Propriétés géométriques du nombre chromatique : polyèdres, structures et algorithmes

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Propriétés géométriques du nombre chromatique : polyèdres, structure et algorithmes

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À mes parents et à mes grands-parents

ABSTRACT

Computing the chromatic number and finding an optimal coloring of a perfect graph can be done efficiently, whereas it is an NP-hard problem in general. Furthermore, testing perfection can be carried-out in polynomial-time.

Perfect graphs are characterized by a minimal structure of their stable set polytope: the non-trivial facets are defined by clique inequalities only.

Conversely, does a similar facet-structure for the stable set polytope imply nice combinatorial and algorithmic properties of the graph ?

A graph is h-perfect if its stable set polytope is completely described by non-negativity, clique and odd-circuit inequalities.

Statements analogous to the results on perfection are far from being understood for h-perfection, and negative results are missing. For example, testing h-perfection and determining the chromatic number of an h-perfect graph are unsolved. Besides, no upper bound is known on the gap between the chromatic and clique numbers of an h-perfect graph.

Our first main result states that h-perfection is closed under the operations of t-minors (this is a non-trivial extension of a result of Gerards and Shepherd on t-perfect graphs). We also show that the Integer Decomposition Property of the stable set polytope is closed under these operations, and use this to answer a question of Shepherd on 3-colorable h-perfect graphs in the negative.

The study of minimally h-imperfect graphs with respect to t-minors may yield a combinatorial co-NP characterization of h-perfection. We review the currently known examples of such graphs, study their stable set polytope and state several conjectures on their structure.

On the other hand, we show that the (weighted) chromatic number of certain h-perfect graphs can be obtained efficiently by rounding-up its fractional relaxation. This is related to conjectures of Goldberg and Seymour on edge-colorings.

Finally, we introduce a new parameter on the complexity of the matching polytope and use it to give an efficient and elementary algorithm for testing h-perfection in line-graphs.

RÉSUMÉ

Le calcul du nombre chromatique et la détermination d'une coloration optimale des sommets d'un graphe sont des problèmes NP-difficiles en général. Ils peuvent cependant être résolus en temps polynomial dans les graphes parfaits. Par ailleurs, la perfection d'un graphe peut être décidée efficacement.

Les graphes parfaits sont caractérisés par la structure de leur polytope des stables : les facettes non-triviales sont définies exclusivement par des inégalités de cliques. Réciproquement, une structure similaire des facettes du polytope des stables détermine-t-elle des propriétés combinatoires et algorithmiques intéressantes ?

Un graphe est h-parfait si les facettes non-triviales de son polytope des stables sont définies par des inégalités de cliques et de circuits impairs.

On ne connaît que peu de résultats analogues au cas des graphes parfaits pour la h-perfection, et on ne sait pas si les problèmes sont NP-difficiles. Par exemple, les complexités algorithmiques de la reconnaissance des graphes h-parfaits et du calcul de leur nombre chromatique sont toujours ouvertes. Par ailleurs, on ne dispose pas de borne sur la différence entre le nombre chromatique et la taille maximum d'une clique d'un graphe h-parfait.

Dans cette thèse, nous montrons tout d'abord que les opérations de t-mineurs conservent la h-perfection (ce qui fournit une extension non triviale d'un résultat de Gerards et Shepherd pour la t-perfection). De plus, nous prouvons qu'elles préservent la propriété de décomposition entière du polytope des stables. Nous utilisons ce résultat pour répondre négativement à une question de Shepherd sur les graphes h-parfaits 3-colorables.

L'étude des graphes minimalement h-imparfaits (relativement aux t-mineurs) est liée à la recherche d'une caractérisation co-NP combinatoire de la h-perfection. Nous faisons l'inventaire des exemples connus de tels graphes, donnons une description de leur polytope des stables et énonçons plusieurs conjectures à leur propos.

D'autre part, nous montrons que le nombre chromatique (pondéré) de certains graphes h-parfaits peut être obtenu efficacement en arrondissant sa relaxation fractionnaire à l'entier supérieur. Ce résultat implique notamment un nouveau cas d'une conjecture de Goldberg et Seymour sur la coloration d'arêtes.

Enfin, nous présentons un nouveau paramètre de graphe associé aux facettes du polytope des couplages et l'utilisons pour donner un algorithme simple et efficace de reconnaissance des graphes h-parfaits dans la classe des graphes adjoints.

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1 INTRODUCTION

1.1 CONTEXT

1.1.1 *Perfect graphs*

The theory of Perfect Graphs is one of the most active topics in the fields of Combinatorial Optimization and Graph Theory. It finds its origins around 1950 in the work of Shannon on the zero-error capacity of communication channels, in the seemingly unrelated field of Information Theory.

Consider a communication channel in which some symbolic messages are transmitted with some risk of error (for example, the human voice at some large enough distance). That is, the received message may be altered and different from the one originally sent. Suppose that the message is a single letter chosen from a set L, and that the set F of pairs of letters which may be confused for one-another during the transmission is known. Shannon asked for the maximum number of letters of L which can be used to communicate such that no confusion can arise between the sent and received messages.

In fact, he reformulated the problem in terms of graphs. We write this formulation in the terminology of modern graph theory. A *stable set* of a graph G is a subset of pairwise non-adjacent vertices of G. The *stability number of* G, denoted $\alpha(G)$, is the largest number of elements of a stable set of G.

Consider the graph $G := (L, F)$ (which is called the *confusion graph*). The maximum number of letters which can be used such that no error arises from transmitting a single letter of L is simply $\alpha(G)$.

A similar formulation holds for the problem of transmitting larger chains. If the length of the chains is at most n, then the maximum number of letters of L that can be used to communicate without ambiguity is $\alpha(G^n)$, where G^n denotes the strong product of G by itself n times (the definition of this product is not needed here, the reader may just keep in mind that G^n is a graph). Then, the information-rate per-letter for chains of length n is $\dfrac{\alpha(G^n)}{n}$.

In this context, Shannon [106] defined the *zero-capacity error of a graph* G, denoted $\Theta(G)$, as follows:

$$\Theta(G) = \sup_{n \geq 1} \sqrt[n]{\alpha(G^n)}.$$

This quantity is also know as the *Shannon capacity of* G.

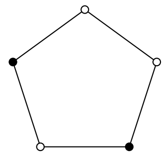


Figure 1.1 – the 5-circuit and a largest stable set (in black). It cannot be covered by 2 cliques

Shannon was concerned with computing this parameter for the 5-circuit C_5 (see Figure 1.1).

He observed that $\Theta(G) = \alpha(G)$ *whenever the set of vertices of G can be covered with at most $\alpha(G)$ cliques* [106], where a *clique of G* is a set of pairwise-adjacent vertices of G. However, this is not the case for C_5, nor for any circuit of odd length at least 5 (see Figure 1.1).

Almost 20 years later, Lovász [73] proved that $\Theta(C_5) = \sqrt{5}$ (Shannon gives this value as a lower bound in [106]). The problem of finding the value of $\Theta(C_7)$ received a lot of attention from the combinatorial community and is still open. Moreover, the computational complexity of determining the Shannon capacity of a graph is not known to this day.

The observation of Shannon (that $\Theta(G) = \alpha(G)$ for each graph G having a clique-cover of cardinality $\alpha(G)$) led Berge to introduce the notion of a *perfect graph* (see [9] for more details).

For every graph G, let $\overline{\chi}(G)$ denote the smallest number of cliques of G whose union is the vertex set of G. A graph G is *perfect* if every induced subgraph H of G satisfies $\alpha(H) = \overline{\chi}(H)$. Several classical results of Combinatorial Optimization can be formulated as the perfection of certain graphs (for example, König's min-max theorems for matchings and edge-colorings in bipartite graphs).

Two conjectures of Berge (around 1960) are mainly responsible for the considerable attention that perfect graphs received. The first one is often refered to as the *Weak Perfect Graph Conjecture*. It is now a theorem and was proved by Lovász [71] in 1972, following a reformulation by Fulkerson [48] in terms of replication of vertices (see Section 3.5 for more details).

THEOREM 1.1.1 (LOVÁSZ [71]) *The complement of a perfect graph is perfect.*

The *chromatic number* of a graph G, denoted $\chi(G)$, is the smallest number of colors needed to color the vertices of G such that adjacent vertices receive different colors; the *clique number of G*, denoted $\omega(G)$ is the largest number of vertices of a clique of G.

While the inequality $\chi(G) \geq \omega(G)$ holds for every graph G, Mycielski [84] built in 1955 a class of graphs with no clique of cardinality 3 and arbitrarily large chromatic number. The Weak Perfect Graph Theorem states that *a graph G is perfect if and only if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$.*

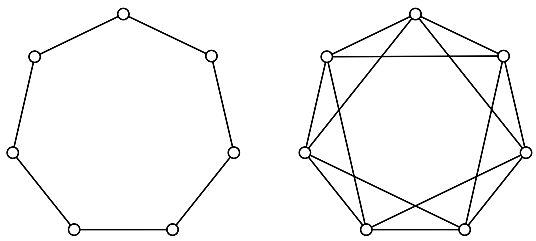


Figure 1.2 – an odd hole of length 7 and its corresponding odd antihole

An *odd hole* of a graph G is an induced circuit of G with an odd number of vertices which is at least 5. An *odd antihole* is the complement graph of an odd hole (see Figure 1.2). It is easy to check that odd holes and odd antiholes are not perfect. Hence a perfect graph cannot have odd holes or odd antiholes.

The second conjecture of Berge (the *Strong Perfect Graph Conjecture*) asserts that this necessary condition is also sufficient. Chudnovsky, Robertson, Seymour and Thomas announced in 2002 that they proved (along almost 150 pages) the following result (known as the *Strong Perfect Graph Theorem*):

Theorem 1.1.2 (Chudnovsky et al. [24]) *A graph is perfect if and only if it does not have an odd hole or odd antihole.*

Furthermore Chudnovsky, Cornuéjols , Liu, Seymour, Vušković [29, 23] obtained *a polynomial-time algorithm to decide perfection*. Results of Grötschel, Lovász and Schrijver [57] imply that *α, $\overline{\chi}$, ω and χ can all be found in polynomial-time in perfect graphs* (as well as their weighted versions) whereas each of these parameters is NP-hard to compute in general.

A surprising aspect of perfect graphs is that they are closely related to polyhedra even though their definition is purely combinatorial.

The *incidence vector* of a subset S of a set V is the 0-1 vector χ^S of $\mathbb{R}^V$ defined for every $v \in V$ by: $\chi^S_v = 1$ if and only if $v \in S$. The *stable set polytope* of a graph G, denoted $\mathsf{STAB}(G)$, is the convex hull of the incidence vectors of the stable sets of G. As a polyhedron, it can be described as the set of solutions of a finite system of linear inequalities. However, deciding whether a vector x belongs to $\mathsf{STAB}(G)$ is an NP-complete problem [64]. Hence, it is unlikely to find a convenient linear system describing $\mathsf{STAB}(G)$ in general, unless $\mathsf{P} = \mathsf{NP}$.

Let V be a set and $S \subseteq V$. For $x \in \mathbb{R}^{V(G)}$, let $x(S) := \sum_{s \in S} x_s$.

It is easy to check that every description of $\mathsf{STAB}(G)$ contains (up to a positive scalar factor) the *non-negativity inequalities* $x_v \geq 0$ for every vertex $v \in V(G)$. Furthermore, Padberg [87] showed that each description of $\mathsf{STAB}(G)$ contains (up to a positive scalar factor) the *clique-inequality* $x(K) \leq 1$ for each inclusion-wise maximal clique K of G. In other words, these inequalities define *facets* of $\mathsf{STAB}(G)$.

Results of Fulkerson [48] and Lovász [71] imply, as stated by Chvátal [26]:

THEOREM 1.1.3 ([26]) *For each graph G, the following statements are equivalent:*

i) G *is perfect,*

ii) $\mathsf{STAB}(G) = \left\{x \in \mathbb{R}^{V(G)} \colon x \geq 0,\, x(K) \leq 1 \text{ for every clique } K \text{ of } G\right\}$.

1.1.2 *Almost-perfect graphs*

The nice structural and algorithmic properties of perfect graphs motivated the study of several variations. The different characterizations of perfect graphs led to distinct notions and problems (for examples, see [25, 58, 59]). In this thesis, we are interested in a notion of "almost perfection" inspired from Theorem 1.1.3.

This result states that the perfection of a graph G induces a minimal structure on the inequalities needed to describe $\mathsf{STAB}(G)$. Conversely, this facet-structure for $\mathsf{STAB}(G)$ implies that the parameters χ, ω (resp. α, $\overline{\chi}$) are equal on each induced subgraph of G and can be computed in polynomial-time (as well as their weighted versions). Hence, it is natural to ask for similar structural properties from classes of graphs which bare an "almost minimal" facet-structure of the stable set polytope.

An important result in studying relations between polyhedra and graphs which are "almost perfect" is due to Padberg.

A graph G is *minimally imperfect* if G is not perfect and for every $v \in V(G)$, the graph $G - v$ is perfect ($G - v$ is the graph obtained from deleting v and every edge incident to it). The Strong Perfect Graph Theorem states that *the minimally imperfect graphs are the odd holes and odd antiholes*. Padberg proved the following:

THEOREM 1.1.4 (PADBERG [89]) *For every minimally imperfect graph G:*

$$\mathsf{STAB}(G) = \left\{ x \in \mathbb{R}^{V(G)} \colon \begin{array}{cc} x \geq 0, & \\ x(K) \leq 1 & \forall K \text{ clique of } G, \\ x(V(G)) \leq \alpha(G) & \end{array} \right\}. \tag{1.1}$$

In this context, Shepherd [107] called a graph G *near-perfect* if adding the *full rank-inequality* $x(V(G)) \leq \alpha(G)$ to the non-negativity and clique-inequalities is enough to completely describe $\mathsf{STAB}(G)$. Hence, perfect graphs and minimally imperfect graphs are near-perfect. Figure 1.3 shows an imperfect near-perfect graph which is not minimally imperfect.

Shepherd [107] gave several conjectures and results on near-perfect graphs. In particular, he proved that *minimally imperfect graphs are*

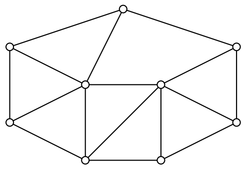


Figure 1.3 – a near-perfect graph which is neither perfect nor minimally imperfect

the near-perfect graphs whose complement is also near-perfect, and showed that their (weighted) chromatic number can be obtained in a way similar to perfect graphs: their stable set polytope has the integer decomposition property (see Section 3.3.3). Wagler [118] characterized near-perfection in the classes of webs and antiwebs. Few other classes of near-perfect graphs are known.

More generally Grötschel, Lovász and Schrijver [57] suggested that other notions of almost perfection can be obtained by:

1. choosing a set of families of valid inequalities for the stable set polytope of a graph (in general) including non-negativity and clique-inequalities,
2. consider the class of graphs whose stable set polytope is completely described by these selected inequalities.

Near-perfect graphs are built as such (the full-rank inequality being the only inequality chosen at step 1).

The topic of this thesis is the study of the structure and properties of the class of *h-perfect graphs*, which is another class of "almost perfect" graphs defined in this way.

1.1.3 *H-perfect graphs*

An *odd-circuit inequality* of a graph G is an inequality over $\mathbb{R}^{V(G)}$ of the form:

$$x(V(C)) \leq \frac{|V(C)|-1}{2},$$

where C is an odd circuit of G. A graph G is *h-perfect* if its stable set polytope can be completely described by non-negativity, clique and odd-circuit inequalities. In other words, if:

$$\mathsf{STAB}(G) = \left\{ x \in \mathbb{R}^{V(G)} : \begin{array}{rl} x \geq 0, & \\ x(K) \leq 1 & \forall K \text{ clique of } G, \\ x(V(C)) \leq \dfrac{|V(C)|-1}{2} & \forall C \text{ odd circuit of } G. \end{array} \right\}. \tag{1.2}$$

We mentioned above that perfect graphs cannot have odd holes. Therefore, Theorem 1.1.3 shows that *perfect graphs are h-perfect*. Furthermore, Theorem 1.1.4 implies that *odd holes are h-perfect* whereas *odd antiholes with at least 7 vertices are not h-perfect*.

The effort to understand h-perfection has been mostly focused on the subclass of h-perfect graphs which do not have cliques with 4 vertices. Such graphs are called *t-perfect*.

The computational complexity of deciding t-perfection is open. T-perfection belongs to co-NP [102] but no combinatorial certificate of t-imperfection is available. Neither an NP-characterization of t-perfection nor a co-NP characterization of h-perfection are known.

For each graph G and non-negative integer weight $c \in \mathbb{Z}_+^{V(G)}$, a *maximum-weight stable set* is a stable set S such that $c(S)$ is maximum. Grötschel, Lovász, Schrijver proved (through the Ellipsoid Method):

THEOREM (GRÖTSCHEL, LOVÁSZ, SCHRIJVER [56]) *A maximum-weight stable set can be found in polynomial-time in h-perfect graphs.*

This is a significant feature of perfection which extends to h-perfection. Eisenbrand et al. [38] gave an *efficient combinatorial algorithm for the cardinality-case in t-perfect graphs*. These algorithms use only the knowledge of the facets of the stable set polytope and do not rely on decomposition results for h-perfect graphs.

Besides, Bruhn and Stein [16] showed that *a maximum clique of an h-perfect graph can be computed in polynomial-time*.

Chvátal defined t-perfection in [26] and conjectured that series-parallel graphs are t-perfect (a graph is *series-parallel* if it does not have the complete graph K_4 as a minor). This was proved by Boulala and Uhry [12] (Mahjoub gave a simpler proof in [77]).

We end this introduction with a condensed overview of the current state of the art on h-perfection.

RECOGNITION OF H-PERFECT GRAPHS Fonlupt and Uhry [44] proved that *almost-bipartite graphs are t-perfect* (a graph is *almost-bipartite* if it has a vertex belonging to every odd circuit).

Sbihi and Uhry [98] showed that under certain assumptions, bipartite graphs could be substituted to edges of series-parallel graphs to obtain t-perfect graphs.

Gerards [50] extended the results of Fonlupt, Boulala and Uhry by proving that *graphs which do not contain (as a subgraph) an odd-K_4 are t-perfect* (an *odd-K_4* is a subdivision of K_4 in which every triangle becomes an odd circuit).

The non-t-perfect subdivisions of K_4 were characterized by Barahona and Mahjoub [3]. Gerards and Shepherd [51] proved that *graphs which do not contain such subdivisions (as a subgraph) are t-perfect*. In fact these graphs, also known as *hereditary t-perfect graphs*, are exactly the graphs which have only t-perfect subgraphs. They can be recognized in polynomial-time. Hence, the result of Gerards and Shepherd extends all previous results on excluding certain subdivisions of K_4 and is maximal with respect to obtaining t-perfection under subgraph-exclusion assumptions.

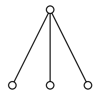


Figure 1.4 – the claw

A graph is *claw-free* if it does not have an induced claw (see Figure 1.4). Bruhn and Stein [16] provided a *characterization of t-perfect claw-free graphs* in terms of minimally t-imperfect graphs with respect to vertex-deletion and t-contraction (the latter is an operation preserving t-perfection defined by Gerards and Shepherd in [51]). Using this result, Bruhn and Schaudt [14] showed a polynomial-time algorithm which decides t-perfection in the class of claw-free graphs. On the other hand, Shepherd [108] characterized t-perfection in the class of *complements of line graphs*.

H-perfection was defined by Sbihi and Uhry in [98]. Fonlupt and Hadjar [43] gave conditions under which certain operations keep h-perfection (*identification of two vertices, addition of an edge,...*). Cao and Nemhauser [19] gave a *forbidden-induced-subgraph characterization of h-perfect line-graphs*. Besides, Arbib and Mosca [2] gave such a characterization (with a single forbidden graph) for the class of *graphs which do not contain an induced path of length 4 nor an induced subgraph isomorphic to K_4 minus an edge*.

STRONG H-PERFECTION It follows from results of Lovász [71] and Fulkerson [48] that *a graph is perfect if and only if the system of non-negativity and clique-inequalities is totally dual-integral* (see definition in Section 3.3.1).

Similarly, is it true that a graph is h-perfect if and only if the system of non-negativity, clique and odd-hole inequalities is totally dual integral ?

A graph G is *strongly h-perfect* if the system of inequalities in Equation (1.2) is *totally dual integral*. It is *strongly t-perfect* if it furthermore has no clique of cardinality 4. Results of Edmonds and Giles [35] show that *every strongly t-perfect graph is t-perfect*. Schrijver conjectures that the converse holds:

CONJECTURE (SCHRIJVER [102]) *Every t-perfect graph is strongly t-perfect.*

In [101], Schrijver proved that *hereditary t-perfect graphs are strongly t-perfect*. Furthermore, Bruhn and Stein [15] showed that *every t-perfect claw-free graph is strongly t-perfect*.

COLORINGS The polyhedral characterization of perfect graphs stated in Theorem 1.1.3 shows that the equality of the chromatic and clique numbers for every induced subgraph implies that the stable set polytope is completely described by non-negativity and clique inequalities.

Since only odd-circuit inequalities are furthermore needed to describe the stable set polytope of an h-perfect graph, one may expect that the chromatic number of an h-perfect graph remains close to its clique number.

It is not known whether there exists a constant c such that every h-perfect graph G satisfies $\chi(G) \leq \omega(G) + c$. Sbihi and Uhry [98] conjectured that every h-perfect graph G with $\omega(G) \geq 3$ is $\omega(G)$-colorable. This was infirmed by Laurent and Seymour, who found *a t-perfect graph with chromatic number 4* [102, pg. 1207]. This graph also disproved a conjecture of Shepherd stating that the stable set polytope of a t-perfect graph has the integer decomposition property (see [63]).

Sebő showed that *the* $(\omega + 1)$*-colorability of h-perfect graphs would follow from the case* $\omega \leq 2$ (see [16]).

Results of Bruhn and Stein [16] imply that *each h-perfect claw-free graph* G *is* $(\omega(G) + 1)$*-colorable* (and an optimal coloring can be found in polynomial-time). Gerards and Shepherd [51] showed that *hereditary t-perfect graphs are 3-colorable* and gave a polynomial-time coloring algorithm.

In this thesis, we investigate the problems of recognizing h-perfect graphs, computing their chromatic number and the related notion of the integer decomposition property of their stable set polytope.

1.2 GENERAL OUTLINE AND CONTRIBUTIONS OF THE THESIS

CHAPTER 3: PRELIMINARIES We give the notations, definitions and results which are needed to understand the rest of the document.

CHAPTER 4: ON OPERATIONS PRESERVING H-PERFECTION In this chapter, we study operations keeping h-perfection and relate some of them to the integer decomposition property.

A *t-contraction* of a graph G is obtained by shrinking a vertex v and its neighbors to a single vertex, when the neighbors of v form a stable set of G. A t-minor of G is a graph obtained from G by a sequence of vertex-deletions and t-contractions. Gerards and Shepherd [51] proved that *t-minors keep t-perfection*.

We first extend this result by showing that *t-minors keep h-perfection*. Furthermore, our proof shows that *perfection is closed under t-minors*.

A polyhedron $P \subseteq \mathbb{R}^n$ has the *integer decomposition property* if for every positive integer k, each integral vector of kP is the sum of k integral vectors of P.

We prove that *t-minors keep the integer decomposition property of the stable set polytope*.

We will use this in Chapter 7 to answer a question of Shepherd on the equivalence of this property with 3-colorability for t-perfect graphs.

Let G, H be graphs and v be a vertex of G. The *substitution of v by H in G* is the graph obtained from the union of disjoint copies of $G - v$ and H by adding the edge uw for each neighbor u of v in G and each vertex w of H. We characterize *the graphs H which can be substituted to a vertex of an h-perfect graph such that the resulting graph is also h-perfect*.

A graph G is *minimally h-imperfect* (resp. *minimally t-imperfect*) if it is h-imperfect (resp. t-imperfect) and every t-minor of G other than itself is h-perfect (resp. t-perfect).

We use our result on substitutions to derive a related property (on homogeneous sets) of minimally h-imperfect and minimally t-imperfect graphs.

CHAPTER 5: MINIMAL H-IMPERFECTION T-perfection is in co-NP but no combinatorial certificate of t-imperfection is known.

Whether h-perfection belongs to NP or co-NP is open. The study of minimally t-imperfect and minimally h-imperfect graphs may hopefully clarify the combinatorial nature of these properties.

We will first review the currently known examples of minimally t-imperfect graphs. We do not provide new ones, but give a *description of their stable set polytope* and formulate a related conjecture. Moreover, we state known and new conjectures and ask further questions on minimally t-imperfect graphs.

It is easy to check that K_4 is the only minimally t-imperfect graph which is not minimally h-imperfect. We determine *the K_4-free graphs which are minimally h-imperfect but not minimally t-imperfect*. They show that some of the questions and conjectures on minimally t-imperfect graphs must be reformulated in order to be extended to minimally h-imperfect graphs.

We present a conjecture of Sebő which states that the minimally h-imperfect graphs with cliques of cardinality at least 4 are odd antiholes and we show that *it holds for planar graphs*.

We characterize *h-perfection and minimal h-imperfection in webs*, and these results hopefully simplify the still open task of proving Sebő's conjecture for the special case of claw-free graphs. If valid, this case would imply (through a theorem of Bruhn and Stein [16]) a forbidden-t-minor characterization of h-perfection in claw-free graphs. The latter would provide a co-NP characterization of h-perfect claw-free graphs.

Finally, we show that the minimally h-imperfect line-graphs can be derived from a theorem of Cao and Nemhauser [19].

CHAPTER 6: INTEGER ROUND-UP PROPERTY FOR THE CHROMATIC NUMBER OF CERTAIN H-PERFECT GRAPHS The chromatic number of a perfect graph is equal to its fractional relaxation. This does not hold further for h-perfect graphs and the gap between the two quantities is unknown. This is related to complexity issues in computing the chromatic number of an h-perfect graph. Most of the content of this chapter is in [7].

For every graph G and every $c \in \mathbb{Z}_+^{V(G)}$, the weighted chromatic number of (G, c) is the minimum cardinality of a multiset $\mathcal{F}$ of stable sets of G such that every $v \in V(G)$ belongs to at least c_v members of $\mathcal{F}$.

We prove that *every h-perfect line-graph and every t-perfect claw-free graph G has the integer round-up property for the chromatic number*: for every non-negative integer weight c on the vertices of G, the weighted chromatic number of (G, c) can be obtained by rounding up its fractional relaxation. This means that *the stable set polytope of G has the integer decomposition property*.

Our results imply the *existence of a polynomial-time algorithm which computes the weighted chromatic number of t-perfect claw-free graphs and h-perfect line-graphs*. They also yield *a new case of a conjecture of Goldberg and Seymour [55, 104]* on edge-colorings.

Sebő [103] proved that projections of polyhedra defined by totally unimodular constraints have the integer decomposition property. We end this chapter by showing that the converse is false, with an example of *a 0-1 polytope which has the integer decomposition property, but is not the projection of a polyhedron defined by totally unimodular constraints*.

CHAPTER 7: ON COLORINGS OF H-PERFECT GRAPHS Using that t-minors keep the integer decomposition property of the stable set polytope (this is proved in CHAPTER 3), we solve a problem raised by Shepherd in [108] by showing *a 3-colorable t-perfect graph which does not have the integer round-up property for the chromatic number*.

Using a theorem of [108], we prove a forbidden-induced-subgraph characterization of *h-perfect complements of line-graphs which have the integer round-up property for the chromatic number for 0-1 weights*. One of the two excluded graphs is *a new example of a non-3-colorable t-perfect graph*.

A graph is *P_6-free* if it does not have an induced path with 6 vertices. After reviewing results and a conjecture of Sebő on the chromatic number of h-perfect graphs, we show that results of Randerath, Schiermeyer and Tewes [93, 94] imply that *each h-perfect P_6-free graph G satisfies* $\chi(G) \le \omega(G) + 1$ (the bound is tight). A corresponding coloring can be found in polynomial-time.

CHAPTER 8: EAR-DECOMPOSITIONS AND H-PERFECTION IN LINE–GRAPHS The complexity of testing h-perfection is not known. This chapter studies the case of line-graphs, in which the problem has a simple combinatorial formulation. It shows connexions with binary spaces, edge-colorings, subdivisions of K_4 and ear-decompositions. The results of this chapter are the subject of [8].

Let C_3^+ denote the graph obtained from the triangle by adding a single parallel edge. An *odd-C_3^+* is a graph obtained by replacing each edge e of C_3^+ with a path of odd length joining the ends of e, such that paths corresponding to different edges do not share inner vertices. A graph is *odd-C_3^+-free* if it does not have a subgraph isomorphic to an odd-C_3^+.

Results of Kawarabayashi, Reed, Wollan [66] (see also Huynh [62]) imply that detecting an odd-C_3^+ subgraph can be done in polynomial-time. Bruhn and Schaudt [14] showed a simpler polynomial-time algorithm for sub-cubic graphs.

We show *a simple and elementary algorithm deciding whether a graph (with arbitrary degrees) is odd-C_3^+-free*. It yields *an efficient algorithm testing h-perfection in line-graphs*.

For each graph G, let $\beta(G)$ denote the largest integer k such that G has a subgraph which has an open odd ear-decomposition with k ears (see Section 3.2.2 for the definition of an ear-decomposition). For example, $\beta(G) \leq 1$ if and only if G is odd-C_3^+-free.

We show that *determining β is fixed-parameter-tractable* and state a conjecture on a round-up property for the chromatic index of graphs for which β is small.

On the other hand, we show *a simpler algorithm for detecting totally odd subdivisions of K_4 in odd-C_3^+-free graphs*. The relation of odd-C_3^+-free graphs and totally odd subdivisions of K_4 is suggested by Cao's thesis [18], which contains structural results and constructions for odd-C_3^+-free simple graphs. We review the related results of the thesis and observe that some of them are incorrect.

CHAPTER 9: CONCLUSION We summarize the questions and conjectures from the preceding chapters and suggest further research directions in the study of h-perfection and related problems.

2 INTRODUCTION (EN FRANÇAIS)

2.1 CONTEXTE

2.1.1 *Graphes parfaits*

La théorie des graphes parfaits est l'un des sujets les plus actifs de l'Optimisation Combinatoire et de la Théorie des Graphes. Elle a débuté dans les années 1950 par le travail de Shannon [106] sur la capacité à zéro-erreur d'un canal de communication, au sein de la Théorie de l'Information.

Considérons un canal de communication dans lequel des messages symboliques sont transmis avec un certain risque d'erreur (l'écoute d'une voix humaine située à une distance assez grande par exemple) : le message reçu peut différer de celui qui a été émis. Supposons qu'un message soit réduit à une seule lettre d'un certain sous-ensemble L de l'alphabet, et que l'ensemble F des paires de lettres pouvant être confondues l'une pour l'autre soit connu. Shannon [106] s'est intéressé au plus grand nombre de lettres de L qui peuvent être utilisées de sorte qu'aucune erreur ne puisse se produire.

En particulier, il a reformulé le problème dans les termes de la théorie des graphes. Nous reprenons ici cette formulation en utilisant les termes actuels de la théorie. Un *stable* d'un graphe G est un sous-ensemble de sommets de G deux-à-deux non-adjacents. La *stabilité de* G, notée $\alpha(G)$, est le plus grand nombre d'éléments d'un stable de G.

Considérons le graphe $G := (L, F)$ (appelé *graphe de confusion*). La stabilité de G représente alors le plus grand nombre de lettres de L utilisables dans la transmission de messages d'une seule lettre de sorte qu'aucune erreur ne puisse se produire.

Le problème de la transmission de messages de plus d'une lettre admet une formulation similaire. Si l'on transmet des chaînes d'au plus n caractères de L, alors le nombre maximum de lettres de L utilisables pour communiquer sans erreur est $\alpha(G^n)$, où G^n est le produit fort de G par lui-même n fois (on ne donnera pas la définition de ce produit, il suffit de retenir que G^n est un graphe). Le taux d'information par lettre pour des chaînes de n lettres est alors $\dfrac{\alpha(G^n)}{n}$.

Ainsi, Shannon [106] a défini la *capacité à zéro-erreur d'un graphe* G, notée $\Theta(G)$, de la façon suivante :

$$\Theta(G) = \sup_{n \geq 1} \sqrt[n]{\alpha(G^n)}.$$

Cette quantité est aussi appelée *capacité de Shannon de* G [106].

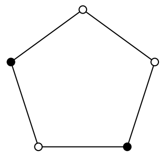


FIGURE 2.1 – le circuit de longueur 5 et un stable de cardinal maximum (en noir). Ce graphe ne peut pas être couvert par 2 cliques

Dans ce contexte, l'un des principaux problèmes de Shannon était le calcul de la capacité du circuit C_5 (voir Figure 2.1).

Il a prouvé que $\Theta(G) = \alpha(G)$ *dès que l'ensemble des sommets du graphe G peut être couvert avec au plus $\alpha(G)$ cliques* [106] (une *clique de G* est un sous-ensemble de sommets deux-à-deux adjacents de G). Cependant, cette dernière propriété n'est pas satisfaite par C_5 (voir Figure 2.1) ni par aucun autre circuit impair de longueur au moins 5.

C'est finalement 20 ans plus tard que Lovász [73] a démontré que $\Theta(C_5) = \sqrt{5}$ (Shannon avait donné cette valeur en tant que borne inférieure [106]). Le calcul de la capacité du cycle de longueur 7 a fait l'objet de nombreuses recherches et cette question est toujours ouverte. De plus, la complexité algorithmique du calcul de la capacité de Shannon d'un graphe n'est pas connue.

Le résultat de Shannon pour les graphes dont les sommets peuvent être couverts par au plus α cliques a conduit Berge à introduire la notion de *graphe parfait* (voir [9]).

Soit G un graphe. Notons $\overline{\chi}(G)$ le plus petit nombre de cliques de G dont l'union est l'ensemble des sommets de G. Un graphe est *parfait* si tout sous-graphe induit H de G satisfait l'égalité $\alpha(H) = \overline{\chi}(H)$. De nombreux résultats classiques de l'Optimisation Combinatoire peuvent être reformulés en termes de perfection de certains graphes, par exemple les théorèmes min-max de König pour les couplages et arête-colorations dans les graphes bipartis.

L'intérêt considérable pour les graphes parfaits provient essentiellement de deux conjectures énoncées par Berge dans les années 1960. En premier lieu, la *conjecture faible des graphes parfaits* énonce que la classe des graphes parfaits est fermée par complémentaire. Elle a été démontrée par Lovász en 1972 [71] suite à une reformulation par Fulkerson [48] en termes de réplication de sommets (voir aussi la Section 3.5) :

THÉORÈME 2.1.1 (LOVÁSZ [71]) *Le complémentaire d'un graphe parfait est parfait.*

Le *nombre chromatique* d'un graphe G, noté $\chi(G)$, est le plus petit nombre de couleurs nécessaires pour colorer les sommets de G de sorte qu'aucune arête n'a ses deux extrémités de la même couleur ; le plus grand nombre d'éléments d'une clique de G est noté $\omega(G)$.

On a évidemment $\chi(G) \geq \omega(G)$ pour tout graphe G. Mycielski [84] a construit une famille de graphes avec $\omega \leq 2$ et un nombre chromatique arbitrairement grand. En comparaison, le théorème faible des graphes parfaits affirme précisément qu'*un graphe G est parfait si et seulement si tout sous-graphe induit H de G satisfait $\chi(H) = \omega(H)$.*

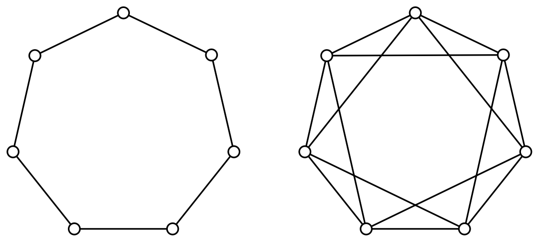


FIGURE 2.2 – un trou impair de longueur 7 et l'anti-trou impair correspondant

Un *trou impair* d'un graphe G est un circuit induit de G qui a un nombre de sommets impair et supérieur ou égal à 5. Un *anti-trou impair* est le complémentaire d'un trou impair (voir Figure 2.2). Il est clair qu'un graphe parfait n'a ni trou impair ni anti-trou impair.

La seconde conjecture de Berge (la *conjecture forte des graphes parfaits*) énonce que cette condition nécessaire est aussi suffisante. Elle a été prouvée par Chudnovsky, Robertson, Seymour et Thomas en 2002 (en environ 150 pages) :

THÉORÈME 2.1.2 (CHUDNOVSKY ET AL. [24]) *Un graphe est parfait si et seulement s'il n'a pas de trou impair ou d'anti-trou impair.*

De plus, Chudnovsky, Cornuéjols, Liu, Seymour, Vušković [29, 23] ont donné un *algorithme polynomial* permettant de déterminer si un graphe est parfait. Par ailleurs, des résultats de Grötschel, Lovász et Schrijver [57] impliquent que *α, $\overline{\chi}$, ω et χ (ainsi que leurs versions pondérées) peuvent être déterminés en temps polynomial dans les graphes parfaits*, alors que leur calcul est NP-difficile en général.

Bien que leur définition soit de nature combinatoire, les graphes parfaits présentent d'étonnantes connexions avec la géométrie de certains polyèdres.

Le *vecteur d'incidence* d'un sous-ensemble S d'un ensemble V est le vecteur 0-1 de $\mathbb{R}^V$, noté χ^S, défini pour tout $v \in V$ par : $\chi^S_v = 1$ si et seulement si $v \in S$. Le *polytope des stables* d'un graphe G, noté $\mathsf{STAB}(G)$, est l'enveloppe convexe des vecteurs d'incidence des stables de G. En tant que polyèdre, il est l'ensemble des solutions d'un système fini d'inégalités linéaires. Cependant, décider l'appartenance d'un vecteur à $\mathsf{STAB}(G)$ est un problème NP-complet [64]. Dès lors, l'obtention d'une description convenable de $\mathsf{STAB}(G)$ pour tout graphe G à l'aide d'inégalités linéaires est improbable, à moins que $\mathsf{P} = \mathsf{NP}$.

Pour un ensemble V, un sous-ensemble fini S de V et $x \in \mathbb{R}^V$, on note $x(S) := \sum_{s\in S} x_s$.

On vérifie aisément que chaque description de $\mathsf{STAB}(G)$ doit contenir les *inégalités de non-négativité* $x_v \geq 0$ pour tout $v \in V(G)$ (à un facteur strictement positif près). D'autre part, Padberg [87] a montré que chaque description contient aussi l'*inégalité de clique* $x(K) \leq 1$ pour toute clique K de G qui est maximale pour l'inclusion (à un facteur strictement positif près). En d'autres termes, ces inégalités définissent toutes des *facettes* de $\mathsf{STAB}(G)$.

Le théorème suivant, énoncé par Chvátal [26], est une conséquence directe de résultats de Fulkerson [48] et Lovász [71] :

THÉORÈME 2.1.3 ([26]) *Pour tout graphe G, les assertions suivantes sont équivalentes :*

i) *G est parfait,*

ii) $\mathsf{STAB}(G) = \left\{x \in \mathbb{R}^{V(G)} : x \geq 0,\ x(K) \leq 1 \text{ pour toute clique } K \text{ de } G\right\}.$

2.1.2 *Graphes presque-parfaits*

Les propriétés remarquables des graphes parfaits ont motivé l'étude de nombreuses variations, et leurs différentes caractérisations ont donné lieu à différents notions et problèmes (voir par exemple [25, 58, 59]). Dans cette thèse, on s'intéresse à une notion de "presque-perfection" issue du Théorème 2.1.3.

Il énonce que la perfection d'un graphe G induit une structure minimale des facettes de $\mathsf{STAB}(G)$: elles sont définies exclusivement par la non-négativité et des cliques. Réciproquement, cette structure géométrique implique que les paramètres combinatoires χ, ω, α, $\overline{\chi}$ peuvent être calculés en temps polynomial. Par conséquent, il convient d'étudier les propriétés combinatoires des graphes dont le polytope des stables a une structure semblable.

Un des premiers résultats dans cette direction est dû à Padberg.

Un graphe G est *minimalement imparfait* si G n'est pas parfait et si pour tout sommet v de G, le graphe $G - v$ est parfait ($G - v$ est le graphe obtenu de G en supprimant v ainsi que les arêtes incidentes à v). Le Théorème Fort des Graphes Parfaits affirme que *les graphes minimalement imparfaits sont les trous impairs et les anti-trous impairs*. Padberg a démontré le théorème suivant :

THÉORÈME 2.1.4 (PADBERG [89]) *Pour tout graphe minimalement imparfait G :*

$$\mathsf{STAB}(G) = \left\{ x \in \mathbb{R}^{V(G)} : \begin{array}{cc} x \geq 0, & \\ x(K) \leq 1 & \forall K \text{ clique de } G, \\ x(V(G)) \leq \alpha(G) & \end{array} \right\}. \tag{2.1}$$

Dans ce contexte, on dit qu'un graphe G est *proche-parfait* s'il suffit d'ajouter *l'inégalité de plein-rang* $x(V(G)) \leq \alpha(G)$ à celles de non-négativité et de cliques pour obtenir une description de STAB(G). En particulier, les graphes parfaits ou minimalement imparfaits sont proche-parfaits. La Figure 2.3 montre un graphe proche-parfait qui n'est ni parfait, ni minimalement imparfait.

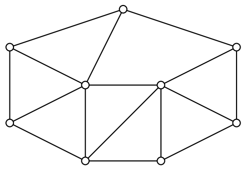


FIGURE 2.3 – un graphe proche-parfait qui n'est ni parfait ni minimalement imparfait

Plusieurs conjectures et résultats sur les graphes proche-parfaits ont été énoncés et obtenus par Shepherd [107]. En particulier, il a montré que *les graphes minimalement imparfaits sont les graphes proche-parfaits dont le complémentaire est aussi proche-parfait*, et que le nombre chromatique (pondéré) d'un graphe proche-parfait s'obtient par une méthode analogue au cas parfait : *le polytope des stables d'un graphe proche-parfait a la Propriété de Décomposition Entière* (voir Section 3.3.3). Wagler [118] a caractérisé la proche-perfection dans les classes des graphes circulants et anti-circulants. On connaît peu d'autres classes de graphes proche-parfaits.

Plus généralement, Grötschel, Lovász et Schrijver [57] ont suggéré que d'autres notions de "presque-perfection" peuvent être obtenues de la façon suivante :

1. on choisit d'abord un ensemble de familles d'inégalités valides pour le polytope des stables qui contient les inégalités de non-négativité et de cliques,
2. on considère la classe des graphes dont le polytope des stables est entièrement décrit par les inégalités choisies.

Les graphes proche-parfaits sont en effet construits de la sorte (l'inégalité de plein-rang étant la seule choisie à l'étape 1 en dehors de la non-négativité et des cliques).

Cette thèse s'attache à l'étude de la structure et des propriétés de la classe des *graphes h-parfaits*, qui est un autre exemple de classe obtenue par cette procédure.

2.1.3 *Graphes h-parfaits*

Une *inégalité de circuit impair* d'un graphe G est une inégalité sur $\mathbb{R}^{V(G)}$ de la forme :

$$x(V(C)) \leq \frac{|V(C)| - 1}{2},$$

où C est un circuit impair de G. Un graphe G est *h-parfait* si son polytope des stables est entièrement décrit par les inégalités de non-négativité, de cliques et de circuits impairs. En d'autres termes, si on a :

$$\mathsf{STAB}(G) = \left\{ x \in \mathbb{R}^{V(G)} : \begin{array}{rr} x \geq 0, & \\ x(K) \leq 1 & \forall K \text{ clique de } G, \\ x(V(C)) \leq \dfrac{|V(C)|-1}{2} & \forall C \text{ circuit impair de } G. \end{array} \right\}. \tag{2.2}$$

On a déjà noté que les graphes parfaits ne peuvent avoir de trou impair. Ainsi, le Théorème 2.1.3 montre que *les graphes parfaits sont h-parfaits*. De plus, le Théorème 2.1.4 implique que les trous impairs sont h-parfaits et que les anti-trous impairs à au moins 7 sommets ne le sont pas.

L'essentiel des résultats sur la h-perfection porte sur le cas des graphes qui n'ont pas de clique à 4 sommets. Un graphe est *t-parfait* s'il est h-parfait et n'a pas de clique de taille 4.

The computational complexity of deciding t-perfection is open.

T-perfection belongs to co-NP [102] but no combinatorial certificate of t-imperfection is available. Neither an NP-characterization of t-perfection nor a co-NP characterization of h-perfection are known.

La complexité algorithmique du problème de la reconnaissance d'un graphe t-parfait est ouverte.

La t-perfection appartient à co-NP [102] mais on ne connaît pas de certificat combinatoire de t-imperfection. Par ailleurs, on ne sait pas si la t-perfection appartient à NP ni si la h-perfection est dans co-NP.

Pour tout graphe G et tout poids $c \in \mathbb{Z}_+^{V(G)}$, *un stable de poids maximum de* (G, c) est un stable S de G pour lequel $c(S)$ est maximum. Grötschel, Lovász, Schrijver ont démontré (par la Méthode des Ellipsoïdes) :

THÉORÈME 2.1.5 (GRÖTSCHEL, LOVÁSZ, SCHRIJVER [56]) *Un stable de poids maximum d'un graphe h-parfait peut être déterminé en temps polynomial.*

Cet résultat donne un exemple d'une propriété majeure de la perfection qui s'étend à la h-perfection. Eisenbrand et al. [38] ont donné *un algorithme combinatoire efficace pour trouver un stable de cardinalité maximum dans un graphe t-parfait*. Par ailleurs, Bruhn et Stein [16] ont montré qu'*une clique de cardinalité maximum d'un graphe h-parfait peut être déterminée efficacement*.

Les graphes t-parfaits ont été introduits par Chvátal [26]. Il a conjecturé que les graphes série-parallèles sont t-parfaits (un graphe est *série-parallèle* s'il n'a pas le graphe complet K_4 pour mineur), ce qui a été prouvé par Boulala et Uhry [12] (Mahjoub a donné une preuve plus simple dans [77]). Nous terminons cette introduction en dressant l'état des principales connaissances à propos des graphes h-parfaits.

RECONNAISSANCE DES GRAPHES H-PARFAITS Fonlupt et Uhry [44] ont démontré que *les graphes presque-bipartis sont t-parfaits* (un graphe est *presque-biparti* s'il a un sommet qui appartient à tous ses circuits impairs).

Sbihi et Uhry [98] ont prouvé que sous certaines conditions, la substitution de graphes bipartis aux arêtes de graphes série-parallèles produit des graphes t-parfaits.

Gerards [50] a étendu les résultats de Fonlupt, Boulala et Uhry en démontrant que *les graphes qui n'ont pas de K_4-impair (comme sous-graphe) sont t-parfaits* (un *K_4-impair* est une subdivision de K_4 dans laquelle chaque triangle de K_4 est transformé en un circuit impair).

Les subdivisions non-t-parfaites de K_4 ont été caractérisées par Barahona et Mahjoub [3]. Gerards et Shepherd [51] ont montré que *les graphes qui ne contiennent pas de subdivision non-t-parfaite de K_4 (comme sous-graphe) sont t-parfaits*. Ces graphes sont exactement ceux dont tous les sous-graphes sont t-parfaits (ils sont dits *t-parfaits héréditaires*) et peuvent être reconnus en temps polynomial. Ainsi, le résultat de Gerards et Shepherd contient tous les précédents énoncés affirmant la t-perfection de graphes ne contenant pas certaines subdivisions de K_4, et est maximal quant à l'obtention de t-perfection par exclusion de sous-graphes.

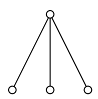


FIGURE 2.4 – la griffe

Un graphe est *sans griffe* s'il n'a pas de griffe induite (voir Figure 2.4). Bruhn et Stein [16] ont prouvé une caractérisation combinatoire co-NP de la t-perfection dans les graphes sans griffe. Elle s'exprime en termes de graphes minimalement t-imparfaits relativement à deux opérations de graphes qui préservent la t-perfection : la suppression d'un sommet et la *t-contraction* (celle-ci a été définie et étudiée par Gerards et Shepherd [51]).

Bruhn et Schaudt [14] ont utilisé cette caractérisation pour donner un algorithme efficace décidant la t-perfection dans la classe des graphes sans griffe. D'autre part, Shepherd [108] a caractérisé les graphes t-parfaits dans la classe des *complémentaires de graphes adjoints*.

Les graphes h-parfaits ont été définis par Sbihi et Uhry dans [98]. Fonlupt et Hadjar [43] ont montré que sous certaines conditions, les opérations d'identification de deux sommets et d'ajout d'une arête conservent la h-perfection. Cao et Nemhauser [19] ont caractérisé la h-perfection dans la classes des graphes adjoints en termes de *sous-graphes induits interdits*. Un résultat similaire a été obtenu par Arbib et Mosca [2] pour la classe des graphes qui n'ont pas de sous-graphe induit isomorphe à un chemin à 5 sommets ou au graphe complet auquel on a retiré une arête.

H-PERFECTION FORTE Des résultats de Lovász [71] et Fulkerson [48] impliquent qu'*un graphe est parfait si et seulement si le système des inégalités de non-négativité et de cliques est totalement dual-entier* (voir la définition dans la Section 3.3.1).

Les graphes h-parfaits admettent-ils une caractérisation analogue ? Un graphe G est *fortement h-parfait* si le système des inégalités de non-négativité, cliques et circuits impairs est totalement dual-entier. Un graphe est *fortement t-parfait* s'il est fortement h-parfait et sans K_4. Des résultats d'Edmonds et Giles [35] impliquent que *tout graphe fortement t-parfait est t-parfait*. Schrijver conjecture la réciproque :

CONJECTURE (SCHRIJVER [102]) *Tout graphe t-parfait est fortement t-parfait.*

Il a démontré que *les graphes t-parfaits héréditaires sont fortement t-parfaits* [101] (on rappelle qu'un graphe est *t-parfait héréditaire* si tous ses sous-graphes sont t-parfaits). Bruhn et Stein [15] ont prouvé que *les graphes t-parfaits sans griffe sont fortement t-parfaits*.

COLORATIONS La caractérisation polyédrale des graphes parfaits (Théorème 1.1.3) montre qu'imposer l'égalité de χ et ω sur tout sous-graphe induit implique que le polytope des stables est entièrement décrit par les inégalités de non-négativité et de cliques.

Étant donné que les inégalités de circuits impairs sont les seules intervenant en plus dans la description du polytope des stables d'un graphe h-parfait, on peut s'attendre à ce que le nombre chromatique d'un tel graphe reste proche de la taille de sa plus grande clique.

L'existence d'une constante c pour laquelle tout graphe h-parfait G satisferait $\chi(G) \leq \omega(G) + c$ n'est pas connue. Laurent et Seymour ont donné *un graphe t-parfait avec* $\chi = 4$ *et* $\omega = 3$ [102, pg. 1207]. Ainsi, c devrait être supérieure ou égale à 1. Par ailleurs, leur exemple montre que le polytope des stables d'un graphe t-parfait n'a pas la propriété de décomposition entière [63].

Sebő a prouvé que *la* $(\omega + 1)$*-colorabilité des graphes h-parfaits découlerait du cas* $\omega \leq 2$ (voir [16]).

Des résultats de Bruhn et Stein [16] impliquent que *tout graphe h-parfait sans griffe est* $(\omega + 1)$*-colorable* (et une coloration optimale peut être trouvée en temps polynomial). Enfin, Gerards et Shepherd [51] ont démontré que *tout graphe t-parfait héréditaire est 3-colorable* (et une 3-coloration peut être trouvée efficacement)

Cette thèse se concentre sur l'étude des problèmes de la reconnaissance des graphes h-parfaits, du calcul de leur nombre chromatique et de la propriété de décomposition entière de leur polytope des stables.

2.2 PLAN DE LA THÈSE ET CONTRIBUTIONS

CHAPITRE 3 : PRÉLIMINAIRES On donne les notations, définitions et résultats nécessaires à la compréhension de la suite du document.

CHAPITRE 4 : SUR LES OPÉRATIONS QUI CONSERVENT LA H-PERFECTION Dans ce chapitre, on étudie des opérations de graphes qui préservent la h-perfection et on montre que certaines conservent de plus la propriété de décomposition entière du polytope des stables.

Une *t-contraction* d'un graphe G est obtenue en identifiant un sommet v à tous ses voisins, lorsque ceux-ci forment un stable de G. Un *t-mineur de* G est un graphe obtenu à partir de G par une suite de suppressions de sommets et de t-contractions (dans n'importe quel ordre). Gerards et Shepherd [51] ont prouvé que *les t-mineurs conservent la t-perfection.*

Nous commençons par étendre ce résultat en démontrant que *les t-mineurs conservent aussi la h-perfection*. La preuve montre de plus que *la perfection est préservée par les t-mineurs*.

Un polyèdre $P \subseteq \mathbb{R}^n$ a la *propriété de décomposition entière* si pour tout entier positif k, chaque vecteur entier de kP est la somme de k vecteurs entiers de P.

On prouve aussi que *la propriété de décomposition entière du polytope des stables est conservée par les t-mineurs*.

Nous utiliserons ce résultat au Chapitre 7 pour répondre négativement à une question de Shepherd sur les graphes t-parfaits 3-colorables.

Soient G et H des graphes et v un sommet de G. La *substitution de* v *par* H *dans* G est le graphe obtenu de l'union de deux copies disjointes de $G - v$ et H en ajoutant une arête uw pour chaque voisin u de v dans G et chaque sommet w de H. Nous caractérisons *les graphes qui peuvent être substitués à un sommet d'un graphe h-parfait de sorte que le graphe obtenu soit h-parfait lui aussi*.

On dit qu'un graphe est *minimalement h-imparfait* (resp. *minimalement t-imparfait*) s'il est h-imparfait (resp. t-imparfait) et si tout t-mineur de G (sauf G lui-même) est h-parfait (resp. t-parfait).

Comme conséquence de notre résultat sur les substitutions, nous obtenons une propriété des *ensembles homogènes* dans les graphes minimalement h-imparfaits et t-imparfaits.

CHAPITRE 5 : SUR LES GRAPHES MINIMALEMENT H-IMPARFAITS La t-perfection est une propriété co-NP mais on ne connaît pas de certificat combinatoire de t-imperfection.

L'appartenance de la h-perfection à NP ou co-NP est toujours ouverte. La caractérisation des graphes minimalement t-imparfaits et minimalement h-imparfaits conduirait à une compréhension combinatoire de ces propriétés.

Nous commencerons par faire l'inventaire des exemples connus de graphes minimalement t-imparfaits. Nous ne donnons pas de nouvel exemple, mais décrivons leur polytope des stables et formulons une conjecture à son sujet. De plus, nous énumérons les conjectures énoncées dans la littérature et en suggérons de nouvelles.

Il est facile de vérifier que K_4 est l'unique graphe minimalement t-imparfait qui n'est pas minimalement h-imparfait. On se propose de déterminer *tous les graphes sans K_4 qui sont minimalement h-imparfaits et pas minimalement t-imparfaits*. Ils montrent que certaines des questions et conjectures formulées pour les graphes minimalement t-imparfaits ne peuvent s'étendre aux minimalement h-imparfaits qu'en excluant certains cas.

On présente une conjecture de Sebő qui énonce que les graphes minimalement h-imparfaits qui ont des cliques d'au moins 4 sommets sont des antitrous impairs, et nous montrons qu'elle est satisfaite par *les graphes planaires*. Nous caractérisons aussi *les graphes h-parfaits et minimalement h-imparfaits dans la classes des graphes circulants*.

Nous expliquons en quoi ces résultats pourraient être utiles dans la recherche d'une preuve de la conjecture de Sebő pour le cas des graphes sans griffe. Combinée aux résultats de Bruhn et Stein [16], une telle preuve fournirait directement une caractérisation combinatoire co-NP de la h-perfection sans griffe.

Nous observons enfin que *les graphes adjoints minimalement h-imparfaits* peuvent être facilement obtenus en utilisant un théorème de Cao et Nemhauser [19].

CHAPITRE 6 : PROPRIÉTÉ D'ARRONDI ENTIER POUR LE NOMBRE CHROMATIQUE DE CERTAINS GRAPHES H-PARFAITS Le nombre chromatique d'un graphe parfait est toujours égal à sa relaxation fractionnaire. Ce n'est pas le cas pour les graphes h-parfaits, et on ne connaît pas de borne sur l'écart maximum de ces deux quantités. La détermination de cet écart est liée à la complexité du calcul du nombre chromatique d'un graphe h-parfait. Les résultats principaux de ce chapitre font l'objet de [7].

Pour tout graphe G et tout $c \in \mathbb{Z}_+^{V(G)}$, le nombre chromatique (pondéré) de (G, c) est le plus petit cardinal d'un multi-ensemble $\mathcal{F}$ de stables de G tel que tout sommet v de G appartient à au moins c_v membres de $\mathcal{F}$.

Nous démontrons que *tout graphe h-parfait adjoint et tout graphe t-parfait sans griffe G a la propriété d'arrondi entier pour le nombre chromatique* : pour tout poids entier positif c sur les sommets, le nombre chromatique de (G, c) s'obtient en arrondissant sa relaxation fractionnaire à l'entier supérieur. En d'autres termes, *le polytope des stables de G a la propriété de décomposition entière*.

Ces résultats impliquent l'existence d'un algorithme polynomial pour *le calcul du nombre chromatique pondéré d'un graphe t-parfait sans griffe ou h-parfait adjoint*. Par ailleurs, ils fournissent *un nouveau cas d'une conjecture de Goldberg et Seymour* [55, 104] sur l'arête-coloration.

Sebő [103] a prouvé que toute projection d'un polyèdre défini par des contraintes totalement unimodulaires à la propriété de décomposition entière. Nous terminons ce chapitre en montrant que la réciproque est fausse, même pour les polytopes 0-1.

CHAPITRE 7 : SUR LA COLORATION DES GRAPHES H-PARFAITS En utilisant le fait que les t-mineurs conservent la propriété de décomposition entière du polytope des stables (prouvé au Chapitre 3), *nous résolvons un problème de Shepherd [108]* sur les graphes t-parfaits 3-colorables.

D'autre part, nous prouvons une caractérisation (par exclusion de sous-graphes induits) de la *propriété d'arrondi entier du nombre chromatique pour les poids 0-1 dans les graphes h-parfaits complémentaires de graphes adjoints*. Un des deux sous-graphes interdits est *un nouvel exemple de graphe t-parfait qui n'est pas 3-colorable*.

Un graphe est *sans P_6* s'il n'a pas de sous-graphe induit isomorphe au chemin à 6 sommets. Après un inventaire des résultats et conjectures sur le nombre chromatique des graphes h-parfaits, nous montrons qu'un résultat de Randerath, Schiermeyer et Tewes [93, 94] implique que *tout graphe h-parfait sans P_6 est $(\omega+1)$-colorable*. Cette borne est serrée et une $(\omega+1)$-coloration peut être trouvée en temps polynomial.

CHAPITRE 8 : DÉCOMPOSITION D'OREILLES ET H-PERFECTION DANS LES GRAPHES ADJOINTS La complexité de la reconnaissance des graphes h-parfaits est ouverte. Dans ce chapitre nous étudions le cas des graphes adjoints, pour lesquels le problème a une formulation combinatoire simple. Notre solution met en évidence des liens entre les espaces binaires, l'arête-coloration, certaines subdivisions de K_4 et les décomposition d'oreilles. Les résultats de ce chapitre font l'objet de [8].

Notons C_3^+ le graphe obtenu du triangle en ajoutant une seule arête parallèle. Un *C_3^+-impair* est un graphe obtenu en remplaçant chaque arête e de C_3^+ par une chaîne ayant un nombre impair d'arêtes, de sorte que les chaînes correspondant à des arêtes différentes ne partagent pas de sommets internes deux-à-deux. Un graphe est dit *sans C_3^+-impair* s'il n'a pas de sous-graphe isomorphe à un C_3^+-impair.

Des résultats de Kawarabayashi, Reed, Wollan [66] (et Huynh [62]) impliquent qu'il est possible de détecter un C_3^+-impair (comme sous-graphe) en temps polynomial. De plus, Bruhn et Schaudt [14] ont donné un algorithme efficace plus simple pour le cas des graphes de degré maximum 3.

Nous proposons *un algorithme simple et élémentaire pour la reconnaissance des graphes sans* C_3^+*-impair* (sans restriction sur les degrés). Il peut être converti en *un algorithme efficace qui décide la h-perfection dans les graphes adjoints*.

Soit G un graphe. On note $\beta(G)$ le plus grand entier k tel que G a un sous-graphe qui admet une décomposition d'oreilles ouverte avec k oreilles (voir Section 3.2.2 pour la définition de ces termes). Par exemple, $\beta(G) \leq 1$ si et seulement si G est sans C_3^+-impair.

Nous prouvons que *le calcul de* β *est résoluble en temps polynomial à paramètre fixé* et énonçons une conjecture sur la propriété d'arrondi entier de l'indice chromatique des graphes pour lesquels β est petit.

D'autre part, nous donnons *un algorithme efficace simple pour la détection de subdivisions totalement impaires de* K_4 *dans les graphes sans* C_3^+*-impair*.

La relation des graphes sans C_3^+-impair aux subdivisions totalement impaires de K_4 a été mise en évidence dans la thèse de Cao [18]. Cette dernière contient des résultats structurels et des constructions pour les graphes simples sans C_3^+-impair. Nous clôturons le chapitre avec l'énumération des principaux résultats de [18] liés aux graphes sans C_3^+-impair et observons que certains énoncés sont incorrects.

CHAPITRE 10 : CONCLUSION On résume les questions et conjectures énoncées dans les chapitres précédents. On suggère enfin diverses perspectives de recherche pour la poursuite de l'étude de la h-perfection et des problèmes liés.

3 PRELIMINARIES

This chapter contains the notations, definitions and results which are necessary in understanding the rest of the document. No new result is given and it is intended as a reference chapter only.

With a few exceptions, we do not provide proofs. Indeed, the content is standard regarding the combinatorial optimization literature and they can be found in the indicated references.

Ce chapitre contient l'ensemble des notations, définitions et résultats nécessaires à la compréhension de la suite de la thèse. Il ne propose pas de nouveau résultat et est conçu en tant que référence pour les chapitres suivants.

Les résultats énoncés sont pour la plupart standards en Optimisation Combinatoire. À quelques exceptions près nous ne fournissons pas leurs preuves et proposons des références qui les contiennent.

3.1 NUMBERS, SETS AND FAMILIES

The sets of integers, rational numbers and real numbers are respectively denoted $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$. The set of non-negative integers is written $\mathbb{Z}_+$ and the set of non-negative rational numbers is denoted $\mathbb{Q}_+$. We write $\varnothing$ for the empty set. For every real number x, let $\lfloor x \rfloor$ (resp. $\lceil x \rceil$) denote the floor (resp. ceiling) of x.

We will use the usual convention: $\max \varnothing = +\infty$ and $\min \varnothing = -\infty$. For each non-negative integer k, we put $[k] := \{1, \ldots, k\}$. Finally, we write $\mathbb{F}_2$ for the field with 2 elements. The all-1 vector of a finite-dimensional vector space is always written $\mathbf{1}$ without further precision (there shall not be any ambiguity).

Let S be a set. The number of elements (or cardinality) of S is written $|S|$. The *incidence vector* of a subset Y of S, denoted χ^Y, is the vector of $\mathbb{R}^S$ defined for every $s \in S$ by: $\chi^Y(s) = 1$ if $s \in Y$ and $\chi^Y(s) = 0$ otherwise. If Y has a single element s, we write χ^s instead of $\chi^{\{s\}}$. For every $x \in \mathbb{R}^S$, let $x(Y)$ denote $\sum_{s \in Y} x_s$.

For each $Y \subseteq S$, we define *the projection of* $\mathbb{R}^S$ *on* $\mathbb{R}^Y$ as the map $\pi : \mathbb{R}^S \to \mathbb{R}^Y$ which sends each vector $x \in \mathbb{R}^S$ to its restriction to Y. That is, the map just deletes the coordinates indexed by the elements of $S \setminus Y$. We often refer to π simply as *the projection* $\mathbb{R}^S \to \mathbb{R}^Y$. The *projection of* $X \subseteq \mathbb{R}^S$ *on* $\mathbb{R}^Y$ is the set $\pi(X)$.

For elements u, v of a set X, we write uv for the set $\{u, v\}$. A *pair* is a set of cardinality 2. The symmetric difference of two sets X and Y is denoted $X \Delta Y$.

A *multiset* is a set in which the same element can occur more than once. The number of occurrences of an element is its *multiplicity*.

The *cardinality* of a multiset $\mathcal{F}$, denoted $|\mathcal{F}|$ (extending the notation for sets), is the sum of the multiplicities of the elements of $\mathcal{F}$.

The *sum* of two multisets $\mathcal{F}_1$ and $\mathcal{F}_2$, denoted $\mathcal{F}_1 + \mathcal{F}_2$, corresponds to their union in which multiplicities are added. If $\mathcal{F}_2$ consists of a single element s with multiplicity 1, we write the sum $\mathcal{F}_1 + s$ instead of $\mathcal{F}_1 + \{s\}$. The *difference* $\mathcal{F}_1 - \mathcal{F}_2$ of $\mathcal{F}_1$ and $\mathcal{F}_2$ is obtained by removing from $\mathcal{F}_1$ each element s of $\mathcal{F}_2$ as many times as its multiplicity in $\mathcal{F}_2$ (if the latter is greater than the multiplicity in $\mathcal{F}_1$, then we simply delete every occurrence of x from $\mathcal{F}_1$). Similarly, we write $\mathcal{F}_1 - x$ instead of $\mathcal{F}_1 - \{x\}$.

A subset (resp. submultiset) of a set S (resp. multiset) is *proper* if it is different from S. An element of a set $\mathcal{F}$ of subsets of a set X is *inclusion-wise maximal* if it is not a proper subset of an element of $\mathcal{F}$. Besides, $\mathcal{F}$ *covers* X if each element belongs to at least one member of $\mathcal{F}$. It is easy to check that if $\mathcal{F}$ is closed by taking subsets, then $\mathcal{F}$ *covers* X *if and only if* X *can be partitioned into members of* $\mathcal{F}$.

The symbol ■ marks the end of the proof of a proposition, lemma, theorem, corollary, whereas □ denotes the end of a proof of a claim inside a larger proof.

For further standard and basic related notations and notions, we refer to Chapter 2 of [102].

3.2 GRAPHS

In this thesis we only consider finite undirected graphs. They can have multiple edges but no loops.

Let G be a graph. The set of vertices and the set of edges of G are respectively denoted $V(G)$ and $E(G)$. The *ends* of an edge e of G are its two elements. We use the notation $e = uv$ to specify that u and v are the ends of e. Two vertices u and v of G are *adjacent* if G has an edge whose ends are u and v.

Edges are *parallel* if they have the same ends. The *multiplicity* of an edge e of G is the number of edges which are parallel to e (including e). Besides, G is *simple* if its edges all have multiplicity 1.

A graph is *complete* if its vertices are pairwise-adjacent. For each $X \subseteq V(G)$, the set of edges which have both ends in X is written $E_G(X)$.

Let v be a vertex of G. A *neighbor of v in G* is a vertex adjacent to u. The *neighborhood* of v in G, denoted $N_G(v)$, is the set of neighbors of v in G. Besides, let $N_G[v] := N_G(v) \cup \{v\}$.

An edge e and a vertex u of G are *incident* if u is an end of e. Two edges are *incident* if they have at least one common end.

3.2.1 *Basic notions*

DEGREES For each vertex u of G, we write $\delta_G(u)$ for the set of edges of G which are incident with u. The *degree of u in G*, denoted $d_G(u)$, is the number $|\delta_G(u)|$. The largest degree of a vertex of G is written $\Delta(G)$. A vertex of G is *isolated* if it has degree 0 (that is it has no neighbor). Unless G is simple, the degree and the number of neighbors of a vertex may differ.

A graph is *k-regular* if its vertices all have the same degree k. In this case, we say that k is the *degree* of the graph.

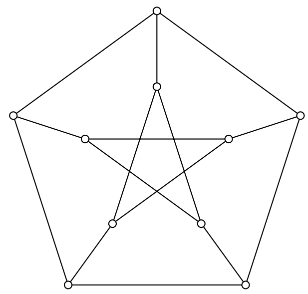


Figure 3.1 – a 3-regular simple graph (*Petersens's graph*)

ISOMORPHISMS Two graphs G_1 and G_2 are *isomorphic* if there exist bijective maps $f : V(G_1) \to V(G_2)$ and $g : E(G_1) \to E(G_2)$ such that for each $uv \in E(G_1)$, the ends of $g(uv)$ are $f(u)$ and $f(v)$.

If G_1 and G_2 are simple, then this is equivalent to state that there exists a bijective map (called an *isomorphism*) $f : V(G_1) \to V(G_2)$ such that for every pair uv of vertices of G_1: the pair $f(u)f(v)$ is an edge of G_2 if and only if uv is an edge of G_1.

A simple graph H is *vertex-transitive* if for each pair of vertices u and v of G, there exists an isomorphism of G onto itself which sends u to v.

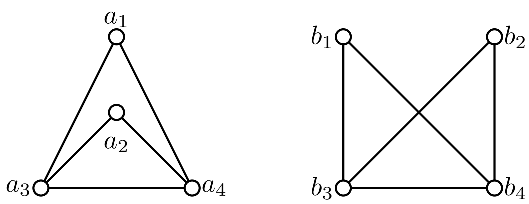


Figure 3.2 – two isomorphic graphs ($a_i \to b_i$ is an isomorphism)

OPERATIONS ON GRAPHS For each $F \subseteq E(G)$, let $G - F$ denote the graph $(V(G), E(G) - F)$. We say that $G - F$ is obtained from G by *deleting* F and we refer to this operation as an *edge-deletion*. If F has a single element e, then we write $G - e$ instead of $G - \{e\}$.

The *underlying simple graph* of G is the graph (unique up to isomorphism) obtained from G by deleting edges until no pair of parallel edges remains.

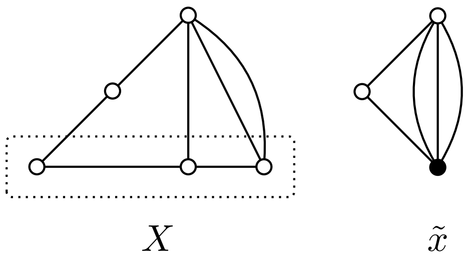
For each set F of pairs of vertices of G, let $G + F$ denote the graph $(V(G), E(G) + F)$. We say that $G + F$ is obtained from G by *adding the pairs of F as edges*. If F has a single element e, then we write $G + e$ instead of $G + \{e\}$.

For each $X \subseteq V(G)$, let $G - X$ denote the graph obtained from G by deleting each vertex of X and each edge of G incident to a vertex of X. We say that $G - X$ is obtained from G by *deleting* X and we refer to this operation as a *vertex-deletion*. If X has a single element v, then we write $G - v$ instead of $G - \{v\}$.

Let $\tilde{x}$ be a new element which does not belong to $V(G)$ and let F be the set of edges of G which have no end in X. We define:

$$\tilde{E} := F + \{u\tilde{x} \colon \forall uv \in E(G) \text{ with } v \in X \text{ and } u \notin X\}\,.$$

The pair $(V(G) - X + \tilde{x}, \tilde{E})$ is a graph, denoted G/X and we say that it is obtained from G by *identifying (or shrinking) the vertices of X to a single vertex* (see Figure 3.3).



(a) G (b) G/X

Figure 3.3 – shrinking a set X of vertices of G to a single vertex $\tilde{x}$

For each edge $e = uv$ of G, we write G/e for $G/\{u, v\}$ and say that G/e is obtained from G by *contracting* e. We refer to this operation as an *edge-contraction*. A *minor* of G is a graph obtained from G by a sequence of vertex or edge-deletions and edge-contractions.

SUBGRAPHS A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. It is *proper* if it is distinct from G, and it is *spanning* if $V(H) = V(G)$. For each graph H, we will say that G *contains* H if G has a subgraph isomorphic to H. For each $X \subseteq V(G)$, *the subgraph of G induced by X*, denoted $G[X]$, is the subgraph $G - (V(G) \setminus X)$. In other words, it is the graph $(X, E_G(X))$. A subgraph H of G is *induced* if it is induced by some subset of vertices of G.

For each simple graph H, we say that G *contains an induced* H (or that G has an induced H) if G has an induced subgraph whose underlying simple graph is isomorphic to H.

If G is simple, then *G has an induced H if and only if there exists $X \subseteq V(G)$ such that $H = G[X]$*.

Two subgraphs are *vertex-disjoint* (resp. *edge-disjoint*) if they do not have a common vertex (resp. edge).

UNION. INTERSECTION. COMPLEMENT The *union* of G_1 and G_2, denoted $G_1 \cup G_2$, is the graph $(V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. Their *intersection*, denoted $G_1 \cap G_2$, is the graph $(V(G_1) \cap V(G_2), E(G_1) \cap E(G_2))$.

The *complement* of a graph G, denoted $\overline{G}$, is the simple graph whose vertex-set is $V(G)$ and whose edges are the pairs of non-adjacent vertices of G.

PATHS, CIRCUITS AND CONNECTIVITY A *path* P of G is a sequence $(u_1, \ldots, u_k)$ (with $k \geq 1$) of pairwise-distinct vertices of G such that for each $i \in [k-1]$: $u_i u_{i+1} \in E(G)$. We will often identify it to the subgraph $(\{u_1, \ldots, u_k\}, \{u_1u_2, \ldots, u_{k-1}u_k\})$.

The vertices u_1 and u_k (which may be the same) are the *ends* of P and $u_2, \ldots, u_{k-1}$ are the *inner* vertices of P. Besides, we say that P *joins* its ends. Two paths are *inner-disjoint* if they do not share inner vertices. For $u, v \in V(G)$, a *uv-path* of G is a path joining u to v.

Let $v \in V(G)$. The *component of v in G* is the subgraph of G induced by the set of vertices which are joined to v by a path in G. The components of the vertices of G define a partition of its vertex-set. A graph is *connected* if it is a component of itself.

A *circuit* is a 2-regular connected graph. A circuit of G is a subgraph of G which is a circuit.

The *length* of a path (or circuit) is the number of its edges. A *k-circuit* is a circuit of length k. A path (or circuit) is *odd* if it has odd length, and *even* otherwise. For $u, v \in V(G)$, the *distance of u and v in G*, denoted $d_G(u, v)$, is the smallest length of a uv-path of G.

A *chord* of a circuit C of G is an edge of G whose ends are two non-adjacent vertices of C.

A *hole of G* is a circuit of G which has length at least 4 and no chord. An *antihole of G* is a subgraph of G which is the complement of a hole of $\overline{G}$.

A hole (or antihole) is *odd* if it has an odd number of vertices.

TREES A graph is a *tree* if it is connected and does not contain a circuit. It is well-known that *each connected graph has a spanning tree*. Besides, an easy induction shows that *each tree with n vertices has $n-1$ edges*. Therefore, *each spanning tree of a connected graph G has $|V(G)|-1$ edges*.

SUBDIVISIONS A graph H is a *subdivision of G* if it is obtained from G by replacing each edge $e \in E(G)$ with a path P_e (of non-zero length) joining the ends of e such that: for every pair of edges $e, f \in E(G)$, the paths P_e and P_f do not share inner vertices.

If each odd circuit of G stays odd in the subdivision, we say that H is an *odd subdivision* of G. If each P_e has odd length, then we furthermore say that H is a *totally odd subdivision of* G (see Figure 3.4).

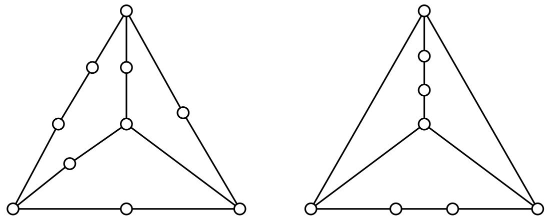


Figure 3.4 – non-totally and totally odd subdivisions of K_4

LINE GRAPH The *line graph of* G, denoted $L(G)$, is the simple graph whose vertices are the edges of G and whose edge-set is the set of pairs of incident edges of G. A graph H is a *line-graph* if there exists a graph G such that $H = L(G)$.

GRAPHS K_n, P_n, C_n Simple complete graphs (or paths, circuits) of the same size are obviously isomorphic to each other. We give notations to represent each of these isomorphism classes.

Let $n \geq 1$ be an integer and $V_n := \{v_1, \ldots, v_n\}$ be a set of n elements.

The graph K_n is the complete graph whose vertex-set is V_n (see Figure 3.5). A graph is K_n*-free* if it does not contain K_n.

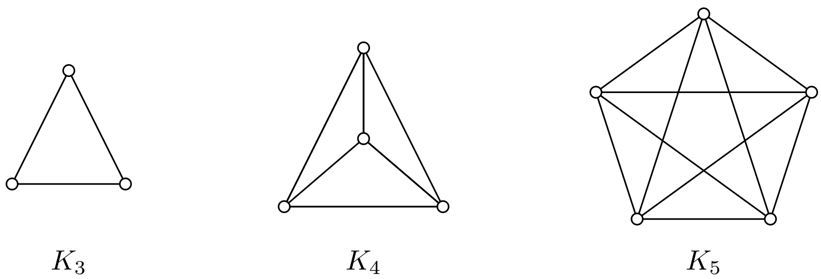


Figure 3.5 – examples of complete graphs

Let P_n denote the graph whose vertex-set is V_n and whose edge-set is $E(P_n) := \{v_1v_2, \ldots, v_{n-1}v_n\}$. A graph is P_n*-free* if it does not have an induced P_n.

Finally, for $n \geq 2$ the graph C_n is defined by $V(C_n) := V_n$ and $E(C_n) := E(P_n) \cup \{v_1v_n\}$ (see Figure 3.6).

WHEELS Let $n \geq 3$ be an integer. Let W_n denote the graph obtained from C_n by adding a new vertex c_n (the *center* of W_n) and

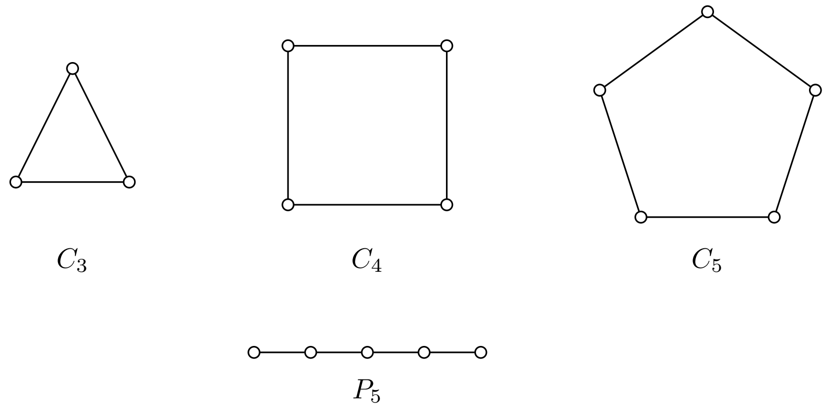


Figure 3.6 – examples of circuits and paths

every edge between C_n and c_n. The graph W_n is the *wheel of size n (or n-wheel)*(see Figure 3.7). The circuit C_n of W_n is the *rim* of W_n.

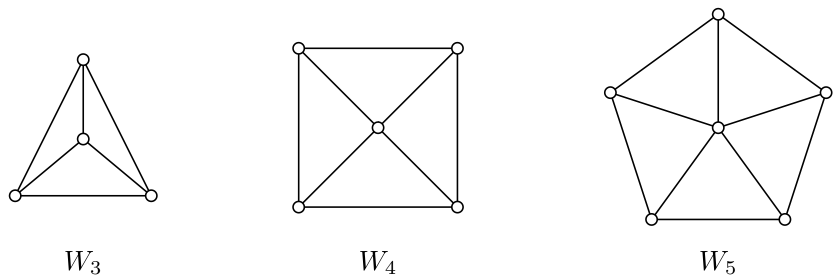


Figure 3.7 – examples of wheels

WEBS AND ANTIWEBS Let $k \geq 1$ and $n \geq 3$ be integers. Let E_k be the set of pairs uv of vertices of C_n such that $d_{C_n}(u, v) \leq k$. The graph C_n^k is defined as $C_n + E_k$. For example, if $n \leq 2k + 1$ then $C_n^k = K_n$. Besides, $C_n = C_n^1$.

A graph is a *web* if it is isomorphic to a C_n^k. An *antiweb* is the complement of a web.

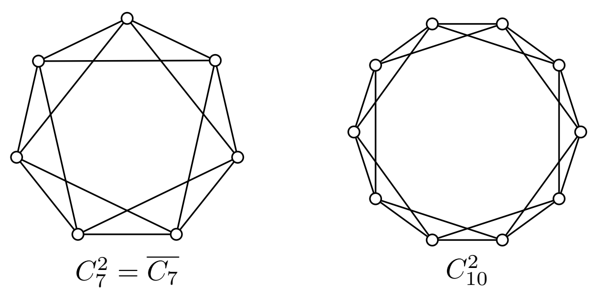


Figure 3.8 – examples of webs

3.2.2 *Higher connectivity. Menger's theorem. Ear-decompositions*

DISJOINT PATHS We will use Menger's theorem several times in this thesis. We follow the exposition of Chapter 9.1 in [102].

Let G be a graph and S, T be subsets of $V(G)$. An $\{S, T\}$*-path* is a path joining a vertex of S to a vertex of T. An $\{S, T\}$*-separator* of G is a subset of vertices of G meeting every $\{S, T\}$-path.

THEOREM 3.2.1 (MENGER [80]) *Let G be a graph and $S, T \subseteq V(G)$. The maximum number of vertex-disjoint $\{S, T\}$-paths is the minimum cardinality of an $\{S, T\}$-separator of G.*

A maximum number of vertex-disjoint $\{S, T\}$-paths can be found in polynomial-time (by a reduction to the max-flow min-cut algorithm of Ford and Fulkerson [45, 46]. When $\max(|S|, |T|) \leq 2$, the recent and faster algorithm of Tholey [114] can also be used.

CUTS AND CONNECTIVITY Let G be a graph. A *vertex-cut* of G is a set $X \subseteq V(G)$ whose deletion increases the number of components of G. A vertex-cut having k elements is called a *k-vertex-cut*. The unique element of a 1-cut is a *cut-vertex* of G.

For each positive integer k, a graph is *k-connected* if it has more than k vertices and no k-cut. A graph is *k-edge-connected* if deleting less than k edges cannot increase the number of components.

BLOCKS A *bridge* of a graph is an edge whose deletion increases the number of components. A *block* of a graph G is either an isolated vertex or the subgraph induced by the two ends of a bridge or a 2-connected subgraph of G which is maximal for the subgraph relation.

Two blocks cannot intersect on more than one vertex, which must be a cut-vertex.

EAR-DECOMPOSITIONS An *ear-decomposition* of a graph G is a sequence $(C, P_1, \ldots, P_k)$ of a circuit C and paths (or circuits) $P_1, \ldots, P_k$ of G (with $k \geq 0$) such that $G = C \cup P_1 \cup \cdots \cup P_k$ and for each $i \in [k]$: if P_i is a path, then exactly its ends are vertices of $C \cup (P_1 \cup \cdots \cup P_{i-1})$ and if P_i is a circuit, then P_i has exactly one vertex in $C \cup (P_1 \cup \cdots \cup P_{i-1})$. The paths or circuits $C, P_1, \ldots, P_k$ are the *ears* of the decomposition.

The decomposition is *open* if $P_1, \ldots, P_k$ are all paths and C has length at least 3. Besides, it is *odd* if all the ears have odd length.

THEOREM 3.2.2 (WHITNEY [120]) *A graph has an open ear-decomposition if and only if it is 2-connected.*

THEOREM 3.2.3 (ROBBINS [95]) *A graph has an ear-decomposition if and only if it is 2-edge-connected.*

The following shows that all ear-decompositions of a 2-edge-connected graph have the same number of ears:

Proposition 3.2.4 *Let G be a 2-edge-connected graph.*

The ear-decompositions of G have the same number of ears, which is:

$$|E(G)| - |V(G)| + 1.$$

Proof – Let $(C, P_1, \ldots, P_k)$ be an ear-decomposition of G.

Let $e \in E(C)$ and for each $1 \leq i \leq k$, let $e_i \in E(P_i)$. It is easy to check that $T := G - e - e_1 - \cdots - e_k$ is a spanning tree of G. Hence, the number of edges of G which do not belong to T is $k+1$. Since T is a tree: $|E(T)| = |V(G)| - 1$.

Therefore, $k = |E(G)| - |V(G)| + 1$.

■

Hence, we may speak of the *number of ears* of a 2-edge-connected graph. This number is also known as the *cyclomatic number* of the graph.

Using Menger's theorem (Theorem 3.2.1), it is straightforward to check the following:

Proposition 3.2.5 *Let G be a 2-edge-connected graph.*

Each ear-decomposition of a 2-edge-connected subgraph H of G can be completed in an ear-decomposition of G.

Furthermore if G is 2-connected, then the ear-decomposition of H can be completed with open ears.

3.2.3 *Cliques, stable sets, matchings.*

Let G be a graph. A subset of vertices of G is a *clique* if it induces a complete subgraph of G. In other words, it is a set of pairwise-adjacent vertices of G. A *triangle* is a clique of 3 vertices.

A *stable set* of G is a set of pairwise-non-adjacent vertices of G. That is a clique of $\overline{G}$.

A *matching* of G is a set of pairwise-non-adjacent edges of G, that is a stable set of $L(G)$. A matching is *perfect* if it covers every vertex of G.

Let $c \in \mathbb{Z}_+^{V(G)}$. The *stability number of* (G, c), denoted $\alpha(G, c)$, is the largest value of $c(S)$ over all stable sets S of G. Similarly, the *clique number of* (G, c), denoted $\omega(G, c)$, is defined by $\omega(G, c) := \alpha(\overline{G}, c)$. It is the largest value of $c(K)$ over all cliques K of G.

A stable set (resp. clique) of G is of *maximum-weight* in (G, c) if it attains the value $\alpha(G, c)$ (resp. $\omega(G, c)$). We say that a stable set (or clique) of G is *maximum* if it has maximum cardinality in G.

Let $d \in \mathbb{Z}_+^{E(G)}$. The *matching number of* (G, d), denoted $\nu(G, d)$, is the maximum of $d(M)$ over all matchings M of G. In other words, $\nu(G, d) = \alpha(L(G), d)$.

In particular, the *stability number*, *clique number* and *matching number* of G are defined respectively as $\alpha(G) := \alpha(G, \mathbf{1})$, $\omega(G) := \omega(G, \mathbf{1})$ and $\nu(G) := \nu(G, \mathbf{1})$. That is, they are respectively the largest cardinalities of a stable set, clique or matching of G.

3.2.4 *Graph colorings*

CHROMATIC NUMBER Let G be a graph and k be a non-negative integer. A *k-coloring* of G is a map from $V(G) \to [k]$ which assigns distinct numbers (called *colors*) to adjacent vertices. We say that G is *k-colorable* if it admits a k-coloring. Clearly, *G is k-colorable if and only if $V(G)$ can be covered by (equivalently, partitioned into) k stable sets*.

The *chromatic number of G*, denoted $\chi(G)$, is the smallest integer such that G is k-colorable.

BIPARTITE GRAPHS A graph is *bipartite* if it is 2-colorable. That is, it has a *bipartition*: a partition of its vertex-set into two stable sets. It is easy to show that *a graph is bipartite if and only if it does not contain an odd circuit*.

CLIQUE-COVER NUMBER The *clique-cover number of G* is denoted $\overline{\chi}(G)$ and is defined as $\overline{\chi}(G) := \chi(\overline{G})$. In other words, it is the smallest number of cliques needed to cover (equivalently, partition) the vertex-set of G.

CHROMATIC INDEX A *k-edge-coloring of G* is a map from $E(G) \to [k]$ which assigns distinct numbers to incident edges: it is a k-coloring of $L(G)$. We say that G is *k-edge-colorable* if it has a k-edge-coloring. Hence, G is k-edge-colorable if and only if its edge-set can be covered (or partitioned) with k matchings.

The *chromatic index of G*, denoted $\chi'(G)$, is the smallest integer k such that G is k-edge-colorable. Equivalently, $\chi'(G) = \chi(L(G))$.

WEIGHTED COLORINGS Let $c \in \mathbb{Z}_+^{V(G)}$ and k be a positive integer. A *k-coloring* of (G, c) is a multiset $\mathcal{F}$ of cardinality k formed by stable sets of G such that each $v \in V(G)$ belongs to at least c_v members of $\mathcal{F}$. We say that (G, c) is *k-colorable* if it admits a k-coloring. The *chromatic number of (G, c)*, denoted $\chi(G, c)$, is the smallest integer k such that (G, c) is k-colorable.

Let $\mathcal{S}(G)$ denote the set of stable sets of G. It is easy to check that we have:

$$\chi(G,c) = \min\left\{ \sum_{S\in\mathcal{S}(G)} y_S \colon\ y \in \mathbb{Z}_+^{\mathcal{S}(G)};\ \sum_{S\in\mathcal{S}(G)\colon v\in S} y_S \geq c_v,\ \forall v \in V(G) \right\}. \tag{3.1}$$

Clearly, $\chi(G, \mathbf{1}) = \chi(G)$. The *clique-cover number of* (G, c), denoted $\overline{\chi}(G, c)$, is defined by $\overline{\chi}(G, c) := \overline{\chi}(G, c)$ and we have $\overline{\chi}(G, \mathbf{1}) = \overline{\chi}(G)$.

In general, we will often speak of a *coloring* (or clique-cover) without referring to its cardinality. A coloring (resp. clique-cover) of (G, c) is *optimal* if it uses $\chi(G, c)$ stable sets (resp. $\overline{\chi}(G, c)$ cliques).

3.3 LINEAR PROGRAMMING AND POLYHEDRA

We refer to Chapter 7 and 8 of [100] for the proofs of the results stated in this section.

3.3.1 *Linear programming*

MATRICES AND INEQUALITIES Vectors are column-matrices by default and matrices are always of finite size. For each matrix M, we write $M^\top$ for the transpose of M. We use only the standard product of matrices. A matrix of is *rational* if its coefficients are all rational numbers. It is furthermore *integral* if its coefficients are all integers. For a positive integer k, we say that a rational matrix M is $\frac{1}{k}$*-integral* if kM is integral.

A *linear inequality* over $\mathbb{R}^n$ is an inequality of the form $a^\top x \leq b$ ($x \in \mathbb{R}^n$), where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Such an inequality is *rational* if a and b are rational. It is *tight* for $y \in \mathbb{R}^n$ if $a^\top y = b$.

In this thesis, we only consider systems of *finite* number of rational inequalities.

THE DUALITY THEOREM Several times throughout this document, we will use the duality theorem of linear programming (due to Von Neumann [117]) in the following form. We refer to Chapter 7 in [100] for further details on this theorem and the complementary slackness corollary.

THEOREM 3.3.1 (DUALITY THEOREM OF LINEAR PROGRAMMING) *For every rational matrix A with m rows and n columns and each $c \in \mathbb{Q}^n$ and $b \in \mathbb{Q}^m$:*

$$\begin{array}{cccccc} \max & c^\top x & = & \min & \min y^\top b & \\ s.t & Ax \leq b & & & y^\top A \geq c^\top & \\ & x \geq 0 & & & y \geq 0 & \end{array} \tag{3.2}$$

If the maximum is finite, then both the maximum and the minimum are attained by rational vectors.

We will also use the following corollary:

COROLLARY 3.3.2 (COMPLETEMENTARY SLACKNESS) *In Theorem 3.3.1, if the maximum is finite then: for every pair of respective optimal solutions x and y for the maximum and the minimum, we have $y^\top(Ax - b) = 0$.*

TOTAL DUAL-INTEGRALITY Let A be a rational matrix with m rows and n columns and $b \in \mathbb{Q}^m$. The system of linear inequalities $\{Ax \leq b, x \geq 0\}$ over $\mathbb{R}^n$ is *totally dual-integral* if for every $c \in \mathbb{Z}^n$ such that:

$$\min \left\{ y^\top b \colon y^\top A \geq c^\top ; y \geq 0 \right\}$$

is finite, this minimum is attained by an integral vector.

We use this notion through the following result due to Edmonds and Giles:

THEOREM 3.3.3 (EDMONDS, GILES [35]) *If* $\{Ax \leq b, x \geq 0\}$ *is a total dual-integral system over* $\mathbb{R}^n$ *and if* b *is integral then, for every* $c \in \mathbb{Z}^n$ *such that:*

$$\max\left\{c^\top x\colon Ax \leq b, x \geq 0\right\}$$

is finite, this maximum is attained on an integral vector.

3.3.2 *Affine spaces and maps*

A set A is an *affine subspace* of $\mathbb{R}^n$ it is of the form $A = u + F$, where $u \in \mathbb{R}^n$ and F is a linear subspace of $\mathbb{R}^n$. The linear space F is uniquely determined by A and is called the *direction of* A.

An *affine combination* of a finite number of vectors $x_1, \ldots, x_k$ of $\mathbb{R}^n$ is a vector of the form $\sum_{i=1}^k \lambda_i x_i$, where $\lambda_1, \ldots, \lambda_k$ are real numbers satisfying $\sum_{i=1}^k \lambda_i = 1$. The vectors $x_1, \ldots, x_k$ are *affinely independent* if for every $(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$ such that $\sum_{i=1}^k \lambda_i x_i = 0$ and $\sum_{i=1}^k \lambda_i = 0$, we have $\lambda_1 = \cdots = \lambda_k = 0$. Linear independent vectors are obviously affinely independent too.

Let $X \subseteq \mathbb{R}^n$. The set of all affine combinations of the elements of X is an affine subspace of $\mathbb{R}^n$. It is the *affine hull* of X and is denoted $\mathsf{aff}(X)$.

The *dimension* of an affine subspace A of $\mathbb{R}^n$ is the dimension of its direction (as a linear subspace). It is easy to check that is is *the largest number of affinely independent vectors of* A *minus one*. The *dimension of a subset* X *of* $\mathbb{R}^n$ is the dimension of $\mathsf{aff}(X)$.

3.3.3 *Polyhedra*

A set $P \subseteq \mathbb{R}^n$ is a *polyhedron* if there exists a finite number of linear inequalities $a_1^\top x \leq b_1, \ldots, a_k^\top x \leq b_k$ $(x \in \mathbb{R}^n)$ such that:

$$P = \left\{x \in \mathbb{R}^n\colon a_i^\top x \leq b_i \, \forall i \in [k]\right\}.$$

Such a set of inequalities is a *description* of P. We also say that P *is described* by these inequalities. Besides, P is *rational* if the inequalities $a_1^\top x \leq b_1, \ldots, a_k^\top x \leq b_k$ are rational.

A polyhedron of $\mathbb{R}^n$ is *full-dimensional* if it has dimension n. That is, P contains $n+1$ affinely independent vectors.

In this thesis, we only consider rational polyhedra and almost always deal with full-dimensional ones.

FACES A linear inequality $a^\top x \leq b$ over $\mathbb{R}^n$ is *valid* for P if every $x \in P$ satisfies $a^\top x \leq b$.

A non-empty set $F \subseteq P$ is a *face* of P if there exists a valid inequality $a^\mathsf{T} x \leq b$ for P such that $F = \{x \in P \colon a^\mathsf{T} x = b\}$. We say that the inequality $a^\mathsf{T} x \leq b$ *defines* F. It is straightforward to check that the faces of P are polyhedra themselves and that a non-empty intersection of faces is also a face.

INTEGRALITY P is *integral* if each face of P contains an integral point. Since the set of optimal solutions of a linear program is clearly a face of the underlying polyhedron, we have the following straightforward characterization:

PROPOSITION 3.3.4 *Let $P \subseteq \mathbb{R}^n$ be a polyhedron. The following statements are equivalent:*

i) P is integral,

ii) for every $c \in \mathbb{Z}^n$ such that $\max_{x \in P} c^\mathsf{T} x$ is finite, this maximum is attained by an integral point.

FACETS A *facet* of a polyhedron P is an inclusion-wise maximal proper face of P. The following characterization can be obtained as a consequence of the duality theorem of linear programming and basic linear algebra arguments:

PROPOSITION 3.3.5 *Let $P \subseteq \mathbb{R}^n$ be a full-dimensional polyhedron and F be a face of P. The following statements are equivalent:*

i) F is a facet of P,

ii) F contains n affinely independent vectors.

This directly implies that for each facet F of a full-dimensional polyhedron P, the valid inequalities for P which define F are positive multiples of each other. That is, *F is defined by a unique valid inequality (up to a positive scalar factor)*.

Let P be a polyhedron and S be a set of inequalities describing P. An inequality $e \in S$ is *redundant* if $S - e$ also describes P. A description of P is *irredundant* if its has no redundant inequality. The following result links inequalities of irredundant descriptions and facets:

PROPOSITION 3.3.6 *For every full-dimensional polyhedron $P \subseteq \mathbb{R}^n$ and each irredundant description $a_1^\mathsf{T} x \leq b_1, \dots, a_k^\mathsf{T} x \leq b_k$ of P, the following statements hold:*

i) Each inequality $a_i^\mathsf{T} x \leq b_i$ ($i \in [k]$) defines a facet of P,

ii) Each facet of P is defined by a unique inequality $a_i^\mathsf{T} x \leq b_i$ ($i \in [k]$).

VERTICES Let P be a polyhedron. A *vertex* of P is a $u \in P$ such that for every $v, w \in P$: $u = \frac{v+w}{2}$ implies $u = v = w$. Equivalently, *$\{u\}$ is a face of P*. We will frequently use the following characterization of vertices of a polyhedron:

Proposition 3.3.7 *Let $P \subseteq \mathbb{R}^n$ be a polyhedron described by the inequalities $a_1^\mathsf{T} x \leq b_1, \ldots, a_k^\mathsf{T} x \leq b_k$ and let $u \in P$. The following statements are equivalent:*

i) u is a vertex of P,

ii) there exist n inequalities $a_{i_1}^\mathsf{T} x \leq b_{i_1}, \ldots, a_{i_n}^\mathsf{T} x \leq b_{i_n}$ which are tight for u and such that the vectors $a_{i_1}, \ldots, a_{i_k}$ are linearly independent (or span $\mathbb{R}^n$).

It is easy to check that *if P is an integral polyhedron contained in $[0,1]^n$, then each integral point of P is a vertex of P.*

POLYTOPES A *convex combination* of vectors $a_1, \ldots, a_k$ of $\mathbb{R}^n$ is a vector of the form $\sum_{i=1}^k \lambda_i a_i$, where $\lambda_1, \ldots, \lambda_k$ are real non-negative numbers such that $\sum_{i=1}^k \lambda_i = 1$. The *convex hull* of a set $X \subseteq \mathbb{R}^n$ is the set of all convex combinations of elements of X.

A *polytope* of $\mathbb{R}^n$ is the convex hull of a finite set $X \subseteq \mathbb{R}^n$. The finite basis theorem for polytopes (attributed to Minkowski [82], Steinitz [110] and Weyl [119]) states that:

Theorem 3.3.8 (Finite basis theorem for polytopes) *A set $P \subseteq \mathbb{R}^n$ is a polytope if and only if it is a bounded polyhedron.*

Hence, the terminology given for polyhedra also applies to polytopes. In particular, it is straightforward to check that *a polytope is the convex hull of its vertices*. This implies easily that *a polytope is integral if and only if its vertices are integral*.

INTEGER DECOMPOSITION PROPERTY For each subset $X \subseteq \mathbb{R}^n$ and each $\lambda \in \mathbb{R}$, let $\lambda X := \{\lambda x \colon x \in X\}$.

A polyhedron $P \subseteq \mathbb{R}^n$ has the *integer decomposition property* (abbreviated IDP) if for every positive integer k, each integral vector of kP is the sum of k integral vectors of P.

3.4 THE STABLE SET POLYTOPE

3.4.1 *Definitions. General properties*

STABLE SET POLYTOPE Let G be a graph. The *stable set polytope of G*, denoted $\mathsf{STAB}(G)$, is the convex hull of the incidence vectors of the stable sets of G. In particular, for every $c \in \mathbb{Z}_+^{V(G)}$:

$$\alpha(G,c) = \max\left\{c^\mathsf{T} x \colon x \in \mathsf{STAB}(G)\right\}.$$

Clearly, $\mathsf{STAB}(G)$ is a full-dimensional polytope. Hence, Proposition 3.3.6 shows that $\mathsf{STAB}(G)$ can be uniquely described as the set of solutions of a system of linear inequalities (up to positive scalar multiplication of these). *It is not possible to test in polynomial-time if a given vector belongs to* $\mathsf{STAB}(G)$*, unless* $\mathsf{P} = NP$ (see [64]).

It is easy to check that for each $v \in V(G)$, *the inequality* $x_v \geq 0$ *defines a facet of* $\mathsf{STAB}(G)$. They are the *non-negativity inequalities* and define the *trivial facets* of $\mathsf{STAB}(G)$.

FRACTIONAL COLORINGS Let $\mathcal{S}(G)$ denote the set of stable sets of G and let $c \in \mathbb{Z}_+^{V(G)}$. The *fractional chromatic number of* (G,c), denoted $\chi_f(G,c)$, is defined as the optimum value of the linear relaxation of the integer program (3.1) for $\chi(G,c)$:

$$\chi_f(G,c) := \min\left\{\sum_{S\in\mathcal{S}(G)} y_S\colon\; y \in \mathbb{R}_+^{\mathcal{S}(G)};\; \sum_{S\in\mathcal{S}(G)\colon v\in S} y_S \geq c_v,\, \forall v \in V(G)\right\}. \tag{3.3}$$

In particular, *we always have* $\chi(G,c) \geq \chi_f(G,c)$. A *fractional coloring of* (G,c) is a feasible solution for the minimum above. Theorem 3.3.1 implies directly that (G,c) *always has a rational optimal fractional coloring*.

The *fractional chromatic number of* G, denoted $\chi_f(G)$, is defined by $\chi_f(G) := \chi_f(G,\mathbf{1})$.

Clearly, an inequality $a^\mathsf{T}x \leq b$ defining a non-trivial facet of $\mathsf{STAB}(G)$ must satisfy $a \geq 0$ and $b > 0$. The duality theorem of linear programming implies the following:

PROPOSITION 3.4.1 *Let* G *be a graph and* $c \in \mathbb{Z}_+^{V(G)}$.

If $\mathsf{STAB}(G)$ *is completely described by non-negativity inequalities and* $a_1^\mathsf{T}x \leq b_1, \dots, a_k^\mathsf{T}x \leq b_k$ *(where* $a_1,\dots,a_k$ *are non-negative and* $b_1,\dots,b_k$ *are positive), then:*

$$\chi_f(G,c) = \max_{i\in[k]} \frac{c^\mathsf{T}a_i}{b_i}. \tag{3.4}$$

PROOF – Let $M := \max_{i\in[k]} \frac{c^\mathsf{T}a_i}{b_i}$. By the duality theorem of linear programing (Theorem 3.3.1), we have:

$$\chi_f(G,c) = \max\left\{c^\mathsf{T}x\colon\; x \geq 0;\; x(S) \leq 1 \text{ for all } S \in \mathcal{S}(G)\right\}. \tag{3.5}$$

Clearly, $\frac{a_i}{b_i}$ belongs to the polytope defined by the inequalities $x \geq 0$ and $x(S) \leq 1$ (for all $S \in \mathcal{S}(G)$). This shows that $\chi_f(G,c) \geq M$.

Let $d \in \mathbb{R}^{V(G)}$ attaining the maximum in Equation (3.5). By the duality theorem of linear programming:

$$\max\left\{d^\mathsf{T}x\colon\; x \in \mathsf{STAB}(G)\right\} = \min\left\{\sum_{i=1}^k y_i b_i\colon\; y \geq 0;\; \sum_{i=1}^k y_i a_i^\mathsf{T} \geq d^\mathsf{T}\right\}.$$

Let y be a vector attaining the minimum in this equation. Since the inequality $d^\mathsf{T} x \leq 1$ is valid for $\mathsf{STAB}(G)$, we have $\sum_{i=1}^{k} y_i b_i \leq 1$. Hence:

$$\chi_f(G,c) = c^\mathsf{T} d \leq \sum_{i=1}^{k} y_i b_i \frac{c^\mathsf{T} a_i}{b_i} \leq (\sum_{i=1}^{k} y_i b_i) M \leq M,$$

as required.

■

COMPUTING χ_f The *maximum-weight stable set problem* is as follows: for a graph G and a weight $c \in \mathbb{Z}_+^{V(G)}$, find a stable set S of G such that $c(S)$ is maximum. This problem is NP-hard in general [64].

The problem of the computation of the (weighted) fractional chromatic number of a graph and the maximum-weight stable set problem are related as follows (through the Ellipsoid method):

THEOREM 3.4.2 (GRÖTSCHEL, LOVÁSZ AND SCHRIJVER [57]) *Let $\mathcal{C}$ be a class of graphs. The following statements are equivalent:*

i) the fractional chromatic number of (G, c) can be computed in polynomial time for every $G \in \mathcal{C}$ and every $c \in \mathbb{Z}_+^{V(G)}$,

ii) the maximum-weight stable set problem can be solved in polynomial time in $\mathcal{C}$.

INTEGER DECOMPOSITION PROPERTY Results of Baum and Trotter imply the following characterization for the integer decomposition property of the stable set polytope:

THEOREM 3.4.3 (BAUM, TROTTER[4]) *Let G be a graph. The following statements are equivalent:*

i) $\mathsf{STAB}(G)$ has the integer decomposition property,

ii) for every $c \in \mathbb{Z}_+^{V(G)}$: $\chi(G,c) = \lceil \chi_f(G,c) \rceil$.

3.5 PERFECT GRAPHS

A graph G is *perfect* if each induced subgraph H of G satisfies $\chi(H) = \omega(H)$. It is *imperfect* otherwise.

For example, bipartite graphs are perfect. Besides, it is easy to check that perfect graphs cannot have odd holes or odd antiholes (see Section 3.2.1 for the definition of holes and antiholes).

REPLICATION For every graph G and every $c \in \mathbb{Z}_+^{V(G)}$, let G^c denote the graph obtained as follows: replace each vertex v of G with a new complete graph K_v of cardinality c_v (in particular, delete v if $c_v = 0$) and for every $u, v \in V(G)$: put every edge between K_u and K_v if $uv \in E(G)$ and no edge otherwise.

It is straightforward to check the following relations:

PROPOSITION 3.5.1 *For every graph G and every $c \in \mathbb{Z}_+^{V(G)}$:*

$$\chi(G,c) = \chi(G^c),\ \omega(G,c) = \omega(G^c) \text{ and } \chi_f(G,c) = \chi_f(G^c).$$

WEAK PERFECT GRAPH THEOREM The following theorem is due to Lovász and solved Berge's Weak Perfect Graph Conjecture [9]:

THEOREM 3.5.2 (LOVÁSZ [71]) *Let G be a graph. The following statements are equivalent:*

i) G is perfect,
ii) $\overline{G}$ is perfect,
iii) for every $c \in \mathbb{Z}_+^{V(G)}$: $\chi(G,c) = \omega(G,c)$,
iv) for every $c \in \mathbb{Z}_+^{V(G)}$: $\overline{\chi}(G,c) = \alpha(c,G)$.

NON-NEGATIVITY AND CLIQUE INEQUALITIES The *non-negativity inequalities* for G are the $x_v \geq 0$, for $v \in V(G)$. Each of them defines a facet of $\mathsf{STAB}(G)$, and these facets are said to be *trivial*.

A *clique-inequality* of G is of the form $x(K) \leq 1$, where K is a clique of G. Padberg [87] showed that $x(K) \leq 1$ *defines a facet of* $\mathsf{STAB}(G)$ *if and only if K is an inclusion-wise maximal clique of G* (this can also be directly proved using Proposition 3.3.5).

Let:

$$\mathsf{QSTAB}(G) := \left\{x \in \mathbb{R}^{V(G)} \colon x \geq 0;\ x(K) \leq 1 \text{ for every clique } K \text{ of } G\right\}.$$

POLYHEDRAL CHARACTERIZATION Clearly, $\mathsf{QSTAB}(G)$ is a polytope containing $\mathsf{STAB}(G)$. In general, this inclusion is strict (consider C_5 for example). Fulkerson [48] pointed out that the assertion of Theorem 3.5.2 would imply that the equality of STAB and QSTAB characterizes perfection. This is stated by Chvátal in [26]:

THEOREM 3.5.3 ([26]) *Let G be graph. The following statements are equivalent:*

i) G is perfect,
ii) $\mathsf{STAB}(G) = \mathsf{QSTAB}(G)$.

By Proposition 3.4.1, this implies that *for every perfect graph G and every $c \in \mathbb{Z}_+^{V(G)}$: $\chi_f(G,c) = \omega(G,c)$.*

MINIMALLY IMPERFECT GRAPHS An imperfect graph G is *minimally imperfect* if all its proper induced subgraphs are perfect. It is straightforward to check that odd holes and odd antiholes are minimally imperfect.

A cornerstone-result of graph theory is the Strong Perfect Graph Theorem, whose statement was conjectured by Berge [9].

It was proved by Chudnovsky, Robertson, Seymour and Thomas:

THEOREM 3.5.4 (CHUDNOVSKY ET AL. [24]) *A graph is minimally imperfect if and only if it is an odd hole or an odd antihole.*

In other words, *perfect graphs are the graphs which do not have an odd hole or odd antihole*. The proof of this result is long and elaborated but several special cases do have a simpler proof (for example, for claw-free graphs [90]). We shall not use this statement in this thesis.

By Theorem 3.5.3, the stable set polytope of a minimally imperfect graph must have a facet which is not defined by a non-negativity or clique inequality. Padberg proved that there is only one such facet and that it is defined by the *full-rank inequality* $x(V(G)) \leq \alpha(G)$:

THEOREM 3.5.5 (PADBERG [89]) *For each minimally imperfect graph G, the inequality $x(V(G)) \leq \alpha(G)$ defines a facet of* $\mathsf{STAB}(G)$ *and:*

$$\mathsf{STAB}(G) = \left\{ x \in \mathbb{R}^{V(G)} : \begin{array}{cr} x \geq 0, & \\ x(K) \leq 1 & \forall K \textit{ clique of } G, \\ x(V(G)) \leq \alpha(G). & \end{array} \right\}. \tag{3.6}$$

3.6 H-PERFECT GRAPHS

3.6.1 *Definitions and basic results*

ODD-CIRCUIT INEQUALITIES Let G be a graph. An *odd-circuit inequality* is of the form $x(V(C)) \leq \frac{|V(C)|-1}{2}$, where C is an odd circuit of G. Since $\alpha(C) = \frac{|V(C)|-1}{2}$, it is valid for $\mathsf{STAB}(G)$.

Let $\mathsf{HSTAB}(G)$ denote the polytope defined by the non-negativity, clique and odd-circuit inequalities of G. In other words:

$$\mathsf{HSTAB}(G) := \left\{ x \in \mathbb{R}^{V(G)} : \begin{array}{cr} x \geq 0, & \\ x(K) \leq 1 & \forall K \text{ clique of } G, \\ x(V(C)) \leq \dfrac{|V(C)|-1}{2} & \forall C \text{ odd circuit of } G. \end{array} \right\}. \tag{3.7}$$

A clique-inequality $x(K) \leq 1$ with $|K| = 2$ is an *edge-inequality*. Let $\mathsf{TSTAB}(G)$ denote the polytope defined by the non-negativity, edge and odd-circuit inequalities.

Clearly, if G has no clique of cardinality 4 then $\mathsf{HSTAB}(G)$ and $\mathsf{TSTAB}(G)$ coincide. Furthermore, we always have:

$$\mathsf{STAB}(G) \subseteq \mathsf{HSTAB}(G) \subseteq \mathsf{TSTAB}(G).$$

Each inclusion is strict in general. For example, Theorem 3.5.5 easily shows that $\mathsf{STAB}(\overline{C_7}) \subsetneq \mathsf{HSTAB}(\overline{C_7})$.

Besides: $\mathsf{HSTAB}(K_4) \subsetneq \mathsf{TSTAB}(K_4)$, as shows the vector $\frac{1}{3}$.

H-PERFECTION A graph G is *h-perfect* if $\mathsf{STAB}(G) = \mathsf{HSTAB}(G)$. It is *h-imperfect* otherwise.

A graph G is *t-perfect* if $\mathsf{STAB}(G) = \mathsf{TSTAB}(G)$, and *t-imperfect* otherwise. For each graph G, cliques of cardinality 4 of G are contained in inclusion-wise maximal cliques whose inequalities define facets of $\mathsf{STAB}(G)$ (see Section 3.5). These inequalities cannot be defined by edge or odd-circuit inequalities, hence:

PROPOSITION 3.6.1 *A graph is t-perfect if and only if it is h-perfect and has no clique of cardinality 4.*

Since both $\mathsf{HSTAB}(G)$ and $\mathsf{TSTAB}(G)$ are contained in $[0,1]^{V(G)}$:

PROPOSITION 3.6.2 *A graph G is h-perfect (resp. t-perfect) if and only if* $\mathsf{HSTAB}(G)$ *(resp.* $\mathsf{TSTAB}(G)$*) is integral.*

It is easy to check that for every $v \in V(G)$: $\mathsf{HSTAB}(G - v)$ is the projection of the face $x_v = 0$ of $\mathsf{HSTAB}(G)$ on $\mathbb{R}^{V(G-v)}$. This directly implies:

PROPOSITION 3.6.3 *Each induced subgraph of an h-perfect graph is h-perfect.*

This also shows that *t-perfection is closed under taking induced subgraphs*.

FACETS OF HSTAB Some of the clique and odd-circuit inequalities defining $\mathsf{HSTAB}(G)$ are redundant. In fact, the following well-known statement holds:

PROPOSITION 3.6.4 *For every graph G:*

$$\mathsf{HSTAB}(G) := \left\{ x \in \mathbb{R}^{V(G)} : \begin{array}{rl} x_v \geq 0 & \forall v \in V(G), \\ x(K) \leq 1 & \forall K \text{ inclusion-wise} \\ & \text{maximal clique of } G, \\ x(V(C)) \leq \dfrac{|V(C)| - 1}{2} & \forall C \text{ odd hole of } G. \end{array} \right\}, \tag{3.8}$$

and this description is irredundant for $\mathsf{HSTAB}(G)$*. In particular, each inequality defines a facet of* $\mathsf{HSTAB}(G)$*.*

PROOF – Consider the description $\mathcal{D}$ defining $\mathsf{HSTAB}(G)$, and keep exactly one occurrence of each inequality (odd-circuit inequalities may appear more than once).

The clique-inequalities which do not correspond to inclusion-wise maximal cliques are clearly redundant.

Furthermore if C is an odd circuit which has a chord, then the odd-circuit inequality defined by C is implied by the inequality of a proper odd circuit C' of C and the edge-inequalities corresponding to a perfect matching of $C - V(C')$.

Hence we may only keep the inequalities corresponding to maximal cliques and odd holes, and still have a description $\mathcal{D}'$ of $\mathsf{HSTAB}(G)$. Now, we show that this description is irredundant. By Proposition 3.3.6, this will imply the proposition.

Let K be an inclusion-wise maximal clique and $u \in K$. Clearly, $\frac{1}{|K|}(\chi^K + \chi^u)$ satisfies all the inequalities of $\mathcal{D}'$ except $x(K) \leq 1$. Hence, this inequality is not redundant.

Finally, let C be an odd hole of G, $v \in V(C)$ and let $k := \frac{|V(C)|-1}{2}$. It is straightforward to check that $\frac{1}{|V(C)|}(k\chi^{V(C)} + \chi^v)$ satisfies every inequality of $\mathcal{D}'$ except $x(V(C)) \leq k$. Therefore, this inequality is not redundant and we are done.

■

In particular since perfect graphs cannot have odd holes, this and Theorem 3.5.3 directly imply that *perfect graphs are h-perfect*.

Odd holes form the most basic class of imperfect h-perfect graphs. Besides, Proposition 3.6.4 and Theorem 3.5.3 easily show that *each imperfect h-perfect graph has an odd hole*.

We end this section with another relation between perfection and h-perfection. It follows directly from the following (see Section 3.2.1 for the definition of wheels):

PROPOSITION 3.6.5 *For each $k \geq 1$, the graph W_{2k+1} is t-imperfect.*

PROOF – Let v be the center of $G := W_{2k+1}$. Let $x \in \mathbb{R}^{V(G)}$ be defined for every $u \in V(G)$ by: $x_u := \frac{1}{2k+1}$ if $u = v$ and by $x_u := \frac{k}{2k+1}$ otherwise. Clearly, $x \in \mathsf{TSTAB}(G)$ and $x(V(G)) > \alpha(G)$. Therefore x shows that $\mathsf{TSTAB}(G) \neq \mathsf{STAB}(G)$ and G is t-imperfect.

■

Hence, except $W_3 = K_4$, the odd wheels are h-imperfect. Since each induced subgraph of an h-perfect graph is perfect or has an induced C_{2k+1} with $k \geq 2$, this propositions implies:

PROPOSITION 3.6.6 *The neighborhood of each vertex of an h-perfect graph induces a perfect graph.*

3.6.2 *H-covers and strong h-perfection*

H-COVERS Let G be a graph and $c \in \mathbb{Z}_+^{V(G)}$. An *integral h-cover of* (G, c) is a multiset $\mathcal{F}$ of cliques and odd circuits of G such that each vertex v of G belongs to at least c_v elements of $\mathcal{F}$.

The *cost* of a clique in an integral h-cover is 1, and the cost of an odd circuit C is $\frac{|V(C)|-1}{2}$. The *cost* of an integral h-cover of G is the sum of the costs of its members. The minimum cost of an integral h-cover of (G, c) is denoted $\rho^h(G, c)$.

Let $\mathcal{K}(G)$ be the set of cliques of G and $\mathcal{O}(G)$ be the set of its odd circuits. Clearly integral h-covers of G correspond to integral solutions of the following linear program, and the cost of the cover is its value in the program:

$$\begin{aligned} \min \quad & \sum_{K \in \mathcal{K}(G)} y_K + \sum_{C \in \mathcal{O}(G)} \frac{|V(C)|-1}{2} y_C \\ \text{s.t} \quad & \sum_{K \in \mathcal{K}(G)} y_K \chi^K + \sum_{C \in \mathcal{O}(G)} y_C \chi^{V(C)} \geq c, \\ & y_K \geq 0, y_C \geq 0 \qquad\qquad \forall K \in \mathcal{K}(G), \forall C \in \mathcal{O}(G) \end{aligned} \tag{3.9}$$

A solution of this linear program is a *fractional h-cover of* (G, c) and its *cost* is defined as its value in the program. The minimum cost of a fractional h-cover of (G, c) is written $\rho_f^h(G, c)$.

The second part of Theorem 3.3.1 shows that *there always exists a rational optimal fractional h-cover of* (G, c). Furthermore, it is straightforward to check that there exists such an h-cover whose odd circuits of length at least 5 are all odd holes.

Clearly, (3.9) is the dual program of maximizing $c^\mathsf{T}x$ over the inequalities defining $\mathsf{HSTAB}(G)$. As $\mathsf{STAB}(G) \subseteq \mathsf{HSTAB}(G)$, *we always have* $\alpha(G, c) \leq \rho_f^h(G, c)$.

The duality theorem of linear programming (Theorem 3.3.1) and Proposition 3.3.4 imply the following statement:

PROPOSITION 3.6.7 *For every graph G, the following statements are equivalent:*

i) G is h-perfect,

ii) for every $c \in \mathbb{Z}_+^{V(G)}$: $\alpha(G, c) = \rho_f^h(G, c)$.

STRONG H-PERFECTION A graph G is *strongly h-perfect* if the system of non-negativity, clique and odd-circuit inequalities of G is totally dual-integral (see Section 3.3.1).

Theorem 3.3.3 directly implies:

PROPOSITION 3.6.8 *For every graph G, the following statements are equivalent:*

i) G is strongly h-perfect,

ii) for every $c \in \mathbb{Z}_+^{V(G)}$: $\alpha(G, c) = \rho^h(G, c)$.

Besides, Theorem 3.3.3 and Proposition 3.3.4 show:

PROPOSITION 3.6.9 *Every strongly h-perfect graph is h-perfect.*

A graph is *strongly t-perfect* if the system of non-negativity, edge and odd-circuit inequalities is totally dual-integral. Similarly to Proposition 3.6.1: *a graph is strongly t-perfect if and only if it is strongly h-perfect* *and has no clique of cardinality 4*. In particular, Proposition 3.6.9 shows that *strongly t-perfect graphs are t-perfect*.

The main problem on strong t-perfection is the following conjecture, due to Schrijver:

CONJECTURE 3.6.10 (SCHRIJVER [102]) *Every t-perfect graph is strongly t-perfect.*

3.6.3 *Some sufficient conditions of h-perfection*

It is not easy in general to show that a specific graph is h-perfect. In this section, we state useful conditions of h-perfection for this thesis.

CLIQUE-SUMS Let G be a graph with a vertex-cut X which is a clique, and let $C_1, \ldots, C_k$ be the components of $G - X$. For each $i \in [k]$, put $G_i := G[C_i \cup X]$. We say that G is the *clique-sum* of the G_i (along X). Chvátal proved:

THEOREM 3.6.11 (CHVÁTAL [26]) *Let G and $G_1, \ldots, G_k$ be graphs and for each $i \in [k]$, let D_i be a description of $\mathsf{STAB}(G_i)$. If G is a clique-sum of $G_1, \ldots, G_k$, then the union of $D_1, \ldots, D_k$ is a description of $\mathsf{STAB}(G)$.*

This directly implies:

COROLLARY 3.6.12 *A clique-sum of h-perfect graphs is h-perfect.*

In fact, the proof of Theorem 3.6.11 implicitly shows that this statement still holds if h-perfection is replaced with strong h-perfection.

NICE SUBGRAPHS A subgraph H of a graph G is *nice* if every inclusion-wise maximal stable set of G meets H on $\alpha(H)$ vertices. In particular, a clique (as a complete subgraph) is nice if it meets every inclusion-wise maximal stable set of G.

The following is implicit in [102, Section 68.4 pg. 1194] and is not difficult to check:

PROPOSITION 3.6.13 *Let G be a graph and H be a nice clique or odd circuit of G. If $G - v$ is (strongly) h-perfect for every $v \in V(H)$, then G is (strongly) h-perfect too.*

PROOF – By Proposition 3.6.7, we need only to prove that for every $c \in \mathbb{Z}_+^{V(G)}$: $\alpha(G, c) = \rho_f^h(G, c)$.

Let $c \in \mathbb{Z}_+^{V(G)}$. We proceed by induction on $c(V(G))$.

If there exists $v \in V(H)$ such that $c_v = 0$, then the required equality follows from the assumption on $G - v$. Therefore, we may suppose that every $v \in V(H)$ satisfies $c_v \geq 1$.

Let $d := c - \chi^{V(H)}$. By induction, we have $\alpha(G, d) = \rho_f^h(G, d)$. Let S be a stable set of G with $d(S) = \alpha(G, d)$. Since d is non-negative, we may suppose that S is inclusion-wise maximal. By assumption, S meets H on $\alpha(H)$ vertices. Hence:

$$\alpha(G,c) \geq c(S) = \alpha(G,d) + \alpha(H) \geq \rho_f^h(G,d) + \alpha(H).$$

Now, $\rho_f^h(G,d) + \alpha(H)$ is clearly the cost of a fractional h-cover of (G,c) obtained by adding H to an optimal fractional h-cover of (G,d). This shows that $\alpha(G,c) \geq \rho_f^h(G,c)$ and we obtain the required equality (as the converse inequality is always satisfied).

The proof directly shows that the statement remains true if we replace h-perfection with strong h-perfection.

■

almost-bipartite graphs A graph is *almost-bipartite* if it has a vertex whose deletion yields a bipartite graph. Fonlupt and Uhry proved:

Theorem 3.6.14 (Fonlupt, Uhry [44]) *Every almost-bipartite graph is t-perfect.*

Further sufficient conditions of h-perfection (involving subdivisions of K_4) are available but we shall not use them in this thesis.

edge deletion and contraction We conclude this section with two graphs showing that *h-perfection (or t-perfection) is not closed under edge-deletion nor edge-contraction*.

Using the sufficient conditions given above, it is straightforward to check (using Proposition 3.6.13 on a triangle and Corollary 3.6.12) that the graph of Figure 3.9 is h-perfect. Contracting the edge e yields the t-imperfect graph W_5^{--} (see Proposition 4.3.2).

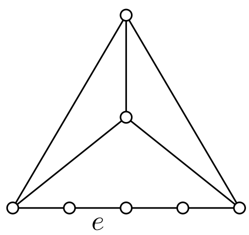


Figure 3.9 – contracting an edge does not keep h-perfection

Similarly, it is straightforward to show that the graph of Figure 3.10 is h-perfect. Consider the vector $x := \frac{1}{3}(\chi^{V(G)} + \chi^u)$.

Clearly, $x \in \mathsf{HSTAB}(G-uv)$. Besides, $x(V(G)) > 2 = \alpha(G-uv)$. Hence, $x \notin \mathsf{STAB}(G-uv)$ and $G-uv$ is h-imperfect. This example is due to Schrijver [102, pg. 1195].

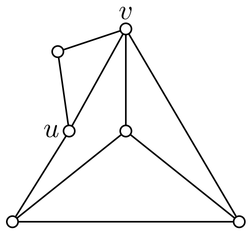


Figure 3.10 – deleting an edge does not keep h-perfection

3.6.4 *The fractional chromatic number of h-perfect graphs*

Since h-perfection is defined by a specific description of the stable set polytope, Proposition 3.4.1 directly yields a formula for the fractional chromatic number of an h-perfect graph [98].

For each graph G and $c \in \mathbb{Z}_+^{V(G)}$, let:

$$\Gamma(G,c) := \max\left\{\frac{2}{|V(C)|-1}c(V(C)) \colon C \text{ odd hole of } G\right\}.$$

Besides, let $\Gamma(G) := \Gamma(G, \mathbf{1})$.

Proposition 3.6.15 *For every h-perfect graph G and every $c \in \mathbb{Z}_+^{V(G)}$:*

$$\chi_f(G,c) = \max(\omega(G,c), \Gamma(G,c)).$$

Since $\Gamma(G) \leq 3$ (and as observed in [98]), this implies:

Corollary 3.6.16 *Every h-perfect graph G with $\omega(G) \geq 3$ satisfies:*

$$\chi_f(G) = \omega(G).$$

In particular, the fractional chromatic number of a t-perfect graph is at most 3.

This bound on the fractional chromatic number motivates several questions studied in this thesis.

Since the clique and odd circuit inequalities are valid for $\mathsf{STAB}(G)$ (whether G is h-perfect or not), Proposition 3.4.1 shows that *we always have* $\chi_f(G,c) \geq \max(\omega(G,c), \Gamma(G,c))$.

3.6.5 *Algorithms*

Grötschel, Lovász and Schrijver showed that finding a maximum-weight stable set of a perfect graph can be done in polynomial-time. They proved that this property extends to h-perfect graphs:

Theorem 3.6.17 (Grötschel, Lovász, Schrijver [56]) *A maximum-weight stable set of an h-perfect graph can be found in polynomial-time.*

Through the Ellipsoid method (see Section 3.4), this implies the following :

THEOREM 3.6.18 (GRÖTSCHEL, LOVÁSZ, SCHRIJVER [56, 57]) *The weighted fractional chromatic number of an h-perfect graph can be computed in polynomial time.*

Bruhn and Stein used this to show:

THEOREM 3.6.19 (BRUHN, STEIN [16]) *A maximum clique of an h-perfect graph can be found in polynomial-time.*

By contrast, it is not known whether a maximum-weight clique can be efficiently found in h-perfect graphs.

Schrijver [102, pg. 1194] showed that for each graph G and $x \in \mathbb{Q}^{V(G)}$, testing whether $x \in \mathsf{TSTAB}(G)$ (or finding a separating hyperplane certifying the contrary) reduces to a shortest path problem in some auxiliary graph, and can be easily done in polynomial-time. Through the Ellipsoid method, this implies that a maximum-weight stable set of a t-perfect graph can be found efficiently (the case of h-perfect graphs is more difficult, see Theorem 3.6.17). A combinatorial algorithm for finding a maximum-cardinality stable set in t-perfect graphs was given by Eisenbrand et al. [38].

Besides, it directly implies that showing a non-integral vertex of $\mathsf{TSTAB}(G)$ can be carried out efficiently. Hence:

THEOREM 3.6.20 (SCHRIJVER [102]) *T-perfection is in co-NP.*

It is not known whether h-perfection belongs to one of the classes NP or co-NP. Besides, no combinatorial certificate of t-imperfection is known.

3.7 THE MATCHING POLYTOPE

Let G be a graph. The matching polytope of G, denoted $\mathsf{MATCH}(G)$, is the convex hull of the incidence vectors of the matchings of G. That is: $\mathsf{MATCH}(G) = \mathsf{STAB}(L(G))$.

Edmonds [34] gave an efficient algorithm to find a matching of maximum weight in a graph. As a by-product, he obtained a description of the matching polytope. Padberg and Rao [88] gave a *combinatorial* polynomial-time algorithm to test if a given vector belongs to this polytope.

A graph H is *factor-critical* if for every vertex $v \in V(H)$, the graph $H - v$ has a perfect matching. In [36], Edmonds and Pulleyblank described the facets of $\mathsf{MATCH}(G)$. We need only the following part of their result:

Theorem 3.7.1 (Edmonds, Pulleyblank [36]) *For every graph G:*

$$\mathsf{MATCH}(G) := \left\{ x \in \mathbb{R}^{E(G)} : \begin{array}{cc} x \geq 0, & \\ x(\delta_G(v)) \leq 1 & \forall v \in V(G), \\ x(E(H)) \leq \dfrac{|V(H)|-1}{2} & \forall H \text{ 2-connected induced} \\ & \text{factor-critical subgraph of } G. \end{array} \right\}. \tag{3.10}$$

The chromatic index of G is denoted $\chi'(G)$ and is the smallest number of matchings needed to cover (or, partition) the edge-set of G (see Section 3.2.4). In other words, $\chi'(G) = \chi(L(G))$.

FRACTIONAL CHROMATIC INDEX The *fractional chromatic index of* G, denoted $\chi'_f(G)$, is defined by $\chi'_f(G) = \chi_f(L(G))$.

Since (3.10) holds for every graph, Proposition 3.4.1 yields a formula *in general* for the fractional chromatic index of graph. For every graph G, let:

$$\sigma(G) := \max\left\{ \frac{2|E(H)|}{|V(H)|-1} : H \text{ 2-connected factor-critical subgraph of } G \right\}.$$

Theorem 3.7.1 and Proposition 3.4.1 directly imply:

Theorem 3.7.2 (Stahl [109], Seymour [104]) *For every graph G:*

$$\chi'_f(G) = \max(\Delta(G), \sigma(G)).$$

Padberg and Rao [88] showed a combinatorial polynomial-time algorithm to determine the fractional chromatic index of a graph. An efficient algorithm also follows, through the Ellipsoid method, from Edmonds' polynomial-time algorithm to find a maximum-weight matching (see also Section 3.4).

FACTOR-CRITICAL GRAPHS Lovász showed the following characterization (see Section 3.2.2 for the related terminology):

Theorem 3.7.3 (Lovász [72, 76]) *A 2-connected graph is factor-critical if and only if it has an open odd ear-decomposition.*

This result plays a key-role in the characterization by Cao and Nemhauser of h-perfect line-graphs.

3.8 H-PERFECT LINE-GRAPHS

Various notions of perfection in line graphs were studied by Cao and Nemhauser in [19].

In particular, they characterized the h-perfection of a line-graph $L(H)$ in terms of forbidden subgraphs for H. This characterization is stated and proved in [16, 18, 19]. The proof consists in combining Theorem 3.7.1 with Theorem 3.7.3. We will use it in Chapter 6 and Chapter 8.

Let C_3^+ be the graph of Figure 3.11. An *odd-*C_3^+ is a totally odd subdivision of C_3^+ (see Section 3.2.1 for the definition of these subdivisions). It is *strict* if it is not C_3^+.

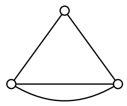


Figure 3.11 – the graph C_3^+

Theorem 3.8.1 (Cao, Nemhauser [19]) *For every graph H, the following statements are equivalent:*

i) $L(H)$ is h-perfect,

ii) H does not contain a strict odd-C_3^+.

The corresponding characterization for t-perfection follows directly from Proposition 3.6.1: *$L(H)$ is t-perfect if and only if $L(H)$ is odd-C_3^+-free and* $\Delta(H) \leq 3$.

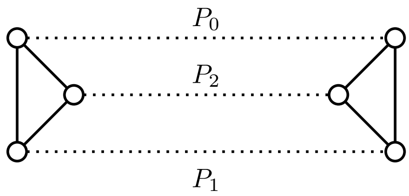


Figure 3.12 – a graph is a *skewed prism* if it is not isomorphic to K_4 and is formed by two vertex-disjoint triangles joined by three vertex-disjoint paths P_0, P_1 and P_2 (drawn dotted) such that: both P_0 and P_2 are even, and P_1 is odd. There are no other edges.

Skewed prisms are the line-graphs of the strict odd-C_3^+ graphs. An equivalent definition of those graphs is given in Figure 3.12. Taking line-graphs, Theorem 3.8.1 directly yields:

Theorem 3.8.2 (Cao, Nemhauser [19]) *Let G be a line-graph. The following statements are equivalent:*

i) G is h-perfect,

ii) G does not have an induced skewed prism.

4 ON OPERATIONS PRESERVING H-PERFECTION

A vertex of a graph is *contractible* if its neighborhood is a stable set. The *t-contraction of G at a contractible $v \in V(G)$* is the graph obtained from G by shrinking v and its neighbors to a single vertex.

H-perfection is closed under vertex-deletion (Proposition 3.6.3). Gerards and Shepherd [51] proved that t-perfection is closed under t-contraction.

A *t-minor* of a graph G is a graph obtained from G by a sequence of vertex-deletions and t-contractions.

In this chapter we first observe that taking t-minors corresponds to taking faces of the stable set polytope and use this to show that *the integer decomposition property of the stable set polytope is closed for t-minors*. This allows us to answer negatively (in Chapter 7) a question of Shepherd on the equivalence of this property and 3-colorability for t-perfect graphs.

Furthermore, we characterize pairs of graphs G and contractible vertices $v \in V(G)$ such that $\mathsf{HSTAB}(G/N_G[v])$ is a face of $\mathsf{HSTAB}(G)$. We use this to extend the result of Gerards and Shepherd by proving that *h-perfection is closed under t-contraction*. Furthermore, our proof shows that *perfection is closed under t-minors* (this is also implied by a proof of [16]).

Finally, we characterize *which graphs can be substituted to a vertex of an h-perfect graph such that the resulting graph remains h-perfect*. This implies related results on *homogeneous sets in minimally h-imperfect graphs*.

Un sommet d'un graphe est *contractible* si ses voisins forment un stable. La t-contraction d'un graphe G en un sommet contractible $v \in V(G)$ est le graphe obtenu de G en identifiant v à tous ses voisins.

La h-perfection est clairement conservée par la suppression de sommets (Proposition 3.6.3). Gerards et Shepherd [107] ont montré que la t-contraction préserve la h-perfection.

Un *t-mineur* d'un graphe G est un graphe obtenu de G par une suite de suppressions de sommets et de t-contractions (dans n'importe quel ordre).

Dans ce chapitre, nous remarquons d'abord que le polytope des stables d'un t-mineur d'un graphe G correspond à une face de $\mathsf{STAB}(G)$. Cette observation nous permet de prouver que *les t-mineurs conservent la propriété de décomposition entière du polytope des stables*. Nous utiliserons ce résultat au Chapitre 7 pour répondre négativement à une question de Shepherd sur les graphes t-parfaits 3-colorables.

D'autre part, nous caractérisons *les paires d'un graphe G et d'un sommet contractible v de G telles que $\mathsf{HSTAB}(G/N_G[v])$ est une face de $\mathsf{HSTAB}(G)$*. Cette caractérisation implique directement que *les t-mineurs conservent la h-perfection* et étend ainsi le résultat de Gerards et Shepherd. Notre preuve montre de plus que *les t-mineurs conservent la perfection* (ce qui est aussi impliqué par une preuve de [15]).

Enfin, nous caractérisons *les graphes pouvant être substitués à un sommet d'un graphe h-parfait de sorte que le graphe obtenu reste h-parfait*. On en déduit un théorème sur les *ensembles homogènes* dans les graphes minimalement h-imparfaits.

4.1 INTRODUCTION

Let G be a graph and $v \in V(G)$. We say that v is *contractible* in G if $N_G(v)$ is a stable set. A *t-contraction* of G is the graph $G/N_G[v]$, where v is a contractible vertex of G (recall that $G/N_G[v]$ denotes the graph obtained by shrinking v and its neighbors in G to a single vertex and deleting the loops which may arise, see Figure 4.1). If v is isolated, then $G/N_G[v]$ and G are obviously identical. Hence, we need only to consider non-isolated contractible vertices.

A *t-minor* of G is a graph obtained from a sequence of either vertex-deletions or t-contractions in any order. It is *proper* if it is different from G.

We say that a graph class $\mathcal{C}$ (or a property defining such a class) is *closed under an operation* $\mathcal{O}$ if each graph obtained from a member of $\mathcal{C}$ by using $\mathcal{O}$ belongs to $\mathcal{C}$.

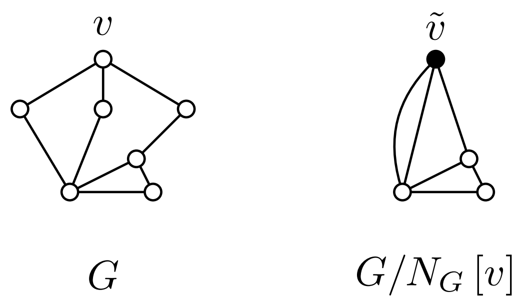


Figure 4.1 – a t-contraction at vertex v

Recall that $\mathsf{STAB}(G)$ denotes the stable set polytope of G and that a polyhedron $P \subseteq \mathbb{R}^n$ has the *integer decomposition property* (abbreviated IDP) if for every positive integer k: each integral vector of kP is the sum of k integral vectors of P.

Clearly, $\mathsf{STAB}(G-v)$ is the projection of the face $x_v = 0$ of $\mathsf{STAB}(G)$ on $\mathbb{R}^{V(G)-v}$. In this chapter, we first observe that $\mathsf{STAB}(G/N_G[v])$ can be interpreted in the same way and prove:

THEOREM 4.1.1 *The integer decomposition property of the stable set polytope is closed under t-minors.*

If G is a t-perfect graph and $\mathsf{STAB}(G)$ has the IDP, then G is 3-colorable (see Theorem 3.4.3 and Corollary 3.6.16). Conversely, *is it true that each 3-colorable t-perfect graph G is such that* $\mathsf{STAB}(G)$ *has the IDP ?* This problem was raised by Shepherd in [108]. Using Theorem 4.1.1, we will show in Chapter 7 that the answer is negative.

H-perfection is closed under taking induced subgraphs (Proposition 3.6.3). Gerards and Shepherd showed the following result (which implies that t-perfection is closed under t-minors):

THEOREM 4.1.2 (GERARDS, SHEPHERD [51]) *Let G be a graph and v be a contractible vertex of G. If G is t-perfect, then $G/N_G[v]$ is t-perfect.*

Developing further arguments to handle cliques with more than 3 vertices, we extend this result to h-perfect graphs:

THEOREM 4.1.3 *The class of h-perfect graphs is closed under t-minors.*

In the process of proving this, we also characterize the *pairs of graphs G and contractible vertices $v \in V(G)$ such that* $\mathsf{HSTAB}(G/N_G[v])$ *can be identified to a face of* $\mathsf{HSTAB}(G)$ (recall that $\mathsf{HSTAB}(G)$ denotes the polytope defined by non-negativity, clique and odd-circuit inequalities of G). We obtain as a byproduct:

THEOREM 4.1.4 *The class of perfect graphs is closed under t-minors.*

This is also implied by a proof of [16] which shows (with trivial modifications) that *strong h-perfection is closed under t-minors*. We do not know whether it could be useful in simplifying proofs of statements on perfect graphs.

An h-imperfect graph G is *minimally h-imperfect* (abbreviated MHI) if every proper t-minor of G is h-perfect. Similarly, a t-imperfect graph G is *minimally t-imperfect* (abbreviated MTI) if every proper t-minor of G is t-perfect.

T-perfection is in co-NP (Theorem 3.6.20) but a combinatorial certificate of t-imperfection is not known. We do not know however whether h-perfection belongs to NP or co-NP.

Since h-perfection and t-perfection are both closed under t-minors (Theorems 4.1.2 and 4.1.3), *a graph is t-perfect (resp. h-perfect) if and only if it does not have a t-minor which is MTI (resp. MHI).*

Hence, an approach to designing a co-NP certificate for t-perfection (resp. h-perfection) would be by using a certificate of minimal t-imperfection (resp. minimal h-imperfection). For instance, Bruhn and Stein [16] determined all the MTI claw-free graphs. The currently known MTI and MHI graphs are reviewed in Chapter 5.

We now discuss substitutions in h-perfect graphs. Let G, H be graphs and $v \in V(G)$. The *substitution of v by H in G*, denoted $G^{v \leftarrow H}$, is the graph obtained from the union of disjoint copies of $G - v$ and H by adding the edge uw for each $u \in N_G(v)$ and $w \in V(H)$.

Substitutions of vertices of perfect graphs by complete graphs play a key-role in the proof of Lovász of the Weak Perfect Graph Theorem (Theorem 1.1.1). A proof of the theorem which does not use substitutions was given in [49]. We prove the following statement, which hopefully clarifies the scope of using substitutions in studying h-perfect graphs:

Theorem 4.1.5 *Let G be an h-perfect graph, v be a non-isolated vertex of G and H be a graph. The graph $G^{v \leftarrow H}$ is h-perfect if and only if at least one of the following statements holds:*

i) H has no edge,

ii) H is perfect and v does not belong to an odd hole of G.

Our proof also shows the same result for strong h-perfection and (strong) t-perfection.

A *module of a graph* G is a subset $X \subseteq V(G)$ such that every $v \in V(G) \setminus X$ satisfies $N_G(v) \supseteq X$ or $N_G(v) \cap X = \varnothing$. The *trivial* modules of G are $\varnothing$, $V(G)$ and the singletons of $V(G)$ (they are clearly modules of G). An *homogeneous set* of G is a non-trivial module of G.

A graph is *prime* if it has no homogeneous set. Prime graphs are involved in decomposition results of certain classes of graphs (see [2] for an example related to h-perfection).

We use Theorem 4.1.5 to derive the following (see Figure 4.2 for the definition of W_5^{--}):

Theorem 4.1.6 *Except W_5^{--}, every minimally h-imperfect graph is prime.*

Specializing to t-perfect graphs, this implies that K_4 *is the only non-prime MTI graph*.

We end this introduction by reviewing related results. A pair of vertices u and v of a graph G is *even* if G does not have an induced odd uv-path. These pairs were introduced by Fonlupt and Uhry [44] who showed that *perfection is closed under shrinking the two vertices of an even pair*. Even pairs of perfect graphs are closely related to designing efficient combinatorial coloring-algorithms for perfect graphs (see [22, 41]). This result was extended by Fonlupt and Hadjar [43]: *shrinking an even pair keeps h-perfection*. Taking a t-minor and shrinking an even pair are the only currently known operations which keep h-perfection and reduce the size of the graph.

Other operations considered in the literature include the union of two graphs [50, 43] and the addition of an edge [43]. It does not seem that these results imply directly Theorem 4.1.5.

Gerards and Shepherd [51] characterized the graphs whose subgraphs are all t-perfect: they are precisely *the graphs which do not contain a t-imperfect subdivision of* K_4 (these subdivisions have a combinatorial characterization [3]).

OUTLINE In Section 4.2, we observe that taking t-minors corresponds to taking faces of the stable set polytope and prove Theorem 4.1.1.

In Section 4.3, we prove a characterization of pairs of graphs G and contractible vertices $v \in V(G)$ such that $\mathsf{HSTAB}(G/N_G[v])$ is a face of $\mathsf{HSTAB}(G)$. We use this to show Theorem 4.1.3 and Theorem 4.1.4. In Section 4.4, we explain that a proof of [16] shows that strong-h-perfection (and perfection) is closed under t-minors.

In Section 4.5, we prove Theorem 4.1.5 and derive Theorem 4.1.6.

4.2 T-MINORS AND THE STABLE SET POLYTOPE

In this section, we show that taking a t-minor corresponds to taking (a projection of) a face of the stable set polytope in general. We deduce that the IDP of the stable set polytope is closed under t-minors.

Let G be a graph and $v \in V(G)$. Let $p_{G,v}$ be the projection $\mathbb{R}^{V(G)} \to \mathbb{R}^{V(G-v)}$. Obviously:

$$\mathsf{STAB}(G-v) = p_{G,v}(\mathsf{STAB}(G) \cap \left\{x \in \mathbb{R}^{V(G)} : x_v = 0\right\}).$$

The situation is similar for t-contractions. Suppose that v is a contractible and non-isolated vertex of G and let $\tilde{v}$ be the new vertex of $G/N_G[v]$. Let:

$$F(G,v) := \mathsf{STAB}(G) \cap \left\{x \in \mathbb{R}^{V(G)} : x_u + x_v = 1\ \forall uv \in E(G)\right\}.$$

Clearly, $F(G,v)$ is a face of $\mathsf{STAB}(G)$ (it is a non-empty intersection of faces).

Let $x \in F(G,v)$ and c be the common value of x over the neighbors of v in G. Let $\gamma_{G,v}(x)$ be the vector of $\mathbb{R}^{V(G/N_G[v])}$ defined for each $u \in V(G/N_G[v])$ by:

$$(\gamma_{G,v}(x))_u = \begin{cases} c & \text{if } u = \tilde{v}, \\ x_u & \text{otherwise.} \end{cases}$$

Hence, $\gamma_{G,v}$ defines a map $F(G,v) \to \mathbb{R}^{V(G/N_G[v])}$. We have the following relation:

Proposition 4.2.1 *Let G be a graph and v be a non-isolated vertex of G. If v is contractible, then:*

$$\mathsf{STAB}(G/N_G[v]) = \gamma_{G,v}(F(G,v)).$$

Proof – Let $\mathcal{T}$ be the set of stable sets of G which contain $\{v\}$ or $N_G(v)$. Clearly, $F(G,v)$ is the convex hull of the incidence vectors of the elements of $\mathcal{T}$.

For every $S \in \mathcal{T}$, let $f(S) := S - v$ if $v \in S$, and $f(S) := S - N_G(v) + \tilde{v}$ otherwise. Since v is contractible, f is a bijection between $\mathcal{T}$ and the set of stable sets of $G/N_G[v]$. Obviously, $\gamma_{G,v}$ sends χ^T to $\chi^{f(T)}$ for every $T \in \mathcal{T}$ and this directly implies the stated equality

■

Let G be a graph, v be a contractible and non-isolated vertex of G and $\tilde{v}$ be the new vertex of $G/N_G[v]$. Let u be a neighbor of v in G.

The identification of $\mathbb{R}^{V(G/N_G[v])}$ with $\mathbb{R}^{V(G-(N_G[v]-u))}$ (that is, $\tilde{v}$ is replaced with u) shows that $\gamma_{G,v}$ *is the restriction to* $F(G,v)$ *of the projection* $\mathbb{R}^{V(G)} \to \mathbb{R}^{V(G-(N_G[v]-u))}$.

We use this observation and Proposition 4.2.1 to prove Theorem 4.1.1. The last ingredient is the following straightforward fact. For each positive integer k, a vector x of $\mathbb{R}^n$ is $\frac{1}{k}$*-integral* if kx is integral.

Proposition 4.2.2 *Let* $P \subseteq \mathbb{R}^n$ *be a polyhedron which has the integer decomposition property and* π *be the projection* $\mathbb{R}^n \to \mathbb{R}^m$.

If for every $k \geq 1$*, each* $\frac{1}{k}$*-integral vector of* $\pi(P)$ *has a* $\frac{1}{k}$*-integral preimage in* P *by* π*, then* $\pi(P)$ *has the integer decomposition property.*

The pre-image condition of the proposition is crucial. Indeed *the IDP of polytopes is not closed under taking projections in general*, as shown by a simple 0-1 example due to Sebő: let G be the union of two vertex-disjoint copies of K_4, and let v_1 and v_2 be respective vertices of the two copies. Let $P \subseteq \mathbb{R}^{V(G)}$ be the convex hull of the incidence vectors of the cliques of size 3 of G containing v_1 or v_2. It is straightforward to check that P has the IDP. Let π be the projection $\mathbb{R}^{V(G)} \to \mathbb{R}^{V(G)-v_1-v_2}$ and $Q := \pi(P)$. Clearly, the all-1 vector $\mathbf{1}$ of $\mathbb{R}^{V(G)}$ belongs to $3P$, thus $\pi(\mathbf{1})$ is an integral vector of $3Q$. It is easy to check however that $\pi(\mathbf{1})$ cannot be written as the sum of 3 integral vectors of Q and therefore Q does not have the IDP.

We now prove:

Theorem 4.1.1 *The integer decomposition property of the stable set polytope is closed under t-minors.*

Proof – Let G be a graph such that $\mathsf{STAB}(G)$ has the IDP and H be a proper t-minor of G. By induction, we may assume without loss of generality that H is obtained from G by a single vertex-deletion or t-contraction.

The discussion above shows that in both cases, $\mathsf{STAB}(H)$ is the image of a face F of $\mathsf{STAB}(G)$ by a projection π such that the pair (F,π) obviously satisfies the pre-image assumption of Proposition 4.2.2.

It is straightforward to check that F has the IDP (as a face of a polyhedron which has the IDP). Therefore, Proposition 4.2.2 shows that $\mathsf{STAB}(H)$ also has the IDP.

■

4.3 T-MINORS AND H-PERFECTION

In Section 4.3.1, we state a characterization of pairs of graphs G and contractible vertices $v \in V(G)$ such that $\mathsf{HSTAB}(G/N_G[v])$ can be identified to a face of $\mathsf{HSTAB}(G)$ (through the map $\gamma_{G,v}$ of Section 4.2).

We use this to show that h-perfection is closed under t-contractions (and thus t-minors).

The proof of the characterization itself is postponed to Section 4.3.2, and involves a structural lemma on cliques and t-contractions in certain graphs. Since we use this latter result also in Chapter 5, we present it separately in Section 4.3.3.

4.3.1 $\mathsf{HSTAB}(G/N_G[v])$ *as a face of* $\mathsf{HSTAB}(G)$

Let G be a graph and v be a contractible and non-isolated vertex of G. We recall the notations of the previous section. Let:

$$F(G,v) := \mathsf{STAB}(G) \cap \left\{ x \in \mathbb{R}^{V(G)} \colon x_u + x_v = 1 \ \forall uv \in E(G) \right\}.$$

Let $x \in F(G,v)$ and c be the common value of x over the neighbors of v in G. Let $\gamma_{G,v}(x)$ be the vector of $\mathbb{R}^{V(G/N_G[v])}$ defined for each $u \in V(G/N_G[v])$ by:

$$(\gamma_{G,v}(x))_u = \begin{cases} c & \text{if } u = \tilde{v}, \\ x_u & \text{otherwise.} \end{cases}$$

Furthermore, let:

$$F_h(G,v) := \mathsf{HSTAB}(G) \cap \left\{ x \in \mathbb{R}^{V(G)} \colon x_u + x_v = 1 \ \forall uv \in E(G) \right\}.$$

In this section, we state a characterization of the equality:

$$\mathsf{HSTAB}(G/N_G[v]) = \gamma_{G,v}(F_h(G,v)),$$

in terms of forbidden-induced subgraphs of G. We use this and one more result to show Theorem 4.1.3 (which states that h-perfection is closed under t-minors).

Let K_4^*, W_5^- and W_5^{--} be the graphs shown in Figure 4.2. The black-filled vertices are called *special*.

Theorem 4.3.1 *Let G be a graph and v be a contractible and non-isolated vertex of G. The following statements are equivalent:*

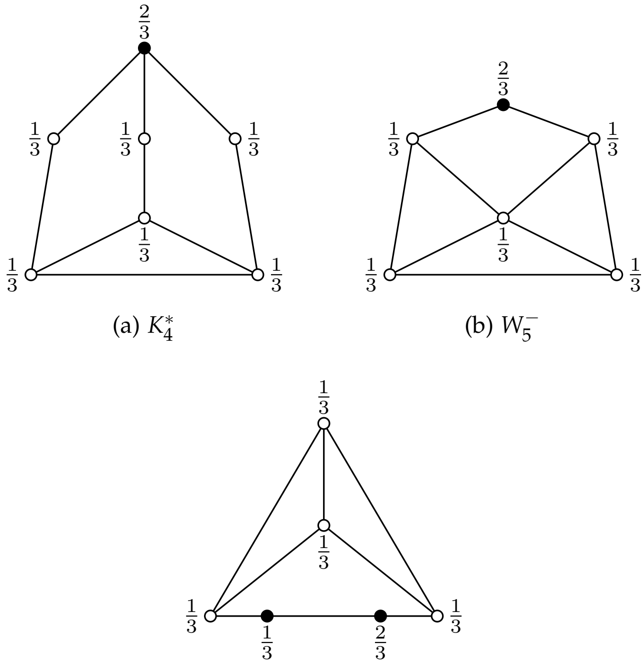
i) $\mathsf{HSTAB}(G/N_G[v]) = \gamma_{G,v}(F_h(G,v))$,

ii) v is not a special vertex of an induced K_4^*, W_5^- or W_5^{--} of G.

The other result used in the proof of Theorem 4.1.3 is the h-imperfection of K_4^*, W_5^- and W_5^{--}. We will use the following statement also in Section 4.3.2.

Proposition 4.3.2 *If G is one of the graphs K_4^*, W_5^-, W_5^{--} and if v is a special vertex of G, then* $\mathsf{HSTAB}(G)$ *has a non-integral vertex which belongs to $F_h(G,v)$. In particular, G is h-imperfect.*

Proof – For each such graph G and special vertex v, the non-integral vector x over $V(G)$ given in Figure 4.2 belongs to $F_h(G,v)$ and it is straightforward to find $|V(G)|$ linearly independent clique and odd-circuit inequalities which are tight at x. Hence, x is a non-integral vertex of $\mathsf{HSTAB}(G)$. In particular, G is h-imperfect.



(a) K_4^* (b) W_5^- (c) W_5^{--}

Figure 4.2 – the graphs K_4^*, W_5^- and W_5^{--}, their respective special vertices (shown black-filled) and a non-integral vertex of F_h at a special vertex.

■

We now use this and ii)=>i) of Theorem 4.3.1 to prove that h-perfection is closed under t-minors:

Theorem 4.1.3 *The class of h-perfect graphs is closed under t-minors.*

Proof – The case of vertex-deletion is straightforward (see Proposition 3.6.3) Hence we need only to prove that h-perfection is closed under t-contractions.

Let G be an h-perfect graph and $v \in V(G)$ be a contractible vertex. We may obviously assume that v is non-isolated. We will prove that $\mathsf{HSTAB}(G/N_G[v]) = \mathsf{STAB}(G/N_G[v])$ and this means that $G/N_G[v]$ is h-perfect.

H-perfection is closed under vertex-deletion and K_4^*, W_5^- and W_5^{--} are h-imperfect (Proposition 4.3.2). Hence, G cannot have an induced K_4^*, W_5^- or W_5^{--} and condition ii) of Theorem 4.3.1 is satisfied. Thus:

$$\mathsf{HSTAB}(G/N_G[v]) = \gamma_{G,v}(F_h(G,v)).$$

Since G is h-perfect, $F_h(G,v) = F(G,v)$. Therefore, Proposition 4.2.1 shows that $\mathsf{HSTAB}(G/N_G[v]) = \mathsf{STAB}(G/N_G[v])$ as required.

■

We will see in Chapter 5 that the graphs K_4^*, W_5^- and W_5^{--} are the only minimally h-imperfect K_4-free graphs which are not minimally t-imperfect (Theorem 5.1.4). Here, we merely needed their h-imperfection.

Finally, we observe that Theorem 4.3.1 implies furthermore that the class of perfect graphs is closed under t-minors. Recall that $\mathsf{QSTAB}(G)$ denotes the polyhedron defined by non-negativity and clique inequalities of a graph G.

THEOREM 4.1.4 *The class of perfect graphs is closed under t-minors.*

PROOF – The case of vertex-deletion is obvious. Let G be a perfect graph and suppose that v is a contractible and non-isolated vertex of G.

Put $H := G/N_G[v]$. We prove that $\mathsf{QSTAB}(H) = \mathsf{STAB}(H)$ and this will imply the perfection of H (Theorem 3.5.3).

Since G is perfect, it does not have an odd hole. Clearly, this implies that H does not have an odd hole and $\mathsf{QSTAB}(H) = \mathsf{HSTAB}(H)$.

Furthermore, G cannot have an induced K_4^*, W_5^- or W_5^{--} (they all have odd holes). Hence, Theorem 4.3.1 shows that:

$$\mathsf{HSTAB}(H) = \gamma_{G,v}(F_h(G,v)).$$

Now, G is h-perfect (as it is perfect). Thus, $F_h(G,v) = F(G,v)$ and Proposition 4.2.1 implies that $\mathsf{QSTAB}(H) = \mathsf{STAB}(H)$ as stated.

■

This is also shown by a proof of Bruhn and Stein [16] (see Section 4.4).

4.3.2 *Proof of Theorem 4.3.1*

This section is devoted to the proof of Theorem 4.3.1 (see the preceding section for the definition of γ and F_h).

THEOREM 4.3.1 *Let G be a graph and v be a contractible and non-isolated vertex of G. The following statements are equivalent:*

i) $\mathsf{HSTAB}(G/N_G[v]) = \gamma_{G,v}(F_h(G,v))$,

ii) v is not a special vertex of an induced K_4^*, W_5^- or W_5^{--} of G.

First, the implication i)=>ii) follows from Proposition 4.3.2:

PROOF (OF THEOREM 4.3.1, I)=>II)) – Let H be an induced K_4^*, W_5^- or W_5^{--} of G such that v is a special vertex of H. To the contrary, suppose that $\mathsf{HSTAB}(G/N_G[v]) = \gamma_{G,v}(F_h(G,v))$.

Let x be the non-integral vector of $F_h(H,v)$ shown in Figure 4.2 (see Proposition 4.3.2). Let $y \in \mathbb{R}^{V(G)}$ be defined for all $u \in V(G)$ by: $y_u = x_u$ if $u \in V(H)$ and $y_u = 0$ otherwise.

Clearly, $y \in F_h(G, v)$. Put $z = \gamma_{G,v}(y)$ and let K be the set of vertices u of $G/N_G[v]$ such that $z_u \neq 0$. Clearly, K is a clique of cardinality 4.

By assumption, $z \in \mathsf{HSTAB}(G/N_G[v])$ and thus $z(K) \leq 1$. However, Figure 4.2 shows (in each case) that $z(K) > 1$: a contradiction.

■

We prove ii)=>i) by showing both inclusions of sets. We first observe that "$\subseteq$" holds in general.

By Proposition 3.6.4, for every graph G and $x \in \mathbb{R}^{V(G)}$, we may prove that $x \in \mathsf{HSTAB}(G)$ by showing that *x satisfies the non-negativity, clique and the odd-circuit inequalities corresponding to odd holes of G only.*

Proposition 4.3.3 *For every graph G and every contractible non-isolated vertex v of G:*

$$\mathsf{HSTAB}(G/N_G[v]) \subseteq \gamma_{G,v}(F_h(G, v)).$$

Proof – Let $G' := G/N_G[v]$, $F := F_h(G, v)$ and let $\tilde{v}$ be the new vertex of G'.

Let $y \in \mathsf{HSTAB}(G')$. Let $x \in \mathbb{R}^{V(G)}$ be defined for each $u \in V(G)$ as follows:

$$x_u = \begin{cases} 1 - y_{\tilde{v}} & \text{if } u = v, \\ y_{\tilde{v}} & \text{if } u \in N_G(v), \\ y_u & \text{otherwise.} \end{cases}$$

For each $u \in N_G(v)$, $x_v + x_u = 1$. We will show that $x \in \mathsf{HSTAB}(G)$. This will imply that $x \in F_h(G, v)$ and, as $y = \gamma_{G,v}(x)$ (obviously), end the proof.

Clearly, x is non-negative and satisfies the clique inequalities.

Let C be an odd hole of G and $l := |V(C)|$. We show:

$$x(V(C)) \leq \frac{l-1}{2}.$$

This is obvious if C contains at most one neighbor of v, so let us assume the contrary.

If $v \in V(C)$, then $|N_G(v) \cap V(C)| = 2$ (because C has no chord). Hence, $C' := G'[V(C) - N_G[v] + \tilde{v}]$ is an odd circuit of G'. Since $x \in \mathsf{HSTAB}(G')$, we have $x(V(C')) \leq \frac{|V(C')|-1}{2}$. Thus:

$$x(V(C)) = x(V(C')) + 1 \leq \frac{l-1}{2}.$$

So we may assume that $v \notin V(C)$ and put $s := |N_G(v) \cap V(C)|$. Observe that $H := G'[V(C) - N_G[v] + \tilde{v}]$ is the union of circuits $C_1, \ldots, C_r, C_{r+1}, \ldots, C_s$ which pairwise-intersect in $\tilde{v}$ and such that $C_1, \ldots, C_r$ are odd and $C_{r+1}, \ldots, C_s$ are even.

If C_i is odd, then by assumption $x(V(C_i)) \leq \frac{|V(C_i)|-1}{2}$. Moreover, if C_j is even then any perfect matching of C_j shows that $x(V(C_j)) \leq \frac{|C_j|}{2}$. Clearly $|V(C)| = \sum_{i=1}^{s} |V(C_i)|$. Since C is odd, we have $r \geq 1$ and obtain:

$$x(V(C)) = \sum_{i=1}^{s} x(V(C_i)) \leq \sum_{i=1}^{r} \frac{|V(C_i)|-1}{2} + \sum_{j=r+1}^{s} \frac{|V(C_j)|}{2} \leq \frac{|V(C)|-1}{2}.$$

Therefore, $x \in \mathsf{HSTAB}(G)$.

■

Finally, we use the following lemma whose proof is postponed to the next section. A *2-neighbor* of a vertex v of a graph is a vertex at distance exactly 2 of v (in the sense of shortest paths).

LEMMA 4.3.4 *Let G be a graph and let v be a vertex of G which is not a special vertex of an induced K_4^*, W_5^- or W_5^{--} of G. If K is a clique of G of cardinality at least 3 which satisfies both following conditions:*

i) the vertices of K are 2-neighbors of v,

ii) each $w \in K$ has a neighbor $v_w \in N_G(v)$ such that $\{v_w \colon w \in K\}$ is a stable set,

then v has a neighbor u such that $K + u$ is a clique of G.

We now use this and Proposition 4.3.3 to prove ii)=>i) of Theorem 4.3.1:

PROOF (OF THEOREM 4.3.1, II)=>I)) – Let G be a graph and v be a contractible vertex of G which is not a special vertex of an induced K_4^*, W_5^- or W_5^{--} of G. By Proposition 4.3.3, we need only show:

$$\gamma_{G,v}(F_h(G,v)) \subseteq \mathsf{HSTAB}(G/N_G[v]).$$

Let $x \in F_h(G,v)$. Put $y := \gamma_{G,v}(x)$. That is, for every $u \in V(G/N_G[v])$:

$$y_u = \begin{cases} 1 - x_v & \text{if } u = \tilde{v}, \\ x_u & \text{otherwise.} \end{cases}$$

We prove that $y \in \mathsf{HSTAB}(G/N_G[v])$. Clearly, y is non-negative.

Let C be an odd circuit of $G/N_G[v]$ and $l := |V(C)|$. We first show:

$$y(V(C)) \leq \frac{l-1}{2}.$$

It is obvious if $\tilde{v} \notin V(C)$ so let us assume the contrary. If there exists $u \in N_G(v)$ such that $C' := G[V(C) - \tilde{v} + u]$ is an odd circuit of G then, since $x \in \mathsf{HSTAB}(G)$, we have: $y(V(C)) = x(V(C')) \leq \frac{l-1}{2}$.

Otherwise, there must exist two neighbors u and w of v in G such that $C' := G[V(C) - \tilde{v} + u + w + v]$ is an odd circuit of G. Therefore:

$$y(V(C)) = x(V(C')) - 1 \leq \frac{l-1}{2}.$$

Now, let K be a clique of $G/N_G[v]$. We need to prove that $y(K) \leq 1$. If $\tilde{v} \notin K$ or if $|K| \leq 2$ then the result is obvious. If K is a triangle, then it follows from the odd-circuit case.

So we may assume that $|K| \geq 4$ and $\tilde{v} \in K$. Let $K' := K - \tilde{v}$. The clique K' of G has at least 3 vertices and is obviously formed by 2-neighbors of v only. Besides, each $w \in K'$ has a neighbor v_w in $N_G(v)$ (because w is adjacent to $\tilde{v}$ in $G/N_G[v]$) and since v is contractible, the set $\{v_w \colon w \in K'\}$ must be a stable set.

Thus, Lemma 4.3.4 implies that v has a neighbor u such that $K'' := K' + u$ is a clique of G. Since $x \in F_h(G, v)$, we have $x_u = x_{\tilde{v}}$ and:

$$y(K) = x(K'') \leq 1,$$

as required.

Therefore, $y \in \mathsf{HSTAB}(G/N_G[v])$ and we are done.

■

4.3.3 *A lemma for cliques and t-contractions*

In this section, we prove Lemma 4.3.4. Besides, we show that it implies a result on the cardinality of cliques in t-contractions which will be useful in Chapter 5.

The lemma is derived from the following basic result on bipartite graphs.

Let G be a bipartite graph with bipartition $\{A, B\}$. A *$P_{2,3}$ of G with ends in A* is a subgraph of G formed by the vertex-disjoint union of a K_2 and a P_3 whose ends belong to A. Let $3K_2$ denote the graph formed by 3 vertex-disjoint edges.

Proposition 4.3.5 *Let G be a bipartite simple graph with bipartition $\{A, B\}$ such that $|A| \geq 3$ and without isolated vertices.*

If G has no induced $3K_2$, no induced $P_{2,3}$ with ends in A and no induced P_5 with ends in A, then there exists $b \in B$ such that $N_G(b) = A$.

Proof – Let $k = \max_{b \in B} d_G(b)$ and let $b \in B$ with $d_G(b) = k$. We show that $N_G(b) = A$.

We first observe that $k \geq 2$: since G has no isolated vertex and $|A| \geq 3$, having $k = 1$ would yield an induced $3K_2$ of G contradicting our assumption on G.

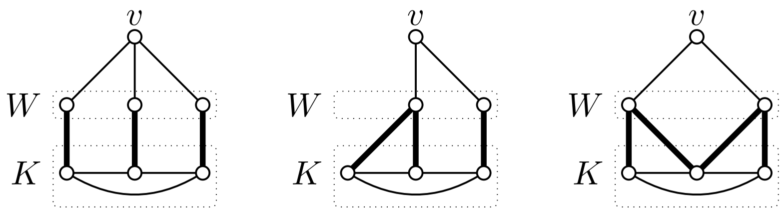
Now suppose by contradiction that there exists a vertex $a \in A$ which is not adjacent to b. Since a is not isolated, there exists $b' \in B \setminus \{b\}$ such that $ab' \in E(G)$.

$k \geq 2$. Let $X := N_G(b) \setminus N_G(b')$. We have $|X| \geq 1$: otherwise $d_G(b') > d_G(b)$, contradicting the choice of b.

Furthermore, $|X| \leq 1$. Otherwise since $k \geq 2$, b must have two neighbors a_1 and a_2 which are not adjacent to b and $\{a, a_1, a_2, b, b'\}$ would induce a $P_{2,3}$ with ends in A: a contradiction.

Therefore, X has a unique element a'. Since $k \geq 2$, the vertex b has a neighbor a'' different from a and therefore (a', b, a'', b', a) is an induced P_5 of G with ends in A: a contradiction again.

■



(a) case of $3K_2$ (b) case of $P_{2,3}$ (c) case of P_5

Figure 4.3 – obtaining an induced K_4^*, W_5^{--} and W_5^- of G from subgraphs of H

Recall that a *2-neighbor* of a vertex v of a graph is a vertex at distance exactly 2 of v. We now prove Lemma 4.3.4:

LEMMA 4.3.4 *Let G be a graph and let v be a vertex of G which is not a special vertex of an induced K_4^*, W_5^- or W_5^{--} of G. If K is a clique of G of cardinality at least 3 which satisfies both following conditions:*

i) the vertices of K are 2-neighbors of v,

ii) each $w \in K$ has a neighbor $v_w \in N_G(v)$ such that $\{v_w : w \in K\}$ is a stable set,

then v has a neighbor u such that $K + u$ is a clique of G.

PROOF – Let $W := \{v_w : w \in K\}$. Consider the underlying simple graph H of the bipartite graph $G[W \cup K] - E_G(K)$ with bipartition $\{W, K\}$. The graph H does not have an induced $3K_2$, since G would otherwise have an induced K_4^* with special vertex v (see Figure 4.3a).

Similarly, H cannot have an induced $P_{2,3}$ (resp. P_5) with ends in K since it would yield an induced W_5^{--} (resp. W_5^-) of G with special vertex v (see Figures 4.3b and 4.3c).

Hence, Proposition 4.3.5 shows (taking $A := K$ and $B := W$) that there exists $u \in W$ such that $N_G(u) = K$. Therefore, $K + u$ is a clique of G.

■

We will use the following immediate consequence of this lemma in Chapter 5:

PROPOSITION 4.3.6 *Let G be a graph and v be a contractible vertex of G.*

If $\omega(G/N_G[v]) > \omega(G) \geq 3$, then v is a special vertex of an induced K_4^*, W_5^- or W_5^{--} of G.

PROOF – Suppose that $\omega(G/N_G[v]) > \omega(G) \geq 3$.

Let $\tilde{v}$ be the new vertex of $G/N_G[v]$ and let K be a maximum clique of $G/N_G[v]$.

We have $\tilde{v} \in K$. Otherwise K would be a clique of G showing $\omega(G) \geq \omega(G/N_G[v])$: a contradiction.

Put $K' := K - \tilde{v}$. Obviously K' is a clique in G with at least 3 vertices and formed by 2-neighbors of v. For each $w \in K'$, let v_w be a common neighbor of v and w. Since v is contractible, $\{v_w : w \in K'\}$ is a stable set.

Furthermore, no neighbor u of v can form a clique with K' since $K' + u$ would otherwise be a clique of G of cardinality $\omega(G/N_G[v])$: a contradiction.

Therefore, Lemma 4.3.4 implies that v must be a special vertex of an induced K_4^*, W_5^- or W_5^{--} of G.

■

This result does not hold if we omit the assumption $\omega(G) \geq 3$. For example, the t-contraction of a vertex of C_5 yields K_3.

4.4 T-MINORS AND STRONG H-PERFECTION

It is not clear whether Theorem 4.3.1 implies that every t-minor of a strongly h-perfect graph is strongly h-perfect too. However, it is well-known that t-minors of strongly t-perfect graphs are strongly t-perfect (see [102, pg. 1195]). The first published proof of this fact appears in [15] and it can be directly used (with only trivial modifications) to obtain the same result for strong h-perfection:

THEOREM 4.4.1 (BRUHN, STEIN [15]) *Strong h-perfection is closed under t-minors.*

PROOF (BRUHN, STEIN [15]) – It is straightforward to show that strong h-perfection is closed under vertex-deletion, so only t-contractions need to be checked. Furthermore by induction, it suffices to prove that a single t-contraction keeps strong h-perfection.

Let G be a strongly h-perfect graph, v be a contractible vertex of G and $H := G/N_G[v]$. Let $\tilde{v}$ denote the new vertex of H. Obviously, G may be assumed simple and v non-isolated.

Let $c \in \mathbb{Z}_+^{V(H)}$. The aim of the proof is to show: $\alpha(H, c) = \rho^h(H, c)$ (see Section 3.6.2 for the definition of ρ^h).

Put $\beta := c(V(H)) + 1$ and let $b \in \mathbb{Z}^{V(G)}$ be defined for every $t \in V(G)$ by:

$$b_t = \begin{cases} d_G(v)\beta - c_{\tilde{v}} & \text{if } t = v, \\ \beta & \text{if } t \in N_G(v), \\ c_t & \text{otherwise.} \end{cases}$$

Let S be a stable set of (G, b) with $b(S) = \alpha(G, b)$. Clearly, either $v \in S$ or $N_G(v) \subseteq S$.

If $v \in S$, then $b(S) \leq \alpha(H, c) + b_v$. Otherwise, $S - N_G(v) + \tilde{v}$ is a stable set of H and this implies that $b(S) \leq d(v)\beta + \alpha(H, c) - c_{\tilde{v}} = \alpha(H, c) + b_v$. Hence, in both cases we get :

$$\alpha(G, b) \leq \alpha(H, c) + b_v. \tag{4.1}$$

Since G is strongly h-perfect, there exists an h-cover $\mathcal{F}$ of G of cost $\alpha(G, b)$. We may assume that every odd circuit of $\mathcal{F}$ whose length is at least 5 is an odd hole of G (see Section 3.6.2). Moreover, we can suppose that the multiset $\mathcal{F}_v$ of elements of $\mathcal{F}$ containing v has exactly b_v elements which are different from $\{v\}$.

Now, we build an h-cover $\mathcal{D}$ of (H, c) from $\mathcal{F}$ with cost $\alpha(G, b) - b_v$. By Equation (4.1), this will prove $\alpha(H, c) \geq \rho^h(H, c)$ and therefore the theorem.

Since v is contractible, $\mathcal{F}_v$ is the union of a multiset $\mathcal{K}_v$ of cliques of cardinality at most 2 and a multiset $\mathcal{C}$ of odd holes. For every odd hole C in $\mathcal{F}_v$, let $C' := H[V(C) - N_G(v) + \tilde{v}]$. Clearly, C' is a triangle or odd hole of H and $|V(C')| = |V(C)| - 2$. Let $\mathcal{C}' := \{C' \colon C \in \mathcal{C}\}$.

We start the construction of $\mathcal{D}$ with $\mathcal{D} := \mathcal{C}'$. Then, add to $\mathcal{D}$ every clique and every odd hole of $\mathcal{F}$ which does not meet $N_G[v]$. Also, for every clique K in $\mathcal{F} \setminus \mathcal{F}_v$ meeting $N_G(v)$, add the clique $(K \setminus N_G(v)) + \tilde{v}$ to $\mathcal{D}$.

Now, let C be an odd hole in $\mathcal{F} \setminus \mathcal{F}_v$ meeting $N_G(v)$. The graph $H[V(C) - N_G(v) + \tilde{v}]$ is the union of odd circuits $C_1, \ldots, C_r$ and even circuits $C_{r+1}, \ldots, C_s$ (possibly of length 2) which pairwise-intersect in $\tilde{v}$ only. For $i = 1, \ldots, r$, add C_i to $\mathcal{D}$. For $j = r+1, \ldots, s$, choose a perfect matching M_j of C_j and add every edge of M_j to $\mathcal{D}$.

By construction, $\mathcal{D}$ is a multiset of cliques and odd holes of H.

$\mathcal{D}$ is an h-cover of (H, c) of cost $\alpha(G, b) - b_v$.

As mentioned above, this will end the proof of the theorem.

We first examine the cost of $\mathcal{D}$: it is straightforward to check that it is the sum of the costs of $\mathcal{F} \setminus \mathcal{F}_v$ and $\mathcal{C}'$. By assumption, $b_v = |\mathcal{K}_v| + |\mathcal{C}|$. Hence, the cost of $\mathcal{C}'$ is

$$\sum_{C \in \mathcal{C}} \frac{|V(C)| - 1}{2} - |\mathcal{C}| = \sum_{C \in \mathcal{C}} \frac{|V(C)| - 1}{2} + |\mathcal{K}_v| - b_v,$$

that is the cost of $\mathcal{F}_v$ minus b_v. Thus the cost of $\mathcal{D}$ is indeed $\alpha(H, c) - b_v$.

We now prove that $\mathcal{D}$ is an h-cover of (H, c). Obviously, every $t \in V(H) \setminus \{\tilde{v}\}$ is covered at least c_t times by $\mathcal{D}$ (since $b_t = c_t$). So it remains only to check that $\tilde{v}$ is covered at least $c_{\tilde{v}}$ times by $\mathcal{D}$.

It is straightforward to check that each $F \in \mathcal{F} \setminus \mathcal{F}_v$ which meets $N_G(v)$ gives rise to $|V(F) \cap N_G|$ elements of $\mathcal{D}$ (with repetitions) containing $\tilde{v}$. Also, each $C \in \mathcal{C}$ covers two neighbors of v while the associated circuit $C' \in \mathcal{C}'$ covers $\tilde{v}$ only once. Finally, each clique in $\mathcal{K}_v$ is different from $\{v\}$, thus it covers exactly one neighbor of v. Since those cliques are not counted in $\mathcal{D}$, we get that the number of times $\mathcal{D}$ covers $\tilde{v}$ is at least: $d_G(v)\beta - |\mathcal{K}_v| - |\mathcal{C}| = b_v$. Indeed, $b_v = |\mathcal{K}_v| + |\mathcal{C}|$ because $\mathcal{F}$ covers v exactly c_v times.

■

By Theorem 3.5.2, a graph G is perfect if and only if every induced subgraph H of G has a clique-cover of cardinality $\alpha(H)$.

Clearly, a clique-cover $\mathcal{K}$ is an integral h-cover of cost $|\mathcal{K}|$. Furthermore, in the proof above: if $\mathcal{F}$ contains cliques only, then $\mathcal{D}$ will be a clique-cover.

Therefore, this proof also shows that *perfection is closed under t-minors* (Theorem 4.1.4).

4.5 SUBSTITUTIONS IN H-PERFECT GRAPHS

Let G, H be graphs and $v \in V(G)$. The *substitution of v by H in G*, denoted $G^{v \leftarrow H}$, is the graph obtained from the union of disjoint copies of $G - v$ and H by adding the edge uw for each $u \in N_G(v)$ and $w \in V(H)$.

The following lemma, formulated by Fulkerson [48] and proved by Lovász [71], plays a key-role in the proof of the Weak Perfect Graph Theorem (Theorem 3.5.2).

Theorem 4.5.1 (Lovász [71]) *Let G be a perfect graph and $v \in V(G)$. If H is a perfect graph, then the graph $G^{v \leftarrow H}$ is perfect.*

In this section, we characterize the substitutions which keep h-perfection.

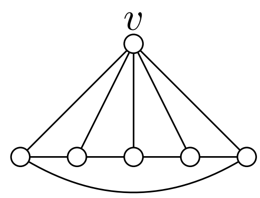


Figure 4.4 – substituting a vertex of K_2 by C_5

It is not true in general that substitutions by perfect graphs keep h-perfection. Indeed, substituting a K_2 for a vertex of C_5 (which is h-perfect) yields the h-imperfect graph W_5^{--} (see Proposition 4.3.2). Furthermore, substituting an odd hole to a vertex of K_2 yields an odd wheel W_{2k+1} with $k \geq 2$ (see Figure 4.4) and we have (see Section 3.6.1 for a proof):

PROPOSITION 3.6.5 *For each $k \geq 1$, the graph W_{2k+1} is t-imperfect.*

Further details on the stable set polytope of odd wheels can be found in Chapter 5.

We use only the polyhedral characterization of perfect graphs to prove our substitution theorem.

Obviously, if G is h-perfect and v is an isolated vertex of G, then $G^{v \leftarrow H}$ is h-perfect if and only if H is h-perfect too.

THEOREM 4.1.5 *Let G be an h-perfect graph, v be a non-isolated vertex of G and H be a graph. The graph $G^{v \leftarrow H}$ is h-perfect if and only if at least one of the following statements holds:*

i) H has no edge,

ii) H is perfect and v does not belong to an odd hole of G.

PROOF – Let $G' := G^{v \leftarrow H}$. We first prove the "only if" part of the statement.

ONLY IF: Suppose that G' is h-perfect and that H has an edge.

Clearly, an odd hole of H would yield an induced W_{2k+1} of G' with $k \geq 2$, contradicting the h-perfection of G'.

Hence, H has no odd hole. Since H is an induced subgraph of G', it is h-perfect and this implies that H is perfect (see Section 3.6.1).

To see that v does not belong to an odd hole of G, observe that G' would otherwise have an induced subgraph which can be t-contracted to W_5^{--}, contradicting the h-perfection of G (Proposition 4.3.2).

We now show the "if" part of the theorem.

IF: The result is trival if H has at most one vertex, so we may henceforth assume that $|V(H)| \geq 2$.

Let $d \in \mathbb{Z}_+^{V(G')}$. We prove:

$$\alpha(G', d) = \rho_f^h(G', d). \tag{4.2}$$

By Proposition 3.6.7, this will imply the h-perfection of G'. Since the inequality $\leq$ always holds, we need only to show the converse.

Let d^H be the restriction of d to $V(H)$. Now, let $c \in \mathbb{Z}_+^{V(G)}$ be defined as follows: for every $t \in V(G)$, $c_t := \alpha(H, d^H)$ if $t = v$ and $c_t := d_t$ otherwise. It is straightforward to check that $\alpha(G', d) = \alpha(G, c)$.

Since G is h-perfect, Proposition 3.6.7 implies:

$$\alpha(G', d) = \rho_f^h(G, c). \tag{4.3}$$

Let y be a fractional rational h-cover of (G,c) of cost $\rho_f^h(G,c)$ (see Section 3.6.2) and let k be a positive integer such that ky is integral. Let $\mathcal{F}$ be the integral h-cover of (G,kc) corresponding to ky.

Depending on whether condition i) or ii) of the statement of the theorem holds, we build an integral h-cover $\mathcal{F}'$ of (G',kd) of the cost of $\mathcal{F}$.

This construction will indeed end the proof of the "if" part of the theorem: it implies $k\rho_f^h(G,c) \geq \rho_f^h(G',kd)$ and since $\rho_f^h(G',kd) \geq k\rho_f^h(G',d)$ (this is straightforward), we obtain $\rho_f^h(G,c) \geq \rho_f^h(G',d)$. By (4.3), this shows $\alpha(G',d) \geq \rho_f^h(G',d)$ and we finally get (4.2) as required.

Case 1. H has no edge. Clearly, by induction on $|V(H)|$ we only need to check the case $|V(H)| = 2$, so put $V(H) = \{u,w\}$.

Since v belongs to kc_v elements of $\mathcal{F}$ and $kc_v = kd_u + kd_w$, we can choose a sub-multiset $\mathcal{F}_u$ of $\mathcal{F}$ with kd_u elements all containing v. Then, there is a sub-multiset $\mathcal{F}_w$ of $\mathcal{F} - \mathcal{F}_u$ of kd_w elements all containing v.

Let $\mathcal{F}'_u := \{F - v + u \colon F \in \mathcal{F}_u\}$ and $\mathcal{F}'_w := \{F - v + w \colon F \in \mathcal{F}_w\}$. The multiset $\mathcal{F}' := \mathcal{F}'_u + \mathcal{F}'{}_w + (\mathcal{F} - \mathcal{F}_u - \mathcal{F}_w)$ is clearly an integral h-cover of (G',kd) of the same cost as $\mathcal{F}$.

Case 2. H is perfect and v does not belong to an odd hole of G. Here, $\mathcal{F}_v$ only contains cliques. By definition, it has at least $k\alpha(H,d^H)$ elements.

Let $r := \alpha(H,d^H)$. Since H is perfect, (H,d^H) has a clique-cover formed by r cliques $K_1,\dots,K_r$ (Theorem 3.5.2).

Let $\mathcal{K} := \{K'_1,\dots,K'_r\}$ be a sub-multiset of $\mathcal{F}_v$ with kr elements and let $\mathcal{F}'$ be obtained from $\mathcal{F}$ by replacing in $\mathcal{K}$ the clique K'_i by the set $K'_i \cup K_i$ of G' for every $i \in [r]$.

In G', every vertex of H is adjacent to every neighbor of v in G. Thus, the sets $K_i \cup K'_i$ are cliques of G' and $\mathcal{F}'$ is an integral h-cover of (G',kd) which has the same cost as $\mathcal{F}$.

■

A general result of Chvátal [26] shows that the stable set polytope of $G^{v\leftarrow H}$ can be obtained from descriptions for $\mathsf{STAB}(G)$ and $\mathsf{STAB}(H)$. This result can be used to give another proof of the "if" part of our result.

The proof above directly shows however that *the theorem remains true if we replace h-perfection with strong h-perfection*. Specializing Theorem 4.1.5 to K_4-free graphs, it is straightforward to prove:

COROLLARY 4.5.2 *Let G be a (strongly) t-perfect graph, v be a non-isolated vertex of G and H be a graph. The graph $G^{v\leftarrow H}$ is (strongly) t-perfect if and only if at least one of the following statements holds:*

i) H has no edge,

ii) v does not belong to an induced odd circuit of G and H is bipartite.

4.6 HOMOGENEOUS SETS IN MINIMALLY H-IMPERFECT GRAPHS

In this section, we show that Theorem 4.1.5 has a simple consequence for the structure of h-imperfect graphs which are minimal with respect to t-minors.

Recall that a graph G is *minimally h-imperfect* (resp. *minimally t-imperfect*) if it is h-imperfect (resp. t-imperfect) and every proper t-minor of G is h-perfect (resp. t-perfect).

A *module of a graph* G is a subset $X \subseteq V(G)$ such that every $v \in V(G) \setminus X$ satisfies $N_G(v) \supseteq X$ or $N_G(v) \cap X = \varnothing$. The *trivial* modules of G are $\varnothing$, $V(G)$ and the singletons of $V(G)$ (they clearly are modules of G). An *homogeneous set* of G is a non-trivial module of G. A graph is *prime* if it has no homogeneous set.

The graph W_5^{--} is minimally h-imperfect (see Theorem 5.1.4) and has an homogeneous clique of cardinality two. We prove that it is the only minimally h-imperfect graph which is not prime.

THEOREM 4.1.6 *Except W_5^{--}, every minimally h-imperfect graph is prime.*

PROOF – Let G be a minimally h-imperfect graph and suppose that G has an homogeneous set X. We show that W_5^{--} is a t-minor of G and this will obviously prove the theorem.

Put $H := G[X]$. Let $v \in X$ and put $X' := X - v$. Since X is homogeneous, G is obtained from $G - X'$ by substituting v by H. Besides, G is minimally h-imperfect thus $G - X'$ is h-perfect.

Therefore, H must have an edge: otherwise Theorem 4.1.5 would imply that G is h-perfect, contradicting our assumption.

Let $u \in V(G) \setminus X$. The graph $G - u$ is h-perfect and is obtained from $G - u - X'$ by substituting v by H, which has at least one edge. Thus, another application of Theorem 4.1.5 shows that H is perfect and v does not belong to an odd hole of $G - u - X'$.

On the other hand, G is h-imperfect. Therefore, Theorem 4.1.5 implies that v must belong to an odd hole C of $G - X'$.

Since H has at least an edge, v has a neighbor w in X. As C is induced, it contains exactly two neighbors of v in $G - X'$. Now, every neighbor of v is adjacent to w and therefore $G[V(C) + w]$ can be t-contracted to W_5^{--}. This shows that W_5^{--} is a t-minor of G.

■

Specializing our result to minimally t-imperfect graph, we directly obtain:

COROLLARY 4.6.1 *K_4 is the only non-prime minimally t-imperfect graph.*

5 MINIMAL H-IMPERFECTION

T-perfection is in co-NP (Theorem 3.6.20) but no combinatorial certificate of t-imperfection is known.

Gerards and Shepherd [51] showed that t-perfection is closed for t-minors. A t-imperfect graph is *minimally t-imperfect* if all its proper t-minors are t-perfect. Hence, a graph is t-imperfect if and only if it has a t-minor which is minimally t-imperfect.

Therefore , studying minimally t-imperfect graphs may hopefully yield a combinatorial certificate of t-imperfection and provide a solid basis towards understanding the complexity of deciding t-perfection.

For example, Bruhn and Stein [16] determined every minimally t-imperfect claw-free graphs. Bruhn and Schaudt [14] used this to show a polynomial-time algorithm testing t-perfection in claw-free graphs.

Several minimally t-imperfect graphs have been identified by Shepherd [108] and Bruhn, Stein [13, 16].

In this chapter, we will first review the *known examples of minimally t-imperfect graphs*. We do not provide new ones, but give a description of their *stable set polytope* and formulate a related conjecture. Moreover, we state *known and new conjectures and further questions on minimally t-imperfect graphs*.

Clearly, K_4 is the only minimally t-imperfect graph which is not minimally h-imperfect. In a second part, we prove that *there are exactly three minimally h-imperfect K_4-free graphs which are not minimally t-imperfect: K_4^*, W_5^- and W_5^{--}*. These graphs show that conjectures stated for minimally t-imperfect graphs have to be reformulated in order to extended to minimally h-imperfect graphs.

It is not known whether h-perfection is in NP or co-NP. We state a conjecture of Sebő (personal communication) whose validity would imply that odd antiholes with at least 9 vertices are the only minimally h-imperfect graphs having cliques of cardinality larger than 3. In particular, it would put h-perfection in co-NP.

We prove that the conjecture holds for a proper superclass of *planar graphs*. Webs form a class of claw-free graphs containing the odd antiholes. We determine the *h-perfect and minimally h-imperfect webs*. Our results hopefully simplify the still open task of characterizing minimally h-imperfect claw-free graphs.

Finally, we use the forbidden-induced-subgraph characterization of Cao and Nemhauser (Theorem 3.8.2) to determine the minimally h-imperfect line-graphs.

La t-perfection est une propriété co-NP (Théorème 3.6.20) mais on ne connaît pas de certificat combinatoire de t-imperfection.

Gerards et Shepherd [51] ont montré que les t-mineurs conservent la t-perfection. Un graphe t-imparfait est *minimalement t-imparfait* si tous ses t-mineurs propres sont t-parfaits. Ainsi, un graphe est t-imparfait si et seulement s'il a un t-mineur qui est minimalement t-imparfait.

Dès lors, la caractérisation des graphes minimalement t-imparfaits fournit une approche possible pour obtenir un certificat combinatoire de t-imperfection. Elle donnerait aussi une base solide pour l'étude de la complexité de la reconnaissance des graphes t-parfaits.

Par exemple, Bruhn et Stein [16] ont déterminé les graphes minimalement t-imparfaits sans griffe. Bruhn et Schaudt [14] ont utilisé ce résultat dans leur algorithme polynomial de reconnaissance des t-parfaits dans la classe des graphes sans griffe.

Plusieurs autres exemples de graphes minimalement t-imparfaits ont été donnés par Shepherd [108] et Bruhn, Stein [13, 16].

Dans ce chapitre, nous faisons d'abord l'inventaire des *exemples connus de graphes minimalement t-imparfaits* et décrivons leur polytope des stables. Nous ne proposons pas de nouvel exemple mais sommes conduits à énoncer une conjecture sur le polytope des stables des graphes minimalement t-imparfaits. Par ailleurs, nous énumérons plusieurs conjectures connues et en suggérons de nouvelles sur les propriétés combinatoires de ces graphes.

Il est facile de vérifier que K_4 est l'unique graphe minimalement t-imparfait qui n'est pas minimalement h-imparfait. Dans la seconde partie du chapitre, nous prouvons qu'*il y a exactement trois graphes sans K_4 minimalement h-imparfaits qui ne sont pas minimalement t-imparfaits : K_4^*, W_5^- et W_5^{--}*. Ces graphes montrent que les conjectures formulées pour les minimalement t-imparfaits doivent être modifiées pour être étendues aux minimalement h-imparfaits.

L'appartenance de la h-perfection à NP ou co-NP est toujours ouverte. Nous énonçons une conjecture de Sebő (communication personnelle) affirmant que les anti-trous impairs à au moins 9 sommets sont exactement les graphes minimalement h-imparfaits qui contiennent K_4. La validité de cette conjecture placerait directement la h-perfection dans co-NP.

Nous prouvons que cette conjecture est satisfaite par une classe contenant (strictement) *les graphes planaires*. Les graphes circulants forment une sous-classe des graphes sans griffe qui contient tous les anti-trous impairs. On caractérise les graphes circulants h-parfaits et minimalement h-imparfaits. Nous expliquons en quoi ces résultats pourraient être utiles dans la recherche d'une preuve de la conjecture de Sebő pour le cas des graphes sans griffe.

Enfin, nous utilisons la caractérisation par sous-graphes-induits interdits des graphes h-parfaits adjoints due à Cao et Nemhauser (Théorème 3.8.2) pour déterminer tous les graphes adjoints minimalement t-imparfaits.

5.1 INTRODUCTION

T-perfection belongs to co-NP but a combinatorial certificate of t-imperfection is not known (Theorem 3.6.20). Whether t-perfection belongs to NP is open.

The Strong Perfect Graph Theorem (Theorem 1.1.2) shows that imperfection can be certified by showing an odd hole or odd antihole. It is easy to check that a graph is an odd hole or odd antihole, and this places perfection in co-NP (a simpler certificate is provided by a theorem of Lovász [102, pg. 1109]). Odd holes and antiholes are the minimally imperfect graphs: they are imperfect and their proper induced subgraphs are all perfect.

This suggests using operations which keep t-perfection and reduce the size of the graph in the process of seeking a combinatorial co-NP characterization of t-perfection.

Recall that a *t-minor* of a graph G is any graph obtained from G by a sequence of vertex-deletions and t-contractions. It is *proper* if it is different from G.

Gerards and Shepherd [51] proved that *t-perfection is closed under t-minors*. A t-imperfect graph is *minimally t-imperfect* (abbreviated *MTI*) if all its proper t-minors are t-perfect. Hence, a graph is t-imperfect if and only if it has an MTI t-minor.

A combinatorial algorithm recognizing minimally t-imperfect (or minimally h-imperfect) graphs would directly imply a combinatorial certificate of t-imperfection (or h-imperfection).

The *claw* is the graph shown in Figure 5.1. A graph is *claw-free* if it does not have an induced claw. It is straightforward to check that the class of claw-free graphs is closed under t-minors.

THEOREM 5.1.1 (BRUHN, STEIN [16]) *The minimally t-imperfect claw-free graphs are K_4, W_5, C_7^2 and C_{10}^2.*

Therefore, *a claw-free graph is t-perfect if and only if it does not have one of these graphs as a t-minor*. This directly puts t-perfection of claw-free graphs in co-NP and was used by Bruhn and Schaudt [14] to show *a polynomial-time algorithm testing t-perfection in claw-free graphs*.

In this chapter, *we will first review the currently known MTI graphs* which can be found in [13, 15]. There are several *infinite* families of such graphs. No new example is provided and we do not know if there are more.

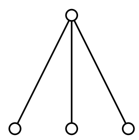


Figure 5.1 – the claw

An imperfect graph G is *minimally imperfect* if every proper induced subgraph of G is perfect. By the polyhedral characterization of perfection (Theorem 3.5.3), $\mathsf{STAB}(G)$ must have a non-trivial facet which is not defined by a clique-inequality of G. Furthermore, the polytope $\mathsf{QSTAB}(G)$ described by the non-negativity and clique-inequalities of G (see Section 3.5) must have a non-integral vertex. Padberg proved the following:

THEOREM 5.1.2 (PADBERG, [89]) *For every minimally imperfect graph G, the inequality $x(V(G)) \leq \alpha(G)$ defines a facet of* $\mathsf{STAB}(G)$ *and:*

$$\mathsf{STAB}(G) = \left\{ x \in \mathbb{R}^{V(G)} : \begin{array}{cc} x \geq 0, & \\ x(K) \leq 1 & \forall K \text{ clique of } G, \\ x(V(G)) \leq \alpha(G). & \end{array} \right\}.$$

Furthermore, $\frac{1}{\omega(G)}\mathbf{1}$ is the unique non-integral vertex of $\mathsf{QSTAB}(G)$.

In this context, Shepherd [107] called a graph G *near-perfect* if the non-trivial facets of $\mathsf{STAB}(G)$ are defined by clique-inequalities or the *full-rank inequality* $x(V(G)) \leq \alpha(G)$. Hence, both perfect graphs and minimally imperfect graphs are near-perfect. In general, it is not true that the complement of a near-perfect graph is near-perfect (see Figure 1.3). In [107], Shepherd showed that *a graph G is minimally imperfect if and only if G and $\overline{G}$ are near-perfect*.

This motivates the definition of an analogous notion for h-perfection. We say that a graph G is *near-h-perfect* if $\mathsf{STAB}(G)$ is described by non-negativity, clique and odd-circuit inequalities and the full-rank inequality of G. Near-h-perfect graphs include near-perfect and h-perfect graphs.

An h-imperfect graph is *minimally h-imperfect* (abbreviated *MHI*) if all its proper t-minors are h-imperfect. In Chapter 4, we showed that h-perfection is closed under t-minors. Therefore, a graph is h-perfect if and only if it does not have an MHI t-minor.

In this chapter, we will show that *except odd wheels W_{2n+1} (with $n \geq 2$) and K_4^* (see Figure 5.2), the known examples of MHI graphs are near-h-perfect*. Hence, we conjecture the following:

CONJECTURE 5.1.3 *Except K_4^* and the odd wheels W_{2n+1} with $n \geq 2$, every minimally h-imperfect graph is near-h-perfect.*

We will survey questions of Bruhn and Stein [16] (and ask new ones) related to MTI graphs.

Clearly, K_4 is the only MTI graph which is not MHI. There are K_4-free MHI graph which are not MTI (they have K_4 as a t-minor) and we determine all of them (see Figure 5.2 for the definitions of K_4^*, W_5^-, W_5^{--}):

THEOREM 5.1.4 *The only K_4-free minimally h-imperfect graphs are K_4^*, W_5^- and W_5^{--}.*

These graphs are precisely those involved in Theorem 4.3.1 of Chapter 4. They also show that some of the conjectures formulated for MTI graphs do not hold for MHI graphs in general.

On the other hand, we study minimally h-imperfect graphs with cliques of cardinality at least 4. Odd antiholes with at least 7 vertices form a class of minimally imperfect graphs with arbitrarily large

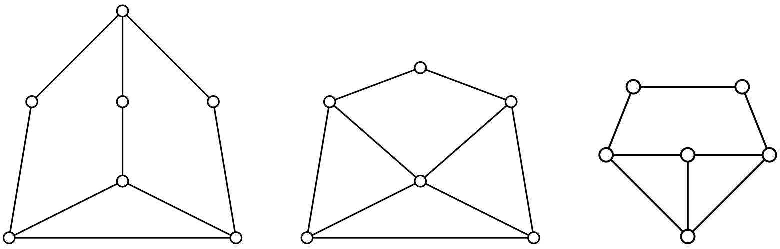


Figure 5.2 – the graphs K_4^*, W_5^- and W_5^{--}

clique number. Theorem 5.1.2 shows that they are also MHI (see Proposition 5.4.4 for a direct proof).

Can a characterization of h-perfect graphs be reduced to one for t-perfect graphs ? Sebő (personal communication) conjectures the following: *a graph G is h-perfect if and only if every K_4-free induced subgraph of G is t-perfect and G has no odd antihole with at least 9 vertices.*

An h-imperfect graph G is *critically h-imperfect* if every proper induced subgraph of G is h-perfect. Clearly, this conjecture states:

CONJECTURE 5.1.5 (SEBŐ) *Every critically h-imperfect graph with $\omega \geq 4$ is an odd antihole.*

Since MHI graphs are critically h-imperfect, it would imply that *the MHI graphs are the MTI graphs (except K_4), W_5^-, W_5^{--}, K_4^* and the $\overline{C_{2n+1}}$ with $n \geq 4$.*

We say that a graph G is *4-clique-separated* if each non-complete connected induced subgraph H of G with $\omega(H) \geq 4$ has a vertex-cut which is a clique. It is easy to check that the class of 4-clique-separated graphs is closed under taking induced subgraphs and contains no odd antihole $\overline{C_{2n+1}}$ with $n \geq 4$.

Tucker [116] proved that *planar graphs are 4-clique-separated*. We observe that Conjecture 5.1.5 holds for the class of 4-clique-separated graphs in the simplest possible way:

THEOREM 5.1.6 *Critically h-imperfect 4-clique-separated graphs are K_4-free.*

In other words: *a 4-clique-separated graph is h-perfect if and only if its K_4-free induced subgraphs are t-perfect.*

Recall that for integers $n \geq 3$ and $k \geq 1$: C_n^k denotes the graph obtained from C_n by adding an edge between two vertices whose distance (in the sense of shortest paths) on C_n is at most k.

The graphs C_n^k are called *webs* (see Section 3.2.1). We prove:

THEOREM 5.1.7 *A web is h-perfect if and only if it is a circuit or a complete graph or a C_{2k+2}^k with $k \geq 1$.*

THEOREM 5.1.8 *A web is critically h-imperfect if and only if it is C_{10}^2 or an odd antihole $\overline{C_{2n+1}}$ with $n \geq 3$.*

In particular, the critically h-imperfect webs which have cliques of cardinality 4 are odd antiholes. Webs form a proper subclass of claw-free graphs and we do not know if Conjecture 5.1.5 further holds for the latter. Still, our results on webs show that the conjecture for claw-free graphs is implied by the following statement (see Section 5.2.4 for the definition of partitionable graphs):

CONJECTURE 5.1.9 *Each critically h-imperfect claw-free graph with* $\omega \geq 4$ *is partitionable.*

Antiwebs (the complements of webs) also include odd antiholes, but we do not know if critically h-imperfect antiwebs which have cliques of cardinality 4 are odd antiholes. Antiwebs form a class of *nearly-bipartite graphs* (that is each vertex has a neighbor in each odd circuit). These graphs were studied by Shepherd in [108] (see also Section 5.2.2).

We end this chapter by observing that the characterization of h-perfect line-graphs by Cao and Nemhauser (Theorem 3.8.2) implies a characterization of the minimal (under t-minors) h-imperfect line-graphs:

THEOREM 5.1.10 *The minimally h-imperfect line-graphs are* W_5^- *and* W_5^{--}*.*

Hence, *a line-graph is h-perfect if and only if it does not have* W_5^- *or* W_5^{--} *as t-minor*.

OUTLINE In Section 5.2, we review the known examples of MTI graphs, state related properties and give a description of their respective stable set polytope. In particular we show that, except odd wheels W_{2n+1} with $n \geq 2$, all these examples are near-h-perfect.

In Section 5.3, we mention questions and conjectures of Bruhn and Stein for MTI graphs and give new ones. We also discuss the problem of finding more examples of MTI graphs.

In Section 5.4 we show that K_4^*, W_5^- and W_5^{--} are the K_4-free MHI graphs which are not MTI. We describe the stable set polytope of these graphs and observe that the conjectures given for MTI graphs must be reformulated to be extended to MHI graphs. Besides, we show that Conjecture 5.1.5 holds for 4-clique-separated graphs.

In Section 5.5, we determine the h-perfect and critically h-imperfect webs. We use these results to show that Conjecture 5.1.5 for claw-free graphs would follow from Conjecture 5.1.9. We conclude the chapter by determining the MHI line-graphs.

5.2 A REVIEW OF KNOWN MINIMALLY T-IMPERFECT GRAPHS. CONSEQUENCES AND CONJECTURES

Since t-perfection is closed under t-minors (Theorem 4.1.2), *a graph* G *is MTI if and only if for every* $v \in V(G)$*:* $G - v$ *is t-perfect, and if* v *is contractible then* $G/N_G[v]$ *is t-perfect.*

In this section, we review the currently known examples of MTI graphs which can be found in [13, 15]. We do not provide new examples.

Our contribution consists in finding a description of the stable set polytope for each of these examples (when such a description is not known). This led us to the formulation of the notion of near-h-perfection and a related conjecture. We will often use the following result:

PROPOSITION 5.2.1 *Let G be a critically h-imperfect graph and F be a non-trivial facet of* $\mathsf{STAB}(G)$ *which is not defined by a clique or odd-circuit inequality.*

If $a^{\mathsf{T}}x \leq b$ is valid for $\mathsf{STAB}(G)$ *and defines F, then $a > 0$. Furthermore, if every inclusion-wise maximal stable set is maximum, then*

$$F = \{x \in \mathsf{STAB}(G) \colon x(V(G)) \leq \alpha(G)\}\,,$$

and G is near-h-perfect.

PROOF – We first prove $a > 0$. Let G be a critically h-imperfect graph and suppose to the contrary that G has a vertex v such that $a_v = 0$. Let a' be the restriction of a to $V(G) \setminus \{v\}$. It is straightforward to check that $a'^{\mathsf{T}}x \leq b$ (over $\mathbb{R}^{V(G-v)}$) defines a facet of $\mathsf{STAB}(G-v)$. Since G is critically h-imperfect, $a'^{\mathsf{T}}x \leq b$ is a clique or odd-circuit inequality of $G - v$ (up to a positive scalar factor). Hence F is defined by a clique or odd circuit inequality of G: a contradiction.

Now, let us assume furthermore that every inclusion-wise maximal stable set of G is maximum and let F_α be the face of $\mathsf{STAB}(G)$ defined by the inequality $x(V(G)) \leq \alpha(G)$. Since $a > 0$, every stable set S of G satisfying $a(S) = b$ is inclusion-wise maximal and thus has $|S| = \alpha(G)$.

In other words, every vertex of F belongs to F_α and thus $F \subseteq F_\alpha$. Since F is a facet of $\mathsf{STAB}(G)$, we must have $F = F_\alpha$.

■

The currently known MTI graphs can be grouped into four families.

5.2.1 *Odd wheels*

Let $k \geq 3$ and C_k be the circuit of length k. The *wheel graph of order k*, denoted W_k, is obtained by adding to C_k a new vertex c which is adjacent to every vertex of C_k (see Figure 5.3 for an example). The *rim* of W_k is the circuit corresponding to C_k and c is the *center* of W_k. Notice that W_k has $k + 1$ vertices.

Clearly, if k is even then W_k is almost-bipartite and Theorem 3.6.14 shows that it is t-perfect. By contrast, the following straightforward statement holds [102, pg. 1194]:

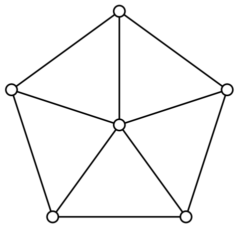


Figure 5.3 – the graph W_5

Proposition 5.2.2 *For each integer $n \geq 1$, W_{2n+1} is minimally t-imperfect.*

Since $K_4 = W_3$ is h-perfect, W_{2n+1} *is MHI if and only if $n \geq 2$.* Proposition 5.2.2 and Proposition 5.2.1 yield the following well-known result:

Proposition 5.2.3 *Let $n \geq 2$ be an integer and W be the wheel of size $2n+1$. Let C be the rim and c be the center of W. The following holds:*

$$\mathsf{STAB}(W) = \left\{ x \in \mathbb{R}^{V(W)} : \begin{array}{cc} x_v \geq 0 & \forall v \in V(W), \\ x(K) \leq 1 & \textit{for each triangle } K \textit{ of } W, \\ nx_c + x(V(C)) \leq \alpha(W). & \end{array} \right\}.$$

Furthermore, each of the above inequalities defines a facet of $\mathsf{STAB}(W)$.

Proof – The non-negativity and inclusion-wise maximal clique inequalities define facets of $\mathsf{STAB}(W)$ (see Section 3.4).

Clearly, $\alpha(W) = n$ and the inequality $nx_c + x(V(C)) \leq n$ is valid for $\mathsf{STAB}(W)$. The $2n+1$ stable sets of cardinality n of C form, together with $\{c\}$, a set of $2n+2$ affinely independent stable sets of W which are tight for this inequality. Hence it defines a facet of $\mathsf{STAB}(W)$.

Suppose by contradiction that $a^{\mathsf{T}}x \leq b$ defines a non-trivial facet F of $\mathsf{STAB}(W)$ which is not defined by a clique or odd-circuit inequality or $nx_c + x(V(C)) \leq n$.

It is easy to check that since F is not defined by a triangle inequality of W, a must be constant on the rim.

Since G is critically t-imperfect, Proposition 5.2.1 shows that $a > 0$ and thus we may assume without loss of generality that $a_v = 1$ for every $v \in V(C)$.

Furthermore, there must exists a stable set S of W such that $a(S) = b$ and $n|S \cap \{c\}| + |S \setminus \{c\}| \leq n-1$. Hence, $c \notin S$ and $b = |S| \leq n-1$. Since $a^{\mathsf{T}}x \leq b$ is valid for $\mathsf{STAB}(W)$, we also have $b \geq \alpha(C) = n$: a contradiction.

This shows that $\mathsf{STAB}(G)$ is described by non-negativity, clique-inequalities, $x(V(C)) \leq n$ and $nx_c + x(V(C)) \leq n$. Besides, it is easy to check that there are only $2n$ stable sets of G which are tight for $x(V(C)) \leq n$. Therefore, it does not define a facet of G.

■

In particular, K_4 is the only near-h-perfect odd wheel. We end this section with the following consequence of Proposition 5.2.2 for critically h-imperfect graphs.

Corollary 5.2.4 *A critically h-imperfect graph which is not an odd wheel W_{2n+1} (with $n \geq 2$) is such that each neighborhood induces a perfect graph.*

Proof – Suppose that G has a vertex v such that $G[N_G(v)]$ is imperfect. Since this graph is a proper induced subgraph of G, it is h-perfect and must have an odd hole. Hence, G has an induced W_{2n+1} with $n \geq 2$ and, by criticality, must be isomorphic to it.

■

This result further holds for MHI graphs as they are obviously critically h-imperfect.

5.2.2 *Even Möbius Ladders*

Let $k \geq 2$, $P := (u_1, \ldots, u_k)$ and $Q := (v_1, \ldots, v_k)$ be two vertex-disjoint copies of P_k. Let M_k be the graph obtained by taking the union of P and Q and adding the edges $u_1 v_k$, $v_1 u_k$ and $u_i v_i$ for each $i \in [k]$. The graph M_k is *the Möbius ladder of size* k and has $2k$ vertices (see Figure 5.4). A Möbius ladder M_k is *even* if k is even. For example, $M_2 = K_4$. These graphs were introduced in the context of t-perfection by Shepherd [108].

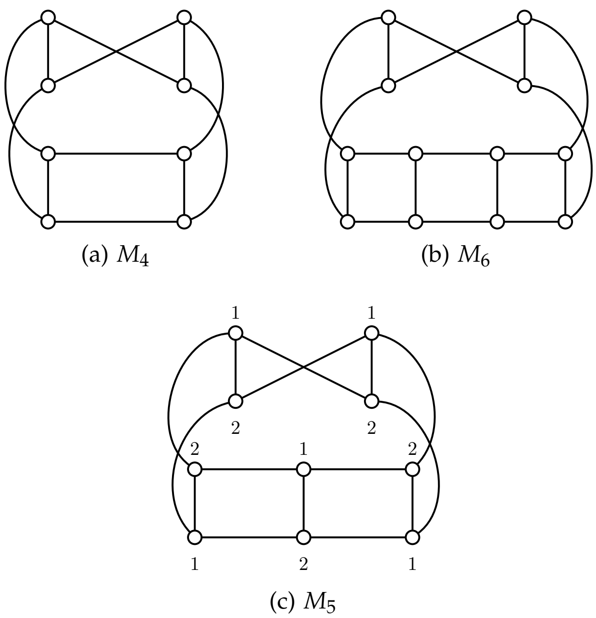


Figure 5.4 – examples of Möbius ladders and a 2-vertex-coloring of M_5

When k is odd, the graph M_k is bipartite (see Figure 5.4.c)). The description of the stable set polytope of even Möbius ladders follows from a general result of Shepherd for nearly-bipartite graphs.

A graph G is *nearly-bipartite* if for every vertex v of G, the graph $G - N_G[v]$ is bipartite. Clearly, every Möbius ladder is nearly-bipartite.

An *antiweb* is the complement of a web, and $\overline{C_n^k}$ denotes the complement of the web C_n^k (for integers $k \geq 1$ and $n \geq 3$). See Section 3.2.1 for the definition of webs. Obviously, antiwebs are regular graphs.

An antiweb $\overline{C_n^k}$ is *prime* if $n \geq 2k+2$, and $k+1$ and n are relatively prime. For each odd integer $k \geq 3$, the even Möbius ladder M_{k+1} is the prime antiweb $\overline{C_{2k+2}^{k-1}}$ (see [108] for details). Besides, C_{2l+1} (with $l \geq 2$) is clearly isomorphic to the prime antiweb $\overline{C_{2k+1}^{k-1}}$.

Let G be a graph. A *set-join of* G is a set $\{X_1, \ldots, X_l\}$ of pairwise-disjoint (possibly empty) subsets of vertices of G such that for all $1 \leq i < j \leq l$ and for each $(u, v) \in X_i \times X_j$, we have $uv \in E(G)$. It is furthermore *prime* if each X_i is either a clique or induces a prime antiweb. Finally, the *inequality of G associated to a set-join* $\{X_1, \ldots, X_l\}$ is:

$$\sum_{i=1}^{l} \frac{1}{\alpha(G[X_i])} x(X_i) \leq 1.$$

Clearly, this inequality is valid for $\mathsf{STAB}(G)$. Shepherd proved the following:

THEOREM 5.2.5 (SHEPHERD, [108]) *If G is a nearly-bipartite graph, then the non-trivial facets of* $\mathsf{STAB}(G)$ *are defined by inequalities of prime set-joins.*

We use this result to simultaneously prove that the even Möbius ladders are MTI (their critical t-imperfection is stated by Shepherd in [108]) and give a description of their stable set polytope.

COROLLARY 5.2.6 *The even Möbius ladders are minimally t-imperfect and near-h-perfect.*

PROOF – Put $M := M_{2k}$ with $k \geq 2$ (M_2 is just K_4). Clearly, M is 3-regular and connected. Hence, its regular proper induced connected subgraphs must be 2-regular. Besides, it is easy to check that 2-regular prime antiwebs are odd holes.

Therefore, the proper induced subgraphs of M which are prime antiwebs are its odd holes.

Obviously, M does not contain odd wheels. Thus, the prime set-joins of M are the cliques, the odd holes of M and M itself.

Hence, Theorem 5.2.5 shows that the non-trivial facets of $\mathsf{STAB}(M)$ are defined by cliques, odd-circuit inequalities and $x(V(M)) \leq \alpha(M)$. In other words, M is near-h-perfect.

Since being nearly-bipartite is clearly closed under taking induced subgraphs, Theorem 5.2.5 also shows that M is a critically t-imperfect graph. Hence, it remains only to prove that every t-contraction of M is t-perfect.

Since M is clearly vertex-transitive and every vertex of M is contractible, we need only to show that $M/N_M[v]$ is t-perfect for some arbitrary $v \in V(M)$. Let $\tilde{v}$ be the new vertex of $M/N_M[v]$. It is straightforward to check that $(M/N_M[v]) - \tilde{v}$ is bipartite. Hence $M/N_M[v]$ is almost-bipartite and thus Theorem 3.6.14 shows that it is t-perfect. ■

Shepherd showed as a corollary of Theorem 5.2.5 that:

THEOREM 5.2.7 (SHEPHERD [108]) *A nearly-bipartite graph is t-perfect if and only if it does not have an induced odd wheel or an induced prime antiweb which is not an odd hole.*

Hence, the MTI nearly-bipartite graphs are odd wheels and MTI prime antiwebs. As we have seen above, prime antiwebs include the even Möbius ladders. In Section 5.2.4, we will see that there is at least one other example of an MTI prime antiweb: $\overline{C_{10}^2}$. We do not know if there are more.

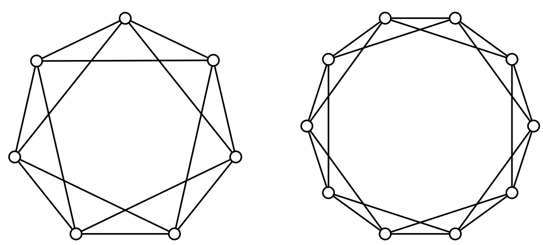
5.2.3 *Squares of circuits*

Dahl [32] gave a complete description of $\mathsf{STAB}(C_n^2)$ for each integer $n \geq 3$. Bruhn and Stein [16] proved that K_4 is a t-minor of C_n^2 for each $n \geq 5$ which is not 6, 7 or 10.

It is straightforward to check that C_6^2 is perfect (and thus t-perfect) and that C_7^2 is MTI (the latter also follows from Theorem 5.1.2 by observing that $C_7^2 = \overline{C_7}$).

THEOREM 5.2.8 (BRUHN, STEIN [16]) *Let $n \geq 3$ be an integer. The following statements hold:*

a) C_n^2 is t-perfect if and only if $n \in \{3, 6\}$,
b) C_n^2 is minimally t-imperfect if and only if $n \in \{4, 7, 10\}$.



(a) C_7^2 (b) C_{10}^2

Figure 5.5 – the minimally t-imperfect graphs C_7^2 and C_{10}^2

Since $C_7^2 = \overline{C_7}$, Theorem 5.1.2 shows:

COROLLARY 5.2.9 *C_7^2 is near-perfect.*

Finally, the near-h-perfection of C_{10}^2 is a direct consequence of results of Dahl [32, Lemma 3.2 and Theorem 3.3, pg. 9-10].

PROPOSITION 5.2.10 *C_{10}^2 is near-h-perfect.*

5.2.4 *Partitionable graphs*

A graph G is *partitionable* if $|V(G)| = \alpha(G)\omega(G) + 1$ and for every $v \in V(G)$: $G - v$ has a partition into $\omega(G)$ stable sets which are all of cardinality $\alpha(G)$ and a partition into $\alpha(G)$ cliques which are all of cardinality $\omega(G)$. It is easy to check that each partitionable graph G satisfies $\alpha(G) \geq 2$ and $\omega(G) \geq 2$.

For integers $p \geq 2$ and $q \geq 2$, a *(p, q)-graph* is a partitionable graph with $\alpha(G) = p$ and $\omega(G) = q$.

In this section, we review known examples of MTI partitionable graphs and observe that they are near-h-perfect.

Clearly, partitionability only depends on the underlying simple graph. Hence, we need only to consider simple graphs.

The class of partitionable graphs plays a key-role in the study of perfect graphs. By a theorem of Lovász [71], *every minimally imperfect graph is partitionable*.

It is an easy exercise to check that *the $(p, 2)$-graphs are the odd holes*. Hence, they are all t-perfect. By contrast:

PROPOSITION 5.2.11 *Each partitionable graph G with $\omega(G) \geq 3$ is h-imperfect.*

PROOF – Clearly, the vector $x := \frac{\mathbb{1}}{\omega(G)}$ belongs to $\mathsf{HSTAB}(G)$ and

$$x(V(G)) = \frac{1}{3}(\alpha(G)\omega(G) + 1) > \alpha(G).$$

Hence, $x \notin \mathsf{STAB}(G)$ and G is h-imperfect.

■

Obviously, each $(p, 4)$-graph is not MTI. Hence, *the MTI partitionable graphs are $(p, 3)$-graphs.*

Shepherd showed an algorithm testing partitionability [92, chap. 12], but it is still difficult in general to produce explicit examples of partitionable graphs [11]. The $(p, 3)$-graphs with $p \leq 3$ are well-known however and we review these graphs below.

Clearly, *the only MTI $(2, 3)$-graph is $\overline{C_7}$* (that is C_7^2). We now review $(3, 3)$-graphs.

An edge of a (p, q)-graph G is *undetermined* if it is not an edge of a maximum clique of G (that is a clique of cardinality q).

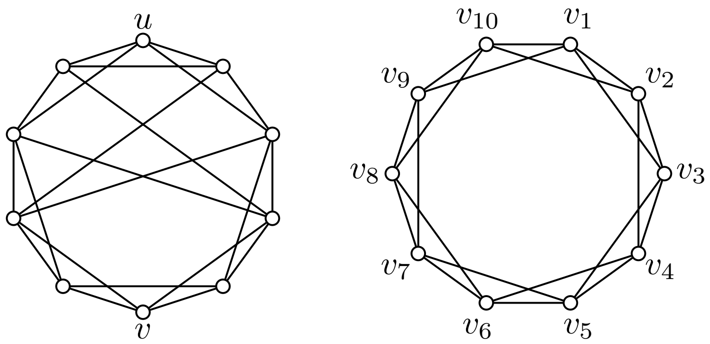
A (p, q)-graph is *normalized* if it has no undetermined edge. For each normalized partitionable graph H, let $U(H)$ denote the set of non-adjacent pairs of vertices uv of H such that each maximum clique of $H + uv$ is a clique of H.

Chvátal, Graham, Whitesides and Perold [27] observed that *for each* (p, q)*-graph* G *and each undetermined* $e \in E(G)$*: the maximum stable sets of* $G - e$ *are maximum stable sets of* G.

In particular, $G - e$ *is a* (p, q)*-graph*. This shows that *each* (p, q)*-graph is of the form* $H + F$*, where* H *is a normalized* (p, q)*-graph and* $F \subseteq U(H)$.

Let D be the graph of Figure 5.6. It is straightforward to check that D is a normalized $(3, 3)$-graph. This graph was found by Huang [10] and Chvátal et al. [27].

Theorem 5.2.12 (Chvátal et al. [27]) *The normalized* $(3, 3)$*-graphs are* C_{10}^2 *and* D.



(a) D (b) C_{10}^2

Figure 5.6 – the two normalized $(3, 3)$-graphs

Clearly, $U(D) = \{uv\}$ (see Figure 5.6) and $U(C_{10}^2)$ is the set of *diagonals* $v_i v_{i+5}$ with $i \in [5]$. The graphs obtained by adding (as edges) pairs from $U(D)$ and $U(C_{10}^2)$ to D and C_{10}^2 respectively are shown in Figure 5.7 and Figure 5.8.

Clearly, they all are $(3, 3)$-graphs. Besides, it is straightforward to check that $D^+ = \overline{D}$, $C_{10}^2(5) = \overline{C_{10}^2}$, $C_{10}^2(3) = \overline{C_{10}^2(2)'}$, $C_{10}^2(3)' = \overline{C_{10}^2(2)}$ and $C_{10}^2(4) = \overline{C_{10}^2(1)}$.

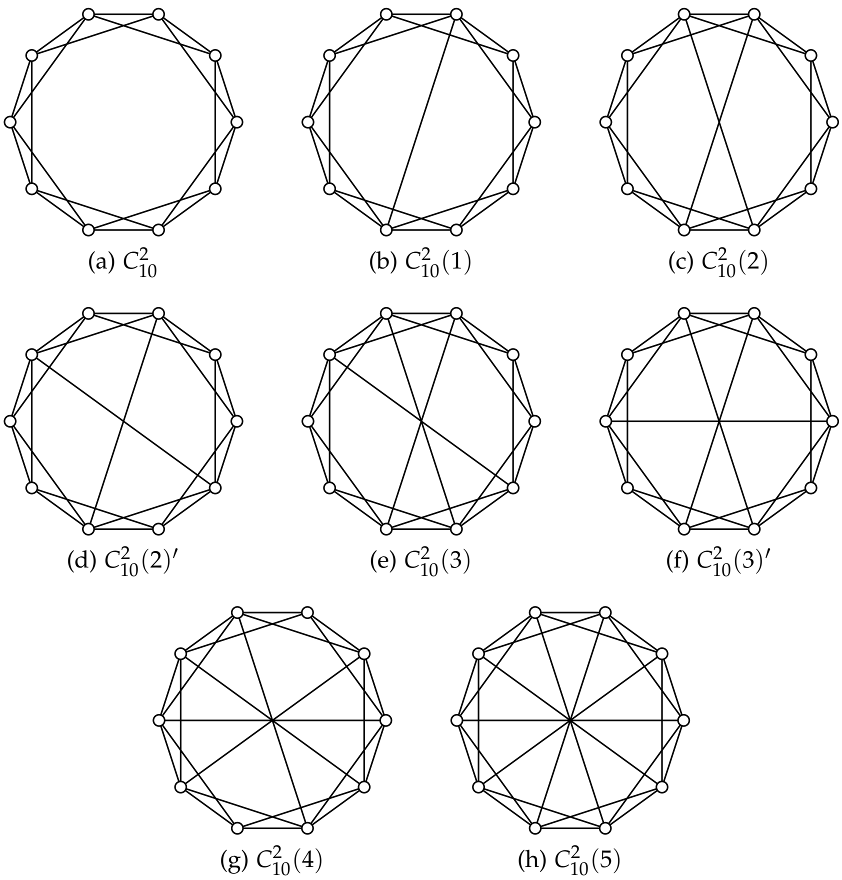
Therefore:

Corollary 5.2.13 (Chvátal et al. [27]) *The* $(3, 3)$*-graphs are* D*,* C_{10}^2*,* $C_{10}^2(1)$*,* $C_{10}^2(2)$*,* $C_{10}^2(2)'$ *and their complements.*

Bruhn and Stein proved that *all these graphs are* MTI and showed that they are *minimally strongly t-imperfect*: they are t-imperfect and their proper t-minors are strongly t-perfect.

Theorem 5.2.14 (Bruhn, Stein [13, 15]) *The* $(3, 3)$*-graphs are minimally (strongly) t-imperfect.*

The situation for $(4, 3)$-graphs is different. Indeed, a normalized $(4, 3)$-graph found by Chvátal et al. [27] has a proper induced W_7 [27, graph of Figure 6 pg. 89]. Hence, it is not MTI.

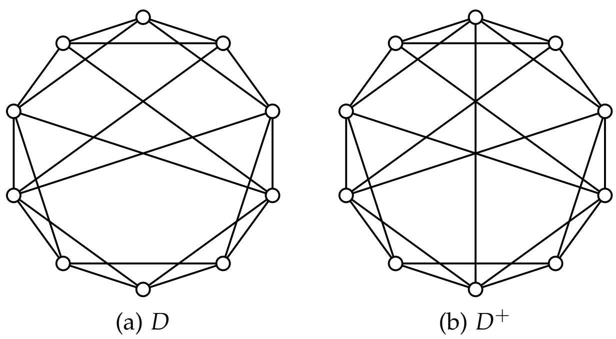


(a) C_{10}^2 (b) $C_{10}^2(1)$ (c) $C_{10}^2(2)$

(d) $C_{10}^2(2)'$ (e) $C_{10}^2(3)$ (f) $C_{10}^2(3)'$

(g) $C_{10}^2(4)$ (h) $C_{10}^2(5)$

Figure 5.7 – the $(3,3)$-graphs obtained from C_{10}^2



(a) D (b) D^+

Figure 5.8 – the $(3,3)$-graphs obtained from D

We show that the $(3,3)$-graphs are near-h-perfect. Since the inclusion-wise maximal stable sets of $\overline{C_{10}^2}$ and $\overline{D}$ are maximum, Proposition 5.2.1 directly implies:

Proposition 5.2.15 *$\overline{C_{10}^2}$ and $\overline{D}$ are near-h-perfect.*

Besides, the near-h-perfection of C_{10}^2 follows from results of Dahl [32] (see Proposition 5.2.10).

The other $(3,3)$-graphs obtained from C_{10}^2 and D have maximal stable sets which are not maximum. Even though our proofs for each of these graphs are similar, we do not know of a more general argument which would imply their near-h-perfection all-at-once. As an example, we present here the proof for $\overline{C_{10}^2(1)}$.

PROPOSITION 5.2.16 *The graph $\overline{C_{10}^2(1)}$ is near-h-perfect.*

PROOF – Put $G := \overline{C_{10}^2(1)}$. We will use the vertex-numbering of Figure 5.9. By contradiction, suppose that $\mathsf{STAB}(G)$ has a non-trivial facet F which is not defined by a clique or odd-circuit inequality or $x(V(G)) \leq \alpha(G)$. Let $a^T x \leq b$ be a valid inequality for $\mathsf{STAB}(G)$ defining F.

Since G is MTI (by Theorem 5.2.14), Proposition 5.2.1 shows that $a > 0$. Put $M := \max_{v \in V(G)} a_v$ and let $T := \{v \in V(G) \colon a_v = M\}$. We will prove that $T = V(G)$. This will imply that $b = M \cdot \alpha(G)$ and contradict that F is not defined by $x(V(G)) \leq \alpha(G)$.

Since F is not defined by $x(V(G)) \leq \alpha(G)$ and as $\alpha(G) = 3$, G has a stable set S_0 such that $|S_0| \leq 2$ and $a(S_0) = b$. Since $a > 0$, S_0 is inclusion-wise maximal. Obviously, we must have $S_0 = \{v_1, v_6\}$ (that is the only non-adjacent diagonal pair).

We claim that $S_0 \cap T \neq \varnothing$. Indeed, let $v \in V(G)$ such that $a_v = M$. Clearly v is a neighbor of v_1 or v_6, say v_6 (by horizontal symmetry). Then $\{v_1, v\}$ is a stable set and $a_{v_1} + a_v \geq a(S_0) = b$, thus $a_{v_1} + a_v = b$. Therefore, $a_{v_6} = a_v = M$ and $v_6 \in T$.

By symmetry, we may henceforth assume that $v_6 \in T$. Since F is not defined by the clique inequality of $K := \{v_6, v_7, v_8\}$, G has an inclusion-wise maximal stable set S_1 such that $a(S_1) = b$ and $S_1 \cap \{v_6, v_7, v_8\} = \varnothing$. This implies that $S_1 = \{v_2, v_5, v_9\}$. Now, $S_1 - v_5 + v_6$ is a stable set thus: $a(S_1 - v_5 + v_6) \leq b$, which implies that $a_{v_5} \geq a_{v_6} = M$. Hence, $v_5 \in T$.

The same argument used successively with the clique $\{v_5, v_6, v_7\}$ (resp. $\{v_4, v_5, v_6\}$) instead of K yields $v_4, v_7 \in T$ (resp. $v_3 \in T$). Now, we also have $v_1 \in T$. Indeed, $\{v_6, v_3\}$ is a stable set and since $v_3 \in T$: $a_{v_6} + a_{v_3} \geq a_{v_6} + a_{v_1} = b$, thus $a_{v_1} = a_{v_3} = M$.

Finally, $b = a(S_0) = a_{v_1} + a_{v_6} = 2M$ because $v_1, v_6 \in T$. However $\{v_1, v_4, v_7\}$ is a stable set contained in T, hence $a(\{v_1, v_4, v_7\}) = 3M \leq 2M$. This is absurd.

■

The proofs for the other $(3,3)$-graphs follow the same plan. We do not include them as they do not clearly provide further information on how to build a general argument.

THEOREM 5.2.17 *The $(3,3)$-graphs are near-h-perfect.*

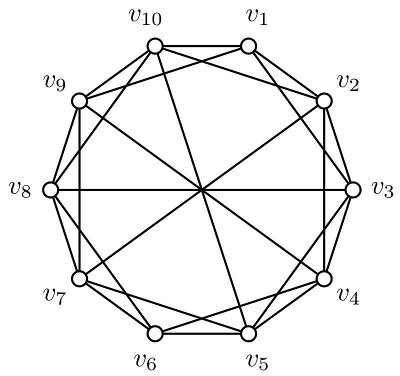


Figure 5.9 – denoting the vertices of $\overline{C^2_{10}(1)}$ for the proof of Proposition 5.2.16

5.3 CONJECTURES AND QUESTIONS ON MINIMALLY T-IMPERFECT GRAPHS

The currently known MTI graphs are (see the previous section):

odd wheels, even Möbius ladders, C^2_7 *and the* $(3,3)$*-graphs* $(*)$

We do not know if this list is complete. More examples of MTI graphs may be found by investigating t-perfection in particular families of graphs which are closed under t-minors (see the next section). For example: K_4, W_5, C^2_7 *and* C^2_{10} *are the MTI claw-free graphs [16].*

In this section, we list known and new conjectures, and questions which may motivate further research on MTI graphs.

CONNECTIVITY Recall that a *vertex-cut* of a connected graph G is a set $X \subseteq V(G)$ such that $G - X$ is not connected. Clearly, each MTI graph is connected and has no vertex-cut which is a clique (see Corollary 3.6.12). However, *can two non-adjacent vertices of an MTI graph form a vertex-cut ?* In other words:

QUESTION 5.3.1 (BRUHN, STEIN [16]) *Is every minimally t-imperfect graph 3-connected ?*

In this context, it is worth mentioning the following result:

THEOREM 5.3.2 (BRUHN, STEIN [16]) *Let G be a minimally t-imperfect graph.*

If G has a vertex-cut $\{u, v\}$ formed by two non-adjacent vertices, then $G - u - v$ has exactly two connected components C_1, C_2. Furthermore, for exactly one $i \in \{1, 2\}$: $G[C_i \cup \{u, v\}]$ is a path between u and v.

This directly implies that if an MTI graph has no vertex of degree 2, then it must be 3-connected. Therefore, Bruhn and Stein also asked the following:

QUESTION 5.3.3 (BRUHN, STEIN [16]) *Does every minimally t-imperfect graph have minimum degree 3 ?*

All known examples obviously have minimum degree 3. In Section 5.4, we will see that this is not true for MHI graphs.

NEAR-H-PERFECTION In the preceding sections, we gave a description of the stable set polytope of each MTI graph of the list $(*)$. We observed that, except odd wheels W_{2n+1} with $n \geq 2$, these graphs are near-h-perfect. Hence, we conjecture the following:

CONJECTURE 5.3.4 *Except the odd wheels* W_{2n+1} *with* $n \geq 2$*, every minimally t-imperfect graph is near-h-perfect.*

If valid, this would provide an analog of Theorem 5.1.2 for minimal t-imperfection. Furthermore, it would imply (through Proposition 3.4.1) the following formula for the fractional chromatic number of an MTI graph G which is not a W_{2n+1} with $n \geq 2$ (see Section 3.6.4 for the notation Γ):

$$\chi_f(G) = \max\left(\omega(G), \Gamma(G), \frac{|V(G)|}{\alpha(G)}\right).$$

In Section 5.4, we will see that the statement of Conjecture 5.3.4 does not hold for MHI graphs in general.

FRACTIONAL CHROMATIC NUMBER As clique and odd circuit inequalities of a graph G are always valid for $\mathsf{STAB}(G)$, Proposition 3.4.1 shows that every graph G satisfies $\chi_f(G) \geq \max(\omega(G), \Gamma(G))$.

Besides, Proposition 3.6.15 states that equality holds when G is t-perfect. Bruhn (personal communication) conjectures that this property characterizes t-perfection.

CONJECTURE 5.3.5 (BRUHN) *Every minimally t-imperfect graph G satisfies:*

$$\chi_f(G) > \max(\omega(G), \Gamma(G)).$$

It is straightforward to check that this holds for every graph in $(*)$.

Now, if Conjecture 5.3.4 is valid then Conjecture 5.3.5 is equivalent to the following statement: *except the* W_{2n+1} *with* $n \geq 2$*, every minimally t-imperfect graph G satisfies:*

$$|V(G)| > \alpha(G) \cdot \max(\omega(G), \Gamma(G)).$$

This statement may be compared to the following characterization of perfect graphs due to Lovász:

THEOREM 5.3.6 (LOVÁSZ [71]) *For every graph G, the following statements are equivalent:*

i) G is perfect,

ii) every induced subgraph H of G satisfies $|V(H)| \leq \alpha(H)\omega(H)$.

VERTICES OF TSTAB By Theorem 5.1.2, every minimally imperfect graph G is such that $\mathrm{QSTAB}(G)$ has a unique non-integral vertex which is $\frac{1}{\omega(G)}\mathbf{1}$.

Proposition 3.6.2 shows that if G is an MTI graph, then $\mathrm{TSTAB}(G)$ has at least one non-integral vertex. Bruhn and Stein asked whether there could be more non-integral vertices of $\mathrm{TSTAB}(G)$:

QUESTION 5.3.7 (BRUHN, STEIN [16]) *For a minimally t-imperfect graph G, does $\mathrm{TSTAB}(G)$ have precisely one non-integral vertex ?*

It is not difficult to check that the answer is positive at least for the odd wheels, C_7^2 and C_{10}^2. We do not know if it holds for every MTI graph of the list ($*$). We will see in Section 5.4 that it does not hold for two K_4-free MHI graphs which are not MTI.

Finally, trying to extend the proof of Theorem 5.1.2 by Padberg led us to ask:

QUESTION 5.3.8 *Can a minimally t-imperfect graph have both contractible and non-contractible vertices ?*

QUESTION 5.3.9 *Do the odd holes of a minimally t-imperfect graph all have the same length ?*

The answers to these questions are obviously positive for the graphs of the list ($*$). In Section 5.4.1, we will see that the answer to Question 5.3.8 is positive for each K_4-free MHI graph which is not MTI.

MORE MTI GRAPHS We end this section by discussing how other examples of MTI graphs might be found.

CLearly, the class of planar graphs is closed under t-minors. Among the known examples of MTI graphs, only the odd wheels (including K_4) and C_{10}^2 are planar. Bruhn (personal communication) asks the following:

QUESTION 5.3.10 (BRUHN) *Are there minimally t-imperfect planar graphs other than C_{10}^2 and the odd wheels ?*

We will show in Section 5.4.2 that the MHI planar graphs are K_4-free and that characterizing h-perfect planar graphs can be easily reduced to determining the t-perfect ones.

Let $k \geq 1$ be an integer. A graph is *P_k-free* if it does not have an induced P_k (see Section 3.2.1). It is straightforward to check that the class of P_k-free graphs is closed under t-minors. Hence, the class of P_k-free graphs may be of interest for seeking new examples of MTI graphs.

Esperet et al. [40] proved that *every $\{P_5, K_4\}$-free graph is 5-colorable*. Besides, *a maximum-weight stable set can be found in polynomial-time in P_5-free graphs* [70].

The known MTI P_5-free graphs are K_4, W_5, C_7^2 and $\overline{C_{10}^2}$.

QUESTION 5.3.11 *Are there minimally t-imperfect P_5-free graphs other than K_4, W_5, C_7^2 and $\overline{C_{10}^2}$?*

Sumner proved the following theorem:

THEOREM 5.3.12 (SUMNER, [111]) *Each triangle-free P_5-free graph is either bipartite, or obtained from C_5 by substituting its vertices by stable sets.*

Since substituting stable sets for vertices of a t-perfect graph keeps t-perfection (Theorem 4.1.5), this implies:

COROLLARY 5.3.13 *Every triangle-free P_5-free graph is t-perfect.*

5.4 MINIMALLY H-IMPERFECT GRAPHS IN GENERAL

In this section, we first determine the K_4-free MHI graphs which are not MTI. Then, we give a description of their stable set polytope. Our results show that some of the conjectures and questions given for MTI graphs need to be reformulated for MHI graphs. Moreover, we state a conjecture of Sebő for critically h-imperfect graphs with cliques of cardinality 4 and prove it for 4-clique-separated graphs (which include planar graphs).

5.4.1 *K_4-free minimally h-imperfect graphs. Near-h-perfection and conjectures*

We first show:

THEOREM 5.1.4 *The only K_4-free minimally h-imperfect graphs are K_4^*, W_5^- and W_5^{--}.*

PROOF – The h-imperfection of K_4^*, W_5^- and W_5^{--} is stated in Proposition 4.3.2. Furthermore, it is straightforward to check that every proper t-minor of each of these graphs is formed by clique-sums of odd circuits and complete graphs, thus it is h-perfect (Corollary 3.6.12). Therefore, they are MHI.

Conversely, let G be a K_4-free MHI graph which is not MTI. Clearly, G is t-imperfect. Moreover for each $v \in V(H)$: $G - v$ is h-perfect and K_4-free, thus it is t-perfect. Since G is not MTI, G must have a contractible vertex u such that $G/N_G[u]$ is t-imperfect. Since h-perfection is closed under t-contractions (Theorem 4.1.3), $G/N_G[u]$ is h-perfect. Hence, $G/N_G[u]$ has a K_4.

In particular, $\omega(G/N_G[u]) > \omega(G)$ and Proposition 4.3.6 implies that u must be a special vertex of an induced K_4^*, W_5^- or W_5^{--} of G. Since G is MHI and each of these graphs is MHI, H must coincide with one of them.

■

These graphs are precisely those appearing in our characterization of graphs and contractible vertices for which t-contraction can be interpreted as taking a face of HSTAB (Theorem 4.3.1). Besides, we proved that W_5^{--} *is the only MHI graph which has an homogeneous set* (see Theorem 4.1.6 in Chapter 4). Since the inclusion-wise maximal stable sets of W_5^- and W_5^{--} are maximum, Proposition 5.2.1 implies:

PROPOSITION 5.4.1 *W_5^- and W_5^{--} are near-h-perfect.*

Let c denote the vertex of K_4^* which is not adjacent to its triangle (see Figure 5.10).

PROPOSITION 5.4.2 *The non-trivial facets of* $\mathsf{STAB}(K_4^*)$ *are defined by clique, odd-circuit inequalities and the inequality*

$$2x_c + x(V(K_4^*) \setminus \{c\}) \leq 3, \tag{5.1}$$

which defines a facet of $\mathsf{STAB}(K_4^*)$.

PROOF – Put $G := K_4^*$. In this proof, we speak of a *maximal* stable set to mean "inclusion-wise maximal".

The inequality (5.1) is obviously valid for $\mathsf{STAB}(G)$ and it is easy to find 7 tight linearly independent stable sets of G. Hence, it defines a facet of $\mathsf{STAB}(G)$.

Now, let F be a non-trivial facet of $\mathsf{STAB}(G)$ which is not defined by a clique or odd circuit-inequality. Let $a^T x \leq b$ be a valid inequality of $\mathsf{STAB}(G)$ defining F. We will show that $a^T x \leq b$ is a positive multiple of (5.1).

By Theorem 5.1.4, K_4^* is MHI. Hence, Proposition 5.2.1 shows that $a > 0$. We use the numbering of the vertices of G given in Figure 5.10.

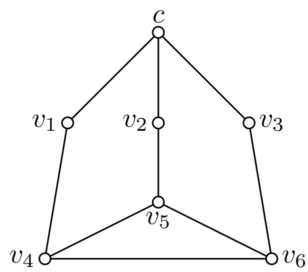


Figure 5.10 – notation for the vertices of K_4^*

Since F is not defined by the inequality of the odd hole $C := cv_2v_5v_4v_1$, there exists a stable set S of G such that $a(S) = b$ and $|S \cap V(C)| < 2$. Since $a > 0$, S is maximal and this implies that $S = \{c, v_6\}$. Hence $a_c + a_{v_6} = b$. Using the same argument with the other induced odd holes of G (all have length 5), we obtain: $a_c + a_{v_4} = a_c + a_{v_5} = b$. Therefore: $a_{v_4} = a_{v_5} = a_{v_6}$.

Since F is not defined by the clique-inequality of $v_4v_5v_6$, there exists a maximal stable set S' such that $a(S') = b$ and $S' \cap \{v_4, v_5, v_6\} = \varnothing$.

This implies that $S' = \{v_1, v_2, v_3\}$. Now $S' - v_1 + v_4$ is a stable set of G and $a^T x \leq b$ is valid for $\mathsf{STAB}(G)$. Therefore, $a(S' - v_1 + v_4) \leq b$ and thus we get $a_{v_4} \leq a_{v_1}$. Similarly, we obtain $a_{v_5} \leq a_{v_2}$ and $a_{v_6} \leq a_{v_3}$.

Again, F is not defined by the inequality associated to the clique cv_1 so there is a maximal stable set S'' such that $a(S'') = b$ and which is disjoint from cv_1. Clearly, $v_4 \in S''$ and $S'' - v_4 + v_1$ is a stable set of G. Hence $a(S'' - a_{v_4} + a_{v_1}) \leq a(S'')$, that is $a_{v_1} \leq a_{v_4}$.

We similarly show $a_{v_2} \leq a_{v_5}$ and $a_{v_3} \leq a_{v_6}$. Therefore: $a_{v_1} = a_{v_4}$, $a_{v_2} = a_{v_5}$ and $a_{v_3} = a_{v_6}$.

The equalities obtained yield that: $a_{v_i} = a_{v_j}$ for every $1 \leq i, j \leq 6$. Hence, we can assume without loss of generality that $a_v = 1$ for every $v \in V(G) \setminus \{c\}$ and thus $b = a(S') = 3$. Besides, $a(S) = a_c + 1 = 3$ thus $a_c = 2$. This shows that $a^\mathsf{T} x \leq b$ is a positive multiple of (5.1) and ends the proof.

■

Finally, we discuss possible generalizations of the conjectures stated in Section 5.3. First, since K_4^* is not near-h-perfect, we must exclude it in extending Conjecture 5.3.4 to MHI graphs:

Conjecture 5.1.3 *Except K_4^* and the odd wheels W_{2n+1} with $n \geq 2$, every minimally h-imperfect graph is near-h-perfect.*

Furthermore, notice that the non-integral vertex of $\mathsf{HSTAB}(W_5^{--})$ given in Figure 4.2 yields another non-integral vertex by vertical symmetry (this is observed in [16]). Hence, $\mathsf{HSTAB}(W_5^{--})$ has more than one non-integral vertex and this implies that the answer to Question 5.3.7 is negative for MHI graphs in general.

With Question 5.3.3 in mind and observing that K_4^*, W_5^- and W_5^{--} are the only known minimally h-imperfect graphs which have vertices of degree 2, we ask:

Question 5.4.3 *Are K_4^*, W_5^- and W_5^{--} the only minimally h-imperfect graphs which have vertices of degree 2 (or, which are not 3-connected) ?*

Besides, each of these three graphs both have contractible and non-contractible vertices (see Question 5.3.8). We do not know if they are the only MHI graphs which have this property.

5.4.2 *A conjecture of Sebő for critically h-imperfect graphs with $\omega \geq 4$*

In this section, we investigate the structure of MHI graphs which have cliques of cardinality larger than 3. Odd antiholes form a class of minimally imperfect graphs with clique numbers taking any integer value larger than 2. Except $\overline{C_5} = C_5$, they are also MHI. This is well-known and follows from Theorem 5.1.2. It can also be proved directly:

PROPOSITION 5.4.4 *For every $n \geq 3$, the odd antihole $\overline{C_{2n+1}}$ is minimally h-imperfect.*

PROOF – Since $\overline{C_{2n+1}}$ (with $n \geq 3$) is minimally imperfect and not an odd hole, it is critically h-imperfect. It has no contractible vertex and thus it is MHI.

■

Actually, odd antiholes with at least 9 vertices are the only known examples of critically h-imperfect graphs which have cliques of cardinality larger than 4. Sebő (personal communication) conjectures the following:

CONJECTURE (SEBŐ) *Let G be a graph. The following statements are equivalent:*

i) G is h-perfect,

ii) each induced K_4-free subgraph of G is t-perfect and G has no induced $\overline{C_{2n+1}}$ with $n \geq 4$.

This means that a characterization of h-perfection by forbidden induced-subgraphs could be reduced to one for t-perfection. As t-perfection is in co-NP, *the conjecture would imply that h-perfection is also in co-NP*.

It is straightforward to check that the conjecture has the following equivalent form:

CONJECTURE 5.1.5 (SEBŐ) *Every critically h-imperfect graph with $\omega \geq 4$ is an odd antihole.*

Since MHI graphs are obviously critically h-imperfect, this would directly imply that *the MHI graphs are: the MTI graphs (except K_4), K_4^*, W_5^-, W_5^{--} and the odd antiholes $\overline{C_{2n+1}}$ with $n \geq 4$.*

We observe that *Conjecture 5.1.5 would imply that the Strong Perfect Graph Theorem* (Theorem 1.1.2) *can be easily proved from the K_4-free case due to Tucker [116]*, and the latter has a considerably simpler proof than the general case (due to Chudnovsky et al. [24]).

Indeed, a minimally imperfect graph G which is not an odd hole is obviously critically h-imperfect and Conjecture 5.1.5 would imply that G is either an odd antihole or is K_4-free. Then, Tucker's theorem for K_4-free minimally imperfect graphs [116] shows that G must be an odd antihole.

This indicates that using the Strong Perfect Graph Theorem may be crucial in showing Conjecture 5.1.5. For example, it would suffice to prove that *every critically h-imperfect graph is minimally imperfect*.

If every MTI graph has minimum degree 3 (that is, the answer to Question 5.3.3 is *yes*) and Conjecture 5.1.5 is valid, then Question 5.4.3 would also have a positive answer.

By Theorem 5.1.2 odd antiholes are near-perfect and thus near-h-perfect. Hence, since W_5^- and W_5^{--} are near-h-perfect: the validity of both Conjecture 5.3.4 and Conjecture 5.1.5 would imply Conjecture 5.1.3.

We end this section by proving that Conjecture 5.1.5 holds for a certain class of graphs which contains planar graphs. We say that a graph G is *4-clique-separated* if every non-complete connected induced subgraph H of G with $\omega(H) \geq 4$ has a vertex-cut which is a clique. It is easy to check that the $\overline{C_{2n+1}}$ with $n \geq 4$ are not 4-clique-separated.

THEOREM 5.1.6 *Critically h-imperfect 4-clique-separated graphs are K_4-free.*

PROOF – To the contrary, suppose that H is a critically h-imperfect 4-clique-separated graph with $\omega(H) \geq 4$. Clearly, it is connected and non-complete. Hence it has a vertex-cut K which is a clique. Since H is critically h-imperfect, this implies that H is a clique-sum of h-perfect graphs and Corollary 3.6.12 shows that H must be h-perfect: a contradiction.

■

Since the class of 4-clique-separated graphs is closed under taking induced subgraphs, this shows that *a 4-clique-separated graph is h-perfect if and only if all its K_4-free induced subgraphs are t-perfect*. Tucker proved the following result:

THEOREM 5.4.5 (TUCKER [116]) *Let G be a non-complete connected planar graph. If G has a clique K of cardinality 4, then K contains a triangle which is a cut.*

Since planarity is closed under taking induced subgraphs, this implies that *planar graphs are 4-clique-separated*. There are non-planar 4-clique-separated graphs (for example the complete bipartite graph $K_{3,3}$).

Hence, Conjecture 5.1.5 directly follows for planar graphs:

COROLLARY 5.4.6 *A planar graph is h-perfect if and only if each of its K_4-free induced subgraphs is t-perfect.*

If Question 5.3.10 has a negative answer, then this would imply that *a planar graph is h-perfect if and only if does not have an induced C_{10}^2 or W_{2n+1} with $n \geq 2$.*

A graph is *chordal* if each circuit of length greater than 3 has a chord. Dirac [33] proved that *a graph is chordal if and only if each inclusion-wise minimal vertex-cut is a clique*. It is easy to check that the class of 4-clique-separated graphs and the class of chordal graphs are incomparable with respect to inclusion.

5.5 ON MINIMALLY H-IMPERFECT CLAW-FREE GRAPHS

A graph is *claw-free* if it does not have an induced claw (see Figure 5.1). It is straightforward to check that the class of claw-free graphs form a proper superclass of line-graphs and that it is closed under t-minors.

Bruhn and Schaudt [14] gave a polynomial-time algorithm deciding t-perfection in claw-free graphs. The complexity of testing h-perfection in claw-free graphs is not known.

THEOREM 5.1.1 (BRUHN, STEIN [16]) *The minimally t-imperfect claw-free graphs are K_4, W_5, C_7^2 and C_{10}^2.*

Theorem 5.1.4 shows that W_5^- and W_5^{--} are the K_4-free claw-free graphs which are MHI but not MTI.

Hence, to obtain an excluded-t-minor characterization of h-perfect claw-free graphs, it suffices to find the MHI claw-free graphs which have cliques of cardinality 4. In this case, Conjecture 5.1.5 states that these graphs are odd antiholes:

CONJECTURE 5.5.1 *Every critically h-imperfect claw-free graph with $\omega \geq 4$ is an odd antihole.*

If valid, this would imply that *the MHI claw-free graphs are K_4, W_5, C_{10}^2, W_5^-, W_5^{--} and the odd antiholes $\overline{C_{2n+1}}$ with $n \geq 3$* ($C_7 = C_7^2$). In particular, it would put h-perfection of claw-free graphs in co-NP and hopefully help in designing an efficient algorithm.

Webs (see definition in Section 3.2.1) form a class of claw-free graphs which contains all odd antiholes (they are the C_{2k+3}^k with $k \geq 1$).

In Section 5.5.1, we characterize h-perfect and critically h-imperfect webs. We prove and use a basic property of odd holes in claw-free graphs (Proposition 5.5.3) that will play an important role in the next chapter.

We observe in Section 5.5.3 that our results for webs and a theorem of Giles, Trotter and Tucker [54] show that Conjecture 5.5.1 follows from Conjecture 5.1.9.

On the other hand, in Section 5.5.4 we observe that the characterization by Cao and Nemhauser of h-perfect line-graphs (Theorem 3.8.2) implies that *the MHI line-graphs are W_5^- and W_5^{--}.*

5.5.1 *H-perfect and critically h-imperfect webs*

Let $k \geq 1$ and $n \geq 3$ be integers. Clearly, if $k = 1$ or $n \leq 2k + 1$ then C_n^k is a circuit or a complete graph and thus it is h-perfect. Furthermore, C_{2k+2}^k is isomorphic to K_{2k+2} minus a perfect matching and it is straightforward to check that this graph is perfect (and thus h-perfect). In this section, we prove that there are no other h-perfect webs.

On the other hand, if $k \geq 2$ then C_{2k+3}^k is the odd antihole $\overline{C_{2k+3}}$, which is critically h-imperfect. We will show that C_{10}^2 is the only other critically h-imperfect web.

Both results are implied by the following lemma, which gives certain h-imperfect t-minors of C_n^k when $n \geq 2k+4$. Its proof is postponed to the next section.

Lemma 5.5.2 *Let $k \geq 2$ and n be two integers such that $n \geq 2k+4$. If C_n^k does not have an odd antihole with at least 7 vertices, then at least one of the following statements holds:*

i) C_n^k has an induced C_l^2 with $l \geq 8$,

ii) C_n^k has a t-minor among W_5, W_5^- or W_5^{--}.

We use this to determine the h-perfect webs:

Theorem 5.1.7 *A web is h-perfect if and only if it is a circuit or a complete graph or a C_{2k+2}^k with $k \geq 1$.*

Proof – See the first paragraph of this section for the h-perfection of circuits, complete graphs and webs C_{2k+2}^k with $k \geq 1$.

Let $k \geq 1$ and $n \geq 3$ be integers. Put $G := C_n^k$ and suppose that G is neither a circuit nor a complete graph, and that $n \neq 2k+2$. In particular, $k \geq 2$ and $n > 2k+2$. We prove that G is h-imperfect.

If $n = 2k+3$ then G is isomorphic to $\overline{C_{2k+3}}$, which is h-imperfect (by Proposition 5.4.4).

If $n \geq 2k+4$, then Lemma 5.5.2 shows that G has a t-minor which is one of W_5, W_5^-, W_5^{--}, the webs C_l^2 with $l \geq 8$ or the odd antiholes $\overline{C_{2m+1}}$ with $m \geq 3$. By Theorem 5.1.4, Theorem 5.2.8 and Proposition 5.4.4, none of these graphs are h-perfect. Since h-perfection is closed under t-minors (Theorem 4.1.3), this shows that G is h-imperfect.

■

The critical h-imperfection of C_{10}^2 and the odd antiholes $\overline{C_{2l+1}}$ with $l \geq 3$ follows respectively from Theorem 5.2.8 and Proposition 5.4.4. We use Lemma 5.5.2 to prove that these are the only critically h-imperfect webs:

Theorem 5.1.8 *A web is critically h-imperfect if and only if it is C_{10}^2 or an odd antihole $\overline{C_{2n+1}}$ with $n \geq 3$.*

Proof – Let $k \geq 1$ and $n \geq 3$ be integers and let $G := C_n^k$ be a critically h-imperfect web which is not an odd antihole. By Theorem 5.1.7, we have $k \geq 2$ and $n \geq 2k+3$. In particular, no vertex of G is contractible and thus it is MHI. Moreover C_{2k+3}^k is an odd antihole, thus $n \geq 2k+4$.

By criticality, G cannot contain an induced $\overline{C_{2l+1}}$ with $l \geq 3$. We may apply Lemma 5.5.2 to G.

Suppose first that G has a t-minor among W_5, W_5^- or W_5^{--}. Since G is MHI, G is isomorphic to one of these three graphs. However, none of them is a web: a contradiction.

Hence, G has an induced C_l^2 for an integer $l \geq 8$. By Theorem 5.2.8, the graphs C_l^2 with $l \geq 8$ are h-imperfect and among them, only C_{10}^2 is critically h-imperfect.

As G is critically h-imperfect itself, this implies that G must be isomorphic to C_{10}^2.

■

5.5.2 *Proof of Lemma 5.5.2*

To prove Lemma 5.5.2, we use the following basic property of odd holes in claw-free graphs. It will also play a key-role in Chapter 6:

Proposition 5.5.3 *Let G be a claw-free graph. If G has a vertex v and an odd hole C such that v has at least 3 neighbors in C, then G has a t-minor among W_5, W_5^- and W_5^{--}. In particular, G is h-imperfect.*

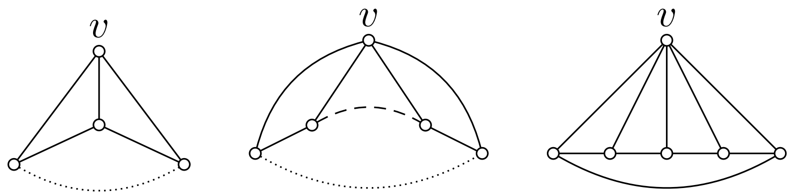
Proof – Let $N = |N_G(v) \cap V(C)|$, $H = G[V(C) \cup \{v\}]$ and suppose that $N \geq 3$. Since G is claw-free, we have $N \leq 5$.

If $N \leq 4$ then H is of the form shown in Figures 5.11a and 5.11b, and can hence be t-contracted respectively to W_5^{--} (if $N = 3$) and W_5^- (if $N = 4$).

Otherwise, if $N = 5$ then H is isomorphic to W_5 (Figure 5.11c).

By Theorem 5.1.4 and Proposition 5.2.2, G has an h-imperfect t-minor in each case. Since h-perfection is closed under t-minors (Theorem 4.1.3), G is h-imperfect too.

■



(a) $N = 3$ (b) $N = 4$ (c) $N = 5$

Figure 5.11 – The different possibilities for H: dotted and dashed lines denote pairwise-disjoint paths. Each ordinary line denotes an edge, dotted lines correspond to odd paths and the dashed one to a non-trivial even path. There is no other edge.

We need the two following results of Trotter: Notice that if $k \geq \lfloor n/2 \rfloor$ (that is, $n \leq 2k + 1$), then C_n^k is just a clique. Hence, we often exclude this trivial case.

Theorem 5.5.4 (Trotter [115]) *Let $n, n' \geq 3$ and $k, k' \geq 1$ be integers such that $k \leq \lfloor n/2 \rfloor$ and $k' \leq \lfloor n'/2 \rfloor$. The following statements are equivalent:*

i) $C_{n'}^{k'}$ *is an induced subgraph of* C_n^k
ii) $n(k'+1) \geq n'(k+1)$ *and* $nk' \leq n'k$.

COROLLARY 5.5.5 (TROTTER, [115]) *Let* $n \geq 3$ *and* $k \geq 1$. *If* $n \geq 2k+4$, *then* C_n^k *contains an odd hole or odd antihole.*

The following proposition is a straightforward application of Theorem 5.5.4:

PROPOSITION 5.5.6 *Let* $n \geq 8$ *and* $k \geq 2$. *The following conditions are equivalent:*

i) C_n^k *contains an induced* C_l^2 *with* $l \geq 8$,

ii) $\lceil \frac{2n}{k} \rceil \leq \lfloor \frac{3n}{k+1} \rfloor$ *and* $8 \leq \frac{3n}{k+1}$.

We can now prove Lemma 5.5.2:

LEMMA 5.5.2 *Let* $k \geq 2$ *and* n *be two integers such that* $n \geq 2k+4$. *If* C_n^k *does not have an odd antihole with at least 7 vertices, then at least one of the following statements holds:*
i) C_n^k *has an induced* C_l^2 *with* $l \geq 8$,
ii) C_n^k *has a t-minor among* W_5, W_5^- *or* W_5^{--}.

PROOF (OF LEMMA 5.5.2) – Let $k \geq 2$ and n be integers such that $n \geq 2k+4$ and suppose that $G := C_n^k$ does not have an odd antihole with at least 7 vertices. Furthermore, we assume that G does not have a t-minor among W_5, W_5^- or W_5^{--}. We will show that G has an induced C_l^2 for some integer $l \geq 8$.

By definition, G is obtained from C_n by adding pairs uv of non-adjacent vertices of C_n which have distance at most k on C_n (in the sense of shortest paths). Let H be the circuit of C_n^k corresponding to C_n and for every $u, v \in V(H)$, let $d_H(u,v)$ denote the distance of u and v in H.

Claim: There exists an integer $r \geq 5$ *such that* $n = kr$.

Proof – By Corollary 5.5.5, G contains an odd hole or an odd antihole. Since G does not contain an odd antihole with at least 7 vertices and as $\overline{C_5}$ is isomorphic to C_5, G must contain an induced odd hole. Let C be an odd hole of G. Let $v \in V(C)$ and let $N_C(v) = \{u, w\}$. Clearly, $d_H(u,v) \leq k$. We now prove:

$$d_H(u,v) = k. \tag{5.2}$$

Suppose to the contrary that $d_H(u,v) \leq k-1$. This implies that $d_H(v,w) \geq 2$. Otherwise, we would have $d_H(u,w) \leq k$ and thus uw would be a chord of C contradicting that it is a hole.

Let t be the neighbor of v in H which belongs to the shortest path of H joining v and w. Since $d_H(v,w) \geq 2$, we have $t \neq w$. Besides, t is adjacent to w. As $d_H(u,v) \leq k-1$, we have $d_H(t,u) \leq k$ and thus

$tu \in E(G)$. Furthermore, $t \notin V(C)$ since vt would otherwise be a chord of C.

Therefore, t is a vertex of G which does not belong to C and which has at least 3 neighbors on C. Since G is claw-free, Proposition 5.5.3 shows that $G[V(C) \cup \{t\}]$ must have a t-minor among W_5, W_5^- or W_5^{--}. This contradicts our assumptions and ends the proof of (5.2).

In particular, each pair of adjacent vertices $u, v \in V(C)$ satisfies $d_H(u,v) = k$. This easily implies that $|V(G)| = k|E(C)|$ and proves the claim.

□

By assumption, G has no odd antihole with at least 7 vertices. Hence (by Proposition 5.5.6), showing that G contains an induced C_l^2 with $l \geq 8$ only requires checking: i) $\lceil \frac{2n}{k} \rceil \leq \lfloor \frac{3n}{k+1} \rfloor$ and ii) $8 \leq \frac{3n}{k+1}$.

Since $n = kr$, i) can be rewritten: $2r \leq \lceil \frac{3kr}{k+1} \rceil$ which holds because $k \geq 2$. Similarly ii) is equivalent to $8 \leq (3r-8)k$, which is true because $k \geq 2$ and $r \geq 5$.

■

5.5.3 *Perspectives*

In this section, we use Theorem 5.1.8 to hopefully simplify Conjecture 5.5.1. We mentioned already that proving this conjecture only requires showing that critically h-imperfect claw-free graphs with $\omega \geq 4$ are minimally imperfect (see Section 5.4.2). It makes sense to try to prove it this way since the Strong Perfect Graph Theorem (Theorem 3.5.4) has a considerably simpler proof for the class of claw-free graphs [90].

Still, claw-freeness can be used to replace the minimal-imperfection condition with the weaker condition of *partitionability* (see Section 5.2.4). Giles, Trotter and Tucker showed:

THEOREM 5.5.7 (GILES, TROTTER, TUCKER [54]) *Each partitionable claw-free graph is a web.*

Combining Theorem 5.1.8 and Theorem 5.5.7 directly yields:

PROPOSITION 5.5.8 *Let G be a critically h-imperfect claw-free graph with $\omega(G) \geq 4$. If G is partitionable, then G is an odd antihole.*

Therefore, Conjecture 5.5.1 would follow from:

CONJECTURE 5.1.9 *Each critically h-imperfect claw-free graph with $\omega \geq 4$ is partitionable.*

Since odd antiholes are obviously partitionable, this must hold if Conjecture 5.1.5 is valid.

In their series of papers on claw-free graphs, Chudnovsky and Seymour [21] gave a decomposition theorem for quasi-line graphs. Eisenbrand et al. [39] used this to prove Ben Rebea's conjecture, which gives a description of the stable set polytope of quasi-line graphs. We do not know if these results could be used to prove Conjecture 5.1.9.

5.5.4 *An excluded-t-minor characterization of h-perfect line-graphs*

It is straightforward to check that *the class of line graphs is closed under t-minors*. Cao and Nemhauser [19] gave a forbidden-induced-subgraph characterization of h-perfection in line-graphs (Theorem 3.8.2). Bruhn and Stein [16] observed that this result implies that K_4 *is the only MTI line-graph*. Hence:

Theorem 5.5.9 (Bruhn, Stein [16]) *A line-graph is t-perfect if and only if it does not have K_4 as t-minor.*

In this section, we observe that Theorem 3.8.2 also yields a similar result for h-perfect line-graphs. We recall this theorem here (see Figure 5.12 for the definition of skewed prisms):

Theorem 3.8.2 (Cao, Nemhauser [19]) *Let G be a line-graph. The following statements are equivalent:*

i) G is h-perfect,

ii) G does not have an induced skewed prism.

It is straightforward to check that W_5^- and W_5^{--} are the respective line-graphs of the graphs obtained from C_5 by adding either a single chord or a single parallel edge. By Theorem 5.1.4, they are MHI.

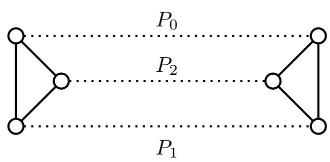


Figure 5.12 – a graph is a *skewed prism* if it is not isomorphic to K_4 and is formed by two vertex-disjoint triangles joined by three vertex-disjoint paths P_0, P_1 and P_2 (drawn dotted) such that: both P_0 and P_2 are even, and P_1 is odd. There are no other edges.

We prove:

Theorem 5.1.10 *The minimally h-imperfect line-graphs are W_5^- and W_5^{--}.*

Proof – Suppose that G is an h-imperfect line-graph. We prove that W_5^- or W_5^{--} is a t-minor of G.

By Theorem 3.8.2, G must have an induced skewed prism H. We use the notation of Figure 5.12.

If P_1 has only one edge, then one of P_0 and P_2 must be of length at least 2 (because $H \neq K_4$). Without loss of generality, we may assume that $|E(P_0)| \geq 2$. Now, we perform t-contractions in H at the internal vertices of P_2 (if any) to reduce it to a single vertex. Besides, we t-contract the internal vertices of P_0 in H such that exactly two edges of P_0 remain. Clearly, the graph obtained is isomorphic to W_5^-. Therefore, W_5^- is a t-minor of G.

Hence, we may assume that $|E(P_1)| > 1$. Since P_1 is odd, we have $|E(P_1)| \geq 3$. In H, we perform t-contractions at the internal vertices of the paths P_0, P_2 (if any) until each of them has a single vertex. Moreover, we do the same for P_1 until exactly 3 edges of P_1 remain. Clearly, the graph obtained is isomorphic to W_5^{--}, thus it is a t-minor of G.

■

Therefore:

COROLLARY 5.5.10 *A line-graph is h-perfect if and only if it does not have W_5^- or W_5^{--} as a t-minor.*

6 INTEGER ROUND-UP PROPERTY FOR THE CHROMATIC NUMBER OF SOME H-PERFECT GRAPHS

For every graph G and every $c \in \mathbb{Z}_+^{V(G)}$, the weighted chromatic number of (G, c) is the minimum cardinality of a multiset $\mathcal{F}$ of stable sets of G such that every $v \in V(G)$ belongs to at least c_v members of $\mathcal{F}$.

In this chapter, we prove that *every h-perfect line-graph and every t-perfect claw-free graph G has the integer round-up property for the chromatic number*: for every non-negative integer weight c on the vertices of G, the weighted chromatic number of (G, c) can be obtained by rounding up its fractional relaxation. In other words, *the stable set polytope of G has the integer decomposition property*.

Another occurrence of this property was recently obtained by Eisenbrand and Niemeier for fuzzy circular interval graphs (extending results of Niessen, Kind and Gijswijt). These graphs form another proper subclass of claw-free graphs.

Our results imply the *existence of a polynomial-time algorithm which computes the weighted chromatic number of t-perfect claw-free graphs and h-perfect line-graphs*. They also yield *a new case of a conjecture of Goldberg and Seymour* on edge-colorings.

Results of Gerards [50] show that the stable set polytope of certain t-perfect graphs is the projection of a polyhedron defined by totally unimodular constraints. Hence, it has the integer decomposition property (through a theorem of Sebő [103]). Laurent and Seymour [102] found a t-perfect graph which does not have this property.

In general, *is it true that each polytope which has the integer decomposition property is the projection of a polyhedron defined by totally unimodular constraints ?* We explain that an example of Gisjwijt and Regts [53] implies that *the answer is negative* even for 0-1 polytopes.

Pour tout graphe G et tout $c \in \mathbb{Z}_+^{V(G)}$, le nombre chromatique pondéré de (G, c) est le cardinal minimum d'un multi-ensemble $\mathcal{F}$ de stables de G tel que tout sommet v de G appartient à au moins c_v membres de $\mathcal{F}$.

Nous prouvons dans ce chapitre que *tout graphe h-parfait adjoint et tout graphe t-parfait sans griffe G a la propriété d'arrondi entier pour le nombre chromatique* : pour tout poids entier positif c sur les sommets de G, le nombre chromatique pondéré de (G, c) s'obtient en arrondissant sa relaxation fractionnaire à l'entier supérieur. En d'autres termes, *le polytope des stables de G a la propriété de décomposition entière*.

Cette propriété a été récemment obtenue par Eisenbrand et Niemeier pour les graphes circulaires flous (ce qui étend des résultats de Niessen, Kind et Gijswijt). Ces graphes forment une autre sous-classe propre des graphes sans griffe.

On déduit de nos résultats l'existence d'*un algorithme polynomial pour le calcul du nombre chromatique pondéré d'un graphe h-parfait adjoint ou t-parfait sans griffe*. Ils impliquent aussi *un nouveau cas d'une conjecture de Goldberg et Seymour* sur l'arête-coloration.

Des résultats de Gerards [50] impliquent que le polytope des stables de certains graphes t-parfaits est la projection d'un polyèdre défini par des contraintes totalement unimodulaires. En particulier, le polytope des stables a la propriété de décomposition entière (par un théorème de Sebő [103]). Laurent et Seymour [102] ont donné un graphe t-parfait qui n'a pas cette propriété.

En général, *est-il vrai que tout polytope qui a la propriété de décomposition entière est la projection d'un polyèdre défini par des contraintes totalement unimodulaires ?* Nous expliquons en quoi un exemple de Gijswijt et Regts implique que la réponse est non, même pour les polytopes 0-1.

6.1 INTRODUCTION

We first recall the definitions of the weighted (fractional or integral) chromatic number of a graph. See Section 3.2.4 and Section 3.4 for further notions and related results.

Let G be a graph and let $\mathcal{S}(G)$ denote the set of stable sets of G. For every $c \in \mathbb{Z}_+^{V(G)}$, the *weighted chromatic number of* (G,c), denoted $\chi(G,c)$, is the minimum cardinality of a multiset $\mathcal{F}$ of stable sets of G such that every $v \in V(G)$ belongs to at least c_v members of $\mathcal{F}$. In other words:

$$\chi(G,c) = \min \left\{ \sum_{S \in \mathcal{S}(G)} y_S \colon\ y \in \mathbb{Z}_+^{\mathcal{S}(G)};\ \sum_{S \in \mathcal{S}(G)} y_S \chi^S \geq c \right\}. \qquad (6.1)$$

The *chromatic number* $\chi(G)$ is equal to $\chi(G,\mathbf{1})$, where $\mathbf{1}$ is the all-1 vector of $\mathbb{Z}^{V(G)}$. We will speak of the *unweighted case* when considering the weight function $\mathbf{1}$.

Replacing $\mathbb{Z}$ with $\mathbb{R}$ in (6.1), we obtain a linear program whose optimum value is the *weighted fractional chromatic number of* (G,c). We write it $\chi_f(G,c)$ (and simply $\chi_f(G)$ in the unweighted case). Hence, the inequality $\lceil \chi_f(G,c) \rceil \leq \chi(G,c)$ always holds.

The chromatic number of a graph has been extensively studied in various contexts of discrete optimization and graph theory (see for example [63]). Karp [64] proved that it is NP-hard to compute and several inapproximability results were later obtained (see Huang [61] for a recent example). Finding its fractional counterpart is also an NP-hard problem in general since it is equivalent to the maximum-weight stable set problem, through the ellipsoid method [57] (a proof of this fact using a Karp-reduction is not known).

It follows from results of Grötschel, Lovász, Schrijver [57] that *the (integer or fractional) weighted chromatic number of a perfect graph can be found in polynomial-time*. Furthermore:

THEOREM 3.6.18 (GRÖTSCHEL, LOVÁSZ, SCHRIJVER [56, 57]) *The weighted fractional chromatic number of an h-perfect graph can be computed in polynomial time.*

The complexity of determining $\chi(G, c)$ in h-perfect graphs is unknown. Hence, the study of the gap between the weighted chromatic number of (G, c) and its fractional version may help to design (either exact or approximation) polynomial-time algorithms for this problem. We do not know of any result giving a bound on this gap for every h-perfect graph and every weight.

A graph G has the *integer round-up property for the chromatic number* (abbreviated IRCN) if for every $c \in \mathbb{Z}_+^{V(G)}$: $\chi(G, c) = \lceil \chi_f(G, c) \rceil$. By Theorem 3.4.3, this is equivalent to state that $\mathsf{STAB}(G)$ *has the integer decomposition property*, that is: for every positive integer k, each integral vector of $k\mathsf{STAB}(G)$ is the sum of k incidence vectors of stable sets.

Therefore, Theorem 3.6.18 implies that if every graph of a subclass $\mathcal{G}$ of h-perfect graphs has this property, then their weighted chromatic number can be computed in polynomial-time (for every weight).

A graph is *claw-free* if it does not have an induced claw (shown in Figure 6.1). Claw-free graphs form a proper superclass of line graphs.

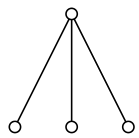


Figure 6.1 – the claw

The class of h-perfect claw-free graphs was investigated by Bruhn and Stein in [16]. In particular, they proved the unweighted case of the IRCN for these graphs:

THEOREM 6.1.1 (BRUHN, STEIN [16]) *Every h-perfect claw-free graph G satisfies* $\chi(G) = \lceil \chi_f(G) \rceil$.

The line graph of the Petersen graph shows that this is not true for line graphs in general. In this chapter, we extend this result to arbitrary weights for two subclasses of h-perfect claw-free graphs. First, we will show:

THEOREM 6.1.2 *Every h-perfect line-graph has the integer round-up property for the chromatic number.*

The corresponding unweighted result was obtained by Bruhn and Stein [16] and serves as a lemma for Theorem 6.1.1. The proof consists in coloring the edges of the source graph, whose structure is described by a result of Cao and Nemhauser (Theorem 3.8.1). We follow the same idea to show Theorem 6.1.2. Further arguments are needed to handle phenomena which occur only in the weighted case.

The *chromatic index* of a graph G, denoted $\chi'(G)$, is the minimum number of colors needed to assign to each edge of G a color such that two incident edges receive different colors. Conjectures of Goldberg [55] and Seymour [104] imply that *for every graph G, $\chi'(G)$ is equal to $\lceil \chi'_f(G) \rceil$ or $\lceil \chi'_f(G) \rceil + 1$.*

As an intermediate step towards Theorem 6.1.2, we prove in Section 6.2 a new case of these conjectures. Let C_5^+ be the graph of Figure 6.2. We say that a graph is *odd-C_5^+-free* if it does not contain a totally odd subdivision of C_5^+ (odd-C_5^+-free graphs are the graphs which do not contain a simple odd-C_3^+, see Section 3.8).

Theorem 6.1.3 *Every odd-C_5^+-free graph H satisfies: $\chi'(H) = \lceil \chi'_f(H) \rceil$.*

There are not many known classes of graphs that are defined by an excluded-subgraph (or minor) assumption and for which $\chi'(G) = \lceil \chi_f(L(G)) \rceil$ holds for every member G of the class. Seymour [105] showed that this holds for graphs without a subdivision of the complete graph K_4 (that is series-parallel graphs; Fernandes and Thomas [42] later found a shorter proof) and Marcotte [78] proved it for graphs which do not have a minor isomorphic to K_5 minus an edge. These edge-coloring results do not imply one another and are all examples of the IRCN for subclasses of line graphs.

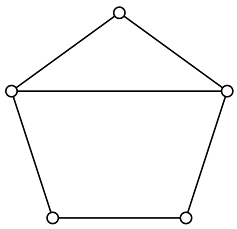


Figure 6.2 – the graph C_5^+, also known as the *house*

The other main result of this chapter is the following:

Theorem 6.1.4 *Every t-perfect claw-free graph has the integer round-up property for the chromatic number.*

The unweighted case is implied by Theorem 6.1.1 and is obtained by a reduction to the line-graph case (through decompositions along vertex-cuts). We had to follow a new approach in proving Theorem 6.1.4: if G is a t-perfect claw-free graph and $c \in \mathbb{Z}_+^{V(G)}$, then we can either reduce the size of c using certain subgraphs (and use induction) or apply Theorem 6.1.2.

We do not know if every h-perfect claw-free graph has the IRCN. It does not hold for h-perfect graphs in general as shown by an example of Laurent and Seymour [102, pg. 1207].

To our knowledge, there are only two known other results on the IRCN for h-perfect (imperfect) graphs: Kilakos and Marcotte [67] proved it for series-parallel graphs. They developed a general method to prove the IRCN. We do not see how to apply this method to t-perfect claw-free graphs.

Besides, Gerards (unpublished, [102, pg. 1207]) showed the IRCN for graphs which do not contain an odd-K_4 (an *odd-K_4* is a subdivision of K_4 in which the triangles become odd circuits).

We say that a polyhedron $P \subseteq \mathbb{R}^n$ is *totally unimodular* if there exist a totally unimodular matrix A (that is, A is integral and each determinant of a square submatrix of A is 0,-1 or 1) and an integral vector b such that $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$.

Results of Gerards [50] imply that the stable set polytope of certain graphs which do not contain an odd-K_4 (including almost-bipartite graphs) is a projection of a totally unimodular polyhedron. A theorem of Sebő [103] shows that such projections have the integer decomposition property.

Is it true that each polytope which has the integer decomposition property is a projection of a totally unimodular polyhedron ? This question is related to *extended formulations*, which recently received considerable attention (see [28]). We observe that an example due to Gisjwijt and Regts [53] shows that *the answer is negative* even for 0-1 polytopes (that is which have 0-1 vertices only):

Proposition 6.1.5 *There exists a 0-1 polytope which has the integer decomposition property and is not the projection of a totally unimodular polyhedron.*

We end this section with related results and the outline of the chapter.

Circular arc graphs form another class of claw-free graphs which have the IRCN. This was obtained by Niessen and Kind [86], Gijswijt [52] and later extended to fuzzy circular interval graphs by Eisenbrand et al. [37] (both these classes are incomparable with the class of h-perfect claw-free graph in terms of inclusion). These graphs appear in the context of the problem of finding a nice description of the stable set polytope of claw-free graphs.

A graph is *quasi-line* if the neighborhood of each vertex is covered with two cliques. In other words, quasi-line graphs are the complements of nearly-bipartite graphs. Quasi-line graphs form a proper superclass of line graphs and a proper subclass of claw-free graphs.

In [16], Bruhn and Stein observed that h-perfect claw-free graphs are quasi-line. Chudnovsky and Seymour [21] gave a decomposition theorem for quasi-line graphs (as a particular case of a more general result for claw-free graphs [23]). We do not use this result in our proofs and we do not know if it could be applied to understand h-perfect claw-free graphs in general (see also Section 5.5). Using the results of [21], King and Reed [69] proved that χ and χ_f agree asymptotically in quasi-line graphs.

OUTLINE Section 6.2 contains the proof of Theorem 6.1.2. We use it to prove Theorem 6.1.4 in Section 6.3.

In Section 6.4, we derive an explicit formula for the weighted chromatic number of h-perfect line-graphs and t-perfect claw-free graphs as a consequence of Theorems 6.1.2 and 6.1.4. Finally, we state a related formula for the chromatic index of odd-C_5^+-free graphs and discuss the algorithmic aspects of our results.

We end this chapter by observing that Proposition 6.1.5 follows from results of Gijswijt and Regts.

6.2 H-PERFECT LINE-GRAPHS

The purpose of this section is to prove Theorem 6.1.2. In Section 6.2.1, we state an edge-coloring result (Theorem 6.1.3) and show that it easily implies Theorem 6.1.2 (through Theorem 3.8.1 by Cao and Nemhauser).

Section 6.2.2 is devoted to the proof of this edge-coloring statement. It relies on an auxiliary result (Lemma 6.2.2) whose proof is postponed to Section 6.2.3.

6.2.1 *Reduction to an edge-coloring result*

The graph C_5^+ is shown in Figure 6.2. An *odd-C_5^+ of H* is a subgraph of H which is isomorphic to a totally odd subdivision of C_5^+. A graph is *odd-C_5^+-free* if it does not have an odd-C_5^+.

Various notions of perfection in line graphs were studied by Cao and Nemhauser in [19]. They characterized the h-perfection of $L(H)$ in terms of the exclusion of certain graphs as subgraphs of H (Theorem 3.8.1). We will use this result again in Chapter 8 to obtain a polynomial-time algorithm deciding h-perfection in the class of line graphs. Here we need only the following part of their result, which directly follows from observing that each odd-C_5^+ is an odd-C_3^+ (see Section 3.8).

THEOREM 6.2.1 (CAO, NEMHAUSER [19]) *Let H be a graph. If $L(H)$ is h-perfect, then H is odd-C_5^+-free.*

For the required terminology and notations of (integral or fractional) edge-coloring, we refer the reader to Section 3.2.1.

THEOREM 6.1.3 *Every odd-C_5^+-free graph H satisfies: $\chi'(H) = \lceil \chi'_f(H) \rceil$.*

We first show that this result and Theorem 6.2.1 imply Theorem 6.1.2. For each graph G and $c \in \mathbb{Z}_+^{V(G)}$, let G^c be the graph obtained by substituting each vertex $v \in V(G)$ by a stable set of cardinality c_v (see also the paragraph of Proposition 3.5.1).

THEOREM 6.1.2 *Every h-perfect line-graph has the integer round-up property for the chromatic number.*

PROOF (OF THEOREM 6.1.2) – Let G be an h-perfect line-graph, $c \in \mathbb{Z}_+^{V(G)}$ and H be a graph such that $G = L(H)$. Let H' be the graph obtained from H by replacing each edge $e = uv \in E(H)$ by c_e parallel edges between u and v. It is straightforward to check that $G^c = L(H')$. By Proposition 3.5.1, $\chi(G,c) = \chi(G^c) = \chi'(H')$ and similarly, $\chi_f(G,c) = \chi'_f(H')$.

By Theorem 6.2.1, H is odd-C_5^+-free. Therefore, H' is also odd-C_5^+-free (it only depends on its underlying simple graph, which is the same as H) and the conclusion follows from Theorem 6.1.3.

■

6.2.2 *Proof of Theorem 6.1.3*

Let H be a graph. An *odd ring* of a graph H is an induced subgraph of H whose *underlying simple graph* is an odd circuit (hence an odd ring may have parallel edges). Let:

$$\Gamma'(H) = \max\left\{\frac{2}{|V(R)|-1}|E(R)| : R \text{ is an odd ring of } H\right\},$$

An edge e of a graph H is *critical* if $\chi'(H-e) < \chi'(H)$ (that is, $\chi'(H-e) = \chi'(H) - 1$).

The main ingredient of the proof of Theorem 6.1.3 is the following "concentration" lemma:

LEMMA 6.2.2 *Let H be a graph such that $\chi'(H) > \Delta(H)$ and let $e \in E(H)$. If e is critical and is not an edge of an odd-C_5^+ of H, then there exists an odd ring R of H such that $e \in E(R)$ and:*

$$|E(R)| = r \cdot \chi'(H-e) + 1,$$

where $r = \frac{|V(R)|-1}{2}$.

We need one more result on the fractional chromatic index of a graph. In Section 6.4, we will see that equality actually holds in the statement i) below for every odd-C_5^+-free graph. As a byproduct, we will obtain a new formula for the chromatic index of these graphs. We defer this formula for the sake of clarity, since only the lower bound is needed in the proof of Theorem 6.1.3.

Since rings are obviously 2-connected and factor-critical graphs (see definition in Section 3.7), the statement i) below is a straightforward consequence of Theorem 3.7.2. We give a direct proof:

PROPOSITION 6.2.3 *Let H be a graph. The following statements hold:*

i) $\chi'_f(H) \geq \max(\Delta(H), \Gamma'(H))$,

ii) for every subgraph K of H, we have $\chi'_f(K) \leq \chi'_f(H)$.

Proof – Let $\mathcal{M}(H)$ be the set of all matchings of H. By the duality theorem of linear programming, we have:

$$\chi'_f(H) = \max\left\{x(E(H))\colon x \in \mathbb{Q}_+^{E(H)};\ x(M) \leq 1, \text{ for every } M \in \mathcal{M}(H)\right\}.$$

Let $v \in V(H)$ and R be an odd ring of H. Let $\delta(v)$ denote the set of edges of H incident with v. Clearly, each matching of H contains at most one edge of $\delta(v)$ and at most $\frac{2}{|V(R)|-1}|E(R)|$ edges of R. Hence, both $\chi^{\delta(v)}$ and $\frac{2}{|V(R)|-1}\chi^{E(R)}$ are feasible solutions of the linear program above and this implies i).

Statement ii) follows from the fact that any optimal solution x of the linear program above for $\chi'_f(K)$ can be extended to a feasible solution of the program for $\chi'_f(H)$, by setting $x_e = 0$ for every $e \in E(H) \setminus E(K)$.

■

We are now ready to prove Theorem 6.1.3 (the proof of Lemma 6.2.2 being postponed to the next part):

Theorem 6.1.3 *Every odd-C_5^+-free graph H satisfies: $\chi'(H) = \lceil \chi'_f(H) \rceil$.*

Proof – For every graph G, let $\kappa(G) := \lceil \chi'_f(G) \rceil$.

By contradiction, let H be an odd-C_5^+-free graph with $\chi'(H) \neq \kappa(H)$ and choose $|E(H)|$ minimum. We actually have $\chi'(H) > \kappa(H)$ (since $\chi'(G) \geq \kappa(G)$ clearly holds for every graph G).

Let $e \in E(H)$ and $H' = H - e$. Since H is minimal, $\chi'(H') = \kappa(H')$. Besides, Proposition 6.2.3.ii) implies that $\kappa(H') \leq \kappa(H)$. Hence:

$$\chi'(H) - 1 \leq \chi'(H') \leq \kappa(H') \leq \kappa(H) \leq \chi'(H) - 1,$$

and these inequalities are in fact equalities. In particular, e is a critical edge of H and $\kappa(H) = \chi'(H')$.

Since H is odd-C_5^+-free, Lemma 6.2.2 can be applied to H and e. Hence, H has an odd ring R such that $e \in E(R)$ and:

$$|E(R)| = r \cdot \chi'(H') + 1,$$

where $r = \frac{|V(R)|-1}{2}$. By Proposition 6.2.3.i), $\kappa(H) \geq \frac{|E(R)|}{r}$. Therefore, $\kappa(H) > \chi'(H')$: a contradiction.

■

6.2.3 *Proof of Lemma 6.2.2*

A matching M of a graph H *covers* a vertex v of H if M has an edge incident with v, and that it *misses* v otherwise.

Proposition 6.2.4 *Let H be a graph such that $\chi'(H) > \Delta(H)$ and let $e = uv$ be a critical edge of H. If Λ is an optimal edge-coloring of $H - e$, then:*

i) *every matching $M \in \Lambda$ covers at least one of u and v ,*
ii) *there exist two matchings $A, B \in \Lambda$ such that A covers u and misses v, whereas B covers v and misses u.*

Proof – If i) did not hold, then Λ could be extended to an edge-coloring Λ' of H (by adding e to a matching which misses both u and v). This would contradict $\chi'(H-e) < \chi(H)$.

We now prove ii). By the symmetry between u and v, it is enough to prove that there is a matching in Λ which covers u and misses v. Suppose to the contrary that every matching in Λ covering u covers v too. By i), every $M \in \Lambda$ covers v, so $\chi'(H-e) = d_H(v) - 1 \leq \Delta(H) - 1$. Thus $\chi'(H) \leq \Delta(H)$, which contradicts the assumption on H.

■

For every graph H and each $F \subseteq E(H)$, we write $H[F]$ for the graph $(V(H), F)$ (ambiguity with the notation for subgraphs induced by sets of vertices should not occur). We will often use the following basic recoloring-argument, which corresponds to switching the colors on a bi-edge-colored component of a graph.

Proposition 6.2.5 *Let H be a graph, Λ be an edge-coloring of H and A, B be distinct elements of Λ.*

If K is a component of $H[A \Delta B]$, then switching A and B on $E(K)$ in Λ yields an edge coloring of H which uses $|\Lambda|$ colors.

We give a few more definitions for the proof of Lemma 6.2.2. Let H be a graph.

Let R be an odd ring of H. A matching M of H is an *R-matching* if $|E(R) \cap M| = \frac{|V(R)|-1}{2}$ and M misses a (necessarily unique) vertex of C. The *end-edges* of a path P of H are the edges of P (if it has any) which are incident to its ends.

We shall detect odd-C_5^+ subgraphs of H using the following basic remark: *the odd-C_5^+ subgraphs of H are the simple graphs formed by an odd circuit C and an odd path P of H such that $V(C) \cap V(P)$ is the set of ends of P.* In other words, an odd-C_5^+ is a simple graph which has an open-ear decomposition with exactly two ears which are both odd. In particular, an odd circuit with a chord forms an odd-C_5^+.

Lemma 6.2.2 *Let H be a graph such that $\chi'(H) > \Delta(H)$ and let $e \in E(H)$. If e is critical and is not an edge of an odd-C_5^+ of H, then there exists an odd ring R of H such that $e \in E(R)$ and:*

$$|E(R)| = r \cdot \chi'(H-e) + 1,$$

where $r = \frac{|V(R)|-1}{2}$.

Proof – Let H be a graph (with possibly multiple edges) such that $\chi'(H) > \Delta(H)$ and let e be a critical edge of G which is not an edge of an odd-C_5^+ of H.

Let u and v be the ends of e and Λ be an optimal edge-coloring of $H - e$. By Proposition 6.2.4, there exist matchings $A, B \in \Lambda$ such that A covers u and misses v, whereas B covers v and misses u.

Consider the component P of u in the graph $H[A \Delta B]$. It is either a path or a circuit, but B misses u so it must be a path. We have $v \in V(P)$: otherwise, $(\Lambda \setminus \{A, B\}) \cup \{A \Delta E(P), B \Delta E(P)\}$ would be an optimal edge-coloring of $H - e$ (by Proposition 6.2.5) in which $A \Delta E(P)$ misses both u and v. It could therefore be extended to an edge-coloring of H, contradicting $\chi'(H - e) < \chi'(H)$.

So P is a uv-path and the circuit L of H obtained by adding e to P is odd. If L had a chord f, then L and f would form an odd-C_5^+ of H containing e: a contradiction.

Thus, $V(L)$ induces an odd ring R of H. Let $r = \frac{|V(R)|-1}{2}$. We claim:

$$\textit{R contains exactly } r \textit{ edges of each matching } M \textit{ of } \Lambda. \tag{6.2}$$

Let us immediately show that this claim implies the theorem: except e, every edge of R belongs to a matching of Λ and these matchings are pairwise-disjoint. Therefore, the number of edges of R is $|\Lambda| r + 1 = r \cdot \chi'(H - e) + 1$ and the conclusion follows.

We now prove (6.2). Let $M \in \Lambda$. If $M \in \{A, B\}$, then M has r edges in R because the edges of P alternate between A and B. So let us henceforth assume that $M \notin \{A, B\}$. Using the symmetry between u and v, we may suppose without loss of generality that M covers u. Let K be the component of u in $H[M \Delta B]$. The graph K is a path since B misses u. We have:

$$K \cap R \textit{ is an even path.} \tag{6.3}$$

Suppose to the contrary that $K \cap R$ has more than one component. Then, there exists a (non-zero length) path Q of K which is edge-disjoint from R and whose ends belong to $V(R)$. Since B is an R-matching, both end-edges of Q must belong to M and Q is odd. Hence, the graph L and Q together form an odd-C_5^+ of H containing e: a contradiction with our assumption. So $K \cap R$ is connected and it joins u to some vertex w of R.

Suppose $|E(K \cap R)| > 0$. Since B is an R-matching missing u, the vertex w is covered by B in the graph $K \cap R$. Therefore, the path $K \cap R$ has exactly one end-edge in B and it must be even as stated above. This proves (6.3).

Let $M' = M \Delta E(K)$. By (6.3) and since the edges of $K \cap R$ alternate between M and B, we obtain:

$$|M \cap E(R)| = |M' \cap E(R)|,$$

We now show that $|M' \cap E(R)| = r$. This will end the proof of (6.2).

Let $B' = B \Delta E(K)$ and $\Lambda' = (\Lambda \setminus \{M, B\}) \cup \{M', B'\}$. By Proposition 6.2.5, Λ' is an optimal edge-coloring of $H - e$. Notice that M'

misses u. Therefore, M' must cover v: otherwise, Λ' could be extended to an edge-coloring of H by adding e to M' and this would contradict $\chi'(H-e) < \chi'(H)$.

Let K' be the component of v in $H\left[M'\Delta A\right]$ and $T = K' \cap R$. Since A is an R-matching of H which misses v, we can repeat the argument of the proof of (6.3) to show that T is a path. Now, we have:

$$u \in V(T).$$

Indeed, suppose that $u \notin V(T)$ and let $M'' = M\Delta E(K')$, $A' = A\Delta E(K')$ and $\Lambda'' = (\Lambda' \setminus \{M', A\}) \cup \{M'', A'\}$. As above, Proposition 6.2.5 implies that Λ'' is an optimal edge-coloring of $H-e$. However, M'' misses both ends of e so Λ'' can be extended to an edge-coloring of H by adding e to M'': a contradiction.

Since M' does not contain an edge parallel to e (it misses u) and since $u \in V(T)$, the only way for T to be a path is that it coincides with P in the underlying simple graph of H. But T alternates between M' and A, hence $|M' \cap E(R)| = |M' \cap E(R)| = r$.

■

6.3 T-PERFECT CLAW-FREE GRAPHS

Our purpose is to prove Theorem 6.1.4. Section 6.3 gives an outline of the proof and hopefully clarifies that we have to take a new approach compared to the unweighted case. The proofs of the two main lemmas are postponed to Sections 6.3.2 and 6.3.3.

6.3.1 *How the proof works ?*

The unweighted case of Theorem 6.1.4 was obtained by Bruhn and Stein in [16] and appears as a preliminary result of Theorem 6.1.1. It is not difficult to see that it means that t-perfect claw-free graphs are 3-colorable (see Corollary 3.6.16).

THEOREM 6.3.1 (BRUHN, STEIN) *Each t-perfect claw-free graph is 3-colorable.*

Let us briefly recall the approach of the proof: the result is first proved for line graphs (using edge-colorings). Then, considering a t-perfect claw-free graph they show: *if G is 3-connected then it is either a line graph or one of a few exceptional graphs which can be easily 3-colored.* Otherwise, they use a 2-vertex-cut (a subset of at most two vertices of G whose deletion disconnects G) to decompose the graph into smaller pieces and apply induction (which is not direct).

Unfortunately, it is not straightforward how two weighted colorings with a small number of colors can be combined along a 2-vertex-cut such that the number of colors remains small. Kilakos and Marcotte [67] gave general sufficient conditions under which this opera-

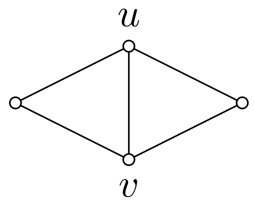


Figure 6.3 – a diamond with central vertices u, v

tion can be performed. However, it is not clear how to apply it directly to t-perfect claw-free graphs.

Therefore, we follow a different approach. We proceed by an induction where the line graphs form the base case. In the presence of certain subgraphs, we reduce the weight function. If no such subgraph appears, then we show that the graph considered is a line graph and apply Theorem 6.1.2.

A *diamond* of a simple graph G is an induced subgraph D of G which is isomorphic to the complete graph K_4 minus an edge (see Figure 6.3). A vertex u of D is *central* if $d_D(u) = 3$.

A central vertex u of D is *small* if $d_G(u) = 3$. We say that D is *small* if it has a small central vertex and that it is *large* otherwise. Notice that if D is large, then both of its central vertices have degree at least 4 in G.

Lemma 6.3.2 *Let G be a t-perfect claw-free simple graph and $c \in \mathbb{Z}_+^{V(G)}$. Suppose that G has a small diamond D with a small central vertex v such that $c_v \geq 1$. Put $c' = c - \chi^v$.*

If $\chi(G, c') = \lceil \chi_f(G, c') \rceil$, then $\chi(G, c) = \lceil \chi_f(G, c) \rceil$.

So an induction on the size of the weight function can be performed as long as there is a small diamond in the graph. Now, the following result shows that the remaining case falls in the scope of Theorem 6.1.2.

Lemma 6.3.3 *Let G be a t-perfect claw-free simple graph. If every diamond of G is large, then G is a line graph.*

The starting point of the approach of Bruhn and Stein for Theorem 6.1.1 is the use of Harary's characterization of line graphs of simple graphs (in terms of triangles). It plays a similar role in the proof of Lemma 6.3.3. The other key-ingredient is the characterization of t-perfection among squares of circuits. These two results are stated in Section 6.3.3.

We now prove that Theorem 6.1.4 follows from Lemmas 6.3.2, 6.3.3 and Theorem 6.1.2.

Proof (of Theorem 6.1.4) – To the contrary, let (G, c) be a counterexample which is minimum with respect to $|V(G)| + c(V(G))$.

Clearly, we can assume that G is simple and that no coordinate of c is equal to zero. Thus, G cannot have a small diamond because of

Lemma 6.3.2. Therefore, Lemma 6.3.3 shows that G is a line graph and the conclusion follows from Theorem 6.1.2.

■

6.3.2 *Proof of Lemma 6.3.2*

Let G be a graph and $v \in V(G)$. We write $N_G(v)$ for the set of neighbors of v. In particular, if G is simple then $d_G(v) = |N_G(v)|$ and $\Delta(G)$ is the largest number of neighbors of a vertex of G. We will use the following direct consequence of Proposition 5.5.3 page 98:

Proposition 6.3.4 *Let G be a t-perfect claw-free graph. If $v \in V(G)$ and C is an odd hole of G, then v has at most 2 neighbors in C.*

We will use again color exchanges on bi-colored components but for colorings of the vertices (the statement of Proposition 6.2.5 can be easily translated using line graphs). We can now prove Lemma 6.3.2:

Proof (of Lemma 6.3.2) – We start with an argument similar to the proof of Proposition 6.2.4. Let $D = G[\{x, v, w, y\}]$, where x and y are the two vertices of degree 2 in D. Hence, the neighbors of v in G are x, y and w. Let $\mathcal{F}$ be a coloring of (G, c'). Without loss of generality, we can assume that every $u \in V(G)$ belongs to exactly c'_u members of $\mathcal{F}$.

If there exists an $S \in \mathcal{F}$ such that $S \cap (N_G(v) \cup \{v\}) = \varnothing$, then $(\mathcal{F} \setminus \{S\}) \cup \{S \cup \{v\}\}$ is a coloring of (G, c) with $\chi(G, c')$ colors. Clearly, $\chi_f(G, c') \leq \chi_f(G, c)$ and the result of the lemma follows.

Hence we will assume that every member of $\mathcal{F}$ meets $N_G(v) \cup \{v\}$. For every $u \in V(G)$, let $\mathcal{F}_u$ denote the set of members of $\mathcal{F}$ containing u. First, suppose that $\mathcal{F}_x \subseteq \mathcal{F}_y$. Then, the number of members of $\mathcal{F}$ intersecting $N_G(v) \cup \{v\}$ is:

$$\chi(G, c') = |\mathcal{F}_y \cup \mathcal{F}_w \cup \mathcal{F}_v| = |\mathcal{F}_y| + |\mathcal{F}_w| + |\mathcal{F}_v| \leq \omega(G, c) - 1,$$

as $\{v, w, y\}$ is a clique. By Proposition 3.6.15, we obtain that $\chi(G, c') \leq \lceil \chi_f(G, c) \rceil - 1$. So adding $\{v\}$ to $\mathcal{F}$ gives a coloring of (G, c) with $\lceil \chi_f(G, c) \rceil$ colors and we are done.

Therefore, we may assume that $\mathcal{F}_x \nsubseteq \mathcal{F}_y$ and by symmetry, that $\mathcal{F}_y \nsubseteq \mathcal{F}_x$. Let $S \in \mathcal{F}_x \setminus \mathcal{F}_y$ and $T \in \mathcal{F}_y \setminus \mathcal{F}_x$. Consider $H = G[S \Delta T]$. Let K be the component of x in H. We claim:

$$y \notin V(K).$$

Suppose to the contrary that $y \in V(K)$ and let P be a shortest (thus induced) path of K joining x and y. The vertices of P alternate between S and T, hence P has odd length. As G does not have a clique of cardinality 4 (it is t-perfect), the length of P is at least 3. Let $L = G[V(P) \cup \{v\}]$. By assumption, $d_G(v) = 3$ so L is an odd hole of G.

Now, w does not belong to L (as it is adjacent to x and y) and it has at least 3 neighbors in L. By Proposition 6.3.4, this contradicts the t-perfection of G.

The lemma now easily follows: $(\mathcal{F} \setminus \{S, T\}) \cup \{S\Delta E(K), T\Delta E(K)\}$ is a coloring of (G, c) with $\chi(G, c')$ colors and $\chi(G, c') = \lceil \chi_f(G, c') \rceil \leq \lceil \chi_f(G, c) \rceil$.

■

6.3.3 *Proof of Lemma 6.3.3*

In Section 5.2.3, we stated a theorem of Bruhn and Stein (Theorem 5.2.8) characterizing t-perfection among the webs C_n^2. We repeat part i) of this statement below, which is the only one needed in this chapter:

THEOREM 6.3.5 (BRUHN, STEIN [16]) *Let $n \geq 3$ be an integer. The graph C_n^2 is t-perfect if and only if $n \in \{3, 6\}$.*

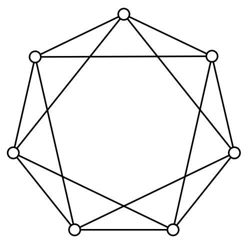


Figure 6.4 – the graph C_7^2

We will use that t-perfect claw-free simple graphs have small degree. This is shown in [16]. We give a proof here.

PROPOSITION 6.3.6 *Every t-perfect claw-free simple graph has maximum degree at most 4.*

PROOF – Let G be a t-perfect claw-free simple graph and let $v \in V(G)$. Since G is simple, we have $d_G(v) = |N_G(v)|$. First, notice that $d_G(v) \leq 5$: otherwise, by Ramsey's theorem, $N_G(v)$ would contain either a triangle or a stable set of cardinality 3. So G would contain a K_4 or a claw: a contradiction.

Now, if $d_G(v) = 5$ then $G[N_G(v)]$ is a graph with 5 vertices having no stable set of cardinality 3 and no triangle. Hence it is an odd circuit of length 5 and this contradicts Proposition 6.3.4 (in Section 6.3.2).

■

The last ingredient needed to prove Lemma 6.3.3 is the following proposition on claw-free simple graphs. Recall that $\omega(G)$ denotes the maximum cardinality of a clique of a graph G.

Proposition 6.3.7 *Let G be a connected claw-free simple graph such that $\Delta(G) \leq 4$ and $\omega(G) \leq 3$. If every diamond of G is large, then at least one of the following statements holds:*

i) G is a line graph,

ii) there exists an integer $k \geq 7$ such that G is isomorphic to C_k^2.

The proof of this proposition is postponed to the end of this section. We first show that these results imply Lemma 6.3.3:

Proof (of Lemma 6.3.3) – Let G be a t-perfect claw-free simple graph such that every diamond of G is large. Clearly, we need only to prove that each component of G is a line graph.

Let H be a component of G. Since H is an induced subgraph of G, we have that H is claw-free and that it is t-perfect too. By Proposition 6.3.6, this implies $\Delta(H) \leq 4$. Furthermore, $\omega(H) \leq 3$ and every diamond of H is large.

By Theorem 6.3.5, the graphs C_k^2 with $k \geq 7$ are not t-perfect. Hence, H cannot be isomorphic to one of them. Therefore, Proposition 6.3.7 shows that H is a line graph.

■

We end this section with the proof of Proposition 6.3.7. We need a characterization of line graphs by Harary in terms of diamonds. A triangle T of a graph G is *odd* if G contains a vertex $v \notin T$ which has an odd number of neighbors in T. A diamond of G is *odd* if both of its triangles are odd triangles of G. The implication ii)⇒i) of the following result is the key to obtain line graphs:

Theorem 6.3.8 (Harary [60]) *Let G be a claw-free simple graph. The following statements are equivalent:*

i) G is the line graph of a simple graph,

ii) G does not have an odd diamond.

Actually, we do not use the simplicity of the graph H whose line graph is G since Theorem 6.1.2 holds for line graphs of non-necessarily simple graphs.

Proof (of Proposition 6.3.7) – Let G be a connected claw-free simple graph with $\Delta(G) \leq 4$, $\omega(G) \leq 3$ and such that every diamond of G is large. Furthermore, let us assume that G is not a line graph. We have to prove:

there exists an integer $k \geq 7$ such that G is isomorphic to the graph C_k^2.

By Theorem 6.3.8, G has an odd diamond D. Put $D = G[v_1, v_2, v_3, v_4]$, where v_2 and v_3 are the central vertices of D. Since D is large, we have $d_G(v_2) = 4$. Hence, v_2 has a neighbor $v_5 \in V(G) \setminus \{v_1, v_3, v_4\}$. As G is claw-free, v_5 is adjacent to at least one of v_1 and v_4. Using the symmetry between v_1 and v_4, we can assume without loss of generality that $v_4 v_5 \in E(G)$.

Again, D is large so $d_G(v_3) = 4$. Since v_5 cannot be a neighbor of v_3 (otherwise $\{v_2, v_3, v_4, v_5\}$ would be a clique of cardinality 4), there exists $v_6 \in V(G) \setminus \{v_1, \dots, v_5\}$ such that $v_3v_6 \in E(G)$. But G is claw-free so at least one of v_1v_6 and v_4v_6 is an edge of G. However:

$$v_4v_6 \notin E(G). \tag{6.4}$$

Otherwise, v_1, v_5 and v_6 would be the only vertices of G having a neighbor in the triangle $v_2v_3v_4$ (because $\Delta(G) = 4$). But each of these vertices have exactly two neighbors on $v_2v_3v_4$. Thus $v_2v_3v_4$ would not be an odd triangle of G. This contradicts that D is odd.

Therefore, $v_1v_6 \in E(G)$. Since v_1 is a central vertex of the diamond induced by $\{v_1, v_2, v_3, v_6\}$ (and every diamond of G is large), we must have $d_G(v_1) = 4$. Furthermore:

$$v_1v_5 \notin E(G). \tag{6.5}$$

Else, the same argument used to prove (6.4) shows that $v_1v_2v_3$ would not be an odd triangle, contradicting that D is odd.

So v_1 must have a neighbor $v_7 \notin \{v_2, \dots, v_6\}$ and as G is claw-free, v_7 is adjacent to at least one of v_2 and v_6. But v_2 already has 4 neighbors among $\{v_1, v_3, v_4, v_5, v_6\}$, thus $v_2v_7 \notin E(G)$ and $v_6v_7 \in E(G)$.

Let $H = G[\{v_1, \dots, v_7\}]$ (see Figure 6.5).

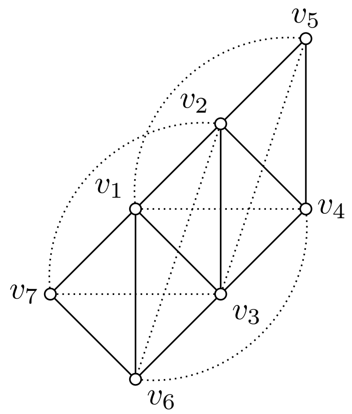


Figure 6.5 – the construction of $v_1, \dots, v_7$. Dotted lines indicate pairs of non-adjacent vertices.

Case 1. $N_G(v_4)$ and $N_G(v_6)$ are contained in $V(H)$.

Since v_4 and v_6 are both central vertices of diamonds of G, we have $d_G(v_6) = d_G(v_4) = 4$. Recall that $v_4v_6 \notin E(G)$ (Equation (6.4)). Thus, $v_5v_6 \in E(G)$ and $v_4v_7 \in E(G)$. In particular, the vertices v_3, v_5 and v_7 are neighbors of v_4. But G has no clique of cardinality 4 so $v_3v_7 \notin E(G)$ and $v_3v_5 \notin E(G)$. Since G is claw-free, this implies that $v_5v_7 \in E(G)$ (see Figure 6.6).

Now, the map $1 \to v_7$, $2 \to v_6$, $3 \to v_1$, $4 \to v_3$, $5 \to v_2$, $6 \to v_4$, $7 \to v_5$ defines an isomorphism from C_7^2 to a subgraph of H. Since

C_7^2 is 4-regular and $\Delta(G) = 4$, the graph H is in fact isomorphic to C_7^2 and is a component of G. Since G is connected, $G = H$ and the conclusion follows.

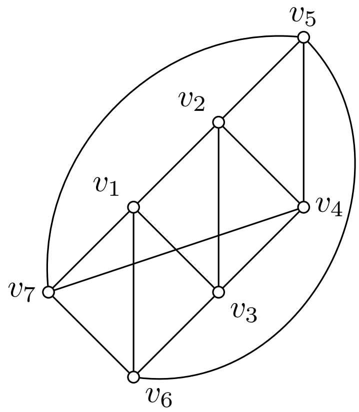


Figure 6.6 – CASE 1. building a C_7^2 from $v_1, \dots, v_7$ when $N_G(v_4) \cup N_G(v_6) \subseteq V(H)$.

Case 2. At least one of v_4 and v_6 has a neighbor outside of $V(H)$.

Using the symmetry between v_4 and v_6, we can assume without loss of generality that v_6 has a neighbor $v_8 \notin V(H)$. Now, the vertices v_3, v_7 and v_8 are three neighbors of v_6. Since G has no clique of cardinality 4, we have $v_3v_7 \notin E(G)$. Furthermore, v_3 already has 4 neighbors among the vertices of H so $v_3v_8 \notin E(G)$ and, since G is claw-free, we have $v_7v_8 \in E(G)$. Define $w_1 = v_8$, $w_2 = v_7$, $w_3 = v_6$, $w_4 = v_1$ $w_5 = v_3$, $w_6 = v_2$, $w_7 = v_4$, $w_8 = v_5$. Notice that $(w_1, \dots, w_8)$ is a path such that $w_iw_{i+2} \in E(G)$ for every $1 \leq i \leq 6$.

Let $Q = (z_1, \dots, z_k)$ be a path of G such that $z_iz_{i+2} \in E(G)$ for every $1 \leq i \leq k-2$, and choose k maximum. The path P shows that $k \geq 8$. Let $L = G[z_1, \dots, z_k]$. We claim:

$$L \text{ is isomorphic to } C_k^2. \tag{6.6}$$

Since $\Delta(G) = 4$ and C_k^2 is 4-regular, this implies that L is a component of G. As G is connected, we have $G = L$ and this ends the proof of Proposition 6.3.7. We now prove (6.6).

Since z_2 is a central vertex of the diamond induced by $\{z_1, z_2, z_3, z_4\}$, we have $d_G(z_2) = 4$. Furthermore, $N_G(z_2) \subseteq V(L)$: we could otherwise use a vertex $z_0 \in N_G(z_2) \setminus V(L)$ to extend Q (z_0 must be adjacent to z_1 because G is claw-free) and contradict the maximality of k. For the same reason, $N_G(z_{k-1}) \subseteq V(L)$.

Now, notice that for every $i \in \{3, \dots, k-2\}$: $d_L(z_i) = 4$. Hence, at least one of z_k and z_{k-1} is a neighbor of z_2. However, z_2z_{k-1} cannot be an edge of G: the vertices z_1, z_4 and z_{k-1} would otherwise be three pairwise non-adjacent neighbors of z_2 and G would contain an induced claw: a contradiction.

Therefore, $z_2 z_k \in E(G)$. Similarly, $d_G(z_{k-1}) = 4$ and $z_1 z_{k-1} \in E(G)$. Since G is claw-free, this implies that $z_1 z_k \in E(G)$ (see Figure 6.7). Finally, the map $i \to z_i$ defines an isomorphism from C_k^2 to a subgraph of L. Since C_k^2 is 4-regular, L is in fact isomorphic to C_k^2.

■

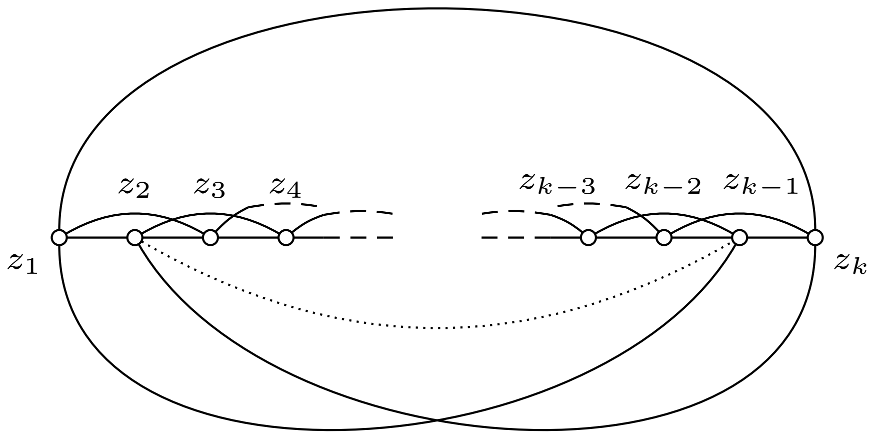


Figure 6.7 – CASE 2. building an induced C_k^2 (with $k \geq 8$) from Q. The dotted line indicates that $z_2 z_{k-1} \notin E(G)$

6.4 MINMAX FORMULAE AND ALGORITHMIC REMARKS

In this section, we first obtain an explicit formula for the weighted chromatic number of t-perfect claw-free graphs and h-perfect line-graphs. We also give a corresponding formula for the chromatic index of an odd-C_5^+-free graph without referring to its line graph. Finally, we discuss the algorithmic aspects of our results.

Using Proposition 3.6.15, Theorems 6.1.2 and 6.1.4, we obtain:

COROLLARY 6.4.1 *Let G be a graph and $c \in \mathbb{Z}_+^{V(G)}$. If G is a t-perfect claw-free graph or an h-perfect line-graph, then:*

$$\chi(G,c) = \max(\omega(G,c), \lceil \Gamma(G,c) \rceil).$$

It is easy to check that $\Gamma(G,\mathbf{1}) \leq 3$, thus we obtain the 3-coloring result of Bruhn and Stein (Theorem 6.3.1) in another way. We expect that a similar result holds for h-perfect claw-free graphs in general.

In Section 6.2.3, we used the fact that every graph H satisfies the inequality $\chi'_f(H) \geq \max(\Delta(H), \Gamma'(H))$. In fact, equality holds for each odd-C_5^+-free graph:

PROPOSITION 6.4.2 *Every odd-C_5^+-free graph H satisfies:*

$$\chi'_f(H) = \max(\Delta(H), \Gamma'(H)).$$

PROOF – By Theorem 3.7.2, $\chi'_f(H) = \max(\Delta(H), \sigma(H))$ where:

$$\sigma(H) := \max\left\{\frac{2|E(F)|}{|V(F)|-1} : F \text{ 2-connected factor-critical subgraph of } G\right\}.$$

Let F be a 2-connected factor-critical subgraph of G and let F' be the underlying simple graph of F. By Theorem 3.7.3, F' has an open odd ear-decomposition D. Clearly, if D has at least two ears, then F' contains an odd-C_5^+. Hence D has exactly one ear, that is F' is a circuit and F is a ring.

Therefore, $\sigma(H) \leq \Gamma'(H)$. The converse inequality follows since rings are 2-connected factor-critical graphs.

■

Using this and Theorem 6.1.3, we get:

COROLLARY 6.4.3 *Every odd-C_5^+-free graph H satisfies:*

$$\chi'(H) = \max(\Delta(H), \lceil\Gamma'(H)\rceil).$$

We now discuss how the terms of the formula of Corollary 6.4.3 are related. The difference $\Gamma' - \Delta$ can be arbitrarily large for odd-C_5^+-free graphs. Indeed, let m be a positive integer and let the graph H_m be obtained as follows (see also Figure 6.8): start with a circuit C of length 5, replace every edge of C with m parallel edges and add a new vertex $v \notin V(C)$ adjacent to exactly two non-adjacent vertices of C.

Clearly, H_m is odd-C_5^+-free and $\Delta(H_m) = 2m+1$ whereas $\Gamma'(H_m) = \frac{5m}{2}$.

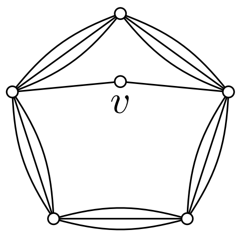


Figure 6.8 – the graph H_3

As a first algorithmic remark, notice that it is straightforward to turn the proofs of Proposition 6.2.4 and Lemma 6.2.2 into a polynomial-time algorithm which computes an optimal edge-coloring of an odd-C_5^+-free graph.

The *maximum-weight stable set problem* can be formulated as follows: considering a graph G and a weight $c \in \mathbb{Z}_+^{V(G)}$, find a stable set

S of G such that $c(S)$ is maximum. Grötschel, Lovász and Schrijver [56] proved that *this problem is solvable in polynomial-time for h-perfect graphs*. Since we only consider claw-free graphs, we can use a specific algorithm (whose construction is more elementary):

THEOREM 6.4.4 (MINTY [83], SBIHI [97], NAKAMURA AND TAMURA [85]) *The maximum-weight stable set problem can be solved in polynomial-time in the class of claw-free graphs.*

By results of Grötschel, Lovász and Schrijver [57], this implies that *there exists a polynomial-time algorithm to find the weighted fractional chromatic number of (G, c) for every claw-free graph G and every $c \in \mathbb{Z}_+^{V(G)}$*. Adding a rounding-step to this algorithm and using theorems 6.1.2 and 6.1.4, we obtain:

COROLLARY 6.4.5 *There exists a polynomial-time algorithm which computes $\chi(G, c)$ for every graph G and every weight $c \in \mathbb{Z}_+^{V(G)}$ such that G is either an h-perfect line-graph or a t-perfect claw-free graph.*

We do not know of a *combinatorial* polynomial-time algorithm which computes the fractional chromatic number of a t-perfect graph. Furthermore, contrarily to the case of perfect graphs, our results do not directly yield an efficient combinatorial algorithm to find an *optimal coloring* of the input graph.

6.5 A QUESTION ON THE INTEGER DECOMPOSITION PROPERTY

An integral matrix is *totally unimodular* if its square submatrices all have determinant 0,-1 or 1. A polyhedron $P \subseteq \mathbb{R}^n$ is *totally unimodular* if there exist a totally unimodular matrix A and an integral vector b such that $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$. As in Section 3.1, we say that a map $\mathbb{R}^n \to \mathbb{R}^m$ with $n \geq m$ is a *projection* if it only deletes some of the coordinates. A *projection of a polyhedron* is its image by a projection.

Recall that a polyhedron $P \subseteq \mathbb{R}^n$ has the *integer decomposition property* (abbreviated *IDP*) if for each positive integer k, every integral vector of kP is the sum of k integral vectors of P.

Baum and Trotter [5] showed that *totally unimodular polyhedra have the IDP*. Furthermore, Sebő proved:

THEOREM 6.5.1 (SEBŐ [103]) *Each projection of a totally unimodular polyhedron has the integer decomposition property.*

Results of Gerards [50] imply that *for certain graphs which do not contain an odd subdivision of K_4 (including almost-bipartite graphs), the stable set polytope of G is the projection of a totally unimodular polyhedron*. Besides, these graphs are t-perfect [50].

By Theorem 6.5.1, this cannot hold for all t-perfect graphs. Indeed, some of them do not have the IRCN (see Section 6.1) and thus their stable set polytope cannot have the IDP (Theorem 3.4.3).

Does the converse statement of Theorem 6.5.1 hold? In other words, *is it true that each polyhedron which has the integer decomposition property is the projection of a totally unimodular polyhedron ?*

In this section, we observe that a result and an example due to Gijswijt and Regts [53] directly imply that the answer is negative. Then, we show the useful properties of this example (they are stated without proof in [53]).

The example of [53] is built from a counter-example by Bruns et al. [17] to a conjecture of Sebő on Hilbert bases of rational cones (see below). Consider the following vectors of $\mathbb{R}^6$:

$$\begin{array}{ll} y^1 = (0,1,1,0,0,0) & y^6 = (1,0,0,1,0,1) \\ y^2 = (0,1,1,1,0,0) & y^7 = (1,0,0,0,1,0) \\ y^3 = (0,1,0,1,1,0) & y^8 = (1,0,1,0,0,1) \\ y^4 = (0,1,0,0,1,1) & y^9 = (1,0,0,1,0,0) \\ y^5 = (0,1,0,0,0,1) & y^{10} = (1,0,1,0,1,0) \end{array} .$$

For each $i \in [10]$, let z^i be obtained from y^i by deleting the first coordinate. Let Q be the convex hull of $z^1, \dots, z^{10}$.

A polyhedron $P \subseteq \mathbb{R}^n$ has the *integer Carathéodory property* (abbreviated *ICP*) if: for every positive integer k and each integral vector w of kP, there exist non-negative integers $\lambda_1, \dots, \lambda_{n+1}$ and affinely independent integral vectors $w^1, \dots, w^{n+1}$ of P such that $\lambda_1 + \dots + \lambda_{n+1} = k$ and $w = \sum_{i=1}^{n+1} \lambda_i w^i$.

Gijswijt and Regts state (without proof) in [53] that Q *has the IDP and not the ICP*. Furthermore, they extended Theorem 6.5.1 as follows:

THEOREM 6.5.2 (GIJSWIJT, REGTS [53]) *Each projection of a totally unimodular polyhedron has the integer Carathéodory property.*

Therefore, this and the polytope Q directly show:

PROPOSITION 6.1.5 *There exists a 0-1 polytope which has the integer decomposition property and is not the projection of a totally unimodular polyhedron.*

We now give a proof that the polytope Q has the claimed properties. We need definitions and a result of [17].

Let $w^1, \dots, w^k$ be rational vectors of $\mathbb{R}^n$. A *non-negative combination* of $w^1, \dots, w^k$ is a vector of the form $\sum_{i=1}^k \lambda_k w^k$, where $\lambda_1, \dots, \lambda_k$ are non-negative real numbers. It is *integral* if $\lambda_1, \dots, \lambda_k$ are integers.

Let X be a finite set of rational vectors. The *cone generated by* X is the set of non-negative combinations of the elements of X.

The set X is a *Hilbert basis* if each integral vector of the cone generated by X is an integral non-negative combination of the elements of X.

It is straightforward to check that *the convex hull of X has the IDP if and only if $\{(1\ x)\colon\ x \in X\}$ is a Hilbert basis* (where $(1\ x)$ denotes the vector of $\mathbb{R}^{n+1}$ obtained by adding a new first coordinate equal to 1).

The cone of $\mathbb{R}^6$ generated by $y^1, \ldots, y^{10}$ is denoted C_6.

THEOREM 6.5.3 (BRUNS ET AL. [17]) *The set $\{y^1, \ldots, y^{10}\}$ is a Hilbert basis, and C_6 contains an integral vector y^* which cannot be expressed as a non-negative integer combination of at most 6 vectors y^i ($i \in [10]$).*

We can now prove the stated properties of Q:

PROPOSITION 6.5.4 *Q has the integer decomposition property.*

PROOF – For each $i \in [10]$, let $x^i := (1\ z^i)$. We need only to show that $\{x^1, \ldots, x^{10}\}$ is a Hilbert basis (see above).

Let w be an integral vector of the cone generated by $x^1, \ldots, x^{10}$. There exist non-negative reals $\lambda_1, \ldots, \lambda_{10}$ such that $w = \sum_{i=1}^{10} \lambda_i x^i$.

Put $w' := \sum_{i=1}^{10} \lambda_i y^i$. Clearly, $w' = (w_1 - w_2, w_2, ..., w_6)$ and thus $w' \in C_6 \cap \mathbb{Z}^6$.

By Theorem 6.5.3, the y^i form a Hilbert basis of C_6 and thus there exist non-negative integers $\mu_1, \ldots, \mu_{10}$ such that $w' = \sum_{i=1}^{10} \mu_i y^i$. Hence, $w = \sum_{i=1}^{10} \mu_i x^i$ and we are done.

■

PROPOSITION 6.5.5 *Q does not have the integer Carathéodory property.*

PROOF – Suppose to the contrary that Q has the ICP. Let $\lambda_1, \ldots, \lambda_{10}$ be non-negative reals such that $y^* = \sum_{i=1}^{10} \lambda_i y^i$. Put $z^* := (y_2^*, \ldots, y_6^*)$ and $k := y_1^* + y_2^*$.

Clearly, $z^* \in kQ \cap \mathbb{Z}^5$. Hence, there exist non-negative integers $\mu_1, \ldots, \mu_6$ and affinely independent integral vectors $u_1, \ldots, u_6$ of Q such that $\sum_{i=1}^{6} \mu_i = k$ and $z^* = \sum_{i=1}^{6} \mu_i u_i$.

Since $Q \subseteq [0,1]^5$, each u_i is a vertex of Q and is among the z_j. Therefore, $y^* = \sum_{i=1}^{6} y^i$ and this contradicts Theorem 6.5.3.

■

7 ON COLORINGS OF H-PERFECT GRAPHS

The integer round-up property for the chromatic number (abbreviated IRCN) was conjectured to hold for t-perfect graphs by Shepherd [63, pg. 144]. This was disproved by Laurent and Seymour [102, pg. 1207] who showed that the complement of the line graph of the prism is t-perfect and 4-chromatic.

Indeed, it is straightforward to check that every t-perfect graph which has the IRCN must be 3-colorable (Corollary 3.6.16).

In [108], Shepherd raised the problem of the converse: does every 3-colorable t-perfect graph have the IRCN? In this chapter, we first show that *the answer is negative in general*.

Our construction uses the complement of the line graph of the prism and the fact that the IRCN is closed under t-contractions (this is a straightforward reformulation of Theorem 4.1.1).

We say that a graph is *complement-line* if it is the complement of a line graph. In [108], Shepherd described the stable set polytope of complement-line graphs. We use this description to obtain *an excluded-induced-subgraph characterization of h-perfect complement-line graphs G such that for every induced subgraph H of G:* $\chi(H) = \lceil \chi_f(H) \rceil$. We show that deciding this property (including the construction of an optimal coloring) can be done in polynomial-time. These results involve a *new example of a t-perfect 4-chromatic graph*: the complement of the line-graph of W_5.

Sebő showed that the $(\omega + 1)$-colorability of h-perfect graphs would follow from the case $\omega \leq 2$ (see [16]). Partial results on the chromatic number of triangle-free t-perfect graphs are in Marcus' thesis [79] but no constant bound is known. Using t-contractions, we first observe that *studying the chromatic number of t-perfect triangle-free graphs can be restricted to graphs whose vertices are covered by 5-circuits*.

On the other hand, we show that a result of Randerath, Schiermeyer and Tewes [94] implies that *every h-perfect P_6-free graph G is $(\omega(G) + 1)$-colorable* (this bound is tight). Besides, an algorithm of Randerath and Schiermeyer [93] can be used to build in polynomial-time an $(\omega + 1)$-coloring of h-perfect P_6-free graphs.

La propriété d'arrondi entier du nombre chromatique (abrégé AENC) des graphes t-parfaits a été conjecturée par Shepherd [63, pg. 144], et a été infirmée par Laurent et Seymour ([102, pg. 1207]) : ils ont montré que le complémentaire du graphe adjoint du prisme est t-parfait mais n'est pas 3-colorable.

On vérifie en effet facilement qu'un graphe t-parfait qui a la propriété AENC est nécessairement 3-colorable (Corollaire 3.6.16).

Shepherd a posé le problème de la réciproque dans [108] : *est-il vrai que tout graphe t-parfait et 3-colorable a la propriété AENC ?*

Dans ce chapitre, nous montrons d'abord que la réponse est *négative*.

Nous construisons un graphe t-parfait 3-colorable qui n'a pas la propriété AENC à partir du complémentaire du graphe adjoint du prisme et en utilisant que les t-contractions conservent la propriété AENC (cette dernière assertion est une reformulation triviale du Théorème 4.1.1).

On dit qu'un graphe est *complémentaire-adjoint* s'il est le complémentaire d'un graphe adjoint. Shepherd a donné dans [108] une description complète du polytope des stables des complémentaire-adjoints. Nous utilisons ce résultat pour caractériser *les graphes complémentaire-adjoints t-parfaits dont chaque sous-graphe induit H satisfait $\chi(H) = \lceil \chi_f(H) \rceil$*. Nous montrons que cette propriété peut être décidée (et la construction d'une coloration optimale trouvée) en temps polynomial. Ces résultats mettent en jeu un nouvel exemple de graphe t-parfait 4-chromatique : *le complémentaire du graphe adjoint de W_5*.

Sebő a prouvé que la $(\omega+1)$-colorabilité des graphes h-parfaits découlerait du cas $\omega \leq 2$ [16]. Plusieurs résultats partiels sur le nombre chromatique des graphes t-parfaits sans triangle ont été obtenus dans la thèse de Marcus [79] mais on ne sait pas s'il est borné. Nous remarquons que les t-contractions peuvent être utilisées pour réduire le problème au cas des *graphes sans triangle et dont les sommets sont couverts par les circuits de longueur 5*.

D'autre part, nous montrons qu'un résultat de Randerath, Schiermeyer et Tewes [94] implique que *les graphes h-parfaits sans P_6 (induit) sont $(\omega+1)$-colorables* (la bornée est serrée). Enfin, un algorithme de Randerath et Schiermeyer [93] peut être utilisé pour construire en temps polynomial une $(\omega+1)$-coloration d'un graphe h-parfait sans P_6.

7.1 INTRODUCTION

We first repeat the definitions of the coloring parameters to accommodate the reader.

A graph G has the *integer round-up property for the chromatic number* (IRCN) if for every $c \in \mathbb{Z}_+^{V(G)}$: $\chi(G,c) = \lceil \chi_f(G,c) \rceil$ (see Section 6.1 or Chapter 3 for the definition of the involved parameters). By Theorem 3.4.3, *G has the IRCN if and only if* $\mathsf{STAB}(G)$ *has the integer decomposition property*.

We proved in Chapter 6 that *h-perfect line-graphs and t-perfect claw-free graphs have the IRCN* (Theorem 6.1.2 and Theorem 6.1.4).

The fractional chromatic number of an h-perfect graph satisfies the following formula (see Section 3.6 for the definitions of the parameters):

PROPOSITION 3.6.15 *For every h-perfect graph G and every $c \in \mathbb{Z}_+^{V(G)}$:*

$$\chi_f(G,c) = \max(\omega(G,c), \Gamma(G,c)).$$

Hence, every t-perfect graph G satisfies $\chi_f(G) \leq 3$ and *t-perfect graphs which have the IRCN must be 3-colorable.*

In fact, Proposition 3.6.15 implies:

PROPOSITION 7.1.1 *Let G be a t-perfect graph. The following statements are equivalent:*

i) G is 3-colorable,
ii) every induced subgraph H of G satisfies: $\chi(H) = \lceil \chi_f(H) \rceil$.

Clearly, ii) means that the equality of the IRCN is satisfied for each $c \in \{0,1\}^{V(G)}$. The IRCN (and hence the 3-colorability) of t-perfect graphs was conjectured by Shepherd [63, pg. 144]. Laurent and Seymour ([102], pg. 1207) gave an example of a t-perfect 4-chromatic graph: the complement of the line graph of the prism Π (see Figure 7.1). In particular, *$\overline{L(\Pi)}$ is a t-perfect graph which does not have the IRCN.*

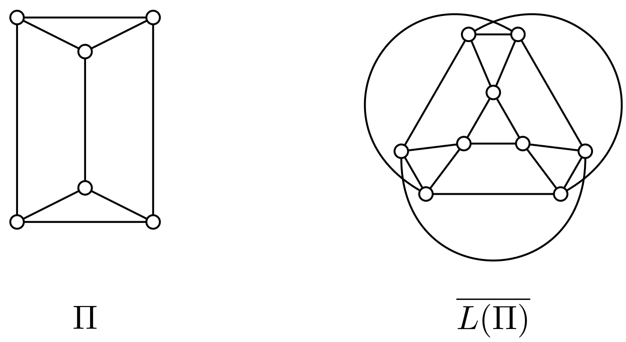


Figure 7.1 – the prism Π and the complement of its line graph

Shepherd [108] raised the following problem: does every 3-colorable t-perfect graph G have the IRCN? By Proposition 7.1.1, this is equivalent to ask whether the IRCN of t-perfect graphs is implied by the 0-1 case for the weights on the vertices.

In this chapter, we prove that the answer is negative.

THEOREM 7.1.2 *The graph of Figure 7.2 is a 3-colorable t-perfect graph which does not have the integer round-up property for the chromatic number.*

The *replication lemma* for perfect graphs can be stated as follows (see Section 3.5 for further details):

THEOREM 7.1.3 (LOVÁSZ [71]) *Let G be a graph. The following statements are equivalent:*
i) for every induced subgraph H of G: $\chi(H) = \chi_f(H)$,
ii) for every $c \in \mathbb{Z}_+^{V(G)}$, $\chi(G,c) = \chi_f(G,c)$.

The graph of Figure 7.2 shows (through Proposition 7.1.1) that there is no statement analogous to the replication lemma for the IRCN, even for t-perfect graphs. It has a triangle and we do not know whether each triangle-free 3-colorable t-perfect graph has the IRCN.

Kilakos and Marcotte [67] gave an example of a graph G and a weight $c \in \mathbb{Z}_+^{V(G)}$ satisfying $\chi(G,c) > \lceil \chi_f(G,c) \rceil$ whereas every induced subgraph H of G satisfies $\chi(H) = \lceil \chi_f(H) \rceil$. Their example is not h-perfect.

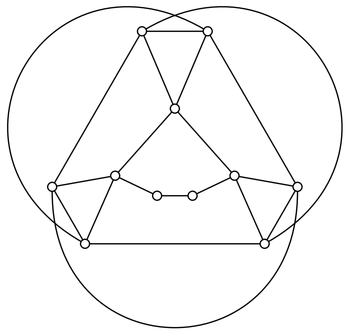


Figure 7.2 – an example of a 3-colorable t-perfect graph which does not have the IRCN

We do not know if there exists a constant $c \in \mathbb{R}$ such that every h-perfect graph G satisfies $\chi(G) - \chi_f(G) \leq c$. The graph $\overline{L(\Pi)}$ shows that such a constant must be at least 1. We will see below that a conjecture of Sebő [16] would imply that $c = 1$ is true. Besides, the complexity of determining the chromatic number of a t-perfect graph is not known. This motivated us to examine the chromatic number of h-perfect complements of line graphs.

We say that a graph is *complement-line* if it is the complement of a line graph. In [108], Shepherd gave a complete description of the stable set polytope of complement-line graphs (it is a direct corollary of Theorem 5.2.5). As a consequence, he characterized h-perfection for those graphs.

We use this result to show that $\overline{L(W_5)}$ (shown in Figure 7.3) is another 4-chromatic t-perfect graph and:

THEOREM 7.1.4 *For every h-perfect complement-line graph G, the following statements are equivalent:*

i) every induced subgraph H of G satisfies $\chi(H) = \lceil \chi_f(H) \rceil$*,*

ii) G has no induced $\overline{L(\Pi)}$ *or* $\overline{L(W_5)}$*.*

Our proof directly yields *a polynomial-time algorithm which checks i) in the class of h-perfect complement line-graphs* (and finds an optimal coloring if it holds).

By Proposition 7.1.1, Theorem 7.1.4 implies that *a t-perfect complement-line graph is 3-colorable if and only if it does not have an induced* $\overline{L(\Pi)}$ *or* $\overline{L(W_5)}$. Furthermore, the thesis of Marcus [79] shows that *every t-perfect complement-line graph is 4-colorable.* Hence, the algorithm can be used to *compute the chromatic number of a t-perfect complement-line graph in polynomial-time*: it finds either a 3-coloring or an induced $\overline{L(\Pi)}$ or $\overline{L(W_5)}$.

A graph G is *nearly-bipartite* if for every $v \in V(G)$, the graph $G - N_G[v]$ is bipartite It is straightforward to check that complement-

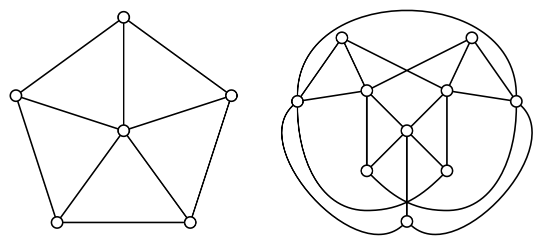


Figure 7.3 – the 5-wheel and the complement of its line-graph

line graphs form a subclass of nearly-bipartite graphs. Shepherd [108] characterized t-perfection in nearly-bipartite graphs in terms of forbidden induced subgraphs. We do not know if Theorem 7.1.4 can be extended to nearly-bipartite graphs nor if part i) of Theorem 7.1.4 can be replaced by the IRCN under the same assumptions.

The only known examples of non-3-colorable t-perfect graphs which are minimal for vertex-deletion are $\overline{L(\Pi)}$ and $\overline{L(W_5)}$. They both have a triangle. Sebő conjectures the following:

CONJECTURE 7.1.5 (SEBŐ, IN [16]) *Each t-perfect triangle-free graph is 3-colorable.*

Moreover, he observed that *if $\mathcal{C}$ is a class of graphs closed under taking induced subgraphs and if every t-perfect triangle-free graph of $\mathcal{C}$ is 3-colorable, then every h-perfect graph G of $\mathcal{C}$ must satisfy* $\chi(G) \leq \omega(G) + 1$. In this case, it is easy to see that each graph G of $\mathcal{C}$ has $\chi(G) \leq \lceil \chi_f(G) \rceil + 1$.

Marcus' thesis [79] contains partial results on Conjecture 7.1.5. We observe that *the conjecture may be restricted to graphs whose vertex set is covered by 5-circuits*.

Finally, we observe that a result of [94] implies that *Conjecture 7.1.5 holds for the class of P_6-free graphs* (a graph is *P_6-free* if it does not have an induced path of length 5). Since this class is closed under taking induced subgraphs, we obtain:

THEOREM 7.1.6 *Each h-perfect P_6-free graph G satisfies* $\chi(G) \leq \omega(G) + 1$.

Results of [93] imply a polynomial-time algorithm for finding an $(\omega + 1)$-coloring, and the graph $\overline{L(\Pi)}$ (or $\overline{L(W_5)}$) shows that this bound is tight (see Section 7.5.3).

We end this introduction with a summary of related results and the outline of the chapter.

Sbihi and Uhry [98] proved that a t-perfect graph whose odd holes are all of the same length is 3-colorable. A graph G is *hereditary t-perfect* if each subgraph of G is t-perfect. Gerards and Shepherd [51] proved that *hereditary t-perfect graphs are the graphs which do not contain* *a non-t-perfect subdivision of* K_4 (these subdivisions were characterized by Barahona and Mahjoub [3]). They showed that these graphs can be recognized in polynomial-time and that they are 3-colorable (an optimal coloring can be built efficiently). These graphs include series-parallel graphs [12] and odd-K_4-free graphs [50].

On the other hand, Bruhn and Stein [16] proved that *every h-perfect claw-free graph* G *satisfies* $\chi(G) = \lceil \chi_f(G) \rceil$ and gave a polynomial-time algorithm to compute the chromatic number of these graphs (the algorithm is combinatorial in the t-perfect case and uses the Ellipsoid Method otherwise).

OUTLINE We prove Theorem 7.1.2 in Section 7.2.

In Section 7.3, we state Shepherd's description of the stable set polytope of complement-line graphs and the consequent characterization of the h-perfection of $\overline{L(G)}$ in terms of the structure of G. We also give the definitions of the parameters which translate the coloring problem on $\overline{L(G)}$ to a covering problem for the edge set of G with stars and triangles. We check that $\overline{L(\Pi)}$ and $\overline{L(W_5)}$ are 4-chromatic t-perfect graphs (this is new for W_5) and give a min-max theorem for the covering problem for graphs whose complement-line graph is h-perfect.

In Section 7.4, we prove Theorem 7.1.4 and give a polynomial-time algorithm which builds the corresponding coloring. We also discuss possible extensions of to the weighted case.

Section 7.5 surveys previous results on Conjecture 7.1.5. We remark that it can be reduced to graphs whose vertex-set is covered by 5-circuits. Finally, we show that it holds for P_6-free graphs and ask a few related questions.

7.2 INTEGER ROUND-UP PROPERTY AND 3-COLORINGS

In this section, we first clarify the relation between 3-colorability and the IRCN for t-perfect graphs and then show an example of a 3-colorable t-perfect graph which *does not* have the IRCN.

For every graph G , let $\Gamma(G)$ denote the maximum of the ratio $\frac{2|V(C)|}{|V(C)|-1}$ over every odd hole C of G. Specialized to the unweighted case, the formula for the weighted fractional chromatic number of an h-perfect graph (Proposition 3.6.15) gives:

PROPOSITION 7.2.1 *Every h-perfect graph* G *satisfies:*

$$\chi_f(G) = \max(\omega(G), \Gamma(G)).$$

This easily shows that *every t-perfect graph* G *satisfies* $\chi_f(G) \leq 3$. In particular, *if a graph is t-perfect and has the IRCN, then it is 3-colorable*. In fact, it is straightforward to check that Proposition 7.2.1 implies:

Proposition 7.1.1 *Let G be a t-perfect graph. The following statements are equivalent:*

i) G is 3-colorable,
ii) every induced subgraph H of G satisfies: $\chi(H) = \lceil \chi_f(H) \rceil$.

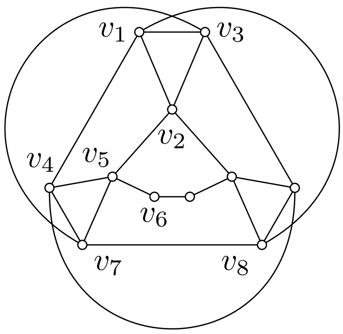
That is, the 3-colorability of a t-perfect graph is equivalent to the equality of the IRCN for 0-1 weights.

Let Q be the graph of Figure 7.4a. It is 3-colorable (as shows Figure 7.4b).

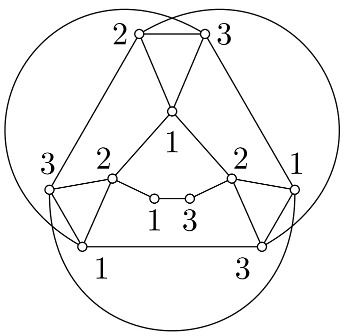
Theorem *Q is a 3-colorable t-perfect graph which does not have the integer round-up property for the chromatic number.*

Clearly, Proposition 7.1.1 is an NP-characterization of the 0-1 case of the IRCN for t-perfect graphs. The graph Q shows that it does not extend to arbitrary weights. Still, we do not know if the IRCN for t-perfect graphs is in NP.

We now prove Theorem 7.1.2:



(a) The graph Q



(b) A 3-coloring of Q

Figure 7.4

Theorem 4.1.1 states that *the integer decomposition property of the stable set polytope is closed under t-minors* (see Section 4.1 for the definition of t-minors). Besides, Theorem 3.4.3 by Baum and Trotter shows that for every graph G, $\mathsf{STAB}(G)$ *has the integer decomposition property if and only if G has the integer round-up property for the chromatic number*. Therefore, Theorem 4.1.1 is equivalent to the following statement:

THEOREM 7.2.2 *The integer round-up property for the chromatic number is closed under t-minors.*

To show that Q does not have the IRCN, we will use the following observation due to Laurent and Seymour :

PROPOSITION 7.2.3 (LAURENT, SEYMOUR [102]) *The graph $\overline{L(\Pi)}$ is a non-3-colorable t-perfect graph.*

A proof of this is given in Section 7.3. We now show:

PROPOSITION 7.2.4 *Q does not have the integer round-up property for the chromatic number.*

PROOF – As $\overline{L(\Pi)}$ is a non-3-colorable t-perfect graph (Proposition 7.2.3), it cannot have the IRCN.

Besides, $\overline{L(\Pi)}$ is obtained from Q by t-contracting v_6 (or v_7). Thus, Theorem 7.2.2 shows that Q cannot have the IRCN.

■

Since $\overline{L(\Pi)}$ is not 3-colorable (Proposition 7.2.3), the graph Q shows that *3-colorability is not closed under taking t-contractions for t-perfect graphs*.

The t-perfection of Q follows from the following two results (the first is a special case of Proposition 3.6.13). A clique of a graph G is *nice* if it meets every inclusion-wise maximal stable set of G.

PROPOSITION 7.2.5 *Let G be a t-perfect graph and let K be a clique of G. If K is nice and $G - v$ is t-perfect for every $v \in K$, then G is also t-perfect.*

A graph is *almost-bipartite* if it has a vertex whose deletion yields a bipartite graph.

THEOREM 3.6.14 (FONLUPT, UHRY [44]) *Every almost-bipartite graph is t-perfect.*

We now show:

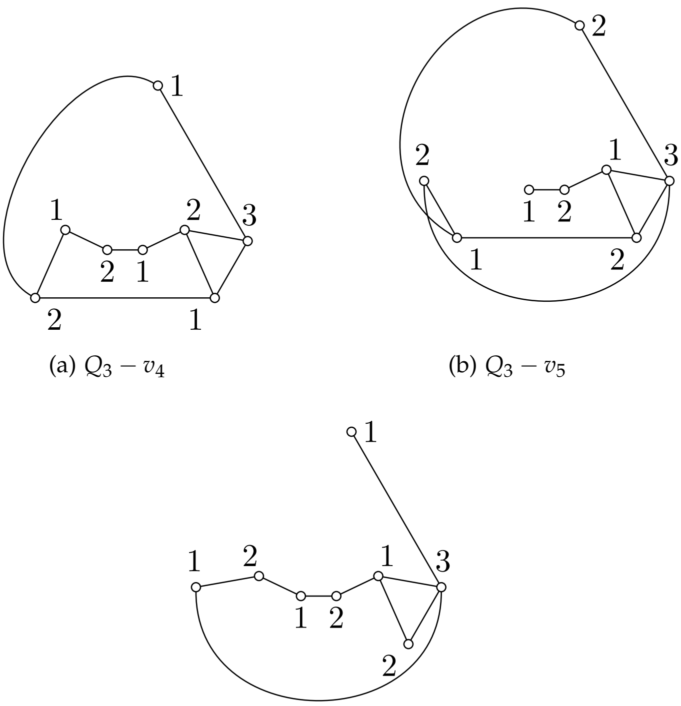
PROPOSITION 7.2.6 *The graph Q is t-perfect.*

PROOF – We use the numbering of the vertices of Q given in Figure 7.4.

It is straightforward to check that $\{v_1, v_2, v_3\}$ is a nice clique of Q. We prove that the graphs $Q - v_1$, $Q - v_2$ and $Q - v_3$ are t-perfect. By

Proposition 7.2.5, this will imply the t-perfection of Q. Since $Q - v_1$ and $Q - v_3$ are obviously isomorphic, we only need to show the t-perfection of $Q - v_1$ and $Q - v_2$.

We first show that $Q - v_2$ is t-perfect. Put $Q_2 := Q - v_2$. Clearly, $v_1 v_3$ is a nice clique of Q_2. Moreover, $Q_2 - v_1$ and $Q_2 - v_3$ are isomorphic. Hence by Proposition 7.2.5, the t-perfection of Q_2 follows from the t-perfection of $Q_2 - v_1$. Put $Q_3 := Q_2 - v_1$. It is easy to check that $\{v_4, v_5, v_{10}\}$ is a nice clique of Q_3. Using Proposition 7.2.5 again, the t-perfection of Q_2 follows from the t-perfection of $Q_3 - v_4$, $Q_3 - v_5$ and $Q_3 - v_{10}$. Figure 7.5 shows that these graphs are almost-bipartite and Theorem 3.6.14 implies that they are t-perfect.



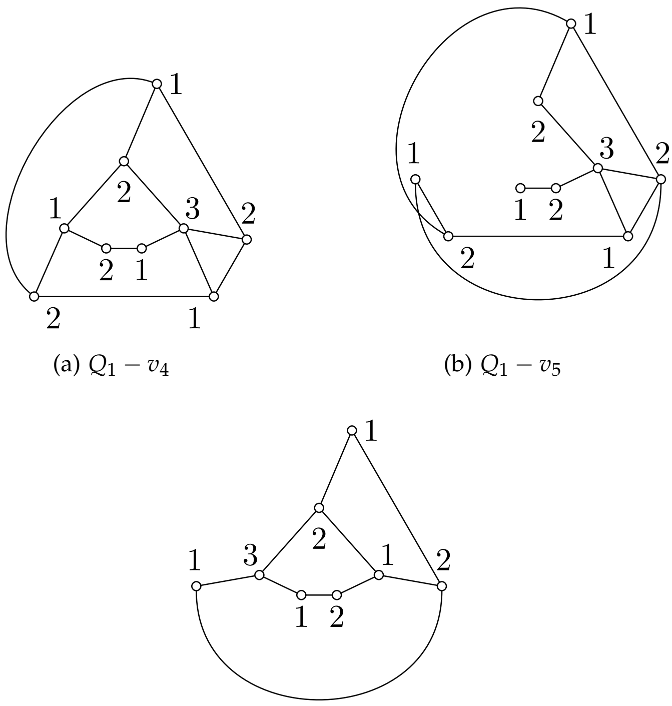
(a) $Q_3 - v_4$ (b) $Q_3 - v_5$ (c) $Q_3 - v_{10}$

Figure 7.5 – a 3-coloring with one color used exactly once

We now prove that $Q - v_1$ is t-perfect. Put $Q_1 := Q - v_1$. It is easy to show that $\{v_4, v_5, v_{10}\}$ is a nice clique of Q_1. Figure 7.6 shows that $Q_1 - v_4$ and $Q_1 - v_5$ are almost-bipartite, hence they are t-perfect. Furthermore, $Q_1 - v_{10} - v_{11}$ is almost-bipartite (see Figure 7.6) and thus t-perfect. Since $Q_1 - v_{10}$ is a clique-sum of this graph and a triangle, Corollary 3.6.12 is also t-perfect.

Therefore, Proposition 7.2.5 implies that Q_1 is t-perfect.

■



(a) $Q_1 - v_4$ (b) $Q_1 - v_5$

(c) $Q_1 - v_{10} - v_{11}$

Figure 7.6

7.3 THE STRUCTURE OF H-PERFECT COMPLEMENT-LINE GRAPHS

7.3.1 *Small graphs and* ST*-covers*

In [108], Shepherd gave a complete description for the stable set polytope of complement-line graphs (as a corollary of Theorem 5.2.5).

A *set-join* of a graph G is a set $\{X_1, \ldots, X_l\}$ of pairwise-disjoint (possibly empty) subsets of vertices of G such that for all $1 \leq i < j \leq l$ and for each $(u, v) \in X_i \times X_j$, we have $uv \in E(G)$. The inequality associated to this set-join is:

$$\sum_{i=1}^{k} \frac{1}{\alpha(G[X_i])} x(X_i) \leq 1.$$

It is obviously valid for $\mathsf{STAB}(G)$.

Theorem 7.3.1 (Shepherd [108]) *Let G be a complement-line graph. The non-trivial facets of* $\mathsf{STAB}(G)$ *are defined by inequalities of set-joins of cliques and odd antiholes of G.*

A set-join of a clique and $\overline{C_5}$ (that is C_5) forms obviously a W_5, which is h-imperfect (Proposition 5.2.2). Moreover, the odd antiholes $\overline{C_{2n+1}}$ with $n \geq 3$ are h-imperfect (Proposition 5.4.4).

Since h-perfection is closed under taking induced subgraphs (Proposition 3.6.3), this theorem directly implies:

COROLLARY 7.3.2 (SHEPHERD [108]) *A complement-line graph is h-perfect if and only if it does not have an induced W_5 or $\overline{C_{2n+1}}$ with $n \geq 3$.*

This characterization can be directly translated in terms of the source graph. We say that a graph H is *small* if each odd circuit of H has length at most 5 and every edge of H is incident to every 5-circuit of H.

COROLLARY 7.3.3 *Let H be a graph. The following statements are equivalent:*

i) $\overline{L(H)}$ *is h-perfect,*

ii) H *is small.*

We will use this result in Section 7.4 to characterize the h-perfect complement-line graphs G such that every induced subgraph H of G satisfies $\chi(H) = \lceil \chi_f(H) \rceil$.

We now give the definitions of the parameters involved in coloring the vertices of h-perfect complement-line graphs. For our purposes, we need only to consider simple graphs for the source-graphs of complement-line graphs (see Section 7.4). Therefore, the definitions and results only involve *simple graphs*.

Let H be a simple graph. A *full star* is a set $\delta_H(v)$, where $v \in V(H)$. A *star* is a subset of a full star.

A set $\mathcal{S}$ of stars and triangles of H (here triangles are identified to their edge-sets) is an *ST-cover* of H if each edge of H belongs to at least (equivalently, to exactly) one element of $\mathcal{S}$. Let $\gamma(H)$ denote the minimum cardinality of an ST-cover of H. Furthermore, let $\nu(H)$ denote the largest number of edges of a matching of H. Clearly, $\omega(\overline{L(H)}) = \nu(H)$. The following equality is also straightforward:

PROPOSITION 7.3.4 *For every simple graph H: $\chi(\overline{L(H)}) = \gamma(H)$.*

This does not hold for non-simple graphs. Indeed, consider the graph H obtained from a triangle by adding a single parallel edge. Clearly, $\chi(\overline{L(H)}) = 1$ whereas $\gamma(H) = 2$.

We use these notions and Corollary 7.3.3 to give a proof of the following:

PROPOSITION 7.3.5 (SEYMOUR, LAURENT [102]) *The graph $\overline{L(\Pi)}$ is t-perfect and 4-chromatic.*

PROOF – The prism Π has 6 vertices so it is obviously small and $\omega(\overline{L(\Pi)}) = \nu(\Pi) = 3$. Hence, Corollary 7.3.3 shows that $\overline{L(\Pi)}$ is t-perfect.

We prove that $\gamma(\Pi) = 4$. Let T_1 and T_2 be the two triangles of Π. Clearly, $\{E(T_1)\} \cup \{\delta_\Pi(v) : v \in T_2\}$ is an ST-cover of Π and thus

$\gamma(\Pi) \leq 4$. To see that $\gamma(\Pi) \geq 4$, consider the perfect matching M of Π formed by the edges joining its two triangles and let $\mathcal{F}$ be an ST-cover of Π. Since the edges of M do not belong to triangles, $\mathcal{F}$ must cover M with three distinct stars. It is easy to check that for any choice of such stars, there is always an edge of a triangle of Π which remains uncovered. Hence $\mathcal{F}$ must contain one more star or triangle and $|\mathcal{F}| \geq 4$. Therefore, $\gamma(\Pi) = 4$.

■

Finally, we show that $\overline{L(W_5)}$ is a new example of a 4-chromatic t-perfect graph. In Section 7.4, we will obtain that these two graphs are the only t-perfect complement-line 4-chromatic graphs whose proper induced subgraphs are 3-colorable.

PROPOSITION 7.3.6 *The graph $\overline{L(W_5)}$ is t-perfect and 4-chromatic.*

PROOF – The t-perfection is shown as for $\overline{L(\Pi)}$, hence we need only to show that $\gamma(W_5) = 4$.

Let T be a triangle of W_5. Clearly, $\{T\} \cup \{\delta_{W_5}(v) \colon v \notin T\}$ is an ST-cover of W_5, hence $\gamma(W_5) \leq 4$.

It is straightforward to check that W_5 cannot be covered by 3 stars. Thus, each ST-cover of W_5 has size at least 4 or contains a triangle. Besides, for each triangle T of W_5, the graph $W_5 - E(T)$ has a matching of cardinality 3 and thus $\gamma(W_5 - E(T)) \geq 3$. This shows $\gamma(W_5) \geq 4$ and the proposition.

■

7.3.2 *The Cunningham-Marsh formula for small graphs*

In this section, we specialize to small simple graphs the theorem of Cunningham and Marsh [31] stating the total dual-integrality of the matching polytope (see Section 3.3.1 and Section 3.7 for the definitions of total dual-integrality and the matching polytope of a graph). We will use the formula obtained in this way to prove Theorem 7.1.4 in Section 7.4.

For every graph G and $c \in \mathbb{Z}_+^{E(G)}$, let $\nu(G,c)$ denote the maximum of $c(M)$ over all matchings M of G. In particular, $\nu(G) = \nu(G, \mathbf{1})$. A graph G is *factor-critical* if for every $v \in V(G)$, the graph $G - v$ has a perfect matching.

THEOREM 7.3.7 (CUNNINGHAM, MARSH [31]) *For every graph H and each $c \in \mathbb{Z}_+^{E(H)}$, the number $\nu(H,c)$ is the minimum of*

$$|\mathcal{U}| + \sum_{F \in \mathcal{F}} \frac{|V(F)| - 1}{2}$$

over all pairs of a multiset $\mathcal{U}$ of vertices of H and a multiset $\mathcal{F}$ of 2-connected factor-critical induced subgraphs of H such that:

$$\sum_{u\in\mathcal{U}} \chi^{\delta_H(u)} + \sum_{F\in\mathcal{F}} \chi^{E(F)} \geq c.$$

Furthermore, a pair $(\mathcal{U}, \mathcal{F})$ attaining the minimum can be found in polynomial-time.

We use the following theorem of Lovász to precise the structure of 2-connected factor-critical subgraphs of small graphs (see Section 3.2.2 for definitions of ear-decompositions):

THEOREM 3.7.3 (LOVÁSZ [72, 76]) *A 2-connected graph is factor-critical if and only if it has an open odd ear-decomposition.*

This implies:

PROPOSITION 7.3.8 *A 2-connected factor-critical subgraph of a small simple graph is a triangle or has a spanning 5-circuit.*

PROOF – Let H be a 2-connected factor-critical subgraph of a small simple graph and suppose that H is not a triangle.

Since H is small, simple and not a triangle, Theorem 3.7.3 easily shows that H must have exactly 5 vertices.

Besides, an open odd ear-decomposition of H starts with either a 5-circuit, or a triangle followed by an ear of length 3. In both cases, this shows a spanning 5-circuit of H.

■

Let H be a simple graph. We say that an induced subgraph of H is a *full-C_5* if it has a spanning 5-circuit. An *odd cover* of a graph H is a set $\mathcal{S}$ of stars, triangles and full-C_5 subgraphs of H such that each $e \in E(G)$ is an edge of at least one element of $\mathcal{S}$. The *cost* of a star or triangle is 1, and the cost of a full-C_5 is 2.

The *cost of an odd cover* is the sum of the costs of its members. Let $\gamma_o(H)$ denote the minimum cost of an odd cover of H. An ST-cover is clearly an odd cover, hence: $\gamma(H) \geq \gamma_o(H)$.

Combining Theorem 7.3.7 with Proposition 7.3.8 yields directly the following min-max formula for small-graphs:

COROLLARY 7.3.9 *For every small simple graph H: $\nu(H) = \gamma_o(H)$.*

7.4 COLORINGS OF H-PERFECT COMPLEMENT-LINE GRAPHS

7.4.1 *The main result and a corollary*

Proposition 7.2.1 shows that $\overline{L(\Pi)}$ and $\overline{L(W_5)}$ satisfy $\chi = \lceil \chi_f \rceil + 1$ (see Figures 7.1 and 7.3 for Π and W_5).

By contrast, *every h-perfect line-graph G satisfies $\chi(G) = \lceil \chi_f(G) \rceil$* [16].

The thesis of Marcus [79] shows that *every h-perfect complement-line graph G satisfies* $\chi(G) \leq \lceil \chi_f(G) \rceil + 1$. The aim of this section is to prove the following:

Theorem 7.1.4 *For every h-perfect complement-line graph G, the following statements are equivalent:*

i) every induced subgraph H of G satisfies $\chi(H) = \lceil \chi_f(H) \rceil$,

ii) G has no induced $\overline{L(\Pi)}$ or $\overline{L(W_5)}$.

The graphs Π and W_5 are not subgraphs of each other, hence they are both necessary in this statement.

A possible extension of Theorem 7.1.4 to arbitrary weights (that is to the IRCN) is discussed in Section 7.4.3.

To prove Theorem 7.1.4 for a graph G, we work in a graph H such that $G = \overline{L(H)}$. The structure of H is given by Corollary 7.3.3.

Hence, we consider small graphs and use the parameter γ (see Section 7.3 for their definitions). For every graph H, let $\gamma_f(H) := \chi_f(\overline{L(H)})$.

The main argument is stated in the following lemma, whose proof is postponed to the next section.

Lemma 7.4.1 *Let H be a small simple graph. If H does not contain Π or W_5, then $\gamma(H) = \lceil \gamma_f(H) \rceil$.*

We now show that this lemma implies Theorem 7.1.4.

Let G be a graph. *Adding a sibling to G* means substituting a vertex of G with two non-adjacent vertices (see Section 4.5). It is easy to check that if G' is obtained by adding siblings to G, then $\chi(G') = \chi(G)$ and $\chi_f(G') = \chi_f(G)$.

Proof (of Theorem 7.1.4) – The implication i)=>ii) is immediate as both $\overline{L(\Pi)}$ and $\overline{L(W_5)}$ have a fractional chromatic number equal to 3 and are 4-chromatic.

Conversely, let G be an h-perfect complement-line graph which does not have an induced $\overline{L(\Pi)}$ or $\overline{L(W_5)}$.

We show that $\chi(G) = \lceil \chi_f(G) \rceil$ and this will prove ii)=>i). Let H be a graph such that $G = \overline{L(H)}$ and let H' be the underlying simple graph of H. Put $G' := \overline{L(H')}$. Clearly, G is obtained from G' by adding siblings. Hence, $\chi(G) = \chi(G')$ and $\chi_f(G) = \chi_f(G')$. Thus we need only to prove that $\chi(G') = \lceil \chi_f(G') \rceil$.

By Corollary 7.3.3, H is small. Since being small does not depend on the multiplicity of edges, H' is small too.

As G has no induced $\overline{L(\Pi)}$ or $\overline{L(W_5)}$, H' cannot contain the prism or the 5-wheel. Hence, Lemma 7.4.1 implies that $\gamma(H') = \lceil \gamma_f(H') \rceil$. Since H' is simple, Proposition 7.3.4 shows that $\chi(G') = \gamma(H')$. Besides, $\gamma_f(H') = \chi_f(G')$ so we finally obtain $\chi(G') = \lceil \chi_f(G') \rceil$ as required.

■

We end this section with corollaries of Theorem 7.1.4. First, Proposition 7.1.1 directly implies:

COROLLARY 7.4.2 *For every t-perfect complement-line graph G, the following statements are equivalent:*

i) $\chi(G) \leq 3$,

ii) *G does not have an induced $\overline{L(\Pi)}$ or $\overline{L(W_5)}$.*

As we shall see in the next section, the proof of Lemma 7.4.1 can be easily converted into a polynomial-time algorithm which finds an optimal ST-cover of a small simple graph which does not contain Π or W_5. Since a source-graph of a line-graph can be computed in polynomial-time [96], we obtain:

COROLLARY 7.4.3 *The chromatic number and an optimal coloring of an h-perfect complement-line graph which does not have an induced $\overline{L(\Pi)}$ or $\overline{L(W_5)}$ can be computed in polynomial-time (with a combinatorial algorithm).*

The thesis of Marcus [79] shows that every t-perfect complement-line graph is 4-colorable. Since deciding whether a graph contains Π or W_5 can be obviously carried out in polynomial-time, Corollary 7.4.2 implies:

COROLLARY 7.4.4 *The chromatic number a t-perfect complement-line graph can be obtained in polynomial-time.*

7.4.2 *Proof of Lemma 7.4.1*

Recall that $\gamma(H)$ (resp. $\gamma_o(H)$) respectively denote the minimum-cost of an ST-cover (resp. odd cover) of a graph H. Clearly, every graph H satisfies $\gamma(H) \geq \gamma_o(H)$.

The following is straightforward:

PROPOSITION 7.4.5 *γ is non-decreasing for the subgraph relation.*

We now prove Lemma 7.4.1:

LEMMA 7.4.1 *Let H be a small simple graph. If H does not contain Π or W_5, then $\gamma(H) = \lceil \gamma_f(H) \rceil$.*

PROOF – Clearly, we may assume without loss of generality that H has no isolated vertex. It is straightforward to check that $\gamma(H) \geq \lceil \gamma_f(H) \rceil$. We prove the converse inequality.

First, suppose that $\gamma(H) = \gamma_o(H)$. By Corollary 7.3.9, $\gamma(H) = \nu(H)$. On the other hand, $\overline{L(H)}$ is h-perfect (Corollary 7.3.3). Therefore, Proposition 7.2.1 implies:

$$\gamma_f(H) = \chi_f(\overline{L(H)}) \geq \omega(\overline{L(H)}) = \nu(H)$$

and $\gamma(H) \leq \lceil \gamma_f(H) \rceil$ as required.

Hence, we may assume from now on that $\gamma(H) > \gamma_o(H)$. In particular, H contains a 5-circuit and thus $\overline{L(H)}$ has an induced C_5. Using Proposition 7.2.1 for $\overline{L(H)}$, we obtain $\lceil \gamma_f(H) \rceil \geq \lceil \Gamma(\overline{L(H)}) \rceil \geq 3$.

We will prove:

$$H \text{ has an } ST\text{-cover of cardinality } 3. \tag{7.1}$$

This will imply $\gamma(H) \leq 3$ and end the proof of the lemma.

Let $\mathcal{S}$ be an odd cover of minimum cost. Since $\gamma(H) > \gamma_o(H)$, $\mathcal{S}$ must contain a full-C_5 K. First, if $\mathcal{S} = \{K\}$, then H is a subgraph of K_5 and it is easy to check that $\gamma(K_5) \leq 3$. Thus, Proposition 7.4.5 shows that $\gamma(H) \leq 3$ and we are done.

Therefore, we may henceforth assume that $\mathcal{S} \neq \{K\}$.

Claim 1. The cost of $\mathcal{S}$ is 3.

Proof – Obviously, the cost of $\mathcal{S}$ is at least 3. Let M be a maximum matching of H. By Theorem 7.3.7, $\mathcal{S}$ has cost $|M|$ and this implies (by complementary slackness (Corollary 3.3.2)) that $|M \cap E(K)| = 2$.

Since H is small, M cannot have more than one edge which does not belong to $E(K)$. Hence, $|M| \leq 3$ and the claim follows.

□

In particular, M has an edge e which has exactly one end in K. Let u and v be the ends of e, with $u \in V(K)$. Put $u_1 := u$ and let $C = (u_1, \ldots, u_5)$ be a spanning 5-circuit of K.

Claim 2. $V(H) \setminus V(K) = \{v\}$.

Proof – Since $\mathcal{S}$ has cost 3, it has exactly one element U other than K, and U must have cost 1. Thus U is either a star or the edge-set of a triangle. Let $N := \{e, u_2u_3, u_4u_5\}$. Since N is a maximum matching of H and $e \notin E(K)$, Corollary 7.3.9 implies (through complementary slackness) that $e \in U$.

Therefore, if U is a star then $U \subseteq \delta_H(u_1)$ or $U \subseteq \delta_H(v)$. Else, U is a triangle which has two vertices on K (otherwise it would have an edge with no end in K contradicting that H is small).

Therefore, in both cases every edge of H which is not an edge of K is incident to at least one of u_1 and v. The claim follows as no vertex of H is isolated by assumption.

□

We are now ready to build an ST-cover of H with 3 elements, as required by (7.1). Our construction depends on the degree of v in H.

Since H has no isolated vertex, $d_H(v) \geq 1$. As H has no W_5 subgraph, we also have $d_H(v) \leq 4$.

Case 1. $d_H(v) = 1$.

If $u_2u_5 \notin E(H)$, then we take the full stars at u_1, u_3 and u_4 to form a convenient ST-cover of H. Otherwise, we simply replace the full star at u_1 by the triangle $u_1u_2u_5$ to get the required ST-cover.

Case 2. $d_H(v) = 2$.

First, suppose that the neighbors of v on C are consecutive. Without loss of generality, we may assume that $N_G(v) = \{u_1, u_2\}$. If $u_3u_5 \notin E(H)$, then the full stars at u_1, u_2 and u_4 form a convenient ST-cover. Otherwise, we replace u_4 with the triangle $u_3u_4u_5$.

Now, suppose that the neighbors of v on C are not consecutive. By symmetry, we may assume that $N_G(v) = \{u_1, u_3\}$. If both u_2u_5 and u_2u_4 are edges of H, then the triangle $u_2u_4u_5$ and the full stars at u_1, u_3 form an ST-cover cover with 3 elements. Hence, we may assume that one of these two edges do not belong to H. By symmetry again, we may suppose that $u_2u_5 \notin E(H)$. Then, the full stars at u_1, u_3 and u_4 form an ST-cover of H as required.

Case 3. $d_H(v) = 3$.

Similarly, we first suppose that the neighbors of v are consecutive on C. Without loss of generality, we have $N_H(v) = \{u_1, u_2, u_5\}$. Since H does not contain W_5, at most one of u_1u_3 and u_1u_4 is an edge of H. By symmetry, we may assume that $u_1u_4 \notin E(H)$. Furthermore, since H does not contain Π, we have $u_2u_4 \notin E(H)$. Thus the triangle vu_1u_2 together with the two full stars at u_3 and u_5 form an ST-cover of H as required.

Now, suppose that the neighbors of v are not consecutive on C. Without loss of generality, we may assume that $N_H(v) = \{u_1, u_3, u_4\}$. Since H does not contain Π, we have $u_2u_5 \notin E(H)$. Hence, the triangle vu_3u_4 and the full stars at u_3 and u_4 form a convenient ST-cover at v.

Case 4. $d_H(v) = 4$.

Without loss of generality, we may assume that u_2 is the vertex of C which is not a neighbor of v. Since H does not contain Π, both u_2u_4 and u_2u_5 cannot be edges of H. Hence, the triangle vu_4u_5 together with the two full stars at u_1 and u_3 form an ST-cover as required.

■

It is straightforward to transform our proof into a combinatorial polynomial-time algorithm which finds an optimal ST-cover for each small simple graph H which does not contain Π or W_5.

First, find a maximum matching M and a minimum-cost odd cover $\mathcal{S}$ of H (using the algorithm of Theorem 7.3.7). If $\mathcal{S}$ has no full-C_5, then it is the required ST-cover and M certifies its optimality. Otherwise if H has at most 5 vertices, then it is easy to build an optimal ST-cover of H from one for K_5. Else, let K be the full-C_5 of H, v be the unique vertex of H which does not belong to it and follow the cases of the proof above to build an ST-cover of H with 3 elements (a certificate of optimality being a 5-circuit).

By contrast, *finding an optimal ST-cover of a graph is NP-hard in general*. Indeed, in triangle-free graphs it reduces to finding a minimum-cardinality vertex-cover. It is well-known that the latter is NP-hard in this class [91].

7.4.3 *Towards the integer round-up property*

In this section, we discuss a possible extension of Theorem 7.1.4 to the weighted case. In Chapter 6, we proved that every h-perfect line-graph has the IRCN.

In Section 7.2, we gave an example of a 3-colorable t-perfect graph which does not have the IRCN. This example is not complement-line. Hence, the following is still open:

Question 7.4.6 *Is it true that each 3-colorable t-perfect complement-line graph has the integer round-up property for the chromatic number?*

More generally, is it true that each h-perfect complement-line graph which does not have an induced $\overline{L(\Pi)}$ or $\overline{L(W_5)}$ has the integer round-up property for the chromatic number?

We now formulate this question in terms of the source graphs. We will use the following well-known fact, which is a straightforward consequence of the forbidden-induced-subgraph characterization of line-graphs due to Beineke [6] (it can also be proved directly using Theorem 6.3.8):

Proposition 7.4.7 *C_5 is the only odd antihole which is a line-graph.*

Let H be a simple graph and $c \in \mathbb{Z}_+^{E(H)}$. An ST-cover of (H, c) is a multiset $\mathcal{F}$ of stars and edge-sets of triangles of H such that each edge e of H belongs to at least c_e members of $\mathcal{F}$.

Let $\gamma(H, c)$ denote the minimum cardinality of an ST-cover of (H, c). It is straightforward to check that $\gamma(H, c) = \chi(\overline{L(H)}, c)$. As before, we put $\gamma_f(H, c) := \chi_f(\overline{L(H)}, c)$.

Besides, let $\Gamma'_5(H, c)$ denote the maximum ratio of $\frac{c(E(C))}{2}$ over all 5-circuits C of H. The maximum value of $c(M)$ over all matchings M of H is denoted $\nu(H, c)$ as usual.

Since the complement of the line-graph of a small graph is h-perfect (Corollary 7.3.3), Proposition 3.6.15 and Proposition 7.4.7 straightforwardly imply:

Proposition 7.4.8 *For every small simple graph H and every $c \in \mathbb{Z}_+^{E(H)}$:*

$$\gamma_f(H, c) = \max(\nu(H, c), \Gamma'_5(H, c)).$$

Hence, the second part of Question 7.4.6 asks:

Question 7.4.9 *Is it true that every small simple graph H which does not contain Π or W_5 satisfies:*

$$\gamma(H, c) = \max(\nu(H, c), \lceil \Gamma'_5(H, c) \rceil) \text{ ?}$$

We do not know whether the proof techniques developed in Section 7.4.2 could be extended to answer this.

Recall that a graph is *nearly-bipartite* if every vertex has a neighbor on each odd circuit. Clearly, every complement-line graph is nearly-bipartite. In [108], Shepherd showed that *a nearly-bipartite graph is t-perfect if and only if it does not have an induced odd wheel or prime antiweb other than an odd hole* (see also Section 5.2.2). We do not know if Theorem 7.1.4 further holds for nearly-bipartite graphs.

7.5 ON COLORINGS OF H-PERFECT GRAPHS IN GENERAL

No constant bound is currently known for the chromatic number of a t-perfect graph. The graphs $\overline{L(\Pi)}$ and $\overline{L(W_5)}$ show the largest known value of the chromatic number of t-perfect graphs, which is 4. In particular, the chromatic number of a t-perfect graph cannot always be obtained by rounding-up its fractional chromatic number.

In Section 7.5.1, we survey linear-programming arguments to obtain bounds for the chromatic number of an h-perfect graph.

In Section 7.5.2, we state a related conjecture of Sebő for triangle-free graphs and show that it can be reduced to the case of C_5-covered graphs.

Finally in Section 7.5.3, we observe that the conjecture holds for the class of P_6-free graphs.

7.5.1 *A survey of results on the chromatic number of h-perfect graphs*

Using decomposition techniques along vertex-cuts, Gerards and Shepherd [51] showed the 3-colorability of t-perfect graphs whose subgraphs are all t-perfect and gave a combinatorial polynomial-time algorithm to optimally color those graphs.

In this section, we survey results on bounds for the chromatic number of h-perfect graphs which arise from linear programming (as in [16].

Let $\mathcal{C}$ be a class of graphs. We say that $\mathcal{C}$ is *hereditary* if it is closed under taking induced subgraphs. Sebő showed that a bound for the chromatic number of an hereditary class of h-perfect graphs can be obtained from a bound for the *triangle-free case* (that is when $\omega \leq 2$). We first state this result and give a proof.

It is obtained by a repeated application of the following well-known and simple observation:

Proposition 7.5.1 *For every h-perfect graph G:*

i) If $\omega(G) \geq 3$, G has a stable set intersecting each maximum clique of G.

ii) If $\omega(G) \leq 2$ and G is non-bipartite, then G has a stable set intersecting each odd circuit C of minimum length on $\frac{|V(C)|-1}{2}$ vertices.

Proof – Suppose that G is h-perfect. Let $\mathcal{S}$ be the set of stable sets of G. By the duality theorem of linear programming (Theorem 3.3.1), we have:

$$\max\left\{\mathbf{1}^T x\colon x \geq 0;\ x(S) \leq 1\,\forall S \in \mathcal{S}\right\} = \min\left\{\sum_{S\in\mathcal{S}} y_S\colon\ y \geq 0;\ \sum_{S\in\mathcal{S}} y_S \chi^S \geq \mathbf{1}\right\}.$$

Clearly, the right-hand side is the fractional chromatic number of G. Let y be an optimal solution for the minimum and let $S \in \mathcal{S}$ such that $y_S > 0$. By complementary slackness, for every optimal solution x for the maximum:

$$x(S) = 1. \tag{7.2}$$

If $\omega(G) \geq 3$, then Proposition 7.2.1 shows $\chi_f(G) = \omega(G)$. Hence every maximum clique of G is an optimal solution for the maximum and i) follows from (7.2).

Otherwise, suppose that $\omega(G) \leq 2$. Clearly, we may assume that G is non-bipartite. Let l be the minimum length of an odd circuit of G. By Proposition 7.2.1, $\chi_f(G) = \frac{l}{\frac{l-1}{2}}$. Hence for each odd circuit C of length l, the vector $\frac{2}{l-1}\chi^{V(C)}$ is an optimal solution for the maximum and (7.2) implies ii).

■

The most general form of Sebő's result is the following statement (this is essentially in [79]):

Theorem 7.5.2 (Sebő [13, 79]) *Let $\mathcal{C}$ be an hereditary class of h-perfect graphs, $f : \mathcal{C} \to \mathbb{Z}_+$ be non-increasing for the induced-subgraph relation and $k \geq 3$ be an integer.*

If every graph H of $\mathcal{C}$ with no clique of cardinality k has $\chi(H) \leq f(H)$, then each $G \in \mathcal{C}$ with $\omega(G) \geq k$ satisfies:

$$\chi(G) \leq \omega(G) + f(G) - k + 1.$$

Proof – Let G be a graph of $\mathcal{C}$ and put $l := \omega(G)$. Since $\mathcal{C}$ is hereditary, Proposition 7.5.1.i) shows that there exist pairwise-disjoint stable sets $S_1, \ldots, S_{l-k+1}$ of G such that $H := G - S_1 - \cdots - S_{l-k+1}$ has no clique of cardinality k. By assumption, $\chi(H) \leq f(H)$. Therefore: $\chi(G) \leq f(H) + l - k + 1$. As f is non-increasing for the induced-subgraph relation, we have $f(H) \leq f(G)$ and the conclusion follows.

■

Sbihi and Uhry used i) of Proposition 7.5.1 to show:

Theorem 7.5.3 (Sbihi, Uhry [98]) *Every t-perfect simple graph whose induced odd circuits all have the same length is 3-colorable.*

Using Theorem 7.5.2 and the formula for the fractional chromatic number of an h-perfect graph (Proposition 7.2.1), this directly yields:

COROLLARY 7.5.4 *Every h-perfect simple graph G whose induced odd circuits all have the same length satisfies* $\chi(G) = \lceil \chi_f(G) \rceil$.

We end this section with an algorithmic remark on Theorem 7.5.2. Using a well-known technique, Bruhn and Stein [16] obtained the following algorithmic version of Proposition 7.5.1.i):

THEOREM 7.5.5 *For every h-perfect graph G with* $\omega(G) \geq 3$, *a stable set intersecting every maximum clique of G can be computed in polynomial-time.*

For a class $\mathcal{C}$ of graphs and an integer $k \geq 1$, let $\mathcal{C}_{\geq k}$ (resp. $\mathcal{C}_{<k}$) denote the class of graphs of $\mathcal{C}$ which have (resp. do not have) a clique of cardinality k. The theorem above directly implies the following algorithmic form of Theorem 7.5.2 (this is implicit in [16]):

COROLLARY 7.5.6 *Let* $\mathcal{C}$ *be an hereditary class of h-perfect graphs,* $f : \mathcal{C} \to \mathbb{Z}_+$ *be non-increasing for the induced-subgraph relation and* $k \geq 3$ *be an integer.*

If a coloring of H with $f(H)$ *colors can be computed efficiently for each graph* $H \in \mathcal{C}_{<k}$, *then for each* $G \in \mathcal{C}_{\geq k}$: *an* $(\omega(G) + f(G) - k + 1)$*-coloring of G can be found in polynomial-time.*

This is used in [16] to derive a polynomial-time optimal algorithm for coloring h-perfect claw-free graphs. We also apply this result in the next section to show that an $(\omega + 1)$-coloring of an h-perfect P_6-free graph can be built efficiently.

7.5.2 *A conjecture for triangle-free graphs*

The two known examples of non-3-colorable t-perfect graphs have triangles. Sebő conjectures the following:

CONJECTURE 7.1.5 (SEBŐ, IN [16]) *Each t-perfect triangle-free graph is 3-colorable.*

Theorem 7.5.2 directly shows that the validity of this conjecture would imply the following bound for h-perfect graphs (equivalent to Conjecture 7.1.5):

CONJECTURE 7.5.7 (SEBŐ [16, 79]) *Each h-perfect graph G satisfies:*

$$\chi(G) \leq \omega(G) + 1.$$

Partial results on this conjecture were given by Marcus in her thesis. In particular, she proved that it holds for h-perfect complement-line graphs (see Section 7.4).

PROPOSITION 7.5.8 (MARCUS [79]) *Every t-perfect triangle-free graph with at most 14 vertices and at least one 5-circuit is 3-colorable.*

PROPOSITION 7.5.9 (MARCUS [79]) *Every t-perfect triangle-free graph with at most 28 vertices is 4-colorable.*

In this section, we first show that Conjecture 7.1.5 can be reduced to graphs whose vertices all belong to 5-circuits. This was suggested by Sebő (personal communication). We refer the reader to Section 4.1 for the definitions of contractible vertices and t-contractions.

PROPOSITION 7.5.10 *Let G be a t-perfect graph and v be a contractible vertex of G.*

If $G/N_G[v]$ has at least one edge, then $\chi(G) \leq \chi(G/N_G[v])$.

PROOF – Let $k := \chi(G/N_G[v])$ and $\tilde{v}$ be the new vertex of $G/N_G[v]$. Let f be a k-coloring of $G/N_G[v]$. Put $x := f(\tilde{v})$.

We color each vertex of G with colors used by f as follows: each neighbor of v is colored with x, and v receives another color (which exists since $k \geq 2$). The other vertices keep the color received from f in $G/N_G[v]$. It is straightforward to check that we obtain a k-coloring of G and thus $\chi(G) \leq k$.

■

We say that a graph G is C_5*-covered* if each vertex of G belongs to at least one induced C_5 of G. Using t-contractions, we obtain:

PROPOSITION 7.5.11 *Let $\mathcal{C}$ be an hereditary class of t-perfect triangle-free graphs which is closed under t-contractions.*

If every triangle-free C_5-covered graph of $\mathcal{C}$ is 3-colorable, then every graph of $\mathcal{C}$ is 3-colorable.

PROOF – Suppose that every C_5-covered graph of $\mathcal{C}$ is 3-colorable.

Seeking a contradiction, let G be a non-3-colorable graph of $\mathcal{C}$ with $|V(G)|$ minimum. By assumption, G is not C_5-covered and thus it has a vertex v which does not belong to an induced C_5 of G. In particular, the t-contraction $G/N_G[v]$ is triangle-free.

As $\mathcal{C}$ is closed under t-contractions, $G/N_G[v]$ belongs to $\mathcal{C}$.

The minimality of G shows that $\chi(G/N_G[v]) \leq 3$. Obviously, G has at least one edge and therefore Proposition 7.5.10 implies that $\chi(G) \leq 3$: a contradiction.

■

This directly implies that Conjecture 7.1.5 is equivalent to the following slightly stronger conjecture:

CONJECTURE 7.5.12 *Each t-perfect triangle-free C_5-covered graph is 3-colorable.*

Still, we do not know how to use this additional structure to eventually prove Conjecture 7.1.5.

7.5.3 *The case of P_6-free graphs*

Let k be a positive integer. A graph is *P_k-free* if it does not have an induced path with k vertices. We end this chapter by observing that a result of Randerath, Schiermeyer and Tewes implies the validity of Conjecture 7.1.5 for P_6-free graphs.

Classes of graphs defined by excluding long induced paths received considerable attention in graph theory and combinatorial optimization: several NP-complete problems have been proven to be polynomial-time-solvable in the class of P_k-free graphs for small values of k. A recent example is the polynomial-time algorithm of Lokshtanov, Vatshelle and Villanger [70] for the maximum-weight stable set problem in P_5-free graphs.

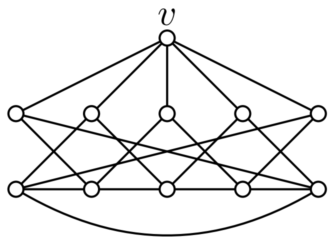


Figure 7.7 – the Mycielski-Grötzsch graph

The *Mycielski-Grötzsch graph* is shown in Figure 7.7. It is the image of the 5-circuit in the construction due to Mycielski to obtain triangle-free graphs with arbitrary large chromatic number [84].

It is straightforward to check that it is triangle-free, 4-chromatic, P_6-free and that a t-contraction at v yields W_5 (after deleting loops and parallel edges). Since W_5 is t-imperfect and as t-perfection is closed under t-contractions (Theorem 4.1.2): *the Mycielski-Grötzsch graph is t-imperfect*.

Two vertices u and v of a graph G are *similar* if $N_G(v) \subseteq N_G(u)$ or $N_G(u) \subseteq N_G(v)$. Randerath, Schiermeyer and Tewes proved the following:

Theorem 7.5.13 (Randerath, Schiermeyer, Tewes [94]) *Let G be a connected triangle-free and P_6-free non-3-colorable graph.*

If G does not have a pair of similar vertices, then it has an induced Mycielski-Grötzsch graph.

We show that this implies Conjecture 7.1.5 for P_6-free graphs:

Corollary 7.5.14 *Every t-perfect triangle-free P_6-free graph is 3-colorable.*

Proof – Let G be a t-perfect triangle-free and P_6-free graph which is not 3-colorable and with $|V(G)|$ minimum.

Clearly, G is connected: otherwise its components are t-perfect, triangle-free, P_6-free and the minimality of G implies that they are all 3-colorable. Hence, G must be 3-colorable: a contradiction.

Furthermore, G has no pair u and v of similar vertices: otherwise, a 3-coloring of $G-u$ could be straightforwardly extended into a 3-coloring of G (similar vertices cannot be adjacent) and this would contradict $\chi(G) > 3$.

Hence Theorem 7.5.13 shows that G has an induced Mycielski-Grötzsch graph. As we observed above, it is not t-perfect and this contradicts the t-perfection of G.

■

Since the class of P_6-free graphs is obviously hereditary, Theorem 7.5.2 implies directly:

Corollary 7.5.15 *Every h-perfect P_6-free graph G is $(\omega(G)+1)$-colorable. This bound is tight.*

It is straightforward to check that line graphs cannot have an induced $\overline{P_6}$ (which is shown in Figure 7.8). Hence, *complement-line graphs are P_6-free* and *$\overline{L(\Pi)}$ and $\overline{L(W_5)}$ show the tightness of the bound stated above.*

Besides, this corollary clearly implies that every h-perfect P_6-free graph G satisfies $\chi(G) \leq \lceil \chi_f(G) \rceil + 1$. Hence, it extends the result of Marcus' thesis for h-perfect complement-line graphs (see Section 7.4.1).

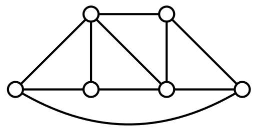


Figure 7.8 – the graph $\overline{P_6}$

By contrast, the current best upper bound known for the chromatic number of P_6-free graphs is given by a result of Gyárfás [58]: *every P_6-free graph G is $5^{\omega(G)-1}$-colorable.*

In [93], Randerath and Schiermeyer gave a combinatorial polynomial-time algorithm which decides whether a P_6-free graph is 3-colorable and finds a 3-coloring if it exists. Combining this algorithm with Corollary 7.5.6, we get:

Corollary 7.5.16 *An $(\omega+1)$-coloring of an h-perfect P_6-free graphs can be computed in polynomial-time.*

We do not know if h-perfection can be tested in polynomial-time in the class of P_5-free graphs (see Section 5.3 for further details on this problem). Furthermore, both $\overline{L(\Pi)}$ and $\overline{L(W_5)}$ have an induced P_5. Hence, we ask the following:

Question 7.5.17 *Does each h-perfect P_5-free graph G has:*

$$\chi(G) = \lceil \chi_f(G) \rceil ?$$

Notice that a positive answer would not directly imply the IRCN for h-perfect P_5-free graphs. Indeed, the class of h-perfect P_5-free graphs is not closed under substitutions of vertices by complete graphs (as shows W_5^-, which is t-imperfect and obtained from C_5 by substituting a vertex by a K_2).

Esperet et al. [40] proved that *every K_4-free and P_5-free graph is 5-colorable*, improving a bound of Gyárfás [58].

Besides, Corollary 7.5.15 implies that every t-perfect P_5-free graph is 4-colorable. A positive answer to the question above would imply that this bound can be reduced to 3.

8 EAR-DECOMPOSITIONS AND H-PERFECTION IN LINE-GRAPHS

The results of this chapter are the subject of [8].

Let C_3^+ denote the graph obtained from the triangle by adding a single parallel edge. An *odd-C_3^+* is a totally odd subdivision of C_3^+. A graph is *odd-C_3^+-free* if it does not contain an odd-C_3^+.

Using Theorem 3.8.1 by Cao and Nemhauser, it is not difficult to show that testing h-perfection in line-graphs reduces to deciding whether a graph is odd-C_3^+-free.

Results of Kawarabayashi, Reed, Wollan [66] (see also Huynh [62]) imply that *recognizing odd-C_3^+-free graphs can be done in polynomial-time*. Still, they build upon the general techniques of the Graph Minor Project of Robertson and Seymour. Hence, it is natural to ask for a more adapted algorithm.

Bruhn and Schaudt [14] showed *a simpler polynomial-time algorithm which checks whether a graph with maximum degree 3 is odd-C_3^+-free*. By Theorem 3.8.1, this algorithm tests t-perfection in line-graphs.

In this chapter, we first prove *a good characterization of odd-C_3^+-free graphs in terms of bases of their cycle space*. Through ear-decompositions, it implies *a rather simple and elementary polynomial-time algorithm deciding whether a graph (with arbitrary degrees) is odd-C_3^+-free*. We use this algorithm to test efficiently h-perfection in line-graphs.

The study of odd-C_3^+-free graphs motivates the introduction of a new graph parameter. For each graph G, let $\beta(G)$ denote the largest integer k such that G has a subgraph which has an open odd ear-decomposition with k ears. For example, G is odd-C_3^+-free if and only if $\beta(G) \leq 1$.

The computational complexity of determining β is not known. We observe that the results of Kawarabayashi, Reed, Wollan and Huynh imply that the problem is *Fixed-Parameter-Tractable*. We also state a conjecture relating β to edge-colorings and the integer decomposition property of the matching polytope.

For each 2-connected graph G, $\beta(G)$ is the largest number of odd ears starting an open ear-decomposition of G. *Is $\beta(G)$ always near the largest number of odd ears in an open ear-decomposition of G ?* The latter parameter, denoted $\overline{\varphi}(G)$, was introduced by Frank [47] (in the equivalent form of the smallest number of even ears).

We answer this question negatively by showing a sequence $(H_k)_{k\geq 1}$ of 2-connected graphs such that $\beta(H_k) = 2$ whereas $\overline{\varphi}(H_k) \to \infty$.

Kawarabayashi, Lee and Reed [65] gave a polynomial-time algorithm deciding whether a graph contains a totally odd subdivision of K_4. This algorithm uses techniques of the Graph Minor Project. We show that $\overline{\varphi}$ can be used to build *a simpler algorithm for this problem in the class of graphs satisfying* $\beta \leq 1$.

Cao's thesis [18] contains several results and statements on simple odd-C_3^+-free graphs (that is the *odd-C_5^+-free* graphs of Chapter 6). They suggest the relation with totally odd subdivisions of K_4 and motivated the use of $\overline{\varphi}$ in our study of odd-C_3^+-free graphs. We give counter-examples to some of these statements.

Les résultats de ce chapitre font l'objet de [8].

Notons C_3^+ le graphe obtenu du triangle en ajoutant une seule arête parallèle. Un C_3^+*-impair* est une subdivision totalement impaire de C_3^+. Un graphe est *sans* C_3^+*-impair* s'il n'a pas de sous-graphe isomorphe à un C_3^+-impair.

Il n'est pas difficile de voir que le Théorème 3.8.1 de Cao et Nemhauser implique que tester la h-perfection du graphe adjoint d'un graphe G se réduit à décider si G est sans C_3^+-impair.

Des résultats de Kawarabayashi, Reed, Wollan [66] (voir aussi Huynh [62]) impliquent que *les graphes sans* C_3^+*-impair peuvent être reconnus en temps polynomial*. Cependant, ces résultats se fondent sur les techniques générales du Graph Minor Project de Robertson et Seymour et il convient donc de chercher un algorithme plus adapté.

Bruhn et Schaudt [14] ont donné un algorithme plus simple pour décider si un graphe de degré maximum 3 est sans C_3^+-impair. D'après le Théorème 3.8.1, cet algorithme reconnaît efficacement la t-perfection dans les graphes adjoints.

Dans ce chapitre, nous prouvons d'abord une bonne caractérisation des graphes sans C_3^+-impair en termes de bases de l'espace des circuits modulo 2. Par les décompositions d'oreilles, ce résultat implique un algorithme plus simple et élémentaire pour la reconnaissance des graphes sans C_3^+-impair (sans restriction sur leurs degrés). Nous utilisons cet algorithme pour *tester efficacement la h-perfection dans la classe des graphes adjoints*.

L'étude des graphes sans C_3^+-impair nous invite à introduire un nouveau paramètre de graphe. Pour tout graphe G, notons $\beta(G)$ le plus grand entier k tel que G a un sous-graphe admettant une décomposition d'oreilles ouvertes à k oreilles. Par exemple, G est sans C_3^+-impair si et seulement si $\beta(G) \leq 1$.

La complexité algorithmique du calcul de β n'est pas connue. Cependant, nous observons que les résultats de Kawarabayashi, Reed, Wollan et Huynh impliquent que *le problème est résoluble en temps polynomial à paramètre fixé*. Nous énonçons de plus une conjecture qui relie β à l'arête-coloration et à la propriété de décomposition entière du polytope des couplages.

Il est clair que lorsque G est un graphe 2-connexe, $\beta(G)$ est simplement le plus grand nombre d'oreilles impaires au début d'une décomposition d'oreilles ouvertes de G. *Le paramètre $\beta(G)$ est-il toujours proche du plus grand nombre d'oreilles impaires dans une décomposition d'oreilles ouvertes de G ?* Cette dernière quantité, notée $\overline{\varphi}(G)$, a été introduite et étudiée par Frank [47] (sous la forme équivalente du plus petit nombre d'oreilles paires).

Nous montrons que la réponse à cette question est *négative* en construisant une suite $(H_k)_{k\geq 1}$ de graphes 2-connexes tels que $\beta(H_k) = 2$ tandis que $\overline{\varphi}(H_k) \to \infty$.

Kawarabayashi, Lee et Reed [65] ont donné un algorithme polynomial pour reconnaître les graphes sans subdivision totalement impaire de K_4. Il utilise des techniques du Graph Minor Project. Nous montrons que $\overline{\varphi}$ peut être utilisée pour construire un algorithme plus simple pour ce problème dans la classe des graphes qui satisfont $\beta \leq 1$.

La thèse de Cao [18] contient plusieurs résultats et énoncés sur les graphes simples sans C_3^+-impair (qui sont les graphes sans C_5^+-impair vus au Chapitre 6). Ces résultats suggèrent que les graphes sans C_3^+-impair et les subdivisions totalement impaires de K_4 sont liées, et ont motivé l'utilisation de $\overline{\varphi}$ dans notre étude. Par ailleurs, nous proposons des contre-exemples à certains énoncés de [18] portant sur les graphes sans C_3^+-impair.

8.1 INTRODUCTION

The graph C_3^+ is shown in Figure 8.1. An *odd-C_3^+* is a totally odd subdivision of C_3^+ (see Section 3.2.1 for the definition of subdivisions). These graphs are also called *skewed thetas* [14]. An odd-C_3^+ is *strict* if it is not C_3^+.

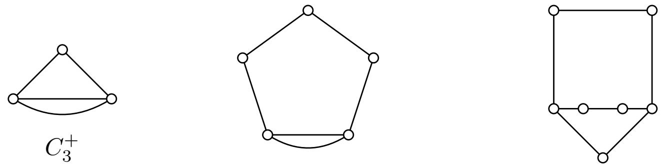


Figure 8.1 – C_3^+ and two odd-C_3^+ graphs

Cao and Nemhauser [19] proved the following characterization of h-perfection in line-graphs:

THEOREM 3.8.1 (CAO, NEMHAUSER [19]) *For every graph H, the following statements are equivalent:*

i) $L(H)$ is h-perfect,

ii) H does not contain a strict odd-C_3^+.

Deciding whether a graph G is a line-graph (and building a graph H such that $G = L(H)$ if it exists) can be done in polynomial-time [96]. Hence, testing h-perfection in line-graphs reduces to detecting strict odd-C_3^+ subgraphs.

A general algorithm of Kawarabayashi, Reed and Wollan (and independently of Huynh in his thesis) implies the following:

THEOREM 8.1.1 (KAWARABAYASHI, REED, WOLLAN [66] AND HUYNH [62]) *Let H be a graph. Deciding whether a graph contains a totally odd subdivision of H can be done in polynomial-time.*

This obviously yields *an efficient algorithm deciding whether a graph contains an odd-C_3^+.*

We will show in Section 8.2.2 that *any polynomial-time algorithm which tests whether a graph is odd-C_3^+-free can be easily used to efficiently detect strict odd-C_3^+ subgraphs*. Hence, Theorem 8.1.1 implies:

COROLLARY 8.1.2 *H-perfection can be tested in polynomial-time in the class of line-graphs.*

By contrast, the computational complexity of deciding t-perfection (and thus h-perfection) in general is unknown.

The algorithms of Kawarabayashi, Reed, Wollan and Huynh are built upon elaborated techniques of the Graph Minor Project of Robertson and Seymour and are oriented towards generality. Therefore, it is natural to ask for simpler and more adapted algorithms testing whether a graph is odd-C_3^+-free.

Bruhn and Schaudt [13] provided *a simpler algorithm for detecting odd-C_3^+ subgraphs in graphs with maximum degree at most 3* and it does not use the techniques of the Graph Minor Project. The degree assumption is crucial and it is not clear whether the method can be extended to arbitrary degrees. Using this algorithm, they proved that t-perfection can be tested in polynomial-time in the class of claw-free graphs (which forms a proper superclass of line-graphs).

In this chapter, we give *a new polynomial-time algorithm for the recognition of odd-C_3^+-free graphs with arbitrary degrees*. It does not use the Graph Minor Project and is rather elementary and simple.

A *cycle* of G is the union of edge-disjoint circuits of G. The *cycle space* of a graph G, denoted $\mathcal{C}(G)$, is the subspace of (the vector space) $\mathbb{F}_2^{E(G)}$ generated by the incidence vectors of edge-sets of cycles of G.

A *cycle basis of* G is a set of cycles whose incidence vectors form a basis of $\mathcal{C}(G)$. A cycle basis is *odd* if all its elements have odd cardinality, and it is *totally odd* if they furthermore pairwise-intersect in an odd number of edges.

The first main result of this chapter is the following characterization of odd-C_3^+-free graphs. We clearly need only to consider 2-connected non-bipartite graphs:

THEOREM 8.1.3 *Let G be a 2-connected non-bipartite graph. The following statements are equivalent:*

i) G is odd-C_3^+-free,
ii) G has a totally odd cycle basis,
iii) each odd cycle basis of G is totally odd.

Clearly, the property of being odd-C_3^+-free is co-NP. The equivalence of i) and ii) provides *an NP-characterization of odd-C_3^+-free graphs*.

It is well-known that bases of the cycle space of a 2-connected graph can be built from ear-decompositions. We use this fact and Theorem 8.1.3 in our polynomial-time algorithm for the recognition of odd-C_3^+-free simple graphs.

Our algorithm can be straightforwardly extended to test also *whether the odd circuits of a binary matroid pairwise-intersect in an odd number of elements* using only a polynomial number of calls to an independence oracle (through the generalization of ear-decompositions to binary matroids by Coullard and Hellerstein [30]). This is detailed in [8].

The rest of this chapter is concerned with statements and questions on a new parameter, which is motivated by the nice properties of odd-C_3^+-free graphs.

For each graph G, let $\beta(G)$ denote the largest integer k such that G contains a graph H having an open odd ear-decomposition with k ears.

For example, a graph G is odd-C_3^+-free if and only if $\beta(G) \leq 1$, and G is bipartite if and only if $\beta(G) = 0$.

We will observe that Theorem 8.1.1 immediately implies: *for each fixed k, deciding whether a graph G satisfies $\beta(G) = k$ can be done in polynomial-time*. In other words:

THEOREM 8.1.4 *Determining β is Fixed-Parameter-Tractable.*

The computational complexity of determining β is unknown:

QUESTION 8.1.5 ([8]) *Can β can be computed in polynomial-time ?*

In fact, we do not know if the property $\beta(G) \geq k$ (for each graph G and integer k) admits a co-NP-characterization (the definition clearly shows that it belongs to NP).

An *odd-C_5^+* is a totally odd subdivision of C_5^+ (the graph C_5^+ is shown in Figure 8.2). Clearly, odd-C_5^+ graphs and simple odd-C_3^+ graphs are the same.

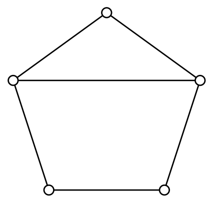


Figure 8.2 – C_5^+

In Chapter 6, we proved that *each graph G which does not contain an odd-C_5^+ satisfies $\chi'(G) = \lceil \chi'_f(G) \rceil$* (Theorem 6.1.3).

Through Theorem 3.4.3 by Baum and Trotter, this easily shows that *for each graph G whose underlying simple graph satisfies $\beta \leq 1$, $\mathsf{MATCH}(G)$ has the integer decomposition property*.

We conjecture that this can be extended as follows:

CONJECTURE *For each graph G whose underlying simple graph satisfies $\beta \leq 3$, the polytope $\mathsf{MATCH}(G)$ has the integer decomposition property.*

If valid, the bound 3 would be optimal. Indeed, let $\mathbf{T}$ be the Petersen graph minus a vertex (see Figure 8.3). It is straightforward to check that $\beta(\mathbf{T}) = 4$, whereas $\chi'_f(\mathbf{T}) = 3$ and $\chi'(\mathbf{T}) = 4$ (see Section 8.3.2.2 for further details). Hence, Theorem 3.4.3 implies that $\mathsf{MATCH}(\mathbf{T})$ does not have the integer decomposition property.

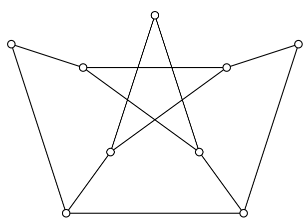


Figure 8.3 – the Petersen graph minus a vertex

For each 2-connected graph G, $\beta(G)$ is obviously the largest number of odd ears which start an open ear-decomposition of G. The other results of this chapter concern the relation between β and the largest number of odd ears (not necessarily at start) in an ear-decomposition. The latter was introduced and studied by Frank [47] (in the equivalent form of the minimum number of even ears).

For each 2-connected graph G, let $\overline{\varphi}(G)$ denote the largest number of odd ears in an open ear-decomposition of G. Clearly, $\overline{\varphi}(G) \geq \beta(G)$.

We show *a sequence of 2-connected graphs* $(H_k)_{k\geq 1}$ *such that* $\beta(H_k) = 2$ *for all k, whereas* $\overline{\varphi}(H_k) \to \infty$. In other words, a large number of odd ears in ear-decompositions does not certify a large value of β in general.

Cao's thesis [18] suggests that totally odd subdivisions of K_4 are related to odd-C_3^+ subgraphs (see Figure 8.4). We show that *totally odd subdivisions of K_4 can be easily detected in polynomial-time in odd-C_3^+-free graphs* (using an algorithm of Frank [47]).

The currently known algorithms for detecting totally odd subdivisions of K_4 in arbitrary graphs are not elementary: Theorem 8.1.1 directly provides one, and Kawarabayashi, Li and Reed [65] gave a simpler and more adapted algorithm. Both use the techniques of the Graph Minor Project.

Our simplification for odd-C_3^+-free graphs is rather specific and does not directly extend to larger values of β. Still, β could possibly be useful in detecting totally odd subdivisions of K_4 in general.

Finally, we review the results of Cao's thesis concerning odd-C_3^+-free graphs and observe that some of the statements are incorrect. In particular, the construction procedure given for odd-C_5^+-free graphs does not work. Still, it motivated the use of $\overline{\varphi}$ in our study odd-C_3^+-free graphs. The final product of this is our algorithm for detecting totally odd subdivisions of K_4 in odd-C_3^+-free graphs.

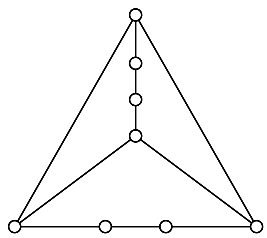


Figure 8.4 – a totally odd subdivision of K_4. Each edge of K_4 is replaced with an odd path

OUTLINE In Section 8.2, we prove Theorem 8.1.3 and build our algorithm for the recognition of odd-C_3^+-free graphs. Furthermore, we show that any efficient algorithm deciding whether a simple graph is odd-C_3^+-free can be used to test h-perfection in polynomial-time in the class of line-graphs.

In Section 8.3, we give the definition and equivalent formulations of β and discuss related problems.

In Section 8.4 we show that $\overline{\varphi}$ can get arbitrarily large while β remains constant, and use results of Frank to design a simpler efficient algorithm to detect totally odd subdivisions of K_4 in odd-C_3^+-free graphs.

Finally, we discuss related statements of Cao's thesis in Section 8.5.

8.2 A NEW ALGORITHM FOR THE RECOGNITION OF ODD-C_3^+-FREE GRAPHS

In Section 8.2.1, we state and prove a good characterization of odd-C_3^+-free graphs after a few preliminary results (whose proofs are postponed to Section 8.2.3). In Section 8.2.2, we use it to show a new polynomial-time algorithm for their recognition and, through Theorem 3.8.1, derive an efficient algorithm testing h-perfection in line-graphs.

8.2.1 *A binary characterization of odd-C_3^+-free graphs*

Let $\mathbb{F}_2$ denote the field of two elements.

Let G be a graph. Clearly, the sum in the vector space $\mathbb{F}_2^{E(G)}$ of the incidence vectors of $F_1 \subseteq E(G)$ and $F_2 \subseteq E(G)$ is the incidence vector of $F_1 \Delta F_2$.

The *cycle space of G*, denoted $\mathcal{C}(G)$, is the subspace of (the vector space) $\mathbb{F}_2^{E(G)}$ spanned by the incidence vectors of edge-sets of circuits of G.

A *cycle* is the union of edge-disjoint circuits of G. It is well-known that $\mathcal{C}(G)$ *is the set of incidence vectors of edge-sets of cycles of G*.

A *cycle basis* of G is a set of cycles $\{C_1, \ldots, C_k\}$ of G such that $\left\{\chi^{E(C_1)}, \ldots, \chi^{E(C_k)}\right\}$ is basis of $\mathcal{C}(G)$.

After stating a few useful results, we prove our characterization of odd-C_3^+-free graphs (Theorem 8.1.3). The proofs of these preliminary statements are postponed to Section 8.2.3.

Since an odd-C_3^+ is 2-connected and non-bipartite, *a graph is odd-C_3^+-free if and only if its non-bipartite blocks are odd-C_3^+-free*. Hence, we need only to consider 2-connected non-bipartite graphs.

Cao's thesis [18] shows that *the odd circuits of a 2-connected odd-C_3^+-free simple graph pairwise-intersect in an odd number of edges*. We first observe that this property characterizes 2-connected odd-C_3^+-free graphs:

LEMMA 8.2.1 *Let G be a 2-connected graph. The following statements are equivalent:*

i) G is odd-C_3^+-free,

ii) for each pair of odd circuits C_1 and C_2 of G: $|E(C_1) \cap E(C_2)|$ is odd.

The proof (given in Section 8.2.3) easily implies that *an odd-C_3^+ of a 2-connected graph can be built efficiently from two odd circuits which meet on an even number of edges*.

Besides, this results shows that deciding whether a graph is odd-C_3^+-free amounts to check that its odd circuits pairwise-intersect in an odd number of edges. Still, this is not an NP-characterization of odd-C_3^+-free graphs as the number of odd circuits in a 2-connected graph may be exponential in the number of vertices and edges.

A cycle of a graph is *odd* if it has an odd number of edges, and a cycle basis of a graph is *odd* if it has only odd elements. The following lemma is crucial in the proof of Theorem 8.1.3. The cycle basis is built using a carefully chosen ear-decomposition of the graph:

Lemma 8.2.2 *Each 2-connected non-bipartite graph has an odd cycle basis formed by circuits only.*

Furthermore, such a cycle basis can be found in polynomial-time.

In general, a cycle basis is not necessarily formed by circuits only. Indeed, each incidence vector of an non-empty cycle obviously belongs to a basis of the cycle space.

An odd cycle basis of a graph is *totally odd* if its elements pairwise-intersect in an odd number of edges. For example, each set of 3 circuits of a totally odd subdivision T of K_4 form a totally odd cycle basis of T (hence the chosen terminology).

The following statement shows that the existence of a totally odd cycle basis constrains odd cycles to pairwise-intersect in an odd number of edges:

Proposition 8.2.3 *If a 2-connected graph has a totally odd cycle basis, then its odd cycles pairwise-intersect in an odd number of edges.*

We now use these results to prove Theorem 8.1.3:

Theorem 8.1.3 *Let G be a 2-connected non-bipartite graph. The following statements are equivalent:*

i) G is odd-C_3^+-free,
ii) G has a totally odd cycle basis,
iii) each odd cycle basis of G is totally odd.

Proof – We first show that i)=>ii). Suppose that G is odd-C_3^+-free. Since G is 2-connected and non-bipartite, Lemma 8.2.2 shows that G has an odd cycle basis $\{C_1, \dots, C_k\}$ such that each C_i ($i \in [k]$) is a circuit.

As G is odd-C_3^+-free, Lemma 8.2.1 shows that the odd circuits $C_1, \dots, C_k$ pairwise-intersect in an odd number of edges. Therefore, the basis $\{C_1, \dots, C_k\}$ is totally odd.

We now prove ii)=>iii). Suppose that G has a totally odd cycle basis $\mathcal{B}$. By Proposition 8.2.3, odd cycles of G pairwise-intersect in an odd number of edges. Since the elements of an odd cycle basis of G are odd cycles, this implies that all odd cycle bases of G must be totally odd.

Finally, we show iii)=>i). Suppose that each odd cycle basis of G is totally odd. Since G is 2-connected and non-bipartite, Lemma 8.2.2 shows that G has an odd cycle basis $\mathcal{B}$. By assumption, $\mathcal{B}$ is totally odd. Hence, Proposition 8.2.3 implies that odd cycles, and in particular odd circuits, pairwise-intersect in an odd number of edges. By Lemma 8.2.1, this shows that G is odd-C_5^+-free.

■

8.2.2 *The algorithm. Recognition of h-perfect line-graphs*

Proving that a graph is odd-C_3^+-free obviously amounts to showing that its non-bipartite blocks are as such. By Theorem 8.1.3, proving that a 2-connected non-bipartite graph G is odd-C_3^+-free can be carried out by providing a set of circuits which forms a totally odd cycle basis of G.

Besides, the following well-known result shows that each cycle basis of G has polynomial-size in the number of vertices and edges of G:

PROPOSITION 8.2.4 *Each connected graph G satisfies:*

$$\dim \mathcal{C}(G) = |E(G)| - |V(G)| + 1.$$

Since checking that a polynomial number of circuits form a totally odd cycle basis of G can be obviously done in polynomial-time, the equivalence of i) and ii) in Theorem 8.1.3 is indeed *an NP-characterization for odd-C_3^+-free graphs*.

We now use Theorem 8.1.3 to show the following new and simpler polynomial-time algorithm for the recognition of odd-C_3^+-free graphs.

Obviously, we may assume that the input graphs are 2-connected (the algorithm can be applied to each block in general).

Algorithm 1: Deciding whether a graph is odd-C_3^+-free

Input: *A 2-connected graph G*
Output: "TRUE" *if G is odd-C_3^+-free, and an odd-C_3^+ of G otherwise*

1 **if** *G is bipartite* **then**
2   **return** TRUE;
3 **else**
4   Find an odd cycle basis $\mathcal{B}$ of G (using Lemma 8.2.2);
5   **for** *each pair of elements of C and D of $\mathcal{B}$* **do**
6   **if** *$|E(C) \cap E(D)|$ is even* **then**
7   Build an odd-C_3^+ H from C and D (see Lemma 8.2.1);
8   **return** H;
9   **return** TRUE;

The correctness of this algorithm is a straightforward corollary of Theorem 8.1.3.

By Lemma 8.2.2, line 4 can be carried out in polynomial-time. Besides, Proposition 8.2.4 shows that there is only a polynomial number of pairs to check at line 5. Finally, an odd-C_3^+ can be built efficiently from two odd circuits meeting on an even number of edges (see Lemma 8.2.1 and its proof). Therefore, *the overall execution-time of the algorithm is polynomial in the size of the input graph.*

We end this section by observing that any algorithm which decides whether a simple graph is odd-C_3^+-free (for example the algorithm above) can be easily used to test h-perfection in line-graphs.

Recall the characterization by Cao and Nemhauser of h-perfect line-graphs. An odd-C_3^+ is *strict* if it is not C_3^+:

THEOREM 3.8.1 (CAO, NEMHAUSER [19]) *For every graph H, the following statements are equivalent:*

i) $L(H)$ is h-perfect,
ii) H does not contain a strict odd-C_3^+.

Since checking whether a graph is a line-graph (and building a corresponding source-graph if it exists) can be done in polynomial-time [96], the problem of deciding h-perfection in line-graphs reduces to the detection of strict odd-C_3^+ subgraphs.

It is easy to check that Theorem 3.8.1 implies that *a line graph $L(H)$ is t-perfect if and only if H is odd-C_3^+-free and* $\Delta(H) \leq 3$. An elementary algorithm for the recognition of odd-C_3^+-free graphs with maximum degree at most 3 was shown by Bruhn and Schaudt [14].

Clearly, the only strict odd-C_3^+ graphs which are not simple are obtained by adding a single parallel edge to an odd circuit of length at least 5. The following proposition shows that it is easy to detect these once simple odd-C_3^+ subgraphs have been cleared out (using ALGORITHM 1 for example):

PROPOSITION 8.2.5 *Let G be a graph whose underlying simple graph is odd-C_3^+-free. The following statements are equivalent:*

i) G does not contain a strict odd-C_3^+,

ii) for each edge $e = uv$ of G which has at least one other parallel edge, the graph $G - (N_G(u) \cap N_G(v))$ does not have a non-bipartite block containing both u and v.

PROOF – Let $e = uv$ be an edge of G which has at least one other parallel edge f. If u and v belong to the same non-bipartite block B of $G - (N_G(u) \cap N_G(v))$, then G contains an even uv-path P which has length at least 4 and $P + e + f$ is a strict odd-C_3^+ of G. This proves i)=>ii).

Conversely, suppose that G contains a strict odd-C_3^+ F. Since the underlying simple graph of G is odd-C_3^+-free by assumption, F must be formed by an odd circuit C of length at least 5 with a single parallel edge $e = uv$.

As G does not contain a simple odd-C_3^+, C cannot have a chord and this shows that no vertex of C is a common neighbor of u and v. In particular, C is entirely contained in a non-bipartite block of $G - (N_G(u) \cap N_G(v))$ and this proves ii)=>i).

■

Hence, testing efficiently whether a graph G (with underlying simple graph H) contains a strict odd-C_3^+ can be done as follows: first, we use an efficient algorithm for the recognition of odd-C_3^+-free simple graphs on H. Then, we test whether property ii) of Proposition 8.2.5 holds (this can be obviously carried out in polynomial-time).

By Theorem 3.8.1, using ALGORITHM 1 in this procedure yields in a new and simpler way:

COROLLARY 8.1.2 *H-perfection can be tested in polynomial-time in the class of line-graphs.*

8.2.3 *Proofs of the preliminary results*

We first show Lemma 8.2.1:

LEMMA 8.2.1 *Let G be a 2-connected graph. The following statements are equivalent:*

i) G is odd-C_3^+-free,

ii) for each pair of odd circuits C_1 and C_2 of G: $|E(C_1) \cap E(C_2)|$ is odd.

PROOF – Clearly, an odd-C_3^+ has exactly two odd circuits which have an even number of common edges. This shows ii)=>i).

Conversely, suppose that G has odd circuits C_1 and C_2 such that $|E(C_1) \cap E(C_2)|$ is even. We show that G contains an odd-C_3^+.

First, let us assume that $|V(C_1) \cap V(C_2)| \leq 1$. Since G is 2-connected, Menger's theorem (Theorem 3.2.1) shows that there exist two vertex-disjoint $\{V(C_1), V(C_2)\}$-paths P and Q (one may have length 0 if C_1

and C_2 meet). Let p and q be the respective ends of P and Q on C_1 and let R be the unique pq-path of C_1 whose parity is distinct from $|E(P)| + |E(Q)|$. Clearly, $R \cup P \cup Q \cup C_2$ is an odd-C_3^+ subgraph of G.

Now, suppose that C_1 and C_2 have at least two vertices in common. Since both are odd, $C_1 \neq C_2$ and $V(C_1) \cap V(C_2)$ defines a partition of C_1 into edge-disjoint paths $P_1, \dots, P_k$ ($k \geq 1$) which have exactly their ends in $V(C_2)$. Since $|E(C_1) \cap E(C_2)|$ is even and as C_1 is odd, at least one of these paths (say P_1) must be odd and $C_2 \cup P_1$ is an odd-C_3^+ of G.

■

The proof easily shows how to build efficiently an odd-C_3^+ of a 2-connected graph from two odd circuits meeting on an even number of edges.

We now prove Lemma 8.2.2:

Lemma 8.2.2 *Each 2-connected non-bipartite graph has an odd cycle basis formed by circuits only.*

Furthermore, such a cycle basis can be found in polynomial-time.

Proof – Let G be a 2-connected non-bipartite graph and C be an odd circuit of G. By Proposition 3.2.5, G has an open ear-decomposition $(C, P_1, \dots, P_k)$.

For each $i \in [k]$, the graph $C \cup P_1 \cdots \cup P_{i-1}$ is 2-connected and non-bipartite. Hence, Menger's theorem (Theorem 3.2.1) straightforwardly shows that it contains a path Q_i which joins the ends of P_i and such that the circuit $P_i \cup Q_i$ is odd (consider two vertex-disjoint paths joining the ends of P_i to C).

For each $i \in [k]$, let $C_i := P_i \cup Q_i$. Put $\mathcal{B} := \{C, C_1, \dots, C_k\}$. We show that $\mathcal{B}$ is a cycle basis of G and this will prove the statement.

It follows from a standard argument that we give here. Proposition 3.2.4 and Proposition 8.2.4 show that $|\mathcal{B}| = \dim \mathcal{C}(G)$. Hence, it suffices to prove that the elements of $\mathcal{B}$ are linearly independent in $\mathbb{F}_2^{E(G)}$. Put $C_0 := C$. By contradiction, suppose that there exist $0 \leq i_1 < \ldots < i_l \leq k$ (with $l \geq 1$) such that $\sum_{j=1}^{l} \chi^{E(C_{i_j})} = 0$.

This means: $E(C_{i_1}) \Delta \cdots \Delta E(C_{i_l}) = \varnothing$. Taking the symmetric difference with $E(C_{i_l})$ on both sides, we obtain:

$$E(C_{i_1}) \Delta \cdots \Delta E(C_{i_{l-1}}) = E(C_{i_l}).$$

Clearly, at least one edge of C_{i_l} does not belong to $E(C_{i_1}) \cup \cdots \cup E(C_{i_{l-1}})$ and this contradicts the equality above.

■

Several polynomial-time algorithms are available for finding an open ear-decomposition of a 2-connected graph (see [99] for a recent example). Also, parallel algorithms were given by Lovász [74] and Miller, Ramachandran [81].

As the paths Q_i can be found in polynomial-time (with an efficient algorithm for finding two vertex-disjoint paths, see Section 3.2.2), the cycle basis of the proof can easily be built in polynomial-time.

Finally, it remains only to prove Proposition 8.2.3:

PROPOSITION 8.2.3 *If a 2-connected graph has a totally odd cycle basis, then its odd cycles pairwise-intersect in an odd number of edges.*

PROOF – Let $\cdot$ denote the standard bilinear form on $\mathbb{F}_2^{E(G)}$. That is, for subsets F_1 and F_2 of $E(G)$: $\chi^{F_1} \cdot \chi^{F_2}$ is the parity of $|F_1 \cap F_2|$. *Until the end of this proof, all equalities take place in* $\mathbb{F}_2^{E(G)}$.

Suppose that G has a totally odd cycle basis $\mathcal{B}$ and let C_1 and C_2 be odd cycles of G. We show that $\chi^{E(C_1)} \cdot \chi^{E(C_2)} = 1$, as stated.

Since $\mathcal{B}$ is a cycle basis of G, there exists $\mathcal{B}_1 \subseteq \mathcal{B}$ and $\mathcal{B}_2 \subseteq \mathcal{B}$ such that:

$$\chi^{E(C_1)} = \sum_{C \in \mathcal{B}_1} \chi^{E(C)} \text{ and } \chi^{E(C_2)} = \sum_{D \in \mathcal{B}_2} \chi^{E(D)}.$$

Since C_1 and $\mathcal{B}$ are odd, multiplying by the all-1 vector $\mathbf{1}$ on both sides of the first equality yields: $|\mathcal{B}_1| = 1$ (that is, $\mathcal{B}_1$ has odd cardinality). Similarly, $|\mathcal{B}_2| = 1$. Since $\mathcal{B}$ is totally odd, we obtain by linearity:

$$\chi^{E(C_1)} \cdot \chi^{E(C_2)} = \sum_{C \in \mathcal{B}_1,\, D \in \mathcal{B}_2} \chi^{E(C)} \cdot \chi^{E(D)} = \sum_{C \in \mathcal{B}_1,\, D \in \mathcal{B}_2} 1 = |\mathcal{B}_1||\mathcal{B}_2| = 1,$$

and this ends the proof of the proposition.

■

8.3 STARTING WITH ODD EARS

In Section 8.3.1, we define β and review its interpretations. In Section 8.3.2, we discuss the complexity of its computation and observe that Theorem 8.1.1 implies that determining β is a fixed-parameter-tractable problem. We also state a related conjecture on the matching polytope.

8.3.1 *Definition and interpretations of β*

Clearly, an odd-C_3^+ is a graph which has an open odd ear-decomposition of two ears.

For each graph G, let $\beta(G)$ denote the largest integer k such that G has a subgraph which has an open odd ear-decomposition with k ears.

For example, *a graph G is odd-C_3^+-free if and only if* $\beta(G) \leq 1$, and *G is bipartite if and only if $\beta(G) = 0$.*

Proposition 3.2.5 easily implies that for each 2-connected graph G: *$\beta(G)$ is the largest number of odd ears starting an open ear-decomposition of G.*

A graph G is *factor-critical* if for each $v \in V(G)$, the graph $G - v$ has a perfect matching.

THEOREM 3.7.3 (LOVÁSZ [72, 76]) *A 2-connected graph is factor-critical if and only if it has an open odd ear-decomposition.*

Proposition 3.2.4 states that *the ear-decompositions of a 2-edge-connected graph G all have the same number of ears*, which is $|E(G)| - |V(G)| + 1$. Thus, we may speak of the *number of ears* of a 2-edge-connected graph, and for each graph G: *$\beta(G)$ is the largest number of ears of a 2-connected factor-critical subgraph of G.*

Hence, Theorem 3.7.3 states that *a 2-connected graph G is factor-critical if and only if $\beta(G) = |E(G)| - |V(G)| + 1$.*

A graph which has a spanning factor-critical subgraph is obviously factor-critical. Therefore, each 2-connected factor-critical subgraph of a graph G with $\beta(G)$ ears is necessarily induced.

Edmonds and Pulleyblank proved:

THEOREM 3.7.1 (EDMONDS, PULLEYBLANK [36]) *For every graph G:*

$$\mathsf{MATCH}(G) := \left\{ x \in \mathbb{R}^{E(G)} : \begin{array}{cc} x \geq 0, & \\ x(\delta_G(v)) \leq 1 & \forall v \in V(G), \\ x(E(H)) \leq \dfrac{|V(H)| - 1}{2} & \forall H \text{ 2-connected induced} \\ & \text{factor-critical subgraph of } G. \end{array} \right\}. \tag{3.10}$$

Furthermore they showed that for each 2-connected factor-critical subgraph H of G, *the inequality $x(E(H)) \leq \frac{|V(H)|-1}{2}$ defines a facet of* $\mathsf{MATCH}(G)$ *if and only if H is an induced subgraph of G.*

This shows that, as the largest number of ears of such subgraphs, $\beta(G)$ can be used as a parameter to separate on, for questions related to the matching polytope (for example the integer decomposition property, see Section 8.3.2.2).

For each graph G, let $\beta'(G)$ denote the largest number of ears of a (non-necessarily 2-connected) factor-critical subgraph of G. Clearly, $\beta'(G) \geq \beta(G)$ and the inequality may obviously be strict in general. We do not know whether equality always holds when G is 2-connected.

8.3.2 *Problems*

8.3.2.1 *Computational complexity*

Clearly, the property $\beta(G) \geq k$ belongs to the class NP: it can be proved by providing an open odd ear-decomposition of a subgraph of G with k ears.

We do not know if it admits a co-NP-characterization and the computational complexity of determining β is open:

QUESTION 8.1.5 ([8]) *Can β can be computed in polynomial-time ?*

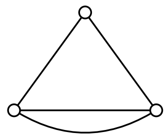


Figure 8.5 – the graph C_3^+

We now observe that Theorem 8.1.1 of Kawarabayashi, Reed, Wollan and Huynh immediately implies that determining β is fixed-parameter-tractable. That is: *for each non-negative integer k, there exists a polynomial-time algorithm deciding whether a graph G satisfies $\beta(G) = k$.*

Let k be a non-negative integer. For each $p \in \{0,1\}^{E(K_{2k})}$, let K_{2k}^p be the graph obtained from K_{2k} by subdividing exactly once each edge e of K_{2k} such that $p(e) = 0$.

Obviously, for each graph G: $\beta(G) \geq k$ *if and only if it contains a totally odd subdivision of a factor-critical K_{2k}^p for some $p \in \{0,1\}^{E(K_{2k})}$.*

Therefore, this and Theorem 8.1.1 directly imply that $\beta(G) \geq k$ can be checked in polynomial-time in the number of edges and vertices of G (and in at least exponential-time in k). In other words:

THEOREM 8.1.4 *Determining β is Fixed-Parameter-Tractable.*

In Section 8.2, we gave a simpler algorithm which tests $\beta \geq 2$. We do not know a similar algorithm testing $\beta \geq k$ for a larger constant k.

8.3.2.2 *β and the integer decomposition property*

It is well-known that the chromatic index of a graph cannot always be obtained by rounding-up its fractional chromatic index (see Section 3.2.4 for the definition of these parameters). The smallest known example is given by the *Petersen graph minus a vertex*, (denoted $\mathbf{T}$ and shown in Figure 8.3). Indeed, Theorem 3.7.2 easily shows that $\chi'_f(\mathbf{T}) = 3$, whereas $\chi'(\mathbf{T}) = 4$.

Recall that a graph is *odd-C_5^+-free* if it does not contain a totally odd subdivision of C_5^+ (the graph C_5^+ is shown in Figure 8.2).

For each graph G, let $\hat{G}$ denote the underlying simple graph of G. Clearly, *a graph G is odd-C_5^+-free if and only if* $\beta(\hat{G}) \leq 1$ (that is, $\hat{G}$ is odd-C_3^+-free).

Theorem 6.1.3 of Chapter 6 states that *each odd-C_5^+-free graph G satisfies $\chi'(G) = \lceil \chi'_f(G) \rceil$*. We conjecture that this result can be extended as follows:

CONJECTURE 8.3.1 *Each graph G with $\beta(\hat{G}) \leq 3$ satisfies $\chi'(G) = \lceil \chi'_f(G) \rceil$.*

The bound 3 would be best possible. Indeed, $\mathbf{T}$ does not satisfy this rounding equality (see above) and $\beta(\mathbf{T}) = 4$ ($\mathbf{T}$ is factor-critical and 2-connected). It is not clear whether the "Kempe-chains argument" used in the proof for odd-C_5^+-free graphs could be extended.

Recall that a polytope $P \subseteq \mathbb{R}^n$ has the *integer decomposition property* if for every positive integer k, each integral vector of kP is the sum of k integral vectors of P (see Chapter 6 and Chapter 7 for further details).

An *inflation* of a graph G is a graph obtained from G by adding parallel edges (possibly none). Theorem 3.4.3 directly shows that for each graph G: $\mathsf{MATCH}(G)$ *has the integer decomposition property if and only if each inflation H of G satisfies $\chi'(H) = \lceil \chi'_f(H) \rceil$.*

Clearly, the class of graphs whose underlying simple graph satisfies $\beta \leq 3$ is closed under inflations. Therefore, Conjecture 8.3.1 would imply:

CONJECTURE 8.3.2 *Let G be a graph. If $\beta(\hat{G}) \leq 3$, then $\mathsf{MATCH}(G)$ has the integer decomposition property.*

This is related to conjectures of Goldberg and Seymour which state:

CONJECTURE 8.3.3 (GOLDBERG [55], SEYMOUR [104]) *Each graph G satisfies $\chi'(G) \leq \lceil \chi'_f(G) \rceil + 1$.*

The *perfect matching polytope* of a graph is the convex hull of the incidence vectors of perfect matchings. Obviously, it is a face of the matching polytope. Lovász [75] conjectures that *the perfect matching polytope of a graph without a Petersen minor has the integer decomposition property*. This implies the 4-color theorem (through planar duality).

Besides, Shepherd and Kilakos [68] conjecture that *every graph G which does not have $\mathbf{T}$ as a minor satisfies $\chi'(G) = \lceil \chi'_f(G) \rceil$*. This would imply that the matching polytope (and thus the perfect matching polytope) of such graphs has the integer decomposition property.

This and Conjecture 8.3.1 do not clearly imply one another. Indeed, it is easy to find graphs without $\mathbf{T}$ as a minor and with an arbitrarily large value of β. Also, the graph obtained from $\mathbf{T}$ by subdividing each edge exactly once is bipartite (that is $\beta = 0$) and has obviously $\mathbf{T}$ as a minor.

8.4 RELATIONS WITH THE LARGEST NUMBER OF ODD EARS

In Section 8.4.1, we show that the largest number of odd ears $\overline{\varphi}$ in an ear-decomposition may be arbitrarily large whereas β remains constant.

In Section 8.4, we use state and prove a characterization of odd-C_3^+-free graphs which do not contain a totally odd subdivision of K_4. We use a few preliminary propositions and lemmas whose proofs are postponed to Section 8.4.3.

8.4.1 *Frank's parameter φ*

For each 2-edge-connected graph G, let $\varphi(G)$ denote the smallest number of even ears in an ear-decomposition of G. We say that an ear-decomposition of G is *optimal* if it has exactly $\varphi(G)$ even ears.

The parameter φ was introduced by Frank in [47]. In particular, he showed *a polynomial-time algorithm to compute φ and an optimal ear-decomposition.*

In this section, we deal with open ear-decompositions only. The results in [47] are stated for 2-edge-connected graphs and non-necessarily open ear-decompositions. Still, it is not difficult to check that *the smallest number of even ears in an open ear-decomposition of a 2-connected graph G is also $\varphi(G)$*. Besides, each result of [47] for 2-edge-connected graphs and ear decompositions stays valid if "2-edge-connected" and "ear-decomposition" are respectively replaced with "2-connected" and "open ear-decomposition" (see [20, Section 3] for a proof).

For each 2-edge-connected graph G, put:

$$\overline{\varphi}(G) := |E(G)| - |V(G)| + 1 - \varphi(G).$$

Since the ear-decompositions of G all have the same number $|E(G)| - |V(G)| + 1$ of ears (Proposition 3.2.4): *$\overline{\varphi}(G)$ is the largest number of odd ears in an ear-decomposition (open or not) of G*. Hence, optimal ear-decompositions of G are those which have $\overline{\varphi}(G)$ odd ears.

Every 2-connected graph G obviously satisfies $\overline{\varphi}(G) \geq \beta(G)$. We build a sequence of graphs showing that $\beta(G)$ may remain constant while $\overline{\varphi}$ gets arbitrarily large (even in simple graphs which have an edge whose deletion yields a bipartite graph).

Let $k \geq 3$ be an integer and $T_1, \ldots, T_k$ be k vertex-disjoint copies of the graph C_5^+ (see Figure 8.2). Let v_i be the unique vertex of degree 2 in the triangle of T_i and let u_i be one of its neighbors.

Now, let H_k be the graph obtained by identifying all the v_i to a single vertex v, all the u_i to a single vertex u and keeping only one copy of the edge uv (see Figure 8.6).

It is straightforward to check the following:

PROPOSITION 8.4.1 *For each $k \geq 2$:*

$$\beta(H_k) = 2 \text{ and } \overline{\varphi}(H_k) \geq k.$$

PROOF – We first observe that $\overline{\varphi}(H_k) \geq k$. Start an ear-decomposition of H_k with a triangle T and continue by adding all the ears of length 2 corresponding to the other triangles. Now, only odd ears can be

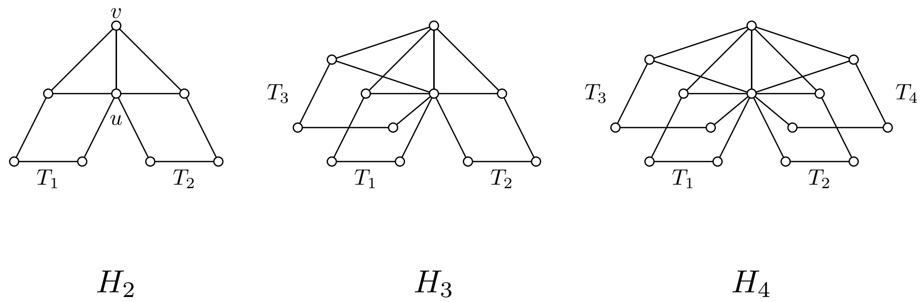


Figure 8.6 – the graphs H_2, H_3 and H_4

added to complete this ear-decomposition into one for H_k and there are k such ears. Hence, $\overline{\varphi}(H_k) \geq k$.

We now prove that $\beta(H_k) = 2$. Any T_i shows that $\beta(G_k) \geq 2$. Let $(C, P_1, \ldots, P_l)$ be an open ear-decomposition of H_k starting with two odd ears C and P_1. We prove that P_2 is even and this will end the proof.

Clearly, $H_k - uv$ is bipartite and therefore each odd circuit of H_k contains uv. In particular, $uv \in E(C)$.

Now, this directly implies that (C, P_1) must be an open odd ear-decomposition of a T_i. Hence, the ends of P_2 must be u and v, which are in the same class in a bipartition of $H_k - uv$. Therefore P_2 is even.

■

In [47], Frank showed a min-max theorem for φ in terms of maximum-cardinality joins: a *join* of a graph G is a set $F \subseteq E(G)$ such that each circuit C of G satisfies $|E(C) \cap F| \leq |E(C) \setminus F|$ (see also [1]). We do not know of a similar min-max result for β.

8.4.2 *Totally odd subdivisions of K_4 in odd-C_3^+-free graphs*

Cao's thesis [18] suggests that totally odd subdivisions of K_4 are related to odd-C_3^+-free graphs (see Section 8.5 for further details). In this section, we show a simple polynomial-time algorithm for detecting totally odd subdivision of K_4 in odd-C_3^+-free graphs.

Finding such subdivisions of K_4 is not elementary in general: the simplest algorithm available for their detection in arbitrary graphs uses techniques of the Graph Minor Project [65].

Our algorithm is based on the following characterization. We prove it after stating a few preliminary results whose proofs are postponed to the next section.

Clearly, we need only to consider simple 2-connected graphs (and the following statement is false for non-simple graphs in general, as shows the graph obtained from C_4 by adding two parallel edges).

Theorem 8.4.2 *Let G be a 2-connected odd-C_3^+-free simple graph. The following statements are equivalent:*

i) *G does not contain a totally odd subdivision of* K_4*,*
ii) $\overline{\varphi}(G) \leq 1$.

Using the algorithm of Frank [47] to compute $\overline{\varphi}$ and an optimal ear-decomposition, our proof can be easily converted into a polynomial-time algorithm for the detection of totally odd subdivisions of K_4 in odd-C_3^+-free simple graphs.

The proof of this result uses a few preliminary results (that we prove in the next section).

An *odd theta* is a graph formed by three inner-disjoint odd paths with the same ends (each path may have one edge only, see Figure 8.7). First, we will use the following facts:

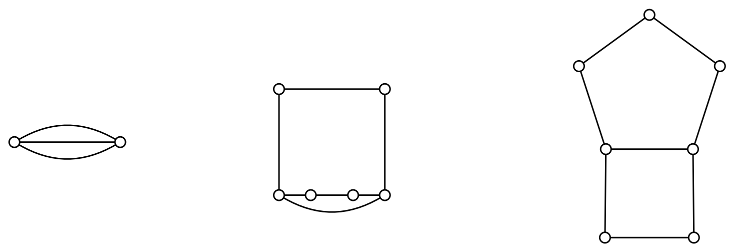


Figure 8.7 – examples of odd thetas

Proposition 8.4.3 *Let G be a 2-connected bipartite graph. If G has an ear-decomposition with an open odd ear, then each vertex of G belongs to an odd theta subgraph of G.*

The proof will directly show that such an odd theta can be built efficiently from an ear-decomposition having an open odd ear.

Proposition 8.4.4 *Let G be a 2-connected non-bipartite graph, C an odd circuit of G and* $v \in V(G) \setminus V(C)$*. If G contains three inner-disjoint odd paths* $\{v, V(C)\}$*-paths, then G contains an odd-*C_3^+ *or a totally odd subdivision of* K_4*.*

The proof of this (given in the next section) will easily show that an odd-C_3^+ or a totally odd subdivision of K_4 can be found in polynomial-time under the corresponding assumptions.

The other main ingredient is the following lemma, which may be of independent interest:

Lemma 8.4.5 *Each 2-connected non-bipartite graph has an open optimal ear-decomposition which starts with an odd ear.*

Using the algorithm of Frank [47], the proof shows that such a decomposition can be found in polynomial-time.

The last tool is the following easy part of Lemma 8.2.1 (in Section 8.2):

Proposition 8.4.6 *If a 2-connected graph G has two odd circuits which have at most one common vertex, then G contains an odd-*C_3^+*.*

The *ends* of an odd-C_3^+ (or an odd theta) are its two vertices of degree 3. We now prove Theorem 8.4.2.

Proof – Clearly, any open ear-decomposition of a totally odd subdivision of K_4 which starts with an odd circuit has two odd ears. This shows that ii)=>i).

We now prove the converse. Since a totally odd subdivision of K_4 has an odd circuit, we may assume that G is non-bipartite. Suppose that $\overline{\varphi}(G) \geq 2$. We will show a totally odd subdivision of K_4 in G.

Since G is 2-connected and non-bipartite, Lemma 8.4.5 shows that G has an open optimal ear-decomposition $(C, P_1, \ldots, P_k)$ such that C is odd.

Let $H := G/V(C)$ (and delete the created loops). Let c be the new vertex.

Claim 1. H is bipartite.

Proof – Suppose to the contrary that H contains an odd circuit D. In G, the graph D is either an odd circuit meeting C in at most one vertex or an odd path which has exactly its ends in C.

If D is an odd circuit in D, Proposition 8.4.6 directly shows an odd-C_3^+ and this contradicts the assumption on G.

Hence, D is an odd path which has exactly its ends in C. Therefore, $D \cup C$ is an odd-C_3^+: a contradiction.

□

Claim 2. G/H contains an odd theta T containing c.

Proof – Since $\overline{\varphi}(G) \geq 2$, there exists $1 \leq i \leq k$ such that P_i is odd. Since G is simple and odd-C_3^+-free, P_i cannot be an edge with both ends in C. Hence, P_i is not deleted as a loop of H and corresponds to a path or a circuit of H with the same length.

As H is bipartite, P_i cannot be a circuit of H. Besides, the ends of P_i must clearly belong to the same block B of H. It is straightforward to check that the ears of $(C, P_1, \ldots, P_k)$ which are contained in B define an ear-decomposition of B in which P_i is an open odd ear.

Since G is 2-connected, B must contain c. Therefore, Proposition 8.4.3 shows that B contains an odd theta T containing c.

□

We now show:

Claim 3. c is an end of T.

Proof – Suppose to the contrary that c is not an end of T. Let u and v be the ends of T and Q_1, Q_2 and Q_3 be the three (odd) uv-paths of T. Without loss of generality, we may assume that c is an inner-vertex of Q_1.

First, suppose that Q_1 is not a path of G. In this case, Q_1 corresponds in G to two vertex-disjoint paths Q_1' and Q_1'' joining respectively u and v to vertices s and t of C. Since C is odd, the two st-paths of C have distinct parities. Using these paths, it is straightforward to check that $T \cup C$ always contains an odd-C_3^+ with ends u and v. This contradicts that G is odd-C_3^+-free.

Hence, we may assume that Q_1 remains a path in G. Then, T is an odd theta of G which has exactly one vertex w in common with C in G.

Since G is 2-connected, Menger's theorem (Theorem 3.2.1) shows that $G - w$ contains a path P which joins a vertex x of C to vertex y of T and which has no other vertex in $C \cup T$.

If $y \in V(Q_1)$, then (using that C contains xw-paths of both parities) it is easy to find an odd-C_3^+ in G with ends u and v, contradicting that G is odd-C_3^+-free. Therefore, we may assume without loss of generality that $y \in Q_2$ and that the uy-path of Q_2 is odd. Again, it is straightforward to build an odd-C_3^+ of G (with ends u and y): a contradiction.

□

Let c' be the other end of T. The three paths of T in H correspond to three inner-disjoint odd $\{c', V(C)\}$-paths of G. Since G is 2-connected and odd-C_3^+-free, Proposition 8.4.4 shows that G contains a totally odd subdivision of K_4, as required.

■

It is straightforward to convert this proof into *a polynomial-time algorithm deciding whether an odd-C_3^+-free simple graph contains a totally odd subdivision of K_4*.

Recall that a graph G is odd-C_3^+-free if and only if $\beta(G) \leq 1$. *Is it true that graphs with $\beta = 2$ must contain a totally odd subdivision of K_4 whenever $\overline{\varphi}$ is large ?* The graphs H_k given in Section 8.4.1 show that the answer is *negative*.

Indeed, $\beta(H_k) = 2$ and each H_k has an edge whose deletion yields a bipartite graph. This shows that H_k does not contain a totally odd subdivision of K_4.

8.4.3 *Proofs of the propositions and lemmas*

We first prove Proposition 8.4.3:

Proposition 8.4.3 *Let G be a 2-connected bipartite graph. If G has an ear-decomposition with an open odd ear, then each vertex of G belongs to an odd theta subgraph of G.*

Proof – Suppose that G has an ear-decomposition $(C, P_1, \ldots, P_k)$ which has an open odd ear.

We first show that G contains an odd theta. Without loss of generality, we may assume that P_k is an open odd ear in the decomposition. Let u_1 and u_2 be the ends of P_k. Let $H := C \cup P_1 \cup \cdots \cup P_{k-1}$.

Since H has an ear-decomposition, it is 2-edge-connected. In particular, Menger's theorem shows that H contains two edge-disjoint u_1u_2-paths Q and R. Since P_k is odd and G is bipartite, both Q and R are odd. Clearly, $V(Q) \cap V(R)$ defines a partition of the edge-set of Q into paths $Q_1, \ldots, Q_l$. Since Q is odd, one of those paths, say Q_1, must be odd. It is easy to check that $R \cup Q_1 \cup P_k$ is an odd theta of G.

Finally, we prove that every vertex of G belongs to an odd theta. Let T be an odd theta of G and let $s \in V(G) \setminus V(T)$. Since G is 2-connected, Menger's theorem shows that there are two $\{s, V(T)\}$-paths Q_1 and Q_2 whose only common vertex is s. A straightforward and short case-checking shows that $Q_1 \cup Q_2 \cup T$ always has an odd theta containing s.

■

It is straightforward to convert this proof into a polynomial-time algorithm which finds an odd theta containing a prescribed vertex under these assumptions.

We now show the second ingredient of the proof of Theorem 8.4.2:

PROPOSITION 8.4.4 *Let G be a 2-connected non-bipartite graph, C an odd circuit of G and $v \in V(G) \setminus V(C)$. If G contains three inner-disjoint odd paths $\{v, V(C)\}$-paths, then G contains an odd-C_3^+ or a totally odd subdivision of K_4.*

PROOF – Let P_1, P_2, P_3 be three inner-disjoint odd $\{v, V(C)\}$-path and let $k := |(\cup_{i=1}^3 V(P_i)) \cap V(C)|$.

Case 1. $k = 1$. Let u be the unique vertex of $(\cup_{i=1}^3 V(P_i)) \cap V(C)$.

Since G is 2-connected, $G - u$ contains a path Q which has an end s in C, an end t in $\cup_{i=1}^3 V(P_i)$ and no other vertex in these two graphs. Without loss of generality, we may assume that $t \in P_1$.

Let P be the tv-path of P_1 and let R be the us-path of C whose parity is the one of $|E(P)| + |E(Q)|$. It is easy to check that $P \cup Q \cup R \cup P_2 \cup P_3$ is an odd-C_3^+ of G (with ends u and v), which shows that $\beta(G) \geq 2$.

Case 2. $k = 2$. Without loss of generality, we may assume that P_2 and P_3 intersect C at the same vertex u and that P_1 meets C at a vertex $s \neq u$. Let Q be the odd su-path of C. Clearly, $Q \cup (\cup_{i=1}^3 P_i)$ is an odd-C_3^+ (with ends u and v), and this shows $\beta(G) \geq 2$.

Case 3. $k = 3$. Let Q_1, Q_2 and Q_3 be the three paths partitioning (the edge-set of) C defined by the respective ends of P_1, P_2 and P_3 on C. If one of the Q_i is even, then it is straightforward to check that $C \cup P_1 \cup P_2 \cup P_3$ contains an odd-C_3^+ showing $\beta(G) \geq 2$.

Therefore, we may assume that Q_1, Q_2 and Q_3 are odd. Hence, $C \cup P_1 \cup P_2 \cup P_3$ is a totally odd subdivision of K_4.

■

Using an efficient algorithm for finding two vertex-disjoint paths, it is easy to convert this proof into a polynomial-time algorithm which finds an odd-C_3^+ or a totally odd subdivision of K_4 as stated in the proposition.

The proof of Lemma 8.4.5 uses the following theorem of Frank:

THEOREM 8.4.7 (FRANK [47]) *Let G be a 2-connected graph. For each edge e of G, there exists an open optimal ear-decomposition of G whose first ear contains e.*

Furthermore, such a decomposition can be found in polynomial-time.

We show:

LEMMA 8.4.5 *Each 2-connected non-bipartite graph has an open optimal ear-decomposition which starts with an odd ear.*

PROOF – Let $(C, P_1, \ldots, P_k)$ be an open optimal ear-decomposition of G. If C is odd, then we are done.

Hence, we may assume that C is even. Let i be the smallest integer of $[k]$ such that $C \cup P_1 \cup \cdots \cup P_i$ is non-bipartite.

Put $H := C \cup P_1 \cup \cdots \cup P_i$ and let $e \in E(P_i)$.

Since H has an open ear-decomposition, it is 2-connected. Hence, Theorem 8.4.7 shows that H has an open optimal ear-decomposition $(D, Q_1, \ldots, Q_i)$ whose first ear contains e (the number of ears is indeed $i + 1$ as all ear-decompositions of H have the same number of ears).

Clearly, $H - e$ is bipartite. Hence, every circuit of H containing e is odd. In particular, D is odd.

Since $(C, P_1, \ldots, P_k)$ is an optimal ear-decomposition of G, the decomposition $(C, P_1, \ldots, P_i)$ must be optimal for H.

Hence, the ear-decomposition $(D, Q_1, \ldots, Q_i, P_{i+1}, \ldots, P_k)$ is open and optimal for G. This proves the lemma.

■

This proof and Theorem 8.4.7 directly show that *such a decomposition can be found in polynomial-time.*

8.5 CAO'S THESIS AND MOTIVATIONS

Recall that an *odd-C_5^+* is a totally odd subdivision of C_5^+ (the graph C_5^+ is shown in Figure 8.2). A graph is *odd-C_5^+-free* if it does not contain an odd-C_5^+.

Clearly, a graph is odd-C_5^+-free if and only if its underlying simple graph is odd-C_3^+-free.

Cao's thesis [18] contains several results and statements on odd-C_5^+-free graphs. For example, it shows that two odd circuits of an odd-C_5^+-free graph must intersect on an odd number of edges (see

Lemma 8.2.1, this is also used in [14]). Furthermore, the thesis states a construction procedure for these graphs.

In this section, we state (with correction) a result of the thesis [18] which motivated the use of the parameter φ of Frank in our study of odd-C_3^+-free graphs (see Section 8.4). The final product of this motivation is Theorem 8.4.2.

Besides, we show that some statements on odd-C_5^+-free graphs of [18] and the procedure for their construction are incorrect (with explicit counter-examples).

We first recall the definitions of [18] to keep the same terminology. A graph is *critical non-bipartite* if it is non-bipartite and each pair of odd circuits has at least one common edge. A critical non-bipartite graph is furthermore *elementary* if it has an edge whose deletion yields a bipartite graph.

A graph H is *basic* if it is obtained from a graph G by subdividing each edge of G exactly once. The vertices of G in H are the *basic vertices of H*. A graph is *critical non-basic* if it is not basic and has an edge whose deletion yields a basic graph.

CRITICAL GRAPHS It is straightforward to check that each critical non-basic graph is odd-C_5^+-free and elementary critical non-bipartite. Lemma 4.5 pg. 70 in [18] states that the converse holds: *each 2-connected odd-C_5^+-free and elementary critical non-bipartite graph is critical non-basic*. The graph of Figure 8.8 shows that this is false: it is obviously 2-connected, odd-C_5^+-free and elementary critical non-bipartite (deleting uv yields a bipartite graph) but it is not critical non-basic.

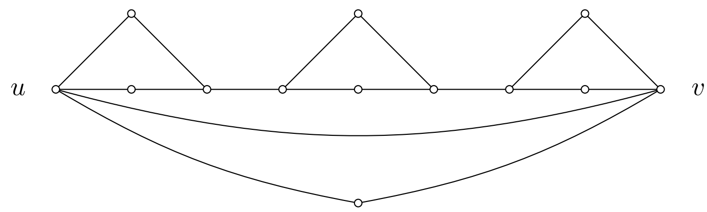


Figure 8.8 – an odd-C_5^+-free 2-connected elementary critical non-bipartite graph which is not critical non-bipartite

TOTALLY ODD SUBDIVISIONS OF K_4 The following result links totally odd subdivisions of K_4 with odd-C_5^+-free graphs. In [18], it is stated with "critical non-basic" in place of "elementary critical non-bipartite" and the graph of Figure 8.8 shows that it is incorrect as such. Still, exchanging these two properties corrects the statement:

THEOREM 8.5.1 (CAO [18]) *Let G be a non-bipartite graph and C be an odd circuit of G. If G does not contain a totally odd subdivision of K_4, then for each component K of $G - E(C)$: the graph $C \cup K$ is elementary critical non-bipartite.*

CONSTRUCTION OF ODD-C_5^+-FREE GRAPHS The conclusion of the section of [18] devoted to odd-C_5^+-free graphs is a construction procedure for odd-C_5^+-free graphs. We observe that it is incorrect. For this purpose, we need only to state a special case of the procedure.

The *sides* of a totally odd subdivision of K_4 are the paths corresponding to the original edges of K_4.

Let F be a totally odd subdivision of K_4. Let P_1 and P_2 be two vertex-disjoint paths and for each $i \in \{1, 2\}$, let u_i and v_i be the ends of P_i. Let G be a graph obtained by identifying u_1, v_1, u_2, v_2 to distinct vertices of F such that for each $i \in \{1, 2\}$: u_i and v_i are identified to vertices which are on sides of F which have a common end w and have even distance to w in F.

Cao's thesis states that *each graph obtained in this way is odd-C_5^+-free*. The graph of Figure 8.9 shows that this is false: it is obviously built as in the procedure, but the thick edges show an odd-C_5^+.

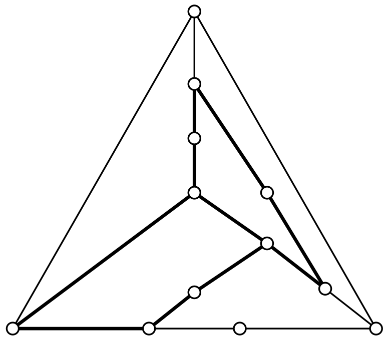


Figure 8.9 – a counter-example to the construction of [18] for odd-C_5^+-free graphs

9 CONCLUSION

Perfect graphs show that combinatorial conditions on graphs may imply a nice characterization of the facets of the stable set polytope. Conversely, how does a nice facet structure of this polytope act on the combinatorial parameters of the graph ? Very little is known on this "meta-problem" and the study of h-perfect graphs represents one of the most elementary approaches: the only facets not defined by non-negativity or cliques are given by odd-circuit inequalities.

Compared to perfect graphs, is there only a small number of families of minimally t-imperfect (or minimally h-imperfect) graphs for t-minors operations ? Do they have a nice characterization ? Can h-perfection be tested in polynomial-time ? Is the chromatic number of an h-perfect graph always close to its clique number ?

None of these questions is answered in general. We hopefully clarified several aspects in particular cases.

In this final chapter, we review questions and conjectures of the previous chapters and provide a few more.

COMBINATORIAL CHARACTERIZATION The complexity of deciding t-perfection is open. It is not difficult to check that t-perfection is in co-NP (see [102, pg. 1194]). However, no combinatorial certificate of t-imperfection is known.

Using operations which keep t-perfection and reduce the size of the graph may yield an "excluded-minor characterization" which would provide such a certificate. Taking a t-minor and shrinking an even pair are the only known operations of this nature.

In Chapter 5, we reviewed the currently known minimally t-imperfect graphs which can be found in [15, 13] (see Chapter 5 for their definitions):

odd wheels, even Möbius ladders, C_7^2 *and the* $(3,3)$*-graphs* $(*)$

We do not know if there are other minimally t-imperfect graphs. Seeking excluded-t-minor characterizations for particular cases may yield further examples. In particular (see Section 5.2.4 for the definition of partitionable graphs):

QUESTION 5.3.11 *Are there minimally t-imperfect* P_5*-free graphs other than* K_4, W_5, C_7^2 *and* $\overline{C_{10}^2}$ *?*

QUESTION 9.0.1 *Are there minimally t-imperfect partitionable graphs* G *with* $\alpha(G) \geq 4$ *?*

It is not known whether h-perfection belongs to NP or co-NP. Recall that an h-imperfect graph is *critically h-imperfect* if all its proper induced subgraphs are h-perfect.

CONJECTURE 5.1.5 (SEBŐ) *Every critically h-imperfect graph with $\omega \geq 4$ is an odd antihole.*

This would imply that an h-imperfect graph has either an induced $\overline{C_{2k+1}}$ with $k \geq 4$ or a K_4-free minimally h-imperfect graph. By Theorem 5.1.4, the latter is either a minimally t-imperfect graph or one of K_4^*, W_5^- and W_5^{--}. Since t-perfection is in co-NP (see above), the conjecture would place h-perfection in co-NP.

We explained in Chapter 5 that if valid, Conjecture 5.1.5 would easily imply the Strong Perfect Graph Theorem (through the K_4-free case due to Tucker [116]). Hence, one may have to use the theorem in trying to prove Conjecture 5.1.5. For example, it would suffice to show that *every critically h-imperfect graph with $\omega \geq 4$ is minimally imperfect*.

We observed in Chapter 4 that *perfection is closed under t-minors*. It is not clear whether this operation could simplify any part of the theory of perfect graphs.

Besides, we may expect that super-classes of h-perfect graphs (for example, a-perfect graphs [118]) are also closed under t-minors.

CLAW-FREE H-PERFECT GRAPHS We showed that Conjecture 5.1.5 holds trivially for planar graphs. It is still open for claw-free graphs.

Clearly, each odd antihole is claw-free and partitionable (see Section 5.2.4 for the definition of partitionable graphs). In Section 5.5, we showed that a theorem of [116] and Theorem 5.1.8 imply that *the odd antiholes $\overline{C_{2k+1}}$ with $k \geq 4$ are the only critically h-imperfect claw-free partitionable graphs with $\omega \geq 4$.*

Hence, proving Conjecture 5.1.5 for claw-free graphs amounts to showing:

CONJECTURE 5.1.9 *Each critically h-imperfect claw-free graph with $\omega \geq 4$ is partitionable.*

By Theorem 5.1.1 and Theorem 5.1.4, if valid this would imply that *the minimally h-imperfect claw-free graphs are: W_5, W_5^-, W_5^{--}, C_{10}^2 and the $\overline{C_{2k+1}}$ with $k \geq 3$.* Thus, h-perfection of claw-free graphs would belong to co-NP.

We do not know whether the decomposition theorem for claw-free graphs by Chudnovsky and Seymour [23] can be used to prove this.

Bruhn and Schaudt [14] showed that *t-perfect claw-free graphs can be recognized in polynomial-time*. In Chapter 8, we gave a simple and efficient algorithm *testing h-perfection in line-graphs*. We hope that a proof of Conjecture 5.1.9 could be combined with this algorithm to obtain an efficient algorithm for the recognition of h-perfect claw-free graphs.

QUESTION 9.0.2 *Can h-perfection be tested in polynomial-time in the class of claw-free graphs ?*

STRUCTURE OF MINIMALLY H-IMPERFECT GRAPHS Bruhn and Stein [16] asked (see Section 5.3):

QUESTION 9.0.3 *Is it true that every minimally t-imperfect graph is 3-connected ?*

By Theorem 5.3.2 of [16], this is equivalent to: *does each minimally t-imperfect graph have minimum degree 3 ?*

Recall that $\mathsf{TSTAB}(G)$ denotes the polyhedron of $\mathbb{R}^{V(G)}$ described by non-negativity, edge and odd-circuits inequalities of a graph G.

QUESTION 5.3.7 (BRUHN, STEIN [16]) *For a minimally t-imperfect graph G, does $\mathsf{TSTAB}(G)$ have precisely one non-integral vertex ?*

Our attempts at answering these questions using the techniques of the proof of Theorem 5.1.2 by Padberg [89] suggested the two following problems on the combinatorial structure of minimally t-imperfect graphs:

QUESTION 5.3.8 *Can a minimally t-imperfect graph have both contractible and non-contractible vertices ?*

QUESTION 5.3.9 *Do the odd holes of a minimally t-imperfect graph all have the same length ?*

We proved that *the only K_4-free minimally h-imperfect graphs which are not minimally t-imperfect are K_4^*, W_5^{--} and W_5^-* (Theorem 5.1.4). They are not 3-connected and have both contractible and non-contractible vertices.

Besides, $\mathsf{TSTAB}(W_5^{--})$ has exactly two non-integral vertices. Hence, extending the questions above to minimally h-imperfect graphs requires adding certain exceptions:

QUESTION 5.4.3 *Are K_4^*, W_5^- and W_5^{--} the only minimally h-imperfect graphs which have vertices of degree 2 (or, which are not 3-connected) ?*

Recall that $\mathsf{HSTAB}(G)$ denotes the polyhedron of $\mathbb{R}^{V(G)}$ described by non-negativity, clique and odd-circuits inequalities of a graph G.

QUESTION 9.0.4 *Except W_5^{--}, is it true that each minimally h-imperfect graph G is such that $\mathsf{HSTAB}(G)$ has a unique non-integral vertex ?*

Notice that W_5^{--} (see Figure 4.2.c) is also the only minimally h-imperfect graph which is not prime (Theorem 4.1.6).

The results of Padberg [89] on minimally imperfect graphs and of Shepherd [108] on near-perfect graphs motivates the study of the stable set polytope of minimally h-imperfect graphs.

A graph G is *near-h-perfect* if its stable set polytope is described by non-negativity, clique, odd circuit inequalities and the full-rank inequality $x(V(G)) \leq \alpha(G)$. The study of the currently known examples of minimally t-imperfect and minimally h-imperfect graphs (see Chapter 5) suggests the following:

CONJECTURE 5.3.4 *Except the odd wheels W_{2n+1} with $n \geq 2$, every minimally t-imperfect graph is near-h-perfect.*

Besides, Bruhn (personal communication) conjectures:

CONJECTURE 5.3.5 (BRUHN) *Every minimally t-imperfect graph G satisfies:*

$$\chi_f(G) > \max(\omega(G), \Gamma(G)).$$

The validity of Conjecture 5.3.4 would imply that Conjecture 5.3.5 is equivalent to stating: *except the W_{2n+1} with $n \geq 2$, every minimally t-imperfect graph G satisfies: $|V(G)| > \alpha(G) \cdot \max(\omega(G), \Gamma(G))$*. Therefore, this would provide an analogous statement to the following result of Lovász [71]: *every minimally imperfect graph G satisfies $|V(G)| > \alpha(G)\omega(G)$*.

The results of Section 5.4.1 show that extending Conjecture 5.3.4 to minimally h-imperfect graphs again requires more exceptions:

CONJECTURE 5.1.3 *Except K_4^* and the odd wheels W_{2n+1} with $n \geq 2$, every minimally h-imperfect graph is near-h-perfect.*

CHROMATIC NUMBER It is not known whether there exists $k \in \mathbb{Z}$ such that every t-perfect graph is k-colorable. The graphs $\overline{L(\Pi)}$ shows the largest known value of the chromatic number of a t-perfect graph, which is 4 (see Chapter 7).

CONJECTURE 7.1.5 (SEBŐ, IN [16]) *Each t-perfect triangle-free graph is 3-colorable.*

He showed that this would imply that *every h-perfect graph G satisfies $\chi(G) \leq \omega(G) + 1$* (see Chapter 7) and thus that *t-perfect graphs are 4-colorable*.

A graph is *C_5-covered* if every vertex belongs to at least one induced 5-circuit. We showed that Conjecture 7.1.5 would follow from the C_5-covered case (this reduction was suggested by Sebő).

CONJECTURE 7.5.12 *Each t-perfect triangle-free C_5-covered graph is 3-colorable.*

A graph is *k-critical* if it has chromatic number k and all its proper induced subgraphs are $(k-1)$-colorable. Theorem 7.1.4 shows that $\overline{L(\Pi)}$ and $\overline{L(W_5)}$ are the only t-perfect complements of line-graphs which are 4-critical.

QUESTION 9.0.5 *Are $\overline{L(\Pi)}$ and $\overline{L(W_5)}$ the only 4-critical t-perfect graphs ?*

Besides, we proved that *each h-perfect P_6-free graph G satisfies $\chi(G) \leq \omega(G)+1$* (Theorem 7.1.6). In particular, each P_6-free t-perfect graph is 4-colorable. The graphs $\overline{L(\Pi)}$ and $\overline{L(W_5)}$ are t-perfect and P_6-free. Hence they certify that this bound is tight.

However, these two graphs are not P_5-free and we do not know if the bound 4 could be improved for the subclass of P_5-free graphs

QUESTION 9.0.6 *Is it true that each t-perfect P_5-free graph is 3-colorable ?*

By Theorem 7.5.2, this would imply that each h-perfect P_5-free graph G satisfies $\chi(G) = \lceil \chi_f(G) \rceil$.

INTEGER DECOMPOSITION In Chapter 6, we proved that each t-perfect claw-free graph and each h-perfect line-graph has the IRCN. We conjecture that it also holds for h-perfect claw-free graphs:

CONJECTURE 9.0.7 *Each h-perfect claw-free graph has the integer round-up property for the chromatic number.*

The proof of the t-perfect case merely needs the t-imperfection of certain graphs. Hence, we expect that this conjecture can be proved without knowing whether Conjecture 5.1.5 holds for claw-free graphs.

Let G be the complement of a line-graph. We proved that each induced subgraph H of G satisfies $\chi(H) = \lceil \chi_f(H) \rceil$ if and only if G does not have an induced $\overline{L(\Pi)}$ or $\overline{L(W_5)}$. We conjecture that this 0-1 case of the IRCN extends as follows:

CONJECTURE 9.0.8 *Let G be an h-perfect complement of a line-graph. The following statements are equivalent:*

i) G has the integer round-up property for the chromatic number,
ii) G does not have an induced $\overline{L(\Pi)}$ or $\overline{L(W_5)}$.

By Proposition 3.6.15, *a t-perfect graph G is 3-colorable if and only if the round-up equality of the IRCN holds for 0-1 weights* (see Section 3.6.4 and Chapter 7). This is an NP-characterization of the unweighted case of the IRCN.

In Chapter 7, we showed a t-perfect graph which is 3-colorable and does not have the IRCN. Hence this characterization does not extend to arbitrary weights. We do not know if the IRCN of t-perfect graphs is in NP.

Our example has a triangle and we do not know if 3-colorability certifies the IRCN for t-perfect triangle-free graphs. If this and Conjecture 7.1.5 hold, then each triangle-free t-perfect graph would have the IRCN.

QUESTION 9.0.9 (SEBŐ) *Does each triangle-free t-perfect graph have the integer round-up property for the chromatic number?*

Except t-perfect claw-free graphs and h-perfect line-graphs, the only class of t-perfect graphs for which the IRCN is known is the class of odd-K_4-free graphs [102] (that is graphs which do not contain an odd subdivision of K_4).

A graph is *hereditary t-perfect* if all its subgraphs are t-perfect. Obviously, odd-K_4-free graphs are hereditary t-perfect. Gerards and Shepherd [51] proved that *a graph is hereditary t-perfect if and only if it does not contain a t-imperfect subdivision of* K_4, and that these graphs are 3-colorable. The t-imperfect subdivisions of K_4 are characterized by Barahona and Mahjoub in [3].

QUESTION 9.0.10 *Do hereditary t-perfect graphs have the integer round-up property for the chromatic number?*

A decomposition result for hereditary t-perfect graphs along vertex-cuts with at most two vertices is given in [51]. The elementary bricks of the decomposition are the odd-K_4-free graphs and a few other basic graphs. Hence, it is tempting to try proving the IRCN by showing it for these bricks and lifting the property along the vertex-cuts.

It is not clear whether the method of Kilakos and Marcotte [67] can be applied.

PARITY AND EAR-DECOMPOSITIONS Let G be a graph. Recall that $\beta(G)$ denotes the largest integer k such that G has a subgraph which has an open odd ear-decomposition with k ears.

We have seen that β is related to h-perfection in line-graphs, the matching polytope and totally odd subdivisions of K_4.

Besides, determining β is a fixed-parameter-tractable problem (see Theorem 8.1.4). Clearly, the property $\beta(G) \geq k$ is in NP (for a graph G and a non-negative integer k). We do not know if it is in co-NP.

QUESTION 8.1.5 ([8]) *Can β can be computed in polynomial-time ?*

For each 2-connected graph G, let $\overline{\varphi}(G)$ denote the largest number of odd ears in an open ear-decomposition of G. Frank [47] showed that $\overline{\varphi}(G)$ and an optimal ear-decomposition can be computed in polynomial-time. Further related results were obtained by Szigeti [113] and Szegedy [112].

Clearly, each 2-connected graph G satisfies $\overline{\varphi}(G) \geq \beta(G)$. In Chapter 8, we showed a sequence of graphs $(H_k)_{k\geq 1}$ such that $\beta(H_k) = 2$ whereas $\overline{\varphi}(H_k) \to +\infty$.

Still, it is not clear whether results on $\overline{\varphi}$ could be useful in computing β.

On the other hand, $\beta(G)$ is the largest number of ears of a 2-connected factor-critical subgraph of a graph G. Hence, Theorem 3.7.1 shows that $\beta(G)$ can be used as a parameter to separate on, for questions on the matching polytope (see Section 8.3.1 for further details).

For each graph G, let $\hat{G}$ denote the underlying simple graph of G. A first result in this direction is Theorem 6.1.3, which states that *each graph G with $\beta(\hat{G}) \leq 1$ is such that* $\mathsf{MATCH}(G)$ *has the integer decomposition property*. We conjecture the following extension:

Conjecture 8.3.1 *Each graph G with $\beta(\hat{G}) \leq 3$ satisfies $\chi'(G) = \lceil \chi'_f(G) \rceil$.*

The bound 3 would be tight, as shows the Petersen graph minus a vertex (see Section 8.3.2.2).

Finally, we proved that totally odd subdivisions of K_4 can be easily detected using $\overline{\varphi}$ in odd-C_3^+-free graphs (that is graphs satisfying $\beta \leq 1$).

As the currently known algorithms for detecting totally odd subdivisions of K_4 in arbitrary graphs use techniques of the Graph Minor Project, it is natural to ask for a rather simple and elementary algorithm. We saw in Section 8.4.2 that our approach for graphs with $\beta \leq 1$ cannot be directly extended to larger values of β. This does not exclude a possible use of β for detecting totally odd subdivisions of K_4 in general.

10 CONCLUSION (EN FRANÇAIS)

Les graphes parfaits montrent qu'imposer des hypothèses combinatoires sur un graphe peut conduire à une structure simple des facettes de son polytope des stables. Réciproquement, une structure similaire des facettes induit-elle des propriétés combinatoires remarquables sur le graphe ? On sait très peu de choses sur ce méta-problème et l'étude de la h-perfection en est une approche élémentaire : les seules facettes qui ne sont pas définies par la non-negativité ou des cliques sont données par des circuits impairs.

En comparaison des graphes parfaits, existe-t-il aussi seulement un petit nombre de familles de graphes minimalement t-imparfaits (ou h-imparfaits) ? Admettent-ils une bonne caractérisation ? La h-perfection peut-elle être testée en temps polynomial ? Le nombre chromatique d'un graphe h-parfait reste-t-il proche de ω ?

Aucune de ces questions n'a encore reçue de réponse définitive. On espère avoir contribué à clarifier certains cas particuliers.

Dans ce dernier chapitre, on se propose d'énumérer les questions et conjectures énoncées dans les chapitres précédents. Nous suggérons aussi quelques perspectives et problèmes supplémentaires.

caractérisations combinatoires On ne connaît pas la complexité de la reconnaissance de la t-perfection. Il n'est pas difficile de voir que le problème est dans co-NP (voir [102, pg. 1194]). Cependant on ne dispose pas d'un certificat combinatoire de t-imperfection.

L'utilisation d'opérations qui préservent la t-perfection et réduisent la taille du graphe pourrait conduire à une caractérisation par mineurs-exclus et un tel certificat. Les t-mineurs et la contraction d'une paire d'amis sont les seules opérations de cet ordre connues à ce jour.

Dans le Chapitre 5, nous avons dressé l'inventaire des exemples connus de graphes minimalement t-imparfaits présentés dans [15, 13] (voir leur définition dans le Chapitre 5)

les roues impaires, les échelles de Möbius paires, C_7^2 et les $(3,3)$-graphes $(*)$

Nous ne connaissons pas d'autres exemples de graphes minimalement t-imparfaits, et l'étude de la t-perfection dans des classes de graphes fermées par t-mineurs pourrait conduire à de tels exemples. En particulier (voir la Section 5.2.4 pour la définition d'un graphe partitionnable) :

Question 5.3.11 *K_4, W_5, C_7^2 et $\overline{C_{10}^2}$ sont-ils les seuls graphes minimalement t-imparfaits sans P_5 (induit) ?*

QUESTION 10.0.1 *Existe-t-il des graphes minimalement t-imparfaits partitionnables de stabilité supérieure ou égale à 4 ?*

L'appartenance de la h-perfection à NP ou co-NP reste ouverte. Rappelons qu'un graphe h-imparfait est *critique* si tous ses sous-graphes induits propres sont h-parfaits.

CONJECTURE 5.1.5 (SEBŐ) *Tout graphe h-imparfait critique avec $\omega \geq 4$ est un anti-trou impair.*

Si cette conjecture est valide, alors les graphes minimalement h-imparfaits sont des anti-trous impairs ou n'ont pas de K_4. D'après le Théorème 5.1.4, les minimalement h-imparfaits sans K_4 sont minimalement t-imparfaits ou l'un des graphes K_4^*, W_5^-, W_5^{--}. Ainsi, puisque la t-perfection est co-NP, la conjecture impliquerait une caractérisation co-NP de la h-perfection.

Au Chapitre 5, on a montré que l'énoncé de la Conjecture 5.1.5 peut être facilement utilisé pour déduire le Théorème Fort des Graphes Parfaits du cas particulier des graphes sans K_4 (considérablement plus simple et dû à Tucker [116]). Ainsi, il paraît nécessaire de faire intervenir le théorème fort pour prouver la Conjecture 5.1.5.

Par exemple, il suffirait de montrer que *tout graphe h-imparfait critique est minimalement imparfait*.

Nous avons observé dans le Chapitre 4 que la perfection est conservée par la t-contraction. On ne sait pas si cette opération pourrait être utilisée avec profit dans certaines parties de la théorie des graphes parfaits.

GRAPHES H-PARFAITS SANS GRIFFE On a montré que la Conjecture 5.1.5 est trivialement satisfaite par les graphes planaires. Cependant, le cas des graphes sans griffe reste ouvert.

Tout anti-trou impair est évidemment sans griffe et partitionnable. Nous avons montré dans la Section 5.5 qu'un théorème de [116] et le Théorème 5.1.8 impliquent que les graphes h-imparfaits critiques sans griffe, partitionnables et avec $\omega \geq 4$ sont des anti-trou impairs. Ainsi, il suffirait de prouver l'assertion suivante pour démontrer la Conjecture 5.1.5 :

CONJECTURE 5.1.9 *Tout graphe h-imparfait critique sans griffe avec $\omega \geq 4$ est partitionnable.*

D'après les Théorèmes 5.1.1 et 5.1.4, la validité de cette assertion impliquerait que *les graphes minimalement h-imparfaits sans griffe sont : W_5, W_5^-, W_5^{--}, C_{10}^2 et les $\overline{C_{2k+1}}$ avec $k \geq 3$*. Ainsi, la h-perfection d'un graphe sans griffe serait une propriété co-NP.

Nous ne savons pas si le théorème de décomposition des graphes sans griffe dû à Chudnovsky et Seymour [23] pourrait être utilisé afin de produire une preuve de la Conjecture 5.1.9.

Bruhn et Schaudt [14] ont montré que les graphes t-parfaits sans griffe peuvent être reconnus en temps polynomial. Nous avons donné au Chapitre 8 un algorithme simple et efficace pour décider la h-perfection d'un graphe adjoint.

QUESTION 10.0.2 *Peut-on décider en temps polynomial la h-perfection d'un graphe sans griffe ?*

STRUCTURE DES GRAPHES MINIMALEMENT H-IMPARFAITS Dans [16], Bruhn et Stein posent la question suivante (voir Section 5.3) :

QUESTION 10.0.3 *Les graphes minimalement t-imparfaits sont-ils tous 3-connexes ?*

D'après le Théorème 5.3.2 de [16], cela équivaut à demander : *les graphes minimalement t-imparfaits ont-ils tous un degré minimum supérieur ou égal à 3 ?*

On rappelle que $\mathsf{TSTAB}(G)$ désigne le polyèdre de $\mathbb{R}^{V(G)}$ décrit par les inégalités de non-négativité, cliques et circuits impairs d'un graphe G.

QUESTION 5.3.7 (BRUHN, STEIN [16]) *Soit G un graphe minimalement t-imparfait. Le polytope $\mathsf{TSTAB}(G)$ a-t-il un unique sommet non-entier ?*

La preuve du théorème analogue pour le cas parfait (Théorème 5.1.2 dû à Padberg [89]) suggère les deux problèmes associés suivants :

QUESTION 5.3.8 *Un graphe minimalement t-imparfait peut-il avoir à la fois des sommets contractibles et non-contractibles ?*

QUESTION 5.3.9 *Les trous impairs d'un graphe minimalement t-imparfait ont-ils tous la même longueur ?*

On a démontré que *K_4^*, W_5^- et W_5^{--} sont les seuls graphes minimalement t-imparfaits sans K_4 qui ne sont pas minimalement t-imparfaits* (Théorème 5.1.4). Ils ne sont pas 3-connexes et ont tous à la fois des sommets contractibles et non-contractibles.

Par ailleurs, $\mathsf{TSTAB}(W_5^{--})$ a exactement deux sommets non-entiers. Ainsi, l'extension des questions ci-dessus aux graphes minimalement t-imparfaits aux h-imparfaits requiert certaines exceptions supplémentaires :

QUESTION 5.4.3 *K_4^*, W_5^- et W_5^{--} sont-ils les seuls graphes minimalement h-imparfaits qui ont des sommets de degré 2 (c'est à dire qui ne sont pas*

3-connexes) ?

On rappelle que $\mathsf{HSTAB}(G)$ est le polyèdre de $\mathbb{R}^{V(G)}$ décrit par les inégalités de non-négativité, cliques et circuits impairs d'un graphe G.

QUESTION 10.0.4 *W_5^{--} est-il le seul graphe minimalement h-imparfait dont le polytope des stables a plus d'un sommet non-entier ?*

Notons que W_5^{--} est le seul graphe minimalement h-imparfait qui n'est pas premier (voir Théorème 4.1.6).

Les résultats de Padberg sur les graphes minimalement imparfaits [89] et ceux de Shepherd sur les proche-parfaits [108] motivent l'étude du polytope des stables des graphes minimalement h-imparfaits.

Un graphe G est *proche-h-parfait* si son polytope des stables est décrit par les inégalités de non-négativité, cliques, circuits impairs et l'inégalité de plein-rang $x(V(G)) \leq \alpha(G)$. L'étude des exemples connus de graphes minimalement t-imparfaits et h-imparfaits nous a conduit à conjecturer l'assertion suivante :

CONJECTURE 5.3.4 *Exceptées les roues impaires W_{2n+1} avec $n \geq 2$, tout graphe minimalement t-imparfait est proche-h-parfait.*

Par ailleurs, Bruhn conjecture (communication personnelle) :

CONJECTURE 5.3.5 (BRUHN) *Tout graphe minimalement t-imparfait G satisfait : $\chi_f(G) > \max(\omega(G), \Gamma(G))$.*

Si la Conjecture 5.3.4 est valide, alors l'assertion de la Conjecture 5.3.5 est équivalente à affirmer qu'*à l'exception des W_{2n+1} avec $n \geq 2$, tout graphe minimalement t-imparfait G satisfait :*

$$|V(G)| > \alpha(G) \cdot \max(\omega(G), \Gamma(G)).$$

Ceci fournirait ainsi un résultat analogue au théorème de Lovász pour les graphes parfaits [71] qui affirme que *tout graphe minimalement imparfait G satisfait $|V(G)| > \alpha(G)\omega(G)$.*

Les résultats de la Section 5.4.1 montrent qu'on ne peut étendre la Conjecture 5.3.4 aux graphes minimalement h-imparfaits qu'en incluant une nouvelle exception :

CONJECTURE 5.1.3 *A l'exception de K_4^* et des roues impaires W_{2n+1} avec $n \geq 2$, tout graphe minimalement h-imparfait est proche-h-parfait.*

NOMBRE CHROMATIQUE On ne sait pas s'il existe une constante k telle que tout graphe t-parfait est k-colorable. La plus grande valeur connue du nombre chromatique d'un graphe t-parfait est 4, comme le montre $\overline{L(\Pi)}$ (voir aussi le Chapitre 7).

CONJECTURE 7.1.5 (SEBŐ, DANS [16]) *Tout graphe t-parfait sans triangle est 3-colorable.*

Sebő a démontré que la validité de cette conjecture impliquerait que tout graphe h-parfait est $(\omega + 1)$-colorable (voir le Chapitre 7). En particulier, les graphes t-parfaits seraient 4-colorables.

Un graphe est C_5*-couvert* si tout sommet appartient à au moins un trou impair de longueur 5. Nous avons montré que la Conjecture 7.1.5 découlerait du cas C_5-couvert (cette réduction est suggérée par Sebő) :

CONJECTURE 7.5.12 *Tout graphe t-parfait sans triangle et C_5-couvert est 3-colorable.*

Un graphe est *k-critique* si son nombre chromatique est k et tous ses sous-graphes induits propres sont $(k-1)$-colorables. Le Théorème 7.1.4 montre que $\overline{L(\Pi)}$ et $\overline{L(W_5)}$ sont les seuls graphes complémentaire-adjoints qui sont t-parfaits et 4-critiques.

QUESTION 10.0.5 $\overline{L(\Pi)}$ *et* $\overline{L(W_5)}$ *sont-ils les seuls graphes t-parfaits 4-critiques ?*

D'autre part, nous avons prouvé que *tout graphe h-parfait sans P_6 (induit) est $(\omega+1)$-colorable* (Théorème 7.1.6). En particulier, tout graphe t-parfait sans P_6 est 4-colorable. Les graphes $\overline{L(\Pi)}$ and $\overline{L(W_5)}$ certifient que la borne $\omega + 1$ est serrée.

Ces graphes contiennent cependant tous deux un P_5 induit et nous ne savons donc pas si la borne supérieure 4 pourrait être diminuée pour la classe des graphes sans P_5 (induit).

QUESTION 10.0.6 *Les graphes t-parfaits sans P_5 sont-ils 3-colorables ?*

D'après le Théorème 7.5.2, une réponse positive impliquerait que tout graphe h-parfait sans P_5 (induit) G satisfait : $\chi(G) = \lceil \chi_f(G) \rceil$.

PROPRIÉTÉ DE DÉCOMPOSITION ENTIÈRE Nous avons prouvé au Chapitre 6 que les graphes t-parfaits sans griffe et les graphes h-parfaits adjoints ont la propriété d'arrondi entier pour le nombre chromatique (abrégée AENC). Nous conjecturons que cette propriété est partagée par les graphes h-parfaits sans griffe :

CONJECTURE 10.0.7 *Tout graphe h-parfait sans griffe a la propriété d'arrondi entier pour le nombre chromatique.*

Notons que la preuve du cas t-parfait n'utilise pas pleinement la caractérisation de [16], mais seulement la t-imperfection de certains graphes. On s'attend ainsi à ce que cette conjecture puisse être prouvée sans connaître la validité de la Conjecture 5.1.5 pour les graphes sans griffe.

Soit G le complémentaire d'un graphe adjoint. Nous avons démontré que tout sous-graphe induit H de G satisfait $\chi(H) = \lceil \chi_f(H) \rceil$ si et seulement si G n'a pas de sous-graphe induit isomorphe à $\overline{L(\Pi)}$ ou $\overline{L(W_5)}$. Nous conjecturons que ce cas 0-1 peut s'étendre à des poids quelconques :

CONJECTURE 10.0.8 *Soit G un graphe h-parfait qui est le complémentaire d'un graphe adjoint. Les assertions suivantes sont équivalentes :*

i) G a la propriété d'arrondi entier pour le nombre chromatique,
ii) G n'a pas de sous-graphe induit isomorphe à $\overline{L(\Pi)}$ ou $\overline{L(W_5)}$.

La Proposition 3.6.15 affirme qu'un graphe t-parfait est 3-colorable si et seulement s'il a la propriété AENC pour les poids 0-1 (voir la Section 3.6.4 et le Chapitre 7). Ce qui est clairement une caractérisation NP du cas 0-1 de la propriété AENC.

Nous avons donné au Chapitre 7 un exemple de graphe t-parfait 3-colorable qui n'a pas la propriété AENC. Ainsi, cette caractérisation ne s'étend pas à tout poids et nous ne savons pas si la propriété AENC pour les graphes t-parfaits appartient à NP.

Notre exemple contient un triangle. Ainsi, l'équivalence de la 3-colorabilité et de la propriété AENC reste ouverte pour les graphes t-parfaits sans triangle. Si celle-ci est vraie et si la Conjecture 7.1.5 est valide, alors tout graphe t-parfait sans triangle a la propriété AENC.

QUESTION 10.0.9 (SEBŐ) *Les graphes t-parfaits sans triangle ont-ils la propriété d'arrondi entier pour le nombre chromatique ?*

À l'exception des graphes t-parfaits sans griffe ou h-parfaits adjoints, la propriété AENC n'est connue que pour une seule autre classe de graphes h-parfaits : les graphes sans subdivision impaire de K_4 [102].

Un graphe est *t-parfait héréditaire* si tous ses sous-graphes sont t-parfaits (par exemple les graphes sans subdivision impaire de K_4). Gerards et Shepherd [51] ont démontré que les graphes t-parfaits héréditaires sont exactement ceux qui ne contiennent pas de subdivision t-imparfaite de K_4, et que ces graphes sont 3-colorables. Les subdivisions de K_4 qui ne sont pas t-parfaites ont été caractérisées par Barahona et Mahjoub dans [3].

QUESTION 10.0.10 *Les graphes t-parfaits héréditaires ont-ils la propriété d'arrondi entier pour le nombre chromatique ?*

Gerards et Shepherd [51] ont montré un résultat de décomposition pour les graphes t-parfaits héréditaires en termes de séparateurs à au plus 2 sommets. Les briques élémentaires de cette décomposition sont les graphes sans K_4-impair et quelques autres graphes. Il est dès lors tentant d'approcher la question ci-dessus en prouvant la propriété AENC pour ces briques d'abord et en transmettant ensuite la propriété au graphe considéré.

Dans ce contexte, il n'est pas évident que la méthode proposée et utilisée par Kilakos et Marcotte [67] pour les graphes série-parallèles s'applique ici.

PARITÉ ET DÉCOMPOSITIONS D'OREILLES Soit G un graphe. Rappelons que $\beta(G)$ désigne le plus grand entier k tel que G a un sous-graphe admettant une décomposition d'oreilles ouvertes à k oreilles.

Le paramètre β est relié à la h-perfection des graphes adjoints, au polytope des couplages et aux subdivisions totalement impaires de K_4.

Par ailleurs, le calcul de β est un problème résoluble en temps polynomial à paramètre fixé (Théorème 8.1.4). La propriété $\beta \geq k$ est évidemment dans NP, mais nous ne savons pas si elle appartient à co-NP. Aussi, nous posons la question suivante :

QUESTION 8.1.5 *β peut-il être calculé en temps polynomial ?*

Pour tout graphe 2-connexe G, $\overline{\varphi}(G)$ désigne le plus grand nombre d'oreilles impaires dans une décomposition d'oreilles ouvertes de G. Frank [47] a prouvé que $\overline{\varphi}$ et une décomposition d'oreilles optimales peuvent être calculées en temps polynomial. Ces résultats ont été étendus par Szigeti [113] et Szegedy [112].

On vérifie aisément que tout graphe 2-connexe G satisfait $\overline{\varphi}(G) \geq \beta(G)$. Nous avons construit au Chapitre 8 une suite de graphes $(H_k)_{k\geq 1}$ telle que $\beta(H_k) = 2$ tandis que $\overline{\varphi}(H_k) \to +\infty$.

On ne peut cependant pas écarter définitivement l'utilité de $\overline{\varphi}$ dans le calcul de β.

Soit G un graphe. On a vu que $\beta(G)$ est le plus grand nombre d'oreilles d'un sous-graphe facteur-critique 2-connexe d'un graphe G. Ainsi, le Théorème 3.7.1 montre que β peut être utilisé comme paramètre critique dans des questions liées au polytope des couplages (voir la Section 8.3.1 pour davantage de détails).

Un premier résultat dans cette direction est le Théorème 6.1.3 : il énonce que *le polytope des couplages de tout graphe G avec $\beta(\hat{G}) \leq 1$ a la propriété de décomposition entière* (où $\hat{G}$ désigne le sous-graphe simple sous-jacent à G). Nous conjecturons l'extension suivante :

CONJECTURE 8.3.1 *Tout graphe G avec $\beta(\hat{G}) \leq 3$ satisfait $\chi'(G) = \lceil \chi'_f(G) \rceil$.*

Le graphe de Petersen montre que la borne 3 serait alors serrée (voir Section 8.3.2.2).

Enfin, nous avons prouvé qu'une subdivision totalement impaire de K_4 peut être facilement détectée dans les graphes sans C_3^+-impair (c'est à dire avec $\beta \leq 1$) en utilisant $\overline{\varphi}$.

Étant donné que les algorithmes généraux disponibles pour la détection d'une subdivision totalement impaire de K_4 utilisent les techniques du Graph Minor Project, la recherche d'algorithmes élémentaires pour ce problème reste pertinente. On a observé à la Section 8.4.2 que notre approche pour les graphes sans C_3^+-impair ne peut être directement étendue à des valeurs plus grandes de β. Ceci n'exclut cependant pas un éventuel usage profitable de β pour la détection des subdivisions totalement impaires de K_4 en général.

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