Control of Hyperbolic Systems by Lyapunov Analysis

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Louis-Ferdinand Céline, *Féerie pour une autre fois*

Contents

Notation

- The set of square real matrices of dimension n is denoted by $\mathbb{R}^{n\times n}$.
- The set $\mathbb{R}^+$ is the set of nonnegative real numbers.
- Given a matrix A, the transpose of the matrix A is denoted by $A^\top$.
- Given N square matrices $A_1, \ldots, A_N$, of respective dimension $k_1, \ldots, k_N$, the block diagonal matrix $A \in \mathbb{R}^{(k_1+\cdots+k_N)\times(k_1+\cdots+k_N)}$ whose block diagonal matrices are $A_1, \ldots, A_N$, is denoted by $\texttt{diag}\,[A_1, \ldots, A_N]$.
- The identity matrix of dimension n is denoted by I_n.
- 0 denoted the zero matrix of suitable dimension according to the context.
- The entry (i,j) of a matrix A is denoted by $A[i,j]$.
- For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, A being positive definite is denoted $A > 0$, while A being positive semi-definite is denoted $A \geq 0$.
- The derivative of a matrix $A(x)$ with respect to variable x is denoted by $A'(x)$.
- The usual Euclidian norm in $\mathbb{R}^n$ is denoted by $|\cdot|$, and the associated matricial norm in $\mathbb{R}^{n\times n}$ is denoted by $\|\cdot\|$.
- Let $I \subseteq \mathbb{R}$ and $J \subseteq \mathbb{R}^p$ for some $p \geq 1$.
- The set of functions $y : I \to J$ such that $|y|^p_{L^p(I,J)} = \int_I |y(x)|^p \, dx < \infty$, is denoted by $L^p(I;J)$.
- The set of functions $y \in L^2(I;J)$ such that there exists a function $g \in L^2(I;J)$ such that $\int_I y\varphi' = -\int_I g\varphi$ for all $\varphi \in C^1_c(I;J) := \left\{h \in C^1(I;J) \,|\, supp(h) \subset I\right\}$ if I is unbounded or $\varphi \in C^1_c(I;J) := \left\{h \in C^1(I;J) \,|\, supp(h) \subseteq I\right\}$ if I is bounded where $supp(h) := \overline{\{x \in I \,|\, h(x) \neq 0\}}$, is denoted by $H^1(I;J)$.
- We denote $y \in L^p_{\mathrm{loc}}(I,J)$ if $y\mathbb{1}_K \in L^p(I,J)$ for all compact subsets K of I, where $\mathbb{1}_K$ is the indicator function of K.
- The scalar product of two functions y_1 and y_2 in $L^2((0,1);\mathbb{R}^n)$ is denoted by (y_1, y_2).
- The restriction of a function $y : I \to J$ on an open interval $(x_1, x_2) \subset I$ is denoted by $y_{|(x_1,x_2)}$.

1. Introduction

In this chapter, we start by introducing the Partial Differential Equations (PDEs) analyzed all along this thesis. More precisely, we introduce hyperbolic systems of *balance laws*. Then, we present the *method of characteristics* which is an important tool to analyze and describe solutions for this type of systems. Three possible applications for hyperbolic PDEs are presented: the *Wave Equation*, the *Saint-Venant Equations*, and the *Aw-Rascle-Zhang Equations* which have been used for the illustration of the different results developed in the other chapters. The first one though a fundamental example, describes the motion of a string deflection, the second one describes the motion of water in a canal, while the third one describes in a macroscopic way the motion of the traffic of vehicles along a road. Therefore, the potential applications given by these two last equations are important from an engineering point of view. In a third part, some of the results for the stability and stabilizability of balance laws are presented. In the last part of this chapter we will present the main contributions of our work.

1.1 Hyperbolic System of Balance Laws

1.1.1 Derivation of the Balance Laws

First of all, we will sketch the “origin” of systems of balance laws. This presentation takes its inspiration from the exposition given in [83].

A wide number of mathematical models (for instance, physical, biological) describe the evolution of physical quantities. Roughly speaking, evolution of quantities means that there is a flow of mass through a physical domain. The flow may depend on the position in the domain, let us denote this variable x.

This notion of *flow* is the key to derive hyperbolic systems of balance laws. Let us consider an open domain Ω and any open regular vicinity $\mathcal{V}$ of a point $\mathbf{x} \in \Omega$ and suppose that a flow of material go through this vicinity, see Figure 1.1. Let us denote $q(t,x)$ this flow. Now consider a surface element of the domain $\mathcal{V}$, denoted ΔS, and denote n the normal of the surface ΔS at the position x. Therefore, the quantity of material passing through the volume during the time interval $[t, t + \Delta t]$ is given by

$$m = n.q(t,x)\Delta S \Delta t\,. \tag{1.1}$$

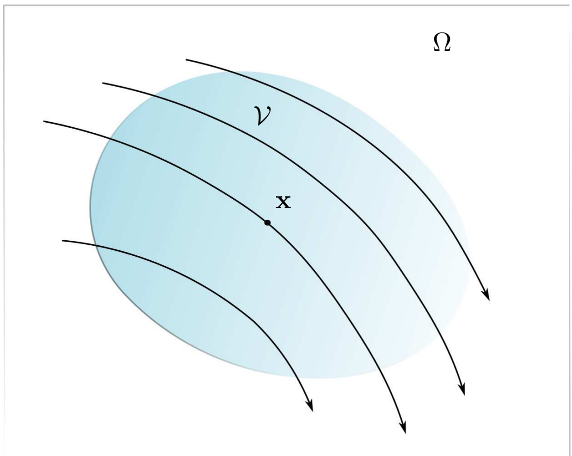

Figure 1.1: Flow through the vicinity $\mathcal{V}$ of $\mathbf{x}$.

The total flow through the surface S is obtained by integrating (1.1) and adding a minus if the scalar product of n and q is negative

$$M = \Delta t \int_S (-n).q(t,s)ds \tag{1.2}$$

which corresponds to contribution of mass in the volume $\mathcal{V}$ during the small time interval $[t, t + \Delta t]$.

Let us consider that at a time t the mass present in the volume is $M_t(\mathcal{V})$, and that at time $t + \Delta t$

the mass is $M_{t+\Delta t}(\mathcal{V})$. Therefore, the variation of mass between t and $t+\Delta t$ is given by

$$\Delta M_t(\mathcal{V}) = M_{t+\Delta t}(\mathcal{V}) - M_t(\mathcal{V})\,.$$

In regards of (1.2) we can write

$$\Delta M_t(\mathcal{V}) = \Delta t \int_{\partial\mathcal{V}} (-n).q(t,s)ds + \Delta t \int_{\mathcal{V}} Q(t,x)dx\,, \tag{1.3}$$

where Q is the rate of mass production per volume. Now dividing (1.3) by Δt and taking the limit as Δt goes to 0 we get

$$\frac{dM_t(\mathcal{V})}{dt} = \int_{\partial\mathcal{V}} (-n).q(t,s)ds + \int_{\mathcal{V}} Q(t,x)dx\,.$$

Moreover let us assume that the measure $\mathcal{V} \mapsto M_t(\mathcal{V})$ has a density ρ we respect to the Lebesgue measure, we get

$$\frac{d}{dt}\int_{\mathcal{V}} \rho(t,x)dx + \int_{\partial\mathcal{V}} n.q(t,s)ds = \int_{\mathcal{V}} Q(t,x)dx\,. \tag{1.4}$$

Let us recall the Stokes formula

$$\int_{\partial\mathcal{V}} n.q(s)ds = \int_{\mathcal{V}} \operatorname{div} q(x)dx\,, \quad \forall q \in H^1(\mathcal{V})\ , \tag{1.5}$$

and let us apply it to the second member of the left-hand side of (1.4), it gives

$$\int_{\mathcal{V}} \left(\frac{\partial\rho}{\partial t}(t,x) + \operatorname{div} q(t,x)\right) dx = \int_{\mathcal{V}} Q(t,x)dx\,.$$

This last relation is valid for any vicinity $\mathcal{V}$ of any point $\mathbf{x}$ in the open set Ω and all time t in a time interval $(0,T)$ then we get a local expression for the balance law

$$\partial_t\rho(t,x) + \operatorname{div} q(t,x) = Q(t,x)\,, \quad (t,x) \in (0,T)\times\Omega\,. \tag{1.6}$$

In this thesis we are interested in system evolving in the one dimensional domain $\Omega = (0,1)$, thus equation (1.6) becomes

$$\partial_t\rho(t,x) + \partial_x q(t,x) = Q(t,x)\,, \quad (t,x) \in (0,T)\times(0,1)\ . \tag{1.7}$$

We are interested in the evolution of n quantities $z : \mathbb{R}^+ \times (0,1) \to \mathbb{R}^n$. We assume there exists a link between them and the density ρ, the flow q, and the rate of mass production Q. More precisely, we have relations

$$\begin{aligned} \rho(t,x) &= f(z(t,x)) \\ q(t,x) &= g(z(t,x)) \\ Q(t,x) &= h(z(t,x))\,. \end{aligned}$$

Hence, systems of balance laws are written as

$$\partial_t f(z(t,x)) + \partial_x g(z(t,x)) = h(z(t,x))\,, \quad (t,x) \in \mathbb{R}^+ \times (0,1)\ .$$

Let $\mathcal{Z}$ a non-empty connected open subset of $\mathbb{R}^n$. Let us make the following assumption.

Assumption 1.1.

1. *$f \in C^2(\mathcal{Z};\mathbb{R}^n)$ is a diffeomorphism on $\mathcal{Z}$;*
2. *$g \in C^2(\mathcal{Z};\mathbb{R}^n)$;*
3. *$h \in C^1(\mathcal{Z};\mathbb{R}^n)$.*

Under Assumption 1.1 the system can be written under the following form

$$\partial_t z(t,x) + \Phi\left(z(t,x)\right)\partial_x z(t,x) = \Upsilon\left(z(t,x)\right)\ , \tag{1.8}$$

where $\Phi : \mathcal{Z} \to \mathbb{R}^{n\times n}$ and $\Upsilon : \mathcal{Z} \to \mathbb{R}^{n\times n}$ with

$$\begin{aligned}\Phi(z) &= \left(\frac{\partial f}{\partial z}\right)^{-1}\left(\frac{\partial g}{\partial z}\right)\\ \Upsilon(z) &= \left(\frac{\partial f}{\partial z}\right)^{-1} h(z)\,.\end{aligned}$$

Definition 1.1. *System (1.8) is said to be hyperbolic if $\Phi(z)$ has n real eigenvalues for all $z \in \mathcal{Z}$.*

Definition 1.2. *System (1.8) is said to be strictly hyperbolic if $\Phi(z)$ has n real distinct eigenvalues for all $z \in \mathcal{Z}$.*

Definition 1.3. *When $\Upsilon \equiv 0$, system (1.8) is said to be a system of conservation laws.*

System (1.8) may be expressed as a diagonal quasi-linear system with a change of variable $y = \chi(z)$, that is

$$\partial_t y(t,x) + \Lambda\left(y(t,x)\right)\partial_x y(t,x) = F(y(t,x))\,, \quad (t,x) \in \mathbb{R}^+ \times [0,1]\ , \tag{1.9}$$

where $\Lambda(y)$ is a diagonal matrix for all $y \in \mathcal{Y} = \chi(\mathcal{Z}) \subset \mathbb{R}^n$. Let us give some properties on the map χ:

1. The function $\chi : \mathcal{Z} \mapsto \mathcal{Y} \subset \mathbb{R}^n$ is a diffeomorphism;
2. The Jacobian matrix $\psi(z)$ of $\chi(z)$ diagonalizes the matrix $\Phi(z)$, that is

$$\psi(z)\Phi(z) = D(z)\psi(z)\,, z \in \mathcal{Z}\,, \tag{1.10}$$

   where $D(z)$ is a diagonal matrix for all $z \in \mathcal{Z}$.

Thus, system (1.8) can be written under the form (1.9) with

$$\Lambda(y) = D\left(\chi^{-1}(y)\right)$$

$$F(y) = \psi\left(\chi^{-1}(y)\right) \Upsilon\left(\chi^{-1}(y)\right) .$$

The change of variables is not always effective, since the equation (1.10) may have no solution, see [6] for references and examples of this property. Nonetheless, as shown in [73] this change of variables always exists for $n = 2$, at least locally.

Remark 1.1. Under suitable compatibility conditions on the initial condition, we may prove (see [75] and Theorem 6.1 of [6]) that there exists a unique classical solutions in $C^1([0,T] \times [0,1])$ of (1.9). Thus imposing $x \in [0,1]$ instead of $x \in (0,1)$ in (1.9) is not abusive. •

Remark 1.2. The spatial domain can take the more general form $[0, L]$. Nonetheless, we will suppose for all our study that $L = 1$. It is not a restriction, since by a change of variable the system may always be put in a dimensionless form, meaning that the spatial domain may be $[0, 1]$. •

1.1.2 Riemann Coordinates, Steady-State, and Linearization

In this thesis, we will work with hyperbolic systems under a linear form, that is

$$\partial_t y(t,x) + \Lambda(x)\partial_x y(t,x) = F(x)y(t,x)\,, \quad (t,x) \in \mathbb{R}^+ \times [0,1]\ , \tag{1.11}$$

where $\Lambda(x)$ is a diagonal matrix for all $x \in [0, 1]$.

Let us show how to arrive formally to a system under the form (1.11). The existence of the change of variables χ, defined above, will not be an issue for our work. Indeed, we will look at system linearized around a steady state. Let us explain the nature of such framework.

Definition 1.4. *A steady state for system (1.8) is a solution z^* of (1.8) for which $\partial_t z^* \equiv 0$.*

Definition 1.4 leads to the Ordinary Differential Equation (ODE) for z^*

$$\Phi\left(z^*\right) \frac{dz^*}{dx} = \Upsilon\left(z^*\right) . \tag{1.12}$$

Then, we define the deviation of the state with respect to the steady state z^* by

$$\tilde{z} = z - z^* . \tag{1.13}$$

Let us define the function $\mathcal{F} : \mathbb{R}^{2n} \to \mathbb{R}^n$ by

$$\mathcal{F}\left(z(t,x), \partial_x z(t,x)\right) = -\Phi(z(t,x))\partial_x z(t,x) + \Upsilon(z(t,x))\,.$$

We have

$$\partial_t z(t,x) = \mathcal{F}\left(z(t,x), \partial_x z(t,x)\right) . \tag{1.14}$$

A Taylor formula for $\mathcal{F}$ around $\left(z^*(x), \frac{dz^*}{dx}(x)\right)$ gives

$$\mathcal{F}\left(z(t,x), \partial_x z(t,x)\right) \approx \mathcal{F}\left(z^*(x), \frac{dz^*}{dx}(x)\right) + \left[\tilde{z}(t,x), \partial_x \tilde{z}(t,x)\right].\nabla_{(z,\partial_x z)}\mathcal{F}\left(z^*(x), \frac{dz^*}{dx}(x)\right).$$

We have

$$\nabla_{(z,\partial_x z)}\mathcal{F}\left(z^*(x), \frac{dz^*}{dx}(x)\right) = \begin{bmatrix} -\left[\nabla_z \Phi\left(z^*(x)\right) \frac{dz^*}{dx}(x)\right] + \nabla_z \Upsilon\left(z^*(x)\right) \\ -\Phi\left(z^*(x)\right) \end{bmatrix},$$

where the notation $\left[\nabla_z \Phi\left(z^*(x)\right) \frac{dz^*}{dx}(x)\right]$ stands for the matrix

$$\left[\nabla_z \Phi\left(z^*(x)\right) \frac{dz^*}{dx}(x)\right] = \left[\partial_{z_1} \Phi\left(z^*\right) \frac{dz^*}{dx}(x) \middle| \dots \middle| \partial_{z_n} \Phi\left(z^*\right) \frac{dz^*}{dx}(x)\right].$$

Hence it comes

$$\begin{aligned} \mathcal{F}\left(z(t,x), \partial_x z(t,x)\right) \approx{}& -\Phi\left(z^*(x)\right) \frac{dz^*}{dx}(x) + \Upsilon\left(z^*\right) \\ &+ \nabla_z \Upsilon\left(z^*(x)\right) \tilde{z}(t,x) - \left[\nabla_z \Phi\left(z^*(x)\right) \frac{dz^*}{dx}(x)\right] \tilde{z}(t,x) \\ &- \Phi\left(z^*(x)\right) \partial_x \tilde{z}(t,x). \end{aligned}$$

Using (1.14) and the fact that z^* is the solution of the ODE (1.12) we get the linearization

$$\partial_t \tilde{z}(t,x) + \Phi\left(z^*(x)\right) \partial_x \tilde{z}(t,x) = \left(\nabla_z \Upsilon\left(z^*(x)\right) - \left[\nabla_z \Phi\left(z^*(x)\right) \frac{dz^*}{dx}(x)\right]\right) \tilde{z}(t,x). \tag{1.15}$$

The linear hyperbolic system is said to be *uniform* if the matrix $\Phi\left(z^*(x)\right)$ is spatially constant, equivalently if $\Phi\left(z^*(x)\right) \equiv \Phi$. Otherwise the system is said to be *non-uniform*. Therefore, the uniformity of the steady state z^* will imply the uniformity of the system.

With the linearized expression (1.15), we can express explicitly the change of variable χ. Since, the system is hyperbolic the matrix $\Phi\left(z^*(x)\right)$ is diagonalizable. Then, there exists an invertible matrix $P(x)$ such that

$$\Lambda\left(z^*(x)\right) P(x) = P(x)\, \Phi\left(z^*(x)\right),$$

where $\Lambda\left(z^*(x)\right) = \mathtt{diag}\left[\lambda_1\left(z^*(x)\right), \dots, \lambda_n\left(z^*(x)\right)\right]$. Multiplying (1.15) by $P(x)$ from the left we get

$$\begin{aligned} \partial_t\left(P(x)\, \tilde{z}(t,x)\right) + \Lambda\left(z^*(x)\right) P(x)\, \partial_x \tilde{z}(t,x) = P(x) \Big(&\nabla_z \Upsilon\left(z^*(x)\right) \\ &- \left[\nabla_z \Phi\left(z^*(x)\right) \frac{dz^*}{dx}(x)\right]\Big) \tilde{z}(t,x). \end{aligned} \tag{1.16}$$

Then, adding the term $\Lambda\left(z^*(x)\right) P'\left(x\right) \tilde{z}(t,x)$ at the left and right-hand side of (1.16) we get

$$\begin{aligned}\partial_t\left(P(x)\tilde{z}(t,x)\right) + \Lambda\left(z^*(x)\right)\partial_x\left(P\left(x\right)\tilde{z}(t,x)\right) = P(x) \Bigg(& \nabla_z \Upsilon\left(z^*(x)\right) \\ & - \left[\nabla_z \Phi\left(z^*(x)\right) \frac{dz^*}{dx}(x)\right]\Bigg) \tilde{z}(t,x) \\ & + \Lambda\left(z^*(x)\right) P'\left(x\right) \tilde{z}(t,x) \,.\end{aligned} \tag{1.17}$$

Thus, making the change of variable

$$y(t,x) = P(x)\tilde{z}(t,x)\,, \tag{1.18}$$

we obtain

$$\begin{aligned}\partial_t y(t,x) + \Lambda\left(z^*(x)\right)\partial_x y(t,x) = P(x) \Bigg(& \nabla_z \Upsilon\left(z^*(x)\right) \\ & - \left[\nabla_z \Phi\left(z^*(x)\right) \frac{dz^*}{dx}(x)\right]\Bigg) P\left(x\right)^{-1} y(t,x) \\ & + \Lambda\left(z^*(x)\right) P'(x) P\left(x\right)^{-1} y(t,x)\,,\end{aligned} \tag{1.19}$$

which is a form as (1.11).

Remark 1.3. In the case where $\Upsilon \equiv 0$, steady-state is any constant value z^*. •

In Subsections 1.2.3, 1.2.2 the linearization for the Saint-Venant equations and the Aw-Rascle-Zhang equations will be derived.

1.1.3 Boundary Conditions and Initial Condition

Additional conditions must be specified at the boundary of the domain $[0,1]$ in order to have a unique and well-defined solution to the system. In the control theory, this boundary conditions will be constructed such that the system shall follow some predefined objectives: stabilization, output tracking, disturbance rejection. The first objective is investigated in Chapters 2 and 4. The two other objectives are investigated in Chapter 3.

The very general form for the boundary conditions is

$$B\left(y(t,0), y(t,1), u(t)\right) = 0\,, \tag{1.20}$$

where $B \in C^1\left(\mathcal{Z}, \mathcal{Z}, \mathbb{R}^q; \mathbb{R}^n\right)$. As noted in [6] the dependence of B on $[y(t,0), y(t,1)]$ refers to physical constraints on the system. The function u is a degree of freedom for controlling the system, otherwise the system would be uncontrolled, which does not exclude that the system might have good "properties".

In order to have a well-posed Cauchy problem, it misses an initial condition. Hence, the initial condition for the system is

$$y(0,x) = y^0(x)\,. \tag{1.21}$$

Definition 1.5. *The initial condition is said to satisfy the zero order compatibility condition with the boundary conditions* (1.20) *if it satisfies*

$$B\left(y^0(0), y^0(1), u(0)\right) = 0\,.$$

The satisfaction of a compatibility condition between the initial condition and the boundary conditions is crucial for the continuity of the solution (see, for instance, [6], [65], and Chapter 3).

To get a better regularity of the solution higher order compatibility conditions are needed. For instance, to hope a C^1 solution in space, besides the previous zero order compatibility condition, the initial condition shall satisfy a one order compatibility condition, that is to satisfy the zero order compatibility condition together with the condition

$$\nabla B\left(y^0(0), y^0(1), u(0)\right) . \begin{bmatrix} -\Lambda(0)\frac{dy^0}{dx}(0) \\ -\Lambda(1)\frac{dy^0}{dx}(1) \\ \frac{du}{dt}(0) \end{bmatrix} = 0\,.$$

Thus, with the boundary conditions and the initial condition we have all the ingredients to write the Cauchy problem and to prove the well-posedness. A method to define solutions is given in the next subsection with the description of the method of characteristics.

1.1.4 Method of Characteristics and Solutions to the Initial Boundary Value Problem

From now on, we assume that the equations considered is under the form (1.11). A fundamental method for hyperbolic PDE is the method of characteristics. The basic idea of this method consists in finding and solving an appropriate set of ODEs which is equivalent to the original PDE. More precisely, let us fix a $(t, x) \in \mathbb{R}^+ \times [0, 1]$ and assume that the solution of y for this point is known. The question is: is it possible to find a curve in the time-space domain $\mathbb{R}^+ \times [0, 1]$ which will connect x to a point $x_0 \in \Gamma = \mathbb{R}^+ \times \{0, 1\}$ along which we are able to compute the solution ?

Let X_i be the solutions of the following set of ODEs

$$\frac{dX_i(s)}{ds} = \lambda_i\left(X_i(s)\right)\,, \quad i = 1, \ldots, n\,, \tag{1.22}$$

with the initial condition

$$X_i(t) = x_0\,,$$

where s lies in a subinterval of $[t, +\infty)$. Formally, the solutions of these ODEs take the form

$$X_i(s) = \int_t^s \lambda_i\left(X_i(\zeta)\right) d\zeta + x_0\,.$$

Letting

$$Y_i(s) = y_i\left(s, X_i(s)\right)\,,$$

where y_i is the i-th component of the solution of (1.11), and differentiating this expression with

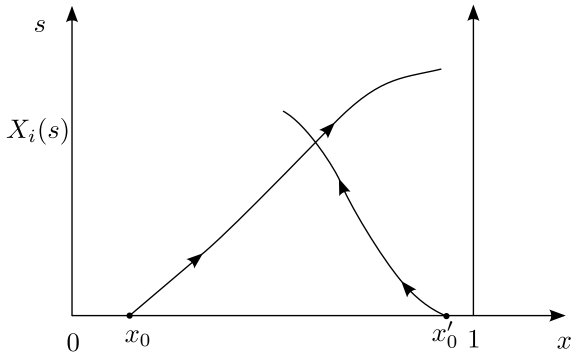

Figure 1.2: Example of crossing characteristic curves $X_i(s)$ in the (s, x)-plane.

respect to s we get

$$\frac{dY_i(s)}{ds} = [1, \lambda_i\left(X_i(s)\right)].\nabla_{(t,x)} y_i\left(s, X_i(s)\right) = F\left(X_i(s)\right) Y_i\left(s\right) . \tag{1.23}$$

The term $[1, \lambda_i\left(X_i(s)\right)].\nabla_{(t,x)} y_i\left(s, X_i(s)\right)$ is a directional derivative in direction $[1, \lambda\left(X_i(s)\right)]^\top$. Hence, relationship (1.23) describes the rate of change of Y_i, equivalently of y_i, along the integral curves of the field $[1, \lambda\left(X_i(s)\right)]$, or in other words along the integral curves solutions of the set of ODEs (1.22). Hence, we get

$$Y_i(s) = \int_0^s F\left(X_i(\zeta)\right) Y_i(\zeta) d\zeta + C\,, \quad C \in \mathbb{R}\,.$$

Let us note that

$$Y_i(t) = y_i\left(t, X_i(t)\right) = y_i\left(t, x_0\right) .$$

It follows that the resolution of the characteristic equations (1.22) is a key to derive solutions for hyperbolic system. The well-known issue for non-linear hyperbolic system is the possibility that characteristic curves cross each other, as illustrated by Figure 1.2, leading to the impossibility to trace back the value of the solution. This issue is out of the scope of this work, for further informations we refer the reader, for instance, to [73], [111], [13], [44]. Obviously this drawback does not exist for linear systems, since the characteristic equations are autonomous ODEs.

Definition 1.6. *In the case where $F(x) \equiv 0$, the characteristics curves X_i are called the Riemann invariants since the solution remains constant along these curves (see (1.23)).*

Nonetheless, the method of characteristics is not the only method available. The question of existence of solutions for (1.11), (1.20), and (1.21) can be given using the semigroup framework. Indeed, it can be proved that the following theorem holds (see, for instance, [6]).

Theorem 1.1. *For every $y^0 \in L^2\left((0,1);\mathbb{R}^n\right)$ there exists a unique (weak) solution $y \in C^0\left(\mathbb{R}^+; L^2\left((0,1);\mathbb{R}^n\right)\right)$ to the Cauchy problem (1.11), (1.20), and (1.21).*

1.2 Some Examples of Hyperbolic Systems

These examples will be studied in the remaining part of the thesis.

1.2.1 A Fundamental Hyperbolic System: The Wave Equation

Let us introduce a fundamental example of hyperbolic systems. It is the wave equation, given by

$$\partial_{tt}w(t,x) = c^2\partial_{xx}w(t,x)\,, \quad t \in \mathbb{R}^+\,,\ x \in [0,1]\ , \tag{1.24}$$

where c is the wave speed. For instance, it can model the displacement of a string deflection. This equation can be expressed as a hyperbolic system thanks to the change of variable

$$\begin{aligned} y_1(t,x) &= \partial_t w(t,x) - c\partial_x w(t,x) \\ y_2(t,x) &= \partial_t w(t,x) + c\partial_x w(t,x)\,. \end{aligned}$$

The states y_1 and y_2 satisfy the system

$$\begin{aligned} \partial_t y_1(t,x) + c\partial_x y_1(t,x) &= 0 \\ \partial_t y_2(t,x) - c\partial_x y_2(t,x) &= 0\,, \end{aligned}$$

with $t \in \mathbb{R}^+$, $x \in [0,1]$. Thus, the solution w of the original wave-equation (1.24) is obtained with

$$w(t,x) = \frac{1}{2}\int_0^t (y_1(s,x) + y_2(s,x))\,ds\,.$$

The wave equation with indefinite in-domain and boundary damping will be used to illustrate the trajectory generation of Chapter 3.

1.2.2 Aw-Rascle-Zhang Equations

The second system considered in this thesis is the Aw-Rascle-Zhang equations modelling the density $\rho(t,x)$ and the velocity $v(t,x)$ at time $t \in \mathbb{R}^+$ and space-location $x \in [0,1]$ of vehicles on a road. It was introduced in the seminal works [4], [113]. The dynamics is given by

$$\partial_t\rho(t,x) + \partial_x(\rho(t,x)v(t,x)) = 0 \tag{1.25}$$

$$\partial_t(v(t,x) - V(\rho(t,x))) + v(t,x)\partial_x(v(t,x) - V(\rho(t,x))) = \frac{1}{\tau}\left(V(\rho(t,x)) - v(t,x)\right)\ . \tag{1.26}$$

The function V is the *desired velocity function* or the *equilibrium velocity function*, it establishes a functional relationship between a density and a velocity $v = V(\rho)$, see Figure 1.3 for an example of the shape of this function. The equation (1.26) takes into account the "deviation" of the velocity to this relationship. The term τ is a *relaxation* which indicates the convergence rate of the velocity v of the cars to the nominal velocity $V(\rho)$. Let us show how the linearized version of the Aw-Rascle-Zhang equations looks like. Let us introduce the steady state $[v^*, \rho^*]$, it satisfies

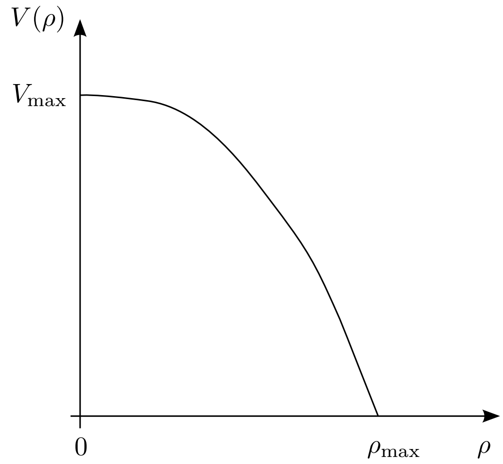

Figure 1.3: Example of equilibrium velocity function.

$$V(\rho^*) = v^* .$$

It follows from this last relationship that the steady state is constant in space. We define the deviation of the state with respect to this steady state by

$$\tilde{\rho}(t,x) = \rho(t,x) - \rho^* \tag{1.27}$$

$$\tilde{v}(t,x) = v(t,x) - v^* . \tag{1.28}$$

The linearized version of the Aw-Rascle-Zhang equations is written as

$$\partial_t \begin{bmatrix} \tilde{\rho}(t,x) \\ \tilde{v}(t,x) \end{bmatrix} + \begin{bmatrix} v^* & \rho^* \\ 0 & v^* + \rho^* V'(\rho^*) \end{bmatrix} \partial_x \begin{bmatrix} \tilde{\rho}(t,x) \\ \tilde{v}(t,x) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\tau}\left(V'(\rho^*)\,\tilde{\rho}(t,x) - \tilde{v}(t,x)\right) \end{bmatrix} \tag{1.29}$$

Let us diagonalize the matrix of the system $\begin{bmatrix} v^* & \rho^* \\ 0 & v^*+\rho^* V'(\rho^*) \end{bmatrix}$. The left eigenvectors are given by

$$l_1 = [-V'(\rho^*), 1]$$
$$l_2 = [0, 1]$$

and the corresponding eigenvalues are

$$\lambda_1 = v^*$$
$$\lambda_2 = v^* + \rho^* V'(\rho^*) .$$

Eigenvalues of opposite sign correspond to a congested mode, while positive eigenvalues correspond to a free flow mode. Obviously eigenvalues of both negative sign is impossible because v^* is a velocity with positive value. Hence, the Riemann coordinates are given by

$$y_1(t,x) = l_1 \begin{bmatrix} \tilde{\rho}(t,x) \\ \tilde{v}(t,x) \end{bmatrix} = \tilde{v}(t,x) - V'(\rho^*)\,\tilde{\rho}(t,x)$$

$$y_2(t,x) = l_2 \begin{bmatrix} \tilde{\rho}(t,x) \\ \tilde{v}(t,x) \end{bmatrix} = \tilde{v}(t,x)\,.$$

Conversely, we have

$$\begin{aligned}\tilde{\rho}(t,x) &= -\frac{1}{V'(\rho^*)}\left(y_1(t,x) - y_2(t,x)\right) \\ \tilde{v}(t,x) &= y_2(t,x)\,.\end{aligned}$$

Finally, the linearized version of the Aw-Rascle-Zhang equations is given by

$$\partial_t \begin{bmatrix} y_1(t,x) \\ y_2(t,x) \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \partial_x \begin{bmatrix} y_1(t,x) \\ y_2(t,x) \end{bmatrix} = \begin{bmatrix} -\frac{1}{\tau} & 0 \\ -\frac{1}{\tau} & 0 \end{bmatrix} \begin{bmatrix} y_1(t,x) \\ y_1(t,x) \end{bmatrix}.$$

As we shall see, the form of the source term for the linearized version is convenient to apply backstepping, see Chapter 3 in which the trajectory generation and tracking is illustrated thanks to these equations.

1.2.3 Saint-Venant Equations

Let us consider an open channel with a constant bottom slope S_b, a rectangular cross-section, and a unit width. The dynamics driven the velocity $V(t,x)$ of the water and the water level $H(t,x)$ in the pool at time $t \in \mathbb{R}^+$ and space-location $x \in [0,L]$, are described by the Saint-Venant equations (or Shallow-Water equations)

$$\partial_t H(t,x) + \partial_x (HV)(t,x) = 0 \tag{1.30}$$

$$\partial_t V(t,x) + \partial_x \left(\frac{V^2}{2} + gH\right)(t,x) + \left(gC\frac{V^2}{H}(t,x) - gS_b\right) = 0\,, \tag{1.31}$$

with C a friction coefficient and g the gravity acceleration. For more general geometry and thorough inspection of these equations we refer the reader to [77] and [6]. System (1.30)–(1.31) is non-linear. We will show how the linearization procedure described in Subsection 1.1.2, works in this case. Equations (1.30) and (1.31) can be rewritten as

$$\begin{aligned}\partial_t H(t,x) + V(t,x)\partial_x H(t,x) + H(t,x)\partial_x V(t,x) &= 0 \\ \partial_t V(t,x) + g\partial_x H(t,x) + V(t,x)\partial_x V(t,x) &= gS_b - gC\frac{V^2}{H}(t,x)\,.\end{aligned}$$

Let us introduce a steady state H^*, V^*, it is any solution to

$$\begin{bmatrix} V^* & H^* \\ g & V^* \end{bmatrix} \frac{d}{dx} \begin{bmatrix} H^* \\ V^* \end{bmatrix} = \begin{bmatrix} 0 \\ gS_b - gC\frac{{V^*}^2}{H^*} \end{bmatrix}.$$

This equation may be rewritten as

$$V^*\frac{dH^*}{dx} = -H^*\frac{dV^*}{dx} = -\frac{gC{V^*}^3 - gS_bV^*H^*}{gH^* - {V^*}^2}\,.$$

Moreover we assume that the steady-state flow is *subcritical* or *fluvial* that is

$$gH^*(x) - {V^*}^2(x) > 0\,, \quad \forall x \in [0, L]\ .$$

This hypothesis guarantees that the matrix of the linearized system is diagonalizable with velocities of opposite sign, as it will be shown now. Defining the deviation of the state with respect to the steady state $H^*(x)$, $V^*(x)$ by

$$\begin{aligned} h(t,x) &= H(t,x) - H^*(x) \\ v(t,x) &= V(t,x) - V^*(x)\,, \end{aligned}$$

and using formula (1.15) we get

$$\begin{aligned} \partial_t \begin{bmatrix} h(t,x) \\ v(t,x) \end{bmatrix} + \begin{bmatrix} V^*(x) & H^*(x) \\ g & V^*(x) \end{bmatrix} \partial_x \begin{bmatrix} h(t,x) \\ v(t,x) \end{bmatrix} \\ = \begin{bmatrix} -\frac{dV^*(x)}{dx} & -\frac{dH^*}{dx}(x) \\ gC\frac{{V^*}^2}{{H^*}^2}(x) & -2gC\frac{V^*}{H^*}(x) - \frac{dV^*}{dx}(x) \end{bmatrix} \begin{bmatrix} h(t,x) \\ v(t,x) \end{bmatrix}\ . \end{aligned}$$

Let us diagonalize the matrix $\left[\begin{smallmatrix} V^*(x) & H^*(x) \\ g & V^*(x) \end{smallmatrix}\right]$ as shown above. The left eigenvectors of this matrix are

$$\begin{aligned} l_1(x) &= \left[\sqrt{\frac{g}{H^*(x)}}, 1\right] \\ l_2(x) &= \left[-\sqrt{\frac{g}{H^*(x)}}, 1\right]\,, \end{aligned}$$

giving the new variable

$$\begin{aligned} y_1(t,x) = l_1(x) \begin{bmatrix} h(t,x) \\ v(t,x) \end{bmatrix} &= \sqrt{\frac{g}{H^*(x)}} h(t,x) + v(t,x) \\ y_2(t,x) = l_2(x) \begin{bmatrix} h(t,x) \\ v(t,x) \end{bmatrix} &= -\sqrt{\frac{g}{H^*(x)}} h(t,x) + v(t,x)\,. \end{aligned}$$

Conversely v and h can be expressed in function of y_1 and y_2

$$\begin{aligned} h(t,x) &= \frac{1}{2}\sqrt{\frac{H^*(x)}{g}}\left(y_1(t,x) - y_2(t,x)\right) \\ v(t,x) &= \frac{1}{2}\left(y_1(t,x) + y_2(t,x)\right)\ . \end{aligned}$$

The eigenvectors of the system are given by

$$\begin{aligned} \lambda_1 &= V^*(x) + \sqrt{gH^*(x)} \\ \lambda_2 &= V^*(x) - \sqrt{gH^*(x)}\,. \end{aligned}$$

The obtained linear system is

$$\partial_t \begin{bmatrix} y_1(t,x) \\ y_2(t,x) \end{bmatrix} + \begin{bmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{bmatrix} \partial_x \begin{bmatrix} y_1(t,x) \\ y_2(t,x) \end{bmatrix} = \begin{bmatrix} \gamma_1(x) & \delta_1(x) \\ \gamma_2(x) & \delta_2(x) \end{bmatrix} \begin{bmatrix} y_1(t,x) \\ y_2(t,x) \end{bmatrix},$$

where

$$\begin{aligned}
\gamma_1(x) &= \frac{gC{V^*}^2(x)}{H^*(x)} \left[\frac{3}{4\left(\sqrt{gH^*(x)} + V^*(x)\right)} - \frac{1}{V^*(x)} + \frac{1}{2\sqrt{gH^*(x)}} \right] \\
&\quad + \frac{3gS_b}{4\left(\sqrt{gH^*(x)} + V^*(x)\right)} \\
\gamma_2(x) &= \frac{gC{V^*}^2(x)}{H^*(x)} \left[\frac{3}{4\left(\sqrt{gH^*(x)} - V^*(x)\right)} - \frac{1}{V^*(x)} + \frac{1}{2\sqrt{gH^*(x)}} \right] \\
&\quad + \frac{gS_b}{4\left(\sqrt{gH^*(x)} + V^*(x)\right)} \\
\delta_1(x) &= \frac{gC{V^*}^2(x)}{H^*(x)} \left[\frac{3}{4\left(\sqrt{gH^*(x)} + V^*(x)\right)} - \frac{1}{V^*(x)} - \frac{1}{2\sqrt{gH^*(x)}} \right] \\
&\quad - \frac{gS_b}{4\left(\sqrt{gH^*(x)} + V^*(x)\right)} \\
\delta_2(x) &= \frac{gC{V^*}^2(x)}{H^*(x)} \left[\frac{3}{4\left(\sqrt{gH^*(x)} - V^*(x)\right)} - \frac{1}{V^*(x)} - \frac{1}{2\sqrt{gH^*(x)}} \right] \\
&\quad - \frac{3gS_b}{4\left(\sqrt{gH^*(x)} + V^*(x)\right)}.
\end{aligned}$$

The last step to obtain a similar form as (1.11) for the Saint-Venant equations consists in making the system dimensionless, for considering an abstract domain $[0,1]$. The new variable

$$\tilde{x} = \frac{x}{L},$$

allows to express the system as

$$\partial_t \begin{bmatrix} \tilde{y}_1(t,\tilde{x}) \\ \tilde{y}_2(t,\tilde{x}) \end{bmatrix} + \frac{1}{L} \begin{bmatrix} \tilde{\lambda}_1(\tilde{x}) & 0 \\ 0 & \tilde{\lambda}_2(\tilde{x}) \end{bmatrix} \partial_{\tilde{x}} \begin{bmatrix} \tilde{y}_1(t,\tilde{x}) \\ \tilde{y}_2(t,\tilde{x}) \end{bmatrix} = \begin{bmatrix} \tilde{\gamma}_1(\tilde{x}) & \tilde{\delta}_1(\tilde{x}) \\ \tilde{\gamma}_2(\tilde{x}) & \tilde{\delta}_2(\tilde{x}) \end{bmatrix} \begin{bmatrix} \tilde{y}_1(t,\tilde{x}) \\ \tilde{y}_2(t,\tilde{x}) \end{bmatrix},$$

where

$$\begin{aligned}
\tilde{y}_i(t,\tilde{x}) &= y_i(t, L\tilde{x}) \\
\tilde{\gamma}_i(\tilde{x}) &= \gamma_i(L\tilde{x}) \\
\tilde{\delta}_i(\tilde{x}) &= \delta_i(L\tilde{x}),
\end{aligned}$$

for $i = 1,2$.

These equations will be used to illustrate results of Chapter 2 and 4.

1.3 Stability and Stabilizability of Balance Laws

In the sequel we denote by E a functional space of functions $y : [0,1] \to \mathbb{R}^n$ with norm $|\cdot|_E$.

The aim of this thesis is to balance laws, and if needed, to control or stabilize them. The aim of this section is to recall some methods for the stability and stabilization analysis of hyperbolic PDEs. The list of references given there is not comprehensive, we cite the works which are the more closely linked to this one.

The book [6] gathers a lot of results in this direction and is, therefore, an excellent reference for detailed informations. The book [22] may also be consulted for more general studies. We can cite also the book [30] which is also an important source of informations for the control, stability, stabilizability of PDEs with the semigroup theory.

For the stability and stabilizability, a powerful analysis due to Alexandr Lyapunov has been extended for distributed systems. It relies on the search of a function, called a *Lyapunov function*, measuring in some sense an energy for a dynamical system.

Let us give several definitions of stability. These notions of stability are strongly related to the norm considered. First, let us define the notion of *Lyapunov Stability* (LS).

Definition 1.7. *Let us denote by $E' \subseteq E$ the set such that for all $y^0 \in E'$ we have $y(t,\cdot) \in E$ for all time t. The system (1.11), (1.20) with $u \equiv 0$, is said to be stable in the sense of Lyapunov for every $y^0 \in E'$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that we have*

$$\left|y^0\right|_E \leq \delta \quad \Rightarrow \quad |y\,(t,\cdot)|_E \leq \varepsilon\,, \quad \forall t \in \mathbb{R}^+\,, \tag{1.32}$$

where y is the solution to (1.11), (1.20), and (1.21).

Let us define the *Global Asymptotic Stability* (GAS).

Definition 1.8. *The system (1.11), (1.20), and (1.21) is said to be Globally Asymptotically Stable in the norm of E if it is Lyapunov Stable and for every initial condition $y^0 \in E$, the solutions to system (1.11), (1.20), and (1.21) satisfy*

$$|y(t,\cdot)|_E \to 0 \quad \textit{as} \quad t \to +\infty\,. \tag{1.33}$$

Let us define the *Global Exponential Stability* (GES).

Definition 1.9. *The system (1.11), (1.20), and (1.21) is said to be Globally Exponentially Stable in the norm of E if there exist $\nu > 0$ and $C > 0$ such that, for every initial condition $y^0 \in E$, the solutions to system (1.11), (1.20), and (1.21) satisfy*

$$|y(t,\cdot)|_E \leq Ce^{-\nu t}\left|y^0\right|_E\,, \quad \forall t \in \mathbb{R}^+\,. \tag{1.34}$$

Let us define the useful class functions $\mathcal{K}_\infty$.

Definition 1.10. *A function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be of* **class** $\mathcal{K}_\infty$ *if it is continuous, strictly increasing, unbounded, and $\alpha(0) = 0$.*

Roughly speaking, the idea of the Lyapunov control theory is to analyze the "dissipativity" of the system through the measure of the energy given by the Lyapunov function. Hence, from this idea the following definition may be stated (see [64] for definitions of Lyapunov function in finite dimension from which the following one is inspired).

Definition 1.11. *A function $V \in C^1(E;\mathbb{R})$ is a strict Lyapunov function for the system (1.11), (1.20), and (1.21) with $u \equiv 0$ if and only if there exist two class functions $\mathcal{K}_\infty$ functions α_1, α_2, and a positive definite function ρ such that V satisfies*

$$\alpha_1\left(|y|_E\right) \leq V(y) \leq \alpha_2\left(|y|_E\right), \quad \forall y \in E,$$

and the time derivative of V along the trajectories of the system satisfies

$$\dot{V}(y) \leq -\rho(y).$$

To show how to use a Lyapunov function, let us assume that V is comparable with the norm $|\cdot|_E$, that is there exist $c > 0$ and $C > 0$ such that

$$c\,|y|_E \leq V(y) \leq C\,|y|_E, \quad \forall y \in E, \tag{1.35}$$

and that it is a strict Lyapunov function with

$$\dot{V}(y) \leq -\alpha V(y), \quad \alpha > 0,$$

along the trajectory of the system (1.11), (1.20), and (1.21). Then, from this latter inequality we can write

$$V\left(y(t,\cdot)\right) \leq e^{-\alpha t} V\left(y^0\right), \quad \forall t \in \mathbb{R}^+.$$

Using the assumption that V and $|\cdot|_E$ are comparable, one gets

$$|y(t,\cdot)|_E \leq \frac{C}{c} e^{-\alpha t} \left|y^0\right|_E, \quad \forall t \in \mathbb{R}^+,$$

hence the system is GES. The derivation above is based on the assumption that V is comparable to $|\cdot|_E$. This assumption always holds in finite dimension, but in the infinite case it may exist system for which the first inequality of (1.35) fails (for an example of such property, see [56]). Nonetheless, even if it does not hold, the Lyapunov analysis is still possible with some other conditions on the system.

The link between stability and Lyapunov function remains an open question for system in infinite dimension. Nonetheless, for special case some converse results exist. For instance, when considering system for which the operator generates a strongly continuous semigroup on a Banach space X, a result states that the system is exponentially stable if and only if there exists a Lyapunov function (see [56]).

1.3.1 Lyapunov Approach for Hyperbolic PDEs

A pioneer work for the use of Lyapunov function for PDEs is [24], where the stabilization of a rotating body beam without damping is realized with the derivation of a Lyapunov function. This analysis is particularly interesting, because of the fact that the Lyapunov function is not strict, that is

$$\dot{V}(y(t,\cdot)) \leq 0\,, \quad \forall t \in \mathbb{R}^+\,,$$

along the trajectories of the system. The precompactness of the trajectories has been proved in order to use the LaSalle's invariance principle and getting the asymptotic stabilization.

To the best of our knowledge one of the first work to analyze the stabilization of a hyperbolic system in term of Lyapunov function is [21]. The Lyapunov function used was

$$V(\omega) = |\omega \exp(-\theta)|_{C^0(\overline{\Omega};\mathbb{R})}\ ,$$

where $w \in C^0\left(I; C^0\left(\overline{\Omega}\right)\right)$ is the state, I a time interval, θ a function with some good properties, and $\overline{\Omega}$ the closure of a non-empty open connected and simply connected subset of $\mathbb{R}^2$ of class C^∞.

A Lyapunov function related to the former one, was introduced in [112] for general hyperbolic systems. In [26], this Lyapunov function was taken back for the control analysis of 2×2 linear and quasi-linear systems of conservation laws for which the boundary condition (1.20) takes the following form

$$B\left(y(t,0), y(t,1)\right) = \begin{bmatrix} y^+(t,0) \\ y^-(t,1) \end{bmatrix} - G \begin{bmatrix} y^+(t,1) \\ y^-(t,0) \end{bmatrix}\ . \tag{1.36}$$

More precisely, this candidate Lyapunov function can be written as

$$V(y) = \int_0^1 y^\top(x)\mathcal{Q}(x)y(x)dx\,, \tag{1.37}$$

where $\mathcal{Q}(x) = \begin{bmatrix} e^{-\mu x}Q^- & 0_{m,n-m} \\ 0_{n-m,m} & e^{\mu x}Q^+ \end{bmatrix}$, with Q^- in $\mathbb{R}^{m\times m}$, Q^+ in $\mathbb{R}^{(n-m)\times(n-m)}$, and $\mu > 0$. Conditions on the matrix G have been given to obtain the stability of the systems when considering different norm:

- L^2-norm in [26], [5], [96], [35], and [99],
- H^2-norm in [26] and [27],
- C^1-norm in [23]

using the Lyapunov analysis. A result of stabilization for a coupled system of gas pipes with a compressor station has been stated in [45] with a Lyapunov analysis for classical solutions. The feedback law takes the form (1.36), nonetheless no conditions are explicitly given for the matrix G. Other boundary conditions have been considered than (1.36). For instance, the PI control of linear systems of conservation laws has been investigated with the Lyapunov analysis in [38]. Boundary control of hyperbolic Lotka-Volterra systems has been investigated with a Lyapunov approach in [85]. In [36], a boundary control embedding time varying delays is analyzed with a Lyapunov function for star-shaped networks of quasi-linear hyperbolic systems of balance laws (see

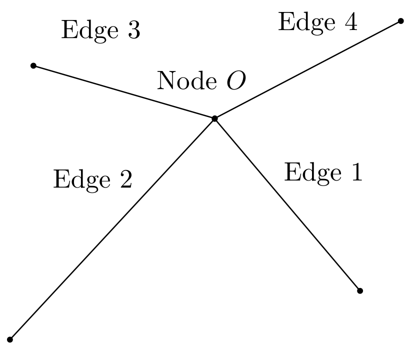

Figure 1.4: Illustration of a star-shaped network with four edges.

Figure 1.4 for an illustration of star-shaped geometry). The construction of full-state feedback law or observers by backstepping uses the Lyapunov analysis to prove the stability of the target system as in [109], [28], [33], [34], [1] (see Chapter 3 for details on backstepping).

This analysis is the one we adopt all along this thesis. However, other approaches can be adopted to prove the stability of such systems.

1.3.2 Time-Delay Systems Approach

Linear hyperbolic systems of conservation laws can be viewed as time-delay systems. Indeed, for all $t \geq 0$, for all $0 \leq x < s \leq 1$, one has

$$y_i(t,x) = y_i\left(t + \frac{s-x}{\lambda_i}, s\right), \quad \forall i = 1,\dots,m, \tag{1.38}$$

$$y_i(t,x) = y_i\left(t + \frac{x-s}{\lambda_i}, s\right), \quad \forall i = m+1,\dots,n. \tag{1.39}$$

In particular, for $x = 0$, $s = 1$, we get

$$y_i(t,1) = y_i(t-\tau_i, 0), \quad \forall i = 1,\dots,m, \tag{1.40}$$

$$y_i(t,0) = y_i(t-\tau_i, 1), \quad \forall i = m+1,\dots,n, \tag{1.41}$$

where $\tau_i = \frac{1}{\lambda_i}$ are the delays. These scalar time-delay systems are interconnected by the boundary condition (1.20). This formulation allows to use results from differential difference equations, as an important source of results in this direction we can cite [54]. This approach has been used in [25] to prove the exponential stability in the Sobolev space $W^{2,p}$.

1.3.3 Characteristics Method Approach

An other approach consists in estimating the norm of the solutions by the characteristics method during the time. It was the approach used in [51] where a quasi-linear wave equation with a boundary damping is investigated. Following the same idea, the boundary feedback for the quasi-linear Saint-Venant equations without slope and friction, is analyzed in [32]. In [37], the friction and the slope are added to the previous work and the analysis is led in the same way. In [95], the method is used to analyze quasi-linear systems of conservation laws subject to boundary errors, as illustration the dynamic of a pipe filled with water is considered. In [98], the robust boundary control of a quasi-linear systems of balance laws with respect to the non-homogeneous term (or source term) is tackled with this estimation technique. In [2], linear switched systems of balance laws are also analyzed. Recently in [87], it has been shown that there exist boundary dissipative conditions such that a finite-time stabilization is achieved for a 2×2 quasi-linear hyperbolic system.

1.3.4 Frequency-Domain Approach

Finally, the last method used for the analysis of linear hyperbolic system is the frequency domain approach. A thorough inspection of this approach is presented in [77] for the Saint-Venant equations. For more general hyperbolic system taking the form of a flow-density conservation laws as

$$\partial_t \rho(t,x) + \partial_x q(t,x) = 0 \tag{1.42}$$

$$\partial_t q(t,x) + \lambda_1 \lambda_2 \partial_x \rho(t,x) + (\lambda_1 - \lambda_2)\,\partial_x q(t,x) = \gamma \rho(t,x) - \delta q(t,x)\,, \tag{1.43}$$

where $\rho(t,x)$ and $q(t,x)$ are respectively the density and the flow density at time t and position x, and $\gamma \geq 0$, $\delta \geq 0$. The frequency domain approach has been led for this type of system. In particular in [78], some conditions have been given in the case of a proportional diagonal boundary controller, that is

$$\begin{bmatrix} q(t,0) \\ q(t,L) \end{bmatrix} = \begin{bmatrix} k_0 & 0 \\ 0 & k_L \end{bmatrix} \begin{bmatrix} h(t,0) \\ h(t,L) \end{bmatrix}. \tag{1.44}$$

The frequency domain approach has been used in [10] for the control of such system, with $\gamma = \delta = 0$, with a Proportional-Integral (PI) control action, see Chapter 3, where PI controller is considered for a tracking issue.

We will use the Lyapunov approach in the following chapters where some additional references will be also given.

1.4 Contributions of the Thesis

In this section, we explain the problems considered in this thesis and the related main results. We consider richer diagonal linear systems of balance laws than (1.11). Indeed, we assume that the velocities may be also time-varying, that is

$$\partial_t y(t,x) + \Lambda(t,x)\partial_x y(t,x) = F(t,x)y(t,x)\,, \quad (t,x) \in \mathbb{R}^+ \times [0,1]\,, \tag{1.45}$$

where $\Lambda(t,x)$ is a diagonal matrix for all $(t,x) \in \mathbb{R}^+ \times [0,1]$.

1.4.1 Chapter 2: Switching Stabilization

In Chapter 2, we are interested in the stabilization of such equation with

$$\begin{aligned} \Lambda(t,x) &= \Lambda(t) \\ F(t,x) &= F(t)\,, \end{aligned}$$

where $F(t)$ is diagonal for all $t \in \mathbb{R}^+$ and $\Lambda(t)$ is positive definite for all $t \in \mathbb{R}^+$, and under switched boundary conditions

$$B_{\sigma(t)}\left(y(t,0), y(t,1)\right) = y(t,0) - G_{\sigma(t)} y(t,1)\,, \tag{1.46}$$

where $\sigma(t)$ is a switching signal taking values in a discrete set $\mathcal{I} := \{1,\ldots,N\}$. It describes the fact that the boundary conditions may change abruptly during the time evolution of the process. The aim of the following chapter is to propose some switching rules σ as an output feedback law

$$\begin{array}{rccc} \sigma[w]: & \mathbb{R}^+ & \to & \mathcal{I} \\ & t & \mapsto & \sigma[w](t)\,, \end{array}$$

where the output w is given by

$$w(t) = y(t,1)\,.$$

Three switching rules are stated. The first one consists in selecting the mode which optimizes the time-derivative of the Lyapunov function (1.37) along the trajectories of the system. In this case we prove that the system is GES (Proposition 2.2, page 40). The second one is a modified version of the first one. We add a hysteresis phenomenon. The third one is a modified version of the second one for which we add a low-pass filter. We prove the well-posedness of the system for the last two switching rules and show a weaker notion of stability with them (Theorems 2.1 and 2.2, pages 43 and 46 respectively).

Besides, the stabilization results, we explore the effect of measurement noise, such that the measured output is

$$\tilde{w}(t) = w(t) + \delta(t)\,,$$

where δ is the measurement noise. We show that modified versions of the two last switching rules guarantee an ISS property (Propositions 2.3 and 2.4, pages 48 and 50 respectively) or a robustness property (Propositions 2.5 and 2.6, pages 52 and 53) which is stronger.

We illustrate our results with an academic example (Subsection 2.7.1, page 54) as well as with a physical one which is the Saint-Venant equations without slope and friction (Subsection 2.7.2, page 58).

1.4.2 Chapter 3: Trajectory Generation and PI Tracking

In Chapter 3, we consider a 2×2 hyperbolic system of balance laws where

$$\Lambda(t,x) = \begin{bmatrix} \lambda_1(x) & 0 \\ 0 & -\lambda_2(x) \end{bmatrix}$$
$$F(t,x) = \begin{bmatrix} c_1(x) & c_2(x) \\ c_3(x) & c_4(x) \end{bmatrix},$$

with λ_1, λ_2 in $C^2([0,1];\mathbb{R})$ and satisfy $\lambda_1(x) > 0$, $\lambda_2(x) > 0$, for all $x \in [0,1]$. The functions c_i, $i = 1,2,3,4$, belong to $C^1([0,1];\mathbb{R})$. The boundary conditions take the form

$$B(y_1(t,0), y_2(t,1), S(t)) = \begin{bmatrix} y_1(t,0) \\ y_2(t,1) \end{bmatrix} - \begin{bmatrix} 0 & q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(t,1) \\ y_2(t,0) \end{bmatrix} - \begin{bmatrix} 0 \\ S(t) \end{bmatrix}.$$

In this work we consider that $y_2(t,0)$ is the measured output of the system. The tracking problem towards a given trajectory by using backstepping method is solved (Theorem 3.1, page 66). Obviously, the initial condition has to satisfy a compatibility condition with the trajectory $z(t)$, which is not always the case. Hence, some stabilization techniques shall be taken to regulate the output to the objective $z(t)$. This regulation is obtained through a PI-controller of the form

$$\tilde{S}(t) = -k_P \tilde{v}(t,0) - k_I \tilde{\eta}(t),$$

with

$$\dot{\tilde{\eta}}(t) = \tilde{v}(t,0),$$

where $\tilde{v}(t,0) = v(t,x) - v^r(t,x)$, $\tilde{S}(t) = S(t) - S^r(t)$ stands for the deviations of the system from the reference $v^r(t,x)$ and from the reference input $S^r(t)$ respectively. The analysis is led through a "novel" quadratic Lyapunov function. Due to the particular form of the system it is shown that a cross term between two components of the augmented system (states of the system, y_1, y_2, and the integrator $\tilde{\eta}$) must be added in order to prove stability by Lyapunov techniques (Theorem 3.2, page 74).

Moreover, we analyze the tracking issue when disturbances are present in the domain as well as at the boundaries. More precisely, the system is assumed to have the following form

$$\partial_t y_1(t,x) + \lambda_1(x)\partial_x y_1(t,x) = c_1(x)y_1(t,x) + c_2(x)y_2(t,x) + d_1(x)$$
$$\partial_t y_2(t,x) - \lambda_2(x)\partial_x y_2(t,x) = c_3(x)y_1(t,x) + c_4(x)y_2(t,x) + d_2(x),$$

and the boundary conditions take the form

$$B\left(y_1(t,0), y_2(t,1), S(t)\right) = \begin{bmatrix} y_1(t,0) \\ y_2(t,1) \end{bmatrix} - \begin{bmatrix} 0 & q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(t,1) \\ y_2(t,0) \end{bmatrix} - \begin{bmatrix} 0 \\ S(t) \end{bmatrix} - \begin{bmatrix} d_3 \\ d_4 \end{bmatrix},$$

where d_1, d_2 are in $C^1\left([0,1]\,;\mathbb{R}\right)$ and d_3, d_4 are in $\mathbb{R}$. The integral action is shown to compensate in the output the distributed and boundary disturbances in the case of solutions in $L^2\left((0,1);\mathbb{R}\right)$ (Proposition 3.1, page 82). In the case of an initial condition in $C^1\left([0,1]\,;\mathbb{R}\right) \times C^1\left([0,1]\,;\mathbb{R}\right)$ and assuming that it satisfies a compatibility condition with the boundary conditions, it is shown that the integral action eliminates these disturbances in a stronger sense (Theorem 3.5, page 84).

The trajectory generation is illustrated with a wave-equation with indefinite in-domain and boundary damping (Section 3.3, page 70). Then, the tracking issue is illustrated through the linearized Aw-Rascle-Zhang equations (Section 3.6, page 85).

1.4.3 Chapter 4: Numerical Techniques for the Lyapunov Analysis

In Chapter 4, we consider system (1.45) with

$$\begin{aligned} \Lambda(t,x) &= \begin{bmatrix} \Lambda^+(x) & 0_{m,n-m} \\ 0_{n-m,m} & \Lambda^-(x) \end{bmatrix} \\ F(t,x) &= F(x)\,, \end{aligned}$$

where Λ^+, Λ^-, and F are positive definite functions in $C^0\left([0,1]\,;\mathbb{R}^{m\times m}\right)$, $C^0\left([0,1]\,;\mathbb{R}^{(n-m)\times(n-m)}\right)$, and $C^0\left([0,1]\,;\mathbb{R}^{n\times n}\right)$ respectively, and under boundary conditions

$$B(y(t,0), y(t,1)) = \begin{bmatrix} y^+(t,0) \\ y^-(t,1) \end{bmatrix} - G \begin{bmatrix} y^+(t,1) \\ y^-(t,0) \end{bmatrix},$$

where G is in $\mathbb{R}^{n\times n}$. We analyze the stability of such system with two Lyapunov functions having the general form

$$V(y) = \int_0^1 y(x)^\top \mathcal{Q}(x) \left|\Lambda(x)\right|^{-1} y(x) dx\,,$$

where the notation $|\Lambda(x)|$ stands for the matrix whose entries are the absolute values of those of $\Lambda(x)$. The kernels $\mathcal{Q}(x)$ used are

$$\begin{aligned} \mathcal{Q}(x) &= \mathtt{diag}\left[e^{-2\mu x}Q^-, e^{2\mu x}Q^+\right] \\ \mathcal{Q}(x) &= \mathtt{diag}\left[(1+\mu x)Q^-, (1-\mu x)Q^+\right]\,, \end{aligned}$$

with μ a real coefficient for the first kernel and a real coefficient in $(-1,1)$ for the second one. The conditions for stability for this system can be written as Matrix Inequalities (MIs) (Propositions 4.1 and 4.3, pages 94 and 96). These MIs depend on the x variable, hence it corresponds to an infinity of MI to solve. In order to make this problem tractable we use a line-search over one of the parameter involving in the MI leading to an infinity of Linear Matrix Inequalities (LMIs) to solve. The difficulty given by the continuum of LMIs is eliminated by an overapproximation by polytopes.

More precisely, we show that the infinity of LMIs can be reduced to a finite number.

Then, we state results to construct boundary controller. In other words, we suppose that the boundary conditions are

$$B(y(t,0), y(t,1)) = \begin{bmatrix} y^+(t,0) \\ y^-(t,1) \end{bmatrix} - (T + LK_B) \begin{bmatrix} y^+(t,1) \\ y^-(t,0) \end{bmatrix} ,$$

where T is in $\mathbb{R}^{n\times n}$, L is in $\mathbb{R}^{n\times q}$ $(n > q)$ are given and the matrix K_B is in $\mathbb{R}^{q\times n}$ has to be designed such that system (1.45), (1.21) with the boundary conditions given above is GES. We construct some MIs conditions for the construction of such controller (Theorems 4.1 and 4.2, pages 99 and 100).

Then, we consider the design of a distributed controller

$$F(x) = H(x) + B(x)K_D(x) , \quad x \in [0,1] ,$$

where matrices $H(x)$ in $\mathbb{R}^{n\times n}$ and $B(x)$ in $\mathbb{R}^{n\times p}$ $(n > p)$ are given and matrix $K_D(x)$ in $\mathbb{R}^{p\times n}$ has to be designed such that system (1.45)–(1.21) is GES with the distributed control as defined above. We assume that $K_D(x)$ is given by

$$K_D(x) = \sum_{i=1}^{\ell} \alpha_i(x) K_i ,$$

where α_i, $i = 1, \dots, \ell$, are some continuous real functions. Again, the conditions for the construction of such controller is written in terms of MIs (Theorems 4.3 and 4.4, pages 101 and 102). Then, we can apply the method described above for the stability checking to the controller design (line-search and overapproximation).

We illustrate our method on academic examples (Subsections 4.5.1 and 4.5.2, pages 113 and 114 respectively) and a physical one given by the Saint-Venant equations with friction and without slope (Subsection 4.5.3, page 116).

1.5 Publications

P.-O. Lamare, A. Girard, and C. Prieur. Switching rules for stabilization of linear systems of conservation laws. *SIAM Journal on Control and Optimization*, 53(3):1599–1624, 2015

P.-O. Lamare and N. Bekiaris-Liberis. Control of 2×2 linear hyperbolic systems: Backstepping-based trajectory generation and PI-based tracking. Provisionally accepted for publication in *Systems & Control Letters*, 2015

P.-O. Lamare, A. Girard, and C. Prieur. Lyapunov techniques for stabilization of switched linear systems of conservation laws. In *IEEE Conference on Decision and Control*, pages 448–453, Florence, Italy, 2013

P.-O. Lamare, N. Bekiaris-Liberis, and A. M. Bayen. Control of 2 × 2 linear hyperbolic systems: Backstepping-based trajectory generation and PI-based tracking. Accepted at the

European Control Conference, Linz, Austria, 2015

P.-O. Lamare, A. Girard, and C. Prieur. Numerical computation of Lyapunov function for hyperbolic PDE using LMI formulation and polytopic embeddings. *IFAC Workshop on Linear Parameter Varying Systems*, Grenoble, France, 2015

2. Switched Hyperbolic PDEs

In this chapter, the exponential convergence in L^2-norm is analyzed for a class of switched linear systems of conservation laws. The boundary conditions are subject to switches. We investigate the problem of synthesizing stabilizing switching controllers. By means of Lyapunov techniques, three control strategies are developed based on steepest descent selection, possibly combined with a hysteresis and a low-pass filter. For the first strategy we show the global exponential stabilizability, but no result for the existence and uniqueness of trajectories can be stated. For the other ones, the problem is shown to be well posed and global exponential convergence can be obtained. Moreover, we consider the ISS and the robustness properties for these switching rules in presence of measurement noise. Some numerical examples illustrate our approach and show the merits of the proposed strategies. Particularly, we have developed a model for a network of open channels, with switching controllers in the gate operations.

The introduction of the three switching rules have been published in the proceeding of the 2013 Conference on Decision and Control (CDC) [68]. The thorough study of the three switching rules with the robustness issue has been published in the SIAM Journal on Control and Optimization [70].

2.1 Problem Statement

We are concerned with $n \times n$ switched linear hyperbolic system of conservation laws of the form

$$\partial_t y(t,x) + \Lambda(t)\partial_x y(t,x) = F(t)y(t,x)\,, \quad t \in \mathbb{R}^+,\, x \in [0,1]\,, \tag{2.1}$$

$$y(t,0) = G_{\sigma(t)} y(t,1)\,, \quad t \in \mathbb{R}^+\,, \tag{2.2}$$

$$y(0,x) = y^0(x)\,, \quad x \in [0,1]\ , \tag{2.3}$$

where $y : \mathbb{R}^+ \times [0,1] \to \mathbb{R}^n$, $\sigma : \mathbb{R}^+ \to \mathcal{I}$ is the switching signal, and y^0 lies in some subspace E of $L^2\left((0,1);\mathbb{R}^n\right)$. This subspace will be clarified in Section 2.3, for the moment we leave it intentionally ambiguous.

For all $i \in \mathcal{I}$, G_i belongs to $\mathbb{R}^{n\times n}$, for all $t \in \mathbb{R}^+$, $\Lambda(t)$ is a diagonal positive definite matrix in $\mathbb{R}^{n\times n}$ and $F(t)$ is a diagonal matrix in $\mathbb{R}^{n\times n}$ i.e. $\Lambda(t) = \mathtt{diag}(\lambda_1(t),\ldots,\lambda_n(t))$, $F(t) = \mathtt{diag}(f_1(t),\ldots,f_n(t))$ where $\lambda_1(t),\ldots,\lambda_n(t), f_1(t),\ldots,f_n(t)$ are in $L^1_{\mathrm{loc}}\left(\mathbb{R}^+,\mathbb{R}\right)$ and there exist n pairs of real positive, non-zero coefficients $\left(\underline{\lambda_d},\overline{\lambda_d}\right)$ and n pairs of real coefficients $\left(\underline{f_d},\overline{f_d}\right)$, $d = 1,\ldots,n$, such that

$$\underline{\lambda_d} \le \lambda_d(t) \le \overline{\lambda_d}\,, \quad \forall t \in \mathbb{R}^+\,, \forall d \in \{1,\ldots,n\}\ , \tag{2.4}$$

$$\underline{f_d} \le f_d(t) \le \overline{f_d}\,, \quad \forall t \in \mathbb{R}^+\,, \forall d \in \{1,\ldots,n\}\ . \tag{2.5}$$

In the sequel, we denote by $\underline{\Lambda}$ and $\underline{F}$ the diagonal matrix whose elements are the lower bounds of the velocities $\Lambda(t)$ and source term $F(t)$ respectively, that is

$$\underline{\Lambda} = \mathtt{diag}\left(\underline{\lambda_1},\ldots,\underline{\lambda_n}\right) \tag{2.6}$$

$$\underline{F} = \mathtt{diag}\left(\underline{f_1},\ldots,\underline{f_n}\right)\ , \tag{2.7}$$

and we denote $\overline{\Lambda}$ and $\overline{F}$ the diagonal matrix whose elements are the upper bounds of the velocities $\Lambda(t)$ and source term $F(t)$ respectively, that is

$$\overline{\Lambda} = \mathtt{diag}\left(\overline{\lambda_1},\ldots,\overline{\lambda_n}\right) \tag{2.8}$$

$$\overline{F} = \mathtt{diag}\left(\overline{f_1},\ldots,\overline{f_n}\right)\ . \tag{2.9}$$

We denote by $\underline{\lambda}$ and $\overline{\lambda}$ respectively the smallest eigenvalues of $\underline{\Lambda}$ and the largest eigenvalues of $\overline{\Lambda}$

$$\underline{\lambda} = \min_{1\le i\le n}\left\{\underline{\lambda_i}\right\}\ , \tag{2.10}$$

$$\overline{\lambda} = \max_{1\le i\le n}\left\{\overline{\lambda_i}\right\}\ . \tag{2.11}$$

The largest eigenvalue of $\overline{F}$ is denoted by $\overline{f}$ that is

$$\overline{f} = \max_{1\le i\le n}\left\{\overline{f_1},\ldots,\overline{f_n}\right\}\ . \tag{2.12}$$

The aim of this work is to design a switching rule which depends only on the measurement at the boundary of the domain, in order to stabilize the system.

Indeed, hyperbolic systems, provided sensors are locally distributed, can be described with measurements at the boundaries. In our case, the system is only observed at the point $x = 1$ at any time. The *output* is thus defined as

$$w(t) = y(t, 1)\,. \tag{2.13}$$

The output function is well defined as soon as the solution y is in a space where the evaluation at the boundary is well-defined. Hence, it is an element to construct our subspace E of $L^2\left((0,1);\mathbb{R}^n\right)$ in which the solution will be considered.

The switching rule σ is given as an output feedback law defined as follows

$$\begin{array}{rccc} \sigma[w]: & \mathbb{R}^+ & \to & \mathcal{I} \\ & t & \mapsto & \sigma[w](t)\,. \end{array} \tag{2.14}$$

To summarize, the process evolves in a mode, a sensor measures the state $w(t)$ of the process at the boundary, then depending on this output, a switching rule imposes the mode in which the system must evolve afterwards. We make the following causality assumption on σ. For all $T \in \mathbb{R}^+$, for all $w\,,\, w' \in C_{rpw}\left([0,T]\,;\mathbb{R}^n\right)$, if

$$w(t) = w'(t)\,, \quad \forall t \in [0,T]\ , \tag{2.15}$$

then we get

$$\sigma[w](t) = \sigma\left[w'\right](t)\,, \quad \forall t \in [0,T]\ . \tag{2.16}$$

Remark 2.1. In the system of equation (2.1), the matrix $\Lambda(t)$ is diagonal positive definite. This assumption is made only for the sake of simplicity in our analysis. Indeed we can consider more general diagonal matrices for $\Lambda(t)$. Suppose that there exists $m > 0$ such that for all $t \in \mathbb{R}^+$, $\Lambda(t)$ is a diagonal matrix satisfying $\Lambda(t) = \mathtt{diag}(\lambda_1(t), \ldots, \lambda_n(t))$ with $\lambda_k \in L^1_{\text{loc}}\left(\mathbb{R}^+;\mathbb{R}\right)$ for $k \in \{1, \ldots, n\}$, and $0 < \underline{\lambda^+} \leq \lambda_k(t) \leq \overline{\lambda^+}$ for $k \in \{1, \ldots, m\}$, $\underline{\lambda^-} \leq \lambda_k \leq \overline{\lambda^-} < 0$ for $k \in \{m+1, \ldots, n\}$. The matrix $\Lambda(t)$ is written as

$$\Lambda(t) = \begin{bmatrix} \Lambda^+(t) & 0_{m,n-m} \\ 0_{n-m,m} & \Lambda^-(t) \end{bmatrix},$$

where $\Lambda^+(t)$ and $\Lambda^-(t)$ are respectively diagonal positive definite matrix and diagonal negative definite matrix. We introduce the notations $y^+ = [y_1, \ldots, y_m]^\top$, $y^- = [y_{m+1}, \ldots, y_n]^\top$ such that $y = [y^+, y^-]^\top$. The system in its general form is

$$\partial_t y(t,x) + \Lambda(t)\partial_x y(t,x) = F(t)y(t,x)\,, \quad t \in \mathbb{R}^+,\, x \in [0,1]\,, \tag{2.17}$$

$$\begin{bmatrix} y^+(t,0) \\ y^-(t,1) \end{bmatrix} = G_{\sigma(t)} \begin{bmatrix} y^+(t,1) \\ y^-(t,0) \end{bmatrix}, \quad t \in \mathbb{R}^+\,, \tag{2.18}$$

$$y(0,x) = y^0(x)\,, \quad x \in [0,1]\ , \tag{2.19}$$

where for all $i \in \mathcal{I}$, $G_i = \left[\begin{smallmatrix} G_i^{++} & G_i^{+-} \\ G_i^{-+} & G_i^{--} \end{smallmatrix}\right]$, such that G_i^{++}, G_i^{--}, G_i^{+-} and G_i^{-+} are matrices respectively in $\mathbb{R}^{m\times m}$, $\mathbb{R}^{(n-m)\times(n-m)}$, $\mathbb{R}^{m\times(n-m)}$ and $\mathbb{R}^{(n-m)\times m}$. In this notation, the output introduced in (2.13) is written as

$$w(t) = \begin{bmatrix} y^+(t,1) \\ y^-(t,0) \end{bmatrix} . \tag{2.20}$$

By the change of variable $z(t,x) = \begin{bmatrix} y^+(t,x) \\ y^-(t,1-x) \end{bmatrix}$ we obtain a new system in the same form as (2.1), (2.2), and (2.3). •

Remark 2.2. At this point we can make a remark about the special behavior of these switched hyperbolic PDEs. First, discontinuities appear and propagate during the time evolution of the system. Indeed, consider the following case with $n = 2$

$$\begin{aligned} \partial_t y_1(t,x) - \partial_x y_1(t,x) &= 0 \\ \partial_t y_2(t,x) + \partial_x y_2(t,x) &= 0\,, \end{aligned}$$

with the following boundary condition

$$\begin{bmatrix} y_1(t,1) \\ y_2(t,0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} y_1(t,0) \\ y_2(t,1) \end{bmatrix} ,$$

for $t \in [0,\tau)$. Let us consider the initial condition

$$\begin{bmatrix} y_1(0,x) \\ y_2(0,x) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

Hence, for $t \in [0,\tau)$ the solution remains $y(t,x) = [1,1]^\top$. Assume that, at $t = \tau$, the boundary condition is

$$\begin{bmatrix} y_1(t,1) \\ y_2(t,0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1(t,0) \\ y_2(t,1) \end{bmatrix} .$$

Then, a discontinuity is introduced at time $t = \tau$ and propagates along the characteristics. The apparition of discontinuities is a special feature of this type of systems. Hence, one cannot expected for regularity in the solution even for linear systems when the switching appears at the boundary. Nonetheless, regularity could be preserved when considering switching with the velocities or in the source term as in [99], but this aspect is not developed in our framework. This special feature is another element to construct the space of solutions. •

In the following, we propose three control strategies. For the first one presented below, we are able to give only a result of global exponential stabilizability in L^2-norm and the existence of solution is not obtained in a general framework. For the next two strategies derived from the first one, we state existence, uniqueness of solution and global convergence in L^2-norm.

Let us define the notion of "convergence" for our switched system.

Definition 2.1. *Given a switching rule σ, the closed-loop system (2.1), (2.2), (2.3), (2.13), and (2.14) is said to be globally exponentially convergent (in L^2-norm), if there exist a positive constant $\alpha > 0$ and a function $g : \mathbb{R}^+ \to \mathbb{R}^+$, such that for all $y^0 \in E$, the solution of (2.1), (2.2), (2.3), (2.13), and (2.14) exists for all $t \in \mathbb{R}^+$ and*

$$|y(t,\cdot)|_{L^2((0,1);\mathbb{R}^n)} \leq e^{-\alpha t} g\left(|y^0|_{L^2((0,1);\mathbb{R}^n)}\right) , \tag{2.21}$$

for all $t \in \mathbb{R}^+$. *System (2.1), (2.2), (2.3), (2.13), and (2.14) is said to be globally exponentially stable (GES) if* g *can be chosen linear.*

With Definition 2.1 at hand, the next step is the analysis and the design of switching rules.

2.2 Motivations for Switching and Existing Results

2.2.1 Motivations

The aim of this subsection is to address the reasons for considering switching for PDEs. The list given there is not comprehensive, other reasons could be imagined.

First, the analysis and stabilization of non-linear PDEs is often too complicated to be tackled directly. Hence, a classical technique consists in linearizing the system around an operating regime as in [8], [77] for the Saint-Venant equations (see also Subsections 1.2.3 and 2.7.2 for the illustration of this aspect), [35] for the Saint-Venant-Exner model, [7] for the Aw-Rascle-Zhang equations (see Subsection 1.2.2 and 3.6 for the illustration of this aspect), and in controlling the linearized system around such operating points. Switched hyperbolic systems can be viewed as a transition between different linearized operating regime. This approach is often called *Multi-Models representation* as in [39].

Then, switching can also be viewed as a way to represent the possibility to change of actuator or sensor configurations. For instance in [41], [42], [43] the switching between N different actuator configurations is investigated for quasi-linear parabolic PDEs. Since the formalism of these latter references is quite different from ours, let us develop the idea for our framework. To be as clear as possible, let us take the simplest situation that we can construct. Let us suppose we are considering a distributed process governed by a hyperbolic equation over a bounded interval $[0, L]$. Moreover, let us assume there are 2 actuators in the domain: one at $x = 0$ and another one at $x_a \in (0, L)$. Then we can consider a 2×2 hyperbolic system for the whole system. The first component of the solution corresponds to the solution for $x \in [0, x_a]$, and the second component of the solution corresponds to the solution for $x \in [x_a, L]$. It is always possible to make the system dimensionless with respect to the space variable. Hence, the intervals $[0, x_a]$ and $[x_a, L]$ become $[0, 1]$, as shown in Subsection 1.2.3 for the Saint-Venant equations. Moreover, we can suppose that the velocity of the material is positive and equal in each subdomain, and that the actuator at $x = x_a$ gives the following boundary condition

$$\begin{bmatrix} y_1(t,0) \\ y_2(t,0) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} y_1(t,1) \\ y_2(t,1) \end{bmatrix}. \tag{2.22}$$

Now let us suppose that at some time $t = \tau$ the actuator at $x = 1$ fails, and that it implies a conservation of materials between the two subdomains. Hence, the boundary condition becomes

$$\begin{bmatrix} y_1(t,0) \\ y_2(t,0) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1(t,1) \\ y_2(t,1) \end{bmatrix}. \tag{2.23}$$

This example illustrates how the switching can be viewed as a way to describe the actuator configuration. In [41], the switching between different actuator configurations to obtain stability of the closed-loop parabolic PDE system is investigated. In [42], this aspect is again investigated when the parabolic equation is subject to actuator failure. In [43], the optimal choice in the actuator configuration is tackled for general PDE described by strongly continuous semigroup.

Finally, performance criteria is an important reason for switching. Indeed, some technologies include switching to give better performance result.

For instance simulated moving bed (SMB) chromatography is one example of such application [9]. This technology consists in extracting different species of a mixture in chromatographic columns. The idea is to switch periodically between different configuration of interconnected-columns in order to obtain better separation performance than in a non-switched case.

Power converters connected to transmission lines [31] are another example of physical system where the switching appears. Due to the complexity of this type of systems, which are represented by a coupled of ODE/PDE, the approach taken in [31] is based on averaging, and considers that the control takes value in the continuous set $[0, 1]$ instead of the discrete set $\{0, 1\}$, which corresponds to the switched configuration. Then, a saturated control is constructed thanks to a Lyapunov function.

Road traffic network is again an application in which the switching phenomena is natural. Indeed, to control this type of system the road speed limit can be changed to avoid the apparition of jam, as in [59] where a Model-Predictive-Control (MPC) is investigated to solve this problem. Moreover, the red and green lights, are by their nature, a switch. In [11], the synthesis of on/off control strategies is analyzed for a highway modelled by the Lighthill-Whitham-Richards (LWR) equation. Recently, the optimal control with respect to the switching times of an on/off control for a nonlinear scalar hyperbolic balance law on $\mathbb{R}$ is investigated in [92] and [90]. The motivation is the application to a red and green lights policy.

The switches can be also used in modelling. For instance, for the road traffic, congested regimes are more accurately described by the LWR model than with the ARZ, while in a free-flow model regime it is the opposite, based on empirical measurements [45]. Hence, some authors propose a model with phase transition depending on the value of the density of traffic, it is the case in [50] and in [20].

2.2.2 Existing Results

The study of switched PDEs is a very new field of research. To the best of our knowledge, one of the first result is given in [100] where the switched system is stated in an abstract form

$$\dot{x}(t) = A_{\sigma(t)} x(t) \,, \quad t \in \mathbb{R}^+ \,, \tag{2.24}$$

$$x(0) = x_0 \,. \tag{2.25}$$

The operators A_i, $i \in \mathcal{I}$ are infinitesimal generators of exponentially stable semigroups on a Hilbert space X. The commutation of the operators A_1 and A_2 for the case of switched system consisting of two modes is a sufficient condition for the stability of the system. Moreover, it is proved the existence of a common quadratic Lyapunov function under this latter condition.

Another result in this framework is given by [56], in which it is given sufficient and necessary conditions for the global exponential stability, uniform with respect to the switching signal, in terms of the existence of a common Lyapunov function to the switched system (2.24), (2.25). The analysis is led on Hilbert and Banach spaces as well.

The switching phenomena has also been studied for the wave equation. The system under consideration is the following

$$\partial_{tt} y(t,x) = c^2 \partial_{xx} y(t,x) \tag{2.26}$$

$$y(t,0) = f(t) \tag{2.27}$$

$$\partial_x y(t,1) = -a(t)\partial_t y(t,1)\,. \tag{2.28}$$

The switching appears with the function $a(t)$ which is piecewise constant. For instance, it can take two values $a_0 > 0$ and 0, as in [82] which is, to our best knowledge, the first work to deal with these type of feedback laws. In [53], the switched feedback appears with a delay that is

$$\partial_x y(t,1) = \frac{\kappa}{c}\partial_t y(t-a(t),1)\,, \tag{2.29}$$

with $\kappa > 0$. Finally, in [52] a star-shaped network in which the wave equation governed each arc, is considered (see Figure 1.4 for an illustration of this geometry). It is proved that if boundary equation (2.28) representing the control at the exterior node is zero from time to time, it is possible to stabilize exponentially the system to zero.

In [74], the switching is used to control quasi-linear hyperbolic systems with some vertical characteristics, meaning that there are some components of the state for which the associated velocity is zero. It is proved that the controllability is obtained thanks to a strategy consisting in using a local internal control which steers the initial condition to an intermediate state and then switch to a boundary control to steer this latter state to the final one.

As mentioned in the previous subsection parabolic PDEs with switching has also been considered in the literature with the already given references [43], [41], [42]. Nonetheless, we can cite the more recent work [114] where the 1-d heat equation is considered

$$\partial_t y(t,x) = \partial_{xx} y(t,x)\,, \quad x \in (0,1)\;, t \in (0,T)\;, \tag{2.30}$$

$$y(0,t) = u_0(t)\,, \tag{2.31}$$

$$y(1,t) = u_1(t)\,, \tag{2.32}$$

$$y(0,x) = y^0(x)\,, \tag{2.33}$$

where for almost every $t \in (0,T)$ one has

$$u_0(t)u_1(t) = 0\,. \tag{2.34}$$

It is shown how to construct u_0 and u_1 satisfying (2.34) such that the system is null-controllable. The construction is based on the minimization of a quadratic functional and the use of the adjoint system of (2.30)– (2.33). The null-controllability with switched pointwise control is also analyzed, that is to consider the control system

$$\partial_t y(t,x) = \partial_{xx} y(t,x) + u_a(t)\delta_a(x) + u_b(t)\delta(x)\,, \quad x \in (0,1)\ ,\ t \in (0,T)\ , \tag{2.35}$$
$$y(t,0) = 0\,, \tag{2.36}$$
$$y(t,1) = 0\,, \tag{2.37}$$
$$y(0,x) = y^0(x)\,, \tag{2.38}$$

where u_a and u_b are such that

$$u_a(t)u_b(t) = 0\,, \tag{2.39}$$

for almost every $t \in (0,T)$.

In [80], the time optimal controls for Schrödinger type systems is considered. It is proved that under some conditions on the operators, the optimal control u^* is bang-bang, which is a switching by nature. Other works are focused on the time optimal controls of PDEs, exhibiting bang-bang property. We refer the reader to the references given in the former cited work.

Finally, [57] is one of the first work in which the study of switched hyperbolic systems is addressed. It analyses a networked transport system in which each arc is governed by an equation of the form

$$\partial_t y(t,x) + \Lambda_{\sigma(t)}(t,x)\partial_x y(t,x) = f_{\sigma(t)}(t,x,y(t,x))\,, \tag{2.40}$$

with a Kirchhoff's law for the boundary conditions. Results of existence of solution are given with a class of piecewise continuous functions multivalued at the point of discontinuities. They are given in the context where the switching signal σ is given as a data and as a feedback. This latter case is the one with which we are concerned in this chapter. Besides, the question of optimal switching for this network is tackled. An analysis based on functions of bounded variation in an extended way is used in [58] to prove well-posedness for system (2.40). In this last two references, the switching modelling is viewed as a tool for systems for which different time and spatial scales interact. It is also the approach used in [101], where a result of well-posedness is obtained for transport equation, in which the velocity term is subject to an abstract closed-loop switching strategy.

The optimality of switching is also considered in [55]. The aim of this work was to approach a given trajectory y_d for the scalar system

$$\partial_t y(t,x) + \partial_x\left(\lambda(t,x)y(t,x)\right) = f_{\sigma(t)}(t,x,y(t,x))\,, \tag{2.41}$$

with binary boundary control, that is

$$y(t,0) = \hat{y}(t;\sigma(t))\,, \tag{2.42}$$

where $\sigma(t) \in \{0,1\}$. In [2], PDEs (2.40) with

$$\Lambda_{\sigma(t)}(t,x) = \Lambda_{\sigma(t)}(x) \tag{2.43}$$
$$f_{\sigma(t)}(t,x,y(t,x)) = F_{\sigma(t)}(x)y(t,x)\,. \tag{2.44}$$

where $\Lambda_{\sigma(t)}(x)$ is a diagonal matrix with a number of positive and negative velocities uniform with respect to the mode, is analyzed. The boundary condition is of the form of (2.2). The norm considered for the analysis is the L^∞-norm. The condition for stability with respect to the set of switching signals σ for which there is a finite number of discontinuities for every compact subset of the time, is proved by a spectral radius condition on the matrices $G_i(x)$, $i \in \mathcal{I}$. The result is proved by the characteristics method and an estimation of the solution along them.

More recently, the analysis of persistently damped transport PDEs for a network of circles has been addressed in [17]. To be more specific the equation on each circles $i = 1, \dots, N$, with N the number of circles, is given by

$$\partial_t y_i(t,x) + \partial_x y_i(t,x) = \alpha_i(t)\chi_i(x) y_i(t,x)\,, \tag{2.45}$$

where $\chi_i(x)$ is a characteristic function of a subdomain of the circle. The equations are mixed through a linear relationship at the central intersecting point. This system can be viewed as a switched system when, for instance, $\alpha_i(t) \in \{0,1\}$. Explicit formulas of the solution allow to prove the exponential stability of the solution under some hypothesis on the boundary condition, damping, and geometrical properties of the circles.

The optimal control with respect to the right boundary data and switching times for a switched nonlinear scalar hyperbolic balance law has been recently addressed in [91].

Finally, switched linear hyperbolic systems have been analyzed with the Lyapunov analysis in [99]. This work is inspired by this last reference for the Lyapunov approach. While the latter reference consider the stability analysis for switched linear hyperbolic systems, we will consider the stabilization of these systems by constructing switching rules.

2.3 Well-Posedness

As explained in Section 2.1, the space of solutions should be constituted by functions well-defined at $x = 1$ and take into account the discontinuities due to the switched boundary condition. Thus, let us introduce the following spaces.

Definition 2.2. *Given an interval $I \subseteq \mathbb{R}$ and a set $J \subseteq \mathbb{R}^n$ for some $n \geq 1$, a* piecewise left-continuous function *(resp. a* piecewise right-continuous function*) $y : I \to J$ is a function continuous on each compact subset of J except maybe on a finite number of points $x_0 < x_1 < \dots < x_p$ such that for all $l \in \{0, \dots, p-1\}$ there exists y_l continuous on $[x_l, x_{l+1}]$ satisfying $y_l = y_{|(x_l,x_{l+1})}$. Moreover at the points $x_1, \dots, x_p$ (resp. $x_0, \dots, x_{p-1}$) the function is continuous from the left (resp. from the right). The set of all piecewise left-continuous functions (resp. piecewise right-continuous functions) is denoted by $C_{lpw}(I;J)$ (resp. $C_{rpw}(I;J)$).*

Note that we have the following inclusions $C_{rpw}([0,1];\mathbb{R}^n)\,,\, C_{lpw}([0,1];\mathbb{R}^n) \subset L^2((0,1);\mathbb{R}^n)$.

In this section we define the solution of the Cauchy problem associated with (2.1), (2.2), and (2.3).

The solution will be defined with the classical method of characteristics (see Subsection 1.1.4). Let us recall the definition of characteristic curves.

Definition 2.3. *Given d in $\{1,\dots,n\}$, the d-th characteristic is an absolutely continuous function $s \mapsto X_d(s;t^\star,x^\star)$ which satisfies $X_d(t^\star;t^\star,x^\star) = x^\star$ and the ordinary differential equation*

$$\frac{d}{ds}X_d(s;t^\star,x^\star) = \lambda_d(s)\,, \tag{2.46}$$

almost everywhere on the domain where $X_d(\cdot;t^\star,x^\star)$ is defined.

Definition 2.4. *Let $y^0 \in C_{lpw}([0,1];\mathbb{R}^n)$ be given with $\sigma \in C_{rpw}(\mathbb{R}^+;\mathcal{I})$. A solution to the Cauchy problem associated with (2.1), (2.2), and (2.3) is a function $y : \mathbb{R}^+ \times [0,1] \to \mathbb{R}^n$ such that, for every $(t^\star,x^\star) \in \mathbb{R}^+ \times [0,1]$, the components of y are satisfying the ordinary differential equations*

$$\frac{d}{dt}y_d(t,X_d(t;t^\star,x^\star)) = f_d(t)y_d(t,X_d(t;t^\star,x^\star))\,, \tag{2.47}$$

for every $t \geq t^\star$, such that $X_d(t;t^\star,x^\star) \in [0,1]$, for all $d = 1,\dots,n$. Moreover the function y satisfies the initial condition for $t = 0$

$$y_d(0,\cdot) = y_d^0\,, \tag{2.48}$$

together with the left boundary condition

$$y_d(\cdot,0) = \sum_{k=1}^{n} G_{\sigma(t)}[d,k]y_k(\cdot,1)\,, \tag{2.49}$$

for all $d = 1,\dots,n$.

Before stating any results on existence and uniqueness let us give some useful notations. Let κ be the time for a wave whose celerity is $\overline{\lambda}$ to cross the spatial domain $[0,1]$

$$\kappa = \frac{1}{\overline{\lambda}}\,.$$

For $p \in \mathbb{N}$, let $\Delta_p \subset \mathbb{R}^+$ be defined by

$$\Delta_p = [p\kappa,(p+1)\kappa]\,. \tag{2.50}$$

Proposition 2.1. *Let $y^0 \in C_{lpw}([0,1];\mathbb{R}^n)$, assume the strategy σ is such that $\sigma[v] \in C_{rpw}(\mathbb{R}^+;\mathcal{I})$ for all $v \in C_{rpw}(\mathbb{R}^+;\mathbb{R}^n)$. Then there exists a unique solution y to the closed-loop switched system (2.1), (2.2), (2.3), (2.13), and (2.14), and $w \in C_{rpw}(\mathbb{R}^+;\mathbb{R}^n)$.*

Proof. Let us start the proof by the existence part. We proceed by induction over the time interval Δ_p. The first step consists in proving that $w \in C_{rpw}(\Delta_0;\mathbb{R}^n)$.

For all $(t,x) \in \Delta_0 \times [0,1]$ such that $\int_0^t \lambda_d(s)ds \leq x$ one gets, by the method of the characteristics,

$$\begin{aligned} y_d(t,x) &= e^{\int_0^t f_d(s)ds} y_d\left(0, x - \int_0^t \lambda_d(s)ds\right) \\ &= e^{\int_0^t f_d(s)ds} y_d^0\left(x - \int_0^t \lambda_d(s)ds\right)\,, \quad d = 1,\dots,n\,. \end{aligned} \tag{2.51}$$

Let $t \in \Delta_0$. Since for all $d = 1, \ldots, n$ and for all $s \in [0, t]$, we have $\lambda_d(s) \leq \bar{\lambda}$, one gets $\int_0^t \lambda_d(s)ds \leq 1$. Thus, the expression of w on Δ_0 is given by

$$w_d(t) = y_d(t, 0) = e^{\int_0^t f_d(s)ds} y_d^0 \left(1 - \int_0^t \lambda_d(s)ds\right), d = 1, \ldots, n. \tag{2.52}$$

Since y^0 is in $C_{lpw}([0, 1]; \mathbb{R}^n)$ it is clear that w is in $C_{rpw}(\Delta_0; \mathbb{R}^n)$, hence by assumption and using the causality property (2.15), (2.16), $\sigma[w] \in C_{rpw}(\Delta_0; \mathcal{I})$ is uniquely defined. The solution on Δ_0 is given by

$$y_d(t, x) := \sum_{k=1}^{n} G_{\sigma[w](t-\tau_d(t,x))}[d, k] w_k(t - \tau_d(t, x)), \forall (t, x) \in \Delta_0 \times [0, 1] \text{ s. t. } \int_0^t \lambda_d(s)ds > x, \tag{2.53}$$

$$y_d(t, x) := e^{\int_0^t f_d(s)ds} y_d^0 \left(x - \int_0^t \lambda_d(s)ds\right), \forall (t, x) \in \Delta_0 \times [0, 1] \text{ s. t. } \int_0^t \lambda_d(s)ds \leq x. \tag{2.54}$$

The function $\tau_d(t, x)$ is uniquely defined by the solution of the following equation

$$\int_{t-\tau_d(t,x)}^{t} \lambda_d(s)ds = x. \tag{2.55}$$

Applying the implicit function theorem we can show that τ_d is continuous w.r.t. the time and space variables. Let $t \in \Delta_0$, since w (resp. $\sigma[w]$, y^0, τ_d) belongs to $C_{rpw}(\Delta_0; \mathbb{R}^n)$ (resp. to $C_{rpw}(\Delta_0; \mathcal{I})$, $C_{lpw}([0, 1]; \mathbb{R}^n)$, and $C^0(\Delta_0 \times [0, 1]; \mathbb{R}^+)$), it follows from (2.53) that $y(t, \cdot)$ is in $C_{lpw}([0, 1]; \mathbb{R}^n)$ for all $t \in \Delta_0$. This concludes the initial step of the induction.

Suppose for $p \geq 0$, $w \in C_{rpw}(\Delta_p; \mathbb{R}^n)$ and $y(t, \cdot) \in C_{lpw}([0, 1]; \mathbb{R}^n)$ for all $t \in \Delta_p$. Taking $y((p+1)\delta, \cdot)$ as the initial condition of the system and applying the same arguments as above we prove that $w \in C_{rpw}(\Delta_{p+1}; \mathbb{R}^n)$ hence by assumption and using the causality property (2.15), (2.16), $\sigma[w] \in C_{rpw}(\Delta_{p+1}; \mathcal{I})$ is uniquely defined, and y exists on Δ_{p+1} with $y(t, \cdot)$ belongs to $C_{lpw}([0, 1]; \mathbb{R}^n)$ for all $t \in \Delta_{p+1}$,. Thus, we proved by induction that, for each $p \in \mathbb{N}$, $w \in C_{rpw}(\Delta_p; \mathbb{R}^n)$ and y exists on Δ_p and $y(t, \cdot)$ belongs to $C_{lpw}([0, 1]; \mathbb{R}^n)$. Therefore, there exists a unique solution to the switched system (2.1), (2.2), (2.3), (2.13), and (2.14), and $w \in C_{rpw}(\mathbb{R}^+; \mathbb{R}^n)$.

The uniqueness part follows directly from the definition of a solution for the system.

It concludes the proof of Proposition 2.1. □

2.4 Switching Rules

2.4.1 Lyapunov Function

In this section, preliminary results on Lyapunov functions are derived. Following [27], the candidate Lyapunov function that is considered in this chapter is written as, for all $y \in C_{lpw}([0, 1]; \mathbb{R}^n)$,

$$V(y) = \int_0^1 y(x)^\top Q y(x) e^{-\mu x} dx, \tag{2.56}$$

for a given diagonal positive definite matrix $Q \in \mathbb{R}^{n\times n}$ and a given $\mu > 0$.

Let y be a solution of system (2.1)–(2.3). We shall denote in the following

$$\forall t \in \mathbb{R}^+ \,,\ V = V(y(t,\cdot)) \text{ and } \dot{V} = \frac{d}{dt}V(y(t,\cdot))\,. \tag{2.57}$$

With this notation we are able to state our first lemma, giving an inequality for the time derivative of V along the solution of the switched system of conservation laws (2.1), (2.2), (2.3), (2.13), and (2.14). This inequality will be useful to design the switching rules, and to give the proof of global exponential convergence of the system with them.

Lemma 2.1. *The time derivative of the candidate Lyapunov function V along a solution to (2.1), (2.2), (2.3), (2.13), and (2.14) satisfies*

$$\dot{V} \leq \left(2\overline{f} - \mu\underline{\lambda}\right) V + q_{\sigma[w](t)}(w(t))\,, \tag{2.58}$$

where $q_i(w) = w^\top \left[G_i^\top Q\overline{\Lambda}G_i - e^{-\mu}Q\underline{\Lambda}\right] w$, with Q a diagonal positive definite matrix in $\mathbb{R}^{n\times n}$.

Proof. The time derivative of V along the solutions of the switched system of conservation laws (2.1), (2.2), (2.3), (2.13), and (2.14) is

$$\begin{aligned}
\dot{V} &= 2\int_0^1 y(t,x)^\top Q\partial_t y(t,x)e^{-\mu x}dx \\
&= -2\int_0^1 y(t,x)^\top Q\Lambda(t)\partial_x y(t,x)e^{-\mu x}dx + 2\int_0^1 y(t,x)^\top F(t)Qy(t,x)e^{-\mu x}dx \\
&= -\left[y(t,x)^\top Q\Lambda(t)y(t,x)e^{-\mu x}\right]_{x=0}^{x=1} - \mu\int_0^1 y(t,x)^\top Q\Lambda(t)y(t,x)e^{-\mu x}dx \\
&\quad + 2\int_0^1 y(t,x)^\top F(t)Qy(t,x)e^{-\mu x}dx \\
&= y(t,0)^\top Q\Lambda(t)y(t,0) - y(t,1)^\top Q\Lambda(t)y(t,1)e^{-\mu} - \mu\int_0^1 y(t,x)^\top Q\Lambda(t)y(t,x)e^{-\mu x}dx \\
&\quad + 2\int_0^1 y(t,x)^\top F(t)Qy(t,x)e^{-\mu x}dx \\
&= y(t,1)^\top \left[G_{\sigma[w](t)}^\top Q\Lambda(t)G_{\sigma[w](t)} - Q\Lambda(t)e^{-\mu}\right] y(t,1) - \mu\int_0^1 y(t,x)^\top Q\Lambda(t)y(t,x)e^{-\mu x}dx \\
&\quad + 2\int_0^1 y(t,x)^\top F(t)Qy(t,x)e^{-\mu x}dx\,.
\end{aligned}$$

Since Q is diagonal positive definite, using (2.10) it holds $Q\Lambda(t) \geq \underline{\lambda}Q$, using (2.12) it holds $\overline{f}Q \geq F(t)Q$. Thus, using (2.6), (2.8), and (2.12) we have

$$\begin{aligned}
\dot{V} &\leq \left(2\overline{f} - \mu\underline{\lambda}\right)\int_0^1 y(t,x)^\top Qy(t,x)e^{-\mu x}dx & (2.59)\\
&\quad + y(t,1)^\top \left[G_{\sigma[w](t)}^\top Q\overline{\Lambda}G_{\sigma[w](t)} - Q\underline{\Lambda}e^{-\mu}\right] y(t,1)\,, & (2.60)
\end{aligned}$$

using the notation in (2.13). This concludes the proof of Lemma 2.1. □

Remark 2.3. In the proof of Lemma 2.1 a technical property has been used. In the computation of the time derivative we have proceeded as the solutions were in C^1. Nonetheless, the calculus remains valid with L^2-solutions. It is due to the density of C^1-solutions in the set of L^2-solutions, see [6] for a developed explanation. •

2.4.2 Argmin

In this section, we consider the closed-loop dynamics of the switched system of conservation laws (2.1), (2.2), and (2.3) when using output controller (2.14). Following the idea developed in [49] and recalling the notation q_i in Lemma 2.1, we define the memoryless switching rule

$$\sigma[w](t) = \arg\min_{i \in \mathcal{I}} q_i(w(t)) . \tag{2.61}$$

The idea of the argmin switching rule is to choose the mode which optimizes the decrease of the Lyapunov function at any time. So we need a condition which ensures that at any time there exists a mode for which the system is decreasing.

To study the convergence of the switched system of conservation laws (2.1), (2.2), (2.3), (2.13), and (2.14) we need the following assumption.

Assumption 2.1. *Let* $\Gamma := \left\{ \gamma \in \mathbb{R}^N \middle| \sum_{i=1}^N \gamma_i = 1, \gamma_i \geq 0 \right\}$. *There exist* $\gamma \in \Gamma$, *a diagonal definite positive matrix* Q *and a parameter* $\mu > 0$ *such that*

$$\frac{2\overline{f}}{\underline{\lambda}} < \mu \tag{2.62}$$

$$\sum_{j=1}^{N} \gamma_j \left(G_j^\top Q \overline{\Lambda} G_j - e^{-\mu} Q \underline{\Lambda} \right) \leq 0 . \tag{2.63}$$

Assumption 2.1 implies that there exists a mode $i \in \mathcal{I}$ such that $q_i(w(t)) \leq 0$. Thus we can give our first result of global exponential stabilizability of the system (2.1), (2.2), (2.3), (2.13), and (2.14) with the argmin switching rule.

Remark 2.4. An important issue is the numerical computation of $\gamma \in \Gamma$, $\mu > 0$ and of a diagonal positive definite matrix Q such that Assumption 2.1 holds. The problem is bilinear in Q, γ and $e^{-\mu}$ and the numerical verification of Assumption 2.1 can be quite complex especially for larger N, since we need to solve a Bilinear Matrix Inequalities (BMI) in γ, $e^{-\mu}$ and Q. A special case is when $N = 2$, a solution consists in performing a line search over the parameters γ and μ and, for each pair (γ, μ), to solve a convex problem in the variables Q written in terms of the Linear Matrix Inequality (LMI) (2.63). This can be done numerically in polynomial time. It is the approach taken for the two examples given in Section 2.7. See [106] for a tutorial on LMI and BMI problems. Moreover, Chapter 3 is dedicated to the resolution of LMIs coming from the Lyapunov analysis of hyperbolic PDEs. •

Proposition 2.2. *Under Assumption 2.1, system (2.1), (2.2), (2.3), (2.13), and (2.14) with switching rule (2.61) is such that, as long as the solution exists,*

$$|y(t,\cdot)|_{L^2((0,1);\mathbb{R}^n)} \leq ce^{-\alpha t} \left|y^0\right|_{L^2((0,1);\mathbb{R}^n)} , \tag{2.64}$$

where $\alpha > 0$ *and* $c > 0$.

Proof. Consider the candidate Lyapunov function (2.56). Thanks to Lemma 2.1 and using the switching rule (2.61) we have, along the solution y,

$$\dot{V} \leq -2\alpha V + q_{\sigma[w](t)}(w(t)) \;=\; -2\alpha V + \min_{i\in\mathcal{I}} q_i(w(t)) \,.$$

By Assumption 2.1 there exists $\gamma \in \Gamma$ such that $\sum_{j=1}^N \gamma_j \left(G_j^\top Q\overline{\Lambda}G_j - e^{-\mu}Q\underline{\Lambda}\right) \leq 0$. Therefore, for all $t \in \mathbb{R}^+$, $\sum_{j=1}^N \gamma_j q_j(w(t)) \leq 0$. Moreover, by Assumption 2.1 there exists $\alpha > 0$ such that $2\overline{f} - \mu\underline{\lambda} < -2\alpha$. Hence for all $t \in \mathbb{R}^+$ there exists $i \in \mathcal{I}$ such that $q_i(w(t)) \leq 0$, which gives $\dot{V} \leq -2\alpha V$. Hence V satisfies $V \leq e^{-2\alpha t} V\left(y^0\right)$. Moreover there exists $\kappa > 0$ (e.g. κ could be the largest eigenvalue of Q) such that $Q \leq \kappa I_n$. Thus, the inequality $V \leq \kappa e^{-2\alpha t}\left|y^0\right|^2_{L^2((0,1);\mathbb{R}^n)}$, holds. In the same manner, there exists $\nu > 0$ (e.g. ν could be the smallest eigenvalue of Q) such that $\nu I_n \leq Q$. Since there exists a $\theta > 0$ such that $\theta \leq e^{-\mu x} \leq 1$ for all $x \in [0,1]$, the inequality $\nu\theta\,|y|^2_{L^2((0,1);\mathbb{R}^n)} \leq V$ holds. Finally, the inequality $|y|_{L^2((0,1);\mathbb{R}^n)} \leq \sqrt{\frac{\kappa}{\nu\theta}} e^{-\alpha t}\left|y^0\right|_{L^2((0,1);\mathbb{R}^n)}$ holds. Therefore the switched hyperbolic system (2.1), (2.2), (2.3), (2.13), and (2.14) is globally exponentially stabilizable (taking $g(r) = \sqrt{\frac{\kappa}{\nu\theta}}r$) with the *argmin* strategy. This concludes the proof of Proposition 2.2. □

Remark 2.5. The major drawback of this switching rule is the possibility of a finite time of existence for the solution, as illustrated by the following example. Let us consider the two transport equations

$$\partial_t \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \partial_x \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0\,,$$

with the boundary matrices

$$G_1 = \begin{bmatrix} 0.99 & -0.99 \\ 0 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 & 0 \\ -0.99 & -0.99 \end{bmatrix}.$$

The initial condition is chosen as

$$y^0(x) = \begin{bmatrix} f(x) \\ x \end{bmatrix}, \quad x \in [0,1]\,,$$

where

$$f(x) = \begin{cases} 0 & \text{if } x = 0\,, \\ x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0\,. \end{cases}$$

Assumption 2.1 is checked with $Q = I_2$, $\mu = 10^{-3}$ and $\gamma_1 = \gamma_2 = 0.5$

$$\sum_{i=1}^{2} 0.5\left(G_i^\top Q\Lambda G_i - e^{-\mu} Q\Lambda\right) = \begin{bmatrix} -0.0189 & 0 \\ 0 & -0.0189 \end{bmatrix}.$$

On Figure 2.1 the conic regions corresponding to $q_i(w) < 0$, $i = 1, 2$ are depicted, giving for the switching rule (2.61),

$$\sigma[w](t) = \begin{cases} 1 & \text{if } w_1 w_2 > 0\,, \\ 2 & \text{if } w_1 w_2 < 0\,. \end{cases}$$

Moreover the system trajectory for $t \in [0, 1)$ is depicted on Figure 2.1.

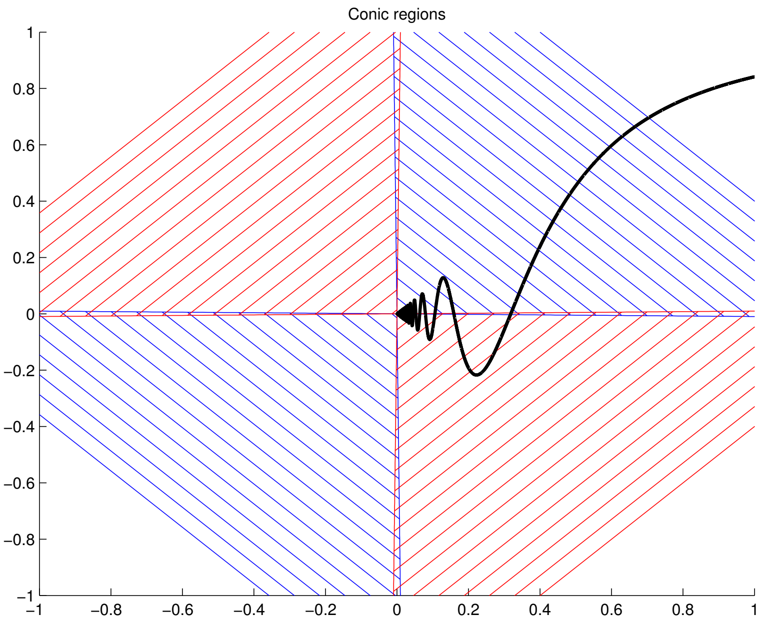

Figure 2.1: Conic regions where each individual system is active, and the trajectory of $w(t)$ for $t \in [0, 1)$

One can see that when the time approaches 1, with the switching rule (2.61) the solution crosses an infinite number of times the thin overlap region, leading to the Zeno behavior (see [76]), that is an accumulation of switching event at a particular instant. Thus the solution is not defined for all positive time but only on $[0, 1)$. •

2.4.3 Hysteresis Switching Rule

The first result shows that under Assumption 2.1, the switched system of conservation laws (2.1), (2.2), (2.3), (2.13), and (2.14) with the argmin switching rule is globally exponentially stabilizable. The limitation of this rule is a possible fast switching behavior (see Subsection 2.7 for an illustration of such behavior), and from a theoretical point of view leading to the absence of solution beyond a given time. From a practical point of view this fast switching is undesirable. So the goal is to use strategies to slow down the switching. The first one is the hysteresis strategy.

First, we will show that under this switching rule and Assumption 2.1 the system is well-posed.

For all $t > 0$ we denote by $\sigma[w](t^-)$ the limit from the left of t of the value of $\sigma[w](t)$. Roughly speaking, it is the value of $\sigma[w]$ "just before t".

The strategy is the following

$$\sigma[w](t) = \begin{cases} \sigma[w](t^-) & \text{if } q_{\sigma[w](t^-)}(w(t)) < \varepsilon(t)\,, \\ \underset{j\in\{1,\dots,N\}}{\arg\min}\; q_j(w(t)) & \text{if } q_{\sigma[w](t^-)}(w(t)) \geq \varepsilon(t)\,, \end{cases} \tag{2.65}$$

$$\sigma[w](0) = \underset{j\in\{1,\dots,N\}}{\arg\min}\; q_j(w(0))\,, \tag{2.66}$$

$$\dot{\varepsilon}(t) = -\eta\varepsilon(t)\,, \quad \varepsilon(0) > 0\,, \tag{2.67}$$

where η is such that $\eta > 2\alpha$, $\alpha = \frac{1}{2}\mu\underline{\lambda} - \overline{f}$.

Lemma 2.2. *Under Assumption 2.1, with the strategy defined in (2.65)–(2.67), if $w \in C_{rpw}(\mathbb{R}^+;\mathbb{R}^n)$ then $\sigma[w] \in C_{rpw}(\mathbb{R}^+;\mathcal{I})$.*

Proof. Let K be a compact subset of $\mathbb{R}^+$. Let us show that the number of discontinuities of $\sigma[w]$ is finite in K. By hypothesis w has a finite number of discontinuities on K. Let $t_1,\dots,t_M \in K$ be these times of discontinuity, and t_0 and t_{M+1} are respectively the lower bound and the upper bound of the interval K.

Pick $i \in \{0,\dots,M\}$. We need to estimate the number of discontinuities of $\sigma[w]$ on each time interval $[t_i,t_{i+1}]$. We can define $\tilde{w}$ as the continuation of w to the time interval $[t_i,t_{i+1}]$ with the left limit of w in t_{i+1}, that is

$$\begin{aligned} \tilde{w}(t) &= w(t)\,, \quad \text{if } t \in [t_i,t_{i+1})\,, \\ \tilde{w}(t_{i+1}) &= \lim_{t\to t_{i+1}^-} w(t)\,. \end{aligned}$$

The definition of $C_{rpw}(\mathbb{R}^+;\mathbb{R}^n)$ insures that the left limit of w exists and that $\tilde{w}$ is continuous on $[t_i,t_{i+1}]$. The number of discontinuities of $\sigma[w]$ on the time interval $[t_i,t_{i+1})$ is less than or possibly equal to the number of discontinuities when considering $\sigma[\tilde{w}]$ on $[t_i,t_{i+1}]$.

Now observe that on $\mathbb{R}^n$, q_k is continuous, as $\tilde{w}$ on $[t_i,t_{i+1}]$, thus the functions $q_k(\tilde{w})$ are continuous on the compact $[t_i,t_{i+1}]$ and therefore uniformly continuous. Using the fact that there is a finite number of functions $q_k(\tilde{w})$ and the uniform continuity, there exists $\tau_i^* > 0$ such that

$$\forall k \in \mathcal{I}, \quad \forall \hat{t}_i, \check{t}_i \in [t_i,t_{i+1}]: \quad \left|\hat{t}_i - \check{t}_i\right| \leq \tau_i^* \Rightarrow \left|q_k\left(\tilde{w}\left(\hat{t}_i\right)\right) - q_k\left(\tilde{w}\left(\check{t}_i\right)\right)\right| \leq \varepsilon\left(t_{i+1}\right)\,.$$

Due to Assumption 2.1, at a switching time t, $q_{\sigma[\tilde{w}](t)}(\tilde{w}(t)) \leq 0$, and therefore the parameter τ_i^* gives a lower bound for the distance between two switches. Thus an upper bound for the maximal number of switches on $[t_i,t_{i+1}]$ is given by

$$\overline{s}_i = \frac{t_{i+1}-t_i}{\tau_i^*} + 1\,.$$

To conclude we get that the number of discontinuities of $\sigma[w]$ on K is bounded by

$$\overline{S} = \sum_{i=1}^{M} \overline{s}_i \,,$$

which is finite. Note that the right continuity of $\sigma[w]$ follows from the strict inequality in the first line of (2.65). It concludes the proof of Lemma 2.2. □

With the above lemma, the following theorem can be stated.

Theorem 2.1. *Under Assumption 2.1, system (2.1), (2.2), (2.3), (2.13), and (2.14) with an initial condition $y^0 \in C_{lpw}\left([0,1]\,;\mathbb{R}^n\right)$ and the switching rule (2.65)–(2.67) is globally exponentially convergent with $\alpha = \frac{1}{2}\mu\underline{\lambda} - \overline{f}$ and $g(r) = c\left(r + \sqrt{\varepsilon(0)}\right)$ with $c > 0$.*

Proof. The existence and uniqueness of a solution to the system (2.1), (2.2), (2.3), (2.13), and (2.14) with the switching rule (2.65)–(2.67) is given by Lemma 2.2 and Proposition 2.1. Now to show that the system is globally exponentially convergent we have to establish relation (2.21). Thanks to Lemma 2.1 and Assumption 2.1 there exists $\alpha > 0$ such that the time derivative of V along the solution of (2.1), (2.2), (2.3), (2.13), and (2.14) is

$$\dot{V} \leq -2\alpha V + q_{\sigma[w](t)}(w(t))\,.$$

Since the invariant in the argmin with hysteresis switching rule is that $q_{\sigma[w](t)}(w(t)) \leq \varepsilon(t)$ at any time $t \in \mathbb{R}^+$, it gives

$$\dot{V} \leq -2\alpha V + \varepsilon(t)\,. \tag{2.68}$$

Then using the Gronwall's Lemma, one gets

$$V \leq e^{-2\alpha t} V\left(y^0\right) + e^{-2\alpha t}\int_0^t e^{2\alpha s}\varepsilon(s)ds\,. \tag{2.69}$$

Using the expression of ε it follows

$$V \leq e^{-2\alpha t} V\left(y^0\right) + \frac{\varepsilon(0)}{2\alpha - \eta} e^{-\eta t} - \frac{\varepsilon(0)}{2\alpha - \eta} e^{-2\alpha t}\,. \tag{2.70}$$

Letting $c = \max\left\{\sqrt{\frac{\kappa}{\nu\theta}}, \frac{1}{\sqrt{\nu\theta(\eta - 2\alpha)}}\right\}$ one gets (2.21) with $g(r) = c\left(r + \sqrt{\varepsilon(0)}\right)$, and thus the exponential convergence of (2.1), (2.2), (2.3), (2.13), and (2.14) with the switching rule (2.65)–(2.67) follows. This concludes the proof of Theorem 2.1. □

Remark 2.6. Let us note that the function $\varepsilon(t) > 0$ is essential to have the existence of solution for all time. Indeed, looking at the example given in Remark 2.5, one can see that the argmin with hysteresis strategy for which $\varepsilon(t) = 0$ leads to the same problem as for the argmin strategy alone. •

2.4.4 Low-Pass Filter Switching Rule

Thanks to Lemma 2.1 and Assumption 2.1 there exists α such that it holds along the solutions to (2.1), (2.2), (2.3), (2.13), and (2.14)

$$\dot{V} \leq -2\alpha V + q_{\sigma[w](t)}(w(t))\,. \tag{2.71}$$

Keeping in mind the objective of decreasing the number of switches, a low-pass filter is added to the switching rule (2.65)–(2.67): instead of imposing that $q_{\sigma[w](t)}(w(t)) \leq \varepsilon(t)$ at any time $t \geq 0$, we just impose that a weighted averaged value of $q_{\sigma[w](s)}(w(s))$ is less than $\varepsilon(t)$.

Let us define a function $m \in C_{rpw}\left(\mathbb{R}^+;\mathbb{R}\right)$ and a switching rule $\sigma[w]$ such that

$$m(0) = 0\,, \tag{2.72}$$

$$\lim_{\tau \to t^+} m(\tau) = \begin{cases} m(t) & \text{if } m(t) < \varepsilon(t)\,, \\ 0 & m(t) = \varepsilon(t)\,, \end{cases} \tag{2.73}$$

$$\sigma[w](t) = \begin{cases} \sigma[w](t^-) & \text{if } m(t) < \varepsilon(t)\,, \\ \underset{j\in\{1,\ldots,N\}}{\arg\min}\; q_j(w(t)) & \text{if } m(t) = \varepsilon(t)\,, \end{cases} \tag{2.74}$$

$$\sigma[w](0) = \underset{j\in\{1,\ldots,N\}}{\arg\min}\; q_j(w(0))\,, \tag{2.75}$$

$$\dot{\varepsilon}(t) = -\eta\varepsilon(t)\,, \quad \varepsilon(0) > 0\,, \tag{2.76}$$

where η is such that $\eta > 2\alpha$, $\alpha = \frac{1}{2}\mu\underline{\lambda} - \overline{f}$. On the intervals where m is continuous, the time derivative of m is the solution of the following Cauchy problem

$$\dot{m}(t) = -2\alpha m(t) + q_{\sigma[w](t_k)}(w(t))\,, \quad t \in [t_k, t_{t_{k+1}})\,, \tag{2.77}$$

$$m(t_k) = 0\,. \tag{2.78}$$

Thus the solution is

$$m(t) = e^{-2\alpha t}\int_{t_k}^{t} e^{2\alpha s} q_{\sigma[w](t_k)}(w(s))ds\,, \quad t \in [t_k, t_{k+1})\,. \tag{2.79}$$

To sum up the control consists in keeping $m(t)$ negative or zero at any time. The motivation for the choice of the function m comes from Gronwall's inequality (see Lemma A in the Appendix).

The following lemma holds for the switching rule (2.72)–(2.76).

Lemma 2.3. *Under Assumption 2.1, with the strategy (2.72)–(2.76), if $w \in C_{rpw}\left(\mathbb{R}^+;\mathbb{R}^n\right)$ then $\sigma[w] \in C_{rpw}\left(\mathbb{R}^+;\mathcal{I}\right)$.*

Proof. Let K be a compact subset of $\mathbb{R}^+$. Let us show that the number of discontinuities of $\sigma[w]$ is finite in K. By hypothesis w has a finite number of discontinuities on K. Let $t_1, \ldots, t_M \in K$ be these times of discontinuity, and t_0 and t_{M+1} are respectively the lower bound and the upper bound of the interval K.

The lines to prove the result are similar to those of the proof of Lemma 2.2. In the same fashion we consider the continuation of w, instead of w itself. But for the sake of simplicity we keep w in our notation.

Pick $i \in \{0, \dots, M\}$. Observe that on $\mathbb{R}^n$, q_k is continuous, as w on $[t_i, t_{i+1}]$, thus the functions $q_k(w)$ are continuous on the compact $[t_i, t_{i+1}]$ and therefore uniformly continuous. Using the fact that there is a finite number of functions $q_k(w)$ and the uniform continuity, there exists $\tau_i^* > 0$ such that

$$\forall k \in \mathcal{I}, \quad \forall \hat{t}_i, \check{t}_i \in [t_i, t_{i+1}] : \quad \left|\hat{t}_i - \check{t}_i\right| \leq \tau_i^* \Rightarrow \left|q_k\left(w\left(\hat{t}_i\right)\right) - q_k\left(w\left(\check{t}_i\right)\right)\right| \leq \varepsilon(t_{i+1}) .$$

Without loss of generality we can choose τ_i^* such that $1 - 2\alpha \leq e^{-2\alpha\tau_i^*}$. Assume $\bar{t} \in [t_i, t_{i+1}]$ is a switching time, hence by Assumption 2.1 it holds $q_{\sigma[w](\bar{t})}(w(\bar{t})) \leq 0$, and one gets

$$\forall \hat{t} \in \left[\bar{t}, t_{i+1}\right] : \quad \left|\bar{t} - \hat{t}\right| \leq \tau_i^* \Rightarrow \left|q_{\sigma[w](\bar{t})}\left(w\left(\bar{t}\right)\right) - q_{\sigma[w](\bar{t})}\left(w\left(\hat{t}\right)\right)\right| \leq \varepsilon(t_{i+1}) .$$

The last inequality is equivalent to

$$-\varepsilon(t_{i+1}) + q_{\sigma[w](\bar{t})}\left(w\left(\bar{t}\right)\right) \leq q_{\sigma[w](\bar{t})}\left(w\left(\hat{t}\right)\right) \leq \varepsilon(t_{i+1}) + q_{\sigma[w](\bar{t})}\left(w\left(\bar{t}\right)\right) .$$

Since $q_{\sigma[w](\bar{t})}\left(w\left(\bar{t}\right)\right) \leq 0$ one gets

$$q_{\sigma[w](\bar{t})}\left(w\left(\hat{t}\right)\right) \leq \varepsilon\left(t_{i+1}\right) . \tag{2.80}$$

With (2.80) one gets

$$e^{-2\alpha\hat{t}} \int_{\bar{t}}^{\hat{t}} e^{2\alpha s} q_{\sigma[w](\bar{t})}\left(w(s)\right) ds \leq e^{-2\alpha\hat{t}} \varepsilon\left(t_{i+1}\right) \int_{\bar{t}}^{\hat{t}} e^{2\alpha s} ds , \quad \forall \hat{t} \in \left[\bar{t}, \bar{t} + \tau_i^*\right] . \tag{2.81}$$

From (2.81) it follows

$$m\left(\hat{t}\right) \leq \frac{\left(1 - e^{-2\alpha\left(\hat{t} - \bar{t}\right)}\right)}{2\alpha} \varepsilon\left(t_{i+1}\right) \leq \varepsilon\left(t_{i+1}\right) , \quad \forall \hat{t} \in \left[t_i, t_{i+1}\right] .$$

Thus, the next switching time after $\bar{t}$ cannot appear before a time τ_i^*. Then an upper bound for the maximal number of switches on $[t_i, t_{i+1}]$ is given by

$$\bar{s}_i = \frac{t_{i+1} - t_i}{\tau_i^*} + 1 .$$

To conclude we get that the number of discontinuities of $\sigma[w]$ on K is bounded by

$$\overline{S} = \sum_{i=1}^{M} \bar{s}_i ,$$

which is finite. The right continuity of $\sigma[w]$ follows from the strict inequality in the first line of (2.74). This concludes the proof of Lemma 2.3. □

As in the last strategy we are able to give a result of global exponential convergence with the switching rule (2.72)–(2.76).

Theorem 2.2. *Under Assumption 2.1, system (2.1), (2.2), (2.3), (2.13), and (2.14) with an initial condition* $y^0 \in C_{lpw}([0,1]; \mathbb{R}^n)$ *and the switching rule (2.72)–(2.76) is globally exponentially convergent with* $\alpha = \frac{1}{2}\mu\underline{\lambda} - \overline{f}$ *and* $g(r) = c\left(r + \sqrt{\varepsilon(0)}\right)$ *with* $c > 0$*.*

Proof. The existence and uniqueness of a solution to the system (2.1), (2.2), (2.3), (2.13), and (2.14) with the switching rule (2.72)–(2.76) is given by Lemma 2.3 and Proposition 2.1. Now to show that the system is globally convergent we have to establish relation (2.21). Thanks to Lemma 2.1 the time derivative of V along the solution of the system (2.1), (2.2), (2.3), (2.13), and (2.14) is given by

$$\dot{V} \leq -2\alpha V + q_{\sigma[w](t)}(w(t))\,.$$

Thanks to differential form of Gronwall's Lemma one gets

$$V \leq e^{-2\alpha t} V\left(y^0\right) + e^{-2\alpha t}\int_0^t e^{2\alpha s} q_{\sigma[w](s)}(w(s))ds\,.$$

Using the linearity of the integral one gets

$$V \leq e^{-2\alpha t} V\left(y^0\right) + e^{-2\alpha t}\left(\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} e^{2\alpha s} q_{\sigma[w](t_i)}(w(s))ds + \int_{t_k}^{t} e^{2\alpha s} q_{\sigma[w](t_k)}(w(s))ds\right)\,, \tag{2.82}$$

where the t_is are the switching times of $\sigma[w]$, t lies in a interswitching interval (t_k, t_{k+1}). Let us point out that the number of them is finite thanks to Lemma 2.3. Since

$$\varepsilon(t_{i+1}) = e^{-2\alpha t_{i+1}}\int_{t_i}^{t_{i+1}} e^{2\alpha s} q_{\sigma[w](t_i)}(w(s))ds\,,$$

one gets

$$V \leq e^{-2\alpha t} V\left(y^0\right) + e^{-2\alpha t}\left(\sum_{i=0}^{k-1} e^{2\alpha t_{i+1}}\varepsilon(t_{i+1}) + \int_{t_k}^{t} e^{2\alpha s} q_{\sigma[w](t_k)}(w(s))ds\right)\,.$$

Using the positivity and the continuity of ε, and the switching rule (2.72)–(2.76) we have the following inequality for all $t \in (t_k, t_{k+1})$

$$V \leq e^{-2\alpha t} V\left(y^0\right) + e^{-2\alpha t}\left(\int_0^{t_k} e^{2\alpha s}\varepsilon(s)ds + e^{2\alpha t}\varepsilon(t)\right)\,.$$

Using the positivity of ε one gets

$$\begin{aligned} V &\leq e^{-2\alpha t} V\left(y^0\right) + e^{-2\alpha t}\left(\int_0^{t_k} e^{2\alpha s}\varepsilon(s)ds + \int_{t_k}^{t} e^{2\alpha s}\varepsilon(s)ds + e^{2\alpha t}\varepsilon(t)\right)\,, \\ V &\leq e^{-2\alpha t} V\left(y^0\right) + \left(1 + \frac{1}{2\alpha - \eta}\right)\varepsilon(0)e^{-\eta t} - \frac{\varepsilon(0)}{2\alpha - \eta}e^{-2\alpha t}\,. \end{aligned}$$

Letting $c = \max\left\{\sqrt{\frac{\kappa}{\nu\theta}}, \sqrt{\frac{\eta - 2\alpha + 1}{\nu\theta(\eta - 2\alpha)}}\right\}$ one gets (2.21) with $g(r) = c\left(r + \sqrt{\varepsilon(0)}\right)$, and thus the global exponential convergence of (2.1), (2.2), (2.3), (2.13), and (2.14) with the switching rule

(2.74) follows.

This concludes the proof of Theorem 2.2. □

2.5 ISS with respect to Measurement Noise

The switching rules (2.61), (2.65), and (2.74) depend on the value $w(t)$. The natural question arising in this context is: how does this switching rule react to a measurement noise ? In this section, we state two results on Input-to-State Stability (ISS). Basically, a system is ISS if for bounded input, as noise, the state of the system stays bounded. This concept has been originally introduced in [104] for system of finite dimension. To the best of our best knowledge the first work dedicated to this notion for hyperbolic PDEs is [96] where an ISS-Lyapunov function has been introduced for time-varying linear hyperbolic PDEs with distributed errors. We will develop ISS results for the argmin with hysteresis switching rule and for the argmin, hysteresis with low-pass filter switching rules.

In order to establish ISS results in presence of measurement noise, let us consider system (2.1), (2.2), and (2.3) with

$$\sigma(t) = \sigma[w + \delta](t)\,, \tag{2.83}$$

where w is given by (2.13) as before and $\delta \in C_{rpw}\left(\mathbb{R}^+; \mathbb{R}^n\right)$ is an unknown disturbance. Let us state the definition of ISS for system (2.1), (2.2), (2.3), (2.13), and (2.83).

Definition 2.5. *The switched system (2.1), (2.2), (2.3), (2.13), and (2.83) is ISS if there exist a positive constant $\alpha > 0$, a function $g : \mathbb{R}^+ \to \mathbb{R}^+$, and a function h of class $\mathcal{K}_\infty$, such that, for all $y^0 \in C_{lpw}\left([0,1]\,; \mathbb{R}^n\right)$ and for all $\delta \in C_{rpw}\left(\mathbb{R}^+; \mathbb{R}\right)$, the solution to (2.1), (2.2), (2.3), (2.13), and (2.83) exists for all $t \in \mathbb{R}^+$ and*

$$|y(t,\cdot)|_{L^2(0,1;\mathbb{R}^n)} \leq e^{-\alpha t} g\left(\left|y^0\right|_{L^2((0,1);\mathbb{R}^n)}\right) + h\left(|\delta|_{C_{rpw}([0,t];\mathbb{R}^n)}\right)\,. \tag{2.84}$$

Let us consider the switching rule adapted from (2.65)–(2.67)

$$\sigma[w+\delta](t) = \begin{cases} \sigma[w+\delta](t^-) & \text{if } q_{\sigma[w+\delta](t^-)}(w(t)+\delta(t)) < -\zeta\,|w\,(t) + \delta\,(t)|^2 + \varepsilon(t)\,, \\ \underset{j\in\{1,\ldots,N\}}{\arg\min}\; q_j(w(t)+\delta(t)) & \text{if } q_{\sigma[w+\delta](t^-)}(w(t)+\delta(t)) \geq -\zeta\,|w\,(t) + \delta\,(t)|^2 + \varepsilon(t)\,, \end{cases} \tag{2.85}$$

$$\sigma[w+\delta](0) = \underset{j\in\{1,\ldots,N\}}{\arg\min}\; q_j(w(0)+\delta(0))\,, \tag{2.86}$$

$$\dot{\varepsilon}(t) = -\eta\varepsilon(t)\,, \quad \varepsilon(0) > 0\,, \tag{2.87}$$

where ζ, α, and η are positive constants $\eta > 2\alpha$, $\alpha = \frac{1}{2}\mu\underline{\lambda} - \overline{f}$. Let us define the parameter

$$\beta = \sup_t \max_{i\in\mathcal{I}} \|G_i^\top Q\Lambda(t)G_i - e^{-\mu}Q\Lambda(t) + \zeta I_n\|\,, \tag{2.88}$$

The following assumption is stated.

Assumption 2.2. *Let* $\Gamma := \left\{\gamma \in \mathbb{R}^N \middle| \sum_{i=1}^N \gamma_i = 1, \gamma_i \geq 0\right\}$. *There exist* $\gamma \in \Gamma$, *a diagonal definite positive matrix* Q *and a parameter* $\mu > 0$ *such that*

$$\frac{2\overline{f}}{\underline{\lambda}} < \mu \tag{2.89}$$

$$\sum_{j=1}^{N} \gamma_j \left(G_j^\top Q \overline{\Lambda} G_j - e^{-\mu} Q \underline{\Lambda}\right) < 0\,. \tag{2.90}$$

It implies that there exists $\zeta > 0$ such that

$$\sum_{j=1}^{N} \gamma_j \left(G_j^\top Q \overline{\Lambda} G_j - e^{-\mu} Q \underline{\Lambda}\right) \leq -\zeta I_n\,. \tag{2.91}$$

This latter Assumption is stronger than Assumption 2.1 by comparing (2.63) with (2.91).

Remark 2.7. Note that β is the supremum of a time-dependent matrix norm. Nonetheless, this value is finite due to the bounds (2.4) on the matrix Λ. For instance, one has

$$\beta \leq \max_{i\in\mathcal{I}} \|Q\|\|\overline{\Lambda}\|\|G_i\|^2 + e^{-\mu}\|Q\|\|\overline{\Lambda}\| + \zeta\,. \tag{2.92}$$

•

2.5.1 ISS Stability with the Hysteresis Switching Rule

Let us state the following result with the adapted switching rule (2.85)–(2.87).

Proposition 2.3. *Under Assumption 2.2, system (2.1), (2.2), (2.3), (2.13), and (2.83) with an initial condition* $y^0 \in C_{lpw}\left([0,1]\,;\mathbb{R}^n\right)$ *and a disturbance* $\delta \in C_{rpw}\left(\mathbb{R}^+;\mathbb{R}^n\right)$ *with the switching rule (2.85)–(2.87) is ISS.*

Proof. First note that Lemma 2.2 is still valid in the context of the switching rule (2.85)–(2.87) instead of (2.65)–(2.67), for any $\delta \in C_{rpw}\left(\mathbb{R}^+;\mathbb{R}^n\right)$. Hence the existence and uniqueness of a solution to the system (2.1), (2.2), (2.3), (2.13), and (2.83) with the switching rule (2.85)–(2.87) is given by Lemma 2.2 and Proposition 2.1. Due to Lemma 2.1 and Assumption 2.2 the time derivative of V is given by

$$\dot{V} \leq -2\alpha V + q_{\sigma[w+\delta](t)}(w(t))\,. \tag{2.93}$$

The quadratic term in the right-hand part of (2.93) can be written as

$$\begin{aligned} q_{\sigma[w+\delta](t)}(w(t)) =& q_{\sigma[w+\delta](t)}\left(w(t)+\delta(t)\right) \\ &- \delta(t)^\top \left(G_{\sigma[w+\delta](t)}^\top Q\Lambda(t) G_{\sigma[w+\delta](t)} - e^{-\mu}Q\Lambda(t)\right)\left(2w(t)+\delta(t)\right)\,. \end{aligned} \tag{2.94}$$

By Assumption 2.2 and (2.90), (2.85) one gets

$$q_{\sigma[w+\delta](t)}(w(t)) \leq -\zeta\left|w(t)+\delta(t)\right|^2 + \varepsilon(t)$$

$$- \delta(t)^\top \left(G^\top_{\sigma[w+\delta](t)} Q\Lambda(t) G_{\sigma[w+\delta](t)} - e^{-\mu} Q\Lambda(t) \right) (2w(t) + \delta(t)) \, .$$

Using that $(a+b)^2 \geq \frac{1}{2}a^2 - b^2$, for all a, b in $\mathbb{R}$, one has

$$\begin{aligned} q_{\sigma[w+\delta](t)}(w(t)) \leq & - \frac{\zeta}{2} \left| w(t) \right|^2 + \zeta \left| \delta(t) \right|^2 + \varepsilon(t) \\ & - \delta(t)^\top \left(G^\top_{\sigma[w+\delta](t)} Q\Lambda(t) G_{\sigma[w+\delta](t)} - e^{-\mu} Q\Lambda(t) \right) (2w(t) + \delta(t)) \, . \end{aligned}$$

Using (2.88) one gets

$$q_{\sigma[w+\delta](t)}(w(t)) \leq -\frac{\zeta}{2} \left| w(t) \right|^2 + \zeta \left| \delta(t) \right|^2 + \varepsilon(t) + \beta \left| \delta(t) \right|^2 + 2\beta \left| \delta(t) \right| \left| w(t) \right| \, . \tag{2.95}$$

Using the Young's inequality with the last term in (2.95) one gets

$$q_{\sigma[w+\delta](t)}(w(t)) \leq \left(\frac{\beta}{\psi} - \frac{\zeta}{2} \right) \left| w(t) \right|^2 + (\zeta + \beta + \beta\psi) \left| \delta(t) \right|^2 + \varepsilon(t) \, , \tag{2.96}$$

with $\psi > 0$. Letting

$$\psi = \frac{2\beta}{\zeta} \, , \tag{2.97}$$

one obtains

$$q_{\sigma[w+\delta](t)}(w(t)) \leq (\zeta + \beta + \beta\psi) \left| \delta(t) \right|^2 + \varepsilon(t) \, . \tag{2.98}$$

Thus, the time-derivative of the candidate Lyapunov function V satisfies

$$\dot{V} \leq -2\alpha V + (\zeta + \beta + \beta\psi) \left| \delta(t) \right|^2 + \varepsilon(t) \, . \tag{2.99}$$

Using the Gronwall's Lemma we get

$$V \leq e^{-2\alpha t} V\left(y^0\right) + e^{-2\alpha t} \int_0^t e^{2\alpha s} \left((\zeta + \beta + \beta\psi) \left| \delta(s) \right|^2 + \varepsilon(s) \right) ds \, . \tag{2.100}$$

Hence, the following inequality holds

$$\begin{aligned} |y|^2_{L^2(0,1;\mathbb{R}^n)} \leq & \frac{\kappa}{\nu\theta} e^{-2\alpha t} \left| y^0 \right|^2_{L^2(0,1;\mathbb{R}^n)} + \frac{\varepsilon(0)}{\nu\theta\,(2\alpha - \eta)} e^{-\eta t} \\ & - \frac{\varepsilon(0)}{\nu\theta\,(2\alpha - \eta)} e^{-2\alpha t} + \frac{\zeta + \beta + \beta\psi}{2\nu\theta\alpha} \sup_{s \in [0,t]} \left| \delta(s) \right| \, . \end{aligned} \tag{2.101}$$

Using the fact that $\eta > 2\alpha$, if we let $c = \max\left\{ \sqrt{\frac{\kappa}{\nu\theta}}, \frac{1}{\sqrt{\nu\theta(\eta - 2\alpha)}} \right\}$ one has (2.84) with $g(r) = c\left(r + \sqrt{\varepsilon(0)} \right)$, and $h(r) = \sqrt{\frac{(\zeta + \beta + \beta\psi)}{2\nu\theta\alpha}}\, r$. This concludes the proof of Proposition 2.3.

□

2.5.2 ISS Stability with the Low-Pass Filter Switching Rule

Motivated by the idea to insure the stability with respect to measurement noise, we consider system (2.1), (2.2), (2.3) , (2.13), and (2.83) with the switching rule

$$m(0) = 0\,, \tag{2.102}$$

$$\lim_{\tau\to t^+} m(\tau) = \begin{cases} m(t) & \text{if } m(t) < -\zeta e^{-2\alpha t}\int_{t_k}^t e^{2\alpha s}\left|w(s)+\delta(s)\right|^2 ds + \varepsilon(t)\,,\\ 0 & m(t) = -\zeta e^{-2\alpha t}\int_{t_k}^t e^{2\alpha s}\left|w(s)+\delta(s)\right|^2 ds + \varepsilon(t)\,, \end{cases} \tag{2.103}$$

$$\sigma[w+\delta](t) = \begin{cases} \sigma[w+\delta](t^-) & \text{if } m(t) < -\zeta e^{-2\alpha t}\int_{t_k}^t e^{2\alpha s}\left|w(s)+\delta(s)\right|^2 ds + \varepsilon(t)\,,\\ \underset{j\in\{1,\dots,N\}}{\arg\min}\; q_j\left(w(t)+\delta(t)\right) & \text{if } m(t) = -\zeta e^{-2\alpha t}\int_{t_k}^t e^{2\alpha s}\left|w(s)+\delta(s)\right|^2 ds + \varepsilon(t)\,, \end{cases} \tag{2.104}$$

$$\sigma[w+\delta](0) = \underset{j\in\{1,\dots,N\}}{\arg\min}\; q_j\left(w(0)+\delta(0)\right)\,, \tag{2.105}$$

$$\dot\varepsilon(t) = -\eta\varepsilon(t)\,,\quad \varepsilon(0) > 0\,,\,, \tag{2.106}$$

where t_k is the last instant before t for which m vanishes and η is such that $\eta > 2\alpha$, $\alpha = \frac{1}{2}\mu\underline{\lambda} - \overline{f}$. On the intervals where m is continuous, the time derivative of m is the solution of the following Cauchy problem

$$\dot m(t) = -2\alpha m(t) + q_{\sigma[w+\delta](t_k)}\left(w(t)+\delta(t)\right)\,, t\in[t_k, t_{k+1})\,, \tag{2.107}$$

$$m\left(t_k\right) = 0\,. \tag{2.108}$$

Thus the solution is

$$m(t) = e^{-2\alpha t}\int_{t_k}^t e^{2\alpha s} q_{\sigma[w+\delta](s)}\left(w(s)+\delta(s)\right) ds\,,\quad t\in[t_k, t_{k+1})\,. \tag{2.109}$$

Let us state the following result.

Proposition 2.4. *Under Assumption 2.2, system (2.1), (2.2), (2.3), (2.13), and (2.83) with an initial condition $y^0 \in C_{lpw}\left([0,1];\mathbb{R}^n\right)$ and a disturbance $\delta \in C_{rpw}\left(\mathbb{R}^+;\mathbb{R}^n\right)$ with the switching rule (2.102)–(2.106) is ISS.*

Proof. First note that Lemma 2.3 is still valid in the context of the switching rule (2.102)–(2.106) instead of (2.72)–(2.76), for any $\delta \in C_{rpw}\left(\mathbb{R}^+;\mathbb{R}^n\right)$. Hence the existence and uniqueness of a solution to the system (2.1), (2.2), (2.3), (2.13), and (2.83) with the switching rule (2.102)–(2.106) is given by Lemma 2.3 and Proposition 2.1. Due to Lemma 2.1 and Assumption 2.2 there exists $\alpha > 0$ such that the time derivative of V is given by

$$\dot V \le -2\alpha V + q_{\sigma[w+\delta](t)}(w(t))\,.$$

Thank to differential form of Gronwall's Lemma one gets

$$V \le e^{-2\alpha t} V\left(y^0\right) + e^{-2\alpha t}\int_0^t e^{2\alpha s} q_{\sigma[w+\delta](s)}(w(s)) ds\,. \tag{2.110}$$

Using (2.94), relation (2.110) becomes

$$V \leq e^{-2\alpha t} V\left(y^0\right) + e^{-2\alpha t} \int_0^t e^{2\alpha s} q_{\sigma[w+\delta](s)}\left(w(s)+\delta(s)\right) ds$$
$$- e^{-2\alpha t} \int_0^t e^{2\alpha s} \delta(s)^\top \left(G^\top_{\sigma[w+\delta](s)} Q\Lambda(t) G_{\sigma[w+\delta](s)} - e^{-\mu} Q\Lambda(t)\right)\left(2w(s)+\delta(s)\right) ds\,. \quad (2.111)$$

Let us denote the second term in the right-hand side of (2.111) by $R(t)$. Using the linearity of the integral one gets

$$R(t) = e^{-2\alpha t}\left(\sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} e^{2\alpha s} q_{\sigma[w+\delta](s)}\left(w(s)+\delta(s)\right) ds + \int_{t_k}^{t} e^{2\alpha s} q_{\sigma[w+\delta](s)}(w(s)+\delta(s)) ds\right) ds\,, \quad (2.112)$$

where the t_is are the switching times of $\sigma[w+\delta]$, t lies in a interswitching interval (t_k, t_{k+1}). Let us point out that the number of them is finite thank to Lemma 2.3. Since

$$m\left(t_{i+1}\right) = -\zeta e^{-2\alpha t_{i+1}} \int_{t_i}^{t_{i+1}} e^{2\alpha s} \left|w(s)+\delta(s)\right|^2 ds + \varepsilon\left(t_{i+1}\right)\,,$$

(2.112) becomes

$$\begin{aligned} R(t) &= e^{-2\alpha t}\left(\sum_{i=0}^{k-1} -\zeta \int_{t_i}^{t_{i+1}} e^{2\alpha s}\left|w(s)+\delta(s)\right|^2 ds + e^{2\alpha t_{i+1}} \varepsilon\left(t_{i+1}\right)\right. \\ &\quad \left. + \int_{t_k}^{t} e^{2\alpha s} q_{\sigma[w+\delta](s)}(w(s)+\delta(s)) ds\right) \\ &= -\zeta e^{-2\alpha t} \int_0^{t_k} e^{2\alpha s}\left|w(s)+\delta(s)\right|^2 ds \\ &\quad + e^{-2\alpha t}\left(\sum_{i=0}^{k-1} e^{2\alpha t_{i+1}} \varepsilon\left(t_{i+1}\right) + \int_{t_k}^{t} e^{2\alpha s} q_{\sigma[w+\delta](s)}(w(s)+\delta(s)) ds\right)\,. \end{aligned}$$

Using the positivity and the continuity of ε, and the switching rule (2.102)–(2.106) we have the following inequality for all $t \in (t_k, t_{k+1})$

$$R(t) \leq -\zeta e^{-2\alpha t} \int_0^t e^{2\alpha s}\left|w(s)+\delta(s)\right|^2 ds + e^{-2\alpha t} \int_0^t e^{2\alpha s} \varepsilon(s) ds\,.$$

Hence

$$V \leq e^{-2\alpha t} V\left(y^0\right) - \zeta e^{-2\alpha t} \int_0^t e^{2\alpha s}\left|w(s)+\delta(s)\right|^2 ds + e^{-2\alpha t} \int_0^t e^{2\alpha s} \varepsilon(s) ds$$
$$- e^{-2\alpha t} \int_0^t e^{2\alpha s} \delta(s)^\top \left(G^\top_{\sigma[w+\delta](s)} Q\Lambda(t) G_{\sigma[w+\delta](s)} - e^{-\mu} Q\Lambda(t)\right)\left(2w(s)+\delta(s)\right) ds\,.$$

Analogously of the proof of Proposition 2.3 we get

$$V \leq e^{-2\alpha t} V\left(y^0\right) + \left(\zeta + \beta + \beta\psi\right) e^{-2\alpha t} \int_0^t e^{2\alpha s}\left|\delta(s)\right|^2 ds + e^{-2\alpha t} \int_0^t e^{2\alpha s} \varepsilon(s) ds\,.$$

Using the end of the proof of Proposition 2.3 we get Proposition 2.4. □

2.6 Robust Stabilization in presence of Measurement Noise

2.6.1 Robust Stabilization with the Hysteresis Switching Rule

As explained in the previous section the switching rule (2.65)–(2.67) depends on the measurements at the boundary. The aim of this section is to find some conditions such that the system is stabilizable with a switching rule despite the measurement noise. The hysteresis properties is used for robustness as in [93], [94], [97] for non-linear finite-dimensional systems. The switching signal is supposed to have the form (2.83). Let us state the following result.

Proposition 2.5. *Under Assumption 2.2, system (2.1), (2.2), (2.3), (2.13), and (2.83) with an initial condition* $y^0 \in C_{lpw}\left([0,1];\mathbb{R}^n\right)$ *and a disturbance* $\delta \in C_{rpw}\left(\mathbb{R}^+;\mathbb{R}^n\right)$ *satisfying*

$$|\delta(t)| \leq \rho\,|w(t)|\,, \quad \forall t \in \mathbb{R}^+\,, \tag{2.113}$$

for a constant ρ *such that*

$$0 < \rho \leq \sqrt{1+\frac{\zeta}{\beta}} - 1\,, \tag{2.114}$$

with the switching rule (2.85)–(2.87) is globally exponentially convergent.

Proof. Once again note that Lemma 2.2 is still valid in the context of the switching rule (2.85)–(2.87) instead of (2.65)–(2.67), for any $\delta \in C_{rpw}\left(\mathbb{R}^+;\mathbb{R}^n\right)$. Hence the existence and uniqueness of a solution to the system (2.1), (2.2),(2.3), (2.13), and (2.83) with the switching rule (2.85)–(2.87) is given by Lemma 2.2 and Proposition 2.1. Due to Lemma 2.1 and Assumption 2.2 there exists $\alpha > 0$ such that the time derivative of V is given by

$$\dot V \leq -2\alpha V + q_{\sigma[w+\delta](t)}(w(t))\,. \tag{2.115}$$

The quadratic term in the right-hand part of (2.115) can be written as

$$\begin{aligned} q_{\sigma[w+\delta](t)}(w(t)) = {} & q_{\sigma[w+\delta](t)}\left(w(t)+\delta(t)\right) \\ & - \delta(t)^\top \left(G^\top_{\sigma[w+\delta](t)} Q\Lambda(t) G_{\sigma[w+\delta](t)} - e^{-\mu} Q\Lambda(t)\right)\left(2w(t)+\delta(t)\right)\,. \end{aligned} \tag{2.116}$$

By Assumption 2.2 and (2.90), (2.85) one gets

$$\begin{aligned} q_{\sigma[w+\delta](t)}(w(t)) \leq {} & -\zeta\,|w(t)+\delta(t)|^2 + \varepsilon(t) \\ & - \delta(t)^\top \left(G^\top_{\sigma[w+\delta](t)} Q\Lambda(t) G_{\sigma[w+\delta](t)} - e^{-\mu} Q\Lambda(t)\right)\left(2w(t)+\delta(t)\right)\,. \end{aligned}$$

Expanding $-\zeta\,|w(t)+\delta(t)|^2$ one has

$$\begin{aligned} q_{\sigma[w+\delta](t)}(w(t)) \leq {} & -\zeta\,|w(t)|^2 + \varepsilon(t) - \zeta\,|\delta(t)|^2 - 2\zeta\delta(t)^\top w(t) \\ & - \delta(t)^\top \left(G^\top_{\sigma[w+\delta](t)} Q\Lambda(t) G_{\sigma[w+\delta](t)} - e^{-\mu} Q\Lambda(t)\right)\left(2w(t)+\delta(t)\right)\,. \end{aligned} \tag{2.117}$$

Rearranging the three last terms of (2.117) as follows

$$
\begin{aligned}
&- \zeta \left|\delta(t)\right|^2 - 2\zeta\delta(t)^\top w(t) - \delta(t)^\top \left(G^\top_{\sigma[w+\delta](t)} Q\Lambda(t) G_{\sigma[w+\delta](t)} - e^{-\mu} Q\Lambda(t)\right)(2w(t)+\delta(t)) \\
&= -\delta(t)^\top \left(G^\top_{\sigma[w+\delta](t)} Q\Lambda(t) G_{\sigma[w+\delta](t)} - e^{-\mu} Q\Lambda(t) + \zeta I_n\right) 2w(t) \\
&\quad - \delta(t)^\top \left(G^\top_{\sigma[w+\delta](t)} Q\Lambda(t) G_{\sigma[w+\delta](t)} - e^{-\mu} Q\Lambda(t) + \zeta I_n\right) \delta(t) \\
&= -\delta(t)^\top \left(G^\top_{\sigma[w+\delta](t)} Q\Lambda(t) G_{\sigma[w+\delta](t)} - e^{-\mu} Q\Lambda(t) + \zeta I_n\right)(2w(t)+\delta(t)) \,,
\end{aligned}
$$

we get

$$
\begin{aligned}
q_{\sigma[w+\delta](t)}(w(t)) \leq &- \zeta \left|w(t)\right|^2 + \varepsilon(t) \\
&- \delta(t)^\top \left(G^\top_{\sigma[w+\delta](t)} Q\Lambda(t) G_{\sigma[w+\delta](t)} - e^{-\mu} Q\Lambda(t) + \zeta I_n\right)(2w(t)+\delta(t)) \,.
\end{aligned} \tag{2.118}
$$

Using (2.118), (2.88), and (2.113) we get

$$
q_{\sigma[w+\delta](t)}(w(t)) \leq \left(\beta\rho^2 + 2\beta\rho - \zeta\right)\left|w(t)\right|^2 + \varepsilon(t) \,.
$$

In order to have $q_{\sigma[w+\delta](t)}(w(t)) \leq \varepsilon(t)$ we need $\beta\rho^2 + 2\beta\rho - \zeta \leq 0$. This expression is a polynomial in ρ. Computing the discriminant one gets

$$
\Delta = 4\beta^2 + 4\beta\zeta > 0 \,. \tag{2.119}
$$

Then there exists two real roots for this polynomial. Since the constant term is negative, the two roots have opposite sign. Hence there exists ρ such that

$$
0 < \rho \leq \sqrt{1 + \frac{\zeta}{\beta}} - 1 \,, \tag{2.120}
$$

for which one has $q_{\sigma[w+\delta](t)}(w(t)) \leq \varepsilon(t)$. The end of the proof follows the proof of Theorem 2.1. This concludes the proof of Proposition 2.5. □

2.6.2 Robust Stabilization with the Low-Pass Filter Switching Rule

As for the hysteresis switching rule a stabilizability result can be stated for the low-pass filter switching rule using the modified version (2.102)–(2.106). Let us state the following result.

Proposition 2.6. *Under Assumption 2.2, system (2.1), (2.2), (2.3), (2.13), and (2.83) with an initial condition $y^0 \in C_{lpw}\left([0,1]\,;\mathbb{R}^n\right)$ and a disturbance $\delta \in C_{rpw}\left(\mathbb{R}^+;\mathbb{R}^n\right)$ satisfying*

$$
\left|\delta(t)\right| \leq \rho \left|w(t)\right| \,, \quad \forall t \in \mathbb{R}^+ \,, \tag{2.121}
$$

for a constant ρ such that

$$
0 < \rho \leq \sqrt{1 + \frac{\zeta}{\beta}} - 1 \,, \tag{2.122}
$$

with the switching rule (2.102)–(2.106) is globally exponentially convergent.

Proof. First note that Lemma 2.3 is still valid in the context of the switching rule (2.102)–(2.106) instead of (2.72)–(2.76), for any $\delta \in C_{rpw}(\mathbb{R}^+;\mathbb{R}^n)$. Hence the existence and uniqueness of a solution to the system (2.1), (2.2), (2.3), (2.13), and (2.83) with the switching rule (2.102)–(2.106) is given by Lemma 2.3 and Proposition 2.1. By Assumption 2.2 and the proof of Proposition 2.4 the following inequality holds.

$$\begin{aligned} V \leq\, & e^{-2\alpha t} V\left(y^0\right) - \zeta e^{-2\alpha t}\int_0^t e^{2\alpha s}\left|w(s)+\delta(s)\right|^2 ds + e^{-2\alpha t}\int_0^t e^{2\alpha s}\varepsilon(s) ds \\ & - e^{-2\alpha t}\int_0^t e^{2\alpha s}\delta(s)^\top\left(G_{\sigma[w+\delta](s)}^\top Q\Lambda(t) G_{\sigma[w+\delta](s)} - e^{-\mu}Q\Lambda(t)\right)(2w(s)+\delta(s))\, ds\,. \end{aligned}$$

By the proof of Proposition 2.5 we know that

$$\begin{aligned} & - e^{-2\alpha t}\int_0^t e^{2\alpha s}\delta(s)^\top\left(G_{\sigma[w+\delta](s)}^\top Q\Lambda(t) G_{\sigma[w+\delta](s)} - e^{-\mu}Q\Lambda(t)\right)(2w(s)+\delta(s))\, ds \\ & - \zeta e^{-2\alpha t}\int_0^t e^{2\alpha s}\left|w(s)+\delta(s)\right|^2 ds \leq \left(\beta\rho^2 + 2\beta\rho - \zeta\right) e^{-2\alpha t}\int_0^t e^{2\alpha s}\left|w(s)\right|^2 ds\,. \end{aligned}$$

Analogously to the proof of Proposition 2.5, the polynomial $\beta\rho^2 + 2\beta\rho - \zeta$ in ρ has a positive root given by

$$\rho = \sqrt{1 + \frac{\zeta}{\beta}} - 1\,.$$

Hence for all choice of ρ such that (2.122) holds we get

$$V \leq e^{-2\alpha t} V\left(y^0\right) + e^{-2\alpha t}\int_0^t e^{2\alpha s}\varepsilon(s) ds\,.$$

The end of the proof follows the proof of Theorem 2.2. This concludes the proof of Proposition 2.6.

□

2.7 Numerical Experiments

2.7.1 Academic Examples

To illustrate the results develop in this chapter let us consider system (2.1)–(2.3) with two modes ($\mathcal{I} = \{1,2\}$). The initial conditions are selected as the first three elements of an orthonormal basis of $L^2\left((0,1);\mathbb{R}^2\right)$. More specifically the following three initial conditions

$$y_k^0(x) = \begin{bmatrix} \sqrt{2}\sin((2k-1)\pi x) \\ \sqrt{2}\sin(2k\pi x) \end{bmatrix},\ k = 1,2,3, \tag{2.123}$$

are considered. The matrix of the system (2.1) is given by

$$\Lambda(t) = \mathtt{diag}\left(0.5 + 0.05\sin(10t), 0.5 + 0.05\cos(10t)\right), \quad t \in \mathbb{R}^+\,. \tag{2.124}$$

Thanks to a result of [27], boundary matrices G_i which destabilize the unswitched system must satisfy $\rho(|G_i|) > 1$[1]. First, let us illustrate Propositions 2.5 and 2.6. The instability of the system with the proposed matrices is checked numerically with a Weighted Essentially Non Oscillatory scheme (see [79]). Boundary matrices G_1 and G_2 are proposed as

$$G_1 = \begin{bmatrix} 1.1 & 0.2 \\ -0.3 & 0.1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & -1.05 \end{bmatrix}. \tag{2.125}$$

The respective spectrum of $|G_1|$ and $|G_2|$ is $\{1.1568; 0.0432\}$ and $\{0.1771; 1.0729\}$.

To illustrate our result of robustness let us set $\zeta = 0.07$, $\mu = 0.01$, $Q = \mathtt{diag}\,(1, 1.0434)$. With $\gamma_1 = 0.45$ (hence $\gamma_2 = 0.55$) it is obtained

$$\sum_{i=1}^{2} \gamma_i \left(G_i^\top Q \overline{\Lambda} G_i - e^{-\mu} Q \underline{\Lambda} + \zeta I_2 \right) = \begin{bmatrix} -0.0375 & 0.0257 \\ 0.0257 & -0.0223 \end{bmatrix}.$$

The eigenvalues of the previous matrix are -0.0567 and -0.0031. Therefore it is a symmetric negative definite matrix, and Assumption 2.2 is satisfied. The computation of β gives 0.4844. One lets the error margin as the maximum margin allowed by Proposition 2.5, that is $\rho = 6.98\%$.

See Figure 2.2 for the time evolution of the solution with constant control input $\bar{i} = 1$ and the initial condition y_2^0 where the instability is observed.

The function ε used in the switching rules (2.85)–(2.87) and (2.102)–(2.106) is chosen to be

$$\varepsilon(t) = 10^{-3} e^{-t}, \quad t \in \mathbb{R}^+. \tag{2.126}$$

The measurement noise is chosen as follows

$$\delta(t) = |w(t)|\, \rho \left[\cos(t), \sin(t)\right]^\top. \tag{2.127}$$

As it was expected the solution is stabilized when switching rules (2.85)–(2.87) and (2.102)–(2.106) are applied to the system under measurement noise. Figure 2.3 shows the time evolution of the two components of the solution with the switching rules (2.85)–(2.87). Numerically the rate of convergence α is equal to 0.4398.

Let us focus on the special behavior of the three switching rules without measurement noise, that is $\delta(t) \equiv 0$. One checks that the rule (2.65)–(2.67) gives more switching times with the optimized $\mu = 0.2175$ than with $\mu = 0.01$, see the third column of Table 2.1. Despite the optimization on μ, the speed of convergence computed numerically along the solution of (2.1), (2.2), (2.3), (2.13), and (2.14) is not better with the two first rules (2.61) and (2.65)–(2.67). However the speed of convergence seems to be larger with the switching rule (2.72)–(2.76) with the optimized $\mu = 0.2175$ than with $\mu = 0.01$.

[1] For the time-invariant velocity case that is $\Lambda(t) \equiv \Lambda$, one has: if $\rho\,(|G|) < 1$ then the system (2.1), (2.2) is exponentially stable (for instance see [27]). Since the velocity perturbation is "small" it is natural to search matrices G which do not satisfy the previous condition in order to destabilize the system. For more discussion about conditions on G see [27].

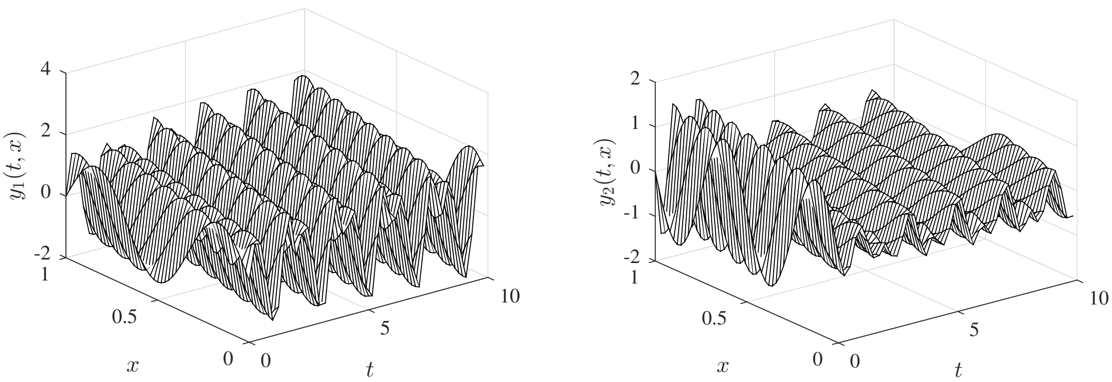

Figure 2.2: Time evolution of the first component y_1 (left) and of the second component y_2 (right) of the solution of the unswitched system (2.1)–(2.3) with the active mode $\bar{i} = 1$.

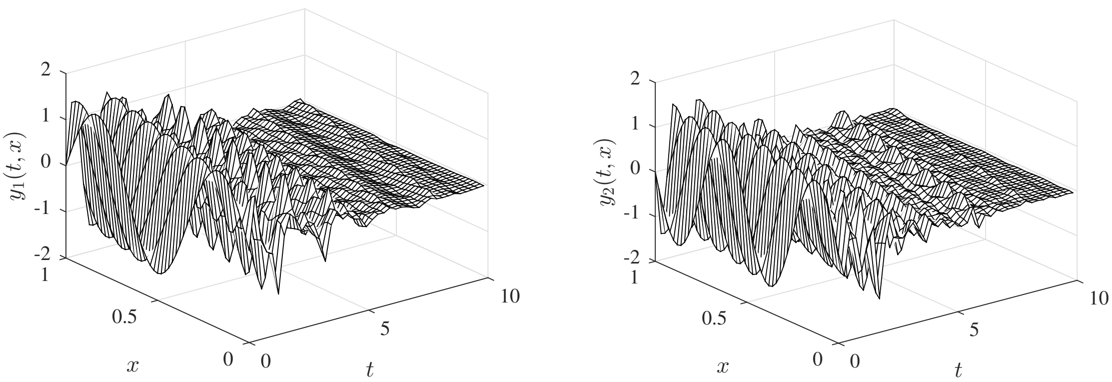

Figure 2.3: Time evolution of the first component y_1 (left) and of the second component y_2 (right) of the solution of (2.1), (2.2), (2.3), (2.13), and (2.14) with the switching rule (2.85)–(2.87) with measurement noise (2.127).

	Argmin	Hysteresis	Low-pass filter
Initial condition y_k^0	Exponential Stability. Lyapunov Function.	Exponential Stability. Lyapunov Function.	Exponential Stability.
Theoretical upper bound on the speed of convergence: 0.0023 (without optimization on μ)			
Number of switches by time unit.			
k = 1	2.5	2.1	0.1
k = 2	4.7	3.9	0.1
k = 3	6.2	4.3	0.1
Speed of convergence			
k = 1	0.4025	0.4166	0.1082
k = 2	0.4282	0.4442	0.1088
k = 3	0.4378	0.4280	0.1088
Theoretical upper bound on the speed of convergence: 0.0489 (with optimization on μ)			
Number of switches by time unit.			
k = 1	3.1	2.5	0.1
k = 2	5	4.6	0.1
k = 3	6.5	5.9	0.1
Speed of convergence			
k = 1	0.4095	0.4159	0.1489
k = 2	0.4328	0.4371	0.1344
k = 3	0.4274	0.4270	0.1405

Table 2.1: Comparison of the different switching strategies for the example with three initial conditions in a $L^2\left((0,1);\mathbb{R}^2\right)$ basis. Performed during 10 units of time.

Finally, let us illustrate Propositions 2.3 and 2.4. Let us assume that the measurement noise is such that

$$\delta(t) = \begin{cases} 1\,, & \text{if } t < 5\,, \\ 0\,, & \text{if } t \geq 5\,. \end{cases} \tag{2.128}$$

As said above, Assumption 2.2 is satisfied, hence the conclusions of Propositions 2.3 and 2.4 hold. Figure 2.4 shows the time evolution of the two components of the solution with the switching rules (2.85)–(2.87). Up to $t = 5$ the solution is bounded, then once the measurement noise disappears the solution goes to zero as it was expected.

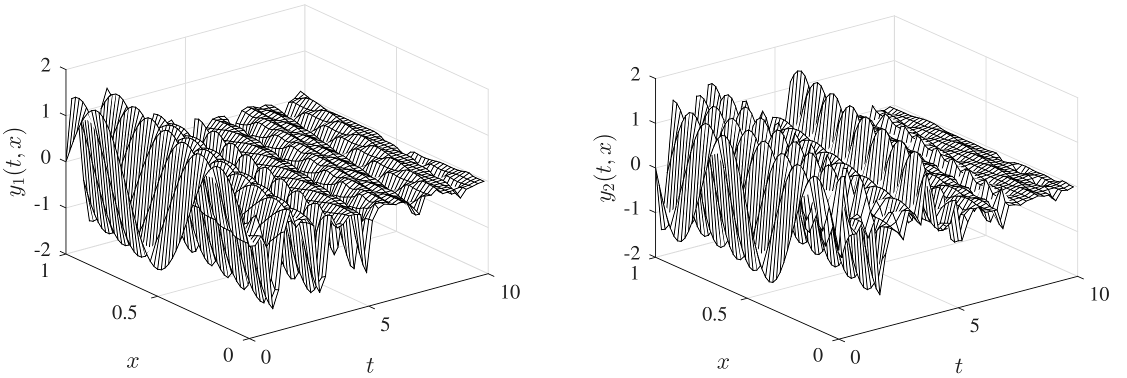

Figure 2.4: Time evolution of the first component y_1 (left) and of the second component y_2 (right) of the solution of (2.1), (2.2), (2.3), (2.13), and (2.14) with the switching rule (2.85)–(2.87) with measurement noise (2.128).

2.7.2 Saint-Venant Equations for a Network of Open Channels

The previous illustrations were motivated by theoretical objectives. In this section, we introduce a more physical system. Let us consider a cascade of $\mathcal{M}$ canal reaches. Each reach is a one-dimensional pool with a rectangular cross-section, a unit width and a zero slope. Moreover we suppose that each reach has the same length L. Besides, the friction effects due to the walls are neglected. The dynamics of the system in each reach is then given by the Saint-Venant equations presented in Subsection 1.2.3.

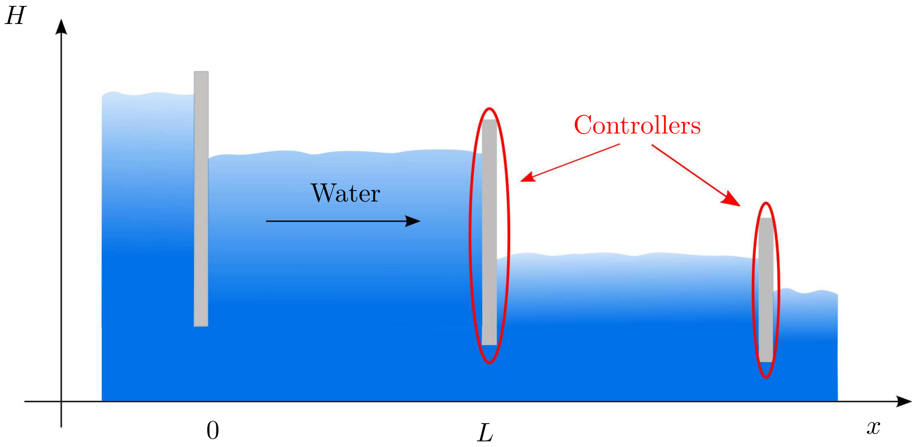

Figure 2.5: Illustration of the multi-canal example.

The equations for each pool are written as

$$\partial_t \begin{bmatrix} H_i(t,x) \\ V_i(t,x) \end{bmatrix} + \partial_x \begin{bmatrix} H_i(t,x)V_i(t,x) \\ \frac{V_i^2(t,x)}{2} + gH_i(t,x) \end{bmatrix} = 0\,, \quad x \in (0,L)\ , \quad i = 1,\dots,\mathcal{M}\,, \tag{2.129}$$

where H_i and V_i denoted respectively the water depth and the velocity in the reach i. Moreover the flow-rate in these pools can be defined by

$$Q_i(t,x) = H_i(t,x)V_i(t,x)\,, \quad i = 1,\ldots,\mathcal{M}\,. \tag{2.130}$$

In order to have a linear hyperbolic system in the form (2.1)–(2.3), we linearized the system around a steady-state and made the system dimensionless in the space variable, see Subsection 1.2.3. A steady-state solution is a constant solution

$$H_i(t,x) = H_i^*\,, \quad V_i(t,x) = V_i^*\,, \quad i = 1,\ldots,\mathcal{M}\,, \quad \forall t \in \mathbb{R}^+\,, \forall x \in [0,L]\ .$$

The steady-state flow is assumed to be *subcritical* or *fluvial* that is

$$gH_i^* - {V_i^*}^2 > 0\,.$$

The system is linearized around these steady states (see Subsection 1.2.3), hence the considered system is a $2\mathcal{M} \times 2\mathcal{M}$ linear hyperbolic system

$$\partial_t y + \Lambda \partial_x y = 0\,, \tag{2.131}$$

where $\Lambda = \mathtt{diag}\,(\lambda_1,\ldots,\lambda_{2\mathcal{M}})$.

2.7.2.1 Boundary Conditions

System (2.131) is subject to a set of boundary conditions, as in [8]. First of all there are $\mathcal{M}-1$ conditions which express the flow conservation between the pools

$$H_i(t,1)V_i(t,1) = H_{i+1}(t,0)V_{i+1}(t,0)\,, \quad \forall t \in \mathbb{R}^+\,, \quad i = 1,\ldots,\mathcal{M}-1\,. \tag{2.132}$$

Then one adds a condition which imposes the inflow rate

$$H_1(t,0)V_1(t,0) = Q_0(t)\,. \tag{2.133}$$

In the following we impose a constant inflow rate, that is $Q_0(t) = Q^* = H_1^* V_1^*$. The last $\mathcal{M}$ boundary conditions are given by the gate operations. We are considering underflow sluice gates with corresponding gate openings u_i^j for reach $i \in \{1,\ldots,\mathcal{M}\}$ in mode $j \in \mathcal{I}$. The discharge relationship is given by

$$H_i(t,1)V_i(t,1) = u_i^j \mu_0 l \sqrt{2g\,(H_i(t,1) - H_{i+1}(t,0))}\,, \quad i = 1,\ldots,\mathcal{M}-1\,, \tag{2.134}$$

where μ_0 is a positive constant coefficient and l is the channel width (here $l = 1$ m). For the last gate we have

$$H_{\mathcal{M}}(t,1)V_{\mathcal{M}}(t,1) = u_{\mathcal{M}}^j \mu_0 l \sqrt{2g\,(H_{\mathcal{M}}(t,1) - H_{down})}\,, \tag{2.135}$$

where $H_{down} > 0$ denotes the constant downstream water level. In the Riemann coordinates, boundary conditions (2.134) and (2.135) are equivalent to

$$y_{i+\mathcal{M}}(t,1) = -k_i^j y_i(t,1)\,, \quad i = 1,\ldots,\mathcal{M}\,,\, j = 1,\ldots,N\,, \tag{2.136}$$

for a suitable choice of the control actions $u_i^j(t)$,

$$u_i^j(t) = \frac{H_i(t,1)\left(\left(\frac{1-k_i^j}{1+k_i^j}\right)\sqrt{\frac{g}{H_i^*}}\left(H_i(t,1)-H_i^*\right)+V_i^*\right)}{\mu_0 l\sqrt{2g\left(H_i(t,1)-H_{i+1}(t,0)\right)}}\,, \quad i = 1,\ldots,\mathcal{M}-1\,, \tag{2.137}$$

$$u_{\mathcal{M}}^j(t) = \frac{H_{\mathcal{M}}(t,1)\left(\left(\frac{1-k_{\mathcal{M}}^j}{1+k_{\mathcal{M}}^j}\right)\sqrt{\frac{g}{H_{\mathcal{M}}^*}}\left(H_{\mathcal{M}}(t,1)-H_{\mathcal{M}}^*\right)+V_{\mathcal{M}}^*\right)}{\mu_0 l\sqrt{2g\left(H_{\mathcal{M}}(t,1)-H_{down}\right)}}\,. \tag{2.138}$$

The values $k_i^j \neq -1$ are tuning parameters. Obviously the control actions are well defined only if $H_i(t,1) > H_{i+1}(t,0)$ for all $i = 1,\ldots,\mathcal{M}-1$ and $H_{\mathcal{M}}(t,1) > H_{down}$. The linearization of the boundary conditions provided N matrices such that

$$\begin{bmatrix} y^+(t,0) \\ y^-(t,1) \end{bmatrix} = \begin{bmatrix} G_j^{++} & G_j^{+-} \\ G_j^{-+} & G_j^{--} \end{bmatrix}\begin{bmatrix} y^+(t,1) \\ y^-(t,0) \end{bmatrix}\,, \quad t \in [t_j, t_{j+1})\,. \tag{2.139}$$

Denote $G_j = \begin{bmatrix} G_j^{++} & G_j^{+-} \\ G_j^{-+} & G_j^{--} \end{bmatrix}$. The matrices G_j^{-+} and G_j^{--} are given by conditions (2.136) as

$$G_j^{-+} = \texttt{diag}\left\{-k_i,\, i = 1,\ldots,\mathcal{M}\right\}\,, \quad G_j^{--} = 0\,, \quad j = 1,\ldots,N\,. \tag{2.140}$$

The conditions (2.132) and (2.133) give the following expression for the matrices G_j^{+-}

$$G_j^{+-} = \texttt{diag}\left\{\frac{\lambda_{i+\mathcal{M}}}{\lambda_i},\, i = 1,\ldots,\mathcal{M}\right\}\,, \quad j = 1,\ldots,N\,. \tag{2.141}$$

Finally, the matrices G_j^{++} are obtained thanks to the conditions (2.132) and (2.136) as

$$G_j^{++}[i+1,i] = \frac{\left(\lambda_i + k_i^j\lambda_{i+\mathcal{M}}\right)}{\lambda_{i+1}}\sqrt{\frac{H_i^*}{H_{i+1}^*}} \text{ and } 0 \text{ elsewhere}\,, \quad i = 1,\ldots,\mathcal{M}\,. \tag{2.142}$$

2.7.2.2 Simulation Experiments

The switched strategies developed in this chapter are now tested with some numerical simulations in the case presented above. To this end, we consider two pools whose parameters are $L = 1000$ m, width $l = 1$ m, $H_{down} = 0.2$ m, $\mu_0 = 0.4$, $(H_1^*, H_2^*) = (2.5, 1)$ m and $Q^* = 1$ m^3.s^{-1}. The initial conditions are, for $x \in [0, L]$, $(H_1(0,x), H_2(0,x)) = (4, 1.4)$ m, $Q(0,x) = 2$ m^3.s^{-1}. There are two modes. The control gains associated to each mode are $\left(k_1^1, k_2^1\right) = (1, 0.2)$ and $\left(k_1^2, k_2^2\right) = (0.2, 1)$.

Assumption 2.1 is satisfied with $\mu = 0.775$, $Q = 10^3\texttt{diag}\,(1.448, 0.001, 2.674, 0.001)$ and $\gamma_1 = 0.95$.

The Saint-Venant equations are integrated numerically using the same scheme as in the Subsection 2.7.1. In Figure 2.6, the evolution of the function V is represented for the argmin and the argmin with hysteresis switching rules, with the constant mode $\sigma(t) = 1$ and with the constant mode $\sigma(t) = 2$. In this figure, we can observe that the switching strategies stabilize the system, and more importantly seem to improve the convergence of the system to the desired steady-state. Moreover, the switching signal for the argmin with hysteresis switching rule is displayed. It can be observed that the system keeps the mode 1 during the first 1200 s, and then starts generating switches. Numerically, we observe that this behavior corresponds to the choice of stabilizing the first pool then the second one. Indeed the mode $i = 1$ is the most efficient to stabilize the first pool. The argmin with hysteresis and filter keeps the mode 2.

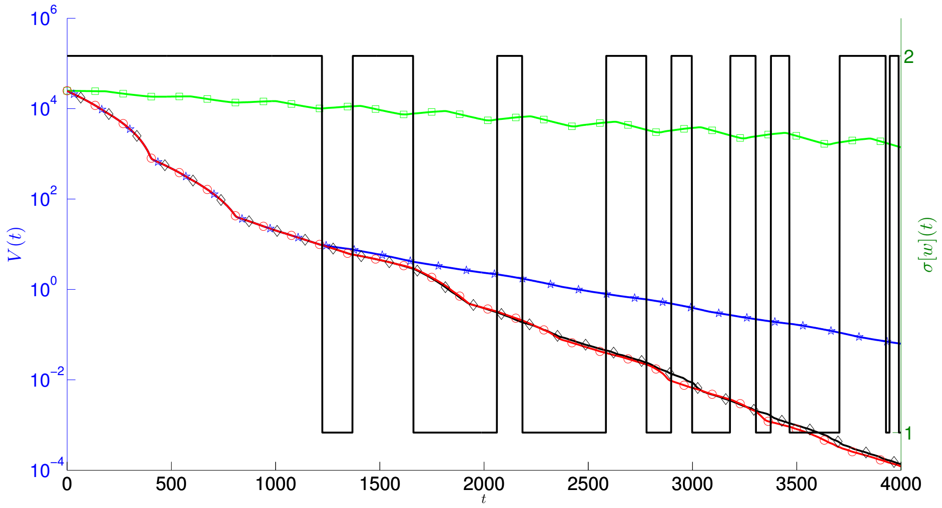

Figure 2.6: Evolution of the function V (semilog scale) and of $\sigma[w]$ for the argmin with hysteresis switching rule. Legend: square marker for constant mode $\sigma(t) = 1$, star marker for constant mode $\sigma(t) = 2$, circle marker for the argmin with hysteresis switching rule and diamond marker for the argmin switching rule.

3. Trajectory Generation and PI Control

IN THIS CHAPTER, we consider the problems of trajectory generation and tracking for general 2×2 systems of first-order linear hyperbolic PDEs with anti-collocated boundary input and output. We solve the trajectory generation problem via backstepping. The reference input, which generates the desired output, incorporates integral operators acting on advanced and delayed versions of the reference output with kernels which were derived by Vazquez, Krstic, and Coron for the backstepping stabilization of 2×2 linear hyperbolic systems. We apply our approach to a wave PDE with indefinite in-domain and boundary damping. For tracking the desired trajectory, we employ a PI control law on the tracking error of the output. We prove exponential stability of the closed-loop system, under the proposed PI control law, when the parameters of the plant and the controller satisfy certain conditions, by constructing a novel "non-diagonal" Lyapunov function. We demonstrate that the proposed PI control law compensates in the output the effect of in-domain and boundary disturbances. We show that in presence of compatibility conditions of the initial condition with the boundary conditions and under assumption on the smoothness of this latter one, the disturbance in the output is rejected in C^0-norm. We illustrate our results with numerical examples.

This work is the result of a collaboration with Nikolaos Bekiaris-Liberis during a research stay at the University of California at Berkeley in the team of Professor Alexandre M. Bayen. A paper related to some of these materials has been accepted for the 2015 European Control Conference (ECC) [69], and a thorough version has been submitted for a publication in Systems & Control Letters [67].

3.1 Problem Statement and Existing Results

In this chapter, we are concerned with the trajectory generation and tracking problems for general 2×2 systems of first-order linear hyperbolic PDEs

$$\partial_t y_1(t,x) + \varepsilon_1(x)\partial_x y_1(t,x) = c_1(x)y_1(t,x) + c_2(x)y_2(t,x) \tag{3.1}$$

$$\partial_t y_2(t,x) - \varepsilon_2(x)\partial_x y_2(t,x) = c_3(x)y_1(t,x) + c_4(x)y_2(t,x)\,, \tag{3.2}$$

with anti-collocated boundary input and output

$$y_1(t,0) = q y_2(t,0) \tag{3.3}$$

$$y_2(t,1) = S(t) \tag{3.4}$$

$$z(t) = y_2(t,0)\,, \tag{3.5}$$

where $t \in \mathbb{R}^+$ is the time variable, $x \in [0,1]$ is the spatial variable, z is the output of the system, and S is the control input. The functions ε_1, ε_2 belong to $C^2\left([0,1]\,;\mathbb{R}\right)$ and satisfy $\varepsilon_1(x) > 0$, $\varepsilon_2(x) > 0$, for all $x \in [0,1]$, the functions c_i, $i = 1,2,3,4$ belong to $C^1([0,1];\mathbb{R})$, and $q \in \mathbb{R}$. The purpose is to construct the control input $S(t)$ such that the output $y_2(t,0)$ follows $z^r(t)$.

The motion planning problem has been solved in [40] and [89] for a water-tank system, that is a moving tank containing a fluid whose dynamics is given by the Saint-Venant equations. In [47], the motion planning is solved for a reaction-advection-diffusion PDE describing a fixed-bed reactor. The problem is tackled by writing the solution X as a serie expansion, that is

$$X(t,x) = \sum_{i=0}^{\infty} a_i(t)\frac{x^i}{i!}\,. \tag{3.6}$$

It is also the approach used in [72] for the heat equation, see also [29] for a more general result. In [88], the motion planning for the wave equation is solved. It described the motion of a trolley carrying a fixed length heavy chain to which a load may be attached. These references have in common to use the flatness approach introduced in [46] for the finite dimensional case and generalized for the infinite dimensional case by the former references. To give an idea of this notion let us describe the finite dimensional case. In [81], we can find this description: given a system with states $a \in \mathbb{R}^n$, and inputs $b \in \mathbb{R}^m$ the system is said to be flat if we can find outputs $c \in \mathbb{R}^m$ written as

$$c = h\left(a, b, \dot{b}, \ldots, b^{(r)}\right)\,, \tag{3.7}$$

such that

$$a = \phi\left(c, \dot{c}, \ldots, c^{(q)}\right)$$

$$b = \alpha\left(c, \dot{c}, \ldots, c^{(q)}\right)\,.$$

We will see later that the construction of the open-loop control to solve our problem follows the flatness description.

The second aspect of this chapter is the disturbance rejection. Algorithms for disturbance rejection in 2×2 hyperbolic systems are recently developed. In [1], backstepping is used to construct the control and reject the disturbance at the boundary. In [3], the previous result is generalized for the disturbance rejection at an arbitrary place in the domain within a finite time. In [10], a spectral analysis is led to derive some conditions for the construction of a PI-control for a density-flow system that is

$$\partial_t \rho(t,x) + \partial_x q(t,x) = 0 \tag{3.8}$$

$$\partial_t q(t,x) + \lambda_1 \lambda_2 \partial_x \rho(t,x) + (\lambda_1 - \lambda_2)\, \partial_x q(t,x) = 0\,, \tag{3.9}$$

where $\rho(t,x)$ and $q(t,x)$ represent respectively the density and the flow at time t and location $x \in [0,L]$, λ_1 and λ_2 are the characteristic velocities of the system. In [105], sliding-mode control is used for a spatially-varying 2×2 linear hyperbolic system. Indeed, sliding-mode is now appearing for infinite dimensional case (see, for instance, [84]). Finally, in [39], PI controller for the Saint-Venant equations is derived by a semi-group approach. The Lyapunov-based output-feedback control with integral action of a 2×2 linear hyperbolic system of the form (3.8), (3.9) is analyzed in [38]. The effectiveness of the method is validated for the regulation of river with experiments on a micro-channel. Our result differs from the results in [10], [39], [38] in that we employ PI control on an output of the system in the Riemann coordinates and we construct a non-diagonal Lyapunov function for proving the closed-loop stability. An integral action is considered for the tracking issue for 2×2 hyperbolic systems of Lotka-Volterra type in [85]. The integral action is "filtered" to prove the exponential stability thanks to a diagonal Lyapunov function. Nonetheless, this filtering does not allow to prove that the disturbances are rejected contrary to what we will do later in this chapter.

We solve the motion planning problem for system (3.1)–(3.5) employing backstepping (Section 3.2). Specifically, we start from a simple transformed system, namely, a cascade of two first-order hyperbolic PDEs, for which the motion planning problem can be trivially solved. We then apply an inverse backstepping transformation to derive the reference trajectory and reference input for the original system.

Our approach is different from the one in [108], [28], in that we use backstepping for trajectory generation rather than stabilization, and from the one in [89], in that we employ a different conceptual idea (backstepping instead of series expansion solution as mentioned above) to a different class of systems. The idea of the backstepping-based trajectory generation for PDEs, which was conceived in [66], is applied to a beam PDE in [103] and the Navier-Stokes equations in [19], and is recently extended to general $n \times n$ linear hyperbolic systems in [63]. We apply this methodology to a wave PDE with indefinite in-domain and boundary damping by transforming (see, for instance, [12]) the wave PDE to a 2×2 linear hyperbolic system coupled with a first-order ODE (Section 3.3).

3.2 Trajectory Generation: Backstepping-based approach

Defining the change of coordinates (see, for example, [5])

$$u(t,x) = \chi_1(x) y_1(t,x) \tag{3.10}$$
$$v(t,x) = \chi_2(x) y_2(t,x)\,, \tag{3.11}$$

where

$$\chi_1(x) = \exp\left(-\int_0^x \frac{c_1(s)}{\varepsilon_1(s)} ds\right) \tag{3.12}$$
$$\chi_2(x) = \exp\left(\int_0^x \frac{c_4(s)}{\varepsilon_2(s)} ds\right) \tag{3.13}$$
$$\chi(x) = \frac{\chi_1(x)}{\chi_2(x)}\,, \tag{3.14}$$

system (3.1)–(3.5) is transformed into the following system

$$\partial_t u(t,x) + \varepsilon_1(x)\partial_x u(t,x) = \gamma_1(x) v(t,x) \tag{3.15}$$
$$\partial_t v(t,x) - \varepsilon_2(x)\partial_x v(t,x) = \gamma_2(x) u(t,x)\,, \tag{3.16}$$

with

$$\gamma_1(x) = \chi(x) c_2(x) \tag{3.17}$$
$$\gamma_2(x) = \chi^{-1}(x) c_3(x)\,. \tag{3.18}$$

The boundary conditions become

$$u(t,0) = q v(t,0) \tag{3.19}$$
$$v(t,1) = U(t)\,, \tag{3.20}$$

where the original control variable satisfies

$$U = \chi_2(1) S\,, \tag{3.21}$$

and the output becomes

$$v(t,0) = z(t)\,. \tag{3.22}$$

The trajectory generation problem is solved by the following theorem.

Theorem 3.1. *Let $y^r \in C^1(\mathbb{R})$ be uniformly bounded. The functions*

$$\begin{aligned} u^r(t,x) &= q y^r\left(t - \Phi_1(x)\right) + \int_0^x \frac{f(\xi)}{\varepsilon_1(\xi)} y^r\left(t - \Phi_1(x) + \Phi_1(\xi)\right) d\xi + \int_0^x L^{\alpha\beta}(x,\xi)\, y^r\left(t + \Phi_2(\xi)\right) d\xi \\ &\quad + q \int_0^x L^{\alpha\alpha}(x,\xi) y^r (t - \Phi_1(\xi)) d\xi \\ &\quad + \int_0^x L^{\alpha\alpha}(x,\xi) \int_0^\xi \frac{f(\zeta)}{\varepsilon_1(\zeta)} y^r\left(t - \Phi_1(\xi) + \Phi_1(\zeta)\right) d\zeta d\xi \end{aligned} \tag{3.23}$$

$$v^r(t,x) = y^r(t+\Phi_2(x)) + q\int_0^x L^{\beta\alpha}(x,\xi)y^r\left(t-\Phi_1(\xi)\right)d\xi + \int_0^x L^{\beta\beta}(x,\xi)y^r\left(t+\Phi_2(\xi)\right)d\xi$$
$$+\int_0^x L^{\beta\alpha}(x,\xi)\int_0^\xi \frac{f(\zeta)}{\varepsilon_1(\zeta)}y^r\left(t-\Phi_1(\xi)+\Phi_1(\zeta)\right)d\zeta d\xi \tag{3.24}$$
$$U^r(t) = y^r\left(t+\Phi_2(1)\right) + q\int_0^1 L^{\beta\alpha}(1,\xi)y^r\left(t-\Phi_1(\xi)\right)d\xi + \int_0^1 L^{\beta\beta}(1,\xi)y^r\left(t+\Phi_2(\xi)\right)d\xi$$
$$+\int_0^1 L^{\beta\alpha}(1,\xi)\int_0^\xi \frac{f(\zeta)}{\varepsilon_1(\zeta)}y^r\left(t-\Phi_1(\xi)+\Phi_1(\zeta)\right)d\zeta d\xi\,, \tag{3.25}$$

where

$$\Phi_1(x) = \int_0^x \frac{1}{\varepsilon_1(s)}ds \tag{3.26}$$
$$\Phi_2(x) = \int_0^x \frac{1}{\varepsilon_2(s)}ds \tag{3.27}$$
$$f(x) = \begin{cases} \varepsilon_2(0)K^{uv}(x,0)\,, & \text{if } q=0 \\ 0\,, & \text{if } q\neq 0\,, \end{cases} \tag{3.28}$$

and $L^{\alpha\alpha}$, $L^{\alpha\beta}$, $L^{\beta\alpha}$, $L^{\beta\beta}$, K^{uv} are the solutions of the following equations

$$\varepsilon_2(x)L_x^{\beta\alpha}(x,\xi) - \varepsilon_1(\xi)L_\xi^{\beta\alpha}(x,\xi) = \varepsilon_1'(\xi)L^{\beta\alpha}(x,\xi) - \gamma_2(x)L^{\alpha\alpha}(x,\xi) \tag{3.29}$$
$$\varepsilon_2(x)L_x^{\beta\beta}(x,\xi) + \varepsilon_2(\xi)L_\xi^{\beta\beta}(x,\xi) = -\varepsilon_2'(\xi)L^{\beta\beta}(x,\xi) - \gamma_2(x)L^{\alpha\beta}(x,\xi) \tag{3.30}$$
$$\varepsilon_1(x)L_x^{\alpha\alpha}(x,\xi) + \varepsilon_1(\xi)L_\xi^{\alpha\alpha}(x,\xi) = -\varepsilon_1'(\xi)L^{\alpha\alpha}(x,\xi) + \gamma_1(x)L^{\beta\alpha}(x,\xi) \tag{3.31}$$
$$\varepsilon_1(x)L_x^{\alpha\beta}(x,\xi) - \varepsilon_2(\xi)L_\xi^{\alpha\beta}(x,\xi) = \varepsilon_2'(\xi)L^{\alpha\beta}(x,\xi) + \gamma_1(x)L^{\beta\beta}(x,\xi) \tag{3.32}$$
$$\varepsilon_1(x)K_x^{uu}(x,\xi) + \varepsilon_1(\xi)K_\xi^{uu}(x,\xi) = -\varepsilon_1'(\xi)K^{uu}(x,\xi) - \gamma_2(x)K^{uv}(x,\xi) \tag{3.33}$$
$$\varepsilon_1(x)K_x^{uv}(x,\xi) - \varepsilon_2(\xi)K_\xi^{uv}(x,\xi) = \varepsilon_2'(\xi)K^{uv}(x,\xi) - \gamma_1(x)K^{uu}(x,\xi)\,, \tag{3.34}$$

with the boundary conditions

$$L^{\beta\alpha}(x,x) = -\frac{\gamma_2(x)}{\varepsilon_1(x)+\varepsilon_2(x)} \tag{3.35}$$
$$L^{\alpha\alpha}(x,0) = \begin{cases} h_1(x)\,, & \text{if } q=0 \\ \frac{\varepsilon_2(0)}{q\varepsilon_1(0)}L^{\alpha\beta}(x,0)\,, & \text{if } q\neq 0 \end{cases} \tag{3.36}$$
$$L^{\beta\beta}(x,0) = \begin{cases} \frac{1}{\varepsilon_2(0)}\int_0^x L^{\beta\alpha}(x,\xi)f(\xi)d\xi\,, & \text{if } q=0 \\ \frac{q\varepsilon_1(0)}{\varepsilon_2(0)}L^{\beta\alpha}(x,0)\,, & \text{if } q\neq 0 \end{cases} \tag{3.37}$$
$$L^{\alpha\beta}(x,x) = \frac{\gamma_1(x)}{\varepsilon_1(x)+\varepsilon_2(x)} \tag{3.38}$$
$$K^{uu}(x,0) = h_2(x) \tag{3.39}$$
$$K^{uv}(x,x) = \frac{\gamma_1(x)}{\varepsilon_1(x)+\varepsilon_2(x)}\,, \tag{3.40}$$

where h_1, $h_2 \in C^1\left([0,1]\right)$ are arbitrary, are uniformly bounded and solve the boundary value problem (3.15), (3.16), (3.19), (3.20). In particular, $v^r(t,0) = y^r(t)$, for $t \geq 0$.

Before proving Theorem 3.1 we make the following observation, which is also helpful in understanding better the proof strategy of Theorem 3.1.

Remark 3.1. The approach for the trajectory generation introduced here is inspired from backstepping. Consider the following system

$$\alpha_t(t,x) + \varepsilon_1(x)\alpha_x(t,x) - f(x)\beta(t,0) = 0 \tag{3.41}$$

$$\beta_t(t,x) - \varepsilon_2(x)\beta_x(t,x) = 0\,, \tag{3.42}$$

with boundary condition

$$\alpha(t,0) = q\beta(t,0)\,, \tag{3.43}$$

which follows by directly applying the backstepping transformation

$$\alpha(t,x) = u(t,x) - \int_0^x K^{uu}(x,\xi)\,u(t,\xi)\,d\xi - \int_0^x K^{uv}(x,\xi)\,v(t,\xi)\,d\xi \tag{3.44}$$

$$\beta(t,x) = v(t,x) - \int_0^x K^{vu}(x,\xi)\,u(t,\xi)\,d\xi - \int_0^x K^{vv}(x,\xi)\,v(t,\xi)\,d\xi\,, \tag{3.45}$$

where the kernels K^{uu}, K^{uv}, K^{vu}, K^{vv} are given in [108], to system (3.15), (3.16), and (3.19). It is shown that the functions

$$\alpha(t,x) = qy^r(t - \Phi_1(x)) + \int_0^x \frac{f(\xi)}{\varepsilon_1(\xi)} y^r(t - \Phi_1(x) + \Phi_1(\xi))\,d\xi \tag{3.46}$$

$$\beta(t,x) = y^r(t + \Phi_2(x))\,, \tag{3.47}$$

where Φ_1 and Φ_2 are defined in (3.26) and (3.27), respectively, satisfy (3.41)–(3.43) with

$$\beta(t,1) = y^r(t + \Phi_2(1)) \tag{3.48}$$

and, in particular, $\beta(t,0) = y^r(t)$. Using the inverse backstepping transformations introduced in [108]

$$u(t,x) = \alpha(t,x) + \int_0^x L^{\alpha\alpha}(x,\xi)\alpha(t,\xi)d\xi + \int_0^x L^{\alpha\beta}(x,\xi)\beta(t,\xi)d\xi \tag{3.49}$$

$$v(t,x) = \beta(t,x) + \int_0^x L^{\beta\alpha}(x,\xi)\alpha(t,\xi)d\xi + \int_0^x L^{\beta\beta}(x,\xi)\beta(t,\xi)d\xi\,, \tag{3.50}$$

and relations (3.46), (3.47) one can conclude that the functions u^r, v^r, and $U^r = v^r(1)$ solve the trajectory generation problem for system (3.15), (3.16), (3.19)–(3.22).

Note that the present approach cannot be directly applied to cases where $\varepsilon_1(x)$ or $\varepsilon_2(x)$ vanish for some $x \in [0,1]$. This is evident, for instance, from (3.37) which would imply that the kernel $L^{\beta\beta}$ of the open-loop control law U^r may become infinity for all $x \in [0,1]$. •

Proof. We first consider the case $q \neq 0$. Note that since ε_1, $\varepsilon_2 \in C^2([0,1])$ with $\varepsilon_1(x)$, $\varepsilon_2(x) > 0$, for all $x \in [0,1]$ and γ_1, $\gamma_2 \in C^1([0,1])$, system (3.29)–(3.38) has a unique solution with $L^{\alpha\alpha}$, $L^{\alpha\beta}$, $L^{\beta\alpha}$, $L^{\beta\beta} \in C^1(\mathcal{T})$ where $\mathcal{T} = \{(x,\xi) : 0 \le \xi \le x \le 1\}$ [28]. Hence, from (3.23)–(3.25) and the uniform boundedness of y^r it follows that u^r, v^r, and U^r are bounded for all $t \ge 0$ and $x \in [0,1]$.

Taking the time and space derivatives of u^r we get

$$\begin{aligned} u_t^r + \varepsilon_1(x) u_x^r = q \int_0^x L^{\alpha\alpha}(x,\xi) y^{r\prime}(t - \Phi_1(\xi)) d\xi + \int_0^x L^{\alpha\beta}(x,\xi) y^{r\prime}(t + \Phi_2(\xi)) d\xi \\ + \varepsilon_1(x) \int_0^x L_x^{\alpha\beta}(x,\xi) y^r\left(t + \Phi_2(\xi)\right) d\xi + q\varepsilon_1(x) \int_0^x L_x^{\alpha\alpha}(x,\xi) y^r\left(t - \Phi_1(\xi)\right) d\xi \\ + \varepsilon_1(x) L^{\alpha\beta}(x,x) y^r\left(t + \Phi_2(x)\right) + q\varepsilon_1(x) L^{\alpha\alpha}(x,x) y^r\left(t - \Phi_1(x)\right). \end{aligned} \tag{3.51}$$

Integrating by parts the first two integrals we get

$$\begin{aligned} u_t^r + \varepsilon_1(x) u_x^r = q \int_0^x \left(\varepsilon_1(x) L_x^{\alpha\alpha}(x,\xi) + \varepsilon_1(\xi) L_\xi^{\alpha\alpha}(x,\xi) + \varepsilon_1'(\xi) L^{\alpha\alpha}(x,\xi)\right) y^r(t - \Phi_1(\xi)) d\xi \\ + \int_0^x \Big(\varepsilon_1(x) L_x^{\alpha\beta}(x,\xi) - \varepsilon_2(\xi) L_\xi^{\alpha\beta}(x,\xi) - \varepsilon_2'(\xi) L^{\alpha\beta}(x,\xi)\Big) y^r(t + \Phi_2(\xi)) d\xi \\ + \left(q\varepsilon_1(0) L^{\alpha\alpha}(x,0) - \varepsilon_2(0) L^{\alpha\beta}(x,0)\right) y^r(t) \\ + \left(\varepsilon_1(x) + \varepsilon_2(x)\right) L^{\alpha\beta}(x,x) y^r(t + \Phi_2(x)). \end{aligned} \tag{3.52}$$

Due to the fact that $L^{\alpha\beta}$ and $L^{\alpha\alpha}$ are the solutions of (3.31) and (3.32) with the boundary conditions (3.36) and (3.38) one gets, by using (3.24), that u^r satisfies (3.15). The proof that v^r satisfies (3.16) follows analogously. Setting $x = 0$ in (3.23), (3.24) and using (3.26), (3.27), we get that u^r and v^r satisfy (3.19). Setting $x = 1$ in (3.24) it follows that (3.25) satisfies (3.20). Setting in (3.24) $x = 0$ and using (3.27) we get $v^r(0,t) = y^r(t)$.

Let us consider next the case $q = 0$. First observe that the PDEs (3.29), (3.31) with boundary conditions (3.35), (3.36), for the kernels $L^{\alpha\alpha}$ and $L^{\beta\alpha}$ are decoupled, and hence, $L^{\alpha\alpha}$ and $L^{\beta\alpha}$ are well-defined [28]. Hence, since f satisfies (3.28) and K^{uv}, K^{uu} are well-defined [28], one can conclude that $L^{\alpha\beta}$ and $L^{\beta\beta}$ are well-defined as well.

Taking the time and space derivatives of u^r we get

$$\begin{aligned} u_t^r + \varepsilon_1(x) u_x^r = f(x) y^r(t) + \int_0^x L^{\alpha\alpha}(x,\xi) \int_0^\xi \frac{f(\zeta)}{\varepsilon_1(\zeta)} y^{r\prime}\left(t - \Phi_1(\xi) + \Phi_1(\zeta)\right) d\zeta d\xi \\ + \int_0^x L^{\alpha\beta}(x,\xi) y^{r\prime}(t + \Phi_2(\xi)) d\xi \\ + \varepsilon_1(x) L^{\alpha\alpha}(x,x) \int_0^x \frac{f(\zeta)}{\varepsilon_1(\zeta)} y^r(t - \Phi_1(x) + \Phi_1(\zeta)) d\zeta \\ + \varepsilon_1(x) L^{\alpha\beta}(x,x) y^r(t + \Phi_2(x)) + \varepsilon_1(x) \int_0^x L_x^{\alpha\beta}(x,\xi) y^r(t + \Phi_2(\xi)) d\xi \\ + \varepsilon_1(x) \int_0^x L_x^{\alpha\alpha}(x,\xi) \int_0^\xi \frac{f(\zeta)}{\varepsilon_1(\zeta)} y^r(t - \Phi_1(\xi) + \Phi_1(\zeta)) d\zeta d\xi. \end{aligned} \tag{3.53}$$

Integrating by parts the first two integrals we get

$$\begin{aligned} u_t^r + \varepsilon_1(x) u_x^r = \int_0^x \left(\varepsilon_1(x) L_x^{\alpha\alpha}(x,\xi) + \varepsilon_1(\xi) L_\xi^{\alpha\alpha}(x,\xi) + \varepsilon_1'(\xi) L^{\alpha\alpha}(x,\xi)\right) \\ \times \int_0^\xi \frac{f(\zeta)}{\varepsilon_1(\zeta)} y^r(t - \Phi_1(\xi) + \Phi_1(\zeta)) d\zeta d\xi \\ + \int_0^x \Big(\varepsilon_1(x) L_x^{\alpha\beta}(x,\xi) - \varepsilon_2(\xi) L_\xi^{\alpha\beta}(x,\xi) - \varepsilon_2'(\xi) L^{\alpha\beta}(x,\xi)\Big) y^r(t + \Phi_2(\xi)) d\xi \end{aligned}$$

$$
\begin{aligned}
&+ (\varepsilon_1(x) + \varepsilon_2(x))\, L^{\alpha\beta}(x,x) y^r(t + \Phi_2(x)) \\
&+ y^r(t) \left(f(x) + \int_0^x L^{\alpha\alpha}(x,\xi) f(\xi) d\xi - \varepsilon_2(0) L^{\alpha\beta}(x,0) \right) . \qquad (3.54)
\end{aligned}
$$

Using (3.24), (3.31), (3.32), and (3.38) one can conclude that u^r satisfies (3.15) if f satisfies

$$
f(x) = \varepsilon_2(0) L^{\alpha\beta}(x,0) - \int_0^x L^{\alpha\alpha}(x,\xi) f(\xi) d\xi\,. \qquad (3.55)
$$

This fact can been shown as follows. The inverse of the backstepping transformation (3.44), (3.45) is uniquely defined and has the form (3.49), (3.50) (see, for example, [62]). Hence, substituting (3.44), (3.45) in (3.49), (3.50) we get

$$
\begin{aligned}
&\int_0^x \left(K^{uu}(x,\xi) - L^{\alpha\alpha}(x,\xi)\right) u(\xi,t) + \left(K^{uv}(x,\xi) - L^{\alpha\beta}(x,\xi)\right) v(t,\xi) d\xi \\
&+ \int_0^x \int_0^\xi \Big(\left(L^{\alpha\alpha}(x,\xi) K^{uu}(\xi,\zeta) + L^{\alpha\beta}(x,\xi) K^{vu}(\xi,\zeta) \right) u(t,\zeta) \\
&+ \left(L^{\alpha\alpha}(x,\xi) K^{uv}(\xi,\zeta) + L^{\alpha\beta}(x,\xi) K^{vv}(\xi,\zeta) \right) v(t,\zeta) \Big) d\zeta d\xi = 0\,. \qquad (3.56)
\end{aligned}
$$

Performing a change in the order of integration in the second integral of (3.56) and using the fact that (3.56) holds for all u and v, one obtains

$$
K^{uv}(x,\xi) = L^{\alpha\beta}(x,\xi) - \int_\xi^x \left(L^{\alpha\alpha}(x,s) K^{uv}(s,\xi) + L^{\alpha\beta}(x,s) K^{vv}(s,\xi) \right) ds\,. \qquad (3.57)
$$

Setting $\xi = 0$ in (3.57), multiplying (3.57) by $\varepsilon_2(0)$, and using the facts that $K^{vv}(x,0) = 0$ for all $x \in [0,1]$ (see relation (31) in [108]) and that f is defined by (3.28), we get that f satisfies (3.55) for $q = 0$. The rest of the proof is similar to the case $q \neq 0$. □

3.3 Application to a Wave PDE with Indefinite In-Domain and Boundary Damping

Let us consider system

$$
\partial_{tt} w(t,x) = \varepsilon(x) \partial_{xx} w(t,x) + h(x) \partial_t w(t,x) + b(x) \partial_x w(t,x) \qquad (3.58)
$$

$$
\partial_x w(t,0) = -g \partial_t w(t,0) \qquad (3.59)
$$

$$
\partial_x w(t,1) = W(t)\,, \qquad (3.60)
$$

with $g \neq \left\{ \frac{1}{\sqrt{\varepsilon(0)}}, -\frac{1}{\sqrt{\varepsilon(0)}} \right\}$, h, $b \in C^1([0,1];\mathbb{R})$, and $\varepsilon \in C^2\left([0,1];\mathbb{R}\right)$ with $\varepsilon(x) > 0$, for all $x \in [0,1]$. The objective is to make $w(t,0)$ to track a reference trajectory, say, $\zeta(t)$, which belongs to $C^2(\mathbb{R};\mathbb{R})$. Let us define the output of the system as

$$
\psi(t) = w(t,0)\,.
$$

With the change of variables

$$
\begin{aligned}
y_1(t,x) &= \frac{1-\sqrt{\varepsilon(0)}g}{1+\sqrt{\varepsilon(0)}g}\left(\partial_t w(t,x) - \sqrt{\varepsilon(x)}\partial_x w(t,x)\right) \\
y_2(t,x) &= \partial_t w(t,x) + \sqrt{\varepsilon(x)}\partial_x w(t,x) \\
S(t) &= \sqrt{\varepsilon(1)}W(t) + \partial_t w(t,1)\,,
\end{aligned}
$$

system (3.58)–(3.60) is rewritten as (3.1)–(3.5) where

$$
\begin{aligned}
z(t) &= \left(1-\sqrt{\varepsilon(0)}g\right)\dot{\psi}(t) \\
\varepsilon_1(x) &= \sqrt{\varepsilon(x)} \\
\varepsilon_2(x) &= \sqrt{\varepsilon(x)} \\
q &= 1 \\
c_1(x) &= \frac{h(x)}{2} - \frac{b(x)}{2\sqrt{\varepsilon(x)}} + \frac{\varepsilon'(x)}{4\sqrt{\varepsilon(x)}} \\
c_2(x) &= mc_4(x) \\
c_3(x) &= \frac{1}{m}c_1(x) \\
c_4(x) &= \frac{h(x)}{2} + \frac{b(x)}{2\sqrt{\varepsilon(x)}} - \frac{\varepsilon'(x)}{4\sqrt{\varepsilon(x)}} \\
m &= \frac{1-\sqrt{\varepsilon(0)}g}{1+\sqrt{\varepsilon(0)}g}\,,
\end{aligned}
$$

together with the integrator $\dot{\psi}(t) = \frac{1}{1-\sqrt{\varepsilon(0)}g}y_2(t,0)$. Applying Theorem 3.1 we get the following reference input

$$
\begin{aligned}
W^r(t) = \frac{1}{2\sqrt{\varepsilon(1)}}\Bigg(&\left(1-\sqrt{\varepsilon(0)}g\right)\exp\left(-\int_0^1 \frac{c_4(s)}{\varepsilon_2(s)}ds\right)\left(\dot{\zeta}\left(t+\Phi_2(1)\right)\right. \\
&\left.+\int_0^1 L^{\beta\alpha}(1,\xi)\dot{\zeta}\left(t-\Phi_1(\xi)\right)d\xi + \int_0^1 L^{\beta\beta}(1,\xi)\dot{\zeta}\left(t+\Phi_2(\xi)\right)d\xi\right) \\
&-\left(1+\sqrt{\varepsilon(0)}g\right)\exp\left(\int_0^1 \frac{c_1(s)}{\varepsilon_1(s)}ds\right)\left(\dot{\zeta}\left(t-\Phi_1(1)\right)\right. \\
&\left.+\int_0^1 L^{\alpha\alpha}(x,\xi)\dot{\zeta}\left(t-\Phi_1(\xi)\right)d\xi + \int_0^1 L^{\alpha\beta}(x,\xi)\dot{\zeta}\left(t+\Phi_2(\xi)\right)d\xi\right)\Bigg)\,.
\end{aligned}
$$

Let us illustrate our trajectory generation methodology with a wave PDE of the form (3.58)–(3.60). We choose the parameters of the system as

$$\varepsilon = 1 \tag{3.61}$$

$$h = 1 \tag{3.62}$$

$$b = -1 \tag{3.63}$$

$$g = 0\,. \tag{3.64}$$

The reference for the output is chosen as $\zeta(t) = \sin(3t)$.

Let us find an explicit form for the kernel $L^{\alpha\beta}$, $L^{\beta\alpha}$, $L^{\alpha\alpha}$, and $L^{\beta\beta}$. Let us proceed by a change of variables (see for instance [107]). Letting

$$\begin{aligned}
G^{\alpha\alpha}(x,\xi) &= e^{\left(\frac{c_1}{\varepsilon_1}+\frac{c_4}{\varepsilon_2}\right)\left(\frac{\varepsilon_2 x-\varepsilon_2\xi}{\varepsilon_1+\varepsilon_2}\right)} L^{\alpha\alpha}(x,\xi) \\
G^{\beta\alpha}(x,\xi) &= e^{-\left(\frac{c_1}{\varepsilon_1}+\frac{c_4}{\varepsilon_2}\right)\left(\frac{\varepsilon_1 x+\varepsilon_2\xi}{\varepsilon_1+\varepsilon_2}\right)} L^{\beta\alpha}(x,\xi) \\
G^{\beta\beta}(x,\xi) &= e^{-\left(\frac{c_1}{\varepsilon_1}+\frac{c_4}{\varepsilon_2}\right)\left(\frac{\varepsilon_1 x-\varepsilon_1\xi}{\varepsilon_1+\varepsilon_2}\right)} L^{\beta\beta}(x,\xi) \\
G^{\alpha\beta}(x,\xi) &= e^{\left(\frac{c_1}{\varepsilon_1}+\frac{c_4}{\varepsilon_2}\right)\left(\frac{\varepsilon_2 x+\varepsilon_1\xi}{\varepsilon_1+\varepsilon_2}\right)} L^{\alpha\beta}(x,\xi)\,,
\end{aligned}$$

one gets

$$\begin{aligned}
\varepsilon_1\partial_x G^{\alpha\alpha}(x,\xi) + \varepsilon_1\partial_\xi G^{\alpha\alpha}(x,\xi) &= c_2 G^{\beta\alpha}(x,\xi) \\
\varepsilon_2\partial_x G^{\beta\alpha}(x,\xi) - \varepsilon_1\partial_\xi G^{\beta\alpha}(x,\xi) &= -c_3 G^{\alpha\alpha}(x,\xi)\,,
\end{aligned}$$

with the boundary conditions

$$\begin{aligned}
G^{\beta\alpha}(x,x) &= -\frac{c_3}{\varepsilon_1+\varepsilon_2} \\
G^{\alpha\alpha}(x,0) &= \frac{\varepsilon_2}{q\varepsilon_1} G^{\alpha\beta}(x,0)\,.
\end{aligned}$$

The second system is

$$\begin{aligned}
\varepsilon_2\partial_x G^{\beta\beta}(x,\xi) + \varepsilon_2\partial_\xi G^{\beta\beta}(x,\xi) &= -c_3 G^{\alpha\beta}(x,\xi) \\
\varepsilon_1\partial_x G^{\alpha\beta}(x,\xi) - \varepsilon_2\partial_\xi G^{\alpha\beta}(x,\xi) &= c_2 G^{\beta\beta}(x,\xi)\,,
\end{aligned}$$

with the boundary conditions

$$\begin{aligned}
G^{\alpha\beta}(x,x) &= \frac{c_2}{\varepsilon_1+\varepsilon_2} \\
G^{\beta\beta}(x,0) &= \frac{q\varepsilon_1}{\varepsilon_2} G^{\beta\alpha}(x,0)\,.
\end{aligned}$$

The choice of parameters gives $c_2 = c_4 = 0$, whence from the above equations we get

$$\begin{aligned}
L^{\alpha\alpha}(x,\xi) &= 0 \\
L^{\alpha\beta}(x,\xi) &= 0 \\
L^{\beta\alpha}(x,\xi) &= -\frac{1}{2}\exp\left(\frac{x+\xi}{2}\right) \\
L^{\beta\beta}(x,\xi) &= -\frac{1}{2}\exp\left(\frac{x-\xi}{2}\right)\,.
\end{aligned}$$

The reference trajectory w^r for system (3.58)–(3.60) is given by

$$\begin{aligned}
w^r(t,x) = \frac{1}{37}&\left(19\sin(3t+3x) - 3\exp(x)\cos(3t-3x) + 3\cos(3t+3x)\right. \\
&\left.+18\exp(x)\sin(3t-3x)\right)\,,
\end{aligned}$$

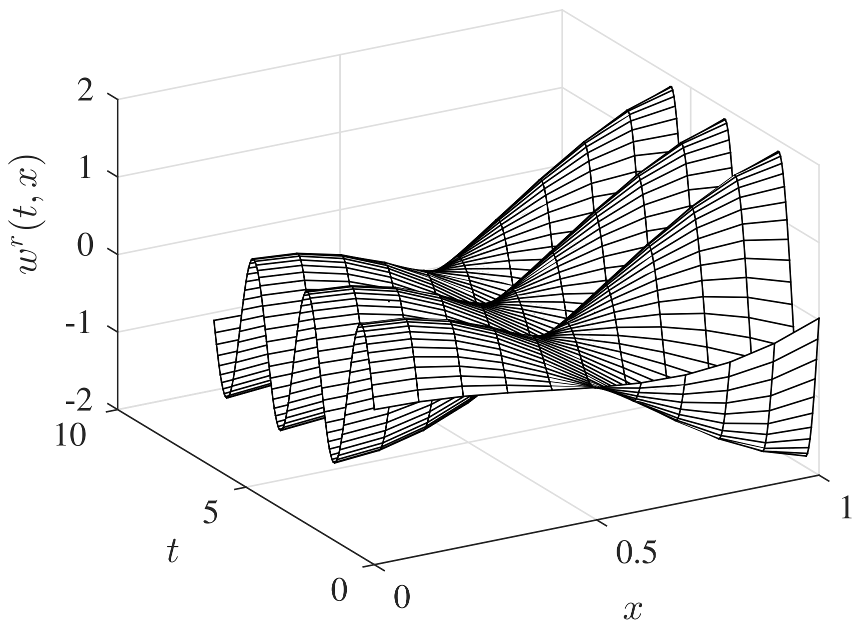

Figure 3.1: Solution to the trajectory generation problem for system (3.58)–(3.60) with parameters (3.61)–(3.64).

which gives the following reference input

$$W^r(t) = \frac{57}{37}\left(\cos(3t+3) - \exp(1)\cos(3t-3)\right) + \frac{9}{37}\left(\exp(1)\sin(3t-3) - \sin(3t+3)\right) . \tag{3.65}$$

Figure 3.1 shows the evolution of the reference trajectory z^r. Figure 3.2 shows the evolution of the spatial derivative of z^r and, in particular, the control effort $W^r(t) = \partial_x w^r(1,t)$ given by (3.65).

3.4 Trajectory Tracking: PI Control, Lyapunov Analysis

Let us define the space E by

$$E = L^2\left((0,1)\,;\mathbb{R}\right) \times L^2\left((0,1)\,;\mathbb{R}\right) \times \mathbb{R}\,. \tag{3.66}$$

For stabilizing the system around the desired trajectory for any initial condition $(u(0,x), v(0,x))$, rather than only for $(u(0,x), v(0,x)) = (u^r(0,x), v^r(0,x))$, we employ a PI-feedback control law. We first write the dynamics of the tracking errors $\tilde{u}(t,x) = u(t,x) - u^r(t,x)$ and $\tilde{v}(t,x) = v(t,x) - v^r(t,x)$ as

$$\partial_t \tilde{u}(t,x) + \varepsilon_1(x)\partial_x \tilde{u}(t,x) = \gamma_1(x)\tilde{v}(t,x) \tag{3.67}$$

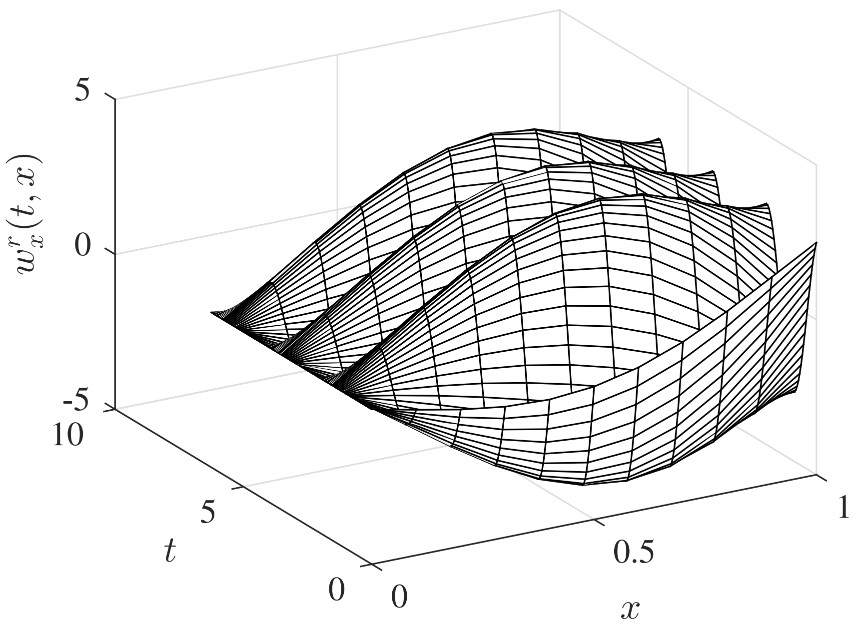

Figure 3.2: The spatial derivative of the reference trajectory of Figure 3.1. Note in particular the reference input $W^r(t) = w_x^r(t,1)$ given by (3.65).

$$\partial_t \tilde{v}(t,x) - \varepsilon_2(x)\partial_x \tilde{v}(t,x) = \gamma_2(x)\tilde{u}(t,x) \tag{3.68}$$

$$\tilde{u}(t,0) = q\tilde{v}(t,0) \tag{3.69}$$

$$\tilde{v}(t,1) = \tilde{U}(t)\,, \tag{3.70}$$

where $\tilde{U} = U - U^r$ and U^r is the reference input generating the desired reference trajectory. We employ the controller

$$\tilde{U}(t) = -k_P\tilde{v}(t,0) - k_I\tilde{\eta}(t)\,, \tag{3.71}$$

with

$$\dot{\tilde{\eta}}(t) = \tilde{v}(t,0)\,, \tag{3.72}$$

and

$$\tilde{\eta}(0) = \tilde{\eta}^0 \in \mathbb{R}\,. \tag{3.73}$$

Theorem 3.2. *Consider system (3.67)–(3.70) together with the control law (3.71), (3.72). Let the positive constants μ, β, ρ, γ, ν, κ, and θ be such that*

$$M_1 = \begin{bmatrix} -q^2 - \beta\left(k_P^2 e^{\mu} - 1\right) - \frac{\kappa\gamma}{2} & -\beta k_P k_I e^{\mu} + \frac{\gamma}{2}\left(e^{\nu}k_P + 1\right) - \frac{\rho}{2} \\ -\beta k_P k_I e^{\mu} + \frac{\gamma}{2}\left(e^{\nu}k_P + 1\right) - \frac{\rho}{2} & -\beta k_I^2 e^{\mu} + \gamma e^{\nu} k_I - \frac{\gamma}{2} \end{bmatrix} > 0 \tag{3.74}$$

$$M_2(x) = \begin{bmatrix} M_{21}(x) & M_{22}(x) \\ M_{23}(x) & M_{24}(x) \end{bmatrix} \geq 0 \tag{3.75}$$

with

$$M_{21}(x) = \left(\mu - \frac{\theta}{\varepsilon_1(x)}\right) e^{-\mu x} + \frac{\gamma^2}{2\,(\theta\rho - \gamma)} \frac{\gamma_2^2(x)}{\varepsilon_2^2(x)} e^{2\nu x} \tag{3.76}$$

$$M_{22}(x) = -\frac{\gamma_1(x)}{\varepsilon_1(x)} e^{-\mu x} - \beta \frac{\gamma_2(x)}{\varepsilon_2(x)} e^{\mu x} - \frac{\gamma^2}{2\,(\theta\rho - \gamma)} \frac{\gamma_2(x)}{\varepsilon_2(x)} \left(\nu - \frac{\theta}{\varepsilon_2(x)}\right) e^{2\nu x} \tag{3.77}$$

$$M_{23}(x) = -\frac{\gamma_1(x)}{\varepsilon_1(x)} e^{-\mu x} - \beta \frac{\gamma_2(x)}{\varepsilon_2(x)} e^{\mu x} - \frac{\gamma^2}{2\,(\theta\rho - \gamma)} \frac{\gamma_2(x)}{\varepsilon_2(x)} \left(\nu - \frac{\theta}{\varepsilon_2(x)}\right) e^{2\nu x} \tag{3.78}$$

$$M_{24}(x) = \beta \left(\mu - \frac{\theta}{\varepsilon_2(x)}\right) e^{\mu x} - \frac{\gamma}{2\kappa} \frac{e^{2\nu x}}{\varepsilon_2^2(x)} + \frac{\gamma^2}{2\,(\theta\rho - \gamma)} \left(\nu - \frac{\theta}{\varepsilon_2(x)}\right)^2 e^{2\nu x}\,, \tag{3.79}$$

for all $x \in [0,1]$, and the inequalities

$$\beta\rho > \frac{\gamma^2 e^{(2\nu-\mu)x}}{2\varepsilon_2(x)}\,, \quad \forall x \in [0,1] \tag{3.80}$$

$$\gamma > \theta\rho\,, \tag{3.81}$$

hold. Then, there exist positive constants λ and Ω such that, for all initial conditions satisfying $\left(\tilde{u}^0(x), \tilde{v}^0(x), \tilde{\eta}^0\right) \in L^2\left((0,1);\mathbb{R}\right) \times L^2\left((0,1);\mathbb{R}\right) \times \mathbb{R}$, the following holds for all $t \geq 0$

$$\int_0^1 \left(\tilde{u}^2(t,x) + \tilde{v}^2(t,x)\right) dx + \tilde{\eta}^2(t) \leq \Omega e^{-\lambda t} \left(\int_0^1 \left(\tilde{u}^2(0,x) + \tilde{v}^2(0,x)\right) dx + \tilde{\eta}^2(0)\right). \tag{3.82}$$

Proof. In order to analyze the stability of system (3.67)–(3.72) we propose the following candidate Lyapunov function, for all $[\tilde{u}, \tilde{v}, \tilde{\eta}] \in E$,

$$\begin{aligned} V(\tilde{u}, \tilde{v}, \tilde{\eta}) &= \int_0^1 \begin{bmatrix} \tilde{u}(x) \\ \tilde{v}(x) \\ \tilde{\eta} \end{bmatrix}^\top P(x) \begin{bmatrix} \tilde{u}(x) \\ \tilde{v}(x) \\ \tilde{\eta} \end{bmatrix} dx \\ &= R_1\left(\tilde{u}\right) + R_2\left(\tilde{v}\right) + R_3\left(\tilde{v}, \tilde{\eta}\right) + R_4\left(\tilde{\eta}\right)\,, \end{aligned} \tag{3.83}$$

with

$$P(x) = \begin{bmatrix} \frac{e^{-\mu x}}{\varepsilon_1(x)} & 0 & 0 \\ 0 & \beta \frac{e^{\mu x}}{\varepsilon_2(x)} & \frac{\gamma e^{\nu x}}{2\varepsilon_2(x)} \\ 0 & \frac{\gamma e^{\nu x}}{2\varepsilon_2(x)} & \frac{\rho}{2} \end{bmatrix}, \tag{3.84}$$

and

$$R_1(\tilde{u}) = \int_0^1 \tilde{u}^2(x) \frac{e^{-\mu x}}{\varepsilon_1(x)} dx \tag{3.85}$$

$$R_2(\tilde{v}) = \beta \int_0^1 \tilde{v}^2(x) \frac{e^{\mu x}}{\varepsilon_2(x)} dx \tag{3.86}$$

$$R_3(\tilde{v}, \tilde{\eta}) = \gamma\tilde{\eta} \int_0^1 \tilde{v}(x) \frac{e^{\nu x}}{\varepsilon_2(x)} dx \tag{3.87}$$

$$R_4(\tilde{\eta}) = \frac{\rho}{2} \tilde{\eta}^2\,. \tag{3.88}$$

Let us introduce the constants

$$\underline{\lambda} = \min_{x\in[0,1]} \lambda_{\min}(P(x)) \tag{3.89}$$

$$\overline{\lambda} = \max_{x\in[0,1]} \lambda_{\max}(P(x))\,. \tag{3.90}$$

Inequality (3.80) ensures that $P(x)$ is positive definite and symmetric for all $x \in [0,1]$, and hence, using the fact that ε_1, $\varepsilon_2 \in C^2([0,1];\mathbb{R})$ with $\varepsilon_1(x)$, $\varepsilon_2(x) > 0$, for all $x \in [0,1]$, one can conclude that, $\overline{\lambda}$, $\underline{\lambda} > 0$. Therefore,

$$\underline{\lambda}\left(\int_0^1 \left(\tilde{u}^2(x) + \tilde{v}^2(x)\right) dx + \tilde{\eta}^2\right) \le V(\tilde{u},\tilde{v},\tilde{\eta}) \le \overline{\lambda}\left(\int_0^1 \left(\tilde{u}^2(x) + \tilde{v}^2(x)\right) dx + \tilde{\eta}^2\right). \tag{3.91}$$

Using (3.85)–(3.88) we get along the solutions of system (3.67)–(3.72) that

$$\begin{aligned}
\dot{R}_1 &= -2\int_0^1 \tilde{u}(t,x)\partial_x\tilde{u}(t,x)e^{-\mu x}dx + 2\int_0^1 \tilde{u}(t,x)\tilde{v}(t,x)\frac{\gamma_1(x)}{\varepsilon_1(x)}e^{-\mu x}dx \\
&= \left(q^2\tilde{v}^2(t,0) - e^{-\mu}\tilde{u}^2(t,1)\right) - \mu\int_0^1 \tilde{u}^2(t,x)e^{-\mu x}dx + 2\int_0^1 \tilde{u}(t,x)\tilde{v}(t,x)\frac{\gamma_1(x)}{\varepsilon_1(x)}e^{-\mu x}dx && (3.92)\\
\dot{R}_2 &= 2\beta\int_0^1 \tilde{v}(t,x)\partial_x\tilde{v}(t,x)e^{\mu x}dx + 2\beta\int_0^1 \tilde{u}(t,x)\tilde{v}(t,x)\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\mu x}dx \\
&= \beta\left(k_P^2e^{\mu}\tilde{v}^2(t,0) + 2k_Pk_Ie^{\mu}\tilde{v}(t,0)\tilde{\eta}(t) + k_I^2e^{\mu}\tilde{\eta}^2(t) - \tilde{v}^2(t,0)\right) - \mu\beta\int_0^1 \tilde{v}^2(t,x)e^{\mu x}dx \\
&\quad + 2\beta\int_0^1 \tilde{u}(t,x)\tilde{v}(t,x)\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\mu x}dx && (3.93)\\
\dot{R}_3 &= \gamma\tilde{\eta}(t)\int_0^1 \partial_x\tilde{v}(t,x)e^{\nu x}dx + \gamma\tilde{v}(t,0)\int_0^1 \tilde{v}(t,x)\frac{e^{\nu x}}{\varepsilon_2(x)}dx + \gamma\tilde{\eta}(t)\int_0^1 \tilde{u}(t,x)\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\nu x}dx \\
&\le \gamma\tilde{\eta}(t)\left(e^{\nu}\left(-k_P\tilde{v}(t,0) - k_I\tilde{\eta}(t)\right) - \tilde{v}(t,0)\right) - \nu\gamma\tilde{\eta}(t)\int_0^1 \tilde{v}(t,x)e^{\nu x}dx \\
&\quad + \frac{\kappa\gamma}{2}\tilde{v}^2(t,0) + \frac{\gamma}{2\kappa}\int_0^1 \tilde{v}^2(t,x)\frac{e^{2\nu x}}{\varepsilon_2^2(x)}dx + \gamma\tilde{\eta}(t)\int_0^1 \tilde{u}(t,x)\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\nu x}dx && (3.94)\\
\dot{R}_4 &= \rho\tilde{v}(t,0)\tilde{\eta}(t)\,, && (3.95)
\end{aligned}$$

where we used integration by parts in the first terms of (3.92)–(3.94) and Young's inequality in the second term of (3.94). Using (3.83), (3.92)–(3.95) we get

$$\begin{aligned}
\dot{V} \le &-\begin{bmatrix}\tilde{v}(t,0)\\ \tilde{\eta}(t)\end{bmatrix}^{\top} M_1 \begin{bmatrix}\tilde{v}(t,0)\\ \tilde{\eta}(t)\end{bmatrix} - \int_0^1 \begin{bmatrix}\tilde{u}(t,x)\\ \tilde{v}(t,x)\\ \tilde{\eta}(t)\end{bmatrix}^{\top} M(x) \begin{bmatrix}\tilde{u}(t,x)\\ \tilde{v}(t,x)\\ \tilde{\eta}(t)\end{bmatrix} dx \\
&- e^{-\mu}\tilde{u}^2(t,1) - \theta V\,, && (3.96)
\end{aligned}$$

where M_1 is given in (3.74) and

$$M(x) = \begin{bmatrix} A(x) & B^{\top}(x) \\ B(x) & C \end{bmatrix}, \tag{3.97}$$

with

$$A(x) = \begin{bmatrix} A_1(x) & A_2(x) \\ A_3(x) & A_4(x) \end{bmatrix}, \tag{3.98}$$

where

$$A_1(x) = \left(\mu - \frac{\theta}{\varepsilon_1(x)}\right) e^{-\mu x} \tag{3.99}$$

$$A_2(x) = -\frac{\gamma_1(x)}{\varepsilon_1(x)} e^{-\mu x} - \beta \frac{\gamma_2(x)}{\varepsilon_2(x)} e^{\mu x} \tag{3.100}$$

$$A_3(x) = -\frac{\gamma_1(x)}{\varepsilon_1(x)} e^{-\mu x} - \beta \frac{\gamma_2(x)}{\varepsilon_2(x)} e^{\mu x} \tag{3.101}$$

$$A_4(x) = \beta \left(\mu - \frac{\theta}{\varepsilon_2(x)}\right) e^{\mu x} - \frac{\gamma}{2\kappa} \frac{e^{2\nu x}}{\varepsilon_2^2(x)} \tag{3.102}$$

$$B(x) = \begin{bmatrix} -\frac{\gamma}{2} \frac{\gamma_2(x)}{\varepsilon_2(x)} e^{\nu x} & \frac{\gamma}{2} \left(\nu - \frac{\theta}{\varepsilon_2(x)}\right) e^{\nu x} \end{bmatrix} \tag{3.103}$$

$$C = \frac{\gamma - \theta \rho}{2}. \tag{3.104}$$

Using the Schur complement of C in $M(x)$ and (3.81), (3.104) one has that $M(x) \geq 0$ for all $x \in [0, 1]$, if and only if

$$M_2(x) = A(x) - B^\top(x) C^{-1} B(x) \geq 0. \tag{3.105}$$

Thus, if $M_1 > 0$ and $M_2(x) \geq 0$, for all $x \in [0, 1]$, one has

$$\dot{V} \leq -e^{-\mu} \tilde{u}^2(t, 1) - \theta V, \tag{3.106}$$

and hence, $V(\tilde{u}, \tilde{v}, \tilde{\eta}) \leq e^{-\theta t} V(0)$, for all $t \geq 0$. Combining this relation with (3.91) the proof is complete. $\square$

Remark 3.2. A control law with an integral action is designed in [38] for 2×2 hyperbolic systems. Stability of the closed-loop system is proved using a diagonal Lyapunov function. Here the non-diagonal term in the Lyapunov function is needed for proving stability using a quadratic Lyapunov function. Indeed, let us assume that the Lyapunov function is diagonal. We can write it as

$$V(\tilde{u}, \tilde{v}, \tilde{\eta}) = \int_0^1 \left(q_1(x) \tilde{u}^2(x) + q_2(x) \tilde{v}^2(x)\right) dx + \frac{\rho}{2} \tilde{\eta}^2, \tag{3.107}$$

where the functions q_1 and q_2 belong to $C^1([0, 1]; \mathbb{R})$ with $q_1(x)$, $q_2(x) > 0$, for all $x \in [0, 1]$. The time derivative of V along the solutions of system (3.67), (3.68) with boundary conditions (3.69)–(3.72) is given by

$$\begin{aligned}\dot{V} = &\begin{bmatrix} \tilde{v}(t, 0) \\ \tilde{\eta}(t) \end{bmatrix}^\top \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix} \begin{bmatrix} \tilde{v}(t, 0) \\ \tilde{\eta}(t) \end{bmatrix} + \int_0^1 \begin{bmatrix} \tilde{u}(t, x) \\ \tilde{v}(t, x) \end{bmatrix}^\top E(x) \begin{bmatrix} \tilde{u}(t, x) \\ \tilde{v}(t, x) \end{bmatrix} dx \\ &- q_1(1) \varepsilon_1(1) \tilde{u}^2(t, 1), \end{aligned} \tag{3.108}$$

where

$$D_1 = q_1(0) \varepsilon_1(0) q^2 - q_2(0) \varepsilon_2(0) + q_2(1) \varepsilon_2(1) k_P^2 \tag{3.109}$$

$$D_2 = \frac{1}{2}\left(q_2(1)\varepsilon_2(1)k_P k_I + \rho\right) \tag{3.110}$$

$$D_3 = \frac{1}{2}\left(q_2(1)\varepsilon_2(1)k_P k_I + \rho\right) \tag{3.111}$$

$$D_4 = q_2(1)\varepsilon_2(1)k_I^2 \tag{3.112}$$

$$E(x) = \begin{bmatrix} \partial_x\left(q_1(x)\varepsilon_1(x)\right) & q_1(x)\gamma_1(x) + q_2(x)\gamma_2(x) \\ q_1(x)\gamma_1(x) + q_2(x)\gamma_2(x) & -\partial_x(q_2(x)\varepsilon_2(x)) \end{bmatrix}. \tag{3.113}$$

Using (3.108) and (3.112) one can conclude that when $k_I \neq 0$ the inequality $\dot{V} \leq 0$ can not be satisfied for any $[\tilde{u}, \tilde{v}, \tilde{\eta}]^\top$.

In [5], it is proved that if there exist two boundary controllers for 2×2 linear hyperbolic systems of the form (3.67), (3.68) such that the function

$$V(\tilde{u}, \tilde{v}, \tilde{\eta}) = \int_0^1 q_1(x)\tilde{u}^2(x) + q_2(x)\tilde{v}^2(x) + q_3(x)\tilde{u}(x)\tilde{v}(x)dx\,, \tag{3.114}$$

along the solutions of the system (3.67), (3.68) with the state $(\tilde{u}, \tilde{v})$ satisfies $\dot{V} < 0$ then the cross term q_3 between $\tilde{u}$ and $\tilde{v}$ is necessarily identically zero. However, in the case of stabilization of 2×2 linear hyperbolic systems of the form (3.67), (3.68) with a PI control law that we consider here, the cross term (3.87) in the Lyapunov function (3.83) between the integral state $\tilde{\eta}$ of the controller and the state of the plant $\tilde{v}$ is necessary (as explained above) for proving stability of the overall closed-loop system consisting of the plant state $(\tilde{u}, \tilde{v})$ and the integral state $\tilde{\eta}$, using the Lyapunov function defined in (3.83) (although a cross term between $\tilde{u}$ and $\tilde{v}$ is not necessary). •

As explained in Remark 3.2 the non-diagonal term in the Lyapunov function is crucial for proving stability using a quadratic Lyapunov function. However, this term adds considerable complexity in verifying analytically that the matrices (3.74), (3.75) are positive definite and that (3.80) holds. Nonetheless, we will show latter that, at least for some applications, it is possible to check these conditions. Moreover, it should be possible to develop some embedding techniques as presented in Chapter 4 in the context of the checking of positive definitiveness of (3.74), (3.75).

Remark 3.3. In [85] the integral action is filtered, that is,

$$\dot{\tilde{\eta}}(t) = -\varepsilon\tilde{\eta}(t) + \tilde{v}(t, 0)$$

where $\varepsilon > 0$ is a small coefficient. This filtering has been introduced to be able to lead a Lyapunov analysis with a diagonal Lyapunov function as (3.107). The same could be done in our case. Nonetheless, the effectiveness of the integral action when disturbances act in the system cannot be analyzed with such filtering, contrary to the integral action that we consider. In the next section, we will show that controller (3.71) under conditions of Theorem 3.2 eliminates disturbance from the output with different meaning depending on the nature of the solution. •

3.5 Tracking Issue in presence In-Domain and Boundary Disturbances

3.5.1 Compensation in the Output of In-Domain and Boundary Disturbances

Let us assume that there exist some disturbances d_1, $d_2 \in C^1([0,1];\mathbb{R})$ on the right-hand side of (3.15), (3.16), respectively and some disturbances d_3, $d_4 \in \mathbb{R}$ on the right-hand side of (3.19), (3.20), respectively. The error system (3.67)–(3.70) becomes

$$\partial_t \tilde{u}(t,x) + \varepsilon_1(x)\partial_x \tilde{u}(t,x) = \gamma_1(x)\tilde{v}(t,x) + d_1(x) \tag{3.115}$$

$$\partial_t \tilde{v}(t,x) - \varepsilon_2(x)\partial_x \tilde{v}(t,x) = \gamma_2(x)\tilde{u}(t,x) + d_2(x) \tag{3.116}$$

$$\tilde{u}(t,0) = q\tilde{v}(t,0) + d_3 \tag{3.117}$$

$$\tilde{v}(t,1) = \tilde{U}(t) + d_4 \tag{3.118}$$

$$\tilde{u}(0,x) = \tilde{u}^0(x) \tag{3.119}$$

$$\tilde{v}(0,x) = \tilde{v}^0(x)\,, \tag{3.120}$$

with $\tilde{U}(t)$ and $\tilde{\eta}(t)$ given by (3.71) and (3.72) respectively. The equilibrium of the perturbed system (3.115)–(3.117) and (3.70) is the solution of the following ordinary differential equation

$$Z'(x) = F(x)Z(x) + G(x), \tag{3.121}$$

where

$$F(x) = \begin{bmatrix} 0 & \frac{\gamma_1(x)}{\varepsilon_1(x)} \\ -\frac{\gamma_2(x)}{\varepsilon_2(x)} & 0 \end{bmatrix}$$

and

$$G(x) = \begin{bmatrix} \frac{d_1(x)}{\varepsilon_1(x)} \\ -\frac{d_2(x)}{\varepsilon_2(x)} \end{bmatrix},$$

with initial conditions

$$Z_1(0) = d_3 \tag{3.122}$$

$$Z_2(0) = 0\,. \tag{3.123}$$

The initial conditions (3.122), (3.123) for the ODE system (3.121) come from the fact that the equilibrium of (3.115)–(3.117) and (3.71) shall follow the reference $z(t)$, meaning that $\tilde{v}(t,0) = 0$. Therefore, replacing $\tilde{v}(t,0)$ by 0 in (3.117) we get the initial condition for Z_1 that is d_3.

The ordinary differential equation (3.121) together with the initial conditions (3.122), (3.123) is a well-posed initial value problem for x. The equilibrium depends on d_1, d_2 and d_3. Let us denote this equilibrium by $\tilde{u}_{ss}(x;d_1,d_2,d_3)$, $\tilde{v}_{ss}(x;d_1,d_2,d_3)$. From (3.118) it follows that the equilibrium value of $\tilde{U}$, namely $\tilde{U}_{ss}$, satisfies

$$\tilde{U}_{ss} = \tilde{v}_{ss}(1;d_1,d_2,d_3) - d_4\,. \tag{3.124}$$

Using (3.71), and (3.123) with $Z = [\tilde{u}_{ss}, \tilde{v}_{ss}]^\top$, it follows from (3.124) that the equilibrium value of $\tilde{\eta}$, namely $\tilde{\eta}_{ss}$, satisfies

$$\tilde{\eta}_{ss} = -\frac{\tilde{v}_{ss}(1; d_1, d_2, d_3) - d_4}{k_I} . \tag{3.125}$$

Let us define

$$\overline{u}(t,x) = \tilde{u}(t,x) - \tilde{u}_{ss}\,(x; d_1, d_2, d_3) \tag{3.126}$$
$$\overline{v}(t,x) = \tilde{v}(t,x) - \tilde{v}_{ss}\,(x; d_1, d_2, d_3) \tag{3.127}$$
$$\overline{\eta}(t) = \tilde{\eta}(t) - \tilde{\eta}_{ss} . \tag{3.128}$$

Using (3.121) with $Z = [\tilde{u}_{ss}, \tilde{v}_{ss}]^\top$ together with (3.115), (3.116) it is shown that the variables $\overline{u}$ and $\overline{v}$ satisfy

$$\partial_t \overline{u}(t,x) + \varepsilon_1(x)\partial_x \overline{u}(t,x) = \gamma_1(x)\overline{v}(t,x) \tag{3.129}$$
$$\partial_t \overline{v}(t,x) - \varepsilon_2(x)\partial_x \overline{v}(t,x) = \gamma_2(x)\overline{u}(t,x) . \tag{3.130}$$

Setting $x = 0$ in (3.126), (3.127), and using (3.117), (3.122), and (3.123) we get that

$$\overline{u}(t,0) = q\overline{v}(t,0) . \tag{3.131}$$

Setting $x = 1$ in (3.127) and using (3.118), (3.71), and (3.125) we get

$$\overline{v}(t,1) = -k_P\tilde{v}(t,0) - k_I\tilde{\eta}(t) + k_I\tilde{\eta}_{ss} .$$

Using (3.127) for $x = 0$ together with (3.123) and (3.128) we arrive at

$$\overline{v}(t,1) = -k_P\overline{v}(t,0) - k_I\overline{\eta}(t) . \tag{3.132}$$

Using (3.128) and the fact that

$$\overline{v}(t,0) = \tilde{v}(t,0) , \tag{3.133}$$

relation (3.72) becomes

$$\dot{\overline{\eta}}(t) = \overline{v}(t,0) . \tag{3.134}$$

Under the assumptions of Theorem 3.2 the zero equilibrium of (3.129)–(3.132) and (3.134) is exponentially stable in the L^2-norm. Nonetheless, the tracking is not achieved strictly speaking. It motivates the title of the present subsection: the presence of disturbances in the system is compensated by the boundary control but not rejected at the output. Nonetheless, let us examine more carefully how the potential disturbance in the output $v(t,0)$ persists. Let us introduce some notations. Let us set

$$\tilde{X} = \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\eta} \end{bmatrix} . \tag{3.135}$$

Let us define the operator $A : D(A) \to E$ by

$$A\tilde{X} = \begin{bmatrix} -\varepsilon_1(x)\frac{d\tilde{u}}{dx} + \gamma_1(x)\tilde{v} \\ \varepsilon_2(x)\frac{d\tilde{v}}{dx} + \gamma_2(x)\tilde{u} \\ \tilde{v}(0) \end{bmatrix} \tag{3.136}$$

with

$$D(A) = \left\{ [\tilde{u}, \tilde{v}, \tilde{\eta}]^\top \in E \mid \tilde{u}, \tilde{v} \in H^1((0,1);\mathbb{R}),\ \tilde{u}(0) = q\tilde{v}(0) + d_3,\ \tilde{v}(1) = -k_P\tilde{v}(0) - k_I\tilde{\eta} + d_4 \right\}. \tag{3.137}$$

Let us define $B \in C^1([0,1];\mathbb{R}^2) \times \mathbb{R}$ by

$$B = \begin{bmatrix} d_1 \\ d_2 \\ 0 \end{bmatrix}. \tag{3.138}$$

Thus, system (3.115)–(3.118) with $\tilde{U}$ and $\tilde{\eta}$ given respectively by (3.71) and (3.72) can be rewritten as

$$\tilde{X}(t) = A\tilde{X}(t) + B. \tag{3.139}$$

In the case where the initial condition $\left[\tilde{u}^0, \tilde{v}^0, \tilde{\eta}^0\right]^\top \in H^1([0,1];\mathbb{R}^2) \times \mathbb{R}$ satisfies the compatibility conditions

$$\tilde{\eta}(0) = \frac{\tilde{v}^0(1) + k_P\tilde{v}^0(0)}{k_I}, \tag{3.140}$$

and

$$\tilde{u}^0(0) = q\tilde{v}^0(0), \tag{3.141}$$

the following theorem holds.

Theorem 3.3. *For every $\left[\tilde{u}^0, \tilde{v}^0, \tilde{\eta}^0\right]^\top \in H^1((0,1);\mathbb{R}) \times H^1((0,1);\mathbb{R}) \times \mathbb{R}$ satisfying the compatibility conditions (3.140), (3.141), there exists a unique solution of (3.115)–(3.120), with $\tilde{U}$ and $\tilde{\eta}$ given respectively by (3.71) and (3.72) such that*

$$[\tilde{u}, \tilde{v}, \tilde{\eta}]^\top \in C([0,T];D(A)) \cap C^1([0,T];E). \tag{3.142}$$

Moreover, this solution satisfies, for all $t \in \mathbb{R}^+$,

$$\begin{aligned}
&|\tilde{u} - \tilde{u}_{ss}, \tilde{v} - \tilde{v}_{ss}, \tilde{\eta} - \tilde{\eta}_{ss}|_E + \int_0^t \left[\varepsilon_1(1)\left(\tilde{u}(s,1) - \tilde{u}_{ss}(1)\right)^2 - \varepsilon_1(0)\left(\tilde{u}(s,0) - \tilde{u}_{ss}(0)\right)^2\right] ds \\
&+ \int_0^t \int_0^1 \varepsilon_1'(x)\left(\tilde{u}(s,x) - \tilde{u}_{ss}(x)\right)^2 dxds - 2\int_0^t \int_0^1 \gamma_1(x)\left(\tilde{u}(s,x) - \tilde{u}_{ss}(x)\right)\left(\tilde{v}(s,x) - \tilde{v}_{ss}(x)\right) dxds \\
&+ \int_0^t \left[\varepsilon_2(0)\tilde{v}^2(s,0) - \varepsilon_2(1)\left(\tilde{v}(s,1) - \tilde{v}_{ss}(1)\right)^2\right] ds \\
&+ \int_0^t \int_0^1 \varepsilon_2'(x)\left(\tilde{v}(s,x) - \tilde{v}_{ss}(x)\right)^2 dxds - 2\int_0^t \int_0^1 \gamma_2(x)\left(\tilde{u}(s,x) - \tilde{u}_{ss}(x)\right)\left(\tilde{v}(s,x) - \tilde{v}_{ss}(x)\right) dxds \\
&- 2\int_0^t \left(\tilde{\eta}(s) - \tilde{\eta}_{ss}\right)\tilde{v}(s,0) ds = \left|\tilde{u}^0 - \tilde{u}_{ss}, \tilde{v}^0 - \tilde{v}_{ss}, \tilde{\eta}^0 - \tilde{\eta}_{ss}\right|_E,
\end{aligned} \tag{3.143}$$

where $[\tilde{u}_{ss}, \tilde{v}_{ss}]^\top$ *is the solution to the ordinary differential equation* (3.121) *with boundary conditions* (3.122), (3.123), *and* $\tilde{\eta}_{ss}$ *is given by* (3.125).

In the more general case where the initial condition does not satisfy the compatibility condition (3.140) and (3.141), and lies in E then the following theorem holds.

Theorem 3.4. *For every* $\left[\tilde{u}^0, \tilde{v}^0, \tilde{\eta}^0\right]^\top \in E$ *there exists a unique solution of* (3.115)–(3.120), *with* $\tilde{U}$ *and* $\tilde{\eta}$ *given respectively by* (3.71) *and* (3.72) *such that*

$$[\tilde{u}, \tilde{v}, \tilde{\eta}] \in C\left([0, T]; E\right). \tag{3.144}$$

Moreover, this solution satisfies (3.143), *for all* $t \in \mathbb{R}^+$.

The proofs of Theorems 3.3, 3.4 rely on the use of the operator (3.136). We refer the reader to [6] and [86] for an insight of the proofs .

Proposition 3.1. *Under conditions of Theorem 3.2, for every* $\left[\tilde{u}^0, \tilde{v}^0, \tilde{\eta}^0\right]^\top \in E$, *and coefficient* $\tau > 0$ *the solution* $\tilde{v}$ *satisfies*

$$\lim_{t\to+\infty} \int_{t-\tau}^{t} \tilde{v}^2(s, 0) ds = 0\,. \tag{3.145}$$

Proof. From (3.91) we have $\underline{\lambda}\, |\tilde{u} - \tilde{u}_{ss}, \tilde{v} - \tilde{v}_{ss}, \tilde{\eta} - \tilde{\eta}_{ss}|_E \leq V\left(\tilde{u} - \tilde{u}_{ss}, \tilde{v}, \tilde{\eta} - \tilde{\eta}_{ss}\right)$ and from (3.106) we have $V \leq e^{-\theta t} V(0)$, together with (3.143) one gets that $\tilde{v}(\cdot, 0) \in L^2\left(\mathbb{R}^+; \mathbb{R}\right)$. This property on $\tilde{v}(\cdot, 0)$ is often related to a "hidden" regularity property (see, for example, [22], Chapter 2, page 32, or [31] for other examples and references on this property). From, this fact we have that, for all $\tau > 0$,

$$\lim_{t\to+\infty} \int_{t-\tau}^{t} \tilde{v}^2(s, 0) ds = 0\,.$$

This concludes the proof of Proposition 3.1. □

Loosely speaking, when conditions of Theorem 3.2 hold the PI controller "rejects on time average" the disturbance in the output.

3.5.2 Disturbances Rejection

Due to the particular nature of our problem, tracking an output given by the solution of a distributed system at a particular point of the space, the L^2-norm is not the better norm to solve it, as mentioned above. Nonetheless, in the case where there exist some compatibility conditions with the initial data which is supposed to be a function in $C^1\left([0,1]; \mathbb{R}\right) \times C^1\left([0,1]; \mathbb{R}\right) \times \mathbb{R}$, it can be proved that the solution lies in $C^1\left(\mathbb{R}^+ \times [0,1]; \mathbb{R}\right) \times C^1\left(\mathbb{R}^+ \times [0,1]; \mathbb{R}\right) \times C^1\left(\mathbb{R}^+; \mathbb{R}\right)$. Using the fact that the trace of a function in this latter space is well-defined and using (3.145) we show that the perturbation in the output is rejected in C^0-norm.

Lemma 3.1. *There exists at most one solution* $[\tilde{u}, \tilde{v}, \tilde{\eta}]^\top$ *to* (3.115)–(3.118) *in* $C^1\left(\mathbb{R}^+ \times [0,1]; \mathbb{R}\right) \times C^1\left(\mathbb{R}^+ \times [0,1]; \mathbb{R}\right) \times C^1\left(\mathbb{R}^+; \mathbb{R}\right)$.

Proof. Let $[\tilde{u}_1, \tilde{v}_1, \tilde{\eta}_1]^\top$ and $[\tilde{u}_2, \tilde{v}_2, \tilde{\eta}_2]^\top$ be solutions in $C^1(\mathbb{R}^+ \times [0,1]; \mathbb{R}) \times C^1(\mathbb{R}^+ \times [0,1]; \mathbb{R}) \times C^1(\mathbb{R}^+; \mathbb{R})$. Their difference

$$[\alpha, \beta, \overline{\eta}]^\top = [\tilde{u}_1 - \tilde{u}_2, \tilde{v}_1 - \tilde{v}_2, \tilde{\eta}_1 - \tilde{\eta}_2]^\top$$

satisfies

$$\partial_t \alpha(t,x) + \varepsilon_1(x)\partial_x \alpha(t,x) = \gamma_1(x)\beta(t,x) \tag{3.146}$$
$$\partial_t \beta(t,x) - \varepsilon_2(x)\partial_x \beta(t,x) = \gamma_2(x)\alpha(t,x) \tag{3.147}$$
$$\alpha(t,0) = q\beta(t,0) \tag{3.148}$$
$$\beta(t,1) = -k_P\beta(t,0) - k_I\check{\eta}(t) \tag{3.149}$$
$$\beta(0,x) = 0 \tag{3.150}$$
$$\alpha(0,x) = 0\,, \tag{3.151}$$

where

$$\dot{\check{\eta}}(t) = \tilde{v}_1(t,0) - \tilde{v}_2(t,0) \tag{3.152}$$
$$\check{\eta}(0) = 0\,, \tag{3.153}$$

and is also a solution in $C^1(\mathbb{R}^+ \times [0,1]; \mathbb{R}) \times C^1(\mathbb{R}^+ \times [0,1]; \mathbb{R}) \times C^1(\mathbb{R}^+; \mathbb{R})$. Due to (3.148) the boundary condition (3.149) can be rewritten as

$$\beta(t,1) = -k'_P\alpha(t,0) - k'_I\check{\eta}(t)\,, \tag{3.154}$$

where

$$\begin{aligned}
\dot{\check{\eta}}(t) &= \tilde{u}_1(t,0) - \tilde{u}_2(t,0) \\
\check{\eta}(0) &= 0 \\
k'_P &= \frac{k_P}{q} \\
k'_I &= \frac{k_I}{q}\,.
\end{aligned} \tag{3.155}$$

Using the change of variables

$$w_1(t,x) = e_1\alpha(t,x) \tag{3.156}$$
$$w_2(t,x) = e_2\beta(t,x)\,, \tag{3.157}$$

equations (3.146), (3.147) become

$$\partial_t w_1(t,x) + \varepsilon_1(x)\partial_x w_1(t,x) = e_1\gamma_1(x)e_2^{-1}w_2(t,x) \tag{3.158}$$
$$\partial_t w_2(t,x) - \varepsilon_2(x)\partial_x w_2(t,x) = e_2\gamma_2(x)e_1^{-1}w_1(t,x)\,, \tag{3.159}$$

and boundary conditions (3.148), (3.149) become

$$w_1(t,0) = e_1 q e_2^{-1} w_2(t,0) \tag{3.160}$$

$$w_2(t,1) = -e_2 k_P' e_1^{-1} w_1(t,0) - e_2 k_I' e_1^{-1} \int_0^1 w_1(s,0)ds\,. \tag{3.161}$$

Hence, the parameters e_1 and e_2 can be chosen such that $e_2 k_P' e_1^{-1}$, $e_2 k_I' e_1^{-1}$, and $e_1 q e_2^{-1}$ are "small". For the sake of simplicity we assume that the parameters q, k_P, and k_I are "small" from the beginning. We compute

$$\begin{aligned}
\frac{d}{dt}\left(\left(\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right],\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]\right)+\frac{\kappa}{2}\check{\eta}(t)^2\right) &= \left(\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right],\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]_t\right)+\left(\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]_t,\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]\right)+\kappa\alpha(t,0)\check{\eta}(t)\\
&= \left(\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right],\left[\begin{smallmatrix}-\varepsilon_1(x)&0\\0&\varepsilon_2(x)\end{smallmatrix}\right]\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]_x\right)+\left(\left[\begin{smallmatrix}-\varepsilon_1(x)&0\\0&\varepsilon_2(x)\end{smallmatrix}\right]\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]_x,\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]\right)\\
&\quad+2\left(\left[\begin{smallmatrix}0&\gamma_1(x)\\\gamma_2(x)&0\end{smallmatrix}\right]\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right],\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]\right)+\kappa\alpha(t,0)\check{\eta}(t)\\
&= \left(\left[\begin{smallmatrix}-\varepsilon_1(x)&0\\0&\varepsilon_2(x)\end{smallmatrix}\right]\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right],\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]_x\right)+\left(\left[\begin{smallmatrix}-\varepsilon_1(x)&0\\0&\varepsilon_2(x)\end{smallmatrix}\right]\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]_x,\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]\right)\\
&\quad+2\left(\left[\begin{smallmatrix}0&\gamma_1(x)\\\gamma_2(x)&0\end{smallmatrix}\right]\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right],\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]\right)+\kappa\alpha(t,0)\check{\eta}(t)\\
&= -\left(\left[\begin{smallmatrix}-\varepsilon_1(x)&0\\0&\varepsilon_2(x)\end{smallmatrix}\right]_x\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right],\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]\right)-\left(\left[\begin{smallmatrix}-\varepsilon_1(x)&0\\0&\varepsilon_2(x)\end{smallmatrix}\right]\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]_x,\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]\right)\\
&\quad+\left(\left[\begin{smallmatrix}-\varepsilon_1(x)&0\\0&\varepsilon_2(x)\end{smallmatrix}\right]\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]_x,\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]\right)+\kappa\alpha(t,0)\check{\eta}(t)\\
&\quad+2\left(\left[\begin{smallmatrix}0&\gamma_1(x)\\\gamma_2(x)&0\end{smallmatrix}\right]\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right],\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]\right)+\left(\left[\begin{smallmatrix}-\varepsilon_1(1)&0\\0&\varepsilon_2(1)\end{smallmatrix}\right]\left[\begin{smallmatrix}\alpha(t,1)\\\beta(t,1)\end{smallmatrix}\right],\left[\begin{smallmatrix}\alpha(t,1)\\\beta(t,1)\end{smallmatrix}\right]\right)\\
&\quad-\left(\left[\begin{smallmatrix}-\varepsilon_1(0)&0\\0&\varepsilon_2(0)\end{smallmatrix}\right]\left[\begin{smallmatrix}\alpha(t,0)\\\beta(t,0)\end{smallmatrix}\right],\left[\begin{smallmatrix}\alpha(t,0)\\\beta(t,0)\end{smallmatrix}\right]\right)\\
&= -\left(\left[\begin{smallmatrix}-\varepsilon_1(x)&0\\0&\varepsilon_2(x)\end{smallmatrix}\right]_x\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right],\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]\right)+2\left(\left[\begin{smallmatrix}0&\gamma_1(x)\\\gamma_2(x)&0\end{smallmatrix}\right]\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right],\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]\right)\\
&\quad-\varepsilon_1(1)\alpha^2(t,0)+\varepsilon_2(1)\left(-k_P\alpha(t,0)-k_I\check{\eta}(t)\right)^2+\varepsilon_1(0)q^2\beta^2(t,0)\\
&\quad-\varepsilon_2(0)\beta^2(t,0)+\kappa\alpha(t,0)\check{\eta}(t)
\end{aligned}$$

Using the assumption on the smallness of q, k_P, and k_I and the fact that $\kappa > 0$ can be chosen arbitrary small we get

$$\frac{d}{dt}\left(\left(\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right],\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]\right)+\frac{\kappa}{2}\check{\eta}(t)^2\right)\le C\left(\left(\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right],\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]\right)+\frac{\kappa}{2}\check{\eta}(t)^2\right)\,,\quad C\in\mathbb{R}^+\,.$$

Using the Gronwall's Lemma, initial condition (3.150), (3.151), and (3.155) we get

$$[\alpha,\beta,\check{\eta}]^\top = [0,0,0]^\top\ .$$

This concludes the proof of Lemma 3.1. □

Theorem 3.5. *Let us assume that the hypothesis of Theorem 3.2 hold and that the initial condition* $\left[\tilde{u}^0,\tilde{v}^0,\tilde{\eta}^0\right]^\top \in C^1([0,1];\mathbb{R})\times C^1([0,1];\mathbb{R})\times\mathbb{R}$ *satisfies the compatibility conditions of order zero* (3.140), (3.141) *together with the compatibility conditions*

$$\tilde{v}^0(0) = \frac{\tilde{v}^{0'}(1)+k_P\tilde{v}^{0'}(0)}{k_I} \tag{3.162}$$

$$\tilde{u}^{0'}(0) = q\tilde{v}^{0'}(0)\,, \tag{3.163}$$

then there exists a unique solution $[\tilde{u}, \tilde{v}, \tilde{\eta}] \in C^1\left(\mathbb{R}^+ \times [0,1]\,;\mathbb{R}\right) \times C^1\left(\mathbb{R}^+ \times [0,1]\,;\mathbb{R}\right) \times C^1\left(\mathbb{R}^+;\mathbb{R}\right)$ *to the system* (3.115)–(3.118) *and it satisfies*

$$\lim_{s\to+\infty} \left|\tilde{v}(\cdot,0)\mathbb{1}_{[s,s+\tau]}(\cdot)\right|_{C^0(\mathbb{R}^+;\mathbb{R})} = 0\,. \tag{3.164}$$

for all $\tau > 0$.

Proof. The proof of existence uses an iteration process and the estimation derived in the proof of Lemma 3.1, the Gronwall Lemma, the Picard Lemma, and finally the Arzelà-Ascoli Theorem (see Appendix on page 125). For the insight of the proof see [65], Section 7.6, pages 253–260, although this reference deals with solution in C^∞, the principle of the proof is the same for C^1. The limit (3.164) is obtained by observing that the trace of a solution in C^1 is well defined and using (3.145).

This concludes the proof of Theorem 3.5. □

Remark 3.4. In the context of Theorem 3.5 we can say that the PI controller rejects the disturbance in the output, but it does not prove that the disturbances are rejected (in C^0-norm) in the rest of the domain. This is not a drawback since our original objective was to track a reference $z^r(t)$ for the output $z(t)$. •

3.6 Application to the Linearized ARZ Equations

We consider the linearized version of ARZ (see Subsection 1.2.2)

$$\partial_t y_1(t,x) + \varepsilon_1 \partial_x y_1(t,x) = -\frac{1}{\tau} y_1(t,x) \tag{3.165}$$

$$\partial_t y_2(t,x) - \varepsilon_2 \partial_x y_2(t,x) = -\frac{1}{\tau} y_1(t,x)\,, \tag{3.166}$$

with boundary conditions

$$y_1(t,0) = q y_2(t,0) \tag{3.167}$$

$$y_2(t,1) = S(t)\,, \tag{3.168}$$

where τ is a positive parameter. The opposite transport velocities in (3.165), (3.166) correspond to traffic flow in a congested mode. The boundary condition (3.167) in the original variables is written as

$$w(t,0) = \frac{\phi'(s^*)s(t,0)}{1-q}\,. \tag{3.169}$$

Hence, the boundary condition (3.167) dictates that there is a static relation, at the entrance of the road, between the density and the velocity similarly to the static relation between the nominal velocity $\phi(s)$ and the density of the cars in the road. The change of variables (3.10), (3.11), (3.17), and (3.18) transform system (3.165)–(3.168) to

$$\partial_t u(t,x) + \varepsilon_1 \partial_x u(t,x) = 0 \tag{3.170}$$

$$\partial_t v(t,x) - \varepsilon_2 \partial_x v(t,x) = -\frac{1}{\tau} \exp\left(-\frac{1}{\tau\varepsilon_1} x\right) u(t,x) \tag{3.171}$$

$$u(t,0) = qv(t,0) \tag{3.172}$$

$$v(t,1) = U(t)\,, \tag{3.173}$$

where $U(t)$ is given by (3.21). Observing that $\gamma_1 = 0$, the computation done in Section 3.3 can be adapted with the parameters of this illustration, hence one obtains

$$L^{\alpha\alpha}(x,\xi) = 0 \tag{3.174}$$

$$L^{\alpha\beta}(x,\xi) = 0 \tag{3.175}$$

$$L^{\beta\alpha}(x,\xi) = \frac{1}{\tau\left(\varepsilon_1 + \varepsilon_2\right)} \exp\left(-\frac{1}{\tau\varepsilon_1}\left(\frac{\varepsilon_1 x + \varepsilon_2 \xi}{\varepsilon_1 + \varepsilon_2}\right)\right) \tag{3.176}$$

$$L^{\beta\beta}(x,\xi) = \frac{q\varepsilon_1}{\tau\varepsilon_2\left(\varepsilon_1 + \varepsilon_2\right)} \exp\left(-\frac{1}{\tau\varepsilon_1}\left(\frac{\varepsilon_1 x - \varepsilon_1 \xi}{\varepsilon_1 + \varepsilon_2}\right)\right)\,. \tag{3.177}$$

Therefore, for system (3.165)–(3.168), the reference input which generates the desired output $z^r(t)$ is

$$\begin{aligned} S^r(t) = {} & z^r\left(t + \frac{1}{\varepsilon_2}\right) + \frac{q}{\tau\left(\varepsilon_1 + \varepsilon_2\right)} \int_0^1 \exp\left(-\frac{1}{\tau\varepsilon_1}\left(\frac{\varepsilon_1 + \varepsilon_2 \xi}{\varepsilon_1 + \varepsilon_2}\right)\right) z^r\left(t - \frac{\xi}{\varepsilon_1}\right) d\xi \\ & + \frac{q\varepsilon_1}{\tau\varepsilon_2\left(\varepsilon_1 + \varepsilon_2\right)} \int_0^1 \exp\left(-\frac{1}{\tau\varepsilon_1}\left(\frac{\varepsilon_1 - \varepsilon_1 \xi}{\varepsilon_1 + \varepsilon_2}\right)\right) z^r\left(t + \frac{\xi}{\varepsilon_2}\right) d\xi\,. \end{aligned} \tag{3.178}$$

Let us present an illustration of Theorem 3.2. Let us set in (3.170)–(3.172)

$$\varepsilon_1 = 6 \tag{3.179}$$

$$\varepsilon_2 = 6 \tag{3.180}$$

$$\tau = 5 \tag{3.181}$$

$$q = 0.2\,, \tag{3.182}$$

and choose U in (3.70) according to (3.71) with

$$k_P = 0.1 \tag{3.183}$$

$$k_I = 1.0583\,, \tag{3.184}$$

in order to stabilize the zero equilibrium of (3.170)–(3.172). We verify numerically that the conditions of Theorem 3.2 are satisfied with

$$(\beta, \kappa, \mu, \nu, \theta, \rho, \gamma) = (0.7, 0.2, 0.5, 0.2, 0.7, 2, 2)\,. \tag{3.185}$$

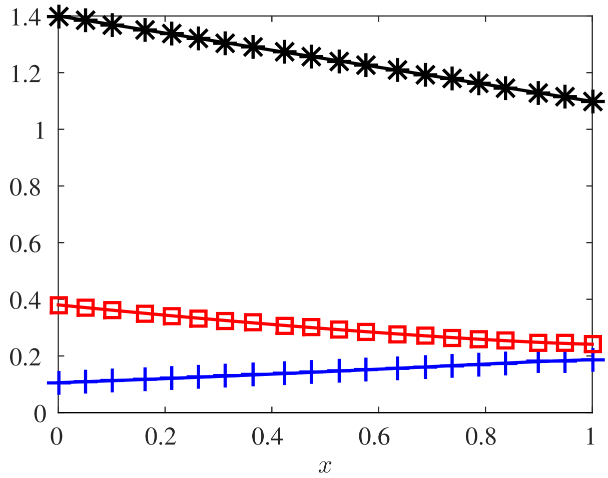

Figure 3.3: Evolution of the eigenvalues of (3.75) as a function of x (square and cross markers), and of the determinant of $P(x)$ in (3.84) (star marker).

From (3.74) we get that

$$M_1 = \begin{bmatrix} 0.4485 & 0 \\ 0 & 0.2926 \end{bmatrix} > 0\,. \tag{3.186}$$

The verification of the positive definiteness of matrix (3.75) is more delicate due to its dependence on x. Figure 3.3 shows the evolution of the eigenvalues of $M_2(x)$ and the determinant of matrix (3.84), which remain positive for all $x \in [0,1]$. The numerical approximation of the solution is computed with a two-step variant of the Lax-Friedrichs (LxF) method [102]. The reference for the output is chosen as $z^r(t) = \cos(t)$. We add disturbances at the right-hand side of (3.170), (3.171) given by

$$d_1(x) = 0.5\exp(x) \tag{3.187}$$

$$d_2(x) = \cos(2x)\,, \tag{3.188}$$

together with constant additive disturbances on the boundary conditions (3.172), (3.173) given by

$$d_3 = 0.5 \tag{3.189}$$

$$d_4 = 0.5\,. \tag{3.190}$$

The initial conditions for u and v are chosen as the reference initial conditions given by (3.23), (3.24) for $t = 0$, perturbed by spatially-varying errors as

$$u(0,x) = u^r(0,x) + \sin(x) \tag{3.191}$$

$$v(0,x) = v^r(0,x) + \cos(x)\,, \tag{3.192}$$

and the initial condition for $\tilde{\eta}$ is chosen such that $U(0) = v(1,0)$, that is,

$$\tilde{\eta}(0) = \frac{U^r(0) - v(0,1) + k_P(v^r(0,0) - v(0,0))}{k_I}\,. \tag{3.193}$$

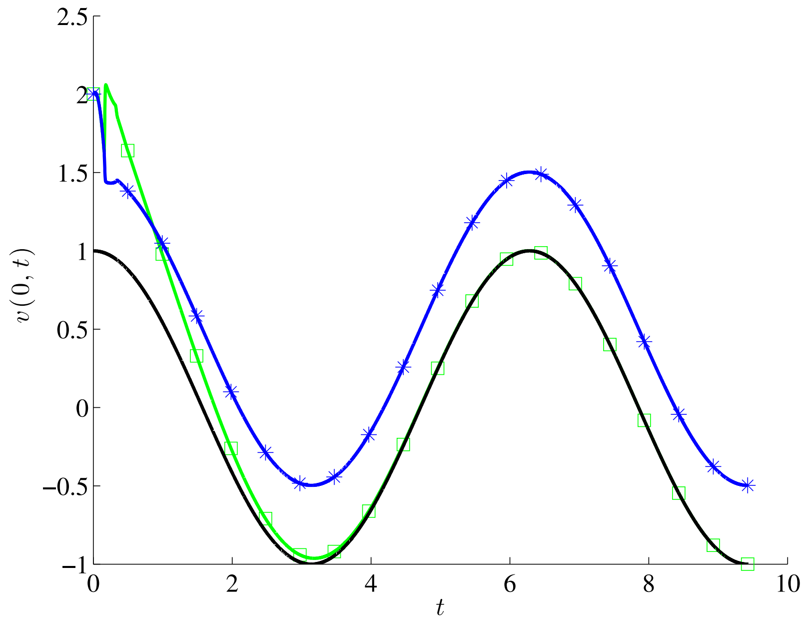

Figure 3.4: The output $v(t,0)$ of system (3.170)–(3.173) with parameters (3.179)–(3.182) under the control law (3.194) with gains (3.183), (3.184) (square marker) and with gains (3.183), $k_I = 0$ (star marker) for the initial conditions (3.191)–(3.193). The single line is the reference output $z^r(t) = \cos(t)$.

Figure 3.4 shows that the output of the system $v(t,0)$ follows the desired trajectory under the PI controller given by

$$\begin{aligned}U(t) = \cos\left(t+\frac{1}{6}\right) &+ \frac{600}{101}\left(\sin\left(t+\frac{1}{6}\right) - \exp\left(-\frac{1}{60}\right)\sin(t)\right)\\ &+ \frac{60}{101}\left(\cos\left(t+\frac{1}{6}\right) - \exp\left(-\frac{1}{60}\right)\cos(t)\right)\\ &+ \frac{120}{101}\left(\exp\left(-\frac{1}{60}\right)\sin(t) - \exp\left(-\frac{1}{30}\right)\sin\left(t-\frac{1}{6}\right)\right)\\ &+ \frac{12}{101}\left(\exp\left(-\frac{1}{60}\right)\cos(t) - \exp\left(-\frac{1}{30}\right)\cos\left(t-\frac{1}{6}\right)\right)\\ &- k_P\left(v(t,0)-\cos(t)\right) - k_I\tilde{\eta}(t)\,,\end{aligned} \tag{3.194}$$

with gains (3.183), (3.184), and $\dot{\tilde{\eta}}(t) = v(t,0) - \cos(t)$. One can also observe that with only a P controller (i.e., when $k_I = 0$ in (3.194)) there is a steady-state tracking error. Figure 3.5 shows the evolution of the state v.

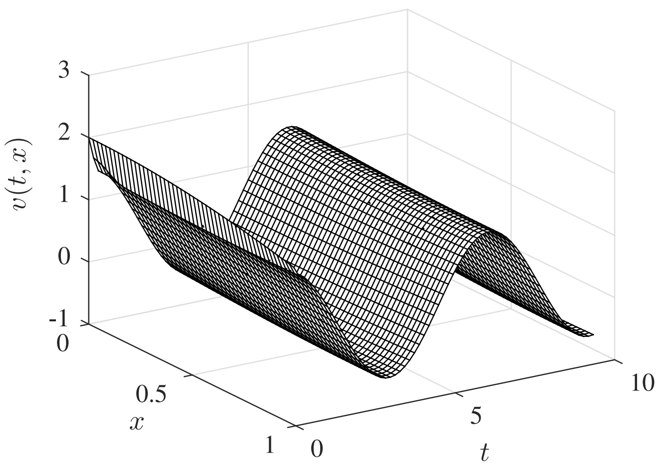

Figure 3.5: Evolution of the state v of system (3.170)–(3.173) with parameters (3.179)–(3.182) under the control law (3.194) with gains (3.183), (3.184) for the initial conditions (3.191)–(3.193).

4. Numerical Methods for Lyapunov Analysis

In this chapter, we consider the problems of stability analysis and control synthesis for first-order hyperbolic linear PDEs over a bounded interval with spatially varying coefficients. We propose LMI-based conditions for the stability and for the design of boundary and distributed control for the system. These LMI-based conditions involve an infinite number of LMI to solve. Hence, we show how to overapproximate these constraints using polytopic embedding to reduce the problem to a finite number of LMI to solve. We show the effectiveness of the overapproximation with several examples, and with the Saint-Venant equations with friction.

Some of this work was submitted as a conference paper for an IFAC meeting [71].

4.1 Problem Statement and Existing Results

We consider the following general system

$$\partial_t y(t,x) + \Lambda(x)\partial_x y(t,x) = F(x)y(t,x)\,, \tag{4.1}$$

where $t \in \mathbb{R}^+$ is the time variable, $x \in [0,1]$ is the spatial variable, $y : \mathbb{R}^+ \times [0,1] \to \mathbb{R}^n$, F and Λ are in $C^0\left([0,1]\,;\mathbb{R}^{n\times n}\right)$. The matrix $\Lambda(x)$ is diagonal and in addition $\Lambda(x) = \mathtt{diag}\left[\lambda_1(x),\dots,\lambda_n(x)\right]$ with $\lambda_k(x) < 0$ for $k \in \{1,\dots,m\}$ and $\lambda_k(x) > 0$ for $k \in \{m+1,\dots,n\}$, for all $x \in [0,1]$. Let us introduce the following notations

$$\begin{aligned} \Lambda(x) &= \mathtt{diag}\left[\Lambda^-(x),\Lambda^+(x)\right]\,, \\ y &= \left[y^-(t,x),y^+(t,x)\right]^\top\,, \end{aligned}$$

where $y^- : \mathbb{R}^+ \times [0,1] \to \mathbb{R}^m$ and $y^+ : \mathbb{R}^+ \times [0,1] \to \mathbb{R}^{(n-m)}$, $\Lambda^-(x) = \mathtt{diag}\left[\lambda_1(x),\dots,\lambda_m(x)\right]$, and $\Lambda^+(x) = \mathtt{diag}\left[\lambda_{m+1}(x),\dots,\lambda_n(x)\right]$. We consider the following boundary conditions

$$\begin{bmatrix} y^-(t,1) \\ y^+(t,0) \end{bmatrix} = G \begin{bmatrix} y^-(t,0) \\ y^+(t,1) \end{bmatrix}\,, \quad t \in \mathbb{R}^+\,, \tag{4.2}$$

where G is a matrix in $\mathbb{R}^{n\times n}$. The initial condition is

$$y(0,x) = y^0(x)\,, \quad x \in (0,1)\,, \tag{4.3}$$

where $y^0 \in L^2\left((0,1);\mathbb{R}^n\right)$.

While for finite dimensional and time-delay systems a large number of numerical techniques for stability analysis exists, for PDEs these tools are mostly lacking. In this chapter, we propose some techniques to verify numerically the existence of quadratic Lyapunov function for first-order hyperbolic PDEs over a bounded interval with spatially varying coefficients. Besides this analysis aspect, we propose some techniques for the synthesis of boundary and distributed controls.

As already mentioned in the Introduction, the use of Lyapunov function is now appearing for hyperbolic systems. In particular, a special attention has been made on quadratic Lyapunov functions. Indeed, this class of functions allows to express conditions for stability as MI as in Chapters 2 and 3. We can cite also [16], [35], [99], and [112] for the linear case. LMI conditions derived by an operators approach is used in [48] for the H^∞ boundary control of parabolic and hyperbolic systems. Quadratic control Lyapunov function has also been used for 2×2 quasi-linear systems [26] and $n \times n$ quasi-linear systems [27]. MI-based conditions derived from a quadratic Lyapunov function were stated in [15] for the construction of boundary observers for linear as well as for quasi-linear hyperbolic systems. However, the approach by a quadratic Lyapunov function is not always effective to prove stability for hyperbolic systems. A result from [5] gives a necessary and sufficient condition for the existence of control Lyapunov function. In [33], an example with a static output feedback has been designed such that this condition is violated and thus that there does not exist a Lyapunov function for this system.

The results in this chapter are related to the resolution of the LMIs proposed in [99] and to the control synthesis. We also propose a Lyapunov function with an "affine" kernel similar to the one used in [31]. The LMI-based conditions involve the spatial variable, hence the number of constraints is infinite. These LMI-based conditions are analogous to stability conditions for finite-dimensional Linear Parameter Varying (LPV) systems. Hence, an approach inspired by this framework is applied to find a candidate Lyapunov function. More precisely, to reduce the numerical complexity, different approximations based on properties of the exponential functions are considered in this chapter. The control synthesis relies on a combination of classical techniques coming from the stabilization for discrete and continuous time finite dimensional systems. Then, the latter overapproximations techniques are used to get a finite number of LMI-based conditions.

4.2 LMI-based Conditions for Stability

In this section, we propose a Lyapunov function, and derive some LMI-based conditions for the solution of system (4.1)–(4.3) to satisfy (1.34).

Let us denote $|\Lambda(x)|$ the matrix whose elements are the absolute value of the elements of the matrix $\Lambda(x)$, that is

$$|\Lambda(x)| = \mathtt{diag}\left[-\Lambda^-(x), \Lambda^+(x)\right] , \tag{4.4}$$

and let us denote by $\tilde{I}_{n,m}$ the matrix

$$\tilde{I}_{n,m} = \begin{bmatrix} -I_m & 0_{m,n-m} \\ 0_{n-m,m} & I_{n-m} \end{bmatrix} . \tag{4.5}$$

For a matrix A in $\mathbb{R}^{n\times n}$, we decompose it in four block matrices A_{--} in $\mathbb{R}^{m\times m}$, A_{-+} $\mathbb{R}^{m\times(n-m)}$, A_{+-} in $\mathbb{R}^{(n-m)\times m}$ and A_{++} in $\mathbb{R}^{(n-m)\times(n-m)}$ such that $A = \left[\begin{smallmatrix} A_{--} & A_{-+} \\ A_{+-} & A_{++} \end{smallmatrix}\right]$.

In [99] sufficient conditions have been given for the stability of (4.1)–(4.3) with $\Lambda(x)$ and $F(x)$ constant. We consider a slightly different Lyapunov function

$$V(y) = \int_0^1 y^\top(t,x) \left|\Lambda(x)\right|^{-1} \mathcal{Q}(x) y(t,x) dx \, , \tag{4.6}$$

where

$$\mathcal{Q}(x) = \mathtt{diag}\left[e^{2\mu x} Q^-, e^{-2\mu x} Q^+\right] , \tag{4.7}$$

with Q^- in $\mathbb{R}^{m\times m}$, Q^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ are two symmetric positive definite matrices and μ a real coefficient.

Proposition 4.1. *If there exist $\nu > 0$, μ in $\mathbb{R}$, and symmetric positive definite matrices Q^- in $\mathbb{R}^{m\times m}$ and Q^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ such that the following conditions hold, for all $x \in [0,1]$,*

$$\mathcal{Q}(x)\Lambda(x) = \Lambda(x)\mathcal{Q}(x)\,, \tag{4.8}$$

$$-2\mu\mathcal{Q}(x) + F^\top(x)\left|\Lambda(x)\right|^{-1}\mathcal{Q}(x) + \mathcal{Q}(x)\left|\Lambda(x)\right|^{-1}F(x) \le -2\nu\left|\Lambda(x)\right|^{-1}\mathcal{Q}(x)\,, \tag{4.9}$$

together with

$$\begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++} \end{bmatrix}^\top \tilde{I}_{n,m}\mathcal{Q}(0)\begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++} \end{bmatrix} \le \begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m} \end{bmatrix}^\top \tilde{I}_{n,m}\mathcal{Q}(1)\begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m} \end{bmatrix}, \tag{4.10}$$

with $\mathcal{Q}(x)$ defined by (4.7), then the linear hyperbolic system (4.1)–(4.3) is GES.

Proof. For the stability analysis we consider the candidate Lyapunov function (4.6), where $\mathcal{Q}(x)$ is given by (4.7). Let us introduce the constants

$$\underline{\lambda} = \min_{x\in[0,1]} \lambda_{\min}\left(\left|\Lambda(x)\right|^{-1}\mathcal{Q}(x)\right)\,, \tag{4.11}$$

$$\overline{\lambda} = \max_{x\in[0,1]} \lambda_{\max}\left(\left|\Lambda(x)\right|^{-1}\mathcal{Q}(x)\right)\,. \tag{4.12}$$

The matrix $\left|\Lambda(x)\right|^{-1}\mathcal{Q}(x)$ being positive definite, one can conclude that $\overline{\lambda}$, $\underline{\lambda} > 0$ and for all $y \in L^2\left((0,1);\mathbb{R}^n\right)$

$$\underline{\lambda}\left|y\right|^2_{L^2((0,1);\mathbb{R}^n)} \le V\left(y\right) \le \overline{\lambda}\left|y\right|^2_{L^2((0,1);\mathbb{R}^n)}\,. \tag{4.13}$$

Let us compute the time-derivative of the candidate Lyapunov function (4.6) along the solutions of system (4.1), (4.2). Using the commutativity condition (4.8), we have

$$\begin{aligned} \dot{V} &= 2\int_0^1 y_t^\top(t,x)\left|\Lambda(x)\right|^{-1}\mathcal{Q}(x)y(t,x)dx \\ &= -2\int_0^1 y^\top(t,x)\tilde{I}_{n,m}\mathcal{Q}(x)y_x(t,x)dx + 2\int_0^1 y^\top(t,x)\mathcal{Q}(x)\left|\Lambda(x)\right|^{-1}F(x)y(t,x)dx\,. \end{aligned} \tag{4.14}$$

Noting that $-2y^\top\tilde{I}_{n,m}\mathcal{Q}y_x = -\left(y^\top\tilde{I}_{n,m}\mathcal{Q}y\right)_x + y^\top\tilde{I}_{n,m}\mathcal{Q}'y$ and $\tilde{I}_{n,m}\mathcal{Q}' = -2\mu\mathcal{Q}$, one has

$$\begin{aligned} \dot{V} = \begin{bmatrix} y^-(t,0) \\ y^+(t,1) \end{bmatrix}^\top &\left[\begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++} \end{bmatrix}^\top \tilde{I}_{n,m}\mathcal{Q}(0)\begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++} \end{bmatrix}\right. \\ &\left.- \begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m} \end{bmatrix}^\top \tilde{I}_{n,m}\mathcal{Q}(1)\begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m} \end{bmatrix}\right]\begin{bmatrix} y^-(t,0) \\ y^+(t,1) \end{bmatrix} \\ &+ 2\int_0^1 y^\top(t,x)\mathcal{Q}(x)\left|\Lambda(x)\right|^{-1}F(x)y(t,x)dx - 2\mu\int_0^1 y^\top(t,x)\mathcal{Q}(x)y(t,x)dx\,. \end{aligned} \tag{4.15}$$

Then, (4.9) and (4.10) imply that $\dot{V} \le -2\nu V$. Hence for all $t \in \mathbb{R}^+$ one has $V\left(y\right) \le e^{-2\nu t}V\left(y^0\right)$. Combining this relation with (4.13), the proof is complete. ☐

Henceforth, for a given μ in $\mathbb{R}$ let us denote $I^e_{n,m}(x)$ the matrix

$$I^e_{n,m}(x) = \mathtt{diag}\left[e^{2\mu x} I_m, e^{-2\mu x} I_{n-m}\right] . \tag{4.16}$$

The relation between the previous stability conditions and LPV systems is done in the following proposition.

Proposition 4.2. *Let μ in $\mathbb{R}$. Conditions (4.9) and (4.10) are satisfied if and only if the continuous time LPV system*

$$\dot{p}(t) = I^e_{n,m}(x) \left(|\Lambda(x)|^{-1} F(x) - \mu I_n \right) p(t) , \quad x \in [0, 1] , \tag{4.17}$$

and the discrete time system

$$h(t+1) = \mathtt{diag}\left[I_m, e^{-\mu} I_{n-m}\right] e^{\mu} G \, \mathtt{diag}\left[I_m, e^{\mu} I_{n-m}\right] h(t) \tag{4.18}$$

share a common block diagonal Lyapunov matrix $\mathtt{diag}[Q^-, Q^+]$, where Q^- and Q^+ are symmetric matrices in $\mathbb{R}^{m\times m}$ and $\mathbb{R}^{(n-m)\times(n-m)}$ respectively.

Proof. LMI-based condition (4.9) describes a condition for the stability of the continuous time LPV system (4.17).

LMI-based condition (4.10) may be developed as

$$P = \begin{bmatrix} P_{--} & P_{-+} \\ P_{+-} & P_{++} \end{bmatrix} \leq 0 , \tag{4.19}$$

with

$$\begin{aligned}
P_{--} &= e^{2\mu} G^\top_{--} Q^- G_{--} + G^\top_{+-} Q^+ G_{+-} - Q^- , \\
P_{-+} &= e^{2\mu} G^\top_{--} Q^- G_{-+} + G^\top_{+-} Q^+ G_{++} , \\
P_{+-} &= P^\top_{-+} , \\
P_{++} &= e^{2\mu} G^\top_{-+} Q^- G_{-+} + G^\top_{++} Q^+ G_{++} - e^{-2\mu} Q^+ .
\end{aligned}$$

The matrix P in (4.19) may be rewritten as

$$P = (e^{\mu} G)^\top \mathtt{diag}\left[Q^-, e^{-2\mu} Q^+\right] e^{\mu} G - \mathtt{diag}\left[Q^-, e^{-2\mu} Q^+\right] . \tag{4.20}$$

Thus, with (4.20) inequality (4.19) leads to establish

$$\begin{aligned}
P \leq 0 \Leftrightarrow\ & e^{\mu} G^\top \mathtt{diag}\left[I_m, e^{-\mu} I_{n-m}\right] \mathtt{diag}\left[Q^-, Q^+\right] \mathtt{diag}\left[I_m, e^{-\mu} I_{n-m}\right] e^{\mu} G \\
& \leq \mathtt{diag}\left[I_m, e^{-\mu} I_{n-m}\right] \mathtt{diag}\left[Q^-, Q^+\right] \mathtt{diag}\left[I_m, e^{-\mu} I_{n-m}\right] \\
\Leftrightarrow\ & \mathtt{diag}\left[I_m, e^{\mu} I_{n-m}\right] e^{\mu} G^\top \mathtt{diag}\left[I_m, e^{-\mu} I_{n-m}\right] \mathtt{diag}\left[Q^-, Q^+\right] \\
& \times \mathtt{diag}\left[I_m, e^{-\mu} I_{n-m}\right] e^{\mu} G \mathtt{diag}\left[I_m, e^{\mu} I_{n-m}\right] \leq \mathtt{diag}\left[Q^-, Q^+\right] .
\end{aligned}$$

Hence, condition (4.10) implies that the discrete time system (4.18) share a common Lyapunov matrix with the continuous time LPV system (4.17). It concludes the proof of Proposition 4.2. $\square$

Remark 4.1. A consequence of Proposition 4.2 is a trade-off in the choice of μ between the satisfaction of (4.9) and (4.10). $\bullet$

Other sufficient conditions for the global exponential stability of system (4.1)–(4.3) are obtained when considering a different kernel $\mathcal{Q}(x)$ in (4.6). The next proposition is built using

$$\mathcal{Q}(x) = \mathtt{diag}\left[(1+\mu x)Q^{-}, (1-\mu x)Q^{+}\right] , \tag{4.21}$$

in (4.6).

Proposition 4.3. *If there exist $\nu > 0$, $\mu \in (-1,1)$ and symmetric positive definite matrices Q^- in $\mathbb{R}^{m\times m}$ and Q^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ such that the following conditions hold, for all $x \in [0,1]$,*

$$\mathcal{Q}(x)\Lambda(x) = \Lambda(x)\mathcal{Q}(x) \tag{4.22}$$

$$-\mu\mathcal{Q}(0) + \left(|\Lambda(x)|^{-1} F(x)\right)^\top \mathcal{Q}(x) + \mathcal{Q}(x)\left(|\Lambda(x)|^{-1} F(x)\right) \leq -2\nu\, |\Lambda(x)|^{-1} \mathcal{Q}(x)\,, \tag{4.23}$$

together with

$$\begin{aligned}\begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++} \end{bmatrix}^\top \begin{bmatrix} -Q^- & 0_{m,n-m} \\ 0_{n-m,m} & Q^+ \end{bmatrix} \begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++} \end{bmatrix} \leq \begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m} \end{bmatrix}^\top \begin{bmatrix} -(1+\mu)Q^- & 0 \\ 0 & (1-\mu)Q^+ \end{bmatrix} \\ \times \begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m} \end{bmatrix} ,\end{aligned} \tag{4.24}$$

then the linear hyperbolic system (4.1)–(4.3) is GES.

Proof. As mentioned earlier, we consider the candidate Lyapunov function (4.6) where $\mathcal{Q}(x)$ is given by (4.21). We used the definitions of $\underline{\lambda}$ and $\overline{\lambda}$ given in (4.11) and (4.12) in the context of $\mathcal{Q}(x)$ given by (4.21). The matrix $|\Lambda(x)|^{-1}\mathcal{Q}(x)$ is positive definite, one can conclude that $\overline{\lambda}$, $\underline{\lambda} > 0$. Therefore,

$$\underline{\lambda}\, |y|^2_{L^2(0,1)} \leq V(y) \leq \overline{\lambda}\, |y|^2_{L^2(0,1)} \,. \tag{4.25}$$

Let us compute the time-derivative of the candidate Lyapunov function along the solutions of system (4.1), (4.2). Using the commutativity condition (4.22), we have

$$\begin{aligned}\dot{V} &= 2\int_0^1 y_t^\top(t,x)\, |\Lambda(x)|^{-1}\, \mathcal{Q}(x) y(t,x) dx \\ &= -2\int_0^1 y_x^\top(t,x) \begin{bmatrix} -I_m & 0_{m,n-m} \\ 0_{n-m,m} & I_{n-m} \end{bmatrix} \mathcal{Q}(x) y(t,x) dx + 2\int_0^1 y^\top(t,x)\mathcal{Q}(x)\, |\Lambda(x)|^{-1} F(x) y(t,x) dx \\ &= -2\int_0^1 y_x^\top(t,x)\mathcal{P}(x) y(t,x) dx + 2\int_0^1 y^\top(t,x)\mathcal{Q}(x)\, |\Lambda(x)|^{-1} F(x) y(t,x) dx\,,\end{aligned}$$

where $\mathcal{P}(x) = \begin{bmatrix} -(1+\mu x)Q^- & 0_{m,n-m} \\ 0_{m,n-m} & (1-\mu x)Q^+ \end{bmatrix}$. Noting that $-2y_x^\top \mathcal{P} y = -\partial_x\left(y^\top \mathcal{P} y\right) + y^\top \mathcal{P}' y$ we get

$$
\begin{aligned}
\dot{V} =& -\left[y^\top(t,x)\mathcal{P}(x)y(t,x)\right]_0^1 - \mu\int_0^1 y^\top(t,x)\mathcal{Q}(0)y(t,x)dx \\
&+ 2\int_0^1 y^\top(t,x)\mathcal{Q}(x)\left|\Lambda(x)\right|^{-1}F(x)y(t,x)dx \\
=& \begin{bmatrix} y^-(t,0) \\ y^+(t,1)\end{bmatrix}^\top \left[\begin{bmatrix} I_m & 0 \\ G_{+-} & G_{++}\end{bmatrix}^\top \mathcal{P}(0)\begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++}\end{bmatrix}\right. \\
&\left. - \begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m}\end{bmatrix}^\top \mathcal{P}(1)\begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m}\end{bmatrix}\right]\begin{bmatrix} y^-(t,0) \\ y^+(t,1)\end{bmatrix} \\
&- \mu\int_0^1 y^\top(t,x)\mathcal{Q}(0)y(t,x)dx + 2\int_0^1 y^\top(t,x)\mathcal{Q}(x)\left|\Lambda(x)\right|^{-1}F(x)y(t,x)dx\,.
\end{aligned}
$$

Then, (4.23) and (4.24) imply that $\dot{V} \leq -2\nu V$. Hence, for all $t \in \mathbb{R}^+$ one has $V(y) \leq e^{-2\nu t}V\left(y^0\right)$. Combining this relation with (4.25) the proof is complete. □

Remark 4.2. The condition

$$\mathcal{Q}(x)\Lambda(x) = \Lambda(x)\mathcal{Q}(x)\,, \tag{4.26}$$

will imply that most of the time the matrices Q^- and Q^+ have to be diagonal. But it may happen that the matrices cannot be diagonal even in the constant case. Let us show it with an example. Let us consider the case $n = 2$, a matrix F spatially constant and Hurwitz, a matrix G marginally stable and $\Lambda = I_2$. Hence, condition (4.9) for stability is rewritten as

$$(F - \mu I_2)^\top Q + Q(F - \mu I_2) < 0\,. \tag{4.27}$$

Let us choose G as

$$G = \frac{1}{2}I_2\,. \tag{4.28}$$

Hence, condition (4.10) is rewritten as

$$\frac{1}{4}Q \leq e^{-2\mu}Q\,, \tag{4.29}$$

which implies

$$\frac{1}{4} \leq e^{-2\mu}\,, \tag{4.30}$$

that is

$$\mu \leq \frac{\ln(4)}{2} = \lambda\,. \tag{4.31}$$

To find a matrix F such that it does exist a diagonal Q we suppose that F is written as

$$F = A + \lambda I_2 = \begin{bmatrix} a+\lambda & b \\ c & d+\lambda\end{bmatrix}\,,$$

where $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. The matrix F has to be Hurwitz, hence one has

$$\begin{aligned} \det(F) > 0 &\Leftrightarrow ad + \lambda(a+d) + \lambda^2 - bc > 0 \\ Tr(F) < 0 &\Leftrightarrow a + d + 2\lambda < 0\,. \end{aligned}$$

Since $\lambda > 0$ we get $Tr(A) < 0$. Without loss of generality, the diagonal matrix Q can be written as $\left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right]$, with $\alpha > 0$. One gets

$$QA + A^\top Q = \begin{bmatrix} 2a & b+\alpha c \\ b+\alpha c & 2\alpha d \end{bmatrix}.$$

Moreover we suppose that $\det(A) = ad - bc > 0$. The determinant of the above matrix is given by $4ad\alpha - (b+\alpha c)^2$, which is a polynomial of degree 2 in α

$$P(\alpha) = -\alpha^2 c^2 + (4ad - 2bc)\alpha - b^2\,. \tag{4.32}$$

The objective is to obtain a matrix A such that the Lyapunov inequality $A^\top Q + QA < 0$ has no diagonal solution. Hence, if the determinant is negative the counter example is obtained. The determinant of P is given by

$$\begin{aligned} (4ad - 2bc)^2 - 4b^2c^2 < 0 &\Leftrightarrow 16a^2d^2 - 16adbc < 0 \\ &\Leftrightarrow a^2d^2 < adbc \\ &\Leftrightarrow ad\,(ad - bc) < 0 \\ &\Leftrightarrow ad < 0\,. \end{aligned}$$

We used the fact that $\det(A) > 0$. Hence, with the different conditions on a, b, c, d we can find a matrix F such that conditions do not hold. For instance,

$$F = \begin{bmatrix} -10 & -9 \\ 10 & 6 \end{bmatrix}, \tag{4.33}$$

is a possible example. •

4.3 Controller Design

4.3.1 Boundary Control Design

We consider next the problem of boundary control design, when boundary condition (4.2) is given by

$$G = T + LK_B\,, \tag{4.34}$$

where matrices T in $\mathbb{R}^{n\times n}$, L in $\mathbb{R}^{n\times q}$ $(n > q)$ are given and the matrix K_B in $\mathbb{R}^{q\times n}$ has to be designed such that system (4.1)–(4.3) with the boundary conditions (4.34) is GES. We propose results based on the use of the Lyapunov function (4.6) in the context of the exponential kernel (4.7) and of the affine kernel (4.21).

Exponential Kernel

Theorem 4.1. *If there exist $\nu > 0$, μ in $\mathbb{R}$, a matrix U in $\mathbb{R}^{q\times n}$, and symmetric matrices S^- in $\mathbb{R}^{m\times m}$, S^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ such that $S(x) = \mathtt{diag}\left[e^{-2\mu x}S^-, e^{2\mu x}S^+\right]$, and such that the following conditions hold, for all $x \in [0,1]$,*

$$S(x)\Lambda(x) = \Lambda(x)S(x)\,, \tag{4.35}$$

$$\begin{bmatrix} \mathtt{diag}\left[S^-, e^{-2\mu}S^+\right] & (TS(0)+LU)^\top \\ TS(0)+LU & \mathtt{diag}\left[e^{-2\mu}S^-, S^+\right] \end{bmatrix} \geq 0\,, \tag{4.36}$$

$$-2\mu S(x) + S(x)F^\top(x)\left|\Lambda(x)\right|^{-1} + \left|\Lambda(x)\right|^{-1}F(x)S(x) \leq -2\nu S(x)\left|\Lambda(x)\right|^{-1}\,, \tag{4.37}$$

then the boundary control given by (4.34) with

$$K_B = US(0)^{-1}\,, \tag{4.38}$$

makes system (4.1)–(4.3) GES.

Proof. Replacing U by $K_BS(0)$ and applying the Schur complement formula in (4.36) one gets

$$\mathtt{diag}\left[S^-, e^{-2\mu}S^+\right] - S(0)\left(T+LK_B\right)^\top \mathtt{diag}\left[e^{-2\mu}S^-, S^+\right]^{-1}\left(T+LK_B\right)S(0) \geq 0\,. \tag{4.39}$$

Reassembling the term in one matrix and multiplying from the left and right with $S(0)^{-1}$ we get a matrix

$$M = \begin{bmatrix} M_{--} & M_{-+} \\ M_{+-} & M_{++} \end{bmatrix} \geq 0\,, \tag{4.40}$$

with

$$\begin{aligned} M_{--} &= \left(S^-\right)^{-1} - e^{2\mu}\left(T+LK_B\right)_{--}^\top\left(S^-\right)^{-1}\left(T+LK_B\right)_{--} \\ &\quad - \left(T+LK_B\right)_{+-}^\top\left(S^+\right)^{-1}\left(T+LK_B\right)_{+-}\,, \end{aligned} \tag{4.41}$$

$$\begin{aligned} M_{-+} &= -e^{2\mu}\left(T+LK_B\right)_{--}^\top\left(S^-\right)^{-1}\left(T+LK_B\right)_{-+} \\ &\quad - \left(T+LK_B\right)_{+-}^\top\left(S^+\right)^{-1}\left(T+LK_B\right)_{++}\,, \end{aligned} \tag{4.42}$$

$$M_{+-} = M_{-+}^\top\,, \tag{4.43}$$

$$\begin{aligned} M_{++} &= e^{-2\mu}\left(S^+\right)^{-1} - e^{2\mu}\left(T+LK_B\right)_{-+}^\top\left(S^-\right)^{-1}\left(T+LK_B\right)_{-+} \\ &\quad - \left(T+LK_B\right)_{++}^\top\left(S^+\right)^{-1}\left(T+LK_B\right)_{++}\,. \end{aligned} \tag{4.44}$$

Letting $Q^- = \left(S^-\right)^{-1}$, $Q^+ = \left(S^+\right)^{-1}$ we get condition (4.8) from (4.35), LMI-based conditions (4.9) from (4.37), (4.10) from the matrix M in (4.40). Indeed, M is equivalent to the matrix $-P$ in (4.19) in the proof of Proposition 4.2, this latter is the derivation of the inequality (4.10)). It concludes the proof of Theorem 4.1. □

Affine Kernel

Theorem 4.2. *If there exist $\nu > 0$, μ in $(-1,1)$, a matrix U in $\mathbb{R}^{q\times n}$, and symmetric matrices S^- in $\mathbb{R}^{m\times m}$, S^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ such that $S(x) = \textbf{\textit{diag}}\left[(1+\mu x)^{-1} S^-, (1-\mu x) S^+\right]$ and such that the following conditions hold, for all $x \in [0,1]$,*

$$S(x)\Lambda(x) = \Lambda(x)S(x)\,, \tag{4.45}$$

$$\begin{bmatrix} \textbf{\textit{diag}}\,[S^-, (1-\mu)S^+] & (TS(0)+LU)^\top \\ TS(0)+LU & \textbf{\textit{diag}}\left[(1+\mu)^{-1} S^-, S^+\right] \end{bmatrix} \geq 0\,, \tag{4.46}$$

$$-2\mu S(0) + S(x)F^\top(x)\,|\Lambda(x)|^{-1} + |\Lambda(x)|^{-1} F(x)S(x) \leq -2\nu S(x)\,|\Lambda(x)|^{-1}\,, \tag{4.47}$$

then the boundary control given by (4.34) with

$$K_B = US(0)^{-1}\,, \tag{4.48}$$

makes system (4.1)–(4.3) GES.

Proof. The proof is similar to the one of Theorem 4.1. Replacing U by $K_B S(0)$ and applying the Schur complement formula in (4.46) one gets

$$\texttt{diag}\left[S^-, (1-\mu) S^+\right] - S(0)\,(T+LK_B)^\top\,\texttt{diag}\left[(1+\mu)^{-1} S^-, S^+\right]^{-1} (T+LK_B)\,S(0) \geq 0\,. \tag{4.49}$$

Reassembling the term in one matrix and multiplying from the left and right with $S(0)^{-1}$ we get a matrix

$$\tilde{M} = \begin{bmatrix} \tilde{M}_{--} & \tilde{M}_{-+} \\ \tilde{M}_{+-} & \tilde{M}_{++} \end{bmatrix} \geq 0\,, \tag{4.50}$$

with

$$\begin{aligned} \tilde{M}_{--} = {} & \left(S^-\right)^{-1} - (1+\mu)\,(T+LK_B)_{--}^\top \left(S^-\right)^{-1} (T+LK_B)_{--} \\ & - (T+LK_B)_{+-}^\top \left(S^+\right)^{-1} (T+LK_B)_{+-}\,, \end{aligned} \tag{4.51}$$

$$\begin{aligned} \tilde{M}_{-+} = {} & -(1+\mu)\,(T+LK_B)_{--}^\top \left(S^-\right)^{-1} (T+LK_B)_{-+} \\ & - (T+LK_B)_{+-}^\top \left(S^+\right)^{-1} (T+LK_B)_{++}\,, \end{aligned} \tag{4.52}$$

$$\tilde{M}_{+-} = \tilde{M}_{-+}^\top\,, \tag{4.53}$$

$$\begin{aligned} \tilde{M}_{++} = {} & (1-\mu)\left(S^+\right)^{-1} - (1+\mu)\,(T+LK_B)_{-+}^\top \left(S^-\right)^{-1} (T+LK_B)_{-+} \\ & - (T+LK_B)_{++}^\top \left(S^+\right)^{-1} (T+LK_B)_{++}\,. \end{aligned} \tag{4.54}$$

Letting $Q^- = (S^-)^{-1}$, $Q^+ = (S^+)^{-1}$ we get condition (4.22) from (4.45), LMI-based conditions (4.9) from (4.47), (4.10) from the matrix M in (4.50). It concludes the proof of Theorem 4.2. ☐

4.3.2 Distributed Control Design

We consider that the right-hand side of (4.1) is of the form

$$F(x) = H(x) + B(x)K_D(x)\,, \quad x \in [0,1]\,, \tag{4.55}$$

where matrices $H(x)$ in $\mathbb{R}^{n\times n}$ and $B(x)$ in $\mathbb{R}^{n\times p}$ $(n > p)$ are given and matrix $K_D(x)$ in $\mathbb{R}^{p\times n}$ has to be designed such that system (4.1)–(4.3) is GES with the distributed control (4.55). In the next we assume that $K_D(x)$ is given by

$$K_D(x) = \sum_{i=1}^{\ell} \alpha_i(x)K_i\,, \tag{4.56}$$

where α_i, $i = 1,\dots,\ell$, are some continuous real functions.

Remark 4.3. Examples of suitable functions α_i, in (4.56) are the Bézier functions basis and spline basis functions of degree 1. •

Exponential Kernel

Theorem 4.3. *Let an integer $\ell > 0$ be given. If there exist $\nu > 0$, μ in $\mathbb{R}$, matrices U_i in $\mathbb{R}^{p\times n}$, $i = 1,\dots,\ell$, and positive definite symmetric matrices S^- in $\mathbb{R}^{m\times m}$, S^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ such that $S(x) = \mathtt{diag}\left[e^{-2\mu x}S^-, e^{2\mu x}S^+\right]$, and such that, the following conditions hold, for all $x \in [0,1]$,*

$$S(x)\Lambda(x) = \Lambda(x)S(x)\,, \tag{4.57}$$

$$\begin{aligned}&\left(|\Lambda(x)|^{-1} H(x) - \mu I_n\right) S(x) + S(x)\left(|\Lambda(x)|^{-1} H(x) - \mu I_n\right)^\top \\ &+ \sum_{i=1}^{\ell}\alpha_i(x)\left(I^e_{n,m}(x)\right)^{-1}U_i^\top B^\top(x)\,|\Lambda(x)|^{-1} + \sum_{i=1}^{\ell}\alpha_i(x)\,|\Lambda(x)|^{-1} B(x)U_i\left(I^e_{n,m}(x)\right)^{-1} \\ &\leq -2\nu\,|\Lambda(x)|^{-1} S(x)\,,\end{aligned} \tag{4.58}$$

$$\begin{bmatrix} \mathtt{diag}\left[S^-, e^{-2\mu}S^+\right] & (GS(0))^\top \\ GS(0) & \mathtt{diag}\left[e^{-2\mu}S^-, S^+\right]\end{bmatrix} \geq 0\,, \tag{4.59}$$

with $I^e_{n,m}(x)$ given in (4.16), then the distributed control given by (4.55) and (4.56) with

$$K_i = U_iS(0)^{-1}\,, \quad i = 1,\dots,\ell\,, \tag{4.60}$$

makes system (4.1)–(4.3) GES.

Proof. We know that system (4.1)–(4.3) is exponentially stable if conditions of Proposition 4.1 hold. To apply this result let us check (4.8), (4.9) and (4.10) successively. Using the Schur complement formula with (4.59), letting $Q^- = (S^-)^{-1}$ and $Q^+ = (S^+)^{-1}$ as in the proof of Theorem 4.1, conditions (4.8) and (4.10) are satisfied. We can rewrite (4.9) as

$$\left(|\Lambda(x)|^{-1} F(x) - \mu I_n\right)^\top \mathcal{Q}(x) + \mathcal{Q}(x)\left(|\Lambda(x)|^{-1} F(x) - \mu I_n\right) \leq -2\nu\,|\Lambda(x)|^{-1}\mathcal{Q}(x)\,.$$

We use the expression of F given by (4.55), (4.56) and get

$$
\begin{aligned}
&\left(|\Lambda(x)|^{-1}\left(H(x)+B(x)\sum_{i=1}^{\ell}\alpha_i(x)K_i\right)-\mu I_n\right)^{\top}\mathcal{Q}(x)\\
&+\mathcal{Q}(x)\left(|\Lambda(x)|^{-1}\left(H(x)+B(x)\sum_{i=1}^{\ell}\alpha_i(x)K_i\right)-\mu I_n\right)\leq -2\nu\,|\Lambda(x)|^{-1}\,\mathcal{Q}(x)\,. \qquad (4.61)
\end{aligned}
$$

This last inequality is not jointly convex in K_i and $\mathcal{Q}(x)$. To overcome this issue we multiply (4.61) at the left and right by $S(x)$ we get

$$
\begin{aligned}
&\left(|\Lambda(x)|^{-1}H(x)-\mu I_n\right)S(x)+S(x)\left(|\Lambda(x)|^{-1}H(x)-\mu I_n\right)^{\top}\\
&+\sum_{i=1}^{\ell}\alpha_i(x)S(x)K_i^{\top}B(x)^{\top}\,|\Lambda(x)|^{-1}+\sum_{i=1}^{\ell}\alpha_i(x)\,|\Lambda(x)|^{-1}\,B(x)K_iS(x)\leq -2\nu\,|\Lambda(x)|^{-1}\,S(x)\,,
\end{aligned}
$$

and we let $K_i = U_iS(0)^{-1}$, $i = 1,\dots,\ell$, giving (4.58). This concludes the proof of Theorem 4.3. □

Affine Kernel

Henceforth, for a given μ in $(-1,1)$ let us denote $I^a_{n,m}(x)$ the matrix

$$
I^a_{n,m}(x) = \mathtt{diag}\left[(1+\mu x)\,I_m,(1-\mu x)\,I_{n-m}\right]\,. \qquad (4.62)
$$

Theorem 4.4. *Let an integer $\ell > 0$ be given. If there exist $\nu > 0$, μ in $(-1,1)$, matrices U_i in $\mathbb{R}^{p\times n}$, $i = 1,\dots,\ell$, and positive definite symmetric matrices S^- in $\mathbb{R}^{m\times m}$, S^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ such that $S(x) = \mathtt{diag}\left[(1+\mu x)^{-1}S^-,(1-\mu x)^{-1}S^+\right]$, and such that, the following conditions hold, for all $x \in [0,1]$,*

$$
\begin{aligned}
&S(x)\Lambda(x)=\Lambda(x)S(x)\,, \qquad (4.63)\\
&\left(|\Lambda(x)|^{-1}H(x)-\mu I_n\right)S(x)+S(x)\left(|\Lambda(x)|^{-1}H(x)-\mu I_n\right)^{\top}\\
&+\sum_{i=1}^{\ell}\alpha_i(x)\left(I^a_{n,m}(x)\right)^{-1}U_i^{\top}B^{\top}(x)\,|\Lambda(x)|^{-1}+\sum_{i=1}^{\ell}\alpha_i(x)\,|\Lambda(x)|^{-1}\,B(x)U_i\left(I^a_{n,m}(x)\right)^{-1}\\
&\leq -2\nu\,|\Lambda(x)|^{-1}\,S(x)\,, \qquad (4.64)\\
&\begin{bmatrix}\mathtt{diag}\left[S^-,(1-\mu)\,S^+\right] & (GS(0))^{\top}\\ GS(0) & \mathtt{diag}\left[(1+\mu)^{-1}S^-,S^+\right]\end{bmatrix}\geq 0\,, \qquad (4.65)
\end{aligned}
$$

with $I^a_{n,m}(x)$ given in (4.62), then the distributed control given by (4.55) and (4.56) with

$$
K_i = U_iS(0)^{-1}\,,\quad i=1,\dots,\ell\,, \qquad (4.66)
$$

makes system (4.1)–(4.3) GES.

Proof. The proof of Theorem 4.4 is immediate with the proof of Theorems 4.2, 4.3. □

Remark 4.4. The simultaneous design of a boundary control and of a distributed control is possible. For the exponential kernel it consists in the computation of matrices S^-, S^+, U, and U_i , $i = 1, \ldots, \ell$, satisfying (4.35), (4.36), and (4.58). For the affine kernel it consists in the computation of matrices S^-, S^+, U, and U_i , $i = 1, \ldots, \ell$, satisfying (4.45), (4.46), and (4.64). •

4.4 Overapproximation Techniques

In this section, we present some practical techniques for the stability analysis and the control design. For each conditions given up to know we propose a result for their overapproximation. We distinguish two cases: spatially varying and non-spatially varying. For the clarity of the exposition we propose a summary table of the results at the end of the section.

4.4.1 Non-Spatially Varying Case

Let us suppose $F(x) = F$, $\Lambda(x) = \Lambda$. The main goal of this section is to provide a way to numerically verify conditions of Propositions 4.1 and 4.3 and of Theorems 4.1, 4.2, 4.3, and 4.4.

Exponential Kernel
For fixed μ in $\mathbb{R}$, Q^- in $\mathbb{R}^{m\times m}$ and Q^+ in $\mathbb{R}^{(n-m)\times(n-m)}$, we write

$$Q_{ij} = \mathtt{diag}\left[e^{2\mu i}Q^-, e^{-2\mu j}Q^+\right] , \quad i,j = 0,1 . \tag{4.67}$$

Lemma 4.1. *For all $x \in [0,1]$, $\mathcal{Q}(x)$ lies in the convex hull formed by Q_{00}, Q_{01}, and Q_{11} if $\mu > 0$ and by Q_{00}, Q_{10}, and Q_{11} if $\mu < 0$.*

Proof. Without loss of generality we assume that $\mu > 0$. $\mathcal{Q} : x \mapsto \mathcal{Q}(x)$ is a parameterized curve in the (Q^-, Q^+) plane. We can express it as an explicit curve. We have $e^{2\mu x}e^{-2\mu x} = 1$, thus the expression of the explicit curve is given by

$$h(X) = \frac{1}{X} , \quad X \in \rho = \left[e^{-2\mu}, 1\right] .$$

This curve is convex on this interval. Then,

$$\frac{1}{\alpha e^{-2\mu} + (1-\alpha)} \leq \alpha g\left(e^{-2\mu}\right) + (1-\alpha)g(1) , \quad \alpha \in (0,1) , \tag{4.68}$$

where $g(X) = \left(\frac{1-e^{2\mu}}{1-e^{-2\mu}}\right) X + e^{2\mu} + 1$. When X lies in ρ, $g(X)$ describes the straight line between Q_{00} and Q_{11}. Hence, for $X \in \rho$ one has $h(X) \leq g(X)$. Thus, $\mathcal{Q}(x)$ lies in the convex hull formed by Q_{00}, Q_{01}, and Q_{11}. □

Proposition 4.4. *If there exist $\nu > 0$, μ in $\mathbb{R}$, and symmetric positive definite matrices Q^- in $\mathbb{R}^{m\times m}$ and Q^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ such that*

$$Q_{ij}\Lambda = \Lambda Q_{ij} , \tag{4.69}$$

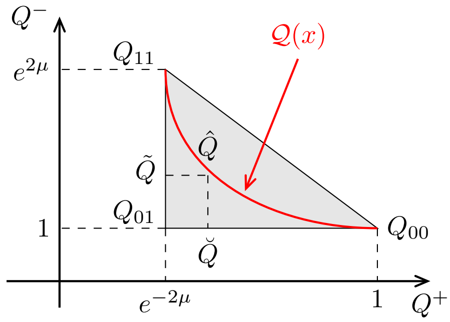

Figure 4.1: Illustration of the elevation of number of points representing the polytope used for the overapproximation in the (Q^-, Q^+)-plane. Case $\mu > 0$.

$$-2\mu Q_{ij} + F^\top |\Lambda|^{-1} Q_{ij} + Q_{ij} |\Lambda|^{-1} F \leq -2\nu |\Lambda|^{-1} Q_{ij} \,, \tag{4.70}$$

hold for all $(i,j) \in \{(0,0),(0,1),(1,1)\}$ *if* $\mu > 0$ *and for all* $(i,j) \in \{(0,0),(1,0),(1,1)\}$ *if* $\mu < 0$, *together with*

$$\begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++} \end{bmatrix}^\top Q_{00} \tilde{I}_{n,m} \begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++} \end{bmatrix} \leq \begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m} \end{bmatrix}^\top Q_{11} \tilde{I}_{n,m} \begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m} \end{bmatrix} \,, \tag{4.71}$$

then conditions (4.8), (4.9), *and* (4.10) *are satisfied for all* $x \in [0,1]$.

Proof. The inequality (4.71) corresponds to (4.10). By Lemma 4.1 the constraint of equality (4.8) and LMI (4.9) are embedded in the polytope formed by the points Q_{00}, Q_{01} and Q_{11}. Thus, conditions (4.8), (4.9), and (4.10) are satisfied. It concludes the proof of Proposition 4.4. □

Remark 4.5. The approximation with the exponential kernel (4.7) can be made tighter by increasing the number of points describing the polytope embedding the constraints given by condition (4.8) and LMIs (4.9), (4.10). For instance, on Figure 4.1 there are 5 points: Q_{11}, $\tilde{Q}$, $\hat{Q}$, $\breve{Q}$ and Q_{00}. The impact of the number of points is explored numerically in the next section. •

The LMI-based conditions for the construction of boundary and distributed controller can be overapproximated in the same manner than the LMI-based conditions for stability. Let us suppose that $H(x) = H$ and $B(x) = B$. For fixed μ in $\mathbb{R}$, S^- in $\mathbb{R}^{m\times m}$ and S^+ in $\mathbb{R}^{(n-m)\times(n-m)}$, we write, for $i,j = 0,1$,

$$\breve{S}_{ij} = \mathtt{diag}\left[e^{-2\mu i} S^-, e^{-2\mu j} S^+\right] \,, \tag{4.72}$$

$$S_{ij} = \mathtt{diag}\left[e^{-2\mu i} S^-, e^{2\mu j} S^+\right] \,, \tag{4.73}$$

and

$$I^{ij}_{n,m} = \mathtt{diag}\left[e^{-2\mu i} I_m, e^{2\mu j} I_{n-m}\right] \,, \quad i,j = 0,1 \,. \tag{4.74}$$

Proposition 4.5. *If there exist* $\nu > 0$, μ *in* $\mathbb{R}$, *a matrix* U *in* $\mathbb{R}^{q\times n}$, *and symmetric matrices* S^- *in* $\mathbb{R}^{m\times m}$, S^+ *in* $\mathbb{R}^{(n-m)\times(n-m)}$ *such that, the following conditions hold,*

$$S_{ij}\Lambda = \Lambda S_{ij} \,, \tag{4.75}$$

$$
\begin{bmatrix} \check{S}_{01} & \left(T\check{S}_{00} + LU\right)^\top \\ T\check{S}_{00} + LU & \check{S}_{10} \end{bmatrix} \geq 0\,, \tag{4.76}
$$

$$
-2\mu S_{ij} + S_{ij} F^\top |\Lambda|^{-1} + |\Lambda|^{-1} F S_{ij} \leq -2\nu S_{ij} |\Lambda|^{-1}\,, \tag{4.77}
$$

for $(i,j) \in \{(0,0),(1,0),(1,1)\}$ *if* $\mu > 0$ *and for* $(i,j) \in \{(0,0),(0,1),(1,1)\}$ *if* $\mu < 0$*, then conditions* (4.35)*,* (4.36)*, and* (4.37) *are satisfied for all* $x \in [0,1]$*.*

Proof. The inequality (4.36) corresponds to (4.76). By Lemma 4.1 the constraint of equality (4.35) and LMIs (4.37) are embedded in the polytope formed by the points S_{00}, S_{10} and S_{11} if $\mu > 0$ or by the points S_{00}, S_{01} and S_{11} if $\mu < 0$. Thus, conditions (4.35), (4.36), and (4.37) are satisfied. It concludes the proof of Proposition 4.5. □

Proposition 4.6. *Let an integer* $\ell > 0$ *be given. If there exist* $\nu > 0$*,* μ *in* $\mathbb{R}$*, matrices* U_i *in* $\mathbb{R}^{p\times n}$*,* $i = 1,\dots,\ell$*, and positive definite symmetric matrices* S^- *in* $\mathbb{R}^{m\times m}$*,* S^+ *in* $\mathbb{R}^{(n-m)\times(n-m)}$ *such that the following conditions hold*

$$
S_{jk}\Lambda = \Lambda S_{jk}\,, \tag{4.78}
$$

$$
\begin{aligned}
&\left(|\Lambda|^{-1} H - \mu I_n\right) S_{jk} + S_{jk}\left(|\Lambda|^{-1} H - \mu I_n\right)^\top \\
&+ \sum_{i=1}^{\ell} \min_{x\in[0,1]} \alpha_i(x) \left(I^{jk}_{n,m}\right)^{-1} U_i^\top B^\top |\Lambda|^{-1} + \sum_{i=1}^{\ell} \min_{x\in[0,1]} \alpha_i(x) |\Lambda|^{-1} B U_i \left(I^{jk}_{n,m}\right)^{-1} \\
&\leq -2\nu |\Lambda|^{-1} S_{jk}\,,
\end{aligned} \tag{4.79}
$$

$$
\begin{aligned}
&\left(|\Lambda|^{-1} H - \mu I_n\right) S_{jk} + S_{jk}\left(|\Lambda|^{-1} H(x) - \mu I_n\right)^\top \\
&+ \sum_{i=1}^{\ell} \max_{x\in[0,1]} \alpha_i(x) \left(I^{jk}_{n,m}\right)^{-1} U_i^\top B^\top |\Lambda|^{-1} + \sum_{i=1}^{\ell} \max_{x\in[0,1]} \alpha_i(x) |\Lambda|^{-1} B U_i \left(I^{jk}_{n,m}\right)^{-1} \\
&\leq -2\nu |\Lambda|^{-1} S_{jk}\,,
\end{aligned} \tag{4.80}
$$

$$
\begin{bmatrix} \check{S}_{01} & \left(G\check{S}_{00}\right)^\top \\ G\check{S}_{00} & \check{S}_{10} \end{bmatrix} \geq 0\,, \tag{4.81}
$$

for $(j,k) \in \{(0,0),(1,0),(1,1)\}$ *if* $\mu > 0$ *and for* $(j,k) \in \{(0,0),(0,1),(1,1)\}$ *if* $\mu < 0$*, then conditions* (4.57)*,* (4.58)*, and* (4.59) *are satisfied for all* $x \in [0,1]$*.*

Proof. First of all the maximum and the minimum of α_i, $i = 1,\dots,\ell$, are well-defined since they are continuous functions. The inequality (4.81) corresponds to (4.59). By Lemma 4.1 the constraint of equality (4.57) and LMI (4.58) are embedded in the polytope formed by the points S_{00}, S_{10}, and S_{11} if $\mu > 0$ or by the points S_{00}, S_{01}, and S_{11} if $\mu < 0$. Noting that any value of a continuous function can be expressed as a convex combination of the maximum and minimum of the function, condition (4.58) holds for all $x \in [0,1]$. It concludes the proof of Proposition 4.6. □

Affine Kernel

Conditions of Proposition 4.3 can be easily verified. For fixed μ in $(-1,1)$, Q^- in $\mathbb{R}^{m\times m}$, Q^+ in $\mathbb{R}^{(n-m)\times(n-m)}$, we write

$$Q_0 = \texttt{diag}\left[Q^-, Q^+\right] , \tag{4.82}$$
$$Q_1 = \texttt{diag}\left[(1+\mu)Q^-, (1-\mu)Q^+\right] , \tag{4.83}$$

and we have the following proposition.

Proposition 4.7. *If there exist $\nu \in \mathbb{R}^+$, $\mu \in (-1,1)$ and symmetric positive definite matrices Q^- in $\mathbb{R}^{m\times m}$ and Q^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ such that*

$$Q_0\Lambda = \Lambda Q_0 , \tag{4.84}$$
$$Q_1\Lambda = \Lambda Q_1 , \tag{4.85}$$
$$-\mu Q_0 + \left(|\Lambda|^{-1} F\right)^\top Q_0 + Q_0\left(|\Lambda|^{-1} F\right) \leq -2\nu\,|\Lambda|^{-1} Q_0 , \tag{4.86}$$
$$-\mu Q_0 + \left(|\Lambda|^{-1} F\right)^\top Q_1 + Q_1\left(|\Lambda|^{-1} F\right) \leq -2\nu\,|\Lambda|^{-1} Q_1 , \tag{4.87}$$
$$\begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++}\end{bmatrix}^\top \begin{bmatrix} -Q^- & 0_{m,n-m} \\ 0_{n-m,m} & Q^+\end{bmatrix}\begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++}\end{bmatrix} \leq \begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m}\end{bmatrix}^\top \begin{bmatrix} -(1-\mu)Q^- & 0_{m,n-m} \\ 0_{n-m,m} & (1+\mu)Q^+\end{bmatrix} \times \begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m}\end{bmatrix} , \tag{4.88}$$

then conditions (4.22), (4.23), and (4.24) are satisfied for all $x \in [0,1]$.

Proof. Note that $\mathcal{Q}(x)$ satisfies

$$\begin{aligned}\mathcal{Q}(x) &= \texttt{diag}\left[(1+\mu x)Q^-, (1-\mu x)Q^+\right] \\ &= (1-x)Q_0 + xQ_1 .\end{aligned} \tag{4.89}$$

Since $1-x$ (resp. x) is positive and Q_0 (resp. Q_1) verifies (4.86) (resp. (4.87)) then $(1-x)Q_0$ (resp. xQ_1) verifies also (4.86) (resp. (4.87)). Hence, condition (4.23) holds. In the same manner condition (4.22) holds. The verification of (4.24) is immediate since (4.88) is equivalent. It concludes the proof of Proposition 4.7. □

Similarly to what have been done with the exponential kernel, we can find some tractable conditions for the checking of conditions of Theorems 4.2, 4.4. For fixed μ in $(-1,1)$, S^- in $\mathbb{R}^{n\times n}$, S^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ we write

$$\mathcal{S}(x) = \texttt{diag}\left[\left[(1+\mu x)^{-1} S^-, (1-\mu x)^{-1} S^+\right] , \quad x \in [0,1] , \tag{4.90}$$

and for $i,j = 0,1$,

$$\tilde{S}_{ij} = \texttt{diag}\left[(1+\mu i)^{-1} S^-, (1-\mu j)^{-1} S^+\right] , \tag{4.91}$$
$$\hat{S}_{ij} = \texttt{diag}\left[(1+\mu i)^{-1} S^-, (1-\mu j)\, S^+\right] , \tag{4.92}$$
$$\hat{I}^j_{n,m} = \texttt{diag}\left[(1+\mu j)\, I_m, (1-\mu j)\, I_{n-m}\right] . \tag{4.93}$$

Since (4.90) is no more affine in the (S^-, S^+)-plane we need the following lemma.

Lemma 4.2. *For all $x \in [0,1]$, $\mathcal{S}(x)$ lies in the convex hull formed by $\tilde{S}_{00}$, $\tilde{S}_{10}$, and $\tilde{S}_{11}$ if $\mu > 0$ and by $\tilde{S}_{00}$, $\tilde{S}_{01}$, and $\tilde{S}_{11}$ if $\mu < 0$.*

Proof. Without loss of generality we assume that $\mu > 0$. The map $\mathcal{S}(x)$ is a parameterized curve in the (S^-, S^+) plane. We can express it as an explicit curve. We have $(1+\mu x)^{-1}(1-\mu x)^{-1} = (1-\mu^2 x^2)^{-1}$, thus the expression of the explicit curve is given by

$$\tilde{h}(X) = \frac{X}{2X-1}\,, \quad X \in \tilde{\rho} = \left[(1+\mu)^{-1}, 1\right].$$

This curve is convex on this interval. Then,

$$\tilde{h}\left(\alpha\,(1+\mu)^{-1} + (1-\alpha)\right) \leq \alpha\tilde{g}\left((1+\mu)^{-1}\right) + (1-\alpha)\tilde{g}(1)\,, \quad \alpha \in (0,1)\;, \tag{4.94}$$

where $\tilde{g}(X) = \left(\frac{(1+\mu)^{-1}-1}{(1-\mu)^{-1}-1}\right) X + (1+\mu)^{-1} + 1$. When X lies in $\tilde{\rho}$, $\tilde{g}(X)$ describes the straight line between $\tilde{S}_{00}$ and $\tilde{S}_{11}$. Hence, for $X \in \tilde{\rho}$ one has $\tilde{h}(X) \leq \tilde{g}(X)$. Thus, $\mathcal{S}$ lies in the convex hull formed by $\tilde{S}_{00}$, $\tilde{S}_{10}$, and $\tilde{S}_{11}$. □

Proposition 4.8. *If there exist $\nu > 0$, μ in $(-1,1)$, a matrix U in $\mathbb{R}^{q\times n}$, and symmetric matrices S^- in $\mathbb{R}^{m\times m}$, S^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ such that, the following conditions hold,*

$$\tilde{S}_{ij}\Lambda = \Lambda\tilde{S}_{ij}\;, \tag{4.95}$$

$$\begin{bmatrix} \hat{S}_{01} & \left(T\hat{S}_{00} + LU\right)^\top \\ T\hat{S}_{00} + LU & \hat{S}_{10} \end{bmatrix} \geq 0\,, \tag{4.96}$$

$$-2\mu\tilde{S}_{ij} + \tilde{S}_{ij}F^\top |\Lambda|^{-1} + |\Lambda|^{-1} F\tilde{S}_{ij} \leq -2\nu\tilde{S}_{ij}\,|\Lambda|^{-1}\;, \tag{4.97}$$

for $(i,j) \in \{(0,0),(1,0),(1,1)\}$ if $\mu > 0$, for $(i,j) \in \{(0,0),(0,1),(1,1)\}$ if $\mu > 0$, then conditions (4.45), (4.46), and (4.47) are satisfied for all $x \in [0,1]$.

Proof. The inequality (4.96) is equivalent to (4.46). By Lemma 4.2 the constraint of equality (4.45) and LMI (4.46) are embedded in the polytope formed by the points $\tilde{S}_{00}$, $\tilde{S}_{10}$ and $\tilde{S}_{11}$ if $\mu > 0$ or by the points $\tilde{S}_{00}$, $\tilde{S}_{01}$ and $\tilde{S}_{11}$ if $\mu < 0$. Thus, conditions (4.45), (4.46), and (4.47) are satisfied for all $x \in [0,1]$. It concludes the proof of Proposition 4.8. □

Proposition 4.9. *Let an integer $\ell > 0$ be given. If there exist $\nu > 0$, μ in $(-1,1)$, matrices U_i in $\mathbb{R}^{p\times n}$, $i = 1,\ldots,\ell$, and positive definite symmetric matrices S^- in $\mathbb{R}^{m\times m}$, S^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ such that the following conditions hold*

$$\tilde{S}_{jk}\Lambda = \Lambda\tilde{S}_{jk}\,, \tag{4.98}$$

$$\begin{aligned} &\left(|\Lambda|^{-1} H - \mu I_n\right)\tilde{S}_{jk} + \tilde{S}_{jk}\left(|\Lambda|^{-1} H - \mu I_n\right)^\top \\ &+ \sum_{i=1}^{\ell} \min_{x\in[0,1]} \alpha_i(x) \left(\hat{I}^j_{n,m}\right)^{-1} U_i^\top B^\top |\Lambda|^{-1} + \sum_{i=1}^{\ell} \min_{x\in[0,1]} \alpha_i(x)\,|\Lambda|^{-1} BU_i \left(\hat{I}^j_{n,m}\right)^{-1} \\ &\leq -2\nu\,|\Lambda|^{-1}\,\tilde{S}_{jk}\,, \end{aligned} \tag{4.99}$$

$$\begin{aligned}
&\left(|\Lambda|^{-1} H - \mu I_n\right) \tilde{S}_{jk} + \tilde{S}_{jk} \left(|\Lambda|^{-1} H(x) - \mu I_n\right)^\top \\
&+ \sum_{i=1}^{\ell} \max_{x\in[0,1]} \alpha_i(x) \left(\hat{I}^j_{n,m}\right)^{-1} U_i^\top B^\top |\Lambda|^{-1} + \sum_{i=1}^{\ell} \max_{x\in[0,1]} \alpha_i(x) |\Lambda|^{-1} BU_i \left(\hat{I}^j_{n,m}\right)^{-1} \\
&\leq -2\nu |\Lambda|^{-1} \tilde{S}_{jk} ,
\end{aligned} \tag{4.100}$$

$$\begin{bmatrix} \hat{S}_{01} & \left(G\hat{S}_{00}\right)^\top \\ G\hat{S}_{00} & \hat{S}_{10} \end{bmatrix} \geq 0 , \tag{4.101}$$

for $(j,k) \in \{(0,0),(1,0),(1,1)\}$ if $\mu > 0$, for $(j,k) \in \{(0,0),(0,1),(1,1)\}$ if $\mu > 0$, then conditions (4.63), (4.64), and (4.65) are satisfied for all $x \in [0,1]$.

Proof. First of all the maximum and the minimum of α_i, $i = 1,\dots,\ell$, are well-defined since they are continuous functions. The inequality (4.101) corresponds to (4.65). By Lemma 4.2 the constraint of equality (4.63) and LMI (4.64) are embedded in the polytope formed by the points $\tilde{S}_{00}$, $\tilde{S}_{10}$, and $\tilde{S}_{11}$ if $\mu > 0$ or by the points $\tilde{S}_{00}$, $\tilde{S}_{01}$, and $\tilde{S}_{11}$ if $\mu < 0$. Thus, conditions (4.63), (4.64), and (4.65) are satisfied for all $x \in [0,1]$. It concludes the proof of Proposition 4.9. □

4.4.2 Spatially-Varying Case

We may generalize the previous results when Λ and F are both spatially varying and lie in a convex hull.

We assume that the parameterized matrix

$$W(x) = |\Lambda(x)|^{-1} F(x) , \tag{4.102}$$

lies for all $x \in [0,1]$ in the convex hull

$$\mathcal{W} := \left\{ W : W = \sum_{i=1}^{N} \alpha_i W_i,\ \sum_{i=1}^{N} \alpha_i = 1 \right\} , \tag{4.103}$$

for given matrices W_i, $i = 1,\dots,N$.

Exponential Kernel

Proposition 4.10. *If there exist $\nu > 0$, μ in $\mathbb{R}$, and diagonal positive definite matrices Q^- in $\mathbb{R}^{m\times m}$ and Q^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ such that*

$$-2\mu Q_{jk} + W_i^\top Q_{jk} + Q_{jk} W_i \leq -2\nu \left|\bar{\Lambda}\right|^{-1} Q_{jk} , \tag{4.104}$$

$$\begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++} \end{bmatrix}^\top \tilde{I}_{n,m} Q_{00} \begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++} \end{bmatrix} \leq \begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m} \end{bmatrix}^\top \tilde{I}_{n,m} Q_{11} \begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m} \end{bmatrix} , \tag{4.105}$$

where

$$\bar{\Lambda} = \boldsymbol{diag}\left[\min_{x\in[0,1]} \lambda_1(x), \dots, \min_{x\in[0,1]} \lambda_m(x), \max_{x\in[0,1]} \lambda_{m+1}(x), \dots, \max_{x\in[0,1]} \lambda_n(x) \right] , \tag{4.106}$$

for all $i = 1, \ldots, N$, *and* $(j,k) \in \{(0,0),(0,1),(1,1)\}$ *if* $\mu > 0$, $(j,k) \in \{(0,0),(1,0),(1,1)\}$ *if* $\mu < 0$, *then conditions* (4.8), (4.9), *and* (4.10) *are satisfied for all* $x \in [0,1]$.

Proof. Multiplying (4.104) by α_i and making the sum for $i = 1, \ldots, N$, we get

$$-2\mu Q_{jk} + W(x)^\top Q_{jk} + Q_{jk} W(x) \leq -2\nu \left|\bar{\Lambda}\right|^{-1} Q_{jk}\,, \tag{4.107}$$

for all $x \in [0,1]$, $(j,k) = \{(0,0),(0,1),(1,1)\}$ if $\mu > 0$ and $(j,k) = \{(0,0),(1,0),(1,1)\}$ if $\mu < 0$. Using Lemma 4.1 and the definition of $\bar{\Lambda}$ in (4.106), one gets (4.9). Condition (4.8) is automatically satisfied because of the diagonal form of Q^- and Q^+. It concludes the proof of Proposition 4.10. $\square$

For the controller design, in the case where H as well as B are spatially-varying we may generalize the above method. We assume that the parameterized matrices

$$|\Lambda(x)|^{-1} H(x)\,, \tag{4.108}$$

and

$$|\Lambda(x)|^{-1} B(x)\,, \tag{4.109}$$

lie, respectively, in

$$\mathcal{R} := \left\{ R : R = \sum_{i=1}^{M_1} \beta_i R_i,\ \sum_{i=1}^{M_1} \beta_i = 1 \right\}, \tag{4.110}$$

and

$$\mathcal{Z} := \left\{ Z : Z = \sum_{i=1}^{M_2} \gamma_i Z_i,\ \sum_{i=1}^{M_2} \gamma_i = 1 \right\}, \tag{4.111}$$

for given matrices R_i, Z_j, $i = 1, \ldots, M_1$, $j = 1, \ldots, M_2$. The following two propositions may be stated.

Proposition 4.11. *If there exist* $\nu > 0$, μ *in* $\mathbb{R}$, *a matrix* U *in* $\mathbb{R}^{q\times n}$, *and diagonal matrices* S^- *in* $\mathbb{R}^{m\times m}$, S^+ *in* $\mathbb{R}^{(n-m)\times(n-m)}$ *such that, the following conditions hold,*

$$\begin{bmatrix} \breve{S}_{01} & \left(T\breve{S}_{00} + LU\right)^\top \\ T\breve{S}_{00} + LU & \breve{S}_{10} \end{bmatrix} \geq 0\,, \tag{4.112}$$

$$-2\mu S_{jk} + S_{jk} W_i^\top + W_i S_{jk} \leq -2\nu S_{jk} \left|\bar{\Lambda}\right|^{-1}\,, \tag{4.113}$$

for all $i = 1, \ldots, N$, *and* $(j,k) \in \{(0,0),(1,0),(1,1)\}$ *if* $\mu > 0$, $(j,k) \in \{(0,0),(0,1),(1,1)\}$ *if* $\mu < 0$, *then conditions* (4.35), (4.36), *and* (4.37) *are satisfied for all* $x \in [0,1]$.

Proof. As in the proof of Proposition 4.10 we multiply (4.113) by α_i, and make the sum for $i = 1, \ldots, N$, getting

$$-2\mu S_{jk} + S_{jk} W(x)^\top + W(x) S_{jk} \leq -2\nu S_{jk} \left|\bar{\Lambda}\right|^{-1}\,, \tag{4.114}$$

for all $x \in [0,1]$, $(j,k) \in \{(0,0),(1,0),(1,1)\}$ if $\mu > 0$, $(j,k) \in \{(0,0),(0,1),(1,1)\}$ if $\mu < 0$. Using Lemma 4.1 and the definition of $\bar{\Lambda}$ in (4.106), one gets (4.37). Condition (4.35) is automatically satisfied because of the diagonal form of S^- and S^+. Condition (4.112) is exactly condition (4.36). It concludes the proof of Proposition 4.11. $\square$

Proposition 4.12. *Let an integer $\ell > 0$ be given. If there exist $\nu > 0$, μ in $\mathbb{R}$, matrices U_i in $\mathbb{R}^{p\times n}$, $i = 1, \dots, \ell$, and positive definite diagonal matrices S^- in $\mathbb{R}^{m\times m}$, S^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ such that the following conditions hold*

$$\begin{aligned}
&(R_{\iota_1} - \mu I_n) S_{jk} + S_{jk} (R_{\iota_1} - \mu I_n)^\top \\
&+ \sum_{i=1}^{\ell} \min_{x\in[0,1]} \alpha_i(x) \left(I_{n,m}^{jk}\right)^{-1} U_i^\top Z_{\iota_2}^\top + \sum_{i=1}^{\ell} \min_{x\in[0,1]} \alpha_i(x) Z_{\iota_2} U_i \left(I_{n,m}^{jk}\right)^{-1} \\
&\leq -2\nu \left|\bar{\Lambda}\right|^{-1} S_{jk}\,, \qquad (4.115)\\
&(R_{\iota_1} - \mu I_n) S_{jk} + S_{jk} (R_{\iota_1} - \mu I_n)^\top \\
&+ \sum_{i=1}^{\ell} \max_{x\in[0,1]} \alpha_i(x) \left(I_{n,m}^{jk}\right)^{-1} U_i^\top Z_{\iota_2}^\top + \sum_{i=1}^{\ell} \max_{x\in[0,1]} \alpha_i(x) Z_{\iota_2} U_i \left(I_{n,m}^{jk}\right)^{-1} \\
&\leq -2\nu \left|\bar{\Lambda}\right|^{-1} S_{jk}\,, \qquad (4.116)\\
&\begin{bmatrix} \breve{S}_{01} & \left(G\breve{S}_{00}\right)^\top \\ G\breve{S}_{00} & \breve{S}_{10} \end{bmatrix} \geq 0\,, \qquad (4.117)
\end{aligned}$$

for all $\iota_1 = 1, \dots, M_1$, $\iota_2 = 1, \dots, M_2$, and $(j,k) \in \{(0,0),(1,0),(1,1)\}$ if $\mu > 0$, $(j,k) \in \{(0,0),(0,1),(1,1)\}$ if $\mu < 0$, then conditions (4.63), (4.64), and (4.65) are satisfied for all $x \in [0,1]$.

Proof. We multiply (4.115) and (4.116) by β_{ι_1}, γ_{ι_2}, and make the sum for $\iota_1 = 1, \dots, M_1$, $\iota_2 = 1, \dots, M_2$ we get

$$\begin{aligned}
&(R(x) - \mu I_n) S_{jk} + S_{jk} (R(x) - \mu I_n)^\top \\
&+ \sum_{i=1}^{\ell} \min_{x\in[0,1]} \alpha_i(x) \left(I_{n,m}^{jk}\right)^{-1} U_i^\top Z(x)^\top + \sum_{i=1}^{\ell} \min_{x\in[0,1]} \alpha_i(x) Z(x) U_i \left(I_{n,m}^{jk}\right)^{-1} \\
&\leq -2\nu \left|\bar{\Lambda}\right|^{-1} S_{jk}\,, \qquad (4.118)\\
&(R(x) - \mu I_n) S_{jk} + S_{jk} (R(x) - \mu I_n)^\top \\
&+ \sum_{i=1}^{\ell} \max_{x\in[0,1]} \alpha_i(x) \left(I_{n,m}^{jk}\right)^{-1} U_i^\top Z(x)^\top + \sum_{i=1}^{\ell} \max_{x\in[0,1]} \alpha_i(x) Z(x) U_i \left(I_{n,m}^{jk}\right)^{-1} \\
&\leq -2\nu \left|\bar{\Lambda}\right|^{-1} S_{jk}\,. \qquad (4.119)
\end{aligned}$$

Using Lemma 4.1, the definition of $\bar{\Lambda}$ in (4.106) and the fact that any value of a continuous function can be expressed as a convex combination of its minimum and maximum, one gets (4.64). Condition (4.63) is automatically satisfied because of the diagonal form of S^- and S^+. Condition (4.117) is exactly condition (4.65). It concludes the proof of Proposition 4.12. $\square$

Affine Kernel

Similar results may be stated for the LMI-based stability conditions and LMI-based design conditions, derived from the Lyapunov function (4.6) with the affine kernel (4.21).

Proposition 4.13. *If there exist $\nu > 0$, μ in $(-1,1)$, and diagonal positive definite matrices Q^- in $\mathbb{R}^{m\times m}$ and Q^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ such that*

$$-\mu Q_0 + W_i^\top Q_0 + Q_0 W_i \leq -2\nu \left|\bar{\Lambda}\right|^{-1} Q_0\,, \tag{4.120}$$

$$-\mu Q_0 + W_i^\top Q_1 + Q_1 W_i \leq -2\nu \left|\bar{\Lambda}\right|^{-1} Q_1\,, \tag{4.121}$$

$$\begin{aligned}\begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++}\end{bmatrix}^\top \begin{bmatrix} -Q^- & 0_{m,n-m} \\ 0_{n-m,m} & Q^+\end{bmatrix} \begin{bmatrix} I_m & 0_{m,n-m} \\ G_{+-} & G_{++}\end{bmatrix} \leq \begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m}\end{bmatrix}^\top \begin{bmatrix} -(1-\mu)Q^- & 0_{m,n-m} \\ 0_{n-m,m} & (1+\mu)Q^+\end{bmatrix} \\ \times \begin{bmatrix} G_{--} & G_{-+} \\ 0_{n-m,m} & I_{n-m}\end{bmatrix}\,,\end{aligned} \tag{4.122}$$

for all $i = 1,\ldots,M$, where $\left|\bar{\Lambda}\right|$ is defined in (4.106) and Q_0, Q_1 are respectively defined in (4.82), (4.83), then conditions (4.22), (4.23), and (4.24) are satisfied for all $x \in [0,1]$.

Proof. The proof is similar than the proof of Proposition 4.10. Multiplying (4.120) and (4.121) by α_i and making the sum for $i = 1,\ldots,M$ we get

$$-\mu Q_0 + W(x)^\top Q_0 + Q_0 W(x) \leq -2\nu \left|\bar{\Lambda}\right|^{-1} Q_0\,, \tag{4.123}$$

$$-\mu Q_0 + W(x)^\top Q_1 + Q_1 W(x) \leq -2\nu \left|\bar{\Lambda}\right|^{-1} Q_1\,, \tag{4.124}$$

for all $x \in [0,1]$. Using the fact that

$$\mathcal{Q}(x) = (1-x)Q_0 + xQ_1\,,$$

and the definition of $\bar{\Lambda}$ in (4.106), one gets (4.23). Condition (4.22) is automatically satisfied because of the diagonal form of Q^- and Q^+. This concludes the proof of Proposition 4.13. □

Proposition 4.14. *If there exist $\nu > 0$, μ in $\mathbb{R}$, a matrix U in $\mathbb{R}^{q\times n}$, and diagonal matrices S^- in $\mathbb{R}^{m\times m}$, S^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ such that, the following conditions hold,*

$$\begin{bmatrix} \hat{S}_{01} & \left(T\hat{S}_{00} + LU\right)^\top \\ T\hat{S}_{00} + LU & \hat{S}_{10}\end{bmatrix} \geq 0\,, \tag{4.125}$$

$$-2\mu\tilde{S}_{jk} + \tilde{S}_{jk}W_i^\top + W_i\tilde{S}_{jk} \leq -2\nu\tilde{S}_{jk}\left|\bar{\Lambda}\right|^{-1}\,, \tag{4.126}$$

for all $i = 1,\ldots,N$, and $(j,k) \in \{(0,0),(1,0),(1,1)\}$ if $\mu > 0$, $(j,k) \in \{(0,0),(0,1),(1,1)\}$ if $\mu < 0$, then conditions (4.45), (4.46), and (4.47) are satisfied for all $x \in [0,1]$.

Proof. As in the proofs of Proposition 4.10, 4.11 we multiply (4.126) by α_i, and make the sum for $i = 1,\ldots,N$, getting

$$-2\mu\tilde{S}_{jk} + \tilde{S}_{jk}W(x)^\top + W(x)\tilde{S}_{jk} \leq -2\nu\tilde{S}_{jk}\left|\bar{\Lambda}\right|^{-1}\,, \tag{4.127}$$

for all $x \in [0,1]$, $(j,k) \in \{(0,0),(1,0),(1,1)\}$ if $\mu > 0$, $(j,k) \in \{(0,0),(0,1),(1,1)\}$ if $\mu < 0$. Using

Lemma 4.2 and the definition of $\bar{\Lambda}$ in (4.106), one gets (4.47). Condition (4.45) is automatically satisfied because of the diagonal form of S^- and S^+. Condition (4.46) is exactly condition (4.125). This concludes the proof of Proposition 4.14. $\square$

Proposition 4.15. *Let an integer $\ell > 0$ be given. If there exist $\nu > 0$, μ in $(-1, 1)$, matrices U_i in $\mathbb{R}^{p\times n}$, $i = 1, \dots, \ell$, and positive definite diagonal matrices S^- in $\mathbb{R}^{m\times m}$, S^+ in $\mathbb{R}^{(n-m)\times(n-m)}$ such that the following conditions hold*

$$
\begin{aligned}
&(R_{\iota_1} - \mu I_n)\, S_{jk} + S_{jk}\, (R_{\iota_1} - \mu I_n)^\top \\
&+ \sum_{i=1}^{\ell} \min_{x\in[0,1]} \alpha_i(x) \left(\hat{I}^{jk}_{n,m}\right)^{-1} U_i^\top Z_{\iota_2}^\top + \sum_{i=1}^{\ell} \min_{x\in[0,1]} \alpha_i(x) Z_{\iota_2} U_i \left(\hat{I}^{jk}_{n,m}\right)^{-1} \\
&\leq -2\nu \left|\bar{\Lambda}\right|^{-1} S_{jk}\,,
\end{aligned} \tag{4.128}
$$

$$
\begin{aligned}
&(R_{\iota_1} - \mu I_n)\, S_{jk} + S_{jk}\, (R_{\iota_1} - \mu I_n)^\top \\
&+ \sum_{i=1}^{\ell} \max_{x\in[0,1]} \alpha_i(x) \left(\hat{I}^{jk}_{n,m}\right)^{-1} U_i^\top Z_{\iota_2}^\top + \sum_{i=1}^{\ell} \max_{x\in[0,1]} \alpha_i(x) Z_{\iota_2} U_i \left(\hat{I}^{jk}_{n,m}\right)^{-1} \\
&\leq -2\nu \left|\bar{\Lambda}\right|^{-1} S_{jk}\,,
\end{aligned} \tag{4.129}
$$

$$
\begin{bmatrix} \hat{S}_{01} & \left(G\hat{S}_{00}\right)^\top \\ G\hat{S}_{00} & \hat{S}_{10} \end{bmatrix} \geq 0\,, \tag{4.130}
$$

for all $\iota_1 = 1, \dots, M_1$, $\iota_2 = 1, \dots, M_2$, and $(j,k) \in \{(0,0),(1,0),(1,1)\}$ if $\mu > 0$, $(j,k) \in \{(0,0),(0,1),(1,1)\}$ if $\mu < 0$, then conditions (4.63), (4.64), and (4.65) are satisfied for all $x \in [0,1]$.

Proof. Using Lemma 4.2 and the proof of Proposition 4.12 the result follows. $\square$

Since this section gathers a lot of results we propose a summary table.

			Exponential kernel	Affine kernel
Analysis		Non Spatially Varying	Proposition 4.4	Proposition 4.7
		Spatially Varying	Proposition 4.10	Proposition 4.13
Controller Design	Boundary	Non Spatially Varying	Proposition 4.5	Proposition 4.8
		Spatially Varying	Proposition 4.11	Proposition 4.14
	Distributed	Non Spatially Varying	Proposition 4.6	Proposition 4.9
		Spatially Varying	Propostion 4.12	Proposition 4.15

Table 4.1: Summary table of the results of Section 4.4.

4.5 Numerical Experiments

In this section, several examples are presented to illustrate the results of the chapter. All the solutions of the LMIs have been computed with the Multi-Parametric Toolbox (MPT) [60].

4.5.1 Stability Analysis, illustrating Proposition 4.4

Example 4.1. Let us consider the following matrices

$$\Lambda = \mathtt{diag}\left[-3, 1\right], \tag{4.131}$$

$$F = \begin{bmatrix} -1 & 0.2 \\ 1 & 0.2 \end{bmatrix}, \tag{4.132}$$

$$G = \begin{bmatrix} 0.2 & -0.3 \\ 0.6 & 0.1 \end{bmatrix}. \tag{4.133}$$

The matrix F in (4.132) is non-Hurwitz and the matrix G in (4.133) is such that $\rho(G) < 1$. This last property is classical for the stability analysis of linear and quasi-linear hyperbolic system [27], [35].

Figure 4.2 shows that the result obtained with only three points for the polytope is optimal. Indeed, the numerical ν obtained with three points is the same than with higher number of points. This result might be expected because all the constraints of the LMI are enclosed by the overapproximation with the polytope described by three points. The lower curve corresponds to the result of the algorithm when the objective is to maximize ν. In order to make this objective tractable, a relaxation on the right-hand side of the inequality (4.9) is made. The upper curve is the result of the algorithm when the objective is to minimize the trace of $\mathcal{Q}(0)$. Unexpectedly, the second algorithm gives a better ν than the first one.

The Lyapunov function (4.6) with the affine kernel (4.21) does not converge, which is not surprising when we look at LMIs (4.86) and (4.87). Indeed, to satisfy these latter LMIs we need a large positive μ, that means a μ near to be one, but in the same time increasing the μ will make the verification of (4.88) complex. Hence, the Lyapunov function (4.6) with the affine kernel (4.21) may not give results when the matrix F is not Hurwitz.

Example 4.2. Let us consider the following matrices

$$\Lambda = \mathtt{diag}\left[-1, 1\right], \tag{4.134}$$

$$F = \begin{bmatrix} -0.3 & 0.1 \\ 0.1 & -0.3 \end{bmatrix}, \tag{4.135}$$

$$G = \begin{bmatrix} 0.1 & -0.8 \\ 0.6 & -0.4 \end{bmatrix}. \tag{4.136}$$

In this example the matrix F in (4.135) is Hurwitz and the matrix G in (4.136) is contractive that is $\rho(G) < 1$.

Figure 4.3 shows that the algorithm for which the objective is to minimize the trace of $\mathcal{Q}(0)$ seems

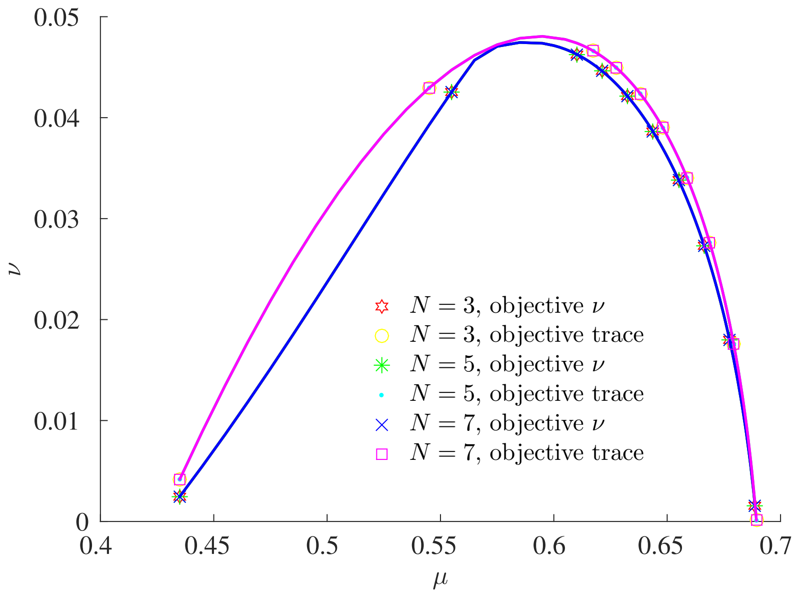

Figure 4.2: Evolution of ν as a function of μ for Example 4.1 for the gridding method depending on the number of points.

to give a better ν than the algorithm for which the objective is to maximize ν. Moreover it shows that the Lyapunov function (4.6) with the affine kernel (4.21) gives a better ν for the first μ than the exponential kernel (4.7), and gives solutions at some μ while the other kernel fails. Moreover, the shape of the curve obtained in this example is not the same as the one presented in Example 4.1. This comes from the fact that the matrix F is Hurwitz, hence increasing μ moves the eigenvalues of $|\Lambda|^{-1} F - \mu I_2$ in the left half-plane of $\mathbb{C}$, so it will increase the parameter ν. The algorithm stops due to LMI (4.10) which is no more solvable for large μ. Thus, this example illustrates also Proposition 4.2.

4.5.2 Controller Design, illustrating Theorems 4.1, 4.2, and 4.3

Example 4.3. Let us consider system (4.1)–(4.3) with

$$\Lambda = \mathtt{diag}\,[-1, 2]\,, \tag{4.137}$$

$$F = \begin{bmatrix} -0.1 & 0.1 \\ 0.5 & -0.8 \end{bmatrix}, \tag{4.138}$$

under the boundary control (4.34) where

$$T = \begin{bmatrix} -0.5 & 1 \\ 0.5 & 1 \end{bmatrix},$$
$$L^\top = \begin{bmatrix} 0.5 & -1 \end{bmatrix}.$$

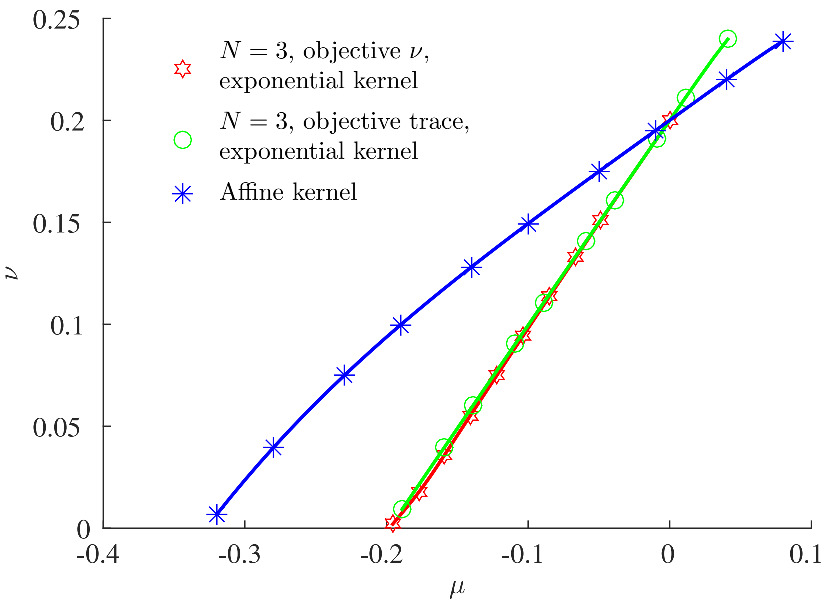

Figure 4.3: Evolution of ν as a function of μ for Example 4.2.

Let us choose $\nu = 0.1$. The design algorithm using the Proposition 4.5 gives

$$\mu = 0.1580\,, \tag{4.139}$$

$$K_B = \begin{bmatrix} 0.5596 & 0.7910 \end{bmatrix}\,, \tag{4.140}$$

which leads to the following boundary control

$$G = \begin{bmatrix} -0.2202 & 1.3955 \\ -0.0596 & 0.2090 \end{bmatrix}\,. \tag{4.141}$$

We check numerically the behavior of the solution of (4.1)–(4.3) with the matrix G given by (4.141), with a two-step variant of the Lax-Friedrichs (LxF) method [102]. The initial condition is chosen as

$$y^0(x) = \begin{bmatrix} \sqrt{2}\sin(\pi x) \\ \sqrt{2}\sin(2\pi x) \end{bmatrix}\,, \quad x \in [0,1]\,. \tag{4.142}$$

Figure 4.4 shows the evolution of the state of the system (4.1)–(4.3) with initial condition given by (4.142) and under the boundary control (4.141).

Example 4.4. Let us consider system (4.1)–(4.3) with

$$\Lambda = \texttt{diag}\,[-2, 1]\,, \tag{4.143}$$

$$G = \begin{bmatrix} 0.5 & -0.4 \\ 0.2 & 0.8 \end{bmatrix}\,, \tag{4.144}$$

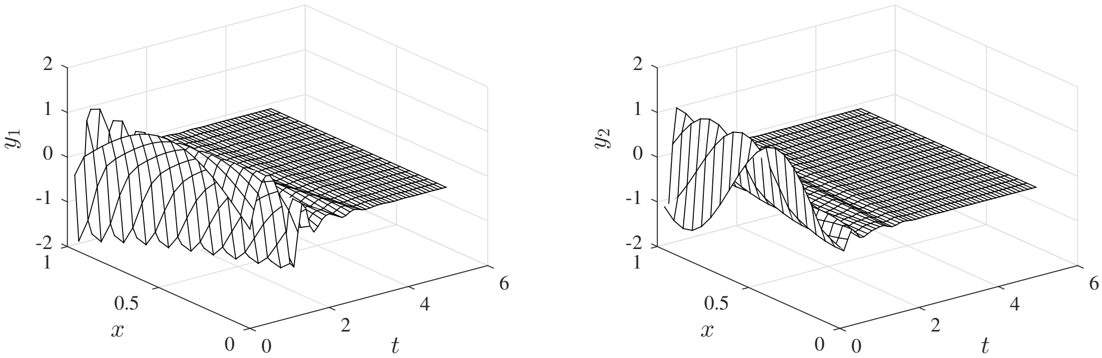

Figure 4.4: Evolution of the first component y_1 (left) and of the second component y_2 (right) of system (4.1)–(4.3) with Λ and F given, respectively, by (4.137) and (4.138) and initial condition given by (4.142) and under the boundary control (4.141).

under the distributed control (4.55) where

$$\begin{aligned} H &= \begin{bmatrix} -0.5 & 0.2 \\ 0.2 & 0.5 \end{bmatrix} , \\ B^\top &= \begin{bmatrix} 0.5 & 1 \end{bmatrix} , \\ (\ell, \alpha) &= (1, 1) . \end{aligned}$$

Numerically, ν is fixed to 0.3. The design algorithm using Proposition 4.6 gives gives

$$\begin{aligned} \mu &= 0.15 , \\ K_D &= \begin{bmatrix} -0.3130 & -1.1485 \end{bmatrix} , \end{aligned}$$

which leads to

$$F = \begin{bmatrix} -0.6565 & -0.3743 \\ -0.1130 & -0.6485 \end{bmatrix} . \tag{4.145}$$

Figure 4.5 shows the evolution of the state of the system (4.1)–(4.3) with initial condition given by (4.142) and under the distributed control (4.145).

4.5.3 Saint-Venant Equations

We illustrate Proposition 4.10 with the Saint-Venant equations for a prismatic horizontal channel, meaning that $S_b = 0$ in (1.31). We consider the linearization of the model around a non-uniform steady-state (H^*, V^*) as introduced in Subsection (1.2.3). The numerical values chosen are $L = 1$ km, $g = 9.81$ m.s^{-2}, $C = 0.002$ s^2.m^{-1}, $Q^* = H^* V^* = 1$ m^3.s^{-1} and the initial condition $V_0^* = 0.5$ m.s^{-1}. We compute numerically the matrices W_i for

$$\mu = 1.4 . \tag{4.146}$$

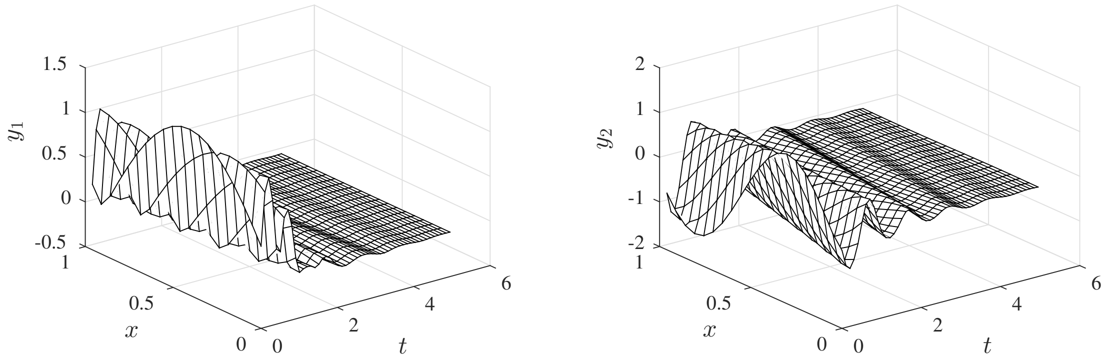

Figure 4.5: Evolution of the first component y_1 (left) and of the second component y_2 (right) of system (4.1)–(4.3) with Λ and G given, respectively, by (4.143) and (4.144) and initial condition given by (4.142) and under the distributed control (4.145).

The set $\mathcal{W}$ defined in (4.102) is described by 16 matrices. Furthermore, the channel is provided with some control devices allowing to assign the flow-rate on both sides of the channel, that is

$$Q_1(t) = H(t,0)V(t,0)\,, \tag{4.147}$$

$$Q_2(t) = H(t,L)V(t,L)\,. \tag{4.148}$$

Let us assume the controllers (4.147) and (4.148) are such that in the characteristic coordinates one has

$$\begin{bmatrix} y_1(t,1) \\ y_2(t,0) \end{bmatrix} = \begin{bmatrix} 0 & 0.2 \\ 0.3 & 0 \end{bmatrix} \begin{bmatrix} y_1(t,0) \\ y_2(t,1) \end{bmatrix}. \tag{4.149}$$

It must be stressed that all of this values may be prone to numerical errors since the explicit expression of V^* is unknown. Next, we compute matrices Q^- and Q^+ such that Proposition 4.10 holds. Conditions (4.104)–(4.105) are checked with

$$Q^- = 1\,, \tag{4.150}$$

$$Q^+ = 10.8171\,, \tag{4.151}$$

$$\nu = 4.8432\,. \tag{4.152}$$

Thus, this numerical method is effective on physical device.

5. Conclusion and Perspectives

5.1 Conclusion

Let us sum up the materials presented in this thesis. We have analyzed different aspects of linear hyperbolic systems in one space dimension.

In a first part, we have considered switching boundary controllers. By a Lyapunov analysis we have derived three switching rules which allow to stabilize in different sense systems of balance laws. The first one is a switching rule based on a steepest descent criteria, we have called it *argmin switching rule*. The drawback of this rule is the possible non-existence of solution globally in time, which have been shown to hold through the construction of a specific example. To encompass this limitation and to limit the number of switches per time unit we have added a hysteresis phenomenon to the previous equation, giving our second switching rule. We have proved that with this modification the system is well-posed (existence of a unique solution for all time). Moreover, we have shown that with a mild modification of this latter switching rule, it has ISS stability and robustness properties with respect to measurement noise. Keeping the idea to reduce the number of switches per time unit we have proposed to add a low-pass filter to hysteresis. The resulting switching rules have been shown to be well-posed. Besides, as for the hysteresis switching rule, a slightly modified version of the *low-pass filter* has been shown to get ISS stability and robustness properties with respect to measurement noise. A main interest of such switching rules is the possible stabilization of system for which every individual system is unstable, as it was illustrated. Moreover, switched boundary controllers seem to improve convergence of the system, as illustrated with the multi-reach governed by the Saint-Venant equations example.

In a second part, we have considered a 2×2 hyperbolic system of balance laws with anti-collocated boundary input and output. The first objective was to generate the trajectory for the system to guarantee that the output follows a desired given trajectory. The problem was solved by back-stepping. Then, we have moved on to the tracking issue which is the natural prolongation of such framework. To insure that the output follows the desired output we have used a PI-controller. Through a Lyapunov analysis we have proved that the L^2-norm of the resulting augmented system (states of the system and integrator) goes to zero. We have investigated the question of presence of distributed and boundary disturbances in the system. It was shown that when considering solution in L^2 the disturbance output are rejected in time-averaged, while considering solution in C^1 the disturbance output is rejected asymptotically for the C^0-norm. Finally, we have illustrated the trajectory generation with a wave equation with indefinite in-domain and boundary damping. The tracking result has been illustrated with the Aw-Rascle-Zhang equations.

In the third and last part of this thesis, we have considered linear hyperbolic systems with spatially varying coefficients. We have given LMI conditions for their stability. These LMIs are infinite due to the dependence on the space variable. Hence, we have reduced this complexity with an embedding of the constraints in polytopes. Then, we have considered the design of boundary and distributed controllers. The conditions are written as the conditions for stability. Hence, the overapproximation techniques have been shown to be effective in the same way. We have illustrated these results through academic examples and the Saint-Venant equations containing a friction coefficient.

5.2 Perspectives

Concerning switched systems of balance laws a lot of things have to be done. Indeed, it is a very new field of research for the control theory of PDEs community.

5.2.1 Other Switched Hyperbolic PDEs

First of all, the generalization of an output feedback law for systems where a switching appears also for the velocities, that is

$$\partial_t y(t,x) + \Lambda_{\sigma(t)} \partial_x y(t,x) = F_{\sigma(t)} y(t,x)\,,$$

is a challenging question from an existence of solution point of view. In the same way designing output feedback laws with a switched source term, when they are non-dissipative individually, seems a challenging question from the stability point of view.

Another idea for switched hyperbolic systems is to investigate systems of the form

$$\partial_t y(t,x) + \Lambda_{\sigma(t)} \left(y(t,x)\right) \partial_x y(t,x) = F_{\sigma(t)} y(t,x)$$

with boundary conditions

$$\begin{bmatrix} y^+(t,0) \\ y^-(t,1) \end{bmatrix} = G\left(\begin{bmatrix} y^+(t,1) \\ y^-(t,0) \end{bmatrix}\right).$$

In other words, it consists in analyzing a non-linear hyperbolic system for which switching appears in the velocities and source term but not at the boundary. Hence, discontinuities may not appear in the solution. The question is: is it possible to find a stabilizing switching for the system ?

Finally, a last idea is to consider the stabilization of a hyperbolic system coupled with a switched system in finite dimension. For instance, let us consider a system similar to the one considered in [1] and [3], that is a 2×2 hyperbolic systems perturbed by a switched finite dimensional linear systems at the left boundary

$$\begin{aligned}
\partial_t y_1(t,x) + \lambda_1(x)\partial_x y_1(t,x) &= \gamma_1(x) y_2(t,x) \\
\partial_t y_2(t,x) - \lambda_2(x)\partial_x y_2(t,x) &= \gamma_2(x) y_1(t,x)\,,
\end{aligned}$$

with

$$\begin{aligned}
y_1(t,0) &= q y_2(t,0) + C_i X(t) \\
y_2(t,1) &= U(t) \\
\dot{X}(t) &= A_i X(t)\,,
\end{aligned}$$

where $i \in \mathcal{I} := \{1,\ldots,N\}$, $A_i \in \mathbb{R}^{n\times n}$, $C_i \in \mathbb{R}^{1\times n}$ for all $i \in \mathcal{I}$. Moreover, we may suppose that the value of X is reset at the time instant of switch t_k, that is

$$X\left(t_k^+\right) = D_i X\left(t_k^-\right)\,,$$

with $D_i \in \mathbb{R}^{n\times n}$.

5.2.2 Supervisory Control and Estimation

In finite dimension, switching is very practical for system where large uncertainties exist. In this case the classical adaptive control is irrelevant or not very effective. Hence, a *supervisory* control is adopted, for instance, see [110], [61], [76], and the references therein. The same idea could be extended for PDEs. For instance, let us consider the 2×2 hyperbolic system

$$\begin{aligned}
\partial_t y_1(t,x) + \varepsilon_1(x)\partial_x y_1(t,x) &= \gamma_1(x) y_2(t,x) \\
\partial_t y_2(t,x) - \varepsilon_2(x)\partial_x y_2(t,x) &= \gamma_2(x) y_1(t,x)
\end{aligned}$$

with boundary conditions

$$\begin{aligned}
y_1(t,x) &= q^* y_2(t,x) \\
y_2(t,x) &= U(t)\,,
\end{aligned}$$

where q^* is an unknown parameter lying in some "large" range $\left[\underline{q},\overline{q}\right]$. The control U may be designed with backstepping for instance, but the solution of the backstepping kernels depends on the value of q^*. Thus, the system may be unstable for large value of $\gamma_1(x)$ and $\gamma_2(x)$. The idea consists in partitioning the range $\left[\underline{q},\overline{q}\right]$ by N values q_i, $i = 1,\ldots,N$. We aim at constructing a supervisory control with the only measurement of $y_2(t,0)$. Let us define the y_1^i, y_2^i as the solution of the following system

$$\begin{aligned}
\partial_t y_1^i(t,x) + \varepsilon_1(x) y_1^i(t,x) &= \left(-1+\gamma_1(x)\right) y_2^i(t,x) \\
\partial_t y_2^i(t,x) - \varepsilon_2(x) y_2^i(t,x) &= \left(-1+\gamma_2(x)\right) y_1^i(t,x)\,,
\end{aligned}$$

with boundary conditions

$$
\begin{aligned}
y_1^i(t,0) &= q_i y_2^i(t,0) \\
y_2^i(t,1) &= U(t)\,.
\end{aligned}
$$

These latter systems are estimator systems. The estimation errors

$$
\begin{aligned}
e_1^{p^*}(t,x) &= y_1^{p^*}(t,x) - y_1(t,x) \\
e_2^{p^*}(t,x) &= y_2^{p^*}(t,x) - y_2(t,x)\,,
\end{aligned}
$$

satisfy

$$
\begin{aligned}
\partial_t e_1^{p^*}(t,x) + \varepsilon_1(x)\partial_x e_1^{p^*}(t,x) &= -e_2^{p^*}(t,x) \\
\partial_t e_2^{p^*}(t,x) - \varepsilon_2(x)\partial_x e_2^{p^*}(t,x) &= -e_1^{p^*}(t,x)\,,
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
e_1^{p^*}(t,0) &= 0 \\
e_2^{p^*}(t,1) &= 0\,.
\end{aligned}
$$

This system is exponentially stable no matter what the control $U(t)$ is. The supervisory control may be defined as taking the value q_i such that the error estimation

$$
\mu_i(t) = \int_0^t \left(e_2^i(s,0)\right)^2 ds
$$

is minimized. To reduce the complexity of the computation of μ_i, a multi-estimator may be designed as in finite dimension, see, for instance, [61], [76]. Let us define two new systems, the first one is

$$
\begin{aligned}
\partial_t z_1(t,x) + \varepsilon_1(x)\partial_x z_1(t,x) &= \left(-1+\gamma_1(x)\right) z_2(t,x) \\
\partial_t z_2(t,x) - \varepsilon_2(x)\partial_x z_2(t,x) &= \left(-1+\gamma_2(x)\right) z_1(t,x)\,,
\end{aligned}
$$

with the boundary conditions

$$
\begin{aligned}
z_1(t,0) &= y_2(t,0) \\
z_2(t,1) &= 0\,.
\end{aligned}
$$

The second one is

$$
\begin{aligned}
\partial_t w_1(t,x) + \varepsilon_1(x)\partial_x w_1(t,x) &= \left(-1+\gamma_1(x)\right) w_2(t,x) \\
\partial_t w_2(t,x) - \varepsilon_2(x)\partial_x w_2(t,x) &= \left(-1+\gamma_2(x)\right) w_1(t,x)\,,
\end{aligned}
$$

with boundary conditions

$$w_1(t,0) = 0$$
$$w_2(t,1) = U(t)\,.$$

We get

$$\begin{bmatrix} y_1^i(t,x) \\ y_2^i(t,x) \end{bmatrix} = \begin{bmatrix} q_i z_1(t,x) + w_1(t,x) \\ q_i z_2(t,x) + w_2(t,x) \end{bmatrix}.$$

A possible control U_i for the mode i would be

$$U_i(t) = \int_0^1 K_i^{\beta\alpha}(1,\xi) y_1^i(t,\xi) d\xi + \int_0^1 K_i^{\beta\beta}(1,\xi) y_2^i(t,\xi) d\xi\,,$$

where $K_i^{\beta\alpha}$ and $K_i^{\beta\beta}$ are the kernels of the backstepping transformation associated with the value q_i, see Chapter 3. The natural questions which may be handled are: the existence of solutions, their regularity, the stabilization, the switching phenomena (does it stop ?), etc.

Recently in [18], the supervisory strategy has been adopted for the parameter and state estimations. In the same idea, a system as the one tackled in [34] could be treated. More precisely, these systems take the following form

$$\partial_t y(t,x) + \Lambda \partial_x y(t,x) = F y(t,x)$$

where $\Lambda = \mathtt{diag}\,[\lambda_1, \dots, \lambda_n, -\mu]$ with $\lambda_i > 0$ for $i = 1, \dots, n$ and $\mu > 0$ and $F[n+1, n+1] = 0$. The boundary conditions at $x = 0$ for state associated to the positive velocities are

$$\begin{bmatrix} y_1(t,0) \\ \dots \\ y_n(t,0) \end{bmatrix} = \begin{bmatrix} q_1 y_{n+1}(t,0) \\ \vdots \\ q_n y_{n+1}(t,0) \end{bmatrix} + \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix},$$

and the last boundary condition, for the state whose velocity is negative, is

$$y_{n+1}(t,1) = \sum_{i=1}^n \rho_i y_i(t,1) + U(t)\,,$$

where $U(t)$ is the control input. The θ_i are supposed to lie in a range $\left[\underline{\theta}_i, \overline{\theta}_i\right]$, $i = 1, \dots, n$ and are uncertain parameters. In [18], the problem is solved with backstepping, and Gradient Descent algorithm. The idea that we have is that a supervisory strategy as described above could be more effective for a large range for the θ_i. Moreover, one could imagine to use methods as developed in [18] with a dynamical gridding of the parameter sets to obtain more accurate estimates.

5.2.3 Perspectives for the Lyapunov Analysis

The perspectives above show that a wide horizon for switched hyperbolic systems exists. Still, for unswitched hyperbolic systems it remains works to be done. For instance, the perspectives for the work of Chapter 3 are to tighten the conditions on k_P and k_I such that the conditions of Theorem 3.2 hold. A possibility would be to pass to a domain frequency approach. Nonetheless, in the optic of applying PI-controller for the non-linear version of the system of Chapter 3, the frequency domain approach would be irrelevant. Moreover, the form of Lyapunov function (diagonal, non-diagonal, degree,...) for system of conservation laws is not yet understood. Some work should be led on these crucial questions.

Appendix

Lemma A (Gronwall's Lemma [65]). *Suppose* $y \in C^1([0,T];\mathbb{R})$, $\psi \in C([0,T];\mathbb{R})$ *satisfy*

$$y'(t) \leq cy(t) + \psi(t)\,, \quad 0 \leq t \leq T\,, \tag{1}$$

for some $c \geq 0$. *Then*

$$y(t) \leq e^{ct}\left(y(0) + \int_0^t |\psi(s)|\,ds\right)\,, \quad 0 \leq t \leq T\,. \tag{2}$$

Lemma B (Picard's Lemma [65]). *Let* $\eta^k(t)$, $k \in \mathbb{N}$ *denote a sequence of nonnegative continuous functions which satisfy the inequalities*

$$\eta^{k+1}(t) \leq a + b\int_0^t \eta^k(s)ds\,, \quad 0 \leq t \leq T\,, \tag{3}$$

with nonnegative constant a, b. *Then*

$$\eta^k(t) \leq a\sum_{i=0}^{k-1}\frac{b^i t^i}{i!} + \frac{b^k t^k}{k!}\max_{0\leq s\leq t}\eta^0(s)\,, \tag{4}$$

for $0 \leq t \leq T$ *and* $k \in \mathbb{N}$. *In particular, the sequence* $\eta^k(t)$, $0 \leq t \leq T$, *is uniformly bounded. If* $a = 0$, *then the sequence converges uniformly to zero.*

Theorem A (Arzelà-Ascoli Theorem [14]). *Let* K *be a compact metric space and let* $\mathcal{H}$ *be a bounded subset of* $C(K;\mathbb{R})$. *Let us assume that* $\mathcal{H}$ *is uniformly equicontinuous that is*

$$\forall\varepsilon > 0 \quad \exists\delta > 0 \quad \text{such that} \quad d(x_1,x_2) < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon \quad \forall f \in \mathcal{H}\,. \tag{5}$$

Then $\mathcal{H}$ *is relatively compact in* $C(K;\mathbb{R})$.

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Control of hyperbolic systems by Lyapunov analysis

Abstract:

In this thesis we have considered different aspects for the control of hyperbolic systems.

First, we have studied switched hyperbolic systems. They contain an interaction between a continuous and a discrete dynamics. Thus, the continuous dynamics may evolve in different modes: these modes are imposed by the discrete dynamics. The change in the mode may be controlled (in case of a closed-loop system), or may be uncontrolled (in case of an open-loop system). We have focused our interest on the former case. We procedeed with a Lyapunov analysis, and construct three switching rules. We have shown how to modify them to get robustness and ISS properties. We have shown their effectiveness with numerical tests.

Then, we have considered the trajectory generation problem for 2×2 linear hyperbolic systems. We have solved it with backstepping. Then, we have considered the tracking problem with a Proportionnal-Integral controller. We have shown that it stabilizes the error system around the reference trajectory with a new non-diagonal Lyapunov function. The integral action has been shown to be able to compensate in-domain, as well as boundary disturbances.

Finally, we have considered numerical aspects for the Lyapunov analysis. The conditions for the stability and design of controllers by quadratic Lyapunov functions involve an infinity of matrix inequalities. We have shown how to reduce this complexity by polytopic embeddings of the constraints.

Many obtained results have been illustrated by academic examples and physically relevant dynamical systems (as Shallow-Water equations and Aw-Rascle-Zhang equations).

Keywords :

Hyperbolic systems; switched systems; Lyapunov function; LMI.