Gradient Schemes for some elliptic and parabolic, linear and non-linear problems

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Schémas gradients appliqués à des problèmes elliptiques et paraboliques, linéaires et non-linéaires

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Résumé

La notion de schémas gradients, conçue pour les équations elliptiques et paraboliques, linéaires et non-linéaires a l'avantage de fournir des résultats de convergence et d'estimations d'erreur valables pour de nombreuses familles de méthodes numériques (éléments finis conformes et non-conformes, éléments finis mixtes, différences finies . . .). Vérifier un ensemble restreint de propriétés suffit pour prouver qu'une méthode numérique donnée rentre dans le cadre de travail des schémas gradients et donc qu'elle sera convergente sur les différents problèmes traités. L'étude du problème de Stefan, celle du problème de Stokes incompressible, ainsi que celle des équations de Navier-Stokes incompressibles sont présentées dans cette thèse, chacune présentant un théorème de convergence établi à l'aide des schémas gradients. Pour Stokes et Navier-Stokes, nous donnerons une preuve de convergence pour les cas stationnaires et transitoires en modifiant certaines hypothèses ce qui aura comme effet de trouver des résultats de convergence différents. Finalement, nous présentons également quatre méthodes (Taylor-Hood, Crouzeix-Raviart, Marker-and-Cell, Hybrid Mixed Mimetic) pour ces deux problèmes et nous vérifions qu'elles rentrent bien dans le cadre des schémas gradients.

Mots clés : Schémas gradients, résultats de convergence, problème de Stefan, Stokes et Navier-Stokes

Abstract

The notion of gradient schemes, designed for linear and nonlinear elliptic and parabolic problems has the benefit of providing common convergence and error estimates results, which hold for a wide variety of numerical methods (finite element methods, nonconforming and mixed finite element methods, hybrid and mixed mimetic finite difference methods . . .). Checking a minimal set of properties for a given numerical method suffices to prove that it belongs to the gradient schemes framework, and therefore that it is convergent on the different problems studied here. The study of the Stefan problem, the incompressible Stokes one and also the incompressible Navier-Stokes equations are presented in this thesis, where each one gets a convergence theorem set up with the gradient schemes framework. For Stokes and Navier-Stokes, we both provide the proof for the steady and the transient case dealing with some variational hypotheses which bring different convergence results. Finally, we also present four methods (Taylor-Hood, Crouzeix-Raviart, Marker-and-Cell, Hybrid Mixed Mimetic) for these two problems and we check that they enter in the gradient schemes framework.

Keywords : Gradient schemes, convergence results, Stefan problem, Stokes problem, Navier-Stokes problem

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Introduction

Schémas gradients

Cette thèse s'inscrit dans le domaine de l'analyse numérique. Plus précisément, nous sommes intéressés par la généralisation des méthodes d'analyse numérique. En effet, de nombreuses méthodes existent pour approcher la solution de problèmes elliptiques ou paraboliques modélisant des phénomènes physiques tels que l'écoulement des fluides (équations de Stokes et Navier-Stokes) ou encore le changement d'état thermodynamique (problème de Stefan). Chacune de ces méthodes possède ses propres avantages et inconvénients qu'il est important de prendre en compte (précision, efficacité, rapidité ...). Ainsi, il est souvent nécessaire de les comparer. Jusqu'à maintenant, il fallait faire l'étude d'un même problème avec chaque méthode pour pouvoir déceler les différences entre deux d'entre elles puisqu'aucun théorème de convergence commun n'existait. Le temps requis pour effectuer l'opération sur plusieurs modélisations peut donc devenir très important. L'idée de mettre en place un cadre de travail généralisant ces différentes méthodes est donc légitime puisqu'il permet non seulement de pouvoir les comparer plus rapidement mais également de ne faire qu'une analyse du problème ce qui amène un gain de temps assez conséquent (voir Figures 1 et 2). Les schémas gradients sont un cadre général de travail qui permet de regrouper un certain nombre de ces méthodes numériques et surtout de donner un théorème de convergence commun pour un problème donné. L'un des objectifs de cette thèse est donc de mettre en place un tel cadre pour différents problèmes. En reprenant l'illustration de la figure 2, nous distinguons deux étapes pour une telle mise en place. La première étant la partie droite qui correspond à l'étude du problème a l'aide des schémas gradients et donc à l'établissement du théorème de convergence commun et la deuxième est de vérifier si

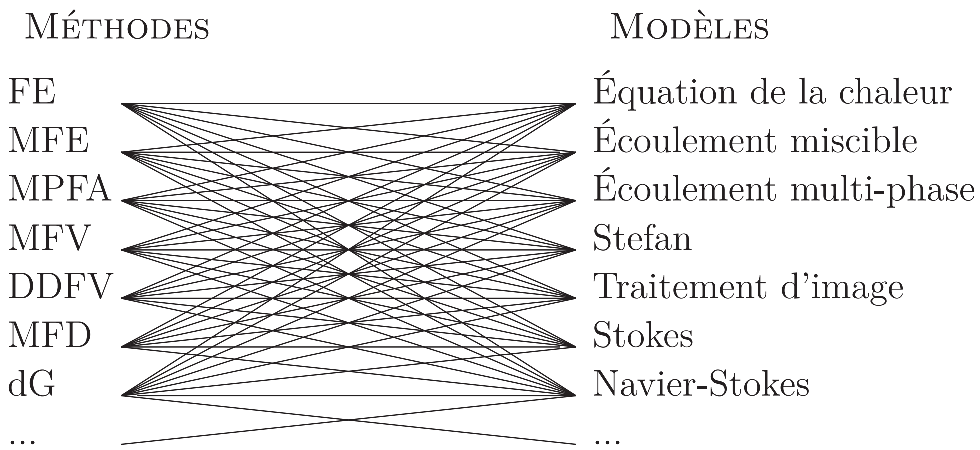


Figure 1 – Illustration de l'intérêt des schémas gradients, avant généralisation : 1 ligne = 1 analyse

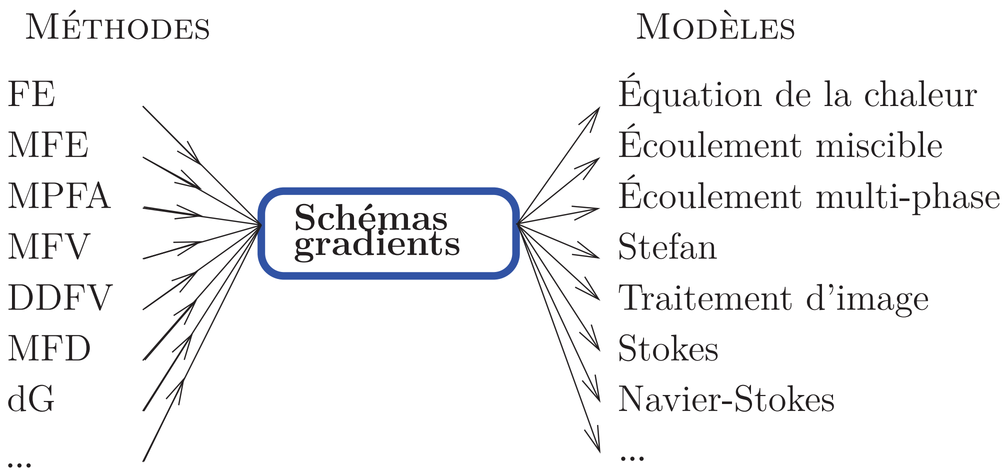


Figure 2 – Illustration de l'intérêt des schémas gradients, après généralisation

les méthodes numériques que nous souhaitons appliquer rentrent dans notre cadre de travail.

L'étude de la convergence du schéma pour des équations elliptiques et paraboliques se fait en plusieurs étapes. La première est de mettre en place plusieurs objets discrets, regroupés sous le nom générique de "discrétisation gradient", permettant de construire le schéma discret de la modélisation. La seconde étape est d'introduire certaines propriétés que devra remplir la discrétisation gradient et qui, à terme, serviront à conclure quant à la convergence de notre schéma. La troisième est évidemment d'introduire le schéma gradient et cela reste assez naturel puisqu'il suffit de remplacer les espaces et opérateurs continus par leurs transcriptions discrètes. La dernière étape est d'établir un théorème de convergence en utilisant uniquement les objets de la discrétisation gradient et les propriétés définies à la seconde étape. Ainsi, nous pouvons affirmer que la solution du schéma approche la solution de notre problème. Par la suite, pour que les méthodes numériques que nous voulons appliquer, et donc comparer, entrent bien dans le cadre de travail des schémas gradients, il suffit de montrer qu'elles vérifient les propriétés de la discrétisation gradient. Bien évidemment, les équations étudiées étant différentes, elles amènent des variations dans les définitions de la discrétisation gradient et des propriétés.

Ce cadre de travail inclut, par exemple, la plupart des méthodes d'éléments finis conformes (voir [4], [47], [92]), mais aussi des éléments finis mixtes [56]. Les schémas gradients ont déjà été étudiés dans [63] pour les problèmes elliptiques linéaires, et dans [44] pour les problèmes elliptiques non-linéaire de type Leray-Lions et les problèmes paraboliques.

Le but de cette thèse est donc de mettre en évidence les différents avantages et inconvénients d'une telle généralisation en l'appliquant à trois modèles mathématiques qui sont le problème de Stefan, les équations de Stokes incompressibles et celles de Navier-Stokes incompressibles. Les différents chapitres de cette thèse on fait l'objet d'articles [51], [40] et [52].

Le problème de Stefan

Présentation du modèle

Le premier chapitre de cette thèse porte sur l'étude d'un problème d'équation parabolique non-linéaire modélisant l'évolution de l'énergie lors d'un changement d'état thermodynamique (principalement la transition solide/liquide) et reprend en grande partie les résultats publiés dans [51]. Un autre exemple est une application en mathématiques financières, où des grandeurs liées au prix des options américaines sont solutions d'un problème de Stefan particulier [5, 12], dont le calcul est équivalent à la résolution d'une inégalité variationnelle.

On note, pour $(x,t) \in \Omega \times (0,T)$, $\Theta(x,t)$ la température et $X(x,t)$ la masse de liquide par unité de volume (si $X(x,t) = 0$, le milieu est donc solide au point (x,t) et si $X(x,t) = 1$, il est liquide), l'énergie interne $\bar{u}(x,t)$ peut être écrite comme suit

$$\bar{u}(x,t) = H_c\Theta(x,t) + L_f X(x,t),$$

avec H_c la capacité calorifique (que nous supposons constante et identique pour les états solide et liquide) et L_f la chaleur latente de fusion à la température Θ_f donnée. L'équation de la chaleur peut alors s'écrire :

$$\partial_t\bar{u} - \mathrm{div}(\lambda\nabla\Theta(\mathrm{x},\mathrm{t})) = \mathrm{f}(\mathrm{x},\mathrm{t}), \text{ sur } \Omega \times (0,\mathrm{T}), \tag{1}$$

où λ est la conductivité de la chaleur (supposée constante, isotrope et identique pour les états solide et liquide). Nous supposons également des conditions au bord de Dirichlet homogènes pour la température Θ, et la donnée de $\bar{u}$ à l'instant initial. L'équilibre thermodynamique s'écrit alors

$$\begin{aligned}(\Theta(x,t) \leq \Theta_f \text{ et } X(x,t) = 0) \text{ ou}\\ (\Theta(x,t) = \Theta_f \text{ et } 0 \leq X(x,t) \leq 1) \text{ ou}\\ (\Theta(x,t) \geq \Theta_f \text{ et } X(x,t) = 1) \text{ pour presque tout } (x,t) \in \Omega \times (0,T).\end{aligned} \tag{2}$$

Nous pouvons remarquer que, sous la condition (2), $X(x,t)$ et $\Theta(x,t)$ peuvent être formulés $X(x,t) = \xi(\bar{u}(x,t))$ et $\Theta(x,t) = \frac{1}{\lambda}\zeta(\bar{u}(x,t))$, avec

$$\xi(s) = \min(\max(\frac{s - H_c\Theta_f}{L_f}, 0), 1) \text{ et } \zeta(s) = \lambda\frac{s - L_f\xi(s)}{H_c},\ \forall s \in \mathbb{R}.$$

Considérer l'expression précédente de $\Theta(x,t)$ comme fonction de $\bar{u}(x,t)$ dans (1) mène à l'équation (3), dans laquelle la fonction ζ est continue, Lipschitzienne, croissante, et constante sur l'intervalle $[H_c\Theta_f, H_c\Theta_f + L_f]$ (elle est en fait continue et constante par morceaux). La modélisation du phénomène s'écrit alors :

$$\partial_t\bar{u} - \Delta\zeta(\bar{u}) = f, \text{ sur } \Omega \times (0,T), \tag{3}$$

avec Ω un ouvert borné de $\mathbb{R}$, de condition initiale :

$$\bar{u}(x,0) = u_{\text{ini}}(x), \text{ for a.e. } x \in \Omega, \tag{4}$$

et de condition de Dirichlet homogène au bord :

$$\zeta(\bar{u}(x,t)) = 0 \text{ sur } \partial\Omega \times (0,T). \tag{5}$$

Beaucoup de résultats sont connus dans cette situation, en particulier le fait que, si $f = 0$ et si la mesure de l'ensemble $\{x \in \Omega, \bar{u}(x,t) \in [H_c\Theta_f, H_c\Theta_f + L_f]\}$ (appelée “mushy region”) est nulle à $t = 0$, alors elle sera nulle pour tout $t > 0$, et une discontinuité de $\bar{u}$ entre les valeurs $H_c\Theta_f$ et $H_c\Theta_f + L_f$ apparaîtra dans le domaine (voir [14]). De plus, le problème (3)-(4)-(5) doit être considéré dans un sens faible, qui inclut une condition Rankine-Hugoniot pour la conservation de $\bar{u}$ dans le cas de discontinuité. Une fonction $\bar{u}$ est dite solution faible du problème (3)-(4)-(5)

si :

$$\begin{aligned}&\bar{u} \in L^2(\Omega \times (0,T)),\ \zeta(\bar{u}) \in L^2(0,T;H_0^1(\Omega)),\\ &\int_0^T \int_\Omega \left(-\bar{u}(x,t)\partial_t\varphi(x,t) + \nabla\zeta(\bar{u})(x,t)\cdot\nabla\varphi(x,t)\right)\,dxdt - \int_\Omega u_{\rm ini}(x)\varphi(\ x,0)\,dx\\ &\qquad = \int_0^T \int_\Omega f(x,t)\varphi(x,t)\,dxdt\ \forall\varphi \in C_c^\infty(\Omega\times[0,T[),\end{aligned} \tag{6}$$

où $C_c^\infty(\Omega \times [0,T[)$ est l'ensemble des restrictions de fonctions de $C_c^\infty(\Omega\times]-\infty,T[)$ à $\Omega \times [0,T[$.

La première preuve d'existence d'une solution du problème (6) a été donnée dans [3]. Cette preuve repose sur la convergence, quand $\varepsilon > 0$ tend vers 0, de la solution $\bar{u}_\varepsilon$ de la régularisation strictement parabolique de (3) :

$$\partial_t\bar{u}_\varepsilon - \Delta(\zeta(\bar{u}_\varepsilon) + \varepsilon\bar{u}_\varepsilon) = f(x,t),\ \text{in } \Omega\times(0,T). \tag{7}$$

En ce qui concerne l'unicité de la solution du problème (6), beaucoup de résultats sont donnés dans la littérature sous différentes hypothèses. Par exemple, un théorème d'unicité a été prouvé dans [62] avec une hypothèse plus restrictive sur Ω, et un théorème d'unicité pour les problèmes non-linéaires de convection-diffusion se trouve dans [23].

Étude du problème

L'objectif de ce premier chapitre est d'établir la convergence des schémas gradients pour l'approximation du problème de Stefan donné sous sa formulation faible (6). Nous commencerons par introduire les outils nécessaires à la discrétisation du problème par la méthode générale, tels que l'espace discret X_D, espace vectoriel réel, et les opérateurs discrets $\Pi_D : X_D \mapsto L^2(\Omega)$ et $\nabla_D : X_D \mapsto L^2(\Omega)^d$. La discrétisation gradient est donc basée sur ces trois outils, il reste à munir X_D d'une norme notée $\|\cdot\|_D$ construite avec les opérateurs discrets. Cette norme dépend des conditions au bord choisies, dans le cas de Dirichlet homogènes, $\|\cdot\|_D = \|\nabla_D\cdot\|_{L^2(\Omega)}$. Puis nous établirons les propriétés utiles à la convergence du problème. Pour Stefan, elles sont au nombre de cinq : la coercivité, la consistance, la conformité à la limite, la compacité et une dernière plus spécifique à ce problème impliquant que la fonction de reconstruction Π_D soit constante par morceaux. Les quatre premières sont plus communes aux autres études impliquant les schémas gradients. La coercivité introduit une inégalité de Poincaré discrète qui permet donc de contrôler la norme L^2 de la fonction de reconstruction Π_D par la norme $\|\cdot\|_D$. La consistance

assure que l'espace discret "remplisse" l'espace continu tandis que la conformité à la limite permet, via une intégration par partie discrète, de conclure que l'opérateur ∇_D tendra bien vers le gradient continu de la solution. La compacité, quand à elle, est nécessaire pour traiter un problème non-linéaire en assurant la convergence forte dans $L^2(\Omega)$ de la reconstruction de la solution discrète vers la solution continue. Le schéma numérique s'obtient en remplaçant les éléments continus de la formulation faible du problème (6) par les objets de la discrétisation gradient définis plus tôt. Nous obtenons donc le schéma suivant,
soit une suite $(u^{(n)})_{n=0,\ldots,N}$ telle que :

$$\begin{cases} u^{(0)} \in X_{D,0}, \\ u^{(n+1)} \in X_{D,0},\ \delta_D^{(n+\frac{1}{2})} u = \Pi_D \dfrac{u^{(n+1)} - u^{(n)}}{\delta t^{(n+\frac{1}{2})}}, \\ \displaystyle\int_\Omega \left(\delta_D^{(n+\frac{1}{2})} u(x) \Pi_D v(x) + \nabla_D \zeta(u^{(n+1)})(x) \cdot \nabla_D v(x) \right) dx = \\ \dfrac{1}{\delta t^{(n+\frac{1}{2})}} \displaystyle\int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f(x,t) \Pi_D v(x)\, dx dt, \quad \forall v \in X_{D,0},\ \forall n = 0, \ldots, N-1. \end{cases}$$

Le théorème de convergence sera établi grâce à un argument de compacité. Certaines estimations sur la solution discrète u_D nécessitent une hypothèse supplémentaire, non restrictive en pratique mais nécessaire à la généralisation :

$$|\zeta(s)| \geq a|s| - b \text{ pour tout } s \in \mathbb{R} \text{ pour } a, b \in (0, +\infty) \text{ donnés.}$$

En effet le problème a déjà été étudié par une méthode d'approximation de volume finis avec des flux à deux points [62]. Cette méthode a pour avantage de pouvoir prendre comme fonction test dans le schéma la solution discrète u_D et d'obtenir une estimation sur sa norme L^2 en espace-temps. Mais la généralisation ne permet pas une telle liberté et nous ne pourrons prendre comme fonction test que $\zeta(u_D)$ dans le schéma. Ainsi l'hypothèse supplémentaire permet de contrôler la norme L^2 de $\Pi_D u_D$ par celle de $\Pi_D \zeta(u_D)$. L'unicité de la solution discrète sera trouvée en montrant l'égalité entre deux solutions discrètes distinctes ; tandis que l'existence de celle-ci viendra d'un argument de degré topologique. Ainsi nous pourrons conclure quant à la convergence faible de la solution discrète $\Pi_D u_D$ vers la solution $\bar{u}$ dans $L^2(\Omega \times (0,T))$, de $\Pi_D \zeta(u_D)$ vers $\zeta(\bar{u})$ dans $L^2(\Omega \times (0,T))$ et de $\nabla_D \zeta(u_D)$ vers une fonction χ dans $L^2(\Omega \times (0,T))^d$. L'astuce de Minty permettra de prouver que $\chi = \nabla \zeta(\bar{u})$, puis des estimations d'énergie, dans les cas continu et discret, permettront d'établir la convergence forte de $\nabla_D \zeta(u_D)$ vers $\nabla \zeta(\bar{u})$, pour enfin trouver la convergence forte de $\zeta(u_D)$ vers $\zeta(\bar{u})$. En suivant les idées présentes dans [39], nous pouvons montrer la convergence uniforme en temps de $\zeta(u_D)$. Par ailleurs, il est aussi prouvé, via l'étude d'un cas particulier, que la multiplication par $\bar{u}$ pour trouver une estimation d'énergie afin d'établir une convergence forte de u_D n'est pas possible. Finalement, des exemples numériques se concentreront sur le comportement du schéma VAG [63] qui présente des caractéristiques intéressantes pour des flux couplés en milieu poreux (voir Section 1.5).

Dans une dernière partie de ce chapitre, nous abordons certains compléments sur le problème de Stefan. En effet, il est présenté un résultat d'unicité de ce problème sous les hypothèses utilisées dans notre étude de convergence. Cette unicité a été prouvée en analysant un problème adjoint régularisé. Les principaux théorèmes utilisés comme celui de compacité d'Alt-Luckhaus discret ou encore l'astuce de Minty sont également présentés.

Le problème de Stokes incompressible

Présentation du modèle

Dans le second chapitre, nous étudions le problème de Stokes stationnaire incompressible (8) qui modélise l'écoulement d'un fluide à faible vitesse en mettant en équation l'évolution de cette vitesse mais également la pression de ce fluide. Puis le problème de Stokes transitoire incompressible (9) où cette fois-ci la vitesse et la pression dépendent du temps. Ce modèle est dérivé du principe fondamental de la dynamique, appliqué à une particule de fluide où l'on aurait négligé le terme de convection $(\bar{u} \cdot \nabla)\bar{u}$ devant les autres termes à cause de la faible vitesse du fluide. En effet, en partant de la formule $m\overrightarrow{a} = \sum \overrightarrow{F}$ avec $\overrightarrow{a}$ l'accélération de la particule, m sa masse et $\sum \overrightarrow{F}$ le bilan de force, nous obtenons alors la formulation forte du problème stationnaire de Stokes incompressible :

$$\begin{cases} \eta\overline{u} - \Delta\overline{u} + \nabla\overline{p} &= f - \mathrm{div}(G) \quad \text{sur } \Omega \\ \mathrm{div}\overline{u} &= 0 \quad \text{sur } \Omega \\ \overline{u} &= 0 \quad \text{sur } \partial\Omega \end{cases} \tag{8}$$

où $\overline{u}$ représente le champ de vitesse, $\overline{p}$ la pression, où Ω est un ouvert borné de $\mathbb{R}^d$, de frontière $\partial\Omega$, $d \geq 1$, $\eta \in \mathbb{R}^+$, et f et G sont deux fonctions données définies sur Ω. La condition $\mathrm{div}\bar{u} = 0$

représente l'incompressibilité du fluide. Dans le cas transitoire, la formulation forte s'écrit :

$$\left\{\begin{aligned} \partial_t \overline{u} - \Delta \overline{u} + \nabla \overline{p} &= f - \text{div}(G) && \text{sur } \Omega \times (0,T) \\ \text{div}\bar{\text{u}} &= 0 && \text{sur } \Omega \times (0,T) \\ \overline{u} &= 0 && \text{sur } \partial\Omega \times (0,T) \\ \overline{u}(\cdot,0) &= \overline{u}_{\text{ini}} && \text{presque partout sur } \Omega, \end{aligned}\right. \tag{9}$$

où les fonctions $\overline{u}$, $\overline{p}$, f et G sont définies sur $\Omega \times (0,T)$ avec $T > 0$, et avec $\overline{u}_{\text{ini}}$ une fonction donnée définie sur Ω.

Pour ce problème, l'existence et l'unicité de la solution $(\bar{u}, \bar{p})$ sont connues. Beaucoup d'approximations numériques différentes de ce modèle sont couramment utilisées pour des applications industrielles et ont toutes fait l'objet d'analyses indépendantes. Nous pouvons, par exemple, citer les résultats suivants : dans [11], on trouve la première estimation d'erreur pour l'approximation de Taylor-Hood pour le problème de Stokes, améliorée quelques années après dans [101]. Dans [30], les auteurs ont établi une estimation d'erreur pour la méthode des éléments finis $\mathbb{P}^1$ non-conforme pour la vitesse, couplée aux éléments finis $\mathbb{P}^0$ pour la pression, cette méthode est aujourd'hui connue sous le nom de Crouzeix-Raviart. Plus tard, dans [90], les auteurs fournissent la première preuve de convergence pour le célèbre schéma Marker-And-Cell (MAC) [80, 94, 103], qui est désormais l'un des plus utilisé en ingénierie et en mécanique des fluides.

Étude du problème

Comme mentionné précédemment, toutes ces études ont été faites indépendamment les unes des autres, malgré la similitude de beaucoup d'idées dans chacune d'entre elles. Le but de ce chapitre sera donc d'établir un cadre de travail le plus général possible à l'aide des schémas gradients. Le principe reste le même que pour l'étude du problème de Stefan, il faut tout d'abord définir les outils nécessaires à la discrétisation gradient. Ici, il y a deux inconnues qui sont la vitesse et la pression et quatre opérateurs différents dans la formulation faible du problème de Stokes incompressible, qu'il soit stationnaire ou transitoire. Il faudra donc deux espaces discrets notés X_D pour la vitesse et Y_D pour la pression et quatre opérateurs pour composer la discrétisation gradient : les opérateurs de reconstruction $\Pi_D : X_D \mapsto L^2(\Omega)^d$ et $\chi_D : Y_D \mapsto L^2(\Omega)$ respectivement pour la vitesse et la pression discrète, mais également l'opérateur de gradient discret $\nabla_D : X_D \mapsto L^2(\Omega)^{d\times d}$ et celui de la divergence discrète $\text{div}_D : \mathrm{X}_D \mapsto \mathrm{L}^2(\Omega)$. Dans le problème de Stokes, aucune non-linéarité n'est présente, les propriétés ne nécessitent donc pas de compacité, elles seront donc au nombre de trois : coercivité, consistance et conformité à la limite. Malgré la même terminologie que pour le problème de Stefan, elles comportent des conditions complémentaires. En effet, l'idée générale de la propriété ne change pas mais nous n'avons pas les mêmes besoins d'un problème à l'autre. Ainsi la coercivité permettra non seulement de contrôler la norme L^2 de la vitesse mais aussi la norme L^2 de sa divergence discrète. De plus elle introduira une condition discrète de Ladyzenuskaja-Babuska-Brezzi (LBB ou aussi appelée condition inf-sup discrète) pour contrôler la norme L^2 de l'approximation de $\bar{p}$. La consistance joue toujours le même rôle qui est d'assurer le "remplissage" des espaces continus par les espaces discrets en tenant compte des nouveaux objets de la discrétisation. La conformité à la limite permet désormais, non seulement de conclure que le gradient discret tend vers le gradient continu, mais aussi que la divergence discrète tend vers la divergence continue.

Pour le cas stationnaire, les résultats de convergence seront basés sur des estimations d'erreurs. Le premier donnera une estimation d'erreur sur la vitesse et la pression et sera établi grâce aux estimations trouvées en étudiant le schéma suivant, obtenue de la même façon que pour le problème de Stefan, c'est-à-dire en partant de la formulation faible du problème et en y incluant les objets discrets :

$$\left\{\begin{array}{l} u \in X_{D,0}\,,\ p \in Y_{D,0}, \\ \eta \displaystyle\int_\Omega \Pi_D u \cdot \Pi_D v \mathrm{d}x + \int_\Omega \nabla_D u : \nabla_D v \mathrm{d}x \\ \qquad\qquad - \displaystyle\int_\Omega \chi_D p \ \mathrm{div}_D v \mathrm{d}x = \int_\Omega \left(f \cdot \Pi_D v + G : \nabla_D v\right) \mathrm{d}x,\ \forall v \in X_{D,0}, \\ \displaystyle\int_\Omega \chi_D q \mathrm{div}_D u \mathrm{d}x = 0,\ \ \forall q \in Y_{D,0}. \end{array}\right.$$

L'un des problèmes de ce résultat est qu'il dépend de la constante de la condition inf-sup discrète et que celle-ci peut devenir très grande voir exploser pour certaines méthodes d'approximation. Le deuxième résultat donne une estimation d'erreur mais sur la vitesse uniquement, en revanche elle ne dépend plus de la condition inf-sup puisque pour un tel résultat, les fonctions test choisies sont à divergence discrète nulle et ainsi le terme introduisant la pression discrète disparaît de la preuve de convergence. Cette hypothèse est malgré cela peu restrictive et beaucoup de schémas permettent une interpolation des espaces continus à divergence nulle vers des espaces discrets également à divergence nulle comme les éléments finis de type Crouzeix-Raviart par exemple. Néanmoins, l'estimation d'erreur dépend toujours de la pression continue, ainsi, même si la vitesse continue est nulle, son approximation peut ne pas l'être. Cette particularité peut parfois être évitée dans le cas de force irrotationnelle et c'est ce qui est proposé dans le troisième résultat de convergence.

Pour le cas transitoire, il faut tout d'abord définir la discrétisation gradient D pour l'espace-temps. Nous rajoutons pour cela une discrétisation de l'intervalle de temps en plusieurs pas de temps δt et nous introduisons un opérateur J_D qui interpolera la condition initiale u_{ini}. Les propriétés de compacité, coercivité et de conformité à la limite ne nécessitent pas de modifications, seule la consistance change. En effet, le nouvel opérateur J_D doit être plongé dans $L^2(\Omega)^d$ et le pas de temps δt_m doit tendre vers 0 quand $m \to \infty$. Nous obtenons alors le schéma suivant :

$$\left\{\begin{array}{l} u_D = (u_D^{(n)})_{n=0,\dots,N}\,,\ p_D = (p_D^{(n)})_{n=1,\dots,N} \text{ telles que } u_D^{(0)} = J_D u_{\text{ini}} \text{ et, } \forall n = 0,\dots,N-1 : \\ u_D^{(n+1)} \in X_{D,0}\,,\ p_D^{(n+1)} \in Y_{D,0}\,, \\ \displaystyle\int_\Omega \Pi_D \delta_D^{n+\frac{1}{2}} u_D \cdot \Pi_D v\, dx + \int_\Omega \nabla_D u_D^{(n+1)} : \nabla_D v\, dx - \int_\Omega \chi_D p_D^{(n+1)} \text{div}_D v\, dx \\ \quad \displaystyle = \frac{1}{\delta t^{n+\frac{1}{2}}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_D v\, dxdt + \frac{1}{\delta t^{n+\frac{1}{2}}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega G \cdot \nabla_D v\, dxdt, \quad \forall v \in X_{D,0}. \\ \displaystyle\int_\Omega \text{div}_D u_D^{(n+1)} \chi_D q \text{d}x = 0, \quad \forall q \in Y_{D,0}. \end{array}\right.$$

Le schéma trouvé permettra tout d'abord de vérifier l'existence et l'unicité de la solution discrète et cette fois-ci, les résultats de convergence ne seront plus sous forme d'estimation d'erreur mais établis par argument de compacité comme pour le problème de Stefan. Nous devrons donc établir d'autres estimations qui permettront de conclure quant à des convergences faibles de la vitesse et de la pression discrète. Une fois de plus, le choix des fonctions test, plus précisément les prendre à divergence discrète nulle ou non, influera sur le résultat de convergence obtenu. Dans le deuxième cas, si nous ajoutons également une certaine condition sur l'interpolation de la condition initiale et si nous appelons (u_D, p_D) la solution discrète, les estimations conduiront aux convergences fortes suivantes :

- $\Pi_{D_m} u_{D_m} \to \bar{u}$ dans $L^2(0, T, H_0^1(\Omega))^d$,
- $\nabla_{D_m} u_{D_m} \to \nabla \bar{u}$ dans $L^2(\Omega \times (0, T))^{d \times d}$,
- $\chi_{D_m} p_{D_m} \to \chi \bar{p}$ dans $L^2(\Omega \times (0, T))$.

En prenant les fonctions test à divergence discrète nulle, nous perdrons la convergence faible de la pression mais nous gagnerons la convergence uniforme en temps de la vitesse en suivant de nouveau les idées introduites dans [39].

Les équations de Navier-Stokes incompressible

Présentation du modèle

Le problème de Stokes désormais traité, nous nous intéressons à un écoulement de fluide plus rapide, ainsi le terme de convection $(\bar{u}\cdot\nabla)\bar{u}$ qui était négligé dans le problème précédent ne peut plus l'être. Dans le troisième chapitre, nous étudions le problème de Navier-Stokes stationnaire incompressible (10) puis transitoire (11). Ci-dessous, nous présentons la formulation forte tout d'abord pour le cas stationnaire puis pour le transitoire.

$$\begin{cases} \eta\overline{u} - \Delta\overline{u} + (\bar{u}\cdot\nabla)\bar{u} + \nabla\overline{p} & = f - \operatorname{div}(G) \quad \text{sur } \Omega \\ \operatorname{div}\overline{u} & = 0 \quad \text{sur } \Omega \\ \overline{u} & = 0 \quad \text{sur } \partial\Omega \end{cases} \tag{10}$$

où $\overline{u}$ représente le champs de vitesse, $\overline{p}$ la pression, le domaine Ω, de frontière $\partial\Omega$, est un ouvert borné de $\mathbb{R}^d$, $d \geq 1$, $\eta \in \mathbb{R}^+$, et f et G sont deux fonctions données définies sur Ω. La condition $\operatorname{div}\bar{u} = 0$ représente l'incompressibilité du fluide.

$$\begin{cases} \partial_t\overline{u} - \Delta\bar{u} + (\bar{u}\cdot\nabla)\bar{u} + \nabla\overline{p} & = f - \operatorname{div}(G) \quad \text{sur } \Omega\times(0,T) \\ \operatorname{div}\bar{u} & = 0 \quad \text{sur } \Omega\times(0,T) \\ \overline{u} & = 0 \quad \text{sur } \partial\Omega\times(0,T) \\ \overline{u}(\cdot,0) & = \overline{u}_{\text{ini}} \quad \text{presque partout sur } \Omega, \end{cases} \tag{11}$$

où les fonctions $\overline{u}$, $\overline{p}$, f et G sont définies sur $\Omega\times(0,T)$ avec $T > 0$, et avec $\overline{u}_{\text{ini}}$ une fonction donnée définie sur Ω.

Étude du problème stationnaire

Pour Navier-Stokes, nous traitons le problème stationnaire différemment du problème transitoire car le fait d'intégrer le terme trilinéaire en espace ou en espace-temps change considérablement l'analyse de celui-ci ; du fait de l'impossibilité de prendre comme fonction test la solution. Une fois de plus, nous commençons par définir la discrétisation gradient. Les six objets définis pour le problème de Stokes seront réutilisés et il faudra en ajouter un septième $b_D : X^2_{D,0} \mapsto \mathbb{R}$, qui est le représentant discret de la formulation faible du terme de convection qui s'écrit alors sous la forme d'un opérateur trilinéaire $b(\bar{u},\bar{u},\bar{v})$ tel que

$$b : (H^1_0(\Omega)^d)^3 \mapsto \mathbb{R} \text{ et } b(\bar{u},\bar{v},\bar{w}) = \int_\Omega (\bar{u}\cdot\nabla)\bar{v}\bar{w}\,dx.$$

Il doit être choisi de sorte que :

- $b_D(u,v)$ est continue par rapport à u,
- $b_D(u,u) \geq 0$,
- il existe une constante $B_D > 0$ telle que $b_D(u,v) \leq B_D \|u\|_D^2 \|v\|_D$,
- $b_D(u,v)$ est linéaire par rapport à v.

Ceci permettra de montrer les estimations nécessaires à la convergence du schéma. Pour les propriétés, la coercivité, la consistance et la conformité à la limite n'ont pas besoin d'adaptation du problème de Stokes à celui de Navier-Stokes. Le terme b_D amenant de la non-linéarité au problème, nous aurons besoin de la propriété de compacité, similaire à celle du problème de Stefan. Ce terme amène également une propriété que l'on nommera conformité à la limite trilinéaire qui consiste à s'assurer que $b_{D_m}(u,v) \to b(\bar{u},\bar{u},\bar{v})$ quand $m \to \infty$, ceci termine les modifications apportées à la discrétisation gradient du problème de Navier-Stokes stationnaire incompressible. Une fois la discrétisation définie, nous pouvons établir le schéma numérique suivant :

$$\begin{cases} u \in X_{D,0}\,,\ p \in Y_{D,0}, \\ \eta \displaystyle\int_\Omega \Pi_D u \cdot \Pi_D v \, dx + \nu \int_\Omega \nabla_D u : \nabla_D v \, dx + b_D(u,v) \\ \qquad\qquad - \displaystyle\int_\Omega \chi_D p \, \mathrm{div}_D v \, dx = \int_\Omega (f \cdot \Pi_D v + G : \nabla_D v) \, dx, \, \forall v \in X_{D,0}, \\ \displaystyle\int_\Omega \chi_D q \, \mathrm{div}_D u \, dx = 0, \ \ \forall q \in Y_{D,0}. \end{cases} \tag{12}$$

Pour ce problème, le résultat de convergence sera de nouveau donné par argument de compacité. Les estimations sur le gradient discret et la pression discrète seront trouvées en prenant des fonctions test particulières dans le schéma. Les propriétés données dans la définition de b_D seront nécessaires lors de cette étape. Nous utiliserons le même argument que pour le problème de Stefan pour établir l'existence de la solution du schéma (12), un argument du degré topologique de Brouwer. Grâce à nos estimations précédemment établies, nous obtiendrons les convergences faibles de nos opérateurs discrets et une fois de plus la consistance et la conformité à la limite assureront que les limites sont bien les opérateurs continus. Pour montrer que l'on est solution du problème faible en passant à la limite, la propriété de compacité permettra d'entrer dans le cadre de la propriété de conformité à la limite trilinéaire et donc d'avoir $b_{D_m}(u_m, v_m) \to b(\bar{u},\bar{u},\bar{v})$ où b est l'opérateur continu trilinéaire de la formulation faible. Puis nous pourrons établir la convergence forte de $\nabla_{D_m} u_m$ en utilisant les mêmes arguments que pour le problème de Stokes transitoire et cela permettra ensuite d'avoir la convergence forte de la pression.

Étude du problème transitoire

Comme dit précédemment, le traitement du terme trilinéaire va poser énormément de problèmes dans l'étude du cas transitoire. En effet, nous n'aurons d'autre choix que de prendre des fonctions test à divergence discrète nulle pour pouvoir trouver les estimations nécessaires au résultat de convergence. La définition de b_D reste malgré tout la même. Les propriétés de coercivité, de compacité et de conformité à la limite ne changent pas et la propriété de consistance subit alors la même adaptation que pour le passage du problème de Stokes stationnaire à Stokes transitoire du fait de la présence d'une condition initiale à prendre en compte. Le principal changement se trouve donc dans la propriété de conformité à la limite trilinéaire en espace-temps puisque les hypothèses nécessaires pour pouvoir passer à la limite sont bien plus exigeantes que pour le cas stationnaire. Le schéma numérique est le suivant :

$$\left\{\begin{array}{l} u_D = (u_D^{(n)})_{n=0,...,N}\ ,\ p_D = (p_D^{(n)})_{n=1,...,N} \text{ tels que } u_D^{(0)} = J_D u_{\text{ini}} \text{ et, } \forall n = 0, ..., N-1 : \\ u_D^{(n+1)} \in X_{D,0}\ ,\ p_D^{(n+1)} \in Y_{D,0}\ , \\ \displaystyle\int_\Omega \Pi_D \delta_D^{n+\frac{1}{2}} u_D \cdot \Pi_D v \mathrm{d}x + \nu \int_\Omega \nabla_D u_D^{(n+1)} : \nabla_D v \mathrm{d}x + b_D(u^{n+1}, v) - \int_\Omega \chi_D p_D^{(n+1)} \mathrm{div}_D v \mathrm{d}x \\ \displaystyle\quad = \frac{1}{\delta t^{n+\frac{1}{2}}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_D v \mathrm{d}x\mathrm{d}t + \frac{1}{\delta t^{n+\frac{1}{2}}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega G \cdot \nabla_D v \mathrm{d}x\mathrm{d}t, \quad \forall v \in X_{D,0}. \\ \displaystyle\int_\Omega \mathrm{div}_D u_D^{(n+1)} \chi_D q \mathrm{d}x = 0, \quad \forall q \in Y_{D,0}. \end{array}\right. \tag{13}$$

Le résultat de convergence sera de nouveau établi grâce à un argument de compacité. Le but étant de pouvoir appliquer la version du théorème de Aubin-Simon discret 3.25 pour obtenir les convergences faibles de la vitesse et de son gradient afin de pouvoir passer à la limite. Pour cela nous aurons besoin prouver quelques estimations. Les premières sont une estimation $L^\infty(0,T;L^2(\Omega))$ de la vitesse reconstruite et une estimation $L^2(\Omega \times (0,T))$ pour le gradient discret. Nous les trouverons en prenant une nouvelle fois comme fonction test la solution discrète u_D dans le schéma (13). L'existence d'au moins une solution sera prouvée de la même manière que pour le problème de Stokes transitoire, c'est-à-dire en remarquant que si nous remplaçons η par $\frac{1}{\delta t}$ dans le schéma stationnaire nous retombons sur le transitoire. En appliquant donc le même argument du degré topologique, on a l'existence d'au moins une solution discrète (u_D, p_D). Les similitudes s'arrêteront malheureusement là, puisque l'on ne pourra pas extraire une estimation sur la pression discrète du fait que $\int_0^T b_D(u_D^{(n+1)}, v)$ ne pourra pas être contrôlé. En effet, un v qui permettrait de majorer ce terme ne permettra pas forcément de rentrer dans le cadre du résultat de Nečas et ainsi contrôler la norme de la pression discrète et inversement. D'où le fait que nos résultats seront exclusivement établis avec des fonctions test à divergence discrète nulle, ce qui élimine le terme de pression. Pour finir d'entrer dans le cadre de travail du lemme de Aubin-Simon, il faut une estimation sur la dérivée discrète en temps, pour cela nous introduisons la semi-norme suivante, $w \in X_{D,0}$

$$|w|_{*,D} = \sup\left\{\int_\Omega \Pi_D w \cdot \Pi_D v \mathrm{d}x \ : v \in E_D, \|v\|_D = 1\right\},$$

avec $E_D = \{v, v \in X_{D,0}, \mathrm{div}_D v = 0\}$.

Nous remarquons une fois de plus que l'espace discret prend en compte uniquement les fonctions à divergence discrète nulle ce qui est primordial pour pouvoir établir notre résultat de convergence. Nous établissons une estimation sur $\int_0^T |\delta_D u_D|_{\star,D} dt$ qui nous permet d'utiliser notre lemme et d'obtenir les convergences faibles de la vitesse et de son gradient. La compacité nous donne la convergence forte de $\Pi_D u$ vers $\bar{u}$ ce qui permettra le passage à la limite en utilisant la conformité à la limite trilinéaire espace-temps pour le terme contenant b_D. C'est tout ce que l'on pourra trouver avec nos hypothèses. En effet, pour avoir la convergence forte du gradient, en utilisant la même méthode que pour Stefan ou Stokes, il faudrait obtenir une équation d'énergie en remplaçant la fonction test de la formulation faible par la solution $\overline{u}$ mais les deux ne vivent pas dans le même espace et un tel remplacement est donc impossible.

Quelques exemples de discrétisations gradients

Le quatrième chapitre sera consacré à la partie gauche de notre Figure 2, c'est-à-dire comment montrer qu'une méthode numérique est une discrétisation gradient. Ceci sera fait pour les problèmes de Stokes et de Navier-Stokes, nous regroupons les deux puisque leurs discrétisations respectives restent proches l'une de l'autre. Ainsi nous montrerons que les éléments finis mixtes de Taylor-Hood et de Crouzeix-Raviart, la méthode Hybrid-Mixed-Mimetic (HMM) que nous voyons comme une extension de Crouzeix-Raviart mais sur un maillage polygonal ou encore la méthode Marker-And-Cell (MAC), sont bien des discrétisations gradients pour nos deux problèmes.

De plus, en prenant garde à ne pas majorer trop brutalement, au moment des démonstrations de conformité à la limite et de consistance, nous pourrons trouver l'ordre de convergence de chaque méthode pour l'estimation d'erreur du problème de Stokes stationnaire. Le schéma gradient une fois ré-écrit avec les opérateurs discrets de chaque méthode permet, dans la plupart des cas, de

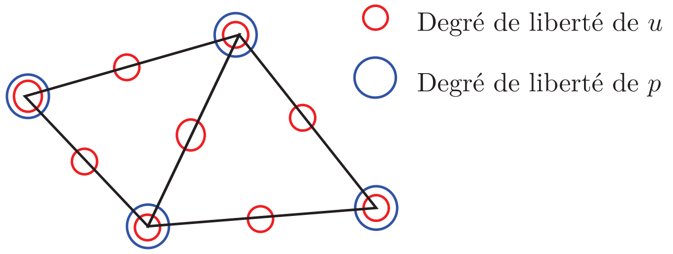


Figure 3 – Représentation des éléments finis de Taylor-Hood en 2D.

retrouver le schéma d'origine de la méthode.

Revenons à la première méthode proposée : les éléments finis mixtes de Taylor-Hood (voir Figure 3), qui sont donc une approximation P_2 conforme pour la vitesse et P_1 conforme pour la pression. Cette méthode sera la plus facile à faire entrer dans le cadre général des schémas gradients de par sa conformité. En effet, pour des méthodes numériques conformes, la propriété de conformité à la limite vaut 0 naturellement puisque dans ce cas $\nabla_D = \nabla \circ \Pi_D$ et de même la divergence discrète $\mathrm{div}_D = \mathrm{div} \circ \Pi_D$. Les autres propriétés ne sont pas difficiles à prouver car la plupart des preuves peuvent aujourd'hui être trouvées dans la littérature pour ce type de méthodes.

Les éléments finis mixtes de Crouzeix-Raviart (voir Figure 4), P_1 non-conforme pour la vitesse et P_0 pour la pression, amènent la première difficulté puisqu'ici le gradient discret est le "broken gradient" classique et donc discontinu sur les arêtes du maillage. Cette particularité est la cause de quelques problèmes à gérer pour démontrer les différentes propriétés de la discrétisation. Heureusement, avec une hypothèse encore une fois non-restrictive sur le maillage, nous pouvons contrôler le saut de cette discontinuité et ainsi faire en sorte qu'il n'explose pas, amenant alors une majoration de ce saut et rendant des intégrations par partie possibles pour prouver les différentes propriétés. Une autre difficulté réside dans la propriété de coercivité, où nous demandons une borne pour la condition inf-sup discrète, nous utiliserons alors fortement un des résultats de Nečas qui permet d'identifier la pression comme divergence d'une fonction particulière tout en contrôlant sa norme. Encore une fois, les éléments finis de Crouzeix-Raviart ont été et sont encore très utilisés et de nombreux résultats existant dans la littérature aident grandement à les faire rentrer dans le cadre des schémas gradients.

Comme dit précédemment, la méthode HMM peut-être considérée comme l'extension des éléments finis de Crouzeix-Raviart sur un maillage polyédrique (voir Figure 5) sur lequel nous faisons tout de même certaines hypothèses pour qu'il soit "acceptable" comme par exemple le fait que la cellule ne doit pas être trop écrasée ou trop petite par rapport aux autres. Les pro-

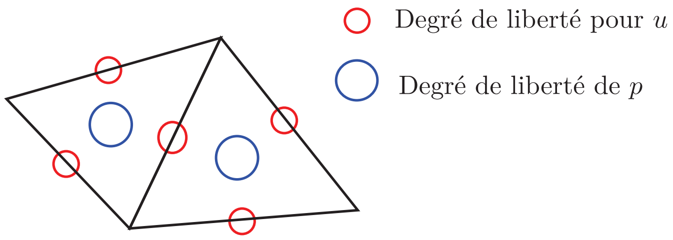


Figure 4 – Représentation des éléments finis de Crouzeix-Raviart en 2D.

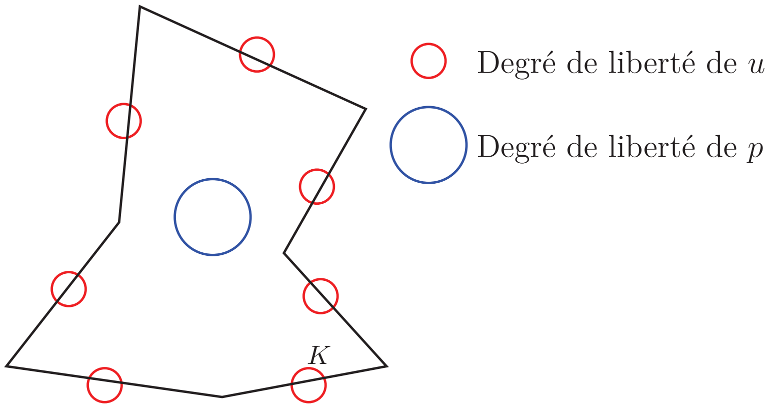


Figure 5 – Représentation du schéma HMM dans une cellule K d'un maillage 2D.

priétés prendront donc en compte la particularité du maillage mais utiliseront principalement les mêmes outils et résultats que pour les éléments finis de Crouzeix-Raviart dont notamment le contrôle du saut du "broken gradient" et le résultat de Nečas pour celui de la condition inf-sup. La particularité d'avoir choisi une méthode HMM est qu'elle avait déjà été utilisée pour Stokes mais avec des résultats de convergence moins complets que ceux donnés par le cadre général de travail.

L'une des méthodes les plus utilisées en ingénierie et en mécanique des fluides aujourd'hui est sans aucun doute le schéma Marker-And-Cell dit schéma MAC. Il peut facilement être mis en place sur un domaine dont les bords sont des parties parallèles aux axes du repère. Les difficultés dans le schéma MAC sont de faire attention non seulement à la définition de notre gradient discret mais aussi aux mailles décalées lorsque nous intégrons par partie pour prouver la conformité à la limite par exemple. Une fois ces difficultés passées, les propriétés se démontrent en s'aidant une fois de plus des résultats de la littérature déjà existants. Pour le problème de Stokes, le schéma gradient appliqué avec la discrétisation MAC du chapitre 4 permet de retrouver les schémas MAC de la littérature ; tandis que pour Navier-Stokes, cela n'a pas été prouvé en raison du terme b_D particulier qui semble poser quelques problèmes.

Dans ce chapitre 4, nous donnerons également un exemple de construction pour b_D remplissant les conditions nécessaires pour être bien défini dans la discrétisation gradient de Navier-Stokes et respectant la propriété de conformité à la limite trilinéaire. Cette application sera en réalité fortement inspirée de celle utilisée dans les éléments finis et aura pour base les opérateurs discrets Π_D et ∇_D.

$$b_D(u,v) = \frac{1}{2}\left(\widetilde{b_D}(u,u,v) - \widetilde{b_D}(u,v,u)\right).$$

Avec $\widetilde{b_D} : X_{D,0}^3 \mapsto \mathbb{R}$ tel que

$$\widetilde{b_D}(u,v,w) = \sum_{i,j=1}^{d} \int_\Omega \Pi_D^{(i)} u \nabla_D^{(i,j)} v \Pi_D^{(j)} w \, dx.$$

D'ailleurs, la généralisation de ce terme n'est venue qu'après avoir réussi les résultats de convergence avec le b_D particulier des méthodes éléments finis. Il nous a permis de mettre en avant les propriétés nécessaires à la convergence du schéma. En effet la propriété de conformité à la limite trilinéaire est née de la volonté de trouver une généralisation de ce b_D particulier. Il amènera également la définition de la propriété de p-coercivité qui suit la même idée que la coercivité mais qui vérifie une inégalité de Sobolev discrète plutôt qu'une inégalité de Poincaré. Elle sera particulièrement nécessaire pour le problème de Navier-Stokes et les méthodes présentées devront être compatibles avec cette p-coercivité pour pouvoir rentrer dans le cadre du théorème de convergence avec le b_D particulier.

Chapitre 1

The Stefan problem

1.1 Introduction

The aim of this chapter is to extend this framework to the Stefan problem and so we are interested in the approximation of $\bar{u}$, solution of the following equations :

$$\partial_t \bar{u} - \Delta\zeta(\bar{u}) = f, \text{ in } \Omega \times (0, T) \tag{1.1}$$

with the following initial condition :

$$\bar{u}(x, 0) = \bar{u}_{\text{ini}}(x), \text{ for a.e. } x \in \Omega, \tag{1.2}$$

together with the homogeneous Dirichlet boundary condition :

$$\zeta(\bar{u}(x, t)) = 0 \text{ on } \partial\Omega \times (0, T), \tag{1.3}$$

under the following assumptions :

$$\Omega \text{ is an open bounded connected polyhedral subset of } \mathbb{R}^d,\ d \in \mathbb{N}^\star \text{ and } T > 0, \tag{1.4a}$$

$$\bar{u}_{\text{ini}} \in L^2(\Omega) \tag{1.4b}$$

$$f \in L^2(\Omega \times (0,T)), \tag{1.4c}$$

$$\zeta \in C^0(\mathbb{R}) \text{ is non–decreasing, Lipschitz continuous with Lipschitz constant } L_\zeta,\ \text{and such that } \zeta(0) = 0, \tag{1.4d}$$

and

$$|\zeta(s)| \geq a|s| - b \text{ for all } s \in \mathbb{R} \text{ for some given values } a, b \in (0, +\infty). \tag{1.4e}$$

Definition 1.1. *A function $\bar{u}$ is said to be a weak solution of Problem (1.1)-(1.2)-(1.3) if the following holds :*

$$\bar{u} \in L^2(\Omega \times (0,T)),\ \zeta(\bar{u}) \in L^2(0,T; H_0^1(\Omega)),$$
$$\int_0^T \int_\Omega \left(-\bar{u}(x,t)\partial_t\varphi(x,t) + \nabla\zeta(\bar{u})(x,t) \cdot \nabla\varphi(x,t)\right)\, dxdt - \int_\Omega u_{\text{ini}}(x)\varphi(x,0)\, dx$$
$$= \int_0^T \int_\Omega f(x,t)\varphi(t)\, dxdt,\ \forall \varphi \in C_c^\infty(\Omega \times [0,T[), \tag{1.5}$$

where we denote by $C_c^\infty(\Omega \times [0,T[)$ the set of the restrictions of functions of $C_c^\infty(\Omega \times]-\infty, T[)$ to $\Omega \times [0,T[$.

1.2 Gradient Discretisation

Definition 1.2 (Gradient Discretisation.)**.** *A gradient discretisation D for a space-dependent second order elliptic problem, with homogeneous Dirichlet boundary conditions, is defined by $D = (X_{D,0}, \Pi_D, \nabla_D)$, where :*

1. *the set of discrete unknowns $X_{D,0}$ is a finite dimensional vector space on $\mathbb{R}$,*
2. *the linear mapping $\Pi_D : X_{D,0} \to L^2(\Omega)$ is the reconstruction of the approximate function,*
3. *the linear mapping $\nabla_D : X_{D,0} \to L^2(\Omega)^d$ is the discrete gradient operator. It must be chosen such that $\|\cdot\|_D := \|\nabla_D \cdot\|_{L^2(\Omega)^d}$ is a norm on $X_{D,0}$.*

Remark 1.3 (Boundary conditions.). *The definition of* $\|\cdot\|_D$ *depends on the considered boundary conditions. Here for simplicity we only consider homogeneous Dirichlet boundary conditions, but other conditions can easily be addressed. For example, in the case of homogeneous Neumann boundary conditions, we will use the notation* X_D *instead of* $X_{D,0}$ *for the discrete space, and define*
$\|\cdot\|_D := (\|\Pi_D \cdot\|^2_{L^2(\Omega)} + \|\nabla_D \cdot\|^2_{L^2(\Omega)^d})^{1/2}$.

Definition 1.4 (Coercivity). *Let* D *be a gradient discretisation in the sense of Definition 1.2, and let* C_D *be the norm of the linear mapping* Π_D*, defined by*

$$C_D = \max_{v \in X_{D,0}\setminus\{0\}} \frac{\|\Pi_D v\|_{L^2(\Omega)}}{\|v\|_D}. \tag{1.6}$$

A sequence $(D_m)_{m\in\mathbb{N}}$ *of gradient discretisations is said to be* ***coercive*** *if there exists* $C_P \in \mathbb{R}_+$ *such that* $C_{D_m} \le C_P$ *for all* $m \in \mathbb{N}$.

Remark 1.5 (Discrete Poincaré inequality.).
Equation (1.6) yields $\|\Pi_D v\|_{L^2(\Omega)} \le C_D \|\nabla_D v\|_{L^2(\Omega)^d}$.

The consistency is ensured by a proper choice of the interpolation operator and discrete gradient.

Definition 1.6 (Consistency.). *Let* D *be a gradient discretisation in the sense of Definition 1.2, and let* $S_D : H^1_0(\Omega) \to [0, +\infty)$ *be defined by*

$$\forall \varphi \in H^1_0(\Omega)\,, \quad S_D(\varphi) = \min_{v\in X_{D,0}} \Big(\|\Pi_D v - \varphi\|_{L^2(\Omega)} + \|\nabla_D v - \nabla\varphi\|_{L^2(\Omega)^d} \Big)\,. \tag{1.7}$$

A sequence $(D_m)_{m\in\mathbb{N}}$ *of gradient discretisations is said to be* ***consistent*** *if, for all* $\varphi \in H^1_0(\Omega)$, $S_{D_m}(\varphi)$ *tends to 0 as* $m \to \infty$.

Since we are dealing with nonconforming methods, we need that the dual of the discrete gradient be "close to" a discrete divergence.

Definition 1.7 (Limit-conformity.). *Let* D *be a gradient discretisation in the sense of Definition 1.2. We let* $H_{\mathrm{div}}(\Omega) = \{\varphi \in L^2(\Omega)^d, \mathrm{div}\varphi \in L^2(\Omega)\}$ *and* $W_D : H_{\mathrm{div}}(\Omega) \to [0, +\infty)$ *be defined by*

$$\begin{aligned} &\forall \varphi \in H_{\mathrm{div}}(\Omega) \\ &W_D(\varphi) = \max_{u\in X_{D,0}\setminus\{0\}} \frac{1}{\|u\|_D} \left| \int_\Omega \left(\nabla_D u(x)\cdot\varphi(x) + \Pi_D u(x)\mathrm{div}\varphi(x) \right)\, dx \right|. \end{aligned} \tag{1.8}$$

A sequence $(D_m)_{m\in\mathbb{N}}$ *of gradient discretisations is said to be* ***limit-conforming*** *if, for all* $\varphi \in H_{\mathrm{div}}(\Omega)$, $W_{D_m}(\varphi)$ *tends to 0 as* $m \to \infty$.

Dealing with generic non-linearity often requires compactness properties on the scheme.

Definition 1.8 (Compactness.)**.** *A sequence $(D_m)_{m\in\mathbb{N}}$ of gradient discretisations is said to be **compact** if, for all sequence $u_m \in X_{D_m,0}$ such that $\|u_m\|_{D_m}$ is bounded, the sequence $(\Pi_{D_m} u_m)_{m\in\mathbb{N}}$ is relatively compact in $L^2(\Omega)$.*

Let us state an important relation between compactness and coercivity.

Lemma 1.9 (Compactness implies coercivity.)**.**

Let $(D_m)_{m\in\mathbb{N}}$ be a compact sequence of gradient discretisations in the sense of Definition 1.8. Then it is coercive in the sense of Definition 1.4.

Proof. Let us assume that the sequence is not coercive. Then there exists a subsequence of $(D_m)_{m\in\mathbb{N}}$ (identically denoted) such that, for all $m \in \mathbb{N}$, there exists $u_m \in X_{D_m,0} \setminus \{0\}$ with

$$\lim_{m\to\infty} \frac{\|\Pi_{D_m} u_m\|_{L^2(\Omega)}}{\|u_m\|_{D_m}} = +\infty.$$

This means that, denoting by $v_m = u_m/\|u_m\|_{D_m}$, $\lim_{m\to\infty} \|\Pi_{D_m} v_m\|_{L^2(\Omega)} = +\infty$.

But we have $\|v_m\|_{D_m} = 1$, and the compactness of the sequence of discretisations implies that the sequence $(\Pi_{D_m} v_m)_{m\in\mathbb{N}}$ is relatively compact in $L^2(\Omega)$. This gives a contradiction.

Thanks to [63, Lemma 2.4], we may check the consistency and limit-conformity properties of given gradient schemes, only using dense subsets of the test functions spaces. The following lemma, useful in Section 1.5, is an immediate consequence of [63, Lemma 2.4] and of Kolmogorov's theorem.

Lemma 1.10 (Sufficient conditions.)**.**

Let Ω be polyhedral and thus locally star-shaped and let $\mathcal{F}$ be a family of gradient discretisations in the sense of Definition 1.2. Assume that there exist $C, \nu \in (0,\infty)$ and, for all $D \in \mathcal{F}$, a real value $h_D \in (0,+\infty)$ such that :

$$S_D(\varphi) \le C h_D \|\varphi\|_{W^{2,\infty}(\Omega)}, \text{ for all } \varphi \in C_c^\infty(\Omega), \tag{1.9a}$$

$$W_D(\varphi) \le C h_D \|\varphi\|_{(W^{1,\infty}(\mathbb{R}^d))^d}, \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^d)^d, \tag{1.9b}$$

$$\max_{v\in X_{D,0}\setminus\{0\}} \frac{\|\Pi_D v(\cdot+\xi) - \Pi_D v\|_{L^p(\mathbb{R}^d)}}{\|v\|_D} \le C|\xi|^\nu, \text{ for all } \xi \in \mathbb{R}^d, \tag{1.9c}$$

where S_D, W_D are defined above.

Then, any sequence $(D_m)_{m\in\mathbb{N}} \subset \mathcal{F}$ such that $h_{D_m} \to 0$ as $m \to \infty$ is consistent, limit-conforming and compact (and therefore coercive).

Remark 1.11. *In several cases, h_D stands for the mesh size : this is the case for the numerical schemes used in Section 1.5.*

Definition 1.12 (Piecewise constant function reconstruction.)**.**

Let $D = (X_{D,0}, \Pi_D, \nabla_D)$ be a gradient discretisation in the sense of Definition 1.2, and I be the finite set of the degrees of freedom, such that $X_{D,0} = \mathbb{R}^I$. We say that Π_D is a piecewise constant function reconstruction if there exists a family of open subsets of Ω, denoted by $(\Omega_i)_{i\in I}$, such that $\bigcup_{i\in I} \overline{\Omega_i} = \overline{\Omega}$, $\Omega_i \cap \Omega_j = \emptyset$ for all $i \neq j$, and $\Pi_D u = \sum_{i\in I} u_i \chi_{\Omega_i}$ for all $u = (u_i)_{i\in I} \in X_{D,0}$, where χ_{Ω_i} is the characteristic function of Ω_i.

Remark 1.13. *Let us notice that $\|\Pi_D \cdot \|_{L^2(\Omega)}$ is not requested to be a norm on $X_{D,0}$. Indeed, in several examples that can be considered, some degrees of freedom are involved in the reconstruction of the gradient of the function, but not in that of the function itself. Hence it can occur that some of the Ω_i are empty.*

Remark 1.14. *An important example of gradient discretisation $D = (X_{D,0}, \Pi_D, \nabla_D)$ in the sense of Definition 1.2, such that Π_D is a piecewise constant function reconstruction in the sense of Definition 1.12, is the case of the mass-lumping of conforming finite elements. Indeed, assuming that $(\xi_i)_{i\in I}$ is the basis of some finite–dimensional space $V_h \subset H^1_0(\Omega)$, we consider a family $(\Omega_i)_{i\in I}$, chosen such that*

$$\| \sum_{i\in I} u_i \chi_{\Omega_i} - \sum_{i\in I} u_i \xi_i \|_{L^2(\Omega)} \leq h \| \sum_{i\in I} u_i \nabla \xi_i \|_{L^2(\Omega)^d}, \ \forall u \in X_{D,0}.$$

We then define Π_D as in Definition 1.12, and $\nabla_D u = \sum_{i\in I} u_i \nabla \xi_i$. This is easily performed, considering P^1 conforming finite element, splitting each simplex in subsets defined by the highest barycentric coordinate, and defining Ω_i by the union of the subsets of the simplices connected to the vertex indexed by i.

Remark 1.15. *Note that we have the two important following properties, in the case of a piecewise constant function reconstruction in the sense of Definition 1.12 :*

$$g(\Pi_D u(x)) = \Pi_D g(u)(x), \ \text{for a.e. } x \in \Omega, \ \forall u \in X_{D,0}, \ \forall g \in C(\mathbb{R}), \tag{1.10}$$

where for any continuous function $g \in C(\mathbb{R})$ and $u = (u_i)_{i\in I} \in X_{D,0}$, we classically denote by $g(u) = (g(u_i))_{i\in I} \in X_{D,0}$ and

$$\Pi_D u(x) \Pi_D v(x) = \Pi_D(uv)(x), \ \text{for a.e. } x \in \Omega, \ \forall u, v \in X_{D,0}, \tag{1.11}$$

where, for $u = (u_i)_{i\in I}$ *and* $v = (v_i)_{i\in I} \in X_{D,0}$*, we denote by* $uv = (u_i v_i)_{i\in I} \in X_{D,0}$.

Definition 1.16 (Space-time gradient discretisation)**.** *Under Hypothesis (1.4a), we say that* $D = (X_{D,0}, \Pi_D, \nabla_D, (t^{(n)})_{n=0,\ldots,N})$ *is a space-time gradient discretisation of* $\Omega \times (0,T)$ *if*

- $(X_{D,0}, \Pi_D, \nabla_D)$ *is a gradient discretisation of* Ω*, in the sense of Definition 1.2,*
- $t^{(0)} = 0 < t^{(1)} \ldots < t^{(N)} = T$.

We then set $\delta t^{(n+\frac{1}{2})} = t^{(n+1)} - t^{(n)}$*, for* $n = 0,\ldots,N-1$*,* $\delta t_D = \max_{n=0,\ldots,N-1} \delta t^{(n+\frac{1}{2})}$ *and we define the dual semi-norm* $|w|_{\star,D}$ *of* $w \in X_{D,0}$ *by*

$$|w|_{\star,D} = \sup\left\{\int_\Omega \Pi_D w(x)\Pi_D z(x)\, dx : z \in X_{D,0}, \|z\|_D = 1\right\}. \tag{1.12}$$

Definition 1.17 (Space-time consistency)**.** *A sequence* $(D_m)_{m\in\mathbb{N}}$ *of space-time gradient discretisations of* $\Omega\times(0,T)$*, in the sense of Definition 1.16, is said to be* ***consistent*** *if it is consistent in the sense of Definition 1.6 and if* δt_{D_m} *tends to 0 as* $m \to \infty$.

1.3 Approximation of the Stefan problem by the Gradient Discretisation

Let $D = (X_{D,0}, \Pi_D, \nabla_D, (t^{(n)})_{n=0,\ldots,N})$ be a space-time discretisation in the sense of Definition 1.16 such that Π_D is a piecewise constant function reconstruction in the sense of Definition 1.12. We define the following (implicit) scheme for the discretisation of Problem (1.5). We consider a sequence $(u^{(n)})_{n=0,\ldots,N}$ such that :

$$\begin{cases} u^{(0)} \in X_{D,0}, \\ u^{(n+1)} \in X_{D,0},\ \delta_D^{(n+\frac{1}{2})} u = \Pi_D \dfrac{u^{(n+1)} - u^{(n)}}{\delta t^{(n+\frac{1}{2})}}, \\ \displaystyle\int_\Omega \Big(\delta_D^{(n+\frac{1}{2})} u(x)\Pi_D v(x) + \nabla_D \zeta(u^{(n+1)})(x)\cdot\nabla_D v(x)\Big)\, dx = \\ \dfrac{1}{\delta t^{(n+\frac{1}{2})}} \displaystyle\int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f(x,t)\Pi_D v(x)\, dxdt, \quad \forall v \in X_{D,0},\ \forall n = 0,\ldots,N-1. \end{cases} \tag{1.13}$$

We again use the notations Π_D and ∇_D for the definition of space-time dependent functions (note that we define these functions for all $t \in [0,T]$) :

$$\begin{aligned}
&\Pi_D u(x,0) = \Pi_D u^{(0)}(x) \text{for a.e. } x \in \Omega,\\
&\Pi_D u(x,t) = \Pi_D u^{(n+1)}(x)\\
&\Pi_D \zeta(u)(x,t) = \Pi_D \zeta(u^{(n+1)})(x)\\
&\nabla_D \zeta(u)(x,t) = \nabla_D \zeta(u^{(n+1)})(x),\\
&\text{for a.e. } x \in \Omega,\ \forall t \in (t^{(n)}, t^{(n+1)}],\ \forall n = 0,\ldots,N-1.
\end{aligned} \tag{1.14}$$

We also denote

$$\delta_D u(x,t) = \delta_D^{(n+\frac{1}{2})} u(x),\ \text{for a.e. } (x,t) \in \Omega \times (t^{(n)}, t^{(n+1)}),\ \forall n = 0,\ldots,N-1. \tag{1.15}$$

We finally introduce the primitive function

$$Z(s) = \int_0^s \zeta(x)\, dx,\ \forall s \in \mathbb{R}. \tag{1.16}$$

which is used several times in the convergence proofs. We then have

$$Z(s) = \int_0^s \zeta(x)\, dx = \int_0^s (\zeta(x) - \zeta(0))\, dx \le L_\zeta \int_0^s x\, dx = L_\zeta \frac{s^2}{2},\ \forall s \in \mathbb{R}, \tag{1.17}$$

and, from Hypotheses (1.4d) and (1.4e),

$$Z(s) \ge \int_0^s \zeta(x) \frac{\zeta'(x)}{L_\zeta}\, dx = \frac{\zeta(s)^2}{2L_\zeta} \ge \frac{a^2 s^2 - 2b^2}{4L_\zeta},\ \forall s \in \mathbb{R}. \tag{1.18}$$

(where we have used the fact that ζ^2 is locally Lipschitz continuous, and therefore locally absolutely continuous).

Lemma 1.18 (A priori estimates and existence of a discrete solution)**.** *Under Hypotheses (1.4), let $D = (X_{D,0}, \Pi_D, \nabla_D, (t^{(n)})_{n=0,\ldots,N})$ be a space-time gradient discretisation in the sense of Definition 1.16 such that Π_D is a piecewise constant function reconstruction in the sense of Definition 1.12. Then there exists at least one solution to Scheme (1.13), which satisfies*

$$\begin{aligned}
&\int_0^T \int_\Omega |\nabla_D \zeta(u)(x,t)|^2\, dxdt\\
&+ \int_\Omega (Z(\Pi_D u^{(N)}(x)) - Z(\Pi_D u^0(x)))\, dx \le \int_0^T \int_\Omega f(x,t) \Pi_D \zeta(u)(x,t)\, dxdt.
\end{aligned} \tag{1.19}$$

Moreover, let $C_P > 0$ such that $C_D \le C_P$, where C_D is the coercivity constant of the discreti-

zation (see Definition 1.4) and let $C_{\text{ini}} > 0$ *be such that* $C_{\text{ini}} \geq \|u_{\text{ini}} - \Pi_D u^{(0)}\|_{L^2(\Omega)}$ *; then there exists* $C_1 > 0$*, only depending on* L_ζ*,* a*,* b*,* C_P*,* C_{ini} *and* f *such that, for any solution* u *to this scheme,*

$$\|\Pi_D \zeta(u)\|_{L^\infty(0,T;L^2(\Omega))} \leq C_1, \text{ and } \|\Pi_D u\|_{L^\infty(0,T;L^2(\Omega))} \leq C_1, \tag{1.20}$$

and

$$\|\nabla_D \zeta(u)\|_{L^2(\Omega\times(0,T))^d} \leq C_1. \tag{1.21}$$

Proof. Before showing the existence of at least one discrete solution to Scheme (1.13), let us first prove if there exists a solution then it satisfies (1.19), (1.20) and (1.21). From the properties of function Z defined by (1.16), and using $\int_a^b \zeta(x)\,dx = Z(b) - Z(a) = \zeta(b)(b-a) - \int_a^b \zeta'(x)(x-a)\,dx$, we get, from Hypothesis (1.4d), that

$$\delta t^{(n+\frac{1}{2})} \delta_D^{(n+\frac{1}{2})} u \;\Pi_D \zeta(u^{(n+1)}) \geq \Pi_D Z(u^{(n+1)}) - \Pi_D Z(u^{(n)}). \tag{1.22}$$

We then let $v = \delta t^{(n+\frac{1}{2})} \zeta(u^{(n+1)})$ in (1.13), we sum the obtained equation for $n = 0, \ldots, m-1$ for a given $m = 1, \ldots, N$, and using (1.22), we get (1.19) replacing T by $t^{(m)}$ and $u^{(N)}$ by $u^{(m)}$. Thanks to the Cauchy-Schwarz inequality, we get that

$$\|\Pi_D Z(u^{(m)})\|_{L^1(\Omega)} + \int_0^{t^{(m)}} \|\nabla_D \zeta(u)(\cdot,t)\|^2_{L^2(\Omega)^d}\,dt \leq \|f\|_{L^2(\Omega\times(0,t^{(m)}))} \|\Pi_D \zeta(u)\|_{L^2(\Omega\times(0,t^{(m)}))} + \|\Pi_D Z(u^{(0)})\|_{L^1(\Omega)},$$

which in turn yields, thanks to the Young inequality, and to (1.17) and (1.18),

$$\frac{1}{2L_\zeta} \|\Pi_D \zeta(u^{(m)})\|^2_{L^2(\Omega)} + \int_0^{t^{(m)}} \|\nabla_D \zeta(u)(\cdot,t)\|^2_{L^2(\Omega)^d}\,dt \leq \frac{C_D^2}{2} \|f\|^2_{L^2(\Omega\times(0,t^{(m)}))} + \frac{1}{2C_D^2} \|\Pi_D \zeta(u)\|^2_{L^2(\Omega\times(0,t^{(m)}))} + \frac{L_\zeta}{2} \|\Pi_D u^{(0)}\|^2_{L^2(\Omega)}.$$

Using the definition (1.6) of C_D, we prove the first estimate of (1.20) and the estimate (1.21). We get the second estimate of (1.20) by using the second part of (1.18).

The existence of a solution follows from these estimates by a now classical topological degree argument. Indeed, let $\theta \in [0,1]$, we introduce $\zeta_\theta(s) = \theta\zeta(s) + (1-\theta)as$, for any $s \in \mathbb{R}$. Replacing ζ by ζ_θ in the scheme, we get the same a priori estimates (1.20) and (1.21) independently of θ. We conclude thanks to the Brouwer topological degree, since setting $\theta = 0$, we obtain the discretization of the heat equation, for which the existence of the solution is well–known.

Lemma 1.19 (Uniqueness results on the discrete solution)**.** *Under Hypotheses* (1.4)*, let* $D = (X_{D,0}, \Pi_D, \nabla_D, (t^{(n)})_{n=0,\ldots,N})$ *be a space-time gradient discretisation in the sense of Definition 1.16 such that* Π_D *is a piecewise constant function reconstruction in the sense of Definition 1.12. Let* $u^{(0)} \in X_{D,0}$ *be given, and, for* $n = 0, \ldots, N-1$*, let* $u^{(n+1)} \in X_{D,0}$ *be such that* (1.13) *holds. Then, for all* $n = 0, \ldots, N-1$*,* $\Pi_D u^{(n+1)} \in L^2(\Omega)$ *and* $\zeta(u^{(n+1)}) \in X_{D,0}$ *are unique.*

Proof. Let us consider two solutions, denoted $u^{(n+1)}, \widetilde{u}^{(n+1)} \in X_{D,0}$, for some $n = 0, \ldots, N-1$, such that (1.13) holds with $\Pi_D u^{(n)}(x) = \Pi_D \widetilde{u}^{(n)}(x)$, for a.e. $x \in \Omega$. We then subtract the corresponding equation with $\widetilde{u}^{(n+1)}$ to that with $u^{(n+1)}$. We get

$$\int_\Omega \Big(\frac{\Pi_D(u^{(n+1)} - \widetilde{u}^{(n+1)})(x)}{\delta t^{(n+\frac{1}{2})}} \Pi_D v(x) + \\ \nabla_D(\zeta(u^{(n+1)}) - \zeta(\widetilde{u}^{(n+1)}))(x) \cdot \nabla_D v(x) \Big) \, dx = 0, \forall v \in X_{D,0}. \quad (1.23)$$

We let $v = \zeta(u^{(n+1)}) - \zeta(\widetilde{u}^{(n+1)})$ in (1.23). Using Hypothesis (1.4d), we may write that

$$(\Pi_D(u^{(n+1)} - \widetilde{u}^{(n+1)})(x))\Pi_D(\zeta(u^{(n+1)}) - \zeta(\widetilde{u}^{(n+1)}))(x) \\ = (\Pi_D u^{(n+1)}(x) - \Pi_D \widetilde{u}^{(n+1)}(x))(\zeta(\Pi_D u^{(n+1)}(x)) - \zeta(\Pi_D \widetilde{u}^{(n+1)}(x))) \geq 0,$$

which implies that

$$\int_\Omega |\nabla_D(\zeta(u^{(n+1)}) - \zeta(\widetilde{u}^{(n+1)}))(x)|^2 \, dx = 0,$$

and therefore that $\zeta(u^{(n+1)}) = \zeta(\widetilde{u}^{(n+1)})$. We then get, from (1.23), that

$$\int_\Omega \frac{\Pi_D(u^{(n+1)} - \widetilde{u}^{(n+1)})(x)}{\delta t^{(n+\frac{1}{2})}} \Pi_D v(x) \, dx = 0, \quad \forall v \in X_{D,0}.$$

It now suffices to let $v = u^{(n+1)} - \widetilde{u}^{(n+1)}$ in the preceding equation, to get that $\Pi_D u^{(n+1)}(x) = \Pi_D \widetilde{u}^{(n+1)}(x)$ for a.e. $x \in \Omega$.

Lemma 1.20 (Estimate on the dual semi-norm of the discrete time derivative)**.**

Under Hypotheses (1.4)*, let* $D = (X_{D,0}, \Pi_D, \nabla_D, (t^{(n)})_{n=0,\ldots,N})$ *be a space-time gradient discretisation in the sense of Definition 1.16 such that* Π_D *is a piecewise constant function reconstruction in the sense of Definition 1.12. Then there exists* $C > 0$*, only depending on* L_ζ*,* a*,* b*,* $C_P > C_D$*,* $C_{\rm ini} > \|u_{\rm ini} - \Pi_D u^{(0)}\|_{L^2(\Omega)}$*,* f *and* T *such that, for any solution* u *to Scheme* (1.13)*,*

$$\int_0^T |\delta_D u(t)|_{\star,D}^2 \, dt \leq C \quad (1.24)$$

Proof. Let us take a generic $v \in X_{D,0}$ as test function in Scheme 1.13. Using the definition of $|\cdot|_{\star,D}$ gives an estimate on $|\delta_D^{(n+\frac{1}{2})}u|_{\star,D}$ depending on $\|\nabla_D \zeta(u^{(n+1)})\|_{L^2(\Omega)}$. Squarring this estimates, multiplying by $\delta t^{(n+\frac{1}{2})}$ and summing over n gives the desired estimate thanks to (1.21).

Lemma 1.21 (Estimate on the time translates)**.**

Under Hypotheses (1.4)*, let* $D = (X_{D,0}, \Pi_D, \nabla_D, (t^{(n)})_{n=0,\ldots,N})$ *be a space-time gradient discretisation in the sense of Definition 1.16 such that* Π_D *is a piecewise constant function reconstruction in the sense of Definition 1.12. Then there exists* $C_2 > 0$*, only depending on* L_ζ*,* a*,* b*,* $C_P > C_D$*,* $C_{\text{ini}} > \|u_{\text{ini}} - \Pi_D u^{(0)}\|_{L^2(\Omega)}$*,* f *such that, for any solution* u *to Scheme* (1.13)*,*

$$\|\Pi_D\zeta(u)(\cdot, \cdot + \tau) - \Pi_D\zeta(u)(\cdot, \cdot)\|^2_{L^2(\Omega\times(0,T-\tau))} \leq C_2(\tau + \delta t), \forall \tau \in (0, T). \tag{1.25}$$

Proof. In order to make the proof clear, let us give its principle, assuming that a solution $\bar{u}$ of the continuous equation (1.1) is regular enough. We write the time translate of this solution in $L^2(\Omega \times (0, T - \tau))$, for a step $\tau \in (0, T)$. We first note that

$$(\zeta(\bar{u}(x, t + \tau)) - \zeta(\bar{u}(x, t)))^2 \leq L_\zeta(\zeta(\bar{u}(x, t + \tau)) - \zeta(\bar{u}(x, t)))(\bar{u}(x, t + \tau) - \bar{u}(x, t)),$$

which gives, using (1.1),

$$\begin{aligned}
&(\zeta(\bar{u}(x, t + \tau)) - \zeta(\bar{u}(x, t)))^2 \\
&\leq L_\zeta(\zeta(\bar{u}(x, t + \tau)) - \zeta(\bar{u}(x, t))) \int_0^\tau \partial_t \bar{u}(x, t + s)\, ds \\
&\leq L_\zeta(\zeta(\bar{u}(x, t + \tau)) - \zeta(\bar{u}(x, t))) \int_0^\tau (\Delta\zeta(\bar{u}(x, t + s)) + f(x, t + s))\, ds.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\int_0^{T-\tau} & \int_\Omega (\zeta(\bar{u}(x, t + \tau)) - \zeta(\bar{u}(x, t)))^2\, dxdt \\
&\leq L_\zeta \int_0^\tau \int_0^{T-\tau} \int_\Omega (\zeta(\bar{u}(x, t + \tau)) - \zeta(\bar{u}(x, t))) \\
&\qquad\qquad (\Delta\zeta(\bar{u}(x, t + s)) + f(x, t + s))\, dxdtds \\
&\leq L_\zeta \int_0^\tau \int_0^{T-\tau} \int_\Omega (-\nabla\zeta(\bar{u}(x, t + \tau)) + \nabla\zeta(\bar{u}(x, t))) \cdot \nabla\zeta(\bar{u}(x, t + s))\, dxdtds \\
&\quad + L_\zeta \int_0^\tau \int_0^{T-\tau} \int_\Omega (\zeta(\bar{u}(x, t + \tau)) - \zeta(\bar{u}(x, t))) f(x, t + s)\, dxdtds.
\end{aligned}$$

Each product ab of the above right hand side is then bounded by $\frac{1}{2}(a^2 + b^2)$, which allows to conclude, thanks to the continuous estimates similar to (1.20), that

$$\int_0^{T-\tau}\int_\Omega (\zeta(\bar{u}(x,t+\tau))-\zeta(\bar{u}(x,t)))^2\, dxdtds \le \tau C.$$

Let us now use the same ideas for the proof of (1.25). Let $\tau\in(0,T)$. Similarly using that L_ζ is a Lipschitz constant of ζ and ζ is nondecreasing, and using (1.10), the following inequality holds :

$$\int_{\Omega\times(0,T-\tau)}\Big(\Pi_D\zeta(u)(x,t+\tau)-\Pi_D\zeta(u)(x,t)\Big)^2\, dxdt \le L_\zeta\int_0^{T-\tau}A(t)\,dt, \tag{1.26}$$

where, for almost every $t\in(0,T-\tau)$,

$$A(t)=\int_\Omega\Big(\Pi_D\zeta(u)(x,t+\tau)-\Pi_D\zeta(u)(x,t)\Big)\Big(\Pi_D u(x,t+\tau)-\Pi_D u(x,t)\Big)\,dx.$$

Let $t\in(0,T-\tau)$. Denoting $n_0(t)$, $n_1(t)=0,\ldots,N-1$ such that $t^{(n_0(t))}\le t<t^{(n_0(t)+1)}$ and $t^{(n_1(t))}\le t+\tau<t^{(n_1(t)+1)}$, we may write

$$\begin{aligned}A(t)=\int_\Omega &\Big(\Pi_D\zeta(u^{(n_1(t)+1)})(x)-\Pi_D\zeta(u^{(n_0(t)+1)})(x)\Big)\\ &\times\Big(\sum_{n=n_0(t)+1}^{n_1(t)}\delta t^{(n+\frac12)}\delta_D^{(n+\frac12)}u(x)\Big)\,dx,\end{aligned}$$

which also reads

$$\begin{aligned}A(t)=\int_\Omega &\Big(\Pi_D\zeta(u^{(n_1(t)+1)})(x)-\Pi_D\zeta(u^{(n_0(t)+1)})(x)\Big)\\ &\times\Big(\sum_{n=1}^{N-1}\chi_n(t,t+\tau)\delta t^{(n+\frac12)}\delta_D^{(n+\frac12)}u(x)\Big)\,dx,\end{aligned}\tag{1.27}$$

with $\chi_n(t,t+\tau)=1$ if $t^{(n)}\in(t,t+\tau]$ and $\chi_n(t,t+\tau)=0$ if $t^{(n)}\notin(t,t+\tau]$. Letting $v=\zeta(u^{(n_1(t)+1)})-\zeta(u^{(n_0(t)+1)})$ in Scheme (1.13), we get from (1.27)

$$\begin{aligned}A(t)=&\\ \sum_{n=1}^{N-1}&\chi_n(t,t+\tau)\\ &\times\int_\Omega\int_{t^{(n)}}^{t^{(n+1)}}f(x,t)\,dt\Big(\Pi_D\zeta(u^{(n_1(t)+1)})(x)-\Pi_D\zeta(u^{(n_0(t)+1)})(x)\Big)\,dx\\ -\sum_{n=1}^{N-1}&\chi_n(t,t+\tau)\delta t^{(n+\frac12)}\\ &\times\int_\Omega\nabla_D\zeta(u^{(n+1)})(x)\cdot\Big(\nabla_D\zeta(u^{(n_1(t)+1)})(x)-\nabla_D\zeta(u^{(n_0(t)+1)})(x)\Big)\,dx.\end{aligned}$$

Using the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$, this yields :

$$A(t) \leq \frac{1}{2} A_0(t) + \frac{1}{2} A_1(t) + A_2(t) + A_3(t), \tag{1.28}$$

with

$$A_0(t) = \sum_{n=1}^{N-1} \chi_n(t, t+\tau) \delta t^{(n+\frac{1}{2})} \int_\Omega |\nabla_D \zeta(u^{(n_0(t)+1)})(x)|^2 \, dx,$$

$$A_1(t) = \sum_{n=1}^{N-1} \chi_n(t, t+\tau) \delta t^{(n+\frac{1}{2})} \int_\Omega |\nabla_D \zeta(u^{(n_1(t)+1)})(x)|^2 \, dx,$$

$$A_2(t) = \sum_{n=1}^{N-1} \chi_n(t, t+\tau) \delta t^{(n+\frac{1}{2})} \int_\Omega |\nabla_D \zeta(u^{(n+1)})(x)|^2 \, dx,$$

and

$$A_3(t) = \sum_{n=1}^{N-1} \chi_n(t, t+\tau) \int_\Omega \int_{t^{(n)}}^{t^{(n+1)}} f(x,t) \, dt \Big(\Pi_D \zeta(u^{(n_1(t)+1)})(x) - \Pi_D \zeta(u^{(n_0(t)+1)})(x) \Big) \, dx.$$

Applying [71, Proposition 9.3] yields

$$\begin{aligned} &\int_0^{T-\tau} A_0(t) \, dt \leq (\tau + \delta t) \|\nabla_D \zeta(u)\|^2_{L^2(\Omega \times (0,T))} \\ \text{and } &\int_0^{T-\tau} A_1(t) \, dt \leq (\tau + \delta t) \|\nabla_D \zeta(u)\|^2_{L^2(\Omega \times (0,T))}, \end{aligned} \tag{1.29}$$

as well as

$$\int_0^{T-\tau} A_2(t) \, dt \leq \tau \|\nabla_D \zeta(u)\|^2_{L^2(\Omega \times (0,T))}, \tag{1.30}$$

and, with again the application of [71, Proposition 9.3], and using the Young inequality as well as (1.20), we obtain

$$\int_0^{T-\tau} A_3(t) \, dt \leq (\tau + \delta t) T C_1^2 + \tau \|f\|^2_{L^2(\Omega \times (0,T))}. \tag{1.31}$$

Using inequalities (1.26), (1.28), (1.29), (1.30) and (1.31), we conclude the proof of (1.25).

1.4 Convergence results

Definition 1.22 (Uniform-in-time $L^2(\Omega)$-weak convergence.). *A sequence of function $u_m : [0,T] \mapsto L^2(\Omega)$ converges weakly in $L^2(\Omega)$ uniformly on $[0,T]$ to a function $u : [0,T] \mapsto L^2(\Omega)$ if, for all $\varphi \in L^2(\Omega)$, the sequence of functions $t \in [0,T] \mapsto \langle u_m(t); \varphi \rangle_{L^2(\Omega)}$ converges uniformly on $[0,T]$ to $t \in [0,T] \mapsto \langle u(t); \varphi \rangle_{L^2(\Omega)}$ as $m \to \infty$, with $\langle \cdot ; \cdot \rangle_{L^2(\Omega)}$ is the inner product in $L^2(\Omega)$.*

Theorem 1.23.

Let Hypotheses (1.4) be fulfilled. Let $(D_m)_{m\in\mathbb{N}}$ be a consistent sequence of space-time gradient discretisations in the sense of Definition 1.17, such that the associated sequence of approximate gradient approximations is limit-conforming (Definition 1.7) and compact (Definition 1.8, it is then coercive in the sense of Definition 1.4), and such that, for all $m \in \mathbb{N}$, Π_{D_m} is a piecewise constant function reconstruction in the sense of Definition 1.12. For any $m \in \mathbb{N}$, let u_m be a solution to Scheme (1.13), such that $\|u_{\text{ini}} - \Pi_{D_m} u_m^{(0)}\|_{L^2(\Omega)} \to 0$ as $m \to \infty$.

Then there exists $u \in L^2(\Omega \times (0,T))$ such that

1. *$\Pi_{D_m} u_m$ converges weakly in $L^2(\Omega)$ uniformly on $[0,T]$ (see Definition 1.22) to $\bar{u}$ as $m \to \infty$,*
2. *$\Pi_{D_m}\zeta(u_m)$ converges in $L^\infty(0,T,L^2(\Omega))$ to $\zeta(\bar{u})$ as $m \to \infty$,*
3. *$\zeta(\bar{u}) \in L^2(0,T;H_0^1(\Omega))$ and $\nabla_{D_m}\zeta(u_m)$ converges in $L^2(\Omega\times(0,T))^d$ to $\nabla\zeta(\bar{u})$ as $m \to \infty$,*

and $\bar{u}$ is the unique weak solution of Problem (1.5).

Proof. This proof follow the work made in [39] putting $\beta = Id$, $\nu = \zeta$ and $a(x,\nu(\bar{u}),\nabla\zeta(\bar{u})) = \nabla\zeta(u)$. **step 1 :** A weak limit of the discrete solution.
We consider, for all $m \in \mathbb{N}$, the spaces $B_m = \Pi_{D_m} X_{D_m,0} \subset L^2(\Omega)$, embedded with the norm

$$\|w\|_{B_m} = \inf\{\|u\|_{D_m},\ \Pi_{D_m} u = w\},\ \forall w \in B_m,\ \forall m \in N.$$

The compactness hypothesis of $(D_m)_{m\in\mathbb{N}}$ allows to enter into the framework of discrete Alt-Luckhaus' theorem 1.33.

Thanks to Lemma 1.18, we get that Hypothesis (h1) of Theorem 1.33 is satisfied. We classically identify $L^2(0,T;L^2(\Omega))$ and $L^2(\Omega\times(0,T))$, and we define, for $\tau \in (0,T)$, $g_m(\tau) = \|\Pi_D\zeta(u)(\cdot,\cdot+\tau) - \Pi_D\zeta(u)(\cdot,\cdot)\|_{L^2(\Omega\times(0,T-\tau))}$ and $g(\tau,t) = (C_2(\tau+\delta t))^{1/2}$. Thanks to Lemma 1.21 and to the continuity in means theorem (which implies that g_m is continuous in 0), we may apply Lemma 1.32 and deduce that hypothesis (h2) of Theorem 1.33 also holds. Therefore, there exists $\chi \in L^2(\Omega\times(0,T))$ such that $\Pi_{D_m}\zeta(u_m)$ converges, up to the extraction of a subsequence, to χ in $L^2(\Omega\times(0,T))$. Again applying Lemma 1.18 and Lemma 1.20 allow us to enter into the framework of [39, Therorem 3.1] , up again to the extraction of a subsequence, we get that there exists $u \in L^2(\Omega\times(0,T))$ such that $\Pi_{D_m} u_m$ converges weakly in $L^2(\Omega)$ uniformly on $[0,T]$ to u. Thanks to Lemma 1.31, we conclude that $\chi(x,t) = \zeta(u(x,t))$ for a.e. $(x,t) \in \Omega\times(0,T)$. Moreover, for any $T_0 \in [0,T]$, since $\Pi_{D_m} u_m(\cdot,T_0) \to \bar{u}(\cdot,T_0)$ weakly in $L^2(\Omega)$, we get that :

$$\int_\Omega \bar{u}(x,T_0)\,dx \le \liminf_{m\to\infty}\int_\Omega \Pi_{D_m} u_m(x,T_0)\,dx. \tag{1.32}$$

step 2 :$\bar{u}$ is the solution to Problem (1.5).

Let $m \in \mathbb{N}$, and let us denote $D = D_m$ (belonging to the above subsequence) and drop some indices m for the simplicity of the notation. Let $\varphi \in C_c^\infty([0,T))$ and $w \in C_c^\infty(\Omega)$, and let $v \in X_{D,0}$ be such that

$$v = \underset{z \in X_{D,0}}{\operatorname{argmin}}\, S_D(w).$$

We take as test function v in (1.13) the function $\delta t^{(n+\frac{1}{2})}\varphi(t^{(n)})v$, and we sum the resulting equation on $n = 0, \ldots, N-1$. we get

$$T_1^{(m)} + T_2^{(m)} = T_3^{(m)}, \tag{1.33}$$

with

$$T_1^{(m)} = \sum_{n=0}^{N-1} \delta t^{(n+\frac{1}{2})}\varphi(t^{(n)}) \int_\Omega \delta_D^{(n+\frac{1}{2})} u(x)\Pi_D v(x)\, dx,$$

$$T_2^{(m)} = \sum_{n=0}^{N-1} \delta t^{(n+\frac{1}{2})}\varphi(t^{(n)}) \int_\Omega \nabla_D \zeta(u^{(n+1)})(x) \cdot \nabla_D v(x)\, dx,$$

and

$$T_3^{(m)} = \sum_{n=0}^{N-1} \varphi(t^{(n)}) \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f(x,t)\Pi_D v(x)\, dxdt.$$

Writing

$$T_1^{(m)} = -\int_0^T \varphi'(t) \int_\Omega \Pi_D u(x,t)\Pi_D v(x)\, dxdt - \varphi(0)\int_\Omega \Pi_D u^{(0)}(x)\Pi_D v(x)\, dx,$$

we get that

$$\lim_{m\to\infty} T_1^{(m)} = -\int_0^T \varphi'(t) \int_\Omega \bar{u}(x,t)w(x)\, dxdt - \varphi(0)\int_\Omega \bar{u}_{\rm ini}(x)w(x)\, dx.$$

We also immediately get that

$$\lim_{m\to\infty} T_2^{(m)} = \int_0^T \varphi(t) \int_\Omega \nabla\zeta(\bar{u})(x,t) \cdot \nabla w(x)\, dxdt,$$

and

$$\lim_{m\to\infty} T_3^{(m)} = \int_0^T \varphi(t) \int_\Omega f(x,t)w(x)\, dxdt.$$

Since the set $\mathcal{T} = \{\sum_{i=1}^{q} \varphi_i(t)w_i(x) : q \in \mathbb{N}, \varphi_i \in C_c^\infty[0,T), w_i \in C_c^\infty(\Omega)\}$ is dense in $C_c^\infty(\Omega \times [0,T))$, we conclude that $\bar{u}$ is the solution to Problem (1.5) thanks to the uniqueness of the limit solution proved in Theorem 1.27 below.

step 3 : Strong convergence of $\nabla_{D_m}\zeta(u_m)$

In fact, the result (1.32) can be improved to the following one : let $T_0 \in [0,T]$ and $(T_m)_{m\in\mathbb{N}}$ be a sequence in $[0,T]$ which converges to T_0. By [39, Lemma 5.1], the uniform-in-time weak convergence of $\Pi_{D_m}u_m$ to $\bar{u}$ and the weak continuity of $\bar{u} : [0,T] \mapsto L^2(\Omega)$, we have $\Pi_{D_m}u_m(T_m) \to \bar{u}(T_0)$ weakly in $L^2(\Omega)$ as $m \to \infty$. This implies, using [39, Lemma 3.2] and the fact that Z is convex, that :

$$\int_\Omega Z(\bar{u}(x,T_0))\, dx \le \liminf_{m\to\infty} \int_\Omega Z(\Pi_{D_m}u_m(x,T_m))\, dx. \tag{1.34}$$

We now put $T = T_m$ in (1.19). We use the convergence of $\Pi_{D_m}\zeta(u_m)$ to $\zeta(\bar{u})$ previously proved and quadratic growth of Z to be able to pass to the superior limit in (1.19), it comes :

$$\limsup_{m\to\infty} \Big(\int_\Omega Z(\Pi_{D_m}u_m(x,T_m))\, dx + \int_0^{T_m}\int_\Omega |\nabla_{D_m}\zeta(u_m)(x,t)|^2\, dxdt \Big) \le \int_\Omega Z(\bar{u}_{\text{ini}}(x))\, dx + \int_0^{T_0}\int_\Omega f(x,t)\zeta(u_m)(x,t)\, dxdt.$$

Combining it with (1.37) of Lemma 1.25 below, we finally get that :

$$\limsup_{m\to\infty} \Big(\int_\Omega Z(\Pi_{D_m}u_m(x,T_m))\, dx + \int_0^{T_m}\int_\Omega |\nabla_{D_m}\zeta(u_m)(x,t)|^2\, dxdt \Big) \le \int_\Omega Z(\bar{u}(x,T_0))\, dx + \int_0^{T_0}\int_\Omega |\nabla\zeta(u_m)(x,t)|^2\, dxdt. \tag{1.35}$$

Let us notice that the right-hand side is finite. We then apply this with $T_m = T_0 = T$ (for any m) and use the property $\limsup_{m\to\infty} a_m \le \limsup_{m\to\infty}(a_m+b_m) - \liminf_{m\to\infty} b_m$ valid whenever $\liminf_{m\to\infty} a_m + b_m)$ is finite. Thanks to 1.34, we get :

$$\limsup_{m\to\infty} \int_0^T\int_\Omega |\nabla_{D_m}\zeta(u_m)(x,t)|^2\, dxdt \le \int_0^{T_0}\int_\Omega |\nabla\zeta(u_m)(x,t)|^2\, dxdt.$$

Hence the strong convergence of $\nabla_{D_m}\zeta(u_m)$ to $\nabla\zeta(\bar{u})$ in $L^2(\Omega\times[0,T])$.

step 4 : Uniform-in-time convergence of $\Pi_{D_m}\zeta(u_m)$.

We come back to a general $(T_m)_{m\in\mathbb{N}}$ converges to T and thanks to the strong convergence of $\nabla_{D_m}\zeta(u_m)$ just been proved, we get :

$$\int_0^{T_m}\int_\Omega |\nabla_{D_m}\zeta(u_m)(x,t)|^2\, dxdt \to \int_0^{T_0}\int_\Omega |\nabla\zeta(\bar{u})(x,t)|^2\, dxdt.$$

Invoking (1.35), this implies that :

$$\limsup_{m\to\infty}\int_\Omega Z(\Pi_{D_m}u_m(x,T_m))\,dx \le \int_\Omega Z(\bar u(x,T_0))\,dx.$$

Combined with (1.34), it gives :

$$\lim_{m\to\infty}\int_\Omega Z(\Pi_{D_m}u_m(x,T_m))\,dx = \int_\Omega Z(\bar u(x,T_0))\,dx. \tag{1.36}$$

Remarking that, thanks to the monotony of ζ, there holds :

$$\begin{aligned}(\zeta(b)-\zeta(a))^2 &\le 2L_\zeta\int_a^b(\zeta(s)-\zeta(a))\,ds\\ &= 2L_\zeta(Z(b)-Z(a)-\zeta(a)(b-a)),\ \text{pour tout } a,b\in\mathbb{R}\end{aligned}$$

Applying it with $a=\bar u(x,T_0)$ and $b=\Pi_{D_m}u_m(x,T_m)$, we get :

$$\begin{aligned}\int_\Omega(\zeta(\Pi_{D_m}u_m(x,T_m))-\zeta(\bar u(x,T_0)))^2&\\ \le 2L_\zeta\int_\Omega(Z(\Pi_{D_m}u_m(x,T_m))-Z(\bar u(x,T_0))\,dx&\\ -2L_\zeta\int_\Omega\zeta(\bar u(x,T_0))(\Pi_{D_m}u_m(x,T_m)-\bar u(x,T_0)))\,dx&\end{aligned}$$

Thanks to (1.36) and to the uniform-in-time $L^2(\Omega)$-weak convergence of $\Pi_{D_m}u_m$ to $\bar u$, the right-hand side of the previous expression tends to 0. This shows that $\zeta(\Pi_{D_m}u_m(\cdot,T_m))$ tends to $\zeta(\bar u(\cdot,T_0))$ strongly in $L^2(\Omega)$. The continuity of $\zeta(\bar u):[0,T]\mapsto L^2(\Omega)$ and [39, Lemma 5.1] allow us to conclude the proof of the strong convergence of $\Pi_{D_m}\zeta(u_m)$ in $L^\infty(0,T,L^2(\Omega))$.

Remark 1.24. *In the case where a two-point flux approximation is used instead of a gradient scheme, one can get with the same arguments that the approximation of u is strongly convergent at all times to the weak solution.*

Lemma 1.25.

Under Hypotheses (1.4), let $\bar u$ be a solution of (1.5). Then the following property holds :

$$\begin{aligned}&\int_0^T\int_\Omega|\nabla\zeta(\bar u)(x,t)|^2\,dxdt+\int_\Omega(Z(\bar u(x,T))-Z(\bar u_{\rm ini}(x)))\,dx\\ &=\int_0^T\int_\Omega f(x,t)\zeta(\bar u(x,t))\,dxdt.\end{aligned} \tag{1.37}$$

Proof. We first notice that (1.5) implies that $\partial_t\bar u\in L^2(0,T;H^{-1}(\Omega))$ (and therefore $\bar u\in$

$C([0,T],H^{-1}(\Omega))$ with $\bar u(0)=\bar u_{\rm ini}$) and that we can write

$$\begin{aligned}&\int_0^T \Big(\langle\partial_t\bar u(t),w(t)\rangle+\int_\Omega\nabla\zeta(\bar u)\cdot\nabla w\,dx\Big)\,dt\\&=\int_0^T\int_\Omega fw\,dxdt,\ \forall w\in L^2(0,T;H_0^1(\Omega)),\end{aligned}\tag{1.38}$$

denoting by $\langle\cdot,\cdot\rangle$ the duality product $(H^{-1}(\Omega),H_0^1(\Omega))$. We prolong $\bar u$ by $\bar u(t)=\bar u_{\rm ini}$ for all $t\le 0$, and by $\bar u(t)=\bar u(T)$ for all $t\ge T$.

Let $h\in(0,T)$. We consider $\alpha_h\in L^2(\mathbb{R};H^{-1}(\Omega))$ defined by

$$\begin{aligned}&\langle\alpha_h(t),w\rangle=\frac1h\int_{t-h}^t\langle\partial_t\bar u(s),w\rangle{\rm d}s\\&=\int_\Omega\frac1h(\bar u(x,t)-\bar u(x,t-h))w(x)\,dx,\ \text{for } t\in\mathbb{R},\ \forall w\in H_0^1(\Omega).\end{aligned}$$

Then α_h tends to $\partial_t\bar u$ in $L^2(\mathbb{R};H^{-1}(\Omega))$ as $h\to0$, which implies that

$$\begin{aligned}&\lim_{h\to0}\int_{\mathbb{R}}\int_\Omega\frac1h(\bar u(x,t)-\bar u(x,t-h))w(x,t)\,dxdt\\&+\int_0^T\int_\Omega\nabla\zeta(\bar u)\cdot\nabla w\,dxdt=\int_0^T\int_\Omega fw\,dxdt,\ \forall w\in L^2(\mathbb{R};H_0^1(\Omega)).\end{aligned}$$

Let us take $w=\zeta(\bar u)$ in the above equation. We get

$$\begin{aligned}&\lim_{h\to0}\int_{\mathbb{R}}\int_\Omega\frac1h(\bar u(x,t)-\bar u(x,t-h))\zeta(\bar u(x,t))\,dxdt\\&+\int_0^T\int_\Omega|\nabla\zeta(\bar u)|^2\,dxdt=\int_0^T\int_\Omega f\ \zeta(\bar u)\,dxdt.\end{aligned}$$

Again observing that $\int_a^b\zeta(x)\,dx=Z(b)-Z(a)=\zeta(b)(b-a)-\int_a^b\zeta'(x)(x-a)\,dx$, which implies $Z(b)-Z(a)\le\zeta(b)(b-a)$, we get that

$$\begin{aligned}&\int_{\mathbb{R}}\int_\Omega\frac1h(\bar u(x,t)-\bar u(x,t-h))\zeta(\bar u(x,t))\,dxdt\\&\ge\int_{\mathbb{R}}\int_\Omega\frac1h(Z(\bar u(x,t))-Z(\bar u(x,t-h)))\,dxdt.\end{aligned}$$

Since

$$\begin{aligned}&\int_{\mathbb{R}}\int_\Omega\frac1h(Z(\bar u(x,t))-Z(\bar u(x,t-h)))\,dxdt\\&=\frac1h\int_T^{T+h}\int_\Omega Z(\bar u(x,T))\,dxdt-\frac1h\int_0^h\int_\Omega Z(\bar u_{\rm ini}(x))\,dxdt\\&=\int_\Omega(Z(\bar u(x,T))-Z(\bar u_{\rm ini}(x)))\,dx.\end{aligned}$$

We may then pass to the limit $h \to 0$. We then obtain

$$\int_\Omega (Z(\bar{u}(x,T)) - Z(\bar{u}_{\rm ini}(x)))\ dx + \int_0^T \int_\Omega |\nabla\zeta(\bar{u})|^2\ dxdt \le \int_0^T \int_\Omega f\ \zeta(\bar{u})\ dxdt. \tag{1.39}$$

We then follow the same reasoning, defining $w = \zeta(\bar{u})$ and $\beta_h \in L^2(\mathbb{R}; H^{-1}(\Omega))$ by

$$\langle \beta_h(t), w\rangle = \frac{1}{h}\int_t^{t+h} \langle \partial_t \bar{u}(s), w\rangle {\rm d}s,\ \text{ for } t \in \mathbb{R},\ \forall w \in H_0^1(\Omega).$$

Remarking that $\int_a^b \zeta(x)\ dx = Z(b) - Z(a) = \zeta(a)(b-a) + \int_a^b \zeta'(x)(b-x)\ dx$, which implies $Z(b) - Z(a) \ge \zeta(a)(b-a)$, we get that

$$\begin{aligned}&\int_{\mathbb{R}}\int_\Omega \frac{1}{h}(\bar{u}(x,t+h) - \bar{u}(x,t))\zeta(\bar{u}(x,t))\ dxdt\\ &\le \int_{\mathbb{R}}\int_\Omega \frac{1}{h}(Z(\bar{u}(x,t+h)) - Z(\bar{u}(x,t)))\ dxdt.\end{aligned}$$

Since

$$\begin{aligned}&\int_{\mathbb{R}}\int_\Omega \frac{1}{h}(Z(\bar{u}(x,t+h)) - Z(\bar{u}(x,t)))\ dxdt\\ &= \frac{1}{h}\int_{T-h}^T \int_\Omega Z(\bar{u}(x,T))\ dxdt - \frac{1}{h}\int_{-h}^0\int_\Omega Z(\bar{u}_{\rm ini}(x))\ dxdt\\ &= \int_\Omega (Z(\bar{u}(x,T)) - Z(\bar{u}_{\rm ini}(x)))\ dx,\end{aligned}$$

we may then pass to the limit $h \to 0$. We thus get

$$\int_\Omega (Z(\bar{u}(x,T)) - Z(\bar{u}_{\rm ini}(x)))\ dx + \int_0^T \int_\Omega |\nabla\zeta(\bar{u})|^2\ dxdt \ge \int_0^T \int_\Omega f\ \zeta(\bar{u})\ dxdt,$$

which, in addition to (1.39), concludes the proof of (1.37).

1.5 Numerical results

1.5.1 The Vertex Approximate Gradient scheme

In the numerical tests proposed in this section, we use the Vertex Approximate Gradient scheme [63]. In this scheme, a primary mesh $\mathcal{M}$ in polyhedra is given. We assume that each element $K \in \mathcal{M}$ is strictly star-shaped with respect to some point x_K. We denote by $\mathcal{E}_K$ the set of all interfaces $\overline{K} \cap \overline{L}$, for all neighbours of K denoted by $L \in \mathcal{M}$ and, for a boundary control volume, $\mathcal{E}_K$ also contains the element $\overline{K} \cap \partial\Omega$. Each $\sigma \in \mathcal{E}_K$ is assumed to be the reunion of $d-1$ simplices (segments if $d = 2$, triangles if $d = 3$) denoted $\tau \in \mathcal{S}_\sigma$. We denote by $\mathcal{V}_\sigma$ the set of all the vertices of σ, located at the boundary of σ, and by $\mathcal{V}_\sigma^0$ the set of all the internal vertices of σ. We assume that, for all $v \in \mathcal{V}_\sigma^0$, there exists coefficients $(\alpha_v^x)_{x\in\mathcal{V}_\sigma}$, such that

$$v = \sum_{x\in\mathcal{V}_\sigma} \alpha_v^x x, \text{ with } \sum_{x\in\mathcal{V}_\sigma} \alpha_v^x = 1.$$

Therefore, the d vertices of any $\tau \in \mathcal{S}_\sigma$ are elements of $\mathcal{V}_\sigma^0 \cup \mathcal{V}_\sigma$. We denote by

$$\mathcal{V} = \bigcup_{\sigma\in\mathcal{E}} \mathcal{V}_\sigma,$$

and by $\mathcal{V}_K$ the set of all elements of $\mathcal{V}$ which are vertices of K. For any $K \in \mathcal{M}$, $\sigma \in \mathcal{E}_K$, $\tau \in \mathcal{S}_\sigma$, we denote by $S_{K,\tau}$ the d-simplex (triangle if $d = 2$, tetrahedron if $d = 3$) with vertex x_K and basis τ.

- We then define X_D as the set of all families $u = ((u_K)_{K\in\mathcal{M}}, (u_v)_{v\in\mathcal{V}})$ and $X_{D,0}$ the set of all families $u \in X_D$ such that $u_v = 0$ for all $v \in \mathcal{V} \cap \partial\Omega$.
- Disjoint arbitrary domains $V_{K,v} \subset \bigcup_{v\in\overline{\tau}} S_{K,\tau}$ are defined for all $v \in \mathcal{V}_K$. Then the mapping Π_D is defined, for any $u \in X_D$, by $\Pi_D u(x) = u_K$, for a.e. $x \in K \setminus \bigcup_{v\in\mathcal{V}_K} V_{K,v}$, and $\Pi_D u(x) = u_v$ for a.e. $x \in V_{K,v}$. It is important to notice that it is not in general necessary to provide a more precise geometric description of $V_{K,v}$ than its measure.
- The mapping ∇_D is defined, for any $u \in X_D$, by $\nabla_D u = \nabla\widehat{\Pi}_D u$, where $\widehat{\Pi}_D u$ is the continuous reconstruction which is affine in all $S_{K,\tau}$, for all $K \in \mathcal{M}$, $\sigma \in \mathcal{E}_K$ and $\tau \in \mathcal{S}_\sigma$, with the values u_K at x_K, u_v at any vertex v of τ which belongs to $\mathcal{V}_\sigma$, and $\sum_{x\in\mathcal{V}_\sigma} \alpha_v^x u_x$ at any vertex v of τ which belongs to $\mathcal{V}_\sigma^0$.

The advantage of this scheme is that it allows to eliminate all values $(u_K)_{K\in\mathcal{M}}$ with respect to the values $(u_v)_{v\in\mathcal{V}}$, leading to linear systems which are well suited to domain decomposition and parallel computing.

We then have the following result.

Lemma 1.26 (Gradient scheme properties of the VAG scheme)**.**
We assume that, for all $m \in \mathbb{N}$, a gradient discretisation $D_m = (X_{D_m}, \Pi_{D_m}, \nabla_{D_m})$ is defined as specified in this section, respecting a uniform bound on the maximum value of the ratio between the diameter of all $K \in \mathcal{M}$ and that of the greatest ball with centre x_K inscribed in K, and the ratio between the diameter of all $S_{K,\tau}$ and that of the greatest ball inscribed in $S_{K,\tau}$, for $K \in \mathcal{M}$, $\sigma \in \mathcal{E}_K$ and $\tau \in \mathcal{S}_\sigma$. We also assume that h_{D_m}, the maximum diameter of all $K \in \mathcal{M}$, tends to 0 as $m \to \infty$. Then the sequence $(D_m)_{m\in\mathbb{N}}$ is consistent, limit-conforming and compact (and therefore coercive).

Proof. For all $u \in X_D$, the following property

$$\|\widehat{\Pi}_D u - \Pi_D u\|_{L^2(\Omega)} \le h_D \|\nabla_D u\|_{L^2(\Omega)^d}, \tag{1.40}$$

is resulting from $\widehat{\Pi}_D u(x) - \Pi_D u(x) = (x - y(x)) \cdot \nabla_D u(x)$, for all $x \in S_{K,\tau}$, where $y(x) \in S_{K,\tau}$ is the point of the mesh $\mathcal{M}$ defined by $y(x) = x_K$ if $x \in S_{K,\tau} \setminus \bigcup_{v \in \mathcal{V}_K} V_{K,v}$, and by $y(x) = v$ if $x \in S_{K,\tau} \cap V_{K,v}$.

Let us check that the hypotheses of Lemma 1.10 are satisfied, for some C only depending on regularity factors specified in the statement of the lemma, for $\widehat{D}_m = (X_{D_m}, \widehat{\Pi}_{D_m}, \nabla_{D_m})$. Then (1.9a) results from the interpolation results on the P^1 finite element under the regularity factor of the mesh, (1.9b) results from

$$\int_\Omega \Big(\nabla_D u(x) \cdot \varphi(x) + \widehat{\Pi}_D u(x) \mathrm{div}\varphi(x)\Big) \, dx = 0,$$

and (1.9c) results from

$$\|\widehat{\Pi}_D u(\cdot + \xi) - \widehat{\Pi}_D u\|_{L^2(\mathbb{R}^d)} \le |\xi| \|\nabla_D u\|_{L^2(\Omega)^d}. \tag{1.41}$$

Therefore we obtain that the sequence $(\widehat{D}_m)_{m\in\mathbb{N}}$ is consistent, limit-conforming and compact. From this result and thanks to (1.40), it is immediate to check that the sequence $(D_m)_{m\in\mathbb{N}}$ is consistent and limit-conforming. We then remark that

$$\begin{aligned}\|\Pi_D u(\cdot + \xi) - \Pi_D u\|_{L^2(\mathbb{R}^d)} \le \; & \|\Pi_D u(\cdot + \xi) - \widehat{\Pi}_D u(\cdot + \xi)\|_{L^2(\mathbb{R}^d)} \\ & + \|\widehat{\Pi}_D u(\cdot + \xi) - \widehat{\Pi}_D u\|_{L^2(\mathbb{R}^d)} + \|\widehat{\Pi}_D u - \Pi_D u\|_{L^2(\mathbb{R}^d)},\end{aligned}$$

which leads, using (1.40) and (1.41), to

$$\|\Pi_D u(\cdot + \xi) - \Pi_D u\|_{L^2(\mathbb{R}^d)} \le (2h_D + |\xi|) \|\nabla_D u\|_{L^2(\Omega)^d}.$$

The application of (1.69) proved in Lemma 1.32 leads to the relative compactness in B of any sequence $(\Pi_{D_m} u_m)_{m\in\mathbb{N}}$, if $u_m \in X_{D_m,0}$ is such that $\|u_m\|_{D_m}$ remains bounded. This completes the proof that the sequence $(D_m)_{m\in\mathbb{N}}$ is compact.

1.5.2 A 2D test case on a variety of meshes

In this 2D test case, we approximate Stefan's problem (1.1) by using the VAG scheme previously described in the domain $\Omega = (0,1)^2$ with the following definition of $\zeta(\bar{u})$,

$$\zeta(\bar{u}) = \begin{cases} \bar{u} & \text{if } \bar{u} < 0, \\ \bar{u} - 1 & \text{if } \bar{u} > 1, \\ 0 & \text{otherwise.} \end{cases}$$

The Dirichlet boundary condition is given by $\bar{u} = -1$ on $\partial\Omega$ and the initial condition (1.2) is given by $\bar{u}(x,0) = 2$. Four grids are used for the computations, a Cartesian grid with $32^2 = 1024$ cells, the same grid randomly perturbed, a triangular grids with 896 cells and a "Kershaw mesh" with 1089 cells as illustrated for example on the Figure 1.3 (such meshes are standard in the framework of underground engineering). The time simulation is 0.1 for a constant given time step of 0.001.

Figures 1.3,1.4, 1.5 and 1.6 represent the discrete solution $u(\cdot,t)$ on all grids for $t = .025, 0.05, 0.075$ and 0.1. For a better comparison we have also plotted the interpolation of u along two lines of the mesh, the first line is horizontal and joins the two points $(0,0.5)$ and $(1,0.5)$, the second one is diagonal and joins points $(0,0)$ and $(1,1)$. The results thus obtained are shown in Figures 1.1 and 1.2.

We can see that the obtained results are weakly dependent on the grid, and that the interface between the regions $u < 0$ and $u > 1$ are located at the same place for all grids. It is worth to notice that this remains true for the very irregular Kershaw mesh, although it presents high ratios between the radii of inscribed balls and the diameter of some internal grid blocks.

1.5.3 A particular solution

In this section, we consider the Stefan's Problem 1.1 in 1D with $\Omega =]-2,2[$ and $T = \log(2)$. Let us define $\bar{u}$ as it follows :

$$\bar{u}(x,t) = \begin{cases} \dfrac{e^{2t} - x^2}{2} + 1 \text{ for } |x| < e^t, \\ 0 \text{ for } |x| > e^t \end{cases} \tag{1.42}$$

And we defined ζ and f such that :

$$\zeta(\bar{u}) = \begin{cases} \bar{u} \text{ if } \bar{u} < 0, \\ \bar{u} - 1 \text{ if } \bar{u} > 1, \\ 0 \text{ if } 0 < \bar{u} < 1 \end{cases} \quad \text{and } f(x,t) = \begin{cases} e^{2t} + 1 \text{ for } |x| < e^t, \\ 0 \text{ for } |x| > e^t \end{cases}. \tag{1.43}$$

To get that $\bar{u}$ is a weak solution in the sense of Definition 1.5, we need to explain with more precision in what sense it will be true. Indeed, we get that $\bar{u}$ is a solution to strong Problem 1.1 on domain of Ω where $\bar{u}$ is continue but we need a compensation to deal with the discontinuity. That is why we introduce the following Rankine-Hugoniot condition (see Lemma 1.35) on $\bar{u}$.

$$V_{\text{disc}}(\bar{u})[\bar{u}] = -(\zeta'(\bar{u}))_{\text{right}}.$$

Where $V_{\text{disc}}(u)$ represents the velocity of the discontinuity of $\bar{u}$, $[\bar{u}]$ represents the value of the jump of the discontinuity and $(\zeta'(\bar{u}))_{\text{right}}$ is the value of the right derivative of ζ. Figure 1.7 shows that the Rankine-Hugoniot condition makes move the discontinuity of $\bar{u}$ by following $|x| = e^t$. We remark that $\bar{u}$ ensure the Rankine-Hugoniot condition and so $\bar{u}$ is a solution of Stefan's problem 1.5. Now we are interested in the convergence of the discrete solution $\Pi_D u$ to $\bar{u}$ strongly in $L^2(\Omega \times [0,T])$. To get it we want to apply the same idea as step 3 of the proof of Theorem 1.23. The real issue is to have an energy equality as in Lemma 1.25 but this time by choosing $w = \bar{u}$ in the equation (1.37), this gives :

$$\int_0^T \langle \partial_t \bar{u}(t), \bar{u}(t) \rangle \, dt + \int_0^T \int_\Omega \nabla\zeta(\bar{u}) \cdot \nabla\bar{u} \, dxdt = \int_0^T \int_\Omega f\bar{u} \, dxdt.$$

First, we can remark that $\bar{u}$ is in $L^2(\Omega \times [0,T])$ and not in $L^2(0,T,H_0^1(\Omega))$ and so $\nabla\bar{u}$ does not make sense on Ω but restricting at $]-e^t, e^t[$ because $\bar{u} = 0$ in $[-2, -e^t[\cup]e^t, 2]$. But even if we find an inequation of energy in the discrete sense taking $v = \Pi_D u$, unfortunately, the equality of energy in the continuous sense is in fact a strict inequality and so passing to the supremum limit and to the infimum limit, as in step 3, does not give us a result of strong convergence. The calculation behind shows the strict inequality :

$$\begin{aligned} \frac{1}{2}\int_{-2}^{2} \bar{u}^2(x,T) - \bar{u}^2(x,0) \, dx &= \frac{441}{45}, \\ \int_0^T \int_{-2}^{2} \nabla\zeta(\bar{u})(x,t) \cdot \nabla\bar{u}(x,t) \, dxdt &= \frac{70}{45}, \\ \int_0^T \int_{-2}^{2} f(x,t)\bar{u}(x,t) \, dxdt &= \frac{556}{45}. \end{aligned}$$

But it is still true for ζ (see the followed calculation) and so the strong convergence of $\Pi_D\zeta(u)$ to $\zeta(\bar{u})$ could be proved in the same way than in Theroem 1.23 :

$$\int_0^T \int_{-2}^2 \partial_t \bar{u}(x,T)\zeta(\bar{u})(x,t)\; dxdt = \frac{186}{45},$$
$$\int_0^T \int_{-2}^2 |\nabla\zeta(\bar{u})(x,t)|^2\; dxdt = \frac{70}{45},$$
$$\int_0^T \int_{-2}^2 f(x,t)\zeta(\bar{u})(x,t)\; dx = \frac{256}{45}.$$

But with a particular scheme as finite difference in 1D, we get the strong convergence (see Figure 1.8a). The Figure 1.8b shows that we don't have the convergence to $\bar{u}$ in $L^\infty(0,T,L^2(\Omega))$ (only uniform-in-time weakly in $L^2(\Omega)$ from the theory).

1.6 Auxilliary results

1.6.1 Proof of uniqueness by a regularised adjoint problem

Let us state and prove the uniqueness theorem, admitting some existence theorem proven below. The method is similar to that of [62], where the existence result for the adjoint problem is given under some regularity hypotheses on Ω which are not done here.

Theorem 1.27. *Under Hypotheses (1.4), there exists at most one solution to (1.5).*

Proof. Let $\bar{u}_1$ and $\bar{u}_2$ be two solutions of Problem (1.5). We set $\bar{u}_d = \bar{u}_1 - \bar{u}_2$. Let us also define, for all $(x,t) \in \Omega\times\mathbb{R}_+^\star$, $q(x,t) = \dfrac{\zeta(\bar{u}_1(x,t)) - \zeta(\bar{u}_2(x,t))}{\bar{u}_1(x,t) - \bar{u}_2(x,t)}$ if $\bar{u}_1(x,t) \neq \bar{u}_2(x,t)$, else $q(x,t) = 0$. For all $T \in \mathbb{R}_+^\star$ and for all $\psi \in L^2(0,T;H_0^1(\Omega))$ with $\partial_t\psi \in L^2(\Omega\times(0,T))$ and $\Delta\psi \in L^2(\Omega\times(0,T))$, we deduce from (1.5), approximating ψ by regular functions $\varphi \in C_c^\infty(\Omega\times[0,T))$, that

$$\int_0^T \int_\Omega \bar{u}_d(x,t)\Big(\partial_t\psi(x,t) + q(x,t)\Delta\psi(x,t)\Big)\; dxdt = 0. \tag{1.44}$$

Let $w \in C_c^\infty(\Omega\times(0,T))$. Let us denote, for $\varepsilon > 0$, $q_\varepsilon = q + \varepsilon$. We have

$$\varepsilon \le q_\varepsilon(x,t) \le L_\zeta + \varepsilon, \text{ for all } (x,t) \in \Omega\times(0,T),$$

and

$$\frac{(q_\varepsilon(x,t) - q(x,t))^2}{q_\varepsilon(x,t)} \le \varepsilon. \tag{1.45}$$

Let ψ_ε be given by lemma 1.28 below, with $g = q_\varepsilon$. Substituting ψ by ψ_ε in (1.44) and using (1.49) give

$$|\int_0^T \int_\Omega \bar{u}_d(x,t)w(x,t)\; dxdt| \le |\int_0^T \int_\Omega \bar{u}_d(x,t)(q_\varepsilon(x,t) - q(x,t))\Delta\psi_\varepsilon(x,t)\; dxdt|. \tag{1.46}$$

The Cauchy-Schwarz inequality, (1.50) and (1.45) imply

$$\begin{aligned}&\Big[\int_0^T \int_\Omega \bar{u}_d(x,t)(q_\varepsilon(x,t) - q(x,t))\Delta\psi_\varepsilon(x,t)|\; dxdt\Big]^2\\ &\le \int_0^T \int_\Omega \bar{u}_d(x,t)^2 \frac{(q(x,t)-q_\varepsilon(x,t))^2}{q_\varepsilon(x,t)}\; dxdt \int_0^T \int_\Omega q_\varepsilon(x,t)\Big(\Delta\psi_\varepsilon(x,t)\Big)^2\; dxdt\\ &\le \varepsilon \int_0^T \int_\Omega \bar{u}_d(x,t)^2\; dxdt \;\; 4T \int_0^T \int_\Omega |\nabla w(x,t)|^2\; dxdt.\end{aligned} \tag{1.47}$$

We deduce that the right hand side of (1.47) tends to zero as $\varepsilon \to 0$. Hence the left hand side of (1.46) also tends to zero as $\varepsilon \to 0$, which gives

$$|\int_0^T \int_\Omega \bar{u}_d(x,t)w(x,t)\; dxdt| = 0. \tag{1.48}$$

Since (1.48) holds for any function $w \in C_c^\infty(\Omega \times (0,T))$, we get that $\bar{u}_d(x,t) = 0$ for a.e. $(x,t) \in \Omega \times (0,T)$, which concludes the proof of Theorem 1.27.

Let us now prove the properties of the function ψ, used in the course of the proof of Theorem 1.27.

Lemma 1.28.

Under Hypothesis (1.4a), let $w \in L^2(0,T;H_0^1(\Omega))$ and $g \in L^\infty(\Omega \times (0,T))$ with $g(x,t) \in [g_{\min}, g_{\max}]$ with given $g_{\max} \ge g_{\min} > 0$ for a.e. $(x,t) \in \Omega \times (0,T)$. Then there exists at least one function ψ such that,

1. *$\psi \in L^\infty(0,T;H_0^1(\Omega))$, $\partial_t\psi \in L^2(\Omega\times(0,T))$, $\Delta\psi \in L^2(\Omega\times(0,T))$ (hence $\psi \in C^0(0,T;L^2(\Omega))$),*
2. *$\psi(\cdot,T) = 0$,*
3. *the following holds*

$$\partial_t\psi(x,t) + g(x,t)\Delta\psi(x,t) = w(x,t), \qquad \textit{for a.e. } (x,t) \in \Omega \times (0,T), \tag{1.49}$$

4. *and*

$$\int_0^T \int_\Omega g(x,t)\Big(\Delta\psi(x,t)\Big)^2\; dxdt \le 4T \int_0^T \int_\Omega |\nabla w(x,t)|^2\; dxdt. \tag{1.50}$$

Proof. We first apply Lemma 1.30, which states the convergence of a gradient scheme to $\psi \in L^\infty(0,T;H_0^1(\Omega))$ with $\partial_t\psi \in L^2(\Omega\times(0,T))$ and $\Delta\psi \in L^2(\Omega\times(0,T))$ such that (1.49) holds, setting $\nu = 1/g$, $f = w/g$, $\mu(s) = s$, $\psi_{\text{ini}} = 0$ and changing t in $-t$ (this ensures that Hypotheses (1.58) are fulfilled). Therefore the existence of ψ satisfying (1.49) follows. Let us prove that it satisfies (1.50). Approximating ψ by a sequence of regular functions and passing to the limit, we get that $\|\nabla\psi(\cdot)\|_{L^2(\Omega)^d} \in C^0([0,T])$ and that

$$\int_s^\tau \int_\Omega \partial_t\psi(x,t)\Delta\psi(x,t)\ dxdt = -\frac{1}{2}\int_\Omega |\nabla\psi(x,\tau)|^2\ dx + \frac{1}{2}\int_\Omega |\nabla\psi(x,s)|^2\ dx,$$

for all $s < \tau \in [0,T]$ and

$$\int_s^\tau \int_\Omega w(x,t)\Delta\psi(x,t)\ dxdt = -\int_s^\tau \int_\Omega \nabla w(x,t)\cdot\nabla\psi(x,t)\ dxdt.$$

We thus obtain, multiplying (1.49) by $\Delta\psi(x,t)$ and integrating on $\Omega\times(0,\tau)$ for any $\tau \in [0,T]$,

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\nabla\psi(x,0)|^2\ dx \ - \ \frac{1}{2}\int_\Omega |\nabla\psi(x,\tau)|^2\ dx \ + \ \int_0^\tau \int_\Omega g(x,t)\Big(\Delta\psi(x,t)\Big)^2\ dxdt = \\ &-\int_0^\tau \int_\Omega \nabla w(x,t)\cdot\nabla\psi(x,t)\ dxdt.\end{aligned} \tag{1.51}$$

Since $\nabla\psi(\cdot,T) = 0$, letting $\tau = T$ in (1.51) leads to

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\nabla\psi(x,0)|^2\ dx + \int_0^T \int_\Omega g(x,t)\Big(\Delta\psi(x,t)\Big)^2\ dxdt = \\ &-\int_0^T \int_\Omega \nabla w(x,t)\cdot\nabla\psi(x,t)\ dxdt.\end{aligned} \tag{1.52}$$

Integrating (1.51) with respect to $\tau \in (0,T)$ leads to

$$\begin{aligned}\frac{1}{2}\int_0^T \int_\Omega |\nabla\psi(x,\tau)|^2\ dxd\tau \leq\ & \frac{T}{2}\int_\Omega |\nabla\psi(x,0)|^2\ dx && + \\ & T\int_0^T \int_\Omega g(x,t)\Big(\Delta\psi(x,t)\Big)^2\ dxdt && + \\ & T\int_0^T \int_\Omega |\nabla w(x,t)\cdot\nabla\psi(x,t)|\ dxdt.\end{aligned} \tag{1.53}$$

Using (1.52) and (1.53), we get

$$\frac{1}{2}\int_0^T \int_\Omega |\nabla\psi(x,\tau)|^2\ dxd\tau \leq 2T\int_0^T \int_\Omega |\nabla w(x,t)\cdot\nabla\psi(x,t)|\ dxdt. \tag{1.54}$$

Thanks to the Cauchy-Schwarz inequality, the right hand side of (1.54) may be estimated as follows :

$$\Big[\int_0^T\int_\Omega |\nabla w(x,t)\cdot\nabla\psi(x,t)|\ dxdt\Big]^2$$
$$\leq \int_0^T\int_\Omega |\nabla\psi(x,t)|^2\ dxdt\int_0^T\int_\Omega |\nabla w(x,t)|^2\ dxdt.$$

With (1.54), this implies

$$\Big[\int_0^T\int_\Omega |\nabla w(x,t)\cdot\nabla\psi(x,t)|\ dxdt\Big]^2$$
$$\leq 4T\int_0^T\int_\Omega |\nabla w(x,t)\cdot\nabla\psi(x,t)|\ dxdt\int_0^T\int_\Omega |\nabla w(x,t)|^2\ dxdt.$$

Therefore,

$$\int_0^T\int_\Omega |\nabla w(x,t)\cdot\nabla\psi(x,t)|\ dxdt \leq 4T\int_0^T\int_\Omega |\nabla w(x,t)|^2\ dxdt,$$

which, together with (1.52), yields (1.50).

In Lemma 1.28, we have used a result of existence of $\bar u\in L^2(0,T;H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega))$, such that $\Delta\bar u\in L^2(\Omega\times(0,T))$, solution to the following problem :

$$\nu(x,t)\partial_t\bar u(x,t)-\Delta\bar u(x,t)=f(x,t),\ \text{ for a.e. } (x,t)\in\Omega\times(0,T) \tag{1.55}$$

with the following initial condition :

$$\bar u(x,0)=\bar u_{\text{ini}}(x),\ \text{ for a.e. } x\in\Omega, \tag{1.56}$$

together with the homogeneous Dirichlet boundary condition :

$$\bar u(x,t)=0 \text{ for a.e. } (x,t)\in\partial\Omega\times(0,T), \tag{1.57}$$

under the following assumptions (which are not exactly the standard ones done in the literature) :

$$\Omega \text{ is an open bounded connected polyhedral subset of } \mathbb{R}^d,\ d\in\mathbb{N}^\star \text{ and } T>0, \tag{1.58a}$$
$$\bar u_{\text{ini}}\in H_0^1(\Omega) \tag{1.58b}$$
$$f\in L^2(\Omega\times(0,T)), \tag{1.58c}$$

and

$$\nu \in L^\infty(\Omega \times (0,T)) \text{ and } \nu(x,t) \in [\nu_{\min}, \nu_{\max}] \text{ with given } \nu_{\max} \geq \nu_{\min} > 0$$
$$\text{for a.e. } (x,t) \in \Omega \times (0,T). \tag{1.58d}$$

This problem, issued from (1.44), is called the regularised adjoint problem to Problem (1.1). In order to prove the existence of a solution to Problem (1.55)-(1.56)-(1.57) under hypotheses (1.58), we consider an approximation of this solution, using a gradient scheme. Let $D = (X_D, \Pi_D, \nabla_D, (t^{(n)})_{n=0,\ldots,N})$ be a space-time discretisation in the sense of Definition 1.16. We define the fully implicit scheme for the discretisation of Problem (1.64) by the sequence $(u^{(n)})_{n=0,\ldots,N} \subset X_{D,0}$ such that :

$$\begin{cases} u^{(0)} \in X_{D,0}, \\ u^{(n+1)} \in X_{D,0},\ \delta_D^{(n+\frac{1}{2})} u = \Pi_D \dfrac{u^{(n+1)} - u^{(n)}}{\delta t^{(n+\frac{1}{2})}}, \\ \displaystyle\int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega \nu(x,t) \delta_D^{(n+\frac{1}{2})} u(x) \Pi_D v(x)\, dxdt \\ \displaystyle +\delta t^{(n+\frac{1}{2})} \int_\Omega \nabla_D u^{(n+1)}(x) \cdot \nabla_D v(x)\, dx = \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f(x,t) \Pi_D v(x)\, dxdt, \\ \forall v \in X_{D,0},\ \forall n = 0, \ldots, N-1. \end{cases} \tag{1.59}$$

We then use the notations Π_D and ∇_D for the definition of space-time dependent functions, defining

$$\begin{aligned} &\Pi_D u(x,0) = \Pi_D u^{(0)}(x) \text{ and } \nabla_D u(x,0) = \nabla_D u^{(0)}(x), \\ &\Pi_D u(x,t) = \Pi_D u^{(n+1)}(x) \text{ and } \nabla_D u(x,t) = \nabla_D u^{(n+1)}(x), \\ &\text{for a.e. } (x,t) \in \Omega \times (t^{(n)}, t^{(n+1)}],\ \forall n = 0, \ldots, N-1. \end{aligned} \tag{1.60}$$

and

$$\delta_D u(x,t) = \delta_D^{(n+\frac{1}{2})} u(x),\ \text{for a.e. } (x,t) \in \Omega \times (t^{(n)}, t^{(n+1)}),\ \forall n = 0, \ldots, N-1. \tag{1.61}$$

Let us state some estimates and the existence and uniqueness of the solution to the scheme.

Lemma 1.29 (Space-time estimates on $\delta_D u$ and u.)**.** *Under Hypotheses* (1.58)*, let D be a space-time gradient discretisation in the sense of Definition* 1.16*. Then, for any solution u to*

Scheme (1.59)*, we have :*

$$\nu_{\min}\int_0^{t^{(m)}}\int_\Omega(\delta_D u(x,t))^2\,dxdt+\|\nabla_D u^m\|^2_{L^2(\Omega)^d}\\ \le\|\nabla_D u^{(0)}\|^2_{L^2(\Omega)^d}+\frac{1}{\nu_{\min}}\|f\|^2_{L^2(\Omega\times(0,T))},\ \forall m=1,\dots,N. \quad (1.62)$$

As a result, there exists one and only one solution u to Scheme (1.59)*.*

Proof. We set $v=u^{(n+1)}-u^{(n)}$ in (1.59) and we sum on $n=0,\dots,N-1$. We can then write

$$\frac12|\nabla_D u^{(n+1)}(x)|^2-\frac12|\nabla_D u^{(n)}(x)|^2\le\nabla_D u^{(n+1)}(x)\cdot(\nabla_D u^{(n+1)}(x)-\nabla_D u^{(n)}(x)).$$

Thanks to the Young inequality applied to the right hand side, we conclude (1.62), which ensures the existence and uniqueness of the solution to the linear Scheme (1.59), which leads to square linear systems.

We then have the following convergence lemma.

Lemma 1.30 (Convergence of the fully implicit scheme)**.**
Let Hypotheses (1.58) *be fulfilled. Let $(D_m)_{m\in\mathbb{N}}$ be a consistent sequence of space-time gradient discretisations in the sense of Definition 1.17, such that the associated sequence of approximate gradient approximations is consistent (Definition 1.6), limit–conforming (Definition 1.7) and compact (Definition 1.8, it is then coercive in the sense of Definition 1.4). For any $m\in\mathbb{N}$, let u_m be the solution to Scheme* (1.59) *for a given $u_m^{(0)}\in X_{D_m,0}$, such that $\|\nabla\bar u_{\rm ini}-\nabla_{D_m}u_m^{(0)})\|_{L^2(\Omega)^d}\to 0$ as $m\to\infty$.*

Then there exist a sub-sequence of $(D_m)_{m\in\mathbb{N}}$, again denoted $(D_m)_{m\in\mathbb{N}}$, and a function $\bar u\in L^2(0,T;H^1_0(\Omega))\cap H^1(0,T;L^2(\Omega))$ such that

1. *for all $t\in[0,T]$, $\Pi_{D_m}u_m(t)$ converges in $L^2(\Omega)$ to $\bar u(t)$ with $\bar u\in L^\infty(0,T;L^2(\Omega))$ as $m\to\infty$,*
2. *$\delta_{D_m}u_m$ weakly converges in $L^2(\Omega\times(0,T))$ to $\partial_t\bar u$ as $m\to\infty$,*
3. *$\nabla_{D_m}u_m$ weakly converges in $L^\infty(0,T;L^2(\Omega)^d)$ to $\nabla\bar u$ as $m\to\infty$.*
4. *$\Delta\bar u\in L^2(\Omega\times(0,T))$,*
5. *(1.55)-(1.56)-(1.57) hold.*

Proof. This proof has a few common points with that of [67, Lemma 4.4]. Thanks to (1.62), $\nabla_{D_m}u_m$ remains bounded in $L^\infty(0,T;L^2(\Omega)^d)$ and $\Pi_{D_m}u_m$ remains bounded in $L^2(\Omega\times(0,T))$.

Since (1.62) also provides an $L^2(\Omega \times (0,T))$ estimate on $\delta_{D_m} u_m$, this immediately provides an $L^\infty(0,T;L^2(\Omega))$ estimate on $(\Pi_{D_m} u_m)_{m\in\mathbb{N}}$, thanks to

$$\begin{aligned}\|\Pi_D u^{(n)} - \Pi_D u^{(p)}\|^2_{L^2(\Omega)} &= \|\sum_{k=p}^{n-1} \delta t^{(k+\frac{1}{2})} \delta_D^{(k+\frac{1}{2})} u\|^2_{L^2(\Omega)} \\ &\le (t^{(n)} - t^{(p-1)}) \|\delta_{D_m} u_m\|^2_{L^2(\Omega\times(0,T))}.\end{aligned}$$

Moreover, the above inequality, and the compactness hypothesis, allow to apply a variant of Ascoli's theorem similar to [67, Theorem 6.1], and whose proof is close to that of Theorem 1.34. We deduce that there exists a function $\bar{u} \in C^0(0,T;L^2(\Omega))$ such that, up to the extraction of a subsequence, $(\Pi_{D_m} u_m(t))_{m\in\mathbb{N}}$ converges to $\bar{u}(t)$ in $L^2(\Omega)$ for all $t \in [0,T]$. Using the limit-conformity of the discretisation, we then get that $\bar{u}$ is such that

$$\begin{aligned}&\bar{u} \in L^\infty(0,T;H^1_0(\Omega)) \cap H^1(0,T;L^2(\Omega)),\\ &\bar{u}(x,0) = \bar{u}_{\text{ini}}(x) \text{ for a.e. } x \in \Omega,\end{aligned} \tag{1.63}$$

and using the consistency of the discretisation in a similar way to the proof of Theorem 1.23, we get that

$$\int_0^T \int_\Omega (\nu\ \partial_t \bar{u}\ v + \nabla \bar{u} \cdot \nabla v)\ dxdt = \int_0^T \int_\Omega f\ v\ dxdt, \forall v \in L^2(0,T;H^1_0(\Omega)). \tag{1.64}$$

Then (1.64) shows that $\Delta \bar{u} \in L^2(\Omega \times (0,T))$ and that (1.55)-(1.56)-(1.57) hold.

1.6.2 Technical results

The next result, which is known in the literature as the Minty trick, is used in the proof of the convergence theorem.

Lemma 1.31 (Minty trick)**.**

Let $\zeta \in C^0(\mathbb{R})$ be a nondecreasing function. Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \ge 1$. Let $(u_n)_{n\in\mathbb{N}} \subset L^2(\Omega)$ such that

(i) there exists $u \in L^2(\Omega)$ such that $(u_n)_{n\in\mathbb{N}}$ weakly converges to u in $L^2(\Omega)$;

(ii) $(\zeta(u_n))_{n\in\mathbb{N}} \subset L^2(\Omega)$ and there exists $w \in L^2(\Omega)$ such that $(\zeta(u_n))_{n\in\mathbb{N}}$ weakly converges to w in $L^2(\Omega)$;

(iii) there holds :

$$\liminf_{n\to\infty} \int_\Omega u_n(x)\zeta(u_n(x)); dx \le \int_\Omega u(x)w(x)\ dx. \tag{1.65}$$

Then $w(x) = \zeta(u(x))$, *for a.e.* $x \in \Omega$.

Proof. We first consider, for any $v \in L^2(\Omega)$ such that $\zeta(v) \in L^2(\Omega)$,

$$A_n = \int_\Omega (\zeta(u_n(x)) - \zeta(v(x)))(u_n(x) - v(x))\, dx.$$

Since ζ is a nondecreasing, we have $A_n \geq 0$. By weak/strong convergence and using (1.65), we get that

$$0 \leq \liminf_{n\to\infty} A_n \leq \int_\Omega (uw - u\zeta(v) - vw + v\zeta(v))\, dx = \int_\Omega (w - \zeta(v))(u - v)\, dx.$$

Hence we get that, for all $v \in L^2(\Omega)$ such that $\zeta(v) \in L^2(\Omega)$,

$$0 \leq \int_\Omega (w - \zeta(v))(u - v)\, dx \leq \int_\Omega (w - \zeta(0) + u - (\zeta(v) - \zeta(0) + v))(u - v)\, dx. \tag{1.66}$$

Since the mapping $\psi : s \to \zeta(s) - \zeta(0) + s$ is continuous from $\mathbb{R}$ to $\mathbb{R}$, strictly increasing and tends to infinity at infinity ($|\psi(s)| \geq |s|$ holds for all $s \in \mathbb{R}$), it is invertible. Let us denote by χ the reciprocal function to ψ, which therefore satisfies

$$|\chi(s)| \leq |s| \text{ and } \zeta(\chi(s)) = s + \zeta(0) - \chi(s),\ \forall s \in \mathbb{R}. \tag{1.67}$$

We then get that, for all $z \in L^2(\Omega)$, the function $v = \chi(z)$ is such that $v \in L^2(\Omega)$ and $\zeta(v) \in L^2(\Omega)$. We then obtain from (1.66) that, denoting by $z_0 = w - \zeta(0) + u \in L^2(\Omega)$,

$$0 \leq \int_\Omega (z_0 - z)(u - \chi(z))\, dx,\ \forall z \in L^2(\Omega). \tag{1.68}$$

We then may take $z = z_0 - t\varphi$, with $t > 0$ and $\varphi \in C_c^\infty(\Omega)$ in (1.68). Dividing by $t > 0$, we obtain

$$\int_\Omega (u(x) - \chi(z_0(x) - t\varphi(x)))\varphi(x)\, dx \geq 0.$$

Letting $t \to 0$ in the above equation, we get, by dominated convergence thanks to (1.67), that

$$\int_\Omega (u(x) - \chi(z_0(x)))\varphi(x)\, dx \geq 0.$$

Since the same inequality holds for $-\varphi$ instead of φ, we get

$$\int_\Omega (u(x) - \chi(z_0(x)))\varphi(x)\, dx = 0.$$

Since the above inequality holds for all $\varphi \in C_c^\infty(\Omega)$, we conclude that $u(x) = \chi(z_0(x))$ for a.e. $x \in \Omega$. This means that $\psi(u(x)) = z_0(x)$ for a.e. $x \in \Omega$, which gives

$$\zeta(u(x)) - \zeta(0) + u(x) = w(x) - \zeta(0) + u(x), \text{ for a.e. } x \in \Omega,$$

and the conclusion of the lemma follows.

The following result is used in the convergence proof, for proving the compactness of a particular scheme.

Lemma 1.32 (Uniform limit.)**.**

Let $N \in \mathbb{N}^\star$ and $(g_m)_{m\in\mathbb{N}}$ be a sequence of functions from $\mathbb{R}^N$ to $\mathbb{R}^+$ such that $g_m(0) = 0$ and g_m is continuous in 0. We assume that there exists a function $g : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}^+$, with $g(0,0) = 0$, continuous in $(0,0)$, and for all $m \in \mathbb{N}$, there exists $\mu_m \in \mathbb{R}^+$ verifying $\lim_{m\to\infty} \mu_m = 0$, such that

$$g_m(\xi) \leq g(\xi, \mu_m), \ \forall m \in \mathbb{N}, \ \forall \xi \in \mathbb{R}^N.$$

Then

$$\lim_{|\xi|\to 0} \sup_{m\in\mathbb{N}} g_m(\xi) = 0. \tag{1.69}$$

Proof. Let $\varepsilon > 0$. Let $\eta > 0$ be such that, for all $(\xi, t) \in B(0,\eta) \times [0,\eta]$, $g(\xi,t) \leq \varepsilon$. Let $m_0 \in \mathbb{N}$ such that, for all $m > m_0$, $\mu_m \leq \eta$. For all $m = 0, \ldots, m_0$, thanks to the continuity of g_m, there exists $\eta_m > 0$ such that, for all ξ verifying $|\xi| \leq \eta_m$, we have $g_m(\xi) \leq \varepsilon$.

We now take $\xi \in \mathbb{R}^N$ such that $|\xi| \leq \min(\eta, (\eta_m)_{m=0,\ldots,m_0})$. We then get that, for all $m = 0, \ldots, m_0$, the inequality $g_m(\xi) \leq \varepsilon$ holds, and for all $m \in \mathbb{N}$ such that $m > m_0$, then $g(\xi, \mu_m) \leq \varepsilon$. Gathering the previous results gives (1.69).

We finally state a discrete version of Alt–Luckhaus theorem [3], whose proof is immediate following [75].

Theorem 1.33 (Discrete Alt–Luckhaus theorem)**.** *Let $T > 0$, let B be a Banach space, and let $p \in [1, +\infty)$. Let $(B_m)_{m\in\mathbb{N}}$ be a sequence of normed subspaces of B such that, for any sequence $(w_m)_{m\in\mathbb{N}}$ such that $w_m \in B_m$ and $(\|w_m\|_{B_m})_{m_in\mathbb{N}}$ is bounded, then the set $\{w_m, m \in \mathbb{N}\}$ is relatively compact in B. Let $(v_m)_{m\in_N}$ such that $v_m \in L^p(0,T;B_m)$ for all $m \in \mathbb{N}$. We assume that*

(h1) the sequence $(\|v_m\|_{L^p(0,T;B_m)})_{m\in\mathbb{N}}$ is bounded,

(h2) $\|v_m(\cdot + h) - v_m\|_{L^p(0,T-h;B)}$ tends to 0 as $h \in (0,T)$ tends to 0, uniformly with respect to $m \in \mathbb{N}$.

Then the set $\{v_m, m \in \mathbb{N}\}$ *is relatively compact in* $L^p(0,T;B)$.

Proof. Our aim is to apply Theorem 2.1 of [75]. We then prolong v_m by 0 on $(-\infty,0)\cup(T,+\infty)$, for all $m_i n\mathbb{N}$. Let us prove that $\|v_m(\cdot+h)-v_m\|_{L^p(\mathbb{R};B)}$ tends to 0 as $h\in(0,T)$ tends to 0, uniformly with respect to $m_i n\mathbb{N}$. Let us first remark that there exists $C_N>0$ such that,

$$\forall m_i n\mathbb{N},\ \forall v\in B_m,\ \|v\|_B\le C_N\|v\|_{B_m}.$$

Indeed, otherwise one could, up to a subsequence of $(B_m)_{m_i n\mathbb{N}}$, construct a sequence such that $\|v_m\|_{B_m}=1$ and $\|v_m\|_B$ tends to infinity, which is in contradiction with the relative compactness in B of $\{v_m, m_i n\mathbb{N}\}$. Hence we can define

$$C_B=\sup_{m_i n\mathbb{N}}\|v_m\|^p_{L^p(0,T;B)}.$$

We have, for all $h\in(0,T)$,

$$\|v_m(\cdot+h)-v_m\|^p_{L^p(\mathbb{R};B)}=\|v_m(\cdot+h)-v_m\|^p_{L^p(0,T-h;B)}+\|v_m\|^p_{L^p(0,h;B)}+\|v_m\|^p_{L^p(T-h,T;B)}.$$

Let us prove that

$$\lim_{h\to0}\sup_{m_i n\mathbb{N}}\|v_m\|^p_{L^p(0,h;B)}=0. \tag{1.70}$$

Let $\varepsilon>0$. We first choose $h_0\in(0,T)$ such that, for all $h\in(0,h_0)$,

$$\|v_m(\cdot+h)-v_m\|^p_{L^p(0,T-h;B)}\le\varepsilon,\ \forall m_i n\mathbb{N}. \tag{1.71}$$

Let $\tau\in(0,T-h_0)$, $h\in(0,h_0)$ and $m_i n\mathbb{N}$ be given. We have

$$\int_0^\tau\|v_m(t)\|^p_B\,dt\le2^{p-1}\left(\int_0^\tau\|v_m(t+h)-v_m(t)\|^p_B\,dt+\int_0^\tau\|v_m(t+h)\|^p_B\,dt\right).$$

Thanks to (1.71), the above inequality gives

$$\int_0^\tau\|v_m(t)\|^p_B\,dt\le2^{p-1}\left(\varepsilon+\int_0^\tau\|v_m(t+h)\|^p_B\,dt\right). \tag{1.72}$$

We then remark that

$$\int_0^{h_0}\int_0^\tau\|v_m(t+h)\|^p_B\,dtdh=\int_0^\tau\int_0^{h_0}\|v_m(t+h)\|^p_B\,dhdt$$
$$\le\int_0^\tau\int_0^T\|v_m(h)\|^p_B\,dhdt\le C_B\tau.$$

This proves that

$$h_0 \inf_{h\in(0,h_0)} \int_0^\tau \|v_m(t+h)\|_B^p \, dt \le C_B\tau.$$

Taking the infimum on h in (1.72), we get, for all $\tau \in (0, T-h_0)$ and $m_i n\mathbb{N}$,

$$\int_0^\tau \|v_m(t)\|_B^p \, dt \le 2^{p-1}\left(\varepsilon + \frac{C_B\tau}{h_0}\right).$$

It now suffices to take $\tau \in (0, \min(T-h_0, \frac{h_0\varepsilon}{C_B}))$ for getting

$$\int_0^\tau \|v_m(t)\|_B^p \, dt \le 2^p\varepsilon, \; \forall m_i n\mathbb{N}.$$

This concludes the proof of (1.70). A similar proof can be done for proving that

$$\lim_{h\to 0} \sup_{m_i n\mathbb{N}} \|v_m\|_{L^p(T-h,T;B)}^p = 0.$$

We thus conclude that

$$\lim_{h\to 0} \sup_{m_i n\mathbb{N}} \|v_m(\cdot + h) - v_m\|_{L^p(\mathbb{R};B)}^p = 0,$$

which enables to apply Theorem 2.1 of [75], hence providing the conclusion of the proof.

Theorem 1.34 (Weak Ascoli theorem). *Let Ω be an open bounded subset of $\mathbb{R}^2$, $a < b \in \mathbb{R}$ and $(u_m)_{m\in\mathbb{N}}$ be a sequence of functions from $[a,b]$ to $L^2(\Omega)$, such that there exists $C_1 > 0$ with*

$$\|u_m(t)\|_{L^2(\Omega)} \le C_1, \; \forall m \in \mathbb{N}, \; \forall t \in [a,b]. \tag{1.73}$$

We also assume that there exists a dense subset R of $L^2(\Omega)$ such that, for all $\varphi \in R$, there exists a function $g_\varphi : \mathbb{R}^+ \times \mathbb{R}^+$ with $g(0,0) = 0$, continuous in $(0,0)$ and a sequence $(h_m^\varphi)_{m\in\mathbb{N}}$ with $h_m^\varphi \ge 0$ and $\lim_{m\to\infty} h_m^\varphi = 0$ and such that

$$\left|\langle u_m(t_2) - u_m(t_1), \varphi\rangle_{L^2(\Omega),L^2(\Omega)}\right| \le g_\varphi(t_2 - t_1, h_m^\varphi), \; \forall m \in \mathbb{N}, \; \forall a \le t_1 \le t_2 \le b. \tag{1.74}$$

Then there exists $u \in L^\infty(a,b;L^2(\Omega))$ with $u \in C_w([a,b], L^2(\Omega))$ (where we denote by $C_w([a,b], L^2(\Omega))$ the set of functions from $[a,b]$ to $L^2(\Omega)$, continuous for the weak topology of $L^2(\Omega)$) and a subsequence of $(u_m)_{m\in\mathbb{N}}$, again denoted $(u_m)_{m\in\mathbb{N}}$, such that, for all $t \in [a,b]$, $u_m(t)$ converges to $u(t)$ for the weak topology of $L^2(\Omega)$.

Proof. The proof follows that of Ascoli's theorem. Let $(t_p)_{p\in\mathbb{N}}$ be a sequence of real numbers, dense in $[a,b]$. Due to (1.73), for each $p \in \mathbb{N}$, we may extract from $(u_m(t_p))_{m\in\mathbb{N}}$ a subsequence which is convergent to some element of $L^2(\Omega)$ for the weak topology of $L^2(\Omega)$. Using a diagonal method, we can choose a sub-sequence, again denoted $(u_m)_{m\in\mathbb{N}}$, such that $(u_m(t_p))_{m\in\mathbb{N}}$ is weakly convergent for all $p \in \mathbb{N}$. For any $t \in [a,b]$ and $v \in L^2(\Omega)$, we then prove that the sequence $(\langle u_m(t), v\rangle_{L^2(\Omega),L^2(\Omega)})_{m\in\mathbb{N}}$ is a Cauchy sequence. Indeed, let $\varepsilon > 0$ be given. We first choose $\varphi \in R$ such that $\|\varphi - v\|_{L^2(\Omega)} \leq \varepsilon$. Let $\eta > 0$ such that, for all $(s,t) \in [0,\eta]^2$, we have $g_\varphi(s,t) \leq \varepsilon$. Then, we choose $p \in \mathbb{N}$ such that $|t - t_p| \leq \eta$. Since $(\langle u_m(t_p), \varphi\rangle_{L^2(\Omega),L^2(\Omega)})_{m\in\mathbb{N}}$ is a Cauchy sequence, we choose $n_0 \in \mathbb{N}$ such that, for $k, l \geq n_0$,

$$\left|\langle u_k(t_p) - u_l(t_p), \varphi\rangle_{L^2(\Omega),L^2(\Omega)}\right| \leq \varepsilon,$$

and such that $h_k^\varphi, h_l^\varphi \leq \eta$. We then get, using (1.74),

$$\left|\langle u_k(t) - u_l(t), \varphi\rangle_{L^2(\Omega),L^2(\Omega)}\right| \leq g_\varphi(|t - t_p|, h_k^\varphi) + g_\varphi(|t - t_p|, h_l^\varphi) + \varepsilon,$$

which gives

$$\left|\langle u_k(t) - u_l(t), \varphi\rangle_{L^2(\Omega),L^2(\Omega)}\right| \leq 3\varepsilon.$$

This proves that the sequence $(\langle u_m(t), v\rangle_{L^2(\Omega),L^2(\Omega)})_{m\in\mathbb{N}}$ converges. Since

$$|\langle u_m(t), v\rangle_{L^2(\Omega),L^2(\Omega)}| \leq C_1\|v\|_{L^2(\Omega)},$$

we get the existence of $u(t) \in L^2(\Omega)$ such that $(u_m(t))_{m\in\mathbb{N}}$ converges to $u(t)$ for the weak topology of $L^2(\Omega)$. Then $u \in C_w([a,b], L^2(\Omega))$ is obtained by passing to the limit $m \to \infty$ in (1.74), and by using the density of R in $L^2(\Omega)$.

Lemma 1.35 (Rankine-Hugoniot condition)**.** *Let Ω be an open bounded subset of $\mathbb{R}^d$, let D be an open domain of $\Omega \times [0,T]$ and let $\chi : [0,T] \mapsto \mathbb{R}^d$ be a continuous function. We introduce $D^+ = \{(x,t); x > \chi(t), (x,t) \in D\}$ and $D^- = \{(x,t); x < \chi(t), (x,t) \in D\}$. Let u be piecewise $C^1(D)$ and F be piecewise $C^1(D)^d$ such that $u \in (C^1(D^+) \cap C^1(D^-))$ and $F \in (C^1(D^+) \cap C^1(D^-))^d$. Let $g \in L^2(D)$ and $\varphi \in C_c^\infty(D)$ then the two following propositions are equivalent :*

1. *u is solution to the weak problem :*

$$-\int_D u\partial_t\varphi \, dxdt - \int_\Omega u_{\text{ini}}\varphi(0)\, dx - \int_D F \cdot \nabla\varphi \, dxdt = \int_D g\varphi \, dxdt.$$

2. *u is solution to :*

$$\partial_t u + \text{div}F = g \text{ on } \overline{D^+} \text{ and on } \overline{D^-},$$

$$u(x,0) = u_{\text{ini}}(x)$$

and

$$\chi'(t)(u^+(t) - u^-(t)) = F^+(t) - F^-(t).$$

Where $u^+(t)$ *(respectively* $F^+(t)$*) is the value of* u *(*F*) as* $x \to \chi(t), x > \chi(t)$ *and* $u^-(t)$ *(*$F^-(t)$*) is the value of* u *(*F*) as* $x \to \chi(t), x < \chi(t)$

Proof. Let $\varphi \in C_c^\infty(D^+)$, we get from 1) and using an integration by part that :

$$\int_{D^+} \partial_t u\varphi \, dxdt + \int_\Omega u(x,0)\varphi(0) - \int_\Omega u_{\text{ini}}\varphi(0) \, dx + \int_{D^+} \text{div}F\varphi \, dxdt = \int_{D^+} g\varphi \, dxdt.$$

Choosing a $t \neq 0$, we immediately have that $\partial_t u + \text{div}F = g$ almost everywhere but associated to the regularity of u and F we get the same result everywhere. Knowing that and choosing $t = 0$, we also get that $u(x,0) = u_{\text{ini}}(x)$. Now it remains to prove the Rankine-Hugoniot condition at the boundary between D^+ and D^- calling Γ. Let $\varphi \in C_c^\infty(D)$ such that it's support is on Γ. Using Stokes formula, we get that :

$$\int_{D^+} u\partial_t\varphi \, dxdt = -\int_{D^+} \partial_t u\varphi \, dxdt + \int_\Gamma u\varphi n_i \, ds.$$

Where $n_i = \overrightarrow{n} \cdot \overrightarrow{e_i}$ with $\overrightarrow{n}$ is the normal vector of Γ going out of D^+, here $\overrightarrow{n} = \begin{pmatrix} \frac{-\chi'}{\sqrt{1+(\chi')^2}} \\ \frac{1}{\sqrt{1+(\chi')^2}} \end{pmatrix}$ and $ds = \dfrac{dt}{\sqrt{1+(\chi')^2}}$. So we find :

$$\int_{D^+} u\partial_t\varphi \, dxdt = -\int_{D^+} \partial_t u\varphi \, dxdt + \int_\Gamma u^+\varphi \frac{-\chi'}{1+(\chi')^2} \, dt.$$

Using the same idea, we get that :

$$\int_{D^+} F\partial_x\varphi \, dxdt = -\int_{D^+} \text{div}F\varphi \, dxdt + \int_\Gamma \frac{F^+}{1+(\chi')^2}\varphi \, dt.$$

The same result is available on D^- taking $\overrightarrow{n} = \begin{pmatrix} \frac{\chi'}{\sqrt{1+(\chi')^2}} \\ \frac{-1}{\sqrt{1+(\chi')^2}} \end{pmatrix}$, this gives :

$$\int_{D^-} u\partial_t\varphi \, dxdt = -\int_{D^-} \partial_t u\varphi \, dxdt + \int_\Gamma u^-\varphi \frac{\chi'}{1+(\chi')^2} \, dt$$

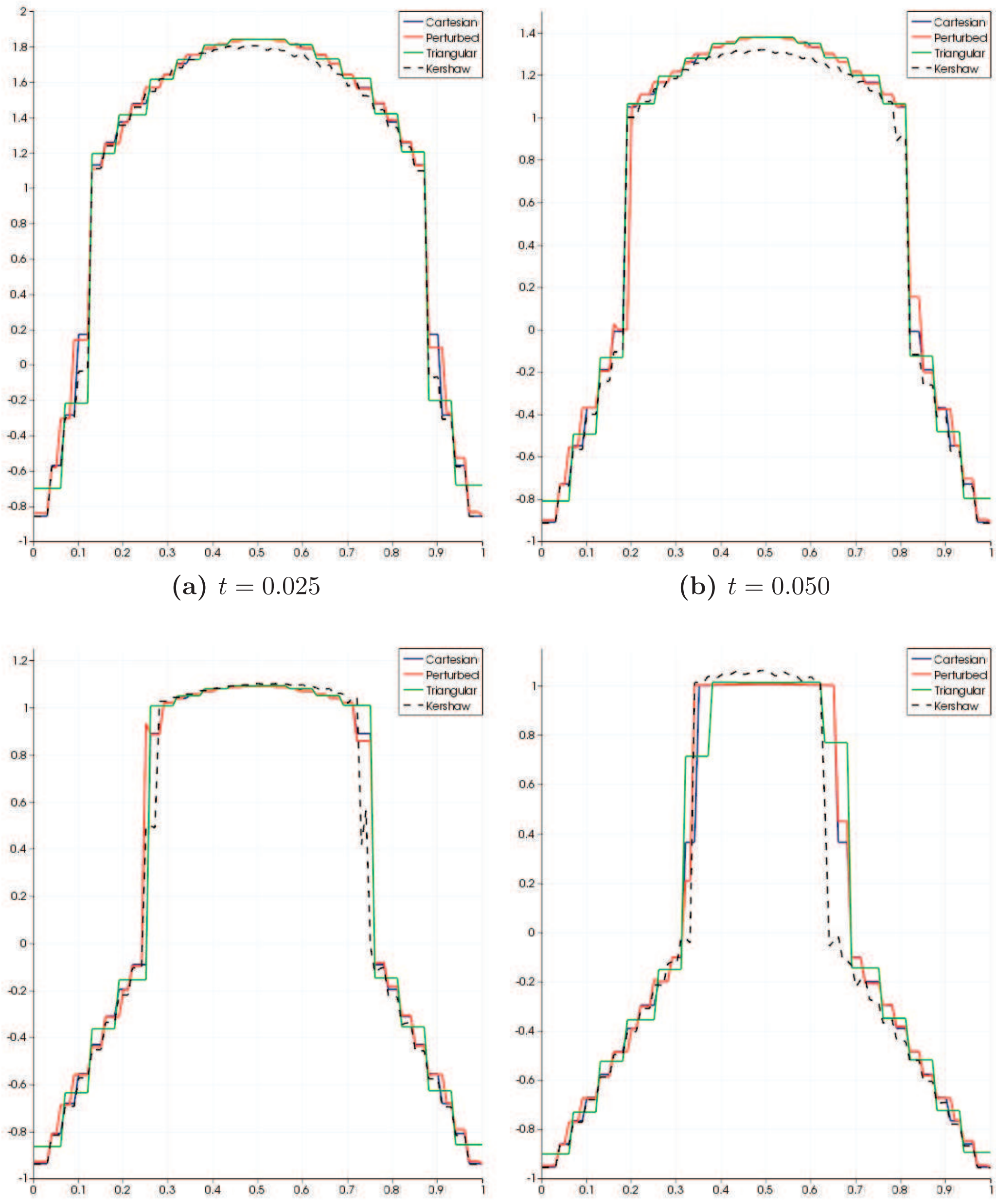
and

$$\int_{D^-} F\partial_x\varphi \, dxdt = -\int_{D^-} \mathrm{div} F\varphi \, dxdt + \int_{\Gamma} \frac{-F^-}{1+(\chi')^2}\varphi \, dt.$$

Since u is solution to the weak problem, by mixing the last 4 equations, it only stands terms on Γ and this gives for all φ :

$$\int_{\Gamma} \frac{\varphi}{1+(\chi')^2}\Big((u^- - u^+)\chi' + F^+ - F^-\Big) \, dt = 0.$$

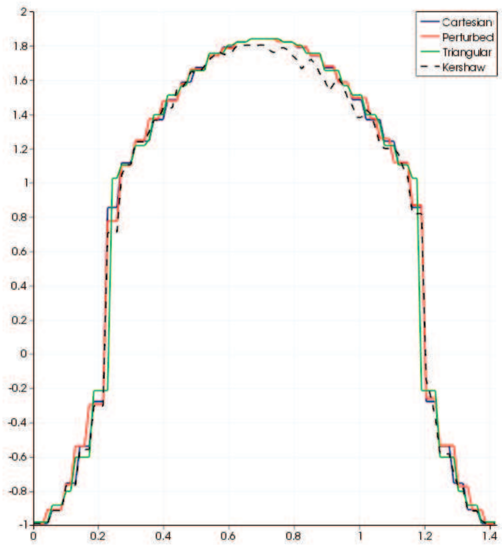
This implies the Rankine-Hugoniot condition expected.



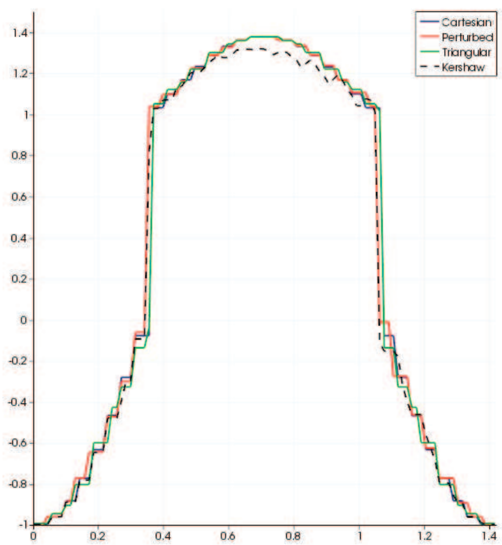
(a) $t = 0.025$ **(b)** $t = 0.050$

(c) $t = 0.075$ **(d)** $t = 0.1$

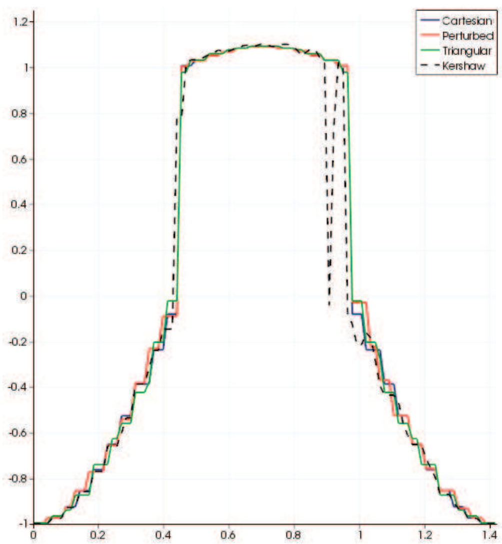
Figure 1.1 – Interpolation of u along the line which joins the two points $(0, 0.5)$ and $(1, 0.5)$ of the mesh for each grids : the Cartesian in blue, the perturbed in red, the triangular in green and the Kershaw in black dashed.



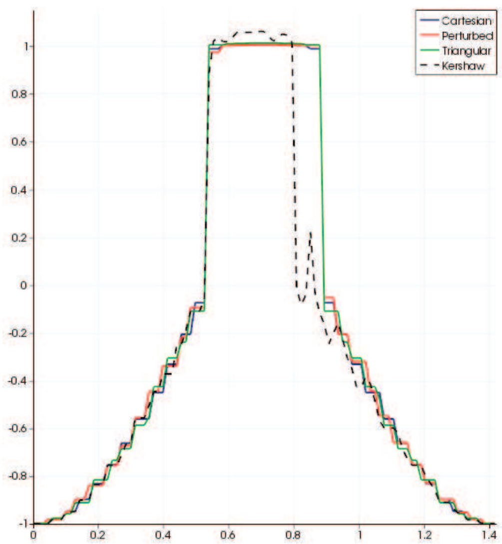
(a) $t = 0.025$



(b) $t = 0.050$

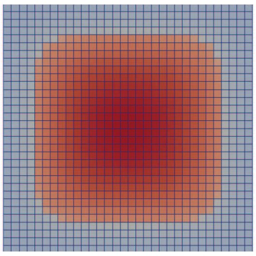
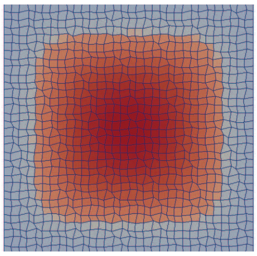


(c) $t = 0.075$

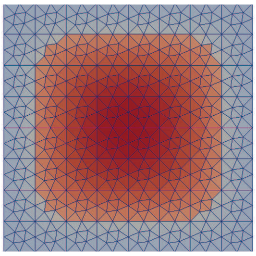
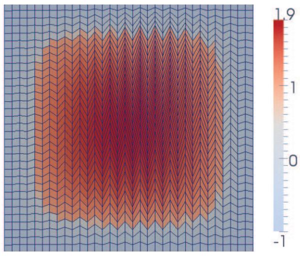


(d) $t = 0.1$

Figure 1.2 – Interpolation of u along a diagonal axe of the mesh for each grids : the Cartesian in blue, the perturbed in red, the triangular in green and the Kershaw in black dashed.

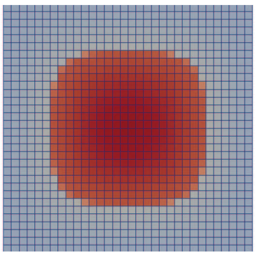
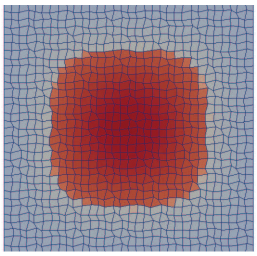



(a) Cartesian **(b)** Perturbed

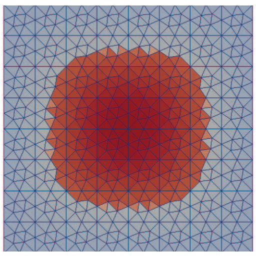
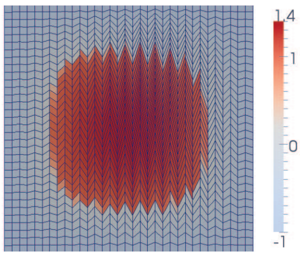



(c) Triangular **(d)** Kershaw

Figure 1.3 – Discrete solution u on all grids at $t = 0.025$.

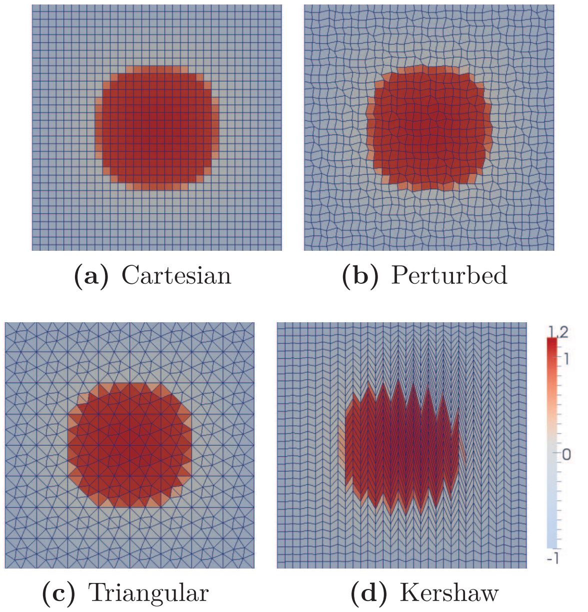



(a) Cartesian **(b)** Perturbed

(c) Triangular **(d)** Kershaw

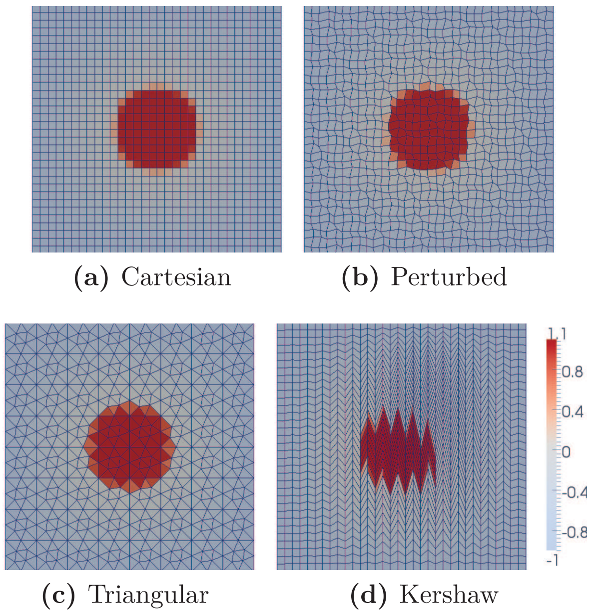
Figure 1.4 – Discrete solution u on all grids at $t = 0.050$.



(a) Cartesian (b) Perturbed

(c) Triangular (d) Kershaw

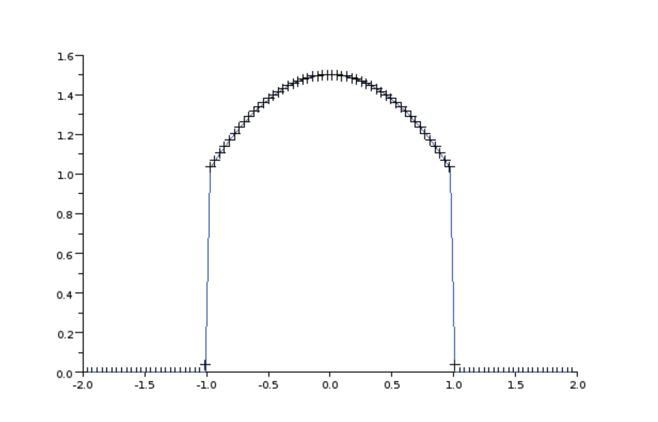
Figure 1.5 – Discrete solution u on all grids at $t = 0.075$.



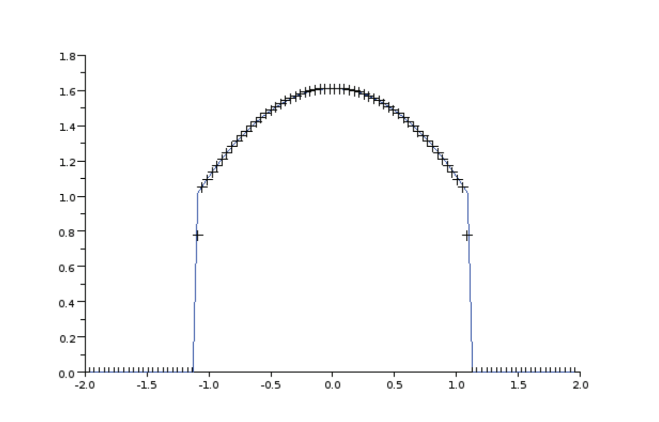
(a) Cartesian (b) Perturbed

(c) Triangular (d) Kershaw

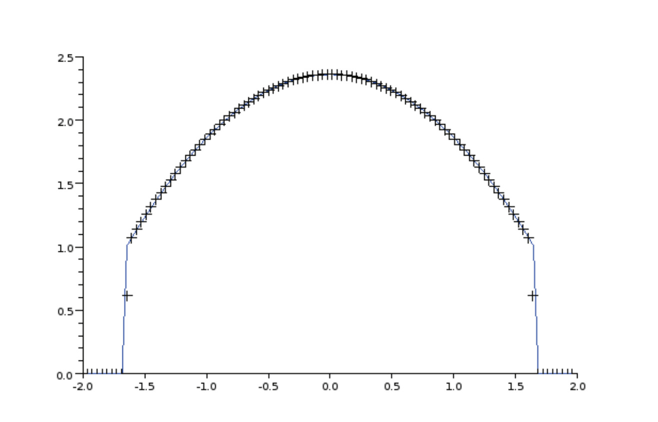
Figure 1.6 – Discrete solution u on all grids at $t = 0.1$.



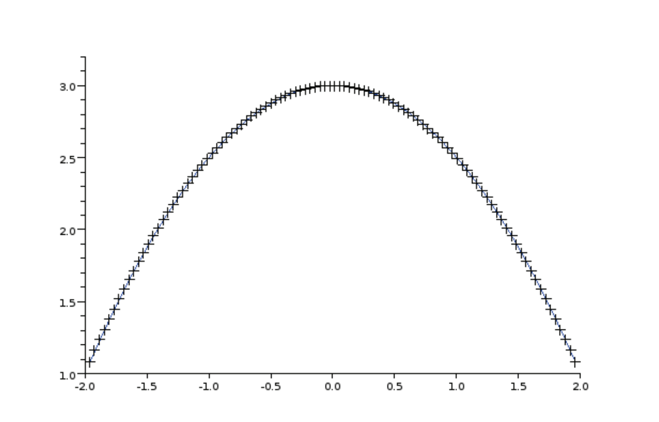
(a) $t = T_{\text{ini}}$



(b) $t = 0.1$

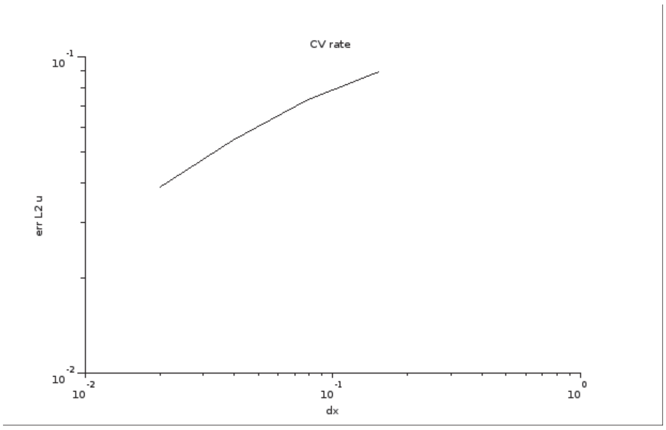


(c) $t = 0.5$

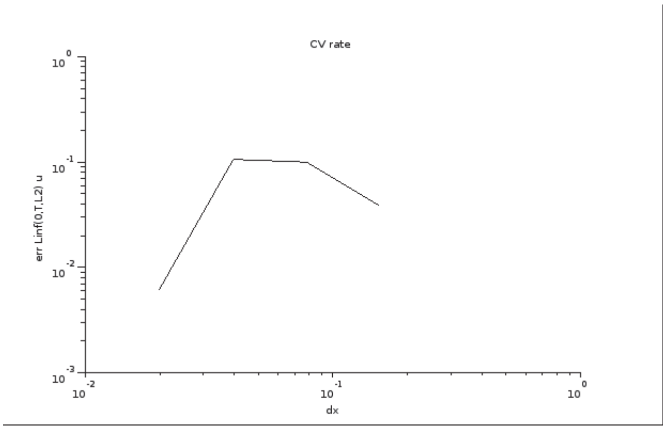


(d) $t = T_{\text{fin}}$

Figure 1.7 – Evolution of the solution $\bar{u}$ in blue and of the discrete solution u in black crux.



(a) Convergence rate of $\Pi_D u \to \bar{u}$ in $L^2(\Omega \times (0,T))$



(b) Convergence rate of $\Pi_D u \to \bar{u}$ in $L^\infty(0,T,L^2(\Omega))$

Figure 1.8 – We use a log / log scale to find the rate for the strong convergence of $\Pi_D u$ to $\bar{u}$ in $L^2(\Omega\times[0,T])$ (see (1.8a)) but not in $L^\infty(0,T,L^2(\Omega))$ (see (1.8b)).

Chapitre 2

The Stokes problem

2.1 Introduction

The aim of this chapter is to extend this framework to the steady and transient Stokes problems :

$$\begin{cases} \eta\overline{u} - \Delta\overline{u} + \nabla\overline{p} & = f - \mathrm{div}(G) & \text{in } \Omega \\ \mathrm{div}\overline{u} & = 0 & \text{in } \Omega \\ \overline{u} & = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

and

$$\begin{cases} \partial_t\overline{u} - \Delta\overline{u} + \nabla\overline{p} & = f - \mathrm{div}(G) & \text{in } \Omega \times (0,T) \\ \mathrm{div}\overline{u} & = 0 & \text{in } \Omega \times (0,T) \\ \overline{u} & = 0 & \text{on } \partial\Omega \times (0,T) \\ \overline{u}(\cdot,0) & = \overline{u}_{\mathrm{ini}} & \text{a.e on } \Omega, \end{cases} \tag{2.2}$$

where $\overline{u}$ represents the velocity field, $\overline{p}$ is the pressure and the domain Ω is a bounded open set in $\mathbb{R}^d$, $d \geq 1$.

The proofs of the results stated in Section 2.3 are then detailed in Section 2.4. One shall notice that these proofs are not straightforward, in particular in the case of the transient problem. Appropriate compactness tools have to be provided, demanding to define suitable discrete norms.

Notations In the following, if F is a vector space we denote by $\boldsymbol{F}$ the space F^d. Thus, $\boldsymbol{L}^2(\Omega) = L^2(\Omega)^d$ and $\boldsymbol{H}_0^1(\Omega) = H_0^1(\Omega)^d$. The space $E(\Omega)$ is the space of fields $v \in \boldsymbol{H}_0^1(\Omega)$ such that $\mathrm{div}(v) = 0$. $L^2_0(\Omega)$ is the space of functions in $L^2(\Omega)$ with a zero mean value over Ω. Finally, $\boldsymbol{H}_{\mathrm{div}}(\Omega)$ is the space of fields $v \in \boldsymbol{L}^2(\Omega)$ such that $\mathrm{div}(v) \in L^2(\Omega)$.

2.2 Gradient discretisations

Gradient discretisations provide the foundations, in terms of discrete spaces, operators and properties, upon which the gradient scheme framework is designed.

2.2.1 Space

Definition 2.1 (Gradient discretisation for the steady Stokes problem)**.** *A gradient discretisation D for the incompressible steady Stokes problem, with homogeneous Dirichlet's boundary conditions, is defined by $D = (X_{D,0}, \Pi_D, \nabla_D, Y_D, \chi_D, \mathrm{div}_D)$, where :*

1. *$X_{D,0}$ is a finite-dimensional vector space on $\mathbb{R}$, we denote $X^*_{D,0} = X_{D,0} \setminus \{0\}$.*
2. *Y_D is a finite-dimensional vector space on $\mathbb{R}$, we denote $Y^*_D = Y_D \setminus \{0\}$.*
3. *The linear mapping $\Pi_D : X_{D,0} \mapsto \boldsymbol{L}^2(\Omega)$ is the reconstruction of the approximate velocity field.*
4. *The linear mapping $\nabla_D : X_{D,0} \mapsto \boldsymbol{L}^2(\Omega)^d$ is the discrete gradient operator. It must be chosen such that $\| \cdot \|_D := \|\nabla_D \cdot \|_{\boldsymbol{L}^2(\Omega)^d}$ is a norm on $X_{D,0}$.*
5. *The linear mapping $\mathrm{div}_D : X_{D,0} \mapsto L^2(\Omega)$ is the discrete divergence operator.*
6. *The linear mapping $\chi_D : Y_D \mapsto L^2(\Omega)$ is the reconstruction of the approximate pressure, and must be chosen such that $\|\chi_D \cdot \|_{L^2(\Omega)}$ is a norm on Y_D. We then set $Y_{D,0} = \{q \in Y_D, \int_\Omega \chi_D q \mathrm{d}x = 0\}$. We assume that the nonnegative quantity β_D, defined by*

$$\beta_D = \min_{q \in Y^*_{D,0}} \max_{v \in X^*_{D,0}} \frac{\int_\Omega \chi_D q \ \mathrm{div}_D v \mathrm{d}x}{\|v\|_D \|\chi_D q\|_{L^2(\Omega)}}. \tag{2.3}$$

   is such that $\beta_D > 0$.

The *coercivity* of a sequence of gradient discretisations ensure that a discrete Poincaré inequality, a control of the discrete divergence and a discrete Ladyzenskaja-Babuska-Brezzi (LBB) conditions can be establish, all uniform along the sequence of discretisations.

Definition 2.2 (Coercivity)**.** *Let D be a discretisation in the sense of Definition 3.6. Let $q \in \mathbb{N}$ and let C_D be defined by*

$$C_D = \max_{v \in X^*_{D,0}} \frac{\|\Pi_D v\|_{\boldsymbol{L}^2(\Omega)}}{\|v\|_D} + \max_{v \in X^*_{D,0}} \frac{\|\mathrm{div}_D v\|_{L^2(\Omega)}}{\|v\|_D}. \tag{2.4}$$

*A sequence $(D_m)_{m\in\mathbb{N}}$ of gradient discretisation is said to be **coercive** if there exist $C_p \geq 0$ and $\beta > 0$ such that $C_{D_m} \leq C_P$ and $\beta_{D_m} \geq \beta$, for all $m \in \mathbb{N}$.*

The *consistency* of a sequence of gradient discretisations states that the discrete space and operators "fill in" the continuous space as the discretisation is refined.

Definition 2.3 (Consistency)**.** *Let D be a gradient discretisation in the sense of Definition 2.1, and let $S_D : \boldsymbol{H}^1_0(\Omega) \to [0,+\infty)$, and $\widetilde{S}_D : L^2_0(\Omega) \to [0,+\infty)$ be defined by*

$$\forall \varphi \in \boldsymbol{H}^1_0(\Omega)\,, \quad S_D(\varphi) = \min_{v \in X_{D,0}} \Big(\|\Pi_D v - \varphi\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla_D v - \nabla\varphi\|_{\boldsymbol{L}^2(\Omega)^d} + \|\mathrm{div}_D v - \mathrm{div}\varphi\|_{L^2(\Omega)} \Big)$$

and

$$\forall \psi \in L^2_0(\Omega)\,, \quad \widetilde{S}_D(\psi) = \min_{z \in Y_{D,0}} \|\chi_D z - \psi\|_{L^2(\Omega)}.$$

*A sequence $(D_m)_{m\in\mathbb{N}}$ of gradient discretisation is said to be **consistent** if, for all $\varphi \in \boldsymbol{H}^1_0(\Omega)$, $S_{D_m}(\varphi)$ tends to 0 as $m \to \infty$ and, for all $\psi \in L^2_0(\Omega)$, $\widetilde{S}_{D_m}(\psi)$ tends to 0 as $m \to \infty$.*

Definition 2.4 (Limit-conformity)**.** *Let D be a gradient discretisation in the sense of Definition 2.1 and let $\overline{W}_D : Z(\Omega) \mapsto [0,+\infty)$, with $Z(\Omega) = \{(\varphi,\psi) \in \boldsymbol{L}^2(\Omega)^d \times L^2(\Omega), \mathrm{div}\varphi - \nabla\psi \in \boldsymbol{L}^2(\Omega)\}$, be defined by*

$$\forall (\varphi,\psi) \in Z(\Omega)\,, \quad \overline{W}_D(\varphi,\psi) = \max_{v \in X^*_{D,0}} \frac{\int_\Omega \Big(\nabla_D v : \varphi + \Pi_D v \cdot (\mathrm{div}\varphi - \nabla\psi) - \psi \mathrm{div}_D v\Big) \mathrm{d}x}{\|v\|_D}.$$

*A sequence $(D_m)_{m\in\mathbb{N}}$ of gradient discretisation is said to be **limit-conforming** if, for all $(\varphi,\psi) \in Z(\Omega)$, $\overline{W}_{D_m}(\varphi,\psi)$ tends to 0 as $m \to \infty$.*

The next lemma states the equivalence with an other definition of limit-conformity used in chapter 4.

Lemma 2.5 (Equivalent formulation for the limit-conformity)**.** *Let D be a gradient discretisation in the sense of Definition 2.1 and let $W_D : \boldsymbol{H}_{\mathrm{div}}(\Omega) \mapsto [0,+\infty)$ be defined by*

$$\forall \varphi \in \boldsymbol{H}_{\mathrm{div}}(\Omega)\,, \quad W_D(\varphi) = \max_{v \in X^*_{D,0}} \frac{\int_\Omega (\nabla_D v : \varphi + \Pi_D v \cdot \mathrm{div}\varphi)\, \mathrm{d}x}{\|v\|_D},$$

and let $\widetilde{W}_D : L^2(\Omega) \mapsto [0, +\infty)$ *be defined by*

$$\forall \psi \in L^2(\Omega)\,, \quad \widetilde{W}_D(\psi) = \max_{v \in X^*_{D,0}} \frac{\int_\Omega \psi \left(\sum_{i=1}^d \nabla_D^{(i,i)} v - \mathrm{div}_D v \right) \mathrm{d}x}{\|v\|_D}.$$

Then a sequence $(D_m)_{m\in\mathbb{N}}$ *of gradient discretisation is said to be* ***limit-conforming*** *in the sense of Definition 2.4 if and only if, for all* $\varphi \in \boldsymbol{H}_{\mathrm{div}}(\Omega)$*,* $W_{D_m}(\varphi)$ *tends to* 0 *and for all* $\psi \in L^2(\Omega)$*,* $\widetilde{W}_{D_m}(\psi)$ *tends to* 0 *as* $m \to \infty$*.*

Proof. Noticing that, for $(\varphi, \psi) \in Z(\Omega)$, we have $\widetilde{\varphi} := \varphi - \psi \mathrm{I}_d \in \boldsymbol{H}_{\mathrm{div}}(\Omega)$, and writing

$$\nabla_D v : \varphi + \Pi_D v \cdot (\mathrm{div}\varphi - \nabla\psi) - \psi \mathrm{div}_D v = \nabla_D v : \widetilde{\varphi} + \Pi_D v \cdot \mathrm{div}\widetilde{\varphi} + \psi \left(\sum_{i=1}^d \nabla_D^{(i,i)} v - \mathrm{div}_D v \right),$$

we get $\overline{W}_D(\varphi, \psi) \le W_D(\widetilde{\varphi}) + \widetilde{W}_D(\psi)$. Reciprocally, for any $\varphi \in \boldsymbol{H}_{\mathrm{div}}(\Omega)$, $(\varphi, 0) \in Z(\Omega)$ and $W_D(\varphi) = \overline{W}_D(\varphi, 0)$, and for any $\psi \in L^2(\Omega)$, $(\psi \mathrm{I}_d, \psi) \in Z(\Omega)$ and $\widetilde{W}_D(\psi) = \overline{W}_D(\psi \mathrm{I}_d, \psi)$. So the above definition of limit-conformity is equivalent to Definition 2.4.

Remark 2.6. *As in [41, 64], the consistency of a coercive sequence of gradient discretisation only needs to be checked on dense subspaces of* $\boldsymbol{H}_0^1(\Omega)$ *and* $L_0^2(\Omega)$*. This is also true for limit-conformity.*

2.2.2 Space-time

The notion of gradient discretisation for transient problems requires the addition of time steps and an interpolation (not necessarily linear) of the initial condition.

Definition 2.7 (Space-time gradient discretisation)**.** *A space-time gradient discretisation* D *for the transient Stokes problem, with homogenous Dirichlet boundary conditions, is defined by a family* $D = (X_{D,0}, \Pi_D, \nabla_D, Y_D, \chi_D, \mathrm{div}_D, (t^{(n)})_{n=0,...,N}, J_D)$ *where :*

- $D^s = (X_{D,0}, \Pi_D, \nabla_D, Y_D, \chi_D, \mathrm{div}_D)$ *a gradient discretisation of* Ω *in the sense of Definition 2.1,*
- $J_D : \boldsymbol{L}^2(\Omega) \mapsto X_{D,0}$ *an interpolation operator ;*
- $t^{(0)} = 0 < t^{(1)} < ... < t^{(N)} = T$*.*

We define $\delta t^{n+\frac{1}{2}} = t^{(n+1)} - t^{(n)}$ *for all* $n = 0, ..., N-1$ *and* $\delta t_D = \max_{n=0,...,N-1} (\delta t^{n+\frac{1}{2}})$*.*

A sequence of space-time gradient discretisation $(D_m)_{m\in\mathbb{N}}$ is coercive (resp. limit-conforming) if its spatial component $(D_m^s)_{m\in\mathbb{N}}$ is coercive (resp. limit-conforming).

Definition 2.8 (Space-time consistency). *A sequence $(D_m)_{m\in\mathbb{N}}$ of space-time gradient discretisations in the sense of Definition 2.7 is said* ***consistent*** *if*

1. $(D^s_m)_{m\in\mathbb{N}}$ *is consistent in the sense of Definition 2.3,*
2. *for all* $\varphi \in \boldsymbol{L}^2(\Omega)$, $\Pi_{D_m} J_{D_m}\varphi \to \varphi$ *in* $\boldsymbol{L}^2(\Omega)$,
3. $\delta t_{D_m} \to 0$ *as* $m \to \infty$.

2.3 Gradient schemes and main results

2.3.1 Steady Stokes problem

Our assumptions for the steady Stokes problem (2.1) are the following :

$$\begin{aligned}&\Omega \text{ is an open bounded Lipschitz domain of } \mathbb{R}^d\ (d\geq 1),\\ &f\in \boldsymbol{L}^2(\Omega),\ G\in \boldsymbol{L}^2(\Omega)^d \text{ and } \eta\in[0,+\infty).\end{aligned} \tag{2.5}$$

Definition 2.9 (Weak solution to the steady Stokes problem). *Under Hypotheses (2.5), $(\overline{u},\overline{p})$ is a weak solution to (2.1) if*

$$\begin{cases} \overline{u}\in \boldsymbol{H}^1_0(\Omega),\ \overline{p}\in L^2_0(\Omega),\\ \eta\displaystyle\int_\Omega \overline{u}\cdot\overline{v}\mathrm{d}x + \int_\Omega \nabla\overline{u}:\nabla\overline{v}\mathrm{d}x - \int_\Omega \overline{p}\mathrm{div}\overline{v}\mathrm{d}x = \int_\Omega (f\cdot\overline{v} + G:\nabla\overline{v})\,\mathrm{d}x,\ \forall \overline{v}\in \boldsymbol{H}^1_0(\Omega),\\ \displaystyle\int_\Omega q\mathrm{div}\overline{u}\mathrm{d}x = 0\,,\ \forall q\in L^2_0(\Omega), \end{cases} \tag{2.6}$$

where "$\cdot$" is the dot product on $\mathbb{R}^d$, and if $\xi = (\xi_{i,j})_{i,j=1,\ldots,d}\in\mathbb{R}^{d\times d}$ and $\chi = (\chi_{i,j})_{i,j=1,\ldots,d}\in\mathbb{R}^{d\times d}$, $\xi:\chi = \sum_{i,j=1}^d \xi_{i,j}\chi_{i,j}$ is the doubly contracted product on $\mathbb{R}^{d\times d}$.

Remark 2.10. *Under Hypotheses (2.5), the existence and uniqueness of a weak solution $(\overline{u},\overline{p})$ to Problem (2.1) in the sense of Definition 2.9 follows from [100, Ch.I, Theorem 2.1].*

The gradient scheme for the steady Stokes problem is based on a discretisation of the weak formulation (2.6), in which the continuous spaces and operators are replaced with discrete ones (in (2.6), we wrote the property "$\mathrm{div}\overline{u} = 0$" using test functions to make clearer this parallel between the weak formulation and the gradient scheme). If D is a gradient discretisation in the sense of Definition 2.1, the scheme is given by :

$$\begin{cases} u \in X_{D,0}\,,\ p \in Y_{D,0}, \\ \eta \displaystyle\int_\Omega \Pi_D u \cdot \Pi_D v \mathrm{d}x + \int_\Omega \nabla_D u : \nabla_D v \mathrm{d}x \\ \qquad\qquad - \displaystyle\int_\Omega \chi_D p \ \mathrm{div}_D v \mathrm{d}x = \int_\Omega (f \cdot \Pi_D v + G : \nabla_D v)\,\mathrm{d}x,\ \forall v \in X_{D,0}, \\ \displaystyle\int_\Omega \chi_D q \mathrm{div}_D u \mathrm{d}x = 0,\ \ \forall q \in Y_{D,0}. \end{cases} \tag{2.7}$$

Our main result on the gradient schemes for steady Stokes problem is the following theorem.

Theorem 2.11 (Error estimates for the steady Stokes problem)**.** *Under Hypotheses* (2.5)*, let* $(\overline{u}, \overline{p})$ *be the unique solution of the incompressible steady Stokes problem* (2.1) *in the sense of Definition 2.9. Let D be a gradient discretisation in the sense of Definition 2.1 such that $\beta_D > 0$ (see Definition 2.2). Then there exists a unique $(u_D, p_D) \in X_{D,0} \times Y_{D,0}$ solution of the gradient scheme* (2.7)*, and there exists $C_e > 0$, non-decreasing w.r.t. η, C_D and $\frac{1}{\beta_D}$, such that*

$$\begin{aligned} &\|\overline{u} - \Pi_D u_D\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla \overline{u} - \nabla_D u_D\|_{\boldsymbol{L}^2(\Omega)^d} + \|\overline{p} - \chi_D p\|_{L^2(\Omega)} \\ &\qquad\qquad \le C_e \left(\overline{W}_D(\nabla \overline{u} - G, \overline{p}) + S_D(\overline{u}) + \widetilde{S}_D(\overline{p}) \right). \end{aligned} \tag{2.8}$$

Remark 2.12. *As a consequence, if $(D_m)_{m\in\mathbb{N}}$ is a coercive, consistent and limit-conforming sequence of gradient discretisations (see Definitions 2.2, 2.3 and 2.4) and if (u_m, p_m) are the solutions to the corresponding gradient schemes, then, as $m \to \infty$, $\Pi_{D_m} u_m \to \overline{u}$ in $\boldsymbol{L}^2(\Omega)$, $\nabla_{D_m} u_m \to \nabla \overline{u}$ in $\boldsymbol{L}^2(\Omega)^d$ and $\chi_{D_m} p_m \to \overline{p}$ in $L^2(\Omega)$.*

The constant C_e in the preceding estimate tends to infinity as β_D tends to zero. For some gradient schemes, we can obtain an estimate on the velocity which is independent on the constant in the inf-sup condition. For a gradient discretisation D, we define the space of discrete divergence-free functions (in the dual sense), discrete version of the space $E(\Omega)$, by

$$E_D = \left\{ v \in X_{D,0} \ :\ \forall q \in Y_{D,0}\,,\ \int_\Omega \chi_D q \, \mathrm{div}_D v \mathrm{d}x = 0 \right\}. \tag{2.9}$$

Theorem 2.13 (Error estimates on the velocity without inf-sup constant)**.** *Under the assumptions of Theorem 2.11, we suppose that*

$$\begin{gathered} \forall v \in X_{D,0}\,,\ \ \textit{if } \int_\Omega \chi_D q \mathrm{div}_D v \mathrm{d}x = 0 \textit{ for all } q \in Y_{D,0},\ \textit{then } \mathrm{div}_D v = 0 \ \textit{a.e. in } \Omega \\ \textit{(that is to say, } E_D = \{ v \in X_{D,0} \ :\ \mathrm{div}_D v = 0 \ \textit{a.e.}\}). \end{gathered} \tag{2.10}$$

Then

$$\|\nabla\overline{u} - \nabla_D u_D\|_{\boldsymbol{L}^2(\Omega)^d} \le (\eta C_D + 2) S_{D,E_D}(\overline{u}) + \overline{W}_D(\nabla\overline{u} - G, \overline{p}) \tag{2.11}$$

$$\|\overline{u} - \Pi_D u_D\|_{\boldsymbol{L}^2(\Omega)} \le \Big(C_D(\eta C_D + 1) + 1\Big) S_{D,E_D}(\overline{u}) + C_D \overline{W}_D(\nabla\overline{u} - G, \overline{p}), \tag{2.12}$$

where

$S_{D,E_D}(\overline{u}) = \min_{v\in E_D} \Big(\|\Pi_D v - \overline{u}\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla_D v - \nabla\overline{u}\|_{\boldsymbol{L}^2(\Omega)^d}\Big).$

Remark 2.14. *Most classical schemes for Stokes problem satisfy (2.10) and have interpolants $E(\Omega) \to E_D$ which ensure that $S_{D,E_D}(\overline{u}) \to 0$ as the mesh size tend to 0. This is for example the case of all schemes presented in Section 4.2.*

Estimates (2.11) and (2.12) on the discrete velocity still depend on the continuous pressure $\overline{p}$. This means that even in the case of purely irrotational forces, with the solution to the Stokes equation $(\overline{u}, \overline{p}) = (0, \overline{p})$, the pressure terms can lead to errors on the velocity [82]. This dependency on the pressure can sometimes be removed.

Theorem 2.15 (Pressure-independent error-estimates on the velocity)**.** *Under the assumptions of Theorem 2.11, we suppose that*

$$\forall v \in E_D\,,\ \forall \psi \in H^1(\Omega)\,,\ \int_\Omega \Pi_D v \cdot \nabla\psi \mathrm{d}x = 0. \tag{2.13}$$

Then, if $\nabla\overline{u} - G \in \boldsymbol{H}_{\mathrm{div}}(\Omega)$ (which amounts to asking that $\overline{p} \in H^1(\Omega)$), we have the following pressure-independent estimates on the velocity :

$$\|\nabla\overline{u} - \nabla_D u_D\|_{\boldsymbol{L}^2(\Omega)^d} \le (\eta C_D + 2) S_{D,E_D}(\overline{u}) + W_D(\nabla\overline{u} - G) \tag{2.14}$$

$$\|\overline{u} - \Pi_D u_D\|_{\boldsymbol{L}^2(\Omega)} \le \Big(C_D(\eta C_D + 1) + 1\Big) S_{D,E_D}(\overline{u}) + C_D W_D(\nabla\overline{u} - G), \tag{2.15}$$

where W_D is given in Definition 1.7.

Remark 2.16. *In case of purely irrotational forces $(f, G) = (\nabla V, 0)$, then the solution to the Stokes problem is $(\overline{u}, \overline{p}) = (0, V)$ and Estimates (2.14) and (2.15) show that the velocity is exactly approximated. In other words, for such irrotational forces, the discrete velocity provided by the scheme is zero.*

Remark 2.17. *Assumption (2.13) is obviously satisfied by conforming methods, such as the Taylor–Hood scheme (cf. Section 4.3). It is not satisfied in general by non-conforming methods such as the Crouzeix-Raviart scheme, when Π_D is the "classical" reconstruction of function (see*

Section 4.4). As suggested in [82], a way to solve these schemes' poor mass conservation (arising from the action at the discrete level of purely irrotational forces on the velocity) is to replace Π_D *with a non-standard reconstruction which satisfies* (2.13)*. The reconstruction proposed in [82] consists in defining* Π_D *as an interpolation in the lowest order Raviart-Thomas space of functions in the Crouzeix-Raviart space : if* v *is a function in the non-conforming* $\mathbb{P}_1^d$ *space and* $(v_\sigma)_{\sigma\in\mathcal{E}}$ *(*$\mathcal{E}$ *being the set of all faces) are its values at the centers of gravity of the faces* $(x_\sigma)_{\sigma\in\mathcal{E}}$*, then* $\Pi_D v = \Pi_D^{\mathrm{RT}} v$ *is the function in the lowest order Raviart-Thomas space which satisfies, for any simplicial cell* T *and any edge* σ *of* T*,*

$$(\Pi_D v)(x_\sigma) \cdot \mathbf{n}_{T,\sigma} = v(x_\sigma) \cdot \mathbf{n}_{T,\sigma},$$

where $\mathbf{n}_{T,\sigma}$ *is the outer unit normal to* T *on* σ*. Then* (2.13) *is satisfied. Indeed,* $\Pi_D^{\mathrm{RT}} v$ *is* H_{div} *conforming and satisfies* $\mathrm{div}(\Pi_D^{\mathrm{RT}} v) = \mathrm{div}_D v$*, where* $\mathrm{div}_D v$ *is the broken piecewise constant divergence of* v*. Hence, if* $v \in E_D$ *we have* $\mathrm{div}(\Pi_D^{\mathrm{RT}} v) = 0$ *in* Ω *and* (2.13) *holds.*

2.3.2 Transient Stokes problem

Weak formulation

We consider the transient Stokes problem (2.2) under the assumptions

$$\begin{aligned}&\Omega \text{ is an open bounded Lipschitz domain of } \mathbb{R}^d\ (d \geq 1),\ T > 0,\\ &\overline{u}_{\mathrm{ini}} \in \boldsymbol{L}^2(\Omega),\ f \in \boldsymbol{L}^2(\Omega \times (0,T)) \text{ and } G \in \boldsymbol{L}^2(\Omega \times (0,T))^d.\end{aligned} \tag{2.16}$$

The solution to (2.2) is initially understood in the following weak sense, in which the pressure is eliminated by the choice of divergence-free test functions. Existence and uniqueness of this solution is proved in [100, Ch.III, Theorem 1.1].

Definition 2.18. *Under Hypothesis* (2.16)*,* $\overline{u}$ *is a weak solution to* (2.2) *if*

$$\left\{\begin{aligned}&\overline{u} \in L^2(0,T,E(\Omega)),\\ &\int_0^T\int_\Omega -\overline{u}\cdot\partial_t\varphi \mathrm{d}x\mathrm{d}t + \int_\Omega \overline{u}_{\mathrm{ini}}\cdot\varphi(\cdot,0)\mathrm{d}x + \int_0^T\int_\Omega \nabla\overline{u} : \nabla\varphi \mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega f\cdot\varphi\mathrm{d}x\mathrm{d}t\\ &\qquad + \int_0^T\int_\Omega G : \nabla\varphi\mathrm{d}x\mathrm{d}t,\ \forall\varphi = \theta w \text{ with } \theta \in C_c^\infty([0,T)) \text{ and } w \in E(\Omega).\end{aligned}\right. \tag{2.17}$$

It can however be seen (see Section 2.3.2), that if $u_{\mathrm{ini}} \in E(\Omega)$ and $\overline{u}$ is the solution to (2.17), then there exists a pressure $\overline{p}$ such that $(\overline{u}, \overline{p})$ is a solution to (2.2) in the following sense.

Proposition 2.19. *Assume Hypotheses* (2.16) *and* $u_{\text{ini}} \in E(\Omega)$ *and let* $\overline{u}$ *be the solution to* (2.17). *Then there exists* $\overline{p}$ *such that* $(\overline{u}, \overline{p})$ *satisfies :*

$$\begin{cases} \overline{u} \in L^2(0,T,E(\Omega)) \cap C([0,T];\boldsymbol{L}^2(\Omega))\,,\ \partial_t\overline{u} \in L^2(0,T,\boldsymbol{H}^{-1}(\Omega))\,, \\ \overline{p} \in L^2(0,T,L^2_0(\Omega))\,, \\ \displaystyle\int_0^T\int_\Omega \langle \partial_t\overline{u},\varphi\rangle \mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega \nabla\overline{u} : \nabla\varphi \mathrm{d}x\mathrm{d}t - \int_0^T\int_\Omega \overline{p}(x,t)\mathrm{div}\varphi \mathrm{d}x\mathrm{d}t \\ \qquad = \displaystyle\int_0^T\int_\Omega f\cdot\varphi \mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega G : \nabla\varphi \mathrm{d}x\mathrm{d}t\,,\ \forall\varphi \in L^2(0,T;\boldsymbol{H}^1_0(\Omega)) \\ \overline{u}(\cdot,0) = \overline{u}_{\text{ini}}\ a.e\ on\ \Omega \end{cases} \tag{2.18}$$

where $\langle\cdot,\cdot\rangle$ *denotes the duality pairing between* $\boldsymbol{H}^{-1}(\Omega)$ *and* $\boldsymbol{H}^1_0(\Omega)$.

An equivalent weak formulation

The theoretical study of the transient Stokes problem, and in particular of the notion of pressure in this model, is significantly more complex than for the steady problem. The initial sense of weak solution for (2.2) given by Definition 2.18 only involves the velocity unknown as the pressure has been eliminated by the choice of divergence-free test functions. The interest of this formulation is that it provides an existence and uniqueness result based on classical variational arguments [100, Ch.III, Theorem 1.1].

An equivalent and useful formulation of (2.17) is the following :

$$\begin{cases} \overline{u} \in L^2(0,T;E(\Omega)) \cap C([0,T];\boldsymbol{L}^2(\Omega))\,,\ \partial_t\overline{u} \in L^2(0,T;E(\Omega)')\,, \\ \displaystyle\int_0^T \langle \partial_t\overline{u},\varphi\rangle \mathrm{d}t + \int_0^T\int_\Omega \nabla\overline{u} : \nabla\varphi \mathrm{d}x\mathrm{d}t \\ \qquad = \displaystyle\int_0^T\int_\Omega f\cdot\varphi \mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega G : \nabla\varphi \mathrm{d}x\mathrm{d}t\,, \quad \forall\varphi \in L^2(0,T;E(\Omega))\,, \\ \overline{u}(\cdot,0) = \overline{u}_{\text{ini}} \text{ in } \boldsymbol{L}^2(\Omega)\,, \end{cases} \tag{2.19}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $E(\Omega)'$ and $E(\Omega)$. See for example the discussion at the end of the proof of Theorem 2.23 on the equivalence between (2.39) and (2.18).

A pressure can be recovered in a very weak sense. Proposition 1.1 in [100, Ch.III] establishes the existence of a distribution $\overline{p}$ on $\Omega \times (0,T)$ such that if $\overline{u}$ is the solution of (2.19) then $(\overline{u}, \overline{p})$ satisfies the PDEs in (2.2) in the sense of distributions. Additional regularity results on $\overline{p}$ can be obtained if we assume that $u_{\text{ini}} \in E(\Omega)$.

Proposition 2.20 (Regularity result)**.** *Let us assume Hypothesis* (2.16), *and that* $\overline{u}_{\text{ini}} \in E(\Omega)$. *We denote by* H *the closure of* $\{\varphi \in C_c^\infty(\Omega)^d : \mathrm{div}(\varphi) = 0\}$ *in* $\boldsymbol{L}^2(\Omega)$. *Then the weak solution*

$(\overline{u},\overline{p})$ to (2.2) in the sense of Definition 2.18 (and $\overline{p}$ as in the discussion above) satisfies : $\overline{u} \in L^2(0,T,E(\Omega)) \cap C([0,T];H)$, $\partial_t\overline{u} \in L^2(0,T,E(\Omega)')$ and $\overline{p} \in L^2(0,T,L^2_0(\Omega))$.

Proof. This is essentially contained in the proof of [100, Ch.III, Theorem 1.1 and Proposition 1.2]. Proposition 1.2 in this reference is proved under more regularity assumption on Ω and the right-hand side, namely G must not be present and f must be divergence-free. This is actually useful just to recover higher regularity on the solution, that is H^2 on $\overline{u}$ and H^1 on $\overline{p}$. Under our assumptions, the proof of [100, Ch.III, Proposition 1.2] gives Proposition 2.20.

These additional regularity results on $(\overline{u},\overline{p})$ make the proof of Proposition 2.19 obvious. Indeed, testing $\partial_t\overline{u} - \Delta\overline{u} + \nabla p = f - \mathrm{div}(G)$ (that is satisfied in the sense of distributions, see above) against $\varphi \in \mathbf{C}_c^\infty(0,T) \times \Omega)$ and using the regularity of $(\overline{u},\overline{p})$, we see that the equation in (2.18) holds for any smooth φ with compact support. The general case is deduced by density of these functions in $L^2(0,T;\boldsymbol{H}^1_0(\Omega))$.

Convergence results

Let D be a space-time gradient discretisation in the sense of Definition (2.7). With the notation

$$\delta_D^{n+\frac{1}{2}} u_D = \frac{u_D^{(n+1)} - u_D^{(n)}}{\delta t^{n+\frac{1}{2}}},$$

the implicit gradient scheme for (2.2) is based on the following approximation of (2.18) :

$$\left\{\begin{array}{l} u_D = (u_D^{(n)})_{n=0,\ldots,N}\,,\ p_D = (p_D^{(n)})_{n=1,\ldots,N} \text{ such that } u_D^{(0)} = J_D u_{\mathrm{ini}} \text{ and, } \forall n = 0, ..., N-1 : \\ u_D^{(n+1)} \in X_{D,0}\,,\ p_D^{(n+1)} \in Y_{D,0}\,, \\ \displaystyle\int_\Omega \Pi_D \delta_D^{n+\frac{1}{2}} u_D \cdot \Pi_D v \mathrm{d}x + \int_\Omega \nabla_D u_D^{(n+1)} : \nabla_D v \mathrm{d}x - \int_\Omega \chi_D p_D^{(n+1)} \mathrm{div}_D v \mathrm{d}x \\ \quad \displaystyle = \frac{1}{\delta t^{n+\frac{1}{2}}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_D v \mathrm{d}x\mathrm{d}t + \frac{1}{\delta t^{n+\frac{1}{2}}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega G \cdot \nabla_D v \mathrm{d}x\mathrm{d}t, \quad \forall v \in X_{D,0}. \\ \displaystyle\int_\Omega \mathrm{div}_D u_D^{(n+1)} \chi_D q \mathrm{d}x = 0, \quad \forall q \in Y_{D,0}. \end{array}\right. \tag{2.20}$$

It is common to use Π_D and ∇_D to denote space-time functions the following way : if $v = (v^n)_{n=0,\ldots,N} \in X_{D,0}$, the functions $\Pi_D v : \Omega \times (0,T) \to \mathbb{R}^d$ and $\nabla_D v : \Omega \times (0,T) \to \mathbb{R}^{d\times d}$ are defined by

$$\begin{array}{l} \forall n = 0, \ldots, N-1\,,\ \forall t \in (t^{(n)}, t^{(n+1)}]\,,\ \forall x \in \Omega\,, \\ \Pi_D v(x,t) = \Pi_D v^{(n+1)}(x)\,,\ \nabla_D v(x,t) = \nabla_D v^{(n+1)}(x) \text{ and } \delta_D v(t) = \delta_D^{n+\frac{1}{2}} v. \end{array} \tag{2.21}$$

Our first convergence result deals only with the velocity.

Theorem 2.21 (Convergence of the velocity for the transient Stokes problem)**.**

Under Hypotheses (2.16)*, let* $\overline{u}$ *the unique weak solution of the incompressible transient Stokes problem* (2.2) *in the sense of Definition 2.18 and let* $(D_m)_{m\in\mathbb{N}}$ *be a sequence of space-time gradient discretisations in the sense of Definition 2.7, which is space-time consistent, limit-conforming and coercive in the sense of Definitions 2.8, 2.4 and 2.2. Then for any* m *there is a unique solution* (u_{D_m}, p_{D_m}) *to* (2.20) *with* $D = D_m$ *and, as* $m \to \infty$*,*

— $\Pi_{D_m} u_{D_m}$ *converges to* $\overline{u}$ *in* $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$*,*

— $\nabla_{D_m} u_{D_m}$ *converges to* $\nabla\overline{u}$ *in* $\boldsymbol{L}^2(\Omega\times(0,T))^d$*.*

Remark 2.22. *Note that since the functions* $\Pi_{D_m} u_{D_m}$ *are piecewise constant in time, their convergence in* $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ *is actually a uniform-in-time convergence (not "uniform a.e. in time").*

Under slightly more restrictive (but usual) conditions on the initial data, we can also prove a convergence result on the pressure.

Theorem 2.23 (Convergence of the pressure for the transient Stokes problem)**.** *Under the assumptions and notations of Theorem 2.21, we suppose that* $G = 0$*,* $u_{\rm ini} \in E(\Omega)$*,* $(||J_{D_m} u_{\rm ini}||_{D_m})_{m\in\mathbb{N}}$ *is bounded and, for all* $m \in \mathbb{N}$*,* $J_{D_m} u_{\rm ini} \in E_{D_m}$ *(where* E_{D_m} *is defined by* (2.9) *with* $D = D^s_m$*, the spatial gradient discretisation corresponding to* D_m*). Then*

— $\Pi_{D_m} u_{D_m}$ *converges to* $\overline{u}$ *in* $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$*,*

— $\nabla_{D_m} u_{D_m}$ *converges to* $\nabla\overline{u}$ *in* $\boldsymbol{L}^2(\Omega\times(0,T))^d$*,*

— $\chi_{D_m} p_{D_m}$ *weakly converges to* $\overline{p}$ *in* $L^2(\Omega\times(0,T))$*,*

where $(\overline{u},\overline{p})$ *is the weak solution to* (2.2) *in the sense* (2.18)*.*

2.4 Proof of the convergence results

2.4.1 Steady problem

Proof (of Theorem 2.11). Once established, Estimate (2.8) shows that if the right-hand of the linear system (2.7) on (u_D, p_D) is zero (i.e. $f = 0$, $G = 0$, which implies $\overline{u} = 0$ and $\overline{p} = 0$), then the solution (u_D, p_D) is also zero. Hence, this square system is invertible, which ensures the existence and uniqueness of its solution for any right-hand side. We now have to show Estimate (2.8). Under the hypotheses of the theorem, since ${\rm div}(\nabla\overline{u}) - \nabla\overline{p} = -f + {\rm div}(G) + \eta\overline{u}$

in the distribution sense, we get $\operatorname{div}(\nabla\overline{u} - G) - \nabla\overline{p} = \eta\overline{u} - f \in \boldsymbol{L}^2(\Omega)$. Using this relation in $\overline{W}_D(\nabla\overline{u} - G, \overline{p})$ we write

$$\overline{W}_D(\nabla\overline{u} - G, \overline{p}) = \max_{\substack{v\in X_{D,0}\\ \|v\|_D=1}} \int_\Omega \Big(\nabla_D v : (\nabla\overline{u} - G) + \Pi_D v \cdot (\eta\overline{u} - f) - \overline{p}\operatorname{div}_D v\Big)\mathrm{d}x.$$

Invoking the gradient scheme (2.7) to replace $\int_\Omega -(\Pi_D v \cdot f + \nabla_D v : G)\mathrm{d}x$, we can write, for any $v \in X_{D,0}$,

$$\int_\Omega \Big(\eta(\overline{u} - \Pi_D u_D) \cdot \Pi_D v + (\nabla\overline{u} - \nabla_D u_D) : \nabla_D v + (\chi_D p_D - \overline{p})\ \operatorname{div}_D v\Big)\mathrm{d}x \le \overline{W}_D(\nabla\overline{u} - G, \overline{p})\|v\|_D. \tag{2.22}$$

Let us introduce $I_D : \boldsymbol{H}^1_0(\Omega) \mapsto X_{D,0}$ and $\widetilde{I}_D : L^2_0(\Omega) \mapsto Y_{D,0}$ defined by

$$\begin{aligned} I_D\varphi &= \operatorname*{argmin}_{v\in X_{D,0}} \Big(\|\Pi_D v - \varphi\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla_D v - \nabla\varphi\|_{\boldsymbol{L}^2(\Omega)^d} + \|\operatorname{div}_D v - \operatorname{div}\varphi\|_{L^2(\Omega)}\Big)\ , \\ \widetilde{I}_D\psi &= \operatorname*{argmin}_{z\in Y_{D,0}} \|\chi_D z - \psi\|_{L^2(\Omega)}. \end{aligned} \tag{2.23}$$

We also define $\varepsilon_D(\overline{u}, \overline{p}) = \overline{W}_D(\nabla\overline{u} - G, \overline{p}) + S_D(\overline{u}) + \widetilde{S}_D(\overline{p})$. We may then write

$$\int_\Omega \Big(\eta(\Pi_D I_D\overline{u} - \Pi_D u_D) \cdot \Pi_D v + (\nabla_D I_D\overline{u} - \nabla_D u_D) : \nabla_D v\Big)\mathrm{d}x \\ + \int_\Omega (\chi_D p_D - \chi_D \widetilde{I}_D\overline{p})\ \operatorname{div}_D v\mathrm{d}x \le (1 + (1+\eta)C_D)\varepsilon_D(\overline{u}, \overline{p})\|v\|_D. \tag{2.24}$$

Thanks to Definition 2.2, let us now take $v \in X_{D,0}$ such that $\|v\|_D = 1$ and

$$\int_\Omega \chi_D(p_D - \widetilde{I}_D\overline{p})\ \operatorname{div}_D v\mathrm{d}x \ge \beta_D\|\chi_D(p_D - \widetilde{I}_D\overline{p})\|_{L^2(\Omega)}.$$

We then get, from (2.24),

$$\|\chi_D(p_D - \widetilde{I}_D\overline{p})\|_{L^2(\Omega)} \le \frac{1 + (1+\eta)C_D}{\beta_D}\varepsilon_D(\overline{u}, \overline{p}) + \frac{1 + \eta C_D}{\beta_D}\|I_D\overline{u} - u_D\|_D. \tag{2.25}$$

Choosing $v = I_D\overline{u} - u_D$ in (2.24) and using $\int_\Omega \operatorname{div}_D u_D\ \chi_D q = 0$ for all $q \in Y_{D,0}$, we can write

$$\|I_D\overline{u} - u_D\|_D^2 + \int_\Omega \chi_D(p_D - \widetilde{I}_D\overline{p})\ \operatorname{div}_D I_D\overline{u}\mathrm{d}x \le (1 + (1+\eta)C_D)\varepsilon_D(\overline{u}, \overline{p})\|I_D\overline{u} - u_D\|_D,$$

which implies, since $\mathrm{div}\overline{u} = 0$,

$$\|I_D\overline{u} - u_D\|_D^2 \le (1 + (1+\eta)C_D)\varepsilon_D(\overline{u},\overline{p})\|I_D\overline{u} - u_D\|_D + S_D(\overline{u})\|\chi_D(p_D - \widetilde{I}_D\overline{p})\|_{L^2(\Omega)}.$$

Thanks to (2.25) and to the Young inequality $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$, the above estimate yields the existence of C_3, non-decreasing w.r.t. $1/\beta_D$, C_D and η, such that $\|I_D\overline{u} - u_D\|_D \le C_3\varepsilon_D(\overline{u},\overline{p})$. The conclusion follows from the definitions of $I_D\overline{u}$, $\widetilde{I}_D\overline{p}$ and C_D, the triangle inequality and (2.25).

Proof (of Theorem 2.13). The proof follows the same idea as the proof of Theorem 2.11, but considering only functions $v \in E_D$. For such functions, owing to (2.10), Equation (2.22) can be written

$$\int_\Omega \Big(\eta(\overline{u} - \Pi_D u_D)\cdot\Pi_D v + (\nabla\overline{u} - \nabla_D u_D) : \nabla_D v\Big)\mathrm{d}x \le \overline{W}_D(\nabla\overline{u} - G, \overline{p})\|v\|_D. \tag{2.26}$$

We introduce $I_{D,E_D} : E(\Omega) \mapsto E_D$ defined by

$$I_{D,E_D}\varphi = \operatorname*{argmin}_{v\in E_D}\Big(\|\Pi_D v - \varphi\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla_D v - \nabla\varphi\|_{\boldsymbol{L}^2(\Omega)^d}\Big)$$

and from (2.26) we deduce, by definition of S_{D,E_D},

$$\int_\Omega \Big(\eta(\Pi_D I_{D,E_D}\overline{u} - \Pi_D u_D)\cdot\Pi_D v + (\nabla_D I_{D,E_D}\overline{u} - \nabla_D u_D) : \nabla_D v\Big)\mathrm{d}x$$
$$\le \Big((\eta C_D + 1)S_{D,E_D}(\overline{u}) + \overline{W}_D(\nabla\overline{u} - G, \overline{p})\Big)\|v\|_D.$$

Choosing $v = I_{D,E_D}\overline{u} - u_D \in E_D$ leads to

$$\|\nabla_D I_{D,E_D}\overline{u} - \nabla_D u_D\|_{\boldsymbol{L}^2(\Omega)^d} \le (\eta C_D + 1)S_{D,E_D}(\overline{u}) + \overline{W}_D(\nabla\overline{u} - G, \overline{p}) \tag{2.27}$$

and the proof of (2.11) is complete since $\|\nabla_D I_{D,E_D}\overline{u} - \nabla\overline{u}\|_{\boldsymbol{L}^2(\Omega)^d} \le S_{D,E_D}(\overline{u})$. Estimate (2.12) follows from the definition of C_D, (2.27) and $\|\Pi_D I_{D,E_D}\overline{u} - \overline{u}\|_{\boldsymbol{L}^2(\Omega)^d} \le S_{D,E_D}(\overline{u})$.

Proof (of Theorem 2.15). The proof starts from the definition of W_D (Definition 2.5). Since $\nabla\overline{u} - G \in \boldsymbol{H}_{\mathrm{div}}(\Omega)$ and $\mathrm{div}(\nabla\overline{u} - G) = \nabla\overline{p} + \eta\overline{u} - f$, we can write, for any $v \in E_D$,

$$\int_\Omega \Big(\nabla_D v : (\nabla\overline{u} - G) + \Pi_D v\cdot(\nabla\overline{p} + \eta\overline{u} - f)\Big)\mathrm{d}x \le W_D(\nabla\overline{u} - G)\|v\|_D.$$

Owing to Assumption (2.13) and since $v \in E_D$, we can remove the term $\nabla\overline{p}$. Using the gradient scheme (2.7) to replace the terms involving f and G, we deduce

$$\int_\Omega (\nabla_D v : (\nabla \overline{u} - \nabla_D u) + \eta \Pi_D v \cdot (\overline{u} - \Pi_D u))\,\mathrm{d}x \leq W_D(\nabla \overline{u} - G)\|v\|_D.$$

Hence, (2.26) is satisfied with $W_D(\nabla\overline{u} - G)$ instead of $\overline{W}_D(\nabla\overline{u} - G, \overline{p})$, and the conclusion follows as in the proof of Theorem 2.13.

2.4.2 Transient problem

The existence and uniqueness of the solution to the gradient scheme for the transient Stokes problem is a straightforward consequence of the study of the gradient scheme for the steady problem.

Lemma 2.24 (Existence and Uniqueness of the discrete solution)**.** *Under Hypothesis (2.16), let D be a space-time discretisation in the sense of Definition 2.7. Then there exists a unique solution (u_D, p_D) to the gradient scheme (2.20).*

Proof. We remark that the equation on $(u^{(n+1)}, p^{(n+1)})$ in (2.20) is the gradient discretisation (2.7) of the steady Stokes problem, with $\eta = \delta t^{n+\frac{1}{2}}$ and a right-hand side depending on $u^{(n)}$. Existence and uniqueness of the solution therefore follows from Theorem 2.11.

Let us now establish some *a priori* estimates on the solution to the scheme.

Lemma 2.25 (Estimates)**.** *Under Hypotheses (2.16), let D be a space-time discretisation in the sense of Definition 2.7 and let (u_D, p_D) be the solution to Scheme (2.20). Then, for all $m = 0, \ldots, N$,*

$$\int_0^{t^{(m)}} \int_\Omega |\nabla_D u_D|^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_\Omega \left(\left|\Pi_D u_D^{(m)}\right|^2 - \left|\Pi_D u_D^{(0)}\right|^2 \right) \mathrm{d}x \leq \int_0^{t^{(m)}} \int_\Omega f \cdot \Pi_D u_D \mathrm{d}x\mathrm{d}t + \int_0^{t^{(m)}} \int_\Omega G : \nabla_D u_D \mathrm{d}x\mathrm{d}t. \quad (2.28)$$

Moreover, if $C_4 > 0$ is such that $C_4 \geq \|\Pi_D u^{(0)}\|_{L^2(\Omega)}$, then there exist $C_5 \geq 0$ only depending on Ω, d, C_4, f, G and C_D such that

$$\|\Pi_D u_D\|_{L^\infty(0,T,\boldsymbol{L}^2(\Omega))} + \|\nabla_D u_D\|_{\boldsymbol{L}^2(\Omega\times(0,T))^d} \leq C_5. \quad (2.29)$$

Proof. Putting $v = \delta t^{n+\frac{1}{2}} u_D^{(n+1)}$ and $q = p_D^{(n+1)}$ in (2.20) we get

$$\int_\Omega \left(\Pi_D u_D^{(n+1)} - \Pi_D u_D^{(n)}\right) \cdot \Pi_D u_D^{(n+1)} \mathrm{d}x + \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega |\nabla_D u_D^{(n+1)}|^2 \mathrm{d}x\mathrm{d}t = \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_D u_D^{(n+1)} \mathrm{d}x\mathrm{d}t + \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega G : \nabla_D u_D^{(n+1)} \mathrm{d}x\mathrm{d}t.$$

Using the inequality $(a-b)\cdot a \geq \frac{1}{2}(|a|^2 - |b|^2)$ (valid for any $a, b \in \mathbb{R}^d$) on the first term, it comes :

$$\frac{1}{2}\int_\Omega \left(\left|\Pi_D u_D^{(n+1)}\right|^2 - \left|\Pi_D u_D^{(n)}\right|^2\right) \mathrm{d}x + \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega |\nabla_D u_D^{(n+1)}|^2 \mathrm{d}x\mathrm{d}t \leq \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_D u_D^{(n+1)} \mathrm{d}x\mathrm{d}t + \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega G : \nabla_D u_D^{(n+1)} \mathrm{d}x\mathrm{d}t.$$

We take $m \in \{0, \ldots, N\}$ and sum the obtained equation over $n = 0, \ldots, m-1$. This gives

$$\frac{1}{2}\|\Pi_D u_D^{(m)}\|^2_{\boldsymbol{L}^2(\Omega)} - \frac{1}{2}\|\Pi_D u^{(0)}\|^2_{\boldsymbol{L}^2(\Omega)} + \int_0^{t^{(m)}} \|\nabla_D u_D\|^2_{\boldsymbol{L}^2(\Omega)^d} \mathrm{d}t \leq \int_0^{t^{(m)}} \int_\Omega f \cdot \Pi_D u_D \mathrm{d}x\mathrm{d}t + \int_0^{t^{(m)}} \int_\Omega G : \nabla_D u_D \mathrm{d}x\mathrm{d}t$$

and the proof of (2.28) is complete. Estimate (2.29) is a straightforward consequence of the definition of C_D and Young's inequality applied to (2.28) with $m = N$.

Definition 2.26. *The semi-norm* $|\cdot|_{*,D}$ *is defined on* $X_{D,0}$ *by*

$$|w|_{*,D} = \sup\left\{\int_\Omega \Pi_D w \cdot \Pi_D v \mathrm{d}x \ : v \in E_D, \|v\|_D = 1\right\},$$

where we recall that E_D *is defined by* (2.9).

Lemma 2.27 (Estimates on $|\delta_D u_D|_{*,D}$). *Under Hypotheses* (2.16)*, let* D *be a space-time discretisation in the sense of Definition 2.7, and let* (u_D, p_D) *be the solution to Scheme* (2.20)*. We take* $C_4 \geq \|\Pi_D u^{(0)}\|_{L^2(\Omega)}$*. Then there exist* $C_6 \geq 0$ *only depending on* Ω*,* d*,* C_4*,* f*,* G *and* C_D *such that*

$$\int_0^T |\delta_D u_D|^2_{*,D} \mathrm{d}t \leq C_6. \tag{2.30}$$

Proof. Taking a generic $v \in E_D$ in Scheme (2.20) and using the definition of $|\cdot|_{*,D}$ gives an estimate on $|\delta_D^{n+\frac{1}{2}} u_D|_{*,D}$ depending on $||\nabla_D u_D^{(n+1)}||_{\boldsymbol{L}^2(\Omega)^d}$. Squaring this estimate, multiplying by $\delta t^{n+\frac{1}{2}}$ and summing over n gives the desired estimate, thanks to (2.29).

We can now prove the first convergence result of gradient schemes for the transient Stokes problem.

Proof (of Theorem 2.21). .
Step 1 : Existence of a weak limit of a subsequence of approximations
Estimate (2.29) gives the existence of $\overline{u} \in L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ and $\zeta \in \boldsymbol{L}^2(\Omega\times(0,T))^d$ such that, up to a subsequence (still indexed by m), $\Pi_{D_m} u_{D_m} \rightharpoonup \overline{u}$ weakly-$*$ in $L^\infty(0,T;\boldsymbol{L}^2(\Omega))$ and $\nabla_{D_m} u_{D_m} \rightharpoonup \zeta$ weakly in $\boldsymbol{L}^2(\Omega\times(0,T))^d$. Taking $\varphi \in C^\infty(\overline{\Omega})^d$ and $\theta \in C_c^\infty(0,T)$, we then see that, for all $n = 0,\ldots,N-1$, and all $t \in (t^{(n)}, t^{(n+1)})$,

$$\int_\Omega \Big(\nabla_{D_m} u_m^{(n+1)} : \varphi\,\theta + \Pi_{D_m} u_m^{(n+1)} \cdot \mathrm{div}\varphi\,\theta\Big)\mathrm{d}x \le \overline{W}_{D_m}(\varphi,0)\,\theta\,||\nabla_{D_m} u_{D_m}||_{\boldsymbol{L}^2(\Omega)^d}.$$

Integrating this over $t \in (t^{(n)}, t^{(n+1)})$, summing on $n = 0,\ldots,N-1$ and using Estimate (2.29), we find C_7 not depending on m such that

$$\int_0^T\int_\Omega \Big(\nabla_{D_m} u_{D_m} : \varphi\,\theta + \Pi_{D_m} u_{D_m} \cdot \mathrm{div}\varphi\,\theta\Big)\mathrm{d}x\mathrm{d}t \le C_7\overline{W}_{D_m}(\varphi,0).$$

We can then pass to the supremum limit as $m \to \infty$ and apply the resulting inequality to $\pm\varphi$ to see that

$$\int_0^T\int_\Omega \Big(\zeta : (\varphi\,\theta) + \overline{u} \cdot \mathrm{div}(\varphi\,\theta)\Big)\mathrm{d}x\mathrm{d}t = 0.$$

This relation first shows, with $\varphi \in C_c^\infty(\Omega)^d$, that $\zeta = \nabla\overline{u}$, and therefore that $\overline{u} \in L^2(0,T;\boldsymbol{H}^1(\Omega))$. Taking then φ which does not vanish on $\partial\Omega$, we also infer that the trace of $\overline{u}$ on $\partial\Omega$ is zero and therefore that $\overline{u} \in L^2(0,T;\boldsymbol{H}_0^1(\Omega))$.
Let us now prove that $\overline{u}$ is divergence-free. From Estimate (2.29) and the coercivity of the sequence of gradient discretisations, we see that $(\mathrm{div}_{D_m} u_{D_m})_{m\in\mathbb{N}}$ (with the same space-time notations as in (2.21)) is bounded in $L^2(\Omega\times(0,T))$ and therefore converges weakly in this space, up to a subsequence, to some λ. Taking $\psi \in H^1(\Omega)$ with zero mean value and $\theta \in C^\infty(0,T)$, we have, for any $n = 0,\ldots,N-1$ and any $t \in (t^{(n)}, t^{(n+1)})$,

$$\int_\Omega \Big(\Pi_{D_m} u_{D_m}^{(n+1)} \cdot \nabla\psi\,\theta + \mathrm{div}_{D_m} u_{D_m}^{(n+1)} \psi\,\theta\Big)\mathrm{d}x \le \overline{W}_{D_m}(0,\psi)\,\theta||\nabla_{D_m} u_{D_m}||_{\boldsymbol{L}^2(\Omega)^d}.$$

As above, we integrate this over t and sum over n to find

$$\int_0^T\int_\Omega \Big(\Pi_{D_m} u \cdot \nabla\psi\,\theta + \mathrm{div}_{D_m} u_{D_m}\,\psi\,\theta\Big)\mathrm{d}x\mathrm{d}t \le C_8\overline{W}_{D_m}(0,\psi)$$

with C_8 not depending on m. Using the last equation in the gradient scheme (2.20), we can introduce $\chi_{D_m}\widetilde{I}_{D_m}\psi$ in the second term of the left-hand side (where $\widetilde{I}_{D_m}$ is defined as in (2.23)) and we get

$$\int_0^T\int_\Omega \Pi_{D_m}u\cdot\nabla\psi\,\theta\mathrm{d}x\mathrm{d}t \leq ||\mathrm{div}_{D_m}u_{D_m}||_{L^2(\Omega\times(0,T))}||\chi_{D_m}\widetilde{I}_{D_m}\psi-\psi||_{L^2_0(\Omega)}||\theta||_{L^2(0,T)}+C_8\overline{W}_{D_m}(0,\psi).$$

Passing to the supremum limit $m\to\infty$, thanks to the limit-conformity and the consistency of the gradient discretisations, and applying the resulting inequality to $\pm\psi$, we deduce that $\int_0^T\int_\Omega \overline{u}\cdot\nabla\psi\theta\mathrm{d}x\mathrm{d}t = 0$. This relation is true for any $\psi\in H^1(\Omega)$ with zero mean value, and hence also for any function in $H^1(\Omega)$. The proof that $\mathrm{div}\overline{u}=0$ is therefore complete.

Step 2 : : $\overline{u}$ is the solution to (2.17).
To simplify notations, we drop the indices m. Let $\theta\in C_c^\infty([0,T))$ and let $w\in E(\Omega)$. As $(w,0)$ is the solution of the incompressible steady Stokes problem (Problem (2.1)) with $f=\eta w$ and $G=\nabla w$, we can find $w_D\in X_{D,0}$ such that $\int_\Omega\chi_D q\mathrm{div}_D w_D=0$ for all $q\in Y_{D,0}$, $\Pi_D w_D\to w$ in $\boldsymbol{L}^2(\Omega)$ and $\nabla_D w_D\to\nabla w$ in $\boldsymbol{L}^2(\Omega)^d$ (Theorem 2.11 and Remark 2.12). We take $v=\delta t^{(n+\frac12)}\theta(t^{(n)})w_D$ as test function in Scheme (2.20) and we sum the resulting equation on $n=0,...,N-1$ to get $T_1+T_2+T_3=T_4$ with

$$\begin{aligned}
T_1 &= \sum_{n=0}^{N-1}\delta t^{n+\frac12}\theta(t^{(n)})\int_\Omega \Pi_D\delta^{(n+\frac12)}u_D\cdot\Pi_D w_D\mathrm{d}x,\\
T_2 &= \sum_{n=0}^{N-1}\delta t^{n+\frac12}\theta(t^{(n)})\int_\Omega \nabla_D u_D^{(n+1)}:\nabla_D w_D\mathrm{d}x,\\
T_3 &= -\sum_{n=0}^{N-1}\delta t^{n+\frac12}\theta(t^{(n)})\int_\Omega \chi_D p_D^{(n+1)}\mathrm{div}_D w_D\mathrm{d}x,\\
T_4 &= \sum_{n=0}^{N-1}\theta(t^{(n)})\int_{t^{(n)}}^{t^{(n+1)}}\int_\Omega f\cdot\Pi_D w_D\mathrm{d}x\mathrm{d}t+\sum_{n=0}^{N-1}\theta(t^{(n)})\int_{t^{(n)}}^{t^{(n+1)}}\int_\Omega G:\nabla_D w_D\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

First, we remark that $T_3=0$ since $\int_\Omega\chi_D q\mathrm{div}_D w_D=0$ for all $q\in Y_{D,0}$. Using discrete integration by parts and writing $\theta(t^{(n+1)})-\theta(t^{(n)})=\int_{t^{(n)}}^{t^{(n+1)}}\theta'$, we find

$$T_1=-\int_0^T\int_\Omega\theta'\Pi_D u_D\cdot\Pi_D w_D\mathrm{d}x\mathrm{d}t-\theta(0)\int_\Omega\Pi_D u_D^{(0)}\cdot\Pi_D w_D\mathrm{d}x.$$

Recall that $u_D^{(0)}=J_D u_{\mathrm{ini}}$, so that the space-time consistency (Definition 2.8) gives $\Pi_D u_D^{(0)}\to u_{\mathrm{ini}}$

in $\boldsymbol{L}^2(\Omega)$ as $m \to \infty$. Thus, by strong convergence in $\boldsymbol{L}^2(\Omega)$ of $\Pi_D w_D$ to w,

$$T_1 \to -\int_0^T \int_\Omega \theta' \overline{u} \cdot w \mathrm{d}x\mathrm{d}t - \theta(0) \int_\Omega \overline{u}_{\mathrm{ini}} \cdot w \mathrm{d}x.$$

It also comes easily, using the regularity of θ and Estimate (2.29), that

$$T_2 \to \int_0^T \theta \int_\Omega \nabla\overline{u} : \nabla w \mathrm{d}x\mathrm{d}t \quad \text{and} \quad T_4 \to \int_0^T \theta \int_\Omega f \cdot w \mathrm{d}x\mathrm{d}t + \int_0^T \theta \int_\Omega G : \nabla w \mathrm{d}x\mathrm{d}t.$$

Passing to the limit in $T_1 + T_2 + T_3 = T_4$ concludes the proof that $\overline{u}$ satisfies (2.17).

Step 3 : convergence in $L^\infty(0, T; \boldsymbol{L}^2(\Omega))$.
The proof of the uniform-in-time convergence follows the same idea as in [39]. Using Lemma 2.28 below and the generalised Ascoli-Arzela theorem [39, Theorem 6.2], we see that for any $\varphi \in E(\Omega)$ the sequence $(\langle \Pi_{D_m} u_{D_m}(\cdot), \varphi \rangle_{\boldsymbol{L}^2})_{m\in\mathbb{N}}$ is relatively compact in $L^\infty(0,T)$. Since this sequence already converges weakly in $L^2(0,T)$ towards $\langle \overline{u}(\cdot), \varphi \rangle_{\boldsymbol{L}^2}$ (because $\Pi_{D_m} u_{D_m} \to \overline{u}$ weakly in $\boldsymbol{L}^2(\Omega \times (0,T))$), we deduce that $\langle \Pi_{D_m} u_{D_m}(\cdot), \varphi \rangle_{\boldsymbol{L}^2} \to \langle \overline{u}(\cdot), \varphi \rangle_{\boldsymbol{L}^2}$ uniformly with respect to t as $m \to \infty$.

Let $\psi \in \boldsymbol{L}^2(\Omega)$ and $\varphi \in E(\Omega)$. By Estimate (2.29), for any $t \in [0,T]$,

$$|\langle \Pi_{D_m} u_{D_m}(t), \psi \rangle_{\boldsymbol{L}^2} - \langle \overline{u}(t), \psi \rangle_{\boldsymbol{L}^2}| \le |\langle \Pi_{D_m} u_{D_m}(t), \varphi \rangle_{\boldsymbol{L}^2} - \langle \overline{u}(t), \varphi \rangle_{\boldsymbol{L}^2}| + C||\varphi - \psi||_{\boldsymbol{L}^2(\Omega)}$$

where C does not depend on m, t, φ or ψ. Assuming that ψ can be approximated in $\boldsymbol{L}^2(\Omega)$ by functions in $E(\Omega)$ (see [100, Ch. I, Theorem 1.4] for a characterisation of such functions ψ), then the preceding estimate and the uniform-in-time convergence of $(\langle \Pi_{D_m} u_{D_m}(\cdot), \varphi \rangle_{\boldsymbol{L}^2})_{m\in\mathbb{N}}$ show that $\langle \Pi_{D_m} u_{D_m}(\cdot), \psi \rangle_{\boldsymbol{L}^2} \to \langle \overline{u}(\cdot), \psi \rangle_{\boldsymbol{L}^2}$ uniformly-in-time as $m \to \infty$. It is known (see Proposition 2.20) that, for any $T_0 \in [0,T]$, $\overline{u}(T_0)$ can be approximated in $\boldsymbol{L}^2(\Omega)$ by functions in $E(\Omega)$; hence, we can apply the preceding result to $\psi = \overline{u}(T_0)$. This allows us to see that, for any $(s_m)_{m\in\mathbb{N}}$ converging to T_0,

$$\|\overline{u}(T_0)\|^2_{\boldsymbol{L}^2(\Omega)} = \lim_{m\to\infty} \langle \overline{u}(s_m), \overline{u}(T_0) \rangle_{\boldsymbol{L}^2} = \lim_{m\to\infty} \langle \Pi_{D_m} u_{D_m}(s_m), \overline{u}(T_0) \rangle_{\boldsymbol{L}^2} \tag{2.31}$$

(we used the continuity of $\overline{u} : [0,T] \mapsto \boldsymbol{L}^2(\Omega)$, see (2.19)). Thus, it comes

$$\|\overline{u}(T_0)\|_{\boldsymbol{L}^2(\Omega)} \le \liminf_{m\to\infty} \|\Pi_{D_m} u_{D_m}(s_m)\|_{\boldsymbol{L}^2(\Omega)}. \tag{2.32}$$

Let $k(m)$ such that $s_m \in (t^{(k(m)-1)}, t^{(k(m))}]$, where $(t^l)_l$ are the time steps of the discretisation. Definition (2.21) gives $\Pi_{D_m} u_{D_m}(s_m) = \Pi_{D_m} u^{(k(m))}_{D_m}$. The discrete energy estimate (2.28) therefore leads to

$$\frac{1}{2}\int_\Omega |\Pi_{D_m}u_{D_m}(s_m)|^2 \mathrm{d}x \leq \frac{1}{2}\int_\Omega |\Pi_{D_m}u^{(0)}_{D_m}|^2 \mathrm{d}x - \int_0^{t^{(k(m))}}\int_\Omega |\nabla_{D_m}u_{D_m}|^2 \mathrm{d}x\mathrm{d}t \\ + \int_0^{t^{(k(m))}}\int_\Omega f\cdot \Pi_{D_m}u_{D_m}\mathrm{d}x\mathrm{d}t + \int_0^{t^{(k(m))}}\int_\Omega G : \nabla_{D_m}u_{D_m}\mathrm{d}x\mathrm{d}t. \quad (2.33)$$

We notice, by weak convergence in $\boldsymbol{L}^2(\Omega\times(0,T))^d$ of $\nabla_{D_m}u_{D_m}$ to $\nabla\overline{u}$ and strong convergence in $\boldsymbol{L}^2(\Omega\times(0,T))^d$ of $\mathbf{1}_{[0,t^{(k(m))}]}\nabla\overline{u}$ toward $\mathbf{1}_{[0,T]}\nabla\overline{u}$ (notice that $t^{(k(m))}\to T_0$), where $\mathbf{1}_A$ is the characteristic function of A,

$$\begin{aligned}
\int_0^{T_0}\int_\Omega |\nabla\overline{u}|^2\mathrm{d}x\mathrm{d}t &= \int_0^T\int_\Omega \mathbf{1}_{[0,T_0]}\nabla\overline{u} : \nabla\overline{u}\mathrm{d}x\mathrm{d}t \\
&= \lim_{m\to\infty}\int_0^T\int_\Omega \mathbf{1}_{[0,t^{(k(m))}]}\nabla\overline{u} : \nabla_{D_m}u_{D_m}\mathrm{d}x\mathrm{d}t \\
&\leq \liminf_{m\to\infty}\left(\left|\left|\mathbf{1}_{[0,t^{(k(m))}]}\nabla\overline{u}\right|\right|_{\boldsymbol{L}^2(\Omega\times(0,T))^d}\left|\left|\mathbf{1}_{[0,t^{(k(m))}]}\nabla_{D_m}u_{D_m}\right|\right|_{\boldsymbol{L}^2(\Omega\times(0,T))^d}\right) \\
&= \left|\left|\mathbf{1}_{[0,T_0]}\nabla\overline{u}\right|\right|_{\boldsymbol{L}^2(\Omega\times(0,T))^d}\liminf_{m\to\infty}\left|\left|\mathbf{1}_{[0,t^{(k(m))}]}\nabla_{D_m}u_{D_m}\right|\right|_{\boldsymbol{L}^2(\Omega\times(0,T))^d}.
\end{aligned}$$

Hence,

$$\int_0^{T_0}\int_\Omega |\nabla\overline{u}|^2\mathrm{d}x\mathrm{d}t \leq \liminf_{m\to\infty}\int_0^{t^{(k(m))}}|\nabla_{D_m}u_{D_m}|^2\mathrm{d}x\mathrm{d}t$$

and we can pass to the limit superior in (2.33), using the weak convergences of $\Pi_{D_m}u_{D_m}$ and $\nabla_{D_m}u_{D_m}$:

$$\limsup_{m\to\infty}\frac{1}{2}\|\Pi_{D_m}u_{D_m}(s_m)\|^2_{\boldsymbol{L}^2(\Omega)} \leq \frac{1}{2}\int_\Omega |\overline{u}_{\mathrm{ini}}|^2\mathrm{d}x - \int_0^{T_0}\int_\Omega |\nabla\overline{u}|^2\mathrm{d}x\mathrm{d}t \\ + \int_0^{T_0}\int_\Omega f\cdot\overline{u}\mathrm{d}x\mathrm{d}t + \int_0^{T_0}\int_\Omega G : \nabla\overline{u}\mathrm{d}x\mathrm{d}t. \quad (2.34)$$

Plugging $\varphi = \overline{u}\mathbf{1}_{[0,T_0]}$ in Problem (2.19) and integrating by parts, we obtain the continuous energy estimate

$$\frac{1}{2}\|\overline{u}(T_0)\|^2_{\boldsymbol{L}^2(\Omega)} = \frac{1}{2}\int_\Omega |\overline{u}_{\mathrm{ini}}|^2\mathrm{d}x - \int_0^{T_0}\int_\Omega |\nabla\overline{u}|^2\mathrm{d}x\mathrm{d}t + \int_0^{T_0}\int_\Omega f\cdot\overline{u}\mathrm{d}x\mathrm{d}t + \int_0^{T_0}\int_\Omega G : \nabla\overline{u}\mathrm{d}x\mathrm{d}t. \quad (2.35)$$

Combined with (2.34), this leads to $\limsup_{m\to\infty}\|\Pi_{D_m}u_{D_m}(s_m)\|^2_{\boldsymbol{L}^2(\Omega)} \leq \|\overline{u}(T_0)|^2_{\boldsymbol{L}^2(\Omega)}$. Using (2.32), we deduce $\lim_{m\to\infty}\|\Pi_{D_m}u_{D_m}(s_m)\|^2_{\boldsymbol{L}^2(\Omega)} = \|\overline{u}(T_0)\|^2_{\boldsymbol{L}^2(\Omega)}$. Recalling (2.31), this allows us to conclude that $||\Pi_{D_m}u_{D_m}(s_m) - \overline{u}(T_0)||^2_{\boldsymbol{L}^2(\Omega)} \to 0$ (just develop the square). Since $\overline{u} : [0,T] \mapsto \boldsymbol{L}^2(\Omega)$ is continuous, we can apply [39, Lemma 5.1] and finally get that $\Pi_{D_m}u_{D_m} \to \overline{u}$ in $\boldsymbol{L}^2(\Omega)$

uniformly-in-time.

Step 4 : strong convergence of $\nabla_{D_m} u_{D_m}$.
We write the discrete energy estimate (2.28) with $t^{(m)} = T$ and use the convergence in $\boldsymbol{L}^2(\Omega)^d$ of $\Pi_{D_m} u_{D_m}(T)$ to $\overline{u}(T)$ to see that

$$\limsup_{m\to\infty} ||\nabla_{D_m} u_{D_m}||^2_{\boldsymbol{L}^2(\Omega\times(0,T))^d} \le \frac{1}{2}\int_\Omega (|\overline{u}(T)|^2 - |\overline{u}_{\rm ini}|^2)\mathrm{d}x - \int_0^T \int_\Omega |\nabla\overline{u}|^2 \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega f\cdot\overline{u}\mathrm{d}x\mathrm{d}t + \int_0^{T_0}\int_\Omega G:\nabla\overline{u}\mathrm{d}x\mathrm{d}t.$$

The energy estimate (2.35) with $T_0 = T$ then shows that

$$\limsup_{m\to\infty} ||\nabla_{D_m} u_{D_m}||^2_{\boldsymbol{L}^2(\Omega\times(0,T))^d} \le ||\nabla\overline{u}||^2_{\boldsymbol{L}^2(\Omega\times(0,T))^d},$$

which allows us to conclude that the weak convergence of $\nabla_{D_m} u_{D_m}$ to $\nabla\overline{u}$ in $\boldsymbol{L}^2(\Omega\times(0,T))^d$ is in fact strong.

The following lemma was the initial key to obtain the uniform-in-time convergence result in the previous proof (Step 3).

Lemma 2.28. *Under the assumptions and notations of Theorem 2.21, for all $\varphi\in E(\Omega)$ the sequence of functions $t\mapsto \langle \Pi_{D_m} u_{D_m}(t),\varphi\rangle_{\boldsymbol{L}^2}$ satisfies the following quasi-equi-continuity property : there exist C_9, not depending on m, and a sequence of real numbers $(\omega_{\varphi,D_m})_{m\in\mathbb{N}}$ converging to 0 such that, for all $t,s\in[0,T]$,*

$$|\langle \Pi_{D_m} u_{D_m}(t) - \Pi_{D_m} u_{D_m}(s),\varphi\rangle_{\boldsymbol{L}^2}| \le C_9|t-s|^{\frac{1}{2}} + \omega_{\varphi,D_m}.$$

Proof. As in the proof of Theorem 2.21, we drop the index m. Let $\varphi\in E(\Omega)$ and, as in Step 2 of the proof of Theorem 2.21, consider the solution φ_D to the steady gradient scheme (2.7) with $f=\eta\varphi$ and $G=\nabla\varphi$; then $\Pi_D\varphi\to\varphi$ in $\boldsymbol{L}^2(\Omega)$ and $\nabla_D\varphi_D\to\nabla\varphi$ in $\boldsymbol{L}^2(\Omega)^d$. Since $\varphi_D\in E_D$, the definition 2.26 of $|\cdot|_{*,D}$ gives

$$|\langle \Pi_D u_D(t) - \Pi_D u_D(s), \Pi_D\varphi_D\rangle_{\boldsymbol{L}^2}| = \left|\sum_{\substack{n \text{ s.t.}\\ s\le t^{(n)}\le t}} \delta t^{(n+\frac{1}{2})}\int_\Omega \Pi_D\delta_D^{n+\frac{1}{2}}u_D\cdot\Pi_D\varphi_D\mathrm{d}x\right| \le \sum_{\substack{n \text{ s.t.}\\ s\le t^{(n)}\le t}} \delta t^{(n+\frac{1}{2})}|\delta_D^{n+\frac{1}{2}}u_D|_{*,D}||\varphi_D||_D.$$

Since $\nabla_D \varphi_D \to \nabla \varphi$ in $\boldsymbol{L}^2(\Omega)^d$, $\|\varphi_D\|_D$ is bounded and so, using Cauchy-Schwarz inequality, we may write

$$|\langle \Pi_D u_D(t) - \Pi_D u_D(s), \Pi_D \varphi_D \rangle_{\boldsymbol{L}^2}| \leq C \left(\int_0^T |\delta_D u_D|_{*D}^2 \mathrm{d}t \right)^{\frac{1}{2}} (|t-s| + \delta t)^{\frac{1}{2}}$$

with C not depending on D. Finally, thanks to Lemma 2.27, we infer

$$|\langle \Pi_D u_D(t) - \Pi_D u_D(s), \Pi_D \varphi_D \rangle_{\boldsymbol{L}^2}| \leq C \left(|t-s| + \delta t\right)^{\frac{1}{2}}.$$

We then write, using Estimate (2.29),

$$\begin{aligned} |\langle \Pi_D u_D(t) - \Pi_D u_D(s), \varphi \rangle_{\boldsymbol{L}^2}| &\leq |\langle \Pi_D u_D(t) - \Pi_D u_D(s), \varphi - \Pi_D \varphi_D \rangle_{\boldsymbol{L}^2}| \\ &\quad + |\langle \Pi_D u_D(t) - \Pi_D u_D(s), \Pi_D \varphi_D \rangle_{\boldsymbol{L}^2}| \\ &\leq C \|\varphi - \Pi_D \varphi_D\|_{\boldsymbol{L}^2(\Omega)} + C \left(|t-s| + \delta t\right)^{\frac{1}{2}} \\ &\leq C_9 |t-s|^{\frac{1}{2}} + C \delta t^{\frac{1}{2}} + C \|\varphi - \Pi_D \varphi_D\|_{\boldsymbol{L}^2(\Omega)}. \end{aligned}$$

The limit-conformity of the sequence of gradient discretisations ensures that $\omega_{\varphi,D} := C\delta t^{\frac{1}{2}} + C\|\varphi - \Pi_D \varphi_D\|_{\boldsymbol{L}^2(\Omega)}$ tends to 0, and the proof is complete.

Let us now turn to the proof of the convergence of the pressure (Theorem 2.23).

Lemma 2.29 (Estimates on discrete time-derivative of velocity)**.** *Under the assumptions of Theorem 2.23, let (u_D, p_D) be the solution to Scheme (2.20). Let $R \geq C_D + \|J_D u_{\mathrm{ini}}\|_D$. Then there exists $C_{10} \geq 0$ only depending on Ω, d, R, f and G such that*

$$\|\Pi_D \delta_D^{n+\frac{1}{2}} u_D\|_{\boldsymbol{L}^2(\Omega \times (0,T))} \leq C_{10}. \tag{2.36}$$

Proof. Put $v = \delta_D^{n+\frac{1}{2}} u_D$ and $q = p_D^{(n+1)}$ in Scheme (2.20). Since $u_D^{(0)} = J_D \overline{u}_{\mathrm{ini}} \in E_D$ we have $v \in E_D$ even if $n = 0$ and therefore

$$\|\Pi_D \delta_D^{n+\frac{1}{2}} u_D\|_{\boldsymbol{L}^2(\Omega)}^2 + \int_\Omega \nabla_D u_D^{(n+1)} : \nabla_D \delta_D^{n+\frac{1}{2}} u_D \mathrm{d}x = \frac{1}{\delta t^{n+\frac{1}{2}}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_D \delta_D^{n+\frac{1}{2}} u_D \mathrm{d}x \mathrm{d}t \tag{2.37}$$

(recall that $G = 0$ here). On the other hand, using $a : (a - b) = \frac{1}{2}(|a|^2 - |b|^2) + \frac{1}{2}|a - b|^2 \geq \frac{1}{2}(|a|^2 - |b|^2)$ for any tensors a, b,

$$\int_\Omega \nabla_D u_D^{(n+1)} : \nabla_D \delta_D^{n+\frac{1}{2}} u_D \mathrm{d}x \geq \frac{1}{2\delta t^{n+\frac{1}{2}}} \int_\Omega \left(|\nabla_D u_D^{(n+1)}|^2 - |\nabla_D u_D^{(n)}|^2 \right) \mathrm{d}x.$$

Plugging this into (2.37) and multiplying by $\delta t^{n+\frac{1}{2}}$, it comes

$$\int_{t^{(n)}}^{t^{(n+1)}} \|\Pi_D \delta_D^{n+\frac{1}{2}} u_D\|^2_{\boldsymbol{L}^2(\Omega)} \mathrm{d}t + \frac{1}{2}\int_\Omega \left(|\nabla_D u_D^{(n+1)}|^2 - |\nabla_D u_D^{(n)}|^2\right)\mathrm{d}x \le \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_D \delta_D^{n+\frac{1}{2}} u_D \mathrm{d}x\mathrm{d}t.$$

We sum on $n = 0, \ldots, m-1$ for a given $m = 1, \ldots, N$ to get

$$\int_0^{t^{(m)}} \|\Pi_D \delta_D^{n+\frac{1}{2}} u_D\|^2_{\boldsymbol{L}^2(\Omega)} \mathrm{d}t \le \frac{1}{2}\int_\Omega |\nabla_D u_D^{(0)}|^2 \mathrm{d}x + \int_0^{t^{(m)}} \int_\Omega f \cdot \Pi_D \delta_D^{n+\frac{1}{2}} u_D \mathrm{d}x\mathrm{d}t.$$

The Cauchy-Schwarz' and Young's inequalities conclude the proof.

Lemma 2.30 (Estimates on discrete pressure)**.** *Under the assumptions of Theorem 2.23, let* (u_D, p_D) *be the solution to Scheme (2.20). Let* $R \ge C_D + \|J_D u_{\mathrm{ini}}\|_D + \beta_D^{-1}$*. Then there exists* $C_{11} \ge 0$ *only depending on* Ω*,* d*,* R*,* f *and* G *such that*

$$\|\chi_D p_D\|_{L^2(\Omega\times(0,T))} \le C_{11} \tag{2.38}$$

Proof. Let $v \in X_{D,0}$ such that $\|v\|_D = 1$ and $\beta_D \|\chi_D p_D^{(n+1)}\|_{L^2(\Omega)} \le \int_\Omega \chi_D p_D^{(n+1)} \mathrm{div}_D v \mathrm{d}x$ (see the definition of β_D in Definition 2.2). Plugging v in Scheme (2.20), we obtain

$$\begin{aligned}&\beta_D \|\chi_D p_D^{(n+1)}\|_{L^2(\Omega)} \\ &\quad \le \left| \int_\Omega \Pi_D \delta_D^{n+\frac{1}{2}} u_D \cdot \Pi_D v \mathrm{d}x + \int_\Omega \nabla_D u_D^{(n+1)} : \nabla_D v \mathrm{d}x - \frac{1}{\delta t^{n+\frac{1}{2}}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_D v \mathrm{d}x\mathrm{d}t \right|.\end{aligned}$$

Using Cauchy-Schwarz and the discrete Poincaré inequalities, we deduce

$$\beta_D \|\chi_D p_D^{(n+1)}\|_{L^2(\Omega)} \le C_D \|\Pi_D \delta_D^{n+\frac{1}{2}} u_D\|_{\boldsymbol{L}^2(\Omega)} + \|u_D^{(n+1)}\|_D + \frac{1}{\delta t^{n+\frac{1}{2}}} \int_{t^{(n)}}^{t^{(n+1)}} C_D \|f\|_{\boldsymbol{L}^2(\Omega)} \mathrm{d}t.$$

We take the square of this estimate, multiply the result by $\delta t^{n+\frac{1}{2}}$ and sum over n. Estimate (2.29) and (2.36) then show that (2.38) holds.

Proof (of Theorem 2.23). We first apply Theorem 2.21 to get the strong convergence of $\Pi_{D_m} u_{D_m}$ to $\overline{u}$ in $\boldsymbol{L}^2(\Omega \times (0,T))$ and of $\nabla_{D_m} u_{D_m}$ to $\nabla \overline{u}$ in $\boldsymbol{L}^2(\Omega \times (0,T))^d$, where $\overline{u} \in L^2(0,T; E(\Omega))$. Thanks to Estimate (2.38), we can find a function $\overline{p} \in L^2(0,T,L^2_0(\Omega))$ such that, up to a subsequence, $\chi_{D_m} p_{D_m}$ weakly converges to $\overline{p}$ in $L^2(\Omega\times(0,T))$. We then take $\theta \in C_c^\infty([0,T))$, $w \in \boldsymbol{H}^1_0(\Omega)$ and use $v = \delta t^{(n+\frac{1}{2})} \theta(t^{(n)}) I_{D_m} w$ as a test function in Scheme (2.20), were I_D is defined by (2.23). Since $\Pi_{D_m} I_{D_m} w \to w$ in $\boldsymbol{L}^2(\Omega)$, $\nabla_{D_m} I_{D_m} w \to \nabla w$ in $\boldsymbol{L}^2(\Omega)^d$ and $\mathrm{div}_{D_m} I_{D_m} w \to \mathrm{div}(w)$

in $L^2(\Omega)$ as $m \to \infty$, we can pass to the limit in all terms $T_1, \ldots, T_4$. Note that T_3 is no longer equal to 0, but it converges thanks to the weak convergence of $\chi_{D_m} p_{D_m}$ to $\overline{p}$. We then see that $(\overline{u}, \overline{p})$ satisfy

$$\begin{cases} \overline{u} \in L^2(0,T,E(\Omega))\,,\ \overline{p} \in L^2(0,T;L^2_0(\Omega)), \\ \displaystyle\int_0^T \int_\Omega -\overline{u}\cdot\partial_t\varphi \mathrm{d}x\mathrm{d}t + \int_\Omega \overline{u}_{\mathrm{ini}}\cdot\varphi(\cdot,0)\mathrm{d}x + \int_0^T\int_\Omega \nabla\overline{u} : \nabla\varphi \mathrm{d}x\mathrm{d}t \\ \qquad - \displaystyle\int_0^T\int_\Omega \overline{p}\,\mathrm{div}\varphi\mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega f\cdot\varphi\mathrm{d}x\mathrm{d}t,\ \forall\varphi = \theta w \text{ with } \theta \in C_c^\infty([0,T))\,,\ w \in \boldsymbol{H}^1_0(\Omega). \end{cases} \tag{2.39}$$

The density of tensorial functions in $L^2(0,T;\boldsymbol{H}^1_0(\Omega))$ ensures that this relation is actually satisfied for any $\varphi \in L^2(0,T;\boldsymbol{H}^1_0(\Omega))$. It therefore shows that $\partial_t u \in L^2(0,T;\boldsymbol{H}^{-1}(\Omega))$. In combination with the fact that $\overline{u} \in L^2(0,T;\boldsymbol{H}^1_0(\Omega))$, this classical implies $u \in C([0,T];\boldsymbol{L}^2(\Omega))$. This regularity of $\overline{u}$ then allows to perform an integration by parts in order to see that (2.39) is equivalent to (2.18).

Chapitre 3

Incompressible Navier-Stokes equations

3.1 Introduction

The aim of this chapter is to extend the framework of gradient schemes to the incompressible Navier-Stokes problem that we present in its stationary and its transient case in the following equations.

$$\left\{\begin{aligned} \eta\overline{u} - \nu\Delta\overline{u} + (\overline{u}\cdot\nabla)\overline{u} + \nabla\overline{p} &= f - \mathrm{div}(G) && \text{in } \Omega\\ \mathrm{div}\overline{u} &= 0 && \text{in } \Omega\\ \overline{u} &= 0 && \text{on } \partial\Omega \end{aligned}\right. \tag{3.1}$$

and to the transient one

$$\left\{\begin{aligned} \partial_t\overline{u} - \nu\Delta\overline{u} + (\overline{u}\cdot\nabla)\overline{u} + \nabla\overline{p} &= f - \mathrm{div}(G) && \text{in } \Omega\times(0,T)\\ \mathrm{div}\overline{u} &= 0 && \text{in } \Omega\times(0,T)\\ \overline{u} &= 0 && \text{on } \partial\Omega\times(0,T)\\ \overline{u}(\cdot,0) &= u_{\mathrm{ini}} && \text{in } \Omega. \end{aligned}\right. \tag{3.2}$$

where $\eta \geq 0$, $\nu > 0$ is the coefficient of kinematic viscosity, $\overline{u}$ represents the velocity field and $\overline{p}$ is the pressure.

Notations In the following, if F is a vector space we denote by $\boldsymbol{F}$ the space F^d. Thus, $\boldsymbol{L}^2(\Omega) = L^2(\Omega)^d$ and $\boldsymbol{H}_0^1(\Omega) = H_0^1(\Omega)^d$. The space $E(\Omega)$ is the space of fields $v \in \boldsymbol{H}_0^1(\Omega)$ such that $\mathrm{div}(v) = 0$. $L_0^2(\Omega)$ is the space of functions in $L^2(\Omega)$ with a zero mean value over Ω. Finally,

$\boldsymbol{H}_{\text{div}}(\Omega)$ is the space of fields $v \in \boldsymbol{L}^2(\Omega)$ such that $\text{div}(v) \in L^2(\Omega)$.

3.1.1 Weak solution of the steady problem

Our assumptions for the steady incompressible Navier-Stokes problem (3.1) are the following :

$$\begin{aligned} &\Omega \text{ is an open bounded Lipschitz domain of } \mathbb{R}^d \ (1 \leq d \leq 3),\\ &f \in \boldsymbol{L}^2(\Omega) \text{ and } G \in \boldsymbol{L}^2(\Omega)^d. \end{aligned} \tag{3.3}$$

Definition 3.1 (Weak solution to the steady Navier-Stokes problem)**.** *Under Hypotheses (3.3), $(\overline{u}, \overline{p})$ is a weak solution to (3.1) if*

$$\begin{cases} \overline{u} \in \boldsymbol{H}_0^1(\Omega), \overline{p} \in L_0^2(\Omega),\\ \eta \displaystyle\int_\Omega \overline{u} \cdot \bar{v}\, dx + \nu \int_\Omega \nabla \overline{u} : \nabla \bar{v}\, dx + b(\overline{u}, \overline{u}, \bar{v})\\ \qquad - \displaystyle\int_\Omega \overline{p}\, \text{div}\bar{v}\, dx = \int_\Omega (f \cdot \bar{v} + G : \nabla \bar{v})\, dx, \ \forall \bar{v} \in \boldsymbol{H}_0^1(\Omega),\\ \displaystyle\int_\Omega q\, \text{div}\overline{u}\, dx = 0, \ \forall q \in L_0^2(\Omega), \end{cases} \tag{3.4}$$

where "$\cdot$" is the dot product on $\mathbb{R}^d$, if $\xi = (\xi_{i,j})_{i,j=1,\ldots,d} \in \mathbb{R}^{d\times d}$ and $\chi = (\chi_{i,j})_{i,j=1,\ldots,d} \in \mathbb{R}^{d\times d}$, $\xi : \chi = \sum_{i,j=1}^d \xi_{i,j}\chi_{i,j}$ is the doubly contracted product on $\mathbb{R}^{d\times d}$ and

$$b(u, v, w) = \sum_{i,j=1}^d \int_\Omega u_i (\partial_i v_j) w_j\, dx. \tag{3.5}$$

Lemma 3.2 (Properties of b)**.** *Under Hypotheses (3.3), b is a trilinear continuous form on $E(\Omega)^3$ and*

$$b(u, v, v) = 0, \forall u \in E(\Omega),\ v \in \boldsymbol{H}_0^1(\Omega), \tag{3.6}$$

$$b(u, v, w) = -b(u, w, v), \forall u \in E(\Omega),\ (v, w) \in \boldsymbol{H}_0^1(\Omega). \tag{3.7}$$

as it is mentioned in [100, Ch.II, Lemma 1.2 and 1.3]

Remark 3.3. *Under Hypothese (3.3), the existence of a weak solution $(\overline{u}, \overline{p})$ to Problem (3.1) in the sense of Definition 3.1 follows from [100, Ch.II, Theorem 1.2]. Moreover, [100, Ch.II, Theorem 1.2] gives us the uniqueness of the weak solution $(\overline{u}, \overline{p})$ dealing with a condition on ν, f and G.*

3.1.2 Weak solution of the transient problem

Our assumptions for the transient Navier-Stokes problem (3.2) are the following :

$$\begin{aligned}&\Omega \text{ is an open bounded Lipschitz domain of } \mathbb{R}^d \ (1 \le d \le 3),\\ &T \in \mathbb{R}, T > 0\\ &f \in \boldsymbol{L}^2(\Omega) \times (0,T) \text{ and } G \in (\boldsymbol{L}^2(\Omega) \times (0,T))^d\\ &u_{\text{ini}} \in \boldsymbol{L}^2(\Omega).\end{aligned} \tag{3.8}$$

Definition 3.4 (Weak solution to the transient Navier-Stokes problem)**.** *Under Hypotheses (3.8), $\overline{u}$ is a weak solution to (3.2) if $\overline{u} \in L^2(0,T,E(\Omega)) \cap L^\infty(0,T,\boldsymbol{L}^2(\Omega))$ and*

$$\begin{cases} -\displaystyle\int_0^T\int_\Omega \overline{u}(x,t)\cdot\partial_t\bar{v}(x,t)\ dxdt - \int_0^T\int_\Omega \overline{u}_{\text{ini}}(x)\cdot\bar{v}(x,0)\ dx\\ \qquad +\nu\displaystyle\int_0^T\int_\Omega \nabla\overline{u}(x,t):\nabla\bar{v}(x,t)\ dxdt + \int_0^T b(\overline{u}(\cdot,t),\overline{u}(\cdot,t),\bar{v}(\cdot,t))\ dt\\ \qquad = \displaystyle\int_0^T\int_\Omega (f(x,t)\cdot\bar{v}(x,t) + G(x,t):\nabla\bar{v}(x,t))\ dxdt,\\ \qquad\qquad \forall \bar{v} \in L^2(0,T,E(\Omega)) \cap \boldsymbol{C}_c^\infty(\Omega\times(-\infty,T)),\end{cases} \tag{3.9}$$

where we recall that $b(u,v,w) = \displaystyle\sum_{i,j=1}^d \int_\Omega u_i(\partial_i v_j)w_j\ dx.$

Remark 3.5. *From (3.9), we get that a weak solution $\overline{u}$ of (3.2) in the sense of Definition 3.4 satisfies $\partial_t\overline{u} \in L^{4/d}(0,T,E'(\Omega))$ and so a weak solution of the problem in a classical way where we do not make the integration by part on $\int_0^T \partial_t\overline{u}\cdot\bar{v}\ dt$.*

3.2 Gradient discretisations

Gradient discretisations gather the discrete spaces, operators and properties, upon which the gradient scheme framework is designed.

Definition 3.6 (Gradient discretisation for the steady Navier-Stokes problem)**.** *A gradient discretisation D for the incompressible steady Navier-Stokes problem, with homogeneous Dirichlet's boundary conditions, is defined by $D = (X_{D,0}, \Pi_D, \nabla_D, B_D, Y_D, \chi_D, \text{div}_D, b_D)$, where :*

1. *$X_{D,0}$ is a finite-dimensional vector space on $\mathbb{R}$, we denote $X_{D,0}^* = X_{D,0} \setminus \{0\}$.*
2. *Y_D is a finite-dimensional vector space on $\mathbb{R}$, we denote $Y_D^* = Y_D \setminus \{0\}$.*
3. *The linear mapping $\Pi_D : X_{D,0} \mapsto \boldsymbol{L}^2(\Omega)$ is the reconstruction of the approximate velocity field.*

4. *The linear mapping* $\nabla_D : X_{D,0} \mapsto \boldsymbol{L}^2(\Omega)^d$ *is the discrete gradient operator. It must be chosen such that* $\|\cdot\|_D := \|\nabla_D \cdot\|_{\boldsymbol{L}^2(\Omega)^d}$ *is a norm on* $X_{D,0}$.
5. *The linear mapping* $\mathrm{div}_D : X_{D,0} \mapsto L^2(\Omega)$ *is the discrete divergence operator.*
6. *The linear mapping* $\chi_D : Y_D \mapsto L^2(\Omega)$ *is the reconstruction of the approximate pressure, and must be chosen such that* $\|\chi_D \cdot\|_{L^2(\Omega)}$ *is a norm on* Y_D. *We then set* $Y_{D,0} = \{q \in Y_D, \int_\Omega \chi_D q \mathrm{d}x = 0\}$. *We assume that the nonnegative quantity* β_D, *defined by*

$$\beta_D = \min_{q\in Y_{D,0}^*} \max_{v\in X_{D,0}^*} \frac{\int_\Omega \chi_D q \ \mathrm{div}_D v \mathrm{d}x}{\|v\|_D \|\chi_D q\|_{L^2(\Omega)}}. \tag{3.10}$$

   is such that $\beta_D > 0$.
7. *The mapping* $b_D : X_{D,0}^2 \mapsto \mathbb{R}$ *is the discrete convection term. It must be chosen such that*
   — b_D *is continuous,*
   — *for all* $(u,v) \in X_{D,0}^2$, $b_D(u,u) \geq 0$,
   — *there exist a constant* $B_D > 0$ *such that* $|b_D(u,v)| \leq B_D \|u\|_D^2 \|v\|_D$,
   — $b_D(u,v)$ *is linear with respect to* v.

The *coercivity* of a sequence of gradient discretisations ensure that a discrete Sobolev inequality, a control of the discrete divergence and a discrete Ladyzenskaja-Babuska-Brezzi (LBB) conditions can be establish, all uniform along the sequence of discretisations.

Definition 3.7 (Coercivity)**.** *Let D be a discretisation in the sense of Definition 3.6. Let $q \in \mathbb{N}$ and let C_D be defined by*

$$C_D = \max_{v\in X_{D,0}^*} \frac{\|\Pi_D v\|_{\boldsymbol{L}^2(\Omega)}}{\|v\|_D} + \max_{v\in X_{D,0}^*} \frac{\|\mathrm{div}_D v\|_{L^2(\Omega)}}{\|v\|_D}. \tag{3.11}$$

A sequence $(D_m)_{m\in\mathbb{N}}$ of gradient discretisation is said to be ***coercive*** *if there exist $C_S \geq 0$ and $\beta > 0$ such that $C_{D_m} \leq C_S$ and $\beta_{D_m} \geq \beta$, for all $m \in \mathbb{N}$.*

The *consistency* of a sequence of gradient discretisations states that the continuous space is approximated as the discretisation is refined.

Definition 3.8 (Consistency)**.** *Let D be a gradient discretisation in the sense of Definition 3.6, and let us define the interpolation operators $I_D : \boldsymbol{H}_0^1(\Omega) \mapsto X_{D,0}$ and $\widetilde{I}_D : L_0^2(\Omega) \mapsto Y_{D,0}$ by*

$$\begin{aligned} I_D\varphi &= \operatorname*{argmin}_{v\in X_{D,0}} \left(\|\Pi_D v - \varphi\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla_D v - \nabla\varphi\|_{\boldsymbol{L}^2(\Omega)^d} + \|\mathrm{div}_D v - \mathrm{div}\varphi\|_{L^2(\Omega)} \right), \\ \widetilde{I}_D\psi &= \operatorname*{argmin}_{z\in Y_{D,0}} \|\chi_D z - \psi\|_{L^2(\Omega)}. \end{aligned} \tag{3.12}$$

Let $S_D : \boldsymbol{H}^1_0(\Omega) \to [0,+\infty)$*, and* $\widetilde{S}_D : L^2_0(\Omega) \to [0,+\infty)$ *be defined by*

$$\forall \varphi \in \boldsymbol{H}^1_0(\Omega)\,, \quad S_D(\varphi) = \|\Pi_D I_D \varphi - \varphi\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla_D I_D \varphi - \nabla\varphi\|_{\boldsymbol{L}^2(\Omega)^d} + \|\mathrm{div}_D I_D\varphi - \mathrm{div}\varphi\|_{L^2(\Omega)},$$

and

$$\forall \psi \in L^2_0(\Omega)\,, \quad \widetilde{S}_D(\psi) = \|\chi_D \widetilde{I}_D \psi - \psi\|_{L^2(\Omega)}.$$

A sequence $(D_m)_{m\in\mathbb{N}}$ *of gradient discretisation is said to be* ***consistent*** *if, for all* $\varphi \in \boldsymbol{H}^1_0(\Omega)$*,* $S_{D_m}(\varphi)$ *tends to* 0 *as* $m \to \infty$ *and, for all* $\psi \in L^2_0(\Omega)$*,* $\widetilde{S}_{D_m}(\psi)$ *tends to* 0 *as* $m \to \infty$*.*

The *limit conformity* of a sequence of gradient discretisations states that the discrete gradient and divergence of bounded sequences whose reconstruction converges, converge to the continuous gradient and divergence of the limit (this property is immediately satisfied by conforming approximations).

Definition 3.9 (Limit-conformity)**.** *Let* D *be a gradient discretisation in the sense of Definition 3.6 and let* $W_D : \boldsymbol{H}_{\mathrm{div}}(\Omega) \mapsto [0,+\infty)$ *be defined by*

$$\forall \varphi \in \boldsymbol{H}_{\mathrm{div}}(\Omega)\,, \quad W_D(\varphi) = \max_{v\in X^*_{D,0}} \frac{\int_\Omega (\nabla_D v : \varphi + \Pi_D v \cdot \mathrm{div}\varphi)\,\mathrm{d}x}{\|v\|_D},$$

and let $\widetilde{W}_D : L^2(\Omega) \mapsto [0,+\infty)$ *be defined by*

$$\forall \psi \in L^2(\Omega)\,, \quad \widetilde{W}_D(\psi) = \max_{v\in X^*_{D,0}} \frac{\int_\Omega \psi \left(\sum_{i=1}^d \nabla_D^{(i,i)} v - \mathrm{div}_D v\right)\mathrm{d}x}{\|v\|_D}.$$

A sequence $(D_m)_{m\in\mathbb{N}}$ *of gradient discretisation is said to be* ***limit-conforming*** *if, for all* $\varphi \in \boldsymbol{H}_{\mathrm{div}}(\Omega)$*,* $W_{D_m}(\varphi)$ *tends to* 0 *and for all* $\psi \in L^2(\Omega)$*,* $\widetilde{W}_D(\psi)$ *tends to* 0 *as* $m \to \infty$*.*

The *compactness* of a sequence of gradient discretisations states that any bounded sequence is relatively compact in the sense that the reconstruction converges up to a subsequence.

Definition 3.10 (Compactness)**.** *Let* D *be a gradient discretisation in the sense of Definition 3.6. A sequence* $(D_m)_{m\in\mathbb{N}}$ *of gradient discretisation is said to be compact if, for all sequence* $(u_m)_{m\in\mathbb{N}} \in X_{D_m,0}$ *such that* $\|u_m\|_{D_m}$ *is bounded, the sequence* $(\Pi_{D_m} u_m)_{m\in\mathbb{N}}$ *is relatively compact in* $\boldsymbol{L}^2(\Omega)$*.*

Definition 3.11 (Trilinear limit-conformity)**.** *A sequence* $(D_m)_{m\in\mathbb{N}}$ *of gradient discretisation in the sense of Definition 3.6 is said to be trilinear limit-conforming if for all sequence* $(u_m, v_m) \in X^2_{D_m,0}$ *such that* $(\|u_m\|_{D_m})_{m\in\mathbb{N}}$ *and* $(\|v_m\|_{D_m})_{m\in\mathbb{N}}$ *are bounded, and such that there exists* $(\overline{u}, \overline{v}) \in \boldsymbol{H}^1_0(\Omega)^2$ *such that*

— $\Pi_{D_m} u_m \to \overline{u}$ *in* $\boldsymbol{L}^2(\Omega)$,
— $\nabla_{D_m} u_m \rightharpoonup \nabla\overline{u}$ *weakly in* $\boldsymbol{L}^2(\Omega)^d$,
— $\Pi_{D_m} v_m \to \overline{v}$ *in* $\boldsymbol{L}^2(\Omega)$,
— $\nabla_{D_m} v_m \rightharpoonup \nabla\overline{v}$ *weakly in* $\boldsymbol{L}^2(\Omega)^d$,

then

$$\lim_{m\to\infty} b_{D_m}(u_m, v_m) = b(\overline{u}, \overline{u}, \overline{v})$$

where b is defined in (3.5).

Lemma 3.12 (Regularity of the limit of bounded sequences)**.** *Let $(D_m)_{m\in\mathbb{N}}$ be a sequence of gradient discretisations which is coercive, limit-conforming and compact, and let for all $m \in \mathbb{N}$, $u_m \in X_{D_m,0}$ be such that the sequence $(\|u_m\|_{D_m})_{m\in\mathbb{N}}$ is bounded. Then there exists $\overline{u} \in H^1_0(\Omega)$ and a subsequence of $(D_m)_{m\in\mathbb{N}}$, again denoted by $(D_m)_{m\in\mathbb{N}}$, such that, as $m \to \infty$,*
— $\Pi_{D_m} u_m \to \overline{u}$ *in* $\boldsymbol{L}^2(\Omega)$,
— $\nabla_{D_m} u_m \rightharpoonup \nabla\overline{u}$ *weakly in* $\boldsymbol{L}^2(\Omega)^d$,
— $\mathrm{div}_{D_m} u_m \rightharpoonup \mathrm{div}\overline{u}$ *weakly in* $L^2(\Omega)$.

Moreover, if the sequence of gradient discretisations $(D_m)_{m\in\mathbb{N}}$ is consistent and if

$$\forall m \in \mathbb{N},\ \forall \varphi \in Y_{D_m,0},\ \int_\Omega \mathrm{div}_{D_m} u_m \chi_{D_m} \varphi \,\mathrm{d}x = 0, \tag{3.13}$$

then $\mathrm{div}\overline{u} = 0$.

Proof. Using the compactness of $(D_m)_{m\in\mathbb{N}}$ gives the existence of $\overline{u} \in \boldsymbol{L}^2(\Omega)$ such that, up to a subsequence, $\Pi_{D_m} u_m \to \overline{u}$ in $\boldsymbol{L}^2(\Omega)$. From this subsequence, and using the fact that $\nabla_{D_m} u_m$ and $\mathrm{div}_{D_m} u_m$ remain bounded (we use here the coercivity of $(D_m)_{m\in\mathbb{N}}$ which provides a bound on $\|\mathrm{div}_{D_m} u_m\|_{L^2(\Omega)}$), we deduce that there exist $\zeta \in \boldsymbol{L}^2(\Omega)^d$ and $\gamma \in L^2(\Omega)$ such that, again up to a subsequence (still indexed by m), $\nabla_{D_m} u_m \rightharpoonup \zeta$ weakly in $\boldsymbol{L}^2(\Omega)^d$ and $\mathrm{div}_{D_m} u_m \rightharpoonup \gamma$ weakly in $L^2(\Omega)$. We extend the definition of all the previous functions by 0 outside Ω. Let $\varphi \in \boldsymbol{C}^\infty(\overline{\mathbb{R}^d})^d$. From Definition 3.9 applied to the restriction of φ to Ω and to its opposite, we have

$$|\int_{\mathbb{R}^d} (\nabla_{D_m} u_m : \varphi + \Pi_{D_m} u_m \cdot \mathrm{div}\varphi) \,\mathrm{d}x| \le W_{D_m}(\varphi|_\Omega)\|u_m\|_{D_m}.$$

Passing to the limit and using the limit-conformity of $(D_m)_{m\in\mathbb{N}}$, we obtain

$$\int_{\mathbb{R}^d} (\zeta : \varphi + \overline{u} \cdot \mathrm{div}\varphi) \,\mathrm{d}x = 0.$$

The last equality shows that $\zeta = \nabla\overline{u}$ on $\mathbb{R}^d$ and therefore that $\overline{u} \in \boldsymbol{H}^1(\mathbb{R}^d)$. Since ζ vanishes outside Ω, we get that $\overline{u} \in \boldsymbol{H}^1_0(\Omega)$. It remains to prove that $\overline{u}$ is free-divergence. Taking

$\psi \in L^2(\Omega)$ with zero mean value, we get that

$$|\int_\Omega \psi(\sum_{i=1}^{d} \nabla_{D_m}^{(i,i)} u_m - \mathrm{div}_{D_m} u_m)\ \mathrm{d}x| \le \widetilde{W}_{D_m}(\psi)\|u_m\|_{D_m}. \tag{3.14}$$

Passing to the limit and again using the limit-conformity of $(D_m)_{m\in\mathbb{N}}$, we obtain

$$\int_\Omega \psi(\mathrm{div}\overline{u} - \gamma)\ \mathrm{d}x = 0,$$

which shows that $\gamma = \mathrm{div}\overline{u}$.

Let us now turn to the last part of the lemma. We then assume the consistency of the sequence of gradient discretisations and that (3.13) holds. Using the interpolation operator defined in Definition 3.8, we get from (3.13) and (3.14) that, for any $\psi \in L_0^2(\Omega)$,

$$|\int_\Omega \psi\ \mathrm{div}_{D_m} u_m\ \mathrm{d}x| \le \|\mathrm{div}_{D_m} u_m\|_{L^2(\Omega)} \|\psi - \chi_{D_m} \widetilde{I}_{D_m}\psi\|_{L^2(\Omega)}.$$

Letting $m \to \infty$, we obtain that $\int_\Omega \psi \mathrm{div}\overline{u}\ \mathrm{d}x = 0$ which implies, since $\mathrm{div}\overline{u} \in L_0^2(\Omega)$, that $\mathrm{div}\overline{u} = 0$ a.e. in Ω.

3.3 Steady Navier-Stokes problem

3.3.1 Gradient Scheme and main result

The gradient scheme for the steady Navier-Stokes problem is based on a discretisation of the weak formulation (3.4), in which the continuous spaces and operators are replaced with discrete ones (in (3.4), we wrote the property "$\mathrm{div}\overline{u} = 0$" using test functions to make clearer this parallel between the weak formulation and the gradient scheme). If D is a gradient discretisation in the sense of Definition 3.6, the scheme is given by :

$$\begin{cases} u \in X_{D,0}\,,\ p \in Y_{D,0}, \\ \eta \int_\Omega \Pi_D u \cdot \Pi_D v + \nu \int_\Omega \nabla_D u : \nabla_D v\ dx + b_D(u,v) \\ \qquad\qquad - \int_\Omega \chi_D p\ \mathrm{div}_D v \mathrm{d}x = \int_\Omega (f \cdot \Pi_D v + G : \nabla_D v)\ \mathrm{d}x,\ \forall v \in X_{D,0}, \\ \int_\Omega \chi_D q\, \mathrm{div}_D u \mathrm{d}x = 0,\ \ \forall q \in Y_{D,0}. \end{cases} \tag{3.15}$$

Our main result for the steady Navier-Stokes problem is the following theorem.

Theorem 3.13 (Convergence of the scheme)**.** *Under Hypotheses (3.3), let $(D_m)_{m\in\mathbb{N}}$ be a sequence of gradient discretisations in the sense of Definition 3.6, which is consistent, limit-conforming, coercive, compact and trilinear limit-conforming in the sense of Definitions 3.8, 3.9, 3.7, 3.10 and 3.11. Then for any m there exists at least one solution (u_{D_m}, p_{D_m}) to (3.15) with $D = D_m$. Moreover, as $m \to \infty$, there exists a subsequence of $(D_m)_{m\in\mathbb{N}}$ again denoted $(D_m)_{m\in\mathbb{N}}$ and there exists $(\overline{u}, \overline{p})$, weak solution of the incompressible steady Navier-Stokes problem (3.1) in the sense of Definition 3.1, such that*

— $\Pi_{D_m} u_{D_m}$ *converges to* $\overline{u}$ *in* $\boldsymbol{L}^2(\Omega)$,

— $\nabla_{D_m} u_{D_m}$ *converges to* $\nabla\overline{u}$ *in* $\boldsymbol{L}^2(\Omega)^d$,

— $\chi_{D_m} p_{D_m}$ *converges to* $\overline{p}$ *in* $L^2(\Omega)$.

3.3.2 Proof of the convergence result

Let us established some estimates on the solution of scheme (3.15).

Lemma 3.14 (Estimates on the discrete velocity)**.** *Under Hypotheses (3.3), let D be a gradient discretisation in the sense of Definition 3.6. If (u_D, p_D) is a solution to Scheme (3.15), then there exists $C_{12} > 0$ only depending on Ω, d, f, G, ν, η and increasingly depending on C_D such that*

$$\eta\|\Pi_D u_D\|^2_{L^2(\Omega)} + \nu\|u_D\|^2_D \leq C_{12} \tag{3.16}$$

Proof. Putting $v = u_D$ and $q = p_D$ in (3.15), we get

$$\eta\|\Pi_D u_D\|^2_{L^2(\Omega)} + \nu\|u_D\|^2_D + \underbrace{b_D(u_D, u_D)}_{\geq 0} - \underbrace{\int_\Omega \chi_D p_D \operatorname{div}_D u_D \, dx}_{=0} = \int_\Omega \left(f \cdot \Pi_D u_D + G : \nabla_D u_D\right) \, dx.$$

Now using the Cauchy-Schwarz inequality in the previous equation, we obtain

$$\eta\|\Pi_D u_D\|^2_{L^2(\Omega)} + \nu\|u_D\|^2_D \leq \|f\|_{\boldsymbol{L}^2(\Omega)}\|\Pi_D u_D\|_{L^2(\Omega)} + \|G\|_{\boldsymbol{L}^2(\Omega)^d}\|u_D\|_D. \tag{3.17}$$

and therefore, using $\|\Pi_D u_D\|_{\boldsymbol{L}^2(\Omega)} \leq C_D\|u_D\|_D$ (see Definition 3.7 of coercivity), we get

$$\nu\|u_D\|_D \leq \|f\|_{\boldsymbol{L}^2(\Omega)} C_D + \|G\|_{\boldsymbol{L}^2(\Omega)^d}.$$

Reporting this inequality in the right-hand side of (3.17), we conclude the proof of (3.16).

Lemma 3.15 (Estimates on the discrete pressure)**.** *Under Hypotheses (3.3), let D be a gradient discretisation in the sense of Definition 3.6. If (u_D, p_D) is a solution to Scheme (3.15), then there* *exists $C_{13} > 0$ only depending on Ω, d, C_{12}, f, G and increasingly depending on $B_D + C_D + \frac{1}{\beta_D}$ such that*

$$\|\chi_D p_D\|_{L^2(\Omega)} \leq C_{13} \tag{3.18}$$

Proof. Let $v \in X_{D,0}$ such that $\|v\|_D = 1$ and $\beta_D \|\chi_D p\|_{L^2(\Omega)} \leq \int_\Omega \chi_D p \operatorname{div}_D v \, dx$ (see Definition 3.7). Putting v in Scheme (3.15), we get

$$\beta_D \|\chi_D p\|_{L^2(\Omega)} \\ \leq \left| \eta \int_\Omega \Pi_D u_D \cdot \Pi_D v \, dx + \nu \int_\Omega \nabla_D u_D : \nabla_D v \, dx + b_D(u_D, v) - \int_\Omega (f \cdot \Pi_D v + G : \nabla_D v) \, dx \right|.$$

Using Cauchy-Schwarz and the discrete Sobolev inequalities, we can deduce :

$$\beta_D \|\chi_D p_D\|_{L^2(\Omega)} \leq \eta C_D \|\Pi_D u_D\|_{\boldsymbol{L}^2(\Omega)} + \nu \|u_D\|_D + |b_D(u_D, v)| + C_D \|f\|_{\boldsymbol{L}^2(\Omega)} + \|G\|_{\boldsymbol{L}^2(\Omega)^d}.$$

Thanks to the assumptions on b_D, we get

$$|b_D(u_D, v)| \leq B_D \|u_D\|_D^2.$$

Estimate (3.16) allow us to conclude the proof of Estimate (3.18).

Lemma 3.16 (Existence of a discrete solution)**.** *Under Hypotheses (3.3), let D be an admissible discretisation of Ω in the sense of Definition 3.6. Then there exists at least one solution (u_D, p_D) to Scheme (3.15).*

Proof. We follow the proof of [69, Theorem 4.3] based on a topological degree argument. Let N (resp. M) be the dimension of $X_{D,0}$ (resp. $Y_{D,0}$) and let $(v^{(i)})_{i=1,\dots,N}$ (respectively $(q^{(j)})_{j=1,\dots,M}$) be a basis of $X_{D,0}$ (respectively $Y_{D,0}$). Let $F : \mathbb{R}^N \times \mathbb{R}^M \times [0,1] \mapsto \mathbb{R}^N \times \mathbb{R}^M$ be the mapping such that, for any $(u = \sum_{i=1}^N u_i v^{(i)}, p = \sum_{j=1}^M p_j q^{(j)}, \lambda) \in X_{D,0} \times Y_{D,0} \times [0,1]$, $F(u,p,\lambda) = (F_i(u,p,\lambda))_{i=1,\dots,N+M}$ with :
for all $i = 1, \dots, N$

$$F_i(u,p,\lambda) = \eta \int_\Omega \Pi_D u \cdot \Pi_D v^{(i)} \, \mathrm{d}x + \nu \int_\Omega \nabla_D u : \nabla_D v^{(i)} \, \mathrm{d}x + \lambda b_D(u_D^{(i)}, v^{(i)}) \\ - \int_\Omega \chi_D p_D \operatorname{div}_D v^{(i)} \, \mathrm{d}x - \lambda \int_\Omega \left(f \cdot \Pi_D v^{(i)} + G : \nabla_D v^{(i)} \right) \, \mathrm{d}x,$$

and for all $j = 1, \dots, M$

$$F_{j+N}(u,p,\lambda) = \int_\Omega \chi_D q^{(j)} \operatorname{div}_D u_D \, \mathrm{d}x.$$

Thanks to the hypotheses on b_D, the mapping F is continuous, and for a given (u, p), such that $F_i(u, p, \lambda) = 0$ for all $i = 1, ..., N + M$, estimates of Lemmas 3.14 and 3.15 hold replacing (b_D, f, G) in Scheme (3.15) by $(\lambda b_D, \lambda f, \lambda G)$. Since $F(u, p, 0)$ is a linear function of (u, p), we deduce from the invariance of the Brouwer topological degree by homotopy that there exists at least one solution (u_D, p_D) to the equation $F(u_D, p_D, 1) = 0$, which is exactly Scheme (3.15).

We will now prove the convergence theorem for the steady Navier-Stokes problem.

Proof (of Theorem 3.13). .

Step 1 : Extraction of a converging subsequence.

Estimate (3.16) allows us to apply Lemma 3.12 and to get the existence of $\overline{u} \in \boldsymbol{H}_0^1(\Omega)$ with $\operatorname{div} \overline{u} = 0$ and, up to a subsequence again indexed by m, $\Pi_{D_m} u_{D_m} \to \overline{u}$ in $\boldsymbol{L}^2(\Omega)$, $\nabla_{D_m} u_{D_m} \to \nabla \overline{u}$ weakly in $\boldsymbol{L}^2(\Omega)^d$ and $\operatorname{div}_{D_m} u_{D_m} \to 0$ weakly in $L^2(\Omega)$. Moreover, thanks to Estimate (3.18), up to a subsequence of the previous one (again indexed by m), we get the existence of $\overline{p} \in L_0^2(\Omega)$ such that $\chi_{D_m} p_{D_m} \to \overline{p}$ weakly in $L^2(\Omega)$.

Step 2 : Proof that $(\overline{u}, \overline{p})$ is solution to (3.4).

Let $\overline{w} \in \boldsymbol{H}_0^1(\Omega)$ be given. Thanks to the consistency hypothesis, we get that $\|I_{D_m} \overline{w}\|_{D_m}$ is bounded, $\Pi_{D_m} I_{D_m} \overline{w} \to \overline{w}$ in $\boldsymbol{L}^2(\Omega)$ and $\nabla_{D_m} I_{D_m} \overline{w} \to \nabla \overline{w}$ in $\boldsymbol{L}^2(\Omega)^d$. Thanks to weak/strong convergence properties, the following holds :

$$\lim_{m\to\infty} \eta \int_\Omega \Pi_{D_m} u_{D_m} \cdot \Pi_{D_m} I_{D_m} \overline{w} \, \mathrm{d}x = \eta \int_\Omega \overline{u} \cdot \overline{w} \, \mathrm{d}x,$$
$$\lim_{m\to\infty} \nu \int_\Omega \nabla_{D_m} u_{D_m} : \nabla_{D_m} I_{D_m} \overline{w} \, \mathrm{d}x = \nu \int_\Omega \nabla \overline{u} : \nabla \overline{w} \, \mathrm{d}x,$$
$$\lim_{m\to\infty} \int_\Omega \chi_{D_m} p_{D_m} \operatorname{div}_{D_m} I_{D_m} \overline{w} \, \mathrm{d}x = \int_\Omega \overline{p} \operatorname{div} \overline{w} \, \mathrm{d}x,$$
$$\lim_{m\to\infty} \int_\Omega \left(f \cdot \Pi_{D_m} I_{D_m} \overline{w} + G : \nabla_{D_m} I_{D_m} \overline{w}\right) \, \mathrm{d}x = \int_\Omega \left(f \cdot \overline{w} + G : \nabla \overline{w}\right) \, \mathrm{d}x,$$

and, thanks to the the trilinear limit-conformity of $(D_m)_{m\in\mathbb{N}}$,

$$\lim_{m\to\infty} b_{D_m}(u_{D_m}, I_{D_m} \overline{w}) = b(\overline{u}, \overline{u}, \overline{w}).$$

Therefore, letting $v = I_{D_m} \overline{w}$ as test function in Scheme (3.15) and passing to the limit, we find that $(\overline{u}, \overline{p})$ is a solution to Problem (3.4).

Step 3 : Proof of strong convergence of $\nabla_{D_m} u_{D_m}$.

Taking $v = u_{D_m}$ as test function in Scheme (3.15), passing to the supremum limit as $m \to$

∞, using the convergence of $\Pi_{D_m} u_{D_m}$ to $\overline{u}$ in $\boldsymbol{L}^2(\Omega)$ and of $\nabla_{D_m} u_{D_m}$ to $\nabla\overline{u}$ in $\boldsymbol{L}^2(\Omega)^d$ and $b_{D_m}(u_{D_m}, u_{D_m}) \geq 0$, we get that :

$$\limsup_{m\to\infty} \left(\eta\|\Pi_{D_m} u_{D_m}\|_{\boldsymbol{L}^2(\Omega)} + \nu\|u_{D_m}\|_{D_m}\right) \leq \int_\Omega (f \cdot \overline{u} + G : \nabla\overline{u})\ \mathrm{d}x.$$

Now choosing $\overline{v} = \overline{u}$ as test function in Problem (3.4), recalling that $b(\overline{u}, \overline{u}, \overline{u}) = 0$, we find

$$\eta\|\overline{u}\|_{\boldsymbol{L}^2(\Omega)} + \nu\|\nabla\overline{u}\|_{\boldsymbol{L}^2(\Omega)^d} = \int_\Omega (f \cdot \overline{u} + G : \nabla\overline{u})\ \mathrm{d}x.$$

Combining the last two equations we get

$$\limsup_{m\to\infty} \left(\eta\|\Pi_{D_m} u_{D_m}\|_{\boldsymbol{L}^2(\Omega)} + \nu\|u_{D_m}\|_{D_m}\right) \leq \eta\|\overline{u}\|_{\boldsymbol{L}^2(\Omega)} + \nu\|\nabla\overline{u}\|_{\boldsymbol{L}^2(\Omega)^d}.$$

Furthermore, thanks again to the convergence of $\Pi_{D_m} u_{D_m}$ to $\overline{u}$ in $\boldsymbol{L}^2(\Omega)$ and to the weak convergence of $\nabla_{D_m} u_{D_m}$ to $\nabla\overline{u}$ in $\boldsymbol{L}^2(\Omega)^d$, we may write that

$$\liminf_{m\to\infty} \left(\eta\|\Pi_{D_m} u_{D_m}\|_{\boldsymbol{L}^2(\Omega)} + \nu\|u_{D_m}\|_{D_m}\right) \geq \eta\|\overline{u}\|_{\boldsymbol{L}^2(\Omega)} + \nu\|\nabla\overline{u}\|_{\boldsymbol{L}^2(\Omega)^d}.$$

Finally we can conclude that $\|u_{D_m}\|_{D_m} \to \|\nabla\overline{u}\|_{\boldsymbol{L}^2(\Omega)^d}$ and this allows us to conclude the proof.

Step 4 : Proof of strong convergence of the approximate pressure in $L^2(\Omega)$.

We select $v_m \in X_{D_m}$ such that $\|v_m\|_{D_m} = 1$ and

$$\beta_{D_m}\|\chi_{D_m}(\widetilde{I}_{D_m}\overline{p} - p_{D_m})\|_{L^2(\Omega)} \leq \int_\Omega \chi_{D_m}(\widetilde{I}_{D_m}\overline{p} - p_{D_m})\mathrm{div}_{D_m} v_m \mathrm{d}x.$$

Letting $v = v_m$ in the scheme, we get

$$\eta\int_\Omega \Pi_{D_m} u_{D_m} \cdot \Pi_{D_m} v_m + \nu\int_\Omega \nabla_{D_m} u_{D_m} : \nabla_{D_m} v_m + b_{D_m}(u_{D_m}, v_m) - \int_\Omega \chi_{D_m} p_{D_m}\ \mathrm{div}_{D_m} v_m = \int_\Omega f \cdot \Pi_{D_m} v_m.$$

Combining the two above relations, we get

$$\begin{aligned}\beta\|\chi_{D_m}(\widetilde{I}_{D_m}\overline{p} - p_{D_m})\|_{L^2(\Omega)} \leq &\int_\Omega f \cdot \Pi_{D_m} v_m + \int_\Omega \chi_{D_m}\widetilde{I}_{D_m}\overline{p}\ \mathrm{div}_{D_m} v_m \\ &- \eta\int_\Omega \Pi_{D_m} u_{D_m} \cdot \Pi_{D_m} v_m - \nu\int_\Omega \nabla_{D_m} u_{D_m} : \nabla_{D_m} v_m - b_{D_m}(u_{D_m}, v_m).\end{aligned}$$

Thanks to the triangle inequality, we deduce

$$\beta\|\overline{p} - \chi_{D_m} p_{D_m}\|_{L^2(\Omega)} \leq \beta\|\chi_{D_m}\widetilde{I}_{D_m}\overline{p} - \overline{p}\|_{L^2(\Omega)} + \int_\Omega f \cdot \Pi_{D_m} v_m + \int_\Omega \chi_{D_m}\widetilde{I}_{D_m}\overline{p}\ \mathrm{div}_{D_m} v_m$$
$$- \eta \int_\Omega \Pi_{D_m} u_{D_m} \cdot \Pi_{D_m} v_m - \nu \int_\Omega \nabla_{D_m} u_{D_m} : \nabla_{D_m} v_m - b_{D_m}(u_{D_m}, v_m).$$

Since $\|v_m\|_{D_m} = 1$, Lemma 3.12 shows the existence of $\overline{v} \in \boldsymbol{H}_0^1(\Omega)$ and of a subsequence, again indexed by m, such that $\Pi_{D_m} v_m$ tends to $\overline{v}$ in $\boldsymbol{L}^2(\Omega)$ and such that $\nabla_{D_m} v_m$ weakly converges to $\nabla\overline{v}$ in $\boldsymbol{L}^2(\Omega)^d$ and $\mathrm{div}_{D_m} v_m$ weakly converges to $\mathrm{div}\overline{v}$ in $L^2(\Omega)$.

Using the (already proved) strong convergence properties for the velocity and the trilinear limit-conformity of $(D_m)_{m\in\mathbb{N}}$, we may now pass to the limit $m \to \infty$, since all integrals involve weak/strong convergence properties. We get

$$\beta \limsup_{m\to\infty} \|\overline{p} - \chi_{D_m} p_{D_m}\|_{L^2(\Omega)} \leq \int_\Omega f \cdot \overline{v} + \int_\Omega \overline{p}\ \mathrm{div}\overline{v} - \eta \int_\Omega \overline{u} \cdot \overline{v} - \nu \int_\Omega \nabla\overline{u} : \nabla\overline{v} - b(\overline{u}, \overline{u}, \overline{v}).$$

It now suffices to use the fact that we already proved that $(\overline{u}, \overline{p})$ is a weak solution to the steady Navier-Stokes equation (3.4). We then get that the right hand side of the previous inequality vanishes, which shows the convergence in $L^2(\Omega)$ for this subsequence. Using a standard uniqueness argument, we deduce that the whole subsequence built at step 1 converges in this sense.

3.4 Transient Navier-Stokes problem

In this section, we are interested in the study of the transient Navier-Stokes problem defined as it follows, its discretisation by our general framework and the convergence of our scheme.

3.4.1 Gradient Scheme and main result

Before introduce the scheme, we need to define a space-time gradient discretisation from an adaptation of the Definition 3.6 of the space one because transient problem requires the addition of time step and an interpolation of initial data.

Definition 3.17 (Space-time gradient discretisation). *A space-time gradient discretisation D for the transient Navier-Stokes problem, with homogeneous Dirichlet boundary conditions, is defined by a family $D = (X_{D,0}, \Pi_D, \nabla_D, B_D, Y_D, \chi_D, \mathrm{div}_D, (t^{(n)})_{n=0,\ldots,N}, J_D)$ where :*

— *$D^s = (X_{D,0}, \Pi_D, \nabla_D, B_D, Y_D, \chi_D, \mathrm{div}_D)$ a gradient discretisation of Ω in the sense of Definition 3.6,*

— $J_D : \boldsymbol{L}^2(\Omega) \mapsto X_{D,0}$ *an interpolation operator ;*
— $t^{(0)} = 0 < t^{(1)} < ... < t^{(N)} = T$.

We define $\delta t^{n+\frac{1}{2}} = t^{(n+1)} - t^{(n)}$ *for all* $n = 0, ..., N-1$ *and* $\delta t_D = \max_{n=0,...,N-1}(\delta t^{n+\frac{1}{2}})$.

A sequence of space-time gradient discretisation $(D_m)_{m\in\mathbb{N}}$ is coercive (resp. limit-conforming and compact) if its spatial component $(D^s_m)_{m\in\mathbb{N}}$ is coercive (resp. limit-conforming and compact).

Definition 3.18 (Space-time consistency)**.** *A sequence $(D_m)_{m\in\mathbb{N}}$ of space-time gradient discretisations in the sense of Definition 3.17 is said* ***consistent*** *if*

1. $(D^s_m)_{m\in\mathbb{N}}$ *is consistent in the sense of Definition 3.8,*

2. *for all* $\varphi \in \boldsymbol{L}^2(\Omega)$, $\Pi_{D_m} J_{D_m}\varphi \to \varphi$ *in* $\boldsymbol{L}^2(\Omega)$,

3. $\delta t_{D_m} \to 0$ *as* $m \to \infty$.

Definition 3.19 (Space-time trilinear limit-conformity)**.** *A sequence $(D_m)_{m\in\mathbb{N}}$ of gradient discretisation in the sense of Definition 3.6 is said to be space-time trilinear limit-conforming if for all sequence $(u_m, v_m)_{m\in\mathbb{N}} \in X^2_{D_m,0}$ such that $(\|\nabla_{D_m} u_m\|_{\boldsymbol{L}^2(\Omega\times(0,T))^d})_{m\in\mathbb{N}}$ and $(\|\nabla_{D_m} v_m\|_{\boldsymbol{L}^2(\Omega\times(0,T))^d})_{m\in\mathbb{N}}$ are bounded, and such that there exists $(\overline{u}, \overline{v}) \in L^2(0, T, \boldsymbol{H}^1_0(\Omega))^2$ such that*

— $\Pi_{D_m} u_m \to \overline{u}$ *in* $L^1(0, T, \boldsymbol{L}^2(\Omega))$ *and* $\|\Pi_{D_m} u_m\|_{L^\infty(0,T,\boldsymbol{L}^2(\Omega))}$ *is bounded,*
— $\nabla_{D_m} u_m \rightharpoonup \nabla\overline{u}$ *weakly in* $\boldsymbol{L}^2(\Omega \times (0, T))^d$,
— $\Pi_{D_m} v_{D_m} \to \overline{v}$ *in* $L^\infty(0, T, \boldsymbol{L}^2(\Omega))$
— $\nabla_{D_m} v_m \to \nabla\overline{v}$ *in* $L^\infty(0, T, \boldsymbol{L}^2(\Omega)^d)$,

then
$$\lim_{m\to\infty} \int_0^T b_{D_m}(u_m, v_m) = \int_0^T b(\overline{u}, \overline{u}, \overline{v}).$$

Let D be a space-time gradient discretisation in the sense of Definition (3.17). With the notation
$$\delta_D^{n+\frac{1}{2}} u_D = \frac{u_D^{(n+1)} - u_D^{(n)}}{\delta t^{n+\frac{1}{2}}},$$

the implicit gradient scheme for (3.2) is based on the following approximation of (3.9) :

$$\left\{\begin{array}{l} u_D = (u_D^{(n)})_{n=0,\dots,N}\,,\ p_D = (p_D^{(n)})_{n=1,\dots,N} \text{ such that } u_D^{(0)} = J_D u_{\rm ini} \text{ and, } \forall n = 0,\dots,N-1 : \\ u_D^{(n+1)} \in X_{D,0}\,,\ p_D^{(n+1)} \in Y_{D,0}\,, \\ \displaystyle\int_\Omega \Pi_D \delta_D^{n+\frac12} u_D \cdot \Pi_D v \mathrm{d}x + \nu \int_\Omega \nabla_D u_D^{(n+1)} : \nabla_D v \mathrm{d}x + b_D(u^{n+1}, v) - \int_\Omega \chi_D p_D^{(n+1)} \mathrm{div}_D v \mathrm{d}x \\ \quad \displaystyle = \frac{1}{\delta t^{n+\frac12}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_D v \mathrm{d}x\mathrm{d}t + \frac{1}{\delta t^{n+\frac12}} \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega G \cdot \nabla_D v \mathrm{d}x\mathrm{d}t, \quad \forall v \in X_{D,0}. \\ \displaystyle\int_\Omega \mathrm{div}_D u_D^{(n+1)} \chi_D q \mathrm{d}x = 0, \quad \forall q \in Y_{D,0}. \end{array}\right. \tag{3.19}$$

It is useful to denote space-time functions Π_D and ∇_D by the following way : if $v = (v^n)_{n=0,\dots,N} \in X_{D,0}$, the functions $\Pi_D v : \Omega \times (0,T) \to \mathbb{R}^d$ and $\nabla_D v : \Omega \times (0,T) \to \mathbb{R}^{d\times d}$ are defined by

$$\begin{aligned} &\forall n = 0,\dots,N-1\,,\ \forall t \in (t^{(n)}, t^{(n+1)}]\,,\ \forall x \in \Omega\,, \\ &\Pi_D v(x,t) = \Pi_D v^{(n+1)}(x)\,,\ \nabla_D v(x,t) = \nabla_D v^{(n+1)}(x) \text{ and } \delta_D v(t) = \delta_D^{n+\frac12} v. \end{aligned} \tag{3.20}$$

Now we get all the tools needed to present our main result for the transient Navier-Stokes problem

Theorem 3.20 (Convergence of the scheme)**.** *Under hypotheses (3.8), let $(D_m)_{m\in\mathbb{N}}$ be a sequence of space-time gradient discretisation in the sense of Definition 3.17 which is consistent, limit-conforming, coercive and compact in the sense of Definition 3.18, 3.9, 3.7 and 3.10. Then for any m, there exists at least a solution (u_{D_m}, p_{D_m}) to Scheme 3.19 with $D = D_m$. Moreover, as $m \to \infty$, there exists a subsequence of $(D_m)_{m\in\mathbb{N}}$ again denoted $(D_m)_{m\in\mathbb{N}}$ such that $\Pi_{D_m} u_{D_m}$ converges in $L^2(0,T,\boldsymbol{L}^2(\Omega))$ to $\overline{u}$, where $\overline{u}$ is a weak solution to the incompressible transient Navier-Stokes problem in the sense of Definition 3.4.*

3.4.2 Proof of the convergence result

First, we will establish some estimates on the discrete velocity.

Lemma 3.21 (Estimates on the discrete velocity)**.** *Under Hypotheses (3.8), let $(D_m)_{m\in\mathbb{N}}$ be a sequence of space-time gradient discretisation in the sense of Definition 3.17. If (u_D, p_D) is a* *solution to Scheme (3.19), then for all $m = 0, ..., N$,*

$$\int_0^{t^{(m)}} \int_\Omega |\nabla_D u_D^{(m)}|^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_\Omega \left(\left|\Pi_D u_D^{(m)}\right|^2 - \left|\Pi_D u_D^{(0)}\right|^2 \right) \mathrm{d}x \leq \int_0^{t^{(m)}} \int_\Omega f \cdot \Pi_D u_D^{(m)} \mathrm{d}x\mathrm{d}t + \int_0^{t^{(m)}} \int_\Omega G : \nabla_D u_D^{(m)} \mathrm{d}x\mathrm{d}t. \quad (3.21)$$

Moreover, if $C_{14} > 0$ is such that $C_{14} \geq \|\Pi_D u^{(0)}\|_{L^2(\Omega)}$, then there exist $C_{15} > 0$ only depending on Ω, d, $\overline{u}_{\mathrm{ini}}$, f, G, ν and increasingly depending on C_D, such that

$$\|\Pi_D u_D\|_{L^\infty(0,T,\boldsymbol{L}^2(\Omega))} + \nu \|\nabla_D u_D\|_{\boldsymbol{L}^2(\Omega\times(0,T))^d} \leq C_{15} + \|\Pi_D J_D u_{\mathrm{ini}}\|^2_{\boldsymbol{L}^2(\Omega)} \quad (3.22)$$

Proof. Putting $v = \delta t^{n+\frac{1}{2}} u_D^{(n+1)}$ and $q = p_D^{(n+1)}$ in (3.19), since $b_D(u_D^{(n+1)}, u_D^{(n+1)}) = 0$, we get

$$\int_\Omega \left(\Pi_D u_D^{(n+1)} - \Pi_D u_D^{(n)}\right) \cdot \Pi_D u_D^{(n+1)} \mathrm{d}x + \nu \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega |\nabla_D u_D^{(n+1)}|^2 \mathrm{d}x\mathrm{d}t = \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_D u_D^{(n+1)} \mathrm{d}x\mathrm{d}t + \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega G : \nabla_D u_D^{(n+1)} \mathrm{d}x\mathrm{d}t.$$

Using the inequality $(a - b) \cdot a \geq \frac{1}{2}(|a|^2 - |b|^2)$ (valid for any $a, b \in \mathbb{R}^d$) on the first term, it comes :

$$\frac{1}{2}\int_\Omega \left[\left|\Pi_D u_D^{(n+1)}\right|^2 - \left|\Pi_D u_D^{(n)}\right|^2 \right] \mathrm{d}x + \nu \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega |\nabla_D u_D^{(n+1)}|^2 \mathrm{d}x\mathrm{d}t \leq \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f \cdot \Pi_D u_D^{(n+1)} \mathrm{d}x\mathrm{d}t + \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega G : \nabla_D u_D^{(n+1)} \mathrm{d}x\mathrm{d}t.$$

We take $n \in \{0, \ldots, N\}$ and sum the obtained equation over $0, \ldots, n-1$. This gives (3.21). Estimate (3.22) is a straightforward consequence of the definition of C_D and Young's inequality applied to (3.21) with $m = N$.

Lemma 3.22 (Existence of a discrete solution)**.** *Under Hypotheses (3.8), let $(D_m)_{m\in\mathbb{N}}$ be a sequence of space-time gradient discretisation in the sense of Definition 3.17, then there exists at least one solution (u_D, p_D) to Scheme (3.19).*

Proof. We remark that, for a given $n = 0, ..., N-1$, taking as unknown $u_D^{(n+1)}$ in Scheme (3.15) with $\eta = \dfrac{1}{\delta t}$, we find the same form as Scheme (3.19). Therefore, the existence of at least one solution of this last scheme follows from Lemma 3.16.

Definition 3.23 (Dual semi-norm)**.** *The semi-norm $|\cdot|_{*,D}$ is defined on $X_{D,0}$ by*

$$|w|_{*,D} = \sup\left\{\int_\Omega \Pi_D w\cdot\Pi_D v\mathrm{d}x\ : v\in E_D, \|v\|_D = 1\right\},$$

where $E_D = \{v, v\in X_{D,0}, \mathrm{div}_D v = 0\}$

An estimate on this semi-norm will allow us to apply theorem 3.25, which is a discrete version of the Aubin-Simon theorem [27, Theorem 7.1].

Lemma 3.24 (Estimate on the dual semi-norm of the discrete time derivative)**.** *Under Hypotheses (3.8), let $(D_m)_{m\in\mathbb{N}}$ be a sequence of space-time gradient discretisation in the sense of Definition 3.17, let (u_D, p_D) be a solution to Scheme (3.19). We take $C_{14}\geq \|\Pi_D u^{(0)}\|_{L^2(\Omega)}$. Then there exist $C_{16}\geq 0$ only depending on Ω, d, C_{14}, f, G and increasingly depending on C_D such that*

$$\int_0^T |\delta_D u_D|_{*,D}\mathrm{d}t \leq C_{16}(1+\|\Pi_D J_D u_{\mathrm{ini}}\|_{\boldsymbol{L}^2(\Omega)}).. \tag{3.23}$$

Proof. Taking any $v\in E_D$ in Scheme (3.19) such that $\|v\|_D = 1$, we have, for $n = 0,\ldots,N-1$:

$$\int_\Omega \Pi_D\delta_D^{n+\frac12}u_D\cdot\Pi_D v\mathrm{d}x \leq \nu\|u^{(n+1)}\|_D + \frac{1}{\delta t^{(n+\frac12)}}\int_{t^{(n)}}^{t^{(n+1)}}\int_\Omega (f\cdot\Pi_D v + G:\nabla_D v)\mathrm{d}x\mathrm{d}t.$$

Using the Cauchy-Schwarz inequality on the last term and the coercivity definition leads to

$$\int_\Omega \Pi_D\delta_D^{n+\frac12}u_D\cdot\Pi_D v\mathrm{d}x \leq \nu\|u^{(n+1)}\|_D + \frac{1}{\delta t^{(n+\frac12)}}\int_{t^{(n)}}^{t^{(n+1)}} (C_D\|f(\cdot,t)\|_{\boldsymbol{L}^2(\Omega)} + \|G(\cdot,t)\|_{\boldsymbol{L}^2(\Omega)^d})\mathrm{d}t,$$

and using the definition of $|\cdot|_{\star,D}$, we get

$$|\delta_D^{n+\frac12}u_D|_{\star,D} \leq \nu\|u^{(n+1)}\|_D + \frac{1}{\delta t^{(n+\frac12)}}\int_{t^{(n)}}^{t^{(n+1)}} (C_D\|f(\cdot,t)\|_{\boldsymbol{L}^2(\Omega)} + \|G(\cdot,t)\|_{\boldsymbol{L}^2(\Omega)^d})\mathrm{d}t.$$

multiplying by $\delta t^{n+\frac12}$ and summing over n gives the desired estimate, thanks to (3.22) and to the Cauchy-Schwarz inequality.

The following theorem is proved in [27, Theorem 7.1] and we need it in the proof of Theorem 3.20.

Theorem 3.25 (Discrete Aubin-Simon theorem)**.** *Let $T > 0$ and let B be a Banach space. Let $(B_\ell)_{\ell\in\mathbb{N}}$ be a sequence of finite dimensional subspaces of B. For any $\ell\in\mathbb{N}$, let $N_\ell\in\mathbb{N}^*$, $t_\ell^{(0)} =$*

$0 < t_\ell^{(1)} < \ldots < t_\ell^{(N_\ell)} = T$ and $\delta t_\ell^{(n)} = t_\ell^{(n)} - t_\ell^{(n-1)}$, $n = 1, \ldots, N_\ell$. Let $\{u_\ell^{(n)}, n = 0, \ldots, N_\ell\} \subset B_\ell$ and let $u_\ell \in L^1(0, T; B_\ell)$ be defined, for a given real family $(\alpha_\ell^{(n)})_{n=1,\ldots,N_\ell}$, by

$$\begin{aligned} u_\ell(\cdot, t) = (1 - \alpha_\ell^{(n)}) u_\ell^{(n-1)} + \alpha_\ell^{(n)} u_\ell^{(n)} \in B_\ell, \\ \text{for a.e. } t \in (t_\ell^{(n-1)}, t_\ell^{(n)}), \text{ and } n \in \{1, \ldots N_\ell\}. \end{aligned} \tag{3.24}$$

Let $\delta_\ell u_\ell$ be the "discrete time derivative", defined by :

$$\delta_\ell u_\ell(\cdot, t) = \delta_\ell^{(n)} u_\ell := \frac{1}{\delta t_\ell^{(n)}} (u_\ell^{(n)} - u_\ell^{(n-1)}) \text{ for a.e. } t \in (t_\ell^{(n-1)}, t_\ell^{(n)}), n \in \{1, \ldots, N_\ell\}.$$

Let $\| \cdot \|_{X_\ell}$ and $\| \cdot \|_{Y_\ell}$ be two norms on B_ℓ. We denote by X_ℓ the space B_ℓ endowed with the norm $\| \cdot \|_{X_\ell}$ and by Y_ℓ the space B_ℓ endowed with the norm $\| \cdot \|_{Y_\ell}$. We assume that

(h1) For any sequence $(w_\ell)_{\ell \in \mathbb{N}}$ such that $w_\ell \in B_\ell$ and $(\|w_\ell\|_{X_\ell})_{\ell \in \mathbb{N}}$ is bounded, then, up to a subsequence, there exists $w \in B$ such that $w_\ell \to w$ in B as $\ell \to +\infty$.

(h2) For any sequence $(w_\ell)_{\ell \in \mathbb{N}}$ such that $w_\ell \in B_\ell$, $(\|w_\ell\|_{X_\ell})_{\ell \in \mathbb{N}}$ is bounded, there exists $w \in B$ such that $w_\ell \to w$ in B and $\|w_\ell\|_{Y_\ell} \to 0$ as $\ell \to +\infty$, then $w = 0$.

(h3) The family $(\alpha_\ell^{(n)})_{n=1,\ldots,N_\ell, \ell \in \mathbb{N}}$ and the sequence $(\|u_\ell\|_{L^1(0,T;X_\ell)})_{\ell \in \mathbb{N}}$ are bounded.

(h4) The sequence $(\|\delta_\ell u_\ell\|_{L^1(0,T;Y_\ell)})_{\ell \in \mathbb{N}}$ is bounded.

Then there exists $u \in L^1(0, T; B)$ such that, up to a subsequence, $u_\ell \to u$ in $L^1(0, T; B)$ as $\ell \to +\infty$.

Now we can prove the convergence result.

Proof (of Theorem 3.20). Since the space-time consistency implies that $\Pi_{D_m} J_{D_m} u_{\text{ini}} \to u_{\text{ini}}$ in $\boldsymbol{L}^2(\Omega)$, we get that $(\|\Pi_{D_m} J_{D_m} u_{\text{ini}}\|_{\boldsymbol{L}^2(\Omega)})_{m \in \mathbb{N}}$ is bounded and therefore that the estimates given by Lemmas 3.21 and 3.24 are independent on $m \in \mathbb{N}$.

Step 1 : Application of 3.25 and consequences.
In our setting, the space B of the theorem is $\boldsymbol{L}^2(\Omega)$ and $B_m = \{\Pi_{D_m} v, v \in E_{D_m}\}$. The norm $\| \cdot \|_{X_m}$ is the norm $\| \cdot \|_{D_m}$ and the norm $\| \cdot \|_{Y_m}$ is defined in Definition 2.26.
The compactness property in the sense of Definition 3.10 of the sequence of discretisations $(D_m)_{m \in \mathbb{N}}$ and Estimate (3.22) give the existence of $\overline{u} \in \boldsymbol{L}^2(\Omega \times (0, T))$ and $\zeta \in \boldsymbol{L}^2(\Omega \times (0, T))^d$ such that, up to a subsequence (still indexed by m), $\Pi_{D_m} u_{D_m} \to \overline{u}$ in $\boldsymbol{L}^2(\Omega \times (0, T))$ and $\nabla_{D_m} u_{D_m} \to \zeta$ weakly in $\boldsymbol{L}^2(\Omega \times (0, T))^d$ and thus assumption (h_1) is satisfied.
The assumption (h_2) of the theorem is a consequence of Definition 2.26. Indeed, let $(\Pi_{D_m} v)_{m \in \mathbb{N}}$

such that $\Pi_{D_m} v \to v$ in B and such that $|v|_{\star,D_m} \to 0$ as $m \to \infty$, we have

$$\int_\Omega \Pi_{D_m} v \cdot \Pi_{D_m} v \, \mathrm{d}x \le |v|_{\star,D_m} \|v\|_{D_m} \to 0 \text{ as } m \to \infty$$

which shows that $v = 0$.
From estimates (3.22) and (3.23), we get that hypotheses (h_3) and (h_4) are satisfied. Therefore, we deduce that there exists $\overline{u} \in L^1(0, T, \boldsymbol{L}^2(\Omega))$ and a subsequence of $(D_m)_{m\in\mathbb{N}}$, denoted in the same way, such that $\Pi_{D_m} u_{D_m} \to \overline{u}$ in $L^1(0, T, \boldsymbol{L}^2(\Omega))$ as $m \to \infty$.

Step 2 : Proof that $\overline{u} \in L^2(0, T, E(\Omega))$.
Let $\varphi \in C^\infty(\overline{\mathbb{R}^d})^d$ and $\theta \in C_c^\infty(0, T)$ be given. We have, for all $n = 0, \ldots, N-1$, and all $t \in (t^{(n)}, t^{(n+1)})$,

$$|\int_{\mathbb{R}^d} \Big(\nabla_{D_m} u_m^{(n+1)} : \varphi\,\theta(t) + \Pi_{D_m} u_m^{(n+1)} \cdot \mathrm{div}\varphi\,\theta(t)\Big)\mathrm{d}x| \le W_{D_m}(\varphi|_\Omega)\,\theta(t)\,||\nabla_{D_m} u_{D_m}||_{\boldsymbol{L}^2(\Omega)^d}.$$

Integrating the above inequality over $t \in (t^{(n)}, t^{(n+1)})$, summing on $n = 0, \ldots, N-1$ and using Estimate (3.22), allows to follow the proof of Lemma 3.12, hence leading to $\overline{u} \in L^2(0, T; \boldsymbol{H}_0^1(\Omega))$ and $\mathrm{div}\overline{u} = 0$.

Step 3 : Proof that $\overline{u}$ is the solution to (3.9).
Let $\theta \in C_c^\infty([0, T))$ and $\overline{w} \in E(\Omega)$. As $(\overline{w}, 0)$ is the solution of the incompressible steady Stokes problem with $f = \eta\overline{w}$ and $G = \nabla\overline{w}$ (Problem (3.1) with $b = 0$), we can find for a given $m \in \mathbb{N}$, an approximation $w_{D_m} \in X_{D_m,0}$ such that $\int_\Omega \chi_{D_m} q \mathrm{div}_{D_m} w_{D_m} = 0$ for all $q \in Y_{D_m,0}$, $\Pi_{D_m} w_{D_m} \to \overline{w}$ in $\boldsymbol{L}^2(\Omega)$ and $\nabla_{D_m} w_{D_m} \to \nabla\overline{w}$ in $\boldsymbol{L}^2(\Omega)^d$. We take $v = \delta t^{(n+\frac{1}{2})}\theta(t^{(n)}) w_{D_m}$ as test function in Scheme (3.19) and we sum the resulting equation on $n = 0, ..., N-1$ to get

$T_1^{(m)} + T_2^{(m)} + T_3^{(m)} + T_4^{(m)} = T_5^{(m)}$ with

$$T_1^{(m)} = \sum_{n=0}^{N-1} \delta t^{n+\frac{1}{2}} \theta(t^{(n)}) \int_\Omega \Pi_{D_m} \delta^{(n+\frac{1}{2})} u_{D_m} \cdot \Pi_{D_m} w_{D_m} \mathrm{d}x,$$
$$T_2^{(m)} = \sum_{n=0}^{N-1} \delta t^{n+\frac{1}{2}} \theta(t^{(n)}) \int_\Omega \nabla_{D_m} u_{D_m}^{(n+1)} : \nabla_{D_m} w_{D_m} \mathrm{d}x,$$
$$T_3^{(m)} = -\sum_{n=0}^{N-1} \delta t^{n+\frac{1}{2}} \theta(t^{(n)}) \int_\Omega \chi_{D_m} p_{D_m}^{(n+1)} \mathrm{div}_{D_m} w_{D_m} \mathrm{d}x,$$
$$T_4^{(m)} = \sum_{n=0}^{N-1} \delta t^{n+\frac{1}{2}} \theta(t^{(n)}) \int_{t^{(n)}}^{t^{(n+1)}} b_{D_m}(u^{(n+1)}, w_{D_m}) \mathrm{d}t = \int_0^T b_{D_m}(u_{D_m}, \theta_{D_m} w_{D_m}) \mathrm{d}t,$$
$$T_5^{(m)} = \sum_{n=0}^{N-1} \theta(t^{(n)}) \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega (f \cdot \Pi_{D_m} w_{D_m} + G : \nabla_D w_{D_m}) \mathrm{d}x \mathrm{d}t.$$

First, we remark that $T_3^{(m)} = 0$ since $\int_\Omega \chi_{D_m} q \mathrm{div}_{D_m} w_{D_m} = 0$ for all $q \in Y_{D_m,0}$. Using discrete integration by parts and writing $\theta(t^{(n+1)}) - \theta(t^{(n)}) = \int_{t^{(n)}}^{t^{(n+1)}} \theta'(t) \mathrm{d}t$, we find

$$T_1^{(m)} = -\int_0^T \int_\Omega \theta' \Pi_{D_m} u_{D_m} \cdot \Pi_{D_m} w_D \mathrm{d}x \mathrm{d}t - \theta(0) \int_\Omega \Pi_{D_m} u_{D_m}^{(0)} \cdot \Pi_{D_m} w_{D_m} \mathrm{d}x.$$

Recall that $u_{D_m}^{(0)} = J_{D_m} u_{\mathrm{ini}}$, so that the space–time consistency (Definition 3.18) gives $\Pi_{D_m} u_{D_m}^{(0)} \to u_{\mathrm{ini}}$ in $\boldsymbol{L}^2(\Omega)$ as $m \to \infty$. Thus, by strong convergence in $\boldsymbol{L}^2(\Omega)$ of $\Pi_{D_m} w_{D_m}$ to $\overline{w}$,

$$T_1^{(m)} \to -\int_0^T \int_\Omega \theta' \overline{u} \cdot \overline{w} \, \mathrm{d}x \mathrm{d}t - \theta(0) \int_\Omega \overline{u}_{\mathrm{ini}} \cdot \overline{w} \, \mathrm{d}x.$$

Using the regularity of θ and the weak convergence of $\nabla_{D_m} u_{D_m}$ to $\nabla \overline{u}$ in $\boldsymbol{L}^2(\Omega \times (0,T))^d$, it easily comes

$$T_2^{(m)} \to \int_0^T \theta \int_\Omega \nabla \overline{u} : \nabla \overline{w} \mathrm{d}x \mathrm{d}t \quad \text{and} \quad T_5^{(m)} \to \int_0^T \theta \int_\Omega (f \cdot \overline{w} + G : \nabla \overline{w}) \mathrm{d}x \mathrm{d}t.$$

For the limit of $T_4^{(m)}$, we remark that the sequences $(\Pi_{D_m} u_{D_m})_{m\in\mathbb{N}}$, $(\nabla_{D_m} u_{D_m})_{m\in\mathbb{N}}$, $(\theta_{D_m} \Pi_{D_m} w_{D_m})_{m\in\mathbb{N}}$ and $(\theta_{D_m} \nabla_{D_m} w_{D_m})_{m\in\mathbb{N}}$ satisfy the required conditions for applying the space-time trilinear limit-conformity in the sense of Definition 3.19. This gives that

$$\lim_{m\to\infty} T_4^{(m)} = \int_0^T b(\overline{u}, \overline{u}, \theta \overline{w}) \, \mathrm{d}t.$$

Finally, passing to the limit in $T_1^{(m)} + T_2^{(m)} + T_3^{(m)} + T_4^{(m)} = T_5^{(m)}$ concludes the proof that $\overline{u}$ satisfies (3.9).

Chapitre 4

Examples of gradient discretisations for the Navier-Stokes problem

In this chapter, we present four examples of gradient discretisations : the MAC scheme on rectangular meshes, the conforming Taylor-Hood scheme, the Crouzeix-Raviart scheme and the HMM scheme which is an extension of Crouzeix-Raviart scheme on polyhedral meshes. For all of this schemes, we will check the consistency, the limit-conformity and the compactness property. To enter into the framework of the gradient discretisation for the Navier-Stokes problem, we need to check not only a discrete Poincare inequality but a discrete Sobolev one, because all our examples use the b_D built with ∇_D and Π_D presented in this chapter (see Section 4.1). That is why we will check for the property of p-coercivity (see Definition 4.3) to ensure that gradient discretisation are acceptable for the Navier-Stokes problem. We then observe that sequences of gradient discretisations which are consistent, limit-conforming and p-coercive for the Navier-Stokes problem are consistent, limit-conforming and coercive for the Stokes problem.

The second part of this chapter presents two possible methods for building the discrete operators div_D and b_D from ∇_D and Π_D. This kind of construction allows to prove the different properties more easily than in the general case.

Although gradient schemes are not necessarily based on meshes, most numerical methods for Stokes' equations are mesh-based. We give here the generic definition of a polyhedral mesh, following [41, 45]. We refer to Figure A.1 for some notations.

Definition 4.1 (Polyhedral mesh of Ω)**.** *Let Ω be a polyhedral open bounded domain of $\mathbb{R}^d$,*

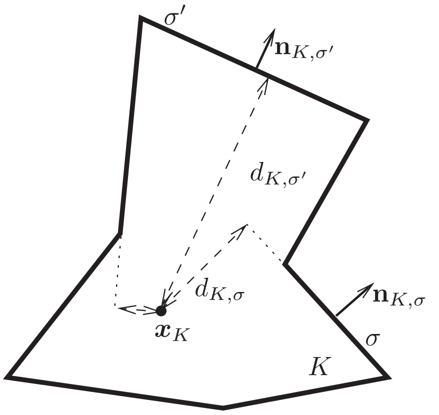


Figure A.1 – A cell K in a polyhedral mesh

$d \geq 1$. A polyhedral mesh of Ω is a triplet $(\mathcal{M}, \mathcal{E}, \mathcal{P})$ where :

1. *$\mathcal{M}$ is a finite family of non empty connected open disjoint subsets of Ω (the "cells") such that $\overline{\Omega} = \cup_{K\in\mathcal{M}}\overline{K}$. For $K \in \mathcal{M}$ we denote by $\partial K = \overline{K} \setminus K$ the boundary of K, by $|K| > 0$ the measure of K, and by h_K the diameter of K.*

2. *$\mathcal{E} = \mathcal{E}_{\rm int} \cup \mathcal{E}_{\rm ext}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the "edges" of the mesh – "faces" in 3D) such that any $\sigma \in \mathcal{E}$ is a non-empty open subset of an hyperplane of $\mathbb{R}^d$. $\mathcal{E}_{\rm int}$ is the set of edges included in Ω, and $\mathcal{E}_{\rm ext}$ is the set of edges included in $\partial\Omega$. The $(d-1)$-dimensional measure and the centre of gravity of $\sigma \in \mathcal{E}$ are respectively denoted by $|\sigma|$ and $\overline{x}_\sigma$.*

   We assume that, for all $K \in \mathcal{M}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \cup_{\sigma\in\mathcal{E}_K}\overline{\sigma}$. We then set $\mathcal{M}_\sigma = \{K \in \mathcal{M} : \sigma \in \mathcal{E}_K\}$ and we assume that, for all $\sigma \in \mathcal{E}$, either $\mathcal{M}_\sigma$ has exactly one element and $\sigma \in \mathcal{E}_{\rm ext}$, or $\mathcal{M}_\sigma$ has exactly two elements and $\sigma \in \mathcal{E}_{\rm int}$. For all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$, we denote by $\mathbf{n}_{K,\sigma}$ the unit vector normal to σ outward to K.

3. *$\mathcal{P} = (x_K)_{K\in\mathcal{M}}$ is a family of points of Ω indexed by $\mathcal{M}$ such that, for all $K \in \mathcal{M}$, K is strictly star-shaped with respect to x_K, meaning that for all $x \in K$ the segment $[x_K, x]$ is included in K. We let $d_{K,\sigma}$ be the signed distance between x_K and σ, that is :*

$$d_{K,\sigma} = (x - x_K) \cdot \mathbf{n}_{K,\sigma}, \quad x \in \sigma. \tag{4.1}$$

   Note that $(x - x_K) \cdot \mathbf{n}_{K,\sigma}$ is constant for $x \in \sigma$, and that it is strictly positive due to K being star-shaped with respect to x_K.

The size of the polyhedral mesh is defined by $h_\mathcal{M} = \max\{h_K : K \in \mathcal{M}\}$.

4.1 Example of b_D built with ∇_D and Π_D

In this section, we are interested in the construction of b_D inspired from the finite elements method and using discrete operators Π_D and ∇_D.

Definition 4.2 (construction of b_D)**.** *Let D be a gradient discretisation in the sense of Definition 3.6. Let $\widetilde{b_D} : X_{D,0}^3 \mapsto \boldsymbol{L}^2(\Omega)$ such that*

$$\widetilde{b_D}(u,v,w) = \sum_{i,j=1}^{d} \int_\Omega \Pi_D^{(i)} u \nabla_D^{(i,j)} v \Pi_D^{(j)} w \, dx.$$

We define b_D by :

$$b_D(u,v) = \frac{1}{2}\left(\widetilde{b_D}(u,u,v) - \widetilde{b_D}(u,v,u)\right). \tag{4.2}$$

The aim is to prove that the above definition of b_D ensures the trilinear limit-conformtity and space-time trilinear limit-conformity properties, as soon as the coercivity property is extended to a discrete Sobolev inequality and not only a discrete Poincaré inequality.

Definition 4.3 (p-coercivity)**.** *Let D be a discretisation in the sense of Definition 3.6. Let $q \in \mathbb{N}$ and let $C_D^{(p)}$ be defined by*

$$C_D^{(p)} = \max_{v \in X_{D,0}^*} \frac{\|\Pi_D v\|_{\boldsymbol{L}^p(\Omega)}}{\|v\|_D} + \max_{v \in X_{D,0}^*} \frac{\|\mathrm{div}_D v\|_{L^2(\Omega)}}{\|v\|_D}. \tag{4.3}$$

*A sequence $(D_m)_{m\in\mathbb{N}}$ of gradient discretisation is said to be **p-coercive** if there exist $C_S \geq 0$, and $\beta > 0$ such that $C_{D_m}^{(p)} \leq C_S$ and $\beta_{D_m} \geq \beta$, for all $m \in \mathbb{N}$.*

Proposition 4.4 (Space trilinear limit-conformity)**.** *Let $(D_m)_{m\in\mathbb{N}}$ be a sequence of gradient discretisations in the sense of Definition 3.6 which is $p-$ coercive for all $p \in [2,6]$ in the sense of Definition 4.3 and such that b_{D_m} is defined by (4.2). Then $(D_m)_{m\in\mathbb{N}}$ is trilinear limit-conforming in the sense of Definition 3.11.*

Proof. First, by definition of b_D, we can remark that, b_D is continuous, for all $u \in X_{D,0}$, $b_D(u,u) = 0$ and that $b_D(u,v)$ is linear with respect to v. Moreover, we may write

$$\widetilde{b}_D(u,u,v) \leq \|\Pi_D u\|_{\boldsymbol{L}^4(\Omega)} \|u\|_D \|\Pi_D v\|_{\boldsymbol{L}^4(\Omega)}.$$

Thanks to the p-coercivity for $p = 4$ of the discretisation, we obtain

$$\widetilde{b}_D(u,u,v) \leq (C_D^{(4)})^2 \|u\|_D^2 \|v\|_D.$$

Using the same idea for $\widetilde{b}_D(u,v,u)$, we finally get the admissibility of b_D in the sense of Definition 3.6, defining $B_D := 2(C_D^{(4)})^2$. Moreover the sequence $(B_{D_m})_{m\in\mathbb{N}}$ is bounded. It remains to prove that, for a sequence $(u_m, v_m)_{m\in\mathbb{N}} \in X^2_{D_m,0}$ with the properties given in Definition 3.11, $b_{D_m}(u_m, v_m) \to b(\overline{u}, \overline{u}, \overline{v})$. We remark that the strong convergence in $\boldsymbol{L}^2(\Omega)$ of $\Pi_{D_m} u_m \to \overline{u}$ and $\Pi_{D_m} v_m \to \overline{v}$ combined with the p-coercivity for $p = 6 > 4$, gives us the convergence in $\boldsymbol{L}^4(\Omega)$ of $\Pi_{D_m} u_m \to \overline{u}$ and $\Pi_{D_m} v_m \to \overline{v}$. Thus, for the first term of the right hand-side of b_D, the weak convergence in $\boldsymbol{L}^2(\Omega)^d$ of $\nabla_{D_m} v_m \to \nabla\overline{v}$ suffices for passing to the limit. Using the same idea for the second term of the right hand-side allows us to write the following result :

$$\lim_{m\to\infty} b_{D_m}(u_m, v_m) = \frac{1}{2}(b(\overline{u}, \overline{u}, \overline{v}) - b(\overline{u}, \overline{v}, \overline{u})).$$

Recalling the property (3.7) of b concludes the proof.

To prove the property of trilinear limit-conformity, we need the two following lemmas.

The two following results of space interpolation are used in the proof of the space-time trilinear limit-conformity of our b_D presented in chapter 4.

Lemma 4.5. *Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ three sequences such that*

1. *$u_n \to u$ in $L^1(0,T;L^2(\Omega))$*
2. *u_n is bounded in $L^2(0,T;L^6(\Omega))$ and in $L^\infty(0,T;L^2(\Omega))$*
3. *$v_n \to v$ weakly in $L^2(0,T;L^2(\Omega))$*
4. *$w_n \to w$ in $L^\infty(0,T;L^2(\Omega))$ and is bounded in $L^\infty(0,T;L^6(\Omega))$.*

then

$$\lim_{n\to\infty} \int_0^T \int_\Omega u_n(x,t) v_n(x,t) w_n(x,t) dx dt = \int_0^T \int_\Omega u(x,t) v(x,t) w(x,t) dx dt.$$

Proof. It suffices to remark that $u_n w_n$ tends to uw in $L^2(0,T;L^2(\Omega))$. Indeed, we have

$$\int_0^T \int_\Omega (u_n(x,t) w_n(x,t) - u(x,t) w(x,t))^2 dx dt \le 2(A_n + B_n)$$

with

$$A_n = \int_0^T \int_\Omega (u_n(x,t) - u(x,t))^2 w_n(x,t)^2 dx dt$$

and

$$B_n = \int_0^T \int_\Omega u(x,t)^2 (w_n(x,t) - w(x,t))^2 dx dt$$

The conclusion follow from the inequality

$$\int_0^T \int_\Omega U(x,t)^2 W(x,t)^2 dxdt \leq \int_0^T \left(\int_\Omega U(x,t)^4 dx\right)^{1/2} \left(\int_\Omega W(x,t)^4 dx\right)^{1/2} dt$$

which leads to

$$\int_0^T \int_\Omega U(x,t)^2 W(x,t)^2 dxdt \leq \|W\|^2_{L^\infty(0,T;L^4(\Omega))} \|U\|^2_{L^2(0,T;L^4(\Omega))}.$$

Lemma 4.6. *Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ three sequences such that*

1. $u_n \to u$ *in* $L^1(0,T;L^2(\Omega))$
2. u_n *is bounded in* $L^2(0,T;L^6(\Omega))$ *and in* $L^\infty(0,T;L^2(\Omega))$
3. $v_n \to v$ *in* $L^1(0,T;L^2(\Omega))$
4. v_n *is bounded in* $L^2(0,T;L^6(\Omega))$ *and in* $L^\infty(0,T;L^2(\Omega))$
5. $w_n \to w$ *in* $L^\infty(0,T;L^2(\Omega))$.

then

$$\lim_{n\to\infty} \int_0^T \int_\Omega u_n(x,t) v_n(x,t) w_n(x,t) dxdt = \int_0^T \int_\Omega u(x,t) v(x,t) w(x,t) dxdt.$$

Proof.

$$\int_0^T \int_\Omega (u_n(x,t) v_n(x,t) w_n(x,t) - u(x,t) v(x,t) w(x,t)) dxdt = A_n + B_n + C_n$$

with

$$A_n = \int_0^T \int_\Omega (u_n(x,t) - u(x,t)) v_n(x,t) w_n(x,t) dxdt$$

$$B_n = \int_0^T \int_\Omega u(x,t)(v_n(x,t) - v(x,t)) w_n(x,t) dxdt$$

and

$$C_n = \int_0^T \int_\Omega u(x,t) v(x,t)(w_n(x,t) - w(x,t)) dxdt$$

The conclusion follows from the fact that

$$\int_0^T \int_\Omega |U(x,t)V(x,t)W(x,t)| dxdt \leq \|W\|_{L^\infty(0,T;L^2(\Omega))} \|U\|_{L^2(0,T;L^4(\Omega))} \|V\|_{L^2(0,T;L^4(\Omega))}$$

Indeed,

$$\int_0^T\int_\Omega |U(x,t)V(x,t)W(x)|dxdt \le \sup_{t\in]0,T[}(\int_\Omega W(x,t)^2dx)^{1/2}(\int_\Omega U(x,t)^2V(x,t)^2dx)^{1/2}dt,$$

so

$$\int_0^T\int_\Omega |U(x,t)V(x,t)W(x)|dxdt \le \sup_{t\in]0,T[}\int_\Omega (W(x,t)^2dx)^{1/2}\int_0^T(\int_\Omega U(x,t)^4dx\int_\Omega V(x,t)^4dx)^{1/4}dt,$$

and therefore

$$\int_0^T\int_\Omega |U(x,t)V(x,t)W(x)|dxdt \le \sup_{t\in]0,T[}(\int_\Omega W(x,t)^2dx)^{1/2}\left(\int_0^T(\int_\Omega U(x,t)^4dx)^{1/2}dt\right)^{1/2} \left(\int_\Omega V(x,t)^4dx)^{1/2}dt\right)^{1/2},$$

which concludes the proof.

Proposition 4.7 (Space-time trilinear-conformity)**.** *Let $(D_m)_{m\in\mathbb{N}}$ be a sequence of space-time gradient discretisations in the sense of Definition 3.17 which is $p-$ coercive in the sense of Definition 4.3 and such that b_{D_m} is defined by (4.2). Then $(D_m)_{m\in\mathbb{N}}$ is space-time trilinear limit-conforming in the sense of Definition 3.19.*

Proof. We consider sequences $(u_m)_{m\in\mathbb{N}}$ and $(v_m)_{m\in\mathbb{N}}$ satisfying that

$\|\nabla_{D_m}u_m\|_{\boldsymbol{L}^2(\Omega\times(0,T))^d}$ and $\|\nabla_{D_m}v_m\|_{\boldsymbol{L}^2(\Omega\times(0,T))^d}$ are bounded and that there exists $(\overline{u},\overline{v})\in L^2(0,T,H_0^1(\Omega))^2$ such that

— $\Pi_{D_m}u_m\to\overline{u}$ in $L^1(0,T,\boldsymbol{L}^2(\Omega))$ and $\|\Pi_{D_m}u_m\|_{L^\infty(0,T,\boldsymbol{L}^2(\Omega))}$ is bounded,

— $\nabla_{D_m}u_m\rightharpoonup\nabla\overline{u}$ weakly in $\boldsymbol{L}^2(\Omega\times(0,T))^d$,

— $\Pi_{D_m}v_{D_m}\to\overline{v}$ in $L^\infty(0,T,\boldsymbol{L}^2(\Omega))$

— $\nabla_{D_m}v_m\to\nabla\overline{v}$ in $L^\infty(0,T,\boldsymbol{L}^2(\Omega))^d$,

Let us check that the four items which are assumed in Lemma 4.5 are satisfied. Items 1 and 3 are assumed on $(\Pi^{(i)}_{D_m}u_m)_{m\in\mathbb{N}}$ and $(\nabla^{(i,j)}_D u_m)_{m\in\mathbb{N}}$. The fact that $(\Pi^{(i)}_{D_m}u_m)_{m\in\mathbb{N}}$ is bounded in $L^2(0,T,\boldsymbol{L}^6(\Omega))$ is a consequence of the p-coercivity assumption thanks to the fact that $\|\nabla_{D_m}u_m\|_{\boldsymbol{L}^2(\Omega\times(0,T))}$ is bounded. Similarly, using the fact that $\|\nabla_{D_m}v_m\|_{L^\infty(0,T,\boldsymbol{L}^2(\Omega))}$ is bounded, we get a bound on $\|\Pi^{(j)}_{D_m}v_m\|_{L^\infty(0,T,\boldsymbol{L}^6(\Omega))}$, hence items 2 and 4 hold. Thus we can apply Lemma

4.5, which leads to

$$\lim_{m\to\infty}\int_0^T \widetilde{b}_{D_m}(u_m,u_m,v_m) = \int_0^T b(\overline{u},\overline{u},\overline{v}).$$

We check as well that the five items assumed in Lemma 4.6 hold, which shows that

$$\lim_{m\to\infty}\int_0^T \widetilde{b}_{D_m}(u_m,v_m,u_m) = \int_0^T b(\overline{u},\overline{v},\overline{u}).$$

Finally, the last two limits combined with Property 3.7 on b conclude the space-time trilinear limit-conformity.

4.2 The MAC scheme

The Marker-And-Cell (MAC) scheme [80, 94, 103] can be easily defined on domains where the boundary is composed of subparts parallel to the axes. Let us show that its basic version fits the gradient scheme framework. For simplicity of presentation, we restrict the presentation to 2D domains and we further assume that $\Omega = (a,b)\times(c,d)$.

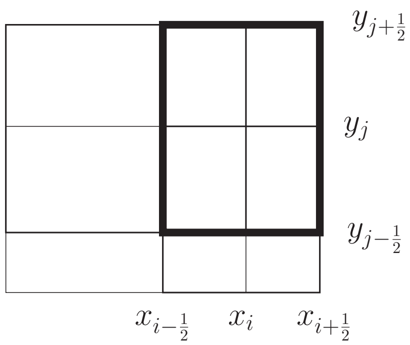


Figure A.2 – Mesh for the MAC method.

We introduce, for given $N,M\in\mathbb{N}^\star$, finite real sequences $x_{\frac{1}{2}} = a < x_{1+\frac{1}{2}} \ldots < b = x_{N+\frac{1}{2}}$ and $y_{\frac{1}{2}} = c < y_{1+\frac{1}{2}} \ldots < d = y_{M+\frac{1}{2}}$. We set $x_0 = a$, $x_i = \frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}})$ for $i = 1,\ldots,N$, $x_{N+1} = b$, $y_0 = c$, $y_j = \frac{1}{2}(y_{j-\frac{1}{2}} + y_{j+\frac{1}{2}})$ for $j = 1,\ldots,M$, $y_{M+1} = d$. We then define a gradient discretization $D = (X_{D,0}, Y_D, \Pi_D, \chi_D, \nabla_D, \mathrm{div}_D)$ as follows.

1. The space of the discrete velocities $X_{D,0}$ is defined by

$$X_{D,0} = \{u = ((u_{i+\frac{1}{2},j})_{i=0,\ldots,N,j=0,\ldots,M+1}, (u_{i,j+\frac{1}{2}})_{i=0,\ldots,N+1,j=0,\ldots,M}) :$$
$$u_{i+\frac{1}{2},j}\in\mathbb{R},\ u_{i,j+\frac{1}{2}}\in\mathbb{R},\ u_{\frac{1}{2},j} = u_{N+\frac{1}{2},j} = u_{i,\frac{1}{2}} = u_{i,M+\frac{1}{2}} = 0 \text{ for all } i,j\}.$$

2. The space of the discrete pressures Y_D is

$$Y_D = \{p = (p_{i,j})_{i=1,\dots,N,j=1,\dots,M} \;:\; p_{i,j} \in \mathbb{R}\}.$$

3. For all $u \in X_{D,0}$, $\Pi_D u = (\Pi_D^{(1)}u, \Pi_D^{(2)}u) \in L^2(\Omega)^2$ with $\Pi_D^{(1)}u$ piecewise constant equal to $u_{i+\frac{1}{2},j}$ in $(x_i, x_{i+1}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$ for $i = 0, \dots, N$ and $j = 1, \dots, M$, and $\Pi_D^{(2)}u$ piecewise constant equal to $u_{i,j+\frac{1}{2}}$ in $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_j, y_{j+1})$ for $i = 1, \dots, N$ and $j = 0, \dots, M$ (this definition accounts for the boundary conditions on the velocity).
4. For all $p \in Y_{D,0}$, $\chi_D p$ is piecewise constant equal to $p_{i,j}$ in $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$ for $i = 1, \dots, N$ and $j = 1, \dots, M$.
5. For all $u \in X_{D,0}$, $\nabla_D u = (\nabla_D^{(a,b)}u)_{a,b=1,2} \in L^2(\Omega)^4$ with $\nabla_D^{(a,b)}u$ the piecewise constant approximation of the b-th derivative of the a-th component defined by :
   (a) $\nabla_D^{(1,1)}u = \frac{u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}$ on $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$ for $i = 1, \dots, N$ and $j = 1, \dots, M$,
   (b) $\nabla_D^{(1,2)}u = \frac{u_{i+\frac{1}{2},j+1} - u_{i+\frac{1}{2},j}}{y_{j+1} - y_j}$ on $(x_i, x_{i+1}) \times (y_j, y_{j+1})$ for $i = 0, \dots, N$ and $j = 0, \dots, M$,
   (c) $\nabla_D^{(2,1)}u = \frac{u_{i+1,j+\frac{1}{2}} - u_{i,j+\frac{1}{2}}}{x_{i+1} - x_i}$ on $(x_i, x_{i+1}) \times (y_j, y_{j+1})$ for $i = 0, \dots, N$ and $j = 0, \dots, M$,
   (d) $\nabla_D^{(2,2)}u = \frac{u_{i,j+\frac{1}{2}} - u_{i,j-\frac{1}{2}}}{y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}}$ on $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$ for $i = 1, \dots, N$ and $j = 1, \dots, M$.
6. $\mathrm{div}_D u = \mathrm{Tr}(\nabla_D u) = \nabla_D^{(1,1)}u + \nabla_D^{(2,2)}u$ (constant in $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$ for $i = 1, \dots, N$ and $j = 1, \dots, M$).

The gradient scheme (2.7) stemming from such a gradient discretisation is identical to the standard MAC scheme, classically written using finite differences. Setting $h_D = \max_{i,j}(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}})$, we have the following result.

Proposition 4.8 (Properties of the MAC gradient discretisation)**.**

For any $m \in \mathbb{N}$ we define a gradient discretisation D_m as above, from some finite sequences $(x_{i+\frac{1}{2}})_{i=0,\dots,N_m}$ and $(y_{j+\frac{1}{2}})_{j=0,\dots,M_m}$ such that h_{D_m} tends to 0 as $m \to \infty$. Then $(D_m)_{m\in\mathbb{N}}$ is p-coercive, consistent, limit-conforming and compact in the sense of Definitions 4.3, 3.8, 3.9 and 3.10.

Proof. We drop the indices m for legibility.

P-COERCIVITY : Since the definition of ∇_D is corresponding to the discrete gradient of a finite volume scheme on a mesh satisfying the usual orthogonality property, the bound on C_D is a consequence of the discrete Sobolev inequality obtained in [29] or [57, Lemma 9.5 p. 790] (the control of div_D by ∇_D is then trivial from its definition). The lower bound on β_D is a consequence of Nečas' result [17, 88]. Indeed, for any $q \in Y_{D,0}$ we can find $w \in \boldsymbol{H}_0^1(\Omega)$ such that $\text{div} w = \chi_D q$ and $\|w\|_{\boldsymbol{H}_0^1(\Omega)} \le C\|\chi_D q\|_{\boldsymbol{L}^2(\Omega)}$. The lower bound on β_D is obtained by considering $v \in X_{D,0}$ defined by averaging this function w on all edges, and by applying [58, Lemma 9.4 p 776].

Consistency : The consistency for the pressure stems from the fact that, given a family of meshes whose size tend to 0, any L^2 function can be approximated by sequences of piecewise constant functions on the meshes. The consistency for the velocity is equally immediate, since Taylor expansions show that for a regular φ the interpolation $u \in X_{D,0}$ defined by $u_{i+\frac{1}{2},j} = \varphi(x_{i+\frac{1}{2}}, y_j)$ and $u_{i,j+\frac{1}{2}} = \varphi(x_i, y_{j+\frac{1}{2}})$ has a reconstruction $\Pi_D u$ and a discrete gradient $\nabla_D u$ close respectively to φ and $\nabla\varphi$ if the mesh size is small.

Limit-conformity : We use Lemma 2.5. We start by taking $\varphi \in \mathbf{C}^\infty(\overline{\Omega})^2$ and we show that $W_D(\varphi) \to 0$ as $h_D \to 0$. The study is simplified by considering each component of the gradient separately. For $u \in X_{D,0}$ we have

$$\begin{aligned}\int_\Omega \nabla_D^{(1,1)} u\, \varphi^{(1,1)} \mathrm{d}x\mathrm{d}y &= \sum_{\substack{i=1,\dots,N\\ j=1,\dots,M}} \int_{(x_{i-\frac{1}{2}},x_{i+\frac{1}{2}})\times(y_{j-\frac{1}{2}},y_{j+\frac{1}{2}})} \frac{(u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j})\varphi^{(1,1)}(x,y)}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} \mathrm{d}x\mathrm{d}y \\ &= \sum_{\substack{i=1,\dots,N\\ j=1,\dots,M}} (u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j})\varphi_{i,j}(y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}) \end{aligned} \tag{4.4}$$

where $\varphi_{i,j}$ is the average of $\varphi^{(1,1)}$ on $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$. Using a discrete integration by parts and the Dirichlet boundary conditions, we may write

$$\begin{aligned}\sum_{\substack{i=1,\dots,N\\ j=1,\dots,M}} (u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j})\varphi_{i,j}(y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}) &= -\sum_{\substack{i=1,\dots,N\\ j=1,\dots,M}} u_{i+\frac{1}{2},j} \frac{\varphi_{i+1,j} - \varphi_{i,j}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}}(y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}})(x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}) \\ &= -\int_\Omega \Pi_D^{(1)} u\, \partial^{(1)}\varphi^{(1,1)} \mathrm{d}x\mathrm{d}y + R_D, \end{aligned} \tag{4.5}$$

where $|R_D| \le Ch_D\|u\|_D$ with C only depending on φ and C_D from (2.4). We used the fact that $\varphi_{i,j} = \varphi^{(1,1)}(x_i, y_j) + O((x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}})(y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}))$, which implies $\frac{\varphi_{i+1,j}-\varphi_{i,j}}{x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}}} = \partial^{(1)}\varphi^{(1,1)}(x_i, y_j) + \mathcal{O}(y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}) = \partial^{(1)}\varphi^{(1,1)} + \mathcal{O}(h_D)$ on $(x_i, x_{i+1}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$. Combining (4.4) and (4.5), we get

$$\left| \int_\Omega (\nabla_D^{(1,1)} u\, \varphi^{(1,1)} + \Pi_D^{(1)} u\, \partial^{(1)}\varphi^{(1,1)}) \mathrm{d}x\mathrm{d}y \right| \le Ch_D||u||_D.$$

Using the same idea for each component of ∇_D, we conclude that $W_D(\varphi) \to 0$ as $m \to \infty$.

Since $\text{div}_D = \nabla_D^{(1,1)} + \nabla_D^{(2,2)}$, we get that $\widetilde{W}_D(\psi) = 0$ for all $\psi \in C^\infty(\overline{\Omega})$ and this completes the proof of the limit-conformity.

COMPACTNESS : The compactness property is resulting from [57, Lemma 9.3 p. 770].

Remark 4.9. *A close inspection of the proof shows that for* $v \in \boldsymbol{H}^2(\Omega)$*,* $\varphi \in \boldsymbol{H}^2(\Omega)^d$ *and* $\psi \in H^2(\Omega)$*, the quantities* $W_D(\varphi)$*,* $S_D(v)$ *and* $\widetilde{S}_D(\psi)$ *are all of order* $\mathcal{O}(h_D)$*. If* $(\overline{u}, \overline{p}) \in \boldsymbol{H}^3(\Omega) \times H^1(\Omega)$*, the rates of convergence provided for the MAC method by the theorems in Section 2.3.1 for the steady Stokes problem are therefore of order one.*

4.3 Conforming Taylor–Hood scheme

We assume here that Ω is a polyhedral bounded open domain of $\mathbb{R}^d$, $d = 2$ or 3, and we take a *simplicial* mesh of Ω, that is a mesh $(\mathcal{M}, \mathcal{E}, \mathcal{P})$ in the sense of Definition 4.1 such that for each cell K we have $\mathrm{Card}(\mathcal{E}_K) = d + 1$. Each cell is therefore a simplex (triangles in 2D, tetrahedra in 3D), and the mesh does not have any hanging node : two neighbouring cells meet along an entire edge/face of each of them.

The set of vertices of the mesh, that is the non-empty intersections of the closures of two edges (in 2D) or 3 faces or more (in 3D), is denoted by $\mathcal{V}$. We define $\theta_{\mathcal{M}} = \inf\{\frac{h_K}{\xi_K} : K \in \mathcal{M}\}$, where ξ_K is the diameter of the largest ball included in K.

The Taylor–Hood scheme [99] on such a simplicial mesh can be seen as the gradient scheme corresponding to the gradient discretisation $D = (X_{D,0}, Y_D, \Pi_D, \chi_D, \nabla_D, \mathrm{div}_D)$ defined as follows.

1. The space of the discrete velocities is $X_{D,0} = \{(v_s)_{s \in V^{(2)}} : v_s \in \mathbb{R}^d\}$, where $V^{(2)} = \mathcal{V} \cup \{\overline{x}_\sigma : \sigma \in \mathcal{E}_K\}$ is the set of nodes of the $\mathbb{P}^2$ finite element discretisation on the simplicial mesh.
2. The space of the discrete pressures is $Y_D = \{(p_s)_{s \in V^{(1)}} : p_s \in \mathbb{R}\}$, where $V^{(1)} = \mathcal{V}$ is the set of nodes of the $\mathbb{P}^1$ finite element discretisation on the mesh.
3. For all $v \in X_{D,0}$, $\Pi_D v = \sum_{s \in V^{(2)}} v_s \varphi_s^{(2)}$, where $\varphi_s^{(2)}$ is the scalar $\mathbb{P}^2$ finite element basis function associated with the node s.
4. For all $p \in Y_D$, $\chi_D p = \sum_{s \in V^{(1)}} p_s \varphi_s^{(1)}$, where $\varphi_s^{(1)}$ is the scalar $\mathbb{P}^1$ finite element basis function associated with the node s.
5. For all $v \in X_{D,0}$, $\nabla_D v = \nabla(\Pi_D v)$.
6. For all $v \in X_{D,0}$, $\mathrm{div}_D v = \mathrm{div}(\Pi_D v)$.

Proposition 4.10 (Properties of the Taylor–Hood gradient discretisation)**.**

Let $(\mathcal{M}_m, \mathcal{E}_m, \mathcal{P}_m)_{m\in\mathbb{N}}$ be a sequence of simplicial meshes of Ω such that $(\theta_{\mathcal{M}_m})_{m\in\mathbb{N}}$ remains bounded, and that $h_{\mathcal{M}_m} \to 0$ as $m \to \infty$. We assume that every cell of every mesh in the sequence has at least d edges in Ω. Let D_m be the gradient discretisation defined as above for $(\mathcal{M}, \mathcal{P}, \mathcal{E}) = (\mathcal{M}_m, \mathcal{P}_m, \mathcal{E}_m)$. Then $(D_m)_{m\in\mathbb{N}}$ is p-coercive, consistant, limit-conforming and compact in the sense of Definitions 4.3, 3.8, 3.9 and 3.10.

Proof. We drop the indices m for legibility.

P-COERCIVITY : Since $\Pi_{D_m}(X_{D_m,0})$ is the set of continuous and piecewise $\mathbb{P}^2$ functions, it is a subset of $H^1_0(\Omega)$. Discrete Sobolev inequality is a consequence of the continuous one because of the conforming scheme. Applying [49, Lemma 4.24] to estimate β_{D_m}, we obtain the coercivity of $(D_m)_{m\in\mathbb{N}}$.

CONSISTENCY : The consistency is proved in [28, Theorem 3.1.6] in the general case of $\mathbb{P}^k$ finite element, thus we just apply this result with $k = 2$ for the discrete velocity and $k = 1$ for the discrete pressure.

LIMIT-CONFORMITY : Because of the definition of ∇_{D_m} and div_{D_m}, W_{D_m} and $\widetilde{W}_{D_m}$ are identically null.

COMPACTNESS : Consequence of the Rellich theorem.

Remark 4.11. *The consistency in [28, Theorem 3.1.6] gives an $\mathcal{O}(h^2_{\mathcal{M}})$ estimate on S_D for functions in $H^3(\Omega)$. Hence, since there is no defect of conformity for the Taylor–Hood method, the rates of convergence provided for this scheme by the theorems in Section 2.3.1 are therefore of order two.*

4.4 The Crouzeix–Raviart scheme

As for the Taylor–Hood method, we consider a simplicial mesh $(\mathcal{M}, \mathcal{E}, \mathcal{P})$ of a bounded polyhedral domain Ω. We still set $\theta_{\mathcal{M}} = \inf\{\frac{h_K}{\xi_K} : K \in \mathcal{M}\}$, where ξ_K is the diameter of the largest ball included in K The Crouzeix–Raviart scheme [30] can be seen as a gradient scheme with the gradient discretisation defined as follows.

1. The space of the discrete velocities is $X_{D,0} = \{v = (v_\sigma)_{\sigma\in\mathcal{E}} : v_\sigma \in \mathbb{R}^d\,,\ v_\sigma = 0 \text{ for all } \sigma \in \mathcal{E}_{\mathrm{ext}}\}$.
2. The space of the discrete pressures is $Y_D = \{p = (p_K)_{K\in\mathcal{M}} : p_K \in \mathbb{R}\}$.

3. The linear mapping $\Pi_D : X_{D,0} \to \boldsymbol{L}^2(\Omega)$ is the nonconforming piecewise affine reconstruction of each component of the velocity defined by

$$\forall v \in X_{D,0},\ \Pi_D v = \sum_{\sigma\in\mathcal{E}} v_\sigma \varphi_\sigma, \tag{4.6}$$

where φ_σ is the non-conforming $\mathbb{P}^1$ basis function associated with the face σ.

4. The linear mapping $\chi_D : Y_D \to L^2(\Omega)$ is defined by : for $p \in Y_D$ and $K \in \mathcal{M}$, $\chi_D p = p_K$ on K.

5. The linear mapping $\nabla_D : X_{D,0} \to \boldsymbol{L}^2(\Omega)^d$ is the piecewise constant "broken gradient" :

$$\forall v \in X_{D,0},\ \forall K \in \mathcal{M}\,,\ (\nabla_D v)_{|K} = (\nabla(\Pi_D v))_{|K}. \tag{4.7}$$

6. The linear mapping $\mathrm{div}_D : X_{D,0} \to L^2(\Omega)$ is defined by

$$\forall v \in X_{D,0}\,,\ \forall K \in \mathcal{M}\,,\ \mathrm{div}_D v = \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}_K} |\sigma| v_\sigma \cdot \mathbf{n}_{K,\sigma} = (\mathrm{div}(\Pi_D v))_{|K}. \tag{4.8}$$

Proposition 4.12 (Properties of the Crouzeix–Raviart gradient discretisation)**.**

Let $(\mathcal{M}_m, \mathcal{E}_m, \mathcal{P}_m)_{m\in\mathbb{N}}$ be a sequence of simplicial meshes of Ω, such that $(\theta_{\mathcal{M}_m})_{m\in\mathbb{N}}$ remains bounded and $h_{\mathcal{M}_m} \to 0$ as $m \to \infty$. We define D_m as above for $(\mathcal{M}, \mathcal{E}, \mathcal{P}) = (\mathcal{M}_m, \mathcal{E}_m, \mathcal{P}_m)$. Then $(D_m)_{m\in\mathbb{N}}$ is p-coercive, consistent, limit-conforming and compact in the sense of Definitions 4.3, 3.8, 3.9 and 3.10.

Proof. We drop the indices m for legibility.

P-COERCIVITY : Direct consequence of [49, Lemmas 4.30 and 4.31], and of [74] for the discrete Sobolev inequality.

CONSISTENCY : The consistency of the operators related to the velocity is shown in [72, Theorem 2.1]. The consistency for the interpolation of the pressure is straightforward since, as $h_{\mathcal{M}_m} \to 0$, any function can be approximated in $L^2(\Omega)$ by piecewise constant functions on $\mathcal{M}_m$.

LIMIT-CONFORMITY : Since Ω is polyhedral and thus locally star-shaped, smooth functions are dense in $\boldsymbol{H}_{\mathrm{div}}(\Omega)$ and in $L^2(\Omega)$, we only need to study the convergence of W_{D_m} and $\widetilde{W}_{D_m}$ on smooth functions (see Remark 2.6).

Let us handle W_{D_m} first. To simplify the notations, we drop the index m. We also only consider one component of the discrete velocity $v \in X_{D,0}$; we therefore treat $\Pi_D v$ as a scalar function and $\nabla_D v$ as a function with values in $\mathbb{R}^d$. Let $\varphi \in \boldsymbol{C}^\infty(\overline{\Omega})$. Since $\nabla_D v = \nabla(\Pi_D v)$ on each $K \in \mathcal{M}$, we have

$$\int_\Omega (\nabla_D v \cdot \varphi + \Pi_D v \mathrm{div}\varphi)\mathrm{d}x = \sum_{K\in\mathcal{M}_m} \int_K (\nabla(\Pi_D v)\cdot\varphi + \Pi_D v \mathrm{div}\varphi)\mathrm{d}x = \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \int_\sigma \varphi\cdot[\Pi_D v\ \mathbf{n}]_\sigma \mathrm{d}\gamma(x) \tag{4.9}$$

where γ is the $(d-1)$-dimensional measure on σ, and $[\Pi_D v\ \mathbf{n}]_\sigma = \Pi_K v\, \mathbf{n}_{K,\sigma} + \Pi_L v\, \mathbf{n}_{L,\sigma}$ with $\{K,L\} = \mathcal{M}_\sigma$ and $\Pi_K v = (\Pi_D v)_{|K}$. For any $x\in\sigma$ we have $\Pi_K v(x) = \Pi_K v(\overline{x}_\sigma) + \nabla_K v\cdot(x-\overline{x}_\sigma)$ where $\nabla_K v = (\nabla_D v)_{|K}$. Since $\Pi_K v(\overline{x}_\sigma) = \Pi_L v(\overline{x}_\sigma)$ we deduce from (4.9) that

$$\int_\Omega (\nabla_D v\cdot\varphi + \Pi_D v \mathrm{div}\varphi)\mathrm{d}x = \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \int_\sigma \varphi\cdot[\mathbf{n}\otimes\nabla_D v]_\sigma (x-\overline{x}_\sigma)\mathrm{d}\gamma(x). \tag{4.10}$$

The smoothness of φ gives, for any $x\in\sigma$, $|\varphi(x)-\varphi(\overline{x}_\sigma)| \le ||\nabla\varphi||_\infty h_K$. Moreover, since $[\mathbf{n}\otimes\nabla_D v]_\sigma$ is constant over σ and $\overline{x}_\sigma$ is the centre of gravity of σ,

$$\int_\sigma \varphi(\overline{x}_\sigma)\cdot[\mathbf{n}\otimes\nabla_D v]_\sigma (x-\overline{x}_\sigma)\mathrm{d}\gamma(x) = \varphi(\overline{x}_\sigma)\cdot[\mathbf{n}\otimes\nabla_D v]_\sigma \int_\sigma (x-\overline{x}_\sigma)\mathrm{d}\gamma(x) = 0.$$

Introducing $\varphi(\overline{x}_\sigma)$ into (4.10) and using the bound on $\theta_\mathcal{M}$ to write $h_K|\sigma| \le C|K|$ with C not depending on $K\in\mathcal{M}$ or $\sigma\in\mathcal{E}_K$, we infer that

$$\begin{aligned}\left|\int_\Omega (\nabla_D v\cdot\varphi + \Pi_D v \mathrm{div}\varphi)\mathrm{d}x\right| &\le ||\nabla\varphi||_\infty \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} |\sigma|(h_K^2|\nabla_K v| + h_L^2|\nabla_L v|)\\ &\le C||\nabla\varphi||_\infty h_{\mathcal{M}_m} \sum_{K\in\mathcal{M}_m} |\nabla_K v| \sum_{\sigma\in\mathcal{E}_K} |K|\\ &= (d+1)C||\nabla\varphi||_\infty h_{\mathcal{M}_m} ||\nabla_D v||_{\boldsymbol{L}^1(\Omega)}.\end{aligned}$$

This shows that $W_{D_m}(\varphi) \to 0$ as $m\to\infty$. Since in each cell $K\in\mathcal{M}$ we have $\mathrm{div}_D v = \mathrm{div}(\Pi_D v) = \mathrm{Tr}(\nabla(\Pi_D v)) = \mathrm{Tr}(\nabla_D v)$ where Tr is the trace of matrices, this gives directly that $\widetilde{W}_D(\psi) = 0$.

Compactness : The compactness property is proved in [73, Theorem 3.3].

Remark 4.13. *As in Remark 4.9, a close inspection of the proof shows for smooth functions estimates $\mathcal{O}(h_\mathcal{M})$ on W_D, S_D and $\widetilde{S}_D$, and the theorems in Section 2.3.1 thus give order one convergence rates on the Crouzeix–Raviart scheme if $(\overline{u},\overline{p}) \in \boldsymbol{H}^3(\Omega)\times H^1(\Omega)$.*

4.5 The HMM extension of the Crouzeix–Raviart scheme

We now turn to a natural extension of the Crouzeix–Raviart scheme to any polyhedral mesh, namely the "Hybrid Mimetic Mixed" (HMM) scheme [42]. A version of this method was developed and studied for Navier–Stokes equations in [38], but with a less efficient stabilisation and with convergence results less detailed than the ones in Section 2.3. The HMM family of schemes contains the Mimetic Finite Difference method of [20]. We note that these mimetic schemes are different from the ones described in [9, 10], where both scalar edge and vector vertex unknowns are used for the velocity. After static condensation to eliminate cell unknowns, the method we present here only uses vector edge unknowns for the velocity and therefore leads to a smaller linear system.

Let $(\mathcal{M}, \mathcal{E}, \mathcal{P})$ be a polyhedral mesh of Ω in the sense of Definition 4.1. For all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$, we denote by $D_{K,\sigma}$ the cone with vertex x_K and basis σ, that is $D_{K,\sigma} = \{x_K + t(y - x_K), y \in \sigma, t \in [0,1]\}$. The HMM gradient discretisation is defined as follows.

1. The space of the discrete velocities is $X_{D,0} = \{v = ((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{E}}) : v_K \in \mathbb{R}^d, v_\sigma \in \mathbb{R}^d\,,\ v_\sigma = 0 \text{ for all } \sigma \in \mathcal{E}_{\rm ext}\}$.
2. The space of the discrete pressures is $Y_D = \{p = (p_K)_{K\in\mathcal{M}} : p_K \in \mathbb{R}\}$.
3. The linear mapping $\Pi_D : X_{D,0} \to \boldsymbol{L}^2(\Omega)$ is the nonconforming piecewise constant reconstruction in the control volumes of each component of the velocity, defined by
$$\forall v \in X_{D,0}\,,\ \forall K \in \mathcal{M},\ \Pi_D v = v_K \text{ on } K. \tag{4.11}$$
4. The linear mapping $\chi_D : Y_D \to L^2(\Omega)$ is defined, as for the Crouzeix–Raviart gradient discretisation, by
$$\forall p \in Y_D\,,\ \forall K \in \mathcal{M},\ \chi_D p = p_K \text{ on } K. \tag{4.12}$$
5. The piecewise constant gradient $\nabla_D : X_{D,0} \to \boldsymbol{L}^2(\Omega)^d$ is defined by
$$\forall K \in \mathcal{M}\,, \forall \sigma \in \mathcal{E}_K\,, \forall x \in D_{K,\sigma},\ \nabla_D v(x) = \nabla_K v + \frac{\sqrt{d}}{d_{K,\sigma}}(\mathcal{A}_K \mathcal{R}_K(V_K))_\sigma \otimes \mathbf{n}_{K,\sigma}, \tag{4.13}$$
   where :
   — $\nabla_K v = \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}_K} |\sigma|(v_\sigma - v_K) \otimes \mathbf{n}_{K,\sigma}$,
   — $V_K = (v_\sigma - v_K)_{\sigma\in\mathcal{E}_K} \in (\mathbb{R}^d)^{\mathcal{E}_K} = (\mathbb{R}^{\mathcal{E}_K})^d$,
   — $\mathcal{R}_K : (\mathbb{R}^{\mathcal{E}_K})^d \to (\mathbb{R}^{\mathcal{E}_K})^d$ is defined component-wise by $\mathcal{R}_K(\boldsymbol{\xi})^{(i)} = R_K(\boldsymbol{\xi}^{(i)})$ for $i = 1, \ldots, d$, where $R_K : \mathbb{R}^{\mathcal{E}_K} \to \mathbb{R}^{\mathcal{E}_K}$ is the linear mapping $R_K(\xi) = (R_{K,\sigma}(\xi))_{\sigma\in\mathcal{E}_K}$ with
$$R_{K,\sigma}(\xi) = \xi_\sigma - \left(\frac{1}{|K|}\sum_{\sigma'\in\mathcal{E}_K} |\sigma'|\xi_{\sigma'}\mathbf{n}_{K,\sigma'}\right) \cdot (\overline{x}_\sigma - x_K).$$

— $\mathcal{A}_K : (\mathbb{R}^{\mathcal{E}_K})^d \to (\mathbb{R}^{\mathcal{E}_K})^d$ is defined component-wise for all $i = 1, \ldots, d$ by $\mathcal{A}_K(\boldsymbol{\xi})^{(i)} = A_K^{(i)}(\boldsymbol{\xi}^{(i)})$ with $A_K^{(i)}$ an isomorphism of the vector space $\mathrm{Im}(R_K) \subset \mathbb{R}^{\mathcal{E}_K}$.

6. The linear mapping $\mathrm{div}_D : X_{D,0} \to L^2(\Omega)$ is the discrete divergence operator, defined by

$$\forall v \in X_{D,0}\,,\ \forall K \in \mathcal{M},\ \mathrm{div}_D v = \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}_K} |\sigma| v_\sigma \cdot \mathbf{n}_{K,\sigma} = \frac{1}{|K|}\int_K \mathrm{Tr}(\nabla_D v)\mathrm{d}x \text{ on } K. \quad (4.14)$$

The last equality in (4.14) is a consequence of [45, Eq. (5.11)] that shows that the average over K of $\nabla_D v$ is $\nabla_K v$.

We define $\theta_D > 0$ as the smallest $\theta > 0$ such that

$$\max\left(\max_{\sigma\in\mathcal{E}_{\mathrm{int}},K,L\in\mathcal{M}_\sigma} \frac{d_{K,\sigma}}{d_{L,\sigma}}, \max_{K\in\mathcal{M},\sigma\in\mathcal{E}_K} \frac{h_K}{d_{K,\sigma}}\right) \le \theta \quad (4.15)$$

and

$$\forall i = 1, \ldots, d\,,\ \forall K \in \mathcal{M}\,,\ \forall \xi \in \mathbb{R}^{\mathcal{E}_K}\,,$$
$$\frac{1}{\theta}\sum_{\sigma\in\mathcal{E}_K} |D_{K,\sigma}| \left|\frac{R_{K,\sigma}(\xi)}{d_{K,\sigma}}\right|^p \le \sum_{\sigma\in\mathcal{E}_K} |D_{K,\sigma}| \left|\frac{(A_K^{(i)} R_K(\xi))_\sigma}{d_{K,\sigma}}\right|^p \le \theta \sum_{\sigma\in\mathcal{E}_K} |D_{K,\sigma}| \left|\frac{R_{K,\sigma}(\xi)}{d_{K,\sigma}}\right|^p. \quad (4.16)$$

As noticed in [45, Remark 5.4], the choice $A_K^{(i)} = \beta_K^{(i)} \mathrm{Id}_{\mathrm{Im}(R_K)}$ for some $\beta_K^{(i)} \in [\frac{1}{\theta}, \theta]$ ensures (4.16). This choice corresponds to the SUSHI method of [60].

We recall that, when using this gradient discretisation in (2.7), the values u_K can be locally eliminated. This is done by taking in (2.7) the test function $v \in X_{D,0}$ such that $v_K^{(i)} = 1$ for all $i = 1, \ldots, d$, and all other degrees of freedom equal to 0. This enables to compute u_K in terms of $(u_\sigma)_{\sigma\in\mathcal{E}_K}$. The global resulting system therefore involves only the velocity unknowns at the faces of the mesh. The HMM methods contain the Mimetic Finite Difference schemes which, for triangular meshes and particular choices of $\beta_K^{(i)}$, are algebraically identical to the lowest order Raviart–Thomas method [83]. The hybridisation of this method gives the same matrix on the edge unknowns as the Crouzeix–Raviart scheme. Since HMM methods are precisely hybrid schemes (with main unknowns on the edges), we can conclude that HMM is indeed an extension to general polyhedral meshes of the Crouzeix–Raviart scheme.

Proposition 4.14 (Properties for the HMM gradient discretisation)**.**

Let $(\mathcal{M}_m, \mathcal{E}_m, \mathcal{P}_m)_{m\in\mathbb{N}}$ be polyhedral meshes of Ω in the sense of Definition 4.1, and let $(D_m)_{m\in\mathbb{N}}$ be the corresponding gradient discretisations, defined as above. We assume that $(\theta_{D_m})_{m\in\mathbb{N}}$

remains bounded and that $h_{\mathcal{M}_m} \to 0$ *as* $m \to \infty$*. Then* $(D_m)_{m\in\mathbb{N}}$ *is p-coercive, consistent, limit-conforming and compact in the sense of Definitions 4.3, 3.8, 3.9 and 3.10.*

Proof. We drop the indices m for legibility.

P-COERCIVITY : The discrete Sobolev inequality is an immediate consequence of [45, Lemma 5.3] (to estimate $\Pi_{D_m} v$) and of the last equality of (4.14) (to estimate $\mathrm{div}_{D_m} v$).

A lower bound on $(\beta_{D_m})_{m\in\mathbb{N}}$ can be obtained as for the MAC and the Crouzeix–Raviart schemes from Nečas' result [17, 88]. It suffices to interpolate in $X_{D,0}$ by edge averages a field $w \in \boldsymbol{H}_0^1(\Omega)$ such that $\mathrm{div} w = \chi_D q$ and $\|w\|_{\boldsymbol{H}_0^1(\Omega)} \le C\|\chi_D q\|_{L^2(\Omega)}$.

CONSISTENCY : The consistency for the operators related to the velocity is shown in [45, Lemma 5.5]. The consistency for the interpolation of the pressure is similar to that of the MAC and the Crouzeix–Raviart schemes.

LIMIT-CONFORMITY : The convergence to 0 of W_{D_m} is proved in [45, Lemma 5.4]. The convergence of $\widetilde{W}_{D_m}$ is obtained by using (4.14) which proves that $\mathrm{div}_D v - \mathrm{Tr}(\nabla_D v)$ converges weakly to 0 and this completes the proof of the limit-conformity.

COMPACTNESS : Compactness property is a consequence of [43, Theorem 4.1].

Remark 4.15. *Similarly to Remarks 4.9 and 4.13, this proof gives estimates on* W_D*,* $\widetilde{W}_D$*,* S_D *and* $\widetilde{S}_D$ *that show that the theorems in Section 2.3.1 provide, for the HMM method,* $\mathcal{O}(h_{\mathcal{M}})$ *error estimates if* $(\overline{u}, \overline{p}) \in \boldsymbol{H}^3(\Omega) \times H^1(\Omega)$*.*

Chapitre 5

Conclusion générale et perspectives

Résultats obtenus

Dans ce manuscrit, nous avons défini pour trois problèmes (Stefan, Stokes, Navier-Stokes) le cadre des schémas gradients, permettant d'obtenir des résultats de convergence et d'estimation d'erreur s'appliquant à toutes les méthodes numériques entrant dans ce cadre (ce qui est notamment le cas des éléments finis conformes, avec condensation de masse pour les problèmes non linéaires, des éléments finis P^1 non conformes, et des méthodes HMM pour les problèmes scalaires, et des schémas Taylor-Hood, Crouzeix-Raviart et MAC pour les problèmes vectoriels). La mise en place de ce cadre générique nous a permis de mettre en évidence les propriétés mathématiques fondamentales, communes à tous ces schémas, permettant ces résultats de convergence. Ce travail avait été entamé bien avant le début de la rédaction de ce manuscrit pour le cas des problèmes elliptiques et paraboliques scalaires. Ce manuscrit apporte des informations nouvelles dans le cadre du problème de Stefan, avec des résultats de convergence uniforme en temps, et des résultats négatifs de convergence de l'équation d'énergie (des résultats de convergence forte pourraient sans doute être obtenus plus aisément en utilisant des schémas de discrétisation en volumes finis avec flux à deux points). Dans le cadre des problèmes de Stokes et de Navier-Stokes, ce manuscrit donne les premiers résultats de convergence s'appliquant simultanément à des schémas semblant au départ très différents. Par ailleurs, la preuve de convergence des schémas HMM dans ce cas est entièrement nouvelle.

Dans le cadre du problème de Stokes, nous avons pu mettre en évidence une famille de résultats d'estimation d'erreur, faisant ou non apparaître la pression, qui s'appliquent donc de manière générique à toutes les méthodes numériques couvertes par les schémas gradients. Ces résultats s'appuient sur des notions clé (consistance, conformité à la limite) dont l'écriture dans le cadre de ce problème s'est révélée beaucoup plus difficile que dans le cadre des problèmes elliptiques scalaires. En particulier, la notion de conformité à la limite a fait l'objet de progrès constants au cours de notre étude. Par ailleurs, nous avons obtenu des résultats de convergence uniforme en temps qui semblent originaux dans ce cadre.

Pour aboutir à un tel cadre commun, il a été nécessaire, dans le cas du problème de Navier-Stokes, d'énoncer les propriétés génériques de l'approximation du terme de convection non-linéaire. Ce travail n'a sans doute pas encore atteint sa forme définitive, car nous n'avons pas encore vérifié sur les différents exemples, que ce cadre couvrait les approximations usuelles de ce terme de convection. Ce travail reste à faire. Nous avons cependant montré que la discrétisation standard dans le cadre des méthodes d'éléments finis conformes, qui s'appuie sur l'antisymétrie de la forme trilinéaire, entrait dans le cadre de notre étude.

Perspectives

La limitation relevée au cours de notre étude sur le traitement des discrétisations usuelles du terme de convection non linéaire devra être levée, en particulier dans le cas du schéma MAC. Ceci pourra permettre ensuite des comparaisons de la précision obtenue selon le choix de ces discrétisations.

Une autre piste d'étude concerne la mise en place d'un nouveau schéma gradient pour les problèmes de Stokes et de Navier-Stokes, aboutissant à la conformité de la divergence discrète mais pas celle du gradient discret, en partant des éléments finis de Raviart-Thomas. Cette piste permettrait ainsi de généraliser le schéma MAC aux triangles et tétraèdres, en assurant une propriété qui semble essentielle lorsque le second membre du problème s'écrit comme un gradient : en effet, dans ce cas, la solution exacte du problème est une vitesse nulle et un champ de pression dont le gradient est égal au second membre. Les défauts d'approximation de la divergence discrète entraînent des solutions de vitesse discrète qui s'écartent de la valeur nulle. L'avantage de cette méthode serait de satisfaire cette propriété, en réduisant autant que possible le nombre d'inconnues, par rapport au schéma de Crouzeix-Raviart (une seule vitesse normale inconnue par face au lieu de toutes les composantes de la vitesse). Des premiers résultats numériques ont été obtenus. Il reste à les étendre dans des cas de plus grande dimension, et à effectuer la comparaison des performances des différentes méthodes.

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