Filtration enlargement and applications to finance

Ricardo Romo Romero

Université Paris-Saclay
École doctorale de mathématiques Hadamard (EDMH, ED 574)

Établissement d'inscription : Université d'Evry-Val d'Essonne

Laboratoire d'accueil : Laboratoire de mathématiques et modélisation d'Évry, UMR 8071 CNRS-INRA

THÈSE DE DOCTORAT EN MATHÉMATIQUES

Spécialité : Mathématiques appliquées

Ricardo ROMO ROMERO

Grossissement de filtrations et applications à la finance

Date de soutenance : 4 Juillet 2016

Après avis des rapporteurs : Ying JIAO (ISFA Lyon)
Mohammed MNIF (ENIT Tunis)

Jury de soutenance :

Christophette BLANCHET-SCALLIET	(Ecole Centrale de Lyon)	Examinateur
Etienne CHEVALIER	(Université d'Evry)	Codirecteur de thèse
Caroline HILLAIRET	(ENSAE)	Président
Monique JEANBLANC	(Université d'Evry)	Directrice de thèse
Ying JIAO	(ISFA Lyon)	Rapporteur
Idris KHARROUBI	(Université Paris Dauphine)	Examinateur
Thomas LIM	(ENSIIE)	Codirecteur de thèse
Mohammed MNIF	(ENIT Tunis)	Rapporteur
Anthony RÉVEILLAC	(INSA Toulouse)	Examinateur

NNT : 2016SACLE012

À la memoire de...
mon père, *Julián ROMO HERRERA* (1942-2015),
mon frère, *Julián ROMO ROMERO (1973-2010)* et
ma sœur *Erika ROMO ROMERO (1975-2008)*.

Remerciements

Après ces quarante-six mois de parcours doctoral, il est temps de remercier tout le monde !

Tout d'abord, Je voudrais remercier infiniment à Monique JEANBLANC, ma directrice de thèse, qui, du début jusqu'à la fin, a suivi ce travail avec une très grande maîtrise et beaucoup d'humanité. Elle a été la meilleure directrice de thèse que je n'aurais jamais imaginé. Elle a été toujours pour moi de bon conseil, me consacrant de son temps Je lui suis entièrement reconnaissant pour le temps conséquent qu'elle a su m'accorder, ses qualités pédagogiques, scientifiques, mais également sa franchise et sa sympathie. Je lui adresse toute ma gratitude pour tout cela.

Je n'ai pas de mots pour exprimer ma gratitude à Etienne CHEVALIER et Thomas LIM, co-encadrants de cette thèse, qui m'ont beaucoup appris et beaucoup apporté, sur le plan scientifique. Qu'ils soient conscients que ce travail ne serait pas là sans eux.

Je voudrais exprimer toute ma gratitude à Ying JIAO et Mohammed MNIF, d'avoir accepté d'être rapporteurs de ce manuscrit.

Je remercie Christophette BLANCHET-SCALLIET, Caroline HILLAIRET, Idris KHARROUBI et Anthony RÉVEILLAC de m'avoir fait l'honneur d'accepter de participer au jury de cette thèse et également d'examiner ce travail.

Merci aux enseignants-chercheurs de l'équipe qui ont été présents pour me conseiller : Abass, Arnaud, Christophe, Claudio, Diego, Libo, Lucilla, Pierre-Gilles, Thomas KRUSE, Stéphane CREPEY, Stéphane MENOZZI, Valeria et Vicent.

Des remerciements amicaux vont aux doctorants et ex-doctorants-docteurs de l'équipe pour avoir partagé ce bout de chemin ensemble : Anna AKSAMIT, Dongli, Florian, Igor, Jing, Kawther, Mai, Nam, Mohammed SEDDIK, Quentin et Yannick.

Je remercie spécialement Oscar, mon colloc, pour son amitié infaillible et pour avoir toujours été là pour moi.

Je porte une attention toute particulière à Valérie GONTIER-PICOT, Thierry MILLANT, El Maouloud OULD BABA, qui m'ont toujours apporté tout leur soutien et témoigné de leur gentillesse.

Je remercie vivement à Carolina, Jeff et Anthony pour leur compréhension et leur soutien.

Je souhaite remercier ma famille, Aimé, Camila, Chapu, Cynthia, Ime, Ivanna, Jaime, July, Lili, Jose Luis, Marcos, Yasmin, mes oncles et cousins pour leur soutien inestimable, en dépit de mon indisponibilité prolongée. Mes remerciements vont tout particulièrement à mon oncle Gil et à mon oncle Fer pour leurs soutien et aide inconditionnels.

Je tiens toutefois à remercier nommément ma maman pour son aide inconditionnelle tout au long de mon parcours scolaire puis universitaire, sans elle je ne serais pas allé aussi loin.

Mes amis (YOLO, Adán, Betty, Blanca, Leslie, Jorge, Paola, Sauri, Ulises...), je m'excuse de ne pas vous nommer tous ici tant la liste serait longue, mais vous savez ma reconnaissance.

Je suis infiniment reconnaissant à celle qui partage ma vie, Anna Pi, pour sa compréhension et sa patience. Merci pour tous ces bons moments passés, et pour ceux à venir. Jeg elsker dig !

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Introduction générale

Cette thèse se compose de quatre parties indépendantes. Le fil conducteur de celle-ci est le grossissement de filtration.

Dans la première partie, nous présentons des résultats classiques de grossissement de filtration en temps discret : le théorème de décomposition des semimartingales de Doob, la décomposition multiplicative pour les semimartingales positives et la décomposition de Kunita-Watanabe. Nous introduisons les processus logarithmique et exponentiel, et on introduit l'arbitrage dans le cadre du grossissement de filtration. Nous étudions quelques exemples dans le cadre du grossissement initial de filtration. Dans le cadre du grossissement progressif nous donnons des conditions pour obtenir la propriété d'immersion des martingales. Nous donnons également diverses caractérisations des pseudo temps d'arrêt et des propriétés pour les temps honnêtes.

Dans la deuxième partie, nous nous intéressons à la détermination du prix de produits à annuités variables dans le cadre de l'assurance vie. Pour cela nous considérons deux modèles, dans ces deux modèles nous considérons que le marché est incomplet et nous adoptons l'approche par prix d'indifférence. Dans le premier modèle nous supposons que l'assuré procède à des retraits aléatoires et nous calculons la prime d'indifférence par des méthodes standards en contrôle stochastique. Nous sommes ramenés à résoudre des équations différentielles stochastiques rétrogrades (EDSR) avec un saut. Nous fournissons un théorème de vérification et nous donnons les stratégies optimales associées à nos problèmes de contrôle. De ceux-ci, nous tirons une méthode de calcul pour obtenir la prime d'indifférence. Dans le second modèle nous proposons la même approche que dans le premier modèle mais nous supposons que l'assuré effectue des retraits qui correspondent au pire cas pour l'assureur. Nous sommes alors amenés à traiter un problème de max-min.

Dans la troisième partie, nous étudions la relation des solutions à des EDSR dans deux filtrations différentes. Nous définissons une EDSR dans une filtration grossie $\mathbb{G}$ ayant pour solution $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, U^{\mathbb{G}}, M^{\mathbb{G}^\perp})$. Nous étudions l'EDSR dans la filtration initiale $\mathbb{F}$ correspondant à la première EDSR définie par la projection de la solution $Y^{\mathbb{G}}$ sur $\mathbb{F}$, nous notons $(Y^{\mathbb{F}}, Z^{\mathbb{F}}, U^{\mathbb{F}})$ la solution de cette deuxième EDSR. Nous étudions alors la relation entre ces deux solutions. Nous appliquons ces résultats au cas du prix d'indifférence dans les deux filtrations. Le but de cette section est de trouver le prix d'indifférence de l'information, à savoir le prix auquel un agent aurait le même niveau d'utilité attendue en utilisant des informations supplémentaires.

Dans la quatrième partie, nous considérons les équations différentielles stochastiques rétrogrades avancées EDSRAs avec un saut. Nous étudions l'existence et l'unicité d'une solution à ces EDSRAs. Pour cela nous utilisons la décomposition des processus à sauts liée au grossissement progressif de filtration pour nous ramener à l'étude d'EDSRAs browniennes avant et après le temps de saut.

Dans la suite de cette introduction, nous allons exposer la problématique de chaque chapitre ainsi que les résultats importants obtenus.

Grossissement de filtration en temps discret

Le premier chapitre de cette thèse est une version étendue de l'article *"Enlargement of filtration in discrete time"* à paraître dans *Risks and Stochastics, Ragnar Norberg at 70*, [BSJRR16]. Dans ce chapitre, nous présentons des résultats sur le grossissement de filtration dans un cadre de temps discret. Il convient de noter que les résultats obtenus en grossissement de filtration dans des modèles en temps continu peuvent être utilisés dans le cadre du temps discret : si X est un processus à temps discret, on introduit un processus càdlàg (continu à droite, pourvu de limites à gauche) $\widehat{X} = (\widehat{X}_t)_{t\in[0,\infty)}$ en posant

$$\widehat{X}_t = \sum_{n=0}^{\infty} X_n \mathbb{1}_{t\in[n,n+1)} \ , \quad \forall t \in [0,\infty) \ .$$

Il est à signaler que de nombreux résultats en temps continu sont obtenus sous l'hypothèse que toutes les $\mathbb{F}$-martingales sont continues (ce qui n'est pas le cas en temps discret) et que la généralisation au cas général nécessite des développements non triviaux et longs.

Dans ce chapitre, nous considérons un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$ muni de deux filtrations en temps discret $\mathbb{F} = (\mathcal{F}_n)_{n\geq 0}$ et $\mathbb{G} = (\mathcal{G}_n)_{n\geq 0}$ telles que $\mathbb{F} \subset \mathbb{G}$. Dans ce contexte, le point important est que toute $\mathbb{F}$-martingale $X = (X_n, n \geq 0)$ est une $\mathbb{G}$-semimartingale (propriété qui demande des conditions particulières, connues sous le nom d'hypothèse $(\mathcal{H}')$, pour être vérifiée en temps continu). En temps discret, ce résultat important est une conséquence immédiate du Théorème de décomposition de Doob qui permet de décomposer tout processus adapté intégrable en la somme d'une martingale et d'un processus prévisible.

L'intérêt de notre étude est de fournir des preuves simples des formules de décomposition des $\mathbb{F}$-martingales vues en tant que $\mathbb{G}$-semimartingales, et de contribuer à comprendre les formules générales obtenues dans la littérature en temps continu. Nous nous intéressons aux deux types de grossissements présents dans la littérature en temps continu : le grossissement initial et le grossissement progressif. Le cas général (grossissement de $\mathbb{F}$ par une filtration $\mathbb{H}$), très peu abordé en temps continu, dépasse le cas de notre étude.

Dans le cadre du grossissement initial de filtration, ξ est une variable aléatoire et on note $\mathbb{F}^{(\xi)}$ la filtration $\mathbb{F}$ grossie initialement par ξ, c'est-à-dire, la plus petite filtration contenant $\mathbb{F}$ et telle que ξ est $\mathcal{F}_0^{(\xi)}$-mesurable.

Le résultat classique en temps continu est le suivant : Sous l'hypothèse d'absolue continuité de Jacod ([Jac85]), i.e. si pour $t \geq 0$ la loi conditionnelle de ξ est absolument continue par rapport à la loi de ξ, soit $\mathbb{P}(\xi \in du|\mathcal{F}_t) = p_t(u)\mathbb{P}(\xi \in du)$, toute $\mathbb{F}$-martingale X est une $\mathbb{G}$-semimartingale, avec la décomposition

$$X_t = X_t^{\mathbb{G}} + \int_0^t \frac{d\langle X, p_s(u)\rangle^{\mathbb{F}}|_{\xi=u}}{p_{s-}(\xi)} \ , \ \forall t \in [0,\infty) \ ,$$

où $X^{\mathbb{G}}$ est une $\mathbb{G}$-martingale.

Dans la Section 1.2.2, en calculant la $\mathbb{F}^{(\xi)}$-décomposition de Doob d'une $\mathbb{F}$-martingale X dans le cas où ξ est une variable aléatoire prenant des valeurs dans $\mathbb{Z}$ avec $p_n(k) = \mathbb{P}(\xi = k|\mathcal{F}_n)$ pour tout $k \in \mathbb{Z}$ et $n \geq 0$, nous obtenons facilement, après avoir défini le crochet prévisible de deux martingales X et Y comme la partie prévisible de la semi-martingale XY, l'analogue de la formule en temps continu : *Toute $\mathbb{F}$-martingale est une $\mathbb{G}$-semimartingale admettant la décomposition*

$$X_0 = X_0^{\mathbb{G}} \quad et \quad X_j = X_j^{\mathbb{G}} + \sum_{n=1}^{j} \frac{\Delta\langle X, p(k)\rangle_n^{\mathbb{F}}}{p_{n-1}(k)} \mathbb{1}_{\{\xi=k\}} \;, \quad \forall j \geq 1 \;,$$

où $X^{\mathbb{G}}$ est une $\mathbb{G}$-martingale.

Nous étudions également un cas semblable aux ponts de Lévy et étudions les possibilités d'arbitrage.

Le grossissement progressif $\mathbb{G}$ de $\mathbb{F}$ par τ, où τ est un temps aléatoire (une variable aléatoire positive) est la plus petite filtration (continue à droite pour le temps continu) contenant $\mathbb{F}$ et faisant de τ un $\mathbb{G}$-temps d'arrêt. La plupart des résultats en temps continu sont établis sous l'hypothèse que toutes les $\mathbb{F}$-martingales sont continues ou que le temps aléatoire évite les $\mathbb{F}$-temps d'arrêt (voir par exemple le *survey* de Nikeghbali et Yor [NY05, Theorem 1.(v)]), et leur généralisation est complexe. En temps discret, on ne peut faire aucune de ces hypothèses, puisque les martingales sont discontinues et les temps aléatoires à valeurs dans l'ensemble des entiers n'évitent pas les $\mathbb{F}$-temps d'arrêt.

Nous rappelons le résultat de décomposition des $\mathbb{F}$-martingales arrêtées au temps τ (voir Jeulin [JY78a]) en temps continu : *Toute $\mathbb{F}$-martingale X arrêtée en τ est une $\mathbb{G}$-semimartingale admettant la décomposition*

$$X_t^\tau = X_t^{\mathbb{G}} + \int_0^{t\wedge\tau} \frac{1}{G_{s-}} d\langle X, \widetilde{m}\rangle_s^{\mathbb{F}} \;, \quad \forall t \geq 0 \;,$$

où $X^{\mathbb{G}}$ est une $\mathbb{G}$-martingale, $G_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$ et $\widetilde{G}_t = \mathbb{P}(\tau \geq t|\mathcal{F}_t)$ pour tout $t \geq 0$. Ici $\widetilde{G} = \widetilde{m} - \widetilde{a}$, où $\widetilde{m}$ est une $\mathbb{F}$-martingale et $\widetilde{a}$ est croissant et continu à gauche.

En temps discret, en notant $G = m - a$ et $\tilde{G} = \widetilde{m} - \widetilde{a}$ les décompositions de Doob des surmartingales d'Azéma G et $\widetilde{G}$, définies par $G_n := \mathbb{P}(\tau > n|\mathcal{F}_n)$ et $\widetilde{G}_n := \mathbb{P}(\tau \geq n|\mathcal{F}_n)$, pour tout $n \geq 0$, nous avons obtenu (voir Prop. 1.3.14) le résultat suivant : *Toute $\mathbb{F}$-martingale X arrêtée au temps τ est une $\mathbb{G}$-semimartingale, admettant la décomposition*

$$X_n^\tau = X_n^{\mathbb{G}} + \sum_{k=1}^{n\wedge\tau} \frac{\Delta\langle X, \widetilde{m}\rangle_k^{\mathbb{F}}}{G_{k-1}} \;, \quad \forall n \geq 1 \;,$$

où $X^{\mathbb{G}}$ est une $\mathbb{G}$-martingale.

Nous étudions ensuite deux types particuliers de temps aléatoires, les temps honnêtes et les pseudo-temps d'arrêt. Nous rappelons la définition de temps honnêtes en temps continu et la formule de décomposition (voir Barlow [Bar78] et Jeulin [Jeu80] pour plus d'information). Un temps aléatoire τ est honnête si pour tout $t \geq 0$, il existe une variable aléatoire $\mathcal{F}_t$-mesurable τ_t, telle que $\tau \mathbb{1}_{\{\tau \leq t\}} = \tau_t \mathbb{1}_{\{\tau \leq t\}}$. Si τ est un temps honnête et X une $\mathbb{F}$-martingale alors,

$$X_t = X_t^{\mathbb{G}} + \int_0^{t\wedge\tau} \frac{1}{G_{s-}} d\langle \widetilde{m}, X\rangle_s - \int_\tau^t \frac{1}{1 - G_{s-}} d\langle \widetilde{m}, X\rangle_s \;, \quad \forall t \geq 0 \;,$$

où $X^{\mathbb{G}}$ est une $\mathbb{G}$-martingale et $\widetilde{m}$ est la partie martingale de $\widetilde{G}$.

En temps discret, la définition de temps honnête est identique : Un temps aléatoire τ qui prend des valeurs dans les nombres entiers positifs est honnête, si pour tout $n \geq 0$, il existe une variable aléatoire $\mathcal{F}_n$-mesurable τ_n, telle que $\tau \mathbb{1}_{\{\tau \leq n\}} = \tau_n \mathbb{1}_{\{\tau \leq n\}}$. Nous avons montré (voir Th. 1.3.18) que : *si τ est un temps honnête et X une $\mathbb{F}$-martingale alors*

$$X_n = X_n^{\mathbb{G}} + \sum_{k=1}^{n\wedge\tau} \frac{1}{G_{k-1}} \Delta\langle \widetilde{m}, X\rangle_k^{\mathbb{F}} - \sum_{k=\tau+1}^{n} \frac{1}{1-G_{k-1}} \Delta\langle \widetilde{m}, X\rangle_k^{\mathbb{F}}, \quad \forall n \geq 1 ,$$

où $X^{\mathbb{G}}$ est une $\mathbb{G}$-martingale.

Un temps aléatoire ρ est un $\mathbb{F}$ pseudo-temps d'arrêt si pour toute $\mathbb{F}$-martingale M bornée, nous avons $E(M_\rho) = M_0$. La notion de pseudo temps d'arrêt a été introduit dans Williams [Wil02] et formalisée par Nikeghbali et Yor [NY05]. Ces deux derniers auteurs ont montré en particulier (voir [NY05, Th. 1]) que si τ est fini, les propriétés suivantes sont équivalentes :

(i) τ est un $\mathbb{F}$-pseudo temps d'arrêt.

(ii) $A_\infty = 1$, où $A = \widetilde{m} - G$ (pour cette égalité en temps continu voir par exemple [Aks14, page 31]).

(iii) $\widetilde{m} = 1$.

(iv) Toute $\mathbb{F}$-martingale locale X arrêtée en τ (c'est-à-dire X^τ) est une $\mathbb{G}$-martingale locale.

Si, en plus, toutes les $\mathbb{F}$-martingales sont continues, chacune des propriétés précédents sont équivalentes à :

(v) $\widetilde{G}$ est un processus décroissant $\mathbb{F}$-prévisible.

Dans le cas de temps discret, en considérant des temps aléatoires pouvant prendre la valeur $+\infty$, nous avons obtenu que *les propriétés suivantes sont équivalentes :*

(i) ρ est un $\mathbb{F}$-pseudo temps d'arrêt.

(ii) $A_{\infty^-} = \mathbb{P}(\tau < \infty | \mathcal{F}_\infty)$ et $A_\infty = 1$ où A est le processus défini par $A = \widetilde{m} - G$.

(iii) $\widetilde{m} = 1$.

(iv) Toute $\mathbb{F}$-martingale locale X arrêtée en τ est une $\mathbb{G}$-martingale locale.

(v) $\widetilde{G}$ est $\mathbb{F}$-prévisible.

La propriété (v), qui est propre au cas discret implique que G (et $\widetilde{G}$) est décroissante.

Nous étudions ensuite le cas où toutes les $\mathbb{F}$ martingales sont des $\mathbb{G}$-martingales (propriété d'immersion, notée $\mathbb{F} \hookrightarrow \mathbb{G}$). Le résultat que nous obtenons reproduit les résultats classiques en temps continu et donne une caractérisation de la propriété d'immersion, propre au temps discret, relative au processus $\widetilde{G}$. *Les assertions suivantes sont équivalentes :*

(i) $\mathbb{F} \hookrightarrow \mathbb{G}$.

(ii) $\widetilde{G}$ est $\mathbb{F}$-prévisible et $\widetilde{G}_{n+1} = \mathbb{P}(\tau \geq n+1 | \mathcal{F}_\infty)$ pour tout $n \geq 0$.

(iii) $G_n = \mathbb{P}(\tau > n | \mathcal{F}_\infty)$ pour tout $n \geq 0$.

(iv) $\mathbb{E}(\mathbb{1}_{\{\tau \leq n\}} | \mathcal{F}_n) = \mathbb{E}(\mathbb{1}_{\{\tau \leq n\}} | \mathcal{F}_m)$ pour tout $0 \leq n \leq m$.

Soit H défini par $H_n = \mathbb{1}_{\{\tau \le n\}}$ le processus croissant associé à τ. Nous calculons le compensateur de H (voir Lemme 1.3.4) : *S*oit Λ le processus croissant $\mathbb{F}$-prévisible donné par

$$\Lambda_0 = \mathbb{P}(\tau = 0|\mathcal{F}_0) \quad \text{et} \quad \Delta\Lambda_n = \lambda_n = \frac{\Delta a_n}{G_{n-1}} \mathbb{1}_{\{G_{n-1}>0\}} \ , \quad \forall n \ge 1 \ .$$

Alors, $N := H - \Lambda^\tau$ (c'est-à-dire $N_n := H_n - \Lambda_{n\wedge\tau}$ pour $n \ge 0$) est une $\mathbb{G}$-martingale. Le processus Λ^τ est le compensateur de H.

Nous introduisons un second processus que nous appelons l'équilibreur de H (voir Lemme 1.3.5) : *S*oit $\widetilde{\Lambda}$ le processus $\mathbb{F}$-adapté donné par

$$\widetilde{\Lambda}_0 = \mathbb{P}(\tau = 0|\mathcal{F}_0) \quad \text{et} \quad \Delta\widetilde{\Lambda}_n = \frac{\Delta A_n}{\widetilde{G}_n} \mathbb{1}_{\{\widetilde{G}_n>0\}} = \frac{\Delta \widetilde{a}_{n+1}}{\widetilde{G}_n} \mathbb{1}_{\{\widetilde{G}_n>0\}} \ , \ \ \forall n \ge 1 \ .$$

Alors, le processus $\widetilde{\Lambda}^\tau$ est l'unique processus $\mathbb{G}$-adapté égal à un processus $\mathbb{F}$-adapté arrêté en τ tel que, pour $\widetilde{N} := H - \widetilde{\Lambda}^\tau$ (c'est-à-dire $\widetilde{N}_n := H_n - \widetilde{\Lambda}_{n\wedge\tau}$ pour $n \ge 0$), tel que le processus $(X \centerdot \widetilde{N})$ est une $\mathbb{G}$-martingale pour tout processus intégrable $\mathbb{F}$-adapté X. En particulier, $\widetilde{N}$ est une $\mathbb{G}$-martingale. Le processus $\widetilde{\Lambda}^\tau$ est appelé l'équilibreur de H.

En utilisant ces définitions, nous avons le théorème suivant qui caractérise le fait que la partie $\mathbb{G}$-martingale de toute $\mathbb{F}$-martingale est orthogonale à N. *Les propositions suivantes sont équivalentes :*

(i) Le compensateur de H est égal à l'équilibreur de H, c'est-à-dire $\Lambda = \widetilde{\Lambda}$ et $N = \widetilde{N}$.

(ii) $\mathbb{E}(\Delta N_n|\mathcal{F}_n) = 0$ pour tout $n \ge 0$.

(iii) $\mathbb{E}\Big((U \centerdot N)_n\big|\mathcal{F}_n\Big) = 0$, pour tout $n \ge 0$ et tout processus $\mathbb{G}$-prévisible U.

(iv) La partie $\mathbb{G}$-martingale $X^{\mathbb{G}}$ de toute $\mathbb{F}$-martingale est orthogonale à N.

Ce résultat n'a pas, à notre connaissance d'équivalent en temps continu.

Un autre but de ce chapitre est d'étudier comment le grossissement de filtration peut introduire des arbitrages.

En temps continu, la théorie de non-arbitrage classique est basée sur les notions d'absence d'arbitrage et de *No Free Lunch with Vanishing Risk (NFLVR)*, développée par Delbaen & Schachermayer [DS94].

Dans notre contexte, nous considérons la définition suivante d'absence d'arbitrage dans une filtration et un processus de prix donné (voir par exemple Jacod et Shiryaev [JS98] pour plus de détails au sujet des arbitrages en temps discret, et Dalang, Morton *et al.* [DMW90] qui montrent qu'en temps discret, il n'y a pas d'arbitrages si et seulement s'il existe une mesure martingale équivalente). Soit X une $\mathbb{A}$-semimartingale. Le modèle $(X, \mathbb{A})$ n'a pas d'arbitrages s'il existe une $\mathbb{A}$-martingale positive L, avec $L_0 = 1$, telle que XL est une $\mathbb{A}$-martingale. Nous montrons que *si X est une $\mathbb{A}$-semimartingale et s'il existe un processus positif $\mathbb{A}$-adapté ψ tel que*

$$\mathbb{E}(X_n\psi_n|\mathcal{A}_{n-1}) = X_{n-1}\mathbb{E}(\psi_n|\mathcal{A}_{n-1}) \ , \ \ \forall n \ge 1 \ ,$$

alors, il existe une $\mathbb{A}$-martingale positive L telle que LX est une $\mathbb{A}$-martingale.

Dans le cadre du grossissement, nous prêtons attention à toutes les $\mathbb{F}$-martingales et donnons une définition du "modèle libre" de l'arbitrage, dans le sens où nous ne précisons pas le processus de prix dans la filtration $\mathbb{F}$. Nous donnons des conditions pour l'existence d'un déflateur pour toutes les $\mathbb{F}$-martingales. L'étude des conditions telles que, pour une $\mathbb{F}$-martingale X donnée, il existe un déflateur, peut être trouvée dans Choulli et Deng [CD14].

Soit $\mathbb{A} \subset \mathbb{B}$, nous disons que le modèle $(\mathbb{A}, \mathbb{B})$ est libre d'arbitrage s'il existe une $\mathbb{B}$-martingale positive L avec $L_0 = 1$ (appelé déflateur) telle que, pour toute $\mathbb{A}$-martingale X, le processus XL est une $\mathbb{B}$-martingale.

En utilisant le résultat précèdent, nous obtenons *qu'il n'y a pas d'arbitrage dans le modèle $(\mathbb{F}, \mathbb{G})$ strictement avant τ.*

Le résultat suivant a été obtenu dans Choulli et Deng [CD14] comme un cas particulier des résultats donnés dans Aksamit *et al.* [ACDJ13]. Nous en donnons une preuve simplifiée (voir Th. 1.3.42). *Supposons que τ n'est pas un $\mathbb{F}$-temps d'arrêt. Alors, il n'y a pas de $\mathbb{G}$-arbitrages avant τ (pour toute $\mathbb{F}$-martingale X, le processus arrêté X^τ admet un déflateur) si et seulement si, pour tout $n \geq 1$, l'ensemble $\{0 = \widetilde{G}_n < G_{n-1}\}$ est vide.*

En cas d'immersion, s'il n'y a pas d'arbitrages dans $\mathbb{F}$ au sens où il existe une $\mathbb{F}$-martingale positive L telle que XL est une $\mathbb{F}$-martingale, il n'y a pas d'arbitrage dans $\mathbb{G}$ au sens où il existe une $\mathbb{G}$-martingale positive $L^{\mathbb{G}}$ telle que XL est une $\mathbb{G}$-martingale (prendre $L^{\mathbb{G}} = L$). Ceci est cohérent avec le résultat précédent, puisque, sous l'hypothèse d'immersion, nous avons $G_{n-1} = \widetilde{G}_n$ pour tous les $n \geq 1$ et par suite $\{0 = \widetilde{G}_n < G_{n-1}\}$ est vide.

Prime d'indifférence de contrats d'assurance vie

Le deuxième chapitre de cette thèse est consacré à la conception de modèles pour l'étude de produits d'assurance vie, souvent désignés par le terme de "contrats à annuités variables".

Introduits dans les années 1970s aux Etats-Unis (voir Sloane [Slo70]) les contrats d'assurance vie à annuités variables sont des contrats, liés à la performance d'actifs financiers formant un portefeuille de référence, entre un assuré et une compagnie d'assurance. L'assuré confie un montant initial d'argent à l'assureur. Ce montant est alors investi dans un portefeuille de référence jusqu'à une date prédéfinie ou jusqu'à une sortie anticipé du contrat par un rachat total de la part de l'assuré ou le décès de celui-ci. À la fin du contrat, l'assurance verse à l'assuré ou à ses héritiers une somme d'argent, fonction de l'évolution de la valeur du portefeuille de référence.

Dans les années 1990s, les assureurs inclurent des garanties protégeant leurs clients contre des fortes baisses des marchés financiers. Les garanties les plus communes sont appelées "Guaranted Minimum Death Benefit" (GMDB) ou "Guaranted Minimum Life Benefits" (GMLB). Pour un contrat de type GMDB (resp. GMLB), si l'assuré décède avant l'échéance du contrat (resp. si l'assuré est toujours vivant à l'échéance du contrat) l'assuré ou ses héritiers obtiennent le montant correspondant au maximum entre la valeur du compte courant et la garantie. Il existe différentes façons de fixer ces garanties et nous nous réfèrerons à Bauer *et al.* [BKR08] pour plus de détails.

Ces produits présentent principalement trois classes de risques pour l'assureur. Premièrement, comme l'assureur propose une garantie, de type option de vente, sur un portefeuille de référence au client, l'assureur est exposé au risque de marché. En plus, un contrat d'assurance vie à annuités variables est un produit à très longue échéance potentielle. Les erreurs de valorisation et de couverture dûes au choix du modèle d'évolution pour la dynamique du portefeuille de référence et celle des taux d'intérêt pourraient être très importantes. Le deuxième risque encouru par l'assureur est le décès de son client. Cela nous conduira à la formulation d'un problème avec maturité aléatoire. Enfin, le client peut à tout moment décider de se retirer, totalement ou partiellement, du contrat. Dans un premier temps, nous supposerons qu'il existe un taux de retrait partiel stochastique, mais nous ne supposerons pas que ce processus résulte d'une stratégie optimale de l'assuré. Dans un deuxième temps, nous étudierons un modèle plus robuste où l'on supposera que l'assuré suit la pire stratégie de retraits possible pour l'assureur.

Notre travail consiste donc à incorporer tous ces risques dans un modèle et à en déduire une valorisation et une stratégie de couverture de ces produits en se plaçant du point de vue de l'assureur. Nous ne ferons pas d'hypothèses restrictives sur la dynamique du portefeuille de référence et celle des taux d'intérêt. Par conséquent, notre problème ne sera pas nécessairement markovien et, contrairement à beaucoup de travaux dans la littérature, nous ne pourrons obtenir de caractérisation de nos fonctions optimisant notre critère par des équations de type Hamilton-Jacobi-Bellman. Nous adopterons donc une approche par EDSRs en nous inspirant des idées d'El Karoui *et al.* [EKPQ97], Hu *et al.* [HIM05] et Rouge et El Karoui [REK00]. Dans notre cas, nous avons à résoudre une EDSR avec un temps terminal aléatoire. Pour cela, nous adapterons de récents résultats sur les EDSR avec saut (voir Ankirchner *et al.* [ABSEL10] et Kharroubi et Lim [KL12]). En outre, nous n'allons pas non plus utiliser des arguments de complétude du marché pour évaluer ces contrats d'assurance vie, nous adopterons donc une approche de valorisation par indifférence d'utilité. Nous supposerons que la prime, caractérisée par un taux de prime prédéfini, est prise par l'assureur en continu sur le compte de l'assuré. Nous définirons ainsi un taux de prime d'indifférence pour l'assureur.

On se place sur un espace de probabilité $(\Omega, \mathcal{G}, \mathbb{P})$ muni d'un mouvement brownien unidimensionnel B et on note $\mathbb{F} := (\mathcal{F}_t)_{0\leq t\leq T}$ la filtration complète et continue à droite générée par B où T est un horizon de temps fini qui correspond à la date d'expiration du contrat.

L'assureur investit sur un marché financier que l'on suppose composé de deux actifs. Soient $\hat{S}^0$, le processus de prix de l'actif sans risque (portefeuille essentiellement composé d'obligations) et $\hat{S}$ le processus de prix de l'actif risqué (portefeuille de référence). Nous supposerons que ces processus sont les solutions des équations différentielles stochastiques linéaires suivantes

$$\begin{aligned} d\hat{S}^0_t &= r_t\hat{S}^0_t dt\ , \quad \forall t\in[0,T]\ , \quad \hat{S}^0_0 = 1\ , \\ d\hat{S}_t &= \hat{S}_t(\mu_t dt + \sigma_t dB_t)\ , \quad \forall t\in[0,T]\ , \quad \hat{S}_0 = s > 0\ , \end{aligned}$$

où μ, σ et r sont des processus $\mathbb{F}$-adaptés qui satisfont certaines hypothèses d'intégrabilité. Nous noterons S_t la valeur actualisée de $\hat{S}_t$ au temps $t\in[0,T]$.

Pour $t\in[0,T]$, on note par π^0_t (resp. π_t) le montant actualisé d'argent investi dans l'actif sans risque (resp. l'actif risqué).

En imposant à l'assureur de ne suivre que des stratégies autofinancées et en notant $X_t^{x,\pi}$ la valeur actualisée du portefeuille de l'assureur au temps t, avec le capital initial $x \in \mathbb{R}^+$ et suivant la stratégie π, nous avons

$$X_t^{x,\pi} \;=\; x + \int_0^t \pi_s(\mu_s - r_s)ds + \int_0^t \pi_s \sigma_s dB_s \;, \quad \forall t \in [0,T] \;.$$

Nous considérons que l'assureur veut maximiser l'espérance de l'utilité de sa richesse terminale $U(X_T^{x,\pi})$ sur les stratégies admissibles, où $U(x) := -\exp(-\gamma x)$ avec $\gamma > 0$ (voir le livre de Borch *et al.* [BSA14] pour plus de détails sur ce choix d'une utilité exponentielle).

Nous considérons les deux temps aléatoires θ^d et θ^w qui représentent respectivement la date du décès de l'assuré et la date de rachat par anticipation de la totalité du contrat de la part de l'assuré. On pose également $\tau = \theta^d \wedge \theta^w$. Le temps aléatoire τ n'est pas supposé être un $\mathbb{F}$-temps d'arrêt. De façon usuelle, nous considérerons $\mathbb{G}$ la filtration grossie progressivement de $\mathbb{F}$ par τ (voir par exemple Bielecki *et al.* [BR04]). Nous supposons que l'hypothèse $(\mathcal{H})$ est vérifiée et qu'il existe un $\mathbb{F}$-compensateur borné λ. On notera M la $\mathbb{G}$-martingale, définie par $M_t := H_t - \int_0^{t\wedge\tau} \lambda_s ds$, pour tout $t \geq 0$.

L'hypothèse $(\mathcal{H})$ peut être vue comme la conséquence d'une structure de dépendance asymétrique entre B et τ. D'un point de vue financier, cela signifie que le temps de sortie τ peut dépendre du caractère aléatoire du marché financier représenté par B. Au contraire, le marché financier n'est pas influencé par τ, les informations sur τ ne changeront pas la dynamique de S.

Soit $\mathbb{T} := (t_i)_{0\leq i\leq n}$ l'ensemble des dates d'anniversaire du contrat, avec $t_0 = 0$ et $t_n = T$. On note aussi $t_{n+1} = +\infty$.

Pour définir un contrat, le premier processus à considérer est la valeur du compte A^p. Le montant total sur le compte investi sur les marchés. Comme les primes et les retraits (dans la première partie de ce chapitre) sont supposés être continuellement pris sur le compte, la dynamique de A^p est donnée par

$$dA_t^p \;=\; A_t^p\big[(\mu_t - r_t - \xi_t - p)dt + \sigma_t dB_t\big] \;, \quad \forall t \in [0,T] \;,$$

avec A_0 la valeur initiale et où $p \geq 0$ est le taux de prime pris par l'assureur et le processus ξ est un processus $\mathbb{G}$-prévisible, positif et borné. ξ_t représente le taux de retrait choisi par l'assuré au moment $t \in [0,T]$. Nous soulignons également que ξ n'est pas nécessairement un processus résultant d'un contrôle optimal de l'assuré, comme par exemple, dans Belanger *et al.* [BFL09], Blanchet-Scalliet *et al.* [BSCKL15], Dai *et al.* [DKZ08] et Milevsky et Salisbury [MS06].

La seconde quantité à définir est le pay-off $F(p)$ du contrat. A la date $T \wedge \tau$, le pay-off est versé par l'assureur à l'assuré ou à ses ayants-droits. Sa valeur est une variable aléatoire $\mathcal{G}_{T\wedge\tau}$-mesurable qui dépend de la garantie choisie. Les plus usuelles sont les garanties constantes, *Ratchet* et *Roll-up* (voir Bauer *et al.* [BKR08] pour plus de détails).

L'objectif de ce chapitre est de déterminer, si il existe, un niveau de taux de prime p^* de telle sorte que si le taux p est supérieur à p^*, l'assureur a intérêt à commercialiser le contrat mais celui-ci ne sera pas intéressant pour un taux de prime p inférieur à ce niveau p^*. Le taux de prime optimal p^* est donc le plus petit nombre positif tel que

$$\sup_{\pi\in\mathcal{A}^{\mathbb{F}}[0,T]} \mathbb{E}\big[U\big(X_T^{x,\pi}\big)\big] \;=\; \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T]} \mathbb{E}\big[U\big(X_T^{x+A_0,\pi} - F(p)\big)\big] \;.$$

La solution sera appelée taux de prime d'indifférence.

La résolution de cette équation passe par l'évaluation des quantités suivantes

$$V_{\mathbb{F}} := \sup_{\pi\in\mathcal{A}^{\mathbb{F}}[0,T]} \mathbb{E}\big[U\big(X_T^\pi\big)\big] \text{ et } V_{\mathbb{G}}(p) := \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T]} \mathbb{E}\big[U\big(A_0 + X_T^\pi - F(p)\big)\big] .$$

En reliant ces problèmes de contrôle régulier à des EDSR, nous fournissons un théorème de vérification et nous donnons des méthodes d'évaluation de ces quantités et des stratégies optimales associées. Nous en déduisons donc une méthode d'approximation du taux de prime d'indifférence p^*.

La première quantité $V_{\mathbb{F}}$ est la valeur d'un problème d'optimisation classique (voir Prop. 2.2.2). Elle est donnée par $V_{\mathbb{F}} = -\exp(\gamma y_0)$, où (y, z) est l'unique solution de l'EDSR suivante :

$$\begin{cases} dy_t &= \Big(\frac{\nu_t^2}{2\gamma} + \nu_t z_t\Big)dt + z_t dB_t , \quad \forall t \in [0, T] , \\ y_T &= 0 , \end{cases}$$

avec $\nu_t = \frac{\mu_t - r_t}{\sigma_t}$. En outre, la stratégie optimale associée à ce problème est définie par

$$\pi_t^* \;:=\; \frac{\nu_t}{\gamma\sigma_t} + \frac{z_t}{\sigma_t} , \quad \forall t \in [0, T] ,$$

(nous nous référerons ici à Hu *et al.* [HIM05] et Rouge et El Karoui [REK00]).

Pour la deuxième quantité $V_{\mathbb{G}}(p)$, nous avons établi dans le Théorème. 2.2.6 que $V_{\mathbb{G}}(p) = -\exp(\gamma(Y_0(p) - A_0))$ où $Y_0(p)$ est défini comme la valeur initiale de la première composante de la solution de l'EDSR

$$\begin{aligned} Y_t(p) \;&=\; \mathfrak{H}(p) + \int_{t\wedge\tau}^{T\wedge\tau} \Big(\lambda_s \frac{e^{\gamma U_s(p)} - 1}{\gamma} - \frac{\nu_s^2}{2\gamma} - \nu_s Z_s(p)\Big)ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s(p)dB_s \\ &\quad - \int_{t\wedge\tau}^{T\wedge\tau} U_s(p)dH_s , \quad \forall t \in [0, T] , \end{aligned}$$

et la stratégie optimale est définie par

$$\pi_t^* \;:=\; \frac{\nu_t}{\gamma\sigma_t} + \frac{Z_t(p)}{\sigma_t}\mathbb{1}_{t\le T\wedge\tau} + \frac{Z_t^{(\tau)}}{\sigma_t}\mathbb{1}_{t>T\wedge\tau} , \quad \forall t \in [0, T] .$$

Notre objectif est de déterminer les taux de prime d'indifférence $p^* > 0$ tel que

$$\sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T]} \mathbb{E}\big[-\exp\big(-\gamma\big(X_T^{A_0,\pi} - F(p^*)\big)\big)\big] \;=\; \sup_{\pi\in\mathcal{A}^{\mathbb{F}}[0,T]} \mathbb{E}\big[-\exp\big(-\gamma X_T^\pi\big)\big] .$$

En utilisant les résultats précédents, on peut reformuler la dernière équation de la manière suivante

$$Y_0(p^*) - A_0 \;=\; y_0 .$$

Pour étudier cette équation, nous introduisons la fonction $\psi : \mathbb{R} \to \mathbb{R}$ définie comme suit

$$\psi(p) \;:=\; Y_0(p) - y_0 - A_0 , \quad \forall p \in \mathbb{R} .$$

Nous concluons cette partie avec des illustrations numériques de la sensibilité des taux de prime d'indifférence aux différents paramètres.

Dans la dernière section de ce chapitre, nous présentons une méthode plus robuste présentée dans Blanchet-Scalliet *et al.* [BSCKL15]. On suppose que l'assuré ne peut effectuer des retraits uniquement aux dates anniversaires $(t_i)_{1\leq i\leq n-1}$ et que la stratégie de retraits anticipés de l'assuré est la pire qui soit pour l'assureur. La fonction valeur de celui-ci est alors

$$V(p) \;:=\; \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T]}\inf_{\xi\in\mathcal{E}}\mathbb{E}\Big[U\Big(A_0+X_T^\pi-\sum_{i=1}^{n-1}\xi_i\mathbb{1}_{t_i\leq\tau}-F(p,\hat{\xi})\Big)\Big]\,.$$

Pour calculer cette fonction valeur, Blanchet-Scalliet *et al.* [BSCKL15] ont prouvé qu'il fallait résoudre un système récursif de problème de contrôle max-min entre les dates anniversaires. Chacun des problèmes de contrôle peut être résolu en utilisant les EDSR et la condition terminale de l'EDSR dépend de la solution de l'EDSR d'après. Dans cette thèse nous avons étudié le problème d'un point de vue numérique et nous donnons des illustrations de la sensibilité aux différents paramètres.

Équations différentielles stochastiques rétrogrades, grossissement de filtration et prix d'indifférence de l'information

Les équations différentielles stochastiques rétrogrades (EDSR) ont été introduites par Bismut dans [Bis73], puis leur étude a été généralisée par Pardoux et Peng dans [PP90]. Ce type d'équations apparait naturellement dans le cadre de problèmes de contrôle stochastique (voir par exemple El Karoui *et al.* [EKPQ97]), en particulier en mathématiques financières. Leur utilisation a permis de résoudre de nombreux problèmes de prix d'indifférence, comme cela est présenté dans l'article fondamental de Rouge et El Karoui [REK00].

Dans ce chapitre, nous travaillons dans un espace de probabilité $(\Omega,\mathcal{F},\mathbb{P})$ muni de deux filtrations $\mathbb{F}\subset\mathbb{G}$. La filtration $\mathbb{F}$ est engendrée par un mouvement brownien W et une mesure aléatoire de Poisson $\mathbf{N}$ telle que $\mathbf{M}=\mathbf{N}-\nu$ avec $\nu(dx,dt):=\lambda(dx)dt$ est une martingale. Nous travaillons sous l'hypothèse $(\mathcal{H}')$ et supposons que $W_t=W_t^{\mathbb{G}}+\int_0^t\mu_s ds$ où $W^{\mathbb{G}}$ est un $\mathbb{G}$-mouvement Brownien et $\mathbf{M}^{\mathbb{G}}=\mathbf{N}-\nu^{\mathbb{G}}$ avec $\nu^{\mathbb{G}}(dx,dt):=\kappa(x)\lambda(dx)dt$ est une $\mathbb{G}$-martingale.

Nous considérons une EDSR dans la filtration $\mathbb{G}$ avec comme générateur f et la condition terminale ξ dans la filtration $\mathbb{G}$

$$\begin{aligned}Y_t^{\mathbb{G}} \;&=\; \xi+\int_t^T f\big(s,Y_s^{\mathbb{G}},Z_s^{\mathbb{G}},U_s^{\mathbb{G}}\big)ds\\ &\quad -\int_t^T Z_s^{\mathbb{G}}dW_s^{\mathbb{G}}-\int_t^T\int_{\mathbb{R}}U_s^{\mathbb{G}}(x)\mathbf{M}^{\mathbb{G}}(ds,dx)-\int_t^T dM_s^{\mathbb{G}^\perp},\ \forall t\in[0,T]\,,\end{aligned}$$

nous notons $(Y^{\mathbb{G}},Z^{\mathbb{G}},U^{\mathbb{G}},M^{\mathbb{G}^\perp})$ la solution à cette EDSR.

Le processus $Y^{\mathbb{F}}$ où $Y_t^{\mathbb{F}}:=\mathbb{E}(Y_t^{\mathbb{G}}|\mathcal{F}_t)$ vérifie une EDSR de la forme

$$Y_t^{\mathbb{F}}=\mathbb{E}(\xi|\mathcal{F}_T)+\int_t^T\widehat{f}_s ds-\int_t^T Z_s^{\mathbb{F}}dW_s-\int_t^T\int_{\mathbb{R}}U_s^{\mathbb{F}}(x)\mathbf{M}(ds,dx)\ ,\ \forall t\in[0,T]\,,$$

où $\widehat{f}_s := \mathbb{E}\big[f(s, Y_s^{\mathbb{G}}, Z_s^{\mathbb{G}}, U_s^{\mathbb{G}})|\mathcal{F}_s\big]$ pour tout $s \in [0,T]$ et nous établissons les relations suivantes entre $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, U^{\mathbb{G}})$ et $(Z^{\mathbb{F}}, U^{\mathbb{F}})$ (voir Th. 3.1.9) :

- $Z_t^{\mathbb{F}} = \mathbb{E}\Big(Z_t^{\mathbb{G}} + \mu_t Y_{t-}^{\mathbb{G}}\Big|\mathcal{F}_t\Big)$, pour tout $t \in [0,T]$.
- $U_t^{\mathbb{F}}(x) = \mathbb{E}\Big(U_t^{\mathbb{G}}(x)\kappa(x) + Y_{t-}^{\mathbb{G}}(\kappa(x)-1)\big|\mathcal{F}_t\Big)$, pour tout $t \in [0,T]$ et $x \in \mathbb{R}$.

Ensuite, nous considérons le cas particulier où le générateur f de la $\mathbb{G}$-EDSR est linéaire dont les coefficients sont $\mathbb{F}$-adaptés et nous définissons la $\mathbb{F}$-EDSR avec le même générateur f et comme condition terminale $\mathbb{E}(\xi|\mathcal{F}_T)$. Nous notons $(\widehat{Y}^{\mathbb{F}}, \widehat{Z}^{\mathbb{F}}, \widehat{U}^{\mathbb{F}})$ la solution à cette $\mathbb{F}$-EDSR. Plus précisément, nous considérons les processus $\mathbb{F}$-adaptés bornés $\alpha, \beta, \gamma(x)$ et δ et nous nous concentrons notre attention sur la $\mathbb{G}$-EDSR suivante :

$$\begin{aligned} Y_t^{\mathbb{G}} &= \xi + \int_t^T \Big(\alpha_s Y_s^{\mathbb{G}} + \beta_s Z_s^{\mathbb{G}} + \int_{\mathbb{R}} \gamma_s(x) U_s^{\mathbb{G}}(x)\lambda^{\mathbb{G}}(dx) + \delta_s\Big)ds \\ &\quad - \int_t^T Z_s^{\mathbb{G}} dW_s^{\mathbb{G}} - \int_t^T \int_{\mathbb{R}} U_s^{\mathbb{G}}(x)\mathbf{M}^{\mathbb{G}}(ds,dx) - \int_t^T dM_s^{\mathbb{G}\perp},\ \forall t \in [0,T] \ , \end{aligned}$$

et nous définissons la $\mathbb{F}$-EDSR suivante :

$$\begin{aligned} \widehat{Y}_t^{\mathbb{F}} &= \mathbb{E}(\xi|\mathcal{F}_T) + \int_t^T \big(\alpha_s \widehat{Y}_s^{\mathbb{F}} + \beta_s \widehat{Z}_s^{\mathbb{F}} + \int_{\mathbb{R}} \gamma_s(x)\widehat{U}_s^{\mathbb{F}}(x)\lambda(dx) + \delta_s\Big)ds \\ &\quad - \int_t^T \widehat{Z}_s^{\mathbb{F}} dW_s - \int_t^T \int_{\mathbb{R}} \widehat{U}_s^{\mathbb{F}}(x)\mathbf{M}(ds,dx) \ , \ \forall t \in [0,T] \, . \end{aligned}$$

Nous montrons (voir Th. 3.1.10) que

$$\widehat{Y}_t^{\mathbb{F}} = Y_t^{\mathbb{F}} + \mathbb{E}\Big(\int_t^T L_{t,s} Y_s^{\mathbb{G}}(\beta_s\mu_s - \int_{\mathbb{R}} \gamma_s(x)(\kappa(x)-1)\lambda(dx))ds\Big|\mathcal{F}_t\Big) \ , \quad \forall t \in [0,T] \ ,$$

où le processus $(L_{t,s})_{s\in[t,T]}$ est l'unique solution de l'équation différentielle stochastique

$$dL_{t,s} = L_{t,s^-}(\alpha_s ds + \beta_s dW_s + \int_{\mathbb{R}} \gamma_s(x)\mathbf{M}(ds,dx))$$

avec $L_{t,t} = 1$.

Dans la deuxième partie du chapitre, nous nous concentrons sur le prix d'indifférence de l'information, c'est-à-dire le prix à payer pour avoir le même niveau d'utilité attendue en utilisant les informations supplémentaires que sans le faire, dans le cas où $\mathbb{F}$ est une filtration brownienne.

Tout d'abord, nous nous concentrons sur les problèmes de maximisation de l'utilité avec une fonction d' utilité exponentielle U de paramètre γ en utilisant les EDSRs. Nous considérons la richesse du portefeuille $X_t^{x,\pi}$ au temps t avec le capital initial x, suivant la stratégie d'investissement π et l'ensemble des stratégies $\mathbb{F}$-prévisibles noté $\mathcal{A}^{\mathbb{F}}$ (resp. l'ensemble des stratégies $\mathbb{G}$-prévisibles noté $\mathcal{A}^{\mathbb{G}}$). Ensuite, nous définissons le prix d'indifférence de l'information comme le nombre réel positif p tel que

$$\sup_{\pi\in\mathcal{A}^{\mathbb{F}}} \mathbb{E}\Big(U\big(X_T^{x,\pi} - \xi\big)\Big) = \sup_{\pi\in\mathcal{A}^{\mathbb{G}}} \mathbb{E}\Big(U\big(X_T^{x-p,\pi} - \xi\big)\Big) \ .$$

Nous pouvons étendre cette définition comme suit : Nous définissons le prix d'indifférence de l'information au temps t comme le processus aléatoire donné par

$$p_t(\xi, \mathbb{G}) := \frac{1}{\gamma}\log\left(\frac{\sup_{\pi\in\mathcal{A}^{\mathbb{F}}[t,T]}\mathbb{E}\Big[U\big(X_T^{t,\pi} + X_t^{\pi_{\mathbb{F}}^*} - \xi\big)\Big|\mathcal{F}_t\Big]}{\sup_{\pi\in\mathcal{A}^{\mathbb{G}}[t,T]}\mathbb{E}\Big[U\big(X_T^{t,\pi} + X_t^{\pi_{\mathbb{G}}^*} - \xi\big)\Big|\mathcal{G}_t\Big]}\right) ,$$

où $X_t^{\pi_{\mathbb{F}}^*}$ (resp. $X_t^{\pi_{\mathbb{G}}^*}$) est le processus de richesse au temps t, avec la $\mathbb{F}$-stratégie optimale $\pi_{\mathbb{F}}^*$ (resp. la $\mathbb{G}$-stratégie optimale $\pi_{\mathbb{G}}^*$) et richesse initial nulle.

Pour trouver ce prix, nous divisons le problème en deux problèmes d'utilité dans différentes filtrations (voir Proposition 3.2.4 à Proposition 3.2.9).

Tout d'abord, nous introduisons le problème classique, où l'ensemble des stratégies admissibles est défini dans $\mathbb{F}$ et le pay-off $\xi^{\mathbb{F}}$ est $\mathcal{F}_T$-mesurable. Pour tout $t \in [0,T]$ et tout capital initial $x \in \mathbb{R}$, on définit la fonction valeur $\overline{V}_t^{\mathbb{F}}(x)$ comme

$$\overline{V}_t^{\mathbb{F}}(x) := \sup_{\pi\in\mathcal{A}^{\mathbb{F}}[t,T]}\mathbb{E}\Big[U\big(X_T^{t,x,\pi} - \xi^{\mathbb{F}}\big)\Big|\mathcal{F}_t\Big] , \ \forall x \in \mathbb{R} .$$

Ce problème a été étudié par Hu *et al.* [HIM05] et Rouge et El Karoui [REK00].

Nous introduisons aussi le problème de maximisation suivant. Nous considérons les stratégies $\mathbb{F}$-prévisibles et le pay-off $\xi^{\mathbb{G}} \in \mathcal{G}_T$. Nous définissons la valeur fonction $V_t^{\mathbb{F}}(x)$ par

$$V_t^{\mathbb{F}}(x) := \sup_{\pi\in\mathcal{A}^{\mathbb{F}}[t,T]}\mathbb{E}\Big[U\big(X_T^{t,x,\pi} - \xi^{\mathbb{G}}\big)\Big|\mathcal{F}_t\Big] , \quad \forall x \in \mathbb{R} \ \text{ et } \ \forall t \in [0,T] .$$

Comme nous le verrons, ce problème se réduit au précédent dans le cas de la fonction d'utilité exponentielle.

Nous considérons également le problème en utilisant des stratégies $\pi \in \mathcal{A}^{\mathbb{G}}[t,T]$ avec pay-off $\xi^{\mathbb{G}} \in \mathcal{G}_T$, et définissons la fonction valeur $V^{\mathbb{G}}(x)$ comme

$$V_t^{\mathbb{G}}(x) := \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[t,T]}\mathbb{E}\Big[U\big(X_T^{t,x,\pi} - \xi^{\mathbb{G}}\big)\Big|\mathcal{G}_t\Big] \quad \forall x \in \mathbb{R} \ \text{ et } \ \forall t \in [0,T] .$$

Ce problème est similaire au premier, travaillant dans un autre filtration. Dans ce cas, nous pouvons également associer un $\mathbb{G}$-EDSR à ce problème.

On peut aussi considérer le problème de maximisation en utilisant des stratégies $\pi \in \mathcal{A}^{\mathbb{G}}[t,T]$ mais avec pay-off $\xi^{\mathbb{F}} \in \mathcal{F}_T$ donnée par la variable aléatoire $\overline{V}_t^{\mathbb{G}}(x)$, définie comme

$$\overline{V}_t^{\mathbb{G}}(x) = \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[t,T]}\mathbb{E}\Big[U\big(X_T^{t,x,\pi} - \xi^{\mathbb{F}}\big)\Big|\mathcal{G}_t\Big] , \quad \forall x \in \mathbb{R} \ \text{ et } \ \forall t \in [0,T] .$$

Ce problème peut être associé à une $\mathbb{G}$-EDSR et est un cas particulier du précédent.

La question naturelle est : Quel est le lien entre $V_t^{\mathbb{F}}$, $\overline{V}_t^{\mathbb{F}}$, $V_t^{\mathbb{G}}$ et $\overline{V}_t^{\mathbb{G}}$? Répondre à cette question pour toute paire de filtrations $\mathbb{F}$ et $\mathbb{G}$ tels que $\mathbb{F} \subset \mathbb{G}$ est très difficile, puisque les EDSR associés dépendent des filtrations considérées. Nous nous restreignons aux cas particuliers où $\mathbb{G}$ est un grossissement initial de $\mathbb{F}$ et où $\mathbb{G}$ est un grossissement progressif de $\mathbb{F}$.

Nous trouvons la solution des problèmes d'utilité dans différentes filtrations en termes des EDSR dans différentes filtrations, et nous utilisons les résultats de la première partie de ce chapitre pour donner le prix d'indifférence de l'information (voir Prop. 3.2.14).

Soit $\mathbb{G}$ une filtration grossie de $\mathbb{F}$ (soit grossissement initial ou progressif) et $\xi \in \mathcal{G}_T$, alors le prix d'indifférence de l'information $p_t(\xi, \mathbb{G})$ est donné en fonction des richesses $X^{\pi_{\mathbb{G}}^*}$ et $X^{\pi_{\mathbb{F}}^*}$ et des solutions des EDSRs associées aux problèmes de maximisation d'utilité dans les filtrations $Y^{\mathbb{F}}$ et $Y^{\mathbb{G}}$, c'est-à-dire :

$$p_t(\xi, \mathbb{G}) = X_t^{\pi_{\mathbb{G}}^*} - X_t^{\pi_{\mathbb{F}}^*} + Y_t^{\mathbb{F}} - Y_t^{\mathbb{G}} , \quad \forall t \in [0, T] .$$

Les stratégies optimales $\pi_{\mathbb{F}}{}^*$ et $\pi_{\mathbb{G}}{}^*$ sont données par

$$\pi_{\mathbb{F}}{}_t^* \quad := \quad \frac{\alpha_t}{\gamma\sigma_t^2} + \frac{Z_t^{\mathbb{F}}}{\sigma_t} , \quad \forall t \in [0, T]$$

et

$$\pi_{\mathbb{G}}{}_t^* \quad := \quad \frac{\alpha_t + \mu_t^{\mathbb{G}}\sigma_t}{\gamma\sigma_t^2} + \frac{Z_t^{\mathbb{G}}}{\sigma_t} , \quad \forall t \in [0, T] ,$$

où $(Y^{\mathbb{F}}, Z^{\mathbb{F}})$ est la solution unique de l'EDSR

$$\begin{cases} -dY_t^{\mathbb{F}} &= -\Big(\frac{\alpha_t^2}{2\gamma\sigma_t^2} + \frac{\alpha_t Z_t^{\mathbb{F}}}{\sigma_t}\Big)dt - Z_t^{\mathbb{F}} dW_t , \quad \forall t \in [0, T] , \\ Y_T^{\mathbb{F}} &= \frac{1}{\gamma}\log\mathbb{E}\big[\exp(\gamma\xi)\big|\mathcal{F}_T\big] . \end{cases}$$

Dans le cas du grossissement initial $\mathbb{G} = \mathbb{G}^{(\zeta)}$ on a que $(Y^{\mathbb{G}^{(\zeta)}}, Z^{\mathbb{G}^{(\zeta)}})$ est la solution unique de l'EDSR

$$\begin{cases} -dY_t^{\mathbb{G}^{(\zeta)}} &= -\Big(\frac{(\alpha_t + \mu_t^{\mathbb{G}^{(\zeta)}}\sigma_t)^2}{2\gamma\sigma_t^2} + \frac{\alpha_t + \mu_t^{\mathbb{G}^{(\zeta)}}\sigma_t}{\sigma_t} Z_t^{\mathbb{G}^{(\zeta)}}\Big)dt - Z_t^{\mathbb{G}^{(\zeta)}} dW_t^{\mathbb{G}^{(\zeta)}} , \quad \forall t \in [0, T] , \\ Y_T^{\mathbb{G}^{(\zeta)}} &= \xi \end{cases}$$

et dans le cadre du grossissement progressif $\mathbb{G} = \mathbb{F}^{(\tau)}$ on a $(Y^{\mathbb{F}^{(\tau)}}, Z^{\mathbb{F}^{(\tau)}}, U^{\mathbb{F}^{(\tau)}})$ est la solution unique de l'EDSR

$$\begin{cases} -dY_t^{\mathbb{F}^{(\tau)}} &= -\Big(\frac{(\alpha_t + \mu_t^{\mathbb{F}^{(\tau)}}\sigma_t)^2}{2\gamma\sigma_t^2} + \frac{\alpha_t + \mu_t^{\mathbb{F}^{(\tau)}}\sigma_t}{\sigma_t} Z_t^{\mathbb{F}^{(\tau)}} - \lambda_t(1 - H_t)\frac{e^{\gamma U_t^{\mathbb{F}^{(\tau)}}} - 1}{\gamma}\Big)dt \\ & \quad - Z_t^{\mathbb{F}^{(\tau)}} dW_t^{\mathbb{F}^{(\tau)}} - U_t^{\mathbb{F}^{(\tau)}} dH_t , \ \forall t \in [0, T] , \\ Y_T^{\mathbb{F}^{(\tau)}} &= \xi . \end{cases}$$

Voir Prop. 3.2.14 pour plus détails.

Équations différentielles stochastiques rétrogrades avancées

Dans la dernière partie de cette thèse nous considérons un type spécial d'équations différentielles stochastiques rétrogrades avec un saut. Plus précisément, nous sommes intéressés par les équations appelées équations différentielles stochastiques rétrogrades avancées (EDSRA). Celles-ci ont été introduites par Peng et Yang [PY09] dans un cadre brownien et par Øksendal et Sulem dans [ØS16] dans un cadre comportant à la fois un mouvement brownien et un processus de sauts poissoniens.

Nous travaillons dans un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$ muni de deux filtrations $\mathbb{F} \subset \mathbb{G}$. La filtration $\mathbb{F}$ est la filtration complète générée par le mouvement brownien B. Soit H le processus défini par $H_t = \mathbb{1}_{\{\tau \leq t\}}$, associé à un temps donné aléatoire τ (une variable aléatoire positive). On définit la filtration $\mathbb{G}$, comme la filtration grossie progressivement de $\mathbb{F}$ par τ. Pour une variable aléatoire intégrable X, nous notons $\mathbb{E}_t^{\mathbb{G}}(X) = \mathbb{E}(X|\mathcal{G}_t)$, où $\mathbb{G}$ est la filtration générée par B et H. Soit $L^2(\mathcal{F}_t)$ l'ensemble des variables aléatoires $\mathcal{F}_t$-mesurables de carré intégrable.

On étudie deux types d'EDSRAs, le premier est une généralisation de Peng et Yang dans [PY09] : nous cherchons un triplet (Y, Z, U) solution de

$$\begin{cases} -dY_t &= f(t, Y_t, \mathbb{E}_t^{\mathbb{G}}[Y_{t+\delta}], Z_t, \mathbb{E}_t^{\mathbb{G}}[Z_{t+\delta}], U_t, \mathbb{E}_t^{\mathbb{G}}[U_{t+\delta}])dt \\ & \quad -Z_t dB_t - U_t dH_t \ , \quad 0 \leq t \leq T \ , \\ Y_{T+t} &= \xi_{T+t} \ , \quad 0 \leq t \leq \delta \ , \\ Z_{T+t} &= P_{T+t} \ , \quad U_{T+t} = Q_{T+t}\mathbb{1}_{\{T+t \leq \tau\}} \ , \quad 0 < t \leq \delta \ . \end{cases}$$

Nous utilisons, dans les preuves un autre résultat des mêmes auteurs sur des équations de la forme

$$\begin{cases} -dY_t &= g(t, Y_t, Y_{t+\delta}, Z_t, Z_{t+\delta})dt - Z_t dB_t \ , \quad 0 \leq t \leq T \\ Y_{T+t} &= \xi_{T+t} \ , \quad 0 \leq t \leq \delta \ , \\ Z_{T+t} &= P_{T+t} \ , \quad 0 < t \leq \delta \ , \end{cases}$$

où $g : \Omega \times [0, T] \times \mathbb{R} \times L^2(\mathcal{F}_{\cdot+\delta}) \times \mathbb{R} \times L^2(\mathcal{F}_{\cdot+\delta}) \to \mathbb{R}$ est tel que $g(\cdot, y, \zeta, z, \eta)$ est $\mathbb{F}$-adapté.

Le deuxième type d'EDSRA que l'on étudie est semblable à celui introduit par Øksendal et Sulem [ØS16] : nous cherchons un triplet (Y, Z, U) solution de

$$\begin{cases} -dY_t &= \mathbb{E}_t^{\mathbb{G}}\big[f(t, Y_t, Y_{t+\delta}, Z_t, Z_{t+\delta}, U_t, U_{t+\delta})\big]dt - Z_t dB_t - U_t dH_t \ , \quad 0 \leq t \leq T \ , \\ Y_{T+t} &= \xi_{T+t} \ , \quad 0 \leq t \leq \delta \ , \\ Z_{T+t} &= P_{T+t} \ , \quad U_{T+t} = Q_{T+t}\mathbb{1}_{\{T+t \leq \tau\}} \ , \quad 0 < t \leq \delta \ . \end{cases}$$

Dans les deux cas, les conditions terminales ξ, P et Q sont des processus donnés, ainsi que le générateur $f : \Omega \times [0, T] \times \mathbb{R}^6 \to \mathbb{R}$. On remarque que le générateur f de ces EDSRAs dépend des valeurs des processus (Y, Z, U) pour le temps présent t, ainsi que pour le temps futur $t + \delta$.

En utilisant la méthode de Kharroubi et Lim [KL12], nous donnons des conditions telles qu'il existe une solution unique pour les EDSRAs dans des espaces adéquats, c'est-à-dire, nous décomposons l'EDSRA définie sur $\mathbb{G}$ en deux EDSRAs sans saut définies sur $\mathbb{F}$.

Nous considérons les décompositions suivantes des conditions terminales

$$\begin{cases} \xi_t &= \xi_t^b \mathbb{1}_{\{t < \tau\}} + \xi_t^a(\tau)\mathbb{1}_{\{t \geq \tau\}} \quad \text{(décomposition optionnelle)} \\ P_t &= P_t^b \mathbb{1}_{\{t \leq \tau\}} + P_t^a(\tau)\mathbb{1}_{\{t > \tau\}} \quad \text{(décomposition prévisible)} \end{cases}$$

et du générateur $f(t, \vec{\mathbf{y}}) = f^b(t, \vec{\mathbf{y}})\mathbb{1}_{\{t < \tau\}} + f^a(t, \tau, \vec{\mathbf{y}})\mathbb{1}_{\{t \geq \tau\}}$.

De la même façon, l'EDSRA se décompose en deux EDSRAs sans saut :

$$\begin{cases} Y_t &= Y_t^b \mathbb{1}_{\{t < \tau\}} + Y_t^a(\tau)\mathbb{1}_{\{t \geq \tau\}} \quad \text{(décomposition optionnelle)} \\ Z_t &= Z_t^b \mathbb{1}_{\{t \leq \tau\}} + Z_t^a(\tau)\mathbb{1}_{\{t > \tau\}} \quad \text{(décomposition prévisible)} \end{cases}$$

Pour les EDSRA du premier type (du type Peng et Yang), nous avons obtenu :

$$\left\{\begin{array}{rcl} -dY_t^a(\theta) & = & f^a\left(t,\theta,Y_t^a(\theta),\mathbb{E}_t^{\mathbb{F}}[Y_{t+\delta}^a(\theta)],Z_t^a(\theta),\mathbb{E}_t^{\mathbb{F}}[Z_{t+\delta}^a(\theta)],0,0\right)dt \\ & & -Z_t^a(\theta)dB_t\ , \quad 0\le t\le T\ , \\ Y_{T+t}^a(\theta) & = & \xi_{T+t}^a(\theta)\ , \quad 0\le t\le\delta\ , \\ Z_{T+t}^a(\theta) & = & P_{T+t}^a(\theta)\ , \quad 0<t\le\delta\ , \end{array}\right.$$

et

$$\left\{\begin{array}{rcl} -dY_t^b & = & g(t,Y_t^b,Y_{t+\delta}^b,Z_t^b,Z_{t+\delta}^b)dt-Z_t^bdB_t\ , \quad 0\le t\le T \\ Y_{T+t}^b & = & \xi_{T+t}^b\ , \quad 0\le t\le\delta\ , \\ Z_{T+t}^b & = & P_{T+t}^b\ , \quad 0<t\le\delta\ , \end{array}\right.$$

où le générateur $g:\Omega\times[0,T]\times\mathbb{R}\times L^2(\mathcal{F}_{\cdot+\delta})\times\mathbb{R}\times L^2(\mathcal{F}_{\cdot+\delta})\to\mathbb{R}$ est défini par

$$\begin{array}{rcl} g(t,y,\zeta,z,\eta) & = & f^b\Big(t,y,\dfrac{1}{G_t}\left(\mathbb{E}_t^{\mathbb{F}}[\zeta G_{t+\delta}]+\displaystyle\int_t^{t+\delta}\mathbb{E}_t^{\mathbb{F}}[Y_{t+\delta}^a(\theta)\alpha_{t+\delta}(\theta)]d\theta\right),\ z, \\ & & \dfrac{1}{G_t}\Big(\mathbb{E}_t^{\mathbb{F}}[\eta G_{t+\delta}]+\displaystyle\int_t^{t+\delta}\mathbb{E}_t^{\mathbb{F}}[Z_{t+\delta}^a(\theta)\alpha_{t+\delta}(\theta)]d\theta\Big),\ Y_t^a(t)-y, \\ & & \dfrac{1}{G_t}\mathbb{E}_t^{\mathbb{F}}[(Y_{t+\delta}^a(t+\delta)-\zeta)G_{t+\delta}]\mathbb{1}_{t+\delta<T}+\dfrac{1}{G_t}\mathbb{E}_t^{\mathbb{F}}[Q_{t+\delta}G_{t+\delta}]\mathbb{1}_{t+\delta\ge T}\Big)\ . \end{array}$$

Nous montrons alors que ces deux équations vérifient les conditions données dans l'article de Peng et Yang et obtenons l'unicité de la solution.

Ls mêmes méthodes s'appliquent au cas de l'EDSRA du type Øksendal et Sulem, que l'on décompose en deux EDSRA dans la filtration brownienne,

$$\left\{\begin{array}{rcl} -dY_t^a(\theta) & = & \mathbb{E}_t^{\mathbb{F}}\big[f^a(t,\theta,Y_t^a(\theta),Y_{t+\delta}^a(\theta),Z_t^a(\theta),Z_{t+\delta}^a(\theta),0,0)\big]dt \\ & & -Z_t^a(\theta)dB_t\ , \quad \theta\le t\le T \\ Y_{T+t}^a(\theta) & = & \xi_{T+t}^a(\theta)\ , \quad 0\le t\le\delta\ , \\ Z_{T+t}^a(\theta) & = & P_{T+t}^a(\theta)\ , \quad 0<t\le\delta\ , \end{array}\right.$$

et la partie avant le saut

$$\left\{\begin{array}{rcl} -dY_t^b & = & g(t,Y_t^b,Y_{t+\delta}^b,Z_t^b,Z_{t+\delta}^b)dt-Z_t^bdB_t\ , \quad 0\le t\le T \\ Y_{T+t}^b & = & \xi_{T+t}^b\ , \quad 0\le t\le\delta\ , \\ Z_{T+t}^b & = & P_{T+t}^b\ , \quad 0<t\le\delta\ , \end{array}\right.$$

d'où

$$\begin{array}{rcl} g(t,y,\zeta,z,\eta) & = & \dfrac{1}{G_t}\displaystyle\int_t^{t+\delta}\mathbb{E}_t^{\mathbb{F}}[f^b(t,y,Y_{t+\delta}^a(\theta),z,Z_{t+\delta}^a(\theta),Y_t^a(t)-y,0)\alpha_{t+\delta}(\theta)]d\theta \\ & & +\dfrac{1}{G_t}\mathbb{E}_t^{\mathbb{F}}[f^b(t,y,\zeta,z,\eta,Y_t^a(t)-y, \\ & & \quad (Y_{t+\delta}^a(t+\delta)-\zeta)\mathbb{1}_{t+\delta<T}+Q_{t+\delta}\mathbb{1}_{t+\delta\ge T})]\ . \end{array}$$

Chapitre 1

Enlargement of filtration in discrete time

Introduction

In this chapter, we present classical results on enlargement of filtration, in a discrete time framework. In such a setting, any $\mathbb{F}$-martingale is a semimartingale for any filtration $\mathbb{G}$ larger than $\mathbb{F}$, and one can think that there are not so many things to do. From our point of view, one interest of our study is that the proofs of the semimartingale decomposition formulae are simple, and give a pedagogical support to understand the general formulae obtained in the literature in continuous time. It can be also noticed that many results are established in continuous time under the hypothesis that all $\mathbb{F}$-martingales are continuous or, in the progressive enlargement case, that the random time avoids the $\mathbb{F}$-stopping times and the extension to the general case is difficult. In discrete time, one can not make any of such assumptions, since martingales are discontinuous and random times valued in the set of integers do not avoid $\mathbb{F}$-stopping times. This chapter is an extended version of [BSJRR16].

In Section 1.1, we recall some well know facts. Section 1.2 is devoted to the case of initial enlargement. The long Section 1.3 presents the case of progressive enlargement with a random time τ. We give a "model-free" definition of arbitrages in the context of enlargement of filtration, we study some examples in initial enlargement and give, in a progressive enlargement setting, necessary and sufficient conditions to avoid arbitrages before τ. We present the particular case of honest times (which are the standard examples in continuous time) and we give conditions to obtain immersion property. We also give also various characterizations of pseudo-stopping times.

After this chapter was quite finished, we discovered the lecture notes of Spreij [Spr15]. We recommend these notes, in which the basic results of Section 1 are given and much more information can be found for discrete time martingales. Also we recommend the Chapter II of the book of Shiryaev [Shi99, Chapter II : Stochastic Models. Discrete Time].

1.1 Definitions, notation and some important results

Let $(\Omega, \mathcal{A}, \mathbb{A}, \mathbb{P})$ be a filtered probability space where $\mathcal{A}$ is a σ-algebra and $\mathbb{A} = \big(\mathcal{A}_n\big)_{n\geq 0}$ is a complete filtration with $\mathcal{A}_\infty = \vee_{n\geq 0}\mathcal{A}_n \subseteq \mathcal{A}$. We also consider a discrete filtration $\mathbb{B} = (\mathcal{B}_n)_{n\geq 0}$ such that $\mathbb{A} \subset \mathbb{B}$ and $\mathcal{B}_\infty \subseteq \mathcal{A}$.

A random variable ξ is positive (resp. non-negative) if $\xi > 0$, (resp. $\xi \geq 0$) and a process X is positive if for all $n \geq 0$, we have that $X_n > 0$. A process is non decreasing (resp. non increasing) if $X_{n-1} \leq X_n$ (resp. $X_n \leq X_{n-1}$) for all $n \geq 1$.

For a discrete time process X, we denote by $\Delta X_n := X_n - X_{n-1}$ its increment at time n, for $n \geq 1$, with the convention that $\Delta X_0 := X_0$. The process X_- is defined as $X_{-_n} = X_{n-1}, n \geq 1$ and $X_{-_0} = 0$. If X_∞ is defined and $\lim_{n\to\infty} X_n$ exists (in a.s. sense), we define the increment at infinity by $\Delta X_\infty := X_\infty - X_{\infty^-}$, where $X_{\infty^-} = \lim_{n\to\infty} X_n$.

Let $n \geq 0$ be fixed and ξ a random variable. We write, with an abuse of notation, $\xi \in \mathcal{A}_n$ to say that ξ is $\mathcal{A}_n$-measurable. If X is a process and ζ a non negative random variable valued in $\{0, 1, \ldots\}$, X^ζ denotes the process X stopped at ζ, i.e. $X^\zeta_n = X_{\zeta\wedge n}$, for all $n \geq 0$.

If $X_n \in \mathcal{A}_n$, for all $n \geq 0$, then we say that the process X is $\mathbb{A}$-adapted (or $\mathbb{A}$-optional). We say that the process X is $\mathbb{A}$-predictable if $X_n \in \mathcal{A}_{n-1}$, for all $n \geq 1$ and X_0 constant.

Remark 1.1.1 In discrete time, there is no distinction between optional and adapted processes. We recall that, in continuous time, the σ-algebra generated by the right-continuous and $\mathbb{A}$-adapted processes is called the $\mathbb{A}$-optional σ-algebra. A process is $\mathbb{A}$-optional if and only if it is measurable w.r.t. the $\mathbb{A}$-optional σ-algebra.

A process X is integrable if $\mathbb{E}|X_n| < \infty$ for all $n \geq 0$ and it is square-integrable if $\mathbb{E}|X_n|^2 < \infty$, for all $n \geq 0$.

A process M is an $\mathbb{A}$-local martingale if there exists a sequence of $\mathbb{A}$-stopping times $(\kappa_n)_{n\geq 0}$ such that :

- $\kappa_n < \kappa_{n+1}$ for all $n \geq 0$;
- $\lim_{n\to\infty} \kappa_n = \infty$;
- the stopped process $M^{\kappa_n} := M_{\cdot\wedge\kappa_n}$ is an $\mathbb{A}$-martingale for every $n \geq 0$.

A process X is an $\mathbb{A}$-semimartingale if it can be decomposed as a sum of a local martingale and a finite variation process.

The following theorem, established in [DM78, Page 89] or [Spr15], is a powerful tool.

Theorem 1.1.2 *If M is an $\mathbb{A}$-local martingale and M is integrable, then M is a martingale. A non-negative local martingale with $E(M_0) < \infty$ is a martingale.*

Let X and Y be two processes, then we denote by $X \centerdot Y$ the discrete stochastic integral, also called martingale transform, in the case where Y is a martingale, defined by

$$(X \centerdot Y)_n := \sum_{k=0}^{n} X_k \Delta Y_k \ , \quad \forall n \geq 0$$

and if the limit exists, we define $(X \centerdot Y)_{\infty^-} := \sum_{k=0}^{\infty} X_k \Delta Y_k = \lim_{n\to\infty}(X \centerdot Y)_n$. In particular, the stochastic integral $X_- \centerdot Y$ is given by

$$(X_- \centerdot Y)_n := \sum_{k=0}^{n} X_{k-1}\Delta Y_k \ , \quad \forall n \geq 0 \ .$$

1.1.1 Basic results

We recall some definitions and results in a discrete time setting that will be crucial for the next sections. We give three important decomposition theorems : Doob's decomposition, multiplicative decomposition and Kunita-Watanabe's decomposition. We recall the definitions of quadratic variation and predictable bracket. We define the exponential and logarithm process. Finally, we present Girsanov's Theorem using the quadratic variation and the predictable bracket.

Decompositions

The first one is Doob's decomposition (see [Doo53]), which allows to decompose explicitly any integrable and adapted process in a sum of a predictable process and a martingale. This result is central in discrete time theory since it implies that any martingale is a semi-martingale in a bigger filtration. The second theorem gives a multiplicative decomposition for any positive integrable adapted process in a martingale and a predictable process. The third theorem is the Kunita-Watanabe decomposition (see [KW67] and [FS04]).

Theorem 1.1.3 ***Doob's decomposition or semimartingale additive decomposition.*** *Let X be any integrable $\mathbb{A}$-adapted process, then X is an $\mathbb{A}$-semimartingale, which can be represented in a unique way as $X = P + M$, where P is an integrable $\mathbb{A}$-predictable process with $P_0 = 0$ and M is an $\mathbb{A}$-martingale. More precisely,*

$$M_0 := X_0 \ , \qquad M_n := X_0 + \sum_{k=1}^{n} \big(X_k - \mathbb{E}(X_k|\mathcal{A}_{k-1})\big) \ , \ \ \forall n \geq 1 \ , \tag{1.1.1}$$

$$P_0 := 0 \ , \qquad P_n := \sum_{k=1}^{n} \big(\mathbb{E}(X_k|\mathcal{A}_{k-1}) - X_{k-1}\big) \ , \ \ \forall n \geq 1 \ . \tag{1.1.2}$$

PROOF: Notice that for each $n \geq 0$ fixed, the random variable M_n, defined by (1.1.1), is integrable, as a sum of integrable random variables and

$$\mathbb{E}(M_n|\mathcal{A}_{n-1}) = X_0 + \mathbb{E}\big(X_n - \mathbb{E}(X_n|\mathcal{A}_{n-1})|\mathcal{A}_{n-1}\big) + \sum_{k=1}^{n-1} \big(X_k - \mathbb{E}(X_k|\mathcal{A}_{k-1})\big) = M_{n-1} \ ,$$

which proves that M is an $\mathbb{A}$-martingale.

In the other hand, P, defined in (1.1.2) is obviously an $\mathbb{A}$-predictable process, satisfies $P = X - M$ and is integrable. The uniqueness of the decomposition follows from the construction since $\mathbb{E}(X_k - X_{k-1}|\mathcal{A}_{k-1}) = P_k - P_{k-1}$ for all $k \geq 1$. Summing over $k = 1, \ldots, n$ and under the condition $P_0 = 0$, give $P_n = \sum_{k=1}^{n} \mathbb{E}(X_k - X_{k-1}|\mathcal{A}_{k-1})$. Consequently, $M_n = X_n - P_n$, for all $n \geq 0$, holds true. $\square$

Remark 1.1.4 The initialization $P_0 = 0$ and $M_0 = X_0$ is the traditional but arbitrary set-up in the Doob decomposition. It is in order to have uniqueness of the decomposition. In general we can set $P_0 = c$ and $M_0 = X_0 - c$ for any constant c.

Remark 1.1.5 Notice that if X is an integrable $\mathbb{A}$-supermartingale (resp. $\mathbb{A}$-submartingale) with Doob's decomposition $X = M + P$, then

$$\begin{aligned} M_{n-1} + P_n = \mathbb{E}(X_n|\mathcal{A}_{n-1}) \leq X_{n-1} = M_{n-1} + P_{n-1} \ , \\ (\text{resp. } M_{n-1} + P_n = \mathbb{E}(X_n|\mathcal{A}_{n-1}) \geq X_{n-1} = M_{n-1} + P_{n-1} \ ) \ , \end{aligned}$$

for all $n \geq 1$, therefore P is a non increasing (resp. non decreasing) process.

Definition 1.1.6 *Two $\mathbb{A}$-martingales X and Y are $\mathbb{A}$-orthogonal if the process XY is an $\mathbb{A}$-martingale.*

This is equivalent to $E(X_n \Delta Y_n|\mathcal{A}_{n-1}) = 0$ or $\mathbb{E}(\Delta X_n \Delta Y_n|\mathcal{A}_{n-1}) = 0$ for all $n \geq 1$.

Proposition 1.1.7 *Let X be an $\mathbb{A}$-adapted square-integrable semimartingale and Y a square-integrable $\mathbb{A}$-martingale, then $X \centerdot Y$ is an $\mathbb{A}$-martingale if and only if the $\mathbb{A}$-martingale part of X is $\mathbb{A}$-orthogonal to Y. In particular, if X is $\mathbb{A}$-predictable, $X \centerdot Y$ is an $\mathbb{A}$-martingale.*

Furthermore, if XY is square-integrable, then

$$\mathbb{E}(|(X \centerdot Y)_n|^2) = \mathbb{E}\Big(\sum_{k=0}^{n} |X_k|^2 |\Delta Y_k|^2\Big) \ , \quad \forall n \geq 0 \ .$$

Proof: Let $X = M + P$ the Doob decomposition of X, with M an $\mathbb{A}$-martingale, and P an $\mathbb{A}$-predictable process. For $n \geq 1$, we have due to the martingale property of Y that $\mathbb{E}(\Delta(X \centerdot Y)_n|\mathcal{A}_{n-1}) = \mathbb{E}(X_n \Delta Y_n|\mathcal{A}_{n-1})$. Then, since $P_n \in \mathcal{A}_{n-1}$ and M is $\mathbb{A}$-orthogonal to Y, we get that $\mathbb{E}(\Delta(X \centerdot Y)_n|\mathcal{A}_{n-1}) = 0$, hence, the martingale property is proved.

For the second part of the proof, since XY are square integrable then we have that $\mathbb{E}(|(X \centerdot Y)_n|^2) < \infty$. In the other hand, for all $n \geq k \geq 1$, it is well known due to the fact that $X \centerdot Y$ is a martingale that $\mathbb{E}\big(\Delta((X \centerdot Y)^2)_k\big|\mathcal{A}_{k-1}\big) = \mathbb{E}\big((\Delta(X \centerdot Y)_k)^2\big|\mathcal{A}_{k-1}\big)$. Then, using $\Delta(X \centerdot Y)_k = X_k \Delta Y_k$ and taking expectations, we get

$$\mathbb{E}\big(\Delta((X \centerdot Y)^2)_k\big) = \mathbb{E}\big(X_k^2 (\Delta Y_k)^2\big) \ . \tag{1.1.3}$$

Finally, taking the sum in (1.1.3) for k from 0 to n, we obtain

$$\mathbb{E}\big(|(X \centerdot Y)_n|^2\big) = \mathbb{E}\Big(\sum_{k=0}^{n} |X_k|^2 |\Delta Y_k|^2\Big) \ , \quad \forall n \geq 0 \ .$$

□

More generally, from Theorem 1.1.2, if Y is a martingale and X a predictable process such that, for any n, the random variable $X_n \Delta Y_n$ is integrable, then $X \centerdot Y$ is an $\mathbb{A}$-martingale. Otherwise, it is a local martingale.

Lemma 1.1.8 *Let ζ be an integrable positive random variable, then $Y_n := \mathbb{E}(\zeta|\mathcal{A}_n) > 0$, i.e., Y is a positive process.*

PROOF: For $n \geq 0$, we have by monotonicity of the conditional expectation that $Y_n \geq 0$. Then, we have to prove that $\mathbb{P}(Y_n = 0) = 0$. Consider the set $A = \{\omega \in \Omega : Y_n(\omega) = 0\} \in \mathcal{A}_n$. Therefore, we obtain that $\mathbb{E}(Y_n \mathbb{1}_A) = 0$. In the other hand, we have by definition of the conditional expectation that $\mathbb{E}(Y_n \mathbb{1}_A) = \mathbb{E}(\zeta \mathbb{1}_A)$, hence $\mathbb{E}(\zeta \mathbb{1}_A) = 0$, but since ζ is positive, we have that necessarily A has measure zero. □

Theorem 1.1.9 ***Semimartingale multiplicative decomposition***. *Let X be a positive integrable $\mathbb{A}$-adapted process, then X is an $\mathbb{A}$-semimartingale which admits the representation*

$$X = \widehat{P}\widehat{M} ,$$

where $\widehat{P}$ is an $\mathbb{A}$-predictable process and $\widehat{M}$ is an $\mathbb{A}$-martingale. Moreover, if we set $\widehat{P}_0 = 1$ the decomposition is unique and is given by

$$\widehat{M}_0 := X_0 , \qquad \widehat{M}_n := X_0 \prod_{k=1}^{n} \frac{X_k}{\mathbb{E}(X_k|\mathcal{A}_{k-1})} , \quad \forall n \geq 1 , \tag{1.1.4}$$

$$\widehat{P}_0 := 1 , \qquad \widehat{P}_n := \prod_{k=1}^{n} \frac{\mathbb{E}(X_k|\mathcal{A}_{k-1})}{X_{k-1}} , \quad \forall n \geq 1 .$$

PROOF: For any $n \geq 1$, the positive random variable $\widehat{M}_n$, given by (1.1.4) is well defined, since the denominator does not vanish by Lemma 1.1.8. By definition, $\widehat{M}_n$ is $\mathcal{A}_n$-measurable and is integrable, since

$$\mathbb{E}(\widehat{M}_n) = X_0 \mathbb{E}\big(\mathbb{E}(\cdots \big(\mathbb{E}\big(\prod_{k=1}^{n} \frac{X_k}{\mathbb{E}(X_k|\mathcal{A}_{k-1})}\big)\big|\mathcal{A}_{n-1}\big) \cdots \big|\mathcal{A}_1)\big) = X_0 < \infty ,$$

with $X_k \in L^1(\mathcal{A}_{n-1}, \Omega)$ for all $k \in \{0, \dots, n-1\}$, also

$$\mathbb{E}(\widehat{M}_n|\mathcal{A}_{n-1}) \quad = X_0 \prod_{k=1}^{n-1} \tfrac{X_k}{\mathbb{E}(X_k|\mathcal{A}_{k-1})} \mathbb{E}\Big(\tfrac{X_n}{\mathbb{E}(X_n|\mathcal{A}_{n-1})}\Big|\mathcal{A}_{n-1}\Big) \quad = \widehat{M}_{n-1} ,$$

then $\widehat{M}$ is an $\mathbb{A}$-martingale.

Moreover, by definition $\widehat{P}$ is an $\mathbb{A}$-predictable process. □

Remark 1.1.10 Notice that if X is a positive integrable $\mathbb{A}$-supermartingale (resp. $\mathbb{A}$-submartingale) with multiplicative decomposition $X = \widehat{M}\widehat{P}$, then $\widehat{P}$ is a non increasing (resp. non decreasing) process,

$$\widehat{M}_{n-1}\widehat{P}_n = \mathbb{E}(X_n|\mathcal{A}_{n-1}) \leq X_{n-1} = \widehat{M}_{n-1}\widehat{P}_{n-1} , \quad \forall n \geq 1 ,$$
$$(\text{resp. } \widehat{M}_{n-1}\widehat{P}_n = \mathbb{E}(X_n|\mathcal{A}_{n-1}) \geq X_{n-1} = \widehat{M}_{n-1}\widehat{P}_{n-1} , \quad \forall n \geq 1) .$$

Let X a positive integrable $\mathbb{A}$-semimartingale with Doob's decomposition $X = P+M$ (Theorem 1.1.3) and with multiplicative decomposition $X = \widehat{P}\widehat{M}$ (Theorem 1.1.9) , one has

$$\widehat{P}_0 = P_0 + 1 \,, \quad \widehat{P}_n = \frac{M_{n-1}+P_n}{X_{n-1}}\widehat{P}_{n-1} = X_0 \prod_{k=1}^{n} \frac{M_{k-1}+P_k}{X_{k-1}} \,, \quad \forall n \geq 1 \,,$$

$$\widehat{M}_0 = M_0 \,, \quad \widehat{M}_n = \frac{X_n}{\widehat{P}_n} = \prod_{k=1}^{n} \frac{X_k}{M_{k-1}+P_k} \,, \quad \forall n \geq 1 \,.$$

Theorem 1.1.11 ***Kunita-Watanabe decomposition***. *Given a square-integrable $\mathbb{A}$-martingale W, every $\mathbb{A}$-martingale X is of the form*

$$X_n = X_0 + \sum_{k=1}^{n} P_k \,\Delta W_k + M_n \,, \quad \forall n \geq 1 \,,$$

or equivalently

$$\Delta X_n = P_n \,\Delta W_n + \Delta M_n \,, \quad \forall n \geq 1 \,, \tag{1.1.5}$$

where P is a square-integrable $\mathbb{A}$-predictable process and M is an $\mathbb{A}$-martingale $\mathbb{A}$-orthogonal to W satisfying $M_0 = 0$.

Moreover, if we define for $n \geq 1$ the set

$$A_{n-1} := \big\{\omega \in \Omega : \mathbb{E}(|\Delta W_n|^2 | \mathcal{A}_{n-1})(\omega) \neq 0\big\}$$

then, on A_{n-1}, the random variable P_n satisfies

$$P_n = \frac{\mathbb{E}\big(\Delta X_n \,\Delta W_n \big| \mathcal{A}_{n-1}\big)}{\mathbb{E}\big(|\Delta W_n|^2 \big| \mathcal{A}_{n-1}\big)} \,. \tag{1.1.6}$$

Proof: For the existence see [FS04, Theorem 10.18].

From (1.1.6) we get that $P_n \in \mathcal{A}_{n-1}$, using this and (1.1.5) we have that

$$\mathbb{E}(\Delta M_n \Delta W_n | \mathcal{A}_{n-1}) \;=\; \mathbb{E}(\Delta X_n \Delta W_n | \mathcal{A}_{n-1}) - P_n \mathbb{E}(|\Delta W_n|^2 | \mathcal{A}_{n-1}) \,.$$

It follows, from the $\mathbb{A}$-orthogonality of M and W, that

$$\mathbb{E}(\Delta X_n \Delta W_n | \mathcal{A}_{n-1}) - P_n \mathbb{E}(|\Delta W_n|^2 | \mathcal{A}_{n-1}) = 0 \,,$$

hence the form of P_n on A_{n-1}. □

Brackets

The quadratic variation and the predictable bracket are fundamental for stochastic calculus. Here, we recall these definitions in a discrete time setting and we give their computation. We have the analogue of the formula for integration by parts. Also, we give Girsanov's Theorem in terms of quadratic variation and predictable bracket.

Definition 1.1.12 ***Quadratic variation***. *The quadratic variation of a process X is defined by*

$$[X]_n := \sum_{k=0}^{n} (\Delta X_k)^2 \ , \quad \forall n \geq 0 \ .$$

The quadratic covariation $[X, Y]$ of processes X and Y is defined by

$$[X, Y]_n := \sum_{k=0}^{n} \Delta X_k \Delta Y_k \ , \quad \forall n \geq 0 \ .$$

Lemma 1.1.13 *If X and Y are two square integrable $\mathbb{A}$-martingales then $XY - [X, Y]$ is an $\mathbb{A}$-martingale. In particular, $X^2 - [X]$ is an $\mathbb{A}$-martingale for a square-integrable $\mathbb{A}$-martingale X.*

Proof: For $n \geq 1$ fixed, we have by definition of the quadratic variation,

$$\mathbb{E}(\Delta(XY - [X, Y])_n | \mathcal{A}_{n-1}) = \mathbb{E}\Big(\Delta(XY)_n - \Delta X_n \Delta Y_n | \mathcal{A}_{n-1}\Big) \tag{1.1.7}$$

then, simplifying (1.1.7) and using the fact that X_{n-1}, $Y_{n-1} \in \mathcal{A}_{n-1}$ and that X and Y are $\mathbb{A}$-martingales, we get that

$$\begin{aligned} \mathbb{E}(\Delta(XY - [X, Y])_n | \mathcal{A}_{n-1}) &= \mathbb{E}(X_n Y_{n-1} - X_{n-1} Y_n | \mathcal{A}_{n-1}) \\ &= Y_{n-1} \mathbb{E}(X_n | \mathcal{A}_{n-1}) - X_{n-1} \mathbb{E}(Y_n | \mathcal{A}_{n-1}) = 0 \ . \end{aligned}$$

□

Proposition 1.1.14 ***Integration by parts formula***. *For any pair of processes X and Y, we have that*

$$\begin{aligned} X_n Y_n &= X_0 Y_0 + (X_- \cdot Y)_n + (Y_- \cdot X)_n + [X, Y]_n \\ &= X_0 Y_0 + (X_- \cdot Y)_n + (Y \cdot X)_n \ , \end{aligned}$$

for all $n \geq 1$.

Proof: For $n \geq 1$ fixed. Notice that

$$\Delta(XY)_n = X_{n-1} Y_n - X_{n-1} Y_{n-1} + X_n Y_{n-1} + X_n Y_n - X_n Y_{n-1} - X_{n-1} Y_n \ ,$$

factorizing properly, we get that

$$\Delta(XY)_n = X_{n-1} \Delta Y_n + Y_{n-1} \Delta X_n + \Delta X_n \Delta Y_n$$

and the result follows. □

Definition 1.1.15 ***The predictable bracket***. *The predictable bracket of two $\mathbb{A}$-semimartingales X and Y in discrete time, denoted by $\langle X, Y \rangle^{\mathbb{A},\mathbb{P}}$, is the unique $\mathbb{A}$-predictable process such that $[X, Y] - \langle X, Y \rangle^{\mathbb{A},\mathbb{P}}$ is an $(\mathbb{A}, \mathbb{P})$-martingale and $\langle X, Y \rangle_0^{\mathbb{A},\mathbb{P}} = 0$. In case of no ambiguity, we denote the predictable bracket of X and Y by $\langle X, Y \rangle$.*

Lemma 1.1.16 *Let X and Y two $\mathbb{A}$-adapted square integrable processes. Then,*

$$\langle X, Y\rangle_n = \sum_{k=1}^{n} \mathbb{E}(\Delta X_k \Delta Y_k | \mathcal{A}_{k-1}) \ , \quad \forall n \geq 1 \ .$$

In particular, if Y is an $\mathbb{A}$-martingale, $\langle X, Y\rangle_n = \sum_{k=1}^{n} \mathbb{E}(X_k \Delta Y_k | \mathcal{A}_{k-1})$.

PROOF: From Doob's decomposition Theorem 1.1.3 applied to the process $[X, Y]$

$$\langle X, Y\rangle_n = \sum_{k=1}^{n} \mathbb{E}\big([X, Y]_k - [X, Y]_{k-1} \big| \mathcal{A}_{k-1}\big) = \sum_{k=1}^{n} \mathbb{E}(\Delta X_k \Delta Y_k | \mathcal{A}_{k-1}) \ , \ \ \forall n \geq 1 \ .$$

□

Exponential and logarithm process

We define the exponential and the logarithm process and we give some important properties.

Definition 1.1.17 ***Exponential process***. *The exponential process of X, denoted by $\mathcal{E}(X)$, is the solution of the following equation in differences :*

$$\begin{cases} \Delta \mathcal{E}(X)_n &= \mathcal{E}(X)_{n-1} \Delta X_n \ , \quad \forall n \geq 1 \ , \\ \mathcal{E}(X)_0 &= 1 \ . \end{cases} \tag{1.1.8}$$

Proposition 1.1.18 *The solution of (1.1.8), is given by*

$$\mathcal{E}(X)_n = \Pi_{k=1}^{n} (1 + \Delta X_k) \ , \quad \forall n \geq 1 \ . \tag{1.1.9}$$

PROOF: The proof is made by induction. First for $n = 1$ from (1.1.8), we have that

$$\Delta \mathcal{E}(X)_1 = \mathcal{E}(X)_0 \Delta X_1 \ , \tag{1.1.10}$$

then, using the initial condition, we have that (1.1.10) is equivalent to

$$\mathcal{E}(X)_1 = 1 + \Delta X_1 \ ,$$

which satisfies the hypothesis of induction (1.1.9) for $n = 1$.

Now, we suppose that (1.1.9) is satisfied for n, and we prove that it is satisfied for $n + 1$. Using (1.1.8) for $n + 1$, we get

$$\Delta \mathcal{E}(X)_{n+1} = \mathcal{E}(X)_n \Delta X_{n+1} \ , \tag{1.1.11}$$

then using the induction hypothesis in (1.1.11), i.e. using that $\mathcal{E}(X)_n = \Pi_{k=1}^{n}(1 + \Delta X_k)$, we have

$$\mathcal{E}(X)_{n+1} - \Pi_{k=1}^{n} (1 + \Delta X_k) = \Pi_{k=1}^{n} (1 + \Delta X_k) \Delta X_{n+1} \ ,$$

which is equivalent to $\mathcal{E}(X)_{n+1} = \Pi_{k=1}^{n+1} (1 + \Delta X_k) = \mathcal{E}(X)_n (1 + \Delta X_{n+1})$. □

Corollary 1.1.19 *$\mathcal{E}(X)$ is positive if and only if $\Delta X > -1$.*

PROOF: The proof follows directly from Proposition 1.1.18 . $\square$

Notice that if there exists $m \leq n$ such that the set $D_m := \{\Delta X_m = -1\}$ is not empty, then $\mathbb{1}_{D_m}\mathcal{E}(X)_n = \mathbb{1}_{D_m}\Pi_{k=1}^n(1+\Delta X_k) = 0$. If the condition $\Delta X > -1$ fails, the exponential process takes negative values.

Proposition 1.1.20 *Let X be an $\mathbb{A}$-martingale, such that $\Delta X > -1$. Then $\mathcal{E}(X)$ is an $\mathbb{A}$-martingale with $\mathbb{E}\big(\mathcal{E}(X)\big) = 1$.*

PROOF: The proof is a consequence of the fact that the local martingale property follows from $\mathcal{E}(X)_n = 1 + \sum_{k=0}^n \mathcal{E}(X)_{k-1}\Delta X_k$ and the martingale property from Theorem 1.1.2, due to the positivity of $\mathcal{E}(X)$. $\square$

In the general case, $\mathcal{E}(X)$ is a local martingale. If it is an integrable process, this is a martingale.

Definition 1.1.21 ***Logarithm process***. *We define the logarithm process of a positive process Y, denoted by $\mathcal{L}og(Y)$, as the solution of the following equation in differences :*

$$\begin{cases} \Delta Y_n &= \Delta\mathcal{L}og(Y)_n Y_{n-1} \ , \quad \forall n \geq 1 \ , \\ \mathcal{L}og(Y)_0 &= 0 \ . \end{cases} \tag{1.1.12}$$

Proposition 1.1.22 *Let Y be a positive process, then*

$$\mathcal{L}og(Y)_n = \sum_{k=1}^{n} \frac{\Delta Y_k}{Y_{k-1}} \ , \quad \forall n \geq 1 \ . \tag{1.1.13}$$

PROOF: The proof follows directly from the definition of $\mathcal{L}og$. $\square$

Proposition 1.1.23 *Let Y be a positive process with $Y_0 = 1$ and define $X := \mathcal{L}og(Y)$, then $\mathcal{E}(X) = Y$.*

PROOF: The proof follows directly by the definitions of $\mathcal{E}$ and $\mathcal{L}og$. $\square$

Remark 1.1.24 Notice that if Y is a positive martingale and $\mathcal{L}og(Y)$ is integrable, then $\mathcal{L}og(Y)$ is a martingale.

Equivalent probability measure

We present the Girsanov Theorem (see [Gir60]) using the concepts of quadratic variation and predictable bracket and give the proof using Doob's decomposition.

Theorem 1.1.25 ***Girsanov's Theorem for discrete martingales***. *Let X be an $(\mathbb{A}, \mathbb{P})$-martingale, $\mathbb{Q}$ a probability measure equivalent to $\mathbb{P}$, and L its Radon-Nikodym* *density* $L_n = \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{A}_n}$. *Assume that* X_n *is* $\mathbb{Q}$*-integrable for all* $n \geq 0$*, then the processes* $M^{\langle X \rangle}$ *and* $M^{[X]}$*, given by*

$$M_0^{\langle X \rangle} = X_0 \ , \quad M_n^{\langle X \rangle} = X_n - \sum_{k=1}^{n} \frac{1}{L_{k-1}} \Delta \langle X, L \rangle_k^{\mathbb{P}} \ , \quad \forall n \geq 1 \ ,$$

$$M_0^{[X]} = X_0 \ , \quad M_n^{[X]} = X_n - \sum_{k=1}^{n} \frac{1}{L_k} \Delta \big[X, L\big]_k \ , \quad \forall n \geq 1 \ ,$$

are $(\mathbb{A}, \mathbb{Q})$*-martingales. The process* $M^{\langle X \rangle}$ *and* $M^{[X]}$ *will be called Girsanov's transformation of* X *given by the predictable bracket and the quadratic variation respectively.*

Proof: In the one hand, by Doob's decomposition (Theorem 1.1.3), we know that

$$\Big(X_n - \sum_{k=1}^{n} \mathbb{E}^{\mathbb{Q}}(\Delta X_k | \mathcal{A}_{k-1}) \Big)_{n \geq 1}$$

is an $(\mathbb{A}, \mathbb{Q})$-martingale, where

$$\mathbb{E}^{\mathbb{Q}}(\Delta X_k | \mathcal{A}_{k-1}) = \frac{1}{L_{k-1}} \mathbb{E}^{\mathbb{P}}(L_k \Delta X_k | \mathcal{A}_{k-1}) = \frac{1}{L_{k-1}} \Delta \langle X, L \rangle_k^{\mathbb{P}} \ , \quad \forall k \geq 1 \ ,$$

thus $M^{\langle X \rangle}$ is an $(\mathbb{A}, \mathbb{Q})$-martingale.
In the other hand, to prove that $M^{[X]}$ is an $(\mathbb{A}, \mathbb{Q})$-martingale, we have that for $n \geq 1$ fixed, by definition of L_n and $\Delta M_n^{[X]}$,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}(\Delta M_n^{[X]} | \mathcal{A}_{n-1}) &= \frac{1}{L_{n-1}} \mathbb{E}(\Delta M_n^{[X]} L_n | \mathcal{A}_{n-1}) = \frac{1}{L_{n-1}} \mathbb{E}\Big(\Big(\Delta X_n - \frac{\Delta [X, L]_n}{L_n} \Big) L_n | \mathcal{A}_{n-1} \Big) \\ &= \frac{1}{L_{n-1}} \mathbb{E}(L_n \Delta X_n - \Delta X_n \Delta L_n | \mathcal{A}_{n-1}) \\ &= \frac{1}{L_{n-1}} \mathbb{E}\big(L_{n-1} \Delta X_n | \mathcal{A}_{n-1}\big) = 0 \ , \end{aligned}$$

where the last equality is due to the $(\mathbb{A}, \mathbb{P})$-martingale property of X. □

Corollary 1.1.26 *Let* Y *be an* $(\mathbb{A}, \mathbb{P})$*-martingale such that* $\Delta Y > -1$*, and define the probability* $\mathbb{Q}$ *as*

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{A}_n} = \mathcal{E}(Y)_n \ , \quad \forall n \geq 0 \ .$$

Then for a $\mathbb{Q}$*-integrable* $(\mathbb{A}, \mathbb{P})$*-martingale* X*, the process* $\widetilde{X} = X - \langle X, Y \rangle^{\mathbb{P}}$ *is a* $(\mathbb{A}, \mathbb{Q})$*-martingale.*

Proof: By definition, $\Delta \langle X, \mathcal{E}(Y) \rangle_k^{\mathbb{P}} = \mathcal{E}(Y)_{k-1} \mathbb{E}^{\mathbb{P}}((1 + \Delta Y_k) \Delta X_k | \mathcal{A}_{k-1})$. Using that X is an $(\mathbb{A}, \mathbb{P})$-martingale, $\mathbb{E}^{\mathbb{P}}((1 + \Delta Y_k) \Delta X_k | \mathcal{A}_{k-1}) = \mathbb{E}^{\mathbb{P}}(\Delta X_k \Delta Y_k | \mathcal{A}_{k-1})$ and the result follows from Girsanov's Theorem. □

Projections

We end this section with the definition of optional and predictable projections and of dual optional and dual predictable projection. The dual projection concepts were introduced originally in continuous time, for more details we refer to [Nik06], [JYC09, Section 5.2] or [HWY92, Chapter V]. In continuous time, dual projections are defined only for integrable finite variation processes ; in discrete time, since any process is with finite variation, we can define dual projections for any integrable process.

Definition 1.1.27 ***Optional and predictable projection***. *Let X be an integrable process (not necessarily $\mathbb{A}$-adapted). We call the $\mathbb{A}$-optional (resp. $\mathbb{A}$-predictable) projection of X, the integrable $\mathbb{A}$-optional (resp. $\mathbb{A}$-predictable) process defined as ${}^{(o)}X_n = \mathbb{E}(X_n|\mathcal{A}_n)$ for all $n \geq 0$ (resp. ${}^{(p)}X_n = \mathbb{E}(X_n|\mathcal{A}_{n-1})$ and ${}^{(p)}X_0 = \mathbb{E}(X_0|\mathcal{A}_0)$) .*

Definition 1.1.28 ***Dual optional projection***. *Let X be an integrable process (not necessarily $\mathbb{A}$-adapted). We call the dual $\mathbb{A}$-optional projection of X, the integrable $\mathbb{A}$-optional ($\mathbb{A}$-adapted) process $X^{(o)}$, defined as $\Delta X^{(o)}_n = \mathbb{E}(\Delta X_n|\mathcal{A}_n)$ for all $n \geq 0$.*

Remark 1.1.29 Notice that the dual $\mathbb{A}$-optional projection of X, satisfies that

$$\mathbb{E}\Big((Y \centerdot X)_{\infty^-}\Big) = \mathbb{E}\Big((Y \centerdot X^{(o)})_{\infty^-}\Big)$$

for any non negative bounded $\mathbb{A}$-optional process Y. Moreover, if X is non decreasing, then $X^{(o)}$ is also non decreasing.

Definition 1.1.30 ***Dual predictable projection***. *Let X be an integrable process (not necessarily $\mathbb{A}$-adapted). We call the dual $\mathbb{A}$-predictable projection of X, the integrable $\mathbb{A}$-predictable process $X^{(p)}$, defined as $X^{(p)}_0 = \mathbb{E}(X_0|\mathcal{A}_0)$ and $\Delta X^{(p)}_n = \mathbb{E}(\Delta X_n|\mathcal{A}_{n-1})$ for all $n \geq 1$.*

Remark 1.1.31 Notice that dual $\mathbb{A}$-predictable projection of X, satisfies that

$$\mathbb{E}\Big((Y \centerdot X)_{\infty^-}\Big) = \mathbb{E}\Big((Y \centerdot X^{(p)})_{\infty^-}\Big)$$

for any non negative bounded $\mathbb{A}$-predictable process Y. Moreover, if X is non decreasing, then $X^{(p)}$ is also non decreasing.

Theorem 1.1.32 *Let X be an integrable process (not necessarily $\mathbb{A}$-adapted). Then, the processes $Y = {}^{(o)}X - X^{(o)}$ and $\widehat{Y} = {}^{(o)}X - X^{(p)}$ are $\mathbb{A}$-martingales.*

PROOF: The integrability of Y and $\widehat{Y}$ is obvious. Then, we have that for all $n \geq 1$,

$$\mathbb{E}(\Delta Y_n|\mathcal{F}_{n-1}) = \mathbb{E}\big(\Delta\mathbb{E}(X_n|\mathcal{F}_n) - \mathbb{E}(\Delta X_n|\mathcal{F}_n)|\mathcal{F}_{n-1}\big) = 0$$

and

$$\mathbb{E}(\Delta\widehat{Y}_n|\mathcal{F}_{n-1}) = \mathbb{E}\big(\Delta\mathbb{E}(X_n|\mathcal{F}_n) - \mathbb{E}(\Delta X_n|\mathcal{F}_{n-1})|\mathcal{F}_{n-1}\big) = 0\ .$$

□

1.1.2 Arbitrage

In continuous time, the classical no-arbitrage theory is based on the notions of Arbitrage Opportunity and Free Lunch with Vanishing Risk, as developed by Delbaen & Schachermayer [DS94]. In our setting, we consider the following definition 1.1.33 for no arbitrage in a filtration, a price process being given (see for example [JS98] for more details about arbitrages in discrete time and [Bjo09] for the continuous setting).

In the enlargement of filtration setting, we pay attention to all $\mathbb{A}$-martingales. We give a "model free" definition of arbitrage, in the sense that we do not specify the price process in the filtration $\mathbb{A}$ and we give conditions for the existence of a deflator for all the $\mathbb{A}$-martingales. The study of conditions so that, for a given $\mathbb{A}$-martingale X, there exists a deflator, can be found in [CD14].

Definition 1.1.33 *Let X be an $\mathbb{A}$-semimartingale. We say that the model $(X, \mathbb{A})$ has no arbitrages if there exists a positive $\mathbb{A}$-martingale L, with $L_0 = 1$, such that XL is an $\mathbb{A}$-martingale.*

We start with a general result, valid for any filtration.

Lemma 1.1.34 *Let X be an $\mathbb{A}$-semimartingale. If there exists a positive $\mathbb{A}$-adapted process ψ such that*

$$\mathbb{E}(X_n\psi_n|\mathcal{A}_{n-1}) = X_{n-1}\mathbb{E}(\psi_n|\mathcal{A}_{n-1}) \ , \quad \forall n \geq 1 \ , \tag{1.1.14}$$

then, there exists a positive $\mathbb{A}$-martingale L such that LX is an $\mathbb{A}$-martingale.

Proof: Let X be a $(\mathbb{P}, \mathbb{A})$-semimartingale with $(\mathbb{P}, \mathbb{A})$-Doob's decomposition $X = M + P$, where $\Delta P_n = \mathbb{E}(\Delta X_n|\mathcal{A}_{n-1})$ for all $n \geq 1$ and M is a $(\mathbb{P}, \mathbb{A})$-martingale. If a process ψ satisfying (1.1.14) exists, then we define the positive martingale L by

$$L_0 = 1 \ \textit{and} \ L_n = \prod_{k=1}^{n} \frac{\psi_k}{\mathbb{E}(\psi_k|\mathcal{A}_{k-1})} = L_{n-1}\frac{\psi_n}{\mathbb{E}(\psi_n|\mathcal{A}_{n-1})} \ , \quad \forall n \geq 1 \ .$$

Setting $d\mathbb{Q} = Ld\mathbb{P}$, the process M has $(\mathbb{Q}, \mathbb{A})$-Doob's decomposition given by $M = \widetilde{M} + \widetilde{P}$ where $\widetilde{M}$ is a $(\mathbb{Q}, \mathbb{A})$-martingale and

$$\Delta\widetilde{P}_n = \mathbb{E}^{\mathbb{Q}}(\Delta M_n|\mathcal{A}_{n-1}) = \frac{1}{L_{n-1}}\mathbb{E}(L_n\Delta M_n|\mathcal{A}_{n-1}) \ , \quad \forall n \geq 1 \ .$$

By definition of $\frac{L_n}{L_{n-1}} = \frac{\psi_n}{\mathbb{E}(\psi_n|\mathcal{A}_{n-1})}$ for all $n \geq 1$, we get that

$$\Delta\widetilde{P}_n = \frac{1}{\mathbb{E}(\psi_n|\mathcal{A}_{n-1})}\mathbb{E}(\psi_n\Delta M_n|\mathcal{A}_{n-1}) \ , \quad \forall n \geq 1 \ .$$

Therefore, the process X is a $(\mathbb{Q}, \mathbb{A})$-martingale if $P + \widetilde{P} = 0$, i.e. if for all $n \geq 1$, we have that

$$\frac{1}{\mathbb{E}(\psi_n|\mathcal{A}_{n-1})}\mathbb{E}(\psi_n\Delta M_n|\mathcal{A}_{n-1}) + \mathbb{E}(\Delta X_n|\mathcal{A}_{n-1}) = 0 \ ,$$

or equivalently

$$\mathbb{E}(\psi_n \Delta M_n|\mathcal{A}_{n-1}) + \mathbb{E}(\psi_n|\mathcal{A}_{n-1})\mathbb{E}(\Delta X_n|\mathcal{A}_{n-1}) = 0 ,$$

then using that $\Delta M_n = X_n - \mathbb{E}(X_n|\mathcal{A}_{n-1})$, we get

$$\mathbb{E}(\psi_n X_n|\mathcal{A}_{n-1}) - \mathbb{E}(\psi_n|\mathcal{A}_{n-1})\mathbb{E}(X_n|\mathcal{A}_{n-1}) + \mathbb{E}(\psi_n|\mathcal{A}_{n-1})\mathbb{E}(\Delta X_n|\mathcal{A}_{n-1}) = 0 .$$

Finally, simplifying we obtain that

$$\mathbb{E}(\psi_n X_n|\mathcal{A}_{n-1}) = \mathbb{E}(\psi_n|\mathcal{A}_{n-1})X_{n-1} ,$$

which concludes the proof. □

In the setting of enlargement of filtration, we introduce the following "model free" definition (see [ACDJ13] for the continuous case).

Definition 1.1.35 *Let $\mathbb{A} \subset \mathbb{B}$, we say that the model $(\mathbb{A}, \mathbb{B})$ is arbitrage free if there exists a positive $\mathbb{B}$-martingale L with $L_0 = 1$ (called a deflator) such that, for any $\mathbb{A}$-martingale X, the process XL is a $\mathbb{B}$-martingale.*

1.1.3 Filtration enlargement

Now we recall general definitions and results of filtration enlargement (see for example [JY78b], [JY78a], [Jeu80] and [AJ16]).

Consider the filtrations $\mathbb{A}$ and $\mathbb{B}$, such that $\mathbb{A} \subset \mathbb{B}$. One of the main problem in continuous time of enlargement of filtration is to give a criterium such that all the $\mathbb{A}$-martingales remain semimartingales in $\mathbb{B}$ and, if it is the case, to give the semimartingale decomposition in $\mathbb{B}$. In discrete time, we have that we can decompose any integrable process as a sum of a martingale and a predictable process.

Remark 1.1.36 *Notice that all the results in continuous time for filtration enlargement can be directly applied to the discrete time setting. For example, If X is a discrete time process and $\mathbb{A}$ the discrete time filtration, then we can set the right continuous filtration $\widehat{\mathbb{A}} = (\mathcal{A}_t)_{t\in[0,\infty)}$, with $\widehat{\mathcal{A}}_t = \mathcal{A}_n \mathbb{1}_{\{t\in[n,n+1)\}}$, and the càdlàg process $\widehat{X} = (X_t)_{t\in[0,\infty)}$ with $\widehat{X}_t = X_n \mathbb{1}_{\{t\in[n,n+1)\}}$.*

Definition 1.1.37 *$(\mathcal{H}')$-hypothesis is satisfied between the filtration $\mathbb{A}$ and a larger filtration $\mathbb{B}$ if any $\mathbb{A}$-martingale is a $\mathbb{B}$-semimartingale.*

In discrete time, we have the following theorem, which shows that $(\mathcal{H}')$-hypothesis is always satisfied. This is a crucial difference between continuous case and discrete case. In continuous case, $(\mathcal{H}')$-hypothesis may fail (see [JY79]).

Theorem 1.1.38 *Every $\mathbb{A}$-martingale is a $\mathbb{B}$-semimartingale.*

Proof: Let M be an $\mathbb{A}$-martingale, then M is $\mathbb{B}$-adapted and by Theorem 1.1.3 M is a $\mathbb{B}$-semimartingale. □

Definition 1.1.39 *($\mathcal{H}$)-hypothesis (immersion) is satisfied between the filtration $\mathbb{A}$ and a larger filtration $\mathbb{B}$, if any $\mathbb{A}$-martingale is a $\mathbb{B}$-martingale. If this property is achieved, we will denote it by $\mathbb{A} \hookrightarrow \mathbb{B}$. In order to specify that the immersion is achieved with the probability measure $\mathbb{P}$, we will denote it by $\mathbb{A} \overset{\mathbb{P}}{\hookrightarrow} \mathbb{B}$.*

The following result is well known and useful (see for example [BY78]) .

Proposition 1.1.40 *($\mathcal{H}$)-hypothesis is equivalent to any of the following properties :*

($\mathcal{H}$1) $\forall n \geq 0$, *the σ-fields $\mathcal{A}_\infty$ and $\mathcal{B}_n$ are conditionally independent given $\mathcal{A}_n$, i.e. if for all $n \geq 0$, for all random variables $B_n \in \mathcal{B}_n$ and $A \in \mathcal{A}_\infty$, with A and B_n square-integrable, $\mathbb{E}(AB_n|\mathcal{A}_n) = \mathbb{E}(A|\mathcal{A}_n)\mathbb{E}(B_n|\mathcal{A}_n)$.*

($\mathcal{H}$2) $\forall n \geq 0$, *$B_n \in \mathcal{B}_n$, with B_n integrable, $\mathbb{E}(B_n|\mathcal{A}_n) = \mathbb{E}(B_n|\mathcal{A}_\infty)$.*

($\mathcal{H}$3) $\forall n \geq 0$, *$A \in \mathcal{A}_\infty$, with A integrable, $\mathbb{E}(A|\mathcal{A}_n) = \mathbb{E}(A|\mathcal{B}_n)$.*

Proof: We recall the proof for the ease of the reader.

- ($\mathcal{H}$) $\Rightarrow$ ($\mathcal{H}1$). Let $A \in \mathcal{A}_\infty$ be a random variable square-integrable. Under ($\mathcal{H}$)-hypothesis, the $\mathbb{A}$-martingale $(A_n)_{n\geq 0}$, defined by $A_n := \mathbb{E}(A|\mathcal{A}_n)$ for all $n \geq 0$, is a $\mathbb{B}$-martingale. Hence, for any $n \geq 0$ and any $B_n \in \mathcal{B}_n$ square integrable,

$$\mathbb{E}(AB_n|\mathcal{A}_n) = \mathbb{E}\big[\mathbb{E}(A|\mathcal{B}_n)B_n\big|\mathcal{A}_n\big] \overset{(\mathcal{H})}{=} \mathbb{E}\big[\mathbb{E}(A|\mathcal{A}_n)B_n\big|\mathcal{A}_n\big] = \mathbb{E}(A|\mathcal{A}_n)\mathbb{E}(B_n|\mathcal{A}_n) ,$$

  which is ($\mathcal{H}1$).
- ($\mathcal{H}1$) $\Rightarrow$ ($\mathcal{H}2$). If ($\mathcal{H}1$) holds for any $A \in \mathcal{A}_\infty$ square-integrable and any $B_n \in \mathcal{B}_n$ square-integrable implies

$$\mathbb{E}\big[A\mathbb{E}(B_n|\mathcal{A}_n)\big] = \mathbb{E}\big[\mathbb{E}(A|\mathcal{A}_n)\mathbb{E}(B_n|\mathcal{A}_n)\big] \overset{(\mathcal{H}1)}{=} \mathbb{E}\big[\mathbb{E}(AB_n|\mathcal{A}_n)\big] = \mathbb{E}(AB_n) ,$$

  which is exactly ($\mathcal{H}2$) for square integrable random variables. The general case follows easily.
- ($\mathcal{H}2$) $\Rightarrow$ ($\mathcal{H}3$). Suppose ($\mathcal{H}2$) and let $A \in \mathcal{A}_\infty$ be bounded and $B_n \in \mathcal{B}_n$ integrable for any $n \geq 0$, then

$$\mathbb{E}\big[\mathbb{E}(A|\mathcal{A}_n)B_n\big] = \mathbb{E}\big[A\mathbb{E}(B_n|\mathcal{A}_n)\big] \overset{(\mathcal{H}2)}{=} \mathbb{E}\big[A\mathbb{E}(B_n|\mathcal{A}_\infty)\big] = \mathbb{E}(AB_n) ,$$

  which implies ($\mathcal{H}3$).
- ($\mathcal{H}3$) $\Rightarrow$ ($\mathcal{H}$). Consider an $\mathcal{A}$-martingale $(A_n)_{n\geq 0}$ of the form $A_n := \mathbb{E}(A|\mathcal{A}_n)$. Then, $(B_n)_{n\geq 0}$ defined by $B_n := \mathbb{E}(A_\infty|\mathcal{B}_n)$ for all $n \geq 0$ is an $\mathbb{B}$-martingale. Then under ($\mathcal{H}3$), we have that $A_n = B_n$ for all $n \geq 0$, therefore ($\mathcal{H}$) is satisfied for uniformly integrable martingales. The extension to all martingales is standard.

□

1.2 Initial enlargement

In this section, we consider the initial filtration enlargement (see Chapter III of [JY78b] or Chapter 6 of [AJ16]). We present the analogues of the Brownian motion bridge, Jacod's criterion and its relation with arbitrages.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space where $\mathbb{F} = \big(\mathcal{F}_n\big)_{n\geq 0}$ is a complete filtration with $\mathcal{F}_0$ the completed σ-algebra trivial and $\mathcal{F}_\infty \subseteq \mathcal{F}$. We consider a random variable ξ. Let $\mathbb{F}^{(\xi)} = \big(\mathcal{F}_n^{(\xi)}\big)_{n\geq 0}$ be the complete enlarged filtration, $\mathcal{F}_n^{(\xi)} = \mathcal{F}_n \vee \sigma(\xi)$ for all $n \geq 0$, where $\sigma(\xi)$ is the σ-algebra generated by the random variable ξ.

For $n \geq 0$ fixed we can describe the events which belongs to the σ-algebra $\mathcal{F}_n^{(\xi)}$ on the set $\{\xi = k\}$. If $E_n \in \mathcal{F}_n^{(\xi)}$, then $E_n \cap \{\xi = k\} = \widehat{E}_n \cap \{\xi = k\}$ for some event $\widehat{E}_n \in \mathcal{F}_n$. Therefore, from the monotone class Theorem, any $\mathcal{F}_n^{(\xi)}$-measurable random variable Y_n satisfies $Y_n \mathbb{1}_{\{\xi=k\}} = y_n \mathbb{1}_{\{\xi=k\}}$, where y_n is an $\mathcal{F}_n$-measurable random variable.

1.2.1 Random walk bridge

Before introducing the random walk bridge, we recall a result of the Brownian bridge in continuous time. We consider $\mathbb{F}^B$ the filtration generated by a Brownian motion B, a terminal time $T \in (0, \infty)$ and the filtration $\mathbb{F}^{(B_T)}$, which is the initial enlargement of $\mathbb{F}^B$ enlarged by B_T. We have the following decomposition for B in $\mathbb{F}^{(B_T)}$,

$$B_t = \beta_t + \int_0^t \frac{B_T - B_s}{T - s} ds \ , \quad \forall t \leq T \ , \tag{1.2.1}$$

where β is an $\mathbb{F}^{(B_T)}$-Brownian motion (see for example Section 5.9.2 in [JYC09]) .

The objective of this subsection is to obtain a similar formula with enlargement of the terminal value of a particular martingale.

Consider the process $X = (X_n)_{n\geq 0}$ given by $X_0 = 0$ and $X_n = \sum_{i=1}^n Y_i$, where $(Y_i)_{i\geq 1}$ are integrable, independent and identically distributed random variables with zero mean. Let $N > 0$ be a terminal fixed time and define $\xi = X_N$. We denote by $\mathbb{F}$ the natural filtration of X, and we notice that X is an $\mathbb{F}$-martingale.

By the Doob decomposition (Theorem 1.1.3), we have that there exists an $\mathbb{F}^{(\xi)}$-predictable process P and an $\mathbb{F}^{(\xi)}$-martingale M, such that $X = M + P$, where P is given by

$$\Delta P_n = \mathbb{E}\big(\Delta X_n | \mathcal{F}_{n-1}^{(\xi)}\big) \ , \quad \forall n \geq 1 \ .$$

By definition of X and using that the random variables $(Y_i)_{i\geq 1}$ are independent and identically distributed, we obtain that

$$\begin{aligned} \Delta P_n &= \mathbb{E}\big(Y_n | \mathcal{F}_{n-1}^{(\xi)}\big) = \frac{1}{N-n+1} \mathbb{E}\Big(\sum_{i=n}^N Y_i | \mathcal{F}_{n-1}^{(\xi)}\Big) \\ &= \frac{1}{N-n+1} \mathbb{E}(X_N - X_{n-1} | \mathcal{F}_{n-1}^{(\xi)}) \ , \end{aligned}$$

then, since $X_N \in \sigma(\xi) = \sigma(X_N)$ and $X_{n-1} \in \mathcal{F}_{n-1}$, we can deduce that

$$\Delta P_n = \frac{X_N - X_{n-1}}{N-n+1} \ .$$

Hence, the $\mathbb{F}^{(\xi)}$-martingale M is given by

$$M_n = X_n - \sum_{k=1}^n \frac{X_N - X_{k-1}}{N-k+1} \ , \quad \forall N \geq n \geq 1 \ .$$

Therefore, the Doob decomposition of X in the filtration $\mathbb{F}^{(\xi)}$ is

$$X_n = M_n + \sum_{k=1}^{n} \frac{X_N - X_{k-1}}{N-(k-1)} \ , \quad \forall N \geq n \geq 1 \ .$$

1.2.2 Initial enlargement with a $\mathbb{Z}$-valued random variable

In continuous time, under Jacod's hypothesis (i.e., if for $t \geq 0$, we have that $\mathbb{P}(\xi \in du|\mathcal{F}_t) = p_t(u)\mathbb{P}(\xi \in du)$, see [Jac85]), any $\mathbb{F}$-martingale is a $\mathbb{G}$-semimartingale, with decomposition

$$X_t = M_t + \int_0^t \frac{d\langle X, p_s(u)\rangle|_{\xi=u}}{p_{s-}(\xi)} \ , \ \forall t \in [0,\infty) \ .$$

Here, we present an analogue formula in discrete time.

Suppose that ξ is a random variable taking values in $\mathbb{Z}$. Denote by $p_n(k) = \mathbb{P}(\xi = k|\mathcal{F}_n)$ for all $k \in \mathbb{Z}$ and $n \geq 0$. Let X be an $\mathbb{F}$-martingale with $\mathbb{F}^{(\xi)}$-Doob's decomposition $X = P + M$.

The $\mathbb{F}^{(\xi)}$-predictable P process is defined by $\Delta P_n = \mathbb{E}(\Delta X_n|\mathcal{F}^{(\xi)}_{n-1})$, for all $n \geq 1$. Now, in order to write ΔP in terms of $p(k)$ when $\{\xi = k\}$, we prove the following lemma.

Lemma 1.2.1 *Let Y be an integrable random variable. Then,*

$$\mathbb{E}(Y|\mathcal{F}^{(\xi)}_n)\mathbb{1}_{\{\xi=k\}} = \frac{\mathbb{E}(Y\mathbb{1}_{\{\xi=k\}}|\mathcal{F}_n)}{\mathbb{P}(\xi = k|\mathcal{F}_n)}\mathbb{1}_{\{\xi=k\}} \ , \quad k \in \mathbb{Z} \quad \text{and} \quad \forall n \geq 0 \ .$$

Proof: For $n \geq 0$ and $k \in \mathbb{Z}$ fixed, we have that $Y_n := \mathbb{E}(Y|\mathcal{F}^{(\xi)}_n)$ is an $\mathcal{F}^{(\xi)}_n$-measurable random variable. Then, there exists $y_n \in \mathcal{F}_n$ such that $Y_n\mathbb{1}_{\{\xi=k\}} = y_n\mathbb{1}_{\{\xi=k\}}$. Taking the conditional expectation w.r.t. $\mathcal{F}_n$ of both members, we get that

$$\mathbb{E}(Y_n\mathbb{1}_{\{\xi=k\}}|\mathcal{F}_n) = y_n\mathbb{E}(\mathbb{1}_{\{\xi=k\}}|\mathcal{F}_n) \ .$$

Note that on the set $\{\xi = k\}$, we have that $p_n(k) > 0$, indeed

$$\mathbb{E}(\mathbb{1}_{\{\xi=k\}}\mathbb{1}_{\{p_n(k)=0\}}) = \mathbb{E}\big(\mathbb{E}(\mathbb{1}_{\{\xi=k\}}|\mathcal{F}_n)\mathbb{1}_{\{p_n(k)=0\}}\big) = \mathbb{E}\big(p_n(k)\mathbb{1}_{\{p_n(k)=0\}}\big) = 0 \ .$$

It follows that

$$Y_n\mathbb{1}_{\{\xi=k\}} = \frac{\mathbb{E}(Y_n\mathbb{1}_{\{\xi=k\}}|\mathcal{F}_n)}{\mathbb{E}(\mathbb{1}_{\{\xi=k\}}|\mathcal{F}_n)}\mathbb{1}_{\{\xi=k\}} \ .$$

Finally, by the definition of Y_n, using that $\{\xi = k\} \in \mathcal{F}^{(\xi)}_n$, and that, on $\{\xi = k\}$, Y_n is $\mathcal{F}_n$-measurable, we obtain

$$\mathbb{E}(Y|\mathcal{F}^{(\xi)}_n)\mathbb{1}_{\{\xi=k\}} = \frac{\mathbb{E}(\mathbb{E}(Y\mathbb{1}_{\{\xi=k\}}|\mathcal{F}^{(\xi)}_n)|\mathcal{F}_n)}{\mathbb{E}(\mathbb{1}_{\{\xi=k\}}|\mathcal{F}_n)}\mathbb{1}_{\{\xi=k\}} = \frac{\mathbb{E}(Y_n\mathbb{1}_{\{\xi=k\}}|\mathcal{F}_n)}{\mathbb{E}(\mathbb{1}_{\{\xi=k\}}|\mathcal{F}_n)}\mathbb{1}_{\{\xi=k\}}$$

which finishes the proof. □

Let $n \geq 1$ and $k \in \mathbb{Z}$ be fixed. Then, from Lemma 1.2.1, we have that

$$\mathbb{E}(\Delta X_n|\mathcal{F}^{(\xi)}_{n-1})\mathbb{1}_{\{\xi=k\}} = \frac{\mathbb{E}(\Delta X_n \mathbb{1}_{\{\xi=k\}}|\mathcal{F}_{n-1})}{\mathbb{P}(\xi = k|\mathcal{F}_{n-1})}\mathbb{1}_{\{\xi=k\}} .$$

Hence, using that $\xi \in \mathcal{F}^{(\xi)}_{n-1}$ and the tower property, we have

$$\Delta P_n \mathbb{1}_{\{\xi=k\}} = \frac{\mathbb{E}(\Delta X_n \mathbb{1}_{\{\xi=k\}}|\mathcal{F}_{n-1})}{\mathbb{P}(\xi = k|\mathcal{F}_{n-1})}\mathbb{1}_{\{\xi=k\}} = \frac{\mathbb{E}(\Delta X_n p_n(k)|\mathcal{F}_{n-1})}{p_{n-1}(k)}\mathbb{1}_{\{\xi=k\}} .$$

Then from definition of predictable brackets, we get

$$\Delta P_n \mathbb{1}_{\{\xi=k\}} = \frac{\Delta\langle X, p(k)\rangle_n}{p_{n-1}(k)}\mathbb{1}_{\{\xi=k\}} .$$

Therefore,

$$X_0 = M_0 \quad \text{and} \quad X_n = M_n + \sum_{j=1}^{n} \frac{\Delta\langle X, p(k)\rangle_j}{p_{j-1}(k)}\mathbb{1}_{\{\xi=k\}} , \ \forall n \geq 1 .$$

1.2.3 Arbitrages

By Definition 1.1.33, we have the following result for initial enlargement of filtration for arbitrages.

Proposition 1.2.2 *If $\xi \in \mathcal{F}_N$ but $\xi \notin \mathcal{F}_0$, then the model $(\mathbb{F}, \mathbb{F}^{(\xi)})$ is not arbitrage free.*

Proof: For $0 \leq n \leq N$ fixed, we define $X_n = \mathbb{E}(\xi|\mathcal{F}_n)$ and we proceed by contradiction. Suppose that the model is arbitrage free, then there exists an $\mathbb{F}^{(\xi)}$-deflator L such that

$$\mathbb{E}(X_N L_N|\mathcal{F}^{(\xi)}_n) = X_n L_n . \tag{1.2.2}$$

In the other hand, $X_N = \xi \in \mathcal{F}^{(\xi)}_n$, therefore

$$\mathbb{E}(X_N L_N|\mathcal{F}^{(\xi)}_n) = X_N L_n . \tag{1.2.3}$$

In particular, by (1.2.2) and (1.2.3), we get that $X_0 L_0 = X_N L_0$ which is a contradiction, since $\xi \in \mathcal{F}_N$ but $\xi \notin \mathcal{F}_0$. □

In a random walk bridge model (Section 1.2.1), we have that if $\xi = X_N$, for some $N > 0$, then the following result is straightforward.

Corollary 1.2.3 *The bridge model is not arbitrage free except if the random variables $(Y_i)_{i\geq 1}$ are null.*

1.3 Progressive enlargement

In this section, we consider the progressive enlargement of filtration (see [JY78b, Chapter IV] or [AJ16, Chapter 7]). We start studying the Azéma supermartingales in discrete time and we give some of its properties. Then, we introduce the concepts of compensator and balancer and their relations with the Azéma supermartingales. We study pseudo stopping times, honest times and some implications in arbitrage and immersion setting. We also study different martingale representation theorems, equivalent probability measures, the Cox process and arbitrages.

1.3.1 Definitions and first results

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space where $\mathbb{F} = \big(\mathcal{F}_n\big)_{n\geq 0}$ is a complete filtration with $\mathcal{F}_0$ the completed trivial σ-algebra and $\mathcal{F}_\infty \subseteq \mathcal{F}$. We consider a random time τ taking values in the set of non negative integers. Define the indicator process H by

$$H_n := \mathbb{1}_{\{\tau \leq n\}} \ , \ \forall n \geq 0 \ .$$

Remark 1.3.1 Notice that we can generalize to a random time τ taking values in $\mathbb{R}^+$. In this case we would have set

$$\Delta H_n = \mathbb{1}_{\{\tau \in (n-1,n]\}} \ , \quad \forall n \geq 0 \ .$$

Then, defining

$$\rho := \lceil \tau \rceil = \begin{cases} n\mathbb{1}_{\{\tau \in (n-1,n]\}} & \text{if } \tau > 0 \\ 0 & \text{if } \tau = 0 \end{cases} \ , \quad \forall n \geq 0 \ ,$$

we obtain a random time ρ taking values in the set of non negative integers with the following property

$$\widetilde{H}_n := \mathbb{1}_{\{\rho \leq n\}} = H_n \ , \quad \forall n \geq 0 \ .$$

Hence, for simplicity we consider a random time taking values in the set of non negative integers.

Dual optional projection and dual predictable projection

We denote by A the dual $\mathbb{F}$-optional projection of H and by a the dual $\mathbb{F}$-predictable projection of H (see Definitions 1.1.28 and 1.1.30). These projections are defined as

$$A_n \ := \ \sum_{k=0}^{n} \mathbb{E}(\Delta H_k | \mathcal{F}_k) \ , \quad \forall n \geq 0 \ , \tag{1.3.1}$$

respectively,

$$a_0 = \mathbb{E}(\mathbb{1}_{\{\tau = 0\}} | \mathcal{F}_0) \qquad \text{and} \qquad a_n \ := \ \sum_{k=1}^{n} \mathbb{E}(\Delta H_k | \mathcal{F}_{k-1}) \ , \quad \forall n \geq 1 \ .$$

It is useful to notice that for all $n \geq 0$, we have that $\mathbb{E}(a_n)$ and $\mathbb{E}(A_n)$ are in $[0,1]$. Indeed, from the tower property and definitions of A and a, we have that

$$\mathbb{E}(\mathbb{1}_{\{\tau \leq n\}}) = \mathbb{E}\Big(\sum_{k=0}^{n} \mathbb{E}(\Delta H_k | \mathcal{F}_k) \Big) = \mathbb{E}(A_n) \ , \quad \forall n \geq 0 \ , \tag{1.3.2}$$

$$\mathbb{E}(\mathbb{1}_{\{0 \leq \tau \leq n\}}) = \mathbb{E}\Big(\sum_{k=0}^{n} \mathbb{1}_{\{\tau = n\}} \Big) = \mathbb{E}\Big(a_0 + \sum_{k=1}^{n} \mathbb{E}(\Delta H_k | \mathcal{F}_{k-1}) \Big) = \mathbb{E}(a_n) \ , \quad \forall n \geq 0 \ , \tag{1.3.3}$$

therefore, $0 \leq \mathbb{E}(A_n) \leq 1$, and $0 \leq \mathbb{E}(a_n) \leq 1$, for all $n \geq 1$.

We deduce that $\lim_{n\to\infty} A_n$ exists, and we introduce for future use

$$A_{\infty^-} := \lim_{n\to\infty} A_n = \sum_{k=0}^{\infty} \mathbb{E}(\Delta H_k|\mathcal{F}_k) \quad \text{and} \tag{1.3.4}$$

$$A_{\infty} := A_{\infty^-} + \mathbb{E}(\Delta H_\infty|\mathcal{F}_\infty) = \sum_{k=0}^{\infty} \mathbb{E}(\Delta H_k|\mathcal{F}_k) + \mathbb{E}(\Delta H_\infty|\mathcal{F}_\infty) \ , \tag{1.3.5}$$

so that, $\Delta A_\infty := A_\infty - A_{\infty^-} = E(\Delta H_\infty|\mathcal{F}_\infty)$.

Consider M a bounded $\mathbb{F}$-martingale, then $M_{\infty-} = \lim_{n\to\infty} M_n < \infty$ is well defined and, from integration by parts formula Proposition 1.1.14, we get

$$A_{\infty^-}M_{\infty^-} = (A_- \centerdot M)_{\infty^-} + (M \centerdot A)_{\infty^-} \ .$$

By Proposition 1.1.7 we have that $A_- \centerdot M$ is an $\mathbb{F}$-martingale with null initial value, then taking the expectation value,

$$\mathbb{E}(A_{\infty^-}M_{\infty^-}) = \mathbb{E}\Big((M \centerdot A)_{\infty^-}\Big) \ ,$$

using the definition of A, we obtain

$$\mathbb{E}(A_{\infty^-}M_{\infty^-}) = \mathbb{E}\Big((M \centerdot H)_{\infty^-}\Big) = \mathbb{E}(M_\tau \mathbb{1}_{\{\tau<\infty\}}) \ . \tag{1.3.6}$$

The enlarged filtration and decompositions of the Azéma supermartingales

Let $\mathbb{G} = \big(\mathcal{G}_n\big)_{n\geq 0}$ be the enlarged filtration, $\mathcal{G}_n = \mathcal{F}_n \vee \mathcal{H}_n$, where

$$\mathcal{H}_n = \sigma(H_0, H_1, H_2, \dots, H_n) \ , \quad \forall n \geq 0 \ .$$

In particular $\{\tau = 0\} \in \mathcal{G}_0$, so that in general $\mathcal{G}_0$ is not trivial.

We introduce two $\mathbb{F}$-supermartingales G and $\widetilde{G}$, defined by

$$G_n := \mathbb{P}(\tau > n|\mathcal{F}_n) \ \text{ and } \ \widetilde{G}_n := \mathbb{P}(\tau \geq n|\mathcal{F}_n) \ , \ \forall n \geq 0 \ ,$$

also, we define for future use $G_{\infty-} := \mathbb{P}(\tau = \infty|\mathcal{F}_\infty)$ and $G_\infty := 0$.

We have the following trivial, but useful relations

$$\Delta A_n = \mathbb{P}(\tau = n|\mathcal{F}_n) = \widetilde{G}_n - G_n = \widetilde{G}_n - \mathbb{E}(\widetilde{G}_{n+1}|\mathcal{F}_n) \ , \quad \forall n \geq 0 \ . \tag{1.3.7}$$

From the Doob decomposition (Theorem 1.1.3), G can be written as $G = m - a$, where a is the dual $\mathbb{F}$-predictable projection of H and m is an $\mathbb{F}$-martingale. In the other hand, we denote by $\widetilde{G} = \widetilde{m} - \widetilde{a}$ the Doob decomposition of $\widetilde{G}$, where $\widetilde{m}$ is an $\mathbb{F}$-martingale and $\widetilde{a}$ is a non decreasing integrable $\mathbb{F}$-predictable process. Notice that those processes are given by

$$\begin{aligned}
\Delta m_n &= G_n - \mathbb{E}(G_n|\mathcal{F}_{n-1}) \ , & \Delta a_n &= G_{n-1} - \mathbb{E}(G_n|\mathcal{F}_{n-1}) \\
& & &= \mathbb{P}(\tau = n|\mathcal{F}_{n-1}) = \mathbb{E}(\Delta H_n|\mathcal{F}_{n-1}) \ , \\
\Delta \widetilde{m}_n &= \widetilde{G}_n - \mathbb{E}(\widetilde{G}_n|\mathcal{F}_{n-1}) \quad \text{and} & \Delta \widetilde{a}_n &= \widetilde{G}_{n-1} - \mathbb{E}(\widetilde{G}_n|\mathcal{F}_{n-1}) \\
& & &= \mathbb{P}(\tau = n-1|\mathcal{F}_{n-1}) = \mathbb{E}(\Delta H_{n-1}|\mathcal{F}_{n-1}) \ ,
\end{aligned}$$

for all $n \geq 1$, with $m_0 = \widetilde{m}_0 = 1$, $a_0 = \mathbb{E}(\mathbb{1}_{\{\tau=0\}}|\mathcal{F}_0)$ and $\widetilde{a}_0 = 0$. Note that this is not the standard Doob's decomposition in the sense that the initial values of the predictable are not zero.

Notice that the predictable part of Doob's decomposition of $\widetilde{G}$ has the following relation with the dual $\mathbb{F}$-optional projection of H :

$$\widetilde{a}_n = A_{n-1} \ , \quad \forall n \geq 1 \ , \tag{1.3.8}$$

and that

$$\widetilde{m}_n = \widetilde{G}_n + A_{n-1} \ , \quad \forall n \geq 1 \ . \tag{1.3.9}$$

In particular, $\widetilde{a}_1 = \mathbb{P}(\tau = 0|\mathcal{F}_0) = A_0$.

Now, we compute the Doob decomposition of $\widetilde{G}$. First, we consider the $\mathbb{F}$-martingale $(\mathbb{E}(A_\infty|\mathcal{F}_n), n \geq 0)$. This martingale is equal to the martingale part of the Doob decomposition of $\widetilde{G}$. Indeed, by (1.3.5) and since $\Delta A_k \in \mathcal{F}_n$ for all $k \leq n$,

$$\mathbb{E}(A_\infty|\mathcal{F}_n) = A_0 + \mathbb{E}\Big(\sum_{k=1}^{\infty} \Delta A_k \Big| \mathcal{F}_n\Big) + \mathbb{E}(\Delta A_\infty|\mathcal{F}_n) = A_n + \mathbb{E}\Big(\sum_{k=n+1}^{\infty} \Delta A_k + \Delta A_\infty \Big| \mathcal{F}_n\Big) \ , \tag{1.3.10}$$

then, by the tower property, we obtain from (1.3.10) that

$$\begin{aligned}
\mathbb{E}(A_\infty|\mathcal{F}_n) &= A_n + \mathbb{E}\Big(\sum_{k=n+1}^{\infty} \mathbb{P}(\tau = k|\mathcal{F}_k) + \mathbb{P}(\tau = \infty|\mathcal{F}_\infty)\Big|\mathcal{F}_n\Big) \\
&= A_n + \sum_{k=n+1}^{\infty} \mathbb{P}(\tau = k|\mathcal{F}_n) + \mathbb{P}(\tau = \infty|\mathcal{F}_n) = A_n + G_n \ ,
\end{aligned}$$

finally, using (1.3.8) and $G_n + \mathbb{P}(\tau = n|\mathcal{F}_n) = \widetilde{G}_n$, we get that

$$\mathbb{E}(A_\infty|\mathcal{F}_n) = A_n + G_n = G_n + \Delta A_n + \widetilde{a}_n = \widetilde{G}_n + \widetilde{a}_n = \widetilde{m}_n \ . \tag{1.3.11}$$

Decompositions after and before τ

Any integrable $\mathcal{G}_n$-measurable random variable ζ_n can be decomposed as

$$\zeta_n = \bar{\zeta}_n \mathbb{1}_{\{\tau > n\}} + \zeta_n(\tau)\mathbb{1}_{\{\tau \leq n\}} \ , \quad \forall n \geq 0 \ , \tag{1.3.12}$$

where $\bar{\zeta}_n \in \mathcal{F}_n$ and $\zeta_n(k) \in \mathcal{F}_n$ for all $0 \leq k \leq n$ (see [Jeu80]).

The following lemmas are classical results and they will be very useful. The proofs in discrete time are very close to the proofs in continuous time.

Lemma 1.3.2 *On the set $\{\tau \geq n\}$, $\widetilde{G}_n$ and G_{n-1} are positive. On the set $\{\tau < n\}$, we have that $\widetilde{G}_n < 1$ and $G_{n-1} < 1$.*

PROOF: In order to show that on the set $\{\tau \geq n\}$, $\widetilde{G}_n$ and G_{n-1} are positive, we use the two following equalities :

$$\begin{aligned}
\mathbb{E}(\mathbb{1}_{\{\tau \geq n\}}\mathbb{1}_{\{G_{n-1}=0\}}) &= \mathbb{E}\big[\mathbb{P}(\tau \geq n|\mathcal{F}_{n-1})\mathbb{1}_{\{G_{n-1}=0\}}\big] = \mathbb{E}\big(G_{n-1}\mathbb{1}_{\{G_{n-1}=0\}}\big) = 0 \ , \\
\mathbb{E}(\mathbb{1}_{\{\tau \geq n\}}\mathbb{1}_{\{\widetilde{G}_n=0\}}) &= \mathbb{E}\big(\mathbb{P}(\tau \geq n|\mathcal{F}_n)\mathbb{1}_{\{\widetilde{G}_n=0\}}\big) = \mathbb{E}\big(\widetilde{G}_n\mathbb{1}_{\{\widetilde{G}_n=0\}}\big) = 0 \ .
\end{aligned}$$

To prove that on the set $\{\tau < n\}$, we have that $\widetilde{G}_n < 1$ and $G_{n-1} < 1$, we use the following equations :

$$\mathbb{E}(\mathbb{1}_{\{\tau<n\}}\mathbb{1}_{\{G_{n-1}=1\}}) = \mathbb{E}\big(\big(1-\mathbb{P}(\tau \geq n|\mathcal{F}_{n-1})\big)\mathbb{1}_{\{G_{n-1}=1\}}\big) = \mathbb{E}\big((1-G_{n-1})\mathbb{1}_{\{G_{n-1}=1\}}\big) = 0 \ ,$$

$$\mathbb{E}(\mathbb{1}_{\{\tau<n\}}\mathbb{1}_{\{\widetilde{G}_n=1\}}) = \mathbb{E}\big(\big(1-\mathbb{P}(\tau \geq n|\mathcal{F}_n)\big)\mathbb{1}_{\{\widetilde{G}_n=1\}}\big) = \mathbb{E}\big((1-\widetilde{G}_n)\mathbb{1}_{\{\widetilde{G}_n=1\}}\big) = 0 \ .$$

□

Lemma 1.3.3 ***Key Lemma.***
a) Let X be an integrable $\mathcal{G}_\infty$-measurable random variable, then

$$\mathbb{E}(X|\mathcal{G}_n)\mathbb{1}_{\{\tau>n\}} = \frac{\mathbb{E}(X\mathbb{1}_{\{\tau>n\}}|\mathcal{F}_n)}{G_n}\mathbb{1}_{\{\tau>n\}} \ , \qquad \forall n \geq 0 \ ,$$

b) Let X_n be an integrable $\mathcal{F}_n$-measurable random variable for a fixed $n \geq 1$, then

$$\mathbb{E}\Big(\frac{X_n}{\widetilde{G}_n}\mathbb{1}_{\{\tau\geq n\}}\big|\mathcal{G}_{n-1}\Big) = \frac{\mathbb{E}(X_n\mathbb{1}_{\{\widetilde{G}_n>0\}}|\mathcal{F}_{n-1})}{G_{n-1}}\mathbb{1}_{\{\tau\geq n\}} \ ,$$

$$\mathbb{E}(X_n|\mathcal{G}_{n-1})\mathbb{1}_{\{\tau\geq n\}} = \frac{\mathbb{E}(X_n\widetilde{G}_n|\mathcal{F}_{n-1})}{G_{n-1}}\mathbb{1}_{\{\tau\geq n\}} \ .$$

Proof: To prove assertion a), we suppose $n \geq 0$ fixed. Then, $\mathbb{E}(X|\mathcal{G}_n)$ is $\mathcal{G}_n$-measurable and by (1.3.12), there exists an $\mathcal{F}_n$-measurable random variable $\bar{X}$, such that

$$\mathbb{E}(X|\mathcal{G}_n)\mathbb{1}_{\{\tau>n\}} = \bar{X}\mathbb{1}_{\{\tau>n\}} \ . \tag{1.3.13}$$

From the facts that $\{\tau > n\} \in \mathcal{G}_n$, $\bar{X} \in \mathcal{F}_n$, taking the $\mathcal{F}_n$-conditional expectation and the tower property, we obtain

$$\mathbb{E}(X\mathbb{1}_{\{\tau>n\}}|\mathcal{F}_n) = \bar{X}G_n \ . \tag{1.3.14}$$

Note that on the set $\{\tau > n\}$ the random variable G_n is positive (Lemma 1.3.2). Then, using (1.3.13) in (1.3.14), we get

$$\mathbb{E}(X|\mathcal{G}_n)\mathbb{1}_{\{\tau>n\}} = \bar{X}\mathbb{1}_{\{\tau>n\}} = \frac{\mathbb{E}(X\mathbb{1}_{\{\tau>n\}}|\mathcal{F}_n)}{G_n}\mathbb{1}_{\{\tau>n\}} \ .$$

To prove assertion b), we fix $n \geq 1$. Then, the first equality of b) is a consequence of a), since taking $X = \frac{X_n}{\widetilde{G}_n}\mathbb{1}_{\{\widetilde{G}_n>0\}}$ we obtain that

$$\mathbb{E}\Big(\frac{X_n}{\widetilde{G}_n}\mathbb{1}_{\{\widetilde{G}_n>0\}}\mathbb{1}_{\{\tau\geq n\}}\big|\mathcal{G}_{n-1}\Big) = \frac{\mathbb{E}(\frac{X_n}{\widetilde{G}_n}\mathbb{1}_{\{\widetilde{G}_n>0\}}\mathbb{1}_{\{\tau\geq n\}}|\mathcal{F}_{n-1})}{G_{n-1}}\mathbb{1}_{\{\tau\geq n\}} \ . \tag{1.3.15}$$

As $\{\tau \geq n\} \subset \{\widetilde{G}_n > 0\}$ in the left hand side and from the tower property in the right hand side of (1.3.15), and the fact that $X_n \in \mathcal{F}_n$, we obtain the desired result.

Finally, for the last equality of b), applying a) for $X = X_n$ and by the tower property, and the fact that $X_n \in \mathcal{F}_n$, we have that

$$\mathbb{E}(X_n|\mathcal{G}_{n-1})\mathbb{1}_{\{\tau\geq n\}} = \frac{\mathbb{E}(X_n\mathbb{1}_{\{\tau\geq n\}}|\mathcal{F}_{n-1})}{G_{n-1}}\mathbb{1}_{\{\tau\geq n\}} = \frac{\mathbb{E}(X_n\widetilde{G}_n|\mathcal{F}_{n-1})}{G_{n-1}}\mathbb{1}_{\{\tau\geq n\}} \ .$$

□

Lemma 1.3.4 *Consider the process R defined by*

$$R_n := \frac{\mathbb{1}_{\{\tau>n\}}}{G_n}\ , \quad \forall n\geq 0\ ,$$

then R is a non negative $\mathbb{G}$-martingale.

Proof: First notice that R is well defined since G does not vanish before τ by Proposition 1.3.2.

Let $n\geq 1$ be fixed. We have that (either considering generalized conditional expectation, or using a localizing procedure and applying Theorem 1.1.2 at the end) $\mathbb{E}(R_n|\mathcal{G}_{n-1}) = \mathbb{E}(R_n|\mathcal{G}_{n-1})\mathbb{1}_{\{\tau>n-1\}}$, since $\mathbb{1}_{\{\tau>n-1\}}\in\mathcal{G}_{n-1}$ and $R_n\mathbb{1}_{\{\tau>n-1\}} = R_n$, then by Lemma 1.3.3, we deduce that

$$\mathbb{E}(R_n|\mathcal{G}_{n-1}) = \frac{\mathbb{E}\big[\frac{\mathbb{1}_{\{\tau>n\}}}{G_n}\big|\mathcal{F}_{n-1}\big]}{G_{n-1}}\mathbb{1}_{\{\tau>n-1\}} = \frac{1}{G_{n-1}}\mathbb{1}_{\{\tau>n-1\}} = R_{n-1}\ ,$$

i.e. R is a $\mathbb{G}$-martingale. □

Compensator and Balancer

We recall that the $\mathbb{G}$-compensator of a non-decreasing $\mathbb{G}$ adapted process K (a sub-martingale) is the predictable part of the Doob decomposition of K, hence it is unique and non decreasing.

Lemma 1.3.5 ***Compensator***. *Let Λ be the non-decreasing $\mathbb{F}$-predictable process given by*

$$\Lambda_0 = \mathbb{P}(\tau=0|\mathcal{F}_0) \quad \textit{and} \quad \Delta\Lambda_n = \lambda_n = \frac{\Delta a_n}{G_{n-1}}\mathbb{1}_{\{G_{n-1}>0\}}\ , \quad \forall n\geq 1\ . \tag{1.3.16}$$

Then, $N := H - \Lambda^\tau$ (i.e. $N_n := H_n - \Lambda_{n\wedge\tau}$ for $n\geq 0$) is a $\mathbb{G}$-martingale.

The process Λ^τ is the compensator of H.

Proof: First, we notice that $0\leq\lambda_n\leq 1$ for all $n\geq 0$ since $\lambda_0 = \mathbb{P}(\tau=0|\mathcal{F}_0)$ by definition and $\{\tau=n\}\subset\{\tau>n-1\}=\{\tau\geq n\}$ which implies that $\Delta a_n\leq G_{n-1}$ for all $n\geq 1$.

The $\mathbb{G}$-martingale part N of the $\mathbb{G}$-semimartingale H is given by

$$\begin{cases} N_n &= H_n - \big(\sum_{k=1}^n \mathbb{E}(\Delta H_k|\mathcal{G}_{k-1}) + \Lambda_0 H_0\big)\ , \quad \forall n\geq 1\ , \\ N_0 &= H_0 - \Lambda_0^\tau\ . \end{cases}$$

Since $\Delta H_k\mathbb{1}_{\{\tau>k-1\}} = \Delta H_k$ and $\mathbb{1}_{\{\tau>k-1\}}\in\mathcal{G}_{k-1}$ for all $1\leq k\leq n$.

$$\begin{cases} N_n &= H_n - \big(\sum_{k=1}^n \mathbb{E}(\Delta H_k|\mathcal{G}_{k-1})\mathbb{1}_{\{\tau>k-1\}} + \Lambda_0^\tau\big)\ , \quad \forall n\geq 1\ , \\ N_0 &= H_0 - \Lambda_0^\tau\ . \end{cases}$$

The random variable $\mathbb{E}(\Delta H_k|\mathcal{G}_{k-1})$ is equal to an $\mathcal{F}_{k-1}$-measurable random variable for $\{\tau>k-1\}$ and

$$\mathbb{1}_{\{\tau>k-1\}}\mathbb{E}(\Delta H_k|\mathcal{G}_{k-1}) = \mathbb{1}_{\{\tau>k-1\}}\lambda_k\ , \tag{1.3.17}$$

where λ_k is $\mathcal{F}_{k-1}$-measurable. By Lemma 1.3.3.a), we can choose

$$\lambda_k = \frac{\Delta a_k}{G_{k-1}} \mathbb{1}_{\{G_{k-1}>0\}} .$$

□

We introduce a second process that we call the balancer of H. The following lemma was obtained in [CD14] in a continuous time setting.

Lemma 1.3.6 ***Balancer***. *Let $\widetilde{\Lambda}$ be the $\mathbb{F}$-adapted process given by*

$$\widetilde{\Lambda}_0 = \mathbb{P}(\tau = 0|\mathcal{F}_0) \quad \textit{and} \quad \Delta\widetilde{\Lambda}_n = \widetilde{\lambda}_n = \frac{\Delta A_n}{\widetilde{G}_n} \mathbb{1}_{\{\widetilde{G}_n>0\}} = \frac{\Delta\widetilde{a}_{n+1}}{\widetilde{G}_n} \mathbb{1}_{\{\widetilde{G}_n>0\}} , \quad \forall n \geq 1 . \tag{1.3.18}$$

Then, $\widetilde{\Lambda}^\tau$ is the unique $\mathbb{G}$-adapted process, equal to an $\mathbb{F}$ adapted process up to time τ such that, for $\widetilde{N} := H - \widetilde{\Lambda}^\tau$ (i.e. $\widetilde{N}_n := H_n - \widetilde{\Lambda}_{n\wedge\tau}$ for $n \geq 0$), the process $(X \centerdot \widetilde{N})$ is a $\mathbb{G}$-martingale for any integrable $\mathbb{F}$-adapted process X. In particular, $\widetilde{N}$ is an $\mathbb{G}$ martingale.

The process $\widetilde{\Lambda}^\tau$ is called the balancer of H.

Proof: First, we notice that $0 \leq \widetilde{\lambda}_n \leq 1$ for all $n \geq 0$ since $\{\tau = n\} \subset \{\tau > n-1\} = \{\tau \geq n\}$ which implies that $\Delta A_n \leq \widetilde{G}_n$ for all $n \geq 0$.

There are many non decreasing $\mathbb{F}$-adapted processes K such that $J = H - K^\tau$ is a martingale. However, the condition $X \centerdot J$ is a $\mathbb{G}$-martingale for any integrable $\mathbb{F}$-adapted process X is satisfied only when K is the balancer.

Let us give conditions on K such that the martingale property of $X \centerdot J$ is satisfied, i.e. $E(X_n \Delta J_n|\mathcal{G}_{n-1}) = 0$. By definition $\Delta J_n = \mathbb{1}_{\{\tau=n\}} - \Delta K_n \mathbb{1}_{\{\tau>n-1\}}$, one has $\Delta J_n \mathbb{1}_{\{\tau<n\}} = 0$. Then,

$$\begin{aligned} E(X_n \Delta J_n|\mathcal{G}_{n-1}) &= \mathbb{E}(X_n \Delta J_n|\mathcal{G}_{n-1})\mathbb{1}_{\{\tau\geq n\}} = \frac{\mathbb{E}(X_n \mathbb{1}_{\{\tau\geq n\}}\Delta J_n|\mathcal{F}_{n-1})}{G_{n-1}} \mathbb{1}_{\{\tau>n-1\}} \\ &= \frac{1}{G_{n-1}} \mathbb{E}(X_n(\mathbb{P}(\tau = n|\mathcal{F}_n) - \widetilde{G}_n \Delta K_n)|\mathcal{F}_{n-1}) . \end{aligned}$$

Hence, the choice $\mathbb{P}(\tau = n|\mathcal{F}_n) = \widetilde{G}_n \Delta K_n$. In particular if $X \equiv 1$, we get that $\widetilde{N}$ is a $\mathbb{G}$-martingale.

Now we give the multiplicative decomposition of G and $\widetilde{G}$, using the exponential process given by Definition 1.1.17.

Proposition 1.3.7 *If G (resp. $\widetilde{G}$) is positive, then the multiplicative decomposition of G (resp. $\widetilde{G}$) is given by*

$$G = M^G \mathcal{E}\big(-\Lambda\big) , \quad \textit{(resp. } \widetilde{G} = M^{\widetilde{G}} \mathcal{E}\big(-\Theta\big) \textit{)} ,$$

where M^G (resp. $M^{\widetilde{G}}$) is an $\mathbb{F}$-martingale and Λ is given by Lemma 1.3.5 (resp. $(\Theta_n)_{n\geq 0}$ is given by $\Theta_n = \sum_{k=0}^n \theta_k$ with $\theta_0 = 0$ and $\theta_n = \frac{\Delta\widetilde{a}_n}{\widetilde{G}_{n-1}} = \frac{\Delta A_{n-1}}{\widetilde{G}_{n-1}}$ for all $n \geq 1$).

PROOF: By Theorem 1.1.9, there exist an $\mathbb{F}$-martingale M^G (resp. $M^{\widetilde{G}}$) and an $\mathbb{F}$-predictable P^G (resp. $P^{\widetilde{G}}$) such that, for all $n \geq 1$ fixed, we have that

$$G_n = M_n^G P_n^G \ , \quad (\text{resp. } \widetilde{G}_n = M_n^{\widetilde{G}} P_n^{\widetilde{G}}) \ ,$$

where

$$P_n^G = \prod_{k=1}^{n} \frac{\mathbb{E}(G_k|\mathcal{F}_{k-1})}{G_{k-1}} \ , \quad (\text{resp. } P_n^{\widetilde{G}} = \prod_{k=1}^{n} \frac{\mathbb{E}(\widetilde{G}_k|\mathcal{F}_{k-1})}{\widetilde{G}_{k-1}} \ )$$

and

$$M_n^G = \prod_{k=1}^{n} \frac{G_k}{\mathbb{E}(G_k|\mathcal{F}_{k-1})} \ , \quad (\text{resp. } M_n^{\widetilde{G}} = \prod_{k=1}^{n} \frac{\widetilde{G}_k}{\mathbb{E}(\widetilde{G}_k|\mathcal{F}_{k-1})} \ ) \ ,$$

which are equivalent to

$$\begin{aligned} P_n^G &= \textstyle\prod_{k=1}^n \left[\frac{\mathbb{E}(G_k|\mathcal{F}_{k-1})-G_{k-1}}{G_{k-1}} + 1\right] , \\ (\text{resp. } P_n^{\widetilde{G}} &= \textstyle\prod_{k=1}^n \left[\frac{\mathbb{E}(\widetilde{G}_k|\mathcal{F}_{k-1})-\widetilde{G}_{k-1}}{\widetilde{G}_{k-1}} + 1\right] \ ) \end{aligned} \tag{1.3.19}$$

and

$$\begin{aligned} M_n^G &= \textstyle\prod_{k=1}^n \frac{G_k}{G_{k-1}+\Delta a_k} \ , \\ (\text{resp. } M_n^{\widetilde{G}} &= \textstyle\prod_{k=1}^n \frac{\widetilde{G}_k}{G_{k-1}} \ ) \ . \end{aligned} \tag{1.3.20}$$

Also, by Definition 1.3.16 and using that $G_{n-1} > 0$ (resp. $\widetilde{G}_n > 0$) we have

$$\Delta\Lambda_n = \lambda_n = \frac{\Delta a_n}{G_{n-1}} \ , \quad (\text{resp. } \Delta\Theta_n = \theta_n = \frac{\Delta\widetilde{a}_n}{\widetilde{G}_{n-1}} = \frac{\Delta A_{n-1}}{\widetilde{G}_{n-1}}) \ , \tag{1.3.21}$$

then using (1.3.21) and Proposition 1.1.18 for (1.3.19), we get that

$$P_n^G = \mathcal{E}\big(-\Lambda\big)_n \ , \quad (\text{resp. } P_n^{\widetilde{G}} = \mathcal{E}\big(-\Theta\big)_n \ ) \ .$$

□

Remark 1.3.8 Note that, we have the following relation between $\widetilde{\lambda}$ (given by Lemma 1.3.6) and θ (given by Proposition 1.3.7), for $\widetilde{G}$ positive

$$\theta_n = \frac{\Delta A_{n-1}}{\widetilde{G}_{n-1}} = \widetilde{\lambda}_{n-1} \ , \quad \forall n \geq 1 \ .$$

Lemma 1.3.9 *The process M^N, defined by*

$$M_0^N = 1\!\!1_{\{\tau=0\}} \quad \textit{and} \quad M_n^N = N_n^2 - \sum_{k=0}^{n\wedge\tau} \lambda_k(1-\lambda_k) \ , \quad \forall n \geq 1$$

is a $\mathbb{G}$-martingale.

PROOF: Notice that $\mathbb{E}(N_n^2) \leq \mathbb{E}(H_n^2) + \mathbb{E}(\Lambda^2_{n\wedge\tau}) \leq 1 + n^2 < \infty$ for all $n \geq 0$.

For $n \geq 1$, we have that $\mathbb{E}[\Delta(N_n^2)|\mathcal{G}_{n-1}] = \Delta\langle N\rangle_n^{\mathbb{G}}$. Then, by definition of N and Lemma 1.3.5, we get

$$\begin{aligned}\Delta\langle N\rangle_n^{\mathbb{G}} &= \mathbb{E}\big[\Delta H_n - 2(\Delta H_n)\lambda_n \mathbb{1}_{\{\tau>n-1\}} + \lambda_n^2 \mathbb{1}_{\{\tau>n-1\}}\big|\mathcal{G}_{n-1}\big] \\ &= \lambda_n(1-\lambda_n)\mathbb{1}_{\{\tau>n-1\}}.\end{aligned}$$

Therefore, $\Big(\sum_{k=0}^n \lambda_k(1-\lambda_k)\Big)_{n\geq 0}$ is the predictable bracket of N^2. □

Lemma 1.3.10 *$\widetilde{G}$ is $\mathbb{F}$-predictable if and only if for all $n \geq 1$, we have that $\widetilde{G}_n = G_{n-1}$.*

PROOF: The proof follows from the tower property and the following equalities :

$$\widetilde{G}_n = \mathbb{E}(\widetilde{G}_n|\mathcal{F}_{n-1}) = \mathbb{E}(\tau \geq n|\mathcal{F}_{n-1}) = \mathbb{E}(\tau > n-1|\mathcal{F}_{n-1}) = G_{n-1}\ .$$

□

Lemma 1.3.11 *If $\widetilde{G}$ is $\mathbb{F}$-predictable, then $N^{(o)} = \Lambda_0(1-\Lambda_0)+1-m$, where m is the martingale part of Doob's decomposition of G and $N^{(o)}$ is the dual optional projection of N.*

PROOF: For n fixed with $n \geq 1$. By definition of N, we have that

$$\mathbb{E}(\Delta N_n|\mathcal{F}_n) = \mathbb{E}(\mathbb{1}_{\{\tau\leq n\}} - \mathbb{1}_{\{\tau\leq n-1\}} - \lambda_n\mathbb{1}_{\{\tau\geq n\}}|\mathcal{F}_n)\ . \tag{1.3.22}$$

Using in (1.3.22) that $\lambda_n \in \mathcal{F}_n$, we obtain

$$\mathbb{E}(\Delta N_n|\mathcal{F}_n) = \mathbb{E}(-\mathbb{1}_{\{\tau>n\}} + \mathbb{1}_{\{\tau\geq n\}}|\mathcal{F}_n) - \lambda_n\mathbb{E}(\mathbb{1}_{\{\tau\geq n\}}|\mathcal{F}_n)\ ,$$

by definition of G_n and $\widetilde{G}_n$, we have that

$$\mathbb{E}(\Delta N_n|\mathcal{F}_n) = -G_n + \widetilde{G}_n - \lambda_n\widetilde{G}_n\ . \tag{1.3.23}$$

Using that $\widetilde{G}$ is $\mathbb{F}$-predictable, we have that $\widetilde{G}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_{n-1}) = G_{n-1}$, then (1.3.23) implies

$$\mathbb{E}(\Delta N_n|\mathcal{F}_n) = -\Delta G_n - \lambda_n G_{n-1}\ . \tag{1.3.24}$$

Note that $\Delta a_n\mathbb{1}_{\{G_{n-1}>0\}} = \Delta a_n$, since

$$\{\Delta a_n > 0\} = \{\mathbb{P}(\tau = n|\mathcal{F}_{n-1}) > 0\} \subset \{\mathbb{P}(\tau \geq n|\mathcal{F}_{n-1}) > 0\} = \{G_{n-1} > 0\}\ .$$

Using that $\Delta G_n + \Delta a_n = \Delta m_n$ and $\lambda_n G_{n-1} = \Delta a_n\mathbb{1}_{\{G_{n-1}>0\}} = \Delta a_n$ in (1.3.24), we get

$$\mathbb{E}(\Delta N_n|\mathcal{F}_n) = -\Delta m_n\ .$$

Finally, since $\mathbb{E}(N_0|\mathcal{F}_0) = \Lambda_0(1-\Lambda_0)$ and $m_0 = 1$, we get the desired result □

The converse is not true. If $N^{(o)} = \Lambda_0(1-\Lambda_0) + 1 - m$, then $\widetilde{G}$ is not necessarily $\mathbb{F}$-predictable. We need the extra hypothesis that $\lambda < 1$.

Proposition 1.3.12 *If $\lambda < 1$, then $\widetilde{G}$ is $\mathbb{F}$-predictable if and only if $N^{(o)} = \Lambda_0(1 - \Lambda_0) + 1 - m$.*

PROOF: $\Rightarrow$) This is a direct consequence of Lemma 1.3.11.
$\Leftarrow$) For $n \geq 1$ fixed. By definition of N, we get that $\mathbb{E}(\Delta N_n|\mathcal{F}_n) = -G_n + \widetilde{G}_n - \lambda_n \widetilde{G}_n$. Also we have that $\Delta m_n = \Delta G_n + \Delta a_n = \Delta G_n + \lambda_n G_{n-1}$. Therefore, by hypothesis

$$\Delta G_n + \lambda_n G_{n-1} = G_n - \widetilde{G}_n + \lambda_n \widetilde{G}_n$$

which implies that

$$(\lambda_n - 1)(\widetilde{G}_n - G_{n-1}) = 0 .$$

Finally, since $\lambda_n < 1$ we deduce that $\widetilde{G}_n = G_{n-1}$, i.e. $\widetilde{G}$ is $\mathbb{F}$-predictable. □

Theorem 1.3.13 *Let X be a $\mathbb{G}$-martingale. Then, there exist a $\mathbb{G}$-predictable process q and a $\mathbb{G}$-martingale $M^{\perp}$ orthogonal to N, such that*

$$\Delta X_n = q_n \Delta N_n + \Delta M^{\perp}_n , \quad \forall n \geq 0 .$$

PROOF: This is a special case from Theorem 1.1.11. □

Decomposition of $\mathbb{F}$-martingales up to time τ

We present a classical result of decomposition of $\mathbb{F}$-martingales up to time τ (see [JY78a]). In continuous time : If $\mathbb{G}$ is the progressive enlargement of $\mathbb{F}$ with a random time τ, any $\mathbb{F}$-martingale M stopped at τ is a $\mathbb{G}$-semimartingale with decomposition

$$M^{\tau}_t = \widehat{M}_t + \int_0^{t\wedge\tau} \frac{1}{G_{s-}} d\langle M, \widetilde{m}\rangle^{\mathbb{F}}_s , \quad \forall t \geq 0 ,$$

where $G_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$ and $\widetilde{G}_t = \mathbb{P}(\tau \geq t|\mathcal{F}_t)$ for all $t \geq 0$. Here $\widetilde{G} = \widetilde{m} - \widetilde{a}$, where $\widetilde{m}$ is an $\mathbb{F}$-martingale and $\widetilde{a}$ is $\mathbb{F}$-predictable. As $\widetilde{G}$ is not càdlàg, this is not the standard Doob-Meyer decomposition established only for càdlàg supermartingales (see [Mer72] for the general case). We present a proof in discrete time, based on Doob's decomposition, which is more direct and simple.

Proposition 1.3.14 *Every $\mathbb{F}$-martingale M stopped at time τ is a $\mathbb{G}$-semimartingale, with canonical decomposition :*

$$M^{\tau}_n = M^{\mathbb{G}}_n + \sum_{k=1}^{n\wedge\tau} \frac{\Delta\langle M, \widetilde{m}\rangle^{\mathbb{F}}_k}{G_{k-1}} , \quad \forall n \geq 1 ,$$

where $M^{\mathbb{G}}$ is a $\mathbb{G}$-martingale and $\widetilde{m} = \widetilde{G} + \widetilde{a}$.

PROOF: For $n \geq 0$ fixed. We compute the predictable part of the $\mathbb{G}$-semimartingale M. Using Lemma 1.3.3.b),

$$\mathbb{E}(\Delta M_{n+1}|\mathcal{G}_n)1\!\!1_{\{\tau>n\}} = \frac{1}{G_n}\mathbb{E}(\widetilde{G}_{n+1}\Delta M_{n+1}|\mathcal{F}_n)1\!\!1_{\{\tau>n\}}.$$

Using now the Doob decomposition of $\widetilde{G}$, and the martingale property of M, we obtain

$$\begin{aligned}\mathbb{E}(\widetilde{G}_{n+1}\Delta M_{n+1}|\mathcal{F}_n) &= \mathbb{E}\big((\widetilde{m}_{n+1}-\widetilde{a}_{n+1})\Delta M_{n+1}|\mathcal{F}_n\big)\\ &= \mathbb{E}(\widetilde{m}_{n+1}\Delta M_{n+1}|\mathcal{F}_n) = \Delta\langle M,\widetilde{m}\rangle^{\mathbb{F}}_{n+1}\end{aligned}$$

and finally

$$1\!\!1_{\{\tau>n\}}\mathbb{E}(\Delta M_{n+1}|\mathcal{G}_n) = 1\!\!1_{\{\tau>n\}}\frac{1}{G_n}\Delta\langle M,\widetilde{m}\rangle^{\mathbb{F}}_{n+1}.$$

□

1.3.2 Some particular random times

We study two special kind of random times, the honest times and the pseudo-stopping times. We give a discrete time proof of Jeulin's formula for honest times. In continuous time, we refer to [Bar78] and [Jeu80] for more information about the honest times. The notion of pseudo-stopping time was introduced in [Wil02] and formalized in [NY05].

Honest times

First, we give the definition of honest times and two important results to recover the Jeulin's formula for honest time in discrete time. In continuous time, it says that if τ is an honest time and X an $\mathbb{F}$-martingale, then,

$$X_t = X^{\mathbb{G}}_t + \int_0^{t\wedge\tau}\frac{1}{G_{s-}}d\langle\widetilde{m},X\rangle_s - \int_\tau^t \frac{1}{1-G_{s-}}d\langle\widetilde{m},X\rangle_s\,,\quad \forall t\geq 0.$$

where $X^{\mathbb{G}}$ is a $\mathbb{G}$-martingale and $\widetilde{m}$ is the martingale part of $\widetilde{G}$. With the convention of $\int_b^c \cdot = 0$ if $c<b$.

Definition 1.3.15 *A random time τ is honest, if for any $n\geq 0$, there exists an $\mathcal{F}_n$-measurable random variable τ_n, such that $\tau 1\!\!1_{\{\tau\leq n\}} = \tau_n 1\!\!1_{\{\tau\leq n\}}$.*

It follows that any $\mathbb{G}$-predictable process P can be written as

$$P_n = P^{(b)}_n 1\!\!1_{\{\tau\geq n\}} + P^{(a)}_n 1\!\!1_{\{\tau<n\}}\,,\quad \forall n\geq 0\,,$$

where $P^{(a)}$ and $P^{(b)}$ are $\mathbb{F}$-predictable processes.

Lemma 1.3.16 *If τ is honest, $G_n = \widetilde{G}_n$ on the set $\{\tau<n\}$ and $\widetilde{G}_\tau = 1$.*
If $\widetilde{G}_\tau = 1$, then τ is honest.

PROOF: First, we suppose that τ is honest, then for any $n \geq 0$,

$$\begin{aligned}\mathbb{P}(\tau = n|\mathcal{F}_n)\mathbb{1}_{\{\tau<n\}} &= \mathbb{P}(\tau = n|\mathcal{F}_n)\mathbb{1}_{\{\tau<n\}}\mathbb{1}_{\{\tau_n<n\}} = \mathbb{E}(\mathbb{1}_{\{\tau=n\}}\mathbb{1}_{\{\tau_n<n\}}|\mathcal{F}_n)\mathbb{1}_{\{\tau<n\}} \\ &= \mathbb{E}(\mathbb{1}_{\{\tau=n\}}\mathbb{1}_{\{\tau_n<n\}}\mathbb{1}_{\{\tau<n\}}|\mathcal{F}_n)\mathbb{1}_{\{\tau<n\}} = 0\,.\end{aligned}$$

Then, since $G_n + \mathbb{P}(\tau = n|\mathcal{F}_n) = \widetilde{G}_n$, it follows that $G_n\mathbb{1}_{\{\tau<n\}} = \widetilde{G}_n\mathbb{1}_{\{\tau<n\}}$.

Furthermore,

$$\begin{aligned}\widetilde{G}_n\mathbb{1}_{\{\tau=n\}} &= \mathbb{P}(\tau \geq n|\mathcal{F}_n)\mathbb{1}_{\{\tau=n\}}\mathbb{1}_{\{\tau_n=n\}} \\ &= \mathbb{1}_{\{\tau=n\}}\mathbb{E}(\mathbb{1}_{\{\tau_n=n\}}\mathbb{1}_{\{\tau\geq n\}}|\mathcal{F}_n) = \mathbb{1}_{\{\tau=n\}}\ ,\end{aligned}$$

which implies $\widetilde{G}_\tau = 1$.

Now suppose that $\widetilde{G}_\tau = 1$. Then, for $n \geq 0$ fixed, we define

$$\tau_n := \sup\{k \leq n :\ \widetilde{G}_k = 1\}\ .$$

Then, for any $n \geq 0$, one has $\tau = \tau_n$ on the set $\{\tau \leq n\}$. □

Lemma 1.3.17 *If τ is an honest time and X is integrable*

$$\mathbb{E}(X|\mathcal{G}_n)\mathbb{1}_{\{\tau\leq n\}} = \frac{\mathbb{E}(X\mathbb{1}_{\{\tau\leq n\}}|\mathcal{F}_n)}{\mathbb{P}(\tau \leq n|\mathcal{F}_n)}\mathbb{1}_{\{\tau\leq n\}}\ , \quad \forall n \geq 0\ .$$

PROOF: For $n \geq 0$ fixed, from (1.3.12) there exists a family of random variables $(\widetilde{X}(k))_{k\geq 0}$, such that $\widetilde{X}(k) \in \mathcal{F}_n$ for all $0 \leq k \leq n$ and

$$\mathbb{E}(X|\mathcal{G}_n)\mathbb{1}_{\{\tau\leq n\}} = \widetilde{X}(\tau)\mathbb{1}_{\{\tau\leq n\}}\ .$$

Then, using that τ is honest, we obtain that

$$\mathbb{E}(X|\mathcal{G}_n)\mathbb{1}_{\{\tau\leq n\}} = \widetilde{X}(\tau_n)\mathbb{1}_{\{\tau\leq n\}}\ , \tag{1.3.25}$$

taking $\mathcal{F}_n$-conditional expectation in (1.3.25), we deduce that

$$\widetilde{X}(\tau_n) = \frac{\mathbb{E}(X\mathbb{1}_{\{\tau\leq n\}}|\mathcal{F}_n)}{\mathbb{P}(\tau \leq n|\mathcal{F}_n)}\ . \tag{1.3.26}$$

Finally, by (1.3.25) and (1.3.26), we arrive at the desired result. □

Theorem 1.3.18 *Let τ be an honest time and X an $\mathbb{F}$-martingale. Then,*

$$X_n = X_n^{\mathbb{G}} + \sum_{k=1}^{n\wedge\tau}\frac{1}{G_{k-1}}\Delta\langle\widetilde{m}, X\rangle_k^{\mathbb{F}} - \sum_{k=\tau+1}^{n}\frac{1}{1-G_{k-1}}\Delta\langle\widetilde{m}, X\rangle_k^{\mathbb{F}}\ ,\ \ \forall n \geq 1\ ,$$

where $X^{\mathbb{G}}$ is a $\mathbb{G}$-martingale.

PROOF: Let $X = M + P$ be the $\mathbb{G}$-Doob's decomposition of X. Then, for $n \geq 0$ fixed, by (1.3.25), there exists $\widetilde{P}$, an $\mathbb{F}$-predictable process, such that

$$P_n \mathbb{1}_{\{\tau \leq n\}} = \widetilde{P}_n \mathbb{1}_{\{\tau \leq n\}} .$$

Also, we have that $\Delta P_{n+1} = \mathbb{E}(\Delta X_{n+1} | \mathcal{G}_n)$, then

$$\mathbb{E}(\Delta X_{n+1} | \mathcal{G}_n) \mathbb{1}_{\{\tau \leq n\}} = \Delta \widetilde{P}_{n+1} \mathbb{1}_{\{\tau \leq n\}} . \tag{1.3.27}$$

We now take the $\mathcal{F}_n$-conditional expectation in (1.3.27). Taking into account that $\widetilde{P}$ is $\mathbb{F}$-predictable, we obtain

$$\mathbb{E}(\Delta X_{n+1} \mathbb{1}_{\{\tau \leq n\}} | \mathcal{F}_n) = \Delta \widetilde{P}_{n+1} \mathbb{E}(\mathbb{1}_{\{\tau \leq n\}} | \mathcal{F}_n) .$$

Now, using that X is an $\mathbb{F}$-martingale, $\mathbb{E}(\mathbb{1}_{\{\tau \leq n\}} | \mathcal{F}_n) = 1 - G_n$ and $\mathbb{E}(\mathbb{1}_{\{\tau \leq n\}} | \mathcal{F}_{n+1}) = 1 - \widetilde{G}_{n+1}$, we obtain, on the set $\{\tau \leq n\}$ that

$$(1 - G_n) \Delta \widetilde{P}_{n+1} = -\mathbb{E}(\widetilde{G}_{n+1} \Delta X_{n+1} | \mathcal{F}_n) = -\langle \widetilde{m}, X \rangle^{\mathbb{F}}_{n+1} .$$

Then, the result follows from Lemma 1.3.17 and Proposition 1.3.14. □

Proposition 1.3.19 *If τ is honest, then G and $\widetilde{G}$ are $\mathbb{F}$-martingales on $\{n \geq \tau\}$.*

PROOF: Note that for any $n \geq 0$, and recalling the Doob decompositions of $G = m - a$ and $\widetilde{G} = \widetilde{m} - \widetilde{a}$, we have that

$$\Delta a_n \mathbb{1}_{\{\tau \leq n\}} = \mathbb{E}(\mathbb{1}_{\{\tau = n\}} | \mathcal{F}_{n-1}) \mathbb{1}_{\{\tau \leq n\}} \quad \text{and} \quad \Delta \widetilde{a}_n \mathbb{1}_{\{\tau \leq n\}} = \mathbb{E}(\mathbb{1}_{\{\tau = n-1\}} | \mathcal{F}_{n-1}) \mathbb{1}_{\{\tau \leq n\}} ,$$

then, using that τ is honest

$$\begin{aligned} \Delta a_n \mathbb{1}_{\{\tau \leq n\}} &= \mathbb{E}(\mathbb{1}_{\{\tau = n\}} | \mathcal{F}_{n-1}) \mathbb{1}_{\{\tau < n-1\}} = \mathbb{1}_{\{\tau_{n-1} = n\}} \mathbb{1}_{\{\tau < n-1\}} \\ &= \mathbb{1}_{\{\tau = n\}} \mathbb{1}_{\{\tau < n-1\}} = 0 \end{aligned}$$

and

$$\begin{aligned} \Delta \widetilde{a}_n \mathbb{1}_{\{\tau \leq n\}} &= \mathbb{E}(\mathbb{1}_{\{\tau = n-1\}} | \mathcal{F}_{n-1}) \mathbb{1}_{\{\tau < n-1\}} = \mathbb{1}_{\{\tau_{n-1} = n-1\}} \mathbb{1}_{\{\tau < n-1\}} \\ &= \mathbb{1}_{\{\tau = n-1\}} \mathbb{1}_{\{\tau < n-1\}} = 0 , \end{aligned}$$

i.e. G and $\widetilde{G}$ are $\mathbb{F}$-martingales after the honest time τ. □

We refer now to [ACDJ13], to a method to compute predictable brackets in $\mathbb{G}$ in terms of $\mathbb{F}$-predictable process for any random time before τ and for honest times after τ in continuous time.

We give the analogous result in discrete time.

Proposition 1.3.20 *Let X and Y be two $\mathbb{F}$-adapted processes. Then*

$$\Delta \langle X, Y \rangle^{\mathbb{G}}_n \mathbb{1}_{\{\tau \geq n\}} = \frac{1}{G_{n-1}} \mathbb{E}(\widetilde{G}_n \Delta [X, Y]_n | \mathcal{F}_{n-1}) \mathbb{1}_{\{\tau \geq n\}} , \ \ \forall n \geq 1 .$$

If τ is an honest time, then, after τ one has

$$\Delta \langle X, Y \rangle^{\mathbb{G}}_n \mathbb{1}_{\{\tau < n\}} = \frac{1}{1 - G_{n-1}} \mathbb{E}((1 - \widetilde{G}_n) \Delta [X, Y]_n | \mathcal{F}_{n-1}) \mathbb{1}_{\{\tau < n\}} , \ \ \forall n \geq 1 .$$

PROOF: The first equality is a consequence of Lemma 1.3.3. For the second equality, let $n \geq 1$ be fixed. If τ is honest, by definition of predictable bracket, and Lemma 1.3.17, we get

$$\begin{aligned}\Delta\langle X,Y\rangle^{\mathbb{G}}_n \mathbb{1}_{\{\tau<n\}} &= \mathbb{1}_{\{\tau<n\}}\mathbb{E}(\Delta[X,Y]_n|\mathcal{G}_{n-1}) \\ &= \mathbb{1}_{\{\tau<n\}}\frac{1}{\mathbb{P}(\tau<n|\mathcal{F}_{n-1})}\mathbb{E}(\mathbb{1}_{\{\tau<n\}}\Delta[X,Y]_n|\mathcal{F}_{n-1})\end{aligned}$$

then, by the tower property

$$\Delta\langle X,Y\rangle^{\mathbb{G}}_n \mathbb{1}_{\{\tau<n\}} = \mathbb{1}_{\{\tau<n\}}\frac{1}{1-G_{n-1}}\mathbb{E}((1-\widetilde{G}_n)\Delta[X,Y]_n|\mathcal{F}_{n-1})\,.$$

□

Pseudo-stopping Times

Definition 1.3.21 *We say that a random time ρ is an $\mathbb{A}$-pseudo-stopping time if for any bounded $\mathbb{A}$-martingale M, we have $\mathbb{E}(M_\rho) = M_0$.*

Theorem 1.3.22 *The following statements are equivalent :*

(i) τ is an $\mathbb{F}$-pseudo-stopping time.

(ii) $A_{\infty^-} = \mathbb{P}(\tau<\infty|\mathcal{F}_\infty)$ and $A_\infty = 1$.

(iii) $\widetilde{m} = 1$.

(iv) $\widetilde{G}$ is $\mathbb{F}$-predictable.

(v) Every $\mathbb{F}$-martingale stopped at τ is a $\mathbb{G}$-martingale.

PROOF:

(i) ⇒ (ii)
Let M be a bounded $\mathbb{F}$-martingale, in this case, $M_\infty = M_{\infty-}$. Then, by (1.3.6) and the tower property,

$$\mathbb{E}(M_\tau) = \mathbb{E}(M_\tau\mathbb{1}_{\{\tau<\infty\}}) + \mathbb{E}(M_\tau\mathbb{1}_{\{\tau=\infty\}}) = \mathbb{E}(A_{\infty^-}M_{\infty^-}) + \mathbb{E}(M_\infty) - \mathbb{E}(M_\infty\mathbb{1}_{\{\tau<\infty\}})\,. \tag{1.3.28}$$

Since $\mathbb{E}(M_\tau) = M_0 = \mathbb{E}(M_\infty)$, by (i) in (1.3.28) and the tower property, we have

$$\mathbb{E}(A_{\infty^-}M_\infty) = \mathbb{E}(M_\infty\mathbb{1}_{\{\tau<\infty\}}) = \mathbb{E}\big[\mathbb{P}(\tau<\infty|\mathcal{F}_\infty)M_\infty\big]\,. \tag{1.3.29}$$

Therefore, by (1.3.29) $A_{\infty^-} = \mathbb{P}(\tau<\infty|\mathcal{F}_\infty)$ and

$$A_\infty = A_{\infty^-} + \Delta A_\infty = \mathbb{P}(\tau<\infty|\mathcal{F}_\infty) + \mathbb{P}(\tau=\infty|\mathcal{F}_\infty) = 1\,.$$

(ii) ⇒ (iii)
We have that $G_\infty = \widetilde{m}_\infty - A_\infty$, then since $G_\infty = 0$ by definition and by (ii), we obtain that $\widetilde{m}_\infty = A_\infty = 1$, i.e. $\widetilde{m} \equiv 1$.

(iii) ⇒ (iv)
By definition of $\widetilde{m}$ (see (1.3.9)), we have that

$$\widetilde{m}_n = A_n + G_n = A_{n-1} + \widetilde{G}_n\,,\quad \forall n \geq 1\,, \tag{1.3.30}$$

therefore, by (iii) and since A is $\mathbb{F}$-adapted, we get from (1.3.30) that $\widetilde{G}_n = 1 - A_{n-1}$ which is $\mathcal{F}_{n-1}$-measurable for all $n \geq 1$, i.e. $\widetilde{G}$ is $\mathbb{F}$-predictable.

(iv) $\Rightarrow$ (v)

This is a consequence of the Proposition 1.3.14. By (1.3.30), we have that $\widetilde{m}_n = \widetilde{G}_n + A_{n-1}$, for all $n \geq 1$, then since A is $\mathbb{F}$-adapted and (iv), we obtain that $\widetilde{m}$ is an $\mathbb{F}$-martingale which is $\mathbb{F}$-predictable, i.e. $\widetilde{m}_n = \widetilde{m}_0 = 1$ for all $n \geq 1$, then $\Delta\langle M, \widetilde{m}\rangle_n = 0$ for all $n \geq 1$.

(v) $\Rightarrow$ (i)

It suffices to consider any bounded $\mathbb{F}$-martingale M, which assuming (v), satisfies that M^ρ is a $\mathbb{G}$-martingale. Then, as a consequence of the optional stopping theorem applied in $\mathbb{G}$ at time τ, we get $E(M_\tau) = E(M_0)$, hence, τ is an $\mathbb{F}$ pseudo-stopping time. □

The statements (i), (ii), (iii) and (v) of Theorem 1.3.22 are also equivalent in continuous time (see [NY05]), and the statement (iv) is exclusive of discrete time thanks to the explicit representation of $\widetilde{m}$ in terms of $\widetilde{G}$.

The following theorem is also exclusive of discrete time set up. This theorem gives useful relations between the predictableness of G, predictableness of $\widetilde{G}$, the projections of H, the compensator Λ and the balancer $\widetilde{\Lambda}$.

Theorem 1.3.23 *If at least two of the following statements hold, then all the statements hold.*

(i) G *is* $\mathbb{F}$*-predictable.*

(ii) $\widetilde{G}$ *is* $\mathbb{F}$*-predictable.*

(iii) $A = a$.

(iv) $\Lambda = \widetilde{\Lambda}$*, where* Λ *and* $\widetilde{\Lambda}$ *are given in Lemma 1.3.5 and Lemma 1.3.6.*

Proof: For $n \geq 1$ fixed. We have that

$$\Delta a_n = \mathbb{E}(\Delta H_n|\mathcal{F}_{n-1}) = \mathbb{E}(\widetilde{G}_n|\mathcal{F}_{n-1}) - \mathbb{E}(G_n|\mathcal{F}_{n-1}) \ , \tag{1.3.31}$$

$$\Delta A_n = \mathbb{E}(\Delta H_n|\mathcal{F}_n) = \widetilde{G}_n - G_n \ , \tag{1.3.32}$$

then subtracting (1.3.31) from (1.3.32), we get

$$\Delta A_n - \Delta a_n = \widetilde{G}_n - \mathbb{E}(\widetilde{G}_n|\mathcal{F}_{n-1}) + \mathbb{E}(G_n|\mathcal{F}_{n-1}) - G_n \ . \tag{1.3.33}$$

Hence, from (1.3.33), we deduce that :

- If G and $\widetilde{G}$ are $\mathbb{F}$-predictable, then $\Delta A_n = \Delta a_n$ for all $n \geq 0$ i.e. $A = a$. Also, $\lambda = \widetilde{\lambda}$, in particular $\widetilde{\lambda}_0 = \lambda_0$, therefore $\Lambda = \widetilde{\Lambda}$.
- If G is $\mathbb{F}$-predictable and $A = a$, then $\widetilde{G}$ is $\mathbb{F}$-predictable, $\widetilde{G}_n = G_{n-1}$ it follows that $\lambda = \widetilde{\lambda}$, in particular $\widetilde{\lambda}_0 = \lambda_0$, i.e. $\Lambda = \widetilde{\Lambda}$.
- If $\widetilde{G}$ is $\mathbb{F}$-predictable and $A = a$, then G is $\mathbb{F}$-predictable and $\Lambda = \widetilde{\Lambda}$.

Then, it follows that if (iv) holds, we have that

$$\lambda_n = \Big[1 - \frac{\mathbb{E}(G_n|\mathcal{F}_{n-1})}{\mathbb{E}(\widetilde{G}_n|\mathcal{F}_{n-1})}\Big]\mathbb{1}_{\{\mathbb{E}(\widetilde{G}_n|\mathcal{F}_{n-1})>0\}} = \Big[1 - \frac{G_n}{\widetilde{G}_n}\Big]\mathbb{1}_{\{\widetilde{G}_n>0\}} = \widetilde{\lambda}_n \ , \ \ \forall n \geq 1 \ , \tag{1.3.34}$$

$$\begin{cases} \lambda_0 = \mathbb{P}(\tau = 0|\mathcal{F}_0) = \widetilde{\lambda}_0 \ , \\ \lambda_n = \frac{\Delta a_n}{\mathbb{E}(\widetilde{G}_n|\mathcal{F}_{n-1})} \mathbb{1}_{\{\mathbb{E}(\widetilde{G}_n|\mathcal{F}_{n-1})>0\}} = \frac{\Delta A_n}{\widetilde{G}_n} \mathbb{1}_{\{\widetilde{G}_n>0\}} = \widetilde{\lambda}_n \ , \quad \forall n \geq 1 \ , \end{cases} \tag{1.3.35}$$

where we can deduce that :

- If $\Lambda = \widetilde{\Lambda}$ and $\widetilde{G}$ is $\mathbb{F}$-predictable, it follows directly from (1.3.34) that G is $\mathbb{F}$-predictable, hence $A = a$.
- If $\Lambda = \widetilde{\Lambda}$ and G is $\mathbb{F}$-predictable, then from (1.3.34), we deduce that $\widetilde{G}$ is $\mathbb{F}$-predictable, therefore it follows that $a = A$.
- If $\Lambda = \widetilde{\Lambda}$ and $A = a$, we get from (1.3.35) that $\widetilde{G}$ is $\mathbb{F}$-predictable. Therefore, G is also $\mathbb{F}$-predictable.

□

The next theorem characterizes when the $\mathbb{G}$-martingale part of any $\mathbb{F}$-martingale is orthogonal to N. This result holds just in discrete time, thanks to the explicit representation of the compensator.

Theorem 1.3.24 *The following statements are equivalent :*

(i) The compensator of H is equal to the balancer of H, i.e. $\Lambda = \widetilde{\Lambda}$ and $N = \widetilde{N}$.

(ii) $\mathbb{E}(\Delta N_n|\mathcal{F}_n) = 0$ for all $n \geq 1$.

(iii) $\mathbb{E}\Big((U \centerdot N)_n|\mathcal{F}_n\Big) = 0$, for all $n \geq 1$ and any $\mathbb{G}$-predictable process U .

(iv) The $\mathbb{G}$-martingale part $M^{\mathbb{G}}$ of any $\mathbb{F}$ martingale is orthogonal to N.

Proof:

(i) ⇒ (ii)

For all $n \geq 1$. By definition of N, using that $\lambda_n \in \mathcal{F}_n$, we get

$$\mathbb{E}(\Delta N_n|\mathcal{F}_n) = \mathbb{E}(\Delta H_n - \lambda_n \mathbb{1}_{\{\tau\geq n\}}|\mathcal{F}_n) = \Delta A_n - \lambda_n \widetilde{G}_n \ ,$$

by (i) and using that $\widetilde{\lambda}_n = \frac{\Delta A_n}{\widetilde{G}_n} \mathbb{1}_{\{\widetilde{G}_n>0\}}$, we obtain

$$\mathbb{E}(\Delta N_n|\mathcal{F}_n) = \Delta A_n - \widetilde{\lambda}_n \widetilde{G}_n = \Delta A_n \mathbb{1}_{\{\widetilde{G}_n=0\}} = 0 \ ,$$

since on the set $\{\widetilde{G}_n = 0\} = \{\mathbb{P}(\tau \geq n|\mathcal{F}_n) = 0\}$, we have that $\Delta A_n = \mathbb{P}(\tau = n|\mathcal{F}_n) = 0$.

(ii) ⇒ (iii)

Let n be fixed with $n \geq 1$. Let $\bar{U}_n \in \mathcal{F}_{n-1}$ be such that $\bar{U}_n \mathbb{1}_{\{\tau>n-1\}} = U_n \mathbb{1}_{\{\tau>n-1\}}$, then using that $\Delta N_n = \Delta N_n \mathbb{1}_{\{\tau>n-1\}}$ and (ii), we get

$$\mathbb{E}(U_n \Delta N_n|\mathcal{F}_n) = \bar{U}_n \mathbb{E}(\Delta N_n|\mathcal{F}_n) = 0 \ . \tag{1.3.36}$$

(iii)⇒ (iv)

By Proposition 1.1.14, we have that

$$\mathbb{E}[\Delta(M^{\mathbb{G}}_n N_n)|\mathcal{G}_{n-1}] = \mathbb{E}(\Delta M^{\mathbb{G}}_n \ \Delta N_n|\mathcal{G}_{n-1}) \ , \ \forall n \geq 1 \ , \tag{1.3.37}$$

then, since $\mathbb{E}(\Delta M^{\mathbb{G}}_n \ \Delta N_n|\mathcal{G}_{n-1}) = \mathbb{E}(\Delta M_n \ \Delta N_n|\mathcal{G}_{n-1}) \ \forall n \geq 1$. the orthogonality of $M^{\mathbb{G}}$ and N will follow from $\mathbb{E}(\Delta M_n \ \Delta N_n|\mathcal{G}_{n-1}) = 0$. Let $n \geq 1$ be fixed, by Lemma 1.3.3 and following the notation of Lemma 1.3.4, we have that

$$\mathbb{E}(\Delta M_n \ \Delta N_n|\mathcal{G}_{n-1}) \mathbb{1}_{\{\tau\geq n\}} = \mathbb{E}(\Delta M_n R_{n-1} \Delta N_n|\mathcal{F}_{n-1}) \mathbb{1}_{\{\tau\geq n\}} \ . \tag{1.3.38}$$

Since $\Delta M_n \in \mathcal{F}_n$, we have that

$$\mathbb{E}(\Delta M_n R_{n-1} \Delta N_n | \mathcal{F}_n) = \Delta M_n \mathbb{E}(R_{n-1} \Delta N_n | \mathcal{F}_n) \ , \tag{1.3.39}$$

by (iii), we have that $\mathbb{E}(R_{n-1}\Delta N_n|\mathcal{F}_n) = 0$, then taking conditional expectation in (1.3.39),

$$\mathbb{E}(\Delta M_n R_{n-1} \Delta N_n | \mathcal{F}_{n-1}) = \mathbb{E}\big[\Delta M_n \mathbb{E}(R_{n-1} \Delta N_n | \mathcal{F}_n) | \mathcal{F}_{n-1}\big] = 0 \ . \tag{1.3.40}$$

Replacing (1.3.40) in (1.3.38), we get

$$\mathbb{E}(\Delta M_n \ \Delta N_n | \mathcal{G}_{n-1}) \mathbb{1}_{\{\tau \geq n\}} = 0 \ . \tag{1.3.41}$$

On the set $\tau < n$, using that $\{\tau < n\} \in \mathcal{G}_{n-1}$ and that $\Delta N_n \mathbb{1}_{\{\tau<n\}} = 0$, we obtain

$$\mathbb{E}(\Delta M_n \ \Delta N_n | \mathcal{G}_{n-1}) \mathbb{1}_{\{\tau < n\}} = 0 \ . \tag{1.3.42}$$

(iv)$\Rightarrow$ (i)
We consider the $\mathbb{F}$-martingale M given by $M_n := \mathbb{E}(N_n|\mathcal{F}_n) = \Delta A_n - \lambda_n \widetilde{G}_n$ for all $n \geq 1$. Then, for n fixed with $n \geq 0$, by the tower property, we have

$$\mathbb{E}(\Delta[M,N]_n) = \mathbb{E}(\mathbb{E}(N_n|\mathcal{F}_n)\Delta N_n) = \mathbb{E}\big[\mathbb{E}(N_n|\mathcal{F}_n)^2\big] \ . \tag{1.3.43}$$

In the other hand, using (iv), we have $\mathbb{E}(\Delta[M,N]_n) = 0$. Therefore, (1.3.43) implies that

$$\mathbb{E}(N_n|\mathcal{F}_n) = \Delta A_n - \lambda_n \widetilde{G}_n = 0 \ . \tag{1.3.44}$$

Thus, (1.3.44) implies that

$$\lambda_n = \frac{\Delta A_n}{\widetilde{G}_n} \mathbb{1}_{\{\widetilde{G}_n > 0\}} = \widetilde{\lambda}_n \ .$$

Also, by definition we have that $\widetilde{\lambda}_0 = \lambda_0$, i.e. $\Lambda = \widetilde{\Lambda}$. $\square$

We finish this section with the following theorem, which is a recapitulation of the last three Theorems and gives the relation between the principal processes involved in this section.

Corollary 1.3.25 *If G is $\mathbb{F}$-predictable, the following assertions are equivalent :*

(i) *τ is an $\mathbb{F}$-pseudo-stopping time.*
(ii) $A_{\infty^-} = \mathbb{P}(\tau < \infty | \mathcal{F}_\infty)$ *and* $A_\infty = 1$.
(iii) $\widetilde{m} = 1$.
(iv) *$\widetilde{G}$ is predictable.*
(v) *Every $\mathbb{F}$-martingale stopped at τ is a $\mathbb{G}$-martingale.*
(vi) $\Lambda = \widetilde{\Lambda}$.
(vii) $\mathbb{E}(\Delta N_n|\mathcal{F}_n) = 0$ *for all* $n \geq 1$.
(viii) $\mathbb{E}\Big((U \centerdot N)_n \big| \mathcal{F}_n\Big) = 0$, *for all $n \geq 1$ and any $\mathbb{G}$-predictable process U.*
(ix) *The $\mathbb{G}$-martingale part $M^{\mathbb{G}}$ of any $\mathbb{F}$-martingale is orthogonal to N.*

PROOF: (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v)
It follows directly from Theorem 1.3.22.

(iv) $\Leftrightarrow$ (vi)
By hypothesis $\mathbb{P}(\tau = 0) = 0$ then by Theorem 1.3.23, we have the result.

(vi) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (viii) $\Leftrightarrow$ (ix)
This is implied by Theorem 1.3.24. □

1.3.3 Immersion setting

We study the case where the $(\mathcal{H})$-hypothesis holds under progressive filtration enlargement for the discrete case. The following theorem characterizes the immersion property, which combines the classical results for progressive enlargement in continuous case and also gives another relation, where $\widetilde{G}$ is $\mathbb{F}$-predictable and $\widetilde{G}_n = \mathbb{P}(\tau \geq n | \mathcal{F}_\infty)$ for all $n \geq 0$, which holds only in discrete case.

Theorem 1.3.26 ***Characterization of immersion***. *The following statements are equivalent :*

(i) $\mathbb{F} \hookrightarrow \mathbb{G}$.

(ii) $\widetilde{G}$ *is* $\mathbb{F}$*-predictable and* $\widetilde{G}_{n+1} = \mathbb{P}(\tau \geq n+1 | \mathcal{F}_\infty)$ *for all* $n \geq 0$.

(iii) $G_n = \mathbb{P}(\tau > n | \mathcal{F}_\infty)$ *for all* $n \geq 0$.

(iv) $\mathbb{E}(H_n | \mathcal{F}_n) = \mathbb{E}(H_n | \mathcal{F}_m)$ *for all* $0 \leq n \leq m$.

PROOF:

(i) $\Rightarrow$ (ii)
If $\mathbb{F} \hookrightarrow \mathbb{G}$ then by Proposition 1.1.40, we have that for any integrable random variable $\mathcal{G}_n$-measurable ζ_n satisfies, $\mathbb{E}(\zeta_n | \mathcal{F}_n) = \mathbb{E}(\zeta_n | \mathcal{F}_{n+1}) = \mathbb{E}(\zeta_n | \mathcal{F}_\infty)$, for all $n \geq 0$, in special taking $\zeta_n = \mathbb{1}_{\{\tau \geq n+1\}}$, we obtain $\mathbb{E}(\widetilde{G}_{n+1} | \mathcal{F}_n) = \widetilde{G}_{n+1} = \mathbb{P}(\tau \geq n+1 | \mathcal{F}_\infty)$ for all $n \geq 0$, therefore (ii) is satisfied.

(ii) $\Rightarrow$ (iii)
Since (ii) is achieved and by Lemma 1.3.10, we have that $G_n = \mathbb{E}(\widetilde{G}_{n+1} | \mathcal{F}_n) = \widetilde{G}_{n+1} = \mathbb{P}(\tau \geq n+1 | \mathcal{F}_\infty) = \mathbb{P}(\tau > n | \mathcal{F}_\infty)$ for all $n \geq 0$.

(iii) $\Rightarrow$ (iv)
By definition of G, (iii) is equivalent to $\mathbb{E}(1 - H_n | \mathcal{F}_n) = \mathbb{E}(1 - H_n | \mathcal{F}_\infty)$, which is equivalent to $\mathbb{E}(H_n | \mathcal{F}_n) = \mathbb{E}(H_n | \mathcal{F}_\infty)$.

(iv) $\Rightarrow$ (i)
From (1.3.12) for all $n \geq 0$ any integrable $\mathcal{G}_n$-measurable random variable ζ_n, can be decomposed as $\zeta_n = \bar{\zeta}_n \mathbb{1}_{\{\tau > n\}} + \zeta_n(\tau) \mathbb{1}_{\{\tau \leq n\}}$, where $\bar{\zeta}_n \in \mathcal{F}_n$ and $\zeta_n(k) \in \mathcal{F}_n$ for all $0 \leq k \leq n$.

For $0 \leq n \leq m$ fixed. Using that $\mathbb{1}_{\{\tau > n\}} = 1 - H_n$ and that $\bar{\zeta}_n \in \mathcal{F}_n$, we have that $\mathbb{E}(\bar{\zeta}_n \mathbb{1}_{\{\tau > n\}} | \mathcal{F}_n) = \bar{\zeta}_n \big[1 - \mathbb{E}(H_n | \mathcal{F}_n)\big]$, then by (iv) and $\bar{\zeta}_n \in \mathcal{F}_m$, we get

$$\mathbb{E}(\bar{\zeta}_n \mathbb{1}_{\{\tau > n\}} | \mathcal{F}_n) = \mathbb{E}(\bar{\zeta}_n \mathbb{1}_{\{\tau > n\}} | \mathcal{F}_m) \ . \tag{1.3.45}$$

In the other hand, $\zeta_n(\tau)H_n = (\zeta \centerdot H)_n$ and since $\zeta_n(k) \in \mathcal{F}_n$ for all $k \le n$, we get that

$$\mathbb{E}(\zeta_n(\tau)H_n|\mathcal{F}_n) = \mathbb{E}\big((\zeta \centerdot H)_n\big|\mathcal{F}_n\big) = \sum_{k=0}^{n} \zeta_n(k)\mathbb{E}(\Delta H_k|\mathcal{F}_n) ,$$

by (iv), we obtain $\mathbb{E}(\zeta_n(\tau)H_n|\mathcal{F}_n) = \sum_{k=0}^{n} \zeta_n(k)\mathbb{E}(\Delta H_k|\mathcal{F}_m)$, then since $\zeta_n(k) \in \mathcal{F}_m$ for all $k \le m$, we get that

$$\mathbb{E}(\zeta_n(\tau)H_n|\mathcal{F}_n) = \mathbb{E}(\zeta_n(\tau)H_n|\mathcal{F}_m) . \tag{1.3.46}$$

Finally, using (1.3.45) and (1.3.46) we obtain $\mathbb{E}(\zeta_n|\mathcal{F}_n) = \mathbb{E}(\zeta_n|\mathcal{F}_m)$ for all $0 \le n \le m$, which is equivalent to immersion. □

In continuous time, if we have that $\mathbb{F} \hookrightarrow \mathbb{G}$ and some continuity hypothesis we get that $A = a$, in discrete time we do not have the continuity, but we can use the predictability of G in order to have the analogous results as in continuous time (see for example [Nik06]).

Theorem 1.3.27 *Suppose $\mathbb{F} \hookrightarrow \mathbb{G}$, then the following statements are equivalent*

(i) *G is $\mathbb{F}$-predictable.*

(ii) $A = a$.

(iii) $\Lambda = \widetilde{\Lambda}$.

(iv) $\mathbb{E}(\Delta N_n|\mathcal{F}_\infty) = 0$ *for all* $n \ge 1$.

(v) $\mathbb{E}\Big((U \centerdot N)_n\big|\mathcal{F}_m\Big) = 0$, *for all* $1 \le n \le m$ *and any $\mathbb{G}$-predictable process U.*

(vi) *Any $\mathbb{F}$-martingale M (which is also a $\mathbb{G}$-martingale by immersion) is $\mathbb{G}$-orthogonal to N.*

Proof:
(i) ⇔ (ii) ⇔ (iii)
The immersion hypothesis implies by Theorem 1.3.26 that $\widetilde{G}$ is predictable, then from Theorem 1.3.23, we deduce that (i), (ii) and (iii) are equivalent.

(iii) ⇔ (iv) ⇔ (v) ⇔ (vi)
This is a corollary from Theorem 1.3.24 and Proposition 1.1.40. □

Remark 1.3.28 In continuous time, we have that if $\mathbb{F} \hookrightarrow \mathbb{G}$ and τ avoids $\mathbb{F}$-stopping times, then $\mathbb{E}\Big(\int_0^t U_s dN_s\big|\mathcal{F}_t\Big) = 0$ for all $t \ge 0$, any $\mathbb{G}$-predictable process U and N the compensated martingale (see [CJN12, Lemma 5.1]).

We now deduce easily from Theorem 1.3.27 the following projection formula :

Lemma 1.3.29 *Suppose $\mathbb{F} \hookrightarrow \mathbb{G}$, $\tau < \infty$ and let X be an $\mathbb{F}$-adapted integrable process, then*

$$\mathbb{E}(X_\tau|\mathcal{F}_n) = \mathbb{E}\big((X \centerdot A)_{\infty^-}\big|\mathcal{F}_n\big) , \quad \forall n \ge 0 .$$

Moreover, if G is $\mathbb{F}$-predictable, then

$$\mathbb{E}(X_\tau|\mathcal{F}_n) = -\mathbb{E}\big((X \centerdot G)_{\infty^-}\big|\mathcal{F}_n\big) .$$

PROOF: Notice that $X_\tau = (X \centerdot H)_{\infty^-}$.Let $n \geq 0$ be fixed. We consider the two events $\{\tau \leq n\}$ and $\{\tau > n\}$, and write

$$\mathbb{E}(X_\tau|\mathcal{F}_n) = \mathbb{E}(X_\tau \mathbb{1}_{\{\tau\leq n\}}|\mathcal{F}_n) + \mathbb{E}(X_\tau \mathbb{1}_{\{\tau>n\}}|\mathcal{F}_n) \ . \tag{1.3.47}$$

Using that $X_k \in \mathcal{F}_n$ for all $k \leq n$ and immersion property we get that Theorem 1.3.26 implies $X_k\mathbb{E}(\Delta H_k|\mathcal{F}_n) = X_k\mathbb{E}(\Delta H_k|\mathcal{F}_k)$, then

$$\mathbb{E}(X_\tau \mathbb{1}_{\{\tau\leq n\}}|\mathcal{F}_n) = \sum_{k=0}^{n} X_k\mathbb{E}(\Delta H_k|\mathcal{F}_n) = \sum_{k=0}^{n} X_k\mathbb{E}(\Delta H_k|\mathcal{F}_k) = (X \centerdot A)_n \ . \tag{1.3.48}$$

In the other hand,

$$\mathbb{E}(X_\tau \mathbb{1}_{\{\tau>n\}}|\mathcal{F}_n) \;=\; \mathbb{E}\Big(\sum_{k=0}^{\infty} X_k\Delta H_k \mathbb{1}_{\{\tau>n\}}\Big|\mathcal{F}_n\Big) = \mathbb{E}\Big(\sum_{k=n+1}^{\infty} X_k\Delta H_k\Big|\mathcal{F}_n\Big) \ ,$$

by the tower property and using that X is $\mathbb{F}$-adapted, we get

$$\mathbb{E}(X_\tau \mathbb{1}_{\{\tau>n\}}|\mathcal{F}_n) = \mathbb{E}\Big(\sum_{k=n+1}^{\infty} \mathbb{E}(X_k\Delta H_k|\mathcal{F}_k)\Big|\mathcal{F}_n\Big) = \mathbb{E}\Big(\sum_{k=n+1}^{\infty} X_k\Delta A_k\Big|\mathcal{F}_n\Big) \ , \tag{1.3.49}$$

replacing (1.3.48) and (1.3.49) in (1.3.47) and using that $(X \centerdot A)_n \in \mathcal{F}_n$ for all $k \leq n$,

$$\mathbb{E}(X_\tau|\mathcal{F}_n) = \mathbb{E}\big((X \centerdot A)_{\infty^-}\big|\mathcal{F}_n\big) \ .$$

Finally, by immersion $\tilde{G}$ is $\mathbb{F}$-predictable. Therefore, since G and $\widetilde{G}$ are $\mathbb{F}$-predictable, from Theorem 1.3.23 $\Delta A_k = \Delta a_k = -\Delta G_k$, i.e.

$$\mathbb{E}(X_\tau|\mathcal{F}_n) = -\mathbb{E}\big((X \centerdot G)_{\infty^-}\big|\mathcal{F}_n\big) \ .$$

□

1.3.4 Representation Theorem

In this subsection, we present a predictable representation theorem.

Theorem 1.3.30 *Suppose $\mathbb{F} \hookrightarrow \mathbb{G}$. Let X be a $\mathbb{G}$-martingale and M an $\mathbb{F}$-martingale. Then, there exist p and q two $\mathbb{G}$-predictable processes and $M^\perp$ a martingale orthogonal to M and N, such that*

$$X_n = (p \centerdot M)_n + (q \centerdot N)_n + M_n^\perp$$

or equivalently

$$\Delta X_n = p_n \ \Delta M_n + q_n\Delta N_n + \Delta M_n^\perp \ .$$

Moreover, the processes p, q and $M^\perp$ are given explicitly on the sets $(B_n)_{n\geq 1}$, where

$$B_{n-1} := \{\omega \ : \ \mathbb{E}(\Delta M_n\Delta N_n|\mathcal{G}_{n-1})^2 - \mathbb{E}(|\Delta M_n|^2|\mathcal{G}_{n-1})\mathbb{E}(|\Delta N_n|^2|\mathcal{G}_{n-1}) \neq 0\} \ , \tag{1.3.50}$$

$$
\begin{aligned}
p_n &:= \frac{\mathbb{E}\{[X_n-\mathbb{E}(X_n|\mathcal{G}_{n-1})]\Delta M_n|\mathcal{G}_{n-1}\}\mathbb{E}\Big(|\Delta N_n|^2|\mathcal{G}_{n-1}\Big)}{\mathbb{E}\Big(|\Delta M_n|^2|\mathcal{G}_{n-1}\Big)\mathbb{E}\Big(|\Delta N_n|^2|\mathcal{G}_{n-1}\Big)-\mathbb{E}(\Delta M_n\Delta N_n|\mathcal{G}_{n-1})^2} \\
&\quad - \frac{\mathbb{E}\{[X_n-\mathbb{E}(X_n|\mathcal{G}_{n-1})]\Delta N_n|\mathcal{G}_{n-1}\}\mathbb{E}(\Delta M_n\Delta N_n|\mathcal{G}_{n-1})}{\mathbb{E}\Big(|\Delta M_n|^2|\mathcal{G}_{n-1}\Big)\mathbb{E}\Big(|\Delta N_n|^2|\mathcal{G}_{n-1}\Big)-\mathbb{E}(\Delta M_n\Delta N_n|\mathcal{G}_{n-1})^2}\ , \\
q_n &:= \frac{\mathbb{E}\{[X_n-\mathbb{E}(X_n|\mathcal{G}_{n-1})]\Delta N_n|\mathcal{G}_{n-1}\}\mathbb{E}(|\Delta M_n|^2|\mathcal{G}_{n-1})}{\mathbb{E}\Big(|\Delta M_n|^2|\mathcal{G}_{n-1}\Big)\mathbb{E}\Big(|\Delta N_n|^2|\mathcal{G}_{n-1}\Big)-\mathbb{E}(\Delta M_n\Delta N_n|\mathcal{G}_{n-1})^2} \\
&\quad - \frac{\mathbb{E}\{[X_n-\mathbb{E}(X_n|\mathcal{G}_{n-1})]\Delta M_n|\mathcal{G}_{n-1}\}\mathbb{E}\Big(\Delta M_n\Delta N_n|\mathcal{G}_{n-1}\Big)}{\mathbb{E}\Big(|\Delta M_n|^2|\mathcal{G}_{n-1}\Big)\mathbb{E}\Big(|\Delta N_n|^2|\mathcal{G}_{n-1}\Big)-\mathbb{E}(\Delta M_n\Delta N_n|\mathcal{G}_{n-1})^2}\ ,
\end{aligned}
\tag{1.3.51}
$$

and the orthogonal martingale $M^\perp$ is given by

$$
\Delta M_n^\perp \ := \ X_n - \mathbb{E}(X_n|\mathcal{G}_{n-1}) - p_n\Delta M_n - q_n\Delta N_n\ ,\forall n \geq 1\ .
$$

PROOF: For the existence, we cite [FS04] Theorem 10.18. We give the explicit values of p, q and $M^\perp$ for the sets $(B_n)_{n\geq 0}$.

For $n \geq 1$ fixed, we compute $\mathbb{E}(\Delta X_n \Delta M_n|\mathcal{G}_{n-1})$ and $\mathbb{E}(\Delta X_n\Delta N_n|\mathcal{G}_{n-1})$,

$$
\begin{cases}
\mathbb{E}(\Delta X_n\Delta M_n|\mathcal{G}_{n-1}) = p_n\mathbb{E}(|\Delta M_n|^2|\mathcal{G}_{n-1}) + q_n\mathbb{E}(\Delta M_n\Delta N_n|\mathcal{G}_{n-1})\ , \\
\mathbb{E}(\Delta X_n\Delta N_n|\mathcal{G}_{n-1}) = p_n\mathbb{E}(\Delta M_n\Delta N_n|\mathcal{G}_{n-1}) + q_n\mathbb{E}(|\Delta N_n|^2|\mathcal{G}_{n-1})\ ,
\end{cases}
\tag{1.3.52}
$$

Solving this system (1.3.52), gives the result over A_{n-1}.

We will prove that K is orthogonal to M and N, where

$$
\Delta K_n := \Delta X_n - p_n\Delta M_n - q_n\Delta N_n\ ,
$$

i.e. we will show that $\mathbb{E}(\Delta K_n\Delta M_n|\mathcal{G}_{n-1}) = 0$ and $\mathbb{E}(\Delta K_n\Delta N_n|\mathcal{G}_{n-1}) = 0$.

For $n \geq 0$ fixed, we have that

$$
\begin{aligned}
\mathbb{E}(\Delta K_n\Delta M_n|\mathcal{G}_{n-1}) &= \mathbb{E}(\Delta X_n\Delta M_n|\mathcal{G}_{n-1}) - p_n\mathbb{E}(|\Delta M_n|^2|\mathcal{G}_{n-1}) \\
&\quad - q_n\mathbb{E}(\Delta M_n\Delta N_n|\mathcal{G}_{n-1})
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}(\Delta K_n\Delta N_n|\mathcal{G}_{n-1}) &= \mathbb{E}(\Delta X_n\Delta N_n|\mathcal{G}_{n-1}) - p_n\mathbb{E}(\Delta M_n\Delta N_n|\mathcal{G}_{n-1}) \\
&\quad - q_n\mathbb{E}(|\Delta N_n|^2|\mathcal{G}_{n-1})\ .
\end{aligned}
$$

By (1.3.52), we have that $\mathbb{E}(\Delta K_n\Delta M_n|\mathcal{G}_{n-1}) = \mathbb{E}(\Delta K_n\Delta N_n|\mathcal{G}_{n-1}) = 0$. □

1.3.5 Equivalent probability measures

We denote by $\mathbb{F} \overset{\mathbb{P}}{\hookrightarrow} \mathbb{G}$ for immersion of $\mathbb{F}$ in $\mathbb{G}$ under the probability $\mathbb{P}$. Let $\mathcal{I}(\mathbb{P})$ be the set of all the probability measures $\mathbb{Q}$ equivalent to $\mathbb{P}$ and such that $\mathbb{F} \overset{\mathbb{Q}}{\hookrightarrow} \mathbb{G}$.

Proposition 1.3.31 *Suppose $\mathbb{F} \overset{\mathbb{P}}{\hookrightarrow} \mathbb{G}$. Let $\mathbb{Q}$ be a probability measure which is equivalent to $\mathbb{P}$ and let L be its Radon-Nikodym density. If L is $\mathbb{F}$-adapted, then*

$$
\mathbb{Q}(\tau > n|\mathcal{F}_n) = \mathbb{P}(\tau > n|\mathcal{F}_n) = G_n\ ,\ \forall n \geq 0
$$

and $\mathbb{F} \overset{\mathbb{Q}}{\hookrightarrow} \mathbb{G}$. Consequently, the predictable compensator of H is unchanged under such equivalent changes of probability measures, i.e. N is a $\mathbb{G}$-martingale under $\mathbb{P}$ and $\mathbb{Q}$.

PROOF: Let X be an $(\mathbb{F},\mathbb{Q})$-martingale, then since L is $\mathbb{F}$-adapted we get thet $(X_nL_n, n\geq 0)$ is an $(\mathbb{F},\mathbb{P})$-martingale, and since $\mathbb{F}$ is immersed in $\mathbb{G}$ under $\mathbb{P}$, we have that $(X_nL_n, n\geq 0)$ is a $(\mathbb{G},\mathbb{P})$-martingale which implies that X is a $(\mathbb{G},\mathbb{Q})$-martingale, i.e. $\mathbb{F}\overset{\mathbb{Q}}{\hookrightarrow}\mathbb{G}$. We have for each $n\leq k$, using Bayes formula

$$\mathbb{Q}(\tau\leq n|\mathcal{F}_k)=\frac{\mathbb{E}^{\mathbb{P}}(L_k\mathbb{1}_{\{\tau\leq n\}}|\mathcal{F}_k)}{\mathbb{E}^{\mathbb{P}}(L_k|\mathcal{F}_k)}=\mathbb{P}(\tau\leq n|\mathcal{F}_k)\ ,$$

in particular, $\mathbb{Q}(\tau\leq n|\mathcal{F}_n)=\mathbb{P}(\tau\leq n|\mathcal{F}_n)$, then by $\mathbb{F}\overset{\mathbb{P}}{\hookrightarrow}\mathbb{G}$, $\mathbb{Q}(\tau\leq n|\mathcal{F}_n)=\mathbb{Q}(\tau\leq n|\mathcal{F}_k)$ and the assertion follows. □

Theorem 1.3.32 *Assume $\mathbb{F}\overset{\mathbb{P}}{\hookrightarrow}\mathbb{G}$ and G is $\mathbb{F}$-predictable. Let M be an $(\mathbb{F},\mathbb{P})$-martingale and let F be an integrable $\mathbb{G}$-predictable process such that $\mathcal{E}(F\centerdot M)$ is a positive $\mathbb{G}$-martingale. Let P be an integrable $\mathbb{F}$-predictable process such that $\mathcal{E}(P\centerdot N)$ is a positive $\mathbb{G}$-martingale. Let*

$$E_n:=\mathcal{E}(F\centerdot M)_n\mathcal{E}(P\centerdot N)_n\ ,\quad\forall n\geq 0\ ,$$

and assume that E is uniformly integrable.

Define

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{G}_n}=E_n\ ,\quad\forall n\geq 0\ .$$

Then, the $\mathbb{Q}$-Azéma supermartingale associated with τ has the following multiplicative decomposition :

$$G^{\mathbb{Q}}_n=\mathbb{Q}(\tau>n|\mathcal{F}_n)=\mathcal{E}\Big((\bar{F}-a^F)\centerdot\widetilde{M}\Big)_n\,\mathcal{E}\Big(-P\centerdot\Lambda\Big)_n G_n\ ,\quad\forall n\geq 0\ ,$$

where

- *a^F is the $\mathbb{F}$-predictable projection of the process F under the probability $\mathbb{Q}$, i.e. $a^F_0:=F_0$ and $a^F_n:=\mathbb{E}^{\mathbb{Q}}(F_n|\mathcal{F}_{n-1})$, for all $n\geq 1$,*
- *$\bar{F}$ is an $\mathbb{F}$-predictable process such that $F_n\mathbb{1}_{\{\tau>n-1\}}=\bar{F}_n\mathbb{1}_{\{\tau>n-1\}}$, for all $n\geq 0$ and*
- *$\widetilde{M}$ defined by $\widetilde{M}_0:=M_0$ and $\widetilde{M}_n:=M_n-\sum_{k=1}^n\frac{\Delta[M,e]_k}{e_k}$ for all $n\geq 0$, is an $(\mathbb{F},\mathbb{Q})$-martingale with $e_k=\mathbb{E}(E_k|\mathcal{F}_k)$, for all $k\geq 0$.*

The process $N^{\mathbb{Q}}$, given by

$$N^{\mathbb{Q}}_0:=H_0\ ,\quad N^{\mathbb{Q}}_n:=H_n-\sum_{k=1}^{n\wedge\tau}\Big(1-\frac{G_k}{G_{k-1}}P_k\Big)\Delta\Lambda_k\ ,\quad\forall n\geq 1,$$

is a $(\mathbb{G},\mathbb{Q})$-martingale. In particular, if the process F is $\mathbb{F}$-predictable, then :

$$G^{\mathbb{Q}}_n=\mathbb{Q}(\tau>n|\mathcal{F}_n)=\mathcal{E}(-P\centerdot\Lambda)_nG_n\ ,\quad\forall n\geq 0$$

and the immersion property holds under $\mathbb{Q}$.

PROOF: Let $n\geq 0$ be fixed. Note that E is a martingale : this is a local martingale by orthogonality of M and N and a martingale by Theorem 1.1.2. First, we compute $e_n:=\mathbb{E}^{\mathbb{P}}(E_n|\mathcal{F}_n)$. By Proposition 1.1.18, we have that for all $k\geq 1$,

$$E_k=E_{k-1}(1+F_k\Delta M_k)(1+P_k\Delta N_k)\ ,$$

which is equivalent to

$$\Delta E_k = E_{k-1}(F_k \Delta M_k + P_k \Delta N_k + F_k P_k \Delta[M,N]_k) \ .$$

Then, taking the sum for $1 \le k \le n$, and since $E_0 = 1$,

$$E_n = 1 + \big((E_- F) \centerdot M)_n + \big((E_- P) \centerdot N\big)_n + \big((E_- FP) \centerdot [M,N]\big)_n \ . \tag{1.3.53}$$

Using that $\Delta M_k \in \mathcal{F}_n$, $E_{k-1}F_kP_k \in \mathcal{G}_{n-1}$ for all $1 \le k \le n$ and G predictable by Theorem 1.3.27, we have that

$$\mathbb{E}\Big(\big((E_- FP) \centerdot [M,N]\big)_n \big| \mathcal{F}_n\Big) = \sum_{k=1}^{n} \Delta M_k \mathbb{E}(E_{k-1}F_kP_k \Delta N_k | \mathcal{F}_n) = 0 \ , \tag{1.3.54}$$

and again by Theorem 1.3.27,

$$\mathbb{E}\big((E_- P) \centerdot N_n \big| \mathcal{F}_n\big) = 0 \ . \tag{1.3.55}$$

Taking the conditional expectation of (1.3.53) and using (1.3.54) and (1.3.55), we obtain

$$e_n = 1 + \mathbb{E}\big((E_- F \centerdot M)_n | \mathcal{F}_n\big) = 1 + \sum_{k=1}^{n} \mathbb{E}\big(E_{k-1}F_k | \mathcal{F}_n\big) \Delta M_k \ .$$

Then by immersion, since $E_{k-1}F_k \in \mathcal{G}_{k-1}$, we have that $\mathbb{E}(E_{k-1}F_k|\mathcal{F}_n) = \mathbb{E}(E_{k-1}F_k|\mathcal{F}_{k-1})$ for all $k \le n$,

$$e_n := 1 + \sum_{k=1}^{n} \mathbb{E}(E_{k-1}F_k | \mathcal{F}_{k-1}) \Delta M_k \ .$$

Since $\mathbb{E}(E_{n-1}F_n|\mathcal{F}_{n-1}) = \mathbb{E}^{\mathbb{Q}}(F_n|\mathcal{F}_{n-1})e_{n-1}$, we obtain

$$e_n = e_{n-1} + e_{n-1}\mathbb{E}^{\mathbb{Q}}(F_n|\mathcal{F}_{n-1})\Delta M_n = \mathcal{E}(a^F \centerdot M)_n \ , \tag{1.3.56}$$

where a^F is the $\mathbb{F}$-predictable projection of F under the probability $\mathbb{Q}$.

Now, we compute $\mathbb{E}(\mathbb{1}_{\{\tau > n\}} E_n | \mathcal{F}_n)$. Let $\bar{F}$ be an $\mathbb{F}$-predictable process, such that $F_k \mathbb{1}_{\{\tau > n\}} = \bar{F}_k \mathbb{1}_{\{\tau > n\}}$ for all $k \le n$, then

$$\mathcal{E}(F \centerdot M)_n \mathbb{1}_{\{\tau > n\}} = \Pi_{k=1}^{n}(1 + \bar{F}_k \Delta M_k)\mathbb{1}_{\{\tau > n\}} \ . \tag{1.3.57}$$

In the other hand, we have that

$$\mathcal{E}(P \centerdot N)_n \mathbb{1}_{\{\tau > n\}} = \Pi_{k=1}^{n}(1 + P_k(\Delta H_k - \Delta\Lambda_k \mathbb{1}_{\{\tau \ge k\}})\mathbb{1}_{\{\tau > n\}} = \Pi_{k=1}^{n}(1 - P_k \Delta\Lambda_k)\mathbb{1}_{\{\tau > n\}} \ . \tag{1.3.58}$$

Then, using (1.3.57), we get that

$$\mathbb{E}(\mathbb{1}_{\{\tau > n\}} E_n | \mathcal{F}_n) = \mathcal{E}(\bar{F} \centerdot M)_n \mathbb{E}\big(\mathcal{E}(P \centerdot N)_n \mathbb{1}_{\{\tau > n\}} \big| \mathcal{F}_n\big) \ ,$$

by (1.3.58) and $P_k \Delta\Lambda_k \in \mathcal{F}_n$ for all $1 \le k \le n$, we have that

$$\mathbb{E}(\mathbb{1}_{\{\tau > n\}} E_n | \mathcal{F}_n) = \mathcal{E}(\bar{F} \centerdot M)_n \mathcal{E}(-P \centerdot \Lambda)_n G_n \ . \tag{1.3.59}$$

Replacing (1.3.56) and (1.3.59) in the formula $G^{\mathbb{Q}}_n = \mathbb{E}^{\mathbb{P}}(\mathbb{1}_{\{\tau>n\}}E_n|\mathcal{F}_n)/e_n$ leads to

$$G^{\mathbb{Q}}_n = \mathcal{E}(-P \centerdot \Lambda)_n \frac{\mathcal{E}(\bar{F} \centerdot M)_n}{\mathcal{E}(a^F \centerdot M)_n} G_n \ .$$

By definition of the exponential, it follows that

$$\frac{\mathcal{E}(\bar{F} \centerdot M)_n}{\mathcal{E}(a^F \centerdot M)_n} = \mathcal{E}\Big(\sum_k^{\cdot}(\bar{F}_k - a^F_k)\Delta M_k \frac{1}{a^F_k \Delta M_k + 1}\Big)_n \ . \tag{1.3.60}$$

From $\frac{\Delta e_k}{e_{k-1}} = a^F_k \Delta M_k$, we have that

$$\frac{1}{a^F_k \Delta M_k + 1} = \frac{e_{k-1}}{e_k} = 1 - \frac{\Delta e_k}{e_k} \ , \ \forall 0 \leq k \leq n \ . \tag{1.3.61}$$

Then, replacing (1.3.61) in (1.3.60), we obtain

$$\frac{\mathcal{E}(\bar{F} \centerdot M)_n}{\mathcal{E}(a^F \centerdot M)_n} = \mathcal{E}\big((\bar{F} - a^F) \centerdot \widetilde{M}\big)_n \ ,$$

where $\widetilde{M}$ defined by $\widetilde{M}_0 = 1$ and $\Delta\widetilde{M}_k = \Delta M_k - \frac{\Delta e_k \Delta M_k}{e_k}$, for all $1 \leq k \leq n$ is an $(\mathbb{F}, \mathbb{Q})$-martingale by Theorem 1.1.25. Therefore

$$G^{\mathbb{Q}}_n = \mathcal{E}(-P \centerdot \Lambda)_n \mathcal{E}\big((\bar{F} - a^F) \centerdot \widetilde{M}\big)_n G_n \ .$$

It follows that for all $k \geq 1$,

$$\frac{G^{\mathbb{Q}}_k}{G^{\mathbb{Q}}_{k-1}} = (1 - P_k \Delta\Lambda_k)\big(1 + (\bar{F}_k - a^F_k)\Delta\widetilde{M}_k\big)\frac{G_k}{G_{k-1}} \ , \tag{1.3.62}$$

then taking conditional expectation under $\mathbb{Q}$ in (1.3.62), and using that a^F, F, P, Λ and G are $\mathbb{F}$-predictable and that $\widetilde{M}$ is an $(\mathbb{F}, \mathbb{Q})$-martingale, we get

$$\mathbb{E}^{\mathbb{Q}}\Big(\frac{G^{\mathbb{Q}}_k}{G^{\mathbb{Q}}_{k-1}}\Big|\mathcal{F}_{k-1}\Big) = (1 - P_k \Delta\Lambda_k)\frac{G_k}{G_{k-1}} \ , \quad \forall k \geq 1 \ . \tag{1.3.63}$$

Computing the compensated $\mathbb{Q}$-martingale of H, we get

$$N^{\mathbb{Q}}_n := H_n - \sum_{k=1}^{n\wedge\tau} \frac{G^{\mathbb{Q}}_{k-1} - \mathbb{E}^{\mathbb{Q}}(G^{\mathbb{Q}}_k|\mathcal{F}_{k-1})}{G^{\mathbb{Q}}_{k-1}} = H_n - \sum_{k=1}^{n\wedge\tau}\Big(1 - \mathbb{E}^{\mathbb{Q}}\Big(\frac{G^{\mathbb{Q}}_k}{G^{\mathbb{Q}}_{k-1}}\Big|\mathcal{F}_{k-1}\Big)\Big) \ .$$

Therefore, using (1.3.63)

$$N^{\mathbb{Q}}_n = H_n - \sum_{k=1}^{n\wedge\tau}\Big(1 - (1 - P_k\Delta\Lambda_k)\frac{G_k}{G_{k-1}}\Big) = H_n - \sum_{k=1}^{n\wedge\tau}\Big(\frac{G_{k-1} - G_k}{G_{k-1}} - \frac{P_k G_k \Delta\Lambda_k}{G_{k-1}}\Big) \ , \tag{1.3.64}$$

since G is predictable, we have that $\Delta\Lambda_k = \frac{G_{k-1}-G_k}{G_{k-1}}\mathbb{1}_{\{G_{k-1}>0\}}$ for all $k \geq 1$, then (1.3.64) is equivalent to

$$N^{\mathbb{Q}}_n = H_n - \sum_{k=1}^{n\wedge\tau}\Big(1 - P_k\frac{G_k}{G_{k-1}}\Big)\Delta\Lambda_k \ .$$

In particular, if F is $\mathbb{F}$-predictable, $\bar{F} = a^F$ then

$$G_n^{\mathbb{Q}} \;=\; \mathcal{E}(-P \centerdot \Lambda)_n G_n \;,\; e_n = 1\,.$$

Let X be an $(\mathbb{F},\mathbb{Q})$-martingale. Since the Azéma supermartingale is predictable, X^τ is a martingale. The change of probability leaves $\mathbb{F}$ invariant, after τ, hence X is a $(\mathbb{Q},\mathbb{G})$-martingale. □

Corollary 1.3.33 *Suppose that $\mathbb{F} \overset{\mathbb{P}}{\hookrightarrow} \mathbb{G}$ and G is $\mathbb{F}$-predictable. Assume further that $G_n > 0$ for all $n \geq 0$. Define $\mathbb{Q}$ on $\mathcal{G}_n$ by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = E_n := \mathcal{E}(F \centerdot M)_n \;, \quad \forall n \geq 0 \;,$$

with F a $\mathbb{G}$-predictable process such that E is a uniformly integrable $\mathbb{G}$-martingale. Then, under $\mathbb{Q}$ the process $N = H - \Lambda^\tau$ remains a G-martingale.

PROOF: It suffices to take $P = 0$ in Theorem 1.3.32. □

Theorem 1.3.27, Theorem 1.3.32 and Corollary 1.3.33 are the discrete version of Lemma 5.1, Theorem 6.4 and Corollary 6.5 in [CJN12]. In continuous time the results holds under Assumption (A) : the random time τ avoids every $\mathbb{F}$ stopping time T, i.e. $\mathbb{P}(\tau = T) = 0$, but in discrete time Assumption (A) does not hold, in order to have the same results we need the hypothesis that G is predictable, instead Assumption (A).

1.3.6 Cox model

The Cox model was introduced for the first time in [Cox72]. In this paper, David Cox proposed a stimulating and pioneering procedure for the regression analysis of censored failure time data.

Definition 1.3.34 *We say that we are in a* (Θ,Γ)*-Cox model if*

$$\tau := \inf\{n : \Gamma_n \geq \Theta\}$$

where

- *Θ is a random variable independent of $\mathcal{F}_\infty$,*
- *Γ is an $\mathbb{F}$-adapted non decreasing process with $\Gamma_0 = 0$.*

Proposition 1.3.35 *Let Θ be a random variable with survival distribution function Ψ. Then for $k \leq n$, we have $\mathbb{P}(\tau > k|\mathcal{F}_n) = \Psi(\Gamma_k)$. In particular, $G_n = \Psi(\Gamma_n)$.*

PROOF: From the independence assumption, it follows that, for $n \geq k$,

$$\mathbb{P}(\tau > k|\mathcal{F}_n) = \mathbb{P}\big(\Gamma_k < \Theta|\mathcal{F}_n\big) = \Psi(\Gamma_k)\,.$$

□

Lemma 1.3.36 *In a Cox Model, any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale, i.e. $\mathbb{F} \hookrightarrow \mathbb{G}$. In particular, $\widetilde{G}$ is $\mathbb{F}$-predictable and $\widetilde{G}_n = G_{n-1}$.*

PROOF: Since Θ is independent from $\mathbb{F}$, it is obvious that any $\mathbb{F}$-martingale is an $\mathbb{F}^\Theta = \mathbb{F} \vee \sigma(\Theta)$-martingale. Since $\mathbb{F} \subset \mathbb{G} \subset \mathbb{F}^\Theta$, it follows that any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale. The predictability of $\widetilde{G}$ and $\widetilde{G}_n = G_{n-1}$ is a general result under immersion. □

Corollary 1.3.37 *If Θ has exponential distribution with parameter 1 in Cox's model, we have*

$$G_n = \exp(-\Gamma_n) \quad \text{and} \quad \widetilde{G}_n = \exp(-\Gamma_{n-1}) \ , \quad \forall n \geq 0 \ .$$

We now compute the quantities λ (resp. $\widetilde{\lambda}$) defined in Lemma 1.3.5 (resp. Lemma 1.3.18).

Corollary 1.3.38 *If Θ has exponential distribution in a Cox Model, we have that*

$$\lambda_n = 1 - \mathbb{E}(e^{-\Delta\Gamma_n}|\mathcal{F}_{n-1}) \ , \quad \forall n \geq 1 \ .$$

In particular, if Γ is $\mathbb{F}$-predictable we have that

$$\lambda_n = \widetilde{\lambda}_n = 1 - e^{-\Delta\Gamma_n} \ , \quad \forall n \geq 1 \ .$$

PROOF: Using Proposition 1.3.37, we get that

$$\Lambda_n = \frac{e^{-\Gamma_{n-1}} - \mathbb{E}(e^{-\Gamma_n}|\mathcal{F}_{n-1})}{e^{-\Gamma_{n-1}}} \ , \quad \forall n \geq 1 \ ,$$

which gives the result. □

The following two lemmas are an easy consequence of immersion.

Lemma 1.3.39 *If Θ has exponential distribution in a Cox model, for $n \leq m$ we have that $\mathbb{P}(\tau > n|\mathcal{F}_m) = \exp(-\Gamma_n)$.*

PROOF: By immersion $\mathbb{P}(\tau > n|\mathcal{F}_m) = \mathbb{P}(\tau > n|\mathcal{F}_n) = G_n = \exp(-\Gamma_n)$ for all $n \leq m$. □

Lemma 1.3.40 *Let Θ be a random variable with exponential distribution with parameter 1 and let X be an $\mathbb{F}$-adapted non negative process. Assume that Θ is independent of the filtration $\mathbb{F}$ and we set*

$$\tau := \inf\{n : X_n \geq \Theta\} \ ,$$

then for $m \leq n$ we have that $\mathbb{P}(\tau > m|\mathcal{F}_n) = \exp(-\sup_{u \leq m} X_u)$.

PROOF: For $m \leq n$ fixed. Notice that for any non negative process X we have that $\{\tau > m\} = \{\sup_{u \leq m} X_u < \Theta\}$, then setting $\Gamma_n = \sup_{u \leq m} X_u$, from Lemma 1.3.39, the result follows. □

1.3.7 Arbitrages

Here we present some of the results of [ACDJ13] in a discrete time setting. We give the following result using the Definition 1.1.35.

Lemma 1.3.41 *There are no arbitrages in the model* $(\mathbb{F}, \mathbb{G})$ *strictly before* τ.

PROOF: Let $\psi = (1/G)\mathbb{1}_{\{G>0\}} + \mathbb{1}_{\{G=0\}}$ be defined on the set $\{1 \leq n < \tau\}$ and let X be an $\mathbb{F}$-martingale. We have that ψ satisfies by Lemma 1.3.3.b), for all $n \geq 1$,

$$\begin{aligned}\mathbb{1}_{\{\tau\geq n-1\}}\mathbb{E}(\psi_n X_n|\mathcal{G}_{n-1}) &= \mathbb{1}_{\{\tau\geq n-1\}}\frac{1}{\widetilde{G}_{n-1}}\mathbb{E}(\psi_n X_n G_n|\mathcal{F}_{n-1}) \\ &= \mathbb{1}_{\{\tau\geq n-1\}}\frac{1}{\widetilde{G}_{n-1}}\mathbb{E}\Big(\big(\frac{1}{G_n}\mathbb{1}_{\{G_n>0\}} + \mathbb{1}_{\{G_n=0\}}\big)X_n G_n|\mathcal{F}_{n-1}\Big) .\end{aligned}$$

Simplifying, and using that $\{G_n > 0\}$ on $\{\tau \geq n-1\}$ and that X is an $\mathbb{F}$-martingale

$$\mathbb{1}_{\{\tau\geq n-1\}}\mathbb{E}(\psi_n X_n|\mathcal{G}_{n-1}) = \mathbb{1}_{\{\tau\geq n-1\}}\frac{1}{\widetilde{G}_{n-1}}\mathbb{E}\Big(X_n\mathbb{1}_{\{G_n>0\}}|\mathcal{F}_{n-1}\Big) = \mathbb{1}_{\{\tau\geq n-1\}}\frac{1}{\widetilde{G}_{n-1}}X_{n-1} . \tag{1.3.65}$$

In the other hand, we have that

$$\mathbb{1}_{\{\tau\geq n-1\}}X_{n-1}\mathbb{E}(\psi_n|\mathcal{G}_{n-1}) = \mathbb{1}_{\{\tau\geq n-1\}}\frac{X_{n-1}}{\widetilde{G}_{n-1}} . \tag{1.3.66}$$

Then, using (1.3.65) and (1.3.66),

$$\mathbb{1}_{\{\tau\geq n-1\}}\mathbb{E}(\psi_n X_n|\mathcal{G}_{n-1}) = \mathbb{1}_{\{\tau\geq n-1\}}X_{n-1}\mathbb{E}(\psi_n|\mathcal{G}_{n-1}) .$$

Hence by Lemma 1.1.34, there exists a positive $\mathbb{G}$-martingale L such that LX is a $\mathbb{G}$-martingale, i.e. there are no arbitrages in $\mathbb{G}$ strictly before τ. □

Theorem 1.3.42 *Assume that* τ *is not an* $\mathbb{F}$*-stopping time. Then, there are no* $\mathbb{G}$*-arbitrages before* τ *if and only, for any* $n \geq 1$, *the set* $\{0 = \widetilde{G}_n < G_{n-1}\}$ *is empty.*

We mean here that, for any $\mathbb{F}$ martingale X, the stopped process X^τ admits a deflator. This result was established in [CD14] and is a particular case of the results obtained in [ACDJ13]. We give here a slightly different proof, using the two following propositions.

Proposition 1.3.43 *Assume that for any* n, *the set* $\{\widetilde{G}_n = 0 < G_{n-1}\}$ *is empty. The process* $L = \mathcal{E}(Y)$, *where* Y *is the* $\mathbb{G}$*-martingale defined by* $\Delta Y_k = \mathbb{1}_{\{\tau\geq k\}}(\frac{G_{k-1}}{\widetilde{G}_k} - 1)$ *for* $k \geq 1$ *and* $Y_0 = 0$, *is a positive* $\mathbb{G}$*-martingale. If* X *is an* $\mathbb{F}$*-martingale, the process* $X^\tau L$ *is a* $(\mathbb{G}, \mathbb{P})$*-martingale.*

PROOF: First, we show that the process Y is a martingale, then for $n \geq 1$ fixed, by Lemma 1.3.3.b),

$$\mathbb{E}(\Delta Y_n|\mathcal{G}_{n-1}) = \mathbb{E}\Big(\mathbb{1}_{\{\tau\geq n\}}\frac{G_{n-1} - \widetilde{G}_n}{\widetilde{G}_n}|\mathcal{G}_{n-1}\Big) = \mathbb{1}_{\{\tau\geq n\}}\frac{1}{G_{n-1}}\mathbb{E}\big(\mathbb{1}_{\{\widetilde{G}_n>0\}}(G_{n-1} - \widetilde{G}_n)|\mathcal{F}_{n-1}\big) ,$$

using $\mathbb{E}(\widetilde{G}_n|\mathcal{F}_{n-1}) = G_{n-1}$ and that, by assumption $\{\widetilde{G}_n = 0\} \subset \{G_{n-1} = 0\}$,

$$\mathbb{E}(\Delta Y_n|\mathcal{G}_{n-1}) = \mathbb{1}_{\{\tau\geq n\}} \frac{1}{G_{n-1}} \mathbb{E}(G_{n-1} - \widetilde{G}_n|\mathcal{F}_{n-1}) = 0\,.$$

Hence L is a martingale. Notice that the fact that $\{G_{n-1} = 0\} \subset \{\widetilde{G}_n = 0\}$ implies that the hypothesis $\{\widetilde{G}_n = 0\} \subset \{G_{n-1} = 0\}$ is equivalent to $\{\widetilde{G}_n = 0\} = \{G_{n-1} = 0\}$, or to $\{\widetilde{G}_n = 0 < G_{n-1}\}$ is empty. On the set $\{\tau \geq k\}$, one has $G_{k-1} > 0$ which implies that $\Delta Y_k = (\frac{G_{k-1}}{\widetilde{G}_k} - 1) > -1$, hence L is positive. Furthermore, for X an $\mathbb{F}$-martingale and definition of L

$$\mathbb{E}\Big(X^\tau_{n+1}\frac{L_{n+1}}{L_n}|\mathcal{G}_n\Big) = \mathbb{E}\Big(X_{(n+1)\wedge\tau}\big(1 + \mathbb{1}_{\{\tau\geq n+1\}}\frac{G_n - \widetilde{G}_{n+1}}{\widetilde{G}_{n+1}}\big)|\mathcal{G}_n\Big)\,,$$

simplifying, using that $X_\tau\mathbb{1}_{\{\tau<n+1\}} \in \mathcal{G}_n$, we get

$$\mathbb{E}\Big(X^\tau_{n+1}\frac{L_{n+1}}{L_n}|\mathcal{G}_n\Big) = \mathbb{E}\Big(X_{n+1}\mathbb{1}_{\{\tau\geq n+1\}}\frac{G_n}{\widetilde{G}_{n+1}}|\mathcal{G}_n\Big) + X_\tau\mathbb{1}_{\{\tau<n+1\}}\,.$$

Then, simplifying

$$\begin{aligned}\mathbb{E}\Big(X^\tau_{n+1}\frac{L_{n+1}}{L_n}|\mathcal{G}_n\Big) &= \mathbb{1}_{\{\tau>n\}}\frac{1}{G_n}\mathbb{E}(X_{n+1}G_n\mathbb{1}_{\{\widetilde{G}_{n+1}>0\}}|\mathcal{F}_n) + X_\tau\mathbb{1}_{\{\tau\leq n\}}\\ &= \mathbb{1}_{\{\tau>n\}}\frac{1}{G_n}\mathbb{E}\big(X_{n+1}G_n(1 - \mathbb{1}_{\{\widetilde{G}_{n+1}=0\}})|\mathcal{F}_n\big) + X_\tau\mathbb{1}_{\{\tau\leq n\}}\,.\end{aligned}$$

By assumption $G_n\mathbb{1}_{\{\widetilde{G}_{n+1}=0\}} = 0$, we have

$$\mathbb{E}\Big(X^\tau_{n+1}\frac{L_{n+1}}{L_n}|\mathcal{G}_n\Big) = \mathbb{1}_{\{\tau>n\}}\frac{1}{G_n}\mathbb{E}(X_{n+1}G_n|\mathcal{F}_n) + X_\tau\mathbb{1}_{\{\tau\leq n\}} = X^\tau_n\,.$$

Therefore L is a deflator. □

Remark 1.3.44 In case of immersion, there are no arbitrages (indeed any e.m.m. in $\mathbb{F}$ will be an e.m.m. in $\mathbb{G}$). This can be also obtained using the previous result, since, under immersion hypothesis, one has $G_{n-1} = \widetilde{G}_n$ for all $n \geq 1$.

The following Proposition is the discrete version of [AFK16].

Proposition 1.3.45 *If there exists $n \geq 1$ such that the set $\{0 = \widetilde{G}_n < G_{n-1}\}$ is not empty, and if τ is not an $\mathbb{F}$-stopping time, there exists an $\mathbb{F}$-martingale X such that X^τ is a $\mathbb{G}$-adapted non decreasing process with $X^\tau_0 = 1$, $\mathbb{P}(X^\tau_\tau > 1) > 0$. Hence, there are arbitrages in $\mathbb{G}$.*

Proof: Let $\vartheta = \inf\{n : 0 = \widetilde{G}_n < G_{n-1}\}$. The random time ϑ is an $\mathbb{F}$-stopping time satisfying $\tau \leq \vartheta$ and $\mathbb{P}(\tau < \vartheta) > 0$. Let $I_n = \mathbb{1}_{\{\vartheta\leq n\}}$ and denote by D the $\mathbb{F}$-predictable process part of the Doob decomposition of I. One has $D_0 = 0$ and $\Delta D_n = \mathbb{P}(\vartheta = n|\mathcal{F}_{n-1})$. We introduce the $\mathbb{F}$-predictable non decreasing process U setting $U_n = \frac{1}{\mathcal{E}(-D)_n}$. Then,

$$\Delta U_n = \frac{1}{\mathcal{E}(-D)_{n-1}}\big(\frac{1}{1-\Delta D_n} - 1\big) = \frac{1}{\mathcal{E}(-D)_{n-1}}\frac{\Delta D_n}{1-\Delta D_n} = U_n\Delta D_n\,.$$

We consider the process $X = UK$, where $K = 1 - I$,

$$\Delta X_n = -U_n \Delta I_n + K_{n-1} \Delta U_n = -U_n \left(\Delta I_n - K_{n-1} \Delta D_n \right)$$

and

$$\mathbb{E}(\Delta X_n | \mathcal{F}_{n-1}) = -U_n \mathbb{E}(\Delta I_n - K_{n-1} \Delta D_n | F_{n-1}) \,.$$

Then using that $K_{n-1} \mathbb{P}(\vartheta = n | \mathcal{F}_{n-1}) = \mathbb{E}(K_{n-1} \mathbb{1}_{\vartheta = n} | \mathcal{F}_{n-1}) = \mathbb{P}(\vartheta = n | \mathcal{F}_{n-1})$, we get

$$\begin{aligned} \mathbb{E}(\Delta X_n | \mathcal{F}_{n-1}) &= U_n \left(\mathbb{P}(\vartheta = n | \mathcal{F}_{n-1}) - K_{n-1} \mathbb{P}(\Delta D_n | F_{n-1}) \right) \\ &= U_n K_{n-1} \left(\mathbb{P}(\vartheta = n | \mathcal{F}_{n-1}) - \mathbb{P}(\Delta D_n | F_{n-1}) \right) = 0 \,, \end{aligned}$$

hence X is an $\mathbb{F}$-martingale.

We now prove that $X_\tau \geq 1$ and $\mathbb{P}(X_\tau > 1) > 0$, equivalently that $D_\tau \geq 0$ and $\mathbb{P}(D_\tau > 0) > 0$. For that, we compute

$$\begin{aligned} \mathbb{E}(D_\tau \mathbb{1}_{\{\tau < \infty\}}) &= \sum_{n=0}^{\infty} \mathbb{E}(D_n \mathbb{1}_{\{\tau = n\}}) = \sum_{n=0}^{\infty} \mathbb{E}(D_n \mathbb{P}(\tau = n | \mathcal{F}_n)) \\ &= \sum_{n=1}^{\infty} \mathbb{E}\big(D_n \mathbb{P}(\tau > n-1 | \mathcal{F}_n)\big) - \sum_{n=1}^{\infty} \mathbb{E}\big(D_n \mathbb{P}(\tau > n | \mathcal{F}_n)\big) + D_0 \mathbb{P}(\tau = 0) \,. \end{aligned}$$

Since D is predictable, we have that

$$\begin{aligned} \mathbb{E}(D_\tau \mathbb{1}_{\{\tau < \infty\}}) &= \sum_{n=1}^{\infty} \mathbb{E}\big(D_n \mathbb{P}(\tau > n-1 | \mathcal{F}_{n-1})\big) - \sum_{n=1}^{\infty} \mathbb{E}\big(D_n \mathbb{P}(\tau > n | \mathcal{F}_n)\big) \\ &= -\sum_{n=1}^{\infty} \mathbb{E}(D_n \Delta G_n) = \mathbb{E}\big((G_- \centerdot D)_\infty\big) = \mathbb{E}(G_{\vartheta-1} \mathbb{1}_{\{\vartheta < \infty\}}) > 0 \,, \end{aligned}$$

where, in the last inequality, we used that $\tau \leq \vartheta$ and $\mathbb{P}(\tau = \vartheta) < 1$. The process X^τ is then a non decreasing process and can not be turned in a martingale by change of probability. □

1.3.8 Construction of τ from a given supermartingale

We now answer the following question. Let G be a supermartingale on $(\Omega, \mathbb{F}, \mathbb{P})$, valued in $[0, 1]$ such that $G_\infty = 0$. Is it possible to construct τ such that G is its Azéma supermartingale. We mimic the general proof of Song [Son14a], extending the probability space. To achieve our goal, we will use randomized stopping times. Note that the construction is not unique.

Randomized stopping times

First, we introduce the notion of a randomized stopping time introduced in [BC77].

Definition 1.3.46 *Randomized stopping time. Define*
– $\overline{\Omega} = \mathbb{N} \times \Omega$,

– $\overline{\mathcal{F}} = \mathcal{B}(\mathbb{N}) \otimes \mathcal{F}$,
– $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_n)_{n\geq 0}$ *with* $\overline{\mathcal{F}}_n = \mathcal{B}(\mathbb{N}) \otimes \mathcal{F}_n$ *and*
– $\mathbb{Q}$, *a probability measure on* $(\overline{\Omega}, \overline{\mathcal{F}})$

where $\mathcal{B}(\mathbb{N})$ *is the Borel* σ*-algebra for* $\mathbb{N}$ *and* $\mathbb{Q}$ *is such that it coincides with* $\mathbb{P}$ *on* $\mathcal{F}$.

Then, $\phi : \mathbb{N} \times \Omega \to \mathbb{N}$ *is called a randomized stopping time if* ϕ *is an* $\overline{\mathbb{F}}$*-stopping time, i.e. for every* $k \in \mathbb{N}$*, we have that* $\phi(k, \cdot)$ *is an* $\mathbb{F}$*-stopping time.*

So, we construct a probability $\mathbb{Q}$ such that the randomized stopping time given by the identity map $\tau(n,\omega) = n$ satisfies $\mathbb{Q}(\tau > n|\mathcal{F}_n) = G_n$ for all $n \in \mathbb{N}$ and $\mathbb{Q}$ coincides with $\mathbb{P}$ on $\mathbb{F}$. To do so, we need to construct a family of martingales $(M^k)_{k\in\mathbb{N}}$ which will represent $\mathbb{Q}(\tau \leq k|\mathcal{F}_n)$.

This family must be valued in $[0,1]$, increasing w.r.t. k, and coincide with the given supermartingale for $n = k$, in particular

$$M^k_k = 1 - G_k \ , \quad M^k_n \leq M^{k+1}_n \ ,$$

and for all $n \leq k$, $M^k_n = \mathbb{E}(M^k_k|\mathcal{F}_n)$.

Consider the multiplicative decomposition of $G = \mathcal{E}(-\Lambda)M^G$ (Proposition 1.3.7), given by

$$\mathcal{E}(-\Lambda)_n = \prod_{k=1}^{n} \frac{\mathbb{E}(G_k|\mathcal{F}_{k-1})}{G_{k-1}} \ , \quad \forall n \geq 1$$

and

$$M^G_n = \prod_{k=1}^{n} \frac{G_k}{\mathbb{E}(G_k|\mathcal{F}_{k-1})} \ , \quad \forall n \geq 1 \ .$$

We assume that $\mathbb{E}(G_n|\mathcal{F}_{n-1}) < 1$, for all $n \geq 1$. Let k be fixed and define

$$M^k_k := 1 - G_k \ , \quad M^k_n = M^k_{n-1}\Big(1 - \frac{\mathcal{E}(-\Lambda)_n}{1 - \mathbb{E}(G_n|\mathcal{F}_{n-1})}\Delta M^G_n\Big) \ , \quad \forall n > k \ .$$

Then, for $n \leq k$, using that $0 \leq 1 - G_k \leq 1$ we have that

$$0 \leq \mathbb{E}(M^k_n|\mathcal{F}_{n-1}) = \mathbb{E}\big(M^k_k|\mathcal{F}_n|\mathcal{F}_{n-1}\big) = M^k_{n-1} \leq 1 \ .$$

If $n > k$, then

$$\begin{aligned}
\mathbb{E}(M^k_n|\mathcal{F}_{n-1}) &= \mathbb{E}\Big(M^k_{n-1}\Big(1 - \frac{\mathcal{E}(-\Lambda)_n}{1 - \mathbb{E}(G_n|\mathcal{F}_{n-1})}\Delta M^G_n\Big)\Big|\mathcal{F}_{n-1}\Big) \\
&= M^k_{n-1} - \frac{\mathcal{E}(-\Lambda)_n}{1 - \mathbb{E}(G_n|\mathcal{F}_{n-1})}\mathbb{E}(\Delta M^G_n|\mathcal{F}_{n-1}) \\
&= M^k_{n-1} \ .
\end{aligned}$$

Now we verify that if $n > k$ then $0 \leq M^k_n \leq 1$, i.e. we show that

$$0 \leq 1 - \frac{\mathcal{E}(-\Lambda)_n}{1 - G_{n-1}}\Delta M^G_n \leq 1 \ .$$

Notice that

$$
\begin{aligned}
1-\frac{\mathcal{E}(-\Lambda)_n}{1-\mathbb{E}(G_n|\mathcal{F}_{n-1})}\Delta M_n^G &= 1-\frac{\mathcal{E}(-\Lambda)_n}{1-\mathbb{E}(G_n|\mathcal{F}_{n-1})}M_n^G+\frac{\mathcal{E}(-\Lambda)_n}{1-\mathbb{E}(G_n|\mathcal{F}_{n-1})}M_{n-1}^G \\
&= 1-\frac{G_n}{1-\mathbb{E}(G_n|\mathcal{F}_{n-1})}+\frac{\mathcal{E}(-\Lambda)_n}{1-\mathbb{E}(G_n|\mathcal{F}_{n-1})}M_n^G\frac{\mathbb{E}(G_n|\mathcal{F}_{n-1})}{G_n} \\
&= 1-\frac{G_n}{1-\mathbb{E}(G_n|\mathcal{F}_{n-1})}+\frac{\mathbb{E}(G_n|\mathcal{F}_{n-1})}{1-\mathbb{E}(G_n|\mathcal{F}_{n-1})} \\
&= \frac{1-G_n}{1-\mathbb{E}(G_n|\mathcal{F}_{n-1})}\ .
\end{aligned}
$$

Therefore, using that G is an $\mathbb{F}$-supermartingale in $[0,1)$ we deduce that

$$
0\le\frac{1-G_n}{1-\mathbb{E}(G_n|\mathcal{F}_{n-1})}\le 1\ .
$$

Then, M^k is an $\mathbb{F}$-martingale valued in $[0,1]$ with $M_n^k\le M_n^{k+1}$.

Chapitre 2

BSDEs and variable annuities

In this chapter, we present different method to price variable annuities. The first model has been introduced and studied in [CLRR16]. We work on indifference valuation of variable annuities and give a computation method for indifference fees. We focus on the guaranteed minimum death benefits and the guaranteed minimum living benefits and allow the policyholder to make withdrawals. We assume that the fees are continuously payed and that the fee rate is fixed at the beginning of the contract. Following indifference pricing theory, we define indifference fee rate for the insurer as a solution of an equation involving two stochastic control problems. Relating these problems to backward stochastic differential equations with jumps, we provide a verification theorem and give the optimal strategies associated to our control problems. From these, we derive a computation method to get indifference fee rates. We conclude this part with numerical illustrations of indifference fees sensibilities with respect to parameters.

In the last section, we present a second problem which is based on a very closed model and is studied in [BSCKL15]. The valuation of variable annuities for an insurer is still the aim of this problem. A utility indifference approach is used to determine this fee but assuming that it is the worst case for the insurer i.e. that the insured makes the withdrawals that minimize the expected utility of the insurer. To compute this indifference fee rate, the utility maximization in the worst case for the insurer has been linked to a sequence of maximization and minimization problems that can be computed recursively. This allows to provide an optimal investment strategy for the insurer when the insured follows the worst withdrawals strategy and to compute the indifference fee. We approximate these quantities numerically and we give numerical illustrations of parameter sensibility.

Introduction

Introduced in the 1970s in the United States (see [Slo70]), variable annuities are equity-linked contracts between a policyholder and an insurance company. The policyholder gives an initial amount of money to the insurer. This amount is then invested in a reference portfolio until a preset date, until the policyholder withdraws from the contract or until he dies. At the end of the contract, the insurance pays to the policyholder or to his dependents a pay-off depending on the performance of the reference portfolio. In the 1990s, insurers included put-like derivatives which provided some guarantees to the policyholder. The most usual are guaranteed minimum death benefits (GMDB) and guaranteed minimum living benefits (GMLB). For a GMDB (resp. GMLB) contract, if the insured dies before the contract maturity (resp. is still alive at the maturity) he or his dependents obtain a benefit corresponding to the maximum of the current account value and of a guaranteed benefit. There exist various ways to fix this guaranteed benefit and we refer to [BKR08] for more details.

These products mainly present three risks for the insurer. First, as the insurer offers a put-like derivative on a reference portfolio to the client, he is considerably exposed to market risk. Moreover, variable annuity policies could have very long maturities so the pricing and hedging errors due to the model choice for the dynamics of the reference portfolio and the interest rates could be very important. The second risk faced by the insurer is the death of his client, this leads to the formulation of a problem with random maturity. Finally, the client may decide at any moment to withdraw, totally or partially, from the contract. Throughout this chapter we shall assume that there is a rate of partial withdrawal that could be stochastic or not but we do not assume that it results from an optimal strategy of the insured. In case of total withdrawal, the insured may pay some penalties and will receive the maximum of the account facial value and of a guaranteed benefit minus the amount of previous partial withdrawals.

With the commercial success of variable annuities, the pricing and hedging of these products have been studied in a growing literature. Following the pioneering work of Boyle and Schwartz (see [BS77]), non-arbitrage models allow to extend the Black-Scholes framework to insurance issues. Milvesky and Posner (see [MP01]) are, up to our knowledge, the first to apply risk neutral option pricing theory to value GMDB in variable annuities. Withdrawal options are studied in [CK03] and [Siu05], and a general framework to define variable annuities is presented in [BKR08]. Milevsky and Salisbury (see [MS06]) focus on the links between American put options and dynamic optimal withdrawal policies. This problem is studied in [DKZ08] where an Hamilton-Jacobi-Bellman (HJB) equation is derived for a singular control problem where the control is the continuous withdrawal rate. The GMDB pricing problem is described as an impulse control problem in [BFL09]. The authors model the GMDB problem as a stochastic control problem, derive an HJB equation and solve it numerically. The assumptions needed to get these formulations are the Markovianity of the stochastic processes involved and the existence of a risk neutral probability. The variable annuity policies with GMDB and GMLB are long term products therefore models for assets and interest rates have to be as rich as possible. Moreover, as we obviously face an incomplete market model, the price obtained strongly depends on the arbitrary choice of a risk neutral probability.

This chapter attempts to get an answer to these issues. We shall not make restrictive assumptions on the reference portfolio and the interest rate dynamics. As a result, our problem is not Markovian and we will not be able to derive HJB equations to characterize our value functions. We overcome this difficulty thanks to backward stochastic differential equations (BSDEs) following ideas from [EKPQ97], [HIM05] and [REK00]. In our case, we have to solve BSDE with random terminal time. For that we apply very recent results on BSDEs with jump (see for example [ABSEL10] and [KLN13]). Moreover, we shall not use non-arbitrage arguments to price and hedge variable annuity policies. We will assume that the fees, characterized by a preset fee rate, are continuously taken by the insurer from the policyholder's account and we will define an indifference fee rate for the insurer. Indifference pricing is a standard approach in mathematical finance to determine the price of a contingent claim in an incomplete market. This is a utility-based approach that can be summarized as follows. On the one hand, the investor may maximize his expected utility under optimal trading, investing only in the financial market. On the other hand, he could sell the contingent claim, optimally invest in the financial market and make a pay-off at the terminal time. The indifference price of this contingent claim is then the price such that the insurer gets the same expected utility in each case. For more details, we refer to the monograph [Car09].

This chapter is organized as follows. In Section 1, we define the market model, the random times of death and total withdraw, then variable annuities with GMDB and/or GMLB are defined. We recall the main examples of guarantees associated to. Section 2 is devoted to indifference fee rates. They are defined as solutions of an equality between two regular stochastic control problems. These consist in maximizing the expected utility of the terminal wealth of the insurer portfolio in two cases : when the insurer has not sold variable annuities and when he has. Value functions of these two problems are respectively characterized as initial values of BSDEs. This characterization is well known for the first problem (see [HIM05] and [REK00]) but demands to solve some technical issues for the second one. We conclude this section with a rigorous study of the existence of indifference fee rates in the usual cases i.e. with roll-up or ratchet guarantee. Finally, in Section 3, we conclude this chapter with numerical illustrations of sensibilities of indifference fees with respect to model and market parameters.

2.1 Model for variable annuities

This section is divided as follows, Subsection 2.1.1 introduces the model for the underlying financial market in which the insurer invests. Subsection 2.1.2 describes the terminal date of a variable annuity policy. This one may be due to a total withdraw or to the death of the insured.

2.1.1 The financial market model

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space. We assume that this space is equipped with a one-dimensional standard Brownian motion B and we denote by $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ the right continuous complete filtration generated by B.

We consider a financial market on the time interval $[0, T]$ where $T > 0$ corresponds to the expiration date of the variable annuities studied. We suppose that the financial market is composed by a riskless bond with an interest rate r and a reference portfolio of risky assets underlying the variable annuity policy. The price processes $\hat{S}^0$ of the riskless bond and $\hat{S}$ of a share of the underlying risky portfolio are assumed to be solution of the following linear stochastic differential equations

$$\begin{aligned} d\hat{S}^0_t &= r_t \hat{S}^0_t dt \ , \quad \forall t \in [0, T] \ , \quad \hat{S}^0_0 = 1 \ , \\ d\hat{S}_t &= \hat{S}_t(\mu_t dt + \sigma_t dB_t) \ , \quad \forall t \in [0, T] \ , \quad \hat{S}_0 = s > 0 \ , \end{aligned}$$

where μ, σ and r are $\mathbb{F}$-adapted processes satisfying the following assumptions.

Hypothesis 2.1.1

(i) The processes μ, σ and r are $\mathbb{P}-a.s.$ bounded.

(ii) The process σ is $\mathbb{P}-a.s.$ lower bounded by a positive constant $\underline{\sigma}$.

We shall denote by S_t the discounted value of $\hat{S}_t$ at time $t \in [0,T]$, i.e.

$$S_t := e^{-\int_0^t r_s ds}\hat{S}_t \ , \quad \forall t \in [0,T] \ .$$

The insurer invests on this financial market. For $t \in [0,T]$, we denote by π_t^0 (resp. π_t) the discounted amount of money invested in the riskless bond (resp. the risky portfolio). We suppose that the process π is $\mathbb{F}$-predictable and satisfies the following integrability condition

$$\int_0^T |\pi_s|^2 ds \ < \ +\infty \ , \quad \mathbb{P}-a.s.$$

Assuming that the strategy of the insurer is self-financed and denoting by $X_t^{x,\pi}$ the discounted value of the insurer portfolio at time t with initial capital $x \in \mathbb{R}^+$ and following the strategy π, we have

$$X_t^{x,\pi} \ = \ x + \int_0^t \pi_s(\mu_s - r_s)ds + \int_0^t \pi_s \sigma_s dB_s \ , \quad \forall t \in [0,T] \ .$$

If the initial capital is null we denote X_t^{π} the wealth instead of $X_t^{0,\pi}$.

We consider that the insurer wants to maximize the expected value of the utility of his terminal wealth $U(X_T^{x,\pi})$ on an admissible strategies set, where $U(x) := -\exp(-\gamma x)$ with $\gamma > 0$. Both theory and practice have shown that it is appropriate to use exponential utility functions. Since the decisions do not depend on the initial wealth of the insurer, it is well adapted to our problem of pricing one set of policies. Moreover an appealing feature of decision making using exponential utility function is that decisions are based on comparisons between moment generating functions. They capture all the characteristics of the random outcomes being compared, so that comparisons are based on a wide range of features. We refer to [BSA14] for more details about this choice.

In the following definition, we define the set of admissible strategies for the insurer, making usual restrictions that ensure some integrability properties for the processes involved.

Definition 2.1.2 *($\mathbb{F}$-admissible strategy). Let u and v be two $\mathbb{F}$-stopping times such that $0 \leq u \leq v \leq T$. The set of admissible trading strategies $\mathcal{A}^{\mathbb{F}}[u,v]$ consists of all $\mathbb{F}$-predictable processes $\pi = (\pi_t)_{u \leq t \leq v}$ which satisfy*

$$\mathbb{E}\Big[\int_u^v \big|\pi_t\big|^2 dt\Big] < \infty$$

and

$$\Big\{\exp(-\gamma X_\theta^{x,\pi}),\ \theta \text{ is an } \mathbb{F}\text{-stopping time such that } u \leq \theta \leq v\Big\}$$

is uniformly integrable.

2.1.2 Exit time of a variable annuity policy

We consider two random times θ^d and θ^w which respectively represent the death time of the insured and the time of early closure of the insured account. We denote by $\tau = \theta^d \wedge \theta^w$. The random time τ is not assumed to be an $\mathbb{F}$-stopping time. We therefore use in the sequel the standard approach of filtration enlargement by considering $\mathbb{G}$ the smallest right continuous extension of $\mathbb{F}$ that turns τ into a $\mathbb{G}$-stopping time (see e.g. [BR04, KLN13]). More precisely $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ is defined by

$$\mathcal{G}_t \ := \ \bigcap_{\varepsilon>0} \tilde{\mathcal{G}}_{t+\varepsilon} \ ,$$

for all $t \geq 0$, where $\tilde{\mathcal{G}}_s := \mathcal{F}_s \vee \sigma(\mathbb{1}_{\tau\leq u} \ , u \in [0,s])$, for all $s \geq 0$.

We impose the following assumptions, which are usual in filtration enlargement theory (see for example [BR04, Section 6.1.1]).

Hypothesis 2.1.3 ($\mathcal{H}$)***-hypothesis***. *The process B remains a $\mathbb{G}$-Brownian motion.*

The interpretation of the ($\mathcal{H}$)-hypothesis (also known as immersion hypothesis) is an asymmetric dependence structure between B and τ. From a financial point of view, it means that the exit time τ may depend on the financial market randomness represented by B. On the contrary, the financial market is not influenced by τ, as having some information on τ will not change the dynamic of S.
In the sequel, we introduce the process H defined by $H = \big(1_{\{\tau\leq t\}}\big)_{0\leq t\leq T}$.

Hypothesis 2.1.4 *The process H admits an $\mathbb{F}$-compensator of the form $\int_0^{\cdot\wedge\tau} \lambda_t dt$, i.e. $H - \int_0^{\cdot\wedge\tau} \lambda_t dt$ is a $\mathbb{G}$-martingale, where λ is a bounded $\mathbb{F}$-adapted process.*

M denotes the $\mathbb{G}$-martingale defined by $M_t := H_t - \int_0^{t\wedge\tau} \lambda_s ds$, for all $t \geq 0$.

If the investment strategy of the insurer depends on this exit time, we shall enlarge the set of admissible strategies through the following definition.

Definition 2.1.5 *($\mathbb{G}$-admissible strategy). Let u and v be two $\mathbb{G}$-stopping times such that $0 \leq u \leq v \leq T$. The set of admissible trading strategies $\mathcal{A}^{\mathbb{G}}[u,v]$ consists of all $\mathbb{G}$-predictable processes $\pi = (\pi_t)_{u\leq t\leq v}$ which satisfy*

$$\mathbb{E}\Big[\int_u^v \big|\pi_t\big|^2 dt\Big] < \infty$$

and

$$\Big\{ \exp(-\gamma X_\theta^{x,\pi}),\ \theta \text{ is a } \mathbb{G}\text{-stopping time with values such that } u \leq \theta \leq v \Big\}$$

is uniformly integrable.

2.2 Indifference fee rate for variable annuities

Let $\mathbb{T} := (t_i)_{0\leq i\leq n}$ be the set of policy anniversary dates, with $t_0 = 0$ and $t_n = T$. We also denote $t_{n+1} = +\infty$.

The first process to consider is the discounted account value A^p. The total amount on the account is invested on the market, fees and withdrawals are assumed to be continuously taken from the account therefore the dynamic of the process A^p is as follow

$$dA_t^p \;=\; A_t^p\big[(\mu_t - r_t - \xi_t - p)dt + \sigma_t dB_t\big]\ , \quad \forall t \in [0,T]\ ,$$

with initial value A_0, p is the fee rate taken by the insurer from the account of the insured and the process ξ is a $\mathbb{G}$-predictable, non-negative and bounded process. ξ_t represents the withdrawal rate chosen by the insured at time $t \in [0,T]$. From the insurer point of view, it seems to us more natural to consider a continuous withdrawal rate as many mutualized contracts might be under consideration. We also emphasize that ξ is not necessarily a process resulting from an optimal control of the insured as, for example, in [BFL09], [BSCKL15], [DKZ08] and [MS06]. Throughout this chapter, ξ is assumed to be an exogenous process and no additional hypothesis on the policyholder behavior has to be made. We may refer to the [BKR08, Section 3.4] for different policyholder behavior models.
For any $\pi \in \mathcal{A}^{\mathbb{G}}[0,T]$, we extend the definition of the process $X^{x,\pi}$ where $X_t^{x,\pi}$ is the discounted wealth of the portfolio invested in the financial market at time $t \in [0,T]$ and we set

$$dX_t^{x,\pi} \;=\; \pi_t(\mu_t - r_t)dt + \pi_t\sigma_t dB_t\ , \quad \forall t \in [0,T]\ ,$$

with $X_0^{x,\pi} = x$.

The second quantity to define is the pay-off of the variable annuities. Let $p \geq 0$, the pay-off is paid at time $T \wedge \tau$ to the insured or his dependents and is equal to the following random variable

$$\hat{F}(p) \;:=\; \hat{F}_T^L(p)\mathbb{1}_{\{T<\tau\}} + \hat{F}_\tau^D(p)\mathbb{1}_{\{\tau=\theta^d\leq T\}} + \hat{F}_\tau^W(p)\mathbb{1}_{\{\tau=\theta^w<\theta^d\,;\,\tau\leq T\}}\ . \tag{2.2.1}$$

$\hat{F}_T^L(p)$ is the pay-off if the policyholder is alive at time T and has not totally withdrawn his money from his account. $\hat{F}_\tau^D(p)$ is the pay-off if the policyholder is dead at time τ. $\hat{F}_\tau^W(p)$ is the pay-off if the policyholder totally withdraws his money from his account at time τ. We suppose that $\hat{F}^L(p)$, $\hat{F}^D(p)$ and $\hat{F}^W(p)$ are bounded, non-negative and $\mathbb{G}$-adapted processes.

Including partial withdrawals in the pay-off, we shall use the following notations

$$F_\tau^{D,W}(p) \;:=\; e^{-\int_0^\tau r_u du}\Big(\hat{F}_\tau^D(p)\mathbb{1}_{\{\tau=\theta^d\leq T\}} + \hat{F}_\tau^W(p)\mathbb{1}_{\{\tau=\theta^w<\theta^d\,;\,\tau\leq T\}}\Big) + \int_0^\tau \xi_s A_s^p\, ds\ , \tag{2.2.2}$$

$$F_T^L(p) \;:=\; e^{-\int_0^T r_u du}\hat{F}_T^L(p) + \int_0^T \xi_s A_s^p\, ds\ , \tag{2.2.3}$$

$$F(p) \;:=\; e^{-\int_0^{T\wedge\tau} r_u du}\hat{F}(p) + \int_0^{T\wedge\tau} \xi_s A_s^p\, ds\ . \tag{2.2.4}$$

Notice that $F(p)$ is $\mathcal{G}_{T\wedge\tau}$-measurable.

Usual examples of variable annuities are GMDB and GMLB. In that case, there exist $\hat{G}^D(p)$ and $\hat{G}^L(p)$ non-negative processes such that, for any $Q \in \{D, L\}$, we have

$$\hat{F}_t^Q(p) = \hat{A}_t^p \vee \hat{G}_t^Q(p)\ , \qquad \text{where} \quad \hat{A}_t^p = e^{\int_0^t r_s\, ds} A_t^p\ .$$

The usual guarantee functions used to define GMDB and GMLB are listed below (see [BKR08] for more details).

- Constant guarantee : we have $\hat{G}^Q_t(p) = A_0 - \int_0^t \xi_s \hat{A}^p_s\, ds$ on $[0,T]$, and

$$F(p) \;=\; A^p_{T\wedge\tau} \vee e^{-\int_0^{T\wedge\tau} r_s ds}\Big(A_0 - \int_0^{T\wedge\tau} \xi_s \hat{A}^p_s\, ds\Big) + \int_0^{T\wedge\tau} \xi_s A^p_s\, ds\ ,$$

  then, setting $A^p_{T\wedge\tau}(0) = A^p_{T\wedge\tau} + \int_0^{T\wedge\tau} \xi_s A^p_s\, ds$ and $\beta_t = 1 - e^{-\int_t^{T\wedge\tau} r_s ds}$ for $t \in [0, T\wedge\tau]$, we get

$$F(p) \;=\; A^p_{T\wedge\tau}(0) \vee \Big(e^{-\int_0^{T\wedge\tau} r_s ds} A_0 + \int_0^{T\wedge\tau} \xi_s A^p_s \beta_s\, ds\Big)\ .$$

- Roll-up guarantee : As an interest rate $\eta > 0$ is paid on the guarantee minus the previous withdrawals, we have $\hat{G}^Q_t(p) = (1+\eta)^t \Big(A_0 - \int_0^t \frac{\xi_s \hat{A}^p_s}{(1+\eta)^s}\, ds\Big)$ on $[0,T]$. We obtain

$$\begin{aligned} F(p) \;&=\; A^p_{T\wedge\tau} \vee e^{-\int_0^{T\wedge\tau} r_s ds} \hat{G}^Q_{T\wedge\tau}(p) + \int_0^{T\wedge\tau} \xi_s A^p_s\, ds \\ &=\; A^p_{T\wedge\tau} \vee e^{-\int_0^{T\wedge\tau} r_s ds}(1+\eta)^{T\wedge\tau}\Big(A_0 - \int_0^{T\wedge\tau} \frac{\xi_s \hat{A}^p_s}{(1+\eta)^s}\, ds\Big) + \int_0^{T\wedge\tau} \xi_s A^p_s\, ds\ , \end{aligned}$$

  setting $r^\eta_t = r_t - \ln(1+\eta)$ for all $t \in [0,T]$ and $\beta^\eta_t = 1 - e^{-\int_t^{T\wedge\tau} r^\eta_s ds}$ for $t \in [0, T\wedge\tau]$, we get

$$F(p) \;=\; A^p_{T\wedge\tau}(0) \vee \Big(e^{-\int_0^{T\wedge\tau} r^\eta_s\, ds} A_0 + \int_0^{T\wedge\tau} \xi_s A^p_s \beta^\eta_s\, ds\Big)\ . \tag{2.2.5}$$

- Ratchet guarantee : The guarantee depends on the path of A in the following way : $\hat{G}^Q_t(p) = \max(\hat{a}^p_0(t), \ldots, \hat{a}^p_k(t))$ on $[t_k, t_{k+1})$, for all $0 \le k \le n$, where we have set $\hat{a}^p_k(t) = \hat{A}^p_{t_k} - \int_{t_k}^t \xi_s \hat{A}^p_s\, ds$. We get

$$F(p) \;=\; A^p_{T\wedge\tau} \vee e^{-\int_0^{T\wedge\tau} r_s ds} \max_{0\le i\le n} \big(\hat{a}^p_i(T\wedge\tau) \mathbb{1}_{\{t_i \le T\wedge\tau\}}\big) + \int_0^{T\wedge\tau} \xi_s A^p_s\, ds\ ,$$

  setting $\hat{A}^p_{t_i}(0) = \hat{A}^p_{t_i} + \int_0^{t_i} \xi_s \hat{A}^p_s\, ds$ for all $i \in \{0, .., n\}$, we get that

$$F(p) = A^p_{T\wedge\tau}(0) \vee \Big(\max_{0\le i\le n} \Big[e^{-\int_0^{T\wedge\tau} r_s ds} \hat{A}^p_{t_i}(0) \mathbb{1}_{\{t_i \le T\wedge\tau\}}\Big] + \int_0^{T\wedge\tau} \xi_s A^p_s \beta_s\, ds\Big)\ . \tag{2.2.6}$$

Figure 2.1 illustrates the dynamics of roll-up and ratchet guarantees on a particular brownian path.

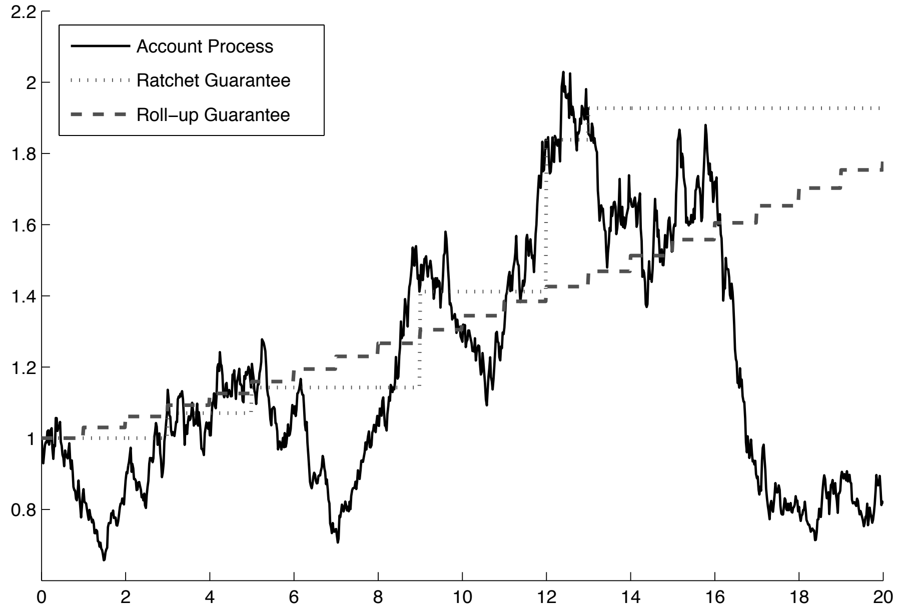

FIGURE 2.1 – Guarantees and Account Value with $A_0 = 1$, $\mu = 0.025$, $r = 0$, $\xi = 0$, $p = 0.02$, $\sigma = 0.4$, $\lambda = 30$ and $\eta = 0.35$.

Remark 2.2.1 *In the usual cases, the terminal pay-off $F(p)$ is non-increasing with respect to p.*
At this point we also notice that, in usual cases, the pay-off $F(p)$ may not be bounded. This Hypothesis is crucial from a mathematical point of view, since it leads to existence and uniqueness of a solution of the BSDEs that we will consider (see Remark 2.2.5). However, our methodology can be applied to such unbounded pay-offs. Indeed, from a numerical point of view, one just has to introduce a positive constant m and replace the pay-off $F(p)$ by $F(p) \wedge m$. For m large enough, we will get a good approximation of the indifference fee rate as $\lim_{m\to+\infty} \mathbb{P}(\sup_{t\in[0,T]} A_t > m) = 0$. We illustrate that in Figure 2.7.

2.2.1 Indifference pricing

The objective of this section is to find, if it exists, a level p^* such that if the fee rate is greater than p^*, the insurer prefers to sell the policy and he has better not to do so if the fee rate is below this level. The optimal fee rate p^* is the smallest p such that

$$\sup_{\pi\in\mathcal{A}^{\mathbb{F}}[0,T]} \mathbb{E}\big[U\big(X_T^{x,\pi}\big)\big] \;=\; \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T]} \mathbb{E}\big[U\big(X_T^{x+A_0,\pi} - F(p)\big)\big]\;. \qquad (2.2.7)$$

Notice that admissible strategies are only $\mathbb{F}$- predictable since the insurer has no information about the random time τ if she has not sold the variable annuities to the policyholder.
A solution of the (2.2.7) will be called an indifference fee rate. Notice that if there exist solutions to the previous equation, they will not depend on the initial wealth invested by the insurer but only on the initial deposit A_0 made by the insured since $U(y) = -\exp(-\gamma y)$. Therefore, solve (2.2.7) is equivalent to solve

$$\sup_{\pi\in\mathcal{A}^{\mathbb{F}}[0,T]} \mathbb{E}\big[U\big(X_T^{\pi}\big)\big] \;=\; \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T]} \mathbb{E}\big[U\big(A_0 + X_T^{\pi} - F(p)\big)\big]\;.$$

To solve this equation, we shall compute the following quantities

$$V_{\mathbb{F}} := \sup_{\pi\in\mathcal{A}^{\mathbb{F}}[0,T]} \mathbb{E}\big[U\big(X_T^{\pi}\big)\big] \text{ and } V_{\mathbb{G}}(p) := \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T]} \mathbb{E}\big[U\big(A_0 + X_T^{\pi} - F(p)\big)\big]\;.$$

$V_{\mathbb{F}}$ is a classical optimization problem, that may be solved thanks to BSDEs like in [HIM05] or [REK00]. We recall the results on this problem in Subsection 2.2.2, then in Subsection 2.2.3 we solve the optimal control problem $V_{\mathbb{G}}(p)$. We will use the tools of BSDEs with respect to the Brownian motion B and to the jump process N to solve it. Finally, in Subsection 2.2.4, we will use the results of Subsections 2.2.2 and 2.2.3 to find indifference fee rates if they exist. An additional difficulty with respect to the classical indifference pricing theory is that fees are continuously paid by the insured. Therefore, one can not use algebraic properties of the utility function to get a semi-explicit formula for the indifference fee rate. We will prove that the function $p \to V_{\mathbb{G}}(p)$ is continuous and monotonic on $\mathbb{R}$, then use the intermediate value theorem to prove that there exists or not a solution of (2.2.7).

2.2.2 Utility maximization without variable annuities

The objective of this part is to compute the value of the maximum expected utility of the terminal wealth at time T when the insurance company has not sold the variable annuity policy. We recall that the maximum expected utility problem is defined by

$$V_{\mathbb{F}} \;:=\; \sup_{\pi\in\mathcal{A}^{\mathbb{F}}[0,T]} \mathbb{E}\big[U(X_T^{\pi})\big]\;.$$

Thanks to Theorem 7 in Hu *et al.* [HIM05], we are able to characterize the value function $V_{\mathbb{F}}$ and the optimal strategy π^* by mean of BSDEs. For that we introduce the following sets.

- $S^\infty_{\mathbb{G}}$ is the subset of $\mathbb{R}$-valued, càd-làg, $\mathbb{G}$-adapted processes $(Y_t)_{t\in[0,T]}$ essentially bounded i.e.

$$\|Y\|_{S^\infty_{\mathbb{G}}} \;:=\; \Big\|\sup_{t\in[0,T]}|Y_t|\Big\|_\infty \;<\; \infty\;.$$

– $L^2_{\mathbb{G}}$ is the subset of $\mathbb{R}$-valued, $\mathbb{G}$-predictable processes $(Z_t)_{t\in[0,T]}$ such that

$$\|Z\|_{L^2_{\mathbb{G}}} \ := \ \Big(\mathbb{E}\Big[\int_0^T |Z_t|^2 dt\Big]\Big)^{1/2} \ < \ \infty \ .$$

– $L^2_{\mathbb{G}}(\lambda)$ is the subset of $\mathbb{R}$-valued, $\mathbb{G}$-predictable processes $(U_t)_{t\in[0,T]}$ such that

$$\|U\|_{L^2_{\mathbb{G}}(\lambda)} \ := \ \Big(\mathbb{E}\Big[\int_0^{T\wedge\tau} \lambda_t |U_t|^2 dt\Big]\Big)^{1/2} \ < \ \infty \ .$$

Proposition 2.2.2 *The value function $V_{\mathbb{F}}$ is given by $V_{\mathbb{F}} = -\exp(\gamma y_0)$, where (y, z) is the unique solution in $S^\infty_{\mathbb{G}} \times L^2_{\mathbb{G}}$ of the following BSDE*

$$\begin{cases} dy_t \ = \ \Big(\frac{\nu_t^2}{2\gamma} + \nu_t z_t\Big) dt + z_t dB_t \ , \quad \forall t \in [0,T] \ , \\ y_T \ = \ 0 \ , \end{cases} \tag{2.2.8}$$

with $\nu_t = \frac{\mu_t - r_t}{\sigma_t}$. Moreover, the optimal strategy associated to this problem is defined by

$$\pi_t^* \ := \ \frac{\nu_t}{\gamma\sigma_t} + \frac{z_t}{\sigma_t} \ , \quad \forall t \in [0,T] \ .$$

For the proof of this proposition we refer to [HIM05] or [REK00].

2.2.3 Utility maximization with variable annuities

We now study the case in which the insurance company proposes the variable annuity policy. We recall that in this case the value function associated to the maximum expected utility problem is given by

$$V_{\mathbb{G}}(p) \ := \ \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T]} \mathbb{E}\big[-\exp\big(-\gamma\big(A_0 + X_T^\pi - F(p)\big)\big)\big] \ , \tag{2.2.9}$$

where $F(p)$ is defined by (2.2.4) and assumed bounded. In practice, $F(p)$ is not always bounded but in that case we shall consider a truncated pay-off, replacing $F(p)$ by $F(p) \wedge m$ (see Remark 2.2.1 and Figure 2.7).

Since we aim at characterizing $V_{\mathbb{G}}(p)$ as a function of the initial value of a BSDE, the first step consists in carefully setting the terminal value of the BSDE. Therefore, we need to deal with the following difficulty : we notice that the random variable X_T^π is $\mathcal{G}_T-$measurable and $F(p)$ is $\mathcal{G}_{T\wedge\tau}-$measurable. The following result allows us to rewrite the problem with a terminal date equal to $T \wedge \tau$.

Lemma 2.2.3 *For any $p \in \mathbb{R}$, we have*

$$V_{\mathbb{G}}(p) \ = \ \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]} \mathbb{E}\big[-\exp\big(-\gamma\big(X_{T\wedge\tau}^{A_0,\pi} - \mathfrak{H}(p)\big)\big)\big] \ , \tag{2.2.10}$$

with

$$\mathfrak{H}(p) := F(p) + \frac{1}{\gamma} \ln\Big\{ \operatorname*{ess\,inf}_{\pi\in\mathcal{A}^{\mathbb{G}}[T\wedge\tau,T]} \mathbb{E}\big[\exp\big(-\gamma\Delta X_{\tau,T}^\pi\big)\big|\mathcal{G}_{T\wedge\tau}\big]\Big\} \ ,$$

where we have set

$$\Delta X_{\tau,T}^\pi \ := \ \int_{T\wedge\tau}^T \pi_s(\mu_s - r_s) ds + \int_{T\wedge\tau}^T \pi_s\sigma_s dB_s \ .$$

PROOF: First we prove that

$$V_{\mathbb{G}}(p) \quad \leq \quad \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]} \mathbb{E}\big[-\exp\big(-\gamma\big(X_T^{A_0,\pi}-\mathfrak{H}(p)\big)\big)\big] \; .$$

Let $\pi'\in\mathcal{A}^{\mathbb{G}}[0,T]$. By the tower property and since $F(p)$ is $\mathcal{G}_{T\wedge\tau}$-measurable, we get

$$\begin{aligned}\mathbb{E}\big[\exp\big(-\gamma\big(X_T^{A_0,\pi'}-F(p)\big)\big)\big] \quad &= \quad \mathbb{E}\Big[\exp\big(-\gamma\big(X_{T\wedge\tau}^{A_0,\pi'}-F(p)\big)\big)\mathbb{E}\big[\exp(-\gamma\Delta X_{\tau,T}^{\pi'})\big|\mathcal{G}_{T\wedge\tau}\big]\Big]\\ &\geq \quad \mathbb{E}\Big[\exp\big(-\gamma\big(X_{T\wedge\tau}^{A_0,\pi'}-F(p)\big)\big)\underline{V}\big|\mathcal{G}_{T\wedge\tau}\big]\Big] \; ,\end{aligned}$$

where we have set

$$\underline{V} \quad := \quad \operatorname*{ess\,inf}_{\pi\in\mathcal{A}^{\mathbb{G}}[T\wedge\tau,T]} \mathbb{E}\big[\exp\big(-\gamma\Delta X_{\tau,T}^{\pi}\big)\big|\mathcal{G}_{T\wedge\tau}\big] \; .$$

Therefore, it follows from the definition of $\mathfrak{H}(p)$ that for any $\pi'\in\mathcal{A}^{\mathbb{G}}[0,T]$, we have

$$\mathbb{E}\big[\exp\big(-\gamma\big(X_T^{A_0,\pi'}-F(p)\big)\big)\big] \quad \geq \quad \inf_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]} \mathbb{E}\big[\exp\big(-\gamma\big(X_{T\wedge\tau}^{A_0,\pi}-\mathfrak{H}(p)\big)\big)\big] \; .$$

This obviously implies that

$$V_{\mathbb{G}}(p) \quad \leq \quad \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]} \mathbb{E}\big[-\exp\big(-\gamma\big(X_{T\wedge\tau}^{A_0,\pi}-\mathfrak{H}(p)\big)\big)\big] \; .$$

Now, we shall prove that

$$V_{\mathbb{G}}(p) \quad \geq \quad \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]} \mathbb{E}\big[-\exp\big(-\gamma\big(X_T^{A_0,\pi}-\mathfrak{H}(p)\big)\big)\big] \; .$$

From Lemma 2.3.2, we deduce that there exists $\pi^{*,\tau}\in\mathcal{A}^{\mathbb{G}}[T\wedge\tau,T]$ such that

$$\mathbb{E}\big[\exp\big(-\gamma\Delta X_{\tau,T}^{\pi^{*,\tau}}\big)\big|\mathcal{G}_{T\wedge\tau}\big] \quad = \quad \operatorname*{ess\,inf}_{\pi\in\mathcal{A}^{\mathbb{G}}[T\wedge\tau,T]} \mathbb{E}\big[\exp\big(-\gamma\Delta X_{\tau,T}^{\pi}\big)\big|\mathcal{G}_{T\wedge\tau}\big] \; .$$

For any $\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]$ we define the strategy $\underline{\pi}$ by

$$\underline{\pi}_t := \begin{cases}\pi_t & \text{if } t\leq T\wedge\tau \; ,\\ \pi_t^{*,\tau} & \text{if } t> T\wedge\tau \; .\end{cases}$$

$\underline{\pi}$ is in $\mathcal{A}^{\mathbb{G}}[0,T]$ since we have $\mathbb{E}[\int_0^T|\underline{\pi}_t\sigma_t|^2dt]<\infty$ and the following family

$$\big\{\exp(-\gamma X_\theta^{x,\underline{\pi}}),\ \theta \text{ is a } \mathbb{G}\text{-stopping time with values in } [0,T]\big\}$$

is uniformly bounded.
Indeed for any $\mathbb{G}$-stopping time θ in $[0,T]$, if $\theta\leq T\wedge\tau$, we have $X_\theta^{x,\underline{\pi}}=X_\theta^{x,\pi}$ and else $X_\theta^{x,\underline{\pi}}=X_{T\wedge\tau}^{x,\pi}+\Delta X_{\tau,\theta}^{*,\tau}$. Moreover, the families

$$\big\{\exp(-\gamma X_\theta^{x,\pi}),\ \theta \text{ is a } \mathbb{G}\text{-stopping time with values in } [0,T\wedge\tau]\big\}$$

and

$$\big\{\exp(-\gamma\Delta X_{\tau,\theta}^{*,\tau}),\ \theta \text{ is a } \mathbb{G}\text{-stopping time with values in } [T\wedge\tau,T]\big\}$$

are uniformly bounded.

We obtain

$$\begin{aligned} V_{\mathbb{G}}(p) &\geq \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]} \mathbb{E}\big[-\exp\big(-\gamma\big(X_T^{A_0,\underline{\pi}} - F(p)\big)\big)\big] \\ &= \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]} \mathbb{E}\big[-\exp\big(-\gamma\big(X_{T\wedge\tau}^{A_0,\pi} + \Delta X_{\tau,T}^{\pi^*,\tau} - F(p)\big)\big)\big] \\ &= \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]} \mathbb{E}\big[-\exp\big(-\gamma\big(X_{T\wedge\tau}^{A_0,\pi} - \mathfrak{H}(p)\big)\big)\big]\ . \end{aligned}$$

Now, we have to solve the optimization problem (2.2.10) and for that we look for a family of processes $\{R^{(\pi)},\ \pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]\}$ satisfying the following conditions

(i) $R_{T\wedge\tau}^{(\pi)} = -\exp(-\gamma(X_{T\wedge\tau}^{A_0,\pi} - \mathfrak{H}(p)))$, for any $\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]$.

(ii) $R_0^{(\pi)} = R_0$ is constant for any $\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]$.

(iii) $R^{(\pi)}$ is a $\mathbb{G}$-supermartingale for any $\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]$.

(iv) There exists $\pi^*\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]$ such that $R^{(\pi^*)}$ is a $\mathbb{G}$-martingale.

If such a family exists, we would have

$$R_0^{(\pi^*)} = \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]} \mathbb{E}\big[-\exp\big(-\gamma\big(X_{T\wedge\tau}^{A_0,\pi} - \mathfrak{H}(p)\big)\big)\big]\ .$$

Indeed, from (i), (ii) and (iii), we might have for any $\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]$,

$$R_0^{(\pi^*)} = R_0^{(\pi)} \geq \mathbb{E}\big[R_{T\wedge\tau}^{(\pi)}\big] = \mathbb{E}\big[-\exp\big(-\gamma\big(X_{T\wedge\tau}^{A_0,\pi} - \mathfrak{H}(p)\big)\big)\big]\ . \tag{2.2.11}$$

Moreover, it would follow from (i) and (iv) that

$$R_0^{(\pi^*)} = \mathbb{E}\big[-\exp\big(-\gamma\big(X_{T\wedge\tau}^{A_0,\pi^*} - \mathfrak{H}(p)\big)\big)\big]\ . \tag{2.2.12}$$

Therefore, from (2.2.11) and (2.2.12), we would get for any $\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]$

$$\mathbb{E}\big[-\exp\big(-\gamma\big(X_{T\wedge\tau}^{A_0,\pi} - \mathfrak{H}(p)\big)\big)\big] \leq R_0^{(\pi^*)} = \mathbb{E}\big[-\exp\big(-\gamma\big(X_{T\wedge\tau}^{A_0,\pi^*} - \mathfrak{H}(p)\big)\big)\big]\ .$$

We can see that it would lead to

$$R_0^{(\pi^*)} = \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]} \mathbb{E}\big[-\exp\big(-\gamma\big(X_{T\wedge\tau}^{A_0,\pi} - \mathfrak{H}(p)\big)\big)\big]\ .$$

Thanks to solutions of BSDEs with jumps, we shall construct a family $\{R^{(\pi)}\ ,\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]\}$ satisfying the previous conditions. Let f be a function defined on $[0,T]\times\Omega\times S_{\mathbb{G}}^\infty\times L_{\mathbb{G}}^2\times L_{\mathbb{G}}^2(\lambda)$ and assume that there exists $(Y(p),Z(p),U(p))$ in $S_{\mathbb{G}}^\infty\times L_{\mathbb{G}}^2\times L_{\mathbb{G}}^2(\lambda)$ solution of the following BSDE : for any $t\in[0,T]$,

$$Y_t(p) = \mathfrak{H}(p) + \int_{t\wedge\tau}^{T\wedge\tau} f(s,Y_s(p),Z_s(p),U_s(p))ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s(p)dB_s - \int_{t\wedge\tau}^{T\wedge\tau} U_s(p)dH_s\ .$$

In this case, for any $\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]$, we set

$$R^{(\pi)} = -\exp\big(-\gamma\big(X^{A_0,\pi} - Y(p)\big)\big)\ , \tag{2.2.13}$$

and look for a function f for which the family $\{R^{(\pi)},\ \pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]\}$ satisfies the conditions (i), (ii), (iii) and (iv). In order to calculate f, we apply Itô's formula and get

$$dR_t^{(\pi)} \;=\; dL_t^\pi + dK_t^\pi\ ,$$

where L^π and K^π are defined by

$$\begin{aligned} dL_t^\pi &:= -\gamma R_t^{(\pi)}\big(\sigma_t\pi_t - Z_t(p)\big)dB_t + R_{t^-}^{(\pi)}\big(e^{\gamma U_t(p)}-1\big)dM_t\ ,\\ dK_t^\pi &:= -\gamma R_t^{(\pi)}\Big[\pi_t(\mu_t-r_t)+f(t,Y_t(p),Z_t(p),U_t(p))-\frac{\gamma}{2}(\sigma_t\pi_t-Z_t(p))^2-\lambda_t\frac{e^{\gamma U_t(p)}-1}{\gamma}\Big]dt\ .\end{aligned}$$

As we hope that $R^{(\pi)}$ is a supermartingale the process K^π must be non-increasing, hence f should satisfy

$$-\gamma R_t^{(\pi)}\Big[\pi_t(\mu_t-r_t)+f(t,Y_t(p),Z_t(p),U_t(p))-\frac{\gamma}{2}(\sigma_t\pi_t-Z_t(p))^2-\lambda_t\frac{e^{\gamma U_t(p)}-1}{\gamma}\Big] \;\le\; 0\ ,$$

and since $-\gamma R_t^{(\pi)}\ge 0$, it would lead to

$$f(t,Y_t(p),Z_t(p),U_t(p)) \;\le\; \frac{\gamma}{2}(\sigma_t\pi_t-Z_t(p))^2+\lambda_t\frac{e^{\gamma U_t(p)}-1}{\gamma}-\pi_t(\mu_t-r_t)\ .$$

Moreover, for some particular $\pi^*\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]$, we hope that $R^{(\pi^*)}$ is a martingale so the process K^{π^*} must be constant, hence f should satisfy

$$f(t,Y_t(p),Z_t(p),U_t(p)) \;=\; \operatorname*{ess\,inf}_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]}\Big\{\frac{\gamma}{2}(\sigma_t\pi_t-Z_t(p))^2+\lambda_t\frac{e^{\gamma U_t(p)}-1}{\gamma}-\pi_t(\mu_t-r_t)\Big\}$$

and π_t^*, such that $dK_t^{\pi^*}=0$, would be defined by

$$\pi_t^* \;:=\; \frac{\nu_t}{\gamma\sigma_t}+\frac{Z_t(p)}{\sigma_t}\ .$$

Hence f would be the following function

$$f(t,y,z,u) \;=\; \lambda_t\frac{e^{\gamma u}-1}{\gamma}-\frac{\nu_t^2}{2\gamma}-\nu_t z\ ,$$

defined on $[0,T]\times\Omega\times S_{\mathbb{G}}^\infty\times L_{\mathbb{G}}^2\times L_{\mathbb{G}}^2(\lambda)$.

The following proposition asserts that the following BSDE with jump

$$\begin{aligned} Y_t(p) &= \mathfrak{H}(p)+\int_{t\wedge\tau}^{T\wedge\tau}\Big(\lambda_s\frac{e^{\gamma U_s(p)}-1}{\gamma}-\frac{\nu_s^2}{2\gamma}-\nu_s Z_s(p)\Big)ds-\int_{t\wedge\tau}^{T\wedge\tau}Z_s(p)dB_s\\ &\quad -\int_{t\wedge\tau}^{T\wedge\tau}U_s(p)dH_s\ ,\quad \forall t\in[0,T]\ ,\end{aligned} \tag{2.2.14}$$

admits a solution in $S_{\mathbb{G}}^\infty\times L_{\mathbb{G}}^2\times L_{\mathbb{G}}^2(\lambda)$.

Proposition 2.2.4 *Recalling notations in (2.2.2), (2.2.3) and (2.2.4), the BSDE (2.2.14) admits a solution $(Y(p), Z(p), U(p)) \in S^\infty_{\mathbb{G}} \times L^2_{\mathbb{G}} \times L^2_{\mathbb{G}}(\lambda)$ given for any $t \in [0, T]$ by*

$$\begin{cases} Y_t(p) &= Y^0_t(p)\mathbb{1}_{t<\tau} + F^{D,W}_\tau(p)\mathbb{1}_{\tau\le t} \;, \\ Z_t(p) &= Z^0_t(p)\mathbb{1}_{t\le\tau} \;, \\ U_t(p) &= \big(F^{D,W}_t(p) - Y^0_t(p)\big)\mathbb{1}_{t\le\tau} \;, \end{cases} \tag{2.2.15}$$

where $(Y^0(p), Z^0(p))$ is the unique solution in $S^\infty_{\mathbb{G}} \times L^2_{\mathbb{G}}$ of the following BSDE

$$\begin{cases} -dY^0_t(p) &= \Big\{\lambda_t \frac{e^{\gamma(F^{D,W}_t(p)-Y^0_t(p))}-1}{\gamma} - \frac{\nu_t^2}{2\gamma} - \nu_t Z^0_t(p)\Big\}dt - Z^0_t(p)dB_t \;, \\ Y^0_T(p) &= F^L_T(p) \;. \end{cases} \tag{2.2.16}$$

Proof: From [BRSM07, Theorem 2.1] and [FJ10, Theorem 1], we know that there is a unique solution $(Y^0(p), Z^0(p)) \in S^\infty_{\mathbb{G}} \times L^2_{\mathbb{G}}$ to the BSDE (2.2.16).

From [KLN13, Theorem 4.3], we know that $(Y(p), Z(p), U(p))$ defined by (2.2.15) is a solution of the BSDE (2.2.14).

Remark 2.2.5 *To apply [BRSM07, Theorem 2.1] and [FJ10, Theorem 1] and get existence result for a solution of the BSDE (2.2.16), the terminal condition $F^L_T(p)$ must be bounded and the process $F^{D,W}(p)$ must be also bounded.*

We conclude this section with its main result which is the following verification theorem.

Theorem 2.2.6 *The value function of the optimization problem (2.2.9) is given by*

$$V_{\mathbb{G}}(p) \;=\; -\exp(\gamma(Y_0(p) - A_0)) \;,$$

where $Y_0(p)$ is defined by the initial value of the first component of the solution of the BSDE (2.2.14) defined in Proposition 2.2.4.

Moreover there exists an optimal strategy $\pi^* \in \mathcal{A}^{\mathbb{G}}[0, T]$ and this one is defined by

$$\pi^*_t \;:=\; \frac{\nu_t}{\gamma\sigma_t} + \frac{Z_t(p)}{\sigma_t}\mathbb{1}_{t\le T\wedge\tau} + \frac{Z^{(\tau)}_t}{\sigma_t}\mathbb{1}_{t>T\wedge\tau} \;, \quad \forall t \in [0, T] \;, \tag{2.2.17}$$

with $Z(p)$ (resp. $Z^{(\tau)}$) defined by the solution of the BSDE (2.2.14) described in Proposition 2.2.4 (resp. see Lemma 2.3.2 in Appendix).

Notice that $Y_0(p) = Y^0_0(p)$ since the insurer can not withdraw his money at time 0.

In the proof of Theorem 2.2.6, the additional space of BMO-martingales intervenes : BMO($\mathbb{P}$) is the subset of $(\mathbb{P}, \mathbb{G})$-martingales m such that

$$\|m\|_{\mathrm{BMO}(\mathbb{P})} \;:=\; \sup_{\theta\in\mathcal{T}_{\mathbb{G}}[0,T]} \Big|\Big|\mathbb{E}\big[\langle m\rangle_T - \langle m\rangle_\theta|\mathcal{G}_\theta\big]^{1/2}\Big|\Big|_\infty \;<\; \infty \;,$$

where $\mathcal{T}_{\mathbb{G}}[0, T]$ is the set of $\mathbb{G}$-stopping times on $[0, T]$.

Before proving Theorem 2.2.6, we need the following lemma.

Lemma 2.2.7 *Let* $(Y^0(p), Z^0(p)) \in S^\infty_{\mathbb{G}} \times L^2_{\mathbb{G}}$ *be the solution of the BSDE (2.2.16), and let* π^* *be the strategy given by* (2.2.17)*. The processes* $\int_0^\cdot Z^0_s(p)dB_s$ *and* $\int_0^\cdot \sigma_s \pi^*_s dB_s$ *are* BMO($\mathbb{P}$)*-martingales.*

The proof of this technical lemma is given in Appendix 2.3.5.

Corollary 2.2.8 *The strategy* π^* *defined in* (2.2.17) *belongs to* $\mathcal{A}^{\mathbb{G}}[0,T]$ *.*

Proof: π^* is $\mathbb{G}$-measurable by definition, now using Hypothesis **A1** and (2.2.4) we have that

$$\begin{aligned}
\mathbb{E}\Big[\int_0^T \big|\pi^*_s\sigma_s\big|^2 ds\Big] &= \mathbb{E}\Big[\int_0^{T\wedge\tau}\Big(\frac{\nu_t}{\gamma} - Z_t(p)\Big)^2 dt\Big] + \mathbb{E}\Big[\int_{T\wedge\tau}^T \Big(\frac{\nu_t}{\gamma} - Z_t^{(\tau)}\Big)^2 dt\Big]\\
&\le c + c\mathbb{E}\Big[\int_0^{T\wedge\tau} \big|Z_t(p)\big|^2 dt\Big] + c\mathbb{E}\Big[\int_{T\wedge\tau}^T \big|Z_t^{(\tau)}\big|^2 dt\Big]\\
&< \infty\,,
\end{aligned}$$

where c is a positive constant.

It follows from Lemma 2.2.7 and properties of BMO-martingales (see for example [Kaz94]) that the family

$$\big\{-\exp\big(-\gamma X^{\pi^*}_\theta\big),\ \theta \text{ is a } \mathbb{G}\text{-stopping time with values in } [0,T]\big\}$$

is uniformly integrable.

Now, we are able to prove Theorem 2.2.6.

Proof: First we check that the family $\big\{R^{(\pi)},\ \pi \in \mathcal{A}^{\mathbb{G}}[0,T]\big\}$ defined in (2.2.13) satisfies properties (i), (ii), (iii) and (iv).

Properties (i) and (ii) directly follow from the definition of $R^{(\pi)}$. To prove that condition (iii) is satisfied, we apply Itô's formula and get

$$\begin{aligned}
dR^{(\pi)}_t = &-\gamma R^{(\pi)}_t\Big[\pi_t(\mu_t - r_t) - \frac{\nu_t^2}{2\gamma} - \nu_t Z_t(p) - \frac{\gamma}{2}(\sigma_t\pi_t - Z_t(p))^2\Big]dt\\
&-\gamma R^{(\pi)}_t(\sigma_t\pi_t - Z_t(p))dB_t + R^{(\pi)}_{t^-}\big(e^{\gamma U_t(p)} - 1\big)dM_t\,.
\end{aligned} \tag{2.2.18}$$

This last equation has an explicit solution given by

$$\begin{aligned}
R^{(\pi)}_t = &\ R_0\mathcal{E}\Big(\int_0^t \gamma\big(Z_s(p) - \pi_s\sigma_s\big)dB_s + \int_0^t \big(e^{\gamma U_s(p)} - 1\big)dM_s\Big)\\
&\times\exp\Big(-\gamma\int_0^t\Big(\pi_s(\mu_s - r_s) - \frac{\nu_s^2}{2\gamma} - \nu_s Z_s(p) - \frac{\gamma}{2}(\sigma_s\pi_s - Z_s(p))^2\Big)ds\Big)\,,
\end{aligned} \tag{2.2.19}$$

where $\mathcal{E}$ denotes the Doléans-Dade exponential. Since $\pi \in \mathcal{A}^{\mathbb{G}}[0,T]$, the process $L^\pi := \mathcal{E}\big(\int_0^\cdot \gamma(Z_t(p) - \sigma_t\pi_t)dB_t + \big(e^{\gamma U_t(p)} - 1\big)dM_t\big)$ is a $\mathbb{G}$-local martingale. Hence, there exists a sequence of $\mathbb{G}$-stopping times $(\theta_n)_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty}\theta_n = T\wedge\tau\ \mathbb{P}-a.s.$ and such that $L^\pi_{\cdot\wedge\theta_n}$ is a positive martingale for each $n \in \mathbb{N}$. Moreover, since

$$f(t, Y_t(p), Z_t(p), U_t(p)) \le \frac{\gamma}{2}(\sigma_t\pi_t - Z_t(p))^2 + \lambda_t\frac{e^{\gamma U_t(p)} - 1}{\gamma} - \pi_t(\mu_t - r_t)\,,$$

the process $\exp\big(-\gamma\int_0^{\cdot}\big(\pi_s(\mu_s-r_s)-\frac{\nu_s^2}{2\gamma}-Z_s(p)\nu_s-\frac{\gamma}{2}(\sigma_s\pi_s-Z_s(p))^2\big)ds\big)$ is non-decreasing. As $R_0<0$, we get that $R^{(\pi)}_{.\wedge\theta_n}$ is a supermartingale and, for any $0\leq s\leq t\leq T$, we have

$$\mathbb{E}[R^{(\pi)}_{t\wedge\theta_n}|\mathcal{G}_s] \;\leq\; R^{(\pi)}_{s\wedge\theta_n}\ .$$

This implies that, for any set $A\in\mathcal{G}_s$, we have the following inequality

$$\mathbb{E}[R^{(\pi)}_{t\wedge\theta_n}\mathbb{1}_A] \;\leq\; \mathbb{E}[R^{(\pi)}_{s\wedge\theta_n}\mathbb{1}_A]\ .$$

Since π is admissible and Y is bounded, we remark that $(R^{(\pi)}_{t\wedge\theta_n})_{n\in\mathbb{N}}$ and $(R^{(\pi)}_{s\wedge\theta_n})_{n\in\mathbb{N}}$ are uniformly integrable, hence we may let n goes to $+\infty$ and get

$$\mathbb{E}[R^{(\pi)}_t\mathbb{1}_A] \;\leq\; \mathbb{E}[R^{(\pi)}_s\mathbb{1}_A]\ ,\quad \forall A\in\mathcal{G}_s\ .$$

This implies the claimed supermartingale property of $R^{(\pi)}$.
Finally, we know from Corollary 2.2.8 that π^* is admissible and from construction of π^*, we have $R^{(\pi^*)}=L^{\pi^*}$, therefore $R^{(\pi^*)}$ is a martingale. This proves that condition (iv) is satisfied.

Hence, for any $\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]$, we obtain that

$$\mathbb{E}\big[-\exp\big(-\gamma\big(X^{A_0,\pi}_{T\wedge\tau}-\mathfrak{H}(p)\big)\big)\big]\leq R^{(\pi)}_0=R^{(\pi^*)}_0=\mathbb{E}\big[-\exp\big(-\gamma\big(X^{A_0,\pi^*}_{T\wedge\tau}-\mathfrak{H}(p)\big)\big)\big]\ .$$

Therefore, $V_{\mathbb{G}}(p)=-\exp(\gamma(Y_0(p)-A_0))$ and π^* is an optimal admissible strategy.

2.2.4 Indifference fee rate

In this section, our goal is to determine indifference fee rates i.e. positive numbers p^* such that

$$\sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T]}\mathbb{E}\big[-\exp\big(-\gamma\big(X^{A_0,\pi}_T-F(p^*)\big)\big)\big] \;=\; \sup_{\pi\in\mathcal{A}^{\mathbb{F}}[0,T]}\mathbb{E}\big[-\exp\big(-\gamma X^{\pi}_T\big)\big]\ .$$

It follows from results of Subsections 2.2.2 and 2.2.3 that the previous equation can be rewritten in the following way

$$Y_0(p^*)-A_0 \;=\; y_0\ .$$

To study this equation we introduce the function $\psi:\mathbb{R}\to\mathbb{R}$ defined as follows

$$\psi(p) \;:=\; Y_0(p)-y_0-A_0\ ,\quad \forall p\in\mathbb{R}\ .$$

There may exist three cases depending on the coefficients values.

(i) For any $p\in\mathbb{R}$, we have $\psi(p)>0$. That means that, for any fee rate p, we have

$$V_{\mathbb{G}}(p) \;<\; V_{\mathbb{F}}\ .$$

Therefore, the insurer's expected utility is always lower if he sells the variable annuities. Thus, he should not sell it.

(ii) For any $p \in \mathbb{R}$, we have $\psi(p) < 0$. That means that, for any fee rate p, we have

$$V_{\mathbb{G}}(p) \quad > \quad V_{\mathbb{F}} \; .$$

Therefore, the insurer's expected utility is always higher if he sells the variable annuities. Thus, he should sell it whatever the fees are.

(iii) There exist p_1 and p_2 such that $\psi(p_1)\psi(p_2) < 0$. In this case, we prove in the remainder of this section that there exist indifference fee rates thanks to the intermediate value theorem applied to the function ψ.

We now give useful analytical properties of the function ψ.

Proposition 2.2.9 *The function ψ is continuous and non-increasing on $\mathbb{R}$.*

PROOF: We first show that ψ is non-increasing. Let $p_1, p_2 \in \mathbb{R}$ with $p_1 \leq p_2$. By definition of the process A^p, for any $t \in [0,T]$, we have

$$A_t^{p_1} \quad \geq \quad A_t^{p_2} \; .$$

It follows from the monotonicity of $\hat{F}^L$, $\hat{F}^D$ and $\hat{F}^W$ that $F(p_1) \geq F(p_2)$ $\mathbb{P}$-a.s. Hence, for any $\pi \in \mathcal{A}^{\mathbb{G}}[0,T]$, we have

$$\mathbb{E}\big[-\exp\big(-\gamma\big(X_T^{A_0,\pi} - F(p_1)\big)\big)\big] \quad \leq \quad \mathbb{E}\big[-\exp\big(-\gamma\big(X_T^{A_0,\pi} - F(p_2)\big)\big)\big] \; .$$

Since this inequality holds for any $\pi \in \mathcal{A}^{\mathbb{G}}[0,T]$, we get

$$V_{\mathbb{G}}(p_1) \quad \leq \quad V_{\mathbb{G}}(p_2) \; .$$

As $V_{\mathbb{G}}(p) = -\exp(\gamma(Y_0(p) - A_0))$ and $\gamma > 0$, it follows that ψ is non-increasing.
We now prove that ψ is continuous on $\mathbb{R}$. For that we prove that the solution of the BSDE (2.2.16) is continuous w.r.t. the terminal condition by adapting a usual approach presented for example in [EKKP⁺97]. Let $p_1 < p_2$ and consider the solutions $(Y^0(p_1), Z^0(p_1))$ and $(Y^0(p_2), Z^0(p_2))$ associated to the BSDE (2.2.16) with respectively parameters p_1 and p_2. We define the processes $\delta Y(p_1,p_2) := Y^0(p_2) - Y^0(p_1)$ and $\delta Z(p_1,p_2) := Z^0(p_2) - Z^0(p_1)$. By applying Itô's formula to the process $\big(e^{\alpha t}|\delta Y_t(p_1,p_2)|^2\big)_{0\leq t\leq T}$, we get that, for any $\alpha > 0$,

$$\begin{aligned} d\big(e^{\alpha t}|\delta Y_t(p_1,p_2)|^2\big) \quad = \quad & \alpha e^{\alpha t}|\delta Y_t(p_1,p_2)|^2 dt + 2e^{\alpha t}\delta Y_t(p_1,p_2) d\big(\delta Y_t(p_1,p_2)\big) \\ & + e^{\alpha t}|\delta Z_t(p_1,p_2)|^2 dt \; . \end{aligned}$$

By usual arguments, we get

$$\begin{aligned} & e^{\alpha t}|\delta Y_t(p_1,p_2)|^2 + \mathbb{E}\Big[\int_t^T e^{\alpha s}|\delta Z_s(p_1,p_2)|^2 ds \Big| \mathcal{F}_t\Big] \leq \\ & \mathbb{E}\Big[e^{\alpha T}|\delta Y_T(p_1,p_2)|^2 - \alpha \int_t^T e^{\alpha s}|\delta Y_s(p_1,p_2)|^2 ds - 2\int_t^T e^{\alpha s}\delta Y_s(p_1,p_2)\delta Z_s(p_1,p_2) ds \Big| \mathcal{F}_t\Big] \\ & \qquad + \frac{2}{\gamma}\mathbb{E}\Big[\int_t^T \lambda_s e^{\alpha s}\delta Y_s(p_1,p_2)\Big(e^{\gamma(F_s^{D,W}(p_2) - Y_s(p_2)} - e^{\gamma(F_s^{D,W}(p_1) - Y_s(p_1))}\Big)\Big| \mathcal{F}_t\Big] \; . \end{aligned}$$

By using Young's inequality, we get

$$\begin{aligned} & e^{\alpha t}|\delta Y_t(p_1,p_2)|^2 \leq \mathbb{E}\big[e^{\alpha T}|\delta Y_T(p_1,p_2)|^2 \big| \mathcal{F}_t\big] + (1-\alpha)\mathbb{E}\Big[\int_t^T e^{\alpha s}|\delta Y_s(p_1,p_2)|^2 ds \Big| \mathcal{F}_t\Big] \\ & \qquad + \frac{2}{\gamma}\mathbb{E}\Big[\int_t^T \lambda_s e^{\alpha s}\delta Y_s(p_1,p_2)\Big(e^{\gamma(F_s^{D,W}(p_2) - Y_s(p_2)} - e^{\gamma(F_s^{D,W}(p_1) - Y_s(p_1))}\Big)\Big| \mathcal{F}_t\Big] \; . \end{aligned}$$

Moreover, we know that $Y(p_1)$ and $Y(p_2)$ are lower bounded, hence there exists a constant k such that $Y_t(p_1) \geq k$ and $Y_t(p_2) \geq k$ for any $t \in [0,T]$. Since the function $\exp(-\gamma(y \vee k))$ is Lipschitz continuous, the process λ is bounded and the processes $F^{D,W}(p_1)$ and $F^{D,W}(p_2)$ are bounded, we can assert that there exists a positive constant C such that

$$e^{\alpha t}|\delta Y_t(p_1,p_2)|^2 \leq \mathbb{E}\big[e^{\alpha T}|\delta Y_T(p_1,p_2)|^2\big|\mathcal{F}_t\big]+(C-\alpha)\mathbb{E}\Big[\int_t^T e^{\alpha s}|\delta Y_s(p_1,p_2)|^2 ds\Big|\mathcal{F}_t\Big] .$$

Hence, for $\alpha = C$, we get

$$|\delta Y_0(p_1,p_2)|^2 \quad \leq \quad \mathbb{E}\big[e^{CT}|\delta Y_T(p_1,p_2)|^2\big] .$$

We conclude the proof by recalling that $Y_T^0(p)$, the terminal condition to the BSDE (2.2.16), is continuous on $\mathbb{R}$ w.r.t. p as we have assumed that the function $\hat{F}^L$ is continuous on $\mathbb{R}$.

We now consider the cases of usual guarantees.

Corollary 2.2.10 *Ratchet guarantee.*
Let $m > A_0$. Recalling notations of (2.2.6), we assume that

$$F(p) = m \wedge \Big[A^p_{T\wedge\tau}(0) \vee \Big(\max_{0\leq i\leq n}\Big[e^{-\int_0^{T\wedge\tau} r_s ds}\hat{A}^p_{t_i}(0)\mathbb{1}_{\{t_i\leq T\wedge\tau\}}\Big] + \int_0^{T\wedge\tau}\xi_s A^p_s\beta_s\, ds\Big)\Big] .$$

There exists $p^* \in \mathbb{R}\cup\{-\infty\}$ such that for $p \geq p^*$ we have $V_{\mathbb{G}}(p) \geq V_{\mathbb{F}}$ and for $p < p^*$ we have $V_{\mathbb{G}}(p) < V_{\mathbb{F}}$.

Proof: From Proposition 2.2.9, we just have to show that $\lim_{p\to+\infty}\psi(p) \leq 0$.
It would follow from the intermediate value theorem and the monotonicity of ψ that there exists $p^* \in \mathbb{R}\cup\{-\infty\}$ such that $\psi(p) \leq 0$ for $p \geq p^*$ and $\psi(p) > 0$ for $p < p^*$. First, notice that we may deduce from Hypothesis 2.1.1 that there exists a positive constant C such that, for any $t \in [0,T]$, $\mathbb{E}[A^p_t] \leq Ce^{-pt}$. Therefore, as $A^p_t \geq 0$, we get

$$\lim_{p\to+\infty} A^p_t = 0 \quad \text{a.s. for any } t \in (0,T] .$$

We now study the limit of ψ at $+\infty$. We have

$$\psi(p) + y_0 \quad = \quad \frac{1}{\gamma}\ln(-V_{\mathbb{G}}(p)) .$$

On the other hand, for any $\pi \in \mathcal{A}^{\mathbb{G}}[0,T]$, we have

$$V_{\mathbb{G}}(p) \quad \geq \quad \mathbb{E}\big[-\exp(-\gamma(X^\pi_T + A_0 - F(p))\big] .$$

Hence, it follows from the monotone convergence theorem that

$$\begin{aligned}
\lim_{p\to+\infty}\psi(p) + y_0 \quad &= \quad \frac{1}{\gamma}\ln\big(-\lim_{p\to+\infty}V_{\mathbb{G}}(p)\big) \\
&\leq \quad \frac{1}{\gamma}\ln\Big(\mathbb{E}\big[\exp(-\gamma(X^\pi_T + A_0 - \lim_{p\to+\infty}F(p))\big]\Big) \\
&= \quad \frac{1}{\gamma}\ln\Big(\mathbb{E}\big[\exp(-\gamma(X^\pi_T + A_0(1-e^{-\int_0^{T\wedge\tau} r_s ds}))\big]\Big) \\
&\leq \quad \frac{1}{\gamma}\ln\Big(\mathbb{E}\big[\exp(-\gamma X^\pi_T)\big]\Big) . \qquad (2.2.20)
\end{aligned}$$

We recall that $y_0 = \frac{1}{\gamma}\ln(-V_{\mathbb{F}})$ and that, from Proposition 2.2.2, there exists $\pi^* \in \mathcal{A}^{\mathbb{F}}[0,T] \subset \mathcal{A}^{\mathbb{G}}[0,T]$, such that $y_0 = \frac{1}{\gamma}\ln(\mathbb{E}[\exp(-\gamma X_T^{\pi^*})])$. Therefore, we obtain that $\lim_{p\to+\infty}\psi(p) \leq 0$ if we choose π^* in (2.2.20).

Corollary 2.2.11 *Roll-up guarantee.*
Let $m > A_0$. Recalling notations of (2.2.5), we assume that

$$F(p) = m \wedge \Big[A^p_{T\wedge\tau}(0) \vee \Big(e^{-\int_0^{T\wedge\tau} r_s^\eta\, ds} A_0 + \int_0^{T\wedge\tau} \xi_s A_s^p \beta_s^\eta\, ds\Big)\Big].$$

There exists $\eta_* \geq 0$ such that for any $\eta \in [0,\eta_*]$, there exists $p^* \in \mathbb{R}\cup\{-\infty\}$ such that for $p \geq p^*$ we have $V_{\mathbb{G}}(p) \geq V_{\mathbb{F}}$ and for $p < p^*$ we have $V_{\mathbb{G}}(p) < V_{\mathbb{F}}$.

PROOF: Let $\eta \geq 0$. From Proposition 2.2.2, there exists $\pi^* \in \mathcal{A}^{\mathbb{F}}[0,T] \subset \mathcal{A}^{\mathbb{G}}[0,T]$, such that $y_0 = \frac{1}{\gamma}\ln(\mathbb{E}[\exp(-\gamma X_T^{\pi^*})])$. Following the proof of Corollary 2.2.10, we deduce from the monotone convergence theorem that

$$\begin{aligned}
\lim_{p\to+\infty}\psi(p) + y_0 &= \frac{1}{\gamma}\ln\big(-\lim_{p\to+\infty}V_{\mathbb{G}}(p)\big)\\
&\leq \frac{1}{\gamma}\ln\Big(\mathbb{E}\big[\exp(-\gamma(X_T^{\pi^*} + A_0 - \lim_{p\to+\infty}F(p)))\big]\Big)\\
&= \frac{1}{\gamma}\ln\big(\Phi(\eta)\big)\ ,
\end{aligned}$$

where we have set

$$\Phi(\eta) \ := \ \mathbb{E}\Big[\exp\big(-\gamma(X_T^{\pi^*} + A_0(1 - e^{-\int_0^{T\wedge\tau} r_s^\eta ds}))\big)\Big]\ .$$

Obviously, Φ is continuous and non-decreasing on $\mathbb{R}^+$. Moreover, we have

$$\Phi(0) \leq \mathbb{E}\big[\exp(-\gamma X_T^{\pi^*})\big] = e^{\gamma y_0} \qquad \text{and} \qquad \lim_{\eta\to+\infty}\Phi(\eta) = +\infty\ .$$

From the intermediate value theorem, we may define $\eta_* \geq 0$ as

$$\eta_* \ := \ \sup\{\eta \geq 0,\ \Phi(\eta) = e^{\gamma y_0}\}\ .$$

We conclude the proof by noticing that for $0 \leq \eta \leq \eta_*$, we have

$$\lim_{p\to+\infty}\psi(p) \ \leq \ \frac{1}{\gamma}\ln\big(\Phi(\eta)\big) - y_0 \leq 0\ .$$

2.2.5 Indifference fee rates for a policyholder

An interesting issue is to define and compute indifference fee rates for the policyholder instead of insurer. The additional difficulty is then to model and determine the withdrawal process ξ. The simplest idea is to consider that the policyholder optimize her withdrawals. In [BSCKL15], this kind of optimization on withdrawal is assumed and a method for computing indifference fee rates for the insurer is given. However, if we consider the policyholder point of view, we have to solve the following problem :

$$\sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]} \mathbb{E}\big[U\big(X^{x,\pi}_{T\wedge\tau}\big)\big] \;=\; \sup_{\xi\in\mathcal{W}}\sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]} \mathbb{E}\big[U\big(X^{x-A_0,\pi}_{T\wedge\tau}+F(p^*)\big)\big]\ ,$$

where $\mathcal{W}$ is the set of admissible withdrawal processes. The first optimization problem to be solved would then be the following

$$\sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]} \mathbb{E}\big[U\big(-A_0+X^{\pi}_{T\wedge\tau}+F(p^*)\big)\big]\ .$$

A first idea would be to use the same techniques that in the previous sections and in this case we must have the condition that $F(p)$ is bounded and consider the following BSDE :

$$\begin{cases} -dY^0_t(p) &= \Big\{\lambda_t\frac{e^{\gamma(-F^{D,W}_t(p)-Y^0_t(p))}-1}{\gamma}-\frac{\nu_t^2}{2\gamma}-\nu_t Z^0_t(p)\Big\}dt - Z^0_t(p)dB_t\ ,\\ Y^0_T(p) &= -F^L_T(p)\ . \end{cases}$$

A second approach is to use the results of [LQ11] and in this case the bound on $F(p)$ is useless. If we suppose that the policyholder can invest only a bounded amount in the risky asset then we have to solve the following BSDE

$$\begin{cases} -dY_t &= \operatorname*{ess\,inf}_{\pi\in\mathcal{C}}\Big\{\frac{\gamma^2}{2}|\pi_t\sigma_t|^2Y_t-\gamma\pi_t(\mu_tY_t+\sigma_tZ_t)\Big\}dt - Z_tdW_t - U_tdM_t\ ,\\ Y_{T\wedge\tau} &= \exp(-\gamma F(p))\ , \end{cases} \tag{2.2.21}$$

where π is the amount invested in the risky asset and $\mathcal{C}$ is a compact set. We shall obtain that

$$Y_0 = -\sup_{\pi\in\mathcal{C}}\mathbb{E}[-\exp(-\gamma(X^{\pi}_{T\wedge\tau}+F(p)))].$$

If we do not assume that the investment strategy π has to respect some constraints, we would have to solve the following BSDE :

$$\begin{cases} -dY_t &= \operatorname*{ess\,inf}_{\pi\in\mathcal{D}}\Big\{\frac{\gamma^2}{2}|\pi_t\sigma_t|^2Y_t-\gamma\pi_t(\mu_tY_t+\sigma_tZ_t)\Big\}dt - dK_t - Z_tdW_t - U_tdM_t\ ,\\ Y_{T\wedge\tau} &= \exp(-\gamma F(p))\ , \end{cases}$$

where K is a non decreasing process which is a part of the solution. The problem with this kind of BSDE is that we have not scheme to approximate the solution since the solution is not unique. It can only be approximated by the sequence Y^k where Y^k is solution to the BSDE (2.2.21) with $\mathcal{C}=[-k,k]$.

It would remain to solve the optimization problem on the withdrawal rate process which, even from a numerical point of view, is, up to our knowledge, an open problem.

2.2.6 Simulations

In this section we present numerical illustrations of parameters sensibility for indifference fee rates. We compute solutions for both optimization problems : $V_{\mathbb{F}}$, the utility maximization problem without variable annuities, and $V_{\mathbb{G}}(p)$, the utility maximization problem with variable annuities. We simulate the BSDEs involved, using the discretization scheme studied in [BT04]. For the computation of the conditional expectations, we use non-parametric regression method with the Gaussian function as kernel. Following a dichotomy method, we find p such that the equality $V_{\mathbb{F}} = V_{\mathbb{G}}(p)$ is satisfied.

We assume that r and μ are Markov chains $\mathbb{F}$-adapted taking values in the states spaces $S^r = \{0,\ 0.01, \dots, 0.25\}$ and $S^\mu = \{0,\ 0.01,\ 0.02, \dots, 0.3\}$. Their respective transitional matrix are $Q^r = \{q^r_{i,j}\}_{1\leq i,j\leq 26}$ and $Q^\mu = \{q^\mu_{i,j}\}_{1\leq i,j\leq 31}$ are given by

$$q^r_{i,j} = \begin{cases} \frac{1}{2} & \text{if} \quad i = j, \\ \frac{1}{2} & \text{if} \quad i = 1 \text{ and } j = 2, \\ \frac{1}{2} & \text{if} \quad i = 27 \text{ and } j = 26, \\ \frac{1}{4} & \text{if} \quad i = j+1 \text{ and } i \leq 26, \\ \frac{1}{4} & \text{if} \quad i = j-1 \text{ and } i \geq 2, \\ 0 & \text{else}, \end{cases} \quad \text{and} \quad q^\mu_{i,j} = \begin{cases} \frac{1}{2} & \text{if} \quad i = j, \\ \frac{1}{2} & \text{if} \quad i = 1 \text{ and } j = 2, \\ \frac{1}{2} & \text{if} \quad i = 32 \text{ and } j = 31, \\ \frac{1}{4} & \text{if} \quad i = j+1 \text{ and } i \leq 31, \\ \frac{1}{4} & \text{if} \quad i = j-1 \text{ and } i \geq 2, \\ 0 & \text{else}, \end{cases}$$

Initial values μ_0 and r_0 will be precised later. For simplicity, we assume that there are no early withdrawals i.e. we set $(\xi_t)_{t\geq 0} \equiv 0$, except for Figure 2.10. We shall give the following numerical values to parameters

$$\gamma = 1.3, \quad \lambda = 0.05, \quad T = 20, \quad A_0 = 1,$$

and, for the financial market parameters

$$r_0 = 0.02, \quad \mu_0 = 0.15, \quad \sigma = 0.3.$$

We divide our numerical study in three parts. First, we consider a product with a ratchet guarantee and describe the dependence with respect to the market parameters : the initial interest rate (see Figure 2.2), the initial drift (see Figure 2.3) and the volatility (see Figure 2.4). In a second part, still with ratchet guarantee, we give illustrations of the dependence with respect to the longevity parameters : the contract maturity and the exit time intensity. In the last part, we consider the case with a roll-up guarantee and compute the sensibilities of indifference fees to variations of the initial value A_0, the roll-up rate η and finally to variations of the withdrawal rate ξ.

Market risk

In this first part, we want to understand the impact of market risks on the indifference fee. For that we consider the case of ratchet guarantees.

Figure 2.2 plots the indifference fee rates when the initial interest rate r_0 ranged from 0.01 to 0.055.

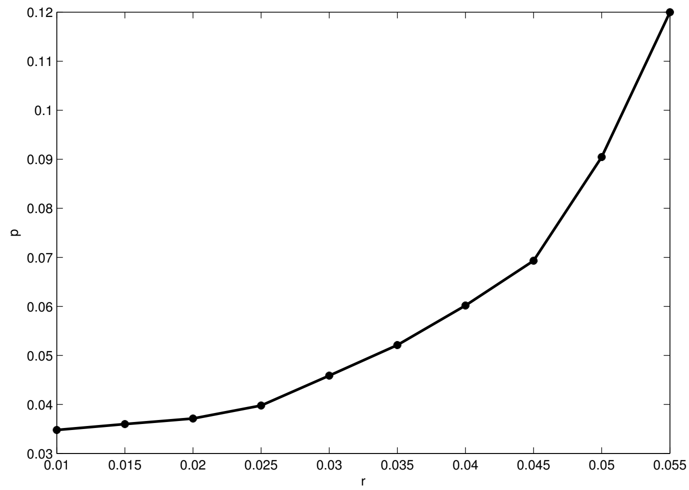

FIGURE 2.2 – Indifference fee rate w.r.t. r_0

We notice that indifference fee rates increase with interest rate. This is due to the guarantee structure of the product : a growth of interest rate will lead to a growth of the quantity $V_{\mathbb{G}}(p)$ with respect to $V_{\mathbb{F}}$ and to compensate this growth we will have to increase p, as $p \to V_{\mathbb{G}}(p)$ is non-increasing.

Figure 2.3 plots the indifference fee rates when the initial drift μ_0 ranged from 0.02 to 0.3.

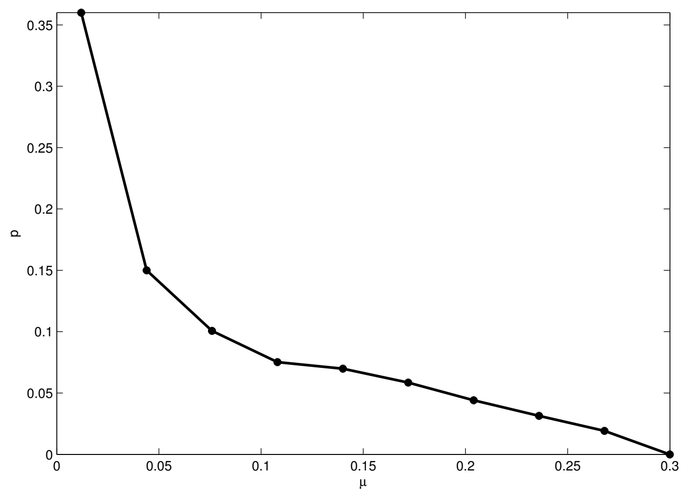

FIGURE 2.3 – Indifference fee rate w.r.t. μ_0

Notice that indifference fee rates decrease with respect to the initial drift. The bigger is the drift the less useful are the guarantees, then the fees payed to get these guarantees have to decrease.

Figure 2.4 plots the indifference fee rates when the volatility σ ranged from 0.1 to 0.4.

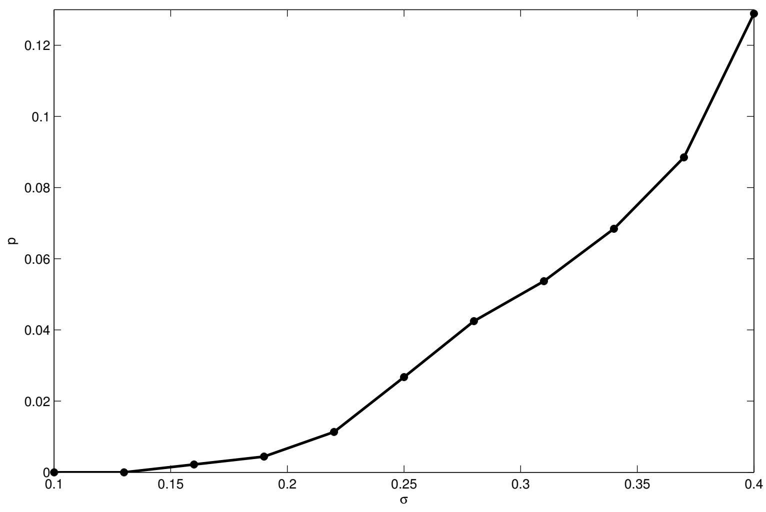

FIGURE 2.4 – Indifference fee rate with respect to σ

Once again, we can get a financial interpretation of the monotonicity of the fees w.r.t. market volatility. The bigger is the volatility the more useful are the guarantees, then the fees payed to get these guarantees have to increase.

Longevity risk

In this second part, we emphasize the impact of longevity risks on indifference fees for ratchet guarantees. Figure 2.5 plots the indifference fee rates when the intensity λ ranged from 0 to 0.25.

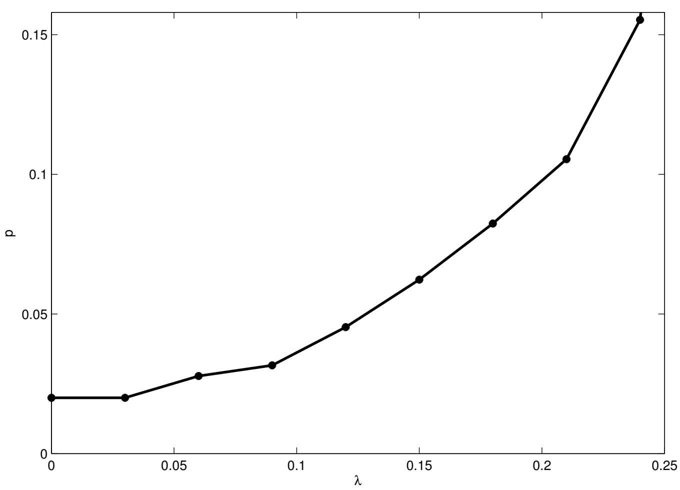

FIGURE 2.5 – Indifference fee rate with respect to λ

Figure 2.6 plots the indifference fee rates when the terminal time of the contract T ranged from 7 to 28.

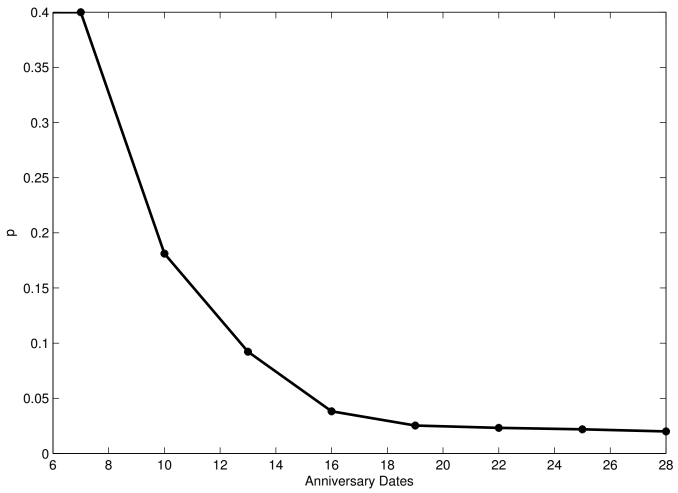

FIGURE 2.6 – Indifference fee rate with respect to the maturity

Notice that the more it remains time or expected time to maturity, the more the insurer will receive fees. Hence, fee rate should decrease when time or expected time to maturity increases.

Truncation and Guarantee risk

To end this numerical section, we consider two cases. First we present here an illustration of Remark 2.2.1 which stated that we could consider truncated payoffs. Indeed, if the truncation goes to infinity, we have a convergence for the fee associated to the payoff. The following figure illustrates this for ratchet guarantees.

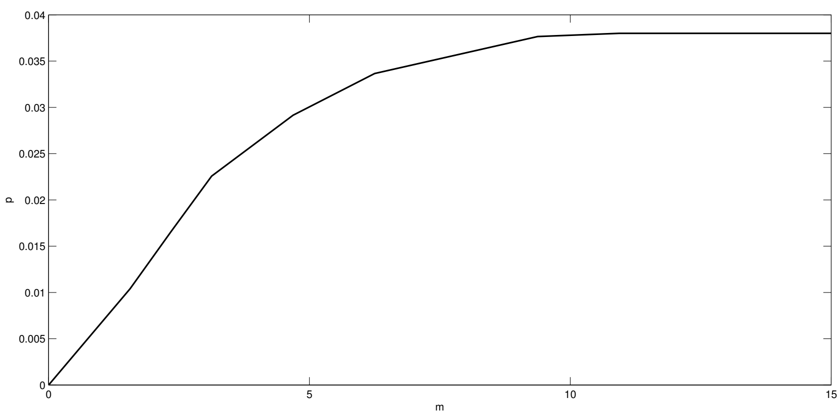

FIGURE 2.7 – Indifference fee rate with respect to the payoff truncation m.

We conclude with investigating the roll-up guarantee case. We present some sensibilities of indifference fee rates to the roll-up rate η, to the initial investment A_0 and to the withdrawal rate ξ.

Figure 2.8 plots the indifference fee rates when the roll-up rate η ranged from 0 to 0.05.

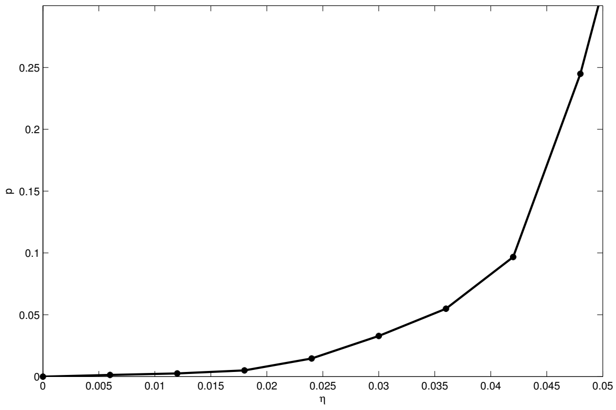

FIGURE 2.8 – Indifference fee rate with respect to η

We remark that the indifference fee rates are increasing with respect to the roll-up rate η with an exponential growth. The insurer has to be careful when he offers a roll up guarantee : if he proposes a rate η too high (for example $\eta > 0.05$), the guarantee could be not profitable to sell, at any price.

Figure 2.9 plots the indifference fee rates when the initial value A_0 ranged from 0.5 to 2.

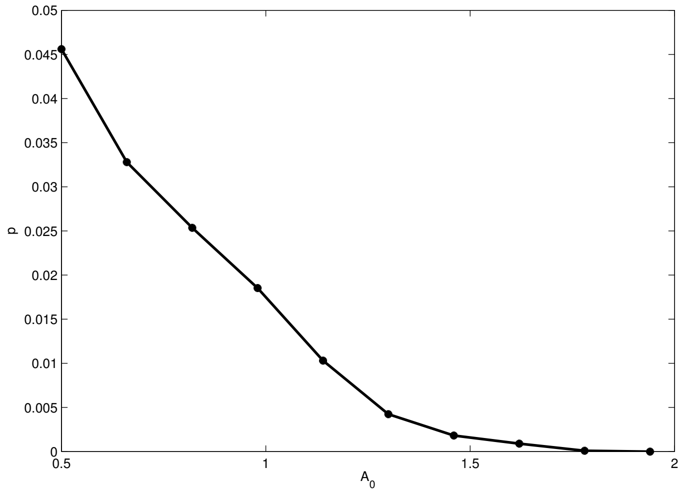

FIGURE 2.9 – Indifference fee rate with respect to A_0

As expected, indifference fee rates are decreasing with respect to the initial investment A_0. If A_0 is too small it could be not interesting for the insurer to sell the product, whatever the fees are.

Figure 2.10 plots the indifference fee rates when the withdrawal rate ξ is constant and ranged from 0 to 0.3. It shows that indifference fee rates are linearly increasing w.r.t. the withdrawal rate ξ.

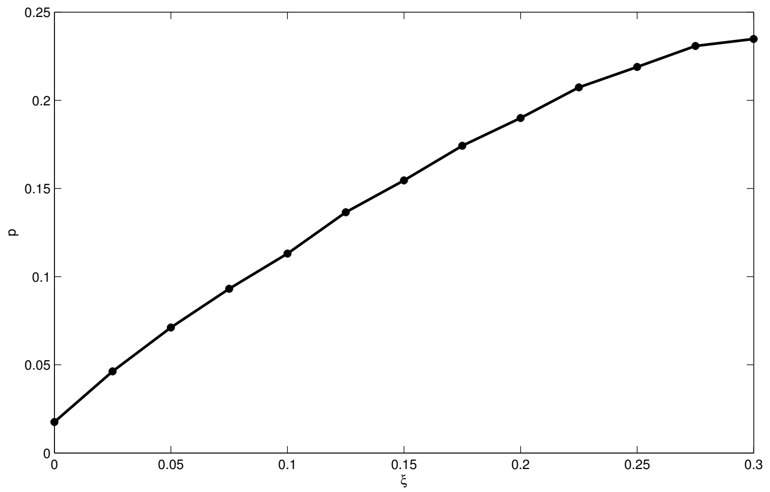

FIGURE 2.10 – Indifference fee rate with respect to ξ

2.3 Variable Annuities pricing in the worst case

This section is devoted to the presentation of a numerical method, very closed to the one presented in the previous section, and applied to compute an indifference price for variable annuities in the worst case for the insurer.

This problem, carefully studied in [BSCKL15], is a min-max optimization problem. To obtain a tractable model and a numerical method to solve this control problem, Blanchet & als have introduced a slightly different model for variable annuities, adding some constraints on withdrawals. We present their model here and then we give some numerical illustrations that we have done.

Blanchet & als make here an additional assumption on the withdrawal process ξ : they assume that the insured aims at minimizing the expected utility of the insurer, that would be the worst case for the insurer and it gives a robust indifference price which has to be taken as an upper bound for prices.

Let $:= (\mathtt{t_i})_{\mathtt{0\leq i\leq n}}$ the set of policy anniversary dates, with $t_0 = 0$ and $t_n = T$. By convention we set $t_{n+1} = +\infty$. At any date t_i, for $i \in \{1, \dots, n-1\}$, the insured, if he is still alive, is allowed to withdraw an amount of money. This should be lower than a bounded non-negative $\mathcal{G}_{t_i}$-measurable random variable $\widehat{G}_i$ which may depend on previous withdrawals, on previous account values and on some guarantees determined in the policy. We define $\widehat{W}$ as a finite subset of $[0, 1]$, which contains 0 and 1, and introduce the set of admissible withdraw policies

$$\widehat{\mathfrak{E}} = \Big\{(\alpha_i \widehat{G}_i)_{1\leq i\leq n-1} \ : \ \alpha_i \text{ is a } \mathcal{G}_{t_i}\text{-measurable random variable such that } \alpha_i \in \widehat{W}, \ \forall i \in \{1, \dots, n-1\}\Big\} .$$

For $\widehat{\xi} \in \widehat{\mathfrak{E}}$ and $i \in \{1, \dots, n-1\}$, $\widehat{\xi}_i$ is the withdrawal made by the insured at time t_i and we introduce the family $(\xi_i)_{1\leq i\leq n-1}$ such that $\xi_i := e^{-\int_0^{t_i} r_s ds}\widehat{\xi}_i$ is the discounted withdrawal made at time t_i. We define by $\mathfrak{E}$ the admissible discounted withdraw policies with $\xi \in \mathfrak{E}$ if and only if the vector $\widehat{\xi} \in \widehat{\mathfrak{E}}$. For any $k \in \{0, \dots, n-2\}$ and $i \in \{1, \dots, n-k-1\}$, we also define the set $\mathfrak{E}_k^i$ by

$$\mathfrak{E}_k^i = \left\{ \xi \in \mathfrak{E} \text{ s. t. } \xi_j = 0 \ \forall j \notin \{k+1, \dots, k+i\} \right\} .$$

$\mathfrak{E}_k^i$ is the set of admissible withdraw policies such that all withdrawals are made between times t_{k+1} and t_{k+i}.

$$\begin{cases} dA_t^p = A_t^p \big[(\mu_t - r_t - \xi_t - p) dt + \sigma_t dB_t \big] \ , & \forall t \notin \ , , \\ A_{t_i}^p = (A_{t_i^-}^p - f_i) \vee 0, & \text{for } 1 \le i \le n-1 \ , \end{cases}$$

where f_i is a $\mathcal{G}_{t_i}$-measurable random variable greater than ξ_i for any $i \in \{1, \dots, n-1\}$ and depending on previous withdrawals, on previous account values and on some guarantees determined in the policy. The simplest case would be to have $f_i = \xi_i$ but variable annuities contracts may be more complex. For instance, for a given withdrawal $\widehat{\xi}_i$ withdraw a larger amount of money from the insured account.

We now focus on the dependencies of f_i and $\widehat{G}_i$. Let $\widehat{g}$ be a bounded non-negative deterministic function, defined on $[0,T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$ such that for any $i \in \{1, \dots, n-1\}$ and $(t, x, e) \in [0,T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$, the function

$$y \mapsto \widehat{g}(t, x, e_1, \dots, e_{i-1}, y, ei+1, \dots, e_{n-1})$$

is non-increasing and for any $j \in \{1, \dots, n+1\}$, the function

$$y \mapsto \widehat{g}(t, x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_{n+1}, e)$$

is non-decreasing. We assume that, for any $i \in \{1, \dots, n-1\}$,

$$f(i, \widehat{A}_{t_0}^p, \dots, \widehat{A}_{t_{i-1}}^p, \widehat{A}_{t_i^-}^p, 0, \dots, 0, \widehat{\xi}_1, \dots, \widehat{\xi}_i, 0, \dots, 0) = \\ e^{\int_0^{t_i} r_s ds} \widehat{f}(i, \widehat{A}_{t_0}^p, \dots, \widehat{A}_{t_{i-1}}^p, \widehat{A}_{t_i^-}^p, 0, \dots, 0, \widehat{\xi}_1, \dots, \widehat{\xi}_i, 0, \dots, 0) \ .$$

The last quantity to define is the pay-off of the variable annuities. Let $\widehat{F}^L$ and $\widehat{F}^D$ be bounded and non-negative functions defined on $[0,T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$ such that for any $Q \in \{L, D\}$, $i \in \{1, \dots, n+1\}$ and $(t, x, e) \in [0,T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$, the following function $y \mapsto \widehat{F}^Q(t, x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_{n+1}, e)$ is non-decreasing and the function $y \mapsto \widehat{F}^Q(t, x, e_1, \dots, e_{i-1}, y, e_{i+1}, \dots, e_{n-1})$ is $widehat F(p, \widehat{\xi})$. non-increasing. The pay-off is paid at time $T \wedge \tau$ to the insured or his dependents and it is equal to the following random variable

$$\widehat{F}(p, \widehat{\xi}) := \widehat{F}^L(T, \widehat{a}^p, \widehat{\xi}) \mathbb{1}_{\{T < \tau\}} + \widehat{F}^D(\tau, \widehat{a}^p, \widehat{\xi}) \mathbb{1}_{\{\tau \| eq T\}} \ , \tag{2.3.1}$$

where $\widehat{a}^p = (\widehat{A}_{t_i \wedge \tau}^p)_{0 \le i \le n}$. $\widehat{F}^L$ is the pay-off if the policyholder is alive at time T and $\widehat{F}^D$ is the pay-off if the policyholder is dead at time τ . Notice that $\widehat{F}(p, \widehat{\xi})$ is $\mathcal{G}_{T \wedge \tau}$-measurable.

In the following, we denote by $F(p, \widehat{\xi})$ the discounted pay-off defined by

$$F(p, \widehat{\xi}) = e^{-\int_0^{T \wedge \tau} r_s ds} \ .$$

2.3.1 GMDB and GMLB contracts

Here we consider the non-negative functions $\widehat{G}^D$, $\widehat{G}^L$ and $\widehat{G}^W$ defined on $[0,T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$ such that for any $Q \in \{L, D, W\}$, $i \in \{1, \ldots, n+1\}$ and $(t, x, e) \in [0,T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$, the function $y \mapsto \widehat{G}^Q(t, x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_{n+1}, e)$ is non-decreasing and the function $y \mapsto \widehat{G}^Q(t, x_1, x, e_1, \ldots, e_{i-1}, y, e_{i+1}, \ldots, e_{n-1})$ is non-increasing . Moreover, for any $Q \in \{D, L\}$, on $[0,T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$, we have $\widehat{F}^Q(t, x, e) = x_{n+1} \vee \widehat{G}^Q(t, x, e)$, and

$$\widehat{g}(t, x, e) = \sum_{i=0}^{n} \left[x_{i+1} \vee \widehat{G}^W(t, x_0, \ldots, x_{i+1}, 0, \ldots, 0, e_1, \ldots, e_{i-1}, 0, \ldots, 0) \right] \mathbb{1}_{t_i \le t < t_{i+1}} .$$

In that case, the penalty function f is often given by

$$f(i, x, e) = \begin{cases} e_i & \text{if } e_i \le G_i \ , \\ G_i + \kappa(e_i - G_i) & \text{if } e_i > G_i \ , \end{cases}$$

where $\kappa > 1$ and $G_i := G^W(t_i, x_0, \ldots, x_{i+1}, 0, \ldots, 0, e_1, \ldots, e_{i-1}, 0, \ldots, 0)$. The insurer takes a fee if the insured withdraws more than the guarantee G_i, this fee is equal to $(\kappa - 1)(e_i - G_i)$.
The usual guarantee functions used to define GMDB and GMLB are listed below (see [BKR08] for more details).

- Constant guarantee : For $i \in \{0, \ldots, n\}$ and $t_i \le t < t_{i+1}$, we se

$$\widehat{G}^Q(t, x, e) = x_1 - \sum_{k=1}^{i} \widehat{f}(k, x, e) \quad \text{on } [0,T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1} .$$

Hence, following the withdraw strategy $\xi \in \mathfrak{E}$, the insured will get

$$F(p, \widehat{\xi}) \quad = \quad A^p_{T\wedge\tau} \vee \left(e^{-\int_0^{T\wedge\tau} r_s ds} \sum_{i=0}^{n} \left(A_0 - \sum_{k=1}^{i} \widehat{f}(k, \widehat{a}^p, \widehat{\xi}) \right) \mathbb{1}_{\{t_i \le T\wedge\tau \le t_{i+1}\}} \right) .$$

- Roll-up guarantee : For $\eta > 0$, $i \in \{0, \ldots, n\}$ and $t_i \le t < t_{i+1}$, we set $\widehat{G}^Q(t, x, e) = x_1(1+\eta)^i - \sum_{k=1}^{i} \widehat{f}(k, x, e)(1+\eta))^{i-k}$ on $[0,T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$, and then if the insured follows the withdraw strategy $\xi \in \mathfrak{E}$, he will get

$$F(p, \widehat{\xi}) = A^p_{T\wedge\tau} \vee \left[e^{-\int_0^{T\wedge\tau} r_s ds} \sum_{i=0}^{n} \left(A_0(1+\eta)^i - \sum_{k=1}^{i} \widehat{f}(k, \widehat{a}^p, \widehat{\xi})(1+\eta)^{i-k} \right) \mathbb{1}_{\{t_i \le T\wedge\tau < t_{i+1}\}} \right] .$$

- Ratchet guarantee : The guarantee depends on the path of A in the following way

$$\widehat{G}^Q(t, x, e) = \sum_{i=0}^{n} \max \left(x_1 - \sum_{k=1}^{i} \widehat{f}(k, x, e), \ldots, x_i - \widehat{f}(i, x, e), x_{i+1} \right) \mathbb{1}_{t_i \le T\wedge\tau < t_{i+1}} ,$$

for any $(t, x, e) \in [0,T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$. The insured will get

$$F(p, \widehat{\xi}) = A^p_{T\wedge\tau} \vee \left(e^{-\int_0^{T\wedge\tau} r_s ds} \sum_{i=0}^{n} \max \widehat{a}^p_0 - \sum_{k=1}^{i} \widehat{f}(k, \widehat{a}^p, \widehat{\xi}), \ldots, \widehat{a}^p_i \right) \mathbb{1}_{t_i \le T\wedge\tau < t_{i+1}} .$$

Remark 2.3.1 We notice that such classical pay-offs are not bounded. Unfortunately, we need to suppose them to be bounded in our approach. From an economical point of view, the boundedness of the pay-offs can be justified by saying that the insurer can provide at the best an amount m which corresponds to her cash account. Therefore, the real pay-off that the insurer can provide is not $F(p,\widehat{\xi})$ but $F(p,\widehat{\xi})\wedge m$.

2.3.2 Utility maximization and indifference pricing

As in the previous sections, if it exists, we look for the fee rate p^* defined as the smallest p such that

$$V^0 := \sup_{\pi\in\mathcal{A}^{\mathbb{F}}[0,T]} \mathbb{E}\big[U\big(X_T^\pi\big)\big] \leq V(p) := \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T]} \inf_{\xi\in\mathfrak{E}} \mathbb{E}\Big[U\Big(A_0+X_T^\pi-\sum_{i=1}^{n-1}\xi_i\mathbb{1}_{\{t_i\leq\tau\}}-F(p,\widehat{\xi})\Big)\Big] \ . \tag{2.3.2}$$

A solution of inequality (2.3.2) will be called an indifference fee rate. Notice that, since the utility function is an exponential function, indifference fee rates will not depend on the initial wealth invested by the insurer but only on the initial deposit A_0 made by the insured. For this reason, we do not consider the initial wealth of the insurer and we assume without loss of generality that her initial endowment is zero.

2.3.3 Simulations

In this section, we present numerical illustrations of parameters sensibility for indifference fee rates. We compute solutions for both optimization problems : V_0, the utility maximization problem without variable annuities, and $V(p)$, the utility maximization problem with variable annuities. We use the numerical method described in [BSCKL15, Section 5] and simulate the BSDEs involved, using the discretization scheme studied in [BT04]. For the computation of the conditional expectations, we use non-parametric regression method with the Gaussian function as kernel. Following a dichotomy method, we find p^* such that

$$p^* = \ \inf\{p\in\mathbb{R}; V_0\leq V(p)\} \ .$$

We consider that the insured can withdraw only every ten years. We shall give the following numerical values to parameters

$$\gamma = 1.3 \ , \quad T = 30 \ , \quad A_0 = 1 \ , \quad r_0 = 0.02 \ , \mu_0 = 0.15 \ , \sigma = 0.3 \ .$$

We describe the dependence with respect to the market parameters : the initial interest rate, the initial drift and the volatility.
Notice that indifference fee rates decrease with respect to the drift up to a certain level. The bigger is the drift the less useful are the guarantees, then the fees payed to get these guarantees decrease. When μ is very big with respect to r and σ then an investor has rather to invest only in the risky assets. Selling guarantees compels the insurer to hedge against interest rates variations. That is why the fees increase when μ is greater than 0.7 in our case.
We notice that indifference fee rates increase with interest rate. This is due to the guarantee structure of the product : a growth of interest rate will lead to a growth of

the quantity $V(p)$ with respect to V^0 and to compensate this growth we will have to increase p, as $p \mapsto V(p)$ is non-increasing.

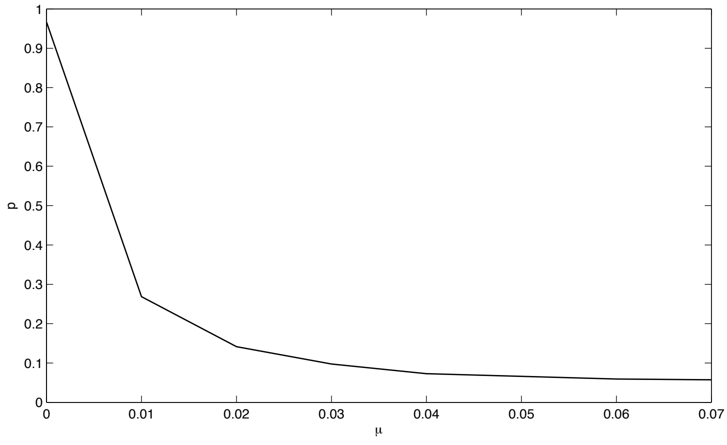

FIGURE 2.11 – Indifference fee rate w.r.t. μ

Notice that indifference fee rates decrease with respect to the drift. The bigger is the drift the less useful are the guarantees, then the fees payed to get these guarantees have to decrease.

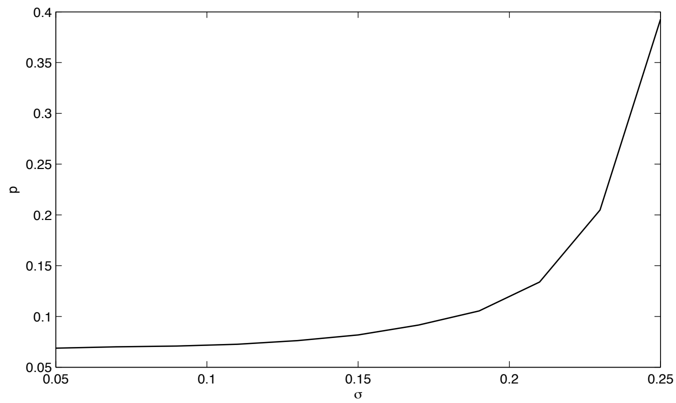

FIGURE 2.12 – Indifference fee rate with respect to σ

Once again, we can get a financial interpretation of the monotonicity of the fees w.r.t. market volatility. The bigger is the volatility the more useful are the guarantees, then the fees payed to get these guarantees have to increase.

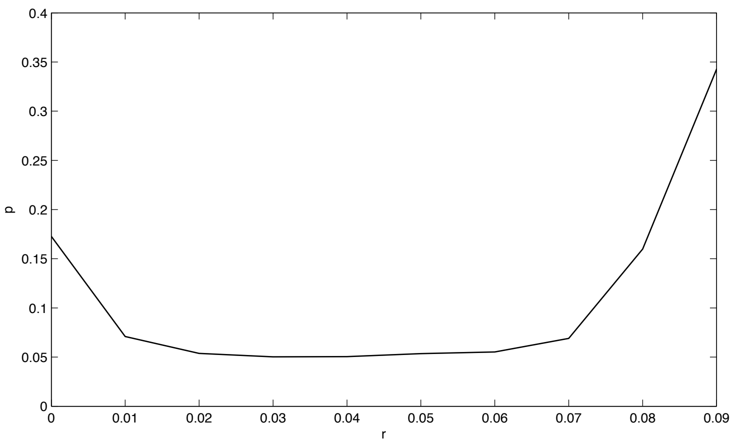

FIGURE 2.13 – Indifference fee rate with respect to r

Appendix

2.3.4 Utility maximization between $T \wedge \tau$ and T

Lemma 2.3.2 *There exists a strategy $\pi^{*,\tau} \in \mathcal{A}^{\mathbb{G}}[T\wedge\tau, T]$ such that*

$$\operatorname*{ess\,inf}_{\pi\in\mathcal{A}^{\mathbb{G}}[T\wedge\tau,T]} \mathbb{E}\big[\exp(-\gamma\Delta X^{\pi}_{\tau,T})|\mathcal{G}_{T\wedge\tau}\big] \quad = \quad \mathbb{E}\big[\exp(-\gamma\Delta X^{\pi^{*,\tau}}_{\tau,T})|\mathcal{G}_{T\wedge\tau}\big] \; .$$

Moreover, there exists a process $Y^{(\tau)}$ such that

$$\operatorname*{ess\,inf}_{\pi\in\mathcal{A}^{\mathbb{G}}[T\wedge\tau,T]} \mathbb{E}\big[\exp(-\gamma\Delta X^{\pi}_{T\wedge\tau,T})|\mathcal{G}_{T\wedge\tau}\big] \quad = \quad \exp(\gamma Y^{(\tau)}_{T\wedge\tau}) \; ,$$

where $(Y^{(\tau)}, Z^{(\tau)})$ is solution of the BSDE

$$\begin{cases} dY^{(\tau)}_t & = \quad \Big[\frac{\nu_t^2}{\gamma} + \nu_t Z^{(\tau)}_t\Big]dt + Z^{(\tau)}_t dB_t \; , \\ Y^{(\tau)}_T & = \quad 0 \; . \end{cases} \tag{2.3.3}$$

PROOF: We look for a process $Y^{(\tau)}$ such that the family of processes $\{J^{(\tau)}(\pi), \pi \in \mathcal{A}^{\mathbb{G}}[T\wedge\tau, T]\}$ defined for any $\pi \in \mathcal{A}^{\mathbb{G}}[T\wedge\tau, T]$ by

$$J^{(\tau)}_t(\pi) \quad := \quad \exp\big(-\gamma(\Delta X^{\pi}_{\tau,t} - Y^{(\tau)}_t)\big)$$

satisfied the following conditions

(i) $J^{(\tau)}_T(\pi) = \exp(-\gamma\Delta X^{\pi}_{\tau,T})$.

(ii) $J^{(\tau)}_{T\wedge\tau}(\pi)$ is a random variable $\mathcal{G}_{T\wedge\tau}$-measurable and independent of π.

(iii) $J^{(\tau)}(\pi)$ is a submartingale for any $\pi \in \mathcal{A}^{\mathbb{G}}[T\wedge\tau, T]$ on the time interval $[T\wedge\tau, T]$.

(iv) There exists a strategy $\pi^{*,\tau}$ such that $J^{(\tau)}(\pi^{*,\tau})$ is a martingale on the time interval $[T\wedge\tau, T]$.

The process $Y^{(\tau)}$ is looked under the form

$$\begin{cases} -dY_t^{(\tau)} &= f(t, Y_t^{(\tau)}, Z_t^{(\tau)})dt - Z_t^{(\tau)}dB_t \,, \\ Y_T^{(\tau)} &= 0 \,, \end{cases}$$

and we are bounded to choose the function f for which $J^{(\tau)}(\pi)$ satisfies the previous conditions. Classically we obtain

$$f(t, y, z) \;=\; -\frac{\nu_t^2}{\gamma} - \nu_t z$$

and the candidate to be $\pi^{*,\tau}$ is given by

$$\pi_t^{*,\tau} \;=\; \frac{1}{\hat{\sigma}_t}\Big[\frac{\nu_t}{\gamma} + Z_t^{(\tau)}\Big] \,, \quad \forall t \in [T \wedge \tau, T] \,.$$

The end of the proof is identical to the one in [HIM05].

2.3.5 Proof of Lemma 2.2.7

We denote the upper bound of the uniformly bounded process $Y^0(p)$ by k. Applying Itô's formula to $(Y^0(p) - k)^2$, we obtain, for any $\mathbb{G}$-stopping times $\theta \leq T$,

$$\big|Y_T^0(p) - k\big|^2 - \big|Y_\theta^0(p) - k\big|^2 \;=\; 2\int_\theta^T (Y_s^0(p) - k)dY_s^0(p) + \int_\theta^T \big|Z_s^0(p)\big|^2 ds \quad (2.3.4)$$

Taking the conditional expected value, we get

$$\begin{aligned} \mathbb{E}\Big[\int_\theta^T \big|Z_s^0(p)\big|^2 ds\Big|\mathcal{G}_\theta\Big] \;&=\; 2\mathbb{E}\Big[\int_\theta^T (k - Y_s^0(p))\Big[\frac{\nu_s^2}{2\gamma} + \nu_s Z_s^0(p) - \lambda_s \frac{e^{\gamma(F_s^{D,W}(p) - Y_s^0(p))} - 1}{\gamma}\Big]\Big|\mathcal{G}_\theta\Big] \\ &\quad + \mathbb{E}\Big[\big|F_T^L(p) - k\big|^2\Big|\mathcal{G}_\theta\Big] - \big|Y_\theta^0(p) - k\big|^2 \,. \end{aligned}$$

Due to Hypothesis **A1** and the fact that $Y^0(p) \in S_{\mathbb{G}}^\infty$, there exist two positive constants c_1 and c_2 such that

$$\begin{aligned} \mathbb{E}\Big[\int_\theta^T \big|Z_s^0(p)\big|^2 ds\Big|\mathcal{G}_\theta\Big] \;&\leq\; c_1 + c_1\mathbb{E}\Big[\int_\theta^T Z_s^0(p)ds\Big|\mathcal{G}_\theta\Big] \\ &\leq\; c_1 + c_1\mathbb{E}\Big[\int_\theta^T \Big(\frac{1}{2c_2}\big|Z_s^0(p)\big|^2 + \frac{c_2}{2}\Big)ds\Big|\mathcal{G}_\theta\Big] \,. \end{aligned}$$

Therefore, there exists a positive constant c such that

$$\mathbb{E}\Big[\int_\theta^T \big|Z_s^0(p)\big|^2 ds\big|\mathcal{G}_\theta\Big] \;\leq\; c \,.$$

Hence $\int_0^\cdot Z_s^0(p)dB_s$ is a BMO($\mathbb{P}$)-martingale. By definition of π^*, Hypothesis **A1** and using the results of [HIM05] for $Z^{(\tau)}$, it follows that $\int_0^\cdot \sigma_s\pi_s^* dB_s$ is a BMO($\mathbb{P}$)-martingale, since the processes μ, σ, γ and r are bounded.

Chapitre 3

BSDEs, filtration enlargement and indifference price of information.

Introduction

Backward Stochastic Differential Equations were introduced by Bismut in [Bis73], then generalized in Pardoux and Peng [PP90]. They have many applications in finance and to optimal control theory [EKPQ97]. Applications to exponential utility problems were introduced by El Karoui and Rouge in [REK00].

The filtration enlargement theory was developed by Jacod, Jeulin and Yor in [JY78a], [JY79], [Jeu80], [Jac85]. These authors give conditions under which, for $\mathbb{F} \subset \mathbb{G}$, any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale (these conditions are named $(\mathcal{H})$- hypothesis) and conditions under which any $\mathbb{F}$-martingale is a $\mathbb{G}$-semimartingale ($(\mathcal{H}')$ hypothesis). They pay attention to the two specific cases of initial and progressive enlargement.

In this chapter, we study the relation of the solutions of BSDEs in two filtrations $\mathbb{F} \subset \mathbb{G}$. First, we define a BSDE in the bigger filtration $\mathbb{G}$ with solution $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, U^{\mathbb{G}}, M^{\mathbb{G}\perp})$. We obtain a BSDE in the smaller filtration $\mathbb{F}$ defined by the projection of $Y^{\mathbb{G}}$ on $\mathbb{F}$ with solution $(Y^{\mathbb{F}}, Z^{\mathbb{F}}, U^{\mathbb{F}})$ and we study their relations. Then, we consider a linear driver and the terminal condition of the $\mathbb{G}$-BSDE and we project them in $\mathbb{F}$ and we define a new BSDE in $\mathbb{F}$ with solution $(\widehat{Y}^{\mathbb{F}}, \widehat{Z}^{\mathbb{F}}, \widehat{U}^{\mathbb{F}})$. We give the relation between $Y^{\mathbb{G}}$ and $\widehat{Y}^{\mathbb{F}}$. The existence of solutions of the BSDEs in the filtration enlarged has been studied for example in [EL05b] or [EL05a] for the case of initial enlargement. For the second part of the chapter, we focus in the indifference price of information. The goal of this section is to find the indifference price of information, i.e. the price at which an agent would have the same expected utility level using extra information as by not doing so. There are other approaches, for example [Laz04] where the author studies the preference for information for a specific class of intertemporal utilities, assuming the existence of a predictable representation property with respect to a standard n-dimensional Brownian motion for each filtration $\mathbb{F} \subset \mathbb{G}$, which is a very restrictive hypothesis. In [CCR14] the authors study if BSDEs under restricted information can be derived from a related problem with BSDEs under full information.

This chapter is organized as follows. In Section 3.1, we present the framework for BSDEs driven by a Brownian motion and a Poisson random measure and the hypo-

theses under which we work. We give also an useful Lemma that will be used to prove the main results of this chapter. We project the solution of a BSDE on a smaller filtration (we refer to [Wu13, Part 1] for more details about projections in different filtrations) and we give its relationship with the original BSDE for a Lipschitz driver. We also project the driver on a smaller filtration, and we define a new BSDE in the smaller filtration and we give the relationship with the original BSDE for the case where the driver is linear.
In Section 3.2, we give the definition of indifference price of information. In order to define and compute the Indifference Price of Information (IPI), we recall the main results of utility optimization. We focus on problems of utility maximization considering an exponential utility. We solve them using BSDEs. Then, we define the Indifference Price of Information (IPI). To find this price, we divide the problem on two utility problems in different filtrations.

Finally, in order to give the indifference price in terms of solutions of BSDEs, we use the results of the first section of this chapter.

Framework

Before introducing BSDEs, we set the probability space and the sets where various parts of the solution of the BSDEs will take values.

Let $(\Omega, \mathbb{P})$ be a probability space and $T \in (0, \infty)$ a fixed terminal time. We define some important spaces associated to a filtration $\mathbb{A}$, or to a σ-algebra $\mathcal{A}$ for two fixed times $0 \leq t_1 < t_2 \leq T$.

– $S^\infty_{\mathbb{A}}[t_1, t_2]$ is the subset of $\mathbb{R}$-valued, càd-làg, $\mathbb{A}$-adapted processes $(a_t)_{t\in[t_1,t_2]}$ essentially bounded i.e.

$$\|a\|_{S^\infty_{\mathbb{G}}} \;:=\; \Big\| \operatorname*{ess\,sup}_{t\in[t_1,t_2]} |a_t| \Big\|_\infty \;<\; \infty\ .$$

– For $i \in \{1, 2\}$, $L^i_{\mathbb{A}}[t_1, t_2]$, the set of $\mathbb{A}$-adapted processes $(a_s)_{s\in[t_1,t_2]}$, such that

$$\int_{t_1}^{t_2} |a_s|^i ds < \infty\ .$$

– $L^i_{\mathbb{A}}\big([t_1, t_2] \times \Omega\big)$, the set of $\mathbb{A}$-adapted processes $(a_s)_{s\in[t_1,t_2]}$, such that

$$\mathbb{E}\Big(\int_{t_1}^{t_2} |a_s|^i ds \Big) < \infty\ .$$

– For $i \in \{1, 2\}$, $L^i(\Omega, \mathcal{A})$, the set of $\mathcal{A}$-measurable random variables ζ, such that

$$\mathbb{E}(|\zeta|^i) < \infty\ .$$

– $\mathcal{M}^2_{\mathbb{A}}\big([t_1, t_2] \times \Omega\big)$, the set of square integrable $\mathbb{A}$-martingales m defined on $[t_1, t_2]$, i.e. $\mathbb{A}$-martingales m such that

$$\sup_{s\in[t_1,t_2]} \mathbb{E}\big(|m_s|^2\big) < \infty\ .$$

3.1 BSDEs in different filtrations

We present the framework for BSDEs driven by a Brownian motion and a Poisson random measure and the hypothesis under which we work. We give also two Lemmas that will be very useful to prove the main results of this chapter. Let $\mathcal{B}(\mathbb{R})$ be the Borel σ-algebra of $\mathbb{R}$. We consider

– a Poisson random measure $\mathbf{N}$ on $([0,T],\mathcal{B}(\mathbb{R}))$ with compensator $\nu(dx,dt) = \lambda(dx)dt$ so that

$$\mathbf{M}([0,t]\times A) := (\mathbf{N}-\nu)([0,t]\times A)\ , \quad t\geq 0$$

is a martingale for all $A\in\mathcal{B}(\mathbb{R})$ satisfying $\lambda(A)<\infty$. Here λ is a σ-finite and positive measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ such that $\int_{\mathbb{R}}(1\wedge x^2)\lambda(dx)<\infty$,

– a Brownian motion $W=(W_t)_{t\geq 0}$ independent of $\mathbf{N}$.

We denote by $\mathbb{F}=(\mathcal{F}_t)_{t\in[0,T]}$ the filtration generated by the Brownian motion W and the Poisson random measure $\mathbf{N}$. Let $\mathbb{G}=(\mathcal{G}_t)_{t\geq 0}$ be a filtration containing $\mathbb{F}$ such that :

Hypotheses 3.1.1 *(i) Every $\mathbb{F}$-martingale is a $\mathbb{G}$-semimartingale. In other terms, the $(\mathcal{H}')$-hypothesis is satisfied.*

(ii) There exists $\mu\in L^2_{\mathbb{G}}([0,T]\times\Omega)$ such that $W_t=W^{\mathbb{G}}_t+\int_0^t\mu_s ds$ for all $t\in[0,T]$, where $W^{\mathbb{G}}$ is a $\mathbb{G}$-Brownian motion.

(iii) There exists a positive σ-finite measure $\lambda^{\mathbb{G}}$ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ satisfying

$$\int_{\mathbb{R}}(1\wedge x^2)\lambda^{\mathbb{G}}(dx)<\infty\,,$$

such that

$$\mathbf{M}^{\mathbb{G}}([0,t]\times A) = (\mathbf{N}-\nu^{\mathbb{G}})([0,t]\times A)\ , \quad t\geq 0$$

is a $\mathbb{G}$-martingale for all $A\in\mathcal{B}(\mathbb{R})$ satisfying $\lambda^{\mathbb{G}}(A)<\infty$ where $\nu^{\mathbb{G}}(dx,dt):=\lambda^{\mathbb{G}}(dx)dt$. Moreover, we assume that $\lambda^{\mathbb{G}}(dx)=\kappa(x)\lambda(dx)$ with $\kappa\in L^2(\Omega,\mathcal{F}_0)$, i.e. an integrable deterministic function.

Notice that we have the following relation between $\mathbf{M}$ and $\mathbf{M}^{\mathbb{G}}$

$$\mathbf{M}^{\mathbb{G}}([0,t]\times A) := \mathbf{M}([0,t]\times A)-(\nu^{\mathbb{G}}-\nu)([0,t]\times A)\ , \quad \forall A\in\mathcal{B}(\mathbb{R}) \quad \text{and} \quad t\geq 0\ .$$

Remark 3.1.2 *The existence of an integrable process μ satisfying (ii) has been studied for example in [Jac85] for the case of initial enlargement of filtration, in [FJS14] for progressive enlargement of filtration, or in the books of [Pro05, Chapter VI], [MY06] and [AJ16]. The hypothesis made over the existence of a square integrable μ in (ii) can be justified in finance to avoid arbitrages of the first kind (see [ACDJ13] and [AJ16]).*

Finally, we introduce a last set where part of the solution of the BSDEs that we are going to study will take values. For $0\leq t_1<t_2\leq T$, we define $H^2_{\mathbb{F}}([t_1,t_2],\mathbf{M})$, the set of all $\mathbb{F}\times\mathcal{B}(\mathbb{R})$-predictable processes U such that

$$\int_{t_1}^{t_2}\int_{\mathbb{R}}U_s^2(x)\lambda(dx)ds<\infty\ ,$$

and $H^2_{\mathbb{G}}([t_1,t_2],\mathbf{M}^{\mathbb{G}})$, the set of all $\mathbb{G}\times\mathcal{B}(\mathbb{R})$-predictable processes U such that

$$\int_{t_1}^{t_2}\int_{\mathbb{R}} U_s^2(x)\lambda^{\mathbb{G}}(dx)ds < \infty \ .$$

The following result will be very useful for the following sections.

Lemma 3.1.3 *Let $\mathbb{F}$ and $\mathbb{G}$ be two filtrations such that $\mathbb{F}\subset\mathbb{G}$. For any process $h\in L^1_{\mathbb{G}}([0,T]\times\Omega)$ we have that $H_t := \int_0^t \mathbb{E}(h_r|\mathcal{F}_r)dr - \mathbb{E}\Big(\int_0^t h_r dr\Big|\mathcal{F}_t\Big)$ for all $t\geq 0$ is an $\mathbb{F}$-martingale.*

Proof: For $t\in[0,T]$ fixed. We have that $H_t\in L^1(\mathcal{F}_t,\Omega)$ and for all $0<s<t$

$$\begin{aligned}\mathbb{E}(H_t|\mathcal{F}_s) &= \mathbb{E}\Big(\int_0^s \mathbb{E}(h_r|\mathcal{F}_r)dr - \mathbb{E}\Big(\int_0^s h_r dr\Big|\mathcal{F}_t\Big)\Big|\mathcal{F}_s\Big)\\ &\quad +\mathbb{E}\Big(\int_s^t \mathbb{E}(h_r|\mathcal{F}_r)dr - \mathbb{E}\Big(\int_s^t h_r dr\Big|\mathcal{F}_t\Big)\Big|\mathcal{F}_s\Big)\ ,\end{aligned}$$

applying Fubini's Theorem and the tower property

$$\mathbb{E}(H_t|\mathcal{F}_s) = \int_0^s \mathbb{E}(h_r|\mathcal{F}_r)dr - \mathbb{E}\Big(\int_0^s h_r dr\Big|\mathcal{F}_s\Big) + \int_s^t \mathbb{E}(h_r|\mathcal{F}_s)dr - \mathbb{E}\Big(\int_s^t h_r dr\Big|\mathcal{F}_s\Big)\ ,$$

finally by Fubini's Theorem, the definition of H and the fact that the two last terms cancel, we obtain

$$\mathbb{E}(H_t|\mathcal{F}_s) = H_s\ .$$

□

3.1.1 Projection of the solution of a BSDE

Let $\mathcal{U}$ be the set of Borelian maps, from $\mathbb{R}$ to $\mathbb{R}$ such that $\int_{\mathbb{R}} u^2(x)\lambda(dx)<\infty$. For a family of $\mathbb{G}$-adapted processes $(f(t,y,z,u),t\geq 0)$ where $y,z\in\mathbb{R},u\in\mathcal{U}$ and a bounded random variable $\xi\in\mathcal{G}_T$, we consider the following $\mathbb{G}$-BSDE

$$Y_t^{\mathbb{G}} = \xi + \int_t^T f\big(s,Y_s^{\mathbb{G}},Z_s^{\mathbb{G}},U_s^{\mathbb{G}}\big)ds - \int_t^T Z_s^{\mathbb{G}}dW_s^{\mathbb{G}} - \int_t^T\int_{\mathbb{R}} U_s^{\mathbb{G}}(x)\mathbf{M}^{\mathbb{G}}(ds,dx) - \int_t^T dM_s^{\mathbb{G}^\perp}\ , \tag{3.1.1}$$

for all $t\in[0,T]$, where the solution, if it exists, is a quadruplet $(Y^{\mathbb{G}},Z^{\mathbb{G}},U^{\mathbb{G}},M^{\mathbb{G}^\perp})\in S^\infty_{\mathbb{G}}[0,T]\times L^2_{\mathbb{G}}([0,T]\times\Omega)\times H^2_{\mathbb{G}}([0,T],\mathbf{M}^{\mathbb{G}})\times\mathcal{M}^2_{\mathbb{G}}([0,T]\times\Omega)$, with $Z^{\mathbb{G}}$ a $\mathbb{G}$-predictable process, $M^{\mathbb{G}^\perp}$ a $\mathbb{G}$-martingale orthogonal to $W^{\mathbb{G}}$ and $\mathbf{M}^{\mathbb{G}}$.

The orthogonal martingale part is needed, unless if we assume that the pair $W^{\mathbb{G}}$ and $\mathbf{M}^{\mathbb{G}}$ enjoys the predictable representation property for the filtration $\mathbb{G}$.

In the case of a BSDE driven by a Brownian motion, if the driver f is linear with respect to z, and satisfies Lipschitz condition, then, using a Girsanov's transformation to remove the part in z, shows that Y is bounded, as soon as ξ is bounded.

In the case with jumps, Becherer [Bec06, Theorem 3.5] gives a general hypothesis which is satisfied in the particular case where the driver satisfies Lipschitz condition and integrability condition and has linear growth with respect to y.

Example 3.1.4 *In the case where $f \equiv 0$ and ξ is bounded in (3.1.1), we are looking for a $\mathbb{G}$-martingale with given terminal value ξ. Consider the bounded $\mathbb{G}$-martingale $Y^{\mathbb{G}} = (Y_t^{\mathbb{G}})_{t\in[0,T]}$ defined by $Y_t^{\mathbb{G}} := \mathbb{E}(\xi|\mathcal{G}_t)$ for all $t \in [0,T]$, and $Y^{\mathbb{G}} \in L^2_{\mathbb{G}}([0,T]\times\Omega)$ then by Galtchouk-Kunita-Watanabe decomposition (see [KW67] and [Gal76]), there exists a triplet of processes $(Z^{\mathbb{G}}, U^{\mathbb{G}}, M^{\mathbb{G}^\perp})$, such that $Z^{\mathbb{G}} \in L^2_{\mathbb{G}}([0,T]\times\Omega)$, $U^{\mathbb{G}} \in H^2_{\mathbb{G}}([0,T], \mathbf{M}^{\mathbb{G}})$ and $M^{\mathbb{G}^\perp} \in \mathcal{M}^2_{\mathbb{G}}([0,T]\times\Omega)$ is a $\mathbb{G}$-martingale orthogonal to $W^{\mathbb{G}}$ and $\mathbf{M}^{\mathbb{G}}$ with $M_0^{\mathbb{G}^\perp} = 0$ and satisfies*

$$Y_t^{\mathbb{G}} = \xi - \int_t^T Z_s^{\mathbb{G}} dW_s^{\mathbb{G}} - \int_t^T \int_{\mathbb{R}} U_s^{\mathbb{G}}(x)\mathbf{M}^{\mathbb{G}}(ds,dx) - \int_t^T dM_s^{\mathbb{G}^\perp} ,$$

for all $t \in [0,T]$.

Assuming that ξ is bounded, that $f(\cdot,y,z,u)$ belongs to $L^2_{\mathbb{G}}([0,T]\times\Omega)$ for all $y,z \in \mathbb{R}, u \in \mathcal{U}$ and satisfies the following Lipschitz condition

$$|f(t,y,z,u) - f(t,\widehat{y},\widehat{z},\widehat{u})| \le C\Big(|y-\widehat{y}| + |z-\widehat{z}| + \Big|\int_{\mathbb{R}} u(x)\lambda(dx) - \int_{\mathbb{R}} \widehat{u}(x)\lambda(dx)\Big|\Big) ,$$

for all $t \in [0,T]$ and that (3.1.1) has a solution $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, U^{\mathbb{G}}, M^{\mathbb{G}^\perp}) \in S^\infty_{\mathbb{G}}[0,T] \times L^2_{\mathbb{G}}([0,T]\times\Omega) \times H^2_{\mathbb{G}}([0,T],\mathbf{M}^{\mathbb{G}}) \times \mathcal{M}^2_{\mathbb{G}}([0,T]\times\Omega)$ with $Z^{\mathbb{G}}$ predictable, one defines $Y_t^{\mathbb{F}} := \mathbb{E}(Y_t^{\mathbb{G}}|\mathcal{F}_t)$ for all $t \in [0,T]$.
Then,

$$\begin{aligned} Y_t^{\mathbb{F}} = Y_0^{\mathbb{F}} &- \mathbb{E}\Big(\int_0^t f\big(s, Y_s^{\mathbb{G}}, Z_s^{\mathbb{G}}, U_s^{\mathbb{G}}\big)ds\Big|\mathcal{F}_t\Big) \\ &+ \mathbb{E}\Big(\int_0^t Z_s^{\mathbb{G}} dW_s^{\mathbb{G}} + \int_0^t U_s^{\mathbb{G}}(x)\mathbf{M}^{\mathbb{G}}(ds,dx) + \int_0^t dM_s^{\mathbb{G}^\perp}\Big|\mathcal{F}_t\Big) , \quad \forall t \in [0,T] . \end{aligned} \tag{3.1.2}$$

Notice that $\Big(\mathbb{E}\Big(\int_0^t Z_s^{\mathbb{G}} dW_s^{\mathbb{G}} + \int_0^t \int_{\mathbb{R}} U_s^{\mathbb{G}}(x)\mathbf{M}^{\mathbb{G}}(ds,dx) + \int_0^t dM_s^{\mathbb{G}^\perp}|\mathcal{F}_t\Big)\Big)_{t\in[0,T]}$ is a square integrable $\mathbb{F}$-martingale since it is the projection of a square integrable $\mathbb{G}$-martingale on $\mathbb{F}$.

Moreover, from Lemma 3.1.3, the process

$$\mathbb{E}\Big(\int_0^t f\big(s, Y_s^{\mathbb{G}}, Z_s^{\mathbb{G}}, U_s^{\mathbb{G}}\big)ds\Big|\mathcal{F}_t\Big) - \int_0^t \mathbb{E}\Big(f\big(s, Y_s^{\mathbb{G}}, Z_s^{\mathbb{G}}, U_s^{\mathbb{G}}\big)\Big|\mathcal{F}_s\Big)ds , \quad \forall t \in [0,T] .$$

is an $\mathbb{F}$-martingale. It is also square integrable, since by the square integrability of f and Jensen's inequality,

$$\begin{aligned} & \mathbb{E}\Big(\mathbb{E}\Big(\int_0^t f\big(s, Y_s^{\mathbb{G}}, Z_s^{\mathbb{G}}, U_s^{\mathbb{G}}\big)ds\Big|\mathcal{F}_t\Big) - \int_0^t \mathbb{E}\Big(f\big(s, Y_s^{\mathbb{G}}, Z_s^{\mathbb{G}}, U_s^{\mathbb{G}}\big)\Big|\mathcal{F}_s\Big)ds\Big)^2 \\ \le\ & \mathbb{E}\Big(\mathbb{E}\Big(\int_0^t f\big(s, Y_s^{\mathbb{G}}, Z_s^{\mathbb{G}}, U_s^{\mathbb{G}}\big)^2 ds\Big|\mathcal{F}_t\Big) + \int_0^t \mathbb{E}\Big(f\big(s, Y_s^{\mathbb{G}}, Z_s^{\mathbb{G}}, U_s^{\mathbb{G}}\big)^2\Big|\mathcal{F}_s\Big)ds\Big) \\ =\ & 2\mathbb{E}\Big(\int_0^t f\big(s, Y_s^{\mathbb{G}}, Z_s^{\mathbb{G}}, U_s^{\mathbb{G}}\big)^2 ds\Big) < \infty . \end{aligned}$$

Then (3.1.2) implies

$$Y_t^{\mathbb{F}} \;=\; Y_0^{\mathbb{F}} - \int_0^t \mathbb{E}\Big(f\big(s, Y_s^{\mathbb{G}}, Z_s^{\mathbb{G}}, U_s^{\mathbb{G}}\big)\Big|\mathcal{F}_s\Big)ds + \mathbb{F}\text{-mtg}\;, \quad \forall t \in [0,T]\;.$$

where $\mathbb{F}$-mtg $\in \mathcal{M}^2_{\mathbb{F}}([0,T]\times\Omega)$. Then, from the predictable representation property (PRP) in $\mathbb{F}$ (see for example [Run03] or [Kun10]), there exists a pair of processes $(Z^{\mathbb{F}}, U^{\mathbb{F}}) \in L^2_{\mathbb{F}}([0,T]\times\Omega)\times H^2_{\mathbb{F}}([0,T],\mathbf{M})$ such that $\mathbb{F}$-mtg $= \int_0^t Z_s^{\mathbb{F}} dW_s + \int_0^t\int_{\mathbb{R}} U_s^{\mathbb{F}}(x)\mathbf{M}(ds,dx)$, therefore

$$Y_t^{\mathbb{F}} = \mathbb{E}(\xi|\mathcal{F}_T) + \int_t^T \widehat{f}_s ds - \int_t^T Z_s^{\mathbb{F}} dW_s - \int_t^T\int_{\mathbb{R}} U_s^{\mathbb{F}}(x)\mathbf{M}(ds,dx)\;, \tag{3.1.3}$$

where $\widehat{f}_s := \mathbb{E}\big[f(s, Y_s^{\mathbb{G}}, Z_s^{\mathbb{G}}, U_s^{\mathbb{G}})|\mathcal{F}_s\big]$ for all $s\in[0,T]$.

The goal is now to give a relationship between $(Z^{\mathbb{F}}, U^{\mathbb{F}})$ and $(Z^{\mathbb{G}}, U^{\mathbb{G}})$.

Theorem 3.1.5 *If the equation (3.1.1) with $\mathbb{G}$-adapted bounded process f and a bounded random variable $\xi\in\mathcal{G}_T$, has a solution $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, U^{\mathbb{G}}) \in S^\infty_{\mathbb{G}}[0,T]\times L^2_{\mathbb{G}}([0,T]\times\Omega)\times H^2_{\mathbb{G}}([0,T],\mathbf{M}^{\mathbb{G}})$, then the processes $Z^{\mathbb{F}}$ and $U^{\mathbb{F}}$ in (3.1.3) are given by*

- $Z_t^{\mathbb{F}} = \mathbb{E}\Big(Z_t^{\mathbb{G}} + \mu_t Y_{t-}^{\mathbb{G}}\Big|\mathcal{F}_t\Big)$, *for all* $t\in[0,T]$.
- $U_t^{\mathbb{F}}(x) = \mathbb{E}\Big(U_t^{\mathbb{G}}(x)\kappa(x) + Y_{t-}^{\mathbb{G}}(\kappa(x)-1)\big|\mathcal{F}_t\Big)$, *for all* $t\in[0,T]$ *and* $x\in\mathbb{R}$.

Proof: We consider a bounded $\mathcal{F}_T$-measurable random variable F_T of the form $F_T = \int_0^T \psi_t dW_t + \int_0^T\int_{\mathbb{R}}\rho_t(x)\mathbf{M}(dt,dx)$ where $(\psi_t)_{t\in[0,T]}$ and for any $x\in\mathbb{R}$ $(\rho_t(x))_{t\in[0,T]}$ are $\mathbb{F}$-predictable processes. Since F_T is square-integrable we have that $\psi\in L^2_{\mathbb{G}}([0,T]\times\Omega)$ and $\rho\in H^2_{\mathbb{G}}([0,T],\mathbf{M}^{\mathbb{G}})$. We define an $\mathbb{F}$-bounded martingale J by $J_t := \mathbb{E}(F_T|\mathcal{F}_t) = \int_0^t\psi_s dW_s + \int_0^t\int_{\mathbb{R}}\rho_s(x)\mathbf{M}(ds,dx)$ for all $t\in[0,T]$.
We now divide the proof in four steps :

Step 1. In a first step we compute $\mathbb{E}(Y_T^{\mathbb{F}} J_T)$. By integration by parts, we get

$$\begin{aligned}\mathbb{E}(Y_T^{\mathbb{F}} J_T) \;&=\; \mathbb{E}\Big(\int_0^T Y_{t-}^{\mathbb{F}}\psi_t dW_t + \int_0^T\int_{\mathbb{R}} Y_{t-}^{\mathbb{F}}\rho_t(x)\mathbf{M}(dt,dx) + \int_0^T J_{t-}dY_t^{\mathbb{F}}\Big)\\ &\quad+\mathbb{E}\Big(\int_0^T Z_t^{\mathbb{F}}\psi_t dt + \int_0^T\int_{\mathbb{R}} U_t^{\mathbb{F}}(x)\rho_t(x)\mathbf{N}(dt,dx)\Big)\;;\end{aligned}$$

using (3.1.3) we get

$$\int_0^T J_{t-}dY_t^{\mathbb{F}} \;=\; -\mathbb{E}\Big(\int_0^T J_t\widehat{f}_t dt - \int_0^T J_{t-}Z_t^{\mathbb{F}} dW_t - \int_0^T\int_{\mathbb{R}} J_{t-}U_t^{\mathbb{F}}(x)\mathbf{M}(dt,dx)\Big)\;,$$

where $\widehat{f}_t := \mathbb{E}(f(t, Y_t^{\mathbb{G}}, Z_t^{\mathbb{G}}, U_t^{\mathbb{G}})|\mathcal{F}_t)$ for all $t\in[0,T]$.
The processes $(Y_{t-}^{\mathbb{F}}\psi_t)_{t\in[0,T]}$, $(J_{t-}Z_t^{\mathbb{F}})_{s\in[0,T]}$ (resp. $(Y_{t-}^{\mathbb{F}}\rho_t(x))_{t\in[0,T]}$ and $(J_{t-}U_t^{\mathbb{F}}(x))_{t\in[0,T]}$) are $\mathbb{F}$-predictable, and since Y and J are bounded, these processes belong to $L^2([0,T]\times\Omega)$ (resp. to $H^2([0,T]\times\mathbf{M})$).
Then, the processes

$$\int_0^t J_{s-}Z_s^{\mathbb{F}} dW_s \quad\text{and}\quad \int_0^t\int_{\mathbb{R}} J_{s-}U_s^{\mathbb{F}}(x)\mathbf{M}(ds,dx)\;,\;\; \forall t\in[0,T]$$

are $\mathbb{F}$-martingales.

Hence,

$$\mathbb{E}(Y^{\mathbb{F}}_T J_T) \;=\; -\mathbb{E}\Big(\int_0^T J_t\widehat{f}_t dt\Big) + \mathbb{E}\Big(\int_0^T Z^{\mathbb{F}}_t\psi_t dt + \int_0^T\int_{\mathbb{R}} U^{\mathbb{F}}_t(x)\rho_t(x)\mathbf{N}(dt,dx)\Big)\ .$$

Using that the boundedness of Y imply that of U, we see that

$$\Big(\int_0^t\int_{\mathbb{R}} U^{\mathbb{F}}_s(x)\rho_s(x)\mathbf{M}(ds,dx)\Big)_{t\in[0,T]}$$

is an $\mathbb{F}$-martingale. Therefore, we obtain

$$\mathbb{E}(Y^{\mathbb{F}}_T J_T) \;=\; -\mathbb{E}\Big(\int_0^T J_t\widehat{f}_t dt\Big) + \mathbb{E}\Big(\int_0^T Z^{\mathbb{F}}_t\psi_t dt + \int_0^T\int_{\mathbb{R}} U^{\mathbb{F}}_t(x)\rho_t(x)\nu(dt,dx)\Big)\ .$$

Step 2. In a second step we compute $\mathbb{E}(Y^{\mathbb{G}}_T J_T)$. Recalling that $dW_t = dW^{\mathbb{G}}_t + \mu_t dt$ and $\mathbf{M}(dt,dx) = \mathbf{M}^{\mathbb{G}}(dt,dx) + (\nu^{\mathbb{G}} - \nu)(dt,dx)$, and using integration by parts, we get

$$\begin{aligned}
\mathbb{E}(Y^{\mathbb{G}}_T J_T) \;=\; & \mathbb{E}\Big(\int_0^T Y^{\mathbb{G}}_{t-}\psi_t dW^{\mathbb{G}}_t + \int_0^T\int_{\mathbb{R}} Y^{\mathbb{G}}_{t-}\rho_t(x)\mathbf{M}^{\mathbb{G}}(dt,dx)\Big)\\
&+\mathbb{E}\Big(\int_0^T Y^{\mathbb{G}}_t\mu_t\psi_t dt + \int_0^T\int_{\mathbb{R}} Y^{\mathbb{G}}_{t-}\rho_t(x)(\nu^{\mathbb{G}} - \nu)(dt,dx)\Big)\\
&+\mathbb{E}\Big(\int_0^T J_{t-}dY^{\mathbb{G}}_t\Big) + \mathbb{E}\Big(\int_0^T Z^{\mathbb{G}}_t\psi_t dt + \int_0^T\int_{\mathbb{R}} U^{\mathbb{G}}_t(x)\rho_t(x)\mathbf{N}(dt,dx)\Big)\ .
\end{aligned}$$

From (3.1.1), we obtain

$$\begin{aligned}
\mathbb{E}\Big(\int_0^T J_{t-}dY^{\mathbb{G}}_t\Big) \;=\; & \mathbb{E}\Big(-\int_0^T J_t f\big(t, Y^{\mathbb{G}}_t, Z^{\mathbb{G}}_t, U^{\mathbb{G}}_t\big)dt + \int_0^T J_{t-}Z^{\mathbb{G}}_t dW^{\mathbb{G}}_t\Big)\\
&+\mathbb{E}\Big(\int_0^T\int_{\mathbb{R}} J_{t-}U^{\mathbb{G}}_t(x)\mathbf{M}^{\mathbb{G}}(dt,dx) + \int_0^T J_{t-}dM^{\perp}_t\Big)\ .
\end{aligned}$$

Note that $Y\psi\mu$ is integrable (we use again that Y is bounded). By definition $(Y^{\mathbb{G}}_{t-}\phi_t)_{t\in[0,T]}$, $(J_{t-}Z^{\mathbb{G}}_t)_{t\in[0,T]}$, and $(J_{t-})_{t\in[0,T]}$ (resp. $(Y^{\mathbb{G}}_{t-}\rho_t(x))_{t\in[0,T]}$ and $(J_{t-}U^{\mathbb{G}}_t(x))_{t\in[0,T]}$), are $\mathbb{G}$-predictable and due to the boundedness property of Y and J, they belong to $L^2_{\mathbb{G}}([0,T]\times\Omega)$ (resp. to $H^2_{\mathbb{G}}([0,T]\times\mathbf{M})$. Then, the local martingales which appear are martingales and

$$\begin{aligned}
\mathbb{E}(Y^{\mathbb{G}}_T J_T) \;=\; & \mathbb{E}\Big(\int_0^T Y^{\mathbb{G}}_t\psi_t\mu_t dt + \int_0^T\int_{\mathbb{R}} Y^{\mathbb{G}}_{t-}\rho_t(x)(\nu^{\mathbb{G}} - \nu)(dt,dx)\Big)\\
&+\mathbb{E}\Big(-\int_0^T J_t f_t dt + \int_0^T Z^{\mathbb{G}}_t\psi_t dt + \int_0^T\int_{\mathbb{R}} U^{\mathbb{G}}_t(x)\rho_t(x)\mathbf{N}(dt,dx)\Big)\\
\;=\; & \mathbb{E}\Big(\int_0^T Y^{\mathbb{G}}_t\psi_t\mu_t dt + \int_0^T\int_{\mathbb{R}} Y^{\mathbb{G}}_{t-}\rho_t(x)(\nu^{\mathbb{G}} - \nu)(dt,dx)\Big)\\
&+\mathbb{E}\Big(-\int_0^T J_t f_t dt + \int_0^T Z^{\mathbb{G}}_t\psi_t dt + \int_0^T\int_{\mathbb{R}} U^{\mathbb{G}}_t(x)\rho_t(x)\nu^{\mathbb{G}}(dt,dx)\Big)\ ,
\end{aligned}$$

where $f_t := f(t, Y^{\mathbb{G}}_t, Z^{\mathbb{G}}_t, U^{\mathbb{G}}_t)$ for all $t\in[0,T]$.

Step 3. In a third step we show that $\mathbb{E}\Big(\int_0^T J_t\widehat{f}_t dt\Big) = \mathbb{E}\Big(\int_0^T J_t f_t dt\Big)$. First, by definition of $\widehat{f}$ and by Fubini's Theorem,

$$\begin{aligned}
\mathbb{E}\Big(\int_0^T J_t\widehat{f}_t dt\Big) &= \mathbb{E}\Big(\int_0^T J_t\mathbb{E}(f(t,Y_t^{\mathbb{G}},Z_t^{\mathbb{G}},U_t^{\mathbb{G}})|\mathcal{F}_t)dt\Big) \\
&= \int_0^T \mathbb{E}\Big(J_t\mathbb{E}(f(t,Y_t^{\mathbb{G}},Z_t^{\mathbb{G}},U_t^{\mathbb{G}})|\mathcal{F}_t)\Big)dt\ , \\
&= \int_0^T \mathbb{E}\Big(J_t f(t,Y_t^{\mathbb{G}},Z_t^{\mathbb{G}},U_t^{\mathbb{G}})\Big)dt = \mathbb{E}\Big(\int_0^T J_t f_t dt\Big)\ .
\end{aligned}$$

Step 4. By definition, we have that $Y_t^{\mathbb{F}} = \mathbb{E}(Y_t^{\mathbb{G}}|\mathcal{F}_t)$, which, using the fact that $J_T \in \mathcal{F}_T$, implies that $\mathbb{E}(Y_T^{\mathbb{F}}J_T) = \mathbb{E}\big(Y_T^{\mathbb{G}}J_T\big)$, then by the previous steps we get

$$\begin{aligned}
&\mathbb{E}\Big(\int_0^T Y_t^{\mathbb{G}}\mu_t\psi_t dt + \int_0^T\int_{\mathbb{R}} Y_{t-}^{\mathbb{G}}\rho_t(x)(\nu^{\mathbb{G}}-\nu)(dt,dx)\Big) \\
&\quad +\mathbb{E}\Big(\int_0^T Z_t^{\mathbb{G}}\psi_t dt + \int_0^T\int_{\mathbb{R}} U_t^{\mathbb{G}}(x)\rho_t(x)\nu^{\mathbb{G}}(dt,dx)\Big) \\
&\quad = \mathbb{E}\Big(\int_0^T Z_t^{\mathbb{F}}\psi_t dt + \int_0^T\int_{\mathbb{R}} U_t^{\mathbb{F}}(x)\rho_t(x)\nu(dt,dx)\Big)\ .
\end{aligned}$$

Using that $\lambda^{\mathbb{G}}(dx) = \kappa(x)\lambda(dx)$, we get

$$\begin{aligned}
&\mathbb{E}\Big(\int_0^T (Z_t^{\mathbb{G}}+\mu_t Y_t^{\mathbb{G}})\psi_t dt\Big) + \mathbb{E}\Big(\int_0^T\int_{\mathbb{R}}\big(U_t^{\mathbb{G}}(x)\kappa(x) + Y_{t-}^{\mathbb{G}}(\kappa(x)-1)\big)\rho_t(x)\lambda(dx)dt\Big) \\
&\qquad = \mathbb{E}\Big(\int_0^T Z_t^{\mathbb{F}}\psi_t dt + \int_0^T\int_{\mathbb{R}} U_t^{\mathbb{F}}(x)\rho_t(x)\lambda(dx)dt\Big)\ .
\end{aligned}$$

Finally, Fubini's Theorem and the tower property lead to

$$\begin{aligned}
&\mathbb{E}\Big(\int_0^T \big(Z_t^{\mathbb{F}} - \mathbb{E}\big(Z_t^{\mathbb{G}} + \mu_t Y_{t-}^{\mathbb{G}}|\mathcal{F}_t\big)\big)\psi_t dt\Big) + \\
&\mathbb{E}\Big(\int_{\mathbb{R}}\int_0^T \big(U_t^{\mathbb{F}}(x) - \mathbb{E}\big(U_t^{\mathbb{G}}(x)\kappa(x) + Y_{t-}^{\mathbb{G}}\big(\kappa(x)-1\big)\big|\mathcal{F}_t\big)\big)\rho_t(x)\lambda(dx)dt\Big) = 0\ .
\end{aligned} \tag{3.1.4}$$

The equation (3.1.4) is true for any $\mathbb{F}$-predictable processes (ψ,ρ) such that J is bounded, and extends to any pair in $L^2_{\mathbb{F}}([0,T]\times\Omega)\times H^2_{\mathbb{F}}([0,T],\mathbf{M})$ (using a localizing procedure if the martingale J is only square integrable), hence if we take

$$\psi_t = \mathbb{1}_{\{Z_t^{\mathbb{F}} - \mathbb{E}\big(Z_t^{\mathbb{G}}+\mu_t Y_{t-}^{\mathbb{G}}\big|\mathcal{F}_t\big)>0\}} - \mathbb{1}_{\{Z_t^{\mathbb{F}} - \mathbb{E}\big(Z_t^{\mathbb{G}}+\mu_t Y_{t-}^{\mathbb{G}}\big|\mathcal{F}_t\big)<0\}}$$

and $\rho_t(x) = 0$ for all $t\in[0,T]$ and $x\in E$, we get

$$\mathbb{E}\Big(\int_0^T \Big|Z_t^{\mathbb{F}} - \mathbb{E}\big(Z_t^{\mathbb{G}} + \mu_t Y_{t-}^{\mathbb{G}}\big|\mathcal{F}_t\big)\Big|dt\Big) = 0\ ,$$

therefore

$$Z_t^{\mathbb{F}} = \mathbb{E}\big(Z_t^{\mathbb{G}} + \mu_t Y_{t-}^{\mathbb{G}}\big|\mathcal{F}_t\big)\ ,\quad \forall t\in[0,T]\ .$$

Analogously, if we take $\psi_t = 0$ and

$$\rho_t(x) = \mathbb{1}_{\{U_t^{\mathbb{F}}(x) - \mathbb{E}(U_t^{\mathbb{G}}(x)\kappa(x) + Y_{t-}^{\mathbb{G}}(\kappa(x)-1)|\mathcal{F}_t) > 0\}} - \mathbb{1}_{\{U_t^{\mathbb{F}}(x) - \mathbb{E}(U_t^{\mathbb{G}}(x)\kappa(x) + Y_{t-}^{\mathbb{G}}(\kappa(x)-1)|\mathcal{F}_t) < 0\}} ,$$

for all $t \in [0,T]$ and for all $x \in E$. Then, we obtain

$$\mathbb{E}\Big(\int_{\mathbb{R}} \int_0^T \Big| U_t^{\mathbb{F}}(x) - \mathbb{E}\big(U_t^{\mathbb{G}}(x)\kappa(x) + Y_{t-}^{\mathbb{G}}\big(\kappa(x) - 1\big)\big|\mathcal{F}_t\big) \Big| \lambda(dx)dt \Big) = 0 ,$$

it follows that

$$U_t^{\mathbb{F}}(x) = \mathbb{E}(U_t^{\mathbb{G}}(x)\kappa(x) + Y_{t-}^{\mathbb{G}}(\kappa(x) - 1)\big|\mathcal{F}_t) , \quad \forall t \in [0,T] \text{ and } \forall x \in \mathbb{R} .$$

□

3.1.2 Projection of the driver of a BSDE

In this subsection, we study a different problem, instead considering the projection of the solution of a $\mathbb{G}$-BSDE to obtain a BSDE in $\mathbb{F}$, we consider the $\mathbb{F}$-BSDE given by the projection of the driver and the terminal condition of a linear $\mathbb{G}$-BSDE. More precisely, we focus to BSDEs with linear driver with $\mathbb{F}$-adapted coefficients i.e. we assume that $\alpha, \beta, \gamma(x)$ and δ are $\mathbb{F}$-adapted bounded processes, and we focus our attention on the following linear BSDEs

$$\begin{cases} -d\widehat{Y}_t^{\mathbb{F}} = & \big(\alpha_t \widehat{Y}_t^{\mathbb{F}} + \beta_t \widehat{Z}_t^{\mathbb{F}} + \int_{\mathbb{R}} \gamma_t(x) \widehat{U}_t^{\mathbb{F}}(x)\lambda(dx) + \delta_t\big)dt \\ & -\widehat{Z}_t^{\mathbb{F}} dW_t - \int_{\mathbb{R}} \widehat{U}_t^{\mathbb{F}}(x)\mathbf{M}(dt,dx) , \ \forall t \in [0,T] \\ \widehat{Y}_T^{\mathbb{F}} = & \mathbb{E}(\xi|\mathcal{F}_T) , \end{cases} \tag{3.1.5}$$

$$\begin{cases} -dY_t^{\mathbb{G}} = \big[\alpha_t Y_t^{\mathbb{G}} + \beta_t Z_t^{\mathbb{G}} + \displaystyle\int_{\mathbb{R}} \gamma_t(x) U_t^{\mathbb{G}}(x)\lambda^{\mathbb{G}}(dx) + \delta_t\big]dt \\ \qquad - Z_t^{\mathbb{G}} dW_t^{\mathbb{G}} - \displaystyle\int_{\mathbb{R}} U_t^{\mathbb{G}}(x)\mathbf{M}^{\mathbb{G}}(dt,dx) - dM_t^{\perp}, \ \ \forall t \in [0,T] \\ Y_T^{\mathbb{G}} = \xi , \end{cases} \tag{3.1.6}$$

where $\xi \in \mathcal{G}_T$ is bounded.

Theorem 3.1.6 *The following relation is satisfied*

$$\widehat{Y}_t^{\mathbb{F}} = Y_t^{\mathbb{F}} + \mathbb{E}\Big(\int_t^T L_{t,s} Y_s^{\mathbb{G}} (\beta_s \mu_s - \int_{\mathbb{R}} \gamma_s(x)(\kappa(x) - 1)\lambda(dx))ds \Big| \mathcal{F}_t \Big) , \quad \forall t \in [0,T] ,$$

where for $t \in [0,T]$, $Y_t^{\mathbb{F}} = \mathbb{E}(Y_t^{\mathbb{G}}|\mathcal{F}_t)$ and the process $(L_{t,s})_{s \in [t,T]}$ is the unique solution of the following stochastic differential equation

$$dL_{t,s} = L_{t,s^-}(\alpha_s ds + \beta_s dW_s + \int_{\mathbb{R}} \gamma_s(x)\mathbf{M}(ds,dx))$$

and $L_{t,t} = 1$.

PROOF: Let $t \in [0,T]$ fixed. We recall that

$$Y_t^{\mathbb{F}} = \mathbb{E}(\xi|\mathcal{F}_T) + \int_t^T \widehat{f}_s ds - \int_t^T Z_s^{\mathbb{F}} dW_s - \int_t^T \int_{\mathbb{R}} U_s^{\mathbb{F}}(x)\mathbf{M}(ds,dx) ,$$

where $\widehat{f}_s := \mathbb{E}\big[\alpha_s Y_s^{\mathbb{G}} + \beta_s Z_s^{\mathbb{G}} + \int_{\mathbb{R}} \gamma_s(x) U_s^{\mathbb{G}}(x)\lambda^{\mathbb{G}}(dx) + \delta_s|\mathcal{F}_s\big]$ for all $s \in [0,T]$. Applying Theorem 3.1.5, we have that $\mathbb{E}(Z_s^{\mathbb{G}}|\mathcal{F}_s) = Z_s^{\mathbb{F}} - \mathbb{E}(\mu_s Y_{s-}^{\mathbb{G}}|\mathcal{F}_s)$ and $U_s^{\mathbb{F}}(x) = \mathbb{E}\Big(U_s^{\mathbb{G}}(x)\kappa(x) + Y_{s-}^{\mathbb{G}}(\kappa(x)-1)\big|\mathcal{F}_s\Big)$, for all $x \in E$ and $s \in [t,T]$, we obtain

$$\begin{aligned} \widehat{f}_s = \alpha_s Y_s^{\mathbb{F}} + \beta_s Z_s^{\mathbb{F}} + \int_{\mathbb{R}} \gamma_s(x) U_s^{\mathbb{F}}(x)\lambda(dx) + \delta_s + \beta_s \mathbb{E}(\mu_s Y_{s-}^{\mathbb{G}}|\mathcal{F}_s) \\ -\mathbb{E}\Big[\int_{\mathbb{R}} \gamma_s(x) Y_{s-}^{\mathbb{G}}(\kappa(x)-1)\lambda(dx)\big|\mathcal{F}_s\Big] , \quad \forall s \in [t,T] . \end{aligned} \tag{3.1.7}$$

Consider $\overline{Y} := Y^{\mathbb{F}} - \widehat{Y}^{\mathbb{F}}$, $\overline{Z} := Z^{\mathbb{F}} - \widehat{Z}^{\mathbb{F}}$ and $\overline{U} := U^{\mathbb{F}} - \widehat{U}^{\mathbb{F}}$, then from (3.1.7) and (3.1.5)

$$\begin{aligned} \overline{Y}_t = \int_t^T \big(\alpha_s \overline{Y}_s + \beta_s \overline{Z}_s + \int_{\mathbb{R}} \gamma_s(x)\overline{U}_s(x)\lambda(dx) + \beta_s \mathbb{E}(\mu_s Y_{s-}^{\mathbb{G}}|\mathcal{F}_s) \\ - \gamma_s \mathbb{E}\Big[\int_{\mathbb{R}} Y_{s-}^{\mathbb{G}}(\kappa(x)-1)\lambda(dx)\big|\mathcal{F}_s\Big]\Big) ds - \int_t^T \overline{Z}_s dW_s - \int_t^T \int_{\mathbb{R}} \overline{U}_s(x)\mathbf{M}(ds,dx) , \end{aligned} \tag{3.1.8}$$

since this is a linear BSDE with terminal solution $\overline{Y}_T = 0$, the solution is unique and explicit see [QS13]), given by

$$\overline{Y}_t = -\mathbb{E}\Big(\int_t^T L_{t,s^-}\Big[- \beta_s \mathbb{E}(\mu_s Y_{s-}^{\mathbb{G}}|\mathcal{F}_s) + \mathbb{E}\Big(\int_{\mathbb{R}} \gamma_s(x) Y_{s-}^{\mathbb{G}}(\kappa(x)-1)\lambda(dx)\big|\mathcal{F}_s\Big)\Big] ds\big|\mathcal{F}_t\Big) .$$

The result follows. □

3.2 Indifference price of information

In this section, we show some applications in finance of the main results of Section 3.1. In Subsection 3.2.1, we set the financial background of our application for financial market, we give also the hypothesis on the probability space where we work. In order to define and compute the Indifference Price of Information (IPI), we present and solve, in Subsection 3.2.2, the utility maximization problems involved using BSDEs theory. In Subsection 3.2.3, we define the IPI, this concept gives a link between the information and its potential value for a particular contingent claim. We study its properties and different extensions.

3.2.1 Financial market and probability space

Let $(\Omega, \mathbb{P})$ be a probability space and W a Brownian motion with natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$.

We consider a financial market on the time interval $[0,T]$, where the terminal state T is a positive constant. In order to simplify our model, we suppose that the financial market is composed by a riskless bond with an interest rate $r=0$, i.e., $S^0_t=1$ and a risky asset. The price process S of the risky asset, is assumed to be solution of the following linear stochastic differential equation

$$dS_t \;=\; S_t(\alpha_t dt+\sigma_t dW_t)\;,\quad \forall t\in[0,T]\;,\quad S_0=s>0\;,$$

where α and σ are $\mathbb{F}$-predictable processes.

Hypotheses 3.2.1 *We suppose the following assumptions :*

1. *The processes α and σ are bounded.*
2. *The process σ is lower bounded by a positive constant $\underline{\sigma}$.*
3. *The filtration $\mathbb{G}$ is larger than $\mathbb{F}$ and $(\mathcal{H}')$ holds, i.e., every $\mathbb{F}$-martingale is a $\mathbb{G}$-semimartingale.*
4. *There exists μ, a bounded $\mathbb{G}$ adapted process such that $W_t=W^{\mathbb{G}}_t+\int_0^t\mu_s ds$ for all $t\in[0,T]$, where $W^{\mathbb{G}}$ is a $\mathbb{G}$-Brownian motion.*

Under these hypotheses, the solution of $dL_t=-L_t\frac{\alpha_t+\mu_t\sigma_t}{\sigma_t}dW^{\mathbb{G}}_t$, $L_0=1$ is a $\mathbb{G}$ martingale. The process SL is a $\mathbb{G}$ martingale, in other terms L is an e.m.m and the No Free lunch with Vanishing Risk condition is satisfied, hence there are no arbitrages in the filtration $\mathbb{G}$. The strong hypothesis on boundedness of μ will allow us to have uniqueness of the solution of the BSDEs we shall study.

For $t\in[0,T]$, we denote by π^0_t (resp. π_t) the amount of money invested in the riskless bond (resp. the risky asset). We study the strategies in the two different filtrations $\mathbb{F}$ and $\mathbb{G}$.

We define the set of admissible strategies on a time interval $[t_1,t_2]$.

Definition 3.2.2 ***Admissible strategies.*** *Let $0\le t_1\le t_2\le T$. The set of admissible trading strategies $\mathcal{A}^{\mathbb{F}}[t_1,t_2]$ (resp. $\mathcal{A}^{\mathbb{G}}[t_1,t_2]$) consists of all $\mathbb{F}$-predictable processes $\pi^{\mathbb{F}}$ (resp. $\mathbb{G}$-predictable processes $\pi^{\mathbb{G}}$) $\pi^{\cdot}=(\pi^{\cdot}_t)_{t_1\le t\le t_2}$, which satisfy*

$$\int_{t_1}^{t_2}|\pi^{\mathbb{F}}_s\alpha_s|ds<\infty\;,\quad (\text{resp. }\int_{t_1}^{t_2}|\pi^{\mathbb{G}}_s(\alpha_s+\mu_s\sigma_s)|ds<\infty\;),\quad \mathbb{E}\Big[\int_{t_1}^{t_2}\big|\pi^{\cdot}_t\sigma_t\big|^2dt\Big]<\infty\;,$$

and

$$\big\{\exp(-\gamma X^{x,\pi}_\theta),\ \theta \text{ is an } \mathbb{F}\text{-stopping (resp. a } \mathbb{G}\text{-stopping time) with values in } [t_1,t_2]\big\}$$

is uniformly integrable. Here, $X^{x,\pi}_t$ is the value at time t of the strategy π, with initial capital $x\in\mathbb{R}^+$

$$X^{x,\pi}_t \;=\; x+\int_0^t\pi_s\alpha_s ds+\int_0^t\pi_s\sigma_s dW_s\;,\quad \forall t\in[0,T]\;,$$

where the stochastic integral has to be understood as with respect to the $\mathbb{G}$-semimartingale W, in case of $\mathbb{G}$-adapted strategy, equivalently

$$X^{x,\pi}_t \;=\; x+\int_0^t\pi_s(\alpha_s+\mu_s\sigma_s)ds+\int_0^t\pi_s\sigma_s dW^{\mathbb{G}}_s\;,\quad \forall t\in[0,T]\;.$$

If $t_1=0$ and $t_2=T$, we will denote by $\mathcal{A}^{\mathbb{F}}$ (resp. $\mathcal{A}^{\mathbb{G}}$) the set of admissible strategies instead of $\mathcal{A}^{\mathbb{F}}[0,T]$ (resp. $\mathcal{A}^{\mathbb{G}}[0,T]$) .

If the initial capital is null we denote X_t^π the wealth instead of $X_t^{0,\pi}$.

We suppose that at the terminal time T, the agent pays an amount ξ, where ξ is a $\mathcal{G}_T$-measurable bounded random variable. Our goal is to find the optimal hedge and price for ξ using the exponential utility.

We consider that the agent wants to maximize the expected value of the utility of his terminal wealth on an admissible strategies set, where the utility function is given by $U(x) := -\exp(-\gamma x)$ for all $x \in \mathbb{R}$ with $\gamma > 0$.

Notice that the uniform integrability condition in Definition 3.2.2 coincides with the notion of class D, which is equivalent to have the Doob-Meyer Decomposition (see [DM78, Chapter VII, 12] or [Pro05, Chapter III]) which is a key element to find the value function of optimization problems with exponential utility (see [HIM05] and [Del13, Chapter 11]).

3.2.2 Utility maximization

Before introducing the definition of indifference price of information, we study the problem of utility exponential maximization over the setup of the financial market of Subsection 3.2.1 using BSDEs tools defined in different filtrations. We use BSDEs for many reasons, the first one is because, we apply this methodology for incomplete markets, for which we do not have a closed form for the optimal terminal wealth (see [REK00] or [HIM05]). We shall face the incomplete market case in progressive enlargement case. The second one, is that we can characterize the value function corresponding to the utility maximization problem as a solution of a BSDE.

We suppose for this section that the Hypotheses 3.2.1 holds.

Let $\mathcal{A}$ be the set of admissible strategies for one of the filtrations $\mathbb{F}$ or $\mathbb{G}$ and a terminal bounded pay-off ξ. The goal is to compute

$$V(x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\Big[U\big(X_T^{x,\pi} - \xi\big)\Big] , \tag{3.2.1}$$

using BSDEs. If we are looking for strategies $\mathbb{F}$-adapted with pay-off $\xi \in \mathcal{F}_T$, the natural BSDE which solves this problem is the BSDE with solutions defined in $\mathbb{F}$. In the other hand, if we are looking for strategies $\mathbb{G}$-adapted with pay-off $\xi \in \mathcal{G}_T$, the natural BSDE to solve this problem is a $\mathbb{G}$-BSDE. But what happens if we are looking for strategies in one filtration, and pay-off in other filtration. Is it possible to define an $\mathbb{F}$-BSDE ? or is it better to define a $\mathbb{G}$-BSDE ?

In order to give answer to these questions, we consider two cases, the one of a pay-off $\xi^{\mathbb{G}} \in \mathcal{G}_T$ bounded, and the case with a pay-off $\xi^{\mathbb{F}} \in \mathcal{F}_T$ bounded.

We denote by $X_{t_2}^{t_1,x,\pi}$ the wealth at time t_2, starting with a capital equal to $x \in \mathbb{R}$ at time t_1 with $0 \le t_1 \le t_2 \le T$, following the strategy π, i.e.

$$X_{t_2}^{t_1,x,\pi} \;\; := \;\; x + \int_{t_1}^{t_2} \pi_s \alpha_s ds + \int_{t_1}^{t_2} \pi_s \sigma_s dW_s \; .$$

We will consider the optimization problem between $[t, T]$ for the cases where we consider $\mathbb{F}$-adapted strategies, $\mathbb{G}$-adapted strategies, pay-off in $\mathcal{F}_T$ and pay-off in $\mathcal{G}_T$.

First we introduce the classical problem, where the set of admissible strategies is defined in $\mathbb{F}$ and the pay-off $\xi^{\mathbb{F}}$ is $\mathcal{F}_T$ measurable. For any initial time $t \in [0,T]$ and capital $x \in \mathbb{R}$, we define the value function $\overline{V}_t^{\mathbb{F}}(x)$ (also denoted by $\overline{V}_t^{\mathbb{F}}$ if the capital $x = 0$) as

$$\overline{V}_t^{\mathbb{F}}(x) := \sup_{\pi \in \mathcal{A}^{\mathbb{F}}[t,T]} \mathbb{E}\Big[U\big(X_T^{t,x,\pi} - \xi^{\mathbb{F}}\big)\Big|\mathcal{F}_t\Big] \ , \ \forall x \in \mathbb{R} \ . \tag{3.2.2}$$

This problem is studied by mean of BSDE in [HIM05] and [REK00]).

Now, we introduce the following maximization problem. We consider $\mathbb{F}$-adapted strategies and pay-off $\xi^{\mathbb{G}}$ in $\mathcal{G}_T$. We define the value function $V_t^{\mathbb{F}}(x)$ (also denoted by $V_t^{\mathbb{F}}$ if the initial capital x is equal to zero) as

$$V_t^{\mathbb{F}}(x) := \sup_{\pi \in \mathcal{A}^{\mathbb{F}}[t,T]} \mathbb{E}\Big[U\big(X_T^{t,x,\pi} - \xi^{\mathbb{G}}\big)\Big|\mathcal{F}_t\Big] \ , \ \ \forall x \in \mathbb{R} \ \text{ and } \ \forall t \in [0,T] \ . \tag{3.2.3}$$

As we shall see, this problem reduces to the previous one, up to a change of the terminal condition in the case of exponential utility function.

We also consider the problem using strategies $\pi \in \mathcal{A}^{\mathbb{G}}[t,T]$ with pay-off $\xi^{\mathbb{G}} \in \mathcal{G}_T$, and define the value function $V_t^{\mathbb{G}}(x)$ (also denoted by $V_t^{\mathbb{G}}$ if the capital $x = 0$) as

$$V_t^{\mathbb{G}}(x) := \sup_{\pi \in \mathcal{A}^{\mathbb{G}}[t,T]} \mathbb{E}\Big[U\big(X_T^{t,x,\pi} - \xi^{\mathbb{G}}\big)\Big|\mathcal{G}_t\Big] \ \ \forall x \in \mathbb{R} \ \text{ and } \ \forall t \in [0,T] \ . \tag{3.2.4}$$

This problem is similar to the first one, working in another filtration. The difficulty, in our setting, is that the first problem take place in a complete market, and this one generaly in an incomplete market. In this case, we can also associate a $\mathbb{G}$-BSDE to this problem.

We can also consider the maximization problem using strategies $\pi \in \mathcal{A}^{\mathbb{G}}[t,T]$ but with pay-off $\xi^{\mathbb{F}} \in \mathcal{F}_T$ given by the random variable $\overline{V}_t^{\mathbb{G}}(x)$ (also denoted by $\overline{V}_t^{\mathbb{G}}$ if the capital $x = 0$), defined as

$$\overline{V}_t^{\mathbb{G}}(x) = \sup_{\pi \in \mathcal{A}^{\mathbb{G}}[t,T]} \mathbb{E}\Big[U\big(X_T^{t,x,\pi} - \xi^{\mathbb{F}}\big)\Big|\mathcal{G}_t\Big] \ , \ \ \forall x \in \mathbb{R} \ \text{ and } \ \forall t \in [0,T] \ . \tag{3.2.5}$$

This can be associated to a $\mathbb{G}$-BSDE and is a particular case of the previous one.

The natural question is : What is the link between $V_t^{\mathbb{F}}$, $\overline{V}_t^{\mathbb{F}}$, $V_t^{\mathbb{G}}$ and $\overline{V}_t^{\mathbb{G}}$? Answering this question for any pair of filtrations $\mathbb{F}$ and $\mathbb{G}$ such that $\mathbb{F} \subset \mathbb{G}$ is very difficult, since the BSDEs associated depend on filtrations. So, we study the particular cases where $\mathbb{G}$ is an initial enlargement of $\mathbb{F}$ and where $\mathbb{G}$ is a progressive enlargement of $\mathbb{F}$.

We start with some (trivial) remarks. For any $\mathbb{F} \subset \mathbb{G}$, $\xi^{\mathbb{G}} \in \mathcal{G}_T$, consider the case where $\xi^{\mathbb{F}} = \mathbb{E}(\xi^{\mathbb{G}}|\mathcal{F}_T)$,

– $V_0^{\mathbb{F}} \leq V_0^{\mathbb{G}}$ and $\overline{V}_0^{\mathbb{F}} \leq \overline{V}_0^{\mathbb{G}}$, since $\mathcal{A}^{\mathbb{F}} \subset \mathcal{A}^{\mathbb{G}}$.

– If $\xi^{\mathbb{G}} \in \mathcal{F}_T$, then $\overline{V}_t^{\mathbb{F}} = V_t^{\mathbb{F}}$, $V_t^{\mathbb{G}} = \overline{V}_t^{\mathbb{G}}$, and $\overline{V}_0^{\mathbb{F}} = V_0^{\mathbb{F}} \leq V_0^{\mathbb{G}} = \overline{V}_0^{\mathbb{G}}$.

We end this subsection with a result on comparison of the values functions in the immersion setting in the case $\xi \in \mathcal{F}_T$ and bounded.

Proposition 3.2.3 *Let U be an exponential utility function and assume that $\mathbb{F} \hookrightarrow \mathbb{G}$ (i.e., if every $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale). Then,*

$$\sup_{\pi \in \mathcal{A}^{\mathbb{F}}} \mathbb{E}\Big[U\big(X_T^{x,\pi}\big)\Big] = \sup_{\pi \in \mathcal{A}^{\mathbb{G}}} \mathbb{E}\Big[U\big(X_T^{x,\pi}\big)\Big] \ ,$$

and if $\xi \in \mathcal{F}_T$ is bounded, then

$$\sup_{\pi \in \mathcal{A}^{\mathbb{F}}} \mathbb{E}\Big[U\big(X_T^{x,\pi} - \xi\big)\Big] = \sup_{\pi \in \mathcal{A}^{\mathbb{G}}} \mathbb{E}\Big[U\big(X_T^{x,\pi} - \xi\big)\Big] .$$

PROOF: It is well known that, in a complete market $X_T^{x,\pi^{\mathbb{F}*}} = (U')^{-1}(\nu L_T)$ where L is the Radon-Nikodym density of the unique equivalent martingale measure and where ν is chosen so that $\mathbb{E}(L_T(U')^{-1}(\nu L_T)) = x$. We give the proof, since, in the literature, we have found proofs only for utility functions defined on $\mathbb{R}^+$. Using that U is concave, we have that, for any $\pi \in \mathcal{A}^{\mathbb{G}}$

$$U(X_T^{x,\pi^{\mathbb{G}}}) - U(X_T^{x,\pi^{\mathbb{F}^*}}) \le U'(X_T^{x,\pi^{\mathbb{F}^*}})(X_T^{x,\pi^{\mathbb{G}}} - X_T^{x,\pi^{\mathbb{F}^*}}) . \qquad (3.2.6)$$

Plugging this to (3.2.6), we get that

$$U(X_T^{x,\pi^{\mathbb{G}}}) - U(X_T^{x,\pi^{\mathbb{F}^*}}) \le (X_T^{x,\pi^{\mathbb{G}}} - X_T^{x,\pi^{\mathbb{F}^*}})\nu L_T . \qquad (3.2.7)$$

It follows by immersion hypothesis that L is also a $\mathbb{G}$-e.m.m. (indeed, L is a $\mathbb{G}$-martingale positive with expectation 1 and SL is a $\mathbb{G}$-martingale). If π is admissible, XL is a martingale and

$$\mathbb{E}((X_T^{x,\pi^{\mathbb{G}}} - X_T^{x,\pi^{\mathbb{F}^*}})L_T) = 0 .$$

Taking the expected value in (3.2.7), and noting that ν is positif, we obtain that

$$\mathbb{E}\big[U(X_T^{x,\pi^{\mathbb{G}}}) - U(X_T^{x,\pi^{\mathbb{F}^*}})\big] \le 0 .$$

Finally, we get

$$\sup_{\pi \in \mathcal{A}^{\mathbb{F}}} \mathbb{E}\Big[U\big(X_T^{x,\pi}\big)\Big] = \sup_{\pi \in \mathcal{A}^{\mathbb{G}}} \mathbb{E}\Big[U\big(X_T^{x,\pi}\big)\Big] .$$

The last part relies on the fact that ξ is hedgeable, and if p is its price $\sup_\pi \mathbb{E}(U(X_T^{x,\pi} - \xi)) = \sup_\pi \mathbb{E}(U(X_T^{x-p,\pi}))$. □

In the following subsection, we characterize the value functions $V^{\mathbb{F}}$, $V^{\mathbb{G}}$, $\overline{V}^{\mathbb{F}}$ and $\overline{V}^{\mathbb{G}}$ as solutions of BSDEs associated.

The classical problem

In this section, we study the maximization problems defined in $\mathbb{F}$, i.e. the value functions $\overline{V}_t^{\mathbb{F}}(x)$ and $V_t^{\mathbb{F}}(x)$ for $t \in [0,T]$ and $x \in \mathbb{R}$ for the exponential utility. For the first case, it does not depend of the larger filtration, so we can solve it directly as follows. The maximization problem given by $\overline{V}_t^{\mathbb{F}}(x)$ is a classical result, for complete markets we can solve it without using BSDEs (see the proof of Proposition 3.2.3). We will make use of BSDE for our particular setup.

Proposition 3.2.4 *The value function $\overline{V}^{\mathbb{F}}$ defined in (3.2.2) is given by $\overline{V}_t^{\mathbb{F}}(x) = -\exp\big[-\gamma(X_t^{t,x,\pi^\star} - \widehat{Y}_t^{\mathbb{F}})\big] = -\exp(-\gamma(x - \widehat{Y}_t^{\mathbb{F}}))$, where $(\widehat{Y}^{\mathbb{F}}, \widehat{Z}^{\mathbb{F}})$ is the unique solution in $S^\infty_{\mathbb{G}}[t,T] \times L^2_{\mathbb{G}}([t,T] \times \Omega)$ of the following BSDE*

$$\begin{cases} -d\widehat{Y}_s^{\mathbb{F}} &= -\Big(\frac{\alpha_s^2}{2\gamma\sigma_s^2} + \frac{\alpha_s \widehat{Z}_s^{\mathbb{F}}}{\sigma_s}\Big)ds - \widehat{Z}_s^{\mathbb{F}} dW_s , \quad \forall s \in [t,T] , \\ \widehat{Y}_T^{\mathbb{F}} &= \xi^{\mathbb{F}} . \end{cases} \qquad (3.2.8)$$

Moreover, the optimal strategy associated to this problem is defined by

$$\pi_s^* \ := \ \frac{\alpha_s}{\gamma\sigma_s^2} + \frac{\widehat{Z}_s^{\mathbb{F}}}{\sigma_s} \ , \quad \forall s \in [t,T] \ ,$$

with π^* *a* $\mathbb{F}$*-strategy admissible.*

Proof: See for example [HIM05] for the form of the BSDE. The existence and uniqueness of the solution is standard, since the driver is Lipschitz. □

The computation of $V^{\mathbb{F}}$ reduces to the previous case, in the case of exponential utility functions since

$$\begin{aligned} V_t^{\mathbb{F}}(x) &= \sup_{\pi\in\mathcal{A}^{\mathbb{F}}[t,T]} \mathbb{E}\Big(\mathbb{E}\Big(-\exp\big(-\gamma(X_T^{t,x,\pi} - \xi^{\mathbb{G}})\big)\Big|\mathcal{F}_T\Big)\Big|\mathcal{F}_t\Big) \\ &= \sup_{\pi\in\mathcal{A}^{\mathbb{F}}[t,T]} \mathbb{E}\Big(-\exp\big(-\gamma X_T^{t,x,\pi}\big)\mathbb{E}\Big(\exp\big(\gamma\xi^{\mathbb{G}}\big)\Big|\mathcal{F}_T\Big)\Big|\mathcal{F}_t\Big) \ , \end{aligned}$$

then by defining

$$\widehat{\xi}^{\mathbb{F}} = \frac{1}{\gamma}\log\mathbb{E}\big[\exp(\gamma\xi^{\mathbb{G}})\big|\mathcal{F}_T\big] \ , \tag{3.2.9}$$

we obtain

$$V_t^{\mathbb{F}}(x) = \sup_{\pi\in\mathcal{A}^{\mathbb{F}}[t,T]} \mathbb{E}\Big(U\big(X_T^{t,x,\pi} - \widehat{\xi}^{\mathbb{F}}\ \big)\big|\mathcal{F}_t\Big) \ ,$$

we can solve it in the same way as $\overline{V}_t^{\mathbb{F}}(x)$, with the BSDE associated with the same dynamic but terminal value $\widehat{\xi}^{\mathbb{F}} \in \mathcal{F}_T$. Proposition 3.2.4 then implies the following :

Proposition 3.2.5 *The value function* $V_t^{\mathbb{F}}(x)$ *is given by* $V_t^{\mathbb{F}}(x) = -\exp[-\gamma(x - Y_t^{\mathbb{F}})]$*, where* $(Y^{\mathbb{F}}, Z^{\mathbb{F}})$ *is the unique solution in* $S_{\mathbb{F}}^{\infty}[t,T] \times L_{\mathbb{F}}^2([t,T]\times\Omega)$ *of the following BSDE*

$$\begin{cases} -dY_s^{\mathbb{F}} &= -\Big(\frac{\alpha_s^2}{2\gamma\sigma_s^2} + \frac{\alpha_s Z_s^{\mathbb{F}}}{\sigma_s}\Big)ds - Z_s^{\mathbb{F}} dW_s \ , \quad \forall s \in [t,T] \ , \\ Y_T^{\mathbb{F}} &= \widehat{\xi}^{\mathbb{F}} \ . \end{cases}$$

Moreover, the optimal strategy associated to this problem is defined by

$$\pi_s^* \ := \ \frac{\alpha_s}{\gamma\sigma_s^2} + \frac{Z_s^{\mathbb{F}}}{\sigma_s} \ , \quad \forall s \in [t,T] \ .$$

with π^* *an* $\mathbb{F}$*-strategy admissible.*

Utility maximization problem with initial enlargement

Consider a random variable ζ and define the right-continuous initially enlarged filtration $\mathbb{G}^{(\zeta)} = (\mathcal{G}_t^{(\zeta)})_{t\geq 0}$ as

$$\mathcal{G}_t^{(\zeta)} = \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon} \vee \sigma(\zeta) \ .$$

We assume that Hypotheses 3.2.1 is satisfied, in particular there exists a bounded $\mathbb{G}^{(\zeta)}$-adapted process $\mu^{\mathbb{G}^{(\zeta)}}$ such that $W_t = W_t^{\mathbb{G}^{(\zeta)}} + \int_0^t \mu_s^{\mathbb{G}^{(\zeta)}} ds$ for all $t \in [0,T]$, where $W^{\mathbb{G}^{(\zeta)}}$ is a $\mathbb{G}^{(\zeta)}$-Brownian motion.

Hypothesis 3.2.6 *We assume that Jacod's equivalence hypothesis is satisfied : the conditional law of ζ is equivalent to the law of ζ. More precisely, we assume that there exists a positive process p such that $\mathbb{P}(\zeta \in dx|\mathcal{F}_t) = p_t(x)\nu(dx)$ where ν is the law of ζ (see [Jac85]).*

The objective is to find the values of $V_t^{\mathbb{G}^{(\zeta)}}(x)$ and $\overline{V}_t^{\mathbb{G}^{(\zeta)}}(x)$ using $\mathbb{G}^{(\zeta)}$-BSDEs for $t \in [0, T]$ and $x \in \mathbb{R}$.

Proposition 3.2.7 *The value function $V_t^{\mathbb{G}^{(\zeta)}}(x)$ is given by*

$$V_t^{\mathbb{G}^{(\zeta)}}(x) = -\exp\big[-\gamma(x - Y_t^{\mathbb{G}^{(\zeta)}})\big] ,$$

where $(Y^{\mathbb{G}^{(\zeta)}}, Z^{\mathbb{G}^{(\zeta)}})$ is the unique solution in $S^\infty_{\mathbb{G}^{(\zeta)}}[t,T] \times L^2_{\mathbb{G}^{(\zeta)}}([t,T] \times \Omega)$ of the following BSDE

$$\begin{cases} -dY_s^{\mathbb{G}^{(\zeta)}} &= -\Big(\frac{(\alpha_s+\mu_s^{\mathbb{G}^{(\zeta)}}\sigma_s)^2}{2\gamma\sigma_s^2} + \frac{\alpha_s+\mu_s^{\mathbb{G}^{(\zeta)}}\sigma_s}{\sigma_s} Z_s^{\mathbb{G}^{(\zeta)}}\Big)ds - Z_s^{\mathbb{G}^{(\zeta)}} dW_s^{\mathbb{G}^{(\zeta)}} , \quad \forall s \in [t,T] , \\ Y_T^{\mathbb{G}^{(\zeta)}} &= \xi^{\mathbb{G}^{(\zeta)}} . \end{cases}$$

Moreover, the optimal strategy associated to this problem is defined by

$$\pi_s^* \ := \ \frac{\alpha_s + \mu_s^{\mathbb{G}^{(\zeta)}}\sigma_s}{\gamma\sigma_s^2} + \frac{Z_s^{\mathbb{G}^{(\zeta)}}}{\sigma_s} , \quad \forall s \in [t,T] ,$$

with π^* a $\mathbb{G}^{(L)}$-strategy admissible.

Proof: The proof is the same as for the proof of Proposition 3.2.4 using the dynamics of S in $\mathbb{G}^{(\zeta)}$. The existence and uniqueness of the solution follows from the fact that the driver is Lipschitz and that under Jacod's equivalence hypothesis, the filtration $\mathbb{G}^{(\zeta)}$ enjoys the PRP with respect to the Brownian motion $W^{\mathbb{G}^{(\zeta)}}$. The fact that the strategy is admissible follows from the boundedness conditions on the coefficients and the fact that $Z^{\mathbb{G}^{(\zeta)}} \in L^2_{\mathbb{G}^{(\zeta)}}([t,T] \times \Omega)$. □
Note that here $\widehat{Y}_0^{\mathbb{G}^{(\zeta)}} \in \mathcal{G}_0$ is, in general, a random variable.

The case where $\xi^{\mathbb{G}^{(\zeta)}} \in \mathcal{F}_T$ is a particular case. Therefore, the value function $\overline{V}_t^{\mathbb{G}^{(\zeta)}}$ is given by

$$\overline{V}_t^{\mathbb{G}^{(\zeta)}} = -\exp\big[-\gamma(x - \widehat{Y}_t^{\mathbb{G}^{(\zeta)}})\big] , \quad \forall t \in [0,T] ,$$

where $(\widehat{Y}^{\mathbb{G}^{(\zeta)}}, \widehat{Z}^{\mathbb{G}^{(\zeta)}})$ is the unique solution in $S^\infty_{\mathbb{G}^{(\zeta)}}[t,T] \times L^2_{\mathbb{G}^{(\zeta)}}([t,T] \times \Omega)$ of the following BSDE

$$\begin{cases} -d\widehat{Y}_s^{\mathbb{G}^{(\zeta)}} &= -\Big(\frac{(\alpha_s+\mu_s^{\mathbb{G}^{(\zeta)}}\sigma_s)^2}{2\gamma\sigma_s^2} + \frac{\alpha_s+\mu_s^{\mathbb{G}^{(\zeta)}}\sigma_s}{\sigma_s} \widehat{Z}_s^{\mathbb{G}^{(\zeta)}}\Big)ds - \widehat{Z}_s^{\mathbb{G}^{(\zeta)}} dW_s^{\mathbb{G}^{(\zeta)}} , \quad \forall s \in [t,T] , \\ \widehat{Y}_T^{\mathbb{G}^{(\zeta)}} &= \xi^{\mathbb{F}} . \end{cases} \tag{3.2.10}$$

Moreover, the optimal strategy associated to this problem is defined by

$$\pi_s^* \ := \ \frac{\alpha_s + \mu_s^{\mathbb{G}^{(\zeta)}}\sigma_s}{\gamma\sigma_s^2} + \frac{\widehat{Z}_s^{\mathbb{G}^{(\zeta)}}}{\sigma_s} , \quad \forall s \in [t,T] ,$$

with π^* a $\mathbb{G}^{(L)}$-strategy admissible.

Utility maximization problem with progressive enlargement

Consider a finite random time τ, i.e. τ is a finite non negative random variable and define

$$\mathcal{F}_t^{(\tau)} = \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon} \vee \sigma(\tau \wedge (t+\epsilon)) \ , \quad \forall t \geq 0 \ ,$$

then we say that $\mathbb{F}^{(\tau)} = (\mathcal{F}_t^{(\tau)})_{t\geq 0}$ is the right-continuous progressively enlarged filtration.

The objective is to find the values of $V_t^{\mathbb{F}^{(\tau)}}$ and $\overline{V}_t^{\mathbb{F}^{(\tau)}}$ using BSDEs. We can consider a BSDE with solution in $\mathbb{F}^{(\tau)}$, since we are looking for a strategy in $\mathbb{F}^{(\tau)}$ and with terminal condition $\mathcal{F}_T^{(\tau)}$-adapted.

We assume Hypotheses 3.2.1, hence there exists a bounded $\mathbb{F}^{(\tau)}$-adapted process $\mu^{\mathbb{F}^{(\tau)}}$ such that $W_t = W^{\mathbb{F}_t^{(\tau)}} + \int_0^t \mu_s^{\mathbb{F}^{(\tau)}} ds$ for all $t \in [0,T]$, where $W^{\mathbb{F}^{(\tau)}}$ is a $\mathbb{F}^{(\tau)}$-Brownian motion. In the sequel, we introduce the process H defined by $H = \big(1_{\{\tau\leq t\}}\big)_{0\leq t\leq T}$ and we suppose that the process H admits a $\mathbb{G}$-compensator of the form $\int_0^{.\wedge\tau} \lambda_t dt$, i.e.

$$M = H - \int_0^{.\wedge\tau} \lambda_t dt$$

is an $\mathbb{F}^{(\tau)}$-martingale. Moreover, we assume that λ is a bounded $\mathbb{F}$-adapted process.

Hypothesis 3.2.8 *We assume that the pair $(W^{\mathbb{F}^{(\tau)}}, M)$ enjoys PRP for the filtration $\mathbb{F}^{(\tau)}$.*

See [JS15] for conditions so that this property holds.

We denote by $H^2_{\mathbb{F}^{(\tau)}}([t,T],\lambda)$, the subset of $\mathbb{R}$-valued, $\mathbb{F}$-predictable processes $(U_s)_{s\in[t,T]}$ such that

$$\|U\|_{H^2_{\mathbb{F}^{(\tau)}}(\lambda)} \ := \ \Big(\mathbb{E}\Big[\int_t^{T\wedge\tau} \lambda_s |U_s|^2 ds\Big]\Big)^{1/2} \ < \ \infty \ .$$

Proposition 3.2.9 *The value function $V_t^{\mathbb{F}^{(\tau)}}(x)$ is given by*

$$V_t^{\mathbb{F}^{(\tau)}}(x) = -\exp\big[-\gamma(x - Y_t^{\mathbb{F}^{(\tau)}})\big]$$

where $(Y^{\mathbb{F}^{(\tau)}}, Z^{\mathbb{F}^{(\tau)}}, U^{\mathbb{F}^{(\tau)}})$ is the unique solution in $L^2_{\mathbb{F}^{(\tau)}}([t,T]\times\Omega) \times H^2_{\mathbb{F}^{(\tau)}}([t,T]\times\Omega) \times H^2_{\mathbb{F}^{(\tau)}}([t,T],\lambda)$ of the following $\mathbb{F}^{(\tau)}$-BSDE

$$\begin{cases} -dY_s^{\mathbb{F}^{(\tau)}} = & -\Big(\frac{(\alpha_s+\mu_s^{\mathbb{F}^{(\tau)}}\sigma_s)^2}{2\gamma\sigma_s^2} + \frac{\alpha_s+\mu_s^{\mathbb{F}^{(\tau)}}\sigma_s}{\sigma_s} Z_s^{\mathbb{F}^{(\tau)}} - \lambda_s(1-H_s)\frac{e^{\gamma U_s^{\mathbb{F}^{(\tau)}}}-1}{\gamma}\Big)ds \\ & -Z_s^{\mathbb{F}^{(\tau)}} dW_s^{\mathbb{F}^{(\tau)}} - U_s^{\mathbb{F}^{(\tau)}} dH_s \ , \quad \forall s \in [t,T] \ , \\ Y_T^{\mathbb{F}^{(\tau)}} = & \xi^{\mathbb{F}^{(\tau)}} \ . \end{cases} \tag{3.2.11}$$

Moreover, the optimal strategy associated to this problem is defined by

$$\pi_s^* \ := \ \frac{\alpha_s + \mu_s^{\mathbb{F}^{(\tau)}}\sigma_s}{\gamma\sigma_s^2} + \frac{Z_s^{\mathbb{F}^{(\tau)}}}{\sigma_s} \ , \quad \forall s \in [t,T] \ ,$$

with π^* a $\mathbb{F}^{(\tau)}$-strategy admissible.

Proof: See [Del13, Chapter 11].

Corollary 3.2.10 *If $\xi^{\mathbb{F}^{(\tau)}} \in \mathcal{F}_T$ and $\mathbb{F}$ is immersed in $\mathbb{G}$, then the solution of (3.2.11) is $(Y^1, Z^1, 0)$ where (Y^1, Z^1) is the solution of (3.2.8).*

Proof: The proof follows from the fact that $\mu^{\mathbb{F}^{(\tau)}} = 0$ and the uniqueness of the solution of (3.2.11). □

Then as a consequence of Proposition 3.2.9 we have that the value function $\overline{V}_t^{\mathbb{F}^{(\tau)}}(x)$ is given by

$$\overline{V}_t^{\mathbb{F}^{(\tau)}}(x) = -\exp\big(-\gamma(x - \overline{Y}_t^{\mathbb{F}^{(\tau)}})\big) \ ,$$

where $(\overline{Y}^{\mathbb{F}^{(\tau)}}, \overline{Z}^{\mathbb{F}^{(\tau)}}, \overline{U}^{\mathbb{F}^{(\tau)}})$ is the unique solution in $L^2_{\mathbb{F}^{(\tau)}}([t,T] \times \Omega) \times H^2_{\mathbb{F}^{(\tau)}}([t,T] \times \Omega) \times H^2_{\mathbb{F}^{(\tau)}}([t,T], \lambda)$ of the following BSDE

$$\begin{cases} -d\overline{Y}_s^{\mathbb{F}^{(\tau)}} &= -\Big(\frac{(\alpha_s + \mu_s^{\mathbb{F}^{(\tau)}}\sigma_s)^2}{2\gamma\sigma_s^2} + \frac{\alpha_s + \mu_s^{\mathbb{F}^{(\tau)}}\sigma_s}{\sigma_s}\overline{Z}_s^{\mathbb{F}^{(\tau)}} - \lambda_s(1 - H_s)\frac{e^{\gamma \overline{U}_s^{\mathbb{F}^{(\tau)}}} - 1}{\gamma}\Big)ds \\ & \quad -\overline{Z}_s^{\mathbb{F}^{(\tau)}} dW_s^{\mathbb{F}^{(\tau)}} - \overline{U}_s^{\mathbb{F}^{(\tau)}} dH_s \ , \ \forall s \in [t,T] \ , \\ \overline{Y}_T^{\mathbb{F}^{(\tau)}} &= \quad \xi^{\mathbb{F}} \ . \end{cases}$$

Moreover, the optimal strategy associated to this problem is defined by

$$\pi_s^* \ := \ \frac{\alpha_s + \mu_s^{\mathbb{F}^{(\tau)}}\sigma_s}{\gamma\sigma_s^2} + \frac{\overline{Z}_s^{\mathbb{F}^{(\tau)}}}{\sigma_s} \ , \quad \forall s \in [t,T] \ ,$$

with π^* an $\mathbb{F}^{(\tau)}$-strategy admissible.

3.2.3 Indifference price of information

We define the Indifference Price of Information (IPI) of a random variable $\xi \in \mathcal{G}_T$ with respect to the filtration $\mathbb{F}$ as the positive real number $p \in \mathbb{R}$ such that

$$\sup_{\pi \in \mathcal{A}^{\mathbb{F}}} \mathbb{E}\Big(U\big(X_T^{x,\pi} - \xi\big)\Big) = \sup_{\pi \in \mathcal{A}^{\mathbb{G}}} \mathbb{E}\Big(U\big(X_T^{x-p,\pi} - \xi\big)\Big) \ .$$

for $U(x) = -e^{-\gamma x}$. This value p depends of the pay-off ξ, the filtrations $\mathbb{F} \subset \mathbb{G}$, and the sets of admissible strategies $\mathcal{A}^{\mathbb{F}}$ and $\mathcal{A}^{\mathbb{G}}$.

We define the following sets :

- $\mathbf{G}$ is the set of filtrations larger than $\mathbb{F}$ such that there are no arbitrages in the filtration $\mathbb{G}$ for $\mathbb{G} \in \mathbf{G}$.
- $\mathbf{\Xi}$ the set of $\mathcal{G}_T$-measurable bounded random variables.

Definition 3.2.11 *Global IPI. We define the global IPI for the exponential utility with parameter γ as the function $p : \mathbf{\Xi} \times \mathbf{G} \mapsto \mathbb{R}$ with*

$$p(\xi, \mathbb{G}) := \frac{1}{\gamma} log\left(\frac{\sup_{\pi \in \mathcal{A}^{\mathbb{F}}} \mathbb{E}\Big(U\big(X_T^{\pi} - \xi\big)\Big)}{\sup_{\pi \in \mathcal{A}^{\mathbb{G}}} \mathbb{E}\Big(U\big(X_T^{\pi} - \xi\big)\Big)}\right) \ .$$

Some (trivial) remarks :

– Invariance of initial capital. Consider $\widehat{p} : \mathbf{\Xi} \times \mathbf{G} \times \mathbb{R} \mapsto \mathbb{R}$ with

$$\begin{aligned} \widehat{p}(\xi, \mathbb{G}, x) &:= \frac{1}{\gamma}\log\left(\frac{\sup_{\pi\in\mathcal{A}^{\mathbb{F}}} \mathbb{E}\Big(U\big(X_T^{x,\pi} - \xi\big)\Big)}{\sup_{\pi\in\mathcal{A}^{\mathbb{G}}} \mathbb{E}\Big(U\big(X_T^{x,\pi} - \xi\big)\Big)}\right) \\ &= \frac{1}{\gamma}\log\left(\frac{\sup_{\pi\in\mathcal{A}^{\mathbb{F}}} \mathbb{E}\Big(-\exp\big(-\gamma(X_T^{\pi} + x - \xi)\big)\Big)}{\sup_{\pi\in\mathcal{A}^{\mathbb{G}}} \mathbb{E}\Big(-\exp\big(-\gamma(X_T^{\pi} + x - \xi)\big)\Big)}\right) \\ &= \frac{1}{\gamma}\log\left(\frac{\sup_{\pi\in\mathcal{A}^{\mathbb{F}}} \mathbb{E}\Big(-\exp\big(-\gamma(X_T^{\pi} - \xi)\big)\Big)}{\sup_{\pi\in\mathcal{A}^{\mathbb{G}}} \mathbb{E}\Big(-\exp\big(-\gamma(X_T^{\pi} - \xi)\big)\Big)}\right) = p(\xi, \mathbb{G}) , \end{aligned}$$

i.e. the initial capital does not affect the IPI.

– Monotonicity with respect to the filtration : If we have two filtrations $\mathbb{G}_1, \mathbb{G}_2 \in \mathbf{G}$ such that $\mathbb{G}_1 \subset \mathbb{G}_2$, a random variable $\xi \in \Xi$ and the IPIs $p_1 := p(\mathbb{G}_1, \xi)$ and $p_2 := p(\mathbb{G}_2, \xi)$, then by definition of p_1 and p_2 we have that

$$\sup_{\pi\in\mathcal{A}^{\mathbb{G}_1}} \mathbb{E}\Big[-\exp\big(-\gamma(X_T^{\pi} - \xi - p_1)\big)\Big] = \sup_{\pi\in\mathcal{A}^{\mathbb{G}_2}} \mathbb{E}\Big[-\exp\big(-\gamma(X_T^{\pi} - \xi - p_2)\big)\Big] \quad (3.2.12)$$

and using that $\mathbb{G}_1 \subset \mathbb{G}_2$ for the *sup* we have that

$$\sup_{\pi\in\mathcal{A}^{\mathbb{G}_1}} \mathbb{E}\Big[-\exp\big(-\gamma(X_T^{\pi} - \xi)\big)\Big] \le \sup_{\pi\in\mathcal{A}^{\mathbb{G}_2}} \mathbb{E}\Big[-\exp\big(-\gamma(X_T^{\pi} - \xi)\big)\Big] , \quad (3.2.13)$$

then from equations (3.2.12) and (3.2.13) we deduce

$$\exp\big(\gamma p_1\big) \le \exp\big(\gamma p_2\big) ,$$

which implies that $p_1 \le p_2$.

Remark 3.2.12 *This property is satisfied for any increasing utility function $\widehat{U} : \mathbb{R} \mapsto \mathbb{R}$. In this case, we consider the IPI $\widehat{p}_1$ associated to $(\xi, \mathbb{G}_1, x) \in \Xi \times \mathbf{G} \times \mathbb{R}$ and the IPI $\widehat{p}_2$ associated to $(\xi, \mathbb{G}_2, x) \in \Xi \times \mathbf{G} \times \mathbb{R}$ both IPIs using the utility function $\widehat{U}$, then using that $\mathbb{G}_1 \subset \mathbb{G}_2$ for the function max we have that for any real number $\widehat{p}_1$, we have*

$$\sup_{\pi\in\mathcal{A}^{\mathbb{G}_1}} \mathbb{E}\Big[\widehat{U}\big(X_T^{x,\pi} - \xi - \widehat{p}_1\big)\Big] \le \sup_{\pi\in\mathcal{A}^{\mathbb{G}_2}} \mathbb{E}\Big[\widehat{U}\big(X_T^{x,\pi} - \xi - \widehat{p}_1\big)\Big] , \quad (3.2.14)$$

also by definition we have

$$\sup_{\pi\in\mathcal{A}^{\mathbb{G}_1}} \mathbb{E}\Big[\widehat{U}\big(X_T^{x,\pi} - \xi - \widehat{p}_1\big)\Big] = \sup_{\pi\in\mathcal{A}^{\mathbb{G}_2}} \mathbb{E}\Big[\widehat{U}\big(X_T^{x,\pi} - \xi - \widehat{p}_2\big)\Big] , \quad (3.2.15)$$

using (3.2.14) and (3.2.15)

$$\sup_{\pi\in\mathcal{A}^{\mathbb{G}_2}} \mathbb{E}\Big[\widehat{U}\big(X_T^{x,\pi} - \xi - \widehat{p}_2\big)\Big] \le \sup_{\pi\in\mathcal{A}^{\mathbb{G}_2}} \mathbb{E}\Big[\widehat{U}\big(X_T^{x,\pi} - \xi - \widehat{p}_1\big)\Big] ,$$

then using the monotonicity property of $\widehat{U}$, we deduce that $\widehat{p}_1 \le \widehat{p}_2$.

Monotonicity with respect to the terminal value. Consider the filtrations $\mathbb{F} \subset \mathbb{G}$ and the IPIs $p_1 := p(\mathbb{G}, \xi_1)$ and $p_2 := p(\mathbb{G}, \xi_2)$ associated to the pay-offs $\xi_1, \xi_2 \in \Xi$ with $\xi_1 \leq \xi_2$ almost surely and $\mathbb{G} \in \mathbf{G}$, then we have that

$$\sup_{\pi \in \mathcal{A}^{\mathbb{G}}} \mathbb{E}\Big[U\big(X_T^\pi - \xi_1 - p_1\big)\Big] = \sup_{\pi \in \mathcal{A}^{\mathbb{G}}} \mathbb{E}\Big[U\big(X_T^\pi - \xi_2 - p_2\big)\Big] ,$$

and by the monotonicity of the function max we have that

$$\sup_{\pi \in \mathcal{A}^{\mathbb{G}}} \mathbb{E}\Big[U\big(X_T^\pi - \xi_1\big)\Big] \geq \sup_{\pi \in \mathcal{A}^{\mathbb{G}}} \mathbb{E}\Big[U\big(X_T^\pi - \xi_2\big)\Big] ,$$

then $\exp\big(\gamma p_1\big) \geq \exp\big(\gamma p_2\big)$, which implies that $p_1 \geq p_2$.
If we have more information, the indifference price of information increases. In the other hand, if the pay-off is bigger, the indifference price of information decreases.

We can extend the definition as follows.

Definition 3.2.13 *We define the IPI at time t as $p : \Omega \times [0,T] \times \mathbf{\Xi} \times \mathbf{G} \mapsto \mathbb{R}$ where*

$$p_t(\xi, \mathbb{G}) := \frac{1}{\gamma} log \left(\frac{\sup_{\pi \in \mathcal{A}^{\mathbb{F}}[t,T]} \mathbb{E}\Big[U\big(X_T^{t,\pi} + X_t^{\pi_{\mathbb{F}}^*} - \xi\big)\Big|\mathcal{F}_t\Big]}{\sup_{\pi \in \mathcal{A}^{\mathbb{G}}[t,T]} \mathbb{E}\Big[U\big(X_T^{t,\pi} + X_t^{\pi_{\mathbb{G}}^*} - \xi\big)\Big|\mathcal{G}_t\Big]} \right) ,$$

where $X_t^{\pi_{\mathbb{F}}^*}$ (resp. $X_t^{\pi_{\mathbb{G}}^*}$) is the wealth process at time t, driven by the optimal strategy ${\pi_{\mathbb{F}}}^* \in \mathcal{A}^{\mathbb{F}}[0,t]$ (resp. ${\pi_{\mathbb{G}}}^* \in \mathcal{A}^{\mathbb{G}}[0,t]$) .

Proposition 3.2.14 *Let $\mathbb{G} \in \{\mathbb{G}^{(\zeta)}, \mathbb{F}^{(\tau)}\}$ and $\xi \in \mathcal{G}_T$, then the IPI $p_t(\xi, \mathbb{G})$ is given by*

$$p_t(\xi, \mathbb{G}) = X_t^{\pi_{\mathbb{G}}^*} - X_t^{\pi_{\mathbb{F}}^*} + Y_t^{\mathbb{F}} - Y_t^{\mathbb{G}} , \quad \forall t \in [0,T] ,$$

with optimal strategies ${\pi_{\mathbb{F}}}^*$ and ${\pi_{\mathbb{G}}}^*$ given by

$${\pi_{\mathbb{F}}}_s^* \ := \ \frac{\alpha_s}{\gamma \sigma_s^2} + \frac{Z_s^{\mathbb{F}}}{\sigma_s} , \quad \forall s \in [0,t]$$

and

$${\pi_{\mathbb{G}}}_s^* \ := \ \frac{\alpha_s + \mu_s^{\mathbb{G}} \sigma_s}{\gamma \sigma_s^2} + \frac{Z_s^{\mathbb{G}}}{\sigma_s} , \quad \forall s \in [0,t] ,$$

where $(Y^{\mathbb{F}}, Z^{\mathbb{F}})$ is the unique solution in $L^2_{\mathbb{F}}([t,T] \times \Omega) \times H^2_{\mathbb{F}}([t,T] \times \Omega)$ of

$$\begin{cases} -dY_s^{\mathbb{F}} &= -\Big(\frac{\alpha_s^2}{2\gamma\sigma_s^2} + \frac{\alpha_s Z_s^{\mathbb{F}}}{\sigma_s}\Big) ds - Z_s^{\mathbb{F}} dW_s , \quad \forall s \in [t,T] , \\ Y_T^{\mathbb{F}} &= \frac{1}{\gamma} log \mathbb{E}\big[\exp(\gamma\xi)\big|\mathcal{F}_T\big] , \end{cases} \tag{3.2.16}$$

$(Y^{\mathbb{G}^{(\zeta)}}, Z^{\mathbb{G}^{(\zeta)}})$ is the unique solution in $L^2_{\mathbb{G}^{(\zeta)}}([t,T] \times \Omega) \times H^2_{\mathbb{G}^{(\zeta)}}([t,T] \times \Omega)$ of

$$\begin{cases} -dY_s^{\mathbb{G}^{(\zeta)}} &= -\Big(\frac{(\alpha_s + \mu_s^{\mathbb{G}^{(\zeta)}}\sigma_s)^2}{2\gamma\sigma_s^2} + \frac{\alpha_s + \mu_s^{\mathbb{G}^{(\zeta)}}\sigma_s}{\sigma_s} Z_s^{\mathbb{G}^{(\zeta)}}\Big) ds - Z_s^{\mathbb{G}^{(\zeta)}} dW_s^{\mathbb{G}^{(\zeta)}} , \quad \forall s \in [t,T] , \\ Y_T^{\mathbb{G}^{(\zeta)}} &= \xi \end{cases} \tag{3.2.17}$$

and $(Y^{\mathbb{F}^{(\tau)}}, Z^{\mathbb{F}^{(\tau)}}, U^{\mathbb{F}^{(\tau)}})$ *is the unique solution in* $L^2_{\mathbb{F}^{(\tau)}}([t,T]\times\Omega)\times H^2_{\mathbb{F}^{(\tau)}}([t,T]\times\Omega)\times H^2_{\mathbb{F}^{(\tau)}}([t,T],\lambda)$ *of the following BSDE*

$$\begin{cases} -dY^{\mathbb{F}^{(\tau)}}_s &= -\Big(\frac{(\alpha_s+\mu^{\mathbb{F}^{(\tau)}}_s\sigma_s)^2}{2\gamma\sigma_s^2} + \frac{\alpha_s+\mu^{\mathbb{F}^{(\tau)}}_s\sigma_s}{\sigma_s}Z^{\mathbb{F}^{(\tau)}}_s - \lambda_s(1-H_s)\frac{e^{\gamma U^{\mathbb{F}^{(\tau)}}_s}-1}{\gamma}\Big)ds \\ & \quad -Z^{\mathbb{F}^{(\tau)}}_s dW^{\mathbb{F}^{(\tau)}}_s - U^{\mathbb{F}^{(\tau)}}_s dH_s\ ,\ \forall s\in[t,T]\ , \\ Y^{\mathbb{F}^{(\tau)}}_T &= \xi\ . \end{cases} \tag{3.2.18}$$

PROOF: Using the notation of Subsection 3.2.2, we can rewrite $p_t(\xi,\mathbb{G})$ as

$$p_t(\xi,\mathbb{G}) = \frac{1}{\gamma}\log\left(\frac{e^{-\gamma X^{\pi_{\mathbb{F}}^*}_t}V^{\mathbb{F}}_t}{e^{-\gamma X^{\pi_{\mathbb{G}}^*}_t}V^{\mathbb{G}}_t}\right)\ ,$$

then the proof follows directly by the Propositions 3.2.5, 3.2.7 and 3.2.9 we have that we can characterize the solution of the optimization problems $V^{\mathbb{F}}_t$ and $V^{\mathbb{G}}_t$ as a solution of BSDEs, also we have the explicit values of ${\pi_{\mathbb{F}}}^*$ and ${\pi_{\mathbb{G}}}^*$ in terms of the solution of a BSDE. □

In the following Proposition, we can simplify if we specify $\mathbb{G}$.

Proposition 3.2.15 *Let* $\mathbb{G}\in\{\mathbb{G}^{(\zeta)},\mathbb{F}^{(\tau)}\}$ *be an enlarged filtration of* $\mathbb{F}$ *and* $\xi\in\mathcal{G}_T$, *then the IPI* $p_t(\xi,\mathbb{G})$ *is given by*

$$\begin{aligned} p_t(\xi,\mathbb{G}) &= p_0 - Y^{\mathbb{G}}_t + \textstyle\int_0^t\left(\frac{\alpha_s\mu^{\mathbb{G}}_s}{\gamma\sigma_s} + \frac{\alpha_s^2}{2\gamma\sigma_s^2} + \frac{\alpha_s}{\sigma_s}Z^{\mathbb{G}}_s + \left(\frac{\mu^{\mathbb{G}}_s}{\gamma} + Z^{\mathbb{G}}_s\right)\mu^{\mathbb{G}}_s\right)ds \\ & \quad + \textstyle\int_0^t\left(\frac{\mu^{\mathbb{G}}_s}{\gamma} + Z^{\mathbb{G}}_s\right)dW^{\mathbb{G}}_s, \end{aligned}$$

for all $t\in[0,T]$, *where, setting* $\overline{\xi} = \frac{1}{\gamma}log\big[\mathbb{E}\big(\exp(\gamma\xi)\big|\mathcal{F}_T\big)\big] - \mathbb{E}(\xi|\mathcal{F}_T)$, *we have that*

$$p_0 = \mathbb{E}(Y^{\mathbb{G}}_0|\mathcal{F}_0) + \mathbb{E}\Big(\Gamma_T\overline{\xi} + \int_0^T\Gamma_s\Big(-\frac{\alpha_s^2}{2\gamma\sigma_s^2} - \frac{\alpha_s}{\sigma_s}Z^{\mathbb{G}}_s + \mu^{\mathbb{G}}_s Y^{\mathbb{G}}_{s-} - f(s,Z^{\mathbb{G}}_s,U^{\mathbb{G}}_s)\Big)ds\Big|\mathcal{F}_0\Big)$$

f *is the driver of the* $\mathbb{G}$*-BSDE (3.2.17) in the case that* $\mathbb{G}=\mathbb{G}^{(L)}$ *or (3.2.18) for* $\mathbb{G}=\mathbb{F}^{(\tau)}$ *and* $\Gamma := (\Gamma_s)_{s\in[0,T]}$ *is defined by* $\Gamma_0 = 1$ *and*

$$d\Gamma_s = -\frac{\alpha_s}{\sigma_s}\Gamma_s dW_s\ ,\quad \forall s\in[0,T]\ .$$

Moreover, if $\mathbb{F}\hookrightarrow\mathbb{G}^{(\zeta)}$, *then*

$$p_t(\xi,\mathbb{G}^{(\zeta)}) = \mathbb{E}(\Gamma_T\overline{\xi}|\mathcal{F}_0) + \mathbb{E}(Y^{\mathbb{G}^{(\zeta)}}_0|\mathcal{F}_0) - Y^{\mathbb{G}^{(\zeta)}}_0,\quad \forall t\in[0,T]\ .$$

PROOF: By Proposition 3.2.14 we can write the IPI as

$$p_t(\xi,\mathbb{G}) = X^{\pi_{\mathbb{G}}^*}_t - X^{\pi_{\mathbb{F}}^*}_t + Y^{\mathbb{F}}_t - Y^{\mathbb{G}}_t\ ,\quad \forall t\in[0,T]\ , \tag{3.2.19}$$

with associated BSDEs.

By Propositions 3.2.5, 3.2.7 and 3.2.9 we have the explicit values of ${\pi_{\mathbb{F}}}^*$ and ${\pi_{\mathbb{G}}}^*$ in terms of the solution of a BSDE. Then $X_t^{{\pi_{\mathbb{G}}}^*} - X_t^{{\pi_{\mathbb{F}}}^*}$ is given by

$$\begin{aligned}
X_t^{{\pi_{\mathbb{G}}}^*} - X_t^{{\pi_{\mathbb{F}}}^*} &= \int_0^t \Big[\Big(\frac{\alpha_s + \mu_s^{\mathbb{G}}\sigma_s}{\gamma\sigma_s^2} + \frac{Z_s^{\mathbb{G}}}{\sigma_s}\Big) - \Big(\frac{\alpha_s}{\gamma\sigma_s^2} + \frac{Z_s^{\mathbb{F}}}{\sigma_s}\Big)\Big]\alpha_s ds \\
&\quad + \int_0^t \Big[\Big(\frac{\alpha_s + \mu_s^{\mathbb{G}}\sigma_s}{\gamma\sigma_s^2} + \frac{Z_s^{\mathbb{G}}}{\sigma_s}\Big) - \Big(\frac{\alpha_s}{\gamma\sigma_s^2} + \frac{Z_s^{\mathbb{F}}}{\sigma_s}\Big)\Big]\sigma_s dW_s \\
&= \int_0^t \Big(\frac{\alpha_s\mu_s^{\mathbb{G}}}{\gamma\sigma_s} + \frac{\alpha_s}{\sigma_s}Z_s^{\mathbb{G}}\Big)ds + \int_0^t \Big(\frac{\mu_s^{\mathbb{G}}}{\gamma} + Z_s^{\mathbb{G}}\Big)dW_s \\
&\quad - \int_0^t \frac{\alpha_s}{\sigma_s}Z_s^{\mathbb{F}}ds - \int_0^t Z_s^{\mathbb{F}}dW_s \\
&= \int_0^t \Big(\frac{\mu_s^{\mathbb{G}}}{\gamma} + Z_s^{\mathbb{G}}\Big)\Big(\frac{\alpha_s}{\sigma_s} + \mu_s^{\mathbb{G}}\Big)ds + \int_0^t \Big(\frac{\mu_s^{\mathbb{G}}}{\gamma} + Z_s^{\mathbb{G}}\Big)dW_s^{\mathbb{G}} \\
&\quad - \int_0^t \frac{\alpha_s}{\sigma_s}Z_s^{\mathbb{F}}ds - \int_0^t Z_s^{\mathbb{F}}dW_s\,.
\end{aligned}$$

Simplifying and using $Y_t^{\mathbb{F}} - Y_0^{\mathbb{F}} = \int_0^t \Big(\frac{\alpha_s^2}{2\gamma\sigma_s^2} + \frac{\alpha_s}{\sigma_s}Z_s^{\mathbb{F}}\Big)ds + \int_0^t Z_s^{\mathbb{F}}dW_s$, we get

$$X_t^{{\pi_{\mathbb{G}}}^*} - X_t^{{\pi_{\mathbb{F}}}^*} = \int_0^t \Big(\frac{\alpha_s\mu_s^{\mathbb{G}}}{\gamma\sigma_s} + \frac{\alpha_s^2}{2\gamma\sigma_s^2} + \frac{\alpha_s}{\sigma_s}Z_s^{\mathbb{G}}\Big)ds + \int_0^t \Big(\frac{\mu_s^{\mathbb{G}}}{\gamma} + Z_s^{\mathbb{G}}\Big)dW_s - (Y_t^{\mathbb{F}} - Y_0^{\mathbb{F}})\,. \tag{3.2.20}$$

In the other hand, we define $\widetilde{Y_t^{\mathbb{F}}} := \mathbb{E}(Y_t^{\mathbb{G}}|\mathcal{F}_t)$ for all $t \in [0,T]$ and $\xi^{\mathbb{F}} := \mathbb{E}(\xi|\mathcal{F}_T)$, then there exists a process $\widetilde{Z^{\mathbb{F}}} \in H_{\mathbb{F}}^2([0,T]\times\Omega)$, such that

$$\widetilde{Y_t^{\mathbb{F}}} = \xi^{\mathbb{F}} + \int_t^T \widetilde{f}_s ds - \int_t^T \widetilde{Z_s^{\mathbb{F}}}dW_s\,, \quad \forall t \in [0,T]\,,$$

where $\widetilde{f}_s := \mathbb{E}\big[f(s, Z_s^{\mathbb{G}}, U_s^{\mathbb{G}}))\big|\mathcal{F}_s\big]$ is the driver of the $\mathbb{G}$-BSDE associated for all $s \in [t,T]$.

Let $t \in [0,T]$ fixed, then using the same idea of the proof of Theorem 3.1.6, we set $\overline{Y} := Y^{\mathbb{F}} - \widetilde{Y^{\mathbb{F}}}$, a linear $\mathbb{F}$-BSDE

$$\overline{Y}_t = \frac{1}{\gamma}\log\mathbb{E}\big(\exp(\gamma\xi)\big|\mathcal{F}_T\big) - \xi^{\mathbb{F}} + \int_t^T \Big(-\frac{\alpha_s^2}{2\gamma\sigma_s^2} - \frac{\alpha_s Z_s^{\mathbb{F}}}{\sigma_s} - \widetilde{f}_s\Big)ds - \int_t^T (Z_s^{\mathbb{F}} - \widetilde{Z_s^{\mathbb{F}}})dW_s\,,$$

then setting $\overline{Z} := Z^{\mathbb{F}} - \widetilde{Z^{\mathbb{F}}}$, $\overline{\xi} := \frac{1}{\gamma}\log\mathbb{E}\big(\exp(\gamma\xi)\big|\mathcal{F}_T\big) - \xi^{\mathbb{F}}$ and using the representation of $\widetilde{Z^{\mathbb{F}}}$ in terms of $(Y^{\mathbb{G}}, Z^{\mathbb{G}})$ given by Theorem 3.1.5, i.e. using that $\widetilde{Z_t^{\mathbb{F}}} = \mathbb{E}\big(Z_t^{\mathbb{G}} + \mu_t^{\mathbb{G}}Y_{t-}^{\mathbb{G}}|\mathcal{F}_t\big)$, we get

$$\overline{Y}_t = \overline{\xi} + \int_t^T \Big(-\frac{\alpha_s^2}{2\gamma\sigma_s^2} - \frac{\alpha_s}{\sigma_s}\mathbb{E}\big(Z_s^{\mathbb{G}} + \mu_s^{\mathbb{G}}Y_{s-}^{\mathbb{G}}|\mathcal{F}_s\big) - \widetilde{f}_s - \frac{\alpha_s}{\sigma_s}\overline{Z}_s\Big)ds - \int_t^T \overline{Z}_s dW_s\,, \tag{3.2.21}$$

with $(Y^{\mathbb{G}}, Z^{\mathbb{G}}, U^{\mathbb{G}})$ given, (3.2.21) is a linear BSDE with explicit solution

$$\overline{Y}_t = \frac{1}{\Gamma_t}\mathbb{E}\Big(\Gamma_T\overline{\xi} + \int_t^T \Gamma_s\Big(-\frac{\alpha_s^2}{2\gamma\sigma_s^2} - \frac{\alpha_s}{\sigma_s}\mathbb{E}\big(Z_s^{\mathbb{G}} + \mu_s^{\mathbb{G}}Y_{s-}^{\mathbb{G}}|\mathcal{F}_s\big) - \widetilde{f}_s\Big)ds\Big|\mathcal{F}_t\Big)\,, \tag{3.2.22}$$

where $\Gamma := (\Gamma_s)_{s\in[0,T]}$ is defined by $\Gamma_0 = 1$ and

$$d\Gamma_s = -\frac{\alpha_s}{\sigma_s}\Gamma_s dW_s \ , \quad \forall s \in [0,T] \ .$$

Using that $\overline{Y}_t = Y_t^{\mathbb{F}} - \mathbb{E}(Y_t^{\mathbb{G}}|\mathcal{F}_t)$ in (3.2.22) we get

$$Y_t^{\mathbb{F}} = \mathbb{E}(Y_t^{\mathbb{G}}|\mathcal{F}_t) + \frac{1}{\Gamma_t}\mathbb{E}\Big(\Gamma_T\overline{\xi} + \int_t^T \Gamma_s\Big(- \frac{\alpha_s^2}{2\gamma\sigma_s^2} - \frac{\alpha_s}{\sigma_s}\mathbb{E}\big(Z_s^{\mathbb{G}} + \mu_s^{\mathbb{G}} Y_{s-}^{\mathbb{G}}|\mathcal{F}_s\big) - \widetilde{f}_s\Big)ds\Big|\mathcal{F}_t\Big) \ .$$

Then, using that α, γ, σ and Γ are $\mathbb{F}$-adapted and the definition of $\widetilde{f}$, we obtain

$$Y_t^{\mathbb{F}} = \mathbb{E}(Y_t^{\mathbb{G}}|\mathcal{F}_t) + \frac{1}{\Gamma_t}\mathbb{E}\Big(\Gamma_T\overline{\xi} + \int_t^T \mathbb{E}\Big(\Gamma_s\Big(-\frac{\alpha_s^2}{2\gamma\sigma_s^2} - \frac{\alpha_s}{\sigma_s}(Z_s^{\mathbb{G}} + \mu_s^{\mathbb{G}} Y_{s-}^{\mathbb{G}}) - f(s, Z_s^{\mathbb{G}}, U_s^{\mathbb{G}})\Big)\Big|\mathcal{F}_s\Big)ds\Big|\mathcal{F}_t\Big) \ , \tag{3.2.23}$$

simplifying (3.2.23), by Lemma 3.1.3 we have

$$Y_t^{\mathbb{F}} = \mathbb{E}(Y_t^{\mathbb{G}}|\mathcal{F}_t) + \frac{1}{\Gamma_t}\mathbb{E}\Big(\Gamma_T\overline{\xi} + \int_t^T \Gamma_s\Big(-\frac{\alpha_s^2}{2\gamma\sigma_s^2} - \frac{\alpha_s}{\sigma_s}(Z_s^{\mathbb{G}} + \mu_s^{\mathbb{G}} Y_{s-}^{\mathbb{G}}) - f(s, Z_s^{\mathbb{G}}, U_s^{\mathbb{G}})\Big)ds\Big|\mathcal{F}_t\Big) \ . \tag{3.2.24}$$

Then, replacing (3.2.20) in (3.2.19)

$$p_t(\xi, \mathbb{G}) \;=\; \int_0^t \Big(\tfrac{\alpha_s\mu_s^{\mathbb{G}}}{\gamma\sigma_s} + \tfrac{\alpha_s^2}{2\gamma\sigma_s^2} + \tfrac{\alpha_s}{\sigma_s}Z_s^{\mathbb{G}}\Big)ds + \int_0^t \Big(\tfrac{\mu_s^{\mathbb{G}}}{\gamma} + Z_s^{\mathbb{G}}\Big)dW_s + Y_0^{\mathbb{F}} - Y_t^{\mathbb{G}} \ . \tag{3.2.25}$$

Then for (3.2.24) evaluated for $t = 0$ we have that

$$Y_0^{\mathbb{F}} = \mathbb{E}(Y_0^{\mathbb{G}}|\mathcal{F}_0) + \mathbb{E}\Big(\Gamma_T\overline{\xi} + \int_0^T \Gamma_s\Big(-\frac{\alpha_s^2}{2\gamma\sigma_s^2} - \frac{\alpha_s}{\sigma_s}(Z_s^{\mathbb{G}} + \mu_s^{\mathbb{G}} Y_{s-}^{\mathbb{G}}) - f(s, Z_s^{\mathbb{G}}, U_s^{\mathbb{G}})\Big)ds\Big|\mathcal{F}_0\Big) \ . \tag{3.2.26}$$

Remplacing (3.2.26) in (3.2.25), we get that

$$\begin{aligned} p_t(\xi, \mathbb{G}) \;=\; & \int_0^t \Big(\tfrac{\alpha_s\mu_s^{\mathbb{G}}}{\gamma\sigma_s} + \tfrac{\alpha_s^2}{2\gamma\sigma_s^2} + \tfrac{\alpha_s}{\sigma_s}Z_s^{\mathbb{G}}\Big)ds + \int_0^t \Big(\tfrac{\mu_s^{\mathbb{G}}}{\gamma} + Z_s^{\mathbb{G}}\Big)dW_s - Y_t^{\mathbb{G}} + \mathbb{E}(Y_0^{\mathbb{G}}|\mathcal{F}_0) \\ & + \mathbb{E}\Big(\Gamma_T\overline{\xi} + \int_0^T \Gamma_s\Big(- \tfrac{\alpha_s^2}{2\gamma\sigma_s^2} - \tfrac{\alpha_s}{\sigma_s}(Z_s^{\mathbb{G}} + \mu_s^{\mathbb{G}} Y_{s-}^{\mathbb{G}}) - f(s, Z_s^{\mathbb{G}}, U_s^{\mathbb{G}})\Big)ds\Big|\mathcal{F}_0\Big) \ , \end{aligned} \tag{3.2.27}$$

which is the desired result.

Moreover, if $\mathbb{F} \hookrightarrow \mathbb{G}^{(\zeta)}$, then $\mu^{\mathbb{G}} \equiv 0$ which implies that the driver of the $\mathbb{G}$-BSDE is

$$f(s, Z_s^{\mathbb{G}^{(\zeta)}}, U_s^{\mathbb{G}^{(\zeta)}}) = -\frac{\alpha_s^2}{2\gamma\sigma_s^2} - \frac{\alpha_s}{\sigma_s}Z_s^{\mathbb{G}^{(\zeta)}} \ , \quad \forall s \in [0,T] \ , \tag{3.2.28}$$

then (3.2.24) and (3.2.28) imply that

$$Y_t^{\mathbb{F}} = \mathbb{E}(Y_t^{\mathbb{G}^{(\zeta)}}|\mathcal{F}_t) + \frac{1}{\Gamma_t}\mathbb{E}(\Gamma_T\overline{\xi}|\mathcal{F}_t) \ . \tag{3.2.29}$$

Hence, using (3.2.20) and (3.2.29), we get

$$X_t^{\pi_{\mathbb{G}^{(\zeta)}}^*} - X_t^{\pi_{\mathbb{F}}^*} \;=\; \int_0^t \Big(\frac{\alpha_s^2}{2\gamma\sigma_s^2} + \frac{\alpha_s}{\sigma_s}Z_s^{\mathbb{G}^{(\zeta)}}\Big)ds + \int_0^t Z_s^{\mathbb{G}^{(\zeta)}}dW_s - (Y_t^{\mathbb{F}} - Y_0^{\mathbb{F}}) \ ,$$

which is equivalent to

$$X_t^{\pi_{\mathbb{G}(\zeta)}{}^*} - X_t^{\pi_{\mathbb{F}}{}^*} = Y_t^{\mathbb{G}(\zeta)} - Y_0^{\mathbb{G}(\zeta)} - [\mathbb{E}(Y_t^{\mathbb{G}(\zeta)}|\mathcal{F}_t) + \frac{1}{\Gamma_t}\mathbb{E}(\Gamma_T\overline{\xi}|\mathcal{F}_t) - \mathbb{E}(Y_0^{\mathbb{G}(\zeta)}|\mathcal{F}_0) - \mathbb{E}(\Gamma_T\overline{\xi})] \ . \tag{3.2.30}$$

Finally, substituting in (3.2.29) and (3.2.30) in (3.2.19)

$$p_t(\xi, \mathbb{G}^{(\zeta)}) = \mathbb{E}(\Gamma_T\overline{\xi}) + \mathbb{E}(Y_0^{\mathbb{G}(\zeta)}|\mathcal{F}_0) - Y_0^{\mathbb{G}(\zeta)} \ , \quad \forall t \in [0, T] \ ,$$

which finishes the proof. □

Chapitre 4

Some existence results for advanced backward stochastic differential equations with a jump

In this chapter, we are interested by backward stochastic differential equations of one of the following forms, called advanced backward stochastic differential equations (in short ABSDE)

$$\begin{cases} -dY_t &= f(t, Y_t, \mathbb{E}_t^{\mathbb{G}}[Y_{t+\delta}], Z_t, \mathbb{E}_t^{\mathbb{G}}[Z_{t+\delta}], U_t, \mathbb{E}_t^{\mathbb{G}}[U_{t+\delta}])dt \\ & \qquad -Z_t dB_t - U_t dH_t \ , \quad 0 \leq t \leq T \ , \\ Y_{T+t} &= \xi_{T+t} \ , \quad 0 \leq t \leq \delta \ , \\ Z_{T+t} &= P_{T+t} \ , \quad U_{T+t} = Q_{T+t} \mathbb{1}_{\{T+t \leq \tau\}} \quad 0 < t \leq \delta \ , \end{cases} \tag{4.0.1}$$

and

$$\begin{cases} -dY_t &= \mathbb{E}_t^{\mathbb{G}}\big[f(t, Y_t, Y_{t+\delta}, Z_t, Z_{t+\delta}, U_t, U_{t+\delta})]dt - Z_t dB_t - U_t dH_t \ , \quad 0 \leq t \leq T \ , \\ Y_{T+t} &= \xi_{T+t} \ , \quad 0 \leq t \leq \delta \ , \\ Z_{T+t} &= P_{T+t} \ , \quad U_{T+t} = Q_{T+t} \mathbb{1}_{\{T+t \leq \tau\}} \quad 0 < t \leq \delta \ , \end{cases} \tag{4.0.2}$$

where B is a Brownian motion and H is the process $H_t = \mathbb{1}_{\{\tau \leq t\}}$ associated with a given random time τ (a positive random variable). In this equation, for an integrable random variable X, we have used the notation $\mathbb{E}_t^{\mathbb{G}}(X) = \mathbb{E}(X|\mathcal{G}_t)$, where $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ is the filtration generated by B and H. The terminal conditions ξ, P and Q are given processes. We remark that the generator f of this ABSDE depends on the values of the processes (Y, Z, U) for present time t as well as for future time $t + \delta$. The ABSDE (4.0.1) was introduced by Peng and Yang in [PY09] in a Brownian case setting (roughly speaking, for $\tau \equiv 0$). Øksendal and Sulem have introduced ABSDEs of the form (4.0.2) in [ØSZ11], taking into account a random Poisson measure, instead of a single jump process.

Using Kharroubi and Lim methodology [KL12], we give conditions such that there exists a unique solution of (4.0.1) and of (4.0.2) in adequate spaces.

4.1 Framework

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space. We assume that this space is equipped with a one-dimensional standard Brownian motion B and we denote by $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ the right-continuous and complete filtration generated by B. We consider on this space a random time τ and we introduce the right-continuous process $H = \mathbb{1}_{\{\tau \leq .\}}$. We therefore use the standard approach of filtration enlargement by considering the smallest right-continuous extension $\mathbb{G}$ of $\mathbb{F}$ that turns τ into a $\mathbb{G}$-stopping time. More precisely $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ is defined by

$$\mathcal{G}_t \ := \ \bigcap_{\varepsilon>0} \tilde{\mathcal{G}}_{t+\varepsilon} \ ,$$

for any $t \geq 0$, where $\tilde{\mathcal{G}}_s := \mathcal{F}_s \vee \sigma(H_u \ , \ u \in [0, s])$, for any $s \geq 0$.
We denote by $\mathcal{P}(\mathbb{F})$ (resp. $\mathcal{P}(\mathbb{G})$) the σ-algebra of $\mathbb{F}$ (resp. $\mathbb{G}$)-predictable subsets of $\Omega \times \mathbb{R}_+$, i.e., the σ-algebra generated by the left-continuous $\mathbb{F}$ (resp. $\mathbb{G}$)-adapted processes.

We impose the following hypothesis, which is classical in the filtration enlargement theory and is called $(\mathcal{H})$-hypothesis or immersion property.

Hypothesis 4.1.1 *The process B remains a $\mathbb{G}$-Brownian motion.*

We observe that, since the filtration $\mathbb{F}$ is generated by the Brownian motion B, Hypothesis 4.1.1 is equivalent to the fact that all $\mathbb{F}$-martingales are also $\mathbb{G}$-martingales. In particular, the stochastic integral $\int_0^t X_s dB_s$ is a well defined $\mathbb{G}$-local martingale for all $\mathcal{P}(\mathbb{G})$-measurable processes X such that $\int_0^t |X_s|^2 ds < \infty$, for all $t \geq 0$.

We also introduce another hypothesis, often called the Jacod equivalence hypothesis, which will allow us to compute conditional expectations w.r.t. $\mathbb{G}$ in terms of conditional expectations w.r.t. $\mathbb{F}$. We denote by $\mathcal{O}(\mathbb{F})$ (resp. $\mathcal{O}(\mathbb{G})$) the σ-algebra of $\mathbb{F}$ (resp. $\mathbb{G}$)-optional subsets of $\Omega \times \mathbb{R}_+$.

Hypothesis 4.1.2 *We assume that the conditional law of τ is equivalent to the law of τ and that τ admits a density w.r.t. Lebesgue's measure. More precisely, we assume that there exists a positive $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable function $(\omega, t, u) \to \alpha_t(\omega, u)$ continuous in t such that*

a) for every $\theta \geq 0$, the process $(\alpha_t(\theta))_{t\geq 0}$ is an $\mathbb{F}$-martingale,

b) for every $t \geq 0$, the measure $\alpha_t(\omega, \theta)d\theta$ is a version of $\mathbb{P}(\tau \in d\theta|\mathcal{F}_t)(\omega)$, that is for any Borel function f such that $f(\tau)$ is integrable, one has

$$\mathbb{E}[f(\tau)|\mathcal{F}_t] \ = \ \int_0^\infty f(\theta)\alpha_t(\theta)d\theta \ ,$$

In particular, the density of τ is $\alpha_0(\theta)$.

In what follows, and in all the chapter, Hypotheses 4.1.1 and 4.1.2 are in force.

We introduce the $\mathbb{F}$-supermartingale G (called Azéma's supermartingale) defined as

$$G_t \ := \ \mathbb{P}(\tau > t \mid \mathcal{F}_t) = \int_t^\infty \alpha_t(\theta)d\theta \ , \quad t \geq 0 \ .$$

The supermartingale G is non-increasing and continuous, and the process M defined by

$$M_t \ := \ H_t - \int_0^{t\wedge\tau} \frac{\alpha_s(s)}{G_s} ds \ , \quad t \geq 0 \ ,$$

is a $\mathbb{G}$-martingale. The $\mathbb{F}$-adapted process λ defined by

$$\lambda_t := \frac{\alpha_t(t)}{G_t} \ , \quad t \geq 0 \ ,$$

is called the $\mathbb{F}$-intensity of τ (see [EKJJ10] for a proof of these results). Under Hypotheses 4.1.1 and 4.1.2, we have, from [EKJJ10, equality (11)], $\alpha_t(\theta) = \alpha_\theta(\theta)$ for any $t \geq \theta$, which implies $G_t = \exp(-\int_0^t \lambda_s ds)$ since $G_t = \int_t^\infty \alpha_t(\theta)d\theta = 1 - \int_0^t \alpha_t(\theta)d\theta = 1 - \int_0^t \alpha_\theta(\theta)d\theta = 1 - \int_0^t G_\theta \lambda_\theta d\theta$ and $G_0 = 1$.

Hypothesis 4.1.3 *We assume that λ is upper bounded by a constant k.*

Lemma 4.1.4 *For any t, G_t is lower bounded by e^{-kt} and, for any t and any θ , $0 < \alpha_t(\theta) \leq k$.*

PROOF: The bound on G is obvious from the equality $G_t = \exp(-\int_0^t \lambda_s ds)$. It follows that $\alpha_t(\theta) = \alpha_\theta(\theta) = \lambda_\theta G_\theta \leq k$ for $t > \theta$. Since $\alpha(\theta)$ is a martingale, $\alpha_t(\theta) \leq k$ for any $t \leq \theta$.

We recall a decomposition result for $\mathcal{P}(\mathbb{G})$-measurable processes, proved in [Jeu80, Lemma 4.4] for bounded processes. It can be easily extended to the case of unbounded processes.

Proposition 4.1.5 *Any $\mathcal{P}(\mathbb{G})$-measurable process $X = (X_t)_{t\geq 0}$ can be represented as*

$$X_t \ = \ X_t^b \mathbb{1}_{\{t\leq\tau\}} + X_t^a(\tau)\mathbb{1}_{\{t>\tau\}} \ ,$$

for all $t \geq 0$, where X^b is $\mathcal{P}(\mathbb{F})$-measurable and $X^a(\cdot)$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable.

In particular, a $\mathbb{G}$-predictable process is equal to an $\mathbb{F}$-predictable process on the interval $\{t \leq \tau\}$.

Song [Son14b] has extended the previous result to the class of optional processes under some hypotheses, which are satisfied under equivalence Jacod's hypothesis. Hence, one can state the following result :

Proposition 4.1.6 *Any $\mathcal{O}(\mathbb{G})$-measurable process $X = (X_t)_{t\geq 0}$ can be represented as*

$$X_t \ = \ X_t^b \mathbb{1}_{\{t<\tau\}} + X_t^a(\tau)\mathbb{1}_{\{t\geq\tau\}} \ ,$$

for all $t \geq 0$, where X^b is $\mathcal{O}(\mathbb{F})$-measurable and $X^a(\cdot)$ is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable.

If X is bounded by a constant K, then the process X^b is bounded by K and one can also choose the process $X^a(\theta)$ bounded by K for any $\theta \geq 0$. We remark that the uniqueness of $X_t^a(\theta)$ is granted for $\theta \le t$.

We have used the same notation as for predictable parts, mainly because we are in a Brownian filtration. In particular, the process X^b satisfies

$$X_t^b \mathbb{E}(\mathbb{1}_{\{t<\tau\}}|\mathcal{F}_t) \;=\; \mathbb{E}(X_t \mathbb{1}_{\{t<\tau\}}|\mathcal{F}_t)\ .$$

Hence, X^b is uniquely determined on $[0,T]$ by $X_t^b = \frac{1}{G_t}\mathbb{E}(X_t \mathbb{1}_{\{t<\tau\}}|\mathcal{F}_t)$, this quantity will be called the pre-default part.

Let $Y_T(\tau)$ be a bounded $\mathcal{F}_T \otimes \sigma(\tau)$-measurable random variable. Then, for $t \le T$, we have from Proposition 4.1.6,

$$E[Y_T(\tau)|\mathcal{G}_t] = Y_t^b \mathbb{1}_{\{t<\tau\}} + Y_t^a(\tau)\mathbb{1}_{\{\tau\le t\}}\ ,$$

where

$$\begin{aligned} Y_t^b &= \frac{\mathbb{E}\big[\int_t^\infty Y_T(u)\alpha_T(u)du|\mathcal{F}_t\big]}{G_t}\ , \qquad (4.1.1)\\ Y_t^a(\theta) &= \frac{\mathbb{E}\big[Y_T(\theta)\alpha_T(\theta)\big|\mathcal{F}_t\big]}{\alpha_t(\theta)}\ . \end{aligned}$$

Since $\alpha_t(u) = \alpha_u(u)$ for any $t \ge u$, we have

$$Y_t^a(\theta) \;=\; \mathbb{E}\big[Y_T(\theta)\big|\mathcal{F}_t\big]\ , \quad \forall \theta \le t$$

or, equivalently

$$Y_t^a(\tau) \;=\; \mathbb{E}\big[Y_T(\theta)\big|\mathcal{F}_t\big]|_{\theta=\tau}\ , \quad \forall \tau \le t\ . \qquad (4.1.2)$$

Therefore, if $Y_T(\tau)$ is bounded by a constant K then the processes Y^b and $Y^a(\theta)$ are bounded by K for any $\theta \ge 0$. Furthermore, if $Y \in \mathcal{G}_T$ is integrable, then $\mathbb{E}(Y|\mathcal{G}_t)\mathbb{1}_{\{t<\tau\}} = \frac{1}{G_t}\mathbb{E}(Y\mathbb{1}_{\{t<\tau\}}|\mathcal{F}_t)$.

If the process X satisfies the following stochastic differential equation

$$dX_t \;=\; \mu(t,X_t,\eta_t)dt + \sigma(t,X_t,\eta_t)dB_t + \varphi(t,X_t,\eta_t)dH_t\ ,$$

where $\mu(\cdot,x,u), \sigma(\cdot,x,u)$ and η are $\mathbb{G}$-optional processes and $\varphi(\cdot,x,u)$ an $\mathbb{F}$-predictable process, then $X^a(\tau)$ and X^b satisfy

$$\begin{aligned} dX_t^a(\tau) &= \mu^a(t,\tau,X_t^a(\tau),\eta_t^a(\tau))dt + \sigma^a(t,\tau,X_t^a(\tau),\eta_t^a(\tau))dB_t\ , \quad t \in [\tau,T]\ ,\\ dX_t^b &= \mu^b(t,X_t^b,\eta_t^b)dt + \sigma^b(t,X_t^b,\eta_t^b)dB_t\ , \quad t \in [0,T]\ ,\\ X_t^a(t) - X_t^b &= \varphi(t,X_t^b,\eta_t^b)\ , \quad t \le \tau\ , \qquad (4.1.3) \end{aligned}$$

where, for the last equality, we have used that if an $\mathbb{F}$-continuous process X satisfies $X_\tau = 0$, then $X_t = 0$ on $\{t \le \tau\}$ (see [Ngo10, Lemma 3, Chapter 1]).

To define solutions to ABSDEs, we introduce the following spaces, where $s,t \in \mathbb{R}_+$ with $s \le t$, and $T < \infty$ is the terminal time and δ is a positive constant :

– $S^2_{\mathbb{G}}[s,t]$ (resp. $S^2_{\mathbb{F}}[s,t]$) is the set of $\mathbb{R}$-valued $\mathcal{O}(\mathbb{G})$ (resp. $\mathcal{O}(\mathbb{F})$)-measurable processes $(Y_u)_{u\in[s,t]}$ such that

$$\|Y\|_{S^2[s,t]} \;:=\; \mathbb{E}[\sup_{u\in[s,t]}|Y_u|^2] \;<\; \infty\ .$$

– $L^2_{\mathbb{G}}[s,t]$ (resp. $L^2_{\mathbb{F}}[s,t]$) is the set of $\mathbb{R}$-valued $\mathcal{P}(\mathbb{G})$ (resp. $\mathcal{P}(\mathbb{F})$)-measurable processes $(Z_u)_{u\in[s,t]}$ such that

$$\|Z\|_{L^2[s,t]} \;:=\; \Big(\mathbb{E}\Big[\int_s^t |Z_u|^2 du\Big]\Big)^{\frac{1}{2}} \;<\; \infty\ .$$

– $L^2(\mathcal{F}_t)$ is the set of $\mathbb{R}$-valued square integrable $\mathcal{F}_t$-measurable random variables.

– L^2_τ is the set of $\mathbb{R}$-valued $\mathcal{P}(\mathbb{F})$-measurable processes U such that $U_t = 0$ for $t > \tau$ and

$$||U||_{L^2_\tau} \;:=\; \Big(\mathbb{E}\Big[\int_0^T |U_s|^2 ds\Big]\Big)^{1/2} \;<\; \infty\ .$$

Peng and Yang [PY09] have proved the following proposition :

Proposition 4.1.7 *The following ABSDE*

$$\begin{cases} -dY_t &= f(t, Y_t, \mathbb{E}^{\mathbb{F}}_t[Y_{t+\delta}], Z_t, \mathbb{E}^{\mathbb{F}}_t[Z_{t+\delta}])\, dt - Z_t dB_t\ , \quad 0\le t\le T\ ,\\ Y_{T+t} &= \xi_{T+t}\ , \quad 0\le t\le\delta\ ,\\ Z_{T+t} &= P_{T+t}\ , \quad 0< t\le\delta\ ,\end{cases} \tag{4.1.4}$$

has a unique solution in $S^2_{\mathbb{F}}[0;T+\delta]\times L^2_{\mathbb{F}}[0;T+\delta]$ if the map $f:\Omega\times[0,T]\times\mathbb{R}^4\to\mathbb{R}$ satisfies :

a) $f(\cdot,\vec{\mathbf{y}})$ is $\mathbb{F}$-optional for any $\vec{\mathbf{y}} = (y,\widehat{y},z,\widehat{z})\in\mathbb{R}^4$,

b) there exists C such that for any $t\in[0,T]$, any $\vec{\mathbf{y}},\vec{\mathbf{y}'}$ in $\mathbb{R}^4$, one has

$$|f(t,\vec{\mathbf{y}}) - f(t,\vec{\mathbf{y}'})| \le C|\vec{\mathbf{y}} - \vec{\mathbf{y}}'|$$

where $|\vec{\mathbf{y}}| = |y| + |\widehat{y}| + |z| + |\widehat{z}|$,

c) $\mathbb{E}(\int_0^T |f(t,\vec{\mathbf{0}})|^2 dt) < \infty$ where $\vec{\mathbf{0}} = (0,0,0,0)$,

and if the terminal condition ξ belongs to $S^2_{\mathbb{F}}[T;T+\delta]$ and P belongs to $L^2_{\mathbb{F}}[T;T+\delta]$.

They extend the proof to more general cases, in particular they obtain :

Proposition 4.1.8 *The following ABSDE*

$$\begin{cases} -dY_t &= f(t, Y_t, Y_{t+\delta}, Z_t, Z_{t+\delta})\, dt - Z_t dB_t\ , \quad 0\le t\le T\ ,\\ Y_{T+t} &= \xi_{T+t}\ , \quad 0\le t\le\delta\ ,\\ Z_{T+t} &= P_{T+t}\ , \quad 0< t\le\delta\ ,\end{cases} \tag{4.1.5}$$

has a unique solution in $S^2_{\mathbb{F}}[0;T+\delta]\times L^2_{\mathbb{F}}[0;T+\delta]$ if the map $f:\Omega\times[0,T]\times\mathbb{R}\times L^2(\mathcal{F}_{\cdot+\delta})\times\mathbb{R}\times L^2(\mathcal{F}_{\cdot+\delta})\to\mathbb{R}$ satisfies :

a) $f(\cdot,\vec{\mathbf{y}})$ is $\mathbb{F}$-optional for any $\vec{\mathbf{y}} = (y,\zeta,z,\eta)\in\mathbb{R}\times L^2(\mathcal{F}_{\cdot+\delta})\times\mathbb{R}\times L^2(\mathcal{F}_{\cdot+\delta})$; in particular $f(t,\vec{\mathbf{y}})$ is $\mathcal{F}_t$-measurable,

b) there exists C such that for any $t\in[0,T]$, any $\vec{\mathbf{y}},\vec{\mathbf{y}'}$ in $\mathbb{R}\times L^2(\mathcal{F}_{t+\delta})\times\mathbb{R}\times L^2(\mathcal{F}_{t+\delta})$, one has

$$|f(t,\vec{\mathbf{y}}) - f(t,\vec{\mathbf{y}'})| \le C\big(|y-y'| + \mathbb{E}_t(|\zeta-\zeta'|) + |z-z'| + \mathbb{E}_t(|\eta-\eta'|)\,,$$

c) $\mathbb{E}(\int_0^T |f(t,\vec{\mathbf{0}})|^2 dt) < \infty$

and if the terminal condition ξ belongs to $S^2_{\mathbb{F}}[T;T+\delta]$ and P belongs to $L^2_{\mathbb{F}}[T;T+\delta]$.

4.2 ABSDE with jump of type (4.0.1)

We assume that Hypotheses 4.1.1, 4.1.2 and 4.1.3 hold. We consider in this section an ABSDE of the following form : find a triple $(Y,Z,U)\in S^2_{\mathbb{G}}[0,T+\delta]\times L^2_{\mathbb{G}}[0,T+\delta]\times L^2_\tau$ satisfying

$$\begin{cases} -dY_t &= f\left(t,Y_t,\mathbb{E}^{\mathbb{G}}_t[Y_{t+\delta}],Z_t,\mathbb{E}^{\mathbb{G}}_t[Z_{t+\delta}],U_t,\mathbb{E}^{\mathbb{G}}_t[U_{t+\delta}]\right)dt \\ & \qquad\qquad -Z_t dB_t - U_t dH_t\ , \quad 0\le t\le T\ , \\ Y_{T+t} &= \xi_{T+t}\ , \quad 0\le t\le\delta\ , \\ Z_{T+t} &= P_{T+t}\ , \quad U_{T+t}=Q_{T+t}\mathbb{1}_{\{T+t\le\tau\}}\ , \quad 0<t\le\delta\ . \end{cases} \tag{4.2.1}$$

From Propositions 4.1.5 and 4.1.6, all the involved processes can be decomposed in two parts, before and after τ. In particular, since ξ will be chosen $\mathbb{G}$-optional and P $\mathbb{G}$-predictable, we have

$$\begin{cases} \xi_t &= \xi^b_t\mathbb{1}_{\{t<\tau\}}+\xi^a_t(\{\tau)\mathbb{1}_{\{t\ge\tau\}} \quad \text{(optional decomposition)} \\ P_t &= P^b_t\mathbb{1}_{\{t\le\tau\}}+P^a_t(\tau)\mathbb{1}_{\{t>\tau\}} \quad \text{(predictable decomposition)}\ . \end{cases} \tag{4.2.2}$$

We also consider the process $\sup_{0\le\theta\le T}\xi^a(\theta)=(\sup_{0\le\theta\le T}\xi^a_t(\theta))_{t\ge 0}$ (similar definition regarding P^a). We work under the following hypotheses :

Hypotheses 4.2.1 *Suppose that :*

(i) The terminal conditions satisfy $\xi\in\mathcal{S}^2_{\mathbb{G}}[T,T+\delta]$, $P\in L^2_{\mathbb{G}}[T,T+\delta]$, $Q\in L^2_{\mathbb{F}}[T,T+\delta]$, *there exists a constant* K *such that* $\mathbb{E}[|\xi^a_u(\theta)|^2]\le K$ *and* $\mathbb{E}[|P^a_u(\theta)|^2]\le K$ *for any* (u,θ).

(ii) The generator $f:\Omega\times[0,T]\times\mathbb{R}^6\to\mathbb{R}$ *of the ABSDE is Lipschitz, that means there exists a constant* C *such that, setting* $\vec{\mathbf{y}}=(y,\widehat{y},z,\widehat{z},u,\widehat{u})$ *and* $|\vec{\mathbf{y}}|=|y|+|\hat{y}|+|z|+|\hat{z}|+|u|+|\hat{u}|$, *we have*

$$|f(t,\vec{\mathbf{y}})-f(t,\vec{\mathbf{y}'})| \le C|\vec{\mathbf{y}}-\vec{\mathbf{y}'}|\ ,$$

for any $t\in[0,T]$, *any* $\vec{\mathbf{y}}\in\mathbb{R}^6$ *and* $\vec{\mathbf{y}'}\in\mathbb{R}^6$.

(iii) For any $\vec{\mathbf{y}}\in\mathbb{R}^6$, *the process* $f(\cdot,\vec{\mathbf{y}})$ *is* $\mathbb{G}$*-optional.*

(iv) There exists a constant C *such that* $|f(s,\vec{\mathbf{0}})|<C$.

From Propositions 4.1.5 and 4.1.6 again, we can write

$$\begin{cases} f(t,\vec{\mathbf{y}}) &= f^b(t,\vec{\mathbf{y}})\mathbb{1}_{\{t<\tau\}}+f^a(t,\tau,\vec{\mathbf{y}})\mathbb{1}_{\{t\ge\tau\}} \quad \text{(optional decomposition)} \\ Y_t &= Y^b_t\mathbb{1}_{\{t<\tau\}}+Y^a_t(\tau)\mathbb{1}_{\{t\ge\tau\}} \quad \text{(optional decomposition)} \\ Z_t &= Z^b_t\mathbb{1}_{\{t\le\tau\}}+Z^a_t(\tau)\mathbb{1}_{\{t>\tau\}} \quad \text{(predictable decomposition)}\ . \end{cases} \tag{4.2.3}$$

It follows, using integration by parts formula and assuming for a moment that Y^a and Y^b are continuous, that

$$dY_t = \mathbb{1}_{\{t<\tau\}}dY^b_t+\mathbb{1}_{\{t\ge\tau\}}dY^a_t(\tau)+(Y^a_t(\tau)-Y^b_t)dH_t\ .$$

Since $(Y^a_t(\tau)-Y^b_t)dH_t=(Y^a_t(t)-Y^b_t)dH_t$, we see from (4.1.3), that (Y,U) is part of a solution if and only if $U_t=\left[(Y^a_t(t)-Y^b_t)\mathbb{1}_{\{t\le T\}}+Q_t\mathbb{1}_{\{T<t\le T+\delta\}}\right]\mathbb{1}_{\{t\le\tau\}}$ (note that

this quantity is predictable if Y^b and Y^a are continuous). Moreover, $(Y^a(\tau), Z^a(\tau))$ satisfies

$$\left\{\begin{array}{lcl} -dY_t^a(\tau) & = & f^a\left(t,\tau,Y_t^a(\tau),\mathbb{E}_t^{\mathbb{G}}[Y_{t+\delta}^a(\tau)],Z_t^a(\tau),\mathbb{E}_t^{\mathbb{G}}[Z_{t+\delta}^a(\tau)],0,0\right)dt \\ & & \qquad -Z_t^a(\tau)dB_t\ , \quad T\wedge\tau\le t\le T\ , \\ Y_{T+t}^a(\tau) & = & \xi_{T+t}^a(\tau)\ , \quad 0\le t\le\delta\ , \\ Z_{T+t}^a(\tau) & = & P_{T+t}^a(\tau)\ , \quad 0<t\le\delta\ , \end{array}\right. \tag{4.2.4}$$

whereas, taking into account that the pre-default parts are unique, (Y^b, Z^b) satisfies

$$\left\{\begin{array}{lcl} -dY_t^b & = & f\left(t,Y_t^b,\mathbb{E}_t^{\mathbb{G}}[Y_{t+\delta}],Z_t^b,\mathbb{E}_t^{\mathbb{G}}[Z_{t+\delta}],U_t^b,\mathbb{E}_t^{\mathbb{G}}[U_{t+\delta}]\right)dt \\ & & \qquad -Z_t^b dB_t\ , \quad 0\le t\le T\ , \\ Y_{T+t}^b & = & \xi_{T+t}^b\ , \quad 0\le t\le\delta\ , \\ Z_{T+t}^b & = & P_{T+t}^b\ , \quad U_{T+t}^b=Q_{T+t}\ , \quad 0<t\le\delta\ . \end{array}\right. \tag{4.2.5}$$

4.2.1 Study of the Equation (4.2.4)

Our aim is to write (4.2.4) as a family of ABSDEs in the filtration $\mathbb{F}$. For that purpose, we note that, on the set $\{t\ge\tau\}$, we have from (4.1.2)

$$\mathbb{E}_t^{\mathbb{G}}[Y_{t+\delta}^a(\tau)] \;=\; \mathbb{E}_t^{\mathbb{F}}[Y_{t+\delta}^a(\theta)]_{\theta=\tau}\ ,$$

where $\mathbb{E}_t^{\mathbb{F}}[X]=\mathbb{E}(X|\mathcal{F}_t)$. The same equality holds for the part involving $Z_{t+\delta}^a(\tau)$. Therefore, we study the family of ABSDEs

$$\left\{\begin{array}{lcl} -dY_t^a(\theta) & = & f^a\left(t,\theta,Y_t^a(\theta),\mathbb{E}_t^{\mathbb{F}}[Y_{t+\delta}^a(\theta)],Z_t^a(\theta),\mathbb{E}_t^{\mathbb{F}}[Z_{t+\delta}^a(\theta)],0,0\right)dt \\ & & \qquad -Z_t^a(\theta)dB_t\ , \quad 0\le t\le T\ , \\ Y_{T+t}^a(\theta) & = & \xi_{T+t}^a(\theta)\ , \quad 0\le t\le\delta\ , \\ Z_{T+t}^a(\theta) & = & P_{T+t}^a(\theta)\ , \quad 0<t\le\delta\ . \end{array}\right. \tag{4.2.6}$$

For any fixed $\theta\in[0,T]$, the map $F:=f^a(\theta)$ defined as $F(t,\vec{\mathbf{y}})=f^a(t,\theta,\vec{\mathbf{y}})$ inherits the Lipschitz conditions of Proposition 4.1.7 from the one of f. Due to the boundedness of $f(\cdot,\vec{\mathbf{0}})$, the map $F(\cdot,\vec{\mathbf{0}})$ is bounded too, and satisfies

$$\sup_{0\le\theta\le T}\mathbb{E}\Big[\int_0^T\left(f^a(t,\theta,\vec{\mathbf{0}})\right)^2dt\Big]<\infty$$

and the existence of a solution follows from Proposition 4.1.7.

Using the same proof than the one in [PY09, Proposition 4.4], we see that there exists a constant C such that

$$\begin{aligned}\mathbb{E}_t^{\mathbb{F}}\left(\sup_{s\le T}(Y_s^a(\theta))^2+\int_t^T(Z_s^a(\theta))^2ds\right) \;\le\; & C\mathbb{E}_t^{\mathbb{F}}\Big(|\xi_T^a(\theta)|^2+\int_T^{T+\delta}\left(|\xi_u^a(\theta)|^2+|P_u^a(\theta)|^2\right)du \\ & +\int_t^T|f^a(u,\theta,\vec{\mathbf{0}})|^2du\Big)\ .\end{aligned} \tag{4.2.7}$$

4.2.2 Study of the Equation (4.2.5)

Our aim is to write (4.2.5) as an ABSDE in the filtration $\mathbb{F}$, that is to get rid of the quantities involving processes after time τ and working only with conditional expectation w.r.t. $\mathbb{F}$. Obviously, $\mathbb{E}_t^{\mathbb{G}}[Y_{t+\delta}]=\mathbb{E}_t^{\mathbb{G}}[Y_{t+\delta}\mathbb{1}_{\{t+\delta<\tau\}}]+\mathbb{E}_t^{\mathbb{G}}[Y_{t+\delta}\mathbb{1}_{\{t+\delta\ge\tau\}}]$.

Furthermore, from (4.1.1), we have

$$\mathbb{E}^{\mathbb{G}}_t[Y_{t+\delta}\mathbb{1}_{\{t+\delta<\tau\}}]\mathbb{1}_{\{t<\tau\}} \;=\; \frac{1}{G_t}\mathbb{E}^{\mathbb{F}}_t[Y^b_{t+\delta}\mathbb{1}_{\{t+\delta<\tau\}}]\mathbb{1}_{\{t<\tau\}} \;=\; \frac{1}{G_t}\mathbb{E}^{\mathbb{F}}_t[Y^b_{t+\delta}G_{t+\delta}]\mathbb{1}_{\{t<\tau\}}\ ,$$

and

$$\begin{aligned}\mathbb{E}^{\mathbb{G}}_t[Y_{t+\delta}\mathbb{1}_{\{t+\delta\geq\tau\}}]\mathbb{1}_{\{t<\tau\}} &= \mathbb{E}^{\mathbb{G}}_t[Y^a_{t+\delta}(\tau)\mathbb{1}_{\{t+\delta\geq\tau\}}]\mathbb{1}_{\{t<\tau\}}\\ &= \frac{1}{G_t}\mathbb{E}^{\mathbb{F}}_t\Big[\int_t^{t+\delta} Y^a_{t+\delta}(\theta)\alpha_{t+\delta}(\theta)d\theta\Big]\mathbb{1}_{\{t<\tau\}}\ .\end{aligned}$$

The same equalities hold for the part involving $Z_{t+\delta}$. We are lead to consider, relying on the uniqueness of pre-default parts, the BSDE

$$\begin{cases} -dY^b_t &= g(t,Y^b_t,Y^b_{t+\delta},Z^b_t,Z^b_{t+\delta})dt - Z^b_t dB_t\ , \quad 0\leq t\leq T,\\ Y^b_{T+t} &= \xi^b_{T+t}\ , \quad 0\leq t\leq\delta\ ,\\ Z^b_{T+t} &= P^b_{T+t}\ , \quad 0<t\leq\delta\ ,\end{cases} \tag{4.2.8}$$

where g is the map $\Omega\times[0,T]\times\mathbb{R}\times L^2(\mathcal{F}_{\cdot+\delta})\times\mathbb{R}\times L^2(\mathcal{F}_{\cdot+\delta})\to\mathbb{R}$ defined as, for ζ and η in $\mathcal{F}_{t+\delta}$, in terms of solution of the (4.2.6)

$$\begin{aligned} g(t,y,\zeta,z,\eta) \;=\; & f^b\Big(t,y,\ \frac{1}{G_t}\Big(\mathbb{E}^{\mathbb{F}}_t[\zeta G_{t+\delta}] + \int_t^{t+\delta}\mathbb{E}^{\mathbb{F}}_t[Y^a_{t+\delta}(\theta)\alpha_{t+\delta}(\theta)]d\theta\Big),\ z,\\ & \frac{1}{G_t}\Big(\mathbb{E}^{\mathbb{F}}_t[\eta G_{t+\delta}] + \int_t^{t+\delta}\mathbb{E}^{\mathbb{F}}_t[Z^a_{t+\delta}(\theta)\alpha_{t+\delta}(\theta)]d\theta\Big),\ Y^a_t(t)-y,\\ & \frac{1}{G_t}\mathbb{E}^{\mathbb{F}}_t[(Y^a_{t+\delta}(t+\delta)-\zeta)G_{t+\delta}]\mathbb{1}_{\{t+\delta\leq T\}} + \frac{1}{G_t}\mathbb{E}^{\mathbb{F}}_t[Q_{t+\delta}G_{t+\delta}]\mathbb{1}_{\{t+\delta>T\}}\Big)\ .\end{aligned}$$

It is straightforward that g is $\mathbb{F}$-optional. We now show that g satisfies Lipschitz conditions recalled in (4.1.5).
Since we have

$$f^b(t,\vec{\mathbf{y}}) \;=\; \frac{1}{G_t}\mathbb{E}(f(t,\vec{\mathbf{y}})\mathbb{1}_{\{t<\tau\}}|\mathcal{F}_t)\ ,$$

we obtain, using the Lipschitz condition for f there exists a constant C such that

$$\begin{aligned}|g(t,y,\zeta,z,\eta)-g(t,y',\zeta',z',\eta')| \;\leq\; & \frac{C}{G_t}\Big((|y-y'|+|z-z'|)\mathbb{E}^{\mathbb{F}}_t(\mathbb{1}_{\{t<\tau\}})\\ & +\mathbb{E}^{\mathbb{F}}_t[(|\zeta-\zeta'|+|\eta-\eta'|)G_{t+\delta}]\ \mathbb{E}^{\mathbb{F}}_t(\mathbb{1}_{\{t<\tau\}})\Big)\end{aligned}$$

which, from the definition of G, leads to

$$|g(t,y,\zeta,z,\eta)-g(t,y',\zeta',z',\eta')|\leq C\Big(|y-y'|+|z-z'|+\mathbb{E}^{\mathbb{F}}_t[(|\zeta-\zeta'|+|\eta-\eta'|)G_{t+\delta}]\Big)\ .$$

Noting that G is upper bounded by 1, the Lipschitz property (4.1.5) for g is satisfied.

We now check the integrability condition on $|g(t,\vec{\mathbf{0}})|^2$. We notice that

$$\begin{aligned} g(t,\vec{\mathbf{0}}) \;=\; & f^b\Big(t,\ 0,\ \frac{1}{G_t}\int_t^{t+\delta}\mathbb{E}^{\mathbb{F}}_t[Y^a_{t+\delta}(\theta)\alpha_{t+\delta}(\theta)]\,d\theta,\ 0,\ \frac{1}{G_t}\int_t^{t+\delta}\mathbb{E}^{\mathbb{F}}_t[Z^a_{t+\delta}(\theta)\alpha_{t+\delta}(\theta)]\,d\theta,\\ & Y^a_t(t),\ \frac{1}{G_t}\mathbb{E}^{\mathbb{F}}_t[Y^a_{t+\delta}(t+\delta)G_{t+\delta}]\mathbb{1}_{\{t+\delta\leq T\}} + \frac{1}{G_t}\mathbb{E}^{\mathbb{F}}_t[Q_{t+\delta}G_{t+\delta}]\mathbb{1}_{\{t+\delta>T\}}\Big)\ .\end{aligned}$$

From Lipschitz property of f, and the fact that $f(t,\vec{\mathbf{0}})$ is bounded and $G_t = \mathbb{E}^{\mathbb{F}}_t(\mathbb{1}_{\{t<\tau\}})$, we have

$$f^b(t,\vec{\mathbf{y}}) \;\leq\; \frac{1}{G_t}\Big(\mathbb{E}^{\mathbb{F}}_t[(f(t,\vec{\mathbf{0}}) + C|\vec{\mathbf{y}}|)\mathbb{1}_{\{t<\tau\}}]\;\Big) \;\leq\; C_1 + C|\vec{\mathbf{y}}|\;.$$

Using the fact that $(\sum_{i=1}^k a_i)^2$ is bounded (up to a constant) by $\sum_{i=1}^k a_i^2$, and using again the fact that G is lower bounded, the integrability condition of $|g(t,\vec{\mathbf{0}})|^2$ will follow from the boundedness of the quantities :

$$\begin{cases} \mathbb{E}\left(\int_0^T \left(\int_t^{t+\delta} \mathbb{E}^{\mathbb{F}}_t(Y^a_{t+\delta}(\theta)\alpha_{t+\delta}(\theta))d\theta\right)^2 dt\right)\,, \\ \mathbb{E}\left(\int_0^T \left(\int_t^{t+\delta} \mathbb{E}^{\mathbb{F}}_t(Z^a_{t+\delta}(\theta)\alpha_{t+\delta}(\theta))d\theta\right)^2 dt\right)\,, \\ \mathbb{E}\left(\int_0^T \left(\mathbb{E}^{\mathbb{F}}_t(Y^a_{t+\delta}(t+\delta))\right)^2 dt\right)\,, \\ \mathbb{E}\left(\int_0^T (Y^a_t(t))^2 dt\right)\,, \\ \mathbb{E}\left(\int_T^{T+\delta} (\mathbb{E}^{\mathbb{F}}_t[Q_t])^2 dt\right)\,. \end{cases}$$

The first quantity is bounded if $\mathbb{E}(\int_0^T \int_t^{t+\delta} \left(\mathbb{E}^{\mathbb{F}}_t(Y^a_{t+\delta}(\theta)\alpha_{t+\delta}(\theta)\right)^2 d\theta dt)$ is bounded, which is satisfied if

$$\mathbb{E}\Big[\int_0^T \int_t^{t+\delta} \mathbb{E}^{\mathbb{F}}_t\left[\left(Y^a_{t+\delta}(\theta)\alpha_{t+\delta}(\theta)\right)^2\right] d\theta dt\Big] \;=\; \mathbb{E}\Big[\int_0^T \int_t^{t+\delta} \left(Y^a_{t+\delta}(\theta)\alpha_{t+\delta}(\theta)\right)^2 d\theta dt\Big]$$

is bounded. This last quantity is equal to $\mathbb{E}(\int_0^{T+\delta} d\theta \int_0^T \mathbb{1}_{\{t<\theta<t+\delta\}} \left(Y^a_{t+\delta}(\theta)\alpha_\theta(\theta)\right)^2 dt$ by using the fact that $\alpha_\theta(\theta) = \alpha_{t+\delta}(\theta)$ since $t+\delta > \theta$. Since $\alpha_\theta(\theta)$ is bounded by k, it remains to see that

$$\int_0^{T+\delta} d\theta \int_0^T \mathbb{1}_{\{t<\theta<t+\delta\}}\mathbb{E}(\left(Y^a_{t+\delta}(\theta)\right)^2)\, dt$$

is bounded. Using the inequality (4.2.7) we see that

$$\begin{aligned} \sup_{0\le\theta\le T} \mathbb{E}((\sup_{0\le s\le T} Y^a_{t+\delta}(\theta))^2) \;&\leq\; C \sup_{0\le\theta\le T} \Big(\mathbb{E}\Big(|\xi^a_T(\theta)|^2 + \int_0^T |f^a(u,\theta,\vec{\mathbf{0}})|^2 du \\ &\quad + \int_T^{T+\delta} \left(|\xi^a_u(\theta)|^2 + |P^a_u(\theta)|^2\right) du\Big)\Big) \\ &\leq\; C\Big[\sup_{0\le\theta\le T} \mathbb{E}\left(|\xi^a_T(\theta)|^2\right) + \sup_{0\le\theta\le T} \mathbb{E}\Big(\int_0^T |f^a(u,\theta,\vec{\mathbf{0}})|^2 du\Big) \\ &\quad + \int_T^{T+\delta} \big(\sup_{0\le\theta\le T} \mathbb{E}|\xi^a_u(\theta)|^2 + \sup_{0\le\theta\le T} \mathbb{E}|P^a_u(\theta)|^2\big) du\Big]\,, \end{aligned}$$

hence, the required boundedness. The other quantities are studied using the same methodology and the fact that $Q \in L^2([T,T+\delta])$.

The existence of a unique solution (Y^b, Z^b) of the ABSDE (4.2.8) follows from Proposition 4.1.8. Moreover we have

$$\begin{aligned} \mathbb{E}^{\mathbb{F}}_t\left(\sup_{s\le T}(Y^b_s)^2 + \int_t^T (Z^b_s)^2 ds\right) \;&\leq\; C\mathbb{E}^{\mathbb{F}}_t\Big(|\xi^b_T|^2 + \int_T^{T+\delta} (|\xi^b_u|^2 + |P^b_u|^2) du \\ &\quad + \int_t^T |f^b(u,\vec{\mathbf{0}})|^2 du\Big)\,. \end{aligned} \tag{4.2.9}$$

4.2.3 Integrability of the solutions

In this part we consider the integrability of the solutions (Y, Z, U) where

$$\begin{array}{rcl} Y_t &=& Y_t^b \mathbb{1}_{\{t<\tau\}} + Y_t^a(\tau)\mathbb{1}_{\{t\geq\tau\}} \ , \\ Z_t &=& Z_t^b \mathbb{1}_{\{t\leq\tau\}} + Z_t^a(\tau)\mathbb{1}_{\{t>\tau\}} \ , \\ U_t &=& (Y_t^a(t) - Y_t^b)\mathbb{1}_{\{t\leq\tau\}} \ . \end{array}$$

From Subsections 4.2.1 and 4.2.2 we know that (Y, Z, U) satisfied the ABSDE (4.2.1).

Proposition 4.2.2 *The process U belongs to L^2_τ.*

PROOF: We use the convention $\int_a^b .ds = 0$ if $b < a$ in this proof.

$$\begin{array}{rcl} \mathbb{E}\Big[\displaystyle\int_0^{(T+\delta)\wedge\tau} |U_s|^2 ds\Big] &=& \mathbb{E}\Big[\displaystyle\int_0^{T\wedge\tau} |Y_s^a(s) - Y_s^b|^2 ds\Big] + \Big[\displaystyle\int_T^{(T+\delta)\wedge\tau} |Q_s|^2 ds\Big] \\ &\leq& 2\mathbb{E}\Big[\displaystyle\int_0^T |Y_s^a(s)|^2 ds\Big] + 2\mathbb{E}\Big[\displaystyle\int_0^T |Y_s^b|^2 ds\Big] + \Big[\displaystyle\int_T^{T+\delta} |Q_s|^2 ds\Big] \\ &\leq& 2\displaystyle\int_0^T \mathbb{E}\Big[|Y_s^a(s)|^2\Big] ds + 2T\mathbb{E}\big[\sup_{0\leq t\leq T} |Y_t^b|^2\big] + \Big[\displaystyle\int_T^{T+\delta} |Q_s|^2 ds\Big] \end{array}$$

and the quantities on the right-hand side are finite.

Proposition 4.2.3 *There exists a positive constant C such that the solution (Y, Z, U) of the ABSDE (4.2.1) satisfies*

$$\begin{array}{rl} & \mathbb{E}_t^{\mathbb{G}}\Big[\sup_{t\leq s\leq T} |Y_s|^2 + \displaystyle\int_t^T |Z_s|^2 ds\Big] \\ \leq & C\mathbb{E}_t^{\mathbb{F}}\Big[|\xi_T^b|^2 + \displaystyle\int_T^{T+\delta} (|\xi_s^b|^2 + |P_s^b|^2) ds + \Big(\displaystyle\int_t^T |f^b(s, \vec{0})| ds\Big)^2\Big] \\ & +C\dfrac{1}{\alpha_t(\tau)}\mathbb{E}_t^{\mathbb{F}}\Big[|\xi_T^a(\theta)|^2 + \displaystyle\int_T^{T+\delta} (|\xi_s^a(\theta)|^2 + |P_s^a(\theta)|^2) ds + \Big(\displaystyle\int_t^T |f^a(s, \theta, \vec{0})| ds\Big)^2\Big]_{\theta=\tau} \mathbb{1}_{\{\tau<t\}} \\ & +C\mathbb{1}_{\{t\leq\tau\}}\mathbb{E}_t^{\mathbb{F}}\Big[\displaystyle\int_t^T \Big\{|\xi_T^a(\theta)|^2 + \displaystyle\int_T^{T+\delta} (|\xi_s^a(\theta)|^2 + |P_s^a(\theta)|^2) ds + \Big(\displaystyle\int_t^T |f^a(s, \theta, \vec{0})| ds\Big)^2\Big\} d\theta\Big] \end{array}$$

for each $t \in [0, T]$.

PROOF: In the proof, the constant C can vary from line to line and we use again the convention $\int_a^b .ds = 0$ if $b < a$.

$$\begin{array}{rl} & \mathbb{E}_t^{\mathbb{G}}\Big[\sup_{t\leq s\leq T} |Y_s|^2 + \displaystyle\int_t^T |Z_s|^2 ds\Big] \\ = & \mathbb{E}_t^{\mathbb{G}}\Big[\sup_{t\leq s\leq T} |Y_s|^2 + \displaystyle\int_t^{T\wedge\tau} |Z_s^b|^2 ds + \displaystyle\int_{\{T\wedge\tau\}}^T |Z_s^a(\tau)|^2 ds\Big] \\ \leq & \mathbb{E}_t^{\mathbb{G}}\Big[\sup_{t\leq s\leq T} |Y_s|^2 + \displaystyle\int_t^T |Z_s^b|^2 ds + \displaystyle\int_{\{T\wedge\tau\}}^T |Z_s^a(\tau)|^2 ds\Big] \end{array}$$

On the set $\{\tau < t\}$, we use the fact that

$$\begin{aligned}\mathbb{E}_t^{\mathbb{G}}\Big[\sup_{t\le s\le T}|Y_s|^2\Big] &= \mathbb{E}_t^{\mathbb{G}}\Big[\sup_{t\le s\le T}|Y_s^a(\tau)|^2\Big]=\frac{1}{G_t}\mathbb{E}_t^{\mathbb{F}}\Big[\sup_{t\le s\le T}|Y_s^a(\theta)|^2\alpha_T(\theta)\Big]\\ &\le ke^{kt}\mathbb{E}_t^{\mathbb{F}}\Big[\sup_{t\le s\le T}|Y_s^a(\theta)|^2\Big]\le C\mathbb{E}_t^{\mathbb{F}}\Big[\sup_{t\le s\le T}|Y_s^a(\theta)|^2\Big]\ .\end{aligned}$$

On the set $\{t \le \tau\}$, we remark

$$\mathbb{E}_t^{\mathbb{G}}\Big[\sup_{t\le s\le T}|Y_s|^2\Big] \le \mathbb{E}_t^{\mathbb{F}}\Big[\sup_{t\le s\le T}|Y_s^b|^2\Big]+\mathbb{E}_t^{\mathbb{G}}\Big[\sup_{T\wedge\tau\le s\le T}|Y_s^a(\tau)|^2\Big]\ .$$

From

$$\mathbb{E}_t^{\mathbb{G}}\Big[\sup_{T\wedge\tau\le s\le T}|Y_s^a(\tau)|^2\Big]=\frac{1}{\alpha_t(\tau)}\mathbb{E}_t^{\mathbb{F}}\Big[\sup_{T\wedge\theta\le s\le T}|Y_s^a(\theta)|^2\alpha_T(\theta)\Big]_{\theta=\tau}$$

and the fact that α is bounded, we have

$$\mathbb{E}_t^{\mathbb{G}}\Big[\sup_{T\wedge\tau\le s\le T}|Y_s^a(\tau)|^2\Big]\le C\frac{1}{\alpha_t(\tau)}\mathbb{E}_t^{\mathbb{F}}\Big[\sup_{T\wedge\theta\le s\le T}|Y_s^a(\theta)|^2\Big]_{\theta=\tau}\ .$$

We proceed in the same way for the part $\int_{T\wedge\tau}^T|Z_s^a(\tau)|^2ds$.
Using (4.2.7)-(4.2.9) we can conclude.

4.2.4 Uniqueness of the solution

In this part we consider the uniqueness of the solution of the ABSDE (4.2.1). Suppose that this ABSDE has two solutions (Y,Z,U) and $(\bar{Y},\bar{Z},\bar{U})$. Each process admits a unique decomposition under the form (Y^b,Z^b,U^b)-$(Y^a(\tau),Z^a(\tau))$ and $(\bar{Y}^b,\bar{Z}^b,\bar{U}^b)$-$(\bar{Y}^a(\tau),\bar{Z}^a(\tau))$. Moreover we know (Y^b,Z^b) and $(\bar{Y}^b,\bar{Z}^b)$ are solution of the ABSDE (4.2.5), thus by uniqueness of the solution of the ABSDE (4.2.5) from [PY09, Theorem 4.2] we get that $Y^b=\bar{Y}^b$ and $Z^b=\bar{Z}^b$. We have with the same arguments $Y^a(\tau)=\bar{Y}^a(\tau)$ and $Z^a(\tau)=\bar{Z}^a(\tau)$. Moreover we have $U_t=(Y_t^a(t)-Y_t^b)\mathbb{1}_{\{t\le\tau\}}$, thus $U=\bar{U}$. Finally we get the uniqueness of the solution of the ABSDE (4.2.1).

4.3 ABSDE with jump of type (4.0.2)

We assume that Hypotheses 4.1.1, 4.1.2 and 4.1.3 hold. We consider in this section an ABSDE of the following form : find a triple $(Y,Z,U)\in\mathcal{S}^2_{\mathbb{G}}[0,T+\delta]\times L^2_{\mathbb{G}}[0,T+\delta]\times L^2\tau$ satisfying

$$\begin{cases}-dY_t &= \mathbb{E}_t^{\mathbb{G}}\big[f(t,Y_t,Y_{t+\delta},Z_t,Z_{t+\delta},U_t,U_{t+\delta})\big]dt-Z_tdB_t-U_tdH_t\ ,\quad 0\le t\le T\ ,\\ Y_{T+t} &= \xi_{T+t}\ ,\quad 0\le t\le\delta\ ,\\ Z_{T+t} &= P_{T+t}\ ,\quad U_{T+t}=Q_{T+t}\mathbb{1}_{\{T+t\le\tau\}}\ ,\quad 0<t\le\delta\ ,\end{cases}\tag{4.3.1}$$

with the following hypotheses

Hypotheses 4.3.1 *Suppose that :*

(i) The terminal conditions satisfy $\xi\in\mathcal{S}^2_{\mathbb{G}}[T,T+\delta]$, $P\in L^2_{\mathbb{G}}[T,T+\delta]$ *and* $Q\in L^2_{\mathbb{F}}[T,T+\delta]$ *and* $\sup_\theta\xi^a(\theta)\in\mathcal{S}^2_{\mathbb{F}}[T,T+\delta]$ *and* $\sup_\theta P^a(\theta)\in L^2_{\mathbb{F}}[T,T+\delta]$.

(ii) *The generator* $f : \Omega \times [0,T] \times \mathbb{R}^6 \to \mathbb{R}$ *is Lipschitz, that means there exists a constant* C *such that for any* $t \in [0,T]$, *for any* $\vec{\mathbf{y}}$ *and* $\vec{\mathbf{y}'}$ *in* $\mathbb{R}^6$, *one has* $|f(t,\vec{\mathbf{y}}) - f(t,\vec{\mathbf{y}'})| \leq C|\vec{\mathbf{y}} - \vec{\mathbf{y}'}|$.

(iii) $f(\cdot,\vec{\mathbf{y}})$ *is* $\mathbb{G}$*-optional.*

(iv) *There exists a constant* C *such that* $|f(s,\vec{\mathbf{0}})| \leq C$.

Proposition 4.3.2 *Under Hypotheses 4.3.1, the ABSDE (4.3.1) has a solution.*

Proceeding as before, on the set $\{\tau \leq t\}$ we consider,

$$\begin{cases} -dY_t^a(\tau) &= \mathbb{E}_t^{\mathbb{G}}\big[f^a(t,\tau,Y_t^a(\tau),Y_{t+\delta}^a(\tau),Z_t^a(\tau),Z_{t+\delta}^a(\tau),0,0)\big]dt \\ & \qquad -Z_t^a(\tau)dB_t\ , \quad \tau \leq t \leq T\ , \\ Y_{T+t}^a(\tau) &= \xi_{T+t}^a(\tau)\ , \quad 0 \leq t \leq \delta\ , \\ Z_{T+t}^a(\tau) &= P_{T+t}^a(\tau)\ , \quad 0 < t \leq \delta\ , \end{cases} \tag{4.3.2}$$

whereas, due to the uniqueness of pre-default parts we consider

$$\begin{cases} -dY_t^b &= \mathbb{E}_t^{\mathbb{G}}\big[f^b(t,Y_t^b,Y_{t+\delta},Z_t^b,Z_{t+\delta},U_t^b,U_{t+\delta})\big]dt - Z_t^b dB_t\ , \quad 0 \leq t \leq T\ , \\ Y_{T+t}^b &= \xi_{T+t}^b\ , \quad 0 \leq t \leq \delta\ , \\ Z_{T+t}^b &= P_{T+t}\ , \quad U_{T+t}^b = Q_{T+t}\ , \quad 0 < t \leq \delta\ . \end{cases} \tag{4.3.3}$$

4.3.1 Study of the Equation (4.3.2)

Using the same arguments as in Subsection 4.2.1 we study the family of ABSDEs :

$$\begin{cases} -dY_t^a(\theta) &= \mathbb{E}_t^{\mathbb{F}}\big[f^a(t,\theta,Y_t^a(\theta),Y_{t+\delta}^a(\theta),Z_t^a(\theta),Z_{t+\delta}^a(\theta),0,0)\big]dt\ , \\ & \qquad -Z_t^a(\theta)dB_t\ , \quad \theta \leq t \leq T\ , \\ Y_{T+t}^a(\theta) &= \xi_{T+t}^a(\theta)\ , \quad 0 \leq t \leq \delta\ , \\ Z_{T+t}^a(\theta) &= P_{T+t}^a(\theta)\ , \quad 0 < t \leq \delta\ . \end{cases}$$

This ABSDE can be written under the following form :

$$\begin{cases} -dY_t^a(\theta) &= g(t,\theta,Y_t^a(\theta),Y_{t+\delta}^a(\theta),Z_t^a(\theta),Z_{t+\delta}^a(\theta))dt - Z_t^a(\theta)dB_t\ , \quad \theta \leq t \leq T\ , \\ Y_{T+t}^a(\theta) &= \xi_{T+t}^a(\theta)\ , \quad 0 \leq t \leq \delta\ , \\ Z_{T+t}^a(\theta) &= P_{T+t}^a(\theta)\ , \quad 0 < t \leq \delta\ , \end{cases} \tag{4.3.4}$$

which is on the form of Proposition 4.1.8. The Lipschitz condition on g follows from the hypothesis on f. The square integrability of $g(t,\vec{\mathbf{0}}) = \mathbb{E}_t^{\mathbb{F}}\big[f^a(t,\theta,\vec{\mathbf{0}})\big]$ follows, as in Subsection 4.2.1 from the boundedness hypothesis of $f(t,\vec{\mathbf{0}})$. Thus from Proposition 4.1.8 we get the existence of a unique solution to this ABSDE satisfying

$$\mathbb{E}_t^{\mathbb{F}}\left(\sup_{s\leq T}(Y_s^a(\theta))^2 + \int_t^T (Z_s^a(\theta))^2 ds\right) \leq C\mathbb{E}_t^{\mathbb{F}}\Big(|\xi_T^a(\theta)|^2 + \int_T^{T+\delta}\big(|\xi_u^a(\theta)|^2 + |P_u^a(\theta)|^2\big)du + \int_t^T |g(u,\theta,\vec{\mathbf{0}})|^2 du\Big)\ . \tag{4.3.5}$$

4.3.2 Study of the Equation (4.3.3)

Using the same arguments as in Subsection 4.2.2 we are lead to consider

$$\left\{ \begin{array}{lcl} -dY_t^b & = & g(t, Y_t^b, Y_{t+\delta}^b, Z_t^b, Z_{t+\delta}^b)dt - Z_t^b dB_t \ , \quad 0 \le t \le T \ , \\ Y_{T+t}^b & = & \xi_{T+t}^b \ , \quad 0 \le t \le \delta \ , \\ Z_{T+t}^b & = & P_{T+t}^b \ , \quad 0 < t \le \delta \ , \end{array} \right. \tag{4.3.6}$$

where

$$g(t, y, \zeta, z, \eta) = \frac{1}{G_t} \int_t^{t+\delta} \mathbb{E}_t^{\mathbb{F}}[f^b(t, y, Y_{t+\delta}^a(\theta), z, Z_{t+\delta}^a(\theta), Y_t^a(t) - y, 0)\alpha_{t+\delta}(\theta)]d\theta$$

$$+ \frac{1}{G_t} \mathbb{E}_t^{\mathbb{F}} \left[f^b \left(t, y, \zeta, z, \eta, \ Y_t^a(t) - y, \ (Y_{t+\delta}^a(t+\delta) - \zeta) G_{t+\delta} \mathbb{1}_{\{t+\delta \le T\}} + Q_{t+\delta} G_{t+\delta} \mathbb{1}_{\{t+\delta > T\}} \right) \right] \ .$$

We show that the hypotheses of Proposition 4.1.8 are satisfied. First, we show that the driver is Lipschitz. Using the fact that f is Lipschitz we get

$$|f^b(t, \vec{\mathbf{y}}) - f^b(t, \vec{\mathbf{y}'})| \ \le \ \frac{1}{G_t} \left| \mathbb{E}_t^{\mathbb{F}}[\,|f(t, \vec{\mathbf{y}}) - f(t, \vec{\mathbf{y}'})| \mathbb{1}_{\{t<\tau\}}] \right| \ \le \ C|\vec{\mathbf{y}} - \vec{\mathbf{y}'}| \ .$$

It follows that, setting $\vec{\mathbf{Y}} = (y, Y_{t+\delta}^a(\theta), z, Z_{t+\delta}^a(\theta), Y_t^a(t) - y)$, there exists a constant C such that

$$\begin{aligned} & \left| \mathbb{E}_t^{\mathbb{F}}[f^b(t, \vec{\mathbf{Y}}, 0)\alpha_{t+\delta}(\theta)] - \mathbb{E}_t^{\mathbb{F}}[f^b(t, \vec{\mathbf{Y}'}, 0)\alpha_{t+\delta}(\theta)] \right| \\ & \qquad \le C(|y - y'| + |z - z'|)\mathbb{E}_t^{\mathbb{F}}(\alpha_{t+\delta}(\theta)) = C(|y - y'| + |z - z'|)\alpha_t(\theta) \ , \end{aligned}$$

where we have used the fact that $\alpha(\theta)$ is a martingale. Hence, using the fact that $\int_0^\infty \alpha_t(\theta)d\theta = 1$, we get

$$\int_t^{t+\delta} \mathbb{E}_t^{\mathbb{F}}[|f^b(t, \vec{\mathbf{Y}}, 0) - f^b(t, \vec{\mathbf{Y}'}, 0)|\,] \alpha_{t+\delta}(\theta)]d\theta \ \le \ C(|y - y'| + |z - z'|) \ .$$

In the other hand, using the Lipschitz property of f^b, and the fact that G is upper bounded and noting

$$\begin{aligned} k(t, y, \zeta, z, \eta) := \ & \mathbb{E}_t^{\mathbb{F}} \big[f^b \big(t, y, \zeta, z, \eta, Y_t^a(t) - y, \ (Y_{t+\delta}^a(t+\delta) - \zeta) G_{t+\delta} \mathbb{1}_{\{t+\delta \le T\}} \\ & \qquad + Q_{t+\delta} G_{t+\delta} \mathbb{1}_{\{t+\delta > T\}} \big) \big] \ , \end{aligned}$$

there exists a constant C such that one has,

$$|k(t, y, \zeta, z, \eta) - k(t, y', \zeta', z', \eta')| \le C \left(|y - y'| + |z - z'| + \mathbb{E}_t^{\mathbb{F}}(|\zeta - \zeta'|) + \mathbb{E}_t^{\mathbb{F}}(|\eta - \eta'|) \right) \ .$$

It follows (using one more time that G is lower bounded) that there exists a constant C such that

$$|g(t, y, \zeta, z, \eta) - g(t, y', \zeta', z', \eta')| \le C \left(|y - y'| + |z - z'| + \mathbb{E}_t^{\mathbb{F}}(|\zeta - \zeta'|) + \mathbb{E}_t^{\mathbb{F}}(|\eta - \eta'|) \right)$$

and the Lipschitz property holds.

The integrability condition of

$$\begin{aligned} g(t, \vec{\mathbf{0}}) \ = \ & \frac{1}{G_t} \int_t^{t+\delta} \mathbb{E}_t^{\mathbb{F}}[f^b(t, 0, Y_{t+\delta}^a(\theta), 0, Z_{t+\delta}^a(\theta), Y_t^a(t), 0)\alpha_{t+\delta}(\theta)]d\theta \\ & + \ \frac{1}{G_t} \mathbb{E}_t^{\mathbb{F}}[f^b(t, 0, 0, 0, 0, Y_t^a(t), \ Y_{t+\delta}^a(t+\delta)\, G_{t+\delta} \mathbb{1}_{\{t+\delta < T\}} + Q_{t+\delta}\, G_{t+\delta} \mathbb{1}_{\{t+\delta \ge T\}})] \end{aligned}$$

follows with the same arguments as in Section 4.2.2.

We can also consider the integrability of the solutions (Y, Z, U) for the ABSDE (4.3.1), where

$$\begin{aligned} Y_t &= Y_t^b \mathbb{1}_{\{t<\tau\}} + Y_t^a(\tau)\mathbb{1}_{\{t\geq\tau\}} \ , \\ Z_t &= Z_t^b \mathbb{1}_{\{t\leq\tau\}} + Z_t^a(\tau)\mathbb{1}_{\{t>\tau\}} \ , \\ U_t &= (Y_t^a(t) - Y_t^b)\mathbb{1}_{\{t\leq\tau\}} \ . \end{aligned}$$

This can be done using the same methodology than the one in the previous section, since [PY09, Proposition 4.4] is valid in the case of ABSDE (4.3.1) and we obtain similar results.

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Grossissement de filtrations et applications à la finance

Mots clés : Grossissement de filtration, équations différentielles stochastiques rétrogrades, mathématiques financières

Cette thèse se compose de quatre parties indépendantes. Le fil conducteur de celle-ci est le grossissement de filtration.
Dans la première partie, nous présentons des résultats classiques de grossissement de filtration en temps discret. Nous étudions quelques exemples dans le cadre du grossissement initial de filtration. Dans le cadre du grossissement progressif nous donnons des conditions pour obtenir la propriété d'immersion des martingales. Nous donnons également diverses caractérisations des pseudo temps d'arrêt et nous énonçons des propriétés pour les temps honnêtes.
Dans la deuxième partie, nous nous intéressons à la détermination du prix de produits à annuités variables dans le cadre de l'assurance vie. Pour cela nous considérons deux modèles, dans ces deux modèles nous considérons que le marché est incomplet et nous adoptons l'approche par prix d'indifférence. Dans le premier modèle nous supposons que l'assuré procède à des retraits aléatoires et nous calculons la prime d'indifférence par des méthodes standards en contrôle stochastique. Nous sommes conduits à résoudre des équations différentielles stochastiques rétrogrades (EDSR) avec un saut. Nous fournissons un théorème de vérification et nous donnons les stratégies optimales associées à nos problèmes de contrôle. De ceux-ci, nous tirons une méthode de calcul pour obtenir la prime d'indifférence. Dans le second modèle nous proposons la même approche que dans le premier modèle mais nous supposons que l'assuré effectue des retraits qui correspondent au pire cas pour l'assureur. Nous sommes alors amenés à traiter un problème de max-min.
Dans la troisième partie, nous étudions la relation des solutions d'EDSR dans deux filtrations différentes. Nous étudions alors la relation entre ces deux solutions. Nous appliquons ces résultats pour obtenir le prix d'indifférence dans les deux filtrations, c'est-à-dire le prix auquel un agent aurait le même niveau d'utilité attendue en utilisant des informations supplémentaires.
Dans la quatrième partie, nous considérons des équations différentielles stochastiques rétrogrades avancées (EDSRAs) avec un saut. Nous étudions l'existence et l'unicité d'une solution à ces EDSRAs. Pour cela nous utilisons la décomposition des processus à sauts liée au grossissement progressif de filtration pour nous ramener à l'étude d'EDSRAs browniennes avant et après le temps de saut.

Université Paris-Saclay — École doctorale de mathématiques Hadamard (EDMH)

Université Evry val-d'essonne

LaMME

Filtration Enlargement with applications to finance

Key words : Filtration enlargement, Backward stochastic differential equations, mathematical finance

This thesis consists of four independent parts. The topic in common is the filtration enlargement.
In the first part, we present classical results for filtration enlargement in discrete time. We study some examples in initial enlargement of filtration. For the progressive enlargement of filtration, we give conditions for immersion martingale property. We also provide various characterizations of pseudo-stopping times and properties for honest times.
In the second part, we are interested in determining the indifference price for variable annuities products. For this we consider two models, in both models we suppose that the market is incomplete and we adopt the approach of indifference price. In the first model we assume that the insured performs random withdrawals. Following indifference pricing theory, we define indifference fee rate for the insurer as a solution of an equation involving two stochastic control problems. Relating these problems to backward stochastic differential equations with a jump, we provide a verification theorem and give the optimal strategies associated to our control problems. From these, we derive a computation method to get indifference fee rates. We conclude this part with numerical illustrations of indifference fees sensibilities with respect to parameters. In the second model we propose the same approach as in the first model but we assume that the insured makes withdrawals that match the worst case for the insurer.
In the third part, we study the relation of the solutions of BSDEs in two filtrations. As an application, one of our goals is to find the indifference price of information, i.e. the price at which an agent would have the same expected utility level using extra information as by not doing so.
In the fourth part, we investigate advanced backward stochastic differential equations (ABSDE) with a jump. We study the existence and uniqueness of the solution to these ABSDEs. For this we relate the solution of the ABSDEs wth jumps to Brownian ABSDEs associated to the original ABSDE before and after the time jump.