ROBUST CONTROL IN STATE SPACE

by

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SUMMARY

We consider the problem of robustness, in particular that of robust stability. Such a problem is amenable to analysis by frequency domain techniques, and also using state space methods. Using some recent state space theory yielding the exact radius of the ball around a nominally stable system within which all additive perturbations retain stability, we show how control action may be implemented to increase the radius of this ball.

We present further some material on how destabilizing perturbations may be constructed from solutions of Riccati equations, and how the above mentioned radii may be found with respect to an alternative norm to the one used above. Finally we give some remarks on the use of Lyapunov functions for systems.

CONTENTS

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TO MY PARENTS

0. INTRODUCTION

There has recently been much work published in the control literature concerned with the problem of measuring system "robustness" (Doyle, Stein, Zames, Athans, Owens et al). Of the many performance criteria for a system, two in particular have received much attention - namely how robust the system is in withstanding disturbances and, secondly, how robust is the stability of the system. Typically the approach by authors has been to analyse these problems using frequency domain techniques - a large percentage of the control literature is devoted to the various characterizations that have arisen. The volume of literature on this topic is so large that to present a complete view of the various schemes that have arisen would be an extremely long task especially as the material is still increasing at a very large rate. We thus, in Chapter 1, give some idea of the progress that has been made in this direction, and outline the now famous H^{∞}-approach to design.

In this thesis we, however, emphasize a different viewpoint to that given above in that we analyse the question of robustness of the stability of a system in a finite-dimensional state space framework, concentrating especially on the concepts of real and complex stability radii that have recently been introduced - to this end we give an overview of the various definitions and results that concern these in Chapter 2. The question of robust stability in the state space is not new and papers have appeared on this subject for a number of years.

A problem of equal importance to the one of measuring system robustness is that of choosing assignable system parameters in order to render the overall system more robust - normally this is to be achieved by the choice of some feedback operator F in the system as is illustrated in the H^{∞}-design approach outlined in Chapter 1. In Chapter 3 we show how one of the measures of robustness defined in Chapter 2 may be enhanced using state feedback - thus showing that robustness improvement may be achieved also in the state space formulation. In general the idea of using some form of feedback to enhance the properties of the system is by no means new, thus the importance of having a "robust" system leads naturally to the use of feedback in improving that robustness, possibly ahead of other design considerations.

In Chapter 4 we make some further remarks concerning the concepts of stability radii introduced in Chapter 2 and their relation to symmetric and nonsymmetric solutions of an algebraic Riccati equation. Furthermore, bearing in mind that the concepts of Chapter 2 are introduced with respect to a particular norm on the space of matrices, we show how these stability radii may be calculated when an alternative norm is used. Of course, all norms on finite-dimensional vector spaces are equivalent however the choice of the particular norm used might be important in practice. As a final topic in this chapter we make some remarks on Lyapunov functions for asymptotic stability and for boundedness of solutions.

Finally, in Chapter 5 we give some conclusions to the foregoing work.

1. FREQUENCY DOMAIN ROBUSTNESS ANALYSIS

1.1 INTRODUCTION

Numerous papers have appeared addressing the robustness issue using frequency domain techniques - [1], [2], [4], [5], [12], [16], [22], [23] are but just a few of these. The classical means by which robustness of system stability was characterized in the single-input/single-output case were the concepts of gain and phase margin. So for the "unity negative feedback" system given by figure 1 with plant $G(s)$, multiplicative uncertainty $L(s)$ (whose nominal value is $L(s) = 1$)

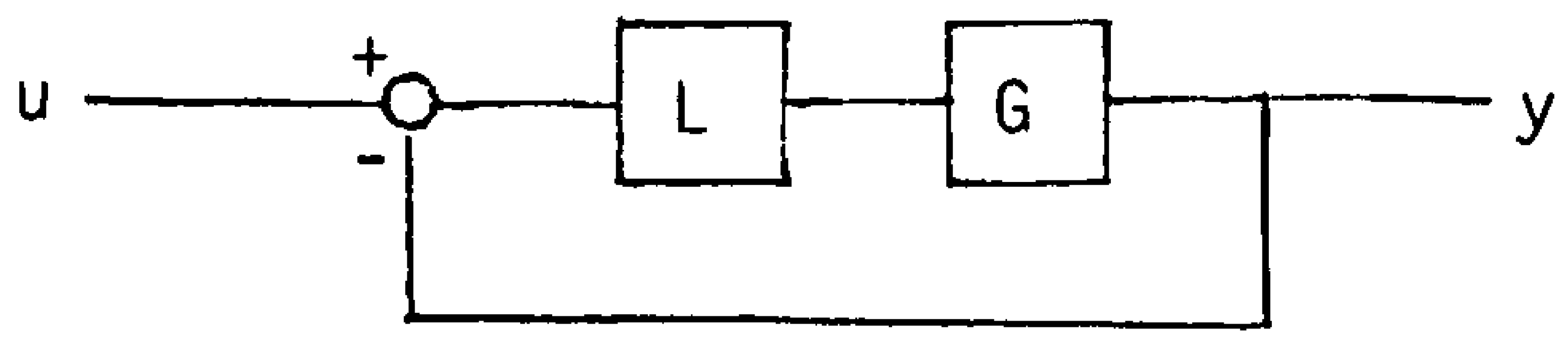


Figure 1

the positive phase margin is the smallest $\phi > 0$ such that the system of figure 1 with $L(i\omega) = e^{i\phi}$ is just unstable. The upward gain margin is the smallest $c > 1$ such that $L(s) = c$ gives rise to instability. (The concepts of negative phase margin and downward gain margin have obvious definitions). These concepts still carry much favour in electrical engineering circles and are usually some of the yardsticks used in the specification for the design of a control system. In section 1.3 we show how the robustness of system stability may be dealt with in the multivariable situation. We first give some preliminary definitions and theory - these will prove to be of great value in section 1.4.

1.2 PRELIMINARIES

In this section we outline some theory that is particularly related to the H^∞-design approach. We follow largely the outline given in [2], [24] - see also the books [3], [25]. A rational function of s is stable if it is analytic in $\text{Re } s \geq 0$, and whenever its coefficients are real it is termed real-rational. (A similar definition holds in the discrete-time system case). The following Hardy spaces for discrete-time systems are used:

H^2 is the space of scalar complex-valued functions $g(z)$ analytic in $|z| < 1$ and with the property

$$\sup_{0 \leq r < 1} \int_0^{2\pi} |g(re^{i\theta})|^2 . d\theta < \infty \quad .$$

Extending $g(z)$ to $|z| = 1$ (as can be done), H^2 becomes a Hilbert space with inner product

$$<g,h> = \frac{1}{2\pi} \int_0^{2\pi} \overline{g(e^{i\theta})}\; h(e^{i\theta}).d\theta \quad .$$

H^∞ is the subspace of H^2 of functions which are (as well) bounded, and norm defined by

$$||g||_\infty = \operatorname*{ess\,sup}_\theta |g(e^{i\theta})| \quad .$$

RH^∞ denotes the subspace of H^∞ of real-rational functions. H^2, H^∞ spaces of vectors and matrices may be defined:

$(H^2)^k$ is the space of k-dimensional vectors with H^2 entries - a Hilbert space with inner product

$$\langle g,h\rangle = \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})^* h(e^{i\theta}).d\theta \quad .$$

The Hilbert space $(H^2)^{k\times\ell}$ is the space of $k\times\ell$ matrices with H^2 entries, its inner product being:

$$\langle G,H\rangle = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}\ \ G(e^{i\theta})^* H(e^{i\theta}).d\theta \ .$$

The norm of G in the subspace $(H^\infty)^{k\times\ell}$ of $(H^2)^{k\times\ell}$ is defined by:

$$||G||_\infty = \{\operatorname*{ess\,sup}_\theta \rho[G(e^{i\theta})^* G(e^{i\theta})]\}^{\frac{1}{2}}$$

where $\rho(M)$ is the spectral radius of the matrix M (that is the largest eigenvalue of M). It is a result that

$$||G||_\infty = \sup\{||Gh||_2 : h \in (H^2)^\ell \ , \ ||h||_2 = 1\} \quad .$$

We introduce next the concepts of inner and outer functions and matrices. A rational function g in H^∞ is inner if $|g(e^{i\theta})| = 1$ for all real θ , and outer if $g(z) \neq 0$ for all $|z| < 1$. For matrices we have that a rational matrix G in $(H^\infty)^{k\times\ell}$ is inner if

$$G(e^{i\theta})^* \ G(e^{i\theta}) = I \quad \text{for all} \quad \theta \quad ,$$

or equivalently $||Gh||_2 = ||h||_2$ for all h in $(H^2)^\ell$. We note that necessarily, from the first definition, $k \geq \ell$. A property arising from this which is important in what follows is that if G is a square inner matrix then $\det G$ is a scalar inner function and $\det G(G)^{-1}$ is a square inner matrix. Similarly a rational matrix G in $(H^\infty)^{k\times\ell}$ is outer if

$$\text{rank } G(z) = k \quad \text{for all } |z| < 1 .$$

Necessarily this means that $k \leq \ell$.

Rational matrices with entries in H^∞ may be factored in terms of inner and outer matrices. If G is a rational matrix in $(H^\infty)^{k\times\ell}$ then

$$G = G_i G_o$$

with G_i inner, and G_o outer with the factorization unique up to multiplication by constant unitary matrices. Furthermore if G' is square it may be factored in the form

$$G' = G_o' G_i' .$$

A square matrix in RH^∞ is said to be unimodular if it is invertible over RH^∞ (it is invertible and its inverse is in RH^∞). All matrices F in RH^∞ can be expressed as

$$F = F_1 F_2 F_3$$

with F_1, F_3 unimodular and F_2 in RH^∞ and of the form

$$F_2 = \left[\begin{array}{ccc|c} \alpha_1 & & & \\ & \ddots & & 0 \\ & & \alpha_k & \\ \hline & 0 & & 0 \end{array}\right]$$

with α_{i+1}/α_i in RH^∞. The α_i are the invariant factors of F and are unique up to multiplication by invertible elements of RH^∞. F_2 is called the Smith form of F.

The H^∞-norm of a proper stable (analytic in the closed right half plane) rational matrix $G(s)$ is defined by

$$||G||_\infty = \{\sup_\omega \rho[G(i\omega)^* G(i\omega)]\}^{\frac{1}{2}} .$$

1.3 ROBUSTNESS OF STABILITY

We show in this section how robustness of system stability may be characterized in the frequency domain. We consider, [1], the system given in figure 2 as a typical representation.

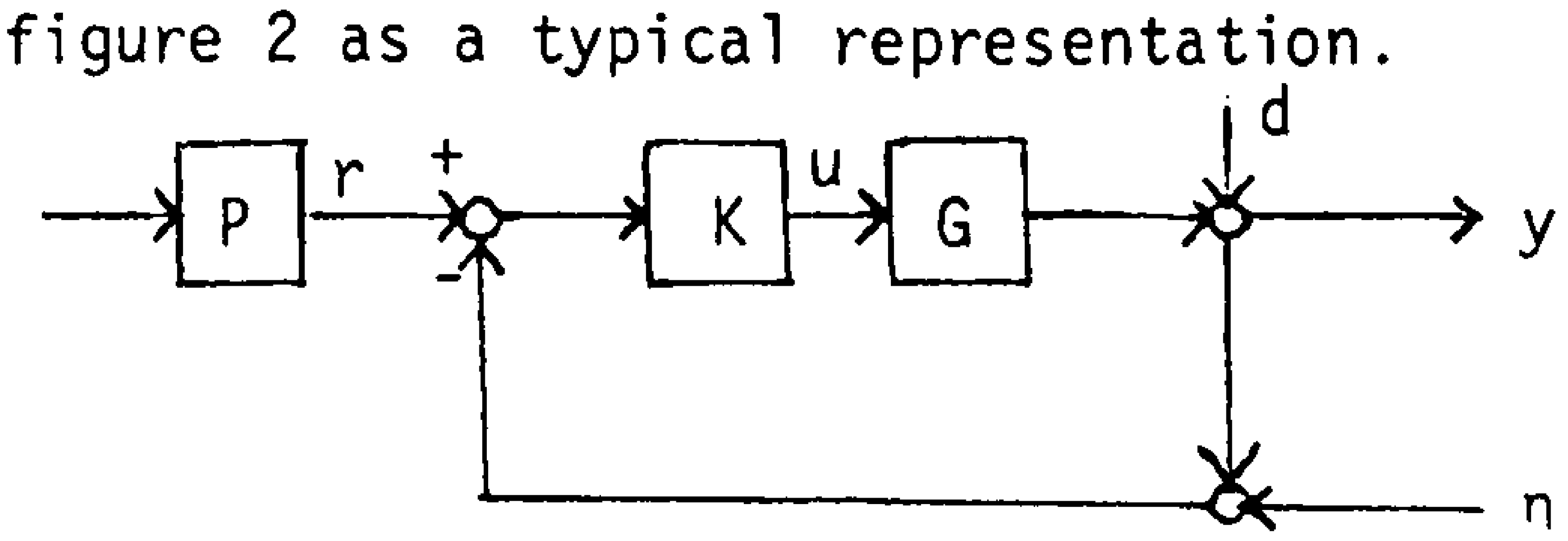


Figure 2

Here G represents the plant, K the controller, r the input, n the measurement noise, d the disturbance, P the precompensator, y the measured output. From the figure we see that

$$y = d + Gu$$

$$u = K(r-\eta-y)$$

so that eliminating u we have

$$y = (I + GK)^{-1}GK(r-\eta) + (I + GK)^{-1}d$$

which by using the identity $(I+M)^{-1}M = M(I+M)^{-1}$ is equivalent to

$$y = GK(I+GK)^{-1}(r-\eta) + (I+GK)^{-1}d \ .$$

Setting now $d = 0$, $\eta = 0$ and assuming the system stable we have

$$y = GK(I + GK)^{-1}r \ . \qquad (1.1)$$

Typically two types of perturbation to the plant G (due to say parameter changes, unmodelled dynamics) are considered, the additive type

$$G'(i\omega) = G(i\omega) + \Delta G(i\omega)$$

with $\bar{\sigma}[\Delta G(i\omega)] < \ell_a(\omega) \quad \forall\ \omega \geq 0$ (1.2)

or the multiplicative type

$$G'(i\omega) = [I + L(i\omega)]G(i\omega)$$

with $\bar{\sigma}[L(i\omega)] < \ell_m(\omega) \quad \forall\ \omega \geq 0$. (1.3)

If we take the definition of "robust stability" given by (1.3) then the authors show that the system is stable to all perturbations of the form (1.3) (assuming G, G', K are strictly proper and G, G' have the same number of unstable modes as each other) iff

$$\bar{\sigma}[GK(I+GK)^{-1}] < 1/\ell_m(\omega) \quad \forall\, \omega \geq 0 \,. \tag{1.4}$$

Here $\bar{\sigma}(\cdot)$ denotes the largest singular value of the matrix - we refer the reader to Chapter 2 where a short discussion on singular values is given.

Clearly an interesting question from our point of view of robustness enhancement is how the LHS of (1.4) may be minimized subject to system "design constraints". The paper [12] considers this question in the following manner - given G and prespecified diagonal elements c_i of the nominal closed loop system find K to minimize the LHS of (1.4) subject to this constraint. In fact, they show the answer is given by choosing K to diagonalize $(I+GK)^{-1}GK$ whenever this is possible. We refer the interested reader to [12] for details of this.

1.4 ROBUSTNESS IN WITHSTANDING DISTURBANCES - H^∞ - OPTIMIZATION

The paper [2] is one of a number showing how to design a feedback controller which minimizes the H^∞-norm of a weighted sensitivity matrix for a linear multivariable plant - thus enhancing disturbance attenuation as this norm is (see section 1.2) the induced norm between the H^2-spaces of disturbances and outputs (see also [24]). This norm may thus be taken as a measure of robustness in this respect. The analysis is done using a theory developed by Ball and Helton. In another paper [5] the authors have used the so called matrix Nevanlinna-Pick theory to solve a similar problem. Such a problem can be extended to include other features - for example in [16] the authors propose the minimization of a combination of the H^∞-norms of weighted sensitivity and complementary sensitivity matrices. This aim is in some sense achieved by the minimization of the H^∞-norm of a composite transfer function matrix which provides an upper bound on the original problem.

For the sake of discussion we outline [2], where the problem of minimizing the H^∞-norm of

$$X = W_1(I + PF)^{-1}W_2 \tag{1.5}$$

is considered. Here P represents the $n \times m$ plant, F represents the $m \times n$ feedback and W_1 , W_2 are square weighting matrices - all being real-rational. The quantity defined by (1.5) is the so called weighted sensitivity matrix and represents the transfer function from v to y in the following diagram.

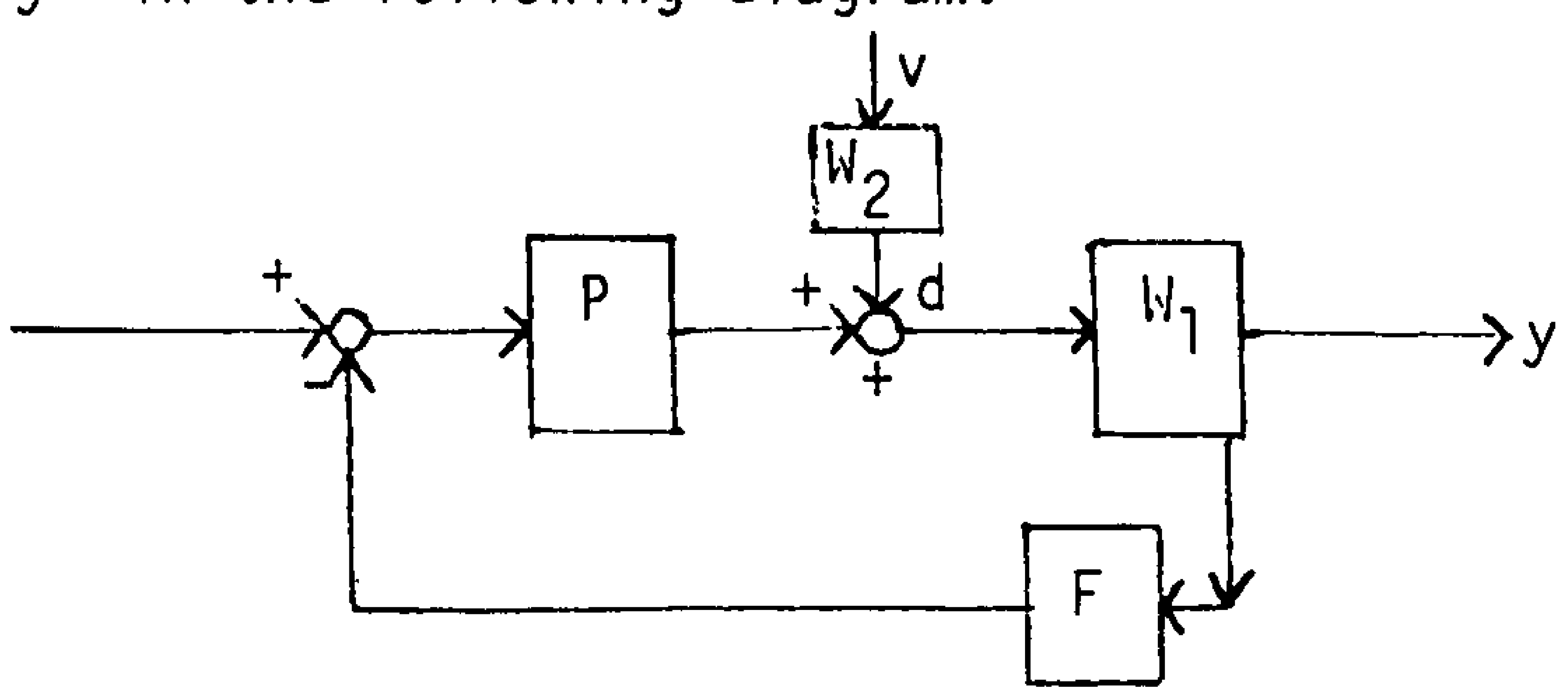


Figure 3

The following assumptions are made on W_1, W_2, P :

1) W_1, W_2, P are all real-rational,

2) P is strictly proper,

3) P has no poles on the imaginary axis and rank $P(i\omega) = n$ for all ω,

4) W_1, W_2 are nonsingular, proper, stable and they have stable inverses.

For (1.5) all real-rational proper matrices F achieving internal stability are firstly characterized. As P is real-rational there exist stable, proper, real-rational matrices A_1, A_2, B_1, B_2, Y_1, Y_2, Z_1, Z_2 such that

$$P = A_1 B_1^{-1} = B_2^{-1} A_2 \,, \tag{1.6}$$

$$Y_1 A_1 + Z_1 B_1 = I \,, \tag{1.7}$$

$$A_2 Y_2 + B_2 Z_2 = I \,, \tag{1.8}$$

and the set of F's is given by

$$F = (Y_2 + B_1 Q)(Z_2 - A_1 Q)^{-1} \tag{1.9}$$

Q stable, proper, real-rational.

Note that in the case where P is stable, we can take $A_1 = A_2 = P$, $B_1 = B_2 = I$, $Z_1 = Z_2 = I$, and $Y_1 = Y_2 = 0$. Using (1.6)-(1.9) in

(1.5) we can transform the problem. We have

$$X = W_1(I+P(Y_2+B_1Q)(Z_2-A_1Q)^{-1})^{-1}W_2$$

$$= W_1[(Z_2-A_1Q+PY_2+PB_1Q)(Z_2-A_1Q)^{-1}]^{-1}W_2 \quad .$$

Using $PB_1Q = A_1Q$ and $PY_2 = B_2^{-1}A_2Y_2 = B_2^{-1}(I-B_2Z_2) = B_2^{-1}-Z_2$ we have $X = W_1[B_2^{-1}(Z_2-A_1Q)^{-1}]^{-1}W_2$, so that

$$X = W_1(Z_2-A_1Q)B_2W_2 \ . \tag{1.10}$$

The original problem is thus converted from a nonlinear optimization problem in F to an affine optimization problem in Q .

We can thus define the original problem as:

$$\mu = \inf\{||W_1(Z_2-A_1Q)B_2W_2||_\infty : Q \text{ is stable, proper, real-rational}\} \ . \tag{1.11}$$

Alongside (1.11) another definition μ_1 is made (by relaxing the restraint that Q be proper),

$$\mu_1 = \inf\{||W_1(Z_2-A_1Q)B_2W_2||_\infty : Q \text{ is stable, real-rational}\} \ . \tag{1.12}$$

We say more regarding the quantity μ_1 in (1.12) subsequently.

The modified problem (1.12) is now transformed from the right half plane to the unit disc via the mapping

$$s \mapsto z = \frac{s - 1}{s + 1} \quad , \tag{1.13}$$

and a further transformation as follows. First set

$$W_1A_1 = (W_1A_1)_i(W_1A_1)_o \quad . \tag{1.14}$$

Now as B_2W_2 is square it can be factored

$$B_2W_2 = (B_2W_2)_o(B_2W_2)_i \quad . \tag{1.15}$$

Setting $\delta = \det(B_2W_2)_i$ then δ is a scalar inner function and $\delta(B_2W_2)_i^{-1}$ is an inner matrix so that multiplication by it preserves norm. So we have from (1.12)

$$||W_1(Z_2-A_1Q)B_2W_2\delta(B_2W_2)_i^{-1}||_\infty \quad ,$$

and from (1.14), (1.15)

$$||W_1Z_2B_2W_2\delta(B_2W_2)_i^{-1}-\delta(W_1A_1)_i(W_1A_1)_oQ(B_2W_2)_o||_\infty \quad .$$

Setting $\quad G = W_1Z_2B_2W_2\delta(B_2W_2)_i^{-1} \quad ,$ (1.16)

$$U = \delta(W_1A_1)_i \quad , \tag{1.17}$$

$$H = (W_1A_1)_oQ(B_2W_2)_o \quad , \tag{1.18}$$

then $G, U \in (RH^\infty)^{n\times n}$ and U is inner and (1.12) reduces to

$$\mu_1 = \inf\{||G-UH||_\infty : H \in (RH^\infty)^{n\times n}\} \quad . \tag{1.19}$$

From every matrix $H \in (RH^\infty)^{n\times n}$, $Q(z)$ may be found from (1.18) as $(W_1A_1)_0(z)$, $(B_2W_2)_0(z)$ have rank n for $|z| \le 1$ - though possibly not $z = 1$ corresponding to $|s| = \infty$. $Q(z)$ has no poles in $|z| \le 1$ though possibly at $z = 1$, i.e. $Q(s)$ is stable, real-rational, but not necessarily proper.

The development continues by showing how to find all H's which achieve the infimum in (1.19) using the Ball-Helton theory.

We note the following alternative characterization of the quantity μ_1 in (1.12). We let $\mathcal{U}$ denote the subspace $U(H^2)^n$ where U is given by (1.17), and $\mathcal{U}^\perp$ denote its (closed) orthogonal complement in $(H^2)^n$ so that

$$(H^2)^n = \mathcal{U} \oplus \mathcal{U}^\perp \quad .$$

(We note that $\mathcal{U}$ is closed under the initial assumptions). Then one can define the bounded linear operator Γ as follows, with $P_{\mathcal{U}^\perp}$ denoting the projection operator from $(H^2)^n$ to $\mathcal{U}^\perp$.

$$\Gamma : (H^2)^n \to \mathcal{U}^\perp \quad ,$$
$$\Gamma f = (P_{\mathcal{U}^\perp}G)(f) \quad . \tag{1.20}$$

Now define (the H^∞-norm of Γ)

$$\nu = ||\Gamma|| = \sup\{||\Gamma f||_2 : f \in (H^2)^n \ , \ ||f||_2 = 1\} \ . \tag{1.21}$$

Then we have for any $f \in (H^2)^n$, $H \in (RH^\infty)^{n\times n}$

$$(G-UH)f = P_{\mathcal{U}}(G-UH)f + P_{\mathcal{U}^\perp}(G-UH)f$$

$$= P_{\mathcal{U}}(G-UH)f + P_{\mathcal{U}^\perp}Gf$$

$$= P_{\mathcal{U}}(G-UH)f + \Gamma f \quad .$$

By orthogonality

$$||(G-UH)f||_2^2 = ||P_{\mathcal{U}}(G-UH)f||_2^2 + ||\Gamma f||_2^2$$
$$\geq ||\Gamma f||_2^2$$

so that

$$||G-UH||_\infty \geq ||\Gamma||$$

and so by definition

$$\mu_1 \geq \nu \quad . \tag{1.22}$$

Furthermore, under the basic assumptions, the equality in (1.22) holds:

Theorem 1.1 (Lemma 1, [2])

The infimum $\mu_1 = \inf\{||G-UH||_\infty : H \in (RH^\infty)^{n\times n}\}$ is achieved, and $\mu_1 = \nu$.

Considering the definition of Γ in equation (1.20), we have for any $f \in (H^2)^n$

$$Gf = \Gamma f + U(\alpha g) \ , \quad \text{for some} \quad \alpha \in \mathbb{C} \ , \ g \in (H^2)^n \ ,$$

whose values we now determine. We recall that U is an inner matrix.

We thus have

$$||\Gamma f||_2^2 = <\Gamma f,\Gamma f> = <Gf-\alpha Ug,Gf-\alpha Ug>$$

$$= ||Gf||_2^2-\alpha<Ug,Gf> - \bar{\alpha}<Gf,Ug> + |\alpha|^2||Ug||_2^2$$

$$= ||Gf||_2^2 - 2\ \mathrm{Re}(\alpha<Ug,Gf>) + |\alpha|^2||g||_2^2 \ .$$

Setting $\alpha = \alpha_R + i\alpha_I$, $<Gf,Ug> = \beta_R + i\beta_I$ we have

$$||Gf||_2^2 - 2(\alpha_R\beta_R+\alpha_I\beta_I) + (\alpha_R^2+\alpha_I^2)||g||_2^2 \ ,$$

so that differentiating w r t α_R, α_I for the minimum

$$\alpha_R = \beta_R \ / \ ||g||_2^2 \ , \ \alpha_I = \beta_I \ / \ ||g||_2^2 \ .$$

Thus

$$||\Gamma f||_2^2 = ||Gf||_2^2 - \frac{1}{||g||_2^2}(\beta_R^2 + \beta_I^2)$$

$$= ||Gf||_2^2 - |<Gf,Ug/||g||_2>|^2 \ .$$

Without loss of generality we can take g s.t. $||g||_2 = 1$, so that

$$||\Gamma f||_2^2 = ||Gf||_2^2 - |<U^x Gf,g>|^2$$

with the latter inner product in L^2 , and the function $U^x(\cdot)$ defined by

$$U^x(z) = U^T(1/z) \ .$$

Thus we have, using orthogonality,

$$||\Gamma f||_2^2 = ||Gf||_2^2 - |\langle P_{(H^2)^n} U^x Gf, g\rangle|^2$$

from which the required projection is obtained by taking

$$g = P_{(H^2)^n} U^x Gf / ||P_{(H^2)^n} U^x Gf||_2 \quad .$$

Thus

$$||\Gamma f||_2^2 = ||Gf||_2^2 - ||P_{(H^2)^n} U^x Gf||_2^2 \; .$$

Hence we have:

<u>Proposition 1.2</u>

$$\nu^2 = \sup_{||f||_2=1} \{||Gf||_2^2 - ||P_{(H^2)^n} U^x Gf||_2^2\}$$

where the map $U^x(\cdot)$ is defined by $U^x(z) = U^T(1/z)$.

We give now an explicit means by which ν of (1.21) may be calculated, following the approach of the paper. We take $g \in H^2$, $a \in \mathbb{C}$ with $|a| < 1$. If $g(a) = \ldots = g^{(k-1)}(a) = 0$ then g has a zero at a of order at least k . The converse of this is true, since by Cauchy's theorem

$$g^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{g(z).dz}{(z-a)^{n+1}} \; , \; n \geq 0 \; , \qquad (1.23)$$

where C is the unit circle covered anticlockwise. The expression (1.23) can be rewritten as an H^2-inner product by setting $z = e^{i\theta}$, so that $dz = ie^{i\theta}.d\theta$, hence

$$g^{(n)}(a) = \frac{1}{2\pi}\int_0^{2\pi} \frac{n!\, e^{-in\theta} g(e^{i\theta}).d\theta}{(1-ae^{-i\theta})^{n+1}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \overline{\frac{n!\, e^{in\theta}}{(1-\bar{a}e^{i\theta})^{n+1}}}\, g(e^{i\theta}).d\theta = \langle \frac{n!\, z^n}{(1-\bar{a}z)^{n+1}}, g(z)\rangle = \langle f_n, g\rangle ,$$

so that g has a zero at a of order at least k iff f_n and g are orthogonal, $n = 0,\ldots,k-1$, where

$$f_n(z) = \frac{n!\, z^n}{(1-\bar{a}z)^{n+1}} . \qquad (1.24)$$

If $U \in RH^\infty$, $U : H^2 \to H^2$, set $\mathcal{U} = U(H^2)^n$. Now an H^2-function belongs to $\mathcal{U}$ iff its set of zeros in the open unit disc contains those of U in the open unit disc. Thus a basis for $\mathcal{U}^\perp$ is given by $f_n(z)$ of (1.24), where a covers all distinct zeros of U in $|z| < 1$ and n goes from zero to one less than the multiplicity of the zero.

In the case where $U \in (RH^\infty)^{p\times m}$ and rank $U(e^{i\theta}) = p$ for all θ (necessarily $p \le m$), we have $[F_2,0]$ (the Smith form of U) satisfying

$$U = F_1[F_2,0]F_3$$

with $F_2 = \mathrm{diag}(\alpha_1,\ldots,\alpha_p)$. Now $\mathcal{U} = U(H^2)^m = F_1F_2(H^2)^p$. Define

$Y = F_2(H^2)^p$, so that $x \in Y$ iff the set of zeros of its i^{th} component in $|z| < 1$ contains those of α_i in $|z| < 1$, $(i = 1,\dots,p)$. As above, a basis for $Y^\perp$ is

$$f_n(z) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{1.25}$$

with the entry 1 in the i^{th} row, and $f_n(z)$ defined by (1.24). The integer i goes from 1 to p , a covers all distinct zeros of α_i in $|z| < 1$, n goes from zero up to one less than the multiplicity of a , i.e. the dimension of $Y^\perp$ is the number of zeros of the product $\alpha_1 \dots \alpha_p$ in $|z| < 1$ (counted according to multiplicity). Now the subspaces U, Y are isomorphic to each other (as F_1 is unimodular), consequently so are $U^\perp$, $Y^\perp$ so that they have the same dimension. A basis for $U^\perp$ is given by vectors of the form

$$P_{(H^2)^p} F_1^{-1x} g \tag{1.26}$$

where the vectors g span $Y^\perp$, and $F_1^{-1x}(z) = F_1^{-1T}(1/z)$.

We then have that if $\{f_i : i = 1,\dots,r\}$ is a basis for $U^\perp$ of (1.20) we set Φ_1, Φ_2 to be the Hermitian matrices whose elements are $\langle f_i, f_j \rangle$, $\langle \Gamma^* f_i, \Gamma^* f_j \rangle$ respectively. If $f \in U^\perp$ then

$$f = \sum_{i=1}^{r} \alpha_i f_i \ .$$

Setting $\alpha = [\alpha_1, \ldots, \alpha_r]^T$ we have $\langle f,f \rangle = \alpha^* \Phi_1 \alpha, \langle \Gamma^* f, \Gamma^* f \rangle = \alpha^* \Phi_2 \alpha$. Now $||\Gamma||^2 = ||\Gamma^*||^2 = \max\{||\Gamma^* f||_2^2 : f \in U^\perp, ||f||_2 = 1\}$ $= \max\{\alpha^* \Phi_2 \alpha : \alpha^* \Phi_1 \alpha = 1\}$, where Γ^* denotes the adjoint of Γ . Setting up the Lagrangian (using ν^2 as a multiplier for reasons which will become evident)

$$L = \alpha^* \Phi_2 \alpha - \nu^2(\alpha^* \Phi_1 \alpha - 1) ,$$

we have the first order conditions

$$\Phi_2 \alpha - \nu^2 \Phi_1 \alpha = 0 ,$$

$$\alpha^* \Phi_1 \alpha = 1 ,$$

so that $\alpha^* \Phi_2 \alpha = \nu^2$. Consequently $||\Gamma|| = \nu$ where ν is the maximum value of the eigenproblem

$$\det(\Phi_2 - \nu^2 \Phi_1) = 0 . \tag{1.27}$$

The relaxation of the requirement that Q be proper in (1.11) under certain conditions has no effect on the value μ . Specifically if $Q(s)$ is any solution of the modified problem (1.12) then we can choose an integer K so that $Q(s)/s^K$ is proper. The following sequence $F_n(s)$ may be defined:

$$F_n = (Y_2 + B_1 Q_n)(Z_2 - A_1 Q_n)^{-1} ,$$

$$Q_n = J_n Q ,$$

$$J_n(s) = [n/(s+n)]^K .$$

The first observation to be made is that $Q_n(s)$ is proper and stable so that from (1.9) $F_n(s)$ is proper and achieves internal stability of the system. The second observation is that for n sufficiently large $Q_n(s)$ approximates $Q(s)$ at low frequencies. This indicates that the sequence so defined should be near optimal if the weighting matrices $W_1(s)$, $W_2(s)$ are "small" at high frequencies. This is in fact the case, as given by the following result:

Theorem 1.3 (Theorem 3, [2])

If the product W_1W_2 is strictly proper, then $\mu = \mu_1$ and

$$\lim_{n\to\infty} ||W_1(I + PF_n)^{-1}W_2||_\infty = \mu \quad .$$

Having outlined the above approach we make some comments on our main problem of interest which will be the subject of the following two chapters, that is of choosing a static feedback controller F so as to stabilize $A-DF$ and simultaneously minimize the quantity

$$||C(sI-A+DF)^{-1}B||_\infty \quad . \tag{1.28}$$

We observe (1.28) can be written as

$$||C[(sI-A)(I+(sI-A)^{-1}DF)]^{-1}B||_\infty$$

$$= ||C(I+(sI-A)^{-1}DF)^{-1}(sI-A)^{-1}B||_\infty \quad ,$$

so that we have:

Proposition 1.4

The quantity $||C(sI-A+DF)^{-1}B||_\infty$ can be written in the form $||W_1(I+PF)^{-1}W_2||_\infty$ by setting $W_1 = C$, $P = (sI-A)^{-1}D$, $W_2 = (sI-A)^{-1}B$.

This means that under certain conditions we may apply the above theory to minimize the value of (1.28). This, in general, will produce dynamic feedbacks which achieve internal stability.

We consider the following simple numerical example with A (asymptotically stable), B, C, D as follows:

$$A = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix} , \quad B = C = I_2 , \quad D = \begin{bmatrix} 1 \\ 0 \end{bmatrix} .$$

So we have $W_1 = I_2$, $W_2 = (sI-A)^{-1}$, $P = (sI-A)^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We note assumption 3) does not hold so that more care need be exercised. As P is stable we set $A_1 = A_2 = P$, $B_2 = Z_2 = I_2$, $B_1 = Z_1 = 1$, $Y_1 = Y_2 = [0,0]$. Thus

$$W_1 = I_2 ,$$

$$W_2 = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s-1 & 2 \\ -3 & s+4 \end{bmatrix} ,$$

$$P = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s-1 \\ -3 \end{bmatrix} .$$

We now transform from the right half s-plane to the unit disc of the z-plane via $s \mapsto z = (s-1)/(s+1)$. This gives

$$W_1 = I_2 ,$$

$$W_2 = \frac{1}{[(1+z+4(1-z))(1+z-(1-z))+6(1-z)^2]}\begin{bmatrix}(1+z)(1-z)-(1-z)^2 & 2(1-z)^2 \\ -3(1-z)^2 & (1+z)(1-z)+4(1-z)^2\end{bmatrix}$$

$$= \frac{1}{6-2z}\begin{bmatrix}2z(1-z) & 2(1-z)^2 \\ -3(1-z)^2 & (1-z)(5-3z)\end{bmatrix} ,$$

$$P = \frac{1}{6-2z}\begin{bmatrix}2z(1-z) \\ -3(1-z)^2\end{bmatrix} .$$

Note $\det W_2 = (1-z)^2/(6-2z)$. We factorize $B_2W_2 = (B_2W_2)_o(B_2W_2)_i$, so can take $(B_2W_2)_i = I_2$,

$$(B_2W_2)_o = \frac{1}{6-2z}\begin{bmatrix}2z(1-z) & 2(1-z)^2 \\ -3(1-z)^2 & (1-z)(5-3z)\end{bmatrix} .$$

Similarly for $W_1A_1 = (W_1A_1)_i(W_1A_1)_o$, we have

$$W_1A_1 = \frac{1}{6-2z}\begin{bmatrix}2z(1-z) \\ -3(1-z)^2\end{bmatrix} = \begin{bmatrix}2az \\ -3a(1-z)\end{bmatrix}\frac{(1-z)}{(6-2z)a}$$

where we choose a so as to make the "vector" part inner, i.e. on $|z| = 1$

$$[2a^*z^*, -3a^*(1-z^*)]\begin{bmatrix}2az \\ -3a(1-z)\end{bmatrix} = 1 ,$$

$$|a|^2[22-9z-9z^*] = 1 .$$

Set $(\alpha-\beta z)(\alpha-\beta z^*) = 22-9z-9z^*$,

so that $\alpha^2 + \beta^2 = 22$,

$\alpha\beta = 9$,

from which we can take $\alpha = 1 + \sqrt{10}$, $\beta = -1 + \sqrt{10}$, and can set

$$a = \frac{1}{(1 + \sqrt{10} + (1-\sqrt{10})z)} .$$

So

$$(W_1A_1)_i = \frac{1}{(1+\sqrt{10}+(1-\sqrt{10})z)} \begin{bmatrix} 2z \\ -3(1-z) \end{bmatrix}, (W_1A_1)_o = \frac{(1+\sqrt{10}+(1-\sqrt{10})z)(1-z)}{(6-2z)} .$$

From this we have from (1.16)-(1.18),

$$G = \frac{1}{6-2z} \begin{bmatrix} 2z(1-z) & 2(1-z)^2 \\ -3(1-z)^2 & (1-z)(5-3z) \end{bmatrix} ,$$

$$U = \frac{1}{(1+\sqrt{10}+(1-\sqrt{10})z)} \begin{bmatrix} 2z \\ -3(1-z) \end{bmatrix} ,$$

$$H = (W_1A_1)_o \, Q(B_2W_2)_o .$$

We have thus to consider a problem of the form

$$\mu_1 = \inf\{||G-UH||_\infty : H \in (RH^\infty)^{m\times(p+m)}\} , \qquad (1.29)$$

where $\mathcal{U}$ is closed (U being inner) so that the infimum is achieved.

We see that $\mathcal{U}^{\perp}$ is no longer finite-dimensional, for putting U $((p+m)\times m)$ in Smith form as before

$$U = F_1 \begin{bmatrix} F_2 \\ 0 \end{bmatrix} F_3 ,$$

where F_1 is $(p+m)\times(p+m)$, F_2 is $m\times m$, F_3 is $m\times m$. Then $\mathcal{U} = U(H^2)^m = F_1 \begin{bmatrix} F_2 \\ 0 \end{bmatrix} (H^2)^m$, and setting $Y = \begin{bmatrix} F_2 \\ 0 \end{bmatrix} (H^2)^m$ we see, as before, that a finite-dimensional subspace of $Y^{\perp}$ is spanned by vectors of the form (1.25) with the index i in (1.25) between 1 and m. However $Y^{\perp}$ also contains all vectors of the form

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \end{bmatrix} \qquad (1.30)$$

with the first m entries all zero. We thus continue by using the approach outlined in the Francis lecture notes. If F is an RH^{∞}-matrix then $||F||_{\infty} \le c$ iff $\bar{\sigma}(F(e^{i\theta})) \le c$ for all θ iff $c^2 I - F(e^{i\theta})^* F(e^{i\theta}) \ge 0$ for all θ iff $c^2 I - F^X F \ge 0$ on the unit circle. We partition $G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$, $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ with $G_1 = p\times(p+m)$, $G_2 = m\times(p+m)$, $U_1 = p\times m$, $U_2 = m\times m$. From (1.29)

$$\mu_1 = \min\{c: ||G-UH||_{\infty} \le c \text{ for some } H \in (RH^{\infty})^{m\times(p+m)}\} ,$$

that is

$$\sum_{k=1}^{2} (G_k - U_k H)^X (G_k - U_k H) \le c^2 I \text{ on the unit circle.}$$

We wish to find matrices U_3 in $(RH^\infty)^{m\times m}$, G_3 in $(RH^\infty)^{m\times(p+m)}$ such that

$$\sum_{k=1}^{2}(G_k-U_kH)^X(G_k-U_kH) = (G_3-U_3H)^X(G_3-U_3H) + Q \qquad (1.31)$$

on the unit circle, where $Q = G_1^XG_1 + G_2^XG_2 - G_3^XG_3$. It is easily shown that (1.31) is an identity in H iff

$$U_1^XG_1 - U_3^XG_3 + U_2^XG_2 = 0 \ , \qquad (1.32)$$

$$U_1^XU_1 + U_2^XU_2 - U_3^XU_3 = 0 \ , \qquad (1.33)$$

on the unit circle. So

$$\mu_1 = \min\{c : \text{for some } H \in (RH^\infty)^{m\times(p+m)} \text{ we have } (G_3-U_3H)^X(G_3-U_3H) + Q \le c^2I \text{ on the unit circle}\} .$$

We observe immediately that

$$\mu_1 \ge ||Q||_\infty^{\frac{1}{2}} \qquad (1.34)$$

where Q is an L^∞-matrix, and evidently also

$$\mu_1 \le ||G||_\infty$$

by taking $H = 0$. If $c > ||Q||_\infty^{\frac{1}{2}}$ then on the unit circle $c^2I-Q > 0$ so we can factorize

$$c^2I-Q = R^XR \ , \ R \text{ unimodular}. \qquad (1.35)$$

If we now define

$$\nu(c) = \min\{||(G_3-U_3H)R^{-1}||_\infty : H \in (RH^\infty)^{m\times(p+m)}\} ,$$

then as R is unimodular

$$\nu(c) = \min\{||G_3R^{-1}-U_3H||_\infty : H \in (RH^\infty)^{m\times(p+m)}\} , \qquad (1.36)$$

which can be computed as $[U_3(H^2)^m]^\perp$ is now finite-dimensional and under the standard assumptions the theory outlined above may be applied. It is easy to see that for $c > ||Q||_\infty^{\frac{1}{2}}$ we have $\mu_1 \leq c$ iff $\nu(c) \leq 1$, thus yielding an algorithm by which μ_1 may be found to any prespecified accuracy. We now continue with our example by finding $||Q||_\infty^{\frac{1}{2}}$. From (1.16), (1.17)

$$G_1 = \frac{1}{6-2z}[2z(1-z),2(1-z)^2] ,$$

$$G_2 = \frac{1}{6-2z}[-3(1-z)^2,(1-z)(5-3z)] ,$$

$$U_1 = 2z/(1+\sqrt{10}+(1-\sqrt{10})z) ,$$

$$U_2 = -3(1-z)/(1+\sqrt{10}+(1-\sqrt{10})z) .$$

So $U_1^x = 2/((1+\sqrt{10})z + (1-\sqrt{10}))$, $U_2^x = 3(1-z)/((1+\sqrt{10})z + (1-\sqrt{10}))$ and from (1.33) it is easily verified that $U_3^xU_3 = 1$, so we can take (in RH^∞)

$$U_3 = \frac{1 - \sqrt{10} + (1+\sqrt{10})z}{1 - \sqrt{10} + (1+\sqrt{10})z^{-1}} .$$

From (1.32) we take in RH^∞

$$G_3 = \frac{z(1-z)}{(6-2z)((1-\sqrt{10})z+(1+\sqrt{10}))} [22z-9-9z^2, 19-28z+9z^2] .$$

We now evaluate $Q = G_1^x G_1 + G_2^x G_2 - G_3^x G_3$. We have

$$G_1^x = \frac{(1-z)}{z(6z-2)} \begin{bmatrix} -2 \\ 2(1-z) \end{bmatrix} , \quad G_2^x = \frac{(1-z)}{z(6z-2)} \begin{bmatrix} -3(1-z) \\ 3-5z \end{bmatrix} ,$$

$$G_3^x = \frac{(1-z)}{z^2(6z-2)((1-\sqrt{10})+(1+\sqrt{10})z)} \begin{bmatrix} 9z^2 + 9 - 22z \\ 28z - 19z^2 - 9 \end{bmatrix} ,$$

and a straightforward calculation gives

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{(1-z)^2}{9z^2 + 9 - 22z} \end{bmatrix} .$$

Now $||Q||_\infty^2 = \max_\theta \rho[Q(e^{i\theta})^* Q(e^{i\theta})]$, so

$$Q^* Q = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{(2-z-z^*)^2}{646-396z-396z^*+81z^2+81z^{*2}} \end{bmatrix} .$$

On the unit circle,

$$Q^* Q = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{(1-\cos\theta)^2}{121-198\cos\theta+81\cos^2\theta} \end{bmatrix} .$$

Differentiating the (2,2) component and setting to zero we have

$$\sin\theta(1-\cos\theta)(11-9\cos\theta) = 0 \quad ,$$

so that $\theta = 0$ or π . The value $\theta = 0$ gives zero and $\theta = \pi$ gives

$$||Q||_\infty^2 = 4/(121 + 198 + 81) \ ,$$

so

$$||Q||_\infty = 1/10 \ .$$

We construct

$$c^2 I - Q = \begin{bmatrix} c^2 & 0 \\ 0 & \dfrac{2(11c^2-1)-(9c^2-1)z-(9c^2-1)z^{-1}}{22 - 9z - 9z^{-1}} \end{bmatrix} ,$$

and factor this according to (1.35). Setting

$$(a-bz)(a-bz^{-1}) = 2(11c^2-1)-(9c^2-1)z-(9c^2-1)z^{-1}$$

we have

$$a = c + \sqrt{10c^2 - 1} \ ,$$

$$b = \sqrt{10c^2 - 1} - c \ .$$

For reasons which will become obvious in Chapter 3 from which the example is taken we take the value $c = 33/100$, so that $c^2 = 1089/10000$ and a,b above become

$$a = \frac{33 + \sqrt{890}}{100} \ , \qquad b = \frac{\sqrt{890} - 33}{100} \ , \qquad (1.37)$$

and $|a| > |b|$. Evidently in (1.35) we may take

$$R = \begin{bmatrix} c & 0 \\ 0 & \dfrac{(a-bz)}{(1+\sqrt{10}+(1-\sqrt{10})z)} \end{bmatrix} , \quad R^{-1} = \begin{bmatrix} 1/c & 0 \\ 0 & \dfrac{(1+\sqrt{10}+(1-\sqrt{10})z)}{a-bz} \end{bmatrix} .$$

We thus have the data matrices G_3R^{-1} , U_3 in (1.36) given by

$$G_3R^{-1} = \frac{z(1-z)}{(6-2z)} \left[\frac{(1+\sqrt{10})z+1-\sqrt{10}}{c}, \quad \frac{19-28z+9z^2}{a-bz} \right] , \tag{1.38}$$

$$U_3 = \frac{z(1-\sqrt{10}+(1+\sqrt{10})z)}{(1-\sqrt{10})z+(1+\sqrt{10})} . \tag{1.39}$$

We now may apply the standard theory. Firstly the zeros of U_3 in $|z| < 1$ are 0 , $(\sqrt{10}-1)/(\sqrt{10}+1)$ and so by (1.24), a basis for $U_3^{\perp}$ is given by

$$f_1 = 1 , \quad f_2 = \frac{\sqrt{10} + 1}{(\sqrt{10}+1+(1-\sqrt{10})z)} . \tag{1.40}$$

We now calculate inner products in order to form the matrix Φ_1 . In general for real constants α,β

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\alpha.d\theta}{\alpha+\beta e^{i\theta}} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\alpha^2+\alpha\beta\cos\theta-i\alpha\beta\sin\theta.d\theta}{\alpha^2+\beta^2 + 2\alpha\beta\cos\theta}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\tfrac{1}{2}(\alpha^2+\beta^2+2\alpha\beta\cos\theta) + \tfrac{1}{2}(\alpha^2-\beta^2)-i\alpha\beta\sin\theta.d\theta}{\alpha^2+\beta^2 + 2\alpha\beta\cos\theta}$$

$$= \tfrac{1}{2} + \frac{1}{2\pi}\int_0^{2\pi} \frac{\tfrac{1}{2}(\alpha^2-\beta^2).d\theta}{\alpha^2+\beta^2 + 2\alpha\beta\cos\theta} .$$

We have $\langle f_1, f_1\rangle = 1$, $\langle f_1, f_2\rangle =$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{(\sqrt{10}+1).d\theta}{(\sqrt{10}+1+(1-\sqrt{10})e^{i\theta})} = \tfrac{1}{2} + \frac{1}{2\pi}\int_0^{2\pi}\frac{2\sqrt{10}.d\theta}{22 - 18\cos\theta} .$$

The latter term is evaluated using the substitution $t = \tan\theta/2$, $dt/d\theta = (1+t^2)/2$, $\cos\theta = (1-t^2)/(1+t^2)$, $\sin\theta = 2t/(1+t^2)$. Performing the integration we have $\langle f_1, f_2\rangle = 1$. Now

$$\langle f_2, f_2\rangle = \frac{1}{2\pi}\int_0^{2\pi} \frac{(\sqrt{10}+1)^2.d\theta}{(\sqrt{10}+1+(1-\sqrt{10})e^{-i\theta})(\sqrt{10}+1+(1-\sqrt{10})e^{i\theta})}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(\sqrt{10}+1)^2.d\theta}{22-18\cos\theta} = \frac{(\sqrt{10}+1)^2}{4\sqrt{10}} .$$

The matrix Φ_1 can now be formed.

We now calculate $(G_3R^{-1})^x f_1$, $(G_3R^{-1})^x f_2$ for f_1, f_2 in (1.40). We have

$$(G_3R^{-1})^x = \frac{(z-1)}{z^2(6z-2)} \begin{bmatrix} \dfrac{(1+\sqrt{10}) + (1-\sqrt{10})z}{c} \\[2ex] \dfrac{19z^2 - 28z + 9}{az-b} \end{bmatrix} . \quad (1.41)$$

Acting (1.41) on f_1 and taking H^2-components we see that, because $|b| < |a|$,

$$\Gamma^* f_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

Similarly

$$(G_3R^{-1})^{x}f_2 = \frac{(z-1)}{z^2(6z-2)} \begin{bmatrix} \dfrac{\sqrt{10}+1}{c} \\ \\ \dfrac{(19z^2-28z+9)(\sqrt{10}+1)}{(az-b)(\sqrt{10}+1+(1-\sqrt{10})z)} \end{bmatrix} ,$$

so that we need to find the H^2-component of the second entry (the H^2-component of the first entry is again 0). We make a partial fraction expansion, so that we want to find E in

$$\frac{(z-1)(19z^2-28z+9)(\sqrt{10}+1)}{z^2(6z-2)(az-b)(\sqrt{10}+1+(1-\sqrt{10})z)} \equiv \frac{A}{z} + \frac{B}{z^2} + \frac{C}{6z-2} + \frac{D}{az-b} + \frac{E}{(\sqrt{10}+1+(1-\sqrt{10})z)} .$$

Setting $z = (\sqrt{10}+1)/(\sqrt{10}-1)$ in

$$(z-1)(19z^2-28z+9)(\sqrt{10}+1) \equiv Ez^2(6z-2)(az-b)$$

we have

$$E = 6/((a-b)\sqrt{10} + (a+b)) ,$$

so that

$$\Gamma^*f_2 = \frac{6}{((a-b)\sqrt{10}+(a+b))(\sqrt{10}+1+(1-\sqrt{10})z)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

Consequently

$$<\Gamma^*f_1,\Gamma^*f_1> = 0 ,$$

$$<\Gamma^*f_1,\Gamma^*f_2> = 0 ,$$

$$\langle\Gamma^*f_2,\Gamma^*f_2\rangle = \frac{1}{2\pi}\int_0^{2\pi} \frac{36.d\theta}{((a-b)\sqrt{10}+(a+b))^2(22-18\cos\theta)}$$

$$= \frac{9}{\sqrt{10}((a-b)\sqrt{10}+(a+b))^2} \equiv \phi_{22} .$$

The matrix Φ_2 can now be formed.

We now calculate the largest value λ s.t.

$$\det(\Phi_2 - \lambda\Phi_1) = 0 ,$$

where

$$\Phi_2 = \begin{bmatrix} 0 & 0 \\ 0 & \phi_{22} \end{bmatrix} , \quad \Phi_1 = \begin{bmatrix} 1 & 1 \\ 1 & \frac{(\sqrt{10}+1)^2}{4\sqrt{10}} \end{bmatrix} .$$

Evaluating the determinant we have

$$\frac{(\sqrt{10}+1)^2\lambda^2}{4\sqrt{10}} - \phi_{22}\lambda - \lambda^2 = 0 .$$

So the required root λ is

$$\lambda = \frac{\phi_{22}\, 4\sqrt{10}}{(\sqrt{10}-1)^2} .$$

Thus $$\nu(0.33) = \sqrt{\lambda} = \frac{6}{(\sqrt{10}-1)(\sqrt{10}.\frac{66}{100} + \frac{2\sqrt{890}}{100})} = \frac{600}{(10-\sqrt{10})(66+2\sqrt{89})}$$

$$= \frac{300}{(10-\sqrt{10})(33+\sqrt{89})} > \frac{300}{(6.9)(42.5)} = \frac{300}{293.25} > 1 .$$

As we remarked earlier $\nu(0.33) > 1$ iff $\mu_1 > 0.33$. Thus we conclude using Theorem 1.3 that the optimal norm of the weighted sensitivity matrix, μ , cannot be reduced below 0.33 which will be an interesting comparison with our work on enhancing the "complex structured stability radius" which we subsequently go on to.

2. STATE SPACE ROBUSTNESS ANALYSIS

2.1 INTRODUCTION

In this chapter we give a brief review of robust stability in the state space - concentrating in particular on the concepts of real and complex stability radii for linear state space systems that were introduced in [6], [7]. Various attempts have been made over the past few years to characterize the robustness of stability of a state space system. In many cases the problem considered is of how an additive perturbation to a nominally stable system destabilizes that system (the nominal system representing the system "model" and together with the perturbation representing the "real world" system), see for example [26]-[28]. In [26] is considered the system

$$\dot{x} = Ax + f(x,t) \qquad (2.1)$$

on $[t_0,\infty)$ where $x \in \mathbb{R}^n$, A is time-invariant and asymptotically stable, $f(x,t)$ is the additive perturbation (in general this will be nonlinear and time-varying) with $f(0,t) = 0$ for all $t \in [t_0,\infty)$. The question to be asked is under which additive perturbation of this form does the system (2.1) retain stability. This is clearly a very open-ended question and one is usually restricted to deriving sufficient conditions under which stability is retained, for example in [26] the following sufficient condition for the system of (2.1) is given and is derived by Lyapunov means, see [35] or [36]:

Theorem 2.1 (Theorem 1, [26])

The system (2.1) above is stable if

$$\frac{||f(z,t)||}{||z||} \leq \frac{\min \lambda(Q)}{\max \lambda(P)} \quad \text{for all} \quad (z,t) \in \mathbb{R}^{n+1} \quad ,$$

where P is the unique positive definite solution of the Lyapunov equation

$$PA + A^T P + 2Q = 0 \quad ,$$

and Q is any positive definite matrix. (Here $||\cdot||$ denotes the standard Euclidean norm on $\mathbb{R}^n$ and $\lambda(\cdot)$ is used to denote any eigenvalue of the matrix).

We note that, in this theorem, Q is any positive definite matrix. Clearly we would like the RHS in the theorem statement to be as large as possible with respect to the choice of Q. An interesting result in this direction is the following:

Theorem 2.2 (Lemma 2, [26])

The RHS in Theorem 2.1 above is maximized by the choice $Q = I$.

The paper goes on, amongst other things, to examine how this bound is related to the matrix A - we refer the reader to [26] for details.

In [27] the author specializes the differential equation (2.1) to the case where the perturbation is linear, so that

$$\dot{x} = (A + E)x \tag{2.2}$$

considering both "structured" perturbations where each element of the matrix E has some bound, and "unstructured" where the perturbation is of unknown structure. We again refer the reader to [27] where bounds are given, thus yielding sufficient conditions.

We turn now to the main topic of this chapter, that is a review of the concepts of real and complex stability radii as introduced in [6], [7]. This discussion is a prerequisite to the following chapter. The basic question considered is to find the minimum radius of the ball around a nominally asymptotically stable time-invariant linear system containing an additive perturbation which destabilizes it. Consequently the result obtained is not just a sufficient condition. To make the development easier to understand we include the following material on singular values of a matrix.

2.2 SINGULAR VALUES

For $K = \mathbb{R}$ or $\mathbb{C}$ if the matrix $A \in K^{m\times n}$ and rank $(A) = r$, then $A^*A \in K^{n\times n}$ is a Hermitian matrix and rank $(A^*A) = r$ so there exists a unitary matrix V (orthogonal matrix in the case $K = \mathbb{R}$) such that

$$V^*A^*AV = \text{diag}(s_1^2, \ldots, s_n^2)$$

where

$$s_1 \geq s_2 \geq \ldots \geq s_r > s_{r+1} = \ldots = s_n = 0 . \qquad (2.3)$$

The non-negative numbers $s_1,\ldots,s_n$ are called the singular values of the matrix A, and the eigenvectors of A^*A are called the singular vectors of A. Commonly s_1 is denoted by $\bar{\sigma}(A)$ and similarly s_n

by $\underline{\sigma}(A)$. We assume the singular values are ordered as in (2.3), then the following well known result gives a description of these quantities (the Minimax Theorem of Courant and Fischer). Here E is a linear subspace in K^n , and $\bar{s}(A|E) \triangleq \max_{0 \neq x \in E} \frac{||Ax||}{||x||}$ where $||\cdot||$ is the standard Euclidean norm on K^n .

Theorem 2.3

The singular values of A in (2.3) have the property

$$s_i = \min_{\text{codim } E = i-1} \bar{s}(A|E) = \min_{\text{codim } E \leq i-1} \bar{s}(A|E)$$

for $i = 1,\ldots, n$.

Immediately from this result it is clear that

$$\bar{\sigma}(A) = ||A||_2 \tag{2.4}$$

where $||\cdot||_2$ denotes the induced Euclidean norm of the matrix.

Denoting $\underline{s}(E) \triangleq \min_{0 \neq x \in E} \frac{||Ax||}{||x||}$ we have the following dual of Theorem 2.3.

Theorem 2.4

The singular values of A have the property

$$s_i = \max_{\dim E = i} \underline{s}(E) = \max_{\dim E \geq i} \underline{s}(E)$$

for $i = 1,\ldots,n$.

An immediate corollary of this is that

$$\underline{\sigma}(A) = \min_{x \neq 0} \frac{||Ax||}{||x||} \quad . \tag{2.5}$$

Combining (2.4), (2.5) it is easy to verify that in the case where the matrix A is square and invertible

$$\underline{\sigma}(A) = \frac{1}{\bar{\sigma}(A^{-1})} = \frac{1}{||A^{-1}||_2} \quad .$$

Singular values also feature prominently in the following decomposition of a matrix, known (not surprisingly) as the Singular Value Decomposition.

<u>Theorem 2.5</u>

Let $A \in K^{m \times n}$ with singular values as in (2.3) and an orthonormal basis $v^1, \ldots, v^n$ of singular vectors. Setting $V = [v^1, \ldots, v^n]$ then there exists a unitary (or orthogonal in the case $K = \mathbb{R}$) $m \times m$ matrix $W = [w_1, \ldots, w_m]$ such that

$$A = \sum_{i=1}^{r} s_i w_i v_i^* \tag{2.6}$$

or equivalently

$$A = W \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} V^* \tag{2.7}$$

where the matrix $\Sigma_r = \text{diag}\,(s_1, \ldots, s_r)$.

Conversely, if W and V are unitary matrices such that equation (2.7) holds then $s_1,\ldots,s_r,s_{r+1} = \ldots = s_n = 0$ are the singular values of A and the columns of V form a basis of corresponding singular vectors of A .

Using the decomposition given by Theorem 2.5 the following important theorem holds:

Theorem 2.6

Let $A \in K^{m\times n}$ have rank r and singular value decomposition of the form (2.7) and denote by $M_k \equiv M_k(m,n)$

$$M_k(m,n) = \{X \in K^{m\times n} : \text{rank } X \le k\} \quad .$$

Then

$$s_{k+1}(A) = \min_{X\in M_k} ||A-X||_2 = ||A-A_k||_2$$

where

$$A_k = \sum_{i=1}^{k} s_i w_i v_i^* \in M_k \quad .$$

The relevance of this result is that it gives the distance of the matrix A from the set of matrices (of the same dimension) of rank less than or equal to k as the $(k+1)^{th}$-singular value of A . Moreover, a construction is given to determine a matrix for which this minimum is achieved. Its important corollary is the following (the distance to invertibility).

Corollary 2.7

If $A \in K^{n\times n}$ is invertible then

$$\underline{\sigma}(A) = \min\{||X||_2 : \det(A+X) = 0\} = \min\{||A-X||_2 : \det X = 0\} \quad .$$

Furthermore $A_{n-1} = A-s_n w_n v_n^*$ is singular, thus yielding a rank one perturbation of norm $s_n \equiv \underline{\sigma}(A)$.

2.3 REAL AND COMPLEX STABILITY RADII

In this section we show how to determine the minimum norm of an unstructured destabilizing perturbation. The term unstructured in this case is to be taken precisely as in [27], and the outline we give is largely a derivation of [6].

For $K = \mathbb{R}$ or $\mathbb{C}$ we denote $\mathcal{U}_n(K)$ as the set of $n\times n$ matrices over K which are not asymptotically stable, that is

$$\mathcal{U}_n(K) = \{U \in K^{n\times n} : \sigma(U) \cap \bar{\mathbb{C}}_+ \neq \phi\} \tag{2.8}$$

with $\bar{\mathbb{C}}_+$ the closed right half plane, and $\sigma(U)$ the spectrum of the matrix U . We take the inner product $\langle\cdot,\cdot\rangle$ and norm $||\cdot||$ on K^n to be the standard Euclidean ones. If $A : K^n \to K^n$ we take as the norm of A the induced Euclidean norm, $||\cdot||_2$, dropping the subscript for convenience - the context of use making clear which norm we are referring to.

For $A \in K^{n\times n}$ we define the distance from instability by $r_K(A)$ as follows:

$$r_K(A) = \inf\{||A-U|| : U \in \mathcal{U}_n(K)\} \quad . \tag{2.9}$$

Clearly the set $\mathcal{U}_n(K)$ is closed and has as its frontier, $\partial\mathcal{U}_n(K)$, the set of matrices with at least one eigenvalue on the imaginary axis but none in $\mathbb{C}_+$ (the open right half plane). Thus there exists, for asymptotically stable $A \in K^{n\times n}$, a minimum norm destabilizing perturbation $P \in K^{n\times n}$ such that $U = A + P \in \mathcal{U}_n(K)$, i.e.

$$||P|| = r_K(A) \quad \text{and} \quad \sigma(A+P) \cap i\mathbb{R} \neq \phi \quad . \tag{2.10}$$

The following properties of $r_K(A)$ are thus immediate:

$r_K(A) = 0$ iff $A \in \mathcal{U}_n(K)$,

$r_K(\alpha A) = \alpha r_K(A)$, $\alpha \geq 0$,

$A \mapsto r_K(A)$ is continuous on $K^{n\times n}$.

If $A \in \mathbb{R}^{n\times n}$ then (2.9) gives two values, namely $r_{\mathbb{C}}(A)$ - the complex unstructured stability radius, and $r_{\mathbb{R}}(A)$ - the real unstructured stability radius. In general these two quantities are distinct. Furthermore the following string of inequalities holds for $A \in \mathbb{R}^{n\times n}$:

$$0 \leq r_{\mathbb{C}}(A) \leq r_{\mathbb{R}}(A) \leq \underline{\sigma}(A) \equiv s_n(A) \tag{2.11}$$

where the last inequality follows from corollary 2.7.

In [6] the following result is proved:

Proposition 2.8 (Proposition 3.1, [6])

Let $\Gamma \subset \mathbb{C}$ be closed, $\partial\Gamma$ its boundary, $A \in \mathbb{C}^{n\times n}$ such that $\sigma(A) \cap \Gamma = \phi$. Then

$$\min_{\substack{X\in\mathbb{C}^{n\times n} \\ \sigma(X)\cap\Gamma\neq\phi}} ||A-X|| = \min_{\gamma\in\Gamma} s_n(\gamma I-A) = \min_{\gamma\in\partial\Gamma} s_n(\gamma I-A) .$$

The consequence of this when $\Gamma = \bar{\mathbb{C}}_+$ is the following characterization of the complex stability radius:

Corollary 2.9 (Corollary 3.2, [6])

$$r_{\mathbb{C}}(A) = \min_{\omega\in\mathbb{R}} s_n(i\omega I-A) = \min_{\omega\in\mathbb{R}} \min_{\substack{z\in\mathbb{C}^n \\ ||z||=1}} ||(i\omega I-A)z|| . \qquad (2.12)$$

We observe two things concerning this result - the first being the fact that (by corollary 2.7) a rank one perturbation may be constructed of norm $r_{\mathbb{C}}(A)$ that destabilizes, the second is that such a perturbation will generally be complex even when A is real. In the case where A is real the expression (2.12) simplifies to give the following result.

Proposition 2.10 (Proposition 3.3, [6])

If $A \in \mathbb{R}^{n\times n}$ then setting $A_u = \frac{1}{2}(A-A^T)$ to be its skew-symmetric part we have

$$r_{\mathbb{C}}^2(A) = \min_{\substack{z\in\mathbb{C}^n \\ ||z||=1}} \{||Az||^2 + \langle A_u z,z\rangle^2\}$$

$$= \min_{\substack{x,y \in \mathbb{R}^n \\ ||x||^2+||y||^2=1}} \{||Ax||^2 + ||Ay||^2 - \langle (A-A^T)x,y\rangle^2\}. \quad (2.13)$$

The usefulness of proposition 2.10 is evident from the following set of inequalities which give bounds on $r_{\mathbb{C}}(A)$ (again $A \in \mathbb{R}^{n\times n}$) with only computations of certain singular values being required. In fact, from (2.11), (2.13) we have

$$s_n^2(A)-||A_u||^2 \le r_{\mathbb{C}}^2(A) \le \min\{s_n^2(A), ||A||^2-||A_u||^2\} . \quad (2.14)$$

The first inequality is immediate using the fact that $\langle A_u z,z\rangle = -2i\langle A_u x,y\rangle$, and the second comes from a maximization over x,y subject to $||x||^2 + ||y||^2 = 1$ of the component $\langle (A-A^T)x,y\rangle^2$ in the second expression of (2.13). From (2.14) it is easy to conclude that in the case where A is symmetric then (as $A_u = 0$)

$$r_{\mathbb{C}}(A) = r_{\mathbb{R}}(A) = s_n(A) , \quad (2.15)$$

thus the set of real symmetric matrices is contained in the set of all real matrices whose complex and real stability radii coincide, we say more about this set subsequently.

In [6] the following numerical example (which we also make reference to later) is given, Example 2.11 (Example 3.6, [6])

$$A = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} .$$

For this matrix the real and complex stability radii differ considerably. The complex stability radius $r_{\mathbb{C}}(A) = 2/3$, whereas the real stability radius $r_{\mathbb{R}}(A) = s_2(A) = \sqrt{15 - \sqrt{200}}$ which is approximately 40% greater than $r_{\mathbb{C}}(A)$.

The following example shows that, for $A \in \mathbb{R}^{n\times n}$, $r_{\mathbb{R}}(A) \neq s_n(A)$ in general.

Example 2.12

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1/3 \end{bmatrix} ,$$

whose eigenvalues are $-\frac{1}{6} \pm i\,\frac{\sqrt{35}}{6}$. It is easily verified that $s_2(A) = \sqrt{\frac{19 - \sqrt{37}}{18}}$. We set the perturbation matrix P to be the following

$$P = \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix} ,$$

then $\quad ||P|| = \frac{1}{2} < \sqrt{\frac{19 - \sqrt{37}}{18}} = s_2(A)$, yet

$$A + P = \begin{bmatrix} 0 & 1 \\ -1 & 1/6 \end{bmatrix} ,$$

whose eigenvalues are $\frac{1}{12} \pm i \frac{\sqrt{143}}{12}$.

Following the lines of the paper we make some comments on the interrelation between the eigenvalues of A and its stability radii, and also how $r_K(A)$ varies under similarity transformations of the matrix A . The first result shows that stability radii may be bounded above in terms of the real parts of the eigenvalues of the matrix A . To be more precise we have:

Proposition 2.13 (Lemma 4.1, [6])

If $A \in K^{n\times n}$ is stable having eigenvalues $\lambda_j = -\alpha_j + i\omega_j$, $\alpha_1 \geq \ldots \geq \alpha_n > 0$. Then $r_K(A) \leq \alpha_n$. Further $r_K(A) = \alpha_n$ iff the matrix $r_K(A)I$ destabilizes A .

This result thus shows that there is always a destabilizing perturbation of norm not exceeding the distance of the spectrum of A from the imaginary axis. In the case where A is normal then we can say more (using the same notation as in proposition 2.13). For $A \in \mathbb{C}^{n\times n}$ we have:

Proposition 2.14 (Proposition 4.2, [6])

(a) ω_n is a minimum of $s_n(i\omega I-A)$. It is unique iff, for all j , $\alpha_j = \alpha_n$ implies $\lambda_j = \lambda_n$;

(b) $r_{\mathbb{C}}(A) = \alpha_n$;

(c) If $A \in \mathbb{R}^{n\times n}$ then $r_{\mathbb{C}}(A) = r_{\mathbb{R}}(A) = \alpha_n$.

Proposition 2.14 shows that for normal matrices the equality in proposition 2.13 holds. Furthermore, the class of real matrices for which real and complex stability radii coincide is thus enlarged from just symmetric real matrices - see (2.15). As regards the behaviour of $r_{K}(\cdot)$ when A is subjected to similarity transformations the value α_n is again prominent, as the paper shows.

We conclude this section with some comments on the real unstructured stability radius, that is for $A \in \mathbb{R}^{n\times n}$

$$r_{\mathbb{R}}(A) = \min\{||P|| : P \in \mathbb{R}^{n\times n} \text{ and } \sigma(A+P) \cap i\mathbb{R} \neq \phi\} \quad .$$

In other words

$$r_{\mathbb{R}}(A) = \inf_{\omega\in\mathbb{R}, x,y\in\mathbb{R}^n, ||x||^2+||y||^2=1} \{||P||:P \in \mathbb{R}^{n\times n} \text{ and}$$

$$\begin{bmatrix} -Ax - \omega y \\ \omega x - Ay \end{bmatrix} = \begin{bmatrix} Px \\ Py \end{bmatrix} \} . \tag{2.16}$$

Firstly the minimum norm of $P \in \mathbb{R}^{n\times n}$ such that $Px = u$, $Py = v$ is determined.

<u>Proposition 2.15 (Lemma 5.1, [6])</u>

If $x,y,u,v \in \mathbb{R}^n$ and x,y are linearly independent then setting

$$\mu = \mu(x,y,u,v) = \min\{||P||:P \in \mathbb{R}^{n\times n} \text{ and } Px = u\ ,\ \ Py = v\}$$

we have

$$\mu^2 = \max_{(\alpha,\beta)\neq 0} \frac{[\alpha,\beta]\begin{bmatrix} ||u||^2 & <u,v> \\ <u,v> & ||v||^2 \end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}}{[\alpha,\beta]\begin{bmatrix} ||x||^2 & <x,y> \\ <x,y> & ||y||^2 \end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}}$$

with μ^2 the largest root of the quadratic equation in λ

$$\det\begin{bmatrix} ||u||^2-\lambda||x||^2 & <u,v>-\lambda<x,y> \\ <u,v>-\lambda<x,y> & ||v||^2-\lambda||y||^2 \end{bmatrix} = 0\ .$$

Using this proposition we may more simply characterize, in the interesting case where the vectors x,y are linearly independent, (2.16) as

$$r_{\mathbb{R}}(A) = \inf_{\omega\in\mathbb{R},x,y\in\mathbb{R}^n,||x||^2+||y||^2=1} \mu(x,y,-Ax-\omega y,\omega x-Ay)\ . \qquad (2.17)$$

The determination of $r_{\mathbb{R}}(A)$ is thus by no means as straightforward as of $r_{\mathbb{C}}(A)$. As we do not deal with $r_{\mathbb{R}}(A)$ subsequently we end this section with one final result on testing whether, for $A \in \mathbb{R}^{n\times n}$, its complex and real stability radii are the same. We have:

<u>Proposition 2.16 (Proposition 5.3, [6])</u>

Let $A \in \mathbb{R}^{n\times n}$, stable. Then $r_{\mathbb{C}}(A) = r_{\mathbb{R}}(A)$ iff there exists

$\omega_0 \in \mathbb{R}$, $z_0 = x_0 + iy_0 \in \mathbb{C}^n$, $||z_0|| = 1$ such that ω_0 minimizes $\omega \mapsto s_n(i\omega I-A)$ and the following equations hold

$$||-Ax_0-\omega_0 y_0|| = ||x_0|| s_n(i\omega_0 I-A) ,$$

$$||\omega_0 x_0-Ay_0|| = ||y_0|| s_n(i\omega_0 I-A) ,$$

$$<-Ax_0-\omega_0 y_0,\omega_0 x_0-Ay_0> = <x_0,y_0> s_n^2(i\omega_0 I-A) .$$

For further details we refer the reader to [6]. In the following section we consider the case where perturbations are "structured".

2.4 STABILITY RADII AND THE ALGEBRAIC RICCATI EQUATION

Here we show how the complex stability radius (in the case of structured perturbations) may be found and how it is related, for example, to solutions of a nonstandard algebraic Riccati equation. The results we outline are, more or less, to be found in [7].

We consider the nominal system

$$\dot{x} = Ax , \tag{2.18}$$

and a perturbed system of the form

$$\dot{x} = (A + BDC)x , \tag{2.19}$$

where $A \in \mathbb{C}^{n\times n}$ is asymptotically stable, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$, and the disturbance matrix $D \in \mathbb{C}^{m\times p}$. In a very simple way the matrices B and C govern the structure of the perturbation, though clearly (2.19) cannot represent totally every possible situation. A good example of what is meant is the second order differential equation

$$\ddot{y} + a_1\dot{y} + a_2 y = 0 \quad ,$$

which can be rewritten in the form (2.18) as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} x \quad . \tag{2.20}$$

From the differential equation we would only expect uncertainty in the values a_1, a_2. This can be modelled in (2.20) by taking $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = I_2$. However if a_2 is known to be correct we could take $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [0,1]$.

In a similar way to the previous section, for $A \in \mathbb{C}^{n\times n}$ asymptotically stable, we may define the complex structured stability radius by

$$r_{\mathbb{C}} = r_{\mathbb{C}}(A;B,C) = \inf\{||D|| : \sigma(A+BDC) \cap \bar{\mathbb{C}}_+ \neq \phi\} \quad , \tag{2.21}$$

with $||\cdot||$ denoting the induced Euclidean norm as before. We thus have

$$r_{\mathbb{C}} = \inf\{||D|| : \sigma(A+BDC) \cap i\mathbb{R} \neq \phi\} \quad , \tag{2.22}$$

and the following result shows how to calculate $r_{\mathbb{C}}$:

<u>Proposition 2.17 (Proposition 2.1, [7])</u>

Setting $G(s) = C(sI-A)^{-1}B$, we have

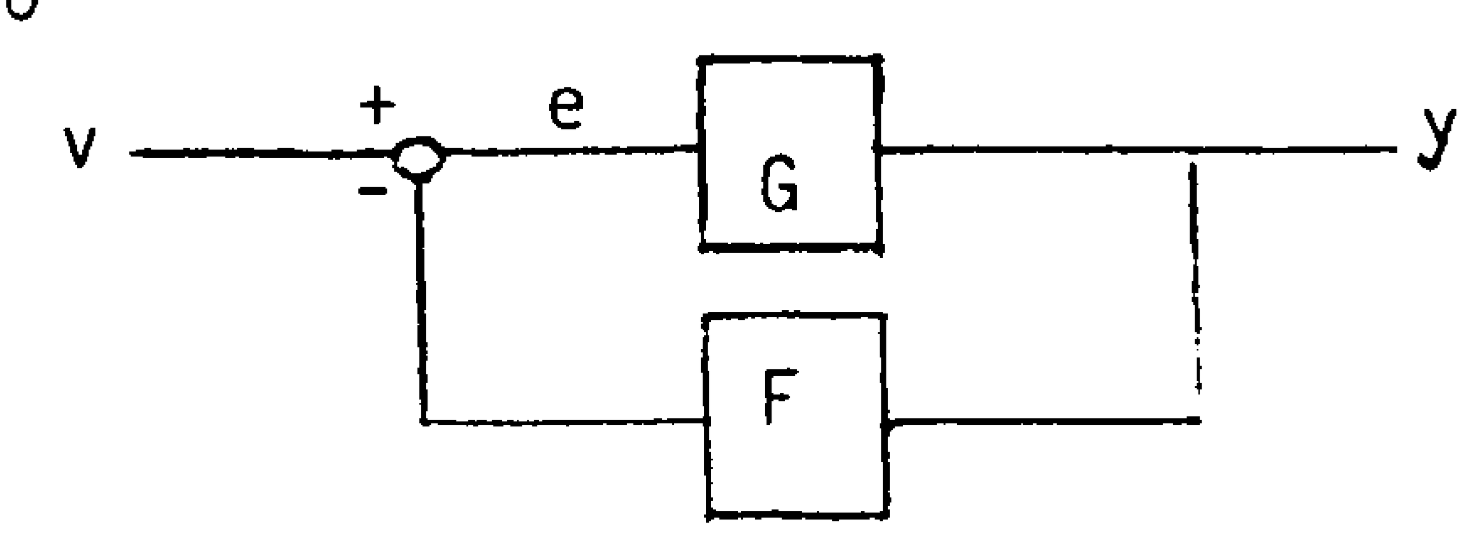
$$r_{\mathbb{C}} = \begin{cases} \dfrac{1}{\max\limits_{\omega\in\mathbb{R}} ||G(i\omega)||} & \text{if } G \not\equiv 0 , \\ \infty & \text{if } G \equiv 0 . \end{cases}$$

In the case $m = p = n$, $B = C = I_n$ this result is in agreement with corollary 2.9 which gives the unstructured complex stability radius - see section 2.2 on singular values.

With this characterization we note the following simple application to input/output stability. By input/output stable we mean BIBO stable - that is bounded inputs produce bounded outputs in the following sense (page 194, [11]). The system

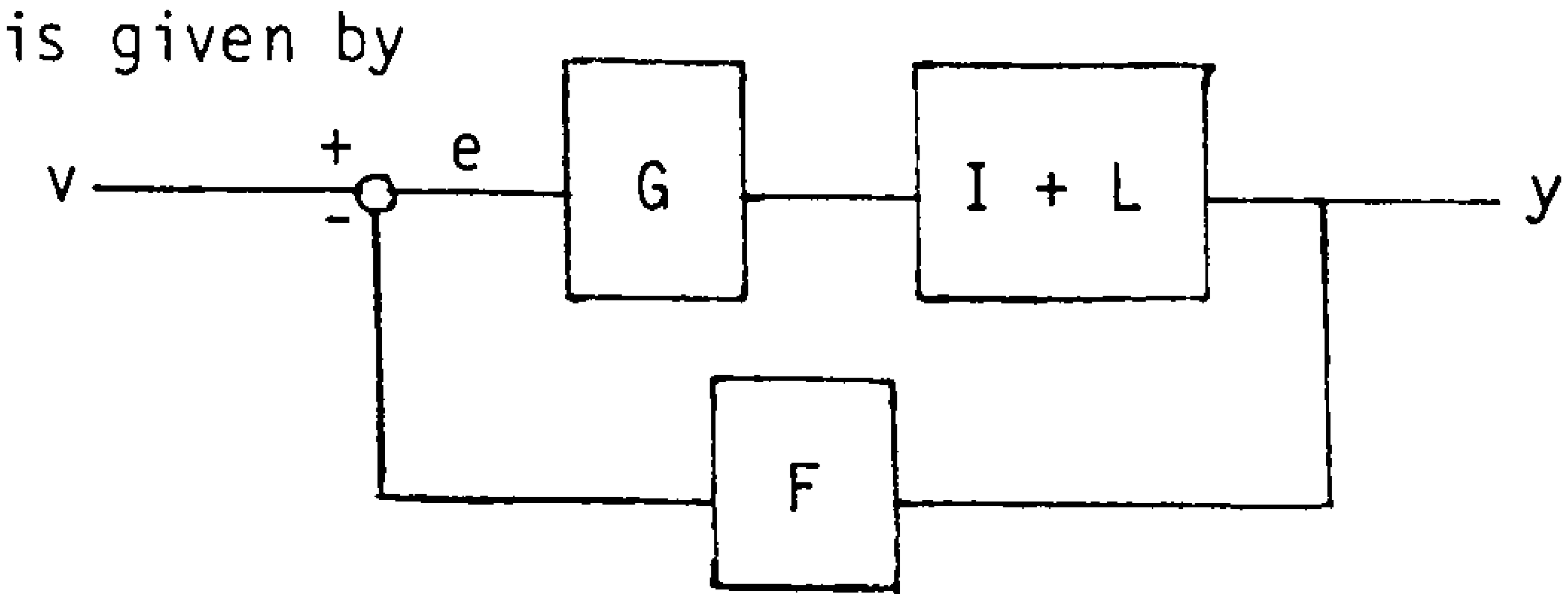
$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} , \; x(t_0) = 0 ,$$

is said to be uniformly BIBO stable if there exists a constant k (independent of t_0) such that for all t_0 we have $||v(t)|| \leq 1$ for all $t \geq t_0$ implies that $||y(t)|| \leq k$ for all $t \geq t_0$. We consider



(N)

with F a constant feedback, G having constant minimal realization $[A,B,C]$, and the nominal system (N) uniformly BIBO stable. The perturbed system (P) (with constant "multiplicative" perturbation) is given by



(P)

so that for (N) we have

$$y = Ge ,$$
$$e = v-Fy$$

with state space realization

$$\dot{x} = Ax + Be , \quad x(t_0) = 0 ,$$
$$y = Cx$$

or

$$\dot{x} = (A-BFC)x + Bv , \; x(t_0) = 0 ,$$
$$y = Cx .$$

So by Theorem 3 (page 197, [11]) Re λ (A-BFC) < 0 as (N) is uniformly BIBO stable and [A-BFC,B,C] is minimal. For the perturbed system (P) we have

$$y = (I + L)Ge ,$$
$$e = v-Fy$$

so that

$$\dot{x} = (A-BFC)x - BFLCx + Bv, \; x(t_0) = 0 ,$$
$$y = (I + L)Cx .$$

From proposition 2.17 we have $\text{Re}\ \lambda(A-BFC-BFLC) < 0$ when

$$||L|| < \frac{1}{\max_{\omega \in \mathbb{R}} ||C(i\omega - A + BFC)^{-1}BF||} \qquad (2.23)$$

and so by Theorem 3 (page 197, [11]) we have uniform BIBO stability of the perturbed system (P) whenever (2.23) holds and the pair $(A-BF(I+L)C,(I+L)C)$ is observable.

Having obtained the characterization of $r_{\mathbb{C}}$ given by proposition 2.17 we continue by giving a number of alternative characterizations. The first is in terms of the induced norm of the following convolution operator between L^2 spaces:

$$L : L^2[0,\infty;\mathbb{C}^m] \to L^2[0,\infty;\mathbb{C}^p] \ ,$$
$$(Lv)(t) = \int_0^t Ce^{A(t-s)}Bv(s).ds \ . \qquad (2.24)$$

There is a simple relation between $r_{\mathbb{C}}$ and $||L||$ as the following result shows:

<u>Proposition 2.18 (Proposition 2.2, [7])</u>

For the operator L of (2.24) we have

$$r_{\mathbb{C}} = 1/||L|| \ .$$

The norm of L has another characterization as we now show. To find $||L||$ we are required to solve the following optimization problem:

Maximize $||Cx||^2_{L^2}$ subject to $\dot{x}-Ax-Bv = 0$, $x(0) = 0$, $||v||^2_{L^2} = 1$.

So we set-up the Lagrangian

$$\mathcal{L} = ||Cx||^2_{L^2} - \mu(||v||^2_{L^2}-1) + 2\langle\lambda,\dot{x}-Ax-Bv\rangle_{L^2} \quad .$$

We note that $\int_0^\infty \lambda^*(t)\dot{x}(t).dt = [\lambda^*(t)x(t)]_0^\infty - \int_0^\infty \dot{\lambda}^*(t)x(t).dt$, so setting $\lambda(\infty) = 0$ this becomes $-\int_0^\infty \dot{\lambda}^*(t)x(t).dt$, and we can rewrite $\mathcal{L}$ as

$$\mathcal{L} = ||Cx||^2_{L^2} - \mu(||v||^2_{L^2}-1) - 2\langle\lambda,Ax+Bv\rangle_{L^2} - 2\langle\dot{\lambda},x\rangle_{L^2} \quad .$$

The first order conditions give

$$\mu v + B^T\lambda = 0 \ ,$$

$$C^TCx - A^T\lambda - \dot{\lambda} = 0 \ .$$

We thus have the following conditions

$$\dot{\lambda} = -A^T\lambda + C^TCx \ ,$$

$$\dot{x} = Ax - \frac{1}{\mu} BB^T\lambda \ ,$$

$$x(0) = 0 \ ,$$

$$\lambda(\infty) = 0 \ ,$$

$$||v||^2_{L^2} = 1 \ .$$

We then have the system

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -\frac{1}{\mu} BB^T \\ C^T C & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}, \; x(0) = 0 \;, \; \lambda(\infty) = 0 \;. \quad (2.25)$$

Furthermore $\langle x, C^T C x\rangle_{L^2} = \langle x, A^T\lambda\rangle_{L^2} + \langle x, \dot{\lambda}\rangle_{L^2}$

$$= \langle Ax, \lambda\rangle_{L^2} - \langle \dot{x}, \lambda\rangle_{L^2}$$

$$= \frac{1}{\mu} \, ||B^T\lambda||^2_{L^2} = \mu \quad .$$

So it is clear that we are required to maximize μ such that the two-point boundary value problem (2.25) has a nontrivial solution. We note that the system matrix in (2.25) is Hamiltonian.

We turn now to the most interesting and important characterizations of the quantity $r_{\mathbb{C}}$ in (2.21). These are in terms of a certain linear-quadratic problem and an associated algebraic Riccati equation. Consider the following linear-quadratic problem parameterized by the real scalar ρ :

$$\text{Minimize} \;\; J(x_0, v) = \int_0^\infty ||v(s)||^2_{\mathbb{C}^m} - \rho||y(s)||^2_{\mathbb{C}^p} . ds, \;\; v \in L^2[0,\infty;\mathbb{C}^m] \;, \quad (2.26)$$

subject to

$$\begin{aligned} \dot{x} &= Ax + Bv \;, \quad x(0) = x_0 \;, \\ y &= Cx \;. \end{aligned} \quad (2.27)$$

We remark that when $\rho \le 0$ then (2.26), (2.27) is a standard linear-quadratic problem. The following two results relate $r_{\mathbb{C}}$ to this optimization problem.

<u>Proposition 2.19 (Proposition 2.3, [7])</u>

In the case where $\sigma(A) \subset \mathbb{C}_-$ we have for $r_{\mathbb{C}} < \infty$

$$\inf_{v \in L^2[0,\infty;\mathbb{C}^m]} J(0,v) \geq 0 \text{ iff } \rho \leq r_{\mathbb{C}}^2 \text{ iff for all } \omega \in \mathbb{R}$$

$$I - \rho G^*(i\omega)G(i\omega) \geq 0 ,$$

where, as before, $G(s) = C(sI-A)^{-1}B$.

The above proposition relates in a straightforward manner the value $r_{\mathbb{C}}$ to the optimization problem starting at zero initial state, whereas the next result gives a nice relation in the case where the system is started from an arbitrary initial state.

<u>Proposition 2.20 (Proposition 3.1, [7])</u>

If $\sigma(A) \subset \mathbb{C}_-$, then for all values $\rho \in (-\infty, r_{\mathbb{C}}^2)$ we have for $r_{\mathbb{C}} < \infty$

$$\left| \inf_{v \in L^2[0,\infty;\mathbb{C}^m]} J(x_0,v) \right| < \infty , \quad \text{for all } x_0 \in \mathbb{C}^n .$$

We note that it is significant in the proof of proposition 2.20 that the parameter ρ is strictly less than $r_{\mathbb{C}}^2$, unlike the characterization we now give in terms of an algebraic Riccati equation.

The algebraic Riccati equation associated with the optimization problem (2.26), (2.27) is

$$PA + A^TP - \rho C^TC - PBB^TP = 0 \ , \qquad (2.28)$$

where in the case $\rho \leq 0$ this is the standard (though parameterized) Riccati equation of optimal control. The following is an extremely important result in the characterization of $r_{\mathbb{C}}$, and we note that no controllability assumptions are placed on the pair (A,B).

Theorem 2.21 (Theorem 3.3, [7])

Suppose $\sigma(A) \subset \mathbb{C}_-$, $r_{\mathbb{C}} < \infty$, $\rho \in (-\infty, r_{\mathbb{C}}^2)$ then there exists a unique solution P_ρ of (2.28) with the property $\sigma(A-BB^TP_\rho) \subset \mathbb{C}_-$. Furthermore when $\rho = r_{\mathbb{C}}^2$ there exists a unique solution $P_{r_{\mathbb{C}}^2}$ of (2.28) with the property $\sigma(A-BB^TP_{r_{\mathbb{C}}^2}) \subset \overline{\mathbb{C}_-}$. For all $\rho \in (-\infty, r_{\mathbb{C}}^2]$, $P_\rho = P_\rho^T$ is real and whenever the pair (A,C) is observable P_ρ is negative definite $(\rho > 0)$.

The characterization is completed by the converse of theorem 2.21, which we now state.

Proposition 2.22 (Proposition 3.4, [7])

We suppose that $\sigma(A) \subset \mathbb{C}_-$. If there exists a real symmetric solution of (2.28) then the parameter ρ satisfies $\rho \leq r_{\mathbb{C}}^2$.

We thus have that $r_{\mathbb{C}}^2$ is the largest value of the parameter ρ such that equation (2.28) has a real symmetric solution. This completes our characterization of the quantity $r_{\mathbb{C}}^2$, though we continue by making some observations on the Riccati equation (2.28). We first, however, make some mention of discrete-time systems, for which the concept of structured stability radius may also be defined in an analogous manner.

For a discrete-time system we define the complex structured stability radius as

$$r_{\mathbb{C}} = r_{\mathbb{C}}(A;B,C) = \inf\{||D||:\sigma(A+BDC) \cap \{z:|z| \geq 1\} \neq \emptyset\} \quad ,$$

then, as before,

$$r_{\mathbb{C}} = \inf\{||D|| : \sigma(A+BDC) \cap e^{i\mathbb{R}} \neq \emptyset\} \quad .$$

Setting $G(z) = C(e^z I-A)^{-1}B$ we have the following analogue of the continuous-time result:

<u>Proposition 2.23</u>

The complex structured stability radius for a discrete-time system, $r_{\mathbb{C}}$, is given by

$$r_{\mathbb{C}} = \begin{cases} \dfrac{1}{\max\limits_{\omega\in[0,2\pi]} ||G(i\omega)||} & \text{if } G \neq 0 , \\ \infty & \text{if } G \equiv 0 . \end{cases}$$

<u>Proof</u>: Similar to the proof in the continuous-time case, see [7].

As in proposition 2.22, we can establish a relationship between $r_{\mathbb{C}}$ and a Riccati equation. The general linear-quadratic problem for a discrete-time system is posed in [32]. He has the linear system

$$x(k+1) = Ax(k) + Bu(k) ,$$
$$k = i,\dots,t , \quad x(i) = \xi , \tag{2.29}$$

$x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^r$. The cost functional is given by

$$J[\xi,u(i,t)] = \sum_{k=i}^{t-1} w[x(k),u(k)] , \tag{2.30}$$

where $w(x,u) = x^TQx + 2x^TSu + u^TRu$, and the control sequence $u(i,t) = (u(i),\dots,u(t-1))$. Associated with (2.29), (2.30) is the Algebraic Riccati Equation

$$K - A^TKA - Q + (S+A^TKB)(R+B^TKB)^+(S^T+B^TKA) = 0 ,$$
$$R + B^TKB \geq 0 , \tag{2.31}$$
$$\mathrm{Ker}(R+B^TKB) \subset \mathrm{Ker}(S+A^TKB) ,$$

where $(\cdot)^+$ denotes the generalized inverse of the matrix - see Chapter 4. In our case we have for (2.31)

$$K - A^TKA + \rho C^TC + A^TKB(I+B^TKB)^+B^TKA = 0 ,$$
$$I + B^TKB \geq 0 , \tag{2.32}$$
$$\mathrm{Ker}(I+B^TKB) \subset \mathrm{Ker}(A^TKB) ,$$

and the following result holds:

Proposition 2.24

If $\sigma(A) \subset \{z: |z| < 1\}$, there exists a real symmetric solution K of (2.32) with the property $I + B^TKB > 0$ only if $\rho \le r_{\mathbb{C}}^2$.

Proof: We have

$$K - A^TKA + A^TKB(B^TKB+I)^{-1}B^TKA = -\rho C^TC \ . \tag{2.33}$$

Set $A_\omega = e^{i\omega}-A$, so from (2.33)

$$I + B^TA_\omega^{*-1}[K-A^TKA+A^TKB(B^TKB+I)^{-1}B^TKA]A_\omega^{-1}B$$
$$= I - \rho B^TA_\omega^{*-1}C^TCA_\omega^{-1}B \ ,$$

whose LHS is

$$I + B^TA_\omega^{*-1}[K-(e^{-i\omega}-A_\omega^*)K(e^{i\omega}-A_\omega) + (e^{-i\omega}-A_\omega^*)KB(B^TKB+I)^{-1} \times B^TK(e^{i\omega}-A_\omega)]A_\omega^{-1}B \ .$$

Expanding, and using the fact that

$$I + B^TKB(B^TKB+I)^{-1}B^TKB-B^TKB = (I+B^TKB)^{-1}$$

we have for the LHS

$$B^TA_\omega^{*-1}e^{-i\omega}KB+B^Te^{i\omega}KA_\omega^{-1}B+B^TA_\omega^{*-1}KB(B^TKB+I)^{-1}B^TKA_\omega^{-1}B$$
$$-B^TA_\omega^{*-1}e^{-i\omega}KB(B^TKB+I)^{-1}B^TKB-B^TKB(B^TKB+I)^{-1} \times B^TKe^{i\omega}A_\omega^{-1}B + (I+B^TKB)^{-1}$$

$$= B^TA_\omega^{*-1}e^{-i\omega}KB(I+B^TKB)^{-1} + (I+B^TKB)^{-1}B^TKe^{i\omega}A_\omega^{-1}B$$
$$+ B^TA_\omega^{*-1}KB(B^TKB+I)^{-1}B^TKA_\omega^{-1}B + (I+B^TKB)^{-1}$$

$$= \quad ((I+B^TKB)^{-\frac{1}{2}} + (I+B^TKB)^{-\frac{1}{2}}B^TKe^{i\omega}A_\omega^{-1}B)^* \times$$
$$((I+B^TKB)^{-\frac{1}{2}} + (I+B^TKB)^{-\frac{1}{2}}B^TKe^{i\omega}A_\omega^{-1}B) \geq 0 \quad .$$

Thus, $I - \rho B^TA_\omega^{*-1}C^TCA_\omega^{-1}B \geq 0$, for all $\omega \in [0,2\pi]$, and the proof is complete.

Turning back now to the algebraic Riccati equation (2.28) the following result shows how the solution parameterized by $\rho \in (-\infty,r_{\mathbb{C}}^2]$ behaves as ρ is varied. In it we set $A_\rho \triangleq A-BB^TP_\rho$, and P_ρ the solution of theorem 2.21.

<u>Proposition 2.25 (Proposition 4.1, [7])</u>

If $\sigma(A) \subset \mathbb{C}_-$ then we have the following,

(a) The maps $\rho \mapsto P_\rho$, $\rho \mapsto A_\rho$ are differentiable on $(-\infty,r_{\mathbb{C}}^2)$, and are continuous on $(-\infty,r_{\mathbb{C}}^2]$.

(b) For values ρ_1,ρ_2 such that $\rho_1 \leq \rho_2 \leq r_{\mathbb{C}}^2$ we have

$$P_{\rho_1} \geq P_{\rho_2} \quad .$$

(c) If $\rho,\rho_1 \leq r_{\mathbb{C}}^2$ then $P \triangleq P_\rho - P_{\rho_1}$ satisfies the algebraic Riccati equation

$$PA_{\rho_1} + A_{\rho_1}^T P - (\rho-\rho_1)C^TC - PBB^TP = 0 \ ,$$

with the additional property $\sigma(A_{\rho_1} - BB^TP) \subset \overline{\mathbb{C}_-}$.

(d) For $\rho \leq r_{\mathbb{C}}^2$ we have the following equality

$$r_{\mathbb{C}}^2(A_\rho;B,C) = r_{\mathbb{C}}^2(A;B,C) - \rho \ .$$

From the above result we see that (from part (b)) as the parameter ρ increases the solution falls in the sense of positive definite matrices (the solution being negative semidefinite for $\rho > 0$) yet (from part (d)) we see that as $\rho \uparrow r_{\mathbb{C}}^2$ the distance to instability of the "feedback" matrix A_ρ goes to zero. On the converse, as the parameter ρ goes to minus infinity the distance to instability of A_ρ becomes arbitrarily large - we note though that the perturbation structure is governed by the matrices B,C and the feedback here is via the B matrix. In the following chapter we show how the stability radius may be improved when the feedback enters via a general control matrix.

Finally the paper also shows the importance of the solution P_ρ of theorem 2.21 in the generation of Lyapunov functions for the asymptotic stability of systems. In Chapter 4 we give a discussion on this subject, and also how such functions in general can establish the boundedness of solutions to systems subject to persistent disturbances.

3. ROBUSTNESS IMPROVEMENT IN STATE SPACE

3.1 INTRODUCTION

Here we show how the measure of robust stability, namely the complex structured stability radius, may be enhanced using state feedback. The use of state feedback in stabilizing systems in the presence of uncertainty has been recognized for some time, [33], [37], [38]. In [33] and [38] feedbacks are constructed using solutions of Riccati equations - in the former case giving robust stability for certain sector bounded perturbations, and in the latter case they are interpreted as yielding certain gain and phase margins. In [37] nonlinear controllers are constructed which stabilize systems for certain classes of uncertainties, and it is shown that in some cases a linear controller may be sufficient. The use of Lyapunov arguments at some stage is common in all of these papers. In [34] it is shown that there is freedom beyond just assigning eigenvalues by using state feedback. In fact it is shown how state feedback may be used in achieving required closed-loop eigenvalues while at the same time classifying all the possible closed-loop eigenvectors that result.

Our problem is to choose F $(r\times n)$ such that

$$r_{\mathbb{C}}(A + DF;B,C) \geq r_{\mathbb{C}}(A;B,C) \tag{3.1}$$

where D $(n\times r)$ is some fixed matrix. The interpretation here is that we wish to improve the complex structured stability radius governed by

(B,C) . The matrices D and F can be interpreted as "input" and "state feedback" matrices, respectively. To do this we make some reference to differential game theory. We recall the characterization of $r_{\mathbb{C}}$ in Chapter 2 in terms of equations (2.26), (2.27). To the cost functional (2.26) we introduce a small penalty term which intuitively allows the value of ρ for which the infimum exists to increase. Correspondingly the dynamics equation (2.27) receives an additional control term. This modification leads naturally to differential game considerations. In the next section, we give a short discussion of linear-quadratic differential games on an infinite time horizon. Section 3.3 then introduces a candidate for our choice of feedback to improve robustness in the sense of (3.1) and we give an analysis of its properties. The section that follows, that is section 3.4, illustrates the preceeding theory showing (with a simple numerical example) what the consequences of implementing such a feedback are. Finally in section 3.5 we give a discussion of necessary and sufficient conditions for an unbounded improvement of the complex structured stability radius, again illustrating the conditions by using numerical examples. For the sake of completeness we include the following (mostly standard) results which will be of use in the sequel.

<u>Lemma 3.1 (Lemma 4, [10]</u>)

If $\dot{x} = Ax + Bu$, $y = C_1x$ is controllable and observable, then there exists a real symmetric solution to

$$A^TK + KA - KBB^TK + C_1^TC_1 = 0$$

which has the property $\text{Re}\,\lambda(A-BB^TK) < 0$ (> 0) . Moreover such a solution is unique and $K > 0$ (< 0) .

<u>Lemma 3.2 (Theorem 23.5, [11])</u>

Let [A,B,C] be a constant minimal realization. Let K be the positive definite solution of

$$A^TK + KA - KBB^TK + C^TC = 0$$

with the property $\text{Re}\,\lambda(A-BB^TK) < 0$. There exists a control which minimizes

$$\eta = \int_0^\infty u^Tu + x^TC^TCx.dt$$

for the system

$$\dot{x} = Ax + Bu \ ; \quad x(0) = x_0 \ .$$

The minimum value of η is $x_0^T Kx_0$. The minimizing control in closed loop form is $u = -B^TKx$.

<u>Lemma 3.3 (Theorem 23.6, [11])</u>

Let A,B and $L = L^T$ be constant matrices. Assume there exists π , a negative definite solution of

$$A^TK + KA - KBB^TK + L = 0$$

such that $\text{Re}\,\lambda(A-BB^T\pi) < 0$. Then there exists a control which minimizes

$$\eta = \int_0^\infty u^T u + x^T Lx.dt$$

for the system $\dot{x} = Ax + Bu$; $x(0) = x_0$. The minimum value of η is $x_0^T \pi x_0$. The minimizing control in closed loop form is $u = -B^T \pi x$.

Lemma 3.4 (Theorem 2.21, Proposition 2.22)

Under the following assumption $\text{Re}\,\lambda(A) < 0$, then there exists a real symmetric solution to

$$A^T K + KA - KBB^T K - C_2^T C_2 = 0$$

having property $\text{Re}\,\lambda(A-BB^T K) \leq 0$ iff

$$I - B^T (i\omega - A)^{*-1} C_2^T C_2 (i\omega - A)^{-1} B \geq 0 \text{ , for all real } \omega \text{ .}$$

Moreover such a solution is unique and $K \leq 0$.

Remarks.

1) Suppose $\text{Re}\,\lambda(A) > 0$. The condition

$$I - B^T (i\omega - A)^{*-1} C_2^T C_2 (i\omega - A)^{-1} B \geq 0$$

is equivalent to

$$I - B^T (i\omega + A)^{*-1} C_2^T C_2 (i\omega + A)^{-1} B \geq 0 \text{ .}$$

So by lemma 3.4 this is equivalent to the existence of a unique Q having the property $\text{Re}\,\lambda(-A-BB^T Q) \leq 0$ satisfying

$$(-A)^T Q + Q(-A) - QBB^T Q - C_2^T C_2 = 0 \text{ .}$$

Such a Q has the property $Q \leq 0$.

Setting $K = -Q \geq 0$ we have a unique K with the property $\mathrm{Re}\,\lambda(A-BB^TK) \geq 0$ satisfying

$$A^TK + KA - KBB^TK - C_2^TC_2 = 0 .$$

2) If (A,C_2) is observable then $K < 0$.

This is so since as $K \leq 0$ $\exists$ $x \neq 0$ s.t. $Kx = 0$. So

$$x^TA^TKx + x^TKAx - x^TK\,BB^TKx - x^TC_2^TC_2x = 0$$

which implies $C_2x = 0$. Multiplying by x on RHS of the Riccati equation

$$A^TKx + KAx - KBB^TKx - C_2^TC_2x = 0$$

which implies $KAx = 0$. Again

$$x^TA^{T2}KAx + x^TA^TKA^2x - x^TA^TKBB^TKAx - x^TA^TC_2^TC_2Ax = 0$$

which implies $C_2Ax = 0$.

Proceeding thus we have $C_2x = 0$, $C_2Ax = 0,\ldots,$ $C_2A^{n-1}x = 0$ and so $x = 0$ by observability. Thus $K < 0$.

<u>Lemma 3.5 (Lemma 3, [10]</u>)

Let K_1 be a real symmetric solution of

$$A^TK_1 + K_1A - K_1BB^TK_1 + Q_1 = 0$$

with Q_1 symmetric and suppose $\mathrm{Re}\,\lambda(A-BB^TK_1) < 0$ (> 0) . Let K_2 be a real symmetric solution of

$$A^TK_2 + K_2A - K_2BB^TK_2 + Q_2 = 0$$

with Q_2 symmetric and $Q_1 \geq Q_2$. Then $K_1 \geq K_2$ $(K_1 \leq K_2)$.

3.2 LINEAR QUADRATIC DIFFERENTIAL GAMES

The infinite duration linear quadratic game is considered in [8] (see also [9], [15]). The finite time zero sum game can be written

$$J_T(u,v;x_0) = \int_0^T [x^TQx + v^TRv - u^TSu].dt \tag{3.2}$$

subject to

$$\dot{x} = Ax + Bv + Cu \ , \quad x(0) = x_0 \ . \tag{3.3}$$

Here u is the maximizer and v the minimizer. A strategy pair (u^0,v^0) is said to be in equilibrium if

$$J_T(u,v^0) \leq J_T(u^0,v^0) \leq J_T(u^0,v) \quad \text{for all} \quad u,v \ . \tag{3.4}$$

Consider the Riccati differential equation

$$\dot{K} + A^TK + KA + Q = K(BR^{-1}B^T - CS^{-1}C^T)K \ , \quad K(T) = 0 \ . \tag{3.5}$$

If this has a solution $K(t;T)$ on $[0,T]$ then the controls

$$\begin{aligned} u^0 &= S^{-1}C^T K(t;T)x \ , \\ v^0 &= -R^{-1}B^TK(t;T)x \ , \end{aligned} \tag{3.6}$$

are in equilibrium and $J_T(u^0,v^0) = x_0^T K(0;T)x_0$. However, there are technical problems with the infinite duration game - if $K^+ = \lim_{T\to\infty} K(0;T)$ then we would expect (as for the standard linear quadratic problem) the strategy pair

$$\begin{aligned} u^0 &= S^{-1}C^TK^+x \ , \\ v^0 &= -R^{-1}B^TK^+x \ , \end{aligned} \tag{3.7}$$

to be in equilibrium in the sense of (3.4). In general, as is shown in [8] with a simple scalar example, this is not true.

We consider the zero sum infinite time linear quadratic differential game

$$J = \int_0^\infty ||v||^2 - \rho||y||^2 - \varepsilon^2||u||^2.dt \tag{3.8}$$

with dynamics given by

$$\begin{aligned} \dot{x} &= Ax + Du + Bv \ , \quad x(0) = x_0 \ , \\ y &= Cx \ . \end{aligned} \tag{3.9}$$

As before u is the maximizing control, v the minimizing control. The associated algebraic Riccati equation is

$$KA + A^TK - \rho C^TC - KBB^TK + \frac{KDD^TK}{\varepsilon^2} = 0 \ . \tag{3.10}$$

We begin with a discussion as to the types of "optimal" solution strategies that may exist.

We first consider the system (3.8), (3.9), and assume there exists a solution $K = K^T < 0$ of (3.10) with the property $\text{Re}\,\lambda(A - BB^TK + \frac{DD^TK}{\varepsilon^2}) < 0$. In this case the strategy pair (3.7) becomes

$$\begin{aligned} u^0 &= \frac{1}{\varepsilon^2} D^TKx , \\ v^0 &= -B^TKx . \end{aligned} \tag{3.11}$$

If the maximizer plays u^0 then the minimizer is faced with the problem

$$\min_v \int_0^\infty ||v||^2 + x^T(-\rho C^TC - \frac{1}{\varepsilon^2} KDD^TK)x.dt$$

and system
$$\dot{x} = (A + \frac{DD^TK}{\varepsilon^2})x + Bv , \quad x(0) = x_0 ,$$
$$y = Cx .$$

Using lemma 3.3, if there exists $\pi < 0$ solving

$$(A + \frac{DD^TK}{\varepsilon^2})^T\pi + \pi(A + \frac{DD^TK}{\varepsilon^2}) - \pi BB^T\pi - \rho C^TC - \frac{1}{\varepsilon^2} KDD^TK = 0$$

and $\text{Re}\,\lambda(A + \frac{DD^TK}{\varepsilon^2} - BB^T\pi) < 0$ then $v = -B^T\pi x$ is minimizing and $J = x_0^T\pi x_0$. Setting $\pi = K$ we see that v^0 is the minimal response to u^0.

Conversely, if the minimizer plays v^0 then the maximizer is faced with

$$\max_u \int_0^\infty - x^T(\rho C^TC - KBB^TK)x - \varepsilon^2||u||^2.dt$$

with
$$\dot{x} = (A - BB^TK)x + Du , \quad x(0) = x_0 ,$$
$$y = Cx .$$

Using lemma 3.2 above (assuming $\rho C^TC > KBB^TK$) if $[A-BB^TK,D,(\rho C^TC-KBB^TK)^{\frac{1}{2}}]$ is minimal then setting $\pi_1 > 0$ as the unique solution of

$$(A-BB^TK)^T\pi_1 + \pi_1(A-BB^TK)-\pi_1 \frac{DD^T}{\varepsilon^2}\pi_1 + \rho C^TC-KBB^TK = 0$$

with the property $\mathrm{Re}\ \lambda(A-BB^TK-\frac{DD^T}{\varepsilon^2}\pi_1) < 0$ the maximizing control is $u = -\frac{D^T\pi_1 x}{\varepsilon^2}$. By uniqueness of solutions $\pi_1 = -K$, and so $u = u^0$. Summarizing this we have:

<u>Proposition 3.6</u>

If there exists a solution $K = K^T < 0$ of (3.10) with the properties $\mathrm{Re}\ \lambda(A-BB^TK + \frac{DD^TK}{\varepsilon^2}) < 0$, $\rho C^TC > KBB^TK$, and $[A-BB^TK,D,(\rho C^TC-KBB^TK)^{\frac{1}{2}}]$ minimal, then the controls u^0,v^0 of (3.11) are in equilibrium in the sense of (3.4).

The quantities $C(i\omega-A)^{-1}$, $C(i\omega-A)^{-1}B$, $C(i\omega-A)^{-1}D$ will be constantly referred to in the following so we denote them by G, G_B, G_D, respectively.

We continue by deriving necessary conditions under which "min max" and "max min" strategies exist in an appropriate sense. The frequency domain condition, (fdc), is important in this respect (and in future, as we shall see):

$$I-\rho G_B^*G_B+\rho^2 G_B^*G_D(\rho G_D^*G_D+\varepsilon^2)^{-1}G_D^*G_B \geq 0 \text{ , for all } \omega \in \mathbb{R} . \qquad (3.12)$$

We have the following propositions which are dual to each other, and a lemma regarding the (fdc), (3.12).

Proposition 3.7

For the cost functional J of (3.8), and dynamics given by (3.9) we have $\inf_v \sup_u J$ is finite only if

$$I-\rho G_B^* G_B + \rho^2 G_B^* G_D(\rho G_D^* G_D + \varepsilon^2)^{-1} G_D^* G_B \geq 0 , \quad \text{for all} \quad \omega \in \mathbb{R} .$$

Proof: Taking the Fourier-Plancherel transform of (3.9) we have (abusing notation and using the same symbols for the transformed variables)

$$i\omega x = Ax + Du + Bv + x_0 ,$$

$$y = Cx .$$

(As u,v are L^2-functions and A is stable we see that x is an L^2-function so that the transforms make sense).

This may be written

$$y = G_D u + G_B v + G x_0. \tag{3.13}$$

By Parseval's theorem (and omitting the factor $1/2\pi$) the cost may be written as, using (3.13),

$$J = \int_{-\infty}^{\infty} v^*(i\omega)v(i\omega)-\rho(G_D u(i\omega)+G_B v(i\omega)+Gx_0)^*(G_D u(i\omega) + G_B v(i\omega)+Gx_0) -\varepsilon^2 u^*(i\omega)u(i\omega).d\omega \tag{3.14}$$

$$= \int_{-\infty}^{\infty} -[u^*(\rho G_D^* G_D+\varepsilon^2)u+2\rho \text{Re }u^* G_D^* G_B v+2\rho \text{Re }u^* G_D^* G x_0]$$

$$+ v^*(I-\rho G_B^* G_B)v-2\rho \text{ Re }v^* G_B^* G x_0 - \rho\, x_0^* G^* G x_0 . d\omega \quad .$$

Set $H = (\rho G_D^* G_D+\varepsilon^2)^{\frac{1}{2}}$, the Hermitian square root, and complete the square in the first term. This yields

$$\int_{-\infty}^{\infty} -[Hu+H^{-1}\rho G_D^*(G_B v+Gx_0)]^*[Hu+H^{-1}\rho G_D^*(G_B v+Gx_0)]$$

$$+ \rho^2(G_B v+Gx_0)^* G_D(\rho G_D^* G_D+\varepsilon^2)^{-1} G_D^*(G_B v + Gx_0)$$

$$+ v^*(I-\rho G_B^* G_B)v-2\rho \text{Re }v^* G_B^* G x_0-\rho x_0^* G^* G x_0 . d\omega$$

$$= \int_{-\infty}^{\infty} -[Hu + H^{-1}\rho G_D^*(G_B v + Gx_0)]^*[\quad]$$

$$+ v^*[I-\rho G_B^* G_B + \rho^2 G_B^* G_D(\rho G_D^* G_D+\varepsilon^2)^{-1} G_D^* G_B]v$$

$$+ 2\rho^2 \text{Re }v^* G_B^* G_D(\rho G_D^* G_D+\varepsilon^2)^{-1} G_D^* G x_0-2\rho \text{Re }v^* G_B^* G x_0$$

$$+ \rho^2 x_0^* G^* G_D(\rho G_D^* G_D+\varepsilon^2)^{-1} G_D^* G x_0 - \rho x_0^* G^* G x_0 . d\omega \quad .$$

From this last expression we see that the conditions in the statement of the proposition imply (3.12), or,

$$I - \rho G_B^* G_B + \rho^2 G_B^* G_D(\rho G_D^* G_D + \varepsilon^2)^{-1} G_D^* G_B \geq 0 \ ,$$

for all $\omega \in \mathbb{R}$, which is the expression in the proposition statement.

The dual of proposition 3.7 is as follows.

Proposition 3.8

For the cost J of (3.8), dynamics (3.9), we have $\sup_u \inf_v J$ is finite only if $I - \rho G_B^* G_B \geq 0$, for all $\omega \in \mathbb{R}$.

Proof: We proceed as in 3.7.

We have $y = G_D u + G_B v + G x_0$, and the cost J can be written

$$J = \int_{-\infty}^{\infty} [v^*(I - \rho G_B^* G_B)v - 2\rho \text{ Re } v^* G_B^* G_D u - 2\rho \text{ Re } v^* G_B^* G x_0]$$

$$- u^*(\rho G_D^* G_D + \varepsilon^2)u - 2\rho \text{ Re } u^* G_D^* G x_0$$

$$- \rho x_0^* G^* G x_0 . d\omega \quad .$$

An examination of this last expression shows that necessarily we must have, under the conditions of the proposition, $I - \rho G_B^* G_B \geq 0$, for all $\omega \in \mathbb{R}$.

We give now an alternative characterization of the (fdc), (3.12). This characterization will turn out to be important in our central aim of improving the complex structured stability radius which we will turn to immediately in the section following this result.

Lemma 3.9

The following frequency domain conditions a), b) are equivalent

a) $I + \rho[\frac{G_D G_D^*}{\varepsilon^2} - G_B G_B^*] \geq 0$, for all $\omega \in \mathbb{R}$; (3.15)

b) $I - \rho G_B^* G_B + \rho^2 G_B^* G_D (\rho G_D^* G_D + \varepsilon^2)^{-1} G_D^* G_B \geq 0$,
for all $\omega \in \mathbb{R}$, see (3.12).

Proof: b) can be rewritten as

$$I - \rho G_B^* G_B + \rho G_B^* G_D \left(\frac{\rho}{\varepsilon^2} G_D^* G_D + I\right)^{-1} \frac{\rho}{\varepsilon^2} G_D^* G_B \geq 0 \ .$$

We now use the well-known identity

$$G(I + FG)^{-1} = (I + GF)^{-1} G$$

to write this as

$$I - \rho G_B^* G_B + \rho G_B^* G_D \frac{\rho}{\varepsilon^2} G_D^* \left(I + \frac{\rho}{\varepsilon^2} G_D G_D^*\right)^{-1} G_B \geq 0 \ ,$$

and then the identity

$$(I + GF)^{-1} = I - GF(I + GF)^{-1}$$

applied to

$$I - \rho G_B^* [I - G_D G_D^* \frac{\rho}{\varepsilon^2} (I + G_D G_D^* \frac{\rho}{\varepsilon^2})^{-1}] G_B \geq 0 ,$$

gives

$$I - \rho G_B^* (I + G_D G_D^* \frac{\rho}{\varepsilon^2})^{-1} G_B \geq 0$$

or

$$I - \rho \varepsilon^2 G_B^* (\varepsilon^2 I + \rho G_D G_D^*)^{-1} G_B \geq 0 \ .$$

As $\varepsilon^2 I + \rho G_D G_D^* > 0$ we can set $\varepsilon^2 I + \rho\, G_D G_D^* = NN$, $N > 0$, $N = N^*$ and N unique. Then $(\varepsilon^2 I + \rho\, G_D G_D^*)^{-1} = N^{-1} N^{-1}$ and the latter condition becomes

$$I \geq \rho \varepsilon^2 G_B^* N^{-1} N^{-1} G_B \ . \tag{3.16}$$

We note that a) can be written

$$NN \geq \varepsilon^2 \rho\, G_B G_B^*$$

or equivalently

$$I \geq \varepsilon^2 \rho N^{-1} G_B G_B^* N^{-1} \quad . \qquad (3.17)$$

Condition (3.16) is equivalent to $\bar{\sigma}(\sqrt{\rho}\varepsilon N^{-1} G_B) \leq 1$, and (3.17) is equivalent to $\bar{\sigma}(\sqrt{\rho}\varepsilon G_B^* N^{-1}) \leq 1$. Using the well known fact that $\bar{\sigma}(M) = \bar{\sigma}(M^*)$, (3.16) and (3.17) are equivalent which in turn implies the equivalence of a), b) in the statement of the lemma.

3.3 ROBUSTNESS IMPROVEMENT VIA STATE FEEDBACK

In this section we show how a particular state feedback may be used to improve the complex stability radius for structured perturbations. In what follows we shall be interested in the situation when there is a real symmetric negative definite solution of (3.10). The following proposition gives a necessary condition for this to be the case. We assume that the triple [A,D,C] is minimal - a standing assumption.

Proposition 3.10

There exists a real symmetric negative definite solution of (3.10) only if

$$I + \rho\left[\frac{G_D G_D^*}{\varepsilon^2} - G_B G_B^*\right] \geq 0 \ , \quad \text{for all } \omega \in \mathbf{R} \ , \quad (3.15).$$

Proof: Equation (3.10) is $KA + A^T K - \rho C^T C - KBB^T K + \frac{KDD^T K}{\varepsilon^2} = 0$. Suppose there exists a negative definite solution K to (3.10). Then it is invertible and, setting $P = K^{-1}$, (3.10) becomes

$$AP + PA^T - \rho PC^T CP + \left[\frac{DD^T}{\varepsilon^2} - BB^T\right] = 0 \ .$$

So adding and subtracting $i\omega P$ to the left hand side

$$(i\omega-A)P + P(i\omega-A)^* - \left[\frac{DD^T}{\varepsilon^2} - BB^T\right] + \rho\, PC^TCP = 0 .$$

Multiplying by $\sqrt{\rho}C(i\omega-A)^{-1}$ on the left, and by its conjugate-transpose on the right,

$$\rho CP(i\omega-A)^{*-1}C^T + \rho C(i\omega-A)^{-1}PC^T - \rho C(i\omega-A)^{-1}\left[\frac{DD^T}{\varepsilon^2} - BB^T\right](i\omega-A)^{*-1}C^T$$
$$+ \rho^2 C(i\omega-A)^{-1}PC^TCP(i\omega-A)^{*-1}C^T = 0 .$$

Completing the square, as usual, we have

$$0 \le [I + \rho CP(i\omega-A)^{*-1}C^T]^*[I + \rho CP(i\omega-A)^{*-1}C^T]$$
$$= I + \rho C(i\omega-A)^{-1}\left[\frac{DD^T}{\varepsilon^2} - BB^T\right](i\omega-A)^{*-1}C^T$$
$$= I + \rho\left[\frac{G_D G_D^*}{\varepsilon^2} - G_B G_B^*\right] , \quad \text{for all} \quad \omega \in \mathbb{R} ,$$

and hence the inequality (3.15). This completes the proof of the proposition.

A derivation of a sufficient condition is an extremely difficult problem, [17]. However, we may make some observations on the "inverse problem":

$$AP + PA^T - \rho PC^TCP + \frac{DD^T}{\varepsilon^2} - BB^T = 0 , \qquad (3.18)$$

in much the same way as in [10].

Proposition 3.11

We assume [A,D,C] is minimal. If the frequency domain condition (3.15) holds then there exist unique solutions P^+, P^- of (3.18) such that $\mathrm{Re}\,\lambda(A^T-\rho C^TCP^+) \le 0$, $\mathrm{Re}\,\lambda(A^T-\rho C^TCP^-) \ge 0$. For any other real symmetric solution P, $P^- \le P \le P^+$. If $I + \rho\left[\frac{G_DG_D^*}{\varepsilon^2} - G_BG_B^*\right] \ge \delta GG^*$ for some $\delta > 0$ then $P^+ > P^-$, $\mathrm{Re}\,\lambda(A^T-\rho C^TCP^+) < 0$, and $\mathrm{Re}\,\lambda(A^T-\rho C^TCP^-) > 0$.

Proof: We start by proving the first part. We may apply lemma 3.1 to the equation

$$AP_1 + P_1A^T - \rho P_1C^TCP_1 + \frac{DD^T}{\varepsilon^2} = 0\,, \qquad (3.19)$$

to give a solution $P_1 < 0$ with the property $\mathrm{Re}\,\lambda(A_1^T) > 0$, where $A_1^T = A^T - \rho C^TCP_1$.

We now set $\Delta P = P-P_1$, then subtracting (3.19) from (3.18)

$$A(P-P_1) + (P-P_1)A^T - \rho(PC^TCP-P_1C^TCP_1) - BB^T = 0\,,$$

or

$$A(P-P_1) + (P-P_1)A^T - \rho((P-P_1)C^TC(P-P_1) + PC^TCP_1 + P_1C^TCP - 2P_1C^TCP_1) - BB^T = 0\,,$$

or using the definition of A_1

$$A_1\Delta P + \Delta PA_1^T - \rho\Delta PC^TC\Delta P - BB^T = 0\,, \qquad (3.20)$$

to be solved for ΔP. Now under the assumption

$$I - \rho C(i\omega-A_1^T)^{*-1}BB^T(i\omega-A_1^T)^{-1}C^T \ge 0\,, \text{ for all } \omega \in \mathbb{R}\,, \qquad (3.21)$$

then (3.20) has a solution $\Delta P \geq 0$ with the property $\operatorname{Re} \lambda(A_1^T - \rho C^T C \Delta P) \geq 0$, by lemma 3.4. (Remark: we note no controllability or observability assumptions are made here, even though the observability of $[A_1, C]$ is a consequence of the observability of $[A, C]$).

If $I + \rho\left[\frac{G_D G_D^*}{\varepsilon^2} - G_B G_B^*\right] \geq 0$, for all $\omega \in \mathbb{R}$, then $I + \rho C(i\omega - A)^{-1}\left[\frac{DD^T}{\varepsilon^2} - BB^T\right](i\omega - A)^{*-1}C^T \geq 0$, for all $\omega \in \mathbb{R}$. Now adding and subtracting $i\omega P_1$ to (3.19) we have

$$(i\omega - A)P_1 + P_1(-i\omega - A^T) + \rho P_1 C^T C P_1 - \frac{DD^T}{\varepsilon^2} = 0 .$$

Pre and post-multiplying this equation by $\sqrt{\rho}C(i\omega - A)^{-1}$, $\sqrt{\rho}(i\omega - A)^{*-1}C^T$ we have

$$\rho C P_1 (i\omega - A)^{*-1} C^T + \rho C (i\omega - A)^{-1} P_1 C^T + \rho^2 C (i\omega - A)^{-1} P_1 C^T C P_1 (i\omega - A)^{*-1} C^T$$
$$- \rho C (i\omega - A)^{-1} \frac{DD^T}{\varepsilon^2} (i\omega - A)^{*-1} C^T = 0 .$$

Completing the square and subtracting the term $[\rho C(i\omega - A)^{-1} BB^T (i\omega - A)^{*-1} C^T]$ on both sides we have

$$[I + \rho C P_1 (i\omega - A)^{*-1} C^T]^* [I + \rho C P_1 (i\omega - A)^{*-1} C^T] - \rho C (i\omega - A)^{-1} BB^T (i\omega - A)^{*-1} C^T$$
$$= I + \rho\left[\frac{G_D G_D^*}{\varepsilon^2} - G_B G_B^*\right] \geq 0 \text{, by assumption.} \qquad (3.22)$$

We now establish an identity to be used in conjunction with (3.22). The following identity is obvious:

$$((i\omega - A)^* + \rho C^T C P_1)(i\omega - A)^{*-1} C^T$$
$$= C^T (I + \rho C P_1 (i\omega - A)^{*-1} C^T) ,$$

and premultiplying by $B^T((i\omega-A)^* + \rho C^T CP_1)^{-1}$, postmultiplying by $(I + \rho CP_1(i\omega-A)^{*-1}C^T)^{-1}$ we obtain

$$B^T(i\omega-A)^{*-1}C^T(I+\rho CP_1(i\omega-A)^{*-1}C^T)^{-1} = B^T((i\omega-A)^*+\rho C^T CP_1)^{-1}C^T . \quad (3.23)$$

We multiply (3.22) on the left by $[I + \rho CP_1(i\omega-A)^{*-1}C^T]^{*-1}$ and on the right by $[I + \rho CP_1(i\omega-A)^{*-1}C^T]^{-1}$ to get the inequality

$$I-\rho[I+\rho CP_1(i\omega-A)^{*-1}C^T]^{*-1}C(i\omega-A)^{-1}BB^T(i\omega-A)^{*-1}C^T[I+\rho CP_1(i\omega-A)^{*-1}C^T]^{-1} \geq 0 ,$$

which, by using (3.23), is equivalent to

$$I-\rho C(i\omega-A + \rho P_1 C^T C)^{-1}BB^T(-i\omega-A^T+\rho C^T CP_1)^{-1}C^T \geq 0 ,$$

for all $\omega \in \mathbb{R}$. Taking account of the definition of A_1 above we see that this last inequality is just (3.21). We thus have a solution $\Delta P \geq 0$ to equation (3.20). Consequently we have a solution P' to (3.18) with property $\text{Re}\ \lambda(A_1^T-\rho C^T C\Delta P) \geq 0$ or $\text{Re}\ \lambda(A^T-\rho C^T CP) \geq 0$, which is P^- . P^+ satisfying (3.18) with the property $\text{Re}\ \lambda(A^T-\rho C^T CP^+) \leq 0$ is proven similarly.

That any other real symmetric solution P satisfies $P^- \leq P \leq P^+$ follows on application of lemma 3.5.

Finally, suppose $I + \rho\left[\frac{G_D G_D^*}{\varepsilon^2} - G_B G_B^*\right] \geq \delta GG^*$ for some $\delta > 0$, then $I + \rho\left[\frac{G_D G_D^*}{\varepsilon^2} - G_B G_B^*\right] \geq 0$. So for P_δ such that $AP_\delta + P_\delta A^T - \rho P_\delta C^T CP_\delta + \frac{DD^T}{\varepsilon^2} - BB^T - \delta I = 0$, and P such that $AP + PA^T - \rho PC^T CP + \frac{DD^T}{\varepsilon^2} - BB^T = 0$ we have by lemma 3.5 $P^- < P_\delta^- \leq P_\delta^+ < P^+$

so that $P^- < P^+$. In turn this implies, for P^+, P^- satisfying (3.18) that

$$(A-\rho P^+C^TC)(P^+-P^-) + (P^+-P^-)(A^T-\rho C^TCP^+) + \rho(P^+-P^-)C^TC(P^+-P^-) = 0 .$$

Using observability of $[A,C]$, we have $\mathrm{Re}\,\lambda(A-\rho P^+C^TC) < 0$. The proof that $\mathrm{Re}\,\lambda(A-\rho P^-C^TC) > 0$ is along similar lines.

We proceed now with our main aim, in other words that of improving the complex stability radius, or more precisely that of choosing F so that $r_{\mathbb{C}}(A + DF;B,C) \geq r_{\mathbb{C}}(A;B,C)$. We suppose that (3.10) has a negative definite real symmetric solution, K . The following theorems show how K may be used to improve the stability radius. We first state the following result on Lyapunov equations.

<u>Lemma 3.12 (Theorem 11.3, [11])</u>

If $\mathrm{Re}\,\lambda(A) < 0$, then $A^TQ + QA + M = 0$ has a unique solution Q . Furthermore the solution, Q , will be given by the convergent integral

$$Q = \int_0^\infty e^{A^Tt}Me^{At}.dt . \tag{3.24}$$

<u>Theorem 3.13</u>

Suppose (3.10) is satisfied by $K < 0$, then the matrix $(A + \frac{DD^TK}{\varepsilon^2})$ is asymptotically stable.

<u>Proof</u>: For convenience we rewrite (3.10) by setting $P = -K > 0$.

Then we have $P > 0$ satisfying

$$PA + A^TP + \rho C^TC + PBB^TP - \frac{PDD^TP}{\varepsilon^2} = 0 \quad ,$$

which by adding and subtracting the term $\frac{PDD^TP}{\varepsilon^2}$ on the LHS becomes

$$P(A - \frac{DD^TP}{\varepsilon^2}) + (A - \frac{DD^TP}{\varepsilon^2})^TP + \rho C^TC+PBB^TP + \frac{PDD^TP}{\varepsilon^2} = 0 \; . \tag{3.25}$$

For the system

$$\dot{x} = (A - \frac{DD^TP}{\varepsilon^2})x \; , \quad x(0) = x_0 \in K^n \; , \tag{3.26}$$

set $v(x) = \langle x,Px\rangle$ ($\langle\cdot,\cdot\rangle$ the standard inner product on K^n) , then

$$\dot{v}(x) = \langle\dot{x},Px\rangle + \langle x,P\dot{x}\rangle = \langle(A- \frac{DD^TP}{\varepsilon^2})x,Px\rangle + \langle x,P(A- \frac{DD^TP}{\varepsilon^2})x\rangle$$

$$= \langle x,[P(A- \frac{DD^TP}{\varepsilon^2}) + (A- \frac{DD^TP}{\varepsilon^2})^TP]x\rangle$$

by (3.25)

$$= -\langle x,(\rho C^TC + PBB^TP + \frac{PDD^TP}{\varepsilon^2})x\rangle$$

$$< 0 \; ,$$ under our standing assumptions on the pair $[A,C]$. So $v(x)$ is a Lyapunov function for (3.26) with $\dot{v}(x) < 0$ so by Lyapunov's asymptotic stability theorem $(A - \frac{DD^TP}{\varepsilon^2})$ is asymptotically stable, i.e. $\mathrm{Re}\,\lambda(A - \frac{DD^TP}{\varepsilon^2}) < 0$ for all eigenvalues λ .

Consequently, by lemma 3.12, using (3.24) we have

$$P = \int_0^\infty e^{(A-\frac{DD^TP}{\varepsilon^2})^Tt} [\rho C^TC+PBB^TP + \frac{PDD^TP}{\varepsilon^2}] e^{(A-\frac{DD^TP}{\varepsilon^2})t} .dt \; .$$

This completes the proof.

We have thus established the stability of $(A + \frac{DD^TK}{\varepsilon^2})$; the next theorem shows that for $F = \frac{D^TK}{\varepsilon^2}$ we have $r_{\mathbb{C}}(A+DF;B,C) \geq \sqrt{\rho}$.

Theorem 3.14

If (3.10) is satisfied by $K < 0$, then for the asymptotically stable matrix $(A + \frac{DD^TK}{\varepsilon^2})$ we have $I - \rho B^T(i\omega - A - \frac{DD^TK}{\varepsilon^2})^{*-1}C^TC(i\omega - A - \frac{DD^TK}{\varepsilon^2})^{-1}B \geq 0$ for all $\omega \in \mathbb{R}$. This implies $\sqrt{\rho} \leq r_{\mathbb{C}}(A+DD^TK/\varepsilon^2;B,C)$.

Proof: We have the existence of K satisfying

$$-KA-A^TK + \rho C^TC + KBB^TK - \frac{KDD^TK}{\varepsilon^2} = 0$$

so adding and subtracting both $i\omega K$ and $\frac{KDD^TK}{\varepsilon^2}$ on the LHS we have

$$K(i\omega - A - \frac{DD^TK}{\varepsilon^2}) + (i\omega - A - \frac{DD^TK}{\varepsilon^2})^*K + \rho C^TC + KBB^TK + \frac{KDD^TK}{\varepsilon^2} = 0 . \tag{3.27}$$

Setting $A_\omega = (i\omega - A - \frac{DD^TK}{\varepsilon^2})$ for notational simplicity, (3.27) becomes

$$KA_\omega + A_\omega^*K + \rho C^TC + KBB^TK + \frac{KDD^TK}{\varepsilon^2} = 0 .$$

We premultiply this by $B^TA_\omega^{*-1}$, and postmultiply by $A_\omega^{-1}B$ to get

$$B^TA_\omega^{*-1}KB + B^TKA_\omega^{-1}B + \rho B^TA_\omega^{*-1}C^TCA_\omega^{-1}B$$
$$+ B^TA_\omega^{*-1}KBB^TKA_\omega^{-1}B + B^TA_\omega^{*-1}\frac{KDD^TK}{\varepsilon^2}A_\omega^{-1}B = 0 .$$

This last expression shows that $I - \rho B^TA_\omega^{*-1}C^TCA_\omega^{-1}B =$
$I + B^TA_\omega^{*-1}KB + B^TKA_\omega^{-1}B + B^TA_\omega^{*-1}KBB^TKA_\omega^{-1}B + B^TA_\omega^{*-1}\frac{KDD^TK}{\varepsilon^2}A_\omega^{-1}B$

and completing the square on the RHS we have

$$= [I + B^T K A_\omega^{-1} B]^* [I + B^T K A_\omega^{-1} B] + \frac{1}{\varepsilon^2} (D^T K A_\omega^{-1} B)^* (D^T K A_\omega^{-1} B)$$

≥ 0 , for all $\omega \in \mathbf{R}$, thus completing the proof.

The picture given by the two previous theorems is that we wish to increase the parameter ρ above $r_{\mathbb{C}}^2(A;B,C)$ whilst maintaining the strict negativity of the solution K of (3.10) - though in this respect we cannot do better than the necessary condition of proposition 3.10.

For $\varepsilon = \infty$, $\rho < r_{\mathbb{C}}^2(A;B,C)$ we have from arguments using theorem 2.21, theorem 5 [10], real symmetric solutions K^+, K^- of (3.10) such that $K^+ \leq K \leq K^- < 0$ (K is any other real symmetric solution). Setting $K = P^{-1}$ we have solutions P^+, P^- of (3.18) such that $P^- \leq P^+ < 0$, and any other real symmetric solution, P, is such that $P \geq P^-$. Thus motivated, we restrict our attention to P^- in (3.18) given by proposition 3.11. Theorem 3.17 below shows that, provided

$I + \rho_1 \left[\frac{G_D G_D^*}{\varepsilon_1^2} - G_B G_B^* \right] \geq \delta G G^*$ for some $\delta > 0$, it is best for increasing ρ to let $\varepsilon \downarrow 0$ in (3.18).

We first give two propositions showing how P^-, P^+, K^-, K^+ behave with respect to ρ, ε where P^+, P^- are the maximal, minimal (respectively) solutions of (3.18) and $K = P^{-1}$.

<u>Proposition 3.15</u>

If $I + \rho \left[\frac{G_D G_D^*}{\varepsilon^2} - G_B G_B^* \right] \geq \delta G G^*$ for some $\delta > 0$ then for P^+, P^-

satisfying (3.18) we have $\frac{dP^+}{d\rho} \le 0$, $\frac{dP^-}{d\rho} \ge 0$, $\frac{dP^+}{d\varepsilon} \le 0$, $\frac{dP^-}{d\varepsilon} \ge 0$.

Moreover, if P is nonsingular then setting $K = P^{-1}$ we have $\frac{dK^+}{d\rho} \ge 0$, $\frac{dK^-}{d\rho} \le 0$, $\frac{dK^+}{d\varepsilon} \ge 0$, $\frac{dK^-}{d\varepsilon} \le 0$.

Proof: Set $\alpha = \frac{1}{\varepsilon^2}$, so $AP + PA^T - \rho PC^TCP + \alpha DD^T - BB^T = 0$, so that $AdP + dPA^T - d\rho PC^TCP - \rho dPC^TCP - \rho PC^TCdP + d\alpha DD^T = 0$. Setting $d\alpha = 0$ we have

$$(A-\rho PC^TC)dP + dP(A-\rho PC^TC)^T - d\rho PC^TCP = 0$$

and so $\frac{dP^+}{d\rho} \le 0$, $\frac{dP^-}{d\rho} \ge 0$. Similarly setting $d\rho = 0$ we have $\frac{dP^+}{d\alpha} \ge 0$, $\frac{dP^-}{d\alpha} \le 0$. This shows the first part.

If P is nonsingular then setting $K = P^{-1}$ we have $dK = -P^{-1}dPP^{-1}$. So $\frac{dK^+}{d\rho} \ge 0$, $\frac{dK^-}{d\rho} \le 0$, $\frac{dK^+}{d\alpha} \le 0$, $\frac{dK^-}{d\alpha} \ge 0$, which completes the proof.

Using the decomposition $P = \Delta P + P_1$ as in proposition 3.11 we may obtain further information by calculating the derivatives of $\Delta P, P_1$. These are not sign definite in general. We take for the sake of discussion the solution $P_1 < 0$ of

$$AP_1 + P_1A^T - \rho P_1C^TCP_1 + \frac{DD^T}{\varepsilon^2} = 0 \tag{3.28}$$

with $\mathrm{Re}\,\lambda(A_1) > 0$, where $A_1 = A-\rho P_1 C^T C$, and $\Delta P \geq 0$ of

$$A_1 \Delta P + \Delta P A_1^T - \rho \Delta P C^T C \Delta P - BB^T = 0 \tag{3.29}$$

with $\mathrm{Re}\,\lambda(A_2) > 0$, where $A_2 = A - \rho P C^T C = A_1 - \rho \Delta P C^T C$.

From (3.28) we have, as in the previous proposition,

$$\frac{dP_1}{d\rho} = \int_0^\infty e^{-A_1 t} P_1 C^T C P_1 \, e^{-A_1^T t} .dt \geq 0 \ . \tag{3.30}$$

From (3.29)

$$(-A_2)d\Delta P + d\Delta P(-A_2)^T + \rho(dP_1 C^T C\Delta P + \Delta P C^T C dP_1) + d\rho(PC^T CP - P_1 C^T CP_1) = 0 \ .$$

Using (3.30) the term $\rho(\frac{dP_1}{d\rho} C^T C\Delta P + \Delta P C^T C \frac{dP_1}{d\rho})$ becomes

$$P_1 C^T CP_1 - \int_0^\infty e^{-A_1 t} P_1 C^T CP_1 e^{-A_1^T t} .dt \, A_2^T - A_2 \int_0^\infty e^{-A_1 t} P_1 C^T CP_1 e^{-A_1^T t} .dt \ .$$

So we have:

<u>Proposition 3.16</u>

If $I + \rho\left[\frac{G_D G_D^*}{\varepsilon^2} - G_B G_B^*\right] \geq \delta GG^*$ for some $\delta > 0$ then for the solution P^- of (3.18)

$$\frac{dP_1}{d\rho} = \int_0^\infty e^{-A_1 t} P_1 C^T CP_1 \, e^{-A_1^T t} .dt \geq 0$$

and $\frac{d\Delta P}{d\rho}$ satisfies the Lyapunov equation

$$(-A_2)\frac{d\Delta P}{d\rho} + \frac{d\Delta P}{d\rho}(-A_2)^T - \int_0^\infty e^{-A_1 t} P_1 C^T C P_1 e^{-A_1^T t}.dt\, A_2^T - A_2 \int_0^\infty e^{-A_1 t} P_1 C^T C P_1 e^{-A_1^T t}.dt + PC^T CP = 0 .$$

Theorem 3.17

We suppose $I + \rho_1 \left[\frac{G_D G_D^*}{\varepsilon_1^2} - G_B G_B^*\right] \geq \delta GG^*$ for some $\delta > 0$. If for equation (3.18) when $\rho = \rho_1$, $\varepsilon = \varepsilon_1$ we have $P_1^- < 0$ then for all $\rho \in [0,\rho_1]$, $\varepsilon < \varepsilon_1$ we have $P^- < 0$.

Proof: We take (3.18) for ρ_1, ε_1, P_1^- and for ρ_2, ε_2, P_2^-. On subtraction of these two equations:

$$(P_1^- - P_2^-)A^T + A(P_1^- - P_2^-) - \rho_1 P_1^- C^T C P_1^- + \rho_2 P_2^- C^T C P_2^- + \left(\frac{1}{\varepsilon_1^2} - \frac{1}{\varepsilon_2^2}\right)DD^T = 0 \text{ , or}$$

$$(P_1^- - P_2^-)(A^T - \rho_2 C^T C P_2^-) + (A - \rho_2 P_2^- C^T C)(P_1^- - P_2^-) - \rho_2 (P_1^- - P_2^-) C^T C (P_1^- - P_2^-)$$
$$- (\rho_1 - \rho_2) P_1^- C^T C P_1^- + \left(\frac{1}{\varepsilon_1^2} - \frac{1}{\varepsilon_2^2}\right)DD^T = 0 .$$

Under the conditions of the theorem and the fact that $\mathrm{Re}\,\lambda(A - \rho_2 P_2^- C^T C) > 0$ we have $P_2^- \leq P_1^- < 0$. The fact that the strict fdc of the theorem statement holds means by proposition 3.11 that P_2^- cannot escape to minus infinity. The proof is thus complete.

Of course this theorem does not answer the question of exactly by how much we may increase ρ. We say some more on this aspect after we have made some observations on how we might solve (3.10) as $\varepsilon \downarrow 0$. The first concerns the behaviour of K as ε falls. For convenience we transform (3.10) to the form (by setting $P = -K$)

$$PA + A^TP + \rho C^TC + PBB^TP - \frac{PDD^TP}{\varepsilon^2} = 0 . \qquad (3.31)$$

By the above we have $(-P^-)^{-1} > 0$ satisfying this latter equation. We consider also the differential Riccati equation

$$\dot{P} + A^TP + PA + \rho C^TC + PBB^TP - \frac{PDD^TP}{\varepsilon^2} = 0 , \quad P(T;T) = 0 , \quad (3.32)$$

and state the following two lemmas regarding it, [8]. We note the dependence of P, $P(t;T)$ on ε by using P_ε, $P_\varepsilon(t;T)$, respectively. (Here P_ε denotes $\lim_{T\to\infty} P_\varepsilon(0;T)$, similarly K_ε for $\lim_{T\to\infty} K_\varepsilon(0;T)$).

Lemma 3.18 (Lemma A2, [8])

For (3.32) $P_\varepsilon(t;T)$ increases in the sense of positive definite matrices with decreasing t (providing it does not "blow-up").

The following lemma ensures the solution cannot escape to infinity.

Lemma 3.19 (Lemma A3, [8])

Let $(-P^-)^{-1} = (-P^-)^{-1T}$ be a nonnegative solution to (3.31). Then $P_\varepsilon(t;T) \leq (-P^-)^{-1}$ for all $t \in [0,T]$.

Now for the finite horizon differential game we have

$$J_T(\varepsilon,u,v) \triangleq J_T = \int_0^T ||v||^2 - \rho||y||^2 - \varepsilon^2||u||^2.dt .$$

If $\bar{\varepsilon} < \varepsilon$ then for all v

$$\max_u J_T(\varepsilon,u,v) \le J_T(\bar{\varepsilon},u,v) \le \max_u J_T(\bar{\varepsilon},u,v)$$

$$\text{i.e.} \quad \max_u J_T(\varepsilon,u,v) \le \max_u J_T(\bar{\varepsilon},u,v) \ . \tag{3.33}$$

Minimizing the RHS of (3.33) with respect to $v(=\bar{v})$,

$$\max_u J_T(\varepsilon,u,\bar{v}) \le \max_u J_T(\bar{\varepsilon},u,\bar{v})$$

whose LHS $\ge \min_v \max_u J_T(\varepsilon,u,v)$. So we have

$$\min_v \max_u J_T(\varepsilon,u,v) \le \min_v \max_u J_T(\bar{\varepsilon},u,v)$$

or from the standard finite time result

$$x_0^T[-P_\varepsilon(0;T)]x_0 \le x_0^T[-P_{\bar{\varepsilon}}(0;T)]x_0 \ ,$$

$$\text{i.e.} \quad P_\varepsilon(0;T) \ge P_{\bar{\varepsilon}}(0;T) \ .$$

Now, $\lim_{T\to\infty} P_\varepsilon(0;T) = P_\varepsilon$ by **definition** of P_ε . So we have that P_ε falls with ε in the sense of positive definite matrices. Now, as $P_\varepsilon(0;T) \ge 0$ for all ε,T so $P_\varepsilon \to P_0(\ge 0)$ as $\varepsilon \downarrow 0$. So we have, in summary:

Proposition 3.20

Consider the solution $K_\varepsilon < 0$ of (3.10). It has the property $K_\varepsilon \to K_0$ (where $K_0 \le 0$) as $\varepsilon \downarrow 0$.

Remark. By lemma A4, [8] the solution K_ε above is in fact $(P^-)^{-1}$.

We consider now how we might find K satisfying (3.10) as $\varepsilon \downarrow 0$. In view of Proposition 3.20 we might set $K = \sum_{i=0}^{\infty} \varepsilon^i K_i$ in (3.10). On substitution it is not difficult to show we are left to solve the pair of equations, as $\varepsilon \downarrow 0$,

$$K_0 A + A^T K_0 - \rho C^T C - K_0 B B^T K_0 + K_1 D D^T K_1 = 0 \ , \qquad (3.34)$$

$$K_0 D = 0 \ . \qquad (3.35)$$

The pair of equations (3.34), (3.35) may be reduced to a Riccati equation for K_0 , as we now show.

By multiplying (3.34) on the left by D^T , and by D on the right and using (3.35) we have

$$\rho(CD)^T(CD) = (D^T K_1 D)^T (D^T K_1 D) \ .$$

The LHS is an $r \times r$ matrix. If the $p \times r$ matrix CD has rank r (so necessarily $p \ge r$) then $\rho(CD)^T(CD) > 0$ and we may take the unique symmetric positive definite square root of $(CD)^T(CD)$, denoted $M^{-\frac{1}{2}} = (M^{-\frac{1}{2}})^T$, $M^{-\frac{1}{2}} > 0$. So we may set

$$D^T K_1 D = \sqrt{\rho} M^{-\frac{1}{2}} \qquad (3.36)$$

then we are left with (3.34) to solve for K_0 , or multiplying on the right by D , and using (3.35)

$$(K_0 A - \rho C^T C) D = -K_1 D D^T K_1 D$$

so that by (3.36)

$$(K_0A - \rho C^TC)\,\frac{DM^{\frac{1}{2}}}{\sqrt{\rho}} = -K_1D\ . \tag{3.37}$$

Substituting in (3.34) we obtain

$$K_0A+A^TK_0-\rho C^TC-K_0BB^TK_0+(K_0A-\rho C^TC)\,\frac{DMD^T}{\rho}\,(A^TK_0-\rho C^TC) = 0$$

or

$$K_0A[I-DMD^TC^TC] + [I-DMD^TC^TC]^TA^TK_0$$
$$- K_0[BB^T - \frac{ADMD^TA^T}{\rho}]K_0 + \rho C^T[CDMD^TC^T-I]C = 0\ . \tag{3.38}$$

So we have:

<u>Proposition 3.21</u>

We suppose the $p\times r$ matrix CD has rank r. Then we can set $M^{-\frac{1}{2}}$ as the unique symmetric positive definite square root of $(CD)^T(CD)$. Then $K = \sum_{i=0}^{\infty} \varepsilon^i K_i$ for ε sufficiently small only if (3.38), (3.37) can be solved for K_0, K_1 respectively.

Before giving examples to illustrate our approach we make a remark concerning a situation where the parameter ρ can be increased no further. We assume an asymptotic analysis is possible in the next result, and for the rest of the section denote P^-, K^- simply by P, K etc. We have the following result:

<u>Proposition 3.22</u>

For our solution K of (3.10) if $\sigma(A-BB^TK + \frac{DD^TK}{\varepsilon^2}) \cap i\mathbb{R} \neq \emptyset$ then there exists a destabilizing perturbation of norm $\sqrt{\rho}$ to $(A + \frac{DD^T K}{\varepsilon^2})$.

Proof: (3.10) can be rewritten

$$K(A-BB^TK + \frac{DD^TK}{\varepsilon^2}) + (A-BB^TK + \frac{DD^TK}{\varepsilon^2})^T K$$
$$- \rho C^TC + KBB^TK - \frac{KDD^TK}{\varepsilon^2} = 0 \ . \qquad (3.39)$$

Under our assumption there exist $\omega \in \mathbb{R}$, $x \in \mathbb{C}^n$ such that

$$(A - BB^TK + \frac{DD^TK}{\varepsilon^2})x = i\omega x \ . \qquad (3.40)$$

Multiplying (3.39) on the left by x^* , and on the right by x and using (3.40) we have

$$x^*Ki\omega x - i\omega x^*Kx - \rho||Cx||^2 + ||B^TKx||^2 - \frac{1}{\varepsilon^2}||D^TKx||^2 = 0 \ , \quad \text{or}$$

$$\rho||Cx||^2 + \frac{1}{\varepsilon^2}||D^TKx||^2 = ||B^TKx||^2 \ . \qquad (3.41)$$

Consider now the perturbation matrix

$$E = \frac{-B^TKx(Cx)^*}{||Cx||^2} \ , \qquad (3.42)$$

then $(A + \frac{DD^TK}{\varepsilon^2} + BEC)x = (A + \frac{DD^TK}{\varepsilon^2})x - BB^TKx = i\omega x$, from (3.40).

Thus E of (3.42) destabilizes $(A + \frac{DD^TK}{\varepsilon^2})$. Furthermore

$$||Ey|| \le \frac{||B^TKx|| \ ||y||}{||Cx||} \quad \text{so that} \quad ||E|| \le \frac{||B^TKx||}{||Cx||} \ ,$$

and for $y = Cx$

$$||ECx||^2 = (Cx)^* E^* E(Cx) = \frac{(Cx)^*(Cx)x^*KBB^TKx(Cx)^*(Cx)}{||Cx||^4} = ||B^TKx||^2 .$$

So $\dfrac{||ECx||^2}{||Cx||^2} = \dfrac{||B^TKx||^2}{||Cx||^2}$, and combining these we have

$$||E||^2 = \frac{||B^TKx||^2}{||Cx||^2} = \rho + \frac{1}{\varepsilon^2} \frac{||D^TKx||^2}{||Cx||^2} \qquad (3.43)$$

by (3.41). Thus $||E||^2 \geq \rho$. We show that as $\varepsilon \downarrow 0$, $||E||^2 \downarrow \rho$.

With $K = K_0 + \varepsilon K_1 + \varepsilon^2 K_2 + \dots$ we have $K_0 D = 0$ by (3.35). Also, setting $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 \dots$, we obtain from (3.40) that

$$DD^T K_1 x_0 = 0 . \qquad (3.44)$$

Using (3.35) it is easy to see that $\frac{1}{\varepsilon^2} ||D^TKx||^2 \to ||D^TK_1x_0||^2$ as $\varepsilon \downarrow 0$. By (3.44) $\frac{1}{\varepsilon^2} ||D^TKx||^2 \to 0$ as $\varepsilon \downarrow 0$, and so from (3.43) we have $||E||^2 \downarrow \rho$ as $\varepsilon \downarrow 0$. This completes the proof.

The following important theorem characterizes the situation where K escapes to minus infinity.

<u>Theorem 3.23</u>

Suppose $I + \rho\left[\dfrac{G_D G_D^*}{\varepsilon^2} - G_B G_B^*\right] \geq \delta GG^*$ for some $\delta > 0$, and $K < 0$

of (3.10). Then the situation where $\sigma(A-BB^TK + \frac{DD^TK}{\varepsilon^2}) \cap i\mathbb{R} \neq \emptyset$ is equivalent to K escaping to minus infinity.

Proof: First suppose we have $\sigma(A-BB^TK + \frac{DD^TK}{\varepsilon^2}) \cap i\mathbb{R} \neq \emptyset$ for $K < 0$ of $KA+A^TK-\rho C^TC-KBB^TK + \frac{KDD^TK}{\varepsilon^2} = 0$. If $\rho < r^2_{\mathbb{C}}(A;B,C)$ then as $\varepsilon \downarrow 0$ we have $(A-BB^TK + \frac{DD^TK}{\varepsilon^2})x = i\omega x$, some $\omega \in \mathbb{R}$, $x \in \mathbb{C}^n$. Furthermore, by theorem 3.17, $K < 0$ and cannot escape to minus infinity or have $0 \in \sigma(K)$ as $\varepsilon \downarrow 0$. So we have $K(A-BB^TK + \frac{DD^TK}{\varepsilon^2})x = i\omega Kx = (\rho C^TC-A^TK)x$. Set $y = Kx$ and so we may set $x = Py$, and so $(\rho C^TCP-A^T)y = i\omega y$ - which contradicts the strict fdc. So we have shownthat for $\rho < r^2_{\mathbb{C}}(A; B,C)$ that as $\varepsilon \downarrow 0$ the situation $\sigma(A-BB^TK+\frac{DD^TK}{\varepsilon^2}) \cap i\mathbb{R} \neq \emptyset$ is impossible. We note, as in theorem 3.17, the strict fdc continues holding as $\varepsilon \downarrow 0$. If we now increase ρ beyond $r^2_{\mathbb{C}}(A;B,C)$ such that $\sigma(A-BB^TK + \frac{DD^TK}{\varepsilon^2}) \cap i\mathbb{R} \neq \emptyset$ then as $\frac{dK}{d\rho} \leq 0$ (by proposition 3.15) we must have $K \downarrow -\infty$ by reasoning the same way as above. This proves the theorem in one direction.

Suppose, on the contrary, that $\sigma(A-BB^TK + \frac{DD^TK}{\varepsilon^2}) \subset \mathbb{C}_-$. Then as $\sigma(A) \subset \mathbb{C}_-$ we may set $K = \int_0^\infty -\rho e^{A^Tt} C^TC e^{(A-BB^TK + \frac{DD^TK}{\varepsilon^2})t} dt$, which is a well defined expression on account of $\sigma(A) \subset \mathbb{C}_-$, $\sigma(A-BB^TK + \frac{DD^TK}{\varepsilon^2}) \subset \mathbb{C}_-$. This completes the proof.

We end this section with a general remark concerning the results presented in it. Using a particular negative definite real symmetric solution K to a Riccati equation

$$KA + A^TK - \rho C^TC - KBB^TK + \frac{KDD^TK}{\varepsilon^2} = 0 \ ,$$

we have constructed a feedback

$$F = D^TK/\varepsilon^2 \ , \qquad (3.45)$$

to our nominally stable system which is stabilizing (Theorem 3.13) and which has certain robustness improving properties (Theorem 3.14). We have shown (Theorem 3.17) that to have greatest effect the weighting parameter ε in the Riccati equation should go to zero - thus creating a high-gain type feedback. The question of how good a robustness improvement actually is achieved is answered by Proposition 3.22 and Theorem 3.23.

We end by noting that for the more general Riccati equation

$$KA + A^TK + Q = K(BR^{-1}B^T - CS^{-1}C^T)K \ , \qquad (3.46)$$

the existence of real symmetric solutions in some closed ball may be investigated by the use of Brouwer's fixed point theorem (see for example page 161, [13]). In fact, such a technique is used by the authors of the paper [14] in order to establish the existence of solutions to coupled Riccati equations.

3.4 SOME NUMERICAL EXAMPLES

In this section we illustrate the ideas of the previous section. We concentrate on a simple case in which $n = 2$, $p = 2$, $m = 2$, $r = 1$, so that

$$A \in \mathbb{R}^{2\times 2} , \tag{3.47}$$

$$B = I_2 , \tag{3.48}$$

$$C = I_2 , \tag{3.49}$$

$$D = \begin{bmatrix} 1 \\ 0 \end{bmatrix} . \tag{3.50}$$

This is the so-called unstructured case studied in [6].

We start by solving (3.10), and seek an asymptotic expansion for the solution $K(\leq 0)$ of the form $K_0 + \varepsilon K_1 + \ldots$. We proceed directly, though we note that in this case the assumptions of proposition 3.2 1 are satisfied. We have:

$$K_0 D = 0 , \tag{3.51}$$

$$K_0 A + A^T K_0 - \rho C^T C - K_0 B B^T K_0 + K_1 D D^T K_1 = 0 . \tag{3.52}$$

Setting $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $K_0 = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix}$, $K_1 = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}$,

we obtain from (3.51) that

$$K_0 = \begin{bmatrix} 0 & 0 \\ 0 & k_3 \end{bmatrix} . \tag{3.53}$$

From (3.52)

$$\begin{bmatrix} 0 & 0 \\ 0 & k_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & k_3 \end{bmatrix} -$$

$$\begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & k_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & k_3 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} = 0 \qquad (3.54)$$

or, on expansion,

$$\begin{aligned} -\rho + k_{11}^2 &= 0 \, , && \text{(a)} \\ a_{21}k_3 + k_{11}k_{12} &= 0 \, , && \text{(b)} \qquad (3.55) \\ 2a_{22}k_3 - \rho - k_3^2 + k_{12}^2 &= 0 \, . && \text{(c)} \end{aligned}$$

For (3.55) (a) we may take $k_{11} = -\sqrt{\rho}$ in which case $k_{12} = a_{21}k_3/\sqrt{\rho}$ in (3.55)(b). Finally to satisfy (3.55)(c) we must solve:

$$\left(\frac{a_{21}^2}{\rho} - 1\right)k_3^2 + 2a_{22}k_3 - \rho = 0 \, ,$$

or

$$k_3 = \frac{-a_{22} \pm \sqrt{a_{22}^2 + a_{21}^2 - \rho}}{\left(\frac{a_{21}^2}{\rho} - 1\right)} \, . \qquad (3.56)$$

In view of (3.53), (3.55) we have

$$K = \begin{bmatrix} 0 & 0 \\ 0 & k_3 \end{bmatrix} + \varepsilon \begin{bmatrix} -\sqrt{\rho} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} + \ldots \, , \qquad (3.57)$$

so that we want $k_3 < 0$ in (3.56). This is achieved as follows (by examination of (3.56)):

$$
\begin{aligned}
&a_{22} < 0 \,,\ \rho < a_{21}^2 \,,\ k_3 < 0 \,; && \text{(a)}\\
&a_{22} < 0 \,,\ a_{21}^2 < \rho < a_{21}^2 + a_{22}^2 \,,\ k_3 < 0 \ \text{(2 distinct solutions)}; && \text{(b)} \quad (3.58)\\
&a_{22} > 0 \,,\ \rho < a_{21}^2 \,,\ k_3 < 0 \,; && \text{(c)}\\
&a_{22} > 0 \,,\ a_{21}^2 < \rho < a_{21}^2 + a_{22}^2 \,,\ \underline{\text{no}}\ \ k_3 < 0 \,. && \text{(d)}
\end{aligned}
$$

It is clear that these values k_3 of (3.58) give solutions K of (3.56) that we seek from the arguments of section 3.3. We see (Appendix A) that in the case $a_{22} < 0$ that as $\rho \to a_{21}^2$ k_3 remains finite. This illustrates the situation in the previous section that K does not escape to minus infinity. On the other hand if $a_{22} > 0$ then as $\rho \uparrow a_{21}^2$ K does indeed escape to minus infinity. In the case (3.58) (b) where there are two distinct negative solutions for k_3, we have:

$$k_3^1 = \frac{-a_{22} - \sqrt{a_{22}^2 + a_{21}^2 - \rho}}{\left(\frac{a_{21}^2}{\rho} - 1\right)} \,, \text{ (least negative solution)} \,,$$

$$k_3^2 = \frac{-a_{22} + \sqrt{a_{22}^2 + a_{21}^2 - \rho}}{\left(\frac{a_{21}^2}{\rho} - 1\right)} \,, \text{ (most negative solution)} \,.$$

We have
$$\frac{dk_3^2}{d\rho} = \frac{-(\frac{a_{21}^2}{\rho}-1)\frac{1}{2}(a_{22}^2+a_{21}^2-\rho)^{-\frac{1}{2}} + (-a_{22}+\sqrt{a_{22}^2+a_{21}^2-\rho})\frac{a_{21}^2}{\rho^2}}{(\frac{a_{21}^2}{\rho}-1)^2} > 0 ,$$

and
$$\frac{dk_3^1}{d\rho} = \frac{(\frac{a_{21}^2}{\rho}-1)\frac{1}{2}(a_{22}^2+a_{21}^2-\rho)^{-\frac{1}{2}} + (-a_{22}-\sqrt{a_{22}^2+a_{21}^2-\rho})\frac{a_{21}^2}{\rho^2}}{(\frac{a_{21}^2}{\rho}-1)^2} < 0 ,$$

for all ρ (Appendix B). Lyapunov arguments in proposition 3.15 show k_3^1 to be the solution we take in order to be consistent with the arguments of the previous section.

In (3.58) we see that we can only have a solution $k_3 < 0$ when $\rho \le a_{21}^2 + a_{22}^2$. This is, in fact, the necessary condition given by proposition 3.10, that as $\varepsilon \downarrow 0$

$$I + \rho\left[\frac{G_D G_D^*}{\varepsilon^2} - G_B G_B^*\right] \ge 0 , \quad \text{for all} \quad \omega \in \mathbb{R} . \tag{3.59}$$

This is easily shown. We have:

$$I + \rho\left[(i\omega-A)^{-1}\left(\begin{bmatrix} 1/\varepsilon^2 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)(i\omega-A)^{*-1}\right] \ge 0$$

or

$$(i\omega-A)(i\omega-A)^* + \rho\begin{bmatrix} \frac{1-\varepsilon^2}{\varepsilon^2} & 0 \\ 0 & -1 \end{bmatrix} \ge 0$$

or

$$\begin{bmatrix} i\omega-a_{11} & -a_{12} \\ -a_{21} & i\omega-a_{22} \end{bmatrix} \begin{bmatrix} -i\omega-a_{11} & -a_{21} \\ -a_{12} & -i\omega-a_{22} \end{bmatrix} + \rho \begin{bmatrix} \frac{1-\varepsilon^2}{\varepsilon^2} & 0 \\ 0 & -1 \end{bmatrix} \geq 0$$

or (for all $\omega \in \mathbb{R}$)

$$a_{21}^2 + a_{22}^2 + \omega^2 \geq \rho$$

or

$$\rho \leq a_{21}^2 + a_{22}^2 \quad .$$

We can characterize (3.59) in another way, as we now show.

We can now obtain an alternative characterization of (3.59), as $\varepsilon \downarrow 0$. We have:

$$\frac{I}{\rho} \geq G_B G_B^* - \frac{G_D G_D^*}{\varepsilon^2} \text{ , for all } \omega \in \mathbb{R} .$$

It is clear that for this to hold as $\varepsilon \downarrow 0$ then equivalently $\frac{1}{\rho} \geq \max ||G_B^* x||^2$ subject to the constraints $||x|| = 1$, $G_D^* x = 0$, or

$$\frac{1}{\rho} \geq \max[\,||G_B^* x||^2 + 2\langle\lambda, G_D^* x\rangle - \nu\,||x||^2\,], \tag{3.60}$$

where $\langle\cdot,\cdot\rangle$ and $||\cdot||$ denote inner products and norms, respectively, on appropriately dimensioned vectors with values in $\mathbb{C}$. The quantities λ,ν are Lagrange multipliers, and without loss of generality $||x|| = 1$.

Performing the differentiation on the RHS of (3.60) we obtain the following necessary conditions

$$\begin{bmatrix} G_B G_B^* - \nu & G_D \\ G_D^* & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = 0 . \qquad (3.61)$$

Considering (3.61), taking inner products in the first equation and using the second equation, it is easy to see that $||G_B^* x||^2 = \nu$. We thus have:

Proposition 3.24

The frequency domain condition

$$\frac{I}{\rho} - G_B G_B^* + \frac{G_D G_D^*}{\varepsilon^2} \geq 0 , \quad \text{for all} \quad \omega \in \mathbb{R} ,$$

is equivalent to $\frac{1}{\rho} \geq \nu$ and the conditions $||x|| = 1$,

$$\begin{bmatrix} G_B G_B^* - \nu & G_D \\ G_D^* & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = 0 .$$

In our example, we have

$$G_B G_B^* x - \nu x + G_D \lambda = 0 ,$$

$$G_D^* x = 0 .$$

So $\quad (i\omega - A)^{-1}(i\omega - A)^{*-1}x - \nu x + (i\omega - A)^{-1}[{}^{1}_{0}]\lambda = 0$,

$$[1,0](i\omega - A)^{*-1}x = 0 .$$

So, if $(i\omega - A)^{*-1}x = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z = \begin{bmatrix} 0 \\ z_2 \end{bmatrix}$, and

$$(i\omega - A)^{-1}z - \nu(i\omega - A)^{*}z + (i\omega - A)^{-1}[{}^{\lambda}_{0}] = 0$$

or $$z^*[I - \nu(i\omega - A)(i\omega - A)^{*}]z + z^*[{}^{\lambda}_{0}] = 0$$

or $$[0, z_2^*]\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \nu\begin{bmatrix} * & * \\ * & \omega^2 + a_{21}^2 + a_{22}^2 \end{bmatrix}\right)\begin{bmatrix} 0 \\ z_2 \end{bmatrix} = 0$$

or $$\nu = \frac{1}{\omega^2 + a_{21}^2 + a_{22}^2} ,$$

from which we obtain the same answer as before.

The question of optimality in this example is easily answered. If $a_{22} > 0$ we can guarantee an unstructured stability radius of at least $|a_{21}|$. Inspection of the A-matrix shows clearly that there exists a perturbation of norm $|a_{21}|$ which is destabilizing, so that our answer is optimal. If $a_{22} < 0$ we can guarantee an unstructured stability radius of at least $\sqrt{a_{21}^2 + a_{22}^2}$. Again we see that there exists a perturbation of norm $\sqrt{a_{21}^2 + a_{22}^2}$ that is destabilizing, hence optimality.

We now give some concrete examples to illustrate the calculations made in this section.

Example 3.25

$A = \begin{bmatrix} -2 & 2 \\ -3 & -1 \end{bmatrix}$, $B = C = I_2$, $D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Here $a_{22} < 0$, and the unstructured stability radius is $\sqrt{2.07} \sim 1.439$. We can increase this to $\sqrt{10} \sim 3.162$.

Example 3.26

$A = \begin{bmatrix} -2 & 3 \\ -17/24 & 1 \end{bmatrix}$, $B = C = I_2$, $D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Here we have $a_{22} > 0$. The uncontrolled stability radius is ~ 0.0328. This can be improved dramatically to $17/24 \sim 0.708$.

Example 3.27

$A = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}$, $B = C = I_2$, $D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Here $a_{22} > 0$. The uncontrolled unstructured stability radius can easily be shownto be $\sqrt{15-\sqrt{221}} \sim 0.366$. By our methods we may increase this radius to 3 with our feedback.

In example 3.27 the uncontrolled complex stability radius is substantially improved by the use of control action. From the results outlined in Chapter 2 on the relation of the complex stability radius to the norm of a "convolution" map, we see that the norm of this map can be made arbitrarily close to the value 1/3 from above. This is indeed an interesting comparison to the H^∞-norm minimization we carried out in Chapter 1 using this same example. Here we showed that no dynamic feedback could get the H^∞-norm below 0.33 (the value 0.33 being chosen to "represent" 1/3 in view of the iterative solution procedure we were required to go through in the case where the "number of outputs" exceeds the "number of inputs"). In the next section we state under what conditions an infinite improvement in the complex structured stability radius may be expected, giving examples to illustrate.

Finally, we note that the complex stability radii in the above examples may be calculated using the formulae for general 2×2 matrices given in the next chapter.

3.5 NECESSARY AND SUFFICIENT CONDITIONS FOR INFINITE ROBUSTNESS

In this section we look at the complex structured stability radius as characterized by the reciprocal of the norm of the "convolution" map (see Chapter 2). The problem of regulating an arbitrarily large stability radius is thus equivalent to that of regulating an arbitrarily small norm for the "convolution" map by choice of state feedback. To show when this is possible we can use those results of [30] relating to a certain "disturbance decoupling problem". In order to be able to do this we introduce certain subspaces of the state space that are essential for the characterization. We start off with the system introduced earlier in this chapter,

$$\dot{x} = Ax + Du + Bv, x(0) = x_0 ,$$

$$y = Cx .$$

If we set $u = Fx$ we obtain

$$\dot{x} = A_F x + Bv \ , \ x(0) = x_0 \ ,$$
$$y = Cx \ . \tag{3.62}$$

We give now some definitions and theory that will be used. For the standard system

$$\dot{x} = Ax + Du \ , \quad x(0) = x_0 \ ,$$

a subspace $V \subset X$ is said to be a (controlled) invariant subspace if for all $x_0 \in V$ there exists an admissible control $u(t)$ such that $x(t) \in V$ for all t . A subspace $V_a \subset X$ is said to be an almost (controlled) invariant subspace if for all $x_0 \in V_a$ and $\varepsilon > 0$ there exists an admissible control $u(t)$ such that $d(x(t),V_a) \leq \varepsilon$ for all t . (Here the distance of the point x to a subspace L is defined by $d(x,L) \triangleq \inf_{y \in L} ||x-y||$) .

A subspace $R \subset X$ is a controllability subspace if for all $x_0, x_1 \in R$, there exists a time $T > 0$ and control $u(t)$ such that $x(t) \in R$ for all t and $x(T) = x_1$. A subspace $R_a \subset X$ is an almost controllability subspace if for all $x_0, x_1 \in R_a$, there exists a time $T > 0$ and a control $u(t)$ such that for all $\varepsilon > 0$ $d(x(t),R_a) \leq \varepsilon$ for all t and $x(T) = x_1$.

We denote $\underline{V}$, $\underline{R}$, $\underline{V_a}$, $\underline{R_a}$ as respectively the set of (controlled) invariant subspaces, controllability subspaces, almost (controlled) invariant subspaces, and almost controllability subspaces using the notation $\underline{V}(K)$, $\underline{R}(K)$, $\underline{V}_a(K)$, $\underline{R}_a(K)$ for those that are contained in a given subspace $K \subset X$. It is clear that $\underline{R} \subset \underline{V} \subset \underline{V}_a$ and $\underline{R} \subset \underline{R}_a \subset \underline{V}_a$. The following result regarding the above holds:

Proposition 3.28 (Theorem 1, [30])

$\underline{V}$, $\underline{R}$, $\underline{V}_a$, $\underline{R}_a$ are closed under subspace addition. Also

$$\sup \underline{V}(K) \triangleq V_K^* \in \underline{V} \quad ,$$

$$\sup \underline{R}(K) \triangleq R_K^* \in \underline{R} \quad ,$$

$$\sup \underline{V}_a(K) \triangleq V_{a,K}^* \in \underline{V}_a \quad ,$$

$$\sup \underline{R}_a(K) \triangleq R_{a,K}^* \in \underline{R}_a \quad .$$

In order to find the subspaces V_K^* , R_K^* , $V_{a,K}^*$, and $R_{a,K}^*$ the following two results are useful. We denote Im D by $\mathcal{D}$ in the following algorithm.

Proposition 3.29 (Proposition 2, [30])

Setting

$$V_K^{k+1} = K \cap A^{-1}(V_K^k + \mathcal{D}) \ ; \quad V_K^0 = X \ ,$$

and

$$R_K^{k+1} = K \cap (AR_K^k + \mathcal{D}) \ ; \ R_K^0 = \{0\} \quad .$$

Then V_K^k is monotone nonincreasing; moreover,

$$V_K^{\dim K+1} = V_K^{\infty} \triangleq \lim_{k\to\infty} V_K^k \ , \quad \text{and}$$

$$\{V_K^{k+1} = V_K^k\} \Rightarrow \{V_K^k = V_K^{\infty}\} \quad .$$

Similarly R_K^k is monotone nondecreasing; moreover,

$$R_K^{\dim K} = R_K^{\infty} \triangleq \lim_{k\to\infty} R_K^k \ , \quad \text{and}$$

$$\{R_K^{k+1} = R_K^k\} \Rightarrow \{R_K^k = R_K^{\infty}\} \quad .$$

In terms of the quantities V_K^{∞}, R_K^{∞} iteratively defined in proposition 3.29, the subspaces V_K^{*}, R_K^{*}, $V_{a,K}^{*}$, $R_{a,K}^{*}$ are given by the next result.

<u>Proposition 3.30 (Theorem 3, [30])</u>

$$V_K^{*} = V_K^{\infty} \ ,$$

$$R_{a,K}^{*} = R_K^{\infty} \ ,$$

$$V_{a,K}^{*} = V_K^{\infty} + R_K^{\infty} \ , \quad \text{and}$$

$$R_K^{*} = V_K^{\infty} \cap R_K^{\infty} = V_{R_K^{\infty}} = R_{V_K^{\infty}} \quad .$$

The distance in the L^p-sense (for $1 \le p < \infty$) from a point $x_0 \in X$ to a subspace $K \subset X$ is defined as follows:

$$d_p(x_0,K) \triangleq \inf_{\substack{u(\cdot) \\ x(0)=x_0}} ||d(x(\cdot),K)||_{L^p}$$

with $||\cdot||_{L^p}$ denoting the standard norm of a function in L^p .

We define further the following two quantities $V^*_{p,K}$, $R^*_{p,K}$. We have

$$V^*_{p,K} \triangleq \{x_0 \in X : d_p(x_0,K) = 0\}$$

as the supremal L^p-almost invariant subspace "contained" in K , and

$$R^*_{p,K} \triangleq R^*_{a,V^*_{p,K}}$$

as the supremal L^p-almost controllability subspace "contained" in K .

The following result shows the invariance of these sets to the integer p .

Proposition 3.31 (Theorem 10, [30])

For $1 \leq p < \infty$, $R^*_{p,K} = AR^*_{a,K} + \Box$, and

$$V^*_{p,K} = R^*_{p,K} + V^*_K = AV^*_{a,K} + \Box + V^*_{a,K} .$$

Proposition 3.31 thus justifies the use of the following notation:

$$R^*_{b,K} \triangleq R^*_{p,K} , \quad V^*_{b,K} \triangleq V^*_{p,K} .$$

These two quantities are crucial in the following development. To compute them we again use an algorithm:

Proposition 3.32 (Proposition 3, [30])

Consider the iterative scheme

$$S_K^{k+1} = \square + A(K \cap S_K^k) \; ; \quad S_K^0 = \{0\} \quad .$$

Then S_K^k is monotone nondecreasing; moreover,

$$S_K^{\dim K+1} = S_K^\infty \triangleq \lim_{k\to\infty} S_K^k \;, \quad \text{and}$$

$$\{S_K^{k+1} = S_K^k\} \Rightarrow \{S_K^k = S_K^\infty\} \quad .$$

Propositions 3.29, 3.32 thus yield algorithms by which V_K^∞, R_K^∞, and S_K^∞ may be computed. These three quantities are exactly what we require for proposition 3.30 and the next result, proposition 3.33.

Proposition 3.33 (Theorem 11, [30])

We have

$$R_{b,K}^* = S_K^\infty ,$$

and

$$V_{b,K}^* = S_K^\infty + V_K^\infty .$$

In the case where $\dim \square \geq \operatorname{codim} K$ we have a more direct result, which we now note.

Proposition 3.34 (Theorem 15, [30])

If $\dim \square \geq \operatorname{codim} K$ we have

$$V_{a,K}^* = K \; ; \quad \text{and}$$

$$V_{b,K}^* = AK + K + \square \quad .$$

The material so far developed can now be used in the characterization of "disturbance decoupling" we require for our purposes. For the system of (3.62) we describe (as in [30]) three types of disturbance decoupling - the third of which will be of particular interest to us. The first type is the standard concept, the system of (3.62) being disturbance decouplable (DDP) if there exists a feedback F such that for

$$y(t) = \int_0^t Ce^{A_F(t-s)} Bv(s).ds \tag{3.63}$$

we have $y(t) = 0$ for all admissible $v(\cdot)$ and $t \geq 0$. It is well known that this will be the case iff

$$\text{im } B \subset V^*_{\ker C} ,$$

where we note that $V^*_{\ker C}$ can be computed using the characterization given in proposition 3.30.

We say that for (3.62) $(ADDP)_p$ - the almost disturbance decoupling problem in the L^p-sense is solvable if for all $\varepsilon > 0$ there exists a feedback F such that $||y||_{L^p} \leq \varepsilon||v||_{L^p}$, and finally if we add the condition that for any given M one has also $\text{Re } \lambda(A_F) \leq M$ then we are speaking of $(ADDPS)_p$ - the almost disturbance decoupling problem with strong stabilization. We thus have the result from [30] which gives necessary and sufficient conditions for both problems:

<u>Theorem 3.35 (Theorem 16, [30])</u>

We have $\{(ADDP)_p$ is solvable$\}$ iff $\{\text{im } B \subset V^*_{b,\ker C}\}$. Also

$\{(ADDPS)_p$ is solvable$\}$ iff $\{\text{im } B \subset R^*_{b,\ker C}$ and (A,D) is a controllable pair$\}$.

The computations required in this result may be achieved using proposition 3.33. We are clearly particularly interested in whether the conditions for $(ADDPS)_p$ are satisfied. The following numerical examples illustrate how this theory may be used to check whether this is the case.

<u>Example 3.36</u>

For A arbitrary, $D = \begin{bmatrix}1\\0\end{bmatrix}$, $B = C = I_2$, we have $\dim \square < \text{codim} \ker C$. We have $\dim \ker C = 0$ so $S^1_{\ker C} = S^\infty_{\ker C}$ and

$$S^1_{\ker C} = sp\begin{bmatrix}1\\0\end{bmatrix} + A(\{0\}) = sp\begin{bmatrix}1\\0\end{bmatrix} \quad .$$

Also

$$V^\infty_{\ker C} = V^1_{\ker C} = \{0\} \quad .$$

Thus

$$V^*_{b,\ker C} = sp\begin{bmatrix}1\\0\end{bmatrix} = R^*_{b,\ker C} \quad .$$

We thus have that $\text{im } B \not\subset R^*_{b,\ker C}$, so that in this example an infinite stability radius cannot be regulated. We note that the main example we have considered so far is therefore of this form.

<u>Example 3.37</u>

We take

$$A = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B = I_2, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Here $\dim \ker C = 1$, and $\ker C = sp[{1 \atop 0}]$. So we have $S^{\infty}_{\ker C} = S^{2}_{\ker C}$.
Using the iterative algorithm we have

$$S^{1}_{\ker C} = sp[{1 \atop 0}] + A(\{0\}) = sp[{1 \atop 0}] ,$$

$$S^{2}_{\ker C} = sp[{1 \atop 0}] + \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}(sp[{1 \atop 0}]) = sp\{[{1 \atop 0}],[{0 \atop 1}]\} .$$

We have further that $V^{\infty}_{\ker C} = V^{2}_{\ker C}$ and

$$V^{1}_{\ker C} = sp[{1 \atop 0}] \cap A^{-1}(sp\{[{1 \atop 0}],[{0 \atop 1}]\}) = sp[{1 \atop 0}] ,$$

$$V^{2}_{\ker C} = sp[{1 \atop 0}] \cap sp \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \{0\} .$$

So

$$V^{*}_{b,\ker C} = sp\{[{1 \atop 0}],[{0 \atop 1}]\} .$$

We therefore have $\operatorname{im} B \subset V^{*}_{b,\ker C}$ so that $(ADDP)_p$ is solvable, also $\operatorname{im} B \subset R^{*}_{b,\ker C}$ and (A,D) is controllable so that the stability radius may be made arbitrarily large for this example. Finally we note that $\operatorname{im} B \not\subset V^{*}_{\ker C}$ so that the system is not disturbance decouplable.

<u>Example 3.38</u>

We consider finally $A = \begin{bmatrix} -1/6 & 1/3 & 2/3 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}$, $D = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$,

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In this case we have $\dim \ker C = 1$, $\ker C = \mathrm{sp}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

A simple calculation gives

$$A^{-1} = \begin{bmatrix} -6 & -6 & -8 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

In this case we have $S^{\infty}_{\ker C} = S^{2}_{\ker C}$, so

$$S^{1}_{\ker C} = \mathrm{sp}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$S^{2}_{\ker C} = \mathrm{sp}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + A(\{0\}) = \mathrm{sp}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

and $V^{\infty}_{\ker C} = V^{2}_{\ker C}$,

$$V^{1}_{\ker C} = \mathrm{sp}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\begin{aligned} V^{2}_{\ker C} &= \mathrm{sp}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cap A^{-1}\left(\mathrm{sp}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \mathrm{sp}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \\ &= \mathrm{sp}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cap \mathrm{sp}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \\ &= \mathrm{sp}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \end{aligned}$$

We conclude that $\mathrm{im}\, B \subset V^{*}_{b,\ker C}$ so that $(ADDP)_p$ is possible,

$\text{im } B \not\subset R^{*}_{b,\ker C}$ so that the stability radius may not be made arbitrarily large. Clearly $\text{im } B \not\subset V^{*}_{\ker C}$ so that disturbance decoupling is not possible.

We have thus shown under what conditions an arbitrarily large stability radius can be regulated - the numerical examples given serving to illustrate how the various conditions can be checked. We note that in [31] the author shows how sequences of feedbacks solving such problems may be constructed - in general (like our feedback constructed from a solution of a certain nonstandard Riccati equation) they are of a "high-gain" nature.

3.6 APPENDICES

APPENDIX A

In the case where $a_{22} < 0$ or $a_{22} > 0$ we have

$$k_3 = \frac{-a_{22} - \sqrt{a_{22}^2 + a_{21}^2 - \rho}}{\left(\frac{a_{21}^2}{\rho} - 1\right)} .$$

If $a_{22} < 0$ we have, when $\rho = a_{21}^2$, both numerator and denominator of k_3 are zero. Application of l'Hospital's Rule shows that as $\rho \to a_{21}^2$ the same limit exists ($\rho \uparrow a_{21}^2$ or $\rho \downarrow a_{21}^2$) and is equal to the solution for k_3 when $\rho = a_{21}^2$, which is

$$a_{21}^2/2a_{22} .$$

In the case $a_{22} > 0$ we see that as $\rho \uparrow a_{21}^2$, $k_3 \downarrow -\infty$.

APPENDIX B

We show $dk_3^1/d\rho < 0$. Considering the numerator

$$\left(\frac{a_{21}^2}{\rho} - 1\right)\tfrac{1}{2}(a_{22}^2+a_{21}^2-\rho)^{-\frac{1}{2}} + \left(-a_{22}-\sqrt{a_{22}^2+a_{21}^2-\rho}\right)\frac{a_{21}^2}{\rho^2} .$$

Set $\theta = \sqrt{a_{21}^2 + a_{22}^2 - \rho}$ then we have

$$\left(\frac{\theta^2-a_{22}^2}{\rho}\right)\frac{1}{2\theta} + (-a_{22}-\theta)\frac{a_{21}^2}{\rho^2}$$

or $$\rho(\theta^2-a_{22}^2) + (-a_{22}-\theta)2\theta a_{21}^2$$

or $$(a_{21}^2+a_{22}^2-\theta^2)(\theta^2-a_{22}^2) - 2\theta a_{21}^2(a_{22}+\theta)$$

or $$(a_{21}^2+a_{22}^2-\theta^2)(\theta-a_{22})(\theta+a_{22}) - 2\theta a_{21}^2(a_{22}+\theta)$$

or $$-(\theta+a_{22})[(\theta^2-a_{21}^2-a_{22}^2)(\theta-a_{22}) + 2\theta a_{21}^2] ,$$

or $$-(\theta+a_{22})[\theta^3-a_{22}\theta^2 + (a_{21}^2-a_{22}^2)\theta + a_{22}(a_{21}^2 + a_{22}^2)]$$

or $$-(\theta+a_{22})^2[\theta^2-2a_{22}\theta + a_{21}^2 + a_{22}^2]$$

or $$-(\theta+a_{22})^2[(\theta-a_{22})^2 + a_{21}^2] < 0 ,$$

as required.

4. FURTHER REMARKS ON STABILITY RADII

4.1 STABILITY RADII AND RICCATI EQUATIONS

In this section we make some further comments on the destabilization of systems via solutions of Riccati equations. We follow largely the approach and notation given in Chapter 2.

In Chapter 2 the Riccati equation

$$PA + A^TP - \rho C^TC - PBB^TP = 0 , \qquad (4.1)$$

was considered and its relation to the complex structured stability radius $r_{\mathbb{C}}$ was shown. In fact proposition 2.22 and theorem 2.21 of Chapter 2 give this relationship. From proposition 2.22 we have for the real symmetric solution P of (4.1) that:

$$(B^TP(i\omega-A)^{-1}B+I)^*(B^TP(i\omega-A)^{-1}B + I) =$$

$$I - \rho B^T(i\omega-A)^{-1*}C^TC(i\omega-A)^{-1}B ,$$

for all $\omega \in \mathbb{R}$. (4.2)

From proposition 2.17 of Chapter 2 we have

$$r_{\mathbb{C}} = \frac{1}{\max\limits_{\omega \in \mathbb{R}} ||C(i\omega-A)^{-1}B||} , \qquad C(i\omega-A)^{-1}B \neq 0 .$$

Let ω_0 be this maximum so that

$$r_{\mathbb{C}} = \frac{1}{||C(i\omega_0-A)^{-1}B||} . \qquad (4.3)$$

If we let $u \in \mathbb{C}^m$ be the corresponding vector such that we have:

$$||u||^2 = r_{\mathbb{C}}^2 u^* B^T (i\omega_0 - A)^{-1*} C^T C (i\omega_0 - A)^{-1} Bu \ . \qquad (4.4)$$

From (4.2), (4.4) we obtain by premultiplying by u^*, and post-multiplying by u and adding

$$||(B^T P(i\omega_0-A)^{-1}B + I)u||^2$$
$$= (r_{\mathbb{C}}^2-\rho)||C(i\omega_0-A)^{-1}Bu||^2 \ ,$$

or

$$||(BB^T P(i\omega_0-A)^{-1} + I)Bu||^2$$
$$\le ||B||^2(r_{\mathbb{C}}^2-\rho)||C(i\omega_0-A)^{-1}Bu||^2 \ ,$$

or

$$||(BB^T P + i\omega_0-A)(i\omega_0-A)^{-1}Bu||^2$$
$$\le ||B||^2(r_{\mathbb{C}}^2-\rho)||C(i\omega_0-A)^{-1}Bu||^2 \ ,$$

so that as $\rho \to r_{\mathbb{C}}^2$, for some eigenvalue λ of $(A-BB^TP)$, $\lambda(A-BB^TP) \to i\omega_0$.

We thus have for some $x \in \mathbb{C}^n$:

$$(A-BB^TP)x = i\omega_0 x \ . \qquad (4.5)$$

The Riccati equation (4.1), for $\rho = r_{\mathbb{C}}^2$, can be rewritten

$$P(A-BB^TP) + (A-BB^TP)^TP - r_{\mathbb{C}}^2 C^T C + PBB^TP = 0 \ ,$$

so by premultiplying by x^*, postmultiplying by x and using (4.5) we have

$$x^*Pi\omega_0 x - i\omega_0 x^*Px - r_{\mathbb{C}}^2||Cx||^2 + ||B^TPx||^2 = 0 . \qquad (4.6)$$

Set $$D = \frac{-B^TPx(Cx)^*}{||Cx||^2} , \qquad (4.7)$$

then

$$(A + BDC)x = Ax - BB^TPx = i\omega_0 x ,$$

from (4.5), so that D of (4.7) is destabilizing.

We proceed by finding $||D||$. We have for all $y \in \mathbb{C}^p$:

$$||Dy|| \le \frac{||B^TPx||\ ||Cx||\ ||y||}{||Cx||^2} = r_{\mathbb{C}}||y|| ,$$

from (4.6), so that

$$||D|| \le r_{\mathbb{C}} . \qquad (4.8)$$

For $y = Cx$,

$$||DCx||^2 = \frac{(Cx)^*(Cx)x^*PBB^TPx(Cx)^*(Cx)}{||Cx||^4}$$

$$= r_{\mathbb{C}}^2||Cx||^2 ,$$ by (4.6). From this and (4.8)

we have $||D|| = r_{\mathbb{C}}$. We may summarize this as:

<u>Proposition 4.1</u>

The perturbation matrix $D = \frac{-B^TPx(Cx)^*}{||Cx||^2}$ is destabilizing and of norm $r_{\mathbb{C}}$. It is, in general, complex.

We next consider the question of how we might construct a real destabilizing perturbation of a certain norm. This may be done under certain conditions as we now show.

Proposition 4.2

Suppose there exists a real (possibly nonsymmetric) solution P of (4.1) with $0 \in \sigma(A-BB^TP)$. There then exists a real destabilizing perturbation of norm $\sqrt{\rho}$.

Proof: We rewrite (4.1) as

$$P(A-BB^TP) + (A-BB^TP)^TP - \rho C^TC + P^TBB^TP = 0 . \qquad (4.9)$$

Under the assumptions of the proposition there exists $x \in \mathbb{R}^n$ such that

$$(A-BB^TP)x = 0 ,$$

so premultiplying (4.9) by x^T , and postmultiplying by x we have

$$\rho||Cx||^2 = ||B^TPx||^2 . \qquad (4.10)$$

Set

$$D = \frac{-B^TPx(Cx)^T}{||Cx||^2} , \qquad (4.11)$$

then

$$(A + BDC)x = Ax - BB^TPx = 0$$

so that D of (4.11) is destabilizing. We also have

$$||Dy|| \leq \frac{||B^T Px|| \; ||y||}{||Cx||} = \sqrt{\rho} \; ||y|| ,$$

from (4.10), so $||D|| \leq \sqrt{\rho}$. For $y = Cx$,

$$\frac{||DCx||}{||Cx||} = \frac{||B^T Px||}{||Cx||} = \sqrt{\rho}$$

from (4.10).

So we have $||D|| = \sqrt{\rho}$. This completes the proof.

The existence of nonsymmetric solutions of (4.1) seems to be a largely uninvestigated area. Indeed, the motivation behind proposition 4.2 was from a general 2×2 matrix case which revealed the existence of nonsymmetric solutions of (4.1) - we say more about this subsequently. We first make some general comments on the existence of nonsymmetric solutions of (4.1). The following general remark gives necessary and sufficient conditions and is found in [18]. The $n \times n$ equation considered is

$$KA + A^T K - KBB^T K + Q = 0 , \tag{4.12}$$

and $Q = Q^T$. From (4.12) we may form the Hamiltonian matrix

$$H = \begin{bmatrix} A & -BB^T \\ -Q & -A^T \end{bmatrix} . \tag{4.13}$$

We then have the following proposition:

Proposition 4.3 (Theorem 1, [18])

There is a one-to-one correspondence between the set of real solutions of (4.12) and the set of n-dimensional H-invariant subspaces which are complementary to the n-dimensional subspace

$$sp\begin{bmatrix} 0 \\ I \end{bmatrix}.$$

(Here, as in the previous chapter, sp denotes the column span of a matrix).

This correspondence assigns the invariant subspace $S(K) \equiv sp\begin{bmatrix} I \\ K \end{bmatrix}$ to the solution K. The matrix of the restriction of H to $S(K)$ with respect to the basis given by the columns of $\begin{bmatrix} I \\ K \end{bmatrix}$ is $A-BB^TK$. Furthermore, K is symmetric if and only if $x^TJy = 0$, for all $x,y \in S(K)$, where J is the $2n \times 2n$ matrix $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$.

We note that a subspace, S, of $\mathbb{R}^{2n}$ such that $x^TJy = 0$, for all $x,y \in S$, is called Lagrangian. Hence proposition 4.3 gives a one-to-one correspondence between the set of real solutions of (4.12) and the set of n-dimensional Lagrangian H-invariant subspaces complementary to $sp\begin{bmatrix} 0 \\ I \end{bmatrix}$.

This proposition gives a classification for all real solutions of (4.1). However, we seek to find nonsymmetric solutions by a more direct approach. We set $P = P_s + P_k$ in (4.1) where $P_s = P_s^T$ (symmetric) and $P_k = -P_k^T$ (skew-symmetric). Equation (4.1) becomes

$$P_sA + P_kA + A^TP_s + A^TP_k - \rho C^TC$$
$$- P_sBB^TP_s - P_sBB^TP_k - P_kBB^TP_s - P_kBB^TP_k = 0 ,$$

or

$$P_sA + A^TP_s - \rho C^TC - P_sBB^TP_s - P_kBB^TP_k$$

$$= -[P_k(A-BB^TP_s) + (A-BB^TP_s)^TP_k] \quad . \qquad (4.14)$$

Inspection of (4.14) shows the LHS to be symmetric and the RHS to be skew-symmetric. Consequently if this equation is to be satisfied then both LHS and RHS must be zero, i.e. (4.14) becomes

$$P_sA + A^TP_s - \rho C^TC - P_sBB^TP_s - P_kBB^TP_k = 0 \, ,$$

$$P_k(A-BB^TP_s) + (A-BB^TP_s)^TP_k = 0 \, ,$$

which can be rewritten as

$$P_sA + A^TP_s - \rho C^TC - P_sBB^TP_s - P_kBB^TP_k = 0 \, , \qquad \text{(a)} \quad (4.15)$$

$$P_k(A-BB^TP) + (A-BB^TP)^TP_k = 0 \quad . \qquad \text{(b)}$$

Using the fact that $\alpha\gamma - \gamma\beta = \delta$, where α,β,δ are constant matrices of suitable dimensions, has a unique solution γ if and only if α and β have no common characteristic roots (Theorem 1.9, [19]) we see that $P_k \neq 0$ in (4.15)(b) **only if** for some eigenvalue λ of $(A-BB^TP)$, $\text{Re}\,\lambda = 0$ or λ_1,λ_2 with $\text{Re}\,\lambda_1 = -\text{Re}\,\lambda_2$ are both eigenvalues of $(A-BB^TP)$. We summarize this discussion as:

Proposition 4.4

Set $P = P_s + P_k$. The equation (4.1)

$$PA + A^TP - \rho C^TC - PBB^TP = 0 \, ,$$

is equivalent to (4.15)

$$P_sA + A^TP_s - \rho C^TC - P_sBB^TP_s - P_kBB^TP_k = 0 ,$$

$$P_k(A-BB^TP) + (A-BB^TP)^TP_k = 0 .$$

Furthermore (4.1) has a real nonsymmetric solution **only if** for some eigenvalue λ of $(A-BB^TP)$, $\mathrm{Re}\,\lambda = 0$ or λ_1, λ_2 with $\mathrm{Re}\,\lambda_1 = -\mathrm{Re}\,\lambda_2$ are both eigenvalues of $(A-BB^TP)$.

We specialize now to the case where $B = C = I$ in (4.1). We have the following lemmas.

Lemma 4.5

Suppose $i\omega \in \sigma(A-P)$ for some $\omega \in \mathbb{R}$ where P is a real nonsymmetric solution to (4.1), then

$$\det \begin{bmatrix} \rho-\omega^2-A^TA & \omega(A-A^T) \\ -\omega(A-A^T) & \rho-\omega^2-A^TA \end{bmatrix} = 0 .$$

Proof: As $i\omega \in \sigma(A-P)$ we have $(A-P)z = i\omega z$, $0 \neq z \in \mathbb{C}^n$, or with $z = x + iy$,

$$\begin{aligned} (A-P)x &= -\omega y , \\ (A-P)y &= \omega x , \end{aligned} \tag{4.16}$$

or

$$\begin{aligned} Px &= Ax + \omega y , \\ Py &= -\omega x + Ay . \end{aligned} \tag{4.17}$$

Equation (4.1) becomes

$$P(A-P) = \rho I - A^TP . \tag{4.18}$$

From (4.16) - (4.18) by eliminating P we arrive at the pair of equations

$$\rho x - A^T(Ax + \omega y) = \omega^2 x - \omega Ay ,$$

$$\rho y - A^T(-\omega x + Ay) = \omega Ax + \omega^2 y ,$$

or

$$\begin{bmatrix} \rho-\omega^2 - A^T A & \omega(A-A^T) \\ -\omega(A-A^T) & \rho-\omega^2-A^T A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 ,$$

from which the lemma follows.

<u>Lemma 4.6</u>

Under the assumptions of the previous lemma, $\rho^{\frac{1}{2}}$ is a singular value of $(i\omega-A)$.

<u>Proof</u>: We have $(A-P)z = i\omega z$ and from (4.18)

$$P(A-P)z = \rho z - A^T Pz = i\omega Pz$$

so that

$$\rho z = (-i\omega+A)^* Pz .$$

Now, as $Pz = (-i\omega+A)z$, substituting in the last equation we have

$$\rho z = (-i\omega+A)^*(-i\omega+A)z ,$$

from which $\rho^{\frac{1}{2}}$ is a singular value of $(i\omega-A)$.

The following proposition is evident from these two lemmas:

Proposition 4.7

If there exists a nonsymmetric solution P of (4.1) as $\omega \to 0$ from $\rho = r_{\mathbb{C}}^2$, $\omega = \omega_0$ then $\rho \to \sigma^2(A)$, where $\sigma(A)$ is a singular value of A .

We remark that our approach of constructing a real destabilizing perturbation is not unlike that of constructing a matrix E which destroys the invertibility of a nonsingular matrix A . It is well known (see section on singular values in Chapter 2) that the distance from singularity of A is given by its lowest singular value, $\underline{\sigma}(A)$. It is easy to construct a rank-one perturbation E such that $\det(A-E) = 0$, as follows. Choose x , $||x|| = 1$, such that $||Ax|| = \underline{\sigma}(A)$. Set $y = Ax$, $E = yx^T$. E has rank one (otherwise the invertibility of A is contradicted) and

$$||E|| \leq ||y|| \; ||x^T|| = \underline{\sigma}(A) .$$

Furthermore

$$||Ex|| = ||yx^Tx|| = ||y|| = \underline{\sigma}(A) , \quad \text{so that} \quad ||E|| = \underline{\sigma}(A) .$$

Now

$$(A-E)x = Ax-Ex = Ax - yx^Tx = 0 ,$$

so that $\det(A-E) = 0$, and $||E|| = \underline{\sigma}(A)$.

We give an illustration of the above ideas by considering the general 2×2 unstructured case $(B = C = I_2)$. We first give a general bound on ω_0 for the $n\times n$ case. We have from corollary 2.9 that

$$\min_{\omega\in\mathbb{R}} \underline{\sigma}(i\omega-A) = \min_{\omega\in\mathbb{R}} \min_{\substack{||z||=1 \\ z\in\mathbb{C}^n}} ||i\omega z-Az|| .$$

Set $f(\omega) = ||i\omega z-Az||^2$, where $z = x + iy$. So $f(\omega) = \langle Ax+\omega y,Ax+\omega y\rangle + \langle \omega x-Ay,\omega x-Ay\rangle$, and using the fact that

$$||z||^2 = ||x||^2 + ||y||^2 = 1 , \tag{4.19}$$

we have

$$f(\omega) = ||Ax||^2 + \omega^2+2\omega(\langle y,Ax\rangle-\langle x,Ay\rangle) + ||Ay||^2 .$$

Setting $df/d\omega = 0$ gives

$$\omega = \langle y,(A^T-A)x\rangle .$$

Using the Cauchy-Schwartz inequality

$$|\omega| \le ||y|| \; ||(A^T-A)x|| \le ||A^T-A|| \; ||x|| \; ||y|| ,$$

and because of (4.19)

$$|\omega_0| \le ||A^T-A|| . \tag{4.20}$$

We consider the general 2×2 matrix case $(B = C = I_2)$. For convenience we transform (4.15) by setting $P = -P$, $\rho = d^2$ to get:

$$P_sA + A^TP_s + P_s^2 + P_k^2 + d^2I = 0 , \tag{4.21}$$

$$P_k(A+P_s) + (A+P_s)^TP_k = 0 . \tag{4.22}$$

We set

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad P_s = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}, \quad P_k = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}. \qquad (4.23)$$

We know $|\omega_0| \le ||A^T - A||$ by (4.20), however in this case we may find it explicitly. We have

$$r_{\mathbb{C}}^2 = [\min_{\omega} \underline{\sigma}(i\omega - A)]^2 , \qquad (4.24)$$

so from (4.23) $r_{\mathbb{C}}^2 = \min_{\omega} \underline{\lambda}$ where $\underline{\lambda}$ is the smallest eigenvalue of

$$\begin{bmatrix} \omega^2 + \alpha^2 + \gamma^2 & \alpha\beta+\gamma\delta - i\omega(\gamma-\beta) \\ \alpha\beta+\gamma\delta + i\omega(\gamma-\beta) & \omega^2 + \beta^2 + \delta^2 \end{bmatrix}. \qquad (4.25)$$

From (4.25)

$$2\underline{\lambda} = 2\omega^2+\alpha^2+\beta^2+\gamma^2+\delta^2 - \surd\{(2\omega^2+\alpha^2+\beta^2+\gamma^2+\delta^2)^2$$
$$-4(\omega^2+\alpha^2+\gamma^2)(\omega^2+\beta^2+\delta^2) + 4(\alpha\beta+\gamma\delta)^2 + 4\omega^2(\gamma-\beta)^2\} \quad .$$

Setting $d(2\underline{\lambda})/d\omega = 0$, $\Delta \triangleq (2\omega^2+\alpha^2+\beta^2+\gamma^2+\delta^2)^2 - 4(\omega^2+\alpha^2+\gamma^2)(\omega^2+\beta^2+\delta^2) + 4(\alpha\beta+\gamma\delta)^2 + 4\omega^2(\gamma-\beta)^2$ we have

$$0 = \omega[1 - \Delta^{-\frac{1}{2}}(\gamma-\beta)^2] \quad .$$

So we have either $\omega_0 = 0$ or $\Delta = (\gamma-\beta)^4$ for ω_0 . If $\omega_0 \neq 0$ then

$$\Delta \equiv (\alpha^2+\beta^2+\gamma^2+\delta^2)^2 - 4(\gamma\beta-\alpha\delta)^2 + 4\omega_0^2(\gamma-\beta)^2 = (\gamma-\beta)^4 , \qquad (4.26)$$

and so using (4.26) for $\omega_0 \neq 0$ we have

$$2\underline{\lambda}_0 = \alpha^2+\beta^2+\gamma^2+\delta^2 - (\gamma-\beta)^2 + \frac{(\gamma-\beta)^2}{2} + 2\,\frac{(\gamma\beta-\alpha\delta)^2}{(\gamma-\beta)^2} - \frac{(\alpha^2+\beta^2+\gamma^2+\delta^2)^2}{2(\gamma-\beta)^2}$$

which reduces to

$$\underline{\lambda}_0 = \frac{-(\alpha+\delta)^2}{4(\beta-\gamma)^2}\,[4\gamma\beta + (\delta-\alpha)^2] \quad .$$

If $\omega_0 = 0$ we have

$$\underline{\lambda}_0 = \tfrac{1}{2}\{\alpha^2+\beta^2+\gamma^2+\delta^2 - \sqrt{[(\alpha^2+\beta^2+\gamma^2+\delta^2)^2-4(\alpha\delta-\beta\gamma)^2]}\} \quad .$$

It is easy to see that the following conclusions may be drawn:

If $(\gamma-\beta)^4 - (\alpha^2+\beta^2+\gamma^2+\delta^2)^2 + 4(\gamma\beta-\alpha\delta)^2 > 0$, (4.27)

then $\underline{\lambda}_0$ occurs at $\pm\omega_0$, where

$$\omega_0 = \frac{1}{2(\gamma-\beta)}\,[(\gamma-\beta)^4-(\alpha^2+\beta^2+\gamma^2+\delta^2)^2 + 4(\gamma\beta-\alpha\delta)^2]^{\frac{1}{2}}$$

and

$$\underline{\lambda}_0 = \frac{-(\alpha+\delta)^2}{4(\beta-\gamma)^2}\,[4\gamma\beta + (\delta-\alpha)^2] \quad . \tag{4.28}$$

If (4.27) ≤ 0 then $\underline{\lambda}_0$ occurs at $\omega_0 = 0$ and

$$\underline{\lambda}_0 = \tfrac{1}{2}[\alpha^2+\beta^2+\gamma^2+\delta^2 - \sqrt{\{(\alpha^2+\beta^2+\gamma^2+\delta^2)^2-4(\alpha\delta-\beta\gamma)^2\}}]. \tag{4.29}$$

We proceed by solving (4.21), (4.22). (4.22) becomes

$$\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}\begin{bmatrix} \alpha+p_{11} & \beta+p_{12} \\ \gamma+p_{12} & \delta+p_{22} \end{bmatrix} + \begin{bmatrix} \alpha+p_{11} & \gamma+p_{12} \\ \beta+p_{12} & \delta+p_{22} \end{bmatrix}\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = 0$$

whose components are

$$b(\gamma + p_{12}) + (\gamma + p_{12})(-b) \equiv 0 \ ,$$

$$-b(\beta + p_{12}) + b(\beta + p_{12}) \equiv 0 \ ,$$

$$b(\delta + p_{22}) + b(\alpha + p_{11}) = 0 \ , \quad b \neq 0 \ ,$$

so that

$$\alpha + \delta + p_{11} + p_{22} = 0 \ . \tag{4.30}$$

(4.21) becomes

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} + \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$+ \begin{bmatrix} p_{11}^2 + p_{12}^2 & p_{12}(p_{11} + p_{22}) \\ p_{12}(p_{11}+p_{22}) & p_{12}^2 + p_{22}^2 \end{bmatrix} + \begin{bmatrix} d^2-b^2 & 0 \\ 0 & d^2-b^2 \end{bmatrix} = 0 \ ,$$

from which we have

$$2(p_{11}\alpha + p_{12}\gamma) + p_{11}^2+p_{12}^2 + d^2-b^2 = 0 \ , \tag{4.31}$$

$$\beta p_{11}+\gamma p_{22}+p_{12}(\alpha+\delta+p_{11}+p_{22}) = 0 \ , \tag{4.32}$$

$$2(\beta p_{12}+\delta p_{22}) + p_{12}^2+p_{22}^2 + d^2-b^2 = 0 \ . \tag{4.33}$$

From (4.30), (4.32) we have $\beta p_{11} + \gamma p_{22} = 0$, so using this in (4.30) we get for all d,b

$$p_{11} = \frac{-\gamma(\alpha+\delta)}{\gamma-\beta} \ , \tag{4.34}$$

$$p_{22} = \frac{\beta(\alpha+\delta)}{\gamma-\beta} \ . \tag{4.35}$$

Using (4.31), (4.33) and (4.34), (4.35) we have

$$2(\beta-\gamma)p_{12} = \frac{2(\alpha\gamma+\delta\beta)(\alpha+\delta)}{(\beta-\gamma)} - \frac{(\gamma+\beta)(\alpha+\delta)^2}{\beta-\gamma}$$

which can be reduced to (for all d,b)

$$p_{12} = \frac{(\alpha+\delta)(\delta-\alpha)}{2(\beta-\gamma)} . \qquad (4.36)$$

We have thus found P_s ; we now find P_k (i.e. b) from (4.31) using (4.34), (4.36). After some manipulation we arrive at:

$$b^2 = \frac{(\alpha+\delta)^2}{4(\beta-\gamma)^2} [4\beta\gamma + (\delta-\alpha)^2] + d^2 . \qquad (4.37)$$

We consider now the cases that arise according as expression (4.27) > 0 , or ≤ 0 . We have for eigenvalues of $A + P_s + P_k$:

$$A + P_s + P_k =$$

$$\begin{bmatrix} \alpha + \dfrac{\gamma(\alpha+\delta)}{\beta-\gamma} & \beta + b + \dfrac{(\alpha+\delta)(\delta-\alpha)}{2(\beta-\gamma)} \\ \gamma - b + \dfrac{(\alpha+\delta)(\delta-\alpha)}{2(\beta-\gamma)} & \delta - \dfrac{\beta(\alpha+\delta)}{\beta-\gamma} \end{bmatrix} . \qquad (4.38)$$

Inspection shows trace $(A + P_s + P_k) = 0$ so that its eigenvalues are of the form $\pm i\omega$ or $\pm \alpha$. So if we set

$$\det(b) = \det \begin{bmatrix} 2(\alpha\beta+\gamma\delta) & 2(\beta+b)(\beta-\gamma)+(\alpha+\delta)(\delta-\alpha) \\ 2(\gamma-b)(\beta-\gamma)+(\alpha+\delta)(\delta-\alpha) & -2(\alpha\beta+\gamma\delta) \end{bmatrix} , \qquad (4.39)$$

then $\det(b) \geq 0$ implies the eigenvalues of $A + P_s + P_k$ are $\pm i\omega$, and $\det(b) \leq 0$ implies they are $\pm \alpha$. Examination of (4.39) shows by differentiation that the quadratic $\det(b)$ has a minimum at $b = \frac{\gamma-\beta}{2}$, whose value is

$$-[4(\alpha\beta+\gamma\delta)^2 + (\beta^2-\gamma^2+\delta^2-\alpha^2)^2] < 0 .$$

As $d(\det(b))/db = 4(\beta-\gamma)^2(2b+\beta-\gamma)$ we also have $d(\det(0))/db = 4(\beta-\gamma)^3$ and we can illustrate graphically $\det(b)$ as follows:

$\beta > \gamma$

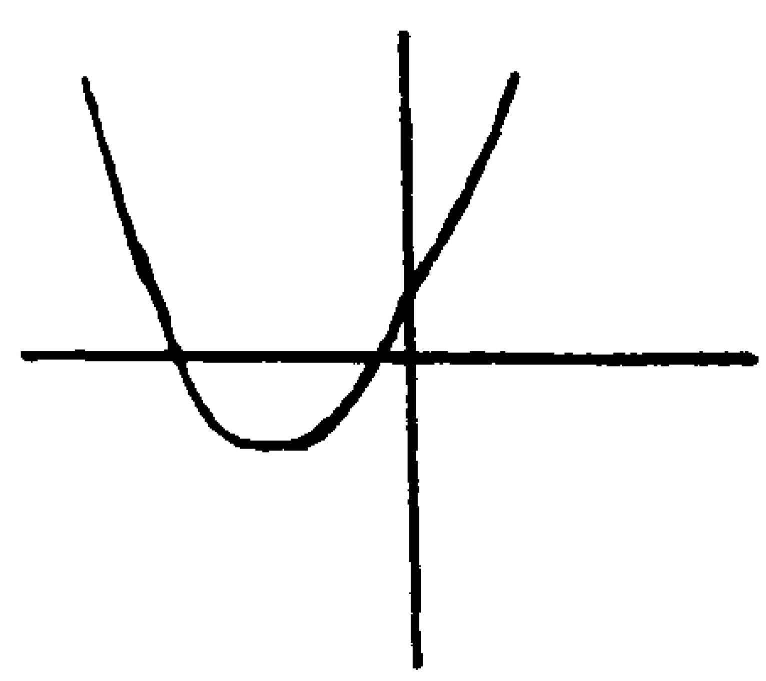


Figure 4

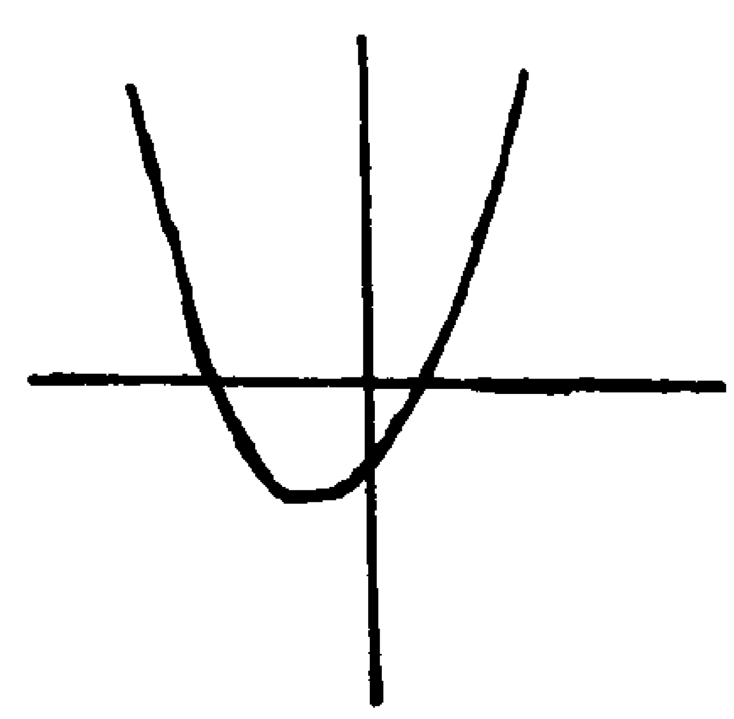


Figure 5

$\beta < \gamma$

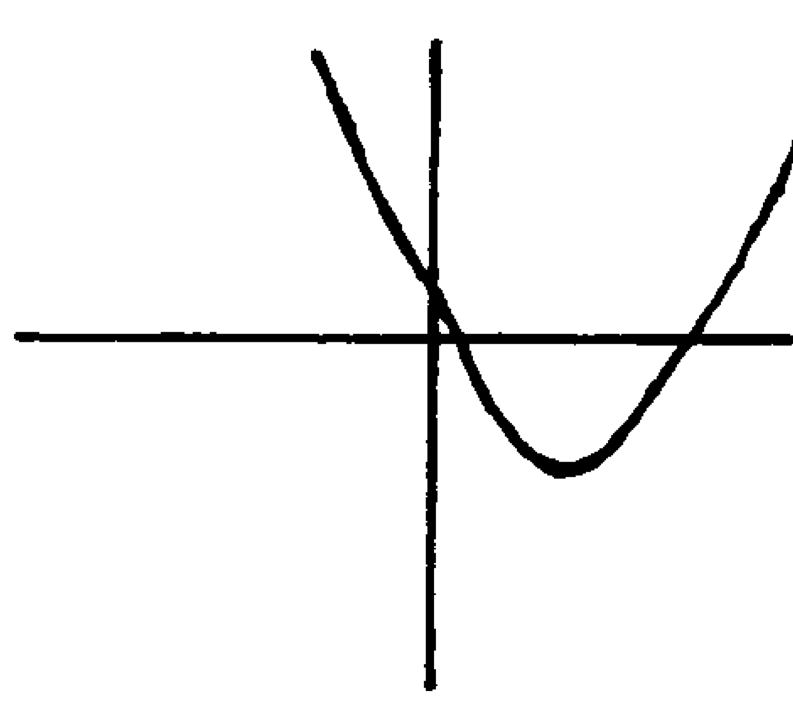


Figure 6

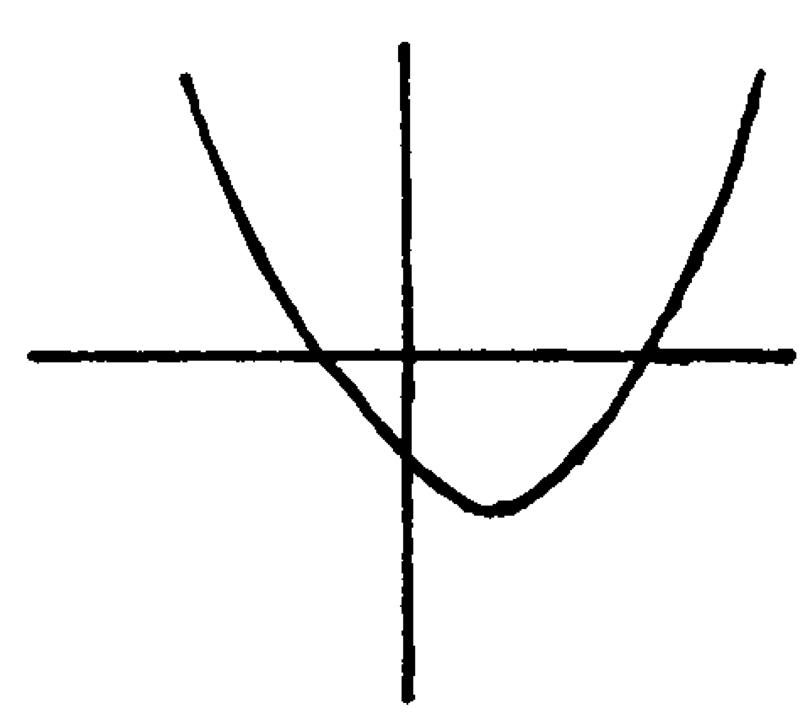


Figure 7

We consider the case where the expression (4.27) > 0 . Simple algebra shows then that $\det(0) > 0$ so that we have either Fig. 4 or Fig. 6. It is then easy to see from (4.28), (4.37) that we may continue the symmetric solution from $\underline{\lambda}_0$, ω_0 in a nonsymmetric way (by inspection of $\det(b)$). Furthermore two nonsymmetric solutions are possible - either $\omega \uparrow \infty$ or $\omega \downarrow 0$.

In the case (4.27) ≤ 0 we have $\det(0) \le 0$ so that we have Fig. 5 or Fig. 7 (in fact (4.27) = 0 gives $\det(0) = 0$ and (4.27) < 0 gives $\det(0) < 0$). By the theory leading to proposition 4.1 as $d^2 \uparrow r_{\mathbb{C}}^2$ then for some eigenvalue λ of A+P, $\lambda \to 0$. We note that in this case $r_{\mathbb{C}}^2 = \underline{\lambda}_0$ given by (4.29). This theory is in fact easily verified. We have

$$2(p_{11}\alpha + p_{12}\gamma) + p_{11}^2 + p_{12}^2 + d^2 = 0 , \tag{4.40}$$

$$\beta p_{11}+\gamma p_{22} + p_{12}(\alpha+\delta+p_{11}+p_{22}) = 0 , \tag{4.41}$$

$$2(\beta p_{12}+\delta p_{22}) + p_{12}^2 + p_{22}^2 + d^2 = 0 , \tag{4.42}$$

and the eigenvalues of A+P are given by

$$\lambda^2-(\alpha+\delta+p_{11}+p_{22})\lambda + (\alpha+p_{11})(\delta+p_{22})-(\beta+p_{12})(\gamma+p_{12}) = 0 , \tag{4.43}$$

so that when $d^2 = \underline{\lambda}_0$ we have from (4.43)

$$(\alpha+p_{11})(\delta+p_{22}) = (\beta+p_{12})(\gamma+p_{12}) . \tag{4.44}$$

From (4.40), (4.42), (4.44) we have (after some manipulation)

$$(p_{11} + \alpha + p_{22} +\delta)^2 = (\alpha+\delta)^2 + 2(\beta\gamma-\alpha\delta)-2\underline{\lambda}_0 . \tag{4.45}$$

That (4.27) < 0 implies the RHS of (4.45) is > 0, and (4.27) = 0 implies the RHS of (4.45) is = 0. So we may set

$$p_{11} + p_{22} + \alpha + \delta = H \le 0 . \tag{4.46}$$

(4.41) then becomes

$$\beta p_{11} + \gamma p_{22} + Hp_{12} = 0 , \tag{4.47}$$

and subtracting (4.40), (4.42) we have

$$2p_{11}\alpha-2\delta p_{22} + 2(\gamma-\beta)p_{12} + (p_{11}-p_{22})(H-\alpha-\delta) = 0 . \qquad (4.48)$$

Our problem is thus reduced to solving (4.46)-(4.48) for p_{11}, p_{12}, p_{22} . After some tedious calculation we obtain:

$$p_{11} = \frac{(H-\alpha-\delta)(H(H+\delta-\alpha)-2\gamma(\beta-\gamma))}{2(H^2+(\beta-\gamma)^2)} ,$$

$$p_{12} = \frac{-(H-\alpha-\delta)((\beta-\gamma)(H+\delta-\alpha)+2\gamma H)}{2(H^2+(\beta-\gamma)^2)} ,$$

$$p_{22} = \frac{(H-\alpha-\delta)(2\beta(\beta-\gamma)-H(-H+\delta-\alpha))}{2(H^2+(\beta-\gamma)^2)} .$$

We see further that (from (4.43), (4.46)) when (4.27) < 0 one eigenvalue of A+P goes to the origin, whereas in the case (4.27) = 0 both eigenvalues do so as $d^2 \uparrow r_{\mathbb{C}}^2$.

The case (4.27) < 0 is not interesting from our point of view - however we make one final comment. The inequality holds

$$\tfrac{1}{2}[\alpha^2+\beta^2+\gamma^2+\delta^2 - \sqrt{\{(\alpha^2+\beta^2+\gamma^2+\delta^2)^2-4(\alpha\delta-\beta\gamma)^2\}}]$$

$$> \frac{-(\alpha+\delta)^2}{4(\beta-\gamma)^2}[4\beta\gamma+(\delta-\alpha)^2] . \qquad (4.49)$$

This is true since (4.49) can be easily manipulated to give

$$(\alpha+\delta)^4[4\beta\gamma+(\delta-\alpha)^2]^2+4(\beta-\gamma)^2(\alpha+\delta)^2(\alpha^2+\beta^2+\gamma^2+\delta^2)[4\beta\gamma+(\delta-\alpha)^2]$$

$$+ 16(\beta-\gamma)^4(\alpha\delta-\beta\gamma)^2 > 0 . \qquad (4.50)$$

However this last inequality (4.50) (it can be shown) is just the perfect square

$$[(\gamma-\beta)^4 - (\alpha^2+\beta^2+\gamma^2+\delta^2)^2 + 4(\gamma\beta-\alpha\delta)^2]^2 > 0 \ .$$

Herein lies the reason why the case (4.27) < 0 is of no interest. We see from (4.29), (4.37), (4.49) that (not surprisingly) there is no continuity to a nonsymmetric solution from the symmetric solution reached when $d^2 \uparrow r_{\mathbb{C}}^2$. This is in contrast to the case (4.27) > 0 .

We return to the case (4.27) > 0 . Firstly, a simple computation from (4.37), (4.39) shows that there is a nonsymmetric solution P from $d = r_{\mathbb{C}}$, $\omega = \omega_0$ such that $i\omega \in \sigma(A+P)$ as $\omega \to 0$ and $d \to \underline{\sigma}(A)$, where

$$2\underline{\sigma}^2(A) = \alpha^2+\beta^2+\gamma^2+\delta^2 - \surd\{(\alpha^2+\beta^2+\gamma^2+\delta^2)^2-4(\alpha\delta-\beta\gamma)^2\} \ .$$

This is in agreement with proposition 4.7. In this light we take a further look at the numerical example considered in [6], see also Chapter 2.

Here $A = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix}$.

In this case (4.27) gives

$$(36)^2-(30)^2 + 4.25 > 0 \ ,$$

and by (4.28)

$$r_{\mathbb{C}}^2 = \frac{-4}{4.36}[-20+4] = \frac{4}{9} \quad ,$$

$$\omega_0 = \frac{1}{2(-6)} [66.6 + 100]^{\frac{1}{2}} = \frac{-\sqrt{31}}{3} .$$

From (4.39)

$$\det(b) = 144b^2 + 864b + 496$$

and $\det(b) = 0$ when $b = \dfrac{-9 \pm 5\sqrt{2}}{3}$.

In this case $\beta > \gamma$ so we take (for the value b such that $\omega \to 0$) $b = \dfrac{-9 + 5\sqrt{2}}{3}$ so that $b^2 = \dfrac{131 - 90\sqrt{2}}{9}$. From (4.37) we have the value of d at which $\omega \to 0$

$$d = \sqrt{\frac{4}{9} + \left(\frac{131 - 90\sqrt{2}}{9}\right)} = \sqrt{15 - \sqrt{200}} ,$$

which is $\underline{\sigma}(A)$ as expected. This is in fact, see Chapter 2, the real unstructured stability radius and we may construct by our approach a real destabilizing perturbation of this norm via the solution $P = P_s + P_k$ of the Riccati equation. This is simply

$$\begin{bmatrix} \frac{5(-2)}{-6} & \frac{-2(-2)}{2.6} \\ \frac{-2(-2)}{2.6} & \frac{1(-2)}{-6} \end{bmatrix} + \begin{bmatrix} 0 & \frac{-9 + 5\sqrt{2}}{3} \\ \frac{9-5\sqrt{2}}{3} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5/3 & \frac{-8+5\sqrt{2}}{3} \\ \frac{10-5\sqrt{2}}{3} & 1/3 \end{bmatrix} .$$

This provides an alternative minimum norm real destabilizing perturbation to that which may be constructed using the comment made following proposition 4.7.

4.2 STABILITY RADII WITH RESPECT TO THE TRACE NORM

We make some comments here on stability radii for continuous time systems when the trace norm is employed on the space of matrices - in [6], [7] the induced Euclidean norm was used in the calculation of stability radii.

In the paper [20] the trace norm is used in the calculation of the distance of a controllable system from the set of uncontrollable systems - that is the minimal norm additive perturbation which makes the system uncontrollable, see also [29].

The trace norm of a matrix X with entries in $\mathbb{C}$ is given by:

$$||X|| = \{\mathrm{tr}\, X^* X\}^{\frac{1}{2}} , \qquad (4.51)$$

where * denotes conjugate - transpose. The matrix X_0 is said to be a best approximate solution of $f(X) = G$ if for all X either

$$||f(X) - G|| > ||f(X_0) - G|| , \text{ or} \qquad (4.52)$$

$$||f(X) - G|| = ||f(X_0) - G|| \text{ and } ||X_0|| \le ||X|| . \qquad (4.53)$$

We then have the following result (see, *for example, [21] Page 68, or [19]*):

Lemma 4.8

The best approximate solution of $AX = C$ is unique and given by A^+C where A^+ denotes the generalized inverse of A .

The following lemma characterizes the generalized inverse, A^+ , of A :

Lemma 4.9 (Theorem 6.1, [19])

Let A be an $m\times n$ matrix. The $n\times m$ matrix X satisfying the four equations

$$AXA = A ,$$
$$XAX = X ,$$
$$(AX)^* = AX ,$$
$$(XA)^* = XA ,$$

exists and is unique. Furthermore, A^+ is a solution to the four equations.

We consider the case $A \in \mathbb{R}^{n\times n}$ with A stable, and the unstructured perturbation matrix P of dimension $n\times n$ (in general $P \in \mathbb{C}^{n\times n}$) . If $A+P$ is unstable then for some ω, z

$$(A + P)z = i\omega z$$

or

$$z^*(A^T+P^*) = -i\omega z^* . \qquad (4.54)$$

Set

$$z^*A^T = \frac{z^*A^Tzz^*}{z^*z} + (z^*A^T \frac{-z^*A^Tzz^*}{z^*z}) = \alpha z^* + z^*_0 , \qquad (4.55)$$

then

$$<z,z_0> = 0$$

so that z,z_0 are orthogonal. We have from (4.54), (4.55)

$$z^*(A^T+P^*) = \alpha z^* + z_0^* + z^*P^* = -i\omega z^*$$

or

$$z^*P^* = -z_0^* - (i\omega+\alpha)z^* . \qquad (4.56)$$

If we set

$$P^* = z(-z_0^* - (i\omega+\alpha)z^*)/z^*z , \qquad (4.57)$$

then P^* satisfies (4.56). We see further that (z/z^*z) satisfies

$$\begin{aligned} z^*Xz^* &= z^* , \\ Xz^*X &= X , \\ (z^*X)^* &= z^*X , \\ (Xz^*)^* &= Xz^* , \end{aligned}$$

so by lemma 4.9 it is the generalized inverse of z^*. So by lemma 4.8 and the fact that it satisfies (4.56), P^* of (4.57) is the minimal norm solution of (4.54). To find this norm we calculate $tr(PP^*)$:

$$\begin{aligned} tr(PP^*) &= tr\left[\frac{(z_0 + (-i\omega+\bar{\alpha})z)z^*z(z_0^* + (i\omega+\alpha)z^*)}{(z^*z)^2}\right] \\ &= \frac{1}{z^*z}\ tr\,[(z_0^*+(i\omega+\alpha)z^*)(z_0+(-i\omega+\bar{\alpha})z)] , \end{aligned}$$

using the fact that $tr(AB) = tr(BA)$. So we have

$$\mathrm{tr}(PP^*) = (z_0^*z_0 + (i\omega+\alpha)z^*z_0 + (-i\omega+\bar{\alpha})z_0^*z + |i\omega+\alpha|^2 z^*z)/z^*z$$

$$= \frac{z_0^*z_0}{z^*z} + |i\omega + \alpha|^2 ,$$

using the fact that z, z_0 are orthogonal. We recall that

$$z_0 = Az - \frac{zz^*Az}{z^*z} = (I - \frac{zz^*}{z^*z})Az$$

so that

$$\mathrm{tr}(PP^*) = \frac{z^*A^T(I - \frac{zz^*}{z^*z})^2 Az}{z^*z} + |i\omega+\alpha|^2$$

$$= \frac{z^*A^T(I - \frac{zz^*}{z^*z})Az}{z^*z} + |i\omega+\alpha|^2 .$$

Summarizing the above, and noting that there is no restriction in taking $||z|| = 1$, we have the following result:

Proposition 4.10

The minimal norm of a destabilizing complex perturbation is given by the square root of

$$\min_{||z||=1} \min_{\omega\in R} z^*A^T(I-zz^*)Az + |i\omega+\alpha|^2 . \qquad (4.58)$$

Remarks. 1) That P of (4.57) is of rank one means that the trace norm gives the same result as the induced Euclidean norm;

2) If A is symmetric, $A = A^T$, then the quantity α in

(4.58) is real and so we take $\omega = 0$. This compares with the findings in [6].

We turn now to the case where $P \in \mathbb{R}^{n\times n}$. We set, in (4.56), $z = x + iy$, $z_0 = x_0 + iy_0$, $-i\omega-\alpha = -i\omega-a-ib = -a-i(\omega+b)$. Then for real P^* we have

$$(x^T-iy^T)P^* = (-x_0^T+iy_0^T) - (a+i(\omega+b))(x^T-iy^T)$$

or

$$\begin{bmatrix} x^T \\ -y^T \end{bmatrix} P^* = \begin{bmatrix} -x_0^T \\ y_0^T \end{bmatrix} + \begin{bmatrix} -ax^T-(\omega+b)y^T \\ -(\omega+b)x^T+ay^T \end{bmatrix} \triangleq \begin{bmatrix} -\tilde{x}_0^T \\ \tilde{y}_0^T \end{bmatrix} . \qquad (4.59)$$

We consider first the case where x,y are LI. Then equivalently, by the Cauchy-Schwartz inequality, $x^Txy^Ty - x^Tyy^Tx \neq 0$. We have then that

$$\begin{bmatrix} x^T \\ -y^T \end{bmatrix}^+ = [x,-y] \begin{bmatrix} x^Tx & -x^Ty \\ -y^Tx & y^Ty \end{bmatrix}^{-1} . \qquad (4.60)$$

This is true since the RHS of (4.60) is well defined and satisfies the four equations of lemma 4.9. In fact, we have:

$$\begin{bmatrix} x^Tx & -x^Ty \\ -y^Tx & y^Ty \end{bmatrix}^{-1} = \frac{1}{x^Txy^Ty-x^Tyy^Tx} \begin{bmatrix} y^Ty & x^Ty \\ y^Tx & x^Tx \end{bmatrix} . \qquad (4.61)$$

The fact that the RHS of (4.60) satisfies lemma 4.9 is easily verified, so we omit the details. If we define

$$P^* = \begin{bmatrix} x^T \\ -y^T \end{bmatrix}^+ \begin{bmatrix} -\tilde{x}_0^T \\ \tilde{y}_0^T \end{bmatrix} . \qquad (4.62)$$

Then, as before, using lemma 4.8 we see that P^* is the best approximate solution of (4.59). Furthermore on substitution of P^* in (4.59) we have

$$\begin{bmatrix} x^T \\ -y^T \end{bmatrix} [x,-y] \begin{bmatrix} x^Tx & -x^Ty \\ -y^Tx & y^Ty \end{bmatrix}^{-1} \begin{bmatrix} -\tilde{x}_0^T \\ \tilde{y}_0^T \end{bmatrix} = \begin{bmatrix} -\tilde{x}_0^T \\ \tilde{y}_0^T \end{bmatrix}$$

so that P^* of (4.62) is the minimal norm solution of (4.59) in the case where x,y are LI. We calculate the norm:

$$||P||^2 = tr(P^*P) =$$

$$tr\left([x,-y] \begin{bmatrix} x^Tx & -x^Ty \\ -y^Tx & y^Ty \end{bmatrix}^{-1} \begin{bmatrix} -\tilde{x}_0^T \\ \tilde{y}_0^T \end{bmatrix} [-\tilde{x}_0, \tilde{y}_0] \begin{bmatrix} x^Tx & -x^Ty \\ -y^Tx & y^Ty \end{bmatrix}^{-1} \begin{bmatrix} x^T \\ -y^T \end{bmatrix} \right)$$

$$= tr\left(\begin{bmatrix} x^Tx & -x^Ty \\ -y^Tx & y^Ty \end{bmatrix}^{-1} \begin{bmatrix} -\tilde{x}_0^T \\ \tilde{y}_0^T \end{bmatrix} [-\tilde{x}_0, \tilde{y}_0] \begin{bmatrix} x^Tx & -x^Ty \\ -y^Tx & y^Ty \end{bmatrix}^{-1} \begin{bmatrix} x^T \\ -y^T \end{bmatrix} [x,-y] \right),$$

by the properties of trace,

$$= tr\left(\begin{bmatrix} x^Tx & -x^Ty \\ -y^Tx & y^Ty \end{bmatrix}^{-1} \begin{bmatrix} \tilde{x}_0^T\tilde{x}_0 & -\tilde{x}_0^T\tilde{y}_0 \\ -\tilde{y}_0^T\tilde{x}_0 & \tilde{y}_0^T\tilde{y}_0 \end{bmatrix} \right) .$$

Using (4.61) this last expression can easily be shown to reduce to the scalar

$$||P||^2 = \frac{y^Ty\tilde{x}_0^T\tilde{x}_0 - 2x^Ty\tilde{y}_0^T\tilde{x}_0 + x^Tx\tilde{y}_0^T\tilde{y}_0}{x^Txy^Ty - x^Tyy^Tx} . \qquad (4.63)$$

In the case where x,y are LD we suppose, for the sake of discussion, that

$$x = ty \ , \quad y \neq 0 \ . \tag{4.64}$$

Then from (4.59) we have

$$x^T P^* = -x_0^T - ax^T - (\omega+b)y^T \ ,$$
$$-y^T P^* = y_0^T - (\omega+b)x^T + ay^T \ .$$

Multiplying the latter equation by t, adding to the former and using (4.64)

$$(\omega+b)(t^2+1)y^T = -x_0^T + ty_0^T \ . \tag{4.65}$$

From (4.64) and the definition of z_0 in (4.55) we obtain

$$x_0 = ty_0 \ . \tag{4.66}$$

So from (4.65), (4.66) we see

$$\omega + b = 0 \ . \tag{4.67}$$

Using (4.66), (4.67) in (4.59) we have

$$y^T P^* = -y_0^T - ay^T \ . \tag{4.68}$$

From the fact that z,z_0 are orthogonal we have $0 = \langle z,z_0\rangle =$
$= \langle x+iy,x_0+iy_0\rangle = (1+t^2)\langle y,y_0\rangle + it\langle y,y_0\rangle - it\langle y,y_0\rangle = (1+t^2)\langle y,y_0\rangle$
so that $y^T y_0 = 0$.

As in the complex perturbation case we see that

$$P^* = \frac{y(-y_0^T - ay^T)}{y^T y} \tag{4.69}$$

is the minimal norm solution of (4.68) in the case where x,y are LD. As before

$$||P||^2 = tr(PP^*) = tr\left[\frac{(y_0+ay)y^Ty(y_0^T + ay^T)}{(y^Ty)^2}\right]$$

$$= \frac{1}{y^Ty}\, tr\,[\, y_0^Ty_0 + ay_0^Ty + ay^Ty_0 + a^2y^Ty\,]$$

$$= \frac{y_0^Ty_0}{y^Ty} + a^2 . \qquad (4.70)$$

In the case where x,y are LD we have by (4.67) that necessarily $\omega+b = 0$. When x,y are LI we have by (4.63)

$$||P||^2 = \frac{y^Ty\tilde{x}_0^T\tilde{x}_0 - 2x^Ty\tilde{y}_0^T\tilde{x}_0 + x^Tx\tilde{y}_0^T\tilde{y}_0}{x^Txy^Ty - x^Tyy^Tx} ,$$

and from (4.59)

$$\tilde{x}_0^T = x_0^T + ax^T + (\omega+b)y^T ,$$

$$\tilde{y}_0^T = y_0^T - (\omega+b)x^T + ay^T .$$

From these last two equations we find that

$$\tilde{x}_0^T\tilde{x}_0 = x_0^Tx_0+a^2x^Tx+(\omega+b)^2y^Ty+2ax_0^Tx+2(\omega+b)x_0^Ty+2a(\omega+b)y^Tx,$$

$$\tilde{y}_0^T\tilde{y}_0 = y_0^Ty_0+(\omega+b)^2x^Tx+a^2y^Ty+2ay_0^Ty-2(\omega+b)y_0^Tx-2a(\omega+b)x^Ty, \qquad (4.71)$$

$$\tilde{y}_0^T\tilde{x}_0 = y_0^Tx_0 + a(y_0^Tx+y^Tx_0) + (\omega+b)(y_0^Ty-x^Tx_0)$$

$$+ a(\omega+b)(y^Ty-x^Tx) - (\omega+b)^2\, x^Ty + a^2y^Tx .$$

From (4.71), (4.63) becomes

$$(||x||^2||y||^2-(x^Ty)^2)||P||^2 = A+2aB+2(\omega+b)C+2a^2D+(\omega+b)^2E , \quad (4.72)$$

where the values of A, B, C, D, E are given by the following:

$$A = ||y||^2||x_0||^2-2x^Tyx_0^Ty_0 + ||x||^2||y_0||^2 ,$$
$$B = ||y||^2x_0^Tx-x^Tyy_0^Tx-x^Tyy^Tx_0 + ||x||^2y_0^Ty , \quad (4.73)$$
$$C = ||y||^2x_0^Ty-x^Tyy_0^Ty + x^Tyx_0^Tx-||x||^2y_0^Tx ,$$
$$D = ||y||^2||x||^2 - (x^Ty)^2 ,$$
$$E = ||y||^4 + 2(x^Ty)^2 + ||x||^4 .$$

To minimize wrt ω we set $\partial(\cdot)/\partial\omega = 0$ in (4.72) giving

$$2C + 2(\omega+b)E = 0 ,$$

or

$$\omega+b = -C/E , \quad (4.74)$$

which is a minimum as $E > 0$.

Setting (4.74) in (4.72) we have

$$||P||^2 = (A + 2aB + 2a^2D - \frac{C^2}{E})/D . \quad (4.75)$$

We may summarize the above discussion on minimum norm real destabilizing perturbations in the following proposition.

<u>Proposition 4.11</u>

In equation (4.56)

$$z^*P^* = -z_0^* -(i\omega+\alpha)z^*$$

we set

$$z = x+iy \ , \ z_0 = x_0 + iy_0 \ , \ \alpha = a + ib \ .$$

Then the minimal norm of a real destabilizing perturbation is given by the square root of

$$\inf_{z \neq 0} f(z) \ , \quad \text{where:}$$

If x,y are LI

$$f(z) = (A + 2aB + 2a^2 D - C^2/E)/D$$

and A, B, C, D, E are given by (4.73);

If $x = ty \ , \ y \neq 0$

$$f(z) = \frac{y_0^T y_0}{y^T y} + a^2 \ .$$

Remarks. 1) The case $y = tx \ , \ x \neq 0$, is handled analogously;

2) As in [20] there may be possible discontinuities in $f(z)$ defined above as the linearly independent pair x,y tend to linear dependence.

We turn now to the case where the perturbation matrix is of a certain structure governed by fixed matrices B, C, as in [7].

As before the stable matrix $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and now D is of dimension $m\times p$ with, in general, entries in $\mathbb{C}^{m\times p}$. We will use the following result (see [21] Page 68, or [19]).

Lemma 4.12

The best approximate solution of $AXB = C$, in the sense of (4.52), (4.53), is unique and given by $A^+ CB^+$ where A^+, B^+ are the generalized inverses of A, B respectively.

If $A + BDC$ is unstable we have, as before,

$$(A + BDC)z = i\omega z$$

or

$$z^*(A^T + C^T D^* B^T) = -i\omega z^* . \tag{4.76}$$

We make the same transformation as in (4.55) for the unstructured case, then we have (as in (4.56))

$$z^* C^T D^* B^T = -z_0^* - (i\omega+\alpha)z^* . \tag{4.77}$$

We make now an assumption concerning the matrices $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ - we assume that both have rank not less than n. If we now set

$$D^* = Cz(z^* C^T Cz)^{-1}(-z_0^* - (i\omega+\alpha)z^*)(BB^T)^{-1}B , \tag{4.78}$$

then it is easy to see that D^* satisfies (4.77). Furthermore using lemma 4.9 we see that $Cz(z^*C^TCz)^{-1}$, $(BB^T)^{-1}B$ are the generalized inverses of z^*C^T, B^T respectively - so by lemma 4.12 we find that D^* of (4.78) is the minimal norm solution of (4.76). We calculate trace(DD^*) :

$$\operatorname{tr}(DD^*) =$$

$$\operatorname{tr}[B^T(BB^T)^{-1}(z_0+(-i\omega+\bar{\alpha})z)(z^*C^TCz)^{-1}(z_0^*+(i\omega+\alpha)z^*)(BB^T)^{-1}B]$$

$$= (z^*C^TCz)^{-1} \; tr[(z_0+(-i\omega+\bar{\alpha})z)(z_0^*+(i\omega+\alpha)z^*)(BB^T)^{-1}]$$

using the properties of trace. So we obtain as a generalization of proposition 4.10:

Proposition 4.13

The minimal norm of a destabilizing complex perturbation is given by square root of

$$\min_{z\neq 0}\min_{\omega\in\mathbb{R}} (z^*C^TCz)^{-1} tr[(z_0+(-i\omega+\bar{\alpha})z)(z_0^*+(i\omega+\alpha)z^*)(BB^T)^{-1}] \quad . \qquad (4.79)$$

In the case where $D \in \mathbb{R}^{m\times p}$, we set, in (4.77), $z = x+iy$, $z_0 = x_0+iy_0$, $\alpha = a+ib$ to give

$$\begin{bmatrix} x^T \\ -y^T \end{bmatrix} C^TD^*B^T \doteq \begin{bmatrix} -x_0^T \\ y_0^T \end{bmatrix} + \begin{bmatrix} -ax^T-(\omega+b)y^T \\ -(\omega+b)x^T+ay^T \end{bmatrix} \triangleq \begin{bmatrix} -\tilde{x}_0^T \\ \tilde{y}_0^T \end{bmatrix} . \qquad (4.80)$$

As before (as rank $C \geq n$) when x,y are LI we have

$$\begin{bmatrix} x^T C^T \\ -y^T C^T \end{bmatrix}^+ = [Cx,-Cy] \begin{bmatrix} x^TC^TCx & -x^TC^TCy \\ -y^TC^TCx & y^TC^TCy \end{bmatrix}^{-1} , \qquad (4.81)$$

with

$$\begin{bmatrix} x^TC^TCx & -x^TC^TCy \\ -y^TC^TCx & y^TC^TCy \end{bmatrix}^{-1} = \frac{1}{x^TC^TCxy^TC^TCy-x^TC^TCyy^TC^TCx} \times \begin{bmatrix} y^TC^TCy & x^TC^TCy \\ y^TC^TCx & x^TC^TCx \end{bmatrix} . \qquad (4.82)$$

We define

$$D^{*} = \begin{bmatrix} x^TC^T \\ -y^TC^T \end{bmatrix}^{+} \begin{bmatrix} -\tilde{x}_0^T \\ \tilde{y}_0^T \end{bmatrix} (BB^T)^{-1}B \,, \tag{4.83}$$

which satisfies (4.80) and so by lemma **4.12** D^* of (4.83) is the minimal norm solution of (4.80) in case x,y LI . We calculate $||D||^2$:

$$||D||^2 = tr(D^*D) =$$

$$tr(\ [Cx,-Cy] \begin{bmatrix} x^TC^TCx & -x^TC^TCy \\ -y^TC^TCx & y^TC^TCy \end{bmatrix}^{-1} \begin{bmatrix} -\tilde{x}_0^T \\ \tilde{y}_0^T \end{bmatrix} (BB^T)^{-1} \times$$

$$BB^T(BB^T)^{-1}[-\tilde{x}_0,\tilde{y}_0] \begin{bmatrix} x^TC^TCx & -x^TC^TCy \\ -y^TC^TCx & y^TC^TCy \end{bmatrix}^{-1} \begin{bmatrix} x^TC^T \\ -y^TC^T \end{bmatrix})$$

$$= \quad tr(\begin{bmatrix} -\tilde{x}_0^T \\ \tilde{y}_0^T \end{bmatrix} (BB^T)^{-1} [-\tilde{x}_0,\tilde{y}_0] \begin{bmatrix} x^TC^TCx & -x^TC^TCy \\ -y^TC^TCx & y^TC^TCy \end{bmatrix}^{-1}) \,.$$

By (4.82),

$$||D||^2 = \frac{1}{x^TC^TCxy^TC^TCy-x^TC^TCyy^TC^TCx} \times$$

$$tr(\begin{bmatrix} -\tilde{x}_0^T \\ \tilde{y}_0^T \end{bmatrix} (BB^T)^{-1}[-\tilde{x}_0y^TC^TCy+\tilde{y}_0y^TC^TCx,-\tilde{x}_0x^TC^TCy+\tilde{y}_0x^TC^TCx]) \,,$$

so that

$$||D||^2 = \frac{1}{x^TC^TCxy^TC^TCy-x^TC^TCyy^TC^TCx}$$

$$\text{tr}\left((BB^T)^{-1} \begin{bmatrix} \tilde{x}_0 y^TC^TCy\tilde{x}_0^T - \tilde{y}_0 y^TC^TCx\tilde{x}_0^T \\ -\tilde{x}_0 x^TC^TCy\tilde{y}_0^T + \tilde{y}_0 x^TC^TCx\tilde{y}_0^T \end{bmatrix} \right) . \qquad (4.84)$$

When x,y are LD , we suppose

$$x = ty , \quad y \neq 0 .$$

We have, as before, $\omega+b = 0$ so that (4.80) reduces to

$$y^TC^TD^*B^T = -y_0^T - ay^T . \qquad (4.85)$$

So we see that

$$D^* = Cy(y^TC^TCy)^{-1}(-y_0^T-ay^T)(BB^T)^{-1}B \qquad (4.86)$$

is the minimal norm solution to (4.85) when x,y are LD . In this case

$$||D||^2 = \text{tr}(DD^*)$$

$$= \text{tr}[B^T(BB^T)^{-1}(y_0+ay)(y^TC^TCy)^{-1}(y_0^T+ay^T)(BB^T)^{-1}B]$$

$$= (y^TC^TCy)^{-1} \; \text{tr}[(y_0+ay)(y_0^T+ay^T)(BB^T)^{-1}] \quad . \qquad (4.87)$$

So we have:

<u>Proposition 4.14</u>

The minimal norm of a real destabilizing perturbation is given by the square root of

$$\inf_{\omega, z \neq 0} f(z,\omega) \ , \quad \text{with:}$$

If x,y are LI

$$f(z,\omega) = \frac{1}{x^TC^TCxy^TC^TCy - x^TC^TCyy^TC^TCx} \times$$

$$\operatorname{tr}\left((BB^T)^{-1}\begin{bmatrix} \tilde{x}_0 y^TC^TCy\tilde{x}_0^T - \tilde{y}_0 y^TC^TCx\tilde{x}_0^T \\ -\tilde{x}_0 x^TC^TCy\tilde{y}_0^T + \tilde{y}_0 x^TC^TCx\tilde{y}_0^T \end{bmatrix}\right) ;$$

If $x = ty$, $y \neq 0$

$$f(z,\omega) = (y^TC^TCy)^{-1} \operatorname{tr}[(y_0+ay)(y_0^T+ay^T)(BB^T)^{-1}] \ .$$

4.3 GENERATION OF LYAPUNOV FUNCTIONS

In this section we make some comments on the existence of Lyapunov functions - firstly for the asymptotic stability of perturbed systems, and secondly for the boundedness of solutions to persistently excited systems. We assume the nominal system to be asymptotically stable.

In the case of a system with structured perturbations as in Chapter 2,

$$\dot{x} = (A + BDC)x \ , \quad x(0) = x_0 \ , \tag{4.88}$$

a Lyapunov function giving asymptotic stability for the set

$$\{A + BDC : ||D||^2 < \rho\} \ ,$$

can only exist when $\rho \leq r_{\mathbb{C}}^2(A;B,C)$. We show that just such a function exists. We consider

$$V(x) = -\langle x,Px\rangle \ , \tag{4.89}$$

where $P \leq 0$ is the solution to the algebraic Riccati equation

$$PA + A^TP - \rho C^TC - PBB^TP = 0 \ , \qquad (4.90)$$

given by theorem 2.21. We then have that for the solution of (4.88)

$$\begin{aligned} \dot{V}(x) &= -\langle\dot{x},Px\rangle - \langle x,P\dot{x}\rangle \\ &= -\langle(A+BDC)x,Px\rangle - \langle x,P(A+BDC)x\rangle \\ &= -\langle BDCx,Px\rangle - \langle x,PBDCx\rangle \\ &\quad -\langle x,(PA+A^TP)x\rangle \ . \end{aligned}$$

Using (4.90) we then have

$$\begin{aligned} \dot{V}(x) &= -\langle BDCx,Px\rangle - \langle x,PBDCx\rangle \\ &\quad -[\rho||Cx||^2 + ||B^TPx||^2] \\ &= -||B^TPx+DCx||^2 - [\rho||Cx||^2 - ||DCx||^2] \ . \end{aligned} \qquad (4.91)$$

So we see that when $||D||^2 < \rho$ we have the inequalities

$$\rho||Cx||^2 - ||DCx||^2 \geq (\rho-||D||^2)||Cx||^2 \geq \varepsilon||Cx||^2 \ , \qquad (4.92)$$

for some $\varepsilon > 0$. From (4.91), (4.92)

$$\dot{V}(x) \leq -\varepsilon||Cx||^2 \ ,$$

and integrating this from 0 to t, using (4.89)

$$-\langle x(t),Px(t)\rangle + \langle x_0,Px_0\rangle \leq -\varepsilon \int_0^t ||Cx||^2 . dt \ ,$$

which, using the fact that $P \leq 0$, shows that for all $t \geq 0$

$$\varepsilon \int_0^t ||Cx||^2.dt \leq -\langle x_0, Px_0 \rangle \quad ,$$

and consequently

$$(Cx)(\cdot) \in L^2 \; . \qquad (4.93)$$

Now the solution of (4.88)

$$x(t) = e^{(A+BDC)t}x_0 \; ,$$

satisfies

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}BDCx(s).ds \; ,$$

so that from (4.93) we have that $x(\cdot) \in L^2$ so that $(A' + BDC)$ is asymptotically stable whenever $||D||^2 < \rho$. Summarizing this, we have shown in an alternative way to that in [7]:

Proposition 4.15

The function $V(x) = -\langle x, Px \rangle$ where P satisfies the algebraic Riccati equation (4.90) is a "best" Lyapunov function for the system $\dot{x} = (A+BDC)x$, $x(0) = x_0$, when we set $\rho = r_{\mathbb{C}}^2(A;B,C)$ in (4.90).

In fact this function is further used in the treatment of time-varying and nonlinear perturbations to the basic asymptotically stable linear system - for details we refer the reader to [7].

We conclude this section with some remarks on Lyapunov functions guaranteeing the boundedness of solutions to persistently disturbed equations.

We consider initially the system on $\mathbb{R}^n$,

$$\dot{x} = f(x) + ug(x) \ , \tag{4.94}$$

where $u(t)$ is a scalar with $|u(t)| \leq k$ for all $t \geq 0$. We suppose there exists some function $V(x)$ with continuous first derivatives and

$$\begin{aligned} &V(x) > 0 \ , \quad x \neq 0 \ , \\ &V(0) = 0 \ . \end{aligned} \tag{4.95}$$

Computing $\dot{V}$ along solutions of (4.94) we have

$$\begin{aligned} \dot{V} &= \langle \nabla V(x), \dot{x} \rangle \\ &= \langle \nabla V(x), f(x) \rangle + \langle \nabla V(x), ug(x) \rangle \ , \end{aligned}$$

where ∇ denotes the gradient vector of partial derivatives. For the worst case disturbance we have

$$\dot{V} \leq \langle \nabla V(x), f(x) \rangle + k |\langle \nabla V(x), g(x) \rangle| \ . \tag{4.96}$$

The following results, not dissimilar in spirit to some in [35], are evident (see figure 8).

<u>Proposition 4.16</u>

Assume there exists $r > 0$ such that for all $||x|| \geq r$

$$\langle \nabla V(x), f(x) \rangle + k|\langle \nabla V(x), g(x) \rangle| \leq 0$$

and $\bar{B}_r(0) \subset \{x : V(x) \leq \ell\} \subset \Omega$ for some $\ell > 0$, Ω closed and bounded. (Here $\bar{B}_r(0)$ denotes the closed ball of radius r about the origin).

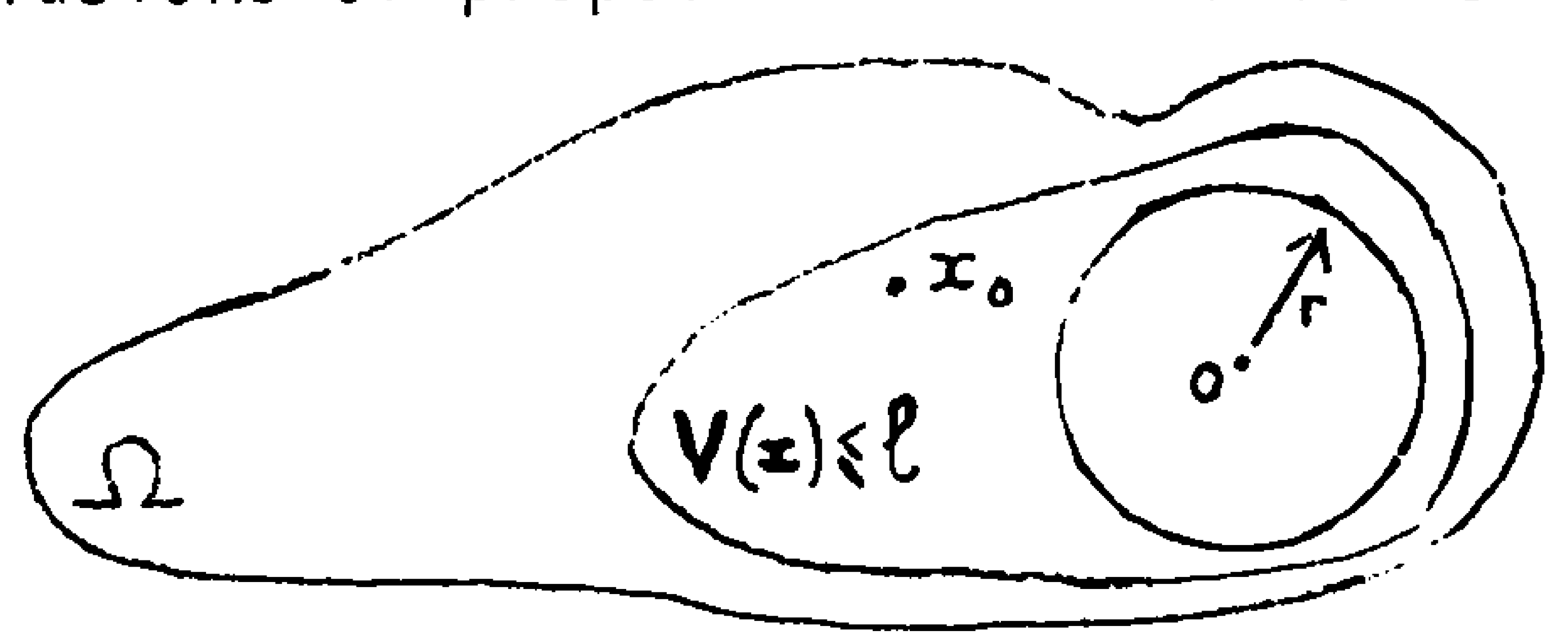
Then if $V(x_0) \leq \ell$ the solution of (4.94) with $x(t_0) = x_0$ is bounded.

<u>Proposition 4.17</u>

Assume for k sufficiently small that for the set S defined as

$$S = \{x : \langle \nabla V(x), f(x) \rangle + k|\langle \nabla V(x), g(x) \rangle| \geq 0\}$$

we have $S \subset \{x : V(x) \leq \ell\} \subset \Omega$ for some $\ell > 0$, Ω closed and bounded, then the conclusions of proposition 4.16 also hold in this case.



<u>Figure 8</u>

Specializing to the linear case we may consider

$$\dot{x} = Ax + Bu \,, \tag{4.97}$$

where $||u(t)|| \leq 1$, say, for all $t \geq 0$. We set

$$V(x) = \langle x, Px \rangle \tag{4.98}$$

where P satisfies

$$PA + A^T P + Q = 0 \tag{4.99}$$

and Q is such that V(x) in (4.98) is of the form (4.95) but otherwise free.

We have then along solutions of (4.97)

$$\begin{aligned}\dot{V} &= \langle \dot{x},Px\rangle + \langle x,P\dot{x}\rangle \\ &= \langle Ax+Bu,Px\rangle + \langle x,P(Ax+Bu)\rangle \\ &= \langle x,(PA+A^TP)x\rangle + 2\langle u,B^TPx\rangle \\ &= -\langle x,Qx\rangle + 2\langle u,B^TPx\rangle\end{aligned}$$

using equation (4.99). Now

$$\begin{aligned}\langle u,B^TPx\rangle \le |\langle u,B^TPx\rangle| &\le ||u||\ ||B^TPx|| \\ &\le ||B^TPx||\end{aligned}$$

as $||u(t)|| \le 1$. So we have

$$\dot{V} \le -\langle x,Qx\rangle + 2||B^TPx|| \ . \tag{4.100}$$

Taking into account (4.100), to obtain our (smallest) region of boundedness for (4.97), we may formulate the problem of maximizing $\langle x,Px\rangle$ subject to the constraint $\langle x,Qx\rangle = 2\,||B^TPx||$. Setting up the Lagrangian

$$L = \langle x,Px\rangle + \lambda[\langle x,Qx\rangle - 2||B^TPx||] \tag{4.101}$$

and using the fact that $||B^TPx|| = \sqrt{\langle B^TPx,B^TPx\rangle}$ we obtain the first order conditions

$$\langle x, Qx \rangle = 2||B^T Px|| \ , \qquad (4.102)$$

$$Px + \lambda Qx - \lambda PBB^T Px/||B^T Px|| = 0 \ , \qquad (4.103)$$

defining the region of boundedness from which solutions can not escape.

For example, we may set $Q = I + \dfrac{PBB^T P}{||B^T Px||}$ in (4.99) to obtain

$$PA + A^T P + I + \frac{PBB^T P}{||B^T Px||} = 0 \ , \qquad (4.104)$$

and then

$$\hat{P} = P/||B^T Px|| \ , \qquad (4.105)$$

to give

$$\hat{P}A + A^T \hat{P} + \frac{I}{||B^T Px||} + \hat{P}BB^T \hat{P} = 0 \ , \qquad (4.106)$$

which is a Riccati equation of the form (4.90) with a change of sign in the solution. So we require that $||B^T Px||^{-1} < r_{\mathbb{C}}^2(A;B,I)$. We have at the optimum $(x = x_1)$

$$\langle x_1, Px_1 \rangle = -\lambda ||B^T Px_1|| \ ,$$

$$B^T Px_1 = -\lambda B^T x_1 \ ,$$

so that

$$\langle x_1, Px_1 \rangle = \lambda^2 ||B^T x_1|| \ .$$

Since

$$||B^T Px_1|| = ||x_1||^2 = -\lambda || B^T x_1|| \ ,$$

$$\langle x_1, Px_1 \rangle = \frac{||x_1||^4}{||B^T x_1||} \ .$$

We thus have as our region of boundedness x such that

$$\langle x, Px \rangle \le \frac{||x_1||^4}{||B^T x_1||} \ .$$

5. CONCLUSIONS

The foregoing work has largely been motivated by the vast quantity of research that has been undertaken on characterizing robustness using frequency domain techniques. With respect to this approach when the H^∞-norm is used as a natural measure of the robustness of a system in withstanding disturbances (being the induced norm between appropriate H^2-spaces of functions) a well developed theory exists for the optimization of this norm involving the determination of a sequence of dynamic feedback operators that is optimal in the limit. The outline we have given is perhaps the most famous of the current approaches. We remark that the basic problem considered has many possible extensions and the solution of these is and will be the subject of much study.

The analysis that we have undertaken is in some sense intended to complement the above approach because for many physical systems it is natural to have state space models as descriptions. The introduction of real and complex stability radii (for a class of structured perturbations) to linear state space systems in [6], [7] is thus a large step in the direction of measuring how robust is the stability of a linear system. Of course the class of perturbations considered here is by no means general but it should be possible to make extensions of the theory to cover more general types of perturbation to the nominally stable system. From our point of view the connection between the complex structured stability radius and a nonstandard Riccati equation is most interesting. By modifying this to give an even more nonstandard (general) Riccati equation we have shown how a control action consisting of a static state feedback (acting through a certain channel) may be used to enhance the complex structured stability radius. Our analysis is thus different from the H^∞-approach where the methods naturally establish dynamic feedbacks which are optimal above all others. The main points to note regarding this feedback are firstly that it is of a high-gain nature in the sense that certain components become large as a weighting parameter goes to zero. Secondly we note that the methods are largely based on linear-quadratic theory, utilizing the solution to a Riccati equation which is, admittedly, of a nonstandard nature.

Perhaps the main drawback of our approach is the solvability of the Riccati equation for a suitable solution. In Chapter 3 we showed how, in a simple case, we could proceed analytically - but it seems likely that computer methods should be resorted to in solving the equation in general. With the advent of more powerful computing systems in one direction and the increased use of personal computers in another, this drawback does not seem so great. Furthermore, the wide availability of specialized subroutines reduce difficulties even further - in this respect we mention the existence of SLICE (Subroutine Library In Control Engineering) due to the Control Systems Research Group, Kingston Polytechnic, London which contains just such a routine RILAC for solving these equations.

In our work we have shown how an improved complex stability radius may be effected. This raises the interesting question of how the real stability radius may be improved or even optimized, though we note in this respect that since

$$r_{\mathbb{R}}(A;B,C) \geq r_{\mathbb{C}}(A;B,C) \quad ,$$

then the feedback improving the complex stability radius will in no way lead to a deterioration of the real stability radius. It does, however, seem likely that a different tact may be needed when trying to improve the real stability radius directly.

In the light of the numerical examples considered, it seems reasonable to conjecture that the methods of feedback design for state space systems given in Chapter 3 are of an optimal nature with regard to robustness (complex stability radius) improvement. We showed in Chapter 1 how to calculate the optimal sensitivity achievable by dynamic feedbacks realizing internal stability. The particular numerical example considered there and in Chapter 3 leads us to conjecture that for a state space system the value of the optimal complex structured stability radius may be calculated via H^{∞}-means.

REFERENCES

1. J.C. Doyle and G. Stein, "Multivariable feedback design: Concepts for a classical/modern synthesis," IEEE Trans. Auto. Contr., vol. AC-26, no. 1, Feb. 1981.

2. B.A. Francis, J.W. Helton, and G. Zames, "H^∞ optimal feedback controllers for linear multivariable systems," IEEE Trans. Auto. Contr., vol. AC-29, no. 10, Oct. 1984.

3. B. Sz.-Nagy and C. Foias, "Harmonic Analysis of Operators on Hilbert Space," New York : Elsevier, 1970.

4. G. Zames, "Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses," IEEE Trans. Auto. Contr., vol. AC-26, no. 2, April 1981.

5. B.-C. Chang and J.B. Pearson, "Optimal disturbance reduction in linear multivariable systems," IEEE Trans. Auto. Contr., vol. AC-29, no. 10, Oct. 1984.

6. D. Hinrichsen and A.J. Pritchard, "Stability radii of linear systems," Systems and Control Letters, Vol. 7, 1986.

7. D. Hinrichsen and A.J. Pritchard, "Stability radius for structured perturbations and the algebraic Riccati equation ," Control Theory Centre Report, 132, University of Warwick, 1986.

8. E.F. Mageirou, "Values and Strategies for Infinite-Time Linear-Quadratic Games," IEEE Trans. Auto. Contr., vol. AC-21, August 1976.

9. D.H. Jacobson, "On Values and Strategies for Infinite-Time Linear-Quadratic Games," IEEE Trans. Auto. Contr., vol. AC-22, June 1977.

10. J.C. Willems, "Least Squares Stationary Optimal Control and the Algebraic Riccati Equation," IEEE Trans. Auto. Contr., vol. AC-16, no. 6, Dec. 1971.

11. R.W. Brockett, "Finite Dimensional Linear Systems," New York: Wiley, 1970.

12. R. Sivan and A. Dickman, "On the Robustness of Multivariable Linear Feedback Systems," IEEE Trans. Auto. Contr., vol. AC-30, April 1985.

13. J.M. Ortega and W.C. Rheinboldt, "Iterative Solution of Nonlinear Equations in Several Variables," New York: Academic Press, 1970.

14. G.P. Papavassilopoulos, J.V. Medanic, and J.B. Cruz, "On the Existence of Nash Strategies and Solutions to Coupled Riccati Equations in Linear-Quadratic Games," Journal of Optimization Theory and Applications, vol. 28, no. 1, May 1979.

15. G. Leitmann, "Stabilization of Dynamical Systems under Bounded Input Disturbance and Parameter Uncertainty," in Differential Games and Control Theory II, Proceedings of the Second Kingston Conference, Rhode Island, 1976, New York: Marcel Dekker, 1976.

16. Yung Kuan Foo and I. Postlethwaite, "An H^{∞}-Minimax Approach to the Design of Robust Control Systems (Revised)," Oxford University Engineering Laboratory Report No. 1537/84, University of Oxford, July 1984.

17. J.C. Willems, "On the Existence of a Nonpositive Solution to the Riccati Equation," IEEE Trans. Auto. Contr., vol. AC-19, Oct. 1974.

18. M.A. Shayman, "Geometry of the Algebraic Riccati Equation, Part I," SIAM J. Control and Optimization, Vol. 21, No. 3, May 1983.

19. S. Barnett, "Matrices in Control Theory, with Applications to Linear Programming," Van Nostrand Reinhold, 1971.

20. R. Eising, "The distance between a system and the set of uncontrollable systems," in: P. Fuhrmann, Ed., Mathematical Theory of Networks and Systems, Proc. MTNS-83, Beer Sheeva, Lecture Notes in Control and Information Sciences 58 (Springer, Berlin-New York, 1984).

21. C.R. Rao and S.K. Mitra, "Generalized Inverse of Matrices and its Applications," Wiley, 1971.

22. D.H. Owens and A. Chotai, "Robust controller design for linear dynamic systems using approximate models," IEE Proc., Vol. 130, Pt. D, No. 2, March 1983.

23. N.A. Lehtomaki, N.R. Sandell, and M. Athans, "Robustness Results in Linear-Quadratic Gaussian Based Multivariable Control Designs," IEEE Trans. Auto. Contr., vol. AC-26, no. 1, Feb. 1981.

24. B.A. Francis, "Notes on H^{∞}-Optimal Linear Feedback Systems," Lectures presented in the Division of Automatic Control, Department of Electrical Engineering, Linköping University, August 22-26, 1983.

25. P.L. Duren, "Theory of H^p Spaces," New York: Academic, 1970.

26. R.V. Patel and M. Toda, "Quantitative Measures of Robustness for Multivariable Systems," Proc. of Joint Automatic Control Conf., San Francisco, TP8-A, 1980.

27. R.K. Yedavalli, "Perturbation bounds for robust stability in linear state space models," Int. J. Control, Vol. 42, No. 6, 1985.

28. M.E. Sezer and D.D. Šiljak, "Robustness of Suboptimal Control : Gain and Phase Margin," IEEE Trans. Auto. Contr., vol. AC-26, no. 4, Aug. 1981.

29. R. Eising, "Between controllable and uncontrollable," Systems and Control Letters, Vol. 4, 1984.

30. J.C. Willems, "Almost Invariant Subspaces: An Approach to High Gain Feedback Design - Part 1 : Almost Controlled Invariant Subspaces," IEEE Trans. Auto. Contr., vol. AC-26, no.1, Feb. 1981.

31. H.L. Trentelman, "Almost invariant subspaces and high gain feedback," Doctoral dissertation, Mathematics Department, Groningen University, The Netherlands.

32. E. Jonckheere, "On the Existence of a Negative Semidefinite, Antistabilizing Solution to the Discrete-Time Algebraic Riccati Equation," IEEE Trans. Auto. Contr., vol. AC-26, no. 3, June 1981.

33. E. Noldus, "Design of robust state feedback laws," Int. J. Control, Vol. 35, No. 6, 1982.

34. B.C. Moore, "On the Flexibility Offered by State Feedback in Multivariable Systems Beyond Closed Loop Eigenvalue Assignment," IEEE Trans. Auto. Contr., vol. AC-21, Oct. 1976.

35. J. La Salle and S. Lefschetz, "Stability by Lyapunov's Direct Method, With Applications," New York : Academic, 1961.

36. S. Barnett and C. Storey, "Matrix methods in stability theory," Nelson, 1970.

37. B.R. Barmish, M. Corless, and G. Leitmann, "A new class of stabilizing controllers for uncertain dynamical systems," SIAM J. Control and Optimization, Vol. 21, No. 2, March 1983.

38. P. Molander and J.C. Willems, "Synthesis of State Feedback Control Laws with a Specified Gain and Phase Margin," IEEE Trans. Auto. Contr., vol. AC-26, no. 5, Oct. 1980.