Extremal Functions for Graph Linkages and Rooted Minors

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Paul Wollan

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Extremal Functions for Graph Linkages and Rooted Minors

Approved by:

Dr. Robin Thomas, Advisor
School of Mathematics
Georgia Institute of Technology

Dr. Richard Duke
School of Mathematics
Georgia Institute of Technology

Dr. Dana Randall
College of Computing
Georgia Institute of Technology

Dr. William Trotter
School of Mathematics
Georgia Institute of Technology

Dr. Xingxing Yu
School of Mathematics
Georgia Institute of Technology

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SUMMARY

A graph G is k-linked if for any $2k$ distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$ there exist k vertex disjoint paths $P_1, \ldots, P_k$ such that the endpoints of P_i are s_i and t_i. Determining the existence of graph linkages is a classic problem in graph theory with numerous applications. In this thesis, we examine sufficient conditions that guarantee a graph to be k-linked and give the following theorems.

(A) Every $2k$-connected graph on n vertices with $5kn$ edges is k-linked.

(B) Every 6-connected graph on n vertices with $5n - 14$ edges is 3-linked.

Theorem (A) improves upon the previously best known bound due to Bollobás and Thomason stating that $11kn$ edges suffice. The edge bound in Theorem (B) is optimal in that there exist 6-connected graphs on n vertices with $5n - 15$ edges that are not 3-linked.

The methods used prove Theorems (A) and (B) extend to a more general structure than graph linkages called *rooted minors*. In the introductory chapter we present the graph theoretical background necessary to state the results, including a brief history of the study of linkages and rooted minors. In Chapters 2 and 3 we proceed to describe the techniques used to prove Theorems (A) and (B) in their full generality for rooted minors. The techniques are placed within the context of previous methods and tools for finding extremal functions for graph linkages and minors. The remaining chapters contain the technical proofs of Theorems (A) and (B), as well as theorems giving edge bounds to contain various families of rooted minors.

We conclude with two graph theoretical applications of graph linkages. The first is to the problem of determining when a small number of vertices can be used to cover all the odd cycles in a graph. The second is a simpler proof of a result of Böhme, Maharry and Mohar on complete minors in huge graphs of bounded tree-width.

CHAPTER 1

INTRODUCTION

We begin this chapter with a brief overview of graph theoretical terms and concepts. In Section 1.2, we define graph linkages and rooted minors and develop some of the special language we will use throughout the proofs in this document. In Section 1.3, we give a broad overview of the extremal theory of graphs, with a particular look to the extremal theory of minors and graph linkages. We conclude Section 1.3 with two different aspects of classical questions that have drawn recent interest. We then state the main results of this thesis.

1.1 Graph Basics

A *graph* G is a pair $(V(G), E(G))$ where $E(G)$ is a set of 2-element subsets of $V(G)$. The elements of $V(G)$ are called the *vertices* of the graph, and the elements of $E(G)$ are called *edges*. Graphs are typically represented by a collection of points corresponding to the vertices with the edges indicated by a line segment connecting the two points in the edge. A common example we will encounter is the *complete graph* on t vertices, denoted K_t, which consists of t vertices where every pair of vertices is an edge of the graph.

Two vertices u and v in a graph are said to be *adjacent* if $\{u, v\}$ is contained in $E(G)$. We will typically write uv or vu for the edge $\{u, v\}$. Given an edge containing u and v, we say that u and v are *neighbors*; the set of all neighbors of a vertex v is called the *neighborhood* of v. The neighborhood of a vertex v will be represented by $N(v)$. Similarly for a set X of vertices, $N(X)$ will denote the set of neighbors of X, in other words, $(\bigcup_{x \in X} N(x)) - X$. The size of the neighborhood of v will be called the *degree* of v. The minimum degree over all the vertices in a graph G will be denoted by $\delta(G)$. Given an edge $e = uv$, we say that u and v are the *endpoints* of e; the edge e is *incident* with u and v. Two edges e and f are adjacent if they share an endpoint. For a given set X of vertices, we will denote by $\partial(X)$

the set of vertices of X with at least one neighbor in $V(G) - X$.

For a graph $G = (V, E)$, if $V' \subseteq V$ and E' is a subset of E such that every edge in E' has both endpoints in V', then the graph defined by the pair (V', E') is a *subgraph* of the graph G. Given a graph $G = (V, E)$, if X is a subset of vertices, we denote by $G[X]$ the subgraph with vertex set X and edge set containing every edge of G with both endpoints contained in X. We say $G[X]$ is *induced* by the set X. An *induced subgraph* of G is a subgraph equal to $G[X]$ for some set of vertices X. For a set of vertices X, we denote by $G - X$ the subgraph $G[V - X]$. Given a subgraph H of G, we will use $G - H$ as shorthand notation for $G - V(H)$. When X consists of a single vertex v, we will write $G - v$ instead of $G - \{v\}$. For a given edge e in G, we will write $G - e$ to denote the subgraph $(V(G), E(G) - e)$.

A *path* in a graph G is a subgraph P of G where we can label the vertices of P as $v_0, \ldots, v_t$ such that the edges of P are $\{v_0v_1, v_1v_2, \ldots, v_{t-1}v_t\}$. The vertices v_0 and v_t then are called the *endpoints* or *ends* of the path and the remaining vertices of P are called the *internal vertices* of P. The *length* of a path is the number of edges in the path. We will consider the graph on one vertex with no edges the *trivial* path of length zero. For notation, given two vertices v_i and v_j on a path P with $i \leq j$, we denote by v_iPv_j the induced subgraph of P on the vertex set $\{v_i, v_{i+1}, \ldots, v_{j-1}, v_j\}$.

Given a path P, we say that the path P *connects* or *links* the endpoints. Extending this notion further, we say a graph G is *connected* if for any two vertices u and v in G there exists a path P in G connecting the two. If a graph is not connected, we call each maximal connected subgraph a *component* of the graph. We quantify the notion of connectivity as follows. A graph G is *k-connected* if for any set X of strictly less than k vertices, $G - X$ is connected. A *separation* of a graph G is a pair (A, B) of subsets of vertices of G such that every vertex is in at least one of A or B and no edge has one endpoint in $A - B$ and the other end in $B - A$. The *order* of a separation (A, B) is the size of $A \cap B$. A separation is *trivial* if $A \subseteq B$ or $B \subseteq A$. Also, often we will be considering separations when we have fixed some set X of vertices. The pair (A, B) is a separation of (G, X) where G is a graph and X a fixed subset of vertices if (A, B) is a separation and $X \subseteq A$. It is an easy exercise to prove that a graph G is k-connected if and only if there exists no non-trivial separation of order at most $k-1$.

We now consider a more general notion of containment than the subgraph relation.

Definition 1 *The graph G contains a graph H as a* minor *if there exist pair-wise disjoint sets $\{S_u \subseteq V(G) | u \in V(H)\}$ such that for every u, $G[S_u]$ is a connected subgraph and for every edge uv in H, there exists an edge of G with one end in S_u and the other end in S_v.*

The S_w will be referred to as the *branch sets* of the H minor in G. There is an equivalent way to think of graph minors. We have already defined the deletion of an edge e in a graph G to be the subgraph $G-e$. Given an edge $e=uv$, we define the *contraction* of e, denoted G/e, to be obtained by deleting the two vertices u and v and adding a new vertex v_e adjacent to every neighbor of u or v. Formally, $G/e = ((V(G) - \{u,v\}) \cup v_e, E(G - \{u,v\}) \cup \{v_e x : x \text{ is adjacent } u \text{ or } v\})$. Then a graph G contains H as a minor if and only if a graph isomorphic to H can be obtained from a subgraph of G by repeatedly contracting and deleting edges. A graph G contains H as a *topological minor* if there exist vertices $\{v_u | u \in V(H)\}$ in G and paths P_e for every edge e in $E(H)$ such that the paths $\{P_e : e \in E(H)\}$ are pairwise internally disjoint and for every edge $e = xy \in E(H)$, the ends of P_e are v_x and v_y, and, further, P_e has no internal vertex equal to v_w for some $w \in V(H)$.

We will generally follow the notation of Diestel [11]. See [11] for any further undefined concepts.

1.2 Graph Linkages and Rooted Minors

A classic Theorem of Menger states that if a graph G is k-connected, then for any two disjoint sets S and T contained in $V(G)$ where $|S| = |T| = k$, there exist k disjoint paths $P_1, \ldots, P_k$ such that P_i has one endpoint in S and one in T. We consider a strengthening of this property.

1.2.1 Definition of k-linked Graphs

Consider the above situation where we are given a graph G and two disjoint sets of k vertices S and T. We label the vertices of S and T and want to find k disjoint paths with the added constraint that the paths $P_1, \ldots, P_k$ connect S to T in some predetermined order.

Definition 2 *A graph is said to be* k-linked *if for every set of* $2k$ *distinct vertices* $\{s_1, \ldots, s_k, t_1, \ldots, t_k\}$ *there exist disjoint paths* $P_1, \ldots, P_k$ *such that the endpoints of* P_i *are* s_i *and* t_i*.*

We now lay out a more exact language for discussing sets of disjoint paths and when these desired disjoint paths exist.

Definition 3 *A* linkage *in a graph* G *is a subgraph* $\mathcal{P}$ *where every connected component of* $\mathcal{P}$ *is a path.*

Given a linkage $\mathcal{P}$, we will use the standard notation $V(\mathcal{P})$ for the set of vertices and $E(\mathcal{P})$ for the set of edges of the linkage. Sometimes we shall regard $\mathcal{P}$ as a set of its components and write $P \in \mathcal{P}$ to mean that the path P is a component of $\mathcal{P}$. If every member of $\mathcal{P}$ has one end in a set X and the other in a set Y, we say that $\mathcal{P}$ is a linkage from X to Y. In that case, we designate, for each $P \in \mathcal{P}$, its end in X the *origin* and its end in Y as the *terminus* of P. If both ends belong to $X \cap Y$, then we make an arbitrary choice.

When attempting to prove that a given graph is k-linked, it will be convenient to restrict ourselves to one specific subset of vertices with a specific labeling.

Definition 4 *Given a graph* G *and a set* $X \subseteq V(G)$*, the pair* (G, X) *is* k-linked *if for any distinct vertices* $s_1, \ldots, s_k, t_1, \ldots, t_k \in X$*, there exist disjoint paths* $P_1, \ldots, P_k$ *such that the ends of* P_i *are* s_i *and* t_i *and no* P_i *has an internal vertex contained in* X*.*

A pair (G, X) is *linked* if it is k-linked for all k.

The following definition formalizes the idea of a particular labeling of vertices for a desired linkage.

Definition 5 *For a set* X *of vertices in a graph* G*, a* linkage problem *on* X *is a set of disjoint subsets of* X *of size* 2.

Given a pair (G, X), a linkage problem $\mathcal{L} = \{\{s_1, t_1\}, \ldots, \{s_k, t_k\}\}$ is *feasible* in (G, X) if there exists a linkage $\mathcal{P}$ such that the components of $\mathcal{P}$ can be labeled $P_1, \ldots, P_k$ such that the ends of P_i are s_i and t_i and P_i has no internal vertex in the set X. Such a linkage is said to *solve* the linkage problem $\mathcal{L}$.

1.2.2 Definition of Rooted Minor and H-Universal Graphs

We will focus on the question where we fix some set X of the vertices of G, and then ask how many edges must G have in order to ensure there exists an H minor where each branch set of the minor contains a pre-specified vertex $x \in X$.

Rigorously, we define a π-rooted H minor as follows.

Definition 6 *Let G and H be graphs and $X \subseteq V(G)$ with $|X| = |V(H)|$. Let $\pi : X \to V(H)$ be an injection. Then the pair (G, X) contains a* π-rooted H minor *if there exist $\{S_u | u \in V(H)\}$ forming the branch sets of an H minor such that for every $x \in X$, $x \in S_{\pi(x)}$.*

We use rooted minors to define a generalization of the property of being k-linked.

Definition 7 *Given a fixed graph H on t vertices, a graph G is H-*universal *if there exists a π-rooted H minor for every set X of vertices with $|X| = t$ and every injective map $\pi : X \to V(H)$.*

To see that this is in fact a generalization of k-linked, fix H to be the graph on $2k$ vertices consisting of k disjoint edges. A graph G is k-linked if and only if it is H-universal.

1.3 Previous and Related Work

In this section we discuss the historical study of graph linkages and rooted minors.

1.3.1 Extremal Graph Theory

The traditional extremal theory of graphs considers some given property $\mathcal{P}$, and then asks how many edges must a graph G have, as a function of the number of vertices in G, to ensure that G has property $\mathcal{P}$. Perhaps the classic theorem in this area is Turán's Theorem. Let $ex(n, K_r)$ be the maximal number of edges a graph on n vertices can have and still have no K_r subgraph. We can construct a family of very dense graphs not containing K_r as follows: Let $T^{r-1}(n)$ be the graph on n vertices with the vertices split as evenly as possible into $r-1$ pairwise disjoint sets $\mathcal{A} = \{A_1, \ldots, A_{r-1}\}$ and add edges between any two vertices x and y if they lie in distinct sets of $\mathcal{A}$, i.e. there does not exist an index i such that $x, y \in A_i$. The graph $T^{r-1}(n)$ has approximately $\binom{n}{2}(1 - \frac{1}{r-1})$ edges. It is a theorem of Turán that $|E(T^{r-1}(n))|$ is the maximum number of edges possible in a graph on n vertices not containing K_r as a subgraph.

Theorem 1.3.1 *[Turán [11]] Let G be a graph on n vertices with $ex(n,r)$ edges and no K_r subgraph. Then G is isomorphic to $T^{r-1}(n)$.*

When seeking a particular subgraph H in a graph G, the number of edges necessary is typically some constant proportion of the total number of edges possible in G. Surprisingly, the required number of edges depends on the the chromatic number of H. The *chromatic number* of a graph H, denoted $\chi(H)$, is the minimum number of colors necessary to assign a color to every vertex of H such that no two adjacent vertices receive the same color. Let $ex(n,H)$ be the maximal number of edges a graph on n vertices can contain without containing H as a subgraph. Then the the fraction of total edges a graph can have and still have no H subgraph is determined by a function of the chromatic number of H.

Theorem 1.3.2 *[Erdős, Simonovits, Stone [11]] Let H be a graph with at least one edge. Then*

$$\lim_{n\to\infty} ex(n,H)\binom{n}{2}^{-1} = \frac{\chi(H)-2}{\chi(H)-1}.$$

A host of other results characterize edge bounds for different subgraphs. See [21] for an overview.

We will focus in this thesis on extremal questions related to the existence of minors and topological minors. These questions have a different feel than the subgraph questions of the previous paragraph. When we consider the number of edges necessary to ensure a graph G contains a particular H as a minor or topological minor, the number is a linear function in the $|V(G)|$. Mader made some of the first results in the area, showing [42] that there exists a function $h(r)$ such that every graph on n vertices with $h(r)n$ edges contains K_r as a topological minor. When considering K_r minors instead of topological minors, Mader proved in a ground breaking result [43] that there exists a constant c such that every graph on n vertices with $cr \log rn$ edges contains K_r as a minor. Random graph examples show that there exist graphs with $cr\sqrt{\log r}n$ edges and no K_r minor. This is in fact the correct edge bound, as shown by Kostochka in 1982 and independently by Thomason in 1984.

Theorem 1.3.3 *[Kostochka [36] and Thomason [58]] There exists a constant c such that every graph on n vertices with $cp\sqrt{\log p}n$ edges contains K_p as a minor.*

The function is optimal in terms of the order of magnitude as a function of p, and in [59], Thomason determined the exact asymptotic value of the constant c. The examples showing the bound is tight are determined by random graphs.

When the parameter p is restricted to small values, then the exact edge bounds for forcing a K_p minor are known.

Theorem 1.3.4 *[Mader [43]] Let p be an integer at most seven. Then every graph on $n \geq p$ vertices with at least $(p-2)n - \binom{p-1}{2} + 1$ edges contains a K_p minor.*

The theorem was first proven for $p \leq 5$ by Dirac [12]. The statement for $p \leq 6$ was later independently discovered by Győri [22]. The theorem is no longer true when $p = 8$; however Jorgensen completely characterized all graphs with $6n - \binom{5}{2} + 1$ edges that do not contain K_8 [26]. Song and Thomas extended this further by completely characterizing the graphs on n vertices with $7n - \binom{8}{2} + 1$ edges and no K_9 minor [55].

We will shortly return to the edge bounds for topological K_p minors, as they are historically intertwined with the study of graph linkages.

1.3.2 Ensuring a Graph is k-linked

A classic problem in the study of k-linked graphs has been an attempt to find the optimal function $f(k)$ such that every $f(k)$-connected graph is k linked. Larman and Mani [39] and independently Jung [28] were the first to show that such a function $f(k)$ exists. They in fact proved that every $2k$ connected graph containing a topological K_{3k} minor is k-linked. Given the result of Mader [42] stating that every graph with sufficiently high average degree contains a topological complete minor, Larman and Mani [39] and Jung [28] conclude the existence of such a function $f(k)$. It is possible, however, to utilize a large complete minor instead of a topological minor. Doing this, Robertson and Seymour proved in [49] that every $2k$-connected graph with a K_{3k} minor is k-linked. As above, using the bound of Theorem 1.3.3, one can conclude that $f(k) \leq ck\sqrt{\log k}$ for some constant c. There exist random graph examples demonstrating that the bound obtained by Kostochka and Thomason is the best possible in terms of the order of magnitude of k. Improvement on bounds for $f(k)$ required further refinements. Bollobás and Thomason proved in [5] that the complete graph in the proof of Robertson and Seymour could be replaced with a non-complete but very dense graph. They proved the necessary average degree to ensure a dense minor satisfying their constraints is linear in k, and in doing so proved that every $22k$-connected graph is k-linked. We will examine in some depth the proof techniques of Robertson and Seymour and Bollobás and Thomason in Chapter 2 showing that linkages can be found using large minors.

A motivation for the study of linkages is that they can be used to give a bound for the extremal function of complete topological minors. It was conjectured by Mader [42] and independently by Erdős and Hajnal [16] that there exists a constant c such that every graph on n vertices with cp^2n edges contains a topological K_p minor. Jung [27] observed that complete bipartite graphs show that this bound would be tight, up to the constant. Probabilistic arguments by Ajtai et al. [41], Erdős and Fajtlowicz [14] and Bollobás and Catlin [4] gave improved lower bounds on the optimal constant c possible in the conjecture of Erdős, Hajnal, and Mader. The conjecture was proven in the affirmative independently by Bollobás and Thomason [6] and Komlós and Szemerédi [35]. The proof of Bollobás and Thomason is a straight-forward application of their theorem that $f(k) \leq 22k$. We will present the proof of the bound on the extremal function by Bollobás and Thomason in Chapter 8.

Lower bounds for the function $f(k)$ are less well developed. Thomassen in [61] conjectured that every $2k + 2$ connected graph is k-linked. In general, this is not true. If we consider a K_{3k-1} complete graph with $2k$ disjoint edges subtracted, the resulting graph is $3k - 4$ connected, but not k-linked. However, this is currently the best known lower bound on $f(k)$. Also, there are no known examples of arbitrarily large graphs with connectivity strictly more than $2k+1$ that are not k-linked. It is possible that for every $k \geq 1$, there exists a constant N_k such that every $(2k+2)$-connected graph on at least N_k vertices is k-linked.

Robertson and Seymour in [50] prove that for any fixed value of k, there exists a polynomial time algorithm for determining whether a given linkage problem is feasible. See [48] for a summary of the argument. When k is part of the input, however, the problem of determining feasibility of a given linkage problem is NP-complete [29]. In fact, the problem is NP-complete even when restricted to planar graphs [40].

1.3.3 k-linked Graphs for Small Values of k

An obstruction to a graph being 2-linked is there exists a planar embedding of the graph with the vertices s_1, s_2, t_1, t_2 on the infinite face in that clockwise order. Then there do not exist disjoint paths connecting s_1 to t_1 and s_2 to t_2. Watkins [64] gave partial results on obstructions to 2-linkages, and Jung showed in [28] that in 4-connected graphs this is the only obstruction.

Theorem 1.3.5 *[Jung [28]] Let G be a 4-connected graph and let s_1, s_2, t_1, t_2 be four distinct vertices of G. Then there do not exist paths linking s_1 to t_1 and s_2 to t_2 if and only if there exists an embedding of G such that s_1, s_2, t_1, t_2 lay on the infinite face in that clockwise order.*

An immediate corollary of the above theorem is that every 4-connected graph on n vertices with $3n-6$ edges is 2-linked. Seymour, Shiloach, and Thomassen [52, 54, 61] all independently give a complete characterization of when a graph is 2-linked based on the above obstruction. They also give an efficient algorithm for solving 2-linkage problems. An immediate consequence of Theorem 1.3.5 is that $f(2)=6$.

The characterization of 2-linked graphs has proven extremely useful in structural graph theory. In addition to several usages in the Graph Minors series of Robertson and Seymour, it was used in the proof of Hadwiger's conjecture in the $k=5$ case [51]. Song and Thomas utilize it in the proof of the extremal function of K_9 minors [55]. We will also use it in finding the extremal function for 3-linked graphs.

In general, when a graph is 3-linked is less well understood. Recent work by Chen et al. [7] shows that the K_9 minor required to apply the theorem of Robertson and Seymour can be relaxed and every $2k$ connected graph containing K_9^- as a minor is 3-linked. The graph K_9^- is obtained by deleting a single edge from K_9. Further, Chen et al. prove an extremal function for K_9 minors proving that every 6-connected graph with minimum degree 18 is 3-linked. It follows that $f(3) \leq 18$. Song and Thomas in [55] showed that every 7-connected graph on n vertices with $7n - 28$ edges contains a K_9 minor. It follows by applying the theorem of Robertson and Seymour that $f(3) \leq 14$. Our Theorem 1.4.3 shows that $f(3) \leq 10$.

1.3.4 General Rooted Minors

Rooted minors have not been extensively studied before. Previous study of rooted minors has often focused on specific rooted minor structures. Robertson et al. in [51] completely characterize pairs (G, X), where G is a graph and X a set of four vertices, for which there exists a π-rooted K_4 minor for an arbitrary π.

Other research has examined what is referred to as a rooted $K_{2,t}(X)$ minors. This is related to our definition of rooted minor, but the structure is slightly different. To avoid any confusion with what we have defined to be rooted minors, we refer to them by the notation of Böhme et al. in [3] and call them labeled $K_{2,t}(X)$ minors.

Definition 8 *Let G be a graph and let X be a set of t vertices in G. Then G contains a* labeled *$K_{2,t}(X)$ minor if there exist pairwise disjoint sets of vertices $A_1, \ldots, A_t, B_1, B_2 \subseteq V(G)$ such that the following hold.*

1. *$A_i \cap X \neq \emptyset$ for all indices $i = 1, \ldots, t$,*
2. *$G[A_i]$ and $G[B_j]$ are connected subgraphs for all $i = 1, \ldots, t$ and $j = 1, 2$, and*
3. *for every $i = 1, \ldots, t$ and $j = 1, 2$, there exists an edge with one end in A_i and the other end in B_j.*

See Figure 1.3.4 for an example.

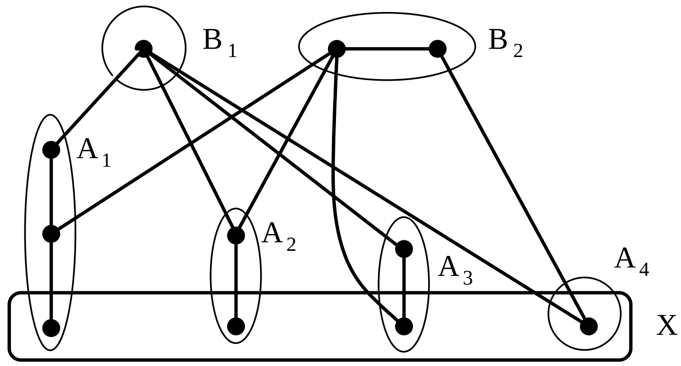

Figure 1: An example of a labeled $K_{2,4}(X)$ minor.

Jorgensen introduced labeled $K_{2,t}(X)$ minors. Specifically, he considered labeled $K_{2,4}(X)$ minors in the proof of the extremal function for $K_{4,4}$ in [25]. Böhme et al. independently studied when there exist large labeled $K_{2,t}(X)$ minors in planar graphs. Later, Jorgensen and Kawarabayashi examined extremal functions for $K_{3,4}(X)$ minors in [31] and Kawarabayashi examined labeled $K_{2,t}(X)$ minors in huge graphs in [30].

1.3.5 Forcing Large Minors with Connectivity

A natural question is how much connectivity, as a function of p, is necessary to force a graph to contain K_p as a minor. However, a classic theorem of Mader says that connectivity alone will not behave dramatically different than edge bounds.

Theorem 1.3.6 *[Mader [44]] Every graph on n vertices with $2tn$ edges contains a t-connected subgraph.*

Thus we see that the random graph example of a graph G with $c'p\sqrt{\log p}|V(G)|$ edges and no K_p minor contains a $(\frac{1}{2}c'p\sqrt{\log p})$-connected subgraph with no K_p minor.

In an effort to understand how connectivity affects the existence of large complete minors, another line of inquiry emerged. The random graph examples showing the optimal edge bound have the number of vertices bounded by a function of p. In fact, Myers [45] showed that any graph with average degree super-linear in p but no K_p minor is essentially a collection of pseudo-random graphs connected by small cut-sets. This has lead to speculation that linear (in p) connectivity may suffice when the graph is assumed to be large. In [1], Böhme, Maharry, and Mohar conjecture that there exists some constant c and a function $N(p)$ such that every cp-connected graph on at least $N(p)$ vertices contains K_p as a minor. Thomason strengthened this as follows.

Conjecture 1 *[Thomason [60]] There exists a function $N = N(p)$ such that every $(p+1)$-connected graph on at least $N(p)$ vertices contains K_p as a minor.*

This question was partially resolved was by Böhme et al. in [2] where they prove:

Theorem 1.3.7 *[Böhme et al. [2]] There exists a function $N = N(p)$ such that every $15p$-connected graph on at least $N(p)$ vertices contains K_p as a minor.*

Recently, Thomas has formulated a strengthening of the conjecture.

Conjecture 2 *[Thomas [56]] There exists a function $N = N(p)$ such that for every p connected graph G on at least $N(p)$ vertices that does not contain K_p as a minor has a set X of $p-5$ vertices such that $G - X$ is planar.*

The conjecture of Thomas extends a previous conjecture due to Jorgensen.

Conjecture 3 *[Jorgensen, [26]] For every 6-connected graph G that does not contain K_6 as a minor, there exists a vertex $v \in V(G)$ such that $G - v$ is planar.*

Thomas' conjecture was recently proven in the case $p = 6$ by DeVos et al. in [10].

Another conjecture considers what happens when we assume both a minimal level connectivity and some edge bound. Recall Theorem 1.3.4 of Mader that says for $p \leq 7$, every graph on n vertices with $(p-2)n - \binom{p-1}{2} + 1$ edges contains K_p as a minor. Seymour and Thomas conjecture that this is the correct edge bound when a basic level of connectivity is assumed.

Conjecture 4 *[Seymour and Thomas [55]] There exists a function $N = N(p)$ such that every $p-2$-connected graph on $n \geq N(P)$ vertices with $(p-2)n - \binom{p-1}{2} + 1$ edges contains K_p as a minor.*

The conjecture was proven in the case $p = 8$ by Jorgensen [26] and in the case $p = 9$ by Song and Thomas [55].

1.3.6 Packing Vertex Disjoint Cycles

The study of families of graphs with the Erdős-Pósa property is a classic problem in graph theory. We will consider an application of graph linkages to problems of this type in Chapter 8. We give a short discussion of the history of these problems here.

A family $\mathcal{F}$ of graphs has the *Erdős-Pósa property*, if for every integer k there exists an integer $f(k, \mathcal{F})$ such that every graph G contains either k vertex-disjoint subgraphs each isomorphic to a graph in $\mathcal{F}$ or a set C of at most $f(k, \mathcal{F})$ vertices such that $G - C$ has no subgraph isomorphic to a graph in $\mathcal{F}$. The term *Erdős-Pósa property* arose because in [15], Erdős and Pósa proved that the family of cycles has this property.

The Erdős-Pósa property has been proven for a wide range of families of graphs including different minor and topological minor families as well as variants on the original cycle problem of Erdős and Pósa. We will focus our attention here on problems of cycles whose lengths satisfy various properties. Thomassen showed in [62] that for any positive integer m, the set of cycles with length congruent to 0 (mod m) has the Erdős-Pósa property. However, there exist examples of d and m due to Dejter and Neumann-Lara [9] and an infinite family of graphs showing that the Erdős-Pósa does not hold for cycles of length congruent to d (mod m).

The family of odd cycles also does not have the Erdős-Pósa property. Thomassen [62] cites a result of Lovász and Schrijver characterize the graphs having no two disjoint odd cycles, relying Seymour's result on regular matroids [53]. One family are non-bipartite graphs embedded in the projective plane with all faces forming even cycles. Thomassen [62] observed that these graphs show that the Erdős-Pósa property does not hold for the family of odd cycles. Reed [47] showed that in a certain sense, these are the only counterexamples in that any graph that does not have a bounded set of vertices covering all odd cycles and does not have a large collection of disjoint odd cycles contains as a topological minor one of these projective planar examples above.

The projective planar examples showing that the Erdős-Pósa property does not hold for odd cycles are not 5-connected, however. One can hope that if a graph is highly connected compared to k, then the integer $F(k, \mathcal{F})$ as above exists. Motivated by this, Thomassen [62] was the first to prove that there exists a function $g(k)$ such that every $g(k)$-connected graph G has either k disjoint odd cycles or a vertex set X of order at most $2k-2$ such that $G - X$ is bipartite. Hence, he showed that the Erdős-Pósa property holds for odd cycles in highly connected graphs. Soon after that, Rautenbach and Reed [46] proved that the function $g(k) = 576k$ suffices. Very recently, Kawarabayashi and Reed [33] further improved the function to $g(k) = 24k$. The bound "$2k-2$" is best possible in a sense since a large bipartite graph with edges of a complete graph on $2k-1$ vertices added to one side of the bipartition set shows that no matter how large the connectivity is, there are no k disjoint odd cycles.

1.3.7 Extensions and Generalizations of Linkages

Several concepts immediately related to k-linkages have been studied. We can generalize the property of being k-linked to directed graphs or alternatively, to consider edge disjoint paths instead of vertex disjoint paths. A directed graph D is *k-directed-linked* if for any $2k$ distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$, there exist directed paths $P_1, \ldots, P_k$ such that the ends of P_i are s_i and t_i. When considering k-directed-linked problems, even the $k = 2$ case is NP-hard, as shown by Fortune et al. [19]. In fact there does not even exist an analogous function $f'(k)$ such that every $f'(k)$ strongly connected graph is k-directed linked by a result of Thomassen [63].

We can also extend the property of being k-linked to consider edge disjoint paths. We say a graph G is *weakly k-linked* if for any $2k$ distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$ there exist edge disjoint paths $P_1, \ldots, P_k$ such that the ends of P_i are s_i and t_i. Unlike in directed graphs, considering edge disjoint paths makes the problem is easier. The problem of solving a weak k-linkage problem can be modeled as a problem of integer multi-commodity flow. A variant of the integer multi-commodity flow problem was shown to be NP-hard by Evan, Itai and Schamir [17]. See [23] for more on the non-integer case of multi-commodity flow problems, but again the problem can be efficiently solved when k is bounded. When we consider the question of the amount of edge connectivity necessary to for a graph to be weakly k-linked, Huck [24] shows that every $(2k + 1)$-edge-connected graph is weakly k-linked, giving a bound only one more than the bound conjectured by Thomassen to be correct.

An alternative generalization of k-linked has recently received attention. While our methods do not directly apply to these problems, for completeness we briefly outline progress in this direction. Given a fixed graph H with vertex set $\{1, \ldots, t\}$, a graph G is *H-linked* if for any specified vertices $v_1, \ldots, v_t$ there exist paths $P_1, \ldots, P_{|E(H)|}$ indexed by the edges of H such that no two paths P_e and P_f have an internal vertex in common and for every edge $e = ij \in E(H)$, the path P_e connects v_i to v_j. If we let H be the graph consisting of $2k$ vertices and k disjoint edges, a graph is H-linked if and only if it is k-linked.

Independently, Kostochka and Yu [38, 37], Kostochka, Gould, and Yu [20], and Ferrara, et al. [18] have independently quantified exact minimal degree conditions to force a graph to be H-linked. Also, while not phrasing it in terms of being H-linked, Bollobás and Thomason prove in [5] that every $(22|E(H)| + |V(H)|)$-connected graph is H-linked.

1.4 Statement of Results

In this section, we outline the main results of this thesis.

1.4.1 Graph Linkages

In the study of general k-linked graphs, we give two main results. The first is an elementary proof of the following.

Theorem 1.4.1 *Every $2k$-connected graph on n vertices with $8kn$ edges is k-linked.*

After this result was written and distributed in 2003, there were two improvements. Kawarabayashi (personal communication) observed that by using the result of Egawa et al [13], the bound in Theorem 1.4.1 could be substantially lowered, possibly improving the edge bound in Theorem 1.4.1 to $12kn$. This was done so by Kawarabayashi et al. in [32].

After the communication with Kawarabayashi and having seen an early version of [32], we improve the edge bound in the following theorem.

Theorem 1.4.2 *Every $2k$-connected graph G on n vertices with $5kn$ edges is k-linked.*

We immediately conclude the following corollary.

Corollary 1.4.1 *Every $10k$-connected graph is k-linked.*

Theorems 1.4.1 and 1.4.2 are proven in Chapter 4. Theorems 1.4.1 and 1.4.2 appear in [57].

When we restrict our attention to small fixed values of k, we are able to tighten the analysis considerably. As remarked after Theorem 1.3.5, every 4-connected graph on $3n-6$ edges is 2-linked. We obtain the optimal edge bound for 3-linked graphs.

Theorem 1.4.3 *Every 6-connected graph on n vertices with $5n-14$ edges is 3-linked.*

Theorem 1.4.3 is proven in Chapter 5

The bound in Theorem 1.4.3 is optimal in that there exist 6-connected graphs on n vertices with $5n-15$ edges that are not 3-linked. When considering 5-connected graphs, there exists a family of graphs that have asymptotically $\binom{n}{2}$ edges that are not 3-linked. This leads to the following conjecture.

Conjecture 5 *Every $2k$-connected graph on n vertices with $(2k-1)n-(3k+1)k/2+1$ edges is k-linked.*

This is in fact the correct value when $k=2$ and $k=3$. In Chapter 6 we construct a family of examples of graphs showing this would be optimal.

1.4.2 Rooted Minors

We extend our proof techniques for linkages to bear on problems of edge bounds for general rooted minors. First, we give a bound for containing an arbitrary H as a given rooted minor.

Theorem 1.4.4 *Let H be a fixed graph and $c \geq 1$ be a constant such that every graph on n vertices with at least cn edges contains H as a minor. If a graph G is $|V(H)|$-connected and has at least $(9c + 395|V(H)|)|V(G)|$ edges, then the graph G is H-universal.*

When we restrict our attention to a fixed family of rooted bipartite minor structures, we are able to get the optimal edge bound. We prove:

Theorem 1.4.5 *Let G be a t-connected graph on n vertices with $tn - \binom{t+1}{2} + 1$ edges. Then for any set $X \subseteq V(G)$ with $|X| = t$, G contains a labeled $K_{2,t}(X)$ minor.*

The edge bound is optimal in that subtracting one from the edge bound makes the theorem no longer true. Theorems 1.4.4 and 1.4.5 are proven in Chapter 7.

1.4.3 Applications of Graph Linkages

We conclude in Chapter 8 with several applications of graph linkages. First, we present the proof of Bollobás and Thomason giving the extremal function for the number of edges required to ensure a graph contains K_p as a topological minor. We follow with two new applications of linkages to previously studied problems. The first is a generalization of odd cycles to a labeling of the edges of a graph with a general group. We postpone formal definitions until Chapter 8. We prove that in this more general setting, in a moderately connected graph, the set of cycles with non-zero weight in this labeling have the Erdős-Pósa property. As a corollary, we conclude the following theorem.

Theorem 1.4.6 *Every $(31/2)k$-connected graph either has k disjoint odd cycles, or there exists a set X of at most $2k - 2$ vertices such that $G - X$ is bipartite.*

Theorem 1.4.6 improves upon the best previous bounds due to [33] with the connectivity bound of $24k$. Theorem 1.4.6 and its generalizations to group labeled graphs originally appeared in [34].

The second new application concerns complete minors in large graphs. We give a new proof of a vital aspect of the proof of Bohme et al. in [2] where they prove that that there exists a positive integer N_t such that every $15t$-connected graph on at least N_t vertices must contain a K_t minor. Their proof proceeds in roughly two halves, where the cases are determined by whether or not there exists a *tree-decomposition* of the graph satisfying certain properties. We explain all necessary background information on the decompositions in Chapter 8 before proceeding with the proof.

Using graph linkages and the tools developed in the proofs, we give a new and shorter proof of an essential component in the proof of the theorem of Böhme et. al, although our proof requires $84p + 5$ connectivity instead of the $15p$ connectivity necessary for the proof of Böhme et al. The bounded tree-width case that we analyze also appeared in an earlier paper by Böhme, Maharry, and Mohar [1].

CHAPTER 2

TRADITIONAL METHODOLOGIES

In this chapter, we examine traditional approaches to finding edge bounds for minors and rooted minors. We begin with a theorem of Mader on the necessary number of edges to find a K_t minor. In Chapter 3, we will see how these methods relate to our techniques.

2.1 Edge Bounds for Graph Minors

A common tool when looking for a particular edge bound to contain a given minor is to first consider a minor minimal graph satisfying the edge bound. Then that graph will both have a vertex of‘ “small” degree v and the neighborhood of v will induce a dense subgraph. Sufficient analysis then yields the desired minor.

As an example of this technique, we return to Theorem 1.3.4 of Mader stating that every graph on $n \geq p$ vertices with $(p-2)n - \binom{p-1}{2} + 1$ edges contains K_p as a minor for any $p \leq 7$. We give a proof of the $p = 5$ case of the theorem. The proof is by induction on $|V(G)| + |E(G)|$. When $|G| = 5$, $|E(G)| \geq 10$, implying G is isomorphic to K_5 and the claim is proved. Assume now that $n = |V(G)| \geq 6$. If we can contract an edge $e = uv$ and still satisfy the edge bound, then by induction, G contains a K_5 minor. Thus we may assume that $|E(G/e)| \leq 3|V(G/e)| - 6$. It follows that upon contracting the edge e, we lost at least four edges. One of those edges was e; thus the ends u and v of e must have at least 3 common neighbors in G. We chose the edge e arbitrarily, so we may assume that the endpoints of any edge have at least three common neighbors.

Now attempt to delete an edge e in G. If $G-e$ satisfied the edge bound, then $G-e$ would contain a K_5 minor by induction, and consequently, G would as well. We may assume that $|E(G-e)| = |E(G)| - 1 \leq 3n - 6$. It follows that $|E(G)| = 3n - 5$, and that the minimum degree of G is strictly less than six.

Let v be a vertex of G with $\deg(v) \leq 5$. For any vertex u contained in the neighborhood

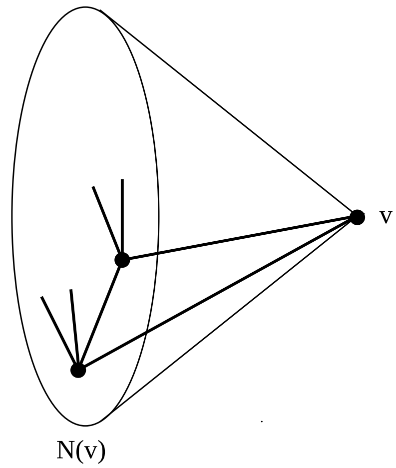

Figure 2: The neighborhood of v induces a subgraph with minimum degree 3.

of v, u and v must have at least three common neighbors. This implies that u has at least three neighbors in $N(v)$. See Figure 2. Since we chose u arbitrarily in the neighborhood of v, we see $N(v)$ induces a subgraph, call it N, with minimum degree 3 on at most 5 vertices. It follows easily that N has a K_4 minor (in fact, this is true without assuming that N has at most five vertices.)

2.2 Previous Methods for Linkages and Rooted Minors

We give here a proof of a more general statement implicit in the proofs of Robertson and Seymour and Bollobás and Thomason.

Theorem 2.2.1 *[Robertson and Seymour [49], Bollobás and Thomason [5]] Let H and G be graphs where G contains H as a minor. Let t be a positive integer such that H is t-connected and further that $|V(H)| \geq 2t$. Assume G is t-connected, and let X be a subset of t vertices in G. Then there exists a subgraph H' of H where $|V(H) - V(H')| \leq t$ and an injection $\pi : X \to V(H')$ such that (G, X) has a π-rooted H' minor.*

This theorem essentially says that given some H minor in a graph G and a set X of t vertices, then if G is t-connected, we can expand some t branch sets of the H minor so that they each contain some vertex of X, and in doing so, we only have to discard at most t other branch sets.

Proof: We give a slight strengthening of the hypothesis that is suitable for induction. Let X be our fixed set in the graph G. A *rooted H semi-minor* is a set $\{S_i \subseteq V(G) : i \in V(H)\}$ where the following properties hold:

1. For any distinct vertices $i, j \in V(H)$, $S_i \cap S_j = \emptyset$.

2. For any vertex $i \in V(H)$, if $S_i \cap X = \emptyset$, then $G[S_i]$ is connected.

3. For any vertex $i \in V(H)$, if $S_i \cap X \neq \emptyset$, then every connected component of $G[S_i]$ intersects X in at least one vertex.

4. For every edge $ij \in E(H)$, if either $S_i \cap X$ or $S_j \cap X$ is empty, then there exists an edge of G with one end in S_i and the other end in S_j.

Our new hypothesis will be the following.

Hypothesis: Let H and G be graphs. Let t be a positive integer such that H is t-connected and further that $|V(H)| \geq 2t$. Let X be a set of t vertices in G and assume G contains H as a rooted H semi-minor. Moreover, assume there does not exist a separation (A, B) of order strictly less than t with $X \subseteq A$ and $S_i \cap B \neq \emptyset$ for all vertices $i = 1, \dots, |V(H)|$.

We will show that if the above hypothesis holds, then the conclusion of the theorem holds as well. This suffices to prove the theorem, since the branch sets of an H minor form a rooted H semi-minor and if our graph is t-connected there can be no separation as forbidden in the hypothesis.

We now proceed by induction on the order of $V(G)$. Let $\{S_i \subseteq V(G) : i \in V(H)\}$ be the branch sets of our rooted H semi-minor.

Claim 1 *There does not exist a nontrivial separation (A, B) of order t with $X \subseteq A$ and $S_i \cap B \neq \emptyset$ for all $i = 1, \dots, |V(H)|$.*

Proof: Assume otherwise and let (A, B) be such a separation. Consider $\{S_i^* : i = 1, \dots, |V(H)|\}$ defined such that $S_i^* = S_i \cap B$. Let $X' = A \cap B$. Then the $\{S_i^* : i =$

$1, \ldots, |V(H)|\}$ form the branch sets of a rooted H semi-minor in $G[B]$ for the set X'. By induction, there exists a π-rooted H' minor in $(G[B], X')$ as in the statement of the theorem. In the graph $G[A]$, there exist t disjoint paths from X to $A \cap B$, lest there exist a separation (A', B') of order strictly less than t with $X \subseteq A'$ and $A \cap B \subseteq B'$. Then $(A', B' \cup B)$ would be a separation of order strictly less than t violating our hypothesis.

Given t disjoint paths from X to $A \cap B$, we can construct a π' rooted H' minor in (G, X) for some map π' where for every vertex in X, we define the branch set including its path to $A \cap B$ and the corresponding branch set of the π rooted H' minor in $G[B]$. See Figure 3. This completes the proof of the claim. □

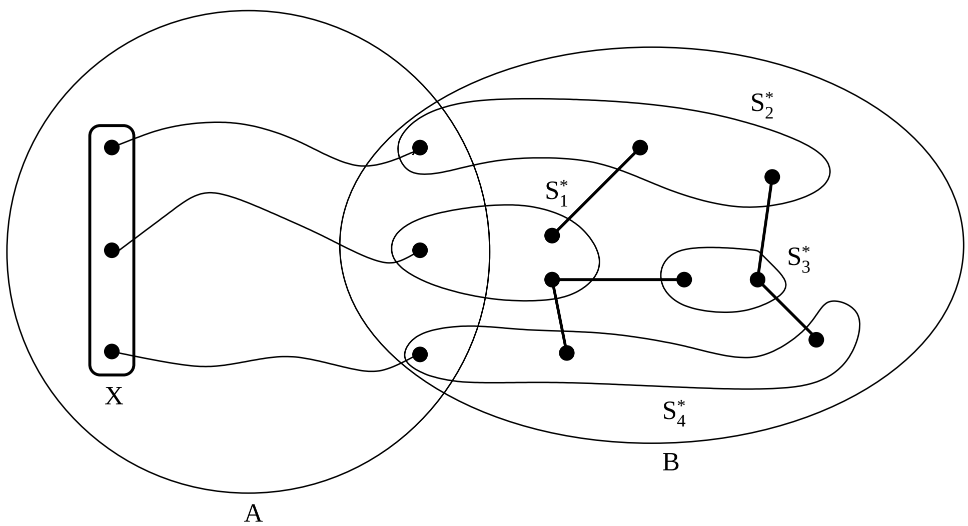

Figure 3: Extending a rooted minor in $G[B]$ to G.

We now prove that we may assume every edge of G either has endpoints in distinct branch sets of the semi-rooted H minor, or has both endpoints in X. Assume otherwise, and let e be an edge either contained in some S_j or have at least one endpoint not contained in any S_i. Further, assume e does not have both endpoints contained in X. Then if we consider G/e, we see that the S_i induce branch sets S_i^* in G/e forming a rooted H semi-minor for X. If G/e satisfied the remainder of the hypothesis, then the pair $(G/e, X)$ would contain a π rooted H' minor for some map π and subgraph H'. Thus we may assume there exists a separation (A^*, B^*) of G/e with $X \subseteq A^*$ and every S_i^* intersecting B^*. Let v_e be the vertex of G/e corresponding to the edge e and let u and v be the endpoints of e in G. The separation (A^*, B^*) induces a separation (A, B) in G where $A = (A^* - \{v_e\}) \cup \{u, v\}$ if $v_e \in A^*$ and $A = A^*$ otherwise. Similarly, we define B. Then the order of the separation (A, B) is at most t. Furthermore, $X \subseteq A$ and every S_i intersects B. But this contradicts the above Claim. The contradiction implies that we may assume no such edge e in fact exists.

We conclude, then, that the only edges of G are edges between two different S_i and S_j, or between two vertices in X. It follows that every S_i not containing a vertex of X is in fact a single vertex. Let U be the set of vertices of H such that $S_i \cap X = \emptyset$ if and only if $i \in U$ and let $\mathcal{S}_U$ be the set of S_i such that $i \in U$. Since we also may assume that G is connected and every vertex of G belongs to some S_i, then every vertex of G either lies in X or some member of $\mathcal{S}_U$.

Consider the bipartite subgraph of G induced by the edges with one endpoint in X and the other endpoint equal to S_i for some $S_i \in \mathcal{S}_U$. Attempt to find a matching from X to $\mathcal{S}_U$ in this bipartite graph. If such a matching exists, then the matching determines an injective mapping π from X to U, and we have found a π-rooted $H[U]$ minor. Since $|V(H) - U| \leq |X|$, we have proven the theorem. If we assume no such matching exists, then by Hall's theorem, there exists a set $\overline{X} \subsetneq X$ such that if we let N be the neighborhood of $\overline{X}$ in $\mathcal{S}_U$, then $|N| < |\overline{X}|$. But then $(\overline{S}_U \cup (X - \overline{X}), X \cup N)$ is a separation of order strictly less than $|X|$. See Figure 4. It is not necessarily a contradiction to our hypothesis, since it is possible that some branch sets of the partial rooted H minor are strictly contained in $\overline{X}$. However, if any such branch set existed, H would have a cut set of order strictly less than t corresponding to the branch sets intersecting $(X - \overline{X}) \cup N$, contrary to the fact that H is t-connected. This final contradiction completes the proof.□

By applying the above theorem, we immediately see that every $2k$ connected graph containing a K_{4k} minor is k-linked, because for any set X of $2k$ vertices, there exists a K_{2k} minor where each branch set contains exactly one vertex of X. Then any desired linkage problem on X can be solved using paths in the K_{2k} minor. This is the essence of the proof

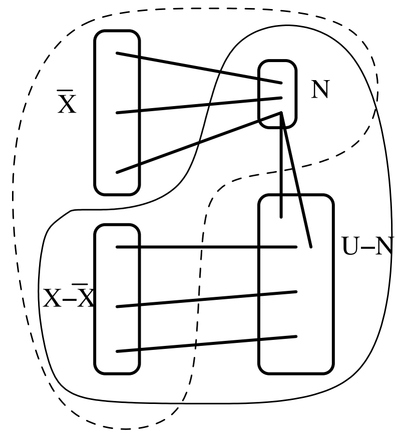

Figure 4: A separation violating the t connectivity of H.

of Robertson and Seymour in [49], although they improved the constants by more stringent analysis.

Consider the following easy lemma.

Lemma 2.2.1 *Let J be a graph such that $2\delta(J) \geq |J| + 3k - 4$. Then J is k-linked.*

Proof Let $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ be a sequence of distinct vertices of J, and let $X = \{s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k\}$. The hypothesis implies that every two nonadjacent vertices of X have at least k common neighbors outside of X, and hence there is a desired linkage consisting of paths of length at most two. $\square$

Assume now that a graph G is $2k$-connected and contains as a minor a graph J where $2\delta(J) \geq |J| + 5k$. Fix a particular set X of $2k$ vertices. By the above theorem, there exists a subgraph J' of J where $t := |V(J)| - |V(J')| \leq 2k$ and every branch set of the J' minor contains exactly one vertex of X. Because J' was obtained by deleting at most $t \leq 2k$ vertices in J, we see that $\delta(J') \geq \delta(J) - t$. Then $2\delta(J') \geq 2\delta(J) - 2t \geq |V(J)| + 5k - 2t = |V(J')| + t + 5k - 2t$. It follows that since $t \leq 2k$, then $2\delta(J') \geq |V(J')| + 3k$. Applying Lemma 2.2.1, we see that J' is k-linked. Thus any linkage problem on X can be solved using paths contained in the J' minor. Since we chose X arbitrarily, we see that G is k-linked.

While improving the analysis to achieve better constants, Bollobás and Thomason [5] essentially follow the above argument for the first half of their proof that every $22k$-connected graph is k-linked. The second step is a probabilistic proof that there exists a constant c such that every graph with minimum degree ck contains as a minor a graph J as in the above argument. We were able to find an argumment that applies the technique of Lemma 2.2.1 directly to the linkage problem without using dense minors as an intermediate step.

CHAPTER 3

TECHNIQUES AND METHODS

In this chapter, we outline a general technique for finding extremal functions for linkages and rooted minors. Our goal will be to avoid the need of the Robertson and Seymour and Bollobás and Thomason arguments to find linkages inside a larger minor. Instead we reduce the problem of finding linkages to an argument more like Theorem 1.3.4 of Mader where we directly analyze a small dense subgraph.

3.1 Weakening Connectivity: λ-massed Graphs

We define a relaxation of k-connectivity that will be suitable to inductive proofs for linkages and rooted minors. Given a graph G and a set $X \subseteq V(G)$, we define $\rho(X)$ to be the number of edges with at least one endpoint in X.

Definition 9 *Let G be a graph, let $X \subseteq V(G)$, and let $\lambda > 0$ be a real number. We say that the pair (G, X) is* λ-massed *if*

(M1) $\rho(V(G) - X) > \lambda|V(G) - X|$, *and*

(M2) *every separation (A, B) of (G, X) of order at most $|X| - 1$ satisfies* $\rho(B - A) \leq \lambda|B - A|$.

The property of being λ-massed suffices when our primary concern is controlling the multiplicative constant in a desired extremal function. In the proof of Theorem 1.4.3, we will need to determine the additive constant as well. Towards that end, we define a refinement of λ-massed.

Definition 10 *Given G a graph, $X \subseteq V(G)$, and α, β two positive integers, (G, X) is* (α, β)-massed *if*

(M1)* $\rho(V(G) - X) \geq \alpha|V(G) - X| + \beta$, *and*

$(M2^*)$ *every separation* (A, B) *of order at most* $|X| - 1$ *with* $X \subseteq A$ *satisfies*

$$\rho(B - A) \leq \alpha|B - A|.$$

3.2 Rigid Separations

When the contraction of an edge e violates the connectivity constraint in an λ- or (α, β)-massed graph, we find a small separation with the additional property that it separates many edges. This will allow us to restrict our attention to a smaller problem and proceed by induction by analyzing what we will refer to as rigid separations.

3.2.1 Definition of Rigid Separation

We formally define rigid separations thus.

Definition 11 *Let* G *be a graph, let* $X \subseteq V(G)$*, and let* (A, B) *be a separation of* G*. We say that* (A, B) *is a* rigid separation of (G, X) *if* $X \subseteq A$*,* $B - A \neq \emptyset$*, and* $(G[B], A \cap B)$ *is linked.*

3.2.2 Definition of Separation Truncation

Separation truncation is the operation whereby we will eliminate rigid separations.

Definition 12 *Given a separation* (A, B) *of a graph* G*, the* truncation *of* (A, B) *is the graph* $G[A]$ *with additional edges connecting every non-adjacent pair of vertices in* $A \cap B$*.*

We will also refer to the separation truncation of (A, B) as simply the truncation of (A, B).

3.2.3 The Truncation of a Rigid Separation

We now see that the property of being linked is preserved under taking separation truncations of rigid separations.

Lemma 3.2.1 *Let* (A, B) *be a rigid separation of the pair* (G, X)*, and let* G' *be the separation truncation of* (A, B)*. Then* (G, X) *is linked if and only if* (G', X) *is linked.*

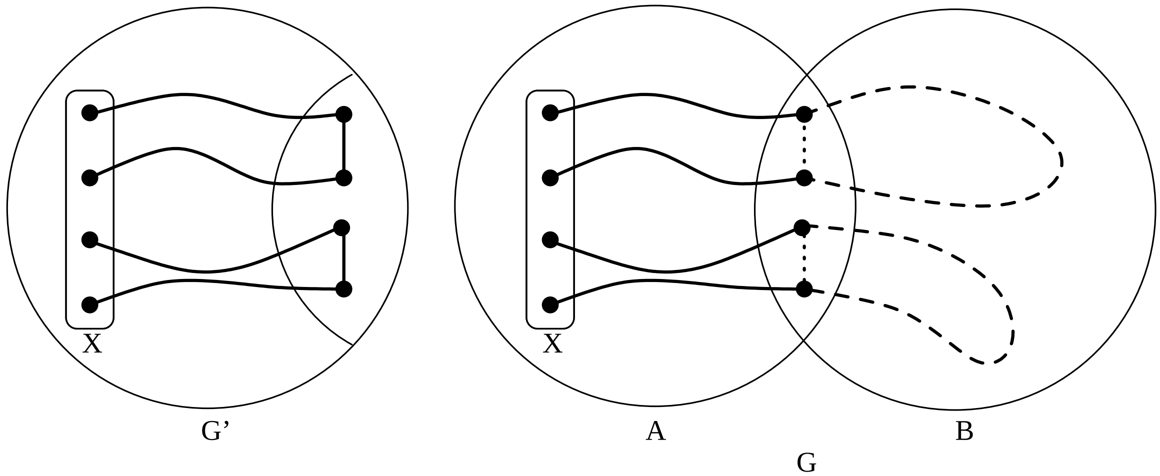

Figure 5: Extending a linkage in G' to a linkage in G.

Proof: Let $\mathcal{L}$ be a linkage problem on X. To prove the lemma, it suffices to show that $\mathcal{L}$ is feasible in G if and only if it is feasible in G'.

First we assume that $\mathcal{L}$ is feasible in G. Let $\mathcal{P}$ be a linkage solving the linkage problem $\mathcal{L}$ in G. Every connected component $P \in \mathcal{P}$ induces a connected subgraph in G' since $G'[A \cap B]$ is a complete subgraph. Thus the subgraph induced by $\mathcal{P}$ in G' contains a linkage solving $\mathcal{L}$, as desired.

Assume now that $\mathcal{P}'$ is a linkage in G' solving the linkage problem $\mathcal{L}$, and moreover, assume we chose $\mathcal{P}'$ to minimize $|V(\mathcal{P}')|$. It follows that every component of $\mathcal{P}'$ uses at most two vertices in $A \cap B$ because $G'[A \cap B]$ is a complete subgraph. If we consider the subgraph in G induced by $V(\mathcal{P}')$, then each component of $\mathcal{P}'$ is missing at most one edge in G corresponding to two non-adjacent vertices in $A \cap B$. The endpoints of any missing edge in $A \cap B$ can be linked by a path in $G[B]$ by the definition of rigid separation. See Figure 5. Thus the linkage $\mathcal{P}'$ extends to a linkage $\mathcal{P}$ in G solving the linkage problem $\mathcal{L}$ in G, proving the lemma. $\square$

Now we extend the above concepts to general rooted minors.

3.2.4 Definition of H-rigid Separations

A rigid separation (A, B) is one where an arbitrary linkage problem on $A \cap B$ is feasible.

Definition 13 *Let G and H be graphs and $X \subseteq V(G)$. Then a non-trivial separation (A, B) is a H-*rigid *separation of the pair (G, X) if*

(R1) *$X \subseteq A$,*

(R2) *the order of (A, B) is at most $|X|$, and*

(R3) *the pair $(G[B], A \cap B)$ contains a π-rooted H' minor for all subgraphs H' of H with $|V(H')| = |A \cap B|$ and for all injections $\pi : A \cap B \hookrightarrow V(H')$.*

We state the extension of Lemma 3.2.1 to rooted minors.

3.2.5 The Separation Truncation of an H-rigid Separation

Lemma 3.2.2 *Let G and H be graphs and let $X \subseteq V(G)$ such that $|X| = |V(H)|$. Assume that (A, B) is an H-rigid separation of (G, X). Then for all injections $\pi : X \to V(H)$, the pair (G, X) contains a π-rooted H minor if and only if (G', X) contains a π-rooted H minor, where G' is the separation truncation of the separation (A, B).*

3.2.6 Proof of Lemma 3.2.2

Before proving Lemma 3.2.2, we first prove a necessary lemma about matchings in bipartite graphs with specific properties.

Definition 14 *Let G be a bipartite graph with bipartition (X, Y). A* neighborhood cover matching *is a pair $(\mathcal{M}, \kappa)$ where $\mathcal{M}$ is a matching in G and κ is a 2 coloring of the edges of $\mathcal{M}$ such that for every vertex $x \in X$ one of the following conditions holds:*

1. *either there exists a y in Y such that xy is in $\mathcal{M}$ and $\kappa(xy) = 1$, or*
2. *for all $y \in N(x)$ there exists a z with $yz \in \mathcal{M}$ and $\kappa(yz) = 2$.*

Lemma 3.2.3 *There exists a neighborhood cover matching for any bipartite graph $G = (X, Y)$.*

Proof: The proof is by induction on $|X|$. Clearly, if $|X| = 1$, then an arbitrary edge incident the vertex of X with that edge colored 1 forms a neighborhood cover matching.

On the other hand, if G has no edge, then the empty set is a matching satisfying the second condition.

If G contains a matching $\mathcal{M}$ covering X, then again, that $(\mathcal{M}, \kappa)$ where $\kappa(e) = 1$ for every edge of $\mathcal{M}$ forms a neighborhood covering matching. If G does not contain such a matching, then there exists a set $B \subseteq X$ violating the condition for Hall's Theorem. Assume B is such a set of minimal size. Then there exists a matching covering $N(B)$ in $G[B \cup N(B)]$, call it $\mathcal{M}_1$. If such a matching did not exist, then there would exist a set $J \subseteq N(B)$ violating Hall's condition in $G[B \cup N(B)]$. But then $B - N(J)$ would be a smaller set than B violating Hall's condition in G, contrary to our choice of B.

Now consider the subgraph G' induced by $(X - B, Y - N(B))$. By induction there exists a neighborhood cover matching $(\mathcal{M}_2, \kappa)$ (possibly the empty matching). Then we define the coloring κ' on $\mathcal{M}_1 \cup \mathcal{M}_2$ to create a neighborhood cover matching in G.

$$\kappa'(e) = \begin{cases} 2 & \text{if } e \in \mathcal{M}_1 \\ \kappa(e) & \text{if } e \in \mathcal{M}_2 \end{cases}$$

Let x be a vertex in $X - B$. Then if there is no edge e in $\mathcal{M}_\in$ incident x with $\kappa'(e) = 1$, then every neighbor of x in $Y - N(B)$ is incident an edge e of $\mathcal{M}_2$ with $\kappa'(e) = 2$. Moreover, since every vertex in $N(B)$ is incident an edge e of $\mathcal{M}_1$ with $\kappa'(e) = 2$, we see that every neighbor of x is incident a matching edge of color 2. Every vertex $x \in B$ has $N(x) \subseteq N(B)$, implying that every neighbor of x is incident an edge e with $\kappa'(e) = 2$. Thus every vertex in X satisfies the conditions of the definition, proving that G contains a neighborhood covering matching. $\square$

We now prove Lemma 3.2.2

Proof: Let G, H, X and (A, B) be given as in the statement of the Lemma. For notation, let the vertex set of H be $\{1, \dots, k\}$. Fix our map $\pi : X \to V(H)$, and let G' be the truncation of the separation (A, B).

First, we see that if (G, X) has a π-rooted H minor, then (G', X) does as well. Let $\{S_1, \dots, S_k\}$ be the branch sets of a π-rooted H minor. Let $S_i' = S_i \cap A$. Clearly, the S_i'

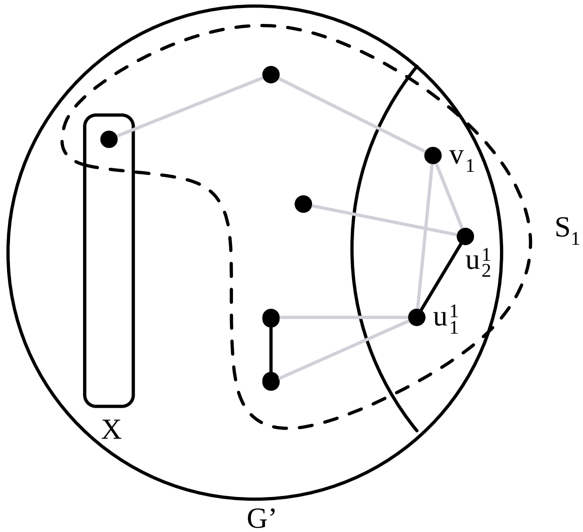

Figure 6: An example of T^1 contained in S_1. The tree T^1 is in grey.

induce connected subgraphs because $G'[A \cap B]$ is complete. Moreover, let uv be an edge connecting S_i and S_j in G. If $uv \subseteq A$, then the edge is present between S_i' and S_j' in G'. Otherwise, both S_i and S_j intersect $A \cap B$, and so because $A \cap B$ induces a complete subgraph of G', there is an edge connecting S_i' and S_j', as desired. Thus the $\{S_1', \ldots, S_k'\}$ do in fact form the branch sets of a π-rooted H minor in (G', X)

Now to prove the other direction, assume that (G', X) contains a π-rooted H minor, and further that we pick such a rooted minor to minimize the number of vertices in the branch sets. Let $\{S_1, \ldots, S_k\}$ be the branch sets of the minor, so that x_i is a member of $S_{\pi(i)}$. For every S_i that intersects $A \cap B$ in at least two vertices, let P_i be a path from $x_{\pi^{-1}(i)}$ to $A \cap B$ in $G'[S_i]$. Let v_i be the end of P_i in $A \cap B$. We may assume that v_i is the only vertex of P_i lying in $A \cap B$. Let $u_1^i, \ldots, u_{t(i)}^i$ be the other vertices of S_i other than v_i in $A \cap B$. For every S_i with $|S_i \cap A \cap B| \geq 2$, let T^i be a spanning tree of $G'[S_i]$ with the following properties:

1. P_i is a subgraph of T^i, and

2. $T^i[A \cap B]$ is a star with root v_i and $u_1^i, \ldots, u_{t(i)}^i$ forming the leaves of the star.

For every S_i that intersects $A \cap B$ in exactly one vertex, let $S_i \cap A \cap B = \{v_i\}$.

To prove that (G, X) contains a π-rooted H minor, it would now suffice to prove that edges of the form $v_i u_j^i$ and $v_i v_k$ for all appropriate i, j, and k can be reconstructed in G by choosing the right H' rooted minor on $(G[B], A \cap B)$ for some subgraph H' of H. Unfortunately, we must proceed more cautiously.

If we remove the edges $v_i u_1^i, \ldots, v_i u_{t(i)}^i$ from T^i, the induced components of T^i partition the vertices of T^i into $t(i)+1$ subtrees. Let $T(v_i)$ be the subtree containing v_i and $T(u_j^i)$ be the subtree containing u_j^i. It is possible that $T(u_j^i)$ will simply be a trivial tree consisting of one vertex.

By the minimality of the number of vertices in branch sets, we know that for every defined $T(u_j^i)$, there is an edge going to some other T^l with l adjacent to i in H and $T^l \cap A \cap B = \emptyset$. There may in fact be several other branch sets of the minor to which T^i connects through $T(u_j^i)$. We define $\mathcal{N}(T(u_j^i))$ to be the set of all such vertices l, or

$$\mathcal{N}(T(u_j^i)) = \left\{ l \in V(H) : \begin{array}{l} i \text{ is adjacent to } l \text{ in } H, \\ \text{there is an edge from } T(u_j^i) \text{ to } T^l, \\ \text{and } T^l \cap A \cap B = \emptyset \end{array} \right\}$$

Note that for any index $l \in \mathcal{N}(T(u_j^i))$, there exists a path from u_j^i to T^l using only vertices of $T(u_j^i)$ and one endpoint in T^l. For notation, denote such a path $P(u_j^i \to l)$.

Now consider the bipartite graph on the vertex set $W \cup Y$, where

$$W = \{u_j^i | i \in V(H) \text{ such that } |S_i \cap A \cap B| \geq 2, 1 \leq j \leq t(i)\},$$

$Y = V(H)$, and (W, Y) is the bipartition of the graph. The edges of the bipartite graph are given by $u_j^i l$ for all $l \in \mathcal{N}(T(u_j^i))$. Then we know from Lemma 3.2.3 that there exists a neighborhood cover matching from W to Y. For notation, we represent the matching as an injective function $\lambda : W \to V(H) \times \{1, 2\}$. Let λ_1 be the value λ takes on $V(H)$, and let λ_2 be the value λ takes on $\{1, 2\}$.

We are now ready to start constructing the branch sets of our π-rooted H minor in (G, X). We first pick an appropriate rooted minor of $(G[B], A \cap B)$. To pick our injective function $\phi : A \cap B \to V(H)$, we define ϕ as follows for every $x \in A \cap B \cap \left(\bigcup_{i \in V(H)} S_i\right)$:

$$\phi(x) = \begin{cases} i & \text{if } x = v_i \text{ for some } i \\ \lambda_1(u^i_j) & \text{if } x = u^j_i \text{ for some } i \text{ and } j \end{cases}$$

To see that ϕ is an injection, assume that we have x and y such that $\phi(x) = \phi(y)$. Since we know that λ_1 is an injection by definition, and since $v_m \neq v_n$ for $n \neq m$, we may assume that $x = v_l$ for some l, and $y = u^i_j$ for some i and j. But this implies that $v_l \in \mathcal{N}(T(u^i_j))$, contrary to the fact that $T^l \cap A \cap B = \emptyset$.

Let H' be the subgraph of H induced on $Im(\phi)$. By definition of a rigid separation, we know there exists a ϕ-rooted H' minor in $(G[B], A \cap B)$. Let $\{U_i | i \in V(H')\}$ be the branch sets of the rooted minor where $x \in A \cap B$ is an element of $U_{\phi(x)}$. There is a slight abuse of notation here in that rooted minors of (G, X) are only defined for injections from X, where as ϕ is not be defined for the vertices of $A \cap B$ not in any S_i. However, in such a case, ϕ could be arbitrarily defined for the remaining vertices of $A \cap B$. We will only need the branch sets of the H' minor rooted on the original domain of ϕ.

We now define the branch sets $\overline{S}$ forming a π-rooted H minor in (G, X). For i with $|S_i \cap A \cap B| \geq 2$, let

$$\overline{S_i} = V\left(T(v_i)\right) \cup U_{\phi(v_i)} \bigcup_{\{j : \lambda_2(u^i_j) = 1\}} U_{\phi(u^i_j)} \cup V\left(T(u^i_j)\right).$$

When S_i intersects $A \cap B$ in exactly one vertex, let

$$\overline{S_i} = S_i \cup U_{\phi(v_i)} = S_i \cup U_i$$

Among the l such that S_l does not intersect $A \cap B$, there are two separate cases: when some u^i_j is mapped to l by λ, or not. For l such that there exists a $u^i_j \in A \cap B$ with $\lambda(u^i_j) = (l, 2)$

$$\overline{S_l} = S_l \cup P(u^i_j \to l) \cup U_{\phi(u^i_j)} = S_l \cup P(u^i_j \to l) \cup U_l.$$

Observe that by the fact that λ is a matching, there is at most one such u^i_j for any index l. Otherwise,

$$\overline{S_i} = S_i.$$

See Figure 7 for examples of how $\overline{S_i}$ can arise.

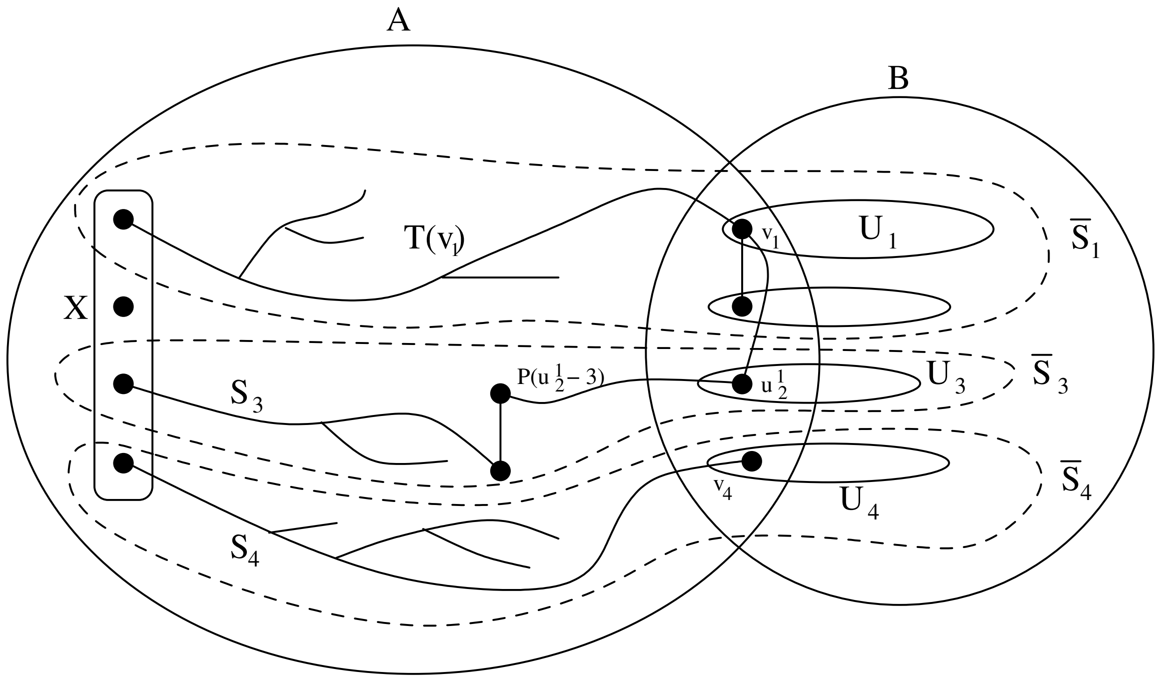

Figure 7: Examples of $\overline{S_i}$

For any $x \in X$, if $|S_{\pi(x)} \cap (A \cap B)| \geq 2$, $T(v_{\pi(x)}) \subseteq \overline{S}_{\pi(x)}$. Otherwise, $S_i \subseteq \overline{S_i}$. Clearly, then, $x \in \overline{S}_{\pi(x)}$. In order to check that the $\overline{S_i}$'s form the branch sets of a π-rooted minor in (G, X), we simply need to verify they induce pair-wise disjoint connected subgraphs and that for any edge xy in H, there is an edge between $\overline{S_x}$ and $\overline{S_y}$.

By construction and the fact that ϕ is an injection, we see that the $\overline{S_i}$'s are pair-wise disjoint. Now we confirm that the $\overline{S_i}$ induce connected subgraphs. Observe that for every i, j, and k, the sets $T(v_i)$, $T(u^i_j)$, $U_{\phi(x)}$, and $P(u^i_j \to l)$ induce connected subgraphs of G. We conclude that if S_l intersects $A \cap B$ in at most 1 vertex, then the sets comprising $\overline{S_l}$ intersect so that their union again induces a connected subgraph.

Instead, assume $|S_l \cap A \cap B| \geq 2$. For every U_x with $U_x \subset \overline{S_l}$ and $x \neq v_l$, we know $U_x = U_{\lambda_1(u^l_j)}$, for some value of j, implying that $T(u^l_j) \cup U_x$ induce a connected subgraph.

Moreover, $\phi(u_j^l) = \lambda_1(u_j^l) \in \mathcal{N}(T(u_j^l))$, implying that there is an edge between U_x and U_l. Every $U_x \cup T(u_j^l)$ induces a connected subgraph and is attached to U_l. U_l contains v_l, connecting it to $T(v_l)$. Thus $\overline{S_l}$ in fact induces a connected subgraph.

Now we prove that every edge of H is present between the appropriate branch sets of our prospective rooted minor. If the edge xy of H is such that S_x and S_y both intersect $A \cap B$ in at most one vertex, then $\overline{S_x} \supseteq S_x$ and $\overline{S_y} \supseteq S_y$. The only possible way that the edge xy could not be present in G is if both S_x and S_y intersect $A \cap B$ and the only edge in the separation truncation between S_x and S_y is the edge in $A \cap B$. However, in this case, $\overline{S_x} \supseteq U_x$ and $\overline{S_y} \supseteq U_y$, and so there is an edge between the two sets of vertices.

We now show that for any i with S_i such that $|S_i \cap A \cap B| \geq 2$ there exists an edge between $\overline{S_i}$ and every $\overline{S_l}$ with i adjacent l in H. Given such an $\overline{S_i}$, first we assume S_l intersects $A \cap B$ in at least one vertex. Then $\overline{S_l} \supseteq U_l$. Since $\overline{S_i} \supseteq U_i$ as well, then there is an edge between $\overline{S_l}$ and $\overline{S_i}$

Assume now that $S_l \cap (A \cap B) = \emptyset$. If the edge in the separation truncation between S_i and S_l is in fact an edge between $T(v_i)$ and S_l, then the edge is an edge of G and given that $\overline{S_i} \supseteq T(v_i)$, we know there is an edge between $\overline{S_i}$ and $\overline{S_l}$. And in fact, if the edge between S_l and S_i is an edge between S_l and some $T(u_j^i) \subseteq \overline{S_i}$, then there is an edge between $\overline{S_i}$ and $\overline{S_l}$. Thus we may assume that there is an edge between S_l and some $T(u_j^i) \nsubseteq \overline{S_i}$. Then $\lambda_2(u_j^i) = 2$ or u_j^i is not matched to any vertex of H in the neighborhood cover matching. However, now for every index in $\mathcal{N}(T(u_j^i))$, there is some other vertex of $A \cap B$ matched to it with an edge colored color 2 by the definition of a neighborhood cover matching. Specifically, there exists some $u_{j'}^{i'}$ with $\lambda(u_{j'}^{i'}) = (l, 2)$. Consequently, $\overline{S_l} \supseteq U_{\phi(u_{j'}^{i'})} = U_l$ and there exists an edge between $\overline{S_i}$ and $\overline{S_l}$, as desired.

This completes the proof that the sets $\{\overline{S_i} | i \in V(H)\}$ form the branch sets of a π-rooted H minor of (G, X). $\square$

3.3 A General Technique for Linkages and Rooted Minors

Now we outline our general approach to finding extremal functions for graph linkages and rooted minors. We will first describe the methods in terms of graph linkages. Assume we are inductively attempting to show that every α-massed pair (G, X) is k-linked.

STEP 1: First we see that we may assume the pair (G, X) does not have a rigid separation (A, B). Assume (A, B) is a rigid separation, and moreover, assume we appropriately chose it from all possible rigid separations. Then if we let G' be the truncation of the separation (A, B), the pair (G', X) is α-massed. Applying induction implies that (G', X) is k-linked, and then Lemma 3.2.1 implies that (G, X) is also k-linked, completing the claim.

STEP 2: Given an edge $e \nsubseteq X$, we see we may assume that the ends of e have many common neighbors. If $(G/e, X)$ were α-massed, then by induction the pair $(G/e, X)$ would be k-linked, implying (G, X) is as well. Thus we may assume contracting the edge e violates one of the conditions to be α-massed. If $(G/e, X)$ were to violate condition (M2) with the separation (A, B), then by examining an extension of the separation (A, B) to G, we find that (G, X) would have a rigid separation contrary to what we have already shown. Otherwise, contracting the edge e violates (M1), implying that the endpoints of e must have many common neighbors, as in the proof of Mader's Theorem.

STEP 3: We now see that we may assume there exists a vertex in $V(G) - X$ of small degree. As above, when we try to delete the edge e, we see that $(G - e, X)$ must violate (M1), implying that (G, X) satisfies (M1) with equality. It follows that there exists a vertex of degree at most 2α.

Now we have captured some of the flavor of the more traditional arguments on the extremal functions for minors as in the proof of Theorem 1.3.4.

STEP 4: Let v be a vertex of degree at most 2α. The neighborhood of the vertex v induces a subgraph N on at most 2α vertices with minimum degree at least α. We verify that our choice of the value α ensures that the subgraph N contains a k-linked subgraph N'.

STEP 5: Now we use the k-linked subgraph N' to solve the linkage problem on X. Attempt to find $|X|$ disjoint paths from X to N'. If they exist, then no matter how the paths land in N', we can link the desired ends by the fact that N' is k-linked. Thus we may assume that the disjoint paths do not exist; but in this case, we easily find a rigid separation contrary to our previous arguments. This completes the proof.

When we generalize to rooted H minors for an arbitrary graph H, the proof follows a similar argument. We eliminate H rigid separations in Step 1., using Lemma 3.2.2 in the place of Lemma 3.2.1. In Step 4, we show that the graph contains an H-universal subgraph in the place of a k-linked subgraph.

3.4 Minimal Graphs Not Containing Rooted Minors

We now expand more explicitly on the method outlined in the previous section. Suppose we are attempting to prove that a particular value of α suffices to imply that every α-massed pair(G, X) contains a π-rooted H minor for an arbitrary choice of π. We define exactly what it means to be a minimal counterexample.

Definition 15 *Let H be some fixed graph and α a positive real number. Let G be a graph and X a subset of $V(G)$ such that*

1. $|X| \leq |V(H)|$,
2. (G, X) *is α-massed,*
3. *there exists an injection $\pi : X \to V(H)$ such that if $H' = H[\pi(X)]$, then (G, X) does not contain a π-rooted H' minor,*
4. *subject to (1.), (2.), (3.), $|V(G)|$ is minimal, and*
5. *subject to (1.), (2.), (3.), and (4.), $\rho(G - X)$ is minimal.*

Then we say that (G, X) is (H, α)-minimal*.*

Following our outline in the previous section, our first goal will be to eliminate H-rigid separations.

Theorem 3.4.1 *Let H and G be graphs and $X \subseteq V(G)$. If (G, X) is (H, α)-minimal, then (G, X) does not contain an H-rigid separation.*

Upon eliminating rigid separations from our pair (G, X), we are now in a position to apply the more traditional tricks for proving extremal functions for graph minors. Specifically, we reduce the problem to examining a small dense neighborhood of the graph G.

Theorem 3.4.2 *Let G and H be graphs and α positive real number with $\alpha \geq |V(H)|$. If (G, X) is (H, α)-minimal for a set of vertices $X \subset V(G)$, then the following hold:*

1. *Let uv be an edge of G not contained in X. Then if neither u nor v is in X, u and v have at least $\lfloor \alpha \rfloor$ common neighbors. If one, say u is an element of X, let t be the number of non-neighbors of u in X. Then u and v have at least $\lfloor \alpha \rfloor - t$ common neighbors.*

2. *There exists a vertex in $V(G) - X$ of degree strictly less than 2α.*

The next step in our outline is to show that

Theorem 3.4.3 *Given graphs G and H and a positive real number α, if (G, X) is (H, α)-minimal for some set of vertices $X \subset V(G)$, then G does not contain an H universal subgraph.*

We now give proofs for the theorems in this section.

3.4.1 Proof: Theorem 3.4.1

Let (G, X), π, H, and H' be given as in the statement of Theorem 3.4.1 and in the definition of (H, α)-minimal. Assume, to reach a contradiction, that (G, X) has an H-rigid separation (A, B). Pick the separation (A, B) over all such H-rigid separations to minimize $|A|$. For notation, let $X = \{x_1, \ldots, x_k\}$. We now proceed with several intermediate claims:

Claim 2 $|A \cap B| < |X|$

Proof: Assume otherwise. First consider when there exist k disjoint paths $P_1, \ldots, P_k$, each with one end in X and the other in $A \cap B$. Without loss of generality, let the ends of P_i be x_i and a_i where $a_i \in A \cap B$. Then by the definition of rigid separation, $(G[B], A \cap B)$ contains a π-rooted H' minor with branch sets $\{S_1, \ldots, S_k\}$ with $a_i \in S_{\pi(i)}$. Then (G, X)

contains a π-rooted H' minor with branch sets $\{S'_1, \ldots, S'_k\}$ where $S'_{\pi(i)} = S_{\pi(i)} \cup P_i$, a contradiction.

Thus no such disjoint paths exist, implying that $G[A]$ contains a separation (A', B') of order strictly less than k, with $X \subseteq A'$ and $A \cap B \subseteq B'$. But if we pick such a separation (A', B') of minimal order, then by the same argument as in the previous paragraph, $(A', B' \cup B)$ is a rigid separation. Moreover, $|A'| < |A|$, contrary to our assumptions. $\square$

Let (G', X) be the separation truncation of the separation (A, B) of the pair (G, X).

Claim 3 *(G', X) is α-massed.*

Proof: By Claim 2 and condition $(M2)$ applied to (G, X), we see that (G', X) must satisfy condition $(M1)$. So assume that (G', X) contains a separation (A', B') violating condition $(M2)$, and assume we pick (A', B') to minimize $|B'|$. Then $(G'[B'], A' \cap B')$ is α-massed, and so by the fact that (G, X) is (H, α)-minimal, we know that in fact $(G'[B'], A' \cap B')$ contains a $\overline{\pi}$-rooted $\overline{H}$ minor for any subgraph $\overline{H}$ of size $|A' \cap B'|$ and any injective map $\overline{\pi}$ from $A' \cap B'$ to $V(\overline{H})$.

The subgraph of G' induced by $A \cap B$ is complete, and so it must be a subset of A' or B'. If $A \cap B \subseteq A'$, then $(A' \cup B, B')$ is a separation in G violating condition $(M1)$.

Thus we know $A \cap B \subseteq B'$. Then (B', B) is a rigid separation of $(G[B \cup B'], A' \cap B')$, and the separation truncation of $G[B' \cup B]$ with respect to the separation (B', B) is simply $G'[B']$. By the (H, α) minimality of (G, X), the pair $(G'[B'], A' \cap B')$ contains a $\overline{\pi}$ rooted $\overline{H}$ minor. Lemma 3.2.2 then implies that $(G[B \cup B'], A' \cap B')$ contains a $\overline{\pi}$ rooted $\overline{H}$ minor. But this was for an arbitrary subgraph $\overline{H}$ of H, and an arbitrary map $\overline{\pi}$. Thus $(A', B' \cup B)$ is a rigid separation, contrary to our choice of (A, B) to minimize $|A|$.$\square$

Now by the definition of (H, α)-minimality, (G', X) contains a π-rooted H' minor. Lemma 3.2.2 implies that (G, X) contains a π-rooted H' minor, contrary to the fact that (G, X) is (H, α) minimal.

This completes the proof of Theorem 3.4.1. $\square$

3.4.2 Proof of Theorems 3.4.2 and 3.4.3

Proof of Theorem 3.4.2 Let G, X, π, H, and H' be given as in the statement of the theorem and the definition of (H, α)-minimality.

Claim 4 *Let uv be an edge of G not contained in X. Then if neither u nor v is in X, u and v have at least $\lfloor \alpha \rfloor$ common neighbors. If one, say u is an element of X, let t be the number of non-neighbors of u in X. Then u and v have at least $\lfloor \alpha \rfloor - t$ common neighbors.*

Proof: Let uv be an edge not contained in X. Then if $(G/uv, X)$ has a π-rooted H' minor, (G, X) would as well. By the (H, α)-minimality of (G, X), we may assume, then, that $(G/uv, X)$ is not α-massed. Let (A, B) be a separation of $(G/uv, X)$ violating (M2), and assume that (A, B) is chosen to minimize $|B|$ from all such separations. Then $(G/uv[B], A \cap B)$ is α-massed. By the (H, α)-minimality of (G, X), the separation (A, B) of $(G/uv, X)$ is rigid. The separation (A, B) induces a separation (A^*, B^*) of (G, X). Let v_e be the vertex of G/uv corresponding to the contracted edge uv. If $v_e \in A$, let $A^* = (A - \{v_e\}) \cup \{u, v\}$, and $A^* = A$ otherwise. Similarly, define B^*. Notice that $\rho(B^* - A^*) \geq \rho(B - A)$. Because (G, X) has no separation violating condition (M2), the vertices u and v must lie in the set B. There are two simple cases now.

Case 1: u, $v \in A^* \cap B^*$ Then the order of (A^*, B^*) is exactly $|X|$. Moreover, $(G[B^*], A^* \cap B^*)$ is α-massed. Thus by the minimality of (G, X), (A^*, B^*) is a rigid separation, contrary to Theorem 3.4.1.

Case 2: Both u and v lie in $B^* - A^*$ We observed above that $(G/uv[B], A \cap B)$ contains a $\overline{\pi}$-rooted $\overline{H}$ minor for any subgraph $\overline{H}$ of H of size $|A \cap B|$. Given that u and v lie in $B^* - A^*$, then $A \cap B = A^* \cap B^*$. Then since $(G[B^*], A^* \cap B^*)$ contains a $\overline{\pi}$-rooted $\overline{H}$ minor,we see that $(G[B], A \cap B)$ does as well. We chose the map $\overline{\pi}$ and subgraph $\overline{H}$ arbitrarily, so the separation (A^*, B^*) is in fact a rigid separation, contradicting Theorem 3.4.1.

Thus contracting the edge uv must violate condition (M1), and the contraction of uv must have removed at least $\lfloor \alpha \rfloor + 1$ edges from G. Those edges either arise as common neighbors of u and v in G, or as edges that originally had only one end in the set X, and after contracting uv, now have two ends in X. Thus if u and v are not contained in X, the claim is proven. If $u \in X$, the number of triangles containing uv is equal to $\lfloor \alpha \rfloor - |\{x \in (X - \{u\}) : x \text{ is not adjacent } u \text{ and } x \text{ is adjacent to } v\}|$. The claim now follows. □

Claim 5 *There exists a vertex v in $V(G) - X$ such that* $\deg(v) < 2\alpha$.

Proof: By the definition of (H, α)-minimality, we know for any edge $e \not\subseteq X$ that $(G-e, X)$ is not (H, α)-minimal. Then $(G - e, X)$ must not be α-massed, implying that $(G - e, X)$ fails to satisfy either (M1) or (M2) in the definition of α-massed. Let $e = uv$ and assume there exists a separation (A, B) violating (M2). To prevent such a separation from violating (M2) in (G, X), we see that, without loss of generality, $u \in A - B$ and $v \in B - A$. By Claim 4, v has $\lfloor \alpha \rfloor$ neighbors that are either common neighbors with u or elements of X. In either case, v has $\lfloor \alpha \rfloor$ neighbors in A (other than the vertex v), implying that the order of (A, B) must be at least $\lfloor \alpha \rfloor$. But this contradicts our choice of α to be at least $|V(H)|$.

Thus we see that $(G - e, X)$ fails to satisfy (M1). This implies that $\rho(G - e, X) = \lfloor \alpha |V(G) - X| \rfloor + 1$. For every vertex $x \in X$, let $d^*(x)$ be the number of neighbors of x in $V(G) - X$. Then

$$2\rho(G - X) = \sum_{x \in X} d^*(x) + \sum_{v \in V(G)-X} \deg(v)$$

Every vertex $x \in X$ must have at least one neighbor y in $V(G) - X$, and by Claim 4, then x must have at least two neighbors in $|V(G) - X|$. Thus if the claim were false and $\deg(v) \geq 2\alpha$ for every $v \in V(G) - X$, we see

$$2\lfloor \alpha |V(G) - X| \rfloor + 2 \geq 2|X| + 2\alpha |V(G) - X|$$

which is false since $|X| \geq 2$ by (H, α)-minimality. □

This completes the proof of Theorem 3.4.2. □

Proof of Theorem 3.4.3. Let G, X, H, H', and π be as in the statement of the theorem. Assume G does contain such an H universal subgraph G'. If G contained $|X|$ disjoint paths from X to the subgraph G', then clearly (G, X) would contain a π-rooted H' minor. Thus there exists a separation (A, B) of G such that $X \subseteq A$, $G' \subseteq B$, and the order of (A, B) is strictly less than $|X|$. But then such a separation chosen of minimal order will be a rigid separation, contrary to Theorem 3.4.1. □

3.5 Minimal Graphs Not Containing Linkages

We translate the results from the previous section into terms of k-linked graphs. First, we consider what it means to be (H, α)-minimal when H is the graph consisting of k disjoint edges.

Definition 16 *Let G be a graph, let $X \subseteq V(G)$, and let $\alpha > 2$ be a real number. We say that the pair (G, X) is (α, k)-*minimal *if*

(1) *(G, X) is αk-massed,*

(2) *$|X| \leq 2k$ and (G, X) is not linked,*

(3) *subject to (1) and (2), $|V(G)|$ is minimum,*

(4) *subject to (1)–(3), $\rho(G - X)$ is minimum, and*

(5) *subject to (1)–(4), the number of edges of G with both ends in X is maximum.*

Now we consider Theorems 3.4.1 and 3.4.2 in terms of (α, k)-minimal pairs.

Corollary 3.5.1 *Let $k \geq 1$ be an integer, let $\alpha \geq 2$ be a real number, let G be a graph, and let $X \subseteq V(G)$ be such that (G, X) is (α, k)-minimal. Then G has no rigid separation of order at most $|X|$, and G has a subgraph L with $|V(L)| \leq \lceil 2\alpha k \rceil$ and minimum degree at least $\lfloor \alpha k \rfloor$.*

Proof: Let (G, X) be a (α, k)-minimal pair. Then if we let H be the graph consisting of $2k$ disjoint edges, (G, X) is also (H, α)-minimal. By Theorem 3.4.2, there exists a vertex v in $V(G) - X$ of degree at most $\lceil 2\alpha k \rceil - 1$ and every vertex adjacent to v shares at least

$\lfloor \alpha k \rfloor - t$ common neighbors with v, where

$$t := \max_{x \in X} |\{y \in (X - \{x\}) : \; y \text{ is not adjacent to } x\}|.$$

However, every vertex in X has at most one non-neighbor in X by (5) in the definition of (α, k)-minimality. Thus the subgraph of G induced by v and $N(v)$ satisfies the conclusion of the Corollary. The fact that (G, X) contains no rigid separation follows immediately from Theorem 3.4.1. □

CHAPTER 4

EXTREMAL FUNCTIONS FOR GENERAL LINKAGES

4.1 Outline

We state a strengthening of Theorem 1.4.2 to apply to α-massed pairs.

Theorem 4.1.1 *Let $k \geq 1$ be an integer, let G be a graph, and let $X \subseteq V(G)$ be such that $|X| \leq 2k$ and (G, X) is $5k$-massed. Then (G, X) is linked.*

In the final section, we will see that Theorems 1.4.1 and 1.4.2 follow easily from Theorem 4.1.1. Following the technique laid out in Chapter 3, we apply Corollary 3.5.1 to an (α, k)-minimal pair (G, X). We will later specify α to be either 8 (to obtain an easy proof) or 5 (to get the best bound).

We conclude that G contains a subgraph L with minimum degree at least αk with the size of L at most $2\alpha k$. The final step in the proof is to prove that L contains a k-linked subgraph L'. This is much easier for $\alpha = 8$, and so we do that first.

Theorem 4.1.2 *Let $k \geq 1$ be an integer, and let H be a graph with minimum degree at least $8k$ on at most $16k$ vertices. Then H has a k-linked subgraph.*

Theorem 4.1.2 will be proved in Section 4.2. By the argument given at the end of this section (with the constant 5 replaced by 8) those two theorems imply Theorem 4.1.1 again with the constant 8 instead of 5. To improve the bound to $5kn$ we need the following strengthening of Theorem 4.1.2, which we prove in Section 4.3.

Theorem 4.1.3 *Let $k \geq 1$ be an integer, and let H be a graph with minimum degree at least $5k$ on at most $10k$ vertices. Then H has a k-linked subgraph.*

4.1.1 Proof of Theorem 4.1.1 (assuming Theorem 4.1.3)

We now deduce Theorem 4.1.1. By changing the the constant 5 to 8 one can avoid using Theorem 4.1.3 and deduce the corresponding weakening of Theorem 4.1.1 using the easier Theorem 4.1.2 instead.

Let (G, X) be as stated in Theorem 4.1.1, and suppose for a contradiction that it is not linked. We may assume that (G, X) is $(5, k)$-minimal, and hence by Corollary 3.5.1 applied with $\alpha = 5$ the graph G has a subgraph H satisfying the hypotheses of Theorem 4.1.2. By Theorem 4.1.2 the graph H, and hence G, has a k-linked subgraph J.

Assume for a moment that G has $|X|$ disjoint paths $P_1, P_2, \ldots$ between X and $V(J)$, and choose them so that they have no internal vertex in J. Since J is k-linked, the ends of P_i in J can be linked as necessary to form a desired set of paths showing that (G, X) is linked, where each of these paths consists of the union of two P_i's with an appropriate subpath of the linkage in J. But this contradicts our assumption that (G, X) is not linked.

Thus the paths $P_1, P_2, \ldots$ of the previous paragraph do not exist, and hence G has a separation (A, B) of order at most $|X| - 1$ with $X \subseteq A$ and $V(J) \subseteq B$. Choose (A, B) of smallest possible order; then there exist $|A \cap B|$ disjoint paths from $A \cap B$ to $V(J)$, and an argument similar to the argument of the previous paragraph shows that (A, B) is rigid, contrary to Corollary 3.5.1. □

4.2 *Proof of Theorem 4.1.2*

Recall that Lemma 2.2.1 states that every graph J with $2\delta(J) \geq |J| + 3k - 4$ is k-linked. We proceed now with the proof of Theorem 4.1.2. Let k and H be as in the statement. We may assume that H is not k-linked, and hence there exists a sequence $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ of distinct vertices of H with no corresponding linkage. Let $X = \{s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k\}$, and let us choose a linkage $\mathcal{P}$ such that for each path $P \in \mathcal{P}$

(a) P has length at most seven,

(b) the ends of P are s_i and t_i for some $i = 1, 2, \ldots, k$,

(c) no internal vertex of P belongs to X,

(d) subject to (a)–(c), $|\mathcal{P}|$ is maximum, and

(e) subject to (a)–(d), the sum of the lengths of the paths in $\mathcal{P}$ is minimum.

Then $|\mathcal{P}| < k$, and so we may assume that s_1 and t_1 belong to no member of $\mathcal{P}$. Let L be the subgraph of H induced on X and the paths in $\mathcal{P}$. Notice that any vertex $v \in V(H) - V(L)$ has at most $3k$ neighbors in L, for otherwise it would have at least four neighbors on some path $P \in \mathcal{P}$, in which case it would have two non-consecutive neighbors on P, and so P could be shortened by using v, contrary to (e). Thus the graph $H - V(L)$ has minimum degree at least $8k - 3k = 5k$. Since L has at most $8(k-1) + 2$ vertices, we see that both s_1 and t_1 have a neighbor in $H - V(L)$.

We now show that $H - V(L)$ is not connected. To this end let S be the set of all vertices of $H - V(L)$ at distance at most two from a neighbor of s_1, where the distance is taken in the graph $H - V(L)$; and let T be defined analogously with t_1 replacing s_1. Then S and T are nonempty; by (d) they are disjoint, and no edge of H has one end in S and the other end in T. See Figure 8. We claim that $S \cup T \cup V(L) = V(H)$. To prove this claim let

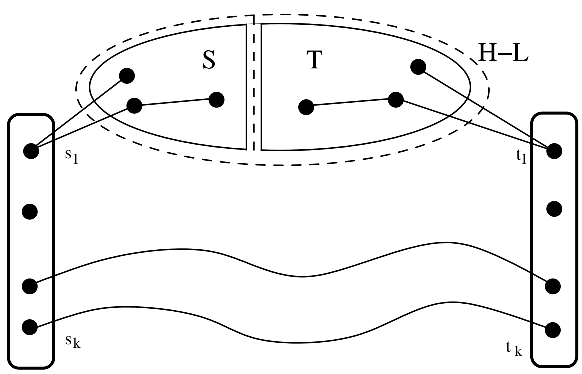

Figure 8: The construction of S and T.

$v \in V(H) - V(L)$, and let x and y be neighbors in $H - V(L)$ of s_1 and t_1, respectively. Then x, y, and v all have at least $5k$ neighbors in $H - V(L)$, but $H - V(L)$ has at most $16k - 2k = 14k$ vertices. Since S and T are disjoint, it follows that v belongs to S or T, as desired. This proves our claim that $S \cup T \cup V(L) = V(H)$, and hence concludes the proof of the fact that $H - V(L)$ is disconnected.

Now let J be the smallest component of $H - V(L)$. Then J has at most $(16k - 2k)/2 = 7k$

vertices and minimum degree at least $5k$. By Lemma 2.2.1 the graph J is k-linked, as desired. This completes the proof of Theorem 4.1.2. □

4.3 Proof of Theorem 4.1.3

We will need the following strengthening of Lemma 2.2.1, due to Egawa et al. [13], and obtained independently by Kawarabayashi, Kostochka and Yu [32]. For $4k \geq n \geq 3k$ the exact numerical bound does not follow from the statement of [13, Theorem 3], but it does follow from the proof.

Theorem 4.3.1 *([13], [32]) Let $k \geq 2$ be an integer, and let H be a graph on $n \geq 3k$ vertices and minimum degree δ. If $n \geq 4k$, then let $2\delta \geq n + 2k - 3$, and otherwise let $3\delta \geq n + 5k - 5$. Then H is k-linked.*

We are now ready to begin the proof of Theorem 4.1.3. Let k and G be as in the statement of the theorem. We may assume that G is not k-linked, and hence there exists a sequence $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ of distinct vertices of G with no corresponding linkage. Let $X = \{s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k\}$. A subgraph L of G is called a *partial linkage* if $X \subseteq V(L)$ and every component P of L satisfies the following conditions:

(a) P is a path of length at most five,

(b) either $V(P)$ consists of one member of X, or the ends of P are s_i and t_i for some $i = 1, 2, \ldots, k$, and

(c) no internal vertex of P belongs to X.

A partial linkage is called *minimal* if

(d) there is no partial linkage with strictly fewer components than L, and

(e) subject to (d), there is no partial linkage with fewer vertices.

By the choice of X, for every partial linkage L there exists an index $i \in \{1, 2, \ldots, k\}$ such that s_i and t_i are not connected by a path of L. Such indices will be called *unresolved* for L.

Claim 6 *Let L be a minimal linkage, let P be a component of L, and let $v \in V(G) - V(L)$. Then any two neighbors of v in P are at distance at most two in P. In particular, v has at most three neighbors on P. Moreover, v has at most $3k-2$ neighbors in $V(L)$.*

Proof To prove the first statement suppose for a contradiction that v has neighbors x and y on P such that the subpath of P from x to y has at least two internal vertices. Then by deleting those internal vertices from L and adding the path xvy we obtain a partial linkage with the same number of components but fewer vertices than L, contrary to the minimality of L. The second statement follows immediately from the first. To prove the third statement notice that if i is an unresolved index for L, then v is adjacent to at most one of s_i, t_i by the minimality of L. □

If L is a partial linkage and $i \in \{1, 2, \ldots, k\}$, then we define $S_i(L)$ to be the set of all neighbors of s_i in $V(G) - V(L)$, and we define $T_i(L)$ analogously.

Claim 7 *Let L be a minimal linkage, let i be unresolved for L, and let $v \in V(G) - V(L)$. Then v has at least five neighbors in $S_i(L) \cup T_i(L)$.*

Proof Let L, i, and v be as stated. For $m = 3, 4, 5, 6$ let l_m be the number of components of L on m vertices, and let l_2 be the number of indices $j \in \{1, 2, \ldots, k\}$ such that s_j and t_j are either adjacent in L, or not connected by a path of L. Let l'_3 be the number of components P of L such that P has three vertices, all adjacent to both s_i and t_i. Clearly $l_2 + l_3 + \cdots + l_6 = k$ and $2l_2 + 3l_3 + \cdots + 6l_6 = |V(L)|$. Let P be a component of L on $m \geq 4$ vertices. Then $|N(s_i) \cap V(P)| + |N(t_i) \cap V(P)| \leq m + 2$, for otherwise s_i and t_i have a common neighbor, say u, in the interior of P. In that case the linkage obtained from L by deleting P and adding the path $s_i u t_i$ has the same number of components as L, but fewer vertices, contrary to the minimality of L. Thus

$$\begin{aligned}|N(s_i) \cap V(L)| + |N(t_i) \cap V(L)| &\leq 4(l_2 - 1) + 6l'_3 + 5(l_3 - l'_3) + 6l_4 + 7l_5 + 8l_6 \\ &\leq |V(L)| + 2k + l'_3 - 4.\end{aligned}$$

From this it follows, since $S_i(L) \cap T_i(L) = \emptyset$ by the minimality of L,

$$|S_i(L) \cup T_i(L)| \geq 5k - |N(s_i) \cap V(L)| + 5k - |N(t_i) \cap V(L)|$$
$$\geq 10k - (|V(L)| + 2k + l_3' - 4) = 8k - |V(L)| - l_3' + 4.$$

Now let P be a component of L, and let $v \in V(G) - V(L)$. Then v has at most three neighbors on P by Claim 6. Moreover, if P has length two and every vertex on P is adjacent to both s_i and t_i, then v has at most two neighbors in P. Indeed, suppose the contrary, and let P have vertex-set $\{s_j, u, t_j\}$; then the partial linkage obtained from L by deleting P and adding the paths $s_i u t_i$ and $s_j v t_j$ contradicts the minimality of L. Thus v has at most two neighbors on P. This implies that for $v \in V(G) - V(L)$

$$|N(v) - V(L)| \geq 5k - 3(k - l_3') - 2l_3' \geq 2k + l_3'.$$

Now let t be the number of neighbors of v in $S_i(L) \cup T_i(L)$. Then

$$10k \geq |S_i(L) \cup T_i(L)| + |V(L)| + |\{v\}| + |N(v) - V(L)| - t$$
$$\geq 8k - |V(L)| - l_3' + 4 + |V(L)| + 1 + 2k + l_3' - t = 10k + 5 - t,$$

and so $t \geq 5$, as desired. □

If L is a partial linkage and i is unresolved for L, then we define $\overline{S}_i(L)$ to be the set of all vertices $v \in V(G) - V(L)$ such that either v belongs to or has a neighbor in $S_i(L)$; and we define $\overline{T}_i(L)$ analogously. See Figure 9. We now prove two fundamental properties of these sets.

Claim 8 *Let L be a minimal linkage, and let i be unresolved for L. Then $\overline{S}_i(L)$ and $\overline{T}_i(L)$ are disjoint, there is no edge between them, and their union is $V(G) - V(L)$.*

Proof If $\overline{S}_i(L)$ and $\overline{T}_i(L)$ are not disjoint, or if there is an edge between them, then there exists a path P between s_i and t_i of length at most five with internal vertices in $\overline{S}_i(L) \cup \overline{T}_i(L)$. But then the linkage $L \cup P$ has fewer components than L, and hence

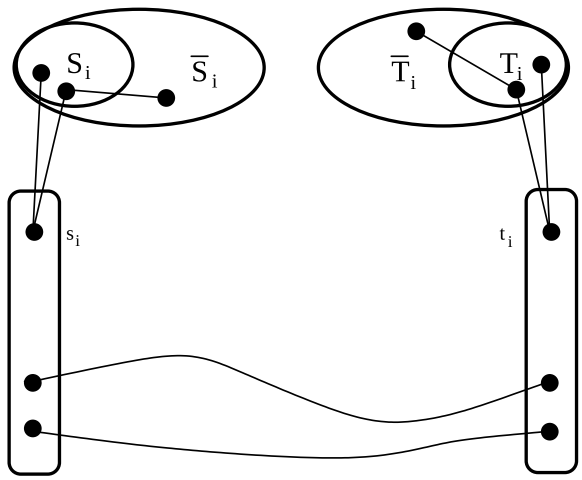

Figure 9: The construction of $\overline{S_i}$ and $\overline{T_i}$

contradicts the minimality of L. Now let $v \in V(G) - V(L)$. By Claim 7 the vertex v has a neighbor in $S_i(L)$ or $T_i(L)$, and so it lies in either $\overline{S}_i(L)$ or $\overline{T}_i(L)$, respectively. □

Claim 9 *Let L be a minimal linkage, and let i be an unresolved index for L. If $\overline{S}_i(L) \neq \emptyset$, then $|\overline{S}_i(L)| \geq 2k+3$. If $\overline{T}_i(L) \neq \emptyset$, then $|\overline{T}_i(L)| \geq 2k+3$.*

Proof From the symmetry it suffices to prove the first statement. By Claim 6, a vertex v in $V(G) - V(L)$ has at least $5k - (3k-2) = 2k+2$ neighbors in $V(G) - V(L)$, implying that if $\overline{S}_i(L)$ is non-empty, then $G[\overline{S}_i(L)]$ has minimum degree at least $2k+2$. Thus $|\overline{S}_i(L)| \geq 2k+3$, as desired. □

Guided by the proof of Theorem 4.1.2 our next objective is to show that a minimal linkage L and an unresolved index i for it can be chosen so that both $\overline{S}_i(L)$ and $\overline{T}_i(L)$ are nonempty. The proof is long, and makes use of further enlargements of the sets $\overline{S}_i(L)$ and $\overline{T}_i(L)$, which we shall denote by $\tilde{S}_i(L)$ and $\tilde{T}_i(L)$, respectively. We now introduce these sets.

Let L be a minimal linkage, let i be an unresolved index for L, and let $v \in \overline{S}_i(L) \cup \overline{T}_i(L)$ have three consecutive neighbors u_1, u_2, u_3, in order, on some component P of L. Let L' be obtained from L by deleting u_2 and adding the vertex v and edges u_1v and u_3v. Then L' is a minimal linkage and i is an unresolved index for L. We say that L' is a v-*flip* of L, and we say that the sequence u_1, u_2, u_3 is the *base* of the flip.

Claim 10 *Let L be a minimal linkage, let i be an unresolved index for L, let $v \in \overline{S}_i(L) \cup \overline{T}_i(L)$, and let L' be a v-flip of L with base u_1, u_2, u_3. Then $\overline{S}_i(L') - \{u_2\} = \overline{S}_i(L) - \{v\}$ and $\overline{T}_i(L') - \{u_2\} = \overline{T}_i(L) - \{v\}$. Moreover, $u_2 \in \overline{S}_i(L')$ if and only if u_2 has a neighbor in $\overline{S}_i(L) - \{v\}$. Similarly, $u_2 \in \overline{T}_i(L')$ if and only if u_2 has a neighbor in $\overline{T}_i(L) - \{v\}$.*

Proof Let $u \in \overline{S}_i(L) - \{v\}$. To prove the first two equalities, it suffices to prove, by symmetry, that $u \in \overline{S}_i(L') - \{u_2\}$. Clearly $u \neq u_2$, because $u \notin V(L)$. By Claim 7 the vertex u has at least five neighbors in $S_i(L) \cup T_i(L)$, but since $u \in \overline{S}_i(L)$, all those neighbors belong to $S_i(L)$ by Claim 8. It follows that u has a neighbor in $S_i(L')$, and hence it belongs to $\overline{S}_i(L')$, as desired. By Claim 7 and Claim 8 the vertex u_2 has at least five neighbors in either $S_i(L')$ or $T_i(L')$. In the former case $u_2 \in \overline{S}_i(L')$ and and it has a neighbor in $S_i(L) - \{v\}$, and in the latter case neither of these statements hold by Claim 8. The last assertion follows by symmetry. □

Let $L, L', i, v, u_1, u_2, u_3$ be as above, and assume now that $v \in \overline{S}_i(L)$. If u_2 has a neighbor $v' \in \overline{S}_i(L) - \{v\}$, then we say that L' is a *proper v-flip* of L. See Figure 10. In that case

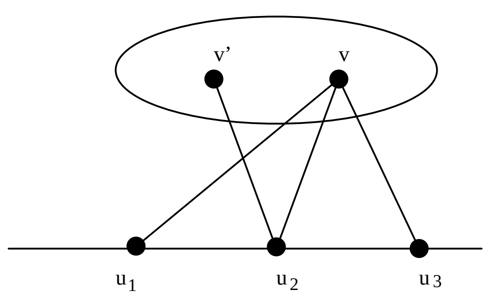

Figure 10: An example of a proper v-flip.

$u_2 \in \overline{S}_i(L')$ by Claim 10 and v has a neighbor in $\overline{S}_i(L') - \{u_2\}$ by Claim 7 and Claim 8. Thus L is a proper u_2-flip of L', and so the relationship is symmetric. We say that L and L' are $\overline{S}_i$-*adjacent*. If $v \in \overline{T}_i(L)$ then we say that the v-flip L' is *proper* if u_2 has a neighbor $v' \in \overline{T}_i(L) - \{v\}$, and say that L and L' are $\overline{T}_i$-*adjacent*. We say that two partial linkages

L and L' are *i-adjacent* if they are $\overline{S}_i$-adjacent or $\overline{T}_i$-adjacent. We say that L and L' are *i-related* if there exists a sequence $L_0, L_1, \ldots, L_n$ of linkages such that $L = L_0$, $L' = L_n$, and L_j is i-adjacent to L_{j-1} for all $j = 1, 2, \ldots, n$. The following is an immediate consequence of Claim 10.

Claim 11 *Let L be a minimal linkage, let i be an unresolved index for L, and let L' be a linkage i-related to L. Then $|\overline{S}_i(L')| = |\overline{S}_i(L)|$ and $|\overline{T}_i(L')| = |\overline{T}_i(L)|$.*

The next claim states that the order of $\overline{S}_i$- and $\overline{T}_i$-adjacencies can be reversed.

Claim 12 *Let L be a minimal linkage with i an unresolved index. Then if the linkage L_1 is $\overline{T}_i$-adjacent to L and L_2 is $\overline{S}_i$-adjacent to L_1, then there exist linkages L'_1 and L'_2 where L'_1 is $\overline{S}_i$-adjacent to L, and L'_2 is $\overline{T}_i$-adjacent to L'_1. Moreover, $L'_2 = L_2$.*

Proof Let L, i, L_1 and L_2 be as in the statement. Let $v_1 \in \overline{S}_i(L)$ be the vertex such that L_1 is a proper v-flip of L and let u_1, u_2, u_3 be the base of the flip. Similarly, let v_2 be the vertex in $\overline{T}_i(L_1)$ such that L_2 is a proper v_2-flip of L_1, and let w_1, w_2, w_3 be the base. By Claim 8 the vertex $v_2 \in \overline{T}_i(L_1) = \overline{T}_i(L)$ is not adjacent to $v_1 \in \overline{S}_i(L)$ or $u_2 \in S_i(L_1)$, where the equality and the last membership hold by Claim 10. Thus we see that $u_2 \notin \{w_1, w_2, w_3\}$. Since L_2 is a proper v_2-flip, the vertex w_2 has at least one other neighbor in $\overline{T}_i(L) = \overline{T}_i(L_1)$ besides the vertex v_2. Thus there exists a linkage L'_1 that is a proper v_2-flip of the linkage L. Moreover, u_1, u_2, u_3 are in some component P'_1 of L'_1, and since $\overline{S}_i(L'_1) = \overline{S}_i(L)$ by Claim 10, we see that there exists a linkage L'_2 that is a proper v_1-flip of L'_1. By construction, $L_2 = L'_2$, as desired. □

We are finally ready to define the promised enlargements of $\overline{S}_i$ and $\overline{T}_i$. Let L_0 be a minimal linkage, and let i be an unresolved index for L_0. We define $\tilde{S}_i(L_0) := \bigcup \overline{S}_i(L)$ and $\tilde{T}_i(L_0) := \bigcup \overline{T}_i(L)$, the unions taken over all linkages L that are i-related to L_0. We now show that these sets satisfy the conclusion of Claim 8.

Claim 13 *Let L_0 be a minimal linkage with i an unresolved index. Then $\tilde{S}_i(L_0)$ and $\tilde{T}_i(L_0)$ are disjoint, and there does not exist an edge with ends u and v such that $u \in \tilde{S}_i(L_0)$ and*

$v \in \tilde{T}_i(L_0)$.

Proof Assume we have $u \in \tilde{S}_i(L_0)$ and $v \in \tilde{T}_i(L_0)$ with u adjacent to v. Then there exists a linkage L i-related to L_0 with $u \in \overline{S}_i(L)$. There also exists a sequence $L = L_0, L_1, \ldots, L_m = L'$ of linkages, where $v \in \overline{T}_i(L')$ and L_j is i-adjacent to L_{j-1} for $j = 1, 2, \ldots, m$. Then by Claim 12, we may assume that there exists $l \leq m$, where for $1 \leq j \leq l$, L_{j-1} is $\overline{T}_i$-adjacent to L_j, and for $l+1 \leq j \leq m$, L_{j-1} is $\overline{S}_i$-adjacent to L_j. By Claim 10 $\overline{S}_i(L_j) = \overline{S}_i(L)$ for every $1 \leq j \leq l$. Importantly, $u \in \overline{S}_i(L_l)$. Similarly, by Claim 10, $v \in \overline{T}_i(L_l)$. But then for the minimal linkage L_l, we have an edge between vertices of $\overline{S}_i(L_l)$ and $\overline{T}_i(L_l)$. This contradicts Claim 8. To see that $\tilde{S}_i(L_0)$ and $\tilde{T}_i(L_0)$ are in fact disjoint, assume $v \in \tilde{S}_i(L_0) \cap \tilde{T}_i(L_0)$. Then there exists a linkage L i-related to L_0 with $v \in \overline{S}_i(L)$. But every vertex in $\overline{S}_i(L)$ has at least five neighbors in $S_i(L)$ by Claim 7 and Claim 8, so v has a neighbor in $\tilde{S}_i(L_0)$. But then there is an edge with one end in $\tilde{S}_i(L_0)$ and the other end in $\tilde{T}_i(L_0)$, contrary to what we have just seen. □

Now we are finally ready to prove that we may assume that there exists a minimal linkage L and an unresolved index i for L such that both $\overline{S}_i(L)$ and $\overline{T}_i(L)$ are nonempty.

Claim 14 *There exists a minimal linkage L and an unresolved index i such that either both $\overline{S}_i(L)$ and $\overline{T}_i(L)$ are nonempty, or one of $\tilde{S}_i(L)$, $\tilde{T}_i(L)$ induces a k-linked subgraph of G.*

Proof Let L_0 be a minimal linkage, and let i be an unresolved index for L_0. If both $\tilde{S}_i(L_0)$ and $\tilde{T}_i(L_0)$ are nonempty, then by Claim 11 we deduce that $\overline{S}_i(L_0)$ and $\overline{T}_i(L_0)$ are both nonempty, and so the claim holds. From the symmetry between $\tilde{S}_i(L_0)$ and $\tilde{T}_i(L_0)$ we may assume therefore that $\tilde{S}_i(L_0) = \emptyset$.

Let $v \in \tilde{T}_i(L_0)$ be a vertex of minimum degree in $G[\tilde{T}_i(L_0)]$, and let L be a linkage related to L_0 such that $v \in \overline{T}_i(L)$. Assume first that there exists a component P of L such that s_i has at least five neighbors on P and v has at least two neighbors on P. Let the ends of P be s_j and t_j. Since P has at least five vertices and v has at least two neighbors in P, Claim 6 implies that v is adjacent to an internal vertex of P. Let us choose such a neighbor, say u, so that it is not adjacent to s_j or t_j, if possible. Since $v \in \bar{T}_i(L)$ there exists a path Q of length at most two with ends v and t_i and internal vertex (if it exists) in $T_i(L)$. If u is adjacent to s_i let P' denote the path $s_i uvQt_i$. If u is not adjacent to s_i, then P has six vertices, and every vertex of $V(P) - \{u\}$ is adjacent to s_i. Let u' be a neighbor of u in P chosen so that u' is not equal or adjacent to s_j or t_j, and let P' denote the path $s_i u' uvQt_i$. Then in either case the length of P' is at most the length of P. Let L' be obtained from L by deleting the internal vertices of P and adding P'; then L' is a minimal linkage and j is an unresolved index for L'. From the symmetry between $S_j(L')$ and $T_j(L')$ we may assume that $S_j(L') = \emptyset$, for if both are nonempty, then the claim holds. In particular, u is adjacent to s_j, for otherwise the neighbor of s_j in P belongs to $S_j(L')$. It follows that there exists a vertex $u'' \in V(P) - V(P')$ not adjacent to s_j or t_j. Then u'' is adjacent to s_i, for otherwise P has length five and u is adjacent to s_i; consequently P' has length at most four, contrary to the minimality of L. By Claim 7 the vertex u'' has at least five neighbors in $S_j(L') \cup T_j(L') = T_j(L')$. Thus u'' has a neighbor $v'' \in T_j(L') - V(P) \subseteq V(G) - V(L) = \bar{T}_i(L)$. Let Q'' be a path of length at most two with ends v'' and t_i and internal vertex in $T_i(L)$. See Figure 11. Let L'' be obtained from L'

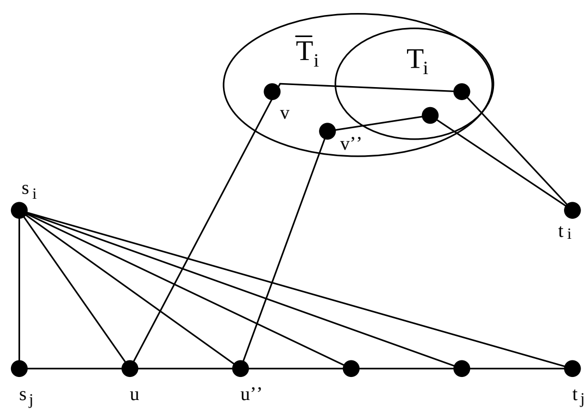

Figure 11: The vertex s_i and it's neighbors on P.

by replacing P' by the path $P'' := s_i u'' v'' Q'' t_i$. Then L'' is a minimal linkage, and by our choice of P'' to include only u'' from P, we see that both s_j's neighbor from P as well as t_j's neighbor from P is not included in L''. Thus j is an unresolved index with both $S_j(L'')$ and $T_j(L'')$ nonempty, proving the claim.

Thus we may assume that if a component P of L includes at least five neighbors of s_i, then it includes at most one neighbor of v. Since $\overline{S}_i(L) = \emptyset$, s_1 has at least $5k$ neighbors in $V(L)$, and hence at least $k/2$ components of L have at least five neighbors of s_1. Those components have at most one neighbor of v. The remaining components have at most two neighbors of v that do not belong to $\tilde{T}_i(L)$, because if v has three neighbors on a component P of L, then those neighbors are consecutive, and by considering a v-flip of L we deduce (using $\tilde{S}_i(L) = \emptyset$ and Claim 8) that the middle of the three neighbors belongs to $\tilde{T}_i(L)$. Thus v has at most $k/2 + 2k/2 = 3k/2$ neighbors outside $\tilde{T}_i(L)$, and hence $G[\tilde{T}_i(L)]$ has minimum degree at least $5k - 3k/2 = 7k/2$. But $\tilde{T}_i(L)$ includes no neighbor s of s_1, for otherwise a linkage L' with $s \in \overline{T}_i(L')$ contradicts Claim 8. Thus $|\tilde{T}_i(L)| \leq 10k - 5k \leq 5k$, and hence $G[\tilde{T}_i(L)]$ is k-linked by Theorem 4.3.1. □

Claim 14 enables us to choose a suitable linkage and an unresolved index for it. A linkage L is called *optimal* if

(O1) L is minimal,

(O2) $i = 1$ is an unresolved index for L, and

(O3) there is no minimal linkage L' with an unresolved index i' for L' such that

$$0 < \min\{|\overline{S}_{i'}(L')|, |\overline{T}_{i'}(L')|\} < \min\{|\overline{S}_1(L)|, |\overline{T}_1(L)|\}.$$

By Claim 14 we may assume (by permuting the elements of X) that there exists an optimal linkage, say L_0, and let L_0 be fixed for the rest of the chapter. Then every linkage 1-related to L_0 is also optimal by Claim 11. From the symmetry between $\overline{S}_1(L_0)$ and $\overline{T}_1(L_0)$ we may assume that $|\overline{S}_1(L_0)| \leq |\overline{T}_1(L_0)|$. Let $\tilde{S} := \tilde{S}_1(L_0)$ and $\tilde{T} := \tilde{T}_1(L_0)$. The following is the main advantage of optimality.

Claim 15 *If L is an optimal linkage and $v \in \overline{S}_1(L)$, then every v-flip is proper.*

Proof Let L' be a v-flip of L with base u_1, u_2, u_3, and suppose for a contradiction that it is not proper. Then $\overline{S}_1(L') = \overline{S}_1(L) - \{v\}$ by Claim 10 and $\overline{S}_1(L') \neq \emptyset$ by Claim 9, contrary to the optimality of L. □

Claim 16 *Either $|\tilde{S}| \geq 4k$ or $G[\tilde{S}]$ is k-linked.*

Proof Let v be a vertex of $\tilde{S}$ such that v is of minimum degree in $G[\tilde{S}]$. Then there exists a linkage L 1-related to L_0 with $v \in \overline{S}_1(L)$. Then, by Claim 6, for each component P of L, v has at most three neighbors in P, and if it has three, then they are consecutive. However, if v has three neighbors on P, say u_1, u_2, u_3, in order, then the v-flip of L is proper by Claim 15, showing that $u_2 \in \tilde{S}$. Thus v has at most 2 neighbors in $V(P) - \tilde{S}$ for each component P of L. Further, v has at most one neighbor among each pair of terminals not connected by a path in L. Thus v has at most $2(k-1)+1$ neighbors not in $\tilde{S}$. But then $G[\tilde{S}]$ has minimum degree at least $5k-(2k-1)=3k-1$. Thus $|\tilde{S}| \geq 3k$. If $|\tilde{S}| \leq 4k-1$, then by Theorem 4.3.1, $G[\tilde{S}]$ is k-linked. Thus the claim holds. $\square$

If Claim 15 held for vertices $v \in \overline{T}_1(L)$, then we would have an analogue of Claim 16 for $\tilde{T}$, and we would be done. Unfortunately, that is not the case, but, luckily, the counterexamples to the analogue of Claim 15 can be managed. Hence the following definition. Let L be an optimal linkage. We say that a vertex $u \in V(L)$ is L-*treacherous* if u is an internal vertex of a component P of L, u has a unique neighbor $v \in \overline{T}_1(L)$, and v is adjacent to both neighbors of u in P. Treacherous vertices are annoying in the sense that if v is as above, then the v-flip of L is not proper. Our intention is to pick two vertices in $\overline{T}_1(L)$ with the most treacherous neighbors, and remove them from $\overline{T}_1(L)$. Actually, we need to be more delicate. We need to not only remove them from $\overline{T}_1(L)$, but we also need to redefine $\tilde{T}$ as if those vertices did not exist. Let us be more precise.

Let L be a linkage, let $v \in \overline{S}_1(L) \cup \overline{T}_1(L)$, let L' be a proper v-flip of L with base u_1, u_2, u_3, and let $V \subseteq \overline{T}_1(L)$ be a set. If $v \notin V$, then we say that L and L' are *adjacent modulo* V. In that case $u_2 \in \overline{S}_1(L') \cup \overline{T}_1(L') - V$ and $V \subseteq \overline{T}_1(L')$ by Claim 10, and so the definition is symmetric in L and L'. We say that two linkages L and L' are *related modulo* V if there exists a sequence $L_0, L_1, \ldots, L_n$ of linkages such that $L = L_0$, $L' = L_n$, and L_i is adjacent to L_{i-1} modulo V for all $i = 1, 2, \ldots, n$. We shall abbreviate "1-adjacent" and "1-related" to "adjacent" and "related", respectively. Thus L and L' are related if and only if they are related modulo $\emptyset$.

Let L_1 be an optimal linkage related to L_0 and a vertex $v_1 \in \overline{T}_1(L_1)$ be chosen to maximize the number of L_1-treacherous neighbors of v_1. Let L_2 be an optimal linkage related to L_1 modulo $\{v_1\}$ and a vertex $v_2 \in \overline{T}_1(L_2) - \{v_1\}$ be chosen to maximize the number of L_2-treacherous neighbors of v_2. Let $\tilde{R} := \bigcup \overline{T}_1(L) - \{v_1, v_2\}$, the union taken over all linkages L related to L_2 modulo $\{v_1, v_2\}$. Then clearly $\tilde{R} \subseteq \tilde{T}$ and $v_1, v_2 \in \overline{T}_1(L)$ for every linkage L related to L_2 modulo $\{v_1, v_2\}$.

Claim 17 *Let L be a linkage related to L_2 modulo $\{v_1, v_2\}$, let $v \in \tilde{R} - V(L)$, and let ξ be the number of L-treacherous neighbors of v that do not belong to $\tilde{R}$. Then v has at least $3k - \xi - 1$ neighbors in $\tilde{R}$.*

Proof Let P be a component of L. We claim that v has at most two neighbors in $V(P) - \tilde{R}$ that are not L-treacherous. If v has three neighbors in $V(P) - \tilde{R}$, then by Claim 6 they are consequtive, say u_1, u_2, u_3, in order. Since $u_2 \notin \tilde{R}$ we deduce that the v-flip of L is not proper, and hence u_2 is L-treacherous. There is at least one index $j \in \{1, 2, \ldots, k\}$ such that s_j, t_j are not joined by a path of L, and the minimality of L implies that v is adjacent to at most one of s_j, t_j. Thus v has at most $2(k-1) + \xi + 1$ neighbors in $V(L) - \tilde{R}$. Hence v has at least $5k - (2k - 1 + \xi) = 3k + 1 - \xi$ neighbors in the complement of $V(L) - \tilde{R}$. Those neighbors belong to $\tilde{R}$, except for v_1 and v_2. Thus the claim holds. □

Claim 18 $|\tilde{R}| \geq 3k$

Proof Each component P of L_2 includes at most two L_2-treacherous vertices, because any two L_2-treacherous vertices on P are at distance at least two on P by the definition of an L_2-treacherous vertex and Claim 6. By Claim 9 and the optimality of L_2 we have $|\overline{T}_1(L_2)| \geq 2k + 3$, and hence there exists a vertex $v \in \overline{T}_1(L_2) - \{v_1, v_2\} \subseteq \tilde{R}$ not adjacent to any L_2-treacherous vertex. By Claim 17 the vertex v has at least $3k - 1$ neighbors in $\tilde{R}$, and the claim follows. □

Let v_3 be a vertex of minimum degree of the graph $G[\tilde{R}]$, and let L_3 be a linkage related to L_2 modulo $\{v_1, v_2\}$ such that $v_3 \in \overline{T}_1(L_3)$. For $i = 1, 2, 3$ let ξ_i denote the number of L_i-treacherous neighbors of v_i that do not belong to $\tilde{R}$.

Claim 19 *Let L be an optimal linkage, let $v \in \overline{T}_1(L)$, and let u be an L-treacherous neighbor of v. Let $w \in \overline{S}_1(L) \cup \overline{T}_1(L) - \{v\}$. Then the base of a w-flip of L does not include u.*

Proof Suppose for a contradiction that the base, say w_1, w_2, w_3, of a w-flip L' includes u. Since u is L-treacherous, v is adjacent to u and both neighbors of u in L. It follows that w_2 is adjacent to v, that u is adjacent to w, and that $w \in \overline{S}_1(L)$. But then the w-flip is proper by Claim 15, and hence $w_2 \in \overline{S}_1(L')$ and $v \in \overline{T}_1(L')$ by Claim 10. But w_2 is adjacent to v, contrary to Claim 8 applied to the linkage L'. □

Claim 20 *Let L be an optimal linkage, let $v \in \overline{T}_1(L)$, let u be an L-treacherous neighbor of v, and let L' be an optimal linkage related to L modulo $\{v\}$. Then $v \in \overline{T}_1(L')$ and u is L'-treacherous.*

Proof We have $v \in \overline{T}_1(L')$ by Claim 10. Let u_1, u_3 be the two neighbors of u in L. It suffices to prove the claim assuming that L' is adjacent to L modulo $\{v\}$. From Claim 19 we deduce that $u_1 u u_3$ is a subpath of L'. Suppose for a contradiction that u is not L'-treacherous. Then u is adjacent to a vertex $v' \in \overline{T}_1(L') - \{v\}$. Let L'' be the v-flip of L' with base u_1, u, u_3. Since u is adjacent to v', this v-flip is proper, and hence L'' is optimal and $u, v' \in \overline{T}_1(L'')$ by Claim 10. The vertex u is adjacent to at least five vertices in $T_1(L'')$ by Claim 7 and Claim 8, and hence it has at least three neighbors in $T_1(L)$, contrary to the fact that it is L-treacherous. □

Claim 21 *Let $i \in \{1, 2, 3\}$, and let u be an L_i-treacherous neighbor of v_i. Then u is not adjacent to v_j for $j \in \{i+1, \dots, 3\}$ and $u \notin \tilde{S}$.*

Proof Since L_j is related to L_i modulo $\{v_i\}$, Claim 20 implies that $v_i \in \overline{T}_1(L_j)$ and u is L_j-treacherous. Thus u is not adjacent to v_j. To prove that $u \notin \tilde{S}$ suppose the contrary. Thus there exists a sequence of linkages $L_i = R_0, R_1, \dots, R_t$ such that R_i is adjacent to R_{i-1} for $i = 1, 2, \dots, t$ and $u \in \overline{S}_1(R_t)$. By Claim 12 we may assume that there is an integer $l \in \{1, 2, \dots, t\}$ such that R_i is $\overline{S}_1$-adjacent to R_{i-1} for $i = 1, 2, \dots, l$ and that R_i is $\overline{T}_1$-adjacent to R_{i-1} for $i = l+1, \dots, t$. Then by Claim 10, $v_i \in \overline{T}_1(L_i) = \overline{T}_1(R_l)$ and $u \in \overline{S}_1(R_t) = \overline{S}_1(R_l)$. The edge uv_i violates Claim 8, a contradiction. □

Claim 22 *For $i = 1, 2$ no L_i-treacherous neighbor of v_i belongs to $\tilde{R}$.*

Proof Let u be an L_i-treacherous neighbor of v_i, and suppose for a contradiction that it belongs to $\tilde{R}$. Thus there exists a linkage L related to L_i modulo $\{v_1, v_2\}$ such that $u \in \overline{T}_1(L)$. By Claim 20 the vertex u is L-treacherous, a contradiction. □

Claim 23 $\xi_1 \geq \xi_2 \geq \xi_3$

Proof Let $i \in \{2, 3\}$. Since L_i is related to L_{i-1} modulo $\{v_1, \dots, v_{i-1}\}$ and L_{i-1} is related to L_{i-2} modulo $\{v_1, \dots, v_{i-2}\}$, we deduce that L_i is related to L_{i-2} modulo $\{v_1, \dots, v_{i-2}\}$. Thus the choice of v_{i-1} implies that v_{i-1} has at least ξ_i neighbors that are L_{i-1}-treacherous; but no treacherous neighbor of v_{i-1} belongs to $\tilde{R}$ by Claim 22, and hence $\xi_{i-1} \geq \xi_i$, as desired. □

Claim 24 *If $|\tilde{R}| < 4k$, then either $|\tilde{R}| \geq 4k - 3\xi_3 + 3$ or the graph $G[\tilde{R}]$ is k-linked.*

Proof The graph $G[\tilde{R}]$ has minimum degree at least $3k - \xi_3 - 1$ by Claim 17, because v_3 is a vertex of minimum degree in that graph. From Claim 18 and Theorem 4.3.1 we deduce that if the first conclusion of the claim does not hold, then the second does, as desired. □

Now we are ready to complete the proof of Theorem 4.1.3. Recall that $X = \{s_1, s_2, \dots, s_k, t_1, t_2, \dots, t_k\}$. By Claim 16 we may assume that $|\tilde{S}| \geq 4k$, for otherwise the theorem holds. But $\tilde{R}$ is disjoint from $\tilde{S} \cup X \cup \{v_1, v_2\}$ by Claim 13, and hence $|\tilde{R}| \leq 10k - 4k - 2k - 2 < 4k$. Similarly, by Claim 24 we may assume that $|\tilde{R}| \geq 4k - 3\xi_3 + 3$. For $i = 1, 2, 3$ let Z_i denote the set of L_i-treacherous neighbors of v_i not in $\tilde{R}$. Thus $|Z_i| = \xi_i$. Since the sets

$\tilde{S}$, $\tilde{R}$, Z_1, Z_2, Z_3 and X are pairwise disjoint by Claim 13 and Claim 21, we have, using Claim 23,

$$10k \geq |\tilde{S}| + |\tilde{R}| + \xi_1 + \xi_2 + \xi_3 + 2k \geq 4k + 4k - 3\xi_3 + 3 + \xi_1 + \xi_2 + \xi_3 + 2k \geq 10k + 3,$$

a contradiction. This proves Theorem 4.1.3.

4.4 Proof of Theorems 1.4.1 and 1.4.2

We prove Theorem 1.4.2. Theorem 1.4.1 follows by the same argument with the constant 8 replacing 5 and utilizing the suitable weakening of Theorem 4.1.1.

Let G be $2k$-connected and assume $|E(G)| \geq 5k|V(G)|$. Let X be a fixed set of $2k$ vertices. Then

$$\begin{aligned}\rho(V(G) - X) &\geq 5k|V(G) - X| + 5k|X| - |E(G[X])| \\ &\geq 5k|V(G) - X| + 5k|X| - \binom{|X|}{2} \\ &\geq 5k|V(G) - X| + 10k^2 - 2k^2.\end{aligned}$$

Thus (G, X) satisfies condition (M1). Further, since G is $2k$-connected, it satisfies (M2), implying that (G, X) is $5k$-massed. Theorem 4.1.1 implies that (G, X) is linked. Since we chose X arbitrarily, it follows that G is k-linked.

CHAPTER 5

THE EXTREMAL FUNCTION FOR 3-LINKAGES

5.1 Introduction

In this chapter, we prove Theorem 1.4.3 and find the optimal edge bound for ensuring a graph is 3-linked. For the proof of the extremal function for 3-linked graphs, we need a strengthening, the following.

Theorem 5.1.1 *Given a graph G on n vertices and $X \subseteq V(G)$ with $|X| = 6$, if (G, X) is $(5, 4)$-massed, then it is linked.*

First, we see that Theorem 1.4.3 follows from Theorem 5.1.1 and a straight forward application of the characterization of 2-linked graphs, Lemma 5.2.2.

Proof of Theorem 1.4.3 assuming Theorem 5.1.1 and Lemma 5.2.2. Let G be a 6-connected graph with $|E(G)| \geq 5|V(G)| - 14$. Fix a set X of six vertices and a linkage problem $\mathcal{L}$ on X. Label the vertices of X such that $\mathcal{L} = \{\{x_1, x_4\}, \{x_2, x_5\}, \{x_3, x_6\}\}$. Let t be the number of edges with both endpoints in X. Then

$$\rho(V(G) - X) = |E(G)| - t \geq 5|V(G)| - 14 - t = |V(G) - X| + 16 - t.$$

If $t = 15$, then the linkage problem $\mathcal{L}$ is feasible because $G[X]$ is a clique. If $t = 13$ or $t = 14$, then $x + i$ is adjacent to x_{i+3} for at least one index $i \in \{1, 2, 3\}$, and hence $\mathcal{L}$ is feasible by Lemma 5.2.2 (iii). Finally, if $t \leq 12$, $\mathcal{L}$ is feasible by Theorem 5.1.1. $\square$

We must slightly refine our notion of (α, k)-minimal pairs (G, X) now that we are considering $(5, 4)$-massed pairs. Towards that end, we define 3-minimal triple $(G, X, \mathcal{L})$ as follows.

Definition 17 *Let G be a graph, $X \subseteq V(G)$ with $|X| = 6$, and let $\mathcal{L}$ be a linkage problem* *on X. Assume the vertices of X are labeled such that $\mathcal{L} = \{\{x_1, x_4\}, \{x_2, x_5\}, \{x_3, x_6\}\}$. Then the triple $(G, X, \mathcal{L})$ is* 3-minimal *if the following hold:*

(A) (G, X) is $(5, 4)$-massed.

(B) The linkage problem $\mathcal{L}$ is not feasible.

(C) Subject to (A) and (B), $|V(G)|$ is minimal.

(D) Subject to (A), (B) and (C), $\rho(V(G) - X)$ is minimal.

(E) Subject to (A), (B), (C) and (D), the number of edges of $G[X]$ is maximal.

In Section 5.2, we use the characterization of 2-linked graphs to find exact edge bounds ensuring that a pair (G, X) is 2-linked when $|X| \leq 6$. A major difficulty in the proof of Theorem 5.1.1 that prevents us from simply applying the methods used to prove Theorem 4.1.1 are separations (A, B) of the following form. Let (A, B) be a separation of order exactly 6 with $X \subseteq A$ and satisfying $5|B - A| < \rho(B - A) < 5|B - A| + 4$. Since $\rho(B - S) < 5|B - A| + 4$, we are unable to find a rigid separation by considering the pair $(G[B], A \cap B)$. However, because $5|B - A| < \rho(B - A)$, the set $B - A$ contains too many edges to simply disregard and delete it. We develop two main tools for handling separations of this form.

We will formulate the first tool in Section 5.3, where we prove a lemma about graphs containing a linkage containing six paths from one set X to another Y, and the necessary number of edges to manipulate that linkage. We will apply it to our separation above by considering 6 paths from X to $A \cap B$ in the graph $G[A]$. The second tool for handling the annoying separations of the previous paragraph is developed in Section 5.5 where we analyze what we call star decompositions of graphs.

5.2 Linking Two Pairs of Vertices

We begin by examining edge bounds to ensure that a pair (G, X) is 2-linked where $|X| = 6$. To achieve this, we will use the following lemma about the number of edges it takes to force the pair (G, X) to be linked for a graph G and a set $X \subseteq V(G)$ when $|X| < 6$. This lemma is proven easily from the characterization of 2-linked graphs. As mentioned in Chapter 1, several researchers independently characterized such graphs (see [28, 52, 54, 61]). We use the formulation from [48].

Lemma 5.2.1 *[48] Let s_1, s_2, t_1, t_2 be distinct vertices of a graph G, such that no separation (A, B) of G of order ≤ 3 has $s_1, s_2, t_1, t_2 \in A \neq V(G)$. Then the following are equivalent:*

1. *There do not exist vertex-disjoint paths P_1, P_2 of G such that P_i links s_i and t_i for $i = 1, 2$.*

2. *G can be drawn in a disc with s_1, s_2, t_1, t_2 on the boundary in order.*

As an easy corollary to the above Lemma, we get the following:

Corollary 5.2.1 *Let G be a graph and $s_1, s_2, t_1, t_2 \in V(G)$. If there do not exist paths linking s_1, t_1 and s_2, t_2, then there exist subsets of vertices $A, B_1, \ldots, B_k$ for some k with the following properties:*

1. *Every edge $e \in E(G)$ either has both ends in A or in B_i for some $i \in \{1, \ldots, k\}$.*

2. *For every i, $|A \cap B_i| \leq 3$ and every $j \neq i$, $B_i \cap B_j \subseteq A$.*

3. *$s_1, s_2, t_1, t_2 \in A$ and $G[A]$ can be drawn in a disc with s_1, s_2, t_1, t_2 on the boundary in that order.*

We use the above corollary to prove the following lemma.

Lemma 5.2.2 *Let G be a graph and $X \subseteq V(G)$ of size at most 6. Let (G, X) be $(5, 1)$-massed. Then*

(i) if $|X| \leq 5$, (G, X) is linked,

(ii) either (G, X) is 2-linked or every pair of adjacent vertices in X have a common neighbor in $V(G) - X$, and

(iii) if (G, X) is $(5, 2)$-massed, then (G, X) is 2-linked.

Proof: The graph $G[V(G)-X]$ must have some connected component with edges to every vertex of X, lest G have some separation violating the definition of $(5,1)$-massed. Thus we may assume there exist distinct vertices s_1, t_1, s_2, t_2 such that there do not exist two disjoint paths P_1 and P_2 with the ends of P_i being s_i and t_i, for otherwise, the lemma holds. Let $X' = \{s_1, t_1, s_2, t_2\}$ and $X'' = X - X'$. By Corollary 5.2.1 applied to $G - X''$, there exist subsets $A, B_1, \ldots, B_k$ of $V(G) - X''$, with the properties stipulated in Corollary 5.2.1.

Then $\rho_G(V(G)-X) = \rho_{G[A\cup X'']}(A-X) + \sum_{i=1}^{k} \rho_G(B_i - A)$. Since $(A \cup (\bigcup_{i\neq k} B_i), B_k)$ is a separation of order at most three in $G - X''$ for every k, we see that $(A \cup X'' \cup (\bigcup_{i\neq k} B_i), B_k \cup X'')$ is a separation of order at most $|X|-1$ in G. Thus, $\rho_G(B_i - A) \leq 5|B_i - A|$ for every i. Moreover, since $G[A]$ is planar and has at least one face of size at least four, we see that $\rho_{G[A]}(A - X') \leq 3|A - X| + 1$ and consequently, $\rho_{G[A\cup X'']}(A - X) \leq (3 + |X''|)|A - X| + 1$. If $|X| \leq 5$, then $\rho_{G[A\cup X'']}(A - X) \leq 5|A - X|$, contrary to the fact that (G, X) is $(5,1)$-massed. This proves the lemma when $|X| \leq 5$; in particular, it proves (i). Thus we may assume that $|X| = 6$. We have $\rho_{G[A\cup X'']}(A - X) \leq 5|A - X| + 1$, and hence (G, X) is not $(5,2)$-massed. Thus (iii) holds. But (G, X) is $(5,1)$-massed, and so the inequalities above hold with equality. In particular, both vertices in X'' are adjacent every vertex of $V(G) - X$, the graph $G[A]$ is a triangulation except for exactly one face of size four (incident with s_1, s_2, t_1, t_2), and the pairs of vertices s_1, t_1 and s_2, t_2 are not adjacent. It follows that every pair of adjacent vertices in X have a common neighbor in $V(G) - X$, as desired by (ii). $\square$

5.3 Extremal Functions for Rerouting Paths

In this section, we focus on graphs where we are given a linkage with components $P_1, \ldots, P_6$ and we want to know how many edges the graph can have before we can find a different linkage $P'_1, \ldots P'_6$ in the graph satisfying various properties.

We are given the following setup: a graph G, a set X of six vertices and a fixed linkage problem $\mathcal{L}$ on X, and six disjoint paths from X to some set X' of six vertices. We want to show that if the graph has enough edges, subject to the graph having a basic amount of connectivity, then either we can reroute the six paths to induce a distinct linkage problem on X', or we can actually find a path linking one pair of the linkage problem $\mathcal{L}$, and still find paths from the remaining four vertices of X to X'. This arises in a natural way when we are attempting to prove the edge bound necessary to force a graph to be 3-linked.

The following will be a common hypothesis of several definitions and lemmas, and therefore it seems worthwhile to give it a name.

Hypothesis H**:** Let G be a graph and $X, X' \subseteq V(G)$ two sets of size six. Let $\mathcal{P} = \{P_1, \ldots, P_6\}$ be 6 disjoint induced paths where the ends of P_i are $x_i \in X$ and $x'_i \in X'$. Let $\mathcal{L}$ be the linkage problem $\{\{x_1, x_4\}, \{x_2, x_5\}, \{x_3, x_6\}\}$, and let $\mathcal{L}'$ be the linkage problem $\{\{x'_1, x'_4\}, \{x'_2, x'_5\}, \{x'_3, x'_6\}\}$.

Before proceeding, we prove a general lemma about desirable linkages from a fixed subgraph to the vertex set of another linkage.

Definition 18 *Let k be an integer and let $\mathcal{P}$ be a linkage with k components from X to X' in a graph G, where $|X| = |X'| = k$. Let the vertices of X and X' and the components $P_1, P_2, \ldots, P_k$ of $\mathcal{P}$ be numbered such that the ends of P_i are $x_i \in X$ and $x'_i \in X'$. Let H be a subgraph of G, and let $\mathcal{Q}$ be a linkage from $V(H)$ to $V(\mathcal{P})$. We say a vertex $v \in V(P_i)$ is* left $\mathcal{Q}$-extremal *if $v \in V(\mathcal{Q})$ and v is the only vertex of x_iP_iv that belongs to $V(\mathcal{Q})$. Similarly, we say $v \in V(P_i)$ is* right $\mathcal{Q}$-extremal *if $v \in V(\mathcal{Q})$, and v is the only vertex of $vP_ix'_i$ that belongs to $V(\mathcal{Q})$. We say a vertex v is* $\mathcal{Q}$-extremal *if it is either left or right $\mathcal{Q}$-extremal. We say that a vertex $v \in V(P_i)$ is* $\mathcal{Q}$-sheltered *if P_i has a $\mathcal{Q}$-extremal vertex and v belongs to the subpath of P_i with ends the left and right $\mathcal{Q}$-extremal vertices.*

*We say that $\mathcal{Q}$ is an H-*comb *if*

1. *for each $Q \in \mathcal{Q}$, its origin is in $V(H)$ and its terminus is a $\mathcal{Q}$-extremal vertex,*

2. *every $\mathcal{Q}$-extremal vertex is the terminus of some component of $\mathcal{Q}$, and*

3. *if some vertex of $V(H) \cap V(P)$ for some $P \in \mathcal{P}$ is not the terminus of any path $Q \in \mathcal{Q}$ and it is not $\mathcal{Q}$-sheltered, then every path of $\mathcal{Q}$ has length zero and P includes the terminus of at most one path in $\mathcal{Q}$.*

Lemma 5.3.1 *Let G be a graph, let $k, t \geq 1$ be integers, and let H be a subgraph of G. Let $X, X' \subseteq V(G)$ with $|X| = |X'| = k$ and let $\mathcal{P}$ be a linkage from X to X' with components $P_1, \dots, P_k$ such that the ends of P_i are $x_i \in X$ and $x_i \in X'$. Then either there exists a separation (A, B) of order strictly less than t with $X \cup X' \subseteq A$ and $V(H) \subseteq B$, or there exists an H-comb with t components.*

Proof: Let there be no separation as stated in the Lemma. By Menger's theorem, there exists a linkage from $V(H)$ to $X \cup X'$ with t components and no internal vertices in $V(H) \cup X \cup X'$. Let us choose such a linkage $\mathcal{Q}$ such that $E(\mathcal{Q}) - E(\mathcal{P})$ is minimal.

Let $Q_1, \dots, Q_t$ be the components of $\mathcal{Q}$. For $j = 1, \dots, t$, let q_j be the origin of Q_j and let $w_j \in V(Q_j) \cap V(\mathcal{P})$. Let Q'_j be defined as $q_j Q_j w_j$, and let $\mathcal{Q}'$ denote the linkage $Q'_1 \cup \dots \cup Q'_t$. Let us pick $w_1, w_2, \dots, w_t$ such that

(i) each w_i is $\mathcal{Q}'$-extremal, and

(ii) subject to (i), $|V(\mathcal{Q}')|$ is minimal.

Such a choice is possible because each terminus of a path in $\mathcal{Q}$ is $\mathcal{Q}$-extremal. We make the following claim.

Claim 25 *Each $\mathcal{Q}'$-extremal vertex is a terminus of a path in $\mathcal{Q}'$.*

Proof: Suppose to the contrary that there exists a $\mathcal{Q}'$-extremal vertex $w \in V(P_i) \cap V(Q'_j)$ for some $i \in \{1, 2, \dots, k\}$ and $j \in \{1, 2, \dots, t\}$, and $w \neq w_j$. Then replacing Q'_j by $q_j Q_j w$ yields a linkage that contradicts (ii). $\square$

It immediately follows that $\mathcal{Q}'$ satisfies conditions 1. and 2. in the definition of H-comb.

To prove that $\mathcal{Q}'$ satisfies Condition 3. in the definition of H-comb, let $x \in V(H) \cap V(P_i)$ be not $\mathcal{Q}'$-sheltered. We may assume from symmetry that the path $x_i P_i x$ is disjoint from $\mathcal{Q}'$. We may also assume that x is the only vertex of $V(H)$ in $x_i P_i x$. Since $x \in V(H) - V(\mathcal{Q}')$ and no internal vertex of a component of $\mathcal{Q}$ belongs to H, we deduce that $x \notin V(\mathcal{Q})$. We claim that $x_i P_i x$ is disjoint from $\mathcal{Q}$. Assume otherwise, and let y be the vertex of $\mathcal{Q}$ in $x_i P_i x$ closest to x, and let j be the index such that $y \in V(Q_j)$, The choice of x implies that

yQ_jq_j includes an edge not in $E(\mathcal{P})$. Thus replacing Q_j by $xP_iy \cup yQ_jw$, where w is the terminus of Q_j, yields a linkage that contradicts the minimality of $\mathcal{Q}$.

Thus x_iP_ix is disjoint from $\mathcal{Q}$. Then the path x_iP_ix could have been chosen for the linkage $\mathcal{Q}$ in lieu of another path. By the minimality of $E(\mathcal{Q}) - E(\mathcal{P})$, we deduce that $\mathcal{Q}$ is a subgraph of $\mathcal{P}$. By (ii), each Q'_j has length zero, and since x_iP_ix is disjoint from $\mathcal{Q}$, we see that P_i includes the terminus of at most one path in $\mathcal{Q}'$. Thus Condition 3. in the definition of H-comb holds. □

We will be looking for conditions to ensure that a graph satisfying Hypothesis H also satisfies one of the following conditions:

(C1) There exist disjoint paths $Q, Q_1, \ldots Q_4$ and an index $j \in \{1,2,3\}$ such that Q links x_j and x_{j+3} and each $Q_1, \ldots, Q_4$ has an end in X and the other end in X'.

(C2) There exist disjoint paths $Q_1, \ldots, Q_6$ with the ends of Q_i being x_i and q_i, where $q_i \in X'$ for all i. Furthermore, the linkage problem $\{\{q_1,q_4\},\{q_2,q_5\},\{q_3,q_6\}\}$ is distinct from $\mathcal{L}'$.

Lemma 5.3.2 *Let G be a graph satisfying Hypothesis H. Let H be an induced subgraph with $|\partial(H)| \geq 5$ and further, assume the following conditions hold:*

1. *At most two of the paths in $\mathcal{P}$ intersect H in more than one vertex, and at most 3 paths total intersect $V(H)$.*

2. *For any distinct vertices $v, s_1, s_2, t_1, t_2 \in \partial(H)$, there exist paths Q_1, Q_2 with ends s_1, t_1 and s_2, t_2 respectively with all internal vertices of the paths in $V(H) - v$.*

Then either (C1) or (C2) holds, or the pair $(G, X \cup X')$ has a rigid separation of order at most four.

While technical, this lemma is saying something fairly intuitive. In Hypothesis H, we are given the six paths in G, and some subgraph H that allows us to cross paths that enter H. By Lemma 5.3.1, if the does not exist an H-comb of with five components, then there exists a small separation separating $X \cup X'$ from H which will necessarily be rigid. Otherwise, we find such an H-comb. Then the H-comb either allows us to cross two of the paths to arrive at X' in a distinct linkage problem, or we can link one pair of terminals in the linkage problem $\mathcal{L}$ and still link the other four vertices in X to X'.

Proof: Assume the statement is false, and let G be as in Hypothesis H forming a counterexample.

If there exists a separation of order at most 4 separating $X \cup X'$ from $V(H)$, then by the assumptions on H, the separation must be rigid. Thus no such separation exists, and by Lemma 5.3.1, there exists an H-comb of $\mathcal{Q}$ with five components. Let the components of $\mathcal{Q}$ be labeled $Q_1, \ldots, Q_5$. Let q_i be the origin of Q_i in H.

We claim that every vertex of $V(H) \cap V(\mathcal{P})$ is $\mathcal{Q}$-sheltered. To see that, let $x \in V(H) \cap V(\mathcal{P})$, and suppose for a contradiction that x is not $\mathcal{Q}$-sheltered. By property 3. in the definition of comb, every path Q_i is trivial and hence at least three paths in $\mathcal{P}$ intersect H, with each intersection corresponding to a trivial path in $\mathcal{Q}$. Then by our assumptions, exactly three paths in $\mathcal{P}$ do, two in at least 2 vertices say P_i and P_j, and one in exactly one vertex, say P_k. By 3. in the definition of H-comb, x cannot lie on P_i or P_j. As a result, either $x \in P_k$ and three paths of $\mathcal{P}$ intersect H in at least 2 vertices, or there is a fourth path of $\mathcal{P}$ intersecting H. Either case is a contradiction to our assumptions. Hence every vertex in $V(H) \cap V(\mathcal{P})$ is $\mathcal{Q}$-sheltered.

Because the five termini of $\mathcal{Q}$ are distributed among the 6 paths of $\mathcal{P}$, there are two cases to consider.

Case 1: *There exists an index i such that P_i and P_{i+3} both include a terminus of a path in $\mathcal{Q}$.*

Without loss of generality, assume that P_1 contains the terminus y_1 of Q_1 and P_4 contains the terminus y_2 of Q_2. Then there is at most one other path containing 2 termini of $\mathcal{Q}$. As a subcase, assume some P_j, $j \neq 1, 4$ contains two termini of $\mathcal{Q}$. Without loss of generality, let P_2 has y_2 the terminus of Q_3 and z_2 the terminus of Q_4. Then there exist disjoint paths R_1, R_2 in H where R_1 links q_1 and q_2 and R_2 links q_3 and q_4. We can pick R_1 and R_2 to avoid q_5, and so by the previous paragraph, we see that R_1 and R_2 have no internal vertices in $V(\mathcal{P}) - (V(y_1P_1z_1) \cup V(y_2P_2z_2))$. Then the linkage

$$x_1P_1y_1Q_1q_1R_1q_2Q_2y_4P_4x_4, \quad x_2P_2y_2Q_3q_3R_2q_4Q_4z_2P_2x_2', \quad P_3, P_5, P_6$$

satisfies (C1).

Otherwise, each Q_i, $i \geq 3$ has its terminus in a different path of $\mathcal{P}$. Then each of P_2, P_3, P_5, and P_6 have at most one vertex in $V(H)$, and any such vertex in H must be equal to q_i for some i. By our assumptions on H, there exists a path R in H linking q_1 and q_2 avoiding q_3, q_4, and q_5. The paths

$$x_1P_1y_1Q_1q_1Rq_2R_2y_4P_4x_4, \quad P_2, P_3, P_5, P_6$$

satisfy (C1).

Case 2: *There exist indices i and $j \neq i, i+3$ such that P_i and P_j each contain at least two termini of $\mathcal{Q}$*

Without loss of generality, let P_1 contain the terminus y_1 of Q_1 and the terminus z_1 of Q_2. Let P_2 contain the terminus y_2 of Q_3 and the terminus z_2 of Q_4. Observe that q_5 is the only possible vertex of P_3, P_4, P_5, P_6 to lie in $V(H)$. By our assumptions on H, H contains disjoint paths R_1 linking q_1 and q_4 and R_2 linking q_2 and q_3 avoiding the vertex q_5. Then the linkage

$$x_1P_1y_1Q_1q_1R_1q_4Q_4z_2P_2x_2' \quad x_2P_2y_2Q_3q_3R_2q_2Q_2z_1P_1x_1', \quad P_3, P_4, P_5, P_6$$

satisfies (C2). This completes the proof of the Lemma. □

Now we immediately apply the previous lemma in proving the following result about the necessary number of edges in a graph to guarantee (C1) or (C2).

Lemma 5.3.3 *Let G be a graph satisfying Hypothesis H. If*

1. $\rho(V(G) - X) \geq 5|V(G) - X| + 1$, *and*

2. every separation (A, B) of order at most 4 with $X, X' \subseteq A$ satisfies

$$\rho(B - A) \leq 5|B - A|,$$

then G satisfies (C1) or (C2).

Proof: Assume the Lemma is false, and let G be a counterexample satisfying Hypothesis H on a minimal number of vertices, and, subject to that, with $\rho(V(G) - X)$ minimal. We assume that X has an edge between all possible pairs of vertices of X except for the pairs $(x_1, x_4), (x_2, x_5), (x_3, x_6)$. Adding these edges if necessary clearly does not change the truth or falsehood of the hypotheses or conclusions of the Lemma.

Claim 26 *$(G, X \cup X')$ has no rigid separation of order at most four.*

Proof: Let (A, B) be such a separation, and assume we have chosen it to minimize $|A|$. Consider the graph G' that is defined to be the graph obtained from $G[A]$ by adding edges between every pair of non-adjacent vertices in $A \cap B$. For notation, let $S := A \cap B$. By Condition 2 in the statement of the lemma, it follows that $\rho(V(G') - X) \geq 5|V(G') - X| + 1$. Also, we know that G' has six disjoint paths from X to X' with the same path ends as in G since any path in G that uses vertices of $B - A$ can be converted to a path in G' because $G'[S]$ is complete. Let the paths be labeled $P'_1, \ldots, P'_6$ with the ends of P'_i being x_i and x'_i. For paths in G' satisfying (C1) or (C2), we may assume that each path uses at most one edge of $G'[S]$. Because edges in S may be extended to paths in G with all internal vertices in $B - A$, we know any paths in G' satisfying (C1) or (C2) extend to paths in G. If G' satisfies Condition 2 in the statement of the lemma, by minimality it follows that G also satisfies (C1) or (C2), a contradiction. Thus we see that G' has a separation violating Condition 2. Let (A', B') be such a separation, and assume (A', B') is chosen to minimize $|B'|$. Then by Lemma 5.2.2, it follows that $(G'[B'], A' \cap B')$ is linked. Because $G'[S]$ is complete, we know that $S \subseteq A'$ or B'. If $S \subseteq A'$, then $(A' \cup B, B')$ is a separation in G violating Condition 2 in the statement of the lemma. Consequently $S \subseteq B'$. As we saw above, disjoint paths in G' linking terminals in $A' \cap B'$ extend to disjoint paths in G, and hence, $(A', B' \cup B)$ is a rigid separation in G violating our choice of (A, B).

This contradiction completes the proof that $(G, X \cup X')$ has no rigid separation of order at most four. $\square$

Claim 27 *G has no nontrivial separation (A, B) of order six with $X \subseteq A$ and $X' \subseteq B$.*

Proof: Assume otherwise, and let (A, B) be such a separation. Then if we consider $G[A]$, X, and $A \cap B$, the linkage $\mathcal{P}$ induces a natural labeling of $A \cap B = \{a_1, \ldots, a_6\}$ with $\{a_i\} = V(P_i) \cap A \cap B$. The graph $G[A]$, the sets X, $A \cap B$, and the paths of $\mathcal{P}$ restricted to A satisfy Hypothesis H. Similarly, $G[B]$, $A \cap B$, and X' also satisfy Hypothesis H. Condition 2 will naturally hold in $G[A]$ and $G[B]$. Moreover, in at least one of $G[A]$ or $G[B]$, Condition 1 will also hold. By the minimality of G as a counterexample, one of $G[A]$ or $G[B]$ has paths satisfying (C1) or (C2), and consequently, G would as well. This contradiction proves the claim. $\square$

Now we attempt to contract an edge e, $e \not\subseteq E(G[X \cup X']) - E(\bigcup_i P_i)$. This may have the effect of merging two vertices, x_j and x'_j into a single vertex, which we will consider to be a member of both X and X' in G/e connected by a path of length zero. Since G has no nontrivial separation of order six separating X from X', we know that G/e has six paths $P^*_1, \ldots P^*_6$ from X to X'. Let the ends of P^*_i be x_i and y'_i. If the linkage problems $\{\{x'_1, x'_4\}$, $\{x'_2, x'_5\}, \{x'_3, x'_6\}\}$ and $\{\{y'_1, y'_4\}, \{y'_2, y'_5\}, \{y'_3, y'_6\}\}$ are distinct, then the paths P^*_i in G/e extend to disjoint paths P'_i with the same endpoints in G satisfying (C2). This implies that $\{\{x'_1, x'_4\}, \{x'_2, x'_5\}, \{x'_3, x'_6\}\} = \{\{y'_1, y'_4\}, \{y'_2, y'_5\}, \{y'_3, y'_6\}\}$, and so for the sake of this paragraph we may assume that by possibly renumbering the vertices of X', the ends of P^*_i are x'_i and x_i. If G/e were to satisfy Conditions 1. and 2. in the statement, then by the minimality of G, G/e has paths $Q^*_1, \ldots Q^*_k$ satisfying (C1) or (C2). Those paths extend to paths $Q'_1, \ldots Q'_k$ in G satisfying (C1) or (C2). Thus we have proven contracting the edge e violates one of the hypotheses of the Lemma.

Claim 28 *G/e violates Condition 1. for every edge $e \not\subseteq X$, $e \not\subseteq X'$.*

Proof: We have seen above that G/e must violate Condition 1. or 2. Assume to reach a contradiction, that G/e has a separation (A', B') violating Condition 2. Pick such a separation to minimize the size of B'. Let v_e be the vertex of G/e corresponding to the contracted edge e, and let P_i^*, P_i' be as in the previous paragraph. Then if $v_e \in A' - B'$, the separation (A', B') induces a separation (A, B) in G violating Condition 2 in the statement of the Lemma. We conclude that $v_e \in B'$. By Lemma 5.2.2, $(G/e[B'], A' \cap B')$ is linked. If $v_e \in B' - A'$, (A', B') induces a rigid separation in G of order at most four, contrary to Claim 26. Thus we may assume in fact that $v_e \in A' \cap B'$, and (A', B') induces a separation (A, B) of order five in G with $X, X' \subseteq A$ and $\rho(B - A) \geq 5|B - A| + 1$. Also, since only one path of the P_i' uses endpoints of e, we know that at most four paths of the P_i' use vertices of B. If exactly four of the paths P_i' use vertices of B, then there exists an index $i = 1, 2, 3$ such that P_i' and P_{i+3}' both use vertices of B. Without loss of generality, assume P_1' and P_4' use vertices of B. It follows that no path can use vertices of $B - A$. The graph $G[B - A]$ must have some connected component with all of $A \cap B$ as neighbors, since $\rho(B - A) \geq 5|B - A| + 1$. Then the pair of terminals x_1 and x_4 can be connected with a path using vertices of B without intersecting the remaing paths P_2', P_3', P_5', P_6' and consequently G satisfies (C1). Thus we may assume that at most three of the paths P_i' use vertices of B, and because $|A \cap B| = 5$, at most two paths use more than one vertex of B. By Lemma 5.2.2, $(G[B], A \cap B)$ is linked. We have shown $G[B]$ satisfies all the conditions of Lemma 5.3.2. Since G has no rigid separation of order at most four, we then know that G would satisfy (C1) or (C2), a contradiction. $\square$

Thus we may assume that contracting the edge e violates Condition 1. in the statement of the lemma. We will show that the endpoints of e have five common neighbors. We refer to these common neighbors as *triangles* containing e. We prove

Claim 29 *Every edge $e \not\subseteq X$, $e \not\subseteq X'$ is contained in at least five triangles.*

Proof: Given such an edge e, by Claim 28 we see that G/e violates Condition 1 in the statement of the Lemma. Since G/e has exactly one fewer vertex in $G/e - X$, the edge count must decrease by at least six. If $e \cap X = \emptyset$, then the decrease in the edge count corresponds to the number of common neighbors of u and v. Thus the endpoints of e have at least five common neighbors, proving the claim. If $e = uv$ and $v \in X$, then upon contracting e, the edge count decreases by the sum the number of common neighbors of u and v and the number of neighbors of u in X besides v. Without loss of generality, assume that $v = x_1$. We know that u is not adjacent to x_4, since by Claim 27 there exist four paths from $X - \{x_1, x_4\}$ to $X' - \{x'_1, x'_4\}$ not containing the vertex u. Moreover, we have already assumed that x_1 is adjacent to all vertices of X besides x_4. Then, in fact, all the neighbors of u in X are common neighbors with v, and u and v consequently have at least five common neighbors. □

Similarly to when we contracted an edge, if $e \nsubseteq X$ and if $G-e$ satisfies the conditions of the Lemma, then by minimality, there exist paths in $G - e$ satisfying (C1) or (C2). Those paths would also exist in G. We conclude that $G - e$ violates Condition 1. or 2. of the Lemma.

Claim 30 *For any edge $e \nsubseteq X$, $e \nsubseteq X'$, $G - e$ violates Condition 1.*

Proof: Assume to reach a contradiction that there exists a separation (A, B) of $G - e$ violating Condition 2. in the Lemma. In order for (A, B) not to induce a separation in G violating Condition 2 in the statement of the Lemma, it must be the case that one end of e belongs to $A - B$ and the other end to $B - A$. But the ends of e must have at least five neighbors in common, and all these common neighbors must lie in $A \cap B$. This contradicts the order of (A, B), proving the Claim. □

Since $G-e$ does not satisfy Condition 1 in the statement of the Lemma, as an immediate consequence we see:

Claim 31 $\rho(V(G) - X) = 5|V(G) - X| + 1$.

We now show that we can find a vertex of small degree outside the sets X and X'. Let

$$A := V(G) - X - X'.$$

First, we see that A is not empty.

Claim 32 $V(G) \neq \bigcup_{i=1}^{6} V(P_i)$.

Proof: Assume that $V(G)$ does in fact consist of the vertices of the paths $P_1, \dots, P_6$. Some path must be non-trivial, since it is not the case that $X = X' = V(G)$. Without loss of generality, assume P_1 is non-trivial, and let uv be an edge on P_1, with x_1, u, v, x_1' occurring on P_1 in the order listed. We may also assume no vertex of P_1 has a neighbor on P_4, lest we satisfy (C1). We see that u and v have five common neighbors on the paths P_2, P_3, P_5, P_6. Then u and v have two common neighbors on the same path, say P_2, call them r and s, and assume r precedes s on the path P_2. Then we get paths

$$x_1 P_1 u s P_2 x_2', \quad x_2 P_2 r v P_1 x_1', \quad P_3, \dots, P_6$$

satisfying (C2), proving the Claim. □

First we prove two facts we will use repeatedly in analyzing the cases to come is the following:

Claim 33 *G contains no K_5 subgraph.*

Proof: The statement follows immediately from Lemma 5.3.2 and the fact that G has no rigid separation of order at most four. □

Claim 34 *For any vertex $v \in A$, there exist six disjoint paths $P_1^*, \dots, P_6^*$ in G where the ends of P_i^* are $x_i \in X$ and $y_i \in X'$ such that the paths avoid v and the linkage problem $\{\{y_1, y_4\}, \{y_2, y_5\}, \{y_3, y_6\}\}$ is equal to $\mathcal{L}$.*

Proof: Given such a vertex $v \in A$, by Claim 27, we know there exist $P_1^*, \dots, P_6^*$ such that the ends of P_i^* are $x_i \in X$ and $y_i \in X'$. To see this, consider $G - v$. If there did not exist six disjoint paths from X to X', then $G - v$ would contain a separation (A, B) of order at most five with $X \subseteq A$ and $X' \subseteq B$. Then $(A \cup \{v\}, B \cup \{v\})$ is a nontrivial separation in G separating X from X' of order at most six, a contradiction to Claim 27.

If the paths $P_1^*, \ldots, P_6^*$ induced a distinct linkage problem on X', this would violate our choice of G as a counterexample. $\Box$

The next claims establish that there exists a vertex in A of small degree.

Claim 35 *Every vertex in A has at most six neighbors in $X \cup X'$.*

Proof: Assume $v \in A$ has strictly more than six neighbors in $X \cup X'$. By Claim 34, we may assume $v \notin \bigcup_i V(P_i)$. Then there exists some index $i \in \{1, 2, 3\}$ such that v has both x_i and x_{i+3} as neighbors, or v has both x'_i and x'_{i+3} as neighbors. Then we are able to link x_i and x_{i+3} through the vertex v and still find paths from the remaining four vertices of X to X'. The graph G would then satisfy (C1), a contradiction. $\Box$

Claim 36 *There exists a vertex in A of degree at most 11.*

Proof: Assume otherwise. If we let $f(x)$ be the number of neighbors a vertex $x \in X$ has in $V(G) - X$, then we see

$$2\rho(V(G) - X) = \sum_{v \in A} \deg(v) + \sum_{x \in X' - X} \deg(x) + \sum_{x \in X} f(x)$$

By assumption, every vertex in A has degree at least 12, and every vertex $v \in X' - X$ has some neighbor u on the path P_i terminating at v. As we saw above, the edge uv is in at least five triangles, implying that v has degree at least six. Thus we see

$$2\rho(V(G) - X) \geq 12|A| + 6|X' - X| + \sum_{x \in X - X'} f(x)$$

Each vertex $v \in X - X'$ has some neighbor u on the path P_i beginning at v, and the edge uv is in at least five triangles. Since we know that v has at most four neighbors in X, $f(v) \geq 2$.

Thus

$$
\begin{aligned}
2\rho(V(G)-X) &\geq 12|A| + 6|X'-X| + 2|X-X'| \\
&= 10|V(G)-X| + 2|V(G)-X| - 4|X'-X| \\
&= 10|V(G)-X| + 2|A| - 2|X'-X|.
\end{aligned}
$$

Then because vertices in A have at most six neighbors in $X \cup X'$, we know that $|A| \geq 7$. But in fact, if $|A| = 7$, $G[A] = K_7$, contradicting the fact that G has no K_5 subgraph. Thus we may assume that $|A| \geq 8$. The above equation then contradicts the fact that $\rho(V(G)-X) = 5|V(G)-X| + 1$. □

Let $v \in A$ be a vertex of degree at most 11. We show that the neighborhood of v, $N(v)$, is sufficiently dense to apply Lemma 5.3.2. By Claim 29, we see that that the minimum degree of $G[N(v)]$ is five. By Claim 34, we may assume that none of the paths $P_1, \ldots, P_6$ uses the vertex v.

Claim 37 *There does not exist an index $i \in \{1, 2, 3\}$ such that P_i and P_{i+3} both intersect $N(v)$.*

Proof: Assume otherwise, and without loss of generality, that P_1 and P_4 intersect $N(v)$. Then P_1 can be linked to P_4 using the vertex v, implying that G would satisfy (C1), a contradiction. □

Claim 38 *At most three paths of $\mathcal{P}$ intersect $N(v)$. If exactly three such paths do intersect $N(v)$, then one of them contains exactly one vertex of $N(v)$.*

Proof: If four or more paths use vertices of $N(v)$, then there exists an index i such that P_i and P_{i+3} both intersect $N(v)$, contradicting Claim 37. Assume exactly three paths do, and further assume that all three paths use at least two vertices of $N(v)$. Again by Claim 37, we may assume the three paths are P_1,P_2, and P_3.

Let $S := N(v) \cap V(\mathcal{P})$ and $T = N(v) - V(\mathcal{P})$. Let s_i and t_i be the first and last vertex of S on P_i, respectively. Then $|S| \geq 6$ and hence $|T| \leq 5$. We claim that for distinct integers $i, j \in \{1, 2, 3\}$

($\star$) There is no path Q from $s \in S \cap V(P_i) - \{t_i\}$ to $t \in S \cap V(P_i) - \{s_j\}$ with interior in T.

Indeed, if such a path Q exists, say for $i = 1$ and $j = 2$, then the paths

$$x_1 P_1 s Q t P_2 x_2', \quad x_2 P_2 s_2 v t_1 P_1 x_1', \quad P_3, P_4, P_5, P_6$$

satisfy (C2), a contradiction.

In particular, ($\star$) implies that every $s \in S$ has at most three neighbors in S because s_1 has at most one neighbor in $V(P_1) \cap S$ (since P_1 is induced) and at most two in $S - V(P_1)$, namely s_2 and s_3. If $s \in V(P_1) - \{s_1, t_1\}$, then s has at most two neighbors in $V(P_1) \cap S$ and none in $S - V(P_1)$. Thus each $s \in S$ has at least two neighbors in T by Claim 29. Also by ($\star$), the neighbors in T of the vertices s_1 and t_2 belong to different components of $G[T]$; thus, in particular, $G[T]$ has at least two components and $|T| \geq 4$. Hence $|S| \leq 7$. Since $|T| \leq 5$, some component of $G[T]$, say J, has at most two vertices. By ($\star$) the neighbors of J that belong to S are contained in one of the following sets: $S \cap V(P_1)$, $S \cap V(P_2)$, $S \cap V(P_3)$, $\{s_1, s_2, s_3\}$, or $\{t_1, t_2, t_3\}$. Since $|S| \leq 7$, each of these sets has at most three vertices. Yet each vertex of J has at least five neighbors in $S \cup T$ by Claim 29, a contradiction. $\square$

In order to apply Lemma 5.3.2 and complete the proof of the lemma, all that remains to show is the following claim.

Claim 39 *Let $S := \{s_1, t_1, s_2, t_2, x\}$ be vertices in $N(v)$. There exist paths in $G[N(v) \cup v]$ linking s_1 to t_1 and s_2 to t_2 that do not contain the vertex x.*

Proof: Consider the vertices s_1 and t_1. We may assume that s_1 is not adjacent to t_1, for otherwise the paths $s_1 t_1$ and $s_2 v t_2$ would satisfy the Claim. Each of s_1 and t_1 must have at least two neighbors each in $N(v) - S$ by Claim 29. We may assume these neighbors are in different components of $G[N(v) - S]$, otherwise, connect s_1 and t_1 with a path in $N(v) - S$

and link s_2 and t_2 with the vertex v. But each vertex in $N(v) - S$ can have at most three neighbors in S, lest s_i and t_i have a common neighbor for one of the values of i. Thus each vertex of $N(v) - S$ has at least two neighbors in $N(v) - S$. Since $G[N(v) - S]$ must have at least two connected components, we see that it in fact consists of two disjoint K_3 subgraphs and every vertex in $N(v) - S$ has exactly three neighbors in S. But then every vertex in $N(v) - S$ is adjacent to x. One of the K_3 subgraphs in $N(v) - S$ along with x and v forms a K_5 subgraph in G, a contradiction to Claim 33. Thus, in fact we are able to link the pairs (s_i, t_i) and avoid the vertex x. □

We have shown that the subgraph $G[N(v) \cup v]$ satisfies all the requirements of Lemma 5.3.2. Because we have shown in Claim 26 that (G, X) does not have any rigid separations of order at most four, we arrive at the final contradiction to our choice of G to not satisfy (C1) or (C2). □

5.4 The Extremal Function for 3-linkages

In the course of the proof, we will ensure that every edge of a 3-minimal triple$(G, X, \mathcal{L})$ not contained in X is in five triangles. This means the neighborhood of a vertex v of minimum degree will induce a subgraph N of minimum degree five. Moreover, we will see that the edge bound in the definition of $(5, 4)$-massed is satisfied with equality. Since the graph then has strictly less than $5|V(G)|$ edges, we know G has a vertex of degree at most nine. Additionally we show that there exists such a vertex v of degree at most nine not contained in the set X. We attempt to find disjoint paths from X to the neighborhood of v. The graph $G[N(v) \cup \{v\}]$ is sufficiently dense that if we consider a set X' of at most five vertices in $N(v)$, the pair $(G[N(v) \cup \{v\}], X')$ is 2-linked. Thus if there exists a small separation separating X from $N(v)$ in the graph G, the pair (G, X) will have a rigid separation. The existence of a rigid separation will provide a contradiction to our choice of a 3-minimal triple $(G, X, \mathcal{L})$.

Given that no small separation exists, by Menger's Theorem there exist six disjoint paths from X to $N(v)$. If we let X' be the set of ends of the paths in $N(v)$ then the linkage problem on X naturally gives a linkage problem $\mathcal{L}'$ on the path ends X'. Ideally, we would link two pairs of $\mathcal{L}'$ in the subgraph N, and link the third pair of terminals using the vertex v. It is not the case, however, that any such two pairs of the linkage problem $\mathcal{L}'$ can be linked in N. This leads us to the following definition.

Definition 19 *Let G be a graph, $X \subseteq V(G)$ with $|X| = 6$, and let $\mathcal{L}$ be a linkage problem on X consisting of three pairs of vertices. Let the vertices of X be labeled such that $\mathcal{L} = \{\{x_1, x_4\}, \{x_2, x_5\}, \{x_3, x_6\}\}$. The triple $(G, X, \mathcal{L})$ is* quasi-firm *if there exist distinct indices i and j in $\{1, 2, 3\}$ and disjoint paths P_i and P_j with all internal vertices in $V(G) - X$ with the ends of P_i equal to x_i and x_{i+3} and the ends of P_j equal to x_j and x_{j+3}.*

In our 3-minimal triple $(G, X, \mathcal{L})$ above, we do not need (N, X') to be 2-linked and that we be able to link any two pairs of vertices; it would suffice that only some two pairs of vertices in $\mathcal{L}$ could be linked. If the triple $(N, X', \mathcal{L}')$ were quasi-firm, we could link the final pair of vertices of $\mathcal{L}'$ using the vertex v adjacent all of $N(v)$. Unfortunately, it is not the case that $(N, X', \mathcal{L}')$ will always be quasi-firm, but the instances where it is not are limited in scope.

First, we prove that a 3-minimal triple cannot contain a rigid separation of order at most six.

Lemma 5.4.1 *Let $(G, X, \mathcal{L})$ be a 3-minimal triple. Then the pair (G, X) does not have a nontrivial rigid separation of order at most six.*

Proof: Assume that (G, X) does have a nontrivial rigid separation, call it (A, B). Assume from all such rigid separations, we pick (A, B) such that $|A|$ is minimized. If $|A \cap B| = 6$, then find six paths from X to $A \cap B$ in $G[A]$. If such paths existed, we could link the endpoints of the paths as prescribed by the linkage problem $\mathcal{L}$ given the fact that $(G[B], A \cap B)$ is linked. But if those six paths did not exist, then $G[A]$ contains a separation (A', B') of order less than six with $X \subseteq A'$ and $A \cap B \subseteq B'$. But such a separation (A', B') chosen of minimal order induces a rigid separation of (G, X), namely $(A', B \cup B')$, violating our choice of (A, B).

Now assume (A, B) has order at most five. Let G' be obtained from G in the following manner. The graph G' is equal to $G[A]$ with additional edges added to every non-adjacent pair of vertices in $A \cap B$. Thus $G'[A \cap B]$ is a complete subgraph. By $(M2^*)$ we deleted at most $5|B - A|$ edges when we deleted the vertices of $B - A$, and as a result, $\rho(V(G') - X) \geq 5|A - X| + 4$. Assume that (G', X) also satisfies condition $(M2^*)$. Then by (C), we know linkage problem $\mathcal{L}$ is feasible in G'. Take three paths linking the pairs of $\mathcal{L}$, and choose them to be as short as possible. Then each path uses at most one edge in $A \cap B$ because $G'[A \cap B]$ is a complete subgraph. These disjoint edges can be extended to disjoint paths in G with every internal vertex in $B - A$ by the fact that $(G[B], A \cap B)$ is linked. Thus the linkage problem $\mathcal{L}$ would be feasible in G, a contradiction.

Consequently, the pair (G', X) has a separation violating $(M2^*)$. Let (A', B') be such a separation, and assume it is picked such that $|B'|$ is minimized. Because $G'[A \cap B]$ is a complete subgraph, $A \cap B \subseteq A'$ or $A \cap B \subseteq B'$. If $A \cap B \subseteq A'$, then $(A' \cup B, B')$ would be a separation in G violating $(M2^*)$. Thus $A \cap B \subseteq B'$. Given our choice of (A', B'), we know $(G'[B'], A' \cap B')$ is $(5, 1)$-massed. By Lemma 5.2.2 (i), we know that $(G'[B'], A' \cap B')$ is linked. Disjoint paths in $G'[B']$ using edges of $A \cap B$ can be extended as in the previous paragraph, so we see that $(A', B' \cup B)$ is a rigid separation of G, violating our choice of (A, B). This proves the lemma. □

The following lemma will be used to show that if $(G, X, \mathcal{L})$ is a 3-minimal triple, $v \in V(G) - X$ has degree at most nine, and $X' \subseteq N(v)$ satisfies $|X'| \leq 5$, then $(G[N(v) \cup \{v\}], X')$ is 2-linked.

Lemma 5.4.2 *Let G be a graph and $X \subseteq V(G)$ with $|X| \leq 5$. Assume that $\delta(G) \geq 6$, $|V(G)| \leq 10$, and moreover, assume there exists a vertex $v \in V(G) - X$ adjacent to every other vertex of G. Then (G, X) is 2-linked.*

Proof: Let $\mathcal{L}$ be a linkage problem on X. Clearly, we may assume that $|X| \geq 4$ and $\mathcal{L}$ consists of two pairs of vertices, otherwise there can be at most one pair of vertices in $\mathcal{L}$ and they can be linked through the vertex v. Assume that the vertices of X are labeled such that $\mathcal{L} = \{\{s_1, t_1\}, \{s_2, t_2\}\}$. Let H be the subgraph induced on $(V(G) - \{v\}) - X$.

Then $|V(H)| \leq 5$. If H is not connected, then it has a component of order at most two. Consequently, there exists a vertex $x \in V(H)$ with at least four neighbors in X, and there exists an index $i = 1$ or 2 such that x is adjacent to both s_i and t_i. Then s_i and t_i can be linked through the vertex x and the other pair of vertices in $\mathcal{L}$ can be linked through the vertex v.

Thus we have shown that H must be a connected subgraph. If s_1 were adjacent to t_1, we could link s_2 and t_2 through the vertex v to show that the linkage problem is feasible. Otherwise, s_1 and t_1 are not neighbors, and consequently they each have at least two neighbors in H. Since H is connected, we can link s_1 and t_1 in the subgraph H and still link s_2 and t_2 through the vertex v. Thus the linkage problem $\mathcal{L}$ is feasible, completing the proof. □

Lemma 5.4.3 *Let G and $X \subseteq V(G)$ with $|X| = 6$ such that $\delta(G) \geq 5$ and $|V(G)| \leq 9$. Let $\mathcal{L} = \{\{x_1, x_4\}, \{x_2, x_5\}, \{x_3, x_6\}\}$ be a linkage problem on X. If $(G, X, \mathcal{L})$ is not quasi-firm, then the following hold:*

1. *For any vertices x_i and x_j in X, there exists a path linking x_i and x_j with no internal vertex in X,*
2. *for any linkage problem $\mathcal{L}'$ on X distinct from $\mathcal{L}$, the triple $(G, X, \mathcal{L}')$ is quasi-firm, and*
3. *for any index $i \in \{1, \ldots, 6\}$ and any vertex $y \in V(G) - X$, if we consider the linkage problem $\mathcal{L}' = \{\{y, x_{i+3}\}, \{x_{i+1}, x_{i+4}\}, \{x_{i+2}, x_{i+5}\}\}$ on $(X - \{x_i\}) \cup \{y\}$ where all index addition is mod 6, then $(G, (X - \{x_i\}) \cup \{y\}, \mathcal{L}')$ is quasi-firm.*

Proof: We prove the lemma by a series of intermediate claims. First, we prove several general observations about the structure of G before we analyze the cases arising from the possible sizes of G. Let H be the induced subgraph on $V(G) - X$, and let the vertices of H be labeled $h_1, \ldots, h_i$ where $i \leq 3$.

Claim 40 *H is a connected subgraph.*

Proof: Assume H is not connected. Because $|V(H)| \leq 3$, one component of H must then consist of an isolated vertex, call it h_1. Then h_1 has at least five neighbors in X, and consequently, there exist distinct indices i and j such that x_i, x_{i+3}, x_j and x_{j+3} all are adjacent to h_1. Also, there exists some h_2 distinct from h_1 that has at most one neighbor in H. Consequently h_2 has at least four neighbors in X, and so there exists an index k such that x_k and x_{k+3} are both adjacent to h_2. The index k must be distinct from i or j, so without loss of generality assume $k \neq i$. Then the paths $x_i h_1 x_{i+3}$ and $x_k h_2 x_{k+3}$ contradict our assumption that $(G, X, \mathcal{L})$ is not quasi-firm. □

Conclusion 1 follows easily now.

Claim 41 *For any x_i and x_j in X, there exists a path linking x_i and x_j with no internal vertex in X.*

Proof: We may assume that x_i is not adjacent x_j. Then x_i and x_j each must have some neighbor in H. By Claim 40, H is connected so the desired path exists. □

Claim 42 *For every $i = 1, 2, 3$, the vertices x_i and x_{i+3} are not adjacent.*

Proof: Assume, without loss of generality, that x_1 is adjacent x_4. Using Claim 41, there exists a path linking x_2 and x_5, contradicting the assumption that $(G, X, \mathcal{L})$ is not quasi-firm. □

We have seen that H is connected, but in fact we can show something stronger. We now prove the following claim.

Claim 43 *H is a complete subgraph.*

Proof: Assume that H is not a complete subgraph. By Claim 40, we may assume that H is connected, forcing H to be a path on three vertices. Without loss of generality, assume that h_1 and h_3 are the endpoints of the path. Then h_1 and h_3 have four neighbors in X, and consequently there exists an index i such that h_1 is adjacent x_i and x_{i+3}. Similarly, there exists an index j such that h_3 is adjacent to x_j and x_{j+3}. We may assume that $i = j$, since otherwise the paths $x_i h_1 x_{i+3}$ and $x_j h_2 x_{j+3}$ contradict our assumption that $(G, X, \mathcal{L})$ is not quasi-firm. Without loss of generality, we assume $i = 1$ and x_1 and x_4 are both adjacent to h_1 and h_3. We know that h_2 must have at least three neighbors in X, so h_2 has some neighbor that is neither x_1 nor x_4. Without loss of generality, assume that x_2 is adjacent to h_2. The vertex x_5 has some neighbor in $V(H)$. If x_5 is adjacent to h_2, we get the linkage $x_1 h_1 x_4$ and $x_2 h_2 x_5$. But otherwise, x_5 is adjacent one of h_1 and h_3. The cases are symmetric, so assume x_5 is adjacent h_1. Then we get the linkage $x_1 h_3 x_4$ and $x_2 h_2 h_1 x_5$. Every case contradicts the assumption that $(G, X, \mathcal{L})$ is not quasi-firm, proving the claim. □

It will be convenient to refer to pairs of vertices we have shown to not be adjacent.

Definition 20 *A set $a = \{x, y\}$ of two distinct vertices x and y is an* anti-edge *if x is not adjacent y.*

To avoid confusion with edges, we will denote anti-edges containing x and y by (x, y). An *anti-matching of size k* is a set of k disjoint anti-edges. A *perfect anti-matching* in a graph H is an anti-matching of size $|V(H)|/2$.

Claim 44 $G[X]$ *does not contain two distinct perfect anti-matchings.*

Proof: We know by Claim 42 that the pairs $x_1 x_4$, $x_2 x_5$, and $x_3 x_6$ form a perfect anti-matching. If another distinct perfect anti-matching on X existed, then there would exist two distinct indices i and j such that x_i, x_{i+3}, x_j and x_{j+3} all have at most three neighbors in X. Thus they each have at least two neighbors in H. Then x_i and x_{i+3} have a common neighbor in H, say h_1. By Claim 43, the subgraph $H - h_1$ is connected. Since x_j and x_{j+3} each have a neighbor in $H - h_1$, we get the linkage consisting of $x_i h_1 x_{i+3}$ and a path from x_j to x_{j+3} with interior in $H - h_1$, a contradiction. □

In other words, if $G[X]$ does not contain a unique perfect anti-matching, then $(G, X, \mathcal{L})$ is quasi-firm. The second conclusion of the lemma now follows easily.

Claim 45 *For any linkage problem $\mathcal{L}'$ on X distinct from $\mathcal{L}$, the triple $(G, X, \mathcal{L}')$ is quasi-firm.*

Proof: Assume that $(G, X, \mathcal{L}')$ is not quasi-firm. Then Claim 42 holds for the triple $(G, X, \mathcal{L}')$. However, then both $\mathcal{L}$ and $\mathcal{L}'$ induce distinct perfect anti-matchings in X, contrary to Claim 44. □

This proves Conclusion 2 of the Lemma. We also can now prove the third point in the Lemma.

Claim 46 *For any index $i \in \{1, \ldots, 6\}$ and any vertex $y \in V(G) - X$, if we consider the linkage problem $\mathcal{L}' = \{\{y, x_{i+3}\}, \{x_{i+1}, x_{i+4}\}, \{x_{i+2}, x_{i+5}\}\}$ on $(X - \{x_i\}) \cup \{y\}$ where all index addition is mod 6, then $(G, (X - \{x_i\}) \cup \{y\}, \mathcal{L}')$ is quasi-firm.*

Proof: Assume the claim is false and that $(G, (X - \{x_i\}) \cup \{y\}, \mathcal{L}')$ is not quasi-firm. Without loss of generality, assume $i = 1$ and $\mathcal{L}' = \{\{y, x_4\}, \{x_2, x_5\}, \{x_3, x_6\}\}$. By the previous claims, we know that H is connected, and that x_1 is not adjacent x_4, which forces x_4 to have at least one neighbor in H. If x_2 and x_5 or x_3 and x_6 had x_1 as a common neighbor, say x_2 and x_5, we would get the path $x_2x_1x_5$ and we can connect y_1 to x_4 using H, contradicting the fact that $(G, (X - \{x_1\}) \cup \{y\}, \mathcal{L}')$ is not quasi-firm. Hence x_1 is adjacent to at most one vertex of x_2 and x_5 and at most one vertex of x_3 and x_6. Without loss of generality assume x_1 is not adjacent x_2 and x_3. By the minimum degree condition of G, it follows then that x_1 has three neighbors in H and that x_1 is adjacent to x_5 and x_6. From Claim 42 applied to the triple $(G, (X - \{x_1\}) \cup \{y\}, \mathcal{L}')$, we deduce that x_4 is not adjacent to y. It follows that x_4 must have a neighbor h_1 in H different from y. Let h_2 be the other vertex of H not equal to h_1 or y. Note that y is adjacent h_1 and h_2 by Claim 43.

If the vertex x_2 is adjacent to h_2, then the linkage $x_2h_2x_1x_5$ and yh_1x_4 contradicts the fact that $(G, (X - \{x_1\}) \cup \{y\}, \mathcal{L}')$ is not quasi firm. Thus x_2 is not adjacent to h_2 and by the minimum degree condition, x_2 is adjacent to y. Similarly, h_2 is not adjacent to x_3 and x_3 is adjacent to y. The vertex h_2 must be adjacent to one of x_5 and x_6, again by the minimum degree condition of G. By symmetry, assume h_2 is adjacent x_5. We get the linkage $x_2yh_2x_5$

and $x_1h_1x_4$, contradicting the fact that the triple $(G, X, \mathcal{L})$ is not quasi-firm. This final contradiction proves the claim. $\square$

This completes the proof of the lemma. $\square$

We now return to the difficulty we introduced in Section 5.1, namely separations (A, B) of order six with $\rho(B - A) < 5|B - A| + 4$ and $\rho(B - A) > 5|B - A|$. Moreover, these unpleasant separation need not be unique. We will have to examine the case when the graph can be decomposed into a large number of non-crossing separations. We explicitly define a decomposition thus:

Definition 21 *Let $X \subseteq V(G)$ with $|X| = 6$ and let $k \geq 1$. A sequence $(A, B_1, \ldots, B_k)$ of subsets of $V(G)$ is a* star decomposition of (G, X) *if the following conditions hold:*

1. *$X \subseteq A$,*
2. *for all distinct indices $i, j \in \{1, \ldots, k\}$, $B_i \cap B_j \subseteq A$,*
3. *for all $i \in \{1, \ldots, k\}$, $(\bigcup_{j \neq i} B_j \cup A, B_i)$ is a separation of order exactly six, and*
4. *for all $i \in \{1, \ldots, k\}$, $(G[B_i], A \cap B_i)$ is 2-linked.*

The separations $(\bigcup_{j \neq i} B_j \cup A, B_i)$ are called the separations *determined* by the star decomposition $(A, B_1, \ldots, B_k)$.

As an easy observation about star decompositions, we give the following lemma.

Lemma 5.4.4 *Let $(G, X, \mathcal{L})$ be a 3-minimal triple, and let $(A, B_1, \ldots, B_k)$ be a star decomposition of (G, X). For all $i = 1, \ldots, k$, there does not exist a separation (C, D) of G of order at most five with $X \subseteq C$ and $B_i \subseteq D$.*

Proof: Assume such a separation (C, D) existed for some index i. Assume we pick such a separation of minimal order. Then there exist disjoint paths from $C \cap D$ to $B_i \cap A$ in $G[D]$. Then any linkage problem on $C \cap D$ extends to a linkage problem on $B_i \cap A$. Moreover, since $|C \cap D| \leq 5$, the induced linkage problem has at most two pairs of vertices.

Consequently, the induced linkage problem will be feasible in $G[B_i]$, implying that (C, D) is a rigid separation of (G, X). This contradicts Lemma 5.4.1. □

Given a pair (G, X) and a star decomposition $(A, B_1, \ldots, B_k)$, let $e = uv$ be a fixed edge of G not contained in X. Then the star decomposition $(A, B_1, \ldots, B_k)$ *induces* the star decomposition $(A^*, B_1^*, \ldots, B_k^*)$ in G/e where v_e, the vertex of G/e corresponding to the contracted edge e, lies in B_i^* or A^* if and only if either u or v is an element of B_i or A, respectively.

Lemma 5.4.5 *Let $(G, X, \mathcal{L})$ be a 3-minimal triple. Let $e = uv$ be a fixed edge in G not contained in X. Let G have a star decomposition $(A, B_1, \ldots, B_k)$ with the added constraint that $e \subseteq B_i \cap A$ for all $i = 1, \ldots k$. Let $(A^*, B_1^*, \ldots B_k^*)$ be the induced star decomposition in G/e. Then $\bigcup_i (B_i^* \cap A^*)$ has at least $3k$ anti-edges.*

The proof of Lemma 5.4.5 is somewhat involved and technical. We postpone the proof until Section 5.5 and proceed with the proof of Theorem 5.1.1.

Proof of Theorem 5.1.1, assuming Lemma 5.4.5. Assume the theorem is false. We let $(G, X, \mathcal{L})$ be a 3-minimal triple.

First, we make the following observation.

Claim 47 *The pairs (x_i, x_{i+3}) are anti-edges for all $i = 1, 2, 3$. Moreover, these are the only anti-edges in $G[X]$.*

Proof: If there exists some index, say $i = 1$, such that x_1 is adjacent to x_4, then by Lemma 5.2.2 (*iii*), the linkage problem $\mathcal{L}$ is feasible in G, a contradiction. Moreover, since adding any edge to $G[X]$ not linking a pair of vertices of $\mathcal{L}$ does not affect the feasibility of $\mathcal{L}$, we see by (E) in the definition of 3-minimality that every anti-edge of $G[X]$ is of the form (x_i, x_{i+3}) for some index i. □

Claim 48 *Every edge e, where $e \nsubseteq X$, the edge e is contained in at least five triangles.*

Proof: Assume $e = uv$ is such an edge but that the endpoints of e do not have five common neighbors. Contract the edge e. If the pair $(G/e, X)$ is $(5,4)$-massed, then by minimality, $\mathcal{L}$ is feasible in G/e. The paths solving $\mathcal{L}$ extend to paths in G, contradicting the fact that $\mathcal{L}$ is not feasible in G. It follows that $(G/e, X)$ fails to satisfy $(M1^*)$ or $(M2^*)$. We claim it fails the latter. To prove this claim, suppose for a contradiction that $(G/e, X)$ satisfies $(M2^*)$; then it does not satisfy $(M1^*)$. Thus $\rho_G(V(G) - X) - \rho_{G/e}(V(G/e) - X) \geq 6$. If e does not have an end in X, the number $\rho_{G/e}(V(G/e) - X)$ decreases by the number of common neighbors of u and v plus one. By our assumptions on e, then, either u or v must be a vertex of X. In this case, $\rho_{G/e}(V(G/e) - X)$ decreases by the number of triangles containing e plus the number of neighbors of v in $X - \{u\}$ not adjacent to u. By Claim 47, the vertex u has at most one non-neighbor in X. It follows that $\rho_{G/e}(V(G/e) - X) \geq 5|V(G/e) - X| + 3$ and if equality holds, there exists an index i such that x_i and x_{i+3} in X are adjacent in G/e. Since the pair $(G/e, X)$ satisfies $(M2^*)$, either $(G/e, X)$ is $(5,4)$-massed, or $(G/e, X)$ is $(5,3)$-massed and x_i is adjacent to x_{i+3}. The linkage problem $\mathcal{L}$ is feasible in G/e by minimality in the first case; $\mathcal{L}$ is feasible by Lemma 5.2.2 (*iii*) in the second case. Either is a contradiction. This proves the claim, and we conclude that $(G/e, X)$ fails to satisfy $(M2^*)$.

Then G/e has a separation (A^*, B^*) of order at most five with $\rho(B^* - A^*) \geq 5|B^* - A^*| + 1$. We will use the separation (A^*, B^*) to construct a star decomposition of (G, X). Note that (A^*, B^*) is a rigid separation of $(G/e, X)$ by Lemma 5.2.2 (*i*). This separation induces a separation (A, B) in G in the following manner. Let $v_e \in V(G/e)$ be the vertex corresponding to the contracted edge, and then $A = (A^* \cup \{u, v\}) - \{v_e\}$ if $v_e \in A^*$ and $A = A^*$ otherwise. Similarly define B. First consider the case when $e \not\subseteq A \cap B$. If the edge $e \subseteq A$, then (A, B) is a separation in G violating $(M2^*)$. Now assume $e \subseteq B$. Then (A, B) is a rigid separation of (G, X), since any paths linking $A^* \cap B^*$ in G/e also exist in G. This is a contradiction to Lemma 5.4.1. We conclude that $e \subseteq A \cap B$. Note that in G, $\rho(B - A) \geq 5|B - A| + 1$. Consequently $|A \cap B| = 6$. By Lemma 5.2.2, we know that (A, B) is a 2-linked separation of (G, X) with $e \subseteq A \cap B$.

A 2-linked separation (A, B) of (G, X) of order six with $e \subseteq A \cap B$ is *maximal* if there does not exist a separation (A', B') of (G, X) of order six with $e \subseteq A' \cap B'$ and $X \subseteq A' \subsetneq A$. Since (G, X) has at least one 2-linked separation with e contained in the intersection, it must have at least one maximal 2-linked separation of order six. Let $(A, B_1, \ldots, B_k)$ be a star decomposition of (G, X), where each separation determined by the star decomposition is maximal and $e \subseteq A_i \cap B_i$ for all i. Assume we have chosen the decomposition such that k is maximum. Let $A^*, B_1^*, \ldots, B_k^*$ be the sets of vertices induced in G/e by $(A, B_1, \ldots, B_k)$, and again let $S_i^* = B_i^* \cap A^*$. We know that $\rho(B_i - S_i) \leq 5|B_i - S_i| + 3$, lest by minimality we find a rigid separation of order six. Thus in G/e, $\rho(B_i^* - S_i^*) \leq 5|B_i^* - S_i^*| + 3$. By Lemma 5.4.5, we see that in G/e that there are a total of $3k$ anti-edges contained in $\bigcup S_i^*$. Moreover, each $(\bigcup_{j \neq i} B_j^* \cup A^*, B_i^*)$ is a rigid separation of G/e.

Consider G^* defined by taking G/e and deleting all vertices in $B_i^* - S_i^*$ for every i and adding edges to all non-adjacent pairs in any S_i^*. First observe that any linkage solving $\mathcal{L}$ in G^* would extend to a linkage in G solving $\mathcal{L}$. That is because if we picked such a linkage to minimize the number of vertices used, each path would use at most one edge in any S_i^* since $G^*[S_i^*]$ is complete. Moreover, since $|S_i^*| \leq 5$, at most two paths in our solution use edges contained in S_i^*. Then looking at the linkage solving $\mathcal{L}$ in G, we are missing at most two edges in S_i for any index i. Because the determined separations of a star decomposition are 2-linked, we can extend the solution of $\mathcal{L}$ in G^* to a solution in G, contradicting the definition of 3-minimality.

We now prove that the pair (G^*, X) satisfies $(M2^*)$. Assume we have a separation (C, D) in (G^*, X) violating $(M2^*)$. Pick such a separation to minimize $|C|$. If $v_e \in C - D$, then every $S_i^* \subseteq C$, and as a consequence, $((C - \{v_e\}) \cup \{u, v\} \cup (\bigcup_i B_i), D)$ is a separation in G violating $(M2^*)$. If $v_e \subseteq D - C$, then $(C, (D - \{v_e\}) \cup \{u, v\} \cup (\bigcup_i B_i))$ is a rigid separation in G because disjoint paths linking $C \cap D$ in G^* extend to disjoint paths in G as in the previous paragraph. Thus we may assume $v_e \in C \cap D$. Then no S_i^* is a subset of D, lest we violate the maximality of the separation $(A \cup (\bigcup_{j \neq i} B_j), B_i)$ or Lemma 5.4.4. Also, we know that $|C \cap D| = 5$, lest (G, X) have a separation violating $(M2^*)$. It follows that $((C - \{v_e\}) \cup \{u, v\}, B_1, \ldots, B_k, (D - \{v_e\}) \cup \{u, v\})$ is a star decomposition of G violating our choice to make k maximum. This completes the proof that (G^*, X) satisfies $(M2^*)$.

We now count $\rho_{G^*}(V(G^*) - X)$ and show that $\mathcal{L}$ must be feasible in G^*, contradicting our earlier observation that a linkage solving $\mathcal{L}$ in G^* extends to a linkage solving $\mathcal{L}$ in G. In our initial observations for this claim, we saw that $\rho_{G/e}(V(G/e) - X) \geq 5|V(G/e) - X| + 3$ with equality holding if and only if there exists an index i such that x_i is adjacent to x_{i+3} in G/e. When we construct G^* and we delete the vertices of $B_i^* - A^*$, we lose at most $5|B_i^* - A^*| + 3$ edges for $i = 1, \ldots, k$. By Lemma 5.4.5, $|E(G^*)| \geq |E(G/e[A^*])| + 3k$, which implies that $\rho_{G^*}(V(G^*) - X) - \rho_{G/e}(A^* - X)$ is at least $3k$ minus the number of edges added to G^* that have both ends in X. We conclude that $\rho_{G^*}(V(G^* - X) \geq 5|V(G^*) - X| + 4 - t$ where t is the number of indices i such that x_i is adjacent x_{i+3} in G^*. By the 3-minimality of $(G, X, \mathcal{L})$ if $t = 0$, or by Lemma 5.2.2 if $t \geq 1$, it follows that $\mathcal{L}$ is feasible in G^*, a contradiction. This completes the proof of the claim. $\square$

Claim 49 $\rho(V(G) - X) = 5|V(G) - X| + 4$.

Proof: Consider an edge $e = uv$ such that $e \not\subseteq X$. If $(G - e, X)$ is $(5, 4)$-massed, then by the definition of 3-minimality, there exist disjoint paths in $G - e$ solving the linkage problem $\mathcal{L}$. Those paths would exist in G as well, a contradiction. We conclude that $G - e$ violates $(M1^*)$ or $(M2^*)$.

Let (A, B) be a separation of $(G - e, X)$ violating $(M2^*)$. Then without loss of generality, we may assume $u \in A - B$ and $v \in B - A$, lest (G, X) have a separation violating $(M2^*)$. By Claim 48, we know u and v have at least five common neighbors. These neighbors must be in A because $u \in A - B$, and these neighbors must also be in B because $v \in B - A$. It follows that u and v are adjacent to every vertex of $A \cap B$, and $|A \cap B| = 5$. By considering the separation $(A, B \cup \{u\})$ in G, we see that $\rho((B \cup \{u\})) - A) \geq 5|(B \cup \{u\}) - A| + 2$, so $(G[B \cup \{u\}], (A \cap B) \cup \{u\})$ is 2-linked by Lemma 5.2.2 (iii). In fact, this is actually a rigid separation because u is adjacent every other vertex in $A \cap B$, so given any linkage problem on $(A \cap B) \cup \{u\}$, we can link u to its paired vertex with an edge and link the remaining two pairs of vertices with paths in $G[B]$. This contradicts Lemma 5.4.1. We conclude that $(G - e, X)$ violates $(M1^*)$, implying the claim. $\square$

Claim 50 *There exists a vertex $v \in V(G) - X$ such that $6 \leq \deg(v) \leq 9$, and in fact if no such vertex of degree at most seven exists, then there exist at least two vertices in $V(G) - X$ with degree either eight or nine.*

Proof: The previous claim states that $\rho(V(G) - X) = 5|V(G) - X| + 4$. Observe that every vertex in X must have at least two neighbors in $V(G) - X$. If $x_i \in X$ had no neighbors in $V(G) - X$, then $(X, V(G) - \{x_i\})$ is a separation of order five violating $(M2^*)$. If x_i had only one neighbor in $V(G) - X$, say the vertex y, then the edge x_iy must be in five triangles. But x_i has no other neighbor in $V(G) - X$, so x_i and y must have five common neighbors in X, and consequently, x_i is adjacent to x_{i+3}, contrary to Claim 47. Hence, every vertex $x_i \in X$ has at least two neighbors in $V(G) - X$.

Define $f(x)$ to be the number of neighbors that $x \in X$ has in $V(G) - X$. Then

$$2\rho(V(G) - X) = \sum_{v \in V(G) - X} \deg(v) + \sum_{x \in X} f(x).$$

Suppose that every vertex of $V(G) - X$ has degree in G at least eight, and let k be the number of vertices of $V(G) - X$ of degree at most nine. Then

$$\sum_{v \in V(G) - X} \deg(v) + \sum_{x \in X} f(x) \geq 10(|V(G) - X| - k) + 8k + 2|X|.$$

Claim 49 implies that the left-hand side is equal to $2(5|V(G) - X| + 4)$, and hence $k \geq 2$, as desired. $\square$

Now we will see that either the linkage problem $\mathcal{L}$ is feasible in G contradicting the fact that $(G, X, \mathcal{L})$ is a 3-minimal pair, or we find a separation violating Lemma 5.4.1.

Claim 51 *There do not exist two vertices in $V(G) - X$ each adjacent to every vertex of X.*

Proof: Assume the claim is false, and let u and v be two such vertices. Then consider connected components $A_1, \ldots, A_t$ of $G - (X \cup \{u, v\})$. Then $\rho_G(V(G) - X) = \rho_{G[X \cup \{v,u\}]}(\{v, u\}) + \sum_{i=1}^{k} \rho(A_i)$. Since $\rho_{G[X \cup \{v,u\}]}(\{v, u\}) \leq 13 = 5(2) + 3$, we see that

$\rho_G(A_i) \geq 5|A_i|+1$ for some index i. Then A_i must have at least six neighbors in $X \cup \{v, u\}$, implying that A_i must have four neighbors among X. Thus there exists an index j such that A_i has a neighbor of x_j and x_{j+3}. Then the linkage problem $\mathcal{L}$ is feasible since we can link one pair with A_i and the other two pairs with u and v, a contradiction. □

Now we examine the neighborhood of a vertex of small degree in $V(G) - X$. Let $v \in V(G) - X$ be such a vertex of degree equal to the minimum of 6 or 7, if possible, and otherwise, pick v to be a vertex of degree at most 9, and if possible, pick it such that it is not adjacent every vertex of X. As we saw above, such a vertex exists. Let N be the subgraph induced on $N(v) \cup \{v\}$.

Claim 52 *There exist six disjoint paths linking X to $N(v)$.*

Proof: Assume the paths do not exist. Then there exists a separation (A, B) of order at most five with $X \subseteq A$ and $N(v) \subseteq B$. Pick such a separation of minimal order. Then there exist a linkage $\mathcal{Q}$ with $|A \cap B|$ components from $A \cap B$ to $N(v)$. We may assume that no component of $\mathcal{Q}$ uses the vertex v. Let X' be the termini of the components of $\mathcal{Q}$ in $N(v)$. By Lemma 5.4.2 and Claim 48, the pair (N, X') is linked. Consequently, the separation (A, B) is rigid contrary to Lemma 5.4.1. This proves the claim. □

Let $\mathcal{P}$ be a linkage from X to $N(v)$. Label the components of $\mathcal{P}$ by $P_1, \ldots, P_6$ and label the termini of $\mathcal{P}$ such that the endpoints of P_i are x_i and x_i'. Let $X' := \{x_1', x_2', \ldots, x_6'\}$ and let $\mathcal{L}'$ be the linkage problem on X' induced by $\mathcal{L}$ and $\mathcal{P}$.

If $(G[N(v)], X', \mathcal{L}')$ were quasi-firm, then $\mathcal{L}$ would be feasible, a contradiction. Thus the conclusions of Lemma 5.4.3 hold for the triple $(G[N(v)], X', \mathcal{L}')$. If there exists a path from $V(\mathcal{P}) - X'$ to $N(v) - X'$, then we can reroute some path P_i to arrive in $N(v)$ in some vertex y not contained in X'. As a result, we have a linkage from X to $N(v)$ with the set of termini being $(X' - \{x_i\}) \cup \{y\}$ such that the linkage problem $\mathcal{L}'$ induced by $\mathcal{L}$ is $\{\{y, x_{i+3}\}, \{x_{i+1}, x_{i+4}\}, \{x_{i+2}, x_{i+5}\}\}$ with all subscript addition mod six. Then by Lemma 5.4.3, $(G[N(v)], (X' - \{x_i\}) \cup \{y\}, \mathcal{L}')$ is quasi-firm, implying that $\mathcal{L}$ is feasible in G, a contradiction.

We conclude that there exists a separation (A, B) with $X \subseteq A$, $V(N) \subseteq B$, and $A \cap B = X'$. Assume that the separation (A, B) is non-trivial. Then we see that $\rho(B - A) \leq 5|B - A| + 3$, lest (A, B) be a rigid separation by the minimality of $(G, X, \mathcal{L})$. Consequently, $\rho_{G[A]}(A - X) \geq 5|A - X| + 1$. We apply Lemma 5.3.3 to the subgraph $G[A]$ and the linkage $\mathcal{P}$ from X to $A \cap B$. If (C1) holds, then we can link the remaining two pairs of vertices in $\mathcal{L}'$ by property 1. in Lemma 5.4.3 and using the vertex v adjacent all of X'. If (C2) holds in the application of Lemma 5.3.3, then there exists a linkage $\mathcal{P}'$ from X to $A \cap B$ inducing a distinct linkage problem on X'. But then by property 2. Lemma 5.4.3, this new linkage problem is feasible in $G[B]$. Either case gives a contradiction to the fact that $\mathcal{L}$ is not feasible in G.

If the separation (A, B) in the previous paragraph were in fact trivial, then the vertex v is adjacent to every vertex of X. If $|N(v)| = 6$, then $G[X]$ is a complete subgraph, a contradiction. If $|N(v)| = 7$, then since there does not exist an index i with x_i adjacent to x_{i+3}, Claim 48 implies that $N(v)$ consists of X and a single vertex adjacent to all of X. This contradicts Claim 51. Thus we see $\deg(v) \geq 8$. But then there is at least one more such vertex u of degree at most nine by Claim 50. By the choice of v, the vertex u is also adjacent to every vertex of X, again contradicting Claim 51. This final contradiction completes the proof of Theorem 5.1.1 demonstrating that no 3-minimal triple exists. $\square$

5.5 Proof of Lemma 5.4.5

Given the star decompositions in the statement, let $S_i := A \cap B_i$ and $S_i^* = B_i^* \cap A^*$. The proof of the lemma will follow from two main arguments. First, since every B_i determines a 2-linked separation, we will see that, every S_i must contain several anti-edges. In fact, we will see that even upon contracting the edge e, S_i will contain three anti-edges. We will show that these anti-edges can be chosen to be pairwise distinct for different values of the index i.

Claim 53 *For all $i = 1, 2, \ldots, k$ we have $\rho(V_i - S_i) \leq 5|V_i - S_i| + 3$.*

Proof: Otherwise the separation $(\bigcup_{j\neq i} B_j \cup A, B_i)$ is rigid by the 3-minimality of $(G, X, \mathcal{L})$, contrary to Lemma 5.4.1. □

Claim 54 *For every value of $i \in \{1, \ldots, k\}$, $G[S_i]$ has two distinct perfect anti-matchings. For any anti-edge (x, y) of either of the two anti-matchings, there exists a linkage $\mathcal{P}$ from X to S_i with six components and an index j such that if we label P_k the component of $\mathcal{P}$ containing x_k, then the termini of P_j and P_{j+3} are x and y.*

Proof: By Lemma 5.4.4, there exists six disjoint paths from X to S_i. Given a linkage from X to S_i, the linkage problem $\mathcal{L}$ induces a linkage problem $\mathcal{L}'$ on S_i. Each pair of $\mathcal{L}'$ must be an anti-edge, lest we link the two remaining pairs in $G[B_i]$ and contradict the fact that $\mathcal{L}$ is not feasible.

Given that $\rho(G - X) \geq 5|G - S| + 4$, Claim 53 implies that $\rho\left(\left(\bigcup_{j\neq i} B_j \cup A\right) - X\right) \geq 5|(\bigcup_{j\neq i} B_j \cup A) - X| + 1$. By Lemma 5.3.3, one of (C1) or (C2) must hold. If (C1) holds, we can link one pair of $\mathcal{L}$ in $G[\bigcup_{j\neq i} B_j \cup A]$ and link the two remaining pairs in $G[B_i]$, making $\mathcal{L}$ feasible, a contradiction. Thus (C2) holds. Through this new linkage from X to S_i, $\mathcal{L}$ induces a linkage problem $\mathcal{L}''$ distinct from $\mathcal{L}'$. As in the previous paragraph, the pairs in $\mathcal{L}''$ form an perfect anti-matching. Thus $G[S_i]$ contains two distinct perfect anti-matchings, and the claim follows. □

We now examine in more depth the properties of $G[S_i]$.

Claim 55 *Fix i and let $x, y \in S_i$. Then $G[S_i]$ has at least two anti-edges not incident with x or y, and moreover, if there are exactly two such edges, then they have a common end point.*

Proof: By Claim 54, the complement of $G[S_i]$ has a subgraph isomorphic to either C_6 or $C_4 \cup K_2$. $G[S_i]$ must contain at least one anti-edge not incident with x or y. We may assume that $G[S_i]$ has at least three anti-edges incident with x or y, for otherwise the conclusion holds.

Assume for every vertex v in $G[S_i]$, there is at most one anti-edge incident with v that does not have x or y as the other endpoint. Let the graph G' obtained from G by deleting the vertices $B_i - S_i$ and for every $z \in S_i - \{x, y\}$, adding the edge xz and yz if it does not already exist and adding the edge xy if it does not already exist. By Claim 53 and the fact that S_i had at least three anti-edges incident with x and y, we know that $\rho_{G'}(V(G') - X) \geq 5|V(G') - X| + 4$. Now if (G', X) had no separation violating $(M2^*)$, then by the 3-minimality of $(G, X, \mathcal{L})$, the pair (G', X) is linked. Let P_1, P_2, and P_3 be paths solving the linkage problem $\mathcal{L}$. At most two of these paths use the vertices x and y, so we may assume P_3 uses only edges present in G. If either the paths P_1 and P_2 contain vertices of S_i, then they have first and last vertices in S_i. Label the vertices w_1, w_2 for P_1, and z_1, z_2 for P_2. In $G[B]$, there exist paths Q_1 and Q_2 with ends w_1, w_2 and z_1, z_2 respectively, with the property that $Q_i \cap V(G') \subseteq S_i$. Then $x_1 P_1 w_1 Q_1 w_2 P_1 x_4$, $x_2 P_2 z_1 Q_2 z_2 P_2 x_5$ and $x_3 P_3 x_6$ is a linkage in G solving $\mathcal{L}$. This contradiction implies that (G', X) has a separation violating $(M2^*)$.

Let (A', B') be a separation in (G', X) violating $(M2^*)$. Then if $S_i \subseteq A'$, then $(A' \cup B_i, B')$ is a separation of (G, X) violating $(M2^*)$. If $S_i \subseteq B'$, then we would have a separation of order at most five separating X from S_i, contradicting Lemma 5.4.4. It follows that there exist some vertices w_1 and w_2 in S_i such that $w_1 \in A' - B'$ and $w_2 \in B' - A'$. Then w_1 is not adjacent w_2, and by our assumptions on $G[S_i]$, we know w_1 and w_2 are each adjacent (in G') to every other vertex in S_i. If the other vertices of S_i are x, y, z_1, z_2, then $x, y, z_1, z_2 \in A' \cap B'$. If $A' \cap B' = \{x, y, z_1, z_2\}$, then $(A' \cup \{w_2\} \cup B_i, B')$ is a separation in G of order five separating X from B_i, a contradiction again to Lemma 5.4.4. We conclude $A' \cap B'$ contains exactly one other vertex not yet defined. Call it a. In the graph G', there exist six disjoint paths from X to $\{x, y, z_1, z_2, a, w_1\}$, lest G have a separation of order at most five separating X from S_i. Label the six paths $P_1, \ldots, P_6$ and let the ends of P_j be $x_j \in X$ and $x_j' \in \{x, y, z_1, z_2, a, w_1\}$. Note P_j may be a trivial path consisting of just one vertex, in which case x_j and x_j' are not distinct.

The linkage problem $\mathcal{L}$ induces the linkage problem $\mathcal{L}' = \{\{x_1', x_4'\}, \{x_2', x_5'\}, \{x_3', x_6'\}\}$ on $\{x, y, z_1, z_2, w_1, a\}$. We now show that the linkage problem $\mathcal{L}'$ is feasible in $G[B' \cup B_i]$, contradicting the fact that $\mathcal{L}$ is not feasible in G. Some pair of vertices in the linkage problem $\mathcal{L}'$ lies in $\{x, y, z_1, z_2\}$. Without loss of generality, say $x_1', x_4' \in \{x, y, z_1, z_2\}$. Then in G there exist paths Q_1, Q_2 with all internal vertices in B_i with the ends of Q_1 being w_1 and w_2 and the ends of Q_2 being x_1' and x_4'. Now there are two cases to consider.

Case 1: w_2 is adjacent to every vertex in $A' \cap B'$. In this case $\rho_G((B' - A') - \{w_2\}) \geq 5|(B' - A') - \{w_2\}| + 1$, and as a consequence, G restricted to $(B' - A') - \{w_2\}$ has some connected component C with $\rho(V(C)) \geq 5|V(C)| + 1$. This implies the vertices of $(A' \cap B') \cup \{w_2\}$ all have a neighbor in the component C, lest (G, X) have a separation violating $(M2^*)$. We can link the path end w_1 with its paired vertex in $\mathcal{L}'$ via w_2 and the path Q_1, x_1' and x_4' via the path Q_2 and the remaining pair of vertices in $\mathcal{L}'$ via the connected component C. This would make $\mathcal{L}'$ feasible in $G[B_i \cup B' \cup \{w_1\}]$, a contradiction.

Case 2: w_2 has at least one non-neighbor in $A' \cap B'$. In this case, $\rho(B' - A' - \{w_2\}) \geq 5|B' - A' - \{w_2\}| + 2$. Then by Lemma 5.2.2, we know any two path problem on $\{x, y, w_2, z_1, z_2, a\}$ can be solved with disjoint paths with all internal vertices in $B' - A' - \{w_2\}$. We can link x_1' and x_4' in B_i with the path Q_1. By linking w_1 to w_2 with Q_2, we can link the remaining two pairs of vertices in $\mathcal{L}'$ in $B' - A' - \{w_2\}$ to show that the linkage problem $\mathcal{L}'$ is feasible.

This completes the proof of Claim 55 that $G[S_i]$ must have at least two anti-edges not incident with e, and if it has exactly two, then they must share a common endpoint. □

Recall, the edge $e = uv$ lies in every S_i of our star decomposition.

Claim 56 *Fix i. If $G[S_i]$ has exactly two anti-edges not incident with e, then we can label the anti-edges a_1 and a_2 and label the underlying vertices $a_1 = (x, y)$, $a_2 = (y, z)$ such that*

1. *There exists a linkage $\mathcal{P}$ with six components from X to S_i and an index j such that if we label P_k the component of $\mathcal{P}$ containing x_k, then a_1 contains the two endpoints of P_j and P_{j+3} in S_i, and*

2. *the vertex z is a common non-neighbor of the ends of e.*

Proof: By Claim 54, the complement of $G[S_i]$ contains a subgraph A isomorphic to C_6 or $C_4 \cup K_2$. If $G[S_i]$ has exactly two anti-edges not incident e, then there are three possible cases, up to isomorphism, for how the edge $e = uv$ intersects with A.

First, assume that A is isomorphic to $C_4 \cup K_2$. Let the vertices of S_i be labeled c_1, c_2, c_3, c_4 corresponding to the C_4 in order and k_1, k_2 corresponding to the K_2.

Case 1: $u = c_1, v = k_1$. In this case, one of the following pairs must be an anti-edge: $(k_1, c_2), (k_1, c_4), (k_2, c_2), (k_2, c_4)$. Otherwise, when we consider the vertices c_3 and c_1, there would not exist at least two incident anti-edges with neither c_3 nor c_1 as an endpoint, contrary to Claim 55. Since $G[S_i]$ has exactly two anti-edges not incident with e, we may assume the pair (k_1, c_2) is an anti-edge. Then let $a_1 = (c_3, c_4)$ and $a_2 = (c_2, c_3)$. By Claim 54, there exists a linkage $\mathcal{P}$ from X to S_i such that if we label P_i the component of $\mathcal{P}$ containing x_i, then there exists an index j such that a_1 contains the two ends of P_j and P_{j+3} in S_i. As we have already seen that c_2 is a common non-neighbor of the ends of e, we have proven the claim.

Case 2: $u = c_2$, $v = c_4$. Again by Claim 55, there must be some other anti-edge not incident with e. Without loss of generality, it's the pair (k_1, c_1). Then if we let $a_1 = (k_1, k_2), a_2 = (k_1, c_1)$ we have the desired labeling of the anti-edges where now c_1 is the common non-neighbor of the ends of e. Again, by Claim 54, there exists a linkage from X to S_i where for some pair of the linkage problem $\mathcal{L}$, the corresponding paths terminate on the anti-edge a_1, as desired.

This completes the analysis when A is isomorphic to $C_4 \cup K_2$. Now we assume A is isomorphic to C_6. Let the vertices of S_i be labeled $c_1, c_2, \ldots, c_6$ in the order determined by A. There is only one possible choice, up to isomorphism, for the edge e such that there are only two anti-edges not incident with e.

Case 3: $u = c_1$, $v = c_3$. By applying Claim 55 to the vertices c_2 and c_5, one of the following pairs must be an anti-edge: $(c_3, c_6), (c_6, c_4), (c_4, c_1)$. And since by assumption $G[S_i]$ has exactly two anti-edges not incident with e, we may assume that either (c_3, c_6)

or (c_1, c_4) is an anti-edge. The two cases are symmetric, so we may assume (c_3, c_6) is an anti-edge, and then if we let $a_1 = (c_4, c_5)$ and $a_2 = (c_5, c_6)$, we have the desired properties. Again the existence of the required linkage follows from Claim 54.

This completes the proof of Claim 56. $\square$

Now we have a solid grip on what the subgraph $G[S_i]$ can look like; $G[S_i]$ must have at least two anti-edges not incident with e. Moreover, if there are exactly two such anti-edges, there is a common non-neighbor of the ends of e. Then clearly, upon contracting the edge e, $G/e[S_i^*]$ has at least three anti-edges. We will first show that if $|S_i \cap S_j| \geq 5$ for some $j \neq i$, then these anti-edges may be chosen so that they belong to now S_l^* for $l \neq i$.

For notation, the next claims will be proven for S_1, S_2, and S_3. Since the labeling of the S_i's is arbitrary, we see that the results will hold for any distinct S_i, S_j, and S_k.

Claim 57 *Given S_1 and S_2 above, $|S_1 \cap S_2| \leq 4$. If $|S_1 \cap S_2| = 4$, then there exists a linkage $\mathcal{P}$ with six components from X to $S_1 \cup S_2$ where if we label P_i the component of $\mathcal{P}$ containing x_i, the following hold.*

1. *There exists an index i such that both P_i and P_{i+3} have their termini in $S_1 \cap S_2$.*

2. *No other component of $\mathcal{P}$ has its terminus in $S_1 \cap S_2$.*

3. *For indices $j \in \{1, \dots, 6\}, j \neq i, i+3$, if the terminus of P_j lies in $S_1 - S_2$, then the terminus of P_{j+3} lies in $S_2 - S_1$, with all index addition mod 6.*

4. *At least one vertex of u and v is not the terminus of a component of $\mathcal{P}$.*

5. *$V(\mathcal{P}) \cap (S_1 \cup S_2)$ consists of the six termini of the components of $\mathcal{P}$.*

Proof: Assume $|S_1 \cap S_2| \geq 4$. Clearly, $S_1 \neq S_2$, lest $(\bigcup_{j \neq 1,2} B_j \cup A, B_1 \cup B_2)$ form a rigid separation. It follows that $|S_1 \cap S_2|$ is at most five. There exists a linkage $\mathcal{P}$ from X to S_1. Let the component of $\mathcal{P}$ containing x_i be labeled P_i. Let the terminus of P_i in S_1 be labeled x_i'. The linkage problem $\mathcal{L}$ induces a linkage problem $\mathcal{L}'$ on S_1. If we can find paths solving $\mathcal{L}'$ that do not use any vertex of $\mathcal{P}$ except for their ends, then clearly we would contradict the fact that $\mathcal{L}$ is not feasible in G. Note that since $|S_1 \cap S_2| \geq 4$, there exists an index i such that P_i and P_{i+3} have their termini in $S_1 \cap S_2$. Without loss of generality, assume that $x_1', x_4' \in S_1 \cap S_2$.

If no path of $\mathcal{P}$ uses vertices of $B_2 - S_2$, then clearly we can link x_1' and x_4' with a path in $B_2 - S_2$ and link the two remaining pairs in $B_1 - S_1$. If at most one path, say P_l, uses vertices of $B_2 - S_2$, let x_l'' be the first vertex of P_l in S_2. Then there are two cases. If $l = 1$ or 4, say $l = 1$, instead of following P_l to x_1', instead find a path in B_2 from x_1'' to x_4'. Link the remaining two pairs of vertices in $\mathcal{L}'$ in B_1. Now assume $l \neq 1$ or 4. Let y be the final vertex of P_l in S_2. Then find paths in B_2 solving the linkage problem $\{\{x_l'', y\}, \{x_1', x_4'\}\}$. Link the remaining two pairs of vertices in $\mathcal{L}'$ in B_1. Either case gives rise to a contradiction. We conclude that $|S_1 \cap S_2| = 4$, and that exactly two paths, say P_l and P_k, use vertices of $B_2 - S_2$. Again, let x_k'' and x_l'' be the first vertices in $S_2 - S_1$ of P_k and P_l, respectively. Then if $k = j + 3$, or $j = k + 3$, then we can link x_k and x_j with a path in B_2 and link the remaining two pairs of terminals in $\mathcal{L}'$ in B_1, a contradiction. Without loss of generality, we assume $k = 1$ and $l = 2$. The paths P_1 and P_2 each use a vertex of $S_2 - S_1$, so it follows that x_1' and x_2' lie in $S_1 \cap S_2$.

Let $\mathcal{P}'$ be the linkage $(\mathcal{P} - \{P_1, P_2\}) \cup \{x_1 P_1 x_1'', x_2 P_2 x_2''\}$. Again, let P_i' be the component of $\mathcal{P}'$ containing x_i. The linkage $\mathcal{P}'$ satisfies the conclusions of the claim. We have proven that P_1' and P_4' have their termini in $S_1 \cap S_2$, and that no other path in $\mathcal{P}'$ has it's terminus in $S_1 \cap S_2$. Thus 2. and 3. follow. Finally, our original linkage $\mathcal{P}$ was such that x_1' and x_4' were not adjacent. When we consider the edge $e = uv \subseteq S_1 \cap S_2$, then at least one vertex of u and v must not be a terminus of a path in $\mathcal{P}'$, proving 4. Condition 5. holds by construction. □

We now want to show that if the two S_1 and S_2 intersect in four vertices, then the other S_i's can only intersect S_1 and S_2 in a very limited manner. Towards this, we prove the following claim.

Claim 58 *If* $|S_1 \cap S_2| = 4$ *and if* S_3 *satisfies* $|S_3 \cap (S_1 \cup S_2)| \geq 3$*, then* $|S_3 \cap (S_1 \cup S_2)| = 3$ *and* $S_3 \cap (S_1 \cup S_2) \subseteq S_1 \cap S_2$.

Proof: Assume $|S_1 \cap S_2| = 4$ and $|S_3 \cap (S_1 \cup S_2)| \geq 3$. We know from Claim 57 that we have a linkage $\mathcal{P}$ with components $P_1, \dots P_6$ from X to $S_1 \cup S_2$ with the path termini as described in the statement of Claim 57. Let x'_i be the terminus of P_i. Without loss of generality, assume x'_3 and x'_6 lie in $S_1 \cap S_2$, x'_1 and x'_2 lie in $S_1 - S_2$ and x'_4 and x'_5 lie in $S_2 - S_1$. Let the vertices w_1 and w_2 be the vertices of $S_1 \cap S_2$ that are not the termini of any path in $\mathcal{P}$. Notice that at least one of $w_1$and w_2 is an endpoint of the edge e, and so without loss of generality, we assume $w_1 \in S_3$. For notation, let $\mathcal{L}'$ be the linkage problem induced by $\mathcal{P}$ and $\mathcal{L}$ on $(S_1 \cup S_2) - \{w_1, w_2\}$.

First assume at most one path P_i uses vertices of $B_3 - S_3$. Let y_1 and y_2 be the first and last vertices of P_i in S_3. Now there are two cases both of which are easily dealt with: either $S_3 \cap (S_1 \cup S_2) \subseteq S_1 \cap S_2$ or $S_3 \cap ((S_1 - S_2) \cup (S_2 - S_1)) \neq \emptyset$.

Case 1: $S_3 \cap (S_1 \cup S_2) \subseteq S_1 \cap S_2$.

If $|S_3 \cap S_1 \cap S_2| = 3$, then the claim is proven. Thus we may assume $|S_3 \cap S_1 \cap S_2| = 4$. Consequently, $x'_3, x'_6 \in S_3$. There exist paths Q_1, Q_2 in $B_3 - S_3$, where the ends of Q_1 are y_1 and y_2 and the ends of Q_2 are x'_3 and x'_6.

Now consider the linkage $\mathcal{P}' = \mathcal{P} - \{P_i\} \cup \{x_i P_i y_1 Q_1 y_2 P_i x'_i\}$. This linkage is disjoint from the sets $B_1 - S_1$, $B_2 - S_2$, and $V(Q_2) - \{x'_3, x'_6\}$. There exist disjoint paths with all internal vertices in $B_1 - S_1$ solving the linkage problem $\{\{x'_1, w_1\}, \{x'_2, w_2\}\}$. Similarly, there exist disjoint paths with all internal vertices in $B_2 - S_2$ solving the linkage problem $\{\{x'_4, w_1\}, \{x'_5, w_2\}\}$. Thus the linkage problem $\{\{x'_1, x'_4\}, \{x'_2, x'_5\}\}$ is feasible in $G[B_1 \cup B_2]$, and we contradict the fact that $\mathcal{L}$ is not feasible in G.

Case 2: $S_3 \cap ((S_1 - S_2) \cup (S_2 - S_1)) \neq \emptyset$.

Then some vertex of $\{x'_1, x'_2, x'_4, x'_5\}$ lies in S_3. Without loss of generality, say $x'_1 \in S_3$. Using the fact that $w_1 \in S_3$, we observe that there exist disjoint paths Q_1, Q_2 with all internal vertices in $B_3 - S_3$ where ends of Q_1 are y_1 and y_2 and the ends of Q_2 are x'_1 and w_1. As above, let $\mathcal{P}'$ be the linkage defined by $\mathcal{P} - \{P_i\} \cup \{x_i P_i y_1 Q_1 y_2 P_i x'_i\}$. There exist disjoint paths R_1, R_2 with all internal vertices in $B_1 - S_1$ where the ends of R_1 are x'_3 and x'_6 and the ends of R_2 are x'_2 and w_2. There exist paths T_1, T_2 with all internal vertices in

$B_2 - S_2$ where the ends of T_1 are x'_4 and w_1 and the ends of T_2 are x'_5 and w_2 respectively. We have the linkage:

$$x'_1 Q_1 w_1 T_1 x'_4, \qquad x'_3 R_1 x'_6, \qquad x'_2 R_2 w_2 T_2 x'_5$$

solving the linkage problem $\mathcal{L}'$ avoiding any non-terminus vertex of $\mathcal{P}'$, a contradiction to the fact that $\mathcal{L}$ is not feasible in G.

The analysis of the cases above shows we may assume at least two paths $P_i, P_j \in \mathcal{P}$ use vertices of $B_3 - S_3$. Assume for the moment that P_i and P_j are the only paths using vertices of $B_3 - S_3$. We may assume that the two paths are not P_3 and P_6, otherwise we could simply link the first vertices of P_3 and P_6 in $B_3 - S_3$ and link the remaining pairs of terminals with paths in $B_2 - S_2$ and $B_1 - S_1$ meeting at the vertices w_1, w_2. Thus we may assume one of the paths P_1, P_2, P_4, P_5 intersects $B_3 - S_3$. Without loss of generality, say P_1. Let x''_1 be P_1's first vertex in S_3. Let P_i be the other path intersecting $B_3 - S_3$, and let y_1 and y_2 be the first and last vertices of P_i in S_3. There exist paths in Q_1, Q_2 in with all internal vertices in $B_3 - S_3$ where the ends of Q_1 are y_1 and y_2 and the ends of Q_2 are x''_1 and w_1. Let $\mathcal{P}'$ be the linkage defined by $P'_i = x_i P_i y_1 Q_1 y_2 P_i x'_i$ and $P'_k = P_k$ for $k \neq i$. There exist paths R_1 and R_2 with all internal vertices in $B_2 - S_2$ where the ends of R_1 are x'_3 and x'_6 and the ends of R_2 are x'_2 and w_2. There exist paths T_1, T_2 with all internal vertices in $B_1 - S_1$ where the endpoints of T_1 are w_1 and x'_4 and the endpoints of T_2 are x'_5 and w_2. We get the following linkage:

$$x_1 P'_1 x''_1 Q_2 w_1 T_1 x'_4 P'_4 x_4 \qquad x_2 P'_2 x'_2 R_2 w_2 T_2 x'_5 P'_5 x_5 \qquad x_3 P'_3 x'_3 R_1 x'_6 P'_6 x_6$$

that contradicts the fact that $\mathcal{L}'$ is not feasible.

Finally three or more components of $\mathcal{P}$ cannot use vertices of $B_3 - S_3$, because each such path must use at least two vertices of S_3 and yet $w_1 \in S_3 - V(\mathcal{P})$. This proves the claim. $\square$

We now will prove that if S_i and S_j intersect in four vertices, then S_i^* (and similarly in

S_j^*), has three anti-edges not contained in any other S_k^*.

Claim 59 *For all distinct indices i, j, if S_i and S_j are such that $|S_i \cap S_j| = 4$, then each $G/e[S_i^*]$ and $G/e[S_j^*]$ have three anti-edges that they do not share with each other or any other $G/e[S_l^*]$.*

Proof: For notation, assume that S_1 and S_2 are as in the statement of the claim and intersect in four vertices. Let $S_1 \cap S_2 = \{u, v, y_1, y_2\}$ where u and v are the endpoints of the edge e specified in the statement of the lemma. Let $\mathcal{P}$ be a linkage as in Claim 57 and let the components of $\mathcal{P}$ be labeled $P_1, \dots, P_6$ such that P_i contains x_i. Let x_i' be the terminus of P_i in $S_1 \cup S_2$. Without loss of generality, assume $x_3', x_6' \in S_1 \cap S_2 = \{u, v, y_1, y_2\}$, and that x_1' and x_2' lie in $S_1 - S_2$. Let $\mathcal{L}'$ be the linkage problem on the appropriate subset of $S_1 \cup S_2$ induced by $\mathcal{L}$ and $\mathcal{P}$. Up to symmetry, there are two cases to consider: $\{x_3', x_6'\} = \{y_1, y_2\}$ or $\{y_1, v\}$.

Case 1: $\{x_3', x_6'\} = \{y_1, y_2\}$

By Claim 55, we know that each of S_1 and S_2 has some anti-edge not incident with e which is not contained in $S_1 \cap S_2$. Call them a_1 and a_2, respectively. Notice that by Claim 58, neither a_1 or a_2 can be contained in S_l^* for any $l \neq 1, 2$.

Consider what happens if u or v were adjacent to any vertex x_1', x_2', x_4', x_5'. Say v is adjacent to x_1'. Then there exist paths Q_1, Q_2 with all internal vertices $B_1 - S_1$ where the ends of Q_1 are x_2' and u and the ends of Q_2 are x_3' and x_6'. Also, there exist paths R_1, R_2 with all internal vertices in $B_2 - S_2$ such that the ends of R_1 are x_4' and v and the ends of R_2 are x_5' and u. We get the linkage

$$x_1' v R_1 x_4' \qquad x_2' Q_1 u R_2 x_5' \qquad x_3' Q_2 x_6'$$

proving that $\mathcal{L}'$ is solvable by paths not intersecting $V(\mathcal{P}) - \{x_1', \dots, x_6'\}$, a contradiction.

Thus we may assume that no such edge exists and then u and v have no neighbor in $S_1 - S_2$ nor in $S_2 - S_1$. If we let v_e be the vertex in G/e coming from the edge e, then in which case, $a_1, (v_e, x_1'), (v_e, x_2')$ are three anti-edges contained in $G/e[S_1^*]$ that are not contained in any other S_l^*. If they were in some S_k^*, S_k would necessarily intersect $S_2 \cup S_1$ in at least three vertices and at least one vertex of $S_1 - S_2$, contrary to what we have seen above in Claim 58. Thus both $G/e[S_1^*]$ and $G/e[S_2^*]$ contain three anti-edges they do not share with each other or any other S_k^*.

Case 2: $\{x_3', x_6'\} = \{y_1, v\}$.

Again, as in the previous case, we may assume that neither y_2 nor u has any neighbor in $S_2 - S_1$ nor in $S_1 - S_2$. Thus if we consider $G[S_1]$, by Claim 55 applied to the vertices y_2 and u, there must exist at least two anti-edges not incident with y_2 or u. We conclude that there exists an anti-edge between either y_1 or v and one of either x_1' and x_2'. Since $x_1'$and x_2' are symmetric here, there are two distinct cases: x_1' is not adjacent v and x_1' is not adjacent to y_1. If x_1' is not adjacent to v, then the anti-edges $(v_e, x_1'), (x_1', y_2), (x_2', y_2)$ are contained in S_1^*. If x_1' is not adjacent y_1, then S_1^* contains the anti-edges $(x_1', y_1), (x_1', y_2)$, and (x_2', y_2). In either case, $G[S_1^*]$ contains three anti-edges that cannot lie in any other S_l^* by Claim 58. This proves the claim. $\square$

Our objective is to show that each S_i^* has at least three anti-edges not shared by S_l^* for $l \neq i$. We have just shown that if $|S_i \cap S_j| \geq 4$, for some $j \neq i$, then the three anti-edges may be chosen so that they belong to no other S_l^* for $l \neq i$. To complete the proof let $i \in \{1, 2, \ldots, k\}$ be such that $|S_i \cap S_j| \leq 3$ for all $j \neq i$. If S_i has at least three anti-edges not incident with u or v, then those are clearly as required. Thus we may assume that S_i has at most two such anti-edges, and hence Claim 55 implies that it has exactly two and they share an end. Let those anti-edges be labeled $a_1^i = (x^i, y^i)$ and $a_2^i = (y^i, z^i)$, consistent with the notation in Claim 55. To complete the proof of Lemma 5.4.5, it suffices to show the following claim.

Claim 60 *If S_i and S_j are as above, then $z^i \neq z^j$.*

We may assume that $i = 1$ and $j = 2$. Suppose for a contradiction that $z^1 = z^2$. We show the linkage problem $\mathcal{L}$ is feasible, a contradiction. The intersection $S_1 \cap S_2 = \{u, v, z^1\}$, where u and v are the ends of e. We know that $a_1^1 \cap (S_1 \cap S_2) = \emptyset$.

Let $\mathcal{P}$ be the linkage in the statement of Claim 56. Let the components of $\mathcal{P}$ and the vertices of $S_1 \cup S_2$ be labeled such that the ends of $P_i \in \mathcal{P}$ are x_i and x_i'. Without loss of generality, assume that the termini of P_1 and P_4 form the anti-edge a_1^1. Let $\mathcal{L}'$ be the linkage problem induced by $\mathcal{L}$ and $\mathcal{P}$ on S_1. Three of the paths P_2, P_3, P_5, P_6 must have their ends in $S_1 \cap S_2$. Again without loss of generality, assume that $x_3', x_5', x_6' \in S_1 \cap S_2$. We will separately consider the possible number of paths that utilize vertices of $B_2 - S_2$.

Case 1: no P_i contains vertices of $B_2 - S_2$.

Then there exists a path Q ends x_3' and x_6' and all internal vertices in $B_2 - S_2$. Then the linkage problem $\{\{x_1', x_4'\}, \{x_2', x_5'\}\}$ is feasible in $G[B_1]$, implying that a solution to the linkage problem $\mathcal{L}'$ exists with no internal vertex intersecting $\mathcal{P}$, a contradiction.

Case 2: exactly one path $P_i \in \mathcal{P}$ contains vertices of $B_2 - S_2$.

Let w_1 be the vertex of P_i in S_2 closest to X on P_i, and w_2 be the vertex of P_i in S_2 closest to S_1 on P_i. There exist paths Q_1 and Q_2 with all internal vertices in $B_2 - S_2$ such that the ends of Q_1 are w_1 and w_2 and the ends of Q_2 are x_3' and x_6'. Then the linkage $\mathcal{P}' = \mathcal{P} - \{P_i\} \cup \{x_i P_i w_1 Q_1 w_2 P_i x_i'\}$ has the same endpoints as $\mathcal{P}$. We can link x_3' and x_6' avoiding all other vertices of $\mathcal{P}'$. As in the previous case, the fact that the linkage problem $\{\{x_1', x_4'\}, \{x_2', x_5'\}\}$ is feasible in $G[B_1]$ implies that $\mathcal{L}$ is feasible, a contradiction.

Case 3: exactly two paths P_i and P_j in $\mathcal{P}$ contain vertices of $B_2 - S_2$.

First, assume $i = j + 3$ or $j = i + 3$. Then we can link x_i to x_j with a path in B_2 avoiding the other paths of $\mathcal{P}$. The other two pairs of vertices in $\mathcal{L}$ can be linked in $G[B_1]$, implying that $\mathcal{L}$ is feasible.

Thus we conclude $i \neq j + 3$ and $j \neq i + 3$. Now assume $i = 3$. Let x_3'' be the vertex of S_2 closest to x_3 on P_i. Let w_1 and w_2 be the vertices of S_2 on P_j closest to x_j and x_j' on P_j, respectively. Then there exist paths Q_1 and Q_2 with all internal vertices in $B_2 - S_2$ such that the ends of Q_1 are x_3'' and x_6' and the ends of Q_2 are w_1 and w_2. Then let $P_j' = x_j P_j w_1 Q_2 w_2 P_j x_j'$ and $P_k' = P_k$ for $k \neq j$. The ends of P_k' are equal to the ends of P_k for all indices k. There exist paths R_1 and R_2 with all internal vertices in $B_1 - S_1$ where the ends of R_1 are x'_1 and x'_4 and the ends of R_2 are x'_2 and x'_5. Then we have the linkage

$$x_1 P'_1 x'_1 R_1 x'_4 P'_4 x_4, \qquad x_2 P'_2 x'_2 R_2 x'_5 P'_5 x_5, \qquad x_3 P'_3 x''_3 Q_1 x'_6 P'_6 x_6$$

solving the linkage problem $\mathcal{L}$, a contradiction.

We conclude that $i \neq 3$, and symmetrically, $i, j \neq 6$. Then at least one of i or j is equal to one or four. Without loss of generality, assume $i = 4$. Then P_i must use two vertices of $S_2 - S_1$. It follows that $j = 5$ since $x'_5 \in S_1 \cap S_2$. Let x''_5 be the unique vertex of P_5 in $S_2 - S_1$ and w_1 the vertex of P_4 in $S_2 - S_1$ closest to x_4 on P_4 and w_2 the other vertex of P_4 in $S_2 - S_1$. There exist disjoint paths Q_1 and Q_2 with all internal vertices in $B_2 - S_2$ such that the ends of Q_1 are x''_5 and w_2 and the ends of Q_2 are w_1 and x'_5. There exist disjoint paths R_1 and R_2 with all internal vertices in $B_1 - S_1$ such that the ends of R_1 are x'_3 and x'_6 and the ends of R_2 are x'_5 and x'_1. Notice by the fact that a^1_1 is the anti-edge (x'_1, x'_4) and the second anti-edge in $G[S_1]$ not incident to e must have z^1 as an endpoint, we conclude that x'_2 is adjacent to x'_1 and x'_4. The linkage

$$x_1 P_1 x'_1 R_2 x'_5 Q_2 w_1 P_4 x_4, \quad x_2 P_2 x'_2 x'_4 P_4 w_2 Q_1 x''_5 P_5 x_5, \quad x_3 P_3 x'_3 R_1 x'_6 P_6 x_6$$

contradicts the fact that $\mathcal{L}$ is not feasible.

Case 4: exactly three paths in $\mathcal{P}$ contain vertices of $B_2 - S_2$

Each of these paths must use at least two vertices in S_2. Since P_3, P_5, P_6 each must use one vertex of S_2, it follows that each of P_3, P_5, P_6 uses vertices of $B_2 - S_2$, and each one uses exactly one vertex of $S_2 - S_1$. Let x''_3, x''_5, x''_6 be the vertices of P_3, P_5, P_6 respectively in $S_2 - S_1$. Then there exists paths Q_1, Q_2 with all interior vertices in $B_2 - S_2$ where the ends of Q_1 are x''_3 and x''_6 and the ends of Q_2 are x''_5 and x'_5. There exist paths R_1 and R_2 with all internal vertices in $B_1 - S_1$ where the ends of R_1 are x'_1 and x'_4 and the ends of R_2 are x'_2 and x'_5. The linkage

$$x_1 P_1 x'_1 R_1 x'_4 P_4 x_4, \qquad x_2 P_2 x'_2 R_2 x'_5 Q_2 x''_5 P_5 x_5, \qquad x_3 P_3 x''_3 Q_1 x''_6 P_6 x_6$$

contradicts the fact that $\mathcal{L}$ is not feasible.

This completes the proof of the claim. $\square$

Now we have completed the proof of Lemma 5.4.5. We have shown that for each S_i, upon contracting the edge e, $G/e[S_i^*]$ contains at least three anti-edges not contained in any other S_j^* implying that if we sum over every such S_i, there is a total of at least $3k$ anti-edges contained in $\bigcup_i G/e[S_i^*]$, as desired. $\square$

CHAPTER 6

LOWER BOUNDS TO THEOREMS 1.4.2 AND 1.4.3

6.1 Outline

We construct a lower bound to Theorems 1.4.2 and 1.4.3 as follows. Start with a linkage $\mathcal{P}$ with $k-1$ components labeled $P_1, \ldots, P_{k-1}$. Let the ends of P_i be s_i and t_i and s_k and t_k be two additional vertices. Add as many edges as possible to the graph maintaining the property that for any linkage $\mathcal{P}'$ with components $P'_1, \ldots, P'_{k-1}$ where the ends of P'_i are s_i and t_i, then $|V(\mathcal{P}')| \geq |V(\mathcal{P})|$. In other words, it is impossible to reroute the paths $P_1, \ldots, P_{k-1}$ to shorten the sum of their lengths. Then add two new vertices s_k and t_k and all the edges from s_k and t_k to every other vertex of the graph. We will see that we can construct such a graph to be $2k$-connected.

The maximum number of edges such a graph can have is determined by the following theorem.

Theorem 6.1.1 *Let G be a graph and $P_1, \ldots, P_k$ be k disjoint paths in G. Assume G has n vertices and $V(G) = \bigcup_i V(P_i)$. Let the ends of P_i be x_i and y_i. Then if $|E(G)| \geq (2k-1)n - 3\binom{k}{2} - k + 1$, then there exist paths $P'_1, \ldots, P'_k$ where the ends of P'_i are x_i and y_i and $\sum_i |V(P_i)| > \sum_i |V(P'_i)|$.*

Such a graph will have the property that any $k-1$ of the pairs s_i and t_i can be linked with paths, but not all k pairs. The proof of Theorem 6.1.1 and the following construction first appeared in [65].

We now give an explicit construction of a $2k$ connected graph G whose vertex set consists of a linkage from $\{s_1, \ldots, s_{k-1}\}$ to $\{t_1, \ldots, t_{k-1}\}$ and two additional vertices s_k and t_k, as above. Let P_1, P_2, P_3, P_4 be four paths of length $l-1$. Let the vertices of P_i be $v_i^1, v_i^2, \ldots, v_i^l$. Furthermore, let $s_1 = v_1^1$, $s_2 = v_2^1$, $t_1 = v_3^1$, and $t_2 = v_4^1$. The vertices of G will be $\{v_i^j : i = 1, \ldots, 4, j = 1, \ldots l\} \cup \{u_1, \ldots, u_k - 1\} \cup \{s_3, t_3, \ldots, s_k, t_k\}$. The edges of G will be given by (with all subscript addition mod 4)

$$
\begin{aligned}
E(G) = &\{E(P_i) : i = 1, 2, 3, 4\} \\
&\cup \{v_i^j v_{i+1}^j : i = 1, 2, 3, 4, j = 1, \ldots, l \\
&\cup \{v_i^j v_{i+1}^{j+1} : i = 1, 2, 3, 4, j = 1, \ldots, l - 1 \\
&\cup \{v_i^l u_n : i = 1, 2, 3, 4, n = 1, \ldots, k - 1\} \\
&\cup \{u_j u_i : i, j = 1, \ldots, k - 1\} \\
&\cup \{s_i x : i = 3, \ldots, k, x \in V(G) - t_i\} \\
&\cup \{t_i x : i = 3, \ldots, k, x \in V(G) - s_i\}
\end{aligned}
$$

See Figure 12 for an example of the above graph when $k = 3$ and $l = 4$. Any graph G

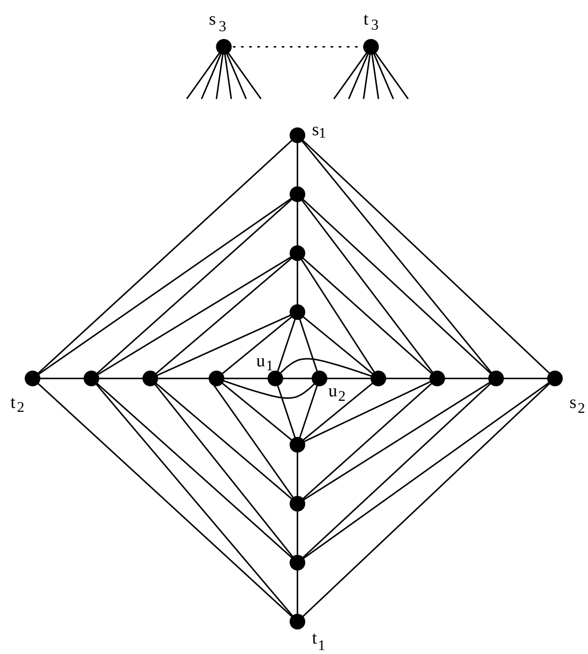

Figure 12: An example showing a lower bound for Theorem 1.4.2

constructed as above will be $2k$-connected, as desired.

6.2 Proof of Theorem 6.1.1

The proof of Theorem 6.1.1 hinges upon the following lemma.

Lemma 6.2.1 *Let G be a graph on n vertices and let P_1 and P_2 be disjoint paths in G with ends x_1, y_1 and x_2, y_2, respectively. Further, assume $V(G) = V(P_1) \cup V(P_2)$. Then if $|E(G)| \geq 3n-4$, there exist disjoint paths P_1' and P_2' with ends x_1, y_1 and x_2, y_2 respectively such that $|V(P_1) \cup V(P_2)| > |V(P_1') \cup V(P_2')|$.*

Proof: First observe that if $n \leq 5$, then the statement is vacuously true. We proceed by induction on n. If upon contracting an edge e with both ends in some P_i, we are able to find shorter paths, then in G as well, we could find shorter paths. Thus we may assume that upon contracting each same edge edge e, we no longer satisfy the edge bound. For that to happen, if $e = uv$, then u and v must have at least 3 common neighbors on the other path.

Now pick some edge $uv \subseteq P_1$ and let w_1, w_2, w_3 be common neighbors on P_2. Assume that one of the pairs (w_1, w_2) and (w_2, w_3) is not an edge of P_2, say (w_2, w_3). Then there exists some $w_4 \in V(P_2)$ adjacent to w_2 with w_4 between w_2 and w_3 on P_2. Then w_2 and w_4 have three common neighbors on P_1 implying w_2 has some neighbor $z \in V(P_1)$, $z \neq u, v$. If $z \in V(vP_1y_1)$, then $x_1P_1uw_2zP_1y_1$ and $x_2P_2w_1vw_3P_2y_2$ are two paths with the desired ends. They are shorter in length than P_1 and P_2 since the new paths do not contain the vertex w_4. The case when $z \in V(x_1P_1u)$ is symmetric. Thus we may assume every edge uv in P_1 has exactly three neighbors in common in P_2 and more over, they are sequential on P_2.

Let u_1u_2 and u_2u_3 be two edges on P_1 such that the common neighbors of u_1 and u_2 are w_1, w_2, w_3 and the common neighbors of u_2 and u_3 are z_1, z_2, z_3 and assume $\{w_1, w_2, w_3\} \neq \{z_1, z_2, z_3\}$. Then there are two very similar cases: w_1 occurs before z_1 on P_2 and z_1 occurs before w_1. In the first case, $x_1P_1u_1w_2P_2z_1u_3P_1y_1$ and $x_2P_2w_1u_2z_3P_2y_2$ are two paths with the desired ends using fewer vertices. We know the sum of the lengths is less because the paths do not include the vertex z_2 since w_2 is adjacent to w_1, and w_1 occurs before z_1. In the second case $x_1P_1u_1w_3P_2z_3u_3P_1y_1$ and $x_2P_2z_1u_2w_3P_2y_2$ are paths with the appropiate ends not utilizing the vertex z_2 in either path.

Thus we may assume that no two consecutive edges on P_1 have a different set of three common neighbors. By beginning with the first edge of P_1, it follows that we may assume all edges of P_1 have the same three common neighbors in P_2. Then P_2 can have no other vertex with a neighbor on P_1, and so in fact $|V(P_2)| = 3$. But the argument symetrically shows $|V(P_1)| = 3$. Given the edge bound, it is impossible that P_1 and P_2 are induced paths, i.e. it must be the case that x_1 is adjacent to y_1 or x_2 is adjacent to y_2, and the statement of the Lemma holds.

We use the above lemma to prove Theorem 6.1.1

Assume the theorem is false, and let G and $P_1, \ldots, P_k$ be a counter example. Then clearly by Lemma 6.2.1 for all i and j, we have $e(G[V(P_i) \cup V(P_j)]) \leq 3(|V(P_i)| + |V(P_j)|) - 5$. Then there are at most $2(|V(P_i)| + |V(P_j)|) - 3$ edges with one end in P_i and the other in P_j. Thus there are at most

$$\sum_{i<j} \left(2\left(|V(P_i)| + |V(P_j)|\right) - 3\right)$$

edges with ends on distinct paths. Adding the edges contained in the paths, we see that

$$\begin{aligned}|E(G)| &\leq \sum_{i<j} \left(2\left(|V(P_i)| + |V(P_j)|\right) - 3\right) + \sum_{i=1}^{k}(|V(P_i)| - 1) \\ &= (2k-1)n - 3\binom{k}{2} - k\end{aligned}$$

contradicting our choice of G and proving Theorem 6.1.1

CHAPTER 7

EXTREMAL FUNCTIONS FOR ROOTED MINORS

Our strategy when considering extremal functions for general rooted minors will be similar to the proofs of extremal functions for linkages. First, we will consider the problem as a problem on α-massed graphs for an appropriate value of α. Then using the techniques of Chapter 3, we reduce the problem to one on small, dense graphs.

7.1 Proof of Theorem 1.4.4

We will prove the stronger statement:

Theorem 7.1.1 *Let G and H be graphs, and let $X \subseteq V(G)$ with $|X| \leq |V(H)|$. Let $t = |V(H)|$ and $c > 1$ be a real number such that every graph on n vertices with cn edges contains H as a minor. If (G, X) is $(9c + 395t)$-massed, (G, X) contains a π-rooted H' minor for all subgraphs H' of H with $|V(H')| = |X|$ and for all bijections $\pi : X \to V(H')$.*

Proof: Let H, t, and c be given as in the statement of the theorem. Assume the theorem is false, and pick (G, X) to be a $(H, (9c + 395t))$-minimal pair. Let H' and π be as in the definition of $(H, (9c + 395t))$ minimality. Notice that by minimality, $t \geq |X| \geq 3$.

Consider a vertex $v \in V(G) - X$ of minimum degree, and let D be the subgraph of G induced by $\{v\} \cup N_v$. Theorem 3.4.2 implies that D has

$$\delta(D) \geq 9c + 394t$$

and if we let $n = |V(D)|$,

$$n \leq 18c + 790t + 1.$$

We now show, contrary to Theorem 3.4.3, that D contains an H-universal subgraph.

First, we will utilize the following observation.

Observation 7.1.1 *Let G be a graph and $d \leq 1$ a positive real number. Then there exists a subgraph H of G with $|V(H)| \leq d|V(G)| + 2$ with $|E(H)| \geq d^2|E(G)|$.*

Proof: The claim follows from a simple probabilistic argument. Let $n := |V(G)|$. If $\lceil d|V(G)|\rceil + 1 \geq n$, then the statement is trivially true with $H = G$. Thus we may assume $\lceil d|V(G)|\rceil + 1 < n$. Choose a set Y of $\lceil d|V(G)|\rceil + 1$ vertices of G uniformly at random. For a given edge $e \in E(G)$, the probability that e has both ends in X is

$$\frac{\binom{n-2}{\lceil dn\rceil - 1}}{\binom{n}{\lceil dn\rceil + 1}} = \frac{(\lceil dn\rceil + 1)\lceil dn\rceil}{n(n-1)} \geq d^2$$

The expected number of edges with both ends in Y is at least $d^2|E(G)|$, and there exists a subgraph achieving the expectation as desired.

Claim 61 *D has a non-trivial separation of order at most $4c + \frac{532}{3}t + 3$.*

Proof: Assume to reach a contradiction that D is at least $4c + \frac{532}{3}t + 3$ connected.

Observe that $|V(D)| \geq \delta(D) > t$. Thus we may fix a set $Y \subseteq V(D)$ with $|Y| = t$. We set $d = \frac{2}{9}$ and utilize Observation 7.1.1. We have a subgraph L of $D - Y$ with

$$\begin{aligned} |V(L)| &\leq \frac{2}{9}(n-t) + 2 \\ &\leq 4c + \frac{526}{3}t + 3 \end{aligned}$$

and at least

$$\begin{aligned} |E(L)| &\geq \left(\frac{2}{9}\right)^2 \frac{1}{2}(\delta(D) - t)(n-t) \\ &\geq \frac{1}{2}\cdot\frac{2}{9}(\delta(D) - t)\left\lceil\frac{2(n-t)}{9}\right\rceil \\ &\geq \frac{1}{9}\left[9c + 393t\right]\cdot\left\lceil\frac{2(n-t)}{9}\right\rceil \\ &\geq c\left\lceil\frac{2(n-t)}{9}\right\rceil + \frac{262}{27}t(n-t) \end{aligned}$$

edges. Utilizing the fact that $n \geq \delta(D) \geq c + t$, we see that

$$|E(L)| \geq c \left\lceil \frac{2(n-t)}{9} \right\rceil + 2c$$
$$\geq c|V(L)|.$$

By the definition of c, the subgraph L contains an H minor with branch sets $\{S_1, \ldots, S_t\}$. Now notice that every vertex of D has at least $\delta(D) - |V(L)| - t$ neighbors in $D - L - Y$.

$$\delta(D) - |V(L)| - t \geq 9c + 394t - \left(4c + \frac{526}{3}t + 3\right) - t$$
$$\geq 10t$$

Clearly, we can then pick distinct vertices $v_1, \ldots, v_t \in V(D) - V(L) - Y$ such that v_i is adjacent a vertex of S_i for all $i = 1, \ldots, t$. Also, by assumption, we know that $D - V(L)$ is $2t$ connected. By Theorem 1.4.2, $D - V(L)$ is t-linked. Then for any bijective map $\phi : X \to V(H)$, there exists disjoint paths P_i where the ends of P_i are x_i and $v_{\phi(i)}$. See Figure 13. We see that (D, Y) has a ϕ rooted H minor with branch sets $P_i \cup S_{\phi(i)}$. Since ϕ

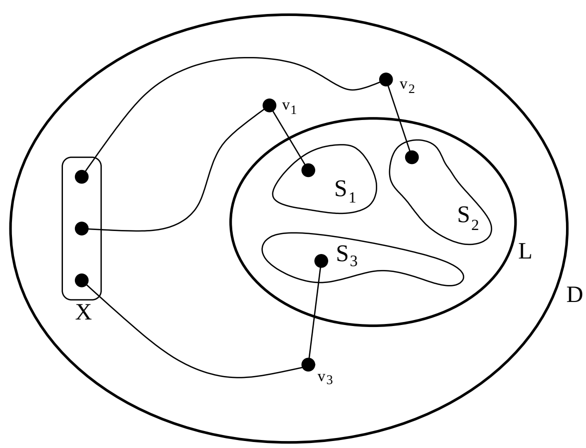

Figure 13: Constructing a π-rooted H minor in D with a linkage from X to L

and Y were chosen arbitrarily, D is H-universal, contrary to Theorem 3.4.3. □

Let (A, B) be a non-trivial separation of exactly order $\left\lfloor 4c + \frac{532}{3}t + 3 \right\rfloor$. Without loss of generality, we may assume that $|A| \leq |B|$. We will roughly iterate the argument above to show that $D[A-B]$ is an H universal subgraph of G.

Let $Z := A \cap B$. Notice that $\delta(D[A-B]) > t$, allowing us to fix a set of t vertices $Y' \subseteq A - Z$. Let D' be the subgraph $D[A-Z-Y']$, and let $n' := |V(D')|$, and $\delta' := \delta(D')$. First we observe that

$$\begin{aligned} n' &\leq \frac{n-|Z|}{2} - t \\ &\leq \frac{1}{2}\left[18c + 790t + 1 - \left(4c + \frac{532}{3}t + 2\right)\right] - t \\ &\leq 7c + \frac{916}{3}. \end{aligned}$$

Moreover,

$$\begin{aligned} \delta' &\geq \delta(D) - |A \cap B| - t \\ &\geq 9c + 394t - \left(4c + \frac{532}{3}t + 3\right) - t \\ &\geq 5c + \frac{647}{3}t \end{aligned}$$

Fix $d' := \frac{2}{5}$, and apply Observation 7.1.1 to D' to find a subgraph L' of D' satisfying the conclusions of the Observation. From Observation 7.1.1, we know that $|V(L')| \leq d'n' + 2$, and $|E(L')| \geq \frac{1}{2}\delta' n'(d')^2$. Then

$$\begin{aligned} \frac{1}{2}\delta' n'(d')^2 &= \left(\frac{\delta'}{2}d'\right) n'd' \\ &\geq \left(c + \frac{647}{15}t\right) n'd' \\ &= c(n'd') + \frac{647}{15}tn'd' \\ &\geq c(n'd' + 2) \\ &= c|V(L')| \end{aligned}$$

where the final inequality follows from the fact that $n' \geq \delta' \geq 5c$. We see that L' contains an H minor with branch sets $\{S'_1, \ldots, S'_t\}$. Now if we consider any two non-adjacent vertices of $D[A-B]$, we see that they have at least $2\delta(D[A-B]) - |A-B| - |L'|$ common neighbors in $A - Z - V(L)$. We will now show that

$$2\delta(D[A-B]) - |A-B| - |V(L')| \geq 2t. \tag{1}$$

This will suffice to complete the proof. Pick a bijective map $\phi : Y' \to V(H)$. Then for any index $i = 1, \ldots, t$, the vertices x'_i and $v'_{\phi(i)}$ are either adjacent or have at least t common neighbors in $A - Y' - V(L')$. We can then link x'_i to $v'_{\phi(i)}$ using a common neighbor for every $i = 1, \ldots, t$. Thus we construct a ϕ-rooted H minor in (D', Y'). Again, ϕ and $Y' \subseteq A - Z$ were chosen arbitrarily, so D' is a universal subgraph, a contradiction.

All that remains is to prove inequality 1 holds.

$$\begin{aligned}
2\delta(D[A-B]) - |A-B| - |V(L')| &\geq 2\delta(D[A-B]) - (n'+t) - (d'n'+2) \\
&\geq 2\delta(D[A-B]) - (d'+1)n' - 2 - t \\
&\geq 2\left[9c + 394t - \left(4c + \frac{532}{3}t + 3\right)\right] - (d'+1)n' - 2 - t \\
&\geq 10c + \frac{1297}{3}t - \frac{7}{5}n' - 8 \\
&\geq 10c - \frac{49}{5}c + \frac{6485}{15}t - \frac{6412}{15}t - 8 \\
&\geq 2t.
\end{aligned}$$

This completes the proof. □

7.1.1 Proof of Theorem 1.4.4.

Let G, H, and c be as in the statement of the theorem, and let $t = |V(H)|$. Consider an arbitrary set $X \subseteq V(G)$ with $|X| = |V(H)|$. The pair (G, X) must satisfy condition (M2) because G is t connected. Also, $\rho(G - X) \geq |E(G)| - |E(G[X])| \geq (9c + 395t)(|V(G) - X|) + (9c+395t)t - \binom{t}{2} \geq (9c+395t)(|V(G)-X|)+1$. Thus (G, X) satisfies condition (M1), and applying Theorem 7.1.1 completes the proof that D is H-universal. □

We conclude with a brief observations on the constants obtained in Theorem 1.4.4. Using the same proof method, the constant 9 could be improved to $6 + 2\sqrt{2} + \epsilon$ for any

$\epsilon > 0$, with a corresponding increase, depending on ϵ, in the constant term multiplying t. We have chosen the constant 9 to smooth the presentation.

7.2 Proof of Theorem 1.4.5

We will actually prove a slightly stronger statement given in terms of α-massed graphs.

Theorem 7.2.1 *Let G be a graph and $X \subseteq V(G)$ with $|X| = s \leq t$. Then if (G, X) is t-massed, G contains a $K_{2,s}(X)$.*

Since the structure $K_{2,s}(X)$ is not exactly a rooted minor according to our definition, we will in effect reprove Theorem 3.4.1 and Theorem 3.4.2. Since the proofs are very similar, we will not go into extensive detail here.

A 2-*bipartite rigid* separation (A, B) of (G, X) will be one where $X \subseteq A$ and $G[B]$ contains a labeled $K_{2,|A\cap B|}(A \cap B)$ minor.

Lemma 7.2.1 *Let (A, B) be a 2-bipartite rigid separation of a pair (G, X) where $|X| = s$, and let G' be the separation truncation of G. If (G', X) contains a labeled $K_{2,s}(X)$ minor, then the pair (G, X) contains a labeled $K_{2,s}(X)$ minor.*

Proof: Let $U_1, \ldots, U_s, W_1, W_2$ be the branch sets of a labeled $K_{2,s}(X)$ minor in (G', X). Define the graph $\overline{G}$ to be the graph with vertex set $A \cup \{z_1, z_2\}$ and edge set $E(G[A]) \cup \{z_1x, z_2x : x \in A \cap B\}$. In other words, $\overline{G}$ is the graph $G[A]$ with two additional vertices z_1 and z_2 adjacent all of the vertices in $A \cap B$. Clearly $\overline{G}$ is a minor of (G, X) and it suffices to prove that $(\overline{G}, X)$ contains a $K_{2,s}(X)$ minor.

For every $1 = 1, \ldots, s$, let $\overline{U_i}$ be a maximal subset of U_i with the following properties:

1. $\overline{U_i} \cap X \neq \emptyset$

2. $\overline{U_i}$ induces a connected subgraph of $\overline{G}$, and

3. $\overline{U_i}$ contains at most one vertex of $A \cap B$.

Note that if U_i does intersect $A \cap B$, then by maximality, $\overline{U_i}$ contains at least one vertex of $A \cap B$. Also, by definition, the $\overline{U_i}$'s are connected and each contains exactly one vertex of X.

If W_i intersects $A \cap B$ for either $i = 1$ or 2, let $\overline{W_i} = (A \cap W_i) \cup \{z_i\}$. In this case, $\overline{W_i}$ induces a connected subgraph of $\overline{G}$ and has an edge going to every $\overline{U_j}$. Thus if both W_1 and W_2 intersect $A \cap B$, we have found a $K_{2,s}(X)$ in $\overline{G}$. If exactly one W_i, say W_1, intersects $A \cap B$, then consider W_2. W_2 induces a connected subgraph in $\overline{G}$ since it cannot use any edges of $A \cap B$. If W_2 has an edge to every $\overline{U_i}$, then $\{\overline{U_1}, \ldots, \overline{U_s}, \overline{W_1}, W_2\}$ form a labeled $K_{2,s}(X)$ minor in $\overline{G}$. Instead assume there is some index, say j, such that W_2 has no edge to $\overline{U_j}$. We know there is an edge in G' between the set W_2 and U_j. Let xy be such an edge, and let x be the end in U_j. Since W_2 does not intersect $A \cap B$, the edge xy is also present in $\overline{G}$. Look at a path in $G'[U_j]$ from x to $\overline{U_j}$. Such a path clearly exists since U_j induces a connected subgraph in G'. Moreover, by the maximality of the $\overline{U_j}$, such a path must intersect $A \cap B$ before reaching $\overline{U_j}$. Let P be a path from x to $A \cap B$ in $\overline{G}[U_j]$ that does not intersect $\overline{U_j}$. Now $\{\overline{U_1}, \ldots, \overline{U_s}, \overline{W_1}, W_2 \cup V(P) \cup \{z_2\}\}$ form a $K_{2,s}(X)$ minor in $\overline{G}$.

We may now assume neither W_i intersects the set $A \cap B$. Let $\mathcal{N}_i$ be the set of indices k such that $\overline{U_k}$ has no edge in $\overline{G}$ to W_i. If both $\mathcal{N}_i$'s are empty, then clearly the sets $\{\overline{U_1}, \ldots \overline{U_s}, W_1, W_2\}$ form a $K_{2,s}(X)$ minor in $\overline{G}$. If exactly one $\mathcal{N}_i$, say $\mathcal{N}_1$ is empty, we construct a labeled $K_{2,s}(X)$ minor as follows. Let i be an index in $\mathcal{N}_2$. Then as in the previous paragraph, let P be a path in $\overline{G}[U_i]$ from W_2 to $A \cap B$. Then $\{\overline{U_1} \ldots, \overline{U_s}, W_1, W_2 \cup V(P) \cup \{z_2\}\}$ form a $K_{2,s}(X)$ minor in $\overline{G}$.

We have now shown that both $\mathcal{N}_i$'s are non-empty. There are two distinct cases:

Case 1: There exist distinct representatives from $\mathcal{N}_1$ and $\mathcal{N}_2$ Let the distinct representatives be $j \in \mathcal{N}_1$ and $k \in \mathcal{N}_2$. As in the previous paragraph, let P_j be a path from W_1 to $A \cap B$ in $\overline{G}[U_j]$ disjoint from $\overline{U_j}$, and similarly define P_k. Then we get a labeled $K_{2,s}(X)$ minor in $\overline{G}$ with branch sets $\{\overline{U_1}, \ldots, \overline{U_s}, W_1 \cup V(P_j) \cup \{z_1\}, W_2 \cup V(P_k) \cup \{z_2\}\}$.

Case 2: No such distinct representatives from $\mathcal{N}_1$ and $\mathcal{N}_2$ exist. In this case, $\mathcal{N}_1 = \mathcal{N}_2 = \{k\}$ for some index k. Now define $U_i^* = \overline{U_i}$ for $i \neq k$ and $U_k^* = U_k \cup \{z_1\}$.

Observe that U_k^* is connected since the vertex z_1 compensates for any missing edges in $A \cap B$. Moreover, U_k^* has an edge to W_1 and W_2 since $U_k^* \supseteq U_k$. Thus $\{U_1^* \ldots, U_s^*, W_1, W_2\}$ form a $K_{2,s}(X)$ minor in $\overline{G}$.

This completes the analysis, proving the lemma. □

Notice that unlike in Lemma 3.2.2, in Lemma 7.2.1 the implication is only in one direction. In fact, the converse is not true. It is possible that a graph G contains a $K_{2,t}(X)$ for some set X, and yet the separation truncation of a 2-bipartite rigid separation does not.

Proof of Theorem 7.2.1. Assume the theorem is false, and let G and X be a counterexample on a minimal number of vertices, and, subject to that, with $\rho(V(G)-X)$ minimized. Also, we may assume that $G[X]$ is a complete graph, since adding any edges to X will not affect the existence of a labeled $K_{2,s}(X)$ minor.

Claim 62 *(G, X) has no 2-bipartite rigid separation.*

Proof: The proof follows the proof of Theorem 3.4.1. Pick such a separation (A, B) to minimize $|A|$. If the separation is of order at least $|X|$, then either there exist $|X|$ disjoint paths from X to $A \cap B$ in which case there exists a labeled $K_{2,s}(X)$ minor, or there exists a separation of smaller order in $G[A]$ separating X from $A \cap B$. Such a separation of minimal order induces a 2-bipartite rigid separation violating our choice of (A, B).

Now assuming that the separation (A, B) is of order strictly less than $|X|$, consider the separation truncation G' of (A, B). If G' is t-massed, then by the minimality of our counterexample, G' would have a labeled $K_{2,s}(X)$ minor. Lemma 7.2.1 implies G would also contain a labeled $K_{2,s}(X)$ minor, a contradiction. Thus we may assume that (G', X) is not t-massed. By the fact that we know that the order of (A, B) is at most $s - 1 \leq |X| - 1$ and the pair (G, X) is t-massed, we know that (G', X) satisfies condition (M1). Thus there exists a separation (A', B') violating condition (M2). Choose such a separation to minimize $|B'|$. Then $(G'[B'], A' \cap B')$ is t-massed, and consequently, $G'[B']$ contains a $K_{2,|A' \cap B'|}(A' \cap B')$ minor. Since $G'[A \cap B]$ is a complete subgraph, $A \cap B$ must be a subset of either A' or B'. Since $(A \cup A', B')$ would be a separation of (G, X) violating condition (M2) in G, we know that $A \cap B$ is a subset of B'. But then (B', B) is a 2-bipartite rigid separation of $(G[B' \cup B], A' \cap B')$. By applying Claim 7.2.1 to this separation, we see that $(A', B' \cup B)$ is a 2-bipartite rigid separation violating our choice of (A, B) to minimize $|A|$. $\square$

Claim 63 *Every edge e of G with at most one end in X is in at least t triangles.*

Proof: Attempt to contract the edge e. By minimality, if $(G/e, X)$ were t-massed, then G/e would contain a labeled $K_{2,s}(X)$ minor implying the existence of a labeled $K_{2,s}(X)$ minor in G. Instead, it must be the case that $(G/e, X)$ is not t-massed. Assume that $(G/e, X)$ contains a separation (A, B) violating (M2), and assume that we chose it to minimize $|B|$. Then $(G/e[B], A \cap B)$ is t-massed. By minimality, $G/e[B]$ contains a $K_{2,|A \cap B|}$ minor. The separation (A, B) induces a separation (A^*, B^*) in G by uncontracting the edge e. If the ends of e lie in $A^* - B^*$, the separation would violate condition (M2). There are now two cases. First, consider when both the endpoints of e are contained in $A^* \cap B^*$. Then $\rho(B^* - A^*) \geq \rho(B - A)$ and $|B^* - A^*| = |B - A|$. Moreover $|A^* \cap B^*| \leq |A \cap B| + 1 \leq |X|$ and by minimality, (A^*, B^*) is a 2-bipartite rigid separation giving us a contradiction. However, in the other case, both ends of e lie in $B^* - A^*$. Then (A^*, B^*) is rigid since $G/e[B]$ contains a labeled $K_{2,|A \cap B|}(A \cap B)$ minor, again a contradiction.

Contracting the edge e must violate condition (M1). But because $G[X]$ is a complete subgraph, the edge e must lie in t triangles, as claimed. $\square$

Then the ends of any edge e not contained in X belong to a $K_{2,t}$ subgraph using the common neighbors of its endpoints. We will now see that it is possible to find disjoint paths from X to one such $K_{2,t}$ subgraph and have all the paths avoid the bipartition of size 2.

Pick a separation (A, B) of order at most $|X|$ with $X \subseteq A$ chosen to minimize $|B - A|$ such that $B \not\subseteq A$. Now choose a separation (A', B') of $G[B]$ of order $|X| + 1$ with $A \cap B \subseteq A'$. Moreover, choose (A', B') to minimize $|B'|$. Notice that by Claim 63, x has degree at least $t + 1$, so there must exist some neighbor of x in $B - A$.

Let x be a vertex of $(A' \cap B') - (A \cap B)$ and y a neighbor of x in $B' - A'$. Let N be t common neighbors of x and y. In $G[B' - \{x, y\}]$, there exist $|X|$ disjoint paths from

$(A' \cap B') - \{x\}$ to N. See Figure 14. Otherwise there would exist a separation of order

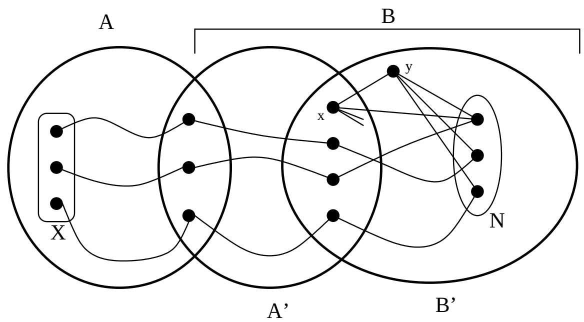

Figure 14: Linking X to a $K_{2,t}$ subgraph in G.

$|X| - 1$, and adding x and y back in we would get a separation violating our choice of (A', B') to minimize $|B'|$. If we now look at $G[A']$, there exist $|X|$ disjoint paths from $A \cap B$ to $(A' \cap B') - \{x\}$, or we would get a separation violating our choice of (A, B) to minimize $|B - A|$. However, now we have found $|X|$ disjoint paths from $A \cap B$ to N avoiding x and y. Thus (A, B) is a 2 - bipartite rigid separation, contrary to Claim 62. This completes the proof. □

CHAPTER 8

APPLICATIONS OF GRAPH LINKAGES

8.1 The Extremal Function for K_p Topological Minors

We present the proof of Bollobás and Thomason giving the optimal (up to a constant) edge bound for the existence of a topological K_p minor

Theorem 8.1.1 *[Bollobás and Thomason [5]] Every graph G with $|E(G)| \geq 10p^2|V(G)|$ contains K_p as a topological minor.*

Proof: Let G satisfy the edge bound. Then by Theorem 1.3.6, G has a $5p^2$-connected subgraph G'. Let $v_1, \ldots, v_p$ be p distinct vertices in $V(G')$. Then because every vertex has degree at least $5p^2$, for every index $i = 1, \ldots, p$, we can find $p-1$ distinct neighbors of v_i and label them $v_i^1, \ldots, v_i^{p-1}$. The graph $G' - \{v_1, \ldots, v_p\}$ is $(5p^2 - p) \geq 10\binom{p}{2}$ connected. It follows from Corollary 1.4.1 that $G' - \{v_1, \ldots, v_p\}$ is $\binom{p}{2}$-linked, and hence there exist disjoint paths $P_{i,j}$ for all $i \neq j$ such that the ends of $P_{i,j}$ are v_i^j and v_j^i. The vertices $v_1, \ldots, v_p$ are the branch vertices of a topological K_p minor where every pair of distinct vertices v_i and v_j is connected by the path $v_i P_{i,j} v_j$. $\square$

8.2 Non-zero Cycles in Group Labeled Graphs

We examine the problem of finding the minimal function $g(k)$ such that every $g(k)$-connected graph G either has k disjoint odd cycles, or there exists a set of $2k-2$ vertices intersecting every odd cycle in G. In this section, we will prove Theorem 1.4.6.

We consider a generalization of this problem to group labelled graphs. Let Γ be an arbitrary group. We will use additive notation for groups, though they need not be abelian. Let G be an oriented graph. For each edge e in G, we assign a weight $\gamma_e \in \Gamma$. The weight γ_e is added when the edge is traversed according to the orientation and subtracted when traversed contrary to the orientation. Rigorously, given an oriented graph G and a group Γ, a Γ-*labelling of* G consists of an assignment of a label $\gamma_e \in \Gamma$ to every edge $e \in E$, and function $\gamma : \{(e, v)|e \in E(G), v \text{ an end of } e\} \rightarrow \Gamma$ such that for every edge $e = (u, v)$ in G, where u is the tail of e and v is the head, $\gamma(e, u) = -\gamma_e = -\gamma(e, v)$. Let $W = (v_0 e_1 v_1 e_2 \ldots e_k v_k = v_0)$ be a (not necessarily directed) walk in the underlying undirected graph $\overline{G}$ of G. Then the *weight* of W, denoted by $w(W)$, is $\sum_{i=1}^{k} \gamma(e_i, v_i)$. We say that a cycle C in $\overline{G}$ is *non-zero* if some (and hence every) walk tracing out C has non-zero weight.

We prove the following theorem.

Theorem 8.2.1 *Let G be an oriented graph and Γ a group. Let the function γ be a Γ labelling of G. Let $\overline{G}$ be the underlying undirected graph. If $\overline{G}$ is $\frac{31}{2}k$-connected, then $\overline{G}$ has either k disjoint non-zero cycles or it has a vertex set Q of order at most $2k - 2$ such that $\overline{G} - Q$ has no non-zero cycles.*

Theorem 1.4.6 follows easily now.

Proof: Theorem 1.4.6 Given a graph G, assign edge directions arbitrarily, and let each edge have weight 1. If Γ is the group on 2 elements, then $\gamma(e, v) = \gamma(e, u)$ for all edges $e = (u, v)$. Thus the non-zero cycles are simply the odd cycles in G. Theorem 8.2.1 finds k disjoint odd cycles or a set of $2k - 2$ vertices intersecting all odd cycles, proving Theorem 1.4.6. $\square$

We briefly introduce a necessary result. For a fixed set of vertices A in a graph G, an *A-path* is a nontrivial path P with both ends in A and no other vertices in A. Chudnovsky, Geelen, Gerards, Goddyn, Lohman, and Seymour examined non-zero A-paths in a group labelled graph G, proving that for any set of vertices $A \subset V(G)$, the Erdős-Pósa property holds for non-zero A paths. Notice that in a group labelled graph, as in the case of non-zero cycles, the weight of an A path will depend on the direction in which the path is traversed. However, whether or not the weight is non-zero will not. Specifically, Chudnovsky et al. proved:

Theorem 8.2.2 *[Chudnovsky et al. [8]] Let Γ be a group, and G be an oriented graph. If γ is a Γ labelling of G, then for any set S of vertices of G and any positive integer k, either*

1. *there are k disjoint non-zero S-paths, or*

2. *there is a vertex set X of order at most $2k-2$ that meets each such non-zero S-path.*

Following the notation of Chudnovsky et al. in [8], consider a vertex $x \in V(G)$ and a value $\alpha \in \Gamma$. Then for each edge e with head v and tail u, we consider a new assignment of weights:

$$\gamma'_e = \begin{cases} \gamma_e + \alpha & \text{if } v = x \\ -\alpha + \gamma_e & \text{if } u = x \\ \gamma_e, & \text{otherwise} \end{cases}$$

We say γ' is obtained by *shifting* γ at x by the value α. Notice that if we shift γ at some vertex $x \in V(G) - A$, then the weight of any A-path remains unchanged. Similarly, the weight of a cycle also remains invariant under shifting γ.

Observation 8.2.1 *If a subgraph H of G contains no non-zero cycles, then there exists a weight function γ' obtained from γ by shifting at various vertices such that every edge $e \in E(H)$ has $\gamma'_e = 0$.*

Proof: Clearly, it suffices to consider each connected component of H separately. Take a spanning tree T of H. We can ensure that each edge of the spanning tree has weight zero by performing a series of shifts. Then every other edge e of H must also have weight 0, since otherwise $\{e\} \cup E(T)$ would contain the edge set of a non-zero cycle. □

Note that if for any edge e in G, we flip the orientation of e and also set $\gamma'_e = -\gamma_e$, we do not change the weight of any cycle or A-path.

8.3 Proof of Theorem 8.2.1

Assume Theorem 8.2.1 is false, and let G be a counterexample with γ a labelling from the group Γ such that there do not exist k disjoint non-zero cycles, nor does there exist a set of $2k-2$ vertices intersecting every non-zero cycle. Moreover, assume that G is a counterexample on a minimal number of vertices.

Take disjoint non-zero cycles $C_1, \ldots, C_l$ such that l is as large as possible (but $G - V(C_1 \cup C_2 \cup \cdots \cup C_l)$ is non-empty), and subject to that, $|V(C_1) \cup V(C_2) \cup \cdots \cup V(C_l)|$ is as small as possible. Clearly $l < k$. Let W be the induced subgraph on $V(C_1 \cup C_2 \cup \cdots \cup C_l)$, and for every vertex $v \in V(G)$, let $d_{C_i}(v)$ be the number of neighbors of v in $V(C_i)$. We proceed with several intermediate claims.

Claim 64 *For any vertex v in $G - V(W)$, $d_{C_i}(v) \leq 3$ for any i with $1 \leq i \leq l$.*

Proof: Suppose for a contradiction that v has four neighbors $v_1, \ldots, v_4$ in C_i. Let P_j be the directed path of C_i with endpoints v_j and v_{j+1} not containing any other vertices among $v_1, \ldots, v_4$ except for v_j and v_{j+1}, where the addition $j+1$ is taken modulo 4. We may assume that each edge (v, v_j) is directed from v to v_j. Let a_j be the weight of the edge (v, v_j) and let b_j be the weight of P_j. See Figure 15

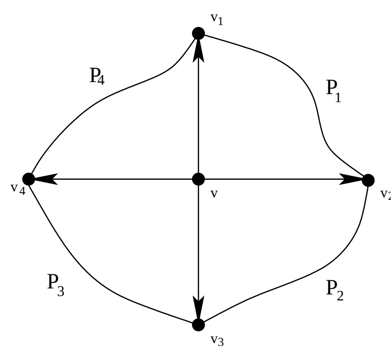

Figure 15: A vertex v with four neighbors on C_i.

Define T_j to be the cycle defined by $vv_jP_jv_{j+1}v$. The weight of T_j is $a_j + b_j - a_{j+1}$. Then

$$\begin{aligned} \sum_{j=1,\ldots,4} w(T_j) &= (a_4 + b_4 - a_1) + (a_1 + b_2 - a_2) + \cdots + (a_3 + b_3 - a_4) \\ &= b_4 + \cdots + b_1 \\ &= w(C_i). \end{aligned}$$

Then since the weight of C_i is non-zero, some T_j must also have non-zero weight. But this contradicts the minimality of the size of C_i, proving the claim. $\square$

Claim 64 implies that the minimum degree of $G - W$ is at least $\frac{31}{2}k - 3(k-1) > \frac{25}{2}k$. Also by the definition of W, $G - W$ has no non-zero cycles. The following result was originally proved in [2].

Lemma 8.3.1 ([2]) *Let G be a graph and k an integer such that*

(a) $|V(G)| \geq \frac{5}{2}k$ *and*

(b) $|E(G)| \geq \frac{25}{4}k|V(G)| - \frac{25}{2}k^2$.

Then $|V(G)| \geq 10k + 2$ and G contains a $2k$-connected subgraph H with at least $5k|V(H)|$ edges.

Lemma 8.3.1 and Theorem 1.4.2 imply that $G - W$ has a k-linked subgraph H. Note that H has minimum degree at least $2k$. As we observed in the previous section, by taking an equivalent weight function, we may assume every edge of H has weight 0.

Utilizing Theorem 8.2.2, we prove the following.

Claim 65 *There exist k vertex disjoint non-zero $V(H)$-paths in G.*

Proof: Assume not. Then by Theorem 8.2.2, there exists a set X of at most $2k - 2$ vertices eliminating all the non-zero H-paths. If $G - X$ contains a non-zero cycle C, then because $G - X$ is still at least 2-connected, there exist two disjoint paths from $V(H) - X$ to C. By routing one way or the other around C, we obtain a non-zero path starting and ending in H. Then there exists a non-zero subpath intersecting H exactly at its endpoints, contradicting our choice of X. $\square$

Now we have proven that there exist k vertex disjoint non-zero H-paths. Clearly these paths can be completed into cycles by linking up their ends in H with paths of weight zero, contradicting our choice of G as a counterexample to Theorem 8.2.1. This completes the proof of Theorem 8.2.1.$\square$

8.4 Clique Minors in Graphs of Bounded Tree-Width

In this section, we consider the problem of forcing K_p minors by requiring the graph to be cp-connected and have at least $N(p)$ vertices for some constant c and some function $N(p)$. Recall from our discussion in Chapter 1 that Böhme et al. proved in Theorem 1.3.7 that such a function $N(p)$ and constant c do in fact exist. The proof of Theorem 1.3.7 proceeds by breaking the problem into two steps. The first considers what is known as a tree-decomposition of the graph and when such a tree-decomposition has bounded width. The second half of the argument utilizes the structure theorem of Robertson and Seymour characterizing graphs with large tree-width and no K_p minor. The case when tree-width is bounded appeared first in [1]. We give an independent and relatively elementary proof of the bounded tree-width case.

Theorem 8.4.1 *Let G be an $(82p+5)$-connected graph of tree-width at most B. If G satisfies*

$$\frac{\log(|V(G)|/B))}{\log\left(p\binom{B}{p}\right)} \geq \left[B(B+1)\left(p\binom{B}{p}+1\right)\right]^B,$$

then G contains K_p as a minor.

We begin by presenting some necessary material on tree-decompositions before proceeding with the proof.

8.4.1 Basics of Tree-Decompositions

First we give the definition of a tree-decomposition.

Definition 22 *A* tree-decomposition *of a graph G is a pair (T, W) where T is a tree and $W = (W_v \subseteq V(G) : v \in V(T))$ is a collection of subsets of vertices of G indexed by the vertices of T with the following properties.*

(T0) $\bigcup_{t\in V(T)} W_t = V(G)$

(T1) *For every edge $e = uv$ in $E(G)$, there exists an $x \in V(T)$ such that $u, v \in W_x$, and*

(T2) *for every vertex $v \in V(G)$, if $v \in W_x$ and $v \in W_y$ for two vertices $x, y \in V(T)$, then for every $z \in V(T)$ lying on the unique path in T connecting x and y, $v \in W_z$.*

Given a particular tree-decomposition (T, W), the *width* of (T, W) is equal to

$$\max_{x \in V(T)} |W_x| - 1.$$

Then the *tree-width* of a graph G is the minimum width of a tree-decomposition of G. Historically, subtracting one in the definition arose as a way to ensure that trees have tree-width one. A tree-decomposition (T, W) is *non–trivial* if $W_x \nsubseteq W_y$ for all distinct vertices x and y in T. We will use the following observation, a common exercise on tree-decompositions.

Observation 8.4.1 *Given a non-trivial tree-decomposition (T, W) of a graph G, then for any $x \in V(T)$, let T' be a component of $T - x$. Then*

$$\left(\bigcup_{y \in V(T')} W_y \cup W_x, \bigcup_{y \in V(T-T')} W_y \right)$$

is a non-trivial separation in G.

Proof: Let the edge uv of G have one end $u \in W_t$ for some $t \in V(T')$ and the other end $v \in W_{t'}$ for some $t' \in V(T - T')$. We will see that either u or v also is contained in W_x, proving that the edge uv does not violate the definition of a separation. By property (T1), the edge uv is contained in W_z for some vertex z in T. If $z \in V(T')$, then the path in T connecting z and t' must intersect x, implying that $v \in W_x$ by (T2). Similarly, if $z \in V(T - T')$, then the path in T connecting u and z must intersect x, implying that $u \in W_x$. This completes the proof of the observation. $\square$

A *path decomposition* is simply a tree-decomposition (T, W), where the tree T is a path. The *length* of a path decomposition (T, W) is the length of the path T. Let P be the path with vertices $v_0, v_1, \ldots, v_t$ such that v_i is adjacent v_{i+1}. Then the *adhesion* of a path decomposition (P, W) of a graph G is equal to

$$\max_{i=0,\ldots,t-1} |W_{v_i} \cap W_{v_{i+1}}|.$$

A path decomposition (P, W) is of *uniform adhesion* t if $|W_{v_i} \cap W_{v_{i+1}}| = t$ for all values $i = 0, \ldots, t-1$.

Definition 23 *Let (P, W) be a path decomposition of a graph G of uniform adhesion t and let the vertices of P be labeled $v_0, \ldots, v_k$. Then (P, W) is a* strongly linked tree-decomposition *if for every $i = 1, \ldots, k-1$, there exists a linkage $\mathcal{Q}$ with t components in $G[W_{v_i}]$ from $W_i \cap W_{i-1}$ to $W_i \cap W_{i+1}$.*

We prove that if a connected graph G has a very long non-trivial path decomposition of bounded width, it must have a reasonably long strongly linked non-trivial path decomposition.

Lemma 8.4.1 *Let (P, W) be a non-trivial path decomposition of adhesion B of a connected graph G. If the length of (P, W) is at least $(Bk)^B$, then G has a strongly linked non-trivial path decomposition (P', W') of adhesion at most B and length k.*

Proof: Let l be the length of (P, W). The proof proceeds by induction on the adhesion of (P, W). Let t be the connectivity of G. If the adhesion of (P, W) equals t, then since $W_i \cap W_{i+1}$ is a cut set in the graph for all $i = 0, \ldots, l-1$, we see that (P, W) is of uniform adhesion t. Moreover, by Menger's Theorem there exist t disjoint paths from $W_{i-1} \cap W_i$ to $W_i \cap W_{i+1}$ in $G[W_i]$ for $i = 1, \ldots, l-1$, implying that (P, W) is a strongly linked path decomposition.

Now assume that the adhesion B of (P, W) is strictly more than t. Let a_i be the sequence determined by $a_i = |W_i \cap W_{i+1}|$ for $i = 0, \ldots, l-1$. Since the values of a_i range between t and B, there exist at least $\lceil l/B \rceil$ repeated values in the series. Let π be the function such that $a_{\pi(1)} = a_{\pi(2)} = \cdots = a_{\pi(\lceil l/B \rceil)}$ and $\pi(0) = 0$. We now define a shorter path decomposition of uniform adhesion. Let U_i be defined such that $U_i = \bigcup_{j=\pi(i)+1,\ldots,\pi(i+1)} W_j$ for $i = 1, \ldots, \lceil l/B \rceil - 1$ and $U_0 = \bigcup_{j=0,\ldots,\pi(1)} W_j$ and $U_{\lceil l/B \rceil} = \bigcup_{j=\pi(\lceil l/B \rceil)+1,\ldots,l} W_j$. The $\{U_i : i = 0, \ldots, \lceil l/B \rceil + 1\}$ give a path decomposition of length $\lceil l/B \rceil + 1$ of uniform adhesion $a_{\pi(1)}$.

If there exists an index i such that for all $j = 1, \ldots, k-1$, there exist $a_{\pi(1)}$ disjoint paths from $U_{i+j-1} \cap U_{i+j}$ to $U_{i+j} \cap U_{i+j+1}$ in $G[U_{i+j}]$, then we have found a strongly linked path decomposition of length k given by the sets $\overline{U_0} = \bigcup_{j=0,\dots,i} U_j$ and $\overline{U_j} = U_{i+j}$ for $j = 1, \dots, k-1$ and $\overline{U_k} = \bigcup_{j=i+k,\dots,l/B} U_j$.

We may then assume that there exist at least $\lceil l/Bk \rceil$ indices i where $G[U_i]$ contains a separation of order strictly less than $a_{\pi(1)}$ separating U_{i-1} from U_{i+1}. Let $\lambda(i)$ for $i = 1, \dots, \lceil l/Bk \rceil$ be a function such that $G[U_{\lambda(i)}]$ contains such a separation, call it (A_i, B_i). Now we define a path decomposition of adhesion b for some $b < a_{\pi(1)} \leq B$ and length at least l/Bk. Let $X_0 = \bigcup_{j=0,\dots\lambda(1)-1} U_j \cup A_1$, and $X_i = \bigcup_{j=\lambda(i)+1,\dots\lambda(i+1)-1} U_j \cup B_i \cup A_{i+1}$ for all $i = 1, \dots, \lceil l/Bk \rceil - 1$. Finally, let $X_{\lceil l/Bk \rceil} = \bigcup_{j=\lambda(\lceil l/Bk \rceil)+1,\dots\lceil l/B \rceil} U_j \cup B_{\lceil l/Bk \rceil}$. The length of this final path decomposition is at least $l/Bk \geq (Bk)^{B-1} \geq (bk)^b$. Thus by induction, G admits a strongly linked path decomposition of length k, proving the claim. □

8.5 *Proof of Theorem 8.4.1*

As a first step in the proof, we see that in fact we may deal with the case when we have a long path decomposition of the graph. Consider a non-trivial tree-decomposition (T, W) of the graph. We will first see that T has bounded degree. Assume there exists a vertex x of T with $\deg_T(x) = D \geq p\binom{B}{p}$. Then if we consider $G - W_x$, the graph must have at least $p\binom{B}{p}$ components, at least one arising from each component of $T - x$ by Observation 8.4.1. Let $v_1, \dots, v_D$ be distinct vertices such that each v_i lies in a distinct component of $G - W_x$. Then for every $i = 1, \dots D$, there exist p internally disjoint paths $P_i^1, \dots, P_i^p$ from v_i to some set of p vertices in W_x. Moreover, since each v_i and v_j lay in distinct components for $i \neq j$, we see that no P_i^l intersects P_j^k for all indices l and k except possibly at their endpoints in W_x. However, since $|W_x| \leq B$, and by our lower bound on D, there exist p different v_i's whose corresponding paths have their ends in the same set of p vertices in W_x. Then the union of those paths gives a $K_{p,p}$ topological minor which clearly contains K_p as a minor. It follows that the degree of every vertex x in T is bounded above by $p\binom{B}{p}$.

The following observation will prove that T contains a long path.

Observation 8.5.1 *Every graph H on n vertices with maximum degree Δ has a path of* *length at least* $\log n / \log \Delta$*.*

Proof: If a graph has degree at most Δ, then there are at most $\Delta(\Delta-1)^{(r-1)}$ vertices at distance exactly r from a given vertex v. It follows that there are at most $(\Delta-1)^r\Delta \leq \Delta^r$ vertices of distance at most r from v. For any vertex x in the graph, there exists a vertex y at distance at least $\log n/\log \Delta$ from x. A path connecting x and y satisfies the claim.□

Returning now to our tree-decomposition (T, W), we see that T must have at least $|V(G)|/B$ vertices. Since the degree of every vertex in T is at most $p\binom{B}{p}$, the graph T must have as a subgraph a path P of length at least

$$\frac{\log(|V(G)|/B))}{\log\left(p\binom{B}{p}\right)}.$$

Let N_1 be the length of P. We get a path decomposition (P, W') of G as follows. Consider the components of $T - E(P)$. Each component contains exactly one vertex on P. For notation, for every vertex v in P, let T_v be the subtree of $T - E(P)$ containing v. Then we define the W' of the path decomposition (P, W') as follows. For every $v \in V(P)$ let $W'_v = \bigcup_{x\in V(T_v)} W_x$. Then (P, W') is a path decomposition. Moreover, for any edge xy of P, any vertex lying in $W'_x \cap W'_y$ must lie in W_x as well. Thus we see that even though the width of the path decomposition (P, W') may have become very large due to merging many W_x from (T, W), the adhesion of (P, W') is bounded by B.

Apply Lemma 8.4.1 to the path decomposition (P, W') to get a strongly linked non-trivial path decomposition of length at least $(B + 1)(p\binom{B}{p} + 1)$. By possibly repeatedly merging together as many as $B + 1$ adjacent vertices of the path, we get a strongly linked path decomposition (Q, U) of length N where $N \geq p\binom{B}{p}+1$ such that if we label the vertices of Q by $v_0, v_1, \dots, v_N$ where v_i is adjacent to v_{i+1} for $i = 0, \dots, N-1$, we may additionally assume that for every $i = 1, \dots, N - 1$, $U_{v_i} - U_{v_{i-1}} - U_{v_{i+1}}$ is non-empty. This is possible by our assumption that G satisfies

$$\frac{\log(|V(G)|/B))}{\log\left(p\binom{B}{p}\right)} \geq \left[B(B+1)\left(p\binom{B}{p}+1\right)\right]^B$$

We index the sets of U to simplify notation so that $U_i = U_{v_i}$. Let the adhesion of (Q, U) be $d \leq B$. There exists a linkage $\mathcal{R}$ with d components from from U_0 to U_N. Let the paths in the linkage be labeled $R_1, \dots, R_d$. The indices on the paths $R_1, \dots, R_d$ induce a labeling of the vertices of $U_i \cap U_{i+1}$ for all i. Let x_i^j be the unique vertex lying in $V(R_i) \cap U_j \cap U_{j+1}$.

We will now see that there exists some set of p paths among $R_1, \dots, R_d$ that we can expand to form the branch sets of a K_p minor.

Given a linkage $\mathcal{L}$ in a graph G, an *anchor* for the linkage $\mathcal{L}$ is a set of vertices Z disjoint from $V(\mathcal{L})$ such that $G[Z]$ is a connected subgraph, and every component of $\mathcal{L}$ has a neighbor in Z.

Definition 24 *We will say a graph H is* p-secured *if for any sets $X, Y \subseteq V(H)$ with $|X| = |Y| = p$, there exists a linkage $\mathcal{L}$ in H from X to Y and an anchor Z for $\mathcal{L}$.*

Observation 8.5.2 *Every $(20p + 1)$-connected graph H is p-secured.*

Proof: Let X and Y be two sets of p vertices, and label the vertices of X and Y as $x_1, \dots, x_p$ and $y_1, \dots, y_p$ respectively such that for all indices i with $x_i \in X \cap Y$, $x_i = y_i$. There exists a vertex $v \in V(H) - X - Y$. Since v has degree at least $20p + 1$, it clearly has p distinct neighbors $z_1, \dots, z_p$ not contained in $X \cup Y$. Again, by the high minimum degree of H, we can pick a distinct neighbor z_i' of z_i for all indices $i = 1, \dots, p$ such that z_i' is also not contained in $X \cup Y$. Let $\mathcal{I}$ be the indices for which $x_i \neq y_i$. Then because $H - v$ is $20p$-connected, it must be $2p$-linked by Corollary 1.4.1. It follows that there exists a linkage with components $\{P_i : i \in \mathcal{I}\} \cup \{Q_i : i \in \mathcal{I}\} \cup \{R_i : i \notin \mathcal{I}\}$ such that the ends of P_i are x_i and z_i for all $i \in \mathcal{I}$. The ends of Q_i are z_i' and y_i for all $i \in \mathcal{I}$. And finally, the ends of R_i are x_i and z_i for all $i \notin \mathcal{I}$.

If we let $Z = \{v\} \cup (\bigcup_{i \notin \mathcal{I}} V(R_i) - \{x_i\})$, then Z anchors the linkage with components $x_i P_i z_i z_i' Q_i y_i$ for all $i \in \mathcal{I}$ and the trivial paths $x_i = y_i$ for $i \notin \mathcal{I}$, proving the claim. $\square$

Definition 25 *Let (Q, U) be our path decomposition above. Let $A \subseteq \{1, \dots, d\}$ be a set of size p. For a set U_i in the path-decomposition (Q, U), we will say that U_i is* A-good *if there* *exists a linkage $\mathcal{L}$ from $\{u_j^i : j \in A\}$ to $\{u_j^{i+1} : j \in A\}$ contained in $G[U_i]$ such that the following hold*

1. *For every component C of $\mathcal{L}$, the endpoints of C are the only vertices of C in $U_{i-1} \cap U_i$ and $U_i \cap U_{i+1}$.*

2. *There exists an anchor Z_i for the linkage such that $Z_i \subseteq U_i - U_{i-1} - U_{i+1}$.*

We now prove that every U_i is A-good for some set A of p vertices except for possibly U_0 and U_N. Let i be an arbitrary index in $\{1, \ldots, N-1\}$. Consider some vertex $v \in U_i - U_{i-1} - U_{i+1}$. If v has $p+1$ neighbors in $U_i \cap U_{i-1}$, then v is adjacent at least p paths among the $R_1, \ldots, R_d$ not containing v. Thus the paths from $\mathcal{R}$ restricted to U_i satisfy the definition of A-good with $\{v\}$ serving as the anchor.

Otherwise, we see that any such vertex v has at most $2p+1$ neighbors in $(U_i \cap U_{i-1}) \cup (U_i \cap U_{i+1})$. Thus the graph $G[U_i - U_{i-1} - U_{i+1}]$ has minimum degree at least $80p+4$. It follows from Theorem 1.3.6 that $G[U_i - U_{i-1} - U_{i+1}]$ contains a $20p+1$ connected subgraph H_i. Applying Observation 8.5.2, we see that H_i is p-secured.

We now apply Lemma 5.3.1 from Chapter 5 to see that there exists an H_i comb with $2p$ components to the linkage $\mathcal{R}$ restricted to U_i. Let $\mathcal{S}$ be the H_i comb. Label the components of $\mathcal{S}$ $S_1, \ldots, S_{2p}$ such that there exists an even integer m where for all indices j satisfying $m < j \leq 2p$, the path S_j has its terminus on R_k for some $R_k \in \mathcal{R}$ and S_j is the only component in $\mathcal{S}$ with a terminus in R_k. For $j \leq m$, if j is odd and S_j has its terminus on R_k, then the terminus of S_j is left extremal, and more over, S_{j+1} also has its terminus on R_k and the terminus of S_{j+1} is the right extremal vertex.

If we look at the origins of the components of $\mathcal{S}$, we can naturally define two sets X' and Y' of p vertices as follows.

$$\begin{aligned} X' :=& \{\text{the origins of } S_j \text{ for } j \leq m \text{ and } j \text{ odd}\} \cup \\ & \cup \{\text{the origins of the } S_j \text{ such that } m < j \leq p - m/2\} \end{aligned}$$

So X' is p origins of paths terminating in left $\mathcal{S}$ extremal vertices. Similarly, we define Y'

as follows.

$$\begin{aligned} Y' :=& \{\text{the origins of } S_j \text{ for } j \leq m \text{ and } j \text{ even}\} \cup \\ & \cup \{\text{the origins of the } S_j \text{ such that } m < j \leq p - m/2\} \end{aligned}$$

By the fact that H_i is p-secured, we see there exists a linkage in H_i from X' to Y' and an anchor for the linkage. We now will extend that linkage to a linkage in $G[U_i]$ and with an anchor.

Let the origin of S_j be s_j and let it's terminus in $\mathcal{R}$ be t_j for all indices j. Let $\mathcal{L}$ be the linkage in H_i from X' to Y' with the component L_{1+2j} connecting s_{1+2j} to s_{2+2j} for $j = 0, \ldots, m/2 - 1$. Let L_j be the trivial path s_j for $j \geq m$. And let Z_i be the anchor for $\mathcal{L}$.

Let the map π be defined such that $\pi(j) = k$ if and only if the terminus of S_j is in $R_k \in \mathcal{R}$. We will define the linkage $\overline{\mathcal{L}}$ with components $\overline{L}_{\pi(1)}, \overline{L}_{\pi(3)}, \ldots, \overline{L}_{(m-1)}, \overline{L}_{\pi(m+1)}$, $\overline{L}_{\pi(m+2)}, \ldots, \overline{L}_{\pi(p-m/2)}$. In other words, the components of $\overline{\mathcal{L}}$ are labeled by the same indices as the paths of $\mathcal{R}$ attached to by the linkage $\mathcal{S}$. Now consider the linkage

$$\bigcup_{j \in \{1,\ldots,m\}, j \text{ odd}} \overline{L}_{\pi(j)} = x^{i-1}_{\pi(j)} R_{\pi(j)} t_j S_j s_j L_j s_{j+1} S_{j+1} t_{j+1} R_{\pi(j)} x^i_{\pi(j)}$$

$$\bigcup_{j = m+1, m+2, \ldots, p-m/2} \overline{L}_{\pi(j)} = x^{i-1}_{\pi(j)} R_{\pi(j)} x^i_{\pi(j)}$$

See Figure 16. Now we let $\overline{Z_i} = Z_i \cup \bigcup_{j=m+1,\ldots,p-m/2} V(S_j) - t_j$. The set of vertices $\overline{Z_i}$ is a anchor for the linkage $\overline{\mathcal{L}}$. Thus we have proven that $G[U_i]$ is A-good for the set $A = \{\pi(1), \pi(3), \ldots, \pi(m-1), \pi(m+1), \pi(m+2), \ldots, \pi(p-m/2)\}$, as desired.

We have seen that every U_i for $i = 1, \ldots, N-1$ is A-good for some set of p indices in $\{1, \ldots, d\}$. Because we have chosen our path decomposition sufficiently long, there exists some particular set A of p indices such that at least p distinct U_i's are A-good. Then focusing on those p paths in $\mathcal{R}$, we can reroute the paths to find p pairwise disjoint anchors for those specific p paths. Thus we have constructed the branch sets of a $K_{p,p}$ in G which contains K_p as a minor. This completes the proof of Theorem 8.4.1.

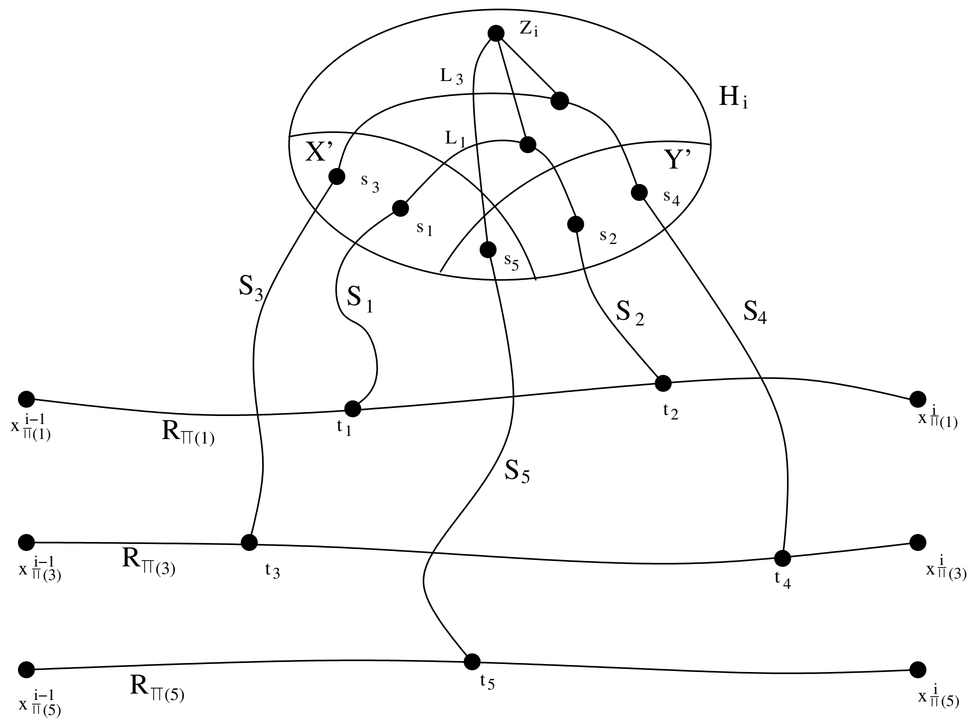

Figure 16: Finding a linkage and an anchor in $G[U_i]$.

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