical variable conjugate to the volume form β. Notice that $\mathfrak{e}$ does not enter into the Euler-Lagrange equations. We can always write $\mathcal{H}(q,p) = \mathcal{H}(q, \mathfrak{e}, p^\star) = \mathfrak{e} + H(q, p^\star)$ and then work on a level set $\mathcal{H}^{-1}(0)$, choosing an appropriate variable $\mathfrak{e}$. If we fix $\mathcal{H}(q,p) = 0$ then $\mathfrak{e} = -H(q, p^\star)$. This is the straightforward analogue for field theory (in the n-dimensional case) of the process described in section (3.4) where we described the Hamiltonian constraint and the *pre*-symplectic formalism in relativistic mechanics. In section (3.4), we considered the relativistic case described as a 1-phase space. The constraint hypersurface $\Sigma_\circ$ is identified with the level set $\mathcal{H}^{-1}(0) = \big\{(q,p) \in \mathcal{M} = \Lambda^n T^\star \mathfrak{Z}^\circ / \mathcal{H}(q,p) = 0\big\}$. Then, $\big(\mathcal{H}^{-1}(0) = \Sigma_\circ, \boldsymbol{\omega}_{|\Sigma_\circ}, (\beta)_{|\Sigma_\circ}\big)$ is a n-phase space characterized by a volume form $(\beta)_{|\Sigma_\circ}$ that does not vanish and $\boldsymbol{\omega}_{|\Sigma_\circ}$ is a $(n+1)$ closed form. The key point is that we may observe a degenerate feature for $\boldsymbol{\omega}_{|\Sigma_\circ}$ and this geometrical setting leads to the following definition. A *pre-multisymplectic manifold* $(\boldsymbol{\mathcal{M}}, \boldsymbol{\omega}^{\mathbf{o}})$ is a smooth manifold endowed with a closed $(n+1)$-form $\boldsymbol{\omega}^{\mathbf{o}}$ $(d\boldsymbol{\omega}^{\mathbf{o}} = 0)$ which may be degenerate. In order to describe the dynamical equations, we need a *volume* n-form β - an everywhere non-vanishing n-form. This indicates the right notion of *n-phase space* as the data $(\boldsymbol{\mathcal{M}}, \boldsymbol{\omega}^{\mathbf{o}}, \beta)$. Therefore, we express *dynamics* on a *level set* of $\mathcal{H}$.[32] The framework which connects relativistic dynamical systems and the treatment of the *Hamiltonian constraint* thus emerges. The *dynamical equations* become in the pre-multisymplectic case (54).

$$\forall \xi \in \mathcal{C}^\infty(\mathcal{M}, T_m\mathcal{M}), \qquad \xi \lrcorner \boldsymbol{\omega}^{\mathbf{o}}|_\Gamma = 0 \qquad \text{and} \qquad \beta|_\Gamma \neq 0. \tag{54}$$

We will encounter these notions later when we briefly discuss the concepts of pre-multisymplectic manifold and *n-phase space* in section (7) - we also offer some reflections on the notion of observables

[32] We can construct canonically a pre-n-multisymplectic manifold. $(\mathcal{M}^\circ, \omega|_{\mathcal{M}^\circ}, \beta = \tau \lrcorner \omega|_{\mathcal{M}^\circ})$. Here the $\mathcal{M}^\circ := \mathcal{H}^{-1}(0) := \{(q,p) \in \mathcal{M} |\ \mathcal{H}(q,p) = 0\}$ and τ is a vector field s.t. $d\mathcal{H}(\tau) = 1$.

in this context. This is strongly connected with the geometrical approach to the covariant phase space. Working in the DW setting and imposing the Hamiltonian constraint gives a geometrization of (52). We consider the n-dimensional submanifold $\Gamma \subset \mathcal{M}^{\mathsf{DW}} = \mathcal{X} \times \mathfrak{Z} \times \mathrm{End}(\mathfrak{Z}^\star, \mathcal{X}^\star)$ such that:

$$\Gamma = \{(x^\mu, y^i, p^\mu_i) \ / \ y^i = \sigma^i(x) \ , \ p^\mu_i = \frac{\partial L}{\partial v^i_\mu}(x^\mu, \sigma^i(x), \partial_\mu \sigma^i(x))\} \tag{55}$$

With the *pre-multisymplectic* $(n+1)$-form $\boldsymbol{\omega}^{\mathbf{o}} = dp^\mu_i \wedge dy^i \wedge \beta_\nu - dH \wedge \beta$ defined on $\mathcal{M}^{\mathsf{DW}}$, the DW system $\forall \xi \in \Lambda^n T_{(x,\sigma(x),p(x))}\Gamma$ takes on the geometrical form $\xi \lrcorner \boldsymbol{\omega}^{\mathbf{o}}|_\Gamma = 0$. Here we work on the level set $\Sigma_\circ \subset \mathcal{M}^{\mathsf{DW}}$.

4.6 Multisymplectic manifolds: the geometrical setting

We do not enter into details here but refer to the work of Hélein and Kouneiher [116, 117] [118] [114] for a deeper exposition. However, we want here briefly to emphasize two aspects: the focus on the geometrical picture by means of the *Grassmanian bundle* and the role of the generalized Legendre correspondence in such a context.

The Grassmanian picture. The idea of a full geometrical setting is to be able to treat more general variational problems. The general study of n-dimensional submanifolds chosen in $\{G^n \subset (\mathfrak{Z}^{\mathbf{o}})^{n+k}\}$ which are critical points of $\mathcal{L}(G) := \int_{\mathbf{G}} L(q, T_q G)\beta$ is one example of such a more general problem. Here β is a volume n-form on $(\mathfrak{Z}^{\mathbf{o}})^{n+k}$. A particular example is when G is the graph in $(\mathfrak{Z}^{\mathbf{o}})^{n+k} = \mathcal{X} \times \mathfrak{Z}$ of some map $\sigma : \mathcal{X} \to \mathfrak{Z}$ as shown earlier. The central object from the geometrical standpoint is the Grassmanian bundle $\mathbf{Gr}^n \mathfrak{Z}^{\mathbf{o}}$. The full geometrical picture describes a Lagrangian density as $L : \mathbf{Gr}^\beta \mathfrak{Z}^{\mathbf{o}} \to \mathbb{R}$. The space $\mathbf{Gr}^\beta \mathfrak{Z}^{\mathbf{o}}$ is the sub-bundle $\mathbf{Gr}^\beta \mathfrak{Z}^{\mathbf{o}} = \big\{(q,T) \in \mathbf{Gr}^n \mathfrak{Z}^{\mathbf{o}} \ / \ \beta_q|_T > 0\big\}$ where $\mathbf{Gr}^n \mathfrak{Z}^{\mathbf{o}}$ is the *Grassmanian* bundle.[33] Finally we introduce the set $G^\beta_{\mathbf{o}}$ as the set of all oriented n-dimensional submanifolds $G \subset \mathfrak{Z}^{\mathbf{o}}$ such that $\forall q \in G$, $T_q G \in \mathbf{Gr}^\beta_q \mathfrak{Z}^{\mathbf{o}}$. The *Grassmanian bundle* is the analogue of the tangent space of classical mechanics. We prefer to describe it by means of an n-vector. So that we construct a map $\mathbf{D}^n_q(\mathfrak{Z}^{\mathbf{o}}) \to \mathbf{Gr}^n_q \mathfrak{Z}^{\mathbf{o}}$ which associates to any $z = z_1 \wedge \cdots z_n \in \mathbf{D}^n_q(\mathfrak{Z}^{\mathbf{o}})$ the vector space $T(z_1, \cdots, z_n)$ which is spanned and oriented by $(z_1, \cdots z_n)$. At this point, we also introduce the following two notations (56) and (57):

$$\mathbf{D}^n_q(\mathfrak{Z}^{\mathbf{o}}) = \{z \in \Lambda^n T_q \mathfrak{Z}^{\mathbf{o}} \ / \ z = z_1 \wedge \cdots \wedge z_n \ / \ \forall 1 \le \mu \le n \ z_\mu \in T_q \mathfrak{Z}^{\mathbf{o}}\}, \tag{56}$$

which is the set of locally decomposable n-vectors on $\mathfrak{Z}^{\mathbf{o}}$. Also, we introduce the set $\mathbf{D}^\beta_q(\mathfrak{Z}^{\mathbf{o}})$ which is the set of decomposable n-vectors with an additional condition. We have thereby fixed a *parametrization* condition.

$$\mathbf{D}^\beta_q(\mathfrak{Z}^{\mathbf{o}}) = \{(q,z) \in \mathbf{D}^n_q(\mathfrak{Z}^{\mathbf{o}}) \ / \ \beta_q(z) = 1\}. \tag{57}$$

Here we recover the set ${}^{\mathfrak{s}}T\mathfrak{Z}^{\mathbf{o}} = {}^{\mathfrak{s}}T(\mathbb{R} \times \mathfrak{Z}) = \big\{(q,z) \in T(\mathbb{R} \times \mathfrak{Z}) \ / \ z \in T_q \mathfrak{Z}^{\mathbf{o}} \ , \ d\tau(z) = 1\big\}$ in relativistic mechanics as well as the set ${}^{\mathfrak{s}}\Lambda^n T\mathfrak{Z}^{\mathbf{o}}$ previously introduced in the variational study of the map $\sigma : \mathcal{X} \to \mathfrak{Z}$. This allows us to construct a diffeomorphism $\mathbf{D}^\beta_q(\mathfrak{Z}^{\mathbf{o}}) \to \mathbf{Gr}^\beta \mathfrak{Z}^{\mathbf{o}}$.

Legendre correspondence, [115, 117] focus mainly on the generalized Legendre correspondence. Notice that the fundamental objects in the full geometrical setting are the *pseudofiber* and the *generalized pseudofiber direction*. We later say more about these in another connection. This concerns the issue of invariance and the prospect for a generalized *non-local* field theory. For the

[33] It is the fiber bundle $\mathbf{Gr}^n \mathfrak{Z}^{\mathbf{o}} \to \mathfrak{Z}^{\mathbf{o}}$ whose fiber over $q \in \mathfrak{Z}^{\mathbf{o}}$ is $\mathbf{Gr}^n_q \mathfrak{Z}^{\mathbf{o}}$, the set of all oriented n-dimensional vector subspace of $T_q \mathfrak{Z}^{\mathbf{o}}$

moment we concentrate on the universal geometrical picture, emphasizing the central role played by the notion of graph.

Universal geometrical picture. The core of the universal geometrical picture is the generalized Legendre correspondence [115, 117]. Here we focus on the central role of graphs which implicitly vindicates Grassman's vision. In the subsequent section, we examine some simple examples to illustrate the relation between the *independence condition* - which allows us to describe some maps as graphs over space-time - and use of the additional dynamical variable $\mathfrak{e}$. The Grassmanian picture allows to treat the following different cases:

[1] Classical mechanics with phase space $T^\star \mathfrak{Z}$.

[2] Relativistic mechanics with the extended phase space $T^\star \mathfrak{Z}^\circ$.

[3] Fields, *i.e* maps $\sigma : \mathcal{X} \to \mathfrak{Z}$, with the multisymplectic phase space $\Lambda^n T^\star(\mathcal{X} \times \mathfrak{Z})$.

Notice that MG is also the natural arena for extra dimensional theories with objects such as strings, membranes or p-branes. Indeed, those different objects are nothing else but maps $\varsigma : \Sigma \to \mathfrak{Z}$ from the parametrization space (a 2-dimensional manifold[34] Σ) to the parametrized space - an arbitrary dimensional manifold $\mathfrak{Z}$. The main example is conformal string theory. The map $\varsigma : \Sigma \to \mathfrak{Z}$ is taken from a pseudo-Riemannian $2D$ manifold $(\Sigma, h_{\alpha\beta})$ into an arbitrary dimensional pseudo-Riemannian: $(\mathfrak{Z}, \mathsf{g}_{\mu\nu})$. We refer to the work of Hélein and Kouneiher [115] and for the more specific example of the Nambu-Goto string, we refer to the work of I.V. Kanatchikov [132, 134].

Graphs as a dynamical entity. Before we study the transition from the phase space of classical mechanics to the extended phase space we make the following observations. We consider variational problem on fields over space-time. Fields are described by a map $\sigma : \mathcal{X} \to \mathfrak{Z}$, which relates two spaces: the parametrization space $\mathcal{X}$ and the image space $\mathfrak{Z}$. In this setting we note the ontological opposition between the *space-time* $\mathcal{X} \leftrightarrows \boxed{\mathrm{S}}$ as opposed to what fills space-time (namely *matter fields*) described by $\mathfrak{Z} \leftrightarrows \boxed{\mathrm{M}}$. We have a structural organization of the ontology: the matter fields are described *on* the space-time. We state that the data of a trajectory or path, - in the n-dimensional case this appears as the data of the map σ - itself necessary implies a *duality* in the ontology: the *parametrized* and the *parametrization*. This is why we speak about the *ontologic motif* (58) for the variational problem:

$$\text{Ontologic motif} \qquad \big[\mathcal{X}\big]\big|\sigma\big|\big[\mathfrak{Z}\big] \leftrightarrows \boxed{\mathrm{S}}\boxed{\mathrm{M}}. \tag{58}$$

Analogously, we are concern with the *dynamical motif* (59),

$$\text{Dynamical motif} \qquad \big[\mathcal{X}\big]\big|\mathfrak{z}_\gamma\big|\big[\mathfrak{Z}^\circ = \mathcal{X} \times \mathfrak{Z}\big] \leftrightarrows \boxed{\mathrm{S}}, \boxed{\mathrm{S}} \times \boxed{\mathrm{M}}. \tag{59}$$

The dynamical aspect is connected to the specific use of *graphs.* The expression (59) gives a condition of type $\beta|_{G[\sigma]} > 0$. This condition means that *locally* the map σ over space-time $\mathcal{X}$ is expressed by the graph $G[\sigma]$. Note that $G[\sigma]$ naturally lives in the following space: $\mathfrak{Z}^\circ = \mathcal{X} \times \mathfrak{Z}$. So that the map of interest is $\mathfrak{z}_\sigma : \mathcal{X} \to \mathfrak{Z}^\circ = \mathcal{X} \times \mathfrak{Z}$. The target space of the map $\mathfrak{z}_\sigma$ is the *product* $\mathfrak{Z}^\circ = \mathcal{X} \times \mathfrak{Z} \leftrightarrows \boxed{\mathrm{S}} \times \boxed{\mathrm{M}}$. The requirement that the graph is locally defined over space-time leads to the symbolic organization $\boxed{\mathrm{S}}, \boxed{\mathrm{S}} \times \boxed{\mathrm{M}}$. Note that the *dynamical motif* (59) is intrinsically tied to the setting of the Lagrangian variational problem. Indeed, the very framework of the Lagrangian functional requires us to choose a *volume* form.[35] Furthermore, to express generalized Hamilton equations, we need to find sufficient and necessary conditions on a map $\mathfrak{z}_\sigma^{\boxdot} : \mathcal{X} \to \Lambda^n T^\star(\mathcal{X} \times \mathfrak{Z})$ so

[34]The fact that the parametrization space Σ is a 2-dimensional manifold is expressed in the *world-sheet* picture

[35]We note a subtle point here. At first it seems that there is an epistemological difficulty in a total democracy between space-time *and* fields. But within a more embracing view we understand that the choice of a specific volume form β (for example a Riemannian volume form $\beta = \mathsf{g}dx^1 \wedge \cdots \wedge dx^n$ or a Minkowskian one $\beta = dx^1 \wedge \cdots \wedge dx^n$) is only a particular choice and that in the light of our further investigation of the notion of observable in MG, that such a *specific* choice is of secondary importance.

that it corresponds to the given variational problem. Hamilton equations are translated, thanks to the multisymplectic form, as geometric conditions on a graph $G[\sigma,\pi]$. The first and fundamental condition amongst these is a *parametrization condition*. This ensures that *locally* $G[\sigma,\pi]$ is the graph of a map $\mathfrak{z}_\sigma^{\boxdot} : \mathcal{X} \to \Lambda^n T^\star(\mathcal{X}\times\mathfrak{Z})$ *over* the space-time. This independence condition appears once again here. We define a *dynamical entity* by reference to this independence condition which allows us to work with a map of type $\mathfrak{z}_\sigma^{\boxdot}$ over space-time and underlies the graph representation. We now focus on the simple case of mechanics in order to give a better feel for the fundamentals of this subject.

[ı] Classical mechanics. The configuration space is a manifold $\mathfrak{Z}$ and the velocity configuration space is the tangent bundle $T\mathfrak{Z}$. Here, *time* is seen only as a *parameter*. The object of study is a path $\gamma : I\subset\mathbb{R}\to\mathfrak{Z}$. In this case, a dynamical entity is nothing than the graph $G[\gamma]$, given by $G[\gamma]=\big\{(t,\gamma(t)), t\in I\big\}\subset I\times\mathfrak{Z}$. In section (3.2), we introduced the following graphs $G[\gamma,\zeta]$, $G[\gamma,\pi]$ and $G[\gamma,\zeta,\pi]$ as the representations of the maps $(\gamma,\zeta) : I\to T\mathfrak{Z}$, $(\gamma,\pi) : I\to T^\star\mathfrak{Z}$ and $(\gamma,\zeta,\pi) : I\to T\mathfrak{Z}\times_{\mathfrak{Z}} T^\star\mathfrak{Z}$. A recurring question is the correspondence between a graph and an application. We consider on $I\times\mathfrak{Z}$ the coordinates $(t,q^i(t))=(t,q^1(t)\cdots q^k(t))$. Then any map $\gamma : I\to\mathfrak{Z}$ is represented by the graph $G[\gamma]\subset I\times\mathfrak{Z}$. Equivalently, any oriented curve Γ in $\mathbb{R}\times\mathfrak{Z}$ such that $dt|_\Gamma>0$ is identified with the graph $G[\gamma]$ of the map $\gamma : I\to\mathfrak{Z}$. The key point is the independence condition $dt|_\Gamma>0$. It is this condition that allows us to picture the graph of γ, denoted $G[\gamma]$ as an oriented curve in $\mathfrak{Z}^{\circ}=I\times\mathfrak{Z}$. This is the same independence condition which allows us to associate to any map $(\gamma,\pi) : t\mapsto(\gamma(t),\pi(t))$ its graph $\Gamma=G[\gamma,\pi]$ in $I\times T^\star\mathfrak{Z}$. We observe that $\Gamma=G[\gamma,\pi]$ is the image of the map $\mathfrak{z}_\gamma^{\boxdot} : t\in I\mapsto(t,\gamma(t),\pi(t))$ which is a map from I to $I\times T^\star\mathfrak{Z}$.

Oriented curve in $I\times\mathfrak{Z}$	Oriented curve in $I\times T^\star\mathfrak{Z}$
$\mathfrak{z}_\gamma : t\mapsto\mathfrak{z}(t)=(t,\gamma(t))$	$\mathfrak{z}_\gamma^{\boxdot} : t\in I\mapsto(t,\gamma(t),\pi(t))$
$G[\gamma]=\big\{(t,\gamma(t)), t\in I\big\}\subset I\times\mathfrak{Z}$	$G[\gamma,\pi]=\big\{(t,\gamma(t),\pi(t)), t\in I\big\}\subset I\times T^\star\mathfrak{Z}$

This condition is equivalent the specification of a *volume* 1-form on I. Notice the connection with the definition of the Lagrangian functional $\mathcal{L}[\gamma]=\int_I L(t,\gamma(t),d\gamma(t))dt$ on the set of maps $\gamma : I\to\mathfrak{Z}$. The natural arena where a *dynamical entity*, seen mathematically as a *graph*, live is the space $\mathfrak{Z}^{\circ}=I\times\mathfrak{Z}$. We observe the distinction between the character of the I (the parametrization space) and $\mathfrak{Z}$ (the parametrized space) on the one hand and that of the *dynamical space* $I\times T^\star\mathfrak{Z}$ where the graph lives on the other. In the present section we distinguish between $(T^\star\mathfrak{Z})^{\circ}=I\times T^\star\mathfrak{Z}$. and $T^\star(\mathfrak{Z}^{\circ})=T^\star(I\times\mathfrak{Z})$. The *ontologic motif* of the mechanics variational problem is given[36] by:

$$\text{Ontologic motif}\qquad \big[I\big]\big|\gamma\big|\big[\mathfrak{Z}\big]=\big[t\big]\big|\gamma\big|\big[\gamma(t)\big]\leftrightarrows\boxed{\mathrm{S}}\boxed{\mathrm{M}}. \tag{60}$$

The ontologic motif emphasizes the ontological distinction between *parametrized* spaces and *parametrization* spaces. By the notion of *dynamical motif* we understand the possibility of expressing graphs locally as maps over the parametrization space. This leads to the concept of *dynamical entity* where the object of interest is $G[\gamma]$ or $\Gamma=G[\gamma,\pi]$:

$$\text{Dynamical motif}\qquad \big[I\big]\big|\mathfrak{z}_\gamma\big|\big[\mathfrak{Z}^{\circ}=I\times\mathfrak{Z}\big]=\big[t\big]\big|\mathfrak{z}_\gamma\big|\big[G[\gamma]\big]\leftrightarrows\boxed{\mathrm{S}},\boxed{\mathrm{S}}\times\boxed{\mathrm{M}} \tag{61}$$

We apply this insight concerning the possibility of expressing graphs as maps over the parametrization space to various maps. We are concerned not only with the path $\gamma:\tau\to\gamma(\tau)$.

[36] Although in the general case $\boxed{\mathrm{S}}$ and $\boxed{\mathrm{M}}$ denote *space-time* and *matter*, we use them here for *time* and *configuration space* in the setting of classical mechanics.

Ontological motif $I \times \mathfrak{Z}$

$$\left| \begin{array}{l} [I]\,|\gamma|\,[\mathfrak{Z}] \\ (\gamma) : \left\{ \begin{array}{lcl} I & \to & \mathfrak{Z} \\ t & \mapsto & \gamma(t) \end{array} \right. \end{array} \right.$$

Dynamical motif $dt|_\Gamma > 0$

$$\left| \begin{array}{l} [I]\,|\mathfrak{z}_\gamma|\,[\mathfrak{Z}^\circ = I \times \mathfrak{Z}] = [t]\,|\mathfrak{z}_\gamma|\,[G[\gamma]] \\ \mathfrak{z}_\gamma : \left\{ \begin{array}{lcl} I & \to & \mathfrak{Z}^\circ \\ t & \mapsto & G[\gamma] = \big\{(t,\gamma(t)), t \in I\big\} \subset \mathfrak{Z}^\circ = I \times \mathfrak{Z} \end{array} \right. \end{array} \right.$$

We can also build the analogous construction for the lifts of the tangent and cotangent spaces:
Ontological motif

$$\left| \begin{array}{l} [I]\,|(\gamma,\zeta)|\,[T\mathfrak{Z}] \\ (\gamma,\zeta) : \left\{ \begin{array}{lcl} I & \to & T\mathfrak{Z} \\ t & \mapsto & (\gamma(t),\zeta(t)) \end{array} \right. \end{array} \right. \qquad \left| \begin{array}{l} [I]\,|(\gamma,\pi)|\,[T^\star\mathfrak{Z}] \\ (\gamma,\pi) : \left\{ \begin{array}{lcl} I & \to & T^\star\mathfrak{Z} \\ t & \mapsto & (\gamma(t),\pi(t)) \end{array} \right. \end{array} \right.$$

Dynamical motif

$$\left| \begin{array}{l} [I]\,|\mathfrak{z}_{(\gamma,\zeta)}|\,[(T\mathfrak{Z})^\circ = I \times T\mathfrak{Z}] = [t]\,|\mathfrak{z}_{(\gamma,\zeta)}|\,[G[\gamma,\zeta]] \\ \mathfrak{z}_{(\gamma,\zeta)} : \left\{ \begin{array}{lcl} I & \to & (T\mathfrak{Z})^\circ = I \times T\mathfrak{Z} \\ t & \mapsto & G[\gamma,\zeta] = \big\{(t,\gamma(t),\zeta(t)), t \in I\big\} \subset (T\mathfrak{Z})^\circ \end{array} \right. \end{array} \right.$$

$$\left| \begin{array}{l} [I]\,|\mathfrak{z}_{(\gamma,\pi)}|\,[(T^\star\mathfrak{Z})^\circ = I \times T^\star\mathfrak{Z}] = [t]\,|\mathfrak{z}_{(\gamma,\pi)}|\,[G[\gamma,\pi]] \\ \mathfrak{z}_{(\gamma,\pi)} : \left\{ \begin{array}{lcl} I & \to & (T^\star\mathfrak{Z})^\circ = I \times T^\star\mathfrak{Z} \\ t & \mapsto & G[\gamma,\pi] = \big\{(t,\gamma(t),\pi(t)), t \in I\big\} \subset (T^\star\mathfrak{Z})^\circ \end{array} \right. \end{array} \right.$$

Let $\mathfrak{j}_\Gamma : \Gamma \to I \times T^\star\mathfrak{Z}$ be the embedding of some oriented submanifold $\Gamma \subset I \times \mathcal{M}$, and let $\mathfrak{j}_\Gamma^\star T\mathcal{M}$ be the pullback image of the tangent bundle $T\mathcal{M}$ by $\mathfrak{j}_\Gamma$. The independence condition $dt|_\Gamma \neq 0$ is equivalent to the fact that *locally* Γ is the graph of some map $\mathfrak{z}_\gamma^{\boxdot}$ over *time*. Notice that $dt|_\Gamma := \mathfrak{j}_\Gamma^\star dt$ where dt is a *volume* form on I. At this level, we describe a non-autonomous Hamiltonian system with a Hamiltonian function H defined on $I \times T^\star\mathfrak{Z}$, $H(t,y,p) = p_i v^i - L(x,y,v)$. We have seen that $\gamma : I \to \mathfrak{Z}$ is a solution of the Euler-Lagrange equations, if and only if the map (γ,π) is a solution of the Hamilton system. The geometrization of the system of Hamiltonian equations focuses on the conditions on the graph $\Gamma = G[\gamma,\pi] = \big\{(t,\gamma(t),\pi(t)), t \in I\big\} \subset I \times T^\star\mathfrak{Z}$.

$$\left| \begin{array}{lcl} \dfrac{d\gamma^i}{dt}(t) & = & \dfrac{\partial H}{\partial p_i}(t,q(t),p(t)) \\ \dfrac{d\pi_i}{dt}(t) & = & -\dfrac{\partial H}{\partial y^i}(t,q(t),p(t)) \end{array} \right. \iff \left| \begin{array}{lcl} dp_i \wedge dy^i(\xi,X) & = & dH(\xi)dt(X) \\ dt|_\Gamma > 0 & \neq & 0 \end{array} \right. \tag{62}$$

The symplectic form $\omega = dp_i \wedge dy^i$ in the right part of (62) and is defined on $T^\star\mathfrak{Z} \otimes_{I\times\mathfrak{Z}} TI \simeq I \times T^\star\mathfrak{Z}$ and $X \lrcorner\, \omega$ is the unique 1-form such that[37] for any $\xi \in T_{(t,\gamma(t),\pi(t))}(I \times T^\star\mathfrak{Z})$ we have $X \lrcorner\, \omega(\xi) = \omega(X,\xi)$. Let $X(t) \in T_{(t,\gamma(t),\pi(t))}G[\gamma,\pi]$ and $\xi \in T_{(t,y^i,p_i)}\big(T^\star\mathfrak{Z} \otimes_{I\times\mathfrak{Z}} TI\big)$ be two vector fields described as follows:

$$\begin{array}{lcl} X(t) & = & \dfrac{\partial}{\partial t} + \dfrac{d\gamma^i(t)}{dt}\dfrac{\partial}{\partial y^i} + \dfrac{d\pi_i(t)}{dt}\dfrac{\partial}{\partial p_i} \\ \xi & = & \Theta_\xi^t \dfrac{\partial}{\partial t} + \Theta_\xi^i \dfrac{\partial}{\partial y^i} + \Upsilon_i^\xi \dfrac{\partial}{\partial p_i} \end{array}$$

[37] Notice that the bundle $I \times T^\star\mathfrak{Z}$ is identified with $T^\star\mathfrak{Z} \otimes_{I\times\mathfrak{Z}} TI$ so that we equivalently describe $\xi \in T_{(t,\gamma(t),\pi(t))}(T^\star\mathfrak{Z} \otimes_{I\times\mathfrak{Z}} TI)$.

The vector field $X : I \to I \times T^\star\mathfrak{Z}$ is a "*basis*" of the tangent curve to $G[\gamma,\pi]$ at $(t,\gamma(t),\pi(t))$ with fixed parametrization $dt(X)=1$. Here, the condition $dt|_\Gamma > 0$ is directly taken into account as $dt|_\Gamma = \mathfrak{j}_\Gamma^\star dt = 1$. We focus on the equation: $dp_i \wedge dy^i(\xi,X) = -dH(\xi)dt(X)$. First,

$$\begin{aligned} X \lrcorner\, \omega &= \Big(\frac{d\gamma^j(t)}{dt}\frac{\partial}{\partial y^j} + \frac{d\pi_j(t)}{dt}\frac{\partial}{\partial p_j}\Big) \lrcorner\, [dp_i \wedge dy^i] \\ &= \Big(dp_i(\tfrac{d\pi_j(t)}{dt}\tfrac{\partial}{\partial p_j})dy^i - dy^j(\tfrac{d\gamma^i(t)}{dt}\tfrac{\partial}{\partial y^j})dp_i\Big) = -\Big(\tfrac{d\gamma^i(t)}{dt}dp_i - \tfrac{d\pi_i(t)}{dt}dy^i\Big) \end{aligned}$$

On the other hand, since $dH(t,q,p) = \frac{\partial H}{\partial t}dt + \frac{\partial H}{\partial y^i}dy^i + \frac{\partial H}{\partial p_i}dp_i$. The Hamilton equations take into account the decomposition on dy^i and dp_i. We have: $X \lrcorner\, \omega = -dH_{(t,\gamma(t),\pi(t))} \bmod dt$.

[2] Relativistic Mechanics. We incorporate a further variable $\mathfrak{e}$. The underlined idea is to see the parametrization condition as a new dynamical variable. We work in the following phase space: $T^\star(\mathfrak{Z}^{\mathbf{o}}) = T^\star(I\times\mathfrak{Z})$. This justifies the previous construction found in section (3.5), where the Legendre correspondence was described: ${}^{\mathfrak{s}}T(I\times\mathfrak{Z}) \leftrightarrow T^\star(I\times\mathfrak{Z})$. Recall that we have the canonical identification ${}^{\mathfrak{s}}T(I\times\mathfrak{Z}) \simeq T^\star\mathfrak{Z}\otimes_{I\times\mathfrak{Z}} TI$. The extended Hamiltonian is $\mathcal{H}(q,p) = \mathcal{H}(\tau,q^i,\mathfrak{e},p_i) = \mathfrak{e} + H(\tau,q^i,p_i)$. On the cotangent space, we describe a vector field $X \in T(T^\star(I\times\mathfrak{Z}))$. The independence condition $d\tau|_\Gamma(X) > 0$ is equivalent to the fact that we can parametrize any Hamiltonian curve Γ by the map $\mathfrak{z}^{\boxdot} : \tau \mapsto (\tau, q(\tau), \mathfrak{e}(\tau), \pi_i(\tau))$. In such a context, we work with the same ontologic motif described by the left part of the following table. However the dynamical motif is slightly different[38]:

Ontological motif $I\times\mathfrak{Z}$ — Dynamical motif

$$\Big[I\Big]\,,_\gamma\Big[\mathfrak{Z}\Big] \qquad (\gamma): \left\{\begin{array}{ccc} I & \to & \mathfrak{Z} \\ t & \mapsto & \gamma(t)\end{array}\right.$$

$$\Big[I\Big]\,,_{\mathfrak{z}(\Gamma,\boldsymbol{\pi})}\Big[T^\star(\mathfrak{Z}^{\mathbf{o}}) = T^\star(I\times\mathfrak{Z})\Big] = \Big[t\Big]\,,_{\mathfrak{z}(\gamma,\pi)}\Big[G[\gamma,\pi]\Big] \qquad \mathfrak{z}_{(\gamma,\mathfrak{e},\pi)}: \left\{\begin{array}{ccc} I & \to & T^\star(\mathfrak{Z}^{\mathbf{o}}) = T^\star(I\times\mathfrak{Z}) \\ \tau & \mapsto & G^{\mathfrak{e}}[\gamma,\mathfrak{e},\pi]\end{array}\right.$$

with $G^{\mathfrak{e}}[\gamma,\mathfrak{e},\pi] = \big\{(\tau,\gamma^i(\tau),\mathfrak{e}(\tau),\pi_i(\tau)), \tau\in I\big\} \subset T^\star(\mathfrak{Z}^{\mathbf{o}})$. Notice that in such a picture the path γ lifts to $T(I\times\mathfrak{Z})$, or to $T^\star(I\times\mathfrak{Z})$, placing I and $\mathfrak{Z}$ on an equal footing. We observe that $\big[I\big]\,,_{(\gamma,\zeta)}\big[T(\mathfrak{Z}^{\mathbf{o}})\big]$ and $\big[I\big]\,,_{(\gamma,\pi)}\big[T^\star(\mathfrak{Z}^{\mathbf{o}})\big]$. We are now interested in the latter lift. We consider a 1-dimensional oriented submanifold pictured by the graph $G^{\mathfrak{e}}[\gamma,\mathfrak{e},\pi] = \big\{(\tau,\gamma^i(\tau),\mathfrak{e}(\tau),\pi_i(\tau)), \tau\in I\big\} \subset I\times I_{\mathfrak{e}}\times T^\star\mathfrak{Z}$. Let us denote $(T^\star\mathfrak{Z})^{\mathbf{o},\mathfrak{e}} = I\times I_{\mathfrak{e}}\times T^\star\mathfrak{Z}$. In such a picture, we have a canonical identification: $(T^\star\mathfrak{Z})^{\mathbf{o},\mathfrak{e}} \simeq \big(T^\star\mathfrak{Z}\otimes_{I\times\mathfrak{Z}} TI\big)\times I_{\mathfrak{e}} \simeq T^\star(\mathfrak{Z}^{\mathbf{o}})$. Then, $G[\gamma,\pi]$ is a solution of the system of Hamiltonian equations *and* respects the independence condition - see (62) - if and only if $G^{\mathfrak{e}}[\gamma,\mathfrak{e},\pi]$ is solution a of:

$$\forall\xi\in T(T^\star(\mathfrak{Z}^{\mathbf{o}})) \qquad \omega(\xi,X) = d\mathcal{H}(\xi)d\tau(X)$$

The symplectic form is defined as: $\omega = d\mathfrak{e}\wedge d\tau + dp_i\wedge dy^i$. Notice that a general vector field $\xi\in T(T^\star(\mathfrak{Z}^{\mathbf{o}}))$ is written as: $\xi = \Theta^\tau_\xi\frac{\partial}{\partial\tau} + \Theta^i_\xi\frac{\partial}{\partial q^i} + \Upsilon^\xi_{\mathfrak{e}}\frac{\partial}{\partial\mathfrak{e}} + \Upsilon^\xi_i\frac{\partial}{\partial p_i}$, whereas a vector field tangent to the Hamiltonian curve $X(t)\in T_{(t,\gamma(t),\mathfrak{e}(t),\pi(t))}G^{\mathfrak{e}}[\gamma,\mathfrak{e},\pi]$ is described as:

$$X(t) = \frac{\partial}{\partial t} + \frac{d\gamma^i}{dt}\frac{\partial}{\partial y^i} + \frac{d\mathfrak{e}}{dt}\frac{\partial}{\partial\mathfrak{e}} + \frac{d\pi_i}{dt}\frac{\partial}{\partial\pi_i} = \frac{\partial}{\partial t} + \Theta_i\frac{\partial}{\partial y^i} + \Upsilon_{\mathfrak{e}}\frac{\partial}{\partial\mathfrak{e}} + \Upsilon_i\frac{\partial}{\partial\pi_i} \tag{63}$$

We emphasize that we have chosen a particular parametrization. In full generality, such vector fields would be written as:

$$X(\tau) = \frac{dt}{d\tau}\frac{d}{dt} + \frac{d\gamma^i}{d\tau}\frac{\partial}{\partial y^i} + \frac{d\mathfrak{e}}{d\tau}\frac{\partial}{\partial\mathfrak{e}} + \frac{d\pi_i}{d\tau}\frac{\partial}{\partial\pi_i} = \Theta\frac{\partial}{\partial t} + \Theta_i\frac{\partial}{\partial y^i} + \Upsilon_{\mathfrak{e}}\frac{\partial}{\partial\mathfrak{e}} + \Upsilon_i\frac{\partial}{\partial\pi_i} \tag{64}$$

[38] We describe here only the example of the ontologic and dynamical motives with the cotangent lift

We do not enter into details of the geometrical approach here but refer to the work of Hélein [111, 113, 114] for a deeper presentation. We should like however, to offer one intuitive remark on the passage from the classical to the extended picture of mechanics. Before doing so, we recall the comparison:

Phase space and Extended Phase space motives. Notice that in this picture the notation $\boxed{\text{S}}$ (usually reserved for space-time) and $\boxed{\text{M}}$ (usually reserved for fields) is given a wider meaning, the aim of which is to preserve a general notion of parametrization space and target space. For the classical picture with path $\gamma : I \to \mathfrak{Z}$ we have:

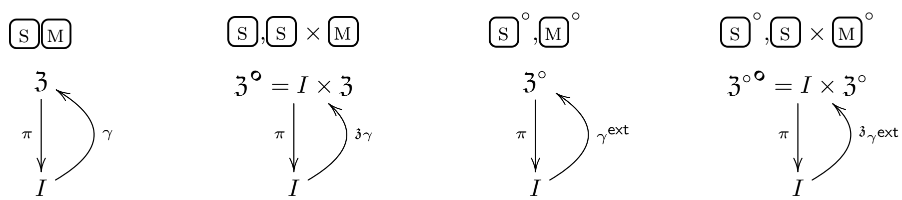
$$\text{Ontologic motif : } [I]|\gamma|[\mathfrak{Z}] = [t]|\gamma|[\gamma(t)]$$

$$\text{Dynamical motif : } [I]|\mathfrak{z}_\gamma|[\mathfrak{Z}^{\circ} = I \times \mathfrak{Z}] = [t]|\mathfrak{z}_\gamma|[G[\gamma]]$$

In this picture, $\boxed{\text{S}}$ is from the beginning treated ambiguously as both *time* and *parametrization variable*. Whereas, for the extended phase space, namely the study of the path $\gamma^{\text{ext}} : I \to \mathfrak{Z}^\circ = \mathbb{R} \times \mathfrak{Z}$ we distinguish from the outset the role of $\boxed{\text{S}}$ as a *parametrization* variable opposed to that of *time*, taken as an additional variable in the ontology $\boxed{\text{M}} = [\mathbb{R} \times \mathfrak{Z}]$

$$\text{Ontologic motif : } [I]|\gamma^{\text{ext}}|[\mathfrak{Z}^\circ] = [\tau]|\gamma^{\text{ext}}|[\gamma^{\text{ext}}(\tau)]$$

$$\text{Dynamical motif : } [I]|\mathfrak{z}_{\gamma^{\text{ext}}}|[\mathfrak{Z}^{\circ\circ} = I \times \mathfrak{Z}^\circ] = [\tau]|\mathfrak{z}_{\gamma^{\text{ext}}}|[G[\gamma^{\text{ext}}]]$$



We see that the important thing is to consider the space of the graph construction. The geometrical picture in the extended case - depicted in the representation of the two bundles to the lower right above - gives non-physical extra relations. The related additional variables play no role in the Euler-Lagrange equations. The key point is to consider the space of graphs over the parametrized space, hence the dynamical picture $\boxed{\text{S}},\boxed{\text{S}} \times \boxed{\text{M}}$ is all that we need for relativistic mechanics. In this case the extended Legendre correspondence is constructed on the spaces ${}^{\mathfrak{s}}T(I \times \mathfrak{Z}) \subset T(I \times \mathfrak{Z})$ and $T^\star(I \times \mathfrak{Z})$. This naturally incorporates the relativistic treatment of mechanics, since symbolically we introduce the differential operation - taking tangent and cotangent space - *after* having fixed the dynamical motif. Hence, symbolically we have: $\boxed{\text{S}},T(\boxed{\text{S}} \times \boxed{\text{M}})$ and $\boxed{\text{S}},T^\star(\boxed{\text{S}} \times \boxed{\text{M}})$. This is the main idea of the mathematical requirement of graphs over the parametrization space $\boxed{\text{S}}$. An heuristic argument for the equivalence of the two cases for *physical* Euler-Lagrange equations is intuitively expressed by the following idea. The ontologic motif in the latter case $\boxed{\text{S}}^\circ,\boxed{\text{M}}^\circ$ is conceptually identified with the dynamical motif in the former case $\boxed{\text{S}},\boxed{\text{S}} \times \boxed{\text{M}}$. The space of interest is the extended configuration space, obtained conceptually from two different constructions. More serious considerations of this issue will be developed with the help of the dimension of the enlarged pseudofibers in section (4.8).

This interplay with parametrization conditions allows us to describe a Hamiltonian curve without the modulo condition - see (43). A Hamiltonian curve $\Gamma \subset T^\star(\mathfrak{Z} \times \mathbb{R})$ is described if for all $m \in \Gamma$, there exists a vector field $X \in T_m\Gamma$ such that $X \lrcorner\, \omega_m = (-1)^n d\mathcal{H}_m$ where we suppose that $d\mathcal{H} \neq 0$. There exists a 1-form β defined on $T^\star(\mathfrak{Z} \times \mathbb{R})$ such that the following properties (65)-(i) and (65)-(ii) are equivalent:

$$\begin{array}{lll} \text{(i)} & \forall m \in \Gamma\ \forall X \in T_m\Gamma & X \lrcorner\, \omega_m = -d\mathcal{H}_m \\ \text{(ii)} & \forall m \in \Gamma\ \forall X \in T_m\Gamma, & X \lrcorner\, \omega_m = \beta_m(X) d\mathcal{H}_m \end{array} \qquad (65)$$

Let X be a Hamiltonian vector field. We have $X \lrcorner \omega_m = -d\mathcal{H}_m$. Since we suppose that $d\mathcal{H} \neq 0$ we can exhibit a vector field $\boldsymbol{\eta}$ such that $d\mathcal{H}(\boldsymbol{\eta}) = 1$. In such a context, β is intrinsically constructed: $\beta = \boldsymbol{\eta} \lrcorner \omega$. Due to the previous considerations concerning the extended phase space, we use traditional notation on $T^\star(\mathbb{R} \times \mathfrak{z})$ with coordinates $(q^\circ, q^i, p_\circ, p_i)$. Let $\omega = dp_\circ \wedge dq^\circ + p_i \wedge dq^i$ be the symplectic form and let $\mathcal{H} = p_\circ + H(\tau, q^i(\tau), p_i(\tau))$ be the extended Hamiltonian. In this case, $\boldsymbol{\eta} = \partial/\partial p_\circ$ since $\boldsymbol{\eta} \lrcorner \omega = \partial/\partial p_\circ \lrcorner \left(dp_\circ \wedge dq^\circ + p_i \wedge dq^i\right) = dq^\circ$ so that $\beta = dq^\circ$. The system of Hamilton equations is equivalent to the condition that $\forall \xi \in T(T^\star \mathfrak{z}^\circ)$

$$dp_\circ \wedge dq^\circ + dp_i \wedge dq^i(\xi, X) = d\mathcal{H}(\xi)\beta(X). \tag{66}$$

With

$$\begin{aligned} \xi &= \Theta^\circ_\xi \frac{\partial}{\partial q^\circ} + \Theta^i_\xi \frac{\partial}{\partial q^i} + \Upsilon^\xi_\circ \frac{\partial}{\partial p_\circ} + \Upsilon^\xi_i \frac{\partial}{\partial p_i} \\ X &= \frac{\partial}{\partial q^\circ} + \Theta_i \frac{\partial}{\partial q^i} + \Upsilon_\circ \frac{\partial}{\partial p_\circ} + \Upsilon_i \frac{\partial}{\partial p_i} \end{aligned}$$

where $\Theta^\circ = 1$. The left part of (66) writes:

$$\begin{aligned} \left(dp_\circ \wedge dq^\circ + dp_i \wedge dq^i\right)(\xi, X) &= X \lrcorner \left(dp_\circ \wedge dq^\circ + dp_i \wedge dq^i\right)(\xi) \\ &= dp_\circ(\xi) dq^\circ(X) - dp_\circ(X) dq^\circ(\xi) \\ &= +dp_i(\xi) dq^i(X) - dp_i(X) dq^i(\xi) \\ &= \Upsilon^\xi_\circ - \Upsilon_\circ \Theta^\circ_\xi + \Upsilon^\xi_i \Theta^i - \Upsilon_i \Theta^i_\xi \end{aligned}$$

We focus now on the right part of (66). Let $X \in T_m\Gamma$ be a Hamiltonian vector field, then:

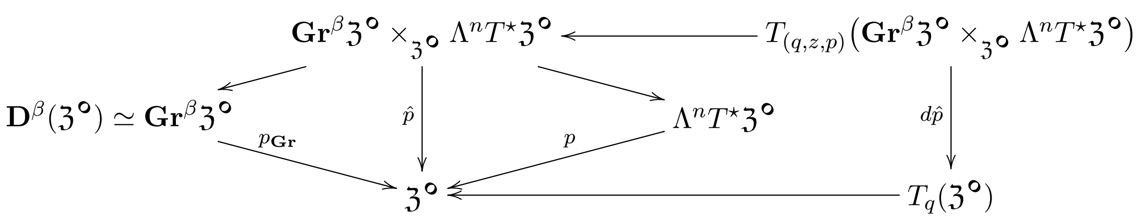
$$\beta(X) = X \lrcorner \beta = X \lrcorner (\boldsymbol{\eta} \lrcorner \omega) = (X \wedge \boldsymbol{\eta}) \lrcorner \omega = -\boldsymbol{\eta} \lrcorner (X \lrcorner \omega) = -\boldsymbol{\eta} \lrcorner (-d\mathcal{H}) = 1$$

$$d\mathcal{H}(\xi)\beta(X) = \left(\Theta^\circ_\xi \frac{\partial \mathcal{H}}{\partial q^\circ} + \Theta^i_\xi \frac{\partial \mathcal{H}}{\partial q^i} + \Upsilon^\xi_\circ \frac{\partial \mathcal{H}}{\partial p_\circ} + \Upsilon^\xi_i \frac{\partial \mathcal{H}}{\partial p_i}\right) = \left(\Theta^\circ_\xi \frac{\partial \mathcal{H}}{\partial q^\circ} + \Theta^i_\xi \frac{\partial \mathcal{H}}{\partial q^i} + \Upsilon^\xi_\circ + \Upsilon^\xi_i \frac{\partial \mathcal{H}}{\partial p_i}\right)$$

Thanks to (66) we recover the Hamilton equations. We refer to the work of Hélein [111, 113, 114] for the geometrical construction in the n-dimensional case and the relation with n-phase space.

4.7 Geometrical construction and Legendre lifts

We develop some points about the geometrical construction and the variational principle treated in [115, 116, 117]. Let $\mathfrak{z}^\circ$ be a $(n+k)$ dimensional manifold. We examine variational problems on n-dimensional submanifolds $\Sigma \subset \mathfrak{z}^\circ$. The diagram under consideration is:



In this diagram, the diamond-shaped part is related to the generalized Legendre correspondence whereas the variational principle is built on the square-shaped diagram. The generalized Legendre correspondence is generated by the function W where the restriction of $d\hat{p}$ on $T_z\mathbf{D}^\beta_q\mathfrak{z}^\circ$ is central. Following [115, 116, 117], we a say few words on Legendre lifts. The construction involves two main spaces. The first is the set of oriented n-dimensional submanifolds $G \subset \mathfrak{Z}^{\circ}$, with the restriction that β on G is positive everywhere. This space is denoted $G^{\beta}_{\circ}$. The other space of interest is the set $\hat{G}^{\beta}_{\circ}$ of oriented n-dimensional submanifolds $\Gamma \subset \mathcal{M} = \Lambda^n T^\star \mathfrak{Z}^{\circ}$ with the restriction that β on Γ is positive everywhere. The construction [115, 116, 117] exhibits the generalized Hamilton equations without the help of contact structure or writing Euler-Lagrange equations. This is accomplished by considering variational formulations on Legendre lifts $\Gamma \in \hat{G}^{\beta}_{\circ}$ of an oriented n-dimensional manifold $G \in G^{\beta}_{\circ}$. We observe that the right mathematical construction, or *parametrization* of the problem is written:

$$\begin{cases} G \in G^{\beta}_{\circ} & \rightarrow \quad \Gamma \in \hat{G}^{\beta}_{\circ} \\ \quad q & \mapsto \quad (q, p(q)). \end{cases}$$

This construction treats the variational problem by means of functionals on the two levels: $\mathcal{L}[G]$ is defined on $G^{\beta}_{\circ}$ whereas $\mathcal{I}[\Gamma]$ is defined on $\hat{G}^{\beta}_{\circ}$. For a Legendre lift, then $\mathcal{I}[\Gamma] = \mathcal{L}[G]$. We apply the general setting to the specific case where G is the graph of some map $\sigma : \mathcal{X} \rightarrow \mathfrak{Z}$ and describe the construction.

Now we develop the spirit of the previous section (4.6) for *fields*, for which the map $\sigma : \mathcal{X} \rightarrow \mathfrak{Z}$. The space of interest is $\mathfrak{Z}^{\circ} = \mathcal{X} \times \mathfrak{Z}$. We associate to the map σ the graph $G[\sigma] = \big\{(x, \sigma(x)), x \in \mathcal{X}\big\}$, the image of the map $\mathfrak{z} : \mathcal{X} \longrightarrow \mathcal{X} \times \mathfrak{Z}$. The dynamical data is described by a point $(q(x), p(x))$ on the multi phase space $\Lambda^n T^\star \mathfrak{Z}^{\circ} = \Lambda^n T^\star(\mathcal{X} \times \mathfrak{Z})$ given by the map $\mathfrak{z}^{\boxdot}(x) = (x, \sigma(x), \mathfrak{e}(x), p(x))$. In the setting of Hamilton geometrization, we focus on necessary and sufficient conditions on maps $\mathfrak{z}^{\boxdot} : x \mapsto (q(x), p(x))$ giving rise the related graph $\Gamma = G[\gamma, \pi] = \mathfrak{z}^{\boxdot}(x)$, which is the image of the map $\mathfrak{z}^{\boxdot}$:

Oriented curve in $\mathcal{X} \times \mathfrak{Z}$	Oriented curve in $\Lambda^n T^\star \mathfrak{Z}^{\circ}$
$\mathfrak{z}_\sigma : x \mapsto \mathfrak{z}(x) = (x, \sigma(x))$ $G[\sigma] = \big\{(x, \sigma(x)), x \in \mathcal{X}\big\} \subset \mathcal{X} \times \mathfrak{Z} = \mathfrak{Z}^{\circ}$	$\mathfrak{z}^{\boxdot}_\sigma : x \mapsto \mathfrak{z}^{\boxdot}_\sigma(x) = (x, \sigma(x), \mathfrak{e}, \pi(x))$ $G[\gamma, \pi] = \big\{(x, \sigma(x), \mathfrak{e}(x), \pi(x)), x \in \mathcal{X}\big\} \subset \Lambda^n T^\star \mathfrak{Z}^{\circ}$

The problem of finding Hamilton equations reduces to that of exhibiting the *Legendre lift* of the graph $G[\sigma]$. The question of the *Legendre lift* connected with both to the variational aspect of the problem and the search of critical points. We see the relation between the graph $G[\sigma]$ and its lift: an n-dimensional oriented submanifold $\Gamma = G[\gamma, \pi] \subset \Lambda^n T^\star \mathfrak{Z}^{\circ}$. Here the graph $G[\gamma, \pi]$ is the image of the map $\mathfrak{z}^{\boxdot}_\sigma : x \mapsto \mathfrak{z}^{\boxdot}_\sigma(x) = (q(x), p(x)) = (x, \sigma(x), \mathfrak{e}(x), p(x))$. Geometrically, the detection of the Legendre lifts involves the parametrization given by the left part in (67):

$$\begin{cases} G^{\beta}_{\circ} & \rightarrow \quad \hat{G}^{\beta}_{\circ} \\ G[\sigma] & \mapsto \quad \Gamma = G[\sigma, \pi] \end{cases} \qquad \begin{cases} \boxed{S}, \boxed{S} \times \boxed{M} & \rightarrow \quad \boxed{S}, \Lambda^n T^\star\big(\boxed{S} \times \boxed{M}\big) \\ G[\sigma] & \mapsto \quad \Gamma = G[\sigma, \pi] \end{cases} \tag{67}$$

This symbolic notation incorporates the dynamical perspective since we work intrinsically via graphs. The dynamical organization is captured by the right part of (67). From the outset in the Lagrangian setting we need the data of a volume form β on $\mathcal{X}$. This is transported on $\mathfrak{Z}^{\circ} = \mathcal{X} \times \mathfrak{Z}$ as the volume n-form $\beta = \pi^\star \beta$ - where $\pi : \mathfrak{Z}^{\circ} \rightarrow \mathcal{X}$ is the canonical projection.

Now we summarize this section by the following tables:

Ontologic motif and data $\boxed{S}\boxed{M}$

Mechanics	**Variational maps**	**General problems**
$\boxed{I}\,\boxed{\gamma}\,\boxed{\mathfrak{Z}} \leftrightarrows \boxed{S}\boxed{M}$	$\boxed{\mathcal{X}}\,\boxed{\sigma}\,\boxed{\mathfrak{Z}} \leftrightarrows \boxed{S}\boxed{M}$	
Paths $\gamma : I \longrightarrow \mathfrak{Z}$	Maps $\sigma : \mathcal{X} \longrightarrow \mathfrak{Z}$	n-dim. sub. $G \subset \mathfrak{Z}^{\circ}$
$G_{\circ}$ Set of paths γ	$G_{\circ}$ Set of maps σ	$G_{\circ} = \{G^n \subset (\mathfrak{Z}^{\circ})^{n+k}\}$

Dynamical motif and data Ⓢ,Ⓢ × Ⓜ

$\mathfrak{z}_\gamma : \begin{cases} I & \to & \mathfrak{Z}^\circ \\ \tau & \mapsto & \mathfrak{z}(\tau) = G[\gamma] \end{cases}$	$\mathfrak{z}_\sigma : \begin{cases} \mathcal{X} & \to & \mathfrak{Z}^\circ = \mathcal{X} \times \mathfrak{Z} \\ x & \mapsto & \mathfrak{z}(x) = G[\sigma] \end{cases}$	G embedding into $\mathfrak{Z}^\circ$
$G[\gamma] = \{(\tau, \gamma(\tau)), \tau \in I\}$	$G[\sigma] = \{(x, \sigma(x)) \ / \ x \in \mathcal{X}\}$	$G \in G^\beta_\circ$

Lagrangian side, functionals and spaces. — In each case, the Lagrangian function, the Lagrangian functional, the analogue of the tangent space for mechanics, he set of decomposable n-multivector fields and finally the set of decomposable n-multivector fields with a parametrization condition are respectively described in the following table:

$L : T\mathfrak{Z} \otimes_{I\times\mathfrak{Z}} T^\star I \to \mathbb{R}$	$L : T\mathfrak{Z} \otimes_{\mathcal{X}\times\mathfrak{Z}} T^\star\mathcal{X} \to \mathbb{R}$	$L : \mathbf{Gr}^\beta \mathfrak{Z}^\circ \to \mathbb{R}$
$\mathcal{L}[\gamma] = \int_I L(\tau, \gamma, d\gamma)d\tau$	$\mathcal{L}[\sigma] = \int_\mathcal{X} L(x, \sigma, d\sigma)\beta$	$\mathcal{L}(G) = \int_G L(q, T_qG)\beta$
$T\mathfrak{Z} \otimes_{I\times\mathfrak{Z}} T^\star I$	$T\mathfrak{Z} \otimes_{\mathcal{X}\times\mathfrak{Z}} T^\star\mathcal{X}$	$\mathbf{Gr}^\beta \mathfrak{Z}^\circ$
$\mathbf{D}^1_q \mathfrak{Z}^\circ \cong \mathfrak{X}(\mathfrak{Z}^\circ)$	$\mathbf{D}^n_q \mathfrak{Z}^\circ$	$\mathbf{D}^n_q \mathfrak{Z}^\circ$
${}^\mathfrak{s}T_q\mathfrak{Z}^\circ$	${}^\mathfrak{s}\Lambda^n T_q\mathfrak{Z}^\circ$	$\mathbf{D}^\beta_q \mathfrak{Z}^\circ$

So that we can define the main canonical diffeomorphism:

${}^\mathfrak{s}T\mathfrak{Z}^\circ \simeq T\mathfrak{Z} \otimes_{I\times\mathfrak{Z}} T^\star I$	${}^\mathfrak{s}\Lambda^n T_q\mathfrak{Z}^\circ \simeq T\mathfrak{Z} \otimes_{I\times\mathfrak{Z}} T^\star I$	$\mathbf{Gr}^\beta_q \mathfrak{Z}^\circ \simeq \mathbf{D}^\beta_q \mathfrak{Z}^\circ$

The link between the right column and the other two is obtained thanks to the following diffeomorphisms: ${}^\mathfrak{s}T_{(\tau,v)}\mathfrak{Z}^\circ \simeq \mathbf{Gr}^{d\tau}\mathfrak{Z}^\circ$ and ${}^\mathfrak{s}T_{(x,v)}\mathfrak{Z}^\circ \simeq \mathbf{Gr}^\beta\mathfrak{Z}^\circ$. These respectively concern mechanics and field theory. $\forall(\tau, y) \in I \times \mathfrak{Z}$ and $\forall(x, y) \in \mathcal{X} \times \mathfrak{Z}$,

$$\begin{cases} T_y\mathfrak{Z} \otimes T^\star_\tau I & \to & \mathbf{Gr}^\beta_{(\tau,y)}(\mathfrak{Z}^\circ) \\ v & \mapsto & T(v) \end{cases} \qquad \begin{cases} T_y\mathfrak{Z} \otimes T^\star_x \mathcal{X} & \to & \mathbf{Gr}^\beta_{(\tau,y)}\mathfrak{Z}^\circ \\ v & \mapsto & T(v), \end{cases}$$

where $T(v)$ is the graph of the linear map $v : T_\tau I \to T_y\mathfrak{Z}$ and $v : T_x\mathcal{X} \to T_y\mathfrak{Z}$. Thus $T(v)$ is identified with $T(z)$ and $T(z_1, \cdots, z_n)$. - the vector space spanned and oriented respectively by z and $(z_1, \cdots, z_n)$. -Recall that $z \in {}^\mathfrak{s}T_{(\tau,v)}\mathfrak{Z}^\circ$ and $z = z_1 \wedge \cdots \wedge z_n \in {}^\mathfrak{s}T_{(\tau,z)}\mathfrak{Z}^\circ$ are given:

$$z = \frac{\partial}{\partial\tau} + \sum_{1\le i\le k} v^i \frac{\partial}{\partial y^i} \qquad \forall 1 \le \nu \le n \ , \ z_\nu = \frac{\partial}{\partial x^\nu} + \sum_{1\le i\le k} v^i_\mu \frac{\partial}{\partial y^i} \otimes dx^\nu$$

Legendre correspondence $(q, z) \leftrightarrow (q, p)$

${}^\mathfrak{s}T\mathfrak{Z}^\circ \times T^\star\mathfrak{Z}^\circ$	${}^\mathfrak{s}\Lambda^n T\mathfrak{Z}^\circ \times \Lambda^n T^\star\mathfrak{Z}^\circ$	$\mathbf{Gr}^\beta\mathfrak{Z}^\circ \times_{\mathfrak{Z}^\circ} \Lambda^n T^\star\mathfrak{Z}^\circ$

Hamiltonian function and spaces. — We note that the analogue of the cotangent space for mechanics, the Hamiltonian function, the map of interest for generalized Hamilton equations and theanalogue of the Hamiltonian curve space for mechanics are summarized in the following table:

$T^\star(\mathbb{R} \times \mathfrak{Z}) = T^\star\mathfrak{Z}^\circ$	$\Lambda^n T^\star(\mathcal{X} \times \mathfrak{Z}) = \Lambda^n T^\star\mathfrak{Z}^\circ$	$\Lambda^n T^\star\mathfrak{Z}^\circ$
$\mathcal{H} : \mathcal{M} = T^\star\mathfrak{Z}^\circ \to \mathbb{R}$	$\mathcal{H} : \mathcal{M} = \Lambda^n T^\star\mathfrak{Z}^\circ \to \mathbb{R}$	$\mathcal{H} : \mathcal{M} \to \mathbb{R}$
$\mathfrak{z}^\boxdot : I \to \mathcal{M} = T^\star\mathfrak{Z}^\circ$	$\mathfrak{z}^\boxdot : \mathcal{X} \to T^\star\mathfrak{Z} \otimes_{\mathcal{X}\times\mathfrak{Z}} T\mathcal{X} \times I_\mathfrak{e}$	$\mathfrak{z}^\boxdot : G \to \mathcal{M}$
$\Gamma[\gamma, \pi]$: $\{(\tau, \gamma^i(\tau), \pi_\circ(\tau), \pi_i(\tau))\}$	$\Gamma[\gamma, \pi]$: $\{(x, \sigma(x), \mathfrak{e}(x), \pi(x))\}$,	$\Gamma \subset \mathcal{M} \in \hat{G}^\beta_\circ$
$\Gamma \subset \mathcal{M} = T^\star\mathfrak{Z}^\circ$	$\Gamma \subset \mathcal{M} = \Lambda^n T^\star\mathfrak{Z}^\circ$	$\Gamma \subset \mathcal{M} = \Lambda^n T^\star\mathfrak{Z}^\circ$

4.8 Infinitesimal symplectomorphism and pseudofiber

The infinitesimal symplectomorphisms are the objects of the *ontologic symmetry*. We consider the geometric multisymplectic setting as the source of fundamental insights into classical field theory where space, time *and* fields are treated on an equal footing. We emphasize that conceptually, we work on the *ontologic* space[39] Note that vector fields $\Xi \in \Gamma(\mathcal{M}, T\mathcal{M})$ for ontologic symmetry, describe the ontologic space - which takes into account the common source of space-time, field coordinates and their dualities: the space of n-forms $\mathcal{M} = \Lambda^n T^\star \mathfrak{Z}^{\mathbf{o}}$.

Definition 4.8.1. *A* symplectomorphism *of* $(\mathcal{M}, \boldsymbol{\omega})$ *is a smooth diffeomorphism* $\boldsymbol{\varkappa} : \mathcal{M} \to \mathcal{M}$ *such that* $\boldsymbol{\varkappa}^\star \boldsymbol{\omega} = \boldsymbol{\omega}$.

Definition 4.8.2. *An* infinitesimal symplectomorphism *of* $(\mathcal{M}, \boldsymbol{\omega})$ *is a vector field* $\Xi \in \Gamma(\mathcal{M}, T\mathcal{M})$ *such that* $\mathcal{L}_\Xi \boldsymbol{\omega} = 0$.

Thanks to the Cartan formula, we obtain: $\mathcal{L}_\Xi \boldsymbol{\omega} = d(\Xi \lrcorner \boldsymbol{\omega}) + \Xi \lrcorner d\boldsymbol{\omega} = 0$. This relation is equivalent to $d(\Xi \lrcorner \boldsymbol{\omega}) = 0$. We denote $\mathfrak{sp}_\circ(\mathcal{M})$ the set of all symplectomorphisms of $(\mathcal{M}, \boldsymbol{\omega})$.

Now we introduce the following fundamental objects: the enlarged pseudofiber, the pseudofiber and the pseudofiber direction. Following [118] the enlarged pseudofiber is defined to be:

$$\mathbf{P}_q(z) = \{p \in \Lambda^n T_q^\star \mathfrak{Z}^{\mathbf{o}} \ / \ \frac{\partial W}{\partial z}(q, z, p) = 0\} \tag{68}$$

The enlarged pseudofiber is understood as the space of n-forms $\mathbf{P}_q(z) \subset \Lambda^n T_q^\star \mathfrak{Z}^{\mathbf{o}}$ such that the generalized Legendre correspondence is satisfied: $(q, z) \leftrightarrow (q, p)$. We refer to [115, 116, 117] for further details. The key point is that $\mathbf{P}_q(z)$ is an affine subspace of $\Lambda^n T_q^\star \mathfrak{Z}^{\mathbf{o}}$ with $\dim(\mathbf{P}_q(z)) = \frac{(n+k)!}{n!k!} - nk$. As noticed by Hélein and Kouneiher we find an *explanation* for the interplay between time and energy in mechanics where $\dim(\mathbf{P}_q(z)) = 1$. Considerations on the dimension of the enlarged pseudofiber emphasize the equivalence previously described for the extended phase space and the reparametrization problem. From this perspective we compare both pictures. The first one is given by the ontologic motif $\boxed{\text{S}}\boxed{\text{M}}$ (variational problems on path $\gamma : I \to \mathfrak{Z}$) whereas the second one is described by the motif $\boxed{\text{S}}^\circ, \boxed{\text{M}}^\circ$ (variational problems on paths $\gamma_{\text{ext}} : I \to \mathfrak{Z}^\circ = \mathbb{R} \times \mathfrak{Z}$). We denote by $\mathbf{P}_q(z)$ and $\mathbf{P}_q^\circ(z)$ respectively the related enlarged pseudofibers.

$$\dim(\mathbf{P}_q(z)) = \frac{(1+k)!}{1!k!} - k = 1 \qquad \dim(\mathbf{P}_q^\circ(z)) = \frac{(1+(k+1))!}{1!(k+1)!} - (k+1) = 1$$

Hence we observe the same freedom in both constructions. Notice that we could think of the extended picture with maps defined on a worldsheet: $\varsigma : \Sigma \to \mathfrak{Z}$. In this case, $\dim(\Sigma) = 2$ and $\dim(\mathfrak{Z}) = k$.

$$\dim({}^\varsigma\mathbf{P}_q(z)) = \frac{(2+k)!}{2!k!} - 2k = \frac{(2+k)(k+1)}{2} - 2k = \frac{k^2 - k + 2}{2}$$

Now we try to picture the reparametrization problem considering the follwing ontologic motif: $\varsigma_{\text{ext}} : \Sigma \to \mathfrak{Z}^{\circ\circ} = \mathbb{R} \times \mathbb{R} \times \mathfrak{Z}$ with $\dim(\Sigma) = 2$ and $\dim(\mathfrak{Z}^{\circ\circ}) = k + 2$.

$$\dim({}^\varsigma\mathbf{P}_q^\circ(z)) = \frac{(2+(k+2))!}{2!(k+2)!} - 2(k+2) = \frac{(k+3)(k+4) - 4(k+2)}{2} = \frac{k^2 + 3k + 4}{2}$$

Note that $\dim({}^\varsigma\mathbf{P}_q^\circ(z)) \neq \dim({}^\varsigma\mathbf{P}_q(z))$, by contrast with $\dim(\mathbf{P}_q^\circ(z)) = \dim(\mathbf{P}_q(z)) = 1$. This is why, if the dimension of the enlarged pseudofiber is perceived as an indication of the interplay of

[39] In the section (6) below we will describe this idea by the symbolic picture $\mathfrak{sp}_\circ(\mathcal{M}) \leftrightarrows [\triangleright]$.

canonical variables, we recover the time-energy relation only in the case $n = 1$. Finally, as noticed in [115, 116, 117], for a given $(q, z) \in \mathbf{D}^\beta \mathfrak{Z}^\circ$, we can find *at the same time* an element $p \in \mathbf{P}_q(z)$ and choose the value of $\mathcal{H}(q, p)$. Therefore, we find the definition of the *pseudofiber* to be the space:

$$\mathbf{P}_q^h(z) = \{p \in \mathbf{P}_q(z) \ / \ \mathcal{H}(q, p) = h\}. \tag{69}$$

Notice that $\dim(\mathbf{P}_q^h(z)) = \dim(\mathbf{P}_q(z)) - 1$ and that $\mathbf{P}_q(z)$ and $\mathbf{P}_q^h(z)$ are affine subspaces parallel to $[T_z\mathbf{D}_q^\beta \mathfrak{Z}^\circ]^\perp$ and $[T_z\mathbf{D}_q^n \mathfrak{Z}^\circ]^\perp$ where the spaces $[T_z\mathbf{D}_q^\beta \mathfrak{Z}^\circ]^\perp, [T_z\mathbf{D}_q^n \mathfrak{Z}^\circ]^\perp \subset \Lambda^n T_q^\star \mathfrak{Z}^\circ$ are respectively defined by (70)

$$\begin{array}{rcl} [T_z\mathbf{D}_q^\beta \mathfrak{Z}^\circ]^\perp & = & \{p \in \Lambda^n T_q^\star \mathfrak{Z}^\circ \ / \ \forall \xi \in T_z\mathbf{D}_q^\beta \mathfrak{Z}^\circ \ , \ p(\xi) = 0\} \\ {[T_z\mathbf{D}_q^n \mathfrak{Z}^\circ]^\perp} & = & \{p \in \Lambda^n T_q^\star \mathfrak{Z}^\circ \ / \ \forall \xi \in T_z\mathbf{D}_q^n \mathfrak{Z}^\circ \ , \ p(\xi) = 0\} \end{array} \tag{70}$$

In the general case we have the following dimension for the involved spaces: $\dim[\mathbf{Gr}^n \mathfrak{Z}] = n + k + nk$, $\dim[\Lambda^n T^\star \mathfrak{Z}] = n + k + \frac{(n+k)!}{n!k!}$, $\dim[\mathbf{D}_q^\beta \mathfrak{Z}] = nk$ and $\dim[\Lambda^n T_q^\star \mathfrak{Z}] = \frac{(n+k)!}{n!k!}$. We also have $\dim(\mathbf{P}_q(z)) = \frac{(n+k)!}{n!k!} - nk$ and $\dim(\mathbf{P}_q^h(z)) = \dim(\mathbf{P}_q(z)) - 1$ for the dimension of the enlarged pseudofiber $\mathbf{P}_q(z)$ and the pseudofiber $\mathbf{P}_q^h(z)$, respectively.

Definition 4.8.3. *The* generalized pseudofiber direction *for a Hamiltonian function* $\mathcal{H} : \mathcal{M} \to \mathbb{R}$ *is described by:*

$$\mathbf{L}_m^{\mathcal{H}} = \{\Xi \in T_m\mathcal{M} \ / \ \forall X \in [X]_m^{\mathcal{H}} \ , \ \forall \delta X \in T_X\mathbf{D}_m^n(\mathcal{M}) \ , \ \Xi \lrcorner \boldsymbol{\omega}(\delta X) = 0\}. \tag{71}$$

The generalized pseudofiber $\mathbf{L}_m^{\mathcal{H}}$ direction is a set of vector fields on $(\mathcal{M}, \boldsymbol{\omega})$. The tangent subspaces to pseudofibers $\mathbf{L}_m^{\mathcal{H}} = [T_{[X]_m^{\mathcal{H}}}\mathbf{D}_m^n \boldsymbol{\mathcal{M}} \lrcorner \boldsymbol{\omega}]^\perp$ can be given an intrinsic characterization. Another key point is the invariance of an observable functional along $\mathbf{L}_m^{\mathcal{H}}$. We do not develop this point further for the moment and refer to [115, 116, 117]. We see later the crucial importance of this concept with the example of canonical forms and observable functionals for gravity. Here we end the introduction to the general framework for MG developed by Hélein, [111, 113, 114] and Hélein and Kouneiher [115, 116, 117, 118]. We next give some remarks about the jet bundle formalism for DW theory where the issue of graded Poisson structure appears. These remarks motivate the construction developed by Hélein and Kouneiher of the distinction between AOF and OF.

5 Traditional Multisymplectic setting: graded standpoint

The traditional DW approach rests on the notions of *contact structure* and *jet bundles* [26, 35, 38, 58, 77, 78, 91, 92, 96, 97, 98, 102, 132, 133, 134, 205, 206, 207, 208]. The Lagrangian density $L : J^1\mathfrak{Z} \to \mathbb{R}$ is described on the first order jet bundle over $\mathfrak{Z}$. However, this (poly)multisymplectic approach involves the triviality of the extended phase space as a bundle over space-time: it contains a duality between the two ontological categories, *space-time* vs *fields*, and exhibits a decomposition of forms and multivectors along with vertical and horizontal components. We refer to the appendix (C) for a basic introduction of jet manifold and contact structures. We discuss the expression of canonical forms in terms of contact structure and finally address the the issue of graded Poisson structure. In this connection we note in particular how observable $(p-1)$-forms raise severe difficulties for quantization and conceptual issues concerning duality.

5.1 Geometrical spaces for first order jet De Donder-Weyl theory

The *Lagrangian density* is described as the mapping $\mathcal{L} : J^1\mathfrak{Z} \to \Lambda^n T^\star \mathcal{X}$. In local coordinates, we write $\mathcal{L}[z] = L(x^\mu, z^i, z^i_\mu)\beta$ where $L(x^\mu, z^i, z^i_\mu)$ is the Lagrangian function - whereas $\beta = dx^1 \wedge \cdots \wedge dx^n$ is the volume form on $\mathcal{X}$. The projection of interest in this trivial picture over space-time $\mathcal{X}$ is the source projection $\pi^1 = \pi^1_\circ \circ \pi : J^1\mathfrak{Z} \to \mathcal{X}$ - see Appendix (C). The Lagrangian density is equivalently described as a π^1-semibasic n-form on $J^1\mathfrak{Z}$ such that $\mathcal{L} = L(\pi^{1\star}\beta)$, thus the Lagrangian function is defined: $L \in C^\infty(J^1\mathfrak{Z})$.

The analogue of the tangent space for mechanics is the first order jet bundle $J^1\mathfrak{Z}$ with local coordinates (x^μ, z^i, z^i_μ). The bundle $J^1\mathfrak{Z} \to \mathfrak{Z}$ is an *affine* bundle. On the other hand, the analogue of the cotangent bundle is the *dual* first order jet bundle, denoted $J^1\mathfrak{Z}^\star$. It is the *vector* bundle over $\mathfrak{Z}$ with fiber at $z \in \mathfrak{Z}_x$ given by the set of *affine* maps $J^1_z\mathfrak{Z} \to \Lambda^n T^\star_x \mathcal{X}$. Fiber coordinates on $J^1\mathfrak{Z}^\star$, defined via the affine map $z^i_\mu \mapsto (\mathfrak{e} + p^\mu_i z^i_\mu)\beta$, are denoted $(\mathfrak{e}, p^\mu_i)$. Notice that $\dim(J^1_z\mathfrak{Z}^\star) = \dim(J^1\mathfrak{Z}) + 1$. We refer to M. Forger and S.V. Romero [80] for a clear introduction to the issue of *affine* space and the concept of *duality* for affine space in the jet landscape - see also M.J. Gotay [98]. The bundle $\mathbf{Z}(\mathfrak{Z}) = \Lambda^n_2 T^\star\mathfrak{Z} \subset \Lambda^n T^\star \mathfrak{Z}$ (see appendix (464)) plays the role of the multiphase space:

$$\begin{array}{c} \mathbf{Z}(\mathfrak{Z}) = \Lambda^n_2 T^\star \mathfrak{Z} \\ {\scriptstyle \pi_{\mathbf{Z}(\mathfrak{Z})}} \downarrow \ \curvearrowright \\ \mathfrak{Z} \end{array} \tag{72}$$

In this context, $\boldsymbol{\varphi} \in \mathbf{Z}(\mathfrak{Z})$ is written as $\boldsymbol{\varphi} = \mathfrak{e}\beta + p^\mu_i dz^i \wedge \beta_\mu$ and local coordinates on $\mathbf{Z}(\mathfrak{Z})$ are $(x^\mu, z^i, \mathfrak{e}, p^\mu_i)$. We introduce the projection $\mathfrak{e} : \mathbf{Z}(\mathfrak{Z}) \longrightarrow \mathbf{P}(\mathfrak{Z}) : (x^\mu, z^i, \mathfrak{e}, p^\mu_i) \mapsto (x^\mu, z^i, p^\mu_i)$. Notice that $\mathfrak{e}\beta \in \omega^n_\circ(\mathfrak{Z})$ is a horizontal n-form whereas $p^\mu_i dz^i \wedge \beta_\mu \in \omega^{n-1}_1(\mathfrak{Z})$ is a $(n-1)$-horizontal n-form. The key point concerns the identification of fiber coordinates between $J^1\mathfrak{Z}^\star$ and $\Lambda^n_2 T^\star\mathfrak{Z}$. In such a context, we have a vector bundle isomorphism $J^1\mathfrak{Z}^\star \cong \Lambda^n_2 T^\star \mathfrak{Z}$ - see [18, 38, 96, 98] - denoted $\alpha : \Lambda^n_2 T^\star \mathfrak{Z} \to J^1\mathfrak{Z}^\star$.
Let $\boldsymbol{\sigma} \in J^1\mathfrak{Z}$ so that $\boldsymbol{\sigma} = j^1\sigma(x)$ for a given section $\sigma : \mathcal{X} \to J^1\mathfrak{Z}$.[40] The canonical isomorphism $\alpha : \mathbf{Z}(\mathfrak{Z}) = \Lambda^n_2 T^\star \mathfrak{Z} \to J^1\mathfrak{Z}^\star$ is given for any $\boldsymbol{\varphi} \in \Lambda^n_2 T^\star(\mathfrak{Z})$ by $\alpha(\boldsymbol{\varphi}) \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma}^\star \boldsymbol{\varphi} \in \Lambda^n T^\star_x \mathcal{X}$. Consider the local expression of a form $\boldsymbol{\varphi} = \mathfrak{e}\beta + p^\mu_i dz^i \wedge \beta_\mu$. Since $\boldsymbol{\sigma}^\star(dx^\mu) = dx^\mu$ and $\boldsymbol{\sigma}^\star(dz^i) = \boldsymbol{\sigma}^i_\mu dx^\mu$, the expression for the pullback of $\boldsymbol{\varphi}$ by the prolongation of the section $\boldsymbol{\sigma}$ is given by: $\boldsymbol{\sigma}^\star\boldsymbol{\varphi} = \boldsymbol{\sigma}^\star(\mathfrak{e}\beta + p^\mu_i dz^i \wedge \beta_\mu) = (\mathfrak{e} + p^\mu_i \boldsymbol{\sigma}^i_\mu)\beta$. Hence we identify the affine dual of the first jet bundle of sections $J^1_z\mathfrak{Z}^\star$ with the previously introduced multisymplectic De Donder-Weyl space $\mathbf{Z}(\mathfrak{Z}) = \Lambda^n_2 T^\star \mathfrak{Z}$.

5.2 Canonical Forms and contact structure

We define the *Poincaré-Cartan canonical* n-form θ (45) and its associated canonical multisymplectic form $\boldsymbol{\omega} = d\theta$ (47) on $\Lambda^n T^\star \mathfrak{Z}$. Now, denoting the inclusion mapping $\mathfrak{i} : \Lambda^n T^\star \mathfrak{Z} \to \mathbf{Z}(\mathfrak{Z}) = \Lambda^n_2 T^\star \mathfrak{Z}$ so that $\boldsymbol{\omega}^{\mathsf{DW}} = \mathfrak{i}^\star \boldsymbol{\omega}$ and $\theta^{\mathsf{DW}} = \mathfrak{i}^\star\theta$, we recover the local coordinates for $\boldsymbol{\omega}^{\mathsf{DW}}$ and for θ^{DW}.

$$\theta^{\mathsf{DW}} = \mathfrak{e}\beta + p^\mu_i dz^i \wedge \beta_\mu \qquad\qquad \boldsymbol{\omega}^{\mathsf{DW}} = d\mathfrak{e} \wedge \beta + dp^\mu_i \wedge dz^i \wedge \beta_\mu \tag{73}$$

The Legendre morphism[41] is denoted $\overline{\mathbb{F}}(\mathcal{L}) : J^1\mathfrak{Z} \longrightarrow \mathbf{P}(\mathfrak{Z})$ and is defined by means of the Legendre transform, defining momenta as functions of fields and their first derivatives, via the mapping:

$$\overline{\mathbb{F}}(\mathcal{L}) : (x^\mu, z^i, z^i_\mu) \mapsto \big(x^\mu, z^i, p^\mu_i\big) = \Big(x^\mu, z^i(x^\mu), \frac{\partial \mathcal{L}}{\partial z^i_\mu}(x^\mu, z^i, z^i_\mu)\Big).$$

[40] The correspondence between $\boldsymbol{\sigma}$ and $T_x\sigma$ is described by an isomorphism of the bundle $J^1\mathfrak{Z} \to \mathfrak{Z}$ with the *affine* bundle whose typical fiber over $\sigma(x)$ is $\big\{\boldsymbol{\sigma} : T_x\mathcal{X} \longrightarrow T_{\sigma(x)}\mathfrak{Z} \ / \ \pi^1_\star \circ \boldsymbol{\sigma} = \mathbf{Id}_{T_x\mathcal{X}}\big\}$

[41] It is sometimes also called the Poincaré-Cartan morphism depending on authors

The *extended* covariant Legendre transform $\mathbb{F}(\mathcal{L}) : J^1\mathfrak{Z} \to \mathbf{Z}(\mathfrak{Z})$ is a fiber-preserving morphism locally described by:

$$\mathbb{F}(\mathcal{L}) = (x^\mu, z^i, z^i_\mu) \mapsto \big(x^\mu, z^i(x^\mu), \mathfrak{e}, p^\mu_i\big) = \Big(x^\mu, z^i, -(z^i_\mu \frac{\partial \mathcal{L}}{\partial z^i_\mu} - \mathcal{L}), \frac{\partial \mathcal{L}}{\partial z^i_\mu}\Big).$$

Thanks to the projection $\mathfrak{e} : \mathbf{Z}(\mathfrak{Z}) \longrightarrow \mathbf{P}(\mathfrak{Z}) : (x^\mu, z^i, \mathfrak{e}, p^\mu_i) \mapsto (x^\mu, z^i, p^\mu_i)$, the bundle $\mathbf{Z}(\mathfrak{Z})$ fibers over $\mathbf{P}(\mathfrak{Z})$. The Poincaré-Cartan n-form on $J^1\mathfrak{Z}$ is defined intrinsically by $\theta_{\mathcal{L}} = (\mathbb{F}\mathcal{L})^\star \theta^{\mathsf{DW}}$:

$$\theta_{\mathcal{L}} = (\mathbb{F}\mathcal{L})^\star\big(\mathfrak{e}\beta + p^\mu_i dz^i \wedge \beta_\mu\big) = -(z^i_\mu \frac{\partial \mathcal{L}}{\partial z^i_\mu} - \mathcal{L})\beta + \frac{\partial \mathcal{L}}{\partial z^i_\mu} dz^i \wedge \beta_\mu = (\mathcal{L} - z^i_\mu \frac{\partial \mathcal{L}}{\partial z^i_\mu})\beta + \frac{\partial \mathcal{L}}{\partial z^i_\mu} dz^i \wedge \beta_\mu$$

We write the Cartan form by means of contact basis forms ϑ^i - see Appendix (C).

Proposition 5.1. *The Cartan canonical form $\theta_{\mathcal{L}} = (\mathbb{F}\mathcal{L})^\star \theta^{DW}$ on $J^1\mathfrak{Z}$ is given by means of the contact forms ϑ^i: $\theta_{\mathcal{L}} = \mathcal{L}\beta + p^\mu_i \vartheta^i \wedge \beta_\mu$.*

⌈ Proof We have the straightforward calculation:

$$\theta_{\mathcal{L}} = \mathcal{L}\beta + (\frac{\partial \mathcal{L}}{\partial z^i_\mu})(dz^i \wedge \beta_\mu - z^i_\mu \beta) = \mathcal{L}\beta + (\frac{\partial \mathcal{L}}{\partial z^i_\mu})(dz^i \wedge \beta_\mu - z^i_\nu \delta^\nu_\mu \beta) = \mathcal{L}\beta + (\frac{\partial \mathcal{L}}{\partial z^i_\mu})(dz^i - z^i_\nu dx^\nu) \wedge \beta_\mu$$

Since $dx^\nu \wedge \beta_\mu = \delta^\nu_\mu \beta$. Then: $\theta_{\mathcal{L}} = \mathcal{L}\beta + (\frac{\partial \mathcal{L}}{\partial z^i_\mu})\vartheta^i \wedge \beta_\mu$. The Poincaré-Cartan form is written as $\theta_{\mathcal{L}} = \mathcal{L}\beta + p^\mu_i \vartheta^i \wedge \beta_\mu$. Notice that, since $d_{\mathbf{V}} z^i = \vartheta^i$ we write $\theta_{\mathcal{L}} = \mathcal{L}\beta + p^\mu_i (d_{\mathbf{V}} z^i) \wedge \beta_\mu$. ⌋

The canonical $(n+1)$-form $\boldsymbol{\omega}_{\mathcal{L}} \in \omega^{n+1}(J^1\mathfrak{Z})$ is defined intrinsically way by $\boldsymbol{\omega}_{\mathcal{L}} = (\mathbb{F}\mathcal{L})^\star \boldsymbol{\omega}^{\mathsf{DW}}$ We have the Cartan form in local coordinates:

$$\theta_{\mathcal{L}} = (\mathcal{L} - z^i_\mu \frac{\partial \mathcal{L}}{\partial z^i_\mu})\beta + \frac{\partial \mathcal{L}}{\partial z^i_\mu} dz^i \wedge \beta_\mu$$

so that we obtain for the canonical $(n+1)$-form:

$$\boldsymbol{\omega}_{\mathcal{L}} = d\theta_{\mathcal{L}} = d\big(\mathcal{L} - z^i_\mu \frac{\partial \mathcal{L}}{\partial z^i_\mu}\big) \wedge \beta + d(\frac{\partial \mathcal{L}}{\partial z^i_\mu}) \wedge dz^i \wedge \beta_\mu \tag{74}$$

We exhibit two subsequent formulations for (74). The first one simply expands (74) whereas the second is given by means of contact forms. The canonical $(n+1)$-form $\boldsymbol{\omega}_{\mathcal{L}}$ is written:

$$\boldsymbol{\omega}_{\mathcal{L}} = \Big(\frac{\partial \mathcal{L}}{\partial z^i} - z^i_\mu \frac{\partial^2 \mathcal{L}}{\partial z^j \partial z^i_\mu} - \frac{\partial^2 \mathcal{L}}{\partial x^\mu \partial z^i_\mu}\Big) dz^i \wedge \beta$$
$$-\Big(z^i_\mu \frac{\partial^2 \mathcal{L}}{\partial z^j_\nu \partial z^i_\mu}\Big) dz^j_\nu \wedge \beta + \Big(\frac{\partial^2 \mathcal{L}}{\partial z^j \partial z^i_\mu}\Big) dz^j \wedge dz^i \wedge \beta_\mu + \Big(\frac{\partial^2 \mathcal{L}}{\partial z^j_\nu \partial z^i_\mu}\Big) dz^j_\nu \wedge dz^i \wedge \beta_\mu$$

⌈ Proof First we just expand (74) this gives the straightforward calculation:

$$\begin{aligned}
\boldsymbol{\omega}_{\mathcal{L}} &= \frac{\partial \mathcal{L}}{\partial z^i} dz^i \wedge \beta + \frac{\partial \mathcal{L}}{\partial z^i_\mu} dz^i_\mu \wedge \beta - \big(dz^i_\mu \frac{\partial \mathcal{L}}{\partial z^i_\mu} + z^i_\mu d(\frac{\partial \mathcal{L}}{\partial z^i_\mu})\big) \wedge \beta + d(\frac{\partial \mathcal{L}}{\partial z^i_\mu}) \wedge dz^i \wedge \beta_\mu \\
&= \frac{\partial \mathcal{L}}{\partial z^i} dz^i \wedge \beta - z^i_\mu d(\frac{\partial \mathcal{L}}{\partial z^i_\mu}) \wedge \beta + d(\frac{\partial \mathcal{L}}{\partial z^i_\mu}) \wedge dz^i \wedge \beta_\mu \\
&= \frac{\partial \mathcal{L}}{\partial z^i} dz^i \wedge \beta - z^i_\mu \Big(\frac{\partial^2 \mathcal{L}}{\partial z^j \partial z^i_\mu} dz^j + \frac{\partial^2 \mathcal{L}}{\partial z^j_\nu \partial z^i_\mu} dz^j_\nu\Big) \wedge \beta \\
&\quad + \Big(\frac{\partial^2 \mathcal{L}}{\partial x^\nu \partial z^i_\mu} dx^\nu + \frac{\partial^2 \mathcal{L}}{\partial z^j \partial z^i_\mu} dz^j + \frac{\partial^2 \mathcal{L}}{\partial z^j_\nu \partial z^i_\mu} dz^j_\nu\Big) \wedge dz^i \wedge \beta_\mu \\
&= \Big(\frac{\partial \mathcal{L}}{\partial z^i} - z^i_\mu \frac{\partial^2 \mathcal{L}}{\partial z^j \partial z^i_\mu} - \frac{\partial^2 \mathcal{L}}{\partial x^\mu \partial z^i_\mu}\Big) dz^i \wedge \beta - \Big(z^i_\mu \frac{\partial^2 \mathcal{L}}{\partial z^j_\nu \partial z^i_\mu}\Big) dz^j_\nu \wedge \beta + \Big(\frac{\partial^2 \mathcal{L}}{\partial z^j \partial z^i_\mu}\Big) dz^j \wedge dz^i \wedge \beta_\mu \\
&\quad + \Big(\frac{\partial^2 \mathcal{L}}{\partial z^j_\nu \partial z^i_\mu}\Big) dz^j_\nu \wedge dz^i \wedge \beta_\mu
\end{aligned}$$
⌋

We also express the canonical $n+1$-form thanks to the help of basis contact form ϑ^i:

$$\boldsymbol{\omega}_{\mathcal{L}} = \vartheta^i \wedge \Big(\frac{\partial \mathcal{L}}{\partial z^i}\beta - d(\frac{\partial \mathcal{L}}{\partial z^i_\mu}) \wedge \beta_\mu\Big)$$

⌈ Proof We have the straightforward calculation:

$$\begin{aligned}
\boldsymbol{\omega}_{\mathcal{L}} &= \frac{\partial \mathcal{L}}{\partial z^i} dz^i \wedge \beta - \delta^\nu_\mu z^i_\nu d(\frac{\partial \mathcal{L}}{\partial z^i_\mu})) \wedge \beta - dz^i \wedge d(\frac{\partial \mathcal{L}}{\partial z^i_\mu}) \wedge \beta_\mu \\
&= \frac{\partial \mathcal{L}}{\partial z^i} dz^i \wedge \beta - z^i_\nu d(\frac{\partial \mathcal{L}}{\partial z^i_\mu})) \wedge (dx^\nu \wedge \beta_\mu) - dz^i \wedge d(\frac{\partial \mathcal{L}}{\partial z^i_\mu}) \wedge \beta_\mu \\
&= \frac{\partial \mathcal{L}}{\partial z^i} dz^i \wedge \beta - \big(dz^i - z^i_\nu dx^\nu\big) \wedge d(\frac{\partial \mathcal{L}}{\partial z^i_\mu}) \wedge \beta_\mu \\
&= \frac{\partial \mathcal{L}}{\partial z^i} dz^i \wedge \beta - \vartheta^i \wedge d(\frac{\partial \mathcal{L}}{\partial z^i_\mu}) \wedge \beta_\mu = \vartheta^i \wedge \Big(\frac{\partial \mathcal{L}}{\partial z^i}\beta - d(\frac{\partial \mathcal{L}}{\partial z^i_\mu}) \wedge \beta_\mu\Big)
\end{aligned}$$

Since, $\vartheta^i \wedge \frac{\partial \mathcal{L}}{\partial z^i}\beta = (dz^i - z^i_\mu dx^\mu) \wedge \frac{\partial \mathcal{L}}{\partial z^i}\beta = \frac{\partial \mathcal{L}}{\partial z^i} dz^i \wedge \beta$. ⌋

For a presentation of the multisymplectic jet bundle setting see [26, 35, 38, 58, 77, 78, 91, 92, 96, 97, 98, 102, 132, 133, 205, 206, 207, 208]. Here the point of interest concerns algebraic graded structure. In the following section, we first describe Hamiltonian $(n-1)$-forms and their related Hamiltonian vector fields, and secondly, Hamiltonian $(p-1)$-forms and related vector fields. Poisson structure for arbitrary degree is not uniquely defined.

Remark. *As in the previous section for symplectic geometry,* Hamiltonian 0*-forms are related to Hamiltonian vector fields, and for the rest of the present section we speak about* Hamiltonian $(n-1)$*-forms. Later, we call such object* algebraic observable $(n-1)$*-forms. This offers insight into the classification of observables, an insight which rests on a clearer epistemological grasp of the status of observables in our general understanding of physical representation.*

5.3 Hamiltonian $(n-1)$-forms and Hamiltonian vector fields

The graded commutative algebra, $\mathfrak{X}^\star(\mathcal{M}) = \bigoplus_{0\leq k\leq \dim(\mathcal{M})} \Lambda^k(\mathfrak{X}(\mathcal{M}))$ is considered. $\mathfrak{X}^\star(\mathcal{M})$ is endowed with a Gerstenhaber algebra when we consider the Schouten bracket[42] structure. Let $\boldsymbol{\varphi} \in \omega^\star(\mathcal{M})$ and let $X = X_1 \wedge \cdots \wedge X_n \in \mathbf{D}^n(\mathcal{M})$, the interior product $X \lrcorner \boldsymbol{\varphi}$ is given by $X \lrcorner \boldsymbol{\varphi} = (X_1 \wedge \cdots \wedge X_n) \lrcorner \boldsymbol{\varphi} = X_n \lrcorner \cdots \lrcorner X_1 \lrcorner \boldsymbol{\varphi}$. Note that we extend the interior product on arbitrary (not necessarily decomposable) multivector fields by $C^\infty(\mathcal{M})$-linearity. Now we introduce the notion of Hamiltonian form.

Definition 5.3.1. *Let $(\mathcal{M},\boldsymbol{\omega})$ be an n-multisymplectic manifold. An $(n-1)$-form $\boldsymbol{\varphi}$ is Hamiltonian if and only if there exist a vector field $\Xi_{\boldsymbol{\varphi}} \in \mathfrak{X}(\mathcal{M})$ such that $d\boldsymbol{\varphi} + \Xi_{\boldsymbol{\varphi}} \lrcorner \boldsymbol{\omega} = 0$.*

The important point is that we use to the adjective *Hamiltonian* when a form is described *together with* an associated vector field $\Xi_{\boldsymbol{\varphi}} \in \Gamma(\mathcal{M}, T\mathcal{M})$[43]. For the moment we keep this terminology from the symplectic setting as a point of departure for understanding general cases. Let $\mathfrak{X}^1_{\mathsf{ham}}(\mathcal{M})$ and $\Omega^{n-1}_{\mathsf{ham}}(\mathcal{M})$ be respectively the set of Hamiltonian vector fields and the set of Hamiltonian $(n-1)$-forms on $(\mathcal{M},\boldsymbol{\omega})$: both are vector spaces. We introduce the Lie derivative $\mathcal{L}_\Xi$ of $\boldsymbol{\varphi} \in \omega^\star(\mathcal{M})$ along a multivector field $\Xi \in \mathfrak{X}^\star(\mathcal{M})$ defined via the graded commutator:

$$\mathcal{L}_\Xi \boldsymbol{\varphi} = d(\Xi \lrcorner \boldsymbol{\varphi}) - (-1)^{|\Xi|}\Xi \lrcorner d\boldsymbol{\varphi}.$$

In the case where $\Xi \in \mathfrak{X}^1(\mathcal{M}) = \Gamma(\mathcal{M}, T\mathcal{M})$ is a 1-vector field $\mathcal{L}_\Xi \boldsymbol{\varphi} = d(\Xi \lrcorner \boldsymbol{\varphi}) + \Xi \lrcorner d\boldsymbol{\varphi}$. The first obvious property is that a Hamiltonian vector field $\Xi_{\boldsymbol{\varphi}} \in \mathfrak{X}^1_{\mathsf{ham}}(\mathcal{M})$ preserve the multisymplectic

[42] a -1 degree Lie bracket which satisfies the graded Leibniz rule with respect to the wedge product.

[43] Notice that this leads to the definition of *algebraic observable* $(n-1)$*-form* in the next section

structure. $\mathcal{L}_{\Xi_{\varphi}}\boldsymbol{\omega} = 0$:

$$\mathcal{L}_{\Xi_{\varphi}}\boldsymbol{\omega} = d(\Xi_{\varphi} \lrcorner \boldsymbol{\omega}) + \Xi_{\varphi} \lrcorner d\boldsymbol{\omega} = d(\Xi_{\varphi} \lrcorner \boldsymbol{\omega}) = -d(d\boldsymbol{\varphi}) = 0$$

So that we have: $\mathfrak{X}^1_{\mathsf{ham}}(\mathcal{M}) \subset \mathfrak{X}_{\mathsf{sym}}(\mathcal{M})$

Definition 5.3.2. *The Poisson Bracket on two Hamiltonian $(n-1)$-forms $\boldsymbol{\varphi}, \boldsymbol{\rho} \in \Omega^{n-1}_{ham}(\mathcal{M})$ is*

$$\big\{\boldsymbol{\varphi}, \boldsymbol{\rho}\big\} = \Xi_{\varphi} \wedge \Xi_{\rho} \lrcorner \boldsymbol{\omega} = \Xi_{\varphi} \lrcorner d\boldsymbol{\rho} = -\Xi_{\rho} \lrcorner d\boldsymbol{\varphi} \tag{75}$$

The bracket (75) satisfies the antisymmetry property as in the symplectic case.

$$\big\{\boldsymbol{\varphi}, \boldsymbol{\rho}\big\} + \big\{\boldsymbol{\rho}, \boldsymbol{\varphi}\big\} = 0 \tag{76}$$

We notice that the bracket $\big\{\boldsymbol{\varphi}, \boldsymbol{\varrho}\big\}$ of two Hamiltonian $(n-1)$-forms $\boldsymbol{\varphi}, \boldsymbol{\varrho} \in \omega^{n-1}_{\mathsf{ham}}(\mathcal{M})$ is Hamiltonian, $\big\{\boldsymbol{\varphi}, \boldsymbol{\varrho}\big\} \in \Omega^{n-1}_{\mathsf{ham}}(\mathcal{M})$. We have the straightforward calculation:

$$d\big\{\boldsymbol{\varphi}, \boldsymbol{\varrho}\big\} = d(\Xi_{\varphi} \lrcorner \Xi_{\varrho} \lrcorner \boldsymbol{\omega}) = \mathcal{L}_{\Xi_{\varrho}}(\Xi_{\varphi} \lrcorner \boldsymbol{\omega}) + \Xi_{\varrho} \lrcorner d(d\boldsymbol{\rho}) = [\Xi_{\varrho}, \Xi_{\varphi}] \lrcorner \boldsymbol{\omega} + \Xi_{\varphi} \lrcorner \mathcal{L}_{\Xi_{\varrho}}\boldsymbol{\omega}$$

Therefore, we obtain:

$$\big\{\boldsymbol{\varphi}, \boldsymbol{\rho}\big\} = -[\Xi_{\varphi}, \Xi_{\varrho}] \lrcorner \boldsymbol{\omega} \tag{77}$$

The relation (77) can equivalently be written $\Xi_{\{\varphi,\rho\}} = [\Xi_{\varphi}, \Xi_{\varrho}]$. However, unlike the case of symplectic geometry ($n = 1$), the bracket satisfies Jacobi structure only modulo an exact term. Hence, $\forall \boldsymbol{\varphi}, \boldsymbol{\varrho}, \boldsymbol{\eta} \in \Omega^{n-1}_{\mathsf{ham}}(\mathcal{M})$ we have:

$$\big\{\{\boldsymbol{\varphi}, \boldsymbol{\varrho}\}, \boldsymbol{\eta}\big\} + \big\{\{\boldsymbol{\varrho}, \boldsymbol{\eta}\}, \boldsymbol{\varphi}\big\} + \big\{\{\boldsymbol{\eta}, \boldsymbol{\varphi}\}, \boldsymbol{\varrho}\big\} = d(\Xi_{\varphi} \wedge \Xi_{\varrho} \wedge \Xi_{\eta} \lrcorner \boldsymbol{\omega}) \tag{78}$$

We refer to the work of Baez, and C.L. Rogers [19] for Lie 2-algebra structure, to the thesis of Rogers [196] for relation with Deligne cohomology and Courant algebroids and to Baez and U. Schreiber [20], for relations with Category theory, to offer some selected examples of connections between MG and more mathematical perspective. One of the essential questions for MG concerns a good choice of Poisson bracket structure in the case of forms of lower degree. Notice that if we focus on what we refer as the *symmetry-algebraic* picture, as reflected in (75) to (77) above, for the case of forms of arbitrary degree the link between Poisson brackets and dynamics is not clear - by contrast to the $(n-1)$-form case. More precisely, for $1 \leq p \leq n$ the $(n+1-p)$ multivector field Ξ_{φ} such that $d\boldsymbol{\varphi} + \Xi_{\varphi} \lrcorner \boldsymbol{\omega} = 0$ is not uniquely defined. It leads to a generalization of the Poisson bracket within the setting of graded structure [134, 188, 225]. The focus is on graded bracket generalization with objects such as the Schouten-Nijenhuis and the Frölicher-Nijenhuis brackets.[44] However the lack of good *dynamical* properties for such forms suggests that we should give much more considerations to the notion of OF developed in [117, 118]. In such a context, working on the basis of the Einstein picture, reflection on the very nature of observable forms led Hélein and Kouneiher [117, 118] to the notion of *copolar forms*. The notion of copolarisation allows us to define observable forms of any degree *collectively*. This feature, as we will see later, is in perfect harmony with the spirit of GR. Before examining this step, we first describe the traditional setting of graded structures in order to understand the legacy of OF.

[44]In those works bracket operations on Hamiltonian multivector fields and Hamiltonian forms capture graded Poisson structure built upon higher-order generalization of Gerstenhaber algebra. We refer to the work of P.W. Michor [173] and P.W. Michor and M. Dubois-Violette [174] for a presentation of the Frölicher-Nijenhuis bracket or the presentation of L.K Norris [185]

5.4 Hamiltonian $(p-1)$-forms and their related Hamiltonian vector fields

Here we follow the *symmetry-algebraic* standpoint. Following a similar underlying theme to that of the previous section *i.e* - namely the study of the notions of infinitesimal symplectomorphism $\mathfrak{X}_{\mathsf{sym}}(\mathcal{M})$, the set of Hamiltonian vector fields $\mathfrak{X}_{\mathsf{ham}}(\mathcal{M})$ and *locally* Hamiltonian vector fields $\mathfrak{X}_{\mathsf{loc}}(\mathcal{M})$ in the symplectic case $(n=1)$ - we now go on to compare the rules for forms of arbitrary degree. With this aim in view, we emphasize two main group of references. The first is found in the work of Kanatchikov [132, 133, 134, 135, 136, 137] where interesting ideas on the graded setting, in particular in connection with dynamical evolution for forms of lower degrees are developed. The second concerns the closely related work of Forger, C. Paufler and H. Römer [77, 78, 79, 80, 188].

Definition 5.4.1. *A p-multivector field* $\overset{p}{\Xi} \in \mathfrak{X}^p(\mathcal{M})$ *on a multisympletic manifold* $(\mathcal{M},\boldsymbol{\omega})$ *is a locally Hamiltonian p-multivector field if* $\overset{p}{\Xi} \lrcorner \boldsymbol{\omega}$ *is closed or equivalently if* $\mathcal{L}_{\overset{p}{\Xi}}\boldsymbol{\omega} = 0$*. We shall denote the set of locally Hamiltonian p-multivector fields by* $\mathfrak{X}^p_{\mathsf{loc}}(\mathcal{M})$*.*

Definition 5.4.2. *A p-multivector field* $\overset{p}{\Xi}$ *on a multisympletic manifold* $(\mathcal{M},\boldsymbol{\omega})$ *is a globally Hamiltonian p-multivector field if* $\overset{p}{\Xi} \lrcorner \boldsymbol{\omega}$ *is exact or equivalently if there exists an* $(n-p)$*-form* $\boldsymbol{\varphi}$ *on* $\mathcal{M}$ *such that* $\overset{p}{\Xi} \lrcorner \boldsymbol{\omega} = d\overset{n\text{-}p}{\boldsymbol{\varphi}}$*. We shall denote the set of Hamiltonian p-multivector fields by* $\mathfrak{X}^p_{\mathsf{ham}}(\mathcal{M})$*.*

The relation between vector field $\Xi \in \mathfrak{X}^{\mathrm{p}}_{\mathsf{ham}}(\mathcal{M})$ and $(n-p)$-form $\boldsymbol{\varphi} \in \Omega^{n-p}(\mathcal{M})$ is summarized:
Locally Hamiltonian p-vector fields

$$\left| \begin{array}{l} \overset{\mathrm{p}}{\Xi} \in \mathfrak{X}^{\mathrm{p}}_{\mathsf{loc}}(\mathcal{M}) \\ \overset{\mathrm{p}}{\Xi} \lrcorner \boldsymbol{\omega} \text{ is closed or } \mathcal{L}_{\overset{\mathrm{p}}{\Xi}}\boldsymbol{\omega} = 0 \end{array} \right.$$

Hamiltonian p-vector fields

$$\left| \begin{array}{l} \overset{\mathrm{p}}{\Xi} \in \mathfrak{X}^{\mathrm{p}}_{\mathsf{ham}}(\mathcal{M}) \\ \overset{\mathrm{p}}{\Xi} \lrcorner \boldsymbol{\omega} \text{ is exact: there exists an } (n-1)\text{-form } \boldsymbol{\varphi} \text{ on } \mathcal{M} \text{ such that } \overset{\mathrm{p}}{\Xi} \lrcorner \boldsymbol{\omega} = d\overset{\text{n-p}}{\boldsymbol{\varphi}} \end{array} \right.$$

Definition 5.4.3. *A* $n-r$*-form* $\overset{n-r}{\boldsymbol{\varphi}}$ *on a multisympletic manifold* $(\mathcal{M},\boldsymbol{\omega})$ *is a Hamiltonian form if there exists a r-multivector field* $\overset{r}{X}_{\boldsymbol{\varphi}}$ *such that* $\overset{r}{X}_{\boldsymbol{\varphi}} \lrcorner \boldsymbol{\omega} = d\overset{n-r}{\boldsymbol{\varphi}}$*.*

The equation under consideration is $\overset{r}{X}_{\boldsymbol{\varphi}} \lrcorner \boldsymbol{\omega} = d\overset{n-r}{\boldsymbol{\varphi}}$ so we say that the Hamiltonian multivector field $\overset{r}{X}_{\boldsymbol{\varphi}}$ is associated with the Hamiltonian form $\overset{n-r}{\boldsymbol{\varphi}}$. Neither $\overset{r}{X}_{\boldsymbol{\varphi}}$ nor $\overset{n-r}{\boldsymbol{\varphi}}$ are uniquely defined. Equivalently, the kernel of $\boldsymbol{\omega}$ on multivector fields is non-trivial. This simple fact reflects a non unique correspondence between Hamiltonian multivector fields and Hamiltonian forms. [77, 78]. The following table exhibits the parallel between the symplectic case $n=1$ and the higher dimensional case:

Symplectic manifold $(\mathcal{M},\omega)$	Multisymplectic manifold $(\mathcal{M},\boldsymbol{\omega})$
$\xi_\varphi \lrcorner \omega = -d\varphi$	$\Xi_{\boldsymbol{\varphi}} \lrcorner \boldsymbol{\omega} = d\boldsymbol{\varphi}$
ξ_φ is uniquely defined by $d\varphi$	$\Xi_{\boldsymbol{\varphi}}$ is not uniquely defined by $d\boldsymbol{\varphi}$
Every $0-$ form φ is Hamiltonian	Not every $(n-r)-$ form $\boldsymbol{\varphi}$ is Hamiltonian

5.5 A glimpse of Poisson structure in the graded standpoint

We follow the setting developed by Kanatchikov [132, 133, 134, 137] and his polymomentum phase space defined on the bundle $\mathfrak{Z} \xrightarrow{\pi} \mathcal{X}$ over the manifold $\mathcal{X}$. In this work we encounter various objects such as the vertical vector $X^{\mathbf{V}}$ - an element of the vertical tangent bundle $\mathbf{V}T\mathfrak{Z}$ - vertical p-multivector fields $\overset{p}{X}_{\boldsymbol{\varphi}}$, vertical $(n-p)$-multivector fields $\overset{n-p}{X}_{\boldsymbol{\varphi}}$. The central algebraic object for graded structure is the Schouten-Nijenhuis bracket $\big[,\big]_{\text{SN}}$, first discover by Schouten [209, 210] and later developed by Nijenhuis [182, 183]. The Schouten-Nijenhuis bracket is a $\mathbb{R}$-bilinear map: $\mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \longrightarrow \mathfrak{X}(\mathcal{M})$ which obeys the following properties:

[1] Graded antisymmetric property: $\big[Y, X\big]_{\text{SN}} = -(-1)^{(p-1)(q-1)}\big[X, Y\big]_{\text{SN}}$.

[2] Graded Leibniz rule: $\big[X, Y \wedge Z\big]_{\text{SN}} = \big[X, Y\big]_{\text{SN}} \wedge Z - (-1)^{(p-1)(q)} Y \wedge \big[X, Z\big]_{\text{SN}}$.

[3] Graded Jacobi identity:

$$(-1)^{(p-1)(r-1)} Y \wedge \big[X, \big[Y, Z\big]_{\text{SN}}\big]_{\text{SN}} + (-1)^{(q-1)(r-1)} Y \wedge \big[Z, \big[X, Y\big]_{\text{SN}}\big]_{\text{SN}} + (-1)^{(p-1)(q-1)} Y \wedge \big[Y, \big[Z, X\big]_{\text{SN}}\big]_{\text{SN}} = 0.$$

The Schouten-Nijenhuis bracket coincides with the standard Lie bracket on vector fields. Let $\mid X \mid = p$ be the degree of the multivector field $X \in \mathfrak{X}^p(\mathcal{M})$ so that we considered two multivector fields $X, Y \in \mathfrak{X}^\star(\mathcal{M})$ with $\mid X \mid = p, \mid Y \mid = q$. Then, $\mid \big[X, Y\big]_{\text{SN}} \mid = p + q - 1$. For detailed studies on the Schouten-Nijenhuis bracket we refer to J.-L. Koszul [152] - who studied it from the cohomological perspective - or C.M. Marle [164] - for its relation with the interior product. The graded structure appears therefore as the bracket of two Hamiltonian forms - here we fix the degree of the Hamiltonian vector fields as: $\overset{p}{X}_{\boldsymbol{\varphi}}$ and $\overset{q}{X}_{\boldsymbol{\varphi}}$ so that, following Kanatchicov we denote $r = n - p$ and $s = n - q$,

$$\big\{\overset{r}{\boldsymbol{\varphi}}, \overset{s}{\boldsymbol{\varrho}}\big\} = (-1)^{n-r} \overset{p}{X}_{\boldsymbol{\varphi}} \lrcorner \overset{q}{X}_{\boldsymbol{\varrho}} \lrcorner \boldsymbol{\omega} = (-1)^{n-r} \overset{p}{X}_{\boldsymbol{\varphi}} \lrcorner d\overset{s}{\boldsymbol{\varrho}}. \tag{79}$$

The Poisson bracket (79) has the following properties:

[1] The first property, that $\mid \boldsymbol{\varphi} \mid = n - 1 - p$, $\mid \boldsymbol{\varrho} \mid = n - 1 - q$ and $\mid \boldsymbol{\varphi} \mid = n - 1 - r$ is satisfied by the graded antisymmetry of the bracket:

$$\big\{\overset{r}{\boldsymbol{\varphi}}, \overset{s}{\boldsymbol{\varrho}}\big\} = -(-1)^{(n-r-1)(n-s-1)}\big\{\overset{s}{\boldsymbol{\varrho}}, \overset{r}{\boldsymbol{\varphi}}\big\}.$$

[2] Graded Jacobi Identity:

$$(-1)^{|\boldsymbol{\varphi}||\boldsymbol{\eta}|}\big\{\overset{p}{\boldsymbol{\varphi}}\big\{\overset{q}{\boldsymbol{\varrho}}, \overset{r}{\boldsymbol{\eta}}\big\}\big\} + (-1)^{|\boldsymbol{\varrho}||\boldsymbol{\varphi}|}\big\{\overset{p}{\boldsymbol{\varrho}}\big\{\overset{q}{\boldsymbol{\eta}}, \overset{r}{\boldsymbol{\varphi}}\big\}\big\} + (-1)^{|\boldsymbol{\eta}||\boldsymbol{\varrho}|}\big\{\overset{p}{\boldsymbol{\eta}}\big\{\overset{q}{\boldsymbol{\varphi}}, \overset{r}{\boldsymbol{\varrho}}\big\}\big\} = 0.$$

As emphasized by Kanatchikov [137] the Poisson bracket of Hamiltonian forms (79) is induced by the Schouten-Nijenhuis bracket $\big[,\big]_{\text{SN}}$ of the Hamiltonian multivector fields $-d\big\{\overset{r}{\boldsymbol{\varphi}}, \overset{s}{\boldsymbol{\varrho}}\big\} = \big[\overset{p}{X}_{\boldsymbol{\varphi}}, \overset{q}{X}_{\boldsymbol{\varrho}}\big]_{\text{SN}} \lrcorner \boldsymbol{\omega}$. The algebraic properties define a Poisson-Gerstenhaber algebra on the space of Hamiltonian forms. A Gerstenhaber algebra is an associative graded commutative algebra with an odd Poisson bracket [89, 90]. See the work of A. Nijenhuis [182, 183] and as emphasized by Y. Kosmann-Schwarzbach [151] - who situates the construction within the general setting of Loday and derived brackets -: *"the Schouten-Nijenhuis bracket of multivectors [...] is the prototypical example of a Gerstenhaber bracket."* We also refer to the work of S.P. Hrabak [123, 124] for a view of this SN bracket structure within the general framework of the Loday bracket. Kanatchikov's work is to be seen as leading to a generalization of the Poisson algebra of observables in classical mechanics to field theory. The natural setting for such a generalization is concerned with graded Leibniz algebra together with a graded derivation. Notice that the exterior product of two Hamiltonian forms $\overset{p}{\boldsymbol{\varphi}} \wedge \overset{q}{\boldsymbol{\varrho}}$ is not in general Hamiltonian. Once again the defect is that this structure is not uniquely defined for such forms of arbitrary degree. This motivates the construction of OF by Hélein and Kouneiher and opens the way for a *good* classification for observables from the viewpoint of our proposed ontologic *vs* dynamical duality.

6 Observables in multisymplectic geometry

We return now to consideration of the approach developed by Hélein and Kouneiher [115, 116, 117, 118] to the treatment of observable and the related issue of the Poisson bracket. While closely following their work, however, we seek to further its epistemological understanding and to encapsulate the guiding ideas in a general symbolic representation. The notion of observable in multisymplectic geometry can be approached from three directions. The first concerns the distinction between *ontologic* and *dynamical* features of the notion. In this first part we emphasize the description of observable forms from the viewpoint of that distinction. This in turn leads to a further distinction between *observable forms* (OF) and *algebraic observable forms* (AOF). The second of those three directions concerns observable functionals. These objects are defined by integration of differential forms on hypersurfaces and we observe that there are two main sorts of such functionals: the kinematical and the dynamical ones. Finally, the last direction concerns the concept of *copolarization*. This concept arises in connection with the definition of observable forms for arbitrary degree $(p-1)$. This is the cornerstone, at the heart of our investigation of the notion of *observable*, that reveals a surprising connection with a deeper understanding of the Relativity Principle. Indeed, the spirit of Einstein relativity leads us to recognize that no observable quantities are measured directly. We emphasize the concept of observable within the GR setting. It is meaningless if referred to an absolute frame. Indeed, observable quantities are only defined by comparison to each other. In this section (6) we discuss the mathematical consequences of this. But this discussion will take us much further. It leads towards deeper *underlying principles* concerning the representation of physical notions. Keeping in mind Dirac's insightful remark about the absence of a well established *theory of observables* in physics, we reach a natural and amazing conclusion. One summarized by this simple statement: *there is no real position of the observer.*

6.1 Ontologic *vs* Dynamical

The first fundamental distinction between observable forms concerns the primordial duality between *ontologic* and *dynamical* symmetry. We oppose two types of observables: *algebraic observables* vs *dynamical observables.* This dual point of view is taken fully into account even at the level of the definition of observable $(n-1)$-forms. This leads on to the distinction between OF and AOF and is also strongly connected to the notion of *copolarization.* Here we offer a broadly heuristic introduction. In the next section we give more mathematical details. First, let focus on *algebraic observables*: they are not dependent on the Hamiltonian function $\mathcal{H}$. These observables have an *ontologic* flavor. Recall that the *ontologic* symmetry $\mathcal{L}_{\Xi}\boldsymbol{\omega} = 0$ is described by the set of infinitesimal symplectomorphisms $\mathfrak{sp}_\circ(\mathcal{M})$. Here we are interested in what were previously called Hamiltonian forms, which we now rename *algebraic observables.* From this new perspective we are interested in the set of $(n-1)$-forms: $\mathfrak{P}^{n-1}_\circ(\mathcal{M}) = \{\boldsymbol{\varphi} \in \Lambda^{n-1}T^\star\mathfrak{Z}/\xi_{\boldsymbol{\varphi}} \lrcorner\, \boldsymbol{\omega} = -d\boldsymbol{\varphi}\}$. We summarize the idea developed here:

Ontologic space	Ontologic representation
$\mathfrak{sp}_\circ(\mathcal{M})$ set of inf. symplectomorphisms	$\mathfrak{P}^{n-1}_\circ(\mathcal{M})$ set of algebraic $(n-1)-$ forms
Symbolic vision: $\mathfrak{sp}_\circ(\mathcal{M})\leftrightarrows[\,\rhd\,]$	Symbolic vision: $\mathfrak{P}^{n-1}_\circ(\mathcal{M})\leftrightarrows[\,\lhd\,]$

Recall that the *dynamical* symmetry $\mathcal{L}_{\Xi}\mathcal{H} = 0$ forms part of the definition the symmetry of an *Hamiltonian system*. Therefore we look for vector fields $\Xi \in \Gamma(\mathcal{M}, T\mathcal{M})$ such that $d\mathcal{H}(\Xi) = 0$. When the infinitesimal symplectomorphisms satisfy the additional property $d\mathcal{H}(\Xi) = 0$[45] we call those vector fields *dynamical.* On the other side we are interested in *dynamical observables*, those which are defined via considerations on the Hamiltonian function $\mathcal{H}$. However the real objects of *purely* dynamical interest are given by Hamiltonian n-vector fields: those are the objects of the so-called *dynamical* space Γ. Here the main objects of interest are the Hamiltonian function and the Hamiltonian n-curves. We consider a Hamiltonian function $\mathcal{H} : \mathcal{M} \to \mathbb{R}$ and the related Hamiltonian n-vector field $X(m) \in [X]^{\mathcal{H}}_m$. We are looking for the set of $(n-1)$-forms $\boldsymbol{\varphi}$ such that $\langle X(m), d\boldsymbol{\varphi}_m\rangle$ does not depend on the choice of $X(m)$ but only on $d\mathcal{H}_m$. Therefore we describe the set of observable forms OF as $\forall X(m) \in [X]^{\mathcal{H}}_m$: $\mathfrak{P}^{n-1}_{\bullet} = \big\{\boldsymbol{\varphi} \in \Lambda^{n-1}T^{\star}\mathfrak{Z},\ /\ \langle X(m), d\boldsymbol{\varphi}_m\rangle \text{ depend only on } d\mathcal{H}_m\big\}$. We summarize the picture:

Dynamical space	Dynamical representation
$X(m) \in [X]^{\mathcal{H}}_m$ Hamiltonian n-vector fields Symbolic vision: $[X]^{\mathcal{H}}_m \leftrightarrows [\rhd]$	$\mathfrak{P}^{n-1}_{\bullet}(\mathcal{M})$ set of observable $(n-1)-$ forms Symbolic vision: $\mathfrak{P}^{n-1}_{\bullet}(\mathcal{M}) \leftrightarrows [\lhd]$

Finally, a *physical* - or *dynamical* observable[46] $\boldsymbol{\varphi}$ is an $(n-1)$-form which involves both aspects. It then respects both symmetries: the ontologic one and the dynamical one. We denote the set of physical observable $(n-1)$-forms by $\boldsymbol{\mathcal{O}}^{\mathcal{H}} = \big\{\boldsymbol{\varphi} \in \Lambda^{n-1}T^{\star}\mathcal{M}/\xi_{\boldsymbol{\varphi}} \lrcorner\, \boldsymbol{\omega} = -d\boldsymbol{\varphi} \quad \text{and} \quad d\mathcal{H}(\xi_{\boldsymbol{\varphi}}) = 0\big\}$.

The opposition of *ontologic* vs *dynamical* considerations points to the opposition of *symmetric* vs *dynamical* aspects for observables. This insight suggests a definition of the brackets between $(n-1)$-forms and later between canonical forms. On the one hand we describe the Poisson bracket between two different $\boldsymbol{\varphi}, \boldsymbol{\rho} \in \mathfrak{P}^{n-1}_{\circ}(\mathcal{M})$ algebraic $(n-1)$-forms whereas, on the other side, we describe the Poisson *pseudobracket* for an observable form $\boldsymbol{\varphi} \in \mathfrak{P}^{n-1}_{\bullet}(\mathcal{M})$. We emphasize once more this feature of the vocabulary: the *ontologic space* and the *dynamical space* are characterized by vector fields whereas the words *ontologic representation* and *dynamical representation* are related to forms. This is the sign of a deeper philosophical perspective, and is connected to the related concept of generalized *topological* and *dynamical* dualities. We need many more details before discussing this connection. They are provided in the works [115, 116, 117]. First we are interested, in section (6.2) in the *symmetry* standpoint and the definition of $(n-1)$-AOF so that we recover exactly the definition of Hamiltonian form $\boldsymbol{\varphi} \in \Omega^{n-1}_{\mathsf{ham}}(\mathcal{M})$ developed in the traditional picture - see the previous section (5.3). In the rest of this text, we adopt the definition of Hélein and Kouneiher [117]. In particular we replace the notation $\Omega^{n-1}_{\mathsf{ham}}(\mathcal{M})$ by $\mathfrak{P}^{n-1}_{\circ}(\mathcal{M})$. This provides a more accurate setting, one which clarifies the distinction between algebraic forms AOF and observable forms OF, denoted $\mathfrak{P}^{n-1}_{\bullet}(\mathcal{M})$.

6.2 Algebraic observable $(n-1)$-forms (AOF)

Definition 6.2.1. *Let* $(\mathcal{M}, \boldsymbol{\omega})$ *be an n-multisymplectic manifold. A* $(n-1)$*-form* $\boldsymbol{\varphi}$ *is called an algebraic observable* $(n-1)$*-form if and only if there exists* $\Xi_{\boldsymbol{\varphi}}$ *such that* $\Xi_{\boldsymbol{\varphi}} \lrcorner\, \boldsymbol{\omega} + d\boldsymbol{\varphi} = 0$.

We denote $\mathfrak{P}^{n-1}_{\circ}(\mathcal{M})$ the set of all algebraic observable $(n-1)$-forms. This reflects the *symmetry* point of view. It is the natural analogue to the question of the Poisson bracket for classical mechanics

[45] We denote $\mathfrak{dy}_{\bullet}(\mathcal{M})$ the set of *dynamical vector fields.*

[46] Note that in the work of Hélein and Kouneiher those objects are called *dynamical observables.* Here we choose the term *physical observable* in order to emphasize the constitutive role of the ontologic vs dynamical duality in the conception of such objects.

(35)(i). Then, $\forall \boldsymbol{\varphi}, \boldsymbol{\rho} \in \mathfrak{P}^{n-1}_{\circ}(\mathcal{M})$, we define the Poisson bracket (80):

$$\{\boldsymbol{\varphi}, \boldsymbol{\varrho}\} = \Xi_{\boldsymbol{\varphi}} \wedge \Xi_{\boldsymbol{\rho}} \lrcorner \boldsymbol{\omega} = \Xi_{\boldsymbol{\varphi}} \lrcorner d\boldsymbol{\varrho} = -\Xi_{\boldsymbol{\rho}} \lrcorner d\boldsymbol{\varphi}. \tag{80}$$

where, $\{\boldsymbol{\varphi}, \boldsymbol{\varrho}\} \in \mathfrak{P}^{n-1}_{\circ}$ and the bracket (80) satisfy the antisymmetry property and Jacobi structure modulo an exact term $\forall \boldsymbol{\varphi}, \boldsymbol{\rho}, \boldsymbol{\eta} \in \mathfrak{P}^{n-1}_{\circ}$:

$$\{\boldsymbol{\varphi}, \boldsymbol{\rho}\} + \{\boldsymbol{\rho}, \boldsymbol{\varphi}\} = 0 \qquad \{\{\boldsymbol{\varphi}, \boldsymbol{\rho}\}\boldsymbol{\eta}\} + \{\{\boldsymbol{\rho}, \boldsymbol{\eta}\}\boldsymbol{\varphi}\} + \{\{\boldsymbol{\eta}, \boldsymbol{\varphi}\}\boldsymbol{\rho}\} = d(\xi_{\boldsymbol{\varphi}} \wedge \xi_{\boldsymbol{\rho}} \wedge \xi_{\boldsymbol{\eta}} \lrcorner \boldsymbol{\omega}).$$

Since we have defined an *infinitesimal symplectomorphism* of $(\mathcal{M}, \boldsymbol{\omega})$ to be a vector field $\Xi \in \Gamma(\mathcal{M}, T\mathcal{M})$ such that $\mathcal{L}_{\Xi}\boldsymbol{\omega} = 0$, using the Cartan formula, we obtain: $\mathcal{L}_{\Xi}\boldsymbol{\omega} = d(\Xi \lrcorner \boldsymbol{\omega}) + \Xi \lrcorner d\boldsymbol{\omega} = 0$.. Now, since the multisymplectic $(n+1)$-form is closed $d\boldsymbol{\omega} = 0$, this relation is equivalent to $d(\Xi \lrcorner \boldsymbol{\omega}) = 0$. We are looking for vector fields $\Xi \in \Gamma(\mathcal{M}, T\mathcal{M})$ such that $d(\Xi \lrcorner \boldsymbol{\omega}) = 0$. Notice that we denote by $\mathfrak{sp}_{\circ}(\mathcal{M})$ the set of infinitesimal symplectomorphisms of the multisymplectic manifold $(\mathcal{M}, \boldsymbol{\omega})$. We define the set of infinitesimal symplectomorphisms:

$$\mathfrak{sp}_{\circ}(\mathcal{M}) = \{\Xi \in \Gamma(\mathcal{M}, T\mathcal{M}) \ / \ d(\Xi \lrcorner \boldsymbol{\omega}) = 0\}. \tag{81}$$

Any infinitesimal symplectomorphism $\Xi \in \mathfrak{sp}_{\circ}(\mathcal{M})$ can be written under the form: $\Xi = \chi + \bar{\zeta}$, see the result of Hélein and Kouneiher [117].

Proposition 6.1. *If $\mathcal{M}$ is an open subset of $\Lambda^n T^\star \mathfrak{Z}$, then the set of all infinitesimal symplectomorphisms Ξ on $\mathcal{M}$ are of the form $\Xi = \chi + \bar{\zeta}$, where*

$$\chi := \sum_{\beta_1 < \cdots < \beta_n} \chi_{\beta_1 \cdots \beta_n}(q) \frac{\partial}{\partial p_{\beta_1 \cdots \beta_n}} \qquad \textit{and} \qquad \bar{\zeta} := \sum_{\alpha} \zeta^\alpha(q) \frac{\partial}{\partial q^\alpha} - \sum_{\alpha, \beta} \frac{\partial \zeta^\alpha}{\partial q^\beta}(q) \Pi^\beta_\alpha, \tag{82}$$

In proposition (6.1) we have:
[1] the coefficients $\chi_{\beta_1 \cdots \beta_n}$ are such that $d(\chi \lrcorner \boldsymbol{\omega}) = 0$.
[2] $\zeta := \sum_\alpha \zeta^\alpha(q) \frac{\partial}{\partial q^\alpha}$ is an arbitrary vector field on $\mathfrak{Z}$.
[3] $\Pi^\beta_\alpha := \sum_{\beta_1 < \cdots < \beta_n} \sum_\mu \delta^\beta_{\beta_\mu} p_{\beta_1 \cdots \beta_{\mu-1} \alpha \beta_{\mu+1} \cdots \beta_n} \frac{\partial}{\partial p_{\beta_1 \cdots \beta_n}}$.

Any algebraic infinitesimal symplectomorphism can always be written $\Xi = \chi + \bar{\zeta}$. [117]. The analogous statement on the form part leads to the observation that any *algebraic observable* $(n-1)$-form $\boldsymbol{\varphi} \in \mathfrak{P}^{n-1}_{\circ}(\mathcal{M})$ can be written as $\boldsymbol{\varphi} = \mathbf{Q}^\xi + \mathbf{P}_\zeta$. Then we introduce the notion of *generalized position* $(n-1)$-forms, denoted $\mathbf{Q}^\xi$

$$\mathbf{Q}^\xi = \sum_{\mu_1 < \cdots < \mu_n} \xi_{\mu_1 \cdots \mu_{n-1}}(q) dq^{\mu_1} \wedge \cdots \wedge dq^{\mu_{n-1}} \quad \text{with} \quad \chi \lrcorner \boldsymbol{\omega} = -d\mathbf{Q}^\xi. \tag{83}$$

and the dual notion of *generalized momenta*, denoted $\mathbf{P}_\zeta$

$$\mathbf{P}_\zeta = \zeta \lrcorner \theta \quad \text{with} \quad \bar{\zeta} \lrcorner \boldsymbol{\omega} = -d\mathbf{P}_\zeta. \tag{84}$$

The decomposition of the set of infinitesimal symplectomorphisms is written as $\mathfrak{sp}_{\circ}(\mathcal{M}) = \mathfrak{sp}_{\mathbf{Q}}(\mathcal{M}) \oplus \mathfrak{sp}_{\mathbf{P}}(\mathcal{M})$. Here $\mathfrak{sp}_{\mathbf{Q}}(\mathcal{M})$ and $\mathfrak{sp}_{\mathbf{P}}(\mathcal{M})$ correspond respectively to the set of infinitesimal symplectomorphisms connected to the position part: $\mathfrak{sp}_{\mathbf{Q}}(\mathcal{M}) = \{\chi \in \Gamma(\mathcal{M}, T\mathcal{M}) \ / \ d(\chi \lrcorner \boldsymbol{\omega}) = 0\}$, where the vector field χ is defined as in proposition (6.1). Equivalently, using the vector field $\bar{\zeta}$ introduced by proposition (6.1) we obtain the set of momenta for which infinitesimal symplectomorphisms are written as: $\mathfrak{sp}_{\mathbf{P}}(\mathcal{M}) = \{\bar{\zeta} \in \Gamma(\mathcal{M}, T\mathcal{M}) \ / \ d(\bar{\zeta} \lrcorner \boldsymbol{\omega}) = 0\}$.. Following [117] we introduce the Lie bracket relations (85). Let $\chi, \chi_\circ, \chi_\bullet \in \mathfrak{sp}_{\mathbf{Q}}(\mathcal{M})$ and let $\bar{\zeta}, \bar{\zeta}_\circ, \bar{\zeta}_\bullet \in \mathfrak{sp}_{\mathbf{P}}(\mathcal{M})$, then we have:

$$\left| \begin{array}{rcl} [\chi_\circ, \chi_\bullet] & = & 0 \\ [\bar{\zeta}_\circ, \bar{\zeta}_\bullet] & = & \overline{[\zeta_\circ, \zeta_\bullet]} \\ [\chi, \zeta] & = & \displaystyle\sum_{\beta_1 < \cdots < \beta_n} \psi_{\beta_1 \cdots \beta_n}(q) \frac{\partial}{\partial p_{\beta_1 \cdots \beta_n}}, \end{array} \right. \tag{85}$$

where $\psi_{\beta_1 \cdots \beta_n} := \sum_\alpha \left(\zeta^\alpha \frac{\partial \chi_{\beta_1 \cdots \beta_n}}{\partial q^\alpha} + \sum_\mu \chi_{\beta_1 \cdots \beta_{\mu-1} \alpha \beta_{\mu+1} \cdots \beta_n} \frac{\partial \zeta^\alpha}{\partial q^{\beta_\mu}} \right)$. The Lie bracket relation (85) involves a semi-direct product structure: $\mathfrak{sp}_\circ(\mathcal{M}) = \mathfrak{sp}_{\mathbf{P}}(\mathcal{M}) \ltimes \mathfrak{sp}_{\mathbf{Q}}$. The application of the general proposition (6.1) allow us to write the generalized *position* $\mathbf{Q}^\xi$ and *momenta* $\mathbf{P}_\zeta$, $(n-1)$-forms:

$$\mathbf{Q}^\xi = \sum_{\mu_1 < \cdots < \mu_n} \xi_{\mu_1 \cdots \mu_{n-1}}(q) dq^{\mu_1} \wedge \cdots \wedge dq^{\mu_{n-1}} \quad \text{and} \quad \mathbf{P}_\zeta = \zeta \lrcorner \theta, \tag{86}$$

and we find related infinitesimal symplectomorphisms $\Xi(\mathbf{Q}^\xi)$ and $\Xi(\mathbf{P}_\zeta)$ of the form:

$$\left| \begin{array}{rcl} \Xi(\mathbf{Q}^\xi) & = & \displaystyle\sum_{\beta_1 < \cdots < \beta_n} \sum_\alpha (-1)^\alpha \frac{\partial \xi_{\beta_1 \cdots \beta_{\alpha-1} \beta_{\alpha+1} \cdots \beta_n}}{\partial q^\alpha} \frac{\partial}{\partial p_{\beta_1 \cdots \beta_n}} \\ \Xi(\mathbf{P}_\zeta) & = & \displaystyle\sum_\alpha \zeta^\alpha(q) \frac{\partial}{\partial q^\alpha} - \sum_{\alpha, \beta} \frac{\partial \zeta^\alpha}{\partial q^\beta}(q) \Pi^\beta_\alpha \end{array} \right. \tag{87}$$

As already understood, $\mathfrak{sp}_{\mathbf{Q}}(\mathcal{M}) \subset \mathfrak{sp}_\circ(\mathcal{M})$ and $\mathfrak{sp}_{\mathbf{P}}(\mathcal{M}) \subset \mathfrak{sp}_\circ(\mathcal{M})$. We also introduce the following notations: $\mathfrak{P}^{n-1}_{\mathbf{Q}}(\mathcal{M})$ and $\mathfrak{P}^{n-1}_{\mathbf{P}}(\mathcal{M})$ are taken to be the set of algebraic observable $(n-1)$-forms such that:

$$\mathfrak{P}^{n-1}_{\mathbf{Q}}(\mathcal{M}) = \{\mathbf{Q}^\xi \in \Lambda^{n-1} T^\star \mathfrak{Z}, / \Xi(\mathbf{Q}^\xi) \lrcorner \boldsymbol{\omega} = -d\mathbf{Q}^\xi.\} \tag{88}$$

Here is the *position* part. On the other side $\mathfrak{P}^{n-1}_{\mathbf{P}}(\mathcal{M})$ is described:

$$\mathfrak{P}^{n-1}_{\mathbf{P}}(\mathcal{M}) = \{\mathbf{P}_\zeta \in \Lambda^{n-1} T^\star \mathfrak{Z}, / \Xi(\mathbf{P}_\zeta) \lrcorner \boldsymbol{\omega} = -d\mathbf{P}_\zeta, \} \tag{89}$$

where the vector field $\Xi(\mathbf{P}_\zeta)$ identified with the infinitesimal symplectomorphism $\bar{\zeta}$ described by proposition (6.1). We therefore consider the Poisson bracket relations constructed on related observable $(n-1)$-forms. Let $\mathbf{P}_\zeta, \mathbf{P}_{\zeta^\circ}, \mathbf{P}_{\zeta^\bullet} \in \mathfrak{P}^{n-1}_{\mathbf{P}}(\mathcal{M})$ be any generalized momenta observable $(n-1)$-form and let $\mathbf{Q}^\xi, \mathbf{Q}^{\xi_\circ}, \mathbf{Q}^{\xi_\bullet} \in \mathfrak{P}^{n-1}_{\mathbf{Q}}(\mathcal{M})$ be any generalized position observable $(n-1)$-form. These Poisson bracket relations are proven by Hélein and Kouneiher [115]:

$$\left| \begin{array}{rcl} \{\mathbf{Q}^{\xi_\circ}, \mathbf{Q}^{\xi_\bullet}\} & = & 0 \\ \{\mathbf{P}_{\zeta^\circ}, \mathbf{P}_{\zeta^\bullet}\} & = & \mathbf{P}_{[\zeta^\circ, \zeta^\bullet]} + d(\zeta^\bullet \lrcorner \zeta^\circ \lrcorner \theta) \\ \{\mathbf{P}_\zeta, \mathbf{Q}^\xi\} & = & \displaystyle\sum_{\beta_1 < \cdots < \beta_n} \sum_\alpha \sum_\mu (-1)^{\alpha+1} \zeta^\mu \frac{\partial \xi_{\beta_1 \cdots \beta_{\alpha-1} \beta_{\alpha+1} \cdots \beta_n}}{\partial q^{\beta_\alpha}} \frac{\partial}{\partial q^\mu} \lrcorner dq^{\beta_1} \wedge \cdots \wedge dq^{\beta_n}. \end{array} \right.$$

6.3 Observable forms (OF)

Observable $(n-1)$-forms OF [117, 118] give the right generalization of the quantum Heisenberg evolution equation. Their underlying motivation is grounded in the *dynamical* aspect. The idea is that given a point $m \in \mathcal{M}$ and a Hamiltonian function $\mathcal{H}$ (with $X(m) \in [X]^{\mathcal{H}}_m$), the object $\langle X(m), d\boldsymbol{\varphi}_m \rangle$ should only depend on $d\mathcal{H}_m$.

Definition 6.3.1. *Let $(\mathcal{M},\boldsymbol{\omega})$ be a n-multisymplectic manifold. A $(n-1)$-form $\boldsymbol{\varphi}$ is called an* observable $(n-1)$-form *if and only if for any $m \in \mathcal{M}$ and $\forall X, \tilde{X} \in \mathbf{D}^n_m(\mathcal{M}) \subset \Lambda^n_m T^\star\mathcal{M}$ such that $X \lrcorner\, \boldsymbol{\omega} = \tilde{X} \lrcorner\, \boldsymbol{\omega}$ then $d\boldsymbol{\varphi}(X) = d\boldsymbol{\varphi}(\tilde{X})$.*

We denote $\mathfrak{P}^{n-1}_{\bullet}(\mathcal{M})$ the set of such observable forms. Then we construct the *pseudobracket* in definition (6.3.2). The evolution of an observable with respect to the Hamiltonian function is connected with the notion of *dynamical potential* to be defined in (20). It is the Hamiltonian n-curve $\Gamma \subset \mathcal{M}$ itself viewed as a space which holds the key to the reconceptualisation of dynamics. We shall work on the class of n-multivector fields $[X]^{\mathcal{H}}_m$.

Definition 6.3.2. *Let $\mathcal{H}$ be an admissible Hamiltonian function $\mathcal{H} : \mathcal{M} \to \mathbb{R}$, such that $[X]^{\mathcal{H}}_m \subset \mathcal{O}_m\mathcal{M} \subset \mathbf{D}^n_m(\mathcal{M})$, then the* pseudobracket *for any $\mathfrak{P}^{n-1}_{\bullet}(\mathcal{M})$ is given by:*

$$\forall [X]^{\mathcal{H}}_m \subset \mathcal{O}_m\mathcal{M} \subset \mathbf{D}^n_m(\mathcal{M}) \qquad \big\{\mathcal{H},\boldsymbol{\varphi}\big\} := X \lrcorner\, d\boldsymbol{\varphi} = d\boldsymbol{\varphi}(X). \tag{90}$$

Then, $\big\{\mathcal{H},\boldsymbol{\varphi}\big\} = d\boldsymbol{\varphi}(X)$ where $X \in \mathbf{D}^n_m(\mathcal{M})$ is such that $X \lrcorner\, \boldsymbol{\omega} = (-1)^n d\mathcal{H}$. We have a relation that involves the evolution of an observable form OF by means of a new object: the Poisson *pseudobracket*. This evolution constitutes the *dynamical* part of the story. It is to be seen as evolution along the Hamiltonian n-curve Γ. In this case we have, $\forall \boldsymbol{\varphi} \in \mathfrak{P}^{n-1}_{\bullet}$:

$$\big\{\mathcal{H},\boldsymbol{\varphi}\big\}\ \beta\big|_{\Gamma} \ = \ d\boldsymbol{\varphi}\big|_{\Gamma}. \tag{91}$$

Now, from (91) we find the relation (92) for all $\boldsymbol{\varphi}, \boldsymbol{\varrho} \in \mathfrak{P}^{n-1}_{\bullet}$ and Γ, a Hamiltonian n-curve.

$$\big\{\mathcal{H},\boldsymbol{\varphi}\big\}\ d\boldsymbol{\varrho}\big|_{\Gamma} = \big\{\mathcal{H},\boldsymbol{\varrho}\big\}\ d\boldsymbol{\varphi}\big|_{\Gamma}. \tag{92}$$

We observe that in the equation (92), no volume form β is singled out: dynamics just prescribes how to compare two observations. This naturally encapsulates the **Relativity Principle**. It also allows us to understand the relationship of *dynamics* and *observables*. Notice that in our general classification of forms, we denote this movement symbolically $\mathfrak{P}^{n-1}_{\bullet}(\mathcal{M})\leftrightarrows[\lhd\!\cdot\,]$. The interplay of both concepts $\mathfrak{P}^{n-1}_{\circ}(\mathcal{M})$ and $\mathfrak{P}^{n-1}_{\bullet}(\mathcal{M})$ leads to the notion of a *pataplectic manifold*: a multisymplectic manifold $(\mathcal{M},\boldsymbol{\omega})$ where the set of observable $(n-1)$-forms coincides with the set of algebraic observable $(n-1)$-forms. For a *pataplectic manifold* we describe:

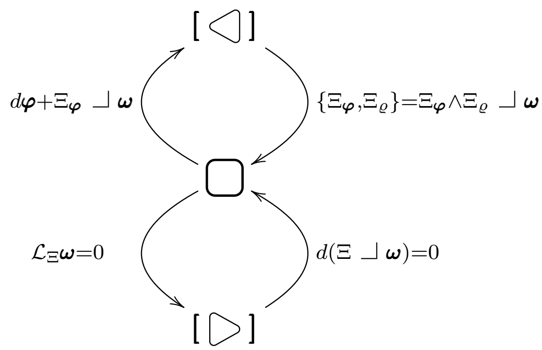
$$\mathfrak{P}^{n-1}_{\circ}(\mathcal{M}) = \mathfrak{P}^{n-1}_{\bullet}(\mathcal{M}) \qquad \text{or in symbolic notation} \qquad [\lhd] \ = \ [\lhd\!\cdot\,]$$

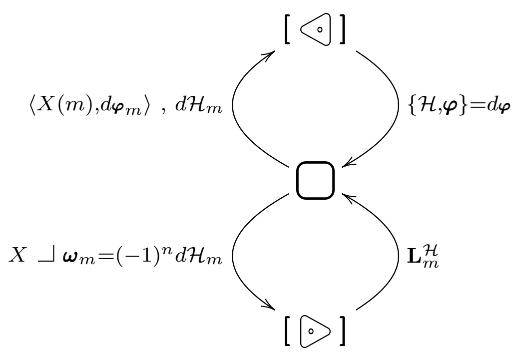
We emphasize the interplay between DW theory and the more general Lepagean theories. Indeed the DW manifold is *not* pataplectic whereas any open subset of the universal multisymplectic manifold $\Lambda^n T^\star \mathfrak{Z}^{\mathbf{o}}$ necessarily is. Therefore we stress below the relationship between the *ontologic* mode of $\mathcal{M} \subset \Lambda^n T^\star \mathfrak{Z}^{\mathbf{o}}$ and the *pataplectic* aspect. We note that the *pataplectic* manifold is an environment in which *ontologic* and *dynamical* observables overlap and therefore a good detector of the physical observables. We enter into more details on this point when we deal with the duality of *ontologic* vs *dynamical* symmetry in a multisymplectic theory - the analogue of the duality symplectomorphisms vs vector fields which left the Hamiltonian function invariant - itself a reflection of the *dual* nature of the multisymplectic form $\boldsymbol{\omega}$, see section (20). We notice two important guiding ideas that informed the previous setting. Firstly on the dynamical side we define $\mathfrak{P}^{n-1}_{\bullet}(\mathcal{M})\leftrightarrows[\lhd\!\cdot\,]$. By this symbol we want to indicate that $(n-1)$-forms provide the possibility of encoding *dynamical* properties. We want to emphasize that $\forall m \in \mathcal{M}, X(m) \in [X]^{\mathcal{H}}_m$, $\langle X(m), d\boldsymbol{\varphi}_m\rangle$ only depend on $d\mathcal{H}_m$. Via the the Hamiltonian function $\mathcal{H} : \mathcal{M} \to \mathbb{R}$, we are led to consider a n-dimensional submanifold $\Gamma \subset \mathcal{M}$, the so-called Hamiltonian n-curve. The *dynamical* symmetry is the *covariance* symmetry, expressed by the generalized Hamilton equations: for any vector field $X_m \in [X]^{\mathcal{H}}_m$ with

$[X]^{\mathcal{H}}_m = \{X \in \mathbf{D}^n_m(\mathcal{M}) \ / \ X \lrcorner \boldsymbol{\omega} = (-1)^n d\mathcal{H}_m\}$. This operation gives a Hamiltonian n-curve which symbolically represent the *dynamical* space. On the kinematical side, $\mathfrak{P}^{n-1}_{\circ}(\mathcal{M}) \leftrightarrows$[◁] symbolically indicates that we consider $(n-1)$-forms that *manifest* the possibility of encoding the *symmetry* standpoint.

6.4 Dynamical vs Ontologic: synthesis on $(n-1)$-forms

This section is devoted to symbolic presentation of the duality of dynamical vs ontologic for multisymplectic theory. We are first interested in the $(n-1)$-forms case, namely AOF and OF.
Symbolic picture ontologic *vs* dynamical:





We call our two Observers who are to be thought of as "traveling through" the symbolic picture above:

◁▷ Ontologic Observer ◁▷ Dynamical Observer

They represent the movement in the search for observables within the *ontologic* area and the *dynamical* area respectively: what we shall term the two sides of the Observer. We first follow the movement from the *ontologic* viewpoint.

- The ontologic search for observables: algebraic observable $(n-1)$-forms (AOF)

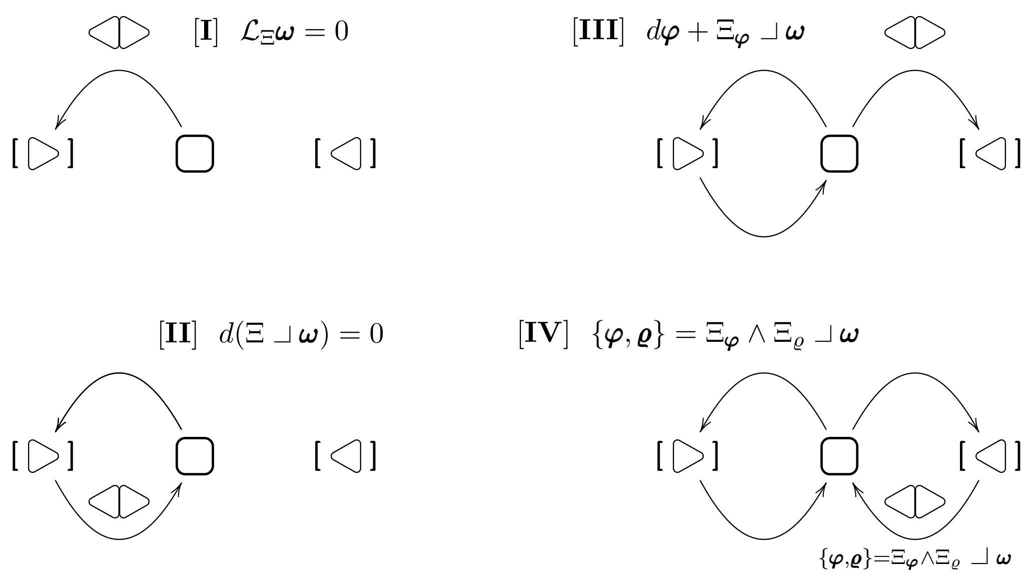
  [**I**] Our first step is the expresses the *ontologic* symmetry. Mathematically it means that we are looking for a vector field $\Xi \in \Gamma(\mathcal{M}, T\mathcal{M})$ such that $\mathcal{L}_{\Xi}\boldsymbol{\omega} = 0$. Here we are only concerned with the preservation of the phase space structure and are looking for a vector field on the ontologic space $(\mathcal{M}, \boldsymbol{\omega})$. This concerns a symmetry issue, more precisely an *invariance*: the preservation of the multisymplectic form. No reference to the Hamiltonian or dynamical evolution is taken into account.

  [**II**] The second step is still concerned with the ontologic space. The Cartan formula is $\mathcal{L}_{\Xi}\boldsymbol{\omega} = \boldsymbol{\Xi} \lrcorner d\boldsymbol{\omega} + d(\boldsymbol{\Xi} \lrcorner \boldsymbol{\omega})$. In addition we assume that the multisymplectic form is closed: $d\boldsymbol{\omega} = 0$.

Due to this condition on the form - the object that shapes our generalized geometry - we obtain for the symmetry condition $\mathcal{L}_{\boldsymbol{\Xi}}\boldsymbol{\omega} = 0$ the following simplification $d(\boldsymbol{\Xi} \lrcorner \boldsymbol{\omega}) = 0$. This mathematical requirement reflect the ontologic symmetry.

[**III**] The third step concerns the ontologic representation, namely the search for the set of algebraic $(n-1)$-forms $\mathfrak{P}^{n-1}_{\circ}(\mathcal{M})$. We are looking for $(n-1)$-forms related to infinitesimal symplectomorphisms determined by the previous step [**II**] - such that $d(\boldsymbol{\Xi} \lrcorner \boldsymbol{\omega}) = 0$. The ontologic representation corresponds to forms $\boldsymbol{\varphi} \in \mathfrak{P}^{n-1}_{\circ}(\mathcal{M})$. Also, we are looking for the following relation $\Xi_{\boldsymbol{\varphi}}$ such that $\Xi_{\boldsymbol{\varphi}} \lrcorner \boldsymbol{\omega} + d\boldsymbol{\varphi} = 0$. This expresses the *topological duality.*

[**IV**] The last step concerns the evolution of algebraic observable $(n-1)$-forms. Here we describe the Poisson bracket structure $\forall \boldsymbol{\varphi}, \boldsymbol{\rho} \in \mathfrak{P}^{n-1}_{\circ}(\mathcal{M})$. We have $\left\{\boldsymbol{\varphi}, \boldsymbol{\varrho}\right\} = \Xi_{\boldsymbol{\varphi}} \wedge \Xi_{\boldsymbol{\rho}} \lrcorner \boldsymbol{\omega} = \Xi_{\boldsymbol{\varphi}} \lrcorner d\boldsymbol{\varrho} = -\Xi_{\boldsymbol{\rho}} \lrcorner d\boldsymbol{\varphi}$. We shall later reformulate steps [**III**] and [**IV**] from a deeper mathematical perspective by appealing to the notion of *copolar* forms. Here we simply draw pictures symbolizing the journey of the ontologic observer ◁▷:



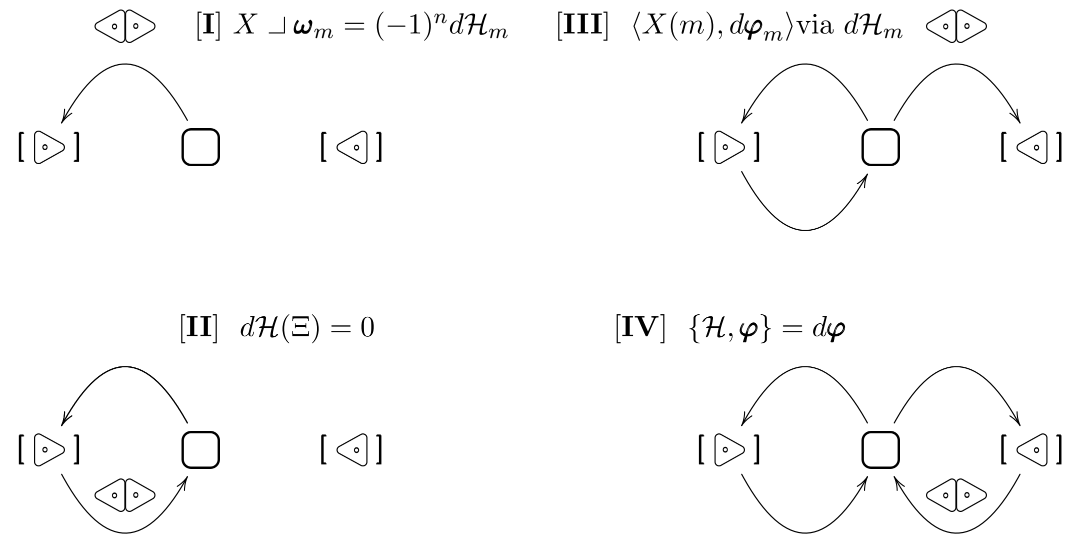
- The dynamical search for observable: observable $(n-1)$-forms (OF)

  [**I**] The first step is an expression of the **pure** *dynamical* symmetry or covariance. We emphasize pure covariance since it concerns the form of the dynamical equations. Mathematically, we are looking for generalized Hamilton equations. $\forall m \in \Gamma, \exists X \in \Lambda^n T_m\Gamma \quad X \lrcorner \boldsymbol{\omega}_m = (-1)^n d\mathcal{H}_m$. We contrast the concept of *pure dynamical symmetry* with that of *dynamical symmetry.* The first is concerned with the *form* of the equations, thanks to the Hamiltonian n-curve $\Gamma \subset \mathcal{M} = \Lambda^n T^{\star}\mathfrak{Z}^{\circ}$ and Hamiltonian n-vector fields. The latter is simply concerned with the invariance of the Hamiltonian function. This means that we are looking for vector fields $\Xi \in \Gamma(\mathcal{M}, T\mathcal{M})$ such that $\mathcal{L}_{\Xi}\mathcal{H} = 0$. This is why we speak only about dynamical symmetry: the object of symmetry - invariance - is the Hamiltonian function.

  [**II**] The second step, still concerning the dynamical area, is strongly connected to the equation $d\mathcal{H}(\Xi) = 0$. The connection here is reflected in the *generalized pseudofiber* $\mathbf{L}^{\mathcal{H}}_m$ for a Hamiltonian function $\mathcal{H} : \mathcal{M} \to \mathbb{R}$. Recall that $\mathbf{L}^{\mathcal{H}}_m = \left\{\Xi \in T_m\mathcal{M} \ / \ \forall X \in [X]^{\mathcal{H}}_m \ , \ \forall \delta X \in T_X\mathbf{D}^n_m(\mathcal{M}) \ , \ \Xi \lrcorner \boldsymbol{\omega}(\delta X) = 0\right\}$ is equivalently described by $\mathbf{L}^{\mathcal{H}}_m = [T_{[X]^{\mathcal{H}}_m}\mathbf{D}^n_m\boldsymbol{\mathcal{M}} \lrcorner \boldsymbol{\omega}]^{\perp}$. See the previous section (4.8).

[**III**] The third step now concerns the *dynamical representation*, namely the search for the set of observable $(n-1)$-forms $\mathfrak{P}^{n-1}_{\bullet}(\mathcal{M})$. We are looking for $(n-1)$-forms such that given a point $m \in \mathcal{M}$ and a Hamiltonian function $\mathcal{H}$ (with $X(m) \in [X]^{\mathcal{H}}_m$), $\langle X(m), d\boldsymbol{\varphi}_m \rangle$ depends only on $d\mathcal{H}_m$. We refer to the development found in the previous section (6.3).

[**IV**] Finally the Poisson pseudobracket gives the evolution of an observable $(n-1)$-form. The pseudobracket for any $\boldsymbol{\varphi} \in \mathfrak{P}^{n-1}_{\bullet}(\mathcal{M})$ is given by: $\forall [X]^{\mathcal{H}}_m \subset \mathcal{O}_m\mathcal{M} \subset \mathbf{D}^n_m(\mathcal{M})$ $\{\mathcal{H}, \boldsymbol{\varphi}\} := X \lrcorner\, d\boldsymbol{\varphi} = d\boldsymbol{\varphi}(X)$. We recall that in this case we have, $\forall \boldsymbol{\varphi} \in \mathfrak{P}^{n-1}_{\bullet}$ $\{\mathcal{H}, \boldsymbol{\varphi}\}\ \beta\big|_{\Gamma} = d\boldsymbol{\varphi}\big|_{\Gamma}$, see section (6.3). Here we draw pictures symbolizing the journey of the dynamical observer $\triangleleft\!\triangleright$:



The right part of the previous two pictures - which concerns the ontologic and dynamical representation - is only an approximate picture. A more accurate vision would take into account the notion of *copolar* form. Before giving it, we would like to describe the construction of brackets from this symbolic viewpoint. The easiest example is the usual Poisson bracket for field theory, namely the construction of the object $\{\boldsymbol{\varphi}, \boldsymbol{\varrho}\} = \Xi_{\boldsymbol{\varphi}} \wedge \Xi_{\boldsymbol{\rho}} \lrcorner\, \boldsymbol{\omega} = -\Xi_{\boldsymbol{\varphi}} \lrcorner\, d\boldsymbol{\varrho}$. This is interpreted as interaction between the ontologic representation with itself, leading to the following diagram:

$$[\triangleleft]\Big|[\triangleleft]$$

$$\left|\begin{array}{ccc} \mathfrak{P}^{n-1}_{\circ}(\mathcal{M}) \times \mathfrak{P}^{n-1}_{\circ}(\mathcal{M}) & \to & \mathfrak{P}^{n-1}_{\circ}(\mathcal{M}) \\ (\boldsymbol{\varphi}, \boldsymbol{\varrho}) & \mapsto & \{\boldsymbol{\varphi}, \boldsymbol{\varrho}\} = \Xi_{\boldsymbol{\varphi}} \wedge \Xi_{\boldsymbol{\rho}} \lrcorner\, \boldsymbol{\omega} = -\Xi_{\boldsymbol{\varphi}} \lrcorner\, d\boldsymbol{\varrho} \end{array}\right|$$

Notice that each algebraic observable $(n-1)$-form $\boldsymbol{\varphi}, \boldsymbol{\varrho} \in \mathfrak{P}^{n-1}_{\circ}(\mathcal{M})$ is obtained by the interplay between vector fields $\Xi_{\boldsymbol{\varphi}}$ and $\Xi_{\boldsymbol{\varrho}}$ forms by means of the equations $\Xi_{\boldsymbol{\varphi}} \lrcorner\, \boldsymbol{\omega} + d\boldsymbol{\varphi} = 0$ and $\Xi_{\boldsymbol{\varrho}} \lrcorner\, \boldsymbol{\omega} + d\boldsymbol{\varrho} = 0$. Here we see the concept of **topological duality** at work. In this symbolism we notice that in order to compare the *ontologic representation* with itself we need previously to have built the two objects $\boldsymbol{\varphi}, \boldsymbol{\varrho} \in \mathfrak{P}^{n-1}_{\circ}(\mathcal{M})$. Each of these is to be understood as the interaction of the *ontologic space* with the *ontologic representation*. We have the following picture:

$$\left|\begin{array}{ccc} [\triangleright]\Big|[\triangleleft] & \mapsto & \Xi_{\boldsymbol{\varphi}} \lrcorner\, \boldsymbol{\omega} + d\boldsymbol{\varphi} = 0 \\ [\triangleright]\Big|[\triangleleft] & \mapsto & \Xi_{\boldsymbol{\varrho}} \lrcorner\, \boldsymbol{\omega} + d\boldsymbol{\varrho} = 0 \end{array}\right|$$

Now we observe that the symbolic picture for the dynamical construction is intrinsically of a different nature. The building of the Poisson *pseudobracket* does not rely on the comparison of the *dynamical representation* with itself. Hence, this construction *intrinsically* compares an observable $\boldsymbol{\varphi} \in \mathfrak{P}^{n-1}_{\bullet}(\mathcal{M})$ with the Hamiltonian function. Recall we are interested in the relation between observables and dynamics. The dynamical space is here taken to be the Hamiltonian n-curve itself while the class of Hamiltonian vector fields as the carrier of the dynamical evolution. It is the interplay between the *dynamical space* and the *dynamical representation* that determines the dynamical evolution for an observable form. All this leads to the following diagram:

$$[\triangleright] \,\Big|\, [\triangleleft]$$

$$\left| \begin{array}{c} \mathcal{H} \text{ admissible} : [X]^{\mathcal{H}}_m \subset \mathcal{O}_m\mathcal{M} \subset \mathbf{D}^n_m(\mathcal{M}) \\ \boldsymbol{\varphi} \in \mathfrak{P}^{n-1}_{\bullet}(\mathcal{M}) \end{array} \right| \leftrightarrow \left| \begin{array}{c} \big\{\mathcal{H}, \boldsymbol{\varphi}\big\} = X \lrcorner\, d\boldsymbol{\varphi} = d\boldsymbol{\varphi}(X) \\ \big\{\mathcal{H}, \boldsymbol{\varphi}\big\}\ \beta\big|_{\Gamma} = d\boldsymbol{\varphi}\big|_{\Gamma} \end{array} \right|$$

The last picture we build depicts the interplay of the *dynamical representation* with itself. We are looking for a relation involving two observable forms $\boldsymbol{\varphi}, \boldsymbol{\varrho} \in \mathfrak{P}^{n-1}_{\bullet}(\mathcal{M})$:

$$[\triangleleft] \,\Big|\, [\triangleleft]$$

$$\left| \begin{array}{c} \mathcal{H} \text{ admissible} : [X]^{\mathcal{H}}_m \subset \mathcal{O}_m\mathcal{M} \subset \mathbf{D}^n_m(\mathcal{M}) \\ (\boldsymbol{\varphi}, \boldsymbol{\varrho}) \in \mathfrak{P}^{n-1}_{\bullet}(\mathcal{M}) \times \mathfrak{P}^{n-1}_{\bullet}(\mathcal{M}) \end{array} \right| \leftrightarrow \left| \begin{array}{c} \big\{\mathcal{H}, \boldsymbol{\varphi}\big\} d\boldsymbol{\varrho}\big|_{\Gamma} = \big\{\mathcal{H}, \boldsymbol{\varrho}\big\} d\boldsymbol{\varphi}\big|_{\Gamma} \\ \big\{\mathcal{H}, \boldsymbol{\varphi}\big\} d\boldsymbol{\varrho}(X) = \big\{\mathcal{H}, \boldsymbol{\varrho}\big\} d\boldsymbol{\varphi}(X) \end{array} \right|$$

Notice that in this context, we recognize what we call the **dynamical duality**. To give a flavour of this idea, we consider *how* we build the respective pictures. In the case of topological duality for algebraic observable $(n-1)$-forms, we are looking for forms such that the exterior derivative $d\boldsymbol{\varphi}$ is given by $d\boldsymbol{\varphi} = -\Xi_{\boldsymbol{\varphi}} \lrcorner\, \boldsymbol{\omega}$. Therefore the map of interest is the following:

$$\mathfrak{U}^1_{\circ} : T_m\mathcal{M} \to \Lambda^n T^{\star}_m\mathcal{M} \simeq (\Lambda^n T_m\mathcal{M})^{\star} \;\; \Xi \mapsto \mathfrak{U}^1_{\circ}(\Xi) = \Xi \lrcorner\, \boldsymbol{\omega}.$$

The topological duality is in this case concerned with the pairing between $T\mathcal{M}$ and $\Lambda^n T^{\star}\mathcal{M}$ via the multisymplectic form $\boldsymbol{\omega}$. By contrast, the dynamical duality, as the relation $\big\{\mathcal{H}, \boldsymbol{\varphi}\big\} = X \lrcorner\, d\boldsymbol{\varphi} = d\boldsymbol{\varphi}(X)$ suggests, is concerned with the pairing of Hamiltonian vector fields $X \in [X]^{\mathcal{H}}_m$, (recall that $[X]^{\mathcal{H}}_m = \big\{X \in \mathbf{D}^n_m(\mathcal{M}) \;/\; X \lrcorner\, \boldsymbol{\omega} = (-1)^n d\mathcal{H}_m\big\}$) with the space of n-forms $\Lambda^n T^{\star}_m\mathcal{M}$. This is the pairing in the main object: $\langle X(m), d\boldsymbol{\varphi}_m\rangle$. In this case, the map of interest is rather:

$$\mathfrak{U}^n_{\bullet} : \mathbf{D}^n_m(\mathcal{M}) \subset \Lambda^n T_m\mathcal{M} \times \Lambda^n T^{\star}_m\mathcal{M} \to \mathbb{R} \;\; (X, d\boldsymbol{\varphi}) \mapsto \mathfrak{U}^n_{\bullet}(X, d\boldsymbol{\varphi}).$$

Now we introduce the notion of *copolar forms*, which allows us to situate the distinction between AOF and OF in a more precise and embracing framework and also provides the right treatment of *polarization* and *copolarization* for forms of arbitrary degree.

6.5 Algebraic copolar and copolar $(n-1)$-forms

We introduce the set of algebraic copolar n-forms $\mathbf{P}^n_{\circ} T^{\star}_m\mathcal{M}$ together with the set of copolar n-forms $\mathbf{P}^n_{\bullet} T^{\star}_m\mathcal{M}$.

Definition 6.5.1. *Let $m \in \mathcal{M}$ and let $\phi \in \Lambda^n T^{\star}_m\mathcal{M}$. The n-form ϕ is called an algebraic copolar n-form if and only if:*

$$\forall X, \tilde{X} \in \Lambda^n T_m\mathcal{M} \qquad X \lrcorner\, \boldsymbol{\omega} = \tilde{X} \lrcorner\, \boldsymbol{\omega} \quad \Longrightarrow \quad \phi(X) = \phi(\tilde{X}),$$

and is called a copolar n-form if and only if:

$$\forall X, \tilde{X} \in \mathcal{O}_m\mathcal{M} \qquad X \lrcorner\, \boldsymbol{\omega} = \tilde{X} \lrcorner\, \boldsymbol{\omega} \quad \Longrightarrow \quad \phi(X) = \phi(\tilde{X}),$$

where there exists an open dense subset $\mathcal{O}_m\mathcal{M} \subset \mathbf{D}^n_m\mathcal{M}$.

The set of algebraic copolar n-forms is denoted $\mathbf{P}^n_\circ T^\star_m\mathcal{M}$ and the set of copolar n-form is denoted $\mathbf{P}^n_\bullet T^\star_m\mathcal{M}$.

Definition 6.5.2. *A $(n-1)$-form $\boldsymbol{\varphi}$ is an observable $(n-1)$-form if and only for any point $m \in \mathcal{M}$, $d\boldsymbol{\varphi}_m \in \mathbf{P}^n_{\bullet m} T^\star_m\mathcal{M}$. The set of observable $(n-1)$-forms on $\mathcal{M}$ is denoted $\mathfrak{P}^{n-1}_\bullet(\mathcal{M})$.*

Definition 6.5.3. *A $(n-1)$-form $\boldsymbol{\varphi}$ is an algebraic observable $(n-1)$-form if and only for any point $m \in \mathcal{M}$, $d\boldsymbol{\varphi}_m \in \mathbf{P}^n_{\circ m} T^\star_m\mathcal{M}$. The set of algebraic observable $(n-1)$-forms on $\mathcal{M}$ is denoted $\mathfrak{P}^{n-1}_\circ(\mathcal{M})$.*

Hence we summarize the following comparison:

Algebraic Observable forms (AOF)

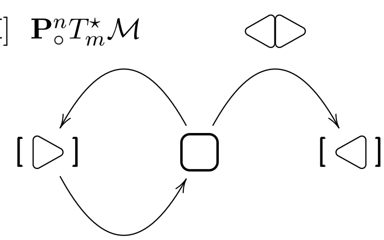
> $\mathfrak{P}^{n-1}_\circ(\mathcal{M})$ set of algebraic observable forms
> *Symmetry* standpoint $\Xi_{\boldsymbol{\varphi}} \in T\mathcal{M} \ / \ d\boldsymbol{\varphi} + \Xi_{\boldsymbol{\varphi}} \lrcorner\, \boldsymbol{\omega} = 0$
> Poisson Bracket $\{\boldsymbol{\varphi}, \boldsymbol{\varrho}\} = \Xi_{\boldsymbol{\varphi}} \wedge \Xi_{\boldsymbol{\rho}} \lrcorner\, \boldsymbol{\omega} = -\Xi_{\boldsymbol{\varphi}} \lrcorner\, d\boldsymbol{\varrho}$
> Copolar feature $\boldsymbol{\varphi} \in \mathfrak{P}^{n-1}_\circ$ iff, for all $m \in \mathcal{M}$, $d\boldsymbol{\varphi}_m \in \mathbf{P}^n_\circ T^\star_m\mathcal{M}$

Observable Forms (OF)

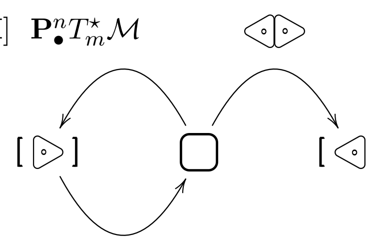
> $\mathfrak{P}^{n-1}_\bullet(\mathcal{M})$ the set of observable forms
> *Dynamical* standpoint $\forall m \in \mathcal{M}, X(m) \in [X]^{\mathcal{H}}_m$, $\langle X(m), d\boldsymbol{\varphi}_m \rangle$ only depend on $d\mathcal{H}_m$
> Pseudobracket $\{\mathcal{H}, \boldsymbol{\varphi}\} := X \lrcorner\, d\varphi = d\boldsymbol{\varphi}(X)$
> Copolar feature $\boldsymbol{\varphi} \in \mathfrak{P}^{n-1}_\bullet$iff, for all $m \in \mathcal{M}$, $d\boldsymbol{\varphi}_m \in \mathbf{P}^n_\bullet T^\star_m\mathcal{M}$

Notice that with the help of those definitions, and considering the movement of the Observers within the symbolic picture for various brackets, we see that the previous symbolic was lacking in precision. For example, we have seen that the bracket and pseudobracket involved different constructions in the ontologic and dynamical , so that the last two steps [**III**] and [**IV**] were actually a mixture of several underlying stages. Now we reformulate the symbolic picture using *copolar* forms. Note that the notion of *copolar forms* can be applied to both the *ontologic* and *dynamical* representation.

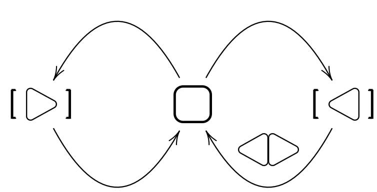
[**III**] $\mathbf{P}^n_\circ T^\star_m\mathcal{M}$



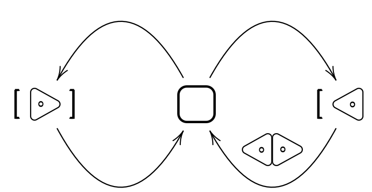
[**III**] $\mathbf{P}^n_\bullet T^\star_m\mathcal{M}$



[**IV**] $m \in \mathcal{M}$, $d\boldsymbol{\varphi}_m \in \mathbf{P}^n_{\circ m} T^\star_m\mathcal{M}$



[**IV**] $m \in \mathcal{M}$, $d\boldsymbol{\varphi}_m \in \mathbf{P}^n_{\bullet m} T^\star_m\mathcal{M}$



The last diagram embodies a more intrinsic picture. Just recall that step [**III**] describes the sets $\mathbf{P}^n_\circ T^\star_m\mathcal{M}$ and $\mathbf{P}^n_\bullet T^\star_m\mathcal{M}$ whereas step [**IV**] describes the sets of observable $(n-1)$-forms and of algebraic observable $(n-1)$-forms on $\mathcal{M}$: $\mathfrak{P}^{n-1}_\bullet(\mathcal{M})$ and $\mathfrak{P}^{n-1}_\circ(\mathcal{M})$ respectively. Now we have the prerequisites for the proper study of the Poisson bracket and Poisson pseudobracket for the dynamical evolution of observables. We delay further symbolic considerations until part [II].

6.6 Frozen or kinematical observable functionals

The other very important objects for the needs of physics are observable functionals. This provides a bridge with the classical or quantum observables of field theory. We describe a multisymplectic manifold $(\mathcal{M},\boldsymbol{\omega})$ together with an Hamiltonian $\mathcal{H}$. We denote by $\mathcal{E}^{\mathcal{H}}$ the set of Hamiltonian n-curves. This picture is the generalization of a Hamiltonian system $(\mathcal{M},\omega,\mathcal{H})$ to the n-dimensional case where the dynamical data are $(\mathcal{M},\boldsymbol{\omega},\mathcal{H})$. Before giving the definition of an observable functional, we introduce the notion of *slice*. The quantities of physical interest are functionals on the set of Hamiltonian n-curves $\mathcal{E}^{\mathcal{H}}$. We construct such observable functionals by integration of an algebraic observable $(n-1)$-form over a submanifold $\Sigma \subset \Gamma$ of codimension 1 of a Hamiltonian n-curve Γ. Here we recover the picture of observable functionals in the classical (quantum) field theory as smeared integrals over a spacelike hypersurface.

Definition 6.6.1. *A slice of codimension* 1 *is a submanifold* $\Sigma \subset \mathcal{M}$ *such that* $T_m\mathcal{M}/T_m\Sigma$ *is smoothly oriented with regard to* m *and, such that for any* $\Gamma \in \mathcal{E}^{\mathcal{H}}$, Σ *is transverse to* Γ.

This definition allows us to give an orientation on $\Sigma \cap \Gamma$. If Σ is a slice of codimension 1 and $\boldsymbol{\varrho}$ is a $(n-1)$-form on $\mathcal{M}$, namely $\boldsymbol{\varrho} \in \Gamma(\mathcal{M}, \Lambda^{n-1}T^\star\mathcal{M})$, we define the concept of functional $\mathfrak{F}_{\boldsymbol{\varrho}} := \int_\Sigma \boldsymbol{\varrho}$. This object is described as $\int_\Sigma \boldsymbol{\varrho} : \mathcal{E}^{\mathcal{H}} \longrightarrow \mathbb{R}$ on the set of Hamiltonian n-curves by means of:

$$\mathfrak{F}_{\boldsymbol{\varrho}} := \int_\Sigma \boldsymbol{\varrho} : \Gamma \mapsto \int_{\Sigma\cap\Gamma} \boldsymbol{\varrho} \tag{93}$$

We can integrate the $(n-1)$-form $\boldsymbol{\varrho}$ on $\Sigma \cap \Gamma$. To reach the object of interest, we pass from those functionals to observable functionals whose form $\boldsymbol{\varrho}$ is an algebraic observable.

Definition 6.6.2. *Let* Σ *be a slice of codimension* 1 *and let be* $\boldsymbol{\varphi}$ *an algebraic observable* $(n-1)$*-form. An observable functional* $\mathfrak{F} = \int_\Sigma \boldsymbol{\varphi}$ *defined on the set of* n*-dimensional submanifolds* $\mathcal{E}^{\mathcal{H}}$ *is given by the map:*

$$\mathfrak{F}_{\boldsymbol{\varphi}} = \int_\Sigma \boldsymbol{\varphi} : \begin{cases} \mathcal{E}^{\mathcal{H}} & \longrightarrow \quad \mathbb{R} \\ \Gamma & \mapsto \quad \mathfrak{F}(\Gamma) = \displaystyle\int_{\Sigma\cap\Gamma} \boldsymbol{\varphi} \end{cases} \tag{94}$$

Then for any $\boldsymbol{\varphi},\boldsymbol{\eta} \in \mathfrak{P}^{n-1}_\circ$ the Poisson bracket - which coincides with the standard bracket for field theory - between two observable functionals $\int_\Sigma \boldsymbol{\varphi}$ and $\int_\Sigma \boldsymbol{\eta}$ is defined such that $\forall\ \Gamma \in \mathcal{E}^{\mathcal{H}}$ we have (95).

$$\left\{\int_\Sigma \boldsymbol{\varphi}, \int_\Sigma \boldsymbol{\eta}\right\}(\Gamma) := \int_{\Sigma\cap\Gamma} \{\boldsymbol{\varphi},\boldsymbol{\eta}\} \tag{95}$$

This Poisson bracket satisfies the Jacobi identity. Let us consider $\boldsymbol{\varphi},\boldsymbol{\rho},\boldsymbol{\eta} \in \mathfrak{P}^{n-1}_\circ$. From previous considerations, we know that $\{\{\boldsymbol{\varphi},\boldsymbol{\rho}\},\boldsymbol{\eta}\} + \{\{\boldsymbol{\rho},\boldsymbol{\eta}\},\boldsymbol{\varphi}\} + \{\{\boldsymbol{\eta},\boldsymbol{\varphi}\},\boldsymbol{\rho}\} = d(\xi_{\boldsymbol{\varphi}} \wedge \xi_{\boldsymbol{\rho}} \wedge \xi_{\boldsymbol{\eta}} \lrcorner\, \boldsymbol{\omega})$ which gives by antisymmetry $\{\boldsymbol{\varphi},\{\boldsymbol{\rho},\boldsymbol{\eta}\}\} + \{\boldsymbol{\eta},\{\boldsymbol{\varphi},\boldsymbol{\rho}\}\} + \{\boldsymbol{\rho},\{\boldsymbol{\eta},\boldsymbol{\varphi}\}\} = -d(\xi_{\boldsymbol{\varphi}} \wedge \xi_{\boldsymbol{\rho}} \wedge \xi_{\boldsymbol{\eta}} \lrcorner\, \boldsymbol{\omega})$.. Therefore, restricting ourselves to the study of functional observables along Hamiltonian n-curves Γ such that $\partial\Gamma = \emptyset$ we have the Jacobi identity as:

$$\left\{\int_\Sigma \boldsymbol{\varphi}, \left\{\int_\Sigma \boldsymbol{\varrho}, \int_\Sigma \boldsymbol{\eta}\right\}\right\} + \left\{\int_\Sigma \boldsymbol{\eta}, \left\{\int_\Sigma \boldsymbol{\varphi}, \int_\Sigma \boldsymbol{\varrho}\right\}\right\} + \left\{\int_\Sigma \boldsymbol{\varrho}, \left\{\int_\Sigma \boldsymbol{\eta}, \int_\Sigma \boldsymbol{\varphi}\right\}\right\} = 0.$$

6.7 Dynamical observable functionals

The question of dynamical observable functionals hold the key to a *fully covariant theory.* In the perspective of a covariant theory, one would like to define a bracket over two different slices Σ°

and $\Sigma^\bullet$. The bracket defined previously in (95) depends on the choice of the given slice Σ. Given, $\boldsymbol{\varphi},\boldsymbol{\eta}\in\mathfrak{P}^{n-1}_\circ$ we define the following bracket:

$$\left\{\int_{\Sigma^\circ}\boldsymbol{\varphi},\int_{\Sigma^\bullet}\boldsymbol{\eta}\right\}(\Gamma):=\int_{\Sigma^\circ\cap\Gamma}\{\boldsymbol{\varphi},\boldsymbol{\eta}\} \tag{96}$$

Therefore, we are interested in dynamical observable functionals. It is precisely for *dynamical observables*[47] that we can construct a fully covariant bracket (96). We consider an algebraic observable $(n-1)$-form $\boldsymbol{\varphi}\in\mathfrak{P}^{n-1}_\circ$ via its related infinitesimal symplectomorphism $\Xi_{\boldsymbol{\varphi}}\in C^\infty(\mathcal{M},T\mathcal{M})$ which is the unique vector field such that $\Xi_{\boldsymbol{\varphi}}\lrcorner\,\boldsymbol{\omega}=-d\boldsymbol{\varphi}$. The algebraic observable $(n-1)$-form becomes a dynamical observable if we have the additional condition:

$$\Xi_{\boldsymbol{\varphi}}\lrcorner\, d\mathcal{H}=d\mathcal{H}(\Xi_{\boldsymbol{\varphi}})=0. \tag{97}$$

This condition reflects a homological feature: if Γ is a Hamiltonian n-curve, then this functional $\mathfrak{F}(\Gamma)$ depends only on the homology class of Σ [117, 118]. More precisely, following Hélein [114, 113] we show that this result follows from:

Proposition 6.2. *Let $\boldsymbol{\varrho}\in\Gamma(\mathcal{M},\Lambda^{n-1}T^\star\mathcal{M})$ be a dynamical $(n-1)$-form. Let Σ° and $\Sigma^\bullet$ be two slices such that there exists an open subset $\mathcal{D}\subset\mathcal{M}$ which verifies $\partial\mathcal{D}=\Sigma^\circ-\Sigma^\bullet$. We have the following equality: $\int_{\Sigma^\circ}\boldsymbol{\varrho}=\int_{\Sigma^\bullet}\boldsymbol{\varrho}$.*

6.8 Copolarization and observables $(p-1)$-forms

For details we refer to Hélein and Kouneiher [116, 117, 118]. Here we set out some key features of the notion of copolarization. In particular we give the general definition of *copolarization.*

Definition 6.8.1. *Let $(\mathcal{M},\boldsymbol{\omega})$ be a multisymplectic manifold. A copolarization on $(\mathcal{M},\boldsymbol{\omega})$ is a smooth vector sub-bundle $\mathbf{P}^*_\bullet T^\star\mathcal{M}\subset\Lambda^*T^\star\mathcal{M}$ which satisfies:*

[1] $\mathbf{P}^*T^\star_\bullet\mathcal{M}=\bigoplus_{1\leq i\leq n}\mathbf{P}^i_\bullet T^\star\mathcal{M}$

[2] *Locally, for any $m\in\mathcal{M}$, $\left(\mathbf{P}^*_\bullet T^\star_m\mathcal{M},+,\wedge\right)$ is a subalgebra of $\left(\Lambda^nT^\star_m\mathcal{M},+,\wedge\right)$*

[3] $\forall m\in\mathcal{M},\forall\phi\in\Lambda^nT^\star_m\mathcal{M},\phi\in\mathbf{P}^n_\bullet T^\star_m\mathcal{M}$ *if and only if* $\forall X,\tilde{X}\in\mathcal{O}_m,\ X\lrcorner\,\boldsymbol{\omega}=\tilde{X}\lrcorner\,\boldsymbol{\omega}\Rightarrow\phi(X)=\phi(\tilde{X})$.

We say that a multisymplectic manifold $(\mathcal{M},\boldsymbol{\omega})$ is equipped with the copolarization $\mathbf{P}^*T^\star\mathcal{M}$. The notion of copolarization intrinsically defines for any $1\leq p\leq n$ the set $\mathfrak{P}^{p-1}(\mathcal{M})$, namely the set of observable $(p-1)$-forms $\boldsymbol{\varphi}$ by $\forall m\in\mathcal{M},d\boldsymbol{\varphi}_m\in\mathbf{P}^pT^\star_m\mathcal{M}$. We refer to [115, 116, 117, 118] for the construction of the *standard copolarization* and example of the Maxwell copolarization. The idea is that this is the natural geometrical setting for canonical forms for field theory based upon potential-like canonical variables. In the case of Maxwell theory, the two canonical forms are the potential 1-form $A=A_\mu dx^\mu$ and its canonical form is the so-called Faraday 2-form (in the 4 dimensional case) $\pi=1/2p^{A_\mu\nu}\beta_{\mu\nu}=1/2\sum_{\mu,\nu}p^{A_\mu\nu}\partial_\mu\lrcorner\,\partial_\nu\lrcorner\,\beta$. In a more general perspective - for gravity or non-abelian theories such as Yang-Mills - we describe a couple of canonical forms as $(\omega,\boldsymbol{\varpi})$. The general setting allows us to construct a well-defined Poisson bracket between observable functionals for such couples of canonical forms $(\omega,\boldsymbol{\varpi})$:

$$\left\{\int_{\Sigma\cap\Gamma_\varkappa}\boldsymbol{\varpi},\int_{\Sigma\cap\Gamma_\varsigma}\omega\right\}(\Gamma)=\sum_{m\in\Sigma\cap\Gamma_\varkappa\cap\Gamma_\varsigma\cap\Gamma}\mathfrak{S}(m) \tag{98}$$

We give more details later about the bracket (98) and the related geometrical objects $\Sigma\cap\Gamma_\varkappa$, $\Sigma\cap\Gamma_\varsigma$ and $\Sigma\cap\Gamma_\varkappa\cap\Gamma_\varsigma\cap\Gamma$, as well as the counting object $\mathfrak{S}(m)$. Notice that the study of $(p-1)$-forms involves analogous definitions for *slices* in this case. We have the following:

[47] Or *physical* observables see some remarks in the previous section (6.1)

Definition 6.8.2. *A slice of codimension $(n-p+1)$ is a submanifold $\Sigma \subset \mathcal{M}$ of codimension $(n-p+1)$ such that $T_m\mathcal{M}/T_m\Sigma$ is smoothly oriented with regard to m and, such that for any $\Gamma \in \mathcal{E}^{\mathcal{H}}$, Σ is transverse to Γ.*

We refer to [117, 118] for the question of orientation for the intersection $\Sigma\cap\Gamma$. The straightforward analogue of definition (6.8.3) for the case of arbitrary $(p-1)$-forms is:

Definition 6.8.3. *Let Σ be a slice of codimension 1 and let $\boldsymbol{\varphi}$ be an algebraic observable $(n-1)$-form. An observable functional $\mathfrak{F} = \int_\Sigma \boldsymbol{\varphi}$ defined on the set of n-dimensional submanifolds $\mathcal{E}^{\mathcal{H}}$ given by the map:*

$$\mathfrak{F}_{\boldsymbol{\varphi}} = \int_\Sigma \boldsymbol{\varphi} : \begin{cases} \mathcal{E}^{\mathcal{H}} & \longrightarrow \quad \mathbb{R} \\ \Gamma & \mapsto \quad \mathfrak{F}(\Gamma) = \displaystyle\int_{\Sigma\cap\Gamma} \boldsymbol{\varphi} \end{cases} \tag{99}$$

Now we emphasize two conceptual points. The first is that the notion of *copolarization* definitively emerges from the philosophy of GR. This highlights the fact that we can not evaluate $d\boldsymbol{\varphi}$ along a Hamiltonian n-vector X. If $1 \leq p < n$ then an arbitrary $(p-1)$-form is necessarily of maximum degree $(n-2)$. Then how are we to understand $d\boldsymbol{\varphi}\big|_X$? It seems we lack a good dynamical duality. Hélein and Kouneiher supply this lack precisely through the notion of copolarization. We construct a set of n 0-forms $\{\boldsymbol{\rho}_i\}_{1\leq i\leq n}$. These n 0-forms are found in the copolarization of the multisymplectic manifold under study $(\mathcal{M},\boldsymbol{\omega})$. These are observables 0-forms: $\forall 1 \leq i \leq n, \boldsymbol{\rho}_i \in \mathfrak{P}^0_\bullet(\mathcal{M})$. Locally we write for $m \in \mathcal{M}$ $\forall 1 \leq i \leq n, d\boldsymbol{\rho}_i \in \mathbf{P}^1_{\bullet m}T^\star_m(\mathcal{M})$. Hence, we reach the full dynamical duality: the evaluation on a Hamiltonian vector field X of the product: $\bigwedge_{1\leq i\leq n} d\boldsymbol{\rho}_i$. This evaluation $\bigwedge_{1\leq i\leq n} d\boldsymbol{\rho}_i(X)$ which only depends on $d\mathcal{H}_m$ which means that $\bigwedge_{1\leq i\leq n} d\boldsymbol{\rho}_i = d\boldsymbol{\rho}_1 \wedge \cdots \wedge \boldsymbol{\rho}_n(X)$ is a copolar form. In the philosophy of GR once again, this is fully acceptable since we never measure an observable *per se* but only compare observable quantities between observables. Now we very briefly discuss the $(p-1)$-bracket found in [115, 116, 117, 118] via the table below. We describe an equivalence class of (decomposable) Hamiltonian vector fields $[X]^{\mathcal{H}}$, observing that if $X \sim \tilde{X}$, we have for any $1 \leq p \leq n$ and $\phi \in \mathbf{P}^p_\bullet T^\star_m\mathcal{M}$, $X \llcorner \phi \sim \tilde{X} \llcorner \phi$ so that we define the equivalence class $[X] \llcorner \phi = [X \llcorner \phi] \in \mathbf{P}^{n-p}_\bullet T\mathcal{M}$. We also have the notion of algebraic copolarization. This involves the same set of rules but with the replacement of $\mathbf{P}^n_\bullet T^\star_m\mathcal{M}$ by $\mathbf{P}^n_\circ T^\star_m\mathcal{M} \subset \mathbf{P}^n_\bullet T^\star_m\mathcal{M}$.

Algebraic Observable forms (AOF)

> Algebraic copolarization, $\mathbf{P}^n_\circ T^\star_m\mathcal{M}$
> $\forall \boldsymbol{\varphi} \in \mathfrak{P}^{p-1}_\circ\mathcal{M}$ $\forall \boldsymbol{\phi} \in \mathbf{P}^{\text{n-p}}_\circ T^\star_m\mathcal{M}$here exists a unique $\Xi_{\boldsymbol{\varphi}}(\boldsymbol{\phi}) \in T_m\mathcal{M}$ such that
> $\boldsymbol{\phi} \wedge d\boldsymbol{\varphi} + \Xi_{\boldsymbol{\varphi}} \lrcorner \boldsymbol{\omega} = 0$
> Pseudobracket: section of $\mathbf{P}^{n-p}_\circ T\mathcal{M}$ $\{\mathcal{H},\boldsymbol{\varphi}\} = -\Xi_{\boldsymbol{\varphi}} \lrcorner d\mathcal{H}$

Observable Forms (OF)

> Copolarization $\mathbf{P}^n_\bullet T^\star_m\mathcal{M}$
> $\boldsymbol{\varphi} \in \mathfrak{P}^{p-1}_\bullet\mathcal{M}$, $\mathcal{H}$ a Hamiltonian function
> Pseudobracket: section of $\mathbf{P}^{n-p}_\bullet T\mathcal{M}$ $\{\mathcal{H},\boldsymbol{\varphi}\} = (-1)^{(n-p)p}[X]^{\mathcal{H}} \llcorner d\boldsymbol{\varphi}$

Hence, rather than using the classical Poisson bracket with graded structure - see section (5.5) - we understand that the copolarization puts the dynamics directly **in** the algebra. We develop this point in part [II] the example of Maxwell multisymplectic theory.

7 Pre-multisymplectic and covariant phase space

The *Covariant Phase Space* approach (CPS) shares many features with MG. The main idea is that we are not working on ordinary phase space but rather on the space of solutions $\mathcal{E}^{\mathcal{H}}$ of a Hamiltonian[48] dynamical problem, namely a functional space. As noticed by G. J. Zuckerman [249] and also though the work of Goldschmidt-Sternberg [100] (1973) and Crnkovic-Witten [49] (1987) the key observation is to define a canonical pre-symplectic structure[49] $\boldsymbol{\varpi} = \delta\Theta^{\Sigma}$ on such a functional space. We will not enter into details here but refer the reader to the ingenious paper of Hélein [111] which draws on the work of Szczyrba-Kijowski (1976) [144], see also the work of Forger and Romero in [80]. See also [111] for the relation of the CPS approach with modern multisymplectic geometry. One observes the same fundamental mathematical entity in both approaches, MG and CPS, clearly manifested in the *n-phase space* notion. We should underline that the *invariant language* provided crucial insights merging from the theory of integral invariants - found by H. Poincaré and further developed by Cartan [39]). Within this approach to dynamical principles, which we may call the *Cartan principle of dynamics*, we find a deep connection between MG and CPS. This relation may be seen as a modern continuation of De Donder's attempt to extend the notion of integral invariants to variational problems with several variables. Finally, we emphasize the strong relation of CPS with the algebraic-topological theory of Vinogradov and the so-called secondary calculus. [178, 231, 232]. See the work of L. Vitaglino [234] for the relation between both approaches. This approach uses tools such as $\mathcal{C}$-spectral sequence or variational bi-complex, see I. Anderson [5] for an introduction on variational bi-complex. Today development of this subject centers mainly on the work of the so-called *diffiety school*. This addresses within a beautiful cohomological setting the idea of a *local* functional differential calculus on the space of solutions of a generic system of *partial differential equations*. We mention some key points very briefly. We introduce the notion of a n-phase space, inspired by Kijowski and Szczyrba [144], and developed further by Hélein [111] [113].

Definition 7.0.4. *A n-multiphase space (or simply an n-phase space) is a triple* $(\mathcal{M}, \boldsymbol{\omega}, \beta)$ *where* $\mathcal{M}$ *is a smooth manifold,* $\boldsymbol{\omega}$ *is a closed (n + 1)-form and* β *is an everywhere non-vanishing n-form.*

7.1 n-phase space: generalized Hamilton equations

Recall that for the multisymplectic case, described by the data $(\mathcal{M}, \boldsymbol{\omega})$ with $\mathcal{H} : \mathcal{M} \to \mathbb{R}$, with $d\mathcal{H} \neq 0$, we have a Hamiltonian n-curve Γ as an oriented n-dimensional submanifold for generalized Hamilton equations:

$$\forall m \in \Gamma, \exists X \in \Lambda^n T_m\Gamma \quad X \lrcorner \boldsymbol{\omega}_m = (-1)^n d\mathcal{H}_m.$$

For a n-phase space $(\mathcal{M}, \boldsymbol{\omega}, \beta)$, a Hamiltonian n-curve is the analogous data. Here it is pictured as an oriented n-submanifold which satisfies:

$$\forall m \in \Gamma, \forall X \in \Lambda^n T_m\Gamma \quad X \lrcorner \boldsymbol{\omega}_m = 0 \qquad \text{and} \qquad \forall m \in \Gamma, \exists X \in \Lambda^n T_m\Gamma \quad X \lrcorner \beta_m \neq 0.$$

The last condition is an independence condition and we recover the general considerations developed in the begining of this section. As emphasized above, we can canonically construct n-phase space data by means of the hypersurface of a multisymplectic manifold. We recall that a premultisymplectic n-form is closed but may be degenerate. In the general picture of a n phase space we express *dynamics* on a *level set* of $\mathcal{H}$. We can construct canonically a pre-n-multisymplectic manifold $(\boldsymbol{\mathcal{M}}^{\mathbf{o}}, \boldsymbol{\omega}|_{\boldsymbol{\mathcal{M}}^{\mathbf{o}}}, \beta = \boldsymbol{\eta} \lrcorner \boldsymbol{\omega}|_{\boldsymbol{\mathcal{M}}^{\mathbf{o}}})$. Here the $\boldsymbol{\omega}|_{\boldsymbol{\mathcal{M}}^{\mathbf{o}}} = \mathcal{H}^{-1}(0) := \{(q,p) \in \boldsymbol{\omega}|_{\boldsymbol{\mathcal{M}}} | \ \mathcal{H}(q,p) = 0\}$

[48] It may also be viewed from a Lagrangian standpoint: the CPS of a Lagrangian field theory is the solution space of the associated Euler-Lagrange equations.

[49] $\boldsymbol{\varpi} = \delta\Theta^{\Sigma}$ is the symplectic structure defined on CPS.

and $\boldsymbol{\eta}$ is a vector field such that $d\mathcal{H}(\boldsymbol{\eta}) = 1$. In this case we clarify the connection between relativistic dynamical systems and the treatment of the *Hamiltonian constraint*. We recover the dynamical equations in the pre-multisymplectic case, written here (123) for the n-dimensional case - see Hélein, [111] [113, 114].

$$\forall \Xi \in C^\infty(\mathcal{M}, T_m\mathcal{M}), \qquad (\Xi \,\lrcorner\, \boldsymbol{\omega})|_\Gamma = 0 \qquad \text{and} \qquad \beta|_\Gamma \neq 0. \tag{100}$$

7.2 Observable and pre-multisymplectic case

The pre-multisymplectic case is described by the n-phase space data. In this connection a very interesting phenomenon concerning observables and observable functionals appears. We discuss only the case of $(n-1)$- algebraic observable forms. The condition that marks the transition from the algebraic to the dynamical setting is expressed, given an infinitesimal symplectomorphism $\Xi_{\boldsymbol{\varphi}} \in C^\infty(\mathcal{M}, T\mathcal{M})$ by:

$$\Xi_{\boldsymbol{\varphi}} \,\lrcorner\, d\mathcal{H} = d\mathcal{H}(\Xi_{\boldsymbol{\varphi}}) = 0.$$

We observe that in the pre-multisymplectic scenario this condition is automatically verified so that any observable $(n-1)$-form is a dynamical one. Once again we refer to [111] [113, 114] for more detailed view.

7.3 Covariant phase space and observable functionals

We refer to the work of Hélein [111, 113] for the relation between CPS and observable functionals. As in the case for dynamical observables - see section (6.7) - there appears once again the central notion of two slices (hypersurfaces) Σ° and $\Sigma^\bullet$ in the same homology class. The central formula in Hélein's work crystalizes two related notions: the canonical pre-symplectic form $\boldsymbol{\varpi}$ for CPS symplectic structure and the theory of integral invariants. [39] We observe the connection between MG and CPS from the point of view of Poincaré-Cartan invariant integrals [39] in the spirit of the DeDonder generalization.

MULTISYMPLECTIC MAXWELL THEORY

The simplest application of MG to field theory is the treatment of the scalar field theory which is free from constraint issue (for the non interacting case). We refer to the work of Kijowski, and Szczyrba [143, 144], Hélein [114, 111], Hélein and Kouneiher [115, 116] and in particular to the thesis of R.D. Harrivel[50] which offers a good introduction to Klein-Gordon multisymplectic theory. In this section, we give a detailed treatment of the multisymplectic technics for Maxwell's theory. We first treat the DW case. The Maxwell theory is the most example for gauge theory corresponding to the non-abelian case. In that case the Dirac primary constraint set is given by $p^{A_\nu \mu} + p^{A_\mu \nu} = 0$. The Maxwell theory is the first example before we treat the case of Palatini gravity in part [III]. The Palatini gravity example is concerned in the study of the constraints with a much complex structure. We recover, in a more embracing view, the results of Kijowski, and Szczyrba [143, 144]. The physical considerations which underline this development are the *gauge potential* and the canonical Poisson bracket structure.

[50] which exposes a much more embracing viewpoint on the interacting picture and quantization,RD-01

Let us recall the expression of the Euler-Lagrange equations for the Maxwell theory. We are first interested with the *vacuum* Maxwell action, given by:

$$(\mathrm{i})\quad \mathcal{L}^{\circ}_{\mathsf{Maxwell}} = \frac{1}{2}\int_{\mathcal{X}} F\wedge\star F, \qquad (\mathrm{ii})\quad \mathcal{L}^{\circ}_{\mathsf{Maxwell}}[x,A,dA] = -\frac{1}{4}\int_{\mathcal{X}} F_{\mu\nu}F^{\mu\nu}\sqrt{-\mathsf{g}}\beta. \tag{101}$$

The Lagrangian density is $L(A) = -1/4F_{\mu\nu}F^{\mu\nu}\sqrt{-\mathsf{g}}$. We denote $\mathrm{vol}_{\mathcal{X}}(\mathsf{g})$ a Riemannian volume form such that $\mathrm{vol}_{\mathcal{X}}(\mathsf{g}) = \sqrt{-\mathsf{g}}d^4x$. In the case where $\mathcal{X}$ is the Minkowski space-time we obtain then $\sqrt{-\mathsf{g}} = 1$ and then $L(A) = -1/4F_{\mu\nu}F^{\mu\nu}$. We have Maxwell vacuum equations:

$$dF = 0 \qquad \text{and} \qquad d\star F = 0 \tag{102}$$

Considering the current of matter $J^{\mu}(x)$ over $\mathcal{X}$ the Lagrangian is written:

$$\mathcal{L}_{\mathsf{Maxwell}}[x,A,dA] = \mathcal{L}^{\circ}_{\mathsf{Maxwell}}[x,A,dA] - J^{\mu}(x)A_{\mu}, \tag{103}$$

and the Euler-Lagrange equations are written:

$$dF = 0 \qquad \text{and} \qquad d\star F = \star J. \tag{104}$$

The curvature in components is : $F = \frac{1}{2}F_{\mu\nu}dx^{\mu}\wedge dx^{\nu}$, with $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$. Thus, we write the Hodge star $\star F$ in components:

$$\begin{aligned}\star F &= \star\big(\frac{1}{2!}F_{\mu\nu}dx^{\mu}\wedge dx^{\nu}\big) = \frac{1}{2!}\big(\star F\big)_{\rho\sigma}dx^{\rho}\wedge dx^{\sigma}, \qquad \text{with} \qquad \big(\star F\big)_{\rho\sigma} = \frac{1}{2!}\boldsymbol{\epsilon}^{\mu\nu}{}_{\rho\sigma}F_{\mu\nu}\\ &= \frac{1}{2!}\frac{1}{2!}\boldsymbol{\epsilon}^{\mu\nu}{}_{\rho\sigma}F_{\mu\nu}dx^{\rho}\wedge dx^{\sigma} = \frac{\sqrt{-\mathsf{g}}}{4}\epsilon^{\mu\nu}{}_{\rho\sigma}F_{\mu\nu}dx^{\rho}\wedge dx^{\sigma} = \frac{\sqrt{-\mathsf{g}}}{4}\mathsf{g}_{\alpha\mu}\mathsf{g}_{\beta\nu}\epsilon^{\mu\nu}{}_{\rho\sigma}F^{\alpha\beta}dx^{\rho}\wedge dx^{\sigma}.\end{aligned}$$

We see the equivalence between (101)(i) and (101)(ii).

⌈ Proof Here we make a straightforward computation which involves the Hodge duality.

$$F\wedge\star F = \Big[\frac{1}{2}F_{\lambda\varsigma}dx^{\lambda}\wedge dx^{\varsigma}\Big]\wedge\Big[\frac{\sqrt{-\mathsf{g}}}{4}\mathsf{g}_{\alpha\mu}\mathsf{g}_{\beta\nu}\epsilon^{\mu\nu}{}_{\rho\sigma}F^{\alpha\beta}dx^{\rho}\wedge dx^{\sigma}\Big] = \Big[\frac{1}{8}F_{\lambda\varsigma}\sqrt{g}\mathsf{g}_{\alpha\mu}\mathsf{g}_{\beta\nu}\epsilon^{\mu\nu}{}_{\rho\sigma}F^{\alpha\beta}\Big]dx^{\lambda}\wedge dx^{\varsigma}\wedge dx^{\rho}\wedge dx^{\sigma}$$

Since $\mathrm{vol}_{\mathcal{X}}(\mathsf{g}) = \sqrt{g}\beta = \frac{1}{4!}\boldsymbol{\epsilon}_{\lambda\varsigma\rho\sigma}dx^{\lambda}\wedge dx^{\varsigma}\wedge dx^{\rho}\wedge dx^{\sigma}$, we obtain

$$\begin{aligned}F\wedge\star F &= \Big[\frac{1}{8}F_{\lambda\varsigma}\sqrt{g}\mathsf{g}_{\alpha\mu}\mathsf{g}_{\beta\nu}\epsilon^{\mu\nu}{}_{\rho\sigma}F^{\alpha\beta}\Big]dx^{\lambda}\wedge dx^{\varsigma}\wedge dx^{\rho}\wedge dx^{\sigma} = \frac{1}{8}F_{\lambda\varsigma}F^{\alpha\beta}\boldsymbol{\epsilon}_{\alpha\beta\rho\sigma}\boldsymbol{\epsilon}^{\lambda\varsigma\rho\sigma}\sqrt{g}\beta\\ &= -\frac{1}{2}\delta^{[\lambda}_{\alpha}\delta^{\varsigma]}_{\beta}F_{\lambda\varsigma}F^{\alpha\beta}\sqrt{g}\beta = -\frac{1}{2}\frac{1}{2}\Big[F_{\alpha\beta}F^{\alpha\beta} - F_{\beta\alpha}F^{\alpha\beta}\Big]\sqrt{g}\beta = -\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta}\sqrt{g}\beta\end{aligned}$$

where in the last line we have used $\boldsymbol{\epsilon}_{\alpha\beta\rho\sigma}\boldsymbol{\epsilon}^{\lambda\varsigma\rho\sigma} = -2!2!\delta^{[\lambda}_{\alpha}\delta^{\varsigma]}_{\beta}$ ⌋.

The Euler-Lagrange equations for the Maxwell theory are written (105)

$$\underbrace{\frac{\partial}{\partial A_{\nu}}\mathcal{L}_{\mathsf{Maxwell}}}_{[\mathbf{I}]} = \underbrace{\partial_{\mu}\Big(\frac{\partial}{\partial(\partial_{\mu}A_{\nu})}\mathcal{L}_{\mathsf{Maxwell}}\Big)}_{[\mathbf{II}]}. \tag{105}$$

We recover the Maxwell's equations:

$$J^{\nu}(x) = -\frac{\partial}{\partial x^{\mu}}F^{\mu\nu}(x). \tag{106}$$

⌜ Proof We compute **[I]** and **[II]**. The first term is: $[\mathbf{I}] = \frac{\partial \mathcal{L}}{\partial A_\nu} = J^\nu(x)$ The second leads to the following calculation:

$$\begin{aligned}
[\mathbf{II}] &= \partial_\mu\Big(-\frac{1}{4}\frac{(F_{\alpha\beta}F^{\alpha\beta})}{\partial(\partial_\mu A_\nu)}\Big) = -\frac{1}{4}\partial_\mu\Big(\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}F^{\alpha\beta} + F_{\alpha\beta}\frac{\partial F^{\alpha\beta}}{\partial(\partial_\mu A_\nu)}\Big) = -\frac{1}{4}\partial_\mu\Big(\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}F^{\alpha\beta} + F_{\alpha\beta}\frac{\partial(\mathsf{g}^{\alpha\sigma}\mathsf{g}^{\beta\rho}F_{\sigma\rho})}{\partial(\partial_\mu A_\nu)}\Big)\\
&= -\frac{1}{4}\partial_\mu\Big(\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}F^{\alpha\beta} + F^{\sigma\rho}\frac{\partial(F_{\sigma\rho})}{\partial(\partial_\mu A_\nu)}\Big) = -\frac{1}{2}\partial_\mu\Big(\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}F^{\alpha\beta}\Big) = -\frac{1}{2}\partial_\mu\Big(\big(\frac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_\mu A_\nu)} - \frac{\partial(\partial_\beta A_\alpha)}{\partial(\partial_\mu A_\nu)}\big)F^{\alpha\beta}\Big)\\
&= -\frac{1}{2}\partial_\mu\Big(\big(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\nu_\alpha\delta^\mu_\beta\big)F^{\alpha\beta}\Big) = -\frac{1}{2}\partial_\mu(F^{\mu\nu} - F^{\nu\mu}) = -\partial_\mu F^{\mu\nu}
\end{aligned}$$

Then we obtain Maxwell's equations (106) ⌟

8 Multisymplectic De Donder-Weyl-Maxwell theory

We describe the geometrical setting and the notations for the four dimensional case. We consider $\mathcal{X}$ to be the *spacetime* manifold with $dim(\mathcal{X}) = n = 4$. Let $A \in T^\star\mathcal{X}$, be the potential 1-form. The space of interested is $\mathfrak{Z} = T^\star\mathcal{X}$. As noticed in [116, 117], the more naive approach is to work in a *local trivialization* of a bundle over $\mathcal{X}$, since a connection is not a section of a bundle. This is the chosen path here. A point (x, A) in $\mathfrak{Z}$ is in the *position* configuration space. Any choice (x, A) is equivalent to the data of an n-dimensional submanifold in $\mathfrak{Z} = T^\star\mathcal{X} \xrightarrow{\pi} \mathcal{X}$ described as a section of the fiber bundle over $\mathcal{X}$. Let us consider the map $\mathfrak{z}_A : \mathcal{X} \to \mathfrak{Z} = T^\star\mathcal{X}$ described by (107),

$$\mathfrak{z}_A : \begin{cases} \mathcal{X} & \to \quad \mathfrak{Z} = T^\star\mathcal{X} \\ x & \mapsto \quad A(x) = A_\mu(x)dx^\mu. \end{cases} \tag{107}$$

which is simply some section of the related bundle. We associate with A, the bundle $\mathcal{P}^A = A^\star T\mathfrak{Z} \otimes_{\mathfrak{Z}} T^\star\mathcal{X}$. The useful quantities to describe dA the differential of the map A as sections of the bundle $\mathcal{P}^A$ over $\mathcal{X}$. We denote the exterior covariant derivative on the 1-form A by $d^{\mathbf{D}}A$:

$$d^{\mathbf{D}}A \;=\; \big[d^{\mathbf{D}}A\big]_{\mu\nu}dx^\mu \wedge dx^\nu \qquad \text{with} \qquad \big[d^{\mathbf{D}}A\big]_{\mu\nu} = \partial_{[\mu}A_{\nu]} \tag{108}$$

We denote $v_{\mu\nu} = \partial_\mu A_\nu$ so that $d^{\mathbf{D}}A = v_{[\mu\nu]}$. The space of interest, the analogue for tangent space is $\Lambda^n T_{(x,e,\omega)}\mathfrak{Z}$ the fiber bundle of n-vector field of $\mathfrak{Z}$ over $\mathcal{X}$. For any $(x^\mu, A_\nu) \in \mathfrak{Z}$, the fiber $\Lambda^n T_{(x,A)}(T^\star\mathcal{X}) = \Lambda^n T_{(x,A)}\mathfrak{Z}$ can be identified with $\mathcal{P} = A^\star T\mathfrak{Z} \otimes_{\mathfrak{Z}} T^\star\mathcal{X}$ via the diffeomorphism:

$$\begin{cases} \mathcal{P} \cong A^\star T\mathfrak{Z} \otimes_{\mathfrak{Z}} T^\star\mathcal{X} & \to \quad \Lambda^n T_{(x,A)}(T^\star\mathcal{X}) \\ \displaystyle\sum_{\mu,\nu} \big[d^{\mathbf{D}}A\big]_{\mu\nu}dx^\mu \otimes dx^\nu & \mapsto \quad z = z_1 \wedge ... \wedge z_n, \end{cases} \tag{109}$$

where $\forall 1 \le \eta \le n$ $z_\eta = \frac{\partial}{\partial x^\alpha} + \sum_{1\le\beta\le n} v_{\alpha\beta}\frac{\partial}{\partial A_\beta}$. In order to fix ideas we stress that we have local coordinates $(x^\mu, A_\mu,)$ for the configuration bundle $\mathfrak{Z}$. The data of the local coordinates $(x^\mu, A_\mu, v_{\mu\nu})$ - or equivalentely $(x^\mu, A_\mu, z_{\mu\nu})$- can be thought as coordinates on $\mathcal{P}$ or $\Lambda^n T_{(x,e,\omega)}(\mathfrak{Z})$ We identify $\mathcal{P} \cong \Lambda^n T_{(x,e,\omega)}(\mathfrak{Z})$.

In this section we first expose the setting of the DW-Maxwell theory. First we exhibit in (8.1) the Dirac primary constraint set and the related Maxwell multisymplectic manifold $\mathcal{M}_{\mathsf{Maxwell}}$, see (116). Then we derive the generalized Hamilton equations in the multisymplectic (8.2) and in the pre-multisymplectic (8.3) settings. In the latter, we observe the connection with the covariant phase space.

8.1 Multisymplectic De Donder-Weyl-Maxwell theory

Generalized Legendre correspondence. the generalized Legendre correspondence is constructed on $\mathcal{M} = \Lambda^n T^\star\mathfrak{Z}$. For all $(q, p) \in \mathcal{M}$ we introduce the local coordinates on the bundle $\mathcal{M}$. Let us denote $(q^{\boldsymbol{\mu}})_{1\leq\boldsymbol{\mu}\leq 2n}$ the local coordinates on $\mathfrak{Z} = T^\star\mathcal{X}$ and $p_{\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_n}$ the local coordinates on $\Lambda^n T^\star_q\mathfrak{Z}$ in the basis $(dq^{\boldsymbol{\mu}_1}\wedge\cdots\wedge dq^{\boldsymbol{\mu}_n})_{1<\boldsymbol{\mu}_1<\cdots<\boldsymbol{\mu}_n<2n}$, completely antisymmetric in $(\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_n)$. The canonical Poincaré-Cartan n-form is written in local coordinates (here $n = k = 4$):

$$\theta = \sum_{1<\boldsymbol{\mu}_1<\cdots<\boldsymbol{\mu}_n<2n} p_{\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_n} dq^{\boldsymbol{\mu}_1}\wedge\ldots\wedge dq^{\boldsymbol{\mu}_n}.$$

We consider the De Donder-Weyl submanifold $\mathcal{M}_{\mathsf{DW}} \subset \mathcal{M}$:

$$\mathcal{M}_{\mathsf{DW}} = \{(x,A,p)/x\in\mathcal{X}, A\in T^\star\mathcal{X}, p\in\Lambda^n T^\star(T^\star\mathcal{X}) \quad \text{such that} \quad \partial_{A_\mu}\wedge\partial_{A_\nu}\lrcorner\, p = 0\}.$$

We restrict and adapt our notations to the case $\mathcal{M}_{\mathsf{DW}}\subset\mathcal{M}$. All the components $p_{\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_n}$ are taken equal to zero, excepted for $p_{1\ldots n} = \mathfrak{e}$ and for the multimomenta $p_{1\ldots(\nu-1)(A_\mu)(\nu+1)\ldots n}$ denoted $p^{A_\mu\nu}$. We define a Legendre correspondence[51]:

$$\Lambda^n T(T^\star\mathcal{X})\times\mathbb{R} = \Lambda^n T\mathfrak{Z}\times\mathbb{R} \leftrightarrow \Lambda^n T^\star(T^\star\mathcal{X}) = \Lambda^n T^\star\mathfrak{Z} : (q,v,w)\leftrightarrow(q,p),$$

which is generated by the function $\mathcal{W} : \Lambda^n T\mathfrak{Z}\times\Lambda^n T^\star\mathfrak{Z} \to \mathbb{R}(q,v,p)\mapsto\langle p,v\rangle - L(q,v)$.

Maxwell multisymplectic manifold $\mathcal{M}_{\mathsf{Maxwell}}$. Let us describe the general construction for the De Donder-Weyl multisymplectic manifold. We consider $\theta^{\mathsf{DW}}_{(q,p)}$, the Poincaré-Cartan n-form:

$$\theta^{\mathsf{DW}}_{(q,p)} := \mathfrak{e}\beta + p^{A_\mu\nu} dA_\mu\wedge\beta_\nu. \tag{110}$$

where $\beta = dx^1\wedge\ldots\wedge dx^n$ is a volume n-form defined on $\mathcal{X}$ and we also denote $\beta_\beta := \partial_\beta\lrcorner\,\beta$. Due to the Legendre correspondence construction, the equivalence relation between (q,v) and (q,p) is written:

$$(q,v)\leftrightarrow(q,p) \qquad\Longleftrightarrow\qquad \frac{\partial\langle p,v\rangle}{\partial v} = \frac{\partial L(q,v)}{\partial v}. \tag{111}$$

The term $\langle p,v\rangle$ is understood as the following expression $\langle p,v\rangle = \theta^{\mathsf{DW}}_p(\mathcal{Z})$. With $\mathcal{Z} = \mathcal{Z}_1\wedge\mathcal{Z}_2\wedge\mathcal{Z}_3\wedge\mathcal{Z}_4$ and where $\forall\alpha\ \mathcal{Z}_\alpha = \frac{\partial}{\partial x^\alpha} + \mathcal{Z}_{\alpha\mu}\frac{\partial}{\partial A_\mu}$. We gives the straightforward calculation with $\mathcal{Z}_{\alpha\mu} = \partial_\alpha A_\mu$:

$$\langle p,v\rangle = \theta^{\mathsf{DW}}_p(\mathcal{Z}) = \mathfrak{e}\beta(\mathcal{Z}) + p^{A_\mu\nu} dA_\mu\wedge\beta_\nu(\mathcal{Z}).$$

We have the expression $\langle p,v\rangle = \theta^{\mathsf{DW}}_p(\mathcal{Z})$:

$$\langle p,v\rangle = \mathfrak{e} + p^{A_\mu\nu}\mathcal{Z}_{\nu\mu} = \mathfrak{e} + p^{A_\mu\nu}\partial_\nu A_\mu. \tag{112}$$

⌜ Proof: We denote

$$\mathcal{Z}_\nu = \frac{\partial}{\partial x^\nu} + \sum_{1\leq\mu\leq n}\mathcal{Z}_{\nu\mu}\frac{\partial}{\partial A_\mu} = \sum_{1\leq\boldsymbol{\mu}\leq 2n}\mathcal{Z}^{\boldsymbol{\mu}}_\nu\frac{\partial}{\partial q^{\boldsymbol{\mu}}},$$

We have $q^{\boldsymbol{\mu}} = x^{\boldsymbol{\mu}} = x^\nu$ if $1\leq\boldsymbol{\mu}=\nu\leq n$ and $q^{\boldsymbol{\mu}} = A_{\boldsymbol{\mu}-n} = A_\mu$ if $1\leq\boldsymbol{\mu}-n=\mu\leq n$. The bold index $1\leq\boldsymbol{\mu}\leq 2n$ is a multi-index such that $\mathcal{Z}^{\boldsymbol{\mu}}_\nu = \delta^{\boldsymbol{\mu}}_\nu$ for $1\leq\boldsymbol{\mu}\leq n$ and $\mathcal{Z}^{\boldsymbol{\mu}}_\nu = \mathcal{Z}_{\nu\mu}$ for $n+1\leq\boldsymbol{\mu}\leq 2n$.

$$\mathcal{Z} = \mathcal{Z}_1\wedge\mathcal{Z}_2\wedge\mathcal{Z}_3\wedge\mathcal{Z}_4 = \sum_{\boldsymbol{\mu}_1<\cdots<\boldsymbol{\mu}_4}\boldsymbol{\mathcal{Z}}^{\boldsymbol{\mu}_1\cdots\boldsymbol{\mu}_4}_{1\cdots 4}\frac{\partial}{\partial q^{\boldsymbol{\mu}_1}}\wedge\cdots\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_4}} = \sum_{\boldsymbol{\mu}_1<\cdots<\boldsymbol{\mu}_4}\begin{vmatrix}\mathcal{Z}^{\boldsymbol{\mu}_1}_1 & \cdots & \mathcal{Z}^{\boldsymbol{\mu}_1}_4\\ \vdots & & \vdots\\ \mathcal{Z}^{\boldsymbol{\mu}_4}_1 & \cdots & \mathcal{Z}^{\boldsymbol{\mu}_4}_4\end{vmatrix}\frac{\partial}{\partial q^{\boldsymbol{\mu}_1}}\wedge\cdots\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_4}}$$

[51] We generally describe a Legendre correspondence by the symbol $\leftrightarrow$. Hence, the analytical duality described by the Legendre correspondence is written $(q,v)\leftrightarrow(q,p)$.

We expand the expression:

$$\mathcal{Z} = \boldsymbol{\mathcal{Z}}^{1234}_{1234}\partial_1\wedge\partial_2\wedge\partial_3\wedge\partial_4 + \underbrace{\sum_{n<\boldsymbol{\mu}_4}\boldsymbol{\mathcal{Z}}^{123\boldsymbol{\mu}_4}_{1234}\partial_1\wedge\partial_2\wedge\partial_3\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_4}}}_{[\mathbf{I}]} + \underbrace{\sum_{n<\boldsymbol{\mu}_4}\boldsymbol{\mathcal{Z}}^{124\boldsymbol{\mu}_4}_{1234}\partial_1\wedge\partial_2\wedge\partial_4\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_4}}}_{[\mathbf{II}]}$$

$$+\underbrace{\sum_{3<\boldsymbol{\mu}_4}\boldsymbol{\mathcal{Z}}^{134\boldsymbol{\mu}_4}_{1234}\partial_1\wedge\partial_3\wedge\partial_4\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_4}}}_{[\mathbf{III}]} + \underbrace{\sum_{3<\boldsymbol{\mu}_4}\boldsymbol{\mathcal{Z}}^{234\boldsymbol{\mu}_4}_{1234}\partial_2\wedge\partial_3\wedge\partial_4\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_4}}}_{[\mathbf{IV}]}$$

Now we detail the different terms involved: $\boldsymbol{\mathcal{Z}}^{1234}_{1234} = 1$

$$\boldsymbol{\mathcal{Z}}^{123\boldsymbol{\mu}_4}_{1234} = \begin{vmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \mathcal{Z}^{\boldsymbol{\mu}_4}_1 & \mathcal{Z}^{\boldsymbol{\mu}_4}_2 & \mathcal{Z}^{\boldsymbol{\mu}_4}_3 & \mathcal{Z}^{\boldsymbol{\mu}_4}_4 \end{vmatrix} = \mathcal{Z}^{\boldsymbol{\mu}_4}_4 \qquad \boldsymbol{\mathcal{Z}}^{124\boldsymbol{\mu}_4}_{1234} = \begin{vmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ \mathcal{Z}^{\boldsymbol{\mu}_4}_1 & \mathcal{Z}^{\boldsymbol{\mu}_4}_2 & \mathcal{Z}^{\boldsymbol{\mu}_4}_3 & \mathcal{Z}^{\boldsymbol{\mu}_4}_4 \end{vmatrix} = -\mathcal{Z}^{\boldsymbol{\mu}_4}_3$$

$$\boldsymbol{\mathcal{Z}}^{134\boldsymbol{\mu}_4}_{1234} = \begin{vmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \mathcal{Z}^{\boldsymbol{\mu}_4}_1 & \mathcal{Z}^{\boldsymbol{\mu}_4}_2 & \mathcal{Z}^{\boldsymbol{\mu}_4}_3 & \mathcal{Z}^{\boldsymbol{\mu}_4}_4 \end{vmatrix} = \mathcal{Z}^{\boldsymbol{\mu}_4}_2 \qquad \boldsymbol{\mathcal{Z}}^{234\boldsymbol{\mu}_4}_{1234} = \begin{vmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \mathcal{Z}^{\boldsymbol{\mu}_4}_1 & \mathcal{Z}^{\boldsymbol{\mu}_4}_2 & \mathcal{Z}^{\boldsymbol{\mu}_4}_3 & \mathcal{Z}^{\boldsymbol{\mu}_4}_4 \end{vmatrix} = -\mathcal{Z}^{\boldsymbol{\mu}_4}_1$$

Therefore we obtain:

$$[\mathbf{I}] = \mathcal{Z}^{\boldsymbol{\mu}_4}_4\partial_1\wedge\partial_2\wedge\partial_3\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_4}} \qquad [\mathbf{II}] = -\mathcal{Z}^{\boldsymbol{\mu}_4}_3\partial_1\wedge\partial_2\wedge\partial_4\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_4}} \qquad [\mathbf{III}] = \mathcal{Z}^{\boldsymbol{\mu}_4}_2\partial_1\wedge\partial_3\wedge\partial_4\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_4}}$$

$[\mathbf{IV}] = -\mathcal{Z}^{\boldsymbol{\mu}_4}_1\partial_2\wedge\partial_3\wedge\partial_4\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_4}}$. Then we obtain the expression for $\mathcal{Z}$:

$$\mathcal{Z} = \partial_1\wedge\partial_2\wedge\partial_3\wedge\partial_4 + \mathcal{Z}_{4\mu}\partial_1\wedge\partial_2\wedge\partial_3\wedge\frac{\partial}{\partial A_\mu} - \mathcal{Z}_{3\mu}\partial_1\wedge\partial_2\wedge\partial_4\wedge\frac{\partial}{\partial A_\mu} + \mathcal{Z}_{2\mu}\partial_1\wedge\partial_3\wedge\partial_4\wedge\frac{\partial}{\partial A_\mu} - \mathcal{Z}_{1\mu}\partial_2\wedge\partial_3\wedge\partial_4\wedge\frac{\partial}{\partial A_\mu}$$

Since, $\langle p, v\rangle = \theta^{\mathsf{DW}}_p(\mathcal{Z}) = \mathfrak{e}\beta(\mathcal{Z}) + p^{A_\mu\nu}dA_\mu\wedge\beta_\nu(\mathcal{Z})$, we expand it[52] as

$$\langle p, v\rangle = \mathfrak{e}\beta(\mathcal{Z}) + p^{A_\mu 1}dA_\mu\wedge\beta_1(\mathcal{Z}) + p^{A_\mu 2}dA_\mu\wedge\beta_2(\mathcal{Z}) + p^{A_\mu 3}dA_\mu\wedge\beta_3(\mathcal{Z}) + p^{A_\mu 4}dA_\mu\wedge\beta_4(\mathcal{Z})$$

$$\begin{aligned}\langle p, v\rangle &= \mathfrak{e}\beta(\partial_1\wedge\partial_2\wedge\partial_3\wedge\partial_4)\\ &\quad + p^{A_\mu 1}dA_\mu\wedge\beta_1(-\mathcal{Z}_{1\mu}\partial_2\wedge\partial_3\wedge\partial_4\wedge\frac{\partial}{\partial A_{\mu_4}}) + p^{A_\mu 2}dA_\mu\wedge\beta_2(\mathcal{Z}_{2\mu}\partial_1\wedge\partial_3\wedge\partial_4\wedge\frac{\partial}{\partial A_\mu})\\ &\quad + p^{A_\mu 3}dA_\mu\wedge\beta_3(-\mathcal{Z}_{3\mu}\partial_1\wedge\partial_2\wedge\partial_4\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_4}}) + p^{A_\mu 4}dA_\mu\wedge\beta_4(\mathcal{Z}_{4\mu}\partial_1\wedge\partial_2\wedge\partial_3\wedge\frac{\partial}{\partial A_\mu})\\ \langle p, v\rangle &= \mathfrak{e} + p^{A_\mu 1}\mathcal{Z}_{1\mu} + p^{A_\mu 2}\mathcal{Z}_{2\mu} + p^{A_\mu 3}\mathcal{Z}_{3\mu} + p^{A_\mu 4}\mathcal{Z}_{4\mu} \quad \lrcorner\end{aligned}$$

Since $\langle p, v\rangle = \mathfrak{e} + p^{A_\mu\nu}\mathcal{Z}_{\nu\mu}$, we obtain:

$$\frac{\partial\langle p, v\rangle}{\partial(\partial_\nu A_\mu)} = \frac{\partial}{\partial(\partial_\nu A_\mu)}\big(\mathfrak{e}\beta + p^{A_\mu\nu}\partial_\nu A_\mu\big) = p^{A_\mu\nu}$$

On the other side, since $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$:

$$\frac{\partial L(q, v)}{\partial(\partial_\nu A_\mu)} = -\frac{1}{4}\frac{\partial}{\partial(\partial_\nu A_\mu)}\big(\mathsf{h}^{\mu\lambda}\mathsf{h}^{\nu\sigma}F_{\mu\nu}F_{\lambda\sigma}\big) = -\frac{1}{4}\mathsf{h}^{\mu\lambda}\mathsf{h}^{\nu\sigma}F_{\lambda\sigma}\frac{\partial}{\partial(\partial_\nu A_\mu)}\big(\partial_\mu A_\nu - \partial_\nu A_\mu\big)$$

The expression of the multimomenta is given by (113).

$$p^{A_\mu\nu} = \mathsf{h}^{\mu\lambda}\mathsf{h}^{\nu\sigma}F_{\lambda\sigma} = F^{\mu\nu} \tag{113}$$

The equivalence (111) is now (114):

$$(q, v)\leftrightarrow(q, p) \qquad \Longleftrightarrow \qquad p^{A_\mu\nu} = \mathsf{h}^{\mu\lambda}\mathsf{h}^{\nu\sigma}F_{\lambda\sigma} \tag{114}$$

[52]Notice that $\beta_1 = \partial_1 \lrcorner\, \beta = (-1)^{1-1}dx^2\wedge dx^3\wedge dx^4 = dx^2\wedge dx^3\wedge dx^4$ as well as $\beta_2 = -dx^1\wedge dx^3\wedge dx^4$, also $\beta_3 = dx^1\wedge dx^2\wedge dx^4$ and finally $\beta_4 = -dx^1\wedge dx^2\wedge dx^3$.

The Legendre transformation is degenerated. We cannot find a unique correspondence between the multivelocities v and the multimomenta p. Given a $v \in T\mathbb{R} \otimes_3 T^\star(T^\star\mathcal{X})$ the equation (111) has a solution $p \in \mathcal{M}_{\mathsf{DW}}$ if and only if $p \in \mathcal{M}_{\mathsf{deg}}$ with:

$$\mathcal{M}_{\mathsf{deg}} = \{(x, A, \mathfrak{e}\beta + \mathsf{h}^{\mu\lambda}\mathsf{h}^{\nu\sigma}F_{\lambda\sigma}dA_\mu \wedge \beta_\nu \ \ / \ \ (x, A) \in T^\star\mathcal{X}, \mathfrak{e} \in \mathbb{R}\} \subset \mathcal{M}_{\mathsf{DW}}. \tag{115}$$

Notice that $\mathcal{M}_{\mathsf{deg}} \subset \mathcal{M}_{\mathsf{DW}}$ is a vector sub-bundle of $\mathcal{M}_{\mathsf{DW}}$. The degenerate feature is related to the contraint $p^{A_\nu\mu} = F^{\nu\mu} = -F^{\mu\nu}$. The Legendre transform is recover if we impose the compatibility conditions: $p^{A_\nu\mu} + p^{A_\mu\nu} = 0$. It is an example of a Dirac *primary constraint set*. Therefore, we restrict ourselves to work with the submanifold:

$$\mathcal{M}_{\mathsf{Maxwell}} = \{(x, A, p) \in \mathcal{M}_{\mathsf{DW}} \ \ / \ \ p^{A_\nu\mu} + p^{A_\mu\nu} = 0 \text{ with } p^{A_\nu\mu} = F^{\nu\mu}\} \subset \mathcal{M}_{\mathsf{DW}}. \tag{116}$$

In the Maxwell case, the Dirac set are compatibility conditions that allows us to recover a Legendre transform. The De Donder-Weyl theory setting is concerned rather with the Legendre correspondence. We introduce two different spaces. The first is the De Donder-Weyl submanifold $\mathcal{M}_{\mathsf{DW}}$ on which we consider the canonical Cartan-Poincaré 4-form (110):

$$\theta^{\mathsf{DW}}_{(q,p)} := \mathfrak{e}\beta + \sum_\mu \sum_\nu p^{A_\mu\nu} dA_\mu \wedge \beta_\nu, \qquad \boldsymbol{\omega}^{\mathsf{DW}} = d\mathfrak{e} \wedge \beta + dp^{A_\mu\nu} \wedge dA_\mu \wedge \beta_\nu.$$

The second is $\mathcal{M}_{\mathsf{Maxwell}}$, defined by (116) (with the imposed constraints $p^{A_\nu\mu} + p^{A_\mu\nu} = 0$)[53].

8.2 Hamilton-Maxwell equations in the De Donder-Weyl framework

We compute the Hamiltonian function of the Maxwell theory in the DW case:

$$\mathcal{H}^{\mathsf{deg}}(q,p) = \langle p, v\rangle - L(q,v) = \langle p, v\rangle + \frac{1}{4}\Big(\mathsf{h}^{\mu\lambda}\mathsf{h}^{\nu\sigma}F_{\mu\nu}F_{\lambda\sigma}\Big)$$

Making use of relation (112) we find:

$$\mathcal{H}^{\mathsf{deg}}(q,p) = \mathfrak{e} + p^{A_\mu\nu}\mathcal{Z}_{\nu\mu} + \frac{1}{4}\Big(\mathsf{h}^{\mu\lambda}\mathsf{h}^{\nu\sigma}F_{\mu\nu}F_{\lambda\sigma}\Big) = \mathfrak{e} + p^{A_\mu\nu}\partial_\nu A_\mu + \frac{1}{4}p^{A_\mu\nu}F_{\mu\nu} = \mathfrak{e} - \frac{1}{4}p^{A_\mu\nu}F_{\mu\nu}$$

Then, the Hamiltonian function (117) is given by:

$$\mathcal{H}^{\mathsf{deg}}(q,p) = \mathfrak{e} - \frac{1}{4}\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}p^{A_\mu\nu}p^{A_\rho\sigma} \qquad \text{with} \qquad p^{A_\mu\nu} = \mathsf{h}^{\mu\lambda}\mathsf{h}^{\nu\sigma}F_{\lambda\sigma} = F^{\mu\nu} \tag{117}$$

In order to obtain the generalized Hamilton equations $X \lrcorner\, \boldsymbol{\omega}^{\mathsf{DW}} = (-1)^n d\mathcal{H}$, we need to compute $d\mathcal{H}^{\mathsf{deg}}(q,p)$, the differential of the Hamiltonian function. Since we work with a degenerate Legendre transform a naive use of the general method leads to incorrect equations of motion. We have: $d\mathcal{H}^{\mathsf{deg}}(q,p) = d\mathfrak{e} - \frac{1}{4}\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}d(p^{A_\mu\nu}p^{A_\rho\sigma}) = d\mathfrak{e} - \frac{1}{2}\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}p^{A_\rho\sigma}dp^{A_\mu\nu}$ which describes the right side of the Hamilton equations (118).

$$X \lrcorner\, \boldsymbol{\omega}^{\mathsf{DW}} = (-1)^n d\mathcal{H} \tag{118}$$

Let us denote $\forall 1 \leq \alpha \leq 4$:

$$X_\alpha = \frac{\partial}{\partial x^\alpha} + \Theta_{\alpha\mu}\frac{\partial}{\partial A_\mu} + \Upsilon_\alpha\frac{\partial}{\partial \mathfrak{e}} + \Upsilon_\alpha^{A_\mu\nu}\frac{\partial}{\partial p^{A_\mu\nu}} \tag{119}$$

Then we consider a n-vector field $X = X_1 \wedge X_2 \wedge X_3 \wedge X_4 \in \Lambda^4 T^\star\mathcal{M}_{\mathsf{DW}}$.

[53] The important point concerns the restriction related to those constraints on the allowed vector fields on the multisymplectic space. In such a context, the vector fields on $\mathcal{M}_{\mathsf{Maxwell}}$ must be written with the term $\Upsilon_\alpha^{A_\mu\nu}(\frac{\partial}{\partial p^{A_\mu\nu}} - \frac{\partial}{\partial p^{A_\nu\mu}})$ rather than with the term $\Upsilon_\alpha^{A_\mu\nu}\frac{\partial}{\partial p^{A_\mu\nu}}$ in the expression (119) of $X \in \Lambda^4 T\mathcal{M}_{\mathsf{Maxwell}}$.

Lemma 8.1. Let X be a $(p-1)$-vector field and let $\{d\boldsymbol{\rho}_i\}_{1\leq i\leq n}$ be a set of n 1-forms. We have:

$$X \lrcorner \,(\bigwedge_{1\leq i\leq n} d\boldsymbol{\rho}_i) = X \lrcorner \, d\boldsymbol{\rho}_1 \wedge \cdots \wedge d\boldsymbol{\rho}_n = \sum_j (-1)^{j+1}\big(d\boldsymbol{\rho}_1 \wedge \cdots \wedge d\boldsymbol{\rho}_{j-1} \wedge d\boldsymbol{\rho}_{j+1} \wedge \cdots \wedge d\boldsymbol{\rho}_n\big)(X) d\boldsymbol{\rho}_j$$

Thanks to lemma (13.2), the left side of the Hamilton equations (118) is written:

$$\begin{array}{rcl} X \lrcorner \, \boldsymbol{\omega}^{\mathsf{DW}} & = & X \lrcorner \, \big(d\mathfrak{e} \wedge \beta + dp^{A_\mu\nu} \wedge dA_\mu \wedge \beta_\nu\big) \\ & = & \beta(X) d\mathfrak{e} - (d\mathfrak{e} \wedge \beta_\rho)(X) dx^\rho + (dA_\mu \wedge \beta_\nu)(X) dp^{A_\mu\nu} \\ & & -(dp^{A_\mu\nu} \wedge \beta_\nu)(X) dA_\mu + (dp^{A_\mu\nu} \wedge dA_\mu \wedge \beta_{\rho\nu})(X) dx^\rho \end{array}$$

So that we obtain:

$$X \lrcorner \, \boldsymbol{\omega}^{\mathsf{DW}} \;=\; d\mathfrak{e} - \Upsilon_\rho dx^\rho + \Theta_{\nu\mu} dp^{A_\mu\nu} - \Upsilon_\nu^{A_\mu\nu} dA_\mu + \big(\Upsilon_\rho^{A_\mu\nu}\Theta_{\nu\mu} - \Upsilon_\nu^{A_\mu\nu}\Theta_{\rho\mu}\big) dx^\rho$$

The decomposition on the different forms $dp^{A_\mu\nu}$, $d\mathfrak{e}$, dA_μ and dx^ρ gives:

$$\begin{array}{rcl} -\Theta_{\nu\mu} & = & -\frac{1}{2}\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}p^{A_\rho\sigma} \\ -\Upsilon_\nu^{A_\mu\nu} & = & 0 \end{array} \qquad \text{and} \qquad -\Upsilon_\rho + \big(\Upsilon_\rho^{A_\mu\nu}\Theta_{\nu\mu} - \Upsilon_\nu^{A_\mu\nu}\Theta_{\rho\mu}\big) \;=\; 0$$

that is equivalent to:

$$\begin{array}{rcl} \partial_\nu A_\nu & = & \frac{1}{2}\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}p^{A_\rho\sigma} \\ \partial_\nu p^{A_\mu\nu} & = & 0 \end{array} \qquad \text{and} \qquad -\partial_\rho\mathfrak{e} + \big((\partial_\rho p^{A_\mu\nu})(\partial_\nu A_\mu) - (\partial_\nu p^{A_\mu\nu})(\partial_\rho A_\mu)\big) \;=\; 0$$

The second line of the previous system gives the half of the Maxwell equations. Notice that the Legendre degenerate transform implies $p^{A_\mu\nu} = F^{\mu\nu}$ so that $\partial_\nu p^{A_\mu\nu} = \partial_\nu F^{\mu\nu} = 0$. However we can not recover the full set of Maxwell's equations since

$$\frac{1}{2}\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}p^{A_\rho\sigma} = \frac{1}{2}F_{\mu\nu} \neq \partial_\mu A_\nu.$$

We are *not* recovering the usual Euler-Lagrange equations precisely because we work on the degenerate space. Now we now work on $\mathcal{M}_{\mathsf{Maxwell}}$. The constraint $p^{A_\mu\nu} + p^{A_\nu\mu} = 0$ selects the authorized directions for the vector fields and the ones we are not allowed to described. In this context, the vector field we shall use to make the contraction with the multisymplectic form is given by (120). We denote $\forall 1 \leq \alpha \leq 4$:

$$X_\alpha = \frac{\partial}{\partial x^\alpha} + \Theta_{\alpha\mu}\frac{\partial}{\partial A_\mu} + \Upsilon_\alpha\frac{\partial}{\partial \mathfrak{e}} + \Upsilon_\alpha^{A_\mu\nu}\Big(\frac{\partial}{\partial p^{A_\mu\nu}} - \frac{\partial}{\partial p^{A_\nu\mu}}\Big) \tag{120}$$

The Hamilton equations (118) becomes: $X \lrcorner \, \boldsymbol{\omega}^{\mathsf{DW}} = (-1)^n d\mathcal{H}$.

$$\begin{array}{rcl} X \lrcorner \, \boldsymbol{\omega}^{\mathsf{DW}} & = & X \lrcorner \, \big(d\mathfrak{e} \wedge \beta + dp^{A_\mu\nu} \wedge dA_\mu \wedge \beta_\nu\big) \\ & = & \beta(X) d\mathfrak{e} - (d\mathfrak{e} \wedge \beta_\rho)(X) dx^\rho + (dA_\mu \wedge \beta_\nu)(X) dp^{A_\mu\nu} \\ & & -(dp^{A_\mu\nu} \wedge \beta_\nu)(X) dA_\mu + (dp^{A_\mu\nu} \wedge dA_\mu \wedge \beta_{\rho\nu})(X) dx^\rho \end{array}$$

Then, we obtain:

$$\begin{array}{rcl} X \lrcorner \, \boldsymbol{\omega}^{\mathsf{DW}} & = & d\mathfrak{e} - \Upsilon_\rho dx^\rho + (\Theta_{\nu\mu} - \Theta_{\mu\nu}) dp^{A_\mu\nu} - (\Upsilon_\nu^{A_\mu\nu} - \Upsilon_\nu^{A_\nu\mu}) dA_\mu \\ & & + \Big(\big(\Upsilon_\rho^{A_\mu\nu}\Theta_{\nu\mu} - \Upsilon_\nu^{A_\mu\nu}\Theta_{\rho\mu}\big) - \big(\Upsilon_\rho^{A_\nu\mu}\Theta_{\mu\nu} - \Upsilon_\mu^{A_\nu\mu}\Theta_{\rho\nu}\big)\Big) dx^\rho \end{array}$$

The decompositions along $dp^{A_\mu\nu}$ and dA_μ, gives:

$$\begin{array}{rcl}(\Theta_{\nu\mu}-\Theta_{\mu\nu}) & = & -\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}p^{A_\rho\sigma} \\ -(\Upsilon_\nu^{A_\mu\nu}-\Upsilon_\nu^{A_\nu\mu}) & = & 0\end{array} \quad\Longrightarrow\quad \begin{array}{rcl}\partial_\mu A_\nu-\partial_\nu A_\mu & = & F_{\mu\nu} \\ \partial_\nu\big(p^{A_\mu\nu}-p^{A_\nu\mu}\big) & = & 0\end{array} \tag{121}$$

Hence, the second line of equation (121) gives Maxwell's equations:

$$\frac{1}{2}\partial_\nu\big(p^{A_\mu\nu}-p^{A_\mu\nu}\big)=\partial_\nu p^{A_\mu\nu}=\partial_\nu F^{\mu\nu}=0. \tag{122}$$

8.3 Maxwell theory as an n-phase space

We refer to the work of Kijowski [141] for the treatment of Maxwell's theory in the setting of a n-phase space. Due to the abelian feature of the Maxwell gauge theory, this treatment is essentially the same that the one exposed in the previous section. the notion of a n-phase space, inspired by Kijowski and Szczyrba [144], and developed further by Hélein [113, 111].

Definition 8.3.1. *A n-multiphase space (or simply an n-phase space) is a triple* $(\mathcal{M},\boldsymbol{\omega},\beta)$ *where* $\mathcal{M}$ *is a smooth manifold,* $\boldsymbol{\omega}$ *is a closed (n + 1)-form and* β *is an everywhere non-vanishing n-form.*

For a n-phase space $(\mathcal{M},\boldsymbol{\omega},\beta)$, a Hamiltonian n-curve is pictured as an oriented n-submanifold which satisfies:

$$\forall m\in\Gamma,\forall X\in\Lambda^nT_m\Gamma\quad X\,\lrcorner\,\boldsymbol{\omega}_m=0\qquad\text{and}\qquad\forall m\in\Gamma,\exists X\in\Lambda^nT_m\Gamma\quad X\,\lrcorner\,\beta_m\neq 0.$$

The last condition is an independence condition. We can canonically construct n-phase space data by means of the hypersurface of a multisymplectic manifold. We recall that a premultisymplectic n-form is closed but may be degenerate. In the general picture of a n phase space we express *dynamics* on a *level set* of $\mathcal{H}$.[54] We recover the dynamical equations in the pre-multisymplectic case (123) - see Hélein [111, 113, 114].

$$\forall\Xi\in C^\infty(\mathcal{M},T_m\mathcal{M}),\qquad(\Xi\,\lrcorner\,\boldsymbol{\omega})|_\Gamma=0\qquad\qquad\text{and}\qquad\qquad\beta|_\Gamma\neq 0. \tag{123}$$

The canonical pre-multisymplectic form is given by:

$$\theta_{(q,p)}^{\text{pre-multi Maxwell}}:=\theta_{(q,p)}^{\text{Maxwell}}\big|_{\mathcal{H}=0}=\mathfrak{e}\beta+p^{A_\mu\nu}dA_\mu\wedge\beta_\nu\big|_{\mathcal{H}=0} \tag{124}$$

We have $\mathcal{H}(q,p)=\mathfrak{e}-1/4\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}p^{A_\mu\nu}p^{A_\rho\sigma}$ with $p^{A_\mu\nu}=\mathsf{h}^{\mu\lambda}\mathsf{h}^{\nu\sigma}F_{\lambda\sigma}=F^{\mu\nu}$. Hence, imposing the Hamiltonian constraint $\mathcal{H}=0$ we are led to consider $\mathfrak{e}=1/4\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}p^{A_\mu\nu}p^{A_\rho\sigma}=-H(x^\mu,A_\nu,p^{A_\mu\nu})$. Hence, the pre-multisymplectic canonical forms $\theta_{(q,p)}^{\text{pre-multi}}$ and $\boldsymbol{\omega}_{(q,p)}^{\text{pre-multi}}$ are respectively written:

$$\theta_{(q,p)}^{\text{pre-multi}}=\frac{1}{4}\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}p^{A_\mu\nu}p^{A_\rho\sigma}\beta+p^{A_\mu\nu}dA_\mu\wedge\beta_\nu=\frac{1}{4}p^{A_\mu\nu}p_{A_\mu\nu}\beta+p^{A_\mu\nu}dA_\mu\wedge\beta_\nu,$$

$$\boldsymbol{\omega}_{(q,p)}^{\text{pre-multi}}=d\theta_{(q,p)}^{\text{pre-multi}}=\frac{1}{2}\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}p^{A_\rho\sigma}dp^{A_\mu\nu}\wedge\beta+dp^{A_\mu\nu}\wedge dA_\mu\wedge\beta_\nu.$$

We denote, to simplify the notations: $\theta_{(q,p)}^{\text{pre-multi}}=\theta^{\circ}_{(q,p)}$ and $\boldsymbol{\omega}_{(q,p)}^{\text{pre-multi}}=\boldsymbol{\omega}^{\circ}_{(q,p)}$. Therefore, we consider the theory on the *pre-multisymplectic* Maxwell space (125):

$$\mathcal{M}^{\circ}_{\text{Maxwell}}=\{(x,A,p)\in\mathcal{M}_{\text{DW}}\ \ /\ \ p^{A_\nu\mu}+p^{A_\mu\nu}=0\text{ and }\mathfrak{e}=\frac{1}{4}\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}p^{A_\mu\nu}p^{A_\rho\sigma}\} \tag{125}$$

[54]We can construct canonically a pre-n-multisymplectic manifold $(\boldsymbol{\mathcal{M}}^{\circ},\boldsymbol{\omega}|_{\boldsymbol{\mathcal{M}}^{\circ}},\beta=\boldsymbol{\eta}\,\lrcorner\,\boldsymbol{\omega}|_{\boldsymbol{\mathcal{M}}^{\circ}})$. Here the $\boldsymbol{\omega}|_{\boldsymbol{\mathcal{M}}^{\circ}}=\mathcal{H}^{-1}(0):=\{(q,p)\in\boldsymbol{\omega}|_{\boldsymbol{\mathcal{M}}}|\ \mathcal{H}(q,p)=0\}$ and $\boldsymbol{\eta}$ is a vector field such that $d\mathcal{H}(\boldsymbol{\eta})=1$. In this case we observe the connection between relativistic dynamical systems and the treatment of the *Hamiltonian constraint.*

We observe the following inclusion of spaces: $\mathcal{M}^{\circ}_{\mathsf{Maxwell}} \subset \mathcal{M}_{\mathsf{Maxwell}} \subset \mathcal{M}_{\mathsf{DW}}$. The generalized Hamilton equations are given with the calculation of $X \lrcorner \boldsymbol{\omega}^{\circ}$:

$$\begin{array}{rcl} X \lrcorner \boldsymbol{\omega}^{\circ} & = & X \lrcorner (1/2\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}p^{A_\rho{}^\sigma} dp^{A_\mu{}^\nu} \wedge \beta) + X \lrcorner (dp^{A_\mu{}^\nu} \wedge dA_\mu \wedge \beta_\nu) \\ & = & 1/2\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}p^{A_\rho{}^\sigma}\beta(X)dp^{A_\mu{}^\nu} - (1/2\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}p^{A_\rho{}^\sigma}dp^{A_\mu{}^\nu} \wedge \beta_\rho)(X)dx^\rho \\ & & +(dA_\mu \wedge \beta_\nu)(X)dp^{A_\mu{}^\nu} - (dp^{A_\mu{}^\nu} \wedge \beta_\nu)(X)dA_\mu + (dp^{A_\mu{}^\nu} \wedge dA_\mu \wedge \beta_{\rho\nu})(X)dx^\rho \end{array}$$

So that:

$$\begin{array}{rcl} X \lrcorner \boldsymbol{\omega}^{\circ} & = & d\mathfrak{e} - \Upsilon_\rho dx^\rho + (\Theta_{\nu\mu} - \Theta_{\mu\nu} + \mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}p^{A_\rho{}^\sigma})dp^{A_\mu{}^\nu} - (\Upsilon_\nu^{A_\mu{}^\nu} - \Upsilon_\nu^{A_\nu{}^\mu})dA_\mu \\ & & +\Big(\big(\Upsilon_\rho^{A_\mu{}^\nu}\Theta_{\nu\mu} - \Upsilon_\nu^{A_\mu{}^\nu}\Theta_{\rho\mu}\big) - \big(\Upsilon_\rho^{A_\nu{}^\mu}\Theta_{\mu\nu} - \Upsilon_\mu^{A_\nu{}^\mu}\Theta_{\rho\nu}\big) - \mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}p^{A_\rho{}^\sigma}\Upsilon_\rho^{A_\mu{}^\nu}\Big)dx^\rho \end{array}$$

Once again, the decompositions along $dp^{A_\mu{}^\nu}$ and dA_μ gives:

$$\begin{array}{rcl} (\Theta_{\nu\mu} - \Theta_{\mu\nu} + \mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}p^{A_\rho{}^\sigma}) & = & 0 \\ -(\Upsilon_\nu^{A_\mu{}^\nu} - \Upsilon_\nu^{A_\nu{}^\mu}) & = & 0 \end{array} \qquad (126)$$

We recover (121) and then the same conclusions.

9 Observables for Maxwell Theory

9.1 Some algebraic observable $(n-1)$-forms

We are interested in the algebraic observable $(n-1)$-forms and their related infinitesimal symplectomorphisms on the multisymplectic manifold $(\mathcal{M}_{\mathsf{Maxwell}}, \boldsymbol{\omega}^{\mathsf{DW}})$. First we take some simple examples and we enter in the larger setting step by step. We find two types of algebraic observable $(n-1)$-forms: the (generalized) position $(n-1)$-forms and the (generalized) momenta observable $(n-1)$-forms. Let us begin with the following algebraic observable $(n-1)$-forms: $\mathbf{P}^\mu = dx^\mu \wedge \pi$, $\mathbf{P}^\mu_\phi = \phi(x)dx^\mu \wedge \pi$, $\mathbf{Q}_{\mu\nu} = A \wedge \beta_{\mu\nu}$ and $\mathbf{Q}^\psi_{\mu\nu} = \psi(x)A \wedge \beta_{\mu\nu}$. We denote the Faraday $(n-2)$-form [115, 132, 134] by:

$$\pi = \frac{1}{2}p^{A_\mu{}^\nu}\beta_{\mu\nu} = \frac{1}{2}\sum_{\mu,\nu} p^{A_\mu{}^\nu}\frac{\partial}{\partial x^\mu} \lrcorner \frac{\partial}{\partial x^\nu} \lrcorner \beta$$

and we denote the potential 1-form $A = A_\mu dx^\mu$. The couple of variables (A, π) depicts the canonical variables for the Maxwell theory [115, 116, 117, 118, 132, 134]. Notice that the Faraday $(n-2)$-form is also written: $\star dA = \mathsf{h}^{\mu\lambda}\mathsf{h}^{\nu\sigma}(\partial_\mu A_\nu - \partial_\nu A_\mu)\beta_{\lambda\sigma}$. First, we focus on $\mathbf{P}^\mu = dx^\mu \wedge \pi$. We have:

$$\mathbf{P}^\rho = dx^\rho \wedge \pi = dx^\rho \wedge \big(\frac{1}{2}p^{A_\mu{}^\nu}\beta_{\mu\nu}\big) = \frac{1}{2}p^{A_\mu{}^\nu}\big(\delta^\rho_\mu\beta_\nu - \delta^\rho_\nu\beta_\mu\big) = \frac{1}{2}\big(p^{A_\rho{}^\nu}\beta_\nu - p^{A_\mu{}^\rho}\beta_\mu\big)$$

Using the constraint $p^{A_\mu{}^\nu} = -p^{A_\nu{}^\mu}$, we obtain: $\mathbf{P}^\mu = p^{A_\mu{}^\nu}\beta_\nu$. Now we compute the exterior differential $d\mathbf{P}^\mu = d\big(dx^\rho \wedge \pi\big) = d\big(p^{A_\mu{}^\nu}\beta_\nu\big) = dp^{A_\mu{}^\nu} \wedge \beta_\nu$: If we consider $\Xi(\mathbf{P}^\mu) = \frac{\partial}{\partial A_\mu}$ we have $d\mathbf{P}^\mu = -\Xi(\mathbf{P}^\mu) \lrcorner \boldsymbol{\omega}^{\mathsf{DW}}$ as shown by the following straightforward calculation:

$$\Xi(\mathbf{P}^\mu) \lrcorner \boldsymbol{\omega}^{\mathsf{DW}} = \frac{\partial}{\partial A_\mu} \lrcorner \big(d\mathfrak{e} \wedge \beta + dp^{A_\mu{}^\nu} \wedge dA_\mu \wedge \beta_\nu\big) = -dp^{A_\mu{}^\nu} \wedge \beta_\nu$$

We prefer to consider $\mathbf{P}_\phi = \phi_\mu(x)p^{A_\mu{}^\nu}\beta_\nu$. The exterior derivative $d\mathbf{P}_\phi$ is given by:

$$\begin{array}{rcl} d\mathbf{P}_\phi & = & d\big(\phi_\mu(x)p^{A_\mu{}^\nu}\big) \wedge \beta_\nu = \big(p^{A_\mu{}^\nu}\dfrac{\partial\phi_\mu}{\partial x^\alpha}(x)dx^\alpha + \phi_\mu(x)dp^{A_\mu{}^\nu}\big) \wedge \beta_\nu \\ & = & p^{A_\mu{}^\nu}\dfrac{\partial\phi_\mu}{\partial x^\nu}\beta + \phi_\mu(x)dp^{A_\mu{}^\nu} \wedge \beta_\nu \end{array}$$

The related infinitesimal symplectomorphism is denoted by $\Xi(\mathbf{P}_\phi)$:

$$\Xi(\mathbf{P}_\phi) = \phi_\mu(x)\frac{\partial}{\partial A_\mu} - \big(\frac{\partial \phi_\mu}{\partial x^\nu}(x)p^{A_\mu\nu}\big)\frac{\partial}{\partial \mathfrak{e}}$$

Let us compute the contraction $\Xi(\mathbf{P}_\phi) \lrcorner\, \boldsymbol{\omega}^{\mathsf{DW}}$:

$$\begin{aligned}\Xi(\mathbf{P}_\phi) \lrcorner\, \boldsymbol{\omega}^{\mathsf{DW}} &= \big(\phi_\mu(x)\frac{\partial}{\partial A_\mu} - \big(\frac{\partial \phi_\mu}{\partial x^\nu}(x)p^{A_\mu\nu}\big)\frac{\partial}{\partial \mathfrak{e}}\big) \lrcorner \big(d\mathfrak{e}\wedge\beta + dp^{A_\mu\nu}\wedge A_\mu \wedge \beta_\nu\big)\\ &= -\big(\frac{\partial \phi_\mu(x)}{\partial x^\nu}p^{A_\mu\nu}\big)\beta - \phi_\mu(x)dp^{A_\mu\nu}\wedge\beta_\nu = -d\mathbf{P}_\phi\end{aligned}$$

Now we focus on some algebraic position $(n-1)$-forms: $\mathbf{Q}^\psi = \frac{1}{2}\psi^{\mu\nu}(x)A\wedge\beta_{\mu\nu}$ with $\psi^{\mu\nu}(x)$ a real function which is antisymmetric in the indices μ,ν.

$$\mathbf{Q}^\psi = \frac{1}{2}\psi^{\mu\nu}(x)A_\rho dx^\rho\wedge\beta_{\mu\nu} = \frac{1}{2}\psi^{\mu\nu}(x)A_\rho\big(\delta^\rho_\mu\beta_\nu - \delta^\rho_\nu\beta_\mu\big) = \frac{1}{2}\psi^{\mu\nu}(x)\big(A_\mu\beta_\nu - A_\nu\beta_\mu\big)$$

Since $\psi^{\mu\nu} = -\psi^{\nu\mu}$ then $\mathbf{Q}^\psi = \psi^{\mu\nu}(x)A_\mu\beta_\nu$. We compute $d\mathbf{Q}^\psi$:

$$\begin{aligned}d\mathbf{Q}^\psi &= = d\big(\psi^{\mu\nu}(x)A_\mu\beta_\nu\big)A_\mu\frac{\partial\psi^{\mu\nu}}{\partial x^\sigma}(x)dx^\sigma\wedge\beta_\nu + \psi^{\mu\nu}(x)dA_\mu\wedge\beta_\nu\\ &= A_\mu\frac{\partial\psi^{\mu\nu}}{\partial x^\nu}(x)\beta + \psi^{\mu\nu}(x)dA_\mu\wedge\beta_\nu\end{aligned}$$

The related infinitesimal symplectomorphisms are denoted $\Xi(\mathbf{Q}^\psi)$ and are given by:

$$\Xi(\mathbf{Q}^\psi) = -\big((A_\mu\frac{\partial\psi^{\mu\nu}}{\partial x^\nu}\big)\frac{\partial}{\partial\mathfrak{e}} + \psi^{\mu\nu}(x)\frac{\partial}{\partial p^{A_\mu\nu}}\big)$$

Let us compute $\Xi(\mathbf{Q}^\psi) \lrcorner\, \boldsymbol{\omega}^{\mathsf{DW}}$:

$$\begin{aligned}\Xi(\mathbf{Q}^\psi) \lrcorner\, \boldsymbol{\omega}^{\mathsf{DW}} &= -\big(\big(A_\mu\frac{\partial\psi^{\mu\nu}}{\partial x^\nu}\big)\frac{\partial}{\partial\mathfrak{e}} + \psi^{\mu\nu}(x)\frac{\partial}{\partial p^{A_\mu\nu}}\big) \lrcorner \big(d\mathfrak{e}\wedge\beta + dp^{A_\mu\nu}\wedge dA_\mu\wedge\beta_\nu\big)\\ &= -\big(A_\mu\frac{\partial\psi^{\mu\nu}}{\partial x^\nu}(x)\big)\beta - \psi^{\mu\nu}(x)dA_\mu\wedge\beta_\nu = -d\mathbf{Q}^\psi\end{aligned}$$

We summarize the results relating the algebraic observables forms $\mathbf{P}_\phi, \mathbf{Q}^\psi$ and their related infinitesimal symplectomorphisms $\Xi(\mathbf{P}_\phi), \Xi(\mathbf{Q}^\psi)$:

$$\begin{aligned}\mathbf{P}_\phi &= \phi_\mu(x)p^{A_\mu\nu}\beta_\nu & \Xi(\mathbf{P}_\phi) &= \phi_\mu(x)\frac{\partial}{\partial A_\mu} - \big\{\frac{\partial\phi_\mu(x)}{\partial x^\nu}p^{A_\mu\nu}\big\}\frac{\partial}{\partial\mathfrak{e}}\\ \mathbf{Q}^\psi &= \frac{1}{2}\psi^{\mu\nu}(x)A\wedge\beta_{\mu\nu} & \Xi(\mathbf{Q}^\psi) &= -\big(A_\mu\frac{\partial\psi^{\mu\nu}}{\partial x^\nu}\big)\frac{\partial}{\partial\mathfrak{e}} + \psi^{\mu\nu}(x)\frac{\partial}{\partial p^{A_\mu\nu}}\end{aligned} \tag{127}$$

Let notice that if we work in the pre-multisymplectic case $(\mathcal{M}_{\mathbf{o}}, \boldsymbol{\omega}^{\mathbf{o}})$ we have:

$$\begin{aligned}\mathbf{P}^{\mathbf{o}}_\phi &= p^{A_\mu\nu}\beta_\nu & \Xi(\mathbf{P}^{\mathbf{o}}_\phi) &= \phi_\mu(x)\frac{\partial}{\partial A_\mu}\\ \mathbf{Q}^\psi_{\mathbf{o}} &= \frac{1}{2}\psi^{\mu\nu}(x)A\wedge\beta_{\mu\nu} & \Xi(\mathbf{Q}^\psi_{\mathbf{o}}) &= -\big[\psi^{\mu\nu}(x)\frac{\partial}{\partial p^{A_\mu\nu}}\big]\end{aligned} \tag{128}$$

We need a more embracing view to better take under consideration the conditions on the functions $\phi_\mu(x)$ and $\psi^{\mu\nu}(x)$ and more general choice of such functions. In doing so we provide a deeper description of the infinitesimal symplectomorphisms $\Xi(\mathbf{Q}^\psi)$, $\Xi(\mathbf{P}_\phi)$, $\Xi(\mathbf{Q}^\psi_{\mathbf{o}})$ and $\Xi(\mathbf{P}^{\mathbf{o}}_\phi)$. It is the subject of the following sections (9.3) and (9.5). Before going to that step, we give in the next section (9.2) the Poisson bracket structure in this simple case. The objects of interest are $\big\{\mathbf{Q}^\psi, \mathbf{Q}^{\tilde\psi}\big\}$, $\big\{\mathbf{P}_\phi, \mathbf{P}_{\phi^\bullet}\big\}$ and $\big\{\mathbf{Q}^\psi, \mathbf{P}_\phi\big\}$.

9.2 Poisson Bracket for algebraic $(n-1)$-forms

First, following the *symmetry-algebraic* standpoint, we describe the diagram is the symbolic picture within we compare the *ontologic representation* with itself $[\lhd]\Big|[\lhd]$. The ontologic reflection for $(\mathcal{M}_{\text{Maxwell}}, \boldsymbol{\omega}^{\text{DW}})$ is given by:

Proposition 9.1. *Let $\phi_\mu(x), \tilde{\phi}_\mu(x)$ and $\psi^{\mu\nu}(x), \tilde{\psi}^{\mu\nu}(x)$ smooth functions with $\psi^{\mu\nu}(x) = -\psi^{\nu\mu}(x)$ and $\tilde{\psi}^{\mu\nu}(x) = -\tilde{\psi}^{\nu\mu}(x)$. for $(\mathcal{M}_{\text{Maxwell}}, \boldsymbol{\omega}^{\text{DW}})$ the set of canonical Poisson bracket is given by:*

$$\{\mathbf{Q}^\psi, \mathbf{Q}^{\tilde{\psi}}\} = \{\mathbf{P}_\phi, \mathbf{P}_{\tilde{\phi}}\} = 0, \qquad \text{and} \qquad \{\mathbf{Q}^\psi, \mathbf{P}_\phi\} = -\psi^{\mu\nu}(x)\phi_\mu(x)\beta_\nu.$$

This corresponds to the mathematical setting of the traditional Poisson bracket for algebraic $(n-1)$-forms: $\mathfrak{P}^{n-1}_\circ(\mathcal{M}) \times \mathfrak{P}^{n-1}_\circ(\mathcal{M}) \to \mathfrak{P}^{n-1}_\circ(\mathcal{M})$. Let us consider two algebraic position observable $(n-1)$-forms $\mathbf{Q}^\psi$ and $\mathbf{Q}^{\tilde{\psi}}$ given by (127):

$$\mathbf{Q}^\psi = \psi^{\mu\nu}(x)A_\mu(x)\beta_\nu, \qquad \text{and} \qquad \mathbf{Q}^{\tilde{\psi}} = \tilde{\psi}^{\mu\nu}(x)A_\mu(x)\beta_\nu.$$

We compute the *internal* bracket:

$$\begin{aligned}
\{\mathbf{Q}^\psi, \mathbf{Q}^{\tilde{\psi}}\} &= \Xi(\mathbf{Q}^\psi) \lrcorner\, \Xi(\mathbf{Q}^{\tilde{\psi}}) \lrcorner\, \boldsymbol{\omega}^{\text{DW}} \\
&= -\Xi(\mathbf{Q}^\psi) \lrcorner \Big((A_\mu \frac{\partial \tilde{\psi}^{\mu\nu}}{\partial x^\nu}) \frac{\partial}{\partial \mathfrak{e}} + \tilde{\psi}^{\mu\nu}(x) \frac{\partial}{\partial p^{A_\mu \nu}} \Big) \lrcorner \big(d\mathfrak{e} \wedge \beta + dp^{A_\mu \nu} \wedge dA_\mu \wedge \beta_\nu \big) \\
&= \big((A_\mu \frac{\partial \psi^{\mu\nu}}{\partial x^\nu}) \frac{\partial}{\partial \mathfrak{e}} + \psi^{\mu\nu}(x) \frac{\partial}{\partial p^{A_\mu \nu}} \big) \lrcorner \big((A_\mu \frac{\partial \tilde{\psi}^{\mu\nu}}{\partial x^\nu}) \beta + \tilde{\psi}^{\mu\nu}(x) dA_\mu \wedge \beta_\nu \big) = 0
\end{aligned}$$

Now we compute $\{\mathbf{P}_\phi, \mathbf{P}_{\phi^\bullet}\}$ where the algebraic $(n-1)$-forms $\mathbf{P}_\phi$ and the related infinitesimal symplectomorphisms $\Xi(\mathbf{P}_\phi)$ are given by (127). Hence we have the internal bracket:

$$\begin{aligned}
\{\mathbf{P}_\phi, \mathbf{P}_{\tilde{\phi}}\} &= \Xi(\mathbf{P}_\phi) \lrcorner\, \Xi(\mathbf{P}_{\tilde{\phi}}) \lrcorner\, \boldsymbol{\omega}^{\text{DW}} = -\Xi(\mathbf{P}_\phi) \lrcorner \big(\tilde{\phi}_\mu(x) \frac{\partial}{\partial A_\mu} - (\frac{\partial \tilde{\phi}_\mu(x)}{\partial x^\nu} p^{A_\mu \nu}) \frac{\partial}{\partial \mathfrak{e}} \big) \lrcorner\, \boldsymbol{\omega}^{\text{DW}} \\
&= -\Xi(\mathbf{P}_\phi) \lrcorner \big(-\tilde{\phi}_\mu(x) dp^{A_\mu \nu} \wedge \beta_\nu - (\frac{\partial \tilde{\phi}_\mu(x)}{\partial x^\nu} p^{A_\mu \nu}) \beta \big) = 0
\end{aligned}$$

Finally we compute the last internal bracket $\{\mathbf{Q}^\psi, \mathbf{P}_\phi\}$:

$$\begin{aligned}
\{\mathbf{Q}^\psi, \mathbf{P}_\phi\} &= -\Xi(\mathbf{Q}^{\psi^\circ}) \lrcorner \big[-\phi_\mu(x) dp^{A_\mu \nu} \wedge \beta_\nu - (\frac{\partial \phi_\mu(x)}{\partial x^\nu} p^{A_\mu \nu}) \beta \big] \\
&= -\big[(A_\mu \frac{\partial \psi^{\mu\nu}}{\partial x^\nu}) \frac{\partial}{\partial \mathfrak{e}} + \psi^{\mu\nu}(x) \frac{\partial}{\partial p^{A_\mu \nu}} \big] \lrcorner \big[-\phi_\mu(x) dp^{A_\mu \nu} \wedge \beta_\nu - (\frac{\partial \phi_\mu(x)}{\partial x^\nu} p^{A_\mu \nu}) \beta \big] \\
&= -\psi^{\mu\nu}(x)\phi_\mu(x)\beta_\nu
\end{aligned}$$

Finally,

$$\{\mathbf{Q}^\psi, \mathbf{P}_\phi\} = \Xi(\mathbf{Q}^\psi) \lrcorner\, d\mathbf{P}_\phi = -\big[(A_\mu \frac{\partial \psi^{\mu\nu}}{\partial x^\nu}) \frac{\partial}{\partial \mathfrak{e}} + \psi^{\mu\nu}(x) \frac{\partial}{\partial p^{A_\mu \nu}} \big] \lrcorner \big[p^{A_\mu \nu} \frac{\partial \phi_\mu}{\partial x^\nu} \beta + \phi_\mu(x) dp^{A_\mu \nu} \wedge \beta_\nu \big]$$

So that $\{\mathbf{Q}^\psi, \mathbf{P}_\phi\} = -\psi^{\mu\nu}(x)\phi_\mu(x)\beta_\nu$. We summarize our results and recover the proposition (9.1):

$$\{\mathbf{Q}^{\psi_\circ}, \mathbf{Q}^{\psi_\bullet}\} = \{\mathbf{P}_\phi, \mathbf{P}_{\phi^\bullet}\} = 0 \qquad \text{and} \qquad \{\mathbf{Q}^\psi, \mathbf{P}_\phi\} = -\psi^{\mu\nu}(x)\phi_\mu(x)\beta_\nu$$

9.3 All algebraic $(n-1)$-forms

the following infinitesimal symplectomorphisms, $\Xi \in \Gamma(\mathcal{M}_{\text{DW}}, T\mathcal{M}_{\text{DW}})$ as:

$$\Xi_{\text{DW}} = \boldsymbol{X}^{\nu}(q,p)\frac{\partial}{\partial x^{\mu}} + \boldsymbol{\Theta}_{\mu}(q,p)\frac{\partial}{\partial A_{\mu}} + \boldsymbol{\Upsilon}(q,p)\frac{\partial}{\partial \mathfrak{e}} + \boldsymbol{\Upsilon}^{A_{\mu}\nu}(q,p)\Big(\frac{\partial}{\partial p^{A_{\mu}\nu}} - \frac{\partial}{\partial p^{A_{\nu}\mu}}\Big) \tag{129}$$

The objects $\boldsymbol{X}^{\nu}(q,p), \boldsymbol{\Theta}_{\mu}(q,p), \boldsymbol{\Upsilon}(q,p)$ and $\boldsymbol{\Upsilon}^{A_{\mu}\nu}(q,p)$ are smooth functions on $\mathcal{M}_{\text{Maxwell}} \subset \mathcal{M}_{\text{DW}} \subset \Lambda^{n}T^{\star}(T^{\star}\mathcal{X})$, with values in $\mathbb{R}$. We evaluate the expression $\Xi \lrcorner \boldsymbol{\omega}^{\text{DW}}$:

Let us consider $\zeta \in \mathfrak{Z}$ namely we introduce the notations:

$$\zeta = X^{\nu}(x,A)\frac{\partial}{\partial x^{\nu}} + \Theta_{\mu}(x,A)\frac{\partial}{\partial A_{\mu}} \tag{130}$$

With $X^{\nu} : \mathfrak{Z} \longrightarrow \mathbb{R}$ and $\Theta_{\mu} : \mathfrak{Z} \longrightarrow \mathbb{R}$ are smooth fonctions on $\mathfrak{Z}$. Hence, we denote the decomposition of $\Xi \in \Gamma(\mathcal{M}_{\text{DW}}, T\mathcal{M}_{\text{DW}})$ as:

$$\Xi = \boldsymbol{\Theta}_{\mu}(q,p)\frac{\partial}{\partial A_{\mu}} + \boldsymbol{\Upsilon}(q,p)\frac{\partial}{\partial \mathfrak{e}} + \boldsymbol{\Upsilon}^{A_{\mu}\nu}(q,p)\frac{\partial}{\partial p^{A_{\mu}\nu}} \tag{131}$$

whereas the decomposition of $\Xi \in \Gamma(\mathcal{M}_{\text{Maxwell}}, T\mathcal{M}_{\text{Maxwell}})$ as:

$$\Xi = \boldsymbol{\Theta}_{\mu}(q,p)\frac{\partial}{\partial A_{\mu}} + \boldsymbol{\Upsilon}(q,p)\frac{\partial}{\partial \mathfrak{e}} + \boldsymbol{\Upsilon}^{A_{\mu}\nu}(q,p)\big(\frac{\partial}{\partial p^{A_{\mu}\nu}} - \frac{\partial}{\partial p^{A_{\nu}\mu}}\big) \tag{132}$$

With $\boldsymbol{X}^{\nu}, \boldsymbol{\Theta}_{\mu}, \boldsymbol{\Upsilon}, \boldsymbol{\Upsilon}^{A_{\mu}\nu}$ are smooth fonctions on $\mathcal{M}_{\text{Maxwell}} \subset \mathcal{M}_{\text{DW}} \subset \Lambda^{n}T^{\star}\mathfrak{Z}$, with values in $\mathbb{R}$. Now we evaluate the expression $\Xi \lrcorner \boldsymbol{\omega}^{\text{DW}}$:

$$\Xi \lrcorner \boldsymbol{\omega}^{\text{DW}} = \boldsymbol{\Upsilon}\beta - \boldsymbol{X}^{\nu}d\mathfrak{e} \wedge \beta_{\nu} + \boldsymbol{\Upsilon}^{A_{\mu}\nu}dA_{\mu} \wedge \beta_{\nu} - \boldsymbol{\Theta}_{\mu}dp^{A_{\mu}\nu} \wedge \beta_{\nu} + \boldsymbol{X}^{\rho}dp^{A_{\mu}\nu} \wedge dA_{\mu} \wedge \beta_{\rho\nu}$$

We rise relations from the definition of a symplectomorphism $d(\Xi \lrcorner \boldsymbol{\omega}^{\text{DW}}) = 0$. We make the following calculation:

$$\begin{aligned} d(\Xi \lrcorner \boldsymbol{\omega}^{\text{DW}}) &= d\boldsymbol{\Upsilon} \wedge \beta - d\boldsymbol{X}^{\nu} \wedge d\mathfrak{e} \wedge \beta_{\nu} \\ &\quad + d\boldsymbol{\Upsilon}^{A_{\mu}\nu}dA_{\mu} \wedge \beta_{\nu} - d\boldsymbol{\Theta}_{\mu} \wedge dp^{A_{\mu}\nu} \wedge \beta_{\nu} + d\boldsymbol{X}^{\rho} \wedge dp^{A_{\mu}\nu} \wedge dA_{\mu} \wedge \beta_{\rho\nu} \end{aligned}$$

Then let us write the expression under process as a sum $d(\Xi \lrcorner \boldsymbol{\omega}^{\text{DW}}) = \sum_{i} \boldsymbol{\iota}_{i}$ with each terms $\boldsymbol{\iota}_{i}$ given by:

$$\begin{aligned} \boldsymbol{\iota}_{1} &= d\boldsymbol{\Upsilon} \wedge \beta \\ \boldsymbol{\iota}_{2} &= -d\boldsymbol{X}^{\nu} \wedge d\mathfrak{e} \wedge \beta_{\nu} \\ \boldsymbol{\iota}_{3} &= d\boldsymbol{\Upsilon}^{A_{\mu}\nu}dA_{\mu} \wedge \beta_{\nu} \\ \boldsymbol{\iota}_{4} &= -d\boldsymbol{\Theta}_{\mu} \wedge dp^{A_{\mu}\nu} \wedge \beta_{\nu} \\ \boldsymbol{\iota}_{5} &= d\boldsymbol{X}^{\rho} \wedge dp^{A_{\mu}\nu} \wedge dA_{\mu} \wedge \beta_{\rho\nu} \end{aligned}$$

Since $d\boldsymbol{\Upsilon} = \frac{\partial \boldsymbol{\Upsilon}}{\partial x^{\alpha}}dx^{\alpha} + \frac{\partial \boldsymbol{\Upsilon}}{\partial A_{\beta}}dA_{\beta} + \frac{\partial \boldsymbol{\Upsilon}}{\partial \mathfrak{e}}d\mathfrak{e} + \frac{\partial \boldsymbol{\Upsilon}}{\partial p^{A_{\beta}\alpha}}dp^{A_{\beta}\alpha}$, the first term $\boldsymbol{\iota}_{1} = d\boldsymbol{\Upsilon} \wedge \beta$ is written:

$$\boldsymbol{\iota}_{1} = \frac{\partial \boldsymbol{\Upsilon}}{\partial A_{\beta}}dA_{\beta} \wedge \beta + \frac{\partial \boldsymbol{\Upsilon}}{\partial \mathfrak{e}}d\mathfrak{e} \wedge \beta + \frac{\partial \boldsymbol{\Upsilon}}{\partial p^{A_{\beta}\alpha}}dp^{A_{\beta}\alpha} \wedge \beta \tag{133}$$

Moreover, since $d\boldsymbol{X}^{\rho} = \frac{\partial \boldsymbol{X}^{\rho}}{\partial x^{\alpha}}dx^{\alpha} + \frac{\partial \boldsymbol{X}}{\partial A_{\beta}}dA_{\beta} + \frac{\partial \boldsymbol{X}^{\rho}}{\partial \mathfrak{e}}d\mathfrak{e} + \frac{\partial \boldsymbol{X}^{\rho}}{\partial p^{A_{\beta}\alpha}}dp^{A_{\beta}\alpha}$, the term $\boldsymbol{\iota}_{2} = -d\mathbf{X}^{\nu} \wedge d\mathfrak{e} \wedge \beta_{\nu}$ and $\boldsymbol{\iota}_{5} = d\mathbf{X}^{\rho} \wedge dp^{A_{\mu}\nu} \wedge dA_{\mu} \wedge \beta_{\rho\nu}$ are written:

$$\boldsymbol{\iota}_{2} = -\frac{\partial \boldsymbol{X}^{\nu}}{\partial x^{\alpha}}dx^{\alpha} \wedge d\mathfrak{e} \wedge \beta_{\nu} - \frac{\partial \boldsymbol{X}^{\nu}}{\partial A_{\beta}}dA_{\beta} \wedge d\mathfrak{e} \wedge \beta_{\nu} - \frac{\partial \boldsymbol{X}^{\nu}}{\partial p^{A_{\beta}\alpha}}dp^{A_{\beta}\alpha} \wedge d\mathfrak{e} \wedge \beta_{\nu} \tag{134}$$

$$\begin{aligned}\boldsymbol{\iota}_5 \;=\;& \frac{\partial \boldsymbol{X}^\rho}{\partial x^\alpha} dx^\alpha \wedge dp^{A_\mu\nu} \wedge dA_\mu \wedge \beta_{\rho\nu} + \frac{\partial \boldsymbol{X}^\rho}{\partial A_\beta} dA_\beta \wedge dp^{A_\mu\nu} \wedge dA_\mu \wedge \beta_{\rho\nu} \\ & + \frac{\partial \boldsymbol{X}^\rho}{\partial \mathfrak{e}} d\mathfrak{e} \wedge dp^{A_\mu\nu} \wedge dA_\mu \wedge \beta_{\rho\nu} + \frac{\partial \boldsymbol{X}^\rho}{\partial p^{A_\beta\alpha}} dp^{A_\beta\alpha} \wedge dp^{A_\mu\nu} \wedge dA_\mu \wedge \beta_{\rho\nu}\end{aligned} \tag{135}$$

Thanks to $d\boldsymbol{\Upsilon}^{A_\mu\nu} = \frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial x^\alpha} dx^\alpha + \frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial A_\beta} dA_\beta + \frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial \mathfrak{e}} d\mathfrak{e} + \frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial p^{A_\beta\alpha}} dp^{A_\beta\alpha}$

$$\begin{aligned}\boldsymbol{\iota}_3 \;=\;& -\Big(\frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial x^\nu} - \frac{\partial \boldsymbol{\Upsilon}^{A_\nu\mu}}{\partial x^\nu}\Big) dA_\mu \wedge \beta_\nu + \Big(\frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial A_\beta} - \frac{\partial \boldsymbol{\Upsilon}^{A_\nu\mu}}{\partial A_\beta}\Big) dA_\beta \wedge dA_\mu \wedge \beta_\nu \\ & + \Big(\frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial \mathfrak{e}} - \frac{\partial \boldsymbol{\Upsilon}^{A_\nu\mu}}{\partial \mathfrak{e}}\Big) d\mathfrak{e} \wedge dA_\mu \wedge \beta_\nu + \Big(\frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial p^{A_\beta\alpha}} - \frac{\partial \boldsymbol{\Upsilon}^{A_\nu\mu}}{\partial p^{A_\beta\alpha}}\Big) dp^{A_\beta\alpha} \wedge dA_\mu \wedge \beta_\nu\end{aligned} \tag{136}$$

and finally $d\boldsymbol{\Theta}_\mu = \frac{\partial \boldsymbol{\Theta}_\mu}{\partial x^\alpha} dx^\alpha + \frac{\partial \boldsymbol{\Theta}_\mu}{\partial A_\beta} dA_\beta + \frac{\partial \boldsymbol{\Theta}_\mu}{\partial \mathfrak{e}} d\mathfrak{e} + \frac{\partial \boldsymbol{\Theta}_\mu}{\partial p^{A_\beta\alpha}} dp^{A_\beta\alpha}$ give us the last term

$$\begin{aligned}\boldsymbol{\iota}_4 \;=\;& \frac{\partial \boldsymbol{\Theta}_\mu}{\partial x^\nu} dp^{A_\mu\nu} \wedge \beta - \frac{\partial \boldsymbol{\Theta}_\mu}{\partial A_\beta} dA_\beta \wedge dp^{A_\mu\nu} \wedge \beta_\nu - \frac{\partial \boldsymbol{\Theta}_\mu}{\partial \mathfrak{e}} d\mathfrak{e} \wedge dp^{A_\mu\nu} \wedge \beta_\nu \\ & - \frac{\partial \boldsymbol{\Theta}_\mu}{\partial p^{A_\beta\alpha}} dp^{A_\beta\alpha} \wedge dp^{A_\mu\nu} \wedge \beta_\nu\end{aligned} \tag{137}$$

The decomposition of the terms (133)-(137) on the different $(n+1)$-forms involves $d\mathfrak{e}\wedge\beta$, $dp^{A_\mu\nu}\wedge\beta$, $dA_\mu\wedge\beta$, $d\mathfrak{e}\wedge dA_\mu\wedge\beta_\nu$, $dp^{A_\beta\alpha}\wedge dA_\mu\wedge\beta_\nu$, $d\mathfrak{e}\wedge dp^{A_\mu\nu}\wedge dA_\mu\wedge\beta_{\rho\nu}$, $dp^{A_\beta\alpha}\wedge dp^{A_\mu\nu}\wedge\beta_\nu$. More precisely we now describe precisely the different terms. First the decomposition involves the following term on $\big[dp^{A_\beta\alpha}\wedge dp^{A_\mu\nu}\wedge\beta_\nu\big]$:

$$-\frac{\partial \boldsymbol{\Theta}_\mu}{\partial p^{A_\beta\alpha}} dp^{A_\beta\alpha} \wedge dp^{A_\mu\nu} \wedge \beta_\nu \tag{138}$$

Let us consider the decomposition on $\big[dp^{A_\beta\alpha}\wedge dp^{A_\mu\nu}\wedge dA_\mu\wedge\beta_{\rho\nu}\big]$

$$\frac{\partial \boldsymbol{X}^\rho}{\partial p^{A_\beta\alpha}} dp^{A_\beta\alpha} \wedge dp^{A_\mu\nu} \wedge dA_\mu \wedge \beta_{\rho\nu} \tag{139}$$

Hence from (138) and (139), we conclude that $\boldsymbol{X}^\rho$ and $\boldsymbol{\Theta}_\mu$ are independent of variables $p^{A_\beta\alpha}$. The next terms involve the decomposition on $\big[d\mathfrak{e}\wedge\beta\big]$,

$$\Big(\frac{\partial \boldsymbol{X}^\nu}{\partial x^\nu} + \frac{\partial \boldsymbol{\Upsilon}}{\partial \mathfrak{e}}\Big) d\mathfrak{e} \wedge \beta \tag{140}$$

on $\big[dp\wedge\beta\big]$:

$$\frac{\partial \boldsymbol{\Theta}_\mu}{\partial x^\nu} dp^{A_\mu\nu} \wedge \beta + \frac{\partial \boldsymbol{\Upsilon}}{\partial p^{A_\beta\alpha}} dp^{A_\beta\alpha} \wedge \beta \tag{141}$$

and on $\big[dA\wedge\beta\big]$

$$\frac{\partial \boldsymbol{\Upsilon}}{\partial A_\beta} dA_\beta \wedge \beta - \Big(\frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial x^\nu} - \frac{\partial \boldsymbol{\Upsilon}^{A_\nu\mu}}{\partial x^\nu}\Big) dA_\mu \wedge \beta, \tag{142}$$

the part of the decomposition on $\big[d\mathfrak{e}\wedge dA\wedge\beta_\nu\big]$

$$\Big(\frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial \mathfrak{e}} - \frac{\partial \boldsymbol{\Upsilon}^{A_\nu\mu}}{\partial \mathfrak{e}}\Big) d\mathfrak{e} \wedge dA_\mu \wedge \beta_\nu - \frac{\partial \boldsymbol{X}^\rho}{\partial A_\beta} dA_\beta \wedge d\mathfrak{e} \wedge \beta \tag{143}$$

decomposition on $\left[dA \wedge dA \wedge \beta_\nu\right]$

$$(\frac{\partial \boldsymbol{\Upsilon}^{A_\mu \nu}}{\partial A_\beta} - \frac{\partial \boldsymbol{\Upsilon}^{A_\nu \mu}}{\partial A_\beta}) dA_\beta \wedge dA_\mu \wedge \beta_\nu \tag{144}$$

decomposition on $\left[dA \wedge dp \wedge \beta_\nu\right]$

$$\frac{\partial \boldsymbol{X}^\rho}{\partial x^\alpha} dx^\alpha \wedge dp^{A_\mu \nu} \wedge dA_\mu \wedge \beta_{\rho\nu} - \frac{\partial \boldsymbol{\Theta}_\mu}{\partial A_\beta} dA_\beta \wedge dp^{A_\mu \nu} \wedge \beta_\nu + \frac{\partial \boldsymbol{\Upsilon}^{A_\mu \nu}}{\partial p^{A_\beta \alpha}} dp^{A_\beta \alpha} \wedge dA_\mu \wedge \beta_\nu \tag{145}$$

decomposition on $\left[d\mathfrak{e} \wedge dp^{A_\mu \nu} \wedge \beta_\nu\right]$

$$-\frac{\partial \boldsymbol{\Theta}_\mu}{\partial \mathfrak{e}} d\mathfrak{e} \wedge dp^{A_\mu \nu} \wedge \beta_\nu \tag{146}$$

the decomposition on $\left[dp^{A_\beta \alpha} \wedge d\mathfrak{e} \wedge \beta_\nu\right]$

$$-\frac{\partial \boldsymbol{X}^\rho}{\partial p^{A_\beta \alpha}} dp^{A_\beta \alpha} \wedge d\mathfrak{e} \wedge \beta_\nu \tag{147}$$

Hence from (146), (147) and (139) we observe that $\boldsymbol{\Theta}_\mu$ are independent of the variable $\mathfrak{e}$. Now we continue the path with the term related to the decomposition on $\left[dA \wedge dp \wedge dA \wedge \beta_{\rho\nu}\right]$

$$\frac{\partial \boldsymbol{X}^\rho}{\partial A_\beta} dA_\beta \wedge dp^{A_\mu \nu} \wedge dA_\mu \wedge \beta_{\rho\nu} \tag{148}$$

Then, from (148) we find that $\mathbf{X}^\rho$ is independent of the variables A_β. The last term is given by the decomposition on $\left[d\mathfrak{e} \wedge dp \wedge dA \wedge \beta_{\rho\nu}\right]$

$$\frac{\partial \boldsymbol{X}^\rho}{\partial \mathfrak{e}} d\mathfrak{e} \wedge dp^{A_\mu \nu} \wedge dA_\mu \wedge \beta_{\rho\nu} \tag{149}$$

From (149) we find again that $\mathbf{X}^\rho$ is independent of the variable $\mathfrak{e}$. The decomposition of $d(\Xi \lrcorner \boldsymbol{\omega}^{\mathsf{DW}})$ gives us information about the dependence of the involved functions. Due to decompositions (138) (139) (146) (147) and (149) we have: $\boldsymbol{X}^\rho = \boldsymbol{X}^\rho(x, A)$ and $\boldsymbol{\Theta}_\mu = \boldsymbol{\Theta}_\mu(x, A)$. From (148), we observe $\boldsymbol{X}^\rho = \boldsymbol{X}^\rho(x)$, so that, due to (143) and (144) we conclude that: $\boldsymbol{\Upsilon}^{A_\mu \nu} = \boldsymbol{\Upsilon}^{A_\mu \nu}(x, p)$. We don't have any extra information on $\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}(x, A, \mathfrak{e}, p)$. The functions $\boldsymbol{X}^\nu, \boldsymbol{\Theta}_\mu, \boldsymbol{\Upsilon}, \boldsymbol{\Upsilon}^{A_\mu \nu}$ are smooth fonctions on $\mathcal{M}_{\mathsf{DW}} \subset \Lambda^n T^\star \mathfrak{Z}$, with values in $\mathbb{R}$ satisfy the following coordinate dependance:

$$\boldsymbol{X}^\nu = \boldsymbol{X}^\nu(x) \quad , \quad \boldsymbol{\Theta}_\mu = \boldsymbol{\Theta}_\mu(x, A) \quad , \quad \boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}(x, A, \mathfrak{e}, p) \quad , \quad \boldsymbol{\Upsilon}^{A_\mu \nu} = \boldsymbol{\Upsilon}^{A_\mu \nu}(x, p) \tag{150}$$

We consider the further condition $\boldsymbol{\Upsilon}^{A_\mu \nu}(q, p) = -\boldsymbol{\Upsilon}^{A_\nu \mu}(q, p)$ so that we are left with equations (140) (141) (142) and (143):

$$\begin{aligned} \frac{\partial \boldsymbol{X}^\nu}{\partial x^\nu} + \frac{\partial \boldsymbol{\Upsilon}}{\partial \mathfrak{e}} &= 0 & \qquad \frac{\partial \boldsymbol{\Upsilon}}{\partial A_\mu} - \frac{\partial \boldsymbol{\Upsilon}^{A_\mu \nu}}{\partial x^\nu} &= 0 \\ \frac{\partial \boldsymbol{\Theta}_\mu}{\partial x^\nu} + \frac{\partial \boldsymbol{\Upsilon}}{\partial p^{A_\mu \nu}} &= 0 & \qquad \frac{\partial \boldsymbol{\Upsilon}^{A_\mu \nu}}{\partial \mathfrak{e}} - \frac{\partial \mathbf{X}^\nu}{\partial A_\mu} &= 0 \end{aligned} \tag{151}$$

together with the set of equations involving more than two terms, namely those witch arise from (145). We have the following proposition:

Proposition 9.2. *Let $\Xi \in \Gamma(\mathcal{M}, T\mathcal{M})$ then Ξ satisfies $d(\Xi \lrcorner \boldsymbol{\omega}^{\mathsf{DW}}) = 0$ if and only if Ξ is written $\Xi = \overline{\zeta} + \chi$ with*

$$\begin{aligned} \overline{\zeta} &= X^\nu \frac{\partial}{\partial x^\nu} + \Theta_\mu \frac{\partial}{\partial A_\mu} - \Big(\mathfrak{e}(\frac{\partial X^\nu}{\partial x^\nu}) + \frac{\partial \Theta_\mu}{\partial x^\nu} p^{A_\mu \nu}\Big) \frac{\partial}{\partial \mathfrak{e}} \\ &\quad - \Big(p^{A_\rho \sigma} \delta^\mu_\rho \big([(\frac{\partial X^\nu}{\partial x^\sigma}) - \delta^\nu_\sigma(\frac{\partial X^\lambda}{\partial x^\lambda})] + (\frac{\partial \Theta_\sigma}{\partial A_\nu})\big) - \mathfrak{e}(\frac{\partial X^\nu}{\partial A_\mu})\Big) \frac{\partial}{\partial p^{A_\mu \nu}} \end{aligned} \tag{152}$$

and $\chi = \Upsilon \frac{\partial}{\partial \mathfrak{e}} + \Upsilon^{A_\mu \alpha} \frac{\partial}{\partial p^{A_\mu \alpha}}$ with $\Upsilon : \mathfrak{Z} \to \mathbb{R}$ and $\Upsilon^{A_\mu \alpha} : \mathfrak{Z} \to \mathbb{R}$ smooth functions on $\mathfrak{Z}$ such that:

$$\frac{\partial \Upsilon}{\partial A_\mu} - \frac{\partial \Upsilon^{A_\mu \nu}}{\partial x^\nu} = 0 \tag{153}$$

The general proposition (6.1) is a result due to Hélein and Kouneiher [117] and it describes the more general search for all algebraic observable $(n-1)$-forms. Any infinitesimal symplectomorphism $\Xi \in \mathfrak{sp}_\circ(\mathcal{M})$ can be written under the form $\Xi = \chi + \overline{\zeta}$.

We decompose the vector field $\Xi \in \Gamma(\mathcal{M}, T\mathcal{M})$ with general coordinates:

$$\Xi = \Xi^\alpha(q,p) \frac{\partial}{\partial q^\alpha} + \sum_{\alpha_1 < \ldots < \alpha_n} \Xi_{\alpha_1 \ldots \alpha_n}(q,p) \frac{\partial}{\partial p_{\alpha_1 \ldots \alpha_n}}$$

Now we adapt our notations for Maxwell, namely we denote and also the only component on multimomenta area as

$$\Xi^\alpha(q,p) = \{\mathbf{X}^\nu(q,p); \boldsymbol{\Theta}_\mu(q,p)\} \qquad \Xi_{\alpha_1 \ldots \alpha_n}(q,p) = \{\boldsymbol{\Upsilon}(q,p); \boldsymbol{\Upsilon}^{A_\mu \nu}(q,p)\} \tag{154}$$

We denote:

$$\chi = \Xi - \overline{\zeta} = \boldsymbol{X}^\nu \frac{\partial}{\partial x^\nu} + \boldsymbol{\Theta}_\mu \frac{\partial}{\partial A_\mu} + \boldsymbol{\Upsilon} \frac{\partial}{\partial \mathfrak{e}} + \boldsymbol{\Upsilon}^{A_\mu \nu} \frac{\partial}{\partial p^{A_\mu \nu}} - \overline{\zeta}$$

We consider the expression (152) so that we obtain a expression for $\chi = \Xi - \overline{\zeta}$

$$\begin{aligned} \chi &= \underbrace{\Big(\boldsymbol{\Upsilon} - \big(\mathfrak{e}(\frac{\partial X^\nu}{\partial x^\nu}) + \frac{\partial \Theta_\mu}{\partial x^\nu} p^{A_\mu \nu}\big)\Big)}_{\chi^{\mathfrak{e}}} \frac{\partial}{\partial \mathfrak{e}} \\ &\quad + \underbrace{\Big(\boldsymbol{\Upsilon}^{A_\mu \nu} - p^{A_\rho \sigma} \delta^\mu_\rho \Big[[(\frac{\partial X^\nu}{\partial x^\sigma}) - \delta^\nu_\sigma(\frac{\partial X^\lambda}{\partial x^\lambda})] + (\frac{\partial \Theta_\sigma}{\partial A_\nu})\Big] - \mathfrak{e}(\frac{\partial X^\nu}{\partial A_\mu})\Big)}_{\chi^p} \frac{\partial}{\partial p^{A_\mu \nu}} \end{aligned} \tag{155}$$

As announced in the proposition (6.1), we have coefficients of χ such that $d(\chi \lrcorner \boldsymbol{\omega}^{\mathsf{DW}}) = 0$. Since $\chi = \chi^{\mathfrak{e}} \frac{\partial}{\partial \mathfrak{e}} + \chi^p \frac{\partial}{\partial p^{A_\mu \nu}}$, the interior product of χ with $\boldsymbol{\omega}^{\mathsf{DW}}$ writes simply:

$$\begin{aligned} \chi \lrcorner \boldsymbol{\omega}^{\mathsf{DW}} &= \big(\chi^{\mathfrak{e}} \frac{\partial}{\partial \mathfrak{e}} + \chi^p \frac{\partial}{\partial p^{A_\mu \nu}}\big) \lrcorner \big(d\mathfrak{e} \wedge \beta + dp^{A_\mu \nu} \wedge dA_\mu \wedge \beta_\nu\big) \\ &= \chi^{\mathfrak{e}} \beta + \big(dp^{A_\mu \nu}(\chi^p \frac{\partial}{\partial p^{A_\mu \nu}}) dA_\mu \wedge \beta_\nu\big) = \chi^{\mathfrak{e}} \beta + \chi^p dA_\mu \wedge \beta_\nu \end{aligned}$$

Now we compute $d(\chi \lrcorner \boldsymbol{\omega}^{\mathsf{DW}}) = d\chi^{\mathfrak{e}} \wedge \beta + d\chi^p \wedge dA_\mu \wedge \beta_\nu$.

$$\begin{aligned} d(\chi \lrcorner \boldsymbol{\omega}^{\mathsf{DW}}) &= d\big(\boldsymbol{\Upsilon} - (\mathfrak{e} \frac{\partial X^\nu}{\partial x^\nu} + \frac{\partial \Theta_\mu}{\partial x^\nu} p^{A_\mu \nu})\big) \wedge \beta \\ &\quad + d\Big(\boldsymbol{\Upsilon}^{A_\mu \nu} - \Big(p^{A_\rho \sigma} \delta^\mu_\rho \big([(\frac{\partial X^\nu}{\partial x^\sigma}) - \delta^\nu_\sigma(\frac{\partial X^\lambda}{\partial x^\lambda})] + (\frac{\partial \Theta_\sigma}{\partial A_\nu})\big] - \mathfrak{e}(\frac{\partial X^\nu}{\partial A_\mu})\big)\Big)\Big) \wedge dA_\mu \wedge \beta_\nu \end{aligned}$$

thanks to the expression of the exterior derivative $d\boldsymbol{\Upsilon}$ and $d\boldsymbol{\Upsilon}^{A_\mu\nu}$, we obtain:

$$\begin{aligned} d(\chi \lrcorner \boldsymbol{\omega}^{\mathsf{DW}}) &= (\frac{\partial \boldsymbol{\Upsilon}}{\partial A_\mu} - \frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial x^\nu}) dA_\mu \wedge \beta_\nu + (\frac{\partial \boldsymbol{\Upsilon}}{\partial \mathfrak{e}} - \frac{\partial X^\nu}{\partial x^\nu}) d\mathfrak{e} \wedge \beta \\ &+ (\frac{\partial \boldsymbol{\Upsilon}}{\partial p^{A_\mu\nu}} - \frac{\partial \Theta_\mu}{\partial x^\nu}) dp^{A_\mu\nu} \wedge \beta + (\frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial A_\beta}) dA_\beta \wedge dA_\mu \wedge \beta_\nu \\ &+ (\frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial \mathfrak{e}} - \frac{\partial X^\nu}{\partial A_\mu}) d\mathfrak{e} \wedge dA_\mu \wedge \beta_\nu \\ &+ \Big((\frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial p^{A_\rho\sigma}}) - \delta^\mu_\rho \big([(\frac{\partial X^\nu}{\partial x^\sigma}) - \delta^\nu_\sigma (\frac{\partial X^\lambda}{\partial x^\lambda})] - (\frac{\partial \Theta_\nu}{\partial A_\sigma}) \big) \Big) dp^{A_\rho\sigma} \wedge dA_\mu \wedge \beta_\nu \end{aligned}$$

Now we are interested in terms in which $\boldsymbol{\Upsilon}$ is involved: the first three terms in the last equation are concerned. Let notice that, if we denote $q = \{x, A\}$ then, $\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}(q, \mathfrak{e}, p)$ and the first two terms in the last equation give:

$$\begin{aligned} \frac{\partial \boldsymbol{\Upsilon}}{\partial \mathfrak{e}}(q, \mathfrak{e}, p) - \frac{\partial X^\nu}{\partial x^\nu}(q) &= 0 \\ \frac{\partial \boldsymbol{\Upsilon}}{\partial p^{A_\mu\nu}}(q, \mathfrak{e}, p) - \frac{\partial \Theta_\mu}{\partial x^\nu}(q) &= 0 \end{aligned} \tag{156}$$

Hence, it exists $\Upsilon(q) = \Upsilon(x, A)$ So that:

$$\boldsymbol{\Upsilon}(q, \mathfrak{e}, p) = \Upsilon(q) + (\frac{\partial \Theta_\mu}{\partial x^\nu}) p^{A_\mu\nu} + \mathfrak{e}(\frac{\partial X^\nu}{\partial x^\nu})$$

On the other side, $\boldsymbol{\Upsilon}^{A_\mu\nu}(q, p) = \boldsymbol{\Upsilon}^{A_\mu\nu}(x, p)$ therefore, the interesting information is contained in the set of equations:

$$\begin{aligned} \frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial \mathfrak{e}}(x, p) - (\frac{\partial X^\nu}{\partial A_\mu})(q) &= 0 \\ \Big[\frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial p^{A_\mu\sigma}}(q, \mathfrak{e}, p) - \delta^\mu_\rho \big([(\frac{\partial X^\nu}{\partial x^\sigma}) - \delta^\nu_\sigma (\frac{\partial X^\lambda}{\partial x^\lambda})] - (\frac{\partial \Theta_\nu}{\partial A_\sigma}) \big) \Big] dp^{A_\mu\sigma} \wedge dA_\mu \wedge \beta_\nu &= 0 \end{aligned} \tag{157}$$

Hence, it exists $\Upsilon^{A_\mu\nu}(q) = \Upsilon^{A_\mu\nu}(x, A)$ So that:

$$\boldsymbol{\Upsilon}^{A_\mu\nu}(q, p) = \Upsilon^{A_\mu\nu}(x, A) - p^{A_\rho\sigma} \delta^\mu_\rho \big\{ \Big[(\frac{\partial X^\nu}{\partial x^\sigma}) - \delta^\nu_\sigma (\frac{\partial X^\lambda}{\partial x^\lambda}) \Big] - (\frac{\partial \Theta_\nu}{\partial A_\sigma}) \big\} + \mathfrak{e}(\frac{\partial X^\nu}{\partial A_\mu})$$

The set of infinitesimal symplectomorphisms $\mathfrak{sp}_\circ(\mathcal{M}_{\mathsf{Maxwell}})$ of $(\mathcal{M}_{\mathsf{Maxwell}}, \boldsymbol{\omega}^{\mathsf{DW}})$ is described by vector fields $\Xi = \Xi\big|_{\mathcal{M}_{\mathsf{Maxwell}}} = \overline{\zeta} + \chi$ with $\overline{\zeta}$ described by (152) and $\chi = \Upsilon \frac{\partial}{\partial \mathfrak{e}} + \Upsilon^{A_\mu\alpha} \frac{\partial}{\partial p^{\omega_\mu\alpha}}$. Here, $X^\nu, \Theta_\mu, \Upsilon, \Upsilon^{A_\mu\alpha}$ are defined on $\mathfrak{Z}$ and not anymore on the full multisymplectic manifold $\mathcal{M}_{\mathsf{Maxwell}}$.

Proposition 9.3. *If we assume that $dx^\mu(\Xi) = 0$ - we throw away the X^μ which correspond to parts of the stress-energy-tensor - the proposition (9.2) gives:*

$$\Xi = \big(\Upsilon^{A_\mu\nu} - p^{A_\mu\sigma} (\frac{\partial \Theta_\sigma}{\partial A_\nu}) \big) \frac{\partial}{\partial p^{A_\mu\nu}} + \big(\Upsilon - \frac{\partial \Theta_\mu}{\partial x^\nu} p^{A_\mu\nu} \big) \frac{\partial}{\partial \mathfrak{e}} + \Theta_\mu \frac{\partial}{\partial A_\mu}$$

$\Upsilon^{A_\mu\nu}$, Υ and Θ_μ are smooth arbitrary functions of (x, A) with $\Upsilon^{A_\mu\nu}(q) = -\Upsilon^{A_\nu\mu}(q)$, they satisfy the condition:

$$\frac{\partial \Upsilon}{\partial A_\mu} - \frac{\partial \Upsilon^{A_\mu\nu}}{\partial x^\nu} = 0$$

Proposition 9.4. *Let $\boldsymbol{\varphi} \in \Gamma(\mathcal{M}_{\textsf{Maxwell}}, \Lambda^{n-1}T^\star\mathcal{M}_{\textsf{Maxwell}})$. The $(n-1)$-form $\boldsymbol{\varphi}$ is an algebraic observable if and only if $\boldsymbol{\varphi}$ is written $\boldsymbol{\varphi} = \boldsymbol{\varphi}_X + \boldsymbol{\varphi}_A + \boldsymbol{\varphi}_\chi$ where*

$$\begin{aligned} \boldsymbol{\varphi}_X &= \mathfrak{e}X^\rho\beta_\rho - p^{A_\mu\nu}X^\rho dA_\mu \wedge \beta_{\rho\nu} \\ \boldsymbol{\varphi}_A &= p^{A_\mu\nu}\Theta_\mu\beta_\nu \end{aligned} \tag{158}$$

where $X^\mu, \Theta_\mu : \mathfrak{Z} \to \mathbb{R}$ are arbitrary smooth function on $\mathfrak{Z}$ and $\boldsymbol{\varphi}_\chi$ is a $(n-1)$-form such that

$$d\boldsymbol{\varphi}_\chi = \Upsilon\beta + \Upsilon^{A_\mu\nu}dA_\mu \wedge \beta_\nu \tag{159}$$

with Υ and $\Upsilon^{A_\mu\nu}$ such that $\Upsilon^{A_\mu\nu} = -\Upsilon^{A_\nu\mu}$. The function Υ and $\Upsilon^{A_\mu\nu}$ satisfy

$$\frac{\partial\Upsilon}{\partial A_\mu} - \frac{\partial\Upsilon^{A_\mu\nu}}{\partial x^\nu} = 0 \tag{160}$$

We notice that $\boldsymbol{\varphi}_X + \boldsymbol{\varphi}_A$ are the so-called generalized algebraic momenta $(n-1)$-forms. Recall that an arbitrary vector field on $\mathfrak{Z}$ is written (130), $\zeta := \sum_\alpha \zeta^\alpha(q)\frac{\partial}{\partial q^\alpha} = X^\nu(x,A)\frac{\partial}{\partial x^\nu} + \Theta_\mu(x,A)\frac{\partial}{\partial A_\mu}$. Let us denote $\mathbf{P}_\zeta = \zeta \lrcorner \theta$ so that

$$\mathbf{P}_\zeta = \zeta \lrcorner \big(\mathfrak{e}\beta + p^{A_\mu\nu}dA_\mu \wedge \beta_\nu\big) = \mathfrak{e}\beta(\zeta) + p^{A_\mu\nu}\big((\zeta \lrcorner dA_\mu) \wedge \beta_\nu - dA_\mu \wedge (\zeta \lrcorner \beta_\nu)\big)$$

Since, $\zeta \lrcorner dA_\mu = \Theta_\mu$ and $\zeta \lrcorner \beta_\nu = (X^\rho\frac{\partial}{\partial x^\rho}) \lrcorner \beta_\nu = X^\rho\beta_{\rho\nu}$ Then we obtain:

$$\mathbf{P}_\zeta = \mathfrak{e}X^\rho\beta_\rho + p^{A_\mu\nu}\Theta_\mu\beta_\nu - p^{A_\mu\nu}X^\rho dA_\mu \wedge \beta_{\rho\nu} = \boldsymbol{\varphi}_X + \boldsymbol{\varphi}_A$$

$\mathbf{P}_\zeta$ are the *generalized momenta* $(n-1)$-form. We have $d\mathbf{P}_\zeta = -\overline{\zeta} \lrcorner \boldsymbol{\omega}^{\textsf{DW}}$. The canonical symplectomorphism associated to $\mathbf{P}_\zeta$ is denoted $\Xi(\mathbf{P}_\zeta) = \bar{\zeta}$. We evaluate the exterior derivative $d\mathbf{P}_\zeta = d[\boldsymbol{\varphi}_X + \boldsymbol{\varphi}_A]$:

$$\begin{aligned} d\mathbf{P}_\zeta &= d\big(\mathfrak{e}\boldsymbol{X}^\nu + p^{A_\mu\nu}\boldsymbol{\Theta}_\mu\big) \wedge \beta_\nu - d\big(p^{A_\mu\nu}\boldsymbol{X}^\rho\big) \wedge dA_\mu \wedge \beta_{\rho\nu} \\ &= \underbrace{\boldsymbol{X}^\nu d\mathfrak{e} \wedge \beta_\nu}_{\boldsymbol{\iota}_1} + \underbrace{\mathfrak{e}d\boldsymbol{X}^\nu \wedge \beta_\nu}_{\boldsymbol{\iota}_2} + \underbrace{p^{A_\mu\nu}d\boldsymbol{\Theta}_\mu \wedge \beta_\nu}_{\boldsymbol{\iota}_3} + \underbrace{\boldsymbol{\Theta}_\mu dp^{A_\mu\nu} \wedge \beta_\nu}_{\boldsymbol{\iota}_4} \\ &\quad - \underbrace{dp^{A_\mu\nu}\boldsymbol{X}^\rho dA_\mu \wedge \beta_{\rho\nu}}_{\boldsymbol{\iota}_5} - \underbrace{p^{A_\mu\nu}d\boldsymbol{X}^\rho dA_\mu \wedge \beta_{\rho\nu}}_{\boldsymbol{\iota}_6} \end{aligned}$$

Now we expand the objects $d\boldsymbol{X}^\rho$, $d\boldsymbol{\Theta}_\mu$ so that:

$$\boldsymbol{\iota}_2 = \mathfrak{e}d\boldsymbol{X}^\nu \wedge \beta_\nu = \mathfrak{e}\big(\frac{\partial\boldsymbol{X}^\nu}{\partial x^\alpha}dx^\alpha + \frac{\partial\boldsymbol{X}^\nu}{\partial A_\beta}dA_\beta + \frac{\partial\boldsymbol{X}^\nu}{\partial\mathfrak{e}}d\mathfrak{e} + \frac{\partial\boldsymbol{X}^\nu}{\partial p^{A_\beta\alpha}}dp^{A_\beta\alpha}\big) \wedge \beta_\nu$$

$$\boldsymbol{\iota}_3 = p^{A_\mu\nu}d\boldsymbol{\Theta}_\mu \wedge \beta_\nu = p^{A_\mu\nu}\big(\frac{\partial\boldsymbol{\Theta}_\mu}{\partial x^\alpha}dx^\alpha + \frac{\partial\boldsymbol{\Theta}_\mu}{\partial A_\beta}dA_\beta + \frac{\partial\boldsymbol{\Theta}_\mu}{\partial\mathfrak{e}}d\mathfrak{e} + \frac{\partial\boldsymbol{\Theta}_\mu}{\partial p^{A_\beta\alpha}}dp^{A_\beta\alpha}\big) \wedge \beta_\nu$$

Then,

$$\begin{aligned} d\mathbf{P}_\zeta = \; & \underbrace{\boldsymbol{X}^\nu d\mathfrak{e} \wedge \beta_\nu}_{\boldsymbol{\iota}_1} + \underbrace{\mathfrak{e}d\boldsymbol{X}^\nu \wedge \beta_\nu}_{\boldsymbol{\iota}_2} + \underbrace{\boldsymbol{\Theta}_\mu dp^{A_\mu\nu} \wedge \beta_\nu}_{\boldsymbol{\iota}_4} - \underbrace{dp^{A_\mu\nu}\boldsymbol{X}^\rho dA_\mu \wedge \beta_{\rho\nu}}_{\boldsymbol{\iota}_5} \\ & + \underbrace{p^{A_\mu\nu}(\frac{\partial\boldsymbol{\Theta}_\mu}{\partial x^\alpha})dx^\alpha \wedge \beta_\nu}_{\boldsymbol{\iota}_7} + \underbrace{p^{A_\mu\nu}(\frac{\partial\boldsymbol{\Theta}_\mu}{\partial A_\beta})dA_\beta \wedge \beta_\nu}_{\boldsymbol{\iota}_8} + \underbrace{p^{A_\mu\nu}(\frac{\partial\boldsymbol{\Theta}_\mu}{\partial\mathfrak{e}})d\mathfrak{e} \wedge \beta_\nu}_{\boldsymbol{\iota}_9} \\ & + \underbrace{p^{A_\mu\nu}(\frac{\partial\boldsymbol{\Theta}_\mu}{\partial p^{A_\beta\alpha}})dp^{A_\beta\alpha} \wedge \beta_\nu}_{\boldsymbol{\iota}_{10}} - \underbrace{p^{A_\mu\nu}\frac{\partial\boldsymbol{X}^\rho}{\partial x^\alpha}dx^\alpha \wedge dA_\mu \wedge \beta_{\rho\nu}}_{\boldsymbol{\iota}_{11}} - \underbrace{p^{A_\mu\nu}\frac{\partial\boldsymbol{X}^\rho}{\partial A_\beta}dA_\beta \wedge dA_\mu \wedge \beta_{\rho\nu}}_{\boldsymbol{\iota}_{12}} \\ & - \underbrace{p^{A_\mu\nu}\frac{\partial\boldsymbol{X}^\rho}{\partial\mathfrak{e}}d\mathfrak{e} \wedge dA_\mu \wedge \beta_{\rho\nu}}_{\boldsymbol{\iota}_{13}} - \underbrace{p^{A_\mu\nu}\frac{\partial\boldsymbol{X}^\rho}{\partial p^{A_\beta\alpha}}dp^{A_\beta\alpha} \wedge dA_\mu \wedge \beta_{\rho\nu}}_{\boldsymbol{\iota}_{14}} \end{aligned}$$

Since, $\boldsymbol{X} = \boldsymbol{X}(x)$ and $\boldsymbol{\Theta}_\mu = \boldsymbol{\Theta}_\mu(x,A)$ - see (150), we obtain vanishing contributions from the terms $\boldsymbol{\iota}_9, \boldsymbol{\iota}_{10}, \boldsymbol{\iota}_{12}, \boldsymbol{\iota}_{13}$ and $\boldsymbol{\iota}_{14}$. Therefore:

$$\begin{aligned} d\mathbf{P}_\zeta &= \underbrace{\boldsymbol{X}^\nu d\mathfrak{e}\wedge\beta_\nu}_{\boldsymbol{\iota}_1} + \underbrace{\mathfrak{e}d\boldsymbol{X}^\nu\wedge\beta_\nu}_{\boldsymbol{\iota}_2} \underbrace{+\boldsymbol{\Theta}_\mu dp^{A_\mu\nu}\wedge\beta_\nu}_{\boldsymbol{\iota}_4} - \underbrace{dp^{A_\mu\nu}\boldsymbol{X}^\rho dA_\mu\wedge\beta_{\rho\nu}}_{\boldsymbol{\iota}_5} \\ &\quad + \underbrace{p^{A_\mu\nu}(\frac{\partial\boldsymbol{\Theta}_\mu}{\partial x^\alpha})dx^\alpha\wedge\beta_\nu}_{\boldsymbol{\iota}_7} \underbrace{+p^{A_\mu\nu}(\frac{\partial\boldsymbol{\Theta}_\mu}{\partial A_\beta})dA_\beta\wedge\beta_\nu}_{\boldsymbol{\iota}_8} - \underbrace{p^{A_\mu\nu}(\frac{\partial\boldsymbol{X}^\rho}{\partial x^\alpha})dx^\alpha\wedge dA_\mu\wedge\beta_{\rho\nu}}_{\boldsymbol{\iota}_{11}} \end{aligned}$$

On the other hand, the general expression for a canonical symplectomorphism is:

$$\begin{aligned} \bar{\zeta}\,\lrcorner\,\boldsymbol{\omega}^{\mathsf{DW}} &= \bar{\zeta}\,\lrcorner\, d\mathfrak{e}\wedge\beta + \bar{\zeta}\,\lrcorner\,\left[dp^{A_\mu\nu}\wedge dA_\mu\wedge\beta_\nu\right] \\ &= -\big(\mathfrak{e}(\frac{\partial X^\nu}{\partial x^\nu}) + \frac{\partial\Theta_\mu}{\partial x^\nu}p^{A_\mu\nu}\big)\beta \underbrace{-\boldsymbol{X}^\nu d\mathfrak{e}\wedge\beta_\nu}_{-\boldsymbol{\iota}_1} \\ &\quad +dp^{A_\mu\nu}(\bar{\zeta})dA_\mu\wedge\beta_\nu - dA_\mu(\bar{\zeta})dp^{A_\mu\nu}\wedge\beta_\nu + dx^\rho(\bar{\zeta})dp^{A_\mu\nu}\wedge dA_\mu\wedge\beta_{\rho\nu} \\ &= \underbrace{-\mathfrak{e}(\frac{\partial\boldsymbol{X}^\nu}{\partial x^\nu})\beta}_{-\boldsymbol{\iota}_2} \underbrace{-\frac{\partial\Theta_\mu}{\partial x^\nu}p^{A_\mu\nu}\beta}_{-\boldsymbol{\iota}_7} \underbrace{-\boldsymbol{X}^\nu d\mathfrak{e}\wedge\beta_\nu}_{-\boldsymbol{\iota}_1} + dp^{A_\mu\nu}(\bar{\zeta})dA_\mu\wedge\beta_\nu \\ &\quad -dA_\mu(\bar{\zeta})dp^{A_\mu\nu}\wedge\beta_\nu \underbrace{+\boldsymbol{X}^\rho dp^{A_\mu\nu}\wedge dA_\mu\wedge\beta_{\rho\nu}}_{-\boldsymbol{\iota}_5} \end{aligned}$$

Since, $dA_\mu(\bar{\zeta}) = \boldsymbol{\Theta}_\mu$ then we compute :

$$-dA_\mu(\bar{\zeta})dp^{A_\mu\nu}\wedge\beta_\nu = -\boldsymbol{\Theta}_\mu dp^{A_\mu\nu}\wedge\beta_\nu = -\boldsymbol{\iota}_4$$

Let us denote, $[\mathbf{I}] = \boldsymbol{\iota}_1 + \boldsymbol{\iota}_2 + \boldsymbol{\iota}_4 + \boldsymbol{\iota}_5 + \boldsymbol{\iota}_7$. We remark that $\bar{\zeta}\,\lrcorner\,\boldsymbol{\omega}^{\mathsf{DW}} = -[\mathbf{I}] + dp^{A_\mu\nu}(\bar{\zeta})dA_\mu\wedge\beta_\nu$ let also notice that $dp^{A_\mu\nu}(\bar{\zeta}) = \bar{\zeta}^p$ with $\bar{\zeta} = \boldsymbol{X}^\nu\frac{\partial}{\partial x^\nu} + \boldsymbol{\Theta}_\rho\frac{\partial}{\partial A_\rho} + \bar{\zeta}^{\mathfrak{e}}\frac{\partial}{\partial\mathfrak{e}} + \bar{\zeta}^p\frac{\partial}{\partial p^{A_\mu\sigma}}$, we are left with the term $\bar{\zeta}^p$:

$$\bar{\zeta}^p = p^{A_\mu\sigma}\Big[\big[(\frac{\partial X^\nu}{\partial x^\sigma}) - \delta^\nu_\sigma(\frac{\partial X^\lambda}{\partial x^\lambda})\big] + (\frac{\partial\Theta_\sigma}{\partial A_\nu})\Big] - \mathfrak{e}(\frac{\partial X^\nu}{\partial A_\mu})$$

so that: $\bar{\zeta}\,\lrcorner\,\boldsymbol{\omega}^{\mathsf{DW}} = -[\mathbf{I}] + \Big[\bar{\zeta}^p\Big]^{\mu\nu} dA_\mu\wedge\beta_\nu$. Finally we shall denote the remaining terms $[\mathbf{II}] = \boldsymbol{\iota}_8 + \boldsymbol{\iota}_{11}$, hence we can write the equality $d[\boldsymbol{\varphi}_X + \boldsymbol{\varphi}_A] = [\mathbf{I}] + [\mathbf{II}]$. Therefore in order to prove the equality $\bar{\zeta}\,\lrcorner\,\boldsymbol{\omega}^{\mathsf{DW}} = -d[\boldsymbol{\varphi}_X + \boldsymbol{\varphi}_A]$, we only need to prove that $dp^{A_\mu\nu}(\bar{\zeta})dA_\mu\wedge\beta_\nu = -[\mathbf{II}]$. Since $dx^\alpha\wedge\beta_{\rho\nu} = \delta^\alpha_\rho\beta_\nu - \delta^\alpha_\nu\beta_\rho$

$$\begin{aligned} \boldsymbol{\iota}_{11} &= -p^{A_\mu\nu}\frac{\partial\mathbf{X}^\rho}{\partial x^\alpha}dx^\alpha\wedge dA_\mu\wedge\beta_{\rho\nu} = p^{A_\mu\nu}\frac{\partial\mathbf{X}^\rho}{\partial x^\alpha}dA_\mu\wedge\Big[\delta^\alpha_\rho\beta_\nu - \delta^\alpha_\nu\beta_\rho\Big] \\ &= p^{A_\mu\nu}\frac{\partial\mathbf{X}^\rho}{\partial x^\rho}dA_\mu\wedge\beta_\nu - p^{A_\mu\nu}\frac{\partial X^\rho}{\partial x^\nu}dA_\mu\wedge\beta_\rho = p^{A_\mu\sigma}\Big[(\frac{\partial X^\nu}{\partial x^\sigma}) - \delta^\nu_\sigma(\frac{\partial X^\lambda}{\partial x^\lambda})\Big] \end{aligned}$$

and also, on the same vein,

$$\boldsymbol{\iota}_8 = p^{A_\mu\sigma}(\frac{\partial\Theta_\nu}{\partial A_\sigma}) - \mathfrak{e}(\frac{\partial X^\nu}{\partial A_\mu})$$

so that we found the wanted result.

9.4 Dynamical equations

In this section we recover the dynamical equations using the following external brackets: $\{\mathcal{H}\beta, \mathbf{P}_\phi\}$ and $\{\mathcal{H}\beta, \mathbf{Q}^\psi\}$. The vocabulary *external bracket* is a reflect of the first work of Hélein and Kouneiher

[115] on the route towards a good picture for the connection between observable $(n-1)$-form and the dynamics. The work of Kanatchicov [132, 134] points also this connection. We introduce briefly this work in the following section (9.6). This *first attempt* in the work of Hélein and Kouneiher is replaced later by the concept of *pseudobracket* which is more deep and gives the good setting for the connection between dynamical properties and the notion of observable forms. We emphasize below in section (9.6) the mathematical structures for the setting which involves Grassman variables - with the underlying framework of graded, super-graded structure and supersymmetric investigations - as opposed to the ones of the copolarization idea. Let us compute the bracket $\{\mathcal{H}\beta, \mathbf{P}_\phi\}$. By definition,

$$\{\mathcal{H}\beta, \mathbf{P}_\phi\} = -\Xi(\mathbf{P}_\phi) \lrcorner\, d\mathcal{H} \wedge \beta = -\big[\phi_\mu(x)\frac{\partial}{\partial A_\mu} - \big(\frac{\partial \phi_\mu}{\partial x^\nu}(x) p^{A_\mu \nu}\big)\frac{\partial}{\partial \mathfrak{e}}\big] \lrcorner\, d\mathcal{H} \wedge \beta$$

$$\{\mathcal{H}\beta, \mathbf{P}_\phi\} = \big[-\phi_\mu(x)\frac{\partial \mathcal{H}}{\partial A_\mu} + \frac{\partial \phi_\mu(x)}{\partial x^\nu} p^{A_\mu \nu}\big]\beta$$

So, along the graph of a solution we have:

$$\{\mathcal{H}\beta, \mathbf{P}_\phi\}\big|_\Gamma = \big[-\phi_\mu J^\mu + \partial_\nu \phi_\mu F^{\mu\nu}\big]\beta.$$

On the other side, since $\mathbf{d}(\mathbf{P}_\phi) = \big[p^{A_\mu \nu}\frac{\partial \phi_\mu}{\partial x^\nu}\beta + \phi_\mu(x) dp^{A_\mu \nu} \wedge \beta_\nu\big]$, we obtain:

$$\mathbf{d}(\mathbf{P}_\phi)\big|_\Gamma = F^{\mu\nu}\frac{\partial \phi_\mu}{\partial x^\nu}\beta + \phi_\mu(x)\frac{\partial F^{\mu\nu}}{dx^\nu}(x)\beta = \big[F^{\mu\nu}\partial_\nu \phi_\mu + \phi_\mu \partial_\nu F^{\mu\nu}\big]\beta$$

So that we finally observe the first set of dynamical evolution equations, along a graph of generalized Hamilton equations:

$$\mathbf{d}(\mathbf{P}_\phi)\big|_\Gamma = \{\mathcal{H}\beta, \mathbf{P}_\phi\}\big|_\Gamma \quad \Longleftrightarrow \quad \partial_\nu F^{\mu\nu} = J^\mu. \tag{161}$$

Now we are intersted in the second bracket:

$$\{\mathcal{H}\beta, \mathbf{Q}^\psi\} = -\Xi(\mathbf{Q}^\psi) \lrcorner\, d\mathcal{H} \wedge \beta = -\big[\big(A_\mu \frac{\partial \psi^{\mu\nu}}{\partial x^\nu}(x)\big)\frac{\partial}{\partial \mathfrak{e}} + \psi^{\mu\nu}(x)\frac{\partial}{\partial p^{A_\mu \nu}}\big] \lrcorner\, d\mathcal{H} \wedge \beta$$

$$\{\mathcal{H}\beta, \mathbf{Q}^\psi\} = \big[\big(A_\mu \frac{\partial \psi^{\mu\nu}}{\partial x^\nu}\big) + \psi^{\mu\nu}(x)\frac{\partial \mathcal{H}}{\partial p^{A_\mu \nu}}\big]\beta.$$

So that along a graph Γ of a solution of the Hamilton equations we have:

$$\{\mathcal{H}\beta, \mathbf{Q}^\psi\}\big|_\Gamma = \big[\big(A_\mu \frac{\partial \psi^{\mu\nu}}{\partial x^\nu}\big) + \psi^{\mu\nu}(x)\frac{\partial \mathcal{H}}{\partial p^{A_\mu \nu}}\big]\beta$$

Whereas the expression of $\mathbf{d}(\mathbf{Q}^\psi)$ is written:

$$\mathbf{d}(\mathbf{Q}^\psi) = \big[A_\mu \frac{\partial \psi^{\mu\nu}}{\partial x^\nu}(x)\big]\beta + \psi^{\mu\nu}(x) dA_\mu \wedge \beta_\nu = \big[A_\mu \frac{\partial \psi^{\mu\nu}}{\partial x^\nu}(x)\big]\beta + \psi^{\mu\nu}(x)\partial_\rho A_\mu dx^\rho \wedge \beta_\nu$$

$$\mathbf{d}(\mathbf{Q}^\psi) = \big[A_\mu \frac{\partial \psi^{\mu\nu}}{\partial x^\nu}(x) + \psi^{\mu\nu}(x)\partial_\nu A_\mu\big]\beta$$

So that we finally observe:

$$\mathbf{d}(\mathbf{Q}^\psi)\big|_\Gamma = \{\mathcal{H}\beta, \mathbf{Q}^\psi\}\big|_\Gamma \quad \Longleftrightarrow \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{162}$$

Notice that in the general underlying symbolic insight we understand this external Poisson bracket as the dynamical evolution, we picture this movement by the comparison of the *dynamical space* with the *dynamical representation*[55]:

$$[\,\rhd\,]\;\Big|\;[\,\lhd\,]:\left\{\begin{array}{lcl}\mathbf{d}(\mathbf{P}_\phi)\big|_\Gamma & = & \{\mathcal{H}\beta,\mathbf{P}_\phi\}\big|_\Gamma\\ \mathbf{d}(\mathbf{Q}^\psi)\big|_\Gamma & = & \{\mathcal{H}\beta,\mathbf{Q}^\psi\}\big|_\Gamma\end{array}\right.$$

9.5 Algebraic $(n-1)$-forms for pre-multisymplectic case

We enter into more details, considering the pre-multisymplectic case. For the DW pure theory without constraint type, and forgetting the decomposition on the space-time variable - so that we forget the stress-energy tensor part - we would focus on the following infinitesimal symplectomorphisms, $\Xi^{\mathbf{o}}\in\Gamma(\mathcal{M}^{\mathbf{o}}_{\mathsf{DW}},T\mathcal{M}^{\mathbf{o}}_{\mathsf{DW}})$ as:

$$\Xi^{\mathbf{o}}_{\mathsf{DW}}=\mathbf{\Theta}_\mu(q,p)\frac{\partial}{\partial A_\mu}+\mathbf{\Upsilon}^{A_\mu\nu}(q,p)\frac{\partial}{\partial p^{A_\mu\nu}}\tag{163}$$

Notice that due to the Dirac primary constraint set, we must consider the following object $\Xi^{\mathbf{o}}\in\Gamma(\mathcal{M}^{\mathbf{o}}_{\mathsf{Maxwell}},T\mathcal{M}^{\mathbf{o}}_{\mathsf{Maxwell}})$ which is given by the interplay of some forbidden directions:

$$\Xi^{\mathbf{o}}=\mathbf{\Theta}_\mu(q,p)\frac{\partial}{\partial A_\mu}+\mathbf{\Upsilon}^{A_\mu\nu}(q,p)\Big(\frac{\partial}{\partial p^{A_\mu\nu}}-\frac{\partial}{\partial p^{A_\nu\mu}}\Big).\tag{164}$$

$\mathbf{\Theta}_\mu(q,p)$ and $\mathbf{\Upsilon}^{A_\mu\nu}(q,p)$ are smooth functions on $\mathcal{M}^{\mathbf{o}}_{\mathsf{Maxwell}}\subset\mathcal{M}_{\mathsf{Maxwell}}\subset\mathcal{M}_{\mathsf{DW}}\subset\Lambda^nT^\star(T^\star\mathcal{X})$, with values in $\mathbb{R}$. We evaluate the expression $\Xi^{\mathbf{o}}\,\lrcorner\,\boldsymbol{\omega}^{\mathbf{o}}$:

$$\Xi^{\mathbf{o}}\,\lrcorner\,\boldsymbol{\omega}^{\mathbf{o}}=\Xi^{\mathbf{o}}\,\lrcorner\,(\frac{1}{2}\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}p^{A_\rho\sigma}dp^{A_\mu\nu}\wedge\beta)+\Xi^{\mathbf{o}}\,\lrcorner\,(dp^{A_\mu\nu}\wedge dA_\mu\wedge\beta_\nu)$$

$$\Xi^{\mathbf{o}}\,\lrcorner\,\boldsymbol{\omega}^{\mathbf{o}}=\big(\mathbf{\Upsilon}^{A_\mu\nu}-\mathbf{\Upsilon}^{A_\nu\mu}\big)dA_\mu\wedge\beta_\nu-\mathbf{\Theta}_\mu dp^{A_\mu\nu}\wedge\beta_\nu+\frac{1}{2}\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}p^{A_\rho\sigma}\big(\mathbf{\Upsilon}^{A_\mu\nu}-\mathbf{\Upsilon}^{A_\nu\mu}\big)\beta$$

Now using the definition of the symplectomorphisms $d(\Xi^{\mathbf{o}}\,\lrcorner\,\boldsymbol{\omega}^{\mathbf{o}})=0$, we make the following calculation:

$$d(\Xi^{\mathbf{o}}\,\lrcorner\,\boldsymbol{\omega}^{\mathbf{o}})=d\big(\mathbf{\Upsilon}^{A_\mu\nu}-\mathbf{\Upsilon}^{A_\nu\mu}\big)\wedge dA_\mu\wedge\beta_\nu-d\mathbf{\Theta}_\mu\wedge dp^{A_\mu\nu}\wedge\beta_\nu+\frac{1}{2}\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}p^{A_\rho\sigma}d\big(\mathbf{\Upsilon}^{A_\mu\nu}-\mathbf{\Upsilon}^{A_\nu\mu}\big)\wedge\beta$$
$$+\frac{1}{2}\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}\big(\mathbf{\Upsilon}^{A_\mu\nu}-\mathbf{\Upsilon}^{A_\nu\mu}\big)dp^{A_\rho\sigma}\wedge\beta$$

Using the decomposition of $d\mathbf{\Theta}_\mu$ and $d\mathbf{\Upsilon}^{A_\mu\nu}$ which are given by:

$$\begin{array}{lcl}d\mathbf{\Theta}_\mu & = & \dfrac{\partial\mathbf{\Theta}_\mu}{\partial x^\alpha}dx^\alpha+\dfrac{\partial\mathbf{\Theta}_\mu}{\partial A_\beta}dA_\beta+\dfrac{\partial\mathbf{\Theta}_\mu}{\partial p^{A_\beta\alpha}}dp^{A_\beta\alpha}\\ d\mathbf{\Upsilon}^{A_\mu\nu} & = & \dfrac{\partial\mathbf{\Upsilon}^{A_\mu\nu}}{\partial x^\alpha}dx^\alpha+\dfrac{\partial\mathbf{\Upsilon}^{A_\mu\nu}}{\partial A_\beta}dA_\beta+\dfrac{\partial\mathbf{\Upsilon}^{A_\mu\nu}}{\partial p^{A_\beta\alpha}}dp^{A_\beta\alpha}\end{array}$$

We write the expression as a sum $d(\Xi^{\mathbf{o}}\,\lrcorner\,\boldsymbol{\omega}^{\mathbf{o}})=\sum_i\boldsymbol{\iota}_i$:

$$\boldsymbol{\iota}_1=(\frac{\partial\mathbf{\Upsilon}^{A_\nu\mu}}{\partial x^\nu}-\frac{\partial\mathbf{\Upsilon}^{A_\mu\nu}}{\partial x^\nu})dA_\mu\wedge\beta+(\frac{\partial\mathbf{\Upsilon}^{A_\mu\nu}}{\partial A_\beta}-\frac{\partial\mathbf{\Upsilon}^{A_\nu\mu}}{\partial A_\beta})dA_\beta\wedge dA_\mu\wedge\beta_\nu+(\frac{\partial\mathbf{\Upsilon}^{A_\mu\nu}}{\partial p^{A_\beta\alpha}}-\frac{\partial\mathbf{\Upsilon}^{A_\nu\mu}}{\partial p^{A_\beta\alpha}})dp^{A_\beta\alpha}\wedge dA_\mu\wedge\beta_\nu$$

[55] In fact we do compare with *ontologic representation* namely we consider algebraic observable $(n-1)$-forms, however since $\mathfrak{P}^{n-1}_\circ(\mathcal{M}_{\mathsf{DW}})\subset\mathfrak{P}^{n-1}_\bullet(\mathcal{M}_{\mathsf{DW}})$ any AOF is an OF.

$$\boldsymbol{\iota}_2 = -d\boldsymbol{\Theta}_\mu \wedge dp^{A_\mu\nu} \wedge \beta_\nu = \frac{\partial \boldsymbol{\Theta}_\mu}{\partial x^\nu} dp^{A_\mu\nu} \wedge \beta - \frac{\partial \boldsymbol{\Theta}_\mu}{\partial A_\beta} dA_\beta \wedge dp^{A_\mu\nu} \wedge \beta_\nu - \frac{\partial \boldsymbol{\Theta}_\mu}{\partial p^{A_\beta\alpha}} dp^{A_\beta\alpha} \wedge dp^{A_\mu\nu} \wedge \beta_\nu$$

$$\boldsymbol{\iota}_3 = \frac{1}{2}\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma} p^{A_\rho\sigma}\big(\frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial A_\beta} - \frac{\partial \boldsymbol{\Upsilon}^{A_\nu\mu}}{\partial A_\beta}\big) dA_\beta \wedge \beta + \frac{1}{2}\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma} p^{A_\rho\sigma}\big(\frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial p^{A_\beta\alpha}} - \frac{\partial \boldsymbol{\Upsilon}^{A_\nu\mu}}{\partial p^{A_\beta\alpha}}\big) dp^{A_\beta\alpha} \wedge \beta$$

$$\boldsymbol{\iota}_4 = \frac{1}{2}\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}\big(\boldsymbol{\Upsilon}^{A_\mu\nu} - \boldsymbol{\Upsilon}^{A_\nu\mu}\big) dp^{A_\rho\sigma} \wedge \beta$$

The different decompositions of $(n+1)$-forms are written:

- decomposition on $dp^{A_\beta\alpha} \wedge dp^{A_\mu\nu} \wedge \beta_\nu$
$$-\frac{\partial \boldsymbol{\Theta}_\mu}{\partial p^{A_\beta\alpha}} dp^{A_\beta\alpha} \wedge dp^{A_\mu\nu} \wedge \beta_\nu \tag{165}$$
So that $\boldsymbol{\Theta}_\mu$ depends only on $\boldsymbol{\Theta}_\mu(x, A)$
- decomposition on $dA \wedge dA \wedge \beta_\nu$
$$\Big(\frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial A_\beta} - \frac{\partial \boldsymbol{\Upsilon}^{A_\nu\mu}}{\partial A_\beta}\Big) dA_\beta \wedge dA_\mu \wedge \beta_\nu \tag{166}$$
Hence $\boldsymbol{\Upsilon}^{A_\mu\nu} = \boldsymbol{\Upsilon}^{A_\mu\nu}(x,p)$
- decomposition on $dA \wedge \beta$
$$\frac{1}{2}\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma} p^{A_\rho\sigma}\Big(\frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial A_\beta} - \frac{\partial \boldsymbol{\Upsilon}^{A_\nu\mu}}{\partial A_\beta}\Big) dA_\beta \wedge \beta + \Big(\frac{\partial \boldsymbol{\Upsilon}^{A_\nu\mu}}{\partial x^\nu} - \frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial x^\nu}\Big) dA_\mu \wedge \beta \tag{167}$$
- decomposition on $dp \wedge \beta$
$$\frac{1}{2}\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma} p^{A_\rho\sigma}\Big(\frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial p^{A_\beta\alpha}} - \frac{\partial \boldsymbol{\Upsilon}^{A_\nu\mu}}{\partial p^{A_\beta\alpha}}\Big) dp^{A_\beta\alpha} \wedge \beta + \frac{\partial \boldsymbol{\Theta}_\mu}{\partial x^\nu} dp^{A_\mu\nu} \wedge \beta + \frac{1}{2}\mathsf{h}_{\mu\rho}\mathsf{h}_{\nu\sigma}\big(\boldsymbol{\Upsilon}^{A_\mu\nu} - \boldsymbol{\Upsilon}^{A_\nu\mu}\big) dp^{A_\rho\sigma} \wedge \beta \tag{168}$$
- decomposition on $dA \wedge dp \wedge \beta_\nu$
$$-\frac{\partial \boldsymbol{\Theta}_\mu}{\partial A_\beta} dA_\beta \wedge dp^{A_\mu\nu} \wedge \beta_\nu + \Big(\frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial p^{A_\beta\alpha}} - \frac{\partial \boldsymbol{\Upsilon}^{A_\nu\mu}}{\partial p^{A_\beta\alpha}}\Big) dp^{A_\beta\alpha} \wedge dA_\mu \wedge \beta_\nu \tag{169}$$

The mathematical requirement on the infinitesimal symplectomorphism $d(\Xi^{\circ} \lrcorner\, \boldsymbol{\omega}^{\circ}) = 0$ allows us to precise the condition on the functions $\boldsymbol{\Theta}_\mu$ and $\boldsymbol{\Upsilon}^{A_\mu\nu}$. The equation (165) gives us that $\boldsymbol{\Theta}_\mu$ is independent of momenta, $\boldsymbol{\Theta}_\mu = \boldsymbol{\Theta}_\mu(x_\rho, A_\rho)$. The equation (166) gives $\boldsymbol{\Upsilon}^{A_\mu\nu} = \boldsymbol{\Upsilon}^{A_\mu\nu}(x,p)$. Since we have equation (167) we obtain the following condition:

$$\Big(\frac{\partial \boldsymbol{\Upsilon}^{A_\nu\mu}}{\partial x^\nu} - \frac{\partial \boldsymbol{\Upsilon}^{A_\mu\nu}}{\partial x^\nu}\Big) = 0 \tag{170}$$

We recover the results of Kijowski [142] and Kijowski and Szczyrba [144].

9.6 Grassman variables vs copolarization

In this section, we heuristically illustrate the tension between the *graded structure* and the *copolarization* process. The fundamental interest for field theory is the search of the good Poisson structure. The modern classification concerning AOF and OF appears in the work of Hélein and Kouneiher . This duality is expressed in the Thesis via the more embracing standpoint: *ontologic* vs *dynamical* - which relates symmetry and observable considerations. The Poisson structures and the brackets are fundamental in QM theory by means of the quantum canonical relations which involve the passage from classical observables to self dual operators. Here we emphasize this tension by means of the example of *superforms* and Grassman variables found in the work of Hélein and Kouneiher . The *tension* is summarized in the following table:

Graded structures

> Supersymmetric algebraic structures, supermanifolds
> Graded generalization of Lie, Schouten-Nijenhuis and Frölicher-Nijenhuis brackets
> MG Graded structure (see I.K. Kanatchicov, F. Forger ...)

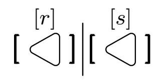
Grassman variables

> Ghosts, anti-ghosts, BRST and BV formalisms
> MG Grassman-odd variables and superforms (see Hélein and Kouneiher [115])
> MG Grassman-odd variables (see S. Hrabak [123, 124])

Copolarization

> Collective definition of observable form from Relativity Principle and dynamics
> Hélein and Kouneiher [116, 117, 118]

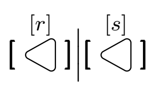
Now we translate the mathematical structure beyond the graded structures with the symbolic picture. In the context of MG, we first focus on the example of I.K. Kanatchicov bracket (79) with main focus on graded antisymmetric bracket $\{\overset{r}{\boldsymbol{\varphi}}, \overset{s}{\boldsymbol{\varrho}}\} = -(-1)^{(n-r-1)(n-s-1)}\{\overset{s}{\boldsymbol{\varrho}}, \overset{r}{\boldsymbol{\varphi}}\}$ (recall that we denote $r = n - p$ and $s = n - q$) so that we draw:



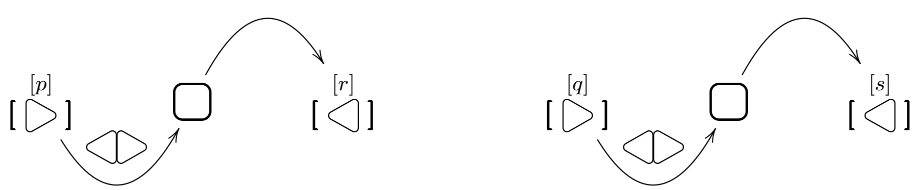
$$\begin{array}{ccc} \mathfrak{X}^{\mathrm{r}}_{\mathsf{ham}}(\mathcal{M}) \times \mathfrak{X}^{\mathrm{s}}_{\mathsf{ham}}(\mathcal{M}) & \to & \mathfrak{X}^{r+s-n+1}_{\mathsf{ham}}(\mathcal{M}) \\ (\overset{r}{\boldsymbol{\varphi}}, \overset{s}{\boldsymbol{\varrho}}) & \mapsto & \{\overset{r}{\boldsymbol{\varphi}}, \overset{s}{\boldsymbol{\varrho}}\} = (-1)^{n-r} \overset{p}{X}_{\boldsymbol{\varphi}} \lrcorner \overset{q}{X}_{\boldsymbol{\varrho}} \lrcorner \boldsymbol{\omega} \end{array}$$

Notice that, as told before, have a fundamental ambiguity in the search for the *ontologic reflection* - which means the compared ontologic representation for form of arbitrary degree p with the ontologic representation for form of degree q - for arbitrary Hamiltonian forms. To be more precise, the ambiguity lays in the choice of the *objects* themselves, namely *in* the choice of the Hamiltonian multivectors fields $\overset{p}{X}_{\boldsymbol{\varphi}}$ and $\overset{q}{X}_{\boldsymbol{\varrho}}$. In the general symbolic picture, we delimit the boundaries of this ambiguity via the previously introduced topological duality. This topological duality, as we briefly described it in section (6.4), consider the pairing of forms and vector fields via the study of the equation: $d\boldsymbol{\varphi} = -\Xi_{\boldsymbol{\varphi}} \lrcorner \boldsymbol{\omega}$. Therefore, for any arbitrary degree, the topological duality is of uncertainty feature expressed by the use of the map:

$$\mathfrak{U}^p_\circ : \Lambda^p T_m\mathcal{M} \to \Lambda^{n+1-p} T^\star_m \mathcal{M} = \Lambda^{r+1} T^\star_m \mathcal{M} \;\; \Xi \mapsto \mathfrak{U}^p_\circ(\Xi) = \Xi \lrcorner \boldsymbol{\omega}$$

So that, the Poisson bracket  is built on two non-uniquely defined prerequisites. In symbolic picture:

$\Xi \mapsto \mathfrak{U}^p_\circ(\Xi)$ $\qquad\qquad$ $\Xi \mapsto \mathfrak{U}^q_\circ(\Xi)$



Hence, the previous drawing is a particular *arbitrary choice*. We symbolically describe with the following the whole issue of the grade scenery and the non-unique Poisson structure. The ontologic Observer is actually "lost": he does not know how to describe the topological duality.
Before discuss the copolarization process, we notice the construction of Hélein and Kouneiher concerning the internal, the external and the $^{\mathfrak{s}}\mathfrak{p}$-bracket - see [115]. These considerations are connected to the expression of the dynamical duality. Notice that we delimitate two directions in connection

with this. The first is the relation with the dynamical area and the construction of the external bracket $\{\mathcal{H}\beta, \boldsymbol{\lambda}\}$ [115] - see the clear relation with the dynamical evolutions equation gvien by I.K. Kanatchicov. The second is in [115] the introduction of Grassman extra variables - which makes connection with the work of S. Hrabak [123, 124]. Notice that all these considerations are *before* the full distinction between AOF and OF. At the end, we write dynamical equations under the form:

$$\mathbf{d}A = \{\mathcal{H}\beta, A\} \qquad \text{and} \qquad \mathbf{d}\pi = \{\mathcal{H}\beta, \pi\} \tag{171}$$

Where $\mathbf{d}$ is the differential along a graph Γ of a solution of the Hamilton equations. This however involves to be able to define such a Poisson bracket between $\mathcal{H}\beta \in \Gamma(\mathcal{M}, \Lambda^n T^\star\mathcal{M})$ and $(p-1)$-forms, with $1 \leq p \leq n-1$. We adopt here the terminology developed in [115] where we find the following different brackets: *external* $\mathfrak{p}$-brackets and *internal* $\mathfrak{p}$-brackets, as well as $^{\mathfrak{s}}\mathfrak{p}$-bracket. First, let us recall the construction of the so called *internal* $\mathfrak{p}$-bracket developped in [115].

(i) Internal $\mathfrak{p}$-bracket. If $\boldsymbol{\lambda}, \boldsymbol{\kappa} \in \mathfrak{P}_\circ^{n-1}(\mathcal{M})$ with $\mathfrak{P}_\circ^{n-1}(\mathcal{M})$ the set of all algebraic observable $(n-1)$-forms, we define the *internal* $\mathfrak{p}$-bracket on $\mathfrak{P}_\circ^{n-1}(\mathcal{M})$ as

$$\{\boldsymbol{\lambda}, \boldsymbol{\kappa}\} = \Xi(\boldsymbol{\kappa}) \lrcorner \Xi(\boldsymbol{\lambda}) \lrcorner \boldsymbol{\omega} \tag{172}$$

The internal bracket is basically defined on algebraic $(n-1)$-forms. Hence we interpret *internal* as a term pointing the fact that we stay in the same degree in the ontologic *reflection*. So that we recover the description found in section (6.2).

(ii) External $\mathfrak{p}$-bracket. Now we extend the previous definition to the case where $\varphi \in \Gamma(\mathcal{M}, \Lambda^p T^\star\mathcal{M})$, with $1 \leq p \leq n$ and $\boldsymbol{\lambda} \in \mathfrak{P}_\circ^{n-1}(\mathcal{M})$ we obtain the *external* $\mathfrak{p}$-bracket:

$$\{\varphi, \boldsymbol{\lambda}\} = -\{\boldsymbol{\lambda}, \varphi\} = -\Xi(\boldsymbol{\lambda}) \lrcorner d\varphi \tag{173}$$

Hence we describe the symbolic picture for external $\mathfrak{p}$-bracket. One part is straightforward understood, since we have $\boldsymbol{\lambda} \in \mathfrak{P}_\circ^{n-1}(\mathcal{M})$ - $\boldsymbol{\lambda}$ belongs to the notion of *ontologic representation.* We have no prescription for the other part: let us consider a general form $\varphi \in \Gamma(\mathcal{M}, \Lambda^p T^\star\mathcal{M})$, with $1 \leq p \leq n$. At this stage, it is not clear whatever we shall termed it *ontologic* or *dynamical* representation. We choose to illustrate this ambiguity by the following picture for $[\![\Gamma(\mathcal{M}, \Lambda^p T^\star\mathcal{M})]\!] \Big| [\![\lhd]\!]$:

$$\begin{array}{ccc} \Gamma(\mathcal{M}, \Lambda^p T^\star\mathcal{M}) \times \mathfrak{P}_\circ^{n-1}(\mathcal{M}) & \to & \Gamma(\mathcal{M}, \Lambda^p T^\star\mathcal{M}) \\ (\varphi, \boldsymbol{\lambda}) & \mapsto & \big\{\boldsymbol{\lambda}, \varphi\big\} = -\Xi(\boldsymbol{\lambda}) \lrcorner d\varphi \end{array}$$

The interesting case for dynamical evolution is when, $\varphi = \mathcal{H}\beta$. Then we notice that for any $\boldsymbol{\lambda}, \in \mathfrak{P}_\circ^{n-1}$ we have the following relation $\{\mathcal{H}\beta, \boldsymbol{\lambda}\} = -\Xi(\boldsymbol{\lambda}) \lrcorner d\mathcal{H} \wedge \beta$.

(iii) $^{\mathfrak{s}}\mathfrak{p}$-bracket. We are first interested by the *first attempt*, developped in [115] which is the construction of a bracket between $(n-1)$-p forms for p of arbitrary degree. Hélein and Kouneiher introduced anticommuting Grassman variables $\boldsymbol{\tau}_1 \cdots \boldsymbol{\tau}_n$ which behave under change of coordinates like $\partial_1 \cdots \partial_n$. Hence, a general form in such a setting depends on the set of variables $(\boldsymbol{\tau}_\alpha, x^\alpha, A_\mu, \mathfrak{e}, p^{A_\mu\alpha})$. For a more detailed presentation of these Grassmannian variables $\boldsymbol{\tau}_\alpha$ - and an intrinsic geometrical picture - see [115]. However, we do not insist on this notions of the $^{\mathfrak{s}}\mathfrak{p}$-bracket since for deeper purpose concerning the treatment of the dynamics, we will choose for adequate bracket a slightly different object.[56]. Here we feel the connection to the *conceptual* setting of the supersymmetric area

[56]The construction rather based on *copolarization* of the multisymplectic space allows us to define observable forms of any degree collectively. Then in the next section we find good bracket described by Hélein and Kouneiher without this *superform artifact.* Notice that the distinction which emphasized the symmetry point of view vs the dynamical point of view - and the related copolarization of a multisymplectic manifold - incorporate this aspect not on *fields* by directly in the *ontologic* space.

- where additional virtual *matter* degree of freedom is find with the notion of ghost - and open the supersymmetric landscape [59, 247] - see Hélein [110] and references therein for a basic introduction. Once again we observe, from the mathematical perspective the graded scenery and the Gerstenhaber algebra [89, 90]. Now we just give some remarks about the $^{\mathfrak{s}}\mathfrak{p}$-bracket to emphasize the difficult road concerning the bracket topic. This philosophy is strongly connected to the one found in the work of S. Hrabak on multisymplectic formulation of the classical BRST symmetry for first order field theories [123, 124].[57] In the work [115] Grassman variables $\boldsymbol{\tau}_\alpha$ make rise the notion of *superform.* For any $\boldsymbol{\lambda} \in \mathfrak{P}^{p-1}_\circ\mathcal{M}$ such that for all $1 \leq \alpha_1 \leq \cdots \leq \alpha_{n-p} \leq n$ we have: $dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{n-p}} \wedge \boldsymbol{\lambda} \in \mathfrak{P}^{n-1}_\circ\mathcal{M}$. We define in this case the superform $^{\mathfrak{s}}\boldsymbol{\lambda} =^{\mathfrak{s}} \boldsymbol{\lambda} = \sum_{\alpha_1<\cdots<\alpha_{n-p}} \boldsymbol{\tau}_{\alpha_1}\cdots\boldsymbol{\tau}_{\alpha_{n-p}} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{n-p}} \wedge \boldsymbol{\lambda}$. We define also a $^{\mathfrak{s}}\mathfrak{p}$-bracket for $\boldsymbol{\varphi} \in \Gamma(\mathcal{M}, \Lambda^n T^\star\mathcal{M})$

$$\left\{\boldsymbol{\varphi},^{\mathfrak{s}}\boldsymbol{\lambda}\right\}_{\mathfrak{s}} = -\Xi(^{\mathfrak{s}}\boldsymbol{\lambda}) \lrcorner\, d\boldsymbol{\varphi} = -\sum_{\alpha_1<\cdots<\alpha_{n-p}} \boldsymbol{\tau}_{\alpha_1}\cdots\boldsymbol{\tau}_{\alpha_{n-p}} \Xi(dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{n-p}} \wedge \boldsymbol{\lambda}) \lrcorner\, d\boldsymbol{\varphi}$$

Let $\boldsymbol{\lambda}$ be an admissible form [115], and let Γ a n-dimensional submanifold of $\mathcal{M}$ which is a graph over $\mathcal{X}$, then for any oriented $\Sigma^p \subset \Gamma$ with $\dim(\Sigma^p) = p$ we have:

$$\int_{\Sigma^p} \left\{\mathcal{H}\beta,\ ^{\mathfrak{s}}\boldsymbol{\lambda}\right\}_{\mathfrak{s}} = \int_{\Sigma^p} \left\{\mathcal{H}\beta, \boldsymbol{\lambda}\right\} \tag{174}$$

In the context of Maxwell theory, the 1-form A and the Faraday $(n-2)$-form π lead to the *superform* $^{\mathfrak{s}}A$ and $^{\mathfrak{s}}\pi$ and to the dynamical equations [115].

$$\text{(i)}\quad \mathbf{d}A = \left\{\mathcal{H}\beta, A\right\} \qquad\qquad \text{(ii)}\quad \mathbf{d}\pi = \left\{\mathcal{H}\beta, \pi\right\}$$

$$\text{(i)}\quad \mathbf{d}A = \sum_{\alpha<\beta} \mathsf{g}_{\alpha\mu}\mathsf{g}_{\beta\nu} p^{A_\mu\nu} dx^\alpha \wedge dx^\beta \qquad\qquad \text{(ii)}\quad \mathbf{d}\pi = J^\alpha\beta_\alpha$$

Canonical bracket is described via the computation of the $^{\mathfrak{s}}\mathfrak{p}$-bracket $\left\{^{\mathfrak{s}}\pi,^{\mathfrak{s}} A\right\}_{\mathfrak{s}}$. The important point is that the additional Grassman variables are only a tool, as in the case of ghost and anti-ghost, and disappear at the end of the calculation. Notice that finally this method is not retained. The good canonical Poisson bracket is now exposed, via the copolarization.

9.7 Copolarization and canonical variables

We recall the result obtained by Hélein and Kouneiher [115], they give a possible copolarization of $(\mathcal{M}^{\mathsf{Maxwell}}, \boldsymbol{\omega})$ for Maxwell theory - with $\boldsymbol{\omega} = d\mathfrak{e} \wedge \beta + d\pi \wedge dA$:

$$\begin{aligned}
\mathbf{P}^1 T^\star\mathcal{M}^{\mathsf{Maxwell}} &= \bigoplus_{0\leq\mu\leq3} dx^\mu \\
\mathbf{P}^2 T^\star\mathcal{M}^{\mathsf{Maxwell}} &= \bigoplus_{0\leq\mu_1<\mu_2\leq3} dx^{\mu_1} \wedge dx^{\mu_2} \oplus dA \\
\mathbf{P}^3 T^\star\mathcal{M}^{\mathsf{Maxwell}} &= \bigoplus_{0\leq\mu_1<\mu_2<\mu_3\leq3} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \oplus \bigoplus_{0\leq\mu\leq3} dx^\mu \wedge dA \oplus d\boldsymbol{\pi} \\
\mathbf{P}^4 T^\star\mathcal{M}^{\mathsf{Maxwell}} &= \beta \oplus \bigoplus_{0\leq\mu\leq3} \frac{\partial}{\partial x^\mu} \lrcorner\, \theta^{\mathsf{DW}} \oplus \bigoplus_{0\leq\mu_1<\mu_2\leq3} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dA \oplus \bigoplus_{0\leq\mu\leq3} dx^\mu \wedge d\boldsymbol{\pi}
\end{aligned}$$

[57] Here lay the connection with the conceptual setting of huge domain of modern investigation of mathematical physics. This concerns the *ghosts* and the *anti-ghosts* in the BRST formalism developed by C. Becchi, A. Rouet, R. Stora, I.V. Tyutin [23, 226] and the related BV setting of I.A. Batalin and G.A. Vilkovisky, [22]. Nevertheless this is another story.

In this case the symbolic picture for *copolarization* process leads to a good setting for structures. The ontologic and the dynamical *Observers* travel freely *in* the framework as opposed to the previous situation of the graded machinery where we have a non uniquely defined topological duality. The notion of copolarization describe the data for forms of various degrees. However, the ambiguity lay now in the choice of copolarization. We can choose several copolarizations for a given theory.
The ontologic (or dynamical) space and its representation are well defined - even if different choices of copolarization are possible. The real gain is in the Relativity essence. Rather to describe ambiguous Poisson structure or to call to virtual matter - see the example of superform - we argue for a more fundamental idea for copolarization process. The forms of interest are described but in a collective manner, implying a consideration on the whole space of forms.

10 Lepage-Dedecker for two dimensional Maxwell theory

10.1 Lepage-Dedecker correspondence

Now we perform a Lepage-Dedecker correspondence for the Maxwell 2D theory. In this section as opposed to the next one, we work with indices notation, in particular with the tedious but straightforward computation of the Hamiltonian. It is just to emphasize the huge amount of calculations for LD theories - even in a simple case $n = 2$ for the simple setting of the Maxwell theory. We refer to H.A. Katstrup [139] or Hélein and Kouneiher [117, 118] for some aspect of the two dimensional Lepage-Dedecker Maxwell theory. First we express the Lagrangian density $L(x, A, dA) = -\frac{1}{4}\mathsf{h}^{\mu\lambda}\mathsf{h}^{\nu\sigma}F_{\mu\nu}F_{\lambda\sigma}$ so that:

$$L(A) = -\frac{1}{4}\big(\mathsf{h}^{1\lambda}\mathsf{h}^{2\sigma}F_{12}F_{\lambda\sigma} + \mathsf{h}^{2\lambda}\mathsf{h}^{1\sigma}F_{21}F_{\lambda\sigma}\big) = -\frac{1}{4}\big(\mathsf{h}^{11}\mathsf{h}^{22}\big(F_{12}\big)^2 + \mathsf{h}^{22}\mathsf{h}^{11}\big(F_{21}\big)^2\big) = \frac{1}{2}\big(F_{12}\big)^2$$

then, the Lagrangian is written:

$$L(x, A, dA) = \frac{1}{2}\big(\partial_1 A_2 - \partial_2 A_1\big)\big(\partial_1 A_2 - \partial_2 A_1\big) = \frac{1}{2}\big(\big(\partial_1 A_2\big)^2 + \big(\partial_2 A_1\big)^2\big) - \big(\partial_1 A_2\big)\big(\partial_2 A_1\big)$$

Now we construct a *non degenerate* Legendre transform in the $2D$ case via the following Poincaré-Cartan form[58] (175) $\theta^{\text{Lepage-Dedecker}}_{(q,p)} = \theta^{[2]\|[2]}_{(q,p)} = \theta^{[2]}_{(q,p)}$.

$$\theta^{[2]}_{(q,p)} := \mathfrak{e}\beta + \pi^{A_\mu\nu} dA_\mu \wedge \beta_\nu + \varsigma dA_1 \wedge dA_2 \tag{175}$$

and the related Multisymplectic 3 form:

$$\boldsymbol{\omega}^{[2]}_{(q,p)} := d\mathfrak{e} \wedge \beta + d\pi^{A_\mu\nu} \wedge dA_\mu \wedge \beta_\nu + d\varsigma \wedge dA_1 \wedge dA_2 \tag{176}$$

Then, we concentrate on the expression of $\langle p, v\rangle$

$$\langle p, v\rangle = \theta^{[2]}_{(q,p)}(\mathcal{Z}) = \mathfrak{e}\beta(\mathcal{Z}) + \pi^{A_\mu\nu} dA_\mu \wedge \beta_\nu(\mathcal{Z}) + \varsigma dA_1 \wedge dA_2(\mathcal{Z}) \tag{177}$$

We demonstrate by direct calculation that:

$$\langle p, v\rangle = \theta^{[2]}_{(q,p)}(\mathcal{Z}) = \pi^{A_\mu\nu}\partial_\nu A_\mu + 2\varsigma\big(\mathcal{Z}_{11}\mathcal{Z}_{22} - \mathcal{Z}_{12}\mathcal{Z}_{21}\big)$$

[58] We use here the following notation $\theta^{[2]\|[2]}_{(q,p)}$ means we specify the canonical Poincaré-Cartan form for Maxwell theory in the 2 dimensional case and taking into account forms that involves 2 fields. (namely forms of the type $\varsigma dA_1 \wedge dA_2$) Following this logic we write the previous canonical form as $\theta^{\text{DW}}_{(q,p)} := \theta^{[1]\|[4]}_{(q,p)}$

⌈ Proof Since $\mathcal{Z}_\nu = \frac{\partial}{\partial x^\nu} + \mathcal{Z}_{\nu\mu}\frac{\partial}{\partial A_\mu}$, we have : $\mathcal{Z}_1 = \partial_1 + \mathcal{Z}_{1\mu_1}\frac{\partial}{\partial A_{\mu_1}}$ and $\mathcal{Z}_2 = \partial_2 + \mathcal{Z}_{2\mu_2}\frac{\partial}{\partial A_{\mu_2}}$ so that we compute $\mathcal{Z} = \mathcal{Z}_1 \wedge \mathcal{Z}_2$ Then we write it as:

$$\mathcal{Z} = \sum_{\mu_1<\mu_2} \mathcal{Z}_{12}^{\mu_1\mu_2}\frac{\partial}{\partial q^{\mu_1}} \wedge \frac{\partial}{\partial q^{\mu_2}} = \sum_{\mu_1<\mu_2} \begin{vmatrix} \mathcal{Z}_1^{\mu_1} & \mathcal{Z}_2^{\mu_1} \\ \mathcal{Z}_1^{\mu_2} & \mathcal{Z}_2^{\mu_2} \end{vmatrix} \frac{\partial}{\partial q^{\mu_1}} \wedge \frac{\partial}{\partial q^{\mu_2}}$$

$$\mathcal{Z} = \mathcal{Z}_{12}^{12}\partial_1 \wedge \partial_2 + \mathcal{Z}_{12}^{1\mu_2}\partial_1 \wedge \frac{\partial}{\partial A_{\mu_2}} + \mathcal{Z}_{12}^{2\mu_2}\partial_2 \wedge \frac{\partial}{\partial A_{\mu_2}} + \mathcal{Z}_{12}^{\mu_1\mu_2}\frac{\partial}{\partial A_{\mu_1}} \wedge \frac{\partial}{\partial A_{\mu_2}}$$

With, the different terms:

$$\begin{cases} \mathcal{Z}_{12}^{12} = 1 & \mathcal{Z}_{12}^{2\mu_2} = \begin{vmatrix} 0 & 1 \\ \mathcal{Z}_1^{\mu_2} & \mathcal{Z}_2^{\mu_2} \end{vmatrix} = -\mathcal{Z}_1^{\mu_2} \\ \mathcal{Z}_{12}^{1\mu_2} = \begin{vmatrix} 1 & 0 \\ \mathcal{Z}_1^{\mu_2} & \mathcal{Z}_2^{\mu_2} \end{vmatrix} = \mathcal{Z}_2^{\mu_2} & \mathcal{Z}_{12}^{\mu_1\mu_2} = \begin{vmatrix} \mathcal{Z}_1^{\mu_1} & \mathcal{Z}_2^{\mu_1} \\ \mathcal{Z}_1^{\mu_2} & \mathcal{Z}_2^{\mu_2} \end{vmatrix} = \left[\mathcal{Z}_1^{\mu_1}\mathcal{Z}_2^{\mu_2} - \mathcal{Z}_1^{\mu_2}\mathcal{Z}_2^{\mu_1}\right] \end{cases} \tag{178}$$

We make the following calculation:

$$\langle p, v\rangle = \mathfrak{e} + \underbrace{\pi^{A_\mu\nu}dA_\mu \wedge \beta_\nu(\mathcal{Z}_{12}^{1\mu_2}\partial_1 \wedge \frac{\partial}{\partial A_{\mu_2}} + \mathcal{Z}_{12}^{2\mu_2}\partial_2 \wedge \frac{\partial}{\partial A_{\mu_2}})}_{[\mathbf{I}]} + \underbrace{\varsigma dA_1 \wedge dA_2(\mathcal{Z}_{12}^{\mu_1\mu_2}\frac{\partial}{\partial A_{\mu_1}} \wedge \frac{\partial}{\partial A_{\mu_2}})}_{[\mathbf{II}]}$$

The first term in the last equation is given by:

$$\begin{aligned} [\mathbf{I}] &= \sum_{\mu,\nu}\pi^{A_\mu\nu}dA_\mu \wedge \beta_\nu(\mathcal{Z}) = \pi^{A_1 1}dA_1 \wedge \beta_1(\mathcal{Z}) + \pi^{A_1 2}dA_1 \wedge \beta_2(\mathcal{Z}) + \pi^{A_2 1}dA_2 \wedge \beta_1(\mathcal{Z}) + \pi^{A_2 2}dA_2 \wedge \beta_2(\mathcal{Z}) \\ &= \pi^{A_1 1}dA_1 \wedge dx^2(\mathcal{Z}_{12}^{2\mu_2}\partial_2 \wedge \frac{\partial}{\partial A_{\mu_2}}) - \pi^{A_1 2}dA_1 \wedge dx^1(\mathcal{Z}_{12}^{1\mu_2}\partial_1 \wedge \frac{\partial}{\partial A_{\mu_2}}) + \pi^{A_2 1}dA_2 \wedge dx^2(\mathcal{Z}_{12}^{2\mu_2}\partial_2 \wedge \frac{\partial}{\partial A_{\mu_2}}) \\ &= -\pi^{A_2 2}dA_2 \wedge dx^1(\mathcal{Z}_{12}^{1\mu_2}\partial_1 \wedge \frac{\partial}{\partial A_{\mu_2}}) = \pi^{A_1 1}\mathcal{Z}_{11} + \pi^{A_1 2}\mathcal{Z}_{21} + \pi^{A_2 1}\mathcal{Z}_{12} + \pi^{A_2 2}\mathcal{Z}_{22} = \pi^{A_\mu\nu}\partial_\nu A_\mu \end{aligned}$$

Whereas the second term is given by:

$$\begin{aligned} [\mathbf{II}] &= \varsigma dA_1 \wedge dA_2(\mathcal{Z}) = \varsigma dA_1 \wedge dA_2(\mathcal{Z}_{12}^{\mu_1\mu_2}\frac{\partial}{\partial A_{\mu_1}} \wedge \frac{\partial}{\partial A_{\mu_2}}) = \varsigma dA_1 \wedge dA_2(\left[\mathcal{Z}_1^{\mu_1}\mathcal{Z}_2^{\mu_2} - \mathcal{Z}_1^{\mu_2}\mathcal{Z}_2^{\mu_1}\right]\frac{\partial}{\partial A_{\mu_1}} \wedge \frac{\partial}{\partial A_{\mu_2}}) \\ &= \varsigma\big(\mathcal{Z}_{11}\mathcal{Z}_{22} - \mathcal{Z}_{12}\mathcal{Z}_{21}\big) \end{aligned}$$

Since

$$2\varsigma\epsilon^{\mu\nu}\mathcal{Z}_{1[\mu}\mathcal{Z}_{2\nu]} = \varsigma\epsilon^{\mu\nu}\mathcal{Z}_{1\mu}\mathcal{Z}_{2\nu} - \mathcal{Z}_{1\nu}\mathcal{Z}_{2\mu} = \varsigma\big(\mathcal{Z}_{11}\mathcal{Z}_{22} - \mathcal{Z}_{12}\mathcal{Z}_{21}\big) - \varsigma\big(\mathcal{Z}_{12}\mathcal{Z}_{21} - \mathcal{Z}_{11}\mathcal{Z}_{22}\big) = 2\varsigma\big(\mathcal{Z}_{11}\mathcal{Z}_{22} - \mathcal{Z}_{12}\mathcal{Z}_{21}\big)$$

we write the term $[\mathbf{II}] = \varsigma\epsilon^{\mu\nu}\mathcal{Z}_{1[\mu}\mathcal{Z}_{2\nu]}$ ⌋

Then we have the expression of $\langle p, v\rangle$

$$\langle p, v\rangle = \pi^{A_1 1}\mathcal{Z}_{11} + \pi^{A_1 2}\mathcal{Z}_{21} + \pi^{A_2 1}\mathcal{Z}_{12} + \pi^{A_2 2}\mathcal{Z}_{22} + \varsigma\big(\mathcal{Z}_{11}\mathcal{Z}_{22} - \mathcal{Z}_{12}\mathcal{Z}_{21}\big) \tag{179}$$

We can equivalently write in more contracted notation: $\langle p, v\rangle = \theta^{[2]}_{(q,p)}(\mathcal{Z}) = \pi^{A_\mu\nu}\partial_\nu A_\mu + \varsigma\epsilon^{\mu\nu}\mathcal{Z}_{1[\mu}\mathcal{Z}_{2\nu]}$. With the notation $\mathcal{Z}_{\nu\mu} = \partial_\nu A_\mu$, we write:

$$\langle p, v\rangle = \theta^{[2]}_{(q,p)}(\mathcal{Z}) = \pi^{A_\mu\nu}\partial_\nu A_\mu + \varsigma\epsilon^{\mu\nu}\partial_1 A_{[\mu}\partial_2 A_{\nu]}$$

Let us denote:

$$\kappa_{\mu\nu} = \kappa^{[2]}_{\mu\nu} = \frac{\partial\langle p, v\rangle}{\partial(\partial_\mu A_\nu)} = \frac{\partial}{\partial(\partial_\mu A_\nu)}\theta^{[2]}_{(q,p)}(\mathcal{Z}) \tag{180}$$

we work in coordinate expression so that we use the expression (179):

$$\theta^{[2]}_{(q,p)}(\mathcal{Z}) = \pi^{A_1 1}\partial_1 A_1 + \pi^{A_1 2}\partial_2 A_1 + \pi^{A_2 1}\partial_1 A_2 + \pi^{A_2 2}\partial_2 A_2 + \varsigma\big(\partial_1 A_1\partial_2 A_2 - \partial_1 A_2\partial_2 A_1\big)$$

Hence, we find the relations (181)(i).

$$(\mathrm{i})\quad \begin{array}{rcl} \kappa_{\mu\nu}|_{\mu=1,\nu=1} &=& \pi^{A_1 1} + \varsigma\partial_2 A_2 \\ \kappa_{\mu\nu}|_{\mu=1,\nu=2} &=& \pi^{A_2 1} - \varsigma\partial_2 A_1 \\ \kappa_{\mu\nu}|_{\mu=2,\nu=1} &=& \pi^{A_1 2} - \varsigma\partial_1 A_2 \\ \kappa_{\mu\nu}|_{\mu=2,\nu=2} &=& \pi^{A_2 2} + \varsigma\partial_1 A_1 \end{array} \qquad (\mathrm{ii})\quad \begin{array}{rcl} \boldsymbol{\lambda}_{\mu\nu}|_{\mu=1,\nu=1} &=& 0 \\ \boldsymbol{\lambda}_{\mu\nu}|_{\mu=1,\nu=2} &=& \partial_1 A_2 - \partial_2 A_1 \\ \boldsymbol{\lambda}_{\mu\nu}|_{\mu=2,\nu=1} &=& \partial_2 A_1 - \partial_1 A_2 \\ \boldsymbol{\lambda}_{\mu\nu}|_{\mu=2,\nu=2} &=& 0 \end{array} \tag{181}$$

On the other side, we denote $\partial L/\partial(\partial_\mu A_\nu) = \boldsymbol{\lambda}_{\mu\nu}$. We use the coordinate expression of $L(x, A, dA)$ and we obtain (181)(ii). The condition for the Legendre transform is:

$$\frac{\partial L}{\partial(\partial_\mu A_\nu)} = \frac{\partial\langle p, v\rangle}{\partial(\partial_\mu A_\nu)}$$

We obtain (182)(i) and, choosing to work in the case $\varsigma = 1$, we then obtain the relations (182)(ii)

$$(\mathrm{i})\quad \begin{array}{ccl} 0 &=& \pi^{A_1 1} + \varsigma\partial_2 A_2 \\ \partial_1 A_2 - \partial_2 A_1 &=& \pi^{A_2 1} - \varsigma\partial_2 A_1 \\ \partial_2 A_1 - \partial_1 A_2 &=& \pi^{A_1 2} - \varsigma\partial_1 A_2 \\ 0 &=& \pi^{A_2 2} + \varsigma\partial_1 A_1 \end{array} \quad \overset{\varsigma=1}{\Longrightarrow} \quad (\mathrm{ii})\quad \begin{array}{rcl} \pi^{A_1 1} &=& -\partial_2 A_2 \\ \pi^{A_2 1} &=& \partial_1 A_2 \\ \pi^{A_1 2} &=& \partial_2 A_1 \\ \pi^{A_2 2} &=& -\partial_1 A_1 \end{array} \tag{182}$$

The generalized Legendre correspondence is non degenerate. It is always possible to invert the multimomenta from multivelocities. Now, we give the expression of the Hamiltonian function. From (182)(i)

$$\begin{array}{rcl} \partial_2 A_2 &=& -(\varsigma^{-1})\pi^{A_1 1} \\ \partial_2 A_1 &=& \big(\varsigma(2-\varsigma)\big)^{-1}\big(\pi^{A_1 2} + (1-\varsigma)\pi^{A_2 1}\big) \\ \partial_1 A_2 &=& \big(\varsigma(2-\varsigma)\big)^{-1}\big(\pi^{A_2 1} + (1-\varsigma)\pi^{A_1 2}\big) \\ \partial_1 A_1 &=& -(\varsigma^{-1})\pi^{A_2 2} \end{array} \tag{183}$$

⌜ Proof Let us explicite the second line in (183), from the second line in (182) we find:

$$\partial_1 A_2 = \pi^{A_2 1} + (1-\varsigma)\partial_2 A_1 \tag{184}$$

The third line of (182) writes:

$$\partial_2 A_1 - \partial_1 A_2 = \pi^{A_1 2} - \varsigma\partial_1 A_2 \quad \Longrightarrow \quad \partial_2 A_1 = \pi^{A_1 2} + (1-\varsigma)\partial_1 A_2 \tag{185}$$

We insert (184) in (185) so that:

$$\partial_2 A_1 = \pi^{A_1 2} + (1-\varsigma)\big(\pi^{A_2 1} + (1-\varsigma)\partial_2 A_1\big)$$

$$\partial_2 A_1\big(1 - (1-\varsigma)^2\big) = \pi^{A_1 2} + (1-\varsigma)\pi^{A_2 1} \quad \Longleftrightarrow \quad \partial_2 A_1\big(2\varsigma - \varsigma^2\big) = \pi^{A_1 2} + (1-\varsigma)\pi^{A_2 1}$$

$\partial_2 A_1 \varsigma(2-\varsigma) = \pi^{A_1 2} + (1-\varsigma)\pi^{A_2 1}$An analogous process holds also for the other relation. ⌟

10.2 Calculation of the Hamiltonian

We are interested in the expression of the Hamiltonian:

$$\mathcal{H} = \theta^{[2]}_{(q,p)}(\mathcal{Z}) - L \tag{186}$$

where $\theta^{[2]}_{(q,p)}(\mathcal{Z}) = [\mathbf{1}] + \cdots + [\mathbf{6}]$ and $-L = [\mathbf{7}] + [\mathbf{8}] + [\mathbf{9}]$ with

$$\begin{array}{rcl} [\mathbf{1}] &=& \pi^{A_1 1}\partial_1 A_1 \\ [\mathbf{2}] &=& \pi^{A_1 2}\partial_2 A_1 \\ [\mathbf{3}] &=& \pi^{A_2 1}\partial_1 A_2 \end{array} \qquad \begin{array}{rcl} [\mathbf{4}] &=& \pi^{A_2 2}\partial_2 A_2 \\ [\mathbf{5}] &=& \varsigma\partial_1 A_1 \partial_2 A_2 \\ [\mathbf{6}] &=& -\varsigma\partial_1 A_2 \partial_2 A_1 \end{array} \qquad \begin{array}{rcl} [\mathbf{7}] &=& -1/2\big(\partial_1 A_2\big)^2 \\ [\mathbf{8}] &=& -1/2\big(\partial_2 A_1\big)^2 \\ [\mathbf{9}] &=& \big(\partial_1 A_2\big)\big(\partial_2 A_1\big) \end{array}$$

⌈ Let us examine each of these terms. We denote by $\boldsymbol{\varsigma} = 1/\varsigma(2-\varsigma)$. The terms [**1**]-[**4**] correspond to the terms $\langle p, v\rangle = \theta^{(\mathrm{DW})}_{(q,p)}(\mathcal{Z})$

$$\begin{array}{rcl} [\mathbf{1}] &=& -(\varsigma^{-1})\pi^{A_1 1}\pi^{A_2 2} \\ [\mathbf{2}] &=& \boldsymbol{\varsigma}\pi^{A_1 2}\big[\pi^{A_1 2} + (1-\varsigma)\pi^{A_2 1}\big] \\ [\mathbf{3}] &=& \boldsymbol{\varsigma}\pi^{A_2 1}\big[\pi^{A_2 1} + (1-\varsigma)\pi^{A_1 2}\big] \\ [\mathbf{4}] &=& -(\varsigma^{-1})\pi^{A_2 2}\pi^{A_1 1} \end{array} \tag{187}$$

Also the two terms which are related to the Lepage-Dedecker part.

$$\begin{array}{rcl} [\mathbf{5}] &=& (\varsigma^{-1})\pi^{A_2 2}\pi^{A_1 1} \\ [\mathbf{6}] &=& -\varsigma\boldsymbol{\varsigma}^2\big[\pi^{A_2 1} + (1-\varsigma)\pi^{A_1 2}\big]\big[\pi^{A_1 2} + (1-\varsigma)\pi^{A_2 1}\big] \end{array} \tag{188}$$

And finally, the three terms which come from the Lagrangian density:

$$\begin{array}{rcl} [\mathbf{7}] &=& -(1/2)\boldsymbol{\varsigma}^2\big[\pi^{A_2 1} + (1-\varsigma)\pi^{A_1 2}\big]\big[\pi^{A_2 1} + (1-\varsigma)\pi^{A_1 2}\big] \\ [\mathbf{8}] &=& -(1/2)\boldsymbol{\varsigma}^2\big[\pi^{A_1 2} + (1-\varsigma)\pi^{A_2 1}\big]\big[\pi^{A_1 2} + (1-\varsigma)\pi^{A_2 1}\big] \\ [\mathbf{9}] &=& \boldsymbol{\varsigma}^2\big[\pi^{A_2 1} + (1-\varsigma)\pi^{A_1 2}\big]\big[\pi^{A_1 2} + (1-\varsigma)\pi^{A_2 1}\big] \end{array} \tag{189}$$

Let us consider the equations [**1**], [**4**] and [**5**] in (187). We denote by (i) = [**1**] + [**4**] + [**5**] so that

$$(\mathrm{i}) = -(\varsigma^{-1})\pi^{A_2 2}\pi^{A_1 1} \tag{190}$$

We denote (ii) = [**2**] + [**3**] so that

$$(\mathrm{ii}) = \boldsymbol{\varsigma}\big[\pi^{A_1 2}\pi^{A_1 2} + \pi^{A_1 2}(1-\varsigma)\pi^{A_2 1} + \pi^{A_2 1}\pi^{A_2 1} + \pi^{A_2 1}(1-\varsigma)\pi^{A_1 2}\big] \tag{191}$$

It remains the following equations [**6**]-[**9**] We have respectively:

$$\begin{array}{rcl} [\mathbf{6}] &=& \big[-\varsigma\boldsymbol{\varsigma}^2\big]\big[\pi^{A_2 1}\pi^{A_1 2} + \pi^{A_2 1}(1-\varsigma)\pi^{A_2 1} + (1-\varsigma)\pi^{A_1 2}\pi^{A_1 2} + (1-\varsigma)\pi^{A_1 2}(1-\varsigma)\pi^{A_2 1}\big] \\ &=& \big[-\varsigma\boldsymbol{\varsigma}^2\big]\big[(1+(1-\varsigma)^2)\pi^{A_2 1}\pi^{A_1 2} + \big[\pi^{A_2 1}\big]^2(1-\varsigma) + (1-\varsigma)\big[\pi^{A_1 2}\big]^2\big] \\ &=& \big[-\varsigma\boldsymbol{\varsigma}^2\big]\big[(2(1-\varsigma)+\varsigma^2)\pi^{A_2 1}\pi^{A_1 2} + \big[\pi^{A_2 1}\big]^2(1-\varsigma) + (1-\varsigma)\big[\pi^{A_1 2}\big]^2\big] \end{array} \tag{192}$$

The second and the third give

$$\begin{array}{rcl} [\mathbf{7}] &=& \big[-(1/2)\boldsymbol{\varsigma}^2\big]\big[\big[\pi^{A_2 1}\big]^2 + (1-\varsigma)^2\big[\pi^{A_1 2}\big]^2 + 2\pi^{A_2 1}(1-\varsigma)\pi^{A_1 2}\big] \\ [\mathbf{8}] &=& \big[-(1/2)\boldsymbol{\varsigma}^2\big]\big[\big[\pi^{A_1 2}\big] + (1-\varsigma)^2\big[\pi^{A_2 1}\big]^2 + 2\pi^{A_1 2}(1-\varsigma)\pi^{A_2 1}\big] \end{array} \tag{193}$$

Now, we denote (iii) = [**6**] + [**9**] so that

$$(\mathrm{iii}) = (1-\varsigma)\boldsymbol{\varsigma}^2\Big[(2(1-\varsigma)+\varsigma^2)\pi^{A_2 1}\pi^{A_1 2} + \big[\pi^{A_2 1}\big]^2(1-\varsigma) + (1-\varsigma)\big[\pi^{A_1 2}\big]^2\Big] \tag{194}$$

and finally we denote ($\mathfrak{iv}$) = [**7**] + [**8**], then

$$(\mathfrak{iv}) = -\frac{1}{2}\boldsymbol{\varsigma}^2\Big[\big[\pi^{A_2 1}\big]^2 + (1-\varsigma)^2\big[\pi^{A_1 2}\big]^2 + 2\pi^{A_2 1}(1-\varsigma)\pi^{A_1 2} + \big[\pi^{A_1 2}\big]^2 + (1-\varsigma)^2\big[\pi^{A_2 1}\big]^2 + 2\pi^{A_1 2}(1-\varsigma)\pi^{A_2 1}\Big] \tag{195}$$

Finally we compute (ii) + (iii) + ($\mathfrak{iv}$). We introduce the following notations:

$$\pi^{\circ\circ} = \pi^{A_1 2}\pi^{A_1 2} = \big[\pi^{A_1 2}\big]^2 \qquad \pi^{\circ\bullet} = \pi^{A_1 2}\pi^{A_2 1} \qquad \overline{\pi}^{\circ\bullet} = \pi^{A_1 1}\pi^{A_2 2} \qquad \pi^{\bullet\bullet} = \pi^{A_2 1}\pi^{A_2 1} = \big[\pi^{A_2 1}\big]^2 \tag{196}$$

So that the equations (191) (194) and (195) are written (197) (ii)-($\mathfrak{iv}$):

$$\begin{array}{rcl} (\mathrm{ii}) &=& \boldsymbol{\varsigma}\big[\pi^{\circ\circ} + 2(1-\varsigma)\pi^{\circ\bullet} + \pi^{\bullet\bullet}\big] \\ (\mathrm{iii}) &=& (1-\varsigma)[\boldsymbol{\varsigma}]^2\Big[(2(1-\varsigma)+\varsigma^2)\pi^{\circ\bullet} + \pi^{\bullet\bullet}(1-\varsigma) + (1-\varsigma)\pi^{\circ\circ}\Big] \\ (\mathfrak{iv}) &=& -1/2[\boldsymbol{\varsigma}]^2\Big[\pi^{\bullet\bullet} + (1-\varsigma)^2\pi^{\circ\circ} + 2(1-\varsigma)\pi^{\circ\bullet} + \pi^{\circ\circ} + (1-\varsigma)^2\pi^{\bullet\bullet} + 2(1-\varsigma)\pi^{\circ\bullet}\Big] \end{array} \tag{197}$$

where we have denoted $\boldsymbol{\varsigma} = \big[\varsigma(2-\varsigma)\big]^{-1}$ So that (197)-(ii) is written:

$$(\mathrm{ii}) = [\boldsymbol{\varsigma}]^2\big[\pi^{\circ\circ} + (2-2\varsigma)\pi^{\circ\bullet} + \pi^{\bullet\bullet}\big]\varsigma(2-\varsigma) = [\boldsymbol{\varsigma}]^2\big[\pi^{\circ\circ} + 2\pi^{\circ\bullet} - 2\varsigma\pi^{\circ\bullet} + \pi^{\bullet\bullet}\big](2\varsigma - \varsigma^2)$$

If we denote $\Phi = \big(2\varsigma^2(2-\varsigma)^2\big)^{-1}$, we obtain:

$$(\mathrm{ii}) = 2\Phi\Big(2\pi^{\circ\circ}\varsigma - \pi^{\circ\circ}\varsigma^2 + 4\pi^{\circ\bullet}\varsigma - 2\pi^{\circ\bullet}\varsigma^2 - 4\pi^{\circ\bullet}\varsigma^2 + 2\pi^{\circ\bullet}\varsigma^3 + 2\pi^{\bullet\bullet}\varsigma - \pi^{\bullet\bullet}\varsigma^2\Big)$$

The equation (197)-(iii) is written:

$$\begin{aligned}(\mathrm{iii}) &= 2\Phi(1-\varsigma)\Big[\big(2(1-\varsigma)+\varsigma^2\big)\pi^{\circ\bullet}+\pi^{\bullet\bullet}\big(1-\varsigma\big)+\big(1-\varsigma\big)\pi^{\circ\circ}\Big]\\ &= 2\Phi(1-\varsigma)\Big[\big(2\pi^{\circ\bullet}-2\varsigma\pi^{\circ\bullet}+\varsigma^2\pi^{\circ\bullet}+\pi^{\bullet\bullet}-\varsigma\pi^{\bullet\bullet}+\pi^{\circ\circ}-\varsigma\pi^{\circ\circ}\Big]\\ &= 2\Phi\Big[2\pi^{\circ\bullet}-2\varsigma\pi^{\circ\bullet}+\varsigma^2\pi^{\circ\bullet}+\pi^{\bullet\bullet}-\varsigma\pi^{\bullet\bullet}+\pi^{\circ\circ}-\varsigma\pi^{\circ\circ}-2\pi^{\circ\bullet}\varsigma\\ &\quad +2\varsigma^2\pi^{\circ\bullet}-\varsigma^3\pi^{\circ\bullet}-\varsigma\pi^{\bullet\bullet}+\varsigma^2\pi^{\bullet\bullet}-\varsigma\pi^{\circ\circ}+\varsigma^2\pi^{\circ\circ}\Big]\end{aligned}$$

and finally (197)-($\mathfrak{iv}$) is written:

$$\begin{aligned}(\mathfrak{iv}) &= -\Phi\Big[\pi^{\bullet\bullet}+\big(1-\varsigma\big)^2\pi^{\bullet\bullet}+\big(1-\varsigma\big)^2\pi^{\circ\circ}+\pi^{\circ\circ}+4\big(1-\varsigma\big)\pi^{\circ\bullet}\Big]\\ &= -\Phi\Big[2\pi^{\bullet\bullet}-2\varsigma\pi^{\bullet\bullet}+\varsigma^2\pi^{\bullet\bullet}+2\pi^{\circ\circ}-2\varsigma\pi^{\circ\circ}+\varsigma^2\pi^{\circ\circ}+4\pi^{\circ\bullet}-4\varsigma\pi^{\circ\bullet}\Big]\end{aligned}$$

We now writes $\mathcal{H} = (\mathfrak{i}) + (\mathfrak{ii}) + (\mathrm{iii}) + (\mathfrak{iv})$

$$\begin{aligned}\mathcal{H} &= (\mathfrak{i})+\Phi\Big[4\pi^{\circ\circ}\varsigma-2\pi^{\circ\circ}\varsigma^2+8\pi^{\circ\bullet}\varsigma-4\pi^{\circ\bullet}\varsigma^2-8\pi^{\circ\bullet}\varsigma^2+4\pi^{\circ\bullet}\varsigma^3+4\pi^{\bullet\bullet}\varsigma-2\pi^{\bullet\bullet}\varsigma^2+4\pi^{\circ\bullet}-4\varsigma\pi^{\circ\bullet}+2\varsigma^2\pi^{\circ\bullet}\\ &\quad +2\pi^{\bullet\bullet}-2\varsigma\pi^{\bullet\bullet}+2\pi^{\circ\circ}-2\varsigma\pi^{\circ\circ}-4\pi^{\circ\bullet}\varsigma+4\varsigma^2\pi^{\circ\bullet}-2\varsigma^3\pi^{\circ\bullet}-2\varsigma\pi^{\bullet\bullet}+2\varsigma^2\pi^{\bullet\bullet}-2\varsigma\pi^{\circ\circ}+2\varsigma^2\pi^{\circ\circ}\\ &\quad -2\pi^{\bullet\bullet}+2\varsigma\pi^{\bullet\bullet}-\varsigma^2\pi^{\bullet\bullet}-2\pi^{\circ\circ}+2\varsigma\pi^{\circ\circ}-\varsigma^2\pi^{\circ\circ}-4\pi^{\circ\bullet}+4\varsigma\pi^{\circ\bullet}\Big]\\ &= \Phi\Big[2\varsigma\pi^{\circ\circ}-\varsigma^2\pi^{\circ\circ}+2\pi^{\bullet\bullet}\varsigma-\varsigma^2\pi^{\bullet\bullet}+2\pi^{\circ\bullet}\varsigma^3-6\pi^{\circ\bullet}\varsigma^2+4\pi^{\circ\bullet}\varsigma\Big]+(\mathfrak{i})\\ &= \Phi\Big[\big(2-\varsigma\big)\pi^{\circ\circ}\varsigma+\big(2-\varsigma\big)\pi^{\bullet\bullet}\varsigma+2\varsigma^2\big(\varsigma-3\big)\pi^{\circ\bullet}+4\pi^{\circ\bullet}\varsigma\Big]+(\mathfrak{i})\\ &= \Phi\Big[\big(2-\varsigma\big)\varsigma\Big[\pi^{\circ\circ}+\pi^{\bullet\bullet}\Big]+2\varsigma^2\big(\varsigma-3\big)\pi^{\circ\bullet}+4\pi^{\circ\bullet}\varsigma\Big]+(\mathfrak{i}) \quad \lrcorner\end{aligned}$$

We finally obtain the expression of the Hamiltonian:

$$\mathcal{H} = -\frac{1}{\varsigma}\overline{\pi}^{\circ\bullet}+\frac{1}{2}\frac{1}{\varsigma\big(2-\varsigma\big)}\Big[\pi^{\circ\circ}+\pi^{\bullet\bullet}\Big]+\frac{\big(\varsigma-3\big)}{\big(2-\varsigma\big)^2}\pi^{\circ\bullet}+\frac{1}{\varsigma\big(2-\varsigma\big)^2}2\pi^{\circ\bullet} \tag{198}$$

If we use the transform with $\varsigma = 1$ then (198)($\mathfrak{ii}$) gives the following Hamiltonian:

$$\mathcal{H} = -\overline{\pi}^{\circ\bullet}+\frac{1}{2}\big(\pi^{\circ\circ}+\pi^{\bullet\bullet}\big) \tag{199}$$

⌜ Proof. We compute in coordinate the straightforward calculation:

$$\begin{aligned}\mathcal{H} &= \pi^{A_\mu\nu}\partial_\nu A_\mu+\varsigma\big[\mathcal{Z}_{11}\mathcal{Z}_{22}-\mathcal{Z}_{12}\mathcal{Z}_{21}\big]-\frac{1}{2}\big(\partial_1A_2\big)^2-\frac{1}{2}\big(\partial_2A_1\big)^2+\big(\partial_1A_2\big)\big(\partial_2A_1\big)\\ &= (\pi^{A_12})^2+(\pi^{A_21})^2-2\pi^{A_11}\pi^{A_22}+\pi^{A_11}\pi^{A_22}-\pi^{A_12}\pi^{A_21}-\frac{1}{2}\big(\pi^{A_21}\big)^2-\frac{1}{2}\big(\pi^{A_12}\big)^2+\pi^{A_12}\pi^{A_21}\\ &= -\pi^{A_11}\pi^{A_22}+\frac{1}{2}\big(\pi^{A_21}\big)^2+\frac{1}{2}\big(\pi^{A_12}\big)^2 \quad \lrcorner\end{aligned}$$

And the Hamiltonian (199) agrees with the general case (198).

10.3 Equations of movement

Now let us derive the generalized Hamilton equations. The general form of a vector field is given:

$$X_\alpha = \frac{\partial}{\partial x^\alpha}+\Theta_{\alpha\mu}\frac{\partial}{\partial A_\mu}+\Upsilon_\alpha\frac{\partial}{\partial\mathfrak{e}}+\Upsilon_\alpha^{A_\mu\nu}\frac{\partial}{\partial\pi^{A_\mu\nu}}+\Upsilon_\alpha^{A_\mu A_\nu}\frac{\partial}{\partial\varsigma^{A_\mu A_\nu}}$$

leads us to $X = X_1 \wedge X_2$

$$
\begin{aligned}
X &= \partial_1 \wedge \partial_2 + \partial_1 \wedge \Theta_{2\mu}\frac{\partial}{\partial A_\mu} + \partial_1 \wedge \Upsilon_2\frac{\partial}{\partial \mathfrak{e}} + \partial_1 \wedge \Upsilon_2^{A_\mu\nu}\frac{\partial}{\partial \pi^{A_\mu\nu}} \\
&+\Theta_{1\mu}\frac{\partial}{\partial A_\mu}\wedge\partial_2 + \Theta_{1\mu}\frac{\partial}{\partial A_\mu}\wedge\Theta_{2\mu}\frac{\partial}{\partial A_\mu} + \Theta_{1\mu}\frac{\partial}{\partial A_\mu}\wedge\Upsilon_2\frac{\partial}{\partial \mathfrak{e}} + \Theta_{1\mu}\frac{\partial}{\partial A_\mu}\wedge\Upsilon_2^{A_\mu\nu}\frac{\partial}{\partial \pi^{A_\mu\nu}} \\
&+\Upsilon_1\frac{\partial}{\partial \mathfrak{e}}\wedge\partial_2 + \Upsilon_1\frac{\partial}{\partial \mathfrak{e}}\wedge\Theta_{2\mu}\frac{\partial}{\partial A_\mu} + \Upsilon_1\frac{\partial}{\partial \mathfrak{e}}\wedge\Upsilon_2\frac{\partial}{\partial \mathfrak{e}} + \Upsilon_1\frac{\partial}{\partial \mathfrak{e}}\wedge\Upsilon_2^{A_\mu\nu}\frac{\partial}{\partial \pi^{A_\mu\nu}} \\
&+\Upsilon_1^{A_\mu\nu}\frac{\partial}{\partial \pi^{A_\mu\nu}}\wedge\partial_2 + \Upsilon_1^{A_\mu\nu}\frac{\partial}{\partial \pi^{A_\mu\nu}}\wedge\Theta_{2\mu}\frac{\partial}{\partial A_\mu} + \Upsilon_1^{A_\mu\nu}\frac{\partial}{\partial \pi^{A_\mu\nu}}\wedge\Upsilon_2\frac{\partial}{\partial \mathfrak{e}} \\
&+\Upsilon_1^{A_\mu\nu}\frac{\partial}{\partial \pi^{A_\mu\nu}}\wedge\Upsilon_2^{A_\mu\nu}\frac{\partial}{\partial \pi^{A_\mu\nu}}
\end{aligned}
$$

We compute the first part of Generalized Hamilton equations, namely $X \lrcorner \boldsymbol{\omega}^{[2]}$

$$
\begin{aligned}
X \lrcorner \boldsymbol{\omega}^{[2]} &= X \lrcorner \left(d\mathfrak{e}\wedge\beta + d\pi^{A_\mu\nu}\wedge dA_\mu\wedge\beta_\nu + d\varsigma\wedge dA_1\wedge dA_2\right) \\
&= d\mathfrak{e} - (d\mathfrak{e}\wedge\beta_\mu)(X)dx^\mu + (dA_\mu\wedge\beta_\nu)(X)d\pi^{A_\mu\nu} - (d\pi^{A_\mu\nu}\wedge\beta_\nu)(X)dA_\mu \\
&\quad +(d\pi^{A_\mu\nu}\wedge dA_\mu\wedge\beta_{\rho\nu})dx^\rho + (dA_1\wedge\beta_2)(X)d\varsigma \\
&\quad -(d\varsigma\wedge dA_2)(X)dA_1 + (d\varsigma\wedge dA_1)dA_2
\end{aligned}
$$

And, in this case, since $d\varsigma = 0$ the multisymplectic form is written $\boldsymbol{\omega}^{[2]}\big|_{\varsigma=1} = d\mathfrak{e}\wedge\beta + d\pi^{A_\mu\nu}\wedge dA_\mu\wedge\beta_\nu$. So that $X \lrcorner \boldsymbol{\omega}^{[2]}\big|_{\varsigma=1}$ is given by:

$$
\begin{aligned}
X \lrcorner \boldsymbol{\omega}^{[2]}\big|_{\varsigma=1} &= d\mathfrak{e} - (d\mathfrak{e}\wedge\beta_\mu)(X)dx^\mu + (dA_\mu\wedge\beta_\nu)(X)d\pi^{A_\mu\nu} - (d\pi^{A_\mu\nu}\wedge\beta_\nu)(X)dA_\mu \\
&\quad +(d\pi^{A_\mu\nu}\wedge dA_\mu\wedge\beta_{\rho\nu})dx^\rho \\
&= d\mathfrak{e} - \Upsilon_\rho dx^\rho + \Theta_{\nu\mu}d\pi^{A_\mu\nu} - \Upsilon_\nu^{A_\mu\nu}dA_\mu + \left(\Upsilon_\rho^{A_\mu\nu}\Theta_{\nu\mu} - \Upsilon_\nu^{A_\mu\nu}\Theta_{\rho\mu}\right)dx^\rho
\end{aligned}
$$

we only keep the interesting part on the decompositions along $d\pi^{A_\mu\nu}$ and dA_μ -

$$
\left\{\begin{array}{rcl} \Theta_{\nu\mu}d\pi^{A_\mu\nu} &=& d\mathcal{H} \\ -\Upsilon_\nu^{A_\mu\nu} &=& 0 \end{array}\right. \qquad \left\{\begin{array}{lrcl} (\mathrm{i}) & \Theta_{1\mu}d\pi^{A_\mu 1} + \Theta_{2\mu}d\pi^{A_\mu 2} &=& d\mathcal{H} \\ (\mathrm{ii}) & -\Upsilon_\nu^{A_\mu\nu} &=& 0 \end{array}\right. \tag{200}
$$

With $d\mathcal{H}\big|_{\varsigma=1} = -\overline{\pi}^{\circ\bullet} + 1/2\left(\pi^{\circ\circ}+\pi^{\bullet\bullet}\right) = -\pi^{A_1 1}d\pi^{A_2 2} - \pi^{A_2 2}d\pi^{A_1 1} + \pi^{A_2 1}d\pi^{A_2 1} + \pi^{A_1 2}d\pi^{A_1 2}$. Finally we obtain from (200)(i) the Legendre transform given by:

$$
\partial_1 A_1 = -\pi^{A_2 2} \qquad \partial_2 A_2 = -\pi^{A_1 1} \qquad \partial_1 A_2 = \pi^{A_2 1} \qquad \partial_2 A_1 = \pi^{A_1 2} \tag{201}
$$

Whereas from (200)(ii) we obtain the Maxwell equations:

$$
\partial_\mu \pi^{A_\nu \mu} = 0 \tag{202}
$$

TOWARDS MULTISYMPLECTIC GRAVITY

11 General Relativity

11.1 Hierarchy of structures beyond mathematical model of space-time

The mathematical model of space-time is involved with a hierarchy of structures, from the topological structure to the causal structure. GR is built on a specific mathematical model[59] of *space-time* denoted: $\boxed{S}\boxed{M}^{\mathsf{GR}}$. The suitable *space-time* entity in GR concerns a *differentiable structure* with a *Riemannian (Lorentzian) metric*.[60] The setting of GR is concerned with a n-dimensional manifold

[59] see the next section (11.2) for the choice of this symbol.

[60] One speaks of a Lorentzian metric when the dimension of the manifold is four and when the signature is of type $(1, 3)$: therefore it is a special case of Riemannian metric.

$\mathcal{X}$ given with a riemannian metric $\mathsf{g}_{\mu\nu}$. We denote symbolically $\boxed{\mathsf{S}}\boxed{\mathsf{M}}^{\mathsf{GR}} = (\mathcal{X}, \mathsf{g}_{\mu\nu})$. This simple statement points the interplay between different layers from mathematical standpoints. We follow C. Isham [128] to emphasize a hierarchy of structures related to the quantization of gravity puzzle. We also denote symbolically the chain rule: $\big(,\big) \subset \big(,\big)^{\mathsf{topo}} \subset \big(,\big)^{\mathsf{diff}} \subset \big(,\big)^{\mathsf{caus}} \subset \boxed{\mathsf{S}}\boxed{\mathsf{M}}^{\mathsf{GR}}$. Let us say few words on the *differential structure* $\big(,\big)^{\mathsf{diff}}$. A *differential structure* is defined via the main concept of smooth *manifold* and we want to use differential calculus for generalized space like the Riemannn-Cartan spaces where curvature and torsion are taken into account. Also we would like to consider different topologies for a given space - or a given model of space-time. After the aspect of the topological structure $\big(,\big)^{\mathsf{topo}}$ the next layer appears as the adequate differential structure. Hence, we addressed this via the notion of a manifold $\mathcal{X}$ perceived as a topological space locally homeomorphic to $\mathbb{R}^n$. Further appears the *causal structure* $\big(,\big)^{\mathsf{caus}}$. The causal structure is defined via the concept of Lorentzian metric. In QFT, the causal structure is fixed and we have a preferred notion of *causality*, or equivalently preferred notion of *locality*. It is one of the Wightman axioms [219], and is related to the existence of a non-dynamical, Minkowski background metric h. Then one observes a well-defined causal structure. However, in General Relativity the situation is drastically different. *Since no prior geometry is given, then what is the meaning of causality relation ?* This point leads us to introduce the spirit of the AQFT approach[61], see [30, 86, 211], toward quantum gravity. At the left side of the chain, we found the symbol $\big(,\big)$ that would be mathematical landscape beyond topology, a kind of *conceptual ground-state*, which represents somehow, from an *abstract* viewpoint, the natural mathematics area on which foundations for *Quantum Gravity* would lay.[62] At the right side of the symbolic chain, we find the mathematical model of *space-time* $\boxed{\mathsf{S}}\boxed{\mathsf{M}}^{\mathsf{GR}}$. In the theory of GR, the topological, differential and causal layers are related to various mathematical settings and obstructions. Then, we observe different specificities depending on witch layer we imposed restrictions. Once again we comprise all these characteristics for the space-time entity of GR with the symbol[63] $\boxed{\mathsf{S}}\boxed{\mathsf{M}}^{\mathsf{GR}}$. For us, and in a broadly first approach, the symbol $\boxed{\mathsf{S}}\boxed{\mathsf{M}}^{\mathsf{GR}}$ means in particular the existence of a Lorentzian metric.

From the Einstein standpoint, the space-time structure $\boxed{\mathsf{S}}\boxed{\mathsf{M}}^{\mathsf{GR}}$ is not defined by points, the space-time entity is rather built on the notion of metric. It means that if we delete the metric structure on the space-time entity, then *nothing* remains: there is no possible Minkowski space-time, there is also no remaining *topology* description. For Einstein, the metric $\mathsf{g}_{\mu\nu}$ describes not only the gravitational field, but also the topological properties of the manifold. Entering in more details later, we will see that, in a modern view, the *gravity* feature is encoded *via* the notion of *solder form*. The solder form

[61]The Algebraic Quantum Field Theory (AQFT) (see [30, 211]) is connected to the study of *formal functional methods* and is to be understood, from a physical standpoint as motivated by the needs of QFT (the path integral approach). A further question concerns the possibility of formulating a consistent axiomatic within arbitrary curved space-time. More technically, following [30] we define a local, generally covariant quantum field theory as a *"covariant functor between the category of globally hyperbolic (four-dimensional) space-time manifolds with isometric embeddings as morphisms and the category of $C^\star$-algebras with invertible endomorphisms as morphisms"*. We mention this approach in order to emphasize a crucial point about *causal structure* since AQFT is based upon two main principles: *covariance* and *locality*. [86] In QFT the causal structure is fixed. Then, one observes a well-defined causal structure: for example, for a scalar field $\varphi(x)$ with spacelike interval between the points x and x', we have causality relations: $[\varphi(x), \varphi(x')] = \varphi(x)\varphi(x') - \varphi(x')\varphi(x) = 0$. Our intuition is that it is the area where MG and CPS overlap, formally and conceptually, that, we shall discover the necessary tools to resolve this very subtle question.

[62]This mathematical area may endow, from *Quantum* viewpoint, the face of intersections like the ones of *algebraic topology* or *quantum-topology*. For example we may try to formulate another chain rule, based on the grounds of Quantum area as $\big(,\big) \subset \big(,\big)^{\mathsf{topo}} \subset \big(,\big)^{\mathsf{quantum}} \subset \cdots \subset \boxed{\mathsf{S}}\boxed{\mathsf{M}}^{\mathsf{GR}}$. However in the statement of some kind of quantum topology, it seems that the first step of the chain, namely $\big(,\big)^{\mathsf{topo}} \subset \big(,\big)^{\mathsf{quantum}}$, destroys the well established differential structures $\big(,\big)^{\mathsf{diff}}$.

[63]We enter slowly into the explanation of the choice of this symbol along the discussion, and, by the way, find full justification for it. It is a manifestation of what later on we will call an *ontologic* mode.

appears as the gravity potential. This is connected to the fact that the disappearance of preferred inertial observer is encoded in the *displacement fields* $\Gamma^{\rho}_{\mu\nu}$ (the Levi-Civita connection) whereas the existence of Riemannian metric $\mathsf{g}_{\mu\nu}$ is of secondary importance in this play. This is the purpose of the following sections where we precise the meaning of close but different related fundamental concepts: the *vierbein* and the *solder form*. Before going on some reflections about the ontology of space-time itself - and its intricate relation with the ontology of *matter*, what *would* be contained in it - and in doing so appreciate the deep vision of relativity rooted in this fundamental problem, we first specify the commonly used model of the general relativistic *space-time*. The general relativistic *space-time structure* $\boxed{\mathrm{S}}\boxed{\mathrm{M}}^{\mathsf{GR}}$ is the data of a manifold $\mathcal{X}$, together with a metric $\mathsf{g}_{\mu\nu}$ of Lorentzian signature and a connection[64] ∇. We forget the connection for the moment, and other artifacts[65] - such as orientation given by the volume element or time orientation - therefore we symbolically describe a space-time structure by:

$$\boxed{\mathrm{S}}\boxed{\mathrm{M}}^{\mathsf{GR}} = \big(\mathcal{X}, \mathsf{g}_{\mu\nu}, \nabla, vol_{\mathcal{X}}(\mathsf{g})...\big) = \big(\mathcal{X}, \mathsf{g}_{\mu\nu}\big)$$

The general setting of metric-compatible connection is connected to the Riemannn-Cartan spaces where the curvature R and torsion T tensors appear as master pieces of the theory. In the general case we not only allow the curvature (Riemannn space) but also the torsion (Riemannn-Cartan space) to be non vanishing objects. Classical GR considers a Riemannn space with the following relations: $\nabla \mathrm{g} = 0$ and $T = 0$, and in this specific case, there exists a unique connection: the Levi-Civita connection, see section (11.9).

11.2 Ontological being of space-time

We are faced to the ontological status of space-time. Now we focus on the space-time manifold $\mathcal{X}$ itself: does a point in space-time $\mathcal{X}$ has an intrinsic existence, as the substantivalism view picture it? The old Newtonian way to capture physical reality is related to the picture that *material content* is defined on *space-time*. On the other side, the Leibniz relational standpoint is a conceptual expression of the underlying impossibility to refer to a space-time structure. This old debate between the *substantivalism* and the *relational* standpoints is perceived from a deeper perspective and leads in a modern view to what Stachel calls *dynamic structural realism* [217]. If we follow[66] Einstein [70], *"On the basis of the general theory of relativity space as opposed to what fills space has no separate existence. If we imagine the gravitational field [...] to be removed, there does not remain a space of the type [of the Minkowski space of SR], but absolutely nothing, not even a topological space [...]*

[64] Here the Levi-Civita, see section (11.9) below, or later a more general Riemannn-Cartan connection.

[65] $\boxed{\mathrm{S}}\boxed{\mathrm{M}}^{\mathsf{GR}}$ roughly means the existence of a Lorentzian metric. We find in the litterature properties such as: *globally hyperbolicity*, *orientation* ... Hence we speak about an oriented, time-oriented, Lorentzian 4D-space-time. Global hyperbolicity is connected to the initial value problem, the definition of Cauchy surface and therefore appears at the intersection of both structures $\big(,\big)^{\mathsf{caus}}$ and $\big(,\big)^{\mathsf{topo}}$. This assumption means that $\mathcal{X}$ is diffeomorphic to $\Sigma \times \mathbb{R}$. As emphasized by Stachel [216, 217], the historical road toward general relativistic Cauchy problem is related to the work of Hilbert, [121] A. Lichnerowicz [162] and G. Darmois [50]. These mathematical investigations led to a discussion on *null* and *space-like* hypersurfaces, the delimitation for initial data on spacelike hypersurface and geometrical objects such as first and second fundamental forms. Finally, appeared also the distinction among the ten field equations into two sectors: on one hand we have four constraints on the initial spacelike hypersurface and, on the other hand, six evolution equations. The interplay between these different structures is really connected to a profound insight revealed by GR. It is worth noticing that, after all this fact isn't mysterious if we follow Einstein himself. From Einstein standpoint, at the heart of GR, space-time structure $\boxed{\mathrm{S}}\boxed{\mathrm{M}}^{\mathsf{GR}}$ is not defined by points, space-time entity is rather built on the notion of metric. What means that if we delete the metric structure on space-time entity, then, *nothing* remains, there is no possible Minkowski space-time, there is also no remaining *topology area*. For Einstein, the metric $\mathsf{g}_{\mu\nu}$ does not only describe the gravitational field, but also the topological properties of the manifold.

[66] The selected citation and comment are from Stachel [217]

There is no such thing as an empty space, i.e., a space without field. Space-time does not claim existence on its own, but only as a structural quality of the field".

This idea leads us to picture the following ontological organization: $\boxed{\mathrm{S}}^{\mathsf{GR}}$ represents the *space-time* itself and $\boxed{\mathrm{M}}^{\mathsf{GR}}$ represents *what fills space-time.* Let notice that the expression *what is contained in space-time* is taken to emphasize the opposition with the ontology of *space-time* itself. From the Einstein viewpoint, $\boxed{\mathrm{S}}^{\mathsf{GR}}$ has no separate existence from $\boxed{\mathrm{M}}^{\mathsf{GR}}$. The abstract symbol, $\boxed{\mathrm{M}}^{\mathsf{GR}}$ represents the *matter area* or more precisely it concerns *what is contained* in space-time: fields. Hence, $\boxed{\mathrm{M}}^{\mathsf{GR}}$ content, actually concerns any interacting entities $\Psi = \{\Psi_{\mathsf{Field}}, \Psi_{\mathsf{Particle}}\}$ defined *on the ground of* the space-time concept - as maps *from* or *into* space-time. The idea of General Relativity is that we cannot separate the two entities. They are solder together: *space-time has no proper existence but as a structural property of the field* [71]. Hence we consider an *interacting entity* $\Psi = \{\Psi_{\mathsf{Field}}, \Psi_{\mathsf{Particle}}\}$ defined abstractly as a map *into* or *from space-time*: *matter.* An interacting entity is either a particle Ψ_{Particle} or a field Ψ_{Field}. A *field* is a map from the space-time manifold $\mathcal{X}$ to a vector space or an affine space, $\Psi_{\mathsf{Field}} : \mathcal{X} \to \mathcal{V}$ whereas a particle Ψ_{Particle} is represented by a map into space-time $\Psi_{\mathsf{Particle}} : \mathbb{R} \to \mathcal{X}$. Then, any field is described as part of an *interacting entity.* This is why we introduce Ψ_{S} and Ψ_{M} where Ψ_{M} denotes abstractly fields - or particles, any *interacting entity* - which represent the *material contents* of space-time whereas Ψ_{S} concerns abstract fields which give the *space-time structure.*

The role of the metric tensor field $\mathsf{g}_{\mu\nu}$ is particular. First, the metric field is a field defined over the manifold, a bilinear symmetric 2-form $\mathsf{g} = \mathsf{g}_{\mu\nu}(x)dx^{\mu} \otimes dx^{\nu}$ and we describe in our picture $\mathsf{g}_{\mu\nu}$ as a field $\Psi_{\mathrm{M}}\leftrightarrows\boxed{\mathrm{M}}$. The Riemannian metric $\mathsf{g}_{\mu\nu}(x)$ is the gravitational field, and deserves the role to be the potentials for the inertio-gravitational field. However, the relativity insight emphasizes also the dual role for $\mathsf{g}_{\mu\nu}$: to set the chrono-geometrical structure of space-time. In that perspective $\mathsf{g}_{\mu\nu}$ is part of fundamental objects that describe our space-time model $\boxed{\mathrm{S}}$. It is then picture by the symbol $\Psi_{\mathrm{S}}\leftrightarrows\boxed{\mathrm{S}}$. Actually, a field $\Psi = \left\{\Psi_{\mathrm{S}}, \Psi_{\mathrm{M}}\right\}$ is either what we would call a dynamical entity Ψ_{S} - namely an object that crystallized the two dual aspects: *inertia* and *gravitation* in a single field (the metric field) - or a *simple* field. It has been expressed by the passage from the special role for the inertial frame in special relativity, to general arbitrary accelerated frame. From physics standpoint, an accelerated frame taken in the setting where we have no gravitational field is equivalent to an inertial frame with gravitational field. Therefore, the mathematical model of space-time in GR is intrinsically related to the *ever present* companion: the field. We denote symbolically:

$$\boxed{\mathrm{S}}\boxed{\mathrm{M}}^{\mathsf{GR}} = (\mathcal{X}\leftrightarrows\boxed{\mathrm{S}}, \mathfrak{Z}\leftrightarrows\boxed{\mathrm{M}}, \Psi_{\mathrm{S}}, \Psi_{\mathrm{M}}) = (\mathcal{X}, \mathfrak{Z}, \Psi_{\mathrm{S}}, \Psi_{\mathrm{M}})$$

to emphasize these consideration on fields and interacting entites. Here $\mathfrak{Z}$ denotes symbolically the target space of matter fields. Again, in the first movement of our journey - that would be transmuted with the multisymplectic relativity insight, see later developments in sections (20) and (20.1) - we see the natural arena for GR can be seen as the mathematical data written $\boxed{\mathrm{S}}\boxed{\mathrm{M}}^{\mathsf{GR}} = \left(\mathcal{X}, \mathsf{g}_{\mu\nu}, \Psi_{\mathrm{M}}\right)$ where Ψ_{M} represent non-dynamical fields. Within the Einstein vision, they represent *"non-gravitational fields and/or matter and acting as sources of the metrical field in the inhomogeneous Einstein equations"* - the citation is from M. Iftime and Stachel [126]. We observe in the following section, the key role of the choice of a *frame of reference.* In that sens, a *point* x^{μ} of the manifold $\mathcal{X}$ is only a label. Associate the choice of coordinate system to any physical content is meaningless. We cite Stachel [217] *"To talk about a principle of relativity only makes sense if one has first defined a frame of reference."* Then the space (*i.e space-time*) in the Einstein viewpoint is described by the choice of a metric. In a more modern language we speak of the *choice* of a section on the linear frame bundle. We would like to emphasize that this is precisely this *choice* (whatever we consider the *metric* or the *solder form*) that highlights Relativity principle.

11.3 Principles of General Relativity

The theory of GR is constructed with two fundamental principles. The first is the so-called *diffeomorphism invariance* whereas the second is the *equivalence principle*. The *diffeomorphism invariance* is the modern expression of the more primitive *principle of covariance*. The proper feature of general relativity is what we call *general covariance*, or *covariance*. Roughly speaking, general covariance means that there is no dependance of physical quantities on the choice of coordinates. This is a mathematical requirement and constrains the way we formulate the theory. The second main principle of GR is the *equivalence principle*. It states that for any *local* region in *space-time* it is equivalent to write equations for physical laws such that the effect of gravitation is not taken into account. The curvature is the geometrical object which encodes the gravitational field. Locally appears this relation with the flat Minkowski space. Hence, the *strong equivalence principle* establishes the local equivalence between *gravitation* and *inertia*.

In the following discussion we concentrate on the idea of background independence. Many misunderstanding about the concept of general covariance trace back to the following concepts: *point* and *events*. Background independence deletes the possibility to identify a point of the manifold $\mathcal{X}$ with any physical content. In this case we formulate a fully dynamical, background-independent space-time theory. In a fully background-independent theory, the global topology of the space-time manifold is not given *a priori*. The topology may depend on solutions of the field equations. This is a crucial aspect of a *dynamical entity* related to the fact that the structure on $\mathcal{X}$ is not fixed *a priori*. We observe that what we usually call a *space-time symmetry*, in a *purely ontological meaning*, is given by means of the diffeomorphisms $\varkappa : \mathcal{X} \to \mathcal{X}$. We realize diffeomorphism invariance via the condition $\varkappa^\star \Psi_{\mathrm{S}} = \Psi_{\mathrm{S}}$ where Ψ_{S} symbolize the typical gravitational field $\mathsf{g}_{\mu\nu}$. The *principle of covariance* is expressed via the *active* view of diffeomophism invariance. If we describe a mathematical model $\boxed{\mathrm{S}}\boxed{\mathrm{M}}^{\mathsf{GR}}$ of the relativistic space-time in correspondence to any precise physical content of a gravitational theory, then for any diffeomorphism $\varkappa$, the abstract mathematical gravitational model $\varkappa^\star\boxed{\mathrm{S}}\boxed{\mathrm{M}}^{\mathsf{GR}}$ equivalently describes the same *physical situation*. This is what we call below an *active view of covariance principle*, [12] symbolically described by:

$$\boxed{\mathrm{S}}\boxed{\mathrm{M}}^{\mathsf{GR}} = (\mathcal{X}, \Psi_{\mathrm{S}}, \Psi_{\mathrm{M}}) \;\equiv\; (\mathcal{X}, \varkappa^\star\Psi_{\mathrm{S}}, \varkappa^\star\Psi_{\mathrm{M}}) = \varkappa^\star\boxed{\mathrm{S}}\boxed{\mathrm{M}}^{\mathsf{GR}}$$

In this *active* standpoint, we speak of general covariance as *diffeomorphism invariance* and we observe a maturation of the old debate concerning the physical content of *general covariance*, crystalized in E. Kretschmann [153] objection - that this principle is empty of any physical content. In this direction, one argue today that *background-independance* is related to the concept of diffeomorphism invariance. As indication of this debate stands the opposition with general coordinate transformations and the *tensorial laws* for physics. Einstein's field equations are invariant under general coordinate transformations. In this story, the pseudo-group of general coordinate transformations $\mathfrak{C}_{\mathrm{oo}}(\mathcal{X})$ is involved and concerns the fact that we allow mathematical possibility of formulating the theory using tensors fields. Tensor fields present good transformation properties under a general coordinate transformation[67] so that they can be defined globally on the manifold.

Let $\mathfrak{D}_{\mathsf{iff}}(\mathcal{X})$ be the diffeomorphism group, the group of *active* transformation of $\mathcal{X}$. If we follow the Einstein point of view, $\mathfrak{D}_{\mathsf{iff}}(\mathcal{X})$, the group of diffeomorphism on a Riemannian *space-time* is seen as the *invariance* group of general relativity.[68] As Einstein emphasized the invariance under

[67] Let $\varsigma \in \mathfrak{C}_{\mathrm{oo}}(\mathcal{M})$ with $\varsigma : x^\mu \longrightarrow \varsigma(x^\mu)$ and let a tensor field $\mathbf{P}(x) = \mathbf{P}^{\mu_1\cdots\mu_q}{}_{\nu_1\ldots\nu_p}(x)\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_q}\otimes dx^{\nu_1}\cdots\otimes dx^{\nu_q}$. Under the general transformation $\varsigma \in \mathfrak{C}_{\mathrm{oo}}(\mathcal{M})$, the components of the tensor field transform as $\mathbf{P}^{\mu'_1\cdots\mu'_q}{}_{\nu'_1\cdots\nu'_p}(x^\mu) = \left(\frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}}\right)\cdots\left(\frac{\partial x^{\mu'_q}}{\partial x^{\mu_q}}\right)\left(\frac{\partial x^{\mu_1}}{\partial x^{\mu'_1}}\right)\cdots\left(\frac{\partial x^{\mu_q}}{\partial x^{\mu'_p}}\right)\mathbf{P}^{\mu_1\cdots\mu_q}{}_{\nu_1\cdots\nu_p}(x^\mu)$

[68] Any theory of nature is (in contrast to a covariant one) is invariant under *passive diffeomorphism* transformations. Also, notice that Einstein's standpoint is historically rooted *before* the *spin* has been discovered.

$\mathfrak{D}_{\mathsf{iff}}(\mathcal{X})$ delete any physical signification to a point of the manifold. If by $\mathfrak{M}_{\mathsf{etrics}}(\mathcal{X})$ we denote the space of all metrics over $\mathcal{X}$, the relevant (*physical*) space is the space of *geometries* of the manifold $\mathfrak{G}_{\mathsf{eom}}(\mathcal{X}) = \mathfrak{M}_{\mathsf{etrics}}(\mathcal{X})/\mathfrak{D}_{\mathsf{iff}}(\mathcal{X})$. The *equivalence principle* is pictured as the existence of a Lorentz metric on our space-time manifold $\mathcal{X}$. One can understand this point as a bundle reduction of the linear frame bundle $F(\mathcal{X})$ to the Lorentz sub-bundle $F_{SO(1,3)}(\mathcal{X})$. This picture appears to be the good landscape for the setting of studying the space $\mathfrak{G}_{\mathsf{eom}}(\mathcal{X})$ - understood as a space of 4-geometries. We refer to the work of M. Iftime [126] for the analyze of the structure of the moduli space $\mathcal{X}_{SO(1,3)}$ of all isomorphism classes of $SO(1,3)$ reduced principle sub-bundles of the linear bundle $F(\mathcal{X})$. We refer to the forthcoming section (12.3) for more comments on the bundle reduction.

The class of *background-independent theory* concerns several aspects. In the most general case, the *background structure* concerns the Lorentzian metric $\mathsf{g}_{\mu\nu}$, - namely $(,)^{\mathsf{caus}}$ previously introduced. However it shall be emphasized that we may also understand the concept of background structure as involved in several of the layers necessary to build the space-time entity $\boxed{\mathsf{S}}\boxed{\mathsf{M}}^{\mathsf{GR}}$. In other words: the topology may be concerned as a proper part of the *background structure* and then the question of its implication in the theory become fundamental. Maybe an acceptable QG theory shall not only try to quantize the gravitational field itself, but also the topology beyond. For example, usual QG approaches - the *covariant* and *canonical* frameworks - drastically suffer on their own way for the application of this idea. On one hand, we consider the *covariant approach*, where we consider the decomposition of the metric $\mathsf{g}_{\mu\nu} = \mathsf{h}_{\mu\nu} + h_{\mu\nu}$. This setting is deeply impregnate with a fixed topology and fixed differential structure modeled on Minkowski background metric. Difficulties of such a *covariant* approach - with preferred causal structure - make not reasonable place for a full *background-independent* meaning. On the other side, in the *canonical approach*, we work with a fixed topology $\Sigma \times \mathbb{R}$ and the *background-independent* prescription is apply on the causal layer. However in this case, we *do* work with *singular* topology. Both approaches,[69] covariant and canonical - the latter leads to the modern LQG program - are part of the same logic that want to realize QG on the basis of a quantification of classical general relativity.

11.4 Canonical framework and topology

One of the conceptual issue of GR faced to the machinery of gauge theory lay in the prism of the following question: it is possible to see GR as a gauge theory and, in this case, how we can implement a gauge vision for the diffeomorphism group $\mathfrak{D}_{\mathsf{iff}}(\mathcal{X})$? The choice of a topology via a given foliation of *space-time* leads to the decomposition $\mathcal{X} = \mathbb{R} \times \Sigma$ and makes more subtile the role of the diffeomorphism group $\mathfrak{D}_{\mathsf{iff}}(\mathcal{X})$ in the story. It is often argued that the symmetry group $\mathfrak{D}_{\mathsf{iff}}(\mathcal{X})$ for Einstein-Hilbert action is *"faithfully implemented in the canonical framework, although in a not very manifest way* [222]. For us however, this question is far from being an artifact. Beyond it, the fundamental issue of a fully covariant Hamiltonian formalism for field theory without the asymmetrical treatment of space and time is related to the perspective of working free from these specific topological constraints. This is one of the main motivations for multisymplectic setting. The classical canonical approach for GR is built on the Hamiltonian setting and is constructed with this *space-time* decomposition. This space-time foliation is the backbone of canonical quantum gravity - historically developed with Wheeler's *Geometrodynamics* [239] and the related work of R. Arnowitt, S. Deser, and C.W. Misner [6]. Space-time entity $\boxed{\mathsf{S}}\boxed{\mathsf{M}}^{\mathsf{GR}}$ is assumed to be globally hyperbolic and

[69]String theory shall be understood to belong to another category: within perturbative tools and with the idea that General Relativity appear as the low energy limit of the theory, this one is build on quantum principles but the classical analogous is not. Then, working within logic of string theory, we have well established quantum part whereas it is commonly admitted that connection to geometry of General Relativity and connection to physical principles is obviously lacking.

thanks to Geroch's theorem [88] it admits a foliation with the singular topology $\mathbb{R} \times \Sigma$. Notice that this picture is related to the initial value of calssical GR formulation and, in this context, to the existence of Cauchy surfaces[70] concerns a topological restriction. Hyperbolic space-time can be foliated by a family of spacelike hypersurfaces Σ_t.[71] Notice that in the classical picture there is no difficulties but in quantum area topological changes *do* have importance. Canonical approach for QG, based on a space-time foliation, suffer from huge global topological restrictions. Nevertheless, these are the roots of traditional *canonical approach* which flourished at the crossroad of the work of Dirac [60, 61, 62, 63] and the one of P.G. Bergmann [24, 25] within the context of Hamiltonian dynamics with constraints. In this setting, the idea is to relate properties defined on a 4-dimensional Riemannian Manifold $(\mathcal{X}, \mathsf{g}_{\mu\nu})$ to an embedded 3-dimensional Riemannian submanifold (Σ, γ_{ij}) via an embedding $\varsigma_t : \Sigma \to \mathcal{X}$. The bilinear form $\mathsf{g}_{\mu\nu}$ defines the induced metric on Σ by means of the pullback $\varsigma^\star g$. We obtain the *induced metric* on $\Sigma \times \mathbb{R}$ denoted $(\varsigma^\star \mathsf{g})_{\mu\nu}(x,t)$. Alternatively, one speak also of the induced metric $\gamma_{\mu\nu}$ on Σ by the diffeomorphsim ς_t. Then, the foliated leaves inherits the structure (Σ, γ) and the canonical variables are the 3-metrics γ_{ij} on Σ, we alternatively describe these objects by $\gamma_{\mu\nu}$ on $\Sigma_t = \varsigma_t(\Sigma)$.

11.5 Basic geometrical features

Dynamics of GR is described by the Einstein equations. They are obtained from variational principle applied on the Einstein-Hilbert action (203) which depends on the metric $\mathsf{g}_{\mu\nu}$. In this approach the metric is the dynamical variable and satisfies Euler Lagrange equations. The fundamental objects - the Levi-Civita connection $\Gamma^\rho_{\mu\nu}$ and the curvature tensor $R^\rho{}_{\mu\nu\sigma}$ - are expressed via the metric. In such a framework, we describe GR as a *metric theory*.

$$\mathcal{L}_{\mathsf{EH}}[\mathsf{g}_{\mu\nu}] = \int_{\mathcal{X}} L[\mathsf{g}_{\mu\nu}]\beta = \int_{\mathcal{X}} R\sqrt{-\mathsf{g}}\beta = \int_{\mathcal{X}} \sqrt{-\mathsf{g}}\mathsf{g}^{\mu\nu} R_{\mu\nu}[\mathsf{g}]\beta. \tag{203}$$

The variational principle is applied to $\mathcal{L}_{\mathsf{EH}}[\mathsf{g}_{\mu\nu}]$. We perform variations with respect to the metric variable $\mathsf{g}_{\mu\nu}$ and we obtain the vacuum Einstein field equations. [69, 70, 71]

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\mathsf{g}_{\mu\nu} R = 0. \tag{204}$$

The action (203) is called the Einstein-Hilbert action since the action principle for gravity was given by Hilbert [120]. Here, $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci curvature tensor and R is the scalar curvature. Besides the curvature tensor, hides the introduction of covariant derivative object ∇_μ. This one turns into the use of an *affine* connection $\Gamma^\rho{}_{\mu\nu}$ (the Christoffel symbols with inhomogeneous local coordinate transformations). Hence, the definition of the Riemannn curvature tensor in terms of commutator of covariant derivative is given by the following expression:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{205}$$

The canonical Riemannian volume form is a coordinate free object, denoted $\mathrm{vol}_{\mathcal{X}}(\mathsf{g})$. Let us denote $\beta = dx^1 \wedge .... \wedge dx^n$ so that $\mathrm{vol}_{\mathcal{X}}(\mathsf{g}) = \sqrt{-\mathsf{g}}\beta = \sqrt{g} dx^1 \wedge .... \wedge dx^n$ so that $\mathrm{vol}_{\mathcal{X}}(\mathsf{g})$ is expressed with the coordinate system $(x^1, ..., x^n)$ where $\{\partial_\mu\}$ is the coordinate basis of the tangent space $T\mathcal{X}$. The dual coordinate basis for the cotangent space $T^\star\mathcal{X}$ is $\{dx^\mu\}$. Let us introduce $\epsilon_{\mu_1 ... \mu_n}$ the Levi-Civita

[70] A Cauchy surface is a spacelike hypersurface such that each causal (i.e. timelike or null) curve without end point intersects it only once. A space-time $(\mathcal{X}, \mathsf{g}_{\mu\nu})$ that admits a Cauchy surface Σ called globally hyperbolic.

[71] The hypersurfaces Σ_t are integral manifold of the distribution. A space-time foliation is given via an embedding $\varsigma_t : \Sigma \to \mathcal{X} : x \in \mathcal{X} \mapsto \varsigma_t(x) = \varsigma(t,x)$, from which the diffeomorphism $\varsigma : \mathbb{R} \times \Sigma \to \mathcal{X} : (t,x) \to \varsigma(t,x) = \varsigma_t(x)$ is constructed.

symbol and $\boldsymbol{\epsilon}_{\mu_1...\mu_n} = \mathrm{vol}_{\mathcal{X}}(\mathrm{g})_{\mu_1...\mu_n}$ the components of the Levi-Civita tensor[72] expressed in the basis $dx^{\mu_1} \otimes ... \otimes dx^{\mu_n}$. We have[73] the relation $\boldsymbol{\epsilon}_{\mu_1...\mu_n} = \sqrt{|g|}\epsilon_{\mu_1...\mu_n}$. In the subsequent sections we use the following property ($\boldsymbol{\sigma}$ is the signature of the metric.)

$$\boldsymbol{\epsilon}^{\mu_1...\mu_p\alpha_1...\alpha_{n-p}}\boldsymbol{\epsilon}_{\mu_1...\mu_p\beta_1...\beta_{n-p}} = (-1)^{\boldsymbol{\sigma}} p!(n-p)!\delta^{[\alpha_1}_{\beta_1}...\delta^{\alpha_{n-p}]}_{\beta_{n-p}}.$$

11.6 Moving frame, holonomic frame

Let us recall some basic facts about moving frames to introduce the deepness of the *tetrad formalism.* Let $\mathcal{X}$ be a n-dimensional manifold. The tangent space $T_x\mathcal{X}$ at any point $x \in \mathcal{X}$ is a vector space so that any vector on it is expanded in a basis or *frame* $\mathbf{e}(x) = \{\mathbf{e}_\mu(x)\}_{1\leq\mu\leq n}$, a set of n linearly independent vector $\mathbf{e}_\mu(x)$ at $x \in \mathcal{X}$. We call a *moving frame* or *repère mobile* - in the spirit of Cartan [40] - a smooth assignment $x \in \mathcal{X} \mapsto \mathbf{e}(x)$ of the frame $\mathbf{e}(x)$ of $T_x\mathcal{X}$ to each point of the manifold. A vector field $\xi : x \in \mathcal{X} \mapsto \xi(x) \in \mathfrak{X}(\mathcal{X})$ is written $\xi(x) = \xi^\mu(x)\mathbf{e}_\mu(x)$ or more simply $\xi = \xi^\mu\mathbf{e}_\mu$. A tangent vector fields on $\mathcal{X}$ is a section of the tangent bundle $T\mathcal{X}$ and $\mathfrak{X}(\mathcal{X})$ is the set of such sections. We introduce the *coframe* as the dual object: an ordered set of n linear 1-form $\theta^\mu(x) = \{\theta^\mu(x)\}_{1\leq\mu\leq n}$ at a given point x of the manifold. The cotangent space $T^\star_x\mathcal{X}$ is also a vector space. A moving coframe is the smooth assignment $\theta : x \in \mathcal{X} \mapsto \theta(x)$ of the coframe $\theta(x)$ of $T^\star_x\mathcal{M}$ to each point $x \in \mathcal{X}$. We observe the duality relation $\theta^\mu(\mathbf{e}_\nu) = \delta^\mu_\nu$. In the section (12.4) we will describe the coframe field as a bundle isomorphism and some considerations on the soldering procedure, which is fundamental for gravity and supergravity theories. Moving frames and moving co-frames do not require the existence of a metric. An *holonomic* frame, or *coordinate* frame[74] is associated to specific coordinates chart x^μ and writes $\mathbf{e}_\mu = \partial_\mu = \frac{\partial}{\partial x^\mu}$, whereas dual object *i.e* the holonomic coframe is given by $\theta^\mu = dx^\mu$. An *holonomic* basis for $T\mathcal{U}$[75], is a set $\{\mathbf{e}_\mu\}$, such that $\forall\mu, \mathbf{e}_\mu \in \Gamma(\mathcal{X}, T\mathcal{X})$ there exists a coordinate chart $(\mathcal{U}, \phi)$ and a coordinate function $x^\mu : \mathcal{U} \to \mathbb{R}$, such that for each differentiable function $\psi : \mathcal{X} \to \mathbb{R}$ with $m \in \mathcal{U} \subset \mathcal{X}$ and $\phi(m) = x \in \mathcal{O} \subset \mathbb{R}^n$

$$\mathbf{e}_\mu(\psi)\big|_m = \frac{\partial}{\partial x^\mu}\psi \circ \phi^{-1}\big|_x$$

This definition justifies the notation $\mathbf{e}_\mu = \partial_\mu$. On the other hand, a set θ^μ where $\theta^\mu \in \Gamma(\mathcal{U}, T^\star\mathcal{U}) \subset \Gamma(\mathcal{X}, T^\star\mathcal{X})$ is called an *holonomic* basis for $T^\star\mathcal{U}$ if there exists a coordinate chart $(\mathcal{U}, \phi)$ and a coordinate function $x^\mu : \mathcal{U} \to \mathbb{R}$ such that $\theta^\mu = dx^\mu$. Since the basis θ^μ is the dual basis of $\mathbf{e}_\mu$ we have for *holonomic* side the relation: $\theta^\mu(\mathbf{e}_\nu) = dx^\mu(\partial_\nu) = \delta^\mu_\nu$. A general coframe $\theta^\mu = dx^\mu$ is holonomic if $d\theta^\mu = 0$, then the coordinate frame is trivially holonomic since $d(dx^\mu) = 0$.

The *structure coefficients* $\mathfrak{c}^\rho_{\mu\nu}$ are the objects which give the indication for *holonomic* feature and are defined via the commutators relations $[\mathbf{e}_\mu,\mathbf{e}_\nu] = \mathfrak{c}^\rho_{\mu\nu}\mathbf{e}_\rho$. The Lie bracket of vector fields is

[72] We have $\mathrm{vol}_{\mathcal{X}}(\mathrm{g}) = \mathrm{vol}_{\mathcal{X}}(\mathrm{g})_{\mu_1...\mu_n} dx^{\mu_1} \otimes ... \otimes dx^{\mu_n} = \boldsymbol{\epsilon}_{\mu_1...\mu_n} dx^{\mu_1} \otimes ... \otimes dx^{\mu_n} = (1/n!)\boldsymbol{\epsilon}_{\mu_1...\mu_n} dx^{\mu_1} \wedge ... \wedge dx^{\mu_n}$

$$\mathrm{vol}_{\mathcal{X}}(\mathrm{g}) = (1/n!)\sqrt{|\mathrm{g}|}\epsilon_{\mu_1...\mu_n} dx^{\mu_1} \wedge ... \wedge dx^{\mu_n} = \sqrt{|\mathrm{g}|}dx^1 \wedge ... \wedge dx^n = \sqrt{|\mathrm{g}|}\beta = \beta^{\mathrm{g}}$$

$\boldsymbol{\epsilon}_{\mu_1...\mu_n}$ represents the components of an n-form, and $\epsilon_{\mu_1...\mu_n}$ are component of a pseudotensor density. From tensorial density $\epsilon_{\mu_1...\mu_n}$ we build up the volume form (*i.e* tensorial invariant) multiplying by $\sqrt{|\mathrm{g}|}$ the Levi-Civita density $\mathrm{vol}_{\mathcal{X}}(\mathrm{g}) = \beta\sqrt{|\mathrm{g}|}$. The volume form is a canonical way to chose a particular continuous and never vanishing n-form on $\mathcal{X}$. Any non degenerate metric provides a canonical volume form.

[73] since $\mathrm{g} = det(\mathrm{g}_{\mu\nu}) = |\mathrm{g}_{\mu\nu}|$ is a tensor density of weight -2. The dual relation is: $\boldsymbol{\epsilon}^{\mu_1...\mu_n} = (1/\sqrt{|g|})\epsilon^{\mu_1...\mu_n}$

[74] The holonomic frame is not orthogonal in general (see later development and the definition of *tetrad* or *vielbein* in section (12.2)). The notion of orthonormal frame is related to the metric structure: $\mathrm{g} = \mathrm{g}_{\mu\nu}dx^\mu \otimes dx^\nu$. The metric distinguishes which frame are orthogonal, namely a change of frame is in this case a linear transformation which preserves orthogonality. By definition the metric acts on basis vector as: $\mathrm{g}(\partial_\mu, \partial_\nu)\big|_x = \mathrm{g}(\partial_\nu, \partial_\mu)\big|_x = \mathrm{g}_{\mu\nu}(x)$

[75] with $\mathcal{U} \subset \mathcal{X}$ an open subset

encoded with the Lie derivative $\mathcal{L}_{\mathbf{e}_\alpha}(\mathbf{e}_\beta) = [\mathbf{e}_\alpha,\mathbf{e}_\beta] = \mathfrak{c}^\rho_{\alpha\beta}\mathbf{e}_\rho = -\mathcal{L}_{\mathbf{e}_\beta}(\mathbf{e}_\alpha) = [\mathbf{e}_\beta,\mathbf{e}_\alpha] = \mathfrak{c}^\rho_{\beta\alpha}\mathbf{e}_\rho$. For an holonomic frame dx^μ, the structure coefficient are zero so that: $\mathfrak{c}^\rho_{\mu\nu} = [\mathbf{e}_\mu, \mathbf{e}_\nu] = \mathcal{L}_{\mathbf{e}_\mu}(\mathbf{e}_\nu) = 0$. The structure equation that encodes the *anholonomicity* for the coframe is given by the Maurer-Cartan relation $d\theta^\rho = -1/2\mathfrak{c}^\rho_{\alpha\beta}\theta^\alpha \wedge \theta^\beta$. We refer to the appendix (B.9) for more details on the Maurer-Cartan form. Finally, a general moving frame is a smooth section of the general linear frame bundle $F(\mathcal{X}) \to \mathcal{X}$. From fiber bundle standpoint the frame bundle $F(\mathcal{X}) = F_{GL(n,\mathbb{R})}(\mathcal{X})$ is a principal bundle associated to the tangent bundle with gauge group $GL(n, \mathbb{R})$. This viewpoint open the road toward G-structure and the bundle reduction. We describe below, in section (12.3), the connection with the Lorentz sub-bundle $F_{SO(1,3)}(\mathcal{X})$.

11.7 Covariant derivatives and connections

We refer to the appendix (B) for a general presentation of covariant derivative and connection son a vector bundle. For classical GR, the *linear* connection on the manifold $\mathcal{X}$ is described as the specific case where the vector bundle is the tangent one. Let $\{\mathbf{e}_\mu\}$ an arbitrary frame on $T\mathcal{X}$, let $\{\theta^\rho\}$ be the *dual* frame. We introduce $\Gamma^\rho_{\mu\nu}$ the *connection coefficients* in the frame $\mathbf{e}_\rho$ defined by means of the action of the covariant derivative $D_{\mathbf{e}_\mu}\mathbf{e}_\nu$ on a vector basis $\mathbf{e}_\nu$, with respect to $\mathbf{e}_\mu$ (206)(i) and the dual relation (206)(ii).

$$\text{(i)} \quad D_{\mathbf{e}_\mu}\mathbf{e}_\nu = \Gamma^\rho_{\mu\nu}\mathbf{e}_\rho \qquad\qquad \text{(ii)} \quad D_{\mathbf{e}_\mu}\theta^\rho = -\Gamma^\rho_{\mu\nu}\theta^\nu \tag{206}$$

Definition 11.7.1. *A linear connection (covariant derivative) on a manifold $\mathcal{X}$ is the mapping D which gives to every pair of vector field $X, Y \in \mathfrak{X}(\mathcal{X}) = \Gamma(\mathcal{X}, T\mathcal{X})$ another vector field*[76] $D_X Y \in \Gamma(\mathcal{X}, T\mathcal{X})$ *such that* $D : \mathfrak{X}(\mathcal{X}) \times \mathfrak{X}(\mathcal{X}) \to \mathfrak{X}(\mathcal{X}) : (X, Y) \mapsto D_X Y$

The notation $\Gamma^\rho_{\mu\nu}$ is reserved for the Christoffel coefficients: the *connection coefficients* in the holonomic moving frame ∂_μ given by relation (207)(i) - and the related dual basis dx^μ given by relation (207)(ii).

$$\text{(i)} \quad D_{\partial_\mu}\partial_\nu = \Gamma^\rho_{\mu\nu}\partial_\rho \qquad\qquad \text{(ii)} \quad D_{\partial_\mu}dx^\nu = -\Gamma^\nu_{\mu\rho}dx^\rho \tag{207}$$

Lemma 11.1. *Let $\mathbf{e}_\mu$ be a moving frame on $\mathcal{X}$ and let θ^μ be the moving co-frame, with the introduction of the connection coefficient $D_{\mathbf{e}_\mu}\mathbf{e}_\nu = \Gamma^\rho_{\mu\nu}\mathbf{e}_\nu$. We express*[77] *the vector field $D_X Y$ in a moving frame $\mathbf{e}_\mu$ as $D_X Y = (X^\mu dY^\rho(\mathbf{e}_\mu) + \Gamma^\rho_{\mu\nu}X^\mu Y^\nu)\mathbf{e}_\rho$*

11.8 Torsion and Curvature Operators

Definition 11.8.1. *The torsion associated to a connection D on $T\mathcal{X}$ is the operator $T : \mathfrak{X}(\mathcal{X}) \times \mathfrak{X}(\mathcal{X}) \to \mathfrak{X}(\mathcal{X})$ such that for any $X, Y \in \mathfrak{X}(\mathcal{X})$ we associate the vector field $T(X, Y)$ given by $T(X, Y) = D_X Y - D_Y X - [X, Y]$.*

Lemma 11.2. *In an arbitrary moving frame $T(X, Y) = X^\mu X^\nu(\Gamma^\rho_{\mu\nu}\mathbf{e}_\rho - \Gamma^\rho_{\nu\mu}\mathbf{e}_\rho - [\mathbf{e}_\mu, \mathbf{e}_\nu])$*

[76] The covariant derivative D_X, in the direction of the vector field $X \in \mathfrak{X}(\mathcal{X})$ satisfy (bi-linearity in X and Y and the Leibniz rule); $\forall f, g \in C^\infty(\mathcal{X})$, $X, Y, Z \in \Gamma(\mathcal{X}, T\mathcal{X})$ (i) $D_{X+Y}Z = D_X Z + D_Y Z$, (ii) $D_{fX}(Y) = fD_X Y$, (iii) $D_X(Y + Z) = D_X Y + D_X Z$ and (iv) $D_X(fZ) = fD_X Z + X(f)Z$

[77] ⌈ Proof Let $X = X^\mu\mathbf{e}_\mu$ and $Y = Y^\nu\mathbf{e}_\nu$ be vector fields on $\mathcal{X}$.

$$D_X Y = D_{X^\mu\mathbf{e}_\mu}(Y^\nu\mathbf{e}_\nu) = X^\mu D_{\mathbf{e}_\mu}(Y^\nu\mathbf{e}_\nu) = X^\mu\left[dY^\nu(\mathbf{e}_\mu)\mathbf{e}_\nu + Y^\nu D_{\mathbf{e}_\mu}\mathbf{e}_\nu\right] = X^\mu dY^\nu(\mathbf{e}_\mu)\mathbf{e}_\nu + X^\mu Y^\nu D_{\mathbf{e}_\mu}\mathbf{e}_\nu$$

$D_{\mathbf{e}_\mu}\mathbf{e}_\nu$ is decomposed on the basis $\mathbf{e}_\mu$, since $\Gamma^\rho_{\mu\nu} = \theta^\rho\left[D_{\mathbf{e}_\mu}\mathbf{e}_\nu\right]$ we have $D_{\mathbf{e}_\mu}\mathbf{e}_\nu = \theta^\rho\left[D_{\mathbf{e}_\mu}\mathbf{e}_\nu\right]\mathbf{e}_\rho = \Gamma^\rho_{\mu\nu}\mathbf{e}_\nu$ ⌋

⌈Proof We explicitly follow Hélein [109] for this proof. From (11.8.1) and Lemma 11.1 we deduce:

$$\begin{aligned} T(X,Y) &= D_XY - D_YX - [X,Y] \\ &= \big[X^\mu dY^\rho(\mathbf{e}_\mu) + \Gamma^\rho_{\mu\nu}X^\mu Y^\nu\big]\mathbf{e}_\rho - \big[Y^\mu dX^\rho(\mathbf{e}_\mu) + \Gamma^\rho_{\mu\nu}Y^\mu X^\nu\big]\mathbf{e}_\rho - [X,Y] \end{aligned} \tag{208}$$

We have,

$$\begin{aligned} \mathcal{L}_{[X,Y]}\varphi &= \mathcal{L}_X(\mathcal{L}_Y\varphi) - \mathcal{L}_Y(\mathcal{L}_X\varphi) = \mathcal{L}_{X^\mu\mathbf{e}_\mu}(\mathcal{L}_{Y^\nu\mathbf{e}_\nu}\varphi) - \mathcal{L}_{Y^\nu\mathbf{e}_\nu}(\mathcal{L}_{X^\mu\mathbf{e}_\mu}\varphi) \\ &= X^\mu Y^\nu\mathcal{L}_{\mathbf{e}_\mu}(\mathcal{L}_{\mathbf{e}_\nu}\varphi) + X^\mu dY^\nu(\mathbf{e}_\mu)\mathcal{L}_{\mathbf{e}_\nu}\varphi - Y^\nu X^\mu\mathcal{L}_{\mathbf{e}_\nu}(\mathcal{L}_{\mathbf{e}_\mu}\varphi) - Y^\nu dX^\mu(\mathbf{e}_\nu)\mathcal{L}_{\mathbf{e}_\mu}\varphi \\ &= X^\mu Y^\nu\big(\mathcal{L}_{\mathbf{e}_\mu}(\mathcal{L}_{\mathbf{e}_\nu}\varphi) - \mathcal{L}_{\mathbf{e}_\nu}(\mathcal{L}_{\mathbf{e}_\mu}\varphi)\big) + \big(X^\mu dY^\nu(\mathbf{e}_\mu) - Y^\mu dX^\nu(\mathbf{e}_\mu)\big)\mathcal{L}_{\mathbf{e}_\nu}\varphi \\ &= X^\mu Y^\nu\mathcal{L}_{[\mathbf{e}_\mu,\mathbf{e}_\nu]}\varphi + \big(X^\mu dY^\nu(\mathbf{e}_\mu) - Y^\mu dX^\nu(\mathbf{e}_\mu)\big)\mathcal{L}_{\mathbf{e}_\nu}\varphi \end{aligned}$$

So that $[X,Y] = X^\mu Y^\nu[\mathbf{e}_\mu,\mathbf{e}_\nu] + \big(X^\mu dY^\nu(\mathbf{e}_\mu) - Y^\mu dX^\nu(\mathbf{e}_\mu)\big)\mathbf{e}_\nu$. We inject the last equation into (208):

$$\begin{aligned} T(X,Y) &= \big[X^\mu dY^\rho(\mathbf{e}_\mu) + \Gamma^\rho_{\mu\nu}X^\mu Y^\nu\big]\mathbf{e}_\rho - \big[Y^\mu dX^\rho(\mathbf{e}_\mu) + \Gamma^\rho_{\nu\mu}Y^\mu X^\nu\big]\mathbf{e}_\rho \\ &\quad -\big\{X^\mu Y^\nu[\mathbf{e}_\mu,\mathbf{e}_\nu] + \big(X^\mu dY^\nu(\mathbf{e}_\mu) - Y^\mu dX^\nu(\mathbf{e}_\mu)\big)\mathbf{e}_\nu\big\} \end{aligned}$$

Then: $T(X,Y) = X^\mu Y^\nu\big(\Gamma^\rho_{\mu\nu}\mathbf{e}_\rho - \Gamma^\rho_{\nu\mu}\mathbf{e}_\rho - [\mathbf{e}_\mu,\mathbf{e}_\nu]\big)$ ⌋

T is identified with a tensor $T = T^\rho_{\mu\nu}\mathbf{e}_\rho\otimes\theta^\mu\otimes\theta^\nu$ where $T(X,Y) = T^\rho_{\mu\nu}\mathbf{e}_\rho X^\mu Y^\nu$. From Lemma (11.2) we set $T^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} - \theta^\rho([\mathbf{e}_\mu,\mathbf{e}_\nu])$. Due to the definition of the structure coefficients we find the expression $T^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} - \mathfrak{c}^\rho_{\mu\nu}$. Notice that the torsion tensor is seen as a section of the fiber $T\mathcal{X}\otimes T^\star\mathcal{X}\otimes T^\star\mathcal{X}$. Since we notice that torsion is antisymmetric we describe T as a section of the fiber $T\mathcal{X}\otimes\Lambda^2T^\star\mathcal{X}$. Let $\mathfrak{X}(\mathcal{X})\otimes\Omega^2(\mathcal{X})$ be the set of sections $\Gamma(\mathcal{X}, T\mathcal{X}\otimes\Lambda^2T^\star\mathcal{X})$, *i.e* the set of $T\mathcal{X}$-valued 2-form.

Definition 11.8.2. *We define the torsion 2-form as a $T\mathcal{X}$-valued 2-form.* $\Theta^\rho = 1/2T^\rho_{\mu\nu}\theta^\mu\wedge\theta^\nu$

We write the torsion tensor $T(X,Y)$ by means of the torsion 2-form Θ^ρ since

$$T(X,Y) = T^\rho_{\mu\nu}\mathbf{e}_\rho X^\mu X^\nu = \frac{1}{2}\mathbf{e}_\rho T^\rho_{\mu\nu}\theta^\mu\wedge\theta^\nu(X,Y) = \mathbf{e}_\rho\Theta^\rho(X,Y).$$

Notice the key following fact. The torsion has only meaning on the tangent bundle but it become *meaningless* on an arbitrary fiber bundle.

Definition 11.8.3. *For any $X, Y\in\mathfrak{X}(\mathcal{X}) = \Gamma(\mathcal{X}, T\mathcal{X})$, we define the curvature operator of a connection D on $T\mathcal{X}$ as the following mapping* (i) *whereas the $T\mathcal{X}$-valued curvature 2-form is given by* (ii)*:*

$$\text{(i)}\quad R(X,Y) = D_XD_Y - D_YD_X - D_{[X,Y]} \qquad\qquad \text{(ii)}\quad \Omega^\sigma_\rho = \frac{1}{2}R^\sigma{}_{\rho\mu\nu}\theta^\mu\wedge\theta^\nu \tag{209}$$

The curvature tensor is a $T\mathcal{M}\otimes T^\star\mathcal{M}$-valued 2-form. The Riemann curvature tensor lives in the tensorial product $T\mathcal{M}\otimes T^\star\mathcal{M}\otimes T^\star\mathcal{M}\otimes T^\star\mathcal{M}$. Then, the components of the curvature tensors are given by:

$$R^\rho{}_{\sigma\mu\nu} = \mathbf{e}_\mu(\Gamma^\rho_{\nu\sigma}) - \mathbf{e}_\nu(\Gamma^\rho_{\mu\sigma}) + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} - \mathfrak{c}^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\sigma}. \tag{210}$$

When we work with *holonomic* frames, we recover the expression previously given (205). We introduce the fundamental Cartan structure equations in modern mathematical physic.

Proposition 11.1. *For a general Riemannian-Cartan space we have the following structure equation*

$$\text{(i)}\quad \Theta^\mu = d\theta^\mu + \omega^\mu_\nu\wedge\theta^\mu \qquad\qquad \text{(ii)}\quad \Omega^\mu_\nu = d\omega^\mu_\nu + \omega^\mu_\rho\wedge\omega^\rho_\nu \tag{211}$$

⌈ Proof. Once again we explicitly follow Hélein [109] for this proof. First we are interested in proving (211)(i). For that purpose, we need to write the torsion 2-form with the help of the connection form $\omega^{\mu}{}_{\nu}$. Let recall that $\omega^{\mu}{}_{\nu} = \Gamma^{\mu}_{\rho\nu}\theta^{\rho}$ then $\omega^{\mu}{}_{\nu}(X) = \omega^{\mu}{}_{\nu}(X_{\rho}\mathbf{e}_{\rho}) = X_{\rho}\Gamma^{\mu}_{\lambda\nu}\theta^{\lambda}(\mathbf{e}_{\rho}) = X_{\rho}\Gamma^{\mu}_{\lambda\nu}\delta^{\lambda}_{\rho}$ Therefore $\omega^{\mu}{}_{\nu}(X) = \Gamma^{\mu}_{\rho\nu}X^{\rho}$.

$$T(X,Y) = X^{\mu}Y^{\nu}(\Gamma^{\rho}_{\mu\nu}\mathbf{e}_{\rho} - \Gamma^{\rho}_{\nu\mu}\mathbf{e}_{\rho} - [\mathbf{e}_{\mu},\mathbf{e}_{\nu}]) = (\Gamma^{\rho}_{\mu\nu}X^{\mu})Y^{\nu}\mathbf{e}_{\rho} - (\Gamma^{\rho}_{\nu\mu}Y^{\nu})X^{\mu}\mathbf{e}_{\rho} - X^{\mu}Y^{\nu}[\mathbf{e}_{\mu},\mathbf{e}_{\nu}])$$

$$T(X,Y) = \big\{\omega^{\rho}_{\nu}(X)\theta^{\nu}(Y) - \omega^{\rho}_{\mu}(Y)\theta^{\mu}(X) - X^{\mu}X^{\nu}\theta^{\rho}([\mathbf{e}_{\mu},\mathbf{e}_{\nu}])\big\}\mathbf{e}_{\rho}$$

Since $d\theta^{\rho}(\mathbf{e}_{\mu},\mathbf{e}_{\nu}) = \mathcal{L}_{\mathbf{e}_{\mu}}(\theta^{\rho}(\mathbf{e}_{\nu})) - \mathcal{L}_{\mathbf{e}_{\nu}}(\theta^{\rho}(\mathbf{e}_{\mu})) - \theta^{\rho}([\mathbf{e}_{\mu},\mathbf{e}_{\nu}]) = \mathcal{L}_{\mathbf{e}_{\mu}}(\delta^{\rho}_{\nu}) - \mathcal{L}_{\mathbf{e}_{\nu}}(\delta^{\rho}_{\mu}) - \theta^{\rho}([\mathbf{e}_{\mu},\mathbf{e}_{\nu}]) = -\theta^{\rho}([\mathbf{e}_{\mu},\mathbf{e}_{\nu}])$

$$T(X,Y) = \big\{\omega^{\rho}_{\nu}(X)\theta^{\nu}(Y) - \omega^{\rho}_{\mu}(Y)\theta^{\mu}(X) - X^{\mu}X^{\nu}d\theta^{\rho}(\mathbf{e}_{\mu},\mathbf{e}_{\nu})\big\}\mathbf{e}_{\rho} = \big\{d\theta^{\rho}(X,Y) + \omega^{\rho}_{\nu}\wedge\theta^{\nu}(X,Y)\big\}\mathbf{e}_{\rho}$$

We finally write $T = \big(d\theta^{\rho} + \omega^{\rho}_{\mu}\wedge\theta^{\mu}\big)\mathbf{e}_{\rho}$ namely Cartan first structure equation $\Theta^{\rho} = d\theta^{\rho} + \omega^{\rho}_{\mu}\wedge\theta^{\mu}$. Now we focus on the proof that concerns (211)(ii). Once again, on the grounds of $D_{\mathbf{e}_{\mu}}\mathbf{e}_{\nu} = \Gamma^{\rho}_{\mu\nu}\mathbf{e}_{\rho}$, the key point is to use the connection form. Now since $\omega^{\mu}{}_{\nu} = \Gamma^{\mu}_{\rho\nu}\theta^{\rho} = \theta^{\mu}(D_{\mathbf{e}_{\rho}}\mathbf{e}_{\nu})\theta^{\rho}$ then it is equivalent to write $D_{\mathbf{e}_{\mu}}\mathbf{e}_{\sigma} = \omega^{\rho}_{\sigma}(\mathbf{e}_{\mu})\mathbf{e}_{\rho}$.

$$R^{\sigma}{}_{\rho\mu\nu}\mathbf{e}_{\sigma} = D_{\mathbf{e}_{\mu}}D_{\mathbf{e}_{\nu}}\mathbf{e}_{\rho} - D_{\mathbf{e}_{\nu}}D_{\mathbf{e}_{\mu}}\mathbf{e}_{\rho} - D_{[\mathbf{e}_{\mu},\mathbf{e}_{\nu}]}\mathbf{e}_{\rho} = D_{\mathbf{e}_{\mu}}(\omega^{\sigma}_{\rho}(\mathbf{e}_{\nu})\mathbf{e}_{\sigma}) - D_{\mathbf{e}_{\nu}}(\omega^{\sigma}_{\rho}(\mathbf{e}_{\mu})\mathbf{e}_{\sigma}) - \omega^{\sigma}_{\rho}([\mathbf{e}_{\mu},\mathbf{e}_{\nu}])\mathbf{e}_{\sigma}$$

$$R^{\sigma}{}_{\rho\mu\nu}\mathbf{e}_{\sigma} = \mathcal{L}_{\mathbf{e}_{\mu}}(\omega^{\sigma}_{\rho}(\mathbf{e}_{\nu}))\mathbf{e}_{\sigma} + \omega^{\sigma}_{\rho}(\mathbf{e}_{\nu})D_{\mathbf{e}_{\mu}}\mathbf{e}_{\sigma} - \big\{\mathcal{L}_{\mathbf{e}_{\nu}}(\omega^{\sigma}_{\rho}(\mathbf{e}_{\mu}))\mathbf{e}_{\sigma} + \omega^{\sigma}_{\rho}(\mathbf{e}_{\mu})D_{\mathbf{e}_{\nu}}\mathbf{e}_{\sigma}\big\} - \omega^{\sigma}_{\rho}([\mathbf{e}_{\mu},\mathbf{e}_{\nu}])\mathbf{e}_{\sigma}$$

$$R^{\sigma}{}_{\rho\mu\nu}\mathbf{e}_{\sigma} = \big\{\mathcal{L}_{\mathbf{e}_{\mu}}(\omega^{\sigma}_{\rho}(\mathbf{e}_{\nu})) - \mathcal{L}_{\mathbf{e}_{\nu}}(\omega^{\sigma}_{\rho}(\mathbf{e}_{\mu})) + \omega^{\sigma}_{\rho}([\mathbf{e}_{\mu},\mathbf{e}_{\nu}])\big\}\mathbf{e}_{\sigma} + \omega^{\sigma}_{\rho}(\mathbf{e}_{\nu})D_{\mathbf{e}_{\mu}}\mathbf{e}_{\sigma} - \omega^{\sigma}_{\rho}(\mathbf{e}_{\mu})D_{\mathbf{e}_{\nu}}\mathbf{e}_{\sigma}$$

Using Cartan formula we notice that $d\omega^{\sigma}_{\rho}(\mathbf{e}_{\mu},\mathbf{e}_{\nu}) = \mathcal{L}_{\mathbf{e}_{\mu}}(\omega^{\sigma}_{\rho}(\mathbf{e}_{\nu})) - \mathcal{L}_{\mathbf{e}_{\nu}}(\omega^{\sigma}_{\rho}(\mathbf{e}_{\mu})) + \omega^{\sigma}_{\rho}([\mathbf{e}_{\mu},\mathbf{e}_{\nu}])$ so that:

$$R^{\sigma}{}_{\rho\mu\nu}\mathbf{e}_{\sigma} = d\omega^{\sigma}_{\rho}(\mathbf{e}_{\mu},\mathbf{e}_{\nu}) + \big\{\omega^{\sigma}_{\rho}(\mathbf{e}_{\nu})D_{\mathbf{e}_{\mu}}\mathbf{e}_{\sigma} - \omega^{\sigma}_{\rho}(\mathbf{e}_{\mu})D_{\mathbf{e}_{\nu}}\mathbf{e}_{\sigma}\big\} = d\omega^{\sigma}_{\rho}(\mathbf{e}_{\mu},\mathbf{e}_{\nu}) + \big\{\omega^{\sigma}_{\rho}(\mathbf{e}_{\nu})\omega^{\lambda}_{\sigma}(\mathbf{e}_{\mu})\mathbf{e}_{\lambda} - \omega^{\rho}_{\sigma}(\mathbf{e}_{\mu})\omega^{\lambda}_{\sigma}(\mathbf{e}_{\nu})\mathbf{e}_{\lambda}\big\}$$

Finally, one writes $R^{\sigma}{}_{\rho\mu\nu}\mathbf{e}_{\sigma} = d\omega^{\sigma}_{\rho}(\mathbf{e}_{\mu},\mathbf{e}_{\nu})\mathbf{e}_{\sigma} + \omega^{\sigma}_{\rho}\wedge\omega^{\lambda}_{\rho}(\mathbf{e}_{\mu},\mathbf{e}_{\nu})\mathbf{e}_{\lambda} = (d\omega^{\lambda}_{\rho} + \omega^{\sigma}_{\rho}\wedge\omega^{\lambda}_{\rho})(\mathbf{e}_{\mu},\mathbf{e}_{\nu})\mathbf{e}_{\lambda}$ ⌋

11.9 Levi Civita connection

We choose on the tangent bundle $T\mathcal{X}$ and the cotangent bundle $T^{\star}\mathcal{X}$ the holonomic moving frame and coframe $\{\partial_{\mu}\}$ and $\{dx^{\mu}\}$. Lemma (11.1) for holonomic case gives the expression $D_XY = (X^{\mu}\partial_{\mu}Y^{\rho} + \Gamma^{\rho}_{\mu\nu}X^{\mu}Y^{\nu})\partial_{\rho}$. In this case we call the connection coefficients the *Christoffel symbols* and we denote by ∇ the associated covariant derivative to the torsion free connection. The Christoffel symbols are given: $\Gamma^{\rho}_{\mu\nu} = dx^{\rho}(\nabla_{\partial_{\mu}}\partial_{\nu})$ and the *holonomic* frame is called integrable if $\forall\mu,\nu\ [\partial_{\mu},\partial_{\nu}] = 0$. In this case, the coefficients of the torsion tensor are simply given by $T^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu}$. The Levi-civita connection, described by means of the Christoffel symbol, appears to be the cornerstone of classical Einstein's theory - see [70, 71]. The Levi-Civita connection satisfy two conditions. The first is the absence of torsion[78] which is related to symmetric feature of the connection coefficients: $\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu}$. The second is the metric compatibility condition $\nabla_{\rho}\mathsf{g}_{\mu\nu} = 0$ (covariant consistency). We construct a covariant derivative operator $\nabla : \mathfrak{X}(\mathcal{X}) \to \Omega^1(\mathcal{X}, T\mathcal{X})$

$$\nabla_{\mu}\xi^{\nu} = \partial_{\mu}\xi^{\nu} + \Gamma^{\nu}_{\mu\lambda}\xi^{\lambda}. \tag{212}$$

The torsion tensor is given by $T^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{[\mu\nu]}$. Then, the Levi Civita connection is the only metric compatible $\nabla_{\mu}\mathsf{g}_{\nu\lambda} = 0$ connection with the torsion free condition $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{(\mu\nu)}$ wich translate $T^{\lambda}_{\mu\nu} = 0$. These conditions allow us to write the expression of the Christoffel $\Gamma^{\lambda}_{\mu\nu} = 1/2\mathsf{g}^{\lambda\alpha}(\partial_{\nu}\mathsf{g}_{\alpha\mu} + \partial_{\mu}\mathsf{g}_{\alpha\nu} - \partial_{\alpha}\mathsf{g}_{\mu\nu})$. Finally, we decompose any connection $\tilde{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + K^{\lambda}_{\mu\nu}$ where $K^{\lambda}_{\mu\nu}$ is the *contorsion* tensor which writes $K^{\lambda}_{\mu\nu} = (1/2)(T^{\rho}_{\mu\nu} + T^{\rho}_{\nu\mu} + T^{\rho}_{\mu\nu})$. Hence, for any affine connection:

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \frac{1}{2}\mathsf{g}^{\lambda\alpha}(\partial_{\nu}\mathsf{g}_{\alpha\mu} + \partial_{\mu}\mathsf{g}_{\alpha\nu} - \partial_{\alpha}\mathsf{g}_{\mu\nu}) + \frac{1}{2}(T^{\rho}_{\mu\nu} + T^{\rho}_{\nu\mu} + T^{\rho}_{\mu\nu}). \tag{213}$$

[78] A connection is said to be torsionless or without torsion if $\forall X, Y \in \mathfrak{X}(\mathcal{X}), T(X,Y) = 0$.

12 General Relativity vs Gauge theory

12.1 Prolegomena

Now we focus on the geometric foundations and we give some remark about the interpretation of gravity as a gauge theory. The key idea is that we may picture Gravity as a gauge theory but with an additional feature: the presence of specific constraints due to the *solder form*.

Cartan [40] gives primacy of the solder form as opposed to the initial Einstein standpoint which emphasizes the metric field. Einstein initially described the metric field as the only dynamically field and in classical GR, the affine connection is determined by the metric via the Christoffel symbols. The Levi-Civita connection - see the fundamental theorem of Riemannian geometry - is the unique *affine* connection which is symmetric and metric compatible. The roots of this issue is related to the correspondence between Einstein and Cartan [41]. Therefore we exhibit two reflections of this question. First, in the Cartan standpoint, there is an unnecessary relation between *metric* considerations and the notion of *absolute parallelism* which is given in the *affinity* of the connection. In a modern view we describe *gravity* with two independent field (e, ω) giving birth to the so called Palatini formalism - or in a more embracing view to what we term the Einstein-Cartan standpoint. The second aspect concerns the fundamental notion of *torsion* and its interplay in the Einstein-Cartan theory [223]. We introduce torsion consideration with the machinery of the *vierbein* or the *tetrad* formalism which allows us to consider *spin* and *matter* fields. However the deepness of Cartan's vision is wider [40]. Gravitational theory is described naturally by Cartan generalized spaces and *Cartan Geometry*, and relates in an astonish way the philosophy of GR. This is the main focus in the following section. Fist let us remark which concerns the form of the Lagrangian. In typical Yang-Mills theory [177], the Lagrangian is quadratic in curvature: $\mathcal{L}_{\mathsf{YM}}(A) = -\frac{1}{4}\int_{\mathcal{X}} \mathrm{tr}\langle F \wedge \star F\rangle \beta$. On the other side, the Einstein-Hilbert Lagrangian (203) is only linear in curvature. This simple fact, is actually deep. However, the difficulty to cast gravity as a gauge theory is related to understand in one single principle the *general covariance*, and the *gauge invariance*.

The first consideration concerns the Kaluza-Klein program[79] [138, 147]. In a second time, arises the Einstein-Palatini formulation which is concerned with local Lorentz symmetry. The modern construction via the use of Einstein-Palatini action - see section (12.5) - is related to the work of [80] given by Utiyama [227], D. Sciama [212] and T. Kibble [140]. The modern Ashtekar program [10, 197, 221, 222] is also fully involved with such a setting. Finally, the *solder form* and the notion of tetrad field appears important ingredients in the geometrical framework of the supergravity program. The simplest supergravity model ($\mathcal{N} = 1, d = 4$) involves the tetrad field e^I_μ, the spin connection ω^{IJ}_μ - considered as an independent field - and the gravitino ψ_μ and is described by the following action: $\mathcal{L}_{\mathsf{Sugra}}[e, \omega, \psi] = \int_{\mathcal{X}} \left(eR + \epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu\mathcal{D}_\rho\psi_\sigma\right)\beta$. The supergravity program [48] [83, 85, 187] proposes to gather the principle of supersymmetry with the theory of GR. In this case, the local symmetry is given by means of the super Poincaré algebra. This direction of mathematical physics has open the road to *string* and *superstring* theories, see the work of J. Polchinski [192], M.B Green, J.H. Schwarz, E. Witten [103] and A.M Polyakov [193]. The common feature of those theories lay on extra dimension and their compactifications, which in turn is perceived as a generalization of the Kaluza-Klein theory. Here, we work with ten, eleven, or twenty-six dimensions wrapped up on

[79] We cite F. Wilczek [243] the Kaluza-Klein idea "seeks to submerge gauge symmetry into general covariance. Its leading idea is that gauge symmetry arises as a reflection in the four familiar macroscopic space-time dimensions of general covariance in a larger number of dimensions, several of which are postulated to be small, presumably for dynamical reasons. [...] In the Kaluza-Klein construction, [...] the gauge symmetries arise only from isometries of the compactified dimensions.

[80] but we would not loose ourselves in the winding historical road that had led to this *gauge gravity* idea.

themselves, and we find the notion of Calabi-Yau spaces and *orbifolds*.

We notice the work of MacDowell and Mansouri [167], and Chamseddine-West [42] (see D.K. Wise [244] for the underlying setting of Cartan generalized space and Cartan connections), which gives a particular focus on the *solder form*. In a modern viewpoint, the fundamental feature that opposes GR theory to a traditional gauge theory is mathematically given by the concept of the solder form. The following section is dedicated to the *clarification* between the tetrad field and the solder form. We have been inspired from the work of W.A. Rodrigues and E.C. Oliveira [195], R. Aldrovandi and J. G. Pereira, [2] and from the book of Baez and J.P. Muniain [14]. Note also the work of Hélein [112] and the related work of[81] N. Kahouadji [131].

12.2 Tetrad field

Remark. *In the subsequent section, the indices $I, J, \cdots$ denote* tetrad *indices, strictly speaking they are related to orthonormal basis grounds. We emphasized this point since in the section devoted to the solder form* (12.4)*, we use the same notation, however there those indices would be taken as Lie algebra (Minkowski vector space) indices.*

An orthonormal moving frame is termed a n-bein or a *vielbein* and in the $4D$ case a *tetrad* or *vierbein*. An orthonormal basis for $T\mathcal{X}$ is a set $\{\mathbf{e}_I\}$ with $\mathbf{e}_I \in \Gamma(\mathcal{X}, T\mathcal{X})$ and we have the relation (214)(i):

$$\text{(i)} \quad \mathsf{g}(\mathbf{e}_I, \mathbf{e}_J)\big|_x = \mathsf{h}_{IJ} \qquad\qquad \text{(ii)} \quad \mathsf{g}(\theta^I, \theta^J)\big|_x = \mathsf{h}^{IJ} \tag{214}$$

where h_{IJ} is the Minkowski metric. We insist here on the *meaning* from greek language. The word $\tau\epsilon\tau\rho\acute{\alpha}\varsigma$ means a group of four or equivalently the number four. Historically, *frame fields* perceived as *tetrad* have been introduced in GR by Weyl [238]. Hence, the *set* $\{\mathbf{e}_I\}$ is a section of the orthonormal frame bundle $F_{SO(1,3)}(\mathcal{X})$, a principal bundle with structural group $SO(1,3)$. As we proceeded for arbitrary moving frame we introduce the set of dual objects as an orthonormal basis for $T^\star\mathcal{X}$ is a set $\{\theta^I\}$ with $\theta^J \in \Gamma(\mathcal{X}, T^\star\mathcal{X})$ and the dual relations (214)(ii). Duality is exhibited thanks to the relation $\theta^I(\mathbf{e}_J) = \delta^I_J$. We expand $\mathbf{e}_I$ in a holonomic frame as $\mathbf{e}_I = e^\nu_I \partial_\nu$ and the dual frame writes $\theta^I = e^I_\mu dx^\mu$. Notice that $e^I_\mu(x), e^\nu_J(x) : \mathcal{X} \to \mathbb{R}$ are functions such that $e^\mu_I e^J_\mu = \delta^J_I$ and $e^\mu_I e^I_\nu = \delta^\mu_\nu$ We have the formula[82].

$$\mathsf{g}_{\mu\nu} = e^I_\mu e^J_\nu \mathsf{h}_{IJ} \qquad\qquad \mathsf{h}_{IJ} = e^\mu_I e^\nu_J \mathsf{g}_{\mu\nu}$$

The notion of tetrad concerns an additional structure: *the metric*. Hence an orthonormal moving frame is pictured as a moving frame that remains orthonormal at each point of the manifold. General frames transform from one to another via local linear transformations associated to the linear group $GL(n, \mathbb{R})$. In the case of the tetrad field the situation involves only the special orthogonal transformation associated to the special orthogonal group $SO(1,3) \subset GL(n, \mathbb{R})$.[83] Hence, the most simple way to picture the tetrad is to see the *tetrad field* $e^I_\mu(x)$ as the components of an orthonormal moving frame given with respect to an *holonomic* one. Once again, we insist on notations: we use either $\{\mathbf{e}_\mu\} = \{\partial_\mu\}$ either $\{\mathbf{e}_I\}$ (respectively an *holonomic* moving frame and an *orthogonal* moving frame) as a basis for $T\mathcal{X}$. Analogously we have for the dual objects: $\{\theta^\mu\} = \{dx^\mu\}$ and $\{\theta^I\}$ which constitute basis for the cotangent space $T^\star\mathcal{X}$. In such context, the objects e^I_ν are seen as the components of a basis of 1-forms θ^I expressed with respect to the canonical *holonomic* 1-forms

[81] which focusses on local conservation laws and generalized isometric embeddings.

[82] From this standpoint, we can write a vector field $\xi \in \Gamma(\mathcal{X}, T\mathcal{X})$ and a 1-form $\alpha \in \Gamma(\mathcal{X}, T^\star\mathcal{X})$ in holonomic basis $\{\mathbf{e}_\mu\} = \{\partial_\mu\}$ and $\{\theta^\mu\} = \{dx^\mu\}$ respectively as $\xi = \xi^\mu \mathbf{e}_\mu = \xi^\mu \partial_\mu$ and $\alpha = \alpha_\mu \theta^\mu = \alpha_\mu dx^\mu$. We may also expand the vector field and the 1-form (covector field) in the associated orthonormal moving frame as: $\xi = \xi^I \mathbf{e}_I$ or $\alpha = \alpha_I \mathbf{e}^I$.

[83] Then a transformed tetrad via such transformation is again a tetrad. In the case of space-time and gravity theory we will see that we are concerned by the Lorentz group and Lorentz transformations.

basis $\{dx^\nu\}$. We develop $\mathbf{e}^I = e^I_\nu dx^\nu = e^I_\nu(x)dx^\nu$ and $\mathbf{e}_I = e^\nu_I(x)\partial_\nu$ so that we express $D_{\partial_\mu}\partial_\nu$ with two different manners. Note that $D_{\partial_\mu}\partial_\nu = \Gamma^\rho_{\mu\nu}\partial_\rho = \Gamma^\rho_{\mu\nu}e^I_\rho\mathbf{e}_I$, so

$$D_{\partial_\mu}\partial_\nu = D_{\partial_\mu}(e^I_\nu\mathbf{e}_I) = (D_{\partial_\mu}e^I_\nu)\mathbf{e}_I + e^I_\nu D_{\partial_\mu}(\mathbf{e}_I) = \partial_\mu(e^I_\nu)\mathbf{e}_I + e^I_\nu\omega^K_{\mu I}\mathbf{e}_K = (\partial_\mu e^I_\nu + e^K_\nu\omega^I_{\mu K})\mathbf{e}_I$$

Equalizing them we obtain: $(\partial_\mu e^I_\nu + e^K_\nu\omega^I_{\mu K})\mathbf{e}_I = \Gamma^\rho_{\mu\nu}e^I_\rho\mathbf{e}_I$. So,

$$\partial_\mu e^I_\nu + e^K_\nu\omega^I_{\mu K} - \Gamma^\rho_{\mu\nu}e^I_\rho = 0 \tag{215}$$

The object e^μ_I are seen as *components* of the vector $\mathbf{e}_I$ in the basis ∂_μ. Equivalently we can think of the object $e^\mu_I(x)$ as matrix components. We see $e^I_\nu(x)$ as the components of a tensor field. Let us consider $\mathbf{e} \in \Gamma(\mathcal{X}, T^\star\mathcal{X} \otimes T\mathcal{X})$ so that $e^I_\nu(x)$ are seen as the components of a tensor field $\mathbf{e} = e^I_\nu(x)dx^\nu \otimes \mathbf{e}_I$ by means of the *mixed* basis. We calculate the covariant derivative $D_{\partial_\mu}\mathbf{e} = D_{\partial_\mu}(e^I_\nu dx^\nu \otimes \mathbf{e}_I)$, for which we introduce the connection coefficients $\omega^J_{\mu I}$ and $\Gamma^\nu_{\mu\lambda}$. They are defined such that $D_{\partial_\mu}\mathbf{e}_I = \omega^J_{\mu I}\mathbf{e}_J$ and $D_{\partial_\mu}dx^\nu = -\Gamma^\nu_{\mu\lambda}dx^\lambda$. Finally we obtain:

$$D_{\partial_\mu}\mathbf{e} = (\partial_\mu e^I_\nu - e^I_\lambda\Gamma^\lambda_{\mu\nu} + e^J_\nu\omega^I_{\mu J})dx^\nu \otimes \mathbf{e}_I$$

⌜ **Proof** We have the straightforward computation:

$$\begin{aligned} D_{\partial_\mu}\mathbf{e} &= D_{\partial_\mu}(e^I_\nu dx^\nu) \otimes \mathbf{e}_I + e^I_\nu dx^\nu \otimes D_{\partial_\mu}(\mathbf{e}_I) = \big((D_{\partial_\mu}e^I_\nu)dx^\nu + (e^I_\nu D_{\partial_\mu}dx^\nu)\big) \otimes \mathbf{e}_I + e^I_\nu dx^\nu \otimes D_{\partial_\mu}(\mathbf{e}_I) \\ &= (\partial_\mu e^I_\nu dx^\nu - e^I_\nu\Gamma^\nu_{\mu\lambda}dx^\lambda) \otimes \mathbf{e}_I + e^I_\nu dx^\nu \otimes \omega^J_{\mu I}\mathbf{e}_J = (\partial_\mu e^I_\nu - e^I_\lambda\Gamma^\lambda_{\mu\nu})dx^\nu \otimes \mathbf{e}_I + e^J_\nu\omega^I_{\mu J}dx^\nu \otimes \mathbf{e}_I \ \lrcorner \end{aligned}$$

Notice that *when* coefficients e^I_μ are defined by $\mathbf{e}^I = e^I_\nu dx^\nu = e^I_\nu(x)dx^\nu$ *then* $D\mathbf{e} = 0$ due to the relation (215). We have:

$$D_\mu e^I_\nu = \big[D_{\partial_\mu}\mathbf{e}\big]^I_\nu = \partial_\mu e^I_\nu + e^K_\nu\omega^I_{\mu K} - \Gamma^\rho_{\mu\nu}e^I_\rho = 0 \tag{216}$$

The notation $\big[D_\mu\mathbf{e}\big]^I_\mu$ stands for the components of $D_{\partial_\mu}\mathbf{e}$ in the basis $\mathbf{e}_I \otimes dx^\mu$. More explicitly the picture gives $D_{\partial_\mu}\mathbf{e} = \big[D_\mu\mathbf{e}\big]^I_\mu\mathbf{e}_I \otimes dx^\mu$. Notice also that the covariant derivative acting on *tensor fields* is the map: $D_X : \mathbf{e} \in \Gamma(\mathcal{X}, T^\star\mathcal{X} \otimes T\mathcal{X}) \times \mathfrak{X}(\mathcal{X}) \to D_X\mathbf{e} \in \Gamma(\mathcal{X}, T^\star\mathcal{X} \otimes T\mathcal{X})$. The important point is the *mixed* basis feature. Let emphasize that the *identity* (216) is a trivial one if we picture the tetrad field as components of an orthonormal moving frame expressed in an *holonomic* basis. As emphasized in [195], for a general tensor field $\mathbf{P} = \mathbf{P}(x) \in \Gamma(\mathcal{X}, T\mathcal{X} \otimes T^\star\mathcal{X})$ described in such a *mixed* basis there is no needs to be the case. For $\mathbf{P} = \mathbf{P}^I_\nu(x)dx^\nu \otimes \mathbf{e}_I$ the components of the covariant derivative $D_{\partial_\mu}\mathbf{P}$ along the vector field ∂_μ writes $D_{\partial_\mu}\mathbf{P} = D_{\partial_\mu}[\mathbf{P}^I_\nu(x)dx^\nu \otimes \mathbf{e}_I] = [D_{\partial_\mu}\mathbf{P}]^I_\nu\mathbf{e}_I \otimes dx^\nu$. We obtain by straightforward calculation:

$$\big[D_{\partial_\mu}\mathbf{P}\big]^I_\nu = \partial_\mu\mathbf{P}^I_\nu - \mathbf{P}^I_\lambda\Gamma^\lambda_{\mu\nu} + \mathbf{P}^J_\nu\omega^I_{\mu J}$$

⌜**Proof** We have $\mathbf{P} = \xi \otimes \eta$ where $\xi \in \Gamma(\mathcal{X}, T\mathcal{X})$ and $\eta \in \Gamma(\mathcal{X}, T^\star\mathcal{X})$. Let $D_\bullet$ and $D_\circ$ be the covariant derivative acting on vector fields $\mathfrak{X}(\mathcal{X})$ and on 1-forms:

(i) $D_\circ : \mathfrak{X}(\mathcal{X}) \times \mathfrak{X}(\mathcal{X}) \to \mathfrak{X}(\mathcal{X}) : (X, Y) \mapsto D_{\circ X}Y$ (ii) $D_\bullet : \mathfrak{X}(\mathcal{X}) \times \Omega^1(\mathcal{X}) \to \Omega^1(\mathcal{X}) : (X, \eta) \mapsto D_{\bullet X}\eta$

we write $D(\mathbf{P}) = D(\xi \otimes \eta) = (D_\bullet\xi) \otimes \eta + \xi \otimes (D_\circ\eta)$ and $D = D_\bullet \otimes \mathbf{Id}_{T^\star\mathcal{X}} + \mathbf{Id}_{T\mathcal{X}} \otimes D_\circ$ Taking $\xi = \mathbf{e}_I$ and $\eta = \mathbf{P}^I_\nu(x)dx^\nu$. We observe then $(D_\bullet\xi) \otimes \eta = D_\bullet(\mathbf{e}_I) \otimes \eta$ and $\xi \otimes (D_\circ\eta) = \xi \otimes D_\circ(\mathbf{P}^I_\nu(x)dx^\nu)$. Connection coefficients $[D_\bullet]_{\partial_\mu}\mathbf{e}_I = \omega^I_{\mu J}\mathbf{e}_J$ and $[D_\circ]_{\partial_\mu}(dx^\nu) = -\Gamma^\nu_{\mu\rho}dx^\rho$ are defined.

$$\begin{aligned} D_{\partial_\mu}(\mathbf{P}) &= \big([D_\bullet]_{\partial_\mu}\mathbf{e}_I\big) \otimes \mathbf{P}^I_\nu(x)dx^\nu + \mathbf{e}_I \otimes ([D_\circ]_{\partial_\mu}(\mathbf{P}^I_\nu(x)dx^\nu)) \\ &= \big(\omega^I_{\mu J}\mathbf{e}_J\big) \otimes \mathbf{P}^I_\nu(x)dx^\nu + \mathbf{e}_I \otimes \big(\mathbf{P}^I_\nu(x)[D_\circ]_{\partial_\mu}(dx^\nu) + dx^\nu\partial_\mu(\mathbf{P}^I_\nu)(x)\big) \\ &= \omega^I_{\mu J}\mathbf{P}^I_\nu(x)\mathbf{e}_J \otimes dx^\nu - \Gamma^\nu_{\mu\rho}\mathbf{P}^I_\nu(x)\mathbf{e}_I \otimes dx^\rho + \mathbf{e}_I \otimes dx^\nu\partial_\mu\mathbf{P}^I_\nu(x) \\ &= (\partial_\mu\mathbf{P}^I_\nu - \mathbf{P}^I_\lambda\Gamma^\lambda_{\mu\nu} + \mathbf{P}^J_\nu\omega^I_{\mu J})\mathbf{e}_I \otimes dx^\nu \ \lrcorner \end{aligned}$$

In the general case that $\big[D_{\partial_\mu}\mathbf{P}\big]^I_\nu$ do not vanish. Notice that $\mathbf{e} \in \Gamma(\mathcal{X}, T^\star\mathcal{X} \otimes T\mathcal{X})$ is pictured as the *identity* tensor on $T\mathcal{X}$. We have, $\mathbf{e} = e^I_\nu(x)dx^\nu \otimes \mathbf{e}_I = \mathbf{e}^I \otimes \mathbf{e}_I = dx^\mu \otimes \partial_\mu$.

12.3 Frame bundle standpoint and G-structure

The *equivalence principle* as one essential feature of GR is related to the framework of G-structures. The *equivalence principle* is related to the reduction of the frame bundle $F(\mathcal{X})$ to the Lorentz sub-bundle $F_{SO(1,3)}(\mathcal{X})$. G. Sardanashvily calls the geometric equivalence *"the existence of Lorentz invariants on a world manifold"*. In the Relativity Theory, geometry of Lorentz invariants are fundamental. The *existence* of a global section on some bundle is equivalent to the *existence* of a pseudo-Riemannian metric which is to be thought of the gravitational field in the Einstein standpoint. The interpretation of the tetrad gravitational field as a Higgs field is given, via a spontaneously symmetry breaking, in the work of D. Ivanenko and G. Sardanashvily [129, 205]. We focus on the concept of *frame bundle*. The frame bundle $F(\mathcal{P})$ is the principal fiber bundle $(\mathcal{P},\mathcal{M},\pi)$ such that the fiber $\mathcal{P}_x$ over $x\in\mathcal{X}$ is the set of all ordered basis - or frames - for $\mathcal{P}_x$.

$$F(\mathcal{P})=\coprod_{x\in\mathcal{M}}\mathcal{P}_x=\{\{\mathbf{e}_\mu\}\big|_{\mathcal{X}}=(x,\{\mathbf{e}_\mu\}_x)\ /\ x\in\mathcal{M},\{\mathbf{e}_\mu\}_x \text{ is an ordered basis of } \mathcal{P}_x\}$$

We have a similar structure to the vector bundle structure, however the fiber in each point $x\in\mathcal{M}$ is not a vector space but is the set of frames on a vector space. $F(\mathcal{P})$ is as a principal fiber bundle. The general linear group $GL(n,\mathbb{R})$ acts on $F(\mathcal{P})$ via a change of basis and gives to the frame bundle the structure of a principal $GL(n,\mathbb{R})$ bundle. In the fiber bundle standpoint, a vector bundle $(\mathcal{P},\mathcal{X},\pi)$ and its frame bundle $F(\mathcal{P})$ are associated bundles - see more comment in appendix (B). If we consider the tangent bundle $T\mathcal{X}$ we call it the *linear frame bundle*, denoted by $F(\mathcal{X})$. A general moving frame is pictured as a section of the frame bundle $F(\mathcal{X})\to\mathcal{X}$, a principal bundle associated to $T\mathcal{X}$, with structural group $GL(n,\mathbb{R})$. An element $\mathbf{\Lambda}\in GL(n,\mathbb{R})$ defines a change of basis whereas an element in $F(\mathcal{X})$ is given by the data $(x,\{\mathbf{e}_\mu\}_x)$ for $x\in\mathcal{X}$. In GR, the Levi-Civita connection ∇ is the central object and we introduce - see the previous section (11.9) - the connection coefficients in an holonomic moving frame[84] $\nabla_\lambda(\partial_\nu)=\nabla_{\partial_\lambda}(\partial_\nu)=\Gamma^\mu_{\lambda\nu}\partial_\mu$. The connection 1-form is given by $\Gamma^\mu{}_\nu=\Gamma^\mu_{\lambda\nu}dx^\lambda$. We focus on the four dimensional space-time, so that we consider the linear group $GL(4,\mathbb{R})$ and the connection takes values in the $\mathfrak{gl}(4,\mathbb{R})$ Lie algebra. The set of connection forms $\Gamma^\mu{}_\nu$ is seen as a $\mathfrak{gl}(4,\mathbb{R})$ Lie-algebra 1-form. The linear connection, denoted $\Gamma_{GL(4,\mathbb{R})}$ is described as a $GL(4,\mathbb{R})$-principal fiber bundle connection which turn to be a Lie algebra $\mathfrak{gl}(4,\mathbb{R})$-valued 1-form defined on the principal frame bundle $F(\mathcal{X})$. We write $\Gamma_{GL(4,\mathbb{R})}\in\Omega^1(F(\mathcal{X}),\mathfrak{gl}(4,\mathbb{R}))$. Let us consider the canonical projection π and $\pi_\star$:

$$\pi:\left\{\begin{array}{ccc} F(\mathcal{X}) & \to & \mathcal{X}\\ (x,\mathbf{e}_x) & \mapsto & \pi\big((x,\mathbf{e}_x)\big)=x\end{array}\right.\qquad\qquad \pi_\star:\left\{\begin{array}{ccc} T_{(x,\mathbf{e}_x)}F(\mathcal{X}) & \to & T_x\mathcal{X}\\ \xi & \mapsto & \pi_\star\xi\end{array}\right.$$

The pullback of the connection form $\Gamma_{GL(4,\mathbb{R})}$ by a section $\sigma:\mathcal{X}\to F(\mathcal{X})$ is denoted by $\Gamma_{GL(4,\mathbb{R})}$ and is the *local* connection form (gauge potential) $\Gamma_{GL(4,\mathbb{R})}=\sigma^\star\big(\Gamma_{GL(4,\mathbb{R})}\big)\in T^\star\mathcal{X}\otimes\mathfrak{gl}(4,\mathbb{R})$. Let $(\mathfrak{I}^\alpha{}_\beta)_{\alpha,\beta=1,\cdots,4}$ be a basis of $\mathfrak{gl}(4,\mathbb{R})$, with some suitable representation.[85] With respect to this basis:

$$\Gamma_{GL(4,\mathbb{R})}=\Gamma^\beta{}_\alpha\mathfrak{I}^\alpha{}_\beta=(\Gamma^\beta_{\mu\alpha}dx^\mu)\mathfrak{I}^\alpha{}_\beta$$

We describe the spin or Lorentz connection. The *vierlbein* or the *tetrad* is given by a section of the orthonormal frame bundle $F_{SO(1,3)}(\mathcal{X})$. Notice that $F_{SO(1,3)}(\mathcal{X})$ is a principal frame bundle with structure group $SO(1,3)$. Let $(\mathfrak{I}^I{}_J)_{I,J=1,\cdots,n}$ be a basis of $\mathfrak{so}(1,3)\subset\mathfrak{gl}(4,\mathbb{R})$ the Lie algebra of the special orthogonal group $SO(1,3)$.

$$\Gamma_{SO(1,3)}=\Gamma^J{}_I\mathfrak{I}^I{}_J=(\Gamma^J_{\mu I}dx^\mu)\mathfrak{I}^I{}_J$$

[84] Notice that the connection coefficients $\Gamma^\mu_{\lambda\nu}$ are frame dependent, they are not components of a tensor.

[85] See appendix (B) for further details on this point.

Note that the two indices I, J are Lorentz (or tetrad) whereas the tensorial index is *holonomic*. In this case, $\Gamma^{IJ} = \mathsf{h}^{JK}\Gamma^I_K$ and $\Gamma^{IJ} = \Gamma^{[IJ]}$ are antisymmetric. A gauge transformation[86] gives the rules relating $(\omega^\bullet), (\omega^\circ) \in \Omega^1(\mathcal{X}) \otimes \mathfrak{g}$ - the connection forms in two different frames $(\mathbf{e}^\circ)$ and $(\mathbf{e}^\bullet)$ related by $(\mathbf{e}^\bullet)_j = (\mathbf{\Lambda}^{-1})^i_j\ (\mathbf{e}^\circ)_i$ as $(\omega^\circ) = \mathbf{\Lambda}^{-1}(\omega^\bullet)\mathbf{\Lambda} + \mathbf{\Lambda}^{-1}d\mathbf{\Lambda}$. Applied to the introduced connections $\Gamma_{GL(4,\mathbb{R})}$ and $\Gamma_{SO(1,3)}$, we are first interested in the general transformation rule, involving $\Gamma_{GL(4,\mathbb{R})} = \Gamma$ relating two holonomic frames and $\Gamma_{SO(1,3)}$ expressed in orthogonal frames: $\mathbf{e}^\bullet_\mu \mapsto \mathbf{e}^\circ_\nu = \mathbf{\Lambda}^\mu_\nu(x)\mathbf{e}^\bullet_\mu$ and $\mathbf{e}^\bullet_I \mapsto \mathbf{e}^\circ_J = \mathbf{\Lambda}^I_J(x)\mathbf{e}^\bullet_I$, with $\mathbf{\Lambda}^\mu{}_\nu(x) \in GL(4,\mathbb{R})$ and $\mathbf{\Lambda}^I{}_J(x) \in \mathrm{SO}(1,3)$. The connection transformation rules are given:

$$(\Gamma^\circ)^\mu_\nu = (\mathbf{\Lambda}^{-1})^\sigma_\lambda(\Gamma^\bullet)^\lambda_\mu\mathbf{\Lambda}^\mu_\nu + (\mathbf{\Lambda}^{-1})^\sigma_\lambda d\mathbf{\Lambda}^\lambda_\nu \qquad (\Gamma^\circ)^I_J = (\mathbf{\Lambda}^{-1})^L_K(\Gamma^\bullet)^K_I\mathbf{\Lambda}^I_J + (\mathbf{\Lambda}^{-1})^L_K d\mathbf{\Lambda}^K_J$$

The gauge transformation for the coframe field is given with the dual relation $\theta^\mu_\bullet \mapsto \theta^\nu_\circ = (\mathbf{\Lambda}^{-1})^\nu{}_\mu(x)\theta^\nu_\circ$. From coordinate transformations standpoint, we observe that two types of indices are involved: the space-time indices μ are related to general coordinate transformations whereas the Lorentz indices I correspond to the *local* action of the pseudo-orthonormal group $SO(1,3)$, with suitable representation. Following [194] and writing $e = e^I_\mu \in GL(4,\mathbb{R})$ the transformation is equivalently given: $\Gamma^\mu{}_\nu \mapsto (\Gamma^I{}_J)^{[e=e^I_\mu]} = e^I_\mu\Gamma^\mu{}_\nu e^\mu_I - d(e^I_\mu)e^\mu_J$. From now, we denote $\omega^I{}_J = (\Gamma^I{}_J)^{[e=e^I{}_\mu]}$ the connection obtained in the orthonormal basis and call it the *spin Lorentz connection*. Then $e \in GL(4,\mathbb{R})$ is seen as a matrix with non vanishing determinant: $\det(e^I_\mu) \neq 0$.

A Riemannian structure is rooted in bundle reduction process from $GL(4,\mathbb{R})$ to $SO(1,3)$ and appears as ground area for classical GR. Reduction of structural group precisely consists to choose a particular frame class. In this case, the pseudo Riemannian metric identify with the global section of the homogeneous space[87] $GL(4,\mathbb{R})/SO(1,3)$. Reduction process pick up the choice of a given metric and exhibits a special sort of geometry - as structure - on the initial manifold. We know already that locally - with respect to a general moving frame $\mathbf{e}_\mu$ - , the metric is written $\mathsf{g} = \mathsf{g}_{\mu\nu}\mathbf{e}^\mu \otimes \mathbf{e}^\nu$ or $\mathsf{g} = \mathsf{g}_{\mu\nu}dx^\mu \otimes dx^\nu$, working locally on the holonomic natural frame. In the case where the frame is orthonormal, we have $\mathsf{g}_{\mu\nu} = \mathsf{h}_{\mu\nu}$. Finally, if a linear connection is given on $F(\mathcal{X})$, and such that this connection is compatible with the reduction $F(\mathcal{X}) \mapsto F_{SO(1,3)}(\mathcal{X})$, it is then equivalent to picture a connection directly on $F_{\mathsf{SO}(1,3)}(\mathcal{X})$, which is called a *metric* connection. If $\mathbf{e}_I$ is such an orthonormal moving frame, then we have the expression $\mathsf{g} = \mathbf{e}_I \otimes \mathbf{e}_J\mathsf{h}^{IJ}$. From geometrical perspective, all this construction is built on the underlying picture on *associated bundles*, described more precisely by means of $T\mathcal{X} = F(\mathcal{X}) \times_{\rho(\mathrm{GL}(n,\mathbb{R}))} \mathcal{V}$ and $T^\star\mathcal{X} = F(\mathcal{X}) \times_{\rho^\star(\mathrm{GL}(n,\mathbb{R}))} \mathcal{V}$.

12.4 Coframe field as a bundle isomorphism

Remark *We describe the notion of solder form, in the subsequent sections, the indices $I, J, \cdots$ are Lie algebra indices, more precisely Lorentz Lie algebra $\mathfrak{so}(1,3)$.*

[86]⌈ Let $\mathbf{D}$ be a connection on a fiber bundle $\mathcal{V} \to \mathcal{X}$ and let $\sigma = \sigma^i\mathbf{e}_i \in \Gamma(\mathcal{V})$ a section on $\mathcal{V}$. Here the indices denoted $i, j\cdots$ refer simply the basis of the fiber: a set of local sections $(\mathbf{e}_1, ..., \mathbf{e}_n)$. We have, $\forall x \in \mathcal{X}, (\mathbf{e}_1(x), ..., \mathbf{e}_n(x))$ is a basis of $\mathcal{V}_x$. Any section σ writes $\sigma : \mathcal{U} \subset \mathcal{X} : x \mapsto \sigma(x) = \sigma^i(x)\mathbf{e}_i(x)$ and, with $\xi = \xi^\mu\partial_\mu \in \Gamma(\mathcal{X}, T\mathcal{X})$ we have for any section $\sigma \in \Gamma(\mathcal{X}, \mathcal{V})$, $\mathbf{D}_\xi\sigma = (\xi^\mu d\sigma^j(\partial_\mu) + \omega^j_{\mu i}\xi^\mu\sigma^i)\mathbf{e}_j$. Let $(\mathbf{e}_i)^\circ$ and $(\mathbf{e}^\bullet)_i$ be two moving frames. It exists a function $\mathbf{\Lambda} : \mathcal{X} \to GL(n,\mathbb{R})$ which describe the matrix transformation $(\mathbf{e}^\circ)_i = (\mathbf{e}^\bullet)_j\mathbf{\Lambda}^j_i$ between $(\mathbf{e}^\circ)_i$ and $(\mathbf{e}^\bullet)_j$, equivalently $(\mathbf{e}^\bullet)_j = (\mathbf{\Lambda}^{-1})^i_j\ (\mathbf{e}^\circ)_i$. Now σ is decomposed on moving frames $(\mathbf{e}^\circ)_I$ and $(\mathbf{e}^\bullet)_I$ as $\sigma = (\mathbf{e}^\bullet)_i(\sigma^\bullet)^i = (\mathbf{e}^\circ)_i(\sigma^\circ)^i = (\mathbf{e}^\bullet)_j\mathbf{\Lambda}^j_i(\sigma^\circ)^I$. The connection forms $(\omega^\circ)^i_j$ and $(\omega^\bullet)^i_j$ are related $\mathbf{D}$ in the different frames. We have $(\mathbf{e}^\circ)_j(\omega^\circ)^j_i = \mathbf{D}(\mathbf{e}^\circ)_i = \mathbf{D}\big((\mathbf{e}^\bullet)_j\mathbf{\Lambda}^j_i\big) = \mathbf{D}(\mathbf{e}^\bullet)_j\mathbf{\Lambda}^j_i + (\mathbf{e}^\bullet)_j d\mathbf{\Lambda}^j_i$, leading to

$$(\mathbf{e}^\circ)_j(\omega^\circ)^j_i = (\mathbf{e}^\bullet)_k(\omega^\bullet)^k_j\mathbf{\Lambda}^j_i + (\mathbf{e}^\bullet)_k d\mathbf{\Lambda}^k_i = (\mathbf{e}^\circ)_L(\mathbf{\Lambda}^{-1})^l_k(\omega^\bullet)^k_j\mathbf{\Lambda}^j_i + (\mathbf{e}^\circ)_l(\mathbf{\Lambda}^{-1})^l_k d\mathbf{\Lambda}^k_i = (\mathbf{e}^\circ)_l\big((\mathbf{\Lambda}^{-1})^l_k(\omega^\bullet)^k_j\mathbf{\Lambda}^j_i + (\mathbf{\Lambda}^{-1})^l_k d\mathbf{\Lambda}^k_i\big)$$

We conclude: $(\omega^\circ)^j_i = (\mathbf{\Lambda}^{-1})^l_k(\omega^\bullet)^k_j\mathbf{\Lambda}^j_i + (\mathbf{\Lambda}^{-1})^l_k d\mathbf{\Lambda}^k_i$ ⌋

[87]Notice that in the general case we have; $\dim(GL(n)) - \dim(\mathsf{SO})(n) = n(n+1)/2$.

Let us cite Hélein [112]: *"The idea of a solder form on a vector bundle is a refinement of the notion of a moving frame"*. This justifies the previous considerations about the *frame field*. Let us notice, following Baez and J.P. Muniain [14] that the *frame field* is thought to be as a trivialization of $T\mathcal{X}$. Before we discuss the concept of the *solder form*, we recall that the frame bundle is a very singular bundle as opposed to any other traditional fiber bundle. In order to clarify this idea, we present gauge gravity as the following setting: *space-time* is represented by an n-dimensional oriented manifold $\mathcal{X}$.

$$\begin{array}{ccc} T\mathcal{X} & \xrightarrow{\;e\;} & \mathcal{V} = \mathcal{X} \times \mathbb{R}^{1,3} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & \mathcal{X} & \end{array}$$

From the Palatini point of view, $\mathcal{X}$ is not directly given with a metric. We recover the metric, by the pullback along the *coframe* field $e : T\mathcal{X} \to \mathcal{V} = \mathcal{X} \times \mathbb{R}^{1,3}$. The trivialized space $\mathcal{V} = \mathcal{X} \times \mathbb{R}^{1,3}$ is called the *internal space*, as bundle over space-time $\mathcal{X}$. In such a context, we work by means of the bundle isomorphism $e : T\mathcal{X} \to \mathcal{V}$ between the tangent bundle $\pi : T\mathcal{X} \to \mathcal{X}$ and the bundle $\mathcal{V} \to \mathcal{X}$. Since $\mathcal{V}$ is locally trivializable, we treat the coframe field e *locally* as an $\mathbb{R}^{(1,3)}$-valued 1-form. Here, we follow [14, 112, 244]. The idea of the coframe field as a bundle isomorphism expresses the notion of *solder form*. If we have a metric h on $\mathcal{V}$ then, we obtain a metric on $\mathcal{X}$ by $\mathsf{g} = e^{\star}\mathsf{h}$, given by

$$\forall m \in \mathcal{X}, \forall \xi, \sigma \in T_m\mathcal{X} \qquad (e^{\star}\mathsf{h})_m(\xi, \sigma) = \mathsf{g}_m(e_m(\xi), e_m(\sigma))$$

In this case, the vector space $\mathcal{V}$ is given with a connection $\mathbf{D}$ so that we obtain the following connection $\nabla = e^{\star}\mathbf{D}$ on $T\mathcal{X}$ - described by: $\forall \xi, \sigma \in \Gamma(\mathcal{X}) = T\mathcal{X}$, $\nabla_{\xi}\sigma = e^{\star}(\mathbf{D}_{\xi} e(\sigma))$. Once again, we refer to appendix (B) for details on connections on vector bundles as the differential operator $\mathbf{D} : \mathcal{X}(\mathcal{X}) \times \Gamma(\mathcal{X}, \mathcal{V}) \longrightarrow \Gamma(\mathcal{X}, \mathcal{V}) : (X, \sigma) \mapsto \mathbf{D}_X\sigma$. We recall the following lemma:

Lemma 12.1. *Let* $s = s^I\mathbf{e}_I \in \Gamma(\mathcal{V})$ *a section of the bundle* $\mathcal{V} \to \mathcal{X}$ *and* $\xi = \xi^{\mu}\partial_{\mu} \in \Gamma(\mathcal{X}, T\mathcal{X})$ *we have* $\mathbf{D}_{\xi}s = (\xi^{\mu} ds^J(\partial_{\mu}) + \omega^J_{\mu I}\xi^{\mu}s^I)\mathbf{e}_J$

⌈Proof, see appendix (**B**). We make the straightforward computation:

$$\mathbf{D}_{\xi}s = \mathbf{D}_{\xi^{\mu}\partial_{\mu}}(s^I\mathbf{e}_I) = \xi^{\mu}\mathbf{D}_{\partial_{\mu}}(s^I\mathbf{e}_I) = \xi^{\mu}\Big[\mathbf{D}_{\partial_{\mu}}(s^I)\mathbf{e}_I + s^I\mathbf{D}_{\partial_{\mu}}(\mathbf{e}_I)\Big] = \xi^{\mu}\Big[ds^I(\partial_{\mu})\mathbf{e}_I + s^I\mathbf{D}_{\partial_{\mu}}(\mathbf{e}_I)\Big]$$

Then we obtain $\mathbf{D}_{\xi}s = \xi^{\mu}ds^I(\partial_{\mu})e_I + \xi^{\mu}s^I\mathbf{D}_{\partial_{\mu}}(\mathbf{e}_I)$ and let denote $\mathbf{D}_{\partial_{\mu}}(\mathbf{e}_I) = \omega^J_{\mu I}\mathbf{e}_J$, where $\omega^J_{\mu I} = \mathbf{e}^J(\mathbf{D}_{\partial_{\mu}}\mathbf{e}_I)$, so that we find the result $\mathbf{D}_{\xi}s = (\xi^{\mu}ds^J(\partial_{\mu}) + \omega^J_{\mu I}\xi^{\mu}s^I)\mathbf{e}_J$. ⌈

The set of 1-forms ω^J_I defined on a open subset $\mathcal{O} \subset \mathcal{X}$ by $\omega^J_I = \omega^J_{\mu I}dx^{\mu}$ and allows us to write $\forall \xi \in \mathcal{X}(\mathcal{O}), \xi^{\mu}\omega^J_{\mu I} = \omega^J_I(\xi)$. Then:

$$\mathbf{D}_{\xi}s = \mathbf{D}_{\xi}(\sigma^I e_I) = ds^I(\xi)e_I + \omega^J_I(\xi)s^I\mathbf{e}_J.$$

We have $(\mathbf{D}_{\mu}\sigma)^I = \partial_{\mu}\sigma^I + \omega^I_{\mu J}\sigma^J$. Now, via the solder form, we obtain a connection on $T\mathcal{X}$. Pulling back the connection on $\mathcal{V}$ via $\nabla_{\xi}\sigma = e^{\star}(\mathbf{D}_{\xi}e(\sigma))$, we get the covariant derivative components:

$$(\nabla_{\mu}\xi)^{\nu} = \partial_{\mu}\xi^{\nu} + (e^{\nu}_I\partial_{\mu}e^I_{\rho} + e^{\nu}_I\omega^I_{\mu J}e^J_{\rho})\xi^{\rho} \tag{217}$$

But we have also: $(\nabla_{\mu}\xi)^{\nu} = \partial_{\mu}\xi^{\nu} + \Gamma^{\nu}_{\mu\rho}\xi^{\rho}$ So that $\Gamma^{\nu}_{\mu\rho} = e^{\nu}_I\partial_{\mu}e^I_{\rho} + e^{\nu}_I\omega^I_{\mu J}e^J_{\rho}$. Therefore, we recover the well known relation between spin connection coefficient and Christoffel symbol $\Gamma^{\rho}_{\mu\nu}$:

$$\Gamma^{\rho}_{\mu\nu} = e^{\rho}_I\partial_{\mu}e^I_{\nu} + e^{\rho}_I\omega^I_{\mu J}e^J_{\nu} \quad \Longrightarrow \quad \partial_{\mu}e^I_{\nu} + e^K_{\nu}\omega^I_{\mu K} - \Gamma^{\rho}_{\mu\nu}e^I_{\rho} = 0$$

The bundle isomorphism gives a correspondence between objects on the tangent bundle $T\mathcal{X}$ and the internal bundle $\mathcal{V}$. The curvature of $\mathbf{D}$ is given by $F^{IJ}_{\mu\nu} = \omega^{IJ}_{[\mu,\nu]} + \omega^I_{[\mu K}\omega^{KJ}_{\nu]}$, written $F^{IJ} =$

$d\omega^{IJ} + \omega^I_K \wedge \omega^{KJ}$. The bundle isomorphism e map the curvature of $\mathbf{D}$ to that of ∇ with the relation $R^{\rho\sigma}_{\mu\nu} = F^{IJ}_{\mu\nu} e^\rho_I e^\sigma_J$. Therefore, we observe the corresponding relation on the Ricci tensor and the curvature tensor (218):

$$R_\mu{}^\nu = F^{IJ}_{\mu\sigma} e^\sigma_I e^\nu_J \qquad \text{and} \qquad R = R^\mu_\mu = e^\sigma_I e^\rho_J F^{IJ}_{\sigma\rho} \tag{218}$$

12.5 Palatini formulation of gravity

Classical general relativity can be formulated in terms of the *vierbein* (tetrad): $e^I = e^I_\mu dx^\mu$ and the spin connection $\omega^{IJ} = \omega^{IJ}_\mu dx^\mu$. The passage from GR seen as a *metric theory* to the first order Palatini gravity based on the use of a *co-frame* and a *spin connection* is built on two steps. The first is the Palatini first order form of the theory: we consider the connection and the metric as independent variables. Symbolically: $\mathcal{L}_{\mathsf{EH}}[\mathsf{g}] \mapsto \mathcal{L}_{\mathsf{Palatini}}[\mathsf{g},\Gamma]$. Therefore, we write $\mathcal{L}_{\mathsf{Palatini}}[\mathsf{g},\Gamma]$ as:

$$\mathcal{L}_{\mathsf{Palatini}}[\mathsf{g},\Gamma] = \int_{\mathcal{X}} \sqrt{-\mathsf{g}}\mathsf{g}^{\mu\nu} R_{\mu\nu}[\Gamma]\beta. \tag{219}$$

We perform respectively the variations $\delta\Gamma$ and $\delta\mathsf{g}$. The variations with respect to the former leads to set the connection Γ to be the Levi-Civita affine connection while variation with the latter give the Einstein vacuum equations (204). The second concerns the use of *tetrad field*. We picture the leap as $\mathcal{L}_{\mathsf{Palatini}}[\mathsf{g},\Gamma] \mapsto \mathcal{L}_{\mathsf{EH}}[e,\omega]$. The following section demonstrates that the Einstein-Palatini first order theory - given by the action (220)(i) - leads to replace Euler-Lagrange system of equations (204) (Einstein equations) by the system (220)(ii).

$$(\mathrm{i}) \quad \mathcal{L}_{\mathsf{EH}} = \int \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL} \qquad (\mathrm{ii}) \quad \begin{cases} d^{\mathbf{D}} e^I = de^I + \omega^I{}_J \wedge e^J & = \quad 0 \\ \epsilon_{IJKL} e^J \wedge F^{KL} & = \quad 0 \end{cases} \tag{220}$$

The tetrad field is a (Minkowski) vector-valued 1-form $e \in \Omega^1(\mathcal{X},\mathcal{V}) = \Gamma(\mathcal{V}) \otimes \Omega^1(\mathcal{X})$. Note that $\mathbf{e}_I$ is a basis on the vector space $\mathcal{V}$ whereas θ^μ is a moving frame defined on $\Omega^1(\mathcal{X})$. We have: $e = e^I_\mu \theta^\mu \otimes \mathbf{e}_I = \mathbf{e}_I e^I_\mu \theta^\mu$. We refer to the appendix (B) for more details. We use the covariant derivative $\mathbf{D} : \Gamma(\mathcal{V}) \longrightarrow \Omega^1(\mathcal{X},\mathcal{V}) = \Gamma(\mathcal{V}) \otimes \Omega^1(\mathcal{X})$. Let σ be a section of the vector bundle $\mathcal{V} \to \mathcal{V}$ so that $\mathbf{D}\sigma$ is a section 1-form, $\mathbf{D}\sigma \in \Gamma(\mathcal{V}) \otimes \Omega^1(\mathcal{X})$. By means of the exterior covariant derivative $d^{\mathbf{D}}$:

$$d^{\mathbf{D}} : \Gamma(\mathcal{V}) \otimes \Omega^n(\mathcal{X}) = \Omega^n(\mathcal{X},\mathcal{V}) \longrightarrow \Gamma(\mathcal{V}) \otimes \Omega^{n+1}(\mathcal{X}) = \Omega^{n+1}(\mathcal{X},\mathcal{V})$$

see (B.4.2), we have:

$$d^{\mathbf{D}} e = d^{\mathbf{D}}[\mathbf{e}_I e^I_\mu \theta^\mu] = d^{\mathbf{D}}[\mathbf{e}_I e^I_\mu] \wedge \theta^\mu + \mathbf{e}_I e^I_\mu d\theta^\mu = (\mathbf{D}\mathbf{e}_I) e^I_\mu \wedge \theta^\mu + \mathbf{e}_I de^I_\mu \wedge \theta^\mu + \mathbf{e}_I e^I_\mu d\theta^\mu$$

Since $\mathbf{D}\mathbf{e}_I = \omega^J_{\nu I} \mathbf{e}_J e^\nu$ and $de^I_\mu = \partial_\nu e^I_\mu e^\mu$ then,

$$d^{\mathbf{D}} e = \omega^J_{\nu I} \mathbf{e}_J \theta^\nu e^I_\mu \wedge \theta^\mu + \mathbf{e}_I \partial_\nu e^I_\mu \theta^\mu \wedge \theta^\nu + \mathbf{e}_I e^I_\mu d\theta^\mu$$

Finally, for an *holonomic* co-frame $\theta^\mu = dx^\mu$ we have $d\theta^\mu = 0$, while $d\theta^\mu = -1/2\mathfrak{c}^\mu_{\rho\nu}\theta^\rho \wedge \theta^\nu$ for a *non holonomic* co-frame. Hence, in this case:

$$d^{\mathbf{D}} e = \mathbf{e}_J \big[\partial_\nu e^J_\mu + \omega^J_{\nu I} e^I_\mu - 1/2 e^J_\rho \mathfrak{c}^\rho_{\nu\mu}\big] \theta^\nu \wedge \theta^\mu$$

For a holonomic frame we obtain the result (221):

$$d^{\mathbf{D}} e = \mathbf{e}_J \big[\partial_\nu e^J_\mu + \omega^J_{\nu I} e^I_\mu\big] dx^\nu \wedge dx^\mu \in \Gamma(\mathcal{V}) \otimes \Omega^1(\mathcal{X}). \tag{221}$$

Once again, we play with notations. Since e we consider $d^{\mathbf{D}}e \in \Gamma(\mathcal{V}) \otimes \Omega^2(\mathcal{M})$. Equivalently, e is decomposed on the basis $\mathbf{e}_I \otimes \mathbf{e}^\mu$ such that (222):

$$e = e^I_\mu \mathbf{e}_I \otimes \mathbf{e}^\mu = \mathbf{e}_I e^I, \tag{222}$$

without making reference to space-time indices. Now we can write the object $d^{\mathbf{D}}e$ decomposed on a basis of $\Gamma(\mathcal{V}) \otimes \Omega^2(\mathcal{M})$, namely $\mathbf{e}_I \otimes \mathbf{e}^\mu \wedge \mathbf{e}^\nu$, then we write it as:

$$d^{\mathbf{D}}e = \frac{1}{2!}\left[d^{\mathbf{D}}e\right]^I_{\mu\nu} \mathbf{e}_I \otimes \mathbf{e}^\mu \wedge \mathbf{e}^\nu. \tag{223}$$

Following notations widely used we may write this object as:

$$d^{\mathbf{D}}e = \mathbf{e}_I \mathcal{D}e^I \qquad \text{with} \qquad \mathcal{D}e^I = de^I + \omega^I_J \wedge e^J \tag{224}$$

We introduce the exterior covariant derivative via the *gauge covariant derivative*. We insist on the underling geometrical treatment via connection form and connection potential. We make explicit reference to the space-time indices $\omega^I{}_J = \omega^I_{\mu J} dx^\mu$ and $de^I = d(e^I_\mu dx^\mu) = de^I_\mu \wedge dx^\mu$. So that:

$$\begin{aligned} \mathcal{D}e^I &= de^I_\mu \wedge dx^\mu + \omega^I_{\mu J} dx^\mu \wedge e^J \\ &= \partial_\mu e^I_\nu dx^\mu \wedge dx^\nu + \omega^I_{\mu J} dx^\mu \wedge e^J_\nu dx^\nu = \left[\partial_\mu e^I_\nu + \omega^I_{\mu J} e^J_\nu\right] dx^\mu \wedge dx^\nu \end{aligned}$$

Then, taking into account the notation exposed in (224) , we find again the formula (221). Let notice that we have used the *holonomic basis* in order to describe the form part. We could equivalently have used a general moving frame θ^μ.

The spin (Lorentz) connection. We refer to the appendix (B), for more detailed treatment about the *gauge potential* ω and the *connection form* $\boldsymbol{\omega}$ on principal and associated bundles. We are interested into $\mathfrak{g}$-valued one-form on $\mathcal{X}$: $\omega \in \Omega^1(\mathcal{X}, \mathfrak{g})$ described as $\omega = \omega_\mu dx^\mu = \mathfrak{b}_{\mathcal{I}} \omega^{\mathcal{I}}_\mu dx^\mu = \omega^{\mathcal{I}}_\mu dx^\mu \otimes \mathfrak{b}_{\mathcal{I}}$. The gauge potential defines a *connection form* (denoted by $\boldsymbol{\omega}$) on the principal fiber bundle $\mathcal{P}(\mathcal{X}, G)$. In this case, $\boldsymbol{\omega} \in \Omega^1(\mathcal{P}, \mathfrak{g})$. In Palatini or Cartan gravity, described by the pair (e, ω) we concentrate now on the second object, the Lorentz connection denoted ω^{IJ}_μ since we implicitly supposed suitable representation for Lorentz group $SO(1,3)$. The curvature is written[88] (225):

$$F^{KL}_{\mu\nu}[\omega] = \partial_{[\mu}\omega^{IJ}_{\nu]} + [\omega_\mu, \omega_\nu]^{IJ} = \partial_\mu \omega^{IJ}_\nu - \partial_\nu \omega^{IJ}_\mu + \omega^I_{\mu K}\omega^{KJ}_\nu - \omega^I_{\nu K}\omega^{KJ}_\mu \tag{225}$$

Notice that the *covariant exterior derivative* $d^{\mathbf{D}}\omega = \mathfrak{b}_{\mathcal{I}}\mathcal{D}\omega^{\mathcal{I}} = \mathfrak{I}_{IJ}\mathcal{D}\omega^{IJ}$ is given by the means of the object $\mathcal{D}\omega^{IJ}$:

$$\begin{aligned} \mathcal{D}\omega^{IJ} &= d\omega^{IJ} + \omega^I_K \wedge \omega^{KJ} + \omega^J_K \wedge \omega^{IK} = d\omega^{IJ} + \omega^I_K \wedge \omega^{KJ} - \omega^J_K \wedge \omega^{KI} \\ (\mathcal{D}\omega)^{IJ}_{\mu\nu} &= \partial_{[\mu}\omega^{IJ}_{\nu]} + \omega_{[\mu}{}^I_K \omega^{KJ}_{\nu]} - \omega_{[\mu}{}^J_K \omega^{KI}_{\nu]} \end{aligned}$$

So that, $\delta F^{IJ}_{\mu\nu} = 2\mathcal{D}_{[\mu}\delta\omega^{IJ}_{\nu]}$. Notice that we also describe the action (220)($\mathfrak{i}$), by means of internal or external hodge operator, see the section (13.3).

[88]Note that in the subsequent section we sometimes use different convention for antisymetrized symbol [,], however the context is usually sufficient to understand if we consider the numerical factor or not.

Remark on notations. In appendix (B) we refer to mathematical treatment of connection, exterior derivative and all that ... Various notations are introduced that may differ from the ones encounter in the body of the manuscript. Let us make the following clarification. For a principal fiber bundle $\mathcal{P}(\mathcal{M}, G)$ we denote the exterior covariant derivative relative to a connection 1-form $\boldsymbol{\omega} \in \Omega^1(\mathcal{P}, \mathfrak{g})$ as $\mathbf{D}^{\boldsymbol{\omega}} : \Omega^n(\mathcal{P}, \mathcal{V}) \to \Omega^{n+1}(\mathcal{P}, \mathcal{V})$. We introduce the connection and exterior derivative on a vector bundle respectively as the operator

$$\mathbf{D} : \Gamma(\mathcal{V}) \to \Omega^1(\mathcal{M}, \mathcal{V}) \qquad d^{\mathbf{D}} : \Omega^n(\mathcal{M}, \mathcal{V}) \to \Omega^{n+1}(\mathcal{M}, \mathcal{V})$$

We introduce the exterior covariant derivative on basic forms $\boldsymbol{\varphi} \in \Omega^n_\rho(\mathcal{P}, \mathcal{V})$ *given by* $\mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\varphi} = d\boldsymbol{\varphi} + \rho(\boldsymbol{\omega}) \wedge \boldsymbol{\varphi} \in \Omega^{n+1}_\rho(\mathcal{P}, \mathcal{V})$. *Finally we have the interplay, between the n-form* $\varphi \in \Omega^n(\mathcal{M}, \mathcal{P} \times_\rho \mathcal{V})$ *and its related basic form* $\boldsymbol{\varphi} \in \Omega^n_G(\mathcal{P}, \mathcal{V})$. *The object* $\mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\varphi}$ *canonically defines* $d^{\mathbf{D}}\varphi \in \Omega^{n+1}(\mathcal{M}, \mathcal{P}\times_\rho\mathcal{V})$ *so that we summarize:*

$$\mathbf{D}^{\boldsymbol{\omega}} : \Omega^n_G(\mathcal{P}, \mathcal{V}) \to \Omega^{n+1}_G(\mathcal{P}, \mathcal{V}) \qquad d^{\mathbf{D}} : \Omega^n(\mathcal{M}, \mathcal{P}\times_\rho\mathcal{V}) \to \Omega^{n+1}(\mathcal{M}, \mathcal{P}\times_\rho\mathcal{V})$$

For **physical** *use, we refer to the connection 1-form* $\boldsymbol{\omega}$ *or to the gauge field* ω *we* alternatively *use the following notations respectively for the exterior covariant derivative on basic forms and on bundle-valued forms:*

$$d_{\boldsymbol{\omega}} : \Omega^n_G(\mathcal{P}, \mathcal{V}) \to \Omega^{n+1}_G(\mathcal{P}, \mathcal{V}) \qquad d_\omega : \Omega^n(\mathcal{M}, \mathcal{P}\times_\rho\mathcal{V}) \to \Omega^{n+1}(\mathcal{M}, \mathcal{P}\times_\rho\mathcal{V})$$

12.6 Topological terms in gauge gravity

In the Palatini-Cartan framework, the gravitational variables constitute a pair $(e^I_\mu, \omega^{IJ}_\mu)$. The Holst action[89] for Gravity writes:

$$\mathcal{L}_{\mathsf{Holst}}[e, \omega] = \frac{1}{2}\int \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL} - \frac{1}{\gamma}\int e^I \wedge e^J \wedge F_{IJ}.$$

We find different topological terms, the Euler invariant $\mathcal{L}_{\mathsf{Euler}}[\omega]$, the Pontrjagin invariant $\mathcal{L}_{\mathsf{Pontrjagin}}[\omega]$ and the Nieh-Yan invariant [181], $\mathcal{L}_{\mathsf{Nieh\text{-}Yan}}[e, \omega]$:

$$\mathcal{L}_{\mathsf{Euler}}[\omega] = \frac{1}{2}\epsilon_{IJKL} F^{IJ} \wedge F^{KL}, \quad \mathcal{L}_{\mathsf{Pontrjagin}}[\omega] = \frac{1}{2} F^{IJ} \wedge F_{IJ}, \quad \mathcal{L}_{\mathsf{Nieh\text{-}Yan}}[e, \omega] = T^I \wedge T_I - F_{IJ} \wedge e^I \wedge e^J.$$

The 4-forms $\mathcal{L}_{\mathsf{Euler}}[\omega], \mathcal{L}_{\mathsf{Pontrjagin}}[\omega]$, $\mathcal{L}_{\mathsf{Nieh\text{-}Yan}}[e, \omega]$ are topological invariants means that they are exact forms, namely it exists 3-forms $\boldsymbol{\eta}_{\mathsf{Euler}}[\omega], \boldsymbol{\eta}_{\mathsf{Pontrjagin}}[\omega]$ and $\boldsymbol{\eta}_{\mathsf{Nieh\text{-}Yan}}[e, \omega]$ such that $\mathcal{L}_{\mathsf{Euler}}[\omega] = d\boldsymbol{\eta}_{\mathsf{Euler}}[\omega], \mathcal{L}_{\mathsf{Pontrjagin}}[\omega] = d\boldsymbol{\eta}_{\mathsf{Pontrjagin}}[\omega]$ and $\mathcal{L}_{\mathsf{Nieh\text{-}Yan}}[e, \omega] = d\boldsymbol{\eta}_{\mathsf{Nieh\text{-}Yan}}[e, \omega]$. Let us say few words about the Nieh-Yan Invariant $\mathcal{L}_{\mathsf{Nieh\text{-}Yan}}[e, \omega]$. It is the only exact 4-form that is invariant under local Lorentz transformations associated with torsion. We show that $\boldsymbol{\eta}_{\mathsf{Nieh\text{-}Yan}}[e, \omega] = e^I \wedge T_I$. We compute:

$$d\boldsymbol{\eta}_{\mathsf{Nieh\text{-}Yan}}[e, \omega] = d\big[e^I \wedge T_I\big] = de^I \wedge T_I - e^I \wedge dT_I.$$

We consider the second Cartan structure equation $T^I = de^I + \omega^I{}_J \wedge e^J$ so that:

$$d\big[e^I \wedge T_I\big] = \big[T^I + \omega_J{}^I \wedge e^J\big] \wedge T_I - e^I \wedge dT_I = T^I \wedge T_I - e^I \wedge \underbrace{\big[\omega_I{}^J \wedge T_J + dT_I\big]}_{[\mathbf{I}]}$$

We recognize in $[\mathbf{I}]$, the covariant derivative $d_\omega T_I = dT_I + \omega_I{}^J \wedge T_J$ so that $d\big[e^I \wedge T_I\big] = T^I \wedge T_I - e^I \wedge d_\omega T_I$. Now with the help of Bianchi identity $d_\omega T_I = F_{IJ} \wedge e^J$ we obtain:

$$d\big[e^I \wedge T_I\big] = T^I \wedge T_I - e^I \wedge F_{IJ} \wedge e^J = T^I \wedge T_I + F_{IJ} \wedge e^J \wedge e^I = T^I \wedge T_I - F_{IJ} \wedge e^I \wedge e^J$$

All the encountered topological terms for gravity are given by: $\star(e^I \wedge e^J) \wedge F_{IJ}$, $e^I \wedge e^J \wedge F_{IJ}$, $\star F^{IJ} \wedge F_{IJ}$ or $F^{IJ} \wedge F_{IJ}$. In the *multisymplectic formalism*, we understand the features connected

[89] γ is the Immirzi-Barbero Parameter, see [21, 127]

to topological terms directly from the principle that gives force to the total democracy of space, time and matter fields. In such a context, we should recover the features of topological terms[90] directly via the interplay of the generalized Lepage-Dedecker correspondence and not by adding those topological terms by hand in the Lagrangian. We refer to this idea as the topological hypothesis. Along the subsequent section we try to put in perspective this idea. More work is need to fully accomplish it. Before we give the set of possible terms involved for first order Palatini theory, we first give the example of the Holst action (see [122]):

$$\mathcal{L}_{\mathsf{Holst}}[e,\omega] = \frac{1}{2}\int_{\mathcal{X}} \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL} - \frac{1}{\gamma}\int_{\mathcal{X}} e^I \wedge e^J \wedge F_{IJ} \tag{226}$$

The Holst action in the Hamiltonian setting leads to recover the Ashtekar-Barbero formalism [8, 9, 21]. Holst action does not modify the *classical* equations of motion. The Einstein-Hilbert action in the Palatini formalism is also seen as a constrained SO(1,3) BF theory given by the action (227):

$$\mathcal{L}_{\mathsf{BF}}[B,F] = \frac{1}{2}\int_{\mathcal{X}} \mathrm{tr}(B \wedge F) = \int_{\mathcal{X}} B^{IJ} \wedge F_{IJ} \tag{227}$$

where the field F is the curvature 2 form which is derived from a connection 1-form ω. In the case of traditional gravity, we consider B as a $\mathrm{Ad}(\mathcal{P})$-valued 2-form - the setting involved a principal $SO(1,3)$-bundle over $\mathcal{X}$, and $\mathrm{Ad}(\mathcal{P})$ an associated vector bundle via the adjoint action of $SO(1,3)$ on its Lie algebra - whereas tr is a trace related to Killing form for of the chosen group. We also find the action (228):

$$\mathcal{L}_{\mathsf{BF}}[B,F] = \frac{1}{2}\int B_{IJ} \wedge F^{IJ} + \frac{1}{\gamma}(\star B)_{IJ} \wedge F^{IJ} \tag{228}$$

One recover the Hilbert Palatini action if we take $B^{IJ} = \star(e^I \wedge e^J)$ this is one example of the so-called *simplicity constraints*. BF theories are topological field theories, we notice in particular the work of J.C Baez [15, 16, 17] or the thesis of Wise [244]. This embracing view and the issue of the simplicity constraints today crystalize efforts for quantum perspective with *spin foam* models - a prolongation of R. Feynman work [76] - within modern LQG program. The path-integral quantization for such theories is considered as understood. The missing point concerns the interpretation of evolution and the full covariant dynamic. It exists now some work that gives force to the picture of GR seen as a symmetry breaking of a BF theory. This emphasize the underlying role of the topological terms as described by S.W. MacDowell and F. Mansouri [167] or L. Smolin and A. Starodubtsev [220]. The general action for first order Palatini gravity is described by a set of finite terms composed of the Holst term, the topological terms and finally the cosmological constant term (229) - see various works [51, 122, 170, 172, 175, 191] which is compatible withthe diffeomorphism invariance and the Lorentz invariance:

$$\mathcal{L}[e,\omega] = \mathcal{L}_{\mathsf{Holst}}[e,\omega] + \mathcal{L}_{\mathsf{Euler}}[\omega] + \mathcal{L}_{\mathsf{Pontrjagin}}[\omega] + \mathcal{L}_{\mathsf{Nieh\text{-}Yan}}[e,\omega] + \mathcal{L}_{\mathsf{Cosmological}}[e] \tag{229}$$

[90] for quantum perspective, mostly

Namely, if we introduce various coefficients $\{\alpha_i\}_{1\leq i\leq 6}$

$$\begin{aligned}\mathcal{L}[e,\omega] \;=\; & \underbrace{\alpha_1\int_{\mathcal{X}}\epsilon_{IJKL}e^I\wedge e^J\wedge F^{KL}+\alpha_2\int_{\mathcal{X}}e^I\wedge e^J\wedge F_{IJ}}_{\mathcal{L}_{\mathsf{Holst}}[e,\omega]}+\underbrace{\alpha_3\int_{\mathcal{X}}\epsilon_{IJKL}F^{IJ}\wedge F^{KL}}_{\mathcal{L}_{\mathsf{Euler}}[\omega]} \\ & +\underbrace{\alpha_4\int_{\mathcal{X}}F^{IJ}\wedge F_{IJ}}_{\mathcal{L}_{\mathsf{Pontrjagin}}[\omega]}+\underbrace{\alpha_5\int_{\mathcal{X}}T^I\wedge T_I-F_{IJ}\wedge e^I\wedge e^J}_{\mathcal{L}_{\mathsf{Nieh\text{-}Yan}}[e,\omega]} \\ & +\underbrace{\alpha_6\int_{\mathcal{X}}\epsilon_{IJKL}e^I\wedge e^J\wedge e^K\wedge e^L}_{\mathcal{L}_{\mathsf{Cosmological}}[e]}\end{aligned} \tag{230}$$

In the next section (12.7), we briefly discuss Cartan Geometry as natural area for gravitational theory. To conclude this section we first say some words about the issue of matter coupling. Notice that to picture in a whole unified picture *matter* fields and space-time one of the main ingredients to understood is the notion of torsion.[91] As already announced, the vision is twofold, in that meaning that now also *torsion*, and not only *curvature* is a cornerstone of the story. We refer to Cartan [40] and F.W. Hehl and *al.* [106, 107, 108] for torsion study.

Einstein-Cartan standpoint and Dirac action. The Einstein-Cartan theory describes a system of fermion fields coupled to gravity via the action $\mathcal{L}_{\mathsf{EC}}=\mathcal{L}_{\mathsf{Holst}}+\mathcal{L}_{\mathsf{Dirac}}$, where $\mathcal{L}_{\mathsf{Dirac}}$ is the Dirac action (231) [4, 29, 170, 172, 191]

$$\mathcal{L}_{\mathsf{Dirac}}[e,\omega,\psi,\overline{\psi}]=\frac{i}{2}\int\star e_I\wedge\left(\bar{\psi}\gamma^I\mathcal{D}\psi-\overline{\mathcal{D}\psi}\gamma^I\psi\right). \tag{231}$$

The covariant derivative operator $\mathcal{D}$ acts on spinor fields with the following definition:

$$\mathcal{D}\psi=d\psi-\frac{i}{4}\omega^{IJ}\Sigma_{IJ}\psi,\qquad\qquad \overline{\mathcal{D}\psi}=d\overline{\psi}+\frac{i}{4}\omega^{IJ}\Sigma_{IJ}\overline{\psi},$$

with Σ_{IJ} the generator of the Lorentz group. This is the minimal coupling for a system of half integer spin fields coupled to gravity. In this case, the usual physical interpretation is the following: spinor (matter field) generate torsion. The variation with respect to the spin connection ω^{IJ} leads to $d_\omega e^I=\star(e^I\wedge e_J J^J)$ and appears the *spinor axial current* $J^I=\overline{\psi}\gamma^I\gamma^5\psi$. [170, 171, 172]. We refer to the work of A. Perez and Rovelli [191], M. Bojowald and R. Das [29], S. Mercuri [170] or other [4, 172] for a detailed treatment on these questions within the prism of $(3+1)$ Hamiltonian setting. The study of non-minimally-coupled fermion field as been studied in particular by S. Mercuri [170, 171, 172] or L. Freidel and *al,* [82]. In this case the non-minimal coupling term proposed writes:

$$\mathcal{L}_{\mathsf{non\text{-}minimal}}[e,\omega,\psi,\overline{\psi}]=\frac{i}{2}\int_{\mathcal{X}}\star e_I\wedge\Big(\bar{\psi}\gamma^I\big(1-\frac{i}{\alpha}\gamma_5\big)\mathcal{D}\psi-\overline{\mathcal{D}\psi}\big(1-\frac{i}{\alpha}\gamma_5\big)\gamma^I\psi\Big)$$

Beyond the scope of *matter field* and Dirac coupling issue this setting is related to the understanding of the Barbero-Immirzi parameter which is an ambiguity quantization parameter. Sometimes is argued the hope to connected future experimental cosmological data. This aspect is related to considerations on parity violation, and an analogy with the θ-angle in QCD, and topological structure of quantum space. The Barbero-Immirzi parameter may be interpreted as an instanton angle, see the work of S. Mercuri and A. Randono [172] - which appears, in the LQG setting, as a modern continuation of the work of J. Zanelli *et al.* or O. Chandia and J. Zanelli [43].

[91] This is related to the leap from traditional Einstein-Hilbert framework, where gravity is addressed as a *metric* dynamical theory, to the general Riemann-Cartan landscape where gravity is described rather as a dynamical theory of a *co-frame* field and a *connection*

12.7 Cartan geometry as ground area for Gravity

Now we briefly discuss the Cartan generalized spaces. Once again, let us notice the specific nature of the gravitational field. Whereas in gauge theory, the principal object is the gauge field - expressed as an Ehresmann connection - in gravity the presence of the *solder form*, - that gives *torsion* - disables such picture. In Yang-Mills theory gauge transformations are vertical automorphisms of $(\mathcal{P}, \mathcal{X}, G)$, the principal G-bundle over $\mathcal{X}$. The case of gravity leads us not only to consider geometry of an abstract *internal* space - picture via the bundle construction - but also geometry of *space-time* itself. Therefore from GR perspective we want to encapsulate a gauge theory which is specific. This gauge theory interact with the geometry of the space it lives in. Gravity itself **is** geometry.

Cartan geometry makes relation between homogeneous space - as the Minkowski space-time - and a non homogenous space - the *space-time*. We understand Cartan geometry as a generalized method to approximate the space-time manifold $\mathcal{X}$, by a more general homogenous space. The *Equivalence Principle* suggests that locally we describe the non homogeneous space-time by the Minkowski space-time. We refer to the work of R.W. Sharpe [214] for the study of Cartan geometry as a generalization of Riemannian and Klein geometry. On this grounds, Gravity is described as Cartan geometry [64] [245] [246] which appears the natural arena for gravitational theory. It provides, by means of the *Cartan connection* - as opposed to the Ehresmann connection - the natural geometric framework for MacDowell and Mansouri [167] theory - see the work of Wise [244, 245, 246] - or supergravity investigation - see the work of M. Egeileh [64, 65]. Those generalizations are intrinsically connected to the solder form.

Modern approaches to gravity share Einstein-Cartan theory which disregards the central role of the *metric field* for replacement of the following two variables: the *tetrad field* together with the *spin connection*. Moreover, we look at the spin connection and the tetrad trapped into a single connection. This connection takes values in the Poincarré group[92] $ISO(1,3) = \mathbb{R}^{1,3} \rtimes SO(1,3)$. This path lays in the geometric framework of Cartan reductive algebra, Cartan geometry [64, 65, 194] [244, 245, 246] and provides the use of Cartan reductive connection $\mathcal{A} = \omega + e$. In the case of zero cosmological constant, gauge gravity is given with the connection $\mathcal{A}$ taking values in the in the Lie algebra $\mathfrak{iso}(1,3)$ of the Poincaré Group[93] $ISO(1,3)$. In such context, the Cartan connection $\mathcal{A} = e + \omega$ is defined on a principal bundle $\pi : \mathcal{P} \to \mathcal{X}$. The Cartan connection is seen as a $\mathfrak{g}$-valued one form $\mathcal{A} : T\mathcal{P} \to \mathfrak{g}$, decomposed as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$. In the case of four dimensional Palatini Gravity, we identify the Lie algebra $\mathfrak{h}$ with the special orthogonal Lie algebra $\mathfrak{so}(3,1)$, and $\mathfrak{g}/\mathfrak{h}$ with $\mathbb{R}^{3,1}$ whereas $\mathfrak{g}$ is the 4D Poincaré Lie algebra. These two objects are the two geometrical ingredients for the Einstein-Cartan gravity. The first one is the *tetrad* field $e \in \Omega^1(\mathcal{X}, \mathfrak{g}/\mathfrak{h}) = \Omega^1(\mathcal{X}, \mathbb{R}^{3,1})$, a vector-valued 1-form taking value in the so-called *internal* space $e : T\mathcal{X} \to \mathfrak{g}/\mathfrak{h}$. The second object is the *spin connection* $\omega : T\mathcal{X} \to \mathfrak{h}$, a $\mathfrak{h}$-valued one-form: $\omega \in \Omega^1(\mathcal{X}, \mathfrak{h})$.

The idea of Klein is to enlarge the Euclidean geometry $(\mathbb{R}^n, ISO(n))$ to arbitrary symmetry group. Klein geometry describes space-time structure of Einstein special relativity. It is a good geometrical construction for a fixed background context. The Klein geometry constructed on $(\mathbb{R}^{1,3} \rtimes SO(1,3), SO(1,3))$ is given as the Minkowski space-time. While Klein geometry generalizes Euclidean geometry, Cartan geometry appears as a generalization of the Klein geometry by taking into account the curvature. Therefore Cartan geometry is understood as a generalization of Riemannian Geometry. More precisely it concerns the generalization of the tangent space geometry. We now see the connection with arbitrary inhomogeneous spaces. Let us cite Wise to emphasize the

[92] In the present case we focus on isometries of Minkowski space, that is the set of transformations preserving the metric: boosts, rotations and translations.

[93] If we consider Λ, a non-zero cosmological constant, comes the interplay with the de Sitter and anti-de Sitter groups, depending on the sign of Λ.

setting of Cartan geometry in this relation to the equivalence principle. *"The Cartan connection - not a connection in the standard sense - is effectively a curved version of flat homogeneous spaces. By "flat", we mean a space for groups; by "curved", we mean a space that infinitesimally approximates this structure. The analogy is with the flat Euclidean space and the curved Riemannian metrics that approximate it."*

13 Actions for Palatini Gravity and equations of movement

13.1 Derivation of Einstein equation from Palatini gravity

We consider the so-called Einstein-Palatini action for General Relativity:

$$\mathcal{L}_{\mathsf{EP}}[e,\omega] = \int_{\mathcal{X}} \beta(e) e^{\mu}_{I} e^{\nu}_{J} F^{IJ}_{\mu\nu}[\omega] = \int_{\mathcal{X}} vol(e) e^{\mu}_{I} e^{\nu}_{J} F^{IJ}_{\mu\nu}[\omega], \tag{232}$$

since $\beta(e)e^{\mu}_{I}e^{\nu}_{J}F^{IJ}_{\mu\nu} = \beta\sqrt{-\mathrm{g}}R^{\mu\nu}_{\mu\nu} = \beta\sqrt{-\mathrm{g}}R = vol(\mathrm{g})R$. We expand the integrand more precisely, involving the $\epsilon_{\mu\nu\rho\sigma}$ symbol (Levi-Civita tensor density):

Lemma 13.1. *The Einstein-Palatini action $\mathcal{L}_{\mathsf{EP}}[e,\omega]$ is described by:*

$$\mathcal{L}_{\mathsf{EP}}[e,\omega] = \frac{1}{4}\int_{\mathcal{X}} \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^{I}_{\mu} e^{J}_{\nu} F^{KL}_{\rho\sigma}[\omega]\beta \tag{233}$$

⌈ Proof We have the straightforward calculation (we directly write the equality with $\mathcal{L}_{\mathsf{EP}}[e,\omega]$ since at the end, we find the result.):

$$\mathcal{L}_{\mathsf{EP}}[e,\omega] = \int_{\mathcal{X}} \sqrt{-\mathrm{g}}R\beta = \int_{\mathcal{X}} \sqrt{-\mathrm{g}}\delta^{\rho}_{[\alpha}\delta^{\sigma}_{\beta]} R^{\alpha\beta}{}_{\rho\sigma}\beta$$

with the notation: $\delta^{\rho}_{[\alpha}\delta^{\sigma}_{\beta]} = \frac{1}{2}[\delta^{\rho}_{\alpha}\delta^{\sigma}_{\beta} - \delta^{\rho}_{\beta}\delta^{\sigma}_{\alpha}]$, since $\delta^{\rho}_{[\alpha}\delta^{\sigma}_{\beta]} = \frac{1}{4}\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu\nu\rho\sigma}$ then, we obtain:

$$\mathcal{L}_{\mathsf{EP}}[e,\omega] = \frac{1}{4}\int_{\mathcal{X}} \sqrt{-\mathrm{g}}\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu\nu\rho\sigma} R^{\alpha\beta}{}_{\rho\sigma}\beta = \frac{1}{4}\int_{\mathcal{X}} \boldsymbol{\epsilon}_{\mu\nu\alpha\beta}\epsilon^{\mu\nu\rho\sigma} R^{\alpha\beta}{}_{\rho\sigma}\beta \tag{234}$$

with $\boldsymbol{\epsilon}_{\mu\nu\alpha\beta} = \sqrt{-\mathrm{g}}\epsilon_{\mu\nu\alpha\beta}$ the Levi-Civita tensor. Since $\boldsymbol{\epsilon}_{\mu\nu\alpha\beta} = \boldsymbol{\epsilon}_{IJKL}e^{I}_{\mu}e^{J}_{\nu}e^{K}_{\alpha}e^{L}_{\beta} = \epsilon_{IJKL}e^{I}_{\mu}e^{J}_{\nu}e^{K}_{\alpha}e^{L}_{\beta}$ we use this decomposition of the Levi-Civita tensor in (234), so that one obtains:

$$\mathcal{L}_{\mathsf{EP}}[e,\omega] = \frac{1}{4}\int_{\mathcal{X}} \epsilon_{IJKL}e^{I}_{\mu}e^{J}_{\nu}e^{K}_{\alpha}e^{L}_{\beta}\epsilon^{\mu\nu\rho\sigma} R^{\alpha\beta}{}_{\rho\sigma}\beta = \frac{1}{4}\int_{\mathcal{X}} \epsilon_{IJKL}e^{I}_{\mu}e^{J}_{\nu}\epsilon^{\mu\nu\rho\sigma}\underbrace{[e^{K}_{\alpha}e^{L}_{\beta}R^{\alpha\beta}{}_{\rho\sigma}]}_{[\mathbf{I}]}\beta \tag{235}$$

One finally notice that the term $[\mathbf{I}]$ in (235) is given by $[\mathbf{I}] = e^{K}_{\alpha}e^{L}_{\beta}R^{\alpha\beta}{}_{\rho\sigma} = F^{KL}_{\rho\sigma}$. Therefore one finds:

$$\mathcal{L}_{\mathsf{EP}}[e,\omega] = \frac{1}{4}\int_{\mathcal{X}} \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^{I}_{\mu} e^{J}_{\nu} F^{KL}_{\rho\sigma}[\omega]\beta \quad ⌋ \tag{236}$$

Apart from the tetrad field e the other fundamental variable for the Palatini action is the connection on $\mathcal{V} \simeq \mathcal{X} \times \mathbb{R}^{1,3}$ given by the object ω^{IJ}_{μ}. The curvature of the connection $\mathbf{D}$ is given by (237):

$$F^{IJ}_{\mu\nu} = \partial_{\mu}\omega^{IJ}_{\nu} - \partial_{\nu}\omega^{IJ}_{\mu} + \omega^{I}_{\mu K}\omega^{KJ}_{\nu} - \omega^{I}_{\nu K}\omega^{KJ}_{\mu} = 2\omega^{IJ}_{[\mu,\nu]} + 2\omega^{I}_{[\mu K}\omega^{KJ}_{\nu]} \tag{237}$$

written from more geometrical standpoint $F^{IJ} = d\omega^{IJ} + \omega^{I}_{K} \wedge \omega^{KJ}$. Torsion is not assumed to vanish a priori. The spin connection is not determined uniquely by the tetrad rather it is seen as an independent variable. Then we calculate the variation $\delta F^{IJ}_{\mu\nu}$ with respect to ω. We refer to Baez and J.P. Muniain [14] for more detailed treatment - from which we takes the following computations.

Lemma 13.2. *If we denote $\mathcal{D}_{\mu}$ the covariant derivative and $F^{IJ}_{\mu\nu}$ the curvature of the spin connection we have $\delta^{[\omega]}F^{IJ}_{\mu\nu} = 2\mathcal{D}_{[\mu}\delta\omega^{IJ}_{\nu]}$*

⌈ Proof. It is straightforward calculation $\delta F^{IJ}_{\mu\nu} = \delta(\partial_\mu \omega^{IJ}_\nu) - \delta(\partial_\nu \omega^{IJ}_\mu) + \delta(\omega^I_{\mu K}\omega^{KJ}_\nu) - \delta(\omega^I_{\nu K}\omega^{KJ}_\mu)$

$$\begin{aligned}\delta F^{IJ}_{\mu\nu} &= 2\delta\omega^{IJ}_{[\mu,\nu]} + \delta(\omega^I_{\mu K})\omega^{KJ}_\nu + \omega^I_{\mu K}\delta(\omega^{KJ}_\nu) - \Big(\delta(\omega^I_{\nu K})\omega^{KJ}_\mu + \omega^I_{\nu K}\delta(\omega^{KJ}_\mu)\Big)\\ &= 2\delta\omega^{IJ}_{[\mu,\nu]} + 2\omega^I_{[\mu K}\delta\omega^{KJ}_{\nu]} + 2\omega^J_{[\mu K}\delta\omega^{KI}_{\nu]} = 2\mathcal{D}_{[\mu}\delta\omega^{IJ}_{\nu]} \quad \rfloor\end{aligned}$$

Then we evaluate: $\delta^{[\omega]}\mathcal{L}[e,\omega] = (\delta\mathcal{L}[e,\omega]/\delta\omega)\delta\omega$ with the Einstein-Palatini action (238).

$$\mathcal{L}[e,\omega] = \int e e^\mu_I e^\nu_J F^{IJ}_{\mu\nu}\beta = \frac{1}{4}\int \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^I_\mu e^J_\nu F^{KL}_{\rho\sigma}\beta \tag{238}$$

Then, we first vary with respect to the variable ω:

$$\begin{aligned}\mathcal{D}\omega^{IJ} &= d\omega^{IJ} + \omega^I_K \wedge \omega^{KJ} + \omega^J_K \wedge \omega^{IK} = d\omega^{IJ} + \omega^I_K \wedge \omega^{KJ} - \omega^J_K \wedge \omega^{KI}\\ (\mathcal{D}\omega)^{IJ}_{\mu\nu} &= \partial_{[\mu}\omega^{IJ}_{\nu]} + \omega_{[\mu}{}^I_K\omega^{KJ}_{\nu]} - \omega_{[\mu}{}^J_K\omega^{KI}_{\nu]}\end{aligned}$$

So that, $\delta F^{IJ}_{\mu\nu} = 2\mathcal{D}_{[\mu}\delta\omega^{IJ}_{\nu]}$.

$$\delta^{[\omega]}\mathcal{L}[e,\omega] = \frac{1}{4}\int e e^\mu_I e^\nu_J \Big(2\mathcal{D}_{[\mu}\delta\omega^{IJ}_{\nu]}\Big)\beta = \frac{1}{2}\int e e^{[\mu}_I e^{\nu]}_J \mathcal{D}_\mu \delta\omega^{IJ}_\nu\beta = -\frac{1}{2}\int \mathcal{D}_\mu\Big(e e^{[\mu}_I e^{\nu]}_J\Big)\delta\omega^{IJ}_\nu\beta$$

The last equality is obtained by integration by part. Since $e e^{[\mu}_I e^{\nu]}_J = 1/2\epsilon^{\mu\nu\sigma\rho}\epsilon_{IJKL}e^K_\sigma e^L_\rho$ we obtain

$$\delta^{[\omega]}\mathcal{L}[e,\omega] = -\frac{1}{4}\int \epsilon^{\mu\nu\sigma\rho}\epsilon_{IJKL}\mathcal{D}_\mu\big(e^K_\sigma e^L_\rho\big)\delta\omega^{IJ}_\nu\beta \tag{239}$$

On the other hand we compute the variation $\delta^{[e]}\mathcal{L}[e,\omega]$ with respect to the variable e:

$$\begin{aligned}\delta^{[e]}\mathcal{L}[e,\omega] &= \delta^{[e]}\Big(\int e e^\mu_I e^\nu_J F^{IJ}_{\mu\nu}\beta\Big) = \int \Big(e\delta e^\mu_I e^\nu_J F^{IJ}_{\mu\nu} + e^\mu_I \delta e^\nu_J F^{IJ}_{\mu\nu} - R e^J_\mu \delta e^\mu_J\Big)\beta\\ &= 2\int \Big(e^\mu_I R^{IJ}_{\mu\nu} - \frac{1}{2}R e^J_\nu\Big)\delta e^\nu_J\beta = 2\int \Big(R_{\mu\nu} - 1/2R\mathrm{g}_{\nu\mu}\Big)e^{K\nu}\delta e^\mu_K\beta\end{aligned} \tag{240}$$

Notice that we can write directly $\delta^{[e]}(\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu e^J_\nu F^{KL}_{\rho\sigma}) = 2\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^J_\nu F^{KL}_{\rho\sigma}\delta e^I_\mu$. From relations (239) and (240) we obtain the equation of movement for Palatini gravity.

$$\begin{aligned}\epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKL}\mathcal{D}_\nu\Big(e^K_\sigma e^L_\rho\Big) &= 0\\ \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^J_\nu F^{KL}_{\rho\sigma} &= 0\end{aligned} \tag{241}$$

13.2 Equivalence of the Einstein-Hilbert action in the language of forms

Now we give the related formulations for the Einstein-Hilbert action.

$$\mathcal{L}_{\mathsf{EH}}[\mathrm{g}] = \int_{\mathcal{X}} \beta\sqrt{-\mathrm{g}}R = \int_{\mathcal{X}} vol(\mathrm{g})R$$

is equivalent to the action:

$$\mathcal{L}_{\mathsf{Palatini}}[e,\omega] = \frac{1}{2}\int \epsilon_{IJKL}e^I \wedge e^J \wedge F^{KL}$$

⌈ Proof. Let us evaluate $vol(\mathrm{g})R = \beta\sqrt{-\mathrm{g}}R$, the integrand of the Einstein Hilbert action. We have, contracting the Riemannn curvature tensor the following equality $R = R^{\alpha\beta}{}_{\rho\sigma}\delta^\rho_{[\alpha}\delta^\sigma_{\beta]}$. Therefore,

$$\mathcal{L}_{\mathsf{EH}} = \int_{\mathcal{X}} vol(\mathrm{g})R = \int_{\mathcal{X}} vol(\mathrm{g})\delta^\rho_{[\alpha}\delta^\sigma_{\beta]}R^{\alpha\beta}{}_{\rho\sigma} = \frac{1}{4}\int_{\mathcal{X}} vol(\mathrm{g})(-1)^s\boldsymbol{\epsilon}_{\mu\nu\alpha\beta}\boldsymbol{\epsilon}^{\mu\nu\rho\sigma}R^{\alpha\beta}{}_{\rho\sigma}$$

We have used (with $n = 4$ and p = 2) the following relation $\delta^{\rho}_{[\alpha}\delta^{\sigma}_{\beta]}p!(n-p)!(-1)^s = \boldsymbol{\epsilon}_{\mu\nu\alpha\beta}\boldsymbol{\epsilon}^{\mu\nu\rho\sigma}$. Then, expanding the volume form written in a holonomic coframe $\{dx^\mu\}$, we get:

$$vol(\mathsf{g}) = \sqrt{-\mathsf{g}}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = \frac{\sqrt{-\mathsf{g}}}{4!}\epsilon_{\lambda\kappa\tau\gamma}dx^\lambda \wedge dx^\kappa \wedge dx^\tau \wedge dx^\gamma = \frac{1}{4!}\boldsymbol{\epsilon}_{\lambda\kappa\tau\gamma}dx^\lambda \wedge dx^\kappa \wedge dx^\tau \wedge dx^\gamma$$

Since $\boldsymbol{\epsilon}^{\mu\nu\rho\sigma}\boldsymbol{\epsilon}_{\lambda\kappa\tau\gamma} = (-1)^s 4!\delta^{[\mu}_{\lambda}\delta^{\nu}_{\kappa}\delta^{\rho}_{\tau}\delta^{\sigma]}_{\gamma}$:

$$\begin{aligned}
\mathcal{L}_{\mathsf{EH}} &= \frac{1}{4}\int_{\mathcal{X}} \frac{(-1)^s}{4!}\boldsymbol{\epsilon}_{\mu\nu\alpha\beta}\boldsymbol{\epsilon}^{\mu\nu\rho\sigma}\boldsymbol{\epsilon}_{\lambda\kappa\tau\gamma}R^{\alpha\beta}{}_{\rho\sigma}dx^\lambda \wedge dx^\kappa \wedge dx^\tau \wedge dx^\gamma \\
&= \frac{1}{4}\int_{\mathcal{X}} \frac{(-1)^s(-1)^s 4!}{4!}\boldsymbol{\epsilon}_{\mu\nu\alpha\beta}\delta^{[\mu}_{\lambda}\delta^{\nu}_{\kappa}\delta^{\rho}_{\tau}\delta^{\sigma]}_{\gamma}R^{\alpha\beta}{}_{\rho\sigma}dx^\lambda \wedge dx^\kappa \wedge dx^\tau \wedge dx^\gamma \\
&= \frac{1}{4}\int_{\mathcal{X}} \boldsymbol{\epsilon}_{\mu\nu\alpha\beta}R^{\alpha\beta}{}_{\rho\sigma}dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \\
&= \frac{1}{2}\int_{\mathcal{X}} \boldsymbol{\epsilon}_{\mu\nu\alpha\beta}dx^\mu \wedge dx^\nu \wedge R^{\alpha\beta}
\end{aligned}$$

where the last equality is obtained since the curvature 2 form as $R^{\alpha\beta} = \frac{1}{2}R^{\alpha\beta}{}_{\rho\sigma}dx^\rho \wedge dx^\sigma$. Finally, using the relation that relation between the volume element $\epsilon_{\mu\nu\alpha\beta}$ of $\mathsf{g}_{\mu\nu} = e^I_\mu e^J_\nu \mathsf{h}_{IJ}$ and the volume element ϵ_{IJKL} of h_{IJ}, namely: $\boldsymbol{\epsilon}_{\mu\nu\alpha\beta} = e^I_\mu e^J_\nu e^K_\alpha e^L_\beta \epsilon_{IJKL}$, then we write:

$$\mathcal{L}_{\mathsf{EH}} = \frac{1}{2}\int_{\mathcal{X}} e^I_\mu e^J_\nu e^K_\alpha e^L_\beta \epsilon_{IJKL}dx^\mu \wedge dx^\nu \wedge R^{\alpha\beta} = \frac{1}{2}\int_{\mathcal{X}} \epsilon_{IJKL}e^I_\mu dx^\mu \wedge e^J_\nu dx^\nu \wedge e^K_\alpha e^L_\beta R^{\alpha\beta} = \frac{1}{2}\int \epsilon_{IJKL}e^I \wedge e^J \wedge F^{KL}$$

Notice that in an tetrad frame, we have: $\epsilon_{IJKL} = \boldsymbol{\epsilon}_{IJKL}$ so that we also may write the integrand of the action as $1/2\boldsymbol{\epsilon}_{IJKL}e^I \wedge e^J \wedge F^{KL}$ $\rfloor$.

13.3 Hodge duality on a vector space

The use of Hodge duality is everywhere in the modern mathematical physics literature - canonical variables, topological terms ... We now explore the relation between Hodge star operator and the Einstein-Cartan equations. First we recall some basic properties about Hodge duality.

Definition 13.1. *Let $\mathcal{V}$ be a n-dimensional vector space with inner product h of signature $(n-m, m)$. We now consider the Hodge dual in $\Lambda\mathcal{V}$ denoted as $\overset{\mathcal{V}}{\star} = \star$. Now we consider $\{\theta_i\}$ an ordered basis for the vector space $\mathcal{V}$ then a p-form $\lambda \in \Lambda^p\mathcal{V}$ is written as $\lambda = (1/p!)\lambda_{i_1...i_p}\theta^{i_1} \wedge ... \wedge \theta^{i_p}$ then the Hodge dual $\star\lambda \in \Lambda^{n-p}\mathcal{V}$ is given by (242):*

$$\star\lambda = \frac{1}{(n-p)!}\big(\star\lambda_{j_1...j_{n-p}}\big)\theta^{j_1} \wedge ... \wedge \theta^{j_{(n-p)}} \tag{242}$$

We denote the components of $\star\lambda$ in the basis $\theta^{j_1} \wedge ... \wedge \theta^{j_{(n-p)}}$ by $\big(\star\lambda_{j_1...j_{n-p}}\big) = \big(\star\lambda\big)_{j_1...j_{n-p}}$. Then, we obtain (243) the expression:

$$\star\lambda_{j_1...j_{n-p}} = \frac{1}{(p)!}\epsilon^{i_1...i_p}{}_{j_1...j_{n-p}}\lambda_{i_1...i_p} \tag{243}$$

From definition (13.1), we consider the case where the considered vector space is **space-time**: $\mathcal{V} = \mathcal{X}$. Let $\{\theta^\mu\} = \{dx^\mu\}$ be a holonomic basis, and let $\forall\lambda \in \Lambda^p\mathcal{X}$ be a p-form on $\mathcal{X}$ we write it as: $\lambda = (1/p!)\lambda_{i_1...i_p}dx^{i_1} \wedge ... \wedge dx^{i_p}$. We consider the Hodge dual in $\Lambda\mathcal{X}$, written $\star\lambda \in \Lambda^{n-p}\mathcal{X}$ and given by:

$$\star\lambda = \frac{1}{(n-p)!}\big(\star\lambda_{\mu_1...\mu_{n-p}}\big)dx^{\mu_1} \wedge ... \wedge dx^{\mu_{n-p}} \qquad \text{with} \qquad \big(\star\lambda_{\mu_1...\mu_{n-p}}\big) = \frac{1}{p!}\boldsymbol{\epsilon}_{\mu_1...\mu_n}\lambda^{\mu_1..\mu_p}$$

We notice that $\lambda^{\mu_1\cdots\mu_p} = g^{\mu_1\nu_1}...g^{\mu_p\nu_p}\lambda_{\nu_1...\nu_p}$ is the contravariant object and we raise and down indices when one contracts with the *metric tensor*.

⌈ **Remark** One may write $\star R = \star(\theta^\mu\wedge\theta^\nu)\wedge R_{\mu\nu}$. Let demonstrate that $\star R = \star(\theta^\mu\wedge\theta^\nu)\wedge R_{\mu\nu}$. The purpose here is to derive Einstein equation thanks to the use of *external Hodge operator*. Let us denote $\Sigma^{\mu\nu} = \star(\theta^\mu\wedge\theta^\nu)$. First, we demonstrate that $\star R = Rd\mathfrak{y} = \Sigma_{\mu\nu}\wedge R^{\mu\nu}$ since:

$$\Sigma^{\mu\nu} = \star(\theta^\mu\wedge\theta^\nu) = \frac{\sqrt{-g}}{2!}\epsilon_{\alpha\beta\rho\sigma}g^{\alpha\mu}g^{\beta\nu}\theta^\rho\wedge\theta^\sigma = \frac{1}{2}\boldsymbol{\epsilon}_{\alpha\beta\rho\sigma}g^{\alpha\mu}g^{\beta\nu}\theta^\rho\wedge\theta^\sigma$$

Therefore we express the covariant quantity $\Sigma_{\alpha\beta}$, contracting two time with the metric tensor: $\Sigma_{\alpha\beta} = g_{\alpha\mu}g_{\beta\nu}\Sigma^{\mu\nu} = 1/2\boldsymbol{\epsilon}_{\alpha\beta\rho\sigma}\theta^\rho\wedge\theta^\sigma$. We obtain:

$$\begin{aligned}\Sigma_{\alpha\beta}\wedge R^{\alpha\beta} &= \big[\frac{1}{2}\boldsymbol{\epsilon}_{\alpha\beta\rho\sigma}\theta^\rho\wedge\theta^\sigma\big]\wedge\big[\frac{1}{2}R^{\alpha\beta}{}_{\mu\nu}\theta^\mu\wedge\theta^\nu\big] = \frac{1}{2}R^{\alpha\beta}{}_{\mu\nu}\big[\frac{1}{2}\boldsymbol{\epsilon}_{\alpha\beta\rho\sigma}\theta^\rho\wedge\theta^\sigma\wedge\theta^\mu\wedge\theta^\nu\big]\\ &= \frac{1}{2}R^{\alpha\beta}{}_{\mu\nu}\big[\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha\big]\beta = R\beta = \star R \quad ⌋\end{aligned}$$

In this section we introduce two operators: the *internal* $\mathring{\star}$ and the *external* $\star$ Hodge operator in the context of vector (Lie algebra)-valued p-form. Let $\lambda\in\Omega^p(\mathcal{X},\mathcal{V}) = \Omega^p(\mathcal{X})\otimes\mathcal{V}$ a $\mathcal{V}$-valued p-form. Naturally we construct the space $\Omega^n(\mathcal{X})\otimes\Lambda^n(\mathcal{V})$ and we consider the case where $\dim(\mathcal{X}) = \dim(\mathcal{V}) = n$. Then, let $0\le p,q\le n$ we describe a $\Lambda^q(\mathcal{V})$-valued p-form on which we apply an Hodge operator either on *internal* indices or on *space-time* indices.

$$\mathring{\star}:\left\{\begin{array}{ccc}\Omega^p(\mathcal{M})\otimes\Lambda^q\mathcal{V} & \to & \Omega^p(\mathcal{M})\otimes\Lambda^{n-q}\mathcal{V}\\ \lambda & \mapsto & \mathring{\star}\lambda\end{array}\right. \qquad \star:\left\{\begin{array}{ccc}\Omega^p(\mathcal{M})\otimes\Lambda^q\mathcal{V} & \to & \Omega^{n-p}(\mathcal{M})\otimes\Lambda^q\mathcal{V}\\ \lambda & \mapsto & \star\lambda\end{array}\right.$$

Definition 13.2. *Hodge duality for vector-valued p-form $\lambda\in\Omega^p(\mathcal{X},\mathcal{V})$. We consider $\{\theta_\mu\}$ a basis of $\mathcal{V}$ and dx^μ the holonomic basis of $T^\star\mathcal{X}$. Therefore, the basis of the space $\Lambda^p(\mathcal{X})\otimes\Lambda^q(\mathcal{V})$ is $dx^1\wedge\cdots\wedge dx^n\otimes\theta_{I_1}\wedge\cdots\wedge\theta_{I_n}$.Then :*

$$\lambda = \frac{1}{p!q!}\Big[\lambda\Big]^{I_1\cdots I_q}_{\mu_1\cdots\mu_p}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\otimes\theta_{I_1}\wedge\cdots\wedge\theta_{I_q} \quad \textit{and} \quad \star\lambda = \frac{1}{(n-p)!}\Big[\star\lambda\Big]^{I_1\cdots I_q}_{\nu_1\cdots\nu_{n-p}}dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{n-p}}\otimes\theta_{I_1}\wedge\cdots\wedge\theta_{I_q}$$

with $\Big[\star\lambda\Big]^{I_1\cdots I_q}_{\nu_1\cdots\nu_{n-p}} = \star\lambda_{\nu_1\cdots\nu_{n-p}} = \frac{1}{p!}\epsilon^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_{n-p}}\lambda_{\mu_1\cdots\mu_p}$

whereas we have analogous for the internal operator, in components:

$$\Big[\mathring{\star}\lambda\Big]^{J_1\cdots J_{n-q}}_{\mu_1\cdots\mu_p} = \mathring{\star}\lambda^{J_1\cdots J_{n-q}} = \frac{1}{q!}\epsilon^{I_1\cdots I_q}{}_{J_1\cdots J_{n-q}}\lambda_{I_1\cdots I_q}$$

We consider now the Minkowski case where $\mathcal{V} = \mathbb{R}^{1,3}$. Let $\mathbf{e}_I$ be a basis of the Minkowski vector space $\mathbb{R}^{1,3}$ and we denote $\sigma = e\wedge e\in\Lambda^2\mathcal{X}\otimes\Lambda^2\mathcal{V}$ a $\Lambda^2\mathcal{V}$-valued 2-form. The related basis $\mathbf{e}_I\wedge\mathbf{e}_J$ of $\Lambda^2\mathcal{V}$ leads us to decompose:

$$\sigma = \frac{1}{2!2!}\sigma^{IJ}{}_{\mu\nu}\big[\mathbf{e}_I\wedge\mathbf{e}_J\big]\otimes\big[dx^\mu\wedge dx^\nu\big] = \frac{1}{4}\big[\sigma\big]^{IJ}_{\mu\nu}\mathbf{e}_I\wedge\mathbf{e}_J\otimes dx^\mu\wedge dx^\nu.$$

If we construct $e = \mathbf{e}_I e^I_\mu dx^\mu$, then $\sigma = (\mathbf{e}_I e^I_\mu dx^\mu)\wedge(\mathbf{e}_J e^J_\nu dx^\nu)$ where the wedge product acts both on vector space *and* space-time part. We can choose to *not* write the Minkowski-like basis (namely $\mathbf{e}_I\wedge\mathbf{e}_J$) on $\Lambda^2\mathcal{V}$ indices so we write:

$$\sigma = \frac{1}{4}\sigma^{IJ}{}_{\mu\nu}dx^\mu\wedge dx^\nu = e^I_\mu e^J_\nu dx^\mu\wedge dx^\nu \qquad \text{with} \qquad \sigma^{IJ}{}_{\mu\nu} = 4e^I_\mu e^J_\nu$$

Lemma 13.3. *With a Hodge operator either on internal indices $\mathring{\star}$ or on space-time indices $\star$ defined above, the tetrad field, seen as a Minkowski vector valued 1-form and $\sigma = e\wedge e$ we have $\mathring{\star}\sigma = \mathring{\star}(e\wedge e) = \star(e\wedge e) = \star\sigma$*

⌈ Proof Notice that we have always the possibility to apply the *external* Hodge operator on σ: we directly apply the definition (13.2)

$$\star\,\sigma = \frac{1}{2}\big[\star\,\sigma\big]_{\rho\sigma}{}^{IJ} dx^\rho\wedge dx^\sigma\otimes \mathbf{e}_I\wedge\mathbf{e}_J \qquad \text{with} \qquad \big[\star\,\sigma\big]^{IJ}_{\rho\sigma} = \frac{1}{2}\boldsymbol{\epsilon}^{\mu\nu}{}_{\rho\sigma}e^I_\mu e^J_\nu \tag{244}$$

Then, the term $\big[\star\,\sigma\big]_{\rho\sigma}{}^{IJ}$ in (244) shall be expand as the following:

$$\big[\star\sigma\big]^{IJ}_{\mu\nu} = \frac{1}{2}\boldsymbol{\epsilon}^{\rho\sigma}{}_{\mu\nu}e^I_\rho e^J_\sigma = \frac{1}{2}\boldsymbol{\epsilon}^{\rho\sigma\alpha\beta}\mathsf{g}_{\alpha\mu}\mathsf{g}_{\beta\nu}e^I_\rho e^J_\sigma = \frac{1}{2}\boldsymbol{\epsilon}^{KLMN}e^\rho_K e^\sigma_L e^\alpha_M e^\beta_N \mathsf{g}_{\alpha\mu}\mathsf{g}_{\beta\nu}e^I_\rho e^J_\sigma \tag{245}$$

Then we also expand in (245) the terms $\mathsf{g}_{\alpha\mu} = \mathsf{h}_{OP}e^O_\alpha e^P_\mu$ and $\mathsf{g}_{\beta\nu} = \mathsf{h}_{QR}e^Q_\beta e^R_\nu$ in order to obtain:

$$\begin{array}{rcl} 2\big[\star\sigma\big]^{IJ}_{\mu\nu} &=& \boldsymbol{\epsilon}^{KLMN}e^\rho_K e^\sigma_L e^\alpha_M e^\beta_N\big[\mathsf{h}_{OP}e^O_\alpha e^P_\mu\big]\big[\mathsf{h}_{QR}e^Q_\beta e^R_\nu\big]e^I_\rho e^J_\sigma = \boldsymbol{\epsilon}^{KLMN}e^\rho_K e^\sigma_L e^P_\mu\big[\delta^O_M\mathsf{h}_{OP}\big]\big[\delta^Q_N\mathsf{h}_{QR}\big]e^R_\nu e^I_\rho e^J_\sigma \\ &=& \boldsymbol{\epsilon}^{KLMN}e^\rho_K e^\sigma_L e^P_\mu\big[\delta^O_M\mathsf{h}_{OP}\big]\big[\delta^Q_N\mathsf{h}_{QR}\big]e^R_\nu e^I_\rho e^J_\sigma = \boldsymbol{\epsilon}^{KLOQ}\mathsf{h}_{OP}\mathsf{h}_{QR}\big[e^\rho_K e^\sigma_L e^I_\rho e^J_\sigma\big]e^P_\mu e^R_\nu \\ &=& \boldsymbol{\epsilon}^{KL}{}_{PR}\delta^I_K\delta^J_L e^P_\mu e^R_\nu = \boldsymbol{\epsilon}^{IJ}{}_{PR}e^P_\mu e^R_\nu \end{array} \tag{246}$$

Now, we apply the *internal* Hodge operator $\overset{\circ}{\star}$ to $e\wedge e$, and once again we directly apply the definition 13.2 hence

$$\text{(i)}\quad \overset{\circ}{\star}\,\sigma = \frac{1}{2}\big[\overset{\circ}{\star}\,\sigma\big]^{KL}_{\mu\nu}dx^\mu\wedge dx^\nu\otimes\mathbf{e}_K\wedge\mathbf{e}_L \qquad\qquad \text{(ii)}\quad \big[\overset{\circ}{\star}\,\sigma\big]^{IJ}_{\mu\nu} = \frac{1}{2}\boldsymbol{\epsilon}^{IJ}{}_{KL}e^K_\mu e^L_\nu \tag{247}$$

Therefore we obtain, from (246) and (247)(ii): $\big[\overset{\circ}{\star}\,\sigma\big]^{IJ}_{\mu\nu} = \big[\star\sigma\big]^{IJ}_{\mu\nu}$ ⌋

Now, we follow the spirit of previous remark, but this time we rather play with Minkowski indices. Then we define: $\Sigma^{IJ} = \star(\mathbf{e}^I\wedge\mathbf{e}^J) = \big[\star\sigma\big]^{IJ}$ and ${}^\circ\Sigma^{KL} = \overset{\circ}{\star}(\mathbf{e}^K\wedge\mathbf{e}^L) = \big[\overset{\circ}{\star}\,\sigma\big]^{KL}$, in more simple notation:

$$\underset{\overset{\circ}{\star}}{}\Sigma^{KL} = \frac{1}{2}\boldsymbol{\epsilon}^{\rho\sigma}{}_{\mu\nu}e^K_\rho e^L_\sigma dx^\mu\wedge dx^\nu = {}_\star\Sigma^{KL}$$

Keeping trace of Minkowski indices I, J is equivalent to describe $\Lambda^2\mathcal{V}$-valued 2-form as $\sigma = \sigma^{IJ}\otimes\mathbf{e}_I\wedge\mathbf{e}_J$ with $\sigma^{IJ} = \sigma^{IJ}{}_{\mu\nu}dx^\mu\wedge dx^\nu$. Therefore we observe the following notation:

$$\overset{\circ}{\star}\lambda^{IJ} = \frac{1}{2}\epsilon^{IJ}{}_{KL}\lambda^{KL}{}_{\mu\nu}dx^\mu\wedge dx^\nu \qquad\qquad \text{and} \qquad\qquad \star\lambda^{IJ} = \frac{1}{2}e\epsilon^{\mu\nu}{}_{\rho\sigma}\lambda^{IJ}{}_{\mu\nu}dx^\rho\wedge dx^\sigma$$

It is then straightforward to find back, with $\Sigma_{IJ} = \star(e_I\wedge e_J)$:

$$\mathcal{L}_{\mathsf{Palatini}}[e,\omega] = \int_{\mathcal{X}}\Sigma_{\alpha\beta}\wedge R^{\alpha\beta} = \int_{\mathcal{X}}(e^\alpha_I e^\beta_J\Sigma_{\alpha\beta})\wedge(e^I_\alpha e^J_\beta R^{\alpha\beta}) = \int_{\mathcal{X}}\Sigma_{IJ}\wedge F^{IJ} = \int_{\mathcal{X}}\star(e_I\wedge e_J)\wedge F^{IJ}$$

Let notice that we may alternatively work with the following action:

$$\mathcal{L}_{\mathsf{Palatini}}[e,\omega] = \int_{\mathcal{X}}\star(e_I\wedge e_J)\wedge F^{IJ} = \int_{\mathcal{X}}\overset{\circ}{\star}(e_I\wedge e_J)\wedge F^{IJ} \tag{248}$$

13.4 Einstein-Cartan field equations, with differential forms

We make some remarks on Einstein-Cartan fields equations, using differential forms. We have the following system of equations:

$$\begin{array}{rcl} \epsilon_{IJKL}e^J\wedge F^{KL} &=& 0 \\ \epsilon_{IJKL}\mathcal{D}(e^K\wedge e^L) &=& 0 \end{array} \tag{249}$$

Here we are dealing with the Einstein-Cartan equations, not in the full Cartan geometry setting. We just want to emphasize to role of the differential form and the possibility to describe *spin* via non-vanishing torsion. Due to the setting of Cartan, a covariant derivative - namely a connection - is described by a Lie algebra-valued 1-form. With the help of Hodge duality, the equations of movement are straightforward derived. Let us compute the variation of the action. For the term $\delta^{[\omega]}\mathcal{L}_{\mathsf{Palatini}}[e,\omega]$

- the full treatment would be the following variation: $\delta\mathcal{L}_{\text{Palatini}}[e,\omega] = \delta^{[e]}\mathcal{L}_{\text{Palatini}}[e,\omega] + \delta^{[\omega]}\mathcal{L}_{\text{Palatini}}[e,\omega]$ we describe variation with respect to ω:

$$\delta^{[\omega]}S = \delta^{[\omega]}\Big(\int_{\mathcal{X}} \star(e^I\wedge e^J)\wedge F_{IJ}\Big) = \delta^{[\omega]}\Big(\int_{\mathcal{X}} \Sigma^{IJ}\wedge F_{IJ}\Big) = \int_{\mathcal{X}} \Sigma^{IJ}\wedge \mathcal{D}\delta\omega_{IJ}$$

Since $\delta^{[\omega]}F_{IJ} = d(\delta\omega_{IJ}) + [\omega,\delta\omega]_{IJ} = \mathcal{D}\delta\omega_{IJ}$ we have:

$$\delta^{[\omega]}S = \int \Sigma^{IJ}\wedge\mathcal{D}\delta\omega_{IJ} = \frac{1}{2}\int \epsilon^{IJ}{}_{KL}\Sigma^{KL}\wedge\mathcal{D}\delta\omega_{IJ} = -\frac{1}{2}\int \epsilon^{IJ}{}_{KL}\mathcal{D}\Sigma^{KL}\wedge\delta\omega_{IJ}$$

Then, $\delta^{[\omega]}S = 0 \iff \mathcal{D}\Sigma^{KL} = \mathcal{D}(e^K\wedge e^L) = 0$. Following Wise [244] we also notice the following equivalent formulation for Palatini gravity. We write the Palatini action as (250):

$$\mathcal{L}_{\text{Palatini}}[e,\omega] = \int \mathrm{tr}(e\wedge e\wedge F) \tag{250}$$

The expression $e\wedge e\wedge F$ is a $\Lambda^4\mathcal{V}$-valued 4-form on $\mathcal{X}$ while tr is a trace - build on the the *internal* Hodge operator $\overset{\circ}{\star}$ -, which turns such a form into an ordinary real-valued 4-form. As already emphasized, - see appendix (B) - the wedge product $\wedge$ acts both on space-time indices and on internal Lorentz indices. F is the curvature of ω, described as a $\Lambda^2\mathcal{V}$-valued 2-form. $\mathrm{tr}: \Omega(\mathcal{X},\Lambda^4\mathcal{V}) \to \Omega(\mathcal{X},\mathbb{R})$ Then performing variation with respect to ω and e, we obtain:

$$\delta\mathcal{L}_{\text{Palatini}} = \int \mathrm{tr}(2\delta e\wedge e\wedge F + d^{\mathbf{D}}(e\wedge e)\wedge\delta\omega)$$

The equations of motion are written by:

$$\begin{aligned} e\wedge F &= 0 \\ d_\omega(e\wedge e) &= 0 \end{aligned} \tag{251}$$

Following Wise [244] we focus on the interplay of *internal* and *external* Hodge duality - the relation $p!\overset{\circ}{\star}\underbrace{\Big[e\wedge\ldots\wedge e\Big]}_{p\ times} = (n-p)!\star\underbrace{\Big[e\wedge\ldots\wedge e\Big]}_{n-p\ times}$. In $4D$ case we found $e\in\mathcal{V}\otimes T^\star\mathcal{X}$ and $e\wedge e\wedge e\in\Lambda^3\mathcal{V}\otimes\Lambda^3T^\star\mathcal{X}$, so that $\overset{\circ}{\star}e$ and $\star(e\wedge e\wedge e)$ are in $\Lambda^3\mathcal{V}\otimes T^\star\mathcal{X}$. In components:

$$\begin{aligned} \overset{\circ}{\star}e &= \Big[\overset{\circ}{\star}e\Big]_\mu^{IJK} dx^\mu\otimes e_I\wedge e_J\wedge e_K \\ \star[e\wedge e\wedge e] &= \Big[\star[e\wedge e\wedge e]\Big]_\mu^{IJK} dx^\mu\otimes e_I\wedge e_J\wedge e_K \end{aligned}$$

we are not enter into details and refer to the work of Wise [244] or Baez [15, 16] for the underlying relation with more general BF theory.

13.5 Einstein-Cartan and the spinorial area

The deepness of Einstein-Cartan formalism reveal two sides of the same mask: either it allow to work with dynamical variables - curvature *and* torsion - or it is the suitable framework for the inclusion of matter field. A system of spin-1/2 fields coupled to gravity is described via the *Einstein-Cartan*-like action (252):

$$\mathcal{L}_{\text{EC}}[e,\omega,\psi,\bar{\psi}] = \frac{1}{2}\int e_I\wedge e_J\wedge\star F^{IJ} + \frac{i}{2}\int \star e_I\wedge\big[\bar{\psi}\gamma^I\mathcal{D}\psi - \mathcal{D}\bar{\psi}\gamma^I\psi\big] \tag{252}$$

Then, we write: $\mathcal{L}_{\text{EC}}[e,\omega,\psi,\bar{\psi}] = \mathcal{L}_{\text{Holst}} + \mathcal{L}_{\text{Dirac}}$ Let notice that:

$$\star\, e = \big[\star\, e\big]_{\nu\rho\sigma}{}^{I} dx^{\nu}\wedge dx^{\rho}\wedge dx^{\sigma}\otimes \mathbf{e}_I \qquad \text{with} \qquad \big[\star\, e\big]_{\nu\rho\sigma}{}^{I} = \frac{1}{3!}\boldsymbol{\epsilon}^{\mu}{}_{\nu\rho\sigma}e^{I}_{\mu} \tag{253}$$

Lemma 13.4. *The interplay of Hodge duality defined in part* (13.3) *leads to:*

$$\frac{i}{2}\int \star e_I \wedge \big[\bar{\psi}\gamma^I\mathcal{D}\psi - \overline{\mathcal{D}\psi}\gamma^I\psi\big] = \frac{i}{2}\int e e^{\mu}_{I}\big[\bar{\psi}\gamma^I\mathcal{D}_{\mu}\psi - \overline{\mathcal{D}_{\mu}\psi}\gamma^I\psi\big]dx^4 \tag{254}$$

$$\mathcal{D}\psi = d\psi + \frac{1}{2}\omega^{IJ}\sigma_{IJ}\psi \qquad \overline{\mathcal{D}\psi} = d\overline{\psi} - \frac{1}{2}\omega^{IJ}\sigma_{IJ}\overline{\psi} \qquad \text{with} \qquad \sigma_{IJ} = \frac{1}{4}[\gamma_I,\gamma_J] = \frac{1}{2}\star e_{IJKL}\gamma^L$$

⌈ Proof The one form $e^I = e^I_{\mu}dx^{\mu}$ is defined on space-time manifold $\mathcal{X}$ and its Hodge dual form is

$$\star e = \frac{1}{3!}\Big[\star\, e_{\nu\rho\sigma}\Big]dx^{\nu}\wedge dx^{\rho}\wedge dx^{\sigma} \qquad \text{and} \qquad \Big[\star\, e_{\nu\rho\sigma}\Big] = \frac{1}{1!}\boldsymbol{\epsilon}_{\mu\nu\rho\sigma}e^{\mu} = \boldsymbol{\epsilon}_{\mu\nu\rho\sigma}e^{\mu}$$

Then

$$\star e = (1/3!)\boldsymbol{\epsilon}_{\mu\nu\rho\sigma}e^{\mu}dx^{\nu}\wedge dx^{\rho}\wedge dx^{\sigma} = (1/3!)e\epsilon_{\mu\nu\rho\sigma}e^{\mu}dx^{\nu}\wedge dx^{\rho}\wedge dx^{\sigma}$$

Taking into account the Lie algebra-valued feature we obtain: $\star e_I = (1/3!)e\epsilon_{\mu\nu\rho\sigma}e^{\mu}_{I}dx^{\nu}\wedge dx^{\rho}\wedge dx^{\sigma}$. Since: $\epsilon^{\nu\rho\sigma\kappa}\beta_{\kappa} = dx^{\nu}\wedge dx^{\rho}\wedge dx^{\sigma}$, we obtain:

$$\star e_I = \frac{1}{3!}e\epsilon_{\mu\nu\rho\sigma}e^{\mu}_{I}\epsilon^{\nu\rho\sigma\kappa}\beta_{\kappa} = \frac{1}{3!}ee^{\mu}_{I}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\nu\rho\sigma\kappa}\beta_{\kappa} = \frac{1}{3!}ee^{\mu}_{I}(3!)\delta^{\kappa}_{\mu}\beta_{\kappa}$$

On the other hand, since $\bar{\psi}\gamma^I\mathcal{D}\psi - \overline{\mathcal{D}\psi}\gamma^I\psi = \big[\bar{\psi}\gamma^I\mathcal{D}_{\lambda}\psi - \overline{\mathcal{D}_{\lambda}\psi}\gamma^I\psi\big]dx^{\lambda}$ we have

$$\star e_I \wedge \big[\bar{\psi}\gamma^I\mathcal{D}\psi - \overline{\mathcal{D}\psi}\gamma^I\psi\big] = ee^{\mu}_{I}\big[\bar{\psi}\gamma^I\mathcal{D}_{\lambda}\psi - \overline{\mathcal{D}_{\lambda}\psi}\gamma^I\psi\big]\beta_{\mu}\wedge dx^{\lambda} = ee^{\mu}_{I}\big[\bar{\psi}\gamma^I\mathcal{D}_{\mu}\psi - \overline{\mathcal{D}_{\mu}\psi}\gamma^I\psi\big]\beta \quad ⌋$$

14 Canonical variable and phase space for canonical gravity

In LQG theory, see A. Ashtekar and J. Lewandowski [10] Rovelli, [197] T. Thiemann [221, 222], we work with basic canonical variables (A^i_a, E^a_i). We observe either the connection representation, see section (14.2), or, via the loop transform, we pass to holonomy-flux algebra, see section (14.3). The latter describes canonical variables as the holonomy $h_{\Gamma}[A]$ and the flux vector $F_{\mathcal{S}}[E]$. We give canonical variables for Ashtekar-Palatini gravity with (A^I_{μ}, E^{μ}_I) - for the *connection representation* - or $(h_{\Gamma}[A], F_{\mathcal{S}}[E])$ - for the *loop representation*, see the seminal work of Rovelli and L. Smolin [201]. The later find natural geometrical setting with the copolarization and the observable functional in MG setting - see (18.3). The main fact related to the issue of the arbitrary foliation of space-time - and beyond the issue of the covariance for gravity - is the following. We realized that in the full multisymplectic setting, we have *directly* an explanation of the right canonical variables $(\boldsymbol{\varpi}_{IJ}, \omega^{IJ})$ as a 2-form and a potential 1-form - for more comments see section (18.3). For an introduction on basic canonical variables and quantum Riemannian geometry in LQG, we refer to [10] [11, 16] [179, 180] - and for more elaborate considerations and generalizations, we refer to [27] [28, 189] [222] and references therein.

14.1 Phase space and canonical variables

We distinguish three distinct periods for the establishment of modern phase space and canonical variable in LQG theory:

[1] Historically, A. Ashtekar [8, 9] developed new variables for general relativity in relation with the work of A. Sen [213] and we focus along this historical perspective on the so-called Ashtekar-Sen connection. The original Ashtekar connection A^i_a is the spatial projection of the self-dual part of the four dimensional spin connection.[94] We have the canonical variables (A^i_a, E^a_i) (255):

$$A^i_a = \Gamma^i_{\ a} + \gamma K^i_a = \frac{1}{2}\epsilon^i_{\ jk}\omega^{jk}_a + \gamma\omega^{\circ i}_a \qquad \text{and} \qquad E^a_i = \frac{1}{2}\epsilon_{abc}\epsilon^{ijk}e^b_j e^c_k \tag{255}$$

We denote equivalently $A^i_a = 1/2\epsilon^{ijk}\omega_{ajk} + \gamma K^i_a$. The usual Hamiltonian formalism process via the $(3+1)$-decomposition gives the set of constraints for Ashtekar phase space are the Gauss $\mathcal{G}_i$, vector $\mathcal{H}_a$ and scalar $\mathcal{H}$ constraints:

$$\mathcal{G}_i = \mathcal{D}_a E^a_i \qquad \mathcal{H} = (E^a_i E^b_j/\sqrt{E})\epsilon^{ij}_{\ \ k}F^k_{ab} \qquad \mathcal{H}_a = E^b_i F^i_{ab}$$

[2] The second step in the development of LQG program has been the switch to the Ashtekar-Holst-Brabero and is now considered as the right generalization for canonical variables in LQG. We denote it as $\left(A^{IJ}_a\right)^{\mathrm{LQG}}$ where now stand the use of the Immirzi-Barbero Parameter - see [21, 127]. This approach is base on the Holst action:

$$\mathcal{L}_{\mathsf{Holst}}[e,\omega] = \frac{1}{2}\int_{\mathcal{X}} e^I \wedge e^J \wedge \big[\star + \frac{1}{\Gamma}\big]F_{IJ} = e^I \wedge e^J \wedge \star^{\Gamma}F_{IJ} \tag{256}$$

The Holst action, via Hamiltonian setting and the time-gauge leads us to recover the feature of the classical ADM phase space. We have the following constraints:

$$^{\mathrm{LQG}}\mathcal{G}_i = \mathcal{D}_a E^a_i \qquad ^{\mathrm{LQG}}\mathcal{H} = \frac{E^a_i E^b_j}{\sqrt{E}}\big(\epsilon^{ij}_{\ \ k}F^k_{ab} - 2(1+\gamma^2)K^i_{[a}K^j_{b]}\big) \qquad ^{\mathrm{LQG}}\mathcal{H}_a = E^b_i F^i_{ab} - (1+\gamma^2)K^i_a\mathcal{G}_i$$

Notice that in the expression of the Gauss constraint we define $\mathcal{D}$ the covariant derivative defined by the connection ${}^{\gamma}A^i_a$ such that: $\mathcal{D}_a E^a_i = \partial_a E^a_i + \epsilon_{ij}^{\ \ k}\,{}^{\gamma}A^j_a E^a_k$. Here, $F^i_{ab} = 2\partial_{[a}{}^{\gamma}A^i_{b]} + \epsilon^i_{\ jk}\,{}^{\gamma}A^j_a\,{}^{\gamma}A^k_b$ is the curvature of the connection ${}^{\gamma}A^j_a$. These constraints are the Gauss law ${}^{\mathrm{LQG}}\mathcal{G}_i$, the vector ${}^{\mathrm{LQG}}\mathcal{H}_a$ and the scalar constraint ${}^{\mathrm{LQG}}\mathcal{H}$ of the loop gravity.

[3] The last modern development is found in the use of new variables (π^a_{IJ}, A^{IJ}_a) and the role of *simplicity constraints* (257). We follow the recent work of the Erlangen group - N. Bodendorfer, T. Thiemann and A. Thurn [27, 28], on the ground of the earlier work of P. Peldan [189]. This approach leads to another extension of the ADM phase space than the usual LQG one. We emphasize the underlined perspective of those authors: higher dimensional supergravity loop quantizations. [28] In this case the new canonical variables are a densitized vielbein π^a_{IJ} in the adjoint representation of $\mathrm{SO}(D+1)$ whereas A^{IJ}_a is the other canonical variable. For us, $\dim(D) = n-1 = 3$ is the number of spatial dimensions. The two canonical variables are: (π^a_{IJ}, A^{IJ}_a) with $\pi^a_{IJ} = \epsilon^{abc}\epsilon_{IJKL}e^K_b e^L_c$. Since π^a_{IJ} has 18 components and e^K_b has 12 components so we need 6 primary constraints $\mathcal{C}^{ab}$. These constraints are called the *simplicity constraints* (257):

$$\mathcal{C}^{ab} = \epsilon^{IJKL}\pi^a_{IJ}\pi^b_{KL} = 0 \tag{257}$$

These constraints play a role in the quantization perspective with different model of *spin foam* and related strategies. We denote π^a_{IJ} the canonical momenta associated to ω^{IJ}_a. In this case, we

[94] In this section we denote $\boldsymbol{\sigma} = 1$ for Riemannian signature, and $\boldsymbol{\sigma} = -1$ for Euclidean one. Indices μ, ν, ρ denote space-time indices whereas a, b, c... denote spatial indices and finaly i, j, k... are related to $\mathfrak{su}(2)$ or $\mathfrak{so}(3)$ Lie algebra indices whereas, I, J, K... denote $\mathfrak{so}(1,3)$ indices. In the literature concerning LQG one distinguish several pairs of canonical variables.

define the momenta $\pi^a_{IJ} = \delta L/\delta\partial_\circ\omega^{IJ}_a = \epsilon_{IJKL}\epsilon^{abc}e^K_b e^L_c$ so that we assume the passage from the Lagrangian $\mathcal{L}_{\mathsf{Holst}}[e,\omega] = \pi^a_{IJ}\dot{\omega}^{IJ}_a + \omega^{IJ}_\circ\mathcal{G}_{IJ} - N\mathcal{H} - N^a\mathcal{H}_a$ - with the Lagrange multiplier $\mathcal{G}_{IJ}, N, N^a$ to Hamiltonian side, via the $(3+1)$-decomposition:

$$\mathcal{H}_{\mathsf{Holst}} = \pi^a_{IJ}\dot{\omega}^{IJ}_a - \mathcal{L}_{\mathsf{Holst}} = \omega^{IJ}_\circ\mathcal{G}_{IJ} - N\mathcal{H} - N^a\mathcal{H}_a + c_{ab}\mathcal{C}^{ab}$$

with the constraints

$$\mathcal{G}_{IJ} = \mathcal{D}_a\pi^a_{IJ} \qquad \mathcal{H} = \pi^a_{IK}\pi^{bK}_J F^{IJ}_{ab} \qquad \mathcal{H}_a = \pi^b_{IJ}F^{IJ}_{ab} \qquad \mathcal{C}^{ab} = \epsilon^{IJKL}\pi^a_{IJ}\pi^b_{KL}$$

In a particular choice of gauge, namely the time-gauge choice $e^\circ_a = 0$, the simplicity constraints does not appear. Now we concentrate on geometrical objects for the canonical variables. Following [27], we define $(n-1)$ linearly independent vectors $\{e^I_a\}_{1\leq a\leq(n-1)}$ so and the common normal vector (258) is written:

$$n_I = \frac{1}{(n-1)}\frac{1}{\sqrt{\mathsf{g}}}\epsilon^{a_1\cdots a_{n-1}}\epsilon_{IJ_1\cdots J_{n-1}}e^{J_1}_{a_1}\cdots e^{J_{n-1}}_{a_{n-1}} \tag{258}$$

which satisfy $e^I_a n_I = 0, \quad n_I n^I = -\boldsymbol{\sigma}$. For example in the case where $\dim(\mathcal{X}) = 4$ - so that the spatial dimension reduce to $\dim(\Sigma) = 3$ - we obtain $n_I = (1/3\sqrt{\mathsf{g}})\epsilon^{abc}\epsilon_{IJKL}e^J_a e^K_b e^L_c$. The *simplicity constraints* is given by the following relation:

$$\pi^a_{IJ} = 2\sqrt{\mathsf{g}}\mathsf{g}^{ab}n^{[I}e^{J]}_b = 2n^{[I}E^{a|J]} \tag{259}$$

14.2 Connection representation: canonical variables (A^I_μ, E^μ_I)

In this section we give the connection representation. We consider the traditional Ashtekar-Sen connection. In the setting of Quantum Riemannian 3-Geometry, we consider basic LQG canonical variables: (A^I_μ, E^μ_I). The first one is a Lie algebra-valued one form $A = A^I_\mu dx^\mu \otimes \mathfrak{g}_I = \mathfrak{g}_I A^I_\mu dx^\mu$. Then, we consider A^I_μ as a $\mathfrak{su}(2)$-valued 1-form. Whereas the second is a densitized triad $E^\mu_I = ee^\mu_I$. It is a Lie algebra-valued vector density of weigh 1. In LQG we find the following expression for the densitized triad E^μ_I:

$$E^\mu_I = \frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}e^J_\rho e^K_\sigma. \tag{260}$$

Following T. Thieman [222], one remarks that the precious consideration on a densitized triad is extend to any dimension. For a general n-bein we define:

$$E^\mu_I = \boldsymbol{\sigma}\det(e^J_\mu)\frac{1}{(n-1)!}\epsilon^{\mu\mu_1\ldots\mu_{n-1}}\epsilon_{II_1\ldots I_{n-1}}e^{I_1}_{\mu_1}\cdots e^{I_{n-1}}_{\mu_{n-1}} = ee^\mu_I \tag{261}$$

Notice that the one form $e^I = e^I_\mu dx^\mu$ is defined on the three dimensional submanifold Σ. We can evaluate the Hodge dual form as:

$$\star e = \frac{1}{2!}\big(\star e_{\mu\nu}\big)dx^\mu\wedge dx^\nu \qquad \text{and} \qquad \big(\star e_{\mu\nu}\big) = \frac{1}{1!}\boldsymbol{\epsilon}_{\mu\nu\rho}e^\rho = \boldsymbol{\epsilon}_{\rho\mu\nu}e^\rho$$

Then, $\star e = 1/2\boldsymbol{\epsilon}_{\rho\mu\nu}e^\rho dx^\mu\wedge dx^\nu = 1/2e\epsilon_{\rho\mu\nu}e^\rho dx^\mu\wedge dx^\nu$. Now we take into account the Lie algebra-valued feature:

$$\star e_I = \frac{1}{2}e\epsilon_{\rho\mu\nu}e^\rho_I dx^\mu\wedge dx^\nu = \frac{1}{2}\epsilon_{\rho\mu\nu}E^\rho_I dx^\mu\wedge dx^\nu \tag{262}$$

We immediately see that the component of the pseudo two from $\star e_I$, denoted as $\big(\star e_{\mu\nu}\big)_I$ are precisely identified with: $\big(\star e_{\mu\nu}\big)_I = \epsilon_{\rho\mu\nu}E^\rho_I$. Since E^μ_I is a vector density of weight one, to it we associate the $\mathfrak{su}(2)$-valued pseudo-two form $\star e_I$ which is often designed by its components $\big(\star E\big)_{\mu\nu I} = E^\rho_I\epsilon_{\mu\nu\rho}$, or without the reference to the Lie algebra indice we write $\star e_{\mu\nu} = \epsilon_{\mu\nu\rho}E^\rho$. The object $\big(\star E\big)_{\mu\nu I}$ is a 2-form of density weight 0.

14.3 Holonomy-flux variables

Now the electric flux $F_I[\mathcal{S}]$ is defined. Let us consider the element $\star e_I = dF_I = 1/2\epsilon_{\mu\rho\sigma}E^\mu_I dx^\rho \wedge dx^\sigma$. Let $\mathcal{S} \subset \Sigma$ br a two dimensional surface, we define the electric flux through $\mathcal{S}$ by the quantity $F_I[\mathcal{S}]$:

$$F_I[\mathcal{S}] = \int_\Sigma dF_I \qquad \text{or} \qquad F_I[\mathcal{S}] = \int_{\mathcal{S}} n_\mu E^\mu_I d^2\varsigma = \int_{\mathcal{S}} n_\mu E^\mu_I d\varsigma^1 \wedge d\varsigma^2 \tag{263}$$

We focus on the geometrical picture beyond the definition (263). We consider an embedded 2-surface $(\mathcal{S}, \mathsf{h})$ into a Riemannian 3-manifold (Σ, g). On Σ we have the 3-metric $\mathsf{g}_{\mu\nu}$. Then, we express the 3-metric in terms of the triad: $\mathsf{g}_{\mu\nu} = e^I_\mu e^J_\nu \delta_{IJ}$. Thanks to the inverse metric g^{-1} we connect the inverse metric components $\mathsf{g}^{\mu\nu}$ to those of the densitized inverse triad via $\mathsf{g}\mathsf{g}^{\mu\nu} = E^\mu_I E^\nu_J \delta^{IJ}$. Riemannian geometry of the three-manifold Σ is encoded in the momenta E^μ_I. We focus on a parametrization of $\mathcal{S}$. Let $\varphi : \mathcal{S} \to \Sigma$ be an embedding, under the hypothesis that we identify the manifold $\mathcal{S}$ with the submanifold $\varphi(\mathcal{S})$ the embedding is the inclusion map $\mathsf{i} : \mathcal{S} \to \Sigma$. The local study of the embedding is perform through the local parametrization $x : \mathcal{U} \to \mathcal{S} \subset \Sigma : \varsigma = (\varsigma^1, \varsigma^2) \mapsto x^\mu(\varsigma)$. Now $(\varsigma^1, \varsigma^2)$ are local coordinates on $\mathcal{S}$. We have the following lemma:

Lemma 14.1. *Let $(\mathcal{S}, h)$ an embedded two surface into a Riemannian 3D manifold (Σ, g) and consider a local parametrization $x : \mathcal{U} \to \mathcal{S} \subset \Sigma : \varsigma = (\varsigma^1, \varsigma^2) \to x^\mu(\varsigma)$. The determinant of the 2D metric h writes:*

$$h = gg^{\alpha\beta}\epsilon_{\mu\nu\alpha}\epsilon_{\rho\sigma\beta}(\frac{\partial x^\mu}{\partial\varsigma^1})(\frac{\partial x^\nu}{\partial\varsigma^1})(\frac{\partial x^\rho}{\partial\varsigma^2})(\frac{\partial x^\sigma}{\partial\varsigma^2})$$

⌈ Proof. Here h is the determinant of the metric so that $\mathsf{h} = \det(\mathsf{h}_{\alpha\beta})$. We have $\mathsf{h}_{\alpha\beta} = \mathsf{g}_{\mu\nu}(\frac{\partial x^\mu}{\partial\varsigma^\alpha})(\frac{\partial x^\nu}{\partial\varsigma^\beta})$, with $\alpha, \beta = 1, 2$. Since $\mathsf{h} = 1/2\epsilon^{\rho\sigma}\epsilon^{\kappa\sigma}\mathsf{h}_{\rho\kappa}\mathsf{h}_{\sigma\lambda}$

$$\mathsf{h} = \det(\mathsf{h}_{\alpha\beta}) = \frac{1}{2}\epsilon^{\eta\gamma}\epsilon^{\kappa\lambda}\mathsf{h}_{\eta\kappa}\mathsf{h}_{\gamma\lambda} = \frac{1}{2}\epsilon^{\eta\gamma}\epsilon^{\kappa\lambda}\Big[\mathsf{g}_{\mu\nu}(\frac{\partial x^\mu}{\partial\varsigma^\eta})(\frac{\partial x^\nu}{\partial\varsigma^\kappa})\Big]\Big[\mathsf{g}_{\rho\sigma}(\frac{\partial x^\rho}{\partial\varsigma^\gamma})(\frac{\partial x^\sigma}{\partial\varsigma^\lambda})\Big]$$

$$\mathsf{h} = \det(\mathsf{h}_{\alpha\beta}) == \frac{1}{2}\epsilon^{\eta\gamma}\epsilon^{\kappa\lambda}\mathsf{g}_{\mu\nu}\mathsf{g}_{\rho\sigma}(\frac{\partial x^\mu}{\partial\varsigma^\eta})(\frac{\partial x^\nu}{\partial\varsigma^\kappa})(\frac{\partial x^\rho}{\partial\varsigma^\gamma})(\frac{\partial x^\sigma}{\partial\varsigma^\lambda})$$

We write explicitly on the local coordinate ς^1, ς^2 on $\mathcal{S}$

$$\mathsf{h} = \frac{1}{2}\mathsf{g}_{\mu\nu}\mathsf{g}_{\rho\sigma}\Big[\epsilon^{12}\epsilon^{12}(\frac{\partial x^\mu}{\partial\varsigma^1})(\frac{\partial x^\nu}{\partial\varsigma^1})(\frac{\partial x^\rho}{\partial\varsigma^2})(\frac{\partial x^\sigma}{\partial\varsigma^2}) + \epsilon^{21}\epsilon^{12}(\frac{\partial x^\mu}{\partial\varsigma^2})(\frac{\partial x^\nu}{\partial\varsigma^1})(\frac{\partial x^\rho}{\partial\varsigma^1})(\frac{\partial x^\sigma}{\partial\varsigma^2})$$

$$+\epsilon^{12}\epsilon^{21}(\frac{\partial x^\mu}{\partial\varsigma^1})(\frac{\partial x^\nu}{\partial\varsigma^2})(\frac{\partial x^\rho}{\partial\varsigma^2})(\frac{\partial x^\sigma}{\partial\varsigma^1}) + \epsilon^{21}\epsilon^{21}(\frac{\partial x^\mu}{\partial\varsigma^2})(\frac{\partial x^\nu}{\partial\varsigma^1})(\frac{\partial x^\rho}{\partial\varsigma^2})(\frac{\partial x^\sigma}{\partial\varsigma^1})\Big]$$

$$\mathsf{h} = \frac{1}{2}\mathsf{g}_{\mu\nu}\mathsf{g}_{\rho\sigma}\Big[(\frac{\partial x^\mu}{\partial\varsigma^1})(\frac{\partial x^\nu}{\partial\varsigma^1})(\frac{\partial x^\rho}{\partial\varsigma^2})(\frac{\partial x^\sigma}{\partial\varsigma^2}) - (\frac{\partial x^\mu}{\partial\varsigma^2})(\frac{\partial x^\nu}{\partial\varsigma^1})(\frac{\partial x^\rho}{\partial\varsigma^1})(\frac{\partial x^\sigma}{\partial\varsigma^2})$$

$$-(\frac{\partial x^\mu}{\partial\varsigma^1})(\frac{\partial x^\nu}{\partial\varsigma^2})(\frac{\partial x^\rho}{\partial\varsigma^2})(\frac{\partial x^\sigma}{\partial\varsigma^1}) + (\frac{\partial x^\mu}{\partial\varsigma^2})(\frac{\partial x^\nu}{\partial\varsigma^1})(\frac{\partial x^\rho}{\partial\varsigma^2})(\frac{\partial x^\sigma}{\partial\varsigma^1})\Big]$$

$$\mathsf{h} = \mathsf{g}_{\mu\nu}\mathsf{g}_{\rho\sigma}\Big[(\frac{\partial x^\mu}{\partial\varsigma^1})(\frac{\partial x^\nu}{\partial\varsigma^1})(\frac{\partial x^\rho}{\partial\varsigma^2})(\frac{\partial x^\sigma}{\partial\varsigma^2}) - (\frac{\partial x^\mu}{\partial\varsigma^1})(\frac{\partial x^\nu}{\partial\varsigma^2})(\frac{\partial x^\rho}{\partial\varsigma^1})(\frac{\partial x^\sigma}{\partial\varsigma^2})\Big] = 2\mathsf{g}_{\mu\nu}\mathsf{g}_{\rho\sigma}(\frac{\partial x^\mu}{\partial\varsigma^1})(\frac{\partial x^{[\nu}}{\partial\varsigma^1})(\frac{\partial x^{\rho]}}{\partial\varsigma^2})(\frac{\partial x^\sigma}{\partial\varsigma^2})$$

$$\mathsf{h} = 2\mathsf{g}_{\mu[\nu}\mathsf{g}_{\rho]\sigma}(\frac{\partial x^\mu}{\partial\varsigma^1})(\frac{\partial x^\nu}{\partial\varsigma^1})(\frac{\partial x^\rho}{\partial\varsigma^2})(\frac{\partial x^\sigma}{\partial\varsigma^2})\text{Since} \quad \mathsf{g}_{\mu[\nu}\mathsf{g}_{\rho]\sigma} = \frac{1}{2}\epsilon_{\mu\nu\alpha}\epsilon_{\rho\sigma\beta}\mathsf{g}\mathsf{g}^{\alpha\beta} \quad \text{, we finally obtain}$$

$$\mathsf{h} = \mathsf{g}\mathsf{g}^{\alpha\beta}\epsilon_{\mu\nu\alpha}\epsilon_{\rho\sigma\beta}(\frac{\partial x^\mu}{\partial\varsigma^1})(\frac{\partial x^\nu}{\partial\varsigma^1})(\frac{\partial x^\rho}{\partial\varsigma^2})(\frac{\partial x^\sigma}{\partial\varsigma^2}) \quad ⌋$$

The previous calculation leads us to construct the *area operator*. We introduce a normal vector $n_\mu = n^{\mathcal{S}}_\mu$ as $n_\mu = \epsilon_{\mu\nu\rho}(\frac{\partial x^\nu}{\partial\varsigma^1})(\frac{\partial x^\rho}{\partial\varsigma^2})$ so that the $2D$ metric h writes:

$$\mathsf{h} = \mathsf{g}\mathsf{g}^{\alpha\beta}\Big[\epsilon_{\mu\nu\alpha}(\frac{\partial x^\mu}{\partial\varsigma^1})(\frac{\partial x^\nu}{\partial\varsigma^1})\Big]\Big[\epsilon_{\rho\sigma\beta}(\frac{\partial x^\rho}{\partial\varsigma^2})(\frac{\partial x^\sigma}{\partial\varsigma^2})\Big] = \mathsf{g}\mathsf{g}^{\alpha\beta}n_\alpha n_\beta \tag{264}$$

If we denote $\forall \varsigma \in \mathcal{U}, x(\varsigma) = X(\varsigma) \in \mathcal{S} \subset \Sigma$, then $E^\mu_I = E^\mu_I(x(\varsigma))$. Hence dF_I and $F_I(\mathcal{S})$ respectively writes:

$$dF_I = \frac{1}{2}\epsilon_{\mu\rho\sigma}E^\mu_I dx^\rho \wedge dx^\sigma = \epsilon_{\mu\rho\sigma}E^\mu_I(x(\varsigma))\frac{\partial x^\rho}{\partial\varsigma^1}(\varsigma)\frac{\partial x^\sigma}{\partial\varsigma^2}(\varsigma)d\varsigma^1 \wedge d\varsigma^2$$

$$F_I(\mathcal{S}) = \int_{\mathcal{S}} \epsilon_{\mu\rho\sigma}\frac{\partial x^\mu}{\partial\varsigma^1}(\varsigma)\frac{\partial x^\rho}{\partial\varsigma^2}(\varsigma)E^\sigma_I(x(\varsigma))d\varsigma^1 \wedge d\varsigma^2$$

With the help of $n_\mu = n^{\mathcal{S}}_\mu = \epsilon_{\mu\nu\rho}(\frac{\partial x^\nu}{\partial\varsigma^1})(\frac{\partial x^\rho}{\partial\varsigma^2})$ we obtain the equivalence of descriptions.

$$dF_I = n_\mu E^\mu_I d\varsigma^1 \wedge d\varsigma^2 \qquad \text{and} \qquad F_I(\mathcal{S}) = \int_{\mathcal{S}} n_\mu E^\mu_I d^2\varsigma = \int_{\mathcal{S}} n_\mu E^\mu_I d\varsigma^1 \wedge d\varsigma^2$$

dF_I is the area element for the spatial submanifold Σ. In term of forms, we equivalently speak about a pseudo 2-form dF_I. We have $dF_I = \epsilon_{IJK}e^K \wedge e^L$. We have the following straightforward calculation:

$$dF_I = \boldsymbol{\epsilon}_{IKL}e^K_\rho e^L_\sigma dx^\rho \wedge dx^\sigma = \boldsymbol{\epsilon}_{JKL}e^J_\mu e^K_\rho e^L_\sigma e^\mu_I dx^\rho \wedge dx^\sigma = \boldsymbol{\epsilon}_{\mu\rho\sigma}e^\mu_I dx^\rho \wedge dx^\sigma = \epsilon_{\mu\rho\sigma}e e^\mu_I dx^\rho \wedge dx^\sigma$$

Then we find $dF_I = \epsilon_{\mu\rho\sigma}E^\mu_I dx^\rho \wedge dx^\sigma$. The link with the traditional classical area picture emerge to the idea of area operator. The area classical area (265) in term of metric is given by:

$$A[\mathcal{S}] = \int_{\mathcal{S}} d\varsigma^1 \wedge d\varsigma^2\sqrt{\mathsf{h}} \tag{265}$$

From this consideration, we derive the area operator (266) in LQG:

$$A[\mathcal{S}] = \int_{\mathcal{S}} \sqrt{E^\mu_I E^\nu_J \delta^{IJ} n_\mu n_\nu}d\varsigma^1 \wedge d\varsigma^2 = \int_{\mathcal{S}} \sqrt{(n_\mu E^\mu_I)(n_\nu E^\nu_J)}d\varsigma^1 \wedge d\varsigma^2 \tag{266}$$

The object $F_I[\mathcal{S}]$ already introduced under the form (263) is now written:

$$F_I[\mathcal{S}] = \int_{\mathcal{S}} (\star e)_I = \frac{1}{2}\int_{\mathcal{S}} e\epsilon_{\rho\mu\nu}e^\rho_I dx^\mu \wedge dx^\nu \tag{267}$$

So that we equivalently write $F[\mathcal{S}] = \int_{\mathcal{S}} n_I \wedge (\star e)^I$. Here n_I is a $\mathfrak{su}(2)$-valued scalar function. We equivalently write $n_I \wedge \star e^I = n_I(\star e)^I$ so that we can write $F[\mathcal{S}] = \int_{\mathcal{S}} n_I \wedge (\star e)^I$ as the classical area. We have:

$$\begin{aligned} F[\mathcal{S}] &= \int_{\mathcal{S}} n_I \wedge (\star e)^I = \int_{\mathcal{S}} \Big(\frac{1}{3}\frac{1}{\sqrt{\mathsf{g}}}\epsilon_{\lambda\kappa}\epsilon^{IJK}e^\lambda_J e^\kappa_K\Big)\Big(\frac{1}{2}\epsilon_{\alpha\mu\nu}e e^\alpha_I dx^\mu \wedge dx^\nu\Big) \\ &= \int_{\mathcal{S}} \frac{1}{3!}\frac{1}{\sqrt{\mathsf{g}}}\epsilon_{\lambda\kappa}\epsilon^{IJK}e e^\alpha_I e^\lambda_J e^\kappa_K \epsilon_{\alpha\mu\nu}dx^\mu \wedge dx^\nu = \int_{\mathcal{S}} \frac{1}{3!}\frac{1}{\sqrt{\mathsf{g}}}e^2(3!)\epsilon_{\lambda\kappa}\epsilon^{\alpha\lambda\kappa}\epsilon_{\alpha\mu\nu}dx^\mu \wedge dx^\nu \\ &= \int_{\mathcal{S}} \sqrt{\mathsf{g}}\epsilon_{\lambda\kappa}(\delta^{[\lambda}_\mu \delta^{\kappa]}_\nu)dx^\mu \wedge dx^\nu = \int_{\mathcal{S}} \sqrt{\mathsf{g}}d\varsigma^1 \wedge d\varsigma^2 \end{aligned}$$

Spin network appears with the concepts of edge Γ, an open curve embedded in spatial manifold Σ and the holonomies, as path oriented exponential object $h_\Gamma[A] = \mathcal{P}\mathrm{exp}\int_\Gamma A_\mu dx^\mu$ - see [27, 28, 179, 180, 197, 221, 222] for details - we describe therefore the holonomy $h_\Gamma[A]$ as a matrix functional which transform SU(2) representations ρ_Γ for given spin j_Γ. Hence the holonomy is spin j_Γ-valued and described as $(\rho_{j_\Gamma}(h_\Gamma[A]))_{\boldsymbol{\alpha\beta}}$, where $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are related to the representation choice. The flux variable is thought to be a smeared version with a test function φ_I: $F[\mathcal{S}, \varphi] = \int_{\mathcal{S}} \varphi_I \epsilon_{\mu\nu\rho}E^{\mu I}dx^\rho \wedge dx^\sigma$.

The canonical Poisson bracket is described as the main object for LQG program with the following relation - given two edge Γ_1 and Γ_2 and related holonomies $h^{\varsigma}_{\Gamma_1}$ and $h^{\varsigma}_{\Gamma_2}$, ς an infinitesimal parameter, [179, 180]:

$$\{(h_\Gamma[A]))_{\boldsymbol{\alpha\beta}}, F_{\mathcal{S}}[E,\varphi]\} = \lim_{\varsigma\to 0}\Big[(h^{\varsigma}_{\Gamma_1}[A])_{\boldsymbol{\alpha\eta}}\{\int_{\Gamma(\varsigma)} A^I_\mu(x)\tau^I_{\boldsymbol{\eta\delta}}dx^\mu, \int_{\mathcal{S}} dx^\mu\wedge dx^\nu \epsilon_{\mu\nu\rho}\varphi_J(x')E^{J\rho}(x')\}(h^{\varsigma}_{\Gamma_2}[A])_{\boldsymbol{\delta\beta}}\Big] \tag{268}$$

From geometrical perspective, this relation founds deeper roots within MG with canonical bracket for forms of arbitrary degrees and the process of copolarization, see some remarks in section (18.3).

15 Chern-Simon Gravity

We discuss Chern-Simon [44] degenerate theory. In doing so, we give the geometrical construction for the covariant derivative and the Killing form. This example of topological field theory is treated in the *degenerate space* in the MG setting. In this case we have an *equivalence* between the Hamilton equations and the Euler-Lagrange equations. This phenomena appears also in the study of Dirac equation [105]. This phenomena appears also in the Chern-Simon theory, Dirac theory and in the Palatini first order theory. As emphasized by Harrivel [105] this is connected to the *first order* linear feature of those theories. The Legendre correspondence is strongly degenerated, however, working on $(\mathcal{M}_{\text{deg}}, \boldsymbol{\omega}^{\text{deg}})$, we derive the Euler-Lagrange equations from the Hamilton equations. This is explained by a geometrical interplay between the two spaces $(\mathcal{M}_{\text{deg}}, \boldsymbol{\omega}^{\text{deg}})$ and $(\mathcal{M}_{\mathsf{CS}}, \boldsymbol{\omega}^{\mathsf{CS}})$ with the projection $\pi : \mathcal{M}_{\mathsf{CS}} \to \mathcal{M}_{\mathsf{deg}}$ and the injection $\mathfrak{i} : \mathcal{M}_{\mathsf{deg}} \to \mathcal{M}_{\mathsf{CS}}$ such that $\pi\circ\mathfrak{i} = \mathbf{Id}_{\mathcal{M}^{\text{deg}}}$. Here we do not enter into details, we give Chern-Simon equations.

15.1 Chern-Simon Lagrangian

The Lagrangian of Chern-Simon theory is described by (269):

$$\mathcal{L}_{\mathsf{CS}}(A) = \frac{1}{2}\int_{\mathcal{X}} \text{tr}\langle A\wedge dA + \frac{2}{3}A\wedge A\wedge A\rangle \tag{269}$$

The Killing form plays the role of a symmetrized trace, $\text{tr}(,) : \mathfrak{g}\otimes\mathfrak{g}\to\mathbb{R}$. We consider a non degenerate Killing form.[95] The symmetric bilinear form is invariant under the adjoint action Ad_g which is the map for any $\xi\in\mathfrak{g}$ from $\mathfrak{g}$ to itself $\text{Ad}_g = \mathfrak{g}\to\mathfrak{g} : \xi\mapsto g\xi\mathrm{g}^{-1}$. We have: $\text{tr}(\text{Ad}_g(\xi), \text{Ad}_g(\zeta)) = \text{tr}(\xi,\zeta)$. Since tr is a two entry object, the object $\text{tr}(A\wedge A\wedge A)$ stands for $1/2A^{\mathcal{I}}\wedge A^{\mathcal{J}}\wedge A^{\mathcal{K}}\text{tr}\langle\mathfrak{b}_{\mathcal{I}}, [\mathfrak{b}_{\mathcal{J}}, \mathfrak{b}_{\mathcal{K}}]\rangle$. Let us notice that via the bracket $[,]$ described[96] in (272) the Chern Simon functional is written:

$$\mathcal{L}_{\mathsf{CS}}(A) = \int_{\mathcal{X}} \text{tr}\langle A\wedge dA + \frac{1}{3}A\wedge[A,A]\rangle = \int_{\mathcal{X}} \text{tr}\langle A\wedge(dA + \frac{1}{3}[A,A])\rangle \tag{270}$$

Chern Simon Lagrangian. We describe A as a $\mathfrak{g}$-valued form $A = A^{\mathcal{I}}\otimes\mathfrak{b}_{\mathcal{I}}\in\Omega^1(\mathcal{M})\otimes\mathfrak{g} = \Omega^1(\mathcal{M},\mathfrak{g})$ where $\{\mathfrak{b}_{\mathcal{I}}\}_{1\le\mathcal{I}\le n}$ denotes a basis of $\mathfrak{g}$. Since A is a connection on a principal G-bundle, it is seen locally as a Lie algebra-valued 1-form. We describe its exterior covariant derivative: $d^{\mathbf{D}}A\in\Omega^2(\mathcal{M},\mathfrak{g})$ and also the exterior differential that acts only on the form part (271). We obtain:

$$dA = dA^{\mathcal{I}}\otimes\mathfrak{b}_{\mathcal{I}} = d(A^{\mathcal{I}}_\mu dx^\mu)\otimes\mathfrak{b}_{\mathcal{I}} = (\partial_\nu A^{\mathcal{I}}_\mu)dx^\nu\wedge dx^\mu\otimes\mathfrak{b}_{\mathcal{I}} \tag{271}$$

[95] $\forall\xi,\zeta\in\mathfrak{g},\ \text{tr}(\xi,\zeta)=0\Rightarrow\xi=0.$
[96] See more details in appendix (B)

Notice that if we use notations introduced in the appendix (B), we can write $A = A^{\mathcal{I}}_{\mu} dx^{\mu} \otimes \mathfrak{b}_{\mathcal{I}} = A^{IJ}_{\mu} \mathfrak{J}_{IJ} dx^{\mu}$. The covariant derivative,[97] which acts on Lie algebra 1-form $\lambda = \lambda^{\mathcal{I}} \otimes \mathfrak{b}_{\mathcal{I}}$.

$$d^{\mathbf{D}}\lambda = \mathfrak{b}_I \mathcal{D}\lambda^I \qquad \text{with} \qquad \mathcal{D}\lambda^I = d\lambda^I + A^I_J \wedge \lambda^J$$

Let us construct the *bracket* (272) which acts on the two parts of the object: it takes the ordinary wedge product on the form part whereas it takes the Lie bracket on the Lie algebra part. Then we have

$$[A, A] = [\mathfrak{b}_{\mathcal{I}}, \mathfrak{b}_{\mathcal{J}}] A^{\mathcal{I}} \wedge A^{\mathcal{J}} = \mathfrak{c}^{\mathcal{K}}_{\mathcal{I}\mathcal{J}} A^{\mathcal{I}} \wedge A^{\mathcal{J}} \otimes \mathfrak{b}_{\mathcal{K}} \tag{272}$$

Notice that $[A, A] \in \Omega^2(\mathcal{X}, \mathfrak{g})$ as well as $dA \in \Omega^2(\mathcal{X}, \mathfrak{g})$. Then we write the curvature (273), locally a $\mathfrak{g}$-valued 2-form, as:

$$F = dA + \frac{1}{2}[A, A] \qquad\qquad F = dA + A \wedge A \tag{273}$$

We define the action of tr on a wedge product of $\mathfrak{g}$ Lie algebra-valued forms (274):

$$\mathrm{tr}\langle \lambda \wedge \sigma \rangle = \mathrm{tr}\langle (\lambda^{\mathcal{I}} \otimes \mathfrak{b}_{\mathcal{I}}) \wedge (\sigma^{\mathcal{J}} \otimes \mathfrak{b}_{\mathcal{J}}) \rangle = \mathrm{tr}\langle \mathfrak{b}_{\mathcal{I}}, \mathfrak{b}_{\mathcal{J}} \rangle \lambda^{\mathcal{I}} \wedge \sigma^{\mathcal{J}} \tag{274}$$

The expression with space-time indices is written: $\mathrm{tr}\langle \lambda \wedge \sigma \rangle = \mathrm{tr}\langle \lambda^{\mathcal{I}}_{\mu} \sigma^{\mathcal{J}}_{\nu} dx^{\mu} \wedge dx^{\nu} \otimes \mathfrak{b}_{\mathcal{I}} \wedge \mathfrak{b}_{\mathcal{J}} \rangle \rangle$

Terms involved in the Chern-Simon Lagrangian The first is the term $\mathrm{tr}\langle A \wedge dA \rangle$.

$$\mathrm{tr}\langle A \wedge dA \rangle = \mathrm{tr}\langle \Big(A^{\mathcal{I}}_{\mu} dx^{\mu} \otimes \mathfrak{b}_{\mathcal{I}} \Big) \wedge \Big(\partial_{\nu} A^{\mathcal{J}}_{\rho} dx^{\nu} \wedge dx^{\rho} \otimes \mathfrak{b}_{\mathcal{J}} \Big) \rangle = \mathrm{tr}\langle \big[\mathfrak{b}_{\mathcal{I}}, \mathfrak{b}_{\mathcal{J}} \big] \rangle A^{\mathcal{I}}_{\mu} (\partial_{\nu} A^{\mathcal{J}}_{\rho}) dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho}$$

For a *semi-simple* Lie algebra $\mathfrak{g}$, in the adjoint representation, we simply denote $\mathrm{tr}_{\mathcal{I}\mathcal{J}} = \mathrm{tr}\langle \mathfrak{b}_{\mathcal{I}}, \mathfrak{b}_{\mathcal{J}} \rangle$, in this case the bilinear form is the Killing metric and we write:

$$\mathrm{tr}\langle A \wedge dA \rangle = \mathrm{tr}_{\mathcal{I}\mathcal{J}} A^{\mathcal{I}}_{\mu} (\partial_{\nu} A^{\mathcal{J}}_{\rho}) dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho}$$

The second term is $\mathrm{tr}\langle A \wedge A \wedge A \rangle$. In this case, we obtain:

$$\begin{aligned}
\mathrm{tr}\langle \frac{2}{3} A \wedge A \wedge A \rangle &= \mathrm{tr}\langle \frac{1}{3} \big(A^{\mathcal{I}}_{\mu} dx^{\mu} \otimes \mathfrak{b}_{\mathcal{I}} \big) \wedge \big(\mathfrak{c}^{\mathcal{L}}_{\mathcal{J}\mathcal{K}} A^{\mathcal{J}} \wedge A^{\mathcal{K}} \otimes \mathfrak{b}_{\mathcal{L}} \big) \rangle \\
&= \frac{1}{3} \mathrm{tr}\langle \mathfrak{b}_{\mathcal{I}}, \mathfrak{b}_{\mathcal{L}} \rangle \mathfrak{c}^{\mathcal{L}}_{\mathcal{J}\mathcal{K}} A^{\mathcal{I}}_{\mu} A^{\mathcal{J}}_{\nu} A^{\mathcal{K}}_{\rho} dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho} \\
&= \frac{1}{3} \mathrm{tr}_{\mathcal{I}\mathcal{J}} \mathfrak{c}^{\mathcal{J}}_{\mathcal{L}\mathcal{K}} A^{\mathcal{I}}_{\mu} A^{\mathcal{L}}_{\nu} A^{\mathcal{K}}_{\rho} dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho}
\end{aligned}$$

We consider the coordinate $(x^{\mu}, A^{\mathcal{I}}_{\mu})$ so that the Lagrangian density is $L(x, A)$. Let $\beta = dx^1 \wedge dx^2 \wedge dx^3$ be a *volume form*. The Chern-Simon Lagrangian is written in component:

$$L(x^{\mu}, A^{\mathcal{I}}_{\mu}) = \frac{1}{2} \mathrm{tr}_{\mathcal{I}\mathcal{J}} \epsilon^{\mu\nu\rho} \big(A^{\mathcal{I}}_{\mu} \partial_{\nu} A^{\mathcal{J}}_{\rho} + \frac{1}{3} A^{\mathcal{I}}_{\mu} [A_{\nu}, A_{\rho}]^{\mathcal{J}} \big) = \frac{1}{2} \mathrm{tr}_{\mathcal{I}\mathcal{J}} \epsilon^{\mu\nu\rho} \big(F^{\mathcal{I}}_{\mu\nu} A^{\mathcal{J}}_{\rho} - \frac{2}{3} A^{\mathcal{I}}_{\mu} [A_{\nu}, A_{\rho}]^{\mathcal{J}} \big)$$

15.2 Chern-Simon multisymplectic manifold

The following De Donder-Weyl Poincaré-Cartan $\theta^{\mathsf{DW}}_{(q,p)}$ n-form is considered:

$$\theta^{\mathsf{DW}}_{(q,p)} := \mathfrak{e}\beta + p^{A_{\mu}\nu}_{\mathcal{I}} dA^{\mathcal{I}}_{\mu} \wedge \beta_{\nu}, \tag{275}$$

[97] Let us recall that if we denote $A^{\mathcal{I}} = A^{\mathcal{I}}_{\mu} dx^{\mu}$ we shall define the *connection matrix*, the image of the *gauge potential* A by the representation ρ. Then we describe $\rho(A)$ in matrix component $\rho(A)^I_J = A^{\mathcal{I}}(\rho(\mathfrak{b}_{\mathcal{I}}))^I_J$. Also we use the following notation: $A^I_J = A^I_{\mu J} dx^{\mu} = A^{\mathcal{I}}_{\mu} (T_{\mathcal{I}})^I_J$.

with $\beta = dx^1 \wedge dx^2 \wedge dx^3$ a volume 3-form on $\mathcal{X}$ and $\beta_\beta := \partial_\beta \lrcorner \beta$. The construction of the Legendre correspondence gives the equivalence relation between (q,v) and (q,p):

$$(q,v) \leftrightarrow (q,p) \qquad \Longleftrightarrow \qquad \frac{\partial \langle p, v\rangle}{\partial v} = \frac{\partial L(q,v)}{\partial v} \tag{276}$$

where the term $\langle p, v\rangle$ is understood as: $\langle p, v\rangle = \theta_p^{\mathsf{DW}}(\mathcal{Z})$ with $\mathcal{Z} = \mathcal{Z}_1 \wedge \mathcal{Z}_2 \wedge \mathcal{Z}_3$ and where $\forall \alpha$ $\mathcal{Z}_\alpha = \frac{\partial}{\partial x^\alpha} + \mathcal{Z}^{\mathcal{I}}_{\alpha\mu}\frac{\partial}{\partial A^{\mathcal{I}}_\mu}$. We make the calculation with $\mathcal{Z}^{\mathcal{I}}_{\alpha\mu} = \partial_\alpha A^{\mathcal{I}}_\mu + \mathfrak{c}^{\mathcal{I}}_{\mathcal{P}\mathcal{Q}} A^{\mathcal{P}}_\alpha A^{\mathcal{Q}}_\mu$:

$$\langle p, v\rangle = \theta_p^{\mathsf{DW}}(\mathcal{Z}) = \mathfrak{e}\beta(\mathcal{Z}) + p_{\mathcal{I}}^{A_\mu\nu} dA^{\mathcal{I}}_\mu \wedge \beta_\nu(\mathcal{Z})$$

The expression $\langle p, v\rangle = \theta_p^{\mathsf{DW}}(\mathcal{Z})$ is written:

$$\langle p, v\rangle = \mathfrak{e}\beta(\mathcal{Z}) + p_{\mathcal{J}}^{A_\rho\nu} dA^{\mathcal{J}}_\rho \wedge \beta_\nu(\mathcal{Z}) = \mathfrak{e} + p_{\mathcal{J}}^{A_\rho\nu}\mathcal{Z}^{\mathcal{J}}_{\nu\rho} = \mathfrak{e} + \frac{1}{2}\mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho}A^{\mathcal{I}}_\mu \mathcal{Z}^{\mathcal{J}}_{\nu\rho} \tag{277}$$

⌜ Proof Let us denote:

$$\mathcal{Z}_\nu = \frac{\partial}{\partial x^\nu} + \sum_\mu \mathcal{Z}^{\mathcal{I}}_{\nu\mu}\frac{\partial}{\partial A^{\mathcal{I}}_\mu} = \frac{\partial}{\partial x^\nu} + \sum_\mu (\partial_\nu A^{\mathcal{I}}_\mu + \mathfrak{c}^{\mathcal{I}}_{\mathcal{P}\mathcal{Q}}A^{\mathcal{P}}_\nu A^{\mathcal{Q}}_\mu)\frac{\partial}{\partial A^{\mathcal{I}}_\mu} = \sum_{1\le\boldsymbol{\mu}\le 2n}\mathcal{Z}^{\boldsymbol{\mu}}_\nu \frac{\partial}{\partial q^{\boldsymbol{\mu}}}$$

We have $q^{\boldsymbol{\mu}} = x^{\boldsymbol{\mu}} = x^\nu$ if $1 \le \boldsymbol{\mu} = \nu \le 3$ and $q^{\boldsymbol{\mu}} = A^{\mathcal{I}}_{\boldsymbol{\mu}-n} = A^{\mathcal{I}}_\mu$ if $1 \le \boldsymbol{\mu} - 3 = \mu \le 3$. Where the bold index $1 \le \boldsymbol{\mu} \le 6$ is a multi-index meaning that $\mathcal{Z}^{\boldsymbol{\mu}}_\nu = \delta^{\boldsymbol{\mu}}_\nu$ for $1 \le \boldsymbol{\mu} \le n$ and $\mathcal{Z}^{\boldsymbol{\mu}}_\nu = \mathcal{Z}^{\mathcal{I}}_{\nu\mu}$ for the other case. Then,

$$\mathcal{Z} = \mathcal{Z}_1\wedge\mathcal{Z}_2\wedge\mathcal{Z}_3 = \sum_{\boldsymbol{\mu}_1<\boldsymbol{\mu}_2<\boldsymbol{\mu}_3} \boldsymbol{\mathcal{Z}}^{\boldsymbol{\mu}_1\boldsymbol{\mu}_2\boldsymbol{\mu}_3}_{123}\frac{\partial}{\partial q^{\boldsymbol{\mu}_1}}\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_2}}\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_3}} = \sum_{\boldsymbol{\mu}_1<\boldsymbol{\mu}_2<\boldsymbol{\mu}_3}\begin{vmatrix} \mathcal{Z}^{\boldsymbol{\mu}_1}_1 & \mathcal{Z}^{\boldsymbol{\mu}_1}_2 & \mathcal{Z}^{\boldsymbol{\mu}_1}_3 \\ \mathcal{Z}^{\boldsymbol{\mu}_2}_1 & \mathcal{Z}^{\boldsymbol{\mu}_2}_2 & \mathcal{Z}^{\boldsymbol{\mu}_2}_3 \\ \mathcal{Z}^{\boldsymbol{\mu}_3}_1 & \mathcal{Z}^{\boldsymbol{\mu}_3}_2 & \mathcal{Z}^{\boldsymbol{\mu}_3}_3 \end{vmatrix}\frac{\partial}{\partial q^{\boldsymbol{\mu}_1}}\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_2}}\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_3}}$$

We therefore expand the expression as:

$$\mathcal{Z} = \boldsymbol{\mathcal{Z}}^{123}_{123}\partial_1\wedge\partial_2\wedge\partial_3 + \underbrace{\sum_{\boldsymbol{\mu}_3}\boldsymbol{\mathcal{Z}}^{12\boldsymbol{\mu}_3}_{123}\partial_1\wedge\partial_2\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_3}}}_{[\mathbf{I}]} + \underbrace{\sum_{\boldsymbol{\mu}_3}\boldsymbol{\mathcal{Z}}^{13\boldsymbol{\mu}_3}_{123}\partial_1\wedge\partial_3\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_3}}}_{[\mathbf{II}]} + \underbrace{\sum_{\boldsymbol{\mu}_3}\boldsymbol{\mathcal{Z}}^{23\boldsymbol{\mu}_3}_{123}\partial_2\wedge\partial_3\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_3}}}_{[\mathbf{III}]}$$

The different terms involved are written $\boldsymbol{\mathcal{Z}}^{123}_{123} = 1$

$$\boldsymbol{\mathcal{Z}}^{12\boldsymbol{\mu}_3}_{123} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathcal{Z}^{\boldsymbol{\mu}_3}_1 & \mathcal{Z}^{\boldsymbol{\mu}_3}_2 & \mathcal{Z}^{\boldsymbol{\mu}_3}_3 \end{vmatrix} = \mathcal{Z}^{\boldsymbol{\mu}_3}_3 \qquad \boldsymbol{\mathcal{Z}}^{13\boldsymbol{\mu}_3}_{123} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ \mathcal{Z}^{\boldsymbol{\mu}_3}_1 & \mathcal{Z}^{\boldsymbol{\mu}_3}_2 & \mathcal{Z}^{\boldsymbol{\mu}_3}_3 \end{vmatrix} = -\mathcal{Z}^{\boldsymbol{\mu}_3}_2$$

$$\boldsymbol{\mathcal{Z}}^{23\boldsymbol{\mu}_3}_{123} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \mathcal{Z}^{\boldsymbol{\mu}_3}_1 & \mathcal{Z}^{\boldsymbol{\mu}_3}_2 & \mathcal{Z}^{\boldsymbol{\mu}_3}_3 \end{vmatrix} = \mathcal{Z}^{\boldsymbol{\mu}_3}_1$$

We obtain:

$$[\mathbf{I}] = \mathcal{Z}^{\boldsymbol{\mu}_3}_3\partial_1\wedge\partial_2\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_3}} \qquad [\mathbf{II}] = -\mathcal{Z}^{\boldsymbol{\mu}_3}_2\partial_1\wedge\partial_3\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_3}} \qquad [\mathbf{III}] = \mathcal{Z}^{\boldsymbol{\mu}_3}_1\partial_2\wedge\partial_3\wedge\frac{\partial}{\partial q^{\boldsymbol{\mu}_3}}$$

$$\mathcal{Z} = \partial_1\wedge\partial_2\wedge\partial_3 + \mathcal{Z}^{\mathcal{I}}_{3\mu}\partial_1\wedge\partial_2\wedge\frac{\partial}{\partial A^{\mathcal{I}}_\mu} - \mathcal{Z}^{\mathcal{I}}_{2\mu}\partial_1\wedge\partial_3\wedge\frac{\partial}{\partial A^{\mathcal{I}}_\mu} + \mathcal{Z}^{\mathcal{I}}_{1\mu}\partial_2\wedge\partial_3\wedge\frac{\partial}{\partial A^{\mathcal{I}}_\mu}$$

Since $\beta_1 = dx^2\wedge dx^3$, $\beta_2 = -dx^1\wedge dx^3$ and $\beta_3 = dx^1\wedge dx^2$, we have:

$$\begin{aligned}
\langle p, v\rangle &= \theta_p^{\mathsf{DW}}(\mathcal{Z}) = \langle p, v\rangle = \mathfrak{e}\beta(\mathcal{Z}) + p_{\mathcal{J}}^{A_\rho\nu}dA^{\mathcal{J}}_\rho\wedge\beta_\nu(\mathcal{Z}) \\
&= \mathfrak{e}\beta(\mathcal{Z}) + p_{\mathcal{J}}^{A_\rho 1}dA^{\mathcal{J}}_\rho\wedge\beta_1(\mathcal{Z}) + p_{\mathcal{J}}^{A_\rho 2}dA^{\mathcal{J}}_\rho\wedge\beta_2(\mathcal{Z}) + p_{\mathcal{J}}^{A_\rho 3}dA^{\mathcal{J}}_\rho\wedge\beta_3(\mathcal{Z}) \\
&= \mathfrak{e}\beta(\partial_1\wedge\partial_2\wedge\partial_3) + p_{\mathcal{J}}^{A_\rho 1}dA^{\mathcal{J}}_\rho\wedge\beta_1(\mathcal{Z}^{\mathcal{I}}_{1\mu}\partial_2\wedge\partial_3\wedge\frac{\partial}{\partial A^{\mathcal{I}}_\mu}) + p_{\mathcal{J}}^{A_\rho 2}dA^{\mathcal{J}}_\rho\wedge\beta_2(-\mathcal{Z}^{\mathcal{I}}_{2\mu}\partial_1\wedge\partial_3\wedge\frac{\partial}{\partial A^{\mathcal{I}}_\mu}) \\
&\quad + p_{\mathcal{J}}^{A_\rho 3}dA^{\mathcal{J}}_\rho\wedge\beta_3(\mathcal{Z}^{\mathcal{I}}_{3\mu}\partial_1\wedge\partial_2\wedge\frac{\partial}{\partial A^{\mathcal{I}}_\mu}) \\
&= \mathfrak{e} + p_{\mathcal{I}}^{A_\mu 1}\mathcal{Z}^{\mathcal{I}}_{1\mu} + p_{\mathcal{I}}^{A_\mu 2}\mathcal{Z}^{\mathcal{I}}_{2\mu} + p_{\mathcal{I}}^{A_\mu 3}\mathcal{Z}^{\mathcal{I}}_{3\mu} \quad ⌟
\end{aligned}$$

We express the Legendre correspondence thanks to the expression $\langle p,v\rangle = \mathfrak{e} + p_{\mathcal{J}}^{A_\rho\nu}\mathcal{Z}_{\nu\rho}^{\mathcal{J}}$, with $\mathcal{Z}_{\nu\rho}^{\mathcal{J}} = \big[\partial_\nu A_\rho^{\mathcal{J}} + \mathfrak{c}_{\mathcal{P}\mathcal{Q}}^{\mathcal{J}} A_\nu^{\mathcal{P}} A_\rho^{\mathcal{Q}}\big]$. Hence:

$$\frac{\partial}{\partial(\partial_\nu A_\rho^{\mathcal{J}})}\langle p,v\rangle = \frac{\partial}{\partial(\partial_\nu A_\rho^{\mathcal{J}})}\big(\mathfrak{e} + p_{\mathcal{J}}^{A_\rho\nu}\mathcal{Z}_{\nu\rho}^{\mathcal{J}}\big) = p_{\mathcal{J}}^{A_\rho\nu}$$

On the other side we have:

$$\frac{\partial L(q,v)}{\partial(\partial_\nu A_\rho^{\mathcal{J}})} = \frac{\partial}{\partial(\partial_\nu A_\rho^{\mathcal{J}})}\Big[\frac{1}{2}\mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho}\big(A_\mu^{\mathcal{I}}\partial_\nu A_\rho^{\mathcal{J}} + \frac{1}{3}A_\mu^{\mathcal{I}}\Big[A_\nu, A_\rho\Big]^{\mathcal{J}}\big)\Big] = -\frac{1}{2}\mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho}A_\mu^{\mathcal{I}}$$

We obtain the expression of the multimomenta (113):

$$p_{\mathcal{J}}^{A_\rho\nu} = \frac{1}{2}\mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho}A_\mu^{\mathcal{I}} \tag{278}$$

The equivalence (276) is now written by (279):

$$(q,v)\leftrightarrow(q,p) \qquad \Longleftrightarrow \qquad p_{\mathcal{J}}^{A_\rho\nu} = \frac{1}{2}\mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho}A_\mu^{\mathcal{I}} \tag{279}$$

Then, the Legendre transformation is highly degenerate. We cannot exhibit a unique correspondence between v, the multivelocities and the multimomenta p. More precisely, given a v the equation (279) has a solution $p \in \mathcal{M}_{\mathsf{DW}}$ if and only if $p \in \mathcal{M}_{\mathsf{deg}}$ with:

$$\mathcal{M}_{\mathsf{deg}} = \{(x, A_\mu^{\mathcal{I}}), \mathfrak{e}\beta + \frac{1}{2}\mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho}A_\mu^{\mathcal{I}}dA_\mu^{\mathcal{I}}\wedge\beta_\nu \ \ / \ \ (x,A)\in\mathfrak{g}\otimes T^\star\mathcal{X}, \mathfrak{e}\in\mathbb{R}\} \subset \mathcal{M}_{\mathsf{DW}} \tag{280}$$

Let notice that $\mathcal{M}_{\mathsf{deg}} \subset \mathcal{M}_{\mathsf{DW}}$ is a vector sub-bundle of $\mathcal{M}_{\mathsf{DW}}$. The degenerate feature leads to the following observation: $p_{\mathcal{J}}^{A_\rho\nu} = -p_{\mathcal{J}}^{A_\nu\rho}$. Then Legendre transform works only provided the compatibility conditions: $p_{\mathcal{J}}^{A_\rho\nu} + p_{\mathcal{J}}^{A_\nu\rho} = 0$. It is an again example of a **Dirac primary constraint set**. Therefore, we work on

$$\mathcal{M}_{\mathsf{CS}} = \{(x, A_\mu^{\mathcal{I}}, p_{\mathcal{J}}^{A_\rho\nu}) \in \mathcal{M}_{\mathsf{DW}} \ \ / \ \ p_{\mathcal{J}}^{A_\rho\nu} + p_{\mathcal{J}}^{A_\nu\rho} = 0\} \subset \mathcal{M}_{\mathsf{DW}} \tag{281}$$

15.3 De Donder-Weyl multisymplectic equations

The next step in our journey is then to consider the Hamiltonian function. Since our aim in this section is to prove that we recover the Chern-Simon feature on the degenerate space $\mathcal{M}_{\mathsf{deg}}$, we consider:

$$\begin{aligned}
\mathcal{H}^{\mathsf{deg}}(q,p) &= \langle p,v\rangle - L(q,v)\\
&= \mathfrak{e} + \frac{1}{2}\mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho}A_\mu^{\mathcal{I}}\big[\partial_\nu A_\rho^{\mathcal{J}} + \mathfrak{c}_{\mathcal{P}\mathcal{Q}}^{\mathcal{J}}A_\nu^{\mathcal{P}}A_\rho^{\mathcal{Q}}\big] - \frac{1}{2}\mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho}\{A_\mu^{\mathcal{I}}\partial_\nu A_\rho^{\mathcal{J}} + \frac{1}{3}A_\mu^{\mathcal{I}}\Big[A_\nu, A_\rho\Big]^{\mathcal{J}}\}\\
&= \mathfrak{e} + \frac{1}{2}\mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho}\mathfrak{c}_{\mathcal{P}\mathcal{Q}}^{\mathcal{J}}A_\mu^{\mathcal{I}}A_\nu^{\mathcal{P}}A_\rho^{\mathcal{Q}} - \frac{1}{2}\mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho}\frac{1}{3}\mathfrak{c}_{\mathcal{P}\mathcal{Q}}^{\mathcal{J}}A_\mu^{\mathcal{I}}A_\nu^{\mathcal{P}}A_\rho^{\mathcal{Q}}\\
&= \mathfrak{e} + \frac{1}{3}\mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho}\mathfrak{c}_{\mathcal{P}\mathcal{Q}}^{\mathcal{J}}A_\mu^{\mathcal{I}}A_\nu^{\mathcal{P}}A_\rho^{\mathcal{Q}}
\end{aligned}$$

We have also

$$d\mathcal{H}^{\mathsf{deg}}(q,p) = d\mathfrak{e} + \frac{1}{2}\mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho}\mathfrak{c}_{\mathcal{P}\mathcal{Q}}^{\mathcal{J}}A_\nu^{\mathcal{P}}A_\rho^{\mathcal{Q}}dA_\mu^{\mathcal{I}}$$

⌜Proof We make the explicit calculation:

$$d\mathcal{H}^{\mathsf{deg}}(q,p) = d\mathfrak{e} + \underbrace{d\Big[\frac{1}{3}\mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho}\mathfrak{c}_{\mathcal{P}\mathcal{Q}}^{\mathcal{J}}A_\mu^{\mathcal{I}}A_\nu^{\mathcal{P}}A_\rho^{\mathcal{Q}}\Big]}_{[\circ]} \tag{282}$$

Let explicite the term [∘] in (282) Then,

$$\begin{aligned}
[\circ] \;=\; & = \frac{1}{3}d\Big[\mathrm{tr}_{\mathcal{IJ}}\big[\epsilon^{123}\mathfrak{c}^{\mathcal{J}}_{\mathcal{PQ}}A^{\mathcal{I}}_1A^{\mathcal{P}}_2A^{\mathcal{Q}}_3 + \epsilon^{132}\mathfrak{c}^{\mathcal{J}}_{\mathcal{PQ}}A^{\mathcal{I}}_1A^{\mathcal{P}}_3A^{\mathcal{Q}}_2 + \epsilon^{213}\mathfrak{c}^{\mathcal{J}}_{\mathcal{PQ}}A^{\mathcal{I}}_2A^{\mathcal{P}}_1A^{\mathcal{Q}}_3 \\
& +\epsilon^{231}\mathfrak{c}^{\mathcal{J}}_{\mathcal{PQ}}A^{\mathcal{I}}_2A^{\mathcal{P}}_3A^{\mathcal{Q}}_1 + \epsilon^{312}\mathfrak{c}^{\mathcal{J}}_{\mathcal{PQ}}A^{\mathcal{I}}_3A^{\mathcal{P}}_1A^{\mathcal{Q}}_2 + \epsilon^{321}\mathfrak{c}^{\mathcal{J}}_{\mathcal{PQ}}A^{\mathcal{I}}_3A^{\mathcal{P}}_2A^{\mathcal{Q}}_1\big]\Big] \\
= \; & \frac{1}{3}\mathrm{tr}_{\mathcal{IJ}}\mathfrak{c}^{\mathcal{J}}_{\mathcal{PQ}}\Big[\big[dA^{\mathcal{I}}_1A^{\mathcal{P}}_2A^{\mathcal{Q}}_3 + A^{\mathcal{I}}_1dA^{\mathcal{P}}_2A^{\mathcal{Q}}_3 + A^{\mathcal{I}}_1A^{\mathcal{P}}_2dA^{\mathcal{Q}}_3\big] - \big[dA^{\mathcal{I}}_1A^{\mathcal{P}}_3A^{\mathcal{Q}}_2 + A^{\mathcal{I}}_1dA^{\mathcal{P}}_3A^{\mathcal{Q}}_2 + A^{\mathcal{I}}_1A^{\mathcal{P}}_3dA^{\mathcal{Q}}_2\big] \\
& -\big[dA^{\mathcal{I}}_2A^{\mathcal{P}}_1A^{\mathcal{Q}}_3 + A^{\mathcal{I}}_2dA^{\mathcal{P}}_1A^{\mathcal{Q}}_3 + A^{\mathcal{I}}_2A^{\mathcal{P}}_1dA^{\mathcal{Q}}_3\big] + \big[dA^{\mathcal{I}}_2A^{\mathcal{P}}_3A^{\mathcal{Q}}_1 + A^{\mathcal{I}}_2dA^{\mathcal{P}}_3A^{\mathcal{Q}}_1 + A^{\mathcal{I}}_2A^{\mathcal{P}}_3dA^{\mathcal{Q}}_1\big] \\
& +\big[dA^{\mathcal{I}}_3A^{\mathcal{P}}_1A^{\mathcal{Q}}_2 + A^{\mathcal{I}}_3dA^{\mathcal{P}}_1A^{\mathcal{Q}}_2 + A^{\mathcal{I}}_3A^{\mathcal{P}}_1dA^{\mathcal{Q}}_2\big] - \big[dA^{\mathcal{I}}_3A^{\mathcal{P}}_2A^{\mathcal{Q}}_1 + A^{\mathcal{I}}_3dA^{\mathcal{P}}_2A^{\mathcal{Q}}_1 + A^{\mathcal{I}}_3A^{\mathcal{P}}_2dA^{\mathcal{Q}}_1\big]\Big] \\
= \; & \frac{2}{3}\mathrm{tr}_{\mathcal{IJ}}\mathfrak{c}^{\mathcal{J}}_{\mathcal{PQ}}\Big[\big[A^{\mathcal{P}}_2A^{\mathcal{Q}}_3dA^{\mathcal{I}}_1 + A^{\mathcal{P}}_3A^{\mathcal{Q}}_1dA^{\mathcal{I}}_2 + A^{\mathcal{P}}_1A^{\mathcal{Q}}_2dA^{\mathcal{I}}_3\big] + \big[A^{\mathcal{I}}_3A^{\mathcal{Q}}_2dA^{\mathcal{P}}_1 + A^{\mathcal{I}}_1A^{\mathcal{Q}}_3dA^{\mathcal{P}}_2 + A^{\mathcal{I}}_2A^{\mathcal{Q}}_1dA^{\mathcal{P}}_3\big] \\
& +\big[A^{\mathcal{I}}_1A^{\mathcal{P}}_2dA^{\mathcal{Q}}_3 + A^{\mathcal{I}}_2A^{\mathcal{P}}_3dA^{\mathcal{Q}}_1 + A^{\mathcal{I}}_3A^{\mathcal{P}}_1dA^{\mathcal{Q}}_2\big]\Big] \\
= \; & \frac{2}{3}\mathrm{tr}_{\mathcal{IJ}}\mathfrak{c}^{\mathcal{J}}_{\mathcal{PQ}}\Big[\big[A^{\mathcal{P}}_2A^{\mathcal{Q}}_3dA^{\mathcal{I}}_1 + A^{\mathcal{I}}_3A^{\mathcal{Q}}_2dA^{\mathcal{P}}_1 + A^{\mathcal{I}}_2A^{\mathcal{P}}_3dA^{\mathcal{Q}}_1\big] + \big[A^{\mathcal{P}}_3A^{\mathcal{Q}}_1dA^{\mathcal{I}}_2 + A^{\mathcal{I}}_1A^{\mathcal{Q}}_3dA^{\mathcal{P}}_2 + A^{\mathcal{I}}_3A^{\mathcal{P}}_1dA^{\mathcal{Q}}_2\big] \\
& +\big[A^{\mathcal{P}}_1A^{\mathcal{Q}}_2dA^{\mathcal{I}}_3 + A^{\mathcal{I}}_2A^{\mathcal{Q}}_1dA^{\mathcal{P}}_3 + A^{\mathcal{I}}_1A^{\mathcal{P}}_2dA^{\mathcal{Q}}_3\big]\Big] \\
= \; & \frac{2}{3}\Big[\mathrm{tr}_{\mathcal{IJ}}\mathfrak{c}^{\mathcal{J}}_{\mathcal{PQ}}A^{\mathcal{P}}_2A^{\mathcal{Q}}_3dA^{\mathcal{I}}_1 + \mathrm{tr}_{\mathcal{PJ}}\mathfrak{c}^{\mathcal{J}}_{\mathcal{IQ}}A^{\mathcal{P}}_3A^{\mathcal{Q}}_2dA^{\mathcal{I}}_1 + \mathrm{tr}_{\mathcal{QJ}}\mathfrak{c}^{\mathcal{J}}_{\mathcal{PI}}A^{\mathcal{Q}}_2A^{\mathcal{P}}_3dA^{\mathcal{I}}_1 + \mathrm{tr}_{\mathcal{IJ}}\mathfrak{c}^{\mathcal{J}}_{\mathcal{PQ}}A^{\mathcal{P}}_3A^{\mathcal{Q}}_1dA^{\mathcal{I}}_2 + \mathrm{tr}_{\mathcal{PJ}}\mathfrak{c}^{\mathcal{J}}_{\mathcal{IQ}}A^{\mathcal{P}}_1A^{\mathcal{Q}}_3dA^{\mathcal{I}}_2 \\
& +\mathrm{tr}_{\mathcal{QJ}}\mathfrak{c}^{\mathcal{J}}_{\mathcal{PI}}A^{\mathcal{Q}}_3A^{\mathcal{P}}_1dA^{\mathcal{I}}_2 + \big[\mathrm{tr}_{\mathcal{IJ}}\mathfrak{c}^{\mathcal{J}}_{\mathcal{PQ}}A^{\mathcal{P}}_1A^{\mathcal{Q}}_2dA^{\mathcal{I}}_3 + \mathrm{tr}_{\mathcal{PJ}}\mathfrak{c}^{\mathcal{J}}_{\mathcal{IQ}}A^{\mathcal{P}}_2A^{\mathcal{Q}}_1dA^{\mathcal{I}}_3 + \mathrm{tr}_{\mathcal{QJ}}\mathfrak{c}^{\mathcal{J}}_{\mathcal{PI}}A^{\mathcal{Q}}_1A^{\mathcal{P}}_2dA^{\mathcal{I}}_3\big] \\
= \; & \frac{2}{3}\Big[\big[\mathrm{tr}_{\mathcal{IJ}}\mathfrak{c}^{\mathcal{J}}_{\mathcal{PQ}} + \mathrm{tr}_{\mathcal{QJ}}\mathfrak{c}^{\mathcal{J}}_{\mathcal{IP}} + \mathrm{tr}_{\mathcal{PJ}}\mathfrak{c}^{\mathcal{J}}_{\mathcal{QI}}\big]A^{\mathcal{P}}_2A^{\mathcal{Q}}_3dA^{\mathcal{I}}_1 + \big[\mathrm{tr}_{\mathcal{IJ}}\mathfrak{c}^{\mathcal{J}}_{\mathcal{PQ}} + \mathrm{tr}_{\mathcal{PJ}}\mathfrak{c}^{\mathcal{J}}_{\mathcal{IQ}} + \mathrm{tr}_{\mathcal{QJ}}\mathfrak{c}^{\mathcal{J}}_{\mathcal{PI}}\big]A^{\mathcal{P}}_3A^{\mathcal{Q}}_1dA^{\mathcal{I}}_2 \\
& +\big[\mathrm{tr}_{\mathcal{IJ}}\mathfrak{c}^{\mathcal{J}}_{\mathcal{PQ}} + \mathrm{tr}_{\mathcal{QJ}}\mathfrak{c}^{\mathcal{J}}_{\mathcal{IP}} + \mathrm{tr}_{\mathcal{PJ}}\mathfrak{c}^{\mathcal{J}}_{\mathcal{QI}}\big]A^{\mathcal{P}}_1A^{\mathcal{Q}}_2dA^{\mathcal{I}}_3\Big]
\end{aligned}$$

In fact, $\mathrm{tr}_{\mathcal{IJ}}$ is invariant under the adjoint action of the Lie algebra $\mathfrak{g}$. It means that we obtain $\mathrm{tr}_{\mathcal{IJ}}[\xi,\chi]^{\mathcal{I}}\zeta^{\mathcal{J}} + \mathrm{tr}_{\mathcal{IJ}}\chi^{\mathcal{I}}[\xi,\zeta]^{\mathcal{J}} = 0$ for any elements $\xi^{\mathcal{I}},\chi^{\mathcal{I}},\zeta^{\mathcal{I}} \in \mathfrak{g}$. We shall define $\mathfrak{c}_{\mathcal{IJK}} = \mathrm{tr}_{\mathcal{IL}}\mathfrak{c}^{\mathcal{L}}_{\mathcal{JK}}$. The invariance property under adjoint action leads to:

$$\mathfrak{c}_{\mathcal{IJK}} = \mathfrak{c}_{[\mathcal{IJK}]} = (1/3!)\sum_{\sigma}(-1)^{\boldsymbol{\sigma}}\mathfrak{c}_{\sigma(\mathcal{I})\sigma(\mathcal{J})\sigma(\mathcal{K})} = (1/3)(\mathfrak{c}_{\mathcal{IJK}} + \mathfrak{c}_{\mathcal{JKI}} + \mathfrak{c}_{\mathcal{KIJ}})$$

$$\begin{aligned}
[\circ] \;=\; & \frac{2}{3}\Big[\big[\mathfrak{c}_{\mathcal{IPQ}} + \mathfrak{c}_{\mathcal{QIP}} + \mathfrak{c}_{\mathcal{PQI}}\big]A^{\mathcal{P}}_2A^{\mathcal{Q}}_3dA^{\mathcal{I}}_1 + \big[\mathfrak{c}_{\mathcal{IPQ}} + \mathfrak{c}_{\mathcal{PIQ}} + \mathfrak{c}_{\mathcal{QPI}}\big]A^{\mathcal{P}}_3A^{\mathcal{Q}}_1dA^{\mathcal{I}}_2 \\
& +\big[\mathfrak{c}_{\mathcal{IPQ}} + \mathfrak{c}_{\mathcal{QIP}} + \mathfrak{c}_{\mathcal{PQI}}\big]A^{\mathcal{P}}_1A^{\mathcal{Q}}_2dA^{\mathcal{I}}_3\Big] \\
= \; & \frac{2}{3}\Big[\big[3\mathfrak{c}_{\mathcal{IPQ}}\big]A^{\mathcal{P}}_2A^{\mathcal{Q}}_3dA^{\mathcal{I}}_1 + \big[3\mathfrak{c}_{\mathcal{IPQ}}\big]A^{\mathcal{P}}_3A^{\mathcal{Q}}_1dA^{\mathcal{I}}_2 + \big[3\mathfrak{c}_{\mathcal{IPQ}}\big]A^{\mathcal{P}}_1A^{\mathcal{Q}}_2dA^{\mathcal{I}}_3\Big] \\
= \; & 2\mathfrak{c}_{\mathcal{IPQ}}\Big[A^{\mathcal{P}}_2A^{\mathcal{Q}}_3dA^{\mathcal{I}}_1 + A^{\mathcal{P}}_3A^{\mathcal{Q}}_1dA^{\mathcal{I}}_2 + A^{\mathcal{P}}_1A^{\mathcal{Q}}_2dA^{\mathcal{I}}_3\Big]
\end{aligned}$$

On the other side:

$$[\circ\circ] = \frac{1}{2}\mathrm{tr}_{\mathcal{IJ}}\epsilon^{\mu\nu\rho}\mathfrak{c}^{\mathcal{J}}_{\mathcal{PQ}}A^{\mathcal{P}}_{\nu}A^{\mathcal{Q}}_{\rho}dA^{\mathcal{I}}_{\mu} = \frac{1}{2}\epsilon^{\mu\nu\rho}\mathfrak{c}_{\mathcal{IPQ}}A^{\mathcal{P}}_{\nu}A^{\mathcal{Q}}_{\rho}dA^{\mathcal{I}}_{\mu} \quad \lrcorner$$

Now we are intersted in the multisymplectic form:

$$\boldsymbol{\omega}^{\mathrm{deg}} = d\mathfrak{e}\wedge\beta - d\big[\frac{1}{2}\mathrm{tr}_{\mathcal{IJ}}\epsilon^{\rho\nu\mu}A^{\mathcal{J}}_{\rho}\big]\wedge dA^{\mathcal{I}}_{\mu}\wedge\beta_{\nu} = d\mathfrak{e}\wedge\beta - \frac{1}{2}\mathrm{tr}_{\mathcal{IJ}}\epsilon^{\mu\nu\rho}dA^{\mathcal{J}}_{\rho}\wedge dA^{\mathcal{I}}_{\mu}\wedge\beta_{\nu}$$

Therefore, to precise better our notation we can write X_1, X_2 and X_3:

$$X_1 = \partial_1 + \Theta^I_{1\mu_1}\frac{\partial}{\partial A^{\mathcal{I}}_{\mu_1}} + \Upsilon_1\frac{\partial}{\partial\mathfrak{e}} \qquad X_2 = \partial_2 + \Theta^I_{2\mu_2}\frac{\partial}{\partial A^{\mathcal{I}}_{\mu_2}} + \Upsilon_2\frac{\partial}{\partial\mathfrak{e}} \quad \text{and} \quad X_3 = \partial_3 + \Theta^I_{3\mu_3}\frac{\partial}{\partial A^{\mathcal{I}}_{\mu_3}} + \Upsilon_3\frac{\partial}{\partial\mathfrak{e}}$$

We obtain, after a straightforward calculation (denoting $\frac{\partial}{\partial x^\mu}$ by ∂_μ) the following expression for $X_1\wedge X_2\wedge X_3$:

$$\begin{aligned}
X_1\wedge X_2\wedge X_3 = \;& \partial_1\wedge\partial_2\wedge\partial_3 + \partial_1\wedge\partial_2\wedge\Theta^I_{3\mu_3}\frac{\partial}{\partial A^{\mathcal{I}}_{\mu_3}} + \partial_1\wedge\Theta^I_{2\mu_2}\frac{\partial}{\partial A^{\mathcal{I}}_{\mu_2}}\wedge\partial_3 + \partial_1\wedge\Theta^I_{2\mu_2}\frac{\partial}{\partial A^{\mathcal{I}}_{\mu_2}}\wedge\Theta^{IJ}_{3\mu_3}\frac{\partial}{\partial A^{\mathcal{I}}_{\mu_3}} \\
& +\Theta^I_{1\mu_1}\frac{\partial}{\partial A^{\mathcal{I}}_{\mu_1}}\wedge\partial_2\wedge\partial_3 + \Theta^I_{1\mu_1}\frac{\partial}{\partial A^{\mathcal{I}}_{\mu_1}}\wedge\partial_2\wedge\Theta^{IJ}_{3\mu_3}\frac{\partial}{\partial A^{\mathcal{I}}_{\mu_3}} + \Theta^I_{1\mu_1}\frac{\partial}{\partial A^{\mathcal{I}}_{\mu_1}}\wedge\Theta^{IJ}_{2\mu_2}\frac{\partial}{\partial A^{\mathcal{I}}_{\mu_2}}\wedge\partial_3 \\
& +\Theta^I_{1\mu_1}\frac{\partial}{\partial A^{\mathcal{I}}_{\mu_1}}\wedge\Theta^{IJ}_{2\mu_2}\frac{\partial}{\partial A^{\mathcal{I}}_{\mu_2}}\wedge\Theta^{IJ}_{3\mu_3}\frac{\partial}{\partial A^{\mathcal{I}}_{\mu_3}} + \;\text{multivectors involving}\;\; \partial/\partial\mathfrak{e}
\end{aligned}$$

Now we can write the multisymplectic Hamilton equations:

$$\begin{aligned} X \lrcorner \boldsymbol{\omega}^{\deg} &= X \lrcorner \big(d\mathfrak{e} \wedge \beta - \frac{1}{2}\mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho} dA^{\mathcal{J}}_{\rho} \wedge dA^{\mathcal{I}}_{\mu} \wedge \beta_{\nu}\big) \\ &= \beta(X) d\mathfrak{e} - (d\mathfrak{e} \wedge \beta_{\rho})(X) dx^{\rho} - \mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho} X \lrcorner dA^{\mathcal{J}}_{\rho} \wedge dA^{\mathcal{I}}_{\mu} \wedge \beta_{\nu} \\ &= d\mathfrak{e} - \Upsilon_{\rho} dx^{\rho} - \mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho}\frac{1}{2}\big((dA^{\mathcal{I}}_{\mu} \wedge \beta_{\nu})(X) dA^{\mathcal{J}}_{\rho} \\ &\quad -(dA^{\mathcal{J}}_{\rho} \wedge \beta_{\nu})(X) dA^{\mathcal{I}}_{\mu} + (dA^{\mathcal{J}}_{\rho} \wedge dA^{\mathcal{I}}_{\mu} \wedge \beta_{\lambda\nu})(X) dx^{\lambda}\big) \end{aligned}$$

Then, we obtain:

$$X \lrcorner \boldsymbol{\omega}^{\deg} = d\mathfrak{e} - \Upsilon_{\rho} dx^{\rho} - \mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho}\big(\Theta^{\mathcal{I}}_{\nu\mu} dA^{\mathcal{J}}_{\rho} + \Theta^{\mathcal{J}}_{\nu\rho} dA^{\mathcal{I}}_{\mu} + \big[\Theta^{\mathcal{I}}_{\nu\mu}\Theta^{\mathcal{J}}_{\lambda\rho} - \Theta^{\mathcal{I}}_{\lambda\mu}\Theta^{\mathcal{J}}_{\nu\rho}\big] dx^{\lambda}\big)$$

So that the decomposition on the term involving dA gives:

$$-\mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho}\big(\Theta^{\mathcal{I}}_{\nu\mu} dA^{\mathcal{J}}_{\rho} + \Theta^{\mathcal{J}}_{\nu\rho} dA^{\mathcal{I}}_{\mu}\big) = -\mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho}\mathfrak{c}^{\mathcal{J}}_{\mathcal{P}\mathcal{Q}} A^{\mathcal{P}}_{\nu} A^{\mathcal{Q}}_{\rho} dA^{\mathcal{I}}_{\mu}$$

$$\big(\mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho}(\Theta^{\mathcal{J}}_{\nu\rho} - \Theta^{\mathcal{J}}_{\rho\nu})\big)^{\mu}_{\mathcal{I}} dA^{\mathcal{I}}_{\mu} = -\big(\mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho}[A_{\nu}, A_{\rho}]^{\mathcal{J}}\big) dA^{\mathcal{I}}_{\mu}$$

Then we obtain Hamilton equations:

$$\mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho}\big(\partial_{\nu} A^{\mathcal{J}}_{\rho} - \partial_{\rho} A^{\mathcal{J}}_{\nu} + [A_{\nu}, A_{\rho}]^{\mathcal{J}}\big) = \mathrm{tr}_{\mathcal{I}\mathcal{J}}\epsilon^{\mu\nu\rho} F^{\mathcal{J}}_{\nu\rho} = 0$$

Since, tr is non degenerate the solutions give a theory of flat connection.

$$\epsilon^{\mu\nu\rho} F^{\mathcal{J}}_{\nu\rho} = 0 \tag{283}$$

16 Multisymplectic Palatini-Hamilton equations

We treat the Palatini-De Donder-Weyl multisymplectic theory. We introduce the multimomenta, respectively related to the tetrad field (284)(i) and to the spin connection (284)(ii):

$$(\mathfrak{i}) \quad p^{e_{\mu}\nu}_{I} = 0 \qquad\qquad (\mathfrak{ii}) \quad p^{\omega_{\mu}\nu}_{IJ} = -e e^{[\mu}_{I} e^{\nu]}_{J} \tag{284}$$

We can write $p^{\omega_{\mu}\nu}_{IJ}$ previously described by (284)(ii) under the following form (285):

$$p^{\omega_{\mu}\nu}_{IJ} = -e e^{[\mu}_{I} e^{\nu]}_{J} = e(e^{\nu}_{I} e^{\mu}_{J} - e^{\mu}_{I} e^{\nu}_{J}) = E^{\nu}_{I} e^{\mu}_{J} - E^{\mu}_{I} e^{\nu}_{J} = -E^{[\mu}_{I} e^{\nu]}_{J} \tag{285}$$

Equivalently we have $p^{\omega_{\mu}\nu}_{IJ} = -e e^{[\mu}_{I} e^{\nu]}_{J} = -1/2\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^{K}_{\rho} e^{L}_{\sigma}$. Let us notice that the multisymplectic treatment of Einstein-Palatini treatment is *rather rare* in the literature. We refer to some work, for example see the work of Rovelli [197] which gives an introduction. We note a covariant treatment for Ashtekar constraints in the work of Esposito and *al,* [72], whereas the work of D. Bruno, R. Cianci and S. Vignolo [31, 32] discusses a more detailed treatment, in the setting of *natural bundle theory* and multisymplectic jet formalism. However, we insist on the observable treatment for Palatini gravity, and on the necessity of a Lepage-Dedecker transform for a full treatment of dynamical observable. We have in several steps. First, following the method developed in [115] we define the general geometrical construction and precise the notations.

16.1 Geometrical setting and notations

We describe the geometrical setting and the notation for the four dimensional case. We consider $\mathcal{X}$ to be the *spacetime* manifold with $dim(\mathcal{X}) = n = 4$. Let $e \in \mathbb{R}^{(1,3)} \otimes T^\star\mathcal{X}$, be the tetrad field, seen as a $\mathbb{R}^{(1,3)}$-valued 1-form - see section (12.4) and (12.5). We denote $\{\mathbf{e}_I\}_{1\leq I\leq n}$ a basis of the Minkowski vector space $\mathbb{R}^{(1,3)}$. In addition we consider the Lorentz spin connection, see section (12.5), as a $\mathfrak{so}(1,3)$-valued Lie algebra 1-form: $\omega \in \mathfrak{so}(1,3) \otimes T^\star\mathcal{X}$ and $\{\mathfrak{b}_{IJ}\}_{1\leq I,J\leq n}$ a basis of the $\mathfrak{so}(1,3)$ lie algebra. We described in section (4.3) variational problems on maps $\sigma : \mathcal{X} \to \mathfrak{Z}$. Here we are concerned with the setting of gauge type theories. Hence, the space of interested is not $\mathfrak{Z}^\circ = \mathcal{X} \times \mathfrak{Z}$. It is rather $\mathfrak{Z} = \mathfrak{iso}(1,3) \otimes T^\star\mathcal{X}$. As noticed in [116, 117], the more naive approach is to work in a local trivialization of a bundle over $\mathcal{X}$, since a connection is not a section of a bundle. This is the chosen path here. A point (x, e, ω) in $\mathfrak{Z}$ is in the *position* configuration space. Any choice (e, ω) is equivalent to the data of an n-dimensional submanifold in $\mathfrak{Z}$ described as a section of the fiber bundle over $\mathcal{X}$.

$$\begin{array}{c} \mathfrak{Z} = \mathfrak{iso}(1,3) \otimes T^\star\mathcal{X} \\ \big\downarrow \pi \\ \mathcal{X} \end{array}$$

Let us consider the following maps, (286), $e : \mathcal{X} \to \mathfrak{Z}^e = \mathbb{R}^{(1,3)} \otimes T^\star\mathcal{X}$ and $\omega : \mathcal{X} \mapsto \mathfrak{Z}^\omega = \mathfrak{so}(1,3) \otimes T^\star\mathcal{X}$.

$$\mathfrak{z}_e : \left\{ \begin{array}{ccc} \mathcal{X} & \to & \mathfrak{Z}^e = \mathbb{R}^{(1,3)} \otimes T^\star\mathcal{X} \\ x & \mapsto & e(x) = e^I_\mu(x) dx^\mu \otimes \mathbf{e}_I \end{array} \right. \quad \text{and} \quad \mathfrak{z}_\omega : \left\{ \begin{array}{ccc} \mathcal{X} & \to & \mathfrak{Z}^\omega = \mathfrak{so}(1,3) \otimes T^\star\mathcal{X} \\ x & \to & \omega(x) = \omega^{IJ}_\mu(x) dx^\mu \otimes \mathfrak{b}_{IJ} \end{array} \right. \tag{286}$$

Also we introduce the maps $\mathfrak{z}_e$ and $\mathfrak{z}_\omega$ which are simply some section of the related bundle. We associate with e, the bundle $\mathcal{P}^e = e^\star T\mathfrak{Z}^e \otimes_{\mathfrak{Z}^e} T^\star\mathcal{X}$. On the other hand, we consider the bundle $\mathcal{P}^\omega = \omega^\star T\mathfrak{Z}^\omega \otimes_{\mathfrak{Z}^\omega} T^\star\mathcal{X}$ or $\mathcal{P} = (e,\omega)^\star T\mathfrak{Z} \otimes_{\mathfrak{Z}^\omega} T^\star\mathcal{X}$. The useful quantities to describe de and $d\omega$ the differential of the map e and ω as sections of the bundle $\mathcal{P}^e$ and $\mathcal{P}^\omega$ over $\mathcal{X}$. For vector-valued or Lie algebra valued n-forms see appendix (B). We denote the exterior covariant derivative on vector valued 1-form e by $d^{\mathbf{D}}e$ and the exterior covariant derivative on $\mathfrak{so}(1,3)$-valued 1-form ω by $d^{\mathbf{D}}\omega$:

$$\begin{array}{rcl} d^{\mathbf{D}}e & = & \left[d^{\mathbf{D}}e\right]^I_{\mu\nu} dx^\mu \wedge dx^\nu \otimes \mathbf{e}_I = \mathcal{D}e^I \otimes \mathbf{e}_I = \mathbf{e}_I \mathcal{D}e^I \\ d^{\mathbf{D}}\omega & = & \left[d^{\mathbf{D}}\omega\right]^{IJ}_{\mu\nu} dx^\mu \wedge dx^\nu \otimes \mathfrak{b}_{IJ} = \mathcal{D}\omega^{IJ} \otimes \mathfrak{b}_{IJ} = \mathfrak{b}_{IJ}\mathcal{D}\omega^{IJ} \end{array} \tag{287}$$

with:

$$\begin{array}{rcl} \left[d^{\mathbf{D}}e\right]^I_{\mu\nu} & = & \partial_{[\mu} e^I_{\nu]} + \omega_{[\mu}{}^I{}_J e^J_{\nu]} \\ \left[d^{\mathbf{D}}\omega\right]^{IJ}_{\mu\nu} & = & \partial_{[\mu} \omega^{IJ}_{\nu]} + \omega_{[\mu}{}^I{}_K \omega^{KJ}_{\nu]} - \omega_{[\mu}{}^J{}_K \omega^{KI}_{\nu]} \end{array} \tag{288}$$

We introduce also another notation:

$$\begin{array}{rcl} {}^e v^I_{\mu\nu} & = & \partial_\mu e^I_\nu + \omega_\mu{}^I{}_J e^J_\nu \\ {}^\omega v^{IJ}_{\mu\nu} & = & \partial_\mu \omega^{IJ}_\nu + \omega_\mu{}^I{}_K \omega_\nu{}^{KJ} - \omega_\mu{}^J{}_K \omega_\nu{}^{KI} \end{array} \tag{289}$$

Then, in our notations, $d^{\mathbf{D}}e = {}^e v^I_{[\mu\nu]}$ and $d^{\mathbf{D}}\omega = {}^\omega v^{IJ}_{[\mu\nu]}$. Following the same underlying theme previously described, with $\mathfrak{Z}^e = \mathbb{R}^{(1,3)} \otimes T^\star\mathcal{X}$ and $\mathfrak{Z}^\omega = \mathfrak{so}(1,2) \otimes T^\star\mathcal{X}$ we denote now:

$$\mathfrak{Z} = \mathfrak{p} \otimes T^\star\mathcal{X} = \mathfrak{iso}(1,3) \otimes T^\star\mathcal{X} \tag{290}$$

The space of interest, the analogue for tangent space is $\Lambda^n T_{(x,e,\omega)}\mathfrak{Z}$ the fiber bundle of n-vector field of $\mathfrak{Z}$ over $\mathcal{X}$. For any $(x^\mu, e^I_\nu, \omega^{IJ}_\nu) \in \mathfrak{Z}$, the fiber $\Lambda^n T_{(x,e,\omega)}(\mathfrak{p} \otimes T^\star\mathcal{X}) = \Lambda^n T_{(x,e,\omega)}\mathfrak{Z}$ can be identified with $\mathcal{P} = (e,\omega)^\star T\mathfrak{Z} \otimes_{\mathfrak{Z}^\omega} T^\star\mathcal{X}$ via the diffeomorphism:

$$\begin{cases} \mathcal{P} \cong (e,\omega)^\star T\mathfrak{Z} \otimes_{\mathfrak{Z}^\omega} T^\star\mathcal{X} & \to \quad \Lambda^n T_{(x,e,\omega)}(\mathfrak{p} \otimes T^\star\mathcal{X}) \\ (\sum_{\mu,\nu}\sum_{I} \left[d^{\mathbf{D}}e\right]^I_{\mu\nu} dx^\mu \otimes dx^\nu \otimes \mathbf{e}_I, \sum_{\mu,\nu}\sum_{I<J} \left[d^{\mathbf{D}}\omega\right]^{IJ}_{\mu\nu} dx^\mu \otimes dx^\nu \otimes \mathfrak{b}_{IJ}) & \mapsto \quad z = z_1 \wedge ... \wedge z_n \end{cases}$$

where $\forall 1 \leq \eta \leq n$:

$$z_\eta = \frac{\partial}{\partial x^\alpha} + \sum_{1\leq\beta\leq n} {}^e v^I_{\alpha\beta} \frac{\partial}{\partial e^I_\beta} + \sum_{1\leq\beta'\leq n} {}^\omega v^{IJ}_{\alpha\beta'} \frac{\partial}{\partial \omega^{IJ}_{\beta'}}$$

In order to fix ideas we stress that we have local coordinates $(x^\mu, e^I_\mu, \omega^{IJ}_\mu)$ for the configuration bundle $\mathfrak{Z}$. The data of the local coordinates $(x^\mu, e^I_\mu, \omega^{IJ}_\mu, v^I_{\mu\nu}, w^{IJ}_{\mu\nu})$ (or equivalentely $(x^\mu, e^I_\mu, \omega^{IJ}_\mu, z^I_{\mu\nu}, z^{IJ}_{\mu\nu})$), can be thought as coordinates on $\mathcal{P}$ or $\Lambda^n T_{(x,e,\omega)}(\mathfrak{Z})$ We identify $\mathcal{P} \cong \Lambda^n T_{(x,e,\omega)}(\mathfrak{Z})$. Alternatively,we understand the set of coordinates as coordinates on the first jet bundle $J^1(\mathfrak{Z})$.

16.2 Legendre correspondence

Let $\mathcal{M} = \Lambda^n T^\star\mathfrak{Z}$. $\forall (q,p) \in \mathcal{M}$, we denote the local coordinate on the bundle $\mathcal{M}$ over $\mathfrak{Z}$ by $q^{\boldsymbol{\mu}}$ and $p_{\boldsymbol{\mu}_1 ... \boldsymbol{\mu}_n}$ completely antisymmetric in $(\boldsymbol{\mu}_1 ... \boldsymbol{\mu}_n)$ and:

$$p = \sum_{\boldsymbol{\mu}_1,...,\boldsymbol{\mu}_n} p_{\boldsymbol{\mu}_1 ... \boldsymbol{\mu}_n} dq^{\boldsymbol{\mu}_1} \wedge ... \wedge dq^{\boldsymbol{\mu}_n}$$

We adapt our notation, the De Donder-Weyl submanifold denoted as $\mathcal{M}_{\mathsf{DW}} \subset \mathcal{M}$ is:

$$\mathcal{M}_{\mathsf{DW}} = \{(x,e,\omega,p)/x \in \mathcal{X}, e \in \mathbb{R}(1,3) \otimes T^\star\mathcal{X}; \omega \in \mathfrak{so}(1,3) \otimes T^\star\mathcal{X}, p \in \Lambda^n T^\star(\mathfrak{p} \otimes T^\star\mathcal{X})\}$$

with $\partial_{e^I_\mu} \wedge \partial_{e^I_\nu} \lrcorner\, p = \partial_{\omega^{IJ}_\mu} \wedge \partial_{\omega^{IJ}_\nu} \lrcorner\, p = \partial_{e^I_\mu} \wedge \partial_{\omega^{IJ}_\nu} \lrcorner\, p = 0$. Therefore, we adapt our notations for $\mathcal{M}_{\mathsf{DW}} \subset \mathcal{M}$ and we consider all the components $p_{\mu_1...\mu_n}$ equal to zero excepted for $p_{1...n} = \mathfrak{e}$ and for the multimomenta related to e and ω written $p_{1...(\nu-1)(e^{IJ}_\mu)(\nu+1)...n} = p^{e_\mu\nu}_{IJ}$ and $p_{1...(\nu-1)(\omega^{IJ}_\mu)(\nu+1),...n} = p^{\omega_\mu\nu}_{IJ}$. We define a Legendre correspondence:

$$\begin{aligned} \Lambda^n T(\mathfrak{p} \otimes T^*\mathcal{X}) \times \mathbb{R} = \Lambda^n T\mathfrak{Z} \times \mathbb{R} \ &\leftrightarrow \Lambda^n T^\star(\mathfrak{p} \otimes T^*\mathcal{X}) = \Lambda^n T^\star\mathfrak{Z} \\ (q,v,w) \ &\leftrightarrow (q,p) \end{aligned} \tag{291}$$

As usual, the Legendre correspondence is generated by the function $W : \Lambda^n T\mathfrak{Z} \times \Lambda^n T^\star\mathfrak{Z} \to \mathbb{R}, (q,v,p) \mapsto \langle p,v\rangle - L(q,v)$.

16.3 Palatini multisymplectic manifold

We consider the following Poincaré-Cartan $\theta^{\mathsf{DW}}_{(q,p)}$ n-form:

$$\theta^{\mathsf{DW}}_{(q,p)} := \mathfrak{e}\beta + p^{e_\mu\nu}_I de^I_\mu \wedge \beta_\nu + p^{\omega_\mu\nu}_{IJ} d\omega^{IJ}_\mu \wedge \beta_\nu,$$

with $\beta = dx^1 \wedge ... \wedge dx^n$ a volume n-form on $\mathcal{X}$ and $\beta_\beta := \partial_\beta \lrcorner\, \beta$. The construction of the Legendre correspondence gives the equivalence relation between (q,v) and (q,p) by:

$$(q,v) \leftrightarrow (q,p) \qquad \Longleftrightarrow \qquad \frac{\partial\langle p,v\rangle}{\partial v} = \frac{\partial L(q,v)}{\partial v} \tag{292}$$

where $\langle p,v\rangle = \theta^{\mathsf{DW}}_{(q,p)}(\mathcal{Z})$ with $\mathcal{Z} = \mathcal{Z}_1 \wedge \mathcal{Z}_2 \wedge \mathcal{Z}_3 \wedge \mathcal{Z}_4$. We have $\forall \mu$

$$\mathcal{Z}_\mu = \frac{\partial}{\partial x^\mu} + \mathcal{Z}^I_{\mu\nu}\frac{\partial}{\partial e^I_\nu} + \mathcal{Z}^{IJ}_{\mu\nu}\frac{\partial}{\partial \omega^{IJ}_\nu} = \frac{\partial}{\partial x^\mu} + \left(\partial_\mu e^I_\nu + \omega_\mu{}^I{}_J e^J_\nu\right)\frac{\partial}{\partial e^I_\nu} + \left(\partial_\mu \omega^{IJ}_\nu + \omega_\mu{}^I{}_K \omega_\nu{}^{KJ} - \omega_\mu{}^J{}_K \omega_\nu{}^{KI}\right)\frac{\partial}{\partial \omega^{IJ}_\nu}$$

We notice that generally, we can write the multivector $\mathcal{Z}$ under the general following form:

$$\mathcal{Z} = \mathcal{Z}_1 \wedge \mathcal{Z}_2 \wedge \mathcal{Z}_3 \wedge \mathcal{Z}_4 = \sum_{\boldsymbol{\mu}_1 < \cdots < \boldsymbol{\mu}_4} \mathcal{Z}_{1\cdots 4}^{\boldsymbol{\mu}_1 \cdots \boldsymbol{\mu}_4} \frac{\partial}{\partial q^{\boldsymbol{\mu}_1}} \wedge \cdots \wedge \frac{\partial}{\partial q^{\boldsymbol{\mu}_4}} = \sum_{\boldsymbol{\mu}_1 < \cdots < \boldsymbol{\mu}_4} \begin{vmatrix} \mathcal{Z}_1^{\boldsymbol{\mu}_1} & \cdots & \mathcal{Z}_4^{\boldsymbol{\mu}_1} \\ \vdots & & \vdots \\ \mathcal{Z}_1^{\boldsymbol{\mu}_4} & \cdots & \mathcal{Z}_4^{\boldsymbol{\mu}_4} \end{vmatrix} \frac{\partial}{\partial q^{\boldsymbol{\mu}_1}} \wedge \cdots \wedge \frac{\partial}{\partial q^{\boldsymbol{\mu}_4}}$$

We now make the straightforward calculation:

$$\langle p, v\rangle = \theta_p^{\mathsf{DW}}(\mathcal{Z}) = \mathfrak{e}\beta(\mathcal{Z}) + p_I^{e_\mu\nu} de_\mu^I \wedge \beta_\nu(\mathcal{Z}) + p_{IJ}^{\omega_\mu\nu} d\omega_\mu^{IJ} \wedge \beta_\nu(\mathcal{Z}) = \mathfrak{e} + p_I^{e_\mu\nu} \mathcal{Z}_{\nu\mu}^I + p_{IJ}^{\omega_\mu\nu} \mathcal{Z}_{\nu\mu}^{IJ}$$

Let us compute the two parts involved in the Legendre correspondence.

$$\begin{array}{lllll} \dfrac{\partial\langle p,v\rangle}{\partial(\partial_\nu\omega_\mu^{IJ})} & = & p_{IJ}^{\omega_\mu\nu} \qquad \dfrac{\partial L(q,v)}{\partial(\partial_\mu\omega_\nu^{IJ})} & = & \dfrac{\partial}{\partial(\partial_\mu\omega_\nu^{IJ})}\Big(ee_I^\mu e_J^\nu(\partial_{[\mu}\omega_{\nu]}^{IJ} + \omega_{[\mu K}^I\omega_{\nu]}^{KJ})\Big) = ee_I^{[\mu}e_J^{\nu]} \\ \dfrac{\partial\langle p,v\rangle}{\partial(\partial_\nu e_\mu^I)} & = & p_I^{e_\mu\nu} \qquad \dfrac{\partial L(q,v)}{\partial(\partial_\mu e_\nu^I)} & = & \dfrac{\partial}{\partial(\partial_\mu e_\nu^I)}\Big(ee_I^\mu e_J^\nu(\partial_{[\mu}\omega_{\nu]}^{IJ} + \omega_{[\mu K}^I\omega_{\nu]}^{KJ})\Big) = 0 \end{array}$$

Therefore the Legendre transform leads us to write:

$$\begin{array}{lll} p_{IJ}^{\omega_\mu\nu} & = & -ee_I^{[\mu}e_J^{\nu]} = -E_I^{[\mu}e_J^{\nu]} \\ p_I^{e_\mu\nu} & = & 0 \end{array} \tag{293}$$

Then Legendre transform works only provided the compatibility conditions: $p_{IJ}^{\omega_\nu\mu} + p_{IJ}^{\omega_\mu\nu} = 0$ and $p_I^{e_\mu\nu} = 0$. It is an example of a Dirac primary constraint set. Therefore, we shall be restricted to the submanifold:

$$\mathcal{M}_{\mathsf{Palatini}} = \{(x,e,\omega,p) \in \mathcal{M}_{\mathsf{DW}} \ / \ p_{IJ}^{\omega_\mu\nu} + p_{IJ}^{\omega_\nu\mu} = 0, \ \ p_{IJ}^{\omega_\mu\nu} = -E_I^{[\mu}e_J^{\nu]} \ \ p_I^{e_\mu\nu} = 0\} \tag{294}$$

The Legendre transformation is strongly degenerated since from momenta one cannot invert any field derivative. Momenta $p_{IJ}^{\omega_\mu\nu}$ are functions of the spin tetrad. A regular Legendre transform (non degenerate) would directly gives the momenta $p^\star = \{p_I^{e_\mu\nu}, p_{IJ}^{\omega_\mu\nu}\}$ as function of the fields derivatives: $p^\star = p^\star(q^\mu; v_{\mu\nu}) = p^\star(x^\mu, e_\nu^I, \omega_\nu^{IJ}, \partial_\mu e_\nu^I, \partial_\mu\omega_\nu^{IJ})$. Here it is impossible to invert any momenta to give all fields derivatives $\partial_\mu e_\nu^I$ or $\partial_\mu\omega_\nu^{IJ}$. This is a general feature for gauge theory: when gauge occurs, the Legendre correspondence degenerates on $\mathcal{M}_{\mathsf{DW}}$ whereas, when it is restricted to the submanifold $\mathcal{M}_{\mathsf{Palatini}} \subset \mathcal{M}_{\mathsf{DW}}$ with the imposition of constraints, it becomes non degenerate. Finally, we introduce the space - see the next section (77):

$$\overline{\mathcal{M}}_{\mathsf{Palatini}} = \{(x,e,\omega,p) \in \mathcal{M}_{\mathsf{DW}} \ / \ p_{IJ}^{\omega_\mu\nu} + p_{IJ}^{\omega_\nu\mu} = 0, \quad p_I^{e_\mu\nu} = 0\} \tag{295}$$

Notice that we have the following inclusion of space:

$$\mathcal{M}_{\mathsf{Palatini}} \subset \overline{\mathcal{M}}_{\mathsf{Palatini}} \subset \mathcal{M}_{\mathsf{DW}}$$

Expression of the Hamiltonian: DW vs Palatini — Now we express the Hamiltonian function for the Palatini degenerate theory. We alternatively term Palatini or degenerate the related submanifold.

$$\begin{array}{lll} \mathcal{H}^{\mathsf{DW}}(q,p) & = & \langle p,v\rangle - L(q,v) = \langle p,v\rangle - ee_I^\mu e_J^\nu(\partial_{[\mu}\omega_{\nu]}^{IJ} + \omega_{[\mu K}^I\omega_{\nu]}^{KJ}) \\ & = & \mathfrak{e} + p_I^{e_\mu\nu}\mathcal{Z}_{\nu\mu}^I + p_{IJ}^{\omega_\mu\nu}\mathcal{Z}_{\nu\mu}^{IJ} - ee_I^{[\mu}e_J^{\nu]}\big(\partial_\mu\omega_\nu^{IJ} + \omega_{\mu K}^I\omega_\nu^{KJ}\big) \\ & = & \mathfrak{e} + p_I^{e_\mu\nu}\mathcal{Z}_{\nu\mu}^I + p_{IJ}^{\omega_\mu\nu}\big(\partial_\nu\omega_\mu^{IJ} + \omega_{\nu}{}_K^I\omega_\mu{}^{KJ} - \omega_{\nu}{}_K^J\omega_\mu{}^{KI}\big) - ee_I^{[\nu}e_J^{\mu]}\big(\partial_\nu\omega_\mu^{IJ} + \omega_{\nu K}^I\omega_\mu^{KJ}\big) \end{array}$$

Let us work on $\mathcal{M}_{\mathsf{Palatini}} \subset \mathcal{M}_{\mathsf{DW}}$ so that the use of the constraint $p_I^{e_\mu\nu} = 0$ leads to

$$\mathcal{H}^{\mathsf{Palatini}}(q,p) = \mathfrak{e} + p_{IJ}^{\omega_\mu\nu}\big(\partial_\nu\omega_\mu^{IJ} + \omega_{\nu}{}_K^I\omega_\mu{}^{KJ} - \omega_{\nu}{}_K^J\omega_\mu{}^{KI}\big) - ee_I^{[\nu}e_J^{\mu]}\big(\partial_\nu\omega_\mu^{IJ} + \omega_{\nu K}^I\omega_\mu^{KJ}\big)$$

The other constraint leads to the identification $p_{IJ}^{\omega_\mu\nu} = -p_{IJ}^{\omega_\nu\mu} = -ee_I^{[\mu}e_J^{\nu]}$, therefore we write $-ee_I^{[\nu}e_J^{\mu]} = -p_{IJ}^{\omega_\mu\nu}$ so we obtain:

$$\mathcal{H}^{\text{Palatini}}(q,p) = \mathfrak{e} + p_{IJ}^{\omega_\mu\nu}\big(\partial_\nu\omega_\mu^{IJ} + \omega_\nu{}^I{}_K\omega_\mu{}^{KJ} - \omega_\nu{}^J{}_K\omega_\mu{}^{KI}\big) - p_{IJ}^{\omega_\mu\nu}\big(\partial_\nu\omega_\mu^{IJ} + \omega_{\nu K}^I\omega_\mu^{KJ}\big)$$

Hence, we keep in mind the Palatini Hamiltonian:

$$\mathcal{H}^{\text{Palatini}}(q,p) = \mathfrak{e} - p_{IJ}^{\omega_\mu\nu}\omega_\nu{}^J{}_K\omega_\mu{}^{KI} \tag{296}$$

Now we introduce the degenerate Hamiltonian $\mathcal{H}^{\text{deg}}(q,p)$ as the following function:

$$\mathcal{H}^{\text{deg}}(q,p) = \mathfrak{e} - ee_I^{[\nu}e_J^{\mu]}\omega_\nu{}^J{}_K\omega_\mu{}^{KI} = \mathfrak{e} - ee_I^{[\mu}e_J^{\nu]}\omega_\mu{}^J{}_K\omega_\nu{}^{KI} = \mathfrak{e} - ee_I^\mu e_J^\nu(\omega_{[\mu K}^J\omega_{\nu]}^{KI}) \tag{297}$$

The Hamiltonian function $\mathcal{H}^{\text{deg}}(q,p) : \mathcal{M}_{\text{deg}} \to \mathbb{R}$ is written $\mathcal{H}^{\text{deg}} = \mathfrak{e} + H^{\text{deg}}$. The constraint $\mathcal{H}^{\text{deg}} = 0$ gives the pre-multisymplectic setting.[98] so that $\mathfrak{e}$ is written $\mathfrak{e}(x) = ee_I^\mu e_J^\nu(\omega_{[\mu K}^J\omega_{\nu]}^{KI})$. It is a singular case of vanishing Hamiltonian found in the context of covariant Hamiltonian theories. We can always choose $\mathfrak{e}(x)$ such that $\mathcal{H}(x, e(x), \omega(x), \mathfrak{e}(x), p^\star(x))$ is constant. The evaluation of $d\mathcal{H}^{\text{deg}}(q,p) = d\mathfrak{e} - d(ee_I^\mu e_J^\nu\omega_{[\mu K}^J\omega_{\nu]}^{KI})$ is given in the Palatini $3D$ and $4D$ cases.

Exterior derivative of the Hamiltonian $d\mathcal{H}^{\text{deg}}$ —

- We are first interest in the three dimensional case, so that $d\mathcal{H}_{3\text{D}}^{\text{deg}}$ is written:

$$d\mathcal{H}_{3\text{D}}^{\text{deg}}(q,p) = d\mathfrak{e}\underbrace{-ee_I^\mu e_J^\nu d\big[\omega_{[\mu K}^J\omega_{\nu]}^{KI})\big]}_{[\mathbf{I}]}\underbrace{-d\big[ee_I^\mu e_J^\nu\big]\omega_{[\mu K}^J\omega_{\nu]}^{KI}}_{[\mathbf{II}]}$$

Since $d(ee_I^\mu e_J^\nu) = d\big[(1/3!)\epsilon_{IJM}\epsilon^{\mu\nu\lambda}e_\lambda^M\big] = (1/3!)\epsilon_{IJM}\epsilon^{\mu\nu\lambda}de_\lambda^M$. Therefore the term $[\mathbf{II}]$ is written:

$$[\mathbf{II}] = (1/3!)\epsilon_{IJM}\epsilon^{\mu\nu\lambda}\omega_{[\mu K}^J\omega_{\nu]}^{KI}de_\lambda^M \tag{298}$$

On the other hand we focus on the other term $[\mathbf{I}]$:

$$\begin{aligned}[\mathbf{I}] &= -ee_I^\mu e_J^\nu d\big[\omega_{[\mu K}^J\omega_{\nu]}^{KI}\big] = -ee_I^\mu e_J^\nu(\omega_{\mu K}^J d\omega_\nu^{KI} - \omega_{\nu K}^J d\omega_\mu^{KI}) = ee_I^{[\mu}e_J^{\nu]}\omega_{\nu K}^J d\omega_\mu^{KI} \\ &= ee_I^{[\mu}e_J^{\nu]}\omega_{\nu M}^J d\omega_\mu^{MI} = (1/3!)\epsilon_{IJK}\epsilon^{\mu\nu\rho}e_\rho^K\omega_{\nu M}^J d\omega_\mu^{MI} - (1/3!)\epsilon_{IJK}\epsilon^{\nu\mu\rho}e_\rho^K\omega_{\nu M}^J d\omega_\mu^{MI}\end{aligned}$$

We introduce the notation (299)

$$\varsigma = \varsigma_{IJL}^{\mu\nu\alpha} = \frac{1}{2}\frac{1}{3!}\epsilon_{IJL}\big(\epsilon^{\mu\nu\alpha} + \epsilon^{\nu\mu\alpha}\big) \tag{299}$$

so that:

$$[\mathbf{I}] = \varsigma_{IJM}^{\mu\nu\lambda}e_\lambda^M\omega_{\nu K}^J d\omega_\mu^{KI} = \varsigma_{IJK}^{\mu\nu\rho}e_\rho^K\omega_{\nu M}^J d\omega_\mu^{MI} \tag{300}$$

See appendix (D) for details, we have the relation (301) in $3D$:

$$\epsilon^{\mu\rho\sigma}\epsilon_{IJK}e_\mu^I\omega_\sigma{}^J{}_M d\omega_\rho^{MK} = -1/2\epsilon^{\mu\rho\sigma}\epsilon_{LJK}e_\mu^I\omega_\sigma{}^L{}_I d\omega_\rho^{JK} \tag{301}$$

$$\epsilon_{IJK}\epsilon^{\mu\nu\rho}e_\rho^K\omega_{\nu M}^J d\omega_\mu^{MI} = -1/2\epsilon^{\mu\nu\rho}\epsilon_{LJI}e_\rho^K\omega_\nu{}^L{}_K d\omega_\mu^{JI} = 1/2\epsilon^{\mu\nu\rho}\epsilon_{LJI}e_\rho^K\omega_\nu{}^L{}_K d\omega_\mu^{IJ}$$

This relation improves the situation, especially in order to compare the different terms in the Hamilton equations. We have the expression of the second term in $d\mathcal{H}^{\text{deg}}$, which is $d\mathcal{H}_\circ = \mathfrak{O}_{LJI}^{\mu\nu\rho}e_\rho^K\omega_\nu{}^L{}_K d\omega_\mu^{IJ}$. Now, we obtain the expression for the 1-form $d\mathcal{H}_{3\text{D}}^{\text{deg}}(q,p)$, thanks to relations (298), (300) and (301):

$$d\mathcal{H}_{3\text{D}}^{\text{deg}}(q,p) = d\mathfrak{e} + \frac{1}{3!}\epsilon_{IJM}\epsilon^{\mu\nu\lambda}\omega_{[\mu K}^J\omega_{\nu]}^{KI}de_\lambda^M + \mathfrak{O}_{LJI}^{\mu\nu\rho}e_\rho^K\omega_\nu{}^L{}_K d\omega_\mu^{IJ} \tag{302}$$

[98] we come back below on this point, leading the construction of dynamical structure of an n-phase space $(\mathcal{M}, \boldsymbol{\omega}, \beta)$, this path open indeed a strong connection with the *covariant phase space* approach.

• In the case of Palatini $4D$ theory, the expression of the Hamiltonian is basically the same, *i.e* the equation (297). For the exterior derivative of the Hamiltonian we obtain $d\mathcal{H}^{\mathsf{deg}}_{\mathsf{4D}}(q,p) = d\mathcal{H}^{\mathsf{deg}}$ such that:

$$\begin{array}{rcl} d\mathcal{H}^{\mathsf{deg}} & = & d\mathfrak{e} - d\Big(ee_I^{[\mu}e_J^{\nu]}(\omega_{\mu K}^{J}\omega_{\nu}^{KI})\Big) = d\mathfrak{e} - d\Big(ee_I^{[\mu}e_J^{\nu]}\Big)(\omega_{\mu K}^{J}\omega_{\nu}^{KI}) - ee_I^{[\mu}e_J^{\nu]}d\Big(\omega_{\mu K}^{J}\omega_{\nu}^{KI}\Big) \\ & = & d\mathfrak{e} - \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^K\omega_{\mu M}^{J}\omega_\nu^{MI}de_\sigma^L + ee_I^{[\mu}e_J^{\nu]}\omega_{\nu M}^{J}d\omega_\mu^{MI} \\ & = & d\mathfrak{e} - \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^K\omega_{\mu M}^{J}\omega_\nu^{MI}de_\sigma^L + \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^K e_\sigma^L\omega_{\nu M}^{J}d\omega_\mu^{MI} \end{array}$$

The counter part of relation (301) for the $4D$ case writes (303):

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKL}e_\mu^I e_\nu^J \omega_\sigma{}^K{}_M d\omega_\rho^{ML} = -\epsilon^{\mu\nu\rho\sigma}\epsilon_{INKL}e_\mu^I e_\nu^J \omega_\sigma{}^N{}_J d\omega_\rho^{KL} \tag{303}$$

Then, after a quick interplay with spacetime indices we obtain:

$$\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^K e_\sigma^L\omega_{\nu M}^{J}d\omega_\mu^{MI} = \epsilon^{\rho\sigma\nu\mu}\epsilon_{KLJI}e_\rho^K e_\sigma^L\omega_{\nu M}^{J}d\omega_\mu^{MI} = -\epsilon^{\mu\nu\rho\sigma}\epsilon_{INKL}e_\rho^K e_\sigma^L\omega_\nu{}^N{}_L d\omega_\mu^{JI}$$

Now we can obtain a suitable expression for later purpose:

$$d\mathcal{H}^{\mathsf{deg}}_{\mathsf{4D}}(q,p) = d\mathfrak{e} - \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^K de_\sigma^L + \epsilon_{INKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^K e_\sigma^L\omega_\nu{}^N{}_L d\omega_\mu^{IJ} \tag{304}$$

16.4 Multisymplectic Hamilton equations on $(\mathcal{M}_{\mathsf{deg}}, \boldsymbol{\omega}^{\mathsf{deg}})$

Let us define the $(n+1)$-form $\boldsymbol{\omega}^{\mathsf{DW}} = d\theta^{\mathsf{DW}}$:

$$\boldsymbol{\omega}^{\mathrm{dDW}} = d\mathfrak{e}\wedge\beta + dp_{IJ}^{\omega_\mu\nu}\wedge d\omega_\mu^{IJ}\wedge\beta_\nu \tag{305}$$

Therefore we can express $\boldsymbol{\omega}^{\mathsf{deg}}$ by:

$$\boldsymbol{\omega}^{\mathsf{deg}} = d\mathfrak{e}\wedge\beta - d\big[ee_I^{[\mu}e_J^{\nu]}\big]\wedge d\omega_\mu^{IJ}\wedge\beta_\nu \tag{306}$$

Thanks to (306) we now enter in the Hamilton equations.

Lemma 16.1. Let X be a $(n-1)$-vector field and let $\{d\boldsymbol{\rho}_i\}_{1\le i\le n}$ be a set of n 1-forms. We have:

$$X \lrcorner \,(\bigwedge_{1\le i\le n} d\boldsymbol{\rho}_i) = X \lrcorner\, d\boldsymbol{\rho}_1\wedge\cdots\wedge d\boldsymbol{\rho}_n = \sum_j (-1)^{j+1}\big(d\boldsymbol{\rho}_1\wedge\cdots\wedge d\boldsymbol{\rho}_{j-1}\wedge d\boldsymbol{\rho}_{j+1}\wedge\cdots\wedge d\boldsymbol{\rho}_n\big)(X)d\boldsymbol{\rho}_j$$

Multisymplectic Hamilton equations for Palatini 3D. — First, let us treat the 3D Palatini theory, one considers $X = X_1\wedge X_2\wedge X_3 \in \Lambda^3 T\mathcal{M}_{\mathrm{Deg}}$. In the present case we write $\forall 1\le\nu\le 3$,

$$X_\nu = \frac{\partial}{\partial x^\nu} + \Theta^I_{\nu\mu}\frac{\partial}{\partial e^I_\mu} + \Theta^{IJ}_{\nu\mu}\frac{\partial}{\partial \omega^I_\mu} + \Upsilon_\nu\frac{\partial}{\partial\mathfrak{e}} \tag{307}$$

Therefore we explicitly write X_1, X_2 and X_3:

$$\left|\begin{array}{rcl} X_1 & = & \dfrac{\partial}{\partial x^1} + \Theta^I_{1\mu_1}\dfrac{\partial}{\partial e^I_{\mu_1}} + \Theta^{IJ}_{1\mu_1}\dfrac{\partial}{\partial\omega^I_{\mu_1}} + \Upsilon_1\dfrac{\partial}{\partial\mathfrak{e}} \\ X_2 & = & \dfrac{\partial}{\partial x^2} + \Theta^I_{2\mu_2}\dfrac{\partial}{\partial e^I_{\mu_2}} + \Theta^{IJ}_{2\mu_2}\dfrac{\partial}{\partial\omega^I_{\mu_2}} + \Upsilon_2\dfrac{\partial}{\partial\mathfrak{e}} \\ X_3 & = & \dfrac{\partial}{\partial x^3} + \Theta^I_{3\mu_3}\dfrac{\partial}{\partial e^I_{\mu_3}} + \Theta^{IJ}_{3\mu_3}\dfrac{\partial}{\partial\omega^I_{\mu_3}} + \Upsilon_3\dfrac{\partial}{\partial\mathfrak{e}} \end{array}\right. \tag{308}$$

Using (308) we obtain, after a straightforward calculation (denoting $\frac{\partial}{\partial x^\mu}$ by ∂_μ) the following expression for $X_1 \wedge X_2 \wedge X_3$:

$$
\begin{aligned}
X_1 \wedge X_2 \wedge X_3 \;=\;& \partial_1 \wedge \partial_2 \wedge \partial_3 + \partial_1 \wedge \partial_2 \wedge \Theta^I_{3\mu_3} \frac{\partial}{\partial e^I_{\mu_3}} + \partial_1 \wedge \partial_2 \wedge \Theta^{IJ}_{3\mu_3} \frac{\partial}{\partial \omega^I_{\mu_3}} \\
& + \partial_1 \wedge \Theta^I_{2\mu_2} \frac{\partial}{\partial e^I_{\mu_2}} \wedge \partial_3 + \partial_1 \wedge \Theta^I_{2\mu_2} \frac{\partial}{\partial e^I_{\mu_2}} \wedge \Theta^{IJ}_{3\mu_3} \frac{\partial}{\partial \omega^I_{\mu_3}} \\
& + \partial_1 \wedge \Theta^{IJ}_{2\mu_2} \frac{\partial}{\partial \omega^I_{\mu_2}} \wedge \partial_3 + \partial_1 \wedge \Theta^{IJ}_{2\mu_2} \frac{\partial}{\partial \omega^I_{\mu_2}} \wedge \Theta^I_{3\mu_3} \frac{\partial}{\partial e^I_{\mu_3}} \\
& + \Theta^I_{1\mu_1} \frac{\partial}{\partial e^I_{\mu_1}} \wedge \partial_2 \wedge \partial_3 + \Theta^I_{1\mu_1} \frac{\partial}{\partial e^I_{\mu_1}} \wedge \partial_2 \wedge \Theta^{IJ}_{3\mu_3} \frac{\partial}{\partial \omega^I_{\mu_3}} + \Theta^I_{1\mu_1} \frac{\partial}{\partial e^I_{\mu_1}} \wedge \Theta^{IJ}_{2\mu_2} \frac{\partial}{\partial \omega^I_{\mu_2}} \wedge \partial_3 \\
& + \Theta^{IJ}_{1\mu_1} \frac{\partial}{\partial \omega^{IJ}_{\mu_1}} \wedge \partial_2 \wedge \partial_3 + \Theta^{IJ}_{1\mu_1} \frac{\partial}{\partial \omega^{IJ}_{\mu_1}} \wedge \partial_2 \wedge \Theta^I_{3\mu_3} \frac{\partial}{\partial e^I_{\mu_3}} + \Theta^{IJ}_{1\mu_1} \frac{\partial}{\partial \omega^{IJ}_{\mu_1}} \wedge \Theta^I_{2\mu_2} \frac{\partial}{\partial e^I_{\mu_2}} \wedge \partial_3 \\
& + \text{multivectors involving} \quad \partial/\partial \mathfrak{e}
\end{aligned}
$$

The first step to obtain the generalized Hamilton equations is to compute the expression $X \lrcorner \boldsymbol{\omega}^{\mathsf{deg}}$. First we re-express $\boldsymbol{\omega}^{\mathsf{deg}}$, we have:

$$
\begin{aligned}
\boldsymbol{\omega}^{\mathsf{deg}} &= d\mathfrak{e} \wedge \beta - \frac{1}{2} d\big(e e^\mu_I e^\nu_J\big) \wedge d\omega^{IJ}_\mu \wedge \beta_\nu + \frac{1}{2} d\big(e e^\nu_I e^\mu_J\big) \wedge d\omega^{IJ}_\mu \wedge \beta_\nu \\
&= d\mathfrak{e} \wedge \beta - \frac{1}{2}\frac{1}{3!} \epsilon_{IJM} \big(\epsilon^{\mu\nu\lambda} + \epsilon^{\nu\mu\lambda}\big) de^M_\lambda \wedge d\omega^{IJ}_\mu \wedge \beta_\nu = d\mathfrak{e} \wedge \beta - \mathfrak{O}^{\mu\nu\lambda}_{IJM} de^M_\lambda \wedge d\omega^{IJ}_\mu \wedge \beta_\nu
\end{aligned}
$$

So that we obtain:

$$
\begin{aligned}
X \lrcorner \boldsymbol{\omega}^{\mathsf{deg}} &= X \lrcorner (d\mathfrak{e} \wedge \beta) - \mathfrak{O}^{\mu\nu\lambda}_{IJM} X \lrcorner (de^M_\lambda \wedge d\omega^{IJ}_\mu \wedge \beta_\nu) \\
&= \beta(X) d\mathfrak{e} - (d\mathfrak{e} \wedge \beta_\rho)(X) dx^\rho - \mathfrak{O}^{\mu\nu\alpha}_{IJL} \Big((d\omega^{IJ}_\mu \wedge \beta_\nu)(X) de^L_\alpha - (de^L_\alpha \wedge \beta_\nu)(X) d\omega^{IJ}_\mu \Big) \\
&\quad - \mathfrak{O}^{\mu\nu\alpha}_{IJL} \Big((de^L_\alpha \wedge d\omega^{IJ}_\mu \wedge \beta_{\rho\nu})(X) dx^\rho \Big) \\
&= d\mathfrak{e} \underbrace{- (d\mathfrak{e} \wedge \beta_\rho)(X) dx^\rho}_{\Delta_\circ} - \underbrace{\mathfrak{O}^{\mu\nu\alpha}_{IJL} (de^L_\alpha \wedge d\omega^{IJ}_\mu \wedge \beta_{\rho\nu})(X) dx^\rho}_{\Delta_\bullet} \\
&\quad - \underbrace{\mathfrak{O}^{\mu\nu\alpha}_{IJL} (d\omega^{IJ}_\mu \wedge \beta_\nu)(X) de^L_\alpha}_{\Delta_e} + \underbrace{\mathfrak{O}^{\mu\nu\alpha}_{IJL} (de^L_\alpha \wedge \beta_\nu)(X) d\omega^{IJ}_\mu}_{\Delta_\omega}
\end{aligned}
$$

Following our notations, the expression of the left side of Hamilton equations becomes:

$$
X \lrcorner \boldsymbol{\omega}^{\mathsf{deg}} = d\mathfrak{e} + \Delta_\circ + \Delta_\bullet + \Delta_e + \Delta_\omega \tag{309}
$$

Each term in (309) writes:

$$
\begin{aligned}
\Delta_\circ &= -(d\mathfrak{e} \wedge \beta_\rho)(X) dx^\rho = -\Upsilon_\rho dx^\rho \\
\Delta_\bullet &= -\mathfrak{O}^{\mu\nu\alpha}_{IJL} (de^L_\alpha \wedge d\omega^{IJ}_\mu \wedge \beta_{\rho\nu})(X) dx^\rho = -\mathfrak{O}^{\mu\nu\alpha}_{IJL} \big(\Theta^{IJ}_{\nu\mu} \Theta^L_{\rho\alpha} - \Theta^{IJ}_{\rho\mu} \Theta^L_{\nu\alpha}\big) dx^\rho \\
\Delta_e &= -\mathfrak{O}^{\mu\nu\alpha}_{IJL} (d\omega^{IJ}_\mu \wedge \beta_\nu)(X) de^L_\alpha = -\frac{1}{2}\frac{1}{3!} \epsilon_{IJL} \big(\epsilon^{\mu\nu\alpha} + \epsilon^{\nu\mu\alpha}\big) \Theta^{IJ}_{\nu\mu} de^L_\alpha \\
\Delta_\omega &= \mathfrak{O}^{\mu\nu\alpha}_{IJL} (de^L_\alpha \wedge \beta_\nu)(X) d\omega^{IJ}_\mu = \frac{1}{2}\frac{1}{3!} \epsilon_{IJL} \big(\epsilon^{\mu\nu\alpha} + \epsilon^{\nu\mu\alpha}\big) \Theta^L_{\nu\alpha} d\omega^{IJ}_\mu
\end{aligned}
$$

All this preparatory work allows us to write the Hamilton equations:

$$
X \lrcorner \boldsymbol{\omega}^{\mathsf{deg}} = (-1)^n d\mathcal{H}^{\mathsf{deg}} \tag{310}
$$

We compare both hand of equation (310). The left hand is given by (309):

$$
X \lrcorner \boldsymbol{\omega}^{\mathsf{deg}} = d\mathfrak{e} - \Upsilon_\rho dx^\rho - \mathfrak{O}^{\mu\nu\alpha}_{IJL} \big(\Theta^{IJ}_{\nu\mu} \Theta^L_{\rho\alpha} - \Theta^{IJ}_{\rho\mu} \Theta^L_{\nu\alpha}\big) dx^\rho - \mathfrak{O}^{\mu\nu\alpha}_{IJL} \Theta^{IJ}_{\nu\mu} de^L_\alpha + \mathfrak{O}^{\mu\nu\alpha}_{IJL} \Theta^L_{\nu\alpha} d\omega^{IJ}_\mu \tag{311}
$$

which is identified with (302):

$$d\mathcal{H}^{\mathrm{deg}}_{3D}(q,p) = d\mathfrak{e} + \frac{1}{3!}\epsilon_{IJM}\epsilon^{\mu\nu\lambda}\omega^{J}_{[\mu K}\omega^{KI}_{\nu]}de^{M}_{\lambda} + \mathfrak{O}^{\mu\nu\rho}_{LJI}e^{K}_{\rho}\omega_{\nu}{}^{L}{}_{K}d\omega^{IJ}_{\mu} \tag{312}$$

For this Palatini $3D$ case, we concentrate only on the decomposition of the 1-form $X \lrcorner \boldsymbol{\omega}^{\mathrm{deg}}$ in the basis de^{L}_{α} and $d\omega^{IJ}_{\mu}$. The identification of the expressions (311) and (312) leads to:

$$\begin{aligned} \frac{1}{2}\epsilon_{IJL}\big(\epsilon^{\mu\nu\alpha}+\epsilon^{\nu\mu\alpha}\big)\Theta^{IJ}_{\nu\mu}\Big|^{L}_{\alpha} &= \epsilon_{IJL}\epsilon^{\mu\nu\alpha}\omega^{J}_{[\mu K}\omega^{KI}_{\nu]}\Big|^{L}_{\alpha} = \frac{1}{2}\epsilon_{IJL}\big(\epsilon^{\mu\nu\alpha}-\epsilon^{\nu\mu\alpha}\big)\omega^{J}_{\mu K}\omega^{KI}_{\nu}\Big|^{L}_{\alpha} \\ -\epsilon_{IJL}\big(\epsilon^{\mu\nu\alpha}+\epsilon^{\nu\mu\alpha}\big)\Theta^{L}_{\nu\alpha}\Big|^{IJ}_{\mu} &= \big(\epsilon^{\mu\nu\alpha}-\epsilon^{\nu\mu\alpha}\big)\epsilon_{LJI}e^{K}_{\alpha}\omega_{\nu}{}^{L}{}_{K}\Big|^{IJ}_{\mu} \end{aligned}$$

So that without making reference to the decompostion on de^{L}_{α} and $d\omega^{IJ}_{\mu}$ we obtain:

$$\begin{aligned} \epsilon_{IJL}\epsilon^{\mu\nu\alpha}\big(\Theta^{IJ}_{\mu\nu}-\Theta^{IJ}_{\nu\mu}\big) &= \epsilon_{IJL}\epsilon^{\mu\nu\alpha}\big(\omega^{J}_{\mu K}\omega^{KI}_{\nu}-\omega^{J}_{\mu K}\omega^{KI}_{\nu}\big) \\ \epsilon_{IJL}\epsilon^{\mu\nu\alpha}\big(\Theta^{L}_{\alpha\nu}-\Theta^{L}_{\nu\alpha}\big) &= -\epsilon_{IJL}\epsilon^{\mu\nu\alpha}\big(e^{K}_{\alpha}\omega_{\nu}{}^{L}{}_{K}-e^{K}_{\alpha}\omega_{\nu}{}^{L}{}_{K}\big) \end{aligned}$$

Finally one obtains the system of Hamilton equations (313):

$$\begin{aligned} \epsilon_{IJL}\epsilon^{\mu\nu\alpha}F^{IJ}_{\mu\nu} &= 0 \\ \epsilon_{IJL}\epsilon^{\mu\nu\alpha}\Big(\Theta^{L}_{[\alpha\nu]}+e^{K}_{[\alpha}\omega_{\nu]}{}^{L}{}_{K}\Big) &= 0 \end{aligned} \tag{313}$$

Then, one looks at the decomposition of the 1-form $X \lrcorner \boldsymbol{\omega}^{\mathrm{deg}}$ on the basis dx^{ρ} so that:

$$-\Upsilon_{\rho} - \mathfrak{O}^{\mu\nu\alpha}_{IJL}\big(\Theta^{IJ}_{\nu\mu}\Theta^{L}_{\rho\alpha}-\Theta^{IJ}_{\rho\mu}\Theta^{L}_{\nu\alpha}\big)\Big|^{\rho} = 0 \tag{314}$$

With (313) and (314), we find the generalized Hamilton equations for the $3D$-Palatini case:

$$\begin{aligned} \epsilon_{IJL}\epsilon^{\mu\nu\alpha}F^{IJ}_{\mu\nu} &= 0 \\ \epsilon_{IJL}\epsilon^{\mu\nu\alpha}\big(\Theta^{L}_{[\alpha\nu]}+e^{K}_{[\alpha}\omega_{\nu]}{}^{L}{}_{K}\big) &= 0 \\ -\Upsilon_{\rho}-\mathfrak{O}^{\mu\nu\alpha}_{IJL}\Big(\Theta^{IJ}_{\nu\mu}\Theta^{L}_{\rho\alpha}-\Theta^{IJ}_{\rho\mu}\Theta^{L}_{\nu\alpha}\Big) &= 0 \end{aligned} \tag{315}$$

Multisymplectic Hamilton equations for Palatini 4D. — We consider:

$$\boldsymbol{\omega}^{\mathrm{deg}} = d\mathfrak{e}\wedge\beta - d\big[ee^{[\mu}_{I}e^{\nu]}_{J}\big]\wedge d\omega^{IJ}_{\mu}\wedge\beta_{\nu} = d\mathfrak{e}\wedge\beta - \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^{K}_{\rho}de^{L}_{\sigma}\wedge d\omega^{IJ}_{\mu}\wedge\beta_{\nu} \tag{316}$$

This time we consider $X = X_1\wedge X_2\wedge X_3\wedge X_4 \in \Lambda^4 T\mathcal{M}_{\mathrm{Palatini}}$ and $\forall\ \nu\in\{1,4\}$:

$$X_{\nu} = \frac{\partial}{\partial x^{\nu}} + \Theta^{I}_{\nu\mu}\frac{\partial}{\partial e^{I}_{\mu}} + \Theta^{IJ}_{\nu\mu}\frac{\partial}{\partial\omega^{I}_{\mu}} + \Upsilon_{\nu}\frac{\partial}{\partial\mathfrak{e}}$$

Then:

$$\begin{aligned} X\lrcorner\boldsymbol{\omega}^{\mathrm{deg}} &= X\lrcorner(d\mathfrak{e}\wedge\beta) - \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^{K}_{\rho}X\lrcorner(de^{L}_{\sigma}\wedge d\omega^{IJ}_{\mu}\wedge\beta_{\nu}) \\ &= \beta(X)d\mathfrak{e} - (d\mathfrak{e}\wedge\beta_{\rho})(X)dx^{\rho} \\ &\quad -\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^{K}_{\rho}\Big((d\omega^{IJ}_{\mu}\wedge\beta_{\nu})(X)de^{L}_{\sigma} - (de^{L}_{\sigma}\wedge\beta_{\nu})(X)d\omega^{IJ}_{\mu}\Big) \\ &\quad -\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^{K}_{\rho}\Big((de^{L}_{\sigma}\wedge d\omega^{IJ}_{\mu}\wedge\beta_{\rho\nu})(X)dx^{\rho}\Big) \end{aligned}$$

We perform the different terms in $X\lrcorner\boldsymbol{\omega}^{\mathrm{deg}}$, namely $(d\omega^{IJ}_{\mu}\wedge\beta_{\nu})(X)$, $(de^{L}_{\sigma}\wedge\beta_{\nu})(X)$ and finally $(de^{L}_{\sigma}\wedge d\omega^{IJ}_{\mu}\wedge\beta_{\rho\nu})(X)$, taking into account the expression of the multivector X, we obtain:

$$X\lrcorner\boldsymbol{\omega}^{\mathrm{deg}} = d\mathfrak{e} - \Upsilon_{\rho}dx^{\rho} - \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^{K}_{\rho}\Big(\Theta^{IJ}_{\nu\mu}de^{L}_{\sigma} - \Theta^{L}_{\nu\sigma}d\omega^{IJ}_{\mu} + \big(\Theta^{IJ}_{\nu\mu}\Theta^{L}_{\rho\sigma}-\Theta^{IJ}_{\rho\mu}\Theta^{L}_{\nu\sigma}\big)dx^{\rho}\Big)$$

Now, we give the Hamilton equations $X \lrcorner \boldsymbol{\omega}^{\mathsf{deg}} = (-1)^n d\mathcal{H}^{\mathsf{deg}}$ by means of the expression of $d\mathcal{H}^{\mathsf{deg}}_{\mathsf{4D}}(q,p)$ (304). Thus we are first interested in the decomposition of the 1-form $X \lrcorner \boldsymbol{\omega}^{\mathsf{deg}}$ in the basis de^L_σ and $d\omega^{IJ}_\mu$:

$$\begin{array}{rcl} \Big(- \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho \Theta^{IJ}_{\nu\mu}\Big)^L_\sigma & = & \Big(- \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho \omega^J_{\mu M}\omega^{MI}_\nu\Big)^L_\sigma \\ \Big(\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho \Theta^L_{\nu\sigma}\Big)^{IJ}_\mu & = & \Big(\epsilon_{INKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho e^L_\sigma \omega_\nu{}^N{}_L\Big)^{IJ}_\mu \end{array} \tag{317}$$

Therefore, we obtain the Hamilton equations:

$$\begin{array}{rcl} \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho \Big(\Theta^{IJ}_{\mu\nu} + \omega^I_{\mu M}\omega^{MJ}_\nu\Big) & = & 0 \\ \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^K_\rho \Big(\Theta^L_{\nu\sigma} + \omega^L_{\nu N} e^N_\sigma\Big) & = & 0 \end{array} \tag{318}$$

We recover the results obtained by D. Bruno, R. Cianci and S. Vignolo [31] or the result obtained by Rovelli [199].

17 Palatini Gravity as an n-phase space

Now let us concentrate on:

$$\mathcal{H}^{\mathsf{deg}}(q,p) = \mathfrak{e} - e e^\mu_I e^\nu_J (\omega^J_{[\mu K}\omega^{KI}_{\nu]}) = 0$$

This relation is equivalent to chose the constraint $\mathfrak{e} = e e^\mu_I e^\nu_J (\omega^J_{[\mu K}\omega^{KI}_{\nu]})$. We are in the framework of a n-phase space. It is precisely the notion developed in [144]. In his seminal paper, Kijowski gives the basic principle of a n-phase space: a n-dimensional submanifold endowed with a closed $(n+1)$-form that may degenerate with the *Hamiltonian constraint* $\mathcal{H} = 0$. We describe two aspects. The first is to describe the intrinsic n-phase space and dynamical structure in the deDonder-Weyl seeting (with the manifold $\mathcal{M}_{\mathsf{DW}}$) but with the additional constraint $\mathcal{H} = 0$. The second apsect is that we have directly a n-phase dynamical structure on the manifold $\mathfrak{Z}$ when given with a closed, degenerate $(n+1)$-form $\boldsymbol{\omega}$. This approach works also for Palatini gravity and non-abelian gauge theory (Yang-Mills fields). See the Palatini action proposed by Hélein and Kouneiher in [116, 117].

17.1 Pre-Multisymplectic treatement of 3D-Palatini Gravity

In this section, we treat the equations of general relativity for $3D$-Palatini action. We consider the Palatini action as the 3-form $1/2\epsilon_{IJK}e^I \wedge F^{JK}$ and the functional:

$$\mathcal{L}_{\mathsf{Palatini}} = \frac{1}{2}\int \epsilon_{IJK} e^I \wedge F^{JK}$$

with $F^{JK} = d\omega^{JK} + \omega^J{}_L \wedge \omega^{LK}$ is the curvature. Since $e^I = e^I_\mu dx^\mu$, $\omega^{JK} = \omega^{JK}_\mu dx^\mu$, we obtain the following expression for the Poincaré-Cartan 3-form:

$$\theta^{\circ} = \frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\Big[e^I_\mu d\omega^{JK}_\sigma \wedge \beta_\rho + e^I_\mu \omega_\rho{}^J{}_L \omega^{LK}_\sigma \beta\Big] \tag{319}$$

⌜ **Proof** We demonstrate (319) by direct calculation:

$$\begin{array}{rcl} \theta^{\circ} & = & \frac{1}{2}\epsilon_{IJK} e^I_\mu dx^\mu \wedge \Big[d(\omega^{JK}_\rho dx^\rho) + \omega_\rho{}^J{}_L dx^\rho \wedge \omega^{LK}_\sigma dx^\sigma\Big] \\ & = & \frac{1}{2}\epsilon_{IJK} e^I_\mu dx^\mu \wedge d\omega^{JK}_\sigma \wedge dx^\sigma + \frac{1}{2}\epsilon_{IJK} e^I_\mu \omega_\rho{}^J{}_L \omega^{LK}_\sigma dx^\mu \wedge dx^\rho \wedge dx^\sigma \end{array}$$

We write the Poincaré-Cartan 3-form as $\theta^{\circ} = \theta^{\circ}_{|\beta} + \theta^{\circ}_{|d\omega\wedge\beta_\sigma}$ With:

$$\begin{cases} \theta^{\circ}_{|d\omega\wedge\beta_\sigma} &= \frac{1}{2}\epsilon_{IJK}e^I_\mu dx^\mu\wedge d\omega^{JK}_\sigma\wedge dx^\sigma \\ \theta^{\circ}_{|\beta} &= \frac{1}{2}\epsilon_{IJK}e^I_\mu \omega_\rho{}^J{}_L\omega^{LK}_\sigma dx^\mu\wedge dx^\rho\wedge dx^\sigma \end{cases}$$

Lemma 17.1. We have the following equality

$$\theta^{\circ}_{|d\omega\wedge\beta_\sigma} = -\frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}e^I_\mu d\omega^{JK}_\rho\wedge\beta_\sigma = \frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\sigma\rho}e^I_\mu d\omega^{JK}_\rho\wedge\beta_\sigma \tag{320}$$

We prove the following equalities (321) and (322):

$$\theta^{\circ}_{|d\omega\wedge\beta_\sigma} = -\frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}e^I_\mu d\omega^{JK}_\rho\wedge\beta_\sigma = \frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\sigma\rho}e^I_\mu d\omega^{JK}_\rho\wedge\beta_\sigma \tag{321}$$

$$\theta^{\circ}_{|\beta} = \frac{1}{2}\epsilon_{IJK}e^I_\mu\omega_\rho{}^J{}_L\omega^{LK}_\sigma dx^\mu\wedge dx^\rho\wedge dx^\sigma = \frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}e^I_\mu\omega_\rho{}^J{}_L\omega^{LK}_\sigma\beta \tag{322}$$

First we are interested in the term $\theta^{\circ}_{|d\omega\wedge\beta_\sigma}$, by direct calculation we decompose the index σ from 1 to $n=3$:

$$\Theta_\circ = \frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}e^I_\mu d\omega^{JK}_\rho\wedge\beta_\sigma = \frac{1}{2}\epsilon_{IJK}\left[\epsilon^{\mu\rho1}e^I_\mu d\omega^{JK}_\rho\wedge\beta_1 + \epsilon^{\mu\rho2}e^I_\mu d\omega^{JK}_\rho\wedge\beta_2 + \epsilon^{\mu\rho3}e^I_\mu d\omega^{JK}_\rho\wedge\beta_3\right]$$

Since $\beta_1 = dx^2\wedge dx^3,\ \beta_2 = -dx^1\wedge dx^3$ and $\beta_3 = dx^1\wedge dx^2$

$$\Theta_\circ = \frac{1}{2}\epsilon_{IJK}\left[\epsilon^{\mu\rho1}e^I_\mu d\omega^{JK}_\rho\wedge dx^2\wedge dx^3 - \epsilon^{\mu\rho2}e^I_\mu d\omega^{JK}_\rho\wedge dx^1\wedge dx^3 + \epsilon^{\mu\rho3}e^I_\mu d\omega^{JK}_\rho\wedge dx^1\wedge dx^2\right]$$

So that if we expend and use properties of the antisymmetric symbol:

$$\Theta_\circ = \frac{1}{2}\epsilon_{IJK}\left[\epsilon^{231}e^I_2 d\omega^{JK}_3\wedge dx^2\wedge dx^3 + \epsilon^{321}e^I_3 d\omega^{JK}_2\wedge dx^2\wedge dx^3 - \epsilon^{132}e^I_1 d\omega^{JK}_3\wedge dx^1\wedge dx^3\right.$$
$$\left. -\epsilon^{312}e^I_3 d\omega^{JK}_1\wedge dx^1\wedge dx^3 + \epsilon^{123}e^I_1 d\omega^{JK}_2\wedge dx^1\wedge dx^2 + \epsilon^{213}e^I_2 d\omega^{JK}_1\wedge dx^1\wedge dx^2\right]$$

$$\Theta_\circ = -\frac{1}{2}\epsilon_{IJK}\left[e^I_2 dx^2\wedge d\omega^{JK}_3\wedge dx^3 + e^I_3 dx^3\wedge d\omega^{JK}_2\wedge dx^2 + e^I_1 dx^1\wedge d\omega^{JK}_3\wedge dx^3\right.$$
$$\left. +e^I_3 dx^3\wedge d\omega^{JK}_1\wedge dx^1 + e^I_1 dx^1\wedge d\omega^{JK}_2\wedge dx^2 + e^I_2 dx^2\wedge d\omega^{JK}_1\wedge dx^1\right]$$

Therefore we finally find:

$$\Theta_\circ = \frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}e^I_\mu d\omega^{JK}_\rho\wedge\beta_\sigma = -\frac{1}{2}\epsilon_{IJK}e^I_\mu dx^\mu\wedge d\omega^{JK}_\sigma\wedge dx^\sigma = -\theta^{\circ}_{|d\omega\wedge\beta_\sigma}$$

which prove (321), also one observes that (322) is right.

Let us express the second term $\Theta_{\circ\circ} = 1/2\epsilon_{IJK}\epsilon^{\mu\rho\sigma}e^I_\mu\omega_\rho{}^J{}_L\omega^{LK}_\sigma\beta$ using the volume form: $\beta = dx^1\wedge dx^2\wedge dx^3$. Since, $\beta = (1/3!)\epsilon_{\alpha\beta\gamma}dx^\alpha\wedge dx^\beta\wedge dx^\gamma$ one writes:

$$\Theta_{\circ\circ} = \frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}e^I_\mu\omega_\rho{}^J{}_L\omega^{LK}_\sigma\beta = \frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}e^I_\mu\omega_\rho{}^J{}_L\omega^{LK}_\sigma\left[\frac{1}{3!}\epsilon_{\alpha\beta\gamma}dx^\alpha\wedge dx^\beta\wedge dx^\gamma\right]$$

$$\Theta_{\circ\circ} = \frac{1}{2}\frac{1}{3!}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\epsilon_{\alpha\beta\gamma}e^I_\mu\omega_\rho{}^J{}_L\omega^{LK}_\sigma\left[dx^\alpha\wedge dx^\beta\wedge dx^\gamma\right]$$

We have the expression $\epsilon^{\mu\rho\sigma}\epsilon_{\alpha\beta\gamma} = \begin{vmatrix}\delta^\mu_\alpha & \delta^\rho_\alpha & \delta^\sigma_\alpha\\ \delta^\mu_\beta & \delta^\rho_\beta & \delta^\sigma_\beta\\ \delta^\mu_\gamma & \delta^\rho_\gamma & \delta^\sigma_\gamma\end{vmatrix} = \delta^\mu_\alpha\begin{vmatrix}\delta^\rho_\beta & \delta^\sigma_\beta\\ \delta^\rho_\gamma & \delta^\sigma_\gamma\end{vmatrix} - \delta^\rho_\alpha\begin{vmatrix}\delta^\mu_\alpha & \delta^\sigma_\alpha\\ \delta^\mu_\gamma & \delta^\sigma_\gamma\end{vmatrix} + \delta^\sigma_\alpha\begin{vmatrix}\delta^\mu_\alpha & \delta^\rho_\alpha\\ \delta^\mu_\beta & \delta^\rho_\beta\end{vmatrix}$

$$\epsilon^{\mu\rho\sigma}\epsilon_{\alpha\beta\gamma} = \delta^\mu_\alpha\delta^\rho_\beta\delta^\sigma_\gamma - \delta^\mu_\alpha\delta^\sigma_\beta\delta^\rho_\gamma - (\delta^\rho_\alpha\delta^\mu_\beta\delta^\sigma_\gamma - \delta^\rho_\alpha\delta^\sigma_\beta\delta^\mu_\gamma) + \delta^\sigma_\alpha\delta^\mu_\beta\delta^\rho_\beta - \delta^\sigma_\alpha\delta^\rho_\beta\delta^\mu_\beta$$

that is equivalently written:

$$\epsilon^{\mu\rho\sigma}\epsilon_{\alpha\beta\gamma} = 3!\delta^{[\mu}_\alpha\delta^\rho_\beta\delta^{\sigma]}_\gamma = \frac{3!}{3}\left[\delta^\mu_\alpha\delta^{[\rho}_\beta\delta^{\sigma]}_\gamma - \delta^\rho_\alpha\delta^{[\mu}_\beta\delta^{\sigma]}_\gamma + \delta^\sigma_\alpha\delta^{[\mu}_\beta\delta^{\rho]}_\gamma\right]$$

So that $\Theta_{\circ\circ}$ is now written:

$$\Theta_{\circ\circ} = \frac{1}{2}\frac{1}{3!}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\epsilon_{\alpha\beta\gamma}e^I_\mu\omega_\rho{}^J{}_L\omega^{LK}_\sigma dx^\alpha\wedge dx^\beta\wedge dx^\gamma = \frac{1}{2}\epsilon_{IJK}e^I_\mu\omega_\rho{}^J{}_L\omega^{LK}_\sigma\Big(dx^\mu\wedge dx^\rho\wedge dx^\sigma - dx^\mu\wedge dx^\sigma\wedge dx^\rho$$
$$-dx^\rho\wedge dx^\mu\wedge dx^\sigma + dx^\rho\wedge dx^\sigma\wedge dx^\mu + dx^\sigma\wedge dx^\mu\wedge dx^\rho - dx^\sigma\wedge dx^\rho\wedge dx^\mu\Big)$$

$$\Theta_{\circ\circ} = \frac{1}{2}\frac{1}{3!}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\epsilon_{\alpha\beta\gamma}e^I_\mu\omega_\rho{}^J{}_L\omega^{LK}_\sigma dx^\alpha\wedge dx^\beta\wedge dx^\gamma = \frac{1}{2}\epsilon_{IJK}e^I_\mu\omega_\rho{}^J{}_L\omega^{LK}_\sigma dx^\mu\wedge dx^\rho\wedge dx^\sigma$$

Similarly we find directly (321), namely $\Theta_\circ$. Notice that:

$$\epsilon^{\mu\rho\sigma}\beta = (1/3!)\epsilon^{\mu\rho\sigma}\epsilon_{\alpha\beta\gamma}dx^\alpha\wedge dx^\beta\wedge dx^\gamma = dx^\mu\wedge dx^\rho\wedge dx^\sigma$$

We recover the Poincaré-Cartan 3-form (319): $\theta^{\circ} = \frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\big(e^I_\mu d\omega^{JK}_\sigma\wedge\beta_\rho + e^I_\mu\omega_\rho{}^J{}_L\omega^{LK}_\sigma\beta\big)$ ⌋

Hence, on one hand, we make the straightforward calculation of the exterior derivative $d\theta^{\circ}_{|d\omega\wedge\beta_\sigma}$ of the first term $\theta^{\circ}_{|d\omega\wedge\beta_\sigma}$ in the Poincaré-Cartan like 3-form θ°:

$$d\theta_{|d\omega\wedge\beta_\sigma} = d\Big[\frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}e^I_\mu d\omega^{JK}_\sigma\wedge\beta_\rho\Big] = \frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}de^I_\mu\wedge d\omega^{JK}_\sigma\wedge\beta_\rho \tag{323}$$

whereas on the other hand, we get for the exterior derivative $d\theta^{\circ}_{|\beta}$ of the second term $\theta^{\circ}_{|\beta}$ in the Poincaré-Cartan 3-form θ°:

$$d\theta^{\circ}_{|\beta} = d\Big[\frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}e^I_\mu\omega_\rho{}^J{}_L\omega^{LK}_\sigma\beta\Big] = \frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}d\Big[e^I_\mu\omega_\rho{}^J{}_L\omega^{LK}_\sigma\Big]\wedge\beta$$

With $d\big[e^I_\mu\omega_\rho{}^J{}_L\omega^{LK}_\sigma\big] = d\big[e^I_\mu\big]\omega_\rho{}^J{}_L\omega^{LK}_\sigma + e^I_\mu d\big[\omega_\rho{}^J{}_L\big]\omega^{LK}_\sigma + e^I_\mu\omega_\rho{}^J{}_L d\big[\omega^{LK}_\sigma\big]$. We obtain the following expression:

$$d\theta^{\circ}_{|\beta} = \frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\omega_\rho{}^J{}_L\omega^{LK}_\sigma de^I_\mu\wedge\beta + \frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}e^I_\mu\Big[d\omega_\rho{}^J{}_L\omega^{LK}_\sigma + \omega_\rho{}^J{}_L d\omega^{LK}_\sigma\Big]\wedge\beta \tag{324}$$

Therefore we consider the multisymplectic 4-form, using (323) and (324):

$$\boldsymbol{\omega}^{\circ} = d\theta^{\circ} = d\theta^{\circ}_{|d\omega\wedge\beta_\sigma} + d\theta^{\circ}_{|\beta} \tag{325}$$

Then, we obtain:

$$\begin{aligned}\boldsymbol{\omega}^{\circ} &= d\theta^{\circ} = \frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\Big[de^I_\mu\wedge d\omega^{JK}_\sigma\wedge\beta_\rho + \omega_\rho{}^J{}_L\omega^{LK}_\sigma de^I_\mu\wedge\beta\Big] + \epsilon_{IJK}\epsilon^{\mu\rho\sigma}\Big[e^I_\mu\omega_\rho{}^J{}_L d\omega^{LK}_\sigma\wedge\beta\Big]\\ &= \frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\Big[de^I_\mu\wedge d\omega^{JK}_\sigma\wedge\beta_\rho + \omega_\rho{}^J{}_L\omega^{LK}_\sigma de^I_\mu\wedge\beta\Big] - \frac{1}{2}\epsilon_{LJK}\epsilon^{\mu\rho\sigma}\Big[e^I_\mu\omega_\sigma{}^L{}_I d\omega^{JK}_\rho\wedge\beta\Big]\end{aligned} \tag{326}$$

We decompose the multisymplectic form (326) in three terms:

$$\boldsymbol{\omega}^{\circ} = \boldsymbol{\omega}^{\circ}_{|de\wedge d\omega\wedge\beta_\sigma} + \boldsymbol{\omega}^{\circ}_{|de\wedge\beta} + \boldsymbol{\omega}^{\circ}_{|d\omega\wedge\beta}$$

namely:

$$\boldsymbol{\omega}^{\circ} = \frac{1}{2}\Big(\underbrace{\epsilon_{IJK}\epsilon^{\mu\rho\sigma}de^I_\mu\wedge d\omega^{JK}_\sigma\wedge\beta_\rho}_{\Omega_{|de\wedge d\omega\wedge\beta_\sigma}} + \underbrace{\epsilon_{IJK}\epsilon^{\mu\rho\sigma}(\omega_\rho{}^J{}_L\omega^{LK}_\sigma)de^I_\mu\wedge\beta}_{\Omega_{|de\wedge\beta}}\underbrace{-\epsilon_{LJK}\epsilon^{\mu\rho\sigma}(e^I_\mu\omega_\sigma{}^L{}_I)d\omega^{JK}_\rho\wedge\beta}_{\Omega_{|d\omega\wedge\beta}}\Big)$$

The Hamilton equations — In the pre-multisymplectic setting, we work with the constraint $\mathcal{H}=0$, so that the Hamilton equations are written $X \lrcorner \boldsymbol{\omega}^{\circ}|_\Gamma = 0$. We evaluate the interior product of the vector field X with each of the term involved in $\boldsymbol{\omega}^{\circ}$. We find three terms $\boldsymbol{\omega}^{\circ}{}_{|de\wedge d\omega\wedge\beta_\sigma}$, $\boldsymbol{\omega}^{\circ}{}_{|de\wedge\beta}$ and $\boldsymbol{\omega}^{\circ}{}_{|d\omega\wedge\beta}$. One is first interested in the first term $X\lrcorner\boldsymbol{\omega}^{\circ}_{|de\wedge d\omega\wedge\beta_\sigma}$:

$$\begin{aligned}X\lrcorner\boldsymbol{\omega}^{\circ}_{|de\wedge d\omega\wedge\beta_\sigma} &= X\lrcorner\big[\frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}de^I_\mu\wedge d\omega^{JK}_\sigma\wedge\beta_\rho\big]\\ &= \frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\big[\big[d\omega^{JK}_\sigma\wedge\beta_\rho\big](X)de^I_\mu - \big[de^I_\mu\wedge\beta_\rho\big](X)d\omega^{JK}_\sigma\big]\\ &= +\frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\big[de^I_\mu\wedge d\omega^{JK}_\sigma\wedge\beta_{\lambda\rho}\big](X)dx^\lambda\big]\end{aligned} \tag{327}$$

The second step is given by:

$$\begin{aligned} X \lrcorner \boldsymbol{\omega}^{\circ}_{|de\wedge\beta} &= X \lrcorner \big[\frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\omega_{\rho}{}^{J}{}_{L}\omega_{\sigma}^{LK}de_{\mu}^{I}\wedge\beta\big] \\ &= \frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\omega_{\rho}{}^{J}{}_{L}\omega_{\sigma}^{LK}\Big[\beta(X)de_{\mu}^{I} - (de_{\mu}^{I}\wedge\beta_{\lambda})(X)dx^{\lambda}\Big] \end{aligned} \tag{328}$$

Finally, the third term is given by:

$$\begin{aligned} X \lrcorner \boldsymbol{\omega}^{\circ}_{|d\omega\wedge\beta} &= X \lrcorner \Big[-\frac{1}{2}\epsilon_{LJK}\epsilon^{\mu\rho\sigma}e_{\mu}^{I}\omega_{\rho}{}^{L}{}_{I}d\omega_{\sigma}^{JK}\wedge\beta\Big] \\ &= -\frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\Big[e_{\mu}^{L}\omega_{\rho}{}^{I}{}_{L}\beta(X)d\omega_{\sigma}^{JK} - (d\omega_{\sigma}^{JK}\wedge\beta_{\lambda})(X)dx^{\lambda}\Big] \end{aligned} \tag{329}$$

Now, thanks to (327), (328), (329) we are left with the following equations:

$$\begin{aligned} \beta(X) &= 1 \\ -(de_{\mu}^{I}\wedge\beta_{\lambda})(X)dx^{\lambda} &= -\Theta^{I}_{\lambda\mu}dx^{\lambda} \\ -(d\omega_{\sigma}^{LK}\wedge\beta_{\lambda})(X)dx^{\lambda} &= -\Theta^{IJ}_{\lambda\mu}dx^{\lambda} \\ \big[de_{\mu}^{I}\wedge d\omega_{\sigma}^{JK}\wedge\beta_{\lambda\rho}\big](X)dx^{\lambda} &= \big[\Theta^{JK}_{\lambda\sigma}\Theta^{I}_{\rho\mu} - \Theta^{JK}_{\rho\sigma}\Theta^{I}_{\lambda\mu}\big] \end{aligned} \tag{330}$$

Then:

$$X \lrcorner \boldsymbol{\omega}^{\circ} = \frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\Big[\big[\Theta^{JK}_{\rho\sigma} + \omega_{\rho}{}^{J}{}_{L}\omega_{\sigma}^{LK}\big]de_{\mu}^{I} - \big[\Theta^{I}_{\rho\mu} + e_{\mu}^{L}\omega_{\rho}{}^{I}{}_{L}\big]d\omega_{\sigma}^{JK}\big] + \big[\boldsymbol{\Upsilon}_{\lambda}\big]dx^{\lambda}\Big]$$

With $\big[\boldsymbol{\Upsilon}_{\lambda}\big] = e_{\mu}^{L}\omega_{\rho}{}^{I}{}_{L}\Theta^{JK}_{\lambda\sigma} - \omega_{\rho}{}^{J}{}_{L}\omega_{\sigma}^{LK}\Theta^{I}_{\lambda\mu} + \big[\Theta^{JK}_{\lambda\sigma}\Theta^{I}_{\rho\mu} - \Theta^{JK}_{\rho\sigma}\Theta^{I}_{\lambda\mu}\big]$ what leads us to the following equations:

$$\begin{aligned} \epsilon_{IJK}\epsilon^{\mu\rho\sigma}\big(\Theta^{JK}_{\rho\sigma} + \omega_{\rho}{}^{J}{}_{L}\omega_{\sigma}^{LK}\big) &= 0 \\ \epsilon_{IJK}\epsilon^{\mu\rho\sigma}\big(\Theta^{I}_{\rho\mu} + e_{\mu}^{L}\omega_{\rho}{}^{I}{}_{L}\big) &= 0 \end{aligned} \tag{331}$$

Together with the following one:

$$\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\Big[\boldsymbol{\Upsilon}\Big] = 0 \tag{332}$$

Remarks:

(i) One observes that if the first two conditions in (331) are satisfy, then the last one (332) $\big[\boldsymbol{\Upsilon}_{\lambda}\big] = 0$ is automatically verified.

$$\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\big[\boldsymbol{\Upsilon}_{\lambda}\big] = \epsilon_{IJK}\epsilon^{\mu\rho\sigma}e_{\mu}^{L}\omega_{\rho}{}^{I}{}_{L}\Theta^{JK}_{\lambda\sigma} - \epsilon_{IJK}\epsilon^{\mu\rho\sigma}\omega_{\rho}{}^{J}{}_{L}\omega_{\sigma}^{LK}\Theta^{I}_{\lambda\mu} + \big[\Theta^{JK}_{\lambda\sigma}\Theta^{I}_{\rho\mu} - \Theta^{JK}_{\rho\sigma}\Theta^{I}_{\lambda\mu}\big]$$

$$\epsilon_{IJK}\epsilon^{\mu\rho\sigma}\big[\boldsymbol{\Upsilon}_{\lambda}\big] = -\Theta^{I}_{\rho\mu}\Theta^{JK}_{\lambda\sigma} + \Theta^{JK}_{\rho\sigma}\Theta^{I}_{\lambda\mu} + \big[\Theta^{JK}_{\lambda\sigma}\Theta^{I}_{\rho\mu} - \Theta^{JK}_{\rho\sigma}\Theta^{I}_{\lambda\mu}\big] = 0$$

(ii) The system (331) gives backs the Einstein 3D equations. It is equivalent to write it under the followings form:

$$\begin{aligned} \epsilon_{IJK}F^{JK} &= 0 \qquad\qquad & F &= 0 \\ \epsilon_{IJK}d^{\mathbf{D}}e^{I} &= 0 \qquad\qquad & d^{\mathbf{D}}e &= 0 \end{aligned} \tag{333}$$

⌜ We have:

$$\epsilon_{IJK}F^{JK} = \epsilon_{IJK}\Big[\partial_{[\rho}\omega_{\sigma]}^{JK} + \omega^{J}_{[\rho_{L}}\omega_{\sigma]}^{LK}\Big]dx^{\rho}\wedge dx^{\sigma} = \epsilon_{IJK}\partial_{[\rho}\omega_{\sigma]}^{JK}dx^{\rho}\wedge dx^{\sigma} + \epsilon_{IJK}\omega^{J}_{[\rho_{L}}\omega_{\sigma]}^{LK}dx^{\rho}\wedge dx^{\sigma}$$

$$\epsilon_{IJK}F^{JK} = \epsilon_{IJK}\Big[\Theta^{JK}_{[\sigma\rho]}\Big]dx^{\rho}\wedge dx^{\sigma} + \epsilon_{IJK}\omega^{J}_{[\rho_{L}}\omega_{\sigma]}^{LK}dx^{\rho}\wedge dx^{\sigma}$$

as we might write $\epsilon^{\rho\sigma\mu}\beta_\mu = dx^\rho \wedge dx^\sigma$

$$\epsilon_{IJK}F^{JK} = \epsilon_{IJK}\Big[\Theta^{JK}_{[\sigma\rho]} + \omega^{J}_{[\rho\,L}\omega^{LK}_{\sigma]}\Big]\epsilon^{\rho\sigma\mu}\beta_\mu = \epsilon_{IJK}\epsilon^{\rho\sigma\mu}\frac{1}{2}\Big[\Theta^{JK}_{\sigma\rho} - \Theta^{JK}_{\rho\sigma} + \omega^{J}_{\rho\,L}\omega^{LK}_{\sigma} - \omega^{J}_{\sigma\,L}\omega^{LK}_{\rho}\Big]\beta_\mu$$

$$\epsilon_{IJK}F^{JK} = \epsilon_{IJK}\epsilon^{\rho\sigma\mu}\Big[\Theta^{JK}_{\sigma\rho} + \omega^{J}_{\rho\,L}\omega^{LK}_{\sigma}\Big]\beta_\mu = \epsilon_{IJK}\epsilon^{\mu\rho\sigma}\Big[\Theta^{JK}_{\sigma\rho} + \omega^{J}_{\rho\,L}\omega^{LK}_{\sigma}\Big]\beta_\mu$$

Hence, having computed the first Hamiltonian equation, we see that as $\beta_\mu \neq 0$ it is equivalent to write it $\epsilon_{IJK}F^{JK} = 0$, namely the first line of (333). Now we are interested in the second equation $\epsilon_{IJK}d^{\mathbf{D}}(e^K) = 0$. We have:

$$\epsilon_{IJK}d^{\mathbf{D}}e^K = \epsilon_{IJK}de^K = \epsilon_{IJK}\Big[\partial_{[\rho}e^K_{\sigma]} + \omega^{K}_{[\rho\,L}e^L_{\sigma]}\Big]dx^\rho\wedge dx^\sigma = \epsilon_{IJK}\partial_{[\rho}e^K_{\sigma]}dx^\rho\wedge dx^\sigma + \epsilon_{IJK}\omega^{K}_{[\rho\,L}e^L_{\sigma]}dx^\rho\wedge dx^\sigma$$

$$\epsilon_{IJK}d^{\mathbf{D}}e^K = \epsilon_{IJK}\Big[\Theta^{K}_{[\sigma\rho]}\Big]dx^\rho \wedge dx^\sigma + \epsilon_{IJK}\omega^{K}_{[\rho\,L}e^L_{\sigma]}dx^\rho \wedge dx^\sigma$$

as we might write $\epsilon^{\rho\sigma\mu}\beta_\mu = dx^\rho \wedge dx^\sigma$

$$\epsilon_{IJK}d^{\mathbf{D}}e^K = \epsilon_{IJK}\epsilon^{\rho\sigma\mu}\Big[\Theta^{K}_{\sigma\rho} + \omega^{K}_{\rho\ L}e^L_{\sigma}\Big]\beta_\mu = \epsilon_{IJK}\epsilon^{\rho\mu\sigma}\Theta^{K}_{\sigma\mu}\beta_\mu + \epsilon^{\mu\sigma\rho}\omega^{J}_{\mu\,L}\omega^{LK}_{\rho}\beta_\rho = \epsilon_{IJK}\epsilon^{\rho\mu\sigma}\Big[\Theta^{K}_{\mu\rho} + \omega^{K}_{\rho\ L}e^L_{\mu}\Big]\beta_\sigma$$

So the result follows. ⌋

17.2 Pre-multisymplectic treatment of Palatini 4D Gravity

Now we are interested in the $4D$ Palatini case, see[99] the work of Rovelli [199]. Let us consider the action (334):

$$S_{\mathsf{Palatini}} = \frac{1}{2}\int \epsilon_{IJKL}e^I \wedge e^J \wedge F^{KL} \tag{334}$$

with $F^{KL} = d\omega^{KL} + \omega^K{}_M \wedge \omega^{ML}$ Then, as, $e^I = e^I_\mu dx^\mu$, $\omega^{KL} = \omega^{KL}_\mu dx^\mu$ we obtain the following expression for the Poincaré-Cartan canonical 4-form:

$$\theta^{\circ} = \frac{1}{2}\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu e^J_\nu d\omega^{KL}_\rho \wedge \beta_\sigma + \frac{1}{2}\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu e^J_\nu \omega_\sigma{}^K{}_M\omega^{ML}_\rho\beta \tag{335}$$

⌈ **Proof** We derive by direct calculation from (334)

$$\theta^{\circ} = \frac{1}{2}\epsilon_{IJKL}e^I_\mu dx^\mu \wedge e^J_\nu dx^\nu \wedge [d(\omega^{KL}_\rho dx^\rho) + \omega_\rho{}^K{}_M dx^\rho \wedge \omega^{ML}_\sigma dx^\sigma]$$

$$\theta^{\circ} = \frac{1}{2}\epsilon_{IJKL}e^I_\mu e^J_\nu dx^\mu \wedge dx^\nu \wedge d\omega^{KL}_\sigma \wedge dx^\sigma + \frac{1}{2}\epsilon_{IJKL}e^I_\mu e^J_\nu \omega_\rho{}^K{}_M\omega^{ML}_\sigma dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$$

So that we write the Poincaré-Cartan form as $\theta = \theta_{|d\omega\wedge\beta_\sigma} + \theta_{|\beta}$ with:

$$\begin{array}{rcl} \theta^{\circ}{}_{|d\omega\wedge\beta_\sigma} & = & \frac{1}{2}\epsilon_{IJKL}e^I_\mu e^J_\nu dx^\mu \wedge dx^\nu \wedge d\omega^{JK}_\sigma \wedge dx^\sigma \\ \theta^{\circ}{}_{|\beta} & = & \frac{1}{2}\epsilon_{IJKL}e^I_\mu e^J_\nu \omega_\rho{}^K{}_M\omega^{ML}_\sigma dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \end{array} \tag{336}$$

Since we have

$$\epsilon^{\mu\nu\rho\sigma}\beta_\sigma = \epsilon^{\mu\nu\rho\sigma}\frac{1}{3!}\epsilon_{\sigma\alpha\beta\gamma}dx^\alpha \wedge dx^\beta \wedge dx^\gamma = \frac{1}{3!}\epsilon^{\sigma\mu\nu\rho}\epsilon_{\sigma\alpha\beta\gamma}dx^\alpha \wedge dx^\beta \wedge dx^\gamma = \frac{1!(4-1)!}{3!}\delta^{[\mu}_\alpha\delta^\nu_\beta\delta^{\rho]}_\gamma dx^\alpha \wedge dx^\beta \wedge dx^\gamma$$

So that $\epsilon^{\mu\nu\rho\sigma}\beta_\sigma = dx^\mu \wedge dx^\nu \wedge dx^\rho$ then, $dx^\mu \wedge dx^\nu \wedge d\omega^{KL}_\sigma \wedge dx^\sigma = d\omega^{KL}_\sigma \wedge dx^\mu \wedge dx^\nu \wedge dx^\sigma = \epsilon^{\mu\nu\rho\sigma}d\omega^{KL}_\rho \wedge \beta_\sigma$

$$\Theta_\circ = \frac{1}{2}\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu e^J_\nu d\omega^{KL}_\rho \wedge \beta_\sigma = \theta^{\circ}{}_{|d\omega\wedge\beta_\sigma} \tag{337}$$

[99] Hélein and Kouneiher [118] propose a slightly different action, $\mathcal{L}_{\mathsf{Palatini}} = (1/4!)\int_{\mathcal{X}}\epsilon_{IJKL}\mathsf{h}^{LN}e^I \wedge e^J \wedge F^K_L$ underlying the account of all possible signatures for the metric.

we recover the first part of the expression (335). For the second part of (335), we find the analogue development for the expression $\Theta_{\circ\circ}$:

$$\Theta_{\circ\circ} = \frac{1}{2}\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^I_\mu e^J_\nu \omega_\sigma{}^K{}_M \omega^{ML}_\rho \beta$$

since $\beta = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$. Since, $\beta = (1/4!)\epsilon_{\alpha\beta\gamma\delta} dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta$ one writes:

$$\Theta_{\circ\circ} = \frac{1}{2}\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^I_\mu e^J_\nu \omega_\sigma{}^K{}_M \omega^{ML}_\rho \beta = \frac{1}{2}\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^I_\mu e^J_\nu \omega_\sigma{}^K{}_M \omega^{ML}_\rho \Big[(1/4!)\epsilon_{\alpha\beta\gamma\delta} dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta\Big]$$

We use $\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta} = \begin{vmatrix} \delta^\mu_\alpha & \delta^\nu_\alpha & \delta^\rho_\alpha & \delta^\sigma_\alpha \\ \delta^\mu_\beta & \delta^\nu_\beta & \delta^\rho_\beta & \delta^\sigma_\beta \\ \delta^\mu_\gamma & \delta^\nu_\gamma & \delta^\rho_\gamma & \delta^\sigma_\gamma \\ \delta^\mu_\delta & \delta^\nu_\delta & \delta^\rho_\delta & \delta^\sigma_\delta \end{vmatrix}$ so that (we drop out the calculation):

$$\Theta_{\circ\circ} = \frac{1}{2}\epsilon_{IJKL} e^I_\mu e^J_\nu \omega_\rho{}^K{}_M \omega^{ML}_\sigma dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = \theta_{|\beta}$$

We have (335) $\quad\rfloor$

Now let us compute the pre-multisymplectic 5-form: $\boldsymbol{\omega}^{\circ} = d\theta^{\circ}$

$$\begin{aligned}
\boldsymbol{\omega}^{\circ} &= \frac{1}{2}\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} d\big[e^I_\mu e^J_\nu\big] \wedge d\omega^{KL}_\rho \wedge \beta_\sigma + \frac{1}{2}\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} d\big[e^I_\mu e^J_\nu \omega_\sigma{}^K{}_M \omega^{ML}_\rho\big] \wedge \beta \\
&= \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^I_\mu de^J_\nu \wedge d\omega^{KL}_\rho \wedge \beta_\sigma + \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^I_\mu \omega_\sigma{}^K{}_M \omega^{ML}_\rho de^J_\nu \wedge \beta \\
&\quad + \frac{1}{2}\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^I_\mu e^J_\nu d\big[\omega_\sigma{}^K{}_M \omega^{ML}_\rho\big] \wedge \beta \\
&= \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^I_\mu \Big[de^J_\nu \wedge d\omega^{KL}_\rho \wedge \beta_\sigma + \omega^K_{\sigma M}\omega^{ML}_\rho de^J_\nu \wedge \beta\Big] \\
&\quad + \frac{1}{2}\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^I_\mu e^J_\nu \omega^{ML}_\rho d\big[\omega^K_{\sigma M}\big] \wedge \beta + \frac{1}{2}\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^I_\mu e^J_\nu \omega_\sigma{}^K{}_M d\big[\omega^{ML}_\rho\big] \wedge \beta
\end{aligned}$$

Finally, we have:

$$\boldsymbol{\omega}^{\circ} = \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^I_\mu \Big[de^J_\nu \wedge d\omega^{KL}_\rho \wedge \beta_\sigma + \omega^K_{\sigma M}\omega^{ML}_\rho de^J_\nu \wedge \beta\Big] + \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^I_\mu e^J_\nu \omega_\sigma{}^K{}_M d\omega^{ML}_\rho \wedge \beta$$

Then, in the $4D$ case[100]

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKL} e^I_\mu e^J_\nu \omega_\sigma{}^K{}_M d\omega^{ML}_\rho = -\epsilon^{\mu\nu\rho\sigma}\epsilon_{INKL} e^I_\mu e^J_\nu \omega_\sigma{}^N{}_J d\omega^{KL}_\rho \tag{338}$$

Then we find the expression of the pre-multisymplectic 5-form:

$$\boldsymbol{\omega}^{\circ} = \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^I_\mu \Big[de^J_\nu \wedge d\omega^{KL}_\rho \wedge \beta_\sigma + \omega_\sigma{}^K{}_M \omega^{ML}_\rho de^J_\nu \wedge \beta\Big] - \epsilon^{\mu\nu\rho\sigma}\epsilon_{INKL} e^I_\mu e^J_\nu \omega_\sigma{}^N{}_J d\omega^{KL}_\rho \wedge \beta$$

We decompose the multisymplectic form given by the last equation into three terms: $\boldsymbol{\omega}^{\circ} = \Omega_{|de\wedge d\omega\wedge\beta_\sigma} + \Omega_{|de\wedge\beta} + \Omega_{|d\omega\wedge\beta}$. namely:

$$\begin{aligned}
\boldsymbol{\omega}^{\circ} = &\underbrace{\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^I_\mu de^J_\nu \wedge d\omega^{KL}_\rho \wedge \beta_\sigma}_{\Omega_{|de\wedge d\omega\wedge\beta_\sigma}} + \underbrace{\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e^I_\mu \omega_\sigma{}^K{}_M \omega^{ML}_\rho de^J_\nu \wedge \beta}_{\Omega_{|de\wedge\beta}} \\
&\underbrace{-\epsilon^{\mu\nu\rho\sigma}\epsilon_{INKL} e^I_\mu e^J_\nu \omega_\sigma{}^N{}_J d\omega^{KL}_\rho \wedge \beta}_{\Omega_{|d\omega\wedge\beta}}
\end{aligned} \tag{339}$$

The Hamilton equation. In the pre-multisymplectic setting, we work with the constraint $\mathcal{H} = 0$, and we recover the Hamilton equations by:

$$X \lrcorner\, \boldsymbol{\omega}^{\circ} = 0 \tag{340}$$

[100]there is a straightforward generalization for n-dimensional gravity.

The idea is to evaluate the interior product of the vector field X with each of the term of $\boldsymbol{\omega}^{\circ}$, just in the interlude below we exhibit each of these three terms $\boldsymbol{\omega}^{\circ}_{|d\mathfrak{e}\wedge d\omega\wedge\beta_\sigma}$, $\boldsymbol{\omega}^{\circ}_{|d\mathfrak{e}\wedge\beta}$ and finally $\boldsymbol{\omega}^{\circ}_{|d\omega\wedge\beta}$ exactly as we did for 3D case. We choose a 4-vecor $X = X_1 \wedge X_2 \wedge X_3 \wedge X_4$, with $\forall\alpha$

$$X_\alpha = \frac{\partial}{\partial x^\alpha} + \Theta^I_{\alpha\mu_\alpha}\frac{\partial}{\partial e^I_{\mu_\alpha}} + \Theta^{IJ}_{\alpha\mu_\alpha}\frac{\partial}{\partial \omega^I_{\mu_\alpha}}$$

The left side of equation (340) is written:

$$X \mathbin{\lrcorner} \boldsymbol{\omega}^{\circ} = X \mathbin{\lrcorner} \boldsymbol{\omega}^{\circ}_{|d\mathfrak{e}\wedge d\omega\wedge\beta_\sigma} + X \mathbin{\lrcorner} \boldsymbol{\omega}^{\circ}_{|d\mathfrak{e}\wedge\beta} + X \mathbin{\lrcorner} \boldsymbol{\omega}^{\circ}_{|d\omega\wedge\beta}$$

Then:

$$X \mathbin{\lrcorner} \boldsymbol{\omega}^{\circ} = \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu\Big[(d\omega^{KL}_\rho\wedge\beta_\sigma)(X)de^J_\nu - (de^J_\nu\wedge\beta_\sigma)(X)d\omega^{KL}_\rho + (de^J_\nu\wedge d\omega^{KL}_\rho\wedge\beta_{\lambda\sigma})(X)dx^\lambda\Big]$$

$$+\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu\omega_\sigma{}^K{}_M\omega^{ML}_\rho\Big[(\beta)(X)de^J_\nu - (de^J_\nu\wedge\beta_\lambda)(X)dx^\lambda\Big]$$

$$-\epsilon_{INKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu e^J_\nu\omega_\sigma{}^N{}_J\Big[(\beta)(X)d\omega^{KL}_\rho + (d\omega^{KL}_\rho\wedge\beta_\lambda)(X)dx^\lambda(X)\Big]$$

$$X \mathbin{\lrcorner} \boldsymbol{\omega}^{\circ} = \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu\Big[(\Theta^{KL}_{\sigma\rho} + \omega_\sigma{}^K{}_M\omega^{ML}_\rho)de^J_\nu - (e^I_\mu e^N_\nu\omega_\sigma{}^J{}_N + \Theta^I_{\sigma\nu})d\omega^{KL}_\rho + [\star\star\star]dx^\lambda\Big]$$

The term $[\star\star\star]$ is not considered for practical purpose. We find the generalized Hamilton Palatini equations (341) in this pre-multisymplectic setting.

$$\begin{aligned}\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu\Big[\Theta^{KL}_{\sigma\rho} + \omega_\sigma{}^K{}_M\omega^{ML}_\rho\Big] &= 0\\ \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^I_\mu\Big[\Theta^J_{\sigma\nu} + e^N_\nu\omega_\sigma{}^J{}_N\Big] &= 0\end{aligned} \tag{341}$$

18 Algebraic observable $(n-1)$-forms and canonical forms

18.1 Prolegomena for the ontologic standpoint

In this section we give the general method for the algebraic observable studies. This example is to emphasize all the complexity of this approach. In particular we focus on the proposition (6.1). We consider the specific case where we are not taking the full constraint set, rather we consider $\overline{\mathcal{M}}_{\mathsf{DW}}$ described by (342). Notice that in the Palatini gravity we have two sets of constraints previously described by (295). We considered $\overline{\mathcal{M}}_{\mathsf{Palatini}}$ by the set of coordinates $(x, e, \omega, p) \in \mathcal{M}_{\mathsf{DW}}$ with $p^{\omega_\mu\nu}_{IJ} + p^{\omega_\nu\mu}_{IJ} = 0$ and $p^{e_\mu\nu}_I = 0$. Now we are not condidering the antisymmetric constraint for the multimomenta $p^{\omega_\mu\nu}_{IJ}$ but we keep the condition $p^{e_\mu\nu}_I = 0$. We denote the related multisymplectic manifold by $(\overline{\mathcal{M}}_{\mathsf{DW}}, \boldsymbol{\omega}^{\mathsf{DW}})$. We focus on the study of the infinitesimal symplectomorphism $\mathfrak{sp}_\circ\overline{\mathcal{M}}_{\mathsf{DW}}$. Hence, we introduce:

$$\overline{\mathcal{M}}_{\mathsf{DW}} = \{(x, e, \omega, p) \in \mathcal{M}_{\mathsf{DW}} \ \ / \ \ p^{\omega_\mu\nu}_{IJ}\ ,\ p^{e_\mu\nu}_I = 0\} \tag{342}$$

Once again, here we do not consider the constraint $p^{\omega_\mu\nu}_{IJ} = ee^{[\nu}_I e^{\mu]}_J = -E^{[\mu}_I e^{\nu]}_J$. The purpose of this approach is to stress the use of geometrical objects beyond the key relation $d(\Xi \mathbin{\lrcorner} \boldsymbol{\omega}) = 0$ and a clear application of the proposition (6.1). We have the following inclusion or injection of multisymplectic manifolds:

$$\mathcal{M}_{\mathsf{Palatini}} \subset \overline{\mathcal{M}}_{\mathsf{Palatini}} \subset \overline{\mathcal{M}}_{\mathsf{DW}} \subset \mathcal{M}_{\mathsf{DW}}$$

We work within the DW setting such that the multisymplectic form of interested is $\boldsymbol{\omega}^{\mathsf{DW}} = d\mathfrak{e}\wedge\beta + dp_{IJ}^{\omega_\mu\nu}\wedge d\omega_\mu^{IJ}\wedge\beta_\nu + dp_{IJ}^{e_\mu\nu}\wedge de_\mu^{IJ}\wedge\beta_\nu$ Let us consider $\zeta\in\mathfrak{Z}$, we introduce the notations:

$$\zeta = X^\nu(x,e,\omega)\frac{\partial}{\partial x^\nu} + \Theta_\lambda^M(x,e,\omega)\frac{\partial}{\partial e_\lambda^M} + \Theta_\mu^{IJ}(x,e,\omega)\frac{\partial}{\partial\omega_\mu^{IJ}} \tag{343}$$

$X^\nu,\Theta_\lambda^M,\Theta_\mu^{IJ} : \mathfrak{Z}^\circ \to \mathbb{R}$ are smooth fonctions on $\mathfrak{Z}$. We also decompose the vector field $\Xi\in\Gamma(\mathcal{M}_{\mathsf{DW}},T\mathcal{M}_{\mathsf{DW}})$. In general case, as introduced in the proposition 3.1 we write:

$$\Xi = \Xi^\alpha(q,p)\frac{\partial}{\partial q^\alpha} + \sum_{\alpha_1<\ldots<\alpha_n}\Xi_{\alpha_1\ldots\alpha_n}(q,p)\frac{\partial}{\partial p_{\alpha_1\ldots\alpha_n}}$$

Now we adapt our notations for the Palatini gravity:

$$\Xi^\alpha(q,p) = \{\boldsymbol{X}^\nu(q,p);\boldsymbol{\Theta}_\lambda^M(q,p);\boldsymbol{\Theta}_\mu^{IJ}(q,p)\}$$

the only component for multimomenta are $\Xi_{\alpha_1\ldots\alpha_n}(q,p) = \{\boldsymbol{\Upsilon}(q,p);\boldsymbol{\Upsilon}_I^{e_\mu\nu}(q,p);\boldsymbol{\Upsilon}_{IJ}^{\omega_\mu\nu}\}$ Hence, we denote the decomposition of $\Xi\in\Gamma(\mathcal{M}_{\mathsf{DW}},T\mathcal{M}_{\mathsf{DW}})$ by:

$$\Xi = \boldsymbol{X}^\nu(q,p)\frac{\partial}{\partial x^\nu} + \boldsymbol{\Theta}_\lambda^M(q,p)\frac{\partial}{\partial e_\lambda^M} + \boldsymbol{\Theta}_\mu^{IJ}(q,p)\frac{\partial}{\partial\omega_\mu^{IJ}} + \boldsymbol{\Upsilon}(q,p)\frac{\partial}{\partial\mathfrak{e}} + \boldsymbol{\Upsilon}_I^{e_\mu\nu}(q,p)\frac{\partial}{\partial p_I^{e_\mu\nu}} + \boldsymbol{\Upsilon}_{IJ}^{\omega_\mu\nu}(q,p)\frac{\partial}{\partial p_{IJ}^{\omega_\mu\nu}}$$

So that $\boldsymbol{X}^\nu,\boldsymbol{\Theta}_\lambda^M,\boldsymbol{\Theta}_\mu^{IJ},\boldsymbol{\Upsilon},\boldsymbol{\Upsilon}_I^{e_\mu\nu},\boldsymbol{\Upsilon}_{IJ}^{\omega_\mu\nu} : \mathcal{M}_{\mathsf{DW}}\to\mathbb{R}$ are smooth fonctions on $\mathcal{M}_{\mathsf{DW}} = \Lambda^n T^\star\mathfrak{Z}^\circ$. We evaluate the expression $\Xi \lrcorner \boldsymbol{\omega}^{\mathsf{DW}}$:

$$\begin{aligned}\Xi \lrcorner \boldsymbol{\omega}^{\mathsf{DW}} = {} & \boldsymbol{\Upsilon}\beta - \boldsymbol{X}^\nu d\mathfrak{e}\wedge\beta_\nu + \boldsymbol{\Upsilon}_{IJ}^{\omega_\mu\nu}d\omega_\mu^{IJ}\wedge\beta_\nu - \boldsymbol{\Theta}_\mu^{IJ}dp_{IJ}^{\omega_\mu\nu}\wedge\beta_\nu + \boldsymbol{X}^\rho dp_{IJ}^{\omega_\mu\nu}\wedge d\omega_\mu^{IJ}\wedge\beta_{\rho\nu}\\ & + \boldsymbol{\Upsilon}_I^{e_\mu\nu}de_\mu^I\wedge\beta_\nu - \boldsymbol{\Theta}_\mu^I dp_I^{e_\mu\nu}\wedge\beta_\nu + \boldsymbol{X}^\rho dp_I^{e_\mu\nu}\wedge de_\mu^I\wedge\beta_{\rho\nu}\end{aligned}$$

We now work with $\overline{\mathcal{M}}_{\mathsf{DW}}$ so that we consider:

$$\Xi = \boldsymbol{X}^\nu(q,p)\frac{\partial}{\partial x^\nu} + \boldsymbol{\Theta}_\lambda^M(q,p)\frac{\partial}{\partial e_\lambda^M} + \boldsymbol{\Theta}_\mu^{IJ}(q,p)\frac{\partial}{\partial\omega_\mu^{IJ}} + \boldsymbol{\Upsilon}(q,p)\frac{\partial}{\partial\mathfrak{e}} + \boldsymbol{\Upsilon}_{IJ}^{\omega_\mu\nu}(q,p)\frac{\partial}{\partial p_{IJ}^{\omega_\mu\nu}} \tag{344}$$

The interior product involve now the following object: $\Xi\in\Gamma(\overline{\mathcal{M}}_{\mathsf{DW}},T\overline{\mathcal{M}}_{\mathsf{DW}})$, also we set $\overline{\boldsymbol{\omega}}^{\mathsf{DW}} = d\mathfrak{e}\wedge\beta + dp_{IJ}^{\omega_\mu\nu}\wedge d\omega_\mu^{IJ}\wedge\beta_\nu$.

$$\Xi \lrcorner \overline{\boldsymbol{\omega}}^{\mathsf{DW}} = \boldsymbol{\Upsilon}\beta - \boldsymbol{X}^\nu d\mathfrak{e}\wedge\beta_\nu + \boldsymbol{\Upsilon}_{IJ}^{\omega_\mu\nu}d\omega_\mu^{IJ}\wedge\beta_\nu - \boldsymbol{\Theta}_\mu^{IJ}dp_{IJ}^{\omega_\mu\nu}\wedge\beta_\nu + \boldsymbol{X}^\rho dp_{IJ}^{\omega_\mu\nu}\wedge d\omega_\mu^{IJ}\wedge\beta_{\rho\nu}$$

The right treatment would consider $\boldsymbol{\omega}^{\mathsf{DW}}$ which makes appear the term $\boldsymbol{\Upsilon}_I^{e_\mu\nu}de_\mu^I\wedge\beta_\nu$. We observe:

$$\Xi \lrcorner \boldsymbol{\omega}^{\mathsf{DW}} = \Xi \lrcorner \overline{\boldsymbol{\omega}}^{\mathsf{DW}} + \boldsymbol{\Upsilon}_I^{e_\mu\nu}de_\mu^I\wedge\beta_\nu$$

Let concentrate on $\Xi \lrcorner \overline{\boldsymbol{\omega}}^{\mathsf{DW}}$. The related infinitesimal symplectomorphism arise from the relation $d(\Xi \lrcorner \overline{\boldsymbol{\omega}}^{\mathsf{DW}}) = 0$. We are led to:

$$d(\Xi \lrcorner \overline{\boldsymbol{\omega}}^{\mathsf{DW}}) = d\boldsymbol{\Upsilon}\wedge\beta - d\boldsymbol{X}^\nu\wedge d\mathfrak{e}\wedge\beta_\nu + d\boldsymbol{\Upsilon}_{IJ}^{\omega_\mu\nu}\wedge d\omega_\mu^{IJ}\wedge\beta_\nu - d\boldsymbol{\Theta}_\mu^{IJ}\wedge dp_{IJ}^{\omega_\mu\nu}\wedge\beta_\nu + d\boldsymbol{X}^\rho\wedge dp_{IJ}^{\omega_\mu\nu}\wedge d\omega_\mu^{IJ}\wedge\beta_{\rho\nu}$$

$$\text{with}\quad\left|\begin{array}{ccc} d\boldsymbol{X}^\rho & = & \dfrac{\partial\boldsymbol{X}^\rho}{\partial x^\alpha}dx^\alpha + \dfrac{\partial\boldsymbol{X}^\alpha}{\partial e_\alpha^I}de_\alpha^I + \dfrac{\partial\boldsymbol{X}^\rho}{\partial\omega_\alpha^{IJ}}d\omega_\alpha^{IJ} + \dfrac{\partial\boldsymbol{X}^\rho}{\partial\mathfrak{e}}d\mathfrak{e} + \dfrac{\partial\boldsymbol{X}^\rho}{\partial p_{IJ}^{\omega_\alpha\beta}}dp_{IJ}^{\omega_\alpha\beta}\\ d\boldsymbol{\Theta}_\lambda^M & = & \dfrac{\partial\boldsymbol{\Theta}_\lambda^M}{\partial x^\alpha}dx^\alpha + \dfrac{\partial\boldsymbol{\Theta}_\lambda^M}{\partial e_\alpha^I}de_\alpha^I + \dfrac{\partial\boldsymbol{\Theta}_\lambda^M}{\partial\omega_\alpha^{IJ}}d\omega_\alpha^{IJ} + \dfrac{\partial\boldsymbol{\Theta}_\lambda^M}{\partial\mathfrak{e}}d\mathfrak{e} + \dfrac{\partial\boldsymbol{\Theta}_\lambda^M}{\partial p_{IJ}^{\omega_\alpha\beta}}dp_{IJ}^{\omega_\alpha\beta}\\ d\boldsymbol{\Theta}_\alpha^{IJ} & = & \dfrac{\partial\boldsymbol{\Theta}_\mu^{IJ}}{\partial x^\alpha}dx^\alpha + \dfrac{\partial\boldsymbol{\Theta}_\mu^{IJ}}{\partial e_\alpha^I}de_\alpha^I + \dfrac{\partial\boldsymbol{\Theta}_\mu^{IJ}}{\partial\omega_\alpha^{IJ}}d\omega_\alpha^{IJ} + \dfrac{\partial\boldsymbol{\Theta}_\mu^{IJ}}{\partial\mathfrak{e}}d\mathfrak{e} + \dfrac{\partial\boldsymbol{\Theta}_\mu^{IJ}}{\partial p_{IJ}^{\omega_\alpha\beta}}dp_{IJ}^{\omega_\alpha\beta}\\ d\boldsymbol{\Upsilon} & = & \dfrac{\partial\boldsymbol{\Upsilon}}{\partial x^\alpha}dx^\alpha + \dfrac{\partial\boldsymbol{\Upsilon}}{\partial e_\alpha^I}de_\alpha^I + \dfrac{\partial\boldsymbol{\Upsilon}}{\partial\omega_\alpha^{IJ}}d\omega_\alpha^{IJ} + \dfrac{\partial\boldsymbol{\Upsilon}}{\partial\mathfrak{e}}d\mathfrak{e} + \dfrac{\partial\boldsymbol{\Upsilon}}{\partial p_{IJ}^{\omega_\alpha\beta}}dp_{IJ}^{\omega_\alpha\beta}\\ d\boldsymbol{\Upsilon}_{IJ}^{\omega_\mu\nu} & = & \dfrac{\partial\boldsymbol{\Upsilon}_{IJ}^{\omega_\mu\nu}}{\partial x^\alpha}dx^\alpha + \dfrac{\partial\boldsymbol{\Upsilon}_{IJ}^{\omega_\mu\nu}}{\partial e_\alpha^I}de_\alpha^I + \dfrac{\partial\boldsymbol{\Upsilon}_{IJ}^{\omega_\mu\nu}}{\partial\omega_\alpha^{IJ}}d\omega_\alpha^{IJ} + \dfrac{\partial\boldsymbol{\Upsilon}_{IJ}^{\omega_\mu\nu}}{\partial\mathfrak{e}}d\mathfrak{e} + \dfrac{\partial\boldsymbol{\Upsilon}_{IJ}^{\omega_\mu\nu}}{\partial p_{IJ}^{\omega_\alpha\beta}}dp_{IJ}^{\omega_\alpha\beta}\end{array}\right.$$

The expression is decomposed:

$$d(\Xi \lrcorner \overline{\boldsymbol{\varpi}}^{\mathsf{DW}}) = \iota_{\mathbf{I}} + \iota_{\mathbf{II}} + \iota_{\mathbf{III}} + \iota_{\mathbf{IV}} + \iota_{\mathbf{V}}$$

with each term ι given by:

$$\iota_{\mathbf{I}} = d\boldsymbol{\Upsilon} \wedge \beta = \Big(\frac{\partial \boldsymbol{\Upsilon}}{\partial e^M_\lambda}\Big) de^M_\lambda \wedge \beta + \Big(\frac{\partial \boldsymbol{\Upsilon}}{\partial \omega^{IJ}_\mu}\Big) d\omega^{IJ}_\mu \wedge \beta + \Big(\frac{\partial \boldsymbol{\Upsilon}}{\partial \mathfrak{e}}\Big) d\mathfrak{e} \wedge \beta + \Big(\frac{\partial \boldsymbol{\Upsilon}}{\partial p^{\omega_\alpha \beta}_{IJ}}\Big) dp^{\omega_\alpha \beta}_{IJ} \wedge \beta$$

$$\iota_{\mathbf{II}} = -d\boldsymbol{X}^\nu \wedge d\mathfrak{e} \wedge \beta_\nu = \Big(\frac{\partial \boldsymbol{X}^\nu}{\partial x^\nu}\Big) d\mathfrak{e} \wedge \beta - \Big(\frac{\partial \boldsymbol{X}^\nu}{\partial e^M_\lambda}\Big) de^M_\lambda \wedge d\mathfrak{e} \wedge \beta_\nu - \Big(\frac{\partial \boldsymbol{X}^\nu}{\partial \omega^{IJ}_\mu}\Big) d\omega^{IJ}_\mu \wedge d\mathfrak{e} \wedge \beta_\nu - \Big(\frac{\partial \boldsymbol{X}^\nu}{\partial p^{\omega_\alpha \beta}_{IJ}}\Big) dp^{\omega_\alpha \beta}_{IJ} \wedge d\mathfrak{e} \wedge \beta_\nu$$

$$\iota_{\mathbf{III}} = d\boldsymbol{\Upsilon}^{\omega_\mu \nu}_{IJ} \wedge d\omega^{IJ}_\mu \wedge \beta_\nu = \Big(-\frac{\partial \boldsymbol{\Upsilon}^{\omega_\mu \nu}_{IJ}}{\partial x^\nu}\Big) d\omega^{IJ}_\mu \wedge \beta + \Big(\frac{\partial \boldsymbol{\Upsilon}^{\omega_\mu \nu}_{IJ}}{\partial \omega^{KL}_\lambda}\Big) d\omega^{KL}_\lambda \wedge d\omega^{IJ}_\mu \wedge \beta_\nu + \Big(\frac{\partial \boldsymbol{\Upsilon}^{\omega_\mu \nu}_{IJ}}{\partial e^M_\lambda}\Big) de^M_\lambda \wedge d\omega^{IJ}_\mu \wedge \beta_\nu$$

$$+\Big(\frac{\partial \boldsymbol{\Upsilon}^{\omega_\mu \nu}_{IJ}}{\partial \mathfrak{e}}\Big) d\mathfrak{e} \wedge d\omega^{IJ}_\mu \wedge \beta_\nu + \Big(\frac{\partial \boldsymbol{\Upsilon}^{\omega_\mu \nu}_{IJ}}{\partial p^{\omega_\alpha \beta}_{KL}}\Big) dp^{\omega_\alpha \beta}_{KL} \wedge d\omega^{IJ}_\mu \wedge \beta_\nu$$

$$\iota_{\mathbf{IV}} = -d\boldsymbol{\Theta}^{IJ}_\mu \wedge dp^{\omega_\mu \nu}_{IJ} \wedge \beta_\nu = \Big(\frac{\partial \boldsymbol{\Theta}^{IJ}_\mu}{\partial x^\nu}\Big) dp^{\omega_\mu \nu}_{IJ} \wedge \beta - \Big(\frac{\partial \boldsymbol{\Theta}^{IJ}_\mu}{\partial e^M_\lambda}\Big) de^M_\lambda \wedge dp^{\omega_\mu \nu}_{IJ} \wedge \beta_\nu - \Big(\frac{\partial \boldsymbol{\Theta}^{IJ}_\mu}{\partial \omega^{KL}_\lambda}\Big) d\omega^{KL}_\lambda \wedge dp^{\omega_\mu \nu}_{IJ} \wedge \beta_\nu$$

$$-\Big(\frac{\partial \boldsymbol{\Theta}^{IJ}_\mu}{\partial \mathfrak{e}}\Big) d\mathfrak{e} \wedge dp^{\omega_\mu \nu}_{IJ} \wedge \beta_\nu - \Big(\frac{\partial \boldsymbol{\Theta}^{IJ}_\mu}{\partial p^{\omega_\alpha \beta}_{KL}}\Big) dp^{\omega_\alpha \beta}_{KL} \wedge dp^{\omega_\mu \nu}_{IJ} \wedge \beta_\nu$$

$$\iota_{\mathbf{V}} = d\boldsymbol{X}^\rho \wedge dp^{\omega_\mu \nu}_{IJ} \wedge d\omega^{IJ}_\mu \wedge \beta_{\rho\nu} = \Big(\frac{\partial \boldsymbol{X}^\rho}{\partial x^\alpha}\Big) dp^{\omega_\mu \nu}_{IJ} \wedge d\omega^{IJ}_\mu \wedge [\delta^\alpha_\nu \beta_\rho - \delta^\alpha_\rho \beta_\nu] + \Big(\frac{\partial \boldsymbol{X}^\alpha}{\partial e^M_\lambda}\Big) de^M_\lambda \wedge dp^{\omega_\mu \nu}_{IJ} \wedge d\omega^{IJ}_\mu \wedge \beta_{\rho\nu}$$

$$\Big(\frac{\partial \boldsymbol{X}^\rho}{\partial \omega^{KL}_\lambda}\Big) d\omega^{KL}_\lambda \wedge dp^{\omega_\mu \nu}_{IJ} \wedge d\omega^{IJ}_\mu \wedge \beta_{\rho\nu} + \Big(\frac{\partial \boldsymbol{X}^\rho}{\partial \mathfrak{e}}\Big) d\mathfrak{e} \wedge dp^{\omega_\mu \nu}_{IJ} \wedge d\omega^{IJ}_\mu \wedge \beta_{\rho\nu} + \Big(\frac{\partial \boldsymbol{X}^\rho}{\partial p^{\omega_\alpha \beta}_{KL}}\Big) dp^{\omega_\alpha \beta}_{KL} \wedge dp^{\omega_\mu \nu}_{IJ} \wedge d\omega^{IJ}_\mu \wedge \beta_{\rho\nu}$$

Now we write the decomposition on the differents $(n+1)$-forms:

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- decomposition on $dp \wedge dp \wedge \beta_\nu$

$$-\Big(\frac{\partial \boldsymbol{\Theta}^{IJ}_\mu}{\partial p^{\omega_\alpha \beta}_{KL}}\Big) dp^{\omega_\alpha \beta}_{KL} \wedge dp^{\omega_\mu \nu}_{IJ} \wedge \beta_\nu \tag{345}$$

- decomposition on $dp \wedge dp \wedge d\omega \wedge \beta_{\rho\nu}$

$$\Big(\frac{\partial \boldsymbol{X}^\rho}{\partial p^{\omega_\alpha \beta}_{KL}}\Big) dp^{\omega_\alpha \beta}_{KL} \wedge dp^{\omega_\mu \nu}_{IJ} \wedge d\omega^{IJ}_\mu \wedge \beta_{\rho\nu} \tag{346}$$

- decomposition on $d\mathfrak{e} \wedge \beta$

$$\Big(\frac{\partial \boldsymbol{X}^\nu}{\partial x^\nu} + \frac{\partial \boldsymbol{\Upsilon}}{\partial \mathfrak{e}}\Big) d\mathfrak{e} \wedge \beta \tag{347}$$

- decomposition on $dp \wedge \beta$

$$\Big(\frac{\partial \boldsymbol{\Theta}^{IJ}_\mu}{\partial x^\nu} + \frac{\partial \boldsymbol{\Upsilon}}{\partial p^{\omega_\mu \nu}_{IJ}}\Big) dp^{\omega_\mu \nu}_{IJ} \wedge \beta \tag{348}$$

- decomposition on $d\omega \wedge \beta$

$$\Big(\frac{\partial \boldsymbol{\Upsilon}}{\partial \omega^{IJ}_\mu} - \frac{\partial \boldsymbol{\Upsilon}^{\omega_\mu \nu}_{IJ}}{\partial x^\nu}\Big) d\omega^{IJ}_\mu \wedge \beta \tag{349}$$

- decomposition on $de \wedge \beta$

$$\Big(\frac{\partial \boldsymbol{\Upsilon}}{\partial e^M_\lambda}\Big) de^M_\lambda \wedge \beta \tag{350}$$

- decomposition on $d\mathfrak{e} \wedge d\omega \wedge \beta_\nu$

$$\Big(\frac{\partial \boldsymbol{X}^\nu}{\partial \omega^{IJ}_\mu} + \frac{\partial \boldsymbol{\Upsilon}^{\omega_\mu \nu}_{IJ}}{\partial \mathfrak{e}}\Big) d\mathfrak{e} \wedge d\omega^{IJ}_\mu \wedge \beta_\nu \tag{351}$$

- decomposition on $d\omega \wedge d\omega \wedge \beta_\nu$

$$\Big(\frac{\partial \boldsymbol{\Upsilon}^{\omega_\mu \nu}_{IJ}}{\partial \omega^{KL}_\lambda}\Big) d\omega^{KL}_\lambda \wedge d\omega^{IJ}_\mu \wedge \beta_\nu \tag{352}$$

- decomposition on $de \wedge d\omega \wedge \beta_\nu$

$$\Big(\frac{\partial \boldsymbol{\Upsilon}^{\omega_\mu \nu}_{IJ}}{\partial e^M_\lambda}\Big) de^M_\lambda \wedge d\omega^{IJ}_\mu \wedge \beta_\nu \tag{353}$$

- decomposition on $d\omega \wedge dp \wedge \beta_\nu$. concerned the two involved terms:

$$-\Big(\frac{\partial \boldsymbol{\Theta}^{IJ}_\mu}{\partial \omega^{KL}_\lambda}\Big) d\omega^{KL}_\lambda \wedge dp^{\omega_\mu \nu}_{IJ} \wedge \beta_\nu + \Big(\frac{\partial \boldsymbol{\Upsilon}^{\omega_\mu \nu}_{IJ}}{\partial p^{\omega_\alpha \beta}_{KL}}\Big) dp^{\omega_\alpha \beta}_{KL} \wedge d\omega^{IJ}_\mu \wedge \beta_\nu \tag{354}$$

- decomposition on $de \wedge dp \wedge \beta_\nu$

$$-\Big(\frac{\partial \boldsymbol{\Theta}^{IJ}_\mu}{\partial e^M_\lambda}\Big) de^M_\lambda \wedge dp^{\omega_\mu \nu}_{IJ} \wedge \beta_\nu \tag{355}$$

- decomposition on $d\mathfrak{e} \wedge dp \wedge \beta_\nu$

$$\Big(\frac{\partial \boldsymbol{X}^\nu}{\partial p^{\omega_\alpha \beta}_{IJ}}\Big) d\mathfrak{e} \wedge dp^{\omega_\alpha \beta}_{IJ} \wedge \beta_\nu - \Big(\frac{\partial \boldsymbol{\Theta}^{IJ}_\mu}{\partial \mathfrak{e}}\Big) d\mathfrak{e} \wedge dp^{\omega_\mu \nu}_{IJ} \wedge \beta_\nu \tag{356}$$

- decomposition on $d\mathfrak{e} \wedge de \wedge \beta_\nu$

$$\Big(\frac{\partial \boldsymbol{X}^\nu}{\partial e^M_\lambda}\Big) d\mathfrak{e} \wedge de^M_\lambda \wedge \beta_\nu \tag{357}$$

- decomposition on $de\wedge dp\wedge d\omega\wedge\beta_{\rho\nu}$

$$\Big(\frac{\partial \boldsymbol{X}^\rho}{\partial e^M_\lambda}\Big)de^M_\lambda\wedge dp^{\omega_\mu\nu}_{IJ}\wedge d\omega^{IJ}_\mu\wedge\beta_{\rho\nu} \tag{358}$$

- decomposition on $d\omega\wedge dp\wedge d\omega\wedge\beta_{\rho\nu}$

$$\Big(\frac{\partial \boldsymbol{X}^\rho}{\partial \omega^{KL}_\lambda}\Big)d\omega^{KL}_\lambda\wedge dp^{\omega_\mu\nu}_{IJ}\wedge d\omega^{IJ}_\mu\wedge\beta_{\rho\nu} \tag{359}$$

- decomposition on $d\mathfrak{e}\wedge dp\wedge d\omega\wedge\beta_{\rho\nu}$

$$\Big(\frac{\partial \boldsymbol{X}^\rho}{\partial \mathfrak{e}}\Big)d\mathfrak{e}\wedge dp^{\omega_\mu\nu}_{IJ}\wedge d\omega^{IJ}_\mu\wedge\beta_{\rho\nu} \tag{360}$$

Finally, it remains only the last decomposition related to (354) on $d\omega\wedge dp\wedge\beta_\nu$.

$$\Big(\frac{\partial \boldsymbol{X}^\rho}{\partial x^\nu}\Big)dp^{\omega_\mu\nu}_{IJ}\wedge d\omega^{IJ}_\mu\wedge\beta_\rho-\Big(\frac{\partial \boldsymbol{X}^\rho}{\partial x^\rho}\Big)dp^{\omega_\mu\nu}_{IJ}\wedge d\omega^{IJ}_\mu\wedge\beta_\nu \tag{361}$$

since $dx^\alpha\wedge\beta_{\rho\nu}=\delta^\alpha_\nu\beta_\rho-\delta^\alpha_\rho\beta_\nu$ we obtain (361).

General conditions lead to determine the dependance coordinates on the following functions: $\boldsymbol{X}^\nu,\boldsymbol{\Theta}^M_\lambda,\boldsymbol{\Theta}^{IJ}_\mu,\boldsymbol{\Upsilon},\boldsymbol{\Upsilon}^{e_\mu\nu}_I,\boldsymbol{\Upsilon}^{\omega_\mu\nu}_{IJ}$ from the analysis of the equation $d(\Xi\,\lrcorner\,\overline{\boldsymbol{\varpi}}^{\mathsf{DW}})$.[101] We conclude that:

$$\begin{array}{rcl}\boldsymbol{X}^\rho &=& \boldsymbol{X}^\rho(x)\\ \boldsymbol{\Theta}^M_\rho(x,e)&=&\boldsymbol{\Theta}^M_\rho(x,e)\\ \boldsymbol{\Theta}^{IJ}_\mu(x,\omega)&=&\boldsymbol{\Theta}^{IJ}_\mu(x,\omega)\end{array}\quad\text{and}\quad\begin{array}{rcl}\boldsymbol{\Upsilon}&=&\boldsymbol{\Upsilon}(x,\omega)\\ \boldsymbol{\Upsilon}^{\omega_\mu\nu}_{IJ}&=&\boldsymbol{\Upsilon}^{\omega_\mu\nu}_{IJ}(x,\mathfrak{e},p)\end{array} \tag{362}$$

We are left with equations (347) (348) (349) (351):

$$\begin{array}{rcl}\dfrac{\partial \boldsymbol{X}^\nu}{\partial x^\nu}+\dfrac{\partial\boldsymbol{\Upsilon}}{\partial\mathfrak{e}}&=&0\\[2ex] \dfrac{\partial\boldsymbol{\Theta}^{IJ}_\mu}{\partial x^\nu}+\dfrac{\partial\boldsymbol{\Upsilon}}{\partial p^{\omega_\mu\nu}_{IJ}}&=&0\end{array}\quad\text{and}\quad\begin{array}{rcl}\dfrac{\partial\boldsymbol{\Upsilon}}{\partial\omega^{IJ}_\mu}-\dfrac{\partial\boldsymbol{\Upsilon}^{\omega_\mu\nu}_{IJ}}{\partial x^\nu}&=&0\\ \dfrac{\partial \boldsymbol{X}^\nu}{\partial\omega^{IJ}_\mu}+\dfrac{\partial\boldsymbol{\Upsilon}^{\omega_\mu\nu}_{IJ}}{\partial\mathfrak{e}}&=&0\end{array} \tag{363}$$

together with the set of equations involving (354) and (361) - decomposition on the term $dp\wedge d\omega\wedge\beta_\rho$:

$$-\Big[\frac{\partial\boldsymbol{\Theta}^{IJ}_\mu}{\partial\omega^{KL}_\lambda}\Big]d\omega^{KL}_\lambda\wedge dp^{\omega_\mu\nu}_{IJ}\wedge\beta_\nu+\Big[\frac{\partial\boldsymbol{\Upsilon}^{\omega_\mu\nu}_{IJ}}{\partial p^{\omega_\alpha\beta}_{KL}}\Big]dp^{\omega_\alpha\beta}_{KL}\wedge d\omega^{IJ}_\mu\wedge\beta_\nu+\frac{\partial \boldsymbol{X}^\rho}{\partial x^\nu}dp^{\omega_\mu\nu}_{IJ}\wedge d\omega^{IJ}_\mu\wedge\beta_\rho-\frac{\partial \boldsymbol{X}^\rho}{\partial x^\rho}dp^{\omega_\mu\nu}_{IJ}\wedge d\omega^{IJ}_\mu\wedge\beta_\nu$$

The conclusion straightforward appears with the equations (345)- (361) and leads to the following conclusion: the set $\mathfrak{sp}_\circ\overline{\mathcal{M}}_{\mathsf{DW}}$ of infinitesimal symplectomorphism of $(\mathfrak{sp}_\circ\overline{\mathcal{M}}_{\mathsf{DW}},\overline{\boldsymbol{\varpi}}^{\mathsf{DW}})$ is described by vector fields $\Xi=\Xi\big|_{\overline{\mathcal{M}}_{\mathsf{DW}}}=\overline{\zeta}+\chi$ with the vector field $\overline{\zeta}$ described by:

$$\overline{\zeta}=X^\nu\frac{\partial}{\partial x^\nu}+\Theta^M_\lambda\frac{\partial}{\partial e^M_\lambda}+\Theta^{IJ}_\mu\frac{\partial}{\partial\omega^{IJ}_\mu}-\underbrace{\{\Big[\mathfrak{e}(\frac{\partial X^\nu}{\partial x^\nu})+\frac{\partial\Theta^{IJ}_\mu}{\partial x^\nu}p^{\omega_\mu\nu}_{IJ}\Big]\}}_{\overline{\zeta}^{\mathfrak{e}}}\frac{\partial}{\partial\mathfrak{e}}$$

$$+\underbrace{\{p^{A_\rho\sigma}_{KL}\delta^\mu_\rho\Big[\delta^K_I\delta^L_J\big[(\frac{\partial X^\nu}{\partial x^\sigma})-\delta^\nu_\sigma(\frac{\partial X^\lambda}{\partial x^\lambda})\big]+(\frac{\partial\Theta^{KL}_\sigma}{\partial\omega^{IJ}_\nu})\Big]-\mathfrak{e}(\frac{\partial X^\nu}{\partial\omega^{IJ}_\mu})\}}_{\overline{\zeta}^{\omega}}\frac{\partial}{\partial p^{\omega_\mu\nu}_{IJ}}$$

and $\chi=\Upsilon\frac{\partial}{\partial\mathfrak{e}}+\Upsilon^{\omega_\mu\alpha}_{IJ}\frac{\partial}{\partial p^{\omega_\mu\alpha}_{KL}}$ with, $\Upsilon,\Upsilon^{\omega_\mu\alpha}_{IJ}$ smooth functions on $\mathfrak{Z}$ such that: $\frac{\partial\Upsilon}{\partial\omega^{IJ}_\mu}-\frac{\partial\Upsilon^{\omega_\mu\nu}_{IJ}}{\partial x^\nu}=0$ ____

[101] We observe in this case that equations (346) (357) (358) (359) (360) give us $\boldsymbol{X}^\rho=\boldsymbol{X}^\rho(x)$. Also from equations (345) (355) (356) we obtain: $\boldsymbol{\Theta}^{IJ}_\mu=\boldsymbol{\Theta}^{IJ}_\mu(x,\omega)$ and from equation (350) $\boldsymbol{\Upsilon}=\boldsymbol{\Upsilon}(x,\omega,\mathfrak{e},p)$. From equation (352) (353) we obtain $\boldsymbol{\Upsilon}^{\omega_\mu\nu}_{IJ}=\boldsymbol{\Upsilon}^{\omega_\mu\nu}_{IJ}(x,\mathfrak{e},p)$ and finally $\boldsymbol{\Theta}^M_\rho=\boldsymbol{\Theta}^M_\rho(x,e,\mathfrak{e})$.

Alternatively we can directly use proposition (6.1). Let us give some details for this approach. $\mathcal{M}_{\mathsf{DW}}$ is an open subset of $\Lambda^n T^\star\mathfrak{Z}$. From the general result, the set of all infinitesimal symplectomorphisms Ξ on $\mathcal{M}_{\mathsf{DW}}$ are of the form $\Xi = \chi + \bar{\zeta}$. Here we denote a vector field $\zeta = \zeta^\alpha(q)\frac{\partial}{\partial q^\alpha}$ with $\zeta \in \Gamma(\mathfrak{Z}, T\mathfrak{Z})$. With this notation $\bar{\zeta}$ is written:

$$\bar{\zeta} = \sum_\alpha \zeta^\alpha(x,e,\omega)\frac{\partial}{\partial q^\alpha} - \sum_{\alpha,\beta}\frac{\partial \zeta^\alpha}{\partial q^\beta}(q)\Pi^\beta_\alpha \quad \text{with} \quad \Pi^\beta_\alpha := \sum_{\beta_1<\cdots<\beta_n}\sum_\mu \delta^\beta_{\beta_\mu} p_{\beta_1\cdots\beta_{\mu-1}\alpha\beta_{\mu+1}\cdots\beta_n}\frac{\partial}{\partial p_{\beta_1\cdots\beta_n}}.$$

$$\bar{\zeta} := \sum_\alpha \zeta^\alpha(q)\frac{\partial}{\partial q^\alpha} - \sum_{\alpha,\beta}\sum_{\beta_1<\cdots<\beta_n}\sum_\mu \delta^\beta_{\beta_\mu}\frac{\partial \zeta^\alpha}{\partial q^\beta}(q) p_{\beta_1\cdots\beta_{\mu-1}\alpha\beta_{\mu+1}\cdots\beta_n}\frac{\partial}{\partial p_{\beta_1\cdots\beta_n}}$$

Notice, that we have slightly modified the notations.[102] In our case of interest, the multisymplectic manifold considered is $\mathcal{M}_{\mathsf{DW}} \subset \Lambda^n T^\star(\mathfrak{p}\otimes T^\star\mathcal{X}) = \Lambda^n T^\star\mathfrak{Z}$ with coordinates denoted as $(q^\mu, p_{\mu_1\ldots\mu_n}) = (x^\mu, e^I_\nu, \omega^{IJ}_\nu, \mathfrak{e}, p^{e_\nu\mu}_{IJ}, p^{\omega_\nu\mu}_{IJ})$ see the notations: $p_{1\ldots n} = \mathfrak{e}$, $p_{1\ldots(\nu-1),e^{IJ}_\mu,(\nu+1),\ldots n} = p^{e_\mu\nu}_{IJ} = 0$ and $p_{1\ldots(\nu-1),\omega^{IJ}_\mu,(\nu+1),\ldots n} = p^{\omega_\mu\nu}_{IJ}$. We translate in these multisymplectic notations, then the expression of the vector field $\bar{\zeta}$ is given by:

$$\bar{\zeta} = X^\nu\frac{\partial}{\partial x^\nu} + \Theta^M_\lambda\frac{\partial}{\partial e^M_\lambda} + \Theta^{IJ}_\mu\frac{\partial}{\partial \omega^{IJ}_\mu} - \underbrace{\Big[\sum_{\nu,\rho}\frac{\partial X^\nu}{\partial x^\rho}\Pi^\rho_\nu + \sum_{\nu,\sigma}\frac{\partial X^\nu}{\partial e^N_\sigma}\Pi^{(e^N_\sigma)}_\nu + \sum_{\nu,\rho}\frac{\partial X^\nu}{\partial \omega^{KL}_\rho}\Pi^{(\omega^{KL}_\rho)}_\nu\Big]}_{\kappa_{\mathbf{I}}}$$

$$-\underbrace{\Big[\sum_{\lambda,\nu}\frac{\partial \Theta^M_\lambda}{\partial x^\nu}\Pi^\nu_{(e^M_\lambda)} + \sum_{\lambda,\rho}\frac{\partial \Theta^M_\lambda}{\partial \omega^{KL}_\rho}\Pi^{(\omega^{KL}_\rho)}_{(e^M_\lambda)} + \sum_{\lambda,\sigma}\frac{\partial \Theta^M_\lambda}{\partial e^N_\sigma}\Pi^{(e^N_\sigma)}_{(e^M_\lambda)}\Big]}_{\kappa_{\mathbf{II}}} - \underbrace{\Big[\sum_{\mu,\rho}\frac{\partial \Theta^{IJ}_\mu}{\partial \omega^{KL}_\rho}\Pi^{(\omega^{KL}_\rho)}_{(\omega^{IJ}_\mu)} + \sum_{\mu,\sigma}\frac{\partial \Theta^{IJ}_\mu}{\partial e^N_\sigma}\Pi^{(e^N_\sigma)}_{(\omega^{IJ}_\mu)} + \sum_{\mu,\nu}\frac{\partial \Theta^{IJ}_\mu}{\partial x^\nu}\Pi^\nu_{(\omega^{IJ}_\mu)}\Big]}_{\kappa_{\mathbf{III}}}$$

The next step, is the expression of the following terms:

$$\Pi^\rho_\nu; \Pi^{(e^N_\sigma)}_\nu; \Pi^{(\omega^{KL}_\rho)}_\nu; \Pi^\nu_{(e^M_\lambda)}; \Pi^{(\omega^{KL}_\rho)}_{(e^M_\lambda)}; \Pi^{(e^N_\sigma)}_{(e^M_\lambda)}; \Pi^{(\omega^{KL}_\rho)}_{(\omega^{IJ}_\mu)}; \Pi^{(e^N_\sigma)}_{(\omega^{IJ}_\mu)}; \Pi^\nu_{(\omega^{IJ}_\mu)}$$

Those appear in the terms $\kappa_{\mathbf{I}}, \kappa_{\mathbf{II}}$ and $\kappa_{\mathbf{III}}$ given by: $\bar{\zeta} = \zeta + \kappa_{\mathbf{I}} + \kappa_{\mathbf{II}} + \kappa_{\mathbf{III}}$. Notice that there is no term in $\kappa_{\mathbf{II}}$, thanks to $p^{e_\mu\nu}_I = 0$. We observe that $\Pi^\nu_{(e^M_\lambda)} = \Pi^{(\omega^{KL}_\rho)}_{(e^M_\lambda)} = \Pi^{(e^N_\sigma)}_{(e^M_\lambda)} = 0$ as well as $\Pi^{(e^N_\sigma)}_\nu = \Pi^{(e^N_\sigma)}_{(\omega^{IJ}_\mu)} = 0$. Hence we set:

$$\sum_{\lambda,\nu}\frac{\partial \Theta^M_\lambda}{\partial x^\nu}\Pi^\nu_{(e^M_\lambda)} = \sum_{\lambda,\rho}\frac{\partial \Theta^M_\lambda}{\partial \omega^{KL}_\rho}\Pi^{(\omega^{KL}_\rho)}_{(e^M_\lambda)} = \sum_{\lambda,\sigma}\frac{\partial \Theta^M_\lambda}{\partial e^N_\sigma}\Pi^{(e^N_\sigma)}_{(e^M_\lambda)} = \sum_{\nu,\sigma}\frac{\partial X^\nu}{\partial e^N_\sigma}\Pi^{(e^N_\sigma)}_\nu = \sum_{\mu,\sigma}\frac{\partial \Theta^{IJ}_\mu}{\partial e^N_\sigma}\Pi^{(e^N_\sigma)}_{(\omega^{IJ}_\mu)} = 0$$

so that the infinitesimal symplectomorphisms are written:

$$\bar{\zeta} = X^\nu\frac{\partial}{\partial x^\nu} + \Theta^M_\lambda\frac{\partial}{\partial e^M_\lambda} + \Theta^{IJ}_\mu\frac{\partial}{\partial \omega^{IJ}_\mu} - \underbrace{\Big[\sum_{\nu,\rho}\frac{\partial X^\nu}{\partial x^\rho}\Pi^\rho_\nu + \sum_{\nu,\rho}\frac{\partial X^\nu}{\partial \omega^{KL}_\rho}\Pi^{(\omega^{KL}_\rho)}_\nu\Big]}_{\kappa_{\mathbf{I}}} - \underbrace{\Big[\sum_{\mu,\rho}\frac{\partial \Theta^{IJ}_\mu}{\partial \omega^{KL}_\rho}\Pi^{(\omega^{KL}_\rho)}_{(\omega^{IJ}_\mu)} + \sum_{\mu,\nu}\frac{\partial \Theta^{IJ}_\mu}{\partial x^\nu}\Pi^\nu_{(\omega^{IJ}_\mu)}\Big]}_{\kappa_{\mathbf{III}}}$$

We refer to (E) for a tedious but straightforward calculation of the terms $\kappa_{\mathbf{I}}$ and $\kappa_{\mathbf{III}}$.

$$\bar{\zeta} = X^\nu\frac{\partial}{\partial x^\nu} + \Theta^M_\lambda\frac{\partial}{\partial e^M_\lambda} + \Theta^{IJ}_\mu\frac{\partial}{\partial \omega^{IJ}_\mu} - \{\Big[\mathfrak{e}(\frac{\partial X^\nu}{\partial x^\nu}) + \frac{\partial \Theta^{IJ}_\mu}{\partial x^\nu}p^{\omega_\mu\nu}_{IJ}\Big]\}\frac{\partial}{\partial \mathfrak{e}}$$

$$+\{p^{\omega_\mu\sigma}_{KL}\Big[\delta^K_I\delta^L_J[(\frac{\partial X^\nu}{\partial x^\sigma}) - \delta^\nu_\sigma(\frac{\partial X^\lambda}{\partial x^\lambda})] - (\frac{\partial \Theta^{KL}_\sigma}{\partial \omega^{IJ}_\nu})\Big] - \mathfrak{e}(\frac{\partial X^\nu}{\partial \omega^{IJ}_\mu})\}\frac{\partial}{\partial p^{\omega_\mu\nu}_{IJ}}$$

[102] In order to not make confusion of the objects Π^β_α with the notation of the multimomenta $p^{\omega_\beta\alpha}$ whereas actually the two notation already differ by the position of the indices and the presence of the greek letter ω or the latin letter e.)

The decomposition of Ξ given by (344) gives us:

$$\Xi - \overline{\zeta} = \boldsymbol{X}^{\nu}\frac{\partial}{\partial x^{\nu}} + \boldsymbol{\Theta}_{\lambda}^{M}\frac{\partial}{\partial e_{\lambda}^{M}} + \boldsymbol{\Theta}_{\mu}^{IJ}\frac{\partial}{\partial \omega_{\mu}^{IJ}} + \boldsymbol{\Upsilon}\frac{\partial}{\partial \mathfrak{e}} + \boldsymbol{\Upsilon}_{IJ}^{\omega_{\mu}\nu}\frac{\partial}{\partial p_{IJ}^{\omega_{\mu}\nu}} - X^{\nu}\frac{\partial}{\partial x^{\nu}} + \Theta_{\lambda}^{M}\frac{\partial}{\partial e_{\lambda}^{M}} + \Theta_{\mu}^{IJ}\frac{\partial}{\partial \omega_{\mu}^{IJ}}$$

$$+\Big[\mathfrak{e}(\frac{\partial X^{\nu}}{\partial x^{\nu}}) + \frac{\partial \Theta_{\mu}^{IJ}}{\partial x^{\nu}}p_{IJ}^{\omega_{\mu}\nu}\Big]\frac{\partial}{\partial \mathfrak{e}} - \{p_{KL}^{\omega_{\mu}\sigma}\Big[\delta_I^K\delta_J^L\big[(\frac{\partial X^{\nu}}{\partial x^{\sigma}}) - \delta_{\sigma}^{\nu}(\frac{\partial X^{\lambda}}{\partial x^{\lambda}})\big] - (\frac{\partial \Theta_{\sigma}^{KL}}{\partial \omega_{\nu}^{IJ}})\Big] - \mathfrak{e}(\frac{\partial X^{\nu}}{\partial \omega_{\mu}^{IJ}})\}\frac{\partial}{\partial p_{IJ}^{\omega_{\mu}\nu}}$$

So that $\Xi - \overline{\zeta} = \chi$ with

$$\chi = \underbrace{\Big[\boldsymbol{\Upsilon} - \Big[\mathfrak{e}(\frac{\partial X^{\nu}}{\partial x^{\nu}}) + \frac{\partial \Theta_{\mu}^{IJ}}{\partial x^{\nu}}p_{IJ}^{\omega_{\mu}\nu}\Big]\Big]}_{\chi^{\mathfrak{e}}}\frac{\partial}{\partial \mathfrak{e}} + \underbrace{\{p_{KL}^{\omega_{\mu}\sigma}\Big[\delta_I^K\delta_J^L\big[(\frac{\partial X^{\nu}}{\partial x^{\sigma}}) - \delta_{\sigma}^{\nu}(\frac{\partial X^{\lambda}}{\partial x^{\lambda}})\big] - (\frac{\partial \Theta_{\sigma}^{KL}}{\partial \omega_{\nu}^{IJ}})\Big] - \mathfrak{e}(\frac{\partial X^{\nu}}{\partial \omega_{\mu}^{IJ}})\}}_{\chi^{\omega}}\frac{\partial}{\partial p_{IJ}^{\omega_{\mu}\nu}}$$

As announced in proposition (6.1) we have coefficients of χ such that $d(\chi \lrcorner \overline{\boldsymbol{\varpi}}^{\mathsf{DW}}) = 0$. The interior product of $\chi \lrcorner \overline{\boldsymbol{\varpi}}^{\mathsf{DW}}$ is written:

$$\chi \lrcorner \overline{\boldsymbol{\varpi}}^{\mathsf{DW}} = \Big[\chi^{\mathfrak{e}}\frac{\partial}{\partial \mathfrak{e}} + \chi^{\omega}\frac{\partial}{\partial p_{IJ}^{\omega_{\rho}\nu}}\Big] \lrcorner \Big[d\mathfrak{e}\wedge\beta + dp_{IJ}^{\omega_{\mu}\nu}\wedge d\omega_{\mu}^{IJ}\wedge\beta_{\nu}\Big]$$

The interior product of $\chi \in \Gamma(\mathcal{M}_{\mathsf{DW}}, T\mathcal{M}_{\mathsf{DW}})$ with $\overline{\boldsymbol{\varpi}}^{\mathsf{DW}}$ is given by $\chi \lrcorner \overline{\boldsymbol{\varpi}}^{\mathsf{DW}} = \chi^{\mathfrak{e}}\beta + \chi^{\omega}d\omega_{\mu}^{IJ}\wedge\beta_{\nu}$ so that:

$$d(\chi \lrcorner \overline{\boldsymbol{\varpi}}^{\mathsf{DW}}) = d\chi^{\mathfrak{e}}\wedge\beta + d\chi^{\omega}\wedge d\omega_{\mu}^{IJ}\wedge\beta_{\nu} = d\Big[\boldsymbol{\Upsilon} - \Big[\mathfrak{e}(\frac{\partial X^{\nu}}{\partial x^{\nu}}) + \frac{\partial \Theta_{\mu}^{IJ}}{\partial x^{\nu}}p_{IJ}^{\omega_{\mu}\nu}\Big]\Big]\wedge\beta$$

$$+d\{\boldsymbol{\Upsilon}_{IJ}^{\omega_{\mu}\nu} - \{p_{KL}^{\omega_{\mu}\sigma}\Big[\delta_I^K\delta_J^L\big[(\frac{\partial X^{\nu}}{\partial x^{\sigma}}) - \delta_{\sigma}^{\nu}(\frac{\partial X^{\lambda}}{\partial x^{\lambda}})\big] - (\frac{\partial \Theta_{\sigma}^{KL}}{\partial \omega_{\nu}^{IJ}})\Big] - \mathfrak{e}(\frac{\partial X^{\nu}}{\partial \omega_{\mu}^{IJ}})\}\}\wedge d\omega_{\mu}^{IJ}\wedge\beta_{\nu}$$

We have in mind the expression of the exterior derivative $d\boldsymbol{\Upsilon}$ and $d\boldsymbol{\Upsilon}_{IJ}^{\omega_{\mu}\nu}$ so:

$$d(\chi \lrcorner \Omega) = \Big(\frac{\partial \boldsymbol{\Upsilon}}{\partial \mathfrak{e}} - \frac{\partial X^{\nu}}{\partial x^{\nu}}\Big)d\mathfrak{e}\wedge\beta + \Big(\frac{\partial \boldsymbol{\Upsilon}}{\partial p_{IJ}^{\omega_{\mu}\nu}} - \frac{\partial \Theta_{\mu}^{IJ}}{\partial x^{\nu}}\Big)dp_{IJ}^{\omega_{\mu}\nu}\wedge\beta + \Big(\frac{\partial \boldsymbol{\Upsilon}}{\partial \omega_{\mu}^{IJ}} - \frac{\partial \boldsymbol{\Upsilon}_{IJ}^{\omega_{\mu}\nu}}{\partial x^{\nu}}\Big)d\omega_{\mu}^{IJ}\wedge\beta$$

$$+\Big(\frac{\partial \boldsymbol{\Upsilon}_{IJ}^{\omega_{\mu}\nu}}{\partial \omega_{\alpha}^{KL}}\Big)d\omega_{\alpha}^{KL}\wedge d\omega_{\mu}^{IJ}\wedge\beta_{\nu} + \Big(\frac{\partial \boldsymbol{\Upsilon}_{IJ}^{\omega_{\mu}\nu}}{\partial \mathfrak{e}} - \frac{\partial X^{\nu}}{\partial \omega_{\mu}^{KL}}\Big)d\mathfrak{e}\wedge d\omega_{\mu}^{IJ}\wedge\beta_{\nu}$$

$$+\Big((\frac{\partial \boldsymbol{\Upsilon}_{IJ}^{\omega_{\mu}\nu}}{\partial p_{KL}^{\omega_{\rho}\sigma}}) - \delta_{\rho}^{\mu}\Big[\delta_I^K\delta_J^L\big[(\frac{\partial X^{\nu}}{\partial x^{\sigma}}) - \delta_{\sigma}^{\nu}(\frac{\partial X^{\lambda}}{\partial x^{\lambda}})\big] - (\frac{\partial \Theta_{\sigma}^{KL}}{\partial \omega_{\nu}^{IJ}})\Big]\Big)dp_{KL}^{\omega_{\rho}\sigma}\wedge d\omega_{\mu}^{IJ}\wedge\beta_{\nu}$$

Now we are interested in terms in which $\boldsymbol{\Upsilon}$ is involved. The first two terms in the equation are concerned. Let us notice that - we denote $q = (x, e, \omega)$ so that $\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}(q, \mathfrak{e}, p)$ - the first two terms in the last equation give:

$$\begin{aligned} \frac{\partial \boldsymbol{\Upsilon}}{\partial \mathfrak{e}}(q, \mathfrak{e}, p) - \frac{\partial X^{\nu}}{\partial x^{\nu}}(q) &= 0 \\ \frac{\partial \boldsymbol{\Upsilon}}{\partial p_{IJ}^{\omega_{\mu}\nu}}(q, \mathfrak{e}, p) - \frac{\partial \Theta_{\mu}^{IJ}}{\partial x^{\nu}}(q) &= 0 \end{aligned}$$

Hence, there exists $\Upsilon(q) = \Upsilon(x, \omega)$ So that:

$$\boldsymbol{\Upsilon}(q, \mathfrak{e}, p) = \Upsilon(q) + (\frac{\partial \Theta_{\mu}^{IJ}}{\partial x^{\nu}})p_{IJ}^{\omega_{\mu}\nu} + \mathfrak{e}(\frac{\partial X^{\nu}}{\partial x^{\nu}})$$

On the other side, we have the function $\boldsymbol{\Upsilon}^{\omega_\mu\nu}_{IJ}(q,p)$ as $\boldsymbol{\Upsilon}^{\omega_\mu\nu}_{IJ}(x,p)$ therefore, the interesting information is contained in the set of equations:

$$\begin{aligned}\frac{\partial \boldsymbol{\Upsilon}^{\omega_\mu\nu}_{IJ}}{\partial \mathfrak{e}}(x,p) - (\frac{\partial X^\nu}{\partial \omega^{IJ}_\mu})(q) &= 0\\ (\frac{\partial \boldsymbol{\Upsilon}^{\omega_\mu\nu}_{IJ}}{\partial p^{\omega_\mu\sigma}_{KL}})(q,\mathfrak{e},p) - \delta^\mu_\rho\{\delta^K_I\delta^L_J\Big[(\frac{\partial X^\nu}{\partial x^\sigma}) - \delta^\nu_\sigma(\frac{\partial X^\lambda}{\partial x^\lambda})\Big] - (\frac{\partial \Theta^{KL}_\nu}{\partial \omega^{IJ}_\sigma})\}dp^{\omega_\mu\sigma}_{KL}\wedge d\omega^{IJ}_\mu\wedge\beta_\nu &= 0\end{aligned}$$

Hence, there exists $\Upsilon^{\omega_\mu\nu}_{IJ}(q) = \Upsilon^{\omega_\mu\nu}_{IJ}(x,\omega)$ such that:

$$\boldsymbol{\Upsilon}^{\omega_\mu\nu}_{IJ}(q,p) = \Upsilon^{\omega_\mu\nu}_{IJ}(x,\omega) + p^{\omega_\rho\sigma}_{KL}\delta^\mu_\rho\big\{\delta^K_I\delta^L_J\Big[(\frac{\partial X^\nu}{\partial x^\sigma}) - \delta^\nu_\sigma(\frac{\partial X^\lambda}{\partial x^\lambda})\Big] - (\frac{\partial \Theta^{KL}_\nu}{\partial \omega^{IJ}_\sigma})\big\}$$

The set of infinitesimal symplectomorphisms $\mathfrak{sp}_\circ\overline{\mathcal{M}}_{\mathsf{DW}}$ of $(\overline{\mathcal{M}}_{\mathsf{DW}},\boldsymbol{\omega})$ is described by vector fields $\Xi = \Xi|_{\overline{\mathcal{M}}_{\mathsf{DW}}} = \overline{\zeta} + \chi$ with $\overline{\zeta}$ described above and $\chi = \Upsilon\frac{\partial}{\partial\mathfrak{e}} + \Upsilon^{\omega_\mu\alpha}_{IJ}\frac{\partial}{\partial p^{\omega_\mu\alpha}_{KL}}$. Here, $X,\Theta,\Theta^{IJ}_\mu,\Upsilon,\Upsilon^{\omega_\mu\alpha}_{IJ}$ are defined on $\mathfrak{Z}$ and not anymore on the full multisymplectic manifold. We can simplify the setting if we assume that $dx^\mu(\Xi)=0$. In doing this we throw away the X^μ which correspond to parts of the stress-energy-tensor. In such a context we obtain:

$$\Xi = \Big\{\Upsilon^{\omega_\mu\nu}_{IJ} - p^{A_\rho\sigma}_{KL}\delta^\mu_\rho(\frac{\partial\Theta^{KL}_\sigma}{\partial\omega^{IJ}_\nu})\Big\}\frac{\partial}{\partial p^{\omega_\mu\nu}_{IJ}} + \Big\{\Upsilon - \frac{\partial\Theta^{IJ}_\mu}{\partial x^\nu}p^{\omega_\mu\nu}_{IJ}\Big\}\frac{\partial}{\partial\mathfrak{e}} + \Theta^{IJ}_\mu\frac{\partial}{\partial\omega^{IJ}_\mu}$$

$\Upsilon^{\omega_\mu\nu}_{IJ}$, Υ and Θ^{IJ}_μ are smooth arbitrary functions of (x,ω) and they satisfy the condition:

$$\frac{\partial\Upsilon}{\partial\omega^{IJ}_\mu} - \frac{\partial\Upsilon^{\omega_\mu\nu}_{IJ}}{\partial x^\nu} = 0$$

18.2 Infinitesimal symplectomorphism $\mathfrak{sp}_0\mathcal{M}_{\mathsf{Palatini}}$ of $(\mathcal{M}_{\mathsf{Palatini}},\boldsymbol{\omega}^{\mathsf{DW}})$

Now we are interested in the application of the constraint set. For Palatini 4D case since we have $p^{\omega_\mu\nu}_{IJ} = -ee^{[\mu}_Ie^{\nu]}_J = \frac{1}{2}\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho e^L_\sigma$ we write:

$$\boldsymbol{\omega}^{\mathsf{Palatini}} = d\mathfrak{e}\wedge\beta - d\big[ee^{[\mu}_Ie^{\nu]}_J\big]\wedge d\omega^{IJ}_\mu\wedge\beta_\nu = d\mathfrak{e}\wedge\beta - d\big[\frac{1}{2}\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho e^L_\sigma\big]\wedge d\omega^{IJ}_\mu\wedge\beta_\nu$$

$$\boldsymbol{\omega}^{\mathsf{Palatini}} = d\mathfrak{e}\wedge\beta - \big[\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho\big]de^L_\sigma\wedge d\omega^{IJ}_\mu\wedge\beta_\nu$$

We focus on the expression $\Xi\,\lrcorner\,\boldsymbol{\omega}^{\mathsf{Palatini}}$. This time we consider a vector field $\Xi\in\Gamma(\mathcal{M}_{\mathsf{Palatini}},T\mathcal{M}_{\mathsf{Palatini}})$:

$$\Xi\big|_{\mathsf{Palatini}} = \boldsymbol{X}^\nu(q,\mathfrak{e})\frac{\partial}{\partial x^\nu} + \boldsymbol{\Theta}^M_\lambda(q,\mathfrak{e})\frac{\partial}{\partial e^M_\lambda} + \boldsymbol{\Theta}^{IJ}_\mu(q,\mathfrak{e})\frac{\partial}{\partial\omega^{IJ}_\mu} + \boldsymbol{\Upsilon}(q,\mathfrak{e})\frac{\partial}{\partial\mathfrak{e}} \tag{364}$$

Therefore we make the computation: $\Xi\,\lrcorner\,\boldsymbol{\omega}$ (we simplify notations). We introduce: $\underset{\circ}{\boldsymbol{X}}^\nu,\boldsymbol{\Theta}^M_\lambda,\boldsymbol{\Theta}^{IJ}_\mu,\boldsymbol{\Upsilon}$ which are smooth functions on $\mathcal{M}_{\mathsf{Palatini}}\to\mathbb{R}\subset\mathcal{M}_{\mathsf{DW}}\subset\mathcal{M} = \Lambda^nT^\star\mathfrak{Z}^{\circ}$.

$$\Xi\,\lrcorner\,\boldsymbol{\omega} = \Big[\boldsymbol{X}^\nu\frac{\partial}{\partial x^\nu} + \boldsymbol{\Theta}^M_\lambda\frac{\partial}{\partial e^M_\lambda} + \boldsymbol{\Theta}^{IJ}_\mu\frac{\partial}{\partial\omega^{IJ}_\mu} + \boldsymbol{\Upsilon}\frac{\partial}{\partial\mathfrak{e}}\Big]\,\lrcorner\,\Big[d\mathfrak{e}\wedge\beta\Big]$$

$$-\Big[\boldsymbol{X}^\nu\frac{\partial}{\partial x^\nu} + \boldsymbol{\Theta}^M_\lambda\frac{\partial}{\partial e^M_\lambda} + \boldsymbol{\Theta}^{IJ}_\mu\frac{\partial}{\partial\omega^{IJ}_\mu} + \boldsymbol{\Upsilon}\frac{\partial}{\partial\mathfrak{e}}\Big]\,\lrcorner\, - \big[\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho\big]\Big[de^L_\sigma\wedge d\omega^{IJ}_\mu\wedge\beta_\nu\Big]$$

The first term $\Xi \lrcorner \Big[d\mathfrak{e} \wedge \beta\Big]$ is straightforward given by:

$$\Xi \lrcorner \Big[d\mathfrak{e} \wedge \beta\Big] = d\mathfrak{e}(\Xi)\beta - \beta(\Xi)d\mathfrak{e} = \boldsymbol{\Upsilon}\beta - \boldsymbol{X}^\nu d\mathfrak{e} \wedge \beta_\nu$$

whereas the second is given by:

$$\Xi \lrcorner -\big[\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^K\big]\Big[de_\sigma^L \wedge d\omega_\mu^{IJ} \wedge \beta_\nu\Big] = -\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^K\Big[\boldsymbol{\Theta}_\sigma^L d\omega_\mu^{IJ} \wedge \beta_\nu - \boldsymbol{\Theta}_\mu^{IJ} de_\sigma^L \wedge \beta_\nu + \boldsymbol{X}^\rho de_\sigma^L \wedge d\omega_\mu^{IJ} \wedge \beta_{\rho\nu}\Big]$$

Then we evaluate the expression $d(\Xi \lrcorner \boldsymbol{\omega})$ so that:

$$d(\Xi \lrcorner \boldsymbol{\omega}) = \underbrace{d\boldsymbol{\Upsilon} \wedge \beta}_{\iota_{\mathbf{I}}} \underbrace{-d\boldsymbol{X}^\nu \wedge d\mathfrak{e} \wedge \beta_\nu}_{\iota_{\mathbf{II}}} \underbrace{-\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^K d\boldsymbol{\Theta}_\sigma^L \wedge d\omega_\mu^{IJ} \wedge \beta_\nu}_{\iota_{\mathbf{III}}} \underbrace{+\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^K d\boldsymbol{\Theta}_\mu^{IJ} \wedge de_\sigma^L \wedge \beta_\nu}_{\iota_{\mathbf{IV}}}$$
$$\underbrace{-\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^K d\boldsymbol{X}^\lambda \wedge de_\sigma^L \wedge d\omega_\mu^{IJ} \wedge \beta_{\lambda\nu}}_{\iota_{\mathbf{V}}}$$

Aslo, we have the expression of $d\boldsymbol{X}^\nu$; $d\boldsymbol{\Theta}_\lambda^M$; $d\boldsymbol{\Theta}_\mu^{IJ}$ and $d\boldsymbol{\Upsilon}$:

$$\begin{aligned}
d\boldsymbol{X}^\rho &= \frac{\partial \boldsymbol{X}^\rho}{\partial x^\alpha}dx^\alpha + \frac{\partial \boldsymbol{X}^\alpha}{\partial e_\alpha^I}de_\alpha^I + \frac{\partial \boldsymbol{X}^\rho}{\partial \omega_\alpha^{IJ}}d\omega_\alpha^{IJ} + \frac{\partial \boldsymbol{X}^\rho}{\partial \mathfrak{e}}d\mathfrak{e} \\
d\boldsymbol{\Theta}_\lambda^M &= \frac{\partial \boldsymbol{\Theta}_\lambda^M}{\partial x^\alpha}dx^\alpha + \frac{\partial \boldsymbol{\Theta}_\lambda^M}{\partial e_\alpha^I}de_\alpha^I + \frac{\partial \boldsymbol{\Theta}_\lambda^M}{\partial \omega_\alpha^{IJ}}d\omega_\alpha^{IJ} + \frac{\partial \boldsymbol{\Theta}_\lambda^M}{\partial \mathfrak{e}}d\mathfrak{e} \\
d\boldsymbol{\Theta}_\mu^{IJ} &= \frac{\partial \boldsymbol{\Theta}_\mu^{IJ}}{\partial x^\alpha}dx^\alpha + \frac{\partial \boldsymbol{\Theta}_\mu^{IJ}}{\partial e_\alpha^I}de_\alpha^I + \frac{\partial \boldsymbol{\Theta}_\mu^{IJ}}{\partial \omega_\alpha^{IJ}}d\omega_\alpha^{IJ} + \frac{\partial \boldsymbol{\Theta}_\mu^{IJ}}{\partial \mathfrak{e}}d\mathfrak{e} \\
d\boldsymbol{\Upsilon} &= \frac{\partial \boldsymbol{\Upsilon}}{\partial x^\alpha}dx^\alpha + \frac{\partial \boldsymbol{\Upsilon}}{\partial e_\alpha^I}de_\alpha^I + \frac{\partial \boldsymbol{\Upsilon}}{\partial \omega_\alpha^{IJ}}d\omega_\alpha^{IJ} + \frac{\partial \boldsymbol{\Upsilon}}{\partial \mathfrak{e}}d\mathfrak{e}
\end{aligned} \tag{365}$$

then, we express the differents $\iota_{\mathbf{i}}$:

$$\iota_{\mathbf{I}} = d\boldsymbol{\Upsilon} \wedge \beta = \Big[\frac{\partial \boldsymbol{\Upsilon}}{\partial e_\alpha^I}de_\alpha^I + \frac{\partial \boldsymbol{\Upsilon}}{\partial \omega_\alpha^{IJ}}d\omega_\alpha^{IJ} + \frac{\partial \boldsymbol{\Upsilon}}{\partial \mathfrak{e}}d\mathfrak{e}\Big] \wedge \beta$$

$$\iota_{\mathbf{II}} = \Big[\frac{\partial \boldsymbol{X}^\nu}{\partial x^\alpha}dx^\alpha + \frac{\partial \boldsymbol{X}^\nu}{\partial e_\alpha^I}de_\alpha^I + \frac{\partial \boldsymbol{X}^\nu}{\partial \omega_\alpha^{IJ}}d\omega_\alpha^{IJ}\Big] \wedge d\mathfrak{e} \wedge \beta_\nu$$

$$\iota_{\mathbf{III}} = -\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^K\Big[d\boldsymbol{\Theta}_\sigma^L\Big] \wedge d\omega_\mu^{IJ} \wedge \beta_\nu$$

$$\iota_{\mathbf{IV}} = \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^K\Big[d\boldsymbol{\Theta}_\mu^{IJ}\Big] \wedge de_\sigma^L \wedge \beta_\nu$$

$$\iota_{\mathbf{V}} = -\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^K\Big[d\boldsymbol{X}^\rho\Big] \wedge de_\sigma^L \wedge d\omega_\mu^{IJ} \wedge \beta_{\rho\nu}$$

The decomposition of $d(\Xi \lrcorner \boldsymbol{\omega})$ is written:

- decomposition on $d\mathfrak{e} \wedge \beta$:

$$\Big[\frac{\partial \boldsymbol{X}^\nu}{\partial x^\nu} + \frac{\partial \boldsymbol{\Upsilon}}{\partial \mathfrak{e}}\Big]d\mathfrak{e} \wedge \beta \tag{366}$$

- decomposition on $de \wedge \beta$:

$$\Big[\frac{\partial \boldsymbol{\Upsilon}}{\partial e_\sigma^L} - \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^K\frac{\partial \boldsymbol{\Theta}_\mu^{IJ}}{\partial x^\nu}\Big]de_\sigma^L \wedge \beta \tag{367}$$

- decomposition on $d\omega \wedge \beta$:

$$\Big[\frac{\partial \boldsymbol{\Upsilon}}{\partial \omega_\mu^{IJ}} + \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^K\frac{\partial \boldsymbol{\Theta}_\sigma^L}{\partial x^\nu}\Big]d\omega_\mu^{IJ} \wedge \beta \tag{368}$$

- decomposition on $d\mathfrak{e} \wedge de \wedge \beta_\nu$:

$$\Big[\frac{\partial \boldsymbol{X}^\nu}{\partial e_\sigma^L} - \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^K\frac{\partial \boldsymbol{\Theta}_\mu^{IJ}}{\partial \mathfrak{e}}\Big]d\mathfrak{e} \wedge de_\sigma^L \wedge \beta_\nu \tag{369}$$

- decomposition on $d\mathfrak{e} \wedge d\omega \wedge \beta_\nu$:

$$\Big[\frac{\partial \boldsymbol{X}^\nu}{\partial \omega_\mu^{IJ}} - \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e_\rho^K\frac{\partial \boldsymbol{\Theta}_\sigma^L}{\partial \mathfrak{e}}\Big]d\mathfrak{e} \wedge d\omega_\mu^{IJ} \wedge \beta_\nu \tag{370}$$

- decomposition on $de \wedge d\omega \wedge \beta_\nu$:

$$\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho\Big[\frac{\partial\mathbf{\Theta}^{IJ}_\mu}{\partial\omega^{IJ}_\alpha}-\frac{\partial\mathbf{\Theta}^L_\sigma}{\partial e^L_\rho}\Big]de^N_\rho\wedge d\omega^{IJ}_\mu\wedge\beta_\nu \tag{371}$$

- decomposition on $de \wedge de \wedge \beta_\nu$:

$$\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho\Big[\frac{\partial\mathbf{\Theta}^{IJ}_\mu}{\partial e^N_\rho}\Big]de^N_\rho\wedge de^L_\sigma\wedge\beta_\nu \tag{372}$$

- decomposition on $d\omega \wedge d\omega \wedge \beta_\nu$:

$$\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho\Big[-\frac{\partial\mathbf{\Theta}^L_\sigma}{\partial\omega^{KL}_\rho}\Big]d\omega^{KL}_\rho\wedge d\omega^{IJ}_\mu\wedge\beta_\nu \tag{373}$$

- decomposition on $de \wedge de \wedge d\omega \wedge \beta_{\rho\nu}$:

$$\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho\Big[\frac{\partial\boldsymbol{X}^\alpha}{\partial e^N_\rho}\Big]de^N_\rho\wedge de^L_\sigma\wedge d\omega^{IJ}_\mu\wedge\beta_{\rho\nu} \tag{374}$$

- decomposition on $dx \wedge de \wedge d\omega \wedge \beta_{\rho\nu}$:

$$\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho\Big[\frac{\partial\boldsymbol{X}^\rho}{\partial x^\alpha}\Big]dx^\alpha\wedge de^L_\sigma\wedge d\omega^{IJ}_\mu\wedge\beta_{\rho\nu} \tag{375}$$

- decomposition on

$$\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho\Big[-\frac{\partial\boldsymbol{X}^\rho}{\partial\omega^{KL}_\rho}\Big]d\omega^{KL}_\rho\wedge de^L_\sigma\wedge d\omega^{IJ}_\mu\wedge\beta_{\rho\nu} \tag{376}$$

- decomposition on

$$\Big[\frac{\partial\boldsymbol{X}^\rho}{\partial x^\alpha}dx^\alpha\Big]d\mathfrak{e}\wedge de^L_\sigma\wedge d\omega^{IJ}_\mu\wedge\beta_{\rho\nu} \tag{377}$$

Equations (374) (376) (377) give us: $\boldsymbol{X}^\rho = \boldsymbol{X}^\rho(x)$. From equations (372) (373) we obtain: $\mathbf{\Theta}^{IJ}_\mu = \mathbf{\Theta}^{IJ}_\mu(x,\omega,\mathfrak{e})$ and $\mathbf{\Theta}^M_\rho = \mathbf{\Theta}^M_\rho(x,e,\mathfrak{e})$. Since, $\boldsymbol{X}^\rho$ is independant of (e,ω) we get from equations (369) (370) $\mathbf{\Theta}^{IJ}_\mu = \mathbf{\Theta}^{IJ}_\mu(x,\omega)$ and $\mathbf{\Theta}^M_\rho = \mathbf{\Theta}^M_\rho(x,e)$. Hence, from this analysis $\boldsymbol{X}^\rho, \mathbf{\Theta}^M_\rho, \mathbf{\Theta}^{IJ}_\mu$ satisfy:

$$\boldsymbol{X}^\rho = \boldsymbol{X}^\rho(x) \qquad \mathbf{\Theta}^M_\rho = \mathbf{\Theta}^M_\rho(x,e) \qquad \mathbf{\Theta}^{IJ}_\mu = \mathbf{\Theta}^{IJ}_\mu(x,\omega) \tag{378}$$

We are left with the equations (366) (367) (368) (371) (375) namely:

$$\begin{array}{rcl}
\dfrac{\partial\boldsymbol{X}^\nu}{\partial x^\nu}+\dfrac{\partial\mathbf{\Upsilon}}{\partial\mathfrak{e}} & = & 0\\
\Big[\dfrac{\partial\mathbf{\Upsilon}}{\partial e^L_\sigma}-\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho\dfrac{\partial\mathbf{\Theta}^{IJ}_\mu}{\partial x^\nu}\Big]de^L_\sigma\wedge\beta & = & 0\\
\Big[\dfrac{\partial\mathbf{\Upsilon}}{\partial\omega^{IJ}_\mu}+\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho\dfrac{\partial\mathbf{\Theta}^L_\sigma}{\partial x^\nu}\Big]d\omega^{IJ}_\mu\wedge\beta & = & 0\\
\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho\Big[\dfrac{\partial\mathbf{\Theta}^{IJ}_\mu}{\partial\omega^{IJ}_\alpha}-\dfrac{\partial\mathbf{\Theta}^L_\sigma}{\partial e^L_\rho}\Big]de^N_\rho\wedge d\omega^{IJ}_\mu\wedge\beta_\nu & = & 0\\
\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma}e^K_\rho\Big[\dfrac{\partial\boldsymbol{X}^\rho}{\partial x^\alpha}\Big]dx^\alpha\wedge de^L_\sigma\wedge d\omega^{IJ}_\mu\wedge\beta_{\rho\nu} & = & 0
\end{array} \tag{379}$$

Equivalently, we apply the proposition (6.1). Since $\mathcal{M}_{\mathsf{Palatini}}$ is an open subset of $\Lambda^n T^\star\mathfrak{Z}$, the set of all infinitesimal symplectomorphisms Ξ on $\mathcal{M}_{\mathsf{Palatini}}$ are of the form $\Xi = \chi + \overline{\zeta}$ where we denote a vector field $\zeta \in \Gamma(\mathfrak{Z}, T\mathfrak{Z})$:

$$\zeta = \boldsymbol{X}^\nu(x,e,\omega)\frac{\partial}{\partial x^\nu}+\mathbf{\Theta}^M_\lambda(x,e,\omega)\frac{\partial}{\partial e^M_\lambda}+\mathbf{\Theta}^{IJ}_\mu(x,e,\omega)\frac{\partial}{\partial\omega^{IJ}_\mu}$$

Then, $\bar\zeta \in \Gamma(\mathcal{M}_{\mathsf{Palatini}}, T\mathcal{M}_{\mathsf{Palatini}})$ $\bar\zeta = \sum_\alpha \zeta^\alpha(x,e,\omega)\frac{\partial}{\partial q^\alpha} - \sum_{\alpha,\beta}\frac{\partial\zeta^\alpha}{\partial q^\beta}(q)\Pi^\beta_\alpha$, with our notations we have for the Palatini multisymplectic manifold:

$$\overline{\zeta} = X^\nu\frac{\partial}{\partial x^\nu}+\Theta^M_\lambda\frac{\partial}{\partial e^M_\lambda}+\Theta^{IJ}_\mu\frac{\partial}{\partial\omega^{IJ}_\mu}-\overline{\zeta}^{\mathfrak{e}}\frac{\partial}{\partial\mathfrak{e}}-\overline{\zeta}^{\omega}\frac{\partial}{\partial p^{\omega_\mu\nu}_{IJ}}$$

with $\overline{\zeta}^{\mathfrak{e}}$ and $\overline{\zeta}^{\omega}$ given by:

$$\left|\begin{array}{lll} \overline{\zeta}^{\mathfrak{e}} & = & \mathfrak{e}(\frac{\partial X^\nu}{\partial x^\nu}) + \frac{\partial \Theta^{IJ}_\mu}{\partial x^\nu} p^{\omega_\mu\nu}_{IJ} = \mathfrak{e}(\frac{\partial X^\nu}{\partial x^\nu}) - E^{[\mu}_I e^{\nu]}_J (\frac{\partial \Theta^{IJ}_\mu}{\partial x^\nu}) \\ \overline{\zeta}^{\omega} & = & -E^{[\rho}_K e^{\sigma]}_L \underbrace{\delta^\mu_\rho \Big[\delta^K_I \delta^L_J [(\frac{\partial X^\nu}{\partial x^\sigma}) - \delta^\nu_\sigma (\frac{\partial X^\lambda}{\partial x^\lambda})] + (\frac{\partial \Theta^{KL}_\sigma}{\partial \omega^{IJ}_\nu})\Big]}_{[[\overline{\zeta}^\omega_\bullet]^{KL}{}_{\rho\sigma}]^{\mu\nu}_{IJ}} - \mathfrak{e}(\frac{\partial X^\nu}{\partial \omega^{IJ}_\mu}) \\ & & -E^{[\rho}_K e^{\sigma]}_L [[\overline{\zeta}^\omega_\bullet]^{KL}{}_{\rho\sigma}]^{\mu\nu}_{IJ} - \mathfrak{e}(\frac{\partial X^\nu}{\partial \omega^{IJ}_\mu}) \end{array}\right.$$

Then we need to compute: $\overline{\zeta}^{\omega} \frac{\partial}{\partial p^{\omega_\mu\nu}_{IJ}}$. Since[103] $\overline{\zeta}^{\omega} \frac{\partial}{\partial p^{\omega_\mu\nu}_{IJ}} = \overline{\zeta}^{\omega} [\epsilon^{IJOP} \epsilon_{\mu\nu\iota\eta} e^\iota_O] \frac{\partial}{\partial e^P_\eta}$ and since $p^{A_\rho\sigma}_{KL} = -E^{[\rho}_K e^{\sigma]}_L = -\frac{1}{2}\epsilon^{\rho\sigma\alpha\beta}\epsilon_{KLQR} e^Q_\alpha e^R_\beta$. We obtain:

$$\overline{\zeta}^{\omega} \frac{\partial}{\partial p^{\omega_\mu\nu}_{IJ}} = \Big[-\frac{1}{2}\epsilon^{\rho\sigma\alpha\beta}\epsilon_{KLQR} e^Q_\alpha e^R_\beta [[\overline{\zeta}^\omega_\bullet]^{KL}{}_{\rho\sigma}]^{\mu\nu}_{IJ} - \mathfrak{e}(\frac{\partial X^\nu}{\partial \omega^{IJ}_\mu})\Big] [\epsilon^{IJOP} 2\epsilon_{\mu\nu\iota\eta} e^\iota_O] \frac{\partial}{\partial e^P_\eta}$$

We simplify the issue, and we focus on the case $dx^\mu(\Xi) = 0$ - we throw away the X^μ which correspond to parts of the stress-energy-tensor:

$$\overline{\boldsymbol{\zeta}}^{\omega} \frac{\partial}{\partial p^{\omega_\mu\nu}_{IJ}} = -[\epsilon^{\mu\sigma\alpha\beta}\epsilon_{\mu\nu\iota\eta}]\epsilon^{IJOP}\epsilon_{KLQR}(\frac{\partial \Theta^{KL}_\sigma}{\partial \omega^{IJ}_\nu}) e^Q_\alpha e^R_\beta e^\iota_O \frac{\partial}{\partial e^P_\eta}$$

A more detailed study of this equation is need to obtain the full set of algebraic observables for Palatini gravity. The idea is to obtain the *classification* of observables and in particular of dynamical observable.

18.3 Canonical forms for Gravity

In this section, we briefly introduce the construction developed by Hélein and Kouneiher for the canonical Poisson bracket on the observable functionals. The most precise mathematical treatment shall appears in a forthcoming paper [229]. However in the spirit of the Ashtekar program - where the metric is not given *a priori* but obtained by pulling back along the tetrad field - it seems natural to focus on the spin connection. We describe the copolarization:

$$\mathbf{P}^1 T^\star \mathcal{M}^{\mathsf{Palatini}} = \bigoplus_{0\le\mu\le 3} dx^\mu$$

$$\mathbf{P}^2 T^\star \mathcal{M}^{\mathsf{Palatini}} = \bigoplus_{0\le\mu_1<\mu_2\le 3} dx^{\mu_1}\wedge dx^{\mu_2} \oplus d\omega^{IJ}, \text{where}, d\omega^{IJ} := d\omega^{IJ}_\mu \wedge dx^\mu.$$

$$\mathbf{P}^3 T^\star \mathcal{M}^{\mathsf{Palatini}} = \bigoplus_{0\le\mu_1<\mu_2<\mu_3\le 3} dx^{\mu_1}\wedge dx^{\mu_2}\wedge dx^{\mu_3} \oplus \bigoplus_{0\le\mu\le 3} dx^\mu \wedge d\omega^{IJ} \oplus d\boldsymbol{\varpi}_{IJ}$$

[103] This is straightforward computation from

$$\frac{\partial e^L_\sigma}{\partial p^{\omega_\rho\nu}_{KL}} = 2\epsilon^{IJKL}\epsilon_{\mu\nu\rho\sigma} e^\rho_K \quad \text{and} \quad \frac{\partial}{\partial p^{\omega_\rho\nu}_{KL}} = (\frac{\partial e^L_\sigma}{\partial p^{\omega_\rho\nu}_{KL}})\frac{\partial}{\partial e^L_\sigma} \quad \Longrightarrow \quad \frac{\partial}{\partial p^{\omega_\rho\nu}_{KL}} = 2\epsilon^{IJKL}\epsilon_{\mu\nu\rho\sigma} e^\rho_K \frac{\partial}{\partial e^L_\sigma}$$

$$\mathbf{P}^4 T^\star \mathcal{M}^{\text{Palatini}} = \beta \oplus \bigoplus_{0\leq\mu_1<\mu_2\leq 3} dx^{\mu_1}\wedge dx^{\mu_2}\wedge d\omega^{IJ} \oplus \bigoplus_{0\leq\mu\leq 3} dx^\mu \wedge d\boldsymbol{\varpi}_{IJ} \oplus \bigoplus_{0\leq\mu\leq 3} \frac{\partial}{\partial x^\mu} \lrcorner\, \theta$$

with $\omega^{IJ} = \omega^{IJ}_\mu dx^\mu$ and $\boldsymbol{\varpi}_{IJ} = \frac{1}{2}\sum_{\mu,\nu} p^{\omega_\mu\nu}_{IJ}\beta_{\mu\nu} = \frac{1}{2}\sum_{\mu,\nu} p^{\omega_\mu\nu}_{IJ}\frac{\partial}{\partial x^\mu}\lrcorner \frac{\partial}{\partial x^\nu}\lrcorner\,\beta$. So that we have:

$$\boldsymbol{\varpi}_{IJ} = \frac{1}{2}\sum_{\mu,\nu} e e^{[\nu}_I e^{\mu]}_J \beta_{\mu\nu} = -\frac{1}{4}\epsilon^{\mu\nu\sigma\rho}\epsilon_{IJKL}e^K_\sigma e^L_\rho \beta_{\nu\mu} \tag{380}$$

We denote $\omega^{IJ} \in \mathfrak{P}^{2-1}\mathcal{M}$ and $\boldsymbol{\varpi}^{IJ} \in \mathfrak{P}^{3-1}\mathcal{M}$. Following the method developed in [118] we make the hypothesis on $\mathbf{P}^1 T^\star_m \mathcal{M}^{\text{Palatini}}$. For all function $\varkappa : \mathcal{M}^{\text{Palatini}} \to \mathbb{R}$ and $\varsigma_1, \varsigma_2 : \mathcal{M}^{\text{Palatini}} \to \mathbb{R}$ whose differentials are proper on $\mathcal{O}_m$ - equivalently $d\varkappa$, $d\varsigma_1$ and $d\varsigma_2$ are in $\mathbf{P}^1 T^\star_m \mathcal{M}^{\text{Palatini}}$[104]. The two canonical forms are $\omega \in \mathfrak{P}^1\mathcal{M}^{\text{Palatini}}$ and $\boldsymbol{\varpi} \in \mathfrak{P}^2\mathcal{M}^{\text{Palatini}}$. We introduce $\varkappa, \varsigma_1, \varsigma_2, t = x^\circ$ on $\mathcal{M}^{\text{Palatini}}$ such that the level sets of t are *slices*. $\forall m \in \mathcal{M}^{\text{Palatini}}, d\varkappa_m, d\varsigma_1, d\varsigma_2$ are proper on $\mathcal{O}_m$ and $d\varkappa_m \wedge d\varsigma^1_m \wedge d\varsigma^2_m \neq 0$. Also let $\underset{\boxdot}{\omega}^{IJ}, \underset{\boxdot}{\boldsymbol{\varpi}}^{IJ} \in \mathfrak{P}^{n-1}\mathcal{M}$ be the forms:

$$\underset{\boxdot}{\boldsymbol{\varpi}}^{IJ} = d\varkappa \wedge \boldsymbol{\varpi}^{IJ} \qquad \text{and} \qquad \underset{\boxdot}{\omega}^{IJ} = d\varsigma^1 \wedge d\varsigma^2 \wedge \omega^{IJ}$$

We assume that $\xi_{[\underset{\boxdot}{\omega}^{IJ}]} \lrcorner\, d\varkappa = 0$ and also $\xi_{[\underset{\boxdot}{\boldsymbol{\varpi}}^{IJ}]} \lrcorner\, d\varsigma^1 = 0$ and $\xi_{[\underset{\boxdot}{\boldsymbol{\varpi}}^{IJ}]} \lrcorner\, d\varsigma^2 = 0$. Finally we denote by Γ a Hamiltonian n-curve and by Σ a level set of t then:

$$\underbrace{\left\{\int_\Sigma \underset{\boxdot}{\boldsymbol{\varpi}}^{IJ}, \int_\Sigma \underset{\boxdot}{\omega}^{IJ}\right\}(\Gamma)}_{[\mathbf{I}]} = \Big(\int_\Sigma \{\underset{\boxdot}{\boldsymbol{\varpi}}^{IJ}, \underset{\boxdot}{\omega}^{IJ}\}\Big)(\Gamma) = \underbrace{\int_{\Sigma\cap\Gamma}\{\underset{\boxdot}{\boldsymbol{\varpi}}^{IJ}, \underset{\boxdot}{\omega}^{IJ}\}}_{[\mathbf{II}]} \tag{381}$$

We introduce the submanifolds $\Gamma_\varkappa$ and Γ_ς of codimension $n-2$ and $n-1$ respectively. For a four dimensional space-time, we obtain $\dim(\Gamma_\varkappa) = 2$ and $\dim(\Gamma_\varsigma) = 1$. These submanifolds $\Gamma_\varkappa$ and Γ_ς are the ones on which the functions $\varkappa$ and $\varsigma = (\varsigma^1, \varsigma^2)$ are defined. Following [117] we suppose that $d\varkappa$ and $d\varsigma^1 \wedge d\varsigma^2$ is set to be zero outside of tubular neighborhood of $\Gamma_\varkappa$ and Γ_ς of width ϵ. Therefore we obtain:

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Sigma\cap\Gamma}\underset{\boxdot}{\boldsymbol{\varpi}}^{IJ} &= \lim_{\varepsilon\to 0}\int_{\Sigma\cap\Gamma} d\varkappa\wedge\boldsymbol{\varpi}^{IJ} = \int_{\Sigma\cap\Gamma_\varkappa\cap\Gamma}\boldsymbol{\varpi}^{IJ}\\ \lim_{\varepsilon\to 0}\int_{\Sigma\cap\Gamma}\underset{\boxdot}{\omega}^{IJ} &= \lim_{\varepsilon\to 0}\int_{\Sigma\cap\Gamma} d\varsigma^1\wedge d\varsigma^2\wedge\omega^{IJ} = \int_{\Sigma\cap\Gamma_\varsigma\cap\Gamma}\omega^{IJ}\end{aligned} \tag{382}$$

We have

$$\lim_{\varepsilon\to 0}[\mathbf{I}] = \lim_{\varepsilon\to 0}\Big[\left\{\int_\Sigma \underset{\boxdot}{\boldsymbol{\varpi}}^{IJ}, \int_\Sigma \underset{\boxdot}{\omega}^{IJ}\right\}(\Gamma)\Big] = \lim_{\varepsilon\to 0}\left\{\int_\Sigma d\varkappa\wedge\boldsymbol{\varpi}^{IJ}, \int_\Sigma d\varsigma^1\wedge d\varsigma^2\wedge\omega^{IJ}\right\}(\Gamma)$$

$$\lim_{\varepsilon\to 0}[\mathbf{I}] = \lim_{\epsilon\to 0}\left\{\int_{\Sigma\cap\Gamma} d\varkappa\wedge\boldsymbol{\varpi}^{IJ}, \int_{\Sigma\cap\Gamma} d\varsigma^1\wedge d\varsigma^2\wedge\omega^{IJ}\right\} = \left\{\int_{\Sigma\cap\Gamma_\varkappa\cap\Gamma}\boldsymbol{\varpi}^{IJ}, \int_{\Sigma\cap\Gamma_\varsigma\cap\Gamma}\omega^{IJ}\right\}$$

We observe that:

$$\lim_{\varepsilon\to 0}[\mathbf{I}] = \left\{\int_{\Sigma\cap\Gamma_\varkappa}\boldsymbol{\varpi}^{IJ}, \int_{\Sigma\cap\Gamma_\varsigma}\omega^{IJ}\right\}(\boldsymbol{\Gamma}) \tag{383}$$

We denote by $\Sigma_\varkappa = \Sigma\cap\Gamma_\varkappa\cap\boldsymbol{\Gamma}$ the surface of interest and $\Sigma_\varsigma = \Sigma\cap\Gamma_\varsigma\cap\boldsymbol{\Gamma}$ the other submanifold, a curve in the three dimensional space $\Sigma\cap\boldsymbol{\Gamma}$. For [**II**], we obtain:

[104] see [118] for details on the notion and hypothesis on proper forms

$$[\mathbf{II}] = \int_{\Sigma\cap\Gamma} \{\boldsymbol{\varpi}^{IJ}_{\boxdot}, \omega^{IJ}_{\boxdot}\} = \int_{\Sigma\cap\Gamma} d\varkappa \wedge d\varsigma^1 \wedge d\varsigma^2 \{\boldsymbol{\varpi}^{IJ}, \omega^{IJ}\}$$

If $\Gamma_\varkappa$ and Γ_ς cross transversaly, we have:

$$\lim_{\varepsilon\to 0}[\mathbf{II}] = \lim_{\varepsilon\to 0} \int_{\Sigma\cap\Gamma} d\varkappa \wedge d\varsigma^1 \wedge d\varsigma^2 \{\boldsymbol{\varpi}^{IJ}, \omega^{IJ}\} = \int_{\Sigma\cap\Gamma_\varkappa\cap\Gamma_\varsigma\cap\Gamma} \{\boldsymbol{\varpi}^{IJ}, \omega^{IJ}\} \tag{384}$$

Therefore from (383) and (384) we have $\lim_{\varepsilon\to 0}[\mathbf{I}] = \lim_{\varepsilon\to 0}[\mathbf{II}]$, then:

$$\left\{\int_{\Sigma\cap\Gamma_\varkappa} \boldsymbol{\varpi}^{IJ}, \int_{\Sigma\cap\Gamma_\varsigma} \omega^{IJ}\right\}(\Gamma) = \int_{\Sigma\cap\Gamma_\varkappa\cap\Gamma_\varsigma\cap\Gamma} \{\boldsymbol{\varpi}^{IJ}, \omega^{IJ}\} \tag{385}$$

We construct the following bracket for the observable functionals:

$$\left\{\int_{\Sigma\cap\Gamma_\varkappa} \boldsymbol{\varpi}, \int_{\Sigma\cap\Gamma_\varsigma} \omega\right\}(\Gamma) = \sum_{m\in\Sigma\cap\Gamma_\varkappa\cap\Gamma_\varsigma\cap\Gamma} \mathfrak{S}(m)$$

We recover the description for the canonical variables used in LQG, see [10, 11, 16, 27, 28, 179, 180, 189, 197, 221, 222] and references therein. We feel the important gain of this approach for the geometrical objects of LQG theory: in particular the Poisson bracket (268) developed in section (14.3). We believe that it corresponds to the natural geometrical setting for the loop variables. The canonical variable is not the densitized triad but the natural canonical variable $\boldsymbol{\varpi}$ (380) which embraces it. Hence $\boldsymbol{\varpi}_{IJ} = \frac{1}{2}\sum_{\mu,\nu} ee_I^{[\nu}e_J^{\mu]}\beta_{\mu\nu}$ appears to be the four dimensional analogue object in analogy with the traditional densitized triad in LQG.

Loop Quantum Gravity	Full covariant Gravity
Canonical variables (E^μ_I, A^I_μ)	Canonical variables $(\boldsymbol{\varpi}_{IJ}, \omega^{IJ})$
Electric field $E^\mu_I = \frac{1}{2}\epsilon_{IJK}\epsilon^{\mu\rho\sigma}e^J_\rho e^K_\sigma$	Electric field $\boldsymbol{\varpi}_{IJ} = \frac{1}{2}\sum_{\mu,\nu} ee_I^{[\nu}e_J^{\mu]}\beta_{\mu\nu}$
Lie algebra connection 1-form A^I_μ	Lie algebra connection 1-form $\omega^{IJ} = \omega^{IJ}_\mu dx^\mu$

19 Topological hypothesis

19.1 Holst and Nieh-Yan terms

In this section we do not give the full treatment we just discuss the context. We consider the following Poincaré-Cartan θ n-form:

$$\theta^{\mathsf{DW}}_{(q,p)} := \mathfrak{e}\beta + p_I^{e_\mu\nu} de^I_\mu \wedge \beta_\nu + p_{IJ}^{\omega_\mu\nu} d\omega^{IJ}_\mu \wedge \beta_\nu$$

with the Holst action functional:

$$\mathcal{L}_{\mathsf{Holst}}[e,\omega] = \frac{1}{2}\int \epsilon_{IJKL} e^I \wedge e^J \wedge \left(F^{KL}[\omega] + \frac{1}{\gamma}\overset{\circ}{\star} F^{IJ}[\omega]\right) \qquad \overset{\circ}{\star} F^{IJ}[\omega] = \frac{1}{2}\boldsymbol{\epsilon}^{IJ}{}_{KL}F^{KL}_{\mu\nu}$$

and the Nieh-Yan term:

$$\mathcal{L}_{\mathsf{Nieh\text{-}Yan}}[e,\omega] \propto T^I \wedge T_I[\omega] = T^I[e,\omega] \wedge T_I[e,\omega] = \alpha\epsilon^{\mu\nu\rho\sigma}T^I_{\mu\nu}[e,\omega]T_{I\rho\sigma}[e,\omega]\beta$$

with $T^I = d_\omega e^I = \frac{1}{2} T^I_{\mu\nu} dx^\mu \wedge dx^\nu \otimes E_I$. The Holst term is:

$$\mathcal{L}_{\mathsf{Holst}}[e,\omega] = \frac{1}{2\gamma} e e^\mu_I e^\nu_J \epsilon^{IJ}{}_{KL} F^{KL}_{\mu\nu} = \frac{1}{2\gamma} e e^\mu_I e^\nu_J \epsilon^{IJ}{}_{KL} \Big(\partial_{[\mu}\omega^{KL}_{\nu]} + \omega^K_{[\mu M}\omega^{ML}_{\nu]} \Big)$$

The Legendre correspondence with the setting of the Holst and the Nieh-Yan terms gives:

$$\begin{aligned} \frac{\partial L(q,v)}{\partial(\partial_\mu \omega^{IJ}_\nu)} &= \frac{\partial}{\partial(\partial_\mu \omega^{IJ}_\nu)} \Big(e e^\mu_I e^\nu_J (\partial_{[\mu}\omega^{IJ}_{\nu]} + \omega^I_{[\mu K}\omega^{KJ}_{\nu]}) + \frac{1}{2\gamma} e e^\mu_I e^\nu_J \epsilon^{IJ}{}_{KL} \Big(\partial_{[\mu}\omega^{KL}_{\nu]} + \omega^K_{[\mu M}\omega^{ML}_{\nu]} \Big) \Big) \\ \frac{\partial L(q,v)}{\partial(\partial_\mu e^I_\nu)} &= \alpha \frac{\partial}{\partial(\partial_\mu e^I_\nu)} [\epsilon^{\mu\nu\rho\sigma} T^I_{\mu\nu} T_{I\rho\sigma}] \end{aligned}$$

$$\begin{aligned} \frac{\partial L(q,v)}{\partial(\partial_\mu \omega^{IJ}_\nu)} &= e e^{[\mu}_I e^{\nu]}_J + \frac{1}{2\gamma} e e^{[\mu}_K e^{\nu]}_L \epsilon^{KL}{}_{IJ} \\ \frac{\partial L(q,v)}{\partial(\partial_\mu e^I_\nu)} &= \alpha \frac{\partial}{\partial(\partial_\mu e^I_\nu)} [\epsilon^{\mu\nu\rho\sigma} [\partial_{[\mu} e^I_{\nu]} + \omega^I_{[\mu J} e^J_{\nu]}] T_{I\rho\sigma}] \end{aligned}$$

Therefore the Legendre transform leads to:

$$\begin{aligned} p^{\omega_\mu\nu}_{IJ} &= -E^{[\mu}_I e^{\nu]}_J (1 + \frac{1}{2\gamma} e e^{[\mu}_K e^{\nu]}_L \epsilon^{KL}{}_{IJ}) \\ p^{e_\mu\nu}_I &= -(\epsilon^{\mu\nu\rho\sigma} T_{I\rho\sigma} - \epsilon^{\nu\mu\rho\sigma} T_{I\rho\sigma}) = -2\epsilon^{\mu\nu\rho\sigma} T_{I\rho\sigma} = -2\epsilon^{\mu\nu\rho\sigma} \mathcal{D}_\rho e_{\sigma I} \end{aligned} \tag{386}$$

The Legendre transform works only provided by the compatibility conditions:

$$\mathcal{M}_{\mathsf{topo}} = \{(x,e,\omega,p) \in \mathcal{M}_{\mathsf{DW}} \ / \ p^{\omega_\mu\nu}_{IJ} = -E^{[\mu}_I e^{\nu]}_J (1 + \frac{1}{2\gamma} e e^{[\mu}_K e^{\nu]}_L \epsilon^{KL}{}_{IJ}), \ \ p^{e_\mu\nu}_I = \epsilon^{\mu\nu\rho\sigma} T_{I\rho\sigma} + \epsilon^{\nu\mu\rho\sigma} T_{I\rho\sigma}\}$$

We should recover various considerations about the topological terms - see the works [51, 122, 170, 172, 175, 191]. However we would like to recover the same equations of movement rather by consideration of additional terms in the Poincaré-Cartan canonical form.

19.2 Lepage-Dedecker transform for first order gravity

The mathematical computation to support the topological hypothesis needs more time for the full setting. We just give here the heuristic and conceptual setting. Within the enlarger topological hypothesis - see some comments below - we want to describe the dynamical equation from higher Lepage-Dedecker geometrizations. Here we simply introduce notation in such a context with additional LD terms. This might seems a notation artifact, however we believe that this proposition is related to a deeper understanding of space-time-matter organization. We consider for example the following Poincarré-Cartan actions:

$$\theta^1_{(q,p)} := \mathfrak{e}\beta + p^{e_\mu\nu}_I de^I_\mu \wedge \beta_\nu + p^{\omega_\mu\nu}_{IJ} d\omega^{IJ}_\mu \wedge \beta_\nu + \underbrace{\varkappa^{e_\rho\omega_\sigma\mu\nu}_{IJK} de^I_\rho \wedge d\omega^{JK}_\sigma \wedge \beta_{\mu\nu}}_{[\mathbf{I}]}$$

$$\theta^2_{(q,p)} := \mathfrak{e}\beta + p^{e_\mu\nu}_I de^I_\mu \wedge \beta_\nu + p^{\omega_\mu\nu}_{IJ} d\omega^{IJ}_\mu \wedge \beta_\nu + \underbrace{\varkappa^{e_\rho e_\sigma\mu\nu}_{IJ} de^I_\rho \wedge de^J_\sigma \wedge \beta_{\mu\nu}}_{[\mathbf{II}]}$$

$$\theta^3_{(q,p)} := \mathfrak{e}\beta + p^{e_\mu\nu}_I de^I_\mu \wedge \beta_\nu + p^{\omega_\mu\nu}_{IJ} d\omega^{IJ}_\mu \wedge \beta_\nu + \underbrace{\varkappa^{\omega_\rho\omega_\sigma\mu\nu}_{IJKL} d\omega^{IJ}_\rho \wedge d\omega^{KL}_\sigma \wedge \beta_{\mu\nu}}_{[\mathbf{III}]}$$

those type of Cartan forms should lead to the *intrinsic* treatment of Holst, Euler, Pontrjagin, Nieh-Yan terms. Then, we obtain the multisymplectic forms:

$$\boldsymbol{\omega}^{1}_{(q,p)} := d\mathfrak{e} \wedge \beta + dp_I^{e_\mu\nu} \wedge de_\mu^I \wedge \beta_\nu + dp_{IJ}^{\omega_\mu\nu} \wedge d\omega_\mu^{IJ} \wedge \beta_\nu + d\varkappa_{IJK}^{e_\rho\omega_\sigma\mu\nu} \wedge de_\rho^I \wedge d\omega_\sigma^{JK} \wedge \beta_{\mu\nu}$$

$$\boldsymbol{\omega}^{2}_{(q,p)} := d\mathfrak{e} \wedge \beta + dp_I^{e_\mu\nu} \wedge de_\mu^I \wedge \beta_\nu + dp_{IJ}^{\omega_\mu\nu} \wedge d\omega_\mu^{IJ} \wedge \beta_\nu + d\varkappa_{IJ}^{e_\rho e_\sigma\mu\nu} \wedge de_\rho^I \wedge de_\sigma^{J} \wedge \beta_{\mu\nu}$$

$$\boldsymbol{\omega}^{3}_{(q,p)} := d\mathfrak{e} \wedge \beta + dp_I^{e_\mu\nu} \wedge de_\mu^I \wedge \beta_\nu + dp_{IJ}^{\omega_\mu\nu} \wedge d\omega_\mu^{IJ} \wedge \beta_\nu + d\varkappa_{IJKL}^{\omega_\rho\omega_\sigma\mu\nu} \wedge d\omega_\rho^{IJ} \wedge d\omega_\sigma^{KL} \wedge \beta_{\mu\nu}$$

The notation we introduce allow to write respectively the terms:

$$[\mathbf{I}] := \theta_{(q,p)}^{[1]^e|[1]^\omega||[4]} \qquad [\mathbf{II}] := \theta_{(q,p)}^{[2]^e|[0]^\omega||[4]} \qquad [\mathbf{III}] := \theta_{(q,p)}^{[0]^e|[2]^\omega||[4]}$$

so that:

$$\theta^1_{(q,p)} = \theta^{\mathsf{DW}}_{(q,p)} + \theta_{(q,p)}^{[1]^e|[1]^\omega||[4]}$$

$$\theta^2_{(q,p)} = \theta^{\mathsf{DW}}_{(q,p)} + \theta_{(q,p)}^{[2]^e|[0]^\omega||[4]}$$

$$\theta^3_{(q,p)} = \theta^{\mathsf{DW}}_{(q,p)} + \theta_{(q,p)}^{[0]^e|[2]^\omega||[4]}$$

Some more work is need to put in better application the topological hypothesis. [230]

SPACE–TIME–MATTER

The last part of this Thesis, attempts to provide insights on the foundations of our thoughts about space, time, matter, symmetry and observables. It must be viewed as a highly speculative, tentative, then provisional, and exploratory discussion. Much more work is needed to put these ideas in a more unified form.

The key feature is an ubiquitous *double duality*. It is at the same time a nested duality each aspect of the duality itself contains a further inner opposition. We first note the opposition between what we term *ontologic space* and *dynamical space*. The former is described by the set of infinitesimal symplectomorphisms $\mathfrak{sp}_\circ(\mathcal{M})$ whereas the latter one is described by the set of Hamiltonian n-vector fields: $X(m) \in [X]^{\mathcal{H}}_m$. Further underpinning each of these respectively are the following geometrical objects: the multisymplectic form $\boldsymbol{\omega}$ and the Hamiltonian function $\mathcal{H}$. The notion of *space* is further involved in understanding this first aspect of the *double duality* via the role of the multisymplectic space $\mathcal{M}$ itself on the ontologic side and the Hamiltonian n-curve $\Gamma \subset \mathcal{M}$ on the dynamical side. This aspect of the *double duality* is further connected to the opposition of the internal or dynamical space vs external or kinematical space. *Symmetry* in such a very general framework is to be understood as carried by vector fields and this provides, in the setting of MG the key to ontological understanding of a much more general notion of dynamical variables. In the former case - that of the *ontologic space* - we note that the central object of study is the $\Xi \in T\mathcal{M} = T\Lambda^n T^\star \mathfrak{Z}^{\circ}$. The concern is here on the side of *ontologic space* with the issue of *invariance*, namely the the *invariance* of the multisymplectic form, see relation $\mathcal{L}_\Xi \boldsymbol{\omega} = 0$ with $X \in T\Gamma \subset \Lambda^n T\mathcal{M}$. On the side of dynamical space our concern is rather with the issue of covariance - what we refer to as the *dynamical symmetry* or *dynamical evolution*. Here the object of focal concern the generalized Hamiltonian equations $\forall m \in \Gamma, \exists X \in \Lambda^n T_m\Gamma, X \lrcorner\, \boldsymbol{\omega}_m = (-1)^n d\mathcal{H}_m$.

We seek to capture the *second* aspect of the duality by the contrast between ontologic and dynamical *representation*. Here our concerns is with the notion of *observables* and the opposition between ontologic and dynamical aspect on the side of representation is connected to the contrast between the concepts of topological duality and dynamical duality. Here the objects on the respective sides are the set of algebraic $(n-1)$-forms $\mathfrak{P}^{n-1}_{\circ}(\mathcal{M})$ and the set of observable forms $(n-1)$-forms $\mathfrak{P}^{n-1}_{\bullet}(\mathcal{M})$. The notion of algebraic $(n-1)$-forms AOF is the core concept on the side of *ontologic representation* see section (6.1), that of observable forms $(n-1)$-forms OF on the side of *dynamical representation* see section (6.3).

The concept of dynamical observables essentially arises from the intersection between the *ontologic* and *dynamical* aspects of the *double duality.* This gives rise to an understanding of dynamics. We are trying to bring to light here nothing less than a radically enlarged conception of physical reality. This is to be thought of as given in the first place through a re-conception and deepening of our notion of dynamical observables. It is not just that this notion of dynamical observables has its source within the opposition of *ontologic* vs *dynamical* aspects of the duality but more exactly that its sources are to be found in a further double movement within that opposition which gives rise to and enlarged notion of the physical on the sides both of space and of representation. The infinitesimal symplectomorphisms must satisfy an dynamical invariance by means of the additional condition $d\mathcal{H}(\Xi) = 0$. The double movement *within* the duality which we are seeking to describe may be briefly set out in the following table. It has emerged how these intuitions can be given mathematical shape in the language of n-forms and *copolar forms* - see section (6.5) - in the setting of MG particularly with respect to the way the notion of the CPS of the theory - see section (7) - re-appears from such a setting.

	Ontologic landscape $(\mathcal{M},\boldsymbol{\omega})$	Dynamical landscape $(\Gamma,\mathcal{H})$
Symmetry	Invariance	Covariance
Observable	Algebraic observables AOF	Observable forms OF

We re-emphasize the key point. In table above, the upper part is concerned with *symmetry* whereas the lower part concerns *observables.* Notice that in the context of variational problems, thanks to the Legendre correspondence, we have available the notion of exact multisymplectic manifold $(\mathcal{M},\boldsymbol{\omega})$. This notion rests on the fact that the multisymplectic form is derived from the canonical Poincaré-Cartan form: $\boldsymbol{\omega} = d\theta$. The canonical Poincaré-Cartan form θ, is what we shall term the (ontologic) potential for $\boldsymbol{\omega}$. Dynamical evolution is described by means of the Hamiltonian function $\mathcal{H}$ but what plays this role in the generalized Hamilton equations is the exterior differential $d\mathcal{H}$. The Hamiltonian function $d\mathcal{H}$ is to be thought of as the *dynamical potential.* Notice that the exterior derivative of the Hamiltonian $d\mathcal{H}$ is the appropriate object, not only to characterize dynamical symmetry or covariance in the setting of the generalized Hamilton equations but also plays a key role in the classification of observables via the *dynamical duality.*

A principal focus of Einstein work in GR [69, 70, 71] was the equality of gravitational and inertial mass. This yielded the celebrated Equivalence Principle. *Inertia* and *gravitation* are essentially the same. From this conclusion appears the idea of a single inertio-gravitational field, the metric field. As emphasized by Stachel, *"The distinction between the two is not absolute - i.e., frame independent -, but depends on the frame of reference adopted."*[217, 218]. In such a picture there is no observable difference between an inertial motion and a motion subject to a gravitational field. In the next section we discuss the double role of the metric field from the multisymplectic viewpoint. The traditional ontological understanding of GR is based upon the categories of *space* and *matter* (see section (11.2) for the our symbolization $\boxed{\text{S}}\boxed{\text{M}}^{\text{GR}}$ in this respect). However GR does *not* encapsulate a *full* intrinsic vision of matter.[105] There are at least two reason for this claim. The first is the issue of the cosmological constant or equivalently, the issue of the ontological status of the objects $G_{\mu\nu}$ and

[105] Einstein himself considered much with his celebrated remark

$T_{\mu\nu}$ that appear in the Einstein fields equations. The second concerns the fermionic form of matter, with the underlying spin and torsion. In its original formulation, GR makes central use of the Levi-Civita connection - which a torsion free connection. Because of the *Equivalence Principle* locally we are dealing with flat space, which involves passing from the Riemannian to the Minkowski metric. The right landscape for further investigations is Cartan geometry and Riemann-Cartan spaces - (12) - where we work not only with curvature but also torsion. However, in this setting the Dirac field in the Lagrangian formulation has to be added by hand and we lack an understanding of how spin arises intrinsically. Roughly speaking the idea of incorporating the interaction of space-time and matter in a unitary ontological picture crystalized by the famous *"space-time tells matter how to move; matter tells space-time how to curve"* [241] has not really succeeded in alleviating the ontological tension between space-time and matter. A principal expression of this simple and central fact of modern physics is the tension between the underlying structure of fiber bundle theory and that of the diffeomorphism group. The ontological aspect of general relativistics space-time, (captured here by the symbol $\boxed{S}\boxed{M}^{\text{GR}}$) is in conflict with the gauge picture which stresses the epistemological aspect. This rests on $\boxed{S}\boxed{M}^{\text{Gauge}}$ which symbolizes the matter fields over space-time and associated bundles. We call these epistemological in the broad sens that they depend on points of space-time.

Very much as Spinoza argued the impossibility of there being two *substances* - a *substance* being that which exists *sui generis* - there cannot be an absolute space-time-fields and matter-fields exiting together, without the theoretical possibility of a further space-time-matter field encompassing them. The setting of MG - as a mathematical framework - together with the idea of Eye-mirror monad, - expressed in the central notion of dynamical figure -holds the key to the conceptual conflict between oneness and plurality. Notice the ubiquity of duality in physical representation - parametrization space vs parametrized space, space-time vs matter etc ... In our view this points to the conclusion that the fundamental category of our ontology should be intrinsically both dynamical and geometrical in its local aspect (abstract points) and its global essence (space).

20 The Double Duality

20.1 Eye-mirror monad

We would like to draw together the previous intuitions concerning the notion of observer into what we call Eye-mirror monad - inspired by Leibniz [158]. The following figure summarizes the whole discussion. Philosophically we may that there are two conceptual setting involved in the notion of observable and symmetry. The first is concerned with the *ontologic landscape* $(\mathcal{M}, \boldsymbol{\omega})$. In this case, the space is the multisymplectic space whose dual nature is described by the multisymplectic form itself and the related ontologic mode. The second is concerned with the *dynamical landscape* described by $(\Gamma, \mathcal{H})$ which is also formed from two objects. The duality is reflected in the Hamiltonian function whereas the carrier of the dynamics is the Hamiltonian curve Γ. The Eye-mirror monad is summarized below in a symbolic *dynamical* figure.

The notion of Eye-mirror monad brings together two key concepts related to symmetries of physics: invariance and covariance. We summarize this unification in the following comparison:

- Invariance

  Concerned with object conservation
  Noether theorem
  Pataplectic invariant Hamiltonian functions $\forall \xi \in \mathbf{L}_m^{\mathcal{H}}, d\mathcal{H}_m(\xi) = 0$
  invariant by deformations parallel to pseudofibers

- Covariance

  Concerned with the form of the equations
  Einstein covariance principle
  The Hamilton equations are invariant by deformations parallel to pseudofibers

The invariance vs covariance opposition is further connected with the contrast between topological duality and dynamical duality:

- Algebraic observable forms – Topological duality
- Observable forms – Dynamical duality

The resulting deeper re-conception of *dynamical observable*, brings together the two dualities. Dynamical observable $(n-1)$-forms correspond to symmetries of the variational problems. More generally we believe that the notion of Eye-mirror monad is an invitation to the use of diagrammatic method in mathematical physics. We have already described in (2.1) how the selected examples of [68, 101, 190, 76] mark the transition from a representational function for diagrams to a conception of the diagram itself as an object of study.

The monad is an invitation to invert this process. The movement within the double duality in its ontologic and dynamical itself forms the diagram. Put differently we may say that the theory in its own development towards grasping its main constitutive objects - in particular in its focus on observables and symmetry as the cornerstone of physical representation - intrinsically describes a *dynamical figure*. Hence, we invert the statement: the order of development of the ideas in the theory, in its essence is the dynamical diagram which itself appears as the primordial conceptual entity or monad. The diagram becomes the essence of the physical representation. It forms a symbolic picture of the physical representation. Representation and Reality always form a duality, the two aspects of the Observer, the ontologic and the dynamical arise at the intersection of observable and symmetry. We remark that in the Leibnizian meaning, there is no part nor figure to described or grasp a monad. Nevertheless we drawn such a figure and call it monad. To say it more simply, it is a paradoxal representation of the *"no name"*.

20.2 Dual Nature of the multisymplectic form

In this section, we do not recapitulate the whole previously presented classification for the dualities around the two "seeds" $(\mathcal{M}, \boldsymbol{\omega})$ and $(\Gamma, \mathcal{H})$. Here we simply explore some possible directions. Recall that we emphasized in section (11.2) the dual nature of the metric field for classical GR. This viewpoint, resting on the work of Einstein [69, 70, 71] gives fundamental insight into GR foundations and underlying principles. We refer here to the work of Stachel [216] for more embracing consideration and a very clear description of the historical and conceptual sources and development for these ideas. Here we propose to take the essence of these ideas and apply them to the multisymplectic setting. For some aspects of the *double duality*:

General Relativity

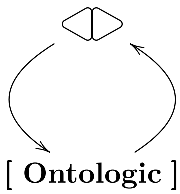
$\mathrm{g}_{\mu\nu}$ set chrono-geometrical structure of space-time.
$\mathrm{g}_{\mu\nu}$ represents the potentials for the inertio-gravitational field.

Multisymplectic Relativity

$\boldsymbol{\omega}$ set multi-chrono-geometrical structure of multi-space-time-matter-hybrid
$\boldsymbol{\omega}$ represents the potentials for the multi-inertio-gravitational field.

We see below that what we call the multi-chrono-geometrical structure for the multisymplectic ontologic space is connected to the concept of ontologic mode $\boxed{S}\boxed{M}\boxed{H}$ which mathematically correspond to the ontologic setting of a Lepage-Dedecker theory - see this section (20.2) and the following (20.3). A second aspect of this structure calls for a generalization of the notion of inertio-gravitational field. This is a much more delicate issue.

Multisymplectic form as a choice of ontologic mode $\boxed{S}\boxed{M}\boxed{H}$. The following diagram expresses the idea that the multisymplectic form, or more exactly the choice of a multisymplectic form $\boldsymbol{\omega}$, corresponds to the choice of a particular *ontologic* mode of our space, denoted therefore $\big(\mathcal{M},\boldsymbol{\omega}\big) = \boxed{S}\boxed{M}\boxed{H}$



$$\big(\mathcal{M},\boldsymbol{\omega}\big) = \boxed{S}\boxed{M}\boxed{H} \longrightarrow \begin{cases} \big(d\boxed{S} = d\mathfrak{e} \wedge \beta\big) \\ \big(d\boxed{M} = dp_i^{\mu} \wedge \beta_{\mu}\big) \\ \big(d\boxed{H} = dp_{i_1\cdots i_j}^{\mu_1\cdots\mu_j} \wedge \beta^{i_1\cdots i_j}_{\mu_1\cdots\mu_j}\big) \end{cases}$$

The pictograph at the top left of the figure is understood as symbolizing the dual nature of the multisymplectic form - or more generally of the dual "seeds" $\big(\mathcal{M},\boldsymbol{\omega}\big)$ and $\big(\Gamma,\mathcal{H}\big)$ - related to the classification of the notions of symmetry and observable in respectively the *ontologic* - kinematic vs *dynamical* - dynamic settings. In the following discussion we introduce these notions.

space-time entity $\boxed{S}$
matter entity $\boxed{M}$
hybrid entity $\boxed{H}$

By way of prelude, we lift arguments which Einstein offered for the metric case to the general multisymplectic setting. Here the structure of interest is no longer the *space-time manifold* but the multisymplectic *space*. We emphasize the key point: we are interested in the ontology of the multisymplectic manifold itself. In parallel with the considerations involved in the choice of metric in the Einstein discussion, we emphasize that, from the *ontologic multisymplectic space* viewpoint, $\mathcal{M}$ - as opposed to what fills $\mathcal{M}$ - has no separate existence. If we imagine the multisymplectic form $\boldsymbol{\omega}$ to be removed, there no longer remains a *space* or a *relativistic space*, not even a *topological space*. In the spirit of Einstein, we say that is no such thing as an empty space: a space without a multisymplectic form is not a multisymplectic ontologic space. In other words if no $\boldsymbol{\omega}$ is given, then we cannot describe any ontologic mode $\big(\mathcal{M},\boldsymbol{\omega}\big) = \boxed{S}\boxed{M}\boxed{H}$. In this case, no such *space-time*, *matter* or *hybrid* entities may be identified. The ontologic space is the more general one, it may be thought as the *ground state* of the theory. We propose to call this concept the *multisymplectic trivial mode*. The multisymplectic trivial mode is understood as the largest *pataplectic manifold* of the theory and is identify with $\mathcal{M} = \Lambda^n T^\star \mathfrak{Z}^{\circ}$.

A particular multisymplectic state $(\mathcal{M}^\circ,\boldsymbol{\omega}^\circ)$ - a multisymplectic manifold with $\mathcal{M}^\circ \subset \mathcal{M}$ - is equivalent to the choice of a particular ontologic mode $\boxed{S}\boxed{M}\boxed{H}^\circ$. Again following the ideas of Einsein in the setting of classical GR, we posit that any such *entity* $\boxed{S}$, $\boxed{M}$ or $\boxed{H}$ has no claim to existence on its own, but only as a structural quality of the form $\boldsymbol{\omega}^\circ$. We insist on the following: to chose a multisymplectic form $\boldsymbol{\omega}^\circ$ and a specific submanifold $\mathcal{M}^\circ \subset \mathcal{M}$ involves two considerations. We can chose to work from the perspective of ontologic space or from the multisymplectic form itself. Hence, in the more general cases - where more than one field is involved, or constraint considerations apply - we see that an ontologic mode is determined by two related aspects: the choice of the space $\mathcal{M}^\circ$ and the choice of the multisymplectic form $\boldsymbol{\omega}^\circ$- which can incorporate mixed terms derived from the

Poincaré-Cartan canonical form - see the example of Dirac theory or Palatini theory. We set this delicate issue to one side. To simplify the description we take only the canonical example of the DW ontologic mode described by the space $\mathcal{M}^{\mathsf{DW}} = \big\{(x,z,\mathfrak{e}\beta + p_i^\mu dz^i \wedge \beta_\mu), (x,z) \in \mathcal{X} \times \mathfrak{Z}, \mathfrak{e}, p_i^\mu \in \mathbb{R}\big\}$ and the DW Poincaré-Cartan form $\theta^{\mathsf{DW}}_{(q,p)} = \mathfrak{e}\beta + p_i^\mu dz^i \wedge \beta_\mu$. Here the multisymplectic canonical form is therefore $\boldsymbol{\omega}^{\mathsf{DW}} = d\mathfrak{e} \wedge \beta + dp_i^\mu \wedge dz^i \wedge \beta_\mu$. We remark that the DW ontologic state differs from the *ontologic trivial mode*. From the perspective of the notion of observable, the key conceptual consideration is the non-pataplectic feature. The ontologic trivial mode corresponds to the whole LD geometrization:

$$\boxed{\mathrm{S}}\boxed{\mathrm{M}}\boxed{\mathrm{H}}^{\mathsf{Trivial}} = \Big(\mathcal{M} = \Lambda^n T^\star \mathfrak{Z} \quad , \quad \boldsymbol{\omega} = \sum_{1 \leq \boldsymbol{\mu}_1 \leq \dots \boldsymbol{\mu}_n < n+k} dp_{\boldsymbol{\mu}_1 \cdots \boldsymbol{\mu}_n} \wedge dq^{\boldsymbol{\mu}_1} \wedge \dots \wedge dq^{\boldsymbol{\mu}_n}\Big)$$

In the universal multisymplectic formalism, the Lepage-Dedecker standpoint leads to the canonical multisymplectic form:

$$\boldsymbol{\omega}^{\mathsf{Trivial}} = \underbrace{d\mathfrak{e} \wedge \beta}_{d\boxed{\mathrm{S}}} + \underbrace{dp_i^\mu \wedge dz^i \wedge \beta_\mu}_{d\boxed{\mathrm{M}}} + \underbrace{\sum_{j=2}^{n} \sum_{\mu_1 < \cdots < \mu_j} \sum_{i_1 < \cdots < i_j} dp^{\mu_1 \cdots \mu_j}_{i_1 \cdots i_j} \wedge \beta^{i_1 \cdots i_j}_{\mu_1 \cdots \mu_j}}_{d\boxed{\mathrm{H}}}$$

We therefore understand the three entities $\boxed{\mathrm{S}}$, $\boxed{\mathrm{M}}$ and $\boxed{\mathrm{H}}$ as the different parts of the $\boldsymbol{\omega}^{\mathsf{Trivial}}$. We observe that the *hybrid* sector $d\boxed{\mathrm{H}}$ can be described only through the Lepage-Dedecker geometrization. The $(n+1)$-forms that constitute the *hybrid* entity from an ontologic standpoint, are denoted $d\boxed{\mathrm{H}}$. We say equivalently that $\boxed{\mathrm{H}}$ is prescribed by forms:

$$d\boxed{\mathrm{H}} := \textstyle\sum_{j=2}^n \sum_{\mu_1 < \cdots < \mu_j} \sum_{i_1 < \cdots < i_j} dp^{\mu_1 \cdots \mu_j}_{i_1 \cdots i_j} \wedge \beta^{i_1 \cdots i_j}_{\mu_1 \cdots \mu_j}$$

Forms that appear in the *hybrid* area are of the type $dp^{\mu_1 \cdots \mu_j}_{i_1 \cdots i_j} \wedge \beta^{i_1 \cdots i_j}_{\mu_1 \cdots \mu_j}$. These do not simply evaluate the interaction within the ontologic mode $(\mathcal{M}, \boldsymbol{\omega}) = \boxed{\mathrm{S}}\boxed{\mathrm{M}}\boxed{\mathrm{H}}$ of the *space-time* entity $\boxed{\mathrm{S}}$ with the *matter* entity $\boxed{\mathrm{M}}$. Rather we see that in addition to the *space-time* entity $\boxed{\mathrm{S}}$ and *matter* entity $\boxed{\mathrm{M}}$, the *hybrid* entity $\boxed{\mathrm{H}}$ is also required has a further primordial "seed" of the ontologic mode. Let us again examine the symbolism. We observe that the *space-time* entity is related to the part $d\boxed{\mathrm{S}} = d\mathfrak{e} \wedge \beta$ of the multisymplectic form $\boldsymbol{\omega}$. The *matter* entity is related to the part $d\boxed{\mathrm{M}} = dp_i^\mu \wedge dz^i \wedge \beta_\mu$ of the multisymplectic form. Notice that the *hybrid* entity *really is* an hybrid object. It correspond to objects that can not be identified purely with either space-time *or* matter. The hybrid entity is constructed on the basis of a "mixture" of "seeds" that appear in $\boxed{\mathrm{S}}$ or $\boxed{\mathrm{M}}$ respectively. It follows that the *hybrid* entity is related to the detection of forms for the description of which classical field theory or classical physics lacks the resources. Indeed, until now there has been no convincing conceptual motivation for the application of higher LD geometrization to physics. The traditional literature only focus on the underlying DW framework. We believe that this application is a natural historical development since the huge number of purportedly unphysical variables and the related mathematical complexity has always been seen as an obstacle *a priori* for physical purposes.

However, we also believe that due to the new insights presented here and in particular the relation to the concept of observable and also due to the status and treatment of the Dirac constraint set - and related obstructions, in particular concerning the quantum formalism - the LD development for variational calculus for field theory deserves careful consideration. Later we suggest a cosmological perspective on the meaning of the interplay of the additional degrees of freedoms which this calculus involves. We suggest that the *hybrid* entity provides a possible framework for the explanation of the concepts of dark matter and dark energy, via the interplay which it permits between the *dynamical* *geometry* prescribed by the *space-time* part $\boxed{S}$ and the *dynamical geometry* prescribed by the *matter* part $\boxed{M}$. Because of the additional degrees of freedom of the dynamics this no longer a matter, as in the old Geometrodynamics program, of bringing matter within the space-time geometry. In this connection, we notice that in the DW ontologic mode, the *hybrid* entity $\boxed{H}_{\mathsf{DW}}$ is empty and therefore plays no role in the story. The DW mode is the multisymplectic manifold $\left(\mathcal{M}_{\mathsf{DW}}, \boldsymbol{\omega}^{\mathsf{DW}}\right)$:

$$\boxed{S}\boxed{M}\boxed{H}_{\mathsf{DW}} = \Big(\mathcal{M}_{\mathsf{DW}} = \Lambda_2^n T^\star \mathfrak{Z} \ \ , \ \ \boldsymbol{\omega}^{\mathsf{DW}} = \underbrace{d\mathfrak{e} \wedge \beta}_{d\boxed{S}_{\mathsf{DW}}} + \underbrace{dp_i^\mu \wedge dz^i \wedge \beta_\mu}_{d\boxed{M}_{\mathsf{DW}}}\Big)$$

We think of the problem in the following way. Notice this is a subtly different viewpoint but the underlying idea is the same. We introduced at the beginning of the discussion the conceptual idea of *ontologic motif* $\boxed{S}\boxed{M}$ and *dynamical motif* $\boxed{S}, \boxed{S} \times \boxed{M}$ as part of the setting of any variational problem.

In connection with the identification of the different parts of the ontologic mode involved in the decomposition of the dynamical variables in the multisymplectic form, we introduce the following variant notions for the space-time, matter and hybrid entities using the color green: $\boxed{S}$, $\boxed{M}$ and $\boxed{H}$. This symbols are intended to keep track of the natural duality of the parametrization vs parametrized space. They lead us to take $n = k$ so that we have the same number of dimensions for space-time and matter degrees of freedom. From the perspective of symmetry, duality and aesthetic consideration this possibility is interesting. It directly addresses the conceptual problem of how to describe scalar or matter fields for arbitrary degrees of freedom.

This idea has been implicitly invoked - for different but related purposes - in the use of the *tetrad field* and related matter representations involving the idea of solder form for gravity e.[106] In the case where $n = k$ we observe the following possibilities: the first is described by a $(n+1)$-form containing only the space-time degrees of freedom: $d\boxed{S} = d\mathfrak{e}_{\boxed{S}} \wedge \beta$ whereas the second is described only by matter degrees of freedom $d\boxed{M} = d\mathfrak{e}_{\boxed{M}} \wedge d\mathfrak{h}$. Notice that we have the following notation: $\mathfrak{e}_{\boxed{S}} = p_{1\cdots n}$, $p_i^\mu = p_{1\cdots(\mu-1)i(\mu+1)\cdots n}$ $\cdots$ $p_{i_1 i_2}^{\mu_1\mu_2} = p_{1\cdots(\mu_1-1)i_1(\mu_1+1)\cdots(\mu_2-1)i_2(\mu_2+1)\cdots n}$ $\cdots$ $\mathfrak{e}_{\boxed{M}} = p_{i_1\cdots i_n}^{\mu_1\cdots\mu_n} = p_{i_1\cdots i_n}$. Also we use the notation $\beta_{\mu_1\cdots\mu_p}^{i_1\cdots i_p} = dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge \left(\partial_{\mu_1} \wedge \cdots \wedge \partial_{\mu_p} \lrcorner \beta\right)$ as well as $\beta_\mu = \partial_\mu \lrcorner \beta$ so that in this specific case, $\beta = dx^1 \wedge \cdots \wedge dx^n$ is a volume n-form on $\mathcal{X}$ whereas the new object $d\mathfrak{h}$ is thought of as a volume n-form on $\mathfrak{Z}$. Hence $d\mathfrak{h} = dz^1 \wedge \cdots \wedge dz^n$. In this case we have the picture:

| space-time entity $\boxed{S}$
| matter entity $\boxed{M}$
| hybrid entity $\boxed{H}$

In such a context, the hybrid entity becomes:

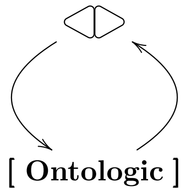
$$d\boxed{H} := \sum_{j=1}^{n-1} \sum_{\mu_1<\cdots<\mu_j} \sum_{i_1<\cdots<i_j} dp_{i_1\cdots i_j}^{\mu_1\cdots\mu_j} \wedge \beta_{\mu_1\cdots\mu_j}^{i_1\cdots i_j}$$

and the multisymplectic form decomposition becomes:

$$\boldsymbol{\omega} = \underbrace{d\mathfrak{e}_{\boxed{S}} \wedge d\mathfrak{h}}_{d\boxed{S}} + \underbrace{\sum_{j=1}^{n-1} \sum_{\mu_1<\cdots<\mu_j} \sum_{i_1<\cdots<i_j} dp_{i_1\cdots i_j}^{\mu_1\cdots\mu_j} \wedge \beta_{\mu_1\cdots\mu_j}^{i_1\cdots i_j}}_{d\boxed{H}} + \underbrace{d\mathfrak{e}_{\boxed{M}} \wedge d\mathfrak{h}}_{d\boxed{M}}$$

We summarize in symbolic picture:

[106] We model the geometry of the tangent space of space-time by the so-called *internal space* $\mathcal{V}$ - see section (12.4).



$$\big(\mathcal{M},\boldsymbol{\omega}\big) = \big(\mathsf{S},\mathsf{M},\mathsf{H}\big) \longrightarrow \begin{cases} \big(d\boxed{S} = d\mathfrak{e}_{\boxed{S}} \wedge \beta\big) \\ \big(d\boxed{M} = d\mathfrak{e}_{\boxed{M}} \wedge d\mathfrak{h}\big) \\ \big(d\boxed{H} = dp^{\mu_1\cdots\mu_j}_{i_1\cdots i_j} \wedge \beta^{i_1\cdots i_j}_{\mu_1\cdots\mu_j}\big) \end{cases}$$

This device of coloring the symbols emphasize how flexible are the concepts $\boxed{S}$, $\boxed{M}$ and $\boxed{H}$. Now we return to the monochromatic world and abandon the green picture.

Relation between ontologic mode and observables. Here we emphasize the role that the *pataplectic* manifold plays in this extended notion of observable. For example, the De Donder-Weyl ontologic mode. $\boxed{S}\boxed{M}\boxed{H}_{\mathsf{DW}}$ is *not* a pataplectic mode, because we find here the example of observable $(n-1)$-forms OF which are not algebraic observable $(n-1)$-forms (AOF). This is a case where we have an inclusion only on one side:

$$\mathfrak{P}^{n-1}_{\circ}(\mathcal{M}_{\mathsf{DW}}) \subset \mathfrak{P}^{n-1}_{\bullet}(\mathcal{M}_{\mathsf{DW}})$$

This remark will play a later role in a search for more embracing for Principles.

20.3 Ontologic groundstate and background independence

In this section we offer some reflections about the ontologic symmetry and a possible notion of multisymplectic metric $\mathfrak{U}\boxed{S}\boxed{M}\boxed{H}$. First we emphasize on background independence. This involves a focus on the ontologic symmetry: invariance of the multisymplectic form and related symplectomorphisms. The ontologic symmetry is described by a symplectomorphism, namely a vector field such that it leaves the multisymplectic form invariant: $\boldsymbol{\varkappa} : \mathcal{M} \longrightarrow \mathcal{M} : \boldsymbol{\varkappa}^\star\boldsymbol{\omega} = \boldsymbol{\omega}$. Hence the symmetry is expressed as $\mathcal{L}_{\Xi}\boldsymbol{\omega} = 0$. The group of infinitesimal symplectomorphism is the analogue of diffeomorphism invariance in General Relativity. In the case of GR the diffeomorphism invariance is fundamentally connected with the notion of *background independence*. We do not want to refer to a particular coordinate system. We have emphasized in section (11.3) that the active view of diffeomorphism invariance pictured symbolically by $\boxed{S}\boxed{M}^{\mathsf{GR}} = \varkappa^\star\boxed{S}\boxed{M}^{\mathsf{GR}}$ assigns a specific role to the metric field $\mathsf{g}_{\mu\nu}$. In GR, it is widely recognized that the observable are endowed with a *non-local* feature due to the role of the diffeomorphism group. We summarize the analogous setting for multisymplectic relativity:

General Relativity	Multisymplectic Relativity
(GR) ontologic space-time $\boxed{S}\boxed{M}^{\mathsf{GR}}$	Structure Trivial mode $\boxed{S}\boxed{M}\boxed{H}^{\mathsf{Trivial}}$
Ontological space is space-time $\mathcal{X}$	Structureal space of n-forms $\mathcal{M} = \Lambda^n T^\star\mathfrak{Z}^{\circ}$
Diffeomorphism $\varkappa : \mathcal{X} \to \mathcal{X}$	Symplectomorphisms $\boldsymbol{\varkappa} : \mathcal{M} \to \mathcal{M}$
Ontologic symmetry $\varkappa^\star\mathsf{g}_{\mu\nu} = \mathsf{g}_{\mu\nu}$	Structure symmetry $\boldsymbol{\varkappa}^\star\boldsymbol{\omega} = \boldsymbol{\omega}$

Before proceeding further, we introduce the concept of *multisymplectic metric* $\mathfrak{U}\boxed{S}\boxed{M}\boxed{H}$. The guiding idea is to grasp the structure of the general ontology in both its static and dynamic aspects. Just as in Riemannian geometry, the metric is the object that determines distances and measures them, we feel the intuitive needs in the extended multisymplectic case for a more embracing analogous arrangement of the broader relationships between the newly introduced entities $\boxed{S}$, $\boxed{M}$ and $\boxed{H}$ which articulate the corresponding ontological MG vision. Thus appears the concept of ontologic mode - the related idea to is take into account all degrees of freedom. Hence we naturally think of the trivial metric, namely the multisymplectic metric associated to the *trivial ontologic mode*. Since the Poincaré-Cartan n-form is the potential for the multisymplectic form we have the related ideas of *potential* metric vs metric. Hence consider respectively the canonical forms:

$$\begin{aligned}\theta^{\mathsf{Trivial}} &= \mathfrak{e}\beta + p_i^\mu dz^i \wedge \beta_\mu + \sum_{j=2}^{n} \sum_{\mu_1<\cdots<\mu_j} \sum_{i_1<\cdots<i_j} p_{i_1\cdots i_j}^{\mu_1\cdots\mu_j} \beta_{\mu_1\cdots\mu_j}^{i_1\cdots i_j} \\ \boldsymbol{\omega}^{\mathsf{Trivial}} &= d\mathfrak{e}\wedge\beta + dp_i^\mu \wedge dz^i \wedge \beta_\mu + \sum_{j=2}^{n} \sum_{\mu_1<\cdots<\mu_j} \sum_{i_1<\cdots<i_j} dp_{i_1\cdots i_j}^{\mu_1\cdots\mu_j} \wedge \beta_{\mu_1\cdots\mu_j}^{i_1\cdots i_j}\end{aligned}$$

We therefore introduce the related potential metric and metric as given by:

$$\mathfrak{U}\boxed{\mathrm{S}}\boxed{\mathrm{M}}\boxed{\mathrm{H}}^{\mathsf{Trivial}} = \big(\boxed{\mathrm{S}} = [\mathfrak{e}], \boxed{\mathrm{M}} = [p_i^\mu], \boxed{\mathrm{H}} = [p_{i_1\cdots i_j}^{\mu_1\cdots\mu_j}]\big)$$

$$d\mathfrak{U}\boxed{\mathrm{S}}\boxed{\mathrm{M}}\boxed{\mathrm{H}}^{\mathsf{Trivial}} = \big(d\boxed{\mathrm{S}} = [1], d\boxed{\mathrm{M}} = [1], d\boxed{\mathrm{H}} = [1]\big)$$

The coefficients of the potential metric $\mathfrak{U}\boxed{\mathrm{S}}\boxed{\mathrm{M}}\boxed{\mathrm{H}}^{\mathsf{Trivial}}$ are simply the multisymplectic manifold coordinates - generalized position and generalized momenta - while the coefficients of the metric $d\mathfrak{U}\boxed{\mathrm{S}}\boxed{\mathrm{M}}\boxed{\mathrm{H}}^{\mathsf{Trivial}}$ are equal to 1. We emphasize that in such a picture we are in the purely ontologic region of the *double duality*. We have not yet spoken of dynamics or the related constraints that appear in the setting of the variational problem - built on the generalized Legendre correspondence. To reflect this ontologic bias, we designate by $\mathfrak{U}^\circ\boxed{\mathrm{S}}\boxed{\mathrm{M}}\boxed{\mathrm{H}}^{\mathsf{Trivial}}$ and $d\mathfrak{U}^\circ\boxed{\mathrm{S}}\boxed{\mathrm{M}}\boxed{\mathrm{H}}^{\mathsf{Trivial}}$ the objects that become respectively the *ontologic potential metric* and the *ontologic metric*. When we pass to the dynamical viewpoint, namely when we taken into account the dynamical duality by means of the Legendre correspondence we introduce two analogous objects: $\mathfrak{U}^\bullet\boxed{\mathrm{S}}\boxed{\mathrm{M}}\boxed{\mathrm{H}}^{\mathsf{Trivial}}$ and $d\mathfrak{U}^\bullet\boxed{\mathrm{S}}\boxed{\mathrm{M}}\boxed{\mathrm{H}}^{\mathsf{Trivial}}$. In such a context, we can not refer to those objects unambiguously without having first imposed the Legendre correspondence. We illustrate these notions by the help of the Palatini example within the DW ontologic mode. Recall that the Palatini multisymplectic form is written (with the imposition of constraints) (316):

$$\boldsymbol{\omega}^{\mathsf{Palatini}} = d\mathfrak{e}\wedge\beta - d\big[e e_I^{[\mu} e_J^{\nu]}\big] \wedge d\omega_\mu^{IJ} \wedge \beta_\nu = d\mathfrak{e}\wedge\beta - \epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e_\rho^K de_\sigma^L \wedge d\omega_\mu^{IJ} \wedge \beta_\nu$$

Hence we write the dynamical metric:

$$d\mathfrak{U}^\bullet\boxed{\mathrm{S}}\boxed{\mathrm{M}}^{\mathsf{Palatini}} = \big(d\boxed{\mathrm{S}} = [1], d\boxed{\mathrm{M}} = [-\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e_\rho^K]\big)$$

From the more general standpoint on variational issues, the Palatini multisymplectic form is seen to come from the more fundamental object of the Poincaré-Cartan form itself. Here we are concerned with an exact multisymplectic form where the Poicaré-Cartan form is taken as a potential - $\boldsymbol{\omega}^{\mathsf{Palatini}} = d\theta^{\mathsf{Palatini}}$. So that the dynamical potential metric is given by the coefficients:

$$\mathfrak{U}^\bullet\boxed{\mathrm{S}}\boxed{\mathrm{M}}^{\mathsf{Palatini}} = \big(\boxed{\mathrm{S}} = [\mathfrak{e}], \boxed{\mathrm{M}} = [-E_I^{[\mu} e_J^{\nu]} = -1/2\epsilon_{IJKL}\epsilon^{\mu\nu\rho\sigma} e_\rho^K e_\sigma^L]\big)$$

Notice that in the previous case we have described the canonical forms with the imposition of constraints. Hence if we apply the previous idea, before describing the constraint set, we have the following case:

$$\mathfrak{U}^\circ\boxed{\mathrm{S}}\boxed{\mathrm{M}}^{\mathsf{Palatini}} = \big(\boxed{\mathrm{S}} = [\mathfrak{e}], \boxed{\mathrm{M}} = [p_{IJ}^{\omega_\mu\nu}]\big) \qquad d\mathfrak{U}^\circ\boxed{\mathrm{S}}\boxed{\mathrm{M}}^{\mathsf{Palatini}} = \big(d\boxed{\mathrm{S}} = [1], d\boxed{\mathrm{M}} = [1]\big)$$

Here we recognize a strong heuristic resonance with the viewpoint of GR. One main aspect of Einstein gravity is the Equivalence Principle. It makes the connection between the curved picture of space-time $\boxed{\mathrm{S}}\boxed{\mathrm{M}}^{\mathsf{GR}}$ and the locally flat Minkowski space $\boxed{\mathrm{S}}\boxed{\mathrm{M}}^{\mathsf{Minkowski}}$. This corresponds to the transition, from metric standpoint:

$$\mathsf{g}_{\mu\nu} = \begin{pmatrix} \mathsf{g}_{11} & \mathsf{g}_{12} & \mathsf{g}_{13} & \mathsf{g}_{14} \\ \mathsf{g}_{21} & \mathsf{g}_{22} & \mathsf{g}_{23} & \mathsf{g}_{24} \\ \mathsf{g}_{31} & \mathsf{g}_{32} & \mathsf{g}_{33} & \mathsf{g}_{34} \\ \mathsf{g}_{41} & \mathsf{g}_{22} & \mathsf{g}_{43} & \mathsf{g}_{44} \end{pmatrix} \qquad \mathsf{h}_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Before going more deeply into the notion of the multi-chrono-geometrical structure of the multi-space-time-matter-hybrid entity, namely *space*, we want to make some remarks about the universal nature of *gravity*. *Gravity* is a singular interaction since it is the only all pervading phenomenon in the Universe. Every piece of matter of the Universe is affected by gravity.[107] The crucial step due to Einstein was to incorporate space, time and matter in a single framework. We have the underlying unifying concept *energy* for whatever we label as part of this Universe. We see the intuitive motive for such claims as: mass *is* energy, gravity *is* geometry ... Energy, in physics is inextricably connected to the concept of Hamiltonian function - which for us, is one of the dual aspects of the multisymplectic form, equivalently to be thought of as the dynamical potential. Also notice that constraints denotes relations between degrees of freedom - they manifest a connection between them. All these previous remarks lead to an interesting suggestion. Perhaps we should think of gravity as the manifestation or as rooted in the interplay of constraints.[108] In this connection we focus on one of Einstein's central insights:

Gravity is Geometry

So that we are face to face with the root question: what *is* Geometry ? We have argued, from a heuristic viewpoint that we would like describe geometry through the constitutive notion of *glyph* (see the forthcoming paper [230]), which is defined by the implementation of symmetries – invariance and covariance, and of observables – (AOF) and (OF). We have a picture which contains the insights - very naturally motivated from the MG context - that glyphs are dynamical glyphs - amongst the symmetries we want not only to describe *invariance* but also *covariance*. We emphasize that this aim is nothing but a continuation of the spirit of the Erlangen program of Klein later greatly developed by Cartan - see section (1) - to perceive geometry as the study of structures on spaces, considered as given via their transformation groups. In this spirit we believe we should reconcieve Geometry in the *glyph* vision and in light of the MG setting, as the study of the organization of the collection of glyph, in particular a collection of pseudofibers. Here we refer to the ideas of Hélein and Kouneiher [116, 117, 118] developing the earlier work of Dedecker [53]. We recall the main conclusion of those work - the following remarks are directly extracted from [116, 117, 118]:

• Two points in the same pseudofiber, $\mathbf{P}_q(z) \subset \Lambda^n T_q^\star \mathfrak{Z}^\circ$ of affine subspace of $\Lambda^n T_q^\star \mathfrak{Z}^\circ$ represent the same physical - infinitesimal - state. Coordinates on $\Lambda^n T^\star \mathfrak{Z}^\circ$, - the Dedecker momentoides - are not themselves physical observables quantities.

• In the case of the DW theory the Legendre transform - as opposed to the generalized Legendre correspondence - transversally intersects all pseudofibers at one point.

• The dynamical structure encoded by a multisymplectic manifold $(\mathcal{M}, \boldsymbol{\omega})$ and a Hamiltonian function $\mathcal{H}$ is invariant by deformation along pseudofibers, [117] *"This situation is similar to gauge theory where two fields which are equivalent through a gauge transformation are supposed to correspond to the same physical state. A slight difference however lies in the fact that pseudofibers*

[107] Light and the related concept of photon may form the only exception. The photon may be the only massless boson - we exclude consideration of gluons since they are never observed as free particles. We notice however that light is central to an understanding of basic principles, as Einstein recognized when he built Special Relativity Theory.

[108] A heuristic argument is the simple fact that even the modern program of LQG can not avoid the needs for a full treatment for the Hamiltonian constraint. We observe the close connection between our choice of description of gravity and the constraint issue

are not fibers in general and can intersect singularly." So that *pseudofiber* are a *reflection* of the interplay of gauge invariance and constraints.

• There is a theorem for invariance of an observable functional along the generalized pseudofiber directions.

• Consider an ontologic vacum mode $(\mathcal{M},\boldsymbol{\omega})$ and $\mathcal{H}$ a Legendre image Hamiltonian. Then all generalized pseudofibers direction $\mathbf{L}_m^{\mathcal{H}}$ are *vertical* $\mathbf{L}_m^{\mathcal{H}} \subset \operatorname{Ker}(d\pi)$ with $\pi : \Lambda^n T^{\star}\mathfrak{Z}^{\circ} \to \mathfrak{Z}^{\circ}$.

Thanks to the previous remarks we can describe the intuitive notion of *flat* as opposed to *curved* geometry. The full expression of invariance and covariance - in the trivial mode - rests on the fact that the generalized pseudofiber directions are *vertical.* In the presence of gauge constraints the pseudofibers (which is equivalently referred as a *glyph* in the section) may intersect. In this case we no longer have a general geometric characterization of the situation. Hence the previous remarks relating the structural organization of the phenomena to the primal glyph $\mathbf{P}_q(z)$ and their possible intrinsic characterization either via subspaces tangent to pseudofibers $\mathbf{L}_m^{\mathcal{H}}$ or via the notion of Hamiltonian pataplectic invariance - lead to a natural and unusual perspective on the opposition flat - curved.

Hence generalized Gravity, is related to the manifestation or absence of a unified geometrical characterization. If we have gauge invariance or a setting where *glyphs* overlap and cross we would speak about *Curved Geometry.* If however the *glyphs* do not overlap (i.e. the generalized pseudofiber direction are parallel), we speak about a flat characterization or *Flat Geometry.* The development of non-Euclidean geometry - in the classic work of C.F. Gauss, B. Riemannn, N.I. Lobatchevski, J. Bolyai, Poincaré, Klein or Cartan - goes back to the initial speculation as to the correct treatment and nature of parallel lines which led to Euclid's fifth postulate.

We observe an analogy here to the ideas presented in this Thesis: the notion of directed line segment in Euclidean space can be thought of as a very primitive kind of glyph - after all the line gives an invariant direction in space. In Euclidean - or *flat* geometry - parallel lines never intersect. Even in such a primitive context we can recognize a connection with the notion of glyph. The lines manifest a global identical *behavior* as glyphs. In curved space the situation is different since parallel lines can intersect and this gives rise to the notion of curvature in geometry.

In GR we find the geodesics - a generalization of the notion of straight line to curved space-time. It is in the light of this analogy that we understand the previous remarks on the distinction between the potential metric and the metric $\mathfrak{U}^{\circ}\boxed{S}\boxed{M}\boxed{H}^{\text{Trivial}}$, $d\mathfrak{U}^{\circ}\boxed{S}\boxed{M}\boxed{H}^{\text{Trivial}}$, $\mathfrak{U}^{\bullet}\boxed{S}\boxed{M}\boxed{H}^{\text{Trivial}}$ and $d\mathfrak{U}^{\bullet}\boxed{S}\boxed{M}\boxed{H}^{\text{Trivial}}$. Here we see the analogue for the metric of GR in its role of setting the chrono-geometrical structure of space-time GR described by the ontology $\boxed{S}\boxed{M}^{\text{GR}}$. Now we able to treat the multi-chrono-geometrical structure of **space** both in its ontologic and dynamical meaning.

The traditional Einstein *equivalence principle*, where we are in the setting of classical space-time geometry, what we call the ontological being $\boxed{S}\boxed{M}^{\text{GR}}$, with locality understood via the notion of a point - where we have no clear distinction between *points*, *glyphs* and *space* - has a conceptual counterpart within Multisymplectic Relativity where the analogue of locality is thought of as given by the **glyph**. We make the analogy between the Einstein Equivalence Principle:

Locally: inertia and gravitation are essentially the same

and the Multisymplectic Relativity Equivalence Principle:

On glyphs, Invariance and Covariance are essentially the same

In the former case we have seen that the distinction between the two notions is frame dependent. Beyond this distinction lies the question of the meaning of the notion of observer. For the multisymplectic relativity equivalence, it is again the notion of observable that allows us to distinguish *invariance* and *covariance* on a glyph. However this notion of observer is much more rich and pervade in the following section as the idea of Eye-mirror monad - see section (20.1). As a last comment we emphasize that we work with the conceptual dualities *inertial mass* vs *gravitational mass* or *movement* vs *geometry*. Hence in the new setting of MG we recover the essence of the above equivalence since the object of dynamics is characterized by the *dynamical glyph* - the generalized pseudofiber direction $\mathbf{L}_m^{\mathcal{H}}$[109] whereas the object of geometry is characterized by the *ontologic glyph* - the pseudofibers $\mathbf{P}_q(z)$. All the previous distinctions within the *double duality* focused on either an ontologic object or an dynamical one (for example the space of n-forms or the Hamiltonian function). But in this contrast of dynamics and geometry we see both the duality and unity of ontologic and dynamical aspect *at the same time*. The meaning of this intertwined nature is the final aim of the *monad* idea - see section (20.1). Before turning to this notion, we want to make some remarks about the "seed" of dynamics $\big(\Gamma, \mathcal{H}\big)$.

20.4 Dual nature of the Hamiltonian function

Here also much more conceptual work is need to grasp the way in which the notion of the Hamiltonian function is connected with the *double duality*. Here, we offer only some intuitions. The first is that the Hamiltonian function should be thought as the dual aspect of the multisymplectic form. As emphasized in the next section (20.1) the two fundamental "seeds" that unite the ideas of symmetry and observables for physical representation are $\big(\mathcal{M}, \boldsymbol{\omega}\big)$ and $\big(\Gamma, \mathcal{H}\big)$.

This is connected with the search for observable forms OF and more generally to the generalized Hamilton equations and the issue of the covariance as distinct from invariance aspect of symmetry. An intrinsic aspect of our notion of dynamical potential - the Hamiltonian function - is the fact that it determines the dynamical space: the Hamiltonian n-curve $\Gamma \subset \mathcal{M}$. This fact is concerned with the dynamical symmetry which appears in connection with the issue of covariance and dynamical evolution. This is why we work with the so-call *Covariant Hamiltonian formalism for field theory*. The Hamiltonian n-curve is the object for the covariance expression of the equation. By the means of this symmetry we are concerned with the *form* of the dynamical equations of motion. Additionally the Hamiltonian function, in the great tradition of physics is related to the notion of energy. Therefore the dynamical potential serves a double role: the first is to structure the data of the evolution space - this idea goes back to Dirac who thought of the dynamical evolution as a transformation in the phase space. Furthermore the dynamical potential concerns the notion of potential energy. We notice that in the work of Hélein and Kouneiher [115] - or remarks found in Hélein [113] - there is a specific treatment of the variable $\mathfrak{e}$ dual to the volume form β, and we have good a representation for the stress energy tensor $\mathfrak{S}_\nu^\mu(x)$. Its Hamiltonian counterpart is the Hamiltonian tensor $\mathfrak{H}_\nu^\mu(q,p)$. More deeply, the concept of graph underlies the whole Grassman calculus - see section (3) (4), especially on the side of its connection with the MG approach. We find in [115, 113] the following definition[110] for the stress energy tensor $\mathfrak{S}_\nu^\mu(x)$ and respectively its Hamiltonian counterpart, $\mathfrak{H}_\nu^\mu(q,p)$:

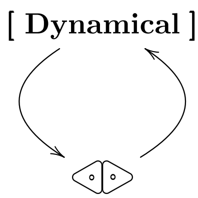
$$\begin{aligned} \mathfrak{S}_\nu^\mu(x) &= \delta_\nu^\mu L(x,\sigma(x),d\sigma(x)) - \frac{\partial L}{\partial v_\mu^i}(x,\sigma(x),d\sigma(x))\frac{\partial \sigma^i}{\partial x^\nu}(x) \\ \mathfrak{H}_\nu^\mu(q,p) &= \delta_\nu^\mu \mathcal{H}(q,p) - \frac{\partial \langle p,z\rangle}{\partial z_\nu^\mu}\Big|_{z=\mathcal{Z}(q,p)} \end{aligned} \tag{387}$$

If $(q,v) \leftrightarrow (q,p)$ we have $\mathfrak{S}_\nu^\mu(x) = -\mathfrak{H}_\nu^\mu(q,p)$. The components of the stress energy-tensor are related,

[109] covariance issue: how to preserve the description of movement.
[110] for a variational problem on map $\sigma : \mathcal{X} \to \mathfrak{Z}$ (with $L : L(x,\sigma(x),d\sigma(x))$)

by means of the Noether theorem, to space-time symmetries - space-time translation, diffeomorphism invariance depending on the context. In the same manner as the emergence of the notion of generalized multisymplectic curvature, viewed as the key to organization of the ontologic space, opened the prospect of new kinds of cosmological models, we believe that a further source for the elaboration of cosmological models can also be found by closer attention to the notion of dynamical potential - the object $\mathfrak{H}^{\mu}_{\nu}(q,p)$ is also holds the promise of a richer landscape of cosmological models within this generalized ontological perspective centered on the entities $\boxed{\text{S}}$, [M] and [H].

This duality for the dynamical "seed" re-appears in the determination of observable forms via the dynamical duality. In the search for dynamical observables, the spirit of the Relativity Principle is expressed via the relation $\{\mathcal{H},\boldsymbol{\varphi}\}d\boldsymbol{\varrho}(X) = \{\mathcal{H},\boldsymbol{\varrho}\}d\boldsymbol{\varphi}(X)$ where no specific volume form is singled out but we can only compare two OF, $\boldsymbol{\varphi},\boldsymbol{\varrho} \in \mathfrak{P}^{n-1}_{\bullet}(\mathcal{M})$ via the relation: $\{\mathcal{H},\boldsymbol{\varphi}\}d\boldsymbol{\varrho}(X) = \{\mathcal{H},\boldsymbol{\varrho}\}d\boldsymbol{\varphi}(X)$. The covariant aspect of dynamical symmetry seen in the Relativity Principle is concentrated in the form of the previous equation. Here we feel the key insight of Relativity lies. We compare observable forms only to one another they have no absolute meaning relative in a fixed background. Notice also that the notion of copolar form and copolarization is grounded in this insight. We summarize:



$(\Gamma,\mathcal{H}) \longrightarrow \forall m \in \Gamma, \exists X \in \Lambda^n T_m\Gamma \quad X \lrcorner\, \boldsymbol{\omega}_m = (-1)^n d\mathcal{H}_m$

$(\Gamma,\mathcal{H}) \longrightarrow \forall m \in \mathcal{M}, X(m) \in [X]^{\mathcal{H}}_m\ ,\ \langle X(m), d\boldsymbol{\varphi}_m\rangle$

$(\Gamma,\mathcal{H}) \longrightarrow X(m) \in [X]^{\mathcal{H}}_m$, Hamiltonian n-vector fields

PERSPECTIVES

We summarize four main ideas and directions for future works. We leave apart the previous discussion to focus on the mathematical and physical aspects. I am now working on the mathematical calculations to describe and support these points [229, 230]. All them lay on an enlarged vision for space, time and matter. This can be summarized in the context of Hamiltonian covariant field theory with the concept of the *ontologic mode* $(\mathcal{M},\boldsymbol{\omega}) = \boxed{\text{S}}\boxed{\text{M}}\boxed{\text{H}}$. In such a context - whatever we refer to $\boxed{\text{S}}\boxed{\text{M}}\boxed{\text{H}}$ or to $\boxed{\text{S}}\boxed{\text{M}}\boxed{\text{H}}$ - we need to further add to the *space-time* entity $\boxed{\text{S}}$ and the *matter* entity $\boxed{\text{M}}$ a third one: the *hybrid* entity $\boxed{\text{H}}$. We enumerate four points:

- The first is the *topological hypothesis*. We discuss it with the help of the topological term in gravity described by the general action: (229) which involved the different part $\mathcal{L}_{\mathsf{Holst}}[e,\omega]$, $\mathcal{L}_{\mathsf{Euler}}[\omega]$, $\mathcal{L}_{\mathsf{Pontrjagin}}[\omega]$, $\mathcal{L}_{\mathsf{Nieh\text{-}Yan}}[e,\omega]$ and $\mathcal{L}_{\mathsf{Cosmological}}[e]$. Notice that the topological hypothesis apply for any theory, - for example the addition of a topological term of type $\int_{\mathcal{X}} F\wedge F$ to the Maxwell vacuum Lagrangian functional $\mathcal{L}^{\circ}_{\mathsf{Maxwell}}[A] = \frac{1}{2}\int_{\mathcal{X}} F\wedge\star F$ The *topological hypothesis* is described by the following idea. Rather than adding by hand the different topological terms - Holst, Euler, Pontrjagin, Nieh-Yan and the Cosmological constant terms - in the original action and *then* proceed to variational study for equation of movement, we would like to directly obtain the *same* equations of movement from higher order Lepage-Dedecker geometrization. It shall be performed making use of Poincaé-Cartan form of various following type:

$$\theta^{[1]^e|[1]^\omega||[4]}_{(q,p)}\ ,\ \theta^{[1]^e|[2]^\omega||[4]}_{(q,p)}\ ,\ \theta^{[2]^e|[1]^\omega||[4]}_{(q,p)}\ ,\ \theta^{[2]^e|[2]^\omega||[4]}_{(q,p)}$$

The case for cosmological constant is heuristically taken into account via:

$$\theta^{[4]^e|[4]}_{(q,p)} = \boldsymbol{\Lambda}^{\mathsf{Cosmological}} e\wedge e\wedge e\wedge e = \boldsymbol{\Lambda}^{\mu\nu\rho\sigma}_{IJKL} de^I_\mu \wedge de^J_\nu \wedge de^K_\rho \wedge de^L_\sigma$$

We write the related Lepage-Dedecker momentum with the notation $\mathbf{\Lambda}^{\mathsf{Cosmological}}$ in relation to question of the cosmological constant. The application of this ideas permit to focus on *one and only one* Lagrangian functional for gravity.

In this direction the case of Dirac field - with the conceptual inclusion of matter fields - is much more delicate. Dirac field makes connection with the issue of *torsion*. However in such a setting, the commonly found statement that the Dirac field generates torsion is naturally inverted. We do not want to find the equations of movement by adding terms by end in the Lagrangian but from consideration of various ontologic modes. Hence we need to understand, after all the subsequent principle underlying the organization of *four* main objects: $e, \omega, \psi, \overline{\psi}$. An indication is the connection between the four degrees of freedom of the axial current $J^I = \overline{\psi}\gamma^I\gamma^5\psi$ and four dimensional tetrad fields e in an embracing view. We refer to the work of MacDowell and Mansouri gravity - and the underlying use of Cartan connection - to stress that we are able to describe gravity with higher action of the type $\mathcal{L}_{\mathsf{MM}}(\mathcal{A}) \propto \int \mathrm{tr}\langle F \wedge \star F\rangle\beta$.

- The second point is the application of the method developed by Hélein and Kouneiher [115, 116, 117, 118] for the natural setting of covariant canonical variables and Poisson bracket to the Einstein-Cartan theory. In particular the Nieh-Yan term gives an interest: we have non vanishing momenta for the tetrad field $p^{e_\mu\nu}$. The idea is to exhibit *good* copolarization in this case.

- The third direction concerns the fundamental issue of *Principles* for physics. We have felt, upon the diagram insight in the search of observable, the idea of *eye-mirror monad*. Maybe would stand a Principle that make central this dual nature for observables and physical representation in what we would call the eye/mirror monad (mirror for the symmetry and eye for the observable) The aim beyond is surely to connect the missing underlying ontology of QM where the notion of observable, one century after the rise of the theory is still involved into paradox and debates. We now give some indication for connection with QM area. We would like to give a quantum picture for MG - some attempt is made in [137, 111]. In pursuing the search for QG we encounter two related issues: *non linearity* and *quantization*. The interplay of these two notions has been revealed in particular by Goldschmidt and Sternberg [100], Kijowski [141], Hélein and Kouneiher, [117, 114]. Their works show that the two questions may be related. Let us cite Hélein [114] *"the quantization procedure works when the classical equation is linear but fails as soon as the problem become nonlinear (interacting fields in the language of physicists)"*. Since some part of the obstruction for quantization can be understood as the fact that we don't get enough dynamical observables when the theory is *non linear*, the presence of gauge symmetry can helps us to construct many more dynamical observables. Actually, beside the gauge sector, one also overcomes these difficulties and finds observable functionals with the tools of perturbations theory. This is the idea developed by Hélein in [114], and applied by Harrivel in the study of interacting Klein-Gordon theory [104]. This is why we shall conceptually think along two directions:

Hence the underlying aim of the thesis shall be picture in a single statement: we shall definitively try to make use of multisymplectic technics for any prolegomena to a future QG theory. This is first due to the LD vision and the treatment of constraint where we can always find a Lepage theory - where the primary Dirac constraint set can be set empty. The idea developed by Hélein and Kouneiher - who have pushed further the classification of observable - definitively appears in relation with deeper foundations and principles for physical representation.

NOTATIONS

Structureal-Dynamicalal consideration

- $[\,\triangleright\,]$ Ontologic space
- $[\,\triangleleft\,]$ Ontologic representation
- $[\,\dot{\triangleright}\,]$ Dynamical space
- $[\,\dot{\triangleleft}\,]$ Dynamical representation
- $\mathfrak{sp}_\circ(\mathcal{M})\leftrightarrows[\,\triangleright\,]$ ontologic vector fields: symplectomorphism
- $[X]^{\mathcal{H}}_m\leftrightarrows[\,\dot{\triangleright}\,]$ dynamical vector fields: Hamiltonian vector fields.
- $\mathfrak{P}^{n-1}_\circ(\mathcal{M})\leftrightarrows[\,\triangleleft\,]$ — $(n-1)$-forms that encodes the *symmetry* standpoint
- $\mathfrak{P}^{n-1}_\bullet(\mathcal{M})\leftrightarrows[\,\dot{\triangleleft}\,]$ — $(n-1)$-forms that encodes the *dynamical* standpoint

Notation for specific theories

- AQFT Algebraic Quantum Field theory
- CPS Covariant Phase Space,
- DW De Donder-Weyl theory,
- GR General Relativity,
- MG Multisymplectic Geometry,
- LD Lepage-Dedecker theory.
- LQG Loop Quantum Gravity,
- QFT Quantum Field Theory,
- QM Quantum Mechanics

Structural modes

- $\boxed{\text{S}}$ **space-time** entity,
- $\boxed{\text{M}}$ **matter** entity,
- $\boxed{\text{H}}$ **hybrid** entity
- $\boxed{\text{S}}\boxed{\text{M}}^{\mathsf{GR}}$: mathematical model of general relativity space-time;
- $\boxed{\text{S}}\boxed{\text{M}}^{\mathsf{Gauge}}$ mathematical model for gauge theory,
- $\boxed{\text{S}}\boxed{\text{M}}\boxed{\text{H}}$ Trivial ontologic mode,
- $\boxed{\text{S}}\boxed{\text{M}}\boxed{\text{H}}_{\mathsf{DW}}$ De Donder-Weyl ontologic mode.

Fiber bundle theory

- $\mathcal{P}(\mathcal{M},G)$ Principal fiber bundle over $\mathcal{M}$ with gauge group G,
- $F(\mathcal{X})$ Linear frame bundle,
- $F_{SO(1,3)}(\mathcal{X})$. Lorentz orthogonal bundle.

General Relativity

- $\mathfrak{D}_{\mathsf{iff}}(\mathcal{X})$ be the diffeomorphism group
- $\mathfrak{M}_{\mathsf{etrics}}(\mathcal{X})$ the space of all metrics over $\mathcal{X}$
- $\mathfrak{G}_{\mathsf{eom}}(\mathcal{X})=\mathfrak{M}_{\mathsf{etrics}}(\mathcal{X})/\mathfrak{D}_{\mathsf{iff}}(\mathcal{X})$ the space of geometries of the manifold

Differential geometry basics

- $T\mathcal{X}=\{(m,\xi)/m\in\mathcal{X},\xi\in T_m\mathcal{X}\}=\bigcup_{m\in\mathcal{X}}T_m\mathcal{X}$ is the tangent bundle of $\mathcal{X}$;
- $T^\star\mathcal{X}=\{(m,\alpha)/m\in\mathcal{X},\alpha\in T^\star_m\mathcal{X}\}=\bigcup_{m\in\mathcal{X}}T^\star_m\mathcal{X}$ is the cotangent bundle of $\mathcal{X}$;
- $\Lambda^nT^\star\mathcal{X}=\{(m,\alpha)/m\in\mathcal{X},\alpha\in\Lambda^nT_m\mathcal{X}\}=\bigcup_{m\in\mathcal{X}}\Lambda^nT^\star_m\mathcal{X}$ is the n-forms bundle of $\mathcal{X}$;
- $\mathfrak{X}(\mathcal{X})=\Gamma(\mathcal{X},T\mathcal{X})$ space of vector fields on a manifold $\mathcal{X}$;
- $\mathfrak{X}^n(\mathcal{X})=\Gamma(\mathcal{X},\Lambda^nT\mathcal{X})$ space of n-vector field on $\mathcal{X}$;
- $\Omega^n(\mathcal{X})=\Gamma(\mathcal{X},\Lambda^nT^\star\mathcal{X})$ the space of differential n-form on $\mathcal{X}$;
- $\Omega^p(\mathcal{X},\mathfrak{g})=\Omega^p(\mathcal{X})\otimes\mathfrak{g}$: the set of $\mathfrak{g}$-valued p-form on $\mathcal{X}$;
- $\Gamma(\mathcal{X},\mathbf{T}\mathcal{X}^{\otimes})$ The space of tensor field on $\mathcal{X}$

$\mathrm{vol}_{\mathcal{X}}(\mathsf{g}) = \beta^{\mathsf{g}}$ Riemannian volume form,
$\mathrm{vol}_{\mathcal{X}}(\mathsf{h}) = \beta$ Minkowski volume form
$\overset{\circ}{\star}$ Internal (usually Minkowski) Hodge star operator,
$\star$ external (space-time) Hodge star operator
$\Gamma^{\rho}_{\mu\nu}$ Connection coefficient in arbitrary frame (holonomic or non holonomic)
$\Gamma^{\rho}_{\mu\nu}$ Christoffel coefficient;

Connections and derivatives
D: arbitrary connection on a vector bundle;
∇ Levi Civita connection on the tangent manifold
$d^{\mathbf{D}}$ exterior covariant derivative;
d_{ω} gauge covariant derivative relative to the connection ω;
$d_{\mathbf{V}}$ Vertical exterior derivative;
$d_{\mathbf{H}}$ Horizontal exterior derivative;
d_{μ} Formal derivative (or total derivative);
d exterior derivative along a solution of the Hamilton equations

First order gauge gravity - Loop Quantum Gravity
e^I_{μ} Tetrad field:
e^{μ}_I Co-tetrad field:
ω^{IJ}_{μ} Lorentz (spin) connection;
E^{μ}_I densitized tetrad
$h_{\Gamma}[A]$ Holonomy of the connection 1-form A.
$F_I[\mathcal{S}]$ Flux variable

Multisymplectic geometry
θ Poincaré-Cartan canonical 1-form (symplectic case);
ω Symplectic canonical 2-form;
θ Poincaré-Cartan canonical n-form;
$\boldsymbol{\omega}$ Multisymplectic canonical $n+1$-form;
$\theta_{\mathcal{L}}$ Cartan canonical n-form (jet bundle formalism);
$\boldsymbol{\omega}_{\mathcal{L}}$ Multisymplectic canonical (jet bundle formalism) $n+1$-form
$\mathfrak{P}^{n-1}_{\circ}(\mathcal{M})$ the set of algebraic observable $(n-1)$-forms;
$\mathfrak{P}^{n-1}_{\bullet}(\mathcal{M})$ the set of observable $(n-1)$-forms.;
$\mathcal{O}^{\mathcal{H}}$ the set of dynamical observables;
$\mathcal{O}_{\text{Dirac}}$ describe a Dirac observable
$\mathfrak{sp}_{\circ}(\mathcal{M}) = \mathfrak{sp}_P(\mathcal{M}) \ltimes \mathfrak{sp}_Q(\mathcal{M})$ the set of algebraic symplectomorphism;
$\mathfrak{sp}_P(\mathcal{M})$ set of symplectomorphism related to $(n-1)$-algebraic generalized momenta;
$\mathfrak{sp}_Q(\mathcal{M})$ set of symplectomorphism related to $(n-1)$-algebraic generalized position
$\mathbf{D}^n_m\mathfrak{Z} := \left\{X_1 \wedge \cdots \wedge X_n \in \Lambda^n T_m\mathfrak{Z} \;/\; X_1, \cdots, X_n \in T_m\mathfrak{Z}\right\}$ The set of decomposable n-vector
$[X]^{\mathcal{H}}_m := \left\{X \in D^n_m\mathfrak{Z} \;/\; X \lrcorner \Omega = (-1)^n d\mathcal{H}_m\right\}$, class of Hamiltonian vector fields.
$\mathbf{P}_q(z)$ Enlarged pseudofiber
$\mathbf{P}^h_q(z)$ Pseudofiber
$\mathbf{L}^{\mathcal{H}}_m$ Generalized pseudofiber direction
$\mathfrak{U}^{\circ}\boxed{\text{S}}\boxed{\text{M}}\boxed{\text{H}}^{\text{Trivial}}$, Potential ontologic multisymplectic metric
$d\mathfrak{U}^{\circ}\boxed{\text{S}}\boxed{\text{M}}\boxed{\text{H}}^{\text{Trivial}}$, Ontologic multisymplectic metric
$\mathfrak{U}^{\bullet}\boxed{\text{S}}\boxed{\text{M}}\boxed{\text{H}}^{\text{Trivial}}$ Potential dynamical multisymplectic metric
$d\mathfrak{U}^{\bullet}\boxed{\text{S}}\boxed{\text{M}}\boxed{\text{H}}^{\text{Trivial}}$. Dynamical multisymplectic metric
$\mathfrak{S}^{\mu}_{\nu}(x)$ Stress-energy tensor
$\mathfrak{H}^{\mu}_{\nu}(q,p)$ Hamiltonian tensor

APPENDIX

A Differential forms and exterior differential calculus

We denote $C^\infty(\mathcal{M})$ the commutative algebra of functions with real values on $\mathcal{M}$. Let $\Lambda^n\mathcal{V}^\star$ be the vector space of n-forms on the real vector space $\mathcal{V}$. (Then a n-form α on $\mathcal{V}$ is a multilinear and alternated application $\alpha : \mathcal{V} \times ... \times \mathcal{V} \to \mathbb{R}$). The exterior product of two forms $\alpha \in \Lambda^p\mathcal{V}^\star$ and $\beta \in \Lambda^q\mathcal{V}^\star$ for $p, q \in \mathbb{N}$ is defined by $\Lambda^p\mathcal{V}^\star \times \Lambda^q\mathcal{V}^\star \to \Lambda^{p+q}\mathcal{V}^\star : (\alpha, \beta) \to \alpha \wedge \beta$ with

$$\alpha \wedge \beta(v_1, ..., v_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in \mathcal{S}_{p+q}} (-1)^{\boldsymbol{\sigma}} \alpha(v_{\sigma_1}, ..., v_{\sigma_p})\beta(v_{\sigma_{p+1}}, ..., v_{\sigma_{p+q}}) \tag{388}$$

We construct the associative exterior algebra $(\Lambda^\star\mathcal{V}^\star = \bigoplus_{0=p}^n \Lambda^p, +, \wedge)$. This is a graded commutative algebra. This property of graded commutativity means that for any $\alpha \in \Lambda^p\mathcal{V}^\star, \beta \in \Lambda^q\mathcal{V}^\star$ then $\alpha \wedge \beta = (-1)^{pq}\beta \wedge \alpha$. We denote respectively $\Omega^n(U)$ and $\Omega^n(\mathcal{M})$ the set of differential n-forms on an open set and on the differential manifold $\mathcal{M}$ *i.e* a regular application $\alpha : U \to \Lambda^n\mathbb{R}^\star$, hence we consider the data at each point $x \in \mathcal{M}$ of a n-form $\alpha_m \in \Lambda^n T^\star\mathcal{M}$. For a coordinate system $(x^1, ..., x^n) : \mathcal{U} \to \mathbb{R}^n$ we decompose α_x on the basis $(dx_m^{i_1} \wedge ... \wedge dx_m^{i_n})_{1 \le i_1 < ... < i_p \le n}$:

$$\alpha_x = \sum_{1 \le i_1 < ... < i_p \le n} \alpha(x)_{i_1,...i_p} dx^{i_1} \wedge ... \wedge dx^{i_n} = \sum_I \alpha(x)_I dx^I \tag{389}$$

We also exhibit the vectorial space generated by graded property. On the manifold $\mathcal{M}$, we denote $\mathcal{X}(\mathcal{M})$ the vector space of tangent vector fields on $\mathcal{M}$. We denote $\mathfrak{X}^n(\mathcal{M})$ the space of multivectors of degree n. It is described as the set of section of the vector bundle $\bigwedge^p T\mathcal{M} \to \mathcal{M}$. In the natural local coordinate system $(x^1, ..., x^n)$, we obtain a natural basis for $T\mathcal{M}$: $(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^p})$. A multivector field $\boldsymbol{\zeta}_x$ at $x \in \mathcal{M}$ - hence $\boldsymbol{\zeta}_x \in \mathfrak{X}^n(\mathcal{M})$ - is written on the basis $(\partial_{i_1} \wedge ... \wedge \partial_{i_n})_{1 \le i_1 < ... < i_p \le n}$. We have:

$$\boldsymbol{\zeta}_x = \sum_{1 \le i_1 < ... < i_p \le n} \boldsymbol{\zeta}(x) i_1, ... i_p \partial_{i_1} \wedge ... \wedge \partial_{i_n} = \sum_I \boldsymbol{\zeta}(x)^I \partial_I \tag{390}$$

Finally we denote the vector space generated by graded process, respectively for differential forms and multivectors fields on a manifold $\mathcal{M}$ with the notations

$$\Omega^\star(\mathcal{M}) = \bigoplus_{0=i}^n = C^\infty(\mathcal{M}) \oplus \Omega^1(\mathcal{M}) \oplus ... \oplus \Omega^n(\mathcal{M}) \quad \text{and} \quad \mathfrak{X}^\star(\mathcal{M}) = \bigoplus_{0=i}^n = C^\infty(\mathcal{M}) \oplus \mathcal{X}^1(\mathcal{M}) \oplus ... \oplus \mathcal{X}^n(\mathcal{M})$$

The exterior differential on the manifold $\mathcal{M}$ is defined as the application on $\Omega^\star(\mathcal{M})$ $d : \Omega^n(\mathcal{M}) \to \Omega^{n+1}(\mathcal{M})$ If we consider the n-form $\alpha \in \Omega^n(\mathcal{M})$

$$\alpha = \sum_{1 \le i_1 < ... < i_p \le n} \alpha_{i_1,...i_p} dx^{i_1} \wedge ... \wedge dx^{i_n} = \sum_I \alpha_I dx^I \tag{391}$$

we have the exterior differential:

$$d\alpha = \sum_{1 \le i_1 < ... < i_p \le n} d\alpha_{i_1,...i_p} dx^{i_1} \wedge ... \wedge dx^{i_n} = \sum_I d\alpha_I dx^I \tag{392}$$

Graded Leibniz property The exterior differential satisfies the graded Leibniz property $\forall \alpha \in \Omega^p(\mathcal{M}), \forall \beta \in \Omega^q(\mathcal{M})$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \tag{393}$$

⌜ We consider $\alpha = \sum_{1 \leq i_1 < \ldots < i_p \leq n} \alpha_{i_1,\ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_n} = \sum_I \alpha_I dx^I$ and $\beta = \sum_{1 \leq i_1 < \ldots < i_p \leq n} \beta_{i_1,\ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_n} = \sum_I \beta_I dx^I$ We write $\alpha \wedge \beta = \sum_I \sum_J \alpha_I \beta_J dx^I \wedge dx^J$ then

$$\begin{aligned} d(\alpha \wedge \beta) &= \sum_I \sum_J d(\alpha_I \beta_J) \wedge dx^I \wedge dx^J = \sum_I \sum_J (d(\alpha_I)\beta_J \wedge dx^I \wedge dx^J) + (\alpha_I d(\beta_J) \wedge dx^I \wedge dx^J) \\ &= (\sum_I d\alpha_I \wedge dx^I) \wedge (\sum_J \beta_J dx^J) + (\sum_I \alpha_I dx^I) \wedge (\sum_J (-1)^p d\beta_J \wedge dx^J) \end{aligned}$$

Interior product and differential forms The interior product. Let $\zeta \in \mathcal{X}(\mathcal{M})$ and let $\alpha \in \Omega^n(\mathcal{M})$ a n-form $\forall x \in \mathcal{M}$ we consider the multilinear and alternated $(n-1)$-form

$$\begin{aligned} &(\zeta \lrcorner \alpha)_x (T_x\mathcal{M})^{n-1} \to \mathbb{R} \\ &\qquad (v_1, \ldots, v_{p-1}) \to \alpha_x(\zeta(x), v_1, \ldots, v_{p-1})) \end{aligned} \tag{394}$$

This application define for any n-form $\alpha \in \Omega^n(\mathcal{M})$ and for any vector field $\zeta \in \mathcal{X}(\mathcal{M})$ the interior product of α by ζ, denoted $\zeta \lrcorner \alpha$ such that $\zeta \lrcorner \alpha \in \Omega^{n-1}(\mathcal{M})$. We have an analogous property to the graded Leibniz rule. We have, $\forall \zeta \in \mathcal{X}(\mathcal{M}), \alpha \in \Omega^p(\mathcal{M}), \beta \in \Omega^q(\mathcal{M})$:

$$\zeta \lrcorner (\alpha \wedge \beta) = (\zeta \lrcorner \alpha) \wedge \beta + (-1)^p \alpha \wedge (\zeta \lrcorner \beta) \tag{395}$$

B Gauge theory, connections, derivatives and all that

B.1 Gauge fields: Fiber bundle framework

In this section we do not give a precise definition and content for most of the introduced objects that appear within the fiber bundle theory and the *connection* on fiber bundle. We refer to classical textbooks for a full description, for example Y. Choquet-Bruhat and C. DeWitt-Morette [45], S. Kobayashi and K. Nomizu [148, 149] or M. Nakahara [176]. This section, also, as been largely inspired from the text about basics of differential geometry and Lie group theory, in french, by R. Coquereaux [46] - see also [47] -, A. Fabretti [73] and T. Masson [165]. The idea is just to emphasize the key points for the modern description of *connection* on principal and associated fiber bundles. The point of interest is therefore twofold. On one hand it allows to grasp the picture of *connection forms* and *local gauge potential*. These are the basis of the theory of connection on fiber bundles that falls in the realm of *Ehresmann* connections. On the other hand, we emphasize the role of associated bundles and clarify the setting for further investigations which concerns the good use of covariant derivative and gauge covariant derivative encounter in various contexts.

We consider a fiber bundle as a fibered space $(\mathcal{P}, \mathcal{M}, \pi)$ where $\mathcal{P}$ is the total space of the bundle, $\mathcal{M}$ is the base space and π is the (surjective) projection. A fiber bundle is called a G-principal fiber bundle $(\mathcal{P}, \mathcal{M}, \pi)$ if it carries the following properties: $(\mathcal{P}, \mathcal{M}, \pi)$ is locally a trivial fiber bundle[111], we allow a Lie group G to act transitively on each fiber of $\mathcal{P}$ and finally the fibers are homeomorphic to G, the structure group. The fiber over $x \in \mathcal{M}$ is by definition $\pi^{-1}(x) = \mathcal{P}_x$, in the case of a G-bundle, the typical fiber diffeomorphic to the group G. We emphasize that the choice of a local section (*a trivialization*) $x \in \mathcal{U} \subset \mathcal{M} \to \sigma(x) \in \mathcal{P}$ allow us to identify the structural group G with the fiber $\mathcal{P}_x$ over x. Then we will speak about a principal[112] G-bundle and we denote it as $\mathcal{P}(\mathcal{M}, \mathcal{G})$.

[111] A locally trivial G-bundle is given by a fiber bundle $(\mathcal{P}, \mathcal{M}, \pi)$ such that for any open set $\mathcal{U} \subset \mathcal{M}$, $\pi^{-1}(\mathcal{U})$ is diffeomorphic to $\mathcal{U} \times G$

[112] Then the set of local trivialization $\varphi_i : \mathcal{U}_i \times G \longrightarrow \pi^{-1}(\mathcal{U}_i)$ is given by $\varphi_i^{-1}(p) = (x, \mathsf{g}_i)$ where $p \in \pi^{-1}(\mathcal{U}_i)$ and $x = \pi(p)$

On such a principal G-bundle, $\mathcal{P}(\mathcal{M},G)$ we associate the canonical *right* action $R:\mathcal{P}\times G\longrightarrow\mathcal{P}$ then, $\forall g\in G, R_g=R(\cdot,g):\mathcal{P}\longrightarrow\mathcal{P}$ for $p=(x,g)\in\mathcal{P}$, we have $R_g(p)=p\cdot g$

First, we introduce the notion of *Ehresmann connection* [67] in the context of principal fiber bundle with G as gauge group. It exists several equivalent definition of a connection form $\boldsymbol{\omega}$. The first one picture the notion of connection as an horizontal equivariant distribution: it lays on the grounds of a separation of the tangent space $T\mathcal{P}$ into vertical $\mathbf{V}\mathcal{P}$ and horizontal $\mathbf{H}\mathcal{P}$ subspaces. Then, equivalently, we picture the notion of connection $\boldsymbol{\omega}$ on the principal fiber bundle as a Lie algebra valued 1-form $\boldsymbol{\omega}\in\Omega^1(\mathcal{P},\mathfrak{g})$ with equivariance property.

(Ehresmann) principal G-connection. Ehresmann [67] defines an infinitesimal connection on the bundle $\mathcal{P}$ as a smooth distribution of horizontal subspaces along with an equivariance property. It is based on the grounds of a decomposition of the tangent space $T\mathcal{P}=\mathbf{H}\mathcal{P}\oplus\mathbf{V}\mathcal{P}$. Thus, a connection in that sens is a way to choose a vector subspace $\mathbf{H}_p\mathcal{P}\subset T_p\mathcal{P}$ such that for any $p\in\mathcal{P}$, one has $T_p\mathcal{P}=\mathbf{H}_p\mathcal{P}\oplus\mathbf{V}_p\mathcal{P}$ and such that the mapping $p\mapsto\mathbf{H}_p\mathcal{P}$ is smooth. One observes that the vertical part is defined in an intrinsic manner. We define the vertical bundle $\mathbf{V}\mathcal{P}$ as the following $\mathbf{V}\mathcal{P}=\mathrm{Ker}(\pi_\star)$ We recall that the linear tangent application $\pi_\star=d\pi$ is express $\forall p\in\mathcal{P}\ /\ (d\pi)_p:T_p\mathcal{P}\longrightarrow T_{\pi(p)}\mathcal{M}$. Therefore $\mathbf{V}\mathcal{P}$ is seen as the kernel[113] of the differential map $d\pi$. A principal Ehresmann connection is a way to choose a complementary sub-bundle $\mathbf{H}\mathcal{P}$ to $\mathbf{V}\mathcal{P}$ in $T\mathcal{P}$. Alternatively, we can define the vertical vector space $\mathbf{V}_p\mathcal{P}$ at p as

$$\mathbf{V}_p\mathcal{P}=\{\zeta\in T_p\mathcal{P}\ /\ \pi_\star(\zeta)=d\pi(\zeta)=0\} \tag{396}$$

We denote $\zeta^{\mathbf{V}}\in\mathbf{V}_p\mathcal{P}$ a vertical vector field at $p\in\mathcal{P}$. We obtain a unic decomposition of any vector field ζ on $\mathcal{P}$: $\zeta=\zeta^{\mathbf{V}}+\zeta^{\mathbf{H}}$ with $\zeta^{\mathbf{H}}\in\mathbf{H}_p\mathcal{P}$. Specific additional property is necessary since we only descibe an arbitrary horizontal distribution up to now. Hence a connection on a principal G-bundle is a smooth distribution of horizontal subspaces $p\mapsto\mathbf{H}_p\mathcal{P}$ with the additional equivariant feature. We denote the *right action*[114] of a group element $g\in G$ on the fiber bundle as $R_g:p\mapsto R_g(p)=p\cdot g$. We also define the associated tangent application $dR_g=(R_g)_\star$. Then the equivariance property means that we want to find equivalence of picking a choice $\mathbf{H}_{p\cdot g}(\mathcal{P})$ or either using the action of G on the fiber namely: $[\mathbf{H}_p\mathcal{P}]\cdot g$

$$\forall p\in\mathcal{P}\ ,\ \forall g\in G\ \ \mathbf{H}_{R_g(p)}\mathcal{P}=\mathbf{H}_{p\cdot g}\mathcal{P}=(dR_g)_p(\mathbf{H}_p\mathcal{P})=((R_g)_\star)_p(\mathbf{H}_p\mathcal{P})\big] \tag{397}$$

Moreover, the canonical right action R_g allows us to describe the vertical subspace $\mathbf{V}\mathcal{P}$ with another way. Hence we picture it as a space generated by *fundamental vector fields* (acting on the right). Since the Lie group G acts on the right on $\mathcal{P}$, we define fundamental vector field $\boldsymbol{\zeta}_\xi$ on $\mathcal{P}$ associated to the any element $\xi\in\mathfrak{g}$. Therefore, we see the vertical space also as the image of the Lie algebra $\mathfrak{g}$ of G under the action R_g. Then, if we fix $p\in\mathcal{P}$ then the action gives a map $G\longrightarrow\mathcal{P}$ whose pushforward at the identity $(R_g)_\star$ define a map $\varsigma_p:\mathfrak{g}\longrightarrow T_p\mathcal{P}$. For an element, let says $\xi\in\mathfrak{g}$, one obtains $\varsigma_p(\xi)\in\mathbf{V}_p\mathcal{P}$ (398), we alternatively denoted then $\varsigma_p(\xi)=\boldsymbol{\zeta}_{\xi_{|p}}$ such a fundamental field generated by ξ.

$$\varsigma_p(\xi)=\frac{d}{dt}\big[p\exp(t\xi)\big]\Big|_{t=0} \tag{398}$$

[113] We can canonically identify the quotient bundle $T\mathcal{P}/\mathbf{V}\mathcal{P}$ with the pullback $\pi^\star T\mathcal{M}$

[114] Generally, a right action R (respectively we may define in an analogous manner the left action L), of a group G on $\mathcal{P}$ is an application: $R:\mathcal{P}\times G\longrightarrow\mathcal{P}:(p,g)\mapsto R(p,g)$ such that $\forall p\in\mathcal{P}$ we have $R(p,e)=p$ and $\forall p\in\mathcal{P},g,h\in G$ such that $R(R(p,g),h)=R(p,gh)$. Then when one considers action of the Lie group G on itself one rather denote the canonical right action as R_g whereas the left action is denoted by L_g

From this standpoint, we shall see the vertical space at $p \in \mathcal{P}$ as the one generated by the vectors fields $(\boldsymbol{\zeta}_\xi)_p$ so that $\mathbf{V}_p\mathcal{P} = \big\{(\boldsymbol{\zeta}_\xi)_{|p}/\xi \in \mathfrak{g}\big\}$, (399)

$$\mathbf{V}_p\mathcal{P} = \Big\{\frac{d}{dt}\big[p\exp(t\xi)\big]\Big|_{t=0} \quad / \quad \xi \in \mathfrak{g}\Big\} \tag{399}$$

Therefore, a connection is defined as the horizontal equivariant subspace, defined via the following

$$\text{(i)} \quad T_p\mathcal{P} = \mathbf{H}_p\mathcal{P} \oplus \mathbf{V}_p\mathcal{P} \qquad \text{and} \qquad \zeta = \zeta^{\mathbf{V}} + \zeta^{\mathbf{H}} \quad \text{with} \quad \zeta^{\mathbf{H}} \in \mathbf{H}_p\mathcal{P} \quad \text{with} \quad \zeta^{\mathbf{H}} \in \mathbf{V}_p\mathcal{P}$$

$$\text{(ii)} \quad \forall p \in \mathcal{P} \quad \text{and} \quad g \in G \quad \mathbf{H}_{p\cdot g}\mathcal{P} = (R_g)_\star(\mathbf{H}_p\mathcal{P})$$

Equivalently we can picture the desired geometrical construction based not any more on the horizontal equivariant distribution, but rather on the connection form $\boldsymbol{\omega}$.

Local gauge potential and connection forms. The idea is that the decomposition of the tangent space $T\mathcal{P} = \mathbf{H}\mathcal{P} \oplus \mathbf{V}\mathcal{P}$ into two subspaces can be picture with the connection form as a 1-form with values in the Lie algebra $\mathfrak{g}$. The main idea is that, if we choose a *trivialization*, we may picture the connection form $\boldsymbol{\omega} \in \Omega^1(\mathcal{P}, \mathfrak{g})$ in relation to a one form on $\mathcal{M}$, described by $\omega \in \Omega^1(\mathcal{M}, \mathfrak{g})$. The underlying idea for connection as a $\mathfrak{g}$-valued 1-form is to encode the projection of $T_p\mathcal{P}$ on its vertical subspace $\mathbf{V}_p\mathcal{P}$, identified with $\mathfrak{g}$. The horizontal subspace $\mathbf{H}_p\mathcal{P} \subset T_p\mathcal{P}$, as a linear subspace, is annihilated by $n = \dim(G)$ linear equations $T_p\mathcal{P} \longrightarrow \mathbb{R}$. It is then equivalent to describe $\mathbf{H}_p\mathcal{P}$ as the kernel of n 1-forms at p, namely an n-dimensional vector (Lie algebra)-valued one form. Hence that picture of a $\mathfrak{g}$-valued one form $\boldsymbol{\omega} \in \Omega^1(\mathcal{P}, \mathfrak{g})$. Then, if $\zeta^{\mathbf{H}}_{|p} \in \mathbf{H}_p\mathcal{P}$ we have $\boldsymbol{\omega}(\zeta^{\mathbf{H}}_{|p}) = 0$ whereas if $\zeta^{\mathbf{V}}_{|p} \in \mathbf{V}_p\mathcal{P}$ we obtain $\boldsymbol{\omega}(\zeta^{\mathbf{V}}_{|p}) = \xi \in \mathfrak{g}$ - where we have described canonically the vector field $\zeta^{\mathbf{V}}_{|p} = \boldsymbol{\zeta}_\xi$. To an element of the Lie algebra $\xi \in \mathfrak{g}$, we associate its fundamental vector field $\boldsymbol{\zeta}_\xi \in \mathbf{V}_p\mathcal{P} \subset T_p\mathcal{P}$ defined by (398). Now we have fully described our horizontal distribution, thanks to $\boldsymbol{\omega} \in \Omega^1(\mathcal{P}, \mathfrak{g})$.

Equivalently, if we denote $\boldsymbol{\zeta}_{\mathcal{I}}$ the fundamental vector field associated to the element $\mathfrak{b}_{\mathcal{I}}$ of the Lie algebra $\mathfrak{g}$ - described by the basis $\{\mathfrak{b}_{\mathcal{I}}\}_{1\leq \mathcal{I} \leq n}$ - , the tangent space decomposition $T_p\mathcal{P} = \mathbf{H}_p\mathcal{P} \oplus \mathbf{V}_p\mathcal{P}$ is generated on its vertical part by the means of vectors fields $\boldsymbol{\zeta}_{\mathcal{I}}$, whereas the horizontal one is describe by the connection $\boldsymbol{\omega}$. Hence, its characterization writes (400)

$$\boldsymbol{\omega}_{|p}(\boldsymbol{\zeta}_{\mathcal{I}|p}) = \mathfrak{b}_{\mathcal{I}} \qquad \text{and} \qquad \ker\,(\boldsymbol{\omega}_{|p}) = \mathbf{H}_p\mathcal{P} \tag{400}$$

In order to give a nice geometrical definition of an *Ehresmann connection*, let introduce quickly the adjoint map and adjoint representation. We recall that $\mathrm{ad} : G \longrightarrow G$ is defined by $\mathrm{ad}_g h = gh\mathrm{g}^{-1}$: this is the adjoint action. We consider the linear tangent application of ad is denoted $\mathrm{Ad}_g : T_h(G) \longrightarrow T_{gh\mathrm{g}^{-1}}(G)$ and is called the adjoint map. When one restrict it to T_eG, and due to the canonical isomorphism $T_eG \cong \mathfrak{g}$ we therefore describe Ad_g as a map from $\mathfrak{g}$ to itself: $\mathrm{Ad}_g = \mathfrak{g} \longrightarrow \mathfrak{g} : \xi \mapsto g\xi\mathrm{g}^{-1}$ for any $\xi \in \mathfrak{g}$.

Definition B.1.1. *An Ehresmann connection on a principal G-bundle $\mathcal{P} \stackrel{\pi}{\longrightarrow} \mathcal{M}$ is a $\mathfrak{g}$-valued 1-form $\boldsymbol{\omega}$ on $\mathcal{P}$ such that:*

$$\text{(i)} \quad \forall g \in G \quad (R_g^\star)\boldsymbol{\omega} = Ad_{\mathrm{g}^{-1}}\boldsymbol{\omega}$$

$$\text{(ii)} \quad \boldsymbol{\omega}(\boldsymbol{\zeta}_\xi) = \xi$$

Here, $(R_g^\star)\boldsymbol{\omega}$ denotes the pullback of $\boldsymbol{\omega}$ by the right action. By definition of the pullback, the first condition (i) in (B.1.1) actually means that for a given $\zeta \in T_pP$ we have $(R_g^\star)\boldsymbol{\omega}_{|pg}(\zeta) = \boldsymbol{\omega}_{|pg}\big((R_g)_\star\zeta\big) = \mathrm{g}^{-1}\boldsymbol{\omega}_{|p}(\zeta)g$. Also, we equivalently write this condition under the following form $\boldsymbol{\omega}_{|pg}(T_pR_g\zeta_{|p}) = \mathrm{g}^{-1}\boldsymbol{\omega}_{|p}(\zeta_{|p})g$. To exhibit (B.1.1), we process the following path. Following (398), we notice that for a $\xi \in \mathfrak{g}$

$$(R_g)_\star\varsigma_p(\xi) = \frac{d}{dt}R_g\big[p\exp(t\xi)\big]\Big|_{t=0} = \frac{d}{dt}\big[p\exp(t\xi)g\big]\Big|_{t=0} = \frac{d}{dt}\big[pg\mathrm{g}^{-1}\exp(t\xi)g\big]\Big|_{t=0}$$

We observe that, if $\xi \in \mathfrak{g}$ generates the one parameter subgroup $H = \{\exp(tX_\xi) \ /t \in \mathbb{R}\}$, then $\mathrm{Ad}_g(\xi)$ generates the subgroup $gH\mathrm{g}^{-1}$ with $\exp(\mathrm{Ad}_g(\xi)) = g(\exp\xi)\mathrm{g}^{-1}$, therefore:

$$(R_g)_\star\varsigma_p(\xi) = \frac{d}{dt}\big[pg\exp(t\mathrm{Ad}_{\mathrm{g}^{-1}}\xi)\big]\Big|_{t=0} = \boldsymbol{\zeta}_{\xi_{|pg}}(\mathrm{Ad}_{\mathrm{g}^{-1}}\xi) = \varsigma_{pg}(\mathrm{Ad}_{\mathrm{g}^{-1}}\xi) \tag{401}$$

Let consider a vertical vector field $\boldsymbol{\zeta}_\xi$ - defined canonically as $\varsigma_p(\xi)$ for a given $\xi \in \mathfrak{g}$ ($\boldsymbol{\zeta}_\xi$ is given by (398)) - then we observe:

$$(R_g^\star)\boldsymbol{\omega}(\varsigma(\xi)) = \boldsymbol{\omega}\big((R_g)_\star\varsigma(\xi)\big) = \boldsymbol{\omega}\big(\varsigma(\mathrm{Ad}_{\mathrm{g}^{-1}}\xi)\big) = \mathrm{Ad}_{\mathrm{g}^{-1}}\xi \tag{402}$$

We choose a local section $\sigma \in \Gamma(\mathcal{P})$ and also we denote, as usual, the linear tangent map by $\sigma_\star = d\sigma : T_{|p}\mathcal{M} \longrightarrow T_{|\sigma(p)}\mathcal{P}$. To chose a particular section is equivalent to perform a gauge choice. We define the *local connection form* as:

$$\forall\xi \in T\mathcal{M} \quad \omega(\xi) = \omega_\sigma(\xi) = \boldsymbol{\omega}_\sigma[\sigma_\star(\xi)] \tag{403}$$

The *local connection form* is thought as 1-form on $\mathcal{M}$ taking value in $\mathfrak{g}$. Later, we make intensively use of such form and write them as $\omega = \omega_\mu^{\mathcal{I}}\mathfrak{b}_{\mathcal{I}}dx^\mu$. We picture the *local connection form*, namely the *gauge potential*, - the pull back of the connection form $\boldsymbol{\omega}$ by a section $\sigma : \mathcal{O} \subset \mathcal{M} \to \mathcal{P}$ - and denoted as $\omega = \sigma^\star(\boldsymbol{\omega}) \in T^\star\mathcal{M} \otimes \mathfrak{g}$. It is worth noticing that the local connection form is only described *in* the local trivialization σ and thereby is a notion that depends on trivialization. In the language of typical Yang-Mills gauge theory, a choice of the section σ is called a choice of *local gauge*. The local connection form is denoted $A = A_\mu^{\mathcal{I}}dx^\mu \otimes \mathfrak{b}_{\mathcal{I}} = A_\mu^{\mathcal{I}}dx^\mu\mathfrak{b}_{\mathcal{I}}$ and is called the local gauge potential. The important point concerns gauge transformation in field theory. Let σ° and $\sigma^\bullet$ be two sections related by the relation $\forall x \in \mathcal{M}, \sigma^\circ(x) = \sigma^\bullet(x)\cdot g(x)$ and $g : \mathcal{M} \longrightarrow G$ at each point $x \in \mathcal{M}$. Then a change of section $\sigma^\bullet \mapsto \sigma^\circ$ is pictured by the action of a group element $g(x) \in G$. Then, we picture the well know transformation of gauge potential (a gauge transformation) as

$$\sigma^\circ = \mathrm{g}^{-1}(\sigma^\bullet)g + \mathrm{g}^{-1}dg \tag{404}$$

B.2 Vector-valued differential forms

$\mathcal{V}$-valued n-forms $\Omega^p(\mathcal{M},\mathcal{V})$ The Lie algebra-valued forms and related bracket structure described in the previous section is a specific case of more general $\mathcal{V}$-valued n-forms $\Omega^p(\mathcal{M},\mathcal{V})$: n-forms with values in a vector space $\mathcal{V}$. Then we write $\varphi \in \Omega^p(\mathcal{M},\mathcal{V})$. We denote $\mathrm{End}(\mathcal{V})$ the vector space of endomorphism of $\mathcal{V}$. Canonically we identify $\mathcal{V}\otimes\mathcal{V}^\star \cong \mathrm{End}(\mathcal{V})$. Any element $\Xi = \Xi^I_J\mathbf{v}_I\otimes\mathbf{v}^J \in \mathcal{V}\otimes\mathcal{V}^\star$ is identified with the endomorphism $\Xi_{\mathcal{V}} \in \mathrm{End}\mathcal{V}$ described as follow:

$$\Xi_{\mathcal{V}} : v = v^I\mathbf{v}_I \mapsto \Xi_{\mathcal{V}}(v) = \Xi v = \Xi^I_J\mathbf{v}_I\mathbf{v}^J(v) = (\Xi^I_Jv^J)\mathbf{v}_I \tag{405}$$

Hence, any endomorphism $\Xi_{\mathcal{V}} \in \mathrm{End}\mathcal{V}$ is given by the matrix Ξ^I_J - in the basis $\mathbf{v}_I$ of the vector space $\mathcal{V}$. Later on, we would be interested in forms of different type, depending on the nature of the vector bundle. It may be the associated bundle $\mathcal{P}\times_\rho\mathcal{V}$ (see below), also it can be a Lie algebra $\mathfrak{g}$ and finally we can consider the vector space where the forms take values as $\mathcal{V}\otimes\mathcal{V}^\star \cong \mathrm{End}\mathcal{V}$. Then, we first consider the product of differential form $\boldsymbol{\varphi} \in \Omega^p(\mathcal{M},\mathfrak{g})$ and $\boldsymbol{\psi} \in \Omega^q(\mathcal{M},\mathcal{V})$. This one is given[115] by $(\rho_{\mathfrak{g}})_\wedge = \rho_{\mathfrak{g}}(\boldsymbol{\varphi}) \wedge \boldsymbol{\psi} \in \Omega^{p+q}(\mathcal{M},\mathcal{V})$ along with the following rule. [3] Let $\zeta_1,...,\zeta_{p+q}$ vectors fields on $\mathcal{M}$,

$$\rho_{\mathfrak{g}}(\boldsymbol{\varphi}) \wedge \boldsymbol{\psi}(\zeta_1...\zeta_{p+q}) = \frac{1}{p!q!}\sum_{\sigma\in\mathcal{S}_{p+q}}(-1)^{\boldsymbol{\sigma}}\rho_{\mathfrak{g}}(\boldsymbol{\varphi}(\zeta_{\sigma(1)}...\zeta_{\sigma(p)}))(\boldsymbol{\psi}(\zeta_{\sigma(p+1)}...\zeta_{\sigma(p+q)})) \tag{406}$$

[115]Following [3] we notice that $(\rho_{\mathfrak{g}})_\wedge : \Omega^\star(\mathcal{M},\mathcal{V}) \longrightarrow \Omega^{\star+p}(\mathcal{M},\mathcal{V})$ is a graded $\Omega(\mathcal{M})$-module homomorphism of degree p.

The product introduce in the following (408) allows us to picture $\Omega^\star(\mathcal{M},\mathfrak{g})$ as a graded Lie algebra structure with the bracket $[,]$. Then, for any $\boldsymbol{\varphi}\in\Omega^p(\mathcal{M},\mathfrak{g})$ and $\boldsymbol{\psi}\in\Omega^q(\mathcal{M},\mathfrak{g})$, we define

$$[\boldsymbol{\varphi},\boldsymbol{\psi}](\zeta_1...\zeta_{p+q})=\frac{1}{p!q!}\sum_{\sigma\in\mathcal{S}_{p+q}}(-1)^{\boldsymbol{\sigma}}[(\boldsymbol{\varphi}(\zeta_{\sigma(1)}...\zeta_{\sigma(p)})),(\boldsymbol{\psi}(\zeta_{\sigma(p+1)}...\zeta_{\sigma(p+q)}))] \tag{407}$$

Finally, the last case of interest is when $\boldsymbol{\varphi}\in\Omega^n(\mathcal{M},\mathrm{End}(\mathcal{V}))=\Omega^n(\mathcal{M},\mathcal{V}\otimes\mathcal{V}^\star)$. For that purpose, considering the tensor algebra generated by $\mathcal{V}$, such forms falls in the category of form in $\Omega^\star(\mathcal{M},\bigotimes\mathcal{V})$ for $\boldsymbol{\varphi}\in\Omega^\star(\mathcal{M},\bigotimes\mathcal{V})$ and $\boldsymbol{\psi}\in\Omega^\star(\mathcal{M},\bigotimes\mathcal{V})$. Thus, we define the associative bigraded product:

$$\boldsymbol{\varphi}\wedge\boldsymbol{\psi}(\zeta_1...\zeta_{p+q})=(\boldsymbol{\varphi}\otimes_\wedge\boldsymbol{\psi})(\zeta_1...\zeta_{p+q})=\frac{1}{p!q!}\sum_{\sigma\in\mathcal{S}_{p+q}}(-1)^{\boldsymbol{\sigma}}(\boldsymbol{\varphi}(\zeta_{\sigma(1)}...\zeta_{\sigma(p)}))\otimes(\boldsymbol{\psi}(\zeta_{\sigma(p+1)}...\zeta_{\sigma(p+q)}))$$

Later on, we will come back on the specific case of $\mathcal{V}$-valued n-form and define an exterior covariant derivative for such form.

$\mathfrak{g}$-valued n-forms $\Omega^p(\mathcal{M},\mathfrak{g})$. Let λ be a $\mathfrak{g}$-valued p-form on $\mathcal{M}$ *i.e* $\lambda\in\Omega^p(\mathcal{M},\mathfrak{g})=\Omega^p(\mathcal{M})\otimes\mathfrak{g}$. Let $\mathfrak{b}_\mathcal{I}$ be a basis on $\mathfrak{g}$. Now, $\forall\lambda\in\Omega^p(\mathcal{M},\mathfrak{g})$ writes as[116]: $\lambda=\lambda^\mathcal{I}\otimes\mathfrak{b}_\mathcal{I}$. If we denote another $\mathfrak{g}$-valued p-form on $\mathcal{M}$ *i.e* $\sigma=\sigma^\mathcal{J}\otimes\mathfrak{b}_\mathcal{J}\in\Omega^q(\mathcal{M},\mathfrak{g})=\Omega^q(\mathcal{M})\otimes\mathfrak{g}$. The bracket of λ,σ denoted as $[\lambda,\sigma]$

$$[\lambda,\sigma]=(\lambda^\mathcal{I}\wedge\sigma^\mathcal{J})\otimes[\mathfrak{b}_\mathcal{I},\mathfrak{b}_J]=\mathfrak{c}^\mathcal{K}_{\mathcal{I}\mathcal{J}}(\lambda^\mathcal{I}\wedge\sigma^\mathcal{J})\otimes\mathfrak{b}_\mathcal{K} \tag{408}$$

We use wedge product on form part and the classical Lie bracket on the Lie algebra part. We observe the following graded property (409)(i) and the graded Jacobi identity (409)(ii)

$$(\mathfrak{i})\quad[\lambda,\sigma]=(-1)^{pg+1}[\sigma,\lambda]\qquad(\mathfrak{ii})\quad[\lambda,[\sigma,\eta]]=[[\lambda,\sigma]\eta]+(-1)^{lp}[\sigma,[\lambda,\eta]] \tag{409}$$

Let $\omega\in\Omega^1(\mathcal{M},\mathfrak{g})$ be a $\mathfrak{g}$-valued 1-form and $\zeta_1,\zeta_2\in\mathcal{X}(\mathcal{M})$ we have:

$$[\omega,\omega](\zeta_1,\zeta_2)=2[\omega(\zeta_1),\omega(\zeta_2)] \tag{410}$$

⌈ Proof Following [46, 176] we decompose $\omega=\omega^I\otimes\mathfrak{b}_\mathcal{I}$

$$[\omega,\omega](\zeta_1,\zeta_2)=[\omega^I\otimes\mathfrak{b}_\mathcal{I},\omega^J\otimes\mathfrak{b}_\mathcal{J}](\zeta_1,\zeta_2)=\omega^I\wedge\omega^\mathcal{J}\otimes[\mathfrak{b}_\mathcal{I},\mathfrak{b}_\mathcal{J}](\zeta_1,\zeta_2)=\mathfrak{c}^\mathcal{K}_{\mathcal{I}\mathcal{J}}(\omega^\mathcal{I}\wedge\omega^\mathcal{J})\otimes\mathfrak{b}_\mathcal{K}(\zeta_1,\zeta_2)$$

Then

$$[\omega,\omega](\zeta_1,\zeta_2)=\big(\mathfrak{c}^\mathcal{K}_{\mathcal{I}\mathcal{J}}(\omega^\mathcal{I}\otimes\omega^\mathcal{J}-\omega^\mathcal{J}\otimes\omega^\mathcal{I})\otimes\mathfrak{b}_\mathcal{K}\big)(\zeta_1,\zeta_2)=2\big(\mathfrak{c}^\mathcal{K}_{\mathcal{I}\mathcal{J}}\omega^\mathcal{I}(\zeta_1)\omega^\mathcal{J}(\zeta_2)\big)\otimes\mathfrak{b}_\mathcal{K}=2\mathfrak{c}^\mathcal{K}_{\mathcal{I}\mathcal{J}}\omega^\mathcal{I}(\zeta_1)\omega^\mathcal{J}(\zeta_2)\mathfrak{b}_\mathcal{K} \tag{411}$$

On the other hand, let $\lambda,\sigma\in\Omega^1(\mathcal{M},\mathfrak{g})$ and $\zeta_1,\zeta_2\in\mathcal{X}(\mathcal{M})$. We define $[\lambda,\sigma](\zeta_1,\zeta_2)=[\lambda(\zeta_1),\sigma(\zeta_2)]$ then :

$$[\omega,\omega](\zeta_1,\zeta_2)=[\omega(\zeta_1),\omega(\zeta_2)]=[\omega^\mathcal{I}(\zeta_1)\otimes\mathfrak{b}_\mathcal{I},\omega^\mathcal{J}(\zeta_2)\otimes\mathfrak{b}_\mathcal{J}]=\omega^\mathcal{I}(\zeta_1)\omega^\mathcal{J}(\zeta_2)[\mathfrak{b}_\mathcal{I},\mathfrak{b}_\mathcal{J}] \tag{412}$$

Hence, compairing (411) and (412) we obtain (410).

The previous lines actually are similar for a more general n-form with value in the vector space $\mathcal{V}$, $\lambda\in\Omega^n(\mathcal{M},\mathcal{V})$. Here, we prefer to directly consider the specific case of $\mathcal{V}$ being a Lie-algebra, allowing us to write the expression of the bracket with (413). Let $\lambda\in\Omega^p(\mathcal{M},\mathfrak{g})$ and $\sigma\in\Omega^q(\mathcal{M},\mathfrak{g})$

$$[\lambda,\sigma]=\lambda^\mathcal{I}\wedge\sigma^\mathcal{J}\otimes[\mathfrak{b}_\mathcal{I},\mathfrak{b}_\mathcal{J}]=\lambda\wedge\sigma-(-1)^{pq}\sigma\wedge\lambda \tag{413}$$

When $\mathfrak{g}$ is a matrix Lie algebra[117] we use the notation $\lambda\wedge\sigma=\mathfrak{b}_\mathcal{I}\mathfrak{b}_\mathcal{J}\otimes\lambda^\mathcal{I}\wedge\lambda^\mathcal{J}$. Therefore, for any odd degree n, let $\lambda\in\Omega^n(\mathcal{M},\mathfrak{g})$, we exhibit $[\lambda,\lambda]=2\lambda\wedge\lambda$. In this case of matrix Lie algebra we often find this notation.

[116] we use the curved capital letters to underlined the fact that we work with the basis of generator of the Lie algebra.
[117] This relation, is generally given for $\mathcal{V}$ being an associative algebra.

B.3 Curvature of principal connection

The idea of curvature of a connection, from mathematical standpoint, is an object that measures the obstruction of integrability of the horizontal distribution. Since we described a connection form $\boldsymbol{\omega} \in \Omega^1(\mathcal{P}, \mathfrak{g})$ - its role is to describe a well given horizontal distribution -, the curvature appears as the covariant derivative $\boldsymbol{\Omega}^{\boldsymbol{\omega}} = \mathbf{h}^\star(d\boldsymbol{\omega}) \in \Omega^2(\mathcal{P}, \mathfrak{g})$. Let go into more details. We denote the horizontal projection $\mathbf{h} : T\mathcal{P} \longrightarrow T\mathcal{P}$ the projection onto the horizontal distribution along the vertical one. It is therefore described as the set of linear maps $\mathbf{h}_p : T_p\mathcal{P} \longrightarrow T_p\mathcal{P}$ such that $\forall p \in \mathcal{P}(\mathcal{M}, G)$ $\forall \zeta \in T\mathcal{P}$ we have: $\mathbf{h}_p(\zeta) = \zeta$ if $\zeta \in \mathbf{H}_p\mathcal{P}$ and $\mathbf{h}_p(\zeta) = 0$ if $\zeta \in \mathbf{V}_p\mathcal{P}$. We extend this notion of *horizontal projection* on forms defined on $\mathcal{P}(\mathcal{M}, G)$. Let $\boldsymbol{\lambda} \in \Omega^n(\mathcal{P})$ such n-form defined on $\mathcal{P}$. We denote: $\mathbf{h}^\star(\boldsymbol{\varphi})(\zeta_1, ..., \zeta_n) = \boldsymbol{\varphi}(\mathbf{h}(\zeta_1), ..., \mathbf{h}(\zeta_n))$. However, notice that *here* $\mathbf{h}^\star$ is *not* the pull-back by any smooth map. It is just a convention of notation to emphasize the *dual* nature of the object: $\mathbf{h}^\star\boldsymbol{\varphi} = \boldsymbol{\varphi} \circ \mathbf{h}$.

Definition B.3.1. *A n-form $\boldsymbol{\varphi} \in \Omega^n(\mathcal{P})$ is called horizontal if $\mathbf{h}^\star\boldsymbol{\varphi} = \boldsymbol{\varphi}$.*

Exterior covariant derivative We define the exterior covariant derivative relative to a connection $\boldsymbol{\omega}$ on the fiber bundle $\mathcal{P}(\mathcal{M}, G)$ - by its action on $\boldsymbol{\varphi}$, a n-form on the principal bundle $\mathcal{P}(\mathcal{M}, G)$ as follows:

$$\mathbf{D}^{\boldsymbol{\omega}} : \Omega^n(\mathcal{P}, \mathcal{V}) \longrightarrow \Omega^{n+1}(\mathcal{P}, \mathcal{V}) : \mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\varphi}(\zeta_1, ..., \zeta_{n+1}) = \mathbf{h}^\star(d\boldsymbol{\varphi}) = d\boldsymbol{\varphi}(\mathbf{h}(\zeta_1), ..., \mathbf{h}(\zeta_{n+1})) \tag{414}$$

We take the ordinary exterior differential on $\mathcal{P}(\mathcal{M}, G)$ but we restrict the form to horizontal part of vector fields. This is a general definition and there is no needs for $\boldsymbol{\varphi}$ to be horizontal or equivariant of type ρ - see below. By construction, $\mathbf{D}^{\boldsymbol{\omega}}$ is dependent of the vector space $\mathcal{V}$ and also of the choice of the connection $\boldsymbol{\omega}$ on $\mathcal{P}(\mathcal{M}, G)$. We emphasize that the horizontal form $\mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\varphi}$ preserves the G-equivariance related to the representation ρ of G on the vector space $\mathcal{V}$. - see below relation (415). We recall that we symbolically denote $\mathbf{h}^\star\boldsymbol{\varphi} = \boldsymbol{\varphi} \circ \mathbf{h}$ following [165]:

$$R_g^\star\mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\varphi} = \mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\varphi}(R_{\mathrm{g}^\star}) = \mathbf{h}^\star(d\boldsymbol{\varphi})(R_{\mathrm{g}^\star}) = (d\boldsymbol{\varphi}) \circ \mathbf{h} \circ R_{\mathrm{g}^\star} = (d\boldsymbol{\varphi}) \circ R_{\mathrm{g}^\star} \circ \mathbf{h} = R_g^\star(d\boldsymbol{\varphi}) \circ \mathbf{h} = d(\rho_{\mathrm{g}^{-1}}\boldsymbol{\varphi}) \circ \mathbf{h}$$

Then:

$$R_g^\star\mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\varphi} = \rho_{\mathrm{g}^{-1}}d(\boldsymbol{\varphi} \circ \mathbf{h}) = \rho_{\mathrm{g}^{-1}}\mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\varphi} \tag{415}$$

Therefore, we obtain the following important result. Considering $\boldsymbol{\varphi} \in \Omega^n(\mathcal{P}, \mathcal{V})$ with the additional property of being a G-equivariant of type $(\rho, \mathcal{V})$, the above relation shows that $\mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\varphi}$ inherits the same equivariance property. Since by construction it is a horizontal form $\mathbf{h}^\star\mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\varphi} = \mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\varphi}$, $\mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\varphi}$ falls in the case of basic form $\Omega_\rho^{n+1}(\mathcal{P}, \mathcal{V})$ described in the section B.5 below. Before performing this step, let us define the curvature of a connection on a principal bundle. In the subsequent section we are partially inspired from notation and logic developed by Rodriguez and al. [195]

Definition B.3.2. Curvature of a connection on a principal bundle *We define the curvature, denoted as $\boldsymbol{\Omega}$, of the connection 1-form $\boldsymbol{\omega} \in \Omega^1(\mathcal{P}, \mathfrak{g})$ as the 2-form $\boldsymbol{\Omega}^{\boldsymbol{\omega}} = \mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\omega} \in \Omega^2(\mathcal{P}, \mathfrak{g})$*

Let notice that $\forall \zeta, \chi \in T\mathcal{P}$, we have:

$$\boldsymbol{\Omega}(\zeta, \chi) = d(\mathbf{h}(\zeta), \mathbf{h}(\chi)) = \mathbf{h}(\zeta)\boldsymbol{\omega}(\mathbf{h}(\chi)) - \mathbf{h}(\chi)\boldsymbol{\omega}(\mathbf{h}(\zeta)) - \boldsymbol{\omega}([\mathbf{h}(\zeta), \mathbf{h}(\chi)]) = -\boldsymbol{\omega}([\mathbf{h}(\zeta), \mathbf{h}(\chi)])$$

Therefore, $\boldsymbol{\Omega}(\zeta, \chi) = 0$ if and only if $[\mathbf{h}(\zeta), \mathbf{h}(\chi)]$ is horizontal. (Frobenius integrability condition). Then, we can also consider the **Cartan structure equation**. The curvature 2-form $\boldsymbol{\Omega} = \mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\omega} \in \Omega^2(\mathcal{P}, \mathfrak{g})$ of the connection 1-form $\boldsymbol{\omega} \in \Omega^1(\mathcal{P}, \mathfrak{g})$. In this case, we surely have:

$$\mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\omega}(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2) = d\boldsymbol{\omega}(\mathbf{h}(\boldsymbol{\zeta}_1), \mathbf{h}(\boldsymbol{\zeta}_2)) = d\boldsymbol{\omega}(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2) + [\boldsymbol{\omega}(\zeta_1), \boldsymbol{\omega}(\zeta_2)] \tag{416}$$

The Cartan structure relation (416) is written:

$$\mathbf{\Omega} = \mathbf{\Omega}^{\boldsymbol{\omega}} = d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}] \tag{417}$$

⌈ Proof Let us demonstrate (416). We consider two vector fields $\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2$ on $\mathcal{P}$. First, we consider that both vector fields are horizontal, *i.e.* for $i = 1, 2$ $\boldsymbol{\zeta}_i \in \mathbf{H}\mathcal{P}$. In this case, the equality is a triviality since $\boldsymbol{\omega}(\boldsymbol{\zeta}_i) = 0$ and $\mathbf{h}(\boldsymbol{\zeta}_i) = \boldsymbol{\zeta}_i$. Now we consider the case where the two vectors are vertical. In this case, using the map defined above (398), we consider their canonical extraction from $\xi_i \in \mathfrak{g}$ such that $\boldsymbol{\zeta}_i = \varsigma(\xi_i)$. In this case we have

$$\underbrace{d\boldsymbol{\omega}(\varsigma(\xi_1), \varsigma(\xi_2)) + [\boldsymbol{\omega}(\varsigma(\xi_1)), \boldsymbol{\omega}(\varsigma(\xi_2))]}_{[\mathbf{I}]} = \mathcal{L}_{\varsigma(\xi_1)}\boldsymbol{\omega}(\varsigma(\xi_2)) - \mathcal{L}_{\varsigma(\xi_2)}\boldsymbol{\omega}(\varsigma(\xi_1)) - \boldsymbol{\omega}([\varsigma(\xi_1), \varsigma(\xi_2)]) + [\boldsymbol{\omega}(\varsigma(\xi_1)), \boldsymbol{\omega}(\varsigma(\xi_2))]$$

Since $\boldsymbol{\omega}(\varsigma(\xi_i)) = \xi_i$ and since $\mathcal{L}_{\varsigma(\xi_i)}\boldsymbol{\omega}(\varsigma(\xi_j)) = \varsigma(\xi_i)\boldsymbol{\omega}(\varsigma(\xi_j)) = \varsigma(\xi_i)\xi_j$, we obtain:

$$[\mathbf{I}] = \varsigma(\xi_1)\xi_2 - \varsigma(\xi_2)\xi_1 - \boldsymbol{\omega}([\varsigma(\xi_1), \varsigma(\xi_2)]) + [\xi_1, \xi_2] = -\boldsymbol{\omega}([\varsigma(\xi_1), \varsigma(\xi_2)]) + [\xi_1, \xi_2]$$

Finally, we notice that $[\varsigma(\xi_1), \varsigma(\xi_2)] = \varsigma([\xi_1, \xi_2])$ so that $[\mathbf{I}] = -\boldsymbol{\omega}(\varsigma([\xi_1), \xi_2])) + [\xi_1, \xi_2] = 0$. Equivalently, we find that $d\boldsymbol{\omega}(\mathbf{h}(\boldsymbol{\zeta}_1), \mathbf{h}(\boldsymbol{\zeta}_2)) = 0$ which clearly gives the relation (416) in this case. Finally, if $\boldsymbol{\zeta} \in \mathbf{H}\mathcal{P}$ and $\boldsymbol{\chi} = \varsigma(\xi) \in \mathbf{V}\mathcal{P}$, we have $d\boldsymbol{\omega}(\mathbf{h}(\boldsymbol{\zeta}), \mathbf{h}(\boldsymbol{\chi})) = d\boldsymbol{\omega}(\boldsymbol{\zeta}, \mathbf{h}(\boldsymbol{\chi})) = 0$ Therefore, we have equivalently $\boldsymbol{\omega}([\boldsymbol{\zeta}, \varsigma(\xi)]) = 0$

Bianchi Identity The Bianchi Identity basically states that

$$\mathbf{D}^{\boldsymbol{\omega}}\mathbf{\Omega}^{\boldsymbol{\omega}} = d\mathbf{\Omega}^{\boldsymbol{\omega}} + [\boldsymbol{\omega}, \mathbf{\Omega}^{\boldsymbol{\omega}}] = 0 \tag{418}$$

⌈ Proof We have:

$$\mathbf{D}^{\boldsymbol{\omega}}\mathbf{\Omega}^{\boldsymbol{\omega}} = \mathbf{h}^\star d\mathbf{\Omega}^{\boldsymbol{\omega}} = \mathbf{h}^\star d(d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}]) = \frac{1}{2}\mathbf{h}^\star([d\boldsymbol{\omega}, \boldsymbol{\omega}] - [\boldsymbol{\omega}, d\boldsymbol{\omega}])$$

Since from bracket properties we notice that $d[\boldsymbol{\omega}, \boldsymbol{\omega}] = [d\boldsymbol{\omega}, \boldsymbol{\omega}] - [\boldsymbol{\omega}, d\boldsymbol{\omega}]$ and also $[d\boldsymbol{\omega}, \boldsymbol{\omega}] = -[\boldsymbol{\omega}, d\boldsymbol{\omega}]$, then $\mathbf{D}^{\boldsymbol{\omega}}\mathbf{\Omega}^{\boldsymbol{\omega}} = \mathbf{h}^\star[d\boldsymbol{\omega}, \boldsymbol{\omega}]$ We notice that: $d\mathbf{\Omega}^{\boldsymbol{\omega}} = [d\boldsymbol{\omega}, \boldsymbol{\omega}]$. We slightly transmute this relation, using (417):

$$d\mathbf{\Omega}^{\boldsymbol{\omega}} = [\mathbf{\Omega}^{\boldsymbol{\omega}} - \frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}], \boldsymbol{\omega}] = [\mathbf{\Omega}^{\boldsymbol{\omega}}, \boldsymbol{\omega}] - \frac{1}{2}[[\boldsymbol{\omega}, \boldsymbol{\omega}], \boldsymbol{\omega}] = [\mathbf{\Omega}^{\boldsymbol{\omega}}, \boldsymbol{\omega}]$$

Then, $d\mathbf{\Omega}^{\boldsymbol{\omega}} = [\mathbf{\Omega}^{\boldsymbol{\omega}}, \boldsymbol{\omega}] \in \Omega^3(\mathcal{P}, \mathfrak{g})$, by definition we have for any vector fields on the principal bundle $\mathcal{P}(\mathcal{M}, G)$, $\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \boldsymbol{\zeta}_3 \in T\mathcal{P}$, we have $\mathbf{D}^{\boldsymbol{\omega}}\mathbf{\Omega}^{\boldsymbol{\omega}}(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \boldsymbol{\zeta}_3) = \mathbf{h}^\star d\mathbf{\Omega}^{\boldsymbol{\omega}}(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \boldsymbol{\zeta}_3) = [\mathbf{\Omega}^{\boldsymbol{\omega}}, \boldsymbol{\omega}](\mathbf{h}(\boldsymbol{\zeta}_1), \mathbf{h}(\boldsymbol{\zeta}_2), \mathbf{h}(\boldsymbol{\zeta}_3))$. We write now $\boldsymbol{\omega} = \boldsymbol{\omega}^{\mathcal{I}} \otimes \mathfrak{b}_{\mathcal{I}}$ and $\mathbf{\Omega}^{\boldsymbol{\omega}} = (\mathbf{\Omega}^{\boldsymbol{\omega}})^{\mathcal{J}} \otimes \mathfrak{b}_{\mathcal{J}}$, therefore $[\mathbf{\Omega}^{\boldsymbol{\omega}}, \boldsymbol{\omega}](\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \boldsymbol{\zeta}_3) = \boldsymbol{\omega}^{\mathcal{I}} \wedge (\mathbf{\Omega}^{\boldsymbol{\omega}})^{\mathcal{J}} \otimes [\mathfrak{b}_{\mathcal{I}}, \mathfrak{b}_{\mathcal{J}}](\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \boldsymbol{\zeta}_3)$. Using the definition of the wedge product on the form part we obtain

$$[\mathbf{\Omega}^{\boldsymbol{\omega}}, \boldsymbol{\omega}](\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \boldsymbol{\zeta}_3) = \mathfrak{c}^{\mathcal{K}}_{\mathcal{I}\mathcal{J}}(\boldsymbol{\omega}^{\mathcal{I}} \wedge (\mathbf{\Omega}^{\boldsymbol{\omega}})^{\mathcal{J}}) \otimes \mathfrak{b}_{\mathcal{K}}(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \boldsymbol{\zeta}_3)\mathfrak{c}^{\mathcal{K}}_{\mathcal{I}\mathcal{J}} = \{\frac{1}{2}\sum_{\sigma \in \mathcal{S}}(-1)^{\boldsymbol{\sigma}}\boldsymbol{\omega}^{\mathcal{I}}(\boldsymbol{\zeta}_{\sigma(1)})(\mathbf{\Omega}^{\boldsymbol{\omega}})^{\mathcal{J}}(\boldsymbol{\zeta}_{\sigma(2)}, \boldsymbol{\zeta}_{\sigma(3)})\} \otimes \mathfrak{b}_{\mathcal{K}}$$

By taking into account only the horizontal part of the vector fields, from the last equation we obtain,

$$[\mathbf{\Omega}^{\boldsymbol{\omega}}, \boldsymbol{\omega}](\mathbf{h}(\boldsymbol{\zeta}_1), \mathbf{h}(\boldsymbol{\zeta}_2), \mathbf{h}(\boldsymbol{\zeta}_3)) = \mathfrak{c}^{\mathcal{K}}_{\mathcal{I}\mathcal{J}}\{\frac{1}{2}\sum_{\sigma \in \mathcal{S}}(-1)^{\boldsymbol{\sigma}}\boldsymbol{\omega}^{\mathcal{I}}\big(\mathbf{h}(\boldsymbol{\zeta}_{\sigma(1)})\big)(\mathbf{\Omega}^{\boldsymbol{\omega}})^{\mathcal{J}}\big(\mathbf{h}(\boldsymbol{\zeta}_{\sigma(2)}), \mathbf{h}(\boldsymbol{\zeta}_{\sigma(3)})\big)\} \otimes \mathfrak{b}_{\mathcal{K}}$$

which clearly vanishes, since $\forall \boldsymbol{\zeta}_i$ we have $\boldsymbol{\omega}^{\mathcal{I}}\big(\mathbf{h}(\boldsymbol{\zeta}_i)\big) = 0$. Therefore we have written the Bianchi Identity.

Curvature form as a basic form of type $(\mathrm{Ad}, \mathfrak{g})$. — Later we will introduce the notion of *basic* form, see (B.5). From this perspective we state that the curvature form, namely $\mathbf{\Omega}^{\boldsymbol{\omega}} = \mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\omega} \in \Omega^2(\mathcal{P}, \mathfrak{g})$ on $\mathcal{P}(\mathcal{M}, \mathfrak{g})$ is a basic 2-form of type $(\mathrm{Ad}, \mathfrak{g})$, indeed, $\boldsymbol{\omega} \in \Omega^1(\mathcal{P}, \mathfrak{g})$ is G-invariant of type $(\mathrm{Ad}, \mathfrak{g})$, namely $R^\star_g\mathbf{\Omega}^{\boldsymbol{\omega}} = \mathrm{Ad}_{\mathfrak{g}^{-1}}\mathbf{\Omega}^{\boldsymbol{\omega}}$. Then we can defined an induced 2-form $\Omega \in \Omega^2(\mathcal{M}, \mathcal{P} \times_{\mathrm{Ad}} \mathfrak{g})$ which take values in the associated bundle $\mathcal{P} \times_{\mathrm{Ad}} \mathfrak{g}$.

Local expression of Curvature form — Now the point is to describe the curvature locally on the base manifold $\mathcal{M}$. Above we have seen $\omega = \sigma^\star(\boldsymbol{\omega}) \in T^\star\mathcal{M} \otimes \mathfrak{g}$. Equivalently we obtain the pullback by a section $\sigma^\star$ of the curvature 2 form $\boldsymbol{\omega} = \mathbf{\Omega}^{\boldsymbol{\omega}} = \mathbf{h}^\star(d\boldsymbol{\omega}) \in \Omega^2(\mathcal{P}, \mathfrak{g})$ which means the object $\Omega = \sigma^\star\boldsymbol{\omega}$. We equivalently may write $\Omega^\omega = \sigma^\star\mathbf{\Omega}^{\boldsymbol{\omega}}$ so that:

$$\Omega^\omega = \sigma^\star(\mathbf{\Omega}^{\boldsymbol{\omega}}) = \sigma^\star(\mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\omega}) = \sigma^\star\big(d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}]\big) = d\omega + \frac{1}{2}[\omega, \omega] \tag{419}$$

The pull-back of the curvature form along with a local section Ω^ω is the field strength in that gauge. This is precisely the object introduced in gauge theory as the curvature F. (Field strength). This object is a 2-form on $\mathcal{M}$ with value in the Lie algebra $\mathfrak{g}$: $\Omega^\omega \in \Omega^2(\mathcal{M},\mathfrak{g})$. Let σ be a local section: $\sigma : x \in \mathcal{M} \longrightarrow \sigma(x) \in \mathcal{P}$ and $\xi_1, \xi_2 \in \mathcal{X}(\mathcal{M})$ two vector fields. We have[118] $\Omega^\omega(\xi_1,\xi_2) = \boldsymbol{\Omega}^{\boldsymbol{\omega}}(\sigma_\star\xi_1, \sigma_\star\xi_2)$ Finally, if we apply this process to two basis vector field $\mathbf{e}_\mu$ and $\mathbf{e}_\nu$, we then denote: $\Omega^\omega_{\mu\nu} = \Omega^\omega(\mathbf{e}_\mu,\mathbf{e}_\nu) \in \mathfrak{g}$. Now, with $\mathfrak{b}_\mathcal{I}$ a basis of the Lie algebra $\mathfrak{g}$, then we write $\Omega^\omega_{\mu\nu} = (\Omega^\omega)^\mathcal{I}_{\mu\nu}\mathfrak{b}_\mathcal{I}$.

Remark When $\mathfrak{g}$ is a matrix Lie algebra, we denote (416) and (419) respectively as (420)(i) and (420)(ii)

$$\text{(i)}\quad \boldsymbol{\Omega}^{\boldsymbol{\omega}} = d\boldsymbol{\omega} + \boldsymbol{\omega}\wedge\boldsymbol{\omega} \qquad\qquad \text{(ii)}\quad \Omega^\omega = d\omega + \omega\wedge\omega \tag{420}$$

Therefore, in a given trivialization, we obtain

$$\omega = \omega^\mathcal{I}\otimes\mathfrak{b}_\mathcal{I} = \omega^\mathcal{I}_\mu dx^\mu \otimes \mathfrak{b}_\mathcal{I} \qquad\qquad \Omega^\omega = (\Omega^\omega)^\mathcal{I}\otimes\mathfrak{b}_\mathcal{I} = \frac{1}{2}\Omega^\mathcal{I}_{\mu\nu} dx^\mu\wedge dx^\nu\otimes\mathfrak{b}_\mathcal{I} \tag{421}$$

with $\Omega^\mathcal{I}_{\mu\nu} = \partial_\mu\omega^\mathcal{I}_\nu - \partial_\nu\omega^\mathcal{I}_\mu + \mathfrak{c}^\mathcal{I}_{\mathcal{J}\mathcal{K}}\omega^\mathcal{J}_\mu\omega^\mathcal{K}_\nu$. Defined on a *principal fiber bundle*, we have proper geometrical picture of a connection. However, we would like to define it on associated bundles, since matter field in gauge theory are described as section of associated bundles. Before going toward the definition of curvature on associated bundle, the underlying right setting for *physical matter field*, we shall turn back to the differential structure on a general vector bundle $\mathcal{V}$.

B.4 Connection, exterior derivative and curvature on a vector bundle.

Definition B.4.1. *The covariant derivative is an operator*

$$\mathbf{D} : \Gamma(\mathcal{V}) \longrightarrow \Omega^1(\mathcal{M},\mathcal{V}) = \Gamma(\mathcal{V})\otimes\Omega^1(\mathcal{M}) \tag{422}$$

that transform the section of $\mathcal{V}$ into 1-form section and verify $\mathbf{D}(\sigma\varphi) = (\mathbf{D}\sigma)\varphi + \sigma\otimes d\varphi$ (the so-called Leibniz rule) for any section $\sigma \in \Gamma(\mathcal{V})$ and any function on $\mathcal{M}$.

Let emphasize that $\mathbf{D}_\xi\sigma$ and σ are sections of the vector bundle $\mathcal{V}$ whereas $\mathbf{D}\sigma$ is a section 1-form, namely $\mathbf{D}\sigma \in \Gamma(\mathcal{V})\otimes\Omega^1(\mathcal{M})$. We come back later on this expression in coordinate components. (see below lemma B.1). Finally, a section $\sigma \in \Omega^0(\mathcal{M},\mathcal{V}) = \Gamma(\mathcal{V})$ is said to be autoparallel (or $\mathbf{D}$-parallel) if we have $\mathbf{D}\sigma = 0$. Alternatively, we call such covariant derivative operator wich satisfy the Leibniz rule a *linear connection.* Notice that the space $\mathcal{A}(\mathcal{V})$ of linear connection on $\mathcal{V}$ is an affine space supported by the director vector space $\Omega^1(\mathcal{M},\mathrm{End}(\mathcal{V}))$[119], see proposition B.1. Hence, a linear connection on $\mathcal{V}$ is the linear application (B.4.1), alternatively describe as $\mathbf{D} : \mathcal{X}(\mathcal{M}) \longrightarrow \mathrm{End}(\Gamma(\mathcal{V}))$ with the following properties:

(i) $\mathbf{D}$ is $C^\infty(\mathcal{M})$-linear $i.e$ $\mathbf{D}_{fX+gY} = f\mathbf{D}_X + g\mathbf{D}_Y$, for all $X, Y \in \mathcal{M}$ and $f, g \in C^\infty(\mathcal{M})$ This is why[120] we can picture the connection as a linear application $\mathbf{D} : \Gamma(\mathcal{V}) \longrightarrow \Omega^1(\mathcal{M},\mathcal{V})$.

(ii) For any $X \in \mathcal{X}(\mathcal{M})$, the application $\mathbf{D}_X : \Gamma(\mathcal{V}) \longrightarrow \Gamma(\mathcal{V})$ is a $C^\infty(\mathcal{M})$-derivation $\mathbf{D}_X(f\sigma) = X(f)\sigma + f\mathbf{D}_X(\sigma)$, for $f \in C^\infty(\mathcal{M})$

[118]Here, we push-forward ξ_1, ξ_2 in order to get vector field $\sigma_\star\xi_1, \sigma_\star\xi_2$ on $\mathcal{P}$ The application of $\boldsymbol{\omega}$ give back an element of $\mathfrak{g}$.

[119]The difference between two linear connections on $\mathcal{V}$ is a $\Omega^0(\mathcal{M},\mathbb{R})$-linear operator from $\Omega^0(\mathcal{M},\mathcal{V})$ to $\Omega^1(\mathcal{M},\mathcal{V})$ and therefore defined by an element of $\Omega^1(\mathcal{M},\mathrm{End}(\mathcal{V}))$.

[120]For any section $\sigma \in \Gamma(\mathcal{V})$ the application $\mathbf{D}(\sigma) : \mathcal{X}(\mathcal{M}) \longrightarrow \Gamma(\mathcal{V}) : X \mapsto \mathbf{D}_X(\sigma)$ is $C^\infty(\mathcal{M})$-linear so that it is seen as an element $\mathbf{D}(\sigma) \in \Omega^1(\mathcal{M}) \otimes_{C^\infty(\mathcal{M})} \Gamma(\mathcal{V})$.

We consider a vector bundle $\mathcal{V} \xrightarrow{\pi} \mathcal{M}$. A local trivialisation $\psi : \pi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U} \times \mathbb{R}^n$ allow us to define un system of local section $(\mathbf{e}_1, ..., \mathbf{e}_n)$ with

$$\mathbf{e}_I : \mathcal{U} \subset \mathcal{M} \longrightarrow \mathcal{V} : x \mapsto \mathbf{e}_I(x) = \psi^{-1}(x, \mathbf{e}_I) \tag{423}$$

$\forall x \in \mathcal{M}, (\mathbf{e}_1(x), ..., \mathbf{e}_n(x))$ is a basis of $\mathcal{V}_x$. Later on, we will emphasize this description and we state that the data of a local trivialization is equivalent to a local frame. In this description, any section σ of $\mathcal{V}$ is written: $\sigma : \mathcal{U} \subset \mathcal{V} : x \mapsto \sigma(x) = \sigma^I(x)\mathbf{e}_I(x)$ where $\sigma^I \in C^\infty(\mathcal{U})$.

Lemma B.1. *Let*[121] $\sigma = \sigma^I \mathbf{e}_I \in \Gamma(\mathcal{V})$ *a section of the bundle* $\mathcal{V} \to \mathcal{M}$ *and* $\xi = \xi^\mu \partial_\mu \in \mathcal{X}(\mathcal{M})$ *we have:*

$$\mathbf{D}_\xi \sigma = \xi^\mu \big(d\sigma^J(\partial_\mu) + \omega^J_{\mu I}\sigma^I\big)\mathbf{e}_J = \xi^\mu \big(\partial_\mu \sigma^J + \omega^J_{\mu I}\sigma^I\big)\mathbf{e}_J \tag{424}$$

⌜Proof We make the straightforward computation:

$$\mathbf{D}_\xi \sigma = \mathbf{D}_{\xi^\mu \partial_\mu}(\sigma^I \mathbf{e}_I) = \xi^\mu \mathbf{D}_{\partial_\mu}(\sigma^I \mathbf{e}_I) = \xi^\mu \Big[\mathbf{D}_{\partial_\mu}(\sigma^I)\mathbf{e}_I + \sigma^I \mathbf{D}_{\partial_\mu}(\mathbf{e}_I)\Big] = \xi^\mu \Big[d\sigma^I(\partial_\mu)\mathbf{e}_I + \sigma^I \mathbf{D}_{\partial_\mu}(\mathbf{e}_I)\Big]$$

Since $\mathbf{D}_\xi(\sigma)$ is a $C^\infty(\mathcal{M})$-derivation in σ. Moreover, since $\mathbf{D}_{\partial_\mu}(\mathbf{e}_I) \in \Gamma(\mathcal{V})$, it exists a local function on $\mathcal{M}$, denoted $\omega^J_{\mu I}$, such that $\mathbf{D}_{\partial_\mu}(\mathbf{e}_I) = \omega^J_{\mu I}\mathbf{e}_J$. Therefore, we obtain $\mathbf{D}_\xi \sigma = (\xi^\mu d\sigma^J(\partial_\mu) + \omega^J_{\mu I}\xi^\mu \sigma^I)\mathbf{e}_J$ which equivalently writes as $\mathbf{D}_\xi \sigma = \xi^\mu \big(\partial_\mu \sigma^J + \omega^J_{\mu I}\sigma^I\big)\mathbf{e}_J$.

Proposition B.1. *The difference* $\mathbf{D}^\circ - \mathbf{D}^\bullet$ *between two connection* $\mathbf{D}^\circ$ *and* $\mathbf{D}^\bullet$ *is an application* $\mathbf{D}^\circ - \mathbf{D}^\bullet : \mathcal{X}(\mathcal{M}) \longrightarrow End(\Gamma(\mathcal{V}))$*, where the target space is* $End(\Gamma(\mathcal{V})) = End_{C^\infty(\mathcal{M})}(\Gamma(\mathcal{V})) = \Gamma(End(\mathcal{V}))$ *is the push-foreward of a bundle morphism. We have:*

$$\mathbf{D}^\circ - \mathbf{D}^\bullet \in Hom_{C^\infty(\mathcal{M})}\big(\mathcal{X}(\mathcal{M}), \Gamma(End\mathcal{V})\big) \cong \Omega^1(\mathcal{M}) \otimes_{C^\infty(\mathcal{M})} \Gamma(End(\mathcal{V})) = \Omega^1(\mathcal{M}, End(\mathcal{V}))$$

We do not prevent the proof of this proposition, however we stress the important point. For a particular connection, let say $\mathbf{D}^\circ$, the space of connections $\mathcal{A}(\mathcal{V})$ on the vector space $\mathcal{V}$ is an *affine* space and isomorphic to $\Omega^1(\mathcal{M}, \mathrm{End}(\mathcal{V}))$

Local form: Connection coefficients and matrix connection The introduced function $\omega^I_{\mu J} \in \Gamma(\mathcal{U})$ are called the *coneection coefficients* and we observe equivalently (425)(i) and (425)(ii)

$$\text{(i)} \quad \mathbf{D}_{\partial_\mu}(\mathbf{e}_I) = \omega^I_{\mu J}\mathbf{e}_I \qquad \text{(ii)} \quad \mathbf{D}(\mathbf{e}_J) = \omega^I_{\mu J} dx^\mu \otimes \mathbf{e}_I \tag{425}$$

Then, the connection matrix, related to the connection $\mathbf{D}$, with respect to the local trivialisation is given by a matrix of 1-forms defined by:

$$\text{(i)} \quad \omega^I{}_J = \omega^I_{\mu J} dx^\mu \qquad \text{(ii)} \quad \mathbf{D}(\mathbf{e}_J) = \omega^I_J \otimes \mathbf{e}_I \tag{426}$$

From these consideration and from lemma B.1, we now can express the section 1-form $\mathbf{D}\sigma \in \Gamma(\mathcal{V}) \otimes \Omega^1(\mathcal{M})$ as:

$$\mathbf{D}\sigma = \big(d\sigma^I + \omega^I_J \sigma^J\big) \otimes \mathbf{e}_I \tag{427}$$

We recall that the *standard flat connection*, denoted as $\mathbf{D}^\circ : \mathcal{X}(\mathcal{M}) \otimes \Gamma(\mathcal{V}) \longrightarrow \Gamma(\mathcal{V})$ such that $\mathbf{D}^\circ_X \sigma = X(\sigma) = X(\sigma^I)\mathbf{e}_I$ is not canonical since it depends on local coordinates on fibers - it is trivialization dependent. However, with the help of proposition B.1, we state that any connection $\mathbf{D}$ on $\mathcal{V}$ writes $\mathbf{D} = \mathbf{D}^\circ + \omega$ with $\omega \in \Omega^1(\mathcal{M}, \mathrm{End}(\mathcal{V}))$ being the *matrix connection*.

Definition B.4.2. *The exterior covariant derivative* $d^{\mathbf{D}}$ *is the operator that allow for prolongation of the operator* $\mathbf{D}$ *as a graded derivation of the algebra* $\bigoplus_n \mathcal{V} \otimes \Omega^n(\mathcal{M})$.

$$d^{\mathbf{D}} : \Gamma(\mathcal{V}) \otimes \Omega^n(\mathcal{M}) = \Omega^n(\mathcal{M}, \mathcal{V}) \longrightarrow \Gamma(\mathcal{V}) \otimes \Omega^{n+1}(\mathcal{M}) = \Omega^{n+1}(\mathcal{M}, \mathcal{V}) \tag{428}$$

[121] More precisely, given a basis of local section $\mathbf{e}_I$, defined in (423) these expressions are dependant of the trivialization ψ so that we shall write $\sigma = \sigma|_{\mathcal{U}}$ and $\xi = \xi|_{\mathcal{U}}$. We speak about the local form of the connection

It is defined, for any $\sigma \in \Gamma(\mathcal{V})$ and for any $\eta \in \Omega^n(\mathcal{M})$ by the following:

$$d^{\mathbf{D}}(\eta \otimes \sigma) = d\eta \otimes \sigma + (-1)^n \eta \wedge \mathbf{D}(\sigma) \tag{429}$$

Thus, it fulfills the Leibniz property (430)(i) and the induced property (430)(ii). For any $\lambda \in \Gamma(\mathcal{V}) \otimes \Omega^p(\mathcal{M}) = \Omega^n(\mathcal{M}, \mathcal{V})$ and $\eta \in \Omega^n(\mathcal{M})$ we define

$$\text{(i)} \quad d^{\mathbf{D}}(f\lambda) = df \wedge \lambda + f d^{\mathbf{D}} \lambda \qquad \text{(ii)} \quad d^{\mathbf{D}}(\lambda \wedge \eta) = d^{\mathbf{D}}\lambda \wedge \eta + (-1)^q \lambda \wedge d\eta \tag{430}$$

The figure below visualize the prolongation of the operator $\mathbf{D}$ as a graded derivation of the algebra $\bigoplus_n \mathcal{V} \otimes \Omega^n(\mathcal{M})$.

$$\begin{array}{ccc} \Gamma(\mathcal{V}) \otimes \Omega^n(\mathcal{M}) & \xrightarrow{d^{\mathbf{D}}} & \Gamma(\mathcal{V}) \otimes \Omega^{n+1}(\mathcal{M}) \\ \updownarrow & & \updownarrow \\ \Gamma(\mathcal{V}) & \xrightarrow[\mathbf{D}]{} & \Gamma(\mathcal{V}) \otimes \Omega^1(\mathcal{M}) \end{array} \qquad \begin{array}{ccc} \Omega^n(\mathcal{M}, \mathcal{V}) & \xrightarrow{d^{\mathbf{D}}} & \Omega^{n+1}(\mathcal{M}, \mathcal{V}) \\ \updownarrow & & \updownarrow \\ \Omega^0(\mathcal{M}, \mathcal{V}) & \xrightarrow[\mathbf{D}]{} & \Omega^1(\mathcal{M}, \mathcal{V}) \end{array}$$

For example, if we denote by $\mathbf{v}_I$ a basis of the vector space $\mathcal{V}$, then, the covariant derivative of $\varphi = \varphi^I \otimes \mathbf{v}_I$ is given by: $d^{\mathbf{D}}\varphi = d^{\mathbf{D}}(\varphi^I \otimes \mathbf{v}_I) = d^{\mathbf{D}}(\varphi^I \mathbf{v}_I) = \mathcal{D}\varphi^I \mathbf{v}_I$. The target space for n-form values, can be the tensor product $\boldsymbol{\mathcal{V}} = \bigotimes \mathcal{V} \otimes \bigotimes \mathcal{V}^\star$. Then, we consider elements $\boldsymbol{\varphi} \in \Gamma(\boldsymbol{\mathcal{V}}) \otimes \Omega^n(\mathcal{M})$.

$$\boldsymbol{\varphi} = \mathbf{v}_{I_1} \otimes ... \otimes \mathbf{v}_{I_p} \otimes \mathbf{v}^{J_1} \otimes ... \otimes \mathbf{v}^{J_q} \frac{1}{n!} (\boldsymbol{\varphi}^{I_1 ... I_p}_{J_1 ... J_q})_{\mu_1 ... \mu_n} dx^{\mu_1} \wedge ... \wedge dx^{\mu_n} \tag{431}$$

Which may alternatively writes $\boldsymbol{\varphi} = \big(\mathbf{v}_{I_1} \otimes ... \otimes \mathbf{v}_{I_p} \otimes \mathbf{v}^{J_1} \otimes ... \otimes \mathbf{v}^{J_q}\big) \boldsymbol{\varphi}^{I_1 ... I_p}_{J_1 ... J_q}$, in this case we forget about the indices related to the form part. Following this line, one shall write the exterior covariant derivative as:

$$d^{\mathbf{D}}\boldsymbol{\varphi} = \mathbf{v}_{I_1} \otimes ... \otimes \mathbf{v}_{I_p} \otimes \mathbf{v}^{J_1} \otimes ... \otimes \mathbf{v}^{J_q} \mathcal{D} \boldsymbol{\varphi}^{I_1 ... I_p}_{J_1 ... J_q} \tag{432}$$

where

$$\mathcal{D}\boldsymbol{\varphi}^{I_1 ... I_p}_{J_1 ... J_q} = d\boldsymbol{\varphi}^{I_1 ... I_p}_{J_1 ... J_q} + \sum_{1 \le \lambda \le p} \omega^{I_\lambda}_K \wedge \boldsymbol{\varphi}^{I_1 ... I_{\lambda-1} K I_{\lambda+1} ... I_p}_{J_1 ... J_q} - \sum_{1 \le \lambda \le q} \omega^K_{J_\lambda} \wedge \boldsymbol{\varphi}^{I_1 ...... I_p}_{J_1 ... J_{\lambda-1} K J_{\lambda+1} J_q}$$

⌜ Let us apply $d^{\mathbf{D}}$ on $\boldsymbol{\varphi}$ then:

$$d^{\mathbf{D}}\boldsymbol{\varphi} = d^{\mathbf{D}}\Big(\big(\bigotimes_{1 \le \lambda \le q} \mathbf{v}_{I_\lambda} \otimes \bigotimes_{1 \le \lambda \le p} \mathbf{v}^{J_\lambda}\big) \wedge \boldsymbol{\varphi}^{I_1 ... I_p}_{J_1 ... J_q}\Big) = \underbrace{d^{\mathbf{D}}\big(\bigotimes_{1 \le \lambda \le q} \mathbf{v}_{I_\lambda} \otimes \bigotimes_{1 \le \lambda \le p} \mathbf{v}^{J_\lambda}\big) \wedge \boldsymbol{\varphi}^{I_1 ... I_p}_{J_1 ... J_q}}_{[\mathbf{I}]} + \big(\bigotimes_{1 \le \lambda \le q} \mathbf{v}_{I_\lambda} \otimes \bigotimes_{1 \le \lambda \le p} \mathbf{v}^{J_\lambda}\big) d\boldsymbol{\varphi}^{I_1 ... I_p}_{J_1 ... J_q}$$

Developing, one gets $[\mathbf{I}]$

$$[\mathbf{I}] = \Big\{ \sum_{1 \le \lambda \le p} \big(\mathbf{v}_{I_1} \otimes ... \otimes \mathbf{v}_{I_{\lambda-1}} \otimes d^{\mathbf{D}}(\mathbf{v}_{I_\lambda}) \otimes \mathbf{v}_{I_{\lambda+1}} ... \otimes \mathbf{v}_{I_p}\big) \otimes \sum_{1 \le \lambda \le q} \big(\mathbf{v}^{J_1} \otimes ... \otimes \mathbf{v}^{J_{\lambda-1}} \otimes d^{\mathbf{D}}(\mathbf{v}^{J_\lambda}) \otimes \mathbf{v}^{J_{\lambda+1}} ... \otimes \mathbf{v}^{J_q}\big) \Big\} \wedge \boldsymbol{\varphi}^{I_1 ... I_p}_{J_1 ... J_q}$$

Since $d^{\mathbf{D}}(\mathbf{v}_{I_\lambda}) = \omega^K_{I_\lambda} \mathbf{v}_K$ and $d^{\mathbf{D}}(\mathbf{v}^{J_\lambda}) = \omega^{J_\lambda}_K \mathbf{v}^K$, therefor one obtains:

$$[\mathbf{I}] = \Big\{ \sum_{1 \le \lambda \le p} \big(\mathbf{v}_{I_1} \otimes ... \otimes \mathbf{v}_{I_{\lambda-1}} \otimes (\omega^K_{I_\lambda} \mathbf{v}_K) \otimes \mathbf{v}_{I_{\lambda+1}} ... \otimes \mathbf{v}_{I_p}\big) \otimes \sum_{1 \le \lambda \le q} \big(\mathbf{v}^{J_1} \otimes ... \otimes \mathbf{v}^{J_{\lambda-1}} \otimes (\omega^{J_\lambda}_K \mathbf{v}^K) \otimes \mathbf{v}^{J_{\lambda+1}} ... \otimes \mathbf{v}^{J_q}\big) \Big\} \wedge \boldsymbol{\varphi}^{I_1 ... I_p}_{J_1 ... J_q}$$

Finally:

$$d^{\mathbf{D}}\boldsymbol{\varphi} = \{ d\boldsymbol{\varphi}^{I_1 ... I_p}_{J_1 ... J_q} + \sum_{1 \le \lambda \le p} \omega^{I_\lambda}_K \wedge \boldsymbol{\varphi}^{I_1 ... I_{\lambda-1} K I_{\lambda+1} ... I_p}_{J_1 ... J_q} - \sum_{1 \le \lambda \le q} \omega^K_{J_\lambda} \wedge \boldsymbol{\varphi}^{I_1 ...... I_p}_{J_1 ... J_{\lambda-1} K J_{\lambda+1} J_q} \} \big(\bigotimes_{1 \le \lambda \le q} \mathbf{v}_{I_\lambda} \otimes \bigotimes_{1 \le \lambda \le p} \mathbf{v}^{J_\lambda}\big)$$

The key point $d^{\mathbf{D}} \circ d^{\mathbf{D}} = (d^{\mathbf{D}})^2$ is non vanishing in general so that, as already emphasized, it exhibits the failure to be a complex: the measure of this obstruction is precisely what is encoded by the *curvature*. Let notice also that we shall consider vector-valued n-form in the case of the endomorphism space $\boldsymbol{\mathcal{V}} = \mathrm{End}(\mathcal{V}) = \mathcal{V} \times \mathcal{V}^\star$. Then, in this case we consider $\boldsymbol{\varphi} = \mathrm{End}(\mathcal{V}) \otimes \Omega^n(\mathcal{M}) = \Omega^n(\mathcal{M}, \mathrm{End}(\mathcal{V}))$. These two ingredients allow us to picture the curvature on a vector bundle $\mathcal{V}$.

Curvature of a connection on a vector bundle. We can picture it as an application[122] $F : \Lambda^2(\mathcal{X}(\mathcal{M})) \otimes \Gamma(\mathcal{V}) \longrightarrow \Gamma(\mathcal{V})$ or equivalently $F : \Lambda^2(\mathcal{X}(\mathcal{M})) \longrightarrow \mathrm{End}(\Gamma(\mathcal{V}))$ defined for all $X, Y \in \mathcal{X}(\mathcal{M})$ by:

$$F(X,Y) = [\mathbf{D}_X, \mathbf{D}_Y] - \mathbf{D}_{[X,Y]} \tag{433}$$

From this standpoint we observe that F satisfy the following linearity properties. (i) The application $F : \Lambda^2(\mathcal{X}(\mathcal{M})) \longrightarrow \mathrm{End}(\Gamma(\mathcal{V}))$ then it is is $C^\infty(\mathcal{M})$-bilinear. (ii) The application $F(X,Y) : \Gamma(\mathcal{V}) \longrightarrow \Gamma(\mathcal{V})$ is $C^\infty(\mathcal{M})$-linear. Linearity property (i) and (ii) leads us to picture the curvature as a $C^\infty(\mathcal{M})$-linear application which takes values in $\mathrm{End}_{C^\infty(\mathcal{M})}\big(\Gamma(\mathcal{V})\big) \cong \Gamma(\mathrm{End}(\mathcal{V}))$. As emphasized in [73], curvature is generated by a bundle morphism, we have:

$$F \in \Omega^2(\mathcal{M}, \mathrm{End}(\mathcal{V})) = \Omega^2(\mathcal{M}) \otimes_{C^\infty(\mathcal{M})} \Gamma(\mathrm{End}(\mathcal{V})) \tag{434}$$

Local coordinates The matrix curvature is the 2-form $\Omega^2(\mathcal{M}, \mathrm{End}(\mathcal{V}))$ is locally a matrix with coefficient $\Omega^I_J = \frac{1}{2} F^I_{\mu\nu J} dx^\mu \wedge dx^\nu$. Finally, we observe that $F(\mathbf{e}_J) = \Omega^I_J \otimes \mathbf{e}_J$. From this standpoint, we find again an avatar of Cartan structure equation (417) with the following proposition.

Proposition B.2. *The connection matrix and the curvature matrix are related by the Cartan structure equation*

$$\Omega^I_J = d\omega^I_J + \omega^I_K \wedge \omega^K_J \qquad \Longleftrightarrow \qquad \Omega = d\omega + \omega \wedge \omega \tag{435}$$

In this case we exhibit the Bianchi identity under the following form

$$d\Omega^I_J = \Omega^I_K \wedge \omega^K_J - \omega^I_K \wedge \Omega^K_J \qquad \Longleftrightarrow \qquad d\Omega = \Omega \wedge \omega - \omega \wedge \Omega \tag{436}$$

Proposition B.3. *The curvature of the connection* $\mathbf{D}$*, seen as an element* $F \in \Omega^2(\mathcal{M}, End(\mathcal{V}))$*, measures the obstruction of the covariant derivative to be a differential, the later being given by* $d^{\mathbf{D}} : \Omega^0(\mathcal{M}, \mathcal{V}) \longrightarrow \Omega^2(\mathcal{M}, \mathcal{V})$*, see definition* (B.4.2)*, . Therefore, for any section* $\sigma \in \Omega^0(\mathcal{M}, \mathcal{V}) = \Gamma(\mathcal{V})$ *we might define the curvature:*

$$F = d^{\mathbf{D}} \circ d^{\mathbf{D}} = F \wedge \sigma = F(\sigma) \tag{437}$$

⌜ Proof Let $\sigma = \sigma^I \mathbf{e}_I \in \Omega^0(\mathcal{M}, \mathcal{V})$ a section of the vector bundle $\mathcal{V}$. We have, in local coordinates:

$$d^{\mathbf{D}} \circ d^{\mathbf{D}}(\sigma) = d^{\mathbf{D}}(d\sigma^I \otimes \mathbf{e}_I) + d^{\mathbf{D}}(\sigma^J \omega^I_J \otimes \mathbf{e}_I) = (d \circ d\sigma^I) \otimes \mathbf{e}_I - d\sigma^J \wedge d^{\mathbf{D}}(\mathbf{e}_J) + d(\sigma^J \omega^I_J) \otimes \mathbf{e}_I - \sigma^J \omega^K_J \wedge d^{\mathbf{D}}(\mathbf{e}_K)$$

Since, $(d \circ d\sigma^I) \otimes \mathbf{e}_I = 0$ and also $d^{\mathbf{D}}(\mathbf{e}_J) = \omega^I_J \otimes \mathbf{e}_I$ as well as $d^{\mathbf{D}}(\mathbf{e}_K) = \omega^I_K \otimes \mathbf{e}_I$, we obtain:

$$d^{\mathbf{D}} \circ d^{\mathbf{D}}(\sigma) = -d\sigma^J \wedge \omega^I_J \otimes \mathbf{e}_I + (d\sigma^J \wedge \omega^I_J + \sigma^J d\omega^I_J) \otimes \mathbf{e}_I - \sigma^J \omega^K_J \wedge \omega^I_J \otimes \mathbf{e}_I = \big(-d\sigma^J \wedge \omega^I_J + d\sigma^J \wedge \omega^I_J + \sigma^J d\omega^I_J - \sigma^J \omega^K_J \wedge \omega^I_K\big) \otimes \mathbf{e}_I$$

Finally we obtain:

$$d^{\mathbf{D}} \circ d^{\mathbf{D}}(\sigma) = \sigma^J \big(d\omega^I_J + \omega^I_J \wedge \omega^K_J\big) \otimes \mathbf{e}_I = \sigma^J \Omega^I_J \otimes \mathbf{e}_I = F(\sigma) \quad ⌟$$

Let notice that with the decomposition $\sigma = \sigma^I \mathbf{e}_I$ we have therefore written $d^{\mathbf{D}} \circ d^{\mathbf{D}}(\sigma)$ under the following form:

$$d^{\mathbf{D}} \circ d^{\mathbf{D}}(\sigma) = \big(d\omega^I_J + \omega^I_K \wedge \omega^K_J \sigma^J\big) \mathbf{e}_I \tag{438}$$

Finally, before we process the next two steps, concerning basics forms (see section B.5) and connections on associated bundle (see section B.6), we first notice that the connection $\mathbf{D}$ on the vector bundle $\mathcal{V}$ induce a connection $\mathbf{D}^{\mathrm{End(V)}} = \mathbf{D}^{\boldsymbol{\mathcal{V}}}$ on $\mathrm{End}(\mathcal{V}) \cong \mathcal{V} \otimes \mathcal{V}^\star = \boldsymbol{\mathcal{V}}$. We have the following proposition:

[122] Let notice that if we would like to be more precise, we denote $F^{\mathbf{D}}$ such curvature application, to remember that it arise from the connection $\mathbf{D}$ on the vector bundle $\mathcal{V}$.

Proposition B.4. *The connection* $\mathbf{D}$ *on* $\mathcal{V}$ *induce a connection* $\mathbf{D}^{End(V)}$ *on* $End(\mathcal{V})$ *alternatively described as* $\mathbf{D}^{End(V)} : \mathcal{X}(\mathcal{M})\otimes\Gamma(End(\mathcal{V})) \longrightarrow \Gamma(End(\mathcal{V}))$ *or as* $\mathbf{D}^{End(V)} : \Gamma(End(\mathcal{V})) \longrightarrow \Omega^1(\mathcal{M}, End(\mathcal{V}))$ *such that for any* $X \in \mathcal{X}(\mathcal{M})$, $\sigma \in \Gamma(\mathcal{V})$ *and* $\varphi \in End(\mathcal{V})$, *we have*

$$\mathbf{D}_X^{End(V)}(\varphi)(\sigma) = \mathbf{D}_X(\varphi(\sigma)) - \varphi(\mathbf{D}_X(\sigma)) \tag{439}$$

The proposition B.4 equivalently states that the operator $\mathbf{D}^{\mathrm{End(V)}} : \Omega^0(\mathcal{M}, \mathrm{End}(\mathcal{V})) \longrightarrow \Omega^0(\mathcal{M}, \mathrm{End}(\mathcal{V}))$, defined by $\mathbf{D}^{\mathrm{End(V)}}\varphi = \mathbf{D}\circ\varphi - \varphi\circ\mathbf{D} = [\mathbf{D}, \varphi]$ is a linear connection on $\mathrm{End}(\mathcal{V})$. Finally, we conclude with three remarks:

(i) Working in local coordinates with a basis of local section of $\boldsymbol{\mathcal{V}}$ given by $\mathbf{e}_I\otimes\mathbf{e}^J$, the connection coefficients related to $\mathbf{D}^{\mathrm{End(V)}} = \mathbf{D}^{\boldsymbol{\mathcal{V}}}$ are given $\left[\omega^{\boldsymbol{\mathcal{V}}}\right]^{KI}_{\mu LJ} = \delta^K_J\omega^I_{\mu L} - \delta^I_L\omega^K_{\mu J}$.

(ii) The connection matrix is given: $\left(\omega^{\boldsymbol{\mathcal{V}}}\right)^{KI}_{LJ} = \delta^K_J\omega^I_L - \delta^I_L\omega^K_J$

(iii) Also, the connection $\mathbf{D}$ on the vector space $\mathcal{V}$ induce a covariant exterior derivative on $\boldsymbol{\mathcal{V}} \cong \mathrm{End}(\mathcal{V})$, denoted $d^{\mathbf{D}^{\mathrm{End}(\mathcal{V})}} : \Omega^n(\mathcal{M}, \mathrm{End}\mathcal{V}) \longrightarrow \Omega^{n+1}(\mathcal{M}, \mathrm{End}\mathcal{V})$. We define it for any $\varphi \in \Gamma(\mathrm{End}(\mathcal{V}))$ and for any $\lambda \in \Omega^n(\mathcal{M})$ via:

$$d^{\mathbf{D}^{\mathrm{End}(\mathcal{V})}}(\lambda\otimes\varphi) = d\lambda\otimes\varphi + (-1)^n\lambda\wedge d^{\mathbf{D}^{\mathrm{End}(\mathcal{V})}}(\varphi) \tag{440}$$

Since $F \in \Omega^2(\mathcal{M}, \mathrm{End}(\mathcal{V}))$, the previous remarks allow us to exhibit the covariant Bianchi identity: $d^{\mathbf{D}^{\mathrm{End}(\mathcal{V})}}F = 0$

B.5 Horizontal, invariant and Basic forms

Now we want to extend the notion of horizontal forms to the notion of *basic forms*. This means that we consider horizontal forms with an additional equivariance property.

Definition B.5.1. *Basic form* $\Omega^n_\rho(\mathcal{P}, \mathcal{V})$. *Let us consider a principal fiber bundle* $\mathcal{P}(\mathcal{M}, G)$ *over* $\mathcal{M}$ *with gauge group* G. *Let* $\mathcal{V}$ *be a vector space and let* $\Omega^n(\mathcal{P}, \mathcal{V}) = \Omega^n(\mathcal{P})\otimes\mathcal{V}$ *the set of* $\mathcal{V}$*-valued* n*-forms on* $\mathcal{P}$. *The vector space endow a representation* $\rho : G \longrightarrow GL(\mathcal{V})$. *A form is called basic if it is horizontal and invariant for all* $g \in G$. *The set of basics forms on* $\mathcal{P}$, *denoted as* $\Omega^n_\rho(\mathcal{P}, \mathcal{V})$, *is then described by:*

$$\Omega^n_\rho(\mathcal{P}, \mathcal{V}) = \{\boldsymbol{\varphi} \in \Omega^n(\mathcal{P}, \mathcal{V}) \quad / \quad \mathbf{h}^\star\boldsymbol{\varphi} = \boldsymbol{\varphi} \quad \textit{and} \quad R^\star_g\boldsymbol{\varphi} = \rho_{(g^{-1})}\boldsymbol{\varphi}\} \tag{441}$$

From this perspective, let notice that the connection 1-form $\boldsymbol{\omega} \in \Omega^1(\mathcal{P}, \mathfrak{g})$ is not a basic form. It satisfies the property of equivariance, as it has been emphasized above, (it is a G-equivariant form of type $(\mathrm{Ad}, \mathfrak{g})$) however it is clearly from the very definition a *vertical* form which vanishes as soon as one vector on which it acts is horizontal. This allows us to picture the notion of curvature from the light of basic form $\Omega^n_\rho(\mathcal{P}, \mathcal{V})$. For any $\boldsymbol{\varphi} \in \Omega^n_\rho(\mathcal{P}, \mathcal{V})$ of type $(\rho, \mathcal{V})$ and ρ is also the induce representation of $\mathfrak{g}$.

$$\left(\mathbf{D}^{\boldsymbol{\omega}}\right)^2\boldsymbol{\varphi} = \mathbf{D}^{\boldsymbol{\omega}}\mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\varphi} = \rho(\boldsymbol{\Omega})\wedge\boldsymbol{\varphi} \tag{442}$$

The key result is that we have a one to one correspondence between form $\boldsymbol{\varphi} \in \Omega^n_\rho(\mathcal{P}, \mathcal{V})$ and differential form defined on $\mathcal{M}$ with values in the associated bundle $\mathcal{P}\times_\rho\mathcal{V}$. This point exhibit the canonical isomorphism $\Omega^n_\rho(\mathcal{P}, \mathcal{V}) \cong \Omega^n(\mathcal{M}, \mathcal{P}\times_\rho\mathcal{V})$. We consider the zero degree step. In this case we have an isomorphism between G-equivariant functions and sections of $\mathcal{P}\times_\rho\mathcal{V}$. In a first place, we are more interested by the covariant exterior derivative on *basic forms*, namely on form $\boldsymbol{\varphi} \in \Omega^n_\rho(\mathcal{P}, \mathcal{V})$.

Exterior covariant derivative on $\boldsymbol{\varphi} \in \Omega^n_\rho(\mathcal{P},\mathcal{V})$ We denote $\rho : G \longrightarrow GL(\mathcal{V})$ a representation of G on the vector space $\mathcal{V}$. Equivalently, we picture a group homomorphism, where $GL(\mathcal{V})$ is the group of invertible endomorphisms of $\mathcal{V}$. We denote $\rho_{\mathfrak{g}} = \rho$ the induced representation on the Lie algebra: $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathcal{V})$[123]. Notice that the expression of $\mathbf{D}^{\boldsymbol{\omega}}$ on $\boldsymbol{\varphi} \in \Omega^n_\rho(\mathcal{P},\mathcal{V})$ is given by:

$$\mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\varphi} = d\boldsymbol{\varphi} + \rho(\boldsymbol{\omega}) \wedge \boldsymbol{\varphi} \in \Omega^{n+1}_\rho(\mathcal{P},\mathcal{V}) \tag{443}$$

The product $\rho(\boldsymbol{\omega}) \wedge \boldsymbol{\varphi}$ is the one defined generally by (406). In this case, we denote the exterior covariant derivative on basic form $\boldsymbol{\varphi} \in \Omega^n_\rho(\mathcal{P},\mathcal{V})$

$$\mathbf{D}^{\boldsymbol{\omega}} : \Omega^n_\rho(\mathcal{P},\mathcal{V}) \longrightarrow \Omega^{n+1}_\rho(\mathcal{P},\mathcal{V}) : \boldsymbol{\varphi} \mapsto \mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\varphi} = \mathbf{h}^\star(d\boldsymbol{\varphi}) \tag{444}$$

For example, let $\boldsymbol{\varphi} = \mathbf{v}_J \otimes \boldsymbol{\varphi}^J$ and $\boldsymbol{\omega} \in \Omega^1(P,\mathfrak{g})$ the connection such that $\boldsymbol{\omega} = \boldsymbol{\omega}^{\mathcal{I}} \otimes \mathfrak{b}_{\mathcal{I}}$, so that the product (406) writes in this case:

$$\rho(\boldsymbol{\omega}) \wedge \boldsymbol{\varphi} = \rho(\boldsymbol{\omega}^{\mathcal{I}} \otimes \mathfrak{b}_{\mathcal{I}}) \wedge (\mathbf{v}_J \otimes \boldsymbol{\varphi}^J) = \rho(\mathfrak{b}_{\mathcal{I}})\mathbf{v}_J \otimes \boldsymbol{\omega}^{\mathcal{I}} \wedge \boldsymbol{\varphi}^J \tag{445}$$

$\rho(\boldsymbol{\omega})$ is the image of the connection form $\boldsymbol{\omega}$ by the induced representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathcal{V})$. The important case for later purpose is the case of forms with values in the Lie algebra $\mathfrak{g}$ of the structure group G. Hence we consider, forms of the type $\boldsymbol{\varphi} \in \Omega_\rho(\mathcal{P},\mathfrak{g})$ which is equivalent to $\boldsymbol{\varphi}$ as a basic form of type $(\mathrm{Ad},\mathfrak{g})$. The relation (443) writes in this case

$$\mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\varphi} = d\boldsymbol{\varphi} + [\boldsymbol{\omega},\boldsymbol{\varphi}] \tag{446}$$

Let us give a glimpse on the important point that will appear in the following. If we consider φ a n-form $\varphi \in \Omega^n(\mathcal{M},\mathcal{P}\times_\rho\mathcal{V})$ we are able to represent it with its related basic form $\boldsymbol{\varphi} \in \Omega^n_G(\mathcal{P},\mathcal{V})$. The object $\mathbf{D}^{\boldsymbol{\omega}}\boldsymbol{\varphi}$ canonically define an element $d^{\mathbf{D}}\varphi \in \Omega^{n+1}(\mathcal{M},\mathcal{P}\times_\rho\mathcal{V})$. The exterior covariant derivative $d^{\mathbf{D}} : \Omega^n(\mathcal{M},\mathcal{P}\times_\rho\mathcal{V}) \longrightarrow \Omega^{n+1}(\mathcal{M},\mathcal{P}\times_\rho\mathcal{V})$ is therefore understood by the commutativity of figure (447)

$$\begin{array}{ccc} \Omega^n_G(\mathcal{P},\mathcal{V}) & \xrightarrow{\mathbf{D}^{\boldsymbol{\omega}} = \mathbf{h}^\star d} & \Omega^{n+1}_G(\mathcal{P},\mathcal{V}) \\ \updownarrow & & \updownarrow \\ \Omega^n(\mathcal{M},\mathcal{P}\times_\rho\mathcal{V}) & \xrightarrow[d^{\mathbf{D}}]{} & \Omega^{n+1}(\mathcal{M},\mathcal{P}\times_\rho\mathcal{V}) \end{array} \tag{447}$$

Later on, we turn back on this equivalence of description later on, and especially within the interpretation of *gauge field* and *gauge group*. For the moment, since in this section we introduced several time the associated bundle $\mathcal{P}\times_\rho\mathcal{V}$ let us explore this notion and the related concept of curvature on associated bundles.

B.6 Connection and Curvature on associated bundle

The key point related to the utilization of associated bundle lay in the description of *matter fields* when stand the central use of a space of linear representation of the Lie group G. We would like that the Lie group now acts on such a linear representation vector space $\mathcal{V}$. The usual theory of connection in the general setting of *fiber bundle* leads us to consider $\mathcal{E} = \mathcal{P}\times_\rho\mathcal{V}$ an associated fiber bundle via the representation ρ on the representation - vector - space $\mathcal{V}$. An associated bundle is defined given a principal bundle $\mathcal{P}(\mathcal{M},G)$, whereas G acts on $\mathcal{V}$ on the left. Then, $\mathcal{P}\times_\rho\mathcal{V}$ is

[123] $\mathfrak{gl}(\mathcal{V})$ is the Lie algebra of $\mathcal{V}$-endomorphism

described by to any $g \in G$ and $(p, v) \in \mathcal{P} \times \mathcal{V}$ it associates the element $(pg, \mathsf{g}^{-1}v) \in \mathcal{P} \times_\rho \mathcal{V}$. Then, $\mathcal{E} = \mathcal{P} \times_\rho \mathcal{V}$ as associated fiber bundle is an equivalence class $\mathcal{P} \times_\rho \mathcal{V}/G$ where the points (p, g) and $(pg, \mathsf{g}^{-1}v)$ are identified. Once again, for a more detailed treatment we refer to [45, 149, 176].

Connection on associated bundle. The section is directly inspired from R. Coquereaux [46]. In this section, we denote $\{\mathfrak{b}_\mathcal{I}\}_{1\leq\mathcal{I}\leq n}$ is a basis of $\mathfrak{g}$. The structure constants of the Lie algebra $\mathfrak{g}$ are $\mathfrak{c}^\mathcal{K}_{\mathcal{I}\mathcal{J}}$ such that: $\left[\mathfrak{b}_\mathcal{I}, \mathfrak{b}_\mathcal{J}\right] = \mathfrak{c}^\mathcal{K}_{\mathcal{I}\mathcal{J}}\mathfrak{b}_\mathcal{K}$. Also, as announced above, let $\rho_G : G \to GL(\mathcal{V}) : g \mapsto \rho_g$ be a representation of the group G associated to Lie algebra $\mathfrak{g}$, on a vector space $\mathcal{V}$. Here $GL(\mathcal{V})$ is the invertible endomorphism on $\mathcal{V}$. Equivalently, one observes it as a Lie algebra homomorphism $\rho_\mathfrak{g} : \mathfrak{g} \to \mathfrak{gl}(\mathcal{V})$ where $\mathfrak{gl}(\mathcal{V})$ is the Lie algebra of $\mathcal{V}$-endomorphism endowed with the Lie bracket operation[124]. We also denote $\rho_\mathfrak{g}(\mathfrak{b}_\mathcal{I}) = \rho(\mathfrak{b}_\mathcal{I}) = (\rho(\mathfrak{b}_\mathcal{I}))^I_J$. It is the matrix viewpoint. In a basis of $\mathcal{V}$: the object $\rho_\mathfrak{g}(\mathfrak{b}_\mathcal{I})$ is the matrix that encodes the endomorphism on $\mathcal{V}$, that is $\rho(\mathfrak{b}_\mathcal{I}) \in \mathrm{End}(\mathcal{V})$. The image $\rho(\omega)$ of the gauge potential ω via the representation ρ gives the matrix connection:

$$\rho(\omega) = \rho(\mathfrak{b}_\mathcal{I}\omega^\mathcal{I}_\mu dx^\mu) = \omega^\mathcal{I}_\mu\rho(\mathfrak{b}_\mathcal{I})dx^\mu = \omega^\mathcal{I}_\mu(\rho(\mathfrak{b}_\mathcal{I}))^I_J dx^\mu \tag{448}$$

Here the object of interested are $(\rho(\mathfrak{b}_\mathcal{I}))^I_J$: it is a matrix of dimension n that characterize an endomorphism on the vector space $\mathcal{V}$. Since we can write $\omega^\mathcal{I} = \omega^\mathcal{I}_\mu dx^\mu$ so that we can alternatively use the notation $\rho(\omega) = \omega^\mathcal{I}(\rho(\mathfrak{b}_\mathcal{I}))^I_J = \rho(\omega)^I{}_J$. The point here is to obseve that $\rho(\omega) \in \mathfrak{gl}(\mathcal{V})$. Then, we denote $\mathfrak{J}^{\mathfrak{gl}(\mathcal{V})}_\mathcal{I} = \mathfrak{J}_\mathcal{I} = \rho(\mathfrak{b}_\mathcal{I})$. $\{\mathfrak{J}_\mathcal{I}\}_{1\leq\mathcal{I}\leq n}$ represents the basis elements of the Lie algebra $\mathfrak{gl}(\mathcal{V})$. Then, following our previous discussion we equivalently write $\omega^I_J = \omega^\mathcal{I}(\rho(\mathfrak{b}_\mathcal{I}))^I_J = \omega^\mathcal{I}(\mathfrak{J}_\mathcal{I})^I_J$. Working in one fixed representation, we now omit the reference to it, writing simply its component $\omega^I{}_J$ rather than the full description given in (448), namely ($\rho(\omega)^I_J = \omega^\mathcal{I}_\mu(\rho(\mathfrak{b}_\mathcal{I}))^I_J dx^\mu$). The matrix elements denoted now by $\omega^I_J = \omega^I_{\mu J}dx^\mu$ with $\omega^I_{\mu J} = \omega^\mathcal{I}_\mu(\mathfrak{J}_\mathcal{I})^I_J$, we can write $\omega^\mathcal{I}_\mu$ as:

$$\omega^\mathcal{I}_\mu = \omega^I_{\mu J}(\mathfrak{J}^\mathcal{I})^J_I \tag{449}$$

Since one writes $\omega = \omega^\mathcal{I}_\mu dx^\mu \otimes \mathfrak{b}_\mathcal{I}$ and making use of (449), we obtain:

$$\omega = \omega^I_{\mu J}(\mathfrak{J}^\mathcal{I})^J_I dx^\mu \otimes \mathfrak{b}_\mathcal{I} = \mathfrak{b}_\mathcal{I}\omega^I_{\mu J}(\mathfrak{J}^\mathcal{I})^J_I dx^\mu \tag{450}$$

Now, in order to lighten the notation, we drop the indices $\mathcal{I}$ which concern the Lie algebra, and the local connection 1-form $\omega \in \Omega^1(\mathcal{M}, \mathfrak{g})$ is given by:

$$\omega = (\omega_\mu{}^I{}_J dx^\mu)\mathfrak{J}^J{}_I = \omega^I{}_J\mathfrak{J}^J{}_I \qquad \text{with } (\mathfrak{J}^J{}_I) \text{ a basis of } \mathfrak{gl}(\mathcal{V})$$

Then, an analogous notation is retained, we equivalently write (450) under the form:

$$\omega = \omega^{IJ}_\mu dx^\mu \otimes \mathfrak{J}_{IJ} = \mathfrak{J}_{IJ}\omega^{IJ}_\mu dx^\mu$$

Curvature on associated bundle. Let us consider $\mathcal{E} = \mathcal{P} \times_\rho \mathcal{V}$ the associated bundle of the principal bundle $\mathcal{P}(\mathcal{M}, G)$, though the representation ρ of G on the vector space $\mathcal{V}$. In this case, we describe $\boldsymbol{\omega} \in \Omega^1(\mathcal{P}, \mathfrak{g})$ now as $\rho(\boldsymbol{\omega})$ whereas the curvature form $\boldsymbol{\omega}$ is now wiritten $\rho(\boldsymbol{\omega})$ with value in the space $\mathrm{End}(\mathcal{V})$ which is written:

$$\rho(\boldsymbol{\omega}) = \rho(d\boldsymbol{\omega}) + \frac{1}{2}\rho(\boldsymbol{\omega}) \wedge \rho(\boldsymbol{\omega}) \tag{451}$$

[124]we have the relation $\forall\ \mathfrak{b}_\mathcal{I}, \mathfrak{b}_\mathcal{J}\ \rho_\mathfrak{g}([\mathfrak{b}_\mathcal{I}, \mathfrak{b}_\mathcal{J}]) = \rho_\mathfrak{g}(\mathfrak{b}_\mathcal{I})\rho_\mathfrak{g}(\mathfrak{b}_\mathcal{J}) - \rho_\mathfrak{g}(\mathfrak{b}_\mathcal{J})\rho_\mathfrak{g}(\mathfrak{b}_\mathcal{I})$

B.7 Group of automorphisms and gauge picture

Vertical automorphisms and gauge group $\mathbf{Gau}(\mathcal{P})$. The best way to grasp this picture is to consider two related diffeomorphisms $\varkappa$ and $\boldsymbol{\varkappa}$, the frist defined on the base m$\omega^I{}_J$ anifold $\mathcal{M}$ whereas the second is defined on the total space of the principal G-bundle $\mathcal{P}(\mathcal{M},\mathcal{G})$. We call a vertical automorphism of the principal fiber bundle $\mathcal{P}(\mathcal{M},G)$ $\boldsymbol{\varkappa}:\mathcal{P}\longrightarrow\mathcal{P}$ if the diffeomorphism $\varkappa:\mathcal{M}\longrightarrow\mathcal{M}$ is the identity. Then, we obtain $(\mathfrak{i})$: $\forall p\in\mathcal{P}$ and $\forall g\in G$, $\boldsymbol{\varkappa}(p)$ and p are in the same fiber and $(\mathfrak{ii})$ $\boldsymbol{\varkappa}(p\cdot g)=\boldsymbol{\varkappa}(p)\cdot g$, this correspond to the compatibility with the G-action on $\mathcal{P}$. This is equivalent to the data of an G-equivariant application $\psi:\mathcal{P}\longrightarrow G$ for the adjoint action ad. Since p and $\boldsymbol{\varkappa}(p)$ are in the same fiber, they are related by $\boldsymbol{\varkappa}(p)=p\cdot\psi(p)$ with $\psi(p)\in G$. After using relation $(\mathfrak{i})$, we now make use of the other relation, namely $(\mathfrak{ii})$. This one gives us $\boldsymbol{\varkappa}(p\cdot g)=(p\cdot g)\psi(p\cdot g)=\boldsymbol{\varkappa}(p)\cdot g=p\cdot\psi(p)\cdot g$ therefore $\psi(p\cdot g)=\mathsf{g}^{-1}\psi(p)g$ and obviously ψ is a G-invariant function. From the previous section, we know that this data is equivalently described by a section of the assiciated bundle $\mathcal{P}\times_{ad}G$.

Definition B.7.1. *The gauge group of the fiber bundle $\mathcal{P}(\mathcal{M},G)$, denoted* $\mathbf{Gau}(\mathcal{P})$ *the set of all vertical automorphisms of $\mathcal{P}(\mathcal{M},G)$.*

As emphasized [165], there is equivalence of the data of: a vertical automorphism $(\mathfrak{i})$ $\boldsymbol{\varkappa}^{\mathbf{V}}=\boldsymbol{\varkappa}:\mathcal{P}\longrightarrow\mathcal{P}$, $(\mathfrak{ii})$ $\psi:\mathcal{P}\longrightarrow G$, a G-equivariant differentiable application $(\mathfrak{iii})$ A differentiable section of the associated bundle $\mathcal{P}\times_{ad}G$. Another point of interested is the action of the *gauge group* on a connection $\boldsymbol{\omega}$. Here we do not enter into details (see [165, 176, 149]), however we drawn the important formula. The question under process is indeed the action of an element $\boldsymbol{\varkappa}\in\mathbf{Gau}(\mathcal{P})$ of the gauge group on the connection $\boldsymbol{\omega}\in\Omega^1(\mathcal{P},\mathfrak{g})$, we have:

$$\boldsymbol{\varkappa}^\star\boldsymbol{\omega}=\psi^{-1}\boldsymbol{\omega}\psi+\psi^{-1}d\psi \tag{452}$$

Later on, we will introduce the Maurer-Cartan form θ on G. We can see [165] that $\psi^{-1}d\psi$ identify with $\psi^\star\theta$ so that we can write (452) as $\boldsymbol{\varkappa}^\star\boldsymbol{\omega}=\mathrm{Ad}_{\psi^{-1}}+\psi^\star\theta$. Finally, we can evaluate the curvature $\boldsymbol{\varkappa}^\star\boldsymbol{\omega}$ of the new connection $\boldsymbol{\varkappa}^\star\boldsymbol{\omega}$. Then the curvature is given by $\boldsymbol{\varkappa}^\star\boldsymbol{\omega}=\mathrm{Ad}_{\psi^{-1}}\boldsymbol{\omega}$

B.8 Yang-Mills theory

With the knowledge of the previous section we are able now to describe briefly the vision of Gauge theory in fiber bundle framework. To introduce the Yang Mills theory, let us set the question in a more mathematical point of view. Yang Mills theory is concerned with the action (in vacuum).

$$S_{\mathsf{YM}}[A]=\frac{1}{2}\int_{\mathcal{X}}\mathrm{tr}(F\wedge\star F)\mathrm{vol}_{\mathcal{X}}(\mathsf{g}) \tag{453}$$

$\mathrm{vol}_{\mathcal{X}}(\mathsf{g})$ is a Riemannian volume form such that $\mathrm{vol}_{\mathcal{X}}(\mathsf{g})=\sqrt{-\mathsf{g}}\beta=\frac{1}{n!}\boldsymbol{\epsilon}_{\mu\nu\rho\sigma}dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma$. Here we consider the space-time manifold $\mathcal{X}=\mathbb{R}\times\Sigma$, a Minkowski manifold endowed with the metric $\mathsf{h}=diag(1,-1,-1,-1)$. Then $P=P(\mathcal{X},G)$ is a principal fiber bundle over $\mathcal{X}$ with group G that has a Lie algebra $\mathfrak{g}$. The theory involve also differential operators[125] such as the de Rham differential $d:\Omega^q(\mathcal{X})\to\Omega^{q+1}(\mathcal{X})$, the Hodge operator $\star:\Omega^q(\mathcal{X})\to\Omega^{4-q}(\mathcal{X})$. The trace operator is actually a map such as: $\mathrm{tr}:\Omega^4(\mathcal{X};\mathfrak{g})\to\Omega^4(\mathcal{X})$. Then we define a principal connection over P: $\nabla=\nabla_0+A$ where ∇_0 is a flat connection. $A\in\Omega^1(\mathcal{X};\mathfrak{g})$ is a $\mathfrak{g}$-valued 1-form over $\mathcal{X}$ that is the gauge potential or what we call *Yang-Mills* field. Following the previous section, it is given as $A(x)=A^{\mathcal{I}}_\mu(x)dx^\mu\mathfrak{b}_{\mathcal{I}}$ and we can derive the curvature of the connection ∇ $F=dA+A\wedge A$.

$$F=\frac{1}{2}F^{\mathcal{I}}_{\mu\nu}dx^\mu\wedge dx^\nu\otimes\mathfrak{b}_{\mathcal{I}}=\frac{1}{2}F_{\mu\nu}dx^\mu\wedge dx^\nu\qquad\text{with}\qquad F^{\mathcal{I}}_{\mu\nu}=\partial_\mu A^{\mathcal{I}}_\nu-\partial_\nu A^{\mathcal{I}}_\mu+[A_\mu,A_\nu]^{\mathcal{I}} \tag{454}$$

[125] For a more beautiful picture, we may use the homology boundary application $\delta:\Omega^q(\mathcal{X})\to\Omega^{q-1}(\mathcal{X})$.

B.9 Maurer-Cartan form

Maurer-Cartan form. In this section we present some basic of differential geometry connected to Lie group and Lie algebra, where we are inspired directly from J. Butterfield [34], M. Nakahara [176] or Marsden and *al.* [169]. Now we consider a Lie group G, with Lie algebra given by $\mathfrak{g}$ and we denote $\{\mathfrak{b}_{\mathcal{I}}\}$ a basis of the Lie algebra. We also denote $\{\theta^{\mathcal{I}}\}$ the dual basis of the dual Lie algebra $\mathfrak{g}^{\star}$. We recall that the Lie structure (Lie bracket) on $\mathfrak{g}$ is given vie the structure coefficients $\mathfrak{c}^{\mathcal{K}}_{\mathcal{I}\mathcal{J}}$ such that $[\mathfrak{b}_{\mathcal{I}},\mathfrak{b}_{\mathcal{J}}]=\mathfrak{c}^{\mathcal{K}}_{\mathcal{I}\mathcal{J}}\mathfrak{b}_{\mathcal{K}}$. Since the fundamental property of a Lie structure is given by antisymmetry (455)(i) and Jacobi identity (455)(ii). Namely, for $\forall X,Y,Z\in\mathfrak{g}$, we have

$$\text{(i)}\quad [X,Y]=-[Y,X]\qquad \text{(ii)}\quad [[X,Y],Z]+[[Z,X],Y]+[[Y,Z],X]=0 \tag{455}$$

Therefore the relation (455)(i) and (455)(ii) transmute in relation on structure constants: $\mathfrak{c}^{\mathcal{K}}_{\mathcal{I}\mathcal{J}}=-\mathfrak{c}^{\mathcal{K}}_{\mathcal{J}\mathcal{I}}$ and $\mathfrak{c}^{\mathcal{L}}_{\mathcal{I}\mathcal{J}}\mathfrak{c}^{\mathcal{M}}_{\mathcal{L}\mathcal{K}}+\mathfrak{c}^{\mathcal{L}}_{\mathcal{K}\mathcal{I}}\mathfrak{c}^{\mathcal{M}}_{\mathcal{L}\mathcal{J}}+\mathfrak{c}^{\mathcal{L}}_{\mathcal{J}\mathcal{K}}\mathfrak{c}^{\mathcal{M}}_{\mathcal{L}\mathcal{I}}=0$. The forms $\theta^{\mathcal{I}}$ are seen in a canonically way, those are the Maurer-Cartan forms associated to the group G. We have the Maurer-Cartan structure equation:

$$d\theta^{\mathcal{K}}=-\frac{1}{2}\mathfrak{c}^{K}_{IJ}\theta^{\mathcal{I}}\wedge\theta^{\mathcal{J}} \tag{456}$$

Definition B.1. *Maurer-Cartan form θ is a canonical 1-form associate to any Lie group G. We describe it as $\theta\in\Omega^1(G,\mathfrak{g})$ so that θ is canonically defined as:*

$$\theta:TG\longrightarrow\mathfrak{g}:\xi\in T_gG\mapsto(L_{\mathfrak{g}^{-1}})_{\star}(\xi) \tag{457}$$

The cornerstone of Lie group theory exhibit the fundamental isomorphism $T_eG\cong\mathfrak{g}$ where we denote by e the identity element of the Lie group G. Hence, the canonical Lie algebra $\mathfrak{g}$ associated to the Lie group G needs the introduction of the left[126] translation: $L_g:G\longrightarrow G:h\mapsto gh$. We denote $e\in G$ the identity element and $\mathbf{Id}_G$ the identity map on G. Then $L_e=\mathbf{Id}_G$ and also $(L_g)^{-1}=L_{\mathfrak{g}^{-1}}$ for any $g\in G$ so that L_g is a diffeomorphism. We consider the linear tangent application $(L_g)_{\star}$. Let emphasize that $(L_g)_{\star}$ is equivalently described such that for any $g,h\in G$ we have $T_hL_g:T_hG\longrightarrow T_{gh}G:X_{|h}\mapsto X_{|gh}$. Then, a left invariant vector field X on G is such that for any $g\in G$, $(L_g)_{\star}X=X$. One equivalently describe *left invariant* vector field X on G such that $\forall g,h\in G$ we have

$$(T_hL_g)X_{|h}=X_{|gh} \tag{458}$$

We observe that the left translation operation L_g via the use of the linear tangent application (or pushforeward) $(L_g)_{\star}$ give a relation between the value taken by a left invariant vector field X at gh to the value taken at h thought the relation (458). We denote by $\mathcal{X}_L(G)\subset\mathcal{X}(G)$ the space of left invariant vector field. If we take $h=e$, we obtain $(T_eL_g)X_{|e}=X_{|g}$. Therefore, in this case a left invariant vector field is totally determined by the value at $e\in G$. This is why for each vector field $\xi\in T_eG$ we associate the left invariant vector field X_ξ on G namely $X_\xi\in\mathcal{X}_L(G)$, which is determined by the vector field $\xi\in T_eG$ and such that for any $g\in G$, we have $X_\xi(g)=(T_eL_g)\xi$. Hence we describe the image of ξ by $(L_g)_{\star}$. Following [169]:

$$X_\xi(gh)=T_eL_{gh}(\xi)=T_e(L_g\circ L_h)(\xi)=T_hL_g(T_eL_h(\xi))=T_hL_g(X_\xi(h)) \tag{459}$$

Then (459) actually means $(X_\xi)_{|gh}=T_hL_g(X_\xi)_{|h}$ is clearly left invariant. Since $[\mathcal{X}_L(G),\mathcal{X}_L(G)]\subset\mathcal{X}_L(G)$ we describe $\mathcal{X}_L(G)$ as a Lie subalgebra of $\mathcal{X}(G)$. From now, we exhibit the main canonical isomorphism $T_eG\cong\mathfrak{g}$. This result is presented as a *"cornerstone of Lie group theory"* by P. Olver

[126] It exists same definition with a right translation, giving actually two Maurer-Cartan forms later on, but in this discussion we only focus on left invariant vector fields.

[186] as emphasized by J. Butterfield [34]. The linear map $\varsigma^{\bullet} : \mathcal{X}_L(G) \longrightarrow T_e(G) : X \mapsto X_{|e}$ and the map [127] $\varsigma_{\bullet} : T_e(G) \longrightarrow \mathcal{X}_L(G) : \xi \mapsto X_\xi$ are such that: $(\varsigma^{\bullet}) \circ (\varsigma_{\bullet}) = \mathbf{Id}_{T_e(G)}$ and $(\varsigma_{\bullet}) \circ (\varsigma^{\bullet}) = \mathbf{Id}_{\mathcal{X}_L(G)}$. Therefore, as noticed in [169], $T_e(G)$ and $\mathcal{X}_L(G)$ are isomorphic as vector spaces. Therefore, the key point is that we picture a natural Lie algebra structure on the space T_eG thanks to the isomorphism $T_eG \cong \mathfrak{g}$ leading to the following Lie bracket definition: $\forall \xi, \eta \in T_eG$, $[\xi, \eta] = [X_\xi, X_\eta](e)$ so that we exhibit the following structure

$$[X_\xi, X_\eta] = [(L_g)_\star X_\xi, (L_g)_\star X_\eta] = (L_g)_\star [X_\xi, X_\eta] = X_{[\xi,\eta]} \tag{460}$$

Remark Another way to picture it using the definition (B.1), is to introduce $\theta_g : T_gG \longrightarrow T_eG : X \mapsto (L_{\mathfrak{g}^{-1}})_\star X$. Then we apply θ_g on left invariant vector field, let say X. Finally, with this notion of Maurer Cartan form, we could have defined an Ehresmann connection [246]

Definition B.9.1. *An Ehresmann connection on a principal G-bundle $\mathcal{P} \xrightarrow{\pi} \mathcal{M}$ is a $\mathfrak{g}$-valued 1-form $\boldsymbol{\omega}$ on $\mathcal{P}$ such that:*

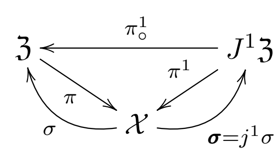
$$\text{(i)} \quad \forall g \in G \quad (R_g^\star)\boldsymbol{\omega} = Ad_{\mathfrak{g}^{-1}}\boldsymbol{\omega}$$

(ii) *$\boldsymbol{\omega}$ restricts to the canonical Maurer-Cartan form $\boldsymbol{\omega} = \boldsymbol{\omega}_G : T\mathcal{P}_\xi$ on fibers of $\mathcal{P}$*

C Jet manifold and contact structure

C.1 Jet manifold

First order jet bundle We refer to D.J. Saunders [215] for a full treatment of first order jet theory. We consider the configuration bundle $\mathfrak{Z} \underset{\pi}{\overset{\sigma}{\rightleftarrows}} \mathcal{X}$, with bundle coordinates (x^μ, z^i). The first order jets of sections at $x \in \mathcal{X}$ is picture as the equivalence classes $j^1_x\sigma$ of its section σ, identified by their values $\sigma^i(x)$ and $\partial_\mu \sigma^i(x)$ at $x \in \mathcal{X}$. Then the set $J^1\mathfrak{Z}$ of first order jets $j^1_x\sigma$ is a smooth manifold with respect to the adapted coordinates (x^μ, z^i, z^i_μ) so that $z^i_\mu(j^1_x\sigma) = \partial_\mu\sigma^i(x)$. The coordinates z^i_μ are referred as the jet coordinates, and the first order jet manifold $J^1\mathfrak{Z}$ carries the natural fibrations [206, 207] $\pi^1 : J^1\mathfrak{Z} \longrightarrow \mathcal{X} : j^1_x\sigma \mapsto x$ and $\pi^1_\circ : J^1\mathfrak{Z} \longrightarrow \mathfrak{Z} : j^1_x\sigma \mapsto \sigma(x)$. Equivalently, we then picture the first order jet prolongation of a section $\sigma : \mathcal{X} \longrightarrow \mathfrak{Z}$ of the bundle $(\mathfrak{Z}, \mathcal{X}, \pi)$ by the section $j^1\sigma$ of $(J^1\mathfrak{Z}, \mathcal{X}, \pi^1)$ which is locally given by $z^i_\mu \circ j^1(\sigma) = \partial_\mu\sigma^i = \partial_\mu(z^i \circ \sigma)$. Let notice that a section $\boldsymbol{\sigma}$ of $(J^1\mathfrak{Z}, \mathcal{X}, \pi^1)$ which is the first order jet prolongation of some section σ of $(\mathfrak{Z}, \mathcal{X}, \pi)$ is called a holonomic section (in this case we have $\boldsymbol{\sigma} = j^1\sigma$). Once again, in order to fix the encountered vocabulary we alternatively call *integrable* such section $\boldsymbol{\sigma}$, pictured as the jet prolongation of some section σ. Here we give an illustration of the first order jet formalism (461).



(461)

Therefore the set $J^1\mathfrak{Z} = \{ j^1_x\sigma \ / \ x \in \mathcal{X}, \sigma \in \Gamma_x(\mathfrak{Z}) \}$ is the first order jet manifold, seen as the total space of a fiber bundle. More precisely, we notice that the bundle $(J^1\mathfrak{Z}, \pi^1_\circ, \mathfrak{Z})$ is an affine bundle whereas $(J^1\mathfrak{Z}, \pi^1, \mathcal{X})$ is a vector bundle. Then σ being a local section of $(\mathfrak{Z}, \mathcal{X}, \pi)$, the prolongation of σ is denoted $\boldsymbol{\sigma} = j^1\sigma$ of the bundle $(J^1\mathfrak{Z}, \pi^1_\circ, \mathfrak{Z})$ express in local adapted coordinates $j^1\sigma(x) = (x^\mu, \sigma^i, \partial_\mu\sigma^i)$.

[127] where actually the leftinvariant vector field is fully describe as $X_\xi = \{g \mapsto X_\xi(g) = (T_eL_g)\xi\}$

k-order jet bundle $J^k\mathfrak{Z}$. The space $(J^k\mathfrak{Z}, \mathcal{M}, \pi^k)$ is defined by k-jets of local section of $(\mathfrak{Z}, \mathcal{M}, \pi)$, the so-called k-order jet prolongation. Coordinates (x^μ, z^i) local coordinates on $\mathfrak{Z}$ induce coordinates on $J^k\mathfrak{Z}$ given by $(x^\mu, z^i_{\boldsymbol{\mu}})$ where $\boldsymbol{\mu}$ is a multi-index of length $\mid \boldsymbol{\mu} \mid$ such that $0 \leq\mid \boldsymbol{\mu} \mid\leq k$. Then $(x^\mu, z^i_{\boldsymbol{\mu}})$ is a short notation for $(x^\mu, z^i, z^i_\mu, z^i_{\mu_1\mu_2} \cdots z^i_{\mu_1\cdots\mu_k})$. The k-order jet of σ (i.e. all partial derivatives of σ of order less than or equal to k), denoted $j^k\sigma$ is a section of $(J^k\mathfrak{Z}, \mathcal{M}, \pi^k)$ given locally:

$$z^i_{\boldsymbol{\mu}} \circ j^k\sigma = \partial_{\boldsymbol{\mu}}\sigma^i = \partial_{\boldsymbol{\mu}}(z^i \circ \sigma) \tag{462}$$

with the notation $\partial_{\boldsymbol{\mu}} = \partial_{\mu_k} \circ \cdots \circ \partial_{\mu_1}$ so that (462) writes equivalently:

$$z^i_{\boldsymbol{\mu}}(j^k\sigma(x)) = \frac{\partial z^i}{\partial x^{\mu_1} \cdots \partial x^{\mu_k}}(x) \tag{463}$$

C.2 Contact structure

We denote $\mathbf{V}\mathfrak{Z}$ the vertical bundle of $\mathfrak{Z}$ the sub-bundle of the tangent bundle $T\mathfrak{Z}$ which carry fiber defined by the null space of the tangent projection $\pi_\star$. Namely as already pictured above, A vector $\zeta^{\mathbf{V}} \in T_z\mathfrak{Z}$ is vertical if and only if $\pi_\star(\zeta^{\mathbf{V}}) = d\pi(\zeta^{\mathbf{V}}) = 0$. From this observation we state that $J^1_z\mathfrak{Z}$ is an affine space modeled on the vector space $\mathbf{V}_z\mathfrak{Z} \otimes T^\star_x\mathcal{M}$. More precisely we write that the affine bundle $(J^1\mathfrak{Z}, \pi^1_\circ, \mathfrak{Z})$ is constructed on the vector bundle $\mathbf{V}\mathfrak{Z} \otimes_\mathfrak{Z} T^\star\mathcal{M}$ - see [215] for further details. Also we introduce the following notation ([18, 19, 124]): we denote $\Lambda^n_2 T^\star\mathfrak{Z}$ the vector sub-bundle of $\Lambda^n T^\star\mathfrak{Z}$ whose fiber at $z \in \mathfrak{Z}$ consists of all $\boldsymbol{\varphi} \in \Lambda^n_z T^\star\mathfrak{Z}$ such that for any $\zeta^{\mathbf{V}}, \chi^{\mathbf{V}} \in \mathbf{V}\mathfrak{Z} = \mathbf{V}\pi$:

$$\Lambda^n_2 T^\star\mathfrak{Z} = \{\boldsymbol{\varphi} \in \Lambda^n_z T^\star\mathfrak{Z} \;/\; \zeta^{\mathbf{V}} \lrcorner\, \chi^{\mathbf{V}} \lrcorner\, \boldsymbol{\varphi} = 0\} \tag{464}$$

Contact form A form $\boldsymbol{\varphi} \in \Omega^p(J^1\mathfrak{Z})$, where $J^1\mathfrak{Z}$ denote the first order jet of the bundle $\mathfrak{Z} \xrightarrow{\pi} \mathcal{X}$ is a contact 1-form if $(j^1\sigma)^\star\boldsymbol{\varphi} = 0$ for any section $\sigma : \mathcal{X} \longrightarrow \mathfrak{Z}$. Contact 1-forms are found to be linear combination of the *basis contact forms* ϑ^i of $J^1\mathfrak{Z}$ given by:

$$\vartheta^i = dz^i - z^i_\mu dx^\mu \tag{465}$$

Horizontal form We call a horizontal p-form $\boldsymbol{\varphi} \in \Omega^p(\mathfrak{Z})$ on $(\mathfrak{Z}, \mathcal{X}, \pi)$ such that for any set of vertical vector fields $\zeta^{\mathbf{V}}_1 ... \zeta^{\mathbf{V}}_p$ we have: $\boldsymbol{\varphi}(\zeta^{\mathbf{V}}_1, ..., \zeta^{\mathbf{V}}_p) = 0$. Locally, we shall write a horizontal p-form on $(\mathfrak{Z}, \mathcal{X}, \pi)$ as

$$\boldsymbol{\varphi} = \frac{1}{p!}\boldsymbol{\varphi}_{\mu_1\dots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \tag{466}$$

We shall denote the space of horizontal p-form as $\Omega^p_\circ(\mathfrak{Z}) \subset \Omega^p(\mathfrak{Z})$. From this clarification and following [166], we obtain the decomposition of any general 1-form $\boldsymbol{\varphi} \in \Omega^1(\mathfrak{Z})$ on $J^1\mathfrak{Z}$ by pulling back of $\boldsymbol{\varphi}$ with $\pi^1_\circ$, the induced natural fibration $\pi^1_\circ : J^1\mathfrak{Z} \longrightarrow \mathfrak{Z}$, leading to:

$$(\pi^1_\circ)^\star\boldsymbol{\varphi} = \boldsymbol{\varphi}_\mu dx^\mu + \boldsymbol{\varphi}_i dz^i = (\boldsymbol{\varphi}_\mu + \boldsymbol{\varphi}_i z^i_\mu) dx^\mu + \boldsymbol{\varphi}_i \vartheta^i \tag{467}$$

Then, the decomposition (467) of $(\pi^1_\circ)^\star\boldsymbol{\varphi}$ exhibit two parts: a part represented by a contact 1-form $\boldsymbol{\varphi}_i\vartheta^i$ and a part identified with a horizontal 1-form on $J^1(\mathfrak{Z})$, that we denote by $\mathbf{h}(\boldsymbol{\varphi}) = [\mathbf{h}(\boldsymbol{\varphi})]_\mu dx^\mu = (\boldsymbol{\varphi}_\mu + \boldsymbol{\varphi}_i z^i_\mu) dx^\mu$. Then $\mathbf{h}(\boldsymbol{\varphi}) \in \Omega^1_\circ(J^1\mathfrak{Z})$ is the horizontal part of $\boldsymbol{\varphi} \in \Omega^1(\mathfrak{Z})$. If we denote $\mathfrak{con}(\boldsymbol{\varphi}) = \boldsymbol{\varphi}_i\vartheta^i$ the contact 1-form that appear in this decomposition, we can write

$$\forall\boldsymbol{\varphi} \in \Omega^1(\mathfrak{Z}) \qquad (\pi^1_\circ)^\star\boldsymbol{\varphi} = \mathbf{h}(\boldsymbol{\varphi}) + \mathfrak{con}(\boldsymbol{\varphi}) \tag{468}$$

Now, the following section, largely inspired from P. Matteucci [166] and we introduce *vertical, horizontal* and *formal derivatives*. Then, the operator $\mathbf{h}$ is actually pictured as $\mathbf{h} : \Omega^1(\mathfrak{Z}) \cong$

$\Omega^1(J^0\mathfrak{Z}) \longrightarrow \Omega^1_\circ(J^1\mathfrak{Z})$. Let notice that, in the case of 0-forms, namely function $\varphi : J^1\mathfrak{Z} \longrightarrow \mathbb{R}$, we observe $\mathbf{h}(\varphi) = \varphi$ for any $\varphi \in \Omega^0(J^1\mathfrak{Z})$. Finally, we can extend the definition of the operator $\mathbf{h}$ acting to any n-form $\boldsymbol{\varphi} \in \Omega^n(\mathfrak{Z})$. In this case, $\mathbf{h} : \Omega^n(J^0\mathfrak{Z}) \to \Omega^n_\circ(J^1\mathfrak{Z})$ reduces to the previous case with $n = 0, 1$ and satisfy the property $\mathbf{h}(\boldsymbol{\varphi} \wedge \boldsymbol{\phi}) = \mathbf{h}(\boldsymbol{\varphi}) \wedge \mathbf{h}(\boldsymbol{\phi})$. From the standpoint, the horizontal operator defined writes for k-jet theory $\mathbf{h} : \Omega^n(J^k\mathfrak{Z}) \longrightarrow \Omega^n_\circ(J^{n+1}\mathfrak{Z})$ whereas the analogous object of contact 1-form and basis contact form, for k-order also arise. We defined a contact n-form $\boldsymbol{\varphi} \in \Omega^n(J^k\mathfrak{Z})$ if $(j^k\sigma)^\star\boldsymbol{\varphi} = 0$ and the basis contact form on $J^k\mathfrak{Z}$ denoted in the k-order case as (this time with $0 \leq |\,\boldsymbol{\mu}\,| \leq k-1$)

$$\vartheta^i_{\boldsymbol{\mu}} = dz^i_{\boldsymbol{\mu}} - z^i_{\boldsymbol{\mu}\nu}dx^\nu \tag{469}$$

These considerations on $J^k\mathfrak{Z}$ allow us to describe for example a 1-form $\boldsymbol{\varphi} \in \Omega^1(J^k\mathfrak{Z})$, ($i.e$ a 1-form on any jet prolongation) we can write it as $\boldsymbol{\varphi} = \boldsymbol{\varphi}_\nu dx^\nu + \boldsymbol{\varphi}^{\boldsymbol{\mu}}_i dz^i_{\boldsymbol{\mu}}$, and it follows that the object $\mathbf{h}(\boldsymbol{\varphi}) \in \Omega^1_\circ(J^{k+1}\mathfrak{Z})$ is given by $\mathbf{h}(\boldsymbol{\varphi}) = (\boldsymbol{\varphi}_\nu + \boldsymbol{\varphi}^{\boldsymbol{\mu}}_i z^i_{\boldsymbol{\mu}\nu})dx^\nu$ where the operator $\mathbf{h}$ is formally defined as $\mathbf{h} : \Omega^1(J^k\mathfrak{Z}) \longrightarrow \Omega^1_\circ(J^{k+1}\mathfrak{Z})$. Finally, drawning inspiration from (468), we observe that the contact part $\mathfrak{con}(\boldsymbol{\varphi})$ of a form $\boldsymbol{\varphi} \in \Omega^n(J^k\mathfrak{Z})$ defined as $\mathfrak{con}(\boldsymbol{\varphi}) = (\pi^{k+1}_k)^\star(\boldsymbol{\varphi}) - \mathbf{h}(\boldsymbol{\varphi})$ is a contact form.[166]

Horizontal differential $d_{\mathbf{H}}$ The horizontal differential $d_{\mathbf{H}} : \Omega^n_\circ(J^k\mathfrak{Z}) \longrightarrow \Omega^{n+1}_\circ(J^{k+1}\mathfrak{Z})$ is given by: $d_{\mathbf{H}} = \mathbf{h}(d\boldsymbol{\varphi})$ for any $\boldsymbol{\varphi} \in \Omega^n_\circ(J^k\mathfrak{Z})$ such that

$$d_{\mathbf{H}}(\boldsymbol{\varphi} \wedge \boldsymbol{\psi}) = d_{\mathbf{H}}(\boldsymbol{\varphi}) \wedge \boldsymbol{\psi} + (-1)^n(\boldsymbol{\varphi} \wedge d_{\mathbf{H}}\boldsymbol{\psi}) \tag{470}$$

Formal derivative denoted $d_\mu : \Omega^0(J^k\mathfrak{Z}) \longrightarrow \Omega^0(J^{k+1}\mathfrak{Z})$. Then, considering a function $\varphi \in \Omega^0(J^k\mathfrak{Z})$, we have (with $0 \leq |\,\boldsymbol{\mu}\,| \leq k$):

$$d_\nu\varphi = \partial_\nu\varphi + z^i_{\boldsymbol{\mu}\nu}\partial^{\boldsymbol{\mu}}_i\varphi = \partial_\nu\varphi + z^i_{\boldsymbol{\mu}\nu}\frac{\partial\varphi}{\partial z^i_{\boldsymbol{\mu}}} \tag{471}$$

In the case we consider only the first jet order setting with $\varphi \in \Omega^0(J^1\mathfrak{Z})$ we obtain:

$$d_\nu\varphi = \partial_\nu\varphi + z^i_\nu\partial_i\varphi + z^i_{\mu\nu}\partial^\mu_i\varphi = \frac{\partial\varphi}{\partial x^\nu} + z^i_\nu\frac{\partial\varphi}{\partial z^i} + z^i_{\mu\nu}\frac{\partial\varphi}{\partial z^i_\mu} = \frac{\partial\varphi}{\partial x^\nu} + \frac{\partial z^i}{\partial x^\nu}\frac{\partial\varphi}{\partial z^i} + \frac{\partial z^i}{\partial x^\mu\partial x^\nu}\frac{\partial\varphi}{\partial z^i_\mu}$$

that we shall denote with the notation $d_\nu\varphi = (\partial_\nu + z^i_\nu\partial_\nu + z^i_{\mu\nu}\partial^\mu_i)\varphi$. For a function $\varphi \in \Omega^0(J^k\mathfrak{Z})$, $d_{\mathbf{H}}\varphi = (d_\mu\varphi)dx^\mu$. By definition of the horizontal differential, we have $\varphi = \varphi(x^\mu, z^i, z^i_{\boldsymbol{\mu}})\big|_{1\leq|\boldsymbol{\mu}|\leq k}$

$$d_{\mathbf{H}}\varphi = \mathbf{h}(d\varphi) = \mathbf{h}\Big(\sum_\mu \underbrace{\frac{\partial\varphi}{\partial x^\mu}(x^\mu, z^i, z^i_{\boldsymbol{\mu}})}_{(d\boldsymbol{\varphi})_\mu}dx^\mu + \sum_i \underbrace{\frac{\partial\varphi}{\partial z^i}(x^\mu, z^i, z^i_{\boldsymbol{\mu}})}_{(d\boldsymbol{\varphi})_i}dz^i + \sum_{1\leq|\boldsymbol{\mu}|\leq k}\sum_i \underbrace{\frac{\partial\varphi}{\partial z^i_{\boldsymbol{\mu}}}(x^\mu, z^i, z^i_{\boldsymbol{\mu}})}_{(d\boldsymbol{\varphi})^{\boldsymbol{\mu}}_i}dz^i_{\boldsymbol{\mu}}\Big)$$

Leading to:

$$d_{\mathbf{H}}\varphi = \mathbf{h}\Big((d\varphi)_\mu dx^\mu + (d\varphi)_i dz^i + \sum_{1\leq|\boldsymbol{\mu}|\leq k}(d\varphi)^{\boldsymbol{\mu}}_i dz^i_{\boldsymbol{\mu}}\Big) = \big((d\varphi)_\nu + \sum_{0\leq|\boldsymbol{\mu}|\leq k}(d\varphi)^{\boldsymbol{\mu}}_i z^i_{\boldsymbol{\mu}\nu}\big)dx^\nu$$

$$d_{\mathbf{H}}\varphi = \big(\partial_\nu\varphi + z^i_{\boldsymbol{\mu}\nu}\partial^{\boldsymbol{\mu}}_i\varphi\big)dx^\nu = (d_\nu\varphi)dx^\nu \tag{472}$$

as well as the property $\forall\varphi \in \Omega^0(J^k\mathfrak{Z})$, $d_\mu d_\nu\varphi = d_\nu d_\mu\varphi \iff d_{[\mu}d_{\nu]} = 0$ and finally for a horizontal n-form $\boldsymbol{\varphi} \in \Omega^n_\circ(J^k\mathfrak{Z})$ on the k-jets: $d_{\mathbf{H}} = (1/n!)d_\mu\boldsymbol{\varphi}_{\mu_1\cdots\mu_n}dx^\mu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$. Therefore if $\varphi \in \Omega^0(J^k\mathfrak{Z})$ or $\boldsymbol{\varphi} \in \Omega^n_\circ(J^k\mathfrak{Z})$ we have then the complex $d_{\mathbf{H}} \circ d_{\mathbf{H}}\varphi = d_{\mathbf{H}} \circ d_{\mathbf{H}}\boldsymbol{\varphi} = 0$.

Vertical differential is generally defined, for any $\boldsymbol{\varphi} \in \Omega^n(J^k\mathfrak{Z})$ as $d_{\mathbf{V}} : \Omega^n(J^k\mathfrak{Z}) \longrightarrow \Omega^{n+1}(J^{k+1}\mathfrak{Z})$ defined as the difference $d_{\mathbf{V}}\boldsymbol{\varphi} = d\boldsymbol{\varphi} - d_{\mathbf{H}}\boldsymbol{\varphi}$. We notice, a perfect analogous in the horizontal case (470), the vertical graded property:

$$d_{\mathbf{V}}(\boldsymbol{\varphi} \wedge \boldsymbol{\psi}) = d_{\mathbf{V}}(\boldsymbol{\varphi}) \wedge \boldsymbol{\psi} + (-1)^n(\boldsymbol{\varphi} \wedge d_{\mathbf{V}}\boldsymbol{\psi}) \tag{473}$$

Also, for any $\boldsymbol{\varphi} \in \Omega^n(J^1\mathfrak{Z})$, we observe the vertical complex since $d_{\mathbf{V}} \circ d_{\mathbf{V}}\boldsymbol{\varphi} = 0$.

D Algebraic identities for Palatini framework

For the $3D$ we have the relation $\epsilon^{\mu\rho\sigma}\epsilon_{IJK}e^I_\mu\omega_\sigma{}^J{}_M d\omega_\rho^{MK} = -\dfrac{1}{2}\epsilon^{\mu\rho\sigma}\epsilon_{LJK}e^I_\mu\omega_\sigma{}^L{}_I d\omega_\rho^{JK}$. If we denote $\mathfrak{J}_{\mu\rho\sigma} = \epsilon_{IJK}e^I_\mu\omega_\rho{}^J{}_M d\omega_\sigma^{MK}$, and $\mathfrak{J} = \epsilon^{\mu\rho\sigma}\Big(\epsilon_{IJK}e^I_\mu\omega_\rho{}^J{}_M d\omega_\sigma^{MK}\Big)$

$$\mathfrak{J}_{\mu\rho\sigma} = \begin{pmatrix} \epsilon_{123}e^1_\mu\omega_\rho{}^2{}_M d\omega_\sigma^{M3} \\ \epsilon_{132}e^1_\mu\omega_\rho{}^3{}_M d\omega_\sigma^{M2} \\ \epsilon_{213}e^2_\mu\omega_\rho{}^1{}_M d\omega_\sigma^{M3} \\ \epsilon_{231}e^2_\mu\omega_\rho{}^3{}_M d\omega_\sigma^{M1} \\ \epsilon_{321}e^3_\mu\omega_\rho{}^2{}_M d\omega_\sigma^{M1} \\ \epsilon_{312}e^3_\mu\omega_\rho{}^1{}_M d\omega_\sigma^{M2} \end{pmatrix} = \begin{pmatrix} e^1_\mu\omega_\rho{}^2{}_M d\omega_\sigma^{M3} \\ -e^1_\mu\omega_\rho{}^3{}_M d\omega_\sigma^{M2} \\ -e^2_\mu\omega_\rho{}^1{}_M d\omega_\sigma^{M3} \\ e^2_\mu\omega_\rho{}^3{}_M d\omega_\sigma^{M1} \\ -e^3_\mu\omega_\rho{}^2{}_M d\omega_\sigma^{M1} \\ e^3_\mu\omega_\rho{}^1{}_M d\omega_\sigma^{M2} \end{pmatrix} = \begin{pmatrix} e^1_\mu\omega_\rho{}^2{}_1 d\omega_\sigma^{13} + e^1_\mu\omega_\rho{}^2{}_2 d\omega_\sigma^{23} + e^1_\mu\omega_\rho{}^2{}_3 d\omega_\sigma^{33} \\ -[e^1_\mu\omega_\rho{}^3{}_1 d\omega_\sigma^{12} + e^1_\mu\omega_\rho{}^3{}_2 d\omega_\sigma^{22} + e^1_\mu\omega_\rho{}^3{}_3 d\omega_\sigma^{32}] \\ -[e^2_\mu\omega_\rho{}^1{}_1 d\omega_\sigma^{13} + e^2_\mu\omega_\rho{}^1{}_2 d\omega_\sigma^{23} + e^2_\mu\omega_\rho{}^1{}_3 d\omega_\sigma^{33}] \\ e^2_\mu\omega_\rho{}^3{}_1 d\omega_\sigma^{11} + e^2_\mu\omega_\rho{}^3{}_2 d\omega_\sigma^{21} + e^2_\mu\omega_\rho{}^3{}_3 d\omega_\sigma^{31} \\ -[e^3_\mu\omega_\rho{}^2{}_1 d\omega_\sigma^{11} + e^3_\mu\omega_\rho{}^2{}_2 d\omega_\sigma^{21} + e^3_\mu\omega_\rho{}^2{}_3 d\omega_\sigma^{31}] \\ e^3_\mu\omega_\rho{}^1{}_1 d\omega_\sigma^{12} + e^3_\mu\omega_\rho{}^1{}_2 d\omega_\sigma^{22} + e^3_\mu\omega_\rho{}^1{}_3 d\omega_\sigma^{32} \end{pmatrix}$$

$$\mathfrak{J} = \epsilon^{\mu\rho\sigma}\begin{pmatrix} e^1_\mu\omega_\rho{}^2{}_1 d\omega_\sigma^{13} \\ -[e^1_\mu\omega_\rho{}^3{}_1 d\omega_\sigma^{12}] \\ -[e^2_\mu\omega_\rho{}^1{}_2 d\omega_\sigma^{23}] \\ e^2_\mu\omega_\rho{}^3{}_2 d\omega_\sigma^{21} \\ -[e^3_\mu\omega_\rho{}^2{}_3 d\omega_\sigma^{31}] \\ e^3_\mu\omega_\rho{}^1{}_3 d\omega_\sigma^{32} \end{pmatrix} \qquad \text{Whereas on the other side we write} \qquad \mathfrak{J}' = -\frac{1}{2}\epsilon^{\mu\rho\sigma}\epsilon_{LJK}e^I_\mu\omega_\sigma{}^L{}_I d\omega_\rho^{JK}$$

$$\mathfrak{J}'_{\mu\rho\sigma} = -\frac{1}{2}\begin{pmatrix} \epsilon_{123}e^I_\mu\omega_\sigma{}^1{}_I d\omega_\rho^{23} \\ \epsilon_{132}e^I_\mu\omega_\sigma{}^1{}_I d\omega_\rho^{32} \\ \epsilon_{213}e^I_\mu\omega_\sigma{}^2{}_I d\omega_\rho^{13} \\ \epsilon_{231}e^I_\mu\omega_\sigma{}^2{}_I d\omega_\rho^{31} \\ \epsilon_{321}e^I_\mu\omega_\sigma{}^3{}_I d\omega_\rho^{21} \\ \epsilon_{312}e^I_\mu\omega_\sigma{}^3{}_I d\omega_\rho^{12} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} e^I_\mu\omega_\sigma{}^1{}_I d\omega_\rho^{23} \\ -e^I_\mu\omega_\sigma{}^1{}_I d\omega_\rho^{32} \\ -e^I_\mu\omega_\sigma{}^2{}_I d\omega_\rho^{13} \\ e^I_\mu\omega_\sigma{}^2{}_I d\omega_\rho^{31} \\ -e^I_\mu\omega_\sigma{}^3{}_I d\omega_\rho^{21} \\ e^I_\mu\omega_\sigma{}^3{}_I d\omega_\rho^{12} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} e^2_\mu\omega_\sigma{}^1{}_2 d\omega_\rho^{23} + e^3_\mu\omega_\sigma{}^1{}_3 d\omega_\rho^{23} \\ -[e^2_\mu\omega_\sigma{}^1{}_2 d\omega_\rho^{32} + e^3_\mu\omega_\sigma{}^1{}_3 d\omega_\rho^{32}] \\ -[e^1_\mu\omega_\sigma{}^2{}_1 d\omega_\rho^{13} + e^3_\mu\omega_\sigma{}^2{}_3 d\omega_\rho^{13}] \\ e^1_\mu\omega_\sigma{}^2{}_1 d\omega_\rho^{31} + e^3_\mu\omega_\sigma{}^2{}_3 d\omega_\rho^{31} \\ -[e^1_\mu\omega_\sigma{}^3{}_1 d\omega_\rho^{21} + e^2_\mu\omega_\sigma{}^3{}_2 d\omega_\rho^{21}] \\ e^1_\mu\omega_\sigma{}^3{}_1 d\omega_\rho^{12} + e^2_\mu\omega_\sigma{}^3{}_2 d\omega_\rho^{12} \end{pmatrix}$$

$$\mathfrak{J}' = \epsilon^{\mu\rho\sigma}\begin{pmatrix} [e^2_\mu\omega_\sigma{}^1{}_2 d\omega_\rho^{23} + e^3_\mu\omega_\sigma{}^1{}_3 d\omega_\rho^{23}] \\ -[e^1_\mu\omega_\sigma{}^2{}_1 d\omega_\rho^{13} + e^3_\mu\omega_\sigma{}^2{}_3 d\omega_\rho^{13}] \\ [e^1_\mu\omega_\sigma{}^3{}_1 d\omega_\rho^{12} + e^2_\mu\omega_\sigma{}^3{}_2 d\omega_\rho^{12}] \end{pmatrix} \qquad \text{so that } \mathfrak{J} = \mathfrak{J}'$$

A similar development leads to the $4D$ case: $\epsilon^{\mu\nu\rho\sigma}\epsilon_{IJKL}e^I_\mu e^J_\nu\omega_\sigma{}^K{}_M d\omega_\rho^{ML} = -\epsilon^{\mu\nu\rho\sigma}\epsilon_{INKL}e^I_\mu e^J_\nu\omega_\sigma{}^N{}_J d\omega_\rho^{KL}$

E Algebraic computation for infinitesimal symplectomorphisms

Let us consider the first term in $\kappa_{\mathbf{I}}$ that involves Π^ρ_ν let us denote $\kappa_{\mathbf{I}} = \kappa_{\mathbf{I}_1} + \kappa_{\mathbf{I}_2} + \kappa_{\mathbf{I}_3}$ with:

$$\kappa_{\mathbf{I}_1} = \sum_{\nu,\rho}\frac{\partial X^\nu}{\partial x^\rho}\Pi^\rho_\nu \qquad \kappa_{\mathbf{I}_2} = \sum_{\nu,\sigma}\frac{\partial X^\nu}{\partial e^N_\sigma}\Pi_\nu^{(e^N_\sigma)} \qquad \kappa_{\mathbf{I}_3} = \sum_{\nu,\rho}\frac{\partial X^\nu}{\partial \omega^{KL}_\rho}\Pi_\nu^{(\omega^{KL}_\rho)}$$

First we are intersted in the calculation of the term:

$$\kappa_{\mathbf{I}_1} = \sum_{\nu,\rho}\frac{\partial X^\nu}{\partial x^\rho}\Pi^\rho_\nu = \sum_{\nu,\rho}(\frac{\partial X^\nu}{\partial x^\rho})\sum_{\beta_1<\cdots<\beta_n}\sum_\mu \delta^\rho_{\beta_\mu}p_{\beta_1\cdots\beta_{\mu-1}\nu\beta_{\mu+1}\cdots\beta_n}\frac{\partial}{\partial p_{\beta_1\cdots\beta_n}}$$

Let us compute the first term in $\kappa_{\mathbf{I}}$ that we denote $\kappa_{\mathbf{I}_1}$ We want to prove that:

$$\kappa_{\mathbf{I}_1} = (\frac{\partial X^\lambda}{\partial x^\lambda})\mathfrak{e}\frac{\partial}{\partial \mathfrak{e}} + p^{\omega_\mu{}^\sigma}_{IJ}\Big[(\frac{\partial X^\nu}{\partial x^\sigma}) - \delta^\nu_\sigma(\frac{\partial X^\lambda}{\partial x^\lambda})\Big]\frac{\partial}{\partial p^{\omega_\mu{}^\nu}_{IJ}} \tag{474}$$

Let us develop first taking values of $1 \leq \rho \leq n$, then $\kappa_{\mathbf{I}_1} = {}^{\mathbf{I}_1}\kappa_1 + {}^{\mathbf{I}_1}\kappa_2 + {}^{\mathbf{I}_1}\kappa_3 + {}^{\mathbf{I}_1}\kappa_4$ We shall expand each of the four terms, the first one writes:

Calculation of the term ${}^{\mathbf{I}_1}\kappa_1 = (\kappa_{\mathbf{I}_1})\Big|_{\rho=1}$

$$ {}^{\mathbf{I}_1}\kappa_1 = (\kappa_{\mathbf{I}_1})\Big|_{\rho=1} = \sum_\nu \frac{\partial X^\nu}{\partial x^1} \sum_{\beta_1<\cdots<\beta_n} \sum_\eta \delta^1_{\beta_\eta} p_{\beta_1\cdots\beta_{\eta-1}\nu\beta_{\eta+1}\cdots\beta_n} \frac{\partial}{\partial p_{\beta_1\cdots\beta_n}} $$

Then, we expand the index η from 1 to n = 4.

$$ \begin{aligned} {}^{\mathbf{I}_1}\kappa_1 &= \sum_\nu \frac{\partial X^\nu}{\partial x^1} \sum_{\beta_1<\beta_2<\beta_3<\beta_4} \delta^1_{\beta_1} p_{\nu\beta_2\beta_3\beta_4} \frac{\partial}{\partial p_{\beta_1\beta_2\beta_3\beta_4}} + \sum_\nu \frac{\partial X^\nu}{\partial x^1} \sum_{\beta_1<\beta_2<\beta_3<\beta_4} \delta^1_{\beta_2} p_{\beta_1\nu\beta_3\beta_4} \frac{\partial}{\partial p_{\beta_1\beta_2\beta_3\beta_4}} \\ &\quad + \sum_\nu \frac{\partial X^\nu}{\partial x^1} \sum_{\beta_1<\beta_2<\beta_3<\beta_4} \delta^1_{\beta_3} p_{\beta_1\beta_2\nu\beta_4} \frac{\partial}{\partial p_{\beta_1\beta_2\beta_3\beta_4}} + \sum_\nu \frac{\partial X^\nu}{\partial x^1} \sum_{\beta_1<\beta_2<\beta_3<\beta_4} \delta^1_{\beta_4} p_{\beta_1\beta_2\beta_3\nu} \frac{\partial}{\partial p_{\beta_1\beta_2\beta_3\beta_4}} \\ &= \sum_\nu \frac{\partial X^\nu}{\partial x^1} \sum_{\beta_1<\beta_2<\beta_3<\beta_4} \Big[\delta^1_{\beta_1} p_{\nu\beta_2\beta_3\beta_4} + \delta^1_{\beta_2} p_{\beta_1\nu\beta_3\beta_4} + \delta^1_{\beta_3} p_{\beta_1\beta_2\nu\beta_4} + \delta^1_{\beta_4} p_{\beta_1\beta_2\beta_3\nu}\Big] \frac{\partial}{\partial p_{\beta_1\beta_2\beta_3\beta_4}} \\ &= \sum_\nu \frac{\partial X^\nu}{\partial x^1} \sum_{1<\beta_2<\beta_3<\beta_4} p_{\nu\beta_2\beta_3\beta_4} \frac{\partial}{\partial p_{1\beta_2\beta_3\beta_4}} + \sum_\nu \frac{\partial X^\nu}{\partial x^1} \sum_{\beta_1<1<\beta_3<\beta_4} p_{\beta_1\nu\beta_3\beta_4} \frac{\partial}{\partial p_{\beta_1 1\beta_3\beta_4}} \\ &\quad + \sum_\nu \frac{\partial X^\nu}{\partial x^1} \sum_{\beta_1<\beta_2<1<\beta_4} p_{\beta_1\beta_2\nu\beta_4} \frac{\partial}{\partial p_{\beta_1\beta_2 1\beta_4}} + \sum_\nu \frac{\partial X^\nu}{\partial x^1} \sum_{\beta_1<\beta_2<\beta_3<1} p_{\beta_1\beta_2\beta_3\nu} \frac{\partial}{\partial p_{\beta_1\beta_2\beta_3 1}} \end{aligned} $$

Let notice that if we consider the condition from the different sum over $\beta_1\cdots\beta_n$, then $\beta_1 < 1$ or $\beta_1 < \beta_3 < 1$ or $\beta_1 < \beta_3 < \beta_3 < 1$ leads us to keep only the first term: Then the only term that remain is:

$$ \begin{aligned} {}^{\mathbf{I}_1}\kappa_1 &= \sum_\nu (\frac{\partial X^\nu}{\partial x^1}) \sum_{1<\beta_2<\beta_3<\beta_4} p_{\nu\beta_2\beta_3\beta_4} \frac{\partial}{\partial p_{1\beta_2\beta_3\beta_4}} \\ &= \sum_\nu (\frac{\partial X^\nu}{\partial x^1}) \Big[p_{\nu 234} \frac{\partial}{\partial p_{1234}} + \sum_{4<\beta_4} p_{\nu 23\beta_4} \frac{\partial}{\partial p_{123\beta_4}} \sum_{4<\beta_4} p_{\nu 24\beta_4} \frac{\partial}{\partial p_{124\beta_4}} + \sum_{4<\beta_4} p_{\nu 34\beta_4} \frac{\partial}{\partial p_{134\beta_4}}\Big] \end{aligned} $$

Since we work on the Weyl manifold $\overline{\mathcal{M}}_{\mathsf{DW}}$ the multimomenta involved is much restrained.

$$ {}^{\mathbf{I}_1}\kappa_1 = (\frac{\partial X^1}{\partial x^1}) \mathfrak{e} \frac{\partial}{\partial \mathfrak{e}} + \underset{\circ}{\kappa} + \underset{\circ\circ}{\kappa} + \underset{\bullet}{\kappa} \quad \text{with} $$

$$ \underset{\circ}{\kappa} = \sum_\nu (\frac{\partial X^\nu}{\partial x^1}) \sum_{4<\beta_4} p_{\nu 23\beta_4} \frac{\partial}{\partial p_{123\beta_4}} \qquad \underset{\circ\circ}{\kappa} = \sum_\nu (\frac{\partial X^\nu}{\partial x^1}) \sum_{4<\beta_4} p_{\nu 24\beta_4} \frac{\partial}{\partial p_{124\beta_4}} \qquad \underset{\bullet}{\kappa} = \sum_\nu (\frac{\partial X^\nu}{\partial x^1}) \sum_{4<\beta_4} p_{\nu 34\beta_4} \frac{\partial}{\partial p_{134\beta_4}} $$

So we are interested first in the term $\underset{\circ}{\kappa}$. This one correspond to the restriction $\underset{\circ}{\kappa} = {}^{\mathbf{I}_1}\kappa_1\Big|_{\beta_2=2,\beta_3=4}$ and from now we translate notation on β indices to the apropriate one with our prefered Palatini notations.

$$ \underset{\circ}{\kappa} = \sum_{1\leq\nu\leq n} \sum_{1\leq\mu\leq n} (\frac{\partial X^\nu}{\partial x^1}) p_{\nu 24(\omega_\mu^{IJ})} \frac{\partial}{\partial p_{124(\omega_\mu^{IJ})}} = -\sum_{1\leq\nu\leq n} \sum_{1\leq\mu\leq n} (\frac{\partial X^\nu}{\partial x^1}) p_{\nu 24(\omega_\mu^{IJ})} \frac{\partial}{\partial p_{12(\omega_\mu^{IJ})4}} $$

Since we work with ${}^{\mathbf{I}_1}\kappa_1\Big|_{\beta_2=2,\beta_3=4}$, the only freedom is $\nu = 1$ or $\nu = 3$

$$ \begin{aligned} \underset{\circ}{\kappa} &= \sum_{1\leq\mu\leq n} (\frac{\partial X^1}{\partial x^1}) p_{12(\omega_\mu^{IJ})4} \frac{\partial}{\partial p_{12(\omega_\mu^{IJ})4}} + \sum_{1\leq\mu\leq n} (\frac{\partial X^3}{\partial x^1}) p_{234(\omega_\mu^{IJ})} \frac{\partial}{\partial p_{12(\omega_\mu^{IJ})4}} \\ &= \sum_{1\leq\mu\leq n} \Big[(\frac{\partial X^1}{\partial x^1}) p_{12(\omega_\mu^{IJ})4} + (\frac{\partial X^3}{\partial x^1}) p_{234(\omega_\mu^{IJ})}\Big] \frac{\partial}{\partial p_{12(\omega_\mu^{IJ})4}} \end{aligned} $$

noticing that in our conventions $p_{12(\omega_\mu^{IJ})4} = p_{IJ}^{\omega_\mu 3}$ and $p_{234(\omega_\mu^{IJ})} = -p_{(\omega_\mu^{IJ})234} = -p_{IJ}^{\omega_\mu 1}$

$$ \underset{\circ}{\kappa} = \sum_{1\leq\mu\leq n} \Big[(\frac{\partial X^1}{\partial x^1}) p_{IJ}^{\omega_\mu 3} - (\frac{\partial X^3}{\partial x^1}) p_{IJ}^{\omega_\mu 1}\Big] \frac{\partial}{\partial p_{IJ}^{\omega_\mu 3}} $$

Then, we write also, $\underset{\bullet}{\kappa} = {}^{\mathbf{I}_1}\kappa_1\Big|_{\beta_2=3,\beta_3=4} \Longrightarrow \nu = 1$ or $\nu = 2$, and: So that:

$$ \begin{aligned} \underset{\bullet}{\kappa} &= \sum_{1\leq\nu\leq n} \sum_{1\leq\mu\leq n} (\frac{\partial X^\nu}{\partial x^1}) p_{\nu 34(\omega_\mu^{IJ})} \frac{\partial}{\partial p_{134(\omega_\mu^{IJ})}} = \sum_{1\leq\nu\leq n} \sum_{1\leq\mu\leq n} (\frac{\partial X^\nu}{\partial x^1}) p_{\nu 34(\omega_\mu^{IJ})} \frac{\partial}{\partial p_{1(\omega_\mu^{IJ})34}} \\ &= \sum_{1\leq\mu\leq n} \Big[(\frac{\partial X^1}{\partial x^1}) p_{134(\omega_\mu^{IJ})} + (\frac{\partial X^2}{\partial x^1}) p_{234(\omega_\mu^{IJ})}\Big] \frac{\partial}{\partial p_{1(\omega_\mu^{IJ})34}} = \sum_{1\leq\mu\leq n} \Big[(\frac{\partial X^1}{\partial x^1}) p_{IJ}^{\omega_\mu 2} - (\frac{\partial X^2}{\partial x^1}) p_{IJ}^{\omega_\mu 1}\Big] \frac{\partial}{\partial p_{IJ}^{\omega_\mu 2}} \end{aligned} $$

Finally we consider the last term $\underset{\circ\circ}{\kappa} = {}^{\mathbf{I}_1}\kappa_1\Big|_{\beta_2=2,\beta_3=3} \Longrightarrow \nu = 1$ or $\nu = 4,$ and:

$$\begin{aligned}\underset{\circ\circ}{\kappa} &= \sum_{1\leq\nu\leq n}\sum_{1\leq\mu\leq n}(\frac{\partial X^\nu}{\partial x^1})p_{\nu 23(\omega_\mu^{IJ})}\frac{\partial}{\partial p_{123(\omega_\mu^{IJ})}} = \sum_{1\leq\nu\leq n}\sum_{1\leq\mu\leq n}(\frac{\partial X^\nu}{\partial x^1})p_{\nu 23(\omega_\mu^{IJ})}\frac{\partial}{\partial p_{IJ}^{\omega_\mu 4}} \\ &= \sum_{1\leq\mu\leq n}\Big[(\frac{\partial X^1}{\partial x^1})p_{123(\omega_\mu^{IJ})} + (\frac{\partial X^4}{\partial x^1})p_{423(\omega_\mu^{IJ})}\Big]\frac{\partial}{\partial p_{IJ}^{\omega_\mu 4}} = \sum_{1\leq\mu\leq n}\Big[(\frac{\partial X^1}{\partial x^1})p_{IJ}^{\omega_\mu 4} - (\frac{\partial X^4}{\partial x^1})p_{IJ}^{\omega_\mu 1}\Big]\frac{\partial}{\partial p_{IJ}^{\omega_\mu 4}}\end{aligned}$$

Calculation of the term ${}^{\mathbf{I}_1}\kappa_2 = (\kappa_{\mathbf{I}_1})\Big|_{\rho=2}$

Now, we continue our path, the second term in $\kappa_{\mathbf{I}_1}$ one writes ${}^{\mathbf{I}_1}\kappa_2$:

$${}^{\mathbf{I}_1}\kappa_2 = (\kappa_{\mathbf{I}_1})\Big|_{\rho=2} = \sum_\nu \frac{\partial X^\nu}{\partial x^2}\sum_{\beta_1<\cdots<\beta_n}\sum_\eta \delta^2_{\beta_\eta} p_{\beta_1\cdots\beta_{\eta-1}\nu\beta_{\eta+1}\cdots\beta_n}\frac{\partial}{\partial p_{\beta_1\cdots\beta_n}}$$

Then, we expand the index η from 1 to n.

$${}^{\mathbf{I}_1}\kappa_2 = \sum_\nu \frac{\partial X^\nu}{\partial x^2}\sum_{\beta_1<\beta_3<\beta_3<\beta_4}\Big[\delta^2_{\beta_1}p_{\nu\beta_2\beta_3\beta_4} + \delta^2_{\beta_2}p_{\beta_1\nu\beta_3\beta_4} + \delta^2_{\beta_3}p_{\beta_1\beta_2\nu\beta_4} + \delta^2_{\beta_4}p_{\beta_1\beta_2\beta_3\nu}\Big]\frac{\partial}{\partial p_{\beta_1\beta_2\beta_3\beta_4}}$$

$$\begin{aligned}{}^{\mathbf{I}_1}\kappa_2 &= \sum_\nu \frac{\partial X^\nu}{\partial x^2}\sum_{2<\beta_2<\beta_3<\beta_4} p_{\nu\beta_2\beta_3\beta_4}\frac{\partial}{\partial p_{2\beta_2\beta_3\beta_4}} + \sum_\nu \frac{\partial X^\nu}{\partial x^2}\sum_{\beta_1<2<\beta_3<\beta_4} p_{\beta_1\nu\beta_3\beta_4}\frac{\partial}{\partial p_{\beta_1 2\beta_3\beta_4}} \\ &+ \sum_\nu \frac{\partial X^\nu}{\partial x^2}\sum_{\beta_1<\beta_2<2<\beta_4} p_{\beta_1\beta_2\nu\beta_4}\frac{\partial}{\partial p_{\beta_1\beta_2 2\beta_4}} + \sum_\nu \frac{\partial X^\nu}{\partial x^2}\sum_{\beta_1<\beta_3<\beta_3<2} p_{\beta_1\beta_2\beta_3\nu}\frac{\partial}{\partial p_{\beta_1\beta_2\beta_3 2}}\end{aligned}$$

Then the only terms that remain are:

$${}^{\mathbf{I}_1}\kappa_2 = \underbrace{\sum_\nu \frac{\partial X^\nu}{\partial x^2}\sum_{2<\beta_2<\beta_3<\beta_4} p_{\nu\beta_2\beta_3\beta_4}\frac{\partial}{\partial p_{2\beta_2\beta_3\beta_4}}}_{\underset{\bullet}{\kappa}} + \underbrace{\sum_\nu \frac{\partial X^\nu}{\partial x^2}\sum_{\beta_1<2<\beta_3<\beta_4} p_{\beta_1\nu\beta_3\beta_4}\frac{\partial}{\partial p_{\beta_1 2\beta_3\beta_4}}}_{\underset{\circ}{\kappa}}$$

Therefore we denote: ${}^{\mathbf{I}_1}\kappa_2 = \underset{\bullet}{\kappa} + \underset{\circ}{\kappa}$. Since we work on the Weyl manifold $\overline{\mathcal{M}}_{\mathsf{DW}}$ we have:

$$\underset{\bullet}{\kappa} = \sum_\nu \frac{\partial X^\nu}{\partial x^2}\sum_{4<\beta_4} p_{\nu 34\beta_4}\frac{\partial}{\partial p_{234\beta_4}} = \sum_{1\leq\mu\leq n}\Big[\frac{\partial X^1}{\partial x^2}p_{134(\omega_\mu^{IJ})} + \frac{\partial X^2}{\partial x^2}p_{234(\omega_\mu^{IJ})} + \frac{\partial X^3}{\partial x^2}p_{334(\omega_\mu^{IJ})} + \frac{\partial X^4}{\partial x^2}p_{434(\omega_\mu^{IJ})}\Big]\frac{\partial}{\partial p_{234(\omega_\mu^{IJ})}}$$

$$\underset{\bullet}{\kappa} = \sum_{1\leq\mu\leq n}\Big[\frac{\partial X^1}{\partial x^2}p_{134(\omega_\mu^{IJ})} + \frac{\partial X^2}{\partial x^2}p_{234(\omega_\mu^{IJ})}\Big]\frac{\partial}{\partial p_{234(\omega_\mu^{IJ})}} = -\sum_{1\leq\mu\leq n}\frac{\partial X^1}{\partial x^2}p_{IJ}^{\omega_\mu 2}\frac{\partial}{\partial p_{IJ}^{\omega_\mu 1}} + \sum_{1\leq\mu\leq n}\frac{\partial X^2}{\partial x^2}p_{IJ}^{\omega_\mu 1}\frac{\partial}{\partial p_{IJ}^{\omega_\mu 1}}$$

On the other hand, we have for the term $\underset{\circ}{\kappa} = \sum_\nu \frac{\partial X^\nu}{\partial x^2}\sum_{\beta_1<2<\beta_3<\beta_4} p_{\beta_1\nu\beta_3\beta_4}\frac{\partial}{\partial p_{\beta_1 2\beta_3\beta_4}}$ the following expression:

$$\underset{\circ}{\kappa} = \sum_{\beta_1<2<\beta_3<\beta_4}\Big[(\frac{\partial X^1}{\partial x^2})p_{\beta_1 1\beta_3\beta_4} + (\frac{\partial X^2}{\partial x^2})p_{\beta_1 2\beta_3\beta_4} + (\frac{\partial X^3}{\partial x^2})p_{\beta_1 3\beta_3\beta_4} + (\frac{\partial X^4}{\partial x^2})p_{\beta_1 4\beta_3\beta_4}\Big]\frac{\partial}{\partial p_{\beta_1 2\beta_3\beta_4}}$$

$$\underset{\circ}{\kappa} = \sum_{2<\beta_3<\beta_4}\Big[(\frac{\partial X^2}{\partial x^2})p_{12\beta_3\beta_4} + (\frac{\partial X^3}{\partial x^2})p_{13\beta_3\beta_4} + (\frac{\partial X^4}{\partial x^2})p_{14\beta_3\beta_4}\Big]\frac{\partial}{\partial p_{12\beta_3\beta_4}}$$

$$\begin{aligned}\underset{\circ}{\kappa} &= (\frac{\partial X^2}{\partial x^2})\{p_{1234}\frac{\partial}{\partial p_{1234}} + \sum_{1\leq\mu\leq n}\Big[p_{123(\omega_\mu^{IJ})}\frac{\partial}{\partial p_{123(\omega_\mu^{IJ})}} + p_{124(\omega_\mu^{IJ})}\frac{\partial}{\partial p_{124(\omega_\mu^{IJ})}}\Big]\} \\ &+ \sum_{1\leq\mu\leq n}\Big[(\frac{\partial X^3}{\partial x^2})p_{134(\omega_\mu^{IJ})}\frac{\partial}{\partial p_{124(\omega_\mu^{IJ})}} + (\frac{\partial X^4}{\partial x^2})p_{143(\omega_\mu^{IJ})}\frac{\partial}{\partial p_{123(\omega_\mu^{IJ})}}\Big]\end{aligned}$$

$$\underset{\circ}{\kappa} = (\frac{\partial X^2}{\partial x^2})\mathfrak{e}\frac{\partial}{\partial\mathfrak{e}} + \sum_{1\leq\mu\leq n}\{(\frac{\partial X^2}{\partial x^2})\Big[p_{IJ}^{\omega_\mu 4}\frac{\partial}{\partial p_{IJ}^{\omega_\mu 4}} - p_{IJ}^{\omega_\mu 3}\frac{\partial}{\partial p_{IJ}^{\omega_\mu 3}}\Big] - (\frac{\partial X^3}{\partial x^2})p_{IJ}^{\omega_\mu 2}\frac{\partial}{\partial p_{IJ}^{\omega_\mu 3}} + (\frac{\partial X^4}{\partial x^2})p_{IJ}^{\omega_\mu 2}\frac{\partial}{\partial p_{IJ}^{\omega_\mu 4}}\}$$

Calculation of the term ${}^{\mathbf{I}_1}\kappa_3 = (\kappa_{\mathbf{I}_1})\Big|_{\rho=3}$

Now, the third term in $\kappa_{\mathbf{I}_1}$ writes ${}^{\mathbf{I}_1}\kappa_3 = (\kappa_{\mathbf{I}_1})\Big|_{\rho=3} = \sum_\nu \frac{\partial X^\nu}{\partial x^3}\sum_{\beta_1<\cdots<\beta_n}\sum_\eta \delta^3_{\beta_\eta}p_{\beta_1\cdots\beta_{\eta-1}\nu\beta_{\eta+1}\cdots\beta_n}\frac{\partial}{\partial p_{\beta_1\cdots\beta_n}}$

$${}^{\mathbf{I}_1}\kappa_3 = \sum_\nu(\frac{\partial X^\nu}{\partial x^3})\sum_{\beta_1<\beta_3<\beta_3<\beta_4}\Big[\delta^3_{\beta_1}p_{\nu\beta_2\beta_3\beta_4} + \delta^3_{\beta_2}p_{\beta_1\nu\beta_3\beta_4} + \delta^3_{\beta_3}p_{\beta_1\beta_2\nu\beta_4} + \delta^3_{\beta_4}p_{\beta_1\beta_2\beta_3\nu}\Big]\frac{\partial}{\partial p_{\beta_1\beta_2\beta_3\beta_4}}$$

$$^{\mathbf{I}_1}\kappa_3 = \sum_\nu (\frac{\partial X^\nu}{\partial x^3})\Big[\sum_{\beta_1<3<\beta_3<\beta_4} p_{\beta_1\nu\beta_3\beta_4}\frac{\partial}{\partial p_{\beta_1 3\beta_3\beta_4}} + \sum_{\beta_1<\beta_2<3<\beta_4} p_{\beta_1\beta_2\nu\beta_4}\frac{\partial}{\partial p_{\beta_1\beta_2 3\beta_4}}\Big]$$

$$^{\mathbf{I}_1}\kappa_3 = \sum_\nu (\frac{\partial X^\nu}{\partial x^3})\{\Big[\sum_{3<\beta_3<\beta_4} p_{1\nu\beta_3\beta_4}\frac{\partial}{\partial p_{13\beta_3\beta_4}} + \sum_{3<\beta_3<\beta_4} p_{2\nu\beta_3\beta_4}\frac{\partial}{\partial p_{23\beta_3\beta_4}}\Big] + \sum_{1\le\mu\le n} p_{12\nu(\omega^{IJ}_\mu)}\frac{\partial}{\partial p_{123(\omega^{IJ}_\mu)}}\}$$

written $^{\mathbf{I}_1}\kappa_3 = \underset{\circ}{\kappa} + \underset{\circ\circ}{\kappa} + \underset{\bullet}{\kappa}$. The first term $\underset{\circ}{\kappa}$ is:

$$\underset{\circ}{\kappa} = \sum_\nu (\frac{\partial X^\nu}{\partial x^3})\sum_{3<\beta_3<\beta_4} p_{1\nu\beta_3\beta_4}\frac{\partial}{\partial p_{13\beta_3\beta_4}} = \sum_\nu (\frac{\partial X^\nu}{\partial x^3})\sum_{1\le\mu\le n} p_{1\nu 3(\omega^{IJ}_\mu)}\frac{\partial}{\partial p_{134(\omega^{IJ}_\mu)}}$$

$$\underset{\circ}{\kappa} = \sum_\nu (\frac{\partial X^\nu}{\partial x^3})\sum_{1\le\mu\le n} p_{1\nu 3(\omega^{IJ}_\mu)}\frac{\partial}{\partial p^{\omega_\mu 2}_{IJ}} = \sum_{1\le\mu\le n}\Big[(\frac{\partial X^2}{\partial x^3})p^{\omega_\mu 4}_{IJ} - (\frac{\partial X^4}{\partial x^3})p^{\omega_\mu 2}_{IJ}\Big]\frac{\partial}{\partial p^{\omega_\mu 2}_{IJ}}$$

whereas the second term $\underset{\circ\circ}{\kappa}$ gives:

$$\underset{\circ\circ}{\kappa} = \sum_\nu (\frac{\partial X^\nu}{\partial x^3})\sum_{3<\beta_3<\beta_4} p_{2\nu\beta_3\beta_4}\frac{\partial}{\partial p_{23\beta_3\beta_4}}$$

$$\underset{\circ\circ}{\kappa} = \sum_\nu (\frac{\partial X^\nu}{\partial x^3})\sum_{1\le\mu\le n} p_{2\nu 4(\omega^{IJ}_\mu)}\frac{\partial}{\partial p_{234(\omega^{IJ}_\mu)}} = -\sum_{1\le\mu\le n}\Big[(\frac{\partial X^1}{\partial x^3})p^{\omega_\mu 3}_{IJ} - (\frac{\partial X^3}{\partial x^3})p^{\omega_\mu 1}_{IJ}\Big]\frac{\partial}{\partial p^{\omega_\mu 1}_{IJ}}$$

and finally the third $\underset{\bullet}{\kappa}$ gives:

$$\underset{\bullet}{\kappa} = \sum_\nu (\frac{\partial X^\nu}{\partial x^3})\sum_{1\le\mu\le n} p_{12\nu(\omega^{IJ}_\mu)}\frac{\partial}{\partial p_{123(\omega^{IJ}_\mu)}} = -\sum_{1\le\mu\le n}\Big[(\frac{\partial X^3}{\partial x^3})p^{\omega_\mu 4}_{IJ} - (\frac{\partial X^4}{\partial x^3})p^{\omega_\mu 3}_{IJ}\Big]\frac{\partial}{\partial p^{\omega_\mu 4}_{IJ}}$$

Calculation of the term $^{\mathbf{I}_1}\kappa_4 = (\kappa_{\mathbf{I}_1})\Big|_{\rho=4}$

Finally we attack the last term in $\kappa_{\mathbf{I}_1}$ one writes $^{\mathbf{I}_1}\kappa_4$:

$$^{\mathbf{I}_1}\kappa_4 = (\kappa_{\mathbf{I}_1})\Big|_{\rho=4} = \sum_\nu \frac{\partial X^\nu}{\partial x^4}\sum_{\beta_1<\cdots<\beta_n}\sum_\eta \delta^4_{\beta_\eta} p_{\beta_1\cdots\beta_{\eta-1}\nu\beta_{\eta+1}\cdots\beta_n}\frac{\partial}{\partial p_{\beta_1\cdots\beta_n}}$$

Once again, the same tedious but straightforward calculation gives:

$$^{\mathbf{I}_1}\kappa_4 = \sum_\nu \frac{\partial X^\nu}{\partial x^4}\sum_{4<\beta_3<\beta_3<\beta_4} p_{\nu\beta_2\beta_3\beta_4}\frac{\partial}{\partial p_{4\beta_2\beta_3\beta_4}} + \sum_\nu \frac{\partial X^\nu}{\partial x^4}\sum_{\beta_1<4<\beta_3<\beta_4} p_{\beta_1\nu\beta_3\beta_4}\frac{\partial}{\partial p_{\beta_1 4\beta_3\beta_4}}$$

$$+\sum_\nu \frac{\partial X^\nu}{\partial x^4}\sum_{\beta_1<\beta_2<4<\beta_4} p_{\beta_1\beta_2\nu\beta_4}\frac{\partial}{\partial p_{\beta_1\beta_2 4\beta_4}} + \sum_\nu \frac{\partial X^\nu}{\partial x^4}\sum_{\beta_1<\beta_2<\beta_3<4} p_{\beta_1\beta_2\beta_3\nu}\frac{\partial}{\partial p_{\beta_1\beta_2\beta_3 4}}$$

It remain only:

$$^{\mathbf{I}_1}\kappa_4 = \sum_\nu \frac{\partial X^\nu}{\partial x^4}\Big[\sum_{\beta_1<\beta_2<4<\beta_4} p_{\beta_1\beta_2\nu\beta_4}\frac{\partial}{\partial p_{\beta_1\beta_2 4\beta_4}} + \sum_{\beta_1<\beta_2<\beta_3<4} p_{\beta_1\beta_2\beta_3\nu}\frac{\partial}{\partial p_{\beta_1\beta_2\beta_3 4}}\Big]$$

$$^{\mathbf{I}_1}\kappa_4 = \sum_\nu (\frac{\partial X^\nu}{\partial x^4})\sum_{1\le\mu\le n}\Big[\sum_{\beta_1<\beta_2<4} p_{\beta_1\beta_2\nu(\omega^{IJ}_\mu)}\frac{\partial}{\partial p_{\beta_1\beta_2 4(\omega^{IJ}_\mu)}}\Big] + (\frac{\partial X^4}{\partial x^4})\mathfrak{e}\frac{\partial}{\partial\mathfrak{e}}$$

We denote $^{\mathbf{I}_1}\kappa_4 = \underset{\circ}{\kappa} + \underset{\circ\circ}{\kappa} + \underset{\bullet}{\kappa}$ with $\underset{\circ}{\kappa} = {}^{\mathbf{I}_1}\kappa_4\Big|_{\beta_1=1,\beta_2=2}$, $\underset{\circ\circ}{\kappa} = {}^{\mathbf{I}_1}\kappa_4\Big|_{\beta_1=1,\beta_2=3}$ and finally $\underset{\bullet}{\kappa} = {}^{\mathbf{I}_1}\kappa_4\Big|_{\beta_1=2,\beta_2=3}$, the first $\underset{\circ}{\kappa} = {}^{\mathbf{I}_1}\kappa_4\Big|_{\beta_1=1,\beta_2=2}$ imply that $\nu=3$ or $\nu=4$

$$\underset{\circ}{\kappa} = (\frac{\partial X^3}{\partial x^4})p_{123(\omega^{IJ}_\mu)}\frac{\partial}{\partial p_{124(\omega^{IJ}_\mu)}} + (\frac{\partial X^4}{\partial x^4})p_{124(\omega^{IJ}_\mu)}\frac{\partial}{\partial p_{124(\omega^{IJ}_\mu)}} = \Big[-(\frac{\partial X^3}{\partial x^4})p^{\omega_\mu 4}_{IJ} + (\frac{\partial X^4}{\partial x^4})p^{\omega_\mu 3}_{IJ}\Big]\frac{\partial}{\partial p^{\omega_\mu 3}_{IJ}}$$

$$^{\mathbf{I}_1}\kappa_4 = \sum_\nu (\frac{\partial X^\nu}{\partial x^4})\sum_{1\le\mu\le n}\Big[\sum_{\beta_1<\beta_2<4} p_{\beta_1\beta_2\nu(\omega^{IJ}_\mu)}\frac{\partial}{\partial p_{\beta_1\beta_2 4(\omega^{IJ}_\mu)}}\Big] + (\frac{\partial X^4}{\partial x^4})\mathfrak{e}\frac{\partial}{\partial\mathfrak{e}}$$

Then, the second gives: $\underset{\circ\circ}{\kappa} = {}^{\mathbf{I}_1}\kappa_4\Big|_{\beta_1=1,\beta_2=3}$ so that $\nu=2$ or $\nu=4$,

$$\underset{\circ\circ}{\kappa} = (\frac{\partial X^2}{\partial x^4})p_{132(\omega^{IJ}_\mu)}\frac{\partial}{\partial p_{134(\omega^{IJ}_\mu)}} + (\frac{\partial X^4}{\partial x^4})p_{134(\omega^{IJ}_\mu)}\frac{\partial}{\partial p_{134(\omega^{IJ}_\mu)}} = \Big[-(\frac{\partial X^2}{\partial x^4})p^{\omega_\mu 4}_{IJ} + (\frac{\partial X^4}{\partial x^4})p^{\omega_\mu 2}_{IJ}\Big]\frac{\partial}{\partial p^{\omega_\mu 2}_{IJ}}$$

and finally the third gives: $\underset{\bullet}{\kappa} = {}^{\mathbf{I}_1}\kappa_4\Big|_{\beta_1=2,\beta_2=3}$ so that $\nu = 1$ or $\nu = 4$,

$$\underset{\bullet}{\kappa} = (\frac{\partial X^1}{\partial x^4})p_{231(\omega_\mu^{IJ})}\frac{\partial}{\partial p_{234(\omega_\mu^{IJ})}} + (\frac{\partial X^4}{\partial x^4})p_{234(\omega_\mu^{IJ})}\frac{\partial}{\partial p_{234(\omega_\mu^{IJ})}} = -\Big[(\frac{\partial X^1}{\partial x^4})p_{IJ}^{\omega_\mu 4} - (\frac{\partial X^4}{\partial x^4})p_{IJ}^{\omega_\mu 1}\Big]\frac{\partial}{\partial p_{IJ}^{\omega_\mu 1}}$$

Finally we can express the term:

$$\kappa_{\mathbf{I}_1} = \sum_{\nu,\rho}\frac{\partial X^\nu}{\partial x^\rho}\overset{\circ\rho}{p_\nu} = \kappa_{\mathbf{I}_1} = {}^{\mathbf{I}_1}\kappa_1 + {}^{\mathbf{I}_1}\kappa_2 + {}^{\mathbf{I}_1}\kappa_3 + {}^{\mathbf{I}_1}\kappa_4$$

$$\kappa_{\mathbf{I}_1} = \Big[(\frac{\partial X^1}{\partial x^1}) + (\frac{\partial X^2}{\partial x^2}) + (\frac{\partial X^3}{\partial x^3}) + (\frac{\partial X^4}{\partial x^4})\Big]\mathfrak{e}\frac{\partial}{\partial\mathfrak{e}}$$
$$-\sum_{1\le\mu\le n}\{\Big[(\frac{\partial X^1}{\partial x^2})p_{IJ}^{\omega_\mu 2} + (\frac{\partial X^1}{\partial x^4})p_{IJ}^{\omega_\mu 4} + (\frac{\partial X^1}{\partial x^3})p_{IJ}^{\omega_\mu 3} - \Big[(\frac{\partial X^2}{\partial x^2}) + (\frac{\partial X^3}{\partial x^3}) + (\frac{\partial X^4}{\partial x^4})\Big]p_{IJ}^{\omega_\mu 1}\}\frac{\partial}{\partial p_{IJ}^{\omega_\mu 1}}$$
$$-\sum_{1\le\mu\le n}\{\Big[(\frac{\partial X^2}{\partial x^1})p_{IJ}^{\omega_\mu 1} + (\frac{\partial X^2}{\partial x^4})p_{IJ}^{\omega_\mu 4} + (\frac{\partial X^2}{\partial x^3})p_{IJ}^{\omega_\mu 3} - \Big[(\frac{\partial X^1}{\partial x^1}) + (\frac{\partial X^3}{\partial x^3}) + (\frac{\partial X^4}{\partial x^4})\Big]p_{IJ}^{\omega_\mu 2}\}\frac{\partial}{\partial p_{IJ}^{\omega_\mu 2}}$$
$$-\sum_{1\le\mu\le n}\{\Big[(\frac{\partial X^3}{\partial x^1})p_{IJ}^{\omega_\mu 1} + (\frac{\partial X^3}{\partial x^2})p_{IJ}^{\omega_\mu 2} + (\frac{\partial X^3}{\partial x^4})p_{IJ}^{\omega_\mu 4} - \Big[(\frac{\partial X^1}{\partial x^1}) + (\frac{\partial X^2}{\partial x^2}) + (\frac{\partial X^4}{\partial x^4})\Big]p_{IJ}^{\omega_\mu 3}\}\frac{\partial}{\partial p_{IJ}^{\omega_\mu 3}}$$
$$-\sum_{1\le\mu\le n}\{\Big[(\frac{\partial X^4}{\partial x^1})p_{IJ}^{\omega_\mu 1} + (\frac{\partial X^4}{\partial x^2})p_{IJ}^{\omega_\mu 2} + (\frac{\partial X^4}{\partial x^3})p_{IJ}^{\omega_\mu 3} - \Big[(\frac{\partial X^1}{\partial x^1}) + (\frac{\partial X^2}{\partial x^2}) + (\frac{\partial X^3}{\partial x^3})\Big]p_{IJ}^{\omega_\mu 4}\}\frac{\partial}{\partial p_{IJ}^{\omega_\mu 4}}$$

$$\kappa_{\mathbf{I}_1} = \sum_\lambda(\frac{\partial X^\lambda}{\partial x^\lambda})\mathfrak{e}\frac{\partial}{\partial\mathfrak{e}} - \sum_{1\le\mu\le n}\{\sum_\rho(\frac{\partial X^1}{\partial x^\rho})p_{IJ}^{\omega_\mu\rho} - \frac{\partial X^1}{\partial x^1}p_{IJ}^{\omega_\mu 1} - \sum_\lambda(\frac{\partial X^\lambda}{\partial x^\lambda})p_{IJ}^{\omega_\mu 1} + \frac{\partial X^1}{\partial x^1}p_{IJ}^{\omega_\mu 1}\Big]\}\frac{\partial}{\partial p_{IJ}^{\omega_\mu 1}}$$
$$-\sum_{1\le\mu\le n}\{\sum_\rho(\frac{\partial X^2}{\partial x^\rho})p_{IJ}^{\omega_\mu\rho} - \frac{\partial X^2}{\partial x^2}p_{IJ}^{\omega_\mu 2} - \sum_\lambda(\frac{\partial X^\lambda}{\partial x^\lambda})p_{IJ}^{\omega_\mu 2} + \frac{\partial X^2}{\partial x^2}p_{IJ}^{\omega_\mu 2}\Big]\}\frac{\partial}{\partial p_{IJ}^{\omega_\mu 2}}$$
$$-\sum_{1\le\mu\le n}\{\sum_\rho(\frac{\partial X^3}{\partial x^\rho})p_{IJ}^{\omega_\mu\rho} - \frac{\partial X^3}{\partial x^3}p_{IJ}^{\omega_\mu 3} - \sum_\lambda(\frac{\partial X^\lambda}{\partial x^\lambda})p_{IJ}^{\omega_\mu 3} + \frac{\partial X^3}{\partial x^3}p_{IJ}^{\omega_\mu 3}\Big]\}\frac{\partial}{\partial p_{IJ}^{\omega_\mu 3}}$$
$$-\sum_{1\le\mu\le n}\{\sum_\rho(\frac{\partial X^4}{\partial x^\rho})p_{IJ}^{\omega_\mu\rho} - \frac{\partial X^4}{\partial x^4}p_{IJ}^{\omega_\mu 4} - \sum_\lambda(\frac{\partial X^\lambda}{\partial x^\lambda})p_{IJ}^{\omega_\mu 4} + \frac{\partial X^4}{\partial x^4}p_{IJ}^{\omega_\mu 4}\Big]\}\frac{\partial}{\partial p_{IJ}^{\omega_\mu 4}}$$

$$\kappa_{\mathbf{I}_1} = \sum_\lambda(\frac{\partial X^\lambda}{\partial x^\lambda})\mathfrak{e}\frac{\partial}{\partial\mathfrak{e}} - \sum_{1\le\mu\le n}\{\sum_\rho(\frac{\partial X^1}{\partial x^\rho})p_{IJ}^{\omega_\mu\rho} - \sum_\lambda(\frac{\partial X^\lambda}{\partial x^\lambda})p_{IJ}^{\omega_\mu 1}\Big]\}\frac{\partial}{\partial p_{IJ}^{\omega_\mu 1}} - \sum_{1\le\mu\le n}\{\sum_\rho(\frac{\partial X^2}{\partial x^\rho})p_{IJ}^{\omega_\mu\rho} - \sum_\lambda(\frac{\partial X^\lambda}{\partial x^\lambda})p_{IJ}^{\omega_\mu 2}\Big]\}\frac{\partial}{\partial p_{IJ}^{\omega_\mu 2}}$$
$$-\sum_{1\le\mu\le n}\{\sum_\rho(\frac{\partial X^3}{\partial x^\rho})p_{IJ}^{\omega_\mu\rho} - \sum_\lambda(\frac{\partial X^\lambda}{\partial x^\lambda})p_{IJ}^{\omega_\mu 3}\Big]\}\frac{\partial}{\partial p_{IJ}^{\omega_\mu 3}} - \sum_{1\le\mu\le n}\{\sum_\rho(\frac{\partial X^4}{\partial x^\rho})p_{IJ}^{\omega_\mu\rho} - \sum_\lambda(\frac{\partial X^\lambda}{\partial x^\lambda})p_{IJ}^{\omega_\mu 4}\Big]\}\frac{\partial}{\partial p_{IJ}^{\omega_\mu 4}}$$

We write it as:

$$\kappa_{\mathbf{I}_1} = (\frac{\partial X^\lambda}{\partial x^\lambda})\mathfrak{e}\frac{\partial}{\partial\mathfrak{e}} - \{\Big[-p_{IJ}^{\omega_\mu\nu}(\frac{\partial X^\lambda}{\partial x^\lambda}) + p_{IJ}^{\omega_\mu\rho}(\frac{\partial X^\nu}{\partial x^\rho})\Big]\}\frac{\partial}{\partial p_{IJ}^{\omega_\mu\nu}} = (\frac{\partial X^\lambda}{\partial x^\lambda})\mathfrak{e}\frac{\partial}{\partial\mathfrak{e}} - \{\Big[-p_{IJ}^{\omega_\mu\rho}\delta_\rho^\nu(\frac{\partial X^\lambda}{\partial x^\lambda}) + p_{IJ}^{\omega_\mu\rho}(\frac{\partial X^\nu}{\partial x^\rho})\Big]\}\frac{\partial}{\partial p_{IJ}^{\omega_\mu\nu}}$$

We conclude that:

$$\kappa_{\mathbf{I}_1} = (\frac{\partial X^\lambda}{\partial x^\lambda})\mathfrak{e}\frac{\partial}{\partial\mathfrak{e}} - \{p_{IJ}^{\omega_\mu\sigma}\Big[(\frac{\partial X^\nu}{\partial x^\sigma}) - \delta_\sigma^\nu(\frac{\partial X^\lambda}{\partial x^\lambda})\Big]\}\frac{\partial}{\partial p_{IJ}^{\omega_\mu\nu}} \tag{475}$$

The term $\kappa_{\mathbf{I}_2}$ identically vanishes, due to the choice of De Donder-Weyl theory.

$$\kappa_{\mathbf{I}_2} = \sum_{\nu,\sigma}\frac{\partial X^\nu}{\partial e_\sigma^N}\Pi_\nu^{(e_\sigma^N)} = 0 \tag{476}$$

Now, one concentrate on

$$\kappa_{\mathbf{I}_3} = \sum_{\nu,\rho}\frac{\partial X^\nu}{\partial\omega_\rho^{KL}}\Pi_\nu^{(\omega_\rho^{KL})} = \sum_{\nu,\rho}\frac{\partial X^\nu}{\partial\omega_\rho^{KL}}\sum_\eta\delta_{\beta_\eta}^{(\omega_\rho^{KL})}p_{\beta_1\cdots\beta_{\mu-1}\nu\beta_{\mu+1}\cdots\beta_n}\frac{\partial}{\partial p_{\beta_1\cdots\beta_n}}$$

$${}^{\mathbf{I}_3}\kappa_1 = (\kappa_{\mathbf{I}_1})\Big|_{\omega_\rho^{KL}=\omega_1^{KL}} = \sum_\nu(\frac{\partial X^\nu}{\partial\omega_1^{KL}})\sum_{\beta_1<\cdots<\beta_n}\sum_\eta\delta_{\beta_\eta}^{\omega_1^{KL}}p_{\beta_1\cdots\beta_{\eta-1}\nu\beta_{\eta+1}\cdots\beta_n}\frac{\partial}{\partial p_{\beta_1\cdots\beta_n}}$$

$$^{\mathbf{I}_3}\kappa_1 = \sum_{\nu}(\frac{\partial X^{\nu}}{\partial \omega_1^{KL}})\{ \sum_{(\omega_1^{KL})<\beta_2<\beta_3<\beta_4} p_{\nu\beta_2\beta_3\beta_4}\frac{\partial}{\partial p_{(\omega_1^{KL})\beta_2\beta_3\beta_4}} \sum_{\beta_1<(\omega_1^{KL})<\beta_3<\beta_4} p_{\beta_1\nu\beta_3\beta_4}\frac{\partial}{\partial p_{\beta_1(\omega_1^{KL})\beta_3\beta_4}}$$
$$\sum_{\beta_1<\beta_2<(\omega_1^{KL})<\beta_4} p_{\beta_1\beta_2\nu\beta_4}\frac{\partial}{\partial p_{\beta_1\beta_2(\omega_1^{KL})\beta_4}} + \sum_{\beta_1<\beta_2<\beta_3<(\omega_1^{KL})} p_{\beta_1\beta_2\beta_3\nu}\frac{\partial}{\partial p_{\beta_1\beta_2\beta_3(\omega_1^{KL})}}\}$$

Then,

$$^{\mathbf{I}_3}\kappa_1 = \mathfrak{e}\Big[(\frac{\partial X^4}{\partial \omega_1^{KL}})\frac{\partial}{\partial p_{KL}^{\omega_1 4}} + (\frac{\partial X^3}{\partial \omega_1^{KL}})\frac{\partial}{\partial p_{KL}^{\omega_1 3}} + (\frac{\partial X^2}{\partial \omega_1^{KL}})\frac{\partial}{\partial p_{KL}^{\omega_1 2}} + (\frac{\partial X^1}{\partial \omega_1^{KL}})\frac{\partial}{\partial p_{KL}^{\omega_1 1}}\Big]$$

We can write for general $1 \leq \rho \leq n$

$$^{\mathbf{I}_3}\kappa_{\mu} = \mathfrak{e}\sum_{1\leq\mu\leq n}(\frac{\partial X^{\nu}}{\partial \omega_{\rho}^{KL}})\frac{\partial}{\partial p_{KL}^{\omega_{\rho}\nu}} \Longrightarrow \kappa_{\mathbf{I}_3} = \mathfrak{e}\sum_{1\leq\nu,\rho\leq n}(\frac{\partial X^{\nu}}{\partial \omega_{\rho}^{KL}})\frac{\partial}{\partial p_{KL}^{\omega_{\rho}\nu}} \tag{477}$$

Then from (475) (476) (477) we obtain the term $\kappa_{\mathbf{I}} = \kappa_{\mathbf{I}_1} + \kappa_{\mathbf{I}_2} + \kappa_{\mathbf{I}_3}$ wich is:

$$\kappa_{\mathbf{I}} = (\frac{\partial X^{\lambda}}{\partial x^{\lambda}})\mathfrak{e}\frac{\partial}{\partial \mathfrak{e}} - p_{IJ}^{\omega_{\mu}\sigma}\{\Big[(\frac{\partial X^{\nu}}{\partial x^{\sigma}}) - \delta_{\sigma}^{\nu}(\frac{\partial X^{\lambda}}{\partial x^{\lambda}})\Big]\} + \mathfrak{e}(\frac{\partial X^{\nu}}{\partial \omega_{\mu}^{IJ}})\frac{\partial}{\partial p_{IJ}^{\omega_{\mu}\nu}} \tag{478}$$

It remain for us to evaluate the terms involved in $\kappa_{\mathbf{III}}$, then the first of it is $\kappa_{\mathbf{III}_1}$

$$\kappa_{\mathbf{III}_1} = \sum_{\mu,\nu}\frac{\partial \Theta_{\mu}^{IJ}}{\partial x^{\nu}}\Pi^{\nu}_{(\omega_{\mu}^{IJ})} = \sum_{\nu,\mu}\frac{\partial \Theta_{\mu}^{IJ}}{\partial x^{\nu}}\sum_{\beta_1<\cdots<\beta_n}\sum_{\eta}\delta^{\nu}_{\beta_\eta}p_{\beta_1\cdots\beta_{\eta-1}(\omega_{\mu}^{IJ})\beta_{\eta+1}\cdots\beta_n}\frac{\partial}{\partial p_{\beta_1\cdots\beta_n}}$$

$$\kappa_{\mathbf{III}_1} = \sum_{\nu,\mu}\frac{\partial \Theta_{\mu}^{IJ}}{\partial x^{\nu}}\sum_{\beta_1<\cdots<\beta_n}\sum_{\eta}\delta^{\nu}_{\beta_\eta}p_{\beta_1\cdots\beta_{\eta-1}(\omega_{\mu}^{IJ})\beta_{\eta+1}\cdots\beta_n}\frac{\partial}{\partial p_{\beta_1\cdots\beta_n}}$$

Then, we have that: $\kappa_{\mathbf{III}_1} = \sum_{\nu}{}^{\mathbf{III}_1}\kappa_{\nu}$ with $\forall 1 \leq \nu \leq n$

$$^{\mathbf{III}_1}\kappa_{\nu} = (\kappa_{\mathbf{III}_1})\Big|_{\nu} = \sum_{\mu}(\frac{\partial \Theta_{\mu}^{IJ}}{\partial x^{\nu}})\sum_{\beta_1<\cdots<\beta_n}\sum_{\eta}\delta^{\nu}_{\beta_\eta}p_{\beta_1\cdots\beta_{\eta-1}(\omega_{\mu}^{IJ})\beta_{\eta+1}\cdots\beta_n}\frac{\partial}{\partial p_{\beta_1\cdots\beta_n}}$$

$$(\kappa_{\mathbf{III}_1})\Big|_{\nu=1} = \sum_{\mu}(\frac{\partial \Theta_{\mu}^{IJ}}{\partial x^{1}})\sum_{\beta_1<\cdots<\beta_n}\sum_{\eta}\delta^{1}_{\beta_\eta}p_{\beta_1\cdots\beta_{\eta-1}(\omega_{\mu}^{IJ})\beta_{\eta+1}\cdots\beta_n}\frac{\partial}{\partial p_{\beta_1\cdots\beta_n}}$$

$$(\kappa_{\mathbf{III}_1})\Big|_{\nu=1} = \sum_{\mu}(\frac{\partial \Theta_{\mu}^{IJ}}{\partial x^{1}})\sum_{\beta_1<\cdots<\beta_n}\Big[\delta^1_{\beta_1} + \delta^1_{\beta_2} + \delta^1_{\beta_3} + \delta^1_{\beta_4}\Big]p_{\beta_1\cdots\beta_{\eta-1}(\omega_{\mu}^{IJ})\beta_{\eta+1}\cdots\beta_n}\frac{\partial}{\partial p_{\beta_1\cdots\beta_n}}$$

$$(\kappa_{\mathbf{III}_1})\Big|_{\nu=1} = \sum_{\mu}(\frac{\partial \Theta_{\mu}^{IJ}}{\partial x^{1}})\sum_{1<\beta_2<\beta_3<\beta_4}\Big[p_{(\omega_{\mu}^{IJ})\beta_2\beta_3\beta_4}\frac{\partial}{\partial p_{1\beta_2\beta_3\beta_4}}\Big] = \sum_{\mu}(\frac{\partial \Theta_{\mu}^{IJ}}{\partial x^{1}})p_{IJ}^{\omega_{\mu}1}\frac{\partial}{\partial \mathfrak{e}}$$

after careful computation for each of the term $^{\mathbf{III}_1}\kappa_{\nu}$ finally, we can write:

$$\kappa_{\mathbf{III}_1} = (\frac{\partial \Theta_{\mu}^{IJ}}{\partial x^{\nu}})p_{IJ}^{\omega_{\mu}\nu}\frac{\partial}{\partial \mathfrak{e}} \tag{479}$$

Finally the only one remaining term is $\kappa_{\mathbf{III}_3}$

$$\kappa_{\mathbf{III}_3} = \sum_{\mu,\rho}\frac{\partial \Theta_{\mu}^{IJ}}{\partial \omega_{\rho}^{KL}}\Pi^{(\omega_{\rho}^{KL})}_{(\omega_{\mu}^{IJ})} = \sum_{\mu,\rho}(\frac{\partial \Theta_{\mu}^{IJ}}{\partial \omega_{\rho}^{KL}})\sum_{\beta_1<\cdots<\beta_n}\sum_{\eta}\delta^{(\omega_{\rho}^{KL})}_{\beta_\eta}p_{\beta_1\cdots\beta_{\eta-1}(\omega_{\mu}^{IJ})\beta_{\eta+1}\cdots\beta_n}\frac{\partial}{\partial p_{\beta_1\cdots\beta_n}}$$

$$\kappa_{\mathbf{III}_3} = \sum_{\mu,\rho}(\frac{\partial \Theta_{\mu}^{IJ}}{\partial \omega_{\rho}^{KL}})\sum_{\beta_1<\cdots<\beta_n}\sum_{\eta}\delta^{(\omega_{\rho}^{KL})}_{\beta_\eta}p_{\beta_1\cdots\beta_{\eta-1}(\omega_{\mu}^{IJ})\beta_{\eta+1}\cdots\beta_n}\frac{\partial}{\partial p_{\beta_1\cdots\beta_n}}$$

$$\kappa_{\mathbf{III}_3}\Big|_{\rho=1} = \sum_{\mu}(\frac{\partial \Theta_{\mu}^{IJ}}{\partial \omega_{1}^{KL}})\sum_{\beta_1<\cdots<\beta_n}\sum_{\eta}\delta^{(\omega_{1}^{KL})}_{\beta_\eta}p_{\beta_1\cdots\beta_{\eta-1}(\omega_{\mu}^{IJ})\beta_{\eta+1}\cdots\beta_n}\frac{\partial}{\partial p_{\beta_1\cdots\beta_n}}$$

$$\kappa_{\mathbf{III}_3}\Big|_{\rho=1} = (\frac{\partial \Theta_{\mu}^{IJ}}{\partial \omega_{1}^{KL}})\sum_{\beta_1<\cdots<\beta_n}\Big[\delta^{(\omega_1^{KL})}_{\beta_1}p_{(\omega_{\mu}^{IJ})\beta_2\beta_3\beta_4} + \delta^{(\omega_1^{KL})}_{\beta_2}p_{\beta_1(\omega_{\mu}^{IJ})\beta_3\beta_4} + \delta^{(\omega_1^{KL})}_{\beta_3}p_{\beta_1\beta_2(\omega_{\mu}^{IJ})\beta_4} + \delta^{(\omega_1^{KL})}_{\beta_4}p_{\beta_1\beta_2\beta_3(\omega_{\mu}^{IJ})}\Big]\frac{\partial}{\partial p_{\beta_1\beta_2\beta_3\beta_4}}$$

$$\kappa_{\mathbf{III}_3}\Big|_{\rho=1} = (\frac{\partial \Theta_{\mu}^{IJ}}{\partial \omega_{1}^{KL}})\sum_{\beta_1<\beta_2<\beta_3<(\omega_1^{KL})}\Big[p_{\beta_1\beta_2\beta_3(\omega_{\mu}^{IJ})}\Big]\frac{\partial}{\partial p_{\beta_1\beta_2\beta_3(\omega_1^{KL})}}$$

$$\kappa_{\mathbf{III}_3}\Big|_{\rho=1} = (\frac{\partial\Theta_\mu^{IJ}}{\partial\omega_1^{KL}})\Big[p_{123(\omega_\mu^{IJ})}\frac{\partial}{\partial p_{123(\omega_1^{KL})}} + p_{134(\omega_\mu^{IJ})}\frac{\partial}{\partial p_{134(\omega_1^{KL})}} + p_{124(\omega_\mu^{IJ})}\frac{\partial}{\partial p_{124(\omega_1^{KL})}} + p_{234(\omega_\mu^{IJ})}\frac{\partial}{\partial p_{234(\omega_1^{KL})}}\Big]$$

$$\kappa_{\mathbf{III}_3}\Big|_{\rho=1} = (\frac{\partial\Theta_\mu^{IJ}}{\partial\omega_1^{KL}})\Big[p_{IJ}^{\omega_\mu 4}\frac{\partial}{\partial p_{KL}^{\omega_1 4}} + p_{IJ}^{\omega_\mu 2}\frac{\partial}{\partial p_{KL}^{\omega_1 2}} + p_{IJ}^{\omega_\mu 3}\frac{\partial}{\partial p_{KL}^{\omega_1 3}} + p_{IJ}^{\omega_\mu 1}\frac{\partial}{\partial p_{KL}^{\omega_1 1}}\Big] = \sum_{\mu,\lambda}(\frac{\partial\Theta_\mu^{IJ}}{\partial\omega_1^{KL}})p_{IJ}^{\omega_\mu\lambda}\frac{\partial}{\partial p_{KL}^{\omega_1\lambda}}$$

And also the similar computation for each $\rho = 2,3,4$ we finally obtain:

$$\kappa_{\mathbf{III}_3} = (\frac{\partial\Theta_\mu^{IJ}}{\partial\omega_\rho^{KL}})p_{IJ}^{\omega_\mu\lambda}\frac{\partial}{\partial p_{KL}^{\omega_\rho\lambda}} = (\frac{\partial\Theta_\lambda^{IJ}}{\partial\omega_\rho^{KL}})p_{IJ}^{\omega_\lambda\mu}\frac{\partial}{\partial p_{KL}^{\omega_\rho\mu}} = (\frac{\partial\Theta_\sigma^{IJ}}{\partial\omega_\rho^{KL}})p_{IJ}^{\omega_\mu\sigma}\frac{\partial}{\partial p_{KL}^{\omega_\mu\rho}} \quad (480)$$

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