# Direct Adaptive Control for Nonlinear Uncertain Dynamical Systems

A Dissertation
Presented to
The Academic Faculty

by

**Tomohisa Hayakawa**

In Partial Fulfillment
of the Requirements for the
Degree of Doctor of Philosophy in Aerospace Engineering

Georgia Institute of Technology
November 2003

# Direct Adaptive Control for Nonlinear Uncertain Dynamical Systems

Approved by:


Dr. Wassim M. Haddad, Chairman
Aerospace Engineering

Dr. J. V. R. Prasad
Aerospace Engineering

Dr. Eric N. Johnson
Aerospace Engineering

Dr. James M. Bailey
Northeast Georgia Medical Center

Dr. David G. Taylor
Electrical and Computer Engineering



Date Approved: November 21, 2003

To my parents
*yume*

# Acknowledgements

There are a number of people I would like to acknowledge in connection with the successful completion of this work. I would like to take this opportunity to express my appreciation to all of them who have influenced, stimulated, expedited, and warmly supported my work in various ways. Specifically, I would like to extend special thanks to the following individuals:

First and foremost, it is an immense pleasure to express my deep and sincere gratitude to my advisor, Dr. Wassim M. Haddad, for his guidance, assistance, encouragement, and hearty support in all phases of my doctoral program at Georgia Tech. He has not only been my academic advisor, but also been a great mentor for my graduate life while I am far away from home in Atlanta. That is not to say that working on state-of-the-art research topics under his guidance is one of the most invaluable experience for my career and the most joyful time I have ever had, but I also have to admit that I have learned even more about life from him. Furthermore, I thank his wife, Mrs. Lydia Haddad, for her emotional support. I will not forget her kindness, tenderness, and the taste of her honey walnut cake.

I am thankful to Dr. J. V. R. Prasad, Dr. Eric N. Johonson, and Dr. David G. Taylor for taking time to serve on my dissertation reading committee and for their useful comments and suggestions to improve this dissertation. I also thank Dr. Anthony J. Calise for being on my proposal committee.

I would also like to thank Dr. James M. Bailey, who has taken the time from his busy schedule to be on my dissertation committee and for providing invaluable insights to the problem of automated anesthesia. The interactions with Dr. Bailey have helped me understand the fundamental principles of clinical pharmacology.

I am greatly indebted to Dr. VijaySekhar Chellaboina, Dr. Alexander Leonessa, and Dr. Naira Hovakimyan for their assistance and shrewd suggestions in our collaborations. To Naira I am additionally thankful for her encouragement. I also acknowledge Dr. Eugene Lavretsky for several stimulating discussions on approximation theory.

I thank all the lab members who shared time, space, and lore with me at Georgia Tech. They made my graduate days wonderful and enjoyable. In particular, Sergei Nersesov also shared rooms with me at conferences and other trips. These trips have given me a lot of memories.

I am unfeignedly grateful to Yasuo Kitane for his friendship from my days at the University at Buffalo. Whenever I talked to him, he soothed my concerns.

Lastly, but by no means least, my heartfelt appreciation and gratitude goes to my family, especially my parents, for their "time-invariant" support, encouragement, and patience.

This work was supported in part by the National Science Foundation under a Presidential Faculty Fellow Award (W. M. Haddad), the Air Force Office of Scientific Research, the Army Research Office, and Georgia Institute of Technology.

# Table of Contents

**Acknowledgements** **iv**

**Summary** **xviii**

**1 Introduction** **1**

1.1. Closed-Loop Adaptive Control . . . . . . . . . . . . . . . . . . . . . . . . 1

1.2. Closed-Loop Control in Clinical Pharmacology . . . . . . . . . . . . . 3

1.3. A Primer on Clinical Pharmacology . . . . . . . . . . . . . . . . . . . . 4

1.4. Clinical Pharmacology and Drug Dosing . . . . . . . . . . . . . . . . . 13

1.5. Closed-Loop Control of Cardiovascular Function . . . . . . . . . . . . 17

1.6. Closed-Loop Control of Anesthesia . . . . . . . . . . . . . . . . . . . 20

1.7. Brief Outline of the Dissertation . . . . . . . . . . . . . . . . . . . . . 27

**2 Direct Adaptive Control for Nonlinear Uncertain Systems with Exogenous Disturbances** **32**

2.1. Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 32

2.2. Adaptive Control for Nonlinear Systems with Exogenous Disturbances 34

2.3. Specialization to Single-Input Systems with Uncertain Dynamics . . . 43

2.4. Adaptive Control for Nonlinear Systems with $L_2$ Disturbances . . . . 47

2.5. Illustrative Numerical Examples . . . . . . . . . . . . . . . . . . . . . 52

2.6. Adaptive Control for Thermoacoustic Combustion Instabilities . . . . 73

2.7. Conclusion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 82

**3 Robust Adaptive Control for Nonlinear Uncertain Systems** **86**

3.1. Introduction . . . . . . . . . . 86

3.2. Robust Adaptive Control for Nonlinear Uncertain Systems . . . . . . 87

3.3. Adaptive Absolute Stabilization for Nonlinear Uncertain Systems . . 95

3.4. Illustrative Numerical Examples . . . . . . . . . . 100

3.5. Conclusion . . . . . . . . . . 104

**4 Adaptive Control for Nonlinear Uncertain Systems with Actuator Amplitude and Rate Saturation Constraints 107**

4.1. Introduction . . . . . . . . . . 107

4.2. Adaptive Tracking for Nonlinear Uncertain Systems . . . . . . . . . . 109

4.3. Adaptive Tracking with Actuator Amplitude and Rate Saturation Constraints . . . . . . . . . . 116

4.4. Illustrative Numerical Examples . . . . . . . . . . 120

4.5. Conclusion . . . . . . . . . . 126

**5 Adaptive Reduced-Order Dynamic Compensation for Nonlinear Uncertain Systems 129**

5.1. Introduction . . . . . . . . . . 129

5.2. Adaptive Dynamic Control for Nonlinear Uncertain Systems . . . . . 130

5.3. Illustrative Numerical Examples . . . . . . . . . . 138

5.4. Conclusion . . . . . . . . . . 142

**6 Direct Adaptive Control for Nonlinear Matrix Second-Order Dynamical Systems with State-Dependent Uncertainty 143**

6.1. Introduction . . . . . . . . . . 143

6.2. Adaptive Control of Nonlinear Matrix Second-Order Dynamical Systems 145

6.3. Polynomial Uncertainty with Unknown Coefficients and Unknown Order 153

6.4. Illustrative Numerical Examples . . . . . . . . . . 159

6.5. Nonlinear Matrix Second-Order Systems with Time-Varying and Sign-Indefinite Damping and Stiffness Operators . . . . . . . . . . 172

6.6. Illustrative Numerical Example . . . . . . . . . . 180

6.7. Applications to Combustion Processes . . . . . . . . . . 181

6.8. Conclusion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 186

**7 Adaptive Control for Nonnegative and Compartmental Dynamical Systems with Applications to General Anesthesia 191**

7.1. Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 191

7.2. Mathematical Preliminaries . . . . . . . . . . . . . . . . . . . . . . . 196

7.3. Adaptive Control for Linear Nonnegative Uncertain Dynamical Systems 203

7.4. Adaptive Control for Linear Nonnegative Dynamical Systems with Nonnegative Control . . . . . . . . . . . . . . . . . . . . . . . . . . 212

7.5. Adaptive Control for General Anesthesia . . . . . . . . . . . . . . . 216

7.6. Conclusion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 223

**8 Direct Adaptive Control of Nonnegative and Compartmental Dynamical Systems with Time Delay 227**

8.1. Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 227

8.2. Mathematical Preliminaries . . . . . . . . . . . . . . . . . . . . . . . 228

8.3. Adaptive Control for Linear Nonnegative Uncertain Dynamical Systems with Time Delay . . . . . . . . . . . . . . . . . . . . . . . . 235

8.4. Adaptive Control for Linear Nonnegative Dynamical Systems with Nonnegative Control and Time Delay . . . . . . . . . . . . . . . . . 243

8.5. Adaptive Control for General Anesthesia . . . . . . . . . . . . . . . 247

8.6. Conclusion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 251

**9 Adaptive Control for Nonlinear Nonnegative and Compartmental Dynamical Systems with Applications to Clinical Pharmacology 254**

9.1. Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 254

9.2. Mathematical Preliminaries . . . . . . . . . . . . . . . . . . . . . . . 256

9.3. Adaptive Control for Nonlinear Nonnegative Uncertain Dynamical Systems . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 259

9.4. Adaptive Control for Nonlinear Nonnegative Dynamical Systems with Nonnegative Control . . . . . . . . . . . . . . . . . . . . . . . . . . 272

9.5. Illustrative Numerical Example . . . . . . . . . . . . . . . . . . . . . 277

9.6. Nonlinear Adaptive Control for General Anesthesia . . . . . . . . . . 278

9.7. Conclusion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 285

**10 Neural Network Adaptive Control for Nonlinear Nonnegative Dynamical Systems 288**

10.1. Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 288

10.2. Mathematical Preliminaries . . . . . . . . . . . . . . . . . . . . . . 291

10.3. Neural Adaptive Control for Nonlinear Nonnegative Uncertain Systems 298

10.4. Neural Adaptive Control for Nonlinear Nonnegative Uncertain Systems with Nonnegative Control . . . . . . . . . . . . . . . . . . . . . . 312

10.5. Neural Adaptive Control for Continuous Stirred Tank Reactors . . . . 321

10.6. Conclusion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 324

**11 Passivity-Based Neural Network Adaptive Output Feedback Control for Nonlinear Nonnegative Dynamical Systems 327**

11.1. Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 327

11.2. Mathematical Preliminaries . . . . . . . . . . . . . . . . . . . . . . 331

11.3. Neural Output Feedback Adaptive Control for Nonlinear Nonnegative Uncertain Systems . . . . . . . . . . . . . . . . . . . . . . . . . . 336

11.4. Neural Adaptive Control for General Anesthesia . . . . . . . . . . . . 347

11.5. Conclusion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 353

**12 Neural Network Adaptive Dynamic Output Feedback Control for Nonlinear Nonnegative Systems using Tapped Delay Memory Units 356**

12.1. Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 356

12.2. Neural Adaptive Output Feedback Control for Nonlinear Nonnegative Uncertain Systems . . . . . . . . . . . . . . . . . . . . . . . . . . 358

12.3. Nonlinear Adaptive Output Feedback Control for General Anesthesia 372

12.4. Conclusion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 377

**13 A Lyapunov-Based Adaptive Control Framework for Discrete-Time Nonlinear Systems with Exogenous Disturbances 381**

13.1. Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 381

13.2. Discrete-Time Adaptive Control for Nonlinear Systems with Exogenous Disturbances . . . . . . . . . . . . . . . . . . . . . . . . . . 384

13.3. Adaptive Control for Nonlinear Systems with $\ell_2$ Disturbances . . . . 394
13.4. Illustrative Numerical Examples . . . . . . . . . . . . . . . . . . . . . . . 400
13.5. Conclusion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 405

**14 Direct Discrete-Time Adaptive Control with Guaranteed Parameter Error Convergence 408**

14.1. Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . 408
14.2. Adaptive Tracking for Nonlinear Uncertain Systems . . . . . . . . . . 409
14.3. Illustrative Numerical Examples . . . . . . . . . . . . . . . . . . . . . . . 416
14.4. Conclusion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 421

**15 Hybrid Adaptive Control for Nonlinear Uncertain Impulsive Dynamical Systems 423**

15.1. Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . 423
15.2. Mathematical Preliminaries . . . . . . . . . . . . . . . . . . . . . . . 425
15.3. Hybrid Adaptive Stabilization for Nonlinear Hybrid Dynamical Systems 428
15.4. Hybrid Adaptive Attraction Control for Nonlinear Hybrid Dynamical Systems . . . . . . . . . . . . . . . . . . . . . . . . . . . . 439
15.5. Illustrative Numerical Examples . . . . . . . . . . . . . . . . . . . . . . . 448
15.6. Conclusion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 453

**16 Concluding Remarks and Recommendations for Future Research 457**

16.1. Conclusions . . . . . . . . . . . . . . . . . . . . . . . . . . . . 457
16.2. Recommendations for Future Research . . . . . . . . . . . . . . . . . . 460

**References 467**

# List of Figures


1.1 $n$-compartment mammillary model . . . . . . . . . . . . . . . . . . . 7
1.2 Bispectral index (BIS) monitor . . . . . . . . . . . . . . . . . . . . . 24
1.3 Adaptive closed-loop control for drug administration . . . . . . . . . 26

2.1 Phase portrait of controlled and uncontrolled Liénard system . . . . . 53
2.2 State trajectories and control signal versus time . . . . . . . . . . . . 54
2.3 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 54
2.4 Phase portrait of controlled and uncontrolled Van der Pol oscillator . 55
2.5 State trajectories and control signal versus time . . . . . . . . . . . . 56
2.6 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 56
2.7 Phase portrait of controlled and uncontrolled Rayleigh system . . . . 58
2.8 State trajectories and control signal versus time . . . . . . . . . . . . 59
2.9 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 59
2.10 Phase portrait of controlled and uncontrolled Duffing system . . . . . 60
2.11 State trajectories and control signal versus time . . . . . . . . . . . . 61
2.12 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 61
2.13 Phase portrait of controlled and uncontrolled Mathieu system . . . . 62
2.14 State trajectories and control signal versus time . . . . . . . . . . . . 63
2.15 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 63
2.16 Position and control signal versus time . . . . . . . . . . . . . . . . . 65
2.17 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 65
2.18 State trajectories versus time . . . . . . . . . . . . . . . . . . . . . . 68
2.19 Error states and control signal versus time . . . . . . . . . . . . . . . 69

2.20 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 69
2.21 Positions and control signals versus time . . . . . . . . . . . . . . . . . 71
2.22 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 72
2.23 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 73
2.24 Angular velocities versus time . . . . . . . . . . . . . . . . . . . . . . 74
2.25 Control signals versus time . . . . . . . . . . . . . . . . . . . . . . . 74
2.26 Open-loop state response versus time . . . . . . . . . . . . . . . . . . 80
2.27 Closed-loop state response versus time . . . . . . . . . . . . . . . . . 82
2.28 Closed-loop state response versus time . . . . . . . . . . . . . . . . . 83
2.29 Closed-loop state response versus time . . . . . . . . . . . . . . . . . 84
2.30 Closed-loop state response versus time . . . . . . . . . . . . . . . . . 84
2.31 Closed-loop state response versus time . . . . . . . . . . . . . . . . . 85

3.1 Adaptive absolute stabilization problem . . . . . . . . . . . . . . . . . 96
3.2 Phase portrait of controlled and uncontrolled system . . . . . . . . . 101
3.3 State trajectories and control signal versus time . . . . . . . . . . . 102
3.4 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . 102
3.5 Phase portraits of controlled and uncontrolled System . . . . . . . . . 105
3.6 State trajectories and control signals versus time . . . . . . . . . . . 106

4.1 Stabilization of the Liénard system with no saturation constraints . . 121
4.2 Stabilization of the Liénard system with amplitude and rate saturation constraints . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 122
4.3 Tracking of the Liénard system with no saturation constraints . . . . 123
4.4 Tracking of the Liénard system with amplitude and rate saturation constraints . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 124
4.5 Tracking of the Liénard system with amplitude and rate saturation constraints . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 125
4.6 Angular velocities versus time . . . . . . . . . . . . . . . . . . . . . 126
4.7 Control signals and rate versus time . . . . . . . . . . . . . . . . . . 128

5.1 Phase portrait of controlled and uncontrolled Van der Pol oscillator . 139

5.2 State trajectories versus time and the control signal versus time . . . 140
5.3 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 141
5.4 Angular velocities, compensator state, and control signal versus time 142

6.1 Representative nonlinearity $n(\cdot) \in \mathcal{N}$ . . . . . . . . . . . . . . . . . . 151
6.2 State trajectories and control signals versus time . . . . . . . . . . . . 160
6.3 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 160
6.4 Phase portraits of controlled and uncontrolled system . . . . . . . . . 162
6.5 State trajectories and control signals versus time . . . . . . . . . . . . 163
6.6 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 163
6.7 Phase portraits of controlled and uncontrolled system . . . . . . . . . 164
6.8 State trajectories and control signals versus time . . . . . . . . . . . . 165
6.9 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 165
6.10 State trajectories, adaptive gains and control signal versus time . . . 167
6.11 State trajectories, adaptive gains and control signal versus time . . . 167
6.12 State trajectories, adaptive gains and control signal versus time . . . 168
6.13 State trajectories, adaptive gains and control signal versus time . . . 168
6.14 State trajectories and control signals versus time . . . . . . . . . . . . 170
6.15 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 170
6.16 State trajectories and control signals versus time . . . . . . . . . . . . 171
6.17 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 171
6.18 Phase portraits of controlled and uncontrolled systems . . . . . . . . 182
6.19 State trajectories and control signals versus time . . . . . . . . . . . . 183
6.20 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 183
6.21 Closed-loop state response versus time . . . . . . . . . . . . . . . . . 188
6.22 Closed-loop state response versus time . . . . . . . . . . . . . . . . . 188
6.23 Closed-loop state response versus time . . . . . . . . . . . . . . . . . 189
6.24 Closed-loop state response versus time . . . . . . . . . . . . . . . . . 189
6.25 Closed-loop state response versus time . . . . . . . . . . . . . . . . . 190

7.1 Three-compartment mammillary model for disposition of propofol . . 217
7.2 Compartmental masses versus time . . . . . . . . . . . . . . . . . . . 219
7.3 Drug concentration in the central compartment and control signal (infusion rate) versus time . . . . . . . . . . . . . . . . . . . . . . . . . . 220
7.4 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 221
7.5 BIS index versus effect site concentration . . . . . . . . . . . . . . . . 222
7.6 Compartmental masses versus time . . . . . . . . . . . . . . . . . . . 224
7.7 BIS index versus time . . . . . . . . . . . . . . . . . . . . . . . . . . 224
7.8 Drug concentration in the central compartment and control signal (infusion rate) versus time . . . . . . . . . . . . . . . . . . . . . . . . . . 225
7.9 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 226

8.1 Three-compartment mammillary model for disposition of propofol . . 248
8.2 Compartmental masses versus time . . . . . . . . . . . . . . . . . . . 251
8.3 BIS Index versus time . . . . . . . . . . . . . . . . . . . . . . . . . . 252
8.4 Drug concentration in the central compartment and control signal (infusion rate) versus time . . . . . . . . . . . . . . . . . . . . . . . . . . 252
8.5 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 253

9.1 State trajectories versus time . . . . . . . . . . . . . . . . . . . . . . 279
9.2 Control signal versus time and adaptive gain history versus time . . . 279
9.3 Pharmacokinetic model for drug distribution during anesthesia . . . . 280
9.4 BIS index versus effect site concentration . . . . . . . . . . . . . . . . 283
9.5 Compartmental masses versus time . . . . . . . . . . . . . . . . . . . 286
9.6 BIS index versus time and control signal (infusion rate) versus time . 286
9.7 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 287

10.1 Visualization of partial boundedness and partial ultimate boundedness 294
10.2 Visualization of sets used in the proof of Theorem 10.4 . . . . . . . . 307
10.3 Block diagram of the closed-loop system . . . . . . . . . . . . . . . . 310
10.4 Exothermic continuously stirred tank reactor . . . . . . . . . . . . . . 322

10.5 State trajectories (reactor temperature and concentration of reactant A) and control signal (jacket temperature) versus time . . . . . . . . 325
10.6 Neural network weighting functions versus time . . . . . . . . . . . . 326

11.1 Visualization of sets used in the proof of Theorem 11.3 . . . . . . . . 345
11.2 Block diagram of the closed-loop system . . . . . . . . . . . . . . . . . 346
11.3 Pharmacokinetic model for drug distribution during anesthesia . . . . 348
11.4 BIS index versus effect site concentration . . . . . . . . . . . . . . . . 350
11.5 Compartmental concentrations versus time . . . . . . . . . . . . . . . 353
11.6 BIS index versus time and control signal (infusion rate) versus time . 354
11.7 neural network weighting functions versus time . . . . . . . . . . . . . 354

12.1 Visualization of sets used in the proof of Theorem 12.2 . . . . . . . . 369
12.2 Combined pharmacokinetic/pharmacodynamic model . . . . . . . . . 375
12.3 BIS index versus effect site concentration . . . . . . . . . . . . . . . . 376
12.4 Compartmental masses versus time . . . . . . . . . . . . . . . . . . . 378
12.5 Concentrations in the central and effect site compartments versus time 378
12.6 Compensator states versus time . . . . . . . . . . . . . . . . . . . . . 379
12.7 BIS index versus time and control signal (infusion rate) versus time . 379

13.1 Phase portrait of controlled and uncontrolled system . . . . . . . . . 402
13.2 State trajectories and control signal versus time . . . . . . . . . . . . 403
13.3 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 403
13.4 Positions and control signals versus time . . . . . . . . . . . . . . . . 404
13.5 State trajectory and control signal versus time . . . . . . . . . . . . . 406
13.6 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 406

14.1 State and reference trajectories and control signal versus time . . . . 418
14.2 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 418
14.3 State trajectory and control signal versus time . . . . . . . . . . . . . 420
14.4 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . 420

14.5 State trajectory and control signal versus time with random reference input . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 421
14.6 Adaptive gain history versus time with random reference input . . . . 422

15.1 Phase portraits of uncontrolled and controlled hybrid system . . . . . 451
15.2 State trajectories versus time . . . . . . . . . . . . . . . . . . . . . . . . 452
15.3 Control signals versus time . . . . . . . . . . . . . . . . . . . . . . . . . 452
15.4 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . . 453
15.5 Phase portraits of uncontrolled and controlled hybrid system . . . . . 454
15.6 State trajectories versus time . . . . . . . . . . . . . . . . . . . . . . . . 455
15.7 Control signals versus time . . . . . . . . . . . . . . . . . . . . . . . . . 455
15.8 Adaptive gain history versus time . . . . . . . . . . . . . . . . . . . . . 456

# List of Tables


7.1 Pharmacokinetic parameters . . . . . . . . . . . . . . . . . . . . . . . . 218

10.1 System parameter values . . . . . . . . . . . . . . . . . . . . . . . . . 324

# Summary

In light of the complex and highly uncertain nature of dynamical systems requiring controls, it is not surprising that reliable system models for many high performance engineering applications are unavailable. In the face of such high levels of system uncertainty, robust controllers may unnecessarily sacrifice system performance whereas adaptive controllers are clearly appropriate since they can tolerate far greater system uncertainty levels to improve system performance. In contrast to fixed-gain robust controllers, which maintain specified constants within the feedback control law to *sustain* robust performance, adaptive controllers directly or indirectly adjust feedback gains to maintain closed-loop stability and *improve* performance in the face of system uncertainties. Specifically, indirect adaptive controllers utilize parameter update laws to identify unknown system parameters and adjust feedback gains to account for system variation, while direct adaptive controllers directly adjust the controller gains in response to plant variations.

To develop a highly robust, adaptive control framework we first develop a general adaptive control framework to address linearly parameterized uncertainties for adaptive stabilization, disturbance rejection, and command following of nonlinear uncertain dynamical systems with exogenous disturbances. In particular, the adaptive control framework is Lyapunov-based and guarantees partial asymptotic stability of the closed-loop system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the system state variables. Building on

this result, we then characterize robust adaptive controllers to account for nonlinear state-dependent uncertainty that is not captured by a finite linear combination of basis functions.

An implicit assumption inherent in most adaptive control frameworks is that the adaptive control law is implemented without any regard to actuator amplitude and rate saturation constraints. As a consequence, actuator nonlinearities arise frequently in practice and can severely degrade closed-loop system performance, and in some cases drive the system to instability. In light of this, we develop an adaptive control framework that guarantees asymptotic stability of the closed-loop tracking error dynamics in the face of amplitude and rate saturation constraints. Specifically, the adaptive control signal to a given reference (governor or supervisor) system is modified to effectively robustify the error dynamics to the saturation constraints.

A novel parametrization-free adaptive control framework is also developed for a class of nonlinear uncertain systems. Specifically, we consider matrix second-order systems that possess sign-varying damping and stiffness operators. All that is required to implement the adaptive controller is that the damping and stiffness operators are continuous and (lower) bounded; otherwise they are unknown. The approach is applied to combustion processes to suppress the effects of thermoacoustic instabilities.

Nonnegative and compartmental dynamical system models are derived from mass and energy balance considerations that involve dynamic states whose values are nonnegative. These models are widespread in engineering and life sciences and typically involve the exchange of nonnegative quantities between subsystems or compartments wherein each compartment is assumed to be kinetically homogeneous. For this class of dynamical systems, we develop adaptive and neural adaptive control frameworks for adaptive set-point regulation of nonlinear uncertain nonnegative and compartmental systems. Based on this result, we apply the adaptive control framework to regulate (maintain) a desired constant level of consciousness for noncardiac surgery.

Even though adaptive control algorithms have been developed in the literature for both continuous-time and discrete-time systems, the majority of the discrete-time results are based on recursive least-squares and least mean squares algorithms with primary focus on state convergence. Alternatively, Lyapunov-based adaptive controllers have been developed for continuous-time systems guaranteeing asymptotic stability of the system states. However, the literature on discrete-time adaptive disturbance rejection control using Lyapunov methods is virtually nonexistent. In light of this, we develop a direct adaptive control framework for adaptive stabilization, disturbance rejection, and command following of multivariable discrete-time nonlinear uncertain systems with exogenous bounded amplitude disturbances and bounded energy (square-summable) $\ell_2$ disturbances. These results are analogous to the continuous-time adaptive disturbance rejection results discussed above for continuous-time nonlinear uncertain systems.

The complexity of modern controlled uncertain nonlinear dynamical systems is often exacerbated by the use of hierarchical abstract decision-making units performing logical checks that identify system mode operation and specify a subcontroller within the feedback control architecture to be activated. These multiechelon systems are classified as *hybrid* systems and involve an *interacting* countable collection of dynamical systems possessing a hierarchical structure characterized by continuous-time dynamics at the lower-level units and logical decision-making units at the higher-level of the hierarchy. In the last part of this dissertation, we develop a hybrid adaptive control framework for hybrid dynamical systems to guarantee asymptotic stability/attraction of the closed-loop system states associated with the hybrid plant dynamics using the hybrid invariance principle.

# Chapter 1

# Introduction

## 1.1. Closed-Loop Adaptive Control

One of the fundamental problems in feedback control design is the ability of the control system to guarantee robust stability and robust performance with respect to system uncertainties in the design model. To this end, adaptive control along with robust control theory have been developed to address the problem of system uncertainty in control-system design. The fundamental differences between adaptive control design and robust control theory can be traced to the modeling and treatment of system uncertainties as well as the controller architecture structures. In particular, adaptive control [12, 121, 176] is based on constant linearly parameterized system uncertainty models of a known structure but unknown variation, while robust control [237, 245] is predicated on structured and/or unstructured linear or nonlinear (possibly time-varying) operator uncertainty models consisting of bounded variation. Hence, for systems with constant real parametric uncertainties with large unknown variations, adaptive control is clearly appropriate, while for systems with time-varying parametric uncertainties and nonparametric uncertainties with norm bounded variations, robust control may be more suitable. Furthermore, in contrast to fixed-gain robust controllers, which maintain specified constants within the feedback control law to *sus-*

*tain* robust performance, adaptive controllers directly or indirectly adjust feedback gains to maintain closed-loop stability and *improve* performance in the face of system uncertainties. Specifically, indirect adaptive controllers utilize parameter update laws to identify unknown system parameters and adjust feedback gains to account for system variation while direct adaptive controllers directly adjust the controller gains in response to plant variation. In either case, the overall process of parameter identification and controller adjustment constitutes a nonlinear control law architecture.

Even though the design of adaptive control for linear plants has evolved tremendously over the past two decades it is only recently with work involving differential geometric methods [120,122,183,232] that has made the design of adaptive controllers for certain classes of nonlinear systems possible using concepts of zero dynamics and feedback linearization [48, 175, 188, 204, 219, 220, 228]. These techniques, however, are limited to low relative degree systems with restrictive matching conditions imposed on the structure of the uncertainty and usually rely on cancelling out system nonlinearities using feedback which may lead to inefficient designs since feedback linearizing controllers may generate unnecessarily large control effort to cancel beneficial system nonlinearities. A major breakthrough in the design of adaptive controllers for a large class of nonlinear cascade systems was introduced with the development of recursive backstepping methods [133,146,147]. The popularity of this adaptive control methodology can be explained in a large part due to the fact that it provides a systematic procedure for finding an adaptive Lyapunov function for the closed-loop system and choosing the adaptive control such that the time derivative of the adaptive Lyapunov function along the trajectories of the closed-loop system is negative. To compensate for estimating the same uncertain system parameters within the recursive backstepping procedure tuning functions were introduced by Krstić *et al.* [145–147] to remove this overparameterization by modifying the recursive update laws. Furthermore, the adaptive controller is obtained in such a way that the nonlinearities of the dynamic system which may be useful in reaching performance objectives need not be cancelled as in state or output feedback linearization.

## 1.2. Closed-Loop Control in Clinical Pharmacology

Control technology is the underpinning for technological advances in fields as diverse as aerospace, chemical, power, manufacturing, electronic, communication, transportation, and network engineering. However, control technology has had less impact on modern medicine. There have been exciting breakthroughs in such areas as robotic surgery, electrophysiological systems (pacemakers and automatic implantable defibrillators), life support (ventilators, artificial hearts), and image-guided therapy and surgery. However, in general, there are steep barriers to the application of modern control theory and technology to medicine. The steepest barriers are the system uncertainties, inherent to biology, that preclude mathematical modeling and application of many of the tools of modern control technology.

One of the exceptions to this generalization is in the area of clinical pharmacology, a discipline in which mathematical modeling has had a prominent role. Some of the most important advances in modern medicine have been in the area of pharmacology. The physician in the 21st century has a broad armamentarium of drugs available for the treatment of disease. This is in contrast to previous generations of physicians, who were largely limited to diagnosis, possible surgery, and often only consolation. But while we have an abundance of therapeutic agents, proper dosing of drugs is often imprecise and may be a significant cause of increased costs and morbidity and mortality. In this dissertation, we develop adaptive control methods for nonlinear uncertain dynamical systems and discuss potential applications of adaptive control to clinical pharmacology, specifically the control of drug dosing.

It is instructive to consider how dose guidelines are derived. Drug development begins with animal experimentation. Promising agents are then taken to human trials, beginning with healthy volunteers and progressing to patients with the disease for which the drug is being developed. Early stages of these trials focus on safety while the final trials usually entail randomized, blinded administration of placebo and different drug doses for the evaluation of efficacy. Efficacy is statistically defined and even when there is a therapeutic effect in the statistical aggregate, there still may be individual patients for whom the drug is either not efficacious or who experience side-effects. If a therapeutic effect is observed, then the drug may be approved by the Federal Drug Administration and, in general, the recommended dose is that found to be efficacious in the "average" patient. And this is the problem. No patient is an "average" patient. There is very substantial variability among patients in the drug concentration at the locus of the effect (the *effect site concentration*) that results from a given dose and there is a very substantial variability among patients in the therapeutic efficacy of any given effect site concentration. Thus, there is large variability among patients in the therapeutic effect of any given dose. In the vast majority of cases, the appropriate dose for a specific patient is found by trial and error. For example, the internist treating a patient with essential hypertension will begin by prescribing the recommended dose and then, in follow-up, will observe the effect of the drug on blood pressure and adjust the dose empirically. This process can be cumbersome, time consuming, and imprecise.

## 1.3. A Primer on Clinical Pharmacology

It has been apparent for some time that dosing of drugs could be put on a more rational basis by using *pharmacokinetic* and *pharmacodynamic* modeling. Pharmacokinetics is the study of the concentration of drugs in various tissues as a function of time and dose schedule. Pharmacodynamics is the study of the relationship between drug concentration and effect. By developing techniques relating dose to resultant drug concentration (pharmacokinetics) and concentration to effect (pharmacodynamics) one can generate a model for drug dosing.

Pharmacokinetic models will be familiar to most control engineers and theorists since they are based on dynamical system theory. The disposition of drugs in the body is a complex interplay of numerous transport and metabolic processes, many of which are still poorly understood [66, 238]. However, *compartmental models* may effectively encapsulate these processes [123]. Common pharmacokinetic models assume that, for the purpose of describing drug disposition, the body is comprised of a few homogenous, well-mixed compartments (so that the drug concentration is constant within the compartment), with linear (proportional to drug concentration) transport to other compartments or elimination from the compartment and the body by metabolic processes. The simplest model, the one-compartment model, assumes that the body is just a single compartment and also typically assumes instantaneous mixing when drug is introduced intravenously, with subsequent linear elimination. The model is characterized by two parameters, the volume of distribution ($V_{\mathrm{d}}$) and the elimination rate constant ($a_{\mathrm{e}}$). With this simple model the concentration ($C$) immediately after a dose of amount of $D$ is equal to $D/V_{\mathrm{d}}$ and drug is subsequently eliminated at a rate equal to $a_{\mathrm{e}}C$ (exponential decay). While the behavior of a few drugs may actually be described by this model, it is too simplistic for most. The assumption of instantaneous mixing, which is clearly unrealistic in the case of drugs that are taken orally, can be remedied by using a two (or more) compartment model in which there is a compartment representing the gastro-intestinal tract that receives the dose and from which drug is transferred irreversibly to a second compartment that represents *intravascular blood* (blood within arteries or veins) and organ systems which receive a large amount of blood flow and hence which equilibrate with intravascular blood rapidly.

For drugs that are administered intravenously, a common model is the two compartment *mammillary* model [123]. This model assumes that there is a central compartment which receives the intravenous dose with instantaneous mixing. Drug is then either transferred to a peripheral compartment or metabolized and eliminated from the body. Drug elimination from the peripheral compartment is ignored since this compartment is identified with tissues such as muscle or fat which are metabolically inert as far as the drug is concerned. (Most drugs are metabolized in the liver or kidney, organs that, along with the heart and brain, equilibrate rapidly with the intravascular blood and are identified with a central compartment that receives the intravenous dose.) Drug in the peripheral compartment transfers back to the central compartment with linear kinetics. The system is then described by the familiar state space model

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \quad t \geq 0, \tag{1.1}$$

where

$$A = \begin{bmatrix} -(a_{21} + a_{01}) & a_{12} \\ a_{21} & -a_{12} \end{bmatrix},$$

$x = [x_1, x_2]^{\mathrm{T}}$ is the state vector representing the masses in the two compartments, $a_{12}$ and $a_{21}$ are the compartment 2 to compartment 1 and the compartment 1 to compartment 2 transfer coefficients, respectively, and $a_{01}$ is the rate at which drug is eliminated out of the system from (the central) compartment 1. The other system parameter is $V_1$, the volume of the central compartment (for a total of four pharmacokinetic parameters). Note that with the assumption of instantaneous mixing, the *concentration* at $t = 0$ after dose $D$ is $D/V_1$. The assumption of instantaneous mixing is unrealistic but has little effect on the predictive accuracy of the model as long as we do not try to model drug concentrations immediately (within 5 minutes)

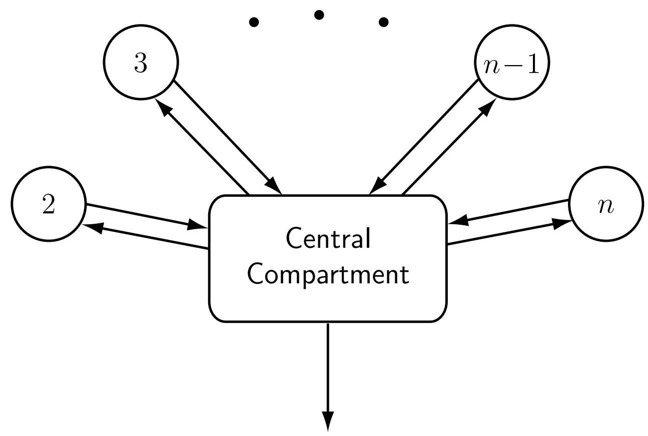



**Figure 1.1**: $n$-compartment mammillary model. The central compartment is the site for drug administration and is generally thought to be comprised of the intravascular blood volume as well as highly perfused organs such as the heart, brain, kidney, and liver. The central compartment exchanges with the peripheral compartments comprised of muscle and fat and which are metabolically inert as far as drug is concerned.

of the initial drug dose. This model, the two-compartment mammillary model, is generally useful for drugs that are administered intravenously although some require an extension of the model to include two distinct peripheral compartments along with the central compartment (the three compartment mammillary model). Other extensions or revisions of the basic model are possible. For example, Figure 1.1 shows an $n$-compartment mammillary model. In most cases the assumption of linear transfer is maintained so that the system equation remains the familiar

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \quad t \geq 0, \tag{1.2}$$

where $x \in \mathbb{R}^n$ represents the system compartmental masses or system compartmental concentrations and $A \in \mathbb{R}^{n \times n}$ is a *compartmental matrix* [123] in the case where $x$ represents compartmental masses and a *nonnegative matrix* [123] in the case where $x$ represents compartmental concentrations. Hence, (1.2) describes a nonnegative, compartmental dynamical system and there is a substantial body of theoretical work which is relevant for analyzing these systems (see [6, 29, 46, 70, 123, 124, 203] and the numerous references therein).

Parenthetically, it is important for the control engineer or theorist who wants to approach the pharmacokinetic literature to realize that the conventions of nomenclature are somewhat different than those used in this dissertation. For example, pharmacokineticists denote the transfer coefficient from compartment $i$ to compartment $j$ as $k_{ij}$ rather than $a_{ji}$. Pharmacokineticists also often parameterize models differently. For example, most pharmacokinetic papers will report the *terminal elimination half-life*, the time required for drug concentration to decrease by 50% if all tissues are equilibrated with the blood concentration. Another commonly reported parameter is the *clearance*, which is the volume of tissue or blood "cleared" of drug per unit time. Many pharmacokinetic investigations will be parameterized in terms of compartment volumes and intercompartmental clearances. These parameters are simply transformations of the basic elements of the system matrix $A$, along with a scale parameter, which in the case of the two-compartment mammillary model is the volume of the central compartment.

The experimental data used for pharmacokinetic modeling is typically collected by administering drug to patients and then drawing blood samples at various times after the initiation of dosing and determining the concentration of drug as a function of time. Consequently most pharmacokinetic investigations focus on blood concentrations and one of the goals of the analysis for drugs administered intravenously is to derive an expression for the *unit disposition function*, the blood concentration that results from a single unit bolus dose (impulse function) of drug. In the case of linear kinetics, if the unit disposition function ($f_{\mathrm{ud}}$) is known then the blood concentration that results from any arbitrary dose schedule is easily calculated by convolution integral

$$C(t) = \int_0^t f_{\mathrm{ud}}(\tau) D(t-\tau) \mathrm{d}\tau, \tag{1.3}$$

where $D(t)$ is the dose as a function of time [206]. Note that it is seldom technically feasible to actually measure drug concentrations in the tissue thought to be the site of the therapeutic effect and it is often assumed that effect site concentration and blood concentration are linearly related, if not equal. The vast majority of drugs are distributed to the site of action by blood flow and in general the effect site rapidly equilibrates with blood. If the finite equilibration time between the central intravascular blood volume and the effect site is clinically relevant, then the pharmacokinetic model should be revised to include a distinct effect site compartment.

Pharmacokinetic parameters (the entries of the system matrix $A$) are estimated by fitting models to the data. The models, of course, are approximations and there are numerous sources of noise in the data, from assay error to human recording error. Thus there is always an offset between the concentration predicted by the model and the observed data, the prediction error. One common method for estimating pharmacokinetic parameters is to use the method of maximum likelihood [51]. In this type of analysis one assumes a specific statistical distribution for the prediction error and then determines the parameter values that would maximize the likelihood of the observed results. For example, suppose we have conducted a study in a single patient in which we have collected blood samples at 10 different points in time after a single bolus intravenous dose of the drug. If we assume that the prediction error has a simple normal or Gaussian distribution, then the likelihood of the observed results will be proportional to

$$\prod_{i=1}^{r} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-PE_i^2/2\sigma^2}, \tag{1.4}$$

where $PE_i$ is the prediction error of the $i$th observation and is given by $PE_i = C_{\mathrm{p}_i} - C_{\mathrm{m}i}$, where $C_{\mathrm{p}_i}$ is the predicted $i$th drug concentration and $C_{\mathrm{m}i}$ is the measured $i$th drug concentration, $\sigma^2$ is the variance of the assumed Gaussian distribution of prediction errors, and $r$ is the number of observations (measured concentrations). We refer to this as the *intrapatient error model*. Note that the above expression is a function of $\sigma$ and the pharmacokinetic parameters (the entries of the system matrix $A$). By maximizing the above expression (or more commonly its logarithm) with respect to the pharmacokinetic parameters and $\sigma$ one may estimate the structural model parameters (the entries of the system matrix $A$) and the error model parameters (in this simple case, $\sigma$) that maximize the likelihood of the observed results. The reader familiar with statistical estimation theory will realize that the above example reduces to simple least squares estimation. However, using a more sophisticated error model (for example, by assuming that prediction error has a normal distribution with variance proportional to the predicted concentration raised to an unknown power) leads to more complex methods of parameter estimation [51].

There are two distinct approaches to estimating mean pharmacokinetic parameters for a population of patients [212, 213]. In the first, models are fitted to data from individual patients and the pharmacokinetic parameters for individual patients are then averaged (*two-stage analysis*) to provide a measure of the pharmacokinetic parameters for the population. The other approach to data analysis involves pooling of the data from individual patients. It is called *mixed-effects modeling* because in this situation the prediction error is determined not only by the stochastic noise of the experiment but also by the fact that *different patients have different pharmacokinetic parameters*. The error model, the analogue of the simple Gaussian distribution used in the example above, must account not only for variability between the observed and predicted concentrations within the same patient but also for variability between patients. The analyst must assume a statistical distribution for both intrapatient variability and interpatient variability. Most commonly it is assumed that pharmacokinetic parameters have a log-normal distribution. This sophisticated method of analysis not only estimates the mean structural pharmacokinetic parameters (the elements of the system matrix $A$) but also the statistical variability of these elements in the population, the *interpatient variability.* Since the total variance is the sum of interpatient and intrapatient variability, the latter is also estimated. This is a very powerful method of analysis for two reasons. First, it gives the clinician not only an estimate of the pharmacokinetic parameters but also an estimate of their variance. This is extremely important for the clinician since no matter how desirable the properties of a drug are, on average, if there is extreme variability in these properties it may not be safe for clinical use. And second, mixed-effects modeling may allow a reduction in the amount of data that is gathered from each individual patient. In a two-stage analysis, one must have enough data points from each patient to estimate their pharmacokinetic parameters. For example, if one adopts a two compartment mammillary model there are 4 pharmacokinetic parameters. It is impossible to estimate these parameters for any one patient with 4 or less data points from that patient. However, with mixed-effects modeling it is possible to use sparse data. This also is an important advantage since pharmacokinetic studies may be expensive and time consuming.

In contrast to pharmacokinetic modeling, pharmacodynamic modeling is more empirical. The molecular mechanism of action of many drugs is reasonably well-understood and most drugs act by binding to some "receptor" on or within target cells [66]. There is a well-developed theory of multiple equilibrium binding of ligands, such as drug molecules, to receptors on larger macromolecules, such as proteins. So in theory pharmacodynamics, the relationship between drug concentration and effect should follow from these models of molecular binding. However, the physiological effect is a complex interplay of numerous factors and it is generally not possible to quantitatively relate the effect at the level of the intact organism to the number of receptors bound by the drug at the molecular level. Empirical models are needed. It could be assumed that drug effect is proportional to the drug concentration at the effect site but this is clearly unrealistic since it admits the possibility of limitless drug effect. For example, consider a drug which lowers heart rate. It is unrealistic to assume that the drug effect is proportional to drug concentration since there is no limit on the drug concentration but there is a limit on the effect (the heart rate cannot be slower than zero). The empirical model should incorporate a ceiling effect. One model that has been quite effective for a variety of drugs is the *Hill equation*

$$E = E_{\max} C^{\gamma}/(C^{\gamma} + C_{50}^{\gamma}), \tag{1.5}$$

where $E$ is the drug effect, $E_{\max}$ is the maximum drug effect, $C$ is the drug concentration, $C_{50}$ is the drug concentration associated with 50% of the maximum effect, and $\gamma$ is a dimensionless parameter that determines the steepness of the concentration-effect relationship [110]. Note that this model reduces the concentration-effect relationship to three parameters, the maximum effect, a measure of the midpoint of the relationship, and a measure of the steepness. It is interesting that this model was first developed in 1906 to describe a *molecular* interaction, the binding of oxygen to hemoglobin. Since that time it has been applied to a wide variety of phenomenon which are far removed from explanations at the molecular level. There are a number of modification of this basic model that have been employed. One important one is when the drug effect is a binary, yes-or-no, variable. An example would be anesthesia, for which the patient is either responsive or not. In this case, the pharmacodynamic model based on the Hill equation becomes

$$P = C^{\gamma}/(C^{\gamma} + C_{50}^{\gamma}), \tag{1.6}$$

where the effect is now the probability $P$ that the patient will not respond to some noxious stimuli (and $E_{\max}$ equals unity) [162, 163].

In typical pharmacodynamic studies, drug is administered and the effect is measured at various points in time. At each point of observation, a blood sample is taken for the determination of the drug concentration at the time of observation of effect. The parameters of the pharmacodynamic model ($E_{\max}$, $C_{50}$, $\gamma$) may then be estimated by the same methods (maximum likelihood, generalized least squares, etc.) described above. Obviously, if blood drug concentrations and effect site concentrations have not equilibrated, then this analysis is invalidated.

It should be noted that pharmacodynamic models are inherently nonlinear, in contrast to pharmacokinetic models, which are usually linear. However, the interplay with pharmacodynamics may lead to nonlinear pharmacokinetics also. For example, some intravenous anesthetics depress *cardiac output*, the volume of blood pumped by the heart per unit of time. Since the basic transport processes that determine pharmacokinetic behavior are fundamentally functions of blood flow, administration of the drug alters its kinetics and since the pharmacodynamic relationship between drug concentration and depression of cardiac output is nonlinear, the pharmacokinetics of the drug are, in reality, also nonlinear.

## 1.4. Clinical Pharmacology and Drug Dosing

In addition to safety and efficacy, the Food and Drug Administration requires pharmacokinetic evaluation before approval of any new drug. The pharmacokinetic profile may be useful in developing dose guidelines. However, this application of basic principles is usually quite simplified. The disposition of most drugs is determined by both metabolic processes that eliminate the drug and distribution processes, i.e., transfer between various tissue groups. The route of distribution is via the intravascular blood volume whether the drug is administered by mouth, intramuscular injection or intravenously. The complexity of these processes implies that the governing dynamical system equation is almost always a vector differential equation. However, the vast majority of drugs are given for chronic conditions and when the time scale of treatment greatly exceeds the time scale of the distributive processes, one can ignore them. Furthermore, very few patients would comply with the complex dosing schemes ("take 3 pills in the morning and then 2 1/2 at 3:00 p.m. and then 2 at 8:00 and 10:00 p.m. and then one the next morning...") needed to account for distributive processes at the onset of therapy. Thus the application of pharmacokinetic principles must be simplified. In terms of the system equation (1.2), we assume that $A$ is a scalar. For example, if we know that a dose of 50 mg of an antihypertensive drug is efficacious in the "average" patient and we also know that the half life in the "average" patient is 12 hours then we may propose a dosing schedule that begins with an initial dose of 50 mg with subsequent dosing of 25 mg every 12 hours. Or, as another example, suppose we know that a blood concentration of an intravenous anesthetic of 100 $\mu$g/ml reliably produces unconsciousness and that we also know that the clearance (the amount of blood cleared of drug per unit time) is 150 ml/minute. Then an infusion of 100 $\mu$g/ml $\times$ 150 ml/min = 15000 $\mu$g/min will maintain this blood concentration, although this concentration will not be achieved until distributive processes have equilibrated. In point of fact, many of the dosing guideline recommended by the manufacturers of drugs are based on simple calculations like these. And although it is often not perceived as such by the clinician, initial drug dosing is a form of open-loop control, that is, control without feedback.

There have been attempts to develop more precise open-loop control in the acute care environment, especially in the area of anesthetic pharmacology. With the increased availability in the 1980s of small computers that could be taken into the operating room, several groups of investigators developed computer-controlled pump systems that continually adjusted the drug infusion rate to achieve and maintain the drug concentration desired by the clinician [5, 14, 210, 211]. These algorithms use the appropriate pharmacokinetic model

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \tag{1.7}$$

with *average* pharmacokinetic parameters taken from previous investigations to calculate the needed dose $u(t)$, $t \geq 0$, usually via the unit disposition function and the assumption of linearity. The output, which is continually updated, drives the infusion pump.

This is clearly open-loop control since, as previously emphasized, no one patient is an "average" patient and there is no mechanism for measuring the concentrations in the individual patient for feedback control. It is technically not feasible to actually measure blood concentrations of intravenous anesthetics in real time. But even with the lack of feedback, numerous studies have demonstrated better control of drug concentrations than the standard empirical dosing used by most clinicians. The clinical relevance of this is unclear. While open-loop control systems have not yet been approved by the Food and Drug Administration for routine clinical use in the United States, several European countries have approved a device for the infusion of the intravenous anesthetic, propofol, and this device is currently in use for clinical delivery of anesthesia.

While initial dosing guidelines may be based on the "average" patient, the very significant interpatient pharmacokinetic and pharmacodynamic variability observed for most drugs leads to the inevitable conclusion that *precise* drug dosing will require closed-loop control. As noted in Section 1.2, in one sense most drug dosing is a form of closed-loop control. Patients are quite familiar with this. The physician prescribes a drug, usually given orally, and an initial dose, observes the response, and adjusts the dose. An experienced physician can be quite adept at this process, but, in general, it is certainly not systematic and is usually time consuming. Most individuals who have been treated for a chronic disease know this well.

The process of dose titration can be made somewhat more precise by the use of mixed-effects pharmacokinetic modeling and *post-hoc* Bayesian estimation of individual patient pharmacokinetic parameters [51, 212, 213]. It will be recalled that mixed-effects modeling provides not only estimates of pharmacokinetic parameters but also their variance within the population. Suppose one has measured one or more drug concentrations in an individual patient. Using Bayesian probability principles, the likelihood of a given value of some pharmacokinetic parameter, $\Theta$, is proportional to $P(C|\Theta)P(\Theta)$. $P(C|\Theta)$ is the probability of the observed concentration(s) as a function of $\Theta$ and is simply the intrapatient error model cited earlier (an example is equation (1.4)). $P(\Theta)$ is the *a priori* probability of a given value of $\Theta$ and is given by the assumed distribution for $\Theta$ (as noted above, usually log-normal) and the variance of $\Theta$ estimated from the mixed-effects analysis. By determining the mode of $P(C|\Theta)P(\Theta)$ with respect to $\Theta$ one can derive a maximum likelihood estimate of $\Theta$ for the specific patient. By estimating patient-specific parameters one can more accurately calculate the necessary dose to achieve a given drug concentration. This process has been demonstrated to improve the precision of drug dosing [168]. But note that it only improves the precision of achieving a given drug *concentration* which may or may not lead to better control of drug *effect*, given pharmacodynamic variability. Also this process requires measurement of drug concentration, something that cannot usually be done quickly (a typical drug assay takes hours, if not more than a day, to complete).

While the process of titrating drug dose to the desired effect may be acceptable (if often frustrating) for chronic outpatient therapy, in the acute care environment, such as the operating room or the intensive care unit, this process may be dangerously slow or imprecise. It is in this environment that control technology has much to offer modern medicine and for the remainder of this article we will restrict ourselves to drugs used in the acute care setting.

In order to implement closed-loop control in an acute care environment one must have a real-time nearly instantaneously measurable performance or control variable. Early attempts at closed-loop control have of necessity focused on control of variables that are conveniently measured. By their very nature, cardiovascular and central nervous system function are critical in the acute care environment, and so mature technologies have evolved for their measurement. Thus, the primary applications of closed-loop control of drug administration have been to hemodynamic management and control of levels of consciousness. Before discussing our investigations of closed-loop control of anesthesia, we will briefly review closed-loop control of cardiovascular function, as it illustrates many of the general problems inherent in the application of control technology to physiological function.

## 1.5. Closed-Loop Control of Cardiovascular Function

After major surgery, especially cardiac surgery, many patients become profoundly hypertensive [158]. While this syndrome is distinct from the essential hypertension well known to both patients and medical professionals, it does require treatment since elevated blood pressure may cause cardiac dysfunction, leading to pulmonary edema or myocardial ischemia, may be a risk factor for stroke, and may exacerbate bleeding from fragile surgical suture lines. There are a number of potent drugs available for the treatment of post-operative hypertension but titrating these drugs to achieve the desired blood pressure may be difficult. Underdosing leaves the patient hypertensive and overdosing can reduce the blood pressure to levels associated with shock. There has been interest since the late 1970s in developing controllers for the administration of sodium nitroprusside (SNP), a commonly used and potent anti-hypertensive. The problems encountered in this endeavor are enlightening. The initial attempts used simple nonadaptive methods such as proportional-derivative or proportional-integral-derivative controllers that assumed a linear relationship between infusion rate and effect [214, 218]. This was a tenuous assumption. While the drug concentration may be the simple convolution of the infusion rate and a transfer function (equation (1.3)), the relationship between effect and infusion rate is not likely to be so simple (see equation (1.5)). Also, one of the significant challenges to the design of a blood pressure controller is the fact that there is a time delay between administration of the drug and the clinical effect. Failure to account for this time delay can lead to significant system oscillations. These early blood pressure controllers included time delays in the system model, however, the delays were assumed to be the same for each patient. While these early controllers were successful in some patients, in general they have not had wide clinical implementation. The barriers to clinical implementation were the nonlinear patient response and significant interpatient differences in drug sensitivity. It was very evident that interpatient variability and also the fact that an individual patient's sensitivity to the drug varies in time made adaptive controllers essential. Subsequently, single model and multiple model adaptive controllers were developed [11,107]. Single model adaptive controllers are based on on-line estimation of system parameters using minimum variance or least squares methodology. These controllers were also not acceptable due to large amplitude transients. Multiple model adaptive controllers represent the system by one of a finite number of models. For each model there is a separate controller. The probabilities that the system is represented by each of the different models are calculated from the relative offsets of the system response and the response predicted by each model. The output of the controller is the probability-weighted sum of the outputs from each model. Multiple model adaptive controllers have proven to be somewhat more satisfactory. Subsequent refinements to blood pressure control have included single model reference adaptive control [189], which appeared promising in simulations, and neural network-based methods [45]. There has also been substantial interest in optimal control since sodium nitroprusside has toxic side effects when the dose is too high [16].

These investigations into control of blood pressure reveal the challenges inherent to biological systems, specifically nonlinearity, interpatient variability (system uncertainty), and time delays. Despite the refinements of closed-loop blood pressure controllers, they are seldom used clinically. While this is due, in part, to the cost of technology acquisition, this is probably not the most important impediment to their clinical use. Blood pressure control is important but cardiovascular function involves several other important variables and all these variables are interrelated [158]. The intensive care unit clinician (nurse or physician) must not only insure that blood pressure is within appropriate limits but that also cardiac output (the amount of blood pumped by the heart per minute) is acceptable and heart rate is within reasonable limits. Mean arterial blood pressure is proportional to cardiac output, with the proportionality constant denoted the systemic vascular resistance, in analogy to Ohm's law. Cardiac output is equal to the product of heart rate and *stroke volume*, the volume of blood pumped with each beat of the heart. Stroke volume, in turn, is a function of *contractility* (the intrinsic strength of the cardiac contraction), *preload* (the volume of blood in the heart at the beginning of the contraction), and *afterload* (the impedance to ejection by the heart). The intensive care unit clinician must balance all these variables. There are drugs (inotropic agents) that increase contractility but they will also have variable effects on heart rate and afterload. There are also drugs which increase (vasopressors) or decrease (vasodilators) afterload. Finally, stroke volume may be increased by increasing preload and this can be accomplished by giving the patients fluid. However, giving too much fluid may be deleterious since it can lead to impaired pulmonary function as fluid builds up in the lungs. The fact that closed-loop control of blood pressure has not widely adopted by clinicians is not too surprising when one considers the complex interrelationships of hemodynamic variables. However, this also indicates an area where future applications of control theory could be invaluable. The technology is currently available to measure heart rate, blood pressure, cardiac output, and measures of preload continuously and in real time. Adaptive and robust optimal controllers which control the administration of multiple drugs (inotropes, vasopressors, vasodilators) and fluids would be a major advance in critical care medicine. There have been some preliminary investigation of the control of multiple hemodynamic drugs [108, 243] but this must still be considered unexplored territory.

## 1.6. Closed-Loop Control of Anesthesia

There has been long-standing interest in closed-loop control of anesthesia. Adequate anesthesia is comprised of several components; *analgesia*, lack of reflex response, such as increased blood pressure or heart rate, to surgical stimulus, lack of movement (which complicates the task of the surgeon), and *hypnosis* or lack of consciousness. In order to implement closed-loop control it is necessary to measure the state and the assessment of consciousness has been challenging. However, two technical innovations have facilitated the development of feedback controllers. The first (historically) is the routine clinical implementation of real-time spectroscopic methods for measuring the concentration of inhaled anesthetic agent in exhaled gases from the lung, in particular end-expiratory (routinely called end-tidal) gases. End-tidal anesthetic gas concentration is a reasonable surrogate for arterial blood anesthetic concentration [57]. Since end-tidal anesthetic agent concentrations can be measured in real-time with this technology, this has allowed closed-loop control of end-tidal anesthetic concentration. However, anesthetic concentration cannot be equated with anesthetic effect. More recently, real-time processed *electroencephalograph* (EEG) measurement has held open the possibility of closed-loop control of anesthetic effect. It has been known for decades that the EEG changes with induction of anesthesia [194]. However, *quantitatively* relating the EEG to anesthetic effect has been challenging. In the last decade, there has been substantial progress in developing processed EEG monitors that provide a measure of the depth of anesthesia and are candidates for performance variables for closed-loop controllers.

Inhaled anesthetic agents have been the mainstay of clinical practice since the first delivery of anesthesia. A fundamental characteristic of every inhaled anesthetic agent is its "MAC" value, for *m*inimum *a*lveolar (*alveoli* are the fundamental units of the lung) *c*oncentration that is associated with a 50% probability of patient movement or no movement in response to surgical stimulus [243]. By maintaining end-tidal concentrations well above MAC, the practitioner is relatively assured of hypnosis. The ready availability of spectroscopic systems for measuring end-tidal anesthetic concentration in real time has led several investigators to develop closed-loop controllers. The earliest of these used proportional-integral-derivative algorithms [195, 197]. As noted above, these share the weaknesses of assuming that all patients are the same. More recently, adaptive model-based controllers have been developed [125,234]. These typically rely on least-squares methods to estimate the specific system parameter for the individual patient. In animal studies, the adaptive controllers have performed, not surprisingly, more robustly than the fixed gain controllers. However, they have not been widely adopted clinically. The primary reason is that because of inter-patient pharmacodynamic variability, control of anesthetic concentration does not translate into control of anesthetic effect, and most clinicians would value control technology only if it prevented the possible overdoses inherent in maintaining end-tidal concentration in each individual patient well above the MAC value, an average from a population of patients. Closed-loop control of anesthesia requires a monitor of anesthetic effect, specifically consciousness.

The development of a monitor of consciousness has been an elusive challenge for anesthesiologists. The EEG, a global measure of electrical activity in the brain, has been an obvious candidate. However, the EEG is a complex of multiple time series and multiple spectra and while there are characteristic changes in the EEG with the induction of anesthesia, it has not been clear which, if any, characteristic of the EEG best reflects the anesthetic state. Building on pioneering work by Bickford [26], Schwilden and his colleagues developed and clinically tested a closed-loop model-based adaptive controller for the delivery of intravenous anesthesia using the median frequency of the EEG power spectrum as the control variable [207]. Their model assumed a two compartment pharmacokinetic model for which the concentration of drug $C(t)$ as a function of time $(t)$ after a single bolus dose was given by

$$C(t) = Ae^{-\alpha t} + Be^{-\beta t}, \tag{1.8}$$

where $A$, $B$, $\alpha$, $\beta$ are patient-specific pharmacokinetic parameters. It was also assumed that the control variable, median EEG frequency (denoted by $E$), was related to the drug concentration by the modified Hill equation

$$E = E_0 - E_{\max} C^{\gamma}/(C^{\gamma} + C_{50}^{\gamma}), \tag{1.9}$$

where $E_0$ is the baseline signal, $E_{\max}$ is the maximum decrease in signal with increasing drug concentration, $C_{50}$ is the drug concentration associated with 50% of the maximum effect, and $\gamma$ is a parameter describing the steepness of the concentration-effect curve. From the above equation it can be seen that the drug effect is a function of the pharmacokinetic parameters ($A$, $B$, $\alpha$, $\beta$) as well as the pharmacodynamic parameters ($E_0$, $E_{\max}$, $C_{50}$, and $\gamma$). If these parameters are known, calculation of the dose regimen needed to achieve the target EEG signal is straightforward. However, these parameters are not known for individual patients. The algorithm developed by Schwilden and his colleagues assumed that each of the pharmacodynamic parameters ($E_0$, $E_{\max}$, $C_{50}$, and $\gamma$) and the pharmacokinetic parameters $\alpha$ and $\beta$ were equal to the mean values reported in prior studies. Then using the mean population values of the pharmacokinetic parameters $A$ and $B$ as starting values, estimates of these parameters were refined by analysis of the difference between the target and observed EEG signal ($\Delta E$). Linearizing $\Delta E$ with respect to $A$ and $B$ we find

$$\Delta E = (\partial E/\partial A)\delta A + (\partial E/\partial B)\delta B, \tag{1.10}$$

where $\delta A$, $\delta B$ represent the updates to the values of $A$ and $B$ in the adaptive control algorithm. In conjunction with minimization of $\delta A^2 + \delta B^2$ this equation was used to solve for $\delta A$ and $\delta B$. It is important to note that this algorithm was only partially adaptive in that the only parameters of the model that were updated were $A$ and $B$. This algorithm was implemented for the intravenous anesthetic agents methohexital and propofol but did not appear to offer great advantage over standard manual control [207, 208]. This may have been due to the approximations of the algorithm or due to the deficiencies of the median EEG frequency as a measure of the depth of anesthesia.

Since the early work by Schwilden *et al.*, other EEG measures of depth of anesthesia have been developed. Possibly the most notable of these is the *bispectral index* or BIS [67, 209]. The BIS is a single composite EEG measure that appears to be closely related to the level of consciousness (see Figure 1.2). Recently, Struys and colleagues have described a closed-loop controller of the delivery of the intravenous anesthetic propofol using a model-based adaptive algorithm with the BIS as the control variable [227]. The algorithm is similar to that of Schwilden and his colleagues in that it is based on a pharmacokinetic model predicting the drug concentration as a function of infusion rate and time and a pharmacodynamic model analogous to that

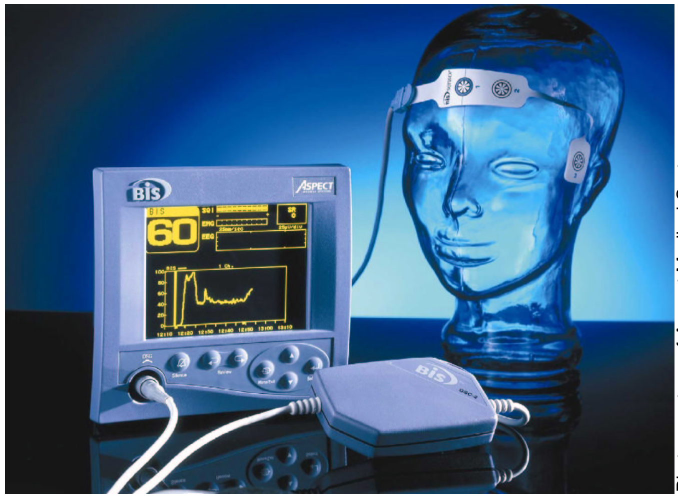



Photo courtesy of Aspect Medical Systems

**Figure 1.2**: Bispectral index (BIS) monitor

used by Schwilden *et al.* [207, 208] relating the BIS signal to concentration. However, in contrast to Schwilden and his colleagues, Struys *et al.* [227] assume that the pharmacokinetic parameters are always correct and that any variability in individual patient response is due to pharmacodynamic variability. More specifically, with induction they calculated a predicted concentration using the pharmacokinetic model and then constructed a BIS-concentration relationship using the observed BIS during induction and the predicted propofol concentration. With each time epoch, the difference between the target BIS signal and the observed BIS signal is used to update the pharmacodynamic parameters relating concentration and BIS signal for the individual patient. Note that this algorithm is only partially adaptive in the sense that there is no adaptive updating of pharmacokinetic parameters. Using this algorithm, Struys *et al.* [227] demonstrated excellent performance as measured by the difference between the target and observed BIS signals. However, as pointed out by Glass and Rampil, the excellent performance of the system may have been because the system was not fully stressed [69]. In their study, Struys *et al.* [227] administered a relatively high fixed dose of the opioid remifentanil, in conjunction with propofol. This blunted the patient response to surgical stimuli and meant that the propofol was needed only to produce unconsciousness in patients who were profoundly analgesic. The result was that only small adjustments in propofol concentrations were necessary. Whether the system would have been robust in the absence of deep narcotization is an open question.

In contrast to these model-based adaptive controllers, Absalom *et al.* have developed a proportional-integral-derivative controller using the BIS signal as the variable to control the infusion of propofol [2]. The median absolute performance error (the median value of the absolute value of $\Delta E/E_{\rm target}$) of this system was good (8.0%) but in 3 of 10 patients oscillations of the BIS signal around the set-point were observed and anesthesia was deemed clinically inadequate in 1 of the 10 patients. This same system has also been used with an auditory evoked potential as the control variable [138]. Intravenous propofol anesthesia has also been delivered by a closed-loop controller that uses both auditory evoked responses and cardiovascular responses as the control variables with a fuzzy-logic algorithm. This system has had only very minimal clinical testing [161]. More recently, Gentilini and his colleagues have described model-based controllers for inhalation anesthetic agents that attempt to control the BIS signal or mean arterial blood pressure, while keeping end-tidal anesthetic concentrations within pre-specified limits [65].

Given the uncertainties of both pharmacokinetic and pharmacodynamic models, and the magnitude of interpatient variability, in this dissertation we investigate parameter-independent adaptive controllers that can be implemented using the processed EEG as a performance variable (see Figure 1.3). Specifically, we develop direct adaptive and neural network adaptive control algorithms for nonnegative and compartmental systems. As mentioned above, nonnegative and compartmental models provide a broad framework for biological and physiological systems, including clinical pharmacology, and are well suited for the problem of closed-loop control of

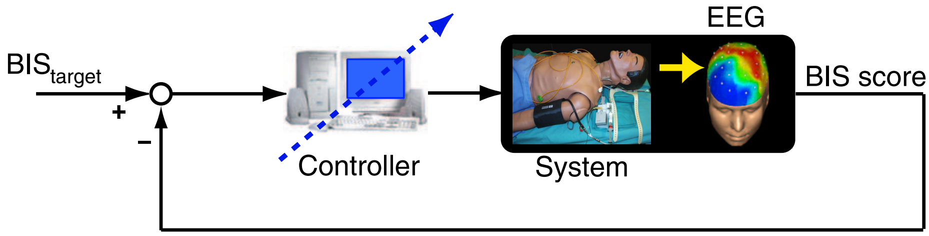



**Figure 1.3**: Adaptive closed-loop control for drug administration

drug administration. Specifically, nonnegative and compartmental dynamical systems [6, 70, 75, 123, 124, 203] are composed of homogeneous interconnected subsystems (or compartments) which exchange variable nonnegative quantities of material with conservation laws describing transfer, accumulation, and elimination between the compartments and the environment. It thus follows from physical considerations that the state trajectory of such systems remains in the nonnegative orthant of the state space for nonnegative initial conditions. Using nonnegative and compartmental model structures, in this dissertation a Lyapunov-based direct adaptive control framework is developed that guarantees partial asymptotic set-point stability of the closed-loop system; that is, asymptotic set-point stability with respect to part of the closed-loop system states associated with the physiological state variables. Furthermore, the remainder of the state associated with the adaptive controller gains is shown to be Lyapunov stable. In addition, the adaptive controllers are constructed *without* requiring knowledge of the system pharmacokinetic and pharmacodynamic parameters while providing a nonnegative control (source) input for robust stabilization with respect to a given set-point in the nonnegative orthant.

Neural network adaptive control algorithms are also developed in this dissertation for addressing closed-loop control of drug administration. Neural networks consist of a weighted interconnection of fundamental elements called *neurons*, which are functions consisting of a summing junction and a nonlinear operation involving an activation function. One of the primary reasons for the large interest in neural networks is their capability to approximate a large class of continuous nonlinear maps from the collective action of very simple, autonomous processing units interconnected in simple ways. In addition, neural networks have attracted attention due to their inherently parallel and highly redundant processing architecture that makes it possible to develop parallel weight update laws. This parallelism makes it possible to effectively update a neural network on line. These properties make neural networks a viable paradigm for adaptive system identification and control in clinical pharmacology. In this dissertation we also present a neural network adaptive control framework that accounts for combined interpatient pharmacokinetic and pharmacodynamic variability. In particular, we develop a neural adaptive output feedback control framework for adaptive set-point regulation of nonlinear uncertain nonnegative and compartmental systems. We emphasize that the formulation addresses adaptive *output feedback* controllers for nonlinear compartmental systems with *unmodeled dynamics* of *unknown dimension* while guaranteing ultimate boundedness of the error signals corresponding to the physical system states as well as the neural network weighting gains. Output feedback controllers are crucial in clinical pharmacology since key physiological (state) variables cannot be measured in practice.

## 1.7. Brief Outline of the Dissertation

In the first part of this dissertation we develop a direct adaptive control framework for adaptive stabilization, adaptive tracking, and disturbance rejection of multivariable nonlinear uncertain dynamical systems with exogenous disturbances. Specifically, in Chapter 2 we develop a Lyapunov-based adaptive control framework that guarantees partial asymptotic stability of the closed-loop system with exogenous bounded disturbances. In the case of bounded energy $L_2$ disturbances the proposed approach guarantees a nonexpansivity constraint on the closed-loop input-output map. Furthermore, if the nonlinear system is represented in normal form with input-to-state stable internal dynamics, then it is shown that nonlinear adaptive controllers can be constructed *without* requiring knowledge of the system dynamics or the system disturbance. This framework provides the basis for the nonlinear adaptive control framework given in the subsequent chapters of this dissertation. In particular, in Chapter 3 we extend the proposed framework to structured (norm-bounded) uncertainties. In addition, we consider an output feedback adaptive control problem for a class of nonlinear uncertain dynamical systems. Specifically, we address an adaptive absolute stabilization problem that is reminiscent to the classical absolute stability problem with the key difference being that the plant dynamics are not assumed to be known nor is the sector assumed to be known.

In Chapter 4, we address the problem of input amplitude and rate saturation constraints. In this research we construct a reference (governor or supervisor) model to derive adaptive update laws that guarantee that the error system dynamics are asymptotically stable in the face of actuator amplitude and rate saturation constraints. In addition, in Chapter 5, we consider an adaptive reduced-order dynamic compensation problem for nonlinear uncertain dynamical systems.

In Chapter 6, we address stabilization problem for a class of time-invariant and time-varying matrix second-order dynamical systems. In this framework no parametrization is required to construct adaptive feedback control laws as long as the generalized damping and stiffness operators are continuous and (lower) bounded. We also extend the result to the case where the system involves unbounded nonlinearities.

In the second part of this dissertation, namely, Chapters 7–12, we characterize adaptive feedback control laws for nonnegative and compartmental dynamical systems with applications to clinical pharmacology. Specifically, in Chapter 7 we develop an adaptive control framework for linear uncertain nonnegative and compartmental systems with specific applications to drug infusion control for general anesthesia. In this framework we present the case where the control input is constrained to be nonnegative as well as the case where such a restriction is not imposed. Based on these results, in Chapter 8 we extend the framework to nonnegative systems involving unknown time delays using a Lyapunov-Krasovskii framework. Our adaptive control laws are designed to guarantee asymptotic set-point regulation in the nonnegative orthant.

In Chapter 9, we further extend the results developed in Chapter 7 to nonlinear nonnegative systems. Analogous results to the results developed in Chapter 7 are obtained for nonlinear nonnegative systems with component decoupled Lyapunov functions.

In Chapters 10–12, we consider neural network adaptive controllers for nonlinear nonnegative dynamical systems to guarantee ultimate boundedness of the physical system states as well as the neural network weighting gains. Specifically, in Chapter 10 we develop neuro adaptive control laws based on the assumption that we have full measurement of the state. On the other hand, in Chapters 11 and 12 we develop neuro adaptive output feedback controllers that require information of part of the system states. In particular, under the assumption of input-to-state stable internal dynamics, the methodology developed in Chapter 11 is based on nonlinear passivity theory, while in Chapter 12 we make use of tapped delay lines to estimate the full states and construct adaptive feedback laws via the estimated controller states. Furthermore, in Chapters 11 and 12 we discuss the notion of partial ultimate boundedness to derive less conservative ultimate bounds for the case where system dynamics possess internal dynamics.

In Chapters 13 and 14 we address adaptive control problems for discrete-time nonlinear dynamical systems. In particular, Lyapunov-based adaptive feedback controllers are developed to achieve adaptive stabilization and tracking as well as adaptive disturbance rejection in Chapter 13. Assuming much weaker conditions on the system, in Chapter 14 we develop adaptive control framework to guarantee state and parameter error convergence when a generic geometric constraint on the adaptive gain matrix function holds. It is shown that this condition is consistent with the notion of persistent excitation in the adaptive control literature.

In Chapter 15, combining the results in the preceding chapters and using the recently developed hybrid invariance principle [35, 80], we construct novel hybrid adaptive control algorithms for impulsive dynamical systems to achieve asymptotic stability. In addition, a less restrictive hybrid adaptive control framework is developed to guarantee attraction of the plant states.

Finally, in Chapter 16 we give concluding remarks and discuss future extensions of the research. Throughout the dissertation numerous illustrative numerical examples as well as specific applications to the problems of thermoacoustic combustion processes and drug delivery systems are provided to demonstrate the efficacy of the proposed approaches.

The notation used in this dissertation is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$ real matrices, $\mathbb{N}^n$ (resp., $\mathbb{P}^n$) denotes the set of $n \times n$ nonnegative (resp., positive) definite matrices, $\mathcal{N}$ denotes the set of nonnegative integers, $(\ )^{\mathrm{T}}$ denotes transpose, $(\ )^{\dagger}$ denotes the Moore-Penrose generalized inverse, and $I_n$ or $I$ denotes the $n \times n$ identity matrix. Furthermore, we write $\mathrm{tr}(\cdot)$ for the trace operator, $\mathrm{spec}(\cdot)$ for the spectrum of a square matrix, $\|\cdot\|$ for the Euclidean vector norm, $\|\cdot\|_{\mathrm{F}}$ for the Frobenius matrix norm, $\ln(\cdot)$ for the natural log operator, $\lambda_{\min}(M)$ (resp., $\lambda_{\max}(M)$) for the minimum (resp., maximum) eigenvalue of the Hermitian matrix $M$, $\sigma_{\max}(M)$ for the maximum singular value of the matrix $M$, $V'(x)$ for the Fréchet derivative of $V$ at $x$, and $\mathrm{dist}(p, \mathcal{M})$ for the smallest distance from a point $p$ to any point in the set $\mathcal{M}$. Finally, $M \otimes N$ denotes the Kronecker product of matrices $M$ and $N$, and $M \geq 0$ (resp., $M > 0$) denotes the fact that the Hermitian matrix $M$ is nonnegative (resp., positive) definite.

# Chapter 2

# Direct Adaptive Control for Nonlinear Uncertain Systems with Exogenous Disturbances

## 2.1. Introduction

In this chapter we develop a direct adaptive control framework for adaptive stabilization, disturbance rejection, and command following of multivariable nonlinear uncertain dynamical systems with exogenous disturbances. In particular, in the first part of the chapter, a Lyapunov-based direct adaptive control framework is developed that requires a matching condition on the system disturbance and guarantees partial asymptotic stability of the closed-loop system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the plant. Furthermore, the remainder of the state associated with the adaptive controller gains is shown to be Lyapunov stable. In the case where the nonlinear system is represented in normal form [122] with input-to-state stable internal dynamics [122, 222], we construct nonlinear adaptive controllers *without* requiring knowledge of the system dynamics or the system disturbance. In addition, the proposed nonlinear adaptive controllers also guarantee asymptotic stability of the system state if the system dynamics are unknown *and* the input matrix function is parameterized by an unknown constant sign-definite matrix. Finally, in the second part of the chapter, we generalize the aforementioned results to uncertain nonlinear systems with exogenous $L_2$ disturbances. In this case, we remove the matching condition on the system disturbance. In addition, the proposed framework guarantees that the closed-loop nonlinear input-output map from uncertain exogenous $L_2$ disturbances to system performance variables is nonexpansive (gain bounded) and the solution of the closed-loop system is partially asymptotically stable. The proposed adaptive controller thus addresses the problem of disturbance rejection for nonlinear uncertain dynamical systems with bounded energy (square-integrable) $L_2$ signal norms on the disturbances and performance variables. This is clearly relevant for uncertain dynamical systems with poorly modeled disturbances which possess significant power within arbitrarily small bandwidths.

We emphasize that the direct adaptive stabilization framework developed in this chapter is distinct from the methods given in [12, 121, 136, 139] predicated on model reference adaptive control. The work of [115, 176] on *linear* direct adaptive control is most closely related to the results presented herein. Specifically, specializing our result to single-input linear systems with no internal dynamics and constant disturbances, we recover the result given in [115].

The contents of the chapter are as follows. In Section 2.2 we present our main direct adaptive control framework for adaptive stabilization, disturbance rejection, and command following of multivariable nonlinear uncertain dynamical systems with matched exogenous bounded disturbances. To further elucidate the proposed approach, in Section 2.3 we specialize the framework developed in Section 2.2 to single-input uncertain dynamical systems in normal form. In Section 2.4 we extend the results of Section 2.2 to nonlinear uncertain dynamical systems with exogenous $L_2$ disturbances without a matching condition requirement. Several illustrative numerical examples are presented in Section 2.5. In Section 2.6 we apply our framework to the control of thermoacoustic combustion instabilities to demonstrate the efficacy of the proposed direct adaptive stabilization and tracking framework. Finally, in Section 2.7 we draw some conclusions.

## 2.2. Adaptive Control for Nonlinear Systems with Exogenous Disturbances

In this section we begin by considering the problem of characterizing adaptive feedback control laws for nonlinear uncertain dynamical systems with exogenous disturbances. Specifically, consider the following controlled nonlinear uncertain system $\mathcal{G}$ given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t) + J(x(t))w(t), \quad x(0) = x_0, \quad t \geq 0, \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $w(t) \in \mathbb{R}^d$, $t \geq 0$, is a known bounded disturbance vector, $f : \mathbb{R}^n \to \mathbb{R}^n$ and satisfies $f(0) = 0$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $J : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ is a disturbance weighting matrix function with *unknown* entries. Note that even though $w(t)$, $t \geq 0$, is assumed to be known, the disturbance signal $J(x(t))w(t)$, $t \geq 0$, is an *unknown* bounded disturbance. The control input $u(\cdot)$ in (2.1) is restricted to the class of *admissible controls* consisting of measurable functions such that $u(t) \in \mathbb{R}^m$, $t \geq 0$. Furthermore, for the nonlinear system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions are satisfied; that is, $f(\cdot)$, $G(\cdot)$, $J(\cdot)$, $u(\cdot)$, and $w(\cdot)$ satisfy sufficient regularity conditions such that (2.1) has a unique solution forward in time.

**Theorem 2.1.** Consider the nonlinear system $\mathcal{G}$ given by (2.1). Assume there exist a matrix $K_{\rm g} \in \mathbb{R}^{m\times s}$ and functions $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, $F : \mathbb{R}^n \to \mathbb{R}^s$, with $F(0) = 0$, and $\ell : \mathbb{R}^n \to \mathbb{R}^t$ such that $V_{\rm s}(\cdot)$ is continuously differentiable, positive definite, radially unbounded, $V_{\rm s}(0) = 0$, $\ell(0) = 0$, and, for all $x \in \mathbb{R}^n$,

$$0 = V_{\rm s}'(x) f_{\rm s}(x) + \ell^{\rm T}(x)\ell(x), \tag{2.2}$$

where

$$f_{\rm s}(x) \triangleq f(x) + G(x)\hat{G}(x)K_{\rm g}F(x). \tag{2.3}$$

Furthermore, assume there exist a matrix $\Psi \in \mathbb{R}^{m\times d}$ and a function $\hat{J} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ such that $G(x)\hat{J}(x)\Psi = J(x)$. Finally, let $Q_1 \in \mathbb{R}^{m\times m}$, $Q_2 \in \mathbb{R}^{m\times m}$, $Y \in \mathbb{R}^{s\times s}$, and $Z \in \mathbb{R}^{d\times d}$ be positive definite. Then the adaptive feedback control law

$$u(t) = \hat{G}(x(t))K(t)F(x(t)) + \hat{J}(x(t))\Phi(t)w(t), \tag{2.4}$$

where $K(t) \in \mathbb{R}^{m\times s}$, $t \geq 0$, and $\Phi(t) \in \mathbb{R}^{m\times d}$, $t \geq 0$, with update laws

$$\dot{K}(t) = -\tfrac{1}{2}Q_1\hat{G}^{\rm T}(x(t))G^{\rm T}(x(t))V_{\rm s}'^{\rm T}(x(t))F^{\rm T}(x(t))Y, \quad K(0) = K_0, \tag{2.5}$$

$$\dot{\Phi}(t) = -\tfrac{1}{2}Q_2\hat{J}^{\rm T}(x(t))G^{\rm T}(x(t))V_{\rm s}'^{\rm T}(x(t))w^{\rm T}(t)Z, \quad \Phi(0) = \Phi_0, \tag{2.6}$$

guarantees that the solution $(x(t), K(t), \Phi(t)) \equiv (0, K_{\rm g}, -\Psi)$ of the closed-loop system given by (2.1), (2.4)–(2.6) is Lyapunov stable and $\ell(x(t)) \to 0$ as $t \to \infty$. If, in addition, $\ell^{\rm T}(x)\ell(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, then $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$.

**Proof.** Note that with $u(t)$, $t \geq 0$, given by (2.4) it follows from (2.1) that

$$\begin{aligned}\dot{x}(t) = {} & f(x(t)) + G(x(t))\hat{G}(x(t))K(t)F(x(t)) + G(x(t))\hat{J}(x(t))\Phi(t)w(t) \\ & +J(x(t))w(t), \quad x(0) = x_0, \quad t \geq 0,\end{aligned} \tag{2.7}$$

or, equivalently, using the fact that $G(x)\hat{J}(x)\Psi = J(x)$,

$$\begin{aligned}\dot{x}(t) = {} & f_{\rm s}(x(t)) + G(x(t))\hat{G}(x(t))(K(t) - K_{\rm g})F(x(t)) \\ & +G(x(t))\hat{J}(x(t))(\Phi(t) + \Psi)w(t), \quad x(0) = x_0, \quad t \geq 0.\end{aligned} \tag{2.8}$$

To show Lyapunov stability of the closed-loop system (2.5), (2.6), and (2.8) consider the Lyapunov function candidate

$$V(x,K,\Phi) = V_{\rm s}(x)+{\rm tr}\,Q_1^{-1}(K-K_{\rm g})Y^{-1}(K-K_{\rm g})^{\rm T}+{\rm tr}\,Q_2^{-1}(\Phi+\Psi)Z^{-1}(\Phi+\Psi)^{\rm T}. \quad (2.9)$$

Note that $V(0,K_{\rm g},-\Psi)=0$ and, since $V_{\rm s}(\cdot)$, $Q_1$, $Q_2$, $Y$, and $Z$ are positive definite, $V(x,K,\Phi)>0$ for all $(x,K,\Phi)\neq(0,K_{\rm g},-\Psi)$. In addition, $V(x,K,\Phi)$ is radially unbounded. Now, letting $x(t)$, $t\geq 0$, denote the solution to (2.8) and using (2.2), (2.5), and (2.6), it follows that the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned}
\dot V(x(t),K(t),\Phi(t)) &= V_{\rm s}'(x(t))\Big[f_{\rm s}(x(t))+G(x(t))\hat G(x(t))(K(t)-K_{\rm g})F(x(t))\\
&\quad +G(x(t))\hat J(x(t))(\Phi(t)+\Psi)w(t)\Big]\\
&\quad +2{\rm tr}\,Q_1^{-1}(K(t)-K_{\rm g})Y^{-1}\dot K^{\rm T}(t)+2{\rm tr}\,Q_2^{-1}(\Phi(t)+\Psi)Z^{-1}\dot\Phi^{\rm T}(t)\\
&= -\ell^{\rm T}(x(t))\ell(x(t))\\
&\quad +{\rm tr}\Big[(K(t)-K_{\rm g})F(x(t))V_{\rm s}'(x(t))G(x(t))\hat G(x(t))\Big]\\
&\quad +{\rm tr}\Big[(\Phi(t)+\Psi)w(t)V_{\rm s}'(x(t))G(x(t))\hat J(x(t))\Big]\\
&\quad -{\rm tr}\Big[(K(t)-K_{\rm g})F(x(t))V_{\rm s}'(x(t))G(x(t))\hat G(x(t))\Big]\\
&\quad -{\rm tr}\Big[(\Phi(t)+\Psi)w(t)V_{\rm s}'(x(t))G(x(t))\hat J(x(t))\Big]\\
&= -\ell^{\rm T}(x(t))\ell(x(t))\\
&\leq 0,\quad t\geq 0, \qquad (2.10)
\end{aligned}$$

which proves that the solution $(x(t),K(t),\Phi(t))\equiv(0,K_{\rm g},-\Psi)$ to (2.5), (2.6), and (2.8) is Lyapunov stable. Furthermore, it follows from Theorem 2 of [42] that $\ell(x(t))\to 0$ as $t\to\infty$. Finally, if $\ell^{\rm T}(x)\ell(x)>0$, $x\in\mathbb{R}^n$, $x\neq 0$, then $x(t)\to 0$ as $t\to\infty$ for all $x_0\in\mathbb{R}^n$. □

**Remark 2.1.** Note that in the case where $\ell^{\rm T}(x)\ell(x)>0$, $x\in\mathbb{R}^n$, $x\neq 0$, the conditions in Theorem 2.1 imply that $x(t)\to 0$ as $t\to\infty$ and hence it follows from

(2.5) and (2.6) that $(x(t), K(t), \Phi(t)) \to \mathcal{M} \triangleq \{(x, K, \Phi) \in \mathbb{R}^n \times \mathbb{R}^{m\times s} \times \mathbb{R}^{m\times d} : x = 0,\ \dot{K} = 0,\ \dot{\Phi} = 0\}$ as $t \to \infty$.

**Remark 2.2.** Theorem 2.1 is also valid for nonlinear *time-varying* uncertain dynamical systems $\mathcal{G}_t$ of the form

$$\dot{x}(t) = f(t, x(t)) + G(t, x(t))u(t) + J(t, x(t))w(t), \quad x(0) = x_0, \quad t \geq 0, \tag{2.11}$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and satisfies $f(t, 0) = 0$, $t \geq 0$, $G : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $J : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$. In particular, replacing $F : \mathbb{R}^n \to \mathbb{R}^s$ by $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^s$, where $F(t, 0) = 0$, $t \geq 0$, $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ by $\hat{G} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{m\times m}$, and requiring $G(t, x)\hat{J}(t, x)\Psi = J(t, x)$, where $\hat{J} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{m\times m}$ and $t \geq 0$, in place of $G(x)\hat{J}(x)\Psi = J(x)$, it follows by using identical arguments as in the proof of Theorem 2.1 that the adaptive feedback control law

$$u(t) = \hat{G}(t, x(t))K(t)F(t, x(t)) + \hat{J}(t, x(t))\Phi(t)w(t), \tag{2.12}$$

with the update laws

$$\dot{K}(t) = -\tfrac{1}{2}Q_1\hat{G}^{\mathrm{T}}(t, x(t))G^{\mathrm{T}}(t, x(t))V_{\mathrm{s}}'^{\mathrm{T}}(x(t))F^{\mathrm{T}}(t, x(t))Y, \quad K(0) = K_0, \tag{2.13}$$

$$\dot{\Phi}(t) = -\tfrac{1}{2}Q_2\hat{J}^{\mathrm{T}}(t, x(t))G^{\mathrm{T}}(t, x(t))V_{\mathrm{s}}'^{\mathrm{T}}(x(t))w^{\mathrm{T}}(t)Z, \quad \Phi(0) = \Phi_0, \tag{2.14}$$

where $V_{\mathrm{s}}'(x)$ satisfies (2.2) with $f_{\mathrm{s}}(x) = f(t, x) + G(t, x)\hat{G}(t, x)K_{\mathrm{g}}F(t, x)$, guarantees that the solution $(x(t), K(t), \Phi(t)) \equiv (0, K_{\mathrm{g}}, -\Psi)$ of the closed-loop system (2.11)–(2.14) is Lyapunov stable and $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$.

**Remark 2.3.** It follows from Remark 2.2 that Theorem 2.1 can also be used to construct adaptive tracking controllers for nonlinear uncertain dynamical systems. Specifically, let $r_{\mathrm{d}}(t) \in \mathbb{R}^n$, $t \geq 0$, denote a command input and define the error state $e(t) \triangleq x(t) - r_{\mathrm{d}}(t)$. In this case, the error dynamics are given by

$$\dot{e}(t) = f_{\mathrm{t}}(t, e(t)) + G_{\mathrm{t}}(t, e(t))u(t) + J_{\mathrm{t}}(t, e(t))w_{\mathrm{t}}(t), \quad e(0) = e_0, \quad t \geq 0, \tag{2.15}$$

where $f_{\rm t}(t,e(t)) = f(e(t)+r_{\rm d}(t)) - n(t)$, with $f(r_{\rm d}(t)) = n(t)$, $G_{\rm t}(t,e(t)) = G(e(t) + r_{\rm d}(t))$, and $J_{\rm t}(t,e(t))w_{\rm t}(t) = n(t) - \dot{r}_{\rm d}(t) + J(e(t)+r_{\rm d}(t))w(t)$. Now, the adaptive tracking control law (2.12)–(2.14), with $x(t)$ replaced by $e(t)$, guarantees that $e(t) \to 0$ as $t \to \infty$ for all $e_0 \in \mathbb{R}^n$.

It is important to note that the adaptive control law (2.4)–(2.6) does *not* require explicit knowledge of the gain matrix $K_{\rm g}$, the disturbance matching matrix $\Psi$, and the disturbance weighting matrix function $J(x)$; even though Theorem 2.1 requires the existence of $K_{\rm g}$ and $\Psi$ along with the construction of $F(x)$, $\hat{G}(x)$, $\hat{J}(x)$, and $V_{\rm s}(x)$ such that $G(x)\hat{J}(x)\Psi = J(x)$ and (2.2) holds. Furthermore, no specific structure on the nonlinear dynamics $f(x)$ is required to apply Theorem 2.1. However, if (2.1) is in normal form [122] with asymptotically stable internal dynamics, then we can always construct functions $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, $F : \mathbb{R}^n \to \mathbb{R}^s$, with $F(0) = 0$, and $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ such that (2.2) holds *without* requiring knowledge of the system dynamics. These facts are exploited below to construct nonlinear adaptive feedback controllers for nonlinear uncertain dynamical systems. For simplicity of exposition in the ensuing discussion we assume that $J(x) = D$, where $D \in \mathbb{R}^{n\times d}$ is a disturbance weighting matrix with unknown entries.

To elucidate the above discussion assume that the nonlinear uncertain system $\mathcal{G}$ is generated by

$$q_i^{(r_i)}(t) = f_{{\rm u}i}(q(t)) + \sum_{j=1}^{m} G_{{\rm s}(i,j)}(q(t))u_j(t) + \sum_{k=1}^{d} \hat{D}_{(i,k)}w_k(t), \quad t \geq 0, \quad i = 1,\cdots,m, \tag{2.16}$$

where $q = [q_1,\cdots,q_1^{(r_1-1)},\cdots,q_m,\cdots,q_m^{(r_m-1)}]^{\rm T}$, $q(0) = q_0$, $q_i^{(r_i)}$ denotes the $r_i^{\rm th}$ derivative of $q_i$, $r_i$ denotes the relative degree with respect to the output $q_i$, $\hat{D}_{(i,k)} \in \mathbb{R}$, $i = 1,\cdots,m$, $k = 1,\cdots,d$, and $w_k(t) \in \mathbb{R}$, $t \geq 0$, $k = 1,\cdots,d$. Here, we assume that the square matrix function $G_{\rm s}(q)$ composed of the entries $G_{{\rm s}(i,j)}(q)$, $i,j = 1,\cdots,m$, is such that $\det G_{\rm s}(q) \neq 0$, $q \in \mathbb{R}^{\hat{r}}$, where $\hat{r} = r_1 + \cdots + r_m$ is the (vector) relative degree of (2.16). Furthermore, since (2.16) is in a form where it does not possess internal dynamics, it follows that $\hat{r} = n$. The case where (2.16) possesses internal dynamics is discussed below.

Next, define $x_i \triangleq \left[ q_i, \cdots, q_i^{(r_i-2)} \right]^{\rm T}$, $i = 1, \cdots, m$, $x_{m+1} \triangleq \left[ q_1^{(r_1-1)}, \cdots, q_m^{(r_m-1)} \right]^{\rm T}$, and $x \triangleq \left[ x_1^{\rm T}, \cdots, x_{m+1}^{\rm T} \right]^{\rm T}$, so that (2.16) can be described by (2.1) with

$$f(x) = \tilde{A}x + \tilde{f}_{\rm u}(x), \quad G(x) = \begin{bmatrix} 0_{(n-m)\times m} \\ G_{\rm s}(x) \end{bmatrix}, \quad J(x) = D = \begin{bmatrix} 0_{(n-m)\times d} \\ \hat{D} \end{bmatrix}, \tag{2.17}$$

where

$$\tilde{A} = \begin{bmatrix} A_0 \\ 0_{m\times n} \end{bmatrix}, \quad \tilde{f}_{\rm u}(x) = \begin{bmatrix} 0_{(n-m)\times 1} \\ f_{\rm u}(x) \end{bmatrix},$$

$A_0 \in \mathbb{R}^{(n-m)\times n}$ is a known matrix of zeros and ones capturing the multivariable controllable canonical form representation [43], $f_{\rm u} : \mathbb{R}^n \to \mathbb{R}^m$ is an unknown function and satisfies $f_{\rm u}(0) = 0$, $G_{\rm s} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, and $\hat{D} \in \mathbb{R}^{m\times d}$. Here, we assume that $f_{\rm u}(x)$ is unknown and is parameterized as $f_{\rm u}(x) = \Theta f_{\rm n}(x)$, where $f_{\rm n} : \mathbb{R}^n \to \mathbb{R}^q$ and satisfies $f_{\rm n}(0) = 0$, $\Theta \in \mathbb{R}^{m\times q}$ is a matrix of uncertain constant parameters. Note that $\hat{J}(x)$ and $\Psi$ in Theorem 2.1 can be taken as $\hat{J}(x) = G_{\rm s}^{-1}(x)$ and $\Psi = \hat{D}$ so that $G(x)\hat{J}(x)\Psi = J(x) = D$ is satisfied.

Next, to apply Theorem 2.1 to the uncertain system (2.1) with $f(x), G(x)$, and $J(x)$ given by (2.17), let $K_{\rm g} \in \mathbb{R}^{m\times s}$, where $s = q + r$, be given by

$$K_{\rm g} = [\, \Theta_{\rm n} - \Theta,\ \Phi_{\rm n} \,], \tag{2.18}$$

where $\Theta_{\rm n} \in \mathbb{R}^{m\times q}$ and $\Phi_{\rm n} \in \mathbb{R}^{m\times r}$ are known matrices, and let

$$F(x) = \begin{bmatrix} f_{\rm n}(x) \\ \hat{f}_{\rm n}(x) \end{bmatrix}, \tag{2.19}$$

where $\hat{f}_{\rm n} : \mathbb{R}^n \to \mathbb{R}^r$, with $\hat{f}_{\rm n}(0) = 0$, is an arbitrary function. In this case, it follows that, with $\hat{G}(x) = G_{\rm s}^{-1}(x)$,

$$f_{\rm s}(x) = f(x) + G(x)\hat{G}(x)K_{\rm g}F(x)$$

$$= \tilde{A}x + \tilde{f}_{\mathrm{u}}(x) + \begin{bmatrix} 0_{(n-m)\times m} \\ G_{\mathrm{s}}(x) \end{bmatrix} G_{\mathrm{s}}^{-1}(x) \left[\ \Theta_{\mathrm{n}} f_{\mathrm{n}}(x) - \Theta f_{\mathrm{n}}(x) + \Phi_{\mathrm{n}} \hat{f}_{\mathrm{n}}(x)\ \right]$$

$$= \tilde{A}x + \begin{bmatrix} 0_{(n-m)\times 1} \\ \Theta_{\mathrm{n}} f_{\mathrm{n}}(x) + \Phi_{\mathrm{n}} \hat{f}_{\mathrm{n}}(x) \end{bmatrix}. \tag{2.20}$$

Now, since $\Theta_{\mathrm{n}} \in \mathbb{R}^{m\times q}$ and $\Phi_{\mathrm{n}} \in \mathbb{R}^{m\times r}$ are arbitrary constant matrices and $\hat{f}_{\mathrm{n}} : \mathbb{R}^n \to \mathbb{R}^r$ is an arbitrary function we can always construct $K_{\mathrm{g}}$, $V_{\mathrm{s}}(x)$, and $F(x)$ without knowledge of $f(x)$ such that (2.2) holds. In particular, choosing $\Theta_{\mathrm{n}} f_{\mathrm{n}}(x) + \Phi_{\mathrm{n}} \hat{f}_{\mathrm{n}}(x) = \hat{A}x$, where $\hat{A} \in \mathbb{R}^{m\times n}$, it follows that (2.20) has the form $f_{\mathrm{s}}(x) = A_{\mathrm{s}}x$, where $A_{\mathrm{s}} = \left[A_0^{\mathrm{T}}, \hat{A}^{\mathrm{T}}\right]^{\mathrm{T}}$ is in multivariable controllable canonical form. Hence, choosing $\hat{A}$ such that $A_{\mathrm{s}}$ is asymptotically stable, it follows from converse Lyapunov theory that there exists a positive-definite matrix $P$ satisfying the Lyapunov equation

$$0 = A_{\mathrm{s}}^{\mathrm{T}} P + P A_{\mathrm{s}} + R, \tag{2.21}$$

where $R$ is positive definite. In this case, with $V_{\mathrm{s}}(x) = x^{\mathrm{T}} P x$, the adaptive feedback controller (2.4) with update laws (2.5), (2.6), or, equivalently,

$$\dot{K}(t) = -Q_1 \hat{G}^{\mathrm{T}}(x(t)) G^{\mathrm{T}}(x(t)) P x(t) F^{\mathrm{T}}(x(t)) Y, \quad K(0) = K_0, \tag{2.22}$$

$$\dot{\Phi}(t) = -Q_2 \hat{J}^{\mathrm{T}}(x(t)) G^{\mathrm{T}}(x(t)) P x(t) w^{\mathrm{T}}(t) Z, \quad \Phi(0) = \Phi_0, \tag{2.23}$$

guarantees global asymptotic stability of the *nonlinear* uncertain dynamical system (2.1) where $f(x)$, $G(x)$, and $J(x)$ are given by (2.17). As mentioned above, it is important to note that it is not necessary to utilize a feedback linearizing function $F(x)$ to produce a linear $f_{\mathrm{s}}(x)$. However, when the system is in normal form, a feedback linearizing function $F(x)$ provides considerable simplification in constructing $V_{\mathrm{s}}'(x)$ necessary in computing the update laws (2.5) and (2.6).

A similar construction as discussed above can be used in the case where (2.1) is in normal form with input-to-state stable internal dynamics [222] and $w(t) \equiv 0$. In this case, (2.16) is given by

$$\dot{z}(t) = f_{\mathrm{z}}(q(t), z(t)), \quad z(0) = z_0, \quad t \geq 0, \tag{2.24}$$

$$q_i^{(r_i)}(t) = f_{\mathrm{u}i}(q(t), z(t)) + \sum_{j=1}^{m} G_{\mathrm{s}(i,j)}(q(t), z(t))u_j(t), \quad i = 1, \cdots, m, \tag{2.25}$$

where $f_{\mathrm{z}} : \mathbb{R}^{\hat{r}} \times \mathbb{R}^{n-\hat{r}} \to \mathbb{R}^{n-\hat{r}}$, $\hat{r} < n$, and where we have assumed for simplicity of exposition that the distribution spanned by the vector fields $\mathrm{col}_1(G(x)), \cdots, \mathrm{col}_m(G(x))$, where $\mathrm{col}_i(G(x))$ denotes the $i^{\mathrm{th}}$ column of $G(x)$, is involutive [122]. Here, we assume that the zero solution $z(t) \equiv 0$ to (2.24) is input-to-state stable with $q$ viewed as the input. Next, define $x \triangleq [z^{\mathrm{T}}, \hat{x}^{\mathrm{T}}]^{\mathrm{T}}$, where $\hat{x} \triangleq [x_1^{\mathrm{T}}, \cdots, x_{m+1}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{\hat{r}}$. Now, since the internal dynamics given by (2.24) are input-to-state stable, it follows from Theorem 2 of [42] and Lemma 5.6 of [139] that the zero solution $x(t) \equiv 0$ to (2.1) with $w(t) \equiv 0$ is globally asymptotically stable.

Next, we consider the case where $f(x)$ and $G(x)$ are both uncertain and $\hat{r} = n$. Specifically, we assume that $G(x)$ is such that $G_{\mathrm{s}}(x)$ is unknown and is parameterized as $G_{\mathrm{s}}(x) = B_{\mathrm{u}}G_{\mathrm{n}}(x)$, where $G_{\mathrm{n}} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is known and satisfies $\det G_{\mathrm{n}}(x) \neq 0$, $x \in \mathbb{R}^n$, and $B_{\mathrm{u}} \in \mathbb{R}^{m\times m}$, with $\det B_{\mathrm{u}} \neq 0$, is an unknown symmetric sign-definite matrix but the sign definiteness of $B_{\mathrm{u}}$ is known; that is, $B_{\mathrm{u}} > 0$ or $B_{\mathrm{u}} < 0$. For the statement of the next result define $B_0 \triangleq \left[0_{m\times(n-m)}, I_m\right]^{\mathrm{T}}$ for $B_{\mathrm{u}} > 0$, and $B_0 \triangleq \left[0_{m\times(n-m)}, -I_m\right]^{\mathrm{T}}$ for $B_{\mathrm{u}} < 0$.

**Corollary 2.1.** Consider the nonlinear system $\mathcal{G}$ given by (2.1) with $f(x)$, $G(x)$, and $J(x)$ given by (2.17) and $G_{\mathrm{s}}(x) = B_{\mathrm{u}}G_{\mathrm{n}}(x)$, where $B_{\mathrm{u}}$ is an unknown symmetric matrix and the sign definiteness of $B_{\mathrm{u}}$ is known. Assume there exist a matrix $K_{\mathrm{g}} \in \mathbb{R}^{m\times s}$ and functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $F : \mathbb{R}^n \to \mathbb{R}^s$, with $F(0) = 0$, and $\ell : \mathbb{R}^n \to \mathbb{R}^t$ such that $V_{\mathrm{s}}(\cdot)$ is continuously differentiable, positive definite, radially unbounded, $V_{\mathrm{s}}(0) = 0$, $\ell(0) = 0$, and (2.2) holds. Finally, let $Y \in \mathbb{R}^{s\times s}$ and $Z \in \mathbb{R}^{d\times d}$ be positive definite. Then the adaptive feedback control law

$$u(t) = G_{\rm n}^{-1}(x(t))K(t)F(x(t)) + G_{\rm n}^{-1}(x(t))\Phi(t)w(t), \tag{2.26}$$

where $K(t) \in \mathbb{R}^{m\times s}$, $t \geq 0$, and $\Phi(t) \in \mathbb{R}^{m\times d}$, $t \geq 0$, with update laws

$$\dot{K}(t) = -\tfrac{1}{2}B_0^{\rm T}V_{\rm s}'^{\rm T}(x(t))F^{\rm T}(x(t))Y, \quad K(0) = K_0, \tag{2.27}$$

$$\dot{\Phi}(t) = -\tfrac{1}{2}B_0^{\rm T}V_{\rm s}'^{\rm T}(x(t))w^{\rm T}(t)Z, \quad \Phi(0) = \Phi_0, \tag{2.28}$$

guarantees that the solution $(x(t), K(t), \Phi(t)) \equiv (0, K_{\rm g}, -\Psi)$, where $\Psi \in \mathbb{R}^{m\times d}$, of the closed-loop system given by (2.1), (2.26)–(2.28) is Lyapunov stable and $\ell(x(t)) \to 0$ as $t \to \infty$. If, in addition, $\ell^{\rm T}(x)\ell(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, then $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$.

**Proof.** The result is a direct consequence of Theorem 2.1. First, let $\hat{G}(x) = \hat{J}(x) = G_{\rm n}^{-1}(x)$ and $\Psi = B_{\rm u}^{-1}\hat{D}$ so that $G(x)\hat{G}(x) = [0_{m\times(n-m)}, B_{\rm u}]^{\rm T}$ and $G(x)\hat{J}(x)\Psi = D$, and let $K_{\rm g} = B_{\rm u}^{-1}[\Theta_{\rm n} - \Theta, \Phi_{\rm n}]$. Next, since $Q_1$ and $Q_2$ are arbitrary positive-definite matrices, $Q_1$ in (2.5) and $Q_2$ in (2.6) can be replaced by $q_1|B_{\rm u}|^{-1}$ and $q_2|B_{\rm u}|^{-1}$, respectively, where $q_1$, $q_2$ are positive constants and $|B_{\rm u}| = (B_{\rm u}^2)^{\frac{1}{2}}$, where $(\cdot)^{\frac{1}{2}}$ denotes the (unique) positive-definite square root. Now, since $B_{\rm u}$ is symmetric and sign definite it follows from the Schur decomposition that $B_{\rm u} = UD_{B_{\rm u}}U^{\rm T}$, where $U$ is orthogonal and $D_{B_{\rm u}}$ is real diagonal. Hence, $|B_{\rm u}|^{-1}\hat{G}^{\rm T}(x)G^{\rm T}(x) = [0_{m\times(n-m)}, \mathcal{I}_m] = B_0^{\rm T}$, where $\mathcal{I}_m = I_m$ for $B_{\rm u} > 0$ and $\mathcal{I}_m = -I_m$ for $B_{\rm u} < 0$. Now, (2.5) and (2.6), with $q_1Y$ and $q_2Z$ replaced by $Y$ and $Z$, imply (2.27) and (2.28), respectively. $\square$

It is important to note that if, as discussed above, $K_{\rm g}$ and $F(x)$ are constructed to give $f_{\rm s}(x) = A_{\rm s}x$ in (2.3), where $A_{\rm s}$ is an asymptotically stable matrix in multivariable controllable canonical form, then considerable simplification occurs in Corollary 2.1. Specifically, in this case $V_{\rm s}(x) = x^{\rm T}Px$, where $P > 0$ satisfies (2.21), and hence (2.27), (2.28) become

$$\dot{K}(t) = -B_0^{\mathrm{T}} P x(t) F^{\mathrm{T}}(x(t)) Y, \quad K(0) = K_0, \tag{2.29}$$

$$\dot{\Phi}(t) = -B_0^{\mathrm{T}} P x(t) w^{\mathrm{T}}(t) Z, \quad \Phi(0) = \Phi_0. \tag{2.30}$$

## 2.3. Specialization to Single-Input Systems with Uncertain Dynamics

In this section we apply the framework developed in Section 2.2 to single-input uncertain dynamical systems in normal form to further elucidate the proposed adaptive stabilization approach. For simplicity of exposition we assume that the system $\mathcal{G}$ has no internal dynamics. The case where $\mathcal{G}$ possesses input-to-state stable internal dynamics can be handled as discussed in Section 2.2. Here, we assume that the nonlinear uncertain system $\mathcal{G}$ is given by (2.1) with

$$f(x) = \begin{bmatrix} x_2 \\ \vdots \\ x_n \\ f_{\mathrm{u}}(x) \end{bmatrix}, \quad G(x) = \begin{bmatrix} 0_{(n-1)\times 1} \\ g_{\mathrm{s}}(x) \end{bmatrix}, \quad J(x) = D = \begin{bmatrix} 0_{(n-1)\times d} \\ \hat{d} \end{bmatrix}, \tag{2.31}$$

where $f_{\mathrm{u}} : \mathbb{R}^n \to \mathbb{R}$ is an unknown function and satisfies $f_{\mathrm{u}}(0) = 0$, $g : \mathbb{R}^n \to \mathbb{R}$ and satisfies $g(x) \neq 0$, $x \in \mathbb{R}^n$, and $\hat{d} \in \mathbb{R}^{1\times d}$. In addition, we assume that $f_{\mathrm{u}}(x)$ is unknown and is parameterized as $f_{\mathrm{u}}(x) = \theta f_{\mathrm{n}}(x)$, where $f_{\mathrm{n}} : \mathbb{R}^n \to \mathbb{R}^q$ and satisfies $f_{\mathrm{n}}(0) = 0$, and $\theta \in \mathbb{R}^{1\times q}$ is a vector of uncertain constant parameters. Note that in the single-input case $m = 1$ and hence $\hat{J}(x)$ and $\Psi$ in Theorem 2.1 can be taken as $\hat{J}(x) = g_{\mathrm{s}}^{-1}(x)$ and $\Psi = \hat{d}$, respectively, so that $G(x)\hat{J}(x)\Psi = J(x) = D$ is satisfied.

Next, to apply Theorem 2.1 to the single-input case with unknown dynamics let $K_{\mathrm{g}} \in \mathbb{R}^{1\times s}$ be given by

$$K_{\mathrm{g}} = [\,\theta_{\mathrm{n}} - \theta,\ \phi_{\mathrm{n}}\,], \tag{2.32}$$

where $\theta_{\mathrm{n}} \in \mathbb{R}^{1\times q}$ and $\phi_{\mathrm{n}} \in \mathbb{R}^{1\times r}$ are known vectors with $s = r + q$, and let $F(x)$ be given by (2.19). In this case, it follows that, with $\hat{G}(x) = g_{\rm s}^{-1}(x)$,

$$\begin{aligned} f_{\rm s}(x) &= f(x) + G(x)\hat{G}(x)K_{\rm g}F(x) \\ &= \begin{bmatrix} x_2 \\ \vdots \\ x_n \\ f_{\rm u}(x) \end{bmatrix} + \begin{bmatrix} 0_{(n-1)\times 1} \\ g_{\rm s}(x) \end{bmatrix} \frac{1}{g_{\rm s}(x)} \left[\ \theta_{\rm n} f_{\rm n}(x) - \theta f_{\rm n}(x) + \phi_{\rm n}\hat{f}_{\rm n}(x)\ \right] \\ &= \begin{bmatrix} x_2 \\ \vdots \\ x_n \\ \theta_{\rm n} f_{\rm n}(x) + \phi_{\rm n}\hat{f}_{\rm n}(x) \end{bmatrix}. \end{aligned} \tag{2.33}$$

Now, since $\theta_{\rm n} \in \mathbb{R}^{1\times q}$ and $\phi_{\rm n} \in \mathbb{R}^{1\times r}$ are arbitrary constant vectors and $\hat{f}_{\rm n} : \mathbb{R}^n \to \mathbb{R}^r$ is an arbitrary function we can always construct $K_{\rm g}$, $V_{\rm s}(x)$, and $F(x)$ without knowledge of $f(x)$ such that (2.2) holds. In particular, choosing $\theta_{\rm n} f_{\rm n}(x) + \phi_{\rm n}\hat{f}_{\rm n}(x) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$, it follows that (2.33) has the form $f_{\rm s}(x) = A_{\rm s}x$, where $A_{\rm s}$ is in controllable canonical form. Hence, choosing $a_i$, $i = 1, \cdots, n$, such that $A_{\rm s}$ is asymptotically stable, it follows that there exists a positive-definite matrix $P$ satisfying (2.21). In this case, with Lyapunov function $V_{\rm s}(x) = x^{\rm T}Px$, the adaptive feedback controller (2.4) with update laws (2.5), (2.6), or, equivalently,

$$\dot{K}(t) = -\hat{G}^{\rm T}(x(t))G^{\rm T}(x(t))Px(t)F^{\rm T}(x(t))Y, \quad K(0) = K_0, \tag{2.34}$$

$$\dot{\Phi}(t) = -\hat{J}^{\rm T}(x(t))G^{\rm T}(x(t))Px(t)w^{\rm T}(t)Z, \quad \Phi(0) = \Phi_0, \tag{2.35}$$

guarantees global asymptotic stability of the nonlinear dynamical system (2.1) where $f(x)$, $G(x)$, and $J(x)$ are given by (2.31).

Next, we consider the case where $f(x)$ and $G(x)$ are both uncertain. Specifically, we assume that $G(x) = [0_{1\times(n-1)}, b_{\rm u}g_{\rm n}(x)]^{\rm T}$, where $g_{\rm n} : \mathbb{R}^n \to \mathbb{R}$ is known and satisfies $g_{\rm n}(x) \neq 0$, $x \in \mathbb{R}^n$, and $b_{\rm u} \neq 0$ is unknown but $\operatorname{sgn} b_{\rm u} \triangleq b_{\rm u}/|b_{\rm u}|$ is known. For the statement of the next result define $B_0 = [0_{1\times(n-1)}, \operatorname{sgn} b_{\rm u}]^{\rm T}$.

**Corollary 2.2.** Consider the nonlinear system $\mathcal{G}$ given by (2.1) with $f(x)$, $G(x)$, and $J(x)$ given by (2.31) and $g_{\rm s}(x) = b_{\rm u} g_{\rm n}(x)$, where $b_{\rm u}$ is unknown but $\operatorname{sgn} b_{\rm u}$ is known. Assume there exists a vector $K_{\rm g} \in \mathbb{R}^{1\times s}$ and functions $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, $F : \mathbb{R}^n \to \mathbb{R}^s$, with $F(0) = 0$, and $\ell : \mathbb{R}^n \to \mathbb{R}^t$ such that $V_{\rm s}(\cdot)$ is continuously differentiable, positive definite, radially unbounded, $V_{\rm s}(0) = 0$, $\ell(0) = 0$, and (2.2) holds. Finally, let $Y \in \mathbb{R}^{s\times s}$ and $Z \in \mathbb{R}^{d\times d}$ be positive definite. Then the adaptive feedback control law

$$u(t) = g_{\rm n}^{-1}(x(t))K(t)F(x(t)) + g_{\rm n}^{-1}(x(t))\Phi(t)w(t), \tag{2.36}$$

where $K(t) \in \mathbb{R}^{1\times s}, t \geq 0$, and $\Phi(t) \in \mathbb{R}^{1\times d}, t \geq 0$, with update laws

$$\dot{K}(t) = -\tfrac{1}{2}B_0^{\rm T} V_{\rm s}'^{\rm T}(x(t))F^{\rm T}(x(t))Y, \quad K(0) = K_0, \tag{2.37}$$

$$\dot{\Phi}(t) = -\tfrac{1}{2}B_0^{\rm T} V_{\rm s}'^{\rm T}(x(t))w^{\rm T}(t)Z, \quad \Phi(0) = \Phi_0, \tag{2.38}$$

guarantees that the solution $(x(t), K(t), \Phi(t)) \equiv (0, K_{\rm g}, -\Psi)$, where $\Psi \in \mathbb{R}^{1\times d}$, of the closed-loop system given by (2.1), (2.36)–(2.38) is Lyapunov stable and $\ell(x(t)) \to 0$ as $t \to \infty$. If, in addition, $\ell^{\rm T}(x)\ell(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, then $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$.

**Proof.** The result is a direct consequence of Theorem 2.1. First, let $\hat{G}(x) = \hat{J}(x) = g_{\rm n}^{-1}(x)$ and note that taking $\Psi = b_{\rm u}^{-1}\hat{d}$ it follows that $G(x)\hat{J}(x)\Psi = D$. Next, since $Q_1$ and $Q_2$ are arbitrary positive-definite matrices, $Q_1$ in (2.5) and $Q_2$ in (2.6) can be replaced by $|b_{\rm u}|^{-1}Q_1$ and $|b_{\rm u}|^{-1}Q_2$, respectively, where in this case $Q_1$ and $Q_2$ are scalars. Hence, (2.5) and (2.6), with $Q_1 Y$ and $Q_2 Z$ replaced by $Y$ and $Z$, imply (2.37) and (2.38), respectively. □

If $K_{\rm g}$ and $F(x)$ are constructed to give $f_{\rm s}(x) = A_{\rm s}x$ in (2.3), where $A_{\rm s}$ is an asymptotically stable matrix in controllable canonical form, then considerable simplification occurs in Corollary 2.2. Specifically, in this case $V_{\rm s}(x) = x^{\rm T}Px$, where $P > 0$ satisfies

(2.21), and hence (2.37), (2.38) become

$$\dot{K}(t) = -B_0^{\mathrm{T}} P x(t) F^{\mathrm{T}}(x(t)) Y, \quad K(0) = K_0, \tag{2.39}$$

$$\dot{\Phi}(t) = -B_0^{\mathrm{T}} P x(t) w^{\mathrm{T}}(t) Z, \quad \Phi(0) = \Phi_0. \tag{2.40}$$

Finally, we specialize Corollary 2.2 to the case where $d = 1$, $w(t) \equiv 1$, $\hat{d} \in \mathbb{R}$, and $f(x) = Ax$, where

$$A = \begin{bmatrix} A_0 \\ \theta \end{bmatrix}, \tag{2.41}$$

$A_0 \in \mathbb{R}^{(n-1)\times n}$ is a known matrix and $\theta \in \mathbb{R}^{1\times n}$ is an unknown vector. In this case, our results specialize to results given in [114, 115].

**Corollary 2.3.** Consider the system $\mathcal{G}$ given by (2.1) with $f(x) = Ax$, $G(x) = B$, $d = 1$, $J(x) = [0_{1\times(n-1)}, \hat{d}]^{\mathrm{T}}$, and $w(t) \equiv 1$. Assume there exists a vector $K_{\mathrm{g}} \in \mathbb{R}^{1\times n}$ such that (2.21) holds with $A_{\mathrm{s}} \triangleq A + BK_{\mathrm{g}}$, $P > 0$, and $R \geq 0$. Furthermore, let $Y \in \mathbb{R}^{n\times n}$ be a positive-definite matrix and $Z \in \mathbb{R}$ be a positive constant. Then the adaptive feedback control law

$$u(t) = K(t)x(t) + \phi(t), \tag{2.42}$$

where $K(t) \in \mathbb{R}^{1\times n}$, $t \geq 0$, and $\phi(t) \in \mathbb{R}$, $t \geq 0$, with update laws

$$\dot{K}(t) = -B_0^{\mathrm{T}} P x(t) x^{\mathrm{T}}(t) Y, \quad K(0) = K_0, \tag{2.43}$$

$$\dot{\phi}(t) = -B_0^{\mathrm{T}} P x(t) Z, \quad \phi(0) = \phi_0, \tag{2.44}$$

guarantees that the solution $(x(t), K(t), \phi(t)) \equiv (0, K_{\mathrm{g}}, -\psi)$, where $\psi \in \mathbb{R}$, of the closed-loop system given by (2.1), (2.42)–(2.44) is Lyapunov stable and $Rx(t) \to 0$ as $t \to \infty$. If, in addition, $R > 0$, then $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$.

**Proof.** The result is a direct consequence of Corollary 2.2 with $s = n$, $d = 1$, $f(x) = Ax$, $w(t) \equiv 1$, $F(x) = x$, and $V_{\mathrm{s}}(x) = x^{\mathrm{T}} P x$. □

Once again, even though Corollary 2.3 requires the existence of $K_{\mathrm{g}}$ such that (2.21) holds, the adaptive feedback controller (2.42)–(2.44) can be constructed without knowledge of $\theta$ in $A$ or $b$ in $B$. However, $\operatorname{sgn} b$ must be known. To see this note that by choosing $K_{\mathrm{g}} = \frac{1}{b}\,[\theta_{\mathrm{n}} - \theta]$ it follows that

$$A + BK_{\mathrm{g}} = \begin{bmatrix} A_0 \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} \frac{1}{b}\,[\theta_{\mathrm{n}} - \theta] = \begin{bmatrix} A_0 \\ \theta_{\mathrm{n}} \end{bmatrix}.$$

Since $\theta_{\mathrm{n}}$ is arbitrary, it follows that $P$ can be determined without knowledge of $\theta$ or $b$.

## 2.4. Adaptive Control for Nonlinear Systems with $\mathrm{L}_2$ Disturbances

In this section we consider the problem of characterizing adaptive feedback control laws for nonlinear uncertain dynamical systems with exogenous $\mathrm{L}_2$ disturbances. Specifically, we consider the following controlled nonlinear uncertain system $\mathcal{G}$ given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t) + J(x(t))w(t), \quad x(0) = x_0, \quad w(\cdot) \in \mathrm{L}_2, \quad t \geq 0, \tag{2.45}$$

with performance variables

$$z(t) = h(x(t)), \tag{2.46}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $w(t) \in \mathbb{R}^d$, $t \geq 0$, is an unknown bounded energy $\mathrm{L}_2$ disturbance, $z(t) \in \mathbb{R}^p$, $t \geq 0$, is a performance variable, $f : \mathbb{R}^n \to \mathbb{R}^n$ and satisfies $f(0) = 0$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $J : \mathbb{R}^n \to \mathbb{R}^{n\times d}$, and $h : \mathbb{R}^n \to \mathbb{R}^p$ is continuous and satisfies $h(0) = 0$. The following theorem generalizes Theorem 2.1 to nonlinear uncertain dynamical systems with exogenous $\mathrm{L}_2$ disturbances.

**Theorem 2.2.** Consider the nonlinear system $\mathcal{G}$ given by (2.45) and (2.46). Assume there exist a matrix $K_{\mathrm{g}} \in \mathbb{R}^{m\times s}$ and functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$,

$F : \mathbb{R}^n \to \mathbb{R}^s$, with $F(0) = 0$, and $\ell : \mathbb{R}^n \to \mathbb{R}^t$ such that $V_{\rm s}(\cdot)$ is continuously differentiable, positive definite, radially unbounded, $V_{\rm s}(0) = 0$, $\ell(0) = 0$, and, for all $x \in \mathbb{R}^n$,

$$0 = V_{\rm s}'(x) f_{\rm s}(x) + \Gamma(x), \tag{2.47}$$

where

$$\Gamma(x) \triangleq \tfrac{1}{4\gamma^2} V_{\rm s}'(x) J(x) J^{\rm T}(x) V_{\rm s}'^{\rm T}(x) + h^{\rm T}(x) h(x) \tag{2.48}$$

and $f_{\rm s}(x)$ is given by (2.3). Finally, let $Q \in \mathbb{R}^{m\times m}$ and $Y \in \mathbb{R}^{s\times s}$ be positive definite. Then the adaptive feedback control law

$$u(t) = \hat{G}(x(t)) K(t) F(x(t)), \tag{2.49}$$

where $K(t) \in \mathbb{R}^{m\times s}$, $t \geq 0$, with update law

$$\dot{K}(t) = -\tfrac{1}{2} Q \hat{G}^{\rm T}(x(t)) G^{\rm T}(x(t)) V_{\rm s}'^{\rm T}(x(t)) F^{\rm T}(x(t)) Y, \quad K(0) = K_0, \tag{2.50}$$

guarantees that the solution $(x(t), K(t)) \equiv (0, K_{\rm g})$ of the undisturbed ($w(t) \equiv 0$) closed-loop system given by (2.45), (2.49), and (2.50) is Lyapunov stable and $h(x(t)) \to 0$ as $t \to \infty$. If, in addition, $h^{\rm T}(x)h(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, then $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$. Furthermore, the solution $x(t)$, $t \geq 0$, to the closed-loop system given by (2.45), (2.49), and (2.50) satisfies the nonexpansivity constraint

$$\int_0^T z^{\rm T}(t) z(t)\,{\rm d}t \leq \gamma^2 \int_0^T w^{\rm T}(t) w(t)\,{\rm d}t + V(x(0), K(0)),$$
$$T \geq 0, \quad \gamma > 0, \quad w(\cdot) \in {\rm L}_2, \tag{2.51}$$

where

$$V(x, K) \triangleq V_{\rm s}(x) + {\rm tr}\, Q^{-1}(K - K_{\rm g}) Y^{-1} (K - K_{\rm g})^{\rm T}. \tag{2.52}$$

**Proof.** Note that with $u(t)$, $t \geq 0$, given by (2.49) it follows from (2.45) that

$$\dot{x}(t) = f(x(t)) + G(x(t)) \hat{G}(x(t)) K(t) F(x(t)) + J(x(t)) w(t), \quad x(0) = x_0,$$
$$w(\cdot) \in {\rm L}_2, \quad t \geq 0, \tag{2.53}$$

or, equivalently, using the definition for $f_{\rm s}(x)$ given in (2.3),

$$\dot{x}(t) = f_{\rm s}(x(t)) + G(x(t))\hat{G}(x(t))(K(t) - K_{\rm g})F(x(t)) + J(x(t))w(t), \quad x(0) = x_0, \quad w(\cdot) \in {\rm L}_2, \quad t \geq 0. \quad (2.54)$$

To show Lyapunov stability of the closed-loop system (2.50) and (2.54) consider the Lyapunov function candidate given by (2.52). Note that $V(0, K_{\rm g}) = 0$ and, since $V_{\rm s}(\cdot)$, $Q$, and $Y$ are positive definite, $V(x, K) > 0$ for all $(x, K) \neq (0, K_{\rm g})$. Furthermore, $V(x, K)$ is radially unbounded. Now, letting $x(t)$, $t \geq 0$, denote the solution to (2.54) and using (2.47) and (2.50), it follows that the Lyapunov derivative along the undisturbed ($w(t) \equiv 0$) closed-loop system trajectories is given by

$$\begin{aligned}
\dot{V}(x(t), K(t)) &= V_{\rm s}'(x(t)) \Big[f_{\rm s}(x(t)) + G(x(t))\hat{G}(x(t))(K(t) - K_{\rm g})F(x(t))\Big] \\
&\quad +2{\rm tr}\, Q^{-1}(K(t) - K_{\rm g})Y^{-1}\dot{K}^{\rm T}(t) \\
&= -\Gamma(x(t)) + {\rm tr}\Big[(K(t) - K_{\rm g})F(x(t))V_{\rm s}'(x(t))G(x(t))\hat{G}(x(t))\Big] \\
&\quad -{\rm tr}\Big[(K(t) - K_{\rm g})F(x(t))V_{\rm s}'(x(t))G(x(t))\hat{G}(x(t))\Big] \\
&= -\Gamma(x(t)) \\
&\leq 0, \quad t \geq 0, \qquad (2.55)
\end{aligned}$$

which proves that the solution $(x(t), K(t)) \equiv (0, K_{\rm g})$ to (2.50) and (2.54) with $w(t) \equiv 0$ is Lyapunov stable. Furthermore, it follows from Theorem 2 of [42] that $h(x(t)) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$. If, in addition, $h^{\rm T}(x)h(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, then $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$.

Finally, to show that the nonexpansivity constraint (2.51) holds, note that, for all $(x, w) \in \mathbb{R}^n \times \mathbb{R}^d$,

$$\begin{aligned}
0 &\leq \left[\tfrac{1}{2\gamma}J^{\rm T}(x)V_{\rm s}'^{\rm T}(x) - \gamma w\right]^{\rm T} \left[\tfrac{1}{2\gamma}J^{\rm T}(x)V_{\rm s}'^{\rm T}(x) - \gamma w\right] \\
&= \Gamma(x) + \gamma^2 w^{\rm T} w - z^{\rm T} z - V_{\rm s}'^{\rm T}(x)J(x)w. \qquad (2.56)
\end{aligned}$$

Now, let $w(\cdot) \in \mathrm{L}_2$ and let $x(t)$, $t \geq 0$, denote the solution of the closed-loop system (2.54). Then, using (2.50), the Lyapunov derivative along the closed-loop system trajectories is given by

$$
\begin{aligned}
\dot{V}(x(t),K(t)) &= V_{\mathrm{s}}'(x(t))\Big[f_{\mathrm{s}}(x(t)) + G(x(t))\hat{G}(x(t))(K(t)-K_{\mathrm{g}})F(x(t)) \\
&\quad + J(x(t))w(t)\Big] + 2\mathrm{tr}\, Q^{-1}(K(t)-K_{\mathrm{g}})Y^{-1}\dot{K}^{\mathrm{T}}(t) \\
&= -\Gamma(x(t)) + \mathrm{tr}\Big[(K(t)-K_{\mathrm{g}})F(x(t))V_{\mathrm{s}}'(x(t))G(x(t))\hat{G}(x(t))\Big] \\
&\quad + V_{\mathrm{s}}'(x(t))J(x(t))w(t) \\
&\quad - \mathrm{tr}\Big[(K(t)-K_{\mathrm{g}})F(x(t))V_{\mathrm{s}}'(x(t))G(x(t))\hat{G}(x(t))\Big] \\
&= -\Gamma(x(t)) + V_{\mathrm{s}}'(x(t))J(x(t))w(t) \\
&\leq \gamma^2 w^{\mathrm{T}}(t)w(t) - z^{\mathrm{T}}(t)z(t), \quad t \geq 0. \qquad (2.57)
\end{aligned}
$$

Now, integrating (2.57) over $[0, T]$ yields

$$
V(x(T),K(T)) \leq \int_0^T \big[\gamma^2 w^{\mathrm{T}}(t)w(t) - z^{\mathrm{T}}(t)z(t)\big]\mathrm{d}t + V(x(0),K(0)),
$$
$$
T \geq 0, \quad \gamma > 0, \quad w(\cdot) \in \mathrm{L}_2, \quad (2.58)
$$

which, by noting that $V(x(T),K(T)) \geq 0$, $T \geq 0$, yields (2.51). □

It is important to note that unlike Theorem 2.1 requiring a matching condition on the disturbance, Theorem 2.2 does not require any such matching condition. Furthermore, as shown in Section 2.2, if (2.45) is in normal form with asymptotically stable internal dynamics, then we can construct functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $F : \mathbb{R}^n \to \mathbb{R}^s$, with $F(0) = 0$, such that (2.47) holds without requiring knowledge of the system dynamics. In addition, in the case where $J(x) = D$ and $h(x) = Ex$, the adaptive controller (2.50) can be constructed to guarantee the nonexpansivity constraint (2.51) using standard *linear* $\mathrm{H}_\infty$ methods. Specifically, choosing $f_{\mathrm{s}}(x) = A_{\mathrm{s}}x$, where $A_{\mathrm{s}}$ is asymptotically stable and in multivariable controllable canonical form, it follows from standard $\mathrm{H}_\infty$ theory [240] that if $(A_{\mathrm{s}}, E)$ is observable, $\|G(s)\|_\infty < \gamma$, where $G(s) = E(sI_n - A_{\rm s})^{-1}D$, if and only if there exists a positive-definite matrix $P$ satisfying the bounded real Riccati equation

$$0 = A_{\rm s}^{\rm T}P + PA_{\rm s} + \gamma^{-2}PDD^{\rm T}P + E^{\rm T}E. \tag{2.59}$$

It is well known that (2.59) has a nonnegative-definite solution if and only if the Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix} A_{\rm s} & \gamma^{-2}DD^{\rm T} \\ -E^{\rm T}E & -A_{\rm s}^{\rm T} \end{bmatrix}, \tag{2.60}$$

has no purely imaginary eigenvalues. In this case, with $V_{\rm s}(x) = x^{\rm T}Px$, the adaptive feedback controller (2.49) with update law (2.50), or, equivalently,

$$\dot{K}(t) = -Q\hat{G}^{\rm T}(x(t))G^{\rm T}(x(t))Px(t)F^{\rm T}(x(t))Y, \quad K(0) = K_0, \tag{2.61}$$

guarantees global asymptotic stability of the nonlinear undisturbed ($w(t) \equiv 0$) dynamical system (2.45), where $f(x)$ and $G(x)$ are given by (2.17). Furthermore, the solution $x(t)$, $t \geq 0$, of the closed-loop *nonlinear* dynamical system (2.45) and (2.49) is guaranteed to satisfy the nonexpansivity constraint (2.51).

Finally, if $f(x)$ and $G(x)$ given by (2.17) are uncertain and $G_{\rm s}(x) = B_{\rm u}G_{\rm n}(x)$, where the sign definiteness of $B_{\rm u}$ is known, then using an identical approach as in Section 2.2, it can be shown that the adaptive feedback control law

$$u(t) = G_{\rm n}^{-1}(x(t))K(t)F(x(t)), \tag{2.62}$$

with update law

$$\dot{K}(t) = -\tfrac{1}{2}B_0^{\rm T}V_{\rm s}'^{\rm T}(x(t))F^{\rm T}(x(t))Y, \quad K(0) = K_0, \tag{2.63}$$

where $B_0$ is defined as in Section 2.2, guarantees asymptotic stability and nonexpansivity of (2.45) and (2.46).

## 2.5. Illustrative Numerical Examples

In this section we present several numerical examples to demonstrate the utility of the proposed direct adaptive control framework for adaptive stabilization, disturbance rejection, and command following.

**Example 2.1.** Consider the uncertain controlled Liénard system given by

$$\ddot{z}(t)+\mu(z^4(t)-\alpha)\dot{z}(t)+\beta z(t)+\gamma\tanh(z(t)) = bu(t), \quad z(0)=z_0, \quad \dot{z}(0)=\dot{z}_0, \quad t\geq 0, \tag{2.64}$$

where $\mu, \alpha, \beta, \gamma, b \in \mathbb{R}$ are unknown. Note that with $x_1 = z$ and $x_2 = \dot{z}$, (2.64) can be written in state space form (2.1) with $x = [x_1, x_2]^{\mathrm{T}}$, $f(x) = [x_2, -\beta x_1 - \gamma \tanh x_1 - \mu(x_1^4 - \alpha)x_2]^{\mathrm{T}}$, and $G(x) = [0, b]^{\mathrm{T}}$. Here, we assume that $f(x)$ is unknown and can be parameterized as $f(x) = [x_2, \theta_1 x_1 + \theta_2 x_2 + \theta_3 \tanh x_1 + \theta_4 x_1^4 x_2]^{\mathrm{T}}$, where $\theta_1, \theta_2, \theta_3$, and $\theta_4$ are unknown constants. Furthermore, we assume that $\operatorname{sgn} b$ is known. Next, let $F(x) = [x_1, x_2, \tanh(x_1), x_1^4 x_2]^{\mathrm{T}}$ and $K_{\mathrm{g}} = \frac{1}{b}\left[\theta_{\mathrm{n}_1} - \theta_1, \theta_{\mathrm{n}_2} - \theta_2, -\theta_3, -\theta_4\right]$, where $\theta_{\mathrm{n}_1}$, $\theta_{\mathrm{n}_2}$ are arbitrary scalars, so that

$$\begin{aligned} f_{\mathrm{s}}(x) &= f(x) + \begin{bmatrix} 0 \\ b \end{bmatrix} \frac{1}{b} \begin{bmatrix} \theta_{\mathrm{n}_1} - \theta_1, \theta_{\mathrm{n}_2} - \theta_2, -\theta_3, -\theta_4 \end{bmatrix} F(x) \\ &= \begin{bmatrix} 0 & 1 \\ \theta_{\mathrm{n}_1} & \theta_{\mathrm{n}_2} \end{bmatrix} x. \end{aligned} \tag{2.65}$$

Now, with the proper choice of $\theta_{\mathrm{n}_1}$ and $\theta_{\mathrm{n}_2}$, it follows from Corollary 2.1 that the adaptive feedback controller (2.26) with $w(t) \equiv 0$ guarantees that $x(t) \to 0$ as $t \to \infty$. Specifically, here we choose $\theta_{\mathrm{n}_1} = -1$, $\theta_{\mathrm{n}_2} = -2$, and $R = 2I_2$, so that $P$ satisfying (2.21) is given by

$$P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}. \tag{2.66}$$

With $\mu = 2$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $b = 3$, $Y = I_4$, and initial conditions $x(0) = [1, 1]^{\mathrm{T}}$ and $K(0) = [0, 0, 0, 0]$, Figure 2.1 shows the phase portrait of the controlled and

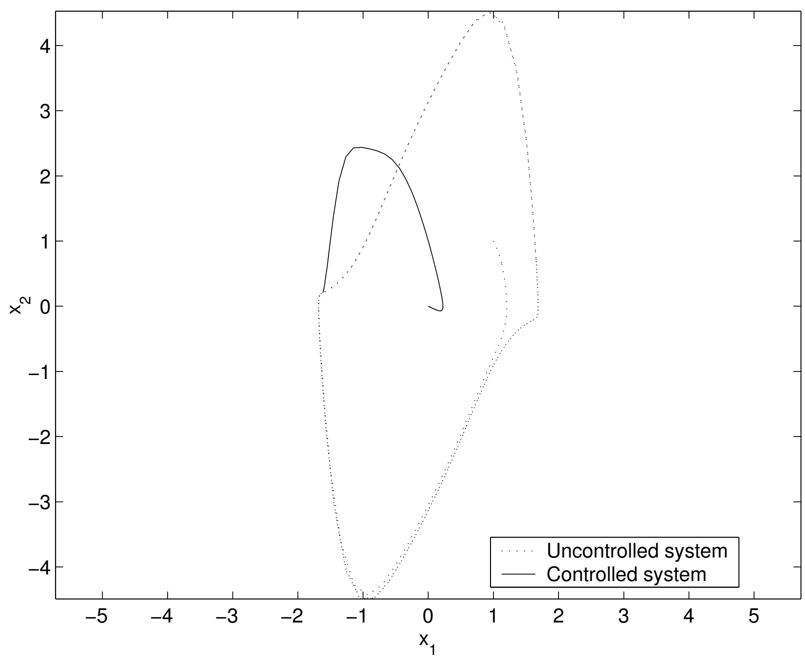



**Figure 2.1**: Phase portrait of controlled and uncontrolled Liénard system

uncontrolled system. Note that the adaptive controller is switched on at $t = 15$ sec. Figure 2.2 shows the state trajectories versus time and the control signal versus time. Finally, Figure 2.3 shows the adaptive gain history versus time.

**Example 2.2.** Consider the uncertain controlled Van der Pol oscillator given by

$$\ddot{z}(t) - \varepsilon(1 - z^2(t))\dot{z}(t) + z(t) = bu(t), \quad z(0) = z_0, \quad \dot{z}(0) = \dot{z}_0, \quad t \geq 0, \tag{2.67}$$

where $\varepsilon, b \in \mathbb{R}$ are unknown. Note that with $x_1 = z$ and $x_2 = \dot{z}$, (2.67) can be written in state space form (2.1) with $x = [x_1, x_2]^{\mathrm{T}}$, $f(x) = [x_2, -x_1 + \varepsilon(1 - x_1^2)x_2]^{\mathrm{T}}$, and $G(x) = [0, b]^{\mathrm{T}}$. Here, we assume that $f(x)$ is unknown and can be parameterized as $f(x) = [x_2, -x_1 + \theta_1 x_2 + \theta_2 x_1^2 x_2]^{\mathrm{T}}$, where $\theta_1$ and $\theta_2$ are unknown constants. Furthermore, we assume that $\operatorname{sgn} b$ is known. Next, let $F(x) = [x_2, x_1^2 x_2]^{\mathrm{T}}$ and $K_{\mathrm{g}} = \frac{1}{b}\left[\theta_{\mathrm{n}_1} - \theta_1, -\theta_2\right]$, where $\theta_{\mathrm{n}_1}$ is an arbitrary scalar, so that

$$\begin{aligned} f_{\mathrm{s}}(x) &= f(x) + \begin{bmatrix} 0 \\ b \end{bmatrix} \frac{1}{b} \begin{bmatrix} \theta_{\mathrm{n}_1} - \theta_1, -\theta_2 \end{bmatrix} F(x) \\ &= \begin{bmatrix} 0 & 1 \\ -1 & \theta_{\mathrm{n}_1} \end{bmatrix} x. \end{aligned}$$

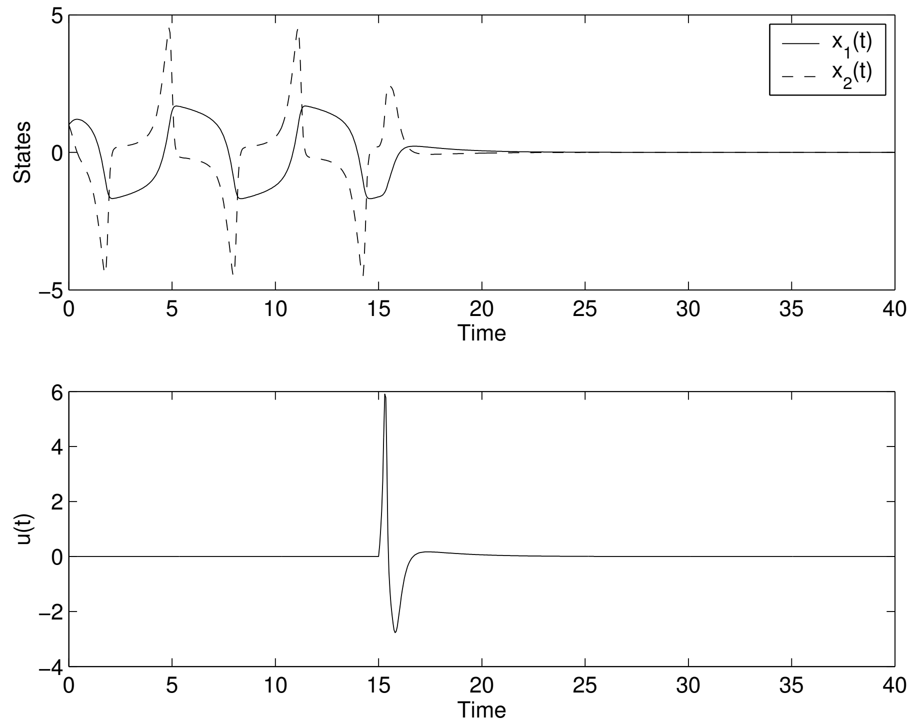



**Figure 2.2**: State trajectories and control signal versus time

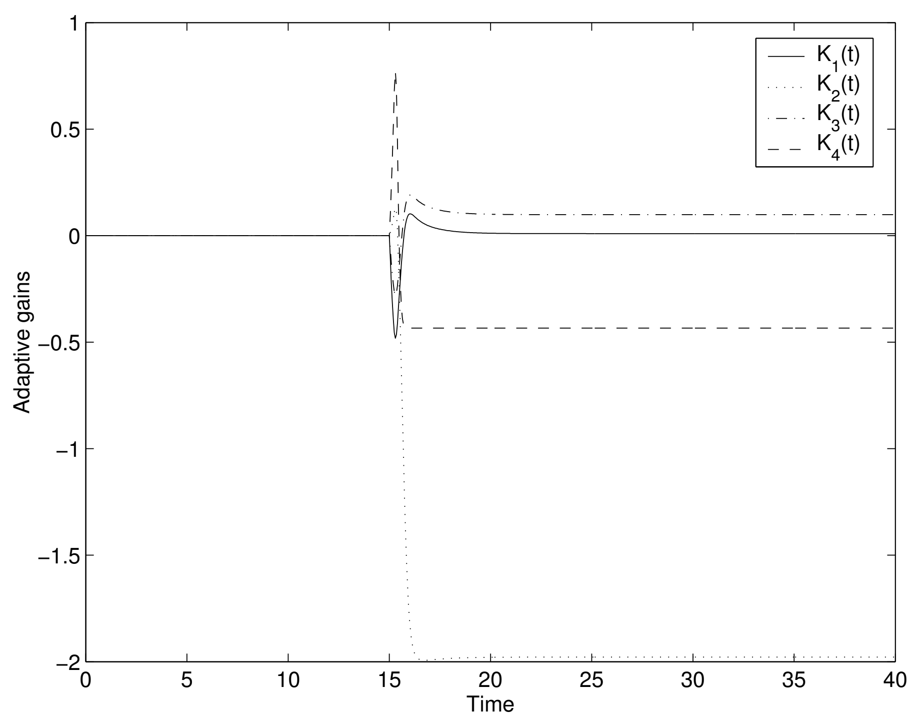



**Figure 2.3**: Adaptive gain history versus time

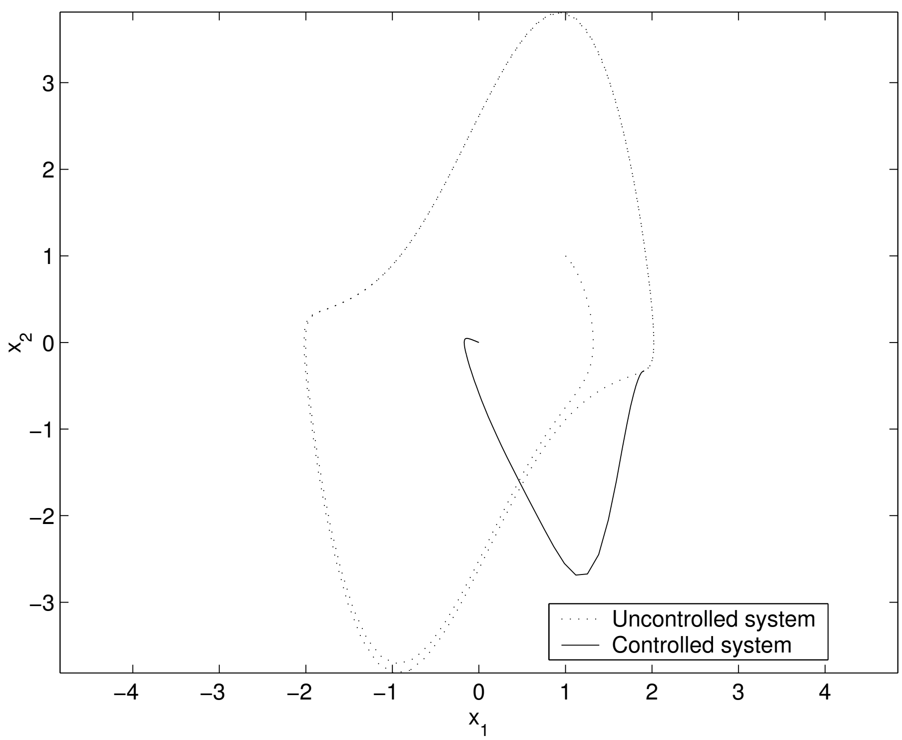



**Figure 2.4**: Phase portrait of controlled and uncontrolled Van der Pol oscillator

Now, with the proper choice of $\theta_{\mathrm{n}_1}$, it follows from Corollary 2.1 that the adaptive feedback controller (2.26) with $w(t) \equiv 0$ guarantees that $x(t) \to 0$ as $t \to \infty$. Specifically, here we choose $\theta_{\mathrm{n}_1} = -2$ and $R = 2I_2$, so that $P$ satisfying (2.21) is given by (2.66). With $\varepsilon = 2$, $b = 3$, $Y = I_2$, and initial conditions $x(0) = [1,1]^{\mathrm{T}}$ and $K(0) = [0,0]$, Figure 2.4 shows that the phase portrait of the controlled and uncontrolled system. Note that the adaptive controller is switched on at $t = 15$ sec. Figure 2.5 shows the state trajectories versus time and the control signal versus time. Finally, Figure 2.6 shows the adaptive gain history versus time.

**Example 2.3.** Consider the uncertain controlled Rayleigh system given by

$$\ddot{z}(t) - \varepsilon(\dot{z}(t) - \alpha\dot{z}^3(t)) + \beta z(t) = bu(t) + \hat{d}, \quad z(0) = z_0, \quad \dot{z}(0) = \dot{z}_0, \quad t \geq 0, \tag{2.68}$$

where $\varepsilon, \alpha, b \in \mathbb{R}$ are unknown and $\hat{d} \in \mathbb{R}$ is an unknown constant disturbance. Note that with $x_1 = z$ and $x_2 = \dot{z}$, (2.68) can be expressed in state space form

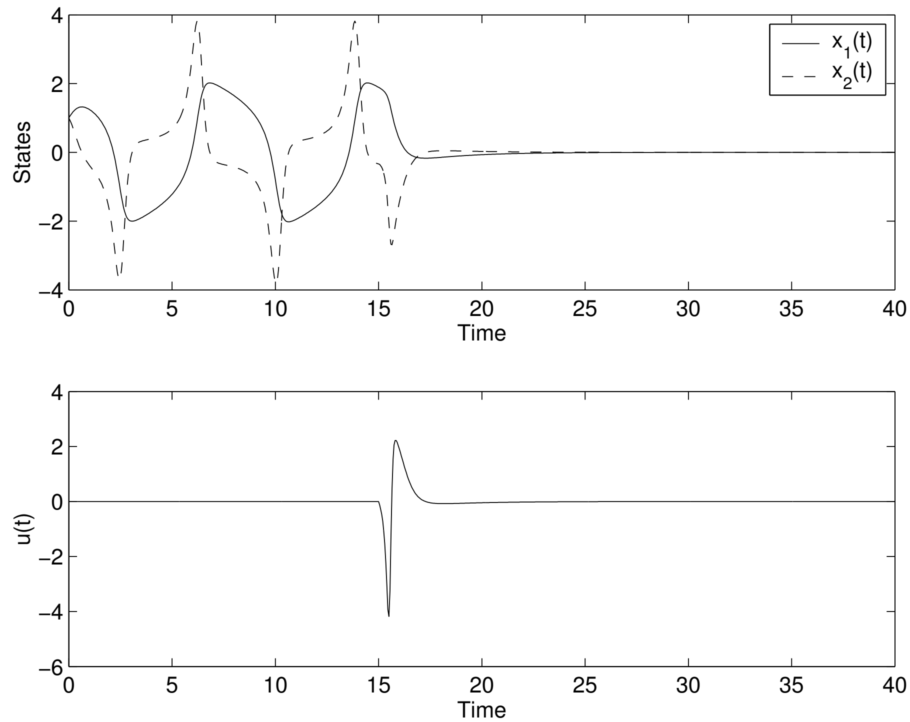



**Figure 2.5**: State trajectories and control signal versus time

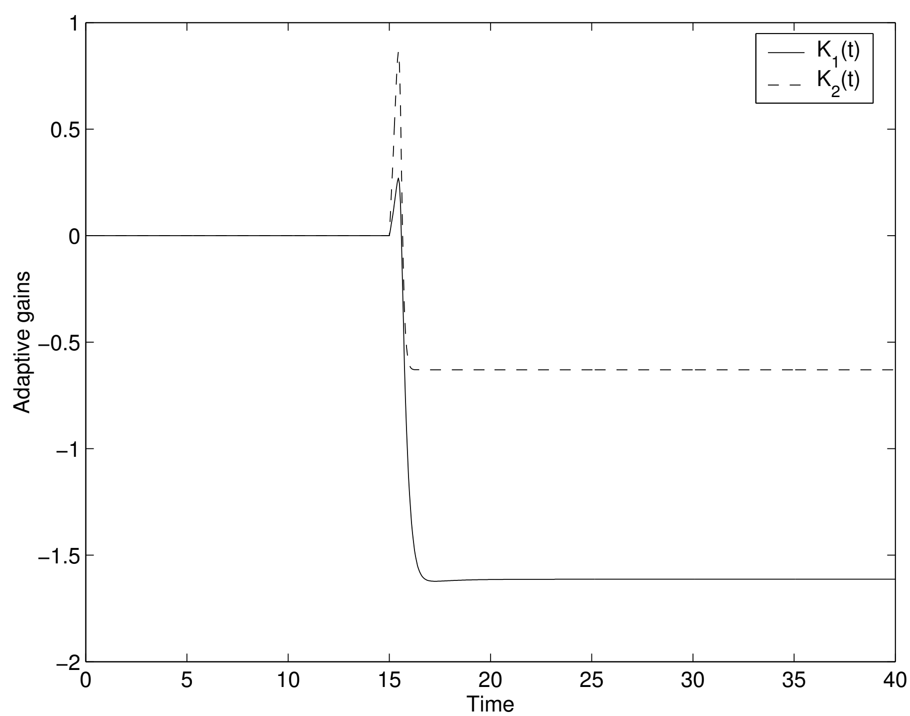



**Figure 2.6**: Adaptive gain history versus time

(2.1) with $x = [x_1, x_2]^{\mathrm{T}}$, $f(x) = [x_2, -\beta x_1 + \varepsilon(x_2 - \alpha x_2^3)]^{\mathrm{T}}$, $G(x) = [0, b]^{\mathrm{T}}$, $J(x) = [0, \hat{d}]^{\mathrm{T}}$, and $w(t) \equiv 1$. Here, we assume $f(x)$ is unknown and can be parameterized as $f(x) = [x_2, \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_2^3]^{\mathrm{T}}$, where $\theta_1, \theta_2$, and $\theta_3$ are unknown constants. Furthermore, we assume that $\operatorname{sgn} b$ is known. Next, let $F(x) = [x_1, x_2, x_2^3]^{\mathrm{T}}$ and $K_{\mathrm{g}} = \frac{1}{b}[\theta_{\mathrm{n}_1} - \theta_1, \theta_{\mathrm{n}_2} - \theta_2, -\theta_3]$, where $\theta_{\mathrm{n}_1}$, $\theta_{\mathrm{n}_2}$ are arbitrary scalars, so that $f_{\mathrm{s}}(x)$ is given by (2.65). Now, with the proper choice of $\theta_{\mathrm{n}_1}$ and $\theta_{\mathrm{n}_2}$, it follows from Corollary 2.1 that the adaptive feedback controller (2.26) guarantees that $x(t) \to 0$ as $t \to \infty$. Specifically, here we choose $\theta_{\mathrm{n}_1} = -1$, $\theta_{\mathrm{n}_2} = -2$, and $R = 2I_2$, so that $P$ satisfying (2.21) is given by (2.66). With $\varepsilon = 1$, $\alpha = \frac{1}{3}$, $\beta = 1$, $b = 3$, $\hat{d} = 3$, $Y = I_3$, $Z = 1$, and initial conditions $x(0) = [1, 1]^{\mathrm{T}}$, $K(0) = [0, 0, 0]$, and $\Phi(0) = 0$, Figure 2.7 shows the phase portrait of the controlled and uncontrolled system. Note that the adaptive controller is switched on at $t = 15$ sec. Figure 2.8 shows the state trajectories versus time and the control signal versus time. Finally, Figure 2.9 shows the adaptive gain history versus time.

**Example 2.4.** The following example considers the utility of the proposed adaptive stabilization framework for systems with time-varying disturbances. Specifically, consider the uncertain controlled Duffing system given by

$$m\ddot{z}(t) + c\dot{z}(t) + kz(t) + ka^2 z^3(t) = bu(t) + A\cos\omega t, \quad z(0) = z_0, \quad \dot{z}(0) = \dot{z}_0, \quad t \geq 0, \tag{2.69}$$

where $m, c, k, a, b$, and $A$ are unknown. Note that with $x_1 = z$ and $x_2 = \dot{z}$, (2.69) can be written in state space form (2.1) with $x = [x_1, x_2]^{\mathrm{T}}$, $f(x) = [x_2, -\frac{k}{m}x_1 - \frac{c}{m}x_2 - \frac{ka^2}{m}x_1^3]^{\mathrm{T}}$, $G(x) = [0, \frac{b}{m}]^{\mathrm{T}}$, $J(x) = [0, \frac{1}{m}A]^{\mathrm{T}}$, and $w(t) = \cos\omega t$. Here, we assume that $\omega$ and $\operatorname{sgn} b$ are known and $f(x)$ can be parameterized as $f(x) = [x_2, \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_1^3]^{\mathrm{T}}$, where $\theta_1, \theta_2$, and $\theta_3$ are unknown constants. Next, let $F(x) = [x_1, x_2, x_1^3]^{\mathrm{T}}$ and $K_{\mathrm{g}} = \frac{m}{b}[\theta_{\mathrm{n}_1} - \theta_1, \theta_{\mathrm{n}_2} - \theta_2, -\theta_3]$, where $\theta_{\mathrm{n}_1}$ and $\theta_{\mathrm{n}_2}$ are arbitrary scalars, so that $f_{\mathrm{s}}(x)$ is given by (2.65). Now, with the proper choice of $\theta_{\mathrm{n}_1}$ and $\theta_{\mathrm{n}_2}$, it follows from

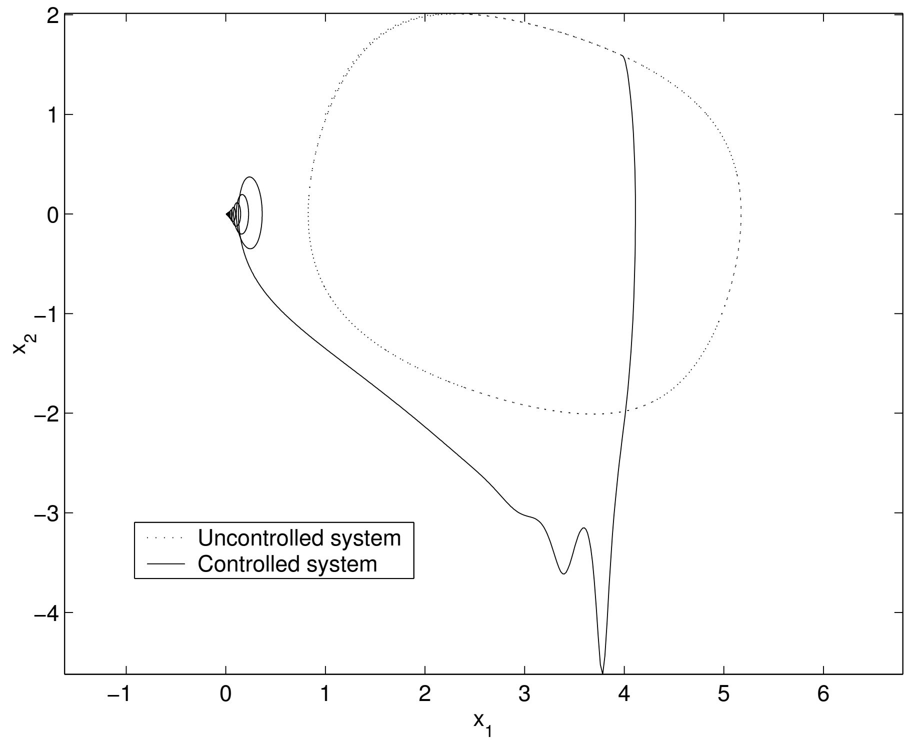



**Figure 2.7**: Phase portrait of controlled and uncontrolled Rayleigh system

Corollary 2.1 that the adaptive feedback controller (2.26) guarantees that $x(t) \to 0$ as $t \to \infty$. Specifically, here we choose $\theta_{\mathrm{n}_1} = -1$, $\theta_{\mathrm{n}_2} = -2$, and $R = 2I_2$, so that $P$ satisfying (2.21) is given by (2.66). With $m = 1, c = 0.1$, $k = 2$, $a = 1$, $A = 4$, $\omega = 1$, $b = 3$, $Y = I_3$, $Z = 1$, and initial conditions $x(0) = [1, 1]^{\mathrm{T}}$, $K(0) = [0, 0, 0]$, and $\Phi(0) = 0$, Figure 2.10 shows the phase portrait of the controlled and uncontrolled system. Once again, the adaptive controller is switched on at $t = 15$ sec. Figure 2.11 shows the state trajectories versus time and the control signal versus time. Finally, Figure 2.12 shows the adaptive gain history versus time.

**Example 2.5.** The following example considers the utility of the proposed adaptive stabilization framework for systems with time-varying dynamics. Specifically, consider the uncertain controlled Mathieu system given by

$$\ddot{z}(t) + \mu(1 + 2\varepsilon \cos 2t)z(t) = bu(t), \quad z(0) = z_0, \quad \dot{z}(0) = \dot{z}_0, \quad t \geq 0, \tag{2.70}$$

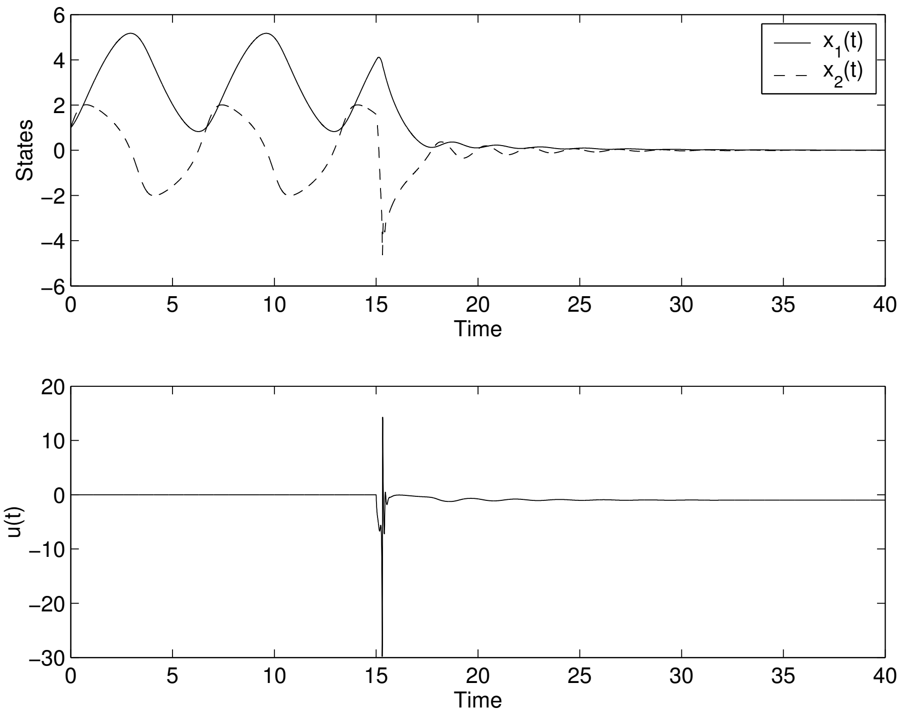



**Figure 2.8**: State trajectories and control signal versus time

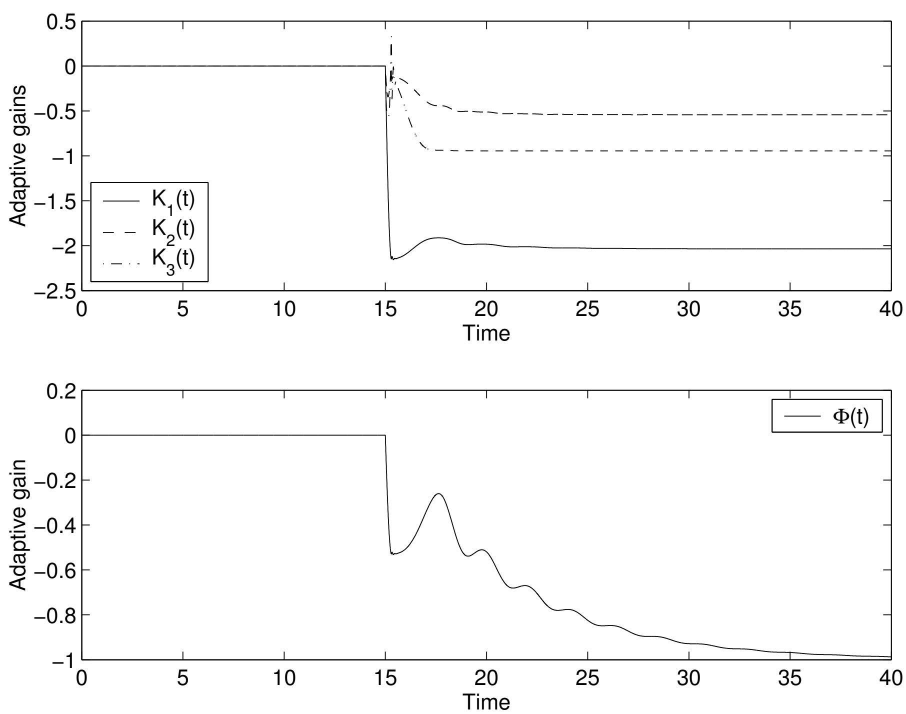



**Figure 2.9**: Adaptive gain history versus time

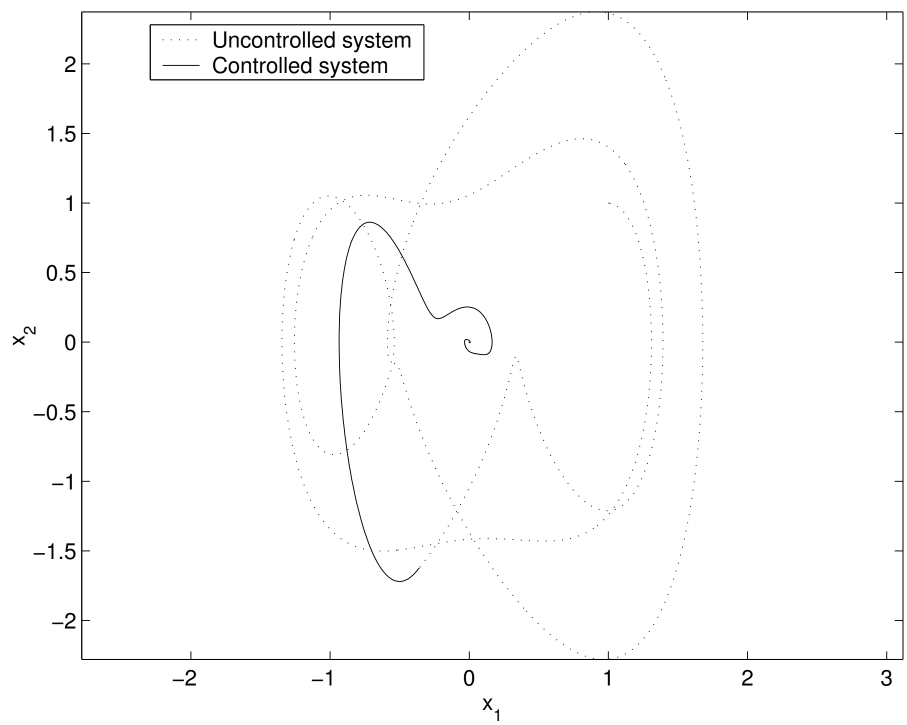



**Figure 2.10**: Phase portrait of controlled and uncontrolled Duffing system

where $\mu, \varepsilon, b \in \mathbb{R}$ are unknown. Note that with $x_1 = z$ and $x_2 = \dot{z}$, (2.70) can be written in state space form (2.11) with $x = [x_1, x_2]^{\mathrm{T}}$, $f(t,x) = [x_2, -\mu(1+2\varepsilon\cos 2t)x_1]^{\mathrm{T}}$, and $G(t,x) = [0, b]^{\mathrm{T}}$. Here, we assume that $\operatorname{sgn} b$ is known and $f(t,x)$ can be parameterized as $f(t,x) = [x_2, \theta_1 x_1 + \theta_2 \cos(2t)x_1]^{\mathrm{T}}$, where $\theta_1$ and $\theta_2$ are unknown constants. Next, let $F(t,x) = [x_1, \cos(2t)x_1, x_2,]^{\mathrm{T}}$ and $K_{\mathrm{g}} = \frac{1}{b}\left[\theta_{\mathrm{n}_1} - \theta_1, -\theta_2, \phi_{\mathrm{n}}\right]$, where $\theta_{\mathrm{n}_1}$ and $\phi_{\mathrm{n}}$ are arbitrary scalars, so that

$$f_{\mathrm{s}}(x) = \begin{bmatrix} 0 & 1 \\ \theta_{\mathrm{n}_1} & \phi_{\mathrm{n}} \end{bmatrix} x.$$

Now, with the proper choice of $\theta_{\mathrm{n}_1}$ and $\phi_{\mathrm{n}}$, it follows from Corollary 2.1 and Remark 2.2 that the adaptive feedback controller (2.26) with $w(t) \equiv 0$ guarantees that $x(t) \to 0$ as $t \to \infty$. Specifically, here we choose $\theta_{\mathrm{n}_1} = -1$, $\phi_{\mathrm{n}} = -2$, and $R = 2I_2$, so that $P$ satisfying (2.21) is given by (2.66). With $\mu = 1$, $\varepsilon = 0.4$, $b = 3$, $Y = I_3$, and initial conditions $x(0) = [1, 1]^{\mathrm{T}}$ and $K(0) = [0, 0, 0]$, Figure 2.13 shows the phase portrait of the controlled and uncontrolled system. Note that the adaptive controller is switched

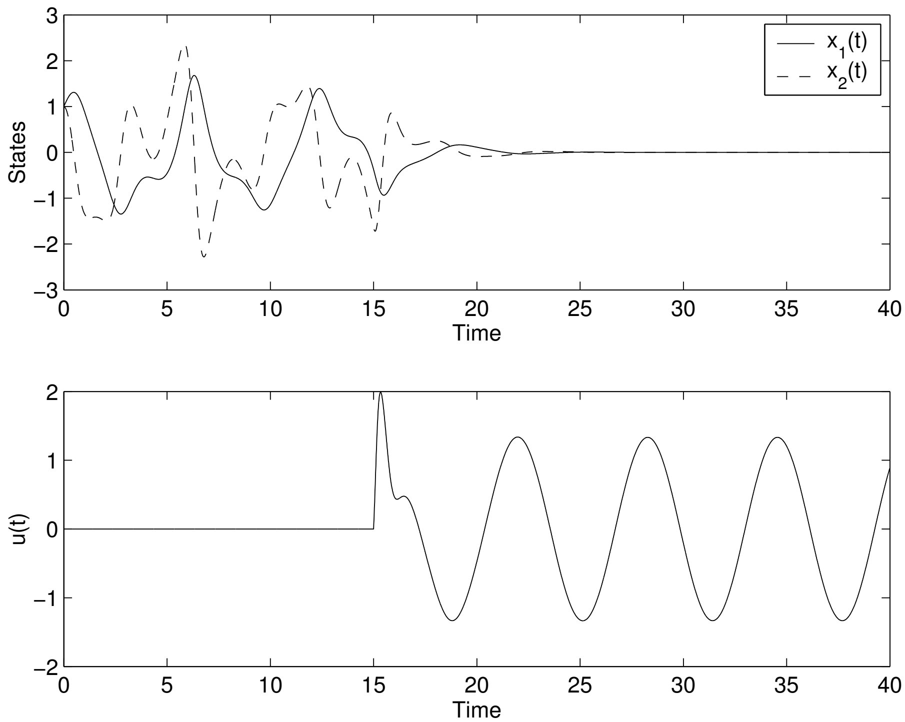



**Figure 2.11**: State trajectories and control signal versus time

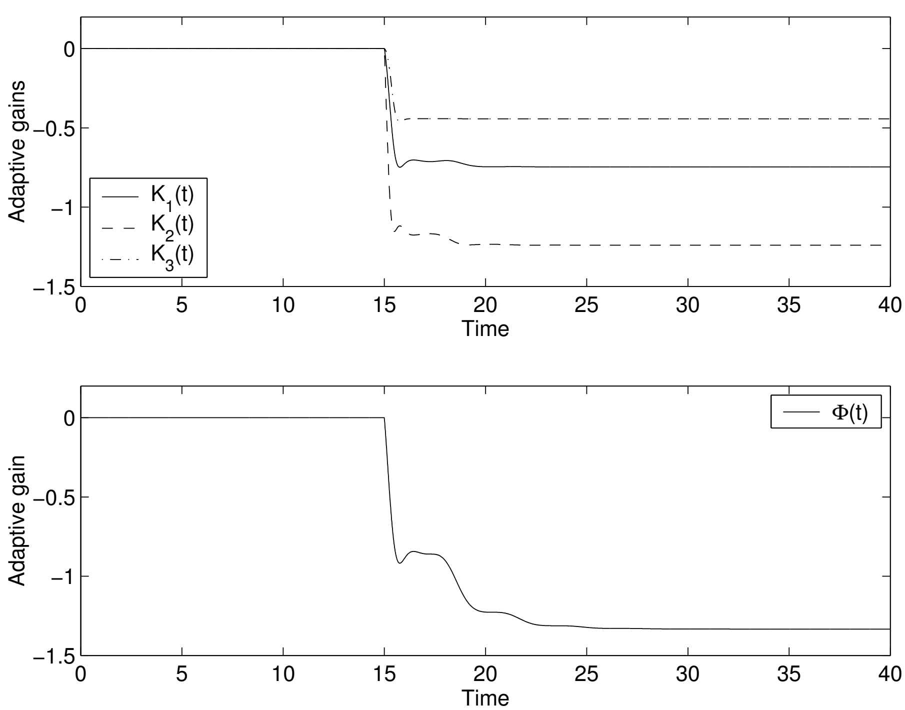



**Figure 2.12**: Adaptive gain history versus time

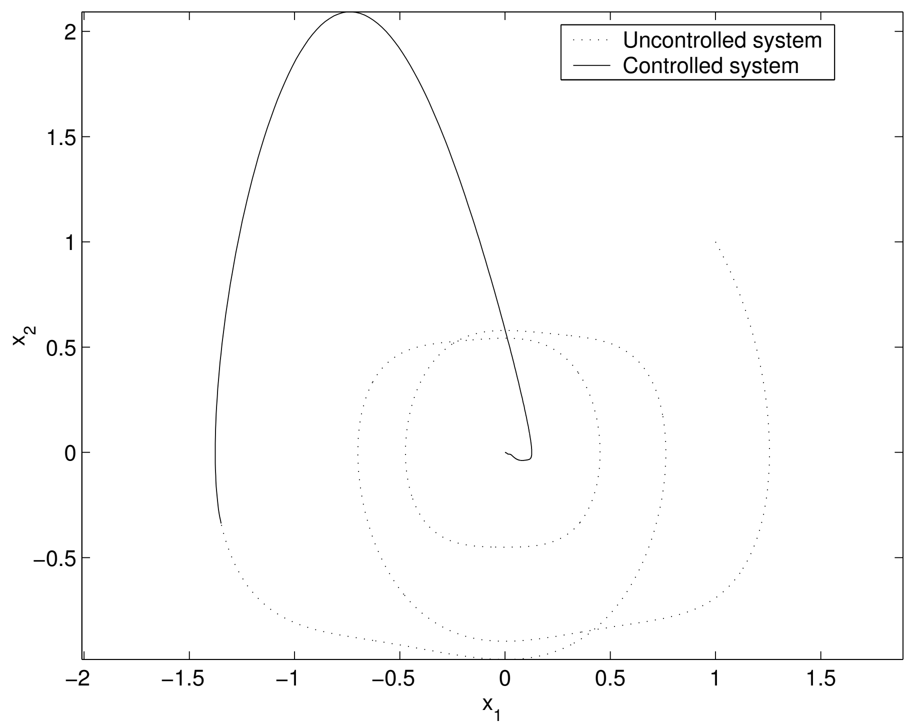



**Figure 2.13**: Phase portrait of controlled and uncontrolled Mathieu system

on at $t = 15$ sec. Figure 2.14 shows the state trajectories versus time and the control signal versus time. Finally, Figure 2.15 shows the adaptive gain history versus time.

**Example 2.6.** The following example considers the utility of the proposed adaptive control framework for command following. Specifically, consider the spring-mass-damper uncertain system with nonlinear stiffness given by

$$m\ddot{x}(t) + c\dot{x}(t) + k_1 x(t) + k_2 x^3(t) = bu(t) + \hat{d}w(t), \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \quad t \geq 0, \tag{2.71}$$

where $m, c, k_1, k_2 \in \mathbb{R}$ are positive unknown constants, and $b$ is unknown but $\operatorname{sgn} b$ is known. Let $r_{\rm d}(t), t \geq 0$, be a desired command signal and define the error state $\tilde{e}(t) \triangleq x(t) - r_{\rm d}(t)$ so that the error dynamics are given by

$$m\ddot{\tilde{e}}(t) + c\dot{\tilde{e}}(t) + (k_1 + k_2(\tilde{e}^2(t) + 3r_{\rm d}(t)\tilde{e}(t) + 3r_{\rm d}^2(t)))\tilde{e}(t) = bu(t) + \hat{d}w(t)$$
$$-(m\ddot{r}_{\rm d}(t) + c\dot{r}_{\rm d}(t) + k_1 r_{\rm d}(t) + k_2 r_{\rm d}^3(t)), \quad \tilde{e}(0) = \tilde{e}_0, \quad \dot{\tilde{e}}(0) = \dot{\tilde{e}}_0, \quad t \geq 0. \tag{2.72}$$

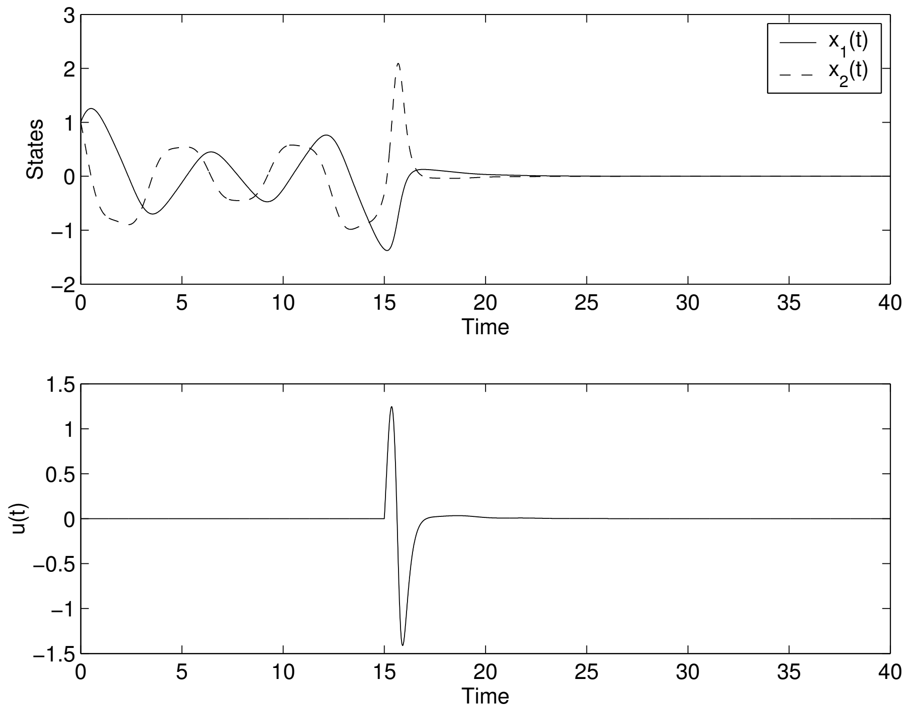



**Figure 2.14**: State trajectories and control signal versus time

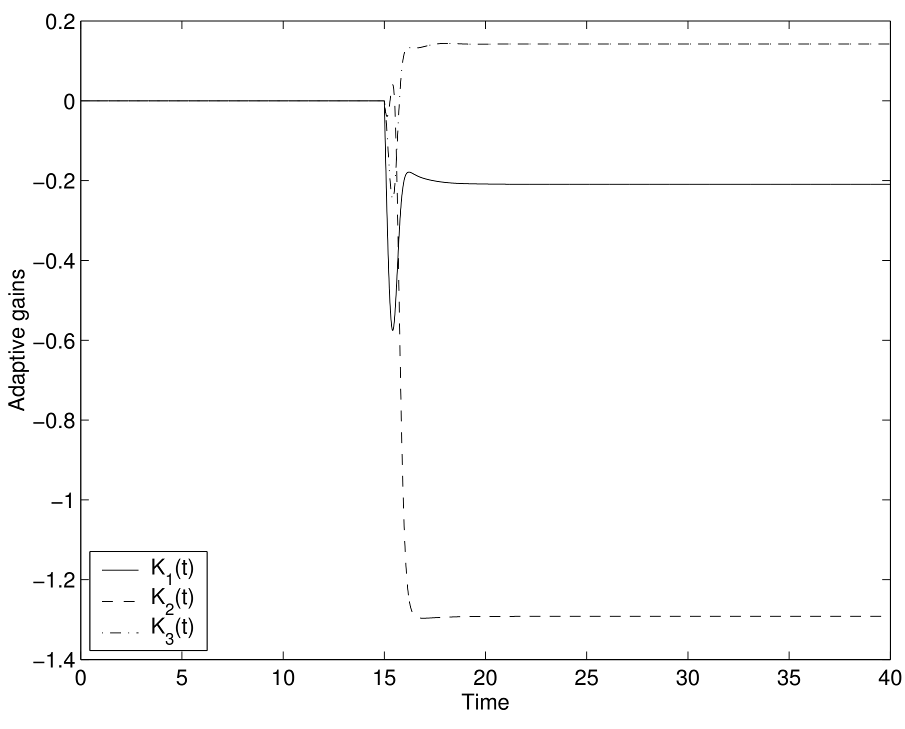



**Figure 2.15**: Adaptive gain history versus time

Here, we assume that the disturbance signal $w(t)$ is a sinusoidal signal with unknown amplitude and phase; that is, $\hat{d}w(t) = \sqrt{A_1^2 + A_2^2}\sin(\omega t + \phi) = A_1 \sin\omega t + A_2 \cos\omega t$, where $\phi = \tan^{-1}(A_2/A_1)$ and $A_1$ and $A_2$ are unknown constants. Furthermore, the desired trajectory is given by

$$r_\mathrm{d}(t) = \tanh\left(\frac{t-20}{5}\right),$$

so that the position of the mass is moved from $-1$ to $1$ at $t = 20$ sec. Note that with $e_1 = \tilde{e}$ and $e_2 = \dot{\tilde{e}}$, (2.71) can be written in state space form (2.15) with $e = [e_1, e_2]^\mathrm{T}$, $f_\mathrm{t}(r_\mathrm{d}, e) = [e_2, -\frac{1}{m}(k_1 + k_2(e_1^2 + 3r_\mathrm{d}e_1 + 3r_\mathrm{d}^2))e_1 - \frac{c}{m}e_2]^\mathrm{T}$, $G(t,e) = [0, \frac{b}{m}]^\mathrm{T}$, $J_\mathrm{t}(t,e) = \frac{1}{m}\left[0_{6\times1}, \hat{d}_\mathrm{t}^\mathrm{T}\right]^\mathrm{T}$, where $\hat{d}_\mathrm{t} = [A_1, A_2, -k_1, -k_2, -c, -m]$, and $w_\mathrm{t}(t) = [\sin\omega t, \cos\omega t, r_\mathrm{d}(t), r_\mathrm{d}^3(t), \dot{r}_\mathrm{d}(t), \ddot{r}_\mathrm{d}(t)]^\mathrm{T}$. Here, we parameterize $f_\mathrm{t}(r_\mathrm{d}, e) = [e_2, \theta_1 e_1 + \theta_2 e_2 + \theta_3 e_1^3 + \theta_4 r_\mathrm{d} e_1^2 + \theta_5 r_\mathrm{d}^2 e_1]^\mathrm{T}$, where $\theta_i$, $i = 1, \cdots, 5$, are unknown constants. Next, let $F(r_\mathrm{d}, e) = [e_1, e_2, e_1^3, r_\mathrm{d}e_1^2, r_\mathrm{d}^2 e_1]^\mathrm{T}$ and $K_\mathrm{g} = \frac{m}{b}\left[\theta_{\mathrm{n}_1} - \theta_1, \theta_{\mathrm{n}_2} - \theta_2, -\theta_3, -\theta_4, -\theta_5\right]$, where $\theta_{\mathrm{n}_1}$, $\theta_{\mathrm{n}_2}$ are arbitrary scalars, so that $f_\mathrm{s}(e)$ is given by (2.65). Now, with the proper choice of $\theta_{\mathrm{n}_1}$ and $\theta_{\mathrm{n}_2}$, it follows from Corollary 2.1 and Remark 2.3 that the adaptive feedback controller (2.26) guarantees that $e(t) \to 0$ as $t \to \infty$. Specifically, here we choose $\theta_{\mathrm{n}_1} = -1$, $\theta_{\mathrm{n}_2} = -2$, and $R = 2I_2$, so that $P$ satisfying (2.21) is given by (2.66). With $m = 1$, $c = 1$, $k_1 = 2$, $k_2 = 0.5$, $\hat{d}w(t) = 2\sin(\omega t + 1)$, $\omega = 2$, $b = 3$, $Y = I_5$, $Z = I_6$, and initial conditions $e(0) = [0, 0]^\mathrm{T}$, $K(0) = 0_{1\times5}$, and $\Phi(0) = 0_{1\times6}$, Figure 2.16 shows the actual position and the reference signal versus time and the control signal versus time. Finally, Figure 2.17 shows the adaptive gain history versus time.

**Example 2.7.** Consider the nonlinear dynamic equations for a single-link manipulator with flexible joints and negligible damping coupled through a gear train to

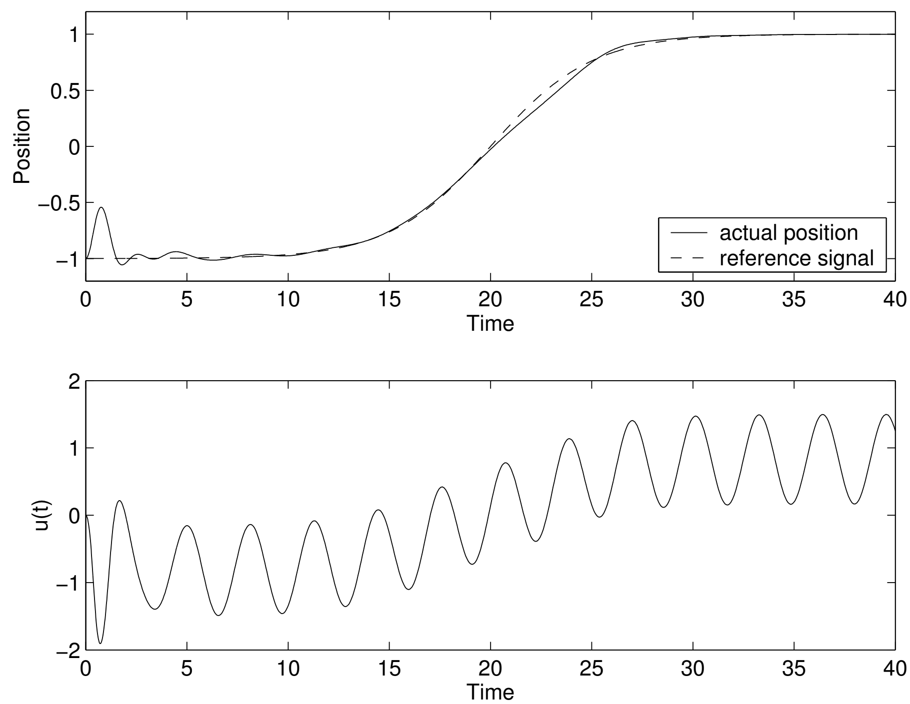



**Figure 2.16**: Position and control signal versus time

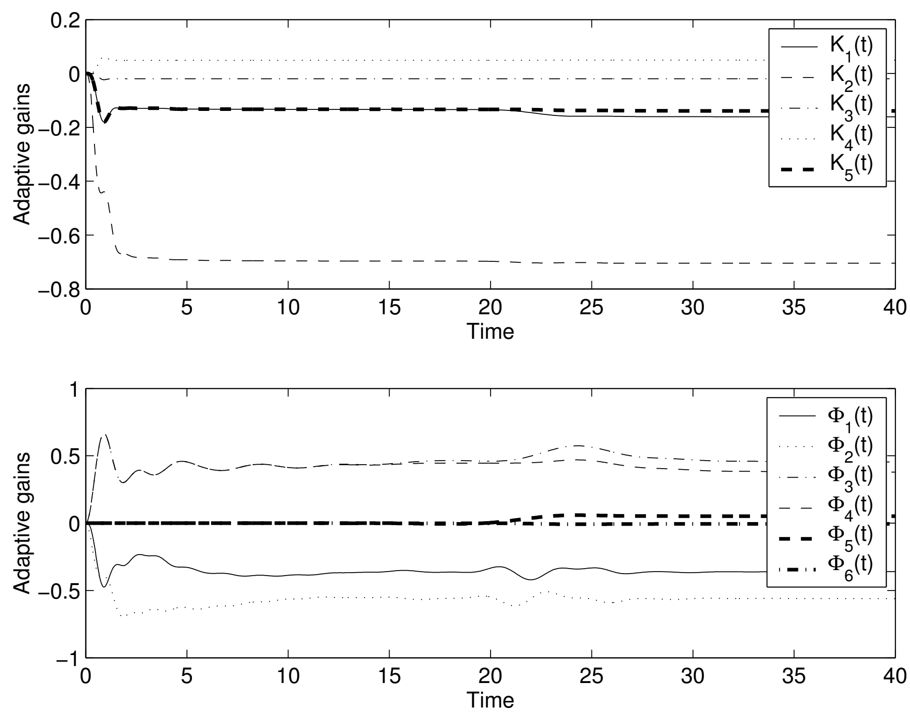



**Figure 2.17**: Adaptive gain history versus time

a DC-motor given by ([225])

$$I_1\ddot{q}_1(t) + MgL\sin q_1(t) + k(q_1(t) - q_2(t)) = 0, \quad q_1(0) = q_{10}, \quad \dot{q}_1(0) = \dot{q}_{10}, \quad t \geq 0, \tag{2.73}$$

$$I_2\ddot{q}_2(t) - k(q_1(t) - q_2(t)) = u(t), \quad q_2(0) = q_{20}, \quad \dot{q}_2(0) = \dot{q}_{20}, \tag{2.74}$$

where $q_1$ and $q_2$ are angular positions, $I_1$ and $I_2$ are mass moments of inertia of the link and the motor, respectively, $k$ is a spring constant, $M$ is the total mass of the link, $L$ is the distance from the joint axis to the link center of mass, $g$ is the gravitational constant, and $u$ is a control torque input. Defining the state variables

$$x_1(t) \triangleq q_1(t), \quad x_2(t) \triangleq \dot{q}_1(t), \quad x_3(t) \triangleq -\frac{MgL}{I_1}\sin q_1(t) - \frac{k}{I_1}(q_1(t) - q_2(t)),$$
$$x_4(t) \triangleq -\frac{MgL}{I_1}\dot{q}_1(t)\cos q_1(t) - \frac{k}{I_1}(\dot{q}_1(t) - \dot{q}_2(t)),$$

(2.73), (2.74) can be written in the form of (2.1) with $x = [x_1, x_2, x_3, x_4]^{\mathrm{T}}$, $G(x) = [0_{1\times 3}, \beta\delta]^{\mathrm{T}}$, $w(t) \equiv 0$, and

$$f(x) = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ -(\alpha\cos x_1 + \beta + \gamma)x_3 + \alpha(x_2^2 - \gamma)\sin x_1 \end{bmatrix},$$

where $\alpha \triangleq \frac{MgL}{I_1}, \beta \triangleq \frac{k}{I_1}, \gamma \triangleq \frac{k}{I_2}, \delta \triangleq \frac{1}{I_2}$. Here, we assume that $\alpha, \beta, \gamma$, and $\delta$ are unknown positive constants. Furthermore, assume that the angular position $q_1(t)$ is required to track the angle $r_{\mathrm{d}}(t) = \sin t$. Next, define the error states $e_i(t) \triangleq \frac{\mathrm{d}^{i-1}}{\mathrm{d}t^{i-1}}(x_1(t) - r_{\mathrm{d}}(t)), i = 1, \cdots, 4$, so that the error dynamics can be written in the form (2.15) with $e \triangleq [e_1, e_2, e_3, e_4]^{\mathrm{T}}$, $f_{\mathrm{t}}(r_{\mathrm{d}}, e) = [e_2, e_3, e_4, f_{\mathrm{tu}}(r_{\mathrm{d}}, \dot{r}_{\mathrm{d}}, \ddot{r}_{\mathrm{d}}, e)]^{\mathrm{T}}$, where

$$f_{\mathrm{tu}}(r_{\mathrm{d}}, \dot{r}_{\mathrm{d}}, \ddot{r}_{\mathrm{d}}, e) = -(\alpha\cos(e_1 + r_{\mathrm{d}}) + \beta + \gamma)e_3 - \alpha\ddot{r}_{\mathrm{d}}(\cos(e_1 + r_{\mathrm{d}}) - \cos r_{\mathrm{d}})$$
$$+\alpha[((e_2 + \dot{r}_{\mathrm{d}})^2 - \gamma)\sin(e_1 + r_{\mathrm{d}}) - (\dot{r}_{\mathrm{d}}^2 - \gamma)\sin r_{\mathrm{d}}],$$

$G(t, e) = [0_{1\times 3}, \beta\delta]$, $J_{\mathrm{t}}(t, e) = [0_{5\times 3}, \hat{d}_{\mathrm{t}}^{\mathrm{T}}]^{\mathrm{T}}$, where $\hat{d}_{\mathrm{t}} = [-1, -(\beta + \gamma), \alpha, -\alpha\gamma, -\alpha]$, and $w_{\mathrm{t}}(t) = [r_{\mathrm{d}}^{(4)}(t),\ \ddot{r}_{\mathrm{d}}(t), \dot{r}_{\mathrm{d}}^2(t)\sin r_{\mathrm{d}}(t), \sin r_{\mathrm{d}}(t), \ddot{r}_{\mathrm{d}}(t)\cos r_{\mathrm{d}}(t)]^{\mathrm{T}}$. Here, we parameterize

$f_{\rm tu}(r_{\rm d}, \dot{r}_{\rm d}, \ddot{r}_{\rm d}, e)$ as

$$f_{\rm tu}(r_{\rm d}, \dot{r}_{\rm d}, \ddot{r}_{\rm d}, e) = \big[\ \theta_1 e_3 + \theta_2 e_3 \cos(e_1 + r_{\rm d}) + \theta_3(\cos(e_1 + r_{\rm d})\ddot{r}_{\rm d} - \cos r_{\rm d}\ddot{r}_{\rm d}) \\ +\theta_4((e_2 + \dot{r}_{\rm d})^2 \sin(e_1 + r_{\rm d}) - \dot{r}_{\rm d}^2 \sin r_{\rm d}) + \theta_5(\sin(e_1 + r_{\rm d}) - \sin r_{\rm d})\ \big]^{\rm T},$$

where, $\theta_i$, $i = 1, \cdots, 5$, are unknown constants. Next, let

$$F(r_{\rm d}, \dot{r}_{\rm d}, \ddot{r}_{\rm d}, e) = \big[\ e_3,\quad e_3 \cos(e_1 + r_{\rm d}),\quad \cos(e_1 + r_{\rm d})\ddot{r}_{\rm d} - \cos r_{\rm d}\ddot{r}_{\rm d}, \\ (e_2 + \dot{r}_{\rm d})^2 \sin(e_1 + r_{\rm d}) - \dot{r}_{\rm d}^2 \sin r_{\rm d},\quad \sin(e_1 + r_{\rm d}) - \sin r_{\rm d},\quad e_1,\quad e_2,\quad e_4\ \big]^{\rm T}$$

and $K_{\rm g} = [\theta_{{\rm n}_1} - \theta_1, -\theta_2, -\theta_3, -\theta_4, -\theta_5, \phi_{{\rm n}_1}, \phi_{{\rm n}_2}, \phi_{{\rm n}_3}]$, where $\theta_{{\rm n}_1}, \phi_{{\rm n}_1}, \phi_{{\rm n}_2}$, and $\phi_{{\rm n}_3}$ are arbitrary scalars, so that $f_{\rm s}(e)$ is given by

$$f_{\rm s}(e) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \phi_{{\rm n}_1} & \phi_{{\rm n}_2} & \theta_{{\rm n}_1} & \phi_{{\rm n}_3} \end{bmatrix} e.$$

Now, with the proper choice of $\theta_{{\rm n}_1}, \phi_{{\rm n}_1}, \phi_{{\rm n}_2}$, and $\phi_{{\rm n}_3}$, it follows from Corollary 2.1 and Remark 2.3 that the adaptive feedback controller (2.26) guarantees that $e(t) \to 0$ as $t \to \infty$. Specifically, here we choose $\theta_{{\rm n}_1} = -24$, $\phi_{{\rm n}_1} = -16$, $\phi_{{\rm n}_2} = -32$, $\phi_{{\rm n}_3} = -8$, and $R = I_4$, so that $P$ satisfying (2.21) is given by

$$P = \begin{bmatrix} 3.0156 & 3.5312 & 1.5977 & 0.0312 \\ 3.5312 & 6.9883 & 3.6719 & 0.1260 \\ 1.5977 & 3.6719 & 3.2861 & 0.1738 \\ 0.0312 & 0.1260 & 0.1738 & 0.0842 \end{bmatrix}.$$

With $\alpha = 10$, $\beta = 2$, $\gamma = 4$, $\delta = 1$, $Y = 20I_8$, $Z = 20I_5$, and initial conditions $e(0) = 0_{4\times1}$, $K(0) = 0_{1\times8}$, and $\Phi(0) = 0_{1\times5}$, Figure 2.18 shows the actual position $q_1(t)$ and the reference signal versus time. Figure 2.19 shows the error signals and the control signal versus time. Finally, Figure 2.20 shows the adaptive gain history versus time.

**Example 2.8.** Consider the two-degree-of-freedom uncertain structural system given by

$$M_{\rm s}\ddot{x}(t) + C_{\rm s}\dot{x}(t) + K_{\rm s}x(t) = u(t), \quad x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \quad t \geq 0, \tag{2.75}$$

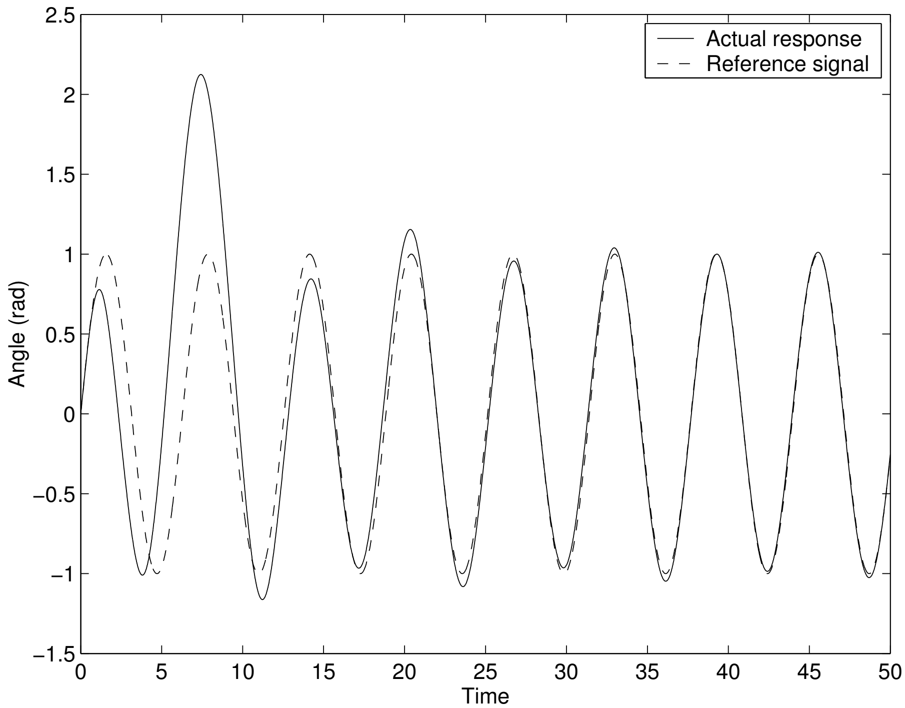



**Figure 2.18**: State trajectories versus time

where $x(t) \in \mathbb{R}^2$, $u(t) \in \mathbb{R}^2$, $t \geq 0$,

$$M_{\rm s} \triangleq \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad C_{\rm s} \triangleq \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}, \quad K_{\rm s} \triangleq \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix},$$

and $m_1, m_2, c_1, c_2, k_1, k_2 \in \mathbb{R}$ are unknown constants such that $m_1, m_2 > 0$. Let $r_{\rm d}(t)$ be a desired command signal and define the error state $\tilde{e}(t) \triangleq x(t) - r_{\rm d}(t)$ so that the error dynamics are given by

$$M_{\rm s}\ddot{\tilde{e}}(t) + C_{\rm s}\dot{\tilde{e}}(t) + K_{\rm s}\tilde{e}(t) = u(t) - M_{\rm s}\ddot{r}_{\rm d}(t) - C_{\rm s}\dot{r}_{\rm d}(t) - K_{\rm s}r_{\rm d}(t),$$
$$\tilde{e}(0) = \tilde{e}_0, \quad \dot{\tilde{e}}(0) = \dot{\tilde{e}}_0, \quad t \geq 0. \quad (2.76)$$

Note that with $e_1 = \tilde{e}$ and $e_2 = \dot{\tilde{e}}$, (2.76) can be written in state space form (2.15) with $e = [e_1^{\rm T}, e_2^{\rm T}]^{\rm T}$, $f_{\rm t}(t,e) = [e_2^{\rm T}, -(M_{\rm s}^{-1}K_{\rm s}e_1 + M_{\rm s}^{-1}C_{\rm s}e_2)^{\rm T}]^{\rm T}$, $G(t,e) = [0_{2\times 2}, M_{\rm s}^{-1}]^{\rm T}$, $J_{\rm t}(t,e) = [0_{6\times 2}, \hat{D}_{\rm t}^{\rm T}]^{\rm T}$, where $\hat{D}_{\rm t} = [-I_2, -M_{\rm s}^{-1}C_{\rm s}, -M_{\rm s}^{-1}K_{\rm s}]$, and $w_{\rm t}(t) = [\ddot{r}_{\rm d}^{\rm T}(t), \dot{r}_{\rm d}^{\rm T}(t), r_{\rm d}^{\rm T}(t)]^{\rm T}$. Note that $M_{\rm s}^{-1}$ is symmetric and positive definite but otherwise unknown. Here, we parameterize $f_{\rm t}(t,e)$ as $f_{\rm t}(t,e) = [e_2^{\rm T}, (\Theta_1 e_1 + \Theta_2 e_2)^{\rm T}]^{\rm T}$, where

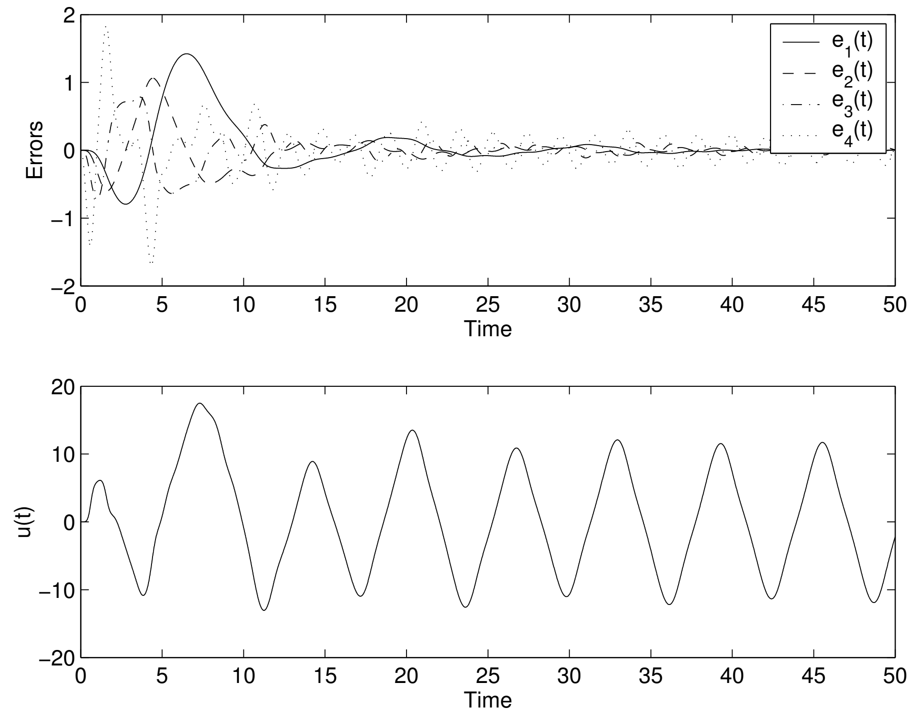



**Figure 2.19**: Error states and control signal versus time

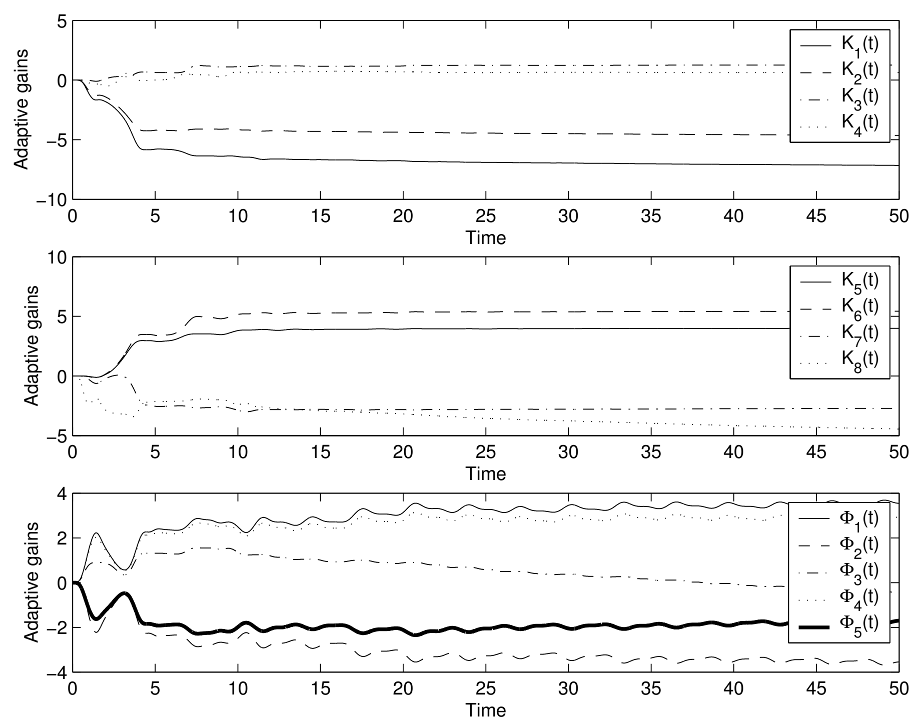



**Figure 2.20**: Adaptive gain history versus time

$\Theta_1 \in \mathbb{R}^{2\times 2}$ and $\Theta_2 \in \mathbb{R}^{2\times 2}$ are unknown constant matrices. Next, let $F(t,e) = e$ and $K_{\rm g} = M_{\rm s}\left[\Theta_{\rm n_1} + M_{\rm s}^{-1}K_{\rm s}, \Theta_{\rm n_2} + M_{\rm s}^{-1}C_{\rm s}\right]$, where $\Theta_{\rm n_1} \in \mathbb{R}^{2\times 2}$, $\Theta_{\rm n_2} \in \mathbb{R}^{2\times 2}$ are arbitrary matrices, so that

$$f_{\rm s}(e) = \begin{bmatrix} 0_2 & I_2 \\ \Theta_{\rm n_1} & \Theta_{\rm n_2} \end{bmatrix} e.$$

Now, with the proper choice of $\Theta_{\rm n_1}$ and $\Theta_{\rm n_2}$, it follows from Corollary 2.1 and Remark 2.3 that the adaptive feedback controller (2.26) guarantees that $e(t) \to 0$ as $t \to \infty$. Specifically, here we choose $\Theta_{\rm n_1} = -I_2$, $\Theta_{\rm n_2} = -I_2$, and $R = 2I_4$, so that $P$ satisfying (2.21) is given by

$$P = \begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

With $m_1 = 3$, $m_2 = 2$, $c_1 = c_2 = 1$, $k_1 = 2$, $k_2 = 1$, $r_{\rm d}(t) = [5\cos(t), 3\cos(\frac{t}{\pi})]^{\rm T}$, $Y = I_4$, $Z = I_6$, and initial conditions $x(0) = 0_{4\times 1}$, $K(0) = 0_{2\times 4}$, and $\Phi(0) = 0_{2\times 6}$, Figure 2.21 shows the actual positions and the reference signals versus time and the control signals versus time. Finally, Figures 2.22 and 2.23 show the adaptive gain history versus time.

**Example 2.9.** The following example considers the utility of the proposed adaptive control framework for $\mathrm{L}_2$ disturbance rejection. Specifically, consider the nonlinear dynamical system representing a controlled rigid spacecraft given by

$$\dot{x}(t) = -I_{\rm b}^{-1}XI_{\rm b}x(t) + I_{\rm b}^{-1}u(t) + Dw(t), \quad x(0) = x_0, \quad w(\cdot) \in \mathrm{L}_2, \quad t \geq 0, \quad (2.77)$$

where $x = [x_1, x_2, x_3]^{\rm T}$ represents the angular velocities of the spacecraft with respect to the body-fixed frame, $I_{\rm b} \in \mathbb{R}^{3\times 3}$ is an unknown positive-definite inertia matrix of the spacecraft, $u = [u_1, u_2, u_3]^{\rm T}$ is a control vector with control inputs providing body-fixed torques about three mutually perpendicular axes defining the body-fixed

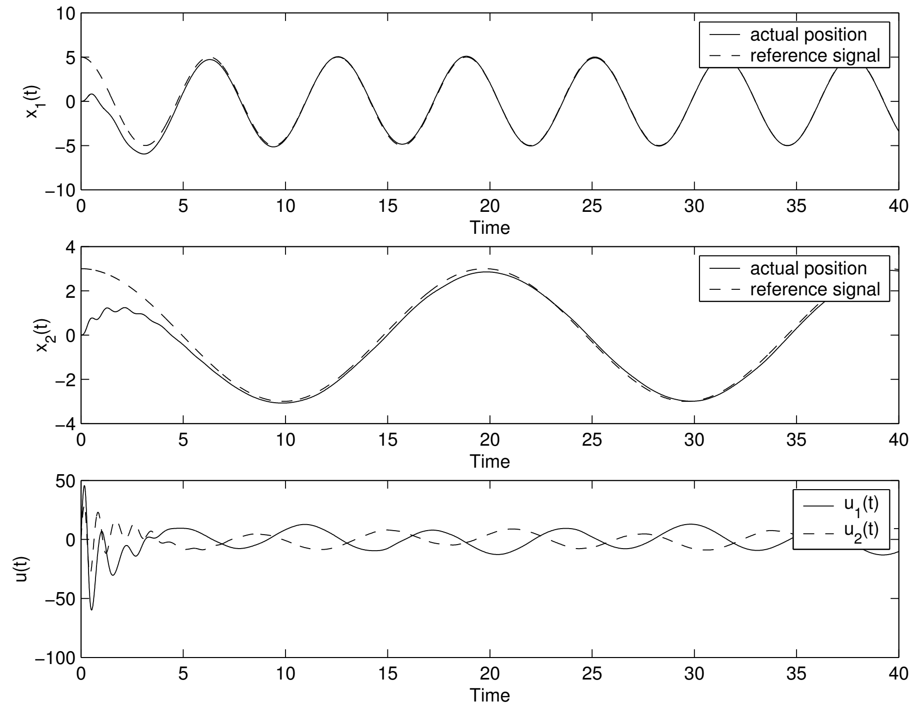



**Figure 2.21**: Positions and control signals versus time

frame of the spacecraft, $D \in \mathbb{R}^{3\times 1}$, and $X$ denotes the skew-symmetric matrix

$$X \triangleq \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

Note that (2.77) can be written in state space form (2.45) with $f(x) = -I_{\rm b}^{-1} X I_{\rm b} x$, $G(x) = I_{\rm b}^{-1}$, and $J(x) = D$. Here, we assume that the inertia matrix $I_{\rm b}$ of the spacecraft is symmetric and positive definite but unknown. Since $f(x)$ is a quadratic function, we parameterize $f(x)$ as $f(x) = \Theta f_{\rm n}(x)$, where $\Theta \in \mathbb{R}^{3\times 6}$ is an unknown matrix and $f_{\rm n}(x) = [x_1^2, x_2^2, x_3^2, x_1x_2, x_2x_3,\ x_3x_1]^{\rm T}$. Next, let $F(x) = \left[f_{\rm n}^{\rm T}(x), x^{\rm T}\right]^{\rm T}$ and $K_{\rm g} = I_{\rm b}\left[-\Theta, \Phi_{\rm n}\right]$, where $\Phi_{\rm n} \in \mathbb{R}^{3\times 3}$, is an arbitrary matrix, so that

$$f_{\rm s}(x) = \Phi_{\rm n} x = A_{\rm s} x.$$

Now, with the proper choice of $\Phi_{\rm n}$, it follows from Theorem 2.2 that the adaptive feedback controller (2.62) with update law (2.63) guarantees that $x(t) \to 0$ as $t \to \infty$ with $w(t) \equiv 0$. Furthermore, the closed-loop nonlinear input-output map from $\mathrm{L}_2$

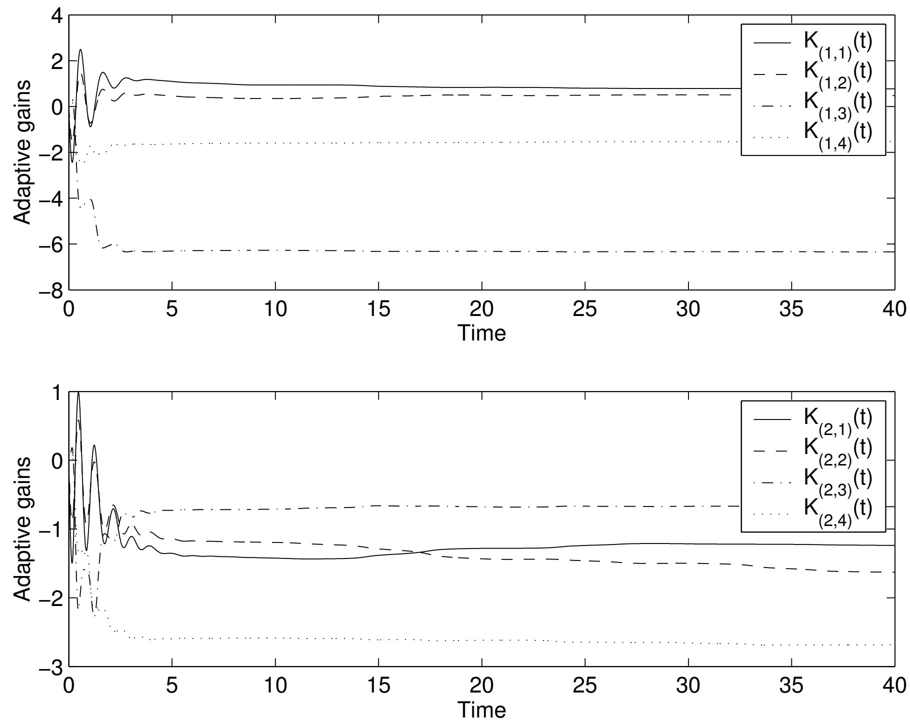



**Figure 2.22**: Adaptive gain history versus time

disturbances $Dw(t)$ to performance variable $z(t) = Ex(t)$ satisfies the nonexpansivity constraint (2.51). Here, we choose $A_{\rm s} = -10I_3$, $E^{\rm T}E = 2I_3$, and $\gamma = 1.4$, so that $P$ satisfying (2.59) is given by

$$P = \begin{bmatrix} 0.1653 & 0.0408 & 0.0245 \\ 0.0408 & 0.1255 & 0.0153 \\ 0.0245 & 0.0153 & 0.1092 \end{bmatrix}.$$

With

$$I_{\rm b} = \begin{bmatrix} 20 & 0 & 0.9 \\ 0 & 17 & 0 \\ 0.9 & 0 & 15 \end{bmatrix}, \quad Y = 10I_9, \quad D = \begin{bmatrix} 8 \\ 5 \\ 3 \end{bmatrix}, \quad w(t) = e^{-0.2t}\sin 1.8t,$$

and initial conditions $x(0) = [0.4, 0.2, -0.2]$, and $K(0) = 0_{3\times 9}$, Figure 2.24 shows the angular velocities versus time. Figure 2.25 shows the control signals versus time. An alternative adaptive feedback controller that also does not require knowledge of the inertia of the spacecraft is presented in [3]. However, unlike the proposed controller, the adaptive controller presented in [3] is tailored to the spacecraft attitude control problem.

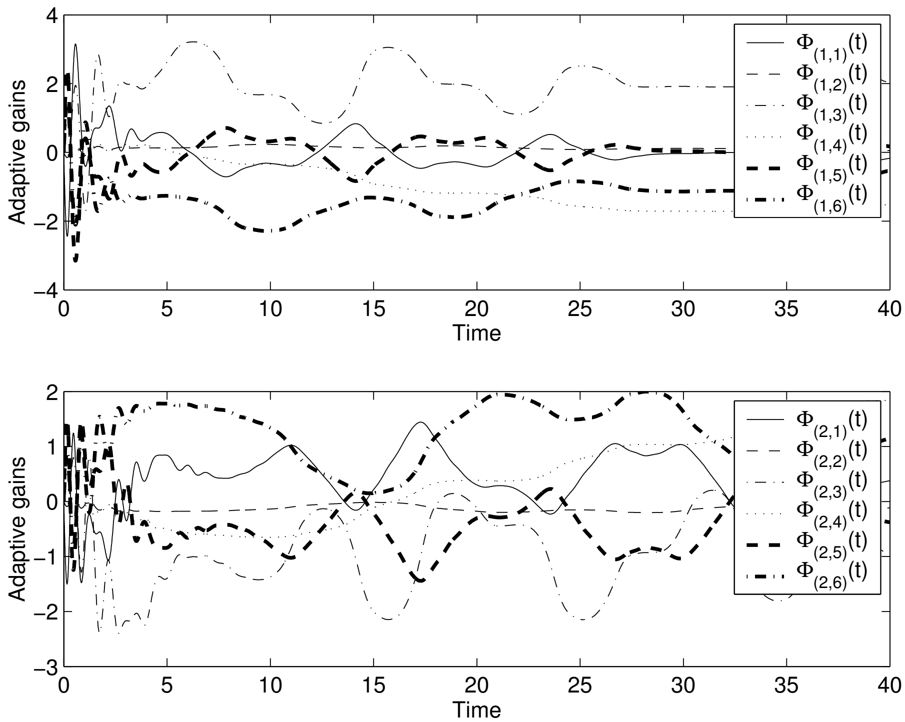



**Figure 2.23**: Adaptive gain history versus time

## 2.6. Adaptive Control for Thermoacoustic Combustion Instabilities

High performance aeroengine afterburners and ramjets often experience combustion instabilities at some operating condition. Combustion in these high energy density engines is highly susceptible to flow disturbances, resulting in fluctuations to the instantaneous rate of heat release in the combustor. This unsteady combustion provides an acoustic source resulting in self-excited oscillations. In particular, unsteady combustion generates acoustic pressure and velocity oscillations which in turn perturb the combustion even further [34, 49]. These pressure oscillations, known as thermoacoustic instabilities, often lead to high vibration levels causing mechanical failures, high levels of acoustic noise, high burn rates, and even component melting. Hence, the need for active control to mitigate combustion induced pressure instabilities is severe.

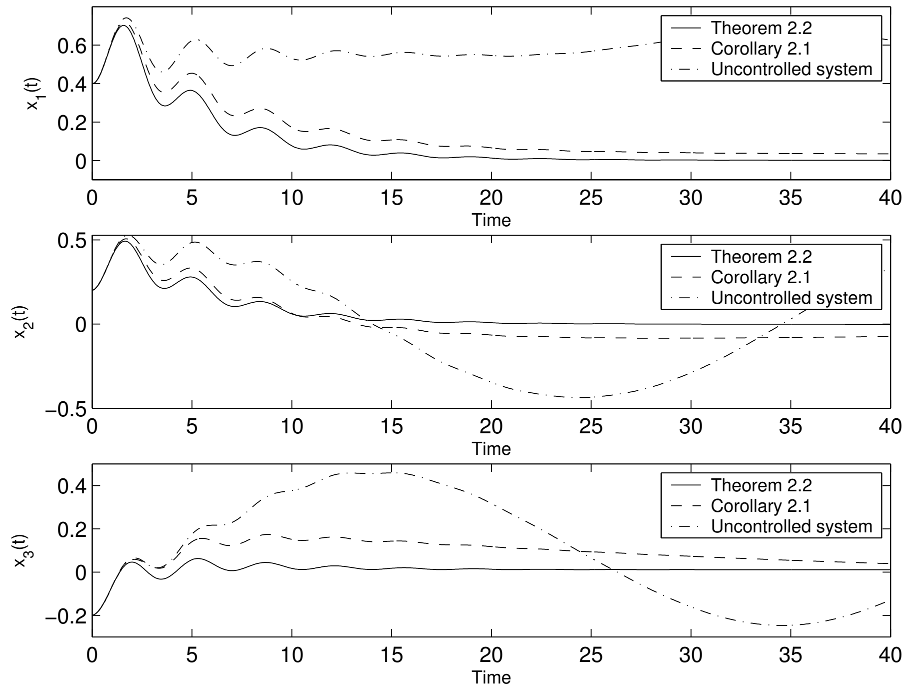



**Figure 2.24**: Angular velocities versus time

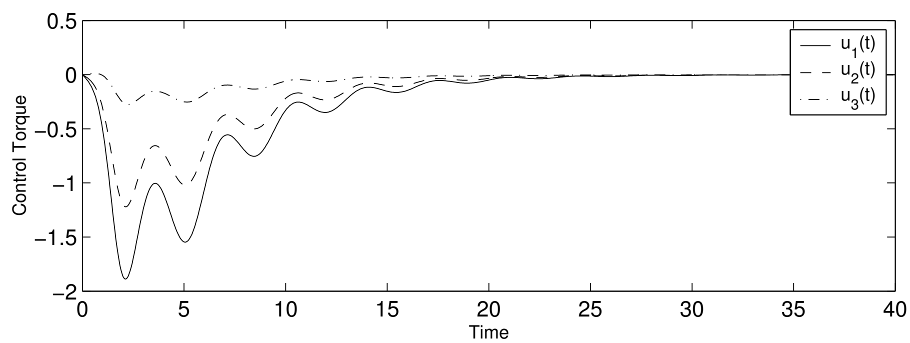



**Figure 2.25**: Control signals versus time

Due to the intricate complex physical phenomena in combustion processes involving acoustics, thermodynamics, fluid mechanics, and chemical kinetics, finite dimensional linear or nonlinear models are unavoidably inaccurate. Basic system data such as damping, frequency, and mode shapes are often poorly known. Furthermore, approximations of pressure and velocity fluctuations involving time averaging in the governing system equations result in further system uncertainty that manifests itself as highly structured constant real parametric uncertainty in modal frequencies and damping [49, 50]. Thus for pressure oscillation suppression in combustion processes, system modeling uncertainty necessitates the need for nonlinear adaptive control.

In this section we apply the Lyapunov-based direct adaptive control framework developed in Section 2.2 to suppress the effects of thermoacoustic instabilities in combustion processes. The overall framework demonstrates that the proposed adaptive controllers provide considerable robustness in suppressing thermoacoustic combustion instabilities in the presence of parametric uncertainties in the model.

In order to develop a state space model for combustion processes that capture the coupling between unsteady combustion and acoustics, we consider the mass, momentum, and energy conservation equations for a two phase mixture in a combustor. Specifically, the conservation equations are given by [50]

$$\frac{\partial \rho}{\partial t} + v_{\rm g} \cdot \nabla p = \mathcal{W}, \tag{2.78}$$

$$\rho \frac{\partial v_{\rm g}}{\partial t} + \rho v_{\rm g} \cdot \nabla v_{\rm g} + \nabla p = \mathcal{F}, \tag{2.79}$$

$$\frac{\partial p}{\partial t} + \gamma p \nabla \cdot v_{\rm g} + v_{\rm g} \cdot \nabla p = \mathcal{P}, \tag{2.80}$$

where $\rho$ is the local density of the mixture, $v_{\rm g}$ is the local velocity of the gas phase, $p$ is the local pressure, $\gamma$ is the mixture ratio of specific heats, $\mathcal{W}$ represents the mass conversion rate of condensed phases to gases per unit volume, $\mathcal{F}$ is the force interaction between the gas and condensed phases, $\mathcal{P}$ is the sum of heat release associated with chemical reactions and energy transfer between the gas-liquid phase, $\nabla$ denotes the nabla operator, and "$\cdot$" denotes the dot product in $\mathbb{R}^n$. In this formulation we assume that droplets are dispersed in the gas, which implies that, if $p_{\rm l}$ and $p_{\rm g}$ are the local pressures of the liquid and gas phase, respectively, $p_{\rm l} \ll p_{\rm g}$, $p \cong p_{\rm g}$ and hence [50]

$$p = \rho \bar{R} T_{\rm g}, \tag{2.81}$$

where $\bar{R}$ is the gas constant for the mixture and $T_{\rm g}$ is the temperature of the gas.

The framework for analyzing combustion instabilities is based on the conservation equations (2.78)–(2.80) for total mass, momentum, and energy, with the energy equation written with the pressure as the dependent variable. Writing all dependent variables as sums of mean $\bar{(\cdot)}$ and fluctuating $(\cdot)'$ parts given by

$$p(r_1, r_2, r_3, t) = \bar{p} + p'(r_1, r_2, r_3, t), \tag{2.82}$$

$$\rho(r_1, r_2, r_3, t) = \bar{\rho}(r_1, r_2, r_3) + \rho'(r_1, r_2, r_3, t), \tag{2.83}$$

$$v_{\rm g}(r_1, r_2, r_3, t) = \bar{v}_{\rm g}(r_1, r_2, r_3) + v'_{\rm g}(r_1, r_2, r_3, t), \tag{2.84}$$

$$T_{\rm g}(r_1, r_2, r_3, t) = \bar{T}_{\rm g}(r_1, r_2, r_3) + T'_{\rm g}(r_1, r_2, r_3, t), \tag{2.85}$$

where $(r_1, r_2, r_3)$ represent generalized coordinates, and assuming that the average values $\bar{p}$, $\bar{\rho}$, $\bar{v}_{\rm g}$, $\bar{T}_{\rm g}$ do not vary with time and the average pressure $\bar{p}$ is uniform inside the combustion chamber, a second-order approximation of (2.78)–(2.81) yields

$$\nabla^2 p' - \frac{1}{\bar{a}^2}\frac{\partial^2 p'}{\partial t^2} = \varphi, \qquad \hat{n}\cdot\nabla p' = -\vartheta, \tag{2.86}$$

where $\bar{a} \triangleq \sqrt{\gamma\frac{\bar{p}}{\bar{\rho}}}$ is the local average sound velocity inside the combustor, $\hat{n}$ is the outward normal vector of the combustor chamber surface, and $\varphi$ and $\vartheta$ are nonlinear terms containing all physical processes of acoustic motions, mean flow, and combustion under conditions with no external forcing [50].

To control combustion instabilities appropriate external forces are needed to influence the unsteady mass, momentum, and energy in the combustion chamber. Hence, control forces are included in the conservation equations by modifying the nonhomogeneous terms of (2.78)–(2.80) to include control input terms of the form $\mathcal{W}_{\rm c}$, $\mathcal{F}_{\rm c}$, and $\mathcal{P}_{\rm c}$, respectively. The specific forms of $\mathcal{W}_{\rm c}$, $\mathcal{F}_{\rm c}$, and $\mathcal{P}_{\rm c}$ depend on the type of control actuation used. In this case, (2.86) becomes

$$\nabla^2 p'(r_1, r_2, r_3, t) - \frac{1}{\bar{a}^2}\frac{\partial^2 p'(r_1, r_2, r_3, t)}{\partial t^2} = \varphi(r_1, r_2, r_3, t) + \varphi_{\rm c}(r_1, r_2, r_3, t), \tag{2.87}$$

$$\hat{n}\cdot\nabla p'(r_1, r_2, r_3, t) = -\vartheta(r_1, r_2, r_3, t) - \vartheta_{\rm c}(r_1, r_2, r_3, t), \tag{2.88}$$

where

$$\varphi_{\rm c} \triangleq \nabla \cdot \mathcal{F}_{\rm c}' - \frac{1}{\bar{a}^2}\frac{\partial \mathcal{P}_{\rm c}'}{\partial t}, \quad \vartheta_{\rm c} \triangleq -\mathcal{F}_{\rm c}' \cdot \hat{n}, \tag{2.89}$$

represent external inputs due to control actuation. Since the input terms in (2.87), (2.88) are treated as small perturbations to the acoustic field, the solution for the unsteady pressure field $p'(r_1, r_2, r_3, t)$ can be approximated by

$$p'(r_1, r_2, r_3, t) = \bar{p} \sum_{i,j,k=0}^{\infty} \eta_{ijk}(t)\psi_{ijk}(r_1, r_2, r_3), \tag{2.90}$$

where $\psi_{ijk}$ are the normal modes of the system forming a complete set of orthogonal basis functions satisfying

$$0 = \nabla^2\psi_{ijk}(r_1, r_2, r_3) + k_{ijk}^2\psi_{ijk}(r_1, r_2, r_3), \tag{2.91}$$

$$0 = \hat{n} \cdot \nabla\psi_{ijk}(r_1, r_2, r_3), \quad i, j, k = 0, 1, 2, \ldots, \tag{2.92}$$

where $k_{ijk}$, $i, j, k = 0, 1, 2, \ldots$, are the wave numbers defined by $k_{ijk} \triangleq \frac{\omega_{ijk}}{\bar{a}}$, and $\omega_{ijk}$, $i, j, k = 0, 1, 2, \ldots$, are the natural frequencies. Now, using a Galerkin decomposition it follows from (2.87), (2.90), and (2.91) that

$$\ddot{\eta}_{ijk} + \omega_{ijk}^2\eta_{ijk} = F_{ijk} + u_{ijk}, \tag{2.93}$$

where

$$F_{ijk} \triangleq -\frac{\bar{a}^2}{\bar{p}E_{ijk}^2}\left(\int_{\mathcal{V}_{\rm m}} \varphi\psi_{ijk}{\rm d}\mathcal{V} + \oint_{\mathcal{S}_{\rm m}} \vartheta\psi_{ijk}{\rm d}\mathcal{S}\right), \tag{2.94}$$

$$u_{ijk} \triangleq -\frac{\bar{a}^2}{\bar{p}E_{ijk}^2}\left(\int_{\mathcal{V}_{\rm m}} \varphi_{\rm c}\psi_{ijk}{\rm d}\mathcal{V} + \oint_{\mathcal{S}_{\rm m}} \vartheta_{\rm c}\psi_{ijk}{\rm d}\mathcal{S}\right), \tag{2.95}$$

$$E_{ijk}^2 \triangleq \int_{\mathcal{V}_{\rm m}} \psi_{ijk}^2{\rm d}\mathcal{V}, \tag{2.96}$$

where $\mathcal{V}_{\rm m}$ is any arbitrary material volume within the continuum, $\mathcal{S}_{\rm m}$ is the surface that encloses $\mathcal{V}_{\rm m}$, and ${\rm d}\mathcal{V}$ and ${\rm d}\mathcal{S}$ are the infinitesimal volume and surface elements, respectively.

Next, considering $m$ point acoustic drivers (actuators) providing control excitation $\hat{u}_a(t)$ at positions $(r_{a1}, r_{a2}, r_{a3})$, $a = 1, \ldots, m$, where we assume mass and momentum are *not* controlled, i.e., $\mathcal{W}'_{\rm c} = 0$ and $\mathcal{F}'_{\rm c} = 0$, we obtain

$$\varphi_{\rm c}(r_1, r_2, r_3, t) = -\sum_{a=1}^{m} \delta_a(r_1, r_2, r_3)\hat{u}_a(t), \tag{2.97}$$

$$\vartheta_{\rm c}(r_1, r_2, r_3, t) = 0, \tag{2.98}$$

where $\delta_a$, $a = 1, \ldots, m$, is the spatial delta function concentrated at $(r_1, r_2, r_3) = (r_{a1}, r_{a2}, r_{a3})$, $a = 1, \ldots, m$, with dimensions $(\text{length})^{-3}$. Using (2.97) and (2.98), (2.95) becomes

$$u_{ijk}(t) = \frac{\bar{a}^2}{\bar{p}E_{ijk}^2} \sum_{a=1}^{m} \hat{u}_a(t)\psi_{ijk}(r_{a1}, r_{a2}, r_{a3}). \tag{2.99}$$

Finally, using a one-dimensional combustor model whose geometry is such that the longitudinal modes are decoupled from the transverse modes, it follows that the index $i$ is the only index in the triple $i, j, k$ that applies. Furthermore, we substitute $x$ for the generalized coordinates $(r_1, r_2, r_3)$ so that $\mathcal{V} = \int_0^L \mathcal{A}_{\rm c}(x){\rm d}x$, where $\mathcal{A}_{\rm c}(x)$ represents the cross sectional area of the combustor and $L$ is the combustor length. In this case, (2.93) becomes

$$\ddot{\eta}_i(t) + \omega_i^2\eta_i(t) + \sum_{p=1}^{\infty}(d_{ip}\dot{\eta}_p(t) + e_{ip}\eta_p(t)) + \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}(a_{ipq}\dot{\eta}_p(t)\dot{\eta}_q(t) + b_{ipq}\eta_p(t)\eta_q(t)) = u_i(t), \tag{2.100}$$

where the constants $d_{ip}$, $e_{ip}$, $a_{ipq}$, and $b_{ipq}$ depend on the unperturbed mode shapes and natural frequencies of the combustor [50], and the control input to the $i^{\rm th}$ mode is given by

$$u_i(t) = \frac{\bar{a}^2}{\bar{p}E_i^2} \sum_{a=1}^{m} \hat{u}_a(t)\psi_i(x_{{\rm s}a}), \tag{2.101}$$

where $E_i^2 = \int_0^L \psi_i(x)\mathcal{A}_{\rm c}(x){\rm d}x$ and $x_{{\rm s}a}$ corresponds to the location of the $a^{\rm th}$ actuator.

To design a direct adaptive controller for combustion systems we concentrate on the nonlinear combustion model developed above, with nonlinearities present due to the second-order gas dynamics. Furthermore, we assume that actuation is provided by loud speakers while we measure pressure fluctuations via pressure-type microphones. Now, using (2.100) and (2.101), a two-mode, nonlinear combustion plant model is given by

$$\dot{x}_1(t) = x_3(t), \quad x_1(0) = x_{10}, \quad t \geq 0, \tag{2.102}$$

$$\dot{x}_2(t) = x_4(t), \quad x_2(0) = x_{20}, \tag{2.103}$$

$$\begin{aligned}\dot{x}_3(t) = {} & 2\alpha_1 x_3(t) - (\omega_1^2 - 2\theta_1\omega_1)x_1(t) - F_{11}x_3(t)x_4(t) - F_{12}x_1(t)x_2(t) \\ & + \tfrac{\bar{a}^2}{\bar{p}E_i^2}(\psi_1(x_{s1})\hat{u}_1(t) + \psi_1(x_{s2})\hat{u}_2(t)), \quad x_3(0) = x_{30},\end{aligned} \tag{2.104}$$

$$\begin{aligned}\dot{x}_4(t) = {} & 2\alpha_2 x_4(t) - (\omega_2^2 - 2\theta_2\omega_2)x_2(t) - F_{21}x_3^2(t) - F_{22}x_1^2(t) \\ & + \tfrac{\bar{a}^2}{\bar{p}E_i^2}(\psi_2(x_{s1})\hat{u}_1(t) + \psi_2(x_{s2})\hat{u}_2(t)), \quad x_4(0) = x_{40},\end{aligned} \tag{2.105}$$

where $x_1(t) = \eta_1(t)$, $x_2(t) = \eta_2(t)$, $x_3(t) = \dot{\eta}_1(t)$, $x_4(t) = \dot{\eta}_2(t)$, $\hat{u}_i(t)$, $i = 1, 2$, are control input signals, $\alpha_i = -\frac{1}{2}d_{ii} \in \mathbb{R}$ represents a growth/decay constant, $\theta_i = -\frac{1}{2}\frac{e_{ii}}{\omega_i} \in \mathbb{R}$ represents a frequency shift constant, $\omega_1$ and $\omega_2$ are the frequencies of the first and second modes, respectively, $F_{11} = \frac{3-2\gamma}{2\gamma}$, $F_{12} = \frac{5(\gamma-1)}{2\gamma}\omega_1^2$, $F_{21} = -\frac{\gamma+3}{2\gamma}$, and $F_{22} = \frac{\gamma-1}{2\gamma}\omega_1^2$. In the case where we consider a cylindrical combustor closed at both ends with pure longitudinal modes, it follows that the first two modes are given by

$$\psi_i(x) = \cos(k_i x), \quad k_i = i\frac{\pi}{L}, \quad i = 1, 2. \tag{2.106}$$

For details of this formulation see [190]. For the nondimensionalized (using the time factor $\tau_t = \pi L/\bar{a}$) data parameters [63] $\alpha_1 = 0.0144$, $\alpha_2 = -0.0559$, $\theta_1 = 0.0062$, $\theta_2 = 0.0178$, $\gamma = 1.2$, $\omega_1 = 1$, $\omega_2 = 2$, and $x_0 = [0.01,\ 0.1,\ 0,\ 0]^{\mathrm{T}}$, the open-loop ($\hat{u}_i(t) \equiv 0$, $i = 1, 2$) dynamics (2.102)–(2.105) result in a limit cycle instability. Figure 2.26 shows the state response versus time of the open-loop system.

To design a direct adaptive controller using Corollary 2.1, note that (2.102)–

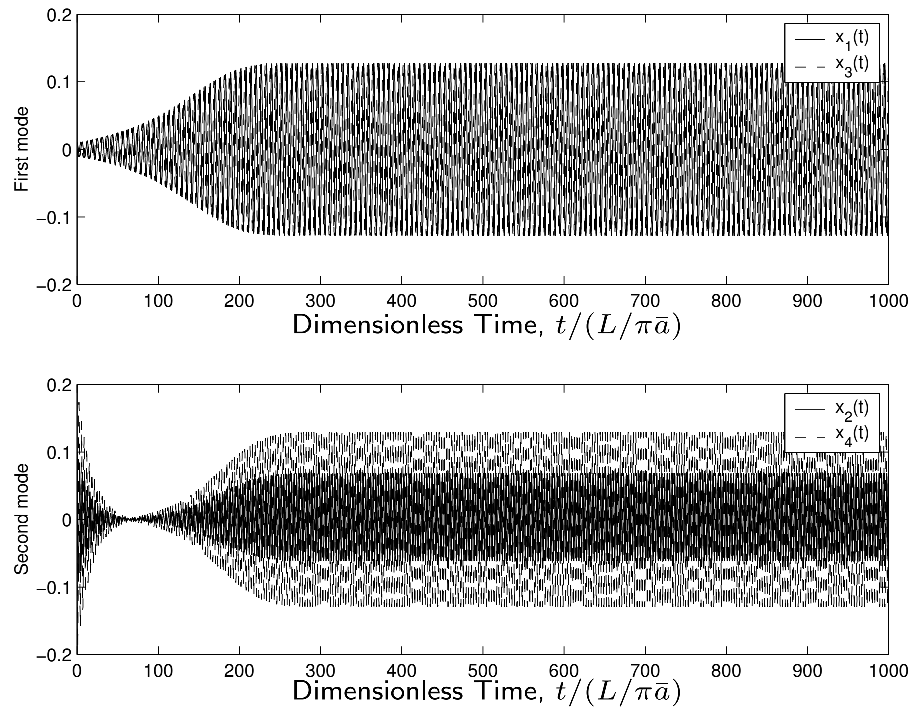



**Figure 2.26**: Open-loop state response versus time

(2.105) can be written in state space form (2.1) with $x = [x_1, x_2, x_3, x_4]^{\mathrm{T}}$,

$$f(x) = \begin{bmatrix} x_3 \\ x_4 \\ 2\alpha_1 x_3 - (\omega_1^2 - 2\theta_1\omega_1)x_1 - F_{11}x_3x_4 - F_{12}x_1x_2 \\ 2\alpha_2 x_4 - (\omega_2^2 - 2\theta_2\omega_2)x_2 - F_{21}x_3^2 - F_{22}x_1^2 \end{bmatrix},$$

$$G(x) = \frac{\bar{a}^2}{\bar{p}E_i^2} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \psi_1(x_{\mathrm{s}1}) & \psi_1(x_{\mathrm{s}2}) \\ \psi_2(x_{\mathrm{s}1}) & \psi_2(x_{\mathrm{s}2}) \end{bmatrix},$$

where $\alpha_i$, $\theta_i$, $\omega_i$, $F_{ij}$, $i, j = 1, 2$, and $\frac{\bar{a}^2}{\bar{p}E_i^2}(> 0)$ are assumed to be unknown. Here, we parameterize $f(x)$ as $f(x) = [x_3,\ x_4,\ (\Theta_\ell x + \Theta_{n\ell} f_{n\ell}(x))^{\mathrm{T}}]^{\mathrm{T}}$, where $f_{n\ell}(x) = [x_1x_2, x_3x_4, x_1^2, x_3^2]^{\mathrm{T}}$ and

$$\Theta_\ell = \begin{bmatrix} -(\omega_1^2 - 2\theta_1\omega_1) & 0 & 2\alpha_1 & 0 \\ 0 & -(\omega_2^2 - 2\theta_2\omega_2) & 0 & 2\alpha_2 \end{bmatrix},$$

$$\Theta_{n\ell} = \begin{bmatrix} -F_{12} & -F_{11} & 0 & 0 \\ 0 & 0 & -F_{22} & -F_{21} \end{bmatrix},$$

are unknown constant matrices. Furthermore, we assume that loud speakers are placed at $x_{s1} = \frac{3}{4}L$ and $x_{s2} = \frac{1}{2}L$ so that $B_u$ and $G_n(x)$ in $G(x) = [0_2,\ G_n^T(x)B_u]^T$ are given by

$$B_u = -\frac{\bar{a}^2}{\bar{p}E_i^2}\begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & 1 \end{bmatrix}, \quad G_n(x) = I_2.$$

Next, let $F(x) = [x^T,\ f_{n\ell}^T(x)]^T$ and $K_g = -\left(\frac{\bar{a}^2}{\bar{p}E_i^2}\right)^{-1}[\Theta_n - \Theta_\ell, -\Theta_{n\ell}]$, where $\Theta_n \in \mathbb{R}^{2\times 4}$ is an arbitrary matrix, so that

$$f_s(x) = \begin{bmatrix} A_0 \\ \Theta_n \end{bmatrix} x = A_s x,$$

where $A_0 \triangleq [0_2,\ I_2]$. Now, with the proper choice of $\Theta_n$ it follows from Corollary 2.1 that the adaptive feedback controller (2.26) with update law (2.27) guarantees that $x(t) \to 0$ as $t \to \infty$. Specifically, here we choose

$$\Theta_n = \begin{bmatrix} -1.5 & 0 & -2.5 & 0 \\ 0 & -5 & 0 & -4.5 \end{bmatrix},$$

and $R = 2I_4$, so that $P$ satisfying (2.21) is given by

$$P = \begin{bmatrix} 2.6667 & 0 & 0.6667 & 0 \\ 0 & 2.2333 & 0 & 0.2000 \\ 0.6667 & 0 & 0.6667 & 0 \\ 0 & 0.2000 & 0 & 0.2667 \end{bmatrix}.$$

To illustrate the dynamic behavior of the closed-loop system, let $\alpha_1 = 0.0144$, $\alpha_2 = -0.0559$, $\theta_1 = 0.0062$, $\theta_2 = 0.0178$, $\gamma = 1.2$, $\omega_1 = 1$, $\omega_2 = 2$, $Y = 2I_8$, and $\frac{\bar{a}^2}{\bar{p}E_i^2} = 0.4$. The response of the controlled system (2.1) with the adaptive feedback control law (2.26), (2.27) and initial conditions $x_0 = [0.01,\ 0.1,\ 0,\ 0]^T$, $K(0) = 0_{2\times 8}$ is shown in Figure 2.27. Stability of the closed-loop system (2.1), (2.26), (2.27) is guaranteed by Corollary 2.1. Note that the adaptive controller is switched on at $t = 300$.

To illustrate the robustness of the proposed adaptive control law, we switch the growth constant of the first mode from $\alpha_1 = 0.0144$ to $\alpha_1 = 0.0720$ at $t = 600$. The closed-loop response is shown in Figure 2.28. Figure 2.29 shows the same change in

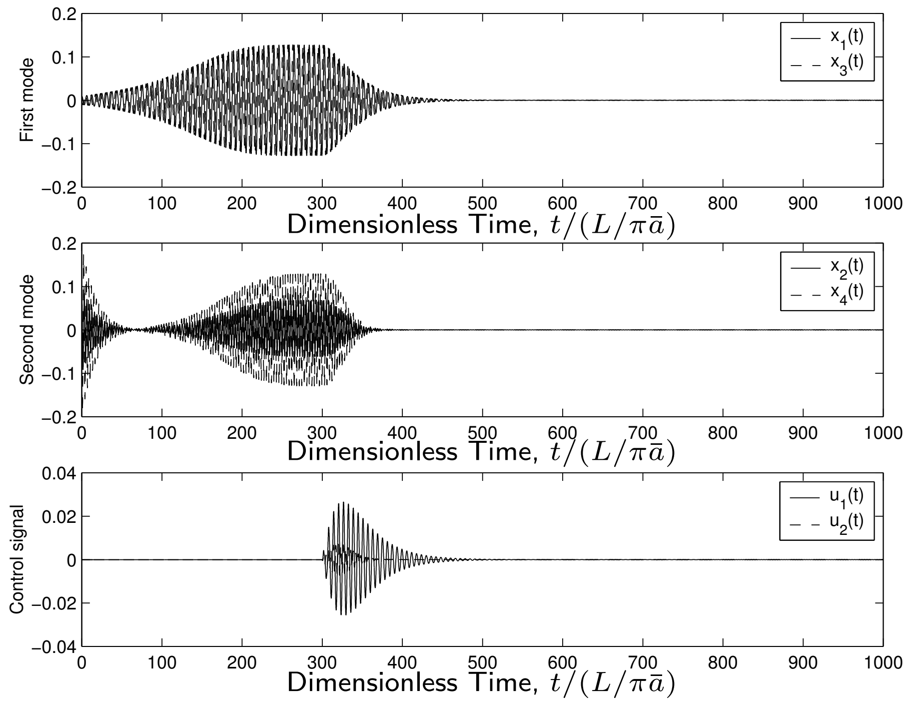



**Figure 2.27**: Closed-loop state response versus time

the growth constant of the first mode with the switch occurring at $t = 350$ while the control law is still in process of adapting.

Finally, we change the transient parameters $\theta_1 = 0.0062$ and $\theta_2 = 0.0178$ to $\theta_1 = 0.54$ and $\theta_2 = 1.006$ at $t = 600$. The closed-loop response is shown in Figure 2.30. Note that this change corresponds to 8709% and 5651%, respectively, of the original values of the parameters. The same change in the transient parameters occurring at $t = 350$ is shown on Figure 2.31.

## 2.7. Conclusion

A direct adaptive nonlinear control framework for adaptive stabilization, disturbance rejection, and command following of multivariable nonlinear uncertain dynamical systems with exogenous bounded disturbances and bounded energy $L_2$ distur-

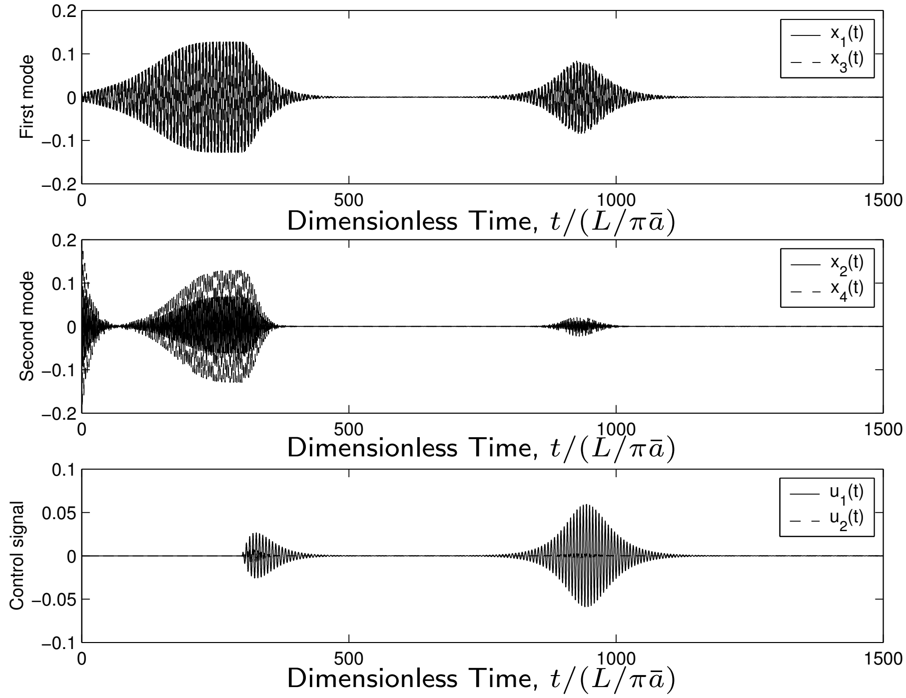



**Figure 2.28**: Closed-loop state response versus time

bances was developed. Using Lyapunov methods the proposed framework was shown to guarantee partial asymptotic stability of the closed-loop system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the plant. Furthermore, in the case where the nonlinear system is represented in normal form with input-to-state stable internal dynamics, the nonlinear adaptive controllers were constructed without knowledge of the system dynamics. Finally, several illustrative numerical examples were presented to show the utility of the proposed adaptive stabilization and tracking scheme.

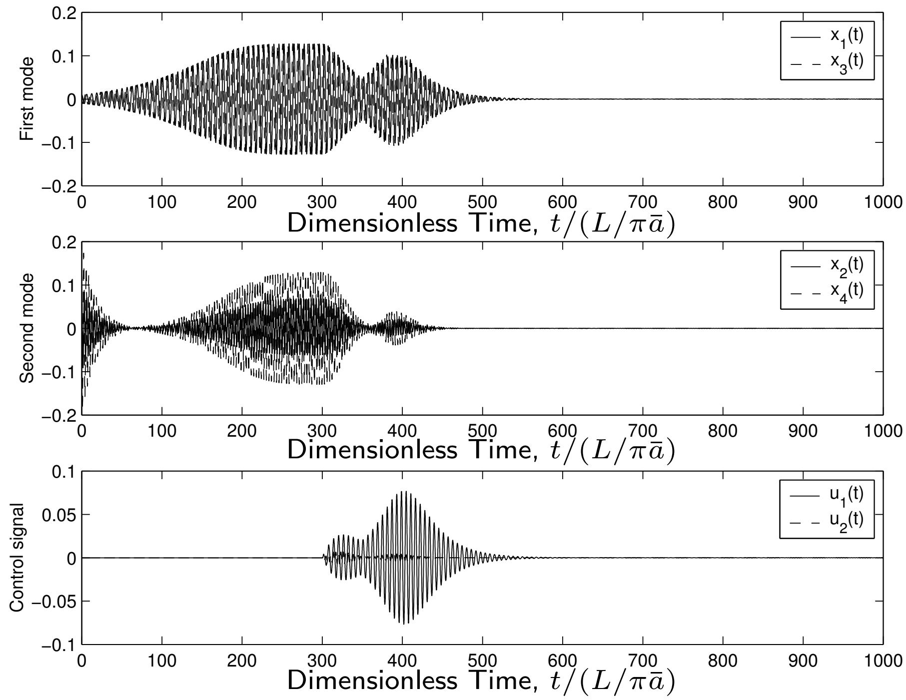



**Figure 2.29**: Closed-loop state response versus time

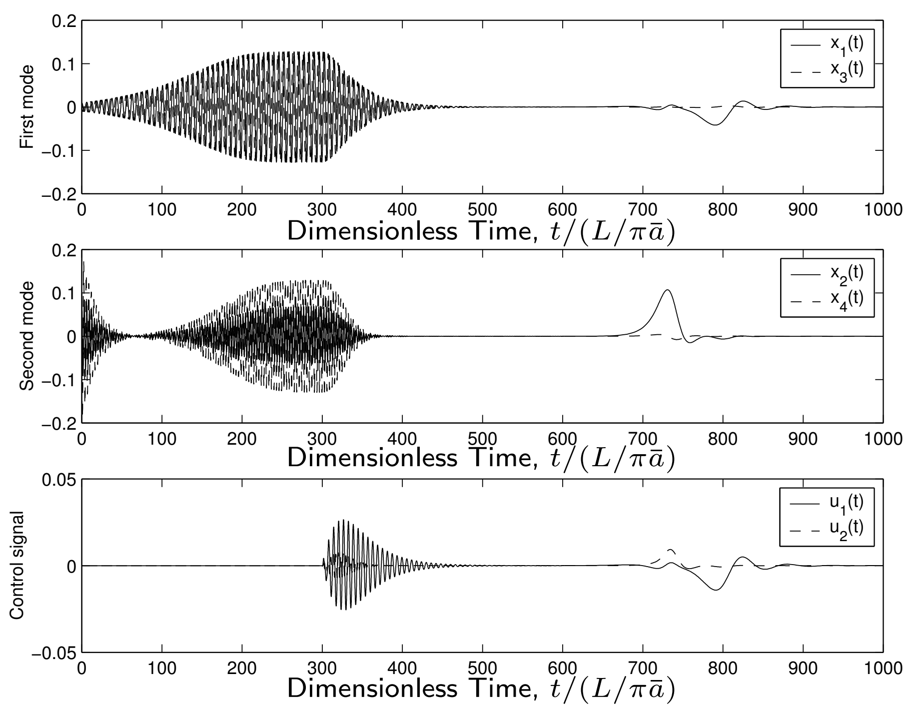



**Figure 2.30**: Closed-loop state response versus time

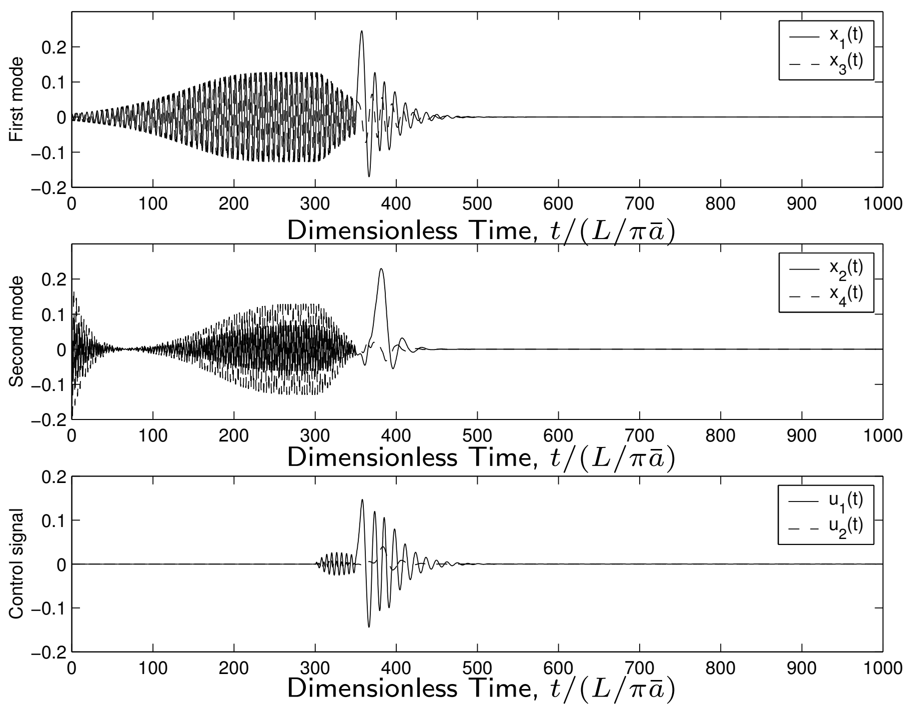



**Figure 2.31**: Closed-loop state response versus time

# Chapter 3

# Robust Adaptive Control for Nonlinear Uncertain Systems

## 3.1. Introduction

In Chapter 2 (see also [84, 90, 91]), a direct nonlinear adaptive control framework for adaptive stabilization, disturbance rejection, and command following was developed. In particular, a Lyapunov-based direct adaptive control framework was developed that guarantees partial asymptotic stability of the closed-loop system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the plant. Furthermore, the remainder of the state associated with the adaptive controller gains was shown to be Lyapunov stable. In the case where the nonlinear system was represented in normal form [122] with input-to-state stable internal dynamics [122, 222], the nonlinear adaptive controller was constructed *without* requiring knowledge of the system dynamics.

As is the case in the adaptive control literature [12, 121, 147, 176], the system errors as characterized in Chapter 2 are captured by a constant linearly parameterized uncertainty model of a known structure but unknown variation. This uncertainty characterization allows the system nonlinearities to be parameterized by a finite linear combination of basis functions within a class of function approximators such as rational functions, spline functions, radial basis functions, sigmoidal functions, and wavelets. However, this linear parametrization of basis functions cannot exactly capture the unknown system nonlinearity. In this Chapter, we generalize the results given in Chapter 2 to nonlinear uncertain dynamical systems with constant linearly parameterized uncertainty and nonlinear state-dependent uncertainty. Specifically, we consider a robust adaptive control problem that guarantees asymptotic robust stability of the system states in the face of structured uncertainty with unknown variation *and* structured (possibly nonlinear) parametric uncertainty with bounded variation. Hence, the overall adaptive control framework captures the residual approximation error inherent in linear parameterizations of system uncertainty via basis functions.

## 3.2. Robust Adaptive Control for Nonlinear Uncertain Systems

In this section we introduce an adaptive feedback control problem for nonlinear uncertain dynamical systems with constant linearly parameterized uncertainty structure and nonlinear state-dependent parametric uncertainty. Specifically, consider the controlled nonlinear uncertain system $\mathcal{G}$ given by

$$\dot{x}(t) = f(x(t)) + \Delta f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad t \geq 0, \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $f : \mathbb{R}^n \to \mathbb{R}^n$ and satisfies $f(0) = 0$, $\Delta f : \mathbb{R}^n \to \mathbb{R}^n$ and satisfies $\Delta f(0) = 0$, and $G : \mathbb{R}^n \to \mathbb{R}^{n \times m}$. Here, we assume that $f(\cdot)$ and $\Delta f(\cdot)$ are uncertain. In particular, the uncertainty in $f(\cdot)$ is captured by a constant linearly parameterized system uncertainty model of a known structure but unknown variation, while $\Delta f(\cdot)$ denotes structured (possibly nonlinear) parametric uncertainty with bounded variation. The structured uncertainty $\Delta f(\cdot)$ can effectively capture the residual approximation error inherent in the linear parameterization of the system uncertainty $f(\cdot)$ as well as capture structured nonlinear (possibly state-dependent) uncertainty in the system of known bounded variation. Here, we assume that $\Delta f(\cdot)$ belongs to the uncertainty set $\mathcal{F}$ given by

$$\mathcal{F} = \{\Delta f : \mathbb{R}^n \to \mathbb{R}^n : \Delta f(x) = G_\delta(x)\delta(h_\delta(x)),\ x \in \mathbb{R}^n,\ \delta(\cdot) \in \mathbf{\Delta}\}, \tag{3.2}$$

where $\mathbf{\Delta}$ satisfies

$$\mathbf{\Delta} = \{\delta : \mathbb{R}^{p_\delta} \to \mathbb{R}^{m_\delta} : \delta(0) = 0,\ \delta^{\mathrm{T}}(y)\delta(y) \leq m^{\mathrm{T}}(y)m(y),\ y \in \mathbb{R}^{p_\delta}\}, \tag{3.3}$$

and where $G_\delta : \mathbb{R}^n \to \mathbb{R}^{n\times m_\delta}$ and $h_\delta : \mathbb{R}^n \to \mathbb{R}^{p_\delta}$, satisfying $h_\delta(0) = 0$, are fixed functions denoting the structure of the uncertainty, $\delta : \mathbb{R}^{p_\delta} \to \mathbb{R}^{m_\delta}$ is an uncertain function, and $m : \mathbb{R}^{p_\delta} \to \mathbb{R}^{m_\delta}$ is a given function such that $m(0) = 0$. The special case $m(y) = \gamma^{-1}y$, where $\gamma > 0$, is worth noting. Specifically, in this case, (3.3) specializes to

$$\mathbf{\Delta} = \{\delta : \mathbb{R}^{p_\delta} \to \mathbb{R}^{m_\delta} : \delta(0) = 0,\ \delta^{\mathrm{T}}(y)\delta(y) \leq \gamma^{-2}y^{\mathrm{T}}y,\ y \in \mathbb{R}^{p_\delta}\}, \tag{3.4}$$

which corresponds to a nonlinear small gain-type norm bounded uncertainty characterization.

The control $u(\cdot)$ in (3.1) is restricted to the class of *admissible controls* consisting of measurable functions such that $u(t) \in \mathbb{R}^m$, $t \geq 0$. Furthermore, for the nonlinear uncertain system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions are satisfied; that is, $f(\cdot)$, $\Delta f(\cdot)$, $G(\cdot)$, and $u(\cdot)$ satisfy sufficient regularity conditions such that (3.1) has a unique solution forward in time.

**Theorem 3.1.** Consider the nonlinear uncertain system $\mathcal{G}$ given by (3.1). Assume there exist a matrix $K_{\mathrm{g}} \in \mathbb{R}^{m\times s}$ and functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, $F : \mathbb{R}^n \to \mathbb{R}^s$, with $F(0) = 0$, and $\ell : \mathbb{R}^n \to \mathbb{R}^t$ such that $V_{\mathrm{s}}(\cdot)$ is continuously differentiable, positive definite, radially unbounded, $V_{\rm s}(0) = 0$, $\ell(0) = 0$, and, for all $x \in \mathbb{R}^n$,

$$0 = V_{\rm s}'(x) f_{\rm s}(x) + \ell^{\rm T}(x)\ell(x) + \Gamma(x), \tag{3.5}$$

where

$$f_{\rm s}(x) \triangleq f(x) + G(x)\hat{G}(x)K_{\rm g}F(x). \tag{3.6}$$

$$\Gamma(x) \triangleq \tfrac{1}{4}V_{\rm s}'(x)G_\delta(x)G_\delta^{\rm T}(x)V_{\rm s}'^{\rm T}(x) + m^{\rm T}(h_\delta(x))m(h_\delta(x)), \tag{3.7}$$

Furthermore, let $Q \in \mathbb{R}^{m\times m}$ and $Y \in \mathbb{R}^{s\times s}$ be positive definite. Then the adaptive feedback control law

$$u(t) = \hat{G}(x(t))K(t)F(x(t)), \tag{3.8}$$

where $K(t) \in \mathbb{R}^{m\times s}$, $t \geq 0$, with update law

$$\dot{K}(t) = -\tfrac{1}{2}Q\hat{G}^{\rm T}(x(t))G^{\rm T}(x(t))V_{\rm s}'^{\rm T}(x(t))F^{\rm T}(x(t))Y, \quad K(0) = K_0, \tag{3.9}$$

guarantees that the solution $(x(t), K(t)) \equiv (0, K_{\rm g})$ of the closed-loop system given by (3.1), (3.8), and (3.9) is Lyapunov stable for all $\Delta f(\cdot) \in \mathcal{F}$ and $\ell(x(t)) \to 0$ as $t \to \infty$. If, in addition, $\ell^{\rm T}(x)\ell(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, then $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$ and $\Delta f(\cdot) \in \mathcal{F}$.

**Proof.** Note that with $u(t)$, $t \geq 0$, given by (3.8) and $\Delta f(\cdot) \in \mathcal{F}$ it follows from (3.1) that

$$\dot{x}(t) = f(x(t)) + G_\delta(x(t))\delta(h_\delta(x(t))) + G(x(t))\hat{G}(x(t))K(t)F(x(t)),$$
$$x(0) = x_0, \quad t \geq 0, \tag{3.10}$$

or, equivalently,

$$\dot{x}(t) = f_{\rm s}(x(t)) + G_\delta(x(t))\delta(h_\delta(x(t))) + G(x(t))\hat{G}(x(t))(K(t) - K_{\rm g})F(x(t)),$$
$$x(0) = x_0, \quad t \geq 0. \tag{3.11}$$

To show Lyapunov stability of the closed-loop system (3.9) and (3.11) consider the Lyapunov function candidate

$$V(x,K) = V_{\rm s}(x) + {\rm tr}\, Q^{-1}(K - K_{\rm g})Y^{-1}(K - K_{\rm g})^{\rm T}. \tag{3.12}$$

Note that $V(0, K_{\rm g}) = 0$ and, since $V_{\rm s}(\cdot)$, $Q$, and $Y$ are positive definite, $V(x,K) > 0$ for all $(x,K) \neq (0, K_{\rm g})$. Furthermore, $V(x,K)$ is radially unbounded. Now, letting $x(t)$, $t \geq 0$, denote the solution to (3.11) and using (3.5) and (3.9), it follows that the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned}
\dot V(x(t), K(t)) &= V_{\rm s}'(x(t))\left[f_{\rm s}(x(t)) + G_\delta(x(t))\delta(h_\delta(x(t)))\right.\\
&\quad \left.+G(x(t))\hat G(x(t))(K(t) - K_{\rm g})F(x(t))\right]\\
&\quad +2{\rm tr}Q^{-1}(K(t) - K_{\rm g})Y^{-1}\dot K^{\rm T}(t)\\
&= -\ell^{\rm T}(x(t))\ell(x(t)) - \Gamma(x(t)) + V_{\rm s}'(x(t))G_\delta(x(t))\delta(h_\delta(x(t)))\\
&\quad +{\rm tr}\Big[(K(t) - K_{\rm g})F(x(t))V_{\rm s}'(x(t))G(x(t))\hat G(x(t))\Big]\\
&\quad -{\rm tr}\Big[(K(t) - K_{\rm g})F(x(t))V_{\rm s}'(x(t))G(x(t))\hat G(x(t))\Big]\\
&= -\ell^{\rm T}(x(t))\ell(x(t)) - \tfrac{1}{4}V_{\rm s}'(x(t))G_\delta(x(t))G_\delta^{\rm T}(x(t))V_{\rm s}'^{\rm T}(x(t))\\
&\quad -m^{\rm T}(h_\delta(x(t)))m(h_\delta(x(t))) + V_{\rm s}'(x(t))G_\delta(x(t))\delta(h_\delta(x(t)))\\
&\leq -\ell^{\rm T}(x(t))\ell(x(t)) - \tfrac{1}{4}V_{\rm s}'(x(t))G_\delta(x(t))G_\delta^{\rm T}(x(t))V_{\rm s}'^{\rm T}(x(t))\\
&\quad -\delta^{\rm T}(h_\delta(x(t)))\delta(h_\delta(x(t))) + V_{\rm s}'(x(t))G_\delta(x(t))\delta(h_\delta(x(t)))\\
&= -\ell^{\rm T}(x(t))\ell(x(t)) - \left[\delta(h_\delta(x(t))) - \tfrac{1}{2}G_\delta^{\rm T}(x(t))V_{\rm s}'^{\rm T}(x(t))\right]^{\rm T}\\
&\quad \cdot\left[\delta(h_\delta(x(t))) - \tfrac{1}{2}G_\delta^{\rm T}(x(t))V_{\rm s}'^{\rm T}(x(t))\right]\\
&= -\ell^{\rm T}(x(t))\ell(x(t))\\
&\leq 0, \quad t \geq 0,
\end{aligned} \tag{3.13}$$

which proves that the solution $(x(t), K(t)) \equiv (0, K_{\rm g})$ to (3.9) and (3.11) is Lyapunov stable for all $\Delta f(\cdot) \in \mathcal{F}$. Furthermore, it follows from Theorem 2 of [42] that

$\ell(x(t)) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$ and $\Delta f(\cdot) \in \mathcal{F}$. If, in addition, $\ell^{\mathrm{T}}(x)\ell(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, then $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$ and $\Delta f(\cdot) \in \mathcal{F}$. $\square$

**Remark 3.1.** Note that in the case where $\ell^{\mathrm{T}}(x)\ell(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, the conditions in Theorem 3.1 imply that $x(t) \to 0$ as $t \to \infty$ for all $\Delta f(\cdot) \in \mathcal{F}$ and hence it follows from (3.9) that $(x(t), K(t)) \to \mathcal{M} \triangleq \{(x, K) \in \mathbb{R}^n \times \mathbb{R}^{m \times s} : x = 0, \dot{K} = 0\}$ as $t \to \infty$.

**Remark 3.2.** Note that $\Gamma(x)$ given by (3.7) serves as a bounding function for the uncertain set $\mathcal{F}$ in the sense that $\Gamma(\cdot)$ bounds $\mathcal{F}$. To see this, note that for all $x \in \mathbb{R}^n$ and $\delta(\cdot) \in \mathbf{\Delta}$,

$$\begin{aligned}
0 &\leq \left[\delta(h_\delta(x)) - \tfrac{1}{2}G_\delta^{\mathrm{T}}(x)V_{\mathrm{s}}'^{\mathrm{T}}(x)\right]^{\mathrm{T}} \left[\delta(h_\delta(x)) - \tfrac{1}{2}G_\delta^{\mathrm{T}}(x)V_{\mathrm{s}}'^{\mathrm{T}}(x)\right] \\
&= \delta^{\mathrm{T}}(h_\delta(x))\delta(h_\delta(x)) + \tfrac{1}{4}V_{\mathrm{s}}'(x)G_\delta(x)G_\delta^{\mathrm{T}}(x)V_{\mathrm{s}}'^{\mathrm{T}}(x) - V_{\mathrm{s}}'(x)G_\delta(x)\delta(h_\delta(x)) \\
&\leq m^{\mathrm{T}}(h_\delta(x))m(h_\delta(x)) + \tfrac{1}{4}V_{\mathrm{s}}'(x)G_\delta(x)G_\delta^{\mathrm{T}}(x)V_{\mathrm{s}}'^{\mathrm{T}}(x) - V_{\mathrm{s}}'(x)\Delta f(x) \\
&\leq \Gamma(x) - V_{\mathrm{s}}'(x)\Delta f(x),
\end{aligned}$$

which shows that $V_{\mathrm{s}}'(x)\Delta f(x) \leq \Gamma(x)$ for all $x \in \mathbb{R}^n$ and $\Delta f(\cdot) \in \mathcal{F}$. For further details see [76].

It is important to note that the adaptive control law (3.8) and (3.9) does *not* require explicit knowledge of the gain matrix $K_{\mathrm{g}}$; even though Theorem 3.1 requires the existence of $K_{\mathrm{g}}$ along with the construction of $F(x)$, $\hat{G}(x)$, and $V_{\mathrm{s}}(x)$ such that (3.5) holds. Furthermore, no specific structure on the nonlinear dynamics $f(x)$ is required to apply Theorem 3.1; all that is required is the existence of $F(x)$ such that (3.5) holds. However, if (3.1) is in normal form with asymptotically stable internal dynamics [122], then we can construct functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $F : \mathbb{R}^n \to \mathbb{R}^s$, with $F(0) = 0$, and $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ such that (3.5) holds *without* requiring knowledge of the system dynamics $f(x)$. To see this assume that the nonlinear uncertain system $\mathcal{G}$ is generated by

$$q_i^{(r_i)}(t) = f_{\mathrm{u}i}(q(t)) + \Delta f_{\mathrm{u}i}(q(t)) + \sum_{j=1}^{m} G_{\mathrm{s}(i,j)}(q(t)) u_j(t), \quad t \geq 0, \quad i = 1, \cdots, m, \tag{3.14}$$

where $q = [q_1, \cdots, q_1^{(r_1-1)}, \cdots, q_m, \cdots, q_m^{(r_m-1)}]^{\mathrm{T}}$, $q(0) = q_0$, $q_i^{(r_i)}$ denotes the $r_i^{\mathrm{th}}$ derivative of $q_i$, and $r_i$ denotes the relative degree with respect to the output $q_i$. Here we assume that the square matrix function $G_{\mathrm{s}}(q)$ composed of the entries $G_{\mathrm{s}(i,j)}(q)$, $i, j = 1, \cdots, m$, is such that $\det G_{\mathrm{s}}(q) \neq 0$, $q \in \mathbb{R}^{\hat{r}}$, where $\hat{r} = r_1 + \cdots + r_m$ is the (vector) relative degree of (3.14). Furthermore, since (3.14) is in a form where it does not possess internal dynamics, it follows that $\hat{r} = n$. The case where (3.14) possesses input-to-state stable internal dynamics can be handled as shown in Section 2.2.

Next, define $x_i \triangleq \left[ q_i, \cdots, q_i^{(r_i-2)} \right]^{\mathrm{T}}$, $i = 1, \cdots, m$, $x_{m+1} \triangleq \left[ q_1^{(r_1-1)}, \cdots, q_m^{(r_m-1)} \right]^{\mathrm{T}}$, and $x \triangleq \left[ x_1^{\mathrm{T}}, \cdots, x_{m+1}^{\mathrm{T}} \right]^{\mathrm{T}}$, so that (3.14) can be described as (3.1) with

$$f(x) = \tilde{A}x + \tilde{f}_{\mathrm{u}}(x), \quad \Delta f(x) = \begin{bmatrix} 0_{(n-m)\times m} \\ \Delta f_{\mathrm{u}}(x) \end{bmatrix}, \quad G(x) = \begin{bmatrix} 0_{(n-m)\times m} \\ G_{\mathrm{s}}(x) \end{bmatrix}, \tag{3.15}$$

where

$$\tilde{A} = \begin{bmatrix} A_0 \\ 0_{m\times n} \end{bmatrix}, \quad \tilde{f}_{\mathrm{u}}(x) = \begin{bmatrix} 0_{(n-m)\times 1} \\ f_{\mathrm{u}}(x) \end{bmatrix},$$

$A_0 \in \mathbb{R}^{(n-m)\times n}$ is a known matrix of zeros and ones capturing the multivariable controllable canonical form representation [43], and $f_{\mathrm{u}} : \mathbb{R}^n \to \mathbb{R}^m$ and $\Delta f_{\mathrm{u}} : \mathbb{R}^n \to \mathbb{R}^m$ are unknown functions. Here, we assume that $f_{\mathrm{u}}(x)$ is unknown and is parameterized as $f_{\mathrm{u}}(x) = \Theta f_{\mathrm{n}}(x)$, where $f_{\mathrm{n}} : \mathbb{R}^n \to \mathbb{R}^q$ and satisfies $f_{\mathrm{n}}(0) = 0$, $\Theta \in \mathbb{R}^{m\times q}$ is a matrix of uncertain constant parameters, and $\Delta f_{\mathrm{u}}(\cdot) \in \mathcal{F}_{\mathrm{u}}$, where $\mathcal{F}_{\mathrm{u}}$ is assumed to be of the form given by (3.2) with $\delta(\cdot) \in \boldsymbol{\Delta}$ satisfying (3.3). More generally, $\Delta f(\cdot)$ need not have the form given by (3.15). In particular, any parametric nonlinear uncertainty can be considered so long as $\Delta f(\cdot) \in \mathcal{F}$.

Next, to apply Theorem 3.1 to the uncertain system (3.1) with $f(x)$ and $G(x)$

given by (3.15) and $\Delta f(\cdot) \in \mathcal{F}$, let $K_{\rm g} \in \mathbb{R}^{m\times s}$, where $s = q + r$, be given by

$$K_{\rm g} = [\,\Theta_{\rm n} - \Theta,\ \Phi_{\rm n}\,], \tag{3.16}$$

where $\Theta_{\rm n} \in \mathbb{R}^{m\times q}$ and $\Phi_{\rm n} \in \mathbb{R}^{m\times r}$ are known matrices, and let

$$F(x) = \begin{bmatrix} f_{\rm n}(x) \\ \hat{f}_{\rm n}(x) \end{bmatrix}, \tag{3.17}$$

where $\hat{f}_{\rm n} : \mathbb{R}^n \to \mathbb{R}^r$ and satisfying $\hat{f}_{\rm n}(0) = 0$ is an arbitrary function. In this case, it follows that, with $\hat{G}(x) = G_{\rm s}^{-1}(x)$,

$$\begin{aligned} f_{\rm s}(x) &= f(x) + G(x)\hat{G}(x)K_{\rm g}F(x) \\ &= \tilde{A}x + \tilde{f}_{\rm u}(x) + \begin{bmatrix} 0_{(n-m)\times m} \\ G_{\rm s}(x) \end{bmatrix} G_{\rm s}^{-1}(x)\left[\ \Theta_{\rm n}f_{\rm n}(x) - \Theta f_{\rm n}(x) + \Phi_{\rm n}\hat{f}_{\rm n}(x)\ \right] \\ &= \tilde{A}x + \begin{bmatrix} 0_{(n-m)\times 1} \\ \Theta_{\rm n}f_{\rm n}(x) + \Phi_{\rm n}\hat{f}_{\rm n}(x) \end{bmatrix}. \end{aligned} \tag{3.18}$$

Now, since $\Theta_{\rm n} \in \mathbb{R}^{m\times q}$ and $\Phi_{\rm n} \in \mathbb{R}^{m\times r}$ are arbitrary constant matrices and $\hat{f}_{\rm n} : \mathbb{R}^n \to \mathbb{R}^r$ is an arbitrary function we can construct $K_{\rm g}$, $V_{\rm s}(x)$, and $F(x)$ without knowledge of $f(x)$ such that the (3.5) holds. In particular, choosing $\Theta_{\rm n}f_{\rm n}(x) + \Phi_{\rm n}\hat{f}_{\rm n}(x) = \hat{A}x$, where $\hat{A} \in \mathbb{R}^{m\times n}$, it follows that (3.18) has the form $f_{\rm s}(x) = A_{\rm s}x$, where $A_{\rm s} = [A_0^{\rm T}, \hat{A}^{\rm T}]^{\rm T}$ is in multivariable controllable canonical form. In addition, in the case where $\Delta f(\cdot)$ is linear; that is, $\Delta f(x) = B_\delta \Delta C_\delta x$, where $\sigma_{\max}(\Delta) \le \gamma^{-1}$, the adaptive controller (3.8) can be constructed to guarantee robustness using linear guaranteed cost robust control theory [23]. Specifically, choosing $f_{\rm s}(x) = A_{\rm s}x$, where $A_{\rm s}$ is asymptotically stable and in multivariable controllable canonical form, it follows from standard robust control theory [23] that if there exists a positive-definite matrix $P$ satisfying the Riccati equation

$$0 = A_{\rm s}^{\rm T}P + PA_{\rm s} + \gamma^{-2}PB_\delta B_\delta^{\rm T}P + C_\delta^{\rm T}C_\delta + R, \tag{3.19}$$

where $R$ is a positive-definite matrix, then the adaptive feedback controller (3.8) guarantees that (3.11) is globally asymptotically stable for all $\Delta f(\cdot) \in \mathcal{F}$ with $G_\delta(x) =$

$B_\delta$, $h_\delta(x) = C_\delta x$, $\delta(y) = \Delta y$, and $m(y) = \gamma^{-1}y$. In this case, with $V_{\rm s}(x) = x^{\rm T}Px$, the update law for the adaptive controller (3.8) is given by

$$\dot{K}(t) = -Q\hat{G}^{\rm T}(x(t))G^{\rm T}(x(t))Px(t)F^{\rm T}(x(t))Y, \quad K(0) = K_0. \tag{3.20}$$

Next, we consider the case where $f(x)$, $\Delta f(x)$, and $G(x)$ are uncertain. Specifically, we assume that $G(x)$ is such that $G_{\rm s}(x)$ is unknown and is parameterized as $G_{\rm s}(x) = B_{\rm u}G_{\rm n}(x)$, where $G_{\rm n} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is known and satisfies $\det G_{\rm n}(x) \neq 0$, $x \in \mathbb{R}^n$, and $B_{\rm u} \in \mathbb{R}^{m\times m}$, with $\det B_{\rm u} \neq 0$, is an unknown symmetric sign-definite matrix but the sign definiteness of $B_{\rm u}$ is known; that is, $B_{\rm u} > 0$ or $B_{\rm u} < 0$. For the statement of the next result define $B_0 \triangleq \left[0_{m\times(n-m)}, I_m\right]^{\rm T}$ for $B_{\rm u} > 0$, and $B_0 \triangleq \left[0_{m\times(n-m)}, -I_m\right]^{\rm T}$ for $B_{\rm u} < 0$.

**Corollary 3.1.** Consider the nonlinear system $\mathcal{G}$ given by (3.1) with $f(x)$ and $G(x)$ given by (3.15), $\Delta f(\cdot) \in \mathcal{F}$, and $G_{\rm s}(x) = B_{\rm u}G_{\rm n}(x)$, where $B_{\rm u}$ is an unknown symmetric matrix and the sign definiteness of $B_{\rm u}$ is known. Assume there exist a matrix $K_{\rm g} \in \mathbb{R}^{m\times s}$ and functions $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, $F : \mathbb{R}^n \to \mathbb{R}^s$, with $F(0) = 0$, and $\ell : \mathbb{R}^n \to \mathbb{R}^t$ such that $V_{\rm s}(\cdot)$ is continuously differentiable, positive definite, radially unbounded, $V_{\rm s}(0) = 0$, $\ell(0) = 0$, and (3.5) holds. Finally, let $Y \in \mathbb{R}^{s\times s}$ be positive definite. Then the adaptive feedback control law

$$u(t) = G_{\rm n}^{-1}(x(t))K(t)F(x(t)), \tag{3.21}$$

where $K(t) \in \mathbb{R}^{m\times s}$, $t \geq 0$, with update law

$$\dot{K}(t) = -\tfrac{1}{2}B_0^{\rm T}V_{\rm s}'^{\rm T}(x(t))F^{\rm T}(x(t))Y, \quad K(0) = K_0, \tag{3.22}$$

guarantees that the solution $(x(t), K(t)) \equiv (0, K_{\rm g})$ of the closed-loop system given by (3.1), (3.21), and (3.22) is Lyapunov stable for all $\Delta f(\cdot) \in \mathcal{F}$ and $\ell(x(t)) \to 0$ as $t \to \infty$. If, in addition, $\ell^{\rm T}(x)\ell(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, then $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$ and $\Delta f(\cdot) \in \mathcal{F}$.

**Proof.** The result is a direct consequence of Theorem 3.1. First, let $\hat{G}(x) = G_{\rm n}^{-1}(x)$ so that $G(x)\hat{G}(x) = [0_{m\times(n-m)}, B_{\rm u}]^{\rm T}$. Next, since $Q$ is an arbitrary positive-definite matrix, $Q$ in (3.9) can be replaced by $q|B_{\rm u}|^{-1}$, where $q$ is a positive constant and $|B_{\rm u}| = (B_{\rm u}^2)^{\frac{1}{2}}$, where $(\cdot)^{\frac{1}{2}}$ denotes the (unique) positive-definite square root. Now, since $B_{\rm u}$ is symmetric and sign definite it follows from the Schur decomposition that $B_{\rm u} = UD_{B_{\rm u}}U^{\rm T}$, where $U$ is orthogonal and $D_{B_{\rm u}}$ is real diagonal. Hence, $|B_{\rm u}|^{-1}\hat{G}^{\rm T}(x)G^{\rm T}(x) = [0_{m\times(n-m)}, \mathcal{I}_m] = B_0^{\rm T}$, where $\mathcal{I}_m = I_m$ for $B_{\rm u} > 0$ and $\mathcal{I}_m = -I_m$ for $B_{\rm u} < 0$. Now, (3.9), with $qY$ replaced by $Y$, implies (3.22). □

## 3.3. Adaptive Absolute Stabilization for Nonlinear Uncertain Systems

In this section we introduce an adaptive absolute stabilization problem. The goal of this problem is to determine an adaptive controller that stabilizes an uncertain system with state-dependent nonlinearities that belong to a given *unknown* sector $\Psi$. Specifically, we consider the controlled nonlinear uncertain system $\mathcal{G}$ given by

$$\dot{x}(t) = Ax(t) + B\psi(y(t)) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \tag{3.23}$$

$$y(t) = Cx(t), \tag{3.24}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $y(t) \in \mathbb{R}^m$, $t \geq 0$, is the system output, and $\psi(\cdot) \in \mathbf{\Psi}$, where

$$\mathbf{\Psi} \triangleq \{\psi : \mathbb{R}^m \to \mathbb{R}^m : \psi(0) = 0,\, y^{\rm T}[\psi(y) + My] \leq 0,\, y \in \mathbb{R}^m\}, \tag{3.25}$$

and $M \in \mathbb{R}^{m\times m}$ is an *unknown* matrix. If $M = \operatorname{diag}[M_1, \cdots, M_m]$ is diagonal, then the sector condition characterizing $\mathbf{\Psi}$ is implied by the scalar sector conditions

$$\psi_i(y)y_i \leq -M_i y_i^2, \quad y_i \in \mathbb{R}, \quad i = 1, \cdots, m, \tag{3.26}$$

where $\psi_i(y)$ and $y_i$ are the $i$th components of $\psi(y)$ and $y$, respectively. Hence, the adaptive absolute stabilization problem is reminiscent to the classical absolute sta-

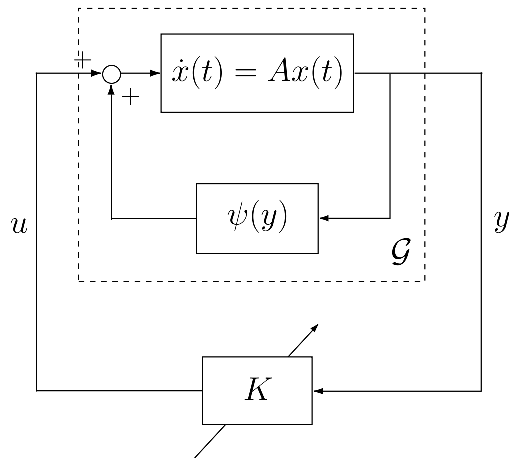



**Figure 3.1**: Adaptive absolute stabilization problem

bility problem [179] with the key difference being that the plant dynamics are not assumed to be known nor is the sector, characterized via $M$, assumed to be known (see Figure 3.1). For the statement of the next result recall the definitions of minimum phase and weakly minimum phase given in [33].

**Theorem 3.2.** Consider the controlled nonlinear uncertain system $\mathcal{G}$ given by (3.23) and (3.24). Assume $\det(CB) \neq 0$ and assume $\mathcal{G}$, with $\psi(y) \equiv 0$, is weakly minimum phase. Furthermore, let $Q \in \mathbb{R}^{m\times m}$ and $Y \in \mathbb{R}^{m\times m}$ be positive definite. Then the adaptive feedback control law

$$u(t) = K(t)y(t), \tag{3.27}$$

where $K(t) \in \mathbb{R}^{m\times m}$, $t \geq 0$, with update law

$$\dot{K}(t) = -Qy(t)y^{\mathrm{T}}(t)Y, \quad K(0) = K_0, \tag{3.28}$$

guarantees that the solution $(x(t), K(t)) \equiv (0, K_{\mathrm{g}} + M)$ of the closed-loop system given by (3.23), (3.27), and (3.28) is Lyapunov stable for all $\psi(\cdot) \in \mathbf{\Psi}$. If, in addition, $\mathcal{G}$, with $\psi(y) \equiv 0$, is minimum phase, then $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$ and $\psi(\cdot) \in \mathbf{\Psi}$.

**Proof.** Since $\det(CB) \neq 0$ and $\mathcal{G}$, with $\psi(y) \equiv 0$, is weakly minimum phase it follows from Theorem 2 of [60] (see also [126]) that there exist matrices $P \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{p\times n}$, and $K_{\rm g} \in \mathbb{R}^{m\times m}$, with $P$ positive definite, such that

$$0 = (A + BK_{\rm g}C)^{\rm T}P + P(A + BK_{\rm g}C) + L^{\rm T}L, \tag{3.29}$$

$$0 = B^{\rm T}P - C. \tag{3.30}$$

Next, with $u(t)$, $t \geq 0$, given by (3.27) and $\psi(\cdot) \in \mathbf{\Psi}$ it follows from (3.23) that

$$\dot{x}(t) = Ax(t) + B\psi(y(t)) + BK(t)y(t), \quad x(0) = x_0, \quad t \geq 0, \tag{3.31}$$

or, equivalently,

$$\dot{x}(t) = (A+BK_{\rm g}C)x(t)+B\psi(y(t))+B(K(t)-K_{\rm g})y(t), \quad x(0) = x_0, \quad t \geq 0. \tag{3.32}$$

To show Lyapunov stability of the closed-loop system (3.28) and (3.32) consider the Lyapunov function candidate

$$V(x, K) = x^{\rm T}Px + {\rm tr}Q^{-1}(K - K_{\rm g} - M)Y^{-1}(K - K_{\rm g} - M)^{\rm T}. \tag{3.33}$$

Note that $V(0, K_{\rm g}+M) = 0$ and, since $P$, $Q$, and $Y$ are positive definite, $V(x, K) > 0$ for all $(x, K) \neq (0, K_{\rm g} + M)$. Furthermore, $V(x, K)$ is radially unbounded. Now, letting $x(t)$, $t \geq 0$, denote the solution to (3.32) and using (3.28)–(3.30), it follows that the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned}\dot{V}(x(t), K(t)) =&\ 2x^{\rm T}(t)P\left[(A + BK_{\rm g}C)x(t) + B\psi(y(t)) + B(K(t) - K_{\rm g})y(t)\right]\\ &+2{\rm tr}Q^{-1}(K(t) - K_{\rm g} - M)Y^{-1}\dot{K}^{\rm T}(t)\\ &+2[y^{\rm T}(t)My(t) - y^{\rm T}(t)My(t)]\\ =&\ -x(t)L^{\rm T}Lx(t) + 2x^{\rm T}(t)C^{\rm T}\psi(y(t)) + 2y^{\rm T}(t)My(t)\\ &+2x^{\rm T}(t)C^{\rm T}(K(t) - K_{\rm g})y(t) - 2y^{\rm T}(t)My(t)\\ &-2{\rm tr}\big[(K(t) - K_{\rm g} - M)y(t)y^{\rm T}(t)\big]\end{aligned}$$

$$
\begin{aligned}
&= -x(t)L^{\mathrm{T}}Lx(t) + 2y^{\mathrm{T}}(x)\left[\psi(y(t)) + My(t)\right] \\
&\quad +2\mathrm{tr}\big[(K(t) - K_{\mathrm{g}} - M)y(t)y^{\mathrm{T}}(t)\big] \\
&\quad -2\mathrm{tr}\big[(K(t) - K_{\mathrm{g}} - M)y(t)y^{\mathrm{T}}(t)\big] \\
&\leq -x^{\mathrm{T}}(t)L^{\mathrm{T}}Lx(t) \\
&\leq 0, \quad t \geq 0,
\end{aligned} \tag{3.34}
$$

which proves that the solution $(x(t), K(t)) \equiv (0, K_{\mathrm{g}} + M)$ to (3.28) and (3.32) is Lyapunov stable for all $\psi(\cdot) \in \boldsymbol{\Psi}$. Furthermore, it follows from Theorem 2 of [42] that $Lx(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$ and $\psi(\cdot) \in \boldsymbol{\Psi}$. If, in addition, $\mathcal{G}$ is minimum phase, then it follows that $L^{\mathrm{T}}L > 0$ and hence $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$ and $\psi(\cdot) \in \boldsymbol{\Psi}$. □

Next, we consider a nonlinear version to Theorem 3.2. Specifically, consider the controlled nonlinear uncertain system $\mathcal{G}$ given by

$$\dot{x}(t) = f(x(t)) + G(x(t))\psi(y(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad t \geq 0, \tag{3.35}$$

$$y(t) = h(x(t)), \tag{3.36}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $y(t) \in \mathbb{R}^m$, $t \geq 0$, is the system output, $f : \mathbb{R}^n \to \mathbb{R}^n$ and satisfies $f(0) = 0$, $G : \mathbb{R}^n \to \mathbb{R}^{n \times m}$, and $\psi(\cdot) \in \boldsymbol{\Psi}$. For the statement of the next result recall the definitions of passivity [33] and exponential passivity [39].

**Theorem 3.3.** Consider the controlled nonlinear uncertain system $\mathcal{G}$ given by (3.35) and (3.36). Assume $\mathcal{G}$, with $\psi(y) \equiv 0$, is exponentially passive with a continuously differentiable storage function $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$ such that $V_{\mathrm{s}}(\cdot)$ is positive definite and radially unbounded. Furthermore, let $Q \in \mathbb{R}^{m \times m}$ and $Y \in \mathbb{R}^{m \times m}$ be positive definite. Then the adaptive feedback control law

$$u(t) = K(t)y(t), \tag{3.37}$$

where $K(t) \in \mathbb{R}^{m\times m}$, $t \geq 0$, with update law

$$\dot{K}(t) = -Qy(t)y^{\mathrm{T}}(t)Y, \tag{3.38}$$

guarantees that the solution $(x(t), K(t)) \equiv (0, M)$ of the closed-loop system given by (3.35), (3.37), and (3.38) is Lyapunov stable for all $\psi(\cdot) \in \mathbf{\Psi}$ and $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$.

**Proof.** Since $\mathcal{G}$, with $\psi(y) \equiv 0$, is exponentially passive with a continuously differentiable storage function $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$ such that $V_{\mathrm{s}}(\cdot)$ is positive definite and radially unbounded, it follows from [111] that, for all $x \in \mathbb{R}^n$, there exist $\ell : \mathbb{R}^n \to \mathbb{R}^t$ and $\varepsilon > 0$ such that

$$0 = V_{\mathrm{s}}'(x)f(x) + \varepsilon V_{\mathrm{s}}(x) + \ell^{\mathrm{T}}(x)\ell(x), \tag{3.39}$$

$$0 = \tfrac{1}{2}V_{\mathrm{s}}'(x)G(x) - h^{\mathrm{T}}(x). \tag{3.40}$$

Next, with $u(t)$, $t \geq 0$, given by (3.37) and $\psi(\cdot) \in \mathbf{\Psi}$ it follows from (3.35) that

$$\dot{x}(t) = f(x(t)) + G(x(t))\psi(h(x(t))) + G(x(t))K(t)h(x(t)), \quad x(0) = x_0, \quad t \geq 0. \tag{3.41}$$

To show Lyapunov stability of the closed-loop system (3.38) and (3.41) consider the Lyapunov function candidate

$$V(x, K) = V_{\mathrm{s}}(x) + \mathrm{tr}Q^{-1}(K - M)Y^{-1}(K - M)^{\mathrm{T}}. \tag{3.42}$$

Note that $V(0, M) = 0$ and, since $V_{\mathrm{s}}(\cdot)$, $Q$, and $Y$ are positive definite, $V(x, K) > 0$ for all $(x, K) \neq (0, M)$. Furthermore, $V(x, K)$ is radially unbounded. Now, Lyapunov stability of the closed-loop system (3.38) and (3.41) as well as $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$ and $\psi(\cdot) \in \Psi$ follows as in the proof of Theorem 3.2. □

## 3.4. Illustrative Numerical Examples

In this section we present two numerical examples to demonstrate the utility of the proposed direct robust adaptive control framework.

**Example 3.1.** Consider the nonlinear uncertain system given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \delta_1 x_1(t) + (1+\delta_2)x_2(t) \\ -\beta x_1(t) + \varepsilon(\alpha - x_1^2(t))x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{b}{1+x_1^2(t)+x_2^2(t)} \end{bmatrix} u(t),$$
$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \quad t \geq 0, \tag{3.43}$$

where $\alpha, \beta, \varepsilon, b, \delta_1, \delta_2 \in \mathbb{R}$ are unknown with $\delta_1 \in [-2, 2]$ and $\delta_2 \in [-1, 1]$. Note that (3.43) can be written in the form of (3.1) with $f(x) = [x_2, -\beta x_1 + \varepsilon(\alpha - x_1^2)x_2]^{\mathrm{T}}$, $\Delta f(x) = [\delta_1 x_1(t) + \delta_2 x_2(t), 0]^{\mathrm{T}}$, and $G(x) = [0, \frac{b}{1+x_1^2+x_2^2}]^{\mathrm{T}}$. Here, we assume that $f(x)$ and $\Delta f(x)$ are unknown and can be parameterized as $f(x) = [x_2, \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_1^2 x_2]^{\mathrm{T}}$ and $\Delta f(x) = B_\delta \Delta C_\delta x$, where $\theta_1$, $\theta_2$, and $\theta_3$ are unknown constants and $B_\delta = [1, 0]^{\mathrm{T}}$, $C_\delta = I_2$, and $\Delta = [\delta_1, \delta_2]$. Note that (3.2) is satisfied with $m(C_\delta x) = \gamma^{-1} C_\delta x = 2x$. Furthermore, we assume that $\operatorname{sgn} b$ is known. Next, let $G_{\mathrm{n}}(x) = 1/(1 + x_1^2 + x_2^2)$, $F(x) = [x_1, x_2, x_1^2 x_2]^{\mathrm{T}}$, and $K_{\mathrm{g}} = \frac{1}{b}[\theta_{\mathrm{n}_1} - \theta_1, \theta_{\mathrm{n}_2} - \theta_2, -\theta_3]$, where $\theta_{\mathrm{n}_1}$, $\theta_{\mathrm{n}_2}$ are arbitrary scalars, so that

$$\begin{aligned} f_{\mathrm{s}}(x) &= f(x) + \begin{bmatrix} 0 \\ \frac{b}{1+x_1^2+x_2^2} \end{bmatrix} (1 + x_1^2 + x_2^2) \frac{1}{b} \begin{bmatrix} \theta_{\mathrm{n}_1} - \theta_1, \theta_{\mathrm{n}_2} - \theta_2, -\theta_3 \end{bmatrix} F(x) \\ &= \begin{bmatrix} 0 & 1 \\ \theta_{\mathrm{n}_1} & \theta_{\mathrm{n}_2} \end{bmatrix} x. \end{aligned} \tag{3.44}$$

Now, with the proper choice of $\theta_{\mathrm{n}_1}$ and $\theta_{\mathrm{n}_2}$, it follows from Corollary 3.1 that if there exists $P > 0$ satisfying (3.19), then the adaptive feedback controller (3.21) guarantees that $x(t) \to 0$ as $t \to \infty$ for all $\Delta f(\cdot) \in \mathcal{F}$. Specifically, here we choose $\theta_{\mathrm{n}_1} = -100$, $\theta_{\mathrm{n}_2} = -10$, and $R = I_2$, so that $P > 0$ satisfying (3.19) is given by

$$P = \begin{bmatrix} 14.6347 & 0.2777 \\ 0.2777 & 0.1287 \end{bmatrix}. \tag{3.45}$$

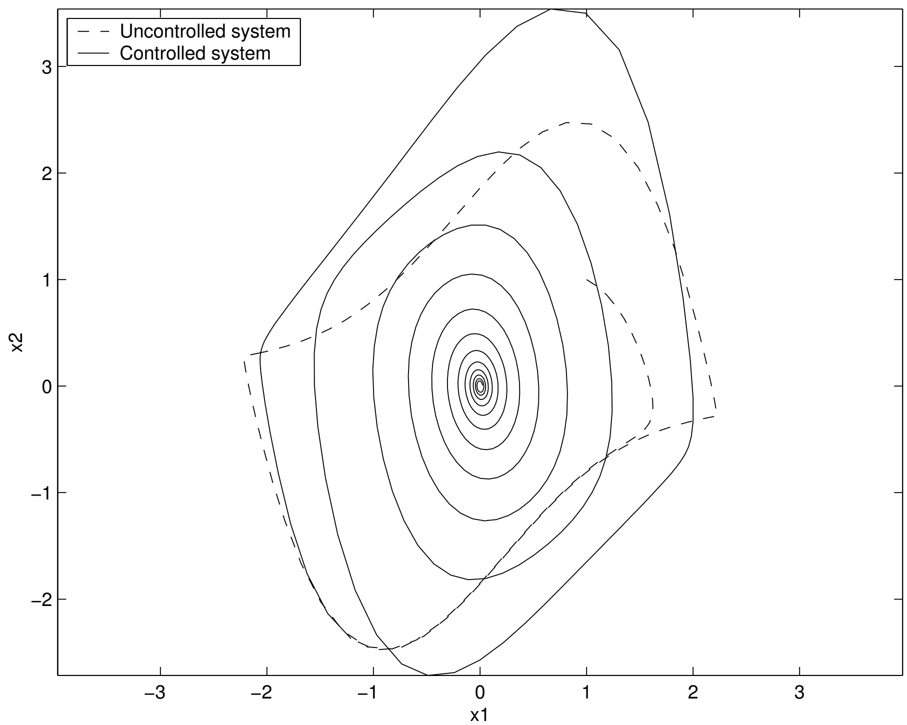



**Figure 3.2**: Phase portrait of controlled and uncontrolled system

With $\alpha = 1$, $\beta = 1$, $\varepsilon = 2$, $b = 3$, $\delta_1 = 0.21$, $\delta_2 = 0.8$, $Y = 0.5I_2$, and initial conditions $x(0) = [1, 1]^{\mathrm{T}}$ and $K(0) = [0, 0, 0]$, Figure 3.2 shows the phase portrait of the controlled and uncontrolled system. Note that the adaptive controller is switched on at $t = 15$ sec. Figure 3.3 shows the state trajectories versus time and the control signal versus time. Finally, Figure 3.4 shows the adaptive gain history versus time.

**Example 3.2.** Consider the nonlinear uncertain system given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} \delta_1 x_1(t) + x_3(t) \\ \delta_2 x_2(t) + \delta_3 x_3(t) + x_4(t) \\ a_1 x_1(t) + a_3 x_3(t) + c_1 x_3(t) x_4(t) + c_3 x_1(t) x_2(t) \\ a_2 x_2(t) + a_4 x_4(t) + c_2 x_1^2(t) + c_4 x_3^2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{b_1}{2+\sin x_3(t)} \\ \frac{b_2}{\cosh x_4(t)} & 0 \end{bmatrix} u(t), \quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \\ x_{40} \end{bmatrix}, \quad t \geq 0, \quad (3.46)$$

where $a_1, \cdots, a_4, c_1, \cdots, c_4, b_1, b_2, \delta_1, \delta_2, \delta_3 \in \mathbb{R}$ are unknown, $b_1$ and $b_2$ are positive, and $\delta_1 \in [-1, 1]$, $\delta_2 \in [-1, 1]$, and $\delta_3 \in [-2, 2]$. Note that (3.46) can be written in the

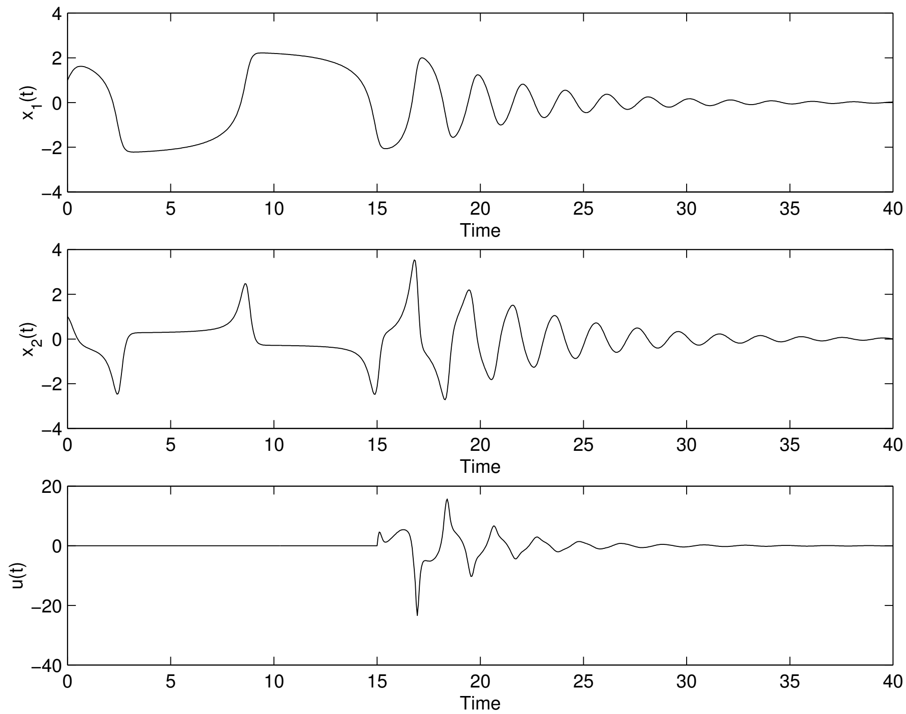



**Figure 3.3**: State trajectories and control signal versus time

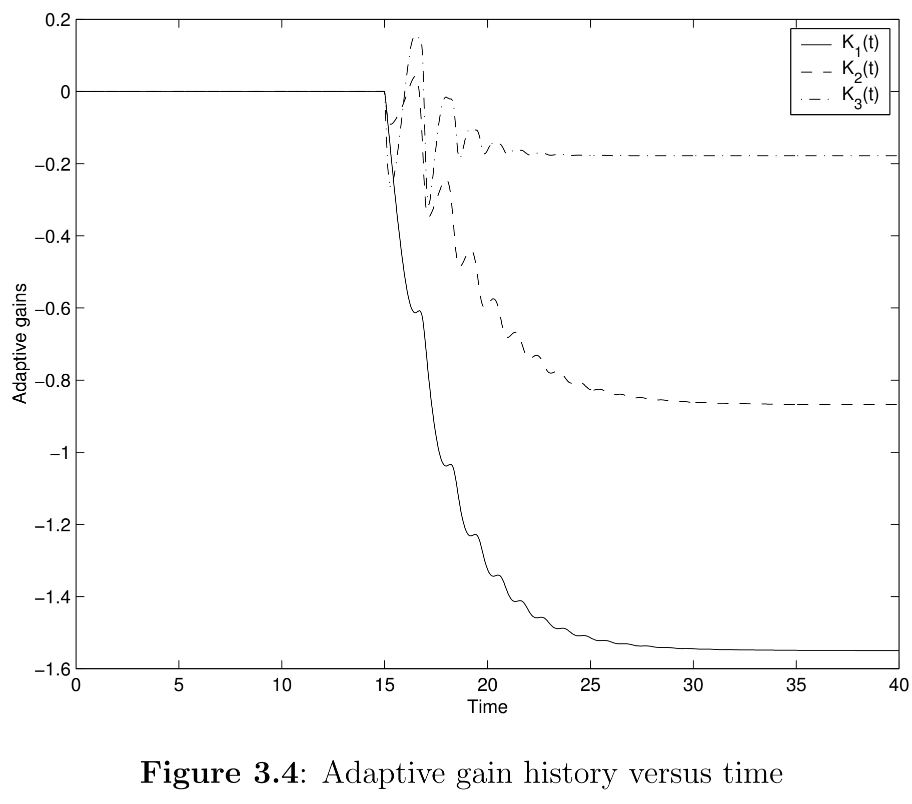



**Figure 3.4**: Adaptive gain history versus time

form of (3.1) with $f(x) = [x_3, x_4, a_1x_1 + a_3x_3 + c_1x_1x_2 + c_3x_3x_4, a_2x_2 + a_4x_4 + c_2x_1^2 + c_4x_3^2]^{\mathrm{T}}$, $\Delta f(x) = [\delta_1 x_1, \delta_2 x_2 + \delta_3 x_3, 0, 0]^{\mathrm{T}}$, and

$$G(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{b_1}{2+\sin x_3} \\ \frac{b_2}{\cosh x_4} & 0 \end{bmatrix}.$$

Here, we assume that $f(x)$, $\Delta f(x)$, and $G(x)$ are unknown and can be parameterized as

$$\begin{aligned} f(x) &= [x_3, x_4, \theta_1 x_1 + \theta_3 x_3 + \theta_5 x_1 x_2 + \theta_6 x_3 x_4, \theta_2 x_2 + \theta_4 x_4 + \theta_7 x_1^2 + \theta_8 x_3^2]^{\mathrm{T}}, \\ \Delta f(x) &= B_\delta \Delta C_\delta x, \quad G(x) = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2+\sin x_3} \\ \frac{1}{\cosh x_4} & 0 \end{bmatrix}, \end{aligned}$$

where $\theta_i$, $i = 1, \cdots, 8$, are unknown constants and $B_\delta = [I_2, 0_2]^{\mathrm{T}}$, $C_\delta = [I_3, 0_{3\times 1}]$, and $\Delta = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & \delta_3 \end{bmatrix}$. Note that (3.2) is satisfied with $m(C_\delta x) = \gamma^{-1} C_\delta x = 2C_\delta x$. Next, let

$$\begin{aligned} G_{\mathrm{n}}(x) &= \begin{bmatrix} 0 & \cosh x_4 \\ 2 + \sin x_3 & 0 \end{bmatrix}, \quad F(x) = \left[x^{\mathrm{T}}, x_1x_2, x_3x_4, x_1^2, x_3^2, \right]^{\mathrm{T}}, \\ K_{\mathrm{g}} &= \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}^{-1} \\ &\quad \cdot \begin{bmatrix} \theta_{\mathrm{n}_{11}} - \theta_1 & \theta_{\mathrm{n}_{12}} & \theta_{\mathrm{n}_{13}} - \theta_3 & \theta_{\mathrm{n}_{14}} & -\theta_5 & -\theta_6 & 0 & 0 \\ \theta_{\mathrm{n}_{21}} & \theta_{\mathrm{n}_{22}} - \theta_2 & \theta_{\mathrm{n}_{23}} & \theta_{\mathrm{n}_{24}} - \theta_4 & 0 & 0 & -\theta_7 & -\theta_8 \end{bmatrix}, \end{aligned}$$

where $\theta_{\mathrm{n}_{ij}}$, $i = 1, 2$, $j = 1, \cdots, 4$, are arbitrary scalars, so that

$$\begin{aligned} f_{\mathrm{s}}(x) &= f(x) + \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2+\sin x_3} \\ \frac{1}{\cosh x_4} & 0 \end{bmatrix} \begin{bmatrix} 0 & \cosh x_4 \\ 2 + \sin x_3 & 0 \end{bmatrix} \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}^{-1} \\ &\quad \cdot \begin{bmatrix} \theta_{\mathrm{n}_{11}} - \theta_1 & \theta_{\mathrm{n}_{12}} & \theta_{\mathrm{n}_{13}} - \theta_3 & \theta_{\mathrm{n}_{14}} & -\theta_5 & -\theta_6 & 0 & 0 \\ \theta_{\mathrm{n}_{21}} & \theta_{\mathrm{n}_{22}} - \theta_2 & \theta_{\mathrm{n}_{23}} & \theta_{\mathrm{n}_{24}} - \theta_4 & 0 & 0 & -\theta_7 & -\theta_8 \end{bmatrix} F(x) \\ &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \theta_{\mathrm{n}_{11}} & \theta_{\mathrm{n}_{13}} & \theta_{\mathrm{n}_{13}} & \theta_{\mathrm{n}_{14}} \\ \theta_{\mathrm{n}_{21}} & \theta_{\mathrm{n}_{23}} & \theta_{\mathrm{n}_{23}} & \theta_{\mathrm{n}_{24}} \end{bmatrix} x. \end{aligned} \tag{3.47}$$

Now, with the proper choice of $\theta_{\mathrm{n}_{ij}}$, $i = 1, 2$, $j = 1, \cdots, 4$, it follows from Corollary 3.1 that if there exists $P > 0$ satisfying (3.19), then the adaptive feedback controller

(3.21) guarantees that $x(t) \to 0$ as $t \to \infty$ for all $\Delta f(\cdot) \in \mathcal{F}$. Specifically, here we choose $A_{\rm s} = \begin{bmatrix} 0_2 & I_2 \\ -100I_2 & -10I_2 \end{bmatrix}$ and $R = I_4$, so that $P > 0$ satisfying (3.19) is given by

$$P = \begin{bmatrix} 14.6347 & 0.0000 & 0.2777 & 0.0000 \\ 0.0000 & 6.0290 & 0.0000 & 0.0554 \\ 0.2777 & -0.0000 & 0.1287 & 0.0000 \\ 0.0000 & 0.0554 & -0.0000 & 0.0556 \end{bmatrix}. \quad (3.48)$$

With $a_1 = -9$, $a_2 = -14$, $a_3 = 0.2$, $a_4 = -6$, $b_1 = 3$, $b_2 = 5$, $c_1 = 0.2$, $c_2 = 0.7$, $c_3 = -1.5$, $c_4 = 0.9$, $\delta_1 = 0.7$, $\delta_2 = 0.6$, $\delta_3 = 1.6$, $Y = I_2$, and initial conditions $x(0) = [1, 0, 0, 0]^{\rm T}$ and $K(0) = 0_{2\times 8}$, Figure 3.5 shows that the phase portraits of the controlled and uncontrolled system. Note that the adaptive controller is switched on at $t = 6$ sec. Figure 3.6 shows the state trajectories versus time and the control signals versus time.

## 3.5. Conclusion

A direct robust adaptive control framework was developed for a class of nonlinear uncertain dynamical systems with constant linearly parameterized uncertainty and nonlinear state-dependent uncertainty. The proposed framework is Lyapunov-based and captures the residual approximation error inherent in standard adaptive control methods predicated on linear parameterizations of system modeling uncertainty.

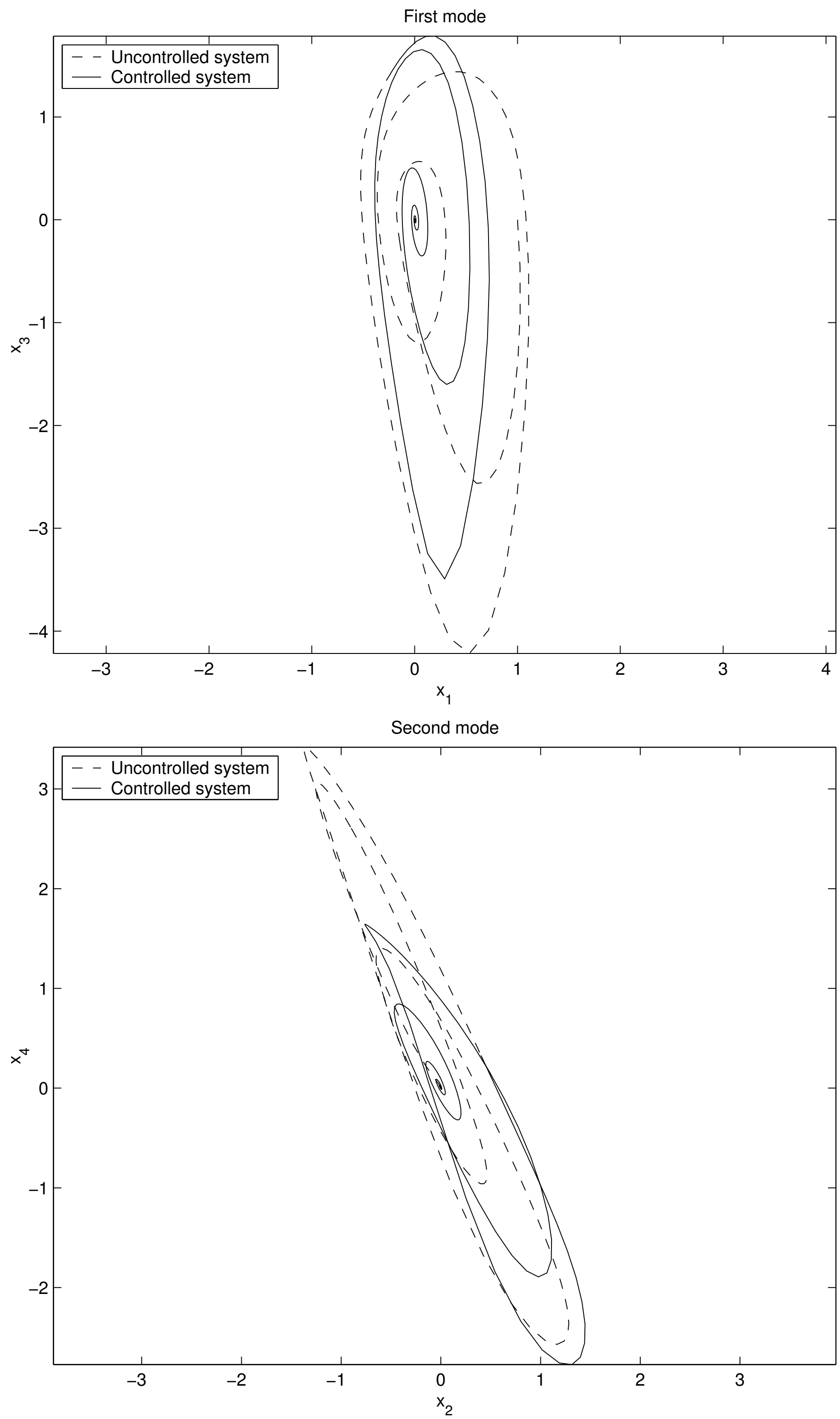



**Figure 3.5**: Phase portraits of controlled and uncontrolled System

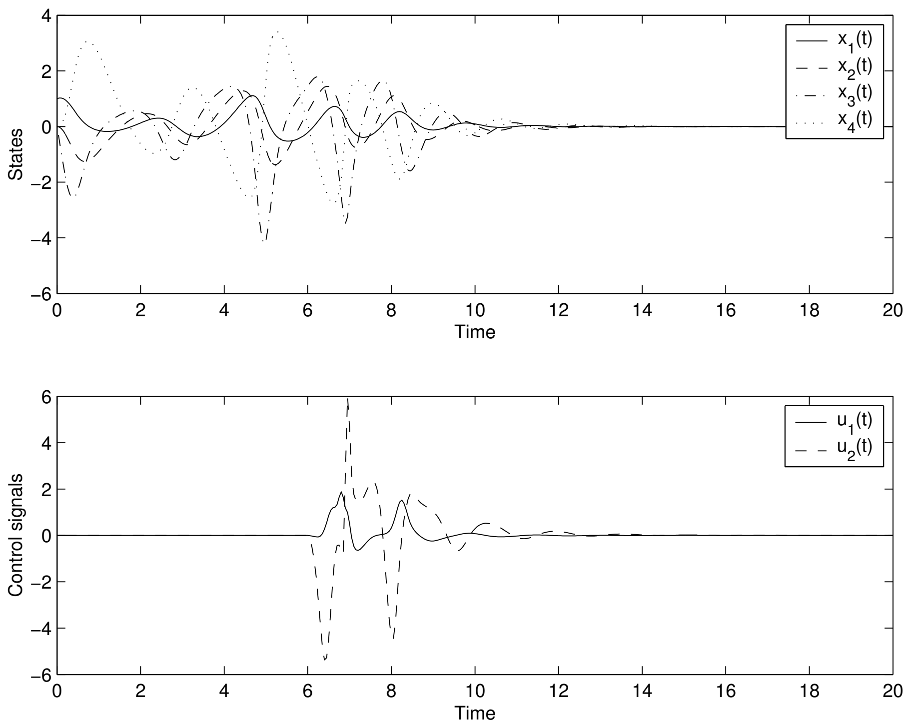



**Figure 3.6**: State trajectories and control signals versus time

# Chapter 4

# Adaptive Control for Nonlinear Uncertain Systems with Actuator Amplitude and Rate Saturation Constraints

## 4.1. Introduction

In light of the increasingly complex and highly uncertain nature of dynamical systems requiring controls, it is not surprising that reliable system models for many high performance engineering applications are unavailable. In the face of such high levels of system uncertainty, robust controllers may unnecessarily sacrifice system performance whereas adaptive controllers are clearly appropriate since they can tolerate far greater system uncertainty levels to improve system performance. However, an implicit assumption inherent in most adaptive control frameworks is that the adaptive control law is implemented without any regard to actuator amplitude and rate saturation constraints. Of course, any electromechanical control actuation device is subject to amplitude and/or rate constraints leading to saturation nonlinearities enforcing limitations on control amplitudes and control rates. As a consequence, actuator nonlinearities arise frequently in practice and can severely degrade closed-loop system performance, and in some cases drive the system to instability. These effects are even more pronounced for adaptive controllers which continue to adapt when the feedback loop has been severed due to the presence of actuator saturation causing unstable controller modes to drift, which in turn leads to severe windup effects.

The research literature on adaptive control with actuator saturation effects is rather limited. Notable exceptions include [1, 7, 129, 134, 186, 191, 244]. However, the results reported in [1, 7, 134, 186, 191, 244] are confined to linear plants with amplitude saturation. Many practical applications involve nonlinear dynamical systems with simultaneous control amplitude and rate saturation. The presence of control rate saturation may further exacerbate the problem of control amplitude saturation. For example, in advanced tactical fighter aircraft with high maneuverability requirements, pilot induced oscillations [109, 171] can cause actuator amplitude and rate saturation in the control surfaces, leading to catastrophic failures.

In this Chapter we develop a direct adaptive control framework for adaptive tracking of multivariable nonlinear uncertain systems with amplitude and rate saturation constraints. In particular, we extend the Lyapunov-based direct adaptive control framework developed in Chapter 2 (see also [84, 90, 91]) to guarantee partial asymptotic stability of the closed-loop tracking system; that is, asymptotic stability with respect to the closed-loop system error states associated with the tracking error dynamics, in the face of actuator amplitude and rate saturation constraints. Specifically, a reference (governor or supervisor) dynamical system is constructed to address tracking and regulation by deriving adaptive update laws that guarantee that the error system dynamics are asymptotically stable and the adaptive controller gains are Lyapunov stable. In the case where the actuator amplitude and rate are limited, the adaptive control signal to the reference system is modified to effectively robustify the error dynamics to the saturation constraints and thus guaranteeing asymptotic stability of the error states.

## 4.2. Adaptive Tracking for Nonlinear Uncertain Systems

In this section we consider the problem of characterizing adaptive feedback tracking control laws for nonlinear uncertain systems. Specifically, we consider the controlled nonlinear uncertain system $\mathcal{G}$ given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad t \geq 0, \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $f : \mathbb{R}^n \to \mathbb{R}^n$, and $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$. The control input $u(\cdot)$ in (4.1) is restricted to the class of *admissible controls* such that (4.1) has a unique solution forward in time. Here, we assume that a *desired* trajectory (command) $x_{\rm d}(t)$, $t \geq 0$, is given and the aim is to determine the control input $u(t)$, $t \geq 0$, so that $\lim_{t\to\infty} \|x(t) - x_{\rm d}(t)\| = 0$. To achieve this, we construct a reference system $\mathcal{G}_{\rm r}$ given by

$$\dot{x}_{\rm r}(t) = A_{\rm r}x_{\rm r}(t) + B_{\rm r}r(t), \quad x_{\rm r}(0) = x_{{\rm r}_0}, \quad t \geq 0, \tag{4.2}$$

where $x_{\rm r}(t) \in \mathbb{R}^n$, $t \geq 0$, is the reference state vector, $r(t) \in \mathbb{R}^m$, $t \geq 0$, is the reference input, and $A_{\rm r} \in \mathbb{R}^{n\times n}$ and $B_{\rm r} \in \mathbb{R}^{n\times m}$ are such that the pair $(A_{\rm r}, B_{\rm r})$ is stabilizable. Now, we design $u(t)$, $t \geq 0$, and a bounded piecewise-continuous reference function $r(t)$, $t \geq 0$, such that $\lim_{t\to\infty} \|x(t) - x_{\rm r}(t)\| = 0$ and $\lim_{t\to\infty} \|x_{\rm r}(t) - x_{\rm d}(t)\| = 0$ so that $\lim_{t\to\infty} \|x(t) - x_{\rm d}(t)\| = 0$. The following result provides a control architecture that achieves tracking error convergence in the case where the dynamics in (4.1) are known. The case where $\mathcal{G}$ is unknown is addressed in Theorem 4.2. For the statement of this result define the tracking error $e(t) \triangleq x(t) - x_{\rm r}(t)$.

**Theorem 4.1.** Consider the nonlinear system $\mathcal{G}$ given by (4.1) and the reference system $\mathcal{G}_{\rm r}$ given by (4.2). Assume there exist gain matrices $\hat{K}_1 \in \mathbb{R}^{m\times m}$ and $\hat{K}_2 \in$

$\mathbb{R}^{m\times s}$, and functions $\hat{G}:\mathbb{R}^n\to\mathbb{R}^{m\times m}$ and $F:\mathbb{R}^n\to\mathbb{R}^s$ such that

$$0 = G(x)\hat{G}(x)\hat{K}_1 - B_{\rm r}, \quad x\in\mathbb{R}^n, \tag{4.3}$$

$$0 = f(x) + B_{\rm r}\hat{K}_2F(x) - A_{\rm r}x, \quad x\in\mathbb{R}^n, \tag{4.4}$$

hold. Furthermore, let $K\in\mathbb{R}^{m\times n}$ be given by

$$K = -R_2^{-1}B_{\rm r}^{\rm T}P, \tag{4.5}$$

where the $n\times n$ positive-definite matrix $P$ satisfies

$$0 = A_{\rm r}^{\rm T}P + PA_{\rm r} - PB_{\rm r}R_2^{-1}B_{\rm r}^{\rm T}P + R_1, \tag{4.6}$$

and $R_1\in\mathbb{R}^{n\times n}$ and $R_2\in\mathbb{R}^{m\times m}$ are arbitrary positive-definite matrices. Then the feedback control law

$$u(t) = \hat{G}(x(t))\hat{K}_1(r(t) + \hat{K}_2F(x(t)) + Ke(t)), \tag{4.7}$$

guarantees that the zero solution $e(t)\equiv 0$ of the error dynamics given by

$$\dot{e}(t) = (f(x(t)) + G(x(t))u(t)) - (A_{\rm r}x_{\rm r}(t) + B_{\rm r}r(t)), \quad e(0) = x_0 - x_{{\rm r}0} \triangleq e_0, \quad t\geq 0, \tag{4.8}$$

is globally asymptotically stable.

**Proof.** Using the feedback control law given by (4.7), (4.8) becomes

$$\begin{aligned}\dot{e}(t) = {} & (A_{\rm r} + G(x(t))\hat{G}(x(t))\hat{K}_1K)e(t)\\ & +(G(x(t))\hat{G}(x(t))\hat{K}_1\hat{K}_2F(x(t)) + f(x(t)) - A_{\rm r}x(t))\\ & +(G(x(t))\hat{G}(x(t))\hat{K}_1 - B_{\rm r})r(t), \quad e(0) = e_0, \quad t\geq 0.\end{aligned} \tag{4.9}$$

Now, using (4.3) and (4.4), it follows from (4.9) that

$$\dot{e}(t) = (A_{\rm r} + B_{\rm r}K)e(t), \quad e(0) = e_0, \quad t\geq 0. \tag{4.10}$$

Finally, since $(A_{\rm r}, B_{\rm r})$ is stabilizable and $R_1 > 0$, it follows from standard linear-quadratic regulator theory that $A_{\rm r} + B_{\rm r}K$, with $K$ given by (4.5), is Hurwitz. Hence, the zero solution $e(t) \equiv 0$ to (4.8) is globally asymptotically stable. $\square$

Theorem 4.1 provides sufficient conditions for characterizing tracking controllers for a given nominal nonlinear dynamical system $\mathcal{G}$. In the next result we show how to construct adaptive gains $K_1(t) \in \mathbb{R}^{m\times m}$, $t \geq 0$, and $K_2(t) \in \mathbb{R}^{m\times s}$, $t \geq 0$, for achieving tracking control in the face of system uncertainty. For this result we do *not* require explicit knowledge of the gain matrices $\hat{K}_1$ and $\hat{K}_2$; all that is required is the existence of $\hat{K}_1$ and $\hat{K}_2$ such that the compatibility relations (4.3) and (4.4) hold. Furthermore, we shall require that $\det \hat{K}_1 \neq 0$.

**Theorem 4.2.** Consider the nonlinear system $\mathcal{G}$ given by (4.1) and the reference system $\mathcal{G}_{\rm r}$ given by (4.2). Assume there exist gain matrices $\hat{K}_1 \in \mathbb{R}^{m\times m}$ and $\hat{K}_2 \in \mathbb{R}^{m\times s}$, with $\det \hat{K}_1 \neq 0$, and functions $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ and $F : \mathbb{R}^n \to \mathbb{R}^s$, with $\det \hat{G}(x) \neq 0$, $x \in \mathbb{R}^n$, such that (4.3) and (4.4) hold. Furthermore, let $K \in \mathbb{R}^{m\times n}$ be given by (4.5), where $P > 0$ satisfies (4.6). In addition, let $Q_1, Q_2 \in \mathbb{R}^{m\times m}$ be positive definite. Then the adaptive feedback control law

$$u(t) = \hat{G}(x(t))K_1(t)(r(t) + K_2(t)F(x(t)) + Ke(t)), \tag{4.11}$$

where $K_1(t) \in \mathbb{R}^{m\times m}$, $t \geq 0$, and $K_2(t) \in \mathbb{R}^{m\times s}$, $t \geq 0$, with update laws

$$\dot{K}_1(t) = -K_1(t)Q_1B_{\rm r}^{\rm T}Pe(t)u^{\rm T}(t)\hat{G}^{-{\rm T}}(x(t))K_1(t), \tag{4.12}$$

$$\dot{K}_2(t) = -Q_2B_{\rm r}^{\rm T}Pe(t)F^{\rm T}(x(t)), \tag{4.13}$$

guarantees that there exists a neighborhood $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^{m\times m} \times \mathbb{R}^{m\times s}$ of $(0, \hat{K}_1, \hat{K}_2)$ such that if $(e(0), K_1(0), K_2(0)) \in \mathcal{D}$, then the solution $(e(t), K_1(t), K_2(t)) \equiv (0, \hat{K}_1, \hat{K}_2)$ of the closed-loop system given by (4.11)–(4.13) is Lyapunov stable and $e(t) \to 0$ as $t \to \infty$.

**Proof.** With $u(t)$, $t \geq 0$, given by (4.11) it follows from (4.3), (4.4), (4.12), and (4.13), that the error dynamics $e(t)$, $t \geq 0$, are given by

$$\dot{e}(t) = (A_{\rm r} + B_{\rm r}K)e(t) + B_{\rm r}(\hat{K}_1^{-1} - K_1^{-1}(t))\hat{G}^{-1}(x(t))u(t) + B_{\rm r}(K_2(t) - \hat{K}_2)F(x(t)),$$
$$e(0) = e_0, \quad t \geq 0. \quad (4.14)$$

To show Lyapunov stability of the closed-loop system (4.11)–(4.14), consider the Lyapunov function candidate

$$V(e, K_1, K_2) = e^{\rm T}Pe + {\rm tr}(\hat{K}_1^{-1} - K_1^{-1})^{\rm T}Q_1^{-1}(\hat{K}_1^{-1} - K_1^{-1}) + {\rm tr}(K_2 - \hat{K}_2)^{\rm T}Q_2^{-1}(K_2 - \hat{K}_2), \quad (4.15)$$

where $P > 0$ satisfies (4.6). Note that $V(0, \hat{K}_1, \hat{K}_2) = 0$ and, since $P$, $Q_1$, and $Q_2$ are positive definite, $V(e, K_1, K_2) > 0$ for all $(e, K_1, K_2) \neq (0, \hat{K}_1, \hat{K}_2)$. Now, letting $e(t)$, $t \geq 0$, denote the solution to (4.14), using (4.6), (4.11)–(4.13), and using the fact that $\frac{\rm d}{{\rm d}t}(K_1^{-1}(t)) = -K_1^{-1}(t)\dot{K}_1(t)K_1^{-1}(t)$, it follows that the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned}
\dot{V}(e(t), K_1(t), K_2(t)) &= e^{\rm T}(t)P\dot{e}(t) + \dot{e}^{\rm T}(t)Pe(t) \\
&\quad +2{\rm tr}(\hat{K}_1^{-1} - K_1^{-1}(t))^{\rm T}Q_1^{-1}\tfrac{\rm d}{{\rm d}t}(-K_1^{-1}(t)) \\
&\quad +2{\rm tr}(K_2(t) - \hat{K}_2)^{\rm T}Q_2^{-1}\dot{K}_2(t) \\
&= e^{\rm T}(t)P(A_{\rm r} + B_{\rm r}K)e(t) + e^{\rm T}(t)(A_{\rm r} + B_{\rm r}K)^{\rm T}Pe(t) \\
&\quad +2e^{\rm T}(t)PB_{\rm r}(\hat{K}_1^{-1} - K_1^{-1}(t))\hat{G}^{-1}(x(t))u(t) \\
&\quad +2e^{\rm T}(t)PB_{\rm r}(K_2(t) - \hat{K}_2)F(x(t)) \\
&\quad -2{\rm tr}(\hat{K}_1^{-1} - K_1^{-1}(t))^{\rm T}B_{\rm r}^{\rm T}Pe(t)u^{\rm T}(t)\hat{G}^{-{\rm T}}(x(t)) \\
&\quad -2{\rm tr}(K_2(t) - \hat{K}_2)^{\rm T}B_{\rm r}^{\rm T}Pe(t)F^{\rm T}(x(t)) \\
&= -e^{\rm T}(t)(R_1 + K^{\rm T}R_2K)e(t) \\
&\quad +2{\rm tr}\hat{G}^{-1}(x(t))u(t)e^{\rm T}(t)PB_{\rm r}(\hat{K}_1^{-1} - K_1^{-1}(t)) \\
&\quad +2{\rm tr}F(x(t))e^{\rm T}(t)PB_{\rm r}(K_2(t) - \hat{K}_2)
\end{aligned}$$

$$
\begin{aligned}
&\quad -2\mathrm{tr}\hat{G}^{-1}(x(t))u(t)e^{\mathrm{T}}(t)PB_{\mathrm{r}}(\hat{K}_1^{-1} - K_1^{-1}(t)) \\
&\quad -2\mathrm{tr}F(x(t))e^{\mathrm{T}}(t)PB_{\mathrm{r}}(K_2(t) - \hat{K}_2) \\
&= -e^{\mathrm{T}}(t)(R_1 + K^{\mathrm{T}}R_2K)e(t) \\
&\leq 0, \quad t \geq 0, \qquad (4.16)
\end{aligned}
$$

which proves that there exists a neighborhood $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^{m\times m} \times \mathbb{R}^{m\times q}$ of $(0, \hat{K}_1, \hat{K}_2)$ such that if $(e(0), K_1(0),\ K_2(0)) \in \mathcal{D}$, then the solution $(e(t), K_1(t), K_2(t)) \equiv (0, \hat{K}_1, \hat{K}_2)$ of the closed-loop system given by (4.11)–(4.14) is Lyapunov stable. Furthermore, since $R_1 + K^{\mathrm{T}}R_2K > 0$, it follows from Theorem 2 of [42] that $e(t) \to 0$ as $t \to \infty$ for all $(e(0), K_1(0),\ K_2(0)) \in \mathcal{D}$. □

**Remark 4.1.** Note that the conditions in Theorem 4.2 imply that $e(t) \to 0$ as $t \to \infty$ and hence it follows from (4.12) and (4.13) that $(e(t), K_1(t), K_2(t)) \to \mathcal{M} \triangleq \{(e, K_1, K_2) \in \mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^{m\times m} \times \mathbb{R}^{m\times s} : e = 0,\ \dot{K}_1 = 0,\ \dot{K}_2 = 0\}$ as $t \to \infty$.

**Remark 4.2.** Note the Lyapunov function candidate (4.15) is not radially unbounded with respect to $K_1$, and hence Theorem 4.2 provides local stability guarantees. However, if $G(x)$, $x \in \mathbb{R}^n$, is known, then $K_1(t)$, $t \geq 0$, can be taken to be the constant gain matrix $\hat{K}_1$ so that (4.12) is superfluous. In this case, the adaptive feedback control law (4.11) with update law (4.13) guarantees that the solution $(e(t), K_2(t)) \equiv (0, \hat{K}_2)$ of the closed-loop system given by (4.11), (4.13), and (4.14) is Lyapunov stable and $e(t) \to 0$ as $t \to \infty$ for all $e_0 \in \mathbb{R}^n$. For further details see Chapter 2 (see also [84, 90, 91]).

It is important to note that the adaptive law (4.11)–(4.13) does *not* require explicit knowledge of the gain matrices $\hat{K}_1$ and $\hat{K}_2$. Furthermore, no specific structure on the nonlinear dynamics $f(x)$ and $G(x)$ are required to apply Theorem 4.2; all that is required is the existence of $F(x)$ and $\hat{G}(x)$ such that the compatibility relations

(4.3) and (4.4) hold for a given reference system $\mathcal{G}_{\mathrm{r}}$. The compatibility conditions (4.3) and (4.4) provide a generalization to the stronger conditions already existing in the literature required for tracking control using feedback linearization techniques. However, if (4.1) is in normal form with asymptotically stable internal dynamics [122], then we can always construct functions $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ and $F : \mathbb{R}^n \to \mathbb{R}^s$, with $\det \hat{G}(x) \neq 0$, $x \in \mathbb{R}^n$, and a stabilizable pair $(A_{\mathrm{r}}, B_{\mathrm{r}})$ such that (4.3) and (4.4) hold *without* requiring knowledge of the system dynamics. To see this assume that the nonlinear uncertain system $\mathcal{G}$ is generated by

$$q_i^{(r_i)}(t) = f_{\mathrm{u}i}(q(t)) + \sum_{j=1}^{m} G_{\mathrm{s}(i,j)}(q(t))u_j(t), \quad t \geq 0, \quad i = 1, \cdots, m, \tag{4.17}$$

where $q = [q_1, \cdots, q_1^{(r_1-1)}, \cdots, q_m, \cdots, q_m^{(r_m-1)}]^{\mathrm{T}}$, $q(0) = q_0$, $q_i^{(r_i)}$ denotes the $r_i^{\mathrm{th}}$ derivative of $q_i$, and $r_i$ denotes the relative degree with respect to the output $q_i$. Here we assume that the square matrix function $G_{\mathrm{s}}(q)$ composed of the entries $G_{\mathrm{s}(i,j)}(q)$, $i, j = 1, \cdots, m$, is such that $\det G_{\mathrm{s}}(q) \neq 0$, $q \in \mathbb{R}^{\hat{r}}$, where $\hat{r} = r_1 + \cdots + r_m$ is the (vector) relative degree of (4.17). Furthermore, since (4.17) is in a form where it does not possess internal dynamics, it follows that $\hat{r} = n$. The case where (4.17) possesses input-to-state stable internal dynamics can be handled as shown in Section 2.2.

Next, define $x_i \triangleq \left[ q_i, \cdots, q_i^{(r_i-2)} \right]^{\mathrm{T}}$, $i = 1, \cdots, m$, $x_{m+1} \triangleq \left[ q_1^{(r_1-1)}, \cdots, q_m^{(r_m-1)} \right]^{\mathrm{T}}$, and $x \triangleq \left[ x_1^{\mathrm{T}}, \cdots, x_{m+1}^{\mathrm{T}} \right]^{\mathrm{T}}$, so that (4.17) can be described as (4.1) with

$$f(x) = \tilde{A}x + \tilde{f}_{\mathrm{u}}(x), \quad G(x) = \begin{bmatrix} 0_{(n-m)\times m} \\ G_{\mathrm{s}}(x) \end{bmatrix}, \tag{4.18}$$

where

$$\tilde{A} = \begin{bmatrix} A_0 \\ 0_{m\times n} \end{bmatrix}, \quad \tilde{f}_{\mathrm{u}}(x) = \begin{bmatrix} 0_{(n-m)\times 1} \\ f_{\mathrm{u}}(x) \end{bmatrix},$$

$A_0 \in \mathbb{R}^{(n-m)\times n}$ is a known matrix of zeros and ones capturing the multivariable controllable canonical form representation [43], and $f_{\mathrm{u}} : \mathbb{R}^n \to \mathbb{R}^m$ and $G_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ are unknown functions. Here, we assume that $f_{\mathrm{u}}(x)$ and $G_{\mathrm{s}}(x)$ are unknown and are parameterized as $f_{\rm u}(x) = \Theta_\ell x + \Theta_{n\ell} f_{n\ell}(x)$, where $f_{n\ell} : \mathbb{R}^n \to \mathbb{R}^q$, $G_{\rm s}(x) = \Phi \bar{G}_{\rm s}(x)$, where $\bar{G}_{\rm s} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ and satisfies $\det \bar{G}_{\rm s}(x) \neq 0$, $x \in \mathbb{R}^n$, and $\Theta_\ell \in \mathbb{R}^{m\times n}$, $\Theta_{n\ell} \in \mathbb{R}^{m\times q}$, and $\Phi \in \mathbb{R}^{m\times m}$, with $\det \Phi \neq 0$, are matrices of uncertain constant parameters.

Next, to apply Theorem 4.2 to the uncertain system (4.1) with $f(x)$ and $G(x)$ given by (4.18), let $B_{\rm r} = \left[0_{(n-m)\times m}, B_{\rm rs}^{\rm T}\right]^{\rm T}$, where $B_{\rm rs} \in \mathbb{R}^{m\times m}$ is invertible, let $A_{\rm r} = \left[A_0^{\rm T}, \Theta_{\rm n}^{\rm T}\right]^{\rm T}$, where $\Theta_{\rm n} \in \mathbb{R}^{m\times n}$ is a known matrix, let $\hat{K}_2 \in \mathbb{R}^{m\times s}$, where $s = n + q$, be given by

$$\hat{K}_2 = B_{\rm rs}^{-1}[\,\Theta_{\rm n} - \Theta_\ell,\ -\Theta_{n\ell}\,], \tag{4.19}$$

and let

$$F(x) = \begin{bmatrix} x \\ f_{n\ell}(x) \end{bmatrix}. \tag{4.20}$$

In this case, it follows that, with $\hat{G}(x) = \bar{G}_{\rm s}^{-1}(x)$ and $\hat{K}_1 = \Phi^{-1} B_{\rm rs}$,

$$G(x)\hat{G}(x)\hat{K}_1 = B_{\rm r} \tag{4.21}$$

and

$$\begin{aligned} f(x) + B_{\rm r}\hat{K}_2 F(x) &= \tilde{A}x + \tilde{f}_{\rm u}(x) + \begin{bmatrix} 0_{(n-m)\times m} \\ B_{\rm rs} \end{bmatrix} B_{\rm rs}^{-1} \left[\ \Theta_{\rm n} x - \Theta_\ell x - \Theta_{n\ell} f_{n\ell}(x)\ \right] \\ &= \tilde{A}x + \begin{bmatrix} 0_{(n-m)\times 1} \\ \Theta_{\rm n} x \end{bmatrix} \\ &= A_{\rm r} x, \end{aligned} \tag{4.22}$$

where $A_{\rm r}$ is in multivariable controllable canonical form. Hence, choosing $A_{\rm r}$ and $B_{\rm r}$ such that $(A_{\rm r}, B_{\rm r})$ is stabilizable and choosing $R_1 > 0$ and $R_2 > 0$, it follows that there exists a positive-definite matrix $P$ satisfying the Riccati equation (4.6).

## 4.3. Adaptive Tracking with Actuator Amplitude and Rate Saturation Constraints

In this section we extend the adaptive control framework presented in Section 4.2 to account for actuator amplitude and rate saturation constraints. Recall that Theorem 4.2 guarantees that the tracking error $e(t)$, $t \geq 0$, converges to zero; that is, the state vector $x(t)$, $t \geq 0$, converges to the reference state vector $x_{\mathrm{r}}(t)$, $t \geq 0$. Furthermore, it is important to note that $x_{\mathrm{r}}(t)$, $t \geq 0$, does not directly appear in the control signal $u(t)$, $t \geq 0$, given by (4.11), which depends on the *reference input* $r(t)$, $t \geq 0$. However, since for a fixed set of initial conditions there exists a one-to-one mapping between the reference input $r(t)$, $t \geq 0$, and the reference state $x_{\mathrm{r}}(t)$, $t \geq 0$, it follows that the control signal (4.11) guarantees convergence of the state $x(t)$, $t \geq 0$, to the reference state $x_{\mathrm{r}}(t)$, $t \geq 0$, corresponding to the specified reference input $r(t)$, $t \geq 0$. Of course, the reference input $r(t)$, $t \geq 0$, should be chosen so as to guarantee asymptotic convergence to a *desired* state vector $x_{\mathrm{d}}(t)$, $t \geq 0$. However, the choice of such a reference input $r(t)$, $t \geq 0$, is not unique since the reference state vector $x_{\mathrm{r}}(t)$, $t \geq 0$, can converge to the desired state vector $x_{\mathrm{d}}(t)$, $t \geq 0$, without matching its transient behavior.

Now, we provide a framework wherein we construct a family of reference inputs $r(t)$, $t \geq 0$, with associated reference state vectors $x_{\mathrm{r}}(t)$, $t \geq 0$, that guarantee that a given reference state vector within this family converges to a desired state vector $x_{\mathrm{d}}(t)$, $t \geq 0$, in the face of actuator amplitude and rate saturation constraints. We begin by solving for $r(t)$, $t \geq 0$, from the control law (4.11) using the assumption that $K_1(t)$, $t \geq 0$, is nonsingular to obtain the reference input as a function of the control input

$$r(t) = K_1^{-1}(t)\hat{G}^{-1}(x(t))u(t) - K_2(t)F(x(t)) - Ke(t), \quad t \geq 0. \tag{4.23}$$

Next, we assume that the control signal is amplitude and rate limited so that $u_{\min,i} \leq u_i(t) \leq u_{\max,i}$ and $\dot{u}_{\min,i} \leq \dot{u}_i(t) \leq \dot{u}_{\max,i}$, $t \geq 0$, $i = 1, \ldots, m$, where $u_i(t)$ and $\dot{u}_i(t)$ denote the $i$th component of $u(t)$ and $\dot{u}(t)$, respectively, and $u_{\min,i}$, $u_{\max,i}$, $\dot{u}_{\min,i}$, and $\dot{u}_{\max,i}$ are given such that $u_{\min,i} \leq u_{\max,i}$, $\dot{u}_{\min,i} \leq \dot{u}_{\max,i}$. For the statement of our main result the following definitions are needed. For $i \in \{1, \cdots, m\}$ define

$$\overline{u}_i(t) \triangleq \min\{u_{\max,i}, u_i(t-\epsilon) + \dot{u}_{\max,i}\epsilon\}, \quad t \geq 0, \tag{4.24}$$

$$\underline{u}_i(t) \triangleq \max\{u_{\min,i}, u_i(t-\epsilon) + \dot{u}_{\min,i}\epsilon\}, \quad t \geq 0, \tag{4.25}$$

where $\epsilon > 0$ is an arbitrary time which can be chosen as small as desired. The introduction of the infinitesimal time $\epsilon > 0$ is necessary since the time derivative of $x(t)$, $t \geq 0$, and the time derivative of $u(t)$, $t \geq 0$, are not available. However, since $x(t)$ and $u(t)$ are known at *all* times, estimates of these derivatives can be obtained with any specified accuracy. Now, define

$$\sigma(u_i(t)) \triangleq \begin{cases} u_i(t), & \text{if } \underline{u}_i(t) \leq u_i(t) \leq \overline{u}_i(t), \\ \overline{u}_i(t), & \text{if } u_i(t) > \overline{u}_i(t), \\ \underline{u}_i(t), & \text{if } u_i(t) < \underline{u}_i(t), \end{cases} \quad i = 1, \cdots, m, \quad t \geq 0. \tag{4.26}$$

It follows from (4.24) and (4.25) that if $\underline{u}_i(t) \leq u_i(t) \leq \overline{u}_i(t)$ at time $t \in [0, \infty)$, then $u_i(t)$ satisfies *both* the amplitude and rate saturation constraints. Finally, for the statement of our main theorem we define the component decoupled diagonal nonlinearity $\Sigma(u)$ by

$$\Sigma(u(t)) \triangleq \operatorname{diag}[\sigma(u_1(t)), \sigma(u_2(t)), \ldots, \sigma(u_m(t))], \quad t \geq 0. \tag{4.27}$$

**Theorem 4.3.** Consider the controlled nonlinear system $\mathcal{G}$ given by (4.1) and the reference system $\mathcal{G}_{\mathrm{r}}$ given by (4.2). Assume there exist gain matrices $\hat{K}_1 \in \mathbb{R}^{m\times m}$ and $\hat{K}_2 \in \mathbb{R}^{m\times s}$, with $\det \hat{K}_1 \neq 0$, and functions $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ and $F : \mathbb{R}^n \to \mathbb{R}^s$, with $\det \hat{G}(x) \neq 0$, $x \in \mathbb{R}^n$, such that (4.3) and (4.4) hold. Furthermore, let $K \in \mathbb{R}^{m\times n}$ be given by (4.5), where $P > 0$ satisfies (4.6). In addition, for a given desired reference input $r_{\rm d}(t)$, $t \geq 0$, let the reference input $r(t)$, $t \geq 0$, be given by

$$r(t) = K_1^{-1}(t)\hat{G}^{-1}(x(t))\Sigma(u^*(t)) - K_2(t)F(x(t)) - Ke(t), \quad t \geq 0, \tag{4.28}$$

where

$$u^*(t) = \hat{G}(x(t))K_1(t)(r^*(t) + K_2(t)F(x(t)) + Ke(t)), \tag{4.29}$$

$$\dot{r}^*(t) = \dot{r}_{\rm d}(t) + \Lambda(r(t) - r_{\rm d}(t)), \quad r^*(0) = r_0^*, \quad t \geq 0, \tag{4.30}$$

and where $\Lambda \in \mathbb{R}^{m\times m}$ is Hurwitz. Then the adaptive feedback control law (4.11), with update laws (4.12) and (4.13) and reference input $r(t)$, $t \geq 0$, satisfying (4.28)–(4.30) guarantees that:

*i)* There exists a neighborhood $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^{m\times m} \times \mathbb{R}^{m\times s}$ of $(0, \hat{K}_1, \hat{K}_2)$ such that if $(e(0), K_1(0), K_2(0)) \in \mathcal{D}$, then the solution $(e(t), K_1(t), K_2(t)) \equiv (0, \hat{K}_1, \hat{K}_2)$ of the closed-loop system given by (4.11)–(4.13) is Lyapunov stable and $e(t) \to 0$ as $t \to \infty$.

*ii)* $u_{\min,i} \leq u_i(t) \leq u_{\max,i}$ for all $t \geq 0$ and $i = 1, \ldots, m$.

*iii)* $\dot{u}_{\min,i} \leq \dot{u}_i(t) \leq \dot{u}_{\max,i}$ for all $t \geq 0$ and $i = 1, \ldots, m$.

*iv)* If there exists $t^* \geq 0$ such that $u^*(t)$, $t \geq t^*$, does not violate the amplitude and rate saturation constraints, then $\lim_{t\to\infty} \|r(t) - r_{\rm d}(t)\| = 0$ and $\lim_{t\to\infty} \|x_{\rm r}(t) - x_{\rm d}(t)\| = 0$.

**Proof.** *i)* is a direct consequence of Theorem 4.2 with $r(t)$, $t \geq 0$, satisfying (4.28)–(4.30). To prove *ii)* and *iii)* note that it follows from (4.11) and (4.28) that

$$\begin{aligned} u(t) &= \hat{G}(x(t))K_1(t)(r(t) + K_2(t)F(x(t)) + Ke(t)) \\ &= \hat{G}(x(t))K_1(t)\left[K_1^{-1}(t)\hat{G}^{-1}(x(t))\Sigma(u^*(t))\right] \\ &= \Sigma(u^*(t)), \quad t \geq 0, \end{aligned} \tag{4.31}$$

which implies $u_i(t) = \sigma(u_i^*(t))$, $i = 1, \cdots, m$. Hence, if the control input $u_i^*(t)$, $i \in \{1, \cdots, m\}$, does not violate the amplitude and rate saturation constraints at time $t \in [0, \infty)$, then it follows from (4.26) that $u_i(t) = u_i^*(t)$ at time $t \in [0, \infty)$. Alternatively, if $u_i^*(t)$, $i \in \{1, \cdots, m\}$, violates one or more of the input amplitude and/or rate constraints at time $t \in [0, \infty)$, then (4.26) and (4.31) imply

1) $u_i(t) = u_{\max,i}$ if $u_i^*(t) > u_{\max,i}$;

2) $u_i(t) = u_{\min,i}$ if $u_i^*(t) < u_{\min,i}$;

3) $\dot{u}_i(t) = \dot{u}_{\max,i}$ if $\frac{u_i^*(t) - u_i^*(t-\epsilon)}{\epsilon} > \dot{u}_{\max,i}$ and $u_{\min,i} \leq u_i^*(t) \leq u_{\max,i}$; and

4) $\dot{u}_i(t) = \dot{u}_{\min,i}$ if $\frac{u_i^*(t) - u_i^*(t-\epsilon)}{\epsilon} < \dot{u}_{\min,i}$ and $u_{\min,i} \leq u_i^*(t) \leq u_{\max,i}$;

which guarantee that $u_{\min,i} \leq u_i(t) \leq u_{\max,i}$ and $\dot{u}_{\min,i} \leq u_i(t) \leq \dot{u}_{\max,i}$ for all $t \geq 0$ and $i = 1, \cdots, m$. Finally, to show *iv)* let $t^* \geq 0$ be such that $\Sigma(u^*(t)) = u^*(t)$, $t \geq t^*$. In this case, $r(t) = K_1^{-1}(t)\hat{G}^{-1}(x(t))u^*(t) - K_2(t)F(x(t)) - Ke(t) = r^*(t)$, $t \geq t^*$. Hence, it follows from (4.30) that $\dot{r}(t) - \dot{r}_{\rm d}(t) = \Lambda(r(t) - r_{\rm d}(t))$, $t \geq t^*$, which, since by assumption $\Lambda \in \mathbb{R}^{m \times m}$ is Hurwitz, guarantees that $\lim_{t\to\infty} \|r(t) - r_{\rm d}(t)\| = 0$ and hence $\lim_{t\to\infty} \|x_{\rm r}(t) - x_{\rm d}(t)\| = 0$. □

Note that it follows from Theorem 4.3 that if the desired reference input $r_{\rm d}(t)$, $t \geq 0$, is such that the actuator amplitude and/or rate saturation constraints are not violated, then $r(t) = r_{\rm d}(t)$, $t \geq 0$, and hence $x(t)$, $t \geq 0$, converges to $x_{\rm d}(t)$, $t \geq 0$. Alternatively, if there exists $t^* > 0$ such that the desired reference input drives one or more of the control inputs to the saturation boundary, then $r(t) \not\equiv r_{\rm d}(t)$, $t \geq t^*$. In this case however, (4.30) guarantees that $\lim_{t\to\infty} \|r(t) - r_{\rm d}(t)\| = 0$ so long as the time interval over which the control input remains saturated is finite. If this is not the case, then our approach cannot guarantee convergence of the reference input $r(t)$, $t \geq 0$, to the desired reference $r_{\rm d}(t)$, $t \geq 0$. Of course, if there exists a solution to the tracking problem wherein the input amplitude and rate saturation constraints are not violated with the proposed controller when the tracking error is within certain bounds, then our approach is guaranteed to always work.

## 4.4. Illustrative Numerical Examples

In this section we present two numerical examples to demonstrate the utility of the proposed direct adaptive control framework for adaptive stabilization and tracking in the face of actuator amplitude and rate saturation constraints.

**Example 4.1.** Consider the uncertain controlled Liénard system given by

$$\ddot{z}(t)+\mu(z^4(t)-\alpha)\dot{z}(t)+\beta z(t)+\gamma\tanh(z(t))=bu(t), \quad z(0)=z_0, \quad \dot{z}(0)=\dot{z}_0, \quad t\geq 0, \tag{4.32}$$

where $\mu,\alpha,\beta,\gamma,b\in\mathbb{R}$ are unknown. Note that with $x_1=z$ and $x_2=\dot{z}$, (4.32) can be written in state space form (4.1) with $x=[x_1,x_2]^{\mathrm{T}}$, $f(x)=[x_2,-\beta x_1-\gamma\tanh x_1-\mu(x_1^4-\alpha)x_2]^{\mathrm{T}}$, and $G(x)=[0,b]^{\mathrm{T}}$. Here, we assume that $f(x)$ and $G(x)$ are unknown and can be parameterized as $f(x)=[x_2,\theta_\ell x+\theta_{n\ell 1}\tanh x_1+\theta_{n\ell 2}x_1^4x_2]^{\mathrm{T}}$ and $G(x)=b[0,\,1]^{\mathrm{T}}$, where $\theta_\ell\in\mathbb{R}^2$, $\theta_{n\ell 1}\in\mathbb{R}$, and $\theta_{n\ell 2}\in\mathbb{R}$ are unknown. Next, let $F(x)=[x^{\mathrm{T}},\tanh(x_1),x_1^4x_2]^{\mathrm{T}}$, $A_{\mathrm{r}}=[A_0^{\mathrm{T}},\theta_{\mathrm{n}}^{\mathrm{T}}]^{\mathrm{T}}$, $B_{\mathrm{r}}=[0,1]^{\mathrm{T}}$, $\hat{G}(x)\equiv 1$, $\hat{K}_1=\frac{1}{b}$, and $\hat{K}_2=[\theta_{\mathrm{n}}-\theta_\ell,-\theta_{n\ell 1},-\theta_{n\ell 2}]$, where $A_0=[0,1]$ and $\theta_{\mathrm{n}}$ is an arbitrary vector, so that

$$G(x)\hat{G}(x)\hat{K}_1=\begin{bmatrix}0\\ b\end{bmatrix}\cdot 1\cdot\frac{1}{b}=\begin{bmatrix}0\\ 1\end{bmatrix}=B_{\mathrm{r}},$$

$$\begin{aligned}f(x)+B_{\mathrm{r}}\hat{K}_2F(x) &= f(x)+\begin{bmatrix}0\\ 1\end{bmatrix}[\theta_{\mathrm{n}}-\theta_\ell,-\theta_{n\ell 1},-\theta_{n\ell 2}]F(x)\\ &=\begin{bmatrix}A_0\\ \theta_{\mathrm{n}}\end{bmatrix}x\\ &=A_{\mathrm{r}}x,\end{aligned}$$

and hence (4.3) and (4.4) hold. Now, it follows from Theorem 4.3 that the adaptive feedback controller (4.11) guarantees that $e(t)\to 0$ as $t\to\infty$ in the face of input

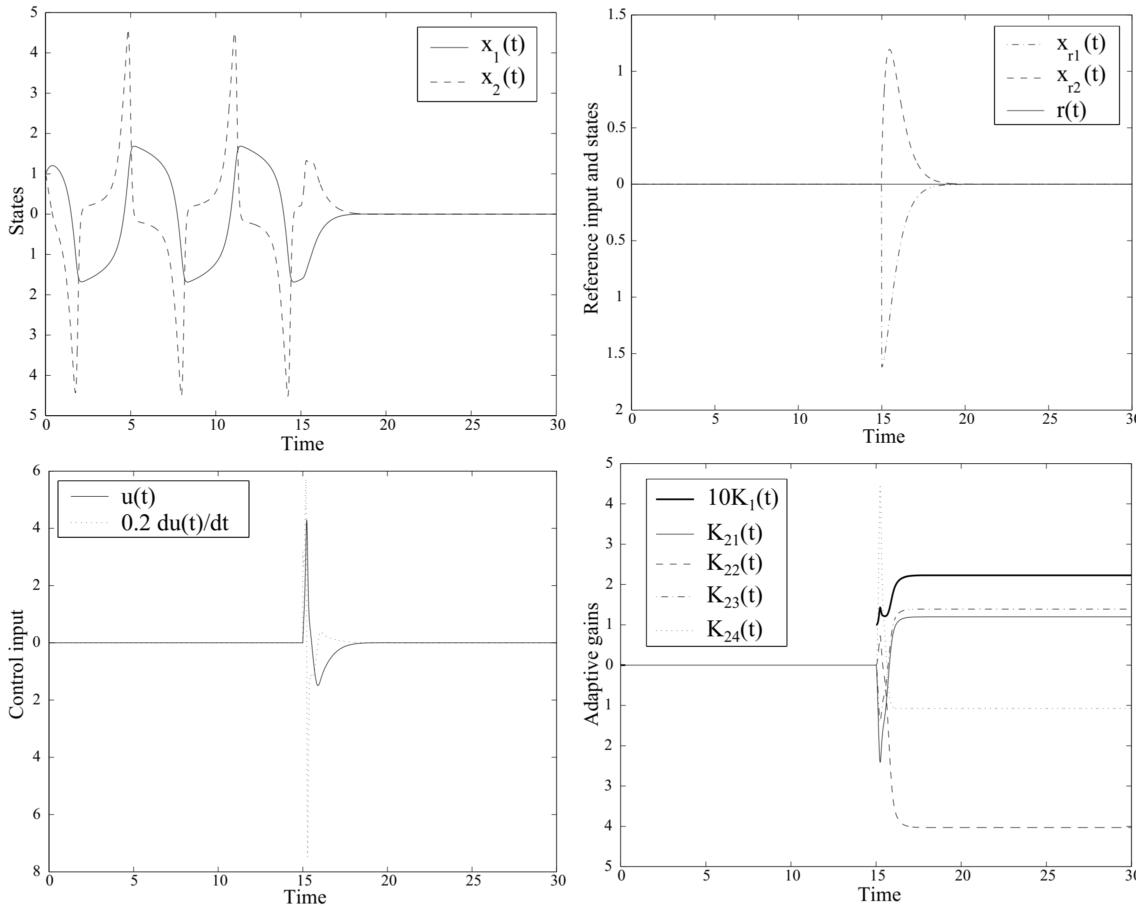



**Figure 4.1**: Stabilization of the Liénard system with no saturation constraints

amplitude and rate saturation constraints. Specifically, here we choose $\theta_{\rm n} = [-4, -4]$, $R_1 = 1000 I_2$, and $R_2 = 1$, so that $K$ and $P$ satisfying (4.5) and (4.6) are given by

$$P = \begin{bmatrix} 1027.5 & 27.9 \\ 27.9 & 28.7 \end{bmatrix}, \quad K = \begin{bmatrix} -27.9 & -28.7 \end{bmatrix}. \tag{4.33}$$

To analyze this design we assume that $\mu = 2$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $b = 3$, $Q_1 = 1$, $Q_2 = I_4$, $\epsilon = 10^{-6}$, with initial condition $x(0) = [1, 1]^{\rm T}$. First, we consider a regulation problem; that is, stabilization to the origin. Figure 4.1 shows the case where no input saturation constraints are considered and Figure 4.2 shows the case where $u_{\max} = -u_{\min} = 0.7$ and $\dot{u}_{\max} = -\dot{u}_{\min} = 2$. Note that the adaptive controller is switched on at $t = 15$ sec with $x_{\rm r}(15) = x(15)$, $K_1(15) = 0.1$, and $K_2(15) = [0, 0, 0, 0]^{\rm T}$.

Next, we consider the case where we seek to track $z_{\rm d}(t) = \sin t$. Figure 4.3 shows

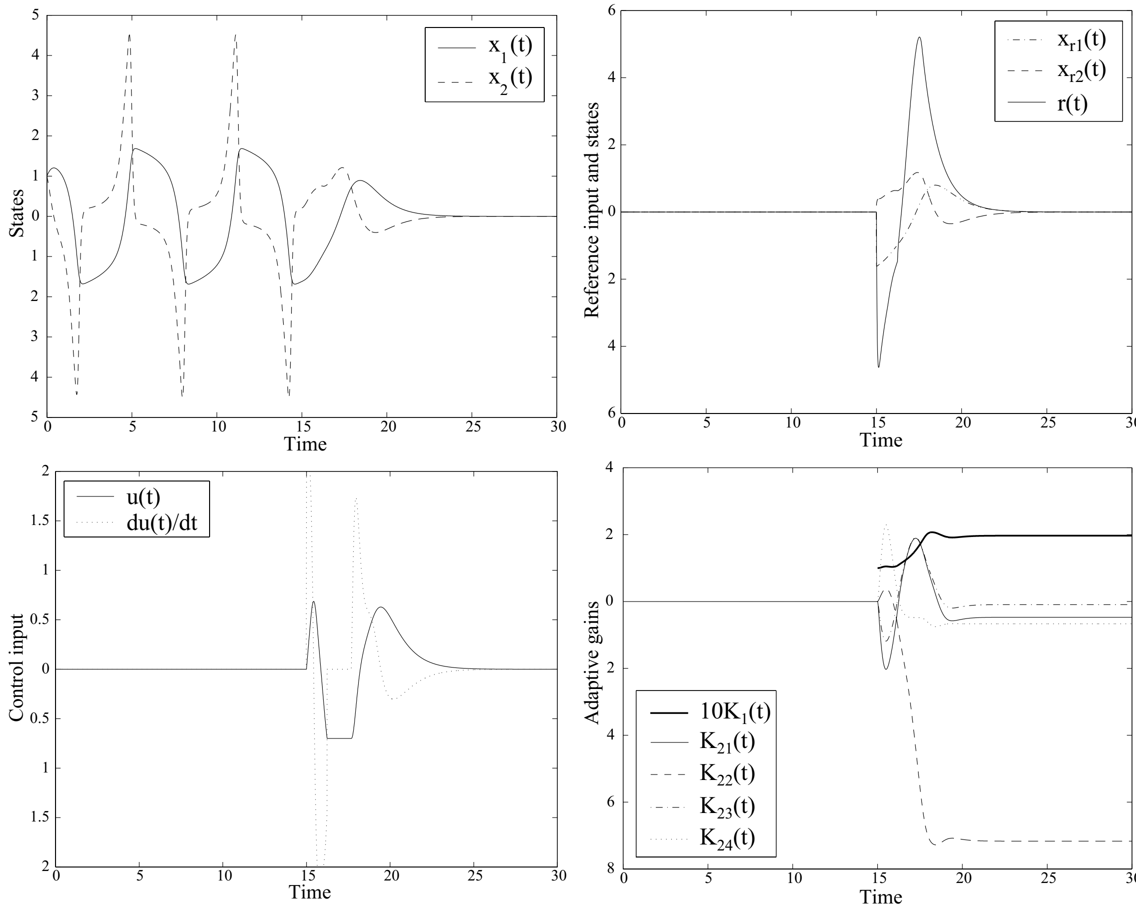



**Figure 4.2**: Stabilization of the Liénard system with amplitude and rate saturation constraints

the case where no input saturation constraints are considered and Figure 4.4 shows the case where $u_{\max} = -u_{\min} = 0.7$ and $\dot{u}_{\max} = -\dot{u}_{\min} = 2$. It is interesting to note that at the given amplitude and rate saturation levels the control signal remains periodically saturated and hence our formulation cannot guarantee that $x_{\mathrm{r}1}(t) \to x_{\mathrm{d}}(t)$ as $t \to \infty$. However, our approach provides a "close" agreement between the desired signal to be tracked and the achieved tracked signal for the given saturation levels. In the case where we slightly relax the saturation levels to $u_{\max} = -u_{\min} = 0.75$ and $\dot{u}_{\max} = -\dot{u}_{\min} = 2$, our approach guarantees perfect tracking (see Figure 4.5). Finally, we note that in the case where $u_{\max} = -u_{\min} = 0.75$ and $\dot{u}_{\max} = -\dot{u}_{\min} = 2$ and the adaptive controller of Theorem 4.2 is used without the reference input as in

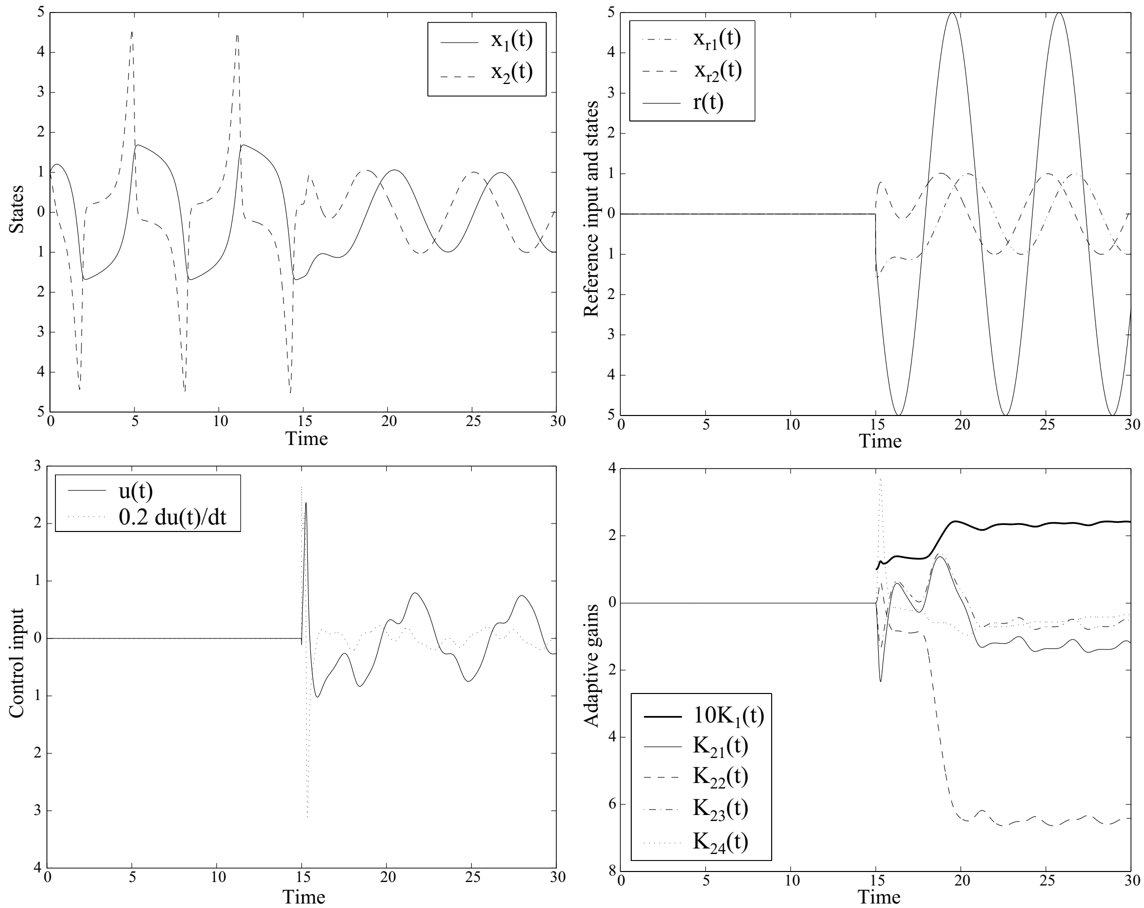



**Figure 4.3**: Tracking of the Liénard system with no saturation constraints

Theorem 4.3, the closed-loop system is unstable and neither regulation nor tracking can be achieved.

**Example 4.2.** Consider the nonlinear dynamical system representing a controlled rigid spacecraft given by

$$\dot{x}(t) = -I_{\mathrm{b}}^{-1} X I_{\mathrm{b}} x(t) + I_{\mathrm{b}}^{-1} u(t), \quad x(0) = x_0, \quad t \geq 0, \tag{4.34}$$

where $x = [x_1, x_2, x_3]^{\mathrm{T}}$ represents the angular velocities of the spacecraft with respect to the body-fixed frame, $I_{\mathrm{b}} \in \mathbb{R}^{3\times 3}$ is an unknown positive-definite inertia matrix of the spacecraft, $u = [u_1, u_2, u_3]^{\mathrm{T}}$ is a control vector with control inputs providing body-fixed torques about three mutually perpendicular axes defining the body-fixed

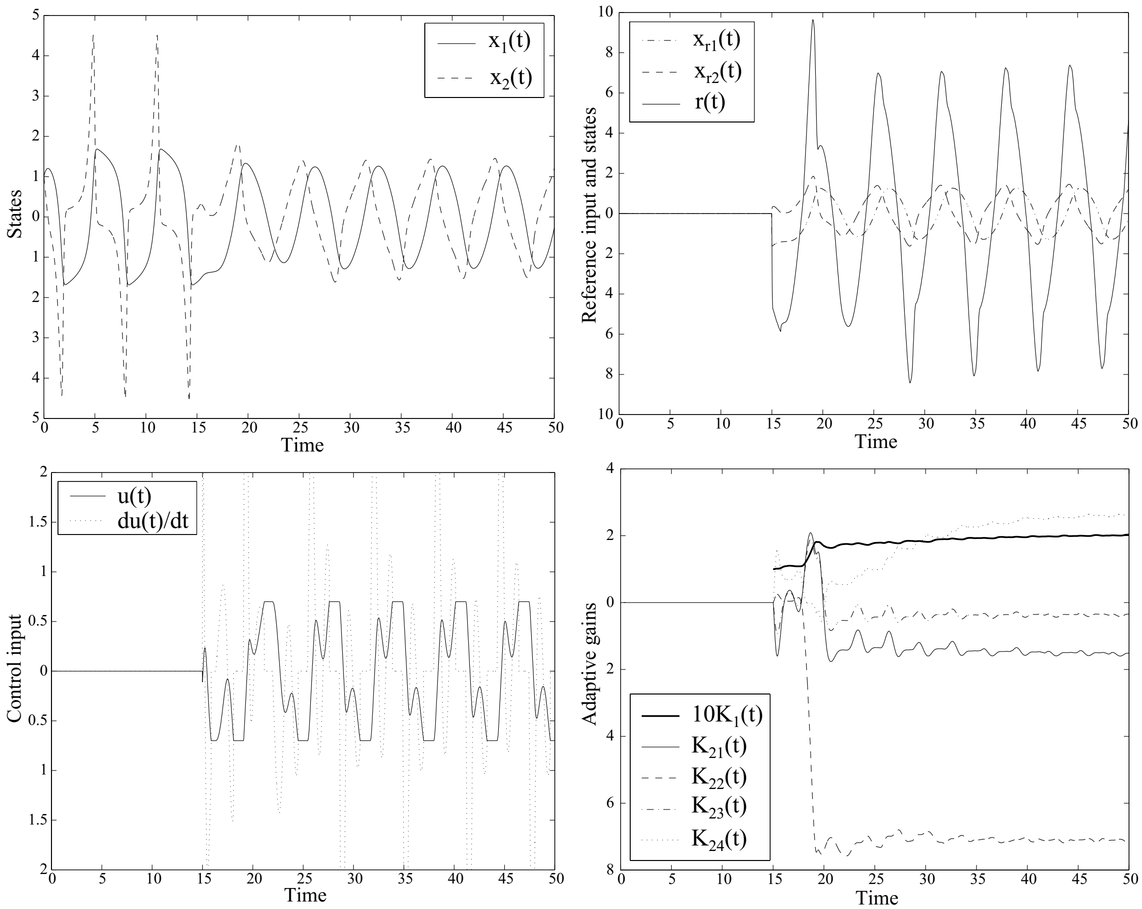



**Figure 4.4**: Tracking of the Liénard system with amplitude and rate saturation constraints

frame of the spacecraft, and $X$ denotes the skew-symmetric matrix

$$X \triangleq \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

Note that (4.34) can be written in state space form (4.1) with $f(x) = -I_{\rm b}^{-1} X I_{\rm b} x$ and $G(x) = I_{\rm b}^{-1}$. Since $f(x)$ is a quadratic function, we parameterize $f(x)$ as $f(x) = \Theta_{n\ell} f_{n\ell}(x)$, where $\Theta_{n\ell} \in \mathbb{R}^{3\times 6}$ is an unknown matrix and $f_{n\ell}(x) = [x_1^2, x_2^2, x_3^2, x_1x_2, x_2x_3, x_3x_1]^{\rm T}$. Next, let $F(x) = [x^{\rm T},\ f_{n\ell}^{\rm T}]^{\rm T}$, $B_{\rm r} = I_3$, $\hat{G}(x) \equiv B_{\rm r} = I_3$, $\hat{G}(x) \equiv I_3$, $\hat{K}_1 = I_{\rm b}$, and $\hat{K}_2[A_{\rm r},\ -\Theta_{n\ell}]$, so that

$$G(x)\hat{G}(x)\hat{K}_1 = I_{\rm b}^{-1} I_3 I_{\rm b} = I_3 = B_{\rm r},$$

$$f(x) + B_{\rm r}\hat{K}_2 F(x) = f(x) + I_3 \left[\ A_{\rm r},\ -\Theta_{n\ell}\ \right] F(x) = A_{\rm r} x,$$

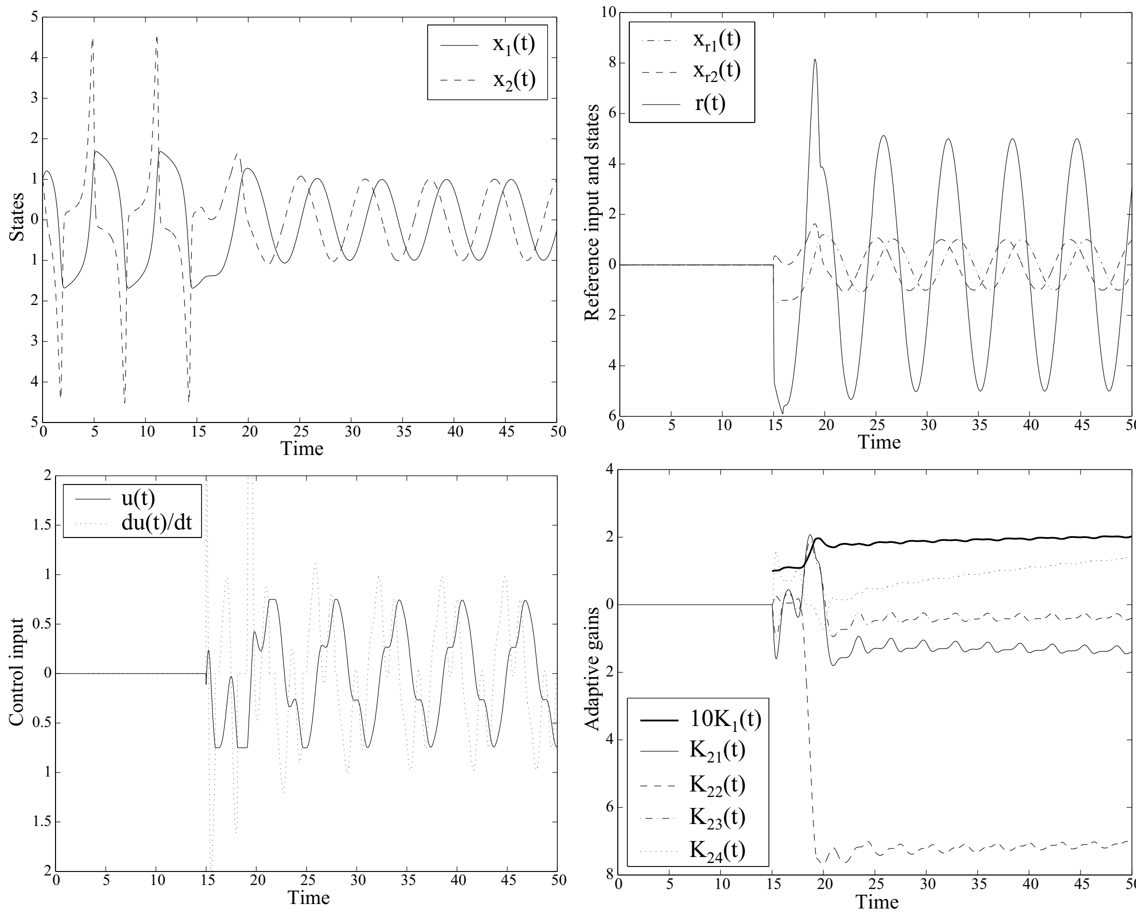



**Figure 4.5**: Tracking of the Liénard system with amplitude and rate saturation constraints

and hence (4.3) and (4.4) hold. Now, it follows from Theorem 4.3 that the adaptive feedback controller (4.11) guarantees that $e(t) \to 0$ as $t \to \infty$ when considering input amplitude and rate saturation constraints. Specifically, here we choose

$$A_{\mathrm{r}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -12 & -6 \end{bmatrix},$$

$R_1 = 100I_3$, and $R_2 = 0.1I_3$, so that $K$ and $P$ satisfying (4.5) are given by

$$P = \begin{bmatrix} 3.2219 & 0.1384 & -0.3097 \\ 0.1384 & 3.2932 & -0.4234 \\ -0.3097 & -0.4234 & 2.5623 \end{bmatrix}, \quad K = \begin{bmatrix} -32.2188 & -1.3837 & 3.0974 \\ -1.3837 & -32.9325 & 4.2344 \\ 3.0974 & 4.2344 & -25.6229 \end{bmatrix}.$$

To analyze this design we assume that

$$I_{\mathrm{b}} = \begin{bmatrix} 20 & 0 & 0.9 \\ 0 & 17 & 0 \\ 0.9 & 0 & 15 \end{bmatrix}, \quad Q_1 = Q_2 = I_3,$$

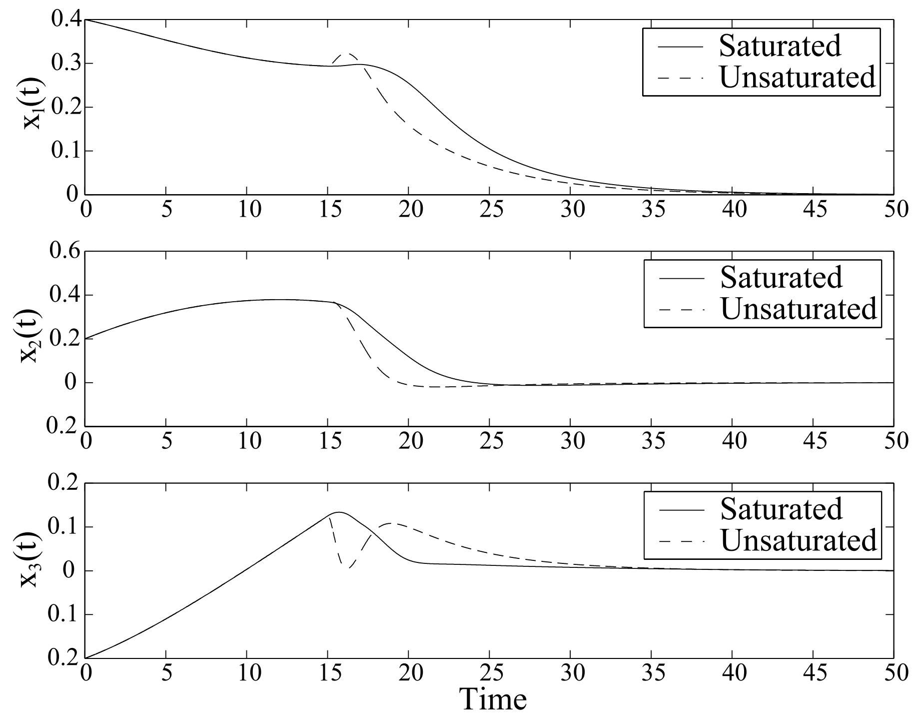



**Figure 4.6**: Angular velocities versus time

$\epsilon = 10^{-6}$, with initial condition $x(0) = [0.4, 0.2, -0.2]^{\mathrm{T}}$. Furthermore, we consider a regulation problem and switch the adaptive controller on at $t = 15$ sec with $x_{\mathrm{r}}(15) = x(15)$, $K_1(15) = 0.1I_3$, and $K_2(15) = 0_{3\times 9}$. Figure 4.6 shows the angular velocities versus time for the case where no saturation constraints are enforced and the case where $u_{\max} = -u_{\min} = 1$ and $\dot{u}_{\max} = -\dot{u}_{\min} = 0.5$. Figure 4.7 shows the corresponding control inputs and their time rate of change.

## 4.5. Conclusion

A direct adaptive nonlinear tracking control framework for multivariable nonlinear uncertain systems with actuator amplitude and rate saturation constraints was developed. By appropriately modifying the adaptive control signal to the reference system dynamics, the proposed approach guarantees asymptotic stability of the error system dynamics in the face of actuator amplitude and rate limitation constraints.

Finally, two numerical examples were presented to show the utility of the proposed adaptive tracking scheme.

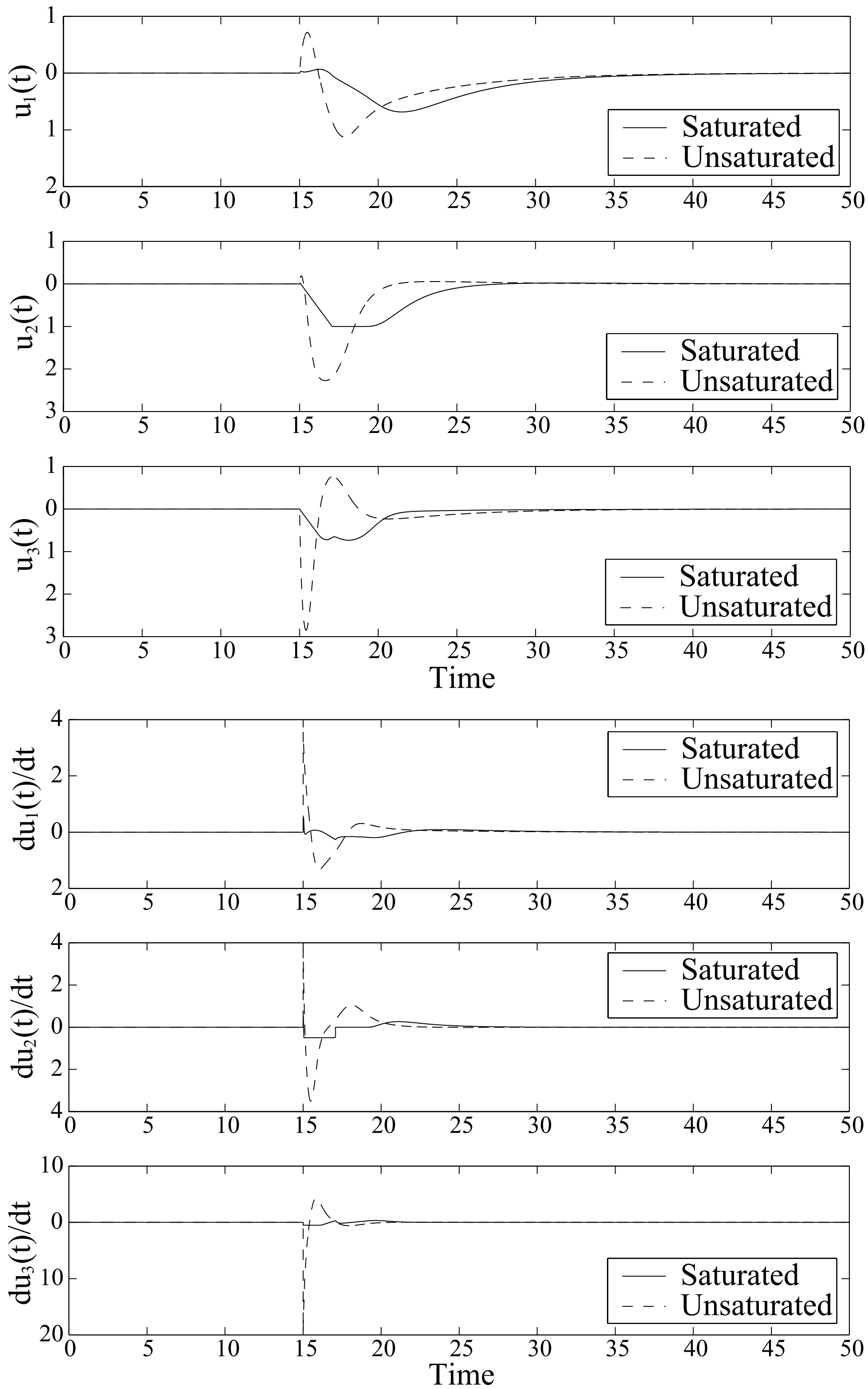



**Figure 4.7**: Control signals and rate versus time

# Chapter 5

# Adaptive Reduced-Order Dynamic Compensation for Nonlinear Uncertain Systems

## 5.1. Introduction

In this chapter a direct adaptive *reduced-order dynamic compensation* framework for nonlinear uncertain dynamical systems is developed. In particular, a Lyapunov-based direct adaptive fixed-order dynamic compensation framework is developed that guarantees partial asymptotic stability of the closed-loop system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the plant and compensator states. Furthermore, the remainder of the states associated with the adaptive dynamic controller gains are shown to be Lyapunov stable. In the case where the controlled nonlinear system is represented in normal form [122] with input-to-state stable internal dynamics [122, 222], the proposed nonlinear adaptive dynamic controller is constructed without requiring knowledge of the system dynamics. Finally, we emphasize that the direct adaptive stabilization framework presented herein builds on the nonlinear adaptive control results developed in Chapter 2 and is distinct from the methods given in [12, 121, 136, 139, 176] predicated on model reference adaptive control.

## 5.2. Adaptive Dynamic Control for Nonlinear Uncertain Systems

In this section we begin by considering the problem of characterizing adaptive reduced-order dynamic feedback control laws for nonlinear uncertain dynamical systems. Specifically, consider the nonlinear uncertain system $\mathcal{G}$ given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad t \geq 0, \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $f : \mathbb{R}^n \to \mathbb{R}^n$ and satisfies $f(0) = 0$, and $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$. Furthermore, consider the $n_{\mathrm{c}}^{\mathrm{th}}$-order adaptive dynamic compensator $\mathcal{G}_{\mathrm{c}}$ given by

$$\dot{x}_{\mathrm{c}}(t) = A_{\mathrm{c}}(t)x_{\mathrm{c}}(t) + B_{\mathrm{c}}(t)x(t), \quad x_{\mathrm{c}}(0) = x_{\mathrm{c}0}, \quad t \geq 0, \tag{5.2}$$

$$u(t) = \hat{G}(x(t))[C_{\mathrm{c}}(t)x_{\mathrm{c}}(t) + D_{\mathrm{c}}(t)F(x(t))], \tag{5.3}$$

where $x_{\mathrm{c}}(t) \in \mathbb{R}^{n_{\mathrm{c}}}$, $t \geq 0$, is the compensator state, $A_{\mathrm{c}} : \mathbb{R} \to \mathbb{R}^{n_{\mathrm{c}}\times n_{\mathrm{c}}}$, $B_{\mathrm{c}} : \mathbb{R} \to \mathbb{R}^{n_{\mathrm{c}}\times n}$, $C_{\mathrm{c}} : \mathbb{R} \to \mathbb{R}^{m\times n_{\mathrm{c}}}$, $D_{\mathrm{c}} : \mathbb{R} \to \mathbb{R}^{m\times s}$, $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, and $F : \mathbb{R}^n \to \mathbb{R}^s$ and satisfies $F(0) = 0$. For the nonlinear system $\mathcal{G}$ and the dynamic compensator $\mathcal{G}_{\mathrm{c}}$ we assume that the required properties for the existence and uniqueness of solutions are satisfied; that is, $f(\cdot)$, $G(\cdot)$, $A_{\mathrm{c}}(\cdot)$, $B_{\mathrm{c}}(\cdot)$, and $u(\cdot)$ satisfy sufficient regularity conditions such that the closed-loop system given by (5.1)–(5.3) has a unique solution forward in time. For the statement of the next result define $\tilde{n} \triangleq n + n_{\mathrm{c}}$.

**Theorem 5.1.** Consider the nonlinear system $\mathcal{G}$ given by (5.1) and the adaptive dynamic compensator $\mathcal{G}_{\mathrm{c}}$ given by (5.2), (5.3). Assume there exist matrices $A_{\mathrm{cg}} \in \mathbb{R}^{n_{\mathrm{c}}\times n_{\mathrm{c}}}$, $B_{\mathrm{cg}} \in \mathbb{R}^{n_{\mathrm{c}}\times n}$, $C_{\mathrm{cg}} \in \mathbb{R}^{m\times n_{\mathrm{c}}}$, $D_{\mathrm{cg}} \in \mathbb{R}^{m\times s}$, and functions $V_{\mathrm{s}} : \mathbb{R}^{\tilde{n}} \to \mathbb{R}$, $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, $F : \mathbb{R}^n \to \mathbb{R}^s$, with $F(0) = 0$, and $\ell : \mathbb{R}^{\tilde{n}} \to \mathbb{R}^t$ such that $V_{\mathrm{s}}(\cdot)$ is continuously differentiable, positive definite, radially unbounded, $V_{\mathrm{s}}(0) = 0$, $\ell(0) = 0$, and, for all $\tilde{x} \in \mathbb{R}^{\tilde{n}}$,

$$0 = V_{\rm s}'(\tilde{x})\tilde{f}_{\rm s}(\tilde{x}) + \ell^{\rm T}(\tilde{x})\ell(\tilde{x}), \tag{5.4}$$

where

$$\tilde{f}_{\rm s}(\tilde{x}) \triangleq \begin{bmatrix} f_{\rm s}(x) + G(x)\hat{G}(x)C_{\rm cg}x_{\rm c} \\ A_{\rm cg}x_{\rm c} + B_{\rm cg}x \end{bmatrix}, \tag{5.5}$$

$$f_{\rm s}(x) \triangleq f(x) + G(x)\hat{G}(x)D_{\rm cg}F(x). \tag{5.6}$$

Furthermore, let $Q_i \in \mathbb{R}^{m\times m}$, $i = 1, \cdots, 4$, $Y_1 \in \mathbb{R}^{n_{\rm c}\times n_{\rm c}}$, $Y_2 \in \mathbb{R}^{n\times n}$, $Y_3 \in \mathbb{R}^{n_{\rm c}\times n_{\rm c}}$, and $Y_4 \in \mathbb{R}^{s\times s}$ be positive definite. Then the adaptive dynamic feedback controller (5.2), (5.3), with update laws

$$\dot{A}_{\rm c}(t) = -\tfrac{1}{2}Q_1\frac{\partial V_{\rm s}^{\rm T}}{\partial x_{\rm c}}(\tilde{x}(t))x_{\rm c}^{\rm T}(t)Y_1, \tag{5.7}$$

$$\dot{B}_{\rm c}(t) = -\tfrac{1}{2}Q_2\frac{\partial V_{\rm s}^{\rm T}}{\partial x_{\rm c}}(\tilde{x}(t))x^{\rm T}(t)Y_2, \tag{5.8}$$

$$\dot{C}_{\rm c}(t) = -\tfrac{1}{2}Q_3\hat{G}^{\rm T}(x(t))G^{\rm T}(x(t))\frac{\partial V_{\rm s}^{\rm T}}{\partial x}(\tilde{x}(t))x_{\rm c}^{\rm T}(t)Y_3, \tag{5.9}$$

$$\dot{D}_{\rm c}(t) = -\tfrac{1}{2}Q_4\hat{G}^{\rm T}(x(t))G^{\rm T}(x(t))\frac{\partial V_{\rm s}^{\rm T}}{\partial x}(\tilde{x}(t))F^{\rm T}(x(t))Y_4, \tag{5.10}$$

guarantees that the solution $(\tilde{x}, A_{\rm c}, B_{\rm c}, C_{\rm c}, D_{\rm c}) \equiv (0, A_{\rm cg}, B_{\rm cg}, C_{\rm cg}, D_{\rm cg})$ of the closed-loop system given by (5.1)–(5.3), (5.7)–(5.10) is Lyapunov stable and $\ell(\tilde{x}(t)) \to 0$ as $t \to \infty$. If, in addition, $\ell^{\rm T}(\tilde{x})\ell(\tilde{x}) > 0$, $\tilde{x} \neq 0$, then $x(t) \to 0$ and $x_{\rm c}(t) \to 0$ as $t \to \infty$ for all $(x_0, x_{\rm c0}) \in \mathbb{R}^n \times \mathbb{R}^{n_{\rm c}}$.

**Proof.** Note that with the dynamic controller (5.2), (5.3), it follows from (5.1) that

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_{\rm c}(t) \end{bmatrix} = \begin{bmatrix} f(x(t)) + G(x(t))\hat{G}(x(t))C_{\rm c}(t)x_{\rm c}(t) + G(x(t))\hat{G}(x(t))D_{\rm c}(t)F(x(t)) \\ A_{\rm c}(t)x_{\rm c}(t) + B_{\rm c}(t)x(t) \end{bmatrix},$$
$$x(0) = x_0, \quad x_{\rm c}(0) = x_{\rm c0}, \quad t \geq 0, \tag{5.11}$$

or, equivalently,

$$\dot{\tilde{x}}(t) = \tilde{f}(\tilde{x}(t)) + \begin{bmatrix} G(x(t))\hat{G}(x(t))(C_{\rm c}(t)-C_{\rm cg})x_{\rm c}(t)+G(x(t))\hat{G}(x(t))(D_{\rm c}(t)-D_{\rm cg})F(x(t)) \\ (A_{\rm c}(t)-A_{\rm cg})x_{\rm c}(t)+(B_{\rm c}(t)-B_{\rm cg})x(t) \end{bmatrix},$$
$$\tilde{x}(0)=\tilde{x}_0,\quad t\geq 0. \quad (5.12)$$

To show Lyapunov stability of the closed-loop system (5.7)–(5.10) and (5.12) consider the Lyapunov function candidate

$$\begin{aligned} V(\tilde{x}, A_{\rm c}, B_{\rm c}, C_{\rm c}, D_{\rm c}) &= V_{\rm s}(\tilde{x}) + {\rm tr}\,Q_1^{-1}(A_{\rm c}-A_{\rm cg})Y_1^{-1}(A_{\rm c}-A_{\rm cg})^{\rm T} + {\rm tr}\,Q_2^{-1}(B_{\rm c}-B_{\rm cg})Y_2^{-1}(B_{\rm c}-B_{\rm cg})^{\rm T} \\ &\quad + {\rm tr}\,Q_3^{-1}(C_{\rm c}-C_{\rm cg})Y_3^{-1}(D_{\rm c}-C_{\rm cg})^{\rm T} + {\rm tr}\,Q_4^{-1}(D_{\rm c}-D_{\rm cg})Y_4^{-1}(D_{\rm c}-C_{\rm cg})^{\rm T}. \quad (5.13) \end{aligned}$$

Note that $V(0, A_{\rm cg}, B_{\rm cg}, C_{\rm cg}, D_{\rm cg}) = 0$ and, since $V_{\rm s}(\cdot)$, $Q_i$, and $Y_i$, $i=1,\cdots,4$, are positive definite, $V(\tilde{x}, A_{\rm c}, B_{\rm c}, C_{\rm c}, D_{\rm c}) > 0$ for all $(\tilde{x}, A_{\rm c}, B_{\rm c}, C_{\rm c}, D_{\rm c}) \neq (0, A_{\rm cg}, B_{\rm cg}, C_{\rm cg}, D_{\rm cg})$. Furthermore, $V(\tilde{x}, A_{\rm c}, B_{\rm c}, C_{\rm c}, D_{\rm c})$ is radially unbounded. Now, letting $\tilde{x}(t)$, $t \geq 0$, denote the solution to (5.12) and using (5.4)–(5.10), it follows that the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned} &\dot{V}(\tilde{x}(t), A_{\rm c}(t), B_{\rm c}(t), C_{\rm c}(t), D_{\rm c}(t)) \\ &\quad = V_{\rm s}'(\tilde{x}(t))f_{\rm s}(\tilde{x}(t)) \\ &\qquad + V_{\rm s}'(\tilde{x}(t)) \begin{bmatrix} \begin{pmatrix} G(x(t))\hat{G}(x(t))(C_{\rm c}(t)-C_{\rm cg})x_{\rm c}(t) \\ +G(x(t))\hat{G}(x(t))(D_{\rm c}(t)-D_{\rm cg})F(x(t)) \end{pmatrix} \\ (A_{\rm c}(t)-A_{\rm cg})x_{\rm c}(t)+(B_{\rm c}(t)-B_{\rm cg})x(t) \end{bmatrix} \\ &\qquad + 2{\rm tr}\,Q_1^{-1}(A_{\rm c}(t)-A_{\rm cg})Y_1^{-1}\dot{A}_{\rm c}^{\rm T}(t) + 2{\rm tr}\,Q_2^{-1}(B_{\rm c}(t)-B_{\rm cg})Y_2^{-1}\dot{B}_{\rm c}^{\rm T}(t) \\ &\qquad + 2{\rm tr}\,Q_3^{-1}(C_{\rm c}(t)-C_{\rm cg})Y_3^{-1}\dot{C}_{\rm c}^{\rm T}(t) + 2{\rm tr}\,Q_4^{-1}(D_{\rm c}(t)-D_{\rm cg})Y_4^{-1}\dot{D}_{\rm c}^{\rm T}(t) \\ &\quad = -\ell^{\rm T}(\tilde{x}(t))\ell(\tilde{x}(t)) \\ &\qquad + \frac{\partial V_{\rm s}^{\rm T}}{\partial x}(\tilde{x}(t))\Big[G(x(t))\hat{G}(x(t))(C_{\rm c}(t)-C_{\rm cg})x_{\rm c}(t) \\ &\qquad + G(x(t))\hat{G}(x(t))(D_{\rm c}(t)-D_{\rm cg})F(x(t))\Big] \end{aligned}$$

$$
\begin{aligned}
&+\frac{\partial V_{\mathrm{s}}^{\mathrm{T}}}{\partial x_{\mathrm{c}}}(\tilde{x}(t))\left[(A_{\mathrm{c}}(t)-A_{\mathrm{cg}})x_{\mathrm{c}}(t)+(B_{\mathrm{c}}(t)-B_{\mathrm{cg}})x(t)\right] \\
&-\mathrm{tr}\left[(A_{\mathrm{c}}(t)-A_{\mathrm{cg}})x_{\mathrm{c}}(t)\frac{\partial V_{\mathrm{s}}^{\mathrm{T}}}{\partial x_{\mathrm{c}}}(\tilde{x}(t))\right]-\mathrm{tr}\left[(B_{\mathrm{c}}(t)-B_{\mathrm{cg}})x(t)\frac{\partial V_{\mathrm{s}}^{\mathrm{T}}}{\partial x_{\mathrm{c}}}(\tilde{x}(t))\right] \\
&-\mathrm{tr}\left[(C_{\mathrm{c}}(t)-C_{\mathrm{cg}})x_{\mathrm{c}}(t)\frac{\partial V_{\mathrm{s}}^{\mathrm{T}}}{\partial x}(\tilde{x}(t))G(x(t))\hat{G}(x(t))\right] \\
&-\mathrm{tr}\left[(D_{\mathrm{c}}(t)-D_{\mathrm{cg}})F(x(t))\frac{\partial V_{\mathrm{s}}^{\mathrm{T}}}{\partial x}(\tilde{x}(t))G(x(t))\hat{G}(x(t))\right] \\
=\;& -\ell^{\mathrm{T}}(\tilde{x}(t))\ell(\tilde{x}(t)) \\
\leq\;& 0, \quad t\geq 0, \qquad (5.14)
\end{aligned}
$$

which proves that the solution $(\tilde{x}(t), A_{\mathrm{c}}(t), B_{\mathrm{c}}(t), C_{\mathrm{c}}(t), D_{\mathrm{c}}(t)) \equiv (0, A_{\mathrm{cg}}, B_{\mathrm{cg}}, C_{\mathrm{cg}}, D_{\mathrm{cg}})$ to (5.7)–(5.10), and (5.12) is Lyapunov stable. Furthermore, it follows from Theorem 2 of [42] that $\ell(\tilde{x}(t)) \to 0$ as $t \to \infty$. Finally, if $\ell^{\mathrm{T}}(\tilde{x})\ell(\tilde{x}) > 0$, $\tilde{x} \in \mathbb{R}^{\tilde{n}}$, $\tilde{x} \neq 0$, then $\tilde{x}(t) \to 0$ as $t \to \infty$ for all $\tilde{x}_0 \in \mathbb{R}^{\tilde{n}}$. □

**Remark 5.1.** Note that in the case where $\ell^{\mathrm{T}}(\tilde{x})\ell(\tilde{x}) > 0$, $\tilde{x} \in \mathbb{R}^{\tilde{n}}$, $\tilde{x} \neq 0$, the conditions in Theorem 5.1 imply $\tilde{x}(t) \to 0$ as $t \to \infty$ and hence it follows from (5.7)–(5.10), using $F(0) = 0$, that $(\tilde{x}(t), A_{\mathrm{c}}(t), B_{\mathrm{c}}(t), C_{\mathrm{c}}(t), D_{\mathrm{c}}(t)) \to \mathcal{M} \triangleq \{(\tilde{x}, A_{\mathrm{c}}, B_{\mathrm{c}}, C_{\mathrm{c}}, D_{\mathrm{c}}) \in \mathbb{R}^{\tilde{n}} \times \mathbb{R}^{n_{\mathrm{c}}\times n_{\mathrm{c}}} \times \mathbb{R}^{n_{\mathrm{c}}\times n} \times \mathbb{R}^{m\times n_{\mathrm{c}}} \times \mathbb{R}^{m\times s} : \tilde{x} = 0,\ \dot{A}_{\mathrm{c}} = 0,\ \dot{B}_{\mathrm{c}} = 0,\ \dot{C}_{\mathrm{c}} = 0,\ \dot{D}_{\mathrm{c}} = 0\}$ as $t \to \infty$.

It is important to note that the adaptive dynamic controller (5.2), (5.3), (5.7)–(5.10), does *not* require explicit knowledge of the gain matrices $A_{\mathrm{cg}}$, $B_{\mathrm{cg}}$, $C_{\mathrm{cg}}$, and $D_{\mathrm{cg}}$; even though Theorem 5.1 requires the existence of $A_{\mathrm{cg}}$, $B_{\mathrm{cg}}$, $C_{\mathrm{cg}}$, and $D_{\mathrm{cg}}$ along with the construction of $F(x)$, $\hat{G}(x)$, and $V_{\mathrm{s}}(\tilde{x})$ such that (5.4) holds. Furthermore, if (5.1) is in normal form with asymptotically stable internal dynamics [122], then we can always construct the gain matrices $A_{\mathrm{cg}}$, $B_{\mathrm{cg}}$, $C_{\mathrm{cg}}$, $D_{\mathrm{cg}}$, and the functions $V_{\mathrm{s}} : \mathbb{R}^{\tilde{n}} \to \mathbb{R}$, $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, and $F : \mathbb{R}^n \to \mathbb{R}^s$, with $F(0) = 0$, such that (5.4) holds *without* requiring knowledge of the system dynamics. To see this assume that the nonlinear uncertain system $\mathcal{G}$ is generated by

$$q_i^{(r_i)}(t) = f_{\mathrm{u}i}(q(t)) + \sum_{j=1}^{m} G_{\mathrm{s}(i,j)}(q(t))u_j(t), \quad t \geq 0, \quad i = 1, \cdots, m, \tag{5.15}$$

where $q = [q_1, \cdots, q_1^{(r_1-1)}, \cdots, q_m, \cdots, q_m^{(r_m-1)}]^{\mathrm{T}}$, $q(0) = q_0$, $q_i^{(r_i)}$ denotes the $r_i^{\mathrm{th}}$ derivative of $q_i$, and $r_i$ denotes the relative degree with respect to the output $q_i$. Here we assume that the square matrix function $G_{\mathrm{s}}(q)$ composed of the entries $G_{\mathrm{s}(i,j)}(q)$, $i, j = 1, \cdots, m$, is such that $\det G_{\mathrm{s}}(q) \neq 0$, $q \in \mathbb{R}^{\hat{r}}$, where $\hat{r} = r_1 + \cdots + r_m$ is the (vector) relative degree of (5.15). Furthermore, since (5.15) is in a form where it does not possess internal dynamics, it follows that $\hat{r} = n$. The case where (5.15) possesses input-to-state stable internal dynamics can be handled as shown in Section 2.2.

Next, define $x_i \triangleq \left[ q_i, \cdots, q_i^{(r_i-2)} \right]^{\mathrm{T}}$, $i = 1, \cdots, m$, $x_{m+1} \triangleq \left[ q_1^{(r_1-1)}, \cdots, q_m^{(r_m-1)} \right]^{\mathrm{T}}$, and $x \triangleq \left[ x_1^{\mathrm{T}}, \cdots, x_{m+1}^{\mathrm{T}} \right]^{\mathrm{T}}$, so that (5.15) can be described as (5.1) with

$$f(x) = Ax + \tilde{f}_{\mathrm{u}}(x), \quad G(x) = \begin{bmatrix} 0_{(n-m)\times m} \\ G_{\mathrm{s}}(x) \end{bmatrix}, \tag{5.16}$$

where

$$A = \begin{bmatrix} A_0 \\ 0_{m\times n} \end{bmatrix}, \quad \tilde{f}_{\mathrm{u}}(x) = \begin{bmatrix} 0_{(n-m)\times 1} \\ f_{\mathrm{u}}(x) \end{bmatrix},$$

$A_0 \in \mathbb{R}^{(n-m)\times n}$ is a known matrix of zeros and ones capturing the multivariable controllable canonical form representation [43], $f_{\mathrm{u}}(x)$ is an unknown function and satisfies $f_{\mathrm{u}}(0) = 0$, and $G_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$. Here, we assume that $f_{\mathrm{u}}(x)$ is unknown and is parameterized as $f_{\mathrm{u}}(x) = \Theta f_{\mathrm{n}}(x)$, where $f_{\mathrm{n}} : \mathbb{R}^n \to \mathbb{R}^q$ and satisfies $f_{\mathrm{n}}(0) = 0$, and $\Theta \in \mathbb{R}^{m\times q}$ is a matrix of uncertain constant parameters.

Next, to apply Theorem 5.1 to the uncertain system (5.1) with $f(x)$ and $G(x)$ given by (5.16), let $D_{\mathrm{cg}} \in \mathbb{R}^{m\times s}$, where $s = q + r$, be given by

$$D_{\mathrm{cg}} = [\,\Theta_{\mathrm{n}} - \Theta,\ \Phi_{\mathrm{n}}\,], \tag{5.17}$$

where $\Theta_{\mathrm{n}} \in \mathbb{R}^{m\times q}$ and $\Phi_{\mathrm{n}} \in \mathbb{R}^{m\times r}$ are known matrices, and let $\hat{G}(x) = G_{\mathrm{s}}^{-1}(x)$ and

$$F(x) = \begin{bmatrix} f_{\mathrm{n}}(x) \\ \hat{f}_{\mathrm{n}}(x) \end{bmatrix}, \tag{5.18}$$

where $\hat{f}_{\rm n} : \mathbb{R}^n \to \mathbb{R}^r$ and satisfies $\hat{f}_{\rm n}(0) = 0$ is an arbitrary function. In this case, it follows that

$$\begin{aligned} f_{\rm s}(x) &= f(x) + G(x)\hat{G}(x)D_{\rm g}F(x) \\ &= Ax + \tilde{f}_{\rm u}(x) \\ &\quad + \begin{bmatrix} 0_{(n-m)\times m} \\ G_{\rm s}(x) \end{bmatrix} G_{\rm s}^{-1}(x) \left[ \begin{array}{c} \Theta_{\rm n} f_{\rm n}(x) - \Theta f_{\rm n}(x) + \Phi_{\rm n}\hat{f}_{\rm n}(x) \end{array} \right] \\ &= Ax + \begin{bmatrix} 0_{(n-m)\times 1} \\ \Theta_{\rm n} f_{\rm n}(x) + \Phi_{\rm n}\hat{f}_{\rm n}(x) \end{bmatrix}. \end{aligned} \tag{5.19}$$

Now, since $\Theta_{\rm n} \in \mathbb{R}^{m\times q}$ and $\Phi_{\rm n} \in \mathbb{R}^{m\times r}$ are arbitrary constant matrices and $\hat{f}_{\rm n} : \mathbb{R}^n \to \mathbb{R}^r$ is an arbitrary function we can always construct $A_{\rm cg}$, $B_{\rm cg}$, $C_{\rm cg}$, $D_{\rm cg}$, $V_{\rm s}(\tilde{x})$, and $F(x)$ without knowledge of $f(x)$ such that (5.4) holds. In particular, choosing $\Theta_{\rm n} f_{\rm n}(x) + \Phi_{\rm n}\hat{f}_{\rm n}(x) = \hat{A}x$, where $\hat{A} \in \mathbb{R}^{m\times n}$, it follows that

$$\tilde{f}_{\rm s}(\tilde{x}) = \begin{bmatrix} A_{\rm s} & \hat{C}_{\rm cg} \\ B_{\rm cg} & A_{\rm cg} \end{bmatrix} \tilde{x} \triangleq \tilde{A}_{\rm s}\tilde{x},$$

where $A_{\rm s} = [A_0^{\rm T}, \hat{A}^{\rm T}]^{\rm T}$ is in multivariable controllable canonical form and $\hat{C}_{\rm cg} = [0_{(n-m)\times n_{\rm c}}, C_{\rm cg}^{\rm T}]^{\rm T}$. Hence, choosing $\hat{A}$, $A_{\rm cg}$, $B_{\rm cg}$, and $C_{\rm cg}$ such that $\tilde{A}_{\rm s}$ is asymptotically stable, it follows from converse Lyapunov theory that there exists a positive-definite matrix $\tilde{P}$ satisfying the Lyapunov equation

$$0 = \tilde{A}_{\rm s}^{\rm T}\tilde{P} + \tilde{P}\tilde{A}_{\rm s} + \tilde{R}, \tag{5.20}$$

where $\tilde{R}$ is positive definite. In this case, with $V_{\rm s}(\tilde{x}) = \tilde{x}^{\rm T}\tilde{P}\tilde{x}$, where

$$\tilde{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^{\rm T} & P_2 \end{bmatrix},$$

$P_1 \in \mathbb{R}^{n\times n}$, $P_{12} \in \mathbb{R}^{n\times n_{\rm c}}$, $P_2 \in \mathbb{R}^{n_{\rm c}\times n_{\rm c}}$, the adaptive dynamic feedback controller (5.2), (5.3), (5.7)–(5.10), or, equivalently,

$$\dot{A}_{\rm c}(t) = -Q_1(P_{12}^{\rm T}x(t) + P_2x_{\rm c}(t))x_{\rm c}^{\rm T}(t)Y_1, \tag{5.21}$$

$$\dot{B}_{\rm c}(t) = -Q_2(P_{12}^{\rm T}x(t) + P_2x_{\rm c}(t))x^{\rm T}(t)Y_2, \tag{5.22}$$

$$\dot{C}_{\rm c}(t) = -Q_3(P_1 x(t) + P_{12} x_{\rm c}(t)) x_{\rm c}^{\rm T}(t) Y_3, \tag{5.23}$$

$$\dot{D}_{\rm c}(t) = -Q_4(P_1 x(t) + P_{12} x_{\rm c}(t)) F^{\rm T}(x(t)) Y_4, \tag{5.24}$$

guarantees global asymptotic stability of the *nonlinear* uncertain dynamical system (5.1) where $f(x)$ and $G(x)$ are given by (5.16).

Next, we consider the case where $f(x)$ and $G(x)$ are uncertain. Specifically, we assume that $G_{\rm s}(x)$ is unknown and is parameterized as $G_{\rm s}(x) = B_{\rm u} G_{\rm n}(x)$, where $G_{\rm n} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is known and satisfies $\det G_{\rm n}(x) \neq 0$, $x \in \mathbb{R}^n$, and $B_{\rm u} \in \mathbb{R}^{m\times m}$, with $\det B_{\rm u} \neq 0$, is a symmetric sign definite matrix but the sign definiteness of $B_{\rm u}$ is known; that is, $B_{\rm u} > 0$ or $B_{\rm u} < 0$. For the statement of the next result define $B_0 \triangleq \left[0_{m\times(n-m)}, I_m\right]^{\rm T}$ for $B_{\rm u} > 0$, and $B_0 \triangleq \left[0_{m\times(n-m)}, -I_m\right]^{\rm T}$ for $B_{\rm u} < 0$.

**Corollary 5.1.** Consider the nonlinear system $\mathcal{G}$ given by (5.1) and the adaptive dynamic compensator $\mathcal{G}_{\rm c}$ given by (5.2) with $f(x)$ and $G(x)$ given by (5.16) and $G_{\rm s}(x) = B_{\rm u} G_{\rm n}(x)$, where $B_{\rm u}$ is an unknown symmetric matrix and the sign definiteness of $B_{\rm u}$ is known. Assume there exist matrices $A_{\rm cg} \in \mathbb{R}^{n_{\rm c}\times n_{\rm c}}$, $B_{\rm cg} \in \mathbb{R}^{n_{\rm c}\times n}$, $C_{\rm cg} \in \mathbb{R}^{m\times n_{\rm c}}$, $D_{\rm cg} \in \mathbb{R}^{m\times s}$, and functions $V_{\rm s} : \mathbb{R}^{\tilde{n}} \to \mathbb{R}$, $\ell : \mathbb{R}^{\tilde{n}} \to \mathbb{R}^t$, and $F : \mathbb{R}^n \to \mathbb{R}^s$, with $F(0) = 0$, such that $V_{\rm s}(\cdot)$ is continuously differentiable, positive definite, radially unbounded, $V_{\rm s}(0) = 0$, $\ell(0) = 0$, and (5.4) holds. Finally, let $Q_i \in \mathbb{R}^{m\times m}$, $i = 1, 2$, $Y_1 \in \mathbb{R}^{n_{\rm c}\times n_{\rm c}}$, $Y_2 \in \mathbb{R}^{n\times n}$, $Y_3 \in \mathbb{R}^{n_{\rm c}\times n_{\rm c}}$, and $Y_4 \in \mathbb{R}^{s\times s}$ be positive definite. Then the adaptive dynamic feedback controller

$$\dot{x}_{\rm c}(t) = A_{\rm c}(t) x_{\rm c}(t) + B_{\rm c}(t) x(t), \quad x_{\rm c}(0) = x_{\rm c0}, \quad t \geq 0, \tag{5.25}$$

$$u(t) = G_{\rm n}^{-1}(x(t))[C_{\rm c}(t) x_{\rm c}(t) + D_{\rm c}(t) F(x(t))], \tag{5.26}$$

with update laws

$$\dot{A}_{\rm c}(t) = -\tfrac{1}{2} Q_1 \frac{\partial V_{\rm s}^{\rm T}}{\partial x_{\rm c}}(\tilde{x}(t)) x_{\rm c}^{\rm T}(t) Y_1, \tag{5.27}$$

$$\dot{B}_{\rm c}(t) = -\tfrac{1}{2} Q_2 \frac{\partial V_{\rm s}^{\rm T}}{\partial x_{\rm c}}(\tilde{x}(t)) x^{\rm T}(t) Y_2, \tag{5.28}$$

$$\dot{C}_{\rm c}(t) = -\tfrac{1}{2}B_0\frac{\partial V_{\rm s}^{\rm T}}{\partial x}(\tilde{x}(t))x_{\rm c}^{\rm T}(t)Y_3, \tag{5.29}$$
$$\dot{D}_{\rm c}(t) = -\tfrac{1}{2}B_0\frac{\partial V_{\rm s}^{\rm T}}{\partial x}(\tilde{x}(t))F^{\rm T}(x(t))Y_4, \tag{5.30}$$

guarantees that the solution $(\tilde{x}, A_{\rm c}, B_{\rm c}, C_{\rm c}, D_{\rm c}) \equiv (0, A_{\rm cg}, B_{\rm cg}, C_{\rm cg}, D_{\rm cg})$ of the closed-loop system given by (5.1), (5.25)–(5.30) is Lyapunov stable and $\ell(\tilde{x}(t)) \to 0$ as $t \to \infty$. If, in addition, $\ell^{\rm T}(\tilde{x})\ell(\tilde{x}) > 0$, $x \neq 0$, then $x(t) \to 0$ and $x_{\rm c}(t) \to 0$ as $t \to \infty$ for all $(x_0, x_{\rm c0}) \in \mathbb{R}^n \times \mathbb{R}^{n_{\rm c}}$.

**Proof.** The result is a direct consequence of Theorem 5.1. First, let $\hat{G}(x) = G_{\rm n}^{-1}(x)$. Next, since $Q_3$ and $Q_4$ are arbitrary positive-definite matrices, $Q_3$ in (5.9) and $Q_4$ in (5.10) can be replaced by $q_3|B_{\rm u}|^{-1}$ and $q_4|B_{\rm u}|^{-1}$, respectively, where $q_3$, $q_4$ are positive constants and $|B_{\rm u}| = (B_{\rm u}^2)^{\frac{1}{2}}$, where $(\cdot)^{\frac{1}{2}}$ denotes the (unique) positive-definite square root. Now, since $B_{\rm u}$ is symmetric and sign definite it follows from the Schur decomposition that $B_{\rm u} = UD_{B_{\rm u}}U^{\rm T}$, where $U$ is orthogonal and $D_{B_{\rm u}}$ is real diagonal. Hence, $|B_{\rm u}|^{-1}B^{\rm T} = [0_{m\times(n-m)}, \mathcal{I}_m] = B_0^{\rm T}$, where $\mathcal{I}_m = I_m$ for $B_{\rm u} > 0$ and $\mathcal{I}_m = -I_m$ for $B_{\rm u} < 0$. Now, (5.9) and (5.10), with $q_3Y_3$ and $q_4Y_4$ replaced by $Y_3$ and $Y_4$, imply (5.29) and (5.30), respectively. □

It is important to note that if, as discussed above, $D_{\rm cg}$ and $F(x)$ are constructed to give $\tilde{f}_{\rm s}(x) = \tilde{A}_{\rm s}\tilde{x}$ in (5.5), then considerable simplification occurs in Corollary 5.1. Specifically, in this case $V_{\rm s}(\tilde{x}) = \tilde{x}^{\rm T}\tilde{P}\tilde{x}$, where $\tilde{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^{\rm T} & P_2 \end{bmatrix} > 0$ satisfies (5.20), and hence (5.27)–(5.30) become

$$\dot{A}_{\rm c}(t) = -Q_1(P_{12}^{\rm T}x(t) + P_2x_{\rm c}(t))x_{\rm c}^{\rm T}(t)Y_1, \tag{5.31}$$
$$\dot{B}_{\rm c}(t) = -Q_2(P_{12}^{\rm T}x(t) + P_2x_{\rm c}(t))x^{\rm T}(t)Y_2, \tag{5.32}$$
$$\dot{C}_{\rm c}(t) = -B_0(P_1x(t) + P_{12}x_{\rm c}(t))x_{\rm c}^{\rm T}(t)Y_3, \tag{5.33}$$
$$\dot{D}_{\rm c}(t) = -B_0(P_1x(t) + P_{12}x_{\rm c}(t))F^{\rm T}(x(t))Y_4. \tag{5.34}$$

## 5.3. Illustrative Numerical Examples

In this section we present two numerical examples to demonstrate the efficacy of the proposed adaptive reduced-order dynamic compensation framework.

**Example 5.1.** Consider the uncertain controlled Van der Pol oscillator given by

$$\ddot{z}(t) - \varepsilon(1 - \alpha z^2(t))\dot{z}(t) + \beta z(t) = bu(t), \quad z(0) = z_0, \quad \dot{z}(0) = \dot{z}_0, \quad t \geq 0, \tag{5.35}$$

where $\varepsilon, \alpha, \beta, b \in \mathbb{R}$ are unknown. Note that with $x_1 = z$ and $x_2 = \dot{z}$, (5.35) can be written in state space form (5.1) with $x = [x_1, x_2]^{\mathrm{T}}$, $f(x) = [x_2, -\beta x_1 + \varepsilon(1-\alpha x_1^2)x_2]^{\mathrm{T}}$, and $G(x) = [0, b]^{\mathrm{T}}$. Here, we assume that $f(x)$ is unknown and can be parameterized as $f(x) = [x_2, \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_1^2 x_2]^{\mathrm{T}}$, where $\theta_1$, $\theta_2$, and $\theta_3$ are unknown constants. Furthermore, we assume that $\operatorname{sgn} b$ is known. Next, let $F(x) = [x_1, x_2, x_1^2 x_2]^{\mathrm{T}}$, $G_{\mathrm{n}}(x) \equiv 1$, and $D_{\mathrm{cg}} = \frac{1}{b}[\theta_{\mathrm{n}_1} - \theta_1, \theta_{\mathrm{n}_2} - \theta_2, -\theta_3]$, where $\theta_{\mathrm{n}_1}, \theta_{\mathrm{n}_2}$ are arbitrary scalars, so that

$$f_{\mathrm{s}}(x) = \begin{bmatrix} 0 & 1 \\ \theta_{\mathrm{n}_1} & \theta_{\mathrm{n}_2} \end{bmatrix} x.$$

Now, with the proper choice of $\theta_{\mathrm{n}_1}$, $\theta_{\mathrm{n}_2}$, $A_{\mathrm{cg}}$, $B_{\mathrm{cg}}$, $C_{\mathrm{cg}}$, it follows from Corollary 5.1 that the adaptive dynamic feedback controller (5.25) and (5.26) guarantees that $\tilde{x}(t) \to 0$ as $t \to \infty$. Specifically, here we choose $n_{\mathrm{c}} = 2$, $\theta_{\mathrm{n}_1} = \theta_{\mathrm{n}_2} = 0$,

$$A_{\mathrm{cg}} = \begin{bmatrix} 0 & 1 \\ -8.75 & -5 \end{bmatrix}, \quad B_{\mathrm{cg}} = \begin{bmatrix} 0 & 0 \\ -1.5 & -6.25 \end{bmatrix}, \quad C_{\mathrm{cg}} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

so that

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1.5 & -6.25 & -8.75 & -5 \end{bmatrix}. \tag{5.36}$$

Furthermore, we choose $\tilde{R} = I_4$ so that $\tilde{P}$ satisfying (5.20) is given by

$$\tilde{P} = \begin{bmatrix} 3.1048 & 3.7560 & 1.9845 & 0.3333 \\ 3.7560 & 7.4708 & 4.3958 & 0.6810 \\ 1.9845 & 4.3958 & 3.9708 & 0.5595 \\ 0.3333 & 0.6810 & 0.5595 & 0.2119 \end{bmatrix}. \tag{5.37}$$

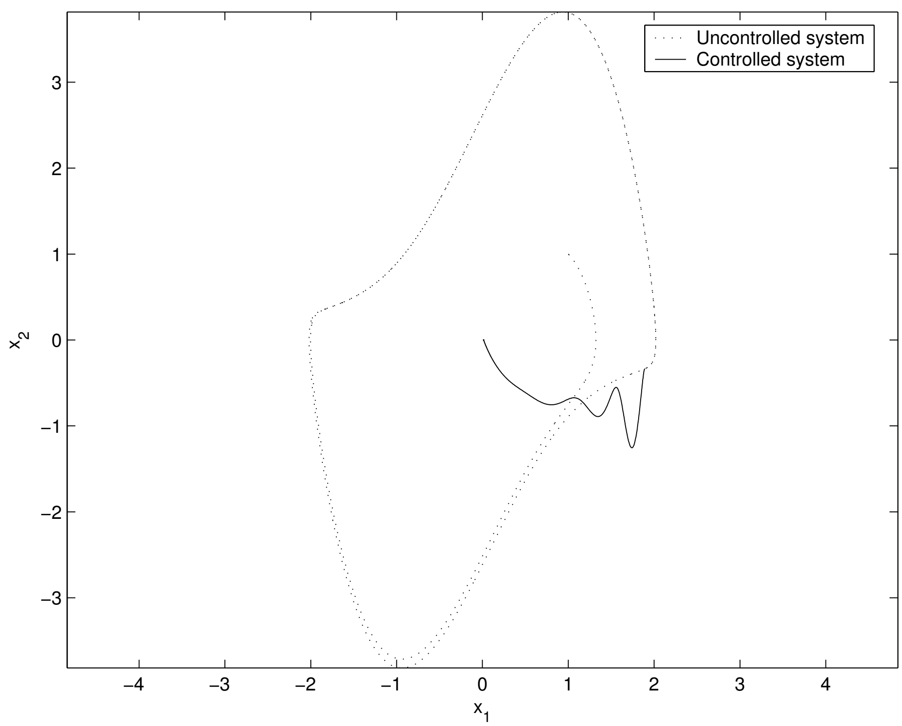



**Figure 5.1**: Phase portrait of controlled and uncontrolled Van der Pol oscillator

With $\varepsilon = 2$, $\alpha = \beta = 1$, $b = 3$, $Q_1 = Q_2 = 1$, $Y_1 = Y_2 = Y_3 = I_2$, $Y_4 = I_3$, and initial conditions $x(0) = [1, 1]^{\mathrm{T}}$, $x_{\mathrm{c}}(0) = [0, 0]^{\mathrm{T}}$, $A_{\mathrm{c}}(0) = 0_2$, $B_{\mathrm{c}}(0) = 0_2$, $C_{\mathrm{c}}(0) = 0_{1\times 2}$, and $D_{\mathrm{c}}(0) = 0_{1\times 3}$, Figure 5.1 shows that the phase portrait of the controlled and uncontrolled system. Note that the adaptive full-order ($n_{\mathrm{c}} = 2$) dynamic controller is switched on at $t = 15$ sec. Figure 5.2 shows the state trajectories versus time and the control signal versus time. Finally, Figure 5.3 shows the adaptive gain history versus time.

**Example 5.2.** Consider the nonlinear dynamical system representing a controlled rigid spacecraft given by

$$\dot{x}(t) = -I_{\mathrm{b}}^{-1} X I_{\mathrm{b}} x(t) + I_{\mathrm{b}}^{-1} u(t), \quad x(0) = x_0, \quad t \geq 0, \tag{5.38}$$

where $x = [x_1, x_2, x_3]^{\mathrm{T}}$ represents the angular velocities of the spacecraft with respect to the body-fixed frame, $I_{\mathrm{b}} \in \mathbb{R}^{3\times 3}$ is an unknown positive-definite inertia matrix

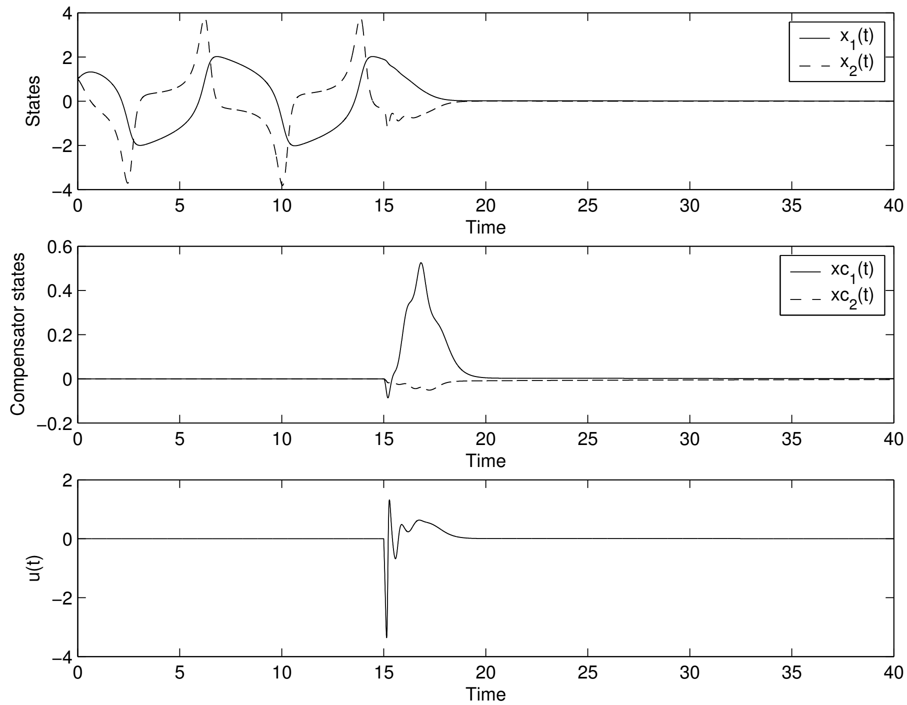



**Figure 5.2**: State trajectories versus time and the control signal versus time

of the spacecraft, $u = [u_1, u_2, u_3]^{\mathrm{T}}$ is a control vector with control inputs providing body-fixed torques about three mutually perpendicular axes defining the body-fixed frame of the spacecraft, and $X$ denotes the skew-symmetric matrix

$$X \triangleq \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

Note that (5.38) can be written in state space form (5.1) with $f(x) = -I_{\mathrm{b}}^{-1} X I_{\mathrm{b}} x$ and $G(x) = I_{\mathrm{b}}^{-1}$. Here, we assume that the inertia matrix $I_{\mathrm{b}}$ of the spacecraft is symmetric and positive definite but unknown. Since $f(x)$ is a quadratic function, we parameterize $f(x)$ as $f(x) = \Theta f_{\mathrm{n}}(x)$, where $\Theta \in \mathbb{R}^{3\times 6}$ is an unknown matrix and $f_{\mathrm{n}}(x) = [x_1^2, x_2^2, x_3^2, x_1x_2, x_2x_3,\ x_3x_1]^{\mathrm{T}}$. Next, let $F(x) = \left[f_{\mathrm{n}}^{\mathrm{T}}(x), x^{\mathrm{T}}\right]^{\mathrm{T}}$, $G_{\mathrm{n}}(x) \equiv 1$, and $D_{\mathrm{cg}} = I_{\mathrm{b}}\left[-\Theta, \Phi_{\mathrm{n}}\right]$, where $\Phi_{\mathrm{n}} \in \mathbb{R}^{3\times 3}$, is an arbitrary matrix, so that

$$f_{\mathrm{s}}(x) = \Phi_{\mathrm{n}} x = A_{\mathrm{s}} x.$$

Now, with the proper choice of $\Phi_{\mathrm{n}}$, $A_{\mathrm{cg}}$, $B_{\mathrm{cg}}$, and $C_{\mathrm{cg}}$, it follows from Corollary 5.1 that

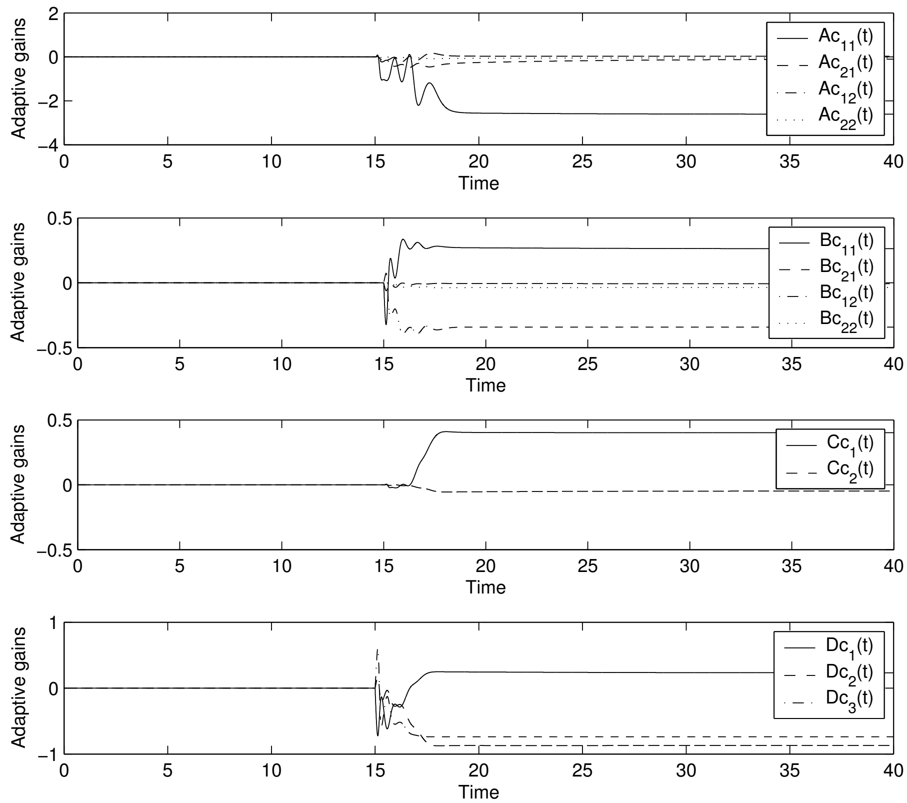



**Figure 5.3**: Adaptive gain history versus time

the dynamic feedback controller (5.25), (5.26) guarantees that $\tilde{x}(t) \to 0$ as $t \to \infty$. Here, we choose $n_{\rm c} = 1$,

$$\Phi_{\rm n} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_{\rm cg} = -5, \quad B_{\rm cg} = \begin{bmatrix} -1.5 & -6.25 & -8.75 \end{bmatrix}, \quad C_{\rm cg} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

so that $\tilde{A}$ is given by (5.36). Furthermore, we choose $\tilde{R} = I_4$ so that $\tilde{P}$ satisfying (5.20) is given by (5.37). With

$$I_{\rm b} = \begin{bmatrix} 20 & 0 & 0.9 \\ 0 & 17 & 0 \\ 0.9 & 0 & 15 \end{bmatrix}, \quad Q_1 = Q_2 = I_3, \quad Y_1 = Y_3 = 30, \quad Y_2 = 30 I_3, \quad Y_4 = 10 I_9,$$

and initial conditions $x(0) = [0.4, 0.2, -0.2]$, $x_{\rm c}(0) = 0$, $A_{\rm c}(0) = 0$, $B_{\rm c}(0) = 0_{1\times 3}$, $C_{\rm c}(0) = 0_{3\times 1}$, and $D_{\rm c}(0) = 0_{3\times 27}$, Figure 5.4 shows the angular velocities, compensator state, and control signal versus time.

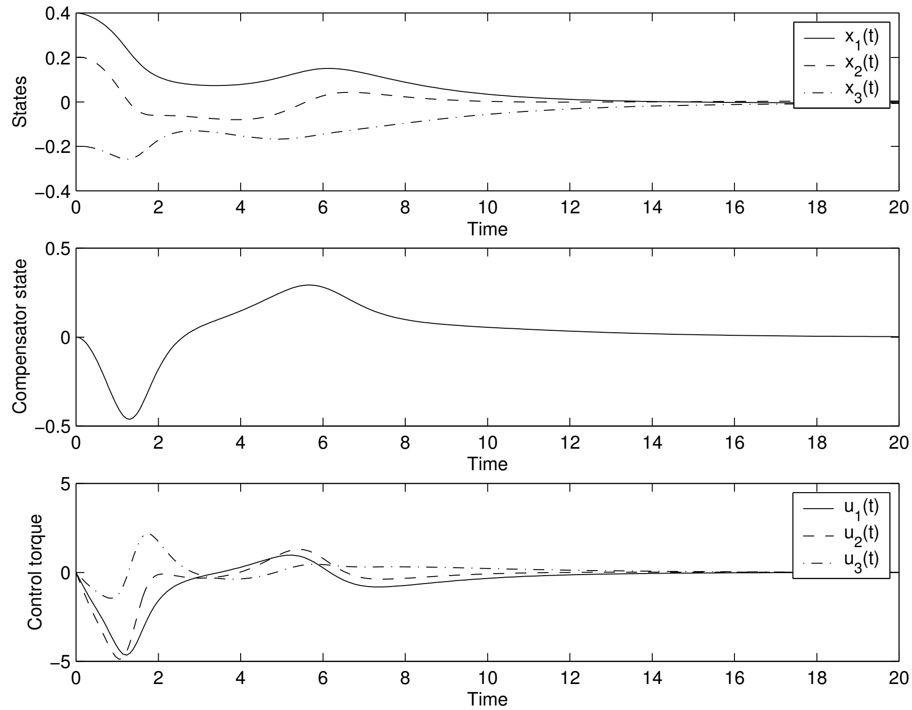



**Figure 5.4**: Angular velocities, compensator state, and control signal versus time

## 5.4. Conclusion

An adaptive reduced-order dynamic compensation framework for nonlinear uncertain dynamical systems was developed. Specifically, using Lyapunov methods the proposed framework was shown to guarantee global asymptotic stability of the closed-loop system states associated with the plant and compensator states. The efficacy of the proposed approach was demonstrated on two representative nonlinear control problems.

# Chapter 6

# Direct Adaptive Control for Nonlinear Matrix Second-Order Dynamical Systems with State-Dependent Uncertainty

## 6.1. Introduction

In light of the increasingly complex and highly uncertain nature of dynamical systems requiring controls, it is not surprising that reliable system models for many high performance engineering applications are unavailable. In the face of such high levels of system uncertainty, adaptive controllers are clearly appropriate since they can tolerate high levels of system errors to improve system performance. However, a fundamental limitation of adaptive control is the fact that system errors are captured by constant linearly parameterized uncertainty models of a known structure but unknown variation [12, 121, 147, 176]. If the system uncertainty is nonlinear in the uncertain parameters or the system uncertainty is nonlinearly dependent on the system states, then adaptive controllers predicated on a constant linearly (over)parameterized model will unnecessarily sacrifice system performance, and in some cases lead to instability.

In [200], the authors present a novel adaptive control framework for *scalar* second-order nonlinear systems that does not require any parametrization of the (partial) state-dependent system uncertainty. In this chapter we generalize the result of [200] in several directions. In particular, for a class of nonlinear multivariable *matrix* second-order uncertain dynamical systems with state-dependent uncertainty we develop a nonlinear adaptive control framework that guarantees global partial asymptotic stability of the closed-loop system; that is, global asymptotic stability with respect to part of the closed-loop system states associated with the plant. This is achieved without requiring any knowledge of the system nonlinearities other than the assumption that they are continuous and lower bounded. The class of systems represented by our framework includes nonlinear vibrational systems, as well as multivariable nonlinear dynamical systems with sign varying; that is, nondissipative, generalized stiffness and damping operators.

Next, we extend our main result to the case where the system nonlinearities are unbounded. Using this result, we provide a universal adaptive controller that guarantees asymptotic stability for the case of matrix second-order systems with polynomial nonlinearities with unknown coefficients and **unknown order**. We note that for the special case of scalar second-order systems this result does not require that the system nonlinearities be lower bounded and hence cannot be obtained using the results of [200]. In addition, we emphasize that the universal adaptive controller for polynomial nonlinearities developed in this section is distinct from standard adaptive controllers involving a parameter estimate update law for the uncertain polynomial coefficients. The proposed adaptive controller is parametrization free and does not require knowledge of the order of the polynomial nonlinearities. Hence, our design methodology yields adaptive controllers that minimize control system complexity by assuring the implementation of the simplest possible controller for achieving system stability in the face of state-dependent system uncertainty. By "simplest" we are referring to the elimination of system parameter estimates within the adaptive controller.

## 6.2. Adaptive Control of Nonlinear Matrix Second-Order Dynamical Systems

In this section we consider the problem of adaptive stabilization of nonlinear matrix second-order dynamical systems with exogenous disturbances. Specifically, consider the controlled nonlinear uncertain matrix second-order dynamical system $\mathcal{G}$ given by

$$M\ddot{q}(t) + C(q(t))\dot{q}(t) + K(q(t))q(t) = u(t) + Dw(t), \quad q(0) = q_0, \quad \dot{q}(0) = \dot{q}_0, \quad t \geq 0, \tag{6.1}$$

where $q(t), \dot{q}(t), \ddot{q}(t) \in \mathbb{R}^n$, $t \geq 0$, represent generalized position, velocity, and acceleration coordinates, respectively, $u(t) \in \mathbb{R}^n$, $t \geq 0$, is the control input, $w(t) \in \mathbb{R}^d$, $t \geq 0$, is a known bounded signal, $M \in \mathbb{R}^{n \times n}$, $C : \mathbb{R}^n \to \mathbb{R}^{n \times n}$, $K : \mathbb{R}^n \to \mathbb{R}^{n \times n}$, and $D \in \mathbb{R}^{n \times d}$. We assume that $M > 0$, $C(\cdot)$ and $K(\cdot)$ are continuous maps, and $C(\cdot), K(\cdot) \in \mathcal{S}$, where

$$\mathcal{S} \triangleq \{F : \mathbb{R}^n \to \mathbb{R}^{n \times n} : F(q) = F^{\mathrm{T}}(q), \sum_{k=1}^{n} q_k \frac{\partial F_{(k,j)}}{\partial q_i}(q) = \sum_{k=1}^{n} q_k \frac{\partial F_{(k,i)}}{\partial q_j}(q), \\ i, j = 1, \ldots, n\}, \tag{6.2}$$

and where $q_i$ denotes the $i$th element of $q$ and $F_{(k,j)}(\cdot)$ denotes the $(k, j)$th element of $F(\cdot)$. Otherwise, we assume that $M$, $C(\cdot)$, $K(\cdot)$, and $D$ are unknown. Hence, even though $w(t)$, $t \geq 0$, is assumed to be known, the disturbance signal $Dw(t)$, $t \geq 0$, is an *unknown* bounded disturbance. The control input $u(\cdot)$ in (6.1) is restricted to the class of *admissible controls* consisting of measurable functions such that $u(t) \in \mathbb{R}^n$, $t \geq 0$. Furthermore, for the uncertain dynamical system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions are satisfied; that is,

$C(\cdot)$, $K(\cdot)$, $u(\cdot)$, and $w(\cdot)$ satisfy sufficient regularity conditions such that (6.1) has a unique solution forward in time.

Next, with $x_1 \triangleq q$, $x_2 \triangleq \dot{q}$, and $x \triangleq [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$, it follows that the state space representation of (6.1) is given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -M^{-1}\Big(K(x_1(t))x_1(t) + C(x_1(t))x_2(t) - u(t) - Dw(t)\Big) \end{bmatrix},$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} q_0 \\ \dot{q}_0 \end{bmatrix}, \quad t \geq 0. \tag{6.3}$$

For the statement of our main result define $B_0 \triangleq [0_n, I_n]^{\mathrm{T}}$.

**Theorem 6.1.** Consider the nonlinear dynamical system (6.3), or, equivalently, the nonlinear matrix second-order dynamical system (6.1). Assume there exists $\varepsilon \in \mathbb{R}$ such that $C(x_1) \geq \varepsilon I_n$ and $K(x_1) \geq \varepsilon I_n$, $q \in \mathbb{R}^n$. Let $Q_1, Q_2 \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{2n\times 2n}$, $Z \in \mathbb{R}^{d\times d}$, and $P \in \mathbb{R}^{2\times 2}$ be positive definite, where $P = \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}$ and $p_{12} > 0$. Then the adaptive feedback control law

$$u(t) = \Psi(t)x(t) + \Phi(t)w(t), \tag{6.4}$$

where $\Psi(t) \in \mathbb{R}^{n\times 2n}$, $t \geq 0$, and $\Phi(t) \in \mathbb{R}^{n\times d}$, $t \geq 0$, with update laws

$$\dot{\Psi}(t) = -Q_1 B_0^{\mathrm{T}}(P \otimes I_n)x(t)x^{\mathrm{T}}(t)Y, \quad \Psi(0) = \Psi_0, \tag{6.5}$$

$$\dot{\Phi}(t) = -Q_2 B_0^{\mathrm{T}}(P \otimes I_n)x(t)w^{\mathrm{T}}(t)Z, \quad \Phi(0) = \Phi_0, \tag{6.6}$$

guarantees that the solution $(x(t), \Psi(t), \Phi(t)) \equiv (0, K_{\mathrm{g}}, -D)$, where $K_{\mathrm{g}} \in \mathbb{R}^{n\times 2n}$, of the closed-loop system given by (6.3)–(6.6) is Lyapunov stable and $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^{2n}$.

**Proof.** Let $\alpha, \beta \in \mathbb{R}$ be such that

$$\alpha M \leq K(q), \quad \beta M \leq C(q), \quad q \in \mathbb{R}^n, \tag{6.7}$$

and let $k_{1\mathrm{g}}, k_{2\mathrm{g}} \in \mathbb{R}$ be such that

$$k_{1\mathrm{g}} < \alpha, \tag{6.8}$$

$$k_{2\mathrm{g}} < \beta - \frac{1}{p_2}\left(\frac{p_1^2}{4p_{12}(\alpha - k_{1\mathrm{g}})} + p_{12}\right). \tag{6.9}$$

Next, let $K_\mathrm{g} = [k_{1\mathrm{g}}M, k_{2\mathrm{g}}M]$ and define $\tilde{\Psi}(t) \triangleq \Psi(t) - K_\mathrm{g}$, $\tilde{\Phi}(t) \triangleq \Phi(t) + D$, $\tilde{C}(x_1) \triangleq C(x_1) - k_{2\mathrm{g}}M$, and $\tilde{K}(x_1) \triangleq K(x_1) - k_{1\mathrm{g}}M$, so that with $u(t)$, $t \geq 0$, given by (6.4), (6.3) becomes

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -M^{-1}\Big(\tilde{K}(x_1(t))x_1(t) + \tilde{C}(x_1(t))x_2(t) - \tilde{\Psi}(t)x(t) - \tilde{\Phi}(t)w(t)\Big) \end{bmatrix},$$
$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} q_0 \\ \dot{q}_0 \end{bmatrix}, \quad t \geq 0. \tag{6.10}$$

To show Lyapunov stability of the closed-loop system (6.5), (6.6), and (6.10) consider the Lyapunov function candidate

$$\begin{aligned} V(x, \Psi, \Phi) = {} & \tfrac{1}{2}x^\mathrm{T}(P \otimes M)x + p_2 \int_{0,\text{path}}^{x_1} \sigma^\mathrm{T}\tilde{K}(\sigma)\mathrm{d}\sigma + p_{12} \int_{0,\text{path}}^{x_1} \sigma^\mathrm{T}\tilde{C}(\sigma)\mathrm{d}\sigma \\ & + \tfrac{1}{2}\mathrm{tr}\, Q_1^{-1}\tilde{\Psi}Y^{-1}\tilde{\Psi}^\mathrm{T} + \tfrac{1}{2}\mathrm{tr}\, Q_2^{-1}\tilde{\Phi}Z^{-1}\tilde{\Phi}^\mathrm{T}, \end{aligned} \tag{6.11}$$

where the path integrals in (6.11) are taken over any path joining the origin to $x_1 \in \mathbb{R}^n$. Note that the path integrals in (6.11) are well defined since $C(\cdot), K(\cdot) \in \mathcal{S}$ and $f_k(x_1) \triangleq x_1^\mathrm{T}\tilde{K}(x_1)$ and $f_c(x_1) \triangleq x_1^\mathrm{T}\tilde{C}(x_1)$ are such that $\frac{\partial f_k}{\partial x_1}$ and $\frac{\partial f_c}{\partial x_1}$ are symmetric and hence gradients of real-valued functions [9, Theorem 10-37]. Thus, using the transformation $\sigma = \theta x_1$, where $\theta \in [0, 1]$, it follows that

$$\int_{0,\text{path}}^{x_1} \sigma^\mathrm{T}\tilde{K}(\sigma)\mathrm{d}\sigma = \int_0^1 \theta x_1^\mathrm{T}\tilde{K}(\theta x_1)x_1\mathrm{d}\theta = \int_0^1 [x_1^\mathrm{T}\tilde{K}(\theta x_1)x_1]\theta\mathrm{d}\theta \geq 0, \quad x_1 \in \mathbb{R}^n. \tag{6.12}$$

An identical analysis shows that

$$\int_{0,\text{path}}^{x_1} \sigma^\mathrm{T}\tilde{C}(\sigma)\mathrm{d}\sigma \geq 0, \quad x_1 \in \mathbb{R}^n. \tag{6.13}$$

Furthermore, note that $V(0, K_\mathrm{g}, -D) = 0$ and, since $P$, $M$, $Q_1$, $Q_2$, $Y$, and $Z$ are positive definite, $V(x, \Psi, \Phi) > 0$ for all $(x, \Psi, \Phi) \neq (0, K_\mathrm{g}, -D)$. In addition, $V(x, \Psi, \Phi)$

is radially unbounded. Now, letting $x(t)$, $t \geq 0$, denote the solution to (6.10) and using (6.5)–(6.7) it follows that the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned}
\dot{V}(x(t), \Psi(t), \Phi(t)) &= x^{\mathrm{T}}(t)(P \otimes M)\dot{x}(t) + x_1^{\mathrm{T}}(t)[p_2\tilde{K}(x_1(t)) + p_{12}\tilde{C}(x_1(t))]\dot{x}_1(t) \\
&\quad + \mathrm{tr}\ Q_1^{-1}\tilde{\Psi}(t)Y^{-1}\dot{\tilde{\Psi}}^{\mathrm{T}}(t) + \mathrm{tr}\ Q_2^{-1}\tilde{\Phi}(t)Z^{-1}\dot{\tilde{\Phi}}^{\mathrm{T}}(t) \\
&= [p_1x_1^{\mathrm{T}}(t) + p_{12}x_2^{\mathrm{T}}(t)]Mx_2(t) \\
&\quad + x_1^{\mathrm{T}}(t)[p_2\tilde{K}(x_1(t)) + p_{12}\tilde{C}(x_1(t))]x_2(t) \\
&\quad - [p_{12}x_1^{\mathrm{T}}(t) + p_2x_2^{\mathrm{T}}(t)][\tilde{K}(x_1(t))x_1(t) + \tilde{C}(x_1(t))x_2(t) - \tilde{\Psi}(t)x(t) \\
&\quad - \tilde{\Phi}(t)w(t)] + \mathrm{tr}\ Q_1^{-1}\tilde{\Psi}(t)Y^{-1}\dot{\Psi}^{\mathrm{T}}(t) + \mathrm{tr}\ Q_2^{-1}\tilde{\Phi}(t)Z^{-1}\dot{\Phi}^{\mathrm{T}}(t) \\
&= -p_{12}x_1^{\mathrm{T}}(t)\tilde{K}(x_1(t))x_1(t) + p_1x_1^{\mathrm{T}}(t)Mx_2(t) - x_2^{\mathrm{T}}(t)(p_2\tilde{C}(x_1(t)) \\
&\quad - p_{12}M)x_2(t) + \mathrm{tr}\ \tilde{\Psi}(t)[x(t)x^{\mathrm{T}}(t)(P \otimes I_n)B_0 + Y^{-1}\dot{\Psi}^{\mathrm{T}}(t)Q_1^{-1}] \\
&\quad + \mathrm{tr}\ \tilde{\Phi}(t)[w(t)x^{\mathrm{T}}(t)(P \otimes I_n)B_0 + Z^{-1}\dot{\Phi}^{\mathrm{T}}(t)Q_2^{-1}] \\
&= -p_{12}x_1^{\mathrm{T}}(t)\tilde{K}(x_1(t))x_1(t) + p_1x_1^{\mathrm{T}}(t)Mx_2(t) \\
&\quad - x_2^{\mathrm{T}}(t)(p_2\tilde{C}(x_1(t)) - p_{12}M)x_2(t) \\
&\leq -p_{12}(\alpha - k_{1\mathrm{g}})x_1^{\mathrm{T}}(t)Mx_1(t) + p_1x_1^{\mathrm{T}}(t)Mx_2(t) \\
&\quad - (p_2(\beta - k_{2\mathrm{g}}) - p_{12})x_2^{\mathrm{T}}(t)Mx_2(t) \\
&= -x^{\mathrm{T}}(t)(R \otimes M)x(t), \quad t \geq 0, \qquad (6.14)
\end{aligned}$$

where

$$R \triangleq \begin{bmatrix} p_{12}(\alpha - k_{1\mathrm{g}}) & -p_1/2 \\ -p_1/2 & p_2(\beta - k_{2\mathrm{g}}) - p_{12} \end{bmatrix}.$$

Now, it follows from (6.8) and (6.9) that $R > 0$ and hence, since $M > 0$, (6.14) implies that $\dot{V}(x(t), \Psi(t), \Phi(t)) \leq 0$, $(x(t), \Psi(t), \Phi(t)) \in \mathbb{R}^{2n} \times \mathbb{R}^{n\times 2n} \times \mathbb{R}^{n\times d}$, $t \geq 0$, which proves that the solution $(x(t), \Psi(t), \Phi(t)) \equiv (0, K_{\mathrm{g}}, -D)$ to (6.5), (6.6), and (6.10) is Lyapunov stable. Furthermore, since $R \otimes M > 0$, it follows from Theorem 2 of [42] that $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$. □

**Remark 6.1.** Note that the conditions in Theorem 6.1 imply that $x(t) \to 0$ as $t \to \infty$ and hence it follows from (6.5) and (6.6) that $(x(t), \Psi(t), \Phi(t)) \to \mathcal{M} \triangleq \{(x, \Psi, \Phi) \in \mathbb{R}^{2n} \times \mathbb{R}^{n\times 2n} \times \mathbb{R}^{n\times d} : x = 0,\ \dot{\Psi} = 0,\ \dot{\Phi} = 0\}$ as $t \to \infty$.

**Remark 6.2.** It is important to note that the bounds for $K(q)$ and $C(q)$, $q \in \mathbb{R}^n$, do not need to be known in order to implement the adaptive controller (6.4)–(6.6). All that is required is that $K(\cdot), C(\cdot) \in \mathcal{S}$, and $K(\cdot), C(\cdot)$ are continuous and lower bounded but otherwise unknown. Furthermore, $M \in \mathbb{R}^{n\times n}$ needs to be positive definite but otherwise unknown.

**Remark 6.3.** It can be seen from the proof of Theorem 6.1 (see the fourth equality in (6.14)) that Theorem 6.1 also holds for the case where $q^{\mathrm{T}}K(q)q \geq \varepsilon q^{\mathrm{T}}q$, $q \in \mathbb{R}^n$. This condition is weaker than requiring $K(q) \geq \varepsilon I_n$, $q \in \mathbb{R}^n$. This observation is key in developing some of the results in Section 6.3. A similar remark, however, does *not* hold for $C(\cdot)$ since the necessary term in (6.14) to be bounded is $\dot{q}^{\mathrm{T}}C(q)\dot{q}$ and not $q^{\mathrm{T}}C(q)q$.

Theorem 6.1 is applicable to the case where $C(q)$ and $K(q)$, $q \in \mathbb{R}^n$, are lower bounded. In practice however, $C(q)$ and $K(q)$, $q \in \mathbb{R}^n$, are often unbounded. Next, we provide a corollary to Theorem 6.1 that addresses the case where $C(\cdot)$ and $K(\cdot)$ can be unbounded operators.

**Corollary 6.1.** Consider the nonlinear dynamical system (6.3), or, equivalently, the nonlinear matrix second-order dynamical system (6.1). Assume there exist known matrix functions $C_{\mathrm{b}}(\cdot), K_{\mathrm{b}}(\cdot) \in \mathcal{S}$ and a scalar $\varepsilon \in \mathbb{R}$ such that $C(x_1) - C_{\mathrm{b}}(x_1) \geq \varepsilon I_n$ and $K(x_1) - K_{\mathrm{b}}(x_1) \geq \varepsilon I_n$, $x_1 \in \mathbb{R}^n$. Let $Q_1, Q_2 \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{2n\times 2n}$, $Z \in \mathbb{R}^{d\times d}$, and $P \in \mathbb{R}^{2\times 2}$ be positive definite, where $P = \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}$ and $p_{12} > 0$. Then the adaptive feedback control law

$$u(t) = K_{\rm b}(x_1(t))x_1(t) + C_{\rm b}(x_1(t))x_2(t) + \Psi(t)x(t) + \Phi(t)w(t), \qquad (6.15)$$

where $\Psi(t) \in \mathbb{R}^{n\times 2n}$, $t \geq 0$, and $\Phi(t) \in \mathbb{R}^{n\times d}$, $t \geq 0$, with update laws (6.5) and (6.6) guarantees that the solution $(x(t), \Psi(t), \Phi(t)) \equiv (0, K_{\rm g}, -D)$, where $K_{\rm g} \in \mathbb{R}^{n\times 2n}$, of the closed-loop system given by (6.3), (6.5), (6.6), and (6.15) is Lyapunov stable and $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^{2n}$.

**Proof.** Rewrite (6.1) as

$$M\ddot{q}(t) + \hat{C}(q(t))\dot{q}(t) + \hat{K}(q(t))q(t) = \hat{u}(t) + Dw(t), \quad q(0) = q_0, \quad \dot{q}(0) = \dot{q}_0, \quad t \geq 0, \qquad (6.16)$$

where $\hat{C}(q) \triangleq C(q) - C_{\rm b}(q)$, $\hat{K}(q) \triangleq K(q) - K_{\rm b}(q)$, and $\hat{u} \triangleq u - C_{\rm b}(q)\dot{q} - K_{\rm b}(q)q$. Now, the result is a direct consequence of Theorem 6.1. □

**Remark 6.4.** Note that Corollary 6.1 gives an adaptive stabilizing controller for a large class of nonlinearities in the generalized damping and stiffness operators $C(q)$ and $K(q)$, $q \in \mathbb{R}^n$. For example, in the special case where $n = 1$, let $C(\cdot)$ and $K(\cdot)$ belong to the set of nonlinearities given by

$$\mathcal{N} \triangleq \{n : \mathbb{R} \to \mathbb{R} : \; n'(q) \to 1 \text{ as } |q| \to \infty\}. \qquad (6.17)$$

See Figure 6.1 for a representative nonlinearity in $\mathcal{N}$. In this case, with $C_{\rm b}(q) = q$ and $K_{\rm b}(q) = q$, Corollary 6.1 can be used to construct "robustly" stabilizing controllers with respect to the class of nonlinearities considered. Note that unlike absolute stability theory, the nonlinearities $n(\cdot) \in \mathcal{N}$ are not required to be sector bounded nor satisfy $n(0) = 0$.

Next, we generalize Theorem 6.1 and Corollary 6.1 to the case where $C(q) - (\theta_c^{\rm T} \otimes I_n)C_{\rm b}(q)$ and $K(q) - (\theta_k^{\rm T} \otimes I_n)K_{\rm b}(q)$ are lower bounded and $\theta_c, \theta_k \in \mathbb{R}^p$ are unknown parameters and $C_{\rm b}, K_{\rm b} : \mathbb{R}^n \to \mathbb{R}^{pn\times n}$ are known functions.

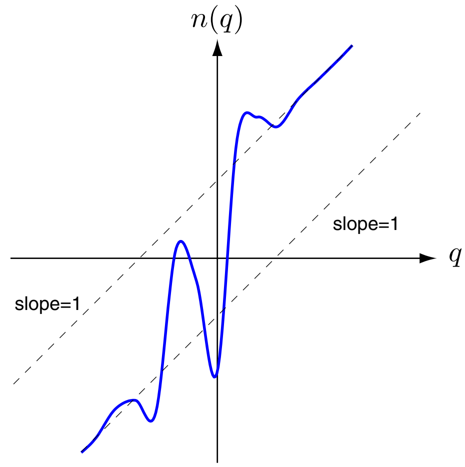



**Figure 6.1**: Representative nonlinearity $n(\cdot) \in \mathcal{N}$

**Theorem 6.2.** Consider the nonlinear dynamical system (6.3), or, equivalently, the nonlinear matrix second-order dynamical system (6.1). Assume there exist a constant $\varepsilon \in \mathbb{R}$, vectors $\theta_c, \theta_k \in \mathbb{R}^p$, and symmetric matrix functions $C_{\rm b}, K_{\rm b} : \mathbb{R}^n \to \mathbb{R}^{pn\times n}$ such that $C(x_1)-(\theta_c^{\rm T}\otimes I_n)C_{\rm b}(x_1) \geq \varepsilon I_n$ and $K(x_1)-(\theta_k^{\rm T}\otimes I_n)K_{\rm b}(x_1) \geq \varepsilon I_n$, $x_1 \in \mathbb{R}^n$. Furthermore, let $C_{{\rm b}i}(\cdot), K_{{\rm b}i}(\cdot) \in \mathcal{S}$, $i = 1,\ldots,p$, where $C_{{\rm b}i}(\cdot)$, $K_{{\rm b}i}(\cdot)$, $i = 1,\ldots,p$, are such that $C_{\rm b}(x_1) = [C_{{\rm b}1}(x_1),\cdots,C_{{\rm b}p}(x_1)]^{\rm T}$ and $K_{\rm b}(x_1) = [K_{{\rm b}1}(x_1),\cdots,K_{{\rm b}p}(x_1)]^{\rm T}$, and let $Q_1, Q_2 \in \mathbb{R}^{n\times n}$, $Q_3, Q_4 \in \mathbb{R}^{p\times p}$, $Y \in \mathbb{R}^{2n\times 2n}$, $Z \in \mathbb{R}^{d\times d}$, and $P \in \mathbb{R}^{2\times 2}$ be positive definite, where $P = \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}$ and $p_{12} > 0$. Then the adaptive feedback control law

$$u(t) = (\Theta_k^{\rm T}(t)\otimes I_n)K_{\rm b}(x_1(t))x_1(t) + (\Theta_c^{\rm T}(t)\otimes I_n)C_{\rm b}(x_1(t))x_2(t) + \Psi(t)x(t) + \Phi(t)w(t), \quad (6.18)$$

where $\Theta_k(t), \Theta_c(t) \in \mathbb{R}^p$, $t \geq 0$, $\Psi(t) \in \mathbb{R}^{n\times 2n}$, $t \geq 0$, and $\Phi(t) \in \mathbb{R}^{n\times d}$, $t \geq 0$, with update laws (6.5), (6.6), and

$$\dot{\Theta}_k(t) = -Q_3(I_p \otimes x_1^{\rm T}(t))K_{\rm b}(x_1(t))B_0^{\rm T}(P\otimes I_n)x(t), \quad \Theta_k(0) = \Theta_{k0}, \quad (6.19)$$

$$\dot{\Theta}_c(t) = -Q_4(I_p \otimes x_2^{\rm T}(t))C_{\rm b}(x_1(t))B_0^{\rm T}(P\otimes I_n)x(t), \quad \Theta_c(0) = \Theta_{c0}, \quad (6.20)$$

guarantees that the solution $(x(t), \Psi(t), \Phi(t), \Theta_c(t), \Theta_k(t)) \equiv (0, K_{\rm g}, -D, \theta_c, \theta_k)$, where $K_{\rm g} \in \mathbb{R}^{n\times 2n}$, of the closed-loop system given by (6.3), (6.5), (6.6), (6.18)–(6.20), is Lyapunov stable and $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^{2n}$.

**Proof.** Let $\alpha, \beta \in \mathbb{R}$ be such that

$$\alpha M \leq K(q) - (\theta_k^{\rm T} \otimes I_n)K_{\rm b}(q), \quad \beta M \leq C(q) - (\theta_c^{\rm T} \otimes I_n)C_{\rm b}(q),$$

and let $k_{1\rm g}, k_{2\rm g} \in \mathbb{R}$ be such that (6.8) and (6.9) hold. Furthermore, define $\tilde{C}(x_1) \triangleq C(x_1) - (\theta_c^{\rm T} \otimes I_n)C_{\rm b}(x_1) - k_{2\rm g}M$, $\tilde{K}(x_1) \triangleq K(x_1) - (\theta_k^{\rm T} \otimes I_n)K_{\rm b}(x_1) - k_{1\rm g}M$, $\tilde{\Theta}_k(t) \triangleq \Theta_k(t) - \theta_k$, and $\tilde{\Theta}_c(t) \triangleq \Theta_c(t) - \theta_c$, and consider the Lyapunov function candidate

$$\begin{aligned} V(x, \Psi, \Phi, \Theta_k, \Theta_c) = {} & \tfrac{1}{2}x^{\rm T}(P \otimes M)x + p_2 \int_{0,\text{path}}^{x_1} \sigma^{\rm T}\tilde{K}(\sigma)\mathrm{d}\sigma + p_{12} \int_{0,\text{path}}^{x_1} \sigma^{\rm T}\tilde{C}(\sigma)\mathrm{d}\sigma \\ & +\tfrac{1}{2}\mathrm{tr}\ Q_1^{-1}\tilde{\Psi}Y^{-1}\tilde{\Psi}^{\rm T} + \tfrac{1}{2}\mathrm{tr}\ Q_2^{-1}\tilde{\Phi}Z^{-1}\tilde{\Phi}^{\rm T} + \tfrac{1}{2}\tilde{\Theta}_k^{\rm T}Q_3^{-1}\tilde{\Theta}_k \\ & +\tfrac{1}{2}\tilde{\Theta}_c^{\rm T}Q_4^{-1}\tilde{\Theta}_c, \end{aligned} \tag{6.21}$$

where the path integrals in (6.21) are taken over any path joining the origin to $x_1 \in \mathbb{R}^n$. Now, the proof is identical to the proof of Theorem 6.1. $\square$

**Remark 6.5.** Theorem 6.2 generalizes Corollary 6.1 in that $C(q) - (\theta_c^{\rm T} \otimes I_n)C_{\rm b}(q)$ and $K(q) - (\theta_k^{\rm T} \otimes I_n)K_{\rm b}(q)$ are lower bounded as opposed to $C(q) - C_{\rm b}(q)$ and $K(q) - K_{\rm b}(q)$ be lower bounded. This gives yet a larger class of nonlinearities that can be considered in $C(\cdot)$ and $K(\cdot)$. To see this, recall Remark 6.4, let $n = 1$, and let $C(\cdot)$ and $K(\cdot)$ belong to the set of nonlinearities given by

$$\mathcal{N}_{\rm e} \triangleq \{n : \mathbb{R} \to \mathbb{R} : \lim_{|q|\to\infty} n'(q) \text{ exists}\}. \tag{6.22}$$

In this case, there exists $\theta_c, \theta_k \in \mathbb{R}$ such that $C(q) - \theta_c q$ and $K(q) - \theta_k q$ are lower bounded. Note that $\mathcal{N}_{\rm e} \supset \mathcal{N}$.

## 6.3. Polynomial Uncertainty with Unknown Coefficients and Unknown Order

In this section we provide special cases to Corollary 6.1 and Theorem 6.2 that address nonlinearities for which there always exist $C_{\mathrm{b}}(\cdot)$ and $K_{\mathrm{b}}(\cdot)$ satisfying the conditions of Corollary 6.1 and Theorem 6.2. Our first result considers scalar second-order systems.

**Proposition 6.1.** Consider the nonlinear dynamical system (6.3), or, equivalently, the nonlinear second-order dynamical system (6.1). Assume $n = 1$ and let $C(\cdot)$ and $K(\cdot)$ be unknown polynomials. Let $Q_1, Q_2 \in \mathbb{R}$, $Y \in \mathbb{R}^{2\times 2}$, $Z \in \mathbb{R}^{d\times d}$, and $P \in \mathbb{R}^{2\times 2}$ be positive definite, where $P = \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}$ and $p_{12} > 0$. Finally, let $\alpha, \beta > 0$ and let $C_{\mathrm{b}} : \mathbb{R} \to \mathbb{R}$ (resp., $K_{\mathrm{b}} : \mathbb{R} \to \mathbb{R}$) be given by one of the following conditions, as appropriate:

*i)* If the order $N$ of the polynomial $C(q)$ (resp., $K(q)$) is known, then choose $C_{\mathrm{b}}(q) = -\alpha q^{N+1}$ (resp., $K_{\mathrm{b}}(q) = -\beta q^{N+1}$) if $N$ is odd, or $C_{\mathrm{b}}(q) = -\alpha q^{N+2}$ (resp., $K_{\mathrm{b}}(q) = -\beta q^{N+2}$) if $N$ is even. If $N$ is even and the sign of the leading coefficient is positive, then choose $C_{\mathrm{b}}(q) = 0$ (resp., $K_{\mathrm{b}}(q) = 0$).

*ii)* If the order $N$ of the polynomial $C(q)$ (resp., $K(q)$) is unknown but the sign, $\sigma \triangleq \operatorname{sgn} a = a/|a|$, of the leading coefficient $a$ is known, then choose $C_{\mathrm{b}}(q) = -\alpha\cosh(\beta q) + \alpha\sigma \sinh(\beta q)$ (resp., $K_{\mathrm{b}}(q) = -\alpha\cosh(\beta q) + \alpha\sigma \sinh(\beta q)$) if $N$ is odd, or $C_{\mathrm{b}}(q) = \alpha(\sigma - 1)\cosh(\beta q)$ (resp., $K_{\mathrm{b}}(q) = \alpha(\sigma - 1)\cosh(\beta q)$) if $N$ is even.

*iii)* If neither the order $N$ of the polynomial $C(q)$ (resp., $K(q)$) nor the sign $\sigma$ of the leading coefficient are known, then choose $C_{\mathrm{b}}(q) = -\alpha\cosh(\beta q)$ (resp., $K_{\mathrm{b}}(q) = -\alpha\cosh(\beta q)$).

Then the adaptive feedback control law

$$u(t) = K_{\mathrm{b}}(x_1(t))x_1(t) + C_{\mathrm{b}}(x_1(t))x_2(t) + \Psi(t)x(t) + \Phi(t)w(t), \tag{6.23}$$

where $\Psi(t) \in \mathbb{R}^{1\times 2}$, $t \geq 0$, and $\Phi(t) \in \mathbb{R}^{1\times d}$, $t \geq 0$, with update laws (6.5) and (6.6) guarantees that the solution $(x(t), \Psi(t), \Phi(t)) \equiv (0, K_{\mathrm{g}}, -D)$, where $K_{\mathrm{g}} \in \mathbb{R}^{1\times 2}$, of the closed-loop system given by (6.3), (6.5), (6.6) and (6.23) is Lyapunov stable and $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^2$.

**Proof.** The proof is a direct consequence of Corollary 6.1 by noting that $C_{\mathrm{b}}(q)$ and $K_{\mathrm{b}}(q)$, $q \in \mathbb{R}$, given by *i*)–*iii*) yield bounds to $C(q)$ and $K(q)$, $q \in \mathbb{R}$, respectively, in each of the three cases. Here we give the proof for $C(q)$ polynomials with $\alpha = \beta = 1$. Identical arguments hold for $K(q)$ polynomials as well as for the case where $\alpha, \beta > 0$ are arbitrary. Specifically, note that any polynomial of even order with leading coefficient as unity is lower bounded. Hence, for all $q \in \mathbb{R}$, $C(q) + q^{N+1}$ is lower bounded if $N$ is odd and $C(q) + q^{N+2}$ is lower bounded if $N$ is even. Furthermore, if $N$ is even and the leading coefficient is positive then $C(q)$, $q \in \mathbb{R}$, is lower bounded which proves the result for condition *i*). To prove the result for condition *ii*), note that for all $q \in \mathbb{R}$, $C(q)$ if lower bounded if $N$ is even and the leading coefficient is positive, $C(q) + 2\cosh(q)$ is lower bounded if $N$ is even and the leading coefficient is negative, $C(q) + e^{q}$ is lower bounded if $N$ is odd and the leading coefficient is positive, and $C(q) + e^{-q}$ is lower bounded if $N$ is odd and the leading coefficient is negative. Finally, to prove the result for condition *iii*) it need only be noted that for all $q \in \mathbb{R}$, $C(q) + \cosh(q)$ is lower bounded irrespective of $N$ and the sign of the leading coefficient. □

**Remark 6.6.** Proposition 6.1 provides a parametrization free universal adaptive controller for scalar second-order systems with polynomial nonlinearities with unknown coefficients and unknown order. We emphasize that (6.23) is distinct from standard adaptive controllers involving an $N$-vector parameter estimate update law for the uncertain polynomial coefficients. In contrast, (6.23) provides a minimal complexity adaptive controller involving the scalar parameters $K_{\rm b}(x_1)$ and $C_{\rm b}(x_1)$.

A multivariable generalization to Proposition 6.1 is not straightforward. To see this let $C(q)$ and $K(q)$ be multivariable polynomial matrix functions in $q = [q_1, \ldots, q_n]^{\rm T}$. Then even though it can be shown that $C(q) - C_{\rm b}(q) \geq \varepsilon I_n$ and $K(q) - K_{\rm b}(q) \geq \varepsilon I_n$, where $C_{\rm b}(q) = K_{\rm b}(q) = -\prod_{i=1}^{n} \cosh(q_i) U$ and $U \in \mathbb{R}^{n\times n}$ is the ones matrix containing all unity elements, $C_{\rm b}(q)$ and $K_{\rm b}(q)$, $q \in \mathbb{R}^n$, do not belong to the set $\mathcal{S}$ given by (6.2). Of course, a trivial extension to Proposition 6.1 is the case where $C(q)$ and $K(q)$, $q \in \mathbb{R}^n$, are diagonal and component decoupled; that is, $C(q) = \operatorname{diag}[C_{(1,1)}(q_1), C_{(2,2)}(q_2), \cdots, C_{(n,n)}(q_n)]$ and similarly for $K(q)$. Next, we consider a partial generalization to Proposition 6.1 for matrix second-order systems. To state this result the following key lemma is needed.

**Lemma 6.1.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function such that $f(q) \to \infty$ as $\|q\| \to \infty$, $f(0) = 0$, and $\frac{\partial f}{\partial q}\Big|_{q=0} = 0$. Then there exists $\varepsilon \in (-\infty, 0]$ such that $f(q) \geq \varepsilon q^{\rm T} q$.

**Proof.** It follows from the radially unbounded condition that there exists $r > 0$ such that

$$\mathcal{F} \triangleq \{q \in \mathbb{R}^n : \ f(q) \leq 0\} \subseteq \{q \in \mathbb{R}^n : \|q\| \leq r\},$$

which implies that $\mathcal{F}$ is compact. Hence, since $f(\cdot)$ is continuous it follows that there exists $\varepsilon_0 \in (-\infty, 0]$ such that $f(q) \geq \varepsilon_0$, $q \in \mathcal{F}$. Now, since $f(q) > 0$, $q \in \mathbb{R}^n \backslash \mathcal{F}$, it follows that

$$f(q) \geq \varepsilon_0, \quad q \in \mathbb{R}^n. \tag{6.24}$$

Next, define $H \triangleq \frac{1}{2}\frac{\partial^2 f}{\partial q^2}\Big|_{q=0}$ and let $\hat{\varepsilon} \in \mathbb{R}$ be such that $H - \hat{\varepsilon} I_n > 0$. Hence, since $f(0) = 0$ and $\frac{\partial f}{\partial q}\Big|_{q=0} = 0$, it follows that

$$f(q) - \hat{\varepsilon} q^{\mathrm{T}} q = q^{\mathrm{T}}(H - \hat{\varepsilon} I_n)q + \mathcal{O}(q),$$

where $\mathcal{O} : \mathbb{R}^n \to \mathbb{R}$ is such that $\frac{\mathcal{O}(q)}{\|q\|^2} \to 0$ as $\|q\| \to 0$. Thus, there exists $\delta_{\hat{\varepsilon}} > 0$ such that

$$f(q) - \hat{\varepsilon} q^{\mathrm{T}} q \geq 0, \quad q \in \mathbb{R}^n, \quad \|q\| \leq \delta_{\hat{\varepsilon}}. \tag{6.25}$$

Next, note that it follows from (6.24) that

$$\frac{f(q)}{q^{\mathrm{T}} q} \geq \frac{\varepsilon_0}{q^{\mathrm{T}} q} > \frac{\varepsilon_0}{\delta_{\hat{\varepsilon}}^2}, \quad q \in \mathbb{R}^n, \quad \|q\| > \delta_{\hat{\varepsilon}}.$$

Hence, $f(q) \geq \varepsilon q^{\mathrm{T}} q$, $q \in \mathbb{R}^n$, where $\varepsilon \triangleq \min\{\hat{\varepsilon}, \frac{\varepsilon_0}{\delta_{\hat{\varepsilon}}^2}\}$. □

**Proposition 6.2.** Consider the nonlinear dynamical system (6.3), or, equivalently, the nonlinear matrix second-order dynamical system (6.1). Let $C(\cdot), K(\cdot)$ be matrix functions with unknown polynomial entries. Furthermore, assume $C(\cdot)$ is such that $C_{(i,j)}(q) = C_{(i,j)}$, $i \neq j$, and $C_{(i,i)}(q) = C_{(i,i)}(q_i)$, where $C_{(i,i)}(q_i)$, $i = 1, \cdots, n$, is an unknown polynomial. Let $Q_1, Q_2 \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{2n\times 2n}$, $Z \in \mathbb{R}^{d\times d}$, and $P \in \mathbb{R}^{2\times 2}$ be positive definite, where $P = \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}$ and $p_{12} > 0$. Finally, let $\alpha_i, \beta_i > 0$, $i = 1, \cdots, n$, and let $C_{\mathrm{b}} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ (resp., $K_{\mathrm{b}} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$) be given by one of the following conditions, as appropriate:

*i)* If the highest order $N$ of the polynomial functions in $C(q)$ (resp., $K(q)$) is known, then choose $C_{\mathrm{b}}(q) = -\mathrm{diag}[\alpha_1 q_1^M, \cdots, \alpha_n q_n^M]$ (resp., $K_{\mathrm{b}}(q) = -\mathrm{diag}[\alpha_1 q_1^M, \cdots, \alpha_n q_n^M]$), where $M$ is the smallest even integer such that $M > N$.

*ii)* If the highest order $N$ of the polynomial functions in $C(q)$ (resp., $K(q)$) is unknown, then choose $C_{\mathrm{b}}(q) = -\mathrm{diag}[\alpha_1 \cosh(\beta_1 q_1), \cdots, \alpha_n \cosh(\beta_n q_n)]$ (resp., $K_{\mathrm{b}}(q) = -\mathrm{diag}[\alpha_1 \cosh(\beta_1 q_1), \cdots, \alpha_n \cosh(\beta_n q_n)]$).

Then the adaptive feedback control law

$$u(t) = K_{\rm b}(x_1(t))x_1(t) + C_{\rm b}(x_1(t))x_2(t) + \Psi(t)x(t) + \Phi(t)w(t), \tag{6.26}$$

where $\Psi(t) \in \mathbb{R}^{n\times 2n}$, $t \geq 0$, and $\Phi(t) \in \mathbb{R}^{n\times d}$, $t \geq 0$, with update laws (6.5) and (6.6) guarantees that the solution $(x(t), \Psi(t), \Phi(t)) \equiv (0, K_{\rm g}, -D)$, where $K_{\rm g} \in \mathbb{R}^{n\times 2n}$, of the closed-loop system given by (6.3), (6.5), (6.6), and (6.26) is Lyapunov stable and $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^{2n}$.

**Proof.** The proof is a direct consequence of Corollary 6.1 and Remark 6.3 by noting that $C_{\rm b}(q)$ and $K_{\rm b}(q)$ given by *i*) and *ii*) belong to $\mathcal{S}$ and there exists $\varepsilon \in \mathbb{R}$ such that $C(q) - C_{\rm b}(q) \geq \varepsilon I_n$ and $q^{\rm T}(K(q) - K_{\rm b}(q))q \geq \varepsilon q^{\rm T}q$, $q \in \mathbb{R}$, for each of the cases. The first inequality is immediate from Proposition 6.1 given the assumed structure of $C(\cdot)$. To show the second inequality, define $f(q) \triangleq f_1(q) + f_2(q)$, where $f_1(q) \triangleq q^{\rm T}K(q)q$ and $f_2(q) \triangleq -q^{\rm T}K_{\rm b}(q)q$. Note that for condition *i*), $f_2(q)$ is a negative definite function for all $q \in \mathbb{R}^n$ and has order $M + 2$. Furthermore, since $-f_2(q)$ is radially unbounded and has a higher order than $f_1(q)$, it follows that $f(q)$ is radially unbounded. Next, note that $f(0) = 0$ and $\left.\frac{\partial f}{\partial q}\right|_{q=0} = 0$ and hence the result follows from Lemma 6.1. To show the result for condition *ii*), note that $f(q)$ is radially unbounded since $-f_2(q) = -q^{\rm T}K_{\rm b}(q)q = \sum_{i=1}^{n} \alpha_i \cosh(\beta_i q_i)q_i^2$ is a hyperbolic function and radially unbounded. Once again, the result now follows from Lemma 6.1. □

**Remark 6.7.** Proposition 6.2 presents a partial generalization to Proposition 6.1 for multivariable matrix second-order systems. It is important to note that $K(\cdot)$ is a general matrix function with unknown polynomial entries and hence no internal structural constraints are imposed on $K(\cdot)$. However, unlike $K(\cdot)$, $C(\cdot)$ is assumed to be a matrix function with constrained structure involving unknown polynomials on the diagonal entries and unknown constants on the off-diagonal entries. As shown in the proof of the proposition, this constraint guarantees the existence of $\varepsilon \in \mathbb{R}$ such that $C(q) - C_{\rm b}(q) \geq \varepsilon I_n$.

Finally, we use Theorem 6.2 to provide a generalization of Proposition 6.1.

**Proposition 6.3.** Consider the nonlinear dynamical system (6.3), or, equivalently, the nonlinear matrix second-order dynamical system (6.1). Assume $n = 1$ and let $C(\cdot)$ and $K(\cdot)$ be unknown polynomials. Let $Q_1, Q_2 \in \mathbb{R}$, $Y \in \mathbb{R}^{2\times 2}$, $Z \in \mathbb{R}^{d\times d}$, and $P \in \mathbb{R}^{2\times 2}$ be positive definite, where $P = \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}$ and $p_{12} > 0$. Finally, let $\alpha, \beta > 0$ and let $C_{\rm b} : \mathbb{R} \to \mathbb{R}^2$ (resp., $K_{\rm b} : \mathbb{R} \to \mathbb{R}^2$) be given by one of the following conditions, as appropriate:

*i)* If the order $N$ of the polynomial $C(q)$ (resp., $K(q)$) is even and known, then choose $C_{\rm b}(q) = [q^{N+2}, q^{N+2}]^{\rm T}$ (resp., $K_{\rm b}(q) = [q^{N+2}, q^{N+2}]^{\rm T}$).

*ii)* If the order $N$ of the polynomial $C(q)$ (resp., $K(q)$) is unknown, then choose $C_{\rm b}(q) = [\cosh(\alpha q), \sinh(\alpha q)]^{\rm T}$ (resp., $K_{\rm b}(q) = [\cosh(\beta q), \sinh(\beta q)]^{\rm T}$).

Then the adaptive feedback control law

$$u(t) = \Theta_k^{\rm T}(t) K_{\rm b}(x_1(t)) x_1(t) + \Theta_c^{\rm T}(t) C_{\rm b}(x_1(t)) x_2(t) + \Psi(t) x(t) + \Phi(t) w(t), \quad (6.27)$$

where $\Theta_k(t), \Theta_c(t) \in \mathbb{R}^2$, $t \geq 0$, $\Psi(t) \in \mathbb{R}^{1\times 2}$, $t \geq 0$, and $\Phi(t) \in \mathbb{R}^{1\times d}$, $t \geq 0$, with update laws (6.5), (6.6), (6.19), and (6.20) guarantees that the solution $(x(t), \Psi(t), \Phi(t), \Theta_c(t), \Theta_k(t)) \equiv (0, K_{\rm g}, -D, \theta_c, \theta_k)$, where $K_{\rm g} \in \mathbb{R}^{1\times 2}$, of the closed-loop system given by (6.3), (6.5), (6.6), (6.19), (6.20), and (6.27) is Lyapunov stable and $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^2$.

**Proof.** The result is a direct consequence of Theorem 6.2. Specifically, as noted in the proof of Proposition 6.1, if the order $N$ of the polynomial $C(q)$ (resp., $K(q)$)

is even and known, then there exist $\theta_{c1}, \theta_{c2} \in \{-1, 0, 1\}$ (resp., $\theta_{k1}, \theta_{k2} \in \{-1, 0, 1\}$) such that $C(q) - \theta_{c1} q^{N+2} - \theta_{c2} q^{N+2}$ (resp., $K(q) - \theta_{k1} q^{N+2} - \theta_{k2} q^{N+2}$) is lower bounded. Alternatively, if the order $N$ of the polynomial $C(q)$ (resp., $K(q)$) is unknown, then there exist $\theta_{c1}, \theta_{c2} \in \{-1, 0, 1\}$ (resp., $\theta_{k1}, \theta_{k2} \in \{-1, 0, 1\}$) such that for every $\alpha > 0$, $C(q) - \theta_{c1}\cosh(\alpha q) - \theta_{c2}\sinh(\alpha q)$ (resp., for every $\beta > 0$, $K(q) - \theta_{k1}\cosh(\beta q) - \theta_{k2}\sinh(\beta q)$) is lower bounded. □

## 6.4. Illustrative Numerical Examples

In this section we present two numerical examples to demonstrate the utility of the proposed direct adaptive control framework for adaptive stabilization.

**Example 6.1.** Consider the nonlinear matrix second-order dynamical system with sign varying stiffness and damping matrix functions given by (6.1) where $n = 2$, $M$, $C(q)$, $K(q)$ are unknown, $M > 0$, $C(q)$ and $K(q)$, $q \in \mathbb{R}^2$, are lower bounded, and $w(t) \equiv 0$. Now, with $p_2 > 0$ and $p_{12} > 0$, it follows from Theorem 6.1 that the adaptive feedback controller (6.4) guarantees that $x(t) \to 0$ as $t \to \infty$. With

$$M = \begin{bmatrix} m_1 & m_{12} \\ m_{12} & m_2 \end{bmatrix}, \quad C(q) = \begin{bmatrix} c_1 \sin(q_1) & 0 \\ 0 & c_2 \cos(q_2) \end{bmatrix},$$
$$K(q) = \begin{bmatrix} k_1 \sin(q_1) & 0 \\ 0 & k_2 \cos(q_2) \end{bmatrix},$$

where $m_1 = m_2 = 2$, $m_{12} = 1$, $c_1 = c_2 = k_1 = k_2 = 1$, and $p_{12} = 1$, $p_2 = 2$, $Q_1 = M$, $Y = 2I_4$, and initial conditions $q(0) = [1,\, 2]^{\mathrm{T}}$, $\dot{q}(0) = [3,\, 4]^{\mathrm{T}}$, and $\Psi(0) = 0_{2\times 4}$, Figure 6.2 shows the state trajectories versus time and the control signals versus time. Figure 6.3 shows the adaptive gain history versus time.

**Example 6.2.** Consider the nonlinear matrix second-order dynamical system with nonlinear damping and stiffness matrix functions given by (6.1) where $n = 2$, $M$, $C(q)$, $K(q)$ are unknown, $M > 0$, $C(q)$ and $K(q)$, $q \in \mathbb{R}^2$, are lower bounded,

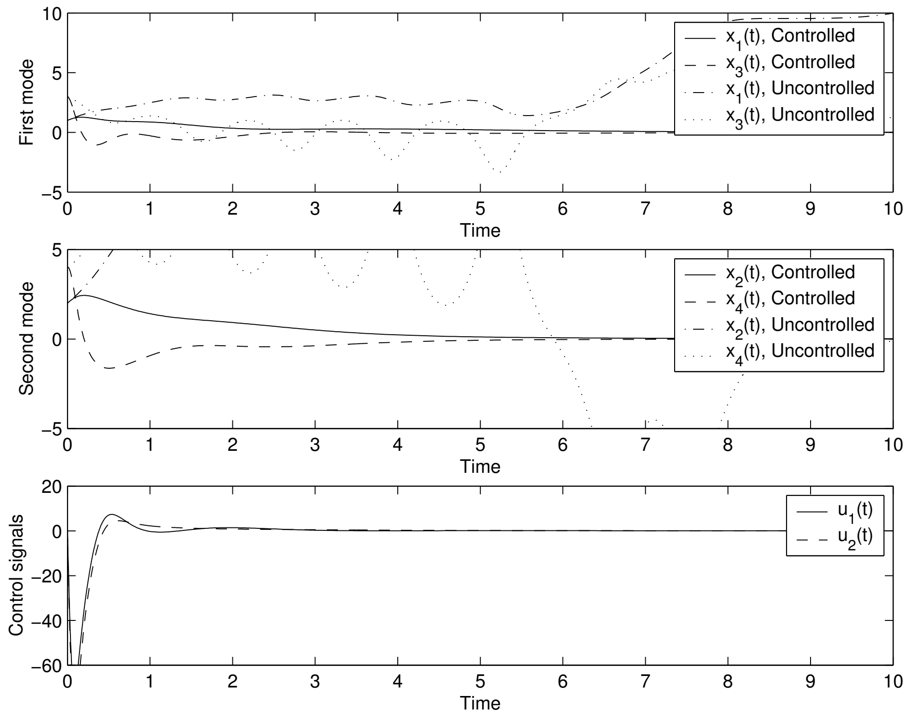



**Figure 6.2**: State trajectories and control signals versus time

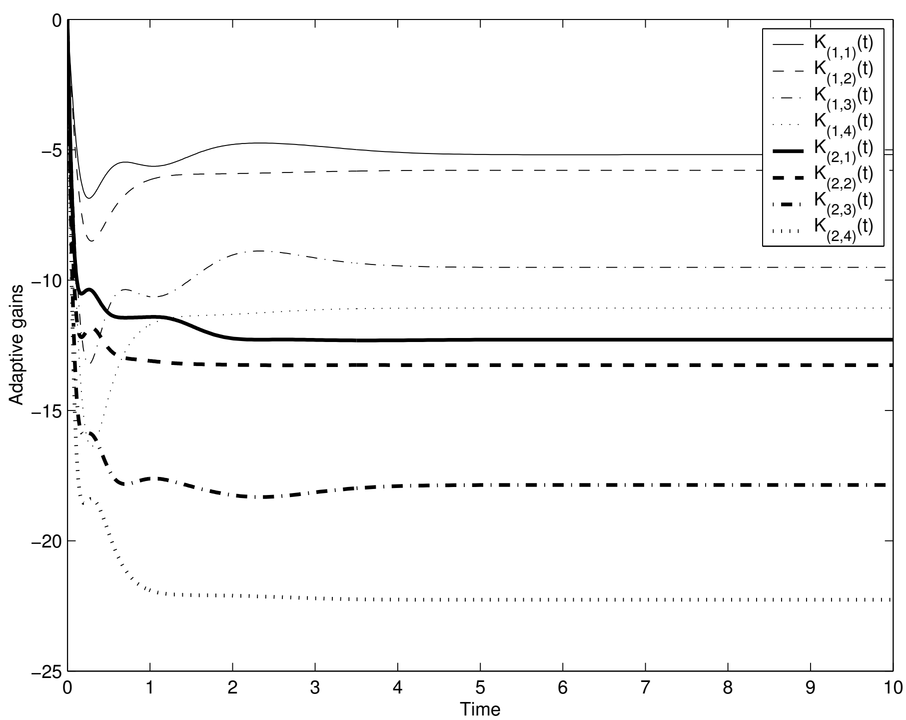



**Figure 6.3**: Adaptive gain history versus time

$C(\cdot), K(\cdot) \in \mathcal{S}$, and $w(t) \equiv 0$. Now, with $p_2 > 0$ and $p_{12} > 0$, it follows from Theorem 6.1 that the adaptive feedback controller (6.4) guarantees that $x(t) \to 0$ as $t \to \infty$. With

$$M = \begin{bmatrix} m_1 & m_{12} \\ m_{12} & m_2 \end{bmatrix}, \quad C(q) = \begin{bmatrix} c_1(q_1^2 - c_2) & 0 \\ 0 & c_3(q_2^2 - c_4) \end{bmatrix}, \quad K(q) = \mathrm{diag}[k_1,\, k_2],$$

where $m_1 = m_2 = 2$, $m_{12} = 1$, $c_1 = c_2 = 2$, $c_3 = c_4 = 1$, $k_1 = k_2 = 20$, and $p_{12} = 1$, $p_2 = 2$, $Q_1 = M$, $Y = 0.5I_4$, and initial conditions $q(0) = [5,\, 0]^{\mathrm{T}}$, $\dot{q}(0) = [0,\, 0]^{\mathrm{T}}$, and $\Psi(0) = 0_{2\times 4}$, Figure 6.4 shows the phase portraits of the controlled and uncontrolled system. Note that the adaptive controller is switched on at $t = 15$ sec. Figure 6.5 shows the state trajectories versus time and the control signals versus time. Finally, Figure 6.6 shows the adaptive gain history versus time.

**Example 6.3.** Consider the nonlinear matrix second-order dynamical system with nonlinear damping and stiffness matrix functions given by (6.1) where $n = 2$, $M$, $C(q)$, $K(q)$ are unknown, $M > 0$, $C(q)$ and $K(q)$, $q \in \mathbb{R}^2$, are lower bounded, $C(\cdot), K(\cdot) \in \mathcal{S}$, and $w(t) \equiv 0$. Now, with $p_2 > 0$ and $p_{12} > 0$, it follows from Theorem 6.1 that the adaptive feedback controller (6.4) guarantees that $x(t) \to 0$ as $t \to \infty$. With

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad C(q) = \begin{bmatrix} c_1(q_1^2 - c_2) & 0 \\ 0 & c_3 \end{bmatrix},$$
$$K(q) = \begin{bmatrix} k_1 + k_3(q_1^2 - 2q_1q_2 + 3q_2^2) & -k_1 - k_3(q_1^2 + q_2^2) \\ -k_1 - k_3(q_1^2 + q_2^2) & k_1 + k_2 + k_3(3q_1^2 - 2q_1q_2 + q_2^2) \end{bmatrix},$$

where $m_1 = 3$, $m_2 = 2$, $c_1 = 3$, $c_2 = 1$, $c_3 = -0.5$, $k_1 = -1$, $k_2 = 5$, $k_3 = 1$, and $p_{12} = 2$, $p_2 = 1$, $Q_1 = I_2$, $Y = I_4$, and initial conditions $q(0) = [0,\, 0]^{\mathrm{T}}$, $\dot{q}(0) = [3,\, 0]^{\mathrm{T}}$, and $\Psi(0) = 0_{2\times 4}$, Figure 6.7 shows the phase portraits of the controlled and uncontrolled system. Note that the adaptive controller is switched on at $t = 10$ sec. Figure 6.8 shows the state trajectories versus time and the control signals versus time. Finally, Figure 6.9 shows the adaptive gain history versus time.

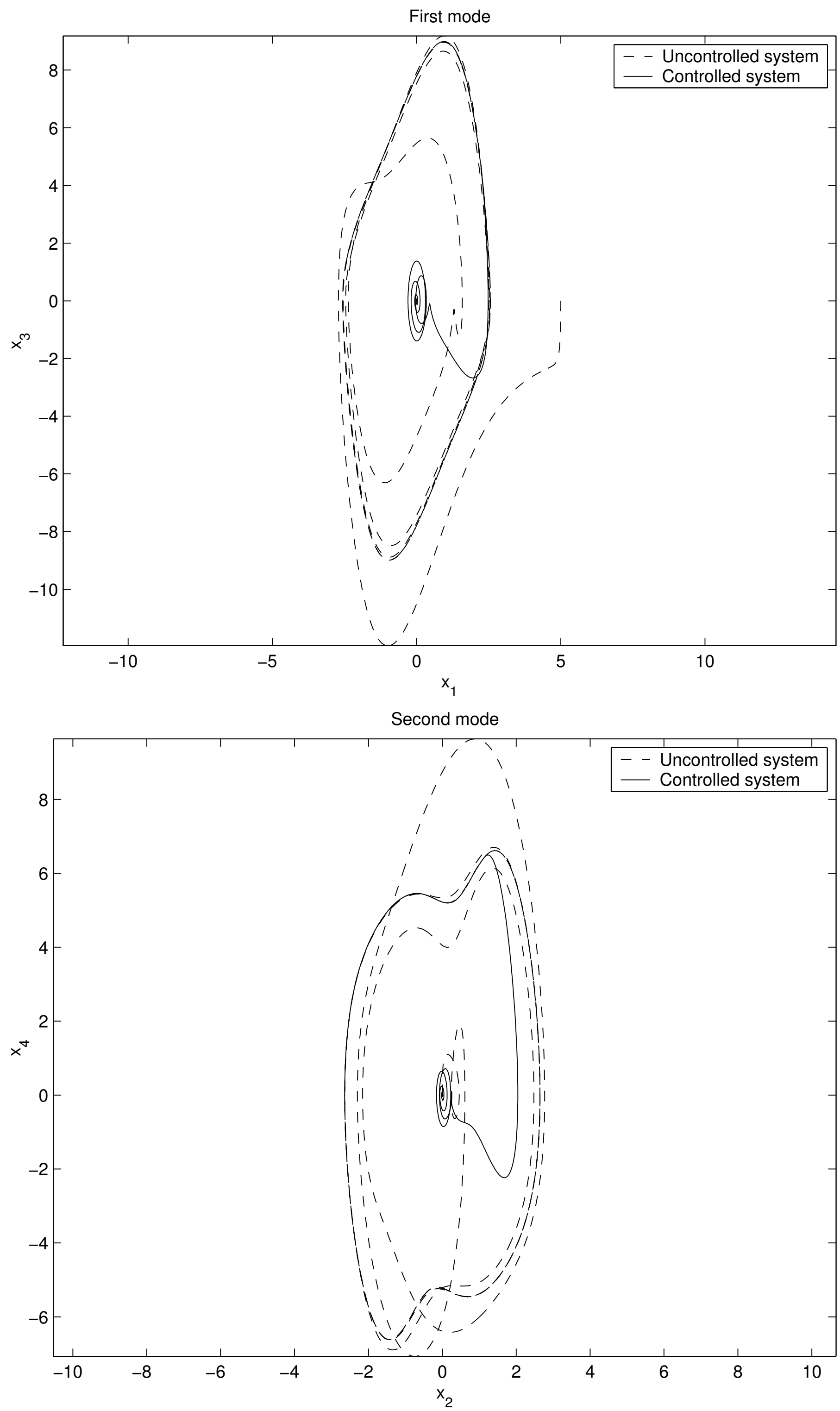



**Figure 6.4**: Phase portraits of controlled and uncontrolled system

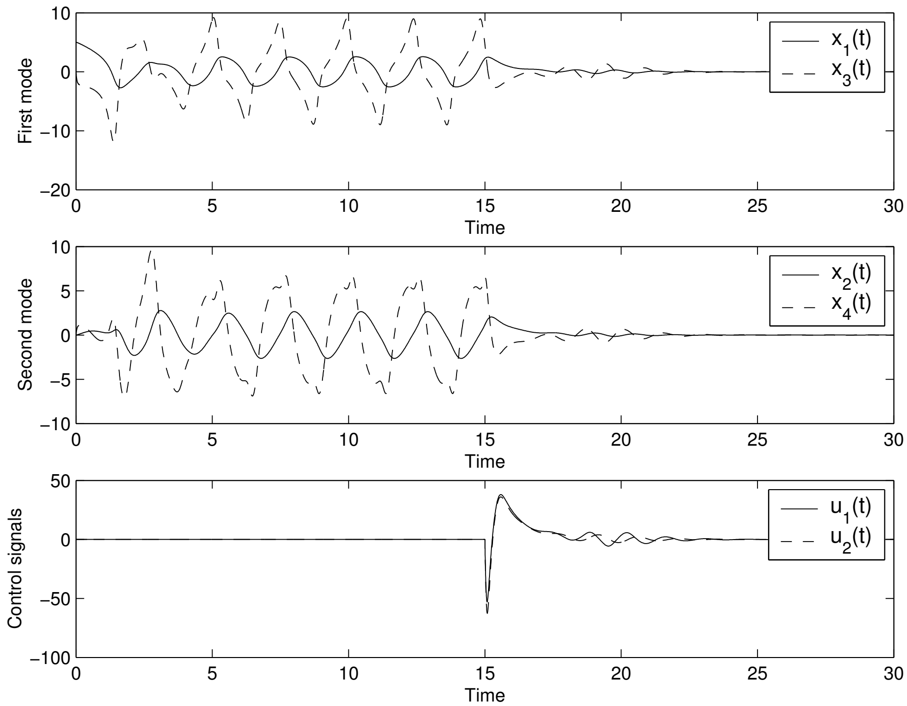



**Figure 6.5**: State trajectories and control signals versus time

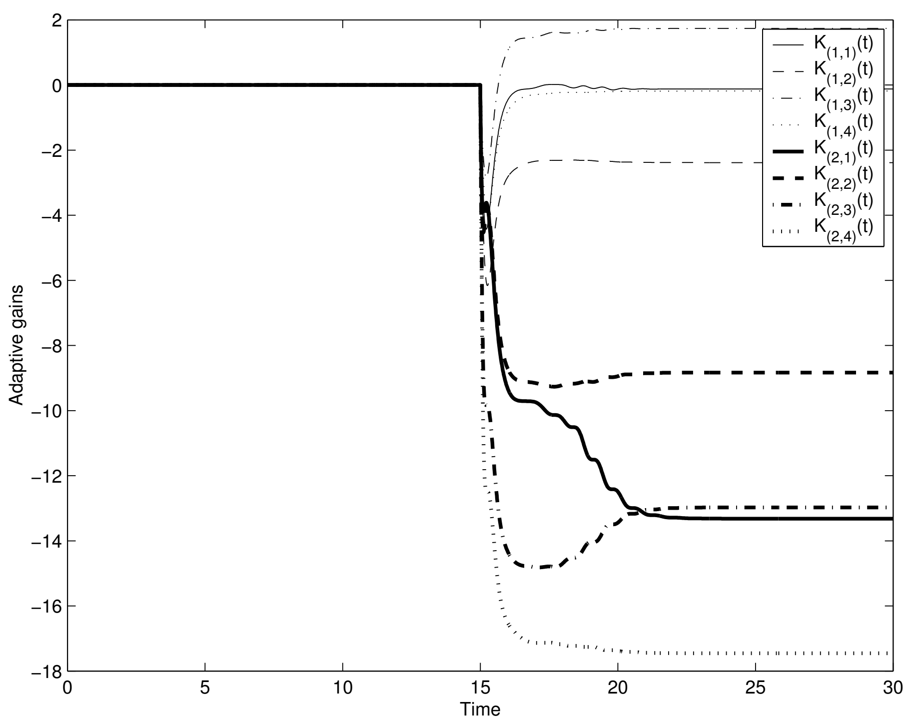



**Figure 6.6**: Adaptive gain history versus time

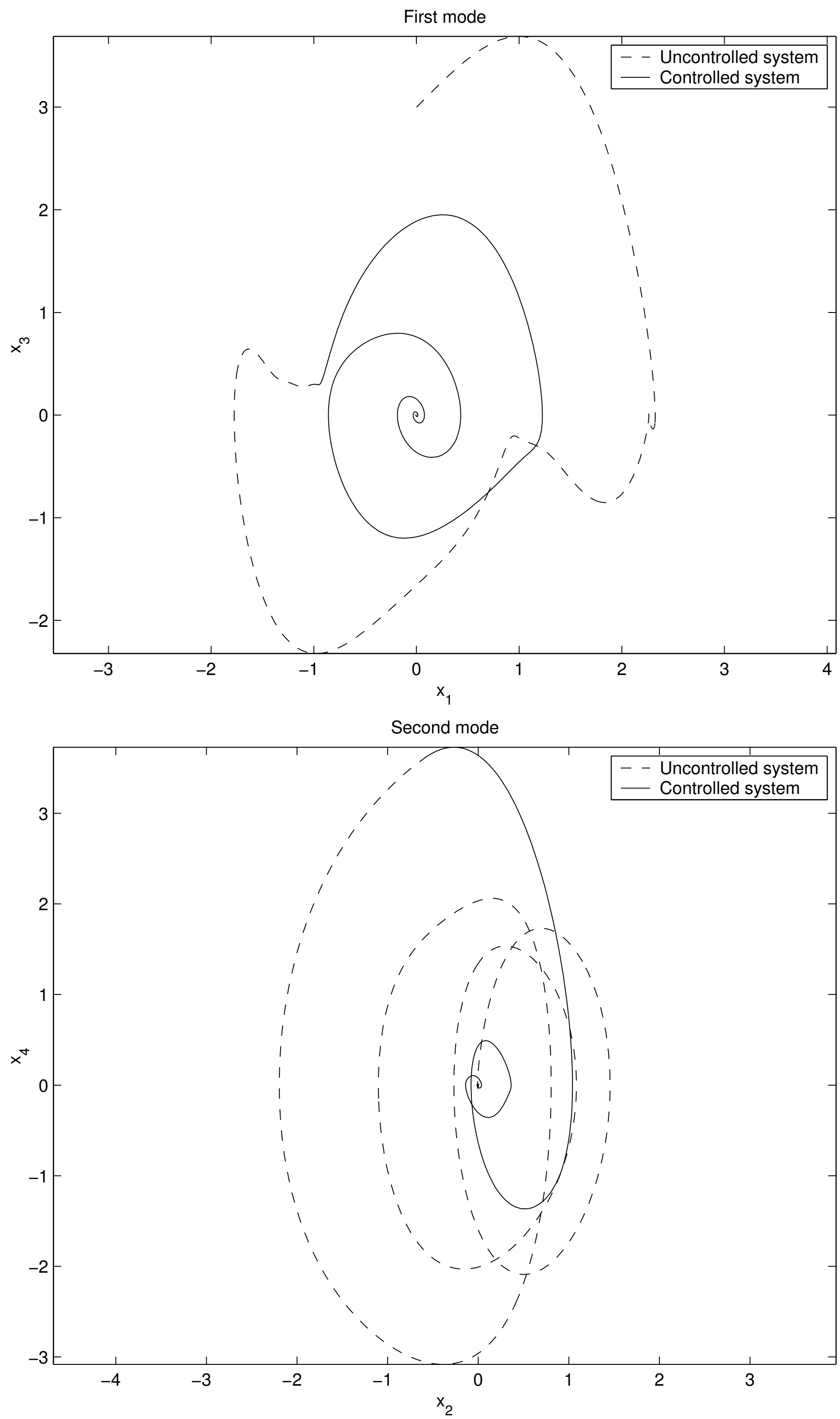



**Figure 6.7**: Phase portraits of controlled and uncontrolled system

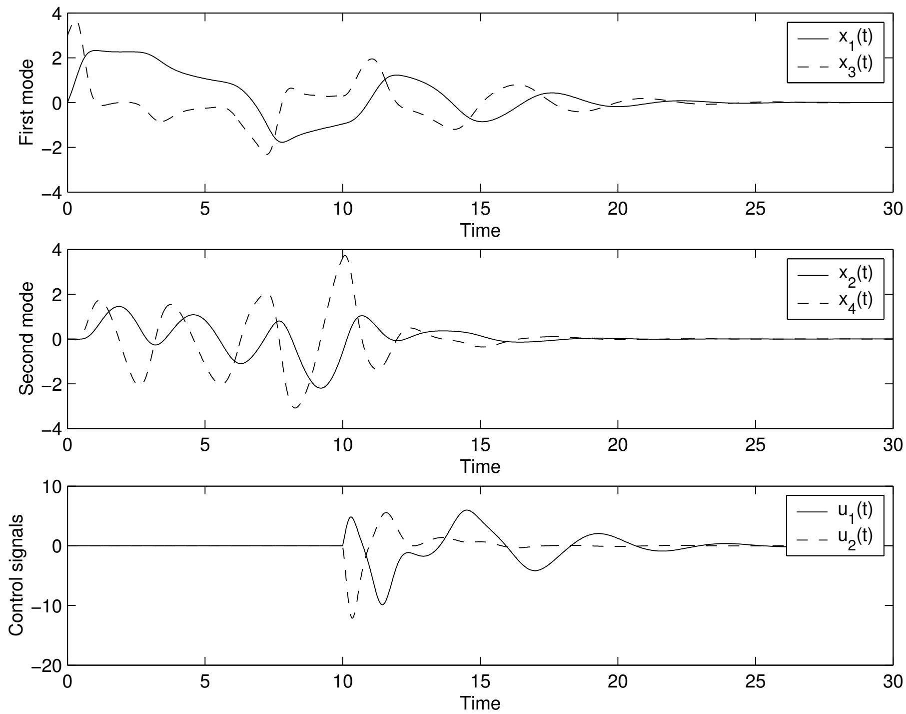



**Figure 6.8**: State trajectories and control signals versus time

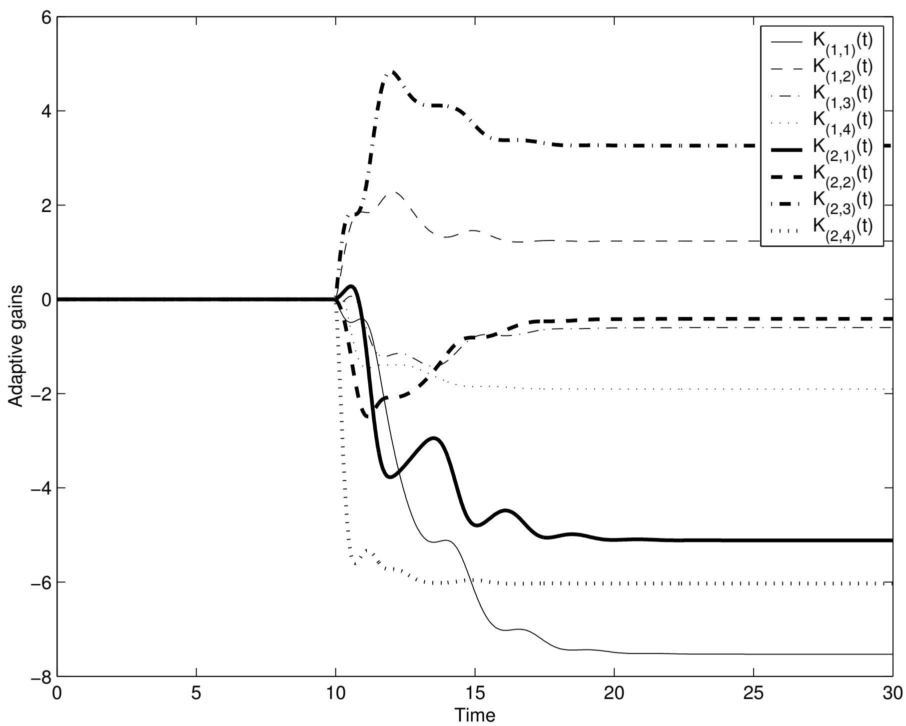



**Figure 6.9**: Adaptive gain history versus time

**Example 6.4.** Consider the nonlinear scalar second-order dynamical system with nonlinear damping and stiffness functions given by (6.1) where $n = 1$, $M$, $C(q)$, $K(q)$ are unknown, $M > 0$, $C(q)$ is known to be a polynomial but otherwise unknown, $K(q)$, $q \in \mathbb{R}$, is lower bounded, and $w(t) \equiv 0$. Now, with $p_2 > 0$ and $p_{12} > 0$, it follows from Proposition 6.1 that the adaptive feedback controller (6.23) with $K_{\rm b}(x_1) \equiv 0$ guarantees that $x(t) \to 0$ as $t \to \infty$. For illustrative purposes we choose $M = 1$, $C(q) = c_1(q^N - c_2)$, and $K(q) = k_1 + k_2 \tanh(q)$, where $c_1 = -2$, $c_2 = 1$, $k_1 = 2$, $k_2 = -1$, and $p_{12} = 2$, $p_2 = 1$, $Q_1 = I_2$, $Y = I_4$. Furthermore, we choose the initial conditions $q(0) = 2$, $\dot{q}(0) = 1$, and $\Psi(0) = 0_{1\times 2}$. First, we consider the case where the order of the polynomial $C(q)$ is odd and known ($N = 3$). In this case, it follows from *i*) of Proposition 6.1 that $C_{\rm b}(q) = -\alpha q^4$. Figure 6.10 shows the state trajectories, adaptive gains, and the control signal versus time for the case where $\alpha = 1$. Next, we assume that the order of the polynomial $C(q)$ is even and known ($N = 4$). In this case, it follows from *i*) of Proposition 6.1 that $C_{\rm b}(q) = -\alpha q^6$. Figure 6.11 shows the state trajectories, adaptive gains, and the control signal versus time for the case where $\alpha = 0.5$.

Next, we consider the case where the order of the polynomial $C(q)$ is odd but unknown. Furthermore, we assume that $c_1$ is known to be negative. In this case, it follows from *ii*) of Proposition 6.1 that $C_{\rm b}(q) = -\alpha(\cosh(\beta q) + \sinh(\beta q))$. Figure 6.12 shows the state trajectories, adaptive gains, and the control signal versus time for the case where $N = 3$, $\alpha = 2$, and $\beta = 1$. Finally, we consider the case where $C(q)$ is an unknown polynomial. In this case, it follows from *iii*) of Proposition 6.1 that $C_{\rm b}(q) = -\alpha \cosh(\beta q)$. Figure 6.13 shows the state trajectories, adaptive gains, and the control signal versus time for the case where $N = 3$, $\alpha = 3$, and $\beta = 1$.

**Example 6.5.** Consider the nonlinear matrix second-order dynamical system with nonlinear damping and stiffness matrix functions given by (6.1) where $n = 2$,

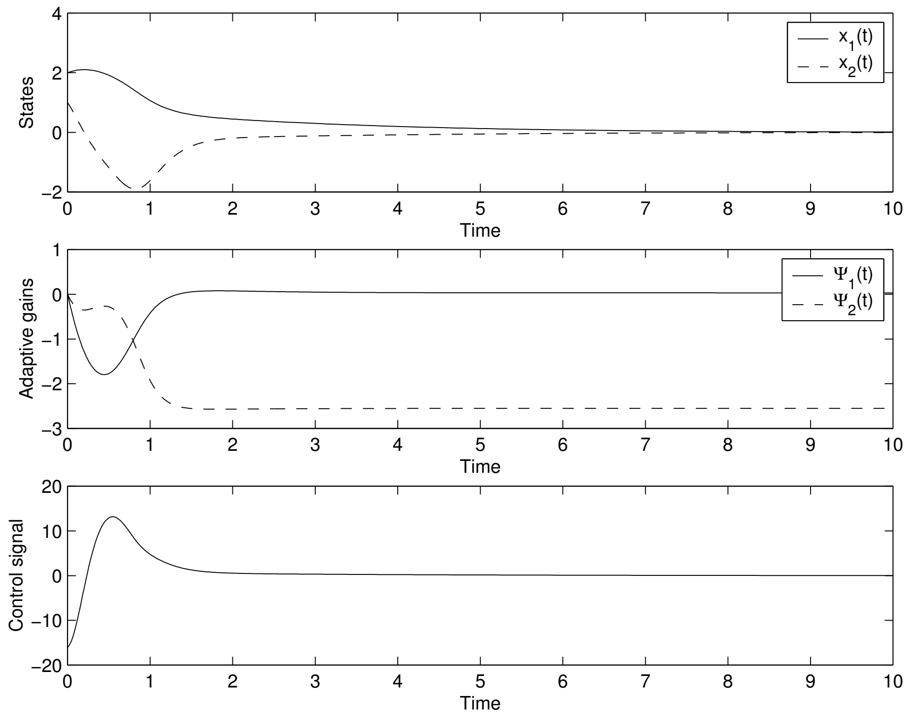



**Figure 6.10**: State trajectories, adaptive gains and control signal versus time

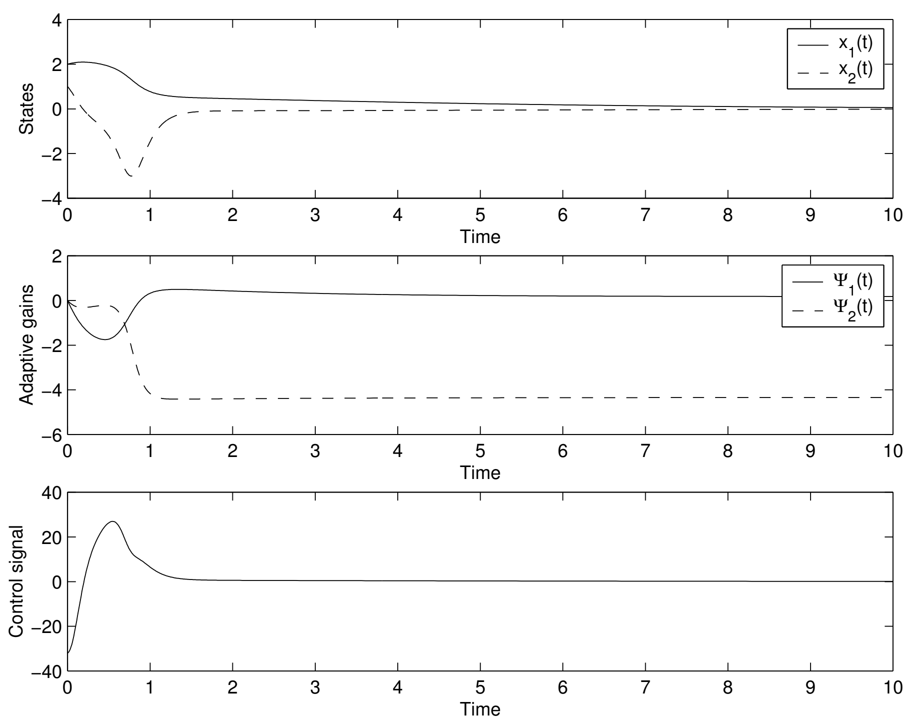



**Figure 6.11**: State trajectories, adaptive gains and control signal versus time

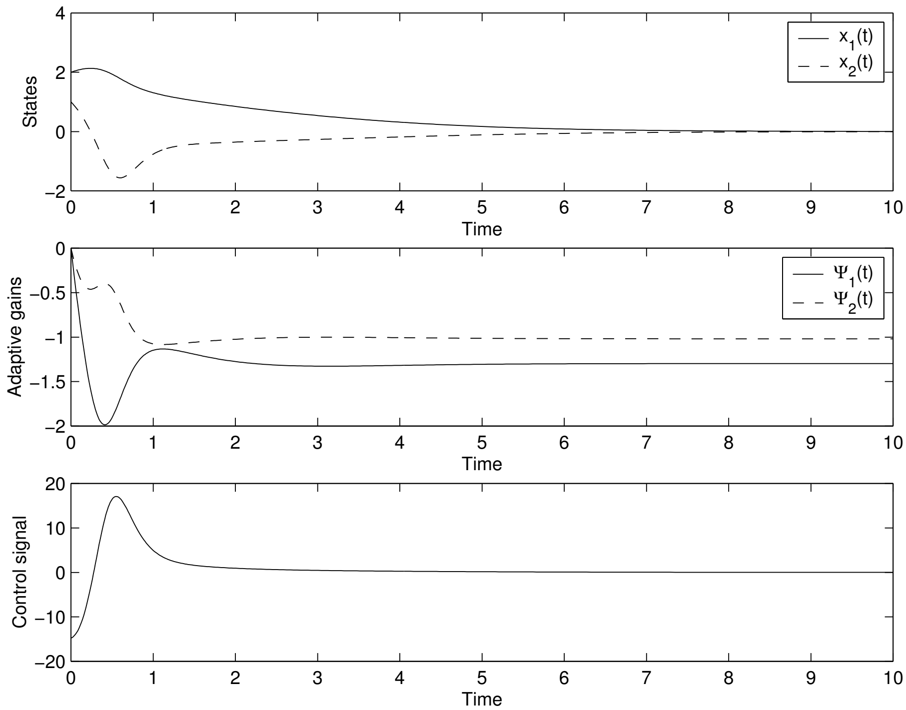



**Figure 6.12**: State trajectories, adaptive gains and control signal versus time

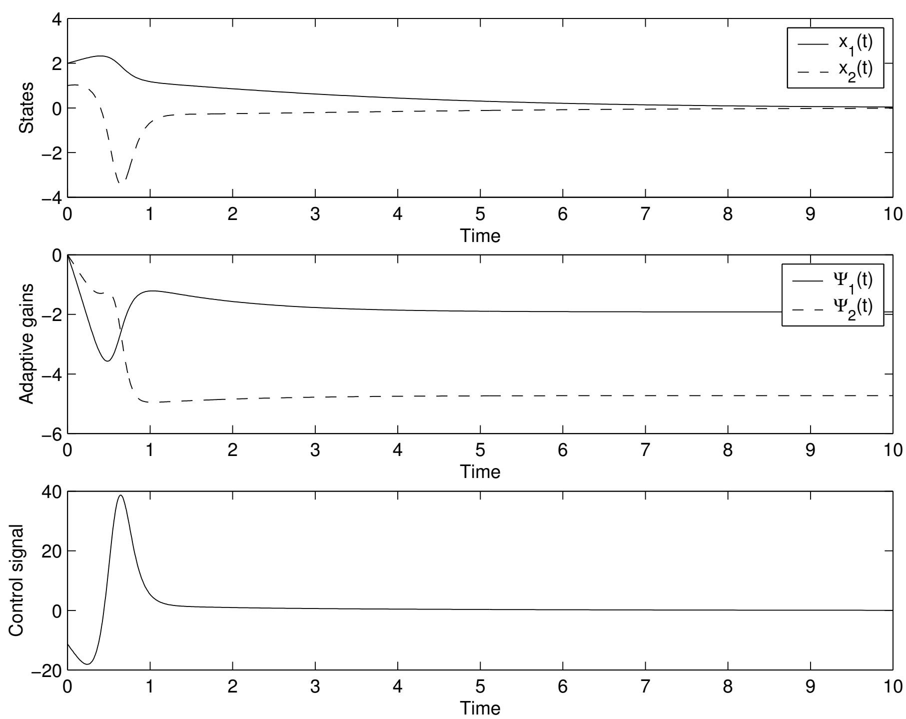



**Figure 6.13**: State trajectories, adaptive gains and control signal versus time

$M$, $K(q)$ are unknown, $M > 0$, $C(q) = 0_2$, $K(q)$, $q \in \mathbb{R}^2$, is known to be a matrix function with unknown polynomial entries but otherwise unknown, $K(\cdot) \in \mathcal{S}$, and $w(t) \equiv 0$. Now, with $p_2 > 0$ and $p_{12} > 0$, it follows from Proposition 6.2 that the adaptive feedback controller (6.26) guarantees that $x(t) \to 0$ as $t \to \infty$. For illustrative purposes we choose

$$M = \mathrm{diag}[m_1, m_2],$$
$$K(q) = \begin{bmatrix} k_1 + k_3(q_1^4 - 3q_1^3 q_2 + 10q_1^2 q_2 - 10q_1 q_2^3 + 4q_2^4) & -k_1 - k_3(q_1^4 + q_2^4) \\ -k_1 - k_3(q_1^4 + q_2^4) & k_1 + k_2 + k_3(4q_1^4 - 10q_1^3 q_2 + 10q_1^2 q_2 - 3q_1 q_2^3 + q_2^4) \end{bmatrix},$$

where $m_1 = 3$, $m_2 = 2$, $k_1 = -1$, $k_2 = 5$, $k_3 = -0.5$, and $p_{12} = 1$, $p_2 = 2$, $Q_1 = I_2$, $Y = I_4$. Furthermore, we choose the initial conditions $q(0) = [0,\ 0]^{\mathrm{T}}$, $\dot{q}(0) = [3,\ 0]^{\mathrm{T}}$, and $\Psi(0) = 0_{2\times 4}$. First, we consider the case where the highest order of the polynomial function $K(q)$ is known ($N = 4$). In this case, it follows from $i$) of Proposition 6.2 that $K_{\mathrm{b}}(q) = -\mathrm{diag}[\alpha_1 q^6, \alpha_2 q_2^6]$. for the case where $\alpha_1 = \alpha_2 = 4$. Figure 6.14 shows the state trajectories versus time and the control signals versus time for the case where $\alpha_1 = \alpha_2 = 4$. Figure 6.15 shows the adaptive gain history versus time. Finally, we consider the case where $C(q)$ is an unknown polynomial matrix function. In this case, it follows from $ii$) of Proposition 6.2 that $K_{\mathrm{b}}(q) = -\mathrm{diag}[\alpha_1 \cosh(\beta_1 q), \alpha_2 \cosh(\beta_2 q)]$. Figure 6.16 shows the state trajectories versus time and the control signals versus time for the case where $\alpha_1 = \alpha_2 = 4$, $\beta_1 = \beta_2 = 1$. Finally, Figure 6.17 shows the adaptive gain history versus time.

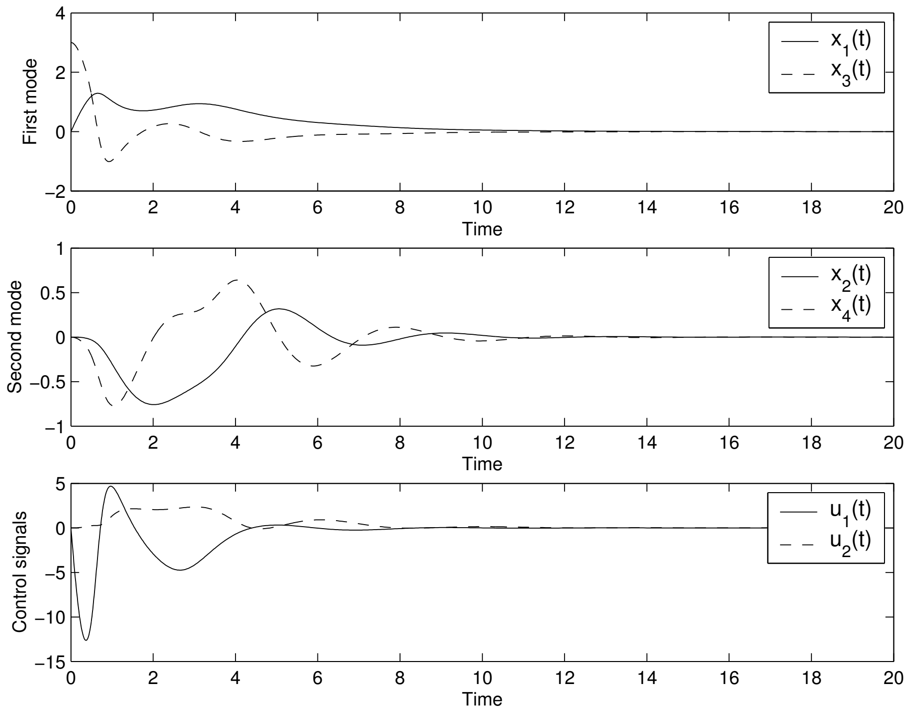



**Figure 6.14**: State trajectories and control signals versus time

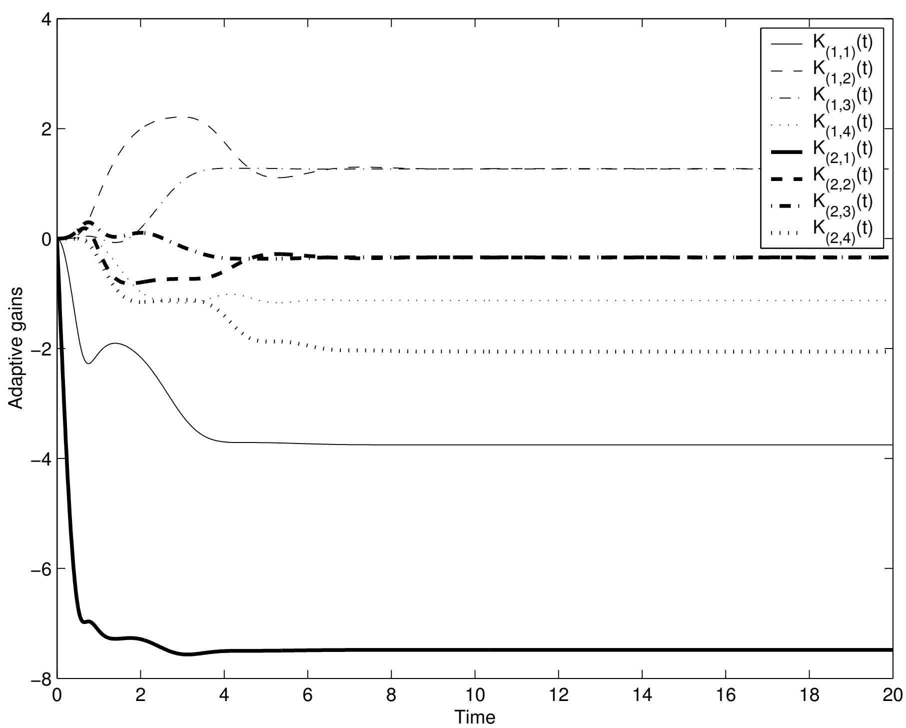



**Figure 6.15**: Adaptive gain history versus time

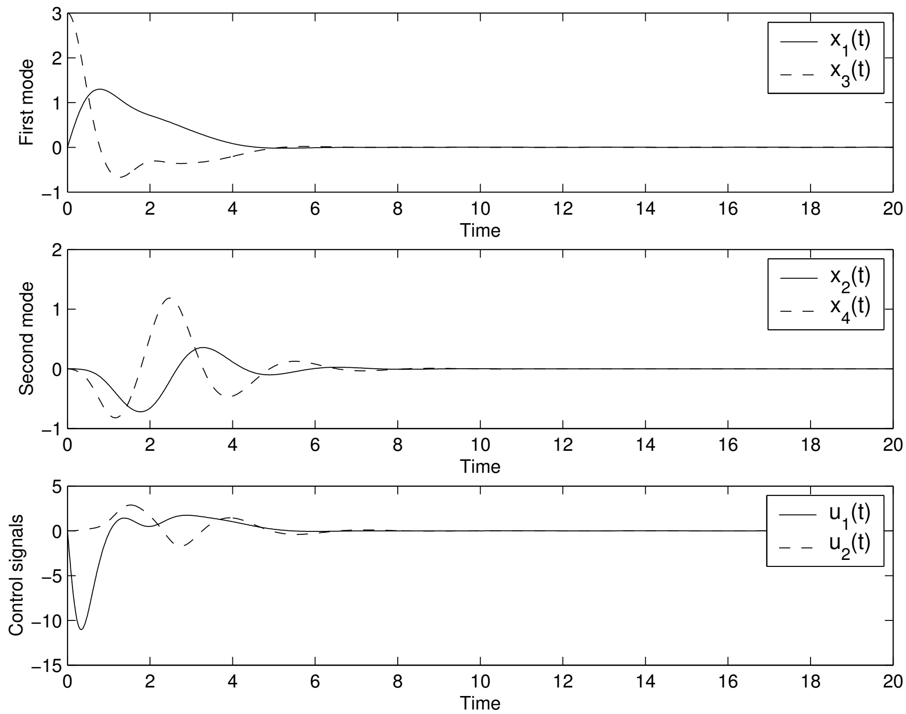



**Figure 6.16**: State trajectories and control signals versus time

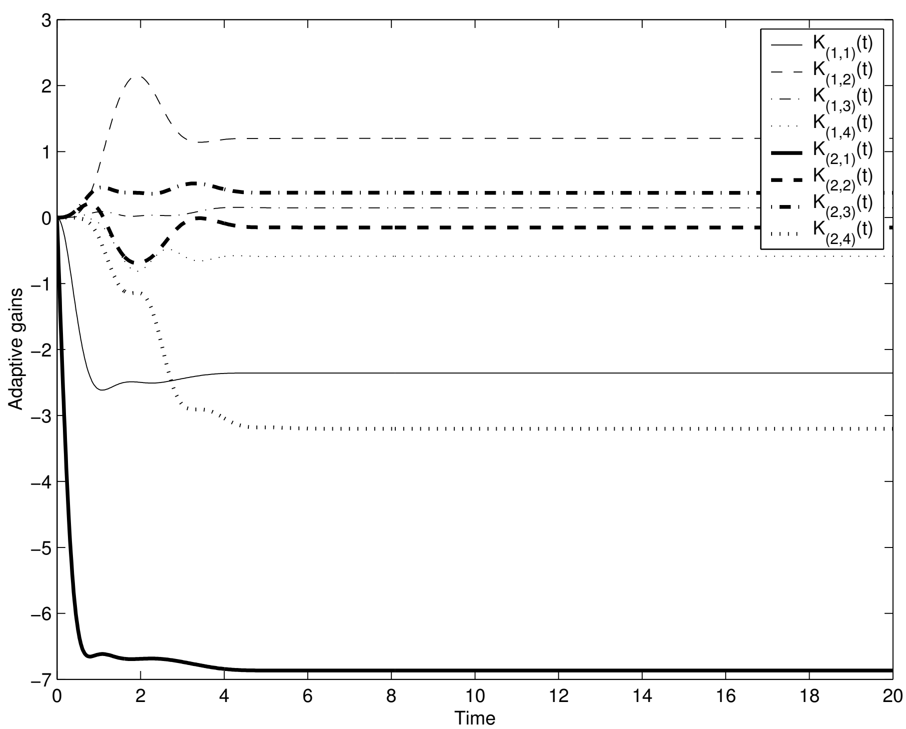



**Figure 6.17**: Adaptive gain history versus time

## 6.5. Nonlinear Matrix Second-Order Systems with Time-Varying and Sign-Indefinite Damping and Stiffness Operators

In this section we further generalize the result of Section 6.2 in several directions. In particular, for a class of *nonlinear* multivariable *matrix* second-order uncertain dynamical systems, with *time-varying* and sign-indefinite damping and stiffness operators, we develop a nonlinear adaptive control framework that guarantees global partial asymptotic stability of the closed-loop system; that is, global asymptotic stability with respect to part of the closed-loop system states associated with the plant. This is achieved without requiring any knowledge of the system nonlinearities other than the assumption that they are continuous and bounded. Hence, unlike standard adaptive control methods [12, 121, 147, 176], the proposed adaptive control framework does not require any parametrization of the state-dependent system uncertainty. The class of systems represented by our framework includes nonlinear vibrational systems, as well as multivariable nonlinear matrix second-order dynamical systems with sign-varying; that is, nondissipative, generalized stiffness and damping time-varying operators. In the special case of scalar second-order systems with linear time-varying coefficients, our results specialize to the results of [199]. Finally, we note that a similar adaptive control framework for nonlinear uncertain matrix second-order systems was considered in Sections 6.2 and 6.3 (see also [37, 38]). The results presented in Sections 6.2 and 6.3 however only address *time-invariant*, sign-indefinite stiffness and damping operator uncertainty, with the damping operator uncertainty being a partial function of the system state. In this case, the unknown system nonlinearities need only be continuous and lower bounded as opposed to continuous and bounded.

In this section we consider the problem of adaptive stabilization of nonlinear time-varying matrix second-order dynamical systems with exogenous disturbances.

Specifically, consider the controlled nonlinear time-varying uncertain matrix second-order dynamical system $\mathcal{G}$ given by

$$M\ddot{q}(t) + C(q(t), \dot{q}(t), t)\dot{q}(t) + K(q(t), t)q(t) = u(t) + Dw(t), \quad q(0) = q_0, \quad \dot{q}(0) = \dot{q}_0, \quad t \geq t_0, \tag{6.28}$$

where $q(t), \dot{q}(t), \ddot{q}(t) \in \mathbb{R}^n$, $t \geq t_0$, represent generalized position, velocity, and acceleration coordinates, respectively, $u(t) \in \mathbb{R}^n$, $t \geq t_0$, is the control input, $w(t) \in \mathbb{R}^d$, $t \geq t_0$, is a known bounded disturbance, $M \in \mathbb{R}^{n\times n}$, $C : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times n}$, $K : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times n}$, and $D \in \mathbb{R}^{n\times d}$. We assume that $M > 0$ and $C(\cdot, \cdot, \cdot)$ and $K(\cdot, \cdot)$ are continuous and symmetric maps. Otherwise, we assume that $M$, $C(\cdot, \cdot, \cdot)$, $K(\cdot, \cdot)$, and $D$ are unknown. Note that even though $w(t)$, $t \geq t_0$, is assumed to be known, the disturbance signal $Dw(t)$, $t \geq t_0$, is an *unknown* bounded disturbance. The control input $u(\cdot)$ in (6.28) is restricted to the class of *admissible controls* consisting of measurable functions such that $u(t) \in \mathbb{R}^n$, $t \geq t_0$. Furthermore, for the uncertain dynamical system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions are satisfied; that is, $C(\cdot, \cdot, \cdot)$, $K(\cdot, \cdot)$, $u(\cdot)$, and $w(\cdot)$ satisfy sufficient regularity conditions such that (6.28) has unique solution forward in time.

Next, with $x_1 \triangleq q$, $x_2 \triangleq \dot{q}$, and $x \triangleq [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$, it follows that the state space representation of (6.28) is given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -M^{-1}\Big(K(x_1(t), t)x_1(t) + C(x_1(t), x_2(t), t)x_2(t) - u(t) - Dw(t)\Big) \end{bmatrix},$$
$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} q_0 \\ \dot{q}_0 \end{bmatrix}, \quad t \geq t_0. \tag{6.29}$$

For the statement of our main result let $P \in \mathbb{R}^{2\times 2}$ be positive definite, where $P = \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}$ and $p_{12} > 0$, and define $B_0 \triangleq [0_n, I_n]^{\mathrm{T}}$.

**Theorem 6.3.** Consider the nonlinear time-varying matrix second-order dynamical system $\mathcal{G}$ given by (6.28), or, equivalently, the nonlinear time-varying dynamical system given by (6.29). Assume there exist scalars $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\alpha_1 M \leq K(x_1, t) \leq \alpha_2 M, \quad \beta_1 M \leq C(x_1, x_2, t) \leq \beta_2 M, \quad (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n, \quad t \geq t_0. \tag{6.30}$$

Let $Q_1, Q_2 \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{2n \times 2n}$, and $Z \in \mathbb{R}^{d \times d}$ be positive definite. Then the adaptive feedback control law

$$u(t) = \Psi(t)x(t) + \Phi(t)w(t), \tag{6.31}$$

where $\Psi(t) \in \mathbb{R}^{n \times 2n}$, $t \geq t_0$, and $\Phi(t) \in \mathbb{R}^{n \times d}$, $t \geq t_0$, with update laws

$$\dot{\Psi}(t) = -Q_1 B_0^{\mathrm{T}} (P \otimes I_n) x(t) x^{\mathrm{T}}(t) Y, \quad \Psi(0) = \Psi_0, \tag{6.32}$$

$$\dot{\Phi}(t) = -Q_2 B_0^{\mathrm{T}} (P \otimes I_n) x(t) w^{\mathrm{T}}(t) Z, \quad \Phi(0) = \Phi_0, \tag{6.33}$$

guarantees that the solution $(x(t), \Psi(t), \Phi(t)) \equiv (0, K_{\mathrm{g}}, -D)$, where $K_{\mathrm{g}} \in \mathbb{R}^{n \times 2n}$, of the closed-loop system given by (6.29), (6.31)–(6.33) is uniformly Lyapunov stable and $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^{2n}$.

**Proof.** Define $\gamma_1 \triangleq p_2\alpha_1 + p_{12}\beta_1$, $\gamma_2 \triangleq p_2\alpha_2 + p_{12}\beta_2$, $\gamma \triangleq \max\{|\gamma_1|^2, |\gamma_2|^2\}$, and define the set $\mathcal{K} \subset \mathbb{R}^{n \times 2n}$ by

$$\begin{aligned}\mathcal{K} \triangleq \{[k_1 M, k_2 M] \in \mathbb{R}^{n \times 2n} : k_1 < \alpha_1,\ k_2 < \beta_1,\ p_1 + p_2 k_1 + p_{12} k_2 < 0,\\ p_{12}(\alpha_1 - k_1)[p_2(\beta_1 - k_2) - p_{12}] \geq 2(\gamma - \gamma_1\sqrt{\gamma}),\\ p_{12}(\alpha_1 - k_1)[p_2(\beta_1 - k_2) - p_{12}] > \gamma\}.\end{aligned} \tag{6.34}$$

Note that since all of the inequalities in (6.34) may be rewritten as upper bounds on $k_1$ and $k_2$, $\mathcal{K}$ is not empty. Next, let $K_{\mathrm{g}} \triangleq [k_{1\mathrm{g}} M, k_{2\mathrm{g}} M] \in \mathcal{K}$ and define $\tilde{\Psi}(t) \triangleq \Psi(t) - K_{\mathrm{g}}$, $\tilde{\Phi}(t) \triangleq \Phi(t) + D$, $\tilde{C}(x, t) \triangleq C(x_1, x_2, t) - k_{2\mathrm{g}} M$, and $\tilde{K}(x_1, t) \triangleq K(x_1, t) - k_{1\mathrm{g}} M$. Furthermore, define $\hat{p}_1 \triangleq -(p_1 + p_2 k_{1\mathrm{g}} + p_{12} k_{2\mathrm{g}})$ and $H(x, t) \triangleq -(p_1 + \hat{p}_1)M + p_{12}\tilde{C}(x, t) + p_2\tilde{K}(x_1, t)$ and note that $\hat{p}_1 > 0$. Now, it follows from the definitions of $\gamma_1$ and $\gamma_2$ that

$$\gamma_1 M \leq H(x, t) \leq \gamma_2 M, \quad x \in \mathbb{R}^{2n}, \quad t \geq t_0. \tag{6.35}$$

Moreover, it follows from (6.35) and the definition of $\gamma$ that

$$0 \le H(x,t)M^{-1}H(x,t) \le \gamma M, \quad x \in \mathbb{R}^{2n}, \quad t \ge t_0. \tag{6.36}$$

Next, with $u(t)$, $t \ge t_0$, given by (6.31), (6.29) becomes

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -M^{-1}\Big(\tilde{K}(x_1(t),t)x_1(t) + \tilde{C}(x(t),t)x_2(t) - \tilde{\Psi}(t)x(t) - \tilde{\Phi}(t)w(t)\Big) \end{bmatrix},$$
$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} q_0 \\ \dot{q}_0 \end{bmatrix}, \quad t \ge t_0. \tag{6.37}$$

To show uniform Lyapunov stability of the closed-loop system (6.32), (6.33), and (6.37), consider the Lyapunov function candidate

$$V(x,\Psi,\Phi) = \frac{1}{2}\Big[x^{\mathrm{T}}(P \otimes M)x + \hat{p}_1 x_1^{\mathrm{T}} M x_1 + \operatorname{tr} Q_1^{-1}\tilde{\Psi}Y^{-1}\tilde{\Psi}^{\mathrm{T}} + \operatorname{tr} Q_2^{-1}\tilde{\Phi}Z^{-1}\tilde{\Phi}^{\mathrm{T}}\Big]. \tag{6.38}$$

Note that $V(0, K_{\mathrm{g}}, -D) = 0$ and, since $P$, $M$, $Q_1$, $Q_2$, $Y$, and $Z$ are positive definite, $V(x,\Psi,\Phi) > 0$ for all $(x,\Psi,\Phi) \ne (0, K_{\mathrm{g}}, -D)$. Furthermore, $V(x,\Psi,\Phi)$ is radially unbounded. Now, letting $x(t)$, $t \ge t_0$, denote the solution to (6.37) and using (6.32) and (6.33) it follows that the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned}
\dot{V}(x(t),\Psi(t),\Phi(t),t) &= x^{\mathrm{T}}(t)(P \otimes M)\dot{x}(t) + \hat{p}_1 x_1^{\mathrm{T}}(t)M\dot{x}_1(t) + \operatorname{tr} Q_1^{-1}\tilde{\Psi}(t)Y^{-1}\dot{\tilde{\Psi}}^{\mathrm{T}}(t) \\
&\quad + \operatorname{tr} Q_2^{-1}\tilde{\Phi}(t)Z^{-1}\dot{\tilde{\Phi}}^{\mathrm{T}}(t) \\
&= p_1 x_1^{\mathrm{T}}(t)Mx_2(t) + p_{12}x_2^{\mathrm{T}}(t)Mx_2(t) - p_{12}x_1^{\mathrm{T}}(t)\tilde{K}(x_1(t),t)x_1(t) \\
&\quad - p_{12}x_1^{\mathrm{T}}(t)\tilde{C}(x(t),t)x_2(t) - p_2 x_2^{\mathrm{T}}(t)\tilde{K}(x_1(t),t)x_1(t) \\
&\quad - p_2 x_2^{\mathrm{T}}(t)\tilde{C}(x(t),t)x_2(t) + p_{12}x_1^{\mathrm{T}}(t)\tilde{\Psi}(t)x(t) + p_2 x_2^{\mathrm{T}}(t)\tilde{\Psi}(t)x(t) \\
&\quad + p_{12}x_1^{\mathrm{T}}(t)\tilde{\Phi}(t)w(t) + p_2 x_2^{\mathrm{T}}(t)\tilde{\Phi}(t)w(t) + \hat{p}_1 x_1^{\mathrm{T}}(t)Mx_2(t) \\
&\quad + \operatorname{tr} Q_1^{-1}\tilde{\Psi}(t)Y^{-1}\dot{\tilde{\Psi}}^{\mathrm{T}}(t) + \operatorname{tr} Q_2^{-1}\tilde{\Phi}(t)Z^{-1}\dot{\tilde{\Phi}}^{\mathrm{T}}(t) \\
&= p_{12}x_2^{\mathrm{T}}(t)Mx_2(t) - p_{12}x_1^{\mathrm{T}}(t)\tilde{K}(x_1(t),t)x_1(t) \\
&\quad - p_2 x_2^{\mathrm{T}}(t)\tilde{C}(x(t),t)x_2(t) - x_1^{\mathrm{T}}(t)H(x(t),t)x_2(t)
\end{aligned}$$

$$
\begin{aligned}
&+\operatorname{tr}\tilde{\Psi}(t)[x(t)x^{\mathrm{T}}(t)(P\otimes I_n)B_0+Y^{-1}\dot{\tilde{\Psi}}^{\mathrm{T}}(t)Q_1^{-1}]\\
&+\operatorname{tr}\tilde{\Phi}(t)[w(t)x^{\mathrm{T}}(t)(P\otimes I_n)B_0+Z^{-1}\dot{\tilde{\Phi}}^{\mathrm{T}}(t)Q_2^{-1}]\\
=\;&-p_{12}x_1^{\mathrm{T}}(t)\tilde{K}(x_1(t),t)x_1(t)-x_1^{\mathrm{T}}(t)H(x(t),t)x_2(t)\\
&-x_2^{\mathrm{T}}(t)(p_2\tilde{C}(x(t),t)-p_{12}M)x_2(t)\\
\le\;&-p_{12}(\alpha_1-k_{1\mathrm{g}})x_1^{\mathrm{T}}(t)Mx_1(t)-x_1^{\mathrm{T}}(t)H(x(t),t)x_2(t)\\
&-(p_2(\beta_1-k_{2\mathrm{g}})-p_{12})x_2^{\mathrm{T}}(t)Mx_2(t)\\
=\;&-x^{\mathrm{T}}(t)\mathcal{R}(x(t),t)x(t),\quad t\ge t_0,
\end{aligned}\tag{6.39}
$$

where

$$
\mathcal{R}(x,t)\triangleq\begin{bmatrix} p_{12}(\alpha_1-k_{1\mathrm{g}})M & H(x,t)/2\\ H(x,t)/2 & (p_2(\beta_1-k_{2\mathrm{g}})-p_{12})M\end{bmatrix}.
$$

Next, define

$$
R\triangleq\frac{1}{2}\begin{bmatrix} p_{12}(\alpha_1-k_{1\mathrm{g}}) & \sqrt{\gamma}\\ \sqrt{\gamma} & p_2(\beta_1-k_{2\mathrm{g}})-p_{12}\end{bmatrix}
$$

and note that, using (6.34), $R>0$. Hence, it follows from (6.34)–(6.36) that $\mathcal{R}(x,t)\ge(R\otimes M)>0$, $t\ge t_0$. Thus, it follows from (6.39) that

$$
\begin{aligned}
\dot{V}(x(t),\Psi(t),\Phi(t),t)&\le -x^{\mathrm{T}}(t)\mathcal{R}(x(t),t)x(t)\\
&\le -x^{\mathrm{T}}(t)(R\otimes M)x(t)\\
&\le 0,\quad t\ge t_0,
\end{aligned}\tag{6.40}
$$

which proves that the solution $(x(t),\Psi(t),\Phi(t))\equiv(0,K_{\mathrm{g}},-D)$ of the closed-loop system (6.32), (6.33), and (6.37) is uniformly Lyapunov stable. Furthermore, since $(R\otimes M)>0$, it follows from Theorem 2 of [42] that $x(t)\to 0$ as $t\to\infty$ for all $x_0\in\mathbb{R}^{2n}$. □

**Remark 6.8.** Note that the conditions in Theorem 6.3 imply that $x(t)\to 0$ as $t\to\infty$ and hence it follows from (6.32) and (6.33) that $(x(t),\Psi(t),\Phi(t))\to\mathcal{M}\triangleq\{(x,\Psi,\Phi)\in\mathbb{R}^n\times\mathbb{R}^{n\times 2n}\times\mathbb{R}^{n\times d}: x=0,\dot{\Psi}=0,\dot{\Phi}=0\}$ as $t\to\infty$.

**Remark 6.9.** It is important to note that the bounds for $K(x_1,t)$, $(x_1,t) \in \mathbb{R}^n \times \mathbb{R}$, and $C(x_1,x_2,t)$, $(x_1,x_2,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, do not need to be known in order to implement the adaptive controller (6.31)–(6.33). All that is required is that $K(\cdot,\cdot)$ and $C(\cdot,\cdot,\cdot)$ are continuous, symmetric, and bounded; otherwise they are unknown. Likewise, $M \in \mathbb{R}^{n\times n}$ needs to be positive definite but is otherwise unknown.

**Remark 6.10.** Although $K(\cdot,\cdot)$ and $C(\cdot,\cdot,\cdot)$ are assumed to be symmetric, Theorem 6.3 also holds for the more general case where $K(\cdot,\cdot)$ and $C(\cdot,\cdot,\cdot)$ are nonsymmetric operators. In this case however, the inequalities in (6.30) involving $K(\cdot,\cdot)$ and $C(\cdot,\cdot,\cdot)$ should be replaced with the symmetric part of $K(\cdot,\cdot)$ and $C(\cdot,\cdot,\cdot)$, respectively. Furthermore, if $M$ is known to be negative definite but otherwise unknown, then Theorem 6.3 holds with $u(t)$ given by (6.31) replaced by $u(t) = -\Psi(t)x(t) - \Phi(t)w(t)$.

Theorem 6.3 is applicable to the case where $C(q,\dot{q},t)$ and $K(q,t)$, $q,\dot{q} \in \mathbb{R}^n$, $t \geq t_0$, are bounded. In practice however, $C(q,\dot{q},t)$ and $K(q,t)$, $q,\dot{q} \in \mathbb{R}^n$, $t \geq t_0$, are often unbounded. Next, we provide a corollary to Theorem 6.3 that addresses the case where $C(\cdot,\cdot,\cdot)$ and $K(\cdot,\cdot)$ can be unbounded operators.

**Corollary 6.2.** Consider the nonlinear time-varying dynamical system $\mathcal{G}$ given by (6.29), or, equivalently, the nonlinear matrix second-order dynamical system $\mathcal{G}$ given by (6.28). Assume there exist known, symmetric matrix functions $K_{\rm b} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times n}$ and $C_{\rm b} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times n}$ and scalars $\alpha_1,\alpha_2,\beta_1,\beta_2 \in \mathbb{R}$ such that $\beta_1 M \leq C(x_1,x_2,t) - C_{\rm b}(x_1,x_2,t) \leq \beta_2 M$ and $\alpha_1 M \leq K(x_1,t) - K_{\rm b}(x_1,t) \leq \alpha_2 M$, $x_1,x_2 \in \mathbb{R}^n$, $t \geq t_0$. Let $Q_1,Q_2 \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{2n\times 2n}$, $Z \in \mathbb{R}^{d\times d}$, and $P \in \mathbb{R}^{2\times 2}$ be positive definite, where $P = \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}$ and $p_{12} > 0$. Then the adaptive feedback control law

$$u(t) = K_{\rm b}(x_1(t),t)x_1(t) + C_{\rm b}(x_1(t),x_2(t),t)x_2(t) + \Psi(t)x(t) + \Phi(t)w(t), \qquad (6.41)$$

where $\Psi(t) \in \mathbb{R}^{n\times 2n}$, $t \geq t_0$, and $\Phi(t) \in \mathbb{R}^{n\times d}$, $t \geq t_0$, with update laws (6.32) and (6.33) guarantees that the solution $(x(t), \Psi(t), \Phi(t)) \equiv (0, K_{\rm g}, -D)$, where $K_{\rm g} \in \mathbb{R}^{n\times 2n}$, of the closed-loop system given by (6.29), (6.32), (6.33), and (6.41) is uniformly Lyapunov stable and $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^{2n}$.

**Proof.** Rewrite (6.28) as

$$M\ddot{q}(t) + \hat{C}(q(t), \dot{q}(t), t)\dot{q}(t) + \hat{K}(q(t), t)q(t) = \hat{u}(t) + Dw(t), \quad q(0) = q_0, \quad \dot{q}(0) = \dot{q}_0, \quad t \geq t_0, \quad (6.42)$$

where $\hat{C}(q, \dot{q}, t) \triangleq C(q, \dot{q}, t) - C_{\rm b}(q, \dot{q}, t)$, $\hat{K}(q) \triangleq K(q, t) - K_{\rm b}(q, t)$, and $\hat{u} \triangleq u - C_{\rm b}(q, \dot{q}, t)\dot{q} - K_{\rm b}(q, t)q$. Now, the result is a direct consequence of Theorem 6.3. $\square$

Finally, we generalize Theorem 6.3 and Corollary 6.2 to the case where $C(q, \dot{q}, t) - (\theta_c^{\rm T} \otimes I_n)C_{\rm b}(q, \dot{q}, t)$ and $K(q, t) - (\theta_k^{\rm T} \otimes I_n)K_{\rm b}(q, t)$ are bounded, where $\theta_c \in \mathbb{R}^{p_c}$ and $\theta_k \in \mathbb{R}^{p_k}$ are unknown parameters and $C_{\rm b} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{p_c n\times n}$ as well as $K_{\rm b} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{p_k n\times n}$ are known functions.

**Theorem 6.4.** Consider the nonlinear time-varying dynamical system $\mathcal{G}$ given by (6.29), or, equivalently, the nonlinear matrix second-order dynamical system $\mathcal{G}$ given by (6.28). Assume there exist scalars $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, vectors $\theta_c \in \mathbb{R}^{p_c}$, $\theta_k \in \mathbb{R}^{p_k}$, and symmetric matrix functions $C_{\rm b} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{p_c n\times n}$ and $K_{\rm b} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{p_k n\times n}$ such that $\alpha_1 M \leq K(x_1, t) - (\theta_k^{\rm T} \otimes I_n)K_{\rm b}(x_1, t) \leq \alpha_2 M$ and $\beta_1 M \leq C(x_1, x_2, t) - (\theta_c^{\rm T} \otimes I_n)C_{\rm b}(x_1, x_2, t) \leq \beta_2 M$, $x_1, x_2 \in \mathbb{R}^n$, $t \geq t_0$. Furthermore, let $C_{{\rm b}i} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times n}$, $i = 1, \ldots, p_c$, and $K_{{\rm b}j} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times n}$, $j = 1, \ldots, p_k$, be symmetric maps such that $C_{\rm b}(x_1, x_2, t) = [C_{\rm b1}(x_1, x_2, t), \cdots, C_{{\rm b}p_c}(x_1, x_2, t)]^{\rm T}$ and $K_{\rm b}(x_1, t) = [K_{\rm b1}(x_1, t), \cdots, K_{{\rm b}p_k}(x_1, t)]^{\rm T}$. Let $Q_1, Q_2 \in \mathbb{R}^{n\times n}$, $Q_3 \in \mathbb{R}^{p_k\times p_k}$, $Q_4 \in \mathbb{R}^{p_c\times p_c}$ $Y \in \mathbb{R}^{2n\times 2n}$, $Z \in \mathbb{R}^{d\times d}$, and $P \in \mathbb{R}^{2\times 2}$ be positive definite, where $P =$

$\begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}$ and $p_{12} > 0$. Then the adaptive feedback control law

$$
\begin{aligned}
u(t) = \; & (\Theta_k^{\mathrm{T}}(t) \otimes I_n) K_{\mathrm{b}}(x_1(t), t) x_1(t) + (\Theta_c^{\mathrm{T}}(t) \otimes I_n) C_{\mathrm{b}}(x_1(t), x_2(t), t) x_2(t) \\
& + \Psi(t) x(t) + \Phi(t) w(t),
\end{aligned} \tag{6.43}
$$

where $\Theta_k(t) \in \mathbb{R}^{p_k}$, $t \geq t_0$, $\Theta_c(t) \in \mathbb{R}^{p_c}$, $t \geq t_0$, $\Psi(t) \in \mathbb{R}^{n \times 2n}$, $t \geq t_0$, and $\Phi(t) \in \mathbb{R}^{n \times d}$, $t \geq t_0$, with update laws (6.32), (6.33), and

$$
\dot{\Theta}_k(t) = -Q_3 (I_{p_k} \otimes x_1^{\mathrm{T}}(t)) K_{\mathrm{b}}(x_1(t), t) B_0^{\mathrm{T}} (P \otimes I_n) x(t), \quad \Theta_k(0) = \Theta_{k0}, \tag{6.44}
$$

$$
\dot{\Theta}_c(t) = -Q_4 (I_{p_c} \otimes x_2^{\mathrm{T}}(t)) C_{\mathrm{b}}(x_1(t), x_2(t), t) B_0^{\mathrm{T}} (P \otimes I_n) x(t), \quad \Theta_c(0) = \Theta_{c0}, \tag{6.45}
$$

guarantees that the solution $(x(t), \Psi(t), \Phi(t), \Theta_c(t), \Theta_k(t)) \equiv (0, K_{\mathrm{g}}, -D, \theta_c, \theta_k)$, where $K_{\mathrm{g}} \in \mathbb{R}^{n \times 2n}$, of the closed-loop system given by (6.29), (6.32), (6.33), (6.43)–(6.45) is uniformly Lyapunov stable and $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^{2n}$.

**Proof.** Let $K_{\mathrm{g}} \triangleq [k_{1\mathrm{g}} M, k_{2\mathrm{g}} M] \in \mathcal{K}$. Furthermore, define $\tilde{C}(x_1, x_2, t) \triangleq C(x_1, x_2, t) - (\theta_c^{\mathrm{T}} \otimes I_n) C_{\mathrm{b}}(x_1, x_2, t) - k_{2\mathrm{g}} M$, $\tilde{K}(x_1, t) \triangleq K(x_1, t) - (\theta_k^{\mathrm{T}} \otimes I_n) K_{\mathrm{b}}(x_1, t) - k_{1\mathrm{g}} M$, $\tilde{\Theta}_k(t) \triangleq \Theta_k(t) - \theta_k$, and $\tilde{\Theta}_c(t) \triangleq \Theta_c(t) - \theta_c$, and consider the Lyapunov function candidate

$$
\begin{aligned}
V(x, \Psi, \Phi, \Theta_k, \Theta_c) = \frac{1}{2} \Big[ & x^{\mathrm{T}} (P \otimes M) x + \hat{p}_1 x_1^{\mathrm{T}} M x_1 + \operatorname{tr} Q_1^{-1} \tilde{\Psi} Y^{-1} \tilde{\Psi}^{\mathrm{T}} \\
& + \operatorname{tr} Q_2^{-1} \tilde{\Phi} Z^{-1} \tilde{\Phi}^{\mathrm{T}} + \tilde{\Theta}_k^{\mathrm{T}} Q_3^{-1} \tilde{\Theta}_k + \tilde{\Theta}_c^{\mathrm{T}} Q_4^{-1} \tilde{\Theta}_c \Big].
\end{aligned} \tag{6.46}
$$

Now, the proof is identical to the proof of Theorem 6.3. □

**Remark 6.11.** Theorem 6.4 generalizes Corollary 6.2 since $C(x_1, x_2, t) - (\theta_c^{\mathrm{T}} \otimes I_n) C_{\mathrm{b}}(x_1, x_2, t)$ and $K(x_1, t) - (\theta_k^{\mathrm{T}} \otimes I_n) K_{\mathrm{b}}(x_1, t)$ are bounded as opposed to $C(x_1, x_2, t) - C_{\mathrm{b}}(x_1, x_2, t)$ and $K(x_1, t) - K_{\mathrm{b}}(x_1, t)$ being bounded. This gives yet a larger class of nonlinearities that can be considered in the operators $C(\cdot, \cdot, \cdot)$ and $K(\cdot, \cdot)$. See Remark 6.4 for further details.

**Remark 6.12.** Once again, as in the case of Theorem 6.3, Corollary 6.2 and Theorem 6.4 also hold for the case where $M$ is only sign definite and $C(\cdot,\cdot,\cdot)$ and $K(\cdot,\cdot)$ are nonsymmetric operators. In this case, $C(\cdot,\cdot,\cdot)$ and $K(\cdot,\cdot)$ should be replaced by their symmetric parts in the expressions $C(x_1, x_2, t) - (\theta_c^{\mathrm{T}} \otimes I_n)C_{\mathrm{b}}(x_1, x_2, t)$ and $K(x_1, t) - (\theta_k^{\mathrm{T}} \otimes I_n)K_{\mathrm{b}}(x_1, t)$, respectively. Furthermore, when $M$ is negative definite but otherwise unknown, the control law (6.41) takes the form

$$\begin{aligned} u(t) = & -(\Theta_k^{\mathrm{T}}(t) \otimes I_n)K_{\mathrm{b}}(x_1(t), t)x_1(t) - (\Theta_c^{\mathrm{T}}(t) \otimes I_n)C_{\mathrm{b}}(x_1(t), x_2(t), t)x_2(t) \\ & -\Psi(t)x(t) - \Phi(t)w(t). \end{aligned} \tag{6.47}$$

## 6.6. Illustrative Numerical Example

In this section we present a numerical example to demonstrate the utility of the proposed direct adaptive control framework for adaptive stabilization. Specifically, consider the nonlinear time-varying matrix second-order dynamical system with nonlinear damping and stiffness matrix functions given by (6.28), where $n = 2$, and $M$, $C(q, \dot{q}, t)$, and $K(q, t)$ are unknown with $M > 0$, and $C(q, \dot{q}, t)$ and $K(q, t)$ bounded for all $q, \dot{q} \in \mathbb{R}^2$ and $t \geq t_0$. Furthermore, assume $w(t) \equiv 0$. Now, with $p_2 > 0$ and $p_{12} > 0$, it follows from Theorem 6.3 that the adaptive feedback controller (6.31) guarantees that $x(t) \to 0$ as $t \to \infty$. For illustrative purposes, consider (6.28) with $n = 2$ and

$$\begin{aligned} M &= \begin{bmatrix} 5 & 0.5 \\ 0.5 & 4 \end{bmatrix}, \\ C(q, \dot{q}, t) &= \begin{bmatrix} \exp(-\dot{q}_1^2)\sin(t) & \sin(\sqrt{t}/4) \\ \sin(\sqrt{t}/4) & \exp(-q_2^2)\sin(t) \end{bmatrix}, \\ K(q, t) &= \begin{bmatrix} \sin(q_1 t) & \sin(t) \\ \sin(t) & \cos(q_2 t) \end{bmatrix}. \end{aligned}$$

Let $p_1 = 2$, $p_2 = p_{12} = 1$, $Q_1 = 2I_2$, $Y = I_4$, and set the initial conditions $q(0) = [1, -2]^{\mathrm{T}}$, $\dot{q}(0) = [0, -1]^{\mathrm{T}}$, and $\Psi(0) = 0_{2\times 4}$. Figure 6.18 shows the phase portraits of the controlled and uncontrolled systems. Note that the adaptive controller is switched on at $t = 10$ sec. Figure 6.19 shows the state trajectories and the control signals versus time. Finally, the adaptive gain history versus time is shown in Figure 6.20.

## 6.7. Applications to Combustion Processes

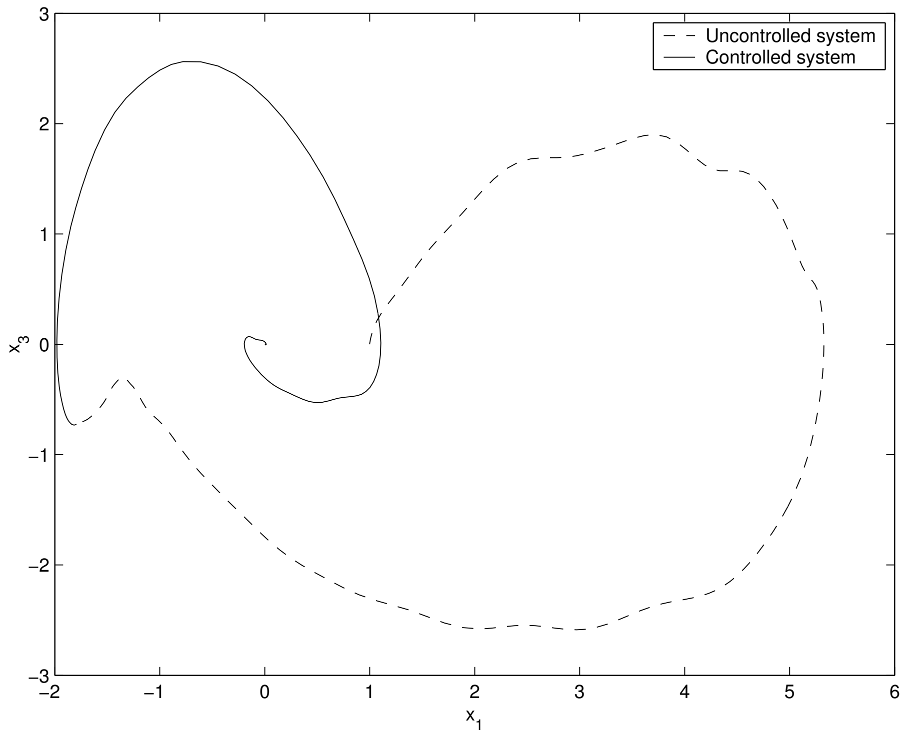
In this section we apply the framework developed in Section 6.5 to suppress the effects of thermoacoustic instabilities in uncertain combustion processes. As shown in Section 2.6, a matrix second-order model with sign-indefinite damping and stiffness operators can be used to capture the coupling between unsteady combustion and acoustics in a combustion process. Specifically, using the mass, momentum, and energy conservation equations for a two phase mixture in a combustor and using a Galerkin decomposition the authors in [50, 98] obtain

$$\ddot{\eta}_i(t)+\omega_i^2\eta_i(t)+\sum_{p=1}^{\infty}(d_{ip}\dot{\eta}_p(t)+e_{ip}\eta_p(t))+\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}(a_{ipq}\dot{\eta}_p(t)\dot{\eta}_q(t)+b_{ipq}\eta_p(t)\eta_q(t)) = u_i(t), \tag{6.48}$$

where $\eta_i$ denotes the $i$th modal combustion pressure, $d_{ip}$, $e_{ip}$, $a_{ipq}$, and $b_{ipq}$, $i = 1, \cdots, n$, are constants depending on the unperturbed mode shapes and natural frequencies of the combustor [50], and $u_i(t)$, $t \geq 0$, $i = 1, \cdots, n$, is the control input to the $i$th mode and is given by

$$u_i(t) = \frac{\bar{a}^2}{\bar{p}E_i^2}\sum_{j=1}^{m}\hat{u}_j(t)\psi_i(x_{\mathrm{s}j}), \tag{6.49}$$

where $\bar{a} \triangleq \sqrt{\gamma\frac{\bar{p}}{\bar{\rho}}}$ is the local average sound velocity inside the combustor, $\bar{\rho}$ is the average density in the two phase mixture, $\gamma$ is the mixture ratio of specific heats, $\bar{p}$ is the average pressure inside the combustor, $\psi_i(\cdot)$, $i = 1, \cdots, n$, are the normal modes of the system, $E_i^2 = \int_0^L \psi_i(x)\mathcal{A}_{\mathrm{c}}(x)\mathrm{d}x$, $\mathcal{A}_{\mathrm{c}}(x)$ is the cross sectional area of the combustor, $L$ is the combustor length, $\hat{u}_{\mathrm{a}}(t)$ is a control excitation through an acoustic driver, and $x_{\mathrm{s}j}$ corresponds to the location of the $j$th actuator.



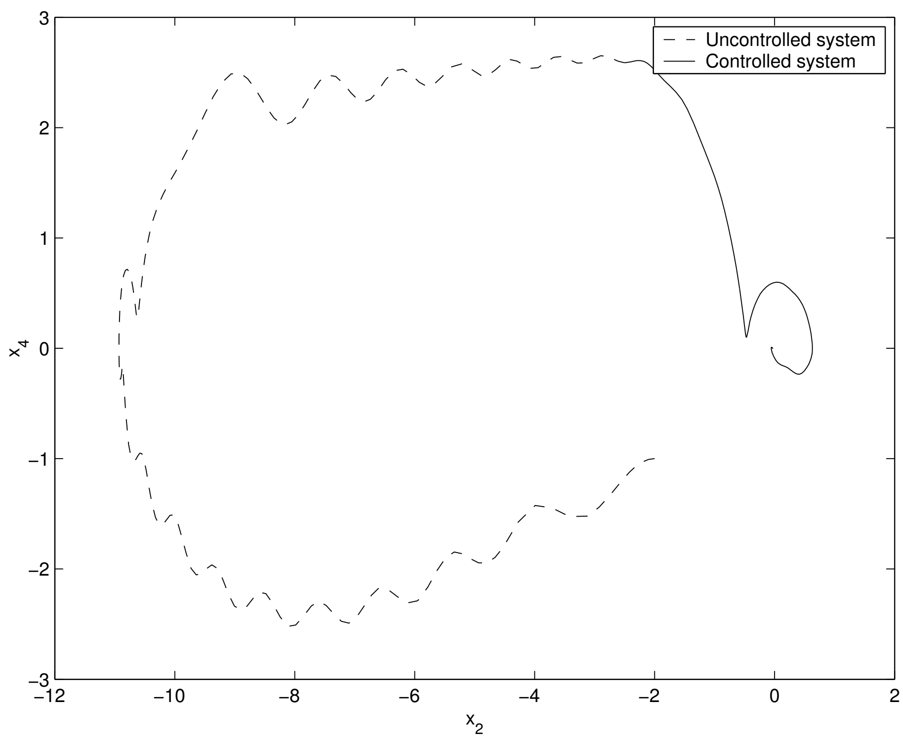




**Figure 6.18**: Phase portraits of controlled and uncontrolled systems

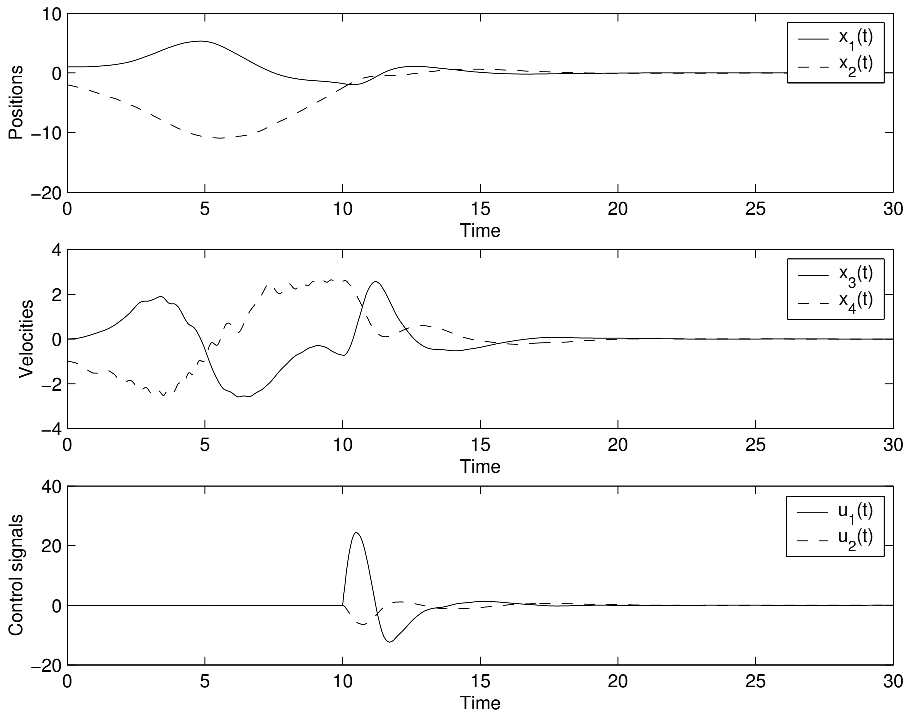



**Figure 6.19**: State trajectories and control signals versus time

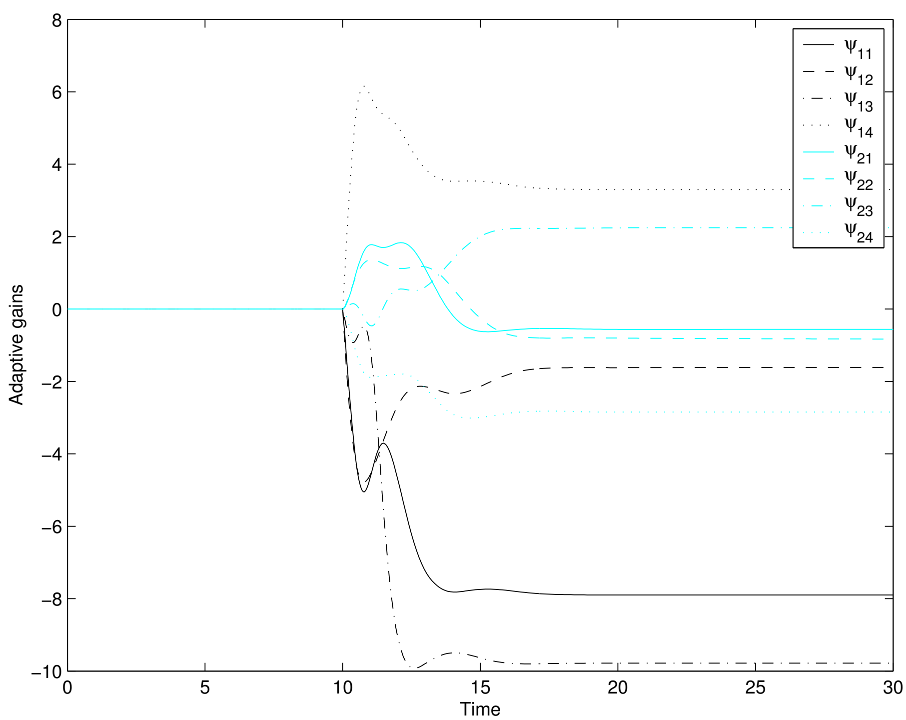



**Figure 6.20**: Adaptive gain history versus time

To design a direct adaptive controller for combustion systems we use the nonlinear combustion model given by the matrix second-order system (6.48) with nonlinearities present due to the second-order gas dynamics. Furthermore, we assume that actuation is provided by loud speakers while we measure pressure fluctuations via pressure-type microphones. Assuming a two-mode, nonlinear combustion plant model, (6.48) and (6.49) yield

$$\begin{aligned}\ddot{\eta}_1(t) = {} & 2\alpha_1\dot{\eta}_1(t) - (\omega_1^2 - 2\theta_1\omega_1)\eta_1(t) - F_{11}\dot{\eta}_1(t)\dot{\eta}_2(t) - F_{12}\eta_1(t)\eta_2(t) \\ & + \tfrac{\bar{a}^2}{\bar{p}E_1^2}(\psi_1(x_{\mathrm{s}1})\hat{u}_1(t) + \psi_1(x_{\mathrm{s}2})\hat{u}_2(t)), \quad \eta_1(0) = \eta_{10},\ \dot{\eta}_1(0) = \dot{\eta}_{10},\ t \geq 0, \end{aligned} \tag{6.50}$$

$$\begin{aligned}\ddot{\eta}_2(t) = {} & 2\alpha_2\dot{\eta}_2(t) - (\omega_2^2 - 2\theta_2\omega_2)\eta_2(t) - F_{21}\dot{\eta}_1^2(t) - F_{22}\eta_1^2(t) \\ & + \tfrac{\bar{a}^2}{\bar{p}E_2^2}(\psi_2(x_{\mathrm{s}1})\hat{u}_1(t) + \psi_2(x_{\mathrm{s}2})\hat{u}_2(t)), \quad \eta_2(0) = \eta_{20}, \quad \dot{\eta}_2(0) = \dot{\eta}_{20}, \end{aligned} \tag{6.51}$$

where $\hat{u}_i(t)$, $i = 1, 2$, are control input signals, $\alpha_i = -\frac{1}{2}d_{ii} \in \mathbb{R}$ represents a growth/decay constant, $\theta_i = -\frac{1}{2}\frac{e_{ii}}{\omega_i} \in \mathbb{R}$ represents a frequency shift constant, $\omega_1$ and $\omega_2$ are the frequencies of the first and second modes, $F_{11} = \frac{3-2\gamma}{2\gamma}$, $F_{12} = \frac{5(\gamma-1)}{2\gamma}\omega_1^2$, $F_{21} = -\frac{\gamma+3}{2\gamma}$, and $F_{22} = \frac{\gamma-1}{2\gamma}\omega_1^2$. In the case where we consider a cylindrical combustor closed at both ends with pure longitudinal modes, it follows that the first two modes are given by

$$\psi_i(x) = \cos(k_i x), \quad k_i = i\frac{\pi}{L}, \quad i = 1, 2. \tag{6.52}$$

For the nondimensionalized (using the time factor $\tau_t = \pi L/\bar{a}$) data parameters [63] $\alpha_1 = 0.0144$, $\alpha_2 = -0.0559$, $\theta_1 = 0.0062$, $\theta_2 = 0.0178$, $\gamma = 1.2$, $\omega_1 = 1$, $\omega_2 = 2$, and $[\eta_0^{\mathrm{T}}\ \dot{\eta}_0^{\mathrm{T}}]^{\mathrm{T}} = [0.01,\ 0.1,\ 0,\ 0]^{\mathrm{T}}$, the open-loop ($\hat{u}_i(t) \equiv 0, i = 1, 2$) dynamics (6.50) and (6.51) result in a limit cycle instability. Figure 2.26 shows the open-loop response versus time of the system.

Next, we assume that loud speakers are placed at $x_{\mathrm{s}1} = \frac{3}{4}L$ and $x_{\mathrm{s}2} = \frac{1}{2}L$. It is important to note that our proposed adaptive controller would stabilize *any* nonlinear time-varying, matrix second-order dynamical system with unknown nonlinear sign-indefinite damping and stiffness operators given by (6.28). Hence, we assume our combustion model is given by (6.50), (6.51) with $n = 2$, $q = [\eta_1, \eta_2]^{\mathrm{T}}$, $u = [\hat{u}_1, \hat{u}_2]^{\mathrm{T}}$,

$$
\begin{aligned}
M &= -\frac{\bar{p}}{\bar{a}^2}\begin{bmatrix} \sqrt{2}E_1^2 & 0 \\ 0 & E_2^2 \end{bmatrix}, \\
C(q,\dot{q},t) &= -\frac{\bar{p}}{\bar{a}^2}\begin{bmatrix} \sqrt{2}E_1^2(F_{11}\dot{q}_2 - 2\alpha_1) & 0 \\ E_2^2 F_{21}\dot{q}_1 & -2E_2^2\alpha_2 \end{bmatrix}, \\
K(q,t) &= -\frac{\bar{p}}{\bar{a}^2}\begin{bmatrix} \sqrt{2}E_1^2(F_{12}q_2 + \omega_1^2 - 2\theta_1\omega_1) & 0 \\ E_2^2 F_{22}q_1 & E_2^2(\omega_2^2 - 2\theta_2\omega_2) \end{bmatrix},
\end{aligned}
$$

where $\alpha_i$, $\theta_i$, $\omega_i$, $F_{ij}$, and $\frac{\bar{a}^2}{\bar{p}E_i^2}(> 0)$, $i,j = 1,2$, are unknown. Next, let $\theta_c = -\frac{\bar{p}}{\bar{a}^2}[\sqrt{2}E_1^2F_{11}, \frac{1}{2}E_2^2F_{21}]^{\mathrm{T}}$, $\theta_k = -\frac{\bar{p}}{\bar{a}^2}[\sqrt{2}E_1^2F_{12}, \frac{1}{2}E_2^2F_{22}]^{\mathrm{T}}$, and

$$
\begin{aligned}
C_{\mathrm{b}1}(q,\dot{q},t) &= \begin{bmatrix} \dot{q}_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_{\mathrm{b}2}(q,\dot{q},t) = \begin{bmatrix} \dot{q}_1 & 0 \\ 0 & \dot{q}_1 \end{bmatrix}, \\
K_{\mathrm{b}1}(q,t) &= \begin{bmatrix} q_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad K_{\mathrm{b}2}(q,t) = \begin{bmatrix} q_1 & 0 \\ 0 & q_1 \end{bmatrix}.
\end{aligned}
$$

Now, it follows from Theorem 6.4 and Remark 6.12 (since the sign-indefinite stiffness and damping operators are not symmetric) that the adaptive feedback controller (6.47) with update laws (6.32), (6.33), (6.44), and (6.45) guarantees that the closed-loop system is uniformly Lyapunov stable and $q(t) \to 0$ as $t \to \infty$.

To illustrate the dynamic behavior of the closed-loop system, let $\alpha_1 = 0.0144$, $\alpha_2 = -0.0559$, $\theta_1 = 0.0062$, $\theta_2 = 0.0178$, $\gamma = 1.2$, $\omega_1 = 1$, $\omega_2 = 2$, $Q_1 = Q_3 = Q_4 = 0.1I_2$, $Y = 0.5I_2$, and $\frac{\bar{a}^2}{\bar{p}E_i^2} = 0.4$, $i = 1,2$. The response of the controlled system (6.28) with the adaptive feedback control law (6.47) and initial conditions $q_0 = [0.01,\ 0.1]^{\mathrm{T}}$, $\dot{q}_0 = [0,\ 0]^{\mathrm{T}}$, $\Psi(0) = 0_{2\times4}$, $\Theta_c(0) = 0_{2\times1}$, and $\Theta_k(0) = 0_{2\times1}$ is shown in Figure 6.21. Uniform Lyapunov stability of the closed-loop system (6.28), (6.32), (6.33), (6.44), (6.45), and (6.47) as well as attraction of $q(t)$ is guaranteed by Theorem 6.4 and Remark 6.12. Note that the adaptive controller is switched on at $t = 300$.

To illustrate the robustness of the proposed adaptive control law, we switch the growth constant of the first mode from $\alpha_1 = 0.0144$ to $\alpha_1 = 0.0720$ at $t = 600$. The closed-loop response is shown in Figure 6.22. Figure 6.23 shows the same change in the growth constant of the first mode with the switch occurring at $t = 350$ while the control law is still in process of adapting.

Finally, we change the transient parameters $\theta_1 = 0.0062$ and $\theta_2 = 0.0178$ to $\theta_1 = 0.4998$ and $\theta_2 = 1.009$ at $t = 600$. The closed-loop response is shown in Figure 6.24. Note that this change corresponds to 8061% and 5669%, respectively, of the original values of the parameters. The same change in the transient parameters occurring at $t = 350$ is shown on Figure 6.25.

To illustrate the robustness of the proposed adaptive control law, we switch the growth constant of the first mode from $\alpha_1 = 0.0144$ to $\alpha_1 = 0.0720$ at $t = 600$. The closed-loop response is shown in Figure 6.22. Figure 6.23 shows the same change in the growth constant of the first mode with the switch occurring at $t = 350$ while the control law is still in process of adapting.

Finally, we change the transient parameters $\theta_1 = 0.0062$ and $\theta_2 = 0.0178$ to $\theta_1 = 0.4998$ and $\theta_2 = 1.009$ at $t = 600$. The closed-loop response is shown in Figure 6.24. Note that this change corresponds to 8061% and 5669%, respectively, of the original values of the parameters. The same change in the transient parameters occurring at $t = 350$ is shown on Figure 6.25.

## 6.8. Conclusion

A direct adaptive control framework for a class of nonlinear matrix second-order systems with state-dependent uncertainty was developed. In particular, using a Lyapunov-based framework, global asymptotic stability of the closed-loop system states associated with the plant dynamics was guaranteed without requiring any knowledge of the system nonlinearities other than the assumption that they are continuous and lower bounded. Generalizations to the case where the system nonlinearities are unbounded were also considered. The efficacy of the proposed approach was demonstrated on several nonlinear systems with sign varying; that is, nondissipative, generalized stiffness and damping operators.

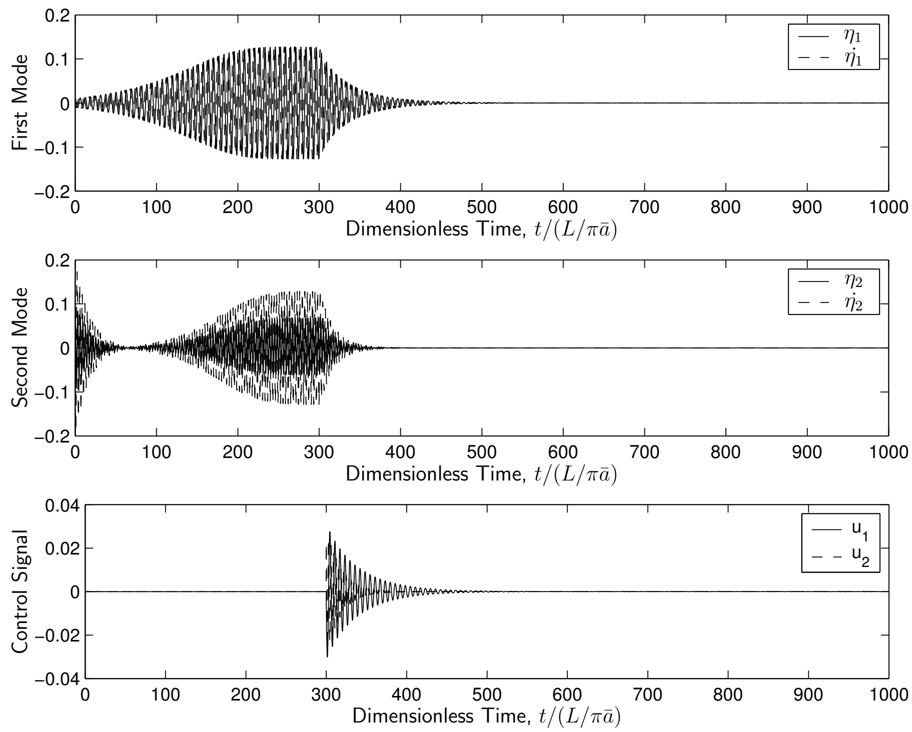



**Figure 6.21**: Closed-loop state response versus time

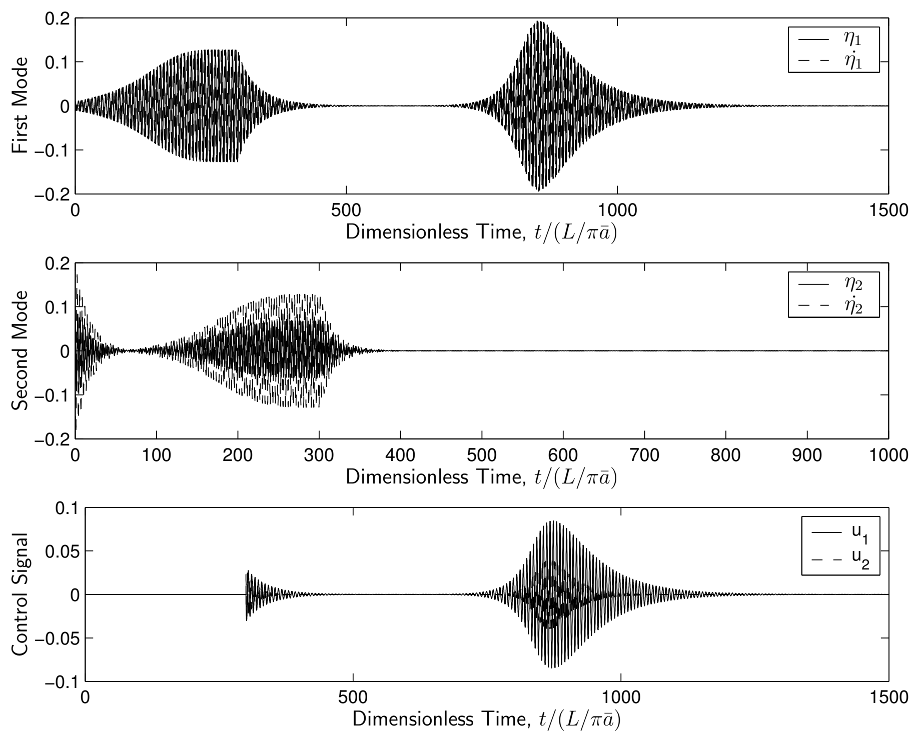



**Figure 6.22**: Closed-loop state response versus time

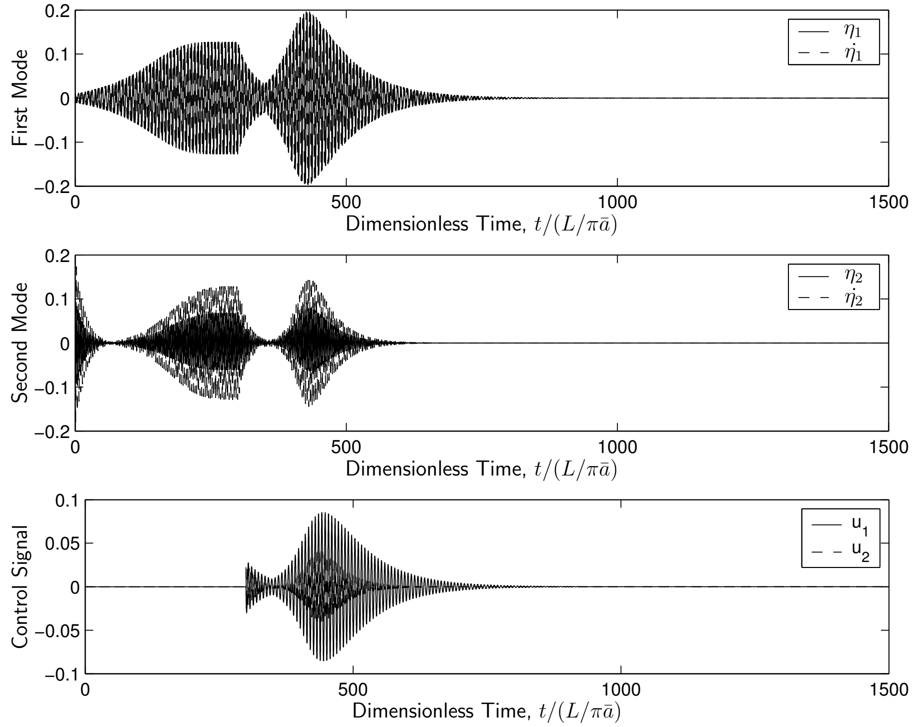



**Figure 6.23**: Closed-loop state response versus time

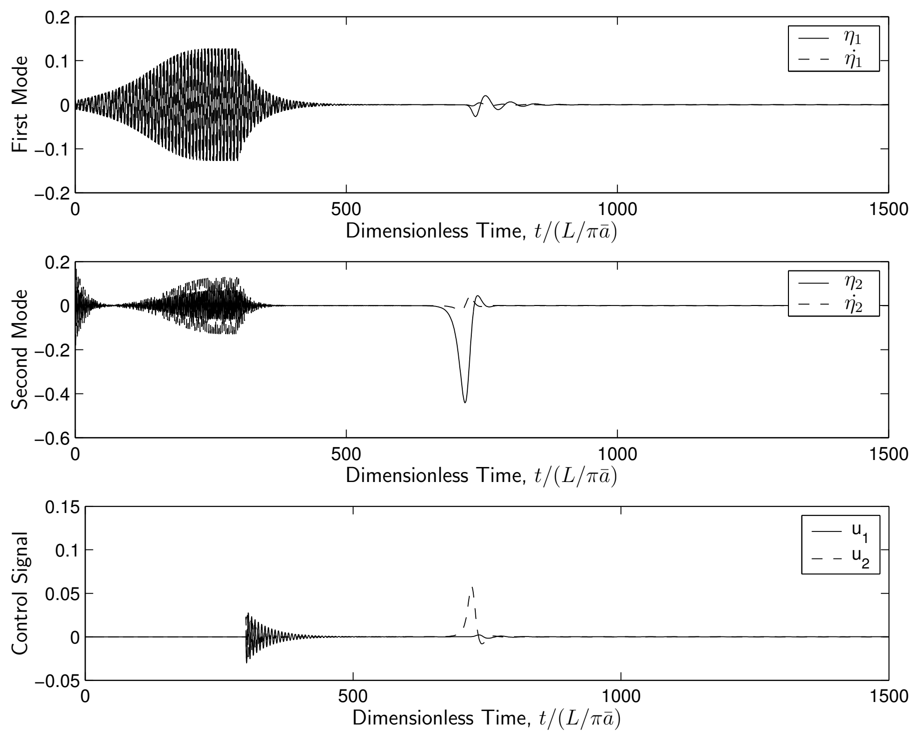



**Figure 6.24**: Closed-loop state response versus time

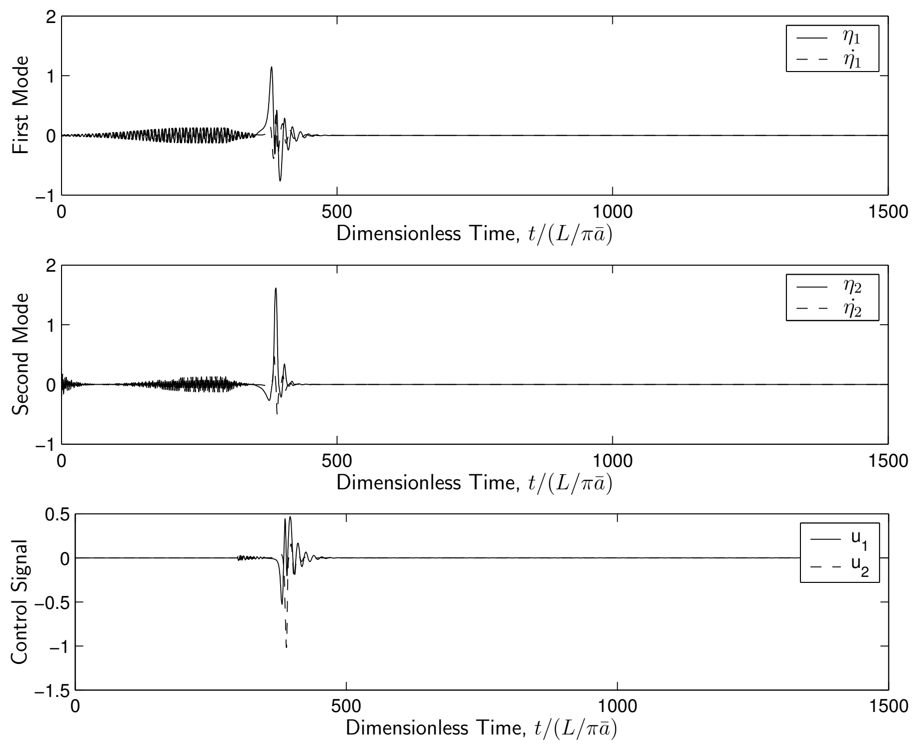



**Figure 6.25**: Closed-loop state response versus time

# Chapter 7

# Adaptive Control for Nonnegative and Compartmental Dynamical Systems with Applications to General Anesthesia

## 7.1. Introduction

Even though advanced robust and adaptive control methodologies have been (and are being) extensively developed for highly complex engineering systems, modern active control technology has received far less consideration in medical systems. The main reason for this state of affairs is the steep barriers to communication between mathematics/control engineering and medicine. However, this is slowly changing and there is no doubt that control-system technology has a great deal to offer medicine. For example, critical care patients, whether undergoing surgery or recovering in intensive care units, require drug administration to regulate key physiological (state) variables (e.g., blood pressure, cardiac output, heart rate, glucose, etc.) within desired levels. The rate of infusion of each administered drug is *critical*, requiring constant monitoring and frequent adjustments. Open-loop control (manual control) by clinical personnel can be very tedious, imprecise, time consuming, and often of poor quality.

Hence, the need for active control (closed-loop control) in medical systems is crucial; with the potential in improving the quality of medical care as well as curtailing the increasing cost of health care.

The complex highly uncertain and hostile environment of surgery places stringent performance requirements for closed-loop set-point regulation of physiological variables. For example, during cardiac surgery, blood pressure control is vital and is subject to numerous highly uncertain exogenous disturbances. Vasoactive and cardioactive drugs are administered resulting in large disturbance oscillations to the system (patient). The arterial line may be flushed and blood may be drawn, corrupting sensor blood pressure measurements. Low anesthetic levels may cause the patient to react to painful stimuli, thereby changing system response characteristics. The flow rate of vasodilator drug infusion may fluctuate causing transient changes in the infusion delay time. Hemorrhage, patient position changes, cooling and warming of the patient, and changes in anesthesia levels will also effect system response characteristics.

In light of the complex and highly uncertain nature of system response characteristics under surgery requiring controls, it is not surprising that reliable system models for many high performance drug delivery systems are unavailable. In the face of such high levels of system uncertainty, robust controllers may unnecessarily sacrifice system performance whereas adaptive controllers can tolerate far greater system uncertainty levels to improve system performance [12, 121, 147, 176]. In contrast to fixed-gain robust controllers, which maintain specified constants within the feedback control law to *sustain* robust performance, adaptive controllers directly or indirectly adjust feedback gains to maintain closed-loop stability and *improve* performance in the face of system uncertainties. Specifically, indirect adaptive controllers utilize parameter update laws to identify unknown system parameters and adjust feedback gains to account for system variation, while direct adaptive controllers directly adjust the controller gains in response to system variations (drug administration).

In this chapter we develop a direct adaptive control framework for adaptive set-point regulation of linear uncertain nonnegative and compartmental systems. Nonnegative and compartmental dynamical systems [6, 19, 24, 62, 70, 75, 123, 124, 164, 166, 172, 182, 187, 203] are composed of homogeneous interconnected subsystems (or compartments) which exchange variable nonnegative quantities of material with conservation laws describing transfer, accumulation, and elimination between the compartments and the environment. It follows from physical considerations that the state trajectory of such systems remains in the nonnegative orthant of the state space for nonnegative initial conditions. Nonnegative and compartmental models thus play a key role in understanding many processes in biological and medical sciences. Using nonnegative and compartmental model structures, a Lyapunov-based direct adaptive control framework is developed that guarantees partial asymptotic set-point stability of the closed-loop system; that is, asymptotic set-point stability with respect to part of the closed-loop system states associated with the physiological state variables. In particular, adaptive controllers are constructed *without* requiring knowledge of the system dynamics while providing a nonnegative control (source) input for robust stabilization with respect to the nonnegative orthant. Furthermore, in certain applications of nonnegative and compartmental systems such as biological systems, population dynamics, and ecological systems involving positive and negative inflows, the nonnegativity constraint on the control input is not natural. In this case, we also develop adaptive controllers that do not place any restriction on the sign of the control signal while guaranteeing that the physical system states remain in the nonnegative orthant of the state space.

Even though the proposed adaptive control framework is applicable to general nonlinear nonnegative and compartmental dynamical systems, in this chapter our application objective is in clinical pharmacology. In particular, we develop adaptive controllers for drug administration for general anesthesia. Adaptive control algorithms in pharmacology are vital since the relationships between drug dose and blood concentration (pharmacokinetics) and between blood concentrations and physiological effect (pharmacodynamics) vary widely among individual patients. Active control for the administration of general anesthesia is not new to this dissertation and has been considered in the literature. Specifically, building on pioneering work of Bickford [26] several groups have developed and clinically tested closed-loop controllers for the delivery of intravenous anesthesia using an electroencephalogram (EEG) signal for the performance and measurement variable. Two model-based control algorithms have been developed using a pharmacokinetic model relating drug concentration to drug dose and a pharmacodynamic model relating drug effect to drug concentration. Unfortunately, biological systems have significant pharmacokinetic and pharmacodynamic variability among individual subjects and using population mean values of pharmacokinetic and pharmacodynamic model parameters may result in very pronounced bias for any specific individual. To simplify one could assume that pharmacodynamics (the relationship between drug concentration and effect) do not vary among individuals and any difference between individual responses is due to pharmacokinetic variability. Alternatively, one could assume that the pharmacokinetic parameters are always correct and all variability is pharmacodynamic. Schwilden *et al.* [207, 208] developed an algorithm which used the former strategy and developed an adaptive control algorithm which progressively refined estimates of individual pharmacokinetic parameters by minimizing the difference between the observed and predicted EEG signal. This algorithm was implemented for the intravenous anesthetic agents methohexital and propofol but did not appear to offer great advantage over standard manual control. This may have been due to the approximations of the algorithm or due to the deficiencies of the median EEG frequency (the EEG signal utilized by the investigators) as a measure of consciousness. In the alternative approach, Struys *et al.* [227] have described a closed-loop controller of the delivery of the intravenous anesthetic propofol using a model-based adaptive control algorithm with the bispectral index (BIS), a derivative of the EEG signal, as the performance and measurement variable that assumes that all variability is pharmacodynamic. More specifically, with induction of anesthesia they calculated a predicted concentration using the pharmacokinetic model and then constructed a BIS-concentration relationship using the observed BIS during induction and the predicted propofol concentration. With each time epoch, the difference between the target BIS signal and the observed BIS signal is used to update the pharmacodynamic parameters relating concentration and BIS signal for the individual patient. Using this algorithm, Struys *et al.* [227] demonstrated excellent performance as measured by the difference between the target and observed BIS signals. However, as pointed out by Glass and Rampil [69], the excellent performance of the system may have been because the system was not fully stressed. In their study, Struys *et al.* [227] administered a relatively high fixed dose of the opioid remifentanil, in conjunction with propofol. This blunted the patient response to surgical stimuli and meant that the propofol was needed only to produce unconsciousness in patients who were profoundly analgesic. The result was that only small adjustments in propofol concentrations were necessary. Whether the system would have been robust in the absence of deep narcotization is an open question.

In contrast, to the above adaptive control algorithms, Absalom *et al.* [2] have described and implemented a proportional-integral-derivative control algorithm that is independent of pharmacokinetic and pharmacodynamic models. While overall precision and bias of this controller was good, the clinical performance was not acceptable due to oscillations observed in 3 of the 10 patients investigated. In this chapter, we present a less restrictive direct adaptive control framework as compared to the existing algorithms discussed above that accounts for interpatient pharmacokinetic and pharmacodynamic variability.

## 7.2. Mathematical Preliminaries

In this section we introduce notation, several definitions, and some key results concerning linear nonnegative dynamical systems [19, 20, 24, 75] that are necessary for developing the main results of this chapter. Specifically, for $x \in \mathbb{R}^n$ we write $x \geq\geq 0$ (resp., $x >> 0$) to indicate that every component of $x$ is nonnegative (resp., positive). In this case we say that $x$ is *nonnegative* or *positive*, respectively. Likewise, $A \in \mathbb{R}^{n\times m}$ is *nonnegative*[1] or *positive* if every entry of $A$ is nonnegative or positive, respectively, which is written as $A \geq\geq 0$ or $A >> 0$, respectively. Let $\overline{\mathbb{R}}_+^n$ and $\mathbb{R}_+^n$ denote the nonnegative and positive orthants of $\mathbb{R}^n$; that is, if $x \in \mathbb{R}^n$, then $x \in \overline{\mathbb{R}}_+^n$ and $x \in \mathbb{R}_+^n$ are equivalent, respectively, to $x \geq\geq 0$ and $x >> 0$. The following definition introduces the notion of a nonnegative (resp., positive) function.

**Definition 7.1.** Let $T > 0$. A real function $u : [0,T] \to \mathbb{R}^m$ is a *nonnegative* (resp., *positive*) *function* if $u(t) \geq\geq 0$ (resp., $u(t) >> 0$) on the interval $[0,T]$.

The next definition introduces the notion of essentially nonnegative matrices.

**Definition 7.2** [24,75]. Let $A \in \mathbb{R}^{n\times n}$. $A$ is *essentially nonnegative* if $A_{(i,j)} \geq 0$, $i,j = 1,\cdots,n$, $i \neq j$.

[1] In this dissertation it is important to distinguish between a square nonnegative (resp., positive) matrix and a nonnegative-definite (resp., positive-definite) matrix.

Next, consider the linear nonnegative dynamical system

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \quad t \geq 0, \tag{7.1}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, and $A \in \mathbb{R}^{n\times n}$ is essentially nonnegative. The solution to (7.1) is standard and is given by $x(t) = e^{At}x(0)$, $t \geq 0$. The following lemma proven in [24] (see also [75]) shows that $A$ is essentially nonnegative if and only if the state transition matrix $e^{At}$ is nonnegative on $[0, \infty)$.

**Proposition 7.1.** Let $A \in \mathbb{R}^{n\times n}$. Then $A$ is essentially nonnegative if and only if $e^{At}$ is nonnegative for all $t \geq 0$. Hence, if $A$ is essentially nonnegative and $x_0 \geq\geq 0$, then $x(t) \geq\geq 0$, $t \geq 0$, where $x(t)$, $t \geq 0$, denotes the solution to (7.1).

The following result shows that, for nonnegative initial conditions, the states of a *time-varying* linear dynamical system $\mathcal{G}$ of the form

$$\dot{x}(t) = A(t)x(t), \quad x(0) = x_0, \quad t \geq t_0, \tag{7.2}$$

where $t_0 \in [0, \infty)$ and $A : [0, \infty) \to \mathbb{R}^{n\times n}$ is continuous and essentially nonnegative pointwise-in-time, remain nonnegative.

**Proposition 7.2.** Consider the time-varying dynamical system (7.2) where $A : [0, \infty) \to \mathbb{R}^{n\times n}$ is continuous. Then $\overline{\mathbb{R}}_+^n$ is an invariant set with respect to (7.2) if and only if $A : [0, \infty) \to \mathbb{R}^{n\times n}$ is essentially nonnegative pointwise-in-time.

**Proof.** The result is a direct consequence of Proposition 6.1 of [75] (see also [22]) by equivalently representing the time-varying system (7.2) as an autonomous nonlinear system by appending another state to represent time. Specifically, defining $y(t - t_0) \triangleq x(t)$ and $y_{n+1}(t - t_0) \triangleq t$, it follows that the solution $x(t)$, $t \geq t_0$, to (7.2) can be equivalently characterized by the solution $y(\tau)$, $\tau \geq 0$, where $\tau \triangleq t - t_0$, to the nonlinear autonomous system

$$\dot{y}(\tau) = A(y_{n+1}(\tau))y(\tau), \quad y(0) = y_0, \quad \tau \geq 0, \tag{7.3}$$

$$\dot{y}_{n+1}(\tau) = 1, \quad y_{n+1}(0) = t_0, \tag{7.4}$$

where $\dot{y}(\cdot)$ and $\dot{y}_{n+1}(\cdot)$ denote differentiation with respect to $\tau$. Now, since $\dot{y}_i(\tau) \geq 0$, $\tau \geq 0$, for $i = 1, \cdots, n+1$, whenever $y_i(\tau) = 0$, the result is a direct consequence of Proposition 6.1 of [75]. □

The following theorems give necessary and sufficient conditions for asymptotic stability of a linear nonnegative dynamical system using *linear* and *quadratic* Lyapunov functions, respectively. For the statement of the first theorem recall that (7.1) is *semistable* if and only if $\lim_{t\to\infty} e^{At}$ exists [24, 25, 75].

**Theorem 7.1** [75]. Consider the linear dynamical system $\mathcal{G}$ given by (7.1) where $A \in \mathbb{R}^{n\times n}$ is essentially nonnegative. Then the following statements hold:

*i)* $\mathcal{G}$ is Lyapunov stable if and only if $\mathcal{G}$ is semistable.

*ii)* If there exist vectors $p, r \in \mathbb{R}^n$ such that $p >> 0$ and $r \geq\geq 0$ satisfy

$$0 = A^{\mathrm{T}}p + r, \tag{7.5}$$

then $\mathcal{G}$ is semistable (and hence Lyapunov stable).

*iii)* If $\mathcal{G}$ is semistable, then there exists vectors $p, r \in \mathbb{R}^n$ such that $p \geq\geq 0$, $p \neq 0$, and $r \geq\geq 0$ satisfy (7.5).

*iv)* If there exist vectors $p, r \in \mathbb{R}^n$ such that $p \geq\geq 0$ and $r \geq\geq 0$ satisfy (7.5) and $(A, r^{\mathrm{T}})$ is observable, then $p >> 0$ and $\mathcal{G}$ is asymptotically stable.

Furthermore, the following statements are equivalent:

*v)* $\mathcal{G}$ is asymptotically stable.

*vi)* There exist vectors $p, r \in \mathbb{R}^n$ such that $p >> 0$ and $r >> 0$ satisfy (7.5).

**Theorem 7.2** [75]. Consider the linear dynamical system $\mathcal{G}$ given by (7.1) where $A \in \mathbb{R}^{n \times n}$ is essentially nonnegative. Then $\mathcal{G}$ is asymptotically stable if and only if there exist a positive diagonal matrix $P \in \mathbb{R}^{n \times n}$ and an $n \times n$ positive-definite matrix $R$ such that

$$0 = A^{\mathrm{T}} P + PA + R. \tag{7.6}$$

Next, we consider a subclass of nonnegative systems; namely, compartmental systems. As discussed in the Introduction, linear compartmental dynamical systems are of major importance in biological and physiological systems. For example, almost the entire field of distribution of tracer labelled materials in steady state systems can be captured by linear compartmental dynamical systems [123].

**Definition 7.3.** Let $A \in \mathbb{R}^{n \times n}$. $A$ is a *compartmental matrix* if $A$ is essentially nonnegative and $\sum_{i=1}^{n} A_{(i,j)} \leq 0$, $j = 1, \cdots, n$.

If $A$ is a compartmental matrix, then the nonnegative system (1) is called an *inflow-closed compartmental system* [75, 123, 124]. As shown in [24, 75], if $A$ is a compartmental matrix, then the entries in $A$ are given by

$$A_{(i,j)} = \begin{cases} -\sum_{k=1}^{n} a_{kj}, & i = j, \\ a_{ij}, & i \neq j, \end{cases} \tag{7.7}$$

where $a_{ii} \geq 0$, $i \in \{1, \cdots, n\}$, denotes the loss coefficient of the $i$th compartment and $a_{ij} \geq 0$, $i \neq j$, $i, j \in \{1, \cdots, n\}$, denotes the transfer coefficient from the $j$th compartment to the $i$th compartment. Note that it follows from (7.7) that $\sum_{i=1}^{n} A_{(i,j)} \leq 0$, $j = 1, \cdots, n$. Recall that an inflow-closed compartmental system possesses a dissipation property and hence is Lyapunov stable since the total mass in the system given by the sum of all components of the state $x(t)$, $t \geq 0$, is nonincreasing along the forward trajectories of (7.1). In particular, with $V(x) = e^{\mathrm{T}}x$, where $e = [1, 1, \cdots, 1]^{\mathrm{T}}$, it follows that $\dot{V}(x) = e^{\mathrm{T}}Ax = \sum_{j=1}^{n}[\sum_{i=1}^{n} A_{(i,j)}]x_j \leq 0$, $x \in \overline{\mathbb{R}}_{+}^{n}$. Furthermore, since $\mathrm{ind}(A) \leq 1$ (see [24] and [75]), where $\mathrm{ind}(A)$ denotes the index of $A$, it follows that $A$ is semistable. Hence, all solutions of inflow-closed linear compartmental systems are convergent. Of course, if $\det A \neq 0$, where $\det A$ denotes the determinant of $A$, then $A$ is asymptotically stable. Alternatively, semistability and asymptotic stability can be deduced from Theorem 7.1. In particular, with $p = e >> 0$ and $r = -A^{\mathrm{T}}e = [-a_{11}, -a_{22}, \cdots, -a_{nn}] \geq\geq 0$, (7.5) is satisfied which implies, by Theorem 3.2 of [75], that an inflow-closed compartmental system is semistable if $A$ is singular and asymptotically stable if $A$ is nonsingular. For details of the above facts see [24, 75].

Next, we show that every asymptotically stable linear nonnegative system is equivalent, modulo a similarity transformation, to a compartmental system.

**Proposition 7.3** [75]. Let $A \in \mathbb{R}^{n\times n}$ be asymptotically stable. Then $A$ is essentially nonnegative if and only if there exists an invertible diagonal matrix $S \in \mathbb{R}^{n\times n}$ such that $SAS^{-1}$ is a compartmental matrix.

Finally, in this chapter we consider controlled dynamical systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \tag{7.8}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, $u(t) \in \mathbb{R}^m$, $t \geq 0$, $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$. The following definition and proposition are needed for the main results of the chapter.

**Definition 7.4.** The linear dynamical system given by (7.8) is *nonnegative* if for every $x(0) \in \overline{\mathbb{R}}_{+}^{n}$ and $u(t) \geq\geq 0$, $t \geq 0$, the solution $x(t)$, $t \geq 0$, to (7.8) is nonnegative.

**Proposition 7.4** [75]. The linear dynamical system given by (7.8) is nonnegative if and only if $A \in \mathbb{R}^{n\times n}$ is essentially nonnegative and $B \in \mathbb{R}^{n\times m}$ is nonnegative.

It follows from Proposition 7.4 that the control input signal $Bu(t)$, $t \geq 0$, needs to be nonnegative to guarantee the nonnegativity of the state of (7.8). This is due to the fact that when the initial state of (7.8) belongs to the boundary of the nonnegative orthant, a negative input can destroy the nonnegativity of the state of (7.8). Alternatively, however, if the initial state is in the interior of the nonnegative orthant, then it follows from continuity of solutions with respect to the system initial conditions that, over a small interval of time, nonnegativity of the state of (7.8) is guaranteed *irrespective* of the sign of each element of the control input $Bu(t)$ over this time interval. However, unlike open-loop control wherein lack of coordination between the input and the state necessitates nonnegativity of the control input, a *feedback* control signal predicated on the system state variables allows for the anticipation of loss of nonnegativity of the state. Hence, state feedback control signals can take negative values while assuring nonnegativity of the system states. For further discussion of the above fact see [53].

Next, we present a time-varying extension to Proposition 7.4 needed for the main theorems of this chapter. Specifically, we consider the time-varying system

$$\dot{x}(t) = A(t)x(t) + Bu(t), \quad x(t_0) = x_0, \quad t \geq t_0, \tag{7.9}$$

where $A : [t_0, \infty) \to \mathbb{R}^{n\times n}$ is continuous. For the following result the definition of nonnegativity holds with (7.8) replaced by (7.9).

**Proposition 7.5.** Consider the time-varying dynamical system (7.9) where $A : [t_0, \infty) \to \mathbb{R}^{n\times n}$ is continuous. If $A : [t_0, \infty) \to \mathbb{R}^{n\times n}$ is essentially nonnegative pointwise-in-time and $B \in \mathbb{R}^{n\times m}$ is nonnegative, then the solution $x(t)$, $t \geq t_0$, to (7.9) is nonnegative.

**Proof.** The result is a direct consequence of Proposition 6.1 of [75] (see also [22]) by equivalently representing the time-varying system (7.9) as an autonomous nonlinear system by appending another state to represent time. Specifically, defining $y(t-t_0) \triangleq x(t)$ and $y_{n+1}(t-t_0) \triangleq t$, it follows that the solution $x(t)$, $t \geq t_0$, to (7.9) can be equivalently characterized by the solution $y(\tau)$, $\tau \geq 0$, where $\tau \triangleq t - t_0$, to the nonlinear autonomous system

$$\dot{y}(\tau) = A(y_{n+1}(\tau))y(\tau), \quad y(0) = y_0, \quad \tau \geq 0, \tag{7.10}$$

$$\dot{y}_{n+1}(\tau) = 1, \quad y_{n+1}(0) = t_0, \tag{7.11}$$

where $\dot{y}(\cdot)$ and $\dot{y}_{n+1}(\cdot)$ denote differentiation with respect to $\tau$. Now, since $\dot{y}_i(\tau) \geq 0$, $\tau \geq 0$, for $i = 1, \cdots, n+1$, whenever $y_i(\tau) = 0$, the result is a direct consequence of Proposition 6.1 of [75]. □

Since stabilization of nonnegative systems naturally deals with equilibrium points in the interior of the nonnegative orthant $\overline{\mathbb{R}}_+^n$, the following proposition provides necessary conditions for the existence of an interior equilibrium point $x_{\rm e} \in \mathbb{R}_+^n$ of (7.8) in terms of the stability properties of the system dynamics matrix $A$.

**Proposition 7.6.** Consider the nonnegative dynamical system (7.8) and assume there exist $x_{\rm e} \in \mathbb{R}_+^n$ and $u_{\rm e} \in \overline{\mathbb{R}}_+^m$ such that

$$0 = Ax_{\rm e} + Bu_{\rm e}. \tag{7.12}$$

Then, $A$ is semistable.

**Proof.** The proof is a direct consequence of *ii*) of Theorem 7.1 with $A$ replaced by $A^{\rm T}$, $p = x_{\rm e}$, and $r = Bu_{\rm e}$. □

It follows from Proposition 7.6 that the existence of an equilibrium point $x_{\rm e} \in \mathbb{R}_+^n$ for (7.8) implies that the system matrix $A$ is semistable. Hence, if (7.12) holds for

$x_e \in \mathbb{R}_+^n$ and $u_e \in \overline{\mathbb{R}}_+^m$, then $A$ is asymptotically stable or $0 \in \operatorname{spec}(A)$, where $\operatorname{spec}(A)$ denotes the spectrum of $A$, is a simple eigenvalue of $A$ and all other eigenvalues of $A$ have negative real parts since $-A$ is an $M$-matrix [20]. In light of the above constraints, it was shown in [53] using Brockett's necessary condition for asymptotic stabilizability [28] that if $0 \in \operatorname{spec}(A)$, then there does *not* exist a *continuous* stabilizing *nonnegative* feedback for set-point regulation in $\mathbb{R}_+^n$ for a nonnegative system. However, that is not to say that asymptotic feedback regulation using *discontinuous* feedback is not possible.

## 7.3. Adaptive Control for Linear Nonnegative Uncertain Dynamical Systems

In this section we consider the problem of characterizing adaptive feedback control laws for nonnegative and compartmental uncertain dynamical systems to achieve *set-point* regulation in the nonnegative orthant. Specifically, consider the following controlled linear uncertain dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \tag{7.13}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $A \in \mathbb{R}^{n \times n}$ is an *unknown* essentially nonnegative matrix, and $B \in \mathbb{R}^{n \times m}$ is an *unknown* nonnegative input matrix. The control input $u(\cdot)$ in (7.13) is restricted to the class of *admissible controls* consisting of measurable functions such that $u(t) \in \mathbb{R}^m$, $t \geq 0$.

As discussed in the Introduction, it follows from physical considerations that the state trajectories of nonnegative and compartmental dynamical systems remain in the nonnegative orthant of the state space for nonnegative initial conditions. However, even though active control of drug delivery systems for physiological applications additionally requires control (source) inputs to be nonnegative, in many applications of nonnegative systems such as biological systems, population dynamics, and ecological systems, the positivity constraint on the control input is not natural. Hence, in this section we do not place any restriction on the sign of the control signal and design an adaptive controller that guarantees that the system states remain in the nonnegative orthant and converge to a desired equilibrium state. Specifically, for a given desired set point $x_{\mathrm{e}} \in \overline{\mathbb{R}}_+^n$, our aim is to design a control input $u(t)$, $t \geq 0$, such that $\lim_{t\to\infty} \|x(t) - x_{\mathrm{e}}\| = 0$. However, since in many applications of nonnegative systems and in particular, compartmental systems, it is often necessary to regulate a subset of the nonnegative state variables which usually include a central compartment, here we require that $\lim_{t\to\infty} x_i(t) = x_{\mathrm{d}i} \geq 0$ for $i = 1, \cdots, m \leq n$, where $x_{\mathrm{d}i}$ is a desired set point for the $i$th state $x_i(t)$. Furthermore, we assume that control inputs are injected directly into $m$ separate compartments so that the input matrix is given by

$$B = \begin{bmatrix} B_{\mathrm{u}} \\ 0_{(n-m)\times m} \end{bmatrix}, \tag{7.14}$$

where $B_{\mathrm{u}} \triangleq \mathrm{diag}[b_1, \cdots, b_m]$ and $b_i \in \mathbb{R}_+$, $i = 1, \cdots, m$. For compartmental systems this assumption is not restrictive since control inputs correspond to control inflows to each individual compartment. Here, we assume that for $i \in \{1, \cdots, m\}$, $b_i$ is *unknown*. For the statement of our main result define $x_{\mathrm{e}} \triangleq [x_{\mathrm{d}}^{\mathrm{T}}, x_{\mathrm{u}}^{\mathrm{T}}]^{\mathrm{T}}$, where $x_{\mathrm{d}} \triangleq [x_{\mathrm{d}1}, \cdots, x_{\mathrm{d}m}]^{\mathrm{T}}$ and $x_{\mathrm{u}} \triangleq [x_{\mathrm{u}1}, \cdots, x_{\mathrm{u}(n-m)}]^{\mathrm{T}}$.

**Theorem 7.3.** Consider the linear uncertain dynamical system $\mathcal{G}$ given by (7.13) where $A$ is essentially nonnegative and $B$ is nonnegative and given by (7.14). For a given $x_{\mathrm{d}}$ assume there exist nonnegative vectors $x_{\mathrm{u}} \in \overline{\mathbb{R}}_+^{n-m}$ and $u_{\mathrm{e}} \in \overline{\mathbb{R}}_+^m$ such that

$$0 = Ax_{\mathrm{e}} + Bu_{\mathrm{e}}. \tag{7.15}$$

Furthermore, assume there exists a diagonal matrix $K_{\mathrm{g}} = \mathrm{diag}[k_{\mathrm{g}_1}, \cdots, k_{\mathrm{g}_m}] \in \mathbb{R}^{m\times m}$ such that $A_{\mathrm{s}} \triangleq A + B\tilde{K}_{\mathrm{g}}$ is asymptotically stable, where $\tilde{K}_{\mathrm{g}} \triangleq [K_{\mathrm{g}}, 0_{m\times(n-m)}]$. Finally, let $q_i$ and $\hat{q}_i$, $i = 1, \cdots, m$, be positive constants. Then the adaptive feedback control law

$$u(t) = K(t)(\hat{x}(t) - x_{\mathrm{d}}) + \phi(t), \tag{7.16}$$

where $K(t) = \mathrm{diag}[k_1(t), \cdots, k_m(t)]$, $\hat{x}(t) = [x_1(t), \cdots, x_m(t)]^{\mathrm{T}}$, and $\phi(t) \in \mathbb{R}^m$, $t \geq 0$, or, equivalently,

$$u_i(t) = k_i(t)(x_i(t) - x_{\mathrm{d}i}) + \phi_i(t), \quad i = 1, \cdots, m, \tag{7.17}$$

where $k_i(t) \in \mathbb{R}$, $t \geq 0$, and $\phi_i(t) \in \mathbb{R}$, $t \geq 0$, $i = 1, \cdots, m$, with update laws

$$\dot{k}_i(t) = -q_i(x_i(t) - x_{\mathrm{d}i})^2, \quad k_i(0) \leq 0, \quad t \geq 0, \quad i = 1, \cdots, m, \tag{7.18}$$

$$\dot{\phi}_i(t) = \begin{cases} 0, & \text{if } \phi_i(t) = 0 \text{ and } x_i(t) \geq x_{\mathrm{d}i}, \\ -\hat{q}_i(x_i(t) - x_{\mathrm{d}i}), & \text{otherwise,} \end{cases}$$
$$\phi_i(0) \geq 0, \quad i = 1, \cdots, m, \tag{7.19}$$

guarantees that the solution $(x(t), K(t), \phi(t)) \equiv (x_{\mathrm{e}}, K_{\mathrm{g}}, u_{\mathrm{e}})$ of the closed-loop system given by (7.13), (7.16), (7.18), (7.19) is Lyapunov stable and $x_i(t) \to x_{\mathrm{d}i}$, $i = 1, \cdots, m$, as $t \to \infty$ for all $x_0 \in \overline{\mathbb{R}}_+^n$. Furthermore, $x(t) \geq\geq 0$ for all $t \geq 0$ and $x_0 \in \overline{\mathbb{R}}_+^n$.

**Proof.** Note that with $u(t)$, $t \geq 0$, given by (7.16) it follows from (7.13) that

$$\dot{x}(t) = Ax(t) + BK(t)(\hat{x}(t) - x_{\mathrm{d}}) + B\phi(t), \quad x(0) = x_0, \quad t \geq 0, \tag{7.20}$$

or, equivalently, using (7.15) and $A_{\mathrm{s}} = A + B\tilde{K}_{\mathrm{g}}$,

$$\dot{x}(t) = A_{\mathrm{s}}(x(t) - x_{\mathrm{e}}) + B(K(t) - K_{\mathrm{g}})(\hat{x}(t) - x_{\mathrm{d}}) + B(\phi(t) - u_{\mathrm{e}}), \quad x(0) = x_0, \quad t \geq 0. \tag{7.21}$$

Furthermore, since $A_{\mathrm{s}}$ is essentially nonnegative and asymptotically stable, it follows from Theorem 7.2 that there exists a positive *diagonal* matrix $P \triangleq \mathrm{diag}[p_1, \cdots, p_n]$ and a positive-definite matrix $R \in \mathbb{R}^{n \times n}$ such that

$$0 = A_{\mathrm{s}}^{\mathrm{T}} P + PA_{\mathrm{s}} + R. \tag{7.22}$$

To show Lyapunov stability of the closed-loop system (7.18), (7.19), and (7.21) consider the Lyapunov function candidate

$$V(x,K,\phi) = (x-x_{\rm e})^{\rm T}P(x-x_{\rm e}) + {\rm tr}(K-K_{\rm g})^{\rm T}Q^{-1}(K-K_{\rm g}) + (\phi-u_{\rm e})^{\rm T}\hat{Q}^{-1}(\phi-u_{\rm e}), \tag{7.23}$$

or, equivalently,

$$V(x,K,\phi) = \sum_{i=1}^{n} p_i(x_i - x_{{\rm e}i})^2 + \sum_{i=1}^{m}\frac{p_ib_i}{q_i}(k_i - k_{{\rm g}_i})^2 + \sum_{i=1}^{m}\frac{p_ib_i}{\hat{q}_i}(\phi_i - u_{{\rm e}i})^2,$$

where $Q = {\rm diag}\left[\frac{q_1}{p_1b_1},\cdots,\frac{q_m}{p_mb_m}\right]$ and $\hat{Q} = {\rm diag}\left[\frac{\hat{q}_1}{p_1b_1},\cdots,\frac{\hat{q}_m}{p_mb_m}\right]$. Note that $V(x_{\rm e}, K_{\rm g}, u_{\rm e}) = 0$ and, since $P$, $Q$, and $\hat{Q}$ are positive definite, $V(x,K,\phi) > 0$ for all $(x,K,\phi) \neq (x_{\rm e}, K_{\rm g}, u_{\rm e})$. Furthermore, $V(x,K,\phi)$ is radially unbounded. Now, letting $x(t)$, $t \geq 0$, denote the solution to (7.21) and using (7.18) and (7.19), it follows that the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned}
\dot{V}(x(t),K(t),\phi(t)) =&\ 2(x(t)-x_{\rm e})^{\rm T}P[A_{\rm s}(x(t)-x_{\rm e}) + B(K(t)-K_{\rm g})(\hat{x}(t)-x_{\rm d}) \\
&+B(\phi(t)-u_{\rm e})] + 2{\rm tr}(K(t)-K_{\rm g})^{\rm T}Q^{-1}\dot{K}(t) \\
&+2(\phi(t)-u_{\rm e})^{\rm T}\hat{Q}^{-1}\dot{\phi}(t) \\
=&\ -(x(t)-x_{\rm e})^{\rm T}R(x(t)-x_{\rm e}) + 2\sum_{i=1}^{m}p_ib_i(k_i(t)-k_{{\rm g}_i})(x_i(t)-x_{{\rm d}i})^2 \\
&+2\sum_{i=1}^{m}p_ib_i(x_i(t)-x_{{\rm d}i})(\phi_i(t)-u_{{\rm e}i}) \\
&+2\sum_{i=1}^{m}\frac{p_ib_i}{q_i}(k_i(t)-k_{{\rm g}_i})\dot{k}_i(t) + 2\sum_{i=1}^{m}\frac{p_ib_i}{\hat{q}_i}(\phi_i(t)-u_{{\rm e}i})\dot{\phi}_i(t) \\
=&\ -(x(t)-x_{\rm e})^{\rm T}R(x(t)-x_{\rm e}) \\
&+2\sum_{i=1}^{m}p_ib_i(\phi_i(t)-u_{{\rm e}i})\left[(x_i(t)-x_{{\rm d}i}) + \frac{1}{\hat{q}_i}\dot{\phi}_i(t)\right].
\end{aligned} \tag{7.24}$$

Now, for each $i \in \{1,\cdots,m\}$ and for the two cases given in (7.19), the last term on the right-hand side of (7.24) gives:

*i)* If $\phi_i(t) = 0$ and $x_i(t) \geq x_{\mathrm{d}i}$, then $\dot{\phi}_i(t) = 0$ and hence

$$p_i b_i(\phi_i(t) - u_{\mathrm{e}i}) \left[ (x_i(t) - x_{\mathrm{d}i}) + \frac{1}{\hat{q}_i} \dot{\phi}_i(t) \right] = -p_i b_i u_{\mathrm{e}i}(x_i(t) - x_{\mathrm{d}i}) \leq 0.$$

*ii)* Otherwise, $\dot{\phi}_i(t) = -\hat{q}_i(x_i(t) - x_{\mathrm{d}i})$ and hence

$$p_i b_i(\phi_i(t) - u_{\mathrm{e}i}) \left[ (x_i(t) - x_{\mathrm{d}i}) + \frac{1}{\hat{q}_i} \dot{\phi}_i(t) \right] = 0.$$

Hence, it follows that in either case

$$\begin{aligned} \dot{V}(x(t), K(t), \phi(t)) &\leq -(x(t) - x_{\mathrm{e}})^{\mathrm{T}} R (x(t) - x_{\mathrm{e}}) \\ &\leq 0, \quad t \geq 0, \end{aligned} \tag{7.25}$$

which proves that the solution $(x(t), K(t), \phi(t)) \equiv (x_{\mathrm{e}}, K_{\mathrm{g}}, u_{\mathrm{e}})$ to (7.18), (7.19), and (7.21) is Lyapunov stable. Furthermore, since $R > 0$ it follows from Theorem 2 of [42] that $x(t) \to x_{\mathrm{e}}$ as $t \to \infty$ for all $x_0 \in \overline{\mathbb{R}}_+^n$.

Finally, to show that $x(t) \geq\geq 0$, $t \geq 0$, for all $x_0 \in \overline{\mathbb{R}}_+^n$, note that the closed-loop system (7.13), (7.16), (7.18), and (7.19) is given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + BK(t)(\hat{x}(t) - x_{\mathrm{d}}) + B\phi(t) \\ &= (A + B[K(t), 0_{m \times (n-m)}])x(t) - BK(t)x_{\mathrm{d}} + B\phi(t) \\ &= \tilde{A}(t)x(t) + v(t) + w(t), \end{aligned} \tag{7.26}$$

where

$$\tilde{A}(t) \triangleq \begin{bmatrix} a_{11} + b_1 k_1(t) & a_{12} & \cdots & a_{1m} & a_{1\,m+1} & \cdots & a_{1n} \\ a_{21} & a_{22} + b_2 k_2(t) & & \vdots & \vdots & \ddots & a_{2n} \\ \vdots & & \ddots & & & & \vdots \\ a_{m1} & \cdots & & a_{mm} + b_m k_m(t) & a_{m\,m+1} & \cdots & a_{mn} \\ a_{m+1\,1} & \cdots & & a_{m+1\,m} & a_{m+1\,m+1} & \cdots & a_{m+1\,n} \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & & a_{nm} & a_{n\,m+1} & \cdots & a_{nn} \end{bmatrix}, \tag{7.27}$$

$$v(t) \triangleq -\begin{bmatrix} b_1 k_1(t) x_{\mathrm{d}1} \\ \vdots \\ b_m k_m(t) x_{\mathrm{d}m} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad w(t) \triangleq \begin{bmatrix} b_1 \phi_1(t) \\ \vdots \\ b_m \phi_m(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{7.28}$$

Now, since, by (7.18) and (7.19), $k_i(t) \leq 0$, $t \geq 0$, $i = 1, \cdots, m$, and $\phi_i(t) \geq 0$, $t \geq 0$, $i = 1, \cdots, m$, it follows that $v(t) \geq\geq 0$, $t \geq 0$, and $w(t) \geq\geq 0$, $t \geq 0$. Hence, since $\tilde{A}(t)$, $t \geq 0$, is essentially nonnegative pointwise-in-time, it follows from Proposition 7.5 that $x(t) \geq\geq 0$ for all $t \geq 0$ and $x_0 \in \overline{\mathbb{R}}_+^n$. $\square$

**Remark 7.1.** Note that the conditions in Theorem 7.3 imply that $x(t) \to x_{\mathrm{e}}$ as $t \to \infty$ and hence it follows from (7.18) and (7.19) that $(x(t), K(t), \phi(t)) \to \mathcal{M} \triangleq \{(x, K, \phi) \in \mathbb{R}^n \times \mathbb{R}^{m\times m} \times \mathbb{R}^m : x = x_{\mathrm{e}},\ \dot{K} = 0,\ \dot{\phi} = 0\}$ as $t \to \infty$.

It is important to note that the adaptive control law (7.16), (7.18), and (7.19) does *not* require the explicit knowledge of the system matrices $A$, $B$, the gain matrix $K_{\mathrm{g}}$, and the nonnegative constant vector $u_{\mathrm{e}}$; even though Theorem 7.3 requires the existence of $K_{\mathrm{g}}$ and nonnegative vectors $x_{\mathrm{u}}$ and $u_{\mathrm{e}}$ such that $A_{\mathrm{s}}$ is essentially nonnegative and asymptotically stable and condition (7.15) holds. Furthermore, in the case where $A$ is semistable and minimum phase with respect to the output $y = \hat{x}$, or $A$ is asymptotically stable, then there always exists a diagonal matrix $K_{\mathrm{g}} \in \mathbb{R}^{m\times m}$ such that $A_{\mathrm{s}}$ is asymptotically stable. Necessary and sufficient conditions for set-point stabilization of the pair $(A, B)$, where $A$ is singular and compartmental are given in [53, 113]. Finally, note that for $i = 1, \cdots, m$, the control input signal $u_i(t)$, $t \geq 0$, can be negative depending on the values of $x_i(t)$, $k_i(t)$, and $\phi_i(t)$, $t \geq 0$. However, as is required in nonnegative and compartmental dynamical systems the closed-loop plant states remain nonnegative.

In the case where our objective is zero set-point regulation, that is, $x_{\mathrm{e}} = 0$, the adaptive controller given in Theorem 7.3 can be considerably simplified. Specifically, since in this case $x(t) \geq\geq x_{\rm e} = 0$, $t \geq 0$, and condition (7.15) is trivially satisfied with $u_{\rm e} = 0$, we can set $\phi(t) \equiv 0$ so that update law (7.19) is superfluous. Furthermore, since (7.15) is trivially satisfied, $A$ can possess eigenvalues in the open right-half plane. Alternatively, exploiting a *linear* Lyapunov function construction for the plant dynamics, an even simpler adaptive controller can be derived. This result is given in the following theorem.

**Theorem 7.4.** Consider the linear uncertain dynamical system $\mathcal{G}$ given by (7.13) where $A$ is essentially nonnegative and $B$ is nonnegative and given by (7.14). Assume there exists a diagonal matrix $K_{\rm g} = \mathrm{diag}[k_{{\rm g}1}, \cdots, k_{{\rm g}m}] \in \mathbb{R}^{m\times m}$ such that $A_{\rm s} \triangleq A + B\tilde{K}_{\rm g}$ is asymptotically stable, where $\tilde{K}_{\rm g} \triangleq [K_{\rm g}, 0_{m\times(n-m)}]$. Furthermore, let $q_i$, $i = 1, \cdots, m$, be positive constants. Then the adaptive feedback control law

$$u(t) = K(t)\hat{x}(t), \tag{7.29}$$

where $K(t) = \mathrm{diag}[k_1(t), \cdots, k_m(t)]$ and $\hat{x}(t) = [x_1(t), \cdots, x_m(t)]^{\rm T}$, or, equivalently,

$$u_i(t) = k_i(t)x_i(t), \quad i = 1, \cdots, m, \tag{7.30}$$

where $k_i(t) \in \mathbb{R}$, $i = 1, \cdots, m$, with update law

$$\dot{K}(t) = -\mathrm{diag}[q_1x_1(t), \cdots, q_mx_m(t)], \tag{7.31}$$

guarantees that the solution $(x(t), K(t)) \equiv (0, K_{\rm g})$ of the closed-loop system given by (7.13), (7.29), (7.31) is Lyapunov stable and $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \overline{\mathbb{R}}_+^n$. Furthermore, $x(t) \geq\geq 0$, $t \geq 0$, for all $x_0 \in \overline{\mathbb{R}}_+^n$.

**Proof.** Note that with $u(t)$, $t \geq 0$, given by (7.29) it follows from (7.13) that

$$\begin{aligned}\dot{x}(t) &= Ax(t) + BK(t)\hat{x}(t), \quad x(0) = x_0, \quad t \geq 0,\\ &= (A + B[K(t), 0_{m\times(n-m)}])x(t)\\ &= \tilde{A}(t)x(t),\end{aligned} \tag{7.32}$$

where $\tilde{A}(t)$, $t \geq 0$, is given by (7.27). Now, since $\tilde{A}(t)$, $t \geq 0$, is essentially nonnegative pointwise-in-time, it follows from Proposition 7.2 that $x(t) \geq\geq 0$, $t \geq 0$, for all $x_0 \in \overline{\mathbb{R}}_+^n$. Next, using $A_{\rm s} = A + B\tilde{K}_{\rm g}$, note that (7.32) can be equivalently written as

$$\dot{x}(t) = A_{\rm s}x(t) + B(K(t) - K_{\rm g})\hat{x}(t), \quad x(0) = x_0, \quad t \geq 0. \tag{7.33}$$

Furthermore, since $A_{\rm s}$ is essentially nonnegative and asymptotically stable, it follows from Theorem 7.1 that there exist $p, r \in \mathbb{R}^n$ such that $p >> 0$ and $r >> 0$ satisfy

$$0 = A_{\rm s}^{\rm T}p + r. \tag{7.34}$$

To show Lyapunov stability of the closed-loop system (7.31) and (7.33) consider the Lyapunov function candidate

$$V(x, K) = p^{\rm T}x + \tfrac{1}{2}{\rm tr}(K - K_{\rm g})^{\rm T}Q^{-1}(K - K_{\rm g}), \tag{7.35}$$

or, equivalently,

$$V(x, K) = p^{\rm T}x + \frac{1}{2}\sum_{i=1}^{m}\frac{p_i b_i}{q_i}(k_i - k_{{\rm g}_i})^2, \tag{7.36}$$

where $Q = {\rm diag}\left[\frac{q_1}{p_1 b_1}, \cdots, \frac{q_m}{p_m b_m}\right]$. Note that $V(0, K_{\rm g}) = 0$ and, since $p >> 0$ and $Q > 0$, $V(x, K) > 0$ for all $(x, K) \neq (0, K_{\rm g})$. Furthermore, $V(x, K)$ is radially unbounded with respect to the nonnegative orthant. Now, letting $x(t)$, $t \geq 0$, denote the solution to (7.33) and using (7.31), it follows that the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned}\dot{V}(x(t), K(t)) &= p^{\rm T}[A_{\rm s}x(t) + B(K(t) - K_{\rm g})\hat{x}(t)] + {\rm tr}(K(t) - K_{\rm g})^{\rm T}Q^{-1}\dot{K}(t) \\ &= -r^{\rm T}x(t) + \sum_{i=1}^{m} p_i b_i (k_i(t) - k_{{\rm g}_i})x_i(t) + \sum_{i=1}^{m}\frac{p_i b_i}{q_i}(k_i(t) - k_{{\rm g}_i})\dot{k}_i(t) \\ &= -r^{\rm T}x(t) \\ &\leq 0, \quad t \geq 0,\end{aligned}$$

which proves that the solution $(x(t), K(t)) \equiv (0, K_{\rm g})$ to (7.31) and (7.33) is Lyapunov stable. Furthermore, since $r >> 0$ it follows from Theorem 2 of [42] that $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \overline{\mathbb{R}}_+^n$. □

Finally, we generalize Theorem 7.3 to the case where the input matrix is not necessarily nonnegative. For the statement of the following result define $\operatorname{sgn} b_i \triangleq b_i/|b_i|$.

**Theorem 7.5.** Consider the linear uncertain dynamical system $\mathcal{G}$ given by (7.13) where $A$ is essentially nonnegative and $B$ is given by (7.14) where $b_i$, $i = 1, \cdots, m$, is an *unknown* constant, but $\operatorname{sgn} b_i$ is known. For a given $x_{\rm d}$ assume there exist a nonnegative vector $x_{\rm u} \in \overline{\mathbb{R}}_+^{n-m}$ and a vector $u_{\rm e} \in \mathbb{R}^m$ such that (7.15) holds with $Ax_{\rm e} \leq\leq 0$. Furthermore, assume there exists a diagonal matrix $K_{\rm g} = \operatorname{diag}[k_{{\rm g}1}, \cdots, k_{{\rm g}m}] \in \mathbb{R}^{m\times m}$ such that $A_{\rm s} \triangleq A + B\tilde{K}_{\rm g}$ is asymptotically stable, where $\tilde{K}_{\rm g} \triangleq [K_{\rm g}, 0_{m\times(n-m)}]$. Finally, let $q_i$ and $\hat{q}_i$, $i = 1, \cdots, m$, be positive constants. Then the adaptive feedback control law (7.16) with update laws

$$\dot{k}_i(t) = -(\operatorname{sgn} b_i)q_i(x_i(t) - x_{{\rm d}i})^2, \quad i = 1, \cdots, m, \tag{7.37}$$

$$\dot{\phi}_i(t) = \begin{cases} 0, & \text{if } \phi_i(t) = 0 \text{ and } x_i(t) \geq x_{\rm d}, \\ -(\operatorname{sgn} b_i)\hat{q}_i(x_i(t) - x_{{\rm d}i}), & \text{otherwise,} \end{cases} \quad i = 1, \cdots, m, \tag{7.38}$$

where $k_i(0)$ and $\phi_i(0)$ are such that $(\operatorname{sgn} b_i)k_i(0) \leq 0$ and $(\operatorname{sgn} b_i)\phi_i(0) \geq 0$, respectively, guarantees that the solution $(x(t), K(t), \phi(t)) \equiv (x_{\rm e}, K_{\rm g}, u_{\rm e})$ of the closed-loop system given by (7.13), (7.16), (7.37), (7.38) is Lyapunov stable and $x_i(t) \to x_{{\rm d}i}$, $i = 1, \cdots, m$, as $t \to \infty$ for all $x_0 \in \overline{\mathbb{R}}_+^n$. Furthermore, $x(t) \geq\geq 0$, $t \geq 0$, for all $x_0 \in \overline{\mathbb{R}}_+^n$.

**Proof.** The proof is similar to that of Theorem 7.3 with $Q$ and $\hat{Q}$ replaced by $Q = \operatorname{diag}\left[\frac{q_1}{p_1|b_1|}, \cdots, \frac{q_m}{p_m|b_m|}\right]$ and $\hat{Q} = \operatorname{diag}\left[\frac{\hat{q}_1}{p_1|b_1|}, \cdots, \frac{\hat{q}_m}{p_m|b_m|}\right]$, respectively. □

Note that the adaptive controller given in Theorem 7.5 does not destroy nonnegativity with respect to the plant states. In particular, the closed-loop system dynamics are given by (7.26). Now, it can be seen that if $b_i$ is negative, then $k_i(t) \geq 0$, $t \geq 0$, and $\phi_i(t) \leq 0$, $t \geq 0$, and hence $v(t) \geq\geq 0$, $t \geq 0$, and $w(t) \geq\geq 0$, $t \geq 0$. Hence, by Proposition 7.5, $x(t) \geq\geq 0$, $t \geq 0$.

## 7.4. Adaptive Control for Linear Nonnegative Dynamical Systems with Nonnegative Control

As discussed in the Introduction, control (source) inputs of drug delivery systems for physiological processes are usually constrained to be nonnegative as are the system states. Hence, in this section we develop adaptive control laws for nonnegative systems with nonnegative control inputs. However, as noted in Section 7.2, since condition (7.12) is required to be satisfied for $x_\mathrm{e} \in \overline{\mathbb{R}}_+^n$ and $u_\mathrm{e} \in \overline{\mathbb{R}}_+^m$, it follows from Brockett's necessary condition for asymptotic stabilizability [53] that there does not exist a continuous stabilizing *nonnegative* feedback if $0 \in \mathrm{spec}(A)$ and $x_\mathrm{e} \in \mathbb{R}_+^n$. Hence, in this section we assume that $A$ is asymptotically stable and hence, without loss of generality, by Proposition 7.3 we further assume that $A$ is an asymptotically stable compartmental matrix. Thus, we proceed with the aforementioned assumptions to design adaptive controllers for uncertain compartmental systems that guarantee that $\lim_{t\to\infty} x_i(t) = x_{\mathrm{d}i} \geq 0$ for $i = 1, \cdots, m \leq n$, where $x_{\mathrm{d}i}$ is a desired set point for the $i$th compartmental state while guaranteeing a nonnegative control input.

**Theorem 7.6.** Consider the linear uncertain dynamical system $\mathcal{G}$ given by (7.13), where $A$ is an asymptotically stable compartmental matrix, and $B$ is nonnegative and given by (7.14). For a given $x_\mathrm{d} \in \mathbb{R}_+^m$ assume there exist vectors $x_\mathrm{u} \in \mathbb{R}_+^{n-m}$ and $u_\mathrm{e} \in \overline{\mathbb{R}}_+^m$ such that (7.15) holds. Furthermore, let $q_i$ and $\hat{q}_i$, $i = 1, \cdots, m$, be positive constants. Then the adaptive feedback control law

$$u_i(t) = \max\{0, \hat{u}_i(t)\}, \quad i = 1, \cdots, m, \tag{7.39}$$

where

$$\hat{u}_i(t) = k_i(t)(x_i(t) - x_{\mathrm{d}i}) + \phi_i(t), \quad i = 1, \cdots, m, \tag{7.40}$$

$k_i(t) \in \mathbb{R}$, $t \geq 0$, and $\phi_i(t) \in \mathbb{R}$, $t \geq 0$, $i = 1, \cdots, m$, with update laws

$$\dot{k}_i(t) = \begin{cases} 0, & \text{if } \hat{u}_i(t) < 0, \\ -q_i(x_i(t) - x_{\mathrm{d}i})^2, & \text{otherwise,} \end{cases} \quad k_i(0) \leq 0, \quad i = 1, \cdots, m, \tag{7.41}$$

$$\dot{\phi}_i(t) = \begin{cases} 0, & \text{if } \phi_i(t) = 0 \text{ and } x_i(t) \geq x_{\mathrm{d}i}, \text{ or if } \hat{u}_i(t) \leq 0, \\ -\hat{q}_i(x_i(t) - x_{\mathrm{d}i}), & \text{otherwise,} \end{cases}$$

$$\phi_i(0) \geq 0, \quad i = 1, \cdots, m, \tag{7.42}$$

guarantees that the solution $(x(t), K(t), \phi(t)) \equiv (x_{\mathrm{e}}, K_{\mathrm{g}}, u_{\mathrm{e}})$, where $K_{\mathrm{g}} = \mathrm{diag}[k_{\mathrm{g}1}, \cdots, k_{\mathrm{g}m}] \leq 0$, of the closed-loop system given by (7.13), (7.39), (7.41), (7.42) is Lyapunov stable and $x_i(t) \to x_{\mathrm{d}i}$, $i = 1, \cdots, m$, as $t \to \infty$ for all $x_0 \in \overline{\mathbb{R}}_+^n$. Furthermore, $u(t) \geq\geq 0$, $t \geq 0$, and $x(t) \geq\geq 0$, $t \geq 0$, for all $x_0 \in \overline{\mathbb{R}}_+^n$.

**Proof.** First, define $K_u(t) \triangleq \mathrm{diag}[k_{u1}(t), \cdots, k_{um}(t)]$ and $\phi_u(t) \triangleq [\phi_{u1}(t), \cdots, \phi_{um}(t)]^{\mathrm{T}}$, where

$$k_{ui}(t) = \begin{cases} 0, & \text{if } \hat{u}_i(t) < 0, \\ k_i(t), & \text{otherwise,} \end{cases} \quad i = 1, \cdots, m, \tag{7.43}$$

$$\phi_{ui}(t) = \begin{cases} 0, & \text{if } \hat{u}_i(t) < 0, \\ \phi_i(t), & \text{otherwise,} \end{cases} \quad i = 1, \cdots, m. \tag{7.44}$$

Now, note that with $u(t)$, $t \geq 0$, given by (7.39) it follows from (7.13) that

$$\dot{x}(t) = Ax(t) + BK_u(t)(\hat{x}(t) - x_{\mathrm{d}}) + B\phi_u(t), \quad x(0) = x_0, \quad t \geq 0, \tag{7.45}$$

or, equivalently, using (7.15),

$$\dot{x}(t) = A(x(t) - x_{\mathrm{e}}) + BK_u(t)(\hat{x}(t) - x_{\mathrm{d}}) + B(\phi_u(t) - u_{\mathrm{e}}), \quad x(0) = x_0, \quad t \geq 0. \tag{7.46}$$

Furthermore, note that since $A$ is essentially nonnegative and asymptotically stable, it follows from Theorem 7.2 that there exists a positive *diagonal* matrix $P \triangleq \mathrm{diag}[p_1, \cdots, p_n]$ and a positive-definite matrix $R \in \mathbb{R}^{n \times n}$ such that

$$0 = A^{\mathrm{T}}P + PA + R. \tag{7.47}$$

To show Lyapunov stability of the closed-loop system (7.41), (7.42), and (7.46) consider the Lyapunov function candidate

$$V(x,K,\phi) = (x-x_{\rm e})^{\rm T}P(x-x_{\rm e}) + {\rm tr}(K-K_{\rm g})^{\rm T}Q^{-1}(K-K_{\rm g}) + (\phi-u_{\rm e})^{\rm T}\hat{Q}^{-1}(\phi-u_{\rm e}), \tag{7.48}$$

or, equivalently,

$$V(x,K,\phi) = \sum_{i=1}^{n} p_i(x_i - x_{{\rm e}i})^2 + \sum_{i=1}^{m}\frac{p_ib_i}{q_i}(k_i - k_{{\rm g}_i})^2 + \sum_{i=1}^{m}\frac{p_ib_i}{\hat{q}_i}(\phi_i - u_{{\rm e}i})^2,$$

where $Q = {\rm diag}\left[\frac{q_1}{p_1b_1}, \cdots, \frac{q_m}{p_mb_m}\right]$ and $\hat{Q} = {\rm diag}\left[\frac{\hat{q}_1}{p_1b_1}, \cdots, \frac{\hat{q}_m}{p_mb_m}\right]$. Note that $V(x_{\rm e}, K_{\rm g}, u_{\rm e}) = 0$ and, since $P$, $Q$, and $\hat{Q}$ are positive definite, $V(x,K,\phi) > 0$ for all $(x,K,\phi) \neq (x_{\rm e}, K_{\rm g}, u_{\rm e})$. Furthermore, $V(x,K,\phi)$ is radially unbounded. Now, letting $x(t)$, $t \geq 0$, denote the solution to (7.46) and using (7.41) and (7.42), it follows that the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned}
\dot{V}(x(t),K(t),\phi(t)) =\ & 2(x(t)-x_{\rm e})^{\rm T}P[A(x(t)-x_{\rm e}) + BK_u(t)(\hat{x}(t)-x_{\rm d}) \\
& +B(\phi_u(t)-u_{\rm e})] + 2{\rm tr}(K(t)-K_{\rm g})^{\rm T}Q^{-1}\dot{K}(t) \\
& +2(\phi(t)-u_{\rm e})^{\rm T}\hat{Q}^{-1}\dot{\phi}(t) \\
=\ & -(x(t)-x_{\rm e})^{\rm T}R(x(t)-x_{\rm e}) + 2\sum_{i=1}^{m} p_ib_ik_{ui}(t)(x_i(t)-x_{{\rm d}i})^2 \\
& +2\sum_{i=1}^{m} p_ib_i(x_i(t)-x_{{\rm d}i})(\phi_{ui}(t)-u_{{\rm e}i}) \\
& +2\sum_{i=1}^{m}\frac{p_ib_i}{q_i}(k_i(t)-k_{{\rm g}_i})\dot{k}_i(t) + 2\sum_{i=1}^{m}\frac{p_ib_i}{\hat{q}_i}(\phi_i(t)-u_{{\rm e}i})\dot{\phi}_i(t) \\
=\ & -(x(t)-x_{\rm e})^{\rm T}R(x(t)-x_{\rm e}) \\
& +2\sum_{i=1}^{m} p_ib_i\left[k_{ui}(t)(x_i(t)-x_{{\rm d}i})^2 + \frac{1}{q_i}(k_i(t)-k_{{\rm g}_i})\dot{k}_i(t)\right] \\
& +2\sum_{i=1}^{m} p_ib_i\left[(x_i(t)-x_{{\rm d}i})(\phi_{ui}(t)-u_{{\rm e}i}) + \frac{1}{\hat{q}_i}(\phi_i(t)-u_{{\rm e}i})\dot{\phi}_i(t)\right].
\end{aligned} \tag{7.49}$$

Now, for each $i \in \{1, \cdots, m\}$ and for the two cases given in (7.41) and (7.42), the last two terms on the right-hand side of (7.49) give:

*i)* If $\hat{u}_i(t) < 0$, then $k_{ui}(t) = 0$, $\phi_{ui}(t) = 0$, $\dot{k}_i(t) = 0$, and $\dot{\phi}_i(t) = 0$. Furthermore, since $\phi_i(t) \geq 0$ and $k_i(t) \leq 0$ for all $t \geq 0$, it follows from (7.40) that $\hat{u}_i(t) < 0$ only if $x_i(t) > x_{\mathrm{d}i}$ and hence

$$\begin{aligned} k_{ui}(t)(x_i(t) - x_{\mathrm{d}i})^2 + \frac{1}{q_i}(k_i(t) - k_{\mathrm{g}i})\dot{k}_i(t) &= 0, \\ (x_i(t) - x_{\mathrm{d}i})(\phi_{ui}(t) - u_{\mathrm{e}i}) + \frac{1}{q_i}(\phi_i(t) - u_{\mathrm{e}i})\dot{\phi}_i(t) &= -(x_i(t) - x_{\mathrm{d}i})u_{\mathrm{e}i} \leq 0. \end{aligned}$$

*ii)* Otherwise, $k_{ui}(t) = k_i(t)$ and $\phi_{ui}(t) = \phi_i(t)$ and hence

$$\begin{aligned} & k_{ui}(t)(x_i(t) - x_{\mathrm{d}i})^2 + \frac{1}{q_i}(k_i(t) - k_{\mathrm{g}i})\dot{k}_i(t) = k_{\mathrm{g}i}(x_i(t) - x_{\mathrm{d}i})^2 \leq 0, \\ & (x_i(t) - x_{\mathrm{d}i})(\phi_{ui}(t) - u_{\mathrm{e}i}) + \frac{1}{\hat{q}_i}(\phi_i(t) - u_{\mathrm{e}i})\dot{\phi}_i(t) \\ & \qquad = \begin{cases} -(x_i(t) - x_{\mathrm{d}i})u_{\mathrm{e}i} \leq 0, & \text{if } \phi_i(t) = 0 \text{ and } x_i(t) \geq x_{\mathrm{d}i}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence, it follows that in either case

$$\begin{aligned} \dot{V}(x(t), K(t), \phi(t)) &\leq -(x(t) - x_{\mathrm{e}})^{\mathrm{T}} R (x(t) - x_{\mathrm{e}}) \\ &\leq 0, \quad t \geq 0, \end{aligned} \tag{7.50}$$

which proves that the solution $(x(t), K(t), \phi(t)) \equiv (x_{\mathrm{e}}, K_{\mathrm{g}}, u_{\mathrm{e}})$ to (7.41), (7.42), and (7.46) is Lyapunov stable. Furthermore, since $R > 0$ it follows from Theorem 2 of [42] that $x(t) \to x_{\mathrm{e}}$ as $t \to \infty$ for all $x_0 \in \overline{\mathbb{R}}_+^n$.

Finally, $u(t) \geq\geq 0$, $t \geq 0$, is a restatement of (7.39). Now, since $B \geq\geq 0$ and $u(t) \geq\geq 0$, $t \geq 0$, it follows from Proposition 7.4 that $x(t) \geq\geq 0$ for all $t \geq 0$ and $x_0 \in \overline{\mathbb{R}}_+^n$. □

**Remark 7.2.** As in the case of Theorem 7.3, the conditions in Theorem 7.6 imply that $x(t) \to x_{\mathrm{e}}$ as $t \to \infty$ and hence it follows from (7.41) and (7.42) that $(x(t), K(t), \phi(t)) \to \mathcal{M} \triangleq \{(x, K, \phi) \in \mathbb{R}^n \times \mathbb{R}^{m\times m} \times \mathbb{R}^m : x = x_{\mathrm{e}},\ \dot{K} = 0,\ \dot{\phi} = 0\}$ as $t \to \infty$.

It is important to note that the adaptive control law (7.39), (7.41), and (7.42) does *not* require the explicit knowledge of the constant vector $u_e$; even though Theorem 7.6 requires the existence of $x_u \in \mathbb{R}_+^{n-m}$ and $u_e \in \mathbb{R}_+^m$ such that condition (7.15) holds. Furthermore, the control input $u_i(t)$, $t \geq 0$, is always nonnegative regardless of the values of $x_i(t)$, $k_i(t)$, and $\phi_i(t)$, $t \geq 0$, $i = 1, \cdots, m$, which ensures that the closed-loop plant states remain nonnegative by Proposition 7.4. Finally, it should be noted that since $A$ is asymptotically stable, the adaptive gains $k_i(t)$, $t \geq 0$, $i = 1, \cdots, m$, only change the performance of the closed-loop system and do not destroy stability even when we set $\dot{k}_i(t) = 0$, $t \geq 0$, with $k_i(0) \leq 0$, $i = 1, \cdots, m$.

## 7.5. Adaptive Control for General Anesthesia

The potential clinical applications of adaptive control for pharmacology in general, and anesthesia and critical care medicine in particular, are clearly apparent. Specifically, monitoring and controlling the levels of consciousness in surgery is of particular importance. Propofol is an intravenous anesthetic that has been used for both induction and maintenance of general anesthesia [54]. A simple yet effective patient model for the disposition of propofol is based on the three-compartment mammillary model shown in Figure 7.1 with the first compartment acting as the central compartment and the remaining two compartments exchanging with the central compartment [68, 161]. The three-compartment mammillary system provides a pharmacokinetic model for a patient describing the distribution of propofol into the central compartment (identified with the intravascular blood volume as well as highly perfused organs) and the other various tissue groups of the body. A mass balance for the whole compartmental system yields

$$\dot{x}_1(t) = -(a_{11} + a_{21} + a_{31})x_1(t) + a_{12}x_2(t) + a_{13}x_3(t) + u(t), \quad x_1(0) = x_{10}, \quad t \geq 0, \tag{7.51}$$

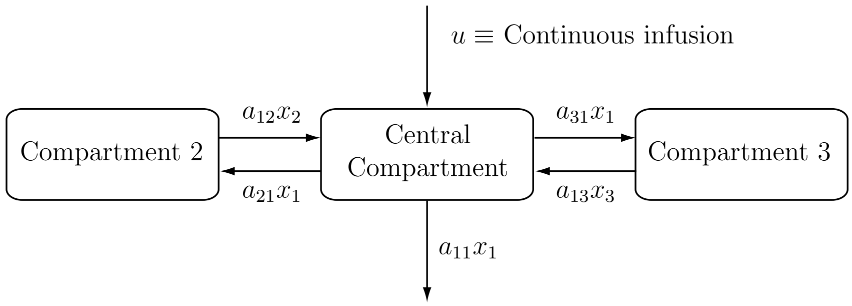



**Figure 7.1**: Three-compartment mammillary model for disposition of propofol

$$\dot{x}_2(t) = a_{21}x_1(t) - a_{12}x_2(t), \quad x_2(0) = x_{20}, \tag{7.52}$$

$$\dot{x}_3(t) = a_{31}x_1(t) - a_{13}x_3(t), \quad x_3(0) = x_{30}, \tag{7.53}$$

where $x_1(t)$, $x_2(t)$, $x_3(t)$, $t \geq 0$, are the masses in grams of propofol in the central compartment and compartments 2 and 3, respectively, $u(t)$, $t \geq 0$, is the infusion rate in grams/min of the anesthetic (propofol) into the central compartment, $a_{ij} > 0$, $i \neq j$, $i, j = 1, 2, 3$, are the rate constants in $\text{min}^{-1}$ for drug transfer between compartments, and $a_{11} > 0$ in $\text{min}^{-1}$ is the rate constant for elimination from the central compartment. Even though these transfer and loss coefficients are positive, they can be uncertain due to patient gender, weight, pre-existing disease, age, and concomitant medication. Hence, adaptive control for propofol regulation during surgery can significantly improve the outcomes for drug administration over manual control.

It has been reported in [239] that a 2.5–6 $\mu$g/m$\ell$ blood concentration level of propofol is required during the maintenance stage in general anesthesia depending on patient fitness and extent of surgical stimulation. Hence, continuous infusion control is required for maintaining this desired level of anesthesia. Here we assume that the transfer and loss coefficients $a_{11}$, $a_{12}$, $a_{21}$, $a_{13}$, and $a_{31}$ are unknown and our objective is to regulate the propofol concentration level of the central compartment to the desired level of 4 $\mu$g/m$\ell$ in the face of system uncertainty. Furthermore, since propofol mass in the blood plasma cannot be measured directly, we measure the concentration of

**Table 7.1**: Pharmacokinetic parameters [68]

| Set | $a_{11}$ ($\text{min}^{-1}$) | $a_{21}$ ($\text{min}^{-1}$) | $a_{12}$ ($\text{min}^{-1}$) | $a_{31}$ ($\text{min}^{-1}$) | $a_{13}$ ($\text{min}^{-1}$) |
|---|---|---|---|---|---|
| A | 0.152 | 0.207 | 0.092 | 0.040 | 0.0048 |
| B | 0.119 | 0.114 | 0.055 | 0.041 | 0.0033 |

propofol in the central compartment; that is, $x_1/V_\text{c}$, where $V_\text{c}$ is the volume in liters of the central compartment. As noted in [161], $V_\text{c}$ can be approximately calculated by $V_\text{c} = (0.159\,\ell/\text{kg})(M\,\text{kg})$, where $M$ is the weight (mass) in kilograms of the patient. In our control design we assume $M = 70$ kg so that the desired level of propofol mass in the central component is given by $x_{\text{d}1} = (4\,\mu\text{g/m}\ell)(0.159\,\ell/\text{kg})(70\,\text{kg}) = 44.52$ mg.

Next, note that (7.51)–(7.53) can be written in state space form (7.13) with $x = [x_1, x_2, x_3]^\text{T}$,

$$A = \begin{bmatrix} -(a_{11} + a_{21} + a_{31}) & a_{12} & a_{13} \\ a_{21} & -a_{12} & 0 \\ a_{31} & 0 & -a_{13} \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{7.54}$$

Now, it can be shown that for $x_{\text{d}1}/V_\text{c} = 4$ $\mu$g/m$\ell$, all the conditions of Theorem 7.6 are satisfied. Hence, it follows from Theorem 7.6 that the adaptive dynamic feedback controller (7.39) with update laws (7.41), (7.42) guarantees that $u(t) \geq 0$ for all $t \geq 0$ and $x_1(t) \to x_{\text{d}1}$ as $t \to \infty$ for any (uncertain) positive values of the transfer and loss coefficients. To illustrate the robustness properties of the proposed adaptive control law, we use the average set of pharmacokinetic parameters given in [68] for 29 patients requiring general anesthesia for noncardiac surgery. For our design we switch from Set A to Set B given in Table 7.1 at $t = 25$ min. With $q_1 = 1000$ g$^{-2}$ min$^{-2}$, $\hat{q}_1 = 0.5$ min$^{-2}$, and initial conditions $x(0) = [0, 0, 0]^\text{T}$ g, $k_1(0) = 0$ min$^{-1}$, and $\phi_1(0) = 0.01$ g/min$^{-1}$, Figure 7.2 shows the masses of propofol in all three compartments versus time. Figure 7.3 shows the propofol concentration in the central compartment and the control signal (propofol infusion rate) versus time. Finally, Figure 7.4 shows the adaptive gain history versus time.

In the above simulations, the adaptive controller was designed using a pharma-

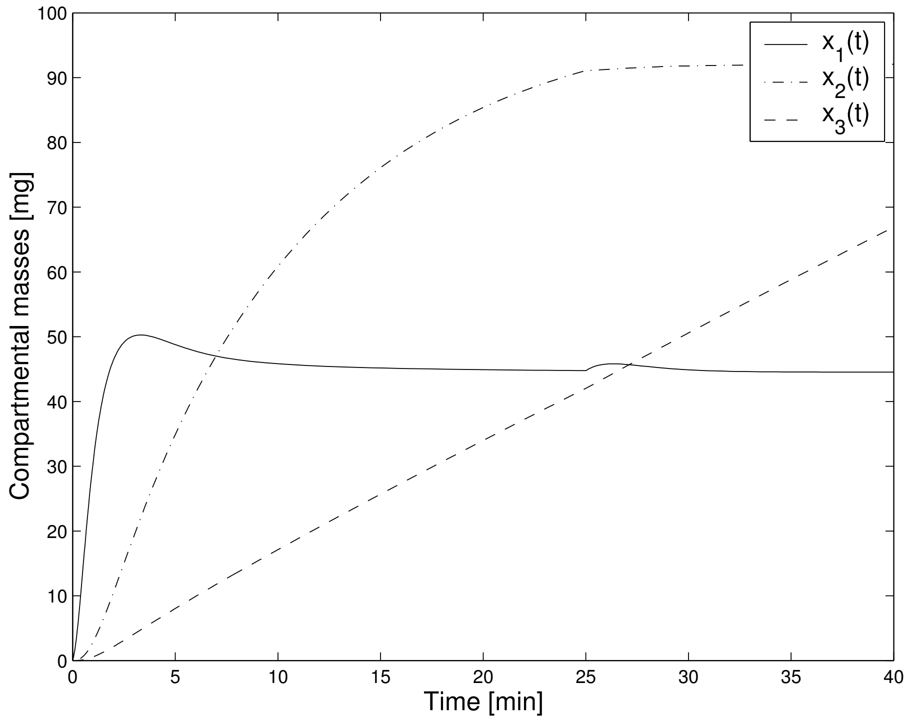



**Figure 7.2**: Compartmental masses versus time

cokinetic model (a model describing drug concentrations as a function of time and dose) for the disposition of propofol. Even though propofol concentration levels in the blood plasma are a good indication of the depth of anesthesia, they cannot be measured in real time during surgery. Furthermore, we are more interested in drug *effect* (depth of hypnosis) rather than drug *concentration*. Hence, we consider a more realistic model involving pharmacokinetics (drug concentration as a function of time) and pharmacodynamics (drug effect as a function of concentration) for control of anesthesia. Specifically, we use an electroencephalogram (EEG) signal as a measure of drug effect of anesthetic compounds on the brain [215]. Since electroencephalography provides real-time monitoring of the central nervous system activity, it can be used to quantify levels of consciousness and hence is amenable for feedback (closed-loop) control in general anesthesia. Recently, a new EEG indicator, the bispectral index (BIS), has been proposed as a measure of anesthetic effect [174]. This index

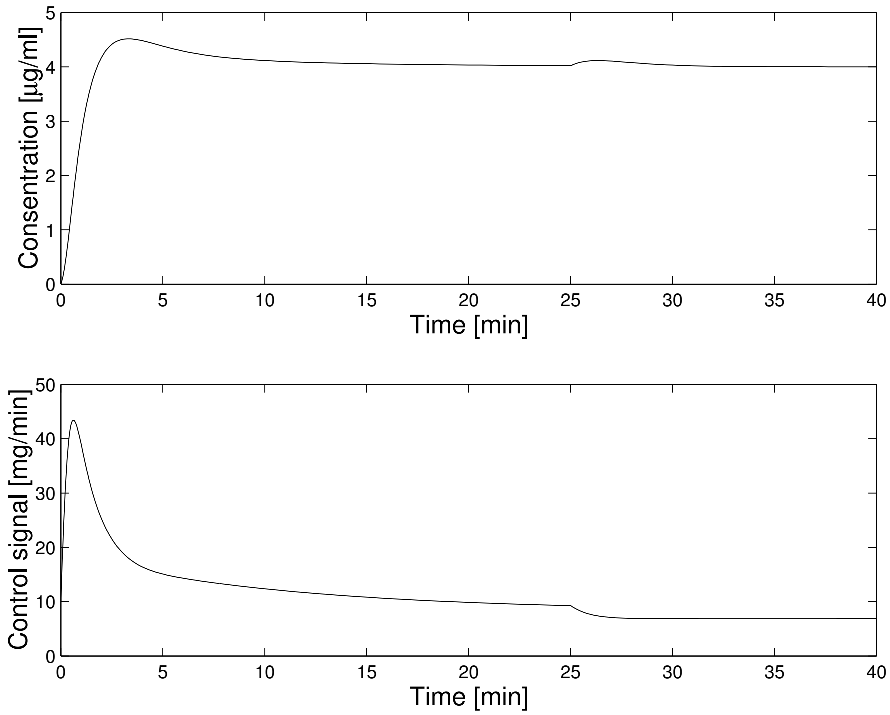



**Figure 7.3**: Drug concentration in the central compartment and control signal (infusion rate) versus time

quantifies the nonlinear relationships between the component frequencies in the electroencephalogram, as well as analyzing their phase and amplitude. The BIS signal is a nonlinear monotonically decreasing function of the level of consciousness and is given by

$$\mathrm{BIS}(c_{\mathrm{eff}}) = \mathrm{BIS}_0 \left(1 - \frac{c_{\mathrm{eff}}^{\gamma}}{c_{\mathrm{eff}}^{\gamma} + \mathrm{EC}_{50}^{\gamma}}\right), \tag{7.55}$$

where $\mathrm{BIS}_0$ denotes the baseline (awake state) value and, by convention, is typically assigned a value of 100, $c_{\mathrm{eff}}$ is the propofol concentration in grams/liter in the effect site compartment (brain), $\mathrm{EC}_{50}$ is the concentration at half maximal effect and represents the patient's sensitivity to the drug, and $\gamma$ determines the degree of nonlinearity in (7.55). Here, the effect site compartment is introduced as a correlate between the central compartment concentration and the central nervous system concentration [205]. The effect site compartment concentration is related to the concentration

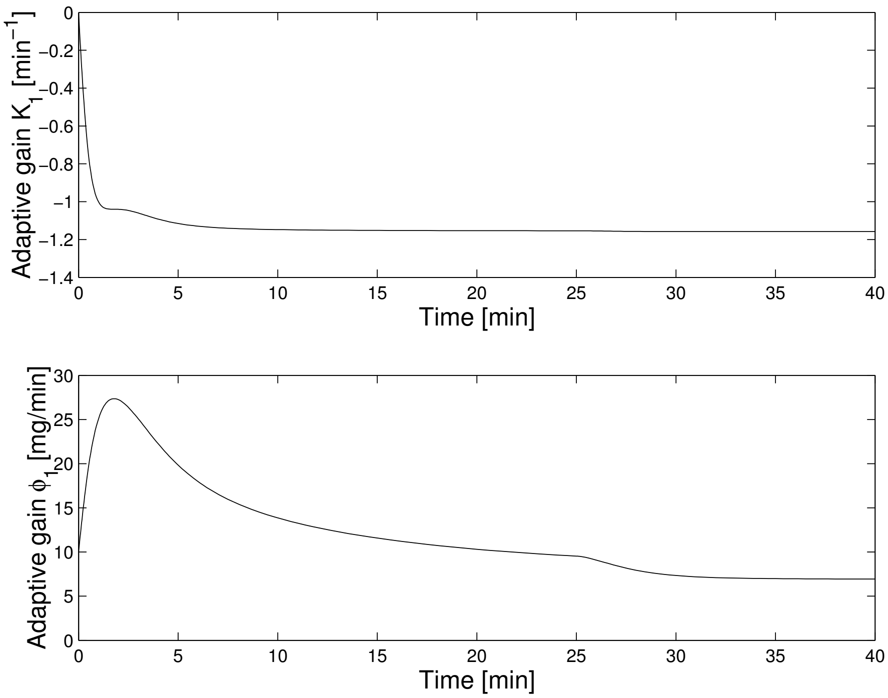



**Figure 7.4**: Adaptive gain history versus time

in the central compartment by the first-order delay model

$$\dot{c}_{\text{eff}}(t) = a_{\text{eff}}(x_1(t)/V_{\text{c}} - c_{\text{eff}}(t)), \quad c_{\text{eff}}(0) = x_1(0), \quad t \geq 0, \tag{7.56}$$

where $a_{\text{eff}}$ in min$^{-1}$ is a positive time constant. Assuming $x_1(0) = 0$, it follows that

$$c_{\text{eff}}(t) = \int_0^t e^{-a_{\text{eff}}(t-s)} a_{\text{eff}} x_1(s)/V_{\text{c}} \, \mathrm{d}s. \tag{7.57}$$

In reality, the effect site compartment equilibrates with the central compartment in a matter of a few minutes. The parameters $a_{\text{eff}}$, $\text{EC}_{50}$, and $\gamma$ are determined by data fitting and vary from patient to patient. BIS index values of 0 and 100 correspond, respectively, to an isoelectric EEG signal and an EEG signal of a fully conscious patient; while the range between 40 and 60 indicates a moderate hypnotic state [65].

In the following numerical simulation we set $\text{EC}_{50} = 3.4$ $\mu$g/m$\ell$, $\gamma = 3$, and $\text{BIS}_0 = 100$, so that the BIS signal is shown in Figure 7.5. The target (desired) BIS value, $\text{BIS}_{\text{target}}$, is set at 50. In this case, the linearized BIS function about the target

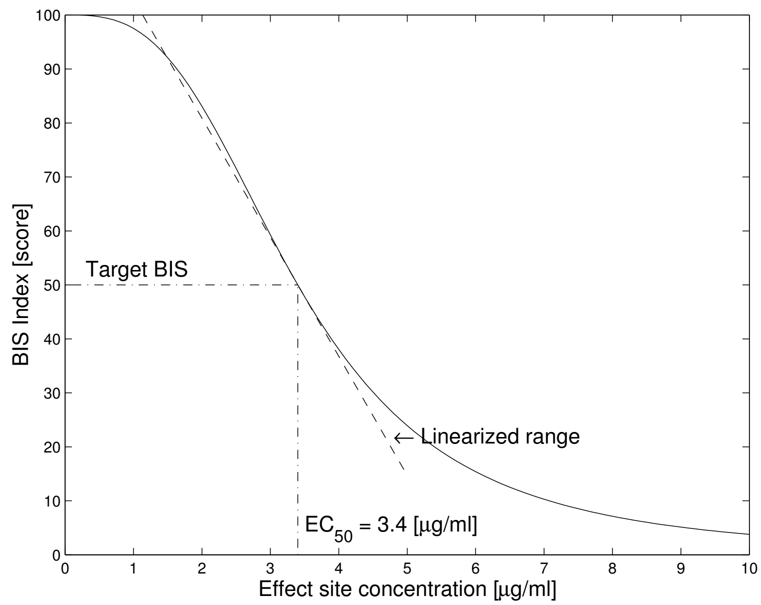



**Figure 7.5**: BIS index versus effect site concentration

BIS value is given by

$$\begin{aligned}\mathrm{BIS}(c_{\mathrm{eff}}) &\simeq \mathrm{BIS}(\mathrm{EC}_{50}) - \mathrm{BIS}_0 \cdot \mathrm{EC}_{50}^{\gamma} \cdot \left.\frac{\gamma c_{\mathrm{eff}}^{\gamma-1}}{(c_{\mathrm{eff}}^{\gamma} + \mathrm{EC}_{50}^{\gamma})^2}\right|_{c_{\mathrm{eff}}=\mathrm{EC}_{50}} \cdot (c_{\mathrm{eff}} - \mathrm{EC}_{50}) \\ &= 125 - 22.06 c_{\mathrm{eff}}. \end{aligned} \tag{7.58}$$

Furthermore, for simplicity of exposition, we assume that the effect site compartment equilibrates instantaneously with the central compartment; that is, we assume that $a_{\mathrm{eff}} \to \infty$, so that (7.57) reduces to $c_{\mathrm{eff}}(t) = x_1(t)/V_{\mathrm{c}}$, $t \geq 0$. Now, using the adaptive feedback controller

$$u(t) = \max\{0, \hat{u}(t)\}, \tag{7.59}$$

where

$$\hat{u}(t) = -k(t)(\mathrm{BIS}(t) - \mathrm{BIS}_{\mathrm{target}}) + \phi(t), \tag{7.60}$$

$k(t) \in \mathbb{R}$, $t \geq 0$, and $\phi(t) \in \mathbb{R}$, $t \geq 0$, with update laws

$$\dot{k}(t) = \begin{cases} 0, & \text{if } \hat{u}(t) < 0, \\ -q_{\mathrm{BIS}}(\mathrm{BIS}(t) - \mathrm{BIS}_{\mathrm{target}})^2, & \text{otherwise,} \end{cases} \quad k(0) \leq 0, \tag{7.61}$$

$$\dot{\phi}(t) = \begin{cases} 0, & \text{if } \phi(t) = 0 \text{ and } \mathrm{BIS}(t) > \mathrm{BIS}_{\mathrm{target}}, \\ & \text{or if } \hat{u}(t) \leq 0, \\ \hat{q}_{\mathrm{BIS}}(\mathrm{BIS}(t) - \mathrm{BIS}_{\mathrm{target}}), & \text{otherwise,} \end{cases} \quad \phi(0) \geq 0,$$

(7.62)

where $q_{\rm BIS}$ and $\hat{q}_{\rm BIS}$ are arbitrary positive constants, it follows from Theorem 7.6 that the control input (anesthetic infusion rate) $u(t)$ is nonnegative for all $t \geq 0$ and $\mathrm{BIS}(t) \to \mathrm{BIS}_{\rm target}$ as $t \to \infty$ for any (uncertain) positive values of the transfer and loss coefficients in the range of $c_{\rm eff}$ where the linearized BIS equation (7.58) is valid. It is important to note that during actual surgery the BIS signal is obtained directly from the EEG and not (7.55). Furthermore, since our adaptive controller only requires the error signal $\mathrm{BIS}(t) - \mathrm{BIS}_{\rm target}$ over the linearized range of (7.55), we do not require knowledge of the slope of the linearized equation (7.58), nor do we require knowledge of the parameters $\gamma$ and $\mathrm{EC}_{50}$. Once again, for our design we assume $M = 70$ kg and we switch from Set A to Set B given in Table 7.1 at $t = 25$ min. Furthermore, we assume that at $t = 25$ min the pharmacodynamic parameters $\mathrm{EC}_{50}$ and $\gamma$ are switched from 3.4 $\mu$g/m$\ell$ and 3 to 4.0 $\mu$g/m$\ell$ and 4, respectively. Here we consider noncardiac surgery since cardiac surgery often utilizes hypothermia which itself changes the BIS signal. With $q_{\rm BIS} = 1 \times 10^{-6}$ g/min$^2$, $\hat{q}_{\rm BIS} = 1 \times 10^{-3}$ g/min$^2$, and initial conditions $x(0) = [0, 0, 0]^{\rm T}$ g, $k(0) = 0$ g/min, and $\phi(0) = 0.01$ g/min, Figure 7.6 shows the masses of propofol in all three compartments versus time. Figure 7.7 shows the BIS index versus time. Figure 7.8 shows the propofol concentration in the central compartment and the control signal (propofol infusion rate) versus time. Finally, Figure 7.9 shows the adaptive gain history versus time.

## 7.6. Conclusion

Nonnegative and compartmental systems are widely used to capture system dynamics involving the interchange of mass and energy between homogeneous subsystems or compartments. Thus, it is not surprising that nonnegative and compartmental models are remarkably effective in describing the dynamical behavior of biological and

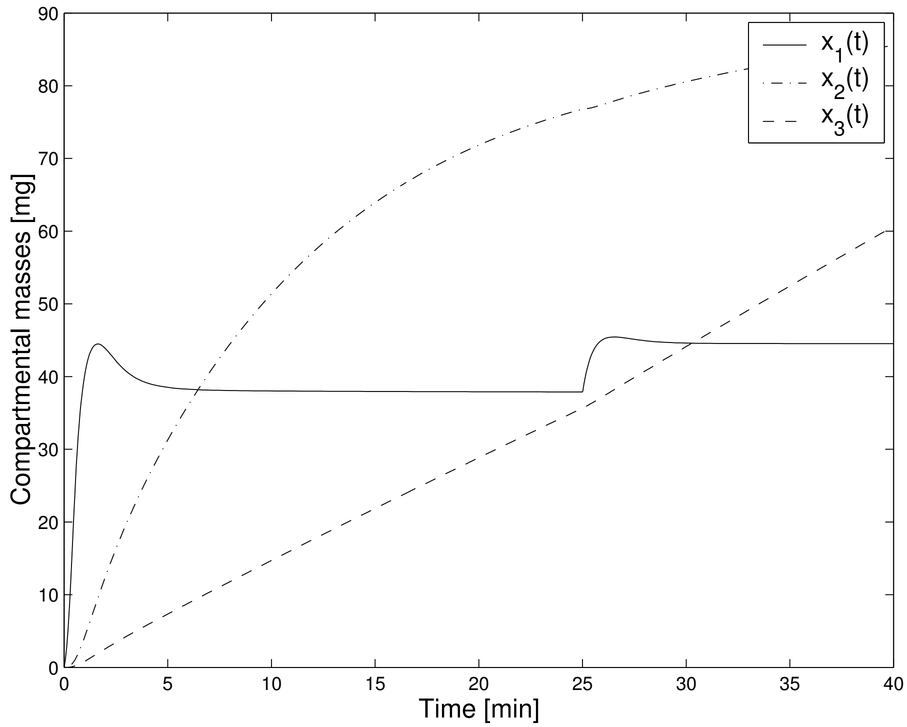



**Figure 7.6**: Compartmental masses versus time

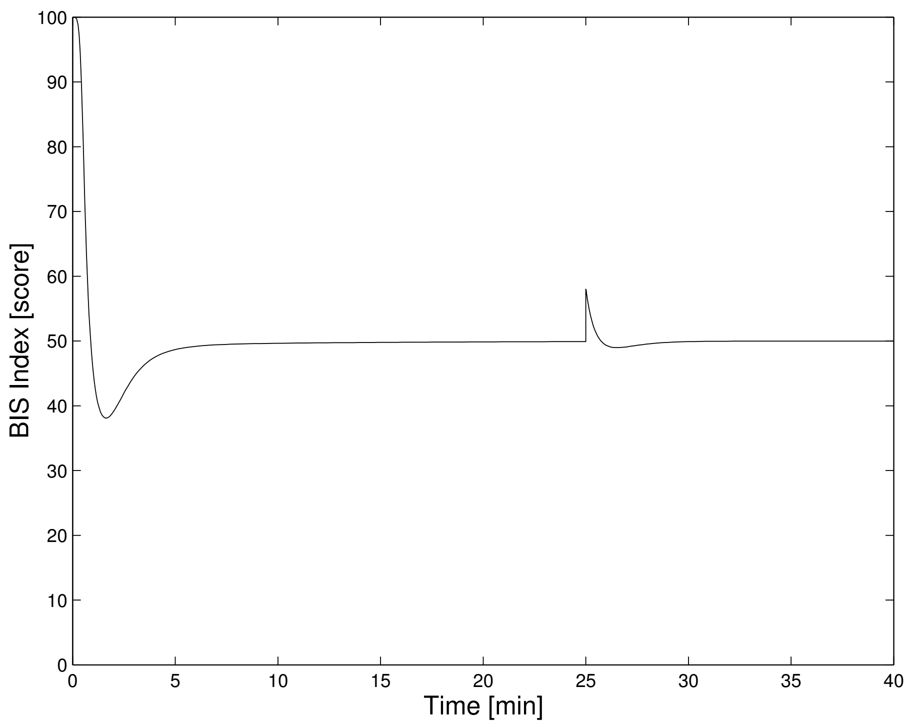



**Figure 7.7**: BIS index versus time

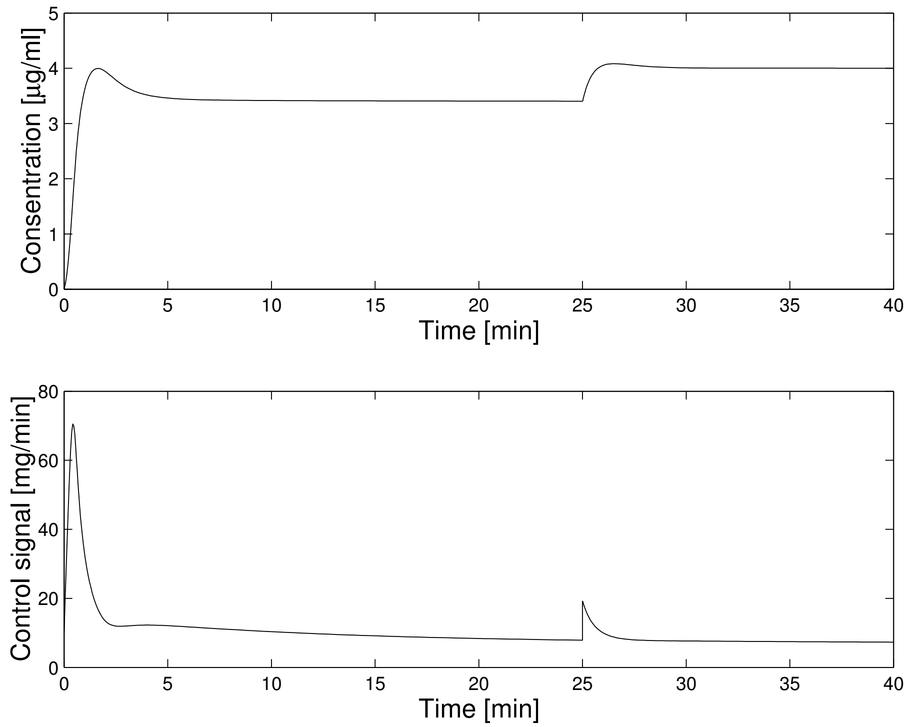



**Figure 7.8**: Drug concentration in the central compartment and control signal (infusion rate) versus time

physiological systems. In this chapter, we developed an adaptive control framework for adaptive set-point regulation of linear uncertain nonnegative and compartmental systems. Using Lyapunov methods the proposed framework was shown to guarantee partial asymptotic set-point stability of the closed-loop system while additionally guaranteeing the nonnegativity of the closed-loop system states associated with the plant dynamics. Finally, using a three-compartment mammillary patient model for the disposition of propofol, the proposed adaptive control framework was used to monitor and control a desired constant level of consciousness for noncardiac surgery. Even though measurement noise was not addressed in our framework, it should be noted that EEG signals may have as much as 10% variation due to noise. In particular, the BIS signal may be corrupted by electromyographic noise; that is, signals emanating from muscle rather than the central nervous system. Clinical implementation of the proposed algorithm would thus have to include muscle paralysis to minimize the

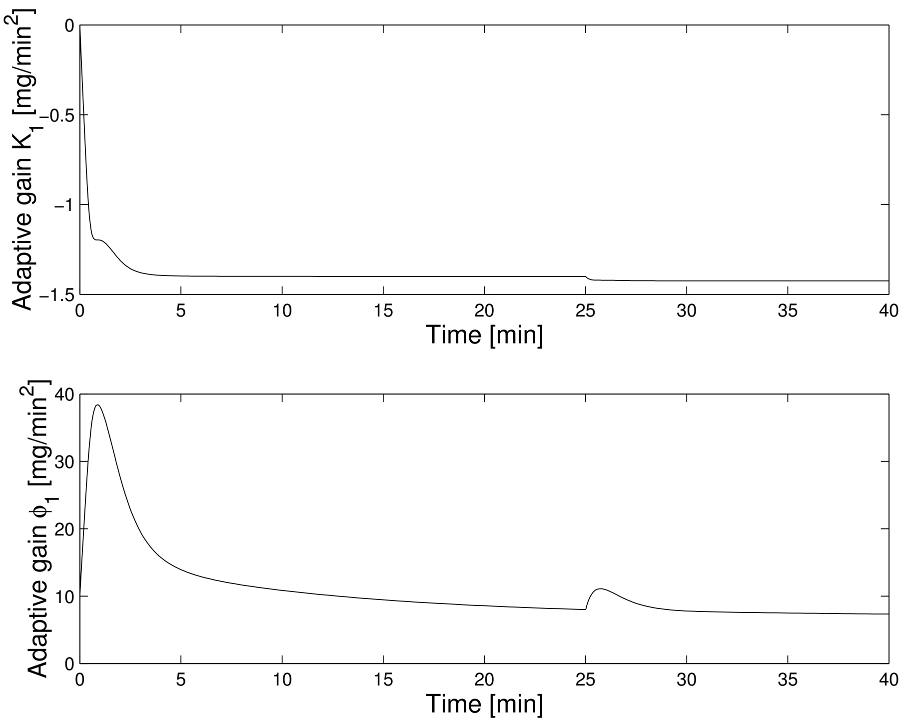



**Figure 7.9**: Adaptive gain history versus time

effects of electromyographic noise.

# Chapter 8

# Direct Adaptive Control of Nonnegative and Compartmental Dynamical Systems with Time Delay

## 8.1. Introduction

As discussed in Chapter 7, nonnegative and compartmental models play a key role in the understanding of many processes in biological and medical sciences [6, 19, 24, 62, 70, 75, 123, 124, 164, 166, 172, 182, 187, 203]. Compartmental systems are modeled by interconnected subsystems (or compartments) which exchange variable nonnegative quantities of material with conservation laws describing transfer, accumulation, and outflows between compartments and the environment. In many compartmental pharmacokinetic system models, transfers between compartments are assumed to be instantaneous; that is, the model does not account for material in transit. Even though this is a valid assumption for certain biological and physiological systems, it is not true in general; especially in pharmacokinetic and pharmacodynamic models. For example, if a bolus of drug is injected into the circulation and we seek its concentration level in the extracellular and intercellular space of some organ, there exists a time lag before it is detected in that organ [74,123,165]. In this case, assuming instantaneous mass transfer between compartments will yield erroneous models. Hence, to accurately describe the distribution of pharmacological agents in the human body, it is necessary to include in any mathematical compartmental pharmacokinetic model some information of the past system states. In this case the state of the system at any given time involves a *piece of trajectories* in the space of continuous functions defined on an interval in the nonnegative orthant. This of course leads to (infinite-dimensional) delay dynamical systems [55, 101, 143, 181].

In Chapter 7 (see also [85,86]), we present a direct adaptive control framework for set-point regulation of linear nonnegative and compartmental systems with applications to clinical pharmacology. In this chapter, we extend the results of Chapter 7 to the case of nonnegative and compartmental dynamical systems with unknown time delay. Specifically, we develop a Lyapunov-Krasovskii-based direct adaptive control framework for guaranteeing set-point regulation for linear uncertain nonnegative and compartmental dynamical systems with unknown time delay. The specific focus of the chapter is on pharmacokinetic models and their applications to drug delivery systems. In particular, we develop direct adaptive controllers with nonnegative control inputs as well as adaptive controllers with the absence of such a restriction. Finally, we demonstrate the framework on a drug delivery model for general anesthesia that involves system time delays.

## 8.2. Mathematical Preliminaries

In this section we introduce some key results concerning linear nonnegative dynamical systems with time delay [81, 83] that are necessary for developing the main results of this chapter. Specifically, consider a controlled linear time-delay dynamical system $\mathcal{G}$ of the form

$$\dot{x}(t) = Ax(t) + A_{\mathrm{d}}x(t-\tau) + Bu(t), \quad x(\theta) = \eta(\theta), \quad -\tau \le \theta \le 0, \quad t \ge 0, \qquad (8.1)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $t \ge 0$, $A \in \mathbb{R}^{n\times n}$, $A_{\mathrm{d}} \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $\tau \ge 0$, $\eta(\cdot) \in \mathcal{C} = \mathcal{C}([-\tau,0],\mathbb{R}^n)$ is a continuous vector-valued function specifying the initial state of the system, and $\mathcal{C}([-\tau,0],\mathbb{R}^n)$ denotes a Banach space of continuous functions mapping the interval $[-\tau,0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. Note that the state of (8.1) at time $t$ is the *piece of trajectories* $x$ between $t-\tau$ and $t$, or, equivalently, the *element* $x_t$ in the space of continuous functions defined on the interval $[-\tau,0]$ and taking values in $\mathbb{R}^n$; that is, $x_t \in \mathcal{C}([-\tau,0],\mathbb{R}^n)$, where $x_t(\theta) \triangleq x(t+\theta)$, $\theta \in [-\tau,0]$. Furthermore, since for a given time $t$ the piece of the trajectories $x_t$ is defined on $[-\tau,0]$, the uniform norm $|||x_t||| = \sup_{\theta\in[-\tau,0]} \|x(t+\theta)\|$, where $\|\cdot\|$ denotes the Euclidean vector norm, is used for the definitions of Lyapunov and asymptotic stability of (8.1) with $u(t) \equiv 0$. For further details see [101, 143]. Finally, note that since $\eta(\cdot)$ is continuous it follows from Theorem 2.1 of [101, p. 14] that there exists a unique solution $x(\eta)$ defined on $[-\tau,\infty)$ that coincides with $\eta$ on $[-\tau,0]$ and satisfies (8.1) for $t \ge 0$.

The following theorem gives necessary and sufficient conditions for asymptotic stability of a linear time-delay nonnegative dynamical system $\mathcal{G}$ given by (8.1) in the case where $u(t) \equiv 0$. For this result, the following definition is needed.

**Definition 8.1.** The linear delay dynamical system given by (8.1) is *nonnegative* if for every $\eta(\cdot) \in \mathcal{C}_+$, and $u(t) \ge\ge 0$, $t \ge 0$, where $\mathcal{C}_+ \triangleq \{\psi(\cdot) \in \mathcal{C} : \psi(\theta) \ge\ge 0,\ \theta \in [-\tau,0]\}$, the solution $x(t)$, $t \ge 0$, to (8.1) is nonnegative.

**Theorem 8.1** [81, 83]. Consider the linear nonnegative dynamical system $\mathcal{G}$ given by (8.1) where $A \in \mathbb{R}^{n\times n}$ is essentially nonnegative, $A_{\mathrm{d}} \in \mathbb{R}^{n\times n}$ is nonnegative, and

$u(t) \equiv 0$. Then, $\mathcal{G}$ is asymptotically stable for all $\tau \in [0, \infty)$ if and only if there exist $p, r \in \mathbb{R}^n$ such that $p >> 0$ and $r >> 0$ satisfy

$$0 = (A + A_{\rm d})^{\rm T} p + r. \tag{8.2}$$

Next, we consider a subclass of nonnegative systems; namely, compartmental systems.

**Definition 8.2** [81, 83]. The linear time-delay dynamical system (8.1) is called a *compartmental dynamical system* if $A$ and $A_{\rm d}$ are given by

$$A_{(i,j)} = \begin{cases} -\sum_{k=1}^{n} a_{ki}, & i = j, \\ 0, & i \neq j, \end{cases} \qquad A_{{\rm d}(i,j)} = \begin{cases} 0, & i = j, \\ a_{ij}, & i \neq j, \end{cases} \tag{8.3}$$

where $a_{ii} \geq 0$, $i \in \{1, \cdots, n\}$, denotes the loss coefficient of the $i$th compartment and $a_{ij} \geq 0$, $i \neq j$, $i, j \in \{1, \cdots, n\}$, denotes the transfer coefficient from the $j$thcompartment to the $i$th compartment.

Note that if (8.1) is a compartmental system, then $A + A_{\rm d}$ is a compartmental matrix. In pharmacokinetic applications, an important subclass of compartmental systems are *mammillary* systems [123]. Mammillary systems are comprised of a *central compartment* from which there is outflow and which exchanges material reversibly with one or more *peripheral compartments*. An *inflow-closed* (i.e., $u(t) \equiv 0$) time-delay mammillary system is given by (8.1) with $A$ and $A_{\rm d}$ given by

$$A = \operatorname{diag}\Big[-\sum_{j=1}^{n} a_{j1}, -a_{12}, \cdots, -a_{1n}\Big], \tag{8.4}$$

$$A_{{\rm d}(i,j)} = \begin{cases} 0, & i = j, \\ 0, & i \neq 1 \text{ and } j \neq 1, \\ a_{ij}, & \text{otherwise}, \end{cases} \tag{8.5}$$

where the transfer coefficients $a_{ij}$, $i, j = 1, \cdots, n$, are positive and the loss coefficient

$a_{11}$ is nonnegative. In this case,

$$A + A_{\mathrm{d}} = \begin{bmatrix} -\sum_{j=1}^{n} a_{j1} & a_{12} & \cdots & a_{1n} \\ a_{21} & -a_{12} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & 0 & \cdots & -a_{1n} \end{bmatrix}. \quad (8.6)$$

The following proposition is needed for the main results of the chapter.

**Proposition 8.1** [81, 83]. The linear delay dynamical system $\mathcal{G}$ given by (8.1) is *nonnegative* if and only if $A \in \mathbb{R}^{n\times n}$ is essentially nonnegative, $A_{\mathrm{d}} \in \mathbb{R}^{n\times n}$ is nonnegative, and $B \in \mathbb{R}^{n\times m}$ is nonnegative.

It follows from Proposition 8.1 that the control input signal $Bu(t)$, $t \geq 0$, needs to be nonnegative to guarantee the nonnegativity of the state of (8.1). This is due to the fact that when the initial state of (8.1) belongs to the boundary of the nonnegative orthant, a negative input can destroy the nonnegativity of the state of (8.1). Alternatively, however, if the initial state is in the interior of the nonnegative orthant, then it follows from continuity of solutions with respect to the system initial conditions that, over a small interval of time, nonnegativity of the state of (8.1) is guaranteed *irrespective* of the sign of each element of the control input $Bu(t)$ over this time interval. However, unlike open-loop control wherein lack of coordination between the input and the state necessitates nonnegativity of the control input, a *feedback* control signal predicated on the system state variables allows for the anticipation of loss of nonnegativity of the state. Hence, state feedback control signals can take negative values while assuring nonnegativity of the system states. For further discussion of the above fact see [53, 86].

Next, we present a time-varying extension to Proposition 8.1 needed for the main theorems of this chapter. Specifically, we consider the linear time-varying delay dynamical system

$$\dot{x}(t) = A(t)x(t) + A_{\rm d}(t)x(t-\tau) + Bu(t), \quad x(\theta) = \eta(\theta), \quad \theta \in [-\tau, 0], \quad t \geq 0, \tag{8.7}$$

where $A : [0,\infty) \to \mathbb{R}^{n\times n}$ and $A_{\rm d} : [0,\infty) \to \mathbb{R}^{n\times n}$ are continuous. For the following result the definition of nonnegativity holds with (8.1) replaced by (8.7).

**Proposition 8.2.** Consider the time-varying delay dynamical system (8.7) where $A : [0,\infty) \to \mathbb{R}^{n\times n}$ and $A_{\rm d} : [0,\infty) \to \mathbb{R}^{n\times n}$ are continuous. If for every $t \in [0,\infty)$, $A : [0,\infty) \to \mathbb{R}^{n\times n}$ is essentially nonnegative, $A_{\rm d} : [0,\infty) \to \mathbb{R}^{n\times n}$ is nonnegative, $B \in \mathbb{R}^{n\times m}$ is nonnegative, and $u(t)$ is nonnegative, then the solution $x(t)$, $t \geq 0$, to (8.7) is nonnegative.

**Proof.** The result is a direct consequence of the nonlinear analogue to Proposition 8.1 by equivalently representing the time-varying delay dynamical system (8.7) as an autonomous nonlinear time-delay system by appending another state to represent time. Specifically, defining $y(t) \triangleq t$ it follows that (8.7) may be rewritten as

$$\dot{x}(t) = A(y(t))x(t) + A_{\rm d}(y(t))x(t-\tau) + Bu(t), \quad x(\theta) = \eta(\theta), \quad \theta \in [-\tau, 0], \quad t \geq 0, \tag{8.8}$$

$$\dot{y}(t) = 1, \quad y(0) = 0, \tag{8.9}$$

or, equivalently,

$$\dot{\tilde{x}}(t) = f(\tilde{x}(t)) + G(\tilde{x}(t))\tilde{x}(t-\tau) + \tilde{B}u(t), \quad \tilde{x}(\theta) = \begin{bmatrix} \eta(\theta) \\ 0 \end{bmatrix}, \quad \theta \in [-\tau, 0], \quad t \geq 0, \tag{8.10}$$

where, $\tilde{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $f(\tilde{x}) = \begin{bmatrix} A(y)x \\ 1 \end{bmatrix}$, $G(\tilde{x}) = \begin{bmatrix} A_{\rm d}(y) & 0 \\ 0 & 0 \end{bmatrix}$, and $\tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$. Now, the result is a direct consequence of the nonlinear analogue to Proposition 3.1 of [83]. □

Since stabilization of nonnegative systems naturally deals with equilibrium points in the interior of the nonnegative orthant $\overline{\mathbb{R}}_+^n$, the following proposition provides necessary conditions for the existence of an interior equilibrium point $x_e \in \mathbb{R}_+^n$ of (8.1) in terms of the stability properties of the system matrices $A$ and $A_d$. For this result, recall that a matrix $M \in \mathbb{R}^{n \times n}$ is *semistable* if and only if $\lim_{t\to\infty} e^{Mt}$ exists.

**Proposition 8.3.** Consider the nonnegative time-delay dynamical system (8.1) and assume there exist $x_e \in \mathbb{R}_+^n$ and $u_e \in \overline{\mathbb{R}}_+^m$ such that

$$0 = (A + A_d)x_e + Bu_e. \tag{8.11}$$

Then, $A + A_d$ is semistable.

**Proof.** The proof is a direct consequence of *ii*) of Theorem 3.2 in [75] with $A$ replaced by $A + A_d$, $p = x_e$, and $r = Bu_e$. □

It follows from Proposition 8.3 that the existence of an equilibrium point $x_e \in \mathbb{R}_+^n$ for (8.1) implies that the matrix $A + A_d$ is semistable. Hence, if (8.11) holds for $x_e \in \mathbb{R}_+^n$ and $u_e \in \overline{\mathbb{R}}_+^m$, then $A + A_d$ is asymptotically stable or $0 \in \mathrm{spec}(A + A_d)$, where $\mathrm{spec}(A+A_d)$ denotes the spectrum of $A+A_d$, is a simple eigenvalue of $A+A_d$ and all other eigenvalues of $A + A_d$ have negative real parts since $-(A + A_d)$ is an $M$-matrix [20].

Finally, the following lemma and proposition are needed for the main results of this chapter.

**Lemma 8.1.** Let $A$, $A_d \in \mathbb{R}^{n \times n}$ be such that $A = A^T < 0$ and $A_d = A_d^T$. If $A + A_d < 0$, then there exists $Q \in \mathbb{R}^{n \times n}$, $Q > 0$, such that

$$2A + Q + A_d Q^{-1} A_d < 0. \tag{8.12}$$

**Proof.** The proof is straightforward with $Q = -A$. Specifically, note that $A + A_{\rm d} < 0$ if and only if $A_{\rm d} < Q$ or, equivalently, $Q^{-\frac{1}{2}} A_{\rm d} Q^{-\frac{1}{2}} < I_n$, where $Q^{-\frac{1}{2}} \in \mathbb{R}^{n\times n}$ is such that $Q^{-\frac{1}{2}} > 0$ and $(Q^{-\frac{1}{2}})^2 = Q^{-1}$. Hence, $A + A_{\rm d} < 0$ if and only if $(Q^{-\frac{1}{2}} A_{\rm d} Q^{-\frac{1}{2}})^2 < I_n$ which is equivalent to $A_{\rm d} Q^{-1} A_{\rm d} < Q$ and which further implies that

$$2A + Q + A_{\rm d} Q^{-1} A_{\rm d} = -Q + A_{\rm d} Q^{-1} A_{\rm d} < 0,$$

proving (8.12). □

**Proposition 8.4.** Consider a linear time-delay mammillary system given by (8.1) where $A$ and $A_{\rm d}$ are given by (8.4) and (8.5), respectively. Then there exist a positive-definite matrix $Q \in \mathbb{R}^{n\times n}$ and a positive diagonal matrix $P \in \mathbb{R}^{n\times n}$ such that

$$0 > A^{\rm T} P + PA + Q + PA_{\rm d} Q^{-1} A_{\rm d}^{\rm T} P. \tag{8.13}$$

**Proof.** Let $\hat{A} \triangleq DAD^{-1}$ and $\hat{A}_{\rm d} = DA_{\rm d}D^{-1}$, where $D = \mathrm{diag}[1,\, d_2, \cdots,\, d_n]$ and $d_j = \sqrt{\frac{a_{1j}}{a_{j1}}}$, $j = 2, \cdots, n$. Now, note that $\hat{A}$ is diagonal and $\hat{A}_{\rm d}$ is symmetric. Since $A + A_{\rm d}$ is similar to $\hat{A} + \hat{A}_{\rm d}$ and $\hat{A} + \hat{A}_{\rm d}$ is symmetric, it follows that the eigenvalues of $A + A_{\rm d}$ are real. Next, it can be shown that $A + A_{\rm d}$ is Hurwitz, which implies that $\hat{A} + \hat{A}_{\rm d} < 0$. Now, it follows from Lemma 8.1 that there exists $\hat{Q} > 0$ such that

$$0 > 2\hat{A} + \hat{Q} + \hat{A}_{\rm d} \hat{Q}^{-1} \hat{A}_{\rm d},$$

or, equivalently,

$$0 > D^{-1} A^{\rm T} D + DAD^{-1} + \hat{Q} + DA_{\rm d} D^{-1} \hat{Q}^{-1} D^{-1} A_{\rm d}^{\rm T} D. \tag{8.14}$$

Next, (8.14) is equivalent to

$$0 > A^{\rm T} D^2 + D^2 A + D\hat{Q}D + D^2 A_{\rm d} D^{-1} \hat{Q}^{-1} D^{-1} A_{\rm d}^{\rm T} D^2. \tag{8.15}$$

Now, the result follows from (8.15) with $P = D^2$ and $Q = D\hat{Q}D$. □

## 8.3. Adaptive Control for Linear Nonnegative Uncertain Dynamical Systems with Time Delay

In this section we consider the problem of characterizing adaptive feedback control laws for nonnegative and compartmental uncertain dynamical systems with time delay to achieve *set-point* regulation in the nonnegative orthant. Specifically, consider the following controlled linear uncertain time-delay dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = Ax(t) + A_{\mathrm{d}}x(t-\tau) + Bu(t), \quad x(\theta) = \eta(\theta), \quad -\tau \le \theta \le 0, \quad t \ge 0, \quad (8.16)$$

where $x(t) \in \mathbb{R}^n$, $t \ge 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \ge 0$, is the control input, $A \in \mathbb{R}^{n\times n}$ is an *unknown* essentially nonnegative matrix, $A_{\mathrm{d}} \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are *unknown* nonnegative matrices, $\eta(\cdot) \in \{\psi(\cdot) \in \mathcal{C}_+([-\tau,0],\mathbb{R}^n) : \psi(\theta) \ge\ge 0, \theta \in [-\tau,0]\}$, and $\tau \ge 0$ is an *unknown* delay amount. The control input $u(\cdot)$ in (8.16) is restricted to the class of admissible controls consisting of measurable functions such that $u(t) \in \mathbb{R}^m$, $t \ge 0$.

It follows from Proposition 8.1 that the state trajectories of nonnegative and compartmental dynamical systems remain in the nonnegative orthant of the state space for nonnegative initial conditions. However, as noted in Chapter 7, even though active control of drug delivery systems for physiological applications additionally requires control (source) inputs to be nonnegative, in many applications of nonnegative systems such as biological systems, population dynamics, and ecological systems, the positivity constraint on the control input is not natural. Hence, in this section we do not place any restriction on the sign of the control signal and design an adaptive controller that guarantees that the system states remain in the nonnegative orthant and converge to a desired equilibrium state. Specifically, for a given desired set point $x_{\mathrm{e}} \in \overline{\mathbb{R}}_+^n$, our aim is to design a control input $u(t)$, $t \ge 0$, such that $\lim_{t\to\infty} \|x(t) - x_{\mathrm{e}}\| = 0$. However, since in many applications of nonnegative systems and in particular, compartmental systems, it is often necessary to regulate a subset of the nonnegative state variables which usually include a central compartment, here we require that $\lim_{t\to\infty} x_i(t) = x_{\mathrm{d}i} \geq 0$ for $i = 1, \cdots, m \leq n$, where $x_{\mathrm{d}i}$ is a desired set point for the $i$th state $x_i(t)$. Furthermore, we assume that control inputs are injected directly into $m$ separate compartments such that the input matrix is given by

$$B = \begin{bmatrix} B_u \\ 0_{(n-m)\times m} \end{bmatrix}, \tag{8.17}$$

where $B_u \triangleq \mathrm{diag}[b_1, \cdots, b_m]$ and $b_i \in \mathbb{R}_+$, $i = 1, \cdots, m$. For compartmental systems this assumption is not restrictive since control inputs correspond to control inflows to each individual compartment. Here, we assume that for $i \in \{1, \cdots, m\}$, $b_i$ is *unknown*. For the statement of our main result define $x_{\mathrm{e}} \triangleq [x_{\mathrm{d}}{}^{\mathrm{T}}, x_{\mathrm{u}}{}^{\mathrm{T}}]^{\mathrm{T}}$, where $x_{\mathrm{d}} \triangleq [x_{\mathrm{d}1}, \cdots, x_{\mathrm{d}m}]^{\mathrm{T}}$ and $x_{\mathrm{u}} \triangleq [x_{\mathrm{u}1}, \cdots, x_{\mathrm{u}(n-m)}]^{\mathrm{T}}$.

**Theorem 8.2.** Consider the linear uncertain time-delay dynamical system $\mathcal{G}$ given by (8.16) where $A$ is essentially nonnegative, $A_{\mathrm{d}}$ is nonnegative, and $B$ is nonnegative and given by (8.17). Assume there exist nonnegative vectors $x_{\mathrm{u}} \in \overline{\mathbb{R}}_+^{n-m}$ and $u_{\mathrm{e}} \in \overline{\mathbb{R}}_+^m$ such that

$$0 = (A + A_{\mathrm{d}})x_{\mathrm{e}} + Bu_{\mathrm{e}}. \tag{8.18}$$

Furthermore, assume there exist a diagonal matrix $K_{\mathrm{g}} = \mathrm{diag}[k_{\mathrm{g}1}, \cdots, k_{\mathrm{g}m}]$, positive *diagonal* matrix $P \triangleq \mathrm{diag}[p_1, \cdots, p_n]$, and positive-definite matrices $\tilde{Q}$, $R \in \mathbb{R}^{n\times n}$ such that

$$0 = A_{\mathrm{s}}^{\mathrm{T}} P + PA_{\mathrm{s}} + \tilde{Q} + PA_{\mathrm{d}}\tilde{Q}^{-1}A_{\mathrm{d}}^{\mathrm{T}}P + R, \tag{8.19}$$

where $A_{\mathrm{s}} \triangleq A + B\tilde{K}_{\mathrm{g}}$ and $\tilde{K}_{\mathrm{g}} \triangleq [K_{\mathrm{g}}\ 0_{m\times(n-m)}]$. Finally, let $q_i$ and $\hat{q}_i$, $i = 1, \cdots, m$, be positive constants. Then the adaptive feedback control law

$$u(t) = K(t)(\hat{x}(t) - x_{\mathrm{d}}) + \phi(t), \tag{8.20}$$

where $K(t) = \mathrm{diag}[k_1(t), \cdots, k_m(t)]$, $\hat{x}(t) = [x_1(t), \cdots, x_m(t)]^{\mathrm{T}}$, and $\phi(t) \in \mathbb{R}^m$, $t \geq 0$, or, equivalently,

$$u_i(t) = k_i(t)(x_i(t) - x_{\mathrm{d}i}) + \phi_i(t), \quad i = 1, \cdots, m, \tag{8.21}$$

where $k_i(t) \in \mathbb{R}$, $t \geq 0$, and $\phi_i(t) \in \mathbb{R}$, $t \geq 0$, $i = 1, \cdots, m$, with update laws

$$\dot{k}_i(t) = -q_i(x_i(t) - x_{\mathrm{d}i})^2, \quad k_i(0) \leq 0, \quad i = 1, \cdots, m, \tag{8.22}$$

$$\dot{\phi}_i(t) = \begin{cases} 0, & \text{if } \phi_i(t) = 0 \text{ and } x_i(t) \geq x_{\mathrm{d}i}, \\ -\hat{q}_i(x_i(t) - x_{\mathrm{d}i}), & \text{otherwise,} \end{cases}$$
$$\phi_i(0) \geq 0, \quad i = 1, \cdots, m, \tag{8.23}$$

guarantees that the solution $(x(t), K(t), \phi(t)) \equiv (x_{\mathrm{e}}, K_{\mathrm{g}}, u_{\mathrm{e}})$ of the closed-loop system given by (8.16), (8.20), (8.22), (8.23) is Lyapunov stable and $x_i(t) \to x_{\mathrm{d}i}$, $i = 1, \cdots, m$ as $t \to \infty$ for all $\eta(\cdot) \in \mathcal{C}_+$. Furthermore, $x(t) \geq\geq 0$ for all $t \geq 0$ and $\eta(\cdot) \in \mathcal{C}_+$.

**Proof.** Note that with $u(t)$, $t \geq 0$, given by (8.20) it follows from (8.16) that

$$\dot{x}(t) = Ax(t) + A_{\mathrm{d}}x(t-\tau) + BK(t)(\hat{x}(t) - x_{\mathrm{d}}) + B\phi(t), \; x(\theta) = \eta(\theta), \; -\tau \leq \theta \leq 0, \; t \geq 0, \tag{8.24}$$

or, equivalently, using (8.18) and $A_{\mathrm{s}} = A + B\tilde{K}_{\mathrm{g}}$,

$$\dot{x}(t) = A_{\mathrm{s}}(x(t) - x_{\mathrm{e}}) + A_{\mathrm{d}}(x(t-\tau) - x_{\mathrm{e}}) + B(K(t) - K_{\mathrm{g}})(\hat{x}(t) - x_{\mathrm{d}}) + B(\phi(t) - u_{\mathrm{e}}),$$
$$x(\theta) = \eta(\theta), \quad -\tau \leq \theta \leq 0, \quad t \geq 0. \tag{8.25}$$

To show Lyapunov stability of the closed-loop system (8.22), (8.23), and (8.25) consider the Lyapunov-Krasovskii functional candidate $V : \mathcal{C}_+ \times \mathbb{R}^{m \times m} \times \mathbb{R}^m \to \mathbb{R}$ given by

$$V(\psi, K, \phi) = (\psi(0) - x_{\mathrm{e}})^{\mathrm{T}} P(\psi(0) - x_{\mathrm{e}}) + \int_{-\tau}^{0} (\psi(\theta) - x_{\mathrm{e}})^{\mathrm{T}} \tilde{Q}(\psi(\theta) - x_{\mathrm{e}}) \mathrm{d}\theta$$
$$+ \mathrm{tr}(K - K_{\mathrm{g}})^{\mathrm{T}} Q^{-1}(K - K_{\mathrm{g}}) + (\phi - u_{\mathrm{e}})^{\mathrm{T}} \hat{Q}^{-1}(\phi - u_{\mathrm{e}}), \tag{8.26}$$

or, equivalently,

$$V(\psi, K, \phi) = \sum_{i=1}^{n} p_i(\psi_i(t) - x_{\mathrm{e}i})^2 + \int_{-\tau}^{0} (\psi(\theta) - x_{\mathrm{e}})^{\mathrm{T}} \tilde{Q}(\psi(\theta) - x_{\mathrm{e}})\mathrm{d}\theta$$
$$+ \sum_{i=1}^{m} \frac{p_i b_i}{q_i}(k_i - k_{\mathrm{g}_i})^2 + \sum_{i=1}^{m} \frac{p_i b_i}{\hat{q}_i}(\phi_i - u_{\mathrm{e}i})^2,$$

where $Q = \mathrm{diag}\left[\frac{q_1}{p_1 b_1}, \cdots, \frac{q_m}{p_m b_m}\right]$ and $\hat{Q} = \mathrm{diag}\left[\frac{\hat{q}_1}{p_1 b_1}, \cdots, \frac{\hat{q}_m}{p_m b_m}\right]$. Note that $V(\psi_{\mathrm{e}}, K_{\mathrm{g}}, u_{\mathrm{e}}) = 0$, where $\psi_{\mathrm{e}}(\theta) = x_{\mathrm{e}}$, $\theta \in [-\tau, 0]$. Furthermore, note that there exist class $\mathcal{K}_\infty$ functions $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot)$ such that

$$V(\psi, K, \phi) \geq \alpha_1(\|\psi(0) - x_{\mathrm{e}}\|) + \alpha_2(\|K - K_{\mathrm{g}}\|_{\mathrm{F}}) + \alpha_3(\|\phi - u_{\mathrm{e}}\|),$$

where $\|\cdot\|$ denotes the Euclidean vector norm and $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius matrix norm. Now, letting $x(t)$, $t \geq 0$, denote the solution to (8.25) and using (8.22) and (8.23), it follows that the Lyapunov-Krasovskii directional derivative along the closed-loop system trajectories is given by

$$\begin{aligned}
\dot{V}(x_t, K(t), \phi(t)) = & -(x(t) - x_{\mathrm{e}})^{\mathrm{T}} R(x(t) - x_{\mathrm{e}}) \\
& -[A_{\mathrm{d}}^{\mathrm{T}} P(x(t) - x_{\mathrm{e}}) - \tilde{Q}(x(t-\tau) - x_{\mathrm{e}})]^{\mathrm{T}} \\
& \cdot \tilde{Q}^{-1}[A_{\mathrm{d}}^{\mathrm{T}} P(x(t) - x_{\mathrm{e}}) - \tilde{Q}(x(t-\tau) - x_{\mathrm{e}})] \\
& +2\sum_{i=1}^{m} p_i b_i (k_i(t) - k_{\mathrm{g}_i})(x_i(t) - x_{\mathrm{d}})^2 \\
& +2\sum_{i=1}^{m} p_i b_i (x_i(t) - x_{\mathrm{e}})(\phi_i(t) - u_{\mathrm{e}i}) \\
& +2\sum_{i=1}^{m} \frac{p_i b_i}{q_i}(k_i(t) - k_{\mathrm{g}_i})\dot{k}_i(t) + 2\sum_{i=1}^{m} \frac{p_i b_i}{\hat{q}_i}(\phi_i(t) - u_{\mathrm{e}i})\dot{\phi}_i(t) \\
\leq & -(x(t) - x_{\mathrm{e}})^{T} R(x(t) - x_{\mathrm{e}}) \\
& +2\sum_{i=1}^{m} p_i b_i (\phi_i(t) - u_{\mathrm{e}i}) \left[(x_i(t) - x_{\mathrm{d}i}) + \frac{1}{\hat{q}_i}\dot{\phi}_i(t)\right], \quad t \geq 0. \quad (8.27)
\end{aligned}$$

Now for the two cases given in (8.23), the last term on the right-hand side of (8.27) gives:

*i)* If $\phi_i(t) = 0$ and $x_i(t) \geq x_{\mathrm{d}i}$, then $\dot{\phi}_i(t) = 0$ and hence

$$p_i b_i(\phi_i(t) - u_{\mathrm{e}i})\Big[(x_i(t) - x_{\mathrm{d}i}) + \frac{1}{\hat{q}_i}\dot{\phi}_i(t)\Big] = -p_i b_i u_{\mathrm{e}i}(x_i(t) - x_{\mathrm{d}i}) \leq 0.$$

*ii)* Otherwise, $\dot{\phi}_i(t) = -\hat{q}_i(x_i(t) - x_{\mathrm{d}i})$ and hence

$$p_i b_i(\phi_i(t) - u_{\mathrm{e}i})\Big[(x_i(t) - x_{\mathrm{d}i}) + \frac{1}{\hat{q}_i}\dot{\phi}_i(t)\Big] = 0.$$

Hence, it follows that in either case

$$\begin{aligned}\dot{V}(x_t, K(t), \phi(t)) &\leq -(x(t) - x_{\mathrm{e}})^{\mathrm{T}} R(x(t) - x_{\mathrm{e}}) \\ &\leq 0, \quad t \geq 0, \end{aligned} \tag{8.28}$$

which proves that the solution $(x(t), K(t), \phi(t)) \equiv (x_{\mathrm{e}}, K_{\mathrm{g}}, u_{\mathrm{e}})$ to (8.22), (8.23), and (8.25) is Lyapunov stable. Furthermore, since the positive orbit $\gamma^+(\eta(\theta), K_0, \phi_0)$ is bounded and $\gamma^+(\eta(\theta), K_0, \phi_0)$ belongs to a compact subset of $\mathcal{C}_+ \times \mathbb{R}^{m\times m} \times \mathbb{R}^m$ [100], and since $R > 0$ it follows from the Krasovskii-LaSalle invariant set theorem for infinite dimensional systems [101, p. 143] that $x(t) \to x_{\mathrm{e}}$ as $t \to \infty$ for all $\eta(\cdot) \in \mathcal{C}_+$.

Finally, to show that $x(t) \geq\geq 0$, $t \geq 0$, for all $\eta(\cdot) \in \mathcal{C}_+$, note that the closed-loop system (8.16), (8.20), (8.22), and (8.23) is given by

$$\begin{aligned}\dot{x}(t) &= Ax(t) + A_{\mathrm{d}}x(t-\tau) + BK(t)(\hat{x}(t) - x_{\mathrm{d}}) + B\phi(t) \\ &= (A + B[K(t),\, 0_{m\times(n-m)}])x(t) + A_{\mathrm{d}}x(t-\tau) - BK(t)x_{\mathrm{d}} + B\phi(t) \\ &= \tilde{A}(t)x(t) + A_{\mathrm{d}}x(t-\tau) + v(t) + w(t), \end{aligned} \tag{8.29}$$

where

$$\tilde{A}(t) \triangleq \begin{bmatrix} a_{11} + b_1 k_1(t) & a_{12} & \cdots & a_{1m} & a_{1\,m+1} & \cdots & a_{1n} \\ a_{21} & a_{22} + b_2 k_2(t) & & \vdots & \vdots & \ddots & a_{2n} \\ \vdots & & \ddots & & & & \vdots \\ a_{m1} & \cdots & & a_{mm} + b_m k_m(t) & a_{m\,m+1} & \cdots & a_{mn} \\ a_{m+1\,1} & \cdots & & a_{m+1\,m} & a_{m+1\,m+1} & \cdots & a_{m+1\,n} \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & & a_{nm} & a_{n\,m+1} & \cdots & a_{nn} \end{bmatrix}, \tag{8.30}$$

$$v(t) \triangleq - \begin{bmatrix} b_1 k_1(t) x_{\mathrm{d}1} \\ \vdots \\ b_m k_m(t) x_{\mathrm{d}m} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad w(t) \triangleq \begin{bmatrix} b_1 \phi_1(t) \\ \vdots \\ b_m \phi_m(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{8.31}$$

Now, since, by (8.22) and (8.23), $k_i(t) \leq 0$, $t \geq 0$, $i = 1, \cdots, m$, and $\phi_i(t) \geq 0$, $t \geq 0$, $i = 1, \cdots, m$, it follows that $v(t) \geq\geq 0$, $t \geq 0$, and $w(t) \geq\geq 0$, $t \geq 0$. Hence, since $\tilde{A}(t)$, $t \geq 0$, is essentially nonnegative pointwise-in-time, it follows from Proposition 8.2 that $x(t) \geq\geq 0$ for all $t \geq 0$ and $\eta(\cdot) \in \mathcal{C}_+$. □

**Remark 8.1.** Note that the conditions in Theorem 8.2 imply that $x(t) \to x_{\mathrm{e}}$ as $t \to \infty$ and hence it follows from (8.22) and (8.23) that $(x(t), K(t), \phi(t)) \to \mathcal{M} \triangleq \{(x, K, \phi) \in \mathbb{R}^n \times \mathbb{R}^{m \times m} \times \mathbb{R}^m : x = x_{\mathrm{e}},\ \dot{K} = 0,\ \dot{\phi} = 0\}$ as $t \to \infty$.

It is important to note that the adaptive control law (8.20), (8.22), and (8.23) does not require the explicit knowledge of the system matrices $A$, $A_{\mathrm{d}}$, and $B$, the gain matrix $K_{\mathrm{g}}$, and the nonnegative constant vector $u_{\mathrm{e}}$; even though Theorem 8.2 requires the existence of $K_{\mathrm{g}}$ and nonnegative vectors $x_{\mathrm{u}}$ and $u_{\mathrm{e}}$ such that the conditions (8.18) and (8.19) hold. Furthermore, in the case where $A + A_{\mathrm{d}}$ is semistable and minimum phase with respect to the output $y = \hat{x}$, or $A + A_{\mathrm{d}}$ is asymptotically stable, then there always exists a diagonal matrix $K_{\mathrm{g}} \in \mathbb{R}^{m \times m}$ such that $A_{\mathrm{s}} + A_{\mathrm{d}}$ is asymptotically stable. In addition, note that for $i = 1, \cdots, m$, the control input signal $u_i(t)$, $t \geq 0$, can be negative depending on the values of $x_i(t)$, $k_i(t)$, and $\phi_i(t)$, $t \geq 0$. However, as is required in nonnegative and compartmental dynamical systems the closed-loop plant states remain nonnegative. Finally, in the case where (8.16) is a mammillary system, $A_{\mathrm{s}}$ is diagonal and hence it follows from Proposition 8.4 there exists a positive diagonal matrix $P$ such that (8.19) holds.

In the case where our objective is zero set-point regulation, that is, $\psi_{\mathrm{e}}(\theta) = x_{\mathrm{e}} =$

$0$, $\theta \in [-\tau, 0]$, the adaptive controller given in Theorem 8.2 can be considerably simplified. Specifically, since in this case $x(t) \geq\geq x_{\mathrm{e}} = 0$, $t \geq 0$, and condition (8.18) is trivially satisfied with $u_{\mathrm{e}} = 0$, we can set $\phi(t) \equiv 0$ so that update law (8.23) is superfluous. Furthermore, since (8.18) is trivially satisfied, $A$ can possess eigenvalues in the open right-half plane. Alternatively, exploiting a *linear* Lyapunov-Krasovskii functional construction for the plant dynamics, an even simpler adaptive controller can be derived. This result is given in the following theorem.

**Theorem 8.3.** Consider the linear uncertain time-delay system $\mathcal{G}$ given by (8.16) where $B$ is nonnegative and given by (8.17). Assume there exists a diagonal matrix $K_{\mathrm{g}} = \mathrm{diag}[k_{\mathrm{g}1}, \cdots, k_{\mathrm{g}m}]$ such that $A_{\mathrm{s}} + A_{\mathrm{d}}$ is asymptotically stable, where $A_{\mathrm{s}} = A + B\tilde{K}_{\mathrm{g}}$ and $\tilde{K}_{\mathrm{g}} = [K_{\mathrm{g}}, 0_{m\times(n-m)}]$. Furthermore, let $q_i$, $i = 1, \cdots, m$, be positive constants. Then the adaptive feedback control law

$$u(t) = K(t)\hat{x}(t), \tag{8.32}$$

where $K(t) = \mathrm{diag}[k_1(t), \cdots, k_m(t)]$ and $\hat{x}(t) = [x_1(t), \cdots, x_m(t)]^{\mathrm{T}}$, or, equivalently,

$$u_i(t) = k_i(t)x_i(t), \quad i = 1, \cdots, m, \tag{8.33}$$

where $k_i(t) \in \mathbb{R}$, $i = 1, \cdots, m$, with update law

$$\dot{K}(t) = -\mathrm{diag}[q_1 x_1(t), \cdots, q_m x_m(t)], \quad K(0) \leq\leq 0, \tag{8.34}$$

guarantees that the solution $(x(t), K(t)) \equiv (0, K_{\mathrm{g}})$ of the closed loop system given by (8.16), (8.32), (8.34) is Lyapunov stable and $x(t) \to 0$ as $t \to \infty$ for all $\eta(\cdot) \in \mathcal{C}_+$.

**Proof.** Note that with $u(t)$, $t \geq 0$, given by (8.32) it follows from (8.16) that

$$\dot{x}(t) = Ax(t) + A_{\mathrm{d}}x(t-\tau) + BK(t)\hat{x}(t), \quad x(\theta) = \eta(\theta), \quad -\tau \leq \theta \leq 0, \quad t \geq 0. \tag{8.35}$$

Now, since for every $t \in [0,\infty)$, $\tilde{A}(t) \triangleq A+B[K(t), 0_{m\times n-m}]$ is essentially nonnegative and $A_{\rm d}$ is nonnegative it follows from Proposition 8.2 that $x(t) \geq\geq 0$, $t \geq 0$, for all $x_0 \in \overline{\mathbb{R}}^n_+$. Next, using $A_{\rm s} = A + B\tilde{K}_{\rm g}$, note that (8.35) can be equivalently written as

$$\dot{x}(t) = A_{\rm s}x(t)+A_{\rm d}x(t-\tau)+B(K(t)-K_{\rm g})\hat{x}(t), \quad x(\theta)=\eta(\theta), \quad -\tau\leq\theta\leq 0, \quad t\geq 0. \tag{8.36}$$

Furthermore, since $A_{\rm s}$ is essentially nonnegative, $A_{\rm d}$ is nonnegative, and $A_{\rm s} + A_{\rm d}$ is asymptotically stable it follows from Theorem 8.1 that there exist vectors $p >> 0$ and $r >> 0$ satisfying

$$0 = (A_{\rm s} + A_{\rm d})^{\rm T}p + r. \tag{8.37}$$

To show Lyapunov stability of the closed-loop system (8.34) and (8.36) consider the Lyapunov-Krasovskii functional candidate $V : \mathcal{C}_+ \times \mathbb{R}^{m\times m} \to \mathbb{R}$ given by

$$V(\psi, K) = p^{\rm T}\psi(0) + \int_{-\tau}^{0} p^{\rm T}A_{\rm d}\psi(\theta){\rm d}\theta + \frac{1}{2}{\rm tr}(K-K_{\rm g})^{\rm T}Q^{-1}(K-K_{\rm g}), \tag{8.38}$$

where $Q = {\rm diag}[\frac{q_1}{p_1b_1}, \cdots, \frac{q_m}{p_mb_m}]$. Furthermore, note that $V(\psi_{\rm e}, K_{\rm g}) = 0$, where $\psi_{\rm e}(\theta) = 0$, $\theta \in [-\tau, 0]$, and, since $x(t) \geq\geq 0$, $t \geq 0$, there exist class $\mathcal{K}$ functions $\alpha_1(\cdot), \alpha_2(\cdot)$ such that

$$V(\psi, K) \geq \alpha_1(\|\psi(0)\|) + \alpha_2(\|K-K_{\rm g}\|_{\rm F}), \quad \psi(0) \in \mathcal{C}_+.$$

Now, letting $x(t)$, $t \geq 0$, denote the solution to (8.36) and using (8.34), it follows that the Lyapunov-Krasovskii directional derivative along the closed-loop system trajectories is given by

$$\begin{aligned}\dot{V}(x_t, K(t)) &= p^{\rm T}A_{\rm s}x(t) + p^{\rm T}A_{\rm d}x(t-\tau) + p^{\rm T}B(K(t)-K_{\rm g})\hat{x}(t) + p^{\rm T}A_{\rm d}x(t)\\ &\quad -p^{\rm T}A_{\rm d}x(t-\tau) + {\rm tr}(K(t)-K_{\rm g})^{\rm T}Q^{-1}\dot{K}(t)\\ &= -r^{\rm T}x(t)\\ &\leq 0, \quad t\geq 0,\end{aligned}$$

which proves that the solution $(x(t), K(t)) \equiv (0, K_{\rm g})$ to (8.34) and (8.36) is Lyapunov stable. Furthermore, since $r >> 0$ it follows from Krasovskii-LaSalle invariant set theorem for infinite dimensional systems [101, p. 143] that $x(t) \to 0$ as $t \to \infty$ for all $\eta(\cdot) \in \mathcal{C}_+$. □

## 8.4. Adaptive Control for Linear Nonnegative Dynamical Systems with Nonnegative Control and Time Delay

In drug delivery systems for physiological processes, control (source) inputs are usually constrained to be nonnegative as are the system states. Hence, in this section we develop adaptive control laws for nonnegative retarded systems with nonnegative control inputs. However, since condition (8.11) is required to be satisfied for $x_\mathrm{e} \in \overline{\mathbb{R}}_+^n$ and $u_\mathrm{e} \in \overline{\mathbb{R}}_+^m$, it follows from Brockett's necessary condition for asymptotic stabilizability [53] that there does not exist a continuous stabilizing *nonnegative* feedback if $0 \in \mathrm{spec}(A+A_\mathrm{d})$ and $x_\mathrm{e} \in \mathbb{R}_+^n$. Hence, in this section we assume that $A+A_\mathrm{d}$ is asymptotically stable compartmental matrix. Thus, we proceed with the aforementioned assumptions to design adaptive controllers for uncertain time-delay compartmental systems that guarantee that $\lim_{t\to\infty} x_i(t) = x_{\mathrm{d}i} \geq 0$ for $i = 1, \cdots, m \leq n$, where $x_{\mathrm{d}i}$ is a desired set point for the $i$th compartmental state while guaranteeing a nonnegative control input.

**Theorem 8.4.** Consider the linear uncertain time-delay system $\mathcal{G}$ given by (8.16), where $A$ is essentially nonnegative, $A_\mathrm{d}$ is nonnegative, and $B$ is nonnegative and given by (8.17). For a given $x_\mathrm{d} \in \mathbb{R}^m$, assume there exist vectors $x_\mathrm{u} \in \mathbb{R}_+^{n-m}$ and $u_\mathrm{e} \in \overline{\mathbb{R}}_+^m$ such that (8.18) holds. Furthermore, assume that there exist a positive *diagonal* matrix $P \triangleq \mathrm{diag}[p_1, \cdots, p_n]$, and positive-definite matrices $\tilde{Q}$, $R \in \mathbb{R}^{n\times n}$ such that

$$0 = A^\mathrm{T} P + PA + \tilde{Q} + PA_\mathrm{d}\tilde{Q}^{-1}A_\mathrm{d}^\mathrm{T} P + R. \tag{8.39}$$

Finally, let $q_i$ and $\hat{q}_i$, $i = 1, \cdots, m$, be positive constants. Then, the adaptive feedback control law

$$u_i(t) = \max\{0, \hat{u}_i(t)\}, \quad i = 1, \cdots, m, \tag{8.40}$$

where

$$\hat{u}_i(t) = k_i(t)(x_i(t) - x_{\mathrm{d}i}) + \phi_i(t), \quad i = 1, \cdots, m, \tag{8.41}$$

$k_i(t) \in \mathbb{R}$, $t \geq 0$, and $\phi_i(t) \in \mathbb{R}$, $t \geq 0$, $i = 1, \cdots, m$, with update laws

$$\dot{k}_i(t) = \begin{cases} 0, & \text{if } \hat{u}_i(t) < 0, \\ -q_i(x_i(t) - x_{\mathrm{d}i})^2, & \text{otherwise,} \end{cases} \quad k_i(0) \leq 0, \quad i = 1, \cdots, m, \tag{8.42}$$

$$\dot{\phi}_i(t) = \begin{cases} 0, & \text{if } \phi_i(t) = 0 \text{ and } x_i(t) > x_{\mathrm{d}i}, \text{ or if } \hat{u}_i(t) \leq 0, \\ -\hat{q}_i(x_i(t) - x_{\mathrm{d}i}), & \text{otherwise,} \end{cases}$$
$$\phi_i(0) \geq 0, \quad i = 1, \cdots, m, \tag{8.43}$$

guarantees that the solution $(x(t), K(t), \phi(t)) \equiv (x_{\mathrm{e}}, 0, u_{\mathrm{e}})$ of the closed-loop system given by (8.16), (8.40), (8.42), (8.43) is Lyapunov stable and $x_i(t) \to x_{\mathrm{d}i}$, $i = 1, \cdots, m$, as $t \to \infty$ for all $\eta(\cdot) \in \mathcal{C}_+$. Furthermore, $u(t) \geq\geq 0$ and $x(t) \geq\geq 0$ for all $t \geq 0$ and $\eta(\cdot) \in \mathcal{C}_+$.

**Proof.** First, define $K_u(t) \triangleq \mathrm{diag}[k_{u1}(t), \cdots, k_{um}(t)]$ and $\phi_u(t) \triangleq [\phi_{u1}(t), \cdots, \phi_{um}(t)]^{\mathrm{T}}$, where

$$k_{ui}(t) = \begin{cases} 0, & \text{if } \hat{u}_i(t) < 0, \\ k_i(t), & \text{otherwise,} \end{cases} \quad i = 1, \cdots, m, \tag{8.44}$$

$$\phi_{ui}(t) = \begin{cases} 0, & \text{if } \hat{u}_i(t) < 0, \\ \phi_i(t), & \text{otherwise,} \end{cases} \quad i = 1, \cdots, m. \tag{8.45}$$

Now, note that with $u(t)$, $t \geq 0$, given by (8.40) it follows from (8.16) that

$$\dot{x}(t) = Ax(t) + A_{\mathrm{d}}x(t-\tau) + BK_u(t)(\hat{x}(t) - x_{\mathrm{d}}) + B\phi_u(t),$$
$$x(\theta) = \eta(\theta), \quad -\tau \leq \theta \leq 0, \quad t \geq 0, \tag{8.46}$$

or, equivalently, using (8.18),

$$\dot{x}(t) = A(x(t) - x_{\mathrm{e}}) + A_{\mathrm{d}}(x(t-\tau) - x_{\mathrm{e}}) + BK_u(t)(\hat{x}(t) - x_{\mathrm{d}}) + B(\phi_u(t) - u_{\mathrm{e}}),$$
$$x(\theta) = \eta(\theta), \quad -\tau \leq \theta \leq 0, \quad t \geq 0. \tag{8.47}$$

To show Lyapunov stability of the closed-loop system (8.42), (8.43), and (8.47) consider the Lyapunov-Krasovskii functional candidate $V : \mathcal{C}_+ \times \mathbb{R}^{m\times m} \times \mathbb{R}^m \to \mathbb{R}$ given by

$$\begin{aligned} V(\psi, K, \phi) = & (\psi(0) - x_\mathrm{e})^\mathrm{T} P(\psi(0) - x_\mathrm{e}) + \int_{-\tau}^{0} (\psi(\theta) - x_\mathrm{e})^\mathrm{T} \tilde{Q}(\psi(\theta) - x_\mathrm{e})\mathrm{d}\theta \\ & + \mathrm{tr}\, K^\mathrm{T} Q^{-1} K + (\phi - u_\mathrm{e})^\mathrm{T} \hat{Q}^{-1}(\phi - u_\mathrm{e}), \end{aligned} \tag{8.48}$$

or, equivalently,

$$\begin{aligned} V(\psi, K, \phi) = & \sum_{i=1}^{n} p_i(\psi_i(0) - x_{\mathrm{e}i})^2 + \int_{-\tau}^{0} (\psi(\theta) - x_\mathrm{e})^\mathrm{T} \tilde{Q}(\psi(\theta) - x_\mathrm{e})\mathrm{d}\theta \\ & + \sum_{i=1}^{m} \frac{p_i b_i}{q_i} k_i^2 + \sum_{i=1}^{m} \frac{p_i b_i}{q_i}(\phi_i - u_{\mathrm{e}i})^2, \end{aligned}$$

where $Q = \mathrm{diag}[\frac{q_1}{p_1 b_1}, \cdots, \frac{q_m}{p_m b_m}]$ and $\hat{Q} = \mathrm{diag}[\frac{\hat{q}_1}{p_1 b_1}, \cdots, \frac{\hat{q}_m}{p_m b_m}]$. Note that $V(\psi_\mathrm{e}, 0, u_\mathrm{e}) = 0$, where $\psi_\mathrm{e}(\theta) = x_\mathrm{e}$, $\theta \in [-\tau, 0]$. Furthermore, there exist class $\mathcal{K}$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$ such that

$$V(\psi, K, \phi) \geq \alpha_1(\|\psi(0) - x_\mathrm{e}\|) + \alpha_2(\|K\|_\mathrm{F}) + \alpha_3(\|\phi - u_\mathrm{e}\|).$$

Now, letting $x(t)$, $t \geq 0$, denote the solution to (8.47) and using (8.42) and (8.43), it follows that the Lyapunov-Krasovskii directional derivative along the closed-loop system trajectories is given by

$$\begin{aligned} \dot{V}(x_t, K(t), \phi(t)) = & -(x(t) - x_\mathrm{e})^\mathrm{T} R(x(t) - x_\mathrm{e}) \\ & -[A_\mathrm{d}^\mathrm{T} P(x(t) - x_\mathrm{e}) - \tilde{Q}(x(t-\tau) - x_\mathrm{e})]^\mathrm{T} \\ & \cdot \tilde{Q}^{-1}[A_\mathrm{d}^\mathrm{T} P(x(t) - x_\mathrm{e}) - \tilde{Q}(x(t-\tau) - x_\mathrm{e})] \\ & +2\sum_{i=1}^{m} p_i b_i k_{ui}(t)(x_i(t) - \dot{x}(t))^2 \\ & +2\sum_{i=1}^{m} p_i b_i (x_i(t) - x_\mathrm{e})(\phi_{ui}(t) - u_{\mathrm{e}i}) \\ & +2\sum_{i=1}^{m} \frac{p_i b_i}{q_i}(k_i(t) - k_{\mathrm{g}i})\dot{k}_i(t) + 2\sum_{i=1}^{m} \frac{p_i b_i}{\hat{q}_i}(\phi_i(t) - u_{\mathrm{e}i})\dot{\phi}_i(t) \end{aligned}$$

$$
\begin{aligned}
&\le -(x(t)-x_{\rm e})^{\rm T}R(x(t)-x_{\rm e})\\
&\quad +2\sum_{i=1}^{m} p_i b_i\Big[k_{ui}(t)(x_i(t)-\dot{x}(t)_i)^2+\frac{1}{q_i}k_i(t)\dot{k}_i(t)\Big]\\
&\quad +2\sum_{i=1}^{m} p_i b_i\Big[(x_i(t)-\dot{x}(t)_i)(\phi_{ui}(t)-u_{{\rm e}i})+\frac{1}{\hat{q}_i}(\phi_i(t)-u_{{\rm e}i})\dot{\phi}_i(t)\Big].
\end{aligned}
\tag{8.49}
$$

Now, for the two cases given in (8.42) and (8.43), the last two terms on the right-hand side of (8.49) give:

*i)* If $\hat{u}_i(t)<0$, then $k_{ui}(t)=0$, $\phi_{ui}(t)=0$, $\dot{k}_i(t)=0$, and $\dot{\phi}_i(t)=0$. Furthermore, since $\phi_i(t)\ge 0$ and $k_i(t)\le 0$ for all $t\ge 0$ and $i=1,\cdots,m$, it follows from (8.41) that $\hat{u}_i(t)<0$ only if $x_i(t)>x_{{\rm d}i}$ and hence

$$
k_{ui}(t)(x_i(t)-x_{{\rm d}i})^2+\frac{1}{q_i}k_i(t)\dot{k}_i(t)=0.
$$
$$
(x_i(t)-x_{{\rm d}i})(\phi_{ui}(t)-u_{{\rm e}i})+\frac{1}{q_i}(\phi_i(t)-u_{{\rm e}i})\dot{\phi}_i(t)=-(x_i(t)-x_{{\rm d}i})u_{{\rm e}i}\le 0.
$$

*ii)* Otherwise, $k_{ui}(t)=k_i(t)$ and $\phi_{ui}(t)=\phi_i(t)$ and hence

$$
k_{ui}(x_i(t)-x_{{\rm d}i})^2+\frac{1}{q_i}k_i(t)\dot{k}_i(t)=0,
$$
$$
\begin{aligned}
&(x_i(t)-x_{{\rm d}i})(\phi_{ui}(t)-u_{{\rm e}i})+\frac{1}{\hat{q}_i}(\phi_i(t)-u_{{\rm e}i})\dot{\phi}_i(t)\\
&\quad=\begin{cases} -(x_i(t)-x_{{\rm d}i})u_{{\rm e}i}\le 0, & \text{if } \phi_i(t)=0 \text{ and } x_i(t)\ge x_{{\rm d}i},\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}
$$

Hence, it follows that in either case

$$
\begin{aligned}
\dot{V}(x_t,K(t),\phi(t)) &\le -(x(t)-x_{\rm e})^{\rm T}R(x(t)-x_{\rm e})\\
&\le 0,\quad t\ge 0,
\end{aligned}
\tag{8.50}
$$

which proves that the solution $(x(t),K(t),\phi(t))\equiv(x_{\rm e},0,u_{\rm e})$ to (8.42), (8.43), and (8.47) is Lyapunov stable. Furthermore, since the positive orbit $\gamma^+(\eta(\theta),K_0,\phi_0)$ is bounded and $\gamma^+(\eta(\theta),K_0,\phi_0)$ belongs to a compact subset of $\mathcal{C}_+\times\mathbb{R}^{m\times m}\times\mathbb{R}^m$ [100], and since $R > 0$ it follows from the Krasovskii-LaSalle invariant set theorem for infinite dimensional systems [101, p. 143] that $x(t) \to x_{\rm e}$ as $t \to \infty$ for all $\eta(\cdot) \in \mathcal{C}_+$.

Finally, $u(t) \geq\geq 0$, $t \geq 0$, is a restatement of (8.40). Now, since $B \geq\geq 0$ and $u(t) \geq\geq 0$, $t \geq 0$, it follows from Proposition 8.1 that $x(t) \geq\geq 0$, $t \geq 0$, for all $x_0 \in \overline{\mathbb{R}}_+^n$. □

As in the case of Theorem 8.2, it is important to note that the adaptive control law (8.40), (8.42), and (8.43) does not require the explicit knowledge of the nonnegative constant vector $u_{\rm e}$; even though Theorem 8.4 requires the existence of nonnegative vectors $x_{\rm u}$ and $u_{\rm e}$ such that the condition (8.18) holds. Furthermore, Theorem 8.4 requires that $A$ and $A_{\rm d}$ are such that there exists a positive diagonal matrix $P$ such that (8.39) holds. However, in the case where (8.16) is a mammillary system the existence of a positive diagonal matrix $P$ satisfying (8.39) is a direct consequence of Proposition 8.4.

## 8.5. Adaptive Control for General Anesthesia

In this section, we illustrate the adaptive control framework developed in this chapter on a model for the disposition of propofol [54, 85, 161] which is based on the three-compartment mammillary model shown in Figure 8.1 with the first compartment acting as the central compartment and the remaining two compartments exchanging with the central compartment. The three-compartment mammillary system with all transfer times between compartments given by $\tau > 0$ provides a pharmacokinetic model for a patient describing the distribution of propofol into the central compartment (identified with the intravascular blood volume as well as highly perfused organs) and other various tissue groups of the body. A mass balance for the

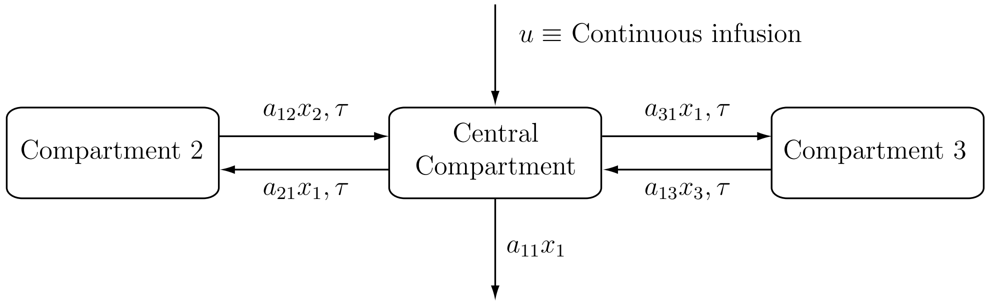



**Figure 8.1**: Three-compartment mammillary model for disposition of propofol

whole compartmental system yields

$$\dot{x}_1(t) = -(a_{11} + a_{21} + a_{31})x_1(t) + a_{12}x_2(t-\tau) + a_{13}x_3(t-\tau) + u(t),$$
$$x_1(\theta) = \eta_1(\theta),\ -\tau \le \theta \le 0,\ t \ge 0, \quad (8.51)$$

$$\dot{x}_2(t) = -a_{12}x_2(t) + a_{21}x_1(t-\tau),\ x_2(\theta) = \eta_2(\theta),\ -\tau \le \theta \le 0, \quad (8.52)$$

$$\dot{x}_3(t) = -a_{13}x_3(t) + a_{31}x_1(t-\tau),\ x_3(\theta) = \eta_3(\theta),\ -\tau \le \theta \le 0, \quad (8.53)$$

where $x_1(t)$, $x_2(t)$, $x_3(t)$, $t \ge 0$, are the masses in grams of propofol in the central compartment and compartments 2 and 3, respectively, $u(t)$, $t \ge 0$, is the infusion rate in grams/min of the anesthetic (propofol) into the central compartment, $a_{ij} > 0$, $i \ne j$, $i, j = 1, 2, 3$, are the rate constants in $\min^{-1}$ for drug transfer between compartments, and $a_{11} > 0$ is the rate constant in $\min^{-1}$ for elimination from the central compartment. Even though these transfer and loss coefficients and the delay amount are positive, they can be uncertain due to patient gender, weight, pre-existing disease, age, and concomitant medication. Hence, adaptive control for propofol set-point regulation can significantly improve the outcome for drug administration over manual control.

It has been reported in [239] that a 2.5–6 $\mu$g/m$\ell$ blood concentration level of propofol is required during the maintenance stage in general anesthesia depending on patient fitness and extent of surgical stimulation. Hence, continuous infusion control is required for maintaining this desired level of anesthesia. Here we assume that the transfer and loss coefficients $a_{11}$, $a_{12}$, $a_{21}$, $a_{13}$, $a_{31}$, and the delay amount $\tau$ are unknown and our objective is to regulate the propofol concentration level of the central compartment to the desired level of 3.4 $\mu$g/m$\ell$ in the face of system uncertainty. Furthermore, since propofol mass in the blood plasma cannot be measured directly, we measure the concentration of propofol in the central compartment; that is, $x_1/V_{\rm c}$, where $V_{\rm c}$ is the volume in liters of the central compartment. As noted in [161], $V_{\rm c}$ can be approximately calculated by $V_{\rm c} = (0.159\,\ell/\text{kg})(M\,\text{kg})$, where $M$ is the weight (mass) in kilograms of the patient.

Next, note that (8.51)–(8.53) can be written in the state space form (8.16) with $x = [x_1, x_2, x_3]^{\rm T}$,

$$A = \begin{bmatrix} -(a_{11}+a_{21}+a_{31}) & 0 & 0 \\ 0 & -a_{12} & 0 \\ 0 & 0 & -a_{13} \end{bmatrix}, \quad A_{\rm d} = \begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{8.54}$$

In the following simulation, we use the bispectral index (BIS) as a measure of anesthetic effect and the BIS signal is given by (7.55), where the effect site compartment concentration is given by the solution of the first-order delay model (7.56) (see Chapter 7 for details). As similarly as in Chapter 7, we set $\text{EC}_{50} = 3.4$ $\mu$g/m$\ell$, $\gamma = 3$, and $\text{BIS}_0 = 100$, so that the BIS signal is shown in Figure 7.5. The target (desired) BIS value, $\text{BIS}_{\text{target}}$, is set at 50. In this case, the linearized BIS function about the target BIS value is given by (7.58). Furthermore, for simplicity of exposition, we assume that the effect site compartment equilibrates instantaneously with the central compartment; that is, we assume that $a_{\text{eff}} \to \infty$ and hence $c_{\text{eff}}(t) = x_1(t)/V_{\rm c}$, $t \geq 0$. Now, using the adaptive feedback controller

$$u(t) = \max\{0, \hat{u}(t)\}, \tag{8.55}$$

where

$$\hat{u}(t) = -k(t)(\text{BIS}(t) - \text{BIS}_{\text{target}}) + \phi(t), \tag{8.56}$$

$k(t) \in \mathbb{R}$, $t \geq 0$, and $\phi(t) \in \mathbb{R}$, $t \geq 0$, with update laws

$$\dot{k}(t) = \begin{cases} 0, & \text{if } \hat{u}(t) < 0, \\ -q_{\mathrm{BIS}}(\mathrm{BIS}(t) - \mathrm{BIS}_{\mathrm{target}})^2, & \text{otherwise}, \end{cases} \quad k(0) \leq 0, \tag{8.57}$$

$$\dot{\phi}(t) = \begin{cases} 0, & \begin{array}{l} \text{if } \phi(t) = 0 \text{ and } \mathrm{BIS}(t) > \mathrm{BIS}_{\mathrm{target}}, \\ \text{or if } \hat{u}(t) \leq 0, \end{array} \\ \hat{q}_{\mathrm{BIS}}(\mathrm{BIS}(t) - \mathrm{BIS}_{\mathrm{target}}), & \text{otherwise}, \end{cases} \quad \phi(0) \geq 0, \tag{8.58}$$

where $q_{\mathrm{BIS}}$ and $\hat{q}_{\mathrm{BIS}}$ are arbitrary positive constants, it follows from Theorem 8.4 that the control input (anesthetic infusion rate) $u(t) \geq 0$ for all $t \geq 0$ and $\mathrm{BIS}(t) \to \mathrm{BIS}_{\mathrm{target}}$ as $t \to \infty$ for any (uncertain) positive values of the transfer and loss coefficients in the range of $c_{\mathrm{eff}}$ where the linearized BIS equation (7.58) is valid. It is important to note that during actual surgery or intensive care unit sedation the BIS signal is obtained directly from the EEG and not (7.55). Furthermore, since our adaptive controller only requires the error signal $\mathrm{BIS}(t) - \mathrm{BIS}_{\mathrm{target}}$ over the linearized range of (7.55), we do not require knowledge of the slope of the linearized equation (7.58), nor do we require knowledge of the parameters $\gamma$ and $\mathrm{EC}_{50}$. To illustrate the robustness properties of the proposed adaptive control law, we use the average set of pharmacokinetic parameters given in [68] for 29 patients requiring general anesthesia for noncardiac surgery. For our design we assume $M = 70$ kg and we switch from Set A to Set B given in Table 7.1 in Chapter 7 at $t = 25$ min. Furthermore, we assume that at $t = 25$ min the pharmacodynamic parameters $\mathrm{EC}_{50}$ and $\gamma$ are switched from 3.4 $\mu$g/m$\ell$ and 3 to 4.0 $\mu$g/m$\ell$ and 4, respectively. Here we consider noncardiac surgery since cardiac surgery often utilizes hypothermia which itself changes the BIS signal. With $q_{\mathrm{BIS}} = 1 \times 10^{-6}$ g/min$^2$, $\hat{q}_{\mathrm{BIS}} = 1 \times 10^{-3}$ g/min$^2$, and initial conditions $x(0) = [0, 0, 0]^{\mathrm{T}}$ g, $k(0) = 0$ min$^{-1}$, and $\phi(0) = 0.01$ g/min$^{-1}$, Figure 8.2 shows the masses of propofol in all three compartments versus time. Figure 8.3 shows the BIS Index versus time. Figure 8.4 shows the propofol concentration in the central compartment and the control signal (propofol infusion rate) versus time. Finally,

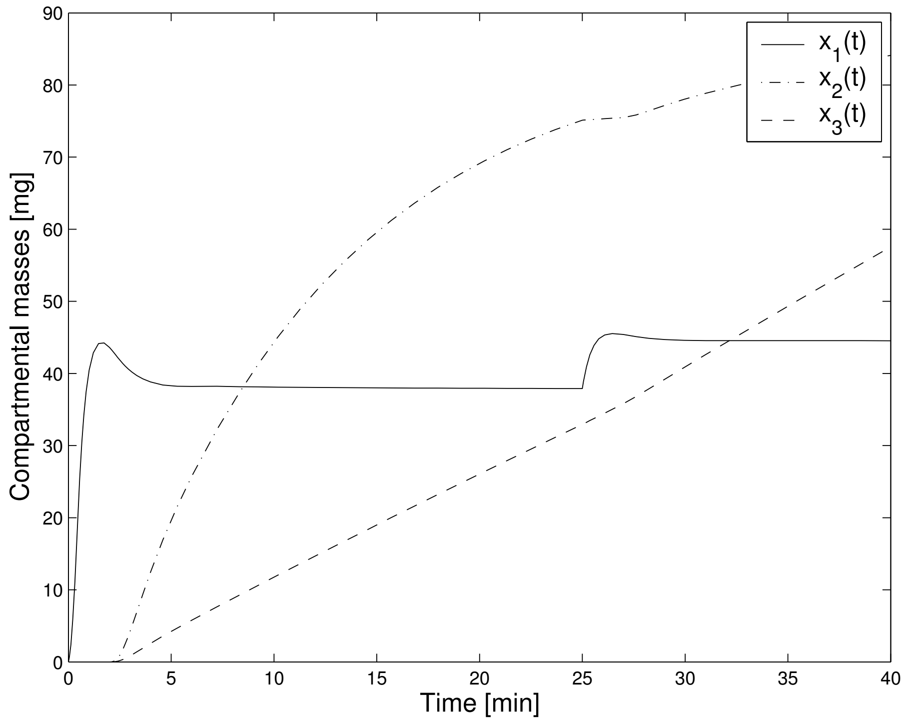



**Figure 8.2**: Compartmental masses versus time

Figure 8.5 shows the adaptive gain history versus time.

## 8.6. Conclusion

In this chapter, we developed a direct adaptive control framework for linear uncertain nonnegative and compartmental dynamical systems with unknown time delay. In particular, a Lyapunov-Krasovskii-based direct adaptive control framework for guaranteeing set-point regulation for nonnegative and compartmental time-delay systems with specific applications to mammillary pharmacokinetic models was developed. Finally, we demonstrated the framework on a drug delivery pharmacokinetic model with time delay. Extensions of the proposed adaptive control framework to nonlinear nonnegative systems as well as to systems with exogenous disturbances will be addressed in future research.

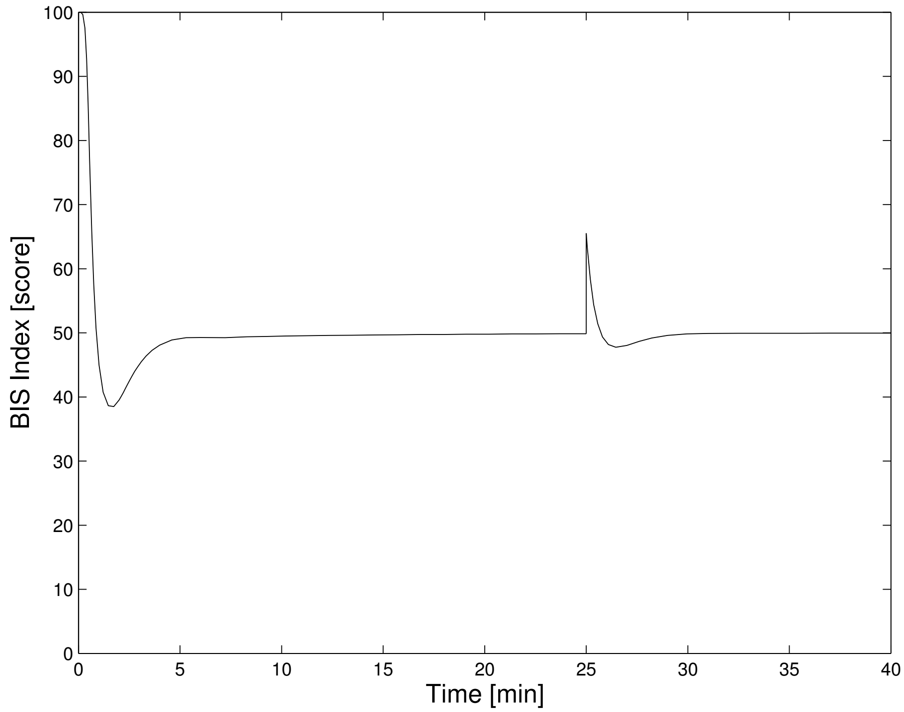



**Figure 8.3**: BIS Index versus time

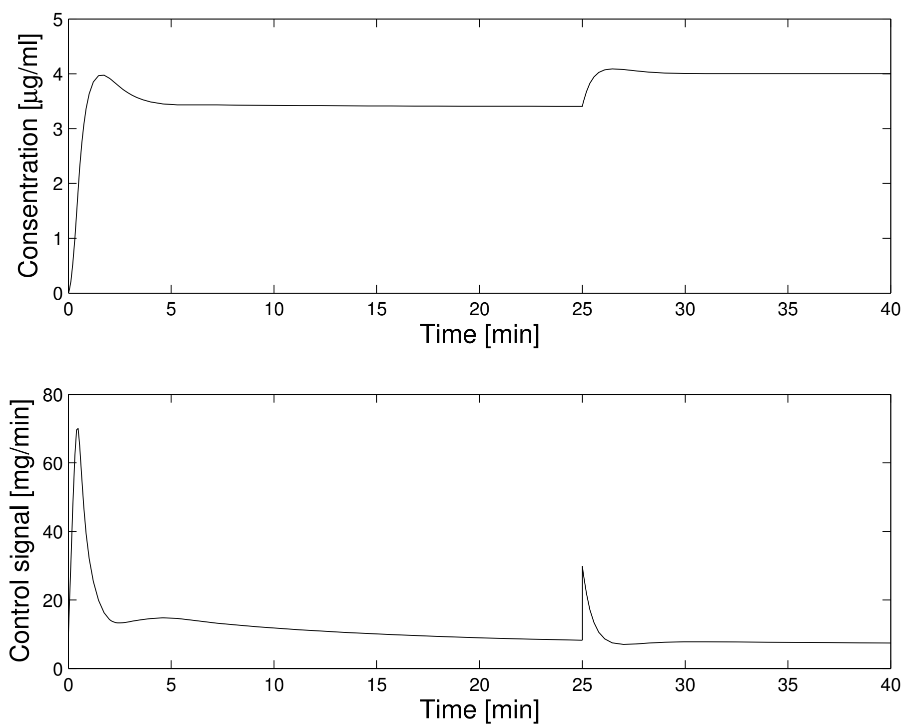



**Figure 8.4**: Drug concentration in the central compartment and control signal (infusion rate) versus time

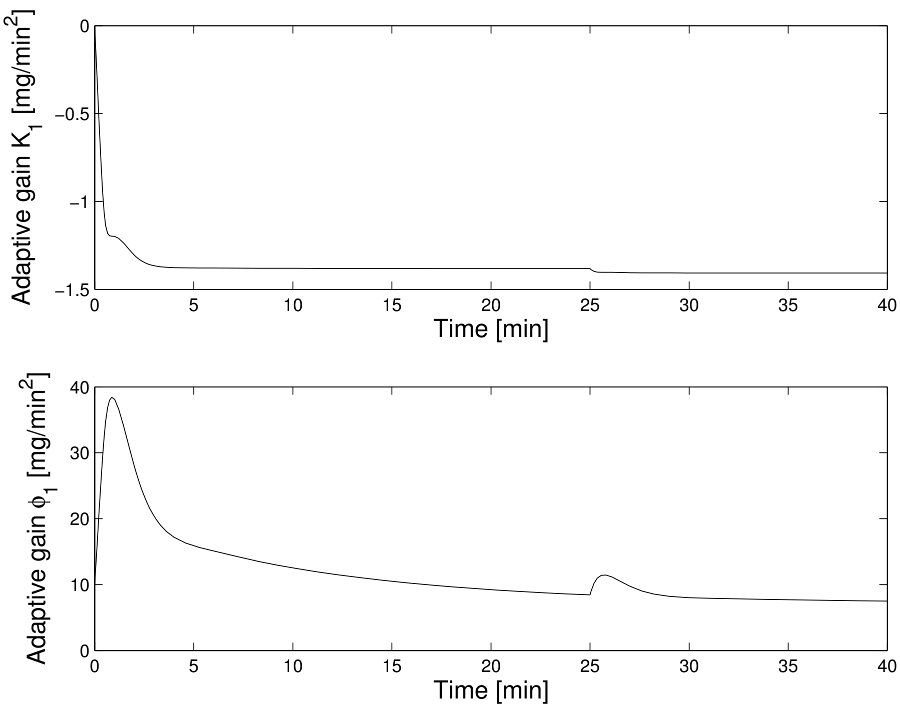



**Figure 8.5**: Adaptive gain history versus time

# Chapter 9

# Adaptive Control for Nonlinear Nonnegative and Compartmental Dynamical Systems with Applications to Clinical Pharmacology

## 9.1. Introduction

Administration of drugs to produce general anesthesia has traditionally been guided by clinical evaluation. However, the clinical measures of depth of anesthesia are imperfect, primarily since the most reliable, purposeful movement in response to noxious stimulus, is masked by the concomitant administration of paralytic agents, given to improve operating conditions for the surgeon. There has been a long-standing interest in the use of the electroencephalogram (EEG) as an objective, quantitative measure of consciousness. Recent work has demonstrated that a derivative of the EEG signal, the Bispectral Index, correlates with changes in consciousness [67, 174, 215]. The Bispectral Index is a scalar measure ranging from 0 to 100, with the upper value of 100 corresponding to the awake state and the lower limit of 0 corresponding to an isoelectrical EEG signal. The ease of Bispectral Index (BIS) monitoring and its ready availability for use in the operating room, opens the possibility of closed-loop control of anesthetic drug administration, using the BIS as the performance and measurement variable. Current standard practice, open-loop control (manual control) by clinical personnel, can be tedious, imprecise, time-consuming, and sometimes of poor quality, depending on the skills and judgment of the clinician. Underdosing can result in patients psychologically traumatized by pain and awareness during surgery, while overdosing, at the very least, may result in delayed recovery from anesthesia and, in the worst case, may result in respiratory and cardiovascular collapse. Closed-loop control may improve the quality of drug administration, lessening the dependence of patient outcome on the skills of the clinician.

Previous efforts to develop closed-loop control of general anesthesia have used either a proportional-integral-derivative control algorithm or linear adaptive control algorithms based on pharmacokinetic/pharmacodynamic models [2, 207, 227]. Adaptive algorithms are vital since the relationships between drug dose and blood concentration (pharmacokinetics) and between blood concentrations and physiological effect (pharmacodynamics) vary widely among individual subjects. Previous model-based algorithms have assumed either a fixed pharmacokinetic or pharmacodynamic model. In this paper, we present a less restrictive direct adaptive control framework that accounts for interpatient pharmacokinetic and pharmacodynamic variability. In particular, we develop a direct adaptive control framework for adaptive set-point regulation of nonlinear uncertain nonnegative and compartmental systems.

Nonnegative and compartmental models provide a broad framework for biological and physiological systems, including clinical pharmacology, and are well suited for the problem of closed-loop control of drug administration. Specifically, nonnegative and compartmental dynamical systems [6, 19, 24, 62, 70, 75, 123, 124, 164, 166, 172, 182, 187, 203] are composed of homogeneous interconnected subsystems (or compartments) which exchange variable nonnegative quantities of material with conservation laws describing transfer, accumulation, and elimination between the compartments and the environment. It thus follows from physical considerations that the state trajectory of such systems remains in the nonnegative orthant of the state space for nonnegative initial conditions. Using nonnegative and compartmental model structures, a Lyapunov-based direct adaptive control framework is developed that guarantees partial asymptotic set-point stability of the closed-loop system; that is, asymptotic set-point stability with respect to part of the closed-loop system states associated with the physiological state variables. In particular, adaptive controllers are constructed *without* requiring knowledge of the system dynamics while providing a nonnegative control (source) input for robust stabilization with respect to the nonnegative orthant. Furthermore, since in certain applications of nonnegative and compartmental systems (e.g., biological systems, population dynamics, and ecological systems involving positive and negative inflows) the nonnegativity constraint on the control input is not natural, we also develop adaptive controllers that do not place any restriction on the sign of the control signal while guaranteeing that the physical system states remain in the nonnegative orthant of the state space. Finally, we emphasize that even though our application objective in this paper is closed-loop adaptive control of drug administration for general anesthesia, the proposed nonlinear adaptive control architecture can be readily applied to deliver sedation to critically ill patients in the intensive care unit, as well as to control glucose, heart rate, and blood pressure during surgery.

## 9.2. Mathematical Preliminaries

In this section we introduce some key results concerning nonlinear nonnegative dynamical systems [19, 20, 24, 75] that are necessary for developing the main results of this chapter. Specifically, consider the nonlinear dynamical system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \in \mathcal{I}_{x_0}, \tag{9.1}$$

where $x(t) \in \mathcal{D}$, $\mathcal{D}$ is an open subset of $\mathbb{R}^n$ with $0 \in \mathcal{D}$, $f : \mathcal{D} \to \mathbb{R}^n$ is locally Lipschitz continuous on $\mathcal{D}$, and $\mathcal{I}_{x_0} = [0, \tau_{x_0})$, $0 < \tau_{x_0} \leq \infty$, is the maximal interval of existence for the solution $x(\cdot)$ of (9.1). Recall that the point $x_{\rm e} \in \mathcal{D}$ is an *equilibrium point* of (9.1) if $f(x_{\rm e}) = 0$. Furthermore, a subset $\mathcal{D}_{\rm c} \subseteq \mathcal{D}$ is an *invariant set* with respect to (9.1) if $\mathcal{D}_{\rm c}$ contains the orbits of all its points. The following definition introduces the notion of essentially nonnegative vector fields [22, 75, 221].

**Definition 9.1.** Let $f = [f_1, \cdots, f_n]^{\rm T} : \mathcal{D} \to \mathbb{R}^n$, where $\mathcal{D}$ is an open subset of $\mathbb{R}^n$ that contains $\overline{\mathbb{R}}_+^n$. Then $f$ is *essentially nonnegative* if $f_i(x) \geq 0$, for all $i = 1, \ldots, n$, and $x \in \overline{\mathbb{R}}_+^n$ such that $x_i = 0$, where $x_i$ denotes the $i$th element of $x$.

Note that if $f(x) = Ax$, where $A \in \mathbb{R}^{n \times n}$, then $f$ is essentially nonnegative if and only if $A$ is essentially nonnegative [22, 75].

**Proposition 9.1** [22, 75]. Suppose $\overline{\mathbb{R}}_+^n \subset \mathcal{D}$. Then $\overline{\mathbb{R}}_+^n$ is an invariant set with respect to (9.1) if and only if $f : \mathcal{D} \to \mathbb{R}^n$ is essentially nonnegative.

In this chapter we consider controlled nonlinear dynamical systems of the form

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad t \geq 0, \tag{9.2}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, $u(t) \in \mathbb{R}^m$, $t \geq 0$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous and satisfies $f(0) = 0$, and $G : \mathbb{R}^n \to \mathbb{R}^{n \times m}$.

The following definition and proposition are needed for the main results of the chapter.

**Definition 9.2.** The nonlinear dynamical system given by (9.2) is *nonnegative* if for every $x(0) \in \overline{\mathbb{R}}_+^n$ and $u(t) \geq\geq 0$, $t \geq 0$, the solution $x(t)$, $t \geq 0$, to (9.2) is nonnegative.

**Proposition 9.2** [75]. The nonlinear dynamical system given by (9.2) is nonnegative if $f : \mathbb{R}^n \to \mathbb{R}^n$ is essentially nonnegative and $G(x) \geq\geq 0$, $x \in \overline{\mathbb{R}}_+^n$.

It follows from Proposition 9.2 that a nonnegative input signal $G(x(t))u(t)$, $t \geq 0$, is sufficient to guarantee the nonnegativity of the state of (9.2).

Finally, we present a time-varying extension to Proposition 9.2 needed for the main theorems of this chapter. Specifically, we consider the time-varying system

$$\dot{x}(t) = f(t, x(t)) + G(x(t))u(t), \quad x(t_0) = x_0, \quad t \geq t_0, \tag{9.3}$$

where $f : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$. For the following result the definition of nonnegativity holds with (9.2) replaced by (9.3).

**Proposition 9.3.** Consider the time-varying dynamical system (9.3) where $f(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous on $\mathbb{R}^n$ for all $t \in [t_0, \infty)$ and $f(\cdot, x) : [t_0, \infty) \to \mathbb{R}^n$ is continuous on $[t_0, \infty)$ for all $x \in \mathbb{R}^n$. If for every $t \in [t_0, \infty)$, $f(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is essentially nonnegative and $G : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ is nonnegative, then the solution $x(t)$, $t \geq t_0$, to (9.3) is nonnegative.

**Proof.** The result is a direct consequence of Proposition 9.2 by equivalently representing the time-varying system (9.3) as an autonomous nonlinear system by appending another state to represent time. Specifically, defining $y(t - t_0) \triangleq x(t)$ and $y_{n+1}(t - t_0) \triangleq t$, it follows that the solution $x(t)$, $t \geq t_0$, to (9.3) can be equivalently characterized by the solution $y(\tau)$, $\tau \geq 0$, where $\tau \triangleq t - t_0$, to the nonlinear autonomous system

$$\dot{y}(\tau) = f(y_{n+1}(\tau), y(\tau)) + G(y(\tau))\hat{u}(\tau), \quad y(0) = y_0, \quad \tau \geq 0, \tag{9.4}$$

$$\dot{y}_{n+1}(\tau) = 1, \quad y_{n+1}(0) = t_0, \tag{9.5}$$

where $\dot{y}(\cdot)$ and $\dot{y}_{n+1}(\cdot)$ denote differentiation with respect to $\tau$ and $\hat{u}(\tau) \triangleq u(\tau + t_0)$. Now, since $\dot{y}_i(\tau) \geq 0$, $\tau \geq 0$, for $i = 1, \cdots, n$, whenever $y_i(\tau) = 0$ and $G(y(\tau))\hat{u}(\tau) \geq\geq 0$, $\tau \geq 0$, the result is a direct consequence of Proposition 9.2. □

## 9.3. Adaptive Control for Nonlinear Nonnegative Uncertain Dynamical Systems

In this section we consider the problem of characterizing adaptive feedback control laws for nonlinear nonnegative and compartmental uncertain dynamical systems to achieve *set-point* regulation in the nonnegative orthant. Specifically, consider the following controlled nonlinear uncertain system $\mathcal{G}$ given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad t \geq 0, \tag{9.6}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $f : \mathbb{R}^n \to \mathbb{R}^n$ is an *unknown* essentially nonnegative function and satisfies $f(0) = 0$, and $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is an *unknown* input matrix function. The control input $u(\cdot)$ in (9.6) is restricted to the class of *admissible controls* consisting of measurable functions such that $u(t) \in \mathbb{R}^m$, $t \geq 0$.

As discussed in the Introduction, it follows from physical considerations that the state trajectories of nonnegative and compartmental dynamical systems remain in the nonnegative orthant of the state space for nonnegative initial conditions. Hence, in this chapter we design adaptive controllers that guarantee that the controlled system states remain in the nonnegative orthant and converge to a desired equilibrium state. Specifically, for a given desired set point $x_{\rm e} \in \overline{\mathbb{R}}_+^n$, our aim is to design a control input

$u(t)$, $t \geq 0$, such that $\lim_{t\to\infty} \|x(t) - x_{\rm e}\| = 0$. We assume that we have $m$ control inputs and the input matrix function is given by

$$G(x) = \begin{bmatrix} B_{\rm u} G_{\rm n}(x) \\ 0_{(n-m)\times m} \end{bmatrix}, \tag{9.7}$$

where $B_{\rm u} = {\rm diag}[b_1, \cdots, b_m]$ is an *unknown* positive diagonal matrix and $G_{\rm n} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is a known nonnegative matrix function such that $\det G_{\rm n}(x) \neq 0$, $x \in \mathbb{R}^n$. Furthermore, for the nonlinear system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions are satisfied; that is, $f(\cdot)$, $G(\cdot)$, and $u(\cdot)$ satisfy sufficient regularity conditions such that (9.6) has a unique solution forward in time.

**Theorem 9.1.** Consider the nonlinear uncertain system $\mathcal{G}$ given by (9.6) where $f : \mathbb{R}^n \to \mathbb{R}^n$ is essentially nonnegative and $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is given by (9.7). For a given $x_{\rm e} \in \overline{\mathbb{R}}_+^n$, assume there exists a vector $u_{\rm e} \in \mathbb{R}^m$ such that

$$0 = f(x_{\rm e}) + \hat{B} u_{\rm e}, \tag{9.8}$$

where $\hat{B} \triangleq [B_{\rm u}, 0_{m\times(n-m)}]^{\rm T}$. Furthermore, assume there exist a rectangular block-diagonal matrix $K_{\rm g} \triangleq \text{block-diag}[k_{\rm g1}^{\rm T}, \cdots, k_{{\rm g}m}^{\rm T}]$, where $k_{{\rm g}i} \in \mathbb{R}^{s_i}$, $i = 1, \cdots, m$, continuously differentiable functions $V_{{\rm s}i} : \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, m$, and $\hat{V}_{\rm s} : \mathbb{R}^{n-m} \to \mathbb{R}$, and continuous functions $\ell : \mathbb{R}^n \to \mathbb{R}^p$ and $F_i : \mathbb{R}^n \to \mathbb{R}^{s_i}$, $i = 1, \cdots, m$, with $F_i(x - x_{\rm e}) \leq\leq 0$ whenever $x_i = 0$ and $F_i(0) = 0$, $i = 1, \cdots, m$, such that $V_{\rm s}(\cdot)$ is positive definite, radially unbounded, $V_{\rm s}(0) = 0$, $\ell(0) = 0$, and, for all $e \in \mathbb{R}^n$,

$$V_{{\rm s}i}'(e_i) F_i(e) \geq\geq 0, \quad i = 1, \cdots, m, \tag{9.9}$$

$$0 = V_{\rm s}'(e)[f_{\rm e}(e) + \hat{B} K_{\rm g} F(e)] + \ell^{\rm T}(e)\ell(e), \tag{9.10}$$

where

$$V_{\rm s}(e) = V_{\rm s1}(e_1) + \cdots + V_{{\rm s}m}(e_m) + \hat{V}_{\rm s}(e_{m+1}, \cdots, e_n), \tag{9.11}$$

$f_{\rm e}(e) \triangleq f(e+x_{\rm e}) - f(x_{\rm e})$, and $F(e) \triangleq [F_1^{\rm T}(e), \cdots, F_m^{\rm T}(e)]^{\rm T}$. Finally, let $q_i$ and $\hat{q}_i$, $i = 1, \cdots, m$, be positive constants. Then the adaptive feedback control law

$$u(t) = G_{\rm n}^{-1}(x(t))K(t)F(x(t)-x_{\rm e}) + G_{\rm n}^{-1}(x(t))\phi(t), \tag{9.12}$$

where $K(t) \triangleq \text{block-diag}[k_1^{\rm T}(t), \cdots, k_m^{\rm T}(t)]$, $k_i(t) \in \mathbb{R}^{s_i}$, $i = 1, \cdots, m$, $t \geq 0$, and $\phi(t) \in \mathbb{R}^m$, $t \geq 0$, with update laws

$$\dot{k}_i^{\rm T}(t) = -\tfrac{q_i}{2} V_{{\rm s}i}'(x_i(t) - x_{{\rm e}i}) F_i^{\rm T}(x(t)-x_{\rm e}), \quad k_i(0) \leq\leq 0, \quad i = 1, \cdots, m, \tag{9.13}$$

$$\dot{\phi}_i(t) = \begin{cases} 0, & \text{if } \phi_i(t) = 0 \text{ and } V_{{\rm s}i}'(x_i(t)-x_{{\rm e}i}) \geq 0, \\ -\tfrac{\hat{q}_i}{2} V_{{\rm s}i}'(x_i(t) - x_{{\rm e}i}), & \text{otherwise,} \end{cases}$$
$$\phi_i(0) \geq 0, \quad i = 1, \cdots, m, \tag{9.14}$$

guarantees that the solution $(x(t), K(t), \phi(t)) \equiv (x_{\rm e}, K_{\rm g}, u_{\rm e})$ of the closed-loop system given by (9.6), (9.12)–(9.14) is Lyapunov stable. If, in addition, $\ell^{\rm T}(e)\ell(e) > 0$, $e \in \mathbb{R}^n$, $e \neq 0$, then $x(t) \to x_{\rm e}$ as $t \to \infty$ for all $x_0 \in \overline{\mathbb{R}}_+^n$. Furthermore, $x(t) \geq\geq 0$ for all $t \geq 0$ and $x_0 \in \overline{\mathbb{R}}_+^n$.

**Proof.** Let $e(t) \triangleq x(t) - x_{\rm e}$ and note that with $u(t)$, $t \geq 0$, given by (9.12) it follows from (9.6) that

$$\dot{x}(t) = f(x(t)) + G(x(t))G_{\rm n}^{-1}(x(t))K(t)F(x(t)-x_{\rm e}) + G(x(t))G_{\rm n}^{-1}(x(t))\phi(t),$$
$$x(0) = x_0, \quad t \geq 0, \tag{9.15}$$

or, equivalently, using (9.7) and (9.8),

$$\begin{aligned}\dot{e}(t) &= f_{\rm e}(e(t)) + f(x_{\rm e}) + \hat{B}K_{\rm g}F(e(t)) + \hat{B}(K(t)-K_{\rm g})F(x(t)-x_{\rm e}) + \hat{B}\phi(t) \\ &= f_{\rm s}(e(t)) + \hat{B}(K(t)-K_{\rm g})F(x(t)-x_{\rm e}) + \hat{B}(\phi(t)-u_{\rm e}), \quad e(0) = x_0 - x_{\rm e}, \quad t \geq 0,\end{aligned} \tag{9.16}$$

where

$$f_{\rm s}(e) \triangleq f_{\rm e}(e) + \hat{B}K_{\rm g}F(e). \tag{9.17}$$

To show Lyapunov stability of the closed-loop system (9.13), (9.14), and (9.16) consider the Lyapunov function candidate

$$V(e,K,\phi) = V_{\rm s}(e) + {\rm tr}(K-K_{\rm g})^{\rm T}Q^{-1}(K-K_{\rm g}) + (\phi-u_{\rm e})^{\rm T}\hat{Q}^{-1}(\phi-u_{\rm e}), \quad (9.18)$$

or, equivalently,

$$V(e,K,\phi) = V_{\rm s}(e) + \sum_{i=1}^{m}\frac{b_i}{q_i}(k_i-k_{{\rm g}_i})^{\rm T}(k_i-k_{{\rm g}_i}) + \sum_{i=1}^{m}\frac{b_i}{\hat{q}_i}(\phi_i-u_{{\rm e}i})^2,$$

where $Q = {\rm diag}\left[\frac{q_1}{b_1},\cdots,\frac{q_m}{b_m}\right]$ and $\hat{Q} = {\rm diag}\left[\frac{\hat{q}_1}{b_1},\cdots,\frac{\hat{q}_m}{b_m}\right]$. Note that $V(0,K_{\rm g},u_{\rm e}) = 0$ and, since $V_{\rm s}(\cdot)$, $Q$, and $\hat{Q}$ are positive definite, $V(e,K,\phi) > 0$ for all $(e,K,\phi) \neq (0,K_{\rm g},u_{\rm e})$. Furthermore, $V(e,K,\phi)$ is radially unbounded. Now, letting $e(t)$, $t \geq 0$, denote the solution to (9.16) and using (9.13) and (9.14), it follows that the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned}\dot{V}(e(t),K(t),\phi(t)) &= V_{\rm s}'(e(t))\Big[f_{\rm s}(e(t)) + \hat{B}(K(t)-K_{\rm g})F(x(t)-x_{\rm e}) + \hat{B}(\phi(t)-u_{\rm e})\Big] \\ &\quad + \sum_{i=1}^{m}\frac{2b_i}{q_i}\dot{k}_i^{\rm T}(t)(k_i(t)-k_{{\rm g}_i}) + \sum_{i=1}^{m}\frac{2b_i}{\hat{q}_i}(\phi_i(t)-u_{{\rm e}i})\dot{\phi}_i(t) \\ &= -\ell^{\rm T}(e(t))\ell(e(t)) + \sum_{i=1}^{m}V_{{\rm s}_i}'(e_i(t))b_i(k_i(t)-k_{{\rm g}_i})^{\rm T}F_i(e(t)) \\ &\quad + \sum_{i=1}^{m}b_iV_{{\rm s}_i}'(e_i(t))(\phi_i(t)-u_{{\rm e}i}) + \sum_{i=1}^{m}\frac{2b_i}{q_i}\dot{k}_i^{\rm T}(t)(k_i(t)-k_{{\rm g}_i}) \\ &\quad + \sum_{i=1}^{m}\frac{2b_i}{\hat{q}_i}(\phi_i(t)-u_{{\rm e}i})\dot{\phi}_i(t) \\ &= -\ell^{\rm T}(e(t))\ell(e(t)) + \sum_{i=1}^{m}b_i(\phi_i(t)-u_{{\rm e}i})\left[V_{{\rm s}_i}'(e_i(t)) + \frac{2}{\hat{q}_i}\dot{\phi}_i(t)\right], \\ &\qquad t \geq 0. \end{aligned} \quad (9.19)$$

Now, for each $i \in \{1,\cdots,m\}$ and for the two cases given in (9.14), the last term on the right-hand side of (9.19) gives:

*i)* If $\phi_i(t) = 0$ and $V_{{\rm s}_i}'(x_i(t)-x_{{\rm e}i}) \geq 0$, then $\dot{\phi}_i(t) = 0$ and hence, since, using

(9.8), $b_i u_{ei} \geq 0$, $i = 1, \cdots, m$, it follows that

$$b_i(\phi_i(t) - u_{ei}) \left[ V'_{si}(e_i(t)) + \frac{2}{\hat{q}_i}\dot{\phi}_i(t) \right] = -b_i u_{ei} V'_{si}(x_i(t) - x_{ei}) \leq 0, \quad i = 1, \cdots, m.$$

*ii*) Otherwise, $\dot{\phi}_i(t) = -\frac{\hat{q}_i}{2} V'_{s\,i}(x_i(t) - x_{ei})$ and hence

$$b_i(\phi_i(t) - u_{ei}) \left[ V'_{si}(e_i(t)) + \frac{2}{\hat{q}_i}\dot{\phi}_i(t) \right] = 0, \quad i = 1, \cdots, m.$$

Hence, it follows that in either case

$$\begin{aligned} \dot{V}(e(t), K(t), \phi(t)) &\leq -\ell^{\mathrm{T}}(e(t))\ell(e(t)) \\ &\leq 0, \quad t \geq 0, \end{aligned} \tag{9.20}$$

which proves that the solution $(e(t), K(t), \phi(t)) \equiv (0, K_{\mathrm{g}}, u_{\mathrm{e}})$ to (9.13), (9.14), and (9.16) is Lyapunov stable. Furthermore, it follows from Theorem 2 of [42] that $\ell(e(t)) \to 0$ as $t \to \infty$. If, in addition, $\ell^{\mathrm{T}}(e)\ell(e) > 0$, $e \in \mathbb{R}^n$, $e \neq 0$, then $x(t) \to x_{\mathrm{e}}$ as $t \to \infty$ for all $x_0 \in \overline{\mathbb{R}}_+^n$.

Finally, to show that $x(t) \geq\geq 0$, $t \geq 0$, for all $x_0 \in \overline{\mathbb{R}}_+^n$ note that the closed-loop system (9.6), (9.12)–(9.14) is given by

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + \hat{B}K(t)F(x(t) - x_{\mathrm{e}}) + \hat{B}\phi(t) \\ &= \tilde{f}(t, x(t)) + \hat{B}K(t)\tilde{F}(x(t) - x_{\mathrm{e}}) + \hat{B}\phi(t) \\ &= \tilde{f}(t, x(t)) + v(t) + w(t), \quad x(0) = x_0, \quad t \geq 0, \end{aligned} \tag{9.21}$$

where $\tilde{F}(x - x_{\mathrm{e}}) \triangleq [\tilde{F}_1^{\mathrm{T}}(x - x_{\mathrm{e}}), \cdots, \tilde{F}_m^{\mathrm{T}}(x - x_{\mathrm{e}})]^{\mathrm{T}}$, $\tilde{F}_i(x - x_{\mathrm{e}}) \triangleq F_i(x - x_{\mathrm{e}})|_{x_i = 0}$, $i = 1, \cdots, m$,

$$\tilde{f}(t, x) \triangleq f(x) + \hat{B}K(t)[F(x - x_{\mathrm{e}}) - \tilde{F}(x - x_{\mathrm{e}})], \tag{9.22}$$

$$v(t) \triangleq \begin{bmatrix} b_1 k_1^{\mathrm{T}}(t)\tilde{F}_1(x(t) - x_{\mathrm{e}}) \\ \vdots \\ b_m k_m^{\mathrm{T}}(t)\tilde{F}_m(x(t) - x_{\mathrm{e}}) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad w(t) \triangleq \begin{bmatrix} b_1\phi_1(t) \\ \vdots \\ b_m\phi_m(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{9.23}$$

Now, since, by (9.9), (9.13), and (9.14), $k_i^{\mathrm{T}}(t) \leq\leq 0$, $t \geq 0$, $i = 1, \cdots, m$, and $\phi_i(t) \geq 0$, $t \geq 0$, $i = 1, \cdots, m$, and since $\tilde{F}_i(x(t) - x_{\mathrm{e}}) \leq\leq 0$, $t \geq 0$, $i = 1, \cdots, m$, it follows that for every $t \in [0, \infty)$, $\tilde{f}(t, x(t))$ is essentially nonnegative, $v(t) \geq\geq 0$, and $w(t) \geq\geq 0$. Hence, it follows from Proposition 9.3 that $x(t) \geq\geq 0$ for all $t \geq 0$ and $x_0 \in \overline{\mathbb{R}}_+^n$. □

**Remark 9.1.** Note that in the case where $\ell^{\mathrm{T}}(e)\ell(e) > 0$, $e \in \mathbb{R}^n$, $e \neq 0$, the conditions in Theorem 9.1 imply that $x(t) \to x_{\mathrm{e}}$ as $t \to \infty$ and hence it follows from (9.13) and (9.14) that $(x(t), K(t), \phi(t)) \to \mathcal{M} \triangleq \{(x, K, \phi) \in \mathbb{R}^n \times \mathbb{R}^{m \times s} \times \mathbb{R}^m : x = x_{\mathrm{e}}, \dot{K} = 0, \dot{\phi} = 0\}$ as $t \to \infty$, where $s \triangleq s_1 + \cdots + s_m$.

It is important to note that the adaptive control law (9.12)–(9.14) does *not* require the explicit knowledge of the gain matrix $K_{\mathrm{g}}$ and the nonnegative vector $u_{\mathrm{e}}$. All that is required is the existence of a vector $u_{\mathrm{e}}$ and a partially component decoupled Lyapunov function $V_{\mathrm{s}}(e)$ along with the construction of $F(e)$ such that (9.10) and the equilibrium condition (9.8) hold. In the case where $f(x)$ in (9.6) is homogeneous, cooperative; that is, the Jacobian matrix $\frac{\partial f(x)}{\partial x}$ is essentially nonnegative for all $x \in \overline{\mathbb{R}}_+^n$, and the Jacobian matrix $\frac{\partial f(x)}{\partial x}$ is irreducible for all $x \in \overline{\mathbb{R}}_+^n$ [20], it follows from Corollary 1 of [52] that there exists an equilibrium point $x_{\mathrm{e}} \in \mathbb{R}_+^n$ if and only if the zero solution $x(t) \equiv 0$ of the undisturbed ($u(t) \equiv 0$) system (9.6) is globally asymptotically stable for all $x_0 \in \overline{\mathbb{R}}_+^n$. In this case, $x_{\mathrm{e}} \in \mathbb{R}_+^n$ is a globally asymptotically stable equilibrium point of (9.6) with a constant control input $\hat{B}u_{\mathrm{e}}$ satisfying (9.8). Finally, it is important to note that for $i = 1, \cdots, m$, the control input signal $u_i(t)$, $t \geq 0$, in Theorem 9.1 can be negative depending on the values of $x(t)$, $k_i(t)$, and $\phi_i(t)$, $t \geq 0$. However, as is required in nonnegative and compartmental dynamical systems the closed-loop plant states remain nonnegative.

Unlike linear asymptotically stable nonnegative systems, the existence of a component decoupled Lyapunov function (see Theorem 7.2) is not necessarily guaranteed for nonlinear asymptotically stable nonnegative systems. Diagonal-type Lyapunov functions that are not necessarily quadratic take the form given by (9.11) with

$$\hat{V}_{\mathrm{s}}(e_{m+1},\cdots,e_n) = V_{\mathrm{s}m+1}(e_{m+1}) + \cdots + V_{\mathrm{s}n}(e_n) \tag{9.24}$$

and hence are component decoupled. Component decoupled Lyapunov functions play a key role in robust stability with structured uncertainty, neural networks, passive circuits, ecological systems, variable structure systems, power systems, and large-scale systems. For details see [135]. Even though the existence of diagonal-type Lyapunov functions for asymptotically stable nonlinear nonnegative systems is not assured, there do exist classes of nonnegative dynamical systems that do admit component decoupled Lyapunov functions. In particular, if the system dynamics given by (9.17) are in the Persidskii form

$$f_{\mathrm{s}}(e) = Af(e), \tag{9.25}$$

where $A$ is essentially nonnegative and asymptotically stable and $f(\cdot)$ belongs to the set $\mathcal{S}$ given by

$$\begin{aligned}\mathcal{S} \triangleq \{f : \mathbb{R}^n \to \mathbb{R}^n : f(0) = 0,\ f_i(e_i)e_i > 0, e_i \neq 0,\\ \text{and } \textstyle\int_0^{e_i} f_i(\sigma)\mathrm{d}\sigma \to \infty \text{ as } |e_i| \to \infty,\ i = 1,\cdots,m\},\end{aligned} \tag{9.26}$$

then the component decoupled Lyapunov function

$$V_{\mathrm{s}}(e) = \frac{1}{2}\sum_{i=1}^{n} p_i \int_0^{e_i} f_i(\sigma)\mathrm{d}\sigma = \frac{1}{2}\int_{0,\mathrm{path}}^{e} f^{\mathrm{T}}(\sigma)P\mathrm{d}\sigma, \tag{9.27}$$

where $P = \mathrm{diag}[p_1,\cdots,p_n]$ satisfies (7.6) and the path integral in (9.27) is taken over any path joining the origin to $e \in \mathbb{R}^n$, guarantees that the zero solution $e(t) \equiv 0$ to (9.17) with $f_{\mathrm{s}}(e)$ given by (9.25) is globally asymptotically stable [135]. Alternatively, if the system dynamics are given by (9.17) with

$$f_{\mathrm{s}}(e) = Ae - Df(e), \tag{9.28}$$

where $A$ is essentially nonnegative and asymptotically stable, $f(\cdot)$ belongs to the set $\mathcal{S}$, and $D$ is a nonnegative diagonal matrix, then the quadratic Lyapunov function $V_{\rm s}(e) = e^{\rm T}Pe$, where $P$ is the diagonal positive definite solution to (7.6), guarantees that the zero solution $e(t) \equiv 0$ to (9.17) with $f_{\rm s}(e)$ given by (9.28) is globally asymptotically stable [135].

In the case where $F : \mathbb{R}^m \to \mathbb{R}^s$ is only a function of $\hat{e} \triangleq [e_1, \cdots, e_m]^{\rm T}$, the adaptive feedback controller given in Theorem 9.1 can be viewed as an adaptive *output* feedback controller with outputs $y = Cx$, where $C \triangleq [I_m, 0_{m\times(n-m)}]$. In this case, it follows from (9.12) that the explicit knowledge of $x_{\rm u} \triangleq [x_{m+1}, \cdots, x_n]^{\rm T}$ and $x_{\rm eu} = [x_{{\rm e}m+1}, \cdots, x_{{\rm e}n}]^{\rm T}$ as well as $u_{\rm e} \in \mathbb{R}^m$ is not required. In addition, if $f(\cdot)$ in (9.6) is such that $f_{\rm e}(\cdot)$ is continuously differentiable, $f_{\rm e}(0) = 0$, and $f_{\rm e}(e)$ is given by

$$f_{\rm e}(e) = \begin{bmatrix} A_{11} + A_{11}(\hat{e}) & A_{12} \\ A_{21} & A_{22} \end{bmatrix} e, \tag{9.29}$$

where $A_{11} : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ is a continuous and essentially nonnegative, $A_{11} \in \mathbb{R}^{m\times m}$ is essentially nonnegative, $A_{12} \in \mathbb{R}^{m\times(n-m)}$ is nonnegative, $A_{21} \in \mathbb{R}^{(n-m)\times m}$ is nonnegative, and $A_{22} \in \mathbb{R}^{(n-m)\times(n-m)}$ is essentially nonnegative, and if (9.6) is stabilizable and feedback linearizable, then there always exists a rectangular block-diagonal matrix $K_{\rm g} \in \mathbb{R}^{m\times s}$ such that (9.10) holds. Furthermore, in this case $V_{\rm s}(\cdot)$ need not be known. To see this, let $A_{11}(\hat{e})\hat{e}$ be parameterized as $A_{11}(\hat{e})\hat{e} = [\theta_1^{\rm T}\hat{F}_1(\hat{e}), \cdots, \theta_m^{\rm T}\hat{F}_m(\hat{e})]^{\rm T}$, where $\hat{F}(\cdot) \triangleq [\hat{F}_1^{\rm T}(\cdot), \cdots, \hat{F}_m^{\rm T}(\cdot)]^{\rm T}$ is a known function such that $\hat{F}_i : \mathbb{R}^m \to \mathbb{R}^{\hat{s}_i}$ satisfies $\hat{F}_i(\hat{x} - \hat{x}_{\rm e}) \leq\leq 0$ whenever $x_i = 0$ and $\hat{F}_i(0) = 0$, $i = 1, \cdots, m$, $\hat{x}_{\rm e} \triangleq [x_{{\rm e}1}, \cdots, x_{{\rm e}m}]^{\rm T}$, and $\theta_i \in \mathbb{R}^{\hat{s}_i}$, $i = 1, \cdots, m$, are unknown constant parameters such that $\theta_i \geq\geq 0$, $i = 1, \cdots, m$. Now, by viewing $\hat{e} = Ce$ as an output, the zero dynamics of $\dot{e}(t) = f_{\rm e}(e(t)) + G(e(t) + x_{\rm e})u(t)$, $e(0) = e_0$, $t \geq 0$, with $f_{\rm e}(e)$ given by (9.29) are given by

$$\dot{z}(t) = A_{22}z(t), \quad z(0) = z_0, \quad t \geq 0, \tag{9.30}$$

where $z \triangleq [e_{m+1}, \cdots, e_m]^{\rm T}$. Since $C\hat{B} = (C\hat{B})^{\rm T} > 0$, it follows from Theorem 2 of [60] (see also [126]) that if $A_{22}$ is asymptotically stable, then there exist matrices $P \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{n\times p}$, and $\hat{\Phi} \in \mathbb{R}^{m\times m}$, with $P$ positive definite, and a positive constant $\varepsilon$ such that

$$0 = (\hat{A} + \hat{B}\hat{\Phi}C)^{\mathrm{T}}P + P(\hat{A} + \hat{B}\hat{\Phi}C) + \varepsilon P + L^{\mathrm{T}}L, \tag{9.31}$$

$$0 = \hat{B}^{\mathrm{T}}P - C, \tag{9.32}$$

where $\hat{A}$ is given by

$$\hat{A} \triangleq \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{9.33}$$

Note that it follows from (9.32) that $P$ has the form $P = \text{block-diag}[P_1, P_2]$, where $P_1 \triangleq \text{diag}[p_1, \cdots, p_m] \in \mathbb{R}^{m\times m}$ and $P_2 \in \mathbb{R}^{(n-m)\times(n-m)}$. Now, defining $\Phi \triangleq \text{diag}[\varphi_1, \cdots, \varphi_m]$ such that $\Phi \leq \frac{1}{2}(\hat{\Phi} + \hat{\Phi}^{\mathrm{T}})$ and $\Phi \leq 0$, it follows that

$$\begin{aligned}(\hat{A} + \hat{B}\Phi C)^{\mathrm{T}}P + P(\hat{A} + \hat{B}\Phi C) &= (\hat{A} + \hat{B}\hat{\Phi}C)^{\mathrm{T}}P + P(\hat{A} + \hat{B}\hat{\Phi}C) \\ &\quad + C^{\mathrm{T}}(\Phi - \hat{\Phi})^{\mathrm{T}}B^{\mathrm{T}}P + PB(\Phi - \hat{\Phi})C \\ &= -\varepsilon P - L^{\mathrm{T}}L + C^{\mathrm{T}}[2\Phi - (\hat{\Phi}^{\mathrm{T}} + \hat{\Phi})]C \\ &\leq -\varepsilon P, \end{aligned} \tag{9.34}$$

and thus $\hat{A} + \hat{B}\Phi C$ is asymptotically stable. Now, with $k_{\mathrm{g}i}^{\mathrm{T}} = [-\theta_i^{\mathrm{T}}/b_i, \varphi_i] \leq\leq 0$, $i = 1, \cdots, m$, and $F_i(e) = [\hat{F}_i^{\mathrm{T}}(\hat{e}), e_i]^{\mathrm{T}}$, it follows that

$$\begin{aligned} f_{\mathrm{s}}(e) &= f_{\mathrm{e}}(e) + \hat{B}K_{\mathrm{g}}F(\hat{e}) \\ &= \begin{bmatrix} A_{11} + A_{11}(\hat{e}) & A_{12} \\ A_{21} & A_{22} \end{bmatrix} e + \begin{bmatrix} B_{\mathrm{u}} \\ 0_{(n-m)\times m} \end{bmatrix} \begin{bmatrix} -\theta_1^{\mathrm{T}}\hat{F}_1(\hat{e})/b_1 + \varphi_1 e_1 \\ \vdots \\ -\theta_m^{\mathrm{T}}\hat{F}_m(\hat{e})/b_m + \varphi_m e_m \end{bmatrix} \\ &= \begin{bmatrix} A_{11} + A_{11}(\hat{e}) & A_{12} \\ A_{21} & A_{22} \end{bmatrix} e - \begin{bmatrix} A_{11}(\hat{e}) & 0 \\ 0 & 0 \end{bmatrix} e + \begin{bmatrix} B_{\mathrm{u}} \\ 0_{(n-m)\times m} \end{bmatrix} \begin{bmatrix} \varphi_1 e_1 \\ \vdots \\ \varphi_m e_m \end{bmatrix} \\ &= (\hat{A} + \hat{B}\Phi C)e. \end{aligned} \tag{9.35}$$

In this case, with $V_{\mathrm{s}}(e) = e^{\mathrm{T}}Pe$, the adaptive feedback controller (9.12) with update laws (9.13), (9.14), or, equivalently,

$$\dot{k}_i^{\mathrm{T}}(t) = -q_i(x_i(t) - x_{\mathrm{e}i})F_i^{\mathrm{T}}(\hat{x}(t) - \hat{x}_{\mathrm{e}}), \quad k_i(0) \leq\leq 0, \quad i = 1, \cdots, m, \tag{9.36}$$

$$\dot{\phi}_i(t) = \begin{cases} 0, & \text{if } \phi_i(t) = 0 \text{ and } x_i(t) \geq x_{\mathrm{e}i}, \\ -\hat{q}_i(x_i(t) - x_{\mathrm{e}i}), & \text{otherwise}, \end{cases} \quad \phi(0) \geq 0, \quad i = 1, \cdots, m, \tag{9.37}$$

with $q_i$ and $\hat{q}_i$ in (9.13) and (9.14) replaced by $\frac{q_i}{p_i}$ and $\frac{\hat{q}_i}{p_i}$, respectively, guarantees global asymptotic stability of the *nonlinear* uncertain dynamical system (9.6) with $f(x) = f_{\mathrm{e}}(e) + f(x_{\mathrm{e}})$, where $f_{\mathrm{e}}(e)$ satisfies (9.29).

It is important to note that the adaptive feedback controller (9.12) with update laws (9.36), (9.37) does *not* require knowledge of the system dynamics (9.29). All that is required is that $A_{22}$ in (9.29) be asymptotically stable. Finally, in the case where $A_{11}(e) = 0$ and $G_{\mathrm{n}}(x) = I_m$, we can simply take $F(e) = \hat{e}$. In this case, the adaptive feedback controller (9.12) with update laws (9.13), (9.14) collapses to

$$u_i(t) = k_i(t)(x_i(t) - x_{\mathrm{e}i}) + \phi_i(t), \quad i = 1, \cdots, m, \tag{9.38}$$

$$\dot{k}_i(t) = -q_i(x_i(t) - x_{\mathrm{e}i})^2, \quad k_i(0) \leq 0, \quad i = 1, \cdots, m, \tag{9.39}$$

$$\dot{\phi}_i(t) = \begin{cases} 0, & \text{if } \phi_i(t) = 0 \text{ and } x_i(t) \geq x_{\mathrm{e}i}, \\ -\hat{q}_i(x_i(t) - x_{\mathrm{e}i}), & \text{otherwise}, \end{cases} \quad \phi(0) \geq 0, \quad i = 1, \cdots, m. \tag{9.40}$$

This is precisely the result given in Chapter 7 (see also [85, 86]).

In the case where our objective is zero set-point regulation, that is, $x_{\mathrm{e}} = 0$, the adaptive controller given in Theorem 9.1 can be simplified. Specifically, since in this case $x(t) \geq x_{\mathrm{e}} = 0$, $t \geq 0$, and condition (9.8) is trivially satisfied with $u_{\mathrm{e}} = 0$, we can set $\phi(t) \equiv 0$ so that the update law (9.14) is superfluous. This result is given in the following theorem.

**Theorem 9.2.** Consider the nonlinear uncertain system $\mathcal{G}$ given by (9.6) where $f : \mathbb{R}^n \to \mathbb{R}^n$ is essentially nonnegative and $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is nonnegative and is given by (9.7). Assume there exist a rectangular block-diagonal matrix $K_{\rm g} \triangleq \text{block-diag}[k_{{\rm g}1}^{\rm T}, \cdots, k_{{\rm g}m}^{\rm T}]$, where $k_{{\rm g}i} \in \mathbb{R}^{s_i}$, $i = 1, \cdots, m$, continuously differentiable functions $V_{{\rm s}i} : \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, m$, and $\hat{V}_{\rm s} : \mathbb{R}^{n-m} \to \mathbb{R}$, and continuous functions $F_i : \mathbb{R}^n \to \mathbb{R}^{s_i}$, with $F_i(0) = 0$, $i = 1, \ldots, m$, and $\ell : \mathbb{R}^n \to \mathbb{R}$ such that $V_{\rm s}(\cdot)$ and $\ell(\cdot)$ are positive definite in the nonnegative orthant and $V_{\rm s}(\cdot)$ is radially unbounded in the nonnegative orthant, $V_{\rm s}(0) = 0$, $\ell(0) = 0$, and, for all $x \in \overline{\mathbb{R}}_+^n$,

$$0 = V_{\rm s}'(x) f_{\rm s}(x) + \ell(x), \tag{9.41}$$

where

$$V_{\rm s}(x) \triangleq V_{{\rm s}1}(x_1) + \cdots + V_{{\rm s}m}(x_m) + \hat{V}_{\rm s}(x_{m+1}, \cdots, x_n), \tag{9.42}$$

$$f_{\rm s}(x) \triangleq f(x) + \hat{B} K_{\rm g} F(x), \tag{9.43}$$

and $F(x) \triangleq [F_1^{\rm T}(x), \cdots, F_m^{\rm T}(x)]^{\rm T}$. Finally, let $q_i$, $i = 1, \cdots, m$, be positive constants. Then the adaptive feedback control law

$$u(t) = G_{\rm n}^{-1}(x(t)) K(t) F(x(t)), \tag{9.44}$$

where $K(t) \triangleq \text{block-diag}[k_1^{\rm T}(t), \cdots, k_m^{\rm T}(t)]$, with $k_i(t) \in \mathbb{R}^{s_i}$, $i = 1, \cdots, m$, satisfying

$$\dot{k}_i^{\rm T}(t) = -\tfrac{q_i}{2} V_{{\rm s}i}'(x_i(t)) F_i^{\rm T}(x(t)), \quad k_i^{\rm T}(0) = k_{i0}^{\rm T}, \quad i = 1, \cdots, m, \tag{9.45}$$

guarantees that the solution $(x(t), K(t)) \equiv (0, K_{\rm g})$ of the closed-loop system given by (9.6), (9.44), and (9.45) is Lyapunov stable and $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \overline{\mathbb{R}}_+^n$. Furthermore, $x(t) \geq\geq 0$ for all $t \geq 0$ and $x_0 \in \overline{\mathbb{R}}_+^n$.

**Proof.** The proof is identical to the proof of Theorem 9.1. □

**Remark 9.2.** Theorem 9.2 provides considerable simplification in the case where (9.6) is feedback linearizable. Specifically, in this case $f_{\rm s}(x) = A_{\rm s} x$ is asymptotically stable and essentially nonnegative and hence it follows from Theorem 3.2 of [75] that there exist $p >> 0$ and $r >> 0$ such that $V_{\rm s}(x) = p^{\rm T}x$ and $\ell(x) = r^{\rm T}x$ satisfy (9.41). In this case, the update law (9.45) can be equivalently written as

$$\dot{k}_i^{\rm T}(t) = -\tfrac{q_i}{2}F_i^{\rm T}(x(t)), \quad k_i^{\rm T}(0) = k_{i0}^{\rm T}, \quad i = 1, \cdots, m, \tag{9.46}$$

with $q_i$ in (9.45) replaced by $\frac{q_i}{p_i}$, where $p_i$ is the $i$th component of $p$. Furthermore, condition (9.9) is not required in Theorem 9.2 and thus $k_i(t)$, $i = 1, \cdots, m$, is not necessarily a nonincreasing function. Thus, the initial values $k_i(0)$, $i = 1, \cdots, m$, can be chosen arbitrarily.

**Remark 9.3.** In the case where $f_{\rm s}(x)$ is compartmental, that is, the $i$th component of $f_{\rm s}(x)$ is given by

$$f_{{\rm s}i}(x) = -\hat{a}_{ii}(x) + \sum_{j=1, i\neq j}^{n} [\hat{a}_{ij}(x) - \hat{a}_{ji}(x)], \tag{9.47}$$

for all $i \in \{1, \cdots, n\}$, where $\hat{a}_{ii}(x) \geq 0$ denotes the instantaneous rate of flow of material loss of the $i$th compartment and $\hat{a}_{ji}(x) \geq 0$ denotes the instantaneous rate of material flow from $i$th compartment to $j$th compartment, it follows that by taking $V_{\rm s}(x) = \hat{\imath}^{\rm T}x$, where $\hat{\imath} \triangleq [1, \cdots, 1]^{\rm T}$, the update law (9.45) can be equivalently written as (9.46).

Finally, we generalize Theorem 9.1 to the case where the input matrix is not necessarily nonnegative. Specifically, here we assume that $b_i$ in Theorem 9.1 is unknown but $\operatorname{sgn} b_i$ is known for all $i = 1, \cdots, m$.

**Theorem 9.3.** Consider the nonlinear uncertain system $\mathcal{G}$ given by (9.6) where $f : \mathbb{R}^n \to \mathbb{R}^n$ is essentially nonnegative and $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is given by (9.7). For a given $x_{\rm e} \in \overline{\mathbb{R}}_+^n$, assume there exists a vector $u_{\rm e} \in \mathbb{R}^m$ such that (9.8) is satisfied with $f(x_{\rm e}) \leq\leq 0$. Furthermore, assume there exist a rectangular block-diagonal matrix

$K_{\rm g} \triangleq$ block-diag$[k_{\rm g1}^{\rm T}, \cdots, k_{{\rm g}m}^{\rm T}]$, where $k_{{\rm g}i} \in \mathbb{R}^{s_i}$, $i = 1, \cdots, m$, continuously differentiable functions $V_{{\rm s}i} : \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, m$, and $\hat{V}_{\rm s} : \mathbb{R}^{n-m} \to \mathbb{R}$, and continuous functions $\ell : \mathbb{R}^n \to \mathbb{R}^p$ and $F_i : \mathbb{R} \to \mathbb{R}^{s_i}$, with $F_i(x - x_{\rm e}) \leq\leq 0$ whenever $x_i = 0$ and $F_i(0) = 0$, $i = 1, \ldots, m$, such that $V_{\rm s}(\cdot)$ is positive definite, radially unbounded, $V_{\rm s}(0) = 0$, $\ell(0) = 0$, and, for all $e \in \mathbb{R}^n$, (9.9) and (9.10) hold. Finally, let $q_i$ and $\hat{q}_i$, $i = 1, \cdots, m$, be positive constants. Then the adaptive feedback control law (9.12), where $K(t) \triangleq$ block-diag$[k_1^{\rm T}(t), \cdots, k_m^{\rm T}(t)]$, $k_i(t) \in \mathbb{R}^{s_i}$, $i = 1, \cdots, m$, $t \geq 0$, with update laws

$$\dot{k}_i^{\rm T}(t) = -(\operatorname{sgn} b_i)\tfrac{q_i}{2} V_{{\rm s}i}'(x_i(t) - x_{{\rm e}i}) F_i^{\rm T}(x(t) - x_{\rm e}), \quad i = 1, \cdots, m, \tag{9.48}$$

$$\dot{\phi}_i(t) = \begin{cases} 0, & \text{if } \phi_i(t) = 0 \text{ and } V_{{\rm s}i}'(x_i(t) - x_{{\rm e}i}) \geq 0, \\ -(\operatorname{sgn} b_i)\tfrac{\hat{q}_i}{2} V_{{\rm s}i}'(x_i(t) - x_{{\rm e}i}), & \text{otherwise}, \end{cases} \qquad i = 1, \cdots, m, \tag{9.49}$$

where $k_i(0)$ and $\phi_i(0)$ are such that $(\operatorname{sgn} b_i)k_i(0) \leq\leq 0$ and $(\operatorname{sgn} b_i)\phi_i(0) \geq 0$, respectively, guarantees that the solution $(x(t), K(t), \phi(t)) \equiv (x_{\rm e}, K_{\rm g}, u_{\rm e})$ of the closed-loop system given by (9.6), (9.12), (9.48), and (9.49) is Lyapunov stable. If, in addition, $\ell^{\rm T}(e)\ell(e) > 0$, $e \in \mathbb{R}^n$, $e \neq 0$, then $x(t) \to x_{\rm e}$ as $t \to \infty$ for all $x_0 \in \overline{\mathbb{R}}_+^n$. Furthermore, $x(t) \geq\geq 0$ for all $t \geq 0$ and $x_0 \in \overline{\mathbb{R}}_+^n$.

**Proof.** The proof is similar to that of Theorem 9.1 with $Q$ and $\hat{Q}$ replaced by $Q = \operatorname{diag}\left[\frac{q_1}{|b_1|}, \cdots, \frac{q_m}{|b_m|}\right]$ and $\hat{Q} = \operatorname{diag}\left[\frac{\hat{q}_1}{|b_1|}, \cdots, \frac{\hat{q}_m}{|b_m|}\right]$, respectively. □

Note that the adaptive controller given in Theorem 9.3 does not destroy nonnegativity with respect to the plant states. In particular, the closed-loop system dynamics are given by (9.21). Now, it can be seen from (9.9), (9.48), and (9.49) that if $b_i$ is negative, then $k_i(t) \geq\geq 0$, $t \geq 0$, and $\phi_i(t) \leq 0$, $t \geq 0$, $i = 1, \cdots, m$, and hence $v(t) \geq\geq 0$, $t \geq 0$, and $w(t) \geq\geq 0$, $t \geq 0$. Hence, by Proposition 9.3, $x(t) \geq\geq 0$, $t \geq 0$, for all $x_0 \in \overline{\mathbb{R}}_+^n$.

## 9.4. Adaptive Control for Nonlinear Nonnegative Dynamical Systems with Nonnegative Control

As discussed in the Introduction, control (source) inputs of drug delivery systems for physiological processes are usually constrained to be nonnegative as are the system states. Hence, in this section we develop adaptive control laws for nonnegative systems with nonnegative control inputs. Specifically, for a given desired set point $x_{\rm e} \in \overline{\mathbb{R}}_+^n$, our aim is to design a control input $u(t)$, $t \geq 0$, such that $\lim_{t\to\infty} \|x(t) - x_{\rm e}\| = 0$. We assume that control inputs are injected directly into $m$ separate compartments and the input matrix function is given by

$$G(x) = \begin{bmatrix} B_{\rm u} G_{\rm n}(x) \\ 0_{(n-m)\times m} \end{bmatrix}, \tag{9.50}$$

where $B_{\rm u} = {\rm diag}[b_1, \cdots, b_m]$ is an *unknown* nonnegative diagonal matrix and $G_{\rm n} = {\rm diag}[g_{{\rm n}1}(x), \cdots, g_{{\rm n}m}(x)]$, where $g_{{\rm n}i} : \overline{\mathbb{R}}_+^n \to \mathbb{R}_+$, $i = 1, \cdots, m$, is a known nonnegative diagonal matrix function. For compartmental systems this assumption is not restrictive since control inputs correspond to control inflows to each individual compartment.

**Theorem 9.4.** Consider the nonlinear uncertain system $\mathcal{G}$ given by (9.6) where $f : \mathbb{R}^n \to \mathbb{R}^n$ is essentially nonnegative and $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is nonnegative and is given by (9.50). For a given $x_{\rm e} \in \overline{\mathbb{R}}_+^n$, assume there exists a nonnegative vector $u_{\rm e} \in \overline{\mathbb{R}}_+^m$ such that (9.8) is satisfied, where $\hat{B} \triangleq [B_{\rm u}, 0_{m\times(n-m)}]^{\rm T}$, and the equilibrium point $x_{\rm e}$ of (9.6) is asymptotically stable for all $x_0 \in \overline{\mathbb{R}}_+^n$ with $u(t) \equiv u_{\rm e}$. Furthermore, assume there exist continuously differentiable functions $V_{{\rm s}i} : \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, m$, and $\hat{V}_{\rm s} : \mathbb{R}^{n-m} \to \mathbb{R}$, and continuous functions $F_i : \mathbb{R}^n \to \mathbb{R}^{s_i}$, with $F_i(0) = 0$, $i = 1, \cdots, m$, and $\ell : \mathbb{R}^n \to \mathbb{R}^p$ such that $V_{\rm s}(\cdot)$ is positive definite, radially unbounded, $V_{\rm s}(0) = 0$, $\ell(0) = 0$, $i = 1, \cdots, m$, and, for all $e \in \mathbb{R}^n$,

$$0 = V_{\rm s}'(e) f_{\rm e}(e) + \ell^{\rm T}(e)\ell(e), \tag{9.51}$$

and (9.9) holds, where $V_{\rm s}(e)$ is given by (9.11), $f_{\rm e}(e) \triangleq f(e+x_{\rm e}) - f(x_{\rm e})$, and $F(e) \triangleq [F_1^{\rm T}(e), \cdots, F_m^{\rm T}(e)]^{\rm T}$. Finally, let $q_i$ and $\hat{q}_i$, $i = 1, \cdots, m$, be positive constants. Then the adaptive feedback control law

$$u_i(t) = \max\{0, \hat{u}_i(t)\}, \quad i = 1, \cdots, m, \tag{9.52}$$

where

$$\hat{u}_i(t) = g_{{\rm n}i}^{-1}(x(t))k_i^{\rm T}(t)F_i(x(t) - x_{\rm e}) + g_{{\rm n}i}^{-1}(x(t))\phi_i(t), \quad i = 1, \cdots, m, \tag{9.53}$$

$k_i(t) \in \mathbb{R}^{s_i}$, $t \geq 0$, $i = 1, \cdots, m$, and $\phi_i(t) \in \mathbb{R}$, $t \geq 0$, $i = 1, \cdots, m$, with update laws

$$\dot{k}_i^{\rm T}(t) = \begin{cases} 0, & \text{if } \hat{u}_i(t) \leq 0, \\ -\frac{q_i}{2}V_{{\rm s}i}'(x_i(t) - x_{{\rm e}i})F_i^{\rm T}(x(t) - x_{\rm e}), & \text{otherwise}, \end{cases}$$
$$k_i(0) \leq\leq 0, \quad i = 1, \cdots, m, \tag{9.54}$$

$$\dot{\phi}_i(t) = \begin{cases} 0, & \text{if } \phi_i(t) = 0 \text{ and } V_{{\rm s}i}'(x_i(t) - x_{{\rm e}i}) \geq 0, \\ & \text{or if } \hat{u}_i(t) \leq 0, \\ -\frac{\hat{q}_i}{2}V_{{\rm s}i}'(x_i(t) - x_{{\rm e}i}), & \text{otherwise}, \end{cases}$$
$$\phi_i(0) = 0, \quad i = 1, \cdots, m, \tag{9.55}$$

guarantees that the solution $(x(t), K(t), \phi(t)) \equiv (x_{\rm e}, K_{\rm g}, u_{\rm e})$, where $K(t) \triangleq$ block-diag$[k_1^{\rm T}(t), \cdots, k_m^{\rm T}(t)]$ and $K_{\rm g} \triangleq$ block-diag$[k_{{\rm g}1}^{\rm T}, \cdots, k_{{\rm g}m}^{\rm T}] \leq\leq 0$, of the closed-loop system given by (9.6), (9.52), (9.54), and (9.55) is Lyapunov stable. If, in addition, $\ell^{\rm T}(e)\ell(e) > 0$, $e \in \mathbb{R}^n$, $e \neq 0$, then $x(t) \to x_{\rm e}$ as $t \to \infty$ for all $x_0 \in \overline{\mathbb{R}}_+^n$. Furthermore, $u(t) \geq\geq 0$ and $x(t) \geq\geq 0$ for all $t \geq 0$ and $x_0 \in \overline{\mathbb{R}}_+^n$.

**Proof.** First, let $e(t) \triangleq x(t) - x_{\rm e}$ and define $K_u(t) \triangleq$ block-diag$[k_{u1}^{\rm T}(t), \cdots, k_{um}^{\rm T}(t)]$ and $\phi_u(t) \triangleq [\phi_{u1}(t), \cdots, \phi_{um}(t)]^{\rm T}$, where

$$k_{ui}(t) = \begin{cases} 0, & \text{if } \hat{u}_i(t) \leq 0, \\ k_i(t), & \text{otherwise}, \end{cases} \quad i = 1, \cdots, m, \tag{9.56}$$

$$\phi_{ui}(t) = \begin{cases} 0, & \text{if } \hat{u}_i(t) \leq 0, \\ \phi_i(t), & \text{otherwise}, \end{cases} \quad i = 1, \cdots, m. \tag{9.57}$$

Now, note that with $u(t)$, $t \geq 0$, given by (9.52) it follows from (9.6) that

$$\dot{x}(t) = f(x(t)) + \hat{B}K_u(t)F(x(t) - x_{\rm e}) + \hat{B}\phi_u(t), \quad x(0) = x_0, \quad t \geq 0, \tag{9.58}$$

or, equivalently, using (9.8),

$$\dot{e}(t) = f_{\rm e}(e(t)) + \hat{B}K_u(t)F(x(t)-x_{\rm e}) + \hat{B}(\phi_u(t)-u_{\rm e}), \quad e(0) = x_0 - x_{\rm e}, \quad t \ge 0. \tag{9.59}$$

To show Lyapunov stability of the closed-loop system (9.54), (9.55), and (9.59) consider the Lyapunov function candidate

$$V(e, K, \phi) = V_{\rm s}(e) + {\rm tr}(K - K_{\rm g})^{\rm T} Q^{-1}(K - K_{\rm g}) + (\phi - u_{\rm e})^{\rm T}\hat{Q}^{-1}(\phi - u_{\rm e}), \tag{9.60}$$

or, equivalently,

$$V(e, K, \phi) = V_{\rm s}(e) + \sum_{i=1}^{m} \frac{b_i}{q_i}(k_i - k_{{\rm g}_i})^{\rm T}(k_i - k_{{\rm g}_i}) + \sum_{i=1}^{m} \frac{b_i}{\hat{q}_i}(\phi_i - u_{{\rm e}i})^2,$$

where $Q = [\frac{q_1}{b_1}, \cdots, \frac{q_m}{b_m}]$ and $\hat{Q} = {\rm diag}[\frac{\hat{q}_1}{b_1}, \cdots, \frac{\hat{q}_m}{b_m}]$. Note that $V(0, K_{\rm g}, u_{\rm e}) = 0$ and, since $V_{\rm s}(\cdot)$, $Q$, and $\hat{Q}$ are positive definite, $V(e, K, \phi) > 0$ for all $(e, K, \phi) \neq (0, K_{\rm g}, u_{\rm e})$. Furthermore, $V(e, K, \phi)$ is radially unbounded. Now, letting $e(t)$, $t \ge 0$, denote the solution to (9.59) and using (9.54) and (9.55), it follows that the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned}
\dot{V}(e(t), K(t), \phi(t)) &= V_{\rm s}'(e(t))\Big[f_{\rm e}(e(t)) + \hat{B}K_u(t)F(x(t) - x_{\rm e}) + \hat{B}(\phi_u(t) - u_{\rm e})\Big] \\
&\quad + 2{\rm tr}(K(t) - K_{\rm g})^{\rm T} Q^{-1}\dot{K}(t) + 2(\phi(t) - u_{\rm e})^{\rm T}\hat{Q}^{-1}\dot{\phi}(t) \\
&= -\ell^{\rm T}(e(t))\ell(e(t)) + \sum_{i=1}^{m} V_{{\rm s}i}'(e_i(t))b_i k_{ui}^{\rm T}(t)F_i(e(t)) \\
&\quad + \sum_{i=1}^{m} b_i V_{{\rm s}i}'(e_i(t))(\phi_{ui}(t) - u_{{\rm e}i}) + \sum_{i=1}^{m} \frac{2b_i}{q_i}\dot{k}_i^{\rm T}(t)(k_i(t) - k_{{\rm g}_i}) \\
&\quad + \sum_{i=1}^{m} \frac{2b_i}{\hat{q}_i}(\phi_i(t) - u_{{\rm e}i})\dot{\phi}_i(t) \\
&= -\ell^{\rm T}(e(t))\ell(e(t)) \\
&\quad + \sum_{i=1}^{m} b_i\Big[V_{{\rm s}i}'(e_i(t))k_{ui}^{\rm T}(t)F_i(e(t)) + \frac{2}{q_i}\dot{k}_i^{\rm T}(t)(k_i(t) - k_{{\rm g}_i})\Big] \\
&\quad + \sum_{i=1}^{m} b_i\Big[V_{{\rm s}i}'(e_i(t))(\phi_{ui}(t) - u_{{\rm e}i}) + \frac{2}{\hat{q}_i}(\phi_i(t) - u_{{\rm e}i})\dot{\phi}_i(t)\Big].
\end{aligned} \tag{9.61}$$

Now, for each $i \in \{1, \cdots, m\}$ and for the two cases given in (9.54) and (9.55), the last two terms on the right-hand side of (9.61) give:

*i)* If $\hat{u}_i(t) \leq 0$, then $k_{ui}(t) = 0$, $\phi_{ui}(t) = 0$, $\dot{k}_i(t) = 0$, and $\dot{\phi}_i(t) = 0$. Furthermore, since $\phi_i(t) \geq 0$ and $k_i(t) \leq 0$ for all $t \geq 0$, it follows from (9.53) that $\hat{u}_i(t) \leq 0$ only if $F_i(x(t) - x_{\rm e}) \geq\geq 0$ which implies $V'_{{\rm s}i}(e_i(t)) \geq 0$ by (9.9) and hence

$$
\begin{aligned}
V'_{{\rm s}i}(e_i(t))k_{ui}^{\rm T}(t)F_i(e(t)) + \frac{2}{q_i}\dot{k}_i^{\rm T}(t)(k_i(t) - k_{{\rm g}i}) &= 0,\\
V'_{{\rm s}i}(e_i(t))(\phi_{ui}(t) - u_{{\rm e}i}) + \frac{2}{\hat{q}_i}(\phi_i(t) - u_{{\rm e}i})\dot{\phi}_i(t) &= -V'_{{\rm s}i}(e_i(t))u_{{\rm e}i} \leq 0.
\end{aligned}
$$

*ii)* Otherwise, $k_{ui}(t) = k_i(t)$ and $\phi_{ui}(t) = \phi_i(t)$ and hence

$$
\begin{aligned}
&V'_{{\rm s}i}(e_i(t))k_{ui}^{\rm T}(t)F_i(e(t)) + \frac{2}{q_i}\dot{k}_i^{\rm T}(t)(k_i(t) - k_{{\rm g}i}) = V'_{{\rm s}i}(e_i(t))k_{{\rm g}i}^{\rm T}(t)F_i(e(t)) \leq 0,\\
&V'_{{\rm s}i}(e_i(t))(\phi_{ui}(t) - u_{{\rm e}i}) + \frac{2}{\hat{q}_i}(\phi_i(t) - u_{{\rm e}i})\dot{\phi}_i(t)\\
&\qquad\qquad = \begin{cases} -V'_{{\rm s}i}(e_i(t))u_{{\rm e}i} \leq 0, & \text{if } \phi_i(t) = 0 \text{ and } V'_{{\rm s}i}(x_i(t) - x_{{\rm e}i}) \geq 0,\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}
$$

Hence, it follows that in either case

$$
\begin{aligned}
\dot{V}(e(t), K(t), \phi(t)) &\leq -\ell^{\rm T}(e(t))\ell(e(t))\\
&\leq 0, \quad t \geq 0,
\end{aligned}
\tag{9.62}
$$

which proves that the solution $(e(t), K(t), \phi(t)) \equiv (0, K_{\rm g}, u_{\rm e})$ to (9.54), (9.55), and (9.59) is Lyapunov stable. Furthermore, it follows from Theorem 2 of [42] that $\ell(e(t)) \to 0$ as $t \to \infty$. If, in addition, $\ell^{\rm T}(e)\ell(e) > 0$, $e \in \mathbb{R}^n$, $e \neq 0$, then $x(t) \to x_{\rm e}$ as $t \to \infty$ for all $x_0 \in \overline{\mathbb{R}}_+^n$.

Finally, $u(t) \geq\geq 0$, $t \geq 0$, is a restatement of (9.52). Now, since $G(x(t)) \geq\geq 0$, $t \geq 0$, and $u(t) \geq\geq 0$, $t \geq 0$, it follows from Proposition 9.2 that $x(t) \geq\geq 0$ for all $t \geq 0$ and $x_0 \in \overline{\mathbb{R}}_+^n$. □

**Remark 9.4.** Note that in the case where $\ell^{\rm T}(e)\ell(e) > 0$, $e \in \mathbb{R}^n$, $e \neq 0$, the conditions in Theorem 9.4 imply that $x(t) \to x_{\rm e}$ as $t \to \infty$ and hence it follows from (9.54) and (9.55) that $(x(t), K(t), \phi(t)) \to \mathcal{M} \triangleq \{(x, K, \phi) \in \mathbb{R}^n \times \mathbb{R}^{m\times s} \times \mathbb{R}^m : x = x_{\rm e}, \dot{K} = 0, \dot{\phi} = 0\}$ as $t \to \infty$.

In Theorem 9.4 we assumed that the equilibrium point $x_{\rm e}$ of (9.6) is globally asymptotically stable with $u(t) \equiv u_{\rm e}$. In general, however, unlike linear nonnegative systems with asymptotically stable plant dynamics, a given set point $x_{\rm e} \in \overline{\mathbb{R}}_+^n$ for the nonlinear nonnegative dynamical system (9.6) may not be asymptotically stabilizable with a constant control $u(t) \equiv u_{\rm e} \in \overline{\mathbb{R}}_+^m$. However, as discussed in Section 9.3, if $f(x)$ is homogeneous, cooperative; that is, the Jacobian matrix $\frac{\partial f(x)}{\partial x}$ is essentially nonnegative for all $x \in \overline{\mathbb{R}}_+^n$ [221], the Jacobian matrix $\frac{\partial f(x)}{\partial x}$ is irreducible for all $x \in \overline{\mathbb{R}}_+^n$ [221], and the zero solution $x(t) \equiv 0$ of the undisturbed ($u(t) \equiv 0$) system (9.6) is globally asymptotically stable, then the set point $x_{\rm e} \in \mathbb{R}_+^n$ satisfying (9.8) is a unique equilibrium point with $u(t) \equiv u_{\rm e} \in \mathbb{R}_+^m$ and is also asymptotically stable for all $x_0 \in \overline{\mathbb{R}}_+^n$ [52]. This implies that the solution $x(t) \equiv x_{\rm e}$ to (9.6) with $u(t) \equiv u_{\rm e}$ is asymptotically stable for all $x_0 \in \overline{\mathbb{R}}_+^n$.

It is important to note that the adaptive control law (9.52), (9.54), and (9.55) does *not* require the explicit knowledge of the nonnegative vector $u_{\rm e}$; all that is required is the existence of the nonnegative constant vector $u_{\rm e}$ and a partially component decoupled Lyapunov function $V_{\rm s}(e)$ along with the construction of $F(e)$ such that (9.9) and (9.51) are satisfied and the equilibrium condition (9.8) holds. Furthermore, note that in the case where $F(e)$ is only a function of $\hat{e} = [e_1, \cdots, e_m]^{\rm T}$ it follows from (9.53) that the adaptive control law (9.52), (9.54), and (9.55) does *not* require the explicit knowledge of the nonnegative constant vectors $x_{\rm eu} = [x_{{\rm e}m+1}, \cdots, x_{{\rm e}n}]^{\rm T}$ and $u_{\rm e} \in \overline{\mathbb{R}}_+^m$; even though Theorem 9.4 requires the existence of $x_{\rm eu} \in \overline{\mathbb{R}}_+^{n-m}$ and $u_{\rm e} \in \overline{\mathbb{R}}_+^m$ such that condition (9.8) holds. Finally, the control input $u(t)$, $t \geq 0$, is always nonnegative regardless of the values of $x_i(t)$, $k_i(t)$, and $\phi_i(t)$, $t \geq 0$, $i = 1, \cdots, m$, which ensures that the closed-loop plant states remain nonnegative by Proposition 9.2.

## 9.5. Illustrative Numerical Example

In this section we present a numerical example to demonstrate the utility of the proposed direct adaptive control framework. Specifically, consider the controlled two-compartment nonnegative dynamical system given by

$$\dot{x}_1(t) = -a_{21}(x_1(t))x_1(t) + a_{12}(x_1(t))x_2(t) + bu(t), \quad x_1(0) = x_{10}, \quad t \geq 0, \tag{9.63}$$

$$\dot{x}_2(t) = a_{21}(x_1(t))x_1(t) - a_{12}(x_1(t))x_2(t), \quad x_2(0) = x_{20}, \tag{9.64}$$

where $a_{21}(x_1) \triangleq c_1 Q(x_1)$, $a_{12}(x_1) \triangleq c_2 + c_3 Q(x_1)$, $Q(x_1) \triangleq \frac{1}{c_4 x_1 + c_5}$, and $c_1, \cdots, c_5$, and $b$ are *unknown* positive constants. Note that with $x = [x_1, x_2]^{\mathrm{T}}$, (9.63) and (9.64) can be written in the form of (9.6) with $f(x) = [-a_{21}(x_1)x_1 + a_{12}(x_1)x_2, a_{21}(x_1)x_1 - a_{12}(x_1)x_2]^{\mathrm{T}}$ and $G(x) = \hat{B} = [b, 0]^{\mathrm{T}}$. Here, our objective is to regulate $x_1$ around the desired value $x_{\mathrm{e}1} \geq 0$. Note that $x_{\mathrm{e}2} = c_1 Q(x_{\mathrm{e}1})x_{\mathrm{e}1}/(c_2 + c_3 Q(x_{\mathrm{e}1}))$ and $u_{\mathrm{e}} = 0$ satisfy the equilibrium condition (9.8) with $x_{\mathrm{e}} = [x_{\mathrm{e}1}, x_{\mathrm{e}2}]^{\mathrm{T}}$. Furthermore, define $e(t) \triangleq x(t) - x_{\mathrm{e}}$ so that $f_{\mathrm{e}}(e)$ is given by

$$f_{\mathrm{e}}(e) = \begin{bmatrix} -[a_{21}(e_1 + x_{\mathrm{e}1}) + a_{12}(e_1 + x_{\mathrm{e}1})(e_2 + x_{\mathrm{e}2}) - [-(a_{21}(x_{\mathrm{e}1}) + a_{12}(x_{\mathrm{e}1})x_{\mathrm{e}2}] \\ a_{21}(e_1 + x_{\mathrm{e}1})(e_1 + x_{\mathrm{e}1}) - a_{12}(e_1 + x_{\mathrm{e}1})(e_2 + x_{\mathrm{e}2}) - [a_{21}(x_{\mathrm{e}1})x_{\mathrm{e}1} - a_{12}(x_{\mathrm{e}1})x_{\mathrm{e}2})] \end{bmatrix}. \tag{9.65}$$

Furthermore, let $K_{\mathrm{g}} = k_{\mathrm{g}}/b$, $F_1(e) = e_1$, and $V_{\mathrm{s}}(e) = e_1^2 + e_2^2$ so that $V_{\mathrm{s}1}'(e)F_1(e) = 2e_1^2 \geq 0$. Next, note that

$$\begin{aligned} V_{\mathrm{s}}'(e)[f_{\mathrm{e}}(e) + \hat{B}K_{\mathrm{g}}F_1(e)] \\ &= e_1[f_{\mathrm{e}1}(e) + k_{\mathrm{g}}e_1] + e_2 f_{\mathrm{e}2}(e) \\ &= -[a_{21}(e_1 + x_{\mathrm{e}1}) + k_{\mathrm{e}}(e_1 + x_{\mathrm{e}1})]e_1^2 + a_{12}(e_1 + x_{\mathrm{e}1})e_1 e_2 \\ &\quad -x_{\mathrm{e}1}[a_{21}(e_1 + x_{\mathrm{e}1}) - a_{21}(x_{\mathrm{e}1})]e_1 - x_{\mathrm{e}2}[a_{12}(e_1 + x_{\mathrm{e}1}) - a_{12}(x_{\mathrm{e}1})]e_1 \\ &\quad +a_{21}(e_1 + x_{\mathrm{e}1})e_1 e_2 - a_{12}(e_1 + x_{\mathrm{e}1})e_2^2 + x_{\mathrm{e}1}[a_{21}(e_1 + x_{\mathrm{e}1}) - a_{21}(x_{\mathrm{e}1})]e_2 \\ &\quad -x_{\mathrm{e}2}[a_{12}(e_1 + x_{\mathrm{e}1}) - a_{12}(x_{\mathrm{e}1})]e_2 + k_{\mathrm{g}}e_1^2, \end{aligned} \tag{9.66}$$

where $f_{\mathrm{e}i}(\cdot)$ denotes the $i$th component of $f_{\mathrm{e}}(\cdot)$, $i = 1, 2$, and $-k_{\mathrm{g}} \in \overline{\mathbb{R}}_+$. Now, since $Q(\cdot)$ is Lipschitz continuous there exist positive constants $\alpha$ and $\beta$ such that $|[Q(e_1 + x_{\mathrm{e}1}) - Q(x_{\mathrm{e}1})]e_1| \le \alpha e_1^2$ and $|[Q(e_1 + x_{\mathrm{e}1}) - Q(x_{\mathrm{e}1})]e_2| \le \beta |e_1||e_2|$, and hence it follows that there exist $\gamma_1, \gamma_2 > 0$ such that

$$\begin{aligned} V_{\mathrm{s}}'(e)[f_{\mathrm{e}}(e) + \hat{B}K_{\mathrm{g}}F_1(e)] &\le \gamma_1 e_1^2 + 2\gamma_2 |e_1||e_2| - c_2 e_2^2 + k_{\mathrm{g}} e_1^2 \\ &= -c_2\Big(\frac{\gamma_2}{c_2}|e_1| - |e_2|\Big)^2 + \Big(\gamma_1 - \frac{\gamma_2^2}{c_2^2} + k_{\mathrm{g}}\Big)e_1^2. \end{aligned}$$

Hence, there exists $k_{\mathrm{g}} < 0$ such that

$$V_{\mathrm{s}}'(e)[f_{\mathrm{e}}(e) + \hat{B}K_{\mathrm{g}}F_1(e)] < 0, \quad e \in \mathbb{R}^2, \quad e \neq 0. \tag{9.67}$$

Now, it follows from Theorem 2.1 that $x(t) \to x_{\mathrm{e}}$ as $t \to \infty$ for any positive constant $c_1, \cdots, c_5$, and $b$. For $x_{\mathrm{e}1} = 2$ and with $c_1 = 2$, $c_2 = 0.1$, $c_3 = 3$, $c_4 = c_5 = 1$, $b = 3$, $q_1 = 0.01$, $\hat{q}_1 = 0.1$, and initial conditions $x(0) = [5, 8]^{\mathrm{T}}$, $k_1(0) = 0$, and $\phi_1(0) = 1$, Figure 9.1 shows the state trajectories versus time. Finally, Figure 9.2 shows the control signal and the adaptive gain history versus time.

## 9.6. Nonlinear Adaptive Control for General Anesthesia

To illustrate the application of our adaptive control framework we consider a hypothetical model for the intravenous anesthetic propofol. The pharmacokinetics of propofol are described by a three compartment model [169]. The model is shown in Figure 9.3. The mass of the drug in the intravascular blood volume as well as the highly perfused organs (organs with high ratios of perfusion to weight) such as the heart, brain, kidney, and liver is denoted by $x_1$. The remainder of the drug in the body is assumed to reside in two peripheral compartments, comprised of muscle and fat, and the masses in these compartments are denoted by $x_2$ and $x_3$.

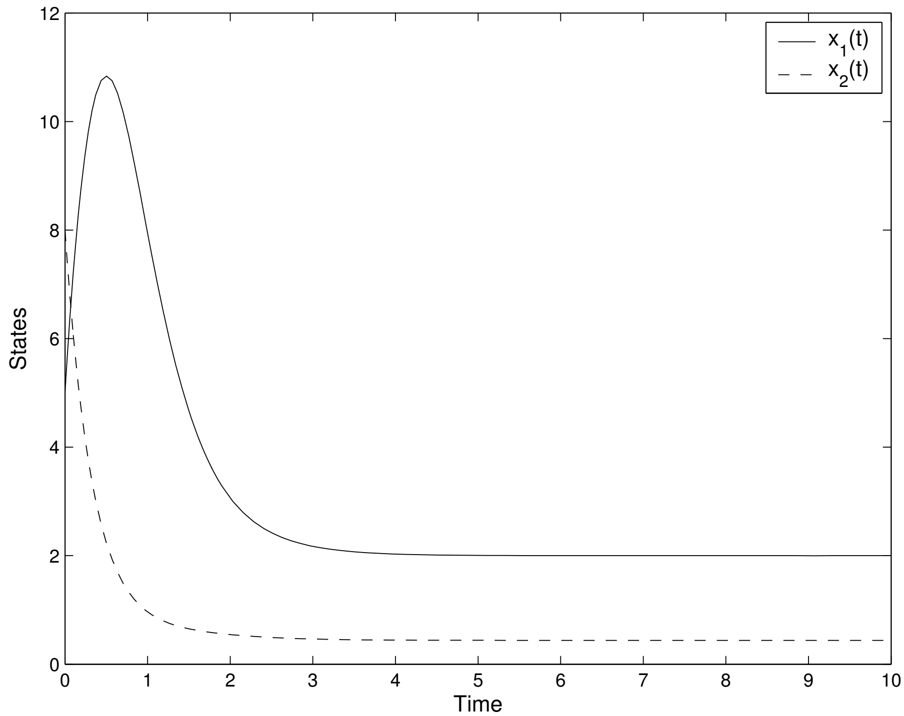



**Figure 9.1**: State trajectories versus time

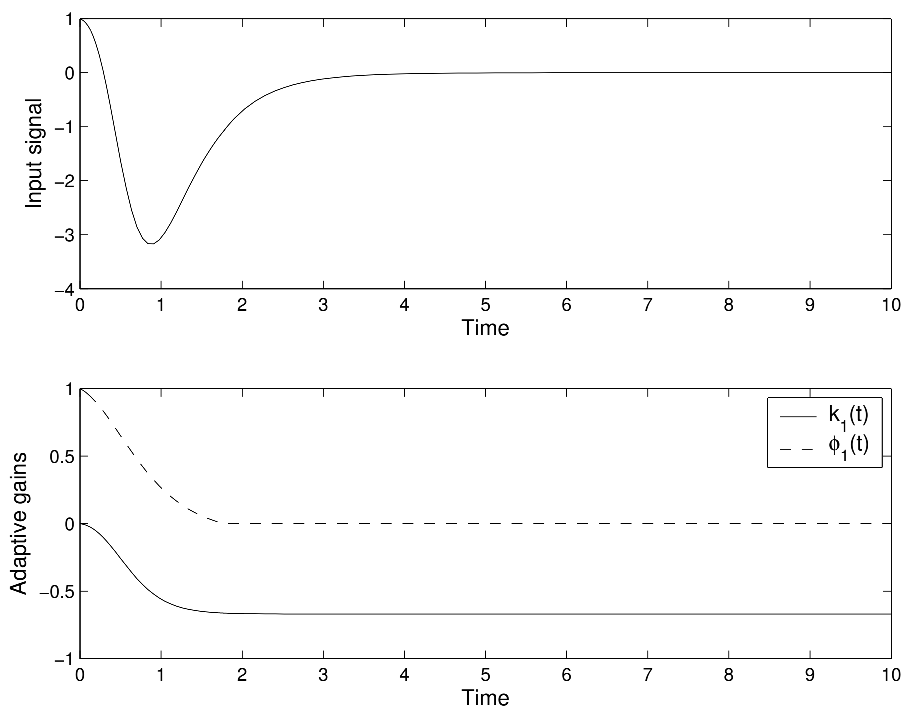



**Figure 9.2**: Control signal versus time and adaptive gain history versus time

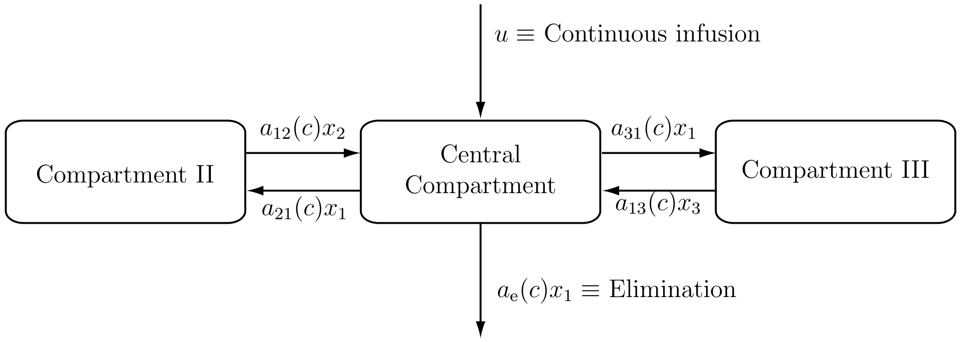



**Figure 9.3**: Pharmacokinetic model for drug distribution during anesthesia

A mass balance of the three-state compartmental model yields

$$\dot{x}_1(t) = -[a_e(c(t)) + a_{21}(c(t)) + a_{31}(c(t))]x_1(t) + a_{12}(c(t))x_2(t) + a_{13}(c(t))x_3(t) + u(t), \quad x_1(0) = x_{10}, \quad t \geq 0, \tag{9.68}$$

$$\dot{x}_2(t) = a_{21}(c(t))x_1(t) - a_{12}(c(t))x_2(t), \quad x_2(0) = x_{20}, \tag{9.69}$$

$$\dot{x}_3(t) = a_{31}(c(t))x_1(t) - a_{13}(c(t))x_3(t), \quad x_3(0) = x_{30}, \tag{9.70}$$

where $c(t) = x_1(t)/V_c$, $V_c$ is the volume of the central compartment, $a_{21}(c)$ is the rate of transfer of drug from the central compartment to Compartment II, $a_{12}(c)$ is the rate of transfer of drug from Compartment II to the central compartment, $a_{31}(c)$ is the rate of transfer of drug from the central compartment to Compartment III, $a_{13}(c)$ is the rate of transfer of drug from Compartment III to the central compartment, $a_e(c)$ is the rate of drug metabolism and elimination (metabolism typically occurs in the liver), and $u(t)$, $t \geq 0$, is the infusion rate of the anesthetic drug propofol into the central compartment. In order to formulate a physiologically realistic nonlinear model we assume that the rate transfers are proportional to the cardiac output. Even though this assumption has not been validated in clinical studies, we make the assumption to develop a nonlinear model to illustrate implementation of our adaptive controller. However, it does have some plausibility since transfer from the central compartment to the peripheral compartments (or *vice versa*) requires physical transport via the blood stream from the heart, brain, etc., to muscle and fat (or *vice versa*). Furthermore, since for many drugs the rate of metabolism (i.e., $a_{\mathrm{e}}(c)$) is proportional to the rate of transport of drug to the liver we assume that $a_{\mathrm{e}}(c)$ is also proportional to the cardiac output. Thus, we assume $a_{21}(c) = A_{21}Q(c)$, $a_{12}(c) = A_{12}Q(c)$, $a_{31}(c) = A_{31}Q(c)$, $a_{13}(c) = A_{13}Q(c)$, and $a_{\mathrm{e}}(c) = A_{\mathrm{e}}Q(c)$, where $A_{12}$, $A_{21}$, $A_{13}$, $A_{31}$, and $A_{\mathrm{e}}$ are positive constants. To develop a nonlinear model we assume a sigmoid relationship between drug concentration in the central compartment and effect so that

$$Q(c) = \frac{Q_0 \mathrm{C}_{50}^{\alpha}}{\mathrm{C}_{50}^{\alpha} + c^{\alpha}}, \tag{9.71}$$

where the effect is related to $c$ (since $c$ is the presumed concentration in the highly perfused myocardium), $Q_0 > 0$ is a constant, $\mathrm{C}_{50} > 0$ is the drug concentration associated with a 50% decrease in the cardiac output, and $\alpha > 1$ determines the steepness of this curve (that is, how rapidly the cardiac output decreases with increasing drug concentration, $c$). Even though the transfer and loss coefficients $A_{12}$, $A_{21}$, $A_{13}$, $A_{31}$, and $A_{\mathrm{e}}$ are positive and $\alpha > 1$, $\mathrm{C}_{50} > 0$, and $Q_0 > 0$, these parameters can be uncertain due to patient gender, weight, pre-existing disease, age, and concomitant medication. Hence, the need for adaptive control to regulate intravenous anesthetics during surgery is crucial.

For set-point regulation define $e(t) \triangleq x(t) - x_{\mathrm{e}}$, where $x_{\mathrm{e}} \in \mathbb{R}^3$ is the set point satisfying the equilibrium condition for (9.68)–(9.70) with $x_1(t) \equiv x_{\mathrm{e}1}$, $x_2(t) \equiv x_{\mathrm{e}2}$, $x_3(t) \equiv x_{\mathrm{e}3}$, and $u(t) \equiv u_{\mathrm{e}}$, so that $f_{\mathrm{e}}(e) = [f_{\mathrm{e}1}(e), f_{\mathrm{e}2}(e), f_{\mathrm{e}3}(e)]^{\mathrm{T}}$ is given by

$$\begin{aligned} f_{\mathrm{e}1}(e) = {} & -[a_{\mathrm{e}}(c) + a_{21}(c) + a_{31}(c)](e_1 + x_{\mathrm{e}1}) + a_{12}(c)(e_2 + x_{\mathrm{e}2}) + a_{13}(c)(e_3 + x_{\mathrm{e}3}) \\ & -[a_{\mathrm{e}}(c_{\mathrm{e}}) + a_{21}(c_{\mathrm{e}}) + a_{31}(c_{\mathrm{e}})]x_{\mathrm{e}1} + a_{12}(c_{\mathrm{e}})x_{\mathrm{e}2} + a_{13}(c_{\mathrm{e}})x_{\mathrm{e}3}, \end{aligned} \tag{9.72}$$

$$f_{\mathrm{e}2}(e) = a_{21}(c)(e_1 + x_{\mathrm{e}1}) - a_{12}(c)(e_2 + x_{\mathrm{e}2}) - [a_{21}(c_{\mathrm{e}})x_{\mathrm{e}1} - a_{12}(c_{\mathrm{e}})x_{\mathrm{e}2}], \tag{9.73}$$

$$f_{\mathrm{e}3}(e) = a_{31}(c)(e_1 + x_{\mathrm{e}1}) - a_{13}(c)(e_3 + x_{\mathrm{e}3}) - [a_{31}(c_{\mathrm{e}})x_{\mathrm{e}1} - a_{13}(c_{\mathrm{e}})x_{\mathrm{e}3}], \tag{9.74}$$

where $c_{\mathrm{e}} \triangleq x_{\mathrm{e}1}/V_{\mathrm{c}}$. Furthermore, let $F(e) = e_1$ and $V_{\mathrm{s}}(e) = e_1^2 + p_2 e_2^2 + p_3 e_3^2$, where $p_2, p_3 > 0$, so that $V_{\mathrm{s}1}'(e)F(e) = 2e_1^2 \geq 0$. Next, linearizing $f_{\mathrm{e}}(e)$ about 0 and computing the eigenvalues of the resulting (compartmental) Jacobian matrix, it can be shown that $x_{\mathrm{e}}$ is asymptotically stable.

Even though propofol concentrations in the blood are known to be correlated with lack of purposeful responsiveness (and presumably consciousness) [137], they cannot be measured in real-time during surgery. Furthermore, we are more interested in drug *effect* (depth of hypnosis) rather than drug *concentration*. Hence, we consider a more realistic model involving pharmacokinetics (drug concentration as a function of time) and pharmacodynamics (drug effect as a function of concentration) for control of anesthesia. Specifically, we use an electroencephalogram (EEG) signal as a measure of drug effect of anesthetic compounds on the brain [67, 174, 215]. Since electroencephalography provides real-time monitoring of the central nervous system activity, it can be used to quantify levels of consciousness and hence is amenable for feedback (closed-loop) control in general anesthesia. As discussed in Chapter 7, a new EEG indicator, the Bispectral Index (BIS), has been proposed as a measure of anesthetic effect [174]. This index quantifies the nonlinear relationships between the component frequencies in the electroencephalogram, as well as analyzing their phase and amplitude. The BIS signal is a nonlinear monotonically decreasing function of the level of consciousness and is given by

$$\mathrm{BIS}(c_{\mathrm{eff}}) = \mathrm{BIS}_0 \left(1 - \frac{c_{\mathrm{eff}}^{\gamma}}{c_{\mathrm{eff}}^{\gamma} + \mathrm{EC}_{50}^{\gamma}}\right), \tag{9.75}$$

where $\mathrm{BIS}_0$ denotes the baseline (awake state) value and, by convention, is typically assigned a value of 100, $c_{\mathrm{eff}}$ is the propofol concentration in micrograms/mililiter in the effect site compartment (brain), $\mathrm{EC}_{50}$ is the concentration at half maximal effect and represents the patient's sensitivity to the drug, and $\gamma$ determines the degree of nonlinearity in (9.75). Here, the effect site compartment is introduced as a correlate between the central compartment concentration and the central nervous system concentration [205]. The effect site compartment concentration is related to the con-

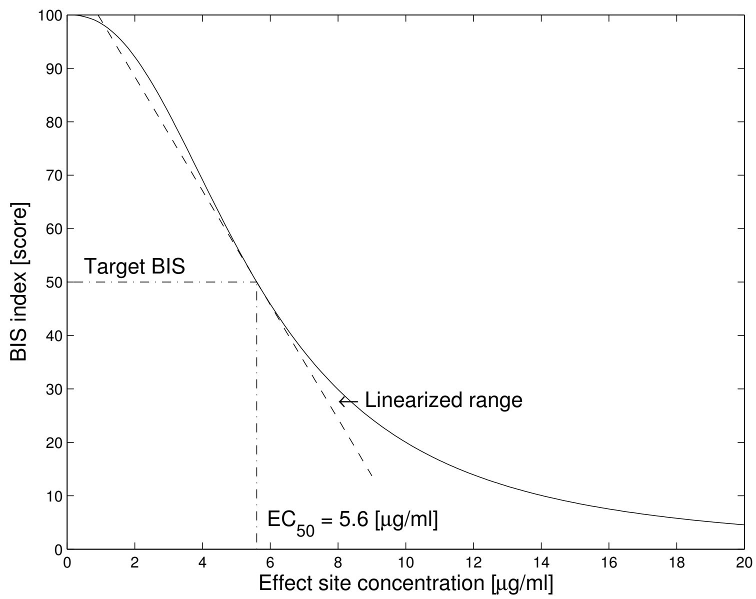



**Figure 9.4**: BIS index versus effect site concentration

centration in the central compartment by the first-order delay model

$$\dot{c}_{\rm eff}(t) = a_{\rm eff}(c(t) - c_{\rm eff}(t)), \quad c_{\rm eff}(0) = c(0), \quad t \geq 0, \tag{9.76}$$

where $a_{\rm eff}$ in $\min^{-1}$ is a positive time constant. Assuming $c(0) = 0$, it follows that

$$c_{\rm eff}(t) = \int_0^t e^{-a_{\rm eff}(t-s)} a_{\rm eff} c(s)\,{\rm d}s. \tag{9.77}$$

In reality, the effect site compartment equilibrates with the central compartment in a matter of a few minutes. The parameters $a_{\rm eff}$, $\mathrm{EC}_{50}$, and $\gamma$ are determined by data fitting and vary from patient to patient. BIS index values of 0 and 100 correspond, respectively, to an isoelectric EEG signal and an EEG signal of a fully conscious patient; while the range between 40 and 60 indicates a moderate hypnotic state [215].

In the following numerical simulation we set $\mathrm{EC}_{50} = 5.6$ $\mu$g/m$\ell$, $\gamma = 2.39$, and $\mathrm{BIS}_0 = 100$, so that the BIS signal is shown in Figure 9.4. The target (desired) BIS value, $\mathrm{BIS}_{\rm target}$, is set at 50. In this case, the linearized BIS function about the target

BIS value is given by

$$\begin{aligned}\mathrm{BIS}(c_{\mathrm{eff}}) &\simeq \mathrm{BIS}(\mathrm{EC}_{50}) - \mathrm{BIS}_0 \cdot \mathrm{EC}_{50}^{\gamma} \cdot \left.\frac{\gamma c_{\mathrm{eff}}^{\gamma-1}}{(c_{\mathrm{eff}}^{\gamma} + \mathrm{EC}_{50}^{\gamma})^2}\right|_{c_{\mathrm{eff}}=\mathrm{EC}_{50}} \cdot (c_{\mathrm{eff}} - \mathrm{EC}_{50}) \\ &= 109.75 - 10.67 c_{\mathrm{eff}}. \end{aligned} \tag{9.78}$$

Furthermore, for simplicity of exposition, we assume that the effect site compartment equilibrates instantaneously with the central compartment; that is, we assume that $a_{\mathrm{eff}} \to \infty$, so that (9.77) reduces to $c_{\mathrm{eff}}(t) = c(t)$, $t \geq 0$. Now, using the adaptive feedback controller

$$u(t) = \max\{0, \hat{u}(t)\}, \tag{9.79}$$

where

$$\hat{u}(t) = -k(t)(\mathrm{BIS}(t) - \mathrm{BIS}_{\mathrm{target}}) + \phi(t), \tag{9.80}$$

$k(t) \in \mathbb{R}$, $t \geq 0$, and $\phi(t) \in \mathbb{R}$, $t \geq 0$, with update laws

$$\dot{k}(t) = \begin{cases} 0, & \text{if } \hat{u}(t) \leq 0, \\ -q_{\mathrm{BIS}}(\mathrm{BIS}(t) - \mathrm{BIS}_{\mathrm{target}})^2, & \text{otherwise}, \end{cases} \quad k(0) \leq 0, \tag{9.81}$$

$$\dot{\phi}(t) = \begin{cases} 0, & \text{if } \phi(t) = 0 \text{ and } \mathrm{BIS}(t) > \mathrm{BIS}_{\mathrm{target}}, \\ & \text{or if } \hat{u}(t) \leq 0, \\ \hat{q}_{\mathrm{BIS}}(\mathrm{BIS}(t) - \mathrm{BIS}_{\mathrm{target}}), & \text{otherwise}, \end{cases} \quad \phi(0) \geq 0, \tag{9.82}$$

where $q_{\mathrm{BIS}}$ and $\hat{q}_{\mathrm{BIS}}$ are arbitrary positive constants, it follows from Theorem 9.4 that the control input (anesthetic infusion rate) $u(t)$ is nonnegative for all $t \geq 0$ and $\mathrm{BIS}(t) \to \mathrm{BIS}_{\mathrm{target}}$ as $t \to \infty$ for any (uncertain) positive values of the pharmacokinetic transfer and loss coefficients $(A_{12}, A_{21}, A_{13}, A_{31}, A_{\mathrm{e}})$ as well as any (uncertain) nonnegative coefficients $\alpha$, $\mathrm{C}_{50}$, and $Q_0$ in the range of $c_{\mathrm{eff}}$ where the linearized BIS equation (9.78) is valid. It is important to note that during actual surgery the BIS signal is obtained directly from the EEG and not (9.75). Furthermore, since our adaptive controller only requires the error signal $\mathrm{BIS}(t) - \mathrm{BIS}_{\mathrm{target}}$ over the linearized range of (9.75), we do not require knowledge of the slope of the linearized equation (9.78), nor do we require knowledge of the pharmacodynamic parameters $\gamma$ and

$\mathrm{EC}_{50}$. For our simulation we assume $V_\mathrm{c} = (0.228\ \ell/\mathrm{kg})(M\ \mathrm{kg})$, where $M = 70$ kg is the weight (mass) of the patient, $A_{21}Q_0 = 0.112\ \mathrm{min}^{-1}$, $A_{12}Q_0 = 0.055\ \mathrm{min}^{-1}$, $A_{31}Q_0 = 0.0419\ \mathrm{min}^{-1}$, $A_{13}Q_0 = 0.0033\ \mathrm{min}^{-1}$, $A_\mathrm{e}Q_0 = 0.119\ \mathrm{min}^{-1}$, $\alpha = 3$, and $\mathrm{C}_{50} = 4\ \mu\mathrm{g/m}\ell$ [169]. Note that the parameter values for $\alpha$ and $\mathrm{C}_{50}$ probably exaggerate the effect of propofol on cardiac output. They have been selected to accentuate nonlinearity but they are not biologically unrealistic. Furthermore, to illustrate the robustness of the proposed adaptive controller we switch the pharmacodynamic parameters $\mathrm{EC}_{50}$ and $\gamma$, respectively, from 5.6 $\mu$g/m$\ell$ and 2.39 to 7.2 $\mu$g/m$\ell$ and 3.39 at $t = 15$ min and back to 5.6 $\mu$g/m$\ell$ and 2.39 at $t = 30$ min. Here we consider noncardiac surgery since cardiac surgery often utilizes hypothermia which itself changes the BIS signal. With $q_\mathrm{BIS} = 1 \times 10^{-6}\ \mathrm{g/min}^2$, $\hat{q}_\mathrm{BIS} = 1 \times 10^{-3}\ \mathrm{g/min}^2$, and initial conditions $x(0) = [0,\ 0,\ 0]^\mathrm{T}$ g, $k(0) = 0$ g/min, and $\phi(0) = 0.01$ g/min, Figure 9.5 shows the masses of propofol in the three compartments versus time. Figure 9.6 shows the BIS index and the control signal (propofol infusion rate) versus time. Finally, Figure 9.7 shows the adaptive gain history versus time.

## 9.7. Conclusion

Nonnegative and compartmental dynamical systems are widely used to capture system dynamics involving the interchange of mass and energy between homogeneous subsystems or compartments. Thus, it is not surprising that nonnegative and compartmental models are remarkably effective in describing the dynamical behavior of biological and physiological systems. While compartmental systems have wide applicability in biology and medicine, their use in the specific field of pharmacology is indispensable for developing models for active control of drug administration. In this chapter, we developed an adaptive control framework for adaptive set-point regulation of nonlinear nonnegative and compartmental systems. Using Lyapunov methods

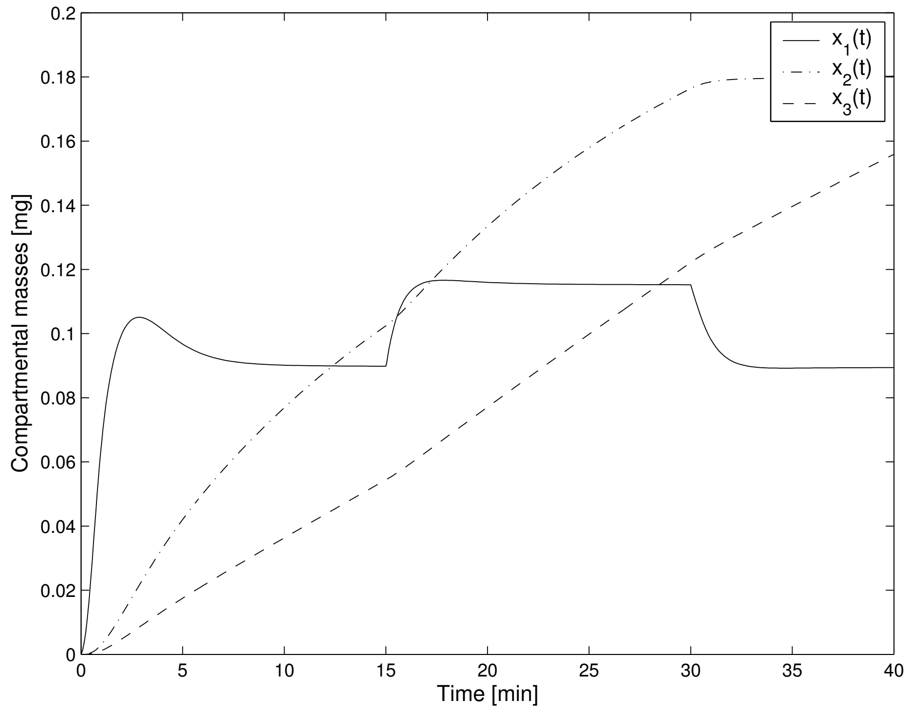



**Figure 9.5**: Compartmental masses versus time

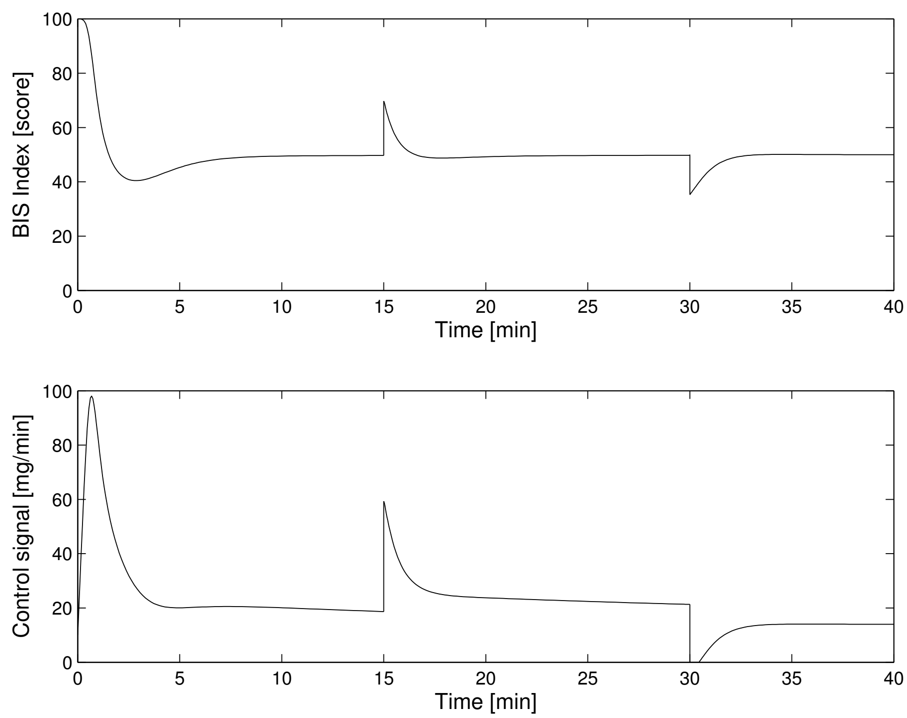



**Figure 9.6**: BIS index versus time and control signal (infusion rate) versus time

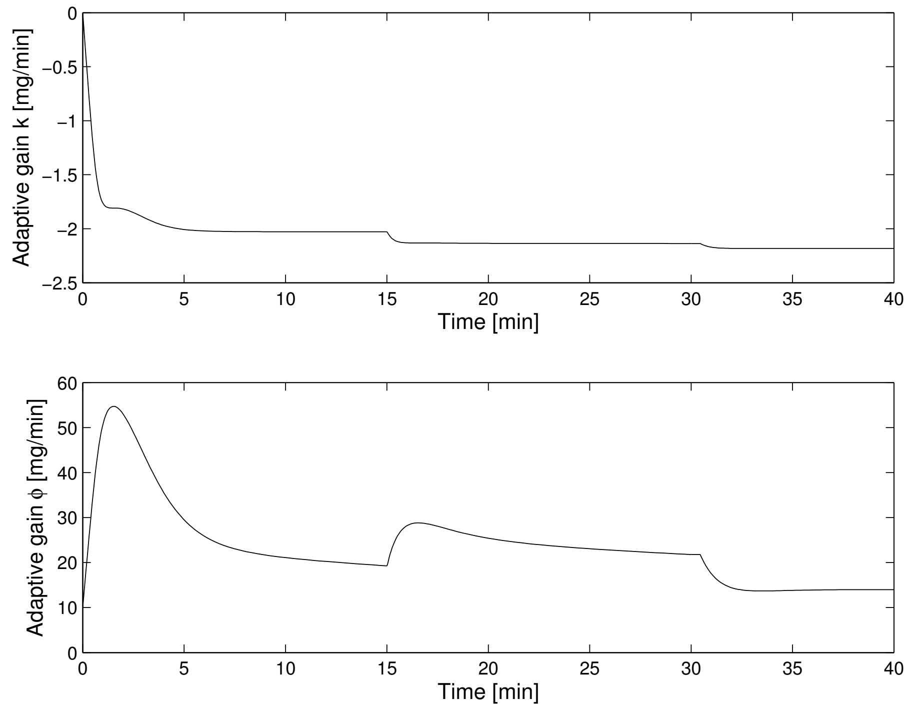



**Figure 9.7**: Adaptive gain history versus time

the proposed framework was shown to guarantee partial asymptotic set-point stability of the closed-loop system while additionally guaranteeing the nonnegativity of the closed-loop system states associated with the plant dynamics. Finally, using a hypothetical nonlinear three-compartment patient model for the disposition of anesthetic drug propofol, the proposed adaptive control framework was illustrated by the control of a desired constant level of consciousness for noncardiac surgery. Even though measurement noise was not addressed in our framework, it should be noted that EEG signals may have as much as 10% variation due to noise. While some of the noise is due to signals emanating from muscle rather than the central nervous system (and hence minimized by muscle paralysis) much of it is stochastic in nature. Extensions of the proposed adaptive control framework that directly address robustness to noise disturbances will be addressed in future research.

# Chapter 10

# Neural Network Adaptive Control for Nonlinear Nonnegative Dynamical Systems

## 10.1. Introduction

Neural networks consist of a weighted interconnection of fundamental elements called neurons, which are functions consisting of a summing junction and a nonlinear operation involving an activation function. One of the primary reasons for the large interest in neural networks is their capability to approximate a large class of continuous nonlinear maps from the collective action of very simple, autonomous processing units interconnected in simple ways. In addition, neural networks have attracted attention due to their inherently parallel and highly redundant processing architecture that makes it possible to develop parallel weight update laws. This parallelism makes it possible to effectively update a neural network on line. These properties make neural networks a viable paradigm for adaptive system identification and control of complex highly uncertain dynamical systems, and as a consequence the use of neural networks for identification and control has become an active area of research (see [44, 119, 159, 160, 178, 226] and the numerous references therein).

Modern complex engineering systems as well as biological and physiological systems are highly interconnected and mutually interdependent, both physically and through a multitude of information and communication networks. By properly formulating these systems in terms of subsystem interaction and energy/mass transfer, the dynamical models of many of these systems can be derived from mass, energy, and information balance considerations that involve dynamic states whose values are nonnegative. Hence, it follows from physical considerations that the state trajectory of such systems remains in the nonnegative orthant of the state space for nonnegative initial conditions. Such systems are commonly referred to as *nonnegative dynamical systems* in the literature [58, 75, 131, 135]. A subclass of nonnegative dynamical systems are *compartmental systems* [6, 24, 62, 70, 75, 123, 124, 164, 166, 172, 203]. Compartmental systems involve dynamical models that are characterized by conservation laws (e.g., mass and energy) capturing the exchange of material between coupled macroscopic subsystems known as compartments. Each compartment is assumed to be kinetically homogeneous; that is, any material entering the compartment is instantaneously mixed with the material of the compartment. The range of applications of nonnegative systems and compartmental systems includes pharmacological systems [17, 229], chemical reaction systems [21, 47, 59, 150, 235], queuing systems [236], large-scale systems [216, 217], stochastic systems (whose state variables represent probabilities) [236], ecological systems [27, 112, 144, 166, 184], economic systems [20], demographic systems [123], telecommunication systems [64], transportation systems, power systems, heat transfer systems, and structural vibration systems [140–142], to cite but a few examples. Due to the severe complexities, nonlinearities, and uncertainties inherent in these systems, neural networks provide an ideal framework for on-line adaptive control because of their parallel processing flexibility and adaptability.

In this chapter we develop a full-state feedback neural adaptive control framework for set-point regulation of nonlinear uncertain nonnegative and compartmental systems. Nonzero set-point regulation for nonnegative dynamical systems is a key design requirement since stabilization of nonnegative systems naturally deals with equilibrium points in the interior of the nonnegative orthant. The proposed framework is Lyapunov-based and guarantees ultimate boundedness of the error signals corresponding to the physical system states as well as the neural network weighting gains. The neuro adaptive controllers are constructed *without* requiring knowledge of the system dynamics while guaranteeing that the physical system states remain in the nonnegative orthant of the state space. The proposed neuro control architecture is modular in the sense that if a linear design model is available, the neuro adaptive controller can be augmented to the nominal design to account for system nonlinearities and system uncertainty. Furthermore, since in certain applications of nonnegative and compartmental systems (e.g., pharmacological systems for active drug administration) control (source) inputs as well as the system states need to be nonnegative, we also develop neuro adaptive controllers that guarantee the control signal as well as the physical system states remain nonnegative for nonnegative initial conditions. We note that neuro adaptive controllers for nonnegative dynamical systems have not been addressed in the literature. Our approach however, is related to the neuro adaptive control methods developed in [116–118]. Finally, the proposed neuro adaptive control framework is used to regulate the temperature of a continuously stirred tank reactor involving exothermic irreversible reactions.

The contents of this chapter are as follows. In Section 10.2 we provide mathematical preliminaries on nonnegative dynamical systems that are necessary for developing the main results of this paper. Furthermore, we develop *new* Lyapunov-like theorems for partial boundedness and partial ultimate boundedness for nonlinear dynamical systems necessary for obtaining less conservative ultimate bounds for neuro adaptive controllers as compared to ultimate bounds derived using classical boundedness and ultimate boundedness notions in Section 10.2. In Section 10.3 we present our main neuro adaptive control framework for adaptive set-point regulation of nonlinear uncertain nonnegative and compartmental systems. In Section 10.4 we extend the results of Section 10.3 to the case where control inputs are constrained to be nonnegative. To demonstrate the efficacy of the proposed neuro adaptive control framework, in Section 10.5 we apply our framework to control a continuously stirred tank reactor involving exothermic irreversible reactions. Finally, in Section 10.6 we draw some conclusions.

## 10.2. Mathematical Preliminaries

In this section we introduce notation, several definitions, and some key results concerning linear and nonlinear nonnegative dynamical systems [19, 20, 24, 75] that are necessary for developing the main results of this chapter. Specifically, consider the controlled linear dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \tag{10.1}$$

where

$$B = \begin{bmatrix} \hat{B} \\ 0_{(n-m)\times m} \end{bmatrix}, \tag{10.2}$$

$A \in \mathbb{R}^{n\times n}$ is essentially nonnegative and $\hat{B} \in \mathbb{R}^{m\times m}$ is nonnegative such that rank $\hat{B} = m$. The following theorem shows that linear stabilizable nonnegative systems possess asymptotically stable zero dynamics with $\hat{x} \triangleq [x_1, \cdots, x_m]$ viewed as the output. For the statement of this result let spec$(A)$ denote the spectrum of $A$, let $\overline{\mathbb{C}}_+ \triangleq \{s \in \mathbb{C} : \mathrm{Re}[s] \geq 0\}$, and let $A \in \mathbb{R}^{n\times n}$ in (10.1) be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{10.3}$$

where $A_{11} \in \mathbb{R}^{m\times m}$ is essentially nonnegative, $A_{12} \in \mathbb{R}^{m\times(n-m)}$ is nonnegative, $A_{21} \in \mathbb{R}^{(n-m)\times m}$ is nonnegative, and $A_{22} \in \mathbb{R}^{(n-m)\times(n-m)}$ is essentially nonnegative.

**Theorem 10.1.** Consider the linear dynamical system $\mathcal{G}$ given by (10.1) where $A \in \mathbb{R}^{n\times n}$ is essentially nonnegative and partitioned as in (10.3), and $B \in \mathbb{R}^{n\times m}$ is nonnegative and is partitioned as in (10.2) with rank $\hat{B} = m$. Then there exists a gain matrix $K \in \mathbb{R}^{m\times n}$ such that $A + BK$ is essentially nonnegative and asymptotically stable if and only if $A_{22}$ is asymptotically stable.

**Proof.** First, let $K \triangleq [K_1, K_2]$, where $K_1 \in \mathbb{R}^{m\times m}$ and $K_2 \in \mathbb{R}^{m\times(n-m)}$, and note that

$$(A+BK)^{\mathrm{T}} = \begin{bmatrix} (A_{11}+\hat{B}K_1)^{\mathrm{T}} & A_{21}^{\mathrm{T}} \\ (A_{12}+\hat{B}K_2)^{\mathrm{T}} & A_{22}^{\mathrm{T}} \end{bmatrix}.$$

Assume that $A + BK$ is essentially nonnegative and asymptotically stable and suppose, *ad absurdum*, $A_{22}$ is not asymptotically stable. Then, it follows from Theorem 7.1 that there does not exist a positive vector $p_2 \in \mathbb{R}_+^{n-m}$ such that $A_{22}^{\mathrm{T}} p_2 << 0$. Next, since $A_{12} + \hat{B}K_2$ is nonnegative it follows that $(A_{12} + \hat{B}K_2)^{\mathrm{T}} p_1 \geq\geq 0$ for any positive vector $p_1 \in \mathbb{R}_+^m$. Thus, there does not exist a positive vector $p \triangleq [p_1^{\mathrm{T}}, p_2^{\mathrm{T}}]^{\mathrm{T}}$ such that $(A+BK)^{\mathrm{T}} p << 0$ and hence it follows from Theorem 7.1 that $A+BK$ is not asymptotically stable leading to a contradiction. Hence, $A_{22}$ is asymptotically stable. Conversely, suppose $A_{22}$ is asymptotically stable. Then taking $K_1 = \hat{B}^{-1}(A_{\mathrm{s}} - A_{11})$ and $K_2 = -\hat{B}^{-1}A_{12}$, where $A_{\mathrm{s}}$ is essentially nonnegative and asymptotically stable, it follows that $\mathrm{spec}(A + BK) \cap \overline{\mathbb{C}}_+ = [\mathrm{spec}(A_{\mathrm{s}}) \cup \mathrm{spec}(A_{22})] \cap \overline{\mathbb{C}}_+ = \emptyset$ and hence $A + BK$ is essentially nonnegative and asymptotically stable. $\square$

Next, consider the nonlinear dynamical system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \in \mathcal{I}_{x_0}, \tag{10.4}$$

where $x(t) \in \mathcal{D}$, $\mathcal{D}$ is an open subset of $\mathbb{R}^n$ with $0 \in \mathcal{D}$, $f : \mathcal{D} \to \mathbb{R}^n$ is locally Lipschitz continuous on $\mathcal{D}$, and $\mathcal{I}_{x_0} = [0, \tau_{x_0})$, $0 < \tau_{x_0} \leq \infty$, is the maximal interval of existence for the solution $x(\cdot)$ of (10.4). The following definition introduces the notion of essentially nonnegative vector fields [22, 75].

**Definition 10.1.** Let $f = [f_1, \cdots, f_n]^{\mathrm{T}} : \mathcal{D} \to \mathbb{R}^n$, where $\mathcal{D}$ is an open subset of $\mathbb{R}^n$ that contains $\overline{\mathbb{R}}_+^n$. Then $f$ is *essentially nonnegative with respect to* $\hat{x} \triangleq [x_1, \cdots, x_m]^{\mathrm{T}}$, $m \leq n$, if $f_i(x) \geq 0$, for all $i = 1, \ldots, m$, and $x \in \overline{\mathbb{R}}_+^n$ such that $x_i = 0$, where $x_i$ denotes the $i$th element of $x$. $f$ is *essentially nonnegative* if $f(x)$ is essentially nonnegative with respect to $x$.

Next, we present Lyapunov-like theorems for *partial boundedness* and *partial ultimate boundedness* of nonlinear dynamical systems. These notions allow us to develop less conservative ultimate bounds for neuro adaptive controllers as compared to ultimate bounds derived using classical boundedness and ultimate boundedness notions. Specifically, consider the nonlinear autonomous interconnected dynamical system

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t)), \quad x_1(0) = x_{10}, \quad t \in \mathcal{I}_{x_{10}, x_{20}}, \tag{10.5}$$

$$\dot{x}_2(t) = f_2(x_1(t), x_2(t)), \quad x_2(0) = x_{20}, \tag{10.6}$$

where $x_1 \in \mathcal{D}$, $\mathcal{D} \subseteq \mathbb{R}^{n_1}$ is an open set such that $0 \in \mathcal{D}$, $x_2 \in \mathbb{R}^{n_2}$, $f_1 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ is such that, for every $x_2 \in \mathbb{R}^{n_2}$, $f_1(0, x_2) = 0$ and $f_1(\cdot, x_2)$ is locally Lipschitz in $x_1$, $f_2 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ is such that, for every $x_1 \in \mathcal{D}$, $f_2(x_1, \cdot)$ is locally Lipschitz in $x_2$, and $\mathcal{I}_{x_{10}, x_{20}} \triangleq [0, \tau_{x_{10}, x_{20}})$, $0 < \tau_{x_{10}, x_{20}} \leq \infty$, is the maximal interval of existence for the solution $(x_1(t), x_2(t))$, $t \in \mathcal{I}_{x_{10}, x_{20}}$, to (10.5), (10.6). Note that under the above assumptions the solution $(x_1(t), x_2(t))$ to (10.5), (10.6) exists and is unique over $\mathcal{I}_{x_{10}, x_{20}}$. For the following definition we assume that $\mathcal{I}_{x_{10}, x_{20}} = [0, \infty)$.

**Definition 10.2.** *i*) The nonlinear dynamical system (10.5), (10.6) is *bounded with respect to* $x_1$ *uniformly in* $x_{20}$ if there exists $\gamma > 0$ such that, for every $\delta \in$

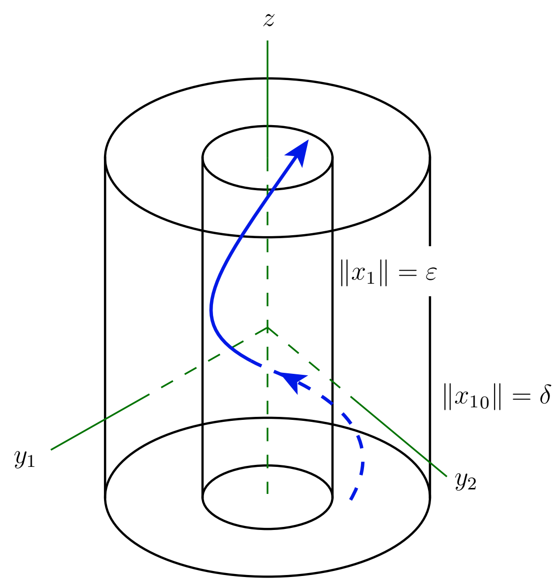



**Figure 10.1**: Visualization of partial boundedness and partial ultimate boundedness with respect to $x_1$ with $x_1 = [y_1, y_2]^{\mathrm{T}}$, $x_2 = z$, and $x = [x_1^{\mathrm{T}}, x_2]^{\mathrm{T}}$

$(0, \gamma)$, there exists $\varepsilon = \varepsilon(\delta) > 0$ such that $\|x_{10}\| < \delta$ implies $\|x_1(t)\| < \varepsilon$, $t \geq 0$. The nonlinear dynamical system (10.5), (10.6) is *globally bounded with respect to* $x_1$ *uniformly in* $x_{20}$ if, for every $\delta \in (0, \infty)$, there exists $\varepsilon = \varepsilon(\delta) > 0$ such that $\|x_{10}\| < \delta$ implies $\|x_1(t)\| < \varepsilon$, $t \geq 0$ (see Figure 10.1).

*ii*) The nonlinear dynamical system (10.5), (10.6) is *ultimately bounded with respect to* $x_1$ *uniformly in* $x_{20}$ *with ultimate bound* $\varepsilon$ if there exists $\gamma > 0$ such that, for every $\delta \in (0, \gamma)$, there exists $T = T(\delta, \varepsilon) > 0$ such that $\|x_{10}\| < \delta$ implies $\|x_1(t)\| < \varepsilon$, $t \geq T$. The nonlinear dynamical system (10.5), (10.6) is *globally ultimately bounded with respect to* $x_1$ *uniformly in* $x_{20}$ *with ultimate bound* $\varepsilon$ if, for every $\delta \in (0, \infty)$, there exists $T = T(\delta, \varepsilon) > 0$ such that $\|x_{10}\| < \delta$ implies $\|x_1(t)\| < \varepsilon$, $t \geq T$.

Note that if a nonlinear dynamical system is (globally) bounded with respect to $x_1$ uniformly in $x_{20}$, then there exists $\varepsilon > 0$ such that it is (globally) ultimately bounded with respect to $x_1$ uniformly in $x_{20}$ with an ultimate bound $\varepsilon$. Conversely, if a nonlinear dynamical system is (globally) ultimately bounded with respect to

$x_1$ uniformly in $x_{20}$ with an ultimate bound $\varepsilon$, then it is (globally) bounded with respect to $x_1$ uniformly in $x_{20}$. The following results present Lyapunov-like theorems for partial boundedness and partial ultimate boundedness. For these results define $\dot{V}(x_1, x_2) \triangleq V'(x_1, x_2)f(x_1,\ x_2)$, where $f(x_1, x_2) \triangleq [f_1^{\mathrm{T}}(x_1, x_2), f_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}}$ and $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ is a given continuously differentiable function. Furthermore, let $\mathcal{B}_\delta(x)$, $x \in \mathbb{R}^n$, $\delta > 0$, denote the open ball centered at $x$ with radius $\delta$ and let $\overline{\mathcal{B}}_\delta(x)$ denote the closure of $\mathcal{B}_\delta(x)$.

**Theorem 10.2.** Consider the nonlinear dynamical system (10.5), (10.6). Assume there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$ such that

$$\alpha(\|x_1\|) \leq V(x_1, x_2) \leq \beta(\|x_1\|), \quad x_1 \in \mathcal{D}, \quad x_2 \in \mathbb{R}^{n_2}, \tag{10.7}$$

$$\dot{V}(x_1, x_2) \leq 0, \quad x_1 \in \mathcal{D}, \quad \|x_1\| \geq \mu, \quad x_2 \in \mathbb{R}^{n_2}, \tag{10.8}$$

where $\mu > 0$ is such that $\mathcal{B}_{\alpha^{-1}(\eta)}(0) \subset \mathcal{D}$ with $\eta \geq \beta(\mu)$. Then the nonlinear dynamical system (10.5), (10.6) is bounded with respect to $x_1$ uniformly in $x_{20}$. Furthermore, for every $\delta \in (0, \gamma)$, $x_{10} \in \overline{\mathcal{B}}_\delta(0)$ implies that $\|x_1(t)\| \leq \varepsilon$, where

$$\varepsilon(\delta) \triangleq \begin{cases} \alpha^{-1}(\beta(\delta)), & \delta \in (\mu, \gamma), \\ \alpha^{-1}(\eta), & \delta \in (0, \mu], \end{cases} \tag{10.9}$$

and $\gamma \triangleq \sup\{r > 0 :\ \mathcal{B}_{\alpha^{-1}(\beta(r))}(0) \subset \mathcal{D}\}$. If, in addition, $\mathcal{D} = \mathbb{R}^{n_1}$ and $\alpha(\cdot)$ is a class $\mathcal{K}_\infty$ function, then the nonlinear dynamical system (10.5), (10.6) is globally bounded with respect to $x_1$ uniformly in $x_{20}$ and for every $x_{10} \in \mathbb{R}^{n_1}$, $\|x_1(t)\| \leq \varepsilon$, $t \geq 0$, where $\varepsilon$ is given by (10.9) with $\delta = \|x_{10}\|$.

**Proof.** First, let $\delta \in (0, \mu]$ and assume $\|x_{10}\| \leq \delta$. If $\|x_1(t)\| \leq \mu$, $t \geq 0$, then it follows from (10.7) that $\|x_1(t)\| \leq \mu \leq \alpha^{-1}(\beta(\mu)) \leq \alpha^{-1}(\eta)$, $t \geq 0$. Alternatively, if there exists $T > 0$ such that $\|x_1(T)\| > \mu$, then it follows from the continuity of $x_1(\cdot)$

that there exists $\tau < T$ such that $\|x_1(\tau)\| = \mu$ and $\|x_1(t)\| \geq \mu$, $t \in [\tau, T]$. Hence, it follows from (10.7) and (10.8) that

$$\alpha(\|x_1(T)\|) \leq V(x_1(T), x_2(T)) \leq V(x_1(\tau), x_2(\tau)) \leq \beta(\mu) \leq \eta,$$

which implies that $\|x_1(T)\| \leq \alpha^{-1}(\eta)$. Next, let $\delta \in (\mu, \gamma)$ and assume $x_{10} \in \overline{\mathcal{B}}_\delta(0)$ and $\|x_{10}\| > \mu$. Now, for every $\hat{t} > 0$ such that $\|x_1(t)\| \geq \mu$, $t \in [0, \hat{t}]$, it follows from (10.7) and (10.8) that

$$\alpha(\|x_1(t)\|) \leq V(x_1(t), x_2(t)) \leq V(x_{10}, x_{20}) \leq \beta(\delta), \quad t \geq 0,$$

which implies that $\|x_1(t)\| \leq \alpha^{-1}(\beta(\delta))$, $t \in [0, \hat{t}]$. Next, if there exists $T > 0$ such that $\|x_1(T)\| \leq \mu$, then it follows as in the proof of the first case given above that $\|x_1(t)\| \leq \alpha^{-1}(\eta)$, $t \geq T$. Hence, if $x_{10} \in \mathcal{B}_\delta(0) \backslash \mathcal{B}_\mu(0)$, then $\|x_1(t)\| \leq \alpha^{-1}(\beta(\delta))$, $t \geq 0$. Finally, if $\mathcal{D} = \mathbb{R}^{n_1}$ and $\alpha(\cdot)$ is a class $\mathcal{K}_\infty$ function it follows that $\beta(\cdot)$ is a class $\mathcal{K}_\infty$ function and hence $\gamma = \infty$. Hence, the nonlinear dynamical system (10.5), (10.6) is globally bounded with respect to $x_1$ uniformly in $x_{20}$. □

**Theorem 10.3.** Consider the nonlinear dynamical system (10.5), (10.6). Assume there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$ such that (10.7) holds. Furthermore, assume that there exists a continuous, positive-definite function $W : \mathcal{D} \to \mathbb{R}$ such that $W(x_1) > 0$, $\|x_1\| > \mu$, and

$$\dot{V}(x_1, x_2) \leq -W(x_1), \quad x_1 \in \mathcal{D}, \quad \|x_1\| > \mu, \quad x_2 \in \mathbb{R}^{n_2}, \tag{10.10}$$

where $\mu > 0$ is such that $\mathcal{B}_{\alpha^{-1}(\eta)}(0) \subset \mathcal{D}$ with $\eta > \beta(\mu)$. Then the nonlinear dynamical system (10.5), (10.6) is ultimately bounded with respect to $x_1$ uniformly in $x_{20}$ with ultimate bound $\varepsilon \triangleq \alpha^{-1}(\eta)$. Furthermore, $\limsup_{t\to\infty} \|x_1(t)\| \leq \alpha^{-1}(\beta(\mu))$. If, in addition, $\mathcal{D} = \mathbb{R}^n$ and $\alpha(\cdot)$ is a class $\mathcal{K}_\infty$ function, then the nonlinear dynamical system (10.5), (10.6) is globally ultimately bounded with respect to $x_1$ uniformly in $x_{20}$ with ultimate bound $\varepsilon$.

**Proof.** First, let $\delta \in (0, \mu]$ and assume $\|x_{10}\| \leq \delta$. As in the proof of Theorem 10.2, it follows that $\|x_1(t)\| \leq \alpha^{-1}(\eta) = \varepsilon$, $t \geq 0$. Next, let $\delta \in (\mu, \gamma)$, where $\gamma \triangleq \sup\{r > 0 : \mathcal{B}_{\alpha^{-1}(\beta(r))}(0) \subset \mathcal{D}\}$ and assume $x_{10} \in \mathcal{B}_\delta(0)$ and $\|x_{10}\| > \mu$. In this case, it follows from Theorem 10.2 that $\|x_1(t)\| \leq \alpha^{-1}(\beta(\delta))$, $t \geq 0$. Suppose, *ad absurdum*, $\|x_1(t)\| \geq \beta^{-1}(\eta)$, $t \geq 0$, or, equivalently, $x_1(t) \in \mathcal{O} \triangleq \mathcal{B}_{\alpha^{-1}(\beta(\delta))}(0) \backslash \mathcal{B}_{\beta^{-1}(\eta)}(0)$, $t \geq 0$. Since $\overline{\mathcal{O}}$ is compact and $W(\cdot)$ is continuous and $W(x_1) > 0$, $\|x_1\| \geq \beta^{-1}(\eta) > \mu$, it follows from Weierstrass' theorem [201, p. 154] that $k \triangleq \min_{x_1 \in \overline{\mathcal{O}}} W(x_1) > 0$ exists. Hence, it follows from (10.10) that

$$V(x_1(t), x_2(t)) \leq V(x_{10}, x_{20}) - kt, \quad t \geq 0, \tag{10.11}$$

which implies that

$$\alpha(\|x_1(t)\|) \leq \beta(\|x_{10}\|) - kt \leq \beta(\delta) - kt, \quad t \geq 0. \tag{10.12}$$

Now, letting $t > \beta(\delta)/k$ it follows that $\alpha(\|x_1(t)\|) < 0$ which is a contradiction. Hence, there exists $T = T(\delta, \eta) > 0$ such that $\|x_1(T)\| < \beta^{-1}(\eta)$. Thus, it follows from Theorem 10.2 that $\|x_1(t)\| \leq \alpha^{-1}(\beta(\beta^{-1}(\eta))) = \alpha^{-1}(\eta)$, $t \geq T$, which proves that the nonlinear dynamical system (10.5), (10.6) is ultimately bounded with respect to $x_1$ uniformly in $x_{20}$ with ultimate bound $\varepsilon = \alpha^{-1}(\eta)$. Furthermore, $\limsup_{t\to\infty} \|x_1(t)\| \leq \alpha^{-1}(\beta(\mu))$. Finally, if $\mathcal{D} = \mathbb{R}^{n_1}$ and $\alpha(\cdot)$ is a class $\mathcal{K}_\infty$ function it follows that $\beta(\cdot)$ is a class $\mathcal{K}_\infty$ function and hence $\gamma = \infty$. Hence, the nonlinear dynamical system (10.5), (10.6) is globally ultimately bounded with respect to $x_1$ uniformly in $x_{20}$ with ultimate bound $\varepsilon$. □

The following result on ultimate boundedness of interconnected systems is needed for the main theorems in this chapter.

**Proposition 10.1.** Consider the nonlinear interconnected dynamical system (10.5), (10.6). If (10.6) is input-to-state stable with $x_1$ viewed as the input and

(10.5), (10.6) is ultimately bounded with respect to $x_1$ uniformly in $x_{20}$, then the solution $(x_1(t), x_2(t))$, $t \geq 0$, of the interconnected dynamical system (10.5), (10.6) is ultimately bounded.

**Proof.** Since (10.5), (10.6) is ultimately bounded with respect to $x_1$ (uniformly in $x_{20}$), there exist positive constants $\varepsilon$ and $T = T(\delta, \varepsilon)$ such that $\|x_1(t)\| < \varepsilon$, $t \geq T$. Furthermore, since (10.6) is input-to-state stable with $x_1$ viewed as the input, it follows that $x_2(T)$ is finite and hence there exist a class $\mathcal{KL}$ function $\eta(\cdot,\cdot)$ and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that

$$\begin{aligned} \|x_2(t)\| &\leq \eta(\|x_2(T)\|, t-T) + \gamma\Big(\sup_{T \leq \tau \leq t} \|x_1(\tau)\|\Big) \\ &= \eta(\|x_2(T)\|, t-T) + \gamma(\varepsilon) \\ &\leq \eta(\|x_2(T)\|, 0) + \gamma(\varepsilon), \quad t \geq T, \end{aligned} \tag{10.13}$$

which proves that the solution $(x_1(t), x_2(t))$, $t \geq 0$, to (10.5), (10.6) is ultimately bounded. □

## 10.3. Neural Adaptive Control for Nonlinear Nonnegative Uncertain Systems

In this section we consider the problem of characterizing neural adaptive feedback control laws for nonlinear nonnegative and compartmental uncertain dynamical systems to achieve *set-point* regulation in the nonnegative orthant. Specifically, consider the controlled nonlinear uncertain dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = f_x(x(t), z(t)) + G(x(t), z(t))u(t), \quad x(0) = x_0, \quad t \geq 0, \tag{10.14}$$

$$\dot{z}(t) = f_z(x(t), z(t)), \quad z(0) = z_0, \tag{10.15}$$

where $x(t) \in \mathbb{R}^{n_x}$, $t \geq 0$, and $z(t) \in \mathbb{R}^{n_z}$, $t \geq 0$, are the state vectors, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $f_x : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_x}$ is essentially nonnegative with respect to $x$ but otherwise unknown and satisfies $f_x(0,z) = 0$, $z \in \mathbb{R}^{n_z}$, $f_z : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_z}$ is essentially nonnegative with respect to $z$ but otherwise unknown and satisfies $f_z(x,0) = 0$, $x \in \mathbb{R}^{n_x}$, and $G : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_x \times m}$ is a known nonnegative input matrix function. Here, we assume that we have $m$ control inputs so that the input matrix function is given by

$$G(x,z) = \begin{bmatrix} B_{\mathrm{u}} G_{\mathrm{n}}(x,z) \\ 0_{(n_x - m)\times m} \end{bmatrix}, \tag{10.16}$$

where $B_{\mathrm{u}} = \mathrm{diag}[b_1, \cdots, b_m]$ is a positive diagonal matrix and $G_{\mathrm{n}} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{m\times m}$ is a nonnegative matrix function such that $\det G_{\mathrm{n}}(x,z) \neq 0$, $(x,z) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$. The control input $u(\cdot)$ in (10.14) is restricted to the class of *admissible controls* consisting of measurable functions such that $u(t) \in \mathbb{R}^m$, $t \geq 0$. In this section we do not place any restriction on the sign of the control signal and design a neuro adaptive controller that guarantees that the system states remain in the nonnegative orthant of the state space for nonnegative initial conditions and are ultimately bounded in the neighborhood of a desired equilibrium point.

In this chapter, we assume that $f_x(\cdot,\cdot)$ and $f_z(\cdot,\cdot)$ are unknown functions with $f_x(\cdot,\cdot)$ given by

$$f_x(x,z) = Ax + \Delta f(x,z), \tag{10.17}$$

where $A \in \mathbb{R}^{n_x \times n_x}$ is a known essentially nonnegative matrix and $\Delta f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_x}$ is an unknown essentially nonnegative function with respect to $x$ and belongs to the uncertainty set $\mathcal{F}$ given by

$$\mathcal{F} = \{\Delta f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_x} : \Delta f(x,z) = B\delta(x,z),\ (x,z) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}\}, \tag{10.18}$$

where $B \triangleq [B_{\mathrm{u}}, 0_{m\times(n-m)}]^{\mathrm{T}}$ and $\delta : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^m$ is an uncertain continuous function such that $\delta(x,z)$ is essentially nonnegative with respect to $x$ and $\delta'(x,z)$, $(x,z) \in \mathcal{D}_{\mathrm{c}x} \times \mathcal{D}_{\mathrm{c}z}$, is bounded. Furthermore, we assume that for a given $x_{\mathrm{e}} \in \mathbb{R}_+^{n_x}$

there exist $z_{\rm e} \in \overline{\mathbb{R}}_+^{n_z}$ and $u_{\rm e} \in \overline{\mathbb{R}}_+^{m}$ such that

$$0 = Ax_{\rm e} + \Delta f(x_{\rm e}, z_{\rm e}) + G(x_{\rm e}, z_{\rm e})u_{\rm e}, \tag{10.19}$$

$$0 = f_z(x_{\rm e}, z_{\rm e}). \tag{10.20}$$

In addition, we assume that (10.15) is input-to-state stable at $z(t) \equiv z_{\rm e}$ with $x(t) - x_{\rm e}$ viewed as the input; that is, there exist a class $\mathcal{KL}$ function $\eta(\cdot, \cdot)$ and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that

$$\|z(t) - z_{\rm e}\| \leq \eta(\|z_0 - z_{\rm e}\|, t) + \gamma\left(\sup_{0\leq\tau\leq t} \|x(\tau) - x_{\rm e}\|\right), \quad t \geq 0, \tag{10.21}$$

where $\|\cdot\|$ denotes the Euclidean vector norm. Unless otherwise stated, henceforth we use $\|\cdot\|$ to denote the Euclidean vector norm. Note that $(x_{\rm e}, z_{\rm e}) \in \mathbb{R}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$ is an equilibrium point of (10.14), (10.15) if and only if there exists $u_{\rm e} \in \overline{\mathbb{R}}_+^{m}$ such that (10.19), (10.20) hold. Furthermore, we assume that, for a given $\varepsilon_i^* > 0$, the $i$th component of the vector function $\delta(x, z) - \delta(x_{\rm e}, z_{\rm e}) - G_{\rm n}(x_{\rm e}, z_{\rm e})u_{\rm e}$ can be approximated over a compact set $\mathcal{D}_{{\rm c}x} \times \mathcal{D}_{{\rm c}z} \subset \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$ by a linear in the parameters neural network up to a desired accuracy so that for $i = 1, \cdots, m$, there exists $\varepsilon_i(\cdot, \cdot)$ such that $|\varepsilon_i(x, z)| < \varepsilon_i^*$, $(x, z) \in \mathcal{D}_{{\rm c}x} \times \mathcal{D}_{{\rm c}z}$, and

$$\delta_i(x, z) - \delta_i(x_{\rm e}, z_{\rm e}) - [G_{\rm n}(x_{\rm e}, z_{\rm e})u_{\rm e}]_i = W_i^{\rm T}\sigma_i(x, z) + \varepsilon_i(x, z), \quad (x, z) \in \mathcal{D}_{{\rm c}x} \times \mathcal{D}_{{\rm c}z}, \tag{10.22}$$

where $W_i \in \mathbb{R}^{s_i}$, $i = 1, \cdots, m$, are optimal *unknown* (constant) weights that minimize the approximation error over $\mathcal{D}_{{\rm c}x} \times \mathcal{D}_{{\rm c}z}$, $\sigma_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{s_i}$, $i = 1, \cdots, m$, are a set of basis functions such that each component of $\sigma_i(\cdot, \cdot)$ takes values between 0 and 1 and $\sigma'(x, z)$, $(x, z) \in \mathcal{D}_{{\rm c}x} \times \mathcal{D}_{{\rm c}z}$, is bounded, $\varepsilon_i : \mathcal{D}_{{\rm c}x} \times \mathcal{D}_{{\rm c}z} \to \mathbb{R}$, $i = 1, \cdots, m$, are the modeling errors, and $\|W_i\| \leq w_i^*$, where $w_i^*$, $i = 1, \cdots, m$, are bounds for the optimal weights $W_i$, $i = 1, \cdots, m$. Since $f_x(\cdot, \cdot)$ is continuous, we can choose $\sigma_i(\cdot, \cdot)$, $i = 1, \cdots, m$, from a linear space $\mathcal{X}$ of continuous functions that forms an algebra and separates points in $\mathcal{D}_{{\rm c}x} \times \mathcal{D}_{{\rm c}z}$. In this case, it follows from the Stone-Weierstrass theorem [201, p. 212] that $\mathcal{X}$ is a dense subset of the set of continuous functions on $\mathcal{D}_{\mathrm{c}x} \times \mathcal{D}_{\mathrm{c}z}$. Now, as is the case in the standard neuro adaptive control literature [159], we can construct the signal $u_{\mathrm{ad}i} = \hat{W}_i^{\mathrm{T}} \sigma_i(x, z)$ involving the estimates of the optimal weights as our adaptive control signal. However, even though $\hat{W}_i^{\mathrm{T}} \sigma_i(x, z)$, $i = 1, \cdots, m$, provide adaptive cancellation of the system uncertainty, it does not necessarily guarantee that the state trajectory of the closed-loop system remains in the nonnegative orthant of the state space for nonnegative initial conditions. To ensure nonnegativity of the closed-loop plant states, the adaptive control signal is assumed to be of the form $\hat{W}_i^{\mathrm{T}} \hat{\sigma}_i(x, z)$, $i = 1, \cdots, m$, where $\hat{\sigma}_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{s_i}$ is such that each component of $\hat{\sigma}_i(\cdot, \cdot)$ takes values between 0 and 1, $\hat{\sigma}_i'(x, z)$, $(x, z) \in \mathcal{D}_{\mathrm{c}x} \times \mathcal{D}_{\mathrm{c}z}$, is bounded, and $\hat{\sigma}_i(x, z) = 0$ whenever $x_i = 0$ for all $i = 1, \cdots, m$. This set of functions do not generate an algebra in $\mathcal{X}$ and hence if used as an approximator for $\delta_i(\cdot, \cdot)$, $i = 1, \cdots, m$, will generate additional conservatism in the ultimate bound guarantees provided by the neural network controller. In particular, since each component of $\sigma_i(\cdot, \cdot)$ and $\hat{\sigma}_i(\cdot, \cdot)$ takes values between 0 and 1, it follows that

$$\|\sigma_i(x, z) - \hat{\sigma}_i(x, z)\| \le \sqrt{s_i}, \quad (x, z) \in \mathcal{D}_{\mathrm{c}x} \times \mathcal{D}_{\mathrm{c}z}, \quad i = 1, \cdots, m. \tag{10.23}$$

This upper bound will be used in the analysis of Theorem 10.4 below.

For the remainder of the chapter we assume that there exists a gain matrix $K \in \mathbb{R}^{m \times n_x}$ such that $A + BK$ is essentially nonnegative and asymptotically stable, where $A$ and $B$ have the forms of (10.3) and (10.2), respectively. Now, partitioning the state in (10.14) as $x = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$, where $x_1 \in \mathbb{R}^m$ and $x_2 \in \mathbb{R}^{n_x - m}$, and using (10.16), it follows that (10.14), (10.15) can be written as

$$\dot{x}_1(t) = A_{11} x_1(t) + A_{12} x_2(t) + \Delta f(x_1(t), x_2(t), z(t)) + B_{\mathrm{u}} G_{\mathrm{n}}(x_1(t), x_2(t), z(t)) u(t),$$
$$x_1(0) = x_{10}, \quad t \ge 0, \tag{10.24}$$

$$\dot{x}_2(t) = A_{21} x_1(t) + A_{22} x_2(t), \quad x_2(0) = x_{20}, \tag{10.25}$$

$$\dot{z}(t) = f_z(x_1(t), x_2(t), z(t)), \quad z(0) = z_0. \tag{10.26}$$

Thus, since $A + BK$ is essentially nonnegative and asymptotically stable, it follows from Theorem 10.1 that the solution $x_2(t) \equiv x_{2\mathrm{e}} \in \mathbb{R}_+^{n_x - m}$ of (10.25) with $x_1(t) \equiv x_{1\mathrm{e}} \in \mathbb{R}_+^m$, where $x_{1\mathrm{e}}$ and $x_{2\mathrm{e}}$ satisfy $0 = A_{21}x_{1\mathrm{e}} + A_{22}x_{2\mathrm{e}}$, is globally exponentially stable and hence (10.25) is input-to-state stable at $x_2(t) \equiv x_{2\mathrm{e}}$ with $x_1(t) - x_{1\mathrm{e}}$ viewed as the input. Thus, in this chapter we assume that the dynamics (10.25) can be included in (10.15) so that $n_x = m$. In this case, the input matrix (10.16) is given by

$$G(x, z) = B_\mathrm{u} G_\mathrm{n}(x, z) \tag{10.27}$$

so that $B = B_\mathrm{u}$. Now, for a given desired set point $(x_\mathrm{e}, z_\mathrm{e}) \in \mathbb{R}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$ and for given $\epsilon_1, \epsilon_2 > 0$, our aim is to design a control input $u(t)$, $t \geq 0$, such that $\|x(t) - x_\mathrm{e}\| < \epsilon_1$ and $\|z(t) - z_\mathrm{e}\| < \epsilon_2$ for all $t \geq T$, where $T \in [0, \infty)$, and $x(t) \geq\geq 0$ and $z(t) \geq\geq 0$ for all $t \geq 0$ and $(x_0, z_0) \in \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$. However, since in many applications of nonnegative systems and in particular, compartmental systems, it is often necessary to regulate a subset of the nonnegative state variables which usually include a central compartment, here we only require that $\|x(t) - x_\mathrm{e}\| < \epsilon_1$, $t \geq T$.

**Theorem 10.4.** Consider the nonlinear uncertain dynamical system $\mathcal{G}$ given by (10.14) and (10.15) where $f_x(\cdot,\cdot)$ and $G(\cdot,\cdot)$ are given by (10.17) and (10.27), respectively, $f_x(\cdot,\cdot)$ is essentially nonnegative with respect to $x$, $f_z(\cdot,\cdot)$ is essentially nonnegative with respect to $z$, and $\Delta f(\cdot,\cdot)$ is essentially nonnegative with respect to $x$ and belongs to $\mathcal{F}$. For a given $x_\mathrm{e} \in \mathbb{R}_+^{n_x}$ assume there exist nonnegative vectors $z_\mathrm{e} \in \overline{\mathbb{R}}_+^{n_z}$ and $u_\mathrm{e} \in \overline{\mathbb{R}}_+^{n_x}$ such that (10.19) and (10.20) hold. Furthermore, assume that (10.15) is input-to-state stable at $z(t) \equiv z_\mathrm{e}$ with $x(t) - x_\mathrm{e}$ viewed as the input. Finally, let $K \in \mathbb{R}^{n_x \times n_x}$ be such that $-K$ is nonnegative and $A_\mathrm{s} \triangleq A + B_\mathrm{u}K$ is essentially nonnegative and asymptotically stable, and let $q_i$ and $\gamma_i$, $i = 1, \cdots, n_x$, be positive constants. Then the neural adaptive feedback control law

$$u(t) = G_{\mathrm{n}}^{-1}(x(t), z(t))\Big[K(x(t) - x_{\mathrm{e}}) - \hat{W}^{\mathrm{T}}(t)\hat{\sigma}(x(t), z(t))\Big], \tag{10.28}$$

where

$$\hat{W}^{\mathrm{T}}(t) \triangleq \text{block-diag}[\hat{W}_1^{\mathrm{T}}(t), \cdots, \hat{W}_{n_x}^{\mathrm{T}}(t)], \tag{10.29}$$

$\hat{W}_i(t) \in \mathbb{R}^{s_i}$, $t \geq 0$, $i = 1, \cdots, n_x$, and $\hat{\sigma}(x, z) \triangleq [\hat{\sigma}_1^{\mathrm{T}}(x, z), \cdots, \hat{\sigma}_{n_x}^{\mathrm{T}}(x, z)]^{\mathrm{T}}$ with $\hat{\sigma}_i(x, z) = 0$ whenever $x_i = 0$, $i = 1, \cdots, n_x$, with update law

$$\dot{\hat{W}}_i(t) = q_i\Big[(x_i(t) - x_{\mathrm{e}i})\hat{\sigma}_i(x(t), z(t)) - \gamma_i\|P^{1/2}(x(t) - x_{\mathrm{e}})\|\hat{W}_i(t)\Big], \quad \hat{W}_i(0) = \hat{W}_{i0},$$
$$i = 1, \cdots, n_x, \tag{10.30}$$

where $P \triangleq \operatorname{diag}[p_1, \cdots, p_{n_x}] > 0$ satisfies

$$0 = A_{\mathrm{s}}^{\mathrm{T}}P + PA_{\mathrm{s}} + R \tag{10.31}$$

for a positive definite $R \in \mathbb{R}^{n_x \times n_x}$, guarantees that there exists a compact positively invariant set $\mathcal{D}_\alpha \subset \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z} \times \mathbb{R}^{s \times n_x}$ such that $(x_{\mathrm{e}}, z_{\mathrm{e}}, W) \in \mathcal{D}_\alpha$, where $W \in \mathbb{R}^{s \times n_x}$, and the solution $(x(t), z(t), \hat{W}(t))$, $t \geq 0$, of the closed-loop system given by (10.14), (10.15), (10.28), and (10.30) is ultimately bounded for all $(x(0), z(0), \hat{W}(0)) \in \mathcal{D}_\alpha$ with ultimate bound $\|P^{1/2}(x(t) - x_{\mathrm{e}})\| < \varepsilon$, $t \geq T$, where

$$\varepsilon > \sqrt{\left(\frac{\nu}{\lambda_{\min}(RP^{-1})}\right)^2 + \sum_{i=1}^{n_x}\left(\frac{w_i^*}{2\sqrt{\hat{q}_i}} + \sqrt{\frac{\nu}{2q_i\gamma_i}}\right)^2}, \tag{10.32}$$

$\hat{q}_i = q_i/p_ib_i$, and

$$\nu \triangleq 2\left[\sum_{i=1}^{n_x} p_ib_i^2(\varepsilon_i^* + \sqrt{s_i}w_i^*)^2\right]^{1/2} + \sum_{i=1}^{n_x}\frac{1}{2}p_ib_i\gamma_iw_i^{*2}. \tag{10.33}$$

Furthermore, $x(t) \geq\geq 0$ and $z(t) \geq\geq 0$, $t \geq 0$, for all $(x_0, z_0) \in \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$.

**Proof.** First, note that with $u(t)$, $t \geq 0$, given by (10.28) it follows from (10.14), (10.17), and (10.27) that

$$\dot{x}(t) = Ax(t) + \Delta f(x(t), z(t)) + B_{\mathrm{u}}K(x(t) - x_{\mathrm{e}}) - B_{\mathrm{u}}\hat{W}^{\mathrm{T}}(t)\hat{\sigma}(x(t), z(t)),$$
$$x(0) = x_0, \quad t \geq 0. \tag{10.34}$$

Now, defining $e_x(t) \triangleq x(t) - x_{\rm e}$ and $e_z(t) \triangleq z(t) - z_{\rm e}$, using (10.18)–(10.20), and noting that $A_{\rm s} = A + B_{\rm u}K$, it follows from (10.15) and (10.34) that

$$\begin{aligned}\dot{e}_x(t) =&\ A_{\rm s}e_x(t) + Ax_{\rm e} + \Delta f(x(t), z(t)) - B_{\rm u}\hat{W}^{\rm T}(t)\hat{\sigma}(x(t), z(t))\\ =&\ A_{\rm s}e_x(t) + B_{\rm u}[\delta(x(t), z(t)) - \delta(x_{\rm e}, z_{\rm e}) - G_{\rm n}(x_{\rm e}, z_{\rm e})u_{\rm e} - \hat{W}^{\rm T}(t)\sigma(x(t), z(t))]\\ &+B_{\rm u}\hat{W}^{\rm T}(t)[\sigma(x(t), z(t)) - \hat{\sigma}(x(t), z(t))], \quad e_x(0) = x_0 - x_{\rm e}, \quad t \geq 0, \quad (10.35)\end{aligned}$$

and

$$\dot{e}_z(t) = \tilde{f}_z(e_x(t), e_z(t)), \quad e_z(0) = z_0 - z_{\rm e}, \quad (10.36)$$

where $\tilde{f}_z(e_x, e_z) \triangleq f_z(e_x + x_{\rm e}, e_z + z_{\rm e}) - f_z(x_{\rm e}, z_{\rm e})$ and $\sigma(x, z)$ is a basis function satisfying (10.22). Furthermore, since $A_{\rm s}$ is essentially nonnegative and asymptotically stable, it follows from Theorem 7.2 that there exists a positive *diagonal* matrix $P = \mathrm{diag}[p_1, \cdots, p_{n_x}]$ and a positive-definite matrix $R \in \mathbb{R}^{n_x \times n_x}$ such that (10.31) holds.

Next, to show ultimate boundedness of the closed-loop system (10.30), (10.35), and (10.36) consider the Lyapunov-like function

$$V(e_x, e_z, \tilde{W}) = e_x^{\rm T}Pe_x + \mathrm{tr}\,\tilde{W}Q^{-1}\tilde{W}^{\rm T}, \quad (10.37)$$

where $Q \triangleq \mathrm{diag}\Big[\hat{q}_1, \cdots, \hat{q}_{n_x}\Big] = \mathrm{diag}\Big[\frac{q_1}{p_1b_1}, \cdots, \frac{q_{n_x}}{p_{n_x}b_{n_x}}\Big]$, $\tilde{W}(t) \triangleq \hat{W}(t) - W$, and $W^{\rm T} \triangleq$ block-diag$[W_1^{\rm T}, \cdots, W_{n_x}^{\rm T}]$. Note that (10.37) satisfies (10.7) with $x_1 = [e_x^{\rm T}, \tilde{W}_1^{\rm T} \cdots, \tilde{W}_{n_x}^{\rm T}]^{\rm T}$, $x_2 = e_z$, $\alpha(\|x_1\|) = \beta(\|x_1\|) = \|x_1\|^2$, where $\|x_1\|^2 \triangleq e_x^{\rm T}Pe_x + \mathrm{tr}\,\tilde{W}Q^{-1}\tilde{W}^{\rm T}$. Furthermore, $\alpha(\|x_1\|)$ is a class $\mathcal{K}_\infty$ function. Now, letting $e_x(t)$, $t \geq 0$, denote the solution to (10.35) and using (10.22), (10.23), and (10.30), it follows that the time derivative of $V(e_x, e_z, \tilde{W})$ along the closed-loop system trajectories is given by

$$\begin{aligned}\dot{V}(e_x(t), e_z(t), \tilde{W}(t)) =&\ 2e_x^{\rm T}(t)P\Big[A_{\rm s}e_x(t) + B_{\rm u}[\delta(x(t), z(t)) - \delta(x_{\rm e}, z_{\rm e}) - G_{\rm n}(x_{\rm e}, z_{\rm e})u_{\rm e}\\ &-\hat{W}^{\rm T}(t)\sigma(x(t), z(t))] + B_{\rm u}\hat{W}^{\rm T}(t)[\sigma(x(t), z(t)) - \hat{\sigma}(x(t), z(t))]\Big]\\ &+2\mathrm{tr}\,\tilde{W}^{\rm T}(t)Q^{-1}\dot{\hat{W}}(t)\end{aligned}$$

$$
\begin{aligned}
&= -e_x^{\mathrm{T}}(t)Re_x(t) \\
&\quad + \sum_{i=1}^{n_x} 2p_i b_i e_{xi}(t)\Big[-\tilde{W}_i^{\mathrm{T}}(t)\sigma_i(x(t),z(t)) + \varepsilon_i(x(t),z(t))\Big] \\
&\quad + \sum_{i=1}^{n_x} 2p_i b_i e_{xi}(t)\hat{W}_i^{\mathrm{T}}(t)[\sigma_i(x(t),z(t)) - \hat{\sigma}_i(x(t),z(t))] \\
&\quad + \sum_{i=1}^{n_x} 2p_i b_i \tilde{W}_i^{\mathrm{T}}(t)\Big[e_{xi}(t)\hat{\sigma}_i(x(t),z(t)) \\
&\qquad -\gamma_i\|P^{1/2}(x(t)-x_{\mathrm{e}})\|\hat{W}_i(t)\Big] \\
&= -e_x^{\mathrm{T}}(t)Re_x(t) + \sum_{i=1}^{n_x} 2p_i b_i \varepsilon_i(x(t),z(t))e_{xi}(t) \\
&\quad + \sum_{i=1}^{n_x} 2p_i b_i e_{xi}(t)W_i^{\mathrm{T}}[\sigma_i(x(t),z(t)) - \hat{\sigma}_i(x(t),z(t))] \\
&\quad - \sum_{i=1}^{n_x} 2p_i b_i \gamma_i\|P^{1/2}e_x(t)\|\tilde{W}_i^{\mathrm{T}}(t)\hat{W}_i(t). \qquad (10.38)
\end{aligned}
$$

Next, completing squares yields

$$
\begin{aligned}
\dot{V}(e_x(t),e_z(t),\tilde{W}(t)) &\leq -e_x^{\mathrm{T}}(t)Re_x(t) + 2\|P^{1/2}e_x(t)\|\left(\sum_{i=1}^{n_x} p_i b_i^2(\varepsilon_i^* + \sqrt{s_i}w_i^*)^2\right)^{1/2} \\
&\quad - \sum_{i=1}^{n_x} 2p_i b_i \gamma_i\|P^{1/2}e_x(t)\|\tilde{W}_i^{\mathrm{T}}(t)\tilde{W}_i(t) \\
&\quad - \sum_{i=1}^{n_x} 2p_i b_i \gamma_i\|P^{1/2}e_x(t)\|\tilde{W}_i^{\mathrm{T}}(t)W_i \\
&\leq -\lambda_{\min}(RP^{-1})\|P^{1/2}e_x(t)\|^2 \\
&\quad +2\left(\sum_{i=1}^{n_x} p_i b_i^2(\varepsilon_i^* + \sqrt{s_i}w_i^*)^2\right)^{1/2}\|P^{1/2}e_x(t)\| \\
&\quad - \sum_{i=1}^{n_x} 2p_i b_i \hat{q}_i\gamma_i\|P^{1/2}e_x(t)\|\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\|^2 \\
&\quad + \sum_{i=1}^{n_x} 2p_i b_i\sqrt{\hat{q}_i}\gamma_i w_i^*\|P^{1/2}e_x(t)\|\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\| \\
&= \|P^{1/2}e_x(t)\|\Big[-\lambda_{\min}(RP^{-1})\|P^{1/2}e_x(t)\| \\
&\quad +2\left(\sum_{i=1}^{n_x} p_i b_i^2(\varepsilon_i^* + \sqrt{s_i}w_i^*)^2\right)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\quad -\sum_{i=1}^{n_x} 2q_i\gamma_i\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\|^2 + \sum_{i=1}^{n_x} 2q_i\hat{q}_i^{-1/2}\gamma_i w_i^*\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\|\Big] \\
&= \|P^{1/2}e_x(t)\|\Big[-\lambda_{\min}(RP^{-1})\|P^{1/2}e_x(t)\| \\
&\quad +2\left(\sum_{i=1}^{n_x} p_i b_i^2(\varepsilon_i^* + \sqrt{s_i}w_i^*)^2\right)^{1/2} \\
&\quad -\sum_{i=1}^{n_x} 2q_i\gamma_i\Big[\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\| - \frac{w_i^*}{2\sqrt{\hat{q}_i}}\Big]^2 + \sum_{i=1}^{n_x}\frac{1}{2}p_i b_i\gamma_i w_i^{*2}\Big] \\
&\leq \|P^{1/2}e_x(t)\|\Big[-\lambda_{\min}(RP^{-1})\|P^{1/2}e_x(t)\| \\
&\quad -\sum_{i=1}^{n_x} 2q_i\gamma_i\Big[\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\| - \frac{w_i^*}{2\sqrt{\hat{q}_i}}\Big]^2 + \nu\Big], \qquad (10.39)
\end{aligned}
$$

where $\nu$ is given by (10.33). Now, for

$$\|P^{1/2}e_x\| \geq \frac{\nu}{\lambda_{\min}(RP^{-1})} \triangleq \alpha_x, \qquad (10.40)$$

or

$$\|\hat{q}_i^{-1/2}\tilde{W}_i\| \ \geq \ \frac{w_i^*}{2\sqrt{\hat{q}_i}} + \sqrt{\frac{\nu}{2q_i\gamma_i}} \triangleq \alpha_{\tilde{W}_i}, \quad i = 1, \ldots, n_x, \qquad (10.41)$$

it follows that $\dot{V}(e_x(t), e_z(t), \tilde{W}(t)) \leq 0$ for all $t \geq 0$; that is, $\dot{V}(e_x(t), e_z(t), \tilde{W}(t)) \leq 0$ for all $(e_x(t), e_z(t), \tilde{W}(t)) \in \tilde{\mathcal{D}}_{\rm e}\backslash\tilde{\mathcal{D}}_{\rm r}$ and $t \geq 0$, where

$$
\begin{aligned}
\tilde{\mathcal{D}}_{\rm e} &\triangleq \Big\{(e_x, e_z, \tilde{W}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{s\times n_x} : x \in \mathcal{D}_{{\rm c}x}\Big\}, \qquad (10.42) \\
\tilde{\mathcal{D}}_{\rm r} &\triangleq \Big\{(e_x, e_z, \tilde{W}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{s\times n_x} : \\
&\qquad \|P^{1/2}e_x\| \leq \alpha_x,\ \|\hat{q}_i^{-1/2}\tilde{W}_i\| \leq \alpha_{\tilde{W}_i},\ i = 1, \ldots, n_x\Big\}. \qquad (10.43)
\end{aligned}
$$

Next, define

$$\tilde{\mathcal{D}}_\alpha \triangleq \Big\{(e_x, e_z, \tilde{W}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{s\times n_x} : V(e_x, e_z, \tilde{W}) \leq \alpha\Big\}, \qquad (10.44)$$

where $\alpha$ is the maximum value such that $\tilde{\mathcal{D}}_\alpha \subseteq \tilde{\mathcal{D}}_{\rm e}$, and define

$$\tilde{\mathcal{D}}_\eta \triangleq \Big\{(e_x, e_z, \tilde{W}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{s\times n_x} : V(e_x, e_z, \tilde{W}) \leq \eta\Big\}, \qquad (10.45)$$

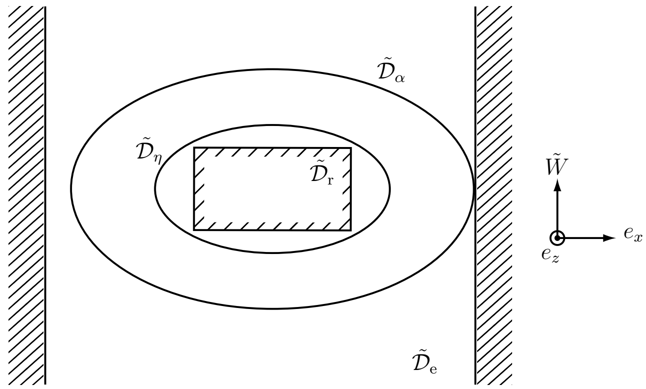



**Figure 10.2**: Visualization of sets used in the proof of Theorem 10.4

where

$$\eta > \beta(\mu) = \mu = \alpha_x^2 + \sum_{i=1}^{n_x} \alpha_{\tilde{W}_i}^2 = \left(\frac{\nu}{\lambda_{\min}(RP^{-1})}\right)^2 + \sum_{i=1}^{n_x}\left[\frac{w_i^*}{2\sqrt{\hat{q}_i}} + \sqrt{\frac{\nu}{2q_i\gamma_i}}\right]^2. \quad (10.46)$$

To show ultimate boundedness of the closed-loop system (10.30), (10.35), and (10.36), assume[2] that $\tilde{\mathcal{D}}_\eta \subset \tilde{\mathcal{D}}_\alpha$ (see Figure 10.2). Now, since $\dot{V}(e_x, e_z, \tilde{W}) \le 0$ for all $(e_x, e_z, \tilde{W}) \in \tilde{\mathcal{D}}_{\rm e} \backslash \tilde{\mathcal{D}}_{\rm r}$ and $\tilde{\mathcal{D}}_{\rm r} \subset \tilde{\mathcal{D}}_\alpha$, it follows that $\tilde{\mathcal{D}}_\alpha$ is positively invariant. Hence, if $(e_x(0), e_z(0), \tilde{W}(0)) \in \tilde{\mathcal{D}}_\alpha$, then it follows from Theorem 10.2 that the solution $(e_x(t), e_z(t), \hat{W}(t))$, $t \ge 0$, to (10.30), (10.35), and (10.36) is bounded with respect to $(e_x, \tilde{W})$ uniformly in $e_z(0)$ and hence ultimately bounded with respect to $(e_x, \tilde{W})$ uniformly in $e_z(0)$. To show that $\|P^{1/2}(x(t) - x_{\rm e})\| < \varepsilon$, $t \ge T$, note that $\tilde{\mathcal{D}}_\eta$ is also positively invariant and hence if there exists $t^* > 0$ such that $(e_x(t^*), e_z(t^*), \hat{W}(t^*)) \in \tilde{\mathcal{D}}_\eta$, then $(e_x(t^*), e_z(t^*), \hat{W}(t^*)) \in \tilde{\mathcal{D}}_\eta$, $t \ge t^*$. Alternatively, suppose the solution $(e_x(t), e_z(t), \hat{W}(t))$, $t \ge 0$, to (10.30), (10.35), and (10.36) remains in $\tilde{\mathcal{D}}_\alpha \backslash \tilde{\mathcal{D}}_\eta$. In this case, the Lyapunov-like function (10.37) is nonincreasing. Furthermore, it follows from (10.38) that

$$\ddot{V}(e_x(t), e_z(t), \tilde{W}(t)) = -2e_x^{\rm T}(t)R\dot{e}_x(t) + 2\dot{e}_x^{\rm T}(t)P\Big[B_{\rm u}[\delta(x(t), z(t)) - \delta(x_{\rm e}, z_{\rm e})$$

[2]This assumption is standard in the neural network literature and ensures that in the error space $\tilde{\mathcal{D}}_{\rm e}$ there exists at least one Lyapunov level set $\tilde{\mathcal{D}}_\eta \subset \tilde{\mathcal{D}}_\alpha$. In the case where the neural network approximation holds in $\mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$, this assumption is automatically satisfied. See Remark 10.1 for further details.

$$
\begin{aligned}
&-G_{\rm n}(x_{\rm e},z_{\rm e})u_{\rm e}-\hat{W}^{\rm T}(t)\hat{\sigma}(x(t),z(t))\Big]\Big]\\
&+2e_x^{\rm T}(t)P\Big[B_{\rm u}[\dot{\delta}(x(t),z(t))-\dot{\hat{W}}^{\rm T}(t)\hat{\sigma}(x(t),z(t))\\
&-\hat{W}^{\rm T}(t)\dot{\hat{\sigma}}(x(t),z(t))]\Big]+\sum_{i=1}^{n_x}2p_ib_i\dot{\tilde{W}}_i^{\rm T}(t)\Big[e_{xi}(t)\hat{\sigma}_i(x(t),z(t))\\
&-\gamma_i\|P^{1/2}e_x(t)\|\hat{W}_i(t)\Big]+\sum_{i=1}^{n_x}2p_ib_i\tilde{W}_i^{\rm T}(t)\Big[\dot{e}_{xi}(t)\hat{\sigma}_i(x(t),z(t))\\
&+e_{xi}(t)\dot{\hat{\sigma}}_i(x(t),z(t))-\gamma_i\Big(\frac{\rm d}{{\rm d}t}\|P^{1/2}e_x(t)\|\Big)\hat{W}_i(t)\\
&-\gamma_i\|P^{1/2}e_x(t)\|\dot{\hat{W}}_i(t)\Big],\quad t\geq 0,
\end{aligned}
\tag{10.47}
$$

where

$$
\dot{\delta}(x(t),z(t)) = \frac{\partial\delta}{\partial x}(x(t),z(t))\dot{x}(t)+\frac{\partial\delta}{\partial z}(x(t),z(t))\dot{z}(t), \tag{10.48}
$$

$$
\dot{\hat{\sigma}}(x(t),z(t)) = \frac{\partial\hat{\sigma}}{\partial x}(x(t),z(t))\dot{x}(t)+\frac{\partial\hat{\sigma}}{\partial z}(x(t),z(t))\dot{z}(t). \tag{10.49}
$$

Note that $\delta'(x,z)$ and $\sigma'(x,z)$ are assumed to be bounded and, since the state trajectory $(e_x(t),e_z(t),\hat{W}(t))$ is bounded, it follows from (10.30), (10.35), (10.36) that $\dot{e}_x(t)$, $\dot{e}_z(t)$, $\dot{\hat{W}}(t)$ are also bounded and hence $\ddot{V}(e_x(t),e_z(t),\tilde{W}(t))$ is bounded. Thus, it follows from Barbalat's lemma [139, p. 192] that $\dot{V}(e_x(t),e_z(t),\tilde{W}(t))\to 0$ as $t\to\infty$. Now, it follows from (10.39) that, since the quantity in the brackets in the right-hand side of (10.39) is strictly positive in $\tilde{\mathcal{D}}_\alpha\backslash\tilde{\mathcal{D}}_\eta$, $\|P^{1/2}e_x(t)\|\to 0$ as $t\to\infty$. Hence, in either case, there exists $T\geq 0$ such that $\|P^{1/2}(x(t)-x_{\rm e})\|<\varepsilon$, $t\geq T$, with $\varepsilon=\alpha^{-1}(\eta)=\sqrt{\eta}$ which yields (10.32).

Next, since (10.36) is input-to-state stable with $e_x$ viewed as the input, it follows from Proposition 10.1 that the solution $e_z(t)$, $t\geq 0$, to (10.36) is ultimately bounded. Furthermore, it follows from Theorem 1 of [223] that there exist a continuously differentiable, radially unbounded, positive-definite function $V_z:\mathbb{R}^{n_z}\to\mathbb{R}$ and class $\mathcal{K}$ functions $\gamma_1(\cdot),\gamma_2(\cdot)$ such that

$$
V_z'(e_z)\tilde{f}_z(e_x,e_z)\leq -\gamma_1(\|e_z\|),\quad \|e_z\|\geq\gamma_2(\|P^{1/2}e_x\|). \tag{10.50}
$$

Since the upper bound for $\|P^{1/2}e_x\|^2$ is given by $\eta$, it follows that the set given by

$$\mathcal{D}_z \triangleq \Big\{ z \in \mathbb{R}^{n_z} : V_z(z - z_{\rm e}) \leq \max_{\|z - z_{\rm e}\| = \gamma_2(\sqrt{\eta})} V_z(z - z_{\rm e}) \Big\}, \quad (10.51)$$

is also positively invariant as long as[3] $\mathcal{D}_z \subset \mathcal{D}_{{\rm c}z}$. Now, since $\tilde{\mathcal{D}}_\alpha$ and $\mathcal{D}_z$ are positively invariant, it follows that

$$\mathcal{D}_\alpha \triangleq \Big\{ (x, z, \hat{W}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{s \times n_x} : (x - x_{\rm e}, z - z_{\rm e}, \hat{W} - W) \in \tilde{\mathcal{D}}_\alpha,\ z \in \mathcal{D}_z \Big\}, \quad (10.52)$$

is also positively invariant. In addition, since (10.14), (10.15), and (10.30) is ultimately bounded with respect to $(x, \hat{W})$ and (10.15) is input-to-state stable at $z(t) \equiv z_{\rm e}$ with $x(t) - x_{\rm e}$ viewed as the input it follows from Proposition 10.1 that the solution $(x(t), z(t), \hat{W}(t))$, $t \geq 0$, of the closed-loop system (10.14), (10.15), (10.28), and (10.30) is ultimately bounded for all $(x(0), z(0), \hat{W}(0)) \in \mathcal{D}_\alpha$.

Finally, to show that $x(t) \geq\geq 0$ and $z(t) \geq\geq 0$ for all $t \geq 0$ and $(x_0, z_0) \in \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$ note that the closed-loop system (10.14), (10.28), and (10.30), is given by

$$\begin{aligned} \dot{x}(t) &= f_x(x(t), z(t)) + B_{\rm u}K(x(t) - x_{\rm e}) - B_{\rm u}\hat{W}^{\rm T}(t)\hat{\sigma}(x(t), z(t)) \\ &= (A + B_{\rm u}K)x(t) + \Delta f(x(t), z(t)) - B_{\rm u}\hat{W}^{\rm T}(t)\hat{\sigma}(x(t), z(t)) - B_{\rm u}Kx_{\rm e} \\ &= \tilde{f}(t, x(t), z(t)) + v, \quad x(0) = x_0, \quad t \geq 0, \end{aligned} \quad (10.53)$$

where

$$\tilde{f}(t, x, z) \triangleq (A + B_{\rm u}K)x + \Delta f(x, z) - B_{\rm u}\hat{W}^{\rm T}(t)\hat{\sigma}(x, z), \quad v \triangleq -B_{\rm u}Kx_{\rm e}. \quad (10.54)$$

Since $\tilde{f}(t, x, z)$, $t \geq 0$, is essentially nonnegative with respect to $x$ pointwise-in-time, $f_z(\cdot, \cdot)$ is essentially nonnegative with respect to $z$, and $v \geq\geq 0$, it follows from Proposition 9.3 that $x(t) \geq\geq 0$, $t \geq 0$, and $z(t) \geq\geq 0$ for all $t \geq 0$ and $(x_0, z_0) \in \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$. □

[3]See Remark 10.1.

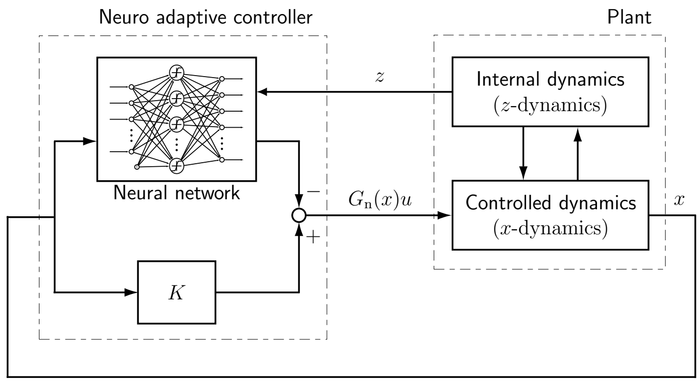



**Figure 10.3**: Block diagram of the closed-loop system

**Remark 10.1.** In the case where the neural network approximation holds in $\mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$, the assumptions $\tilde{\mathcal{D}}_\eta \subset \tilde{\mathcal{D}}_\alpha$ and $\mathcal{D}_z \subset \mathcal{D}_{cz}$ invoked in the proof of Theorem 10.4 are automatically satisfied. Furthermore, in this case the control law (10.28) ensures global ultimate boundedness of the error signals. However, the existence of a global neural network approximator for an uncertain nonlinear map cannot in general be established. Hence, as is common in the neural network literature, for a given arbitrarily large compact set $\mathcal{D}_{cx} \times \mathcal{D}_{cz} \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$, we assume that there exists an approximator for the unknown nonlinear map up to a desired accuracy. This assumption ensures that in the error space $\tilde{\mathcal{D}}_e$ there exists at least one Lyapunov level set such that $\tilde{\mathcal{D}}_\eta \subset \tilde{\mathcal{D}}_\alpha$. In the case where $\delta(\cdot,\cdot)$ is continuous on $\mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$, it follows from the Stone-Weierstrass theorem that $\delta(\cdot,\cdot)$ can be approximated over an arbitrarily large compact set $\mathcal{D}_{cx} \times \mathcal{D}_{cz}$. In this case, our neuro adaptive controller guarantees semiglobal ultimate boundedness; that is, $\mathcal{D}_\alpha$ can be arbitrarily increased. An identical assumption is made in the proof of Theorem 10.6 below.

A block diagram showing the neuro adaptive control architecture given in Theorem 10.4 is shown in Figure 10.3. It is important to note that the adaptive control law (10.28), (10.30) does not require the explicit knowledge of the optimal weighting matrix $W$ and constants $\delta(x_e, z_e)$ and $u_e$. All that is required is the existence of the nonnegative vectors $z_\mathrm{e}$ and $u_\mathrm{e}$ such that the equilibrium conditions (10.19) and (10.20) hold. Furthermore, in the case where $B_\mathrm{u}$ is an *unknown* positive diagonal matrix, we can take the gain matrix $K$ to be diagonal so that $K = \mathrm{diag}[-k_1, \cdots, -k_{n_x}]$, where $k_i$, $i = 1, \cdots, n_x$, are positive. In this case, taking $A$ in (10.17) to be the zero matrix, $A_\mathrm{s}$ is given by $A_\mathrm{s} = \mathrm{diag}[-b_1 k_1, \cdots, -b_{n_x} k_{n_x}]$ which is clearly essentially nonnegative and asymptotically stable. Furthermore, any $P = \mathrm{diag}[p_1, \cdots, p_{n_x}]$ satisfies (10.31). Finally, it is important to note that the control input signal $u(t)$, $t \geq 0$, in Theorem 10.4 can be negative depending on the values of $x(t)$, $z(t)$, and $\hat{W}(t)$, $t \geq 0$. However, as is required for nonnegative and compartmental dynamical systems the closed-loop plant states remain nonnegative.

Next, we generalize Theorem 10.4 to the case where the input matrix is not necessarily nonnegative. For this result $\mathrm{row}_i(K)$ denotes the $i$th row of $K \in \mathbb{R}^{n_x \times n_x}$.

**Theorem 10.5.** Consider the nonlinear uncertain dynamical system $\mathcal{G}$ given by (10.14) and (10.15) where $f_x(\cdot, \cdot)$ and $G(\cdot, \cdot)$ are given by (10.17) and (10.27), respectively, $f_x(\cdot, \cdot)$ is essentially nonnegative with respect to $x$, $f_z(\cdot, \cdot)$ is essentially nonnegative with respect to $z$, and $\Delta f(\cdot, \cdot)$ is essentially nonnegative with respect to $x$ and belongs to $\mathcal{F}$. For a given $x_\mathrm{e} \in \mathbb{R}_+^{n_x}$ assume there exist a nonnegative vector $z_\mathrm{e} \in \overline{\mathbb{R}}_+^{n_z}$ and a vector $u_\mathrm{e} \in \mathbb{R}^{n_x}$ such that (10.19) and (10.20) hold with $f_x(x_\mathrm{e}, z_\mathrm{e}) \leq\leq 0$. Furthermore, assume that (10.15) is input-to-state stable at $z(t) \equiv z_\mathrm{e}$ with $x(t) - x_\mathrm{e}$ viewed as the input. Finally, let $K \in \mathbb{R}^{n_x \times n_x}$ be such that $(\mathrm{sgn}\, b_i)\mathrm{row}_i(K) \leq\leq 0$, $i = 1, \cdots, n_x$, and $A_\mathrm{s} \triangleq A + B_\mathrm{u} K$ is essentially nonnegative and asymptotically stable, and let $q_i$ and $\gamma_i$, $i = 1, \cdots, n_x$, be positive constants. Then the neural adaptive feedback control law (10.28), where $\hat{W}^\mathrm{T}(t) \triangleq \text{block-diag}[\hat{W}_1^\mathrm{T}(t), \cdots, \hat{W}_{n_x}^\mathrm{T}(t)]$, $\hat{W}_i(t) \in \mathbb{R}^{s_i}$, $t \geq 0$, $i = 1, \cdots, n_x$, and $\hat{\sigma}(x, z) \triangleq [\hat{\sigma}_1^\mathrm{T}(x, z), \cdots, \hat{\sigma}_{n_x}^\mathrm{T}(x, z)]^\mathrm{T}$ with $\hat{\sigma}_i(x, z) = 0$ whenever $x_i = 0$, $i = 1, \cdots, n_x$, with update law

$$\dot{\hat{W}}_i(t) = (\operatorname{sgn} b_i) q_i \Big[ (x_i(t) - x_{ei}) \hat{\sigma}_i(x(t), z(t)) - \gamma_i \| P^{1/2}(x(t) - x_e) \| \hat{W}_i(t) \Big],$$
$$\hat{W}_i(0) = \hat{W}_{i0}, \quad i = 1, \cdots, n_x, \quad (10.55)$$

where $P \triangleq \operatorname{diag}[p_1, \cdots, p_{n_x}] > 0$ satisfies (10.31), guarantees that there exists a compact positively invariant set $\mathcal{D}_\alpha \subset \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z} \times \mathbb{R}^{s \times n_x}$ such that $(x_e, z_e, W) \in \mathcal{D}_\alpha$, where $W \in \mathbb{R}^{s \times n_x}$, and the solution $(x(t), z(t), \hat{W}(t))$, $t \geq 0$, of the closed-loop system given by (10.14), (10.15), (10.28), and (10.55) is ultimately bounded for all $(x(0), z(0), \hat{W}(0)) \in \mathcal{D}_\alpha$ with ultimate bound $\|P^{1/2}(x(t) - x_e)\| < \varepsilon$, $t \geq T$, where $\varepsilon$ is given by (10.32). Furthermore, $x(t) \geq\geq 0$ and $z(t) \geq\geq 0$ for all $t \geq 0$ and $(x_0, z_0) \in \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$.

**Proof.** The proof is identical to the proof of Theorem 10.4 with $Q$ replaced by $Q = \operatorname{diag}\left[\frac{q_1}{p_1|b_1|}, \cdots, \frac{q_{n_x}}{p_{n_x}|b_{n_x}|}\right]$. □

Finally, in the case where $B_u$ is an *unknown* diagonal matrix but the sign of each diagonal element is known, we can take the gain matrix $K$ to be diagonal so that $K = \operatorname{diag}[k_1, \cdots, k_{n_x}]$, where $k_i$ is such that $(\operatorname{sgn} b_i)k_i < 0$, $i = 1, \cdots, n_x$. In this case, taking $A$ in (10.17) to be the zero matrix, $A_s$ is given by $A_s = \operatorname{diag}[b_1 k_1, \cdots, b_{n_x} k_{n_x}]$ which is essentially nonnegative and asymptotically stable.

## 10.4. Neural Adaptive Control for Nonlinear Nonnegative Uncertain Systems with Nonnegative Control

As discussed in the Introduction, control (source) inputs of drug delivery systems for physiological and pharmacological processes are usually constrained to be nonnegative as are the system states. Hence, in this section we develop neuro adaptive control laws for nonnegative systems with nonnegative control inputs. Specifically, for a given desired set point $(x_e, z_e) \in \mathbb{R}_+^{n_x} \times \mathbb{R}_+^{n_z}$ and for given $\epsilon_1, \epsilon_2 > 0$, our aim is to design a nonnegative control input $u(t)$, $t \geq 0$, such that $\|x(t) - x_{\rm e}\| < \epsilon_1$ and $\|z(t) - z_{\rm e}\| < \epsilon_2$ for all $t \geq T$, where $T \in [0, \infty)$, and $x(t) \geq\geq 0$ and $z(t) \geq\geq 0$, $t \geq 0$, for all $(x_0, z_0) \in \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$. However, since in many applications of nonnegative systems and in particular, compartmental systems, it is often necessary to regulate a subset of the nonnegative state variables which usually include a central compartment, here we only require that $\|x(t) - x_{\rm e}\| < \epsilon_1$, $t \geq T$. Furthermore, we assume that we have $m$ *independent* control inputs such that the input matrix function is given by $G(x, z) = \operatorname{diag}[g_1(x, z), \cdots, g_m(x, z)]$, where $g_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}_+$, $i = 1, \cdots, m$. For compartmental systems this assumption is not restrictive since control inputs correspond to control inflows to each individual compartment.

**Theorem 10.6.** Consider the nonlinear uncertain dynamical system $\mathcal{G}$ given by (10.14) and (10.15) where $f_x(\cdot, \cdot)$ and $G(\cdot, \cdot)$ are given by (10.17) and (10.27), respectively, $A$ is essentially nonnegative and asymptotically stable, $f_x(\cdot, \cdot)$ is essentially nonnegative with respect to $x$, $f_z(\cdot, \cdot)$ is essentially nonnegative with respect to $z$, and $\Delta f(\cdot, \cdot)$ is essentially nonnegative with respect to $x$ and belongs to $\mathcal{F}$. For a given $x_{\rm e} \in \mathbb{R}_+^{n_x}$ assume there exist positive vectors $z_{\rm e} \in \mathbb{R}_+^{n_z}$ and $u_{\rm e} \in \mathbb{R}_+^{n_x}$ such that (10.19) and (10.20) hold and the equilibrium point $(x_{\rm d}, z_{\rm e}) \in \mathbb{R}_+^{n_x} \times \mathbb{R}_+^{n_z}$ of (10.14), (10.15) is globally asymptotically stable with $u(t) \equiv u_{\rm e}$. Furthermore, assume that (10.15) is input-to-state stable at $z(t) \equiv z_{\rm e}$ with $x(t) - x_{\rm e}$ viewed as the input. Finally, let $q_i$ and $\gamma_i$, $i = 1, \cdots, n_x$, be positive constants and $k_i$, $i = 1, \cdots, n_x$ be nonpositive constants. Then the neural adaptive feedback control law

$$u_i(t) = \max\{0, \hat{u}_i(t)\}, \quad i = 1, \cdots, n_x, \tag{10.56}$$

where

$$\hat{u}_i(t) = -g_{{\rm n}i}^{-1}(x(t), z(t)) \hat{W}_i^{\rm T}(t) \sigma_i(x(t), z(t)) \tag{10.57}$$

and $\hat{W}_i(t) \in \mathbb{R}^{s_i}$, $t \geq 0$, $i = 1, \cdots, n_x$, with update law

$$\dot{\hat{W}}_i(t) = q_i \left[ (x_i(t) - x_{\mathrm{e}i})\sigma_i(x(t), z(t)) - \gamma_i \| P^{1/2}(x(t) - x_{\mathrm{e}}) \| \hat{W}_i(t) \right], \quad \hat{W}_i(0) = \hat{W}_{i0},$$
$$i = 1, \cdots, n_x, \quad (10.58)$$

where $P \triangleq \mathrm{diag}[p_1, \cdots, p_{n_x}] > 0$ satisfies

$$0 = A^{\mathrm{T}} P + PA + R \quad (10.59)$$

for a positive definite $R \in \mathbb{R}^{n_x \times n_x}$, guarantees that there exists a compact positively invariant set $\mathcal{D}_\alpha \subset \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z} \times \mathbb{R}^{s \times n_x}$ such that $(x_{\mathrm{e}}, z_{\mathrm{e}}, W) \in \mathcal{D}_\alpha$, where $W \in \mathbb{R}^{s \times n_x}$, and the solution $(x(t), z(t), \hat{W}(t))$, $t \geq 0$, of the closed-loop system given by (10.14), (10.15), (10.56), and (10.58) is ultimately bounded for all $(x(0), z(0), \hat{W}(0)) \in \mathcal{D}_\alpha$ with ultimate bound $\|P^{1/2}(x(t) - x_{\mathrm{e}})\| < \varepsilon$, $t \geq T$, where

$$\varepsilon > \sqrt{\left( \frac{\nu}{\lambda_{\min}(RP^{-1})} \right)^2 + \sum_{i=1}^{n_x} \left[ \frac{1}{2} \Big( \frac{\sqrt{b_i s_i}}{q_i \gamma_i^2} + \frac{w_i^*}{\sqrt{\hat{q}_i}} \Big) + \sqrt{\frac{\nu}{2 q_i \gamma_i}} \right]^2}, \quad (10.60)$$

$\hat{q}_i = q_i / p_i b_i$, and

$$\nu \triangleq \left( \sum_{i=1}^{n_x} p_i b_i^2 \varepsilon_i^{*2} \right)^{1/2} + \sum_{i=1}^{n_x} \Big[ 2 b_i \sqrt{p_i s_i} w_i^* + \frac{q_i \gamma_i}{2} \Big( \sqrt{\frac{b_i s_i}{q_i \gamma_i^2}} + \frac{w_i^*}{\sqrt{\hat{q}_i}} \Big)^2 \Big]. \quad (10.61)$$

Furthermore, $u(t) \geq\geq 0$, $x(t) \geq\geq 0$, and $z(t) \geq\geq 0$ for all $t \geq 0$ and $(x_0, z_0) \in \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$.

**Proof.** First, define $\hat{W}_u^{\mathrm{T}}(t) \triangleq \text{block-diag}[\hat{W}_{u1}^{\mathrm{T}}(t), \cdots, \hat{W}_{un_x}^{\mathrm{T}}(t)]$ and $K_u \triangleq \mathrm{diag}[k_{u1}, \cdots, k_{un_x}]$, where

$$\hat{W}_{ui}(t) = \begin{cases} 0, & \text{if } \hat{u}_i(t) < 0, \\ \hat{W}_i(t), & \text{otherwise,} \end{cases} \quad i = 1, \cdots, n_x, \quad (10.62)$$

$$k_{ui} = \begin{cases} 0, & \text{if } \hat{u}_i(t) < 0, \\ k_i, & \text{otherwise,} \end{cases} \quad i = 1, \cdots, n_x. \quad (10.63)$$

Next, note that with $u(t)$, $t \geq 0$, given by (10.56) it follows from (10.14), (10.17), and (10.27) that

$$\dot{x}(t) = Ax(t) + \Delta f(x(t), z(t)) + B_{\mathrm{u}}[K_u(x(t) - x_{\mathrm{e}}) - \hat{W}_u^{\mathrm{T}}(t)\sigma(x(t), z(t))],$$
$$x(0) = x_0, \quad t \geq 0. \quad (10.64)$$

Now, defining $e_x(t) \triangleq x(t) - x_{\mathrm{e}}$ and $e_z(t) \triangleq z(t) - z_{\mathrm{e}}$, and using (10.19) and (10.20), it follows from (10.15) and (10.64) that

$$\begin{aligned}\dot{e}_x(t) &= Ae_x(t) + Ax_{\mathrm{e}} + \Delta f(x(t), z(t)) + B_{\mathrm{u}}[K_u e_x(t) - \hat{W}_u^{\mathrm{T}}(t)\sigma(x(t), z(t))] \\ &= Ae_x(t) + B_{\mathrm{u}}[\delta(x(t), z(t)) - \delta(x_{\mathrm{e}}, z_{\mathrm{e}}) - G_{\mathrm{n}}(x_{\mathrm{e}}, z_{\mathrm{e}})u_{\mathrm{e}} + K_u e_x(t) \\ &\quad -\hat{W}^{\mathrm{T}}(t)\sigma(x(t), z(t))] + B_{\mathrm{u}}(\hat{W}(t) - \hat{W}_u(t))^{\mathrm{T}}\sigma(x(t), z(t)),\end{aligned}$$
$$e_x(0) = x_0 - x_{\mathrm{e}}, \quad t \geq 0, \quad (10.65)$$

and

$$\dot{e}_z(t) = \tilde{f}_z(e_x(t), e_z(t)), \quad e_z(0) = z_0 - z_{\mathrm{e}}, \quad (10.66)$$

where $\tilde{f}_z(e_x, e_z) \triangleq f_z(e_x + x_{\mathrm{e}}, e_z + z_{\mathrm{e}}) - f_z(x_{\mathrm{e}}, z_{\mathrm{e}})$. Furthermore, since $A$ is essentially nonnegative and asymptotically stable, it follows from Theorem 7.2 that there exist a positive *diagonal* matrix $P = \mathrm{diag}[p_1, \cdots, p_{n_x}]$ and a positive-definite matrix $R \in \mathbb{R}^{n_x \times n_x}$ such that (10.59) holds.

Next, to show ultimate boundedness of the closed-loop system (10.58), (10.65), and (10.66) consider the Lyapunov-like function

$$V(e_x, e_z, \tilde{W}) = e_x^{\mathrm{T}} P e_x + \mathrm{tr}\, \tilde{W} Q^{-1} \tilde{W}^{\mathrm{T}}, \quad (10.67)$$

where $Q \triangleq \mathrm{diag}\left[\hat{q}_1, \cdots, \hat{q}_{n_x}\right] = \mathrm{diag}\left[\frac{q_1}{p_1 b_1}, \cdots, \frac{q_{n_x}}{p_{n_x} b_{n_x}}\right]$ and $\tilde{W}^{\mathrm{T}}(t) \triangleq \hat{W}^{\mathrm{T}}(t) - W^{\mathrm{T}}$ with $W^{\mathrm{T}}$ given by $W^{\mathrm{T}} = \text{block-diag}[W_1^{\mathrm{T}}, \cdots, W_{n_x}^{\mathrm{T}}]$. Note that (10.67) satisfies (10.7) with $x_1 = [e_x^{\mathrm{T}}, \tilde{W}_1^{\mathrm{T}} \cdots, \tilde{W}_{n_x}^{\mathrm{T}}]^{\mathrm{T}}$, $x_2 = e_z$, $\alpha(\|x_1\|) = \beta(\|x_1\|) = \|x_1\|^2$, where $\|x_1\|^2 \triangleq e_x^{\mathrm{T}} P e_x + \mathrm{tr}\, \tilde{W} Q^{-1} \tilde{W}^{\mathrm{T}}$. Furthermore, $\alpha(\|x_1\|)$ is a class $\mathcal{K}_\infty$ function. Now, letting

$e_x(t)$, $t \geq 0$, denote the solution to (10.65) and using (10.22) and (10.58), it follows that the time derivative of $V(e_x, e_z, \tilde{W})$ along the closed-loop system trajectories is given by

$$\begin{aligned}
\dot{V}(e_x(t), e_z(t), \tilde{W}(t)) = {} & 2e_x^{\mathrm{T}}(t)P\Big[Ae_x(t) + B_{\mathrm{u}}[\delta(x(t), z(t)) - \delta(x_{\mathrm{e}}, z_{\mathrm{e}}) \\
& -G_{\mathrm{n}}(x_{\mathrm{e}}, z_{\mathrm{e}})u_{\mathrm{e}} + K_u e_x(t) - \hat{W}^{\mathrm{T}}(t)\sigma(x(t), z(t))] \\
& +B_{\mathrm{u}}(\hat{W}(t) - \hat{W}_u(t))^{\mathrm{T}}\sigma(x(t), z(t))\Big] + 2\mathrm{tr}\,\tilde{W}^{\mathrm{T}}(t)Q^{-1}\dot{\hat{W}}(t) \\
= {} & -e_x^{\mathrm{T}}(t)Re_x(t) + 2e_x^{\mathrm{T}}(t)PB_{\mathrm{u}}K_u e_x(t) \\
& +\sum_{i=1}^{n_x} 2p_i b_i e_{xi}(t)\Big[-\tilde{W}_i^{\mathrm{T}}(t)\sigma_i(x(t), z(t)) + \varepsilon_i(x(t), z(t))\Big] \\
& +\sum_{i=1}^{n_x} 2p_i b_i e_{xi}(t)(\hat{W}_i(t) - \hat{W}_{ui}(t))^{\mathrm{T}}\sigma_i(x(t), z(t)) \\
& +\sum_{i=1}^{n_x} 2p_i b_i \tilde{W}_i^{\mathrm{T}}(t)\Big[e_{xi}(t)\sigma_i(x(t), z(t)) \\
& -\gamma_i\|P^{1/2}(x(t) - x_{\mathrm{e}})\|\hat{W}_i(t)\Big] \\
\leq {} & -e_x^{\mathrm{T}}(t)Re_x(t) + \sum_{i=1}^{n_x} 2p_i b_i e_{xi}(t)\varepsilon_i(x(t), z(t)) \\
& +\sum_{i=1}^{n_x} 2p_i b_i \Big(e_{xi}(t)(\hat{W}_i(t) - \hat{W}_{ui}(t))^{\mathrm{T}}\sigma_i(x(t), z(t)) \\
& -\gamma_i\|P^{1/2}e_x(t)\|\tilde{W}_i^{\mathrm{T}}(t)\hat{W}_i(t)\Big). \qquad (10.68)
\end{aligned}$$

Now, for each $i \in \{1, \cdots, n_x\}$ and for the two cases given in (10.62), the last term on the right-hand side of (10.68) gives:

*i)* If $\hat{u}_i(t) < 0$, then $\hat{W}_{ui}(t) = 0$ and hence

$$\begin{aligned}
& 2p_i b_i\Big(e_{xi}(t)(\hat{W}_i(t) - \hat{W}_{ui}(t))^{\mathrm{T}}\sigma_i(x(t), z(t)) - \gamma_i\|P^{1/2}e_x(t)\|\tilde{W}_i^{\mathrm{T}}(t)\hat{W}_i(t)\Big) \\
& \quad = 2p_i b_i\Big(e_{xi}(t)(\tilde{W}_i(t) + W_i)^{\mathrm{T}}\sigma_i(x(t), z(t)) - \gamma_i\|P^{1/2}e_x(t)\|\|\tilde{W}_i(t)\|^2 \\
& \qquad -\gamma_i\|P^{1/2}e_x(t)\|\tilde{W}_i^{\mathrm{T}}(t)W_i\Big) \\
& \quad \leq 2b_i\sqrt{p_i\hat{q}_i s_i}\|P^{1/2}e_x(t)\|\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\| + 2b_i\sqrt{p_i s_i}w_i^*\|P^{1/2}e_x(t)\|
\end{aligned}$$

$$-2p_i\hat{q}_ib_i\gamma_i\|P^{1/2}e_x(t)\|\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\|^2$$
$$+2p_ib_i\hat{q}_i^{1/2}\gamma_iw_i^*\|P^{1/2}e_x(t)\|\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\|.$$

*ii)* Otherwise, $\hat{W}_{ui}(t) = \hat{W}_i(t)$ and hence

$$\begin{aligned}
2p_ib_i\Big(&e_{xi}(t)(\hat{W}_i(t) - \hat{W}_{ui}(t))^{\mathrm{T}}\sigma_i(x(t), z(t)) - \gamma_i\|P^{1/2}e_x(t)\|\tilde{W}_i^{\mathrm{T}}(t)\hat{W}_i(t)\Big)\\
&= -2p_ib_i\gamma_i\|P^{1/2}e_x(t)\|\tilde{W}_i^{\mathrm{T}}(t)\hat{W}_i(t)\\
&= -2p_ib_i\gamma_i\|P^{1/2}e_x(t)\|\tilde{W}_i^{\mathrm{T}}(t)(\tilde{W}_i(t) + W_i)\\
&\leq -2p_ib_i\gamma_i\|P^{1/2}e_x(t)\|\tilde{W}_i^{\mathrm{T}}(t)\tilde{W}_i(t)\\
&\quad +2p_ib_i\hat{q}_i^{1/2}\gamma_iw_i^*\|P^{1/2}e_x(t)\|\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\|\\
&\leq 2\sqrt{q_ib_is_i}\|P^{1/2}e_x(t)\|\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\| + 2b_i\sqrt{p_is_i}w_i^*\|P^{1/2}e_x(t)\|\\
&\quad -2q_i\gamma_i\|P^{1/2}e_x(t)\|\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\|^2\\
&\quad +2q_i\hat{q}_i^{-1/2}\gamma_iw_i^*\|P^{1/2}e_x(t)\|\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\|.
\end{aligned}$$

Hence, it follows from (10.68) that in either case

$$\begin{aligned}
\dot{V}(e_x(t), e_z(t), \tilde{W}(t)) \leq\ & -e_x^{\mathrm{T}}(t)Re_x(t) + \sum_{i=1}^{n_x}2p_ib_ie_{xi}(t)\varepsilon_i(x(t), z(t))\\
& + \sum_{i=1}^{n_x}\Big(2\sqrt{q_ib_is_i}\|P^{1/2}e_x(t)\|\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\|\\
& +2b_i\sqrt{p_is_i}w_i^*\|P^{1/2}e_x(t)\| - 2q_i\gamma_i\|P^{1/2}e_x(t)\|\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\|^2\\
& +2q_i\hat{q}_i^{-1/2}\gamma_iw_i^*\|P^{1/2}e_x(t)\|\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\|\Big)\\
\leq\ & -e_x^{\mathrm{T}}(t)Re_x(t) + 2\|P^{1/2}e_x(t)\|\left(\sum_{i=1}^{n_x}p_ib_i^2\varepsilon_i^{*2}\right)^{1/2}\\
& + \sum_{i=1}^{n_x}2\sqrt{q_ib_is_i}\|P^{1/2}e_x(t)\|\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\|\\
& + \sum_{i=1}^{n_x}2b_i\sqrt{p_is_i}w_i^*\|P^{1/2}e_x(t)\|\\
& -2\sum_{i=1}^{n_x}q_i\gamma_i\|P^{1/2}e_x(t)\|\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\|^2
\end{aligned}$$

$$+2\sum_{i=1}^{n_x} q_i\hat{q}_i^{-1/2}\gamma_i w_i^*\|P^{1/2}e_x(t)\|\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\|. \tag{10.69}$$

Next, completing squares yields

$$\begin{aligned}
\dot{V}(e_x(t),e_z(t),\tilde{W}(t)) \leq\ & -\lambda_{\min}(RP^{-1})\|P^{1/2}e_x(t)\|^2 + 2\|P^{1/2}e_x(t)\|\left(\sum_{i=1}^{n_x} p_i b_i^2 \varepsilon_i^{*2}\right)^{1/2} \\
& + \sum_{i=1}^{n_x} 2\sqrt{q_i b_i s_i}\|P^{1/2}e_x(t)\|\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\| \\
& + \sum_{i=1}^{n_x} 2b_i\sqrt{p_i s_i}w_i^*\|P^{1/2}e_x(t)\| \\
& -2\sum_{i=1}^{n_x} q_i\gamma_i\|P^{1/2}e_x(t)\|\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\|^2 \\
& +2\sum_{i=1}^{n_x} q_i\hat{q}_i^{-1/2}\gamma_i w_i^*\|P^{1/2}e_x(t)\|\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\| \\
=\ & \|P^{1/2}e_x(t)\|\Big[-\lambda_{\min}(RP^{-1})\|P^{1/2}e_x(t)\| + 2\left(\sum_{i=1}^{n_x} p_i b_i^2 \varepsilon_i^{*2}\right)^{1/2} \\
& + \sum_{i=1}^{n_x} 2b_i\sqrt{p_i s_i}w_i^* - \sum_{i=1}^{n_x} 2q_i\gamma_i\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\|^2 \\
& + \sum_{i=1}^{n_x} 2q_i\gamma_i\Big(\sqrt{\frac{b_i s_i}{q_i\gamma_i^2}} + \frac{w_i^*}{\sqrt{\hat{q}_i}}\Big)\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\|\Big] \\
=\ & \|P^{1/2}e_x(t)\|\Big[-\lambda_{\min}(RP^{-1})\|P^{1/2}e_x(t)\| + 2\left(\sum_{i=1}^{n_x} p_i b_i^2 \varepsilon_i^{*2}\right)^{1/2} \\
& + \sum_{i=1}^{n_x} 2b_i\sqrt{p_i s_i}w_i^* + \frac{q_i\gamma_i}{2}\Big(\sqrt{\frac{b_i s_i}{q_i\gamma_i^2}} + \frac{w_i^*}{\sqrt{\hat{q}_i}}\Big)^2\Big] \\
& - \sum_{i=1}^{n_x} 2q_i\gamma_i\Big[\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\| - \frac{1}{2}\Big(\sqrt{\frac{b_i s_i}{q_i\gamma_i^2}} + \frac{w_i^*}{\sqrt{\hat{q}_i}}\Big)\Big]^2\Big] \\
=\ & \|P^{1/2}e_x(t)\|\Big[-\lambda_{\min}(RP^{-1})\|P^{1/2}e_x(t)\| + \nu \\
& - \sum_{i=1}^{n_x} 2q_i\gamma_i\Big[\|\hat{q}_i^{-1/2}\tilde{W}_i(t)\| - \frac{1}{2}\Big(\sqrt{\frac{b_i s_i}{q_i\gamma_i^2}} + \frac{w_i^*}{\sqrt{\hat{q}_i}}\Big)\Big]^2\Big],
\end{aligned} \tag{10.70}$$

where $\nu$ is given by (10.61). Now, for

$$\|P^{1/2}e_x\| \geq \frac{\nu}{\lambda_{\min}(RP^{-1})} \triangleq \alpha_x, \tag{10.71}$$

or

$$\|\hat{q}_i^{-1/2}\tilde{W}_i\| \geq \frac{1}{2}\Big(\frac{\sqrt{b_i s_i}}{q_i\gamma_i^2} + \frac{w_i^*}{\hat{q}_i}\Big) + \sqrt{\frac{\nu}{2q_i\gamma_i}} \triangleq \alpha_{\tilde{W}_i}, \quad i = 1, \ldots, n_x, \tag{10.72}$$

it follows that $\dot{V}(e_x(t), e_z(t), \tilde{W}(t)) \leq 0$ for all $t \geq 0$; that is, $\dot{V}(e_x(t), e_z(t), \tilde{W}(t)) \leq 0$ for all $(e_x(t), e_z(t), \tilde{W}(t)) \in \tilde{\mathcal{D}}_{\rm e}\backslash\tilde{\mathcal{D}}_{\rm r}$ and $t \geq 0$, where $\tilde{\mathcal{D}}_{\rm e}$ and $\tilde{\mathcal{D}}_{\rm r}$ are given by (10.42) and (10.43), respectively.

Next, define

$$\tilde{\mathcal{D}}_\eta \triangleq \Big\{(e_x, e_z, \tilde{W}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{s\times n_x} : V(e_x, e_z, \tilde{W}) \leq \eta\Big\}, \tag{10.73}$$

where

$$\begin{aligned}\eta > \beta(\mu) = \mu = {} & \alpha_x^2 + \sum_{i=1}^{n_x} \alpha_{\tilde{W}_i}^2 \\ = {} & \left(\frac{\nu}{\lambda_{\min}(RP^{-1})}\right)^2 + n_x \sum_{i=1}^{n_x}\left[\frac{1}{2}\Big(\frac{\sqrt{b_i s_i}}{q_i\gamma_i^2} + \frac{w_i^*}{\sqrt{\hat{q}_i}}\Big) + \sqrt{\frac{\nu}{2q_i\gamma_i}}\right]^2. \end{aligned} \tag{10.74}$$

To show ultimate boundedness of the closed-loop system (10.58), (10.65), and (10.66), assume that $\tilde{\mathcal{D}}_\eta \subset \tilde{\mathcal{D}}_\alpha$ (see Remark 10.1), where $\tilde{\mathcal{D}}_\alpha$ is given by (10.44) and $\alpha$ is the maximum value such that $\tilde{\mathcal{D}}_\alpha \subseteq \tilde{\mathcal{D}}_{\rm e}$. Now, since $\dot{V}(e_x, e_z, \tilde{W}) \leq 0$ for all $(e_x, e_z, \tilde{W}) \in \tilde{\mathcal{D}}_{\rm e}\backslash\tilde{\mathcal{D}}_{\rm r}$ and $\tilde{\mathcal{D}}_{\rm r} \subset \tilde{\mathcal{D}}_\alpha$, it follows that $\tilde{\mathcal{D}}_\alpha$ is positively invariant. Hence, if $(e_x(0), e_z(0), \tilde{W}(0)) \in \tilde{\mathcal{D}}_\alpha$, then it follows from Theorem 10.3 that the solution $(e_x(t), e_z(t), \hat{W}(t))$, $t \geq 0$, to (10.58), (10.65), and (10.66) is ultimately bounded with respect to $(e_x, \tilde{W})$ uniformly in $e_z(0)$ with ultimate bound given by $\varepsilon = \alpha^{-1}(\eta) = \sqrt{\eta}$ which yields (10.60). In addition, since (10.66) is input-to-state stable with $e_x$ viewed as the input, it follows from Proposition 10.1 that the solution $e_z(t)$, $t \geq 0$, to (10.66) is also ultimately bounded. Furthermore, it follows from Theorem 1 of [223] that there exist a continuously differentiable, radially unbounded, positive-definite function $V_z : \mathbb{R}^{n_z} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\gamma_1(\cdot), \gamma_2(\cdot)$ such that

$$V_z'(e_z)\tilde{f}_z(e_x, e_z) \leq -\gamma_1(\|e_z\|), \quad \|e_z\| \geq \gamma_2(\|P^{1/2}e_x\|). \tag{10.75}$$

Since the upper bound for $\|P^{1/2}e_x\|^2$ is given by $\eta$, it follows that the set given by

$$\mathcal{D}_z \triangleq \left\{ z \in \mathbb{R}^{n_z} : V_z(z - z_\mathrm{e}) \le \max_{\|z - z_\mathrm{e}\| = \gamma_2(\sqrt{\eta})} V_z(z - z_\mathrm{e}) \right\}, \tag{10.76}$$

is also positively invariant as long as $\mathcal{D}_z \subset \mathcal{D}_{\mathrm{c}z}$ (see Remark 10.1). Now, since $\tilde{\mathcal{D}}_\alpha$ and $\mathcal{D}_z$ are positively invariant, it follows that

$$\mathcal{D}_\alpha \triangleq \left\{ (x, z, \hat{W}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{s \times n_x} : (x - x_\mathrm{e}, z - z_\mathrm{e}, \hat{W} - W) \in \tilde{\mathcal{D}}_\alpha,\ z \in \mathcal{D}_z \right\}, \tag{10.77}$$

is also positively invariant. In addition, it follows using identical arguments as in the proof of Theorem 10.4 that the solution $(x(t), z(t), \hat{W}(t))$, $t \ge 0$, of the closed-loop system (10.14), (10.15), (10.56), and (10.58) is ultimately bounded for all $(x(0), z(0), \hat{W}(0)) \in \mathcal{D}_\alpha$.

Finally, $u(t) \ge\ge 0$, $t \ge 0$, is a restatement of (10.56). Now, since $G(x(t)) \ge\ge 0$, $t \ge 0$, and $u(t) \ge\ge 0$, $t \ge 0$, it follows from Proposition 9.2 that $x(t) \ge\ge 0$ and $z(t) \ge\ge 0$ for all $t \ge 0$ and $(x_0, z_0) \in \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$. □

In Theorem 10.6 we assumed that the equilibrium point $(x_\mathrm{e}, z_\mathrm{e})$ of (10.14), (10.15) is globally asymptotically stable with $u(t) \equiv u_\mathrm{e}$. In general, however, unlike linear nonnegative systems with asymptotically stable plant dynamics, a given set point $(x_\mathrm{e}, z_\mathrm{e}) \in \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$ for the nonlinear nonnegative dynamical system (10.14), (10.15) may not be asymptotically stabilizable with a constant control $u(t) \equiv u_\mathrm{e} \in \overline{\mathbb{R}}_+^{n_x}$. However, if $f(\tilde{x}) \triangleq [f_x^\mathrm{T}(x, z), f_z^\mathrm{T}(x, z)]^\mathrm{T}$, where $\tilde{x} \triangleq [x^\mathrm{T}, z^\mathrm{T}]^\mathrm{T}$, is homogeneous, cooperative; that is, the Jacobian matrix $\frac{\partial f(\tilde{x})}{\partial \tilde{x}}$ is essentially nonnegative for all $\tilde{x} \in \overline{\mathbb{R}}_+^{n_x + n_z}$ [221], the Jacobian matrix $\frac{\partial f(\tilde{x})}{\partial \tilde{x}}$ is irreducible for all $\tilde{x} \in \overline{\mathbb{R}}_+^{n_x + n_z}$ [221], and the zero solution $\tilde{x}(t) \equiv 0$ of the undisturbed ($u(t) \equiv 0$) system (10.14), (10.15) is globally asymptotically stable, then the set point $(x_\mathrm{e}, z_\mathrm{e}) \in \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$ satisfying (10.19), (10.20) is a unique equilibrium point with $u(t) \equiv u_\mathrm{e} \in \mathbb{R}_+^{n_x}$ and is also asymptotically stable for all $(x_0, z_0) \in \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$ [52]. This implies that the solution $(x(t), z(t)) \equiv (x_\mathrm{e}, z_\mathrm{e})$ to (10.14), (10.15) with $u(t) \equiv u_\mathrm{e}$ is asymptotically stable for all $(x_0, z_0) \in \overline{\mathbb{R}}_+^{n_x} \times \overline{\mathbb{R}}_+^{n_z}$.

It is important to note that unlike Theorem 10.4, Theorem 10.6 does not require that the set of basis functions $\sigma_i(\cdot,\cdot)$, $i = 1, \cdots, n_x$, be essentially nonnegative nor satisfy $\sigma_i(x, z) = 0$ whenever $x_i = 0$, $i = 1, \cdots, n_x$. This is due to the fact that the control input is constrained to be nonnegative and hence the neuro adaptive controller given by Theorem 10.6 cannot destroy nonnegativity of the closed-loop plant states.

## 10.5. Neural Adaptive Control for Continuous Stirred Tank Reactors

In this section we apply the proposed neuro adaptive control framework to temperature regulation of chemical reactors. In particular, we consider a perfectly mixed, continuously stirred tank reactor shown in Figure 10.4 involving a single, first-order exothermic (i.e., energy releasing) irreversible reaction A → B. The model involves fluid streams that are continuously fed and removed from the reactor. Since we assume perfect mixing in the reactor, the exit stream has the same concentration and temperature as the reactor fluid. Furthermore, the jacket surrounding the reactor is assumed to be perfectly mixed and at a lower temperature than the reactor. In this case, energy (in the form of heat) transfers through the reactor walls into the jacket, removing the heat generated by the reaction. A mass and energy balance of the reactor, assuming constant volume, heat capacity, and density, yields (see [4,18,185,233])

$$\dot{C}_{\rm A}(t) = \frac{F}{\hat{V}}(C_{\rm Af} - C_{\rm A}(t)) - r(T(t), C_{\rm A}(t)), \quad C_{\rm A}(0) = C_{\rm A0}, \quad t \geq 0, \tag{10.78}$$

$$\dot{T}(t) = \frac{F}{\hat{V}}(T_{\rm f} - T(t)) - \left(\frac{-\Delta H}{\rho c_p}\right) r(T(t), C_{\rm A}(t)) - \frac{UA}{\hat{V}\rho c_p}(T(t) - T_{\rm j}(t)), \quad T(0) = T_0, \tag{10.79}$$

where $C_{\rm A}(\cdot)$ is the concentration of reactant A in the reactor effluent in mols/liter, $C_{\rm Af}$ is the concentration of reactant A in the feed stream in mols/liter, $T(\cdot)$ is the reactor temperature in degrees Kelvin, $T_{\rm j}(\cdot)$ is the jacket temperature in degrees Kelvin, $T_{\rm f}$ is

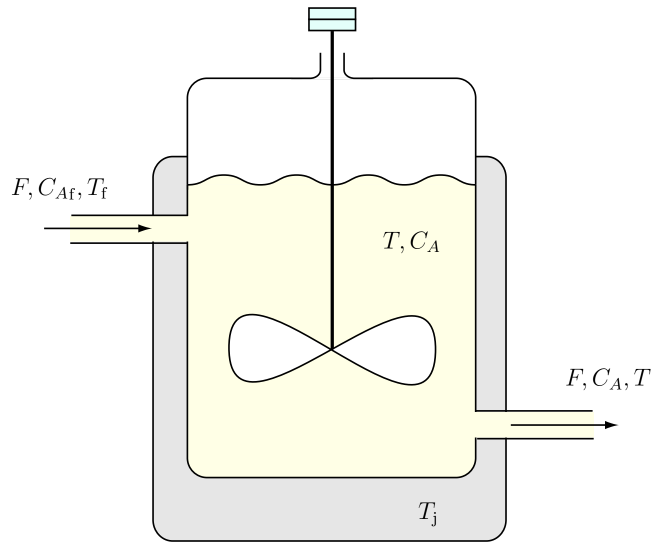



**Figure 10.4**: Exothermic continuously stirred tank reactor

the feed temperature in degrees Kelvin, $F$ is the constant feed flow rate in liters/min, $\hat{V}$ is the reactor volume in liters, $-\Delta H$ is the heat of reaction in Joules/mol, $\rho$ is the density in grams/liter, $c_p$ is the specific heat in Joules/(gram·Kelvin), $UA$ is the heat transfer term in Joules/(min·Kelvin), and $r(T, C_{\mathrm{A}})$ is the rate of reaction satisfying Arrhenius' law given by

$$r(T, C_{\mathrm{A}}) = k_0 C_{\mathrm{A}} e^{-\frac{\Delta E}{RT}}, \tag{10.80}$$

where $k_0$ is the rate constant in $\mathrm{min}^{-1}$, $\Delta E$ is the activation energy in Joules/mol, and $R$ is the ideal gas constant in Joules/(mol·Kelvin).

Due to the exponential nonlinearity in $r(T, C_{\mathrm{A}})$, the nonlinear kinetic equations (10.78), (10.79) can exhibit multiple equilibria, limit cycles, and chaos for fixed jacket temperatures. Here, our control objective is to regulate the reactor temperature $T(\cdot)$ to a prescribed set point $T_{\mathrm{e}}$ by controlling the jacket temperature $T_{\mathrm{j}}(\cdot)$. Note that with $x = T$, $z = C_{\mathrm{A}}$, and $u = T_{\mathrm{j}}$, (10.78) and (10.79) can be written in state-space form (10.14) and (10.15) with

$$f_x(x, z) = -(a_1 + a_3)x + a_4 r(x, z) + a_1 d, \tag{10.81}$$

$$f_z(x, z) = -a_1 z - r(x, z) + a_2, \tag{10.82}$$

$$G(x, z) = b, \tag{10.83}$$

where $a_1 = \frac{F}{\hat{V}}$, $a_2 = \frac{F}{\hat{V}} C_{\mathrm{Af}}$, $a_3 = b = \frac{UA}{\hat{V}\rho c_p}$, $a_4 = \frac{\Delta H}{\rho c_p}$, and $d = T_{\mathrm{f}}$. Note that $f_x(x, z)$ and $f_z(x, z)$ are essentially nonnegative with respect to $x$ and $z$, respectively, and hence it follows from Proposition 9.2 that the state trajectory of (10.78) and (10.79) remain in the nonnegative orthant of the state space for nonnegative initial conditions and a nonnegative input. We assume that there exists an equilibrium point $(x_{\mathrm{e}}, z_{\mathrm{e}}) \in \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+$ so that (10.19) and (10.20) are satisfied (see [185]). Furthermore, we assume that the system kinetics are uncertain with respect to the temperature as well as $a_1, a_2, a_3, a_4, b$, and $d$ are uncertain parameters.

To see that (10.79) is input-to-state stable with $T(\cdot)$ viewed as the input, define $e_x(t) \triangleq x(t) - x_{\mathrm{e}}$ and $e_z(t) \triangleq z(t) - z_{\mathrm{e}}$ so that $\tilde{f}_z(e_x, e_z) \triangleq f_z(e_x + x_{\mathrm{e}}, e_z + z_{\mathrm{e}}) - f_z(x_{\mathrm{e}}, z_{\mathrm{e}})$ is given by

$$\tilde{f}_z(e_x, e_z) = -a_1 e_z - k_0 e^{-\frac{\Delta E}{R(e_x + x_{\mathrm{e}})}} e_z - k_0 z_{\mathrm{e}} \left( e^{-\frac{\Delta E}{R(e_x + x_{\mathrm{e}})}} - e^{-\frac{\Delta E}{R x_{\mathrm{e}}}} \right). \tag{10.84}$$

Now, defining $V_z(e_z) \triangleq \frac{1}{2} e_z^2$ and noting that $\left( e^{-\frac{\Delta E}{R(e_2 + x_{\mathrm{e}2})}} - e^{-\frac{\Delta E}{R x_{\mathrm{e}2}}} \right)$ is bounded, it follows that

$$\begin{aligned} V_z'(e_z)\tilde{f}_z(e_x, e_z) &\le -a_1 e_z^2 - k_0 z_{\mathrm{e}} \left( e^{-\frac{\Delta E}{R(e_x + x_{\mathrm{e}})}} - e^{-\frac{\Delta E}{R x_{\mathrm{e}}}} \right) e_z \\ &\le -\|e_z\| \left[ a_1 \|e_z\| - k_0 z_{\mathrm{e}} \left\| e^{-\frac{\Delta E}{R(e_x + x_{\mathrm{e}})}} - e^{-\frac{\Delta E}{R x_{\mathrm{e}}}} \right\| \right] \\ &\le 0, \quad \|e_z\| \ge \frac{k_0 z_{\mathrm{e}}}{a_1} \left\| e^{-\frac{\Delta E}{R(e_x + x_{\mathrm{e}})}} - e^{-\frac{\Delta E}{R x_{\mathrm{e}}}} \right\|, \end{aligned} \tag{10.85}$$

which shows that $\dot{e}_z(t) = \tilde{f}_z(e_x(t), e_z(t))$, $t \ge 0$, is input-to-state stable with $e_x$ viewed as the input. Hence, it follows from Theorem 10.6 that the adaptive feedback controller (10.28) with update law (10.30) guarantees that there exist positive constants $\varepsilon$ and $T$ such that $|T(t) - T_{\mathrm{e}}| < \varepsilon$, $t \ge T$, for any (uncertain) positive system parameters $a_1, \cdots, a_4, b, d$, and any (uncertain) continuous rate of reaction $r(\cdot, \cdot)$.

For our simulation, we choose the system parameters given in Table 10.1. With

Table 10.1: System parameter values [167]

| Variable | Value |
|---|---|
| $UA$ | $5 \times 10^4$ J/min K |
| $C_{\rm A}$ | 0.5 mol/$\ell$ |
| $C_{\rm Af}$ | 1 mol/$\ell$ |
| $c_p$ | 0.239 J/gK |
| $F$ | 100 $\ell$/min |
| $k_0$ | $7.2 \times 10^{10}$ min$^{-1}$ |
| $T$ | 350 K |
| $T_{\rm f}$ | 350 K |
| $\hat{V}$ | 100 $\ell$ |
| $\Delta E/R$ | 8750 K |
| $(-\Delta H)$ | $5\times10^4$ J/mol |
| $\rho$ | 1000 g/$\ell$ |

$T_{\rm e} = 375$ K, $k_1 = -14$, $\hat{\sigma}_1(x,z) = \eta_1(x)\Big[\frac{1}{1+e^{-a(x-T_{\rm e})}}, \cdots, \frac{1}{1+e^{-6a(x-T_{\rm e})}}, \frac{1}{1+e^{-a(z-0.5)}}, \cdots, \frac{1}{1+e^{-6a(z-0.5)}}\Big]^{\rm T}$, where $\eta_1 : \mathbb{R} \to \mathbb{R}$ is a continuous monotone function such that $\eta_1(0) = 0$ and $\eta_1(x) = 1$, $x \geq \zeta > 0$, $a = 0.5$, $q_1 = 20$, $\gamma_1 = 0.01$, and initial conditions $C_{\rm A}(0) = 0.5$ mol/$\ell$, $T(0) = 350$ K, and $\hat{W}(0) = 0$ K, Figure 10.5 shows the state trajectories (i.e., reactor temperature and concentration of reactant A) versus time and the control signal (i.e., jacket temperature) versus time. Note that $\hat{\sigma}_1(\cdot,\cdot)$ takes values between 0 and 1 and $\hat{\sigma}_1(0,z) = 0$. Finally, Figure 10.6 shows the neural network weight history versus time.

## 10.6. Conclusion

Nonnegative and compartmental systems are widely used to capture system dynamics involving the interchange of mass and energy between homogenous subsystems or compartments. Thus, it is not surprising that nonnegative and compartmental models are remarkably effective in describing the dynamical behavior of complex highly uncertain dynamical systems such as biological systems, physiological systems, pharmacological systems, chemical reaction systems, queuing systems, ecological sys-

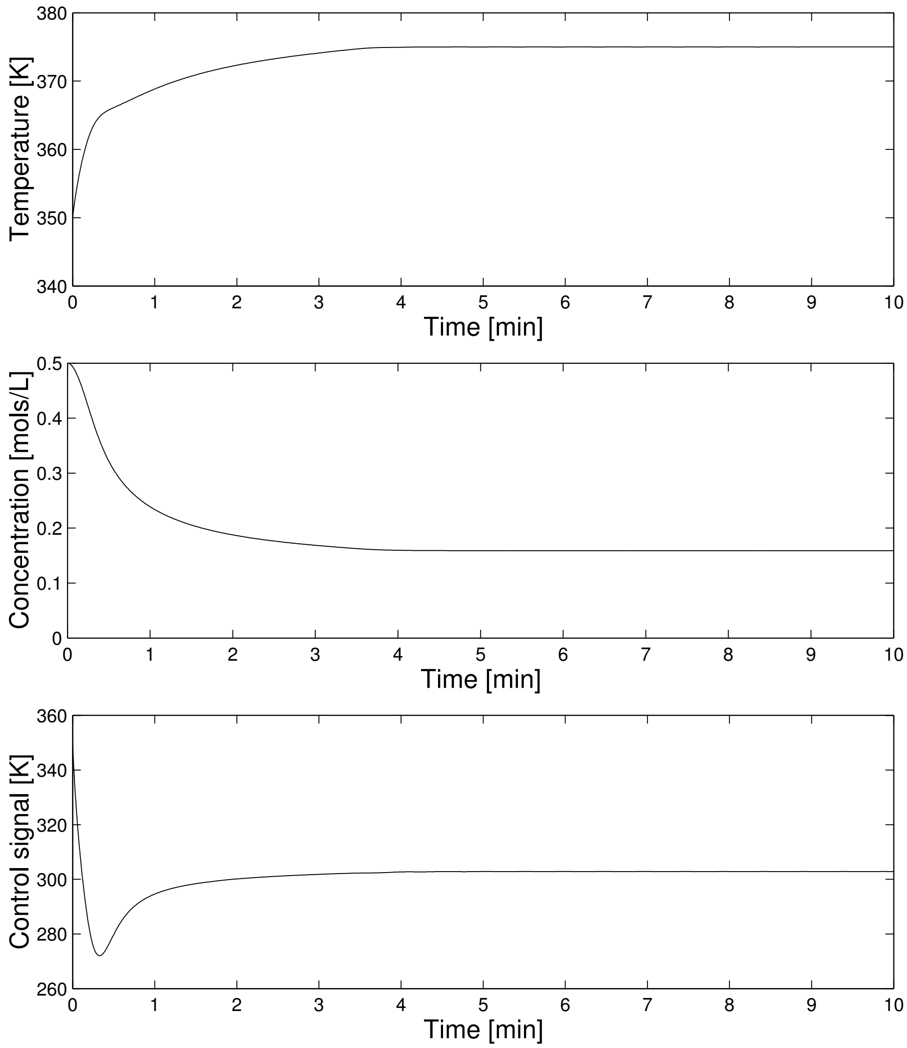



**Figure 10.5**: State trajectories (reactor temperature and concentration of reactant A) and control signal (jacket temperature) versus time

tems, economic systems, telecommunication systems, transportation systems, power systems, and network systems. In this chapter, we developed a neural adaptive control framework for adaptive set-point regulation of nonlinear uncertain nonnegative and compartmental systems. Using Lyapunov-like methods the proposed framework was shown to guarantee ultimate boundedness of the error signals corresponding to the physical system states and the neural network weighting gains while additionally guaranteeing the nonnegativity of the closed-loop system states associated with the plant dynamics. We then generalized our neuro adaptive controller to address the

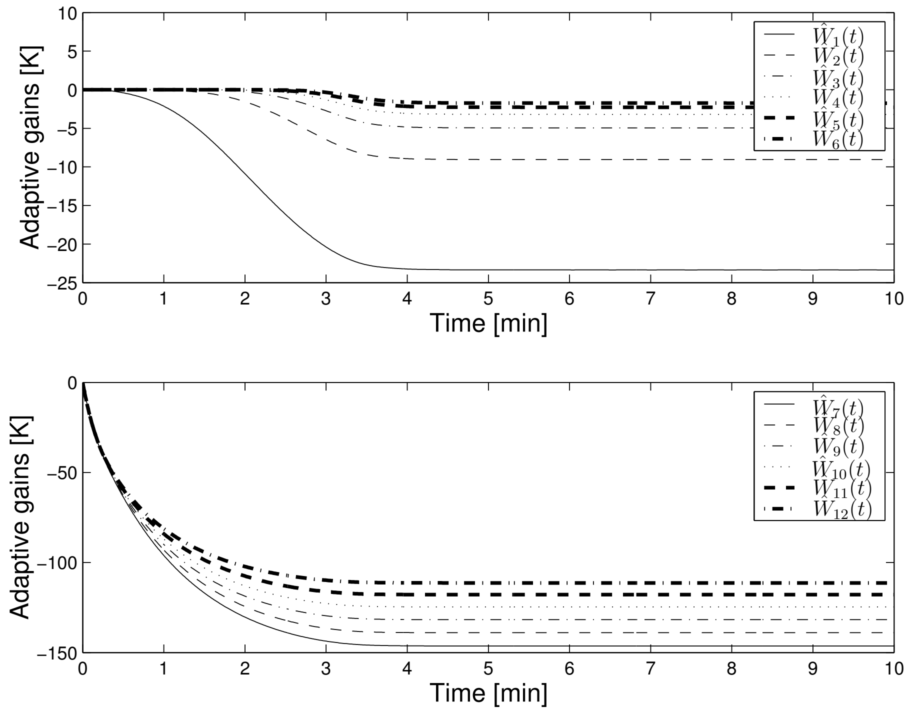



**Figure 10.6**: Neural network weighting functions versus time

problem of nonnegative systems with nonnegative control inputs. This generalization is crucial for physiological, pharmacological, and chemical processes as control inputs are usually constrained to be nonnegative.

# Chapter 11

# Passivity-Based Neural Network Adaptive Output Feedback Control for Nonlinear Nonnegative Dynamical Systems

## 11.1. Introduction

Advanced control methodologies have been (and are being) extensively developed for complex highly uncertain engineering systems. Adaptive control algorithms have been devised that ensure system stability and performance in the face of unavoidable discrepancies between system models and the real physical system. To this end, neural networks have provided an ideal framework for on-line identification and control of many complex uncertain engineering systems because of their great flexibility in approximating a large class of continuous nonlinear maps and their adaptability due to their inherently parallel architecture. However, modern active control technology has received far less consideration in medical systems. The main reason for this state of affairs is the steep barriers to communication between mathematics/control engineering and medicine. However, this is slowly changing and there is no doubt that control-system technology has a great deal to offer medicine. This is particularly true when dealing with critically ill patients in the intensive care unit or operating room. These patients often require administration of drugs to regulate control of key physiological variables, such as level of consciousness, heart rate, blood pressure, ventilatory drive, etc., within desired targets. The rate of administration of these drugs is critical, requiring constant monitoring and frequent adjustments. Open-loop control (manual control) by clinical personnel can be tedious, imprecise, time-consuming, and often of poor quality. Hence, the need for active control (closed-loop control) of drug administration is crucial.

There has been a great deal of interest in the development of algorithms for closed-loop control of intravenous anesthesia. Algorithms for closed-loop control of inhalation anesthesia, using anesthetic concentration as the performance variable, have been developed. However, since it is not possible with current sensor technology to rapidly measure the plasma concentration of intravenously-administered drugs (in contrast to inhalation agents), these algorithms are not useful for intravenous agents. Furthermore, drug concentration, even if it could be measured rapidly, is not the best measurement variable. We are far more interested in drug effect than concentration. More relevant are recently described algorithms for the control of intravenous anesthesia using a processed electroencephalograph (EEG) as the control variable. Building on pioneering work by Bickford [26], Absalom *et al.* [2] developed a proportional-integral-derivative controller using the bispectral index (BIS), a processed EEG signal, as the performance variable to control the infusion of the hypnotic, propofol. While the median performance of the system was good, in 3 of 10 patients oscillations of the BIS signal around the set point were observed and anesthesia was deemed clinically inadequate in 1 of the 10 patients. This would not be acceptable for clinical practice. Alternative algorithms have been devised by both Schwilden *et al.* [207, 208] and Struys *et al.* [227]. Both groups have developed and clinically tested closed-loop, model-based adaptive controllers for the delivery of intravenous anesthesia using a processed EEG signal as the measurement variable. The algorithms are based on a pharmacokinetic model predicting the drug concentration as a function of infusion rate and time and a pharmacodynamic model relating the processed EEG signal to concentration. The pharmacokinetic and pharmacodynamic models are characterized by specific parameters. The two algorithms are similar in assuming that certain model parameters are equal to the mean values from previous pharmacokinetic/pharmacodynamic studies while varying a few select parameters of the models to minimize the difference between the desired and observed processed EEG signal. The primary difference between the two algorithms is in the parameters which are fixed to the mean values from previous studies and the parameters that are chosen for variation. Schwilden *et al.* [207, 208] assume that the pharmacodynamic parameters may be fixed to mean values taken from the literature and vary pharmacokinetic parameters to minimize bias from the target signal. In contrast, Struys *et al.* [227] assume that the pharmacokinetic parameters are always correct and that any variability in individual patient response is due to pharmacodynamic variability. Thus they vary pharmacodynamic parameters to minimize the difference between the observed and target processed EEG signal. Both algorithms have been implemented in the operating room with clinically acceptable performance in small numbers of patients. However, as pointed out by Glass and Rampil [69] in an analysis of the algorithm of Struys *et al.* [227], the systems may not have been fully stressed. For example, in their study, Struys *et al.* [227] administered a relatively high fixed dose of the opioid remifentanil, in conjunction with closed-loop control of the hypnotic, propofol. This blunted the patient response to surgical stimuli and meant that the propofol was needed only to produce unconsciousness in patients who were profoundly analgesic. The result was that only small adjustments in propofol concentrations were necessary.

Whether either system would have been robust in less controlled situations is an open question. And it should be noted that both algorithms are model dependent and only partially adaptive, in the sense that only select pharmacokinetic/pharmacodynamic parameters are varied to minimize the signal bias from the target.

Given the uncertainties in both pharmacokinetic and pharmacodynamic models, and the magnitude of interpatient variability, in this chapter we present a neural network adaptive control framework that accounts for combined interpatient pharmacokinetic and pharmacodynamic variability. In particular, we develop a neural adaptive *output feedback* control framework for adaptive set-point regulation of nonlinear uncertain nonnegative and compartmental systems. Nonnegative and compartmental models provide a broad framework for biological and physiological systems, including clinical pharmacology, and are well suited for the problem of closed-loop control of drug administration. Specifically, nonnegative and compartmental dynamical systems [6, 19, 24, 62, 70, 75, 123, 124, 164, 166, 172, 182, 187, 203] are composed of homogeneous interconnected subsystems (or compartments) which exchange variable nonnegative quantities of material with conservation laws describing transfer, accumulation, and elimination between the compartments and the environment. It thus follows from physical considerations that the state trajectory of such systems remains in the nonnegative orthant of the state space for nonnegative initial conditions. Using nonnegative and compartmental model structures, a Lyapunov-based neural adaptive control framework is developed that guarantees ultimate boundedness of the error signals corresponding to the physical system states as well as the neural network weighting gains. The neuro adaptive controllers are constructed *without* requiring knowledge of the system dynamics while guaranteeing that the physical system states remain in the nonnegative orthant of the state space. Furthermore, since in pharmacological applications involving active drug administration control (source) inputs as well as the system states need to be nonnegative, the proposed neuro adaptive controller also guarantees that the control signal remains nonnegative. We emphasize that even though neuro adaptive *full-state feedback* controllers for nonnegative systems have been recently addressed in [104], our present formulation addresses adaptive *output feedback* controllers for nonlinear systems with *unmodeled dynamics* of *unknown dimension* using the exponential passivity, feedback equivalence, and stabilizability of exponentially minimum phase notions developed in [39, 60]. The framework developed in [104] is limited to full-state feedback controllers and does not address the problem of unmodeled dynamics of unknown dimension. Output feedback controllers are crucial in clinical pharmacology since key physiological (state) variables cannot be measured in practice. Furthermore, the results in [104] are based on the new notions of partial boundedness and partial ultimate boundedness as opposed to the approach of this paper which imposes passivity and positive real requirements on the system dynamics. Thus, the approach of the present paper is related to the neuro adaptive control methods developed in [116, 117].

## 11.2. Mathematical Preliminaries

In this section we introduce some key results concerning passive and exponentially passive dynamical systems [39, 60] that are necessary for developing the main results of this chapter. Specifically, consider the nonlinear dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(t_0) = x_0, \quad t \geq t_0, \tag{11.1}$$

$$y(t) = h(x(t)), \tag{11.2}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, $u(t) \in \mathbb{R}^m$, $t \geq 0$, $y(t) \in \mathbb{R}^m$, $t \geq 0$, $f : \mathbb{R}^n \to \mathbb{R}^n$, and $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$. We assume that $f(\cdot)$, $G(\cdot)$, and $h(\cdot)$ are continuous mappings and $f(\cdot)$ has at least one equilibrium so that, without loss of generality, $f(0) = 0$ and

$h(0) = 0$. Furthermore, for the nonlinear dynamical system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions are satisfied; that is, $f(\cdot)$, $G(\cdot)$, and $u(\cdot)$ satisfy sufficient regularity conditions such that the system (11.1) has a unique solution forward in time. The following definition introduces the notion of exponential passivity.

**Definition 11.1** [39]. A nonlinear dynamical system $\mathcal{G}$ of the form (11.1), (11.2) is *exponentially passive* if there exists a constant $\rho > 0$ such that the *dissipation inequality*

$$0 \leq \int_{t_0}^{t} e^{\rho s} u^{\mathrm{T}}(s) y(s) \mathrm{d}s, \tag{11.3}$$

is satisfied for all $t \geq t_0$ with $x(t_0) = 0$. A nonlinear dynamical system of the form (11.1), (11.2) is *passive* if the dissipation inequality (11.3) is satisfied with $\rho = 0$.

For the statement of the following result recall the definitions of zero-state observability and complete reachability given in [241].

**Theorem 11.1** [39]. Let $\mathcal{G}$ be zero-state observable and completely reachable. $\mathcal{G}$ is exponentially passive if and only if there exist functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$ and $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and a scalar $\rho > 0$ such that $V_{\mathrm{s}}(\cdot)$ is continuously differentiable, positive definite, $V_{\mathrm{s}}(0) = 0$, $\ell(\cdot)$ is continuous, $\ell(0) = 0$, and, for all $x \in \mathbb{R}^n$,

$$0 = V_{\mathrm{s}}'(x) f(x) + \rho V_{\mathrm{s}}(x) + \ell^{\mathrm{T}}(x)\ell(x), \tag{11.4}$$

$$0 = \tfrac{1}{2} V_{\mathrm{s}}'(x) G(x) - h^{\mathrm{T}}(x). \tag{11.5}$$

As shown in [39], an equivalent statement for exponential passivity of $\mathcal{G}$ using (11.4), (11.5) is given by

$$\dot{V}_{\mathrm{s}}(x) = -\rho V_{\mathrm{s}}(x) + u^{\mathrm{T}} y - \ell^{\mathrm{T}}(x)\ell(x), \quad x \in \mathbb{R}^n. \tag{11.6}$$

Hence, if $\mathcal{G}$ is exponentially passive (resp., passive), then the undisturbed ($u(t) \equiv 0$) nonlinear dynamical system (11.1) is asymptotically stable (resp., Lyapunov stable). If, in addition, there exist scalars $\alpha, \beta > 0$ and $p \geq 1$ such that $\alpha\|x\|^p \leq V_{\rm s}(x) \leq \beta\|x\|^p$, $x \in \mathbb{R}^n$, then the undisturbed ($u(t) \equiv 0$) nonlinear dynamical system (11.1) is *exponentially stable*. This leads to the following stronger notion of exponential passivity [60].

**Definition 11.2.** A nonlinear dynamical system $\mathcal{G}$ of the form (11.1), (11.2) is *strongly exponentially passive* if $\mathcal{G}$ is exponentially passive and there exist a continuously differentiable function $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$ and positive scalars $\alpha, \beta > 0$ such that

$$\alpha\|x\|^2 \leq V_{\rm s}(x) \leq \beta\|x\|^2, \quad x \in \mathbb{R}^n. \tag{11.7}$$

Since in this paper we consider nonlinear dynamical systems in *normal form*, for the remainder of this section we restate some of the key results of [60] in a concise and unified format that supports the developments in Section 11.3. Specifically, we consider the normal form characterization of (11.1), (11.2) given by

$$\dot{x}(t) = f_x(x(t), z(t)) + G(x(t), z(t))u(t), \quad x(t_0) = x_0, \quad t \geq t_0, \tag{11.8}$$

$$\dot{z}(t) = f_z(x(t), z(t)), \quad z(t_0) = z_0, \tag{11.9}$$

$$y(t) = x(t), \tag{11.10}$$

where $x(t) \in \mathbb{R}^m$, $t \geq 0$, $z(t) \in \mathbb{R}^{n-m}$, $t \geq 0$, $u(t) \in \mathbb{R}^m$, $t \geq 0$, $y(t) \in \mathbb{R}^m$, $t \geq 0$, $f_x : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ and satisfies $f_x(0, z) = 0$, $z \in \mathbb{R}^{n-m}$, $f_z : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ and satisfies $f_z(x, 0) = 0$, $x \in \mathbb{R}^m$, and $G : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{m \times m}$ with $\det G(x, z) \neq 0$, $(x, z) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$. The following definition introduces the notion of exponentially minimum phase.

**Definition 11.3.** A nonlinear dynamical system $\mathcal{G}$ of the form (11.8)–(11.10) is *exponentially minimum phase* if there exist a continuously differentiable function $V_z : \mathbb{R}^{n-m} \to \mathbb{R}$ and positive constants $\alpha, \beta, \gamma$, and $\delta$ such that

$$\alpha\|z\|^2 \leq V_z(z) \leq \beta\|z\|^2, \tag{11.11}$$

$$V_z'(z)f_z(0,z) \leq -\gamma\|z\|^2, \tag{11.12}$$

$$\|V_z'(z)\| \leq \delta\|z\|. \tag{11.13}$$

It follows from converse Lyapunov theory that if the zero solution $z(t) \equiv 0$ to $\dot{z}(t) = f_z(0, z(t))$, $z(0) = z_0$, $t \geq 0$, is exponentially stable and $f_z(0, \cdot)$ is continuously differentiable, then there exists a continuously differentiable function $V_z : \mathbb{R}^{n-m} \to \mathbb{R}$ such that (11.11)–(11.13) hold. Finally, the following definition and theorem are needed for the main results of this chapter. For the statement of this definition let $\tilde{x} \triangleq [x^{\mathrm{T}}, z^{\mathrm{T}}]^{\mathrm{T}}$, $\tilde{f}(\tilde{x}) \triangleq [f_x^{\mathrm{T}}(x,z), f_z^{\mathrm{T}}(x,z)]^{\mathrm{T}}$, and $\tilde{G}(\tilde{x}) \triangleq [G^{\mathrm{T}}(\tilde{x}), 0_{m\times(n-m)}]^{\mathrm{T}}$.

**Definition 11.4** [60]. A nonlinear dynamical system $\mathcal{G}$ of the form (11.8)–(11.10) is *semiglobally output feedback exponentially passive* if, for any compact set $\mathcal{D}_{\mathrm{c}} \subset \mathbb{R}^n$, there exists a continuous feedback $u : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ of the form

$$u = \alpha_{\mathcal{D}_{\mathrm{c}}}(y) + \beta_{\mathcal{D}_{\mathrm{c}}}(y)v, \tag{11.14}$$

where $\det \beta_{\mathcal{D}_{\mathrm{c}}}(y) \neq 0$, $y \in \mathbb{R}^m$, such that the closed-loop system given by (11.8)–(11.10) and (11.14), or, equivalently,

$$\dot{\tilde{x}}(t) = \tilde{f}_{\mathcal{D}_{\mathrm{c}}}(\tilde{x}(t)) + \tilde{G}_{\mathcal{D}_{\mathrm{c}}}(\tilde{x}(t))v(t), \quad \tilde{x}(0) \in \mathcal{D}_{\mathrm{c}}, \quad t \geq 0, \tag{11.15}$$

$$y(t) = x(t), \tag{11.16}$$

where $\tilde{f}_{\mathcal{D}_{\mathrm{c}}}(\tilde{x}) = \tilde{f}(\tilde{x}) + \tilde{G}(\tilde{x})\alpha_{\mathcal{D}_{\mathrm{c}}}(y)$ and $\tilde{G}_{\mathcal{D}_{\mathrm{c}}}(\tilde{x}) = \tilde{G}(\tilde{x})\beta_{\mathcal{D}_{\mathrm{c}}}(y)$, is strongly exponentially passive from $v$ to $y$ for all $\tilde{x} \in \mathcal{D}_{\mathrm{c}}$.

**Theorem 11.2** [60]. Consider the nonlinear dynamical system $\mathcal{G}$ given by (11.8)–(11.10). Assume that the input matrix function $G(x,z)$, $(x,z) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$, can be factored as

$$G(x,z) = G_{\mathrm{u}}(z)G_{\mathrm{n}}(x), \tag{11.17}$$

where $G_{\mathrm{u}} : \mathbb{R}^{n-m} \to \mathbb{R}^{m\times m}$ and $G_{\mathrm{n}} : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ are continuously differentiable matrix functions such that $G_{\mathrm{u}}(z) = G_{\mathrm{u}}^{\mathrm{T}}(z) > 0$, $z \in \mathbb{R}^{n-m}$, and $\det G_{\mathrm{n}}(x) \neq 0$, $x \in \mathbb{R}^m$. Then $\mathcal{G}$ is semiglobally output feedback exponentially passive if and only if $\mathcal{G}$ is exponentially minimum phase.

**Remark 11.1.** As noted in [60], if $f_z(\cdot,\cdot)$ is globally Lipschitz continuous in $\mathbb{R}^m \times \mathbb{R}^{n-m}$, $G_{\mathrm{u}}(\cdot)$ is uniformly positive definite; that is, there exists $\mu > 0$ such that $G_{\mathrm{u}}(z) = G_{\mathrm{u}}^{\mathrm{T}}(z) \geq \mu I_m$, $z \in \mathbb{R}^{n-m}$, and the zero solution $z(t) \equiv 0$ to $\dot{z}(t) = f_z(0, z(t))$, $z(0) = z_0$, $t \geq 0$, is globally exponentially stable, then the above result holds globally.

**Remark 11.2.** It is important to note that if the conditions in Theorem 11.2 are satisfied, then there exists an *output feedback* control law of the form (11.14) which renders the closed-loop system exponentially passive from $v$ to $y$. Specifically, as shown in [60], the output feedback controller achieving exponential passivity is given by

$$u = -G_{\mathrm{n}}^{-1}(y)[G_{\mathrm{u}}^{-1}(0)f_x(0,y) + \chi y] + G_{\mathrm{n}}^{-1}(y)v, \tag{11.18}$$

where $\chi \in \mathbb{R}$ is a positive constant. Finally, it is important to note that in the case where $G_{\mathrm{u}}(z) \equiv I_m$, $\beta_{\mathcal{D}_{\mathrm{c}}}(\cdot)$ in (11.14) takes the form

$$\beta_{\mathcal{D}_{\mathrm{c}}}(y) = G_{\mathrm{n}}^{-1}(y) = G^{-1}(y). \tag{11.19}$$

This fact will be used for our main result presented in the following section.

## 11.3. Neural Output Feedback Adaptive Control for Nonlinear Nonnegative Uncertain Systems

In this section we consider the problem of characterizing neural adaptive output feedback control laws for nonlinear nonnegative and compartmental uncertain dynamical systems to achieve *set-point* regulation in the nonnegative orthant. Specifically, consider the controlled nonlinear uncertain dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = f_x(x(t), z(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad t \geq 0, \tag{11.20}$$

$$\dot{z}(t) = f_z(x(t), z(t)), \quad z(0) = z_0, \tag{11.21}$$

$$y(t) = x(t), \tag{11.22}$$

where $x(t) \in \mathbb{R}^m$, $t \geq 0$, and $z(t) \in \mathbb{R}^{n-m}$, $t \geq 0$, are the state vectors, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $y(t) \in \mathbb{R}^m$, $t \geq 0$, is the system output, $f_x : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ is essentially nonnegative with respect to $x$ but otherwise unknown and satisfies $f_x(0, z) = 0$, $z \in \mathbb{R}^{n-m}$, $f_z : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ is essentially nonnegative with respect to $z$ but otherwise unknown and satisfies $f_z(x, 0) = 0$, $x \in \mathbb{R}^m$, and $G : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ is an unknown nonnegative input matrix function. Furthermore, the system dimension $n$ need not be known. The control input $u(\cdot)$ in (11.20) is restricted to the class of *admissible controls* consisting of measurable functions such that $u(t) \in \mathbb{R}^m$, $t \geq 0$.

As discussed in the Introduction, control (source) inputs of drug delivery systems for physiological and pharmacological processes are usually constrained to be nonnegative as are the system states. Hence, in this chapter we develop neuro adaptive output feedback control laws for nonnegative systems with nonnegative control inputs. Specifically, for a given desired set point $(y_{\rm d}, z_{\rm e}) \in \mathbb{R}_+^m \times \mathbb{R}_+^{n-m}$ and for given $\epsilon_1, \epsilon_2 > 0$, our aim is to design a nonnegative control input $u(t)$, $t \geq 0$, such that $\|y(t) - y_{\rm d}\| < \epsilon_1$ and $\|z(t) - z_{\rm e}\| < \epsilon_2$ for all $t \geq T$, where $T \in [0, \infty)$, and $x(t) \geq\geq 0$

and $z(t) \geq\geq 0$ for all $t \geq 0$ and $(x_0, z_0) \in \overline{\mathbb{R}}_+^m \times \overline{\mathbb{R}}_+^{n-m}$. However, since in many applications of nonnegative systems and in particular, compartmental systems, it is often necessary to regulate a subset of the nonnegative state variables which usually include a central compartment, here we only require that $\|y(t) - y_{\rm d}\| < \epsilon_1$, $t \geq T$. Furthermore, we assume that we have $m$ *independent* control inputs so that the input matrix function is given by $G(x) = \mathrm{diag}[g_1(x), \cdots, g_m(x)]$, where $g_i : \mathbb{R}^n \to \mathbb{R}_+$, $i = 1, \cdots, m$. For compartmental systems this assumption is not restrictive since control inputs correspond to control inflows to each individual compartment.

In this chapter, we assume that for a given set point $y_{\rm d} \in \mathbb{R}_+^m$ there exist $z_{\rm e} \in \mathbb{R}_+^{n-m}$ and $u_{\rm e} \in \mathbb{R}_+^m$ such that

$$0 = f_x(y_{\rm d}, z_{\rm e}) + G(y_{\rm d})u_{\rm e}, \tag{11.23}$$

$$0 = f_z(y_{\rm d}, z_{\rm e}), \tag{11.24}$$

and the solution $z(t) \equiv z_{\rm e}$ to (11.21) with $x(t) \equiv y_{\rm d}$ is globally exponentially stable so that $\mathcal{G}$ given by (11.20)–(11.22) is exponentially minimum phase at $(y_{\rm d}, z_{\rm e})$ with constant control input $u_{\rm e}$. Note that $(y_{\rm d}, z_{\rm e}) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ is an equilibrium point of (11.20), (11.21) if and only if there exists $u_{\rm e} \in \mathbb{R}_+^m$ such that (11.23), (11.24) hold. Next, defining $e_x(t) \triangleq x(t) - y_{\rm d}$, $e_z(t) \triangleq z(t) - z_{\rm e}$, and $\hat{G}(e_x) \triangleq G(e_x + y_{\rm d})$, and using (11.23), (11.24), it follows that

$$\begin{aligned} \dot{e}_x(t) &= f_x(e_x(t) + y_{\rm d}, e_z(t) + z_{\rm e}) - (f_x(y_{\rm d}, z_{\rm e}) + G(y_{\rm d})u_{\rm e}) + G(x(t))u(t) \\ &= \tilde{f}_x(e_x(t), e_z(t)) - G(y_{\rm d})u_{\rm e} + \hat{G}(e_x(t))u(t), \quad e_x(0) = x_0 - y_{\rm d}, \quad t \geq 0, \end{aligned} \tag{11.25}$$

and

$$\dot{e}_z(t) = \tilde{f}_z(e_x(t), e_z(t)), \quad e_z(0) = z_0 - z_{\rm e}, \tag{11.26}$$

where $\tilde{f}_x(e_x, e_z) \triangleq f_x(e_x + y_{\rm d}, e_z + z_{\rm e}) - f_x(y_{\rm d}, z_{\rm e})$ and $\tilde{f}_z(e_x, e_z) \triangleq f_z(e_x + y_{\rm d}, e_z + z_{\rm e}) - f_z(y_{\rm d}, z_{\rm e})$. Since, by assumption, the solution $z(t) \equiv z_{\rm e}$ to (11.21) with $x(t) \equiv y_{\rm d}$ is globally exponentially stable, it follows from Definition 11.3 and converse Lyapunov theory that $\mathcal{G}$ is exponentially minimum phase and hence it further follows from Theorem 11.2 and Remark 11.2 that for any compact set $\tilde{\mathcal{D}}_{\rm c} \triangleq \tilde{\mathcal{D}}_{{\rm c}x} \times \tilde{\mathcal{D}}_{{\rm c}z}$, where $\tilde{\mathcal{D}}_{{\rm c}x} \subset \mathbb{R}^m$ and $\tilde{\mathcal{D}}_{{\rm c}z} \subset \mathbb{R}^{n-m}$, and for all $\tilde{e} \triangleq [e_x^{\rm T}, e_z^{\rm T}]^{\rm T} \in \tilde{\mathcal{D}}_{\rm c}$, there exist continuous functions $\alpha_{\tilde{\mathcal{D}}_{\rm c}} : \mathbb{R}^m \to \mathbb{R}^m$ and $\beta_{\tilde{\mathcal{D}}_{\rm c}} : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ with $\beta_{\tilde{\mathcal{D}}_{\rm c}}(e_x) = \hat{G}^{-1}(e_x)$, $e_x \in \mathbb{R}^m$, such that, with $u = \alpha_{\tilde{\mathcal{D}}_{\rm c}}(\tilde{y})+\beta_{\tilde{\mathcal{D}}_{\rm c}}(\tilde{y})v$, (11.25), (11.26) is strongly exponentially passive from $v$ to $\tilde{y} \triangleq x - y_{\rm d} = e_x$. Next, adding and subtracting $\hat{G}(e_x)\alpha_{\tilde{\mathcal{D}}_{\rm c}}(e_x)$ to and from (11.25), it follows that (11.25) can be rewritten as

$$\begin{aligned}\dot{e}_x(t) = {} & [\tilde{f}_x(e_x(t), e_z(t)) + \hat{G}(e_x(t))\alpha_{\tilde{\mathcal{D}}_{\rm c}}(e_x(t))] + \hat{G}(e_x(t))[u(t) - \alpha_{\tilde{\mathcal{D}}_{\rm c}}(x(t) - y_{\rm d}) \\ & -G^{-1}(x(t))G(y_{\rm d})u_{\rm e}], \quad e_x(0) = x_0 - y_{\rm d}, \quad t \geq 0. \end{aligned} \tag{11.27}$$

Now, we assume that for a given $\varepsilon_i^* > 0$ the $i$th component of the vector function $\alpha_{\tilde{\mathcal{D}}_{\rm c}}(x - y_{\rm d}) + G^{-1}(x)G(y_{\rm d})u_{\rm e}$ can be approximated over a compact set $\mathcal{D}_{{\rm c}x} \triangleq \{x \in \mathbb{R}^m : x - y_{\rm d} \in \tilde{\mathcal{D}}_{{\rm c}x}\}$ by a linear in parameters neural network up to a desired accuracy so that for $i = 1, \cdots, m$, there exists $\varepsilon_i(\cdot)$ such that $|\varepsilon_i(x)| < \varepsilon_i^*$, $x \in \mathcal{D}_{{\rm c}x}$, and

$$\alpha_{\tilde{\mathcal{D}}_{{\rm c}i}}(x - y_{\rm d}) + g_i^{-1}(x)g_i(y_{\rm d})u_{{\rm e}i} = W_i^{\rm T}\sigma_i(x) + \varepsilon_i(x), \quad x \in \mathcal{D}_{{\rm c}x}, \tag{11.28}$$

where $W_i \in \mathbb{R}^{s_i}$, $i = 1, \cdots, m$, are optimal *unknown* (constant) weights that minimize the approximation error over $\mathcal{D}_{{\rm c}x}$, $\sigma_i : \mathbb{R}^m \to \mathbb{R}^{s_i}$, $i = 1, \cdots, m$, are a set of basis functions such that each component of $\sigma_i(\cdot)$ takes values between 0 and 1, $\varepsilon_i : \mathcal{D}_{{\rm c}x} \to \mathbb{R}$, $i = 1, \cdots, m$, are the modeling errors, and $\|W_i\| \leq w_i^*$, where $w_i^*$, $i = 1, \cdots, m$, are bounds for the optimal weights $W_i$, $i = 1, \cdots, m$. Since $\alpha_{\tilde{\mathcal{D}}_{\rm c}}(\cdot)$ and $G(\cdot)$ are continuous functions, we can choose $\sigma_i(\cdot)$, $i = 1, \cdots, m$, from a linear space $\mathcal{X}$ of continuous functions that forms an algebra and separates points in $\mathcal{D}_{{\rm c}x}$. In this case, it follows from the Stone-Weierstrass theorem [201, p. 212] that $\mathcal{X}$ is a dense subset of the set of continuous functions on $\mathcal{D}_{{\rm c}x}$. Hence, as is the case in the standard neuro adaptive control literature [159], we can construct the signal $u_{{\rm ad}i} = \hat{W}_i^{\rm T}\sigma_i(x)$

involving the estimates of the optimal weights as our adaptive control signal. For the following theorem let $s \triangleq s_1 + \cdots + s_m$ denote the total dimension of the basis functions.

**Theorem 11.3.** Consider the nonlinear uncertain system $\mathcal{G}$ given by (11.20)–(11.22) where $f_x(\cdot,\cdot)$ is essentially nonnegative with respect to $x$, $f_z(\cdot,\cdot)$ is essentially nonnegative with respect to $z$, and $G : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ is nonnegative and given by $G(x) = \mathrm{diag}[g_1(x), \cdots, g_m(x)]$. For a given $y_{\rm d} \in \mathbb{R}_+^m$ assume there exist positive vectors $z_{\rm e} \in \mathbb{R}_+^{n-m}$ and $u_{\rm e} \in \mathbb{R}_+^m$ such that (11.23) and (11.24) hold and the equilibrium point $(y_{\rm d}, z_{\rm e})$ of (11.20), (11.21) is globally asymptotically stable with $u(t) \equiv u_{\rm e}$. In addition, assume that $\mathcal{G}$ is exponentially minimum phase at $(y_{\rm d}, z_{\rm e})$. Finally, let $q_i$ and $\gamma_i$, $i = 1, \cdots, m$, be positive constants. Then the neural adaptive output feedback control law

$$u_i(t) = \max\{0, \hat{u}_i(t)\}, \quad i = 1, \cdots, m, \tag{11.29}$$

where

$$\hat{u}_i(t) = \hat{W}_i^{\rm T}(t)\sigma_i(y(t)), \quad i = 1, \cdots, m, \tag{11.30}$$

and $\hat{W}_i(t) \in \mathbb{R}^{s_i}$, $t \geq 0$, $i = 1, \cdots, m$, with update law

$$\dot{\hat{W}}_i(t) = -\frac{q_i}{2}\left[(y_i(t) - y_{{\rm d}i})\sigma_i(y(t)) + \gamma_i|y_i(t) - y_{{\rm d}i}|\hat{W}_i(t)\right], \quad \hat{W}_i(0) = \hat{W}_{i0},$$
$$i = 1, \cdots, m, \tag{11.31}$$

guarantees that there exists a compact positively invariant set $\mathcal{D}_\alpha \subset \overline{\mathbb{R}}_+^m \times \overline{\mathbb{R}}_+^{n-m} \times \mathbb{R}^{s\times m}$ such that $(y_{\rm d}, z_{\rm e}, W) \in \mathcal{D}_\alpha$, where $W \in \mathbb{R}^{s\times m}$, and the solution $(x(t), z(t), \hat{W}(t))$, $t \geq 0$, of the closed-loop system given by (11.20), (11.21), (11.29), and (11.31) is ultimately bounded for all $(x(0), z(0), \hat{W}(0)) \in \mathcal{D}_\alpha$ with ultimate bound $\|y(t) - y_{\rm d}\|^2 < \epsilon_1$, $t \geq T$, where

$$\epsilon_1 > \frac{1}{\alpha}\left[\beta\left(\frac{\nu}{\rho\alpha}\right)^2 + q_{\min}\left(\sqrt{\sum_{i=1}^{m}\left(\frac{\sqrt{s_i} + \gamma_i w_i^*}{2\gamma_i}\right)^2} + \sqrt{\frac{\nu}{\gamma_{\min}}}\right)^2\right], \tag{11.32}$$

$$\nu \triangleq \sum_{i=1}^{m}(\sqrt{s_i}w_i^* + \varepsilon_i^*)\mu + \sum_{i=1}^{m}\frac{(\sqrt{s_i}+\gamma_i w_i^*)^2}{4\gamma_i}, \tag{11.33}$$

$$\hat{W}(t) \triangleq \text{block-diag}[\hat{W}_1(t), \cdots, \hat{W}_m(t)], \tag{11.34}$$

$\mu \triangleq \max_{x\in\mathcal{D}_{cx}} \lambda_{\max}(G(x)) = \max_{i=\{1,\cdots,m\}} \max_{x\in\mathcal{D}_{cx}}\{g_i(x)\} > 0$, $q_{\min} \triangleq \min_{i\in\{1,\cdots,m\}}$ $\{q_i\}$, $\gamma_{\min} \triangleq \min_{i\in\{1,\cdots,m\}}\{\gamma_i\}$, and $\alpha, \beta$ are positive constants. Furthermore, $u(t) \geq\geq 0$, $x(t) \geq\geq 0$, and $z(t) \geq\geq 0$ for all $t \geq 0$ and $(x_0, z_0) \in \overline{\mathbb{R}}_+^m \times \overline{\mathbb{R}}_+^{n-m}$.

**Proof.** First, since, by assumption, (11.20), (11.21) is exponentially minimum phase at $(y_\mathrm{d}, z_\mathrm{e})$, it follows from Theorem 11.2 and Remark 11.2 that for any compact set $\tilde{\mathcal{D}}_\mathrm{c}$ and for all $\tilde{e} \triangleq [e_x^\mathrm{T}, e_z^\mathrm{T}]^\mathrm{T} \in \tilde{\mathcal{D}}_\mathrm{c}$, there exist continuous functions $\alpha_{\tilde{\mathcal{D}}_\mathrm{c}} : \mathbb{R}^m \to \mathbb{R}^m$ and $\beta_{\tilde{\mathcal{D}}_\mathrm{c}} : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ with $\beta_{\tilde{\mathcal{D}}_\mathrm{c}}(e_x) = \hat{G}^{-1}(e_x)$, $e_x \in \mathbb{R}^m$, such that, with $u = \alpha_{\tilde{\mathcal{D}}_\mathrm{c}}(\tilde{y}) + \beta_{\tilde{\mathcal{D}}_\mathrm{c}}(\tilde{y})v$, (11.25), (11.26) is strongly exponentially passive from $v$ to $\tilde{y} = e_x$. Hence, it follows from Theorem 11.1 that there exist a continuously differentiable function $V_\mathrm{s} : \mathbb{R}^n \to \mathbb{R}$, a continuous function $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and positive constants $\rho, \alpha, \beta$ such that $V_\mathrm{s}(\cdot)$ is positive definite, $V_\mathrm{s}(0) = 0$, $\ell(0) = 0$, and, for all $\tilde{e} \in \mathbb{R}^n$,

$$0 = V_\mathrm{s}'(\tilde{e})[\tilde{f}(\tilde{e}) + \tilde{G}(\tilde{e})\alpha_{\tilde{\mathcal{D}}_\mathrm{c}}(e_x)] + \rho V_\mathrm{s}(\tilde{e}) + \ell^\mathrm{T}(\tilde{e})\ell(\tilde{e}), \tag{11.35}$$

$$0 = \tfrac{1}{2}V_\mathrm{s}'(\tilde{e})\tilde{G}(\tilde{e})\beta_{\tilde{\mathcal{D}}_\mathrm{c}}(e_x) - \tilde{y}, \tag{11.36}$$

and (11.7) hold, where $\tilde{f}(\tilde{e}) \triangleq [\tilde{f}_x^\mathrm{T}(e_x, e_z), \tilde{f}_z^\mathrm{T}(e_x, e_z)]^\mathrm{T}$ and $\tilde{G}(\tilde{e}) \triangleq [\hat{G}(e_x), 0]^\mathrm{T}$.

Next, define

$$\hat{W}_u(t) \triangleq \text{block-diag}[\hat{W}_{u1}(t), \cdots, \hat{W}_{um}(t)], \tag{11.37}$$

$$W \triangleq \text{block-diag}[W_1, \cdots, \hat{W}_m], \tag{11.38}$$

where

$$\hat{W}_{ui}(t) = \begin{cases} 0, & \text{if } \hat{u}_i(t) < 0, \\ \hat{W}_i(t), & \text{otherwise,} \end{cases} \quad i = 1, \cdots, m \tag{11.39}$$

and note that with $u(t)$, $t \geq 0$, given by (11.29) it follows that (11.26) and (11.27) become

$$
\begin{aligned}
\dot{e}_x(t) = {} & [\tilde{f}_x(e_x(t), e_z(t)) + \hat{G}(e_x(t))\alpha_{\tilde{\mathcal{D}}_{\rm c}}(e_x(t))] + [\hat{G}(e_x(t))\beta_{\tilde{\mathcal{D}}_{\rm c}}(e_x(t))]\beta_{\tilde{\mathcal{D}}_{\rm c}}^{-1}(e_x(t)) \\
& \cdot[\hat{W}^{\rm T}(t)\sigma(y(t)) - \alpha_{\tilde{\mathcal{D}}_{\rm c}}(x(t) - y_{\rm d}) - G^{-1}(x(t))G(y_{\rm d})u_{\rm e}] \\
& +[\hat{G}(e_x(t))\beta_{\tilde{\mathcal{D}}_{\rm c}}(e_x(t))]\beta_{\tilde{\mathcal{D}}_{\rm c}}^{-1}(e_x(t))(\hat{W}_u(t) - \hat{W}(t))^{\rm T}\sigma(y(t)), \\
& \qquad e_x(0) = x_0 - y_{\rm d}, \quad t \geq 0, \qquad (11.40)
\end{aligned}
$$

and

$$
\dot{e}_z(t) = \tilde{f}_z(e_x(t), e_z(t)), \quad e_z(0) = z_0 - z_{\rm e}, \qquad (11.41)
$$

or, equivalently,

$$
\begin{aligned}
\dot{\tilde{e}}(t) = {} & [\tilde{f}(\tilde{e}(t)) + \tilde{G}(\tilde{e}(t))\alpha_{\tilde{\mathcal{D}}_{\rm c}}(e_x(t))] \\
& +[\tilde{G}(\tilde{e}(t))\beta_{\tilde{\mathcal{D}}_{\rm c}}(e_x(t))]\beta_{\tilde{\mathcal{D}}_{\rm c}}^{-1}(e_x(t)) \\
& \cdot[\hat{W}^{\rm T}(t)\sigma(y(t)) - \alpha_{\tilde{\mathcal{D}}_{\rm c}}(x(t) - y_{\rm d}) - G^{-1}(x(t))G(y_{\rm d})u_{\rm e}] \\
& +[\tilde{G}(\tilde{e}(t))\beta_{\tilde{\mathcal{D}}_{\rm c}}(e_x(t))]\beta_{\tilde{\mathcal{D}}_{\rm c}}^{-1}(e_x(t))(\hat{W}_u(t) - \hat{W}(t))^{\rm T}\sigma(y(t)), \\
& \qquad e_x(0) = x_0 - y_{\rm d}, \quad t \geq 0. \qquad (11.42)
\end{aligned}
$$

To show ultimate boundedness of the closed-loop system (11.31) and (11.42), consider the Lyapunov-like function

$$
V(\tilde{e}, \tilde{W}) = V_{\rm s}(\tilde{e}) + {\rm tr}\, \tilde{W}Q^{-1}\tilde{W}^{\rm T}, \qquad (11.43)
$$

where $\tilde{W}^{\rm T}(t) \triangleq \hat{W}^{\rm T}(t) - W^{\rm T}$, $W^{\rm T} \triangleq$ block-diag$[W_1^{\rm T}, \cdots, W_m^{\rm T}]$, and $Q \triangleq {\rm diag}[q_1, \cdots, q_m]$. Note that $V(0,0) = 0$ and, since $V_{\rm s}(\cdot)$ and $Q$ are positive definite, $V(\tilde{e}, \tilde{W}) > 0$ for all $(\tilde{e}, \tilde{W}) \neq (0,0)$. Next, letting $\tilde{e}(t)$, $t \geq 0$, denote the solutions to (11.42) and using (11.28), (11.31), (11.35), and (11.36), it follows that the time derivative of $V(\tilde{e}, \tilde{W})$ along the closed-loop system trajectories is given by

$$
\begin{aligned}
\dot{V}(\tilde{e}(t), \tilde{W}(t)) = {} & V_{\rm s}'(\tilde{e}(t))\Big[[\tilde{f}(\tilde{e}) + \tilde{G}(\tilde{e}(t))\alpha_{\tilde{\mathcal{D}}_{\rm c}}(e_x(t))] + [\tilde{G}(\tilde{e}(t))\beta_{\tilde{\mathcal{D}}_{\rm c}}(e_x(t))] \\
& \cdot G(x(t))[\hat{W}^{\rm T}(t)\sigma(y(t)) - \alpha_{\tilde{\mathcal{D}}_{\rm c}}(x(t) - y_{\rm d}) - G^{-1}(x(t))G(y_{\rm d})u_{\rm e}]
\end{aligned}
$$

$$
\begin{aligned}
& +[\tilde{G}(e(t))\beta_{\tilde{\mathcal{D}}_{\mathrm{c}}}(e_x(t))]G(x(t))(\hat{W}_u(t)-\hat{W}(t))^{\mathrm{T}}\sigma(y(t))\Big] \\
& +2\mathrm{tr}\,\tilde{W}^{\mathrm{T}}(t)Q^{-1}\dot{\hat{W}}(t) \\
= \; & -\rho V_{\mathrm{s}}(\tilde{e}(t))-\ell^{\mathrm{T}}(\tilde{e}(t))\ell(\tilde{e}(t)) \\
& +\sum_{i=1}^{m}\tilde{y}_i(t)g_i(x(t))\Big[\tilde{W}_i^{\mathrm{T}}(t)\sigma_i(y(t))-\varepsilon_i(x(t))\Big] \\
& +\sum_{i=1}^{m}\tilde{y}_i(t)g_i(x(t))(\hat{W}_{ui}(t)-\hat{W}_i(t))^{\mathrm{T}}\sigma_i(y(t)) \\
& -\sum_{i=1}^{m}\tilde{W}_i^{\mathrm{T}}(t)\Big[e_{xi}(t)\sigma_i(y(t))+\gamma_i|y_i(t)-y_{\mathrm{d}i}|\hat{W}_i(t)\Big] \\
\leq \; & -\rho V_{\mathrm{s}}(\tilde{e}(t))-\sum_{i=1}^{m}\tilde{y}_i(t)g_i(x(t))\varepsilon_i(x(t)) \\
& +\sum_{i=1}^{m}\Big[\tilde{y}_i(t)g_i(x(t))(\hat{W}_{ui}(t)-W)^{\mathrm{T}}\sigma_i(y(t))-\tilde{y}_i(t)\tilde{W}_i^{\mathrm{T}}(t)\sigma_i(y(t)) \\
& -\gamma_i|\tilde{y}_i(t)|\tilde{W}_i^{\mathrm{T}}(t)\hat{W}_i(t)\Big], \quad t\geq 0. \qquad (11.44)
\end{aligned}
$$

Now, for each $i\in\{1,\cdots,m\}$ and for the two cases given in (11.39), the last term on the right-hand side of (11.44) gives:

*i)* If $\hat{u}_i(t)<0$, then $\hat{W}_{ui}(t)=0$ and hence

$$
\begin{aligned}
& \tilde{y}_i(t)g_i(x(t))(\hat{W}_{ui}(t)-W_i)^{\mathrm{T}}\sigma_i(y(t))-\tilde{y}_i(t)\tilde{W}_i^{\mathrm{T}}(t)\sigma_i(y(t))-\gamma_i|\tilde{y}_i(t)|\tilde{W}_i^{\mathrm{T}}(t)\hat{W}_i(t) \\
& \quad = -\tilde{y}_i(t)g_i(x(t))W_i^{\mathrm{T}}\sigma_i(y(t))-\tilde{y}_i(t)\tilde{W}_i^{\mathrm{T}}(t)\sigma_i(y(t))-\gamma_i|\tilde{y}_i(t)|\tilde{W}_i^{\mathrm{T}}(t)(\tilde{W}_i(t)+W_i) \\
& \quad \leq \sqrt{s_i}\mu w_i^*|\tilde{y}_i(t)|+\sqrt{s_i}|\tilde{y}_i(t)|\|\tilde{W}_i(t)\|-\gamma_i|\tilde{y}_i(t)|\|\tilde{W}_i(t)\|^2+\gamma_i w_i^*|\tilde{y}_i(t)|\|\tilde{W}_i(t)\| \\
& \quad \leq |\tilde{y}_i(t)|\Big[\sqrt{s_i}\mu w_i^*+(\sqrt{s_i}+\gamma_i w_i^*)\|\tilde{W}_i(t)\|-\gamma_i\|\tilde{W}_i(t)\|^2\Big].
\end{aligned}
$$

*ii)* Otherwise, $\hat{W}_{ui}(t)=\hat{W}_i(t)$ and hence

$$
\begin{aligned}
& \tilde{y}_i(t)g_i(x(t))(\hat{W}_{ui}(t)-W_i)^{\mathrm{T}}\sigma_i(y(t))-\tilde{y}_i(t)\tilde{W}_i^{\mathrm{T}}(t)\sigma_i(y(t))-\gamma_i|\tilde{y}_i(t)|\tilde{W}_i^{\mathrm{T}}(t)\hat{W}_i(t) \\
& \quad = -\tilde{y}_i(t)\tilde{W}_i^{\mathrm{T}}(t)\sigma_i(y(t))-\gamma_i|\tilde{y}_i(t)|\tilde{W}_i^{\mathrm{T}}(t)(\tilde{W}_i(t)+W_i) \\
& \quad \leq \sqrt{s_i}|\tilde{y}_i(t)|\|\tilde{W}_i(t)\|-\gamma_i|\tilde{y}_i(t)|\|\tilde{W}_i(t)\|^2+\gamma_i w_i^*|\tilde{y}_i(t)|\|\tilde{W}_i(t)\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sqrt{s_i}\mu w_i^*|\tilde{y}_i(t)| + \sqrt{s_i}|\tilde{y}_i(t)|\|\tilde{W}_i(t)\| - \gamma_i|\tilde{y}_i(t)|\|\tilde{W}_i(t)\|^2 + \gamma_i w_i^*|\tilde{y}_i(t)|\|\tilde{W}_i(t)\| \\
&\leq |\tilde{y}_i(t)|\Big[\sqrt{s_i}\mu w_i^* + (\sqrt{s_i}+\gamma_i w_i^*)\|\tilde{W}_i(t)\| - \gamma_i\|\tilde{W}_i(t)\|^2\Big].
\end{aligned}
$$

Hence, it follows from (11.44) that in either case

$$
\begin{aligned}
\dot{V}(\tilde{e}(t),\tilde{W}(t)) \leq\ & -\rho V_{\mathrm{s}}(\tilde{e}(t)) - \sum_{i=1}^{m}\tilde{y}_i(t)g_i(x(t))\varepsilon_i(x(t)) \\
& +\sum_{i=1}^{m}|\tilde{y}_i(t)|\Big(\sqrt{s_i}\mu w_i^* + (\sqrt{s_i}+\gamma_i w_i^*)\|\tilde{W}_i(t)\| - \gamma_i\|\tilde{W}_i(t)\|^2\Big) \\
\leq\ & -\rho\alpha\|\tilde{e}(t)\|^2 + \sum_{i=1}^{m}\mu\varepsilon_i^*|\tilde{y}_i(t)| \\
& +\sum_{i=1}^{m}|\tilde{y}_i(t)|\Big(\sqrt{s_i}\mu w_i^* + (\sqrt{s_i}+\gamma_i w_i^*)\|\tilde{W}_i(t)\| - \gamma_i\|\tilde{W}_i(t)\|^2\Big) \\
\leq\ & -\rho\alpha\|\tilde{e}(t)\|^2 + \sum_{i=1}^{m}\|\tilde{e}(t)\|\Big((\sqrt{s_i}w_i^* + \varepsilon_i^*)\mu \\
& +(\sqrt{s_i}+\gamma_i w_i^*)\|\tilde{W}_i(t)\| - \gamma_i\|\tilde{W}_i(t)\|^2\Big) \\
\leq\ & \|\tilde{e}(t)\|\Big[-\rho\alpha\|\tilde{e}(t)\| + \sum_{i=1}^{m}(\sqrt{s_i}w_i^* + \varepsilon_i^*)\mu \\
& +\sum_{i=1}^{m}(\sqrt{s_i}+\gamma_i w_i^*)\|\tilde{W}_i(t)\| - \sum_{i=1}^{m}\gamma_i\|\tilde{W}_i(t)\|^2\Big], \quad t\geq 0. \qquad (11.45)
\end{aligned}
$$

Next, completing squares yields

$$
\begin{aligned}
\dot{V}(\tilde{e}(t),\tilde{W}(t)) \leq\ & \|\tilde{e}(t)\|\left[-\rho\alpha\|\tilde{e}(t)\| + \sum_{i=1}^{m}(\sqrt{s_i}w_i^* + \varepsilon_i^*)\mu\right. \\
& \left. -\sum_{i=1}^{m}\gamma_i\Big(\|\tilde{W}_i(t)\| - \frac{\sqrt{s_i}+\gamma_i w_i^*}{2\gamma_i}\Big)^2 + \sum_{i=1}^{m}\frac{(\sqrt{s_i}+\gamma_i w_i^*)^2}{4\gamma_i}\right] \\
=\ & \|\tilde{e}(t)\|\left[-\rho\alpha\|\tilde{e}(t)\| - \sum_{i=1}^{m}\gamma_i\Big(\|\tilde{W}_i(t)\| - \frac{\sqrt{s_i}+\gamma_i w_i^*}{2\gamma_i}\Big)^2 + \nu\right], \quad t\geq 0,
\end{aligned}
\tag{11.46}
$$

where $\nu$ is given by (11.33). Now, for

$$
\|\tilde{e}\| \geq \frac{\nu}{\rho\alpha} \triangleq \alpha_{\tilde{e}}, \tag{11.47}
$$

or

$$\|\tilde{W}\|_{\mathrm{F}} \geq \sqrt{\sum_{i=1}^{m}\Big(\frac{\sqrt{s_i}+\gamma_i w_i^*}{2\gamma_i}\Big)^2} + \sqrt{\frac{\nu}{\gamma_{\min}}} \triangleq \alpha_{\tilde{W}}, \tag{11.48}$$

it follows that $\dot{V}(\tilde{e}(t), \tilde{W}(t)) \leq 0$ for all $t \geq 0$; that is, $\dot{V}(\tilde{e}(t), \tilde{W}(t)) \leq 0$ for all $(e_x(t), e_z(t), \tilde{W}(t)) \in \tilde{\mathcal{D}}_{\mathrm{e}} \backslash \tilde{\mathcal{D}}_{\mathrm{r}}$ and $t \geq 0$, where

$$\tilde{\mathcal{D}}_{\mathrm{e}} \triangleq \Big\{(e_x, e_z, \tilde{W}) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^{s\times m} : x \in \mathcal{D}_{\mathrm{c}x}\Big\}, \tag{11.49}$$

$$\tilde{\mathcal{D}}_{\mathrm{r}} \triangleq \Big\{(e_x, e_z, \tilde{W}) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^{s\times m} : \|\tilde{e}\| \leq \alpha_{\tilde{e}},\ \|\tilde{W}\|_{\mathrm{F}} \leq \alpha_{\tilde{W}}\Big\}. \tag{11.50}$$

Next, define

$$\tilde{\mathcal{D}}_{\alpha} \triangleq \Big\{(e_x, e_z, \tilde{W}) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^{s\times m} : V(\tilde{e}, \tilde{W}) \leq \alpha\Big\}, \tag{11.51}$$

where $\alpha$ is the maximum value such that $\tilde{\mathcal{D}}_{\alpha} \cap \tilde{\mathcal{D}}_{\mathrm{e}} = \tilde{\mathcal{D}}_{\alpha}$, and define

$$\tilde{\mathcal{D}}_{\eta} \triangleq \Big\{(e_x, e_z, \tilde{W}) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^{s\times m} : V(\tilde{e}, \tilde{W}) \leq \eta\Big\}, \tag{11.52}$$

where

$$\eta > \max_{\|\tilde{e}\|=\alpha_{\tilde{e}}} V_{\mathrm{s}}(\tilde{e}) + \max_{\|\tilde{W}\|_{\mathrm{F}}=\alpha_{\tilde{W}}} \operatorname{tr} \tilde{W} Q^{-1} \tilde{W}^{\mathrm{T}} = \beta\alpha_{\tilde{e}}^2 + q_{\min}\alpha_{\tilde{W}}^2. \tag{11.53}$$

To show ultimate boundedness of the closed-loop system (11.31) and (11.42) assume[4] that $\tilde{\mathcal{D}}_{\eta} \subset \tilde{\mathcal{D}}_{\alpha}$ (see Figure 11.1). Now, since $\dot{V}(\tilde{e}, \tilde{W}) \leq 0$ for all $(e_x, e_z, \tilde{W}) \in \tilde{\mathcal{D}}_{\mathrm{e}} \backslash \tilde{\mathcal{D}}_{\mathrm{r}}$ and $\tilde{\mathcal{D}}_{\mathrm{r}} \subset \tilde{\mathcal{D}}_{\alpha}$, it follows that $\tilde{\mathcal{D}}_{\alpha}$ is positively invariant. Hence, if $(e_x(0), e_z(0), \tilde{W}(0)) \in \tilde{\mathcal{D}}_{\alpha}$, then the solution $(e_x(t), e_z(t), \hat{W}(t))$, $t \geq 0$, to (11.31) and (11.42) is ultimately bounded. Furthermore, since $\tilde{\mathcal{D}}_{\alpha}$ is positively invariant, it follows that

$$\mathcal{D}_{\alpha} \triangleq \Big\{(x, z, \hat{W}) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^{s\times m} : (x - y_{\mathrm{d}}, z - z_{\mathrm{e}}, \hat{W} - W) \in \tilde{\mathcal{D}}_{\alpha}\Big\}, \tag{11.54}$$

[4] This assumption is standard in the neural network literature and ensures that in the error space $\tilde{\mathcal{D}}_{\mathrm{e}}$ there exists at least one Lyapunov level set $\tilde{\mathcal{D}}_{\eta} \subset \tilde{\mathcal{D}}_{\alpha}$. In the case where the neural network approximation holds in $\mathbb{R}^m \times \mathbb{R}^{n-m}$, this assumption is automatically satisfied. See Remark 11.3 for further details.

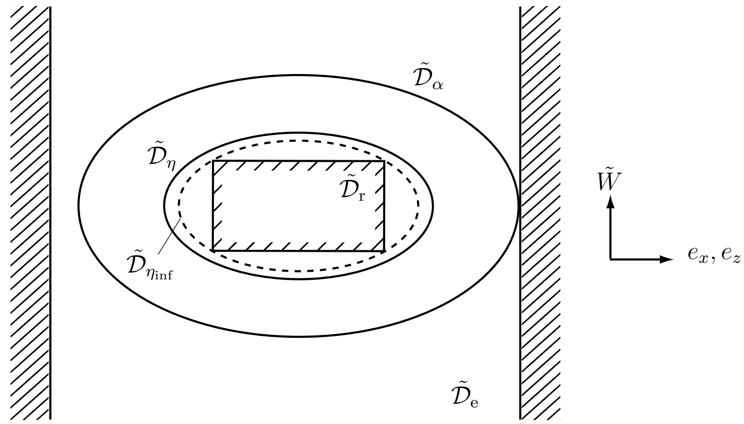



**Figure 11.1**: Visualization of sets used in the proof of Theorem 11.3

is also positively invariant. Now, to show that $\|y(t)-y_{\rm d}\|^2 < \epsilon_1$, $t \geq T = T(x_0, z_0, \hat{W}_0, \epsilon_1)$, suppose there exists $t^* \geq 0$ such that $\tilde{e}(t^*) = 0$ and $\hat{W}(t^*) = 0$. In this case, $\tilde{e}(t) = 0$ and $\hat{W}(t) = 0$ for all $t \geq t^*$ and hence $\|y(t) - y_{\rm d}\|^2 < \epsilon_1$ is trivially satisfied for all $t \geq t^*$. Alternatively, suppose there does not exist $t^* \geq 0$ such that $e(t^*) = 0$ and $\hat{W}(t^*) = 0$. In this case, consider the Lyapunov-like function

$$\tilde{V}(\tilde{e}, \tilde{W}) = \begin{cases} V(\tilde{e}, \tilde{W}) - \eta_{\rm inf}, & (e_x, e_z, \tilde{W}) \in \mathcal{D}_\alpha \backslash \mathcal{D}_{\eta_{\rm inf}}, \\ 0, & (e_x, e_z, \tilde{W}) \in \mathcal{D}_{\eta_{\rm inf}}, \end{cases} \tag{11.55}$$

where $\eta_{\rm inf} \triangleq \beta\alpha_{\tilde{e}}^2 + q_{\rm min}\alpha_{\tilde{W}}^2$ and $\tilde{\mathcal{D}}_{\eta_{\rm inf}} \triangleq \{(e_x, e_z, \tilde{W}) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^{s\times m} : V(\tilde{e}, \tilde{W}) \leq \eta_{\rm inf}\}$. Note that $\tilde{V}(\tilde{e}, \tilde{W})$ is continuous on $\mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^{s\times m}$ and $\tilde{\mathcal{D}}_\eta$ is positively invariant. Furthermore, note that

$$\tilde{V}(\tilde{e}(t), \tilde{W}(t)) \leq \tilde{V}(\tilde{e}(\tau), \tilde{W}(\tau)), \quad 0 \leq \tau \leq t. \tag{11.56}$$

Now, it follows from the generalized Krasovskii-LaSalle invariant set theorem (Theorem 2.3 of [155]) that $(e_x(t), e_z(t), \hat{W}(t)) \to \mathcal{M} \triangleq \cup_{\gamma>0}\mathcal{M}_\gamma$ as $t \to \infty$, where $\mathcal{M}_\gamma$ denotes the largest invariant set contained in $\mathcal{R}_\gamma \triangleq \{(e_x, e_z, \tilde{W}) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^{s\times m} : \tilde{V}(\tilde{e}, \tilde{W}) = \gamma\}$. Hence, since $\mathcal{M}_\gamma = \emptyset$, $\gamma > 0$, and $\mathcal{R}_0 = \tilde{\mathcal{D}}_{\eta_{\rm inf}} \subset \tilde{\mathcal{D}}_\eta$, there exists $T = T(x_0, z_0, \hat{W}_0, \epsilon_1) \geq 0$ such that $(e_x(t), e_z(t), \hat{W}(t)) \in \overset{\circ}{\tilde{\mathcal{D}}}_\eta$ for all $t \geq T$ and hence

$$\|\tilde{e}(t)\|^2 < \max\{\|\tilde{e}\|^2 \in \mathbb{R} : V_{\rm s}(\tilde{e}) = \eta\} = \frac{\eta}{\alpha}, \quad t \geq T. \tag{11.57}$$

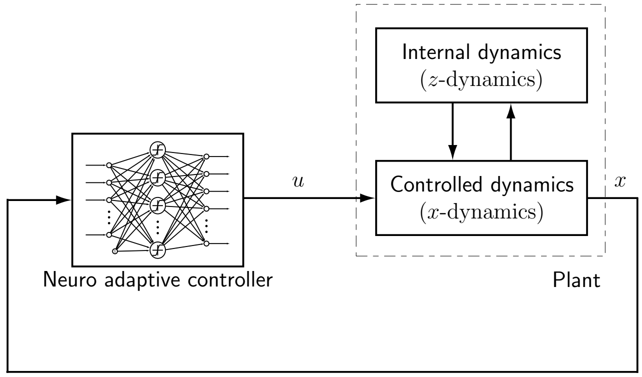



**Figure 11.2**: Block diagram of the closed-loop system

Since $\|e_x\| \leq \|\tilde{e}\|$, (11.57) implies $\|y(t) - y_{\rm d}\|^2 < \epsilon_1$, $t \geq T$.

Finally, $u(t) \geq\geq 0$, $t \geq 0$, is a restatement of (11.29). Now, since $G(x(t)) \geq\geq 0$, $t \geq 0$, and $u(t) \geq\geq 0$, $t \geq 0$, it follows from Proposition 9.2 that $x(t) \geq\geq 0$ and $z(t) \geq\geq 0$ for all $t \geq 0$ and $(x_0, z_0) \in \overline{\mathbb{R}}_+^m \times \overline{\mathbb{R}}_+^{n-m}$. □

**Remark 11.3.** It follows from Theorem 11.2 that if $\mathcal{G}$ given by (11.20)–(11.22) is exponentially minimum phase, then $\mathcal{G}$ is semiglobally output feedback exponentially passive. Hence, for any arbitrarily large compact set $\tilde{\mathcal{D}}_{\rm c}$ there exists an output feedback control law of the form (11.14) that renders the closed-loop system (11.20)–(11.22) exponentially passive. For this compact set $\tilde{\mathcal{D}}_{\rm c}$, as is common in the neural network literature, we assume that there exists an approximator for the unknown nonlinear map $\alpha_{\mathcal{D}_{\rm c}}(x - y_{\rm d}) - G^{-1}(x)G(y_{\rm d})u_{\rm e}$ up to a desired accuracy. Furthermore, we assume that in the error space $\tilde{\mathcal{D}}_{\rm e}$ there exists at least one Lyapunov level set such that $\tilde{\mathcal{D}}_\eta \subset \tilde{\mathcal{D}}_\alpha$.

A block diagram showing the neuro adaptive control architecture given in Theorem 11.3 is shown in Figure 11.2. In Theorem 11.3 we assumed that the equilibrium point $(y_{\rm d}, z_{\rm e})$ of (11.20), (11.21) is globally asymptotically stable with $u(t) \equiv u_{\rm e}$. In general, however, unlike linear nonnegative systems with asymptotically stable plant dynamics, a given set point $(y_{\rm d}, z_{\rm e}) \in \mathbb{R}_+^m \times \mathbb{R}_+^{n-m}$ for the nonlinear nonnegative dynamical system (11.20), (11.21) may not be asymptotically stabilizable with a constant control $u(t) \equiv u_{\rm e} \in \overline{\mathbb{R}}_+^n$. However, if $f(\tilde{x}) \triangleq [f_x^{\rm T}(x,z), f_z^{\rm T}(x,z)]^{\rm T}$ is homogeneous, cooperative; that is, the Jacobian matrix $\frac{\partial f(\tilde{x})}{\partial \tilde{x}}$ is essentially nonnegative for all $\tilde{x} \triangleq [x^{\rm T}, z^{\rm T}]^{\rm T} \in \overline{\mathbb{R}}_+^n$, the Jacobian matrix $\frac{\partial f(\tilde{x})}{\partial \tilde{x}}$ is irreducible for all $\tilde{x} \in \overline{\mathbb{R}}_+^n$ [20], and the zero solution $(x(t), z(t)) \equiv 0$ of the undisturbed $(u(t) \equiv 0)$ system (11.20), (11.21) is globally asymptotically stable, then the set point $(y_{\rm d}, z_{\rm e}) \in \mathbb{R}_+^m \times \mathbb{R}_+^{n-m}$ satisfying (11.23), (11.24) is a unique equilibrium point with $u(t) \equiv u_{\rm e}$ and is also asymptotically stable for all $(y_0, z_0) \in \overline{\mathbb{R}}_+^m \times \overline{\mathbb{R}}_+^{n-m}$ [53]. This implies that the solution $(x(t), z(t)) \equiv (y_{\rm d}, z_{\rm e})$ to (11.20), (11.21) with $u(t) \equiv u_{\rm e}$ is asymptotically stable for all $(y_0, z_0) \in \overline{\mathbb{R}}_+^m \times \overline{\mathbb{R}}_+^{n-m}$.

## 11.4. Neural Adaptive Control for General Anesthesia

Almost all anesthetics are myocardial depressants which lower cardiac output (i.e., the amount of blood pumped by the heart per unit time). As a consequence, decreased cardiac output slows down redistribution kinetics; that is, the transfer of blood from the central compartments (heart, brain, kidney, and liver) to the peripheral compartments (muscle and fat). In addition, decreased cardiac output could increase drug concentrations in the central compartments causing even more myocardial depression and further decrease in cardiac output. To study the effects of pharmacological agents and anesthetics we propose the nonlinear two-compartment model shown in Figure 11.3, where $x_1$ denotes the mass of drug in the central compartment, which is the site for drug administration and is generally thought to be comprised of the intravascular blood volume as well as highly perfused organs such as the heart, brain, kidney, and liver. These organs receive a large fraction of the cardiac output. Alternatively, $x_2$ is the mass of drug in the peripheral compartment, comprised of muscle

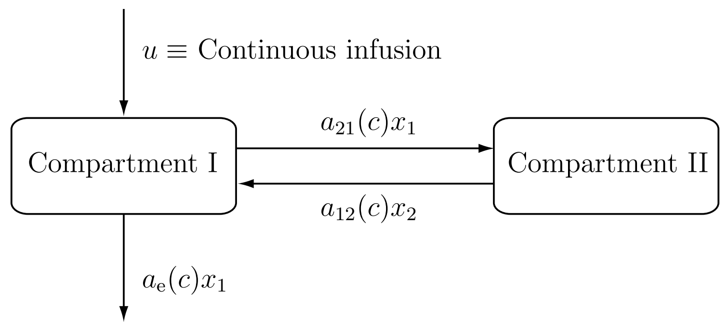



**Figure 11.3**: Pharmacokinetic model for drug distribution during anesthesia

and fat which receive a smaller proportion of the cardiac output.

A mass balance of the two-state compartment model yields

$$\dot{x}_1(t) = -a_{21}(c(t))x_1(t) - a_{\rm e}(c(t))x_1(t) + a_{12}(c(t))x_2(t) + u(t), \quad x_1(0) = x_{10}, \quad t \geq 0, \tag{11.58}$$

$$\dot{x}_2(t) = a_{21}(c(t))x_1(t) - a_{12}(c(t))x_2(t),\ x_2(0) = x_{20}, \tag{11.59}$$

where $c \triangleq x_1/V_{\rm c}$ is the drug concentration in the central compartment, $V_{\rm c}$ is the volume of the central compartment, $a_{21}(c)$ is the rate of transfer of drug from Compartment I to Compartment II, $a_{12}(c)$ is the rate of transfer of drug from Compartment II to Compartment I, $a_{\rm e}(c)$ is the rate of drug metabolism and elimination (metabolism typically occurs in the liver), and $u(t)$, $t \geq 0$, is the infusion rate of an anesthetic drug. As in Section 9.6, we assume $a_{21}(c) = A_1 Q(c)$, $a_{12}(c) = A_2 Q(c)$, and $a_{\rm e}(c) = A_{\rm e} Q(c)$, where $A_1$, $A_2$, and $A_{\rm e}$ are positive constants. Many anesthetics depress the heart, decreasing the cardiac output. Furthermore, the transfer coefficients are functions of the concentration $c$ in the central compartment. Thus, to develop a physiologically plausible model we assume a sigmoid relationship between drug concentration in the central compartment and effect so that $Q(c) = Q_0 {\rm C}_{50}^{\alpha}/({\rm C}_{50}^{\alpha} + c^{\alpha})$, where the effect is related to $c$ (since that is the presumed concentration in the highly perfused myocardium), $Q_0 > 0$ is a constant, ${\rm C}_{50} > 0$ is the drug concentration associated with a 50% decrease in the cardiac output, and $\alpha > 1$ determines the steepness of this curve (that is, how rapidly the cardiac output decreases with increasing drug concentration). Furthermore, this model assumes instantaneous mixing and as $c$ increases, the rate constants decrease through their dependence on the cardiac output. Even though the transfer and loss coefficients $A_1$, $A_2$, and $A_{\rm e}$ are nonnegative, and $\alpha > 1$, $\mathrm{C}_{50} > 0$, and $Q_0 > 0$, these parameters can be uncertain due to patient gender, weight, pre-existing disease, age, and concomitant medication. Hence, the need for adaptive control to regulate intravenous anesthetics during surgery is crucial.

Midazolam is an intravenous anesthetic that has been used for both induction and maintenance of general anesthesia. A simple yet effective patient model for the disposition of midazolam is based on the two-compartment model shown in Figure 11.3 with the first compartment acting as the central compartment. Here, we use the Bispectral Index (BIS) as a measure of anesthetic effect. As discussed in Chapter 7, the BIS signal is a nonlinear monotonically decreasing function of the level of consciousness and is given by

$$\mathrm{BIS}(c_{\rm eff}) = \mathrm{BIS}_0 \left(1 - \frac{c_{\rm eff}^{\gamma}}{c_{\rm eff}^{\gamma} + \mathrm{EC}_{50}^{\gamma}}\right), \tag{11.60}$$

where $\mathrm{BIS}_0$ denotes the baseline (awake state) value and, by convention, is typically assigned a value of 100, $c_{\rm eff}$ is the midazolam concentration in nanograms/liter in the effect site compartment (brain), $\mathrm{EC}_{50}$ is the concentration at half maximal effect and represents the patient's sensitivity to the drug, and $\gamma$ determines the degree of nonlinearity in (11.60). Here, the effect site compartment is introduced as a correlate between the central compartment concentration and the central nervous system concentration. The effect site compartment concentration is related to the concentration in the central compartment by the first-order delay model

$$\dot{c}_{\rm eff}(t) = a_{\rm eff}(c(t) - c_{\rm eff}(t)), \quad c_{\rm eff}(0) = x_1(0), \quad t \geq 0, \tag{11.61}$$

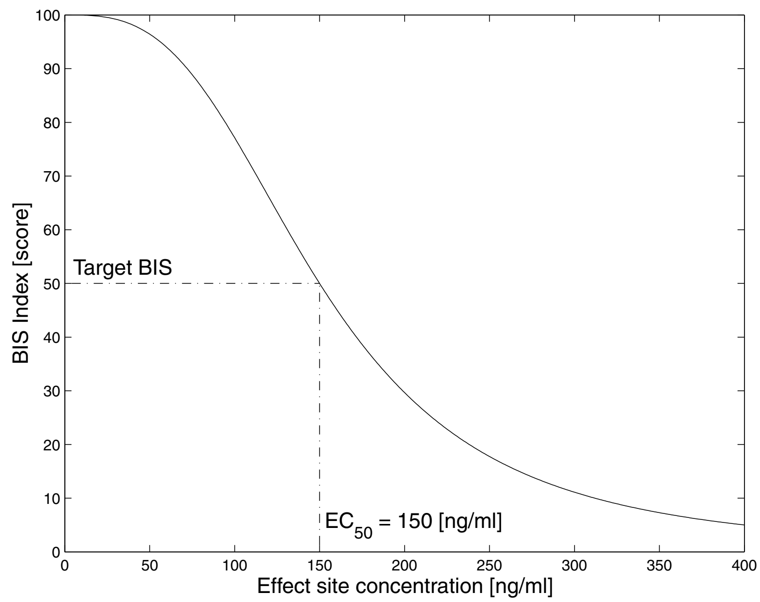



**Figure 11.4**: BIS index versus effect site concentration

where $a_{\rm eff}$ in $\min^{-1}$ is a positive time constant. Assuming $x_1(0) = 0$, it follows that

$$c_{\rm eff}(t) = \int_0^t e^{-a_{\rm eff}(t-s)} a_{\rm eff} c(s)\, {\rm d}s. \tag{11.62}$$

In reality, the effect site compartment equilibrates with the central compartment in a matter of a few minutes. The parameters $a_{\rm eff}$, $\mathrm{EC}_{50}$, and $\gamma$ are determined by data fitting and vary from patient to patient. BIS index values of 0 and 100 correspond, respectively, to an isoelectric EEG signal and an EEG signal of a fully conscious patient; while the range between 40 and 60 indicates a moderate hypnotic state.

In the following numerical simulation we set $\mathrm{EC}_{50} = 150$ ng/m$\ell$, $\gamma = 3$, and $\mathrm{BIS}_0 = 100$, so that the BIS signal is shown in Figure 11.4. The target (desired) BIS value, $\mathrm{BIS}_{\rm target}$, is set at 50. Furthermore, for simplicity of exposition, we assume that the effect site compartment equilibrates instantaneously with the central compartment; that is, we assume that $a_{\rm eff} \to \infty$, so that (11.61) gives $c_{\rm eff}(t) = c(t)$, $t \geq 0$. Now, defining $x \triangleq \mathrm{BIS}_0 - \mathrm{BIS}(c) = h(c)$ and $z \triangleq x_2$, where $h(c) \triangleq \frac{c^\gamma}{c^\gamma + \mathrm{EC}_{50}^\gamma}$, (11.58), (11.59) can be written in form of (11.20)–(11.22) with

$$f_x(x,z) = h'(c)\Big[-a_{21}(c)h^{-1}(x) - a_{\rm e}(c)h^{-1}(x) + a_{12}(c)z/V_{\rm c}\Big], \quad (11.63)$$

$$f_z(x,z) = V_{\rm c}[a_{21}(c)h^{-1}(x) - a_{12}(c)z], \quad (11.64)$$

$$G(x) = h'(c)/V_{\rm c}. \quad (11.65)$$

Note that $f_x(x,z)$ is essentially nonnegative with respect to $x$, $f_z(x,z)$ is essentially nonnegative with respect to $z$, and $G(x)$ is nonnegative. In addition, note that $f_x(x,z)$ is essentially nonnegative with respect to $x$, $f_z(x,z)$ is essentially nonnegative with respect to $z$, and $G(x)$ is nonnegative. In addition, note that since $h(\cdot)$ is a monotonically decreasing function, the mapping $(x_1,x_2)\mapsto(x,z)$ is diffeomorphic. Furthermore, note that since

$$\tilde{f}_z(0,e_z) = -a_{12}(y_{\rm d1}/V_{\rm c})e_z, \quad (11.66)$$

where

$$\tilde{f}_z(e_x,e_z) = f_z(e_x+y_{\rm d}, e_z+z_{\rm e}) - f_z(y_{\rm d}, z_{\rm e}) \quad (11.67)$$

and $a_{12}(y_{\rm d1}/V_{\rm c}) > 0$, it follows that the system zero dynamics are exponentially stable and hence the system given by (11.58), (11.59) is exponentially minimum phase at $(y_{\rm d}, z_{\rm e})$. Thus, since the input matrix function satisfies (11.17), it follows from Theorem 11.2 that (11.58), (11.59) is semiglobally output feedback exponentially passive. Now, using the adaptive output feedback controller

$$u(t) = \max\{0, \hat{u}(t)\}, \quad (11.68)$$

where

$$\hat{u}(t) = \hat{W}^{\rm T}(t)\sigma({\rm BIS}(t)), \quad (11.69)$$

$\hat{W}(t)\in\mathbb{R}^s$, $t\geq 0$, and $\sigma:\mathbb{R}\to\mathbb{R}^s$ is a given basis function, with update law

$$\dot{\hat{W}}(t) = q_{\rm BIS}\Big[(-{\rm BIS}(t)+{\rm BIS}_{\rm target})\sigma({\rm BIS}(t)) - \gamma|{\rm BIS}(t)-{\rm BIS}_{\rm target}|\hat{W}(t)\Big],$$
$$\hat{W}(0) = \hat{W}_{10}, \quad (11.70)$$

where $q_{\text{BIS}}$ is an arbitrary positive constant, it follows from Theorem 11.3 that the control input (anesthetic infusion rate) $u(t)$ is nonnegative for all $t \geq 0$ and there exist positive constants $\varepsilon$ and $T$ such that $|\text{BIS}(t) - \text{BIS}_{\text{target}}| \leq \varepsilon$, $t \geq T$, for any (uncertain) positive values of the transfer and loss coefficients $(A_1, A_2, A_{\text{e}})$ as well as any (uncertain) nonnegative coefficients $\alpha$, $\text{C}_{50}$, and $Q_0$. It is important to note that during actual surgery the BIS signal is obtained directly from the EEG and not (11.60). For our simulation we assume $V_{\text{c}} = 31\ \ell$, $A_1Q_0 = 0.01895\ \text{min}^{-1}$, $A_2Q_0 = 0.01003\ \text{min}^{-1}$, $A_{\text{e}}Q_0 = 0.01651\ \text{min}^{-1}$, $\alpha = 3$, and $\text{C}_{50} = 200\ \text{ng/m}\ell$. Note that these parameter values for $\alpha$ and $\text{C}_{50}$ probably exaggerate the effect of midazolam on cardiac output. They have been selected to accentuate nonlinearity but they are not biologically unrealistic. To illustrate the robustness of the proposed neuro adaptive controller we switch the pharmacodynamic parameters $\text{EC}_{50}$ and $\gamma$, respectively, from 150 ng/m$\ell$ and 3 to 170 ng/m$\ell$ and 2 at $t = 15$ min and back to 150 ng/m$\ell$ and 3 at $t = 30$ min. Furthermore, here we consider noncardiac surgery since cardiac surgery often utilizes hypothermia which itself changes the BIS signal. With $q_{\text{BIS}} = 1 \times 10^4$, $\gamma = 1 \times 10^{-10}$, $s = 6$, $\sigma(\text{BIS}) = \left[\frac{1}{1+e^{-a(\text{BIS}-\text{BIS}_{\text{target}})}}, \cdots, \frac{1}{1+e^{-6a(\text{BIS}-\text{BIS}_{\text{target}})}}\right]^{\text{T}}$, $a = 1$, and initial conditions $x_1(0) = 0$ mg, $x_2(0) = 0$ mg, $\hat{W}(0) = 0_{6\times 1}$ mg/min, Figure 11.5 shows the concentrations of midazolam in the two compartments versus time. Figure 11.6 shows the BIS index and the control signal (midazolam infusion rate) versus time. Finally, Figure 11.7 shows the neural network weight history versus time.

Even though we did not calculate the analytical bounds given by (11.32) due to the fact that one has to solve an optimization problem with respect to (11.28) to obtain $\varepsilon_i^*$ and $w_i^*$, $i = 1, \cdots, 6$, the closed-loop BIS signal response shown in Figure 11.6 is clearly acceptable. Furthermore, the basis functions for $\sigma(\text{BIS})$ are chosen to cover the domain of interest of our pharmacokinetic/pharmacodynamic problem since we know

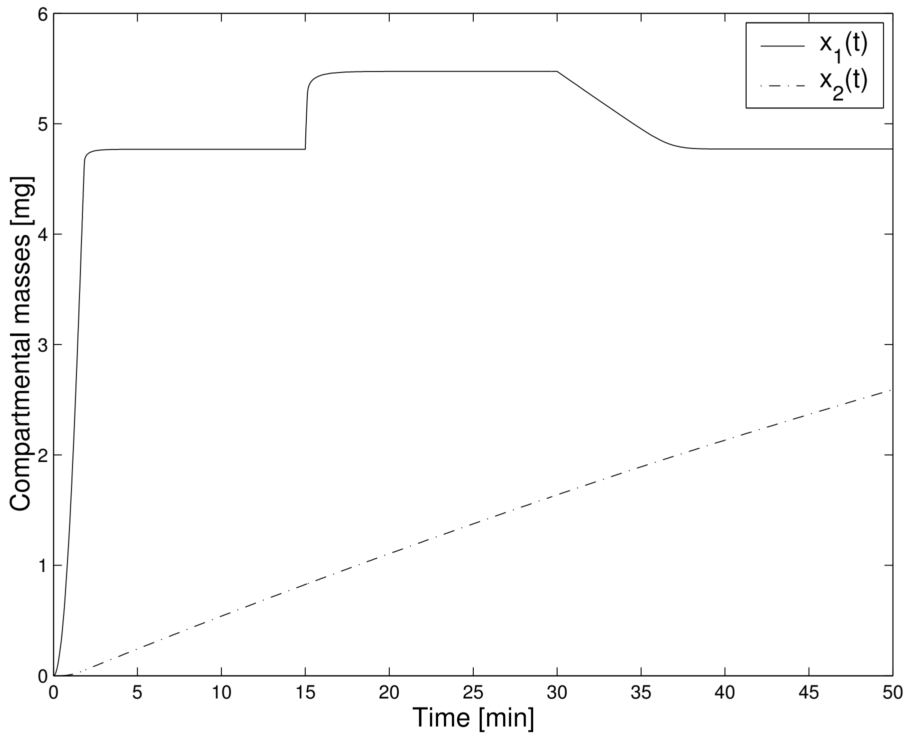



**Figure 11.5**: Compartmental concentrations versus time

that the BIS index varies from 0 to 100. Hence, the basis functions are distributed over that domain. The number of basis functions however is based on trial and error. This goes back to the Stone-Weierstrass theorem which only provides an existence result without any constructive guidelines. Finally, we note that simulations using a larger number of neurons resulted in imperceptible differences in the closed-loop system performance.

## 11.5. Conclusion

Nonnegative and compartmental systems are widely used to capture system dynamics involving the interchange of mass and energy between homogenous subsystems or compartments. Thus, it is not surprising that nonnegative and compartmental models are remarkably effective in describing the dynamical behavior of biological systems, physiological systems, and pharmacological systems. In this chapter, we de-

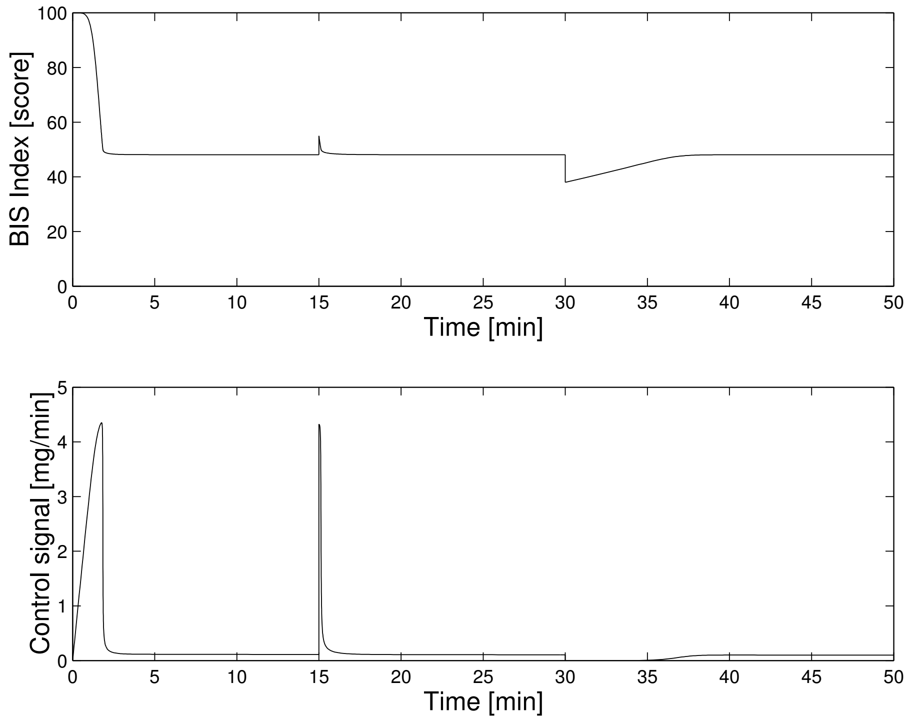



**Figure 11.6**: BIS index versus time and control signal (infusion rate) versus time

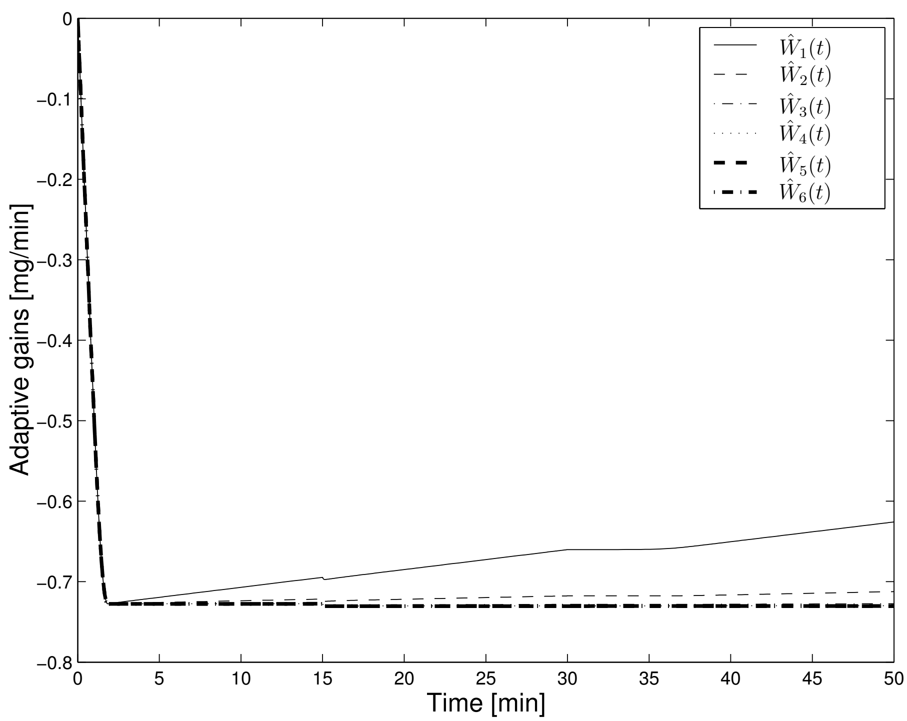



**Figure 11.7**: neural network weighting functions versus time

veloped a neural adaptive output feedback control framework for adaptive set-point regulation of nonlinear uncertain nonnegative and compartmental systems. Using Lyapunov methods the proposed framework was shown to guarantee ultimate boundedness of the error signals corresponding to the physical system states and the neural network weighting gains while additionally guaranteeing the nonnegativity of the closed-loop system states associated with the plant dynamics. Finally, using a nonlinear two-compartment patient model for the disposition of anesthetic drug midazolam, the proposed adaptive control framework was used to monitor and control a desired constant level of consciousness for noncardiac surgery.

# Chapter 12

# Neural Network Adaptive Dynamic Output Feedback Control for Nonlinear Nonnegative Systems using Tapped Delay Memory Units

## 12.1. Introduction

Neural networks offer an ideal framework for on-line system identification and control of many complex uncertain nonlinear dynamical systems. One of the key aspects of neural networks is that a very rich class of continuous nonlinear maps can be approximated from the collective action of very simple, autonomous processing units interconnected in simple ways. This massively parallel and highly redundant processing architecture has resulted in concrete accomplishments in pattern recognition, system identification, and adaptive control (see [44, 119, 159, 160, 178, 226] and the numerous references therein).

Given the complexity, uncertainties, and nonlinearities inherent in pharmacokinetic and pharmacodynamic models needed to capture the wide effects of pharmacological agents and anesthetics in the human body, neural networks can provide an ideal framework for addressing adaptive control for clinical pharmacology [13]. Nonnegative and compartmental models provide a broad framework for biological and physiological systems, including clinical pharmacology, and are well suited for the problem of closed-loop control of drug administration. Specifically, nonnegative and compartmental dynamical systems [6, 70, 75, 123, 124, 203] are composed of homogeneous interconnected subsystems (or compartments) which exchange variable nonnegative quantities of material with conservation laws describing transfer, accumulation, and elimination between the compartments and the environment. It thus follows from physical considerations that the state trajectory of such systems remains in the nonnegative orthant of the state space for nonnegative initial conditions.

In this chapter, we extend the results of [117, 118] to nonnegative and compartmental dynamical systems with applications to the specific problem of automated anesthesia. Specifically, we develop an output feedback neural network adaptive controller that operates over a tapped delay line of available input and output measurements. The neuro adaptive laws for the neural network weights are constructed using a linear observer for the nominal normal form system error dynamics. The approach is applicable to general class of nonlinear nonnegative dynamical systems without imposing a strict positive real requirement on the transfer function of the linear error normal form dynamics. Furthermore, since in pharmacological applications involving active drug administration control inputs as well as the system states need to be nonnegative, the proposed neuro adaptive output feedback controller also guarantees that the control signal remains nonnegative. We emphasize that the proposed framework addresses adaptive *output feedback* controllers for nonlinear compartmental systems with *unmodeled dynamics* of *unknown dimension* while guaranteing ultimate boundedness of the error signals corresponding to the physical system states as well as the neural network weighting gains. Output feedback controllers are crucial in clinical pharmacology since key physiological (state) variables cannot be measured in practice.

## 12.2. Neural Adaptive Output Feedback Control for Nonlinear Nonnegative Uncertain Systems

In this section we consider the problem of characterizing neural adaptive dynamic output feedback control laws for nonlinear nonnegative and compartmental uncertain dynamical systems to achieve *set-point* regulation in the nonnegative orthant. Specifically, consider the controlled square nonlinear uncertain dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad t \geq 0, \tag{12.1}$$

$$y(t) = h(x(t)), \tag{12.2}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $y(t) \in \mathbb{R}^m$, $t \geq 0$, is the system output, $f : \mathbb{R}^n \to \mathbb{R}^n$ is essentially nonnegative but otherwise unknown and satisfies $f(0) = 0$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is an unknown nonnegative input matrix function, and $h : \mathbb{R}^n \to \mathbb{R}^m$ is a nonnegative function and satisfies $h(0) = 0$. We assume that $f(\cdot)$, $G(\cdot)$, and $h(\cdot)$ are smooth (i.e., $\mathrm{C}^\infty$ mappings) and the control input $u(\cdot)$ in (12.1) is restricted to the class of *admissible controls* consisting of measurable functions such that $u(t) \in \mathbb{R}^m$, $t \geq 0$.

As discussed in the Introduction, control (source) inputs of drug delivery systems for physiological and pharmacological processes are usually constrained to be nonnegative as are the system states. Hence, in this chapter we develop neuro adaptive dynamic output feedback control laws for essentially nonnegative systems with nonnegative control inputs. Specifically, for a given desired set point $y_\mathrm{d} \in \overline{\mathbb{R}}_+^m$ and for a given $\varepsilon > 0$, our aim is to design a nonnegative control input $u(t)$, $t \geq 0$, predicated on the system measurement $y(t)$, $t \geq 0$, such that $\|y(t) - y_\mathrm{d}\| < \varepsilon$ for all $t \geq T$, where

$T \in [0, \infty)$, and $x(t) \geq\geq 0$, $t \geq 0$, and $u(t) \geq\geq 0$, $t \geq 0$, for all $x_0 \in \overline{\mathbb{R}}_+^n$.

In this chapter, we assume that for the nonlinear dynamical system (12.1), (12.2), the conditions for the existence of a globally defined diffeomorphism transforming (12.1), (12.2) into normal form [32, 122] are satisfied so that there exist a global diffeomorphism $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathrm{C}^\infty$ functions $f_\xi : \mathbb{R}^r \times \mathbb{R}^{n-r} \to \mathbb{R}^r$ and $f_z : \mathbb{R}^r \times \mathbb{R}^{n-r} \to \mathbb{R}^{n-r}$ such that, in the coordinates

$$\begin{bmatrix} \xi \\ z \end{bmatrix} \triangleq \mathcal{T}(x), \tag{12.3}$$

where $\xi \triangleq [y_1, \dot{y}_1, \cdots, y_1^{(r_1-2)}, \cdots, y_m, \dot{y}_m, \cdots, y_m^{(r_m-2)}; y_1^{(r_1-1)}, \cdots, y_m^{(r_m-1)}] \in \mathbb{R}^r$, $z \in \mathbb{R}^{n-r}$, and $r \triangleq r_1 + \cdots + r_m$ is the (vector) relative degree of $\mathcal{G}$, $\mathcal{G}$ given by (12.1), (12.2) is equivalent to

$$\dot{\xi}(t) = f_\xi(\xi(t), z(t)) + G_\xi(\xi(t), z(t))u(t), \quad \xi(0) = \xi_0, \quad t \geq 0, \tag{12.4}$$

$$\dot{z}(t) = f_z(\xi(t), z(t)), \quad z(0) = z_0, \tag{12.5}$$

$$y(t) = C\xi(t), \tag{12.6}$$

with appropriate initial conditions $\xi_0 \in \mathbb{R}^r$ and $z_0 \in \mathbb{R}^{n-r}$, where

$$f_\xi(\xi, z) = A\xi + \tilde{f}_{\mathrm{u}}(\xi, z), \quad G_\xi(\xi, z) = \begin{bmatrix} 0_{(n-m)\times m} \\ G_{\mathrm{s}}(\tilde{x}) \end{bmatrix}, \tag{12.7}$$

$$A = \begin{bmatrix} A_0 \\ \hat{A} \end{bmatrix}, \quad \tilde{f}_{\mathrm{u}}(\xi, z) = \begin{bmatrix} 0_{(n-m)\times 1} \\ f_{\mathrm{u}}(\tilde{x}) \end{bmatrix}, \tag{12.8}$$

$\tilde{x} \triangleq [\xi^{\mathrm{T}}, z^{\mathrm{T}}]^{\mathrm{T}}$, $A_0 \in \mathbb{R}^{(r-m)\times r}$ is a known matrix of zeros and ones capturing the multivariable controllable canonical form representation [43], $\hat{A} \in \mathbb{R}^{m\times r}$ is such that $A$ is asymptotically stable, $f_{\mathrm{u}} : \mathbb{R}^n \to \mathbb{R}^m$ is an unknown function and satisfies $f_{\mathrm{u}}(0) = 0$, $C \in \mathbb{R}^{m\times r}$ is a known matrix of zeros and ones capturing the system output, and $G_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is an unknown matrix function such that $\det G_{\mathrm{s}}(\tilde{x}) \neq 0$, $\tilde{x} \in \mathbb{R}^n$. Furthermore, we assume that for a given $y_{\mathrm{d}} \in \overline{\mathbb{R}}_+^m$ there exist $z_{\mathrm{e}} \in \mathbb{R}^{n-r}$ and

$u_{\rm e} \in \overline{\mathbb{R}}_+^m$ such that $x_{\rm e} \triangleq \mathcal{T}^{-1}(\tilde{x}_{\rm e}) \geq\geq 0$ and

$$0 = f_\xi(\xi_{\rm e}, z_{\rm e}) + G_\xi(\xi_{\rm e}, z_{\rm e})u_{\rm e}, \tag{12.9}$$

$$0 = f_z(\xi_{\rm e}, z_{\rm e}), \tag{12.10}$$

where $\tilde{x}_{\rm e} \triangleq [\xi_{\rm e}^{\rm T}, z_{\rm e}^{\rm T}]^{\rm T}$ and $\xi_{\rm e}$ is given with $y_i = y_{{\rm d}i}$, $i = 1, \cdots, m$, and $\dot{y}_i = \cdots = y_i^{(r_i-1)} = 0$, $i = 1, \cdots, m$. In addition, we assume that (12.5) is input-to-state stable at $z(t) \equiv z_{\rm e}$ with $\xi(t) - \xi_{\rm e}$ viewed as the input; that is, there exist a class $\mathcal{KL}$ function $\eta(\cdot, \cdot)$ and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that

$$\|z(t) - z_{\rm e}\| \leq \eta(\|z_0 - z_{\rm e}\|, t) + \gamma\left(\sup_{0\leq\tau\leq t} \|\xi(\tau) - \xi_{\rm e}\|\right), \quad t \geq 0, \tag{12.11}$$

where $\|\cdot\|$ denotes the Euclidean vector norm. Unless otherwise stated, henceforth we use $\|\cdot\|$ to denote the Euclidean vector norm. Note that $(\xi_{\rm e}, z_{\rm e}) \in \mathbb{R}^r \times \mathbb{R}^{n-r}$ is an equilibrium point of (12.4), (12.5) if and only if there exists $u_{\rm e} \in \overline{\mathbb{R}}_+^m$ such that (12.9), (12.10) hold. Furthermore, we assume that, for given $\varepsilon^* > 0$, the functions $f_{\rm u}(\mathcal{T}(x)) - f_{\rm u}(\mathcal{T}(x_{\rm e})) - G_{\rm s}(\mathcal{T}(x_{\rm e}))u_{\rm e}$ and $G_{\rm s}(\mathcal{T}(x)) - B_{\rm s}$, where $B_{\rm s} \in \mathbb{R}^{m\times m}$, can be approximated over a compact set $\mathcal{D}_{\rm c} \subset \overline{\mathbb{R}}_+^n$ by a linear in the parameters neural network up to a desired accuracy so that there exist $\varepsilon_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $\varepsilon_2 : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ such that $\|\varepsilon_1(x)\| < \varepsilon^*$ and $\|\varepsilon_2(x)\|_{\rm F} < \varepsilon^*$, $x \in \mathcal{D}_{\rm c}$, and

$$f_{\rm u}(\mathcal{T}(x)) - f_{\rm u}(\mathcal{T}(x_{\rm e})) - G_{\rm s}(\mathcal{T}(x_{\rm e}))u_{\rm e} = W_1^{\rm T}\sigma_1(x) + \varepsilon_1(x), \quad x \in \mathcal{D}_{\rm c}, \tag{12.12}$$

$$G_{\rm s}(\mathcal{T}(x)) - B_{\rm s} = W_2^{\rm T}[I_m \otimes \sigma_2(x)] + \varepsilon_2(x), \quad x \in \mathcal{D}_{\rm c}, \tag{12.13}$$

where $W_1 \in \mathbb{R}^{s_1\times m}$ and $W_2 \in \mathbb{R}^{ms_2\times m}$ are optimal *unknown* (constant) weights that minimize the approximation errors over $\mathcal{D}_{\rm c}$, $\sigma_1 : \mathbb{R}^n \to \mathbb{R}^{s_1}$ and $\sigma_2 : \mathbb{R}^n \to \mathbb{R}^{s_2}$ are sets of basis functions such that each component of $\sigma_1(\cdot)$ and $\sigma_2(\cdot)$ takes values between 0 and 1, and $\varepsilon_1(\cdot)$ and $\varepsilon_2(\cdot)$ are the modeling errors. Since $f_{\rm u}(\cdot)$ and $G_{\rm s}(\cdot)$ are continuous, we can choose $\sigma_1(\cdot)$ and $\sigma_2(\cdot)$ from a linear space $\mathcal{X}$ of continuous functions that forms an algebra and separates points in $\mathcal{D}_{\rm c}$. In this case, it follows from the Stone-Weierstrass theorem [201, p. 212] that $\mathcal{X}$ is a dense subset of the set of continuous functions on $\mathcal{D}_{\mathrm{c}}$. Now, as is the case in the standard neuro adaptive control literature [159], we can construct the signal $u_{\mathrm{ad}} = F(\hat{W}_1, \hat{W}_2, \sigma_1(x), \sigma_2(x))$ involving the estimates of the optimal weights and basis functions as our adaptive control signal. However, in order to develop an *output* feedback neural network, we use the recent approach given in [152] for reconstructing the system states via the system delayed inputs and outputs. Specifically, we use a *memory unit* as a particular form of a tapped delay line that takes a scalar time series input and provides an $mn$-dimensional vector output consisting of the present values of the system outputs and system inputs and their $(mn-2m)$ delayed values. As shown in [152], such a memory unit can be used to characterize an equivalent input-output representation for (12.1), (12.2) in the sense of guaranteeing the existence of a function $g(\cdot)$ and a number $d$ such that the future outputs of (12.1), (12.2) can be determined based on a number of past observations of the inputs and outputs of (12.1), (12.2). The following theorem is given in [152].

**Theorem 12.1** [152]. Consider the nonlinear dynamical system $\mathcal{G}$ given by (12.1), (12.2). Assume that the state vector $x(t)$, $t \geq 0$, of (12.1), (12.2) evolves on $\mathcal{B}_r(0) \triangleq \{x \in \mathbb{R}^n : \|x\| \leq r\}$ and $\mathcal{G}$ is observable. Furthermore, assume that the system output $y(t)$, $t \geq 0$, and its derivatives up to the order $(n-1)$ are bounded for all $t \geq 0$. Then, given an arbitrary $\varepsilon^* > 0$, there exists a set of bounded weights $W$ and a positive scalar $d > 0$ such that any continuous function $g(x, u) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ can be approximated over the compact set $\mathcal{B}_r(0)$ by a linear in the parameters neural network of the form

$$g(x(t), u(t)) = W^{\mathrm{T}}\sigma(\zeta(t)) + \varepsilon(x(t), \zeta(t)), \quad \|\varepsilon(x(t), \zeta(t))\| \leq \varepsilon^*, \quad t \geq 0, \qquad (12.14)$$

where $x(t)$, $t \geq 0$ is the solution to (12.1),

$$\zeta(t) \triangleq [y_1(t), y_1(t-d), \cdots, y_1(t-(r_1-1)d), \cdots,$$
$$y_m(t), y_m(t-d), \cdots, y_m(t-(r_m-1)d); u_1(t), u_1(t-d), \cdots,$$
$$u_1(t-(n-r_1-1)d), \cdots, u_m(t), u_m(t-d), \cdots, u_m(t-(n-r_m-1)d)]^{\mathrm{T}},$$
$$t \geq 0, \quad (12.15)$$

$\|\zeta(t)\| \leq \zeta^*$, $t \geq 0$, and $\zeta^* > 0$ is a uniform bound of $\zeta(\cdot)$ over $\mathcal{B}_r(0)$.

In light of the above theorem, it follows that if the dynamical system $\mathcal{G}$ is observable and its state trajectory $x(t)$, $t \geq 0$, evolves on $\mathcal{D}_{\mathrm{c}}$, then there exist $\varepsilon_1 : \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R}^m$ and $\varepsilon_2 : \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R}^{m\times m}$ such that $\|\varepsilon_1(x(t), \zeta(t))\| < \varepsilon^*$ and $\|\varepsilon_2(x(t), \zeta(t))\|_{\mathrm{F}} < \varepsilon^*$, $t \geq 0$, and

$$f_{\mathrm{u}}(\mathcal{T}(x(t))) - f_{\mathrm{u}}(\mathcal{T}(x_{\mathrm{e}})) - G_{\mathrm{s}}(\mathcal{T}(x_{\mathrm{e}}))u_{\mathrm{e}} = W_1^{\mathrm{T}}\sigma_1(\zeta(t)) + \varepsilon_1(x(t), \zeta(t)), \quad t \geq 0, \quad (12.16)$$

$$G_{\mathrm{s}}(\mathcal{T}(x(t))) - B_{\mathrm{s}} = W_2^{\mathrm{T}}[I_m \otimes \sigma_2(\zeta(t))] + \varepsilon_2(x(t), \zeta(t)), \quad t \geq 0. \quad (12.17)$$

For the statement of the main results of this chapter, define the projection operator $\mathrm{Proj}(\tilde{W}, Y)$ given by

$$\mathrm{Proj}(\tilde{W}, Y) \triangleq \begin{cases} Y & \text{if } \mu(\tilde{W}) < 0, \\ Y & \text{if } \mu(\tilde{W}) \geq 0 \text{ and } \mu'(\tilde{W})Y \leq 0, \\ Y - \frac{\mu'^{\mathrm{T}}(\tilde{W})\mu'(\tilde{W})Y}{\mu'(\tilde{W})\mu'^{\mathrm{T}}(\tilde{W})}\mu(\tilde{W}) & \text{otherwise,} \end{cases} \quad (12.18)$$

where $\tilde{W} \in \mathbb{R}^{s\times m}$, $Y \in \mathbb{R}^{n\times m}$, $\mu(\tilde{W}) \triangleq \frac{\mathrm{tr}\,\tilde{W}^{\mathrm{T}}\tilde{W} - \tilde{w}_{\max}^2}{\varepsilon_{\tilde{W}}}$, $\tilde{w}_{\max} \in \mathbb{R}$ is the norm bound imposed on $\tilde{W}$, and $\varepsilon_{\tilde{W}} > 0$. Note that, given the matrices $\tilde{W} \in \mathbb{R}^{s\times m}$ and $Y \in \mathbb{R}^{s\times m}$, it follows that

$$\begin{aligned} &\mathrm{tr}[(\tilde{W} - W)^{\mathrm{T}}(\mathrm{Proj}(\tilde{W}, Y) - Y)] \\ &\quad = \sum_{i=1}^{n}[\mathrm{col}_i(\tilde{W} - W)]^{\mathrm{T}}(\mathrm{Proj}(\mathrm{col}_i(\tilde{W}), \mathrm{col}_i(Y)) - \mathrm{col}_i(Y)) \\ &\quad \leq 0, \end{aligned} \quad (12.19)$$

where $\operatorname{col}_i(X)$ denotes the $i$th column of the matrix $X$.

**Theorem 12.2.** Consider the nonlinear uncertain dynamical system $\mathcal{G}$ given by (12.1) and (12.2) where $f : \mathbb{R}^n \to \mathbb{R}^n$ is essentially nonnegative and $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is nonnegative. For a given $y_\mathrm{d} \in \overline{\mathbb{R}}_+^m$ assume there exist nonnegative vectors $x_\mathrm{e} \in \overline{\mathbb{R}}_+^n$ and $u_\mathrm{e} \in \overline{\mathbb{R}}_+^m$ such that

$$0 = f(x_\mathrm{e}) + G(x_\mathrm{e})u_\mathrm{e}, \tag{12.20}$$

$$y_\mathrm{d} = h(x_\mathrm{e}). \tag{12.21}$$

Furthermore, assume that the equilibrium point $x_\mathrm{e}$ of (12.1) is globally asymptotically stable with $u(t) \equiv u_\mathrm{e}$. In addition, assume that there exists a global diffeomorphism $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathcal{G}$ can be transformed into the normal form given by (12.4) and (12.5), and (12.5) is input-to-state stable at $z_\mathrm{e}$ with $\xi(t) - \xi_\mathrm{e}$ viewed as the input. Finally, let $Q_1, Q_2 \in \mathbb{R}^{m\times m}$ be positive definite. Then the neural adaptive output feedback control law

$$u(t) = \begin{cases} \hat{u}(t), & \text{if } \hat{u}(t) \geq\geq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{12.22}$$

where

$$\hat{u}(t) = -\left(B_\mathrm{s} + \hat{W}_2^\mathrm{T}(t)[I_m \otimes \sigma_2(\zeta(t))]\right)^{-1} \hat{W}_1^\mathrm{T}(t)\sigma_1(\zeta(t)), \tag{12.23}$$

$B_\mathrm{s} \in \mathbb{R}^{m\times m}$ is nonsingular, $\zeta(t)$, $t \geq 0$, is given by (12.15), $\hat{W}_1(t) \in \mathbb{R}^{s_1\times m}$, $t \geq 0$, and $\hat{W}_2(t) \in \mathbb{R}^{ms_2\times m}$, $t \geq 0$, with update laws

$$\dot{\hat{W}}_1(t) = Q_1 \mathrm{Proj}(\hat{W}_1(t), \sigma_1(\zeta(t))\xi_\mathrm{c}^\mathrm{T}(t)\tilde{P}B_0), \quad \hat{W}_1(0) = \hat{W}_{10}, \tag{12.24}$$

$$\dot{\hat{W}}_2(t) = Q_2 \mathrm{Proj}(\hat{W}_2(t), [I_m \otimes \sigma_2(\zeta(t))]u(t)\xi_\mathrm{c}^\mathrm{T}(t)\tilde{P}B_0), \quad \hat{W}_2(0) = \hat{W}_{20}, \tag{12.25}$$

where $\tilde{P} \in \mathbb{R}^{r\times r}$ is a positive-definite solution of the Lyapunov equation

$$0 = (A - LC)^\mathrm{T}\tilde{P} + \tilde{P}(A - LC) + \tilde{R}, \quad \tilde{R} > 0, \tag{12.26}$$

and $\xi_c(t) \in \mathbb{R}^r$, $t \geq 0$, is the solution to the estimator dynamics

$$\dot{\xi}_c(t) = A\xi_c(t) + L(y(t) - y_c(t) - y_d), \quad \xi_c(0) = \xi_{c0}, \quad t \geq 0, \tag{12.27}$$

$$y_c(t) = C\xi_c(t), \tag{12.28}$$

where $A \in \mathbb{R}^{r\times r}$ is asymptotically stable, $L \in \mathbb{R}^{r\times m}$ is such that $A - LC$ is asymptotically stable, and $B_0 \triangleq [0_{m\times(r-m)}, I_m]^{\mathrm{T}}$, guarantees that there exists a compact positively invariant set $\mathcal{D}_\alpha \subset \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^{s_1\times m} \times \mathbb{R}^{ms_2\times m}$ such that $(x_e, 0, W_1, W_2) \in \mathcal{D}_\alpha$, where $W_1 \in \mathbb{R}^{s_1\times m}$ and $W_2 \in \mathbb{R}^{ms_2\times m}$, and the solution $(x(t), \xi_c(t), \hat{W}_1(t), \hat{W}_2(t))$, $t \geq 0$, of the closed-loop system given by (12.1), (12.22), (12.24), and (12.25) is ultimately bounded for all $(x(0), \xi_c(0), \hat{W}_1(0), \hat{W}_2(0)) \in \mathcal{D}_\alpha$ with ultimate bound $\|y(t) - y_d\|^2 < \varepsilon$, $t \geq T$, where

$$\varepsilon > \Bigg[\left(\sqrt{\frac{\nu}{\lambda_{\min}(RP^{-1})}} + \alpha_1\right)^2 + \left(\sqrt{\frac{\nu}{\lambda_{\min}(\tilde{R}\tilde{P}^{-1})}} + \alpha_2\right)^2 + \lambda_{\max}(Q_1^{-1})\hat{w}_{1\max}^2 + \lambda_{\max}(Q_2^{-1})\hat{w}_{2\max}^2\Bigg]^{\frac{1}{2}} \tag{12.29}$$

$$\nu \triangleq \frac{\alpha_1^2}{\lambda_{\min}(RP^{-1})} + \frac{\alpha_2^2}{\lambda_{\min}(\tilde{R}\tilde{P}^{-1})}, \tag{12.30}$$

$$\alpha_1 \triangleq [\sqrt{s_1}\hat{w}_{1\max} + (b_s + m\sqrt{s_2}\hat{w}_{2\max})u^*]\|P^{-1/2}(P - \tilde{P})B_0\| + (\sqrt{s_1}\hat{w}_{1\max} + (\varepsilon_1^* + \varepsilon_2^* u^*))\|P^{1/2}B_0\|, \tag{12.31}$$

$$\alpha_2 \triangleq [3\sqrt{s_1}\hat{w}_{1\max} + 2(b_s + m\sqrt{s_2}\hat{w}_{2\max})u^* + (\varepsilon_1^* + \varepsilon_2^* u^*)]\|\tilde{P}^{1/2}B_0\|, \tag{12.32}$$

$u^* \triangleq \sup_{t\geq 0}\|u(t)\|$, $b_s \triangleq \lambda_{\max}(B_s)$, $\hat{w}_{i\max}$, $i = 1, 2$, are norm bounds imposed on $\hat{W}_i$, and $P \in \mathbb{R}^{r\times r}$ is a positive-definite solution of the Lyapunov equation

$$0 = A^{\mathrm{T}}P + PA + R, \quad R > 0. \tag{12.33}$$

Furthermore, $u(t) \geq\geq 0$ and $x(t) \geq\geq 0$ for all $t \geq 0$ and $x_0 \in \overline{\mathbb{R}}_+^n$.

**Proof.** First, define

$$\hat{W}_{1u}(t) \triangleq \begin{cases} \hat{W}_1(t), & \text{if } \hat{u}(t) \geq\geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{12.34}$$

Next, defining $e_\xi(t) \triangleq \xi(t) - \xi_{\rm e}$, $e_z(t) \triangleq z(t) - z_{\rm e}$, and $\tilde{\xi}(t) \triangleq \xi_{\rm c}(t) - e_\xi(t)$, and using (12.9), (12.10), (12.16), (12.17), and (12.22) it follows from (12.4), (12.5), and (12.27) that

$$\begin{aligned}
\dot{e}_\xi(t) =\ & Ae_\xi(t) + A\xi_{\rm e} + \tilde{f}_{\rm u}(\xi(t), z(t)) + G_\xi(\xi(t), z(t))u(t) \\
=\ & Ae_\xi(t) + B_0[f_{\rm u}(\mathcal{T}(x(t))) - f_{\rm u}(\mathcal{T}(x_{\rm e})) - G_{\rm s}(\mathcal{T}(x_{\rm e}))] + B_0 G_{\rm s}(\mathcal{T}(x(t)))u(t) \\
& +B_0\left(B_{\rm s} + \hat{W}_2^{\rm T}(t)[I_m \otimes \sigma_2(\zeta(t))]\right) \\
& \cdot\left(-u(t) - \left(B_{\rm s} + \hat{W}_2^{\rm T}(t)[I_m \otimes \sigma_2(\zeta(t))]\right)^{-1}\hat{W}_1^{\rm T}(t)\sigma_1(\zeta(t))\right) \\
=\ & Ae_\xi(t) - B_0\tilde{W}_1^{\rm T}(t)\sigma_1(\zeta(t)) - B_0\tilde{W}_2^{\rm T}(t)[I_m \otimes \sigma_2(\zeta(t))]u(t) \\
& +B_0(\hat{W}_1(t) - \hat{W}_{1u}(t))^{\rm T}\sigma_1(\zeta(t)) + B_0\varepsilon_1(x(t), \zeta(t)) + B_0\varepsilon_2(x(t), \zeta(t))u(t), \\
& \qquad e_\xi(0) = \xi_0 - \xi_{\rm e}, \quad t \geq 0, \qquad (12.35) \\
\dot{e}_z(t) =\ & \tilde{f}_z(e_\xi(t), e_z(t)), \quad e_z(0) = z_0 - z_{\rm e}, \qquad (12.36)
\end{aligned}$$

and

$$\begin{aligned}
\dot{\tilde{\xi}}(t) =\ & \dot{\xi}_{\rm c}(t) - \dot{e}_\xi(t) \\
=\ & \tilde{A}\tilde{\xi}(t) + B_0\tilde{W}_1^{\rm T}(t)\sigma_1(\zeta(t)) + B_0\left(B_{\rm s} + \tilde{W}_2^{\rm T}(t)[I_m \otimes \sigma_2(\zeta(t))]\right)u(t) \\
& -B_0(\hat{W}_1(t) - \hat{W}_{1u}(t))^{\rm T}\sigma_1(\zeta(t)) - B_0\varepsilon_1(x(t), \zeta(t)) - B_0\varepsilon_2(x(t), \zeta(t))u(t), \\
& \qquad \tilde{\xi}(0) = \xi_{\rm c0} - \xi_0 + \xi_{\rm e}, \qquad (12.37)
\end{aligned}$$

where $\tilde{A} \triangleq A - LC$, $\tilde{f}_z(e_\xi, e_z) \triangleq f_z(e_\xi + x_{\rm e}, e_z + z_{\rm e})$, and $\tilde{W}_i(t) \triangleq \hat{W}_i(t) - W_i$, $i = 1, 2$.

To show ultimate boundedness of the closed-loop system (12.24), (12.25), (12.35)–(12.37), consider the Lyapunov-like function

$$V(e_\xi, e_z, \tilde{\xi}, \tilde{W}_1, \tilde{W}_2) = e_\xi^{\rm T} P e_\xi + \tilde{\xi}^{\rm T}\tilde{P}\tilde{\xi} + {\rm tr}\,\tilde{W}_1 Q_1^{-1}\tilde{W}_1^{\rm T} + {\rm tr}\,\tilde{W}_2 Q_2^{-1}\tilde{W}_2^{\rm T}, \quad (12.38)$$

where $P > 0$ and $\tilde{P} > 0$ satisfy (12.26) and (12.33), respectively. Note that (12.38) satisfies (10.7) with $x_1 = [e_\xi^{\rm T}, \tilde{\xi}^{\rm T}, ({\rm vec}\,\hat{W}_1)^{\rm T}, ({\rm vec}\,\hat{W}_2)^{\rm T}]^{\rm T}$, $x_2 = e_z$, $\alpha(\|x_1\|) = \beta(\|x_1\|) = \|x_1\|^2$, where $\|x_1\|^2 \triangleq e_\xi^{\rm T} P e_\xi + \tilde{\xi}^{\rm T}\tilde{P}\tilde{\xi} + {\rm tr}\,\tilde{W}_1 Q_1^{-1}\tilde{W}_1^{\rm T} + {\rm tr}\,\tilde{W}_2 Q_2^{-1}\tilde{W}_2^{\rm T}$. Furthermore,

$\alpha(\|x_1\|)$ is a class $\mathcal{K}_\infty$ function. Now, letting $e_\xi(t)$, $t \geq 0$, and $\xi_{\rm c}(t)$, $t \geq 0$, denote the solution to (12.35) and (12.27), respectively, and using (12.12), (12.19), (12.24), and (12.25), it follows that the time derivative of $V(e_\xi, e_z, \tilde{\xi}, \tilde{W}_1, \tilde{W}_2)$ along the closed-loop system trajectories is given by

$$\begin{aligned}
&\dot{V}(e_\xi(t), e_z(t), \tilde{\xi}(t), \tilde{W}_1(t), \tilde{W}_2(t)) \\
&\quad = 2e_\xi^{\rm T}(t)P\Big[Ae_\xi(t) - B_0\tilde{W}_1^{\rm T}(t)\sigma_1(\zeta(t)) - B_0\left(B_{\rm s} + \tilde{W}_2^{\rm T}(t)[I_m \otimes \sigma_2(\zeta(t))]\right)u(t) \\
&\qquad +B_0(\hat{W}_1(t) - \hat{W}_{1u}(t))^{\rm T}\sigma_1(\zeta(t)) + B_0\varepsilon_1(x(t), \zeta(t)) + B_0\varepsilon_2(x(t), \zeta(t))u(t)\Big] \\
&\qquad +2\tilde{\xi}^{\rm T}(t)\tilde{P}\Big[\tilde{A}\tilde{\xi}(t) + B_0\tilde{W}_1^{\rm T}(t)\sigma_1(\zeta(t)) + B_0\left(B_{\rm s} + \tilde{W}_2^{\rm T}(t)[I_m \otimes \sigma_2(\zeta(t))]\right)u(t) \\
&\qquad -B_0(\hat{W}_1(t) - \hat{W}_{1u}(t))^{\rm T}\sigma_1(\zeta(t)) - B_0\varepsilon_1(x(t), \zeta(t)) - B_0\varepsilon_2(x(t), \zeta(t))u(t)\Big] \\
&\qquad +2{\rm tr}\,\tilde{W}_1^{\rm T}(t)Q_1^{-1}\dot{\hat{W}}_1(t) + 2{\rm tr}\,\tilde{W}_2^{\rm T}(t)Q_2^{-1}\dot{\hat{W}}_2(t) \\
&\quad = -e_\xi^{\rm T}(t)Re_\xi(t) - \tilde{\xi}^{\rm T}(t)\tilde{R}\tilde{\xi}(t) - 2e_\xi^{\rm T}(t)PB_0\tilde{W}_1^{\rm T}(t)\sigma_1(\zeta(t)) \\
&\qquad -2e_\xi^{\rm T}(t)PB_0\left(B_{\rm s} + \tilde{W}_2^{\rm T}(t)[I_m \otimes \sigma_2(\zeta(t))]\right)u(t) \\
&\qquad +2e_\xi^{\rm T}(t)PB_0(\hat{W}_1(t) - \hat{W}_{1u}(t))^{\rm T}\sigma_1(\zeta(t)) \\
&\qquad +2e_\xi^{\rm T}(t)PB_0(\varepsilon_1(x(t), \zeta(t)) + \varepsilon_2(x(t), \zeta(t))u(t)) + 2\tilde{\xi}^{\rm T}(t)\tilde{P}B_0\tilde{W}_1^{\rm T}(t)\sigma_1(\zeta(t)) \\
&\qquad +2\tilde{\xi}^{\rm T}(t)\tilde{P}B_0\left(B_{\rm s} + \tilde{W}_2^{\rm T}(t)[I_m \otimes \sigma_2(\zeta(t))]\right)u(t) \\
&\qquad -2\tilde{\xi}^{\rm T}(t)\tilde{P}B_0(\hat{W}_1(t) - \hat{W}_{1u}(t))^{\rm T}\sigma_1(\zeta(t)) \\
&\qquad -2\tilde{\xi}^{\rm T}(t)\tilde{P}B_0(\varepsilon_1(x(t), \zeta(t)) + \varepsilon_2(x(t), \zeta(t))u(t)) \\
&\qquad +2{\rm tr}\,\tilde{W}_1^{\rm T}(t){\rm Proj}(\hat{W}_1(t), \sigma_1(\zeta(t))\xi_{\rm c}^{\rm T}\tilde{P}B_0) \\
&\qquad +2{\rm tr}\,\tilde{W}_2^{\rm T}(t){\rm Proj}(\hat{W}_2(t), [I_m \otimes \sigma_2(\zeta(t))]\xi_{\rm c}^{\rm T}\tilde{P}B_0) \\
&\quad \leq -\lambda_{\min}(RP^{-1})\|P^{1/2}e_\xi(t)\|^2 - \lambda_{\min}(\tilde{R}\tilde{P}^{-1})\|\tilde{P}^{1/2}\tilde{\xi}(t)\|^2 \\
&\qquad -2e_\xi^{\rm T}(t)(P - \tilde{P})B_0\tilde{W}_1^{\rm T}(t)\sigma_1(\zeta(t)) \\
&\qquad -2e_\xi^{\rm T}(t)(P - \tilde{P})B_0\left(B_{\rm s} + \tilde{W}_2^{\rm T}(t)[I_m \otimes \sigma_2(\zeta(t))]\right)u(t) \\
&\qquad -4\tilde{\xi}^{\rm T}(t)\tilde{P}B_0\tilde{W}_1^{\rm T}(t)\sigma_1(\zeta(t)) - 4\tilde{\xi}^{\rm T}(t)\tilde{P}B_0\left(B_{\rm s} + \tilde{W}_2^{\rm T}(t)[I_m \otimes \sigma_2(\zeta(t))]\right)u(t) \\
&\qquad +2e_\xi^{\rm T}(t)PB_0(\varepsilon_1(x(t), \zeta(t)) + \varepsilon_2(x(t), \zeta(t))u(t))
\end{aligned}$$

$$
\begin{aligned}
&-2\tilde{\xi}^{\mathrm{T}}(t)\tilde{P}B_0(\varepsilon_1(x(t),\zeta(t))+\varepsilon_2(x(t),\zeta(t))u(t)) \\
&+2(e_\xi^{\mathrm{T}}(t)P-\tilde{\xi}^{\mathrm{T}}(t)\tilde{P})B_0(\hat{W}_1(t)-\hat{W}_{1u}(t))^{\mathrm{T}}\sigma_1(\zeta(t)) \\
&+2\mathrm{tr}\,\tilde{W}_1^{\mathrm{T}}(t)\left[\mathrm{Proj}(\hat{W}_1(t),\sigma_1(\zeta(t))\xi_{\mathrm{c}}^{\mathrm{T}}\tilde{P}B_0)-\sigma_1(\zeta(t))\xi_{\mathrm{c}}^{\mathrm{T}}\tilde{P}B_0\right] \\
&+2\mathrm{tr}\,\tilde{W}_2^{\mathrm{T}}(t)\left[\mathrm{Proj}(\hat{W}_2(t),[I_m\otimes\sigma_2(\zeta(t))]\xi_{\mathrm{c}}^{\mathrm{T}}\tilde{P}B_0)-[I_m\otimes\sigma_2(\zeta(t))]\xi_{\mathrm{c}}^{\mathrm{T}}\tilde{P}B_0\right] \\
\le\ &-\lambda_{\min}(RP^{-1})\|P^{1/2}e_\xi(t)\|^2-\lambda_{\min}(\tilde{R}\tilde{P}^{-1})\|\tilde{P}^{1/2}\tilde{\xi}(t)\|^2 \\
&-2e_\xi^{\mathrm{T}}(t)(P-\tilde{P})B_0\tilde{W}_1^{\mathrm{T}}(t)\sigma_1(\zeta(t)) \\
&-2e_\xi^{\mathrm{T}}(t)(P-\tilde{P})B_0\left(B_{\mathrm{s}}+\tilde{W}_2^{\mathrm{T}}(t)[I_m\otimes\sigma_2(\zeta(t))]\right)u(t) \\
&-4\tilde{\xi}^{\mathrm{T}}(t)\tilde{P}B_0\tilde{W}_1^{\mathrm{T}}(t)\sigma_1(\zeta(t))-4\tilde{\xi}^{\mathrm{T}}(t)\tilde{P}B_0\left(B_{\mathrm{s}}+\tilde{W}_2^{\mathrm{T}}(t)[I_m\otimes\sigma_2(\zeta(t))]\right)u(t) \\
&+2e_\xi^{\mathrm{T}}(t)PB_0(\varepsilon_1(x(t),\zeta(t))+\varepsilon_2(x(t),\zeta(t))u(t)) \\
&-2\tilde{\xi}^{\mathrm{T}}(t)\tilde{P}B_0(\varepsilon_1(x(t),\zeta(t))+\varepsilon_2(x(t),\zeta(t))u(t)) \\
&+2(e_\xi^{\mathrm{T}}(t)P-\tilde{\xi}^{\mathrm{T}}(t)\tilde{P})B_0(\hat{W}_1(t)-\hat{W}_{1u}(t))^{\mathrm{T}}\sigma_1(\zeta(t)),\quad t\ge 0.
\end{aligned}
\tag{12.39}
$$

For the two cases given in (12.34), the last term on the right-hand side of (12.39) gives:

*i)* If $\hat{u}(t)\ge\ge 0$, then $\hat{W}_{1u}(t)=\hat{W}_1(t)$ and hence

$$
2(e_\xi^{\mathrm{T}}(t)P-\tilde{\xi}^{\mathrm{T}}(t)\tilde{P})B_0(\hat{W}_1(t)-\hat{W}_{1u}(t))^{\mathrm{T}}\sigma_1(\zeta(t))=0.
$$

*ii)* Otherwise, $\hat{W}_{1u}(t)=0$ and hence

$$
\begin{aligned}
&2(e_\xi^{\mathrm{T}}(t)P-\tilde{\xi}^{\mathrm{T}}(t)\tilde{P})B_0(\hat{W}_1(t)-\hat{W}_{1u}(t))^{\mathrm{T}}\sigma_1(\zeta(t)) \\
&\quad= 2(e_\xi^{\mathrm{T}}(t)P-\tilde{\xi}^{\mathrm{T}}(t)\tilde{P})B_0\hat{W}_1^{\mathrm{T}}(t)\sigma_1(\zeta(t)) \\
&\quad\le 2\sqrt{s_1}\|P^{1/2}B_0\|\hat{W}_{1\max}\|P^{1/2}e_\xi(t)\|+2\sqrt{s_1}\|\tilde{P}^{1/2}B_0\|\hat{W}_{1\max}\|\tilde{P}^{1/2}\tilde{\xi}(t)\|.
\end{aligned}
\tag{12.40}
$$

Hence, it follows from (12.39) that in either case

$$
\begin{aligned}
&\dot{V}(e_\xi(t),e_z(t),\tilde{\xi}(t),\tilde{W}_1(t),\tilde{W}_2(t)) \\
&\quad\le -\lambda_{\min}(RP^{-1})\|P^{1/2}e_\xi(t)\|^2-\lambda_{\min}(\tilde{R}\tilde{P}^{-1})\|\tilde{P}^{1/2}\tilde{\xi}(t)\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+2\sqrt{s_1}\hat{W}_{1\max}\|P^{-1/2}(P-\tilde{P})B_0\|\|P^{1/2}e_\xi(t)\| \\
&+2(b_\mathrm{s}+m\sqrt{s_2}\hat{W}_{2\max})u^*\|P^{-1/2}(P-\tilde{P})B_0\|\|P^{1/2}e_\xi(t)\| \\
&+4\sqrt{s_1}\hat{W}_{1\max}\|\tilde{P}^{1/2}B_0\|\|\tilde{P}^{1/2}\tilde{\xi}(t)\| \\
&+4(b_\mathrm{s}+m\sqrt{s_2}\hat{W}_{2\max})u^*\|\tilde{P}^{1/2}B_0\|\|\tilde{P}^{1/2}\tilde{\xi}(t)\| \\
&+2(\varepsilon_1^*+\varepsilon_2^*u^*)\|P^{1/2}B_0\|\|P^{1/2}e_\xi(t)\|+2(\varepsilon_1^*+\varepsilon_2^*u^*)\|\tilde{P}^{1/2}B_0\|\|\tilde{P}^{1/2}\tilde{\xi}(t)\| \\
&+2\sqrt{s_1}\hat{W}_{1\max}\|P^{1/2}B_0\|\|P^{1/2}e_\xi(t)\|+2\sqrt{s_1}\hat{W}_{1\max}\|\tilde{P}^{1/2}B_0\|\|\tilde{P}^{1/2}\tilde{\xi}(t)\| \\
=\;& -\lambda_{\min}(RP^{-1})(\|P^{1/2}e_\xi(t)\|-\alpha_1)^2-\lambda_{\min}(\tilde{R}\tilde{P}^{-1})(\|\tilde{P}^{1/2}\tilde{\xi}(t)\|-\alpha_2)^2+\nu,
\end{aligned}
\tag{12.41}
$$

where $\nu$, $\alpha_1$, and $\alpha_2$ are given by (12.30) and (12.32), respectively. Now, for

$$
\|P^{1/2}e_\xi\| \geq \sqrt{\frac{\nu}{\lambda_{\min}(RP^{-1})}}+\alpha_1, \tag{12.42}
$$

or

$$
\|\tilde{P}^{1/2}\tilde{\xi}\| \geq \sqrt{\frac{\nu}{\lambda_{\min}(\tilde{R}\tilde{P}^{-1})}}+\alpha_2, \tag{12.43}
$$

it follows that $\dot{V}(e_\xi(t), e_z(t), \tilde{\xi}(t), \tilde{W}_1(t), \tilde{W}_2(t)) \leq 0$ for all $t \geq 0$; that is, $\dot{V}(e_\xi(t), e_z(t), \tilde{\xi}(t), \tilde{W}_1(t), \tilde{W}_2(t)) \leq 0$ for all $(e_\xi(t), e_z(t), \tilde{\xi}(t), \tilde{W}_1(t), \tilde{W}_2(t)) \in \tilde{\mathcal{D}}_\mathrm{e}\backslash\tilde{\mathcal{D}}_\mathrm{r}$ and $t \geq 0$, where (see Figure 12.1)

$$
\tilde{\mathcal{D}}_\mathrm{e} \triangleq \Big\{(e_\xi, e_z, \tilde{\xi}, \tilde{W}_1, \tilde{W}_2) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^r \times \mathbb{R}^{s_1\times m} \times \mathbb{R}^{ms_2\times m} : x \in \mathcal{D}_\mathrm{c}\Big\}, \tag{12.44}
$$

$$
\begin{aligned}
\tilde{\mathcal{D}}_\mathrm{r} \triangleq \Big\{&(e_\xi, e_z, \tilde{\xi}, \tilde{W}_1, \tilde{W}_2) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^r \times \mathbb{R}^{s_1\times m} \times \mathbb{R}^{ms_2\times m} : \\
&\|P^{1/2}e_\xi\| \leq \alpha_{e_\xi},\ \|\tilde{P}^{1/2}\tilde{\xi}\| \leq \alpha_{\tilde{\xi}}\Big\}.
\end{aligned}
\tag{12.45}
$$

Next, define

$$
\begin{aligned}
\tilde{\mathcal{D}}_\alpha \triangleq \Big\{&(e_\xi, e_z, \tilde{\xi}, \tilde{W}_1, \tilde{W}_2) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^r \times \mathbb{R}^{s_1\times m} \times \mathbb{R}^{ms_2\times m} : \\
&V(e_\xi, e_z, \tilde{\xi}, \tilde{W}_1, \tilde{W}_2) \leq \alpha\Big\},
\end{aligned}
\tag{12.46}
$$

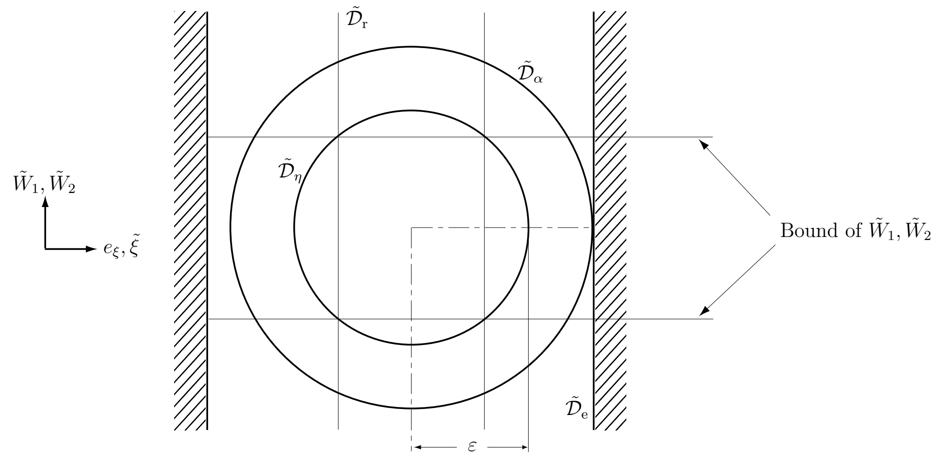



**Figure 12.1**: Visualization of sets used in the proof of Theorem 12.2

where $\alpha$ is the maximum value such that $\tilde{\mathcal{D}}_\alpha \subseteq \tilde{\mathcal{D}}_{\mathrm{e}}$, and define

$$\tilde{\mathcal{D}}_\eta \triangleq \Big\{(e_\xi, e_z, \tilde{\xi}, \tilde{W}_1, \tilde{W}_2) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^r \times \mathbb{R}^{s_1 \times m} \times \mathbb{R}^{ms_2 \times m} : V(e_\xi, e_z, \tilde{\xi}, \tilde{W}_1, \tilde{W}_2) \le \eta\Big\}, \tag{12.47}$$

where

$$\eta > \beta(\mu) = \mu = \alpha_{e_\xi}^2 + \alpha_{\tilde{\xi}}^2 + \lambda_{\max}(Q_1^{-1})\hat{W}_{1\,\max}^2 + \lambda_{\max}(Q_2^{-1})\hat{W}_{2\,\max}^2. \tag{12.48}$$

To show ultimate boundedness of the closed-loop system (12.24), (12.25), (12.35)–(12.37), assume[5] that $\tilde{\mathcal{D}}_\eta \subset \tilde{\mathcal{D}}_\alpha$ (see Remark 12.1 and Figure 12.1). Now, since $\dot{V}(e_\xi, e_z, \tilde{\xi}, \tilde{W}_1,\ \tilde{W}_2) \le 0$ for all $(e_\xi, e_z, \tilde{\xi}, \tilde{W}_1, \tilde{W}_2) \in \tilde{\mathcal{D}}_{\mathrm{e}} \backslash \tilde{\mathcal{D}}_{\mathrm{r}}$ and $\tilde{\mathcal{D}}_{\mathrm{r}} \subset \tilde{\mathcal{D}}_\alpha$, it follows that $\tilde{\mathcal{D}}_\alpha$ is positively invariant. Hence, if $(e_\xi(0), e_z(0), \tilde{\xi}(0), \tilde{W}_1(0), \tilde{W}_2(0)) \in \tilde{\mathcal{D}}_\alpha$, then it follows from Theorem 10.3 that the solution $(e_\xi(t), e_z(t), \tilde{\xi}(t), \hat{W}(t))$, $t \ge 0$, to (12.24), (12.25), (12.35)–(12.37) is ultimately bounded with respect to $(e_\xi, \tilde{\xi}, \tilde{W}_1, \tilde{W}_2)$ uniformly in $e_z(0)$ with ultimate bound given by $\varepsilon = \alpha^{-1}(\eta) = \sqrt{\eta}$ which yields (12.29). In addition, since (12.36) is input-to-state stable with $e_\xi$ viewed as the input, it follows from Proposition 10.1 that the solution $e_z(t)$, $t \ge 0$, to (12.36) is

[5]This assumption is standard in the neural network literature and ensures that in the error space $\tilde{\mathcal{D}}_{\mathrm{e}}$ there exists at least one Lyapunov level set $\tilde{\mathcal{D}}_\eta \subset \tilde{\mathcal{D}}_\alpha$. In the case where the neural network approximation holds in $\mathbb{R}^n$ with delayed values, this assumption is automatically satisfied. See Remark 12.1 for further details.

also ultimately bounded. Furthermore, it follows from Theorem 1 of [223] that there exist a continuously differentiable, radially unbounded, positive-definite function $V_z : \mathbb{R}^{n_z} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\gamma_1(\cdot), \gamma_2(\cdot)$ such that

$$V_z'(e_z)\tilde{f}_z(e_\xi, e_z) \le -\gamma_1(\|e_z\|), \quad \|e_z\| \ge \gamma_2(\|P^{1/2}e_\xi\|). \tag{12.49}$$

Since the upper bound for $\|P^{1/2}e_\xi\|^2$ is given by $\eta$, it follows that the set given by

$$\mathcal{D}_z \triangleq \Big\{ z \in \mathbb{R}^{n-r} : V_z(z - z_\mathrm{e}) \le \max_{\|z - z_\mathrm{e}\| = \gamma_2(\sqrt{\eta})} V_z(z - z_\mathrm{e}) \Big\}, \tag{12.50}$$

is also positively invariant. Now, since $\tilde{\mathcal{D}}_\alpha$ and $\mathcal{D}_z$ are positively invariant, it follows that

$$\begin{aligned} \mathcal{D}_\alpha \triangleq \Big\{ &(x, \tilde{\xi}, \tilde{W}_1, \tilde{W}_2) \in \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^{s_1 \times m} \times \mathbb{R}^{m s_2 \times m} : \\ &V(\xi - y_\mathrm{d}, z - e_z, \tilde{\xi}, \hat{W}_1 - W_1, \hat{W}_2 - W_2) \le \alpha \Big\}, \end{aligned} \tag{12.51}$$

is also positively invariant. In addition, since (12.24), (12.25), (12.35)–(12.37) is ultimately bounded with respect to $(e_\xi, \tilde{\xi}, \tilde{W}_1, \tilde{W}_2)$ and (12.36) is input-to-state stable with $e_\xi$ viewed as the input it follows from Proposition 10.1 that the solution $(e_\xi(t), e_z(t), \tilde{\xi}(t), \tilde{W}_1(t), \tilde{W}_2(t))$, $t \ge 0$, of the closed-loop system (12.24), (12.25), (12.35)–(12.37) is ultimately bounded for all $(e_\xi(0), e_z(0), \tilde{\xi}(0), \tilde{W}_1(0), \tilde{W}_2(0)) \in \tilde{\mathcal{D}}_\alpha$.

Finally, $u(t) \ge\ge 0$, $t \ge 0$, is a restatement of (12.22). Now, since $G(x(t)) \ge\ge 0$, $t \ge 0$, and $u(t) \ge\ge 0$, $t \ge 0$, it follows from Proposition 9.2 that $x(t) \ge\ge 0$ for all $t \ge 0$ and $x_0 \in \overline{\mathbb{R}}_+^n$. □

**Remark 12.1.** It is important to note that the existence of a global neural network approximator for an uncertain nonlinear map using the system outputs and inputs and its delayed values (as in (12.16), (12.17)) cannot in general be established. In the proof of Theorem 12.2, as is common in the neural network literature, we assume that for a given arbitrarily large compact set $\mathcal{D}_{\rm c} \subset \mathbb{R}^n$, there exists an approximator for the unknown nonlinear map up to a desired accuracy. This assumption ensures that in the error space $\tilde{\mathcal{D}}_{\rm e}$ there exists at least one Lyapunov level set such that $\tilde{\mathcal{D}}_\eta \subset \tilde{\mathcal{D}}_\alpha$. In the case where $f_{\rm u}(\cdot)$ and $G_{\rm s}(\cdot)$ are continuous on $\mathbb{R}^n$, it follows from the Stone-Weierstrass theorem that $f_{\rm u}(\cdot)$ and $G_{\rm s}(\cdot)$ can be approximated over an arbitrarily large compact set $\mathcal{D}_{\rm c}$ in the sense of (12.12) and (12.13) and hence (12.16) and (12.17) hold with sufficiently small $d$. In addition, we assume that $\hat{W}_2(0)$ is sufficiently close to the optimal weight $W_2$ so that $B_{\rm s} + \hat{W}_2(t)[I_m \otimes \sigma_2(\zeta(t))]$ is nonsingular for all $t \geq 0$.

**Remark 12.2.** Implementation of (12.23) requires a fixed-point iteration at each integration step; that is, the controller contains an algebraic constraint on $u$. For each choice of $\sigma_1(\cdot)$ and $\sigma_2(\cdot)$ this equation must be examined for solvability in terms of $u$. It is more practical to avoid this iteration by using one-step delayed values of $u$ in calculating $\hat{u}$. Implementations using both approaches result in imperceptible differences in our numerical studies.

**Remark 12.3.** In the case of systems of unknown dimension but with known relative degree, Theorem 12.2 applies with a slight modification to the input vector of the neural network; that is, $n$ in (12.15) should be replaced by a sufficiently large value that is greater than the largest possible system dimension.

In Theorem 12.2 we assumed that the equilibrium point $x_{\rm e}$ of (12.1) is globally asymptotically stable with $u(t) \equiv u_{\rm e}$. In general, however, unlike linear nonnegative systems with asymptotically stable plant dynamics, a given set point $x_{\rm e} \in \mathbb{R}^n_+$ for the nonlinear nonnegative dynamical system (12.1) may not be asymptotically stabilizable with a constant control $u(t) \equiv u_{\rm e} \in \overline{\mathbb{R}}^m_+$. However, if $f(x)$ is homogeneous, cooperative; that is, the Jacobian matrix $\frac{\partial f(x)}{\partial x}$ is essentially nonnegative for all $x \in \overline{\mathbb{R}}^n_+$, the

Jacobian matrix $\frac{\partial f(x)}{\partial x}$ is irreducible for all $x \in \overline{\mathbb{R}}_+^n$ [20], and the zero solution $x(t) \equiv 0$ of the undisturbed ($u(t) \equiv 0$) system (12.1) is globally asymptotically stable, then the set point $x_e \in \mathbb{R}_+^n$ satisfying (12.9), (12.10) is a unique equilibrium point with $u(t) \equiv u_e$ and is also asymptotically stable for all $x_0 \in \overline{\mathbb{R}}_+^n$ [53]. This implies that the solution $x(t) \equiv x_e$ to (12.1) with $u(t) \equiv u_e$ is asymptotically stable for all $x_0 \in \overline{\mathbb{R}}_+^n$.

## 12.3. Nonlinear Adaptive Output Feedback Control for General Anesthesia

To illustrate the application of our adaptive control framework we consider a hypothetical model for the intravenous anesthetic propofol. The pharmacokinetics of propofol are described by the three compartment model given in Section 9.6. The model is shown in Figure 9.3 in Chapter 9 and is given by the three-state compartmental system

$$\begin{aligned}\dot{x}_1(t) = &-[a_e(c(t)) + a_{21}(c(t)) + a_{31}(c(t))]x_1(t) + a_{12}(c(t))x_2(t) + a_{13}(c(t))x_3(t) \\ &+u(t), \quad x_1(0) = x_{10}, \quad t \geq 0, \end{aligned} \tag{12.52}$$

$$\dot{x}_2(t) = a_{21}(c(t))x_1(t) - a_{12}(c(t))x_2(t), \quad x_2(0) = x_{20}, \tag{12.53}$$

$$\dot{x}_3(t) = a_{31}(c(t))x_1(t) - a_{13}(c(t))x_3(t), \quad x_3(0) = x_{30}, \tag{12.54}$$

where $c(t) = x_1(t)/V_c$, $V_c$ is the volume of the central compartment, $a_{21}(c)$ is the rate of transfer of drug from the central compartment to Compartment II, $a_{12}(c)$ is the rate of transfer of drug from Compartment II to the central compartment, $a_{31}(c)$ is the rate of transfer of drug from the central compartment to Compartment III, $a_{13}(c)$ is the rate of transfer of drug from Compartment III to the central compartment, $a_e(c)$ is the rate of drug metabolism and elimination (metabolism typically occurs in the liver), and $u(t)$, $t \geq 0$, is the infusion rate of the anesthetic drug propofol into the central compartment. As in Section 9.6, we assume $a_{21}(c) = A_{21}Q(c)$, $a_{12}(c) = A_{12}Q(c)$, $a_{31}(c) = A_{31}Q(c)$, $a_{13}(c) = A_{13}Q(c)$, and $a_e(c) = A_eQ(c)$, where $A_{12}$, $A_{21}$, $A_{13}$, $A_{31}$, and $A_{\mathrm{e}}$ are positive constants. To develop a nonlinear model we assume a sigmoid relationship between drug concentration in the central compartment and effect so that

$$Q(c) = \frac{Q_0 \mathrm{C}_{50}^{\alpha}}{\mathrm{C}_{50}^{\alpha} + c^{\alpha}}, \tag{12.55}$$

where the effect is related to $c$ (since $c$ is the presumed concentration in the highly perfused myocardium), $Q_0 > 0$ is a constant, and $\mathrm{C}_{50} > 0$ is the drug concentration associated with a 50% decrease in the cardiac output, and $\alpha > 1$ determines the steepness of this curve (that is, how rapidly the cardiac output decreases with increasing drug concentration, $c$). Even though the transfer and loss coefficients $A_{12}$, $A_{21}$, $A_{13}$, $A_{31}$, and $A_{\mathrm{e}}$ are nonnegative, and $\alpha > 1$, $\mathrm{C}_{50} > 0$, and $Q_0 > 0$, these parameters can be uncertain due to patient gender, weight, pre-existing disease, age, and concomitant medication. Hence, the need for neuro adaptive control to regulate intravenous anesthetics during surgery is crucial.

Even though propofol concentrations in the blood are known to be correlated with lack of purposeful responsiveness (and presumably consciousness) [137], they cannot be measured in real-time during surgery. Furthermore, we are more interested in drug *effect* (depth of hypnosis) rather than drug *concentration*. Hence, we consider a more realistic model involving pharmacokinetics (drug concentration as a function of time) and pharmacodynamics (drug effect as a function of concentration) for control of anesthesia. Specifically, we use an electroencephalogram (EEG) signal as a measure of drug effect of anesthetic compounds on the brain [67, 174, 215]. Since electroencephalography provides real-time monitoring of the central nervous system activity, it can be used to quantify levels of consciousness and hence is amenable for feedback (closed-loop) control in general anesthesia. As discussed in Chapter 7, a new EEG indicator, the Bispectral Index (BIS), has been proposed as a measure of anesthetic effect [174]. This index quantifies the nonlinear relationships between the component frequencies in the electroencephalogram, as well as analyzing their phase and amplitude. The BIS signal is a nonlinear monotonically decreasing function of the level of consciousness and is given by

$$\mathrm{BIS}(c_{\mathrm{eff}}) = \mathrm{BIS}_0 \left(1 - \frac{c_{\mathrm{eff}}^{\gamma}}{c_{\mathrm{eff}}^{\gamma} + \mathrm{EC}_{50}^{\gamma}}\right), \tag{12.56}$$

where $\mathrm{BIS}_0$ denotes the baseline (awake state) value and, by convention, is typically assigned a value of 100, $c_{\mathrm{eff}}$ is the propofol concentration in micrograms/mililiter in the effect site compartment (brain), $\mathrm{EC}_{50}$ is the concentration at half maximal effect and represents the patient's sensitivity to the drug, and $\gamma$ determines the degree of nonlinearity in (12.56). Here, the effect site compartment is introduced as a correlate between the central compartment concentration and the central nervous system concentration [205]. The effect site compartment concentration is related to the concentration in the central compartment by the first-order delay model

$$\dot{c}_{\mathrm{eff}}(t) = a_{\mathrm{eff}}(c(t) - c_{\mathrm{eff}}(t)), \quad c_{\mathrm{eff}}(0) = c(0), \quad t \geq 0, \tag{12.57}$$

where $a_{\mathrm{eff}}$ in $\mathrm{min}^{-1}$ is an unknown positive time constant. In reality, the effect site compartment equilibrates with the central compartment in a matter of a few minutes. The parameters $a_{\mathrm{eff}}$, $\mathrm{EC}_{50}$, and $\gamma$ are determined by data fitting and vary from patient to patient. BIS index values of 0 and 100 correspond, respectively, to an isoelectric EEG signal and an EEG signal of a fully conscious patient; while the range between 40 and 60 indicates a moderate hypnotic state [215]. Figure 12.2 shows the combined pharmacokinetic/pharmacodynamic model for propofol distribution.

For set-point regulation define $e(t) \triangleq x(t) - x_{\mathrm{e}}$, where $x_{\mathrm{e}} \in \mathbb{R}^4$ is the set point satisfying the equilibrium condition for (12.52)–(12.54) and (12.57) with $x_1(t) \equiv x_{\mathrm{e}1}$, $x_2(t) \equiv x_{\mathrm{e}2}$, $x_3(t) \equiv x_{\mathrm{e}3}$, $c_{\mathrm{eff}} \equiv \mathrm{EC}_{50}$, and $u(t) \equiv u_{\mathrm{e}}$, so that $f_{\mathrm{e}}(e) = [f_{\mathrm{e}1}(e), f_{\mathrm{e}2}(e),$

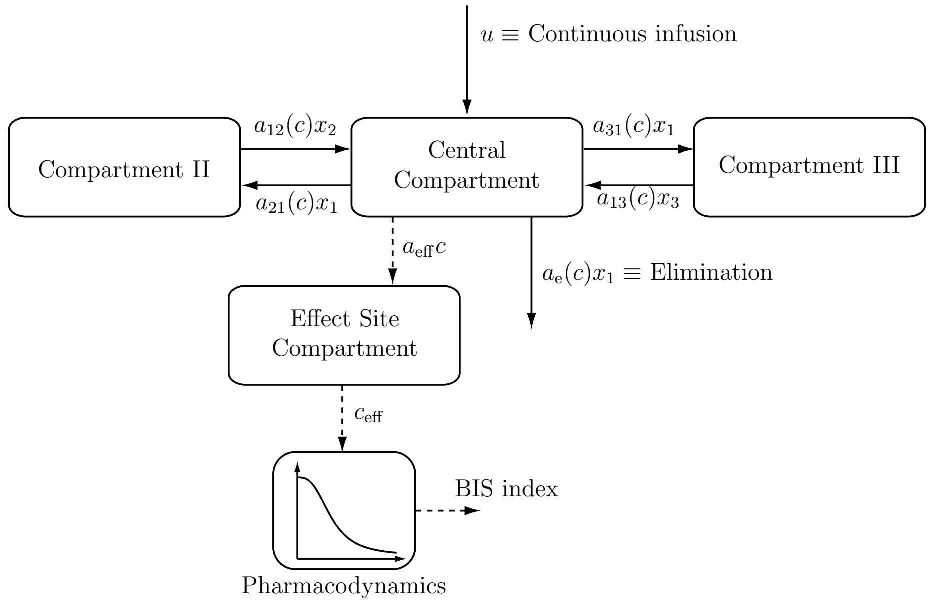



**Figure 12.2**: Combined pharmacokinetic/pharmacodynamic model

$f_{\rm e3}(e), f_{\rm e4}(e)]^{\rm T}$ is given by

$$\begin{aligned} f_{\rm e1}(e) &= -[a_{\rm e}(c) + a_{21}(c) + a_{31}(c)](e_1 + x_{\rm e1}) + a_{12}(c)(e_2 + x_{\rm e2}) + a_{13}(c)(e_3 + x_{\rm e3}) \\ &\quad -[a_{\rm e}(c_{\rm e}) + a_{21}(c_{\rm e}) + a_{31}(c_{\rm e})]x_{\rm e1} + a_{12}(c_{\rm e})x_{\rm e2} + a_{13}(c_{\rm e})x_{\rm e3}, \qquad (12.58) \\ f_{\rm e2}(e) &= a_{21}(c)(e_1 + x_{\rm e1}) - a_{12}(c)(e_2 + x_{\rm e2}) - [a_{21}(c_{\rm e})x_{\rm e1} - a_{12}(c_{\rm e})x_{\rm e2}], \qquad (12.59) \\ f_{\rm e3}(e) &= a_{31}(c)(e_1 + x_{\rm e1}) - a_{13}(c)(e_3 + x_{\rm e3}) - [a_{31}(c_{\rm e})x_{\rm e1} - a_{13}(c_{\rm e})x_{\rm e3}], \qquad (12.60) \\ f_{\rm e4}(e) &= a_{\rm eff}(c - (e_4 + {\rm EC}_{50})) - a_{\rm eff}(e_{\rm e} - {\rm EC}_{50}), \qquad (12.61) \end{aligned}$$

where $c_{\rm e} \triangleq x_{\rm e1}/V_{\rm c}$. Next, linearizing $f_{\rm e}(e)$ about 0 and computing the eigenvalues of the resulting (compartmental) Jacobian matrix, it can be shown that $x_{\rm e}$ is asymptotically stable.

In the following numerical simulation we set ${\rm EC}_{50} = 5.6$ $\mu$g/m$\ell$, $\gamma = 2.39$, and ${\rm BIS}_0 = 100$, so that the BIS signal is shown in Figure 12.3. The target (desired) BIS value, ${\rm BIS}_{\rm target}$, is set at 50. Now, using the adaptive output feedback controller

$$u_1(t) = \max\{0, \hat{u}_1(t)\}, \qquad (12.62)$$

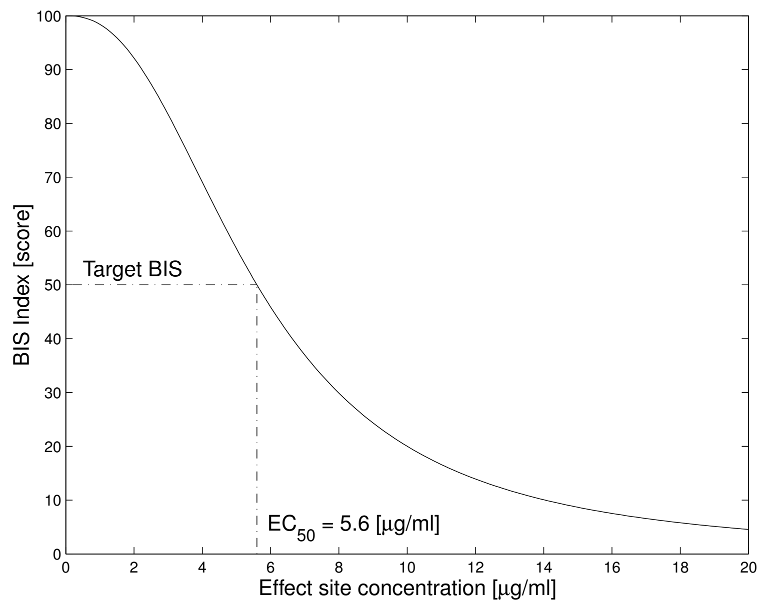



**Figure 12.3**: BIS index versus effect site concentration

where

$$\hat{u}_1(t) = -\frac{\hat{W}_1^{\mathrm{T}}(t)\sigma_1(\zeta(t))}{b_{\mathrm{s}} + \hat{W}_2^{\mathrm{T}}(t)\sigma_2(\zeta(t))}, \tag{12.63}$$

$$\zeta(t) = [\mathrm{BIS}(t), \mathrm{BIS}(t-d), u_1(t), u_1(t-d)]^{\mathrm{T}}, \tag{12.64}$$

$b_{\mathrm{s}} > 0$, with update laws

$$\dot{\hat{W}}_1(t) = Q_{\mathrm{BIS}_1}\mathrm{Proj}(\hat{W}_1(t), \sigma_1(\zeta(t))\xi_{\mathrm{c}}^{\mathrm{T}}(t)\tilde{P}B_0), \quad \hat{W}_1(0) = \hat{W}_{10}, \tag{12.65}$$

$$\dot{\hat{W}}_2(t) = Q_{\mathrm{BIS}_2}\mathrm{Proj}(\hat{W}_2(t), \sigma_2(\zeta(t))u(t)\xi_{\mathrm{c}}^{\mathrm{T}}(t)\tilde{P}B_0), \quad \hat{W}_2(0) = \hat{W}_{20}, \tag{12.66}$$

where $Q_{\mathrm{BIS}_1}$ and $Q_{\mathrm{BIS}_2}$ are arbitrary positive scalars and $\xi_{\mathrm{c}}(t) \in \mathbb{R}^2$, $t \geq 0$, is the solution to the estimator dynamics

$$\dot{\xi}_{\mathrm{c}}(t) = A\xi_{\mathrm{c}}(t) + L(-\mathrm{BIS}(t) - y_{\mathrm{c}}(t) + \mathrm{BIS}_{\mathrm{target}}), \quad \xi_{\mathrm{c}}(0) = \xi_{\mathrm{c}0}, \quad t \geq 0, \tag{12.67}$$

$$y_{\mathrm{c}}(t) = \xi_{\mathrm{c}}(t), \tag{12.68}$$

where $A \in \mathbb{R}^{2\times 2}$ and $L \in \mathbb{R}^{2\times 1}$, it follows from Theorem 12.2 that there exist positive constants $\varepsilon$ and $T$ such that $|\mathrm{BIS}(t) - \mathrm{BIS}_{\mathrm{target}}| \leq \varepsilon$, $t \geq T$, for any (uncertain) positive values of the pharmacokinetic transfer and loss coefficients $(A_{12}, A_{21}, A_{13}, A_{31}, A_{\mathrm{e}})$

as well as any (uncertain) nonnegative coefficients $\alpha$, $\mathrm{C}_{50}$, and $Q_0$. It is important to note that during actual surgery the BIS signal is obtained directly from the EEG and not (12.56). Furthermore, since our adaptive controller only requires the error signal $\mathrm{BIS}(c_{\mathrm{eff}}(t)) - \mathrm{BIS}_{\mathrm{target}}$, we do not require knowledge of the pharmacodynamic parameters $\gamma$ and $\mathrm{EC}_{50}$. For our simulation we assume $V_{\mathrm{c}} = (0.228\ \ell/\mathrm{kg})(M\ \mathrm{kg})$, where $M = 70$ kg is the weight (mass) of the patient, $A_{21}Q_0 = 0.112\ \mathrm{min}^{-1}$, $A_{12}Q_0 = 0.055\ \mathrm{min}^{-1}$, $A_{31}Q_0 = 0.0419\ \mathrm{min}^{-1}$, $A_{13}Q_0 = 0.0033\ \mathrm{min}^{-1}$, $A_{\mathrm{e}}Q_0 = 0.119\ \mathrm{min}^{-1}$, $\alpha = 3$, and $\mathrm{C}_{50} = 4\ \mu\mathrm{g/m}\ell$ [169]. Note that the parameter values for $\alpha$ and $\mathrm{C}_{50}$ probably exaggerate the effect of propofol on cardiac output. They have been selected to accentuate nonlinearity but they are not biologically unrealistic. Furthermore, to illustrate the robustness of the proposed adaptive controller we switch the pharmacodynamic parameters $\mathrm{EC}_{50}$ and $\gamma$, respectively, from $5.6\ \mu\mathrm{g/m}\ell$ and 2.39 to $7.2\ \mu\mathrm{g/m}\ell$ and 3.39 at $t = 15$ min and back to $5.6\ \mu\mathrm{g/m}\ell$ and 2.39 at $t = 30$ min. Here, we consider noncardiac surgery since cardiac surgery often utilizes hypothermia which itself changes the BIS signal. With $A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, $L = [0, 1]^{\mathrm{T}}$, $b_{\mathrm{s}} = 1$, $Q_{\mathrm{BIS}_1} = Q_{\mathrm{BIS}_2} = 8.0 \times 10^{-5}\ \mathrm{g/min}^2$, $d = 0.005$, and initial conditions $x(0) = [0,\ 0,\ 0]^{\mathrm{T}}$ g, $c_{\mathrm{eff}}(0) = 0\ \mathrm{g/m}\ell$, $\xi_{\mathrm{c}}(0) = [0,\ 0]^{\mathrm{T}}$, $\hat{W}_1(0) = 0_{24\times 1}$ g/min, and $\hat{W}_2(0) = 0_{24\times 1}$, Figure 12.4 shows the masses of propofol in the three compartments versus time. Figure 12.5 shows the concentrations in the central and effect site compartments versus time. Figure 12.6 shows the compensator states versus time. Finally, Figure 12.7 shows the BIS index and the control signal (propofol infusion rate) versus time.

## 12.4. Conclusion

Nonnegative and compartmental systems are widely used to capture system dynamics involving the interchange of mass and energy between homogenous subsystems

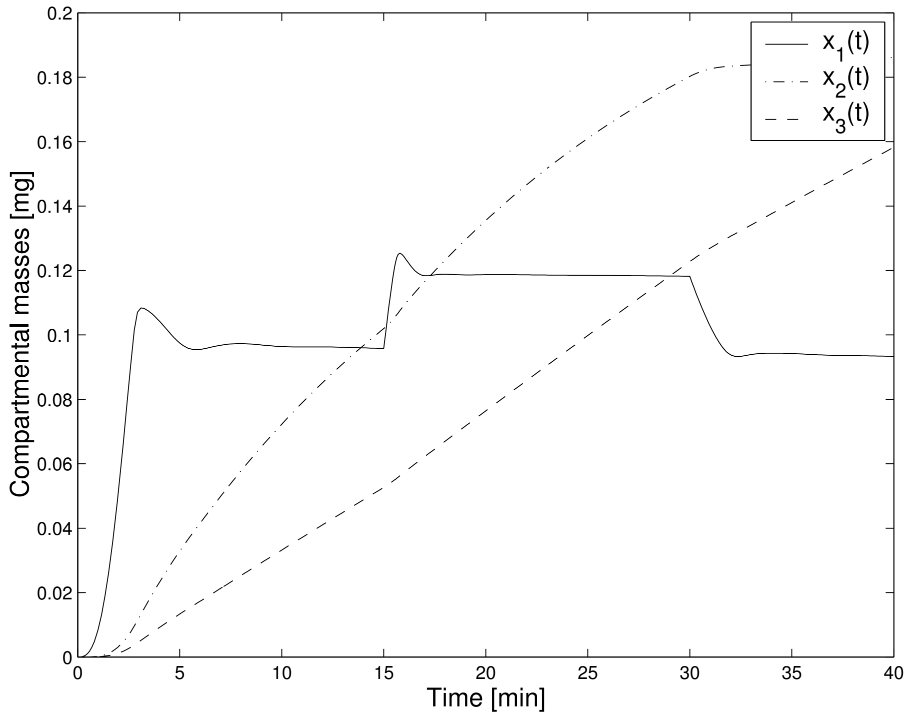



**Figure 12.4**: Compartmental masses versus time

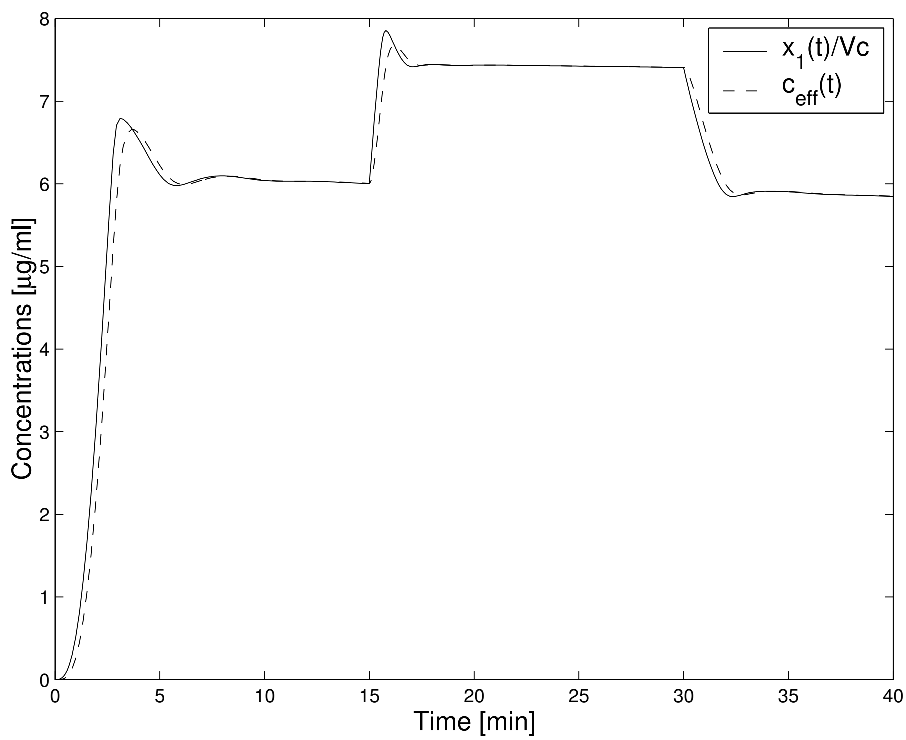



**Figure 12.5**: Concentrations in the central and effect site compartments versus time

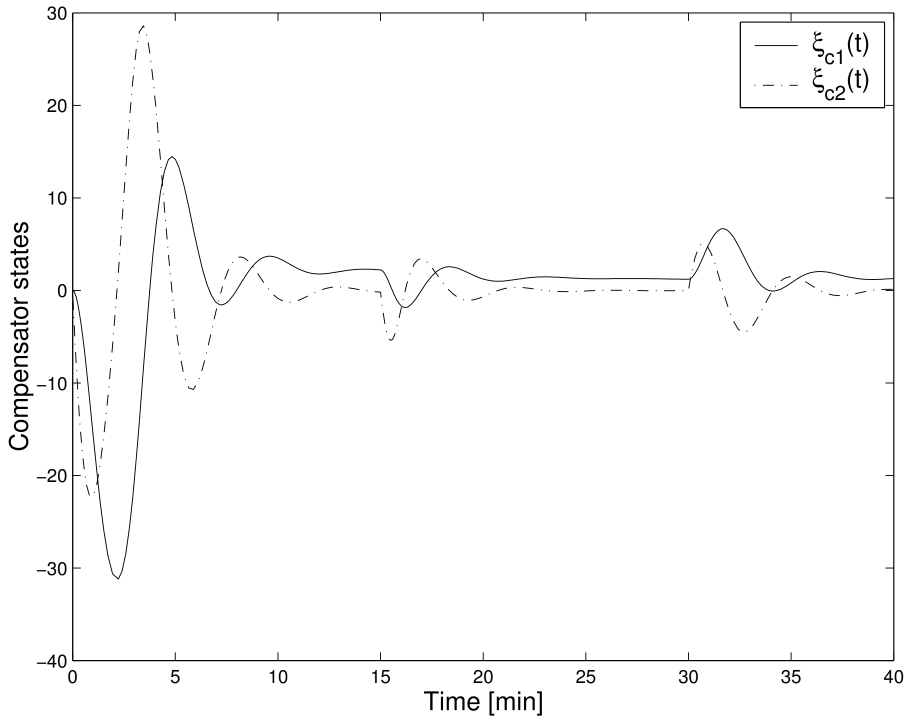



**Figure 12.6**: Compensator states versus time

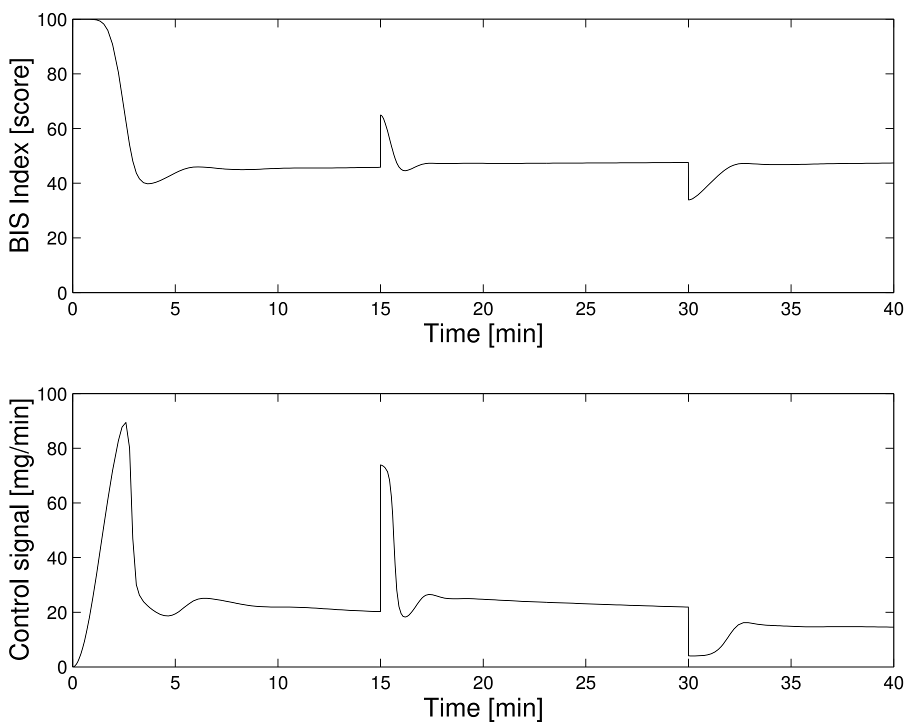



**Figure 12.7**: BIS index versus time and control signal (infusion rate) versus time

or compartments. Thus, it is not surprising that nonnegative and compartmental models are remarkably effective in describing the dynamical behavior of biological systems, physiological systems, and pharmacological systems. In this chapter, we developed a neural adaptive dynamic output feedback control framework for adaptive set-point regulation of nonlinear uncertain nonnegative and compartmental systems. Using Lyapunov methods the proposed framework was shown to guarantee ultimate boundedness of the error signals corresponding to the physical system states and the neural network weighting gains while additionally guaranteeing the nonnegativity of the closed-loop system states associated with the plant dynamics. Finally, using a nonlinear four-compartment pharmacokinetic/pharmacodynamic patient model for the disposition of anesthetic drug propofol, the proposed neuro adaptive control framework was used to monitor and control a desired constant level of consciousness for noncardiac surgery.

# Chapter 13

# A Lyapunov-Based Adaptive Control Framework for Discrete-Time Nonlinear Systems with Exogenous Disturbances

## 13.1. Introduction

The purpose of feedback control is to achieve desirable system performance in the face of system uncertainty and system disturbances. Although system identification can reduce uncertainty to some extent, residual modeling discrepancies always remain. Controllers must therefore be robust to achieve desired disturbance rejection and/or tracking performance requirements in the presence of such modeling uncertainty. To this end, adaptive control along with robust control theory have been developed to address the problem of system performance in the face of system uncertainty in control-system design without excessive reliance on system models.

Adaptive controllers directly or indirectly adjust feedback gains to maintain closed-loop stability and improve performance in the face of system errors. Specifically, indirect adaptive controllers utilize parameter update laws to estimate unknown system parameters and adjust feedback gains to account for system variation, while direct adaptive controllers directly adapt the controller gains in response to system variations. Even though adaptive control algorithms have been developed in the literature for both continuous-time and discrete-time systems, the majority of the discrete-time results are based on recursive least squares and least mean squares algorithms [56, 61, 71, 72, 177] with primary focus on state convergence. Alternatively, Lyapunov-based adaptive controllers have been developed for continuous-time systems guaranteeing asymptotic stability of the system states (see for example [136, 147, 176]). Notable Lyapunov-based adaptive control algorithms for discrete-time systems are given in [128, 196, 230, 242]. However, the literature on discrete-time adaptive disturbance rejection control using Lyapunov methods is virtually nonexistent.

For discrete-time dynamical systems, Lyapunov-based frameworks for adaptive control are quite intricate since the Lyapunov difference does not remove terms involving the model reference stabilizing gain from the resulting Lyapunov difference expression. This leads to asymptotic nonpositivity of the Lyapunov difference and thus Lyapunov stability cannot be guaranteed [230]. This difficulty was first pointed out by [132] and is the main reason why Lyapunov-based discrete-time adaptive control is *not* a straightforward extension of continuous-time adaptive control theory. As a result, most of the discrete-time adaptive model reference and tracking control results are based on the classical key technical lemma which does not guarantee Lyapunov stability.

In this paper, using a logarithmic Lyapunov function we develop a Lyapunov-based direct adaptive control framework for adaptive stabilization, disturbance rejection, and command following of multivariable discrete-time nonlinear uncertain systems with exogenous bounded amplitude disturbances and $\ell_2$ disturbances. These results are analogous to, but by no means a direct extension of, the recent continuous-time adaptive disturbance rejection results in [84] for continuous-time nonlinear uncertain systems. In contrast to the results presented in [84], logarithmic Lyapunov functions are shown to be essential for discrete-time Lyapunov-based adaptive control. Specifically, a logarithmic Lyapunov-based direct adaptive control framework is developed that guarantees partial asymptotic stability of the closed-loop system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the plant. Furthermore, in the case where the nonlinear system is represented in normal form, the nonlinear discrete-time adaptive controller is constructed *without* requiring knowledge of the system dynamics or system disturbances. In the case where the system disturbances are $\ell_2$ disturbances, the proposed framework guarantees that the closed-loop nonlinear input-output map from uncertain exogenous $\ell_2$ disturbances to system performance variables is nonexpansive and the solution of the closed-loop system is partially asymptotically stable. The proposed adaptive controller thus addresses the problem of disturbance rejection for nonlinear uncertain discrete-time systems with bounded energy (square-summable) $\ell_2$ signal norms on the disturbances and performance variables.

The contents of the chapter are as follows. In Section 13.2 we present our main direct adaptive control framework for adaptive stabilization, disturbance rejection, and command following of multivariable nonlinear uncertain discrete-time systems with matched exogenous bounded disturbances. In Section 13.3 we extend the results of Section 13.2 to nonlinear uncertain discrete-time systems with exogenous $\ell_2$ disturbances without a matching condition requirement. Three illustrative numerical examples are presented in Section 13.4 to demonstrate the efficacy of the proposed direct adaptive stabilization and tracking framework. Finally, in Section 13.5 we draw some conclusions.

## 13.2. Discrete-Time Adaptive Control for Nonlinear Systems with Exogenous Disturbances

In this section we consider the problem of characterizing adaptive feedback control laws for nonlinear uncertain discrete-time systems with exogenous disturbances. Specifically, consider the controlled nonlinear uncertain discrete-time system $\mathcal{G}$ given by

$$x(k+1) = f(x(k)) + G(x(k))u(k) + J(x(k))w(k), \quad x(0) = x_0, \quad k \in \mathcal{N}, \tag{13.1}$$

where $x(k) \in \mathbb{R}^n$, $k \in \mathcal{N}$, is the state vector, $u(k) \in \mathbb{R}^m$, $k \in \mathcal{N}$, is the control input, $w(k) \in \mathbb{R}^d$, $k \in \mathcal{N}$, is a known bounded disturbance vector such that $\|w(k)\|_2 \leq \delta$, $k \in \mathcal{N}$, $f : \mathbb{R}^n \to \mathbb{R}^n$ and satisfies $f(0) = 0$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is such that $\operatorname{rank} G(x) = m$, $x \in \mathbb{R}^n$, and $J : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ is a disturbance weighting matrix function with *unknown* entries. Note that even though $w(k)$, $k \in \mathcal{N}$, is assumed to be known, the disturbance signal $J(x(k))w(k)$, $k \in \mathcal{N}$, is an *unknown* bounded disturbance. The control input $u(\cdot)$ in (13.1) is restricted to the class of *admissible controls* consisting of measurable functions such that $u(k) \in \mathbb{R}^m$, $k \in \mathcal{N}$.

**Theorem 13.1.** Consider the nonlinear system $\mathcal{G}$ given by (13.1). Assume there exist a matrix $K_{\rm g} \in \mathbb{R}^{m\times s}$, functions $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, $F : \mathbb{R}^n \to \mathbb{R}^s$, $P_{1u} : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, $\ell : \mathbb{R}^n \to \mathbb{R}^t$, a nonnegative-definite matrix function $P_{2u} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, and positive constants $\bar{\gamma}$, $\varepsilon$, $\mu$, and $\nu$ such that $V_{\rm s}(\cdot)$ and $\ell(\cdot)$ are continuous, $V_{\rm s}(0) = 0$, $\ell(0) = 0$, $\det \hat{G}(x) \neq 0$, $x \in \mathbb{R}^n$, $\hat{G}^{\rm T}(x)P_{2u}(x)\hat{G}(x) \leq \nu I_m$, $x \in \mathbb{R}^n$, and, for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$,

$$V_{\rm s}(f(x) + G(x)u) = V_{\rm s}(f(x)) + P_{1u}(x)u + u^{\rm T}P_{2u}(x)u, \tag{13.2}$$

$$0 \geq V_{\rm s}(f_{\rm s}(x)) - V_{\rm s}(x) + \ell^{\rm T}(x)\ell(x) + \varepsilon P_{1u}(x)\hat{G}(x)\hat{G}^{\rm T}(x)P_{1u}^{\rm T}(x), \tag{13.3}$$

$$F^{\rm T}(x)F(x) \leq \bar{\gamma}x^{\rm T}x, \quad x \in \mathbb{R}^n, \tag{13.4}$$

$$V_{\rm s}(x) \geq \mu x^{\rm T} x, \tag{13.5}$$

where

$$f_{\rm s}(x) \triangleq f(x) + G(x)\hat{G}(x)K_{\rm g}F(x). \tag{13.6}$$

Furthermore, assume there exists a matrix $\Psi \in \mathbb{R}^{m\times d}$ such that $G(x)\hat{G}(x)\Psi = J(x)$. Finally, let $\tilde{x}(k) \triangleq [F^{\rm T}(x(k)), w^{\rm T}(k)]^{\rm T}$, $c > 0$, and $Q \in \mathbb{R}^{m\times m}$ be positive definite such that $\lambda_{\max}(Q) < 2$. Then the adaptive feedback control law

$$u(k) = \hat{G}(x(k))K(k)\tilde{x}(k), \tag{13.7}$$

where $K(k) \in \mathbb{R}^{m\times(s+d)}$, $k \in \mathcal{N}$, with update law

$$K(k+1) = K(k) - \tfrac{1}{c+\tilde{x}^{\rm T}(k)\tilde{x}(k)} Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k))[x(k+1) - f_{\rm s}(x(k))]\tilde{x}^{\rm T}(k), \tag{13.8}$$

guarantees that the solution $(x(k), K(k)) \equiv (0, [K_{\rm g}, -\Psi])$ of the closed-loop system given by (13.1), (13.7), and (13.8) is Lyapunov stable and $\ell(x(k)) \to 0$ as $k \to \infty$. If, in addition, $\ell^{\rm T}(x)\ell(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, then $x(k) \to 0$ as $k \to \infty$ for all $x_0 \in \mathbb{R}^n$.

**Proof.** First, define $\tilde{K}(k) \triangleq K(k) - \hat{K}_{\rm g}$ and $\tilde{u}(k) \triangleq \tilde{K}(k)\tilde{x}(k)$, where $\hat{K}_{\rm g} \triangleq [K_{\rm g}, -\Psi]$. Note that with $u(k)$, $k \in \mathcal{N}$, given by (13.7) it follows from (13.1) that

$$x(k+1) = f(x(k)) + G(x(k))\hat{G}(x(k))K(k)\tilde{x}(k) + J(x(k))w(k), \quad x(0) = x_0, \quad k \in \mathcal{N}, \tag{13.9}$$

or, equivalently, using (13.6) and the fact that $G(x)\hat{G}(x)\Psi = J(x)$,

$$\begin{aligned} x(k+1) &= f_{\rm s}(x(k)) + G(x(k))\hat{G}(x(k))\tilde{K}(k)\tilde{x}(k) \\ &= f_{\rm s}(x(k)) + G(x(k))\hat{G}(x(k))\tilde{u}(k), \quad x(0) = x_0, \quad k \in \mathcal{N}. \end{aligned} \tag{13.10}$$

Furthermore, note that adding and subtracting $\hat{K}_{\rm g}$ to and from (13.8) and using (13.10) it follows that

$$\begin{aligned} \tilde{K}(k+1) &= \tilde{K}(k) - \tfrac{1}{c+\tilde{x}^{\rm T}(k)\tilde{x}(k)} Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k))[G(x(k))\hat{G}(x(k))\tilde{K}(k)\tilde{x}(k)]\tilde{x}^{\rm T}(k) \\ &= \tilde{K}(k) - \tfrac{1}{c+\tilde{x}^{\rm T}(k)\tilde{x}(k)} Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\rm T}(k). \end{aligned} \tag{13.11}$$

To show Lyapunov stability of the closed-loop system (13.10) and (13.11), consider the Lyapunov function candidate

$$V(x,K)=\ln(1+V_{\rm s}(x))+a{\rm tr}(K-\hat{K}_{\rm g})^{\rm T}Q^{-1}(K-\hat{K}_{\rm g}), \tag{13.12}$$

where

$$a\geq\frac{\frac{1}{4\varepsilon}+\nu}{\lambda_{\min}(2I-Q)}\cdot\max\left\{\delta^2+c,\frac{\bar{\gamma}}{\mu}\right\}. \tag{13.13}$$

Note that $V(0,\hat{K}_{\rm g})=0$ and, since $V_{\rm s}(\cdot)$ and $Q$ are positive definite and $a>0$, $V(x,K)>0$ for all $(x,K)\neq(0,\hat{K}_{\rm g})$. Furthermore, $V(x,K)$ is radially unbounded. Now, letting $x(k)$, $k\in\mathcal{N}$, denote the solution to (13.10) and using (13.2), (13.3), and (13.11), it follows that the Lyapunov difference along the closed-loop system trajectories is given by

$$\begin{aligned}
\Delta V(x(k),K(k)) &\triangleq V(x(k+1),K(k+1))-V(x(k),K(k))\\
&= \ln\Big(1+V_{\rm s}(f_{\rm s}(x(k))+G(x(k))\hat{G}(x(k))\tilde{u}(k))\Big)\\
&\quad +a{\rm tr}\left(\tilde{K}(k)-\tfrac{1}{c+\tilde{x}^{\rm T}(k)\tilde{x}(k)}Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\rm T}(k)\right)^{\rm T}Q^{-1}\\
&\quad \cdot\left(\tilde{K}(k)-\tfrac{1}{c+\tilde{x}^{\rm T}(k)\tilde{x}(k)}Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\rm T}(k)\right)-\ln(1+V_{\rm s}(x(k)))\\
&\quad -a{\rm tr}\tilde{K}^{\rm T}(k)Q^{-1}\tilde{K}(k)\\
&= \ln\Big(1+\Big[V_{\rm s}(f_{\rm s}(x(k)))+P_{1u}(x(k))\hat{G}(x(k))\tilde{u}(k)\\
&\quad +\tilde{u}^{\rm T}(k)\hat{G}^{\rm T}(x(k))P_{2u}(x(k))\hat{G}(x(k))\tilde{u}(k)-V_{\rm s}(x(k))\Big]\\
&\quad \cdot[1+V_{\rm s}(x(k))]^{-1}\Big)\\
&\quad +a{\rm tr}\,\tilde{K}^{\rm T}(k)Q^{-1}\tilde{K}(k)-\tfrac{2a}{c+\tilde{x}^{\rm T}(k)\tilde{x}(k)}{\rm tr}\,\tilde{K}^{\rm T}(k)\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\rm T}(k)\\
&\quad +\tfrac{a}{(c+\tilde{x}^{\rm T}(k)\tilde{x}(k))^2}{\rm tr}\,\tilde{x}(k)\tilde{x}^{\rm T}(k)\tilde{K}^{\rm T}(k)Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\rm T}(k)\\
&\quad -a{\rm tr}\tilde{K}^{\rm T}(k)Q^{-1}\tilde{K}(k)\\
&\leq \Big[-\ell^{\rm T}(x(k))\ell(x(k))-\varepsilon P_{1u}(x(k))\hat{G}(x(k))\hat{G}^{\rm T}(x(k))P_{1u}^{\rm T}(x(k))\\
&\quad +P_{1u}(x(k))\hat{G}(x(k))\tilde{u}(k)+\nu\tilde{u}^{\rm T}(k)\tilde{u}(k)\Big]\,[1+V_{\rm s}(x(k))]^{-1}
\end{aligned}$$

$$-\tfrac{2a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}\tilde{x}^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)\tilde{K}(k)\tilde{x}(k)$$
$$+\tfrac{a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}\tilde{x}^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)Q\tilde{K}(k)\tilde{x}(k), \qquad (13.14)$$

where in (13.14) we used $\ln a - \ln b = \ln \frac{a}{b}$ and $\ln(1+c) \le c$ for $a, b > 0$ and $c \ge -1$, respectively, and $\frac{\tilde{x}^{\mathrm{T}}\tilde{x}}{c+\tilde{x}^{\mathrm{T}}\tilde{x}} < 1$. Now, adding and subtracting $\frac{1}{4\varepsilon}\frac{\tilde{u}^{\mathrm{T}}(k)\tilde{u}(k)}{1+V_{\mathrm{s}}(x(k))}$ to and from (13.14) and collecting terms yields

$$\begin{aligned}
&\Delta V(x(k), K(k)) \\
&\quad \le -\tfrac{1}{1+V_{\mathrm{s}}(x(k))}\ell^{\mathrm{T}}(x(k))\ell(x(k)) \\
&\qquad -\tfrac{1}{1+V_{\mathrm{s}}(x(k))}\left[P_{1u}(x(k)),\ \tilde{u}^{\mathrm{T}}(k)\right]\begin{bmatrix} \varepsilon\hat{G}(x(k))\hat{G}^{\mathrm{T}}(x(k)) & -\frac{1}{2}\hat{G}(x(k)) \\ -\frac{1}{2}\hat{G}^{\mathrm{T}}(x(k)) & \frac{1}{4\varepsilon}I_m \end{bmatrix}\begin{bmatrix} P_{1u}^{\mathrm{T}}(x(k)) \\ \tilde{u}(k) \end{bmatrix} \\
&\qquad +\tfrac{1}{1+V_{\mathrm{s}}(x(k))}\left[\tfrac{1}{4\varepsilon}\tilde{u}^{\mathrm{T}}(k)\tilde{u}(k) + \nu\tilde{u}^{\mathrm{T}}(k)\tilde{u}(k)\right] \\
&\qquad -\tfrac{a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}\tilde{x}^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)(2I_m - Q)\tilde{K}(k)\tilde{x}(k) \\
&\quad \le -\frac{\ell^{\mathrm{T}}(x(k))\ell(x(k))}{1+V_{\mathrm{s}}(x(k))} - \frac{\tilde{x}^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)\tilde{R}(x(k), w(k))\tilde{K}(k)\tilde{x}(k)}{(c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k))(1+V_{\mathrm{s}}(x(k)))}, \quad k \in \mathcal{N}, \qquad (13.15)
\end{aligned}$$

where

$$\tilde{R}(x, w) \triangleq a(1+V_{\mathrm{s}}(x))(2I_m - Q) - \left(\frac{1}{4\varepsilon} + \nu\right)(c + \tilde{x}^{\mathrm{T}}\tilde{x})I_m. \qquad (13.16)$$

Noting that $2I_m - Q > 0$, since by assumption $\lambda_{\max}(Q) < 2$, and $a$ satisfies (13.13), it follows that

$$\begin{aligned}
\tilde{R}(x, w) &\ge a(1+\mu x^{\mathrm{T}}x)(2I_m - Q) - \left(\frac{1}{4\varepsilon} + \nu\right)(\delta^2 + c + F^{\mathrm{T}}(x)F(x))I_m \\
&\ge a(1+\mu x^{\mathrm{T}}x)(2I_m - Q) - \left(\frac{1}{4\varepsilon} + \nu\right)\left(\delta^2 + c + \bar{\gamma}x^{\mathrm{T}}x\right)I_m \\
&\ge 0, \quad (x, w) \in \mathbb{R}^n \times \mathbb{R}^d. \qquad (13.17)
\end{aligned}$$

Hence, the Lyapunov difference given by (13.15) yields

$$\begin{aligned}
\Delta V(x(k), K(k)) &\le -\frac{\ell^{\mathrm{T}}(x(k))\ell(x(k))}{1+V_{\mathrm{s}}(x(k))} - \frac{\tilde{x}^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)\tilde{R}(x(k), w(k))\tilde{K}(k)\tilde{x}(k)}{\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)(1+V_{\mathrm{s}}(x(k)))} \\
&\le 0, \quad k \in \mathcal{N}, \qquad (13.18)
\end{aligned}$$

which proves that the solution $(x(k), K(k)) \equiv (0, \hat{K}_{\rm g})$ to (13.10) and (13.11) is Lyapunov stable. Furthermore, it follows from (the discrete-time version of) Theorem 2 of [42] that $\ell(x(k)) \to 0$ as $k \to \infty$. Finally, if $\ell^{\rm T}(x)\ell(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, then $x(k) \to 0$ as $k \to \infty$ for all $x_0 \in \mathbb{R}^n$. □

**Remark 13.1.** Note that in the case where $\ell^{\rm T}(x)\ell(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, the conditions in Theorem 13.1 imply that $x(k) \to 0$ as $k \to \infty$ and hence it follows from (13.8) that $(x(k), K(k)) \to \mathcal{M} \triangleq \{(x, K) \in \mathbb{R}^n \times \mathbb{R}^{m\times(s+d)} : x = 0 \text{ and } K(k+1) = K(k)\}$ as $k \to \infty$.

**Remark 13.2.** Theorem 13.1 is also valid for *time-varying* uncertain dynamical systems $\mathcal{G}_{\rm k}$ of the form

$$x(k+1) = f(k, x(k)) + G(k, x(k))u(k) + J(k, x(k))w(k), \quad x(0) = x_0, \quad k \in \mathcal{N}, \tag{13.19}$$

where $f : \mathcal{N} \times \mathbb{R}^n \to \mathbb{R}^n$ and satisfies $f(k, 0) = 0$, $k \in \mathcal{N}$, $G : \mathcal{N} \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $J : \mathcal{N} \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$. In particular, replacing $F : \mathbb{R}^n \to \mathbb{R}^s$ by $F : \mathcal{N} \times \mathbb{R}^n \to \mathbb{R}^s$ and $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ by $\hat{G} : \mathcal{N} \times \mathbb{R}^n \to \mathbb{R}^{m\times m}$, and requiring $F^{\rm T}(k, x)F(k, x) \leq \bar{\gamma}x^{\rm T}x$, $\bar{\gamma} > 0$, $k \in \mathcal{N}$, in place of (13.4) and $G(k, x)\hat{G}(k, x)\Psi = J(k, x)$ in place of $G(x)\hat{G}(x)\Psi = J(x)$, it follows by using identical arguments as in the proof of Theorem 13.1 that the adaptive feedback control law

$$u(k) = \hat{G}(k, x(k))K(k)\tilde{x}(k), \tag{13.20}$$

where $\tilde{x}(k) \triangleq [F^{\rm T}(k, x(k)),\ w^{\rm T}(k)]^{\rm T}$, with update law

$$K(k+1) = K(k) - \tfrac{1}{c+\tilde{x}^{\rm T}(k)\tilde{x}(k)} Q\hat{G}^{-1}(k, x(k))G^{\dagger}(k, x(k))[x(k+1) - f_{\rm s}(x(k))]\tilde{x}^{\rm T}(k), \tag{13.21}$$

where $f_{\rm s}(x) = f(k, x) + G(k, x)\hat{G}(k, x)K_{\rm g}F(k, x)$, guarantees that the solution $(x(k), K(k)) \equiv (0, [K_{\rm g}, -\Psi])$ of the closed-loop system (13.19)–(13.21) is Lyapunov stable and $x(k) \to 0$ as $k \to \infty$ for all $x_0 \in \mathbb{R}^n$.

**Remark 13.3.** It follows from Remark 13.2 that Theorem 13.1 can also be used to construct adaptive tracking controllers for nonlinear uncertain dynamical systems. Specifically, let $r_{\rm d}(k) \in \mathbb{R}^n$, $k \in \mathcal{N}$, denote a command input and define the error state $e(k) \triangleq x(k) - r_{\rm d}(k)$. In this case, the error dynamics are given by

$$e(k+1) = f_{\rm k}(k, e(k)) + G_{\rm k}(k, e(k))u(k) + J_{\rm k}(k, e(k))w_{\rm k}(k),$$
$$e(0) = e_0, \quad k \in \mathcal{N}, \quad (13.22)$$

where $f_{\rm k}(k, e(k)) = f(e(k) + r_{\rm d}(k)) - n(k)$, with $f(r_{\rm d}(k)) = n(k)$, $G_{\rm k}(k, e(k)) = G(e(k) + r_{\rm d}(k))$, and $J_{\rm k}(k, e(k))w_{\rm k}(k) = n(k) - r_{\rm d}(k+1) + J(e(k) + r_{\rm d}(k))w(k)$. Now, the adaptive tracking control law (13.20) and (13.21), with $x(k)$ replaced by $e(k)$, guarantees that $e(k) \to 0$ as $k \to \infty$ for all $e_0 \in \mathbb{R}^n$.

It is important to note that the adaptive control law (13.7) and (13.8) does *not* require explicit knowledge of the gain matrix $K_{\rm g}$, the disturbance matching matrix $\Psi$, the disturbance weighting matrix function $J(x)$, and the positive constants $\nu$, $\bar{\gamma}$, $\varepsilon$, and $\mu$; even though Theorem 13.1 requires the existence of $K_{\rm g}$ and $\Psi$ along with the construction of $F(x)$, $\hat{G}(x)$, and $V_{\rm s}(x)$ such that $G(x)\hat{G}(x)\Psi = J(x)$ and (13.2)–(13.5) hold. Furthermore, if (13.1) is in normal form with asymptotically stable internal dynamics [122] and if the linear growth condition $f^{\rm T}(x)f(x) \leq \hat{\gamma}x^{\rm T}x$, $x \in \mathbb{R}^n$, $\hat{\gamma} > 0$, holds, then we can always construct functions $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, $F : \mathbb{R}^n \to \mathbb{R}^s$, and $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ such that (13.2)–(13.5) hold *without* requiring knowledge of the system dynamics. For simplicity of exposition in the ensuing discussion we assume that $J(x) = D$, where $D \in \mathbb{R}^{n\times d}$ is a disturbance weighting matrix with unknown entries.

To elucidate the above discussion assume that the nonlinear uncertain system $\mathcal{G}$ is generated by the difference model

$$z_i(k+\tau_i) = f_{\mathrm{u}i}(z(k)) + \sum_{j=1}^{m} G_{\mathrm{s}(i,j)}(z(k))u_j(k) + \sum_{l=1}^{d} \hat{D}_{(i,l)}w_l(k), \quad k \in \mathcal{N}, \quad i = 1, \cdots, m, \quad (13.23)$$

where $\tau_i \in \mathcal{N}$ denotes the time delay (or relative degree) with respect to the output $z_i$, $z(k) = [z_1(k), \cdots, z_1(k+\tau_1-1), \cdots, z_m(k), \cdots, z_m(k+\tau_m-1)]$, $z(0) = z_0$, $\hat{D}_{(i,l)} \in \mathbb{R}$, $i = 1, \cdots, m$, $l = 1, \cdots, d$, and $w_l(k) \in \mathbb{R}$, $k \in \mathcal{N}$, $l = 1, \cdots, d$. Here, we assume that the square matrix function $G_{\mathrm{s}}(z)$ composed of the entries $G_{\mathrm{s}(i,j)}(z)$, $i, j = 1, \cdots, m$, is such that $\det G_{\mathrm{s}}(z) \neq 0$, $z \in \mathbb{R}^{\hat{\tau}}$, where $\hat{\tau} = \tau_1 + \cdots + \tau_m$. Furthermore, since (13.23) is in a form where it does not possess internal dynamics, it follows that $\hat{\tau} = n$. The case where (13.23) possesses input-to-state stable internal dynamics can be analogously handled as shown in Section 2.2.

Next, define $x_i(k) \triangleq [\, z_i(k), \cdots, z_i(k+\tau_i-2)]^{\mathrm{T}}$, $i = 1, \cdots, m$, $x_{m+1}(k) \triangleq [\, z_1(k+\tau_1 -1), \cdots, z_m(k+\tau_m-1)]^{\mathrm{T}}$, and $x(k) \triangleq [\, x_1^{\mathrm{T}}(k), \cdots, x_{m+1}^{\mathrm{T}}(k)]^{\mathrm{T}}$ so that (13.23) can be described by (13.1) with

$$f(x) = \tilde{A}x + \tilde{f}_{\mathrm{u}}(x), \quad G(x) = \begin{bmatrix} 0_{(n-m)\times m} \\ G_{\mathrm{s}}(x) \end{bmatrix}, \quad J(x) = D = \begin{bmatrix} 0_{(n-m)\times d} \\ \hat{D} \end{bmatrix}, \quad (13.24)$$

where

$$\tilde{A} = \begin{bmatrix} A_0 \\ 0_{m\times n} \end{bmatrix}, \quad \tilde{f}_{\mathrm{u}}(x) = \begin{bmatrix} 0_{(n-m)\times 1} \\ f_{\mathrm{u}}(x) \end{bmatrix},$$

$A_0 \in \mathbb{R}^{(n-m)\times n}$ is a known matrix of zeros and ones capturing the multivariable controllable canonical form representation [43], $f_{\mathrm{u}} : \mathbb{R}^n \to \mathbb{R}^m$ is an unknown function and satisfies $f_{\mathrm{u}}^{\mathrm{T}}(x)f_{\mathrm{u}}(x) \leq \gamma_{\mathrm{u}}x^{\mathrm{T}}x$, $x \in \mathbb{R}^n$, where $\gamma_{\mathrm{u}} > 0$, $G_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, and $\hat{D} \in \mathbb{R}^{m\times d}$. Here, we assume that $f_{\mathrm{u}}(x)$ is unknown and is parameterized as $f_{\mathrm{u}}(x) = \Theta f_{\mathrm{n}}(x)$, where $f_{\mathrm{n}} : \mathbb{R}^n \to \mathbb{R}^q$ and satisfies $f_{\mathrm{n}}^{\mathrm{T}}(x)f_{\mathrm{n}}(x) \leq \gamma_{\mathrm{n}}x^{\mathrm{T}}x$, $x \in \mathbb{R}^n$, where $\gamma_{\mathrm{n}} > 0$, and $\Theta \in \mathbb{R}^{m\times q}$ is a matrix of uncertain constant parameters.

Next, to apply Theorem 13.1 to the uncertain system (13.1) with $f(x)$, $G(x)$, and $D$ given by (13.24), let $K_{\mathrm{g}} \in \mathbb{R}^{m\times s}$, where $s = q + r$, be given by

$$K_{\mathrm{g}} = [\,\Theta_{\mathrm{n}} - \Theta,\ \Phi_{\mathrm{n}}\,], \tag{13.25}$$

where $\Theta_{\mathrm{n}} \in \mathbb{R}^{m\times q}$ and $\Phi_{\mathrm{n}} \in \mathbb{R}^{m\times r}$ are known matrices, and let

$$F(x) = \begin{bmatrix} f_{\mathrm{n}}(x) \\ \hat{f}_{\mathrm{n}}(x) \end{bmatrix}, \tag{13.26}$$

where $\hat{f}_{\mathrm{n}} : \mathbb{R}^n \to \mathbb{R}^r$ satisfying $\hat{f}_{\mathrm{n}}^{\mathrm{T}}(x)\hat{f}_{\mathrm{n}}(x) \le \hat{\gamma}_{\mathrm{u}} x^{\mathrm{T}} x$, $x \in \mathbb{R}^n$, $\hat{\gamma}_{\mathrm{u}} > 0$, is an arbitrary function. In this case, it follows that, with $\hat{G}(x) = G_{\mathrm{s}}^{-1}(x)$,

$$\begin{aligned} f_{\mathrm{s}}(x) &= f(x) + G(x)\hat{G}(x)K_{\mathrm{g}}F(x) \\ &= \tilde{A}x + \tilde{f}_{\mathrm{u}}(x) + \begin{bmatrix} 0_{(n-m)\times m} \\ G_{\mathrm{s}}(x) \end{bmatrix} G_{\mathrm{s}}^{-1}(x)\left[\ \Theta_{\mathrm{n}} f_{\mathrm{n}}(x) - \Theta f_{\mathrm{n}}(x) + \Phi_{\mathrm{n}}\hat{f}_{\mathrm{n}}(x)\ \right] \\ &= \tilde{A}x + \begin{bmatrix} 0_{(n-m)\times 1} \\ \Theta_{\mathrm{n}} f_{\mathrm{n}}(x) + \Phi_{\mathrm{n}}\hat{f}_{\mathrm{n}}(x) \end{bmatrix}. \end{aligned} \tag{13.27}$$

Note that, with $\hat{G}(x) = G_{\mathrm{s}}^{-1}(x)$, $\Psi$ in Theorem 13.1 can be taken as $\Psi = \hat{D}$ so that $G(x)\hat{G}(x)\Psi = J(x) = D$ is satisfied, and (13.4) is satisfied with $\bar{\gamma} \ge \gamma_{\mathrm{n}} + \hat{\gamma}_{\mathrm{n}}$.

Now, since $\Theta_{\mathrm{n}} \in \mathbb{R}^{m\times q}$ and $\Phi_{\mathrm{n}} \in \mathbb{R}^{m\times r}$ are arbitrary constant matrices and $\hat{f}_{\mathrm{n}} : \mathbb{R}^n \to \mathbb{R}^r$ is an arbitrary function we can always construct $K_{\mathrm{g}}$, $V_{\mathrm{s}}(x)$, and $F(x)$ without knowledge of $f(x)$ such that (13.2)–(13.5) hold. In particular, choosing $\Theta_{\mathrm{n}} f_{\mathrm{n}}(x) + \Phi_{\mathrm{n}}\hat{f}_{\mathrm{n}}(x) = \hat{A}x$, where $\hat{A} \in \mathbb{R}^{m\times n}$, it follows that (13.27) has the form $f_{\mathrm{s}}(x) = A_{\mathrm{s}}x$, where $A_{\mathrm{s}} = \left[A_0^{\mathrm{T}}, \hat{A}^{\mathrm{T}}\right]^{\mathrm{T}}$ is in multivariable controllable canonical form. Hence, in the case where $G(x) \equiv B$, choosing $\hat{A}$ such that $A_{\mathrm{c}}$ is asymptotically stable it follows that for sufficiently small $\varepsilon > 0$ there exists a positive-definite matrix $P$ satisfying the following Riccati inequality

$$0 \ge A_{\mathrm{s}}^{\mathrm{T}} P A_{\mathrm{s}} - P + R + 4\varepsilon A_{\mathrm{s}}^{\mathrm{T}} P B B^{\mathrm{T}} P A_{\mathrm{s}}, \tag{13.28}$$

where $R$ is a positive-definite matrix. In this case, with $V_{\mathrm{s}}(x) = x^{\mathrm{T}}Px$, (13.2)–(13.5) are satisfied with $\hat{G}(x) \equiv I_m$, $P_{1u}(x) = 2x^{\mathrm{T}}A_{\mathrm{s}}^{\mathrm{T}}PB$, $P_{2u}(x) = B^{\mathrm{T}}PB$, and

$\mu \le \lambda_{\min}(P)$, and hence the adaptive feedback controller (13.7) with update law (13.8) guarantees global asymptotic stability of the nonlinear uncertain discrete-time dynamical system (13.1) where $f(x)$, $G(x)$, and $J(x)$ are given by (13.24) with $G_{\rm s}(x) \equiv B_{\rm s} \in \mathbb{R}^{m\times m}$. As mentioned above, it is important to note that it is not necessary to utilize a feedback linearizing function $F(x)$ to produce a linear $f_{\rm s}(x)$. However, when the system is in normal form, a feedback linearizing function $F(x)$ assures the existence of $V_{\rm s}(x)$ that satisfies the conditions (13.2)–(13.5).

It is important to note that by choosing $\Theta_{\rm n} = \Phi_{\rm n} = 0$ considerable simplification occurs in the update law. Specifically, in this case it follows that

$$G^{\dagger}(x)f_{\rm s}(x) = \left[\begin{array}{cc} 0_{m\times(n-m)}, & G_{\rm s}^{-1}(x) \end{array}\right]\left[\begin{array}{c} A_0 \\ 0_{m\times n} \end{array}\right] x = 0$$

and hence the update law (13.8) can be simplified as

$$K(k+1) = K(k) - \tfrac{1}{c+\tilde{x}^{\rm T}(k)\tilde{x}(k)} Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k))x(k+1)\tilde{x}^{\rm T}(k). \tag{13.29}$$

Finally, it is also important to note that Theorem 13.1 is not restricted to dynamical systems satisfying the linear growth constraint $f^{\rm T}(x)f(x) \le \hat{\gamma}x^{\rm T}x$, $x \in \mathbb{R}^n$, $\hat{\gamma} > 0$. Theorem 13.1 can be used to construct adaptive discrete-time controllers so long as the function $F(x)$ satisfies (13.4) and and we can construct a function $f_{\rm s}(x)$ such that (13.3) holds.

Next, we consider the case where $f(x)$ and $G(x)$ are both uncertain. Specifically, we assume that $G(x)$ is such that $G_{\rm s}(x)$ is unknown and is parameterized as $G_{\rm s}(x) = B_{\rm u}G_{\rm n}(x)$, where $G_{\rm n} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is known and satisfies $\det G_{\rm n}(x) \neq 0$, $x \in \mathbb{R}^n$, and $B_{\rm u} \in \mathbb{R}^{m\times m}$, with $\det B_{\rm u} \neq 0$ and $\sigma_{\max}(B_{\rm u}) \le \alpha$, $\alpha > 0$, is an unknown symmetric sign-definite matrix but a bound $\alpha$ for the maximum singular value of $B_{\rm u}$ and the sign definiteness of $B_{\rm u}$ are known; that is, $B_{\rm u} > 0$ or $B_{\rm u} < 0$. For the statement of the next result define $B_0 \triangleq \left[0_{m\times(n-m)}, I_m\right]^{\rm T}$ for $B_{\rm u} > 0$, and $B_0 \triangleq \left[0_{m\times(n-m)}, -I_m\right]^{\rm T}$ for $B_{\rm u} < 0$.

**Corollary 13.1.** Consider the nonlinear system $\mathcal{G}$ given by (13.1) with $f(x)$, $G(x)$, and $J(x)$ given by (13.24), and $G_{\rm s}(x) = B_{\rm u}G_{\rm n}(x)$, where $B_{\rm u}$, with $\sigma_{\max}(B_{\rm u}) < \alpha$, $\alpha > 0$, is an unknown symmetric sign-definite matrix and the sign definiteness of $B_{\rm u}$ is known. Assume there exist a matrix $K_{\rm g} \in \mathbb{R}^{m\times s}$, functions $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, $F : \mathbb{R}^n \to \mathbb{R}^s$, $P_{1u} : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, $\ell : \mathbb{R}^n \to \mathbb{R}^t$, a nonnegative-definite matrix function $P_{2u} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, and positive constants $\bar{\gamma}$, $\varepsilon$, $\mu$, and $\nu$ such that $V_{\rm s}(\cdot)$ and $\ell(\cdot)$ are continuous, $V_{\rm s}(0) = 0$, $\ell(0) = 0$, $\hat{\alpha}^{-2}G_{\rm n}^{-{\rm T}}(x)P_{2u}(x)G_{\rm n}^{-1}(x) \le \nu I_m$, $x \in \mathbb{R}^n$, $\hat{\alpha} > \frac{\alpha}{2}$, and, for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, (13.2)–(13.5) hold. Then the adaptive feedback control law

$$u(k) = \hat{\alpha}^{-1}G_{\rm n}^{-1}(x(k))K(k)\tilde{x}(k), \tag{13.30}$$

where $K(k) \in \mathbb{R}^{m\times(s+d)}$, $k \in \mathcal{N}$, and $\tilde{x}(k) \triangleq [F^{\rm T}(x(k)),\ w^{\rm T}(k)]^{\rm T}$, with update law

$$K(k+1) = K(k) - \tfrac{1}{c+\tilde{x}^{\rm T}(k)\tilde{x}(k)}B_0^{\rm T}[x(k+1) - f_{\rm s}(x(k))]\tilde{x}^{\rm T}(k), \tag{13.31}$$

guarantees that the solution $(x(k), K(k)) \equiv (0, [K_{\rm g}, -\Psi])$, where $\Psi \in \mathbb{R}^{m\times d}$, of the closed-loop system given by (13.1), (13.30), and (13.31) is Lyapunov stable and $\ell(x(k)) \to 0$ as $k \to \infty$. If, in addition, $\ell^{\rm T}(x)\ell(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, then $x(k) \to 0$ as $k \to \infty$ for all $x_0 \in \mathbb{R}^n$.

**Proof.** The result is a direct consequence of Theorem 13.1. First, let $\hat{G}(x) = \hat{\alpha}^{-1}G_{\rm n}^{-1}(x)$ and $\Psi = \hat{\alpha}B_{\rm u}^{-1}\hat{D}$ so that $G(x)\hat{G}(x) = [0_{m\times(n-m)}, \hat{\alpha}^{-1}B_{\rm u}]^{\rm T}$ and $G(x)\hat{G}(x)\Psi = D$, and let $K_{\rm g} = \hat{\alpha}B_{\rm u}^{-1}[\Theta_{\rm n} - \Theta, \Phi_{\rm n}]$. Next, since $Q$ in (13.8) is an arbitrary positive-definite matrix with $\lambda_{\max}(Q) < 2$, it can be replaced by $\hat{\alpha}^{-1}|B_{\rm u}| = \hat{\alpha}^{-1}(B_{\rm u}^2)^{\frac{1}{2}}$, where $(\cdot)^{\frac{1}{2}}$ denotes the (unique) positive-definite square root. Now, since $B_{\rm u}$ is symmetric and sign definite it follows from the Schur decomposition that $B_{\rm u} = UD_{B_{\rm u}}U^{\rm T}$, where $U$ is orthogonal and $D_{B_{\rm u}}$ is real diagonal. Hence, $\hat{\alpha}^{-1}|B_{\rm u}|\hat{G}^{-1}(x)G^{\dagger}(x) = [0_{m\times(n-m)}, \mathcal{I}_m] = B_0^{\rm T}$, where $\mathcal{I}_m = I_m$ for $B_{\rm u} > 0$ and $\mathcal{I}_m = -I_m$ for $B_{\rm s} < 0$. Now, (13.8) implies (13.31). $\square$

## 13.3. Adaptive Control for Nonlinear Systems with $\ell_2$ Disturbances

In this section we consider the problem of characterizing adaptive feedback control laws for nonlinear discrete-time uncertain dynamical systems with exogenous $\ell_2$ disturbances. Specifically, we consider the controlled nonlinear uncertain system $\mathcal{G}$ given by

$$x(k+1) = f(x(k)) + G(x(k))u(k) + J(x(k))w(k), \quad x(0) = x_0, \quad w(\cdot) \in \ell_2, \quad k \in \mathcal{N}, \tag{13.32}$$

with performance variables

$$z(k) = h(x(k)), \tag{13.33}$$

where $x(k) \in \mathbb{R}^n$, $k \in \mathcal{N}$, is the state vector, $u(k) \in \mathbb{R}^m$, $k \in \mathcal{N}$, is the control input, $w(k) \in \mathbb{R}^d$, $k \in \mathcal{N}$, is an unknown bounded energy $\ell_2$ disturbance, $z(k) \in \mathbb{R}^p$, $k \in \mathcal{N}$, is a performance variable, $f : \mathbb{R}^n \to \mathbb{R}^n$ and satisfies $f(0) = 0$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $J : \mathbb{R}^n \to \mathbb{R}^{n\times d}$, and $h : \mathbb{R}^n \to \mathbb{R}^p$ is continuous and satisfies $h(0) = 0$. The following theorem generalizes Theorem 13.1 to discrete-time nonlinear uncertain dynamical systems with exogenous $\ell_2$ disturbances.

**Theorem 13.2.** Consider the nonlinear system $\mathcal{G}$ given by (13.32) and (13.33). Assume there exist a matrix $K_\mathrm{g} \in \mathbb{R}^{m\times s}$, functions $V_\mathrm{s} : \mathbb{R}^n \to \mathbb{R}$, $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, $F : \mathbb{R}^n \to \mathbb{R}^s$, $P_{1u} : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, $P_{1w} : \mathbb{R}^n \to \mathbb{R}^{1\times d}$, $P_{uw} : \mathbb{R}^n \to \mathbb{R}^{m\times d}$, nonnegative-definite matrix functions $P_{2u} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ and $P_{2w} : \mathbb{R}^n \to \mathbb{R}^{d\times d}$, and positive constants $\bar{\gamma}$, $\hat{\delta}$, $a$, $\varepsilon$, $\mu$, and $\nu$ such that $V_\mathrm{s}(\cdot)$ is continuous and satisfies (13.5), $V_\mathrm{s}(0) = 0$, $\det \hat{G}(x) \neq 0$, $x \in \mathbb{R}^n$, $F(x)$ satisfies (13.4), $(\hat{G}^{-1}(x)G^\dagger(x)J(x))^\mathrm{T}(\hat{G}^{-1}(x)G^\dagger(x)J(x)) \leq \hat{\delta}I_d$, $x \in \mathbb{R}^n$, $\hat{G}^\mathrm{T}(x)P_{2u}(x)\hat{G}(x) < \nu I_m$, $x \in \mathbb{R}^n$, and, for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and

$w \in \mathbb{R}^d$,

$$V_{\rm s}(f(x)+G(x)u+J(x)w) = V_{\rm s}(f(x)) + P_{1u}(x)u + u^{\rm T}P_{2u}(x)u + P_{1w}(x)w + u^{\rm T}P_{uw}(x)w + w^{\rm T}P_{2w}(x)w, \quad (13.34)$$

$$0 \geq V_{\rm s}(f_{\rm s}(x)) - V_{\rm s}(x) + \Gamma(x) + \varepsilon P_{1u}(x)\hat{G}(x)\hat{G}^{\rm T}(x)P_{1u}^{\rm T}(x), \quad (13.35)$$

$$\tfrac{a}{c+F^{\rm T}(x)F(x)}(2I-Q) \geq \frac{\frac{1}{4\varepsilon}+\nu}{1+V_{\rm s}(x)}I_m + \frac{1}{4\tilde{\lambda}}\tilde{P}_{uw}(x)\tilde{P}_{uw}^{\rm T}(x), \quad x \in \mathbb{R}^n, \quad (13.36)$$

where $f_{\rm s}(x)$ is given by (13.6),

$$\Gamma(x) \triangleq \tfrac{1}{4}P_{1w}(x)[(\gamma^2-\tilde{\gamma}^2)I_m - P_{2w}(x)]^{-1}P_{1w}^{\rm T}(x) + h^{\rm T}(x)h(x), \quad (13.37)$$

$$\tilde{P}_{uw}(x) \triangleq \tfrac{1}{1+V_{\rm s}(x)}\hat{G}^{\rm T}(x)P_{uw}(x) - \tfrac{2a}{c+F^{\rm T}(x)F(x)}\hat{G}^{-1}(x)G^{\dagger}(x)J(x) + \tfrac{2aF^{\rm T}(x)F(x)}{(c+F^{\rm T}(x)F(x))^2}Q\hat{G}^{-1}(x)G^{\dagger}(x)J(x), \quad (13.38)$$

$\gamma > 0$, $(\gamma^2-\tilde{\gamma}^2)I_m - P_{2w}(x) > 0$, $Q \in \mathbb{R}^{m\times m}$ is positive definite with $\lambda_{\rm max}(Q) < 2$, and $\tilde{\gamma}$ is such that

$$\frac{\tilde{\gamma}^2}{1+V_{\rm s}(x)} - \frac{a\hat{\delta}\lambda_{\rm max}(Q)}{c+F^{\rm T}(x)F(x)} \geq \tilde{\lambda} > 0, \quad x \in \mathbb{R}^n, \quad (13.39)$$

where $\tilde{\lambda} \in \mathbb{R}$. Then the adaptive feedback control law

$$u(k) = \hat{G}(x(k))K(k)F(x(k)), \quad (13.40)$$

where $K(k) \in \mathbb{R}^{m\times s}$, $k \in \mathcal{N}$, with update law

$$K(k+1) = K(k) - \tfrac{1}{c+F^{\rm T}(x(k))F(x(k))}Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k))[x(k+1)-f_{\rm s}(x(k))]F^{\rm T}(k), \quad (13.41)$$

guarantees that the solution $(x(k),K(k)) \equiv (0,K_{\rm g})$ of the undisturbed $(w(k) \equiv 0)$ closed-loop system given by (13.32), (13.40), and (13.41) is Lyapunov stable and $h(x(k)) \to 0$ as $k \to \infty$. If, in addition, $h^{\rm T}(x)h(x) > 0$, $x \in \mathbb{R}^n$, $x \neq 0$, then $x(k) \to 0$ as $k \to \infty$ for all $x_0 \in \mathbb{R}^n$. Furthermore, the solution $x(k)$, $k \in \mathcal{N}$, to the closed-loop system given by (13.32), (13.40), and (13.41) satisfies the nonexpansivity constraint

$$\sum_{k=0}^{\mathcal{K}-1}\frac{z^{\rm T}(k)z(k)}{1+V_{\rm s}(x(k))} \leq \gamma^2\sum_{k=0}^{\mathcal{K}-1}w^{\rm T}(k)w(k) + V(x(0),K(0)), \quad \mathcal{K} \geq 0, \quad \gamma > 0, \quad w(\cdot) \in \ell_2, \quad (13.42)$$

where

$$V(x,K) \triangleq \ln(1+V_{\rm s}(x)) + a\,{\rm tr}(K-K_{\rm g})^{\rm T}Q^{-1}(K-K_{\rm g}). \tag{13.43}$$

**Proof.** First, define $\tilde{K}(k) \triangleq K(k) - K_{\rm g}$, $\tilde{x}(k) = F(x(k))$, and $\tilde{u}(k) \triangleq \tilde{K}(k)\tilde{x}(k)$. Note that with $u(k)$, $k \in \mathcal{N}$, given by (13.40) it follows from (13.32) that

$$\begin{aligned} x(k+1) = {} & f(x(k)) + G(x(k))\hat{G}(x(k))K(k)F(x(k)) + J(x(k))w(k), \\ & x(0) = x_0, \quad w(\cdot) \in \ell_2, \quad k \in \mathcal{N}, \end{aligned} \tag{13.44}$$

or, equivalently, using the definition for $f_{\rm s}(x)$ given in (13.6),

$$\begin{aligned} x(k+1) &= f_{\rm s}(x(k)) + G(x(k))\hat{G}(x(k))\tilde{K}(k)\tilde{x}(k) + J(x(k))w(k) \\ &= f_{\rm s}(x(k)) + G(x(k))\hat{G}(x(k))\tilde{u}(k) + J(x(k))w(k), \\ & \qquad x(0) = x_0, \quad w(\cdot) \in \ell_2, \quad k \in \mathcal{N}. \end{aligned} \tag{13.45}$$

Furthermore, note that by adding and subtracting $K_{\rm g}$ to and from (13.41) and using (13.45) it follows that

$$\begin{aligned} \tilde{K}(k+1) &= \tilde{K}(k) - \tfrac{1}{c+\tilde{x}^{\rm T}(k)\tilde{x}(k)}Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k))[G(x(k))\hat{G}(x(k))\tilde{K}(k)\tilde{x}(k) \\ &\quad + J(x(k))w(k)]\tilde{x}^{\rm T}(k) \\ &= \tilde{K}(k) - \tfrac{1}{c+\tilde{x}^{\rm T}(k)\tilde{x}(k)}Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\rm T}(k) \\ &\quad - \tfrac{1}{c+\tilde{x}^{\rm T}(k)\tilde{x}(k)}Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k))J(x(k))w(k)\tilde{x}^{\rm T}(k). \end{aligned} \tag{13.46}$$

To show Lyapunov stability of the undisturbed closed-loop system (13.45) and (13.46) consider the Lyapunov function candidate given by (13.43). Note that $V(0,K_{\rm g}) = 0$ and, since $V_{\rm s}(\cdot)$ and $Q$ are positive definite and $a > 0$, $V(x,K) > 0$ for all $(x,K) \neq (0,K_{\rm g})$. Furthermore, $V(x,K)$ is radially unbounded. Now, since (13.34) collapses to (13.2) in the case where $w(k) \equiv 0$, Lyapunov stability of the undisturbed closed-loop system (13.45) and (13.46) as well as $x(k) \to 0$ as $k \to \infty$ for all $x_0 \in \mathbb{R}^n$ follows as in the proof of Theorem 13.1.

To show that the nonexpansivity constraint (13.42) holds, note that, for all $w \in \mathbb{R}^d$,

$$
\begin{aligned}
0 \le\ & \left[\tfrac{1}{2}P_{1w}^{\mathrm{T}}(x) - ((\gamma^2-\tilde{\gamma}^2)I_m - P_{2w}(x))w\right]^{\mathrm{T}} [(\gamma^2-\tilde{\gamma}^2)I_m - P_{2w}(x)]^{-1} \\
& \cdot \left[\tfrac{1}{2}P_{1w}^{\mathrm{T}}(x) - ((\gamma^2-\tilde{\gamma}^2)I_m - P_{2w}(x))w\right] \\
=\ & \Gamma(x) + (\gamma^2-\tilde{\gamma}^2)w^{\mathrm{T}}w - z^{\mathrm{T}}z - P_{1w}(x)w - w^{\mathrm{T}}P_{2w}(x)w. \qquad (13.47)
\end{aligned}
$$

Now, let $w(\cdot) \in \ell_2$ and let $x(k)$, $k \in \mathcal{N}$, denote the solution of the closed-loop system (13.45). Then, using (13.34), (13.35), (13.39), (13.46), and (13.47), the Lyapunov difference along the closed-loop system trajectories is given by

$$
\begin{aligned}
&\Delta V(x(k),K(k)) \\
&\quad = \ln\Big(1 + V_{\mathrm{s}}(f_{\mathrm{s}}(x(k)) + G(x(k))u(k) + J(x(k))w(k))\Big) \\
&\qquad +a\mathrm{tr}\Big(\tilde{K}(k) - \tfrac{1}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k) \\
&\qquad -\tfrac{1}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k))J(x(k))w(k)\tilde{x}^{\mathrm{T}}(k)\Big)^{\mathrm{T}}Q^{-1} \\
&\qquad \cdot\Big(\tilde{K}(k) - \tfrac{1}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k) \\
&\qquad -\tfrac{1}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k))J(x(k))w(k)\tilde{x}^{\mathrm{T}}(k)\Big) \\
&\qquad -\ln(1+V_{\mathrm{s}}(x(k))) - a\mathrm{tr}\tilde{K}^{\mathrm{T}}(k)Q^{-1}\tilde{K}(k) \\
&\quad = \ln\Big(1 + \Big[V_{\mathrm{s}}(f_{\mathrm{s}}(x(k))) + P_{1u}(x(k))\hat{G}(x(k))\tilde{u}(k) \\
&\qquad +\tilde{u}^{\mathrm{T}}(k)\hat{G}^{\mathrm{T}}(x(k))P_{2u}(x(k))\hat{G}(x(k))\tilde{u}(k) + P_{1w}(x(k))w(k) \\
&\qquad +\tilde{u}^{\mathrm{T}}(k)\hat{G}^{\mathrm{T}}(x(k))P_{uw}(x(k))w(k) + w^{\mathrm{T}}(k)P_{2w}(x(k))w(k) - V_{\mathrm{s}}(x(k))\Big] \\
&\qquad \cdot[1+V_{\mathrm{s}}(x(k))]^{-1}\Big) + a\mathrm{tr}\tilde{K}^{\mathrm{T}}(k)Q^{-1}\tilde{K}(k) + \tfrac{a}{(c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k))^2}\mathrm{tr}\,\tilde{x}(k)w^{\mathrm{T}}(k) \\
&\qquad \cdot J^{\mathrm{T}}(x(k)){G^{\dagger}}^{\mathrm{T}}(x(k))\hat{G}^{-\mathrm{T}}Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k))J(x(k))w(k)\tilde{x}^{\mathrm{T}}(k) \\
&\qquad +\tfrac{a}{(c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k))^2}\mathrm{tr}\,\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k) \\
&\qquad -\tfrac{2a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}\mathrm{tr}\,\tilde{K}^{\mathrm{T}}(k)\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k) \\
&\qquad -\tfrac{2a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}\mathrm{tr}\,\tilde{K}^{\mathrm{T}}(k)\hat{G}^{-1}(x(k))G^{\dagger}(x(k))J(x(k))w(k)\tilde{x}^{\mathrm{T}}(k) \\
&\qquad +\tfrac{2a}{(c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k))^2}\mathrm{tr}\,\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k))J(x(k))w(k)\tilde{x}^{\mathrm{T}}(k)
\end{aligned}
$$

$$
\begin{aligned}
&-a\mathrm{tr}\tilde{K}^{\mathrm{T}}(k)Q^{-1}\tilde{K}(k)\\
\leq\ & [-\Gamma(x(k)) - \varepsilon P_{1u}(x(k))\hat{G}(x(k))\hat{G}^{\mathrm{T}}(x(k))P_{1u}^{\mathrm{T}}(x(k)) + P_{1u}(x(k))\hat{G}(x(k))\tilde{u}(k)\\
&+\nu\tilde{u}^{\mathrm{T}}(k)\tilde{u}(k) + P_{1w}(x(k))w(k) + w^{\mathrm{T}}(k)P_{2w}(x(k))w(k)][1+V_{\mathrm{s}}(x(k))]^{-1}\\
&+\tilde{u}^{\mathrm{T}}(k)\Big[\tfrac{1}{1+V_{\mathrm{s}}(x(k))}\hat{G}^{\mathrm{T}}(x(k))P_{uw}(x(k)) - \tfrac{2a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}\hat{G}^{-1}(x(k))G^{\dagger}(x(k))J(x(k))\\
&+\tfrac{2a\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}{(c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k))^2}Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k))J(x(k))\Big]w(k)\\
&+\tfrac{a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}w^{\mathrm{T}}(k)J^{\mathrm{T}}(x(k)){G^{\dagger}}^{\mathrm{T}}(x(k))\hat{G}^{-\mathrm{T}}Q\hat{G}^{-1}(x(k))G^{\dagger}(x(k))J(x(k))w(k)\\
&-\tfrac{a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}\tilde{x}^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)(2I_m-Q)\tilde{K}(k)\tilde{x}(k)\\
\leq\ & \Big[\gamma^2 w^{\mathrm{T}}(k)w(k) - z^{\mathrm{T}}(k)z(k) - \varepsilon P_{1u}(x(k))\hat{G}(x(k))\hat{G}^{\mathrm{T}}(x(k))P_{1u}^{\mathrm{T}}(x(k))\\
&+P_{1u}(x(k))\hat{G}(x(k))\tilde{u}(k) + \nu\tilde{u}^{\mathrm{T}}(k)\tilde{u}(k)\Big][1+V_{\mathrm{s}}(x(k))]^{-1}\\
&+\tilde{u}^{\mathrm{T}}(k)\tilde{P}_{uw}(x(k))w(k) - \tilde{\lambda}w^{\mathrm{T}}(k)w(k)\\
&-\tfrac{a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}\tilde{x}^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)(2I_m-Q)\tilde{K}(k)\tilde{x}(k), \quad k\in\mathcal{N},
\end{aligned}
\tag{13.48}
$$

where in (13.48) we used $\ln a - \ln b = \ln\frac{a}{b}$ and $\ln(1+c)\leq c$ for $a,b>0$ and $c\geq -1$, respectively, and $\frac{\tilde{x}^{\mathrm{T}}\tilde{x}}{c+\tilde{x}^{\mathrm{T}}\tilde{x}}<1$. Now, using (13.36), adding and subtracting $\tilde{u}^{\mathrm{T}}(k)\Big[\frac{1}{4\varepsilon}\frac{1}{1+V_{\mathrm{s}}(x(k))}I_m + \frac{1}{4\tilde{\lambda}}\tilde{P}_{uw}(x(k))\cdot\tilde{P}_{uw}^{\mathrm{T}}(x(k))\Big]\tilde{u}(k)$, $k\in\mathcal{N}$, to and from (13.48), and collecting terms yields

$$
\begin{aligned}
&\Delta V(x(k),K(k))\\
&\leq \tfrac{1}{1+V_{\mathrm{s}}(x(k))}[\gamma^2 w^{\mathrm{T}}(k)w(k) - z^{\mathrm{T}}(k)z(k)]\\
&\quad -\tfrac{1}{1+V_{\mathrm{s}}(x(k))}\left[P_{1u}(x(k)),\ \tilde{u}^{\mathrm{T}}(k)\right]\begin{bmatrix}\varepsilon\hat{G}(x(k))\hat{G}^{\mathrm{T}}(x(k)) & -\frac{1}{2}\hat{G}(x(k))\\ -\frac{1}{2}\hat{G}^{\mathrm{T}}(x(k)) & \frac{1}{4\varepsilon}I_m\end{bmatrix}\begin{bmatrix}P_{1u}^{\mathrm{T}}(x(k))\\ \tilde{u}(k)\end{bmatrix}\\
&\quad -\left[\tilde{u}^{\mathrm{T}}(k),\ w^{\mathrm{T}}(k)\right]\begin{bmatrix}\frac{1}{4\tilde{\lambda}}\tilde{P}_{uw}(x(k))\tilde{P}_{uw}^{\mathrm{T}}(x(k)) & -\frac{1}{2}\tilde{P}_{uw}(x(k))\\ -\frac{1}{2}\tilde{P}_{uw}^{\mathrm{T}}(x(k)) & \tilde{\lambda}I_d\end{bmatrix}\begin{bmatrix}\tilde{u}(k)\\ w(k)\end{bmatrix}\\
&\quad +\tilde{u}^{\mathrm{T}}(k)\left[\tfrac{1}{4\varepsilon}\tfrac{1}{1+V_{\mathrm{s}}(x(k))}I_m + \tfrac{1}{4\tilde{\lambda}}\tilde{P}_{uw}(x(k))\tilde{P}_{uw}^{\mathrm{T}}(x(k))\right]\tilde{u}(k) + \tfrac{\nu}{1+V_{\mathrm{s}}(x(k))}\tilde{u}^{\mathrm{T}}(k)\tilde{u}(k)\\
&\quad -\tfrac{a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}\tilde{x}^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)(2I-Q)\tilde{K}(k)\tilde{x}(k),\\
&\leq \gamma^2 w^{\mathrm{T}}(k)w(k) - \frac{z^{\mathrm{T}}(k)z(k)}{1+V_{\mathrm{s}}(x(k))}\\
&\quad -\tilde{u}^{\mathrm{T}}(k)\left[\tfrac{a}{c+\tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)}(2I-Q) - \frac{\frac{1}{4\varepsilon}+\nu}{1+V_{\mathrm{s}}(x(k))}I_m - \frac{1}{4\tilde{\lambda}}\tilde{P}_{uw}(x(k))\tilde{P}_{uw}^{\mathrm{T}}(x(k))\right]\tilde{u}(k)
\end{aligned}
$$

$$\leq \gamma^2 w^{\mathrm{T}}(k)w(k) - \frac{z^{\mathrm{T}}(k)z(k)}{1+V_{\mathrm{s}}(x(k))}, \quad k \in \mathcal{N}. \tag{13.49}$$

Now, summing (13.49) over $k = 0, \cdots, \mathcal{K}-1$ yields

$$V(x(\mathcal{K}), K(\mathcal{K})) \leq \sum_{k=0}^{\mathcal{K}-1} \left[\gamma^2 w^{\mathrm{T}}(k)w(k) - \frac{z^{\mathrm{T}}(k)z(k)}{1+V_{\mathrm{s}}(x(k))}\right] + V(x(0), K(0)),$$
$$\mathcal{K} \geq 0, \quad w(\cdot) \in \ell_2, \tag{13.50}$$

which, by noting that $V(x(\mathcal{K}), K(\mathcal{K})) \geq 0$, $\mathcal{K} \geq 0$, yields (13.42). □

It is important to note that unlike Theorem 13.1 requiring a matching condition on the disturbance, Theorem 13.2 does not require any such matching condition. Furthermore, as shown in Section 13.2, if (13.32) is in normal form with asymptotically stable internal dynamics and $f^{\mathrm{T}}(x)f(x) \leq \hat{\gamma} x^{\mathrm{T}} x$, $x \in \mathbb{R}^n$, where $\hat{\gamma} > 0$, then we can always construct a function $F : \mathbb{R}^n \to \mathbb{R}^s$ such that $F(\cdot)$ satisfies (13.4) and (13.34)–(13.36) hold without requiring knowledge of the system dynamics. In addition, in the case where $J(x) = D$ and $h(x) = Ex$, the adaptive controller (13.40) can be verified to guarantee the modified nonexpansivity constraint (13.42) using standard *linear* $\mathrm{H}_\infty$ methods. Specifically, choosing $f_{\mathrm{s}}(x) = A_{\mathrm{s}}x$, where $A_{\mathrm{s}}$ is asymptotically stable and in multivariable controllable canonical form, it follows from standard discrete-time $\mathrm{H}_\infty$ theory [73] that if $(A_{\mathrm{s}}, E)$ is observable, $\|G(s)\|_\infty < \sqrt{\gamma^2 - \tilde{\gamma}^2}$, where $G(s) = E(sI_n - A_{\mathrm{s}})^{-1}D$, if and only if there exists a positive-definite matrix $P$ satisfying the discrete-time bounded real Riccati inequality

$$0 > A_{\mathrm{s}}^{\mathrm{T}}PA_{\mathrm{s}} - P + A_{\mathrm{s}}^{\mathrm{T}}PD[(\gamma^2 - \tilde{\gamma}^2)I_m - D^{\mathrm{T}}PD]^{-1}D^{\mathrm{T}}PA_{\mathrm{s}} + E^{\mathrm{T}}E. \tag{13.51}$$

In this case, if $G(x) \equiv B$ is a constant matrix, then there exists a sufficiently small $\varepsilon > 0$ such that

$$0 \geq A_{\mathrm{s}}^{\mathrm{T}}PA_{\mathrm{s}} - P + A_{\mathrm{s}}^{\mathrm{T}}PD[(\gamma^2 - \tilde{\gamma}^2)I_m - D^{\mathrm{T}}PD]^{-1}D^{\mathrm{T}}PA_{\mathrm{s}} + E^{\mathrm{T}}E + 4\varepsilon A_{\mathrm{s}}^{\mathrm{T}}PBB^{\mathrm{T}}PA_{\mathrm{s}}. \tag{13.52}$$

Now, with $V_{\rm s}(x) = x^{\rm T}Px$, there exists $\tilde{\lambda} > 0$ such that (13.34)–(13.36) and (13.39) are satisfied with $\hat{G}(x) = I_m$, $Q = I_m$, $P_{1u}(x) = 2x^{\rm T}A_{\rm s}^{\rm T}PB$, $P_{2u}(x) = B^{\rm T}PB$, $P_{1w}(x) = 2x^{\rm T}A_{\rm s}^{\rm T}PD$, $P_{uw}(x) = 2B^{\rm T}PD$, $P_{2w}(x) = D^{\rm T}PD$, and $a > (\frac{1}{4\varepsilon}+\nu)\cdot\max\{c, \frac{\bar{\gamma}}{\mu}\}$. Hence, the adaptive feedback controller (13.40) with update law (13.41), or, equivalently,

$$K(k+1) = K(k) - \tfrac{1}{c+F^{\rm T}(x(k))F(x(k))}B^{\dagger}[x(k+1) - A_{\rm s}x(k)]F^{\rm T}(k), \quad (13.53)$$

guarantees global asymptotic stability of the nonlinear undisturbed ($w(k) \equiv 0$) dynamical system (13.32), where $f(x)$ and $G(x)$ are given by (13.24) with $G_{\rm s}(x) \equiv B_{\rm s}$. Furthermore, the solution $x(k)$, $k \in \mathcal{N}$, of the closed-loop *nonlinear* dynamical system (13.32), (13.33) is guaranteed to satisfy the nonexpansivity constraint (13.42).

Finally, if $f(x)$ and $G(x)$ given by (13.24) are uncertain and $G_{\rm s}(x) = B_{\rm u}G_{\rm n}(x)$, where a bound for the maximum singular value $\alpha$ of $B_{\rm u}$ and the sign definiteness of $B_{\rm u}$ are known, then using an identical approach as in Section 13.2, it can be shown that the adaptive feedback control law

$$u(k) = \hat{\alpha}^{-1}G_{\rm n}^{-1}(x(k))K(k)F(x(k)), \quad (13.54)$$

where $\hat{\alpha} > \frac{\alpha}{2}$, with update law

$$K(k+1) = K(k) - \tfrac{1}{c+\tilde{x}^{\rm T}(k)\tilde{x}(k)}B_0^{\rm T}[x(k+1) - f_{\rm s}(x(k))]\tilde{x}^{\rm T}(k), \quad (13.55)$$

where $B_0$ is defined as in Section 13.2, guarantees asymptotic stability and nonexpansivity of (13.32).

## 13.4. Illustrative Numerical Examples

In this section we present three numerical examples to demonstrate the utility of the proposed discrete-time adaptive control framework for adaptive stabilization, disturbance rejection, and command following.

**Example 13.1.** Consider the linear uncertain system given by

$$z(k+2) + a_1 z(k+1) + a_0 z(k) = bu(k) + \hat{d}\sin 7k, \quad z(0) = z_0, \quad z(1) = z_1, \quad k \in \mathcal{N}, \tag{13.56}$$

where $z(k) \in \mathbb{R}$, $k \in \mathcal{N}$, $u(k) \in \mathbb{R}$, $k \in \mathcal{N}$, and $a_0, a_1, b, \hat{d} \in \mathbb{R}$ are unknown constants. Note that with $x_1(k) = z(k)$ and $x_2(k) = z(k+1)$, (13.56) can be written in state space form (13.1) with $x = [x_1, x_2]^{\mathrm{T}}$, $f(x) = [x_2, -a_0 x_1 - a_1 x_2]^{\mathrm{T}}$, $G(x) = [0, b]^{\mathrm{T}}$, $J(x) = [0, \hat{d}]^{\mathrm{T}}$, and $w(k) = \sin 7k$. Here, we assume that $f(x)$ is unknown and can be parameterized as $f(x) = [x_2, \theta_1 x_1 + \theta_2 x_2]^{\mathrm{T}}$, where $\theta_1$ and $\theta_2$ are unknown constants. Furthermore, we assume that $\operatorname{sgn} b$ is known and $|b| < \alpha = 2$. Next, let $G_{\mathrm{n}}(x) = 1$, $F(x) = x$, $\hat{\alpha} = 1$, and $K_{\mathrm{g}} = \frac{1}{b}\left[\theta_{\mathrm{n}_1} - \theta_1, \theta_{\mathrm{n}_2} - \theta_2\right]$, where $\theta_{\mathrm{n}_1}$, $\theta_{\mathrm{n}_2}$ are arbitrary scalars, so that

$$\begin{aligned} f_{\mathrm{s}}(x) &= f(x) + \begin{bmatrix} 0 \\ b \end{bmatrix} \frac{1}{b} \begin{bmatrix} \theta_{\mathrm{n}_1} - \theta_1, \theta_{\mathrm{n}_2} - \theta_2 \end{bmatrix} F(x) \\ &= \begin{bmatrix} 0 & 1 \\ \theta_{\mathrm{n}_1} & \theta_{\mathrm{n}_2} \end{bmatrix} x. \end{aligned}$$

Note that since (13.56) is linear all the conditions of Corollary 13.1 are trivially satisfied. Now, with the proper choice of $\theta_{\mathrm{n}_1}$ and $\theta_{\mathrm{n}_2}$, it follows from Corollary 13.1 that the adaptive feedback controller (13.30) guarantees that $x(k) \to 0$ as $k \to \infty$. With $\hat{\alpha} = 1$, $\theta_1 = -1$, $\theta_2 = 0.25$, $b = 0.4$, $\hat{d} = 10$, $c = 1$, $\theta_{\mathrm{n}_1} = -0.02$, $\theta_{\mathrm{n}_2} = 0.3$, and initial conditions $x(0) = [-1, 3]^{\mathrm{T}}$ and $K(0) = [0, 0, 0]$, Figure 13.1 shows the phase portrait of the controlled and uncontrolled system. Note that the adaptive controller is switched on at $k = 30$. Figure 13.2 shows the state trajectories versus time and the control signal versus time. Finally, Figure 13.3 shows the adaptive gain history versus time.

**Example 13.2.** Consider the two-degree of freedom uncertain linear system given by

$$M_{\mathrm{s}} z(k+2) + C_{\mathrm{s}} z(k+1) + K_{\mathrm{s}} z(k) = u(k), \quad z(0) = z_0, \quad z(1) = z_1, \quad k \in \mathcal{N}, \tag{13.57}$$

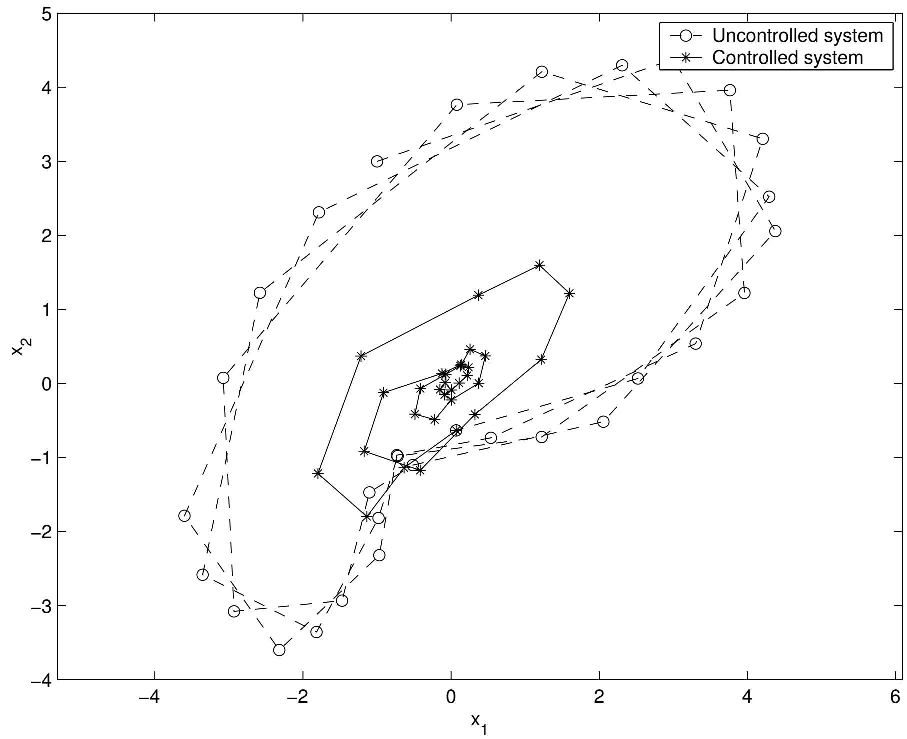



**Figure 13.1**: Phase portrait of controlled and uncontrolled system

where $z(k) \in \mathbb{R}^2$, $u(k) \in \mathbb{R}^2$, $k \in \mathcal{N}$, and $M_\mathrm{s}, C_\mathrm{s}, K_\mathrm{s} \in \mathbb{R}^{2\times 2}$ are unknown matrices. Here we assume that $M_\mathrm{s} = M_\mathrm{s}^\mathrm{T} > 0$ and $\lambda_{\max}(M_\mathrm{s}^{-1}) < \alpha = 2$ but otherwise $M_\mathrm{s}$ is unknown. Let $r_\mathrm{d}(k)$ be a desired command signal and define the error state $\tilde{e}(k) \triangleq z(k) - r_\mathrm{d}(k)$ so that the error dynamics are given by

$$M_\mathrm{s}\tilde{e}(k+2) + C_\mathrm{s}\tilde{e}(k+1) + K_\mathrm{s}\tilde{e}(k) = u(k) - M_\mathrm{s}r_\mathrm{d}(k+2) - C_\mathrm{s}r_\mathrm{d}(k+1) - K_\mathrm{s}r_\mathrm{d}(k),$$
$$\tilde{e}(0) = \tilde{e}_0, \quad \tilde{e}(1) = \tilde{e}_1, \quad k \in \mathcal{N}. \quad (13.58)$$

Note that with $e_1(k) = \tilde{e}(k)$ and $e_2(k) = \tilde{e}(k+1)$, (13.58) can be written in state space form (13.22) with $e = [e_1^\mathrm{T}, e_2^\mathrm{T}]^\mathrm{T}$, $f_\mathrm{k}(k,e) = [e_2^\mathrm{T}, -(M_\mathrm{s}^{-1}K_\mathrm{s}e_1 + M_\mathrm{s}^{-1}C_\mathrm{s}e_2)^\mathrm{T}]^\mathrm{T}$, $G(k,e) = [0_{2\times 2}, M_\mathrm{s}^{-1}]^\mathrm{T}$, $J_\mathrm{k}(k,e) = [0_{6\times 2}, \hat{D}_\mathrm{k}^\mathrm{T}]^\mathrm{T}$, where $\hat{D}_\mathrm{k} = [-I_2, -M_\mathrm{s}^{-1}C_\mathrm{s}, -M_\mathrm{s}^{-1}K_\mathrm{s}]$, and $w_\mathrm{k}(k) = [r_\mathrm{d}^\mathrm{T}(k+2), r_\mathrm{d}^\mathrm{T}(k+1), r_\mathrm{d}^\mathrm{T}(k)]^\mathrm{T}$. Next, let $G_\mathrm{n}(x) = I$, $F(e) = e$, $\hat{\alpha} = 1$, and $K_\mathrm{g} = M_\mathrm{s}[\Theta_{\mathrm{n}_1} + M_\mathrm{s}^{-1}K_\mathrm{s}, \Theta_{\mathrm{n}_2} + M_\mathrm{s}^{-1}C_\mathrm{s}]$, where $\Theta_{\mathrm{n}_1} \in \mathbb{R}^{2\times 2}$, $\Theta_{\mathrm{n}_2} \in \mathbb{R}^{2\times 2}$ are arbitrary

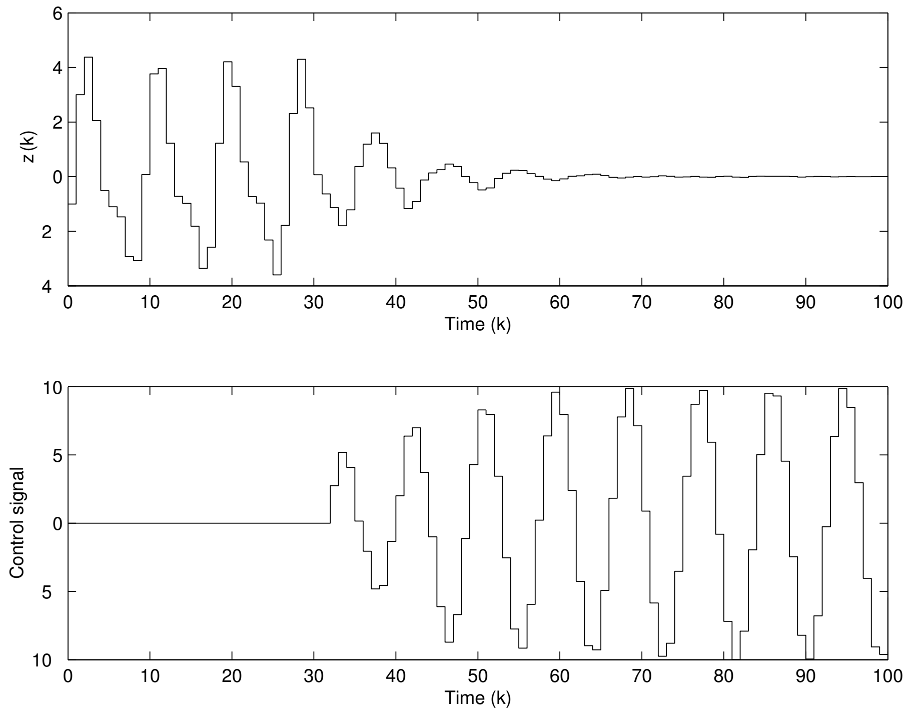



**Figure 13.2**: State trajectories and control signal versus time

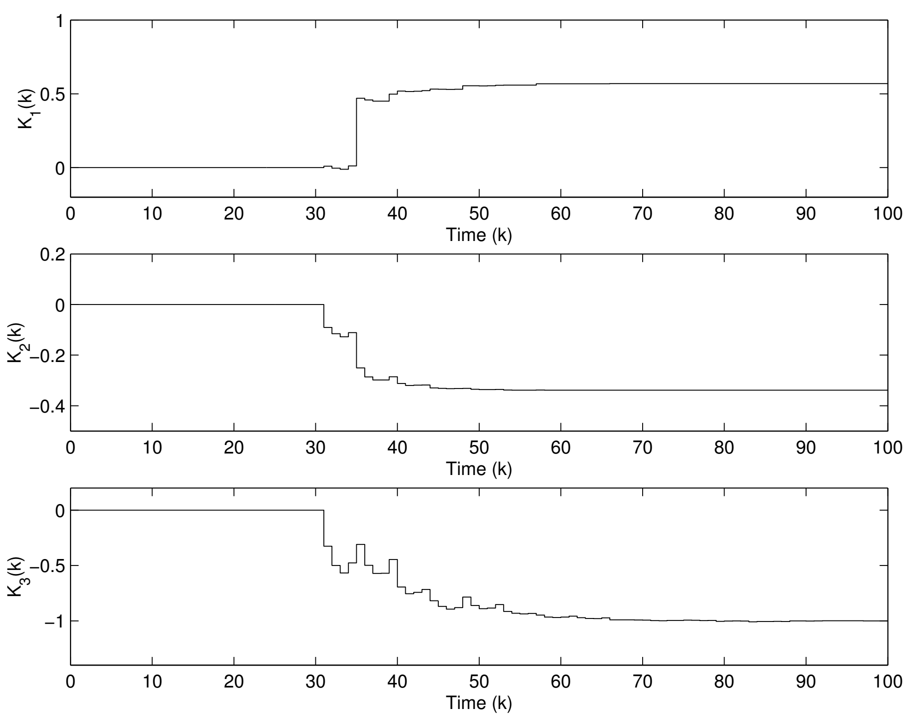



**Figure 13.3**: Adaptive gain history versus time

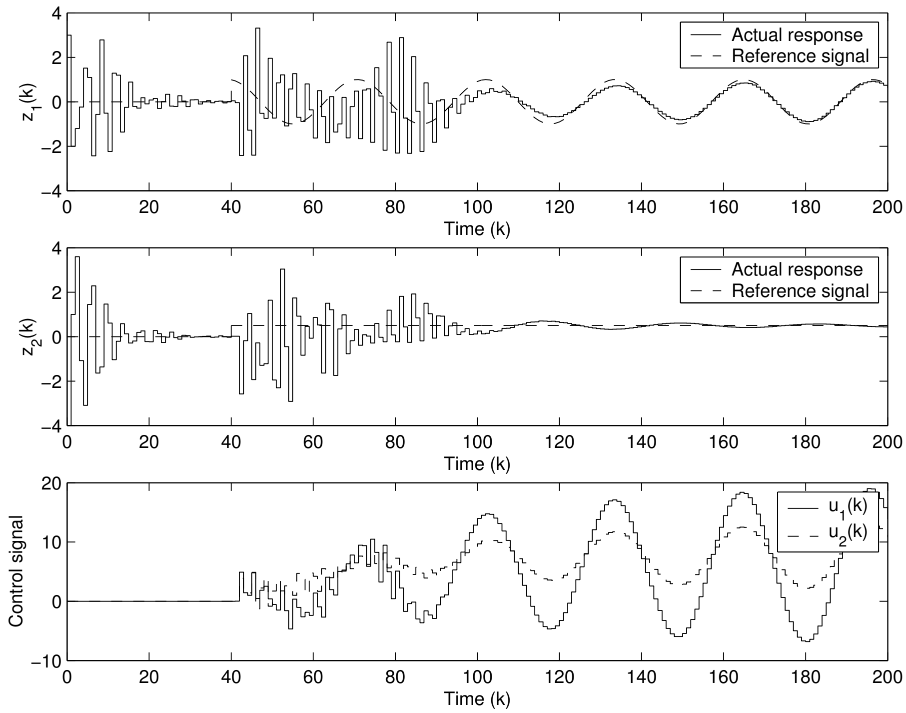



**Figure 13.4**: Positions and control signals versus time

matrices, so that

$$f_{\rm s}(e) = \begin{bmatrix} 0_2 & I_2 \\ \Theta_{\rm n_1} & \Theta_{\rm n_2} \end{bmatrix} e.$$

Note that since (13.57) is linear all the conditions of Corollary 13.1 are trivially satisfied. Now, with the proper choice of $\Theta_{\rm n_1}$ and $\Theta_{\rm n_2}$, it follows from Corollary 13.1 and Remark 13.3 that the adaptive feedback controller (13.30) guarantees that $e(k) \to 0$ as $t \to \infty$. With

$$M_{\rm s} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \quad C_{\rm s} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}, \quad K_{\rm s} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

$r_{\rm d}(k) = [\sin 0.5k, 0.5]^{\rm T}$, $\hat{\alpha} = 1$, $c = 1$, $\Theta_{\rm n_1} = \Theta_{\rm n_2} = 0_2$, and initial conditions $x(0) = [3, -4, -2, 1]^{\rm T}$ and $K(0) = 0_{2\times 10}$, Figure 13.4 shows the actual positions and the reference signals versus time and the control signals versus time. Note that the adaptive controller is switched on at $k = 40$.

**Example 13.3.** Consider the nonlinear uncertain system given by

$$z(k+2)+a_1\frac{z^3(k)}{1+z^2(k)}+a_2\ln(1+|z(k+1)|) = bu(k), \quad z(0) = z_0, \quad z(1) = z_1, \quad k \in \mathcal{N}, \tag{13.59}$$

where $z(k) \in \mathbb{R}$, $k \in \mathcal{N}$, $u(k) \in \mathbb{R}$, $k \in \mathcal{N}$, and $a_1, a_2, b \in \mathbb{R}$ are unknown constants. Note that with $x_1(k) = z(k)$ and $x_2(k) = z(k+1)$, (13.59) can be written in state space form (13.1) with $x = [x_1, x_2]^{\mathrm{T}}$, $f(x) = [x_2, -a_1\frac{x_1^3}{1+x_1^2} - a_2\ln(1+|x_2|)]^{\mathrm{T}}$, $G(x) = [0, b]^{\mathrm{T}}$, and $w(k) \equiv 0$. Here, we assume that $f(x)$ is unknown and can be parameterized as $f(x) = [x_2, \theta_1\frac{x_1^3}{1+x_1^2} + \theta_2\ln(1+|x_2|)]^{\mathrm{T}}$, where $\theta_1$ and $\theta_2$ are unknown constants. Furthermore, we assume that $\operatorname{sgn} b$ is known and $|b| < \alpha = 2$. Next, let $G_{\mathrm{n}}(x) = 1$, $F(x) = [\frac{x_1^3}{1+x_1^2}, \ln(1+|x_2|), x^{\mathrm{T}}]^{\mathrm{T}}$, $\hat{\alpha} = 1$, and $K_{\mathrm{g}} = \frac{1}{b}\left[-\theta_1, -\theta_2, \phi_{\mathrm{n}_1}, \phi_{\mathrm{n}_2}\right]$, where $\phi_{\mathrm{n}_1}$, $\phi_{\mathrm{n}_2}$ are arbitrary scalars, so that

$$\begin{aligned} f_{\mathrm{s}}(x) &= f(x) + \begin{bmatrix} 0 \\ b \end{bmatrix} \frac{1}{b} \begin{bmatrix} -\theta_1, -\theta_2, \phi_{\mathrm{n}_1}, \phi_{\mathrm{n}_2} \end{bmatrix} F(x) \\ &= \begin{bmatrix} 0 & 1 \\ \phi_{\mathrm{n}_1} & \phi_{\mathrm{n}_2} \end{bmatrix} x. \end{aligned}$$

In addition, note that $F^{\mathrm{T}}(x)F(x) = \left(\frac{x_1^2}{1+x_1^2}\right)^2 x_1^2 + \ln^2(1+|x_2|) + x^{\mathrm{T}}x \le 2x^{\mathrm{T}}x$ and thus (13.4) is satisfied with $\bar{\gamma} = 2$. Now, with the proper choice of $\phi_{\mathrm{n}_1}$ and $\phi_{\mathrm{n}_2}$, it follows from Corollary 13.1 that the adaptive feedback controller (13.30) guarantees that $x(k) \to 0$ as $k \to \infty$. With $\hat{\alpha} = 1$, $\theta_1 = 2$, $\theta_2 = -3$, $b = 1.4$, $c = 1$, $\theta_{\mathrm{n}_1} = 0.1$, $\theta_{\mathrm{n}_2} = 0.1$, and initial conditions $x(0) = [1.5, 7.3]^{\mathrm{T}}$ and $K(0) = [0, 0, 0, 0]$, Figure 13.5 shows the state trajectory versus time and the control signal versus time. Finally, Figure 13.6 shows the adaptive gain history versus time.

## 13.5. Conclusion

A discrete-time direct adaptive nonlinear control framework for adaptive stabilization, disturbance rejection, and command following of multivariable nonlinear uncertain dynamical systems with exogenous bounded disturbances and bounded en-

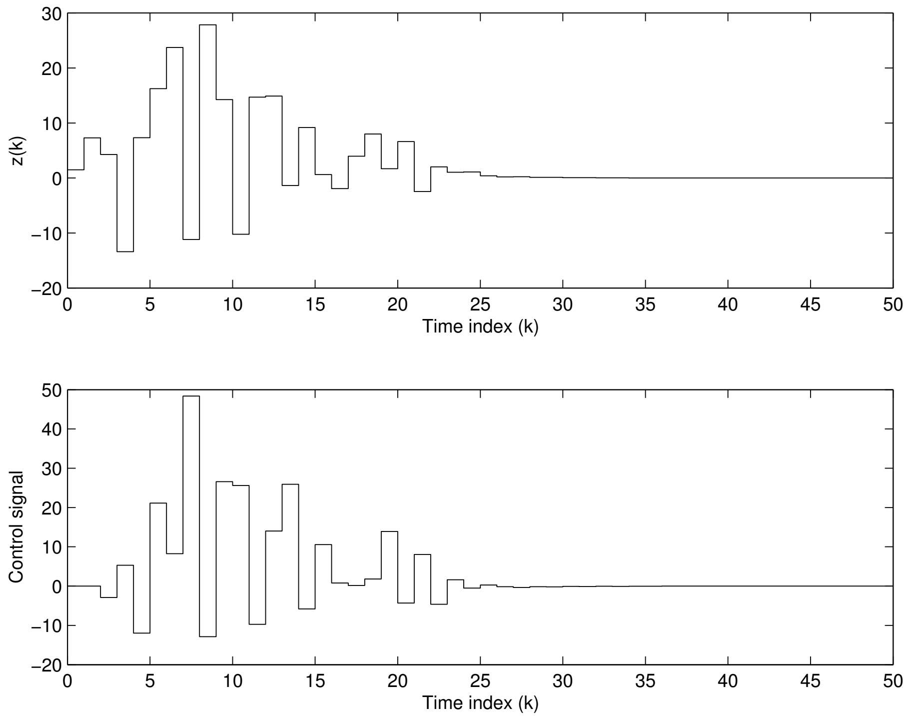



**Figure 13.5**: State trajectory and control signal versus time

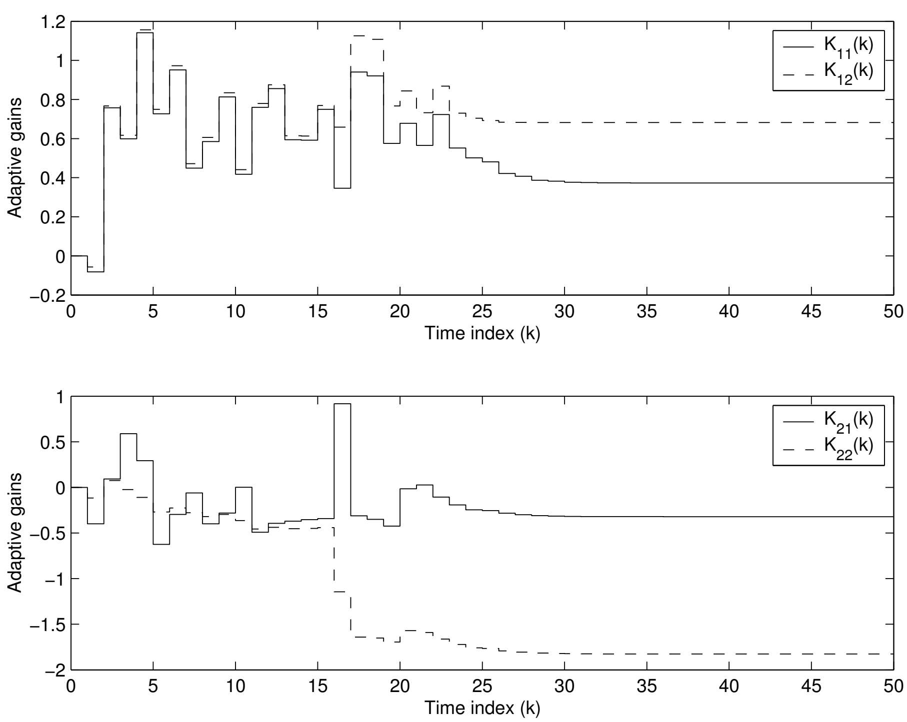



**Figure 13.6**: Adaptive gain history versus time

ergy $\ell_2$ disturbances was developed. This framework is distinct from the standard discrete-time adaptive control methods for model reference and tracking problems developed in the literature predicated on the classical key technical lemma and quadratic Lyapunov functions, which does not guarantee Lyapunov stability. Specifically, using logarithmic Lyapunov functions the proposed framework was shown to guarantee partial asymptotic stability of the closed-loop system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the plant. Hence, unlike continuous-time adaptive control theory based on quadratic Lyapunov functions, logarithmic Lyapunov functions are shown to be essential for discrete-time Lyapunov-based adaptive control. Furthermore, in the case where the nonlinear system is represented in normal form, the nonlinear adaptive controllers were constructed without knowledge of the system dynamics. Future research will involve using logarithmic Lyapunov functions to extend discrete-time adaptive control results based on recursive least squares and least mean squares algorithms to additionally guarantee partial asymptotic stability. Finally, output feedback extensions will also be considered.

# Chapter 14

# Direct Discrete-Time Adaptive Control with Guaranteed Parameter Error Convergence

## 14.1. Introduction

Adaptive control algorithms have been extensively developed in the literature for both continuous-time and discrete-time systems [56, 61, 71, 72, 136, 147, 176, 177]. A salient difference between continuous-time and discrete-time adaptive controllers is that the majority of the discrete-time results are based on recursive least-squares and least mean squares algorithms [56, 61, 71, 72, 177] with primary focus on state convergence. Notable exceptions are given in [92, 128, 196, 231, 242]. In this chapter we develop a direct adaptive nonlinear tracking control framework based on *semidefinite* or *partial* Lyapunov functions for discrete-time nonlinear uncertain dynamical systems. The proposed framework guarantees attraction of the closed-loop tracking error dynamics in the face of parametric system uncertainty. In addition, parameter error convergence is also guaranteed when a generic geometric constraint on the update error gain matrix function holds. This condition is shown to be consistent with the notion of persistent excitation in the adaptive control and system identification literature.

## 14.2. Adaptive Tracking for Nonlinear Uncertain Systems

In this section we consider the problem of characterizing adaptive feedback tracking control laws for nonlinear uncertain discrete-time systems. Specifically, consider the controlled nonlinear uncertain discrete-time system $\mathcal{G}$ given by

$$x(k+1) = f(x(k)) + G(x(k))u(k), \quad x(0) = x_0, \quad k \in \mathcal{N}, \tag{14.1}$$

where $x(k) \in \mathbb{R}^n$, $k \in \mathcal{N}$, is the state vector, $u(k) \in \mathbb{R}^m$, $k \in \mathcal{N}$, is the control input, $f : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $\mathcal{N}$ denotes the set of nonnegative integers. Here, we assume that a *desired* trajectory (command) $x_{\mathrm{d}}(k)$, $k \in \mathcal{N}$, is given and the aim is to determine the control input $u(k)$, $k \in \mathcal{N}$, so that $\lim_{k\to\infty} \|x(k) - x_{\mathrm{d}}(k)\| = 0$. To achieve this, we construct a reference system $\mathcal{G}_{\mathrm{r}}$ given by

$$x_{\mathrm{r}}(k+1) = A_{\mathrm{r}}x_{\mathrm{r}}(k) + B_{\mathrm{r}}r(k), \quad x_{\mathrm{r}}(0) = x_{\mathrm{r}_0}, \quad k \in \mathcal{N}, \tag{14.2}$$

where $x_{\mathrm{r}}(k) \in \mathbb{R}^n$, $k \in \mathcal{N}$, is the reference state vector, $r(k) \in \mathbb{R}^m$, $k \in \mathcal{N}$, is the reference input, $A_{\mathrm{r}} \in \mathbb{R}^{n\times n}$ is Schur, and $B_{\mathrm{r}} \in \mathbb{R}^{n\times m}$. Now, we design $u(k)$, $k \in \mathcal{N}$, and a bounded reference function $r(k)$, $k \in \mathcal{N}$, such that $\lim_{k\to\infty} \|x(k) - x_{\mathrm{r}}(k)\| = 0$ and $\lim_{k\to\infty} \|x_{\mathrm{r}}(k) - x_{\mathrm{d}}(k)\| = 0$, respectively, so that $\lim_{k\to\infty} \|x(k) - x_{\mathrm{d}}(k)\| = 0$. The following result provides a control architecture that achieves tracking error convergence in the case where the dynamics in (14.1) are known. The case where $\mathcal{G}$ is unknown is addressed in Theorem 14.2. For the statement of this result define the tracking error $e(k) \triangleq x(k) - x_{\mathrm{r}}(k)$.

**Theorem 14.1.** Consider the nonlinear dynamical system $\mathcal{G}$ given by (14.1) and the reference system $\mathcal{G}_{\mathrm{r}}$ given by (14.2) with $A_{\mathrm{r}}$ Schur. Assume there exist gain matrices $\hat{K}_1 \in \mathbb{R}^{m\times m}$ and $\hat{K}_2 \in \mathbb{R}^{m\times s}$, and functions $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ and $F : \mathbb{R}^n \to \mathbb{R}^s$ such that

$$0 = G(x)\hat{G}(x)\hat{K}_1 - B_{\rm r}, \quad x \in \mathbb{R}^n, \tag{14.3}$$

$$0 = f(x) + G(x)\hat{G}(x)\hat{K}_2 F(x) - A_{\rm r}x, \quad x \in \mathbb{R}^n, \tag{14.4}$$

hold. Then the feedback control law

$$u(k) = \hat{G}(x(k))(\hat{K}_1 r(k) + \hat{K}_2 F(x(k))), \tag{14.5}$$

guarantees that the zero solution $e(k) \equiv 0$ of the error dynamics given by

$$e(k+1) = f(x(k)) + G(x(k))u(k) - (A_{\rm r}x_{\rm r}(k) + B_{\rm r}r(k)), \quad e(0) = x_0 - x_{{\rm r}0} \triangleq e_0, \tag{14.6}$$

is globally asymptotically stable.

**Proof.** Using the feedback control law given by (14.5), (14.6) becomes

$$\begin{aligned} e(k+1) = {} & A_{\rm r}e(k) + (G(x(k))\hat{G}(x(k))\hat{K}_2 F(x(k)) + f(x(k)) - A_{\rm r}x(k)) \\ & + (G(x(k))\hat{G}(x(k))\hat{K}_1 - B_{\rm r})r(k), \quad e(0) = e_0, \quad k \in \mathcal{N}. \end{aligned} \tag{14.7}$$

Now, using (14.3) and (14.4), it follows from (14.7) that

$$e(k+1) = A_{\rm r}e(k), \quad e(0) = e_0, \quad k \in \mathcal{N}, \tag{14.8}$$

which, since $A_{\rm r}$ is Schur by assumption, proves that the zero solution $e(k) \equiv 0$ to (14.6) is globally asymptotically stable. □

Theorem 14.1 provides sufficient conditions for characterizing tracking controllers for a given nominal nonlinear dynamical system $\mathcal{G}$. In the next result we show how to construct adaptive gains $K_1(k) \in \mathbb{R}^{m\times m}$, $k \in \mathcal{N}$, and $K_2(k) \in \mathbb{R}^{m\times s}$, $k \in \mathcal{N}$, for achieving tracking control in the face of system uncertainty. For this result we do *not* require explicit knowledge of the gain matrices $\hat{K}_1$ and $\hat{K}_2$; all that is required is the existence of $\hat{K}_1$ and $\hat{K}_2$ such that the compatibility relations (14.3) and (14.4) hold. For the statement of the next result $(\ )^\dagger$ denotes the Moore-Penrose generalized inverse.

**Theorem 14.2.** Consider the nonlinear dynamical system $\mathcal{G}$ given by (14.1) and the reference system $\mathcal{G}_\mathrm{r}$ given by (14.2). Assume there exist gain matrices $\hat{K}_1 \in \mathbb{R}^{m\times m}$ and $\hat{K}_2 \in \mathbb{R}^{m\times s}$, and functions $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ and $F : \mathbb{R}^n \to \mathbb{R}^s$ such that $\hat{G}(\cdot)$ is invertible and (14.3) and (14.4) hold. In addition, let $\tilde{x}(k) \triangleq [r^\mathrm{T}(k), F^\mathrm{T}(x(k))]^\mathrm{T}$, $c > 0$, and $Q \in \mathbb{R}^{m\times m}$ be positive definite such that $\lambda_{\max}(Q) < 2$. Then the adaptive feedback control law

$$u(k) = \hat{G}(x(k))K(k)\tilde{x}(k), \tag{14.9}$$

where $K(k) \in \mathbb{R}^{m\times(m+s)}$, $k \in \mathcal{N}$, with update law

$$\begin{aligned} K(k+1) = {} & K(k) - \tfrac{1}{c+\tilde{x}^\mathrm{T}(k)\tilde{x}(k)} Q\hat{G}^{-1}(x(k))G^\dagger(x(k))[e(k+1) - A_\mathrm{r}e(k)]\tilde{x}^\mathrm{T}(k), \\ & K(0) = K_0, \quad k \in \mathcal{N}, \end{aligned} \tag{14.10}$$

guarantees that the solution $(x(k), x_\mathrm{r}(k), K(k))$, $k \in \mathcal{N}$, of the closed-loop system given by (14.1), (14.2), (14.9), and (14.10) satisfies $x(k) \to x_\mathrm{r}(k)$ as $k \to \infty$.

**Proof.** First, note that with $u(k)$, $k \in \mathcal{N}$, given by (14.9) it follows from (14.1)–(14.4) that the error dynamics $e(k)$, $k \in \mathcal{N}$, are given by

$$e(k+1) = A_\mathrm{r}e(k) + w(k), \quad e(0) = e_0, \quad k \in \mathcal{N}, \tag{14.11}$$

where $w(k) \triangleq G(x(k))\hat{G}(x(k))\tilde{K}(k)\tilde{x}(k)$, $k \in \mathcal{N}$, with $\tilde{K}(k) \triangleq K(k) - \hat{K}$ and $\hat{K} \triangleq [\hat{K}_1,\ \hat{K}_2]$. Furthermore, note that adding and subtracting $\hat{K}$ to and from (14.10) and using (14.11) it follows that

$$\begin{aligned} \tilde{K}(k+1) &= \tilde{K}(k) - \tfrac{1}{c+\tilde{x}^\mathrm{T}(k)\tilde{x}(k)} Q\hat{G}^{-1}(x(k))G^\dagger(x(k))[G(x(k))\hat{G}(x(k))\tilde{K}(k)\tilde{x}(k)]\tilde{x}^\mathrm{T}(k) \\ &= \tilde{K}(k) - \tfrac{1}{c+\tilde{x}^\mathrm{T}(k)\tilde{x}(k)} Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^\mathrm{T}(k), \quad K(0) = K_0, \quad k \in \mathcal{N}. \end{aligned} \tag{14.12}$$

To show that $x(k) \to x_{\rm r}(k)$ as $k \to \infty$ or, equivalently, $e(k) \to 0$ as $k \to \infty$, consider the partial Lyapunov function

$$V(K) = {\rm tr}\,(K - \hat{K})^{\rm T} Q^{-1} (K - \hat{K}). \tag{14.13}$$

Note that since $Q$ is positive definite $V(K) > 0$, $K \in \mathbb{R}^{m\times(m+s)}$, $K \neq \hat{K}$. Now, letting $e(k)$, $k \in \mathcal{N}$, denote the solution to (14.11) and using (14.12), it follows that the partial Lyapunov difference $\Delta V(K(k))$ along the closed-loop system trajectories is given by

$$\begin{aligned}
\Delta V(k, x(k), K(k)) &\triangleq V(K(k+1)) - V(K(k)) \\
&= {\rm tr}\,\left(\tilde{K}(k) - \tfrac{1}{c+\tilde{x}^{\rm T}(k)\tilde{x}(k)} Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\rm T}(k)\right)^{\rm T} Q^{-1} \\
&\quad \cdot \left(\tilde{K}(k) - \tfrac{1}{c+\tilde{x}^{\rm T}(k)\tilde{x}(k)} Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\rm T}(k)\right) - {\rm tr}\,\tilde{K}^{\rm T}(k)Q^{-1}\tilde{K}(k) \\
&= {\rm tr}\,\tilde{K}^{\rm T}(k)Q^{-1}\tilde{K}(k) - \tfrac{2}{c+\tilde{x}^{\rm T}(k)\tilde{x}(k)} {\rm tr}\,\tilde{K}^{\rm T}(k)\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\rm T}(k) \\
&\quad + \tfrac{1}{(c+\tilde{x}^{\rm T}(k)\tilde{x}(k))^2} {\rm tr}\,\tilde{x}(k)\tilde{x}^{\rm T}(k)\tilde{K}^{\rm T}(k)Q\tilde{K}(k)\tilde{x}(k)\tilde{x}^{\rm T}(k) \\
&\quad - {\rm tr}\,\tilde{K}^{\rm T}(k)Q^{-1}\tilde{K}(k) \\
&\leq -\tfrac{2}{c+\tilde{x}^{\rm T}(k)\tilde{x}(k)} \tilde{x}^{\rm T}(k)\tilde{K}^{\rm T}(k)\tilde{K}(k)\tilde{x}(k) \\
&\quad + \frac{1}{c+\tilde{x}^{\rm T}(k)\tilde{x}(k)} \tilde{x}^{\rm T}(k)\tilde{K}^{\rm T}(k)Q\tilde{K}(k)\tilde{x}(k) \\
&= -\tfrac{1}{c+\tilde{x}^{\rm T}(k)\tilde{x}(k)} \tilde{x}^{\rm T}(k)\tilde{K}^{\rm T}(k)(2I - Q)\tilde{K}(k)\tilde{x}(k) \\
&\leq 0, \quad k \in \mathcal{N},
\end{aligned} \tag{14.14}$$

where in (14.14) we used $\frac{\tilde{x}^{\rm T}\tilde{x}}{c+\tilde{x}^{\rm T}\tilde{x}} < 1$ and $2I - Q > 0$, since by assumption $\lambda_{\max}(Q) < 2$. Hence, $V(K(k))$, $k \in \mathcal{N}$, is a nonincreasing and bounded function of $k$. Thus, it follows from the monotone convergence theorem (see Theorem 8.6 of [10]) that $\lim_{k\to\infty} V(K(k))$ exists which implies that $\Delta V(k, x(k), K(k)) \to 0$ as $k \to \infty$. Now, it follows from (14.14) that $G(\cdot)$ and $\hat{G}(\cdot)$ are bounded and $\tilde{K}(k)\tilde{x}(k) \to 0$ as $k \to \infty$. Next, to show that $e(k) \to 0$ as $k \to \infty$, note that (14.11) is input-to-state stable [127] with $w(k)$ viewed as the input. Now, it follows from Lemma 3.8 of [127] that (14.11) admits a $\mathcal{K}$-asymptotic gain [127]; that is, there exists a class $\mathcal{K}$ function $\gamma(\cdot)$ such that

$$\lim_{k\to\infty} \|e(k)\| \leq \lim_{k\to\infty} \gamma(\|w(k)\|). \tag{14.15}$$

Hence, since $w(k) \to 0$ as $k \to \infty$, it follows that $e(k) \to 0$ as $k \to \infty$. □

**Remark 14.1.** Note that it was shown in the proof of Theorem 14.2 that $\tilde{K}(k)\tilde{x}(k) \to 0$ as $k \to \infty$. If $\tilde{x}(k) \notin \mathcal{N}(\tilde{K}(k))$, $k \in \mathcal{N}$, where $\mathcal{N}(\cdot)$ denotes the null space operator, it follows that $\tilde{K}(k) \to 0$ as $k \to \infty$ and hence parameter convergence is guaranteed. The condition $\tilde{x}(k) \notin \mathcal{N}(\tilde{K}(k))$, $k \in \mathcal{N}$, is a form of a *persistent excitation* requirement and, within the proposed adaptive control framework, can always be satisfied. Specifically, since $\tilde{x}(k)$, $k \in \mathcal{N}$, contains $r(k)$, which is an arbitrary function, it is always possible to choose $r(k)$, $k \in \mathcal{N}$, to guarantee that $\tilde{x}(k) \notin \mathcal{N}(\tilde{K}(k))$, $k \in \mathcal{N}$.

It is important to note that the adaptive control law (14.9) and (14.10) does *not* require explicit knowledge of the gain matrices $\hat{K}_1$ and $\hat{K}_2$. Furthermore, no specific structure on the nonlinear dynamics $f(x)$ are required to apply Theorem 14.2; all that is required is the existence of $F(x)$ and $\hat{G}(x)$ such that the compatibility relations (14.3) and (14.4) hold for a given reference system $\mathcal{G}_{\mathrm{r}}$. The compatibility conditions (14.3) and (14.4) provide a generalization to the stronger conditions already existing in the literature required for tracking control using feedback linearization techniques. However, if (14.1) is in normal form with asymptotically stable internal dynamics [122], then we can always construct functions $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ and $F : \mathbb{R}^n \to \mathbb{R}^s$, and a pair $(A_{\mathrm{r}}, B_{\mathrm{r}})$ with $A_{\mathrm{r}}$ Schur, such that (14.3) and (14.4) hold *without* requiring knowledge of the system dynamics.

To elucidate the above discussion assume that the nonlinear uncertain system $\mathcal{G}$ is generated by the difference model

$$z_i(k+\tau_i) = f_{\mathrm{u}i}(z(k)) + \sum_{j=1}^{m} G_{\mathrm{s}(i,j)}(z(k))u_j(k), \quad k \in \mathcal{N}, \quad i = 1, \cdots, m, \tag{14.16}$$

where $\tau_i \in \mathcal{N}$ denotes the time delay (or relative degree) with respect to the output $z_i$, $z(k) = [z_1(k), \cdots, z_1(k+\tau_1-1), \cdots, z_m(k), \cdots, z_m(k+\tau_m-1)]$, and $z(0) = z_0$. Here, we assume that the square matrix function $G_{\mathrm{s}}(z)$ composed of the entries $G_{\mathrm{s}(i,j)}(z)$, $i, j = 1, \cdots, m$, is such that $\det G_{\mathrm{s}}(z) \neq 0$, $z \in \mathbb{R}^{\hat{\tau}}$, where $\hat{\tau} = \tau_1 + \cdots + \tau_m$. Furthermore, since (14.16) is in a form where it does not possess internal dynamics, it follows that $\hat{\tau} = n$. The case where (14.16) possesses input-to-state stable internal dynamics can be analogously handled as shown in Section 2.2.

Next, define $x_i(k) \triangleq [\, z_i(k), \cdots, z_i(k+\tau_i-2)]^{\mathrm{T}}$, $i = 1, \cdots, m$, $x_{m+1}(k) \triangleq [\, z_1(k+\tau_1 -1), \cdots, z_m(k+\tau_m-1)]^{\mathrm{T}}$, and $x(k) \triangleq [\, x_1^{\mathrm{T}}(k), \cdots, x_{m+1}^{\mathrm{T}}(k)]^{\mathrm{T}}$ so that (14.16) can be described by (14.1) with

$$f(x) = \tilde{A}x + \tilde{f}_{\mathrm{u}}(x), \quad G(x) = \begin{bmatrix} 0_{(n-m)\times m} \\ G_{\mathrm{s}}(x) \end{bmatrix}, \tag{14.17}$$

where

$$\tilde{A} = \begin{bmatrix} A_0 \\ 0_{m\times n} \end{bmatrix}, \quad \tilde{f}_{\mathrm{u}}(x) = \begin{bmatrix} 0_{(n-m)\times 1} \\ f_{\mathrm{u}}(x) \end{bmatrix}, \tag{14.18}$$

$A_0 \in \mathbb{R}^{(n-m)\times n}$ is a known matrix of zeros and ones capturing the multivariable controllable canonical form representation [43] and $f_{\mathrm{u}} : \mathbb{R}^n \to \mathbb{R}^m$ is an unknown function. Here, we assume that $f_{\mathrm{u}}(x)$ is unknown and is parameterized as $f_{\mathrm{u}}(x) = \Theta_\ell x + \Theta_{n\ell} f_{n\ell}(x)$, where $f_{n\ell} : \mathbb{R}^n \to \mathbb{R}^q$, and $\Theta_\ell \in \mathbb{R}^{m\times n}$ and $\Theta_{n\ell} \in \mathbb{R}^{m\times q}$ are matrices of uncertain constant parameters.

Next, to apply Theorem 14.2 to the uncertain system (14.1) with $f(x)$ and $G(x)$ given by (14.17), let $B_{\mathrm{r}} = \left[0_{(n-m)\times m},\, B_{\mathrm{rs}}^{\mathrm{T}}\right]^{\mathrm{T}}$, let $A_{\mathrm{r}} = \left[A_0^{\mathrm{T}},\, \Theta_{\mathrm{n}}^{\mathrm{T}}\right]^{\mathrm{T}}$, where $\Theta_{\mathrm{n}} \in \mathbb{R}^{m\times n}$ is a known matrix, let $\hat{K}_2 \in \mathbb{R}^{m\times s}$, where $s = n+q$, be given by

$$\hat{K}_2 = [\, \Theta_{\mathrm{n}} - \Theta_\ell,\, -\Theta_{n\ell}\, ], \tag{14.19}$$

and let

$$F(x) = \begin{bmatrix} x \\ f_{n\ell}(x) \end{bmatrix}. \tag{14.20}$$

In this case, it follows that, with $\hat{G}(x) = G_{\rm s}^{-1}(x)$ and $\hat{K}_1 = B_{\rm rs}$,

$$G(x)\hat{G}(x)\hat{K}_1 = B_{\rm r} \tag{14.21}$$

and

$$\begin{aligned} f(x) + G(x)\hat{G}(x)\hat{K}_2 F(x) &= \tilde{A}x + \tilde{f}_{\rm u}(x) + \begin{bmatrix} 0_{(n-m)\times m} \\ I_m \end{bmatrix} \begin{bmatrix} \Theta_{\rm n}x - \Theta_\ell x - \Theta_{n\ell} f_{n\ell}(x) \end{bmatrix} \\ &= \tilde{A}x + \begin{bmatrix} 0_{(n-m)\times 1} \\ \Theta_{\rm n}x \end{bmatrix} \\ &= A_{\rm r}x, \end{aligned} \tag{14.22}$$

where $A_{\rm r}$ is in multivariable controllable canonical form.

Next, we consider the case where $f(x)$ and $G(x)$ are both uncertain. Specifically, we assume that $G(x)$ is such that $G_{\rm s}(x)$ is unknown and is parameterized as $G_{\rm s}(x) = B_{\rm u}G_{\rm n}(x)$, where $G_{\rm n} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is known and satisfies $\det G_{\rm n}(x) \neq 0$, $x \in \mathbb{R}^n$, and $B_{\rm u} \in \mathbb{R}^{m\times m}$, with $\det B_{\rm u} \neq 0$ and $\sigma_{\max}(B_{\rm u}) < \alpha$, $\alpha > 0$, is an unknown symmetric sign-definite matrix but a bound $\alpha$ for the maximum singular value of $B_{\rm u}$ and the sign definiteness of $B_{\rm u}$ are known; that is, $B_{\rm u} > 0$ or $B_{\rm u} < 0$. For the statement of the next result define $B_0 \triangleq \left[0_{m\times(n-m)},\ I_m\right]^{\rm T}$ for $B_{\rm u} > 0$, and $B_0 \triangleq \left[0_{m\times(n-m)},\ -I_m\right]^{\rm T}$ for $B_{\rm u} < 0$.

**Corollary 14.1.** Consider the nonlinear dynamical system $\mathcal{G}$ given by (14.1) with $f(x)$ and $G(x)$ given by (14.17), and $G_{\rm s}(x) = B_{\rm u}G_{\rm n}(x)$, where $B_{\rm u}$, with $\sigma_{\max}(B_{\rm u}) < \alpha$, $\alpha > 0$, is an unknown symmetric sign-definite matrix and the sign definiteness of $B_{\rm u}$ is known. Furthermore, consider the reference system $\mathcal{G}_{\rm r}$ given by (14.2) with $A_{\rm r}$ Schur. Assume there exist gain matrices $\hat{K}_1 \in \mathbb{R}^{m\times m}$ and $\hat{K}_2 \in \mathbb{R}^{m\times s}$, and functions $\hat{G} : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ and $F : \mathbb{R}^n \to \mathbb{R}^s$ such that (14.3) and (14.4) hold. In addition, let

$\tilde{x}(k) \triangleq [r^{\mathrm{T}}(k)\ F^{\mathrm{T}}(x(k))]^{\mathrm{T}}$ and $c > 0$. Then the adaptive feedback control law

$$u(k) = \hat{\alpha}^{-1} G_{\mathrm{n}}^{-1}(x(k)) K(k) \tilde{x}(k), \tag{14.23}$$

where $\hat{\alpha} > \frac{\alpha}{2}$ and $K(k) \in \mathbb{R}^{m \times (m+s)}$, $k \in \mathcal{N}$, with update law

$$K(k+1) = K(k) - \tfrac{1}{c + \tilde{x}^{\mathrm{T}}(k)\tilde{x}(k)} B_0^{\mathrm{T}} [e(k+1) - A_{\mathrm{r}} e(k)] \tilde{x}^{\mathrm{T}}(k), \quad K(0) = K_0, \quad k \in \mathcal{N}, \tag{14.24}$$

guarantees that the solution $(x(k), x_{\mathrm{r}}(k), K(k))$, $k \in \mathcal{N}$, of the closed-loop system given by (14.1), (14.2), (14.23), and (14.24) satisfies $x(k) \to x_{\mathrm{r}}(k)$ as $k \to \infty$.

**Proof.** The result is a direct consequence of Theorem 14.2. First, let $\hat{G}(x) = \hat{\alpha}^{-1} G_{\mathrm{n}}^{-1}(x)$ so that $G(x)\hat{G}(x) = [0_{m \times (n-m)},\ \hat{\alpha}^{-1} B_{\mathrm{u}}]^{\mathrm{T}}$, and let $\hat{K}_1 = \hat{\alpha} B_{\mathrm{u}}^{-1} B_{\mathrm{rs}}$ and $\hat{K}_2 = \hat{\alpha} B_{\mathrm{u}}^{-1} [\, \Theta_{\mathrm{n}} - \Theta_{\ell},\ -\Theta_{n\ell}\,]$. Next, since $Q$ in (14.10) is an arbitrary positive-definite matrix with $\lambda_{\max}(Q) < 2$, it can be replaced by $\hat{\alpha}^{-1}|B_{\mathrm{u}}| = \hat{\alpha}^{-1}(B_{\mathrm{u}}^2)^{\frac{1}{2}}$, where $(\cdot)^{\frac{1}{2}}$ denotes the (unique) positive-definite square root. Now, since $B_{\mathrm{u}}$ is symmetric and sign definite it follows from the Schur decomposition that $B_{\mathrm{u}} = U D_{B_{\mathrm{u}}} U^{\mathrm{T}}$, where $U$ is orthogonal and $D_{B_{\mathrm{u}}}$ is real diagonal. Hence, $\hat{\alpha}^{-1}|B_{\mathrm{u}}|\hat{G}^{-1}(x) G^{\dagger}(x) = [0_{m \times (n-m)},\ \mathcal{I}_m] = B_0^{\mathrm{T}}$, where $\mathcal{I}_m = I_m$ for $B_{\mathrm{u}} > 0$ and $\mathcal{I}_m = -I_m$ for $B_{\mathrm{s}} < 0$. Now, (14.10) implies (14.24).

## 14.3. Illustrative Numerical Examples

In this section we present two numerical examples to demonstrate the utility of the proposed discrete-time adaptive control framework for adaptive tracking and parameter identification. For both of the examples we use the reference system (14.2) with

$$A_{\mathrm{r}} = \begin{bmatrix} 0 & 1 \\ -0.01 & 0.2 \end{bmatrix}, \quad B_{\mathrm{r}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad r(k) = B_{\mathrm{r}}^{\dagger}(x_{\mathrm{d}}(k+1) - A_{\mathrm{r}} x_{\mathrm{d}}(k)), \quad k \in \mathcal{N}, \tag{14.25}$$

where $x_{\mathrm{d}}(k)$, $k \in \mathcal{N}$, is the desired command signal. Note that (14.25) implies that $\Theta_{\mathrm{n}} = [-0.01,\ 0.2]$ and $B_{\mathrm{rs}} = 1$.

**Example 14.1.** Consider the linear uncertain system given by

$$z(k+2)+a_1z(k+1)+a_0z(k)=bu(k),\quad z(0)=z_0,\quad z(1)=z_1,\quad k\in\mathcal{N},\quad (14.26)$$

where $z(k)\in\mathbb{R}$, $k\in\mathcal{N}$, $u(k)\in\mathbb{R}$, $k\in\mathcal{N}$, and $a_0, a_1, b\in\mathbb{R}$ are unknown constants. Note that with $x_1(k)=z(k)$ and $x_2(k)=z(k+1)$, (14.26) can be written in state space form (14.1) with $x=[x_1,\, x_2]^{\mathrm{T}}$, $f(x)=[x_2,\, -a_0x_1-a_1x_2]^{\mathrm{T}}$, and $G(x)=[0,\, b]^{\mathrm{T}}$. Here, we assume that $f(x)$ is unknown and can be parameterized as $f(x)=[x_2,\, \theta_1x_1+\theta_2x_2]^{\mathrm{T}}$, where $\theta_1$ and $\theta_2$ are unknown constants. Furthermore, we assume that $\operatorname{sign} b$ is known and $|b|<\alpha=2$. Next, let $\hat{G}(x)=1$, $G_{\mathrm{n}}(x)=1$, $B_{\mathrm{u}}=b$, $F(x)=x$, $\hat{K}_1=\frac{1}{b}$, and $\hat{K}_2=\frac{1}{b}\begin{bmatrix}\Theta_{\mathrm{n}_1}-\theta_1, & \Theta_{\mathrm{n}_2}-\theta_2\end{bmatrix}$, so that

$$\begin{aligned}
G(x)\hat{G}(x)\hat{K}_1 &= \begin{bmatrix}0\\ b\end{bmatrix}\frac{1}{b}=\begin{bmatrix}0\\ 1\end{bmatrix}=B_{\mathrm{r}},\\
f(x)+G(x)\hat{G}(x)\hat{K}_2F(x) &= \begin{bmatrix}0 & 1\\ \theta_1 & \theta_2\end{bmatrix}x+\begin{bmatrix}0\\ b\end{bmatrix}\frac{1}{b}\begin{bmatrix}\Theta_{\mathrm{n}_1}-\theta_1, & \Theta_{\mathrm{n}_2}-\theta_2\end{bmatrix}x\\
&= \begin{bmatrix}0 & 1\\ \Theta_{\mathrm{n}_1} & \Theta_{\mathrm{n}_2}\end{bmatrix}x\\
&= A_{\mathrm{r}}x.
\end{aligned}$$

Now, with the proper choice of $\Theta_{\mathrm{n}_1}$ and $\Theta_{\mathrm{n}_2}$, it follows from Corollary 14.1 that the adaptive feedback controller (14.23) guarantees that $x(k)\to x_{\mathrm{r}}(k)$ as $k\to\infty$. For our simulations we considered $\theta_1=-1$, $\theta_2=0.25$, $b=0.4$, $c=0.01$, $\hat{\alpha}=1$, initial conditions $x(0)=x_{\mathrm{r}}(0)=[-1,\, 3]^{\mathrm{T}}$ and $K(0)=[0,\, 0,\, 0]$, and $x_{\mathrm{d}}(k)=3\sin(\frac{\pi}{10}k)$. Figure 14.1 shows the state trajectories versus time and the control signal versus time. Figure 14.2 shows the adaptive gain history versus time. Note that the adaptive controller is switched on at $k=30$. The given commanded signal is tracked well and the convergence is fast. However, given the low frequency content of $x_{\mathrm{d}}(k)$ and hence $r(k)$, $r(k)$ is not a rich enough signal to provide persistent excitation and consequently the adaptive gains do not converge to their actual values.

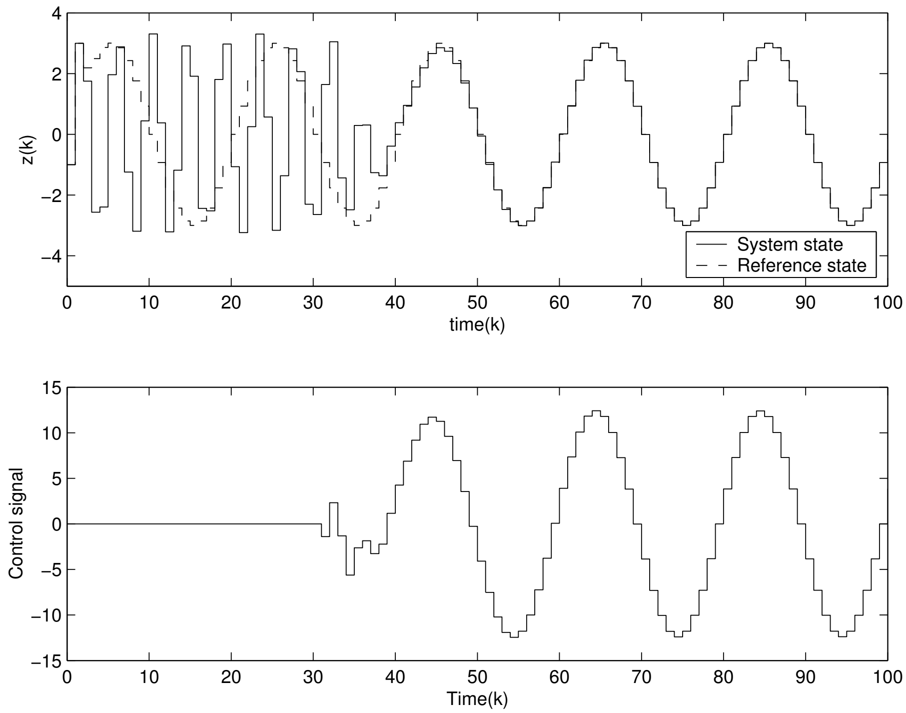



**Figure 14.1**: State and reference trajectories and control signal versus time

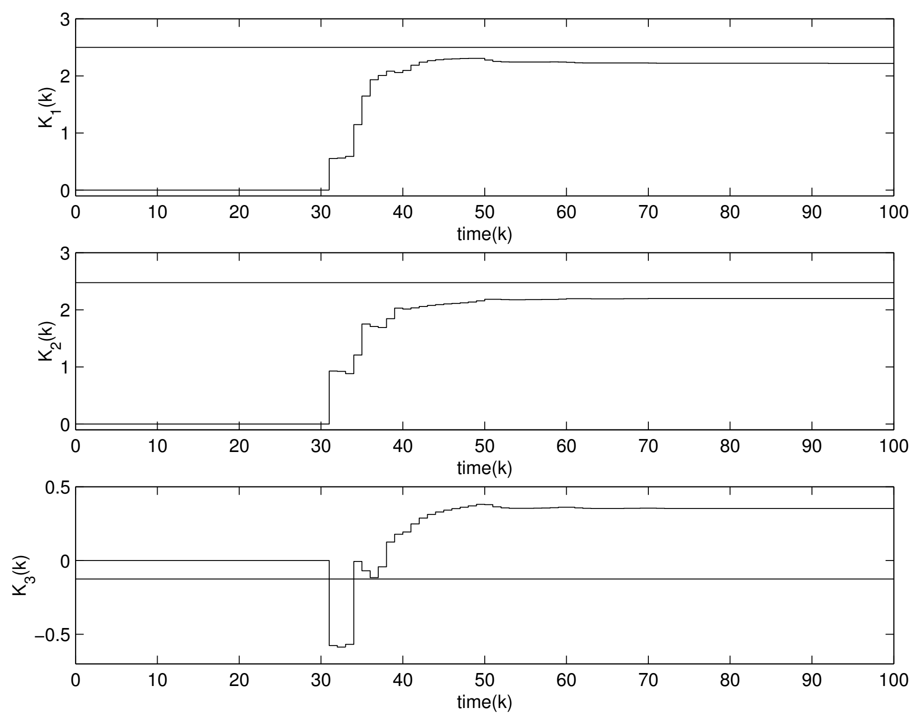



**Figure 14.2**: Adaptive gain history versus time

**Example 14.2.** Consider the nonlinear uncertain system given by

$$z(k+2)+a_1\frac{z^3(k)}{1+z^2(k)}+a_2\ln(1+|z(k+1)|) = bu(k), \quad z(0) = z_0, \quad z(1) = z_1, \quad k \in \mathcal{N}, \tag{14.27}$$

where $z(k) \in \mathbb{R}$, $k \in \mathcal{N}$, $u(k) \in \mathbb{R}$, $k \in \mathcal{N}$, and $a_1, a_2, b \in \mathbb{R}$ are unknown constants. Note that with $x_1(k) = z(k)$ and $x_2(k) = z(k+1)$, (14.27) can be written in state space form (14.1) with $x = [x_1,\, x_2]^{\mathrm{T}}$, $f(x) = [x_2,\, -a_1\frac{x_1^3}{1+x_1^2} - a_2\ln(1+|x_2|)]^{\mathrm{T}}$, and $G(x) = [0,\, b]^{\mathrm{T}}$. Here, we assume that $f(x)$ is unknown and can be parameterized as $f(x) = [x_2,\, \theta_1\frac{x_1^3}{1+x_1^2} + \theta_2\ln(1+|x_2|)]^{\mathrm{T}}$, where $\theta_1$ and $\theta_2$ are unknown constants. Furthermore, we assume that $\operatorname{sign} b$ is known and $|b| < \alpha = 2$. Next, let $\hat{G}(x) = 1$, $G_{\mathrm{n}}(x) = 1$, $B_{\mathrm{u}} = b$, $F(x) = [x^{\mathrm{T}},\, \frac{x_1^3}{1+x_1^2},\, \ln(1+|x_2|)]^{\mathrm{T}}$, $\hat{K}_1 = \frac{1}{b}$, and $\hat{K}_2 = \frac{1}{b}[\Theta_{\mathrm{n}_1}, \Theta_{\mathrm{n}_2}, -\theta_1, -\theta_2]$, so that

$$\begin{aligned}
G(x)\hat{G}(x)\hat{K}_1 &= \begin{bmatrix} 0 \\ b \end{bmatrix}\frac{1}{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = B_{\mathrm{r}}, \\
f(x) + G(x)\hat{G}(x)\hat{K}_2F(x) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}x + \begin{bmatrix} 0 \\ \theta_1\frac{x_1^3}{1+x_1^2} + \theta_2\ln(1+|x_2|) \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 \\ b \end{bmatrix}\frac{1}{b}[\Theta_{\mathrm{n}_1}, \Theta_{\mathrm{n}_2}, -\theta_1, -\theta_2]\,F(x) \\
&= \begin{bmatrix} 0 & 1 \\ \Theta_{\mathrm{n}_1} & \Theta_{\mathrm{n}_2} \end{bmatrix}x \\
&= A_{\mathrm{r}}x.
\end{aligned}$$

Now, with the proper choice of $\Theta_{\mathrm{n}_1}$ and $\Theta_{\mathrm{n}_2}$, it follows from Corollary 14.1 that the adaptive feedback controller (14.23) guarantees that $x(k) \to x_{\mathrm{r}}(k)$ as $k \to \infty$. For our simulations we considered $\theta_1 = 2$, $\theta_2 = -3$, $b = 1.4$, $c = 0.01$, $\hat{\alpha} = 2$, initial conditions $x(0) = x_{\mathrm{r}}(0) = [1.5, 7.3]^{\mathrm{T}}$, $K(0) = [0, 0, 0, 0, 0]$, and $x_{\mathrm{d}}(k) = 3\sin(\frac{\pi}{10}k)$. Figure 14.3 shows the state trajectory versus time and the control signal versus time. Finally, Figure 14.4 shows the adaptive gain history versus time. As in the previous example, the commanded signal is tracked well however the adaptive gains do not converge to their actual values.

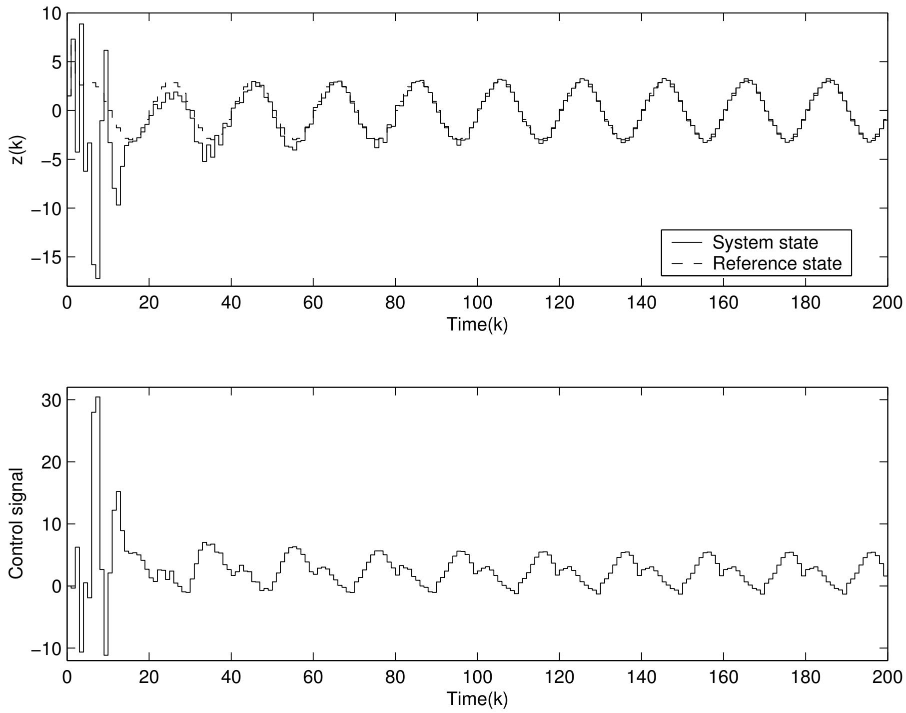



**Figure 14.3**: State trajectory and control signal versus time

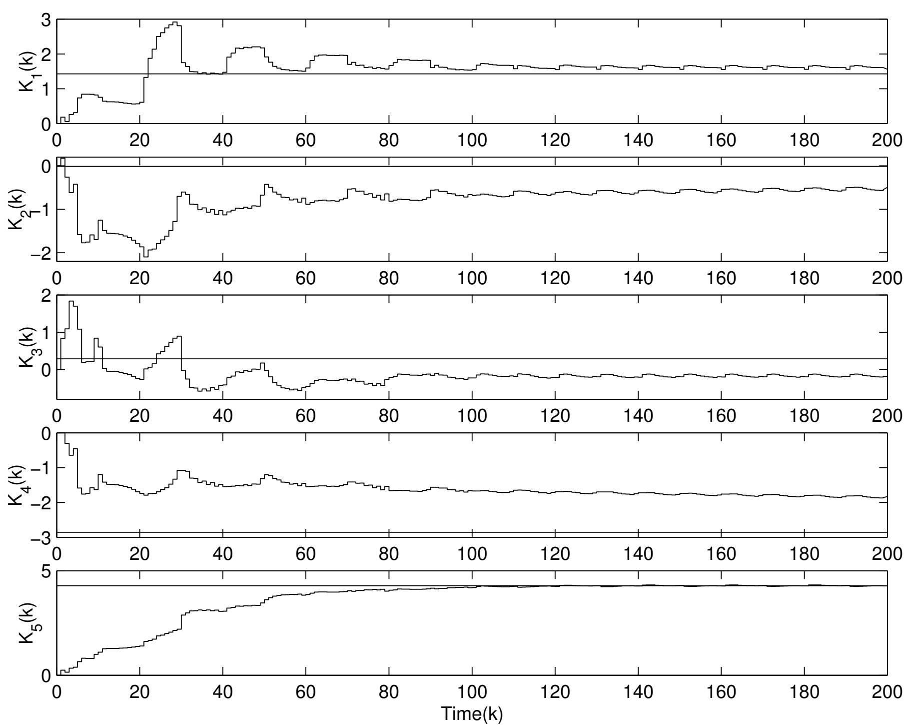



**Figure 14.4**: Adaptive gain history versus time

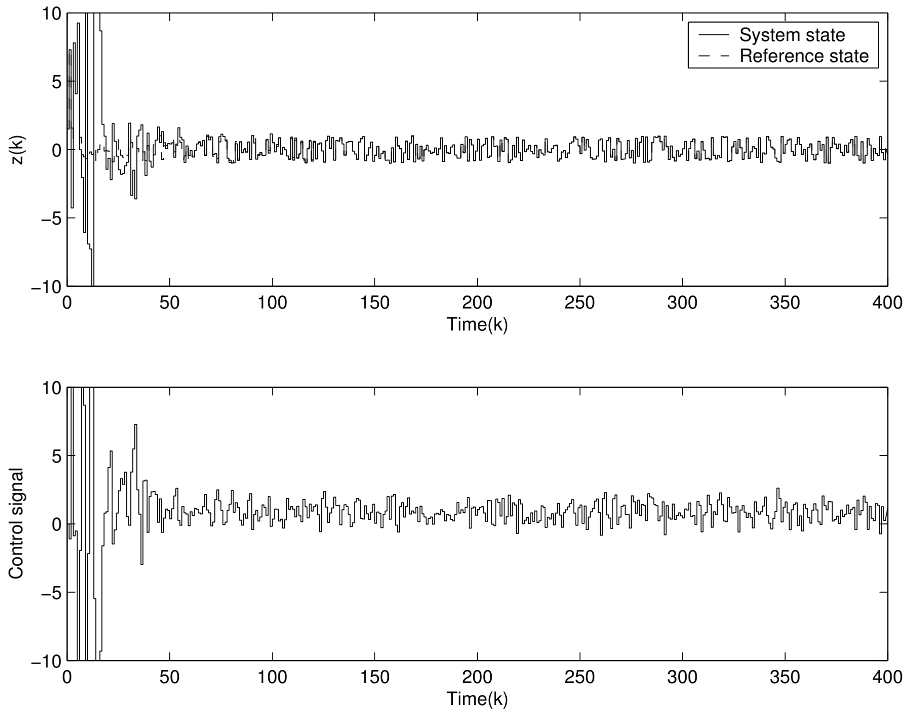



**Figure 14.5**: State trajectory and control signal versus time with random reference input

In order to demonstrate parameter error convergence we consider the same uncertain system but with a richer reference input corresponding to a random signal with values in $[-1, 1]$. Figure 14.5 shows the corresponding state trajectory versus time and the control signal versus time, while Figure 14.6 shows the adaptive gain history versus time. In this case, the persistent excitation condition is satisfied and thus the adaptive gains converge to their actual values.

## 14.4. Conclusion

A direct adaptive nonlinear tracking control framework for discrete-time multivariable nonlinear uncertain dynamical systems was developed. In addition to attraction to a desired trajectory, parameter error convergence was also guaranteed when a generic geometric constraint holds on the update error gain matrix function.

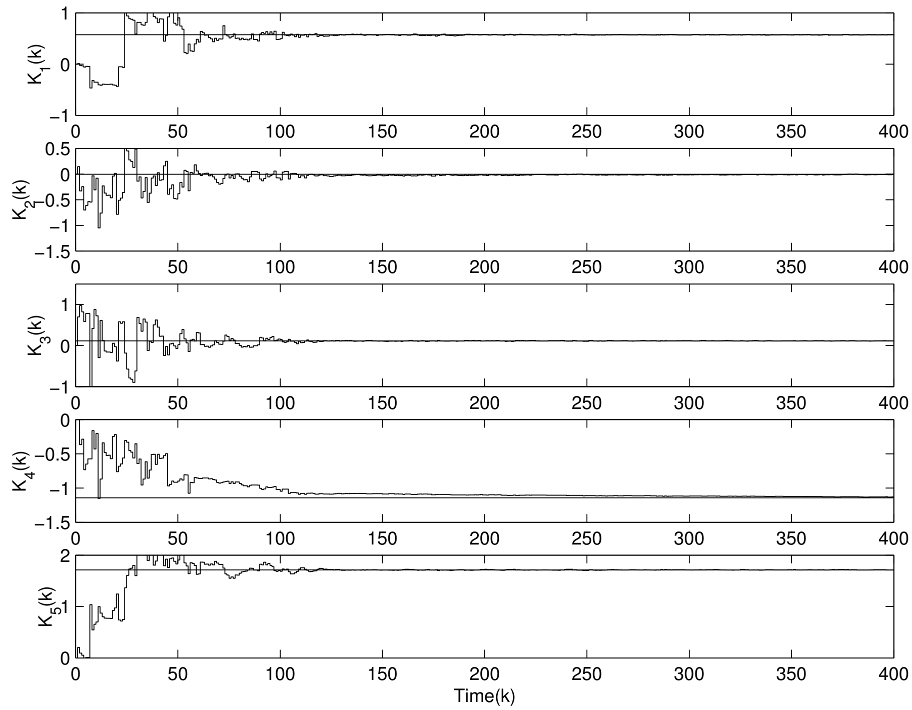



**Figure 14.6**: Adaptive gain history versus time with random reference input

This condition was shown to be consistent with a persistent excitation requirement. Finally, two numerical examples were presented to demonstrate the efficacy of the proposed adaptive tracking scheme.

# Chapter 15

# Hybrid Adaptive Control for Nonlinear Uncertain Impulsive Dynamical Systems

## 15.1. Introduction

Modern complex engineering systems involve multiple modes of operation placing stringent demands on controller design and implementation of increasing complexity. Such systems typically possess a multiechelon hierarchical *hybrid* control architecture characterized by continuous-time dynamics at the lower levels of the hierarchy and discrete-time dynamics at the higher levels of the hierarchy (see [8, 173] and the numerous references therein). The lower-level units directly interact with the dynamical system to be controlled while the higher-level units receive information from the lower-level units as inputs and provide (possibly discrete) output commands which serve to coordinate and reconcile the (sometimes competing) actions of the lower-level units. The hierarchical controller organization reduces processor cost and controller complexity by breaking up the processing task into relatively small pieces and decomposing the fast and slow control functions. Typically, the higher-level units perform logical checks that determine system mode operation, while the lower-level units execute continuous-variable commands for a given system mode of operation.

The mathematical description of many of these systems can be characterized by impulsive differential equations [15, 80, 151, 202].

The ability of developing a hierarchical nonlinear integrated hybrid control-system design methodology for robust, high performance controllers satisfying multiple design criteria and real-world hardware constraints is imperative in light of the increasingly complex nature of dynamical systems requiring controls such as advanced high performance tactical fighter aircraft, variable-cycle gas turbine engines, biological and physiological systems, sampled-data systems, discrete-event systems, intelligent vehicle/highway systems, and flight control systems, to cite but a few examples. The inherent severe nonlinearities and uncertainties of these systems and the increasingly stringent performance requirements required for controlling such modern complex embedded systems necessitates the development of hybrid adaptive nonlinear control methodologies.

Even though adaptive control algorithms have been extensively developed in the literature for both continuous-time and discrete-time systems [56,61,71,72,92,128,136, 147, 176, 177, 196, 230, 242], hybrid adaptive control algorithms for hybrid dynamical systems are nonexistent. In this chapter we develop a direct hybrid adaptive control framework for nonlinear uncertain impulsive dynamical systems. In particular, a Lyapunov-based hybrid adaptive control framework is developed that guarantees partial *asymptotic stability* of the closed-loop hybrid system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the hybrid plant dynamics. Furthermore, the remainder of the state associated with the adaptive controller gains is shown to be Lyapunov stable. Next, using the hybrid invariance principle given in [35, 80], we relax several of the conditions needed for guaranteeing partial asymptotic stabilization to develop an alternative less restrictive hybrid adaptive control framework that guarantees *attraction* of the closed-loop system states associated with the hybrid plant dynamics. In this case, the remainder of the state associated with the hybrid adaptive controller gains is shown to be bounded. In the case where the nonlinear hybrid system is represented in a *hybrid normal form*, the nonlinear hybrid adaptive controllers are constructed *without* requiring knowledge of the hybrid system dynamics. Finally, we note that since impulsive dynamical systems involve a hybrid formulation of continuous-time and discrete-time dynamics, our results build on our adaptive control algorithms for continuous-time and discrete-time systems presented in Chapters 2 and 13 (see also [84, 92, 153]).

## 15.2. Mathematical Preliminaries

In this section we review some basic concepts on impulsive dynamical systems [15,35,80,151,202]. Specifically, we consider controlled *state-dependent* [80] impulsive dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = f_{\mathrm{c}}(x(t)) + G_{\mathrm{c}}(x(t))u_{\mathrm{c}}(t), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}_x, \tag{15.1}$$

$$\Delta x(t) = f_{\mathrm{d}}(x(t)) + G_{\mathrm{d}}(x(t))u_{\mathrm{d}}(t), \quad x(t) \in \mathcal{Z}_x, \tag{15.2}$$

where $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $\Delta x(t) \triangleq x(t^+) - x(t)$, $u_{\mathrm{c}}(t) \in \mathcal{U}_{\mathrm{c}} \subseteq \mathbb{R}^{m_{\mathrm{c}}}$, $u_{\mathrm{d}}(t_k) \in \mathcal{U}_{\mathrm{d}} \subseteq \mathbb{R}^{m_{\mathrm{d}}}$, $t_k$ denotes the $k$th instant of time at which $x(t)$ intersects $\mathcal{Z}_x$ for a particular trajectory $x(t)$, $f_{\mathrm{c}} : \mathcal{D} \to \mathbb{R}^n$ is Lipschitz continuous and satisfies $f_{\mathrm{c}}(0) = 0$, $G_{\mathrm{c}} : \mathcal{D} \to \mathbb{R}^{n \times m_{\mathrm{c}}}$, $f_{\mathrm{d}} : \mathcal{Z}_x \to \mathbb{R}^n$ is continuous, $G_{\mathrm{d}} : \mathcal{Z}_x \to \mathbb{R}^{n \times m_{\mathrm{d}}}$ is such that $\operatorname{rank} G_{\mathrm{d}}(x) = m_{\mathrm{d}}$, $x \in \mathcal{Z}_x$, and $\mathcal{Z}_x \subset \mathcal{D}$ is the *resetting set*. Here, we assume that $u_{\mathrm{c}}(\cdot)$ and $u_{\mathrm{d}}(\cdot)$ are restricted to the class of *admissible* inputs consisting of measurable functions such that $(u_{\mathrm{c}}(t), u_{\mathrm{d}}(t_k)) \in \mathcal{U}_{\mathrm{c}} \times \mathcal{U}_{\mathrm{d}}$ for all $t \geq 0$ and $k \in \mathcal{N}_{[0,t)} \triangleq \{k : 0 \leq t_k < t\}$, where the constrained set $\mathcal{U}_{\mathrm{c}} \times \mathcal{U}_{\mathrm{d}}$ is given with $(0, 0) \in \mathcal{U}_{\mathrm{c}} \times \mathcal{U}_{\mathrm{d}}$. We refer to the differential equation (15.1) as the *continuous-time dynamics*, and we refer to the difference equation (15.2) as the *resetting law*. In this chapter we assume that Assumptions A1 and A2 established in [80] hold for all $u_{\rm d}(\cdot) \in \mathcal{U}_{\rm d}$; that is, the resetting set is such that resetting removes $x(t_k)$ from the resetting set and no trajectory can intersect the interior of $\mathcal{Z}_x$. Hence, as shown in [80], the resetting times are well defined and distinct. Since the resetting times are well defined and distinct and since the solution to (15.1) exists and is unique it follows that the solution of the impulsive dynamical system (15.1), (15.2) also exists and is unique over a forward time interval.

Next, we provide a key result from [35, 80] involving an invariant set stability theorem for hybrid dynamical systems. Specifically, consider the impulsive dynamical system (15.1), (15.2) with hybrid adaptive feedback controllers $u_{\rm c}(\cdot)$ and $u_{\rm d}(\cdot)$ so that the closed-loop hybrid system $\tilde{\mathcal{G}}$ has the form

$$\dot{\tilde{x}}(t) = \tilde{f}_{\rm c}(\tilde{x}(t)), \quad \tilde{x}(0) = \tilde{x}_0, \quad \tilde{x}(t) \notin \mathcal{Z}_{\tilde{x}}, \tag{15.3}$$

$$\Delta\tilde{x}(t) = \tilde{f}_{\rm d}(\tilde{x}(t)), \quad \tilde{x}(t) \in \mathcal{Z}_{\tilde{x}}, \tag{15.4}$$

where $t \geq 0$, $\tilde{x}(t) \in \tilde{\mathcal{D}} \subseteq \mathbb{R}^{\tilde{n}}$, $\tilde{x}(t)$ denotes the closed-loop state involving the system state and the adaptive gains, $\tilde{f}_{\rm c} : \tilde{\mathcal{D}} \to \mathbb{R}^{\tilde{n}}$ and $\tilde{f}_{\rm d} : \tilde{\mathcal{D}} \to \mathbb{R}^{\tilde{n}}$ denote the closed-loop continuous-time and resetting dynamics, respectively, with $\tilde{f}_{\rm c}(\tilde{x}_{\rm e}) = 0$, where $\tilde{x}_{\rm e} \in \tilde{\mathcal{D}} \backslash \mathcal{Z}_{\tilde{x}}$ denotes the closed-loop equilibrium point, and $\tilde{n}$ denotes the dimension of the closed-loop system state. For the statement of the next result the following key assumption is needed.

**Assumption 15.1** [35, 80]. Let $s(t, \tilde{x}_0)$, $t \geq 0$, denote the solution of (15.3), (15.4) with initial condition $\tilde{x}_0 \in \tilde{\mathcal{D}}$. Then for every $\tilde{x}_0 \in \tilde{\mathcal{D}}$, there exists a dense subset $\mathcal{T}_{\tilde{x}_0} \subseteq [0, \infty)$ such that $[0, \infty) \backslash \mathcal{T}_{\tilde{x}_0}$ is (finitely or infinitely) countable and for every $\epsilon > 0$ and $t \in \mathcal{T}_{\tilde{x}_0}$, there exists $\delta(\epsilon, \tilde{x}_0, t) > 0$ such that if $\|\tilde{x}_0 - y\| < \delta(\epsilon, \tilde{x}_0, t)$, $y \in \tilde{\mathcal{D}}$, then $\|s(t, \tilde{x}_0) - s(t, y)\| < \epsilon$.

Assumption 15.1 is a generalization of the standard continuous dependence property for dynamical systems with continuous flows to dynamical systems with left-continuous flows. Specifically, by letting $\mathcal{T}_{\tilde{x}_0} = \overline{\mathcal{T}}_{\tilde{x}_0} = [0, \infty)$, where $\overline{\mathcal{T}}_{\tilde{x}_0}$ denotes the closure of the set $\mathcal{T}_{\tilde{x}_0}$, Assumption 15.1 specializes to the classical continuous dependence of solutions of a given dynamical system with respect to the system's initial conditions $\tilde{x}_0 \in \tilde{\mathcal{D}}$ [232]. Since solutions of impulsive dynamical systems are *not* continuous in time and solutions are *not* continuous functions of the system initial conditions, Assumption 15.1 is needed to apply the hybrid invariance principle developed in [35, 80] to hybrid adaptive systems. Henceforth, we assume that the hybrid adaptive feedback controllers $u_{\mathrm{c}}(\cdot)$ and $u_{\mathrm{d}}(\cdot)$ are such that closed-loop hybrid system (15.3), (15.4) satisfies Assumption 15.1. Necessary and sufficient conditions that guarantee that the nonlinear impulsive dynamical system $\tilde{\mathcal{G}}$ satisfies Assumption 15.1 are given in [35]. A sufficient condition that guarantees that the trajectories of the closed-loop nonlinear impulsive dynamical system (15.3), (15.4) satisfy Assumption 15.1 are Lipschitz continuity of $\tilde{f}_{\mathrm{c}}(\cdot)$ and the existence of a continuously differentiable function $\mathcal{X} : \tilde{\mathcal{D}} \to \mathbb{R}$ such that the resetting set is given by $\mathcal{Z}_{\tilde{x}} = \{\tilde{x} \in \tilde{\mathcal{D}} : \mathcal{X}(\tilde{x}) = 0\}$, where $\mathcal{X}'(\tilde{x}) \neq 0$, $\tilde{x} \in \mathcal{Z}_{\tilde{x}}$, and $\mathcal{X}'(\tilde{x})\tilde{f}_{\mathrm{c}}(\tilde{x}) \neq 0$, $\tilde{x} \in \mathcal{Z}_{\tilde{x}}$. The last condition above insures that the solution of the closed-loop hybrid system is not tangent to the resetting set $\mathcal{Z}_{\tilde{x}}$ for all initial conditions $\tilde{x}_0 \in \tilde{\mathcal{D}}$. For further discussion on Assumption 15.1 see [35, 80].

The following theorem proven in [35, 80] is needed to develop the main results of this chapter.

**Theorem 15.1** [35, 80]. Consider the nonlinear impulsive dynamical system $\tilde{\mathcal{G}}$ given by (15.3), (15.4), assume $\tilde{\mathcal{D}}_{\mathrm{c}} \subset \tilde{\mathcal{D}}$ is a compact positively invariant set with respect to (15.3), (15.4), and assume that there exists a continuously differentiable function $V : \tilde{\mathcal{D}}_{\rm c} \to \mathbb{R}$ such that

$$V'(\tilde{x})\tilde{f}_{\rm c}(\tilde{x}) \le 0, \quad \tilde{x} \in \tilde{\mathcal{D}}_{\rm c}, \quad \tilde{x} \notin \mathcal{Z}_{\tilde{x}}, \tag{15.5}$$

$$V(\tilde{x} + \tilde{f}_{\rm d}(\tilde{x})) \le V(\tilde{x}), \quad \tilde{x} \in \tilde{\mathcal{D}}_{\rm c}, \quad \tilde{x} \in \mathcal{Z}_{\tilde{x}}. \tag{15.6}$$

Let $\mathcal{R} \triangleq \{\tilde{x} \in \tilde{\mathcal{D}}_{\rm c} : \tilde{x} \notin \mathcal{Z}_{\tilde{x}}, V'(\tilde{x})\tilde{f}_{\rm c}(\tilde{x}) = 0\} \cup \{\tilde{x} \in \tilde{\mathcal{D}}_{\rm c} : \tilde{x} \in \mathcal{Z}_{\tilde{x}}, V(\tilde{x} + \tilde{f}_{\rm d}(\tilde{x})) = V(\tilde{x})\}$ and let $\mathcal{M}$ denote the largest invariant set contained in $\mathcal{R}$. If $\tilde{x}_0 \in \tilde{\mathcal{D}}_{\rm c}$, then $\tilde{x}(t) \to \mathcal{M}$ as $t \to \infty$. Finally, if $\tilde{\mathcal{D}} = \mathbb{R}^{\tilde{n}}$ and $V(\tilde{x}) \to \infty$ as $\|\tilde{x}\| \to \infty$, then $\tilde{x}(t) \to \mathcal{M}$ as $t \to \infty$ for all $\tilde{x}_0 \in \mathbb{R}^{\tilde{n}}$.

## 15.3. Hybrid Adaptive Stabilization for Nonlinear Hybrid Dynamical Systems

In this section we consider the problem of hybrid adaptive stabilization for nonlinear uncertain hybrid systems. Specifically, we consider the controlled state-dependent impulsive dynamical system (15.1), (15.2) with $\mathcal{D} = \mathbb{R}^n$, $\mathcal{U}_{\rm c} = \mathbb{R}^{m_{\rm c}}$, and $\mathcal{U}_{\rm d} = \mathbb{R}^{m_{\rm d}}$.

**Theorem 15.2.** Consider the nonlinear uncertain hybrid dynamical system $\mathcal{G}$ given by (15.1), (15.2). Assume there exist a matrix $K_{\rm cg} \in \mathbb{R}^{m_{\rm c} \times s_{\rm c}}$, a continuously differentiable function $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, and continuous functions $\hat{G}_{\rm c} : \mathbb{R}^n \to \mathbb{R}^{m_{\rm c} \times m_{\rm c}}$, $F_{\rm c} : \mathbb{R}^n \to \mathbb{R}^{s_{\rm c}}$, and $\ell_{\rm c} : \mathbb{R}^n \to \mathbb{R}^{p_{\rm c}}$ such that $V_{\rm s}(\cdot)$ is positive definite, radially unbounded, $V_{\rm s}(0) = 0$, $\ell_{\rm c}(0) = 0$, $F_{\rm c}(0) = 0$, and, for all $x \in \mathbb{R}^n \backslash \mathcal{Z}_x$,

$$0 = V_{\rm s}'(x) f_{\rm cs}(x) + \ell_{\rm c}^{\rm T}(x)\ell_{\rm c}(x), \tag{15.7}$$

where

$$f_{\rm cs}(x) \triangleq f_{\rm c}(x) + G_{\rm c}(x)\hat{G}_{\rm c}(x)K_{\rm cg}F_{\rm c}(x). \tag{15.8}$$

Furthermore, assume there exist a matrix $K_{\rm dg} \in \mathbb{R}^{m_{\rm d} \times s_{\rm d}}$, continuous functions $\hat{G}_{\rm d} : \mathcal{Z}_x \to \mathbb{R}^{m_{\rm d} \times m_{\rm d}}$, $F_{\rm d} : \mathcal{Z}_x \to \mathbb{R}^{s_{\rm d}}$, $\ell_{\rm d} : \mathcal{Z}_x \to \mathbb{R}^{p_{\rm d}}$, matrix functions $P_{1u} : \mathcal{Z}_x \to \mathbb{R}^{1 \times m_{\rm d}}$,

$P_{2u} : \mathcal{Z}_x \to \mathbb{N}^{m_{\rm d}}$, and positive constants $\varepsilon$, $\mu$, and $\nu$ such that $\hat{G}_{\rm d}(x)$, $x \in \mathcal{Z}_x$, is invertible, $\hat{G}_{\rm d}^{\rm T}(x)P_{2u}(x)\hat{G}_{\rm d}(x) \le \nu I_{m_{\rm d}}$, $x \in \mathcal{Z}_x$, and, for all $x \in \mathcal{Z}_x$ and $u_{\rm d} \in \mathbb{R}^{m_{\rm d}}$,

$$V_{\rm s}(x + f_{\rm d}(x) + G_{\rm d}(x)u_{\rm d}) = V_{\rm s}(x + f_{\rm d}(x)) + P_{1u}(x)u_{\rm d} + u_{\rm d}^{\rm T}P_{2u}(x)u_{\rm d}, \quad (15.9)$$

$$0 \ge V_{\rm s}(x + f_{\rm ds}(x)) - V_{\rm s}(x) + \ell_{\rm d}^{\rm T}(x)\ell_{\rm d}(x) + \varepsilon P_{1u}(x)\hat{G}_{\rm d}(x)\hat{G}_{\rm d}^{\rm T}(x)P_{1u}^{\rm T}(x), \quad (15.10)$$

$$F_{\rm d}^{\rm T}(x)F_{\rm d}(x) \le \bar{\gamma}x^{\rm T}x, \quad (15.11)$$

$$V_{\rm s}(x) \ge \mu x^{\rm T}x, \quad (15.12)$$

where

$$f_{\rm ds}(x) \triangleq f_{\rm d}(x) + G_{\rm d}(x)\hat{G}_{\rm d}(x)K_{\rm dg}F_{\rm d}(x). \quad (15.13)$$

Finally, let $c > 0$, $Q_{\rm c} \in \mathbb{P}^{m_{\rm c}}$, $Q_{\rm d} \in \mathbb{P}^{m_{\rm d}}$, $Y \in \mathbb{P}^{s_{\rm c}}$, and $\lambda_{\max}(Q_{\rm d}) < 2$. Then the hybrid adaptive feedback control law

$$u_{\rm c}(t) = \hat{G}_{\rm c}(x(t))K_{\rm c}(t)F_{\rm c}(x(t)), \quad x(t) \notin \mathcal{Z}_x, \quad (15.14)$$

$$u_{\rm d}(t) = \hat{G}_{\rm d}(x(t))K_{\rm d}(t)F_{\rm d}(x(t)), \quad x(t) \in \mathcal{Z}_x, \quad (15.15)$$

where $K_{\rm c}(t) \in \mathbb{R}^{m_{\rm c}\times s_{\rm c}}$, $t \ge 0$, and $K_{\rm d}(t) \in \mathbb{R}^{m_{\rm d}\times s_{\rm d}}$, $t \ge 0$, with update laws

$$\dot{K}_{\rm c}(t) = -\tfrac{1}{2(1+V_{\rm s}(x(t)))}Q_{\rm c}\hat{G}_{\rm c}^{\rm T}(x(t))G_{\rm c}^{\rm T}(x(t))V_{\rm s}'^{\rm T}(x(t))F_{\rm c}^{\rm T}(x(t))Y,$$
$$K_{\rm c}(0) = K_{\rm c0}, \quad x(t) \notin \mathcal{Z}_x, \quad (15.16)$$

$$\Delta K_{\rm c}(t) = 0, \quad x(t) \in \mathcal{Z}_x, \quad (15.17)$$

$$\dot{K}_{\rm d}(t) = 0, \quad K_{\rm d}(0) = K_{\rm d0}, \quad x(t) \notin \mathcal{Z}_x, \quad (15.18)$$

$$\Delta K_{\rm d}(t) = -\tfrac{1}{c+F_{\rm d}^{\rm T}(x(t))F_{\rm d}(x(t))}Q_{\rm d}\hat{G}_{\rm d}^{-1}(x(t))G_{\rm d}^{\dagger}(x(t))[\Delta x(t) - f_{\rm ds}(x(t))]F_{\rm d}^{\rm T}(x(t)),$$
$$x(t) \in \mathcal{Z}_x, \quad (15.19)$$

where $\Delta K_{\rm c}(t) \triangleq K_{\rm c}(t^+) - K_{\rm c}(t)$ and $\Delta K_{\rm d}(t) \triangleq K_{\rm d}(t^+) - K_{\rm d}(t)$, guarantees that the solution $(x(t), K_{\rm c}(t), K_{\rm d}(t)) \equiv (0, K_{\rm cg}, K_{\rm dg})$ of the closed-loop hybrid system given by (15.1), (15.2), (15.14)–(15.19) is Lyapunov stable and $\ell_{\rm c}(x(t)) \to 0$ as $t \to \infty$. If, in addition, $\ell_{\rm c}^{\rm T}(x)\ell_{\rm c}(x) > 0$, $x \in \mathbb{R}^n\backslash\mathcal{Z}_x$, $x \ne 0$, and $\mathcal{W} \triangleq \{x \in \mathcal{Z}_x : \ell_{\rm d}^{\rm T}(x)\ell_{\rm d}(x) = 0\} = \emptyset$, then $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$.

**Proof.** First, define $\tilde{K}_{\rm d}(t) \triangleq K_{\rm d}(t) - K_{\rm dg}$ and $\tilde{u}_{\rm d}(t) \triangleq \tilde{K}_{\rm d}(t)F_{\rm d}(x(t))$. Note that with $u_{\rm c}(t)$, $t \geq 0$, and $u_{\rm d}(t_k)$, $k \in \mathcal{N}$, given by (15.14) and (15.15), respectively, it follows that the closed-loop hybrid system (15.1), (15.2) is given by

$$\dot{x}(t) = f_{\rm c}(x(t)) + G_{\rm c}(x(t))\hat{G}_{\rm c}(x(t))K_{\rm c}(t)F_{\rm c}(x(t)), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}_x, \tag{15.20}$$

$$\Delta x(t) = f_{\rm d}(x(t)) + G_{\rm d}(x(t))\hat{G}_{\rm d}(x(t))K_{\rm d}(t)F_{\rm d}(x(t)), \quad x(t) \in \mathcal{Z}_x, \tag{15.21}$$

or, equivalently, using (15.8) and (15.13),

$$\dot{x}(t) = f_{\rm cs}(x(t)) + G_{\rm c}(x(t))\hat{G}_{\rm c}(x(t))(K_{\rm c}(t) - K_{\rm cg})F_{\rm c}(x(t)),$$
$$x(0) = x_0, \quad x(t) \notin \mathcal{Z}_x, \tag{15.22}$$

$$\begin{aligned}\Delta x(t) &= f_{\rm ds}(x(t)) + G_{\rm d}(x(t))\hat{G}_{\rm d}(x(t))(K_{\rm d}(t) - K_{\rm dg})F_{\rm d}(x(t)) \\ &= f_{\rm ds}(x(t)) + G_{\rm d}(x(t))\hat{G}_{\rm d}(x(t))\tilde{u}_{\rm d}(t), \quad x(t) \in \mathcal{Z}_x.\end{aligned} \tag{15.23}$$

Furthermore, note that adding and subtracting $K_{\rm dg}$ to and from (15.19) and using (15.23) it follows that

$$\begin{aligned}\tilde{K}_{\rm d}(t^+) &= \tilde{K}_{\rm d}(t) - \tfrac{1}{c+F_{\rm d}^{\rm T}(x(t))F_{\rm d}(x(t))}Q_{\rm d}\hat{G}_{\rm d}^{-1}(x(t))G_{\rm d}^{\dagger}(x(t)) \\ &\quad \cdot[G_{\rm d}(x(t))\hat{G}_{\rm d}(x(t))\tilde{K}_{\rm d}(t)F_{\rm d}(x(t))]F_{\rm d}^{\rm T}(x(t)) \\ &= \tilde{K}_{\rm d}(t) - \tfrac{1}{c+F_{\rm d}^{\rm T}(x(t))F_{\rm d}(x(t))}Q_{\rm d}\tilde{K}_{\rm d}(t)F_{\rm d}(x(t))F_{\rm d}^{\rm T}(x(t)), \quad x(t) \in \mathcal{Z}_x.\end{aligned} \tag{15.24}$$

To show Lyapunov stability of the closed-loop hybrid system (15.16)–(15.18) and (15.22)–(15.24), consider the Lyapunov function candidate

$$\begin{aligned}V(x, K_{\rm c}, K_{\rm d}) &= \ln(1 + V_{\rm s}(x)) + {\rm tr}\; Q_{\rm c}^{-1}(K_{\rm c} - K_{\rm cg})Y^{-1}(K_{\rm c} - K_{\rm cg})^{\rm T} \\ &\quad + a{\rm tr}\,(K_{\rm d} - K_{\rm dg})^{\rm T}Q_{\rm d}^{-1}(K_{\rm d} - K_{\rm dg}),\end{aligned} \tag{15.25}$$

where

$$a \geq \frac{\frac{1}{4\varepsilon} + \nu}{\lambda_{\min}(2I_{m_{\rm d}} - Q_{\rm d})} \cdot \max\left\{c, \frac{\bar{\gamma}}{\mu}\right\}. \tag{15.26}$$

Note that $V(0, K_{\rm cg}, K_{\rm dg}) = 0$ and, since $V_{\rm s}(\cdot)$, $Q_{\rm c}$, $Q_{\rm d}$, and $Y$ are positive definite and $a > 0$, $V(x, K_{\rm c}, K_{\rm d}) > 0$ for all $(x, K_{\rm c}, K_{\rm d}) \neq (0, K_{\rm cg}, K_{\rm dg})$. In addition, $V(x, K_{\rm c}, K_{\rm d})$

is radially unbounded. Now, letting $x(t)$ denote the solution to (15.22) and using (15.7), (15.16), and (15.18), it follows that the Lyapunov derivative along the closed-loop system trajectories over the time interval $t \in (t_k, t_{k+1}]$, $k \in \mathcal{N}$, is given by

$$
\begin{aligned}
\dot{V}(x(t), K_{\rm c}(t), K_{\rm d}(t)) = {} & \tfrac{V_{\rm s}'(x(t))}{1+V_{\rm s}(x(t))}\Big[f_{\rm cs}(x(t)) + G_{\rm c}(x(t))\hat{G}_{\rm c}(x(t))(K_{\rm c}(t)-K_{\rm cg})F_{\rm c}(x(t))\Big] \\
& +2{\rm tr}\, Q_{\rm c}^{-1}(K_{\rm c}(t)-K_{\rm cg})Y^{-1}\dot{K}_{\rm c}^{\rm T}(t) \\
= {} & -\frac{\ell_{\rm c}^{\rm T}(x(t))\ell_{\rm c}(x(t))}{1+V_{\rm s}(x(t))} \\
& +\tfrac{1}{1+V_{\rm s}(x(t))}{\rm tr}\Big[(K_{\rm c}(t)-K_{\rm cg})F_{\rm c}(x(t))V_{\rm s}'(x(t))G_{\rm c}(x(t))\hat{G}_{\rm c}(x(t))\Big] \\
& -\tfrac{1}{1+V_{\rm s}(x(t))}{\rm tr}\Big[(K_{\rm c}(t)-K_{\rm cg})F_{\rm c}(x(t))V_{\rm s}'(x(t))G_{\rm c}(x(t))\hat{G}_{\rm c}(x(t))\Big] \\
= {} & -\frac{\ell_{\rm c}^{\rm T}(x(t))\ell_{\rm c}(x(t))}{1+V_{\rm s}(x(t))} \\
\le {} & 0, \quad t_k < t \le t_{k+1}.
\end{aligned}
\tag{15.27}
$$

Furthermore, using (15.9), (15.10), (15.17), and (15.24), the Lyapunov difference along the closed-loop system trajectories at the resetting times $t_k$, $k \in \mathcal{N}$, is given by

$$
\begin{aligned}
& \Delta V(x(t_k), K_{\rm c}(t_k), K_{\rm d}(t_k)) \\
& \quad \triangleq V(x(t_k^+), K_{\rm c}(t_k^+), K_{\rm d}(t_k^+)) - V(x(t_k), K_{\rm c}(t_k), K_{\rm d}(t_k)) \\
& \quad = \ln\Big(1 + V_{\rm s}(x(t_k) + f_{\rm ds}(x(t_k)) + G_{\rm d}(x(t_k))\hat{G}_{\rm d}(x(t_k))\tilde{u}_{\rm d}(t_k))\Big) \\
& \qquad + a{\rm tr}\left(\tilde{K}_{\rm d}(t_k) - \tfrac{1}{c+F_{\rm d}^{\rm T}(x(t_k))F_{\rm d}(x(t_k))}Q_{\rm d}\tilde{K}_{\rm d}(t_k)F_{\rm d}(x(t_k))F_{\rm d}^{\rm T}(x(t_k))\right)^{\rm T} \\
& \qquad \cdot Q_{\rm d}^{-1}\left(\tilde{K}_{\rm d}(t_k) - \tfrac{1}{c+F_{\rm d}^{\rm T}(x(t_k))F_{\rm d}(x(t_k))}Q_{\rm d}\tilde{K}_{\rm d}(t_k)F_{\rm d}(x(t_k))F_{\rm d}^{\rm T}(x(t_k))\right) \\
& \qquad - \ln(1+V_{\rm s}(x(t_k))) - a{\rm tr}\,\tilde{K}_{\rm d}^{\rm T}(t_k)Q_{\rm d}^{-1}\tilde{K}_{\rm d}(t_k) \\
& \quad = \ln\Big(1 + \Big[V_{\rm s}(x(t_k) + f_{\rm ds}(x(t_k))) + P_{1u}(x(t_k))\hat{G}_{\rm d}(x(t_k))\tilde{u}_{\rm d}(t_k) \\
& \qquad + \tilde{u}_{\rm d}^{\rm T}(t_k)\hat{G}_{\rm d}^{\rm T}(x(t_k))P_{2u}(x(t_k))\hat{G}_{\rm d}(x(t_k))\tilde{u}_{\rm d}(t_k) - V_{\rm s}(x(t_k))\Big] \\
& \qquad \cdot [1+V_{\rm s}(x(t_k))]^{-1}\Big) + a{\rm tr}\,\tilde{K}_{\rm d}^{\rm T}(t_k)Q_{\rm d}^{-1}\tilde{K}_{\rm d}(t_k) \\
& \qquad - \tfrac{2a}{c+F_{\rm d}^{\rm T}(x(t_k))F_{\rm d}(x(t_k))}{\rm tr}\,\tilde{K}_{\rm d}^{\rm T}(t_k)\tilde{K}_{\rm d}(t_k)F_{\rm d}(x(t_k))F_{\rm d}^{\rm T}(x(t_k)) \\
& \qquad + \tfrac{a}{(c+F_{\rm d}^{\rm T}(x(t_k))F_{\rm d}(x(t_k)))^2}{\rm tr}\,F_{\rm d}(x(t_k))F_{\rm d}^{\rm T}(x(t_k))\tilde{K}_{\rm d}^{\rm T}(t_k)Q_{\rm d}\tilde{K}_{\rm d}(t_k)
\end{aligned}
$$

$$
\begin{aligned}
&\cdot F_{\mathrm{d}}(x(t_k))F_{\mathrm{d}}^{\mathrm{T}}(x(t_k)) - a\mathrm{tr}\,\tilde{K}_{\mathrm{d}}^{\mathrm{T}}(t_k)Q_{\mathrm{d}}^{-1}\tilde{K}_{\mathrm{d}}(t_k) \\
\leq\ &\Big[-\ell_{\mathrm{d}}^{\mathrm{T}}(x(t_k))\ell_{\mathrm{d}}(x(t_k)) \\
&-\varepsilon P_{1u}(x(t_k))\hat{G}_{\mathrm{d}}(x(t_k))\hat{G}_{\mathrm{d}}^{\mathrm{T}}(x(t_k))P_{1u}^{\mathrm{T}}(x(t_k)) \\
&+P_{1u}(x(t_k))\hat{G}_{\mathrm{d}}(x(t_k))\tilde{u}_{\mathrm{d}}(t_k) + \nu\tilde{u}_{\mathrm{d}}^{\mathrm{T}}(t_k)\tilde{u}_{\mathrm{d}}(t_k)\Big]\,[1+V_{\mathrm{s}}(x(t_k))]^{-1} \\
&-\tfrac{2a}{c+F_{\mathrm{d}}^{\mathrm{T}}(x(t_k))F_{\mathrm{d}}(x(t_k))}F_{\mathrm{d}}^{\mathrm{T}}(x(t_k))\tilde{K}_{\mathrm{d}}^{\mathrm{T}}(t_k)\tilde{K}_{\mathrm{d}}(t_k)F_{\mathrm{d}}(x(t_k)) \\
&+\tfrac{a}{c+F_{\mathrm{d}}^{\mathrm{T}}(x(t_k))F_{\mathrm{d}}(x(t_k))}F_{\mathrm{d}}^{\mathrm{T}}(x(t_k))\tilde{K}_{\mathrm{d}}^{\mathrm{T}}(t_k)Q_{\mathrm{d}}\tilde{K}_{\mathrm{d}}(t_k)F_{\mathrm{d}}(x(t_k)), \quad k\in\mathcal{N}, \quad (15.28)
\end{aligned}
$$

where in (15.28) we used $\ln a - \ln b = \ln\frac{a}{b}$ and $\ln(1+d)\leq d$ for $a,b>0$, and $d>-1$, respectively, and $\frac{\tilde{x}^{\mathrm{T}}\tilde{x}}{c+\tilde{x}^{\mathrm{T}}\tilde{x}}<1$. Now, adding and subtracting $\frac{1}{4\varepsilon}\frac{\tilde{u}^{\mathrm{T}}(t_k)\tilde{u}(t_k)}{1+V_{\mathrm{s}}(x(t_k))}$ to and from (15.28) and collecting terms yields

$$
\begin{aligned}
&\Delta V(x(t_k),K_{\mathrm{c}}(t_k),K_{\mathrm{d}}(t_k)) \\
&\quad\leq -\tfrac{1}{1+V_{\mathrm{s}}(x(t_k))}\ell_{\mathrm{d}}^{\mathrm{T}}(x(t_k))\ell_{\mathrm{d}}(x(t_k)) \\
&\quad\quad -\tfrac{1}{1+V_{\mathrm{s}}(x(t_k))}\left[P_{1u}(x(t_k)),\ \tilde{u}_{\mathrm{d}}^{\mathrm{T}}(t_k)\right]\begin{bmatrix}\varepsilon\hat{G}_{\mathrm{d}}(x(t_k))\hat{G}_{\mathrm{d}}^{\mathrm{T}}(x(t_k)) & -\frac{1}{2}\hat{G}_{\mathrm{d}}(x(t_k)) \\ -\frac{1}{2}\hat{G}_{\mathrm{d}}^{\mathrm{T}}(x(t_k)) & \frac{1}{4\varepsilon}I_{m_{\mathrm{d}}}\end{bmatrix} \\
&\quad\quad \cdot\begin{bmatrix}P_{1u}^{\mathrm{T}}(x(t_k)) \\ \tilde{u}_{\mathrm{d}}(t_k)\end{bmatrix} + \tfrac{1}{1+V_{\mathrm{s}}(x(t_k))}[\tfrac{1}{4\varepsilon}\tilde{u}_{\mathrm{d}}^{\mathrm{T}}(t_k)\tilde{u}_{\mathrm{d}}(t_k)+\nu\tilde{u}_{\mathrm{d}}^{\mathrm{T}}(t_k)\tilde{u}_{\mathrm{d}}(t_k)] \\
&\quad\quad -\tfrac{a}{c+F_{\mathrm{d}}^{\mathrm{T}}(x(t_k))F_{\mathrm{d}}(x(t_k))}F_{\mathrm{d}}^{\mathrm{T}}(x(t_k))\tilde{K}_{\mathrm{d}}^{\mathrm{T}}(t_k)(2I_{m_{\mathrm{d}}}-Q_{\mathrm{d}})\tilde{K}_{\mathrm{d}}(t_k)F_{\mathrm{d}}(x(t_k)) \\
&\quad\leq -\frac{\ell_{\mathrm{d}}^{\mathrm{T}}(x(t_k))\ell_{\mathrm{d}}(x(t_k))}{1+V_{\mathrm{s}}(x(t_k))} - \frac{F_{\mathrm{d}}^{\mathrm{T}}(x(t_k))\tilde{K}_{\mathrm{d}}^{\mathrm{T}}(t_k)\tilde{R}(x(t_k))\tilde{K}_{\mathrm{d}}(t_k)F_{\mathrm{d}}(x(t_k))}{(c+F_{\mathrm{d}}^{\mathrm{T}}(x(t_k))F_{\mathrm{d}}(x(t_k)))(1+V_{\mathrm{s}}(x(t_k)))}, \quad k\in\mathcal{N},
\end{aligned} \tag{15.29}
$$

where

$$
\tilde{R}(x) \triangleq a(1+V_{\mathrm{s}}(x))(2I_{m_{\mathrm{d}}}-Q_{\mathrm{d}}) - \left(\frac{1}{4\varepsilon}+\nu\right)(c+F_{\mathrm{d}}^{\mathrm{T}}(x)F_{\mathrm{d}}(x))I_{m_{\mathrm{d}}}. \tag{15.30}
$$

Noting that $2I_{m_{\mathrm{d}}}-Q_{\mathrm{d}}>0$, since by assumption $\lambda_{\max}(Q_{\mathrm{d}})<2$, and $a$ is given by (15.26), it follows that

$$
\begin{aligned}
\tilde{R}(x) &\geq a(1+\mu x^{\mathrm{T}}x)(2I_{m_{\mathrm{d}}}-Q_{\mathrm{d}}) - \left(\frac{1}{4\varepsilon}+\nu\right)(c+F_{\mathrm{d}}^{\mathrm{T}}(x)F_{\mathrm{d}}(x))I_{m_{\mathrm{d}}} \\
&\geq a(1+\mu x^{\mathrm{T}}x)(2I_{m_{\mathrm{d}}}-Q_{\mathrm{d}}) - \left(\frac{1}{4\varepsilon}+\nu\right)\left(c+\bar{\gamma}x^{\mathrm{T}}x\right)I_{m_{\mathrm{d}}} \\
&\geq 0, \quad x\in\mathcal{Z}_x.
\end{aligned} \tag{15.31}
$$

Hence, the Lyapunov difference given by (15.29) yields

$$\begin{aligned}\Delta V(x(t_k), K_{\rm c}(t_k), K_{\rm d}(t_k)) &\leq -\frac{\ell_{\rm d}^{\rm T}(x(t_k))\ell_{\rm d}(x(t_k))}{1+V_{\rm s}(x(t_k))}\\ &\quad -\frac{F_{\rm d}^{\rm T}(x(t_k))\tilde{K}_{\rm d}^{\rm T}(t_k)\tilde{R}(x(t_k))\tilde{K}_{\rm d}(t_k)F_{\rm d}(x(t_k))}{(c+F_{\rm d}^{\rm T}(x(t_k))F_{\rm d}(x(t_k)))(1+V_{\rm s}(x(t_k)))}\\ &\leq -\frac{\ell_{\rm d}^{\rm T}(x(t_k))\ell_{\rm d}(x(t_k))}{1+V_{\rm s}(x(t_k))}\\ &\leq 0, \quad k \in \mathcal{N}. \end{aligned} \tag{15.32}$$

Now, it follows from Theorem 1 of [80] that (15.27) and (15.32) imply that the solution $(x(t), K_{\rm c}(t), K_{\rm d}(t)) \equiv (0, K_{\rm cg}, K_{\rm dg})$ to (15.16)–(15.18) and (15.22)–(15.24) is Lyapunov stable. Furthermore, if $\ell_{\rm c}^{\rm T}(x)\ell_{\rm c}(x) > 0$, $x \in \mathbb{R}^n\backslash\mathcal{Z}_x$, $x \neq 0$, and $\mathcal{W} = \emptyset$, then it follows from Theorem 15.1 with $\mathcal{R} = \mathcal{M} = \{(x, K_{\rm c}, K_{\rm d}) \in \mathbb{R}^n \times \mathbb{R}^{m_{\rm c}\times s_{\rm c}} \times \mathbb{R}^{m_{\rm d}\times s_{\rm d}} : x = 0\}$ that $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$. □

**Remark 15.1.** Note that in the case where $\ell_{\rm c}^{\rm T}(x)\ell_{\rm c}(x) > 0$, $x \in \mathbb{R}^n\backslash\mathcal{Z}_x$, $x \neq 0$, and $\mathcal{W} = \emptyset$, the conditions in Theorem 15.2 imply that $x(t) \to 0$ as $t \to \infty$ and hence it follows from (15.16) that $(x(t), K_{\rm c}(t), K_{\rm d}(t)) \to \mathcal{M} \triangleq \{(x, K_{\rm c}, K_{\rm d}) \in \mathbb{R}^n \times \mathbb{R}^{m_{\rm c}\times s_{\rm c}} \times \mathbb{R}^{m_{\rm d}\times s_{\rm d}} : x = 0,\ \dot{K}_{\rm c} = 0\}$ as $t \to \infty$. Furthermore, if $x(t)$, $t \geq 0$, intersects $\mathcal{Z}_x$ infinitely many times, then $(x(t), K_{\rm c}(t), K_{\rm d}(t)) \to \mathcal{M} \triangleq \{(x, K_{\rm c}, K_{\rm d}) \in \mathbb{R}^n \times \mathbb{R}^{m_{\rm c}\times s_{\rm c}} \times \mathbb{R}^{m_{\rm d}\times s_{\rm d}} : x = 0,\ \dot{K}_{\rm c} = 0,\ K_{\rm d}(t^+) = K_{\rm d}(t)\}$ as $t \to \infty$.

It is important to note that the hybrid adaptive control law (15.14)–(15.19) does *not* require explicit knowledge of the gain matrices $K_{\rm cg}$, $K_{\rm dg}$ and the positive constants $\nu$, $\bar{\gamma}$, $\varepsilon$, and $\mu$. Theorem 15.2 simply requires the existence of $K_{\rm cg}$, $K_{\rm dg}$, $\nu$, $\bar{\gamma}$, $\varepsilon$, and $\mu$ along with the construction of $F_{\rm c}(x)$, $F_{\rm d}(x)$, $\hat{G}_{\rm c}(x)$, $\hat{G}_{\rm d}(x)$, and $V_{\rm s}(x)$ such that (15.7), (15.9)–(15.12) hold. Furthermore, no specific structure on the nonlinear dynamics $f_{\rm c}(x)$ and $f_{\rm d}(x)$ is required to apply Theorem 15.2. However, if (15.1) and (15.2) are such that

$$f_{\rm c}(x) = \tilde{A}_{\rm c}x + \tilde{f}_{\rm cu}(x), \quad G_{\rm c}(x) = \begin{bmatrix} 0_{(n-m_{\rm c})\times m_{\rm c}} \\ G_{\rm cs}(x) \end{bmatrix}, \tag{15.33}$$

$$f_{\rm d}(x) = (\tilde{A}_{\rm d} - I_n)x + \tilde{f}_{\rm du}(x), \quad G_{\rm d}(x) = \begin{bmatrix} 0_{(n-m_{\rm d})\times m_{\rm d}} \\ G_{\rm ds}(x) \end{bmatrix}, \tag{15.34}$$

where

$$\tilde{A}_{\rm c} = \begin{bmatrix} A_{\rm c0} \\ 0_{m_{\rm c}\times n} \end{bmatrix}, \ \tilde{A}_{\rm d} = \begin{bmatrix} A_{\rm d0} \\ 0_{m_{\rm d}\times n} \end{bmatrix}, \ \tilde{f}_{\rm cu}(x) = \begin{bmatrix} 0_{(n-m_{\rm c})\times 1} \\ f_{\rm cu}(x) \end{bmatrix}, \ \tilde{f}_{\rm du}(x) = \begin{bmatrix} 0_{(n-m_{\rm d})\times 1} \\ f_{\rm du}(x) \end{bmatrix},$$

$A_{\rm c0} \in \mathbb{R}^{(n-m_{\rm c})\times n}$ and $A_{\rm d0} \in \mathbb{R}^{(n-m_{\rm d})\times n}$ are known matrices of zeros and ones capturing a multivariable controllable canonical form representation [43], $f_{\rm cu} : \mathbb{R}^n \to \mathbb{R}^{m_{\rm c}}$ and $f_{\rm du} : \mathbb{R}^n \to \mathbb{R}^{m_{\rm d}}$ are unknown functions with $f_{\rm cu}(0) = 0$ and $f_{\rm du}^{\rm T}(x)f_{\rm du}(x) \leq \gamma_{\rm u}x^{\rm T}x$, $x \in \mathcal{Z}_x$, where $\gamma_{\rm u} > 0$, $G_{\rm cs} : \mathbb{R}^n \to \mathbb{R}^{m_{\rm c}\times m_{\rm c}}$, and $G_{\rm ds} : \mathbb{R}^n \to \mathbb{R}^{m_{\rm d}\times m_{\rm d}}$, then we can always construct functions $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, with $V_{\rm s}(0) = 0$, $\hat{G}_{\rm c} : \mathbb{R}^n \to \mathbb{R}^{m_{\rm c}\times m_{\rm c}}$, $\hat{G}_{\rm d} : \mathbb{R}^n \to \mathbb{R}^{m_{\rm d}\times m_{\rm d}}$, $F_{\rm c} : \mathbb{R}^n \to \mathbb{R}^{s_{\rm c}}$, with $F_{\rm c}(0) = 0$, and $F_{\rm d} : \mathbb{R}^n \to \mathbb{R}^{s_{\rm d}}$ such that (15.7), (15.9)–(15.12) hold *without* requiring knowledge of the hybrid system dynamics. To see this assume that $f_{\rm cu}(x)$ and $f_{\rm du}(x)$ are unknown and are parameterized as $f_{\rm cu}(x) = \Theta_{\rm c}f_{\rm cn}(x)$ and $f_{\rm du}(x) = \Theta_{\rm d}f_{\rm dn}(x)$, where $f_{\rm cn} : \mathbb{R}^n \to \mathbb{R}^{q_{\rm c}}$ and $f_{\rm dn} : \mathbb{R}^n \to \mathbb{R}^{q_{\rm d}}$ with $f_{\rm cn}(0) = 0$ and $f_{\rm dn}^{\rm T}(x)f_{\rm dn}(x) \leq \gamma_{\rm n}x^{\rm T}x$, $x \in \mathcal{Z}_x$, where $\gamma_{\rm n} > 0$, and $\Theta_{\rm c} \in \mathbb{R}^{m_{\rm c}\times q_{\rm c}}$ and $\Theta_{\rm d} \in \mathbb{R}^{m_{\rm d}\times q_{\rm d}}$ are matrices of uncertain constant parameters.

Next, to apply Theorem 15.2 to the uncertain nonlinear hybrid system (15.1) and (15.2) with $f_{\rm c}(x)$, $f_{\rm d}(x)$, $G_{\rm c}(x)$, and $G_{\rm d}(x)$ given by (15.33) and (15.34), let $K_{\rm cg} \in \mathbb{R}^{m_{\rm c}\times s_{\rm c}}$ and $K_{\rm dg} \in \mathbb{R}^{m_{\rm d}\times s_{\rm d}}$, where $s_{\rm c} = q_{\rm c} + r_{\rm c}$ and $s_{\rm d} = q_{\rm d} + r_{\rm d}$, be given by

$$K_{\rm cg} = [\,\Theta_{\rm cn} - \Theta_{\rm c},\, \Phi_{\rm cn}\,], \quad K_{\rm dg} = [\,\Theta_{\rm dn} - \Theta_{\rm d},\, \Phi_{\rm dn}\,], \tag{15.35}$$

where $\Theta_{\rm cn} \in \mathbb{R}^{m_{\rm c}\times q_{\rm c}}$, $\Theta_{\rm dn} \in \mathbb{R}^{m_{\rm d}\times q_{\rm d}}$, $\Phi_{\rm cn} \in \mathbb{R}^{m_{\rm c}\times r_{\rm c}}$, and $\Phi_{\rm dn} \in \mathbb{R}^{m_{\rm d}\times r_{\rm d}}$ are known matrices, and let

$$F_{\rm c}(x) = \begin{bmatrix} f_{\rm cn}(x) \\ \hat{f}_{\rm cn}(x) \end{bmatrix}, \quad F_{\rm d}(x) = \begin{bmatrix} f_{\rm dn}(x) \\ \hat{f}_{\rm dn}(x) \end{bmatrix}, \tag{15.36}$$

where $\hat{f}_{\rm cn}:\mathbb{R}^n\to\mathbb{R}^{r_{\rm c}}$ and $\hat{f}_{\rm dn}:\mathbb{R}^n\to\mathbb{R}^{r_{\rm d}}$ satisfying $\hat{f}_{\rm cn}(0)=0$ and $\hat{f}_{\rm du}^{\rm T}(x)\hat{f}_{\rm du}(x)\leq\hat{\gamma}_{\rm n}x^{\rm T}x$, $x\in\mathcal{Z}_x$, $\hat{\gamma}_{\rm n}>0$, are arbitrary functions. In this case, it follows that, with $\hat{G}_{\rm c}(x)=G_{\rm cs}^{-1}(x)$ and $\hat{G}_{\rm d}(x)=G_{\rm ds}^{-1}(x)$,

$$\begin{aligned}
f_{\rm cs}(x) &= f_{\rm c}(x)+G_{\rm c}(x)\hat{G}_{\rm c}(x)K_{\rm cg}F_{\rm c}(x)\\
&= \tilde{A}_{\rm c}x+\tilde{f}_{\rm cu}(x)+\begin{bmatrix}0_{(n-m_{\rm c})\times m_{\rm c}}\\ G_{\rm cs}(x)\end{bmatrix}G_{\rm cs}^{-1}(x)\begin{bmatrix}\Theta_{\rm cn}f_{\rm cn}(x)-\Theta f_{\rm cn}(x)+\Phi_{\rm cn}\hat{f}_{\rm cn}(x)\end{bmatrix}\\
&= \tilde{A}_{\rm c}x+\begin{bmatrix}0_{(n-m_{\rm c})\times 1}\\ \Theta_{\rm cn}f_{\rm cn}(x)+\Phi_{\rm cn}\hat{f}_{\rm cn}(x)\end{bmatrix}
\end{aligned}\tag{15.37}$$

and

$$\begin{aligned}
f_{\rm ds}(x) &= f_{\rm d}(x)+G_{\rm d}(x)\hat{G}_{\rm d}(x)K_{\rm dg}F_{\rm d}(x)\\
&= (\tilde{A}_{\rm d}-I_n)x+\tilde{f}_{\rm du}(x)\\
&\quad+\begin{bmatrix}0_{(n-m_{\rm d})\times m_{\rm d}}\\ G_{\rm ds}(x)\end{bmatrix}G_{\rm ds}^{-1}(x)\left[\Theta_{\rm dn}f_{\rm dn}(x)-\Theta f_{\rm dn}(x)+\Phi_{\rm dn}\hat{f}_{\rm dn}(x)\right]\\
&= (\tilde{A}_{\rm d}-I_n)x+\begin{bmatrix}0_{(n-m_{\rm d})\times 1}\\ \Theta_{\rm dn}f_{\rm dn}(x)+\Phi_{\rm dn}\hat{f}_{\rm dn}(x)\end{bmatrix}.
\end{aligned}\tag{15.38}$$

Now, since $\Theta_{\rm cn}\in\mathbb{R}^{m_{\rm c}\times q_{\rm c}}$, $\Theta_{\rm dn}\in\mathbb{R}^{m_{\rm d}\times q_{\rm d}}$, $\Phi_{\rm cn}\in\mathbb{R}^{m_{\rm c}\times r_{\rm c}}$, and $\Phi_{\rm dn}\in\mathbb{R}^{m_{\rm d}\times r_{\rm d}}$ are arbitrary constant matrices and $\hat{f}_{\rm cn}:\mathbb{R}^n\to\mathbb{R}^{r_{\rm c}}$ and $\hat{f}_{\rm dn}:\mathbb{R}^n\to\mathbb{R}^{r_{\rm d}}$ are arbitrary functions we can always construct $K_{\rm cg}$, $K_{\rm dg}$, $V_{\rm s}(x)$, $F_{\rm c}(x)$, and $F_{\rm d}(x)$ without knowledge of $f_{\rm c}(x)$ and $f_{\rm d}(x)$ such that (15.7), (15.9), (15.10), (15.12) hold, while (15.11) is satisfied with $\bar{\gamma}\geq\gamma_{\rm n}+\hat{\gamma}_{\rm n}$. In particular, choosing $\Theta_{\rm cn}f_{\rm cn}(x)+\Phi_{\rm cn}\hat{f}_{\rm cn}(x)=\hat{A}_{\rm c}x$ and $\Theta_{\rm dn}f_{\rm dn}(x)+\Phi_{\rm dn}\hat{f}_{\rm dn}(x)=\hat{A}_{\rm d}x$, where $\hat{A}_{\rm c}\in\mathbb{R}^{m_{\rm c}\times n}$ and $\hat{A}_{\rm d}\in\mathbb{R}^{m_{\rm d}\times n}$, it follows that (15.37) and (15.38) have the form $f_{\rm cs}(x)=A_{\rm c}x$ and $f_{\rm ds}(x)=(A_{\rm d}-I_n)x$, respectively, where $A_{\rm c}=\left[A_0^{\rm T},\hat{A}_{\rm c}^{\rm T}\right]^{\rm T}$ and $A_{\rm d}=\left[A_0^{\rm T},\hat{A}_{\rm d}^{\rm T}\right]^{\rm T}$ are in multivariable controllable canonical form. Hence, we can choose $\hat{A}_{\rm c}$ and $\hat{A}_{\rm d}$ such that $A_{\rm c}$ is Hurwitz and $A_{\rm d}$ is Schur. Now, it follows from standard converse Lyapunov theory that there exists a positive-definite matrix $P$ satisfying the Lyapunov equation

$$0=A_{\rm c}^{\rm T}P+PA_{\rm c}+R_{\rm c},\tag{15.39}$$

where $R_{\rm c}$ is positive definite. If, in addition, for $G_{\rm d}(x) \equiv B_{\rm d} \in \mathbb{R}^{n\times m_{\rm d}}$, $P$ satisfies the Riccati inequality

$$0 \geq A_{\rm d}^{\rm T} P A_{\rm d} - P + R_{\rm d} + 4\varepsilon A_{\rm d}^{\rm T} P B_{\rm d} B_{\rm d}^{\rm T} P A_{\rm d}, \tag{15.40}$$

where $\varepsilon > 0$ and $R_{\rm d}$ is positive definite, then (15.7), (15.9), (15.10), and (15.12) are satisfied with $V_{\rm s}(x) = x^{\rm T}Px$, $\hat{G}_{\rm d}(x) \equiv I_{m_{\rm d}}$, $P_{1u}(x) = 2x^{\rm T}A_{\rm d}^{\rm T}PB_{\rm d}$, $P_{2u}(x) = B_{\rm d}^{\rm T}PB_{\rm d}$, and $\mu \leq \lambda_{\min}(P)$. Hence, the hybrid adaptive feedback controller (15.14) and (15.15) with update laws (15.16), or, equivalently,

$$\dot{K}_{\rm c}(t) = -\tfrac{1}{2(1+x^{\rm T}(t)Px(t))} Q_{\rm c}\hat{G}_{\rm c}^{\rm T}(x(t))G_{\rm c}^{\rm T}(x(t))Px(t)F_{\rm c}^{\rm T}(x(t))Y, \tag{15.41}$$

and (15.17)–(15.19) guarantees global asymptotic stability of the *nonlinear* hybrid uncertain dynamical system (15.1) and (15.2) where $f_{\rm c}(x)$, $f_{\rm d}(x)$, $G_{\rm c}(x)$, and $G_{\rm d}(x)$ are given by (15.33) and (15.34) with $G_{\rm ds}(x) \equiv B_{\rm ds} \in \mathbb{R}^{m_{\rm d}\times m_{\rm d}}$. Note that since $R_{\rm c}$ and $R_{\rm d}$ are arbitrary, (15.39) and (15.40) can be cast as a linear matrix inequality (LMI) feasibility problem involving $P > 0$, $A_{\rm c}^{\rm T}P + PA_{\rm c} < 0$, and

$$\begin{bmatrix} A_{\rm d}^{\rm T}PA_{\rm d} - P & A_{\rm d}^{\rm T}PB_{\rm d} \\ B_{\rm d}^{\rm T}PA_{\rm d} & -4\varepsilon I_{m_{\rm d}} \end{bmatrix} < 0.$$

Finally, as mentioned above, it is important to note that it is not necessary to utilize a feedback linearizing function $F_{\rm c}(x)$ and $F_{\rm d}(x)$ to produce linear functions $f_{\rm cs}(x)$ and $f_{\rm ds}(x)$. However, as shown above, when the hybrid system is in a *hybrid normal form* given by (15.33), (15.34), the feedback linearizing functions $F_{\rm c}(x)$ and $F_{\rm d}(x)$ provide considerable simplification in constructing $V_{\rm s}(x)$ necessary in computing the hybrid update law (15.16).

Note that by choosing $\Theta_{\rm dn} = \Phi_{\rm dn} = 0$ considerable simplification occurs in the update law (15.19). Specifically, in this case it follows that

$$G_{\rm d}^{\dagger}(x)f_{\rm ds}(x) = \begin{bmatrix} 0_{m\times(n-m)}, & G_{\rm ds}^{-1}(x) \end{bmatrix} \begin{bmatrix} A_0 \\ 0_{m\times n} \end{bmatrix} x = 0, \quad x \in \mathcal{Z}_x,$$

and hence the update law (15.19) can be simplified as

$$\Delta K_{\rm d}(t) = -\tfrac{1}{c+F_{\rm d}^{\rm T}(x(t))F_{\rm d}(x(t))}Q_{\rm d}\hat{G}_{\rm d}^{-1}(x(t))G_{\rm d}^{\dagger}(x(t))\Delta x(t)F_{\rm d}^{\rm T}(x(t)), \quad x(t)\in\mathcal{Z}_x. \tag{15.42}$$

Furthermore, it is also important to note that Theorem 15.2 is not restricted to hybrid dynamical systems satisfying the linear growth constraint $f_{\rm d}^{\rm T}(x)f_{\rm d}(x)\leq\hat{\gamma}x^{\rm T}x$, $x\in\mathcal{Z}_x$, $\hat{\gamma}>0$. Theorem 15.2 can be used to construct hybrid adaptive controllers so long as the function $F_{\rm d}(x)$ satisfies (15.11) and we can construct a function $V_{\rm s}(x)$ such that (15.7), (15.9)–(15.12) hold. Finally, in the case where $\mathcal{Z}_x$ is a bounded set, there always exists $\hat{\gamma}>0$ such that $f_{\rm d}^{\rm T}(x)f_{\rm d}(x)\leq\hat{\gamma}x^{\rm T}x$, $x\in\mathcal{Z}_x$, holds. This implies that in this case we can always construct $F_{\rm d}(x)$ such that (15.11) is satisfied.

Next, we consider the case where $f_{\rm c}(x)$, $f_{\rm d}(x)$, $G_{\rm c}(x)$, and $G_{\rm d}(x)$ are uncertain. Specifically, we assume that $G_{\rm c}(x)$ and $G_{\rm d}(x)$ are such that $G_{\rm cs}(x)$ and $G_{\rm ds}(x)$ are unknown and are parameterized as $G_{\rm cs}(x)=B_{\rm cu}G_{\rm cn}(x)$ and $G_{\rm ds}(x)=B_{\rm du}G_{\rm dn}(x)$, where $G_{\rm cn}:\mathbb{R}^n\to\mathbb{R}^{m_{\rm c}\times m_{\rm c}}$ and $G_{\rm dn}:\mathbb{R}^n\to\mathbb{R}^{m_{\rm d}\times m_{\rm d}}$ are known and satisfy $\det G_{\rm cn}(x)\neq 0$, $x\in\mathbb{R}^n\backslash\mathcal{Z}_x$, $\det G_{\rm dn}(x)\neq 0$, $x\in\mathcal{Z}_x$, and $B_{\rm cu}\in\mathbb{R}^{m_{\rm c}\times m_{\rm c}}$ and $B_{\rm du}\in\mathbb{R}^{m_{\rm d}\times m_{\rm d}}$, with $\det B_{\rm cu}\neq 0$ and $\det B_{\rm du}\neq 0$, are unknown symmetric sign-definite matrices but a bound $\alpha$ for the maximum singular value of $B_{\rm du}$ is known and the sign definiteness of $B_{\rm cu}$ and $B_{\rm du}$ are known. For the statement of the next result define $B_{\rm c0}\triangleq\left[0_{m_{\rm c}\times(n-m_{\rm c})},\ I_{m_{\rm c}}\right]^{\rm T}$ for $B_{\rm cu}>0$, $B_{\rm c0}\triangleq\left[0_{m_{\rm c}\times(n-m_{\rm c})},\ -I_{m_{\rm c}}\right]^{\rm T}$ for $B_{\rm cu}<0$, $B_{\rm d0}\triangleq\left[0_{m_{\rm d}\times(n-m_{\rm d})},\ I_{m_{\rm d}}\right]^{\rm T}$ for $B_{\rm du}>0$, and $B_{\rm d0}\triangleq\left[0_{m_{\rm d}\times(n-m_{\rm d})},\ -I_{m_{\rm d}}\right]^{\rm T}$ for $B_{\rm du}<0$.

**Corollary 15.1.** Consider the nonlinear uncertain hybrid dynamical system $\mathcal{G}$ given by (15.1) and (15.2) with $f_{\rm c}(x)$, $f_{\rm d}(x)$, $G_{\rm c}(x)$, and $G_{\rm d}(x)$ given by (15.33), (15.34), and $G_{\rm cs}(x)=B_{\rm cu}G_{\rm cn}(x)$ and $G_{\rm ds}(x)=B_{\rm du}G_{\rm dn}(x)$, where $B_{\rm cu}\in\mathbb{R}^{m_{\rm c}\times m_{\rm c}}$ and $B_{\rm du}\in\mathbb{R}^{m_{\rm d}\times m_{\rm d}}$ are unknown symmetric matrices and the sign definiteness of $B_{\rm cu}$ and $B_{\rm du}$ are known and $\sigma_{\max}(B_{\rm du})<\alpha$, $\alpha>0$. Assume there exist a matrix

$K_{\mathrm{cg}} \in \mathbb{R}^{m_{\mathrm{c}} \times s_{\mathrm{c}}}$, a continuously differentiable function $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, and continuous functions $\hat{G}_{\mathrm{c}} : \mathbb{R}^n \to \mathbb{R}^{m_{\mathrm{c}} \times m_{\mathrm{c}}}$, $F_{\mathrm{c}} : \mathbb{R}^n \to \mathbb{R}^{s_{\mathrm{c}}}$, and $\ell_{\mathrm{c}} : \mathbb{R}^n \to \mathbb{R}^{p_{\mathrm{c}}}$ such that $V_{\mathrm{s}}(\cdot)$ is positive definite, radially unbounded, $V_{\mathrm{s}}(0) = 0$, $\ell_{\mathrm{c}}(0) = 0$, $F_{\mathrm{c}}(0) = 0$, and, for all $x \in \mathbb{R}^n \backslash \mathcal{Z}_x$, (15.7) holds. Furthermore, assume that there exist a matrix $K_{\mathrm{dg}} \in \mathbb{R}^{m_{\mathrm{d}} \times s_{\mathrm{d}}}$, continuous functions $\hat{G}_{\mathrm{d}} : \mathcal{Z}_x \to \mathbb{R}^{m_{\mathrm{d}} \times m_{\mathrm{d}}}$, $F_{\mathrm{d}} : \mathcal{Z}_x \to \mathbb{R}^{s_{\mathrm{d}}}$, $\ell_{\mathrm{d}} : \mathcal{Z}_x \to \mathbb{R}^{p_{\mathrm{d}}}$, matrix functions $P_{1u} : \mathcal{Z}_x \to \mathbb{R}^{1 \times m_{\mathrm{d}}}$, $P_{2u} : \mathcal{Z}_x \to \mathbb{N}^{m_{\mathrm{d}}}$, and positive constants $\varepsilon$, $\mu$, and $\nu$ such that $\hat{G}_{\mathrm{d}}(x)$, $x \in \mathcal{Z}_x$, is invertible, $\hat{\alpha}^{-2}\hat{G}_{\mathrm{d}}^{\mathrm{T}}(x)P_{2u}(x)\hat{G}_{\mathrm{d}}(x) \leq \nu I_m$, $x \in \mathcal{Z}_x$, where $\hat{\alpha} \geq \alpha/2$, and, for all $x \in \mathcal{Z}_x$ and $u_{\mathrm{d}} \in \mathbb{R}^{m_{\mathrm{d}}}$, (15.9)–(15.12) hold. Finally, let $c > 0$ and $Y \in \mathbb{P}^{s_{\mathrm{c}}}$. Then the hybrid adaptive feedback control law

$$u_{\mathrm{c}}(t) = G_{\mathrm{cn}}^{-1}(x(t))K_{\mathrm{c}}(t)F_{\mathrm{c}}(x(t)), \quad x(t) \notin \mathcal{Z}_x, \tag{15.43}$$

$$u_{\mathrm{d}}(t) = \hat{\alpha}^{-1}G_{\mathrm{dn}}^{-1}(x(t))K_{\mathrm{d}}(k)F_{\mathrm{d}}(x(t)), \quad x(t) \in \mathcal{Z}_x, \tag{15.44}$$

where $K_{\mathrm{c}}(t) \in \mathbb{R}^{m \times s_{\mathrm{c}}}$, $t \geq 0$, and $K_{\mathrm{d}}(t) \in \mathbb{R}^{m \times s_{\mathrm{d}}}$, $t \geq 0$, with update laws

$$\dot{K}_{\mathrm{c}}(t) = -\tfrac{1}{2(1+V_{\mathrm{s}}(x(t)))}B_{\mathrm{c0}}^{\mathrm{T}}V_{\mathrm{s}}'^{\mathrm{T}}(x(t))F_{\mathrm{c}}^{\mathrm{T}}(x(t))Y, \quad K_{\mathrm{c}}(0) = K_{\mathrm{c0}}, \quad x(t) \notin \mathcal{Z}_x, \tag{15.45}$$

$$\Delta K_{\mathrm{c}}(t) = 0, \quad x(t) \in \mathcal{Z}_x, \tag{15.46}$$

$$\dot{K}_{\mathrm{d}}(t) = 0, \quad K_{\mathrm{d}}(0) = K_{\mathrm{d0}}, \quad x(t) \notin \mathcal{Z}_x, \tag{15.47}$$

$$\Delta K_{\mathrm{d}}(t) = -\tfrac{1}{c+F_{\mathrm{d}}^{\mathrm{T}}(x(t))F_{\mathrm{d}}(x(t))}B_{\mathrm{d0}}^{\mathrm{T}}[\Delta x(t) - f_{\mathrm{ds}}(x(t))]F_{\mathrm{d}}^{\mathrm{T}}(x(t)), \quad x(t) \in \mathcal{Z}_x, \tag{15.48}$$

guarantees that the solution $(x(t), K_{\mathrm{c}}(t), K_{\mathrm{d}}(t)) \equiv (0, K_{\mathrm{cg}}, K_{\mathrm{dg}})$ of the closed-loop hybrid system given by (15.1), (15.2), (15.43)–(15.48) is Lyapunov stable. If, in addition, $\ell_{\mathrm{c}}^{\mathrm{T}}(x)\ell_{\mathrm{c}}(x) > 0$, $x \in \mathbb{R}^n \backslash \mathcal{Z}_x$, $x \neq 0$, and $\mathcal{W} \triangleq \{x \in \mathcal{Z}_x : \ell_{\mathrm{d}}^{\mathrm{T}}(x)\ell_{\mathrm{d}}(x) = 0\} = \emptyset$, then $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$.

**Proof.** The result is a direct consequence of Theorem 15.2. First, let $\hat{G}_{\mathrm{c}}(x) = G_{\mathrm{cn}}^{-1}(x)$ and $\hat{G}_{\mathrm{d}}(x) = \hat{\alpha}^{-1}G_{\mathrm{dn}}^{-1}(x)$ so that $G_{\mathrm{c}}(x)\hat{G}_{\mathrm{cn}}(x) = [0_{m \times (n-m)}, B_{\mathrm{cu}}]^{\mathrm{T}}$ and $G_{\mathrm{d}}(x)$ $\cdot\hat{G}_{\mathrm{dn}}(x) = [0_{m \times (n-m)}, \hat{\alpha}^{-1}B_{\mathrm{du}}]^{\mathrm{T}}$, and let $K_{\mathrm{cg}} = B_{\mathrm{cu}}^{-1}[\Theta_{\mathrm{cn}} - \Theta_{\mathrm{c}}, \Phi_{\mathrm{cn}}]$ and $K_{\mathrm{dg}} = \hat{\alpha}B_{\mathrm{du}}^{-1}$ $\cdot[\Theta_{\mathrm{dn}} - \Theta_{\mathrm{d}}, \Phi_{\mathrm{dn}}]$. Next, since $Q_{\mathrm{c}}$ and $Q_{\mathrm{d}}$ are arbitrary positive definite matrices with

$\lambda_{\max}(Q_{\mathrm{d}}) < 2$, $Q_{\mathrm{c}}$ in (15.16) and $Q_{\mathrm{d}}$ in (15.19) can be replaced by $q_{\mathrm{c}}|B_{\mathrm{cu}}|^{-1}$ and $\hat{\alpha}^{-1}|B_{\mathrm{du}}|^{-1}$, respectively, where $q_{\mathrm{c}}$ is a positive constant, $|B_{\mathrm{cu}}| = (B_{\mathrm{cu}}^2)^{\frac{1}{2}}$, and $|B_{\mathrm{du}}| = (B_{\mathrm{du}}^2)^{\frac{1}{2}}$, where $(\cdot)^{\frac{1}{2}}$ denotes the (unique) positive definite square root. Now, since $B_{\mathrm{cu}}$ and $B_{\mathrm{du}}$ are symmetric and sign definite it follows from the Schur decomposition that $B_{\mathrm{cu}} = U_{\mathrm{c}} D_{B_{\mathrm{cu}}} U_{\mathrm{c}}^{\mathrm{T}}$ and $B_{\mathrm{du}} = U_{\mathrm{d}} D_{B_{\mathrm{du}}} U_{\mathrm{d}}^{\mathrm{T}}$, where $U_{\mathrm{c}}$ and $U_{\mathrm{d}}$ are orthogonal and $D_{B_{\mathrm{cu}}}$ and $D_{B_{\mathrm{du}}}$ are real diagonal. Hence, $|B_{\mathrm{cu}}|^{-1}\hat{G}_{\mathrm{c}}^{\mathrm{T}}(x)G_{\mathrm{c}}^{\mathrm{T}}(x) = [0_{m_{\mathrm{c}}\times(n-m_{\mathrm{c}})}, \mathcal{I}_{m_{\mathrm{c}}}] = B_{\mathrm{c}0}^{\mathrm{T}}$ and $\hat{\alpha}^{-1}|B_{\mathrm{du}}|^{-1}\hat{G}_{\mathrm{d}}^{\mathrm{T}}(x)G_{\mathrm{d}}^{\mathrm{T}}(x) = [0_{m_{\mathrm{d}}\times(n-m_{\mathrm{d}})}, \mathcal{I}_{m_{\mathrm{d}}}] = B_{\mathrm{d}0}^{\mathrm{T}}$, where $\mathcal{I}_{m_{\mathrm{c}}} = I_{m_{\mathrm{c}}}$ for $B_{\mathrm{cu}} > 0$, $\mathcal{I}_{m_{\mathrm{c}}} = -I_{m_{\mathrm{c}}}$ for $B_{\mathrm{cu}} < 0$, $\mathcal{I}_{m_{\mathrm{d}}} = I_{m_{\mathrm{d}}}$ for $B_{\mathrm{du}} > 0$, and $\mathcal{I}_{m_{\mathrm{d}}} = I_{m_{\mathrm{d}}}$ for $B_{\mathrm{du}} < 0$. Now, (15.16) and (15.19) imply (15.45) and (15.48), respectively. □

## 15.4. Hybrid Adaptive Attraction Control for Nonlinear Hybrid Dynamical Systems

In this section we relax several of the structural conditions given in Theorem 15.2, needed for guaranteeing partial asymptotic stabilization, to develop hybrid adaptive controllers with less restrictive conditions guaranteeing attraction of the closed-loop system states associated with the hybrid plant dynamics. Specifically, we develop hybrid adaptive attraction controllers without the linear growth assumption (15.11) nor the structural constraints (15.9) and (15.12). Here, once again we consider the controlled state-dependent impulsive dynamical system (15.1), (15.2) with $\mathcal{D} = \mathbb{R}^n$, $\mathcal{U}_{\mathrm{c}} = \mathbb{R}^{m_{\mathrm{c}}}$, and $\mathcal{U}_{\mathrm{d}} = \mathbb{R}^{m_{\mathrm{d}}}$.

**Theorem 15.3.** Consider the nonlinear uncertain hybrid dynamical system $\mathcal{G}$ given by (15.1), (15.2). Assume there exist a matrix $K_{\mathrm{cg}} \in \mathbb{R}^{m_{\mathrm{c}}\times s_{\mathrm{c}}}$, a continuously differentiable function $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, and continuous functions $\hat{G}_{\mathrm{c}} : \mathbb{R}^n \to \mathbb{R}^{m_{\mathrm{c}}\times m_{\mathrm{c}}}$, $F_{\mathrm{c}} : \mathbb{R}^n \to \mathbb{R}^{s_{\mathrm{c}}}$, and $\ell_{\mathrm{c}} : \mathbb{R}^n \to \mathbb{R}^{p_{\mathrm{c}}}$ such that $V_{\mathrm{s}}(\cdot)$ is positive definite, radially unbounded, $V_{\mathrm{s}}(0) = 0$, $\ell_{\mathrm{c}}(0) = 0$, $F_{\mathrm{c}}(0) = 0$, and, for all $x \in \mathbb{R}^n \backslash \mathcal{Z}_x$, (15.7) holds. Furthermore, assume there exist a matrix $K_{\mathrm{dg}} \in \mathbb{R}^{m_{\mathrm{d}}\times s_{\mathrm{d}}}$ and continuous functions

$\hat{G}_{\rm d} : \mathcal{Z}_x \to \mathbb{R}^{m_{\rm d}\times m_{\rm d}}$ and $F_{\rm d} : \mathcal{Z}_x \to \mathbb{R}^{s_{\rm d}}$ such that $\hat{G}_{\rm d}(x)$, $x \in \mathcal{Z}_x$, is invertible and, for all $x \in \mathcal{Z}_x$,

$$0 > V_{\rm s}(x + f_{\rm ds}(x)) - V_{\rm s}(x), \tag{15.49}$$

where $f_{\rm ds}(x)$ is given by (15.13). Finally, let $c > 0$, $Q_{\rm c} \in \mathbb{P}^{m_{\rm c}}$, $Q_{\rm d} \in \mathbb{P}^{m_{\rm d}}$, $Y \in \mathbb{P}^{s_{\rm c}}$, and $\lambda_{\max}(Q_{\rm d}) < 2$. Then the hybrid adaptive feedback control law

$$u_{\rm c}(t) = \hat{G}_{\rm c}(x(t))K_{\rm c}(t)F_{\rm c}(x(t)), \quad x(t) \notin \mathcal{Z}_x, \tag{15.50}$$

$$u_{\rm d}(t) = \hat{G}_{\rm d}(x(t))K_{\rm d}(t)F_{\rm d}(x(t)), \quad x(t) \in \mathcal{Z}_x, \tag{15.51}$$

where $K_{\rm c}(t) \in \mathbb{R}^{m_{\rm c}\times s_{\rm c}}$, $t \geq 0$, and $K_{\rm d}(t) \in \mathbb{R}^{m_{\rm d}\times s_{\rm d}}$, $t \geq 0$, with update laws

$$\dot{K}_{\rm c}(t) = -\tfrac{1}{2}Q_{\rm c}\hat{G}_{\rm c}^{\rm T}(x(t))G_{\rm c}^{\rm T}(x(t))V_{\rm s}'^{\rm T}(x(t))F_{\rm c}^{\rm T}(x(t))Y,$$
$$K_{\rm c}(0) = K_{\rm c0}, \quad x(t) \notin \mathcal{Z}_x, \tag{15.52}$$

$$\Delta K_{\rm c}(t) = 0, \quad x(t) \in \mathcal{Z}_x, \tag{15.53}$$

$$\dot{K}_{\rm d}(t) = 0, \quad K_{\rm d}(0) = K_{\rm d0}, \quad x(t) \notin \mathcal{Z}_x, \tag{15.54}$$

$$\Delta K_{\rm d}(t) = -\tfrac{1}{c+F_{\rm d}^{\rm T}(x(t))F_{\rm d}(x(t))}Q_{\rm d}\hat{G}_{\rm d}^{-1}(x(t))G_{\rm d}^{\dagger}(x(t))[\Delta x(t) - f_{\rm ds}(x(t))]F_{\rm d}^{\rm T}(x(t)),$$
$$x(t) \in \mathcal{Z}_x, \tag{15.55}$$

where $\Delta K_{\rm c}(t) \triangleq K_{\rm c}(t^+) - K_{\rm c}(t)$ and $\Delta K_{\rm d}(t) \triangleq K_{\rm d}(t^+) - K_{\rm d}(t)$, guarantees that the solution $(x(t), K_{\rm c}(t), K_{\rm d}(t))$, $t \geq 0$, of the closed-loop hybrid system given by (15.1), (15.2), (15.50)–(15.55) satisfies $\ell_{\rm c}(x(t)) \to 0$ as $t \to 0$ for all $x_0 \in \mathbb{R}^n$. If, in addition, $\ell_{\rm c}^{\rm T}(x)\ell_{\rm c}(x) > 0$, $x \in \mathbb{R}^n \backslash \mathcal{Z}_x$, $x \neq 0$, then $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$.

**Proof.** First, define $\tilde{K}_{\rm d}(t) \triangleq K_{\rm d}(t) - K_{\rm dg}$ and $\tilde{w}(t) \triangleq G_{\rm d}(x(t))\hat{G}_{\rm d}(x(t))\tilde{K}_{\rm d}(t) \cdot F_{\rm d}(x(t))$. Note that with $u_{\rm c}(t)$, $t \geq 0$, and $u_{\rm d}(t_k)$, $k \in \mathcal{N}$, given by (15.50) and (15.51), respectively, it follows that the closed-loop hybrid system (15.1), (15.2) is given by

$$\dot{x}(t) = f_{\rm c}(x(t)) + G_{\rm c}(x(t))\hat{G}_{\rm c}(x(t))K_{\rm c}(t)F_{\rm c}(x(t)), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}_x, \tag{15.56}$$

$$\Delta x(t) = f_{\rm d}(x(t)) + G_{\rm d}(x(t))\hat{G}_{\rm d}(x(t))K_{\rm d}(t)F_{\rm d}(x(t)), \quad x(t) \in \mathcal{Z}_x, \tag{15.57}$$

or, equivalently, using (15.8) and (15.13),

$$\dot{x}(t) = f_{\rm cs}(x(t)) + G_{\rm c}(x(t))\hat{G}_{\rm c}(x(t))(K_{\rm c}(t) - K_{\rm cg})F_{\rm c}(x(t)), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}_x, \tag{15.58}$$

$$\begin{aligned}\Delta x(t) &= f_{\rm ds}(x(t)) + G_{\rm d}(x(t))\hat{G}_{\rm d}(x(t))(K_{\rm d}(t) - K_{\rm dg})F_{\rm d}(x(t)) \\ &= f_{\rm ds}(x(t)) + \tilde{w}(t), \quad x(t) \in \mathcal{Z}_x.\end{aligned} \tag{15.59}$$

Furthermore, note that adding and subtracting $K_{\rm dg}$ to and from (15.55) and using (15.59) it follows that

$$\begin{aligned}\tilde{K}_{\rm d}(t^+) &= \tilde{K}_{\rm d}(t) - \tfrac{1}{c+F_{\rm d}^{\rm T}(x(t))F_{\rm d}(x(t))}Q_{\rm d}\hat{G}_{\rm d}^{-1}(x(t))G_{\rm d}^{\dagger}(x(t)) \\ &\quad \cdot[G_{\rm d}(x(t))\hat{G}_{\rm d}(x(t))\tilde{K}_{\rm d}(t)F_{\rm d}(x(t))]F_{\rm d}^{\rm T}(x(t)) \\ &= \tilde{K}_{\rm d}(t) - \tfrac{1}{c+F_{\rm d}^{\rm T}(x(t))F_{\rm d}(x(t))}Q_{\rm d}\tilde{K}_{\rm d}(t)F_{\rm d}(x(t))F_{\rm d}^{\rm T}(x(t)), \quad x(t) \in \mathcal{Z}_x.\end{aligned} \tag{15.60}$$

To show convergence of the plant states for the closed-loop hybrid system (15.52)–(15.54) and (15.58)–(15.60) consider the Lyapunov-like function

$$\begin{aligned}V(x, K_{\rm c}, K_{\rm d}) &= V_{\rm s}(x) + {\rm tr}\; Q_{\rm c}^{-1}(K_{\rm c} - K_{\rm cg})Y^{-1}(K_{\rm c} - K_{\rm cg})^{\rm T} \\ &\quad + {\rm tr}\,(K_{\rm d} - K_{\rm dg})^{\rm T}Q_{\rm d}^{-1}(K_{\rm d} - K_{\rm dg}).\end{aligned} \tag{15.61}$$

Note that $V(0, K_{\rm cg}, K_{\rm dg}) = 0$ and, since $V_{\rm s}(\cdot)$, $Q_{\rm c}$, $Q_{\rm d}$, and $Y$ are positive definite, $V(x, K_{\rm c}, K_{\rm d}) > 0$ for all $(x, K_{\rm c}, K_{\rm d}) \neq (0, K_{\rm cg}, K_{\rm dg})$. In addition, $V(x, K_{\rm c}, K_{\rm d})$ is radially unbounded. Now, using (15.7), (15.52), and (15.54), it follows that the time derivative of $V(x, K_{\rm c}, K_{\rm d})$ along the closed-loop system trajectories over the time interval $t \in (t_k, t_{k+1}]$, $k \in \mathcal{N}$, is given by

$$\begin{aligned}\dot{V}(x(t), K_{\rm c}(t), K_{\rm d}(t)) &= V_{\rm s}'(x(t))\left[f_{\rm cs}(x(t)) + G_{\rm c}(x(t))\hat{G}_{\rm c}(x(t))(K_{\rm c}(t) - K_{\rm cg})F_{\rm c}(x(t))\right] \\ &\quad + 2{\rm tr}\, Q_{\rm c}^{-1}(K_{\rm c}(t) - K_{\rm cg})Y^{-1}\dot{K}_{\rm c}^{\rm T}(t) \\ &= -\ell_{\rm c}^{\rm T}(x(t))\ell_{\rm c}(x(t)) \\ &\quad + {\rm tr}\Big[(K_{\rm c}(t) - K_{\rm cg})F_{\rm c}(x(t))V_{\rm s}'(x(t))G_{\rm c}(x(t))\hat{G}_{\rm c}(x(t))\Big]\end{aligned}$$

$$
\begin{aligned}
&-\mathrm{tr}\Big[(K_{\mathrm{c}}(t)-K_{\mathrm{cg}})F_{\mathrm{c}}(x(t))V_{\mathrm{s}}'(x(t))G_{\mathrm{c}}(x(t))\hat{G}_{\mathrm{c}}(x(t))\Big] \\
&= -\ell_{\mathrm{c}}^{\mathrm{T}}(x(t))\ell_{\mathrm{c}}(x(t)) \\
&\le 0, \quad t_k < t \le t_{k+1}.
\end{aligned} \tag{15.62}
$$

Now, suppose there exists $k_{\max} > 0$ such that $k \le k_{\max}$; that is, the closed-loop system trajectory $x(t)$, $t \ge 0$, intersects the resetting set $\mathcal{Z}_x$ a finite number of times. In this case, the closed-loop hybrid system possesses a continuous flow for all $t > t_{k_{\max}}$ and hence it follows from Theorem 2 of [42] that $\ell_{\mathrm{c}}(x(t)) \to 0$ as $t \to \infty$. If, in addition, $\ell_{\mathrm{c}}^{\mathrm{T}}(x)\ell_{\mathrm{c}}(x) > 0$, $x \in \mathbb{R}^n \backslash \mathcal{Z}_x$, $x \neq 0$, then $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$. Alternatively, suppose a trajectory $x(t)$, $t \ge 0$, intersects the resetting set $\mathcal{Z}_x$ infinitely many times. In this case, consider the partial Lyapunov-like function

$$
V_{K_{\mathrm{d}}}(K_{\mathrm{d}}) = \mathrm{tr}\,(K_{\mathrm{d}} - K_{\mathrm{dg}})^{\mathrm{T}} Q_{\mathrm{d}}^{-1}(K_{\mathrm{d}} - K_{\mathrm{dg}}). \tag{15.63}
$$

Note that since $Q_{\mathrm{d}}$ is positive definite, $V_{K_{\mathrm{d}}}(K_{\mathrm{d}}) > 0$, $K_{\mathrm{d}} \in \mathbb{R}^{m_{\mathrm{d}} \times s_{\mathrm{d}}}$, $K_{\mathrm{d}} \neq K_{\mathrm{dg}}$. Now, using (15.60), the difference of $V_{K_{\mathrm{d}}}(K_{\mathrm{d}})$ along the closed-loop system trajectories at the resetting times $t_k$, $k \in \mathcal{N}$, is given by

$$
\begin{aligned}
&\Delta V_{K_{\mathrm{d}}}(x(t_k), K_{\mathrm{d}}(t_k)) \\
&\quad \triangleq V_{K_{\mathrm{d}}}(x(t_k^+), K_{\mathrm{d}}(t_k^+)) - V_{K_{\mathrm{d}}}(x(t_k), K_{\mathrm{d}}(t_k)) \\
&\quad = \mathrm{tr}\left(\tilde{K}_{\mathrm{d}}(t_k) - \tfrac{1}{c+F_{\mathrm{d}}^{\mathrm{T}}(x(t_k))F_{\mathrm{d}}(x(t_k))} Q_{\mathrm{d}}\tilde{K}_{\mathrm{d}}(t_k)F_{\mathrm{d}}(x(t_k))F_{\mathrm{d}}^{\mathrm{T}}(x(t_k))\right)^{\mathrm{T}} Q_{\mathrm{d}}^{-1} \\
&\qquad \cdot \left(\tilde{K}_{\mathrm{d}}(t_k) - \tfrac{1}{c+F_{\mathrm{d}}^{\mathrm{T}}(x(t_k))F_{\mathrm{d}}(x(t_k))} Q_{\mathrm{d}}\tilde{K}_{\mathrm{d}}(t_k)F_{\mathrm{d}}(x(t_k))F_{\mathrm{d}}^{\mathrm{T}}(x(t_k))\right) \\
&\qquad - \mathrm{tr}\ \tilde{K}_{\mathrm{d}}^{\mathrm{T}}(t_k)Q_{\mathrm{d}}^{-1}\tilde{K}_{\mathrm{d}}(t_k) \\
&\quad = \mathrm{tr}\ \tilde{K}_{\mathrm{d}}^{\mathrm{T}}(t_k)Q_{\mathrm{d}}^{-1}\tilde{K}_{\mathrm{d}}(t_k) \\
&\qquad - \tfrac{2}{c+F_{\mathrm{d}}^{\mathrm{T}}(x(t_k))F_{\mathrm{d}}(x(t_k))}\mathrm{tr}\ \tilde{K}_{\mathrm{d}}^{\mathrm{T}}(t_k)\tilde{K}_{\mathrm{d}}(t_k)F_{\mathrm{d}}(x(t_k))F_{\mathrm{d}}^{\mathrm{T}}(x(t_k)) \\
&\qquad + \tfrac{1}{(c+F_{\mathrm{d}}^{\mathrm{T}}(x(t_k))F_{\mathrm{d}}(x(t_k)))^2}\mathrm{tr}\ F_{\mathrm{d}}(x(t_k))F_{\mathrm{d}}^{\mathrm{T}}(x(t_k))\tilde{K}_{\mathrm{d}}^{\mathrm{T}}(t_k)Q_{\mathrm{d}}\tilde{K}_{\mathrm{d}}(t_k) \\
&\qquad \cdot F_{\mathrm{d}}(x(t_k))F_{\mathrm{d}}^{\mathrm{T}}(x(t_k)) - \mathrm{tr}\ \tilde{K}_{\mathrm{d}}^{\mathrm{T}}(t_k)Q_{\mathrm{d}}^{-1}\tilde{K}_{\mathrm{d}}(t_k)
\end{aligned}
$$

$$
\begin{aligned}
&\leq -\tfrac{1}{c+F_{\rm d}^{\rm T}(x(t_k))F_{\rm d}(x(t_k))}F_{\rm d}^{\rm T}(x(t_k))\tilde{K}_{\rm d}^{\rm T}(t_k)(2I_{m_{\rm d}}-Q_{\rm d})\tilde{K}_{\rm d}(t_k)F_{\rm d}(x(t_k)) \\
&\leq 0, \quad k\in\mathcal{N},
\end{aligned} \tag{15.64}
$$

where in (15.64) we used $\frac{F_{\rm d}^{\rm T}(x)F_{\rm d}(x)}{c+F_{\rm d}^{\rm T}(x)F_{\rm d}(x)}<1$ and $2I_{m_{\rm d}}-Q_{\rm d}>0$, since by assumption $\lambda_{\max}(Q_{\rm d})<2$. Hence, $V_{K_{\rm d}}(x(t_k),K(t_k))$, $k\in\mathcal{N}$, is a nonincreasing and bounded function of $k$. Thus, it follows from the monotone convergence theorem (see Theorem 8.6 of [10]) that $\lim_{k\to\infty}V_{K_{\rm d}}(x(t_k),K_{\rm d}(t_k))$ exists which implies that $\Delta V_{K_{\rm d}}(x(t_k),K_{\rm d}(t_k))\to 0$ as $k\to\infty$. Now, it follows from (15.64) that $\tilde{K}_{\rm d}(t_k)F_{\rm d}(x(t_k))\to 0$ as $k\to\infty$ and hence $\tilde{w}(t_k)\to 0$ as $k\to\infty$. Next, to show that $x(t)\to 0$ as $t\to\infty$, note that, since $\tilde{w}(t_k)\to 0$ as $k\to\infty$, there exists $k^*\geq 0$ such that for all $k\geq k^*$,

$$
0\geq V_{\rm s}(x(t_k)+f_{\rm ds}(x(t_k))+\tilde{w}(t_k))-V_{\rm s}(x(t_k)) \tag{15.65}
$$

holds and hence there exist $\hat{\mathcal{Z}}_x\subset\mathcal{Z}_x$ and $\mathcal{K}_{\rm d}\subset\mathbb{R}^{m_{\rm d}\times s_{\rm d}}$ such that

$$
0\geq V_{\rm s}(x+f_{\rm ds}(x)+G_{\rm d}(x)\hat{G}_{\rm d}(x)\tilde{K}_{\rm d}F_{\rm d}(x))-V_{\rm s}(x), \quad (x,K_{\rm d})\in\hat{\mathcal{Z}}_x\times\mathcal{K}_{\rm d}\subset\mathcal{Z}_x\times\mathbb{R}^{m_{\rm d}\times s_{\rm d}}, \tag{15.66}
$$

and $\operatorname{dist}(x(t_k),\hat{\mathcal{Z}}_x)\to 0$ as $k\to\infty$ and $\operatorname{dist}(\tilde{K}_{\rm d}(t_k),\mathcal{K}_{\rm d})\to 0$ as $k\to\infty$. Hence, it follows that the difference of $V(x,K_{\rm c},K_{\rm d})$ along the closed-loop system trajectories at the resetting times $t_k$, $k\geq k^*$, is given by

$$
\begin{aligned}
\Delta V(x(t_k),K_{\rm c}(t_k),K_{\rm d}(t_k)) &\triangleq V(x(t_k^+),K_{\rm c}(t_k^+),K_{\rm d}(t_k^+))-V(x(t_k),K_{\rm c}(t_k),K_{\rm d}(t_k)) \\
&= V_{\rm s}(x(t_k)+f_{\rm ds}(x(t_k))+\tilde{w}(t_k))-V_{\rm s}(x(t_k)) \\
&\quad +\Delta V_{K_{\rm d}}(x(t_k),K_{\rm d}(t_k)) \\
&\leq 0, \quad k\geq k^*.
\end{aligned} \tag{15.67}
$$

Next, for $t\geq t_{k^*}$, define the translated closed-loop hybrid system

$$
\begin{aligned}
\dot{\hat{x}}(\tau) &= f_{\rm c}(\hat{x}(\tau))+G_{\rm c}(\hat{x}(\tau))\hat{G}_{\rm c}(\hat{x}(\tau))\hat{K}_{\rm c}(\tau)F_{\rm c}(\hat{x}(\tau)), \\
&\qquad \hat{x}(0)=x(t_{k^*}^+), \quad \hat{x}(\tau)\notin\mathcal{Z}_x,
\end{aligned} \tag{15.68}
$$

$$\Delta\hat{x}(\tau) = f_{\rm d}(\hat{x}(\tau)) + G_{\rm d}(\hat{x}(\tau))\hat{G}_{\rm d}(\hat{x}(\tau))\hat{K}_{\rm d}(\tau)F_{\rm d}(\hat{x}(\tau)), \quad \hat{x}(\tau) \in \mathcal{Z}_x, \tag{15.69}$$

$$\dot{\hat{K}}_{\rm c}(\tau) = -\tfrac{1}{2}Q_{\rm c}\hat{G}_{\rm c}^{\rm T}(\hat{x}(\tau))G_{\rm c}^{\rm T}(\hat{x}(\tau))V_{\rm s}'^{\rm T}(\hat{x}(\tau))F_{\rm c}^{\rm T}(\hat{x}(\tau))Y, \ \hat{K}_{\rm c}(0) = K_{\rm c}(t_{k^*}^+), \quad \hat{x}(\tau) \notin \mathcal{Z}_x, \tag{15.70}$$

$$\Delta\hat{K}_{\rm c}(\tau) = 0, \quad \hat{x}(\tau) \in \mathcal{Z}_x, \tag{15.71}$$

$$\dot{\hat{K}}_{\rm d}(\tau) = 0, \quad \hat{K}_{\rm d}(0) = K_{\rm d}(t_{k^*}^+), \quad \hat{x}(\tau) \notin \mathcal{Z}_x, \tag{15.72}$$

$$\Delta\hat{K}_{\rm d}(\tau) = -\tfrac{1}{c+F_{\rm d}^{\rm T}(\hat{x}(\tau))F_{\rm d}(\hat{x}(\tau))}Q_{\rm d}\hat{G}_{\rm d}^{-1}(\hat{x}(\tau))G_{\rm d}^{\dagger}(\hat{x}(\tau))[\Delta\hat{x}(\tau) - f_{\rm ds}(\hat{x}(\tau))]F_{\rm d}^{\rm T}(\hat{x}(\tau)), \quad \hat{x}(\tau) \in \mathcal{Z}_x, \tag{15.73}$$

where $\tau \triangleq t - t_{k^*} \geq 0$, $\hat{x}(\tau) \triangleq x(t - t_{k^*})$, $\hat{K}_{\rm c}(\tau) \triangleq K_{\rm c}(t - t_{k^*})$, and $\hat{K}_{\rm d}(\tau) \triangleq K_{\rm d}(t - t_{k^*})$. Furthermore, define $\mathcal{R}_{\rm c} \triangleq \{(\hat{x}, \hat{K}_{\rm c}, \hat{K}_{\rm d}) \in \mathbb{R}^n \times \mathbb{R}^{m_{\rm c}\times s_{\rm c}} \times \mathbb{R}^{m_{\rm d}\times s_{\rm d}} : \hat{x} \notin \mathcal{Z}_x, \dot{V}(\hat{x}, \hat{K}_{\rm c}, \hat{K}_{\rm d}) = 0\} = \{(\hat{x}, \hat{K}_{\rm c}, \hat{K}_{\rm d}) \in \mathbb{R}^n \times \mathbb{R}^{m_{\rm c}\times s_{\rm c}} \times \mathbb{R}^{m_{\rm d}\times s_{\rm d}} : \hat{x} \notin \mathcal{Z}_x, \ell_{\rm c}^{\rm T}(\hat{x})\ell_{\rm c}(\hat{x}) = 0\}$ and $\mathcal{R}_{\rm d} \triangleq \{(\hat{x}, \hat{K}_{\rm c}, \hat{K}_{\rm d}) \in \mathbb{R}^n \times \mathbb{R}^{m_{\rm c}\times s_{\rm c}} \times \mathbb{R}^{m_{\rm d}\times s_{\rm d}} : \hat{x} \in \mathcal{Z}_x, \Delta V(\hat{x}, \hat{K}_{\rm c}, \hat{K}_{\rm d}) = 0\}$. Now, let $\mathcal{M}$ denote the largest invariant set contained in $\mathcal{R} \triangleq \mathcal{R}_{\rm c} \cup \mathcal{R}_{\rm d}$ and note that since $\tilde{w}(t_k) \to 0$ as $k \to \infty$ it follows that for $(\hat{x}, \hat{K}_{\rm c}, \hat{K}_{\rm d}) \in \mathcal{M} \cap (\hat{\mathcal{Z}}_x \times \mathbb{R}^{m_{\rm c}\times s_{\rm c}} \times \mathcal{K}_{\rm d})$, $G_{\rm d}(\hat{x})\hat{G}_{\rm d}(\hat{x})\tilde{K}_{\rm d}F_{\rm d}(\hat{x}) = 0$, $\tilde{K}_{\rm d}F_{\rm d}(\hat{x}) = 0$, and $V_{\rm s}(\hat{x} + f_{\rm ds}(\hat{x})) - V_{\rm s}(\hat{x}) = 0$. However, since (15.66) holds for all $x \in \mathcal{Z}_x$, $\mathcal{M} = \mathcal{R}_{\rm c} \cup \emptyset$ and hence it follows from Theorem 15.1 that the solution $(\hat{x}(\tau), \hat{K}_{\rm c}(\tau), \hat{K}_{\rm d}(\tau))$, $\tau \geq 0$, to (15.68)–(15.73) satisfies $\ell_{\rm c}(\hat{x}(\tau)) \to 0$ as $\tau \to \infty$ and hence $\ell_{\rm c}(x(t)) \to 0$ as $t \to \infty$. Furthermore, if $\ell_{\rm c}^{\rm T}(x)\ell_{\rm c}(x) > 0$, $x \in \mathbb{R}^n \backslash \mathcal{Z}_x$, $x \neq 0$, then $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$. □

**Remark 15.2.** Note that in the case where $\ell_{\rm c}^{\rm T}(x)\ell_{\rm c}(x) > 0$, $x \in \mathbb{R}^n \backslash \mathcal{Z}_x$, $x \neq 0$, the conditions in Theorem 15.3 imply that $x(t) \to 0$ as $t \to \infty$ and hence it follows from (15.52) that $(x(t), K_{\rm c}(t), K_{\rm d}(t)) \to \mathcal{M} \triangleq \{(x, K_{\rm c}, K_{\rm d}) \in \mathbb{R}^n \times \mathbb{R}^{m_{\rm c}\times s_{\rm c}} \times \mathbb{R}^{m_{\rm d}\times s_{\rm d}} : x = 0, \dot{K}_{\rm c} = 0\}$ as $t \to \infty$. Furthermore, if $x(t)$, $t \geq 0$, intersects $\mathcal{Z}_x$ infinitely many times, then $(x(t), K_{\rm c}(t), K_{\rm d}(t)) \to \mathcal{M} \triangleq \{(x, K_{\rm c}, K_{\rm d}) \in \mathbb{R}^n \times \mathbb{R}^{m_{\rm c}\times s_{\rm c}} \times \mathbb{R}^{m_{\rm d}\times s_{\rm d}} : x = 0, \dot{K}_{\rm c} = 0, K_{\rm d}(t^+) = K_{\rm d}(t)\}$ as $t \to \infty$.

**Remark 15.3.** In the case where $u_{\rm d}(t) \equiv 0$, Condition (15.49) can be replaced by

$$0 \geq V_{\rm s}(x + f_{\rm d}(x)) - V_{\rm s}(x). \tag{15.74}$$

Furthermore, taking $F_{\rm d}(x) = 0$, $x \in \mathcal{Z}_x$, and $K_{\rm d}(t) \equiv 0$, (15.65) holds for all $k \in \mathcal{N}$. In this case, since $V(x(t), K_{\rm c}(t), K_{\rm d}(t))$ is nonincreasing for all $t \geq 0$, $V(x, K_{\rm c}, K_{\rm d})$ is a Lyapunov function and hence the closed-loop hybrid system (15.52)–(15.54) and (15.58)–(15.60) is Lyapunov stable and $x(t) \to 0$ as $t \to \infty$.

As shown in Section 15.3, if (15.1) and (15.2) are such that (15.33) and (15.34) hold, then we can always construct functions $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, $F_{\rm c} : \mathbb{R}^n \to \mathbb{R}^{s_{\rm c}}$, and $F_{\rm d} : \mathbb{R}^n \to \mathbb{R}^{s_{\rm d}}$, with $F_{\rm c}(0) = 0$, such that (15.7) and (15.49) hold *without* requiring knowledge of the hybrid system dynamics. Specifically, parameterizing $f_{\rm cu}(x)$ and $f_{\rm du}(x)$ as in Section 15.3 and choosing $\Theta_{\rm cn} f_{\rm cn}(x) + \Phi_{\rm cn} \hat{f}_{\rm cn}(x) = \hat{A}_{\rm c} x$ and $\Theta_{\rm dn} f_{\rm dn}(x) + \Phi_{\rm dn} \hat{f}_{\rm dn}(x) = \hat{A}_{\rm d} x$, where $\hat{A}_{\rm c} \in \mathbb{R}^{m_{\rm c} \times n}$ and $\hat{A}_{\rm d} \in \mathbb{R}^{m_{\rm d} \times n}$, it follows that (15.37) and (15.38) have the form $f_{\rm cs}(x) = A_{\rm c} x$ and $f_{\rm ds}(x) = (A_{\rm d} - I_n)x$, respectively, where $A_{\rm c} = \left[A_0^{\rm T}, \hat{A}_{\rm c}^{\rm T}\right]^{\rm T}$ and $A_{\rm d} = \left[A_0^{\rm T}, \hat{A}_{\rm d}^{\rm T}\right]^{\rm T}$ are in multivariable controllable canonical form. Hence, we can choose $\hat{A}_{\rm c}$ and $\hat{A}_{\rm d}$ such that $A_{\rm c}$ is Hurwitz and $A_{\rm d}$ is Schur. Now, it follows from standard converse Lyapunov theory that there exists a positive-definite matrix $P$ satisfying the Lyapunov equation

$$0 = A_{\rm c}^{\rm T} P + P A_{\rm c} + R_{\rm c}, \tag{15.75}$$

where $R_{\rm c}$ is positive definite. If, in addition, $P$ satisfies

$$0 = A_{\rm d}^{\rm T} P A_{\rm d} - P + R_{\rm d}, \tag{15.76}$$

where $R_{\rm d}$ is positive definite, then (15.7) and (15.49) hold with $V_{\rm s}(x) = x^{\rm T} P x$. Hence, the hybrid adaptive feedback controller (15.50) and (15.51) with update laws (15.52), or, equivalently,

$$\dot{K}_{\rm c}(t) = -Q_{\rm c} \hat{G}_{\rm c}^{\rm T}(x(t)) G_{\rm c}^{\rm T}(x(t)) P x(t) F_{\rm c}^{\rm T}(x(t)) Y, \tag{15.77}$$

and (15.53)–(15.55) guarantees global attraction of the *nonlinear* hybrid uncertain dynamical system (15.1) and (15.2) where $f_{\rm c}(x)$, $f_{\rm d}(x)$, $G_{\rm c}(x)$, and $G_{\rm d}(x)$ are given by (15.33) and (15.34). Note that since $R_{\rm c}$ and $R_{\rm d}$ are arbitrary, (15.75) and (15.76) can be cast as a linear matrix inequality feasibility problem involving $P > 0$, $A_{\rm c}^{\rm T}P + PA_{\rm c} < 0$, and $A_{\rm d}^{\rm T}PA_{\rm d} - P < 0$. Finally, as mentioned in Section 15.3, it is important to note that it is not necessary to utilize a feedback linearizing function $F_{\rm c}(x)$ and $F_{\rm d}(x)$ to produce a linear $f_{\rm cs}(x)$ and $f_{\rm ds}(x)$. However, as shown above, when the hybrid system is in a hybrid normal form given by (15.33), (15.34), the feedback linearizing functions $F_{\rm c}(x)$ and $F_{\rm d}(x)$ provide considerable simplification in constructing $V_{\rm s}(x)$ necessary in computing the hybrid update law (15.52).

Finally, if $f_{\rm c}(x)$, $f_{\rm d}(x)$, $G_{\rm c}(x)$, and $G_{\rm d}(x)$ are uncertain and $G_{\rm c}(x)$ and $G_{\rm d}(x)$ are such that $G_{\rm cs}(x)$ and $G_{\rm ds}(x)$ are unknown and are parameterized as $G_{\rm cs}(x) = B_{\rm cu}G_{\rm cn}(x)$ and $G_{\rm ds}(x) = B_{\rm du}G_{\rm dn}(x)$, where $G_{\rm cn} : \mathbb{R}^n \to \mathbb{R}^{m_{\rm c}\times m_{\rm c}}$ and $G_{\rm dn} : \mathbb{R}^n \to \mathbb{R}^{m_{\rm d}\times m_{\rm d}}$ are known and satisfy $\det G_{\rm cn}(x) \neq 0$, $x \in \mathbb{R}^n\backslash\mathcal{Z}_x$, $\det G_{\rm dn}(x) \neq 0$, $x \in \mathcal{Z}_x$, and $B_{\rm cu} \in \mathbb{R}^{m_{\rm c}\times m_{\rm c}}$ and $B_{\rm du} \in \mathbb{R}^{m_{\rm d}\times m_{\rm d}}$, with $\det B_{\rm cu} \neq 0$ and $\det B_{\rm du} \neq 0$, are unknown symmetric sign-definite matrices but a bound $\alpha$ for the maximum singular value of $B_{\rm du}$ is known and the sign definiteness of $B_{\rm cu}$ and $B_{\rm du}$ are known, then we have the following result. For the statement of this result recall the definitions of $B_{\rm c0}$ for $B_{\rm cu} > 0$ and $B_{\rm cu} < 0$ and $B_{\rm d0}$ for $B_{\rm du} > 0$ and $B_{\rm du} < 0$ given in Section 15.3.

**Corollary 15.2.** Consider the nonlinear uncertain hybrid dynamical system $\mathcal{G}$ given by (15.1) and (15.2) with $f_{\rm c}(x)$, $f_{\rm d}(x)$, $G_{\rm c}(x)$, and $G_{\rm d}(x)$ given by (15.33), (15.34), and $G_{\rm cs}(x) = B_{\rm cu}G_{\rm cn}(x)$ and $G_{\rm ds}(x) = B_{\rm du}G_{\rm dn}(x)$, where $B_{\rm cu} \in \mathbb{R}^{m_{\rm c}\times m_{\rm c}}$ and $B_{\rm du} \in \mathbb{R}^{m_{\rm d}\times m_{\rm d}}$ are unknown symmetric matrices and the sign definiteness of $B_{\rm cu}$ and $B_{\rm du}$ are known and $\sigma_{\max}(B_{\rm du}) < \alpha$, $\alpha > 0$. Assume there exist a matrix $K_{\rm cg} \in \mathbb{R}^{m_{\rm c}\times s_{\rm c}}$, a continuously differentiable function $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, and continuous functions $\hat{G}_{\rm c} : \mathbb{R}^n \to \mathbb{R}^{m_{\rm c}\times m_{\rm c}}$, $F_{\rm c} : \mathbb{R}^n \to \mathbb{R}^{s_{\rm c}}$, and $\ell_{\rm c} : \mathbb{R}^n \to \mathbb{R}^{p_{\rm c}}$ such that $V_{\rm s}(\cdot)$

is positive definite, radially unbounded, $V_{\rm s}(0) = 0$, $\ell_{\rm c}(0) = 0$, $F_{\rm c}(0) = 0$, and, for all $x \in \mathbb{R}^n \backslash \mathcal{Z}_x$, (15.7) holds. Furthermore, assume that there exist a matrix $K_{\rm dg} \in \mathbb{R}^{m_{\rm d} \times s_{\rm d}}$ and continuous functions $\hat{G}_{\rm d} : \mathcal{Z}_x \to \mathbb{R}^{m_{\rm d} \times m_{\rm d}}$ and $F_{\rm d} : \mathcal{Z}_x \to \mathbb{R}^{s_{\rm d}}$ such that $\hat{G}_{\rm d}(x)$, $x \in \mathcal{Z}_x$, is invertible and, for all $x \in \mathcal{Z}_x$, (15.49) holds. Finally, let $c > 0$ and $Y \in \mathbb{P}^{s_{\rm c}}$. Then the hybrid adaptive feedback control law

$$u_{\rm c}(t) = G_{\rm cn}^{-1}(x(t))K_{\rm c}(t)F_{\rm c}(x(t)), \quad x(t) \notin \mathcal{Z}_x, \tag{15.78}$$

$$u_{\rm d}(t) = \hat{\alpha}^{-1}G_{\rm dn}^{-1}(x(t))K_{\rm d}(k)F_{\rm d}(x(t)), \quad x(t) \in \mathcal{Z}_x, \tag{15.79}$$

where $K_{\rm c}(t) \in \mathbb{R}^{m \times s_{\rm c}}$, $t \geq 0$, $K_{\rm d}(t) \in \mathbb{R}^{m \times s_{\rm d}}$, $t \geq 0$, and $\hat{\alpha} \geq \alpha/2$, with update laws

$$\dot{K}_{\rm c}(t) = -B_{\rm c0}^{\rm T} V_{\rm s}'^{\rm T}(x(t))F_{\rm c}^{\rm T}(x(t))Y, \quad K_{\rm c}(0) = K_{\rm c0}, \quad x(t) \notin \mathcal{Z}_x, \tag{15.80}$$

$$\Delta K_{\rm c}(t) = 0, \quad x(t) \in \mathcal{Z}_x, \tag{15.81}$$

$$\dot{K}_{\rm d}(t) = 0, \quad K_{\rm d}(0) = K_{\rm d0}, \quad x(t) \notin \mathcal{Z}_x, \tag{15.82}$$

$$\Delta K_{\rm d}(t) = -\tfrac{1}{c + F_{\rm d}^{\rm T}(x(t))F_{\rm d}(x(t))} B_{\rm d0}^{\rm T}[\Delta x(t) - f_{\rm ds}(x(t))]F_{\rm d}^{\rm T}(x(t)), \quad x(t) \in \mathcal{Z}_x, \tag{15.83}$$

guarantees that the solution $(x(t), K_{\rm c}(t), K_{\rm d}(t))$ of the closed-loop hybrid system given by (15.1), (15.2), (15.78)–(15.83) satisfies $\ell_{\rm c}(x(t)) \to 0$ as $t \to \infty$. If, in addition, $\ell_{\rm c}^{\rm T}(x)\ell_{\rm c}(x) > 0$, $x \in \mathbb{R}^n \backslash \mathcal{Z}_x$, $x \neq 0$, then $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$.

**Proof.** The result is a direct consequence of Theorem 15.3. First, let $\hat{G}_{\rm c}(x) = G_{\rm cn}^{-1}(x)$ and $\hat{G}_{\rm d}(x) = \hat{\alpha}^{-1}G_{\rm dn}^{-1}(x)$ so that $G_{\rm c}(x)\hat{G}_{\rm cn}(x) = [0_{m \times (n-m)}, B_{\rm cu}]^{\rm T}$ and $G_{\rm d}(x) \cdot\hat{G}_{\rm dn}(x) = [0_{m \times (n-m)}, \hat{\alpha}^{-1}B_{\rm du}]^{\rm T}$, and let $K_{\rm cg} = B_{\rm cu}^{-1}[\Theta_{\rm cn} - \Theta_{\rm c}, \Phi_{\rm cn}]$ and $K_{\rm dg} = \hat{\alpha}B_{\rm du}^{-1} \cdot[\Theta_{\rm dn} - \Theta_{\rm d}, \Phi_{\rm dn}]$. Next, since $Q_{\rm c}$ and $Q_{\rm d}$ are arbitrary positive definite matrices with $\lambda_{\max}(Q_{\rm d}) < 2$, $Q_{\rm c}$ in (15.52) and $Q_{\rm d}$ in (15.55) can be replaced by $q_{\rm c}|B_{\rm cu}|^{-1}$ and $\hat{\alpha}^{-1}|B_{\rm du}|^{-1}$, respectively, where $q_{\rm c}$ is a positive constant, $|B_{\rm cu}| = (B_{\rm cu}^2)^{\frac{1}{2}}$, and $|B_{\rm du}| = (B_{\rm du}^2)^{\frac{1}{2}}$, where $(\cdot)^{\frac{1}{2}}$ denotes the (unique) positive definite square root. Now, since $B_{\rm cu}$ and $B_{\rm du}$ are symmetric and sign definite it follows from the Schur decomposition that $B_{\rm cu} = U_{\rm c}D_{B_{\rm cu}}U_{\rm c}^{\rm T}$ and $B_{\rm du} = U_{\rm d}D_{B_{\rm du}}U_{\rm d}^{\rm T}$, where $U_{\rm c}$ and $U_{\rm d}$ are orthogonal and $D_{B_{\rm cu}}$ and

$D_{B_{\rm du}}$ are real diagonal. Hence, $|B_{\rm cu}|^{-1}\hat{G}_{\rm c}^{\rm T}(x)G_{\rm c}^{\rm T}(x) = [0_{m_{\rm c}\times(n-m_{\rm c})}, \mathcal{I}_{m_{\rm c}}] = B_{\rm c0}^{\rm T}$ and $\hat{\alpha}^{-1}|B_{\rm du}|^{-1}\hat{G}_{\rm d}^{\rm T}(x)G_{\rm d}^{\rm T}(x) = [0_{m_{\rm d}\times(n-m_{\rm d})}, \mathcal{I}_{m_{\rm d}}] = B_{\rm d0}^{\rm T}$, where $\mathcal{I}_{m_{\rm c}} = I_{m_{\rm c}}$ for $B_{\rm cu} > 0$, $\mathcal{I}_{m_{\rm c}} = -I_{m_{\rm c}}$ for $B_{\rm cu} < 0$, $\mathcal{I}_{m_{\rm d}} = I_{m_{\rm d}}$ for $B_{\rm du} > 0$, and $\mathcal{I}_{m_{\rm d}} = I_{m_{\rm d}}$ for $B_{\rm du} < 0$. Now, (15.52) and (15.55) imply (15.80) and (15.83), respectively. □

## 15.5. Illustrative Numerical Examples

In this section we present two numerical examples to demonstrate the utility of the proposed hybrid adaptive control framework for hybrid adaptive stabilization and hybrid adaptive attraction, respectively.

**Example 15.1.** Consider the nonlinear uncertain controlled hybrid system given by (15.1), (15.2) with $n = 2$, $x = [x_1, x_2]^{\rm T}$,

$$f_{\rm c}(x) = \begin{bmatrix} x_2 \\ -\beta x_1 - \mu(x_1^2 - \alpha)x_2 \end{bmatrix}, \quad G_{\rm c}(x) = \begin{bmatrix} 0 \\ b_{\rm c} \end{bmatrix}, \tag{15.84}$$

$$f_{\rm d}(x) = \begin{bmatrix} -x_1 + x_2 \\ -x_2 - a_1\frac{x_1^3}{1+x_1^2} - a_2\frac{x_2^3}{1+x_2^2} - a_3\ln(1+|x_2|) \end{bmatrix}, \quad G_{\rm d}(x) = \begin{bmatrix} 0 \\ b_{\rm d} \end{bmatrix}, \tag{15.85}$$

where $\mu, \alpha, \beta, a_1, a_2, a_3, b_{\rm c}, b_{\rm d} \in \mathbb{R}$ are unknown. Furthermore, assume that the resetting set $\mathcal{Z}_x$ is given by

$$\mathcal{Z}_x = \{x \in \mathbb{R}^2 : \mathcal{X}(x) = 0,\ x_2 > 0\}, \tag{15.86}$$

where $\mathcal{X} : \mathbb{R}^2 \to \mathbb{R}$ is a continuously differentiable function given by $\mathcal{X}(x) = x_1$. It can be easily verified that the resetting set $\mathcal{Z}_x$ satisfies Assumptions A1 and A2 given in [80]. Furthermore, $\mathcal{X}'(x) \neq 0$, $x \in \mathcal{Z}_x$, and for the closed-loop hybrid system corresponding to the continuous-time dynamics given by (15.1) and (15.14), $\mathcal{X}'(x)\dot{x} = x_2 \neq 0$, $x \in \mathcal{Z}_x$, and hence the closed-loop hybrid system satisfies Assumption 15.1. Here, we assume that $f_{\rm c}(x)$ and $f_{\rm d}(x)$ are unknown and can be parameterized as $f_{\rm c}(x) = [x_2,\ \theta_{\rm c1}x_1 + \theta_{\rm c2}x_2 + \theta_{\rm c3}x_1^2x_2]^{\rm T}$ and $f_{\rm d}(x) = \Big[-x_1 +$

$x_2, -x_2 + \theta_{\rm d1}\frac{x_1^3}{1+x_1^2} + \theta_{\rm d2}\frac{x_2^3}{1+x_2^2} + \theta_{\rm d3}\ln(1+|x_2|)\Big]^{\rm T}$, where $\theta_{\rm c1}$, $\theta_{\rm c2}$, $\theta_{\rm c3}$, $\theta_{\rm d1}$, $\theta_{\rm d2}$, and $\theta_{\rm d3}$ are unknown constants. Furthermore, we assume that $\operatorname{sign} b_{\rm c}$ and $\operatorname{sign} b_{\rm d}$ are known and $|b_{\rm d}| < 2$. Next, let $\hat{G}_{\rm c}(x) = 1$, $\hat{G}_{\rm d}(x) = 1$, $F_{\rm c}(x) = [x_1, x_2, x_1^2 x_2]^{\rm T}$, $F_{\rm d}(x) = \Big[\frac{x_1^3}{1+x_1^2}, \frac{x_2^3}{1+x_2^2}, \ln(1+|x_2|), x_1, x_2\Big]^{\rm T}$, $K_{\rm cg} = \frac{1}{b_{\rm c}}[\theta_{\rm cn_1} - \theta_{\rm c1}, \theta_{\rm cn_2} - \theta_{\rm c2}, -\theta_{\rm c3}]$, and $K_{\rm dg} = \frac{1}{b_{\rm d}}[-\theta_{\rm d1}, -\theta_{\rm d2}, -\theta_{\rm d3}, \phi_{\rm dn_1}, \phi_{\rm dn_2}]$, where $\theta_{\rm n_1}$, $\theta_{\rm n_2}$, $\phi_{\rm dn_1}$, $\phi_{\rm dn_2}$ are arbitrary scalars, so that

$$\begin{aligned} f_{\rm cs}(x) &= f_{\rm c}(x) + \begin{bmatrix} 0 \\ b_{\rm c} \end{bmatrix} \frac{1}{b_{\rm c}} \begin{bmatrix} \theta_{\rm cn_1} - \theta_{\rm c1}, \theta_{\rm cn_2} - \theta_{\rm c2}, -\theta_{\rm c3} \end{bmatrix} F_{\rm c}(x) \\ &= \begin{bmatrix} 0 & 1 \\ \theta_{\rm cn_1} & \theta_{\rm cn_2} \end{bmatrix} x \end{aligned} \tag{15.87}$$

and

$$\begin{aligned} x + f_{\rm ds}(x) &= x + f_{\rm d}(x) + \begin{bmatrix} 0 \\ b_{\rm d} \end{bmatrix} \frac{1}{b_{\rm d}} \begin{bmatrix} -\theta_{\rm d1}, -\theta_{\rm d2}, -\theta_{\rm d3}, \phi_{\rm dn_1}, \phi_{\rm dn_2} \end{bmatrix} F_{\rm d}(x) \\ &= \begin{bmatrix} 0 & 1 \\ \phi_{\rm dn_1} & \phi_{\rm dn_2} \end{bmatrix} x. \end{aligned} \tag{15.88}$$

In addition, note that $F_{\rm d}^{\rm T}(x)F_{\rm d}(x) = \left(\frac{x_1^2}{1+x_1^2}\right)^2 x_1^2 + \left(\frac{x_2^2}{1+x_2^2}\right)^2 x_2^2 + \ln^2(1+|x_2|) + x^{\rm T}x \le 3x^{\rm T}x$, $x \in \mathbb{R}^2$, and thus (15.11) is satisfied with $\bar{\gamma} = 3$. Now, with the proper choice of $\theta_{\rm cn_1}$, $\theta_{\rm cn_2}$, $\phi_{\rm dn_1}$, and $\phi_{\rm dn_2}$, it follows from Corollary 15.1 that the hybrid adaptive feedback controller (15.43) and (15.44) guarantees that $x(t) \to 0$ as $t \to \infty$. Specifically, here we choose $\theta_{\rm cn_1} = -1$, $\theta_{\rm cn_2} = -2$, $\phi_{\rm dn_1} = -0.1$, $\phi_{\rm dn_2} = -0.1$, so that (15.7) and (15.10) are satisfied with

$$V_{\rm s}(x) = x^{\rm T}Px, \quad P = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}, \quad \ell_{\rm c}(x) = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} x, \quad \ell_{\rm d}(x) = L_{\rm d}x, \tag{15.89}$$

where $L_{\rm d} \in \mathbb{R}^{2\times 2}$ is such that $L_{\rm d}^{\rm T}L_{\rm d} \le 0.3433 I_2$.

With $\mu = 2$, $\alpha = 1$, $\beta = 1$, $a_1 = -5$, $a_2 = -2$, $a_3 = 3$, $\gamma = 1$, $b_{\rm c} = 3$, $b_{\rm d} = 1.4$, $\hat{\alpha} = 1$, $Y = I_3$, and initial conditions $x(0) = [1, 1]^{\rm T}$, $K_{\rm c}(0) = [0, 0, 0]$, and $K_{\rm d}(0) = 0_{1\times 5}$, Figure 15.1 shows the phase portraits of the uncontrolled and controlled hybrid system. Figures 15.2 and 15.3 show the state trajectories versus time and the control signals versus time, respectively. Finally, Figure 15.4 shows the adaptive gain history versus time.

**Example 15.2.** Consider the nonlinear uncertain controlled hybrid system given by (15.1), (15.2) with $n = 2$, $x = [x_1, x_2]^{\mathrm{T}}$,

$$f_{\mathrm{c}}(x) = \begin{bmatrix} x_2 \\ -\beta x_1 - \mu(x_1^2 - \alpha)x_2 \end{bmatrix}, \quad G_{\mathrm{c}}(x) = \begin{bmatrix} 0 \\ b_{\mathrm{c}} \end{bmatrix}, \tag{15.90}$$

$$f_{\mathrm{d}}(x) = \begin{bmatrix} -x_1 + x_2 \\ -x_2 - a_1 x_1^2 - a_2 \frac{x_2^3}{1+x_2^2} - a_3 x_2^3 \end{bmatrix}, \quad G_{\mathrm{d}}(x) = \begin{bmatrix} 0 \\ b_{\mathrm{d}} \end{bmatrix}, \tag{15.91}$$

where $\mu, \alpha, \beta, a_1, a_2, a_3, b_{\mathrm{c}}, b_{\mathrm{d}} \in \mathbb{R}$ are unknown. Furthermore, assume that the resetting set $\mathcal{Z}_x$ is given by (15.86). Here, we assume that $f_{\mathrm{c}}(x)$ and $f_{\mathrm{d}}(x)$ are unknown and can be parameterized as $f_{\mathrm{c}}(x) = [x_2,\ \theta_{\mathrm{c}1}x_1 + \theta_{\mathrm{c}2}x_2 + \theta_{\mathrm{c}3}x_1^2 x_2]^{\mathrm{T}}$ and $f_{\mathrm{d}}(x) = \left[-x_1 + x_2,\ -x_2 + \theta_{\mathrm{d}1}x_1^2 + \theta_{\mathrm{d}2}\frac{x_2^3}{1+x_2^2} + \theta_{\mathrm{d}3}x_2^3\right]^{\mathrm{T}}$, where $\theta_{\mathrm{c}1}$, $\theta_{\mathrm{c}2}$, $\theta_{\mathrm{c}3}$, $\theta_{\mathrm{d}1}$, $\theta_{\mathrm{d}2}$, and $\theta_{\mathrm{d}3}$ are unknown constants. Furthermore, we assume that $\operatorname{sign} b_{\mathrm{c}}$ and $\operatorname{sign} b_{\mathrm{d}}$ are known and $|b_{\mathrm{d}}| < 2$. Next, let $\hat{G}_{\mathrm{c}}(x) = 1$, $\hat{G}_{\mathrm{d}}(x) = 1$, $F_{\mathrm{c}}(x) = [x_1, x_2, x_1^2 x_2]^{\mathrm{T}}$, $F_{\mathrm{d}}(x) = \left[x_1^2, \frac{x_2^3}{1+x_2^2}, x_2^3, x_1, x_2\right]^{\mathrm{T}}$, $K_{\mathrm{cg}} = \frac{1}{b_{\mathrm{c}}}[\theta_{\mathrm{cn}_1} - \theta_{\mathrm{c}1}, \theta_{\mathrm{cn}_2} - \theta_{\mathrm{c}2}, -\theta_{\mathrm{c}3}]$, and $K_{\mathrm{dg}} = \frac{1}{b_{\mathrm{d}}}[-\theta_{\mathrm{d}1}, -\theta_{\mathrm{d}2}, -\theta_{\mathrm{d}3}, \phi_{\mathrm{dn}_1}, \phi_{\mathrm{dn}_2}]$, where $\theta_{\mathrm{n}_1}$, $\theta_{\mathrm{n}_2}$, $\phi_{\mathrm{dn}_1}$, $\phi_{\mathrm{dn}_2}$ are arbitrary scalars, so that

$$\begin{aligned} f_{\mathrm{cs}}(x) &= f_{\mathrm{c}}(x) + \begin{bmatrix} 0 \\ b_{\mathrm{c}} \end{bmatrix} \frac{1}{b_{\mathrm{c}}} \begin{bmatrix} \theta_{\mathrm{cn}_1} - \theta_{\mathrm{c}1}, \theta_{\mathrm{cn}_2} - \theta_{\mathrm{c}2}, -\theta_{\mathrm{c}3} \end{bmatrix} F_{\mathrm{c}}(x) \\ &= \begin{bmatrix} 0 & 1 \\ \theta_{\mathrm{cn}_1} & \theta_{\mathrm{cn}_2} \end{bmatrix} x \end{aligned} \tag{15.92}$$

and

$$\begin{aligned} x + f_{\mathrm{ds}}(x) &= x + f_{\mathrm{d}}(x) + \begin{bmatrix} 0 \\ b_{\mathrm{d}} \end{bmatrix} \frac{1}{b_{\mathrm{d}}} \begin{bmatrix} -\theta_{\mathrm{d}1}, -\theta_{\mathrm{d}2}, -\theta_{\mathrm{d}3}, \phi_{\mathrm{dn}_1}, \phi_{\mathrm{dn}_2} \end{bmatrix} F_{\mathrm{d}}(x) \\ &= \begin{bmatrix} 0 & 1 \\ \phi_{\mathrm{dn}_1} & \phi_{\mathrm{dn}_2} \end{bmatrix} x. \end{aligned} \tag{15.93}$$

Note that $F_{\mathrm{d}}(x)$ need not satisfy the linear growth condition (15.11). Now, with the proper choice of $\theta_{\mathrm{cn}_1}$, $\theta_{\mathrm{cn}_2}$, $\phi_{\mathrm{dn}_1}$, and $\phi_{\mathrm{dn}_2}$, it follows from Corollary 15.2 that the hybrid adaptive feedback controller (15.78) and (15.79) guarantees that $x(t) \to 0$ as $t \to \infty$. Specifically, here we choose $\theta_{\mathrm{cn}_1} = -1$, $\theta_{\mathrm{cn}_2} = -2$, $\phi_{\mathrm{dn}_1} = -0.1$, $\phi_{\mathrm{dn}_2} = -0.1$,

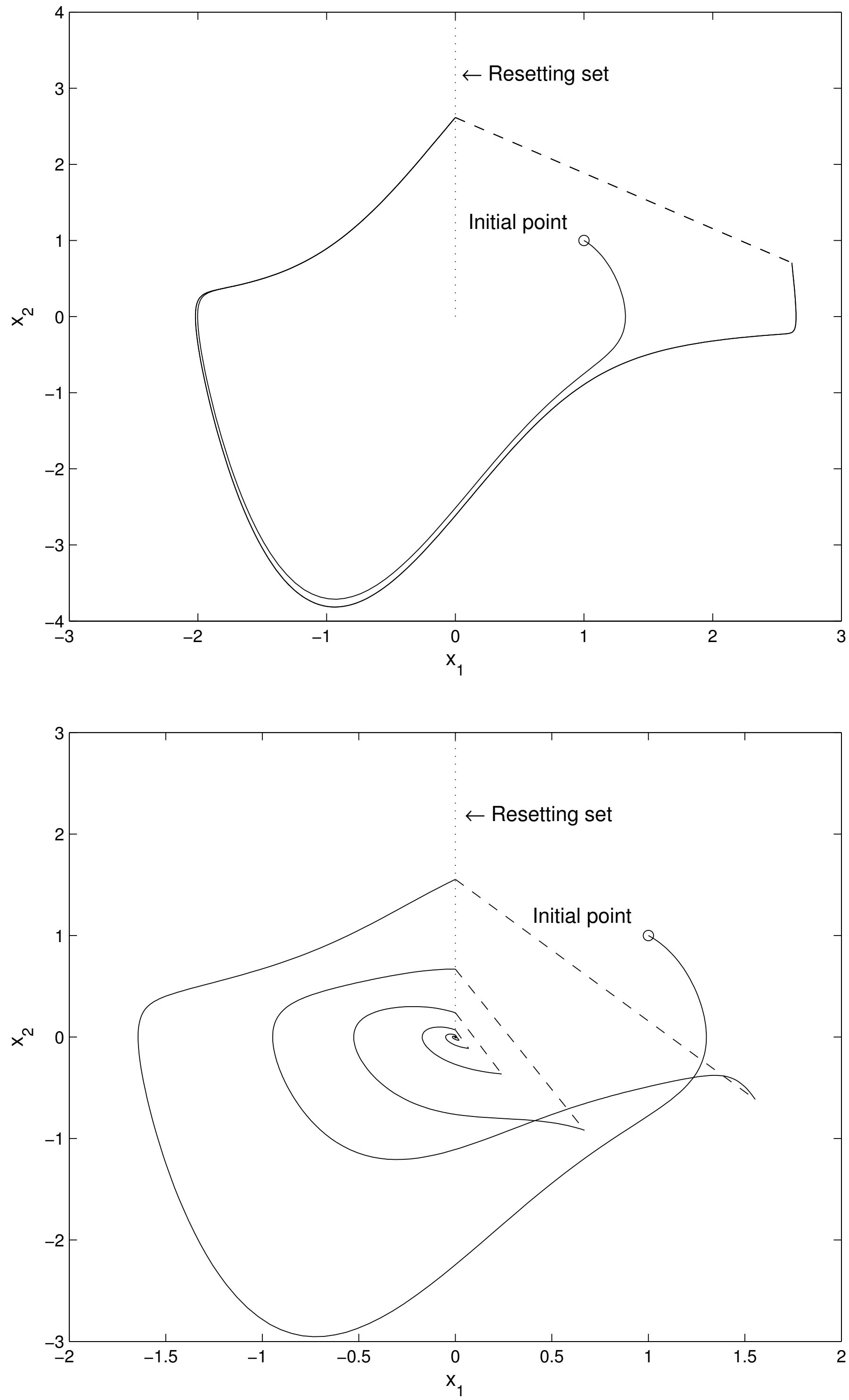



**Figure 15.1**: Phase portraits of uncontrolled and controlled hybrid system

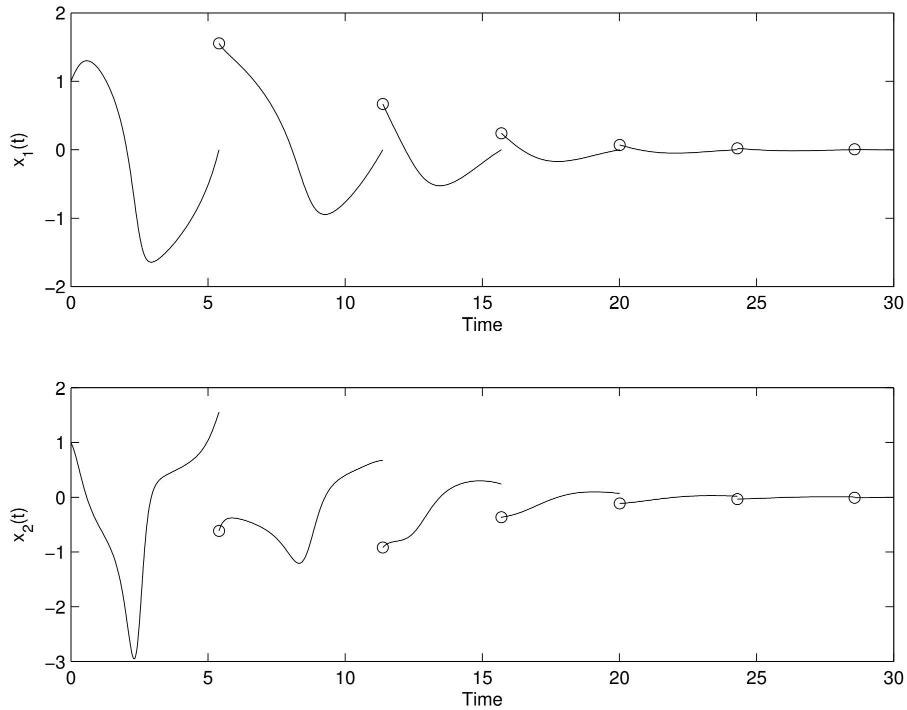



**Figure 15.2**: State trajectories versus time

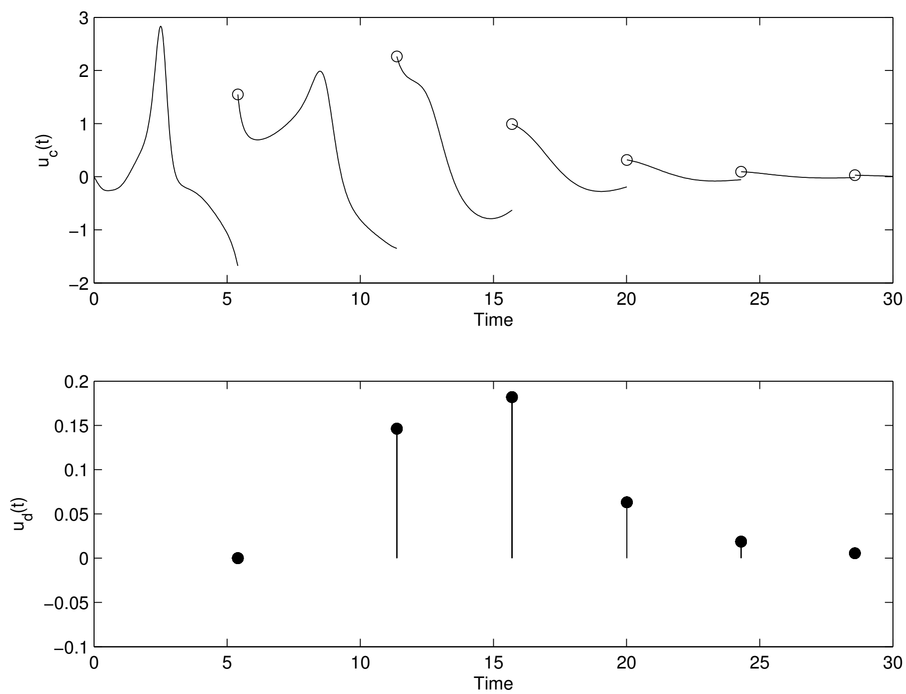



**Figure 15.3**: Control signals versus time

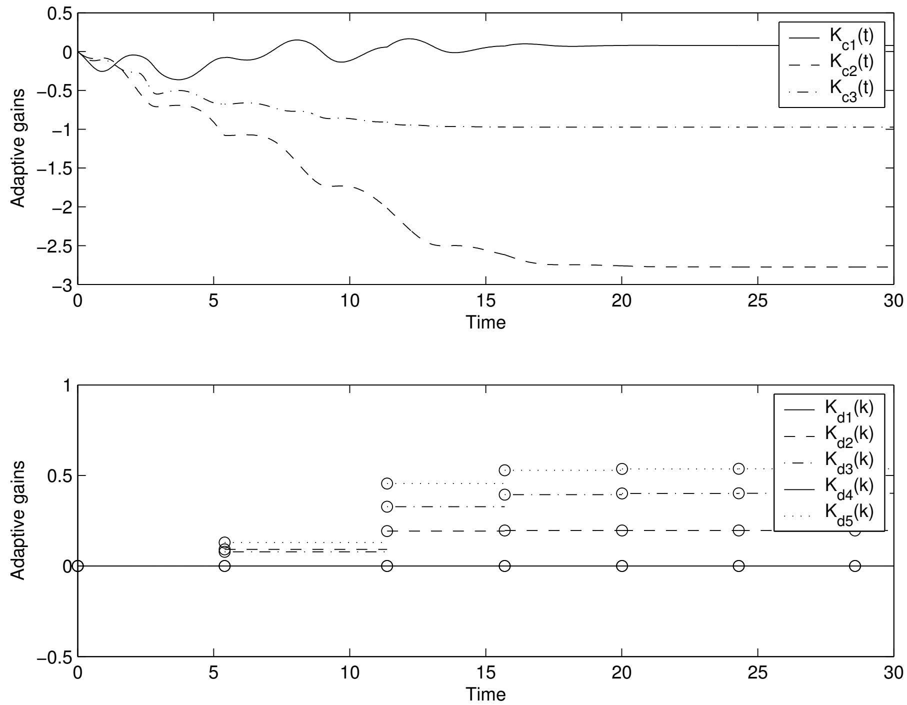



**Figure 15.4**: Adaptive gain history versus time

so that (15.7) and (15.49) are satisfied with

$$V_{\rm s}(x) = x^{\rm T}Px, \quad P = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}, \quad \ell_{\rm c}(x) = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} x. \tag{15.94}$$

With $\mu = 2$, $\alpha = 1$, $\beta = 1$, $a_1 = -5$, $a_2 = -2$, $a_3 = 3$, $\gamma = 1$, $b_{\rm c} = 3$, $b_{\rm d} = 1.4$, $\hat{\alpha} = 1$, $Y = 0.1I_3$, and initial conditions $x(0) = [1,1]^{\rm T}$, $K_{\rm c}(0) = [0,0,0]$, and $K_{\rm d}(0) = 0_{1\times5}$, Figure 15.5 shows the phase portraits of the uncontrolled and controlled hybrid system. Figures 15.6 and 15.7 show the state trajectories versus time and the control signals versus time, respectively. Finally, Figure 15.8 shows the adaptive gain history versus time.

## 15.6. Conclusion

A direct hybrid adaptive nonlinear control framework for hybrid nonlinear uncertain dynamical systems was developed. Using Lyapunov methods the proposed

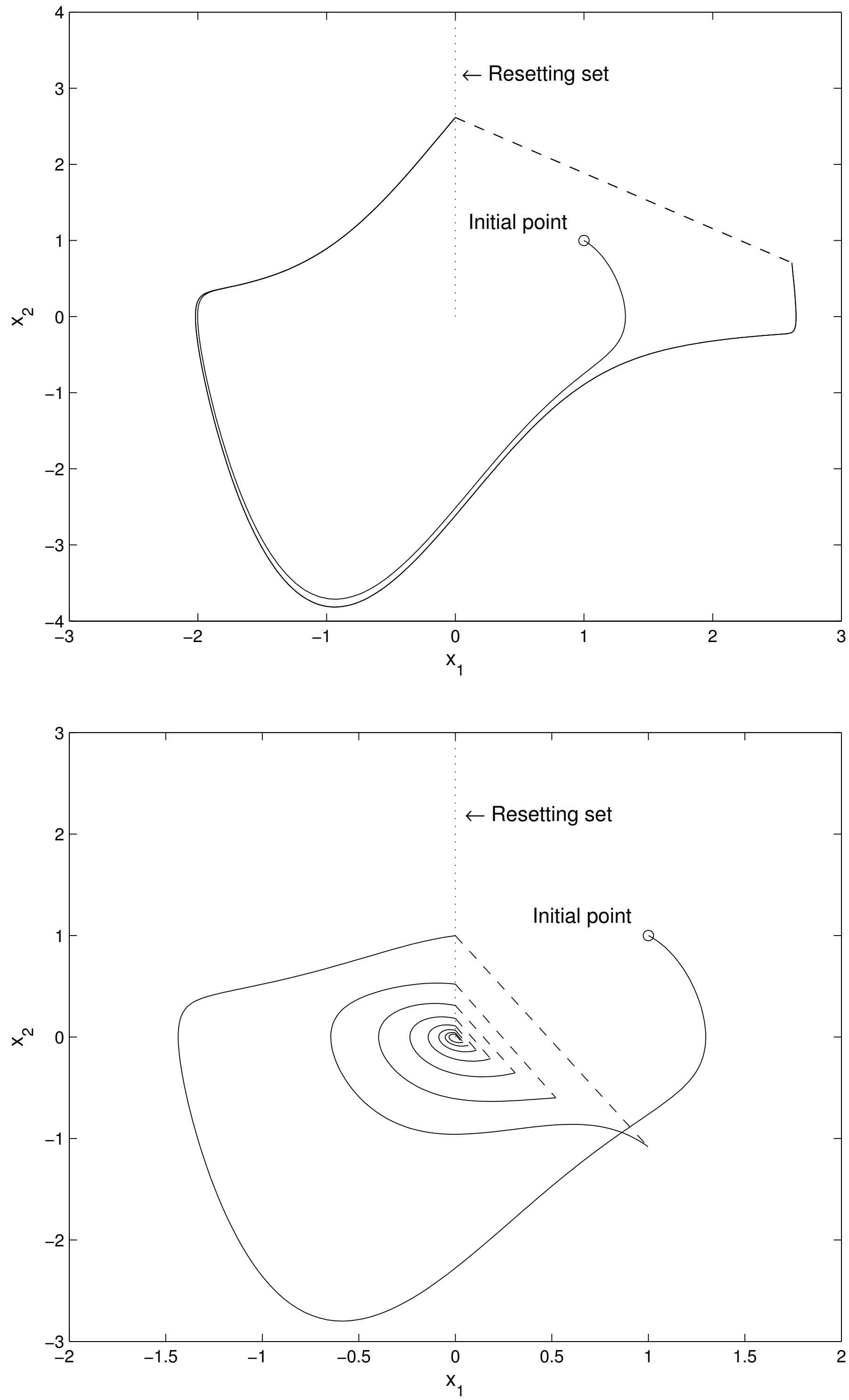



**Figure 15.5**: Phase portraits of uncontrolled and controlled hybrid system

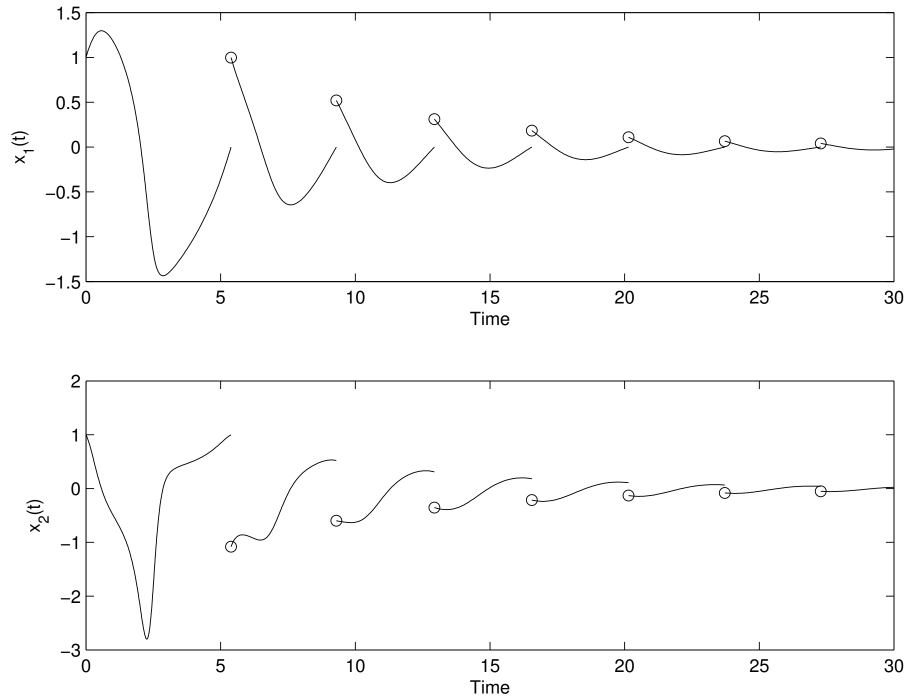



**Figure 15.6**: State trajectories versus time

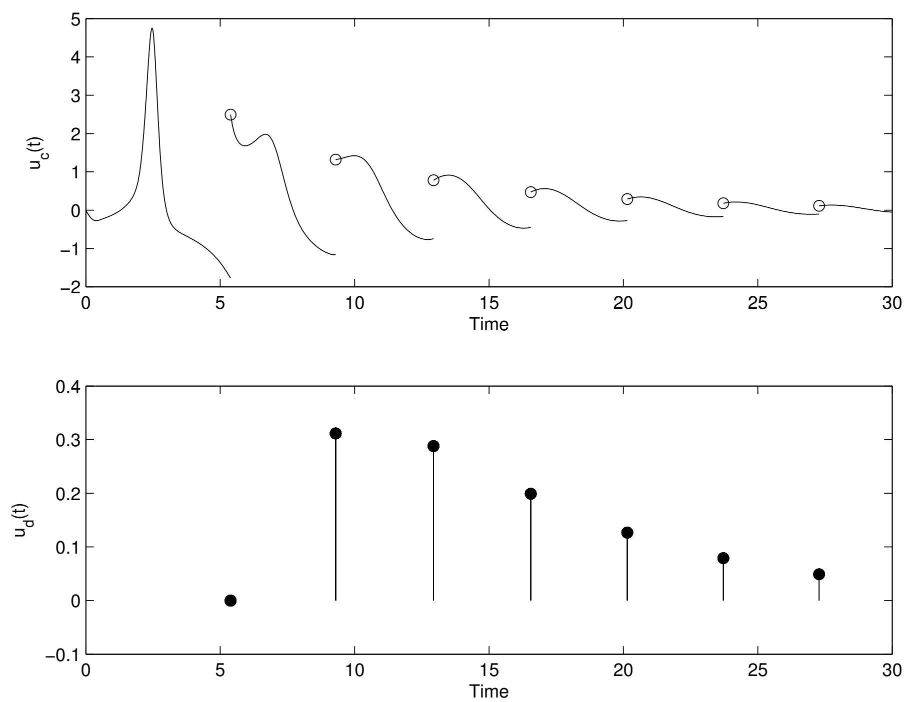



**Figure 15.7**: Control signals versus time

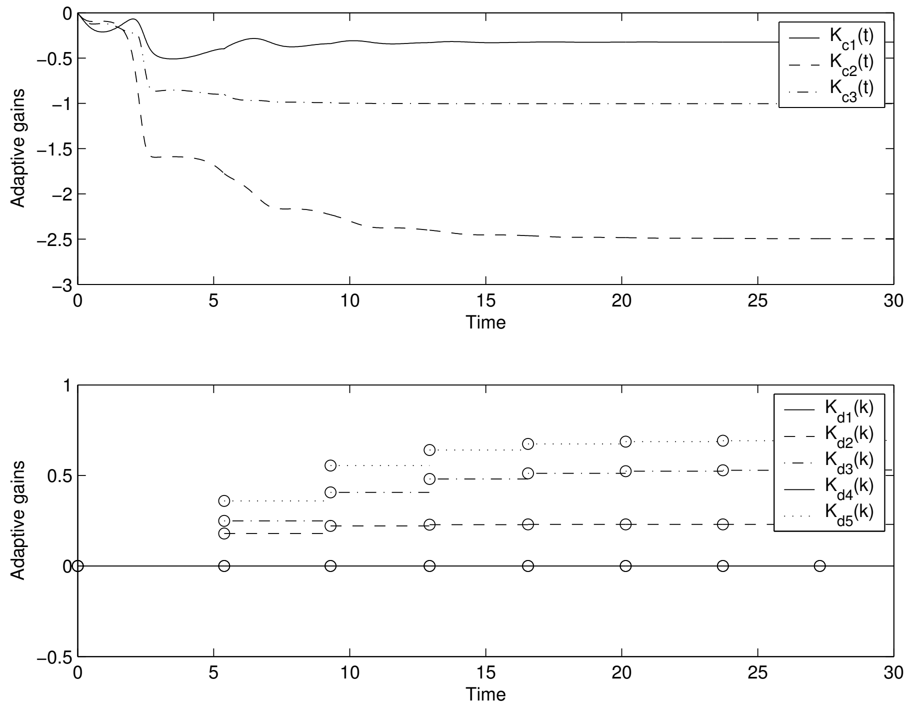



**Figure 15.8**: Adaptive gain history versus time

framework was shown to guarantee partial asymptotic stability of the closed-loop hybrid system; that is, asymptotic stability with respect to part of the closed-loop system states associated with the hybrid plant dynamics. Furthermore, hybrid adaptive controllers guaranteeing attraction of the closed-loop system plant states were also developed. In the case where the nonlinear hybrid system is represented in a hybrid normal form, the nonlinear hybrid adaptive controllers were constructed without knowledge of the system dynamics. Finally, two numerical examples were presented to show the utility of the proposed hybrid adaptive stabilization and attraction schemes.

# Chapter 16

# Concluding Remarks and Recommendations for Future Research

## 16.1. Conclusions

The focus of this dissertation was to address several outstanding issues in direct adaptive control of nonlinear uncertain dynamical systems with exogenous disturbances. The adaptive control laws were predicated on Lyapunov (resp., Lyapunov-like) functions and guaranteed partial asymptotic stability (resp., partial ultimate boundedness) with respect to part of the closed-loop system states associated with the plant. Furthermore, adaptive controller gains (resp., weights) were shown to be bounded. Even though it is not necessary to utilize the notion of feedback linearization, throughout the dissertation it was shown that the adaptive control architecture is considerably simplified if we make use of feedback linearizing functions so that update laws can be constructed by solving Lyapunov/Riccati equations. Furthermore, we have shown that feedback linearization is always possible in the case where the nonlinear system is represented in normal form with input-to-state stable internal dynamics.

To arrive at a tractable control design formulation in spite of extreme complexity of modern engineering systems, we first developed a direct adaptive control framework for general nonlinear uncertain dynamical systems with bounded amplitude and bounded energy disturbances. In the case where the system disturbances are $L_2$ disturbances, the proposed framework guaranteed that the closed-loop nonlinear input-output map from uncertain exogenous $L_2$ disturbances to system performance variables is nonexpansive. Based on this result, the framework was extended for nonlinear uncertain systems with constant linearly parameterized uncertainty and nonlinear state-dependent uncertainty. It was shown that this framework captures the residual approximation error inherent in linear parameterizations of system uncertainty via basis functions. In addition, a direct adaptive tracking control framework with actuator amplitude and rate saturation constraints was also developed. To guarantee asymptotic stability of the closed-loop tracking error dynamics in the face of amplitude and rate saturation constraints, the adaptive control signal to a given reference (governor or supervisor) system was modified to effectively robustify the error dynamics to the saturation constraints.

Next, we developed a novel parametrization-free adaptive control framework for a class of nonlinear matrix second-order dynamical systems with state-dependent uncertainty. The proposed framework guaranteed global asymptotic stability without requiring any knowledge of the system nonlinearities other than the assumption that they are continuous and (lower) bounded. Generalizations to the case where the system nonlinearities are unbounded were also considered. In the special case of matrix second-order systems with polynomial nonlinearities with unknown coefficients and unknown order, we provided a universal adaptive controller that guarantees closed-loop stability of the plant states.

Nonnegative and compartmental models provide a broad framework for biological and physiological systems, including clinical pharmacology, and are well suited for developing models for closed-loop control of drug administration. Motivated by the potential clinical applications of adaptive control for pharmacology in general, and anesthesia and critical care unit medicine in particular, we proposed adaptive control frameworks for linear and nonlinear nonnegative and compartmental uncertain dynamical systems. In particular, we focused on achieving set-point stabilization in the nonnegative orthant of the state space as well as zero-point stabilization. Furthermore, we developed neural network adaptive controllers that guarantee ultimate boundedness of the closed-loop system states. For both cases, the adaptive controllers were shown to guarantee that the physical system states remain in the nonnegative orthant of the state space. In addition, we constructed adaptive controllers that constrain their inputs to be nonnegative, which is necessary to account for nonnegative control inputs (infusion pumps) of drug delivery systems. The proposed approaches were used to control the infusion of the anesthetic drug propofol and midazolam for maintaining a desired constant level of depth of anesthesia for noncardiac surgery in the face of combined interpatient pharmacokinetic and pharmacodynamic variability.

We then turn our attention to addressing the adaptive control problem for discrete-time nonlinear uncertain systems. Specifically, we developed a direct adaptive control framework for adaptive stabilization, disturbance rejection, and command following of multivariable discrete-time nonlinear uncertain systems with exogenous bounded amplitude disturbances and bounded energy (square-summable) $\ell_2$ disturbances. These results are analogous to the continuous-time adaptive disturbance rejection results discussed in Chapter 2 for continuous-time nonlinear uncertain systems. The proposed adaptive controller addresses the problem of disturbance rejection for nonlinear uncertain discrete-time systems with $\ell_2$ signal norms on the disturbances and performance variables. An adaptive control framework, via partial or semi-definite

Lyapunov functions, that guarantees convergence of plant state and parameter errors under a generic geometric constraints was also developed. The generic condition was shown to be consistent with the notion of persistent excitation in the adaptive control and system identification literature required for parameter error convergence.

Finally, we characterized hybrid adaptive control laws for nonlinear uncertain impulsive dynamical systems. Using the the hybrid invariance principle the proposed framework was shown to guarantee asymptotic stability of the closed-loop system states associated with the hybrid plant dynamics. Furthermore, in the case where the nonlinear hybrid system is represented in a hybrid normal form, the nonlinear hybrid adaptive controllers were constructed without knowledge of the system dynamics. Using less restrictive conditions, we also provide adaptive controllers that guarantee attraction of the closed-loop hybrid plant states.

## 16.2. Recommendations for Future Research

Based on the results given in Chapter 2, we developed a robust adaptive control framework in Chapter 3 that captures the residual approximation error inherent in linear parameterizations of system uncertainty via basis functions. In this framework, we assumed that the structured parametric uncertainty with bounded variation is norm bounded with a *known* bound. The knowledge of the bound is required to solve a Hamilton-Jacobi (bounded real Riccati) equation in order to construct $V_{\rm s}'(\cdot)$. However, the norm bound for the bounded variation is not always known. Hence, it is plausible if we can characterize cases where we do not require knowledge of the norm bound. A class of such cases would be matrix second-order systems.

In Chapter 6 we developed an adaptive control framework for a class of nonlinear matrix second-order dynamical systems with exogenous disturbances. The adaptive controller does not require a parametrization of uncertain nonlinear system parameters. The framework is, however, applicable to the case where the generalized damping matrix $C(\cdot)$ is only a function of the generalized position coordinates for the case $C(\cdot)$ is lower bounded. Furthermore, in the case where damping and stiffness operators are time-varying functions, the damping and stiffness operators have to be lower and upper bounded. To address a more general matrix second-order dynamical system it would be of interest to develop an adaptive feedback controller that allows for $C(t, q, \dot{q})$ and $K(t, q)$ to be only lower bounded. This may be achieved via a time-varying Lyapunov function which does not have to be decrescent [198, Theorem 6.23].

A direct adaptive control framework for linear nonnegative and compartmental systems with unknown time delay was developed in Chapter 8. In particular, when the compartmental dynamics are mammillary, we showed that the proposed controller can always stabilize the closed-loop system without knowing the system parameters nor the delay amount. The framework can be extended to *nonlinear* mammillary systems with unknown time delay. Furthermore, an adaptive control framework for general nonlinear systems with *unknown* time delay is virtually nonexistent in the literature.

In Chapters 10–12, we consider neural network adaptive controllers for nonlinear nonnegative dynamical systems to guarantee *ultimate boundedness* of the physical system states as well as the neural network weighting gains. In the literature, there are numerous results on neural network adaptive control framework for general nonlinear systems, but virtually none of them proves attraction of the plant states to the equilibrium point. It may be possible to show state convergence by making use of the robust adaptive ideas presented in Chapter 3; that is, update laws can be constructed assuming that the neural network approximation error is *sector* bounded instead of *amplitude* bounded. In this case, the neural network may not require $\sigma$- or $e$-modification terms in the update laws which can simplify stability proofs.

In Chapter 15 we developed a hybrid adaptive control framework for impulsive (mixed continuous/discrete-time) dynamical systems. Closed-loop stabilization/attraction of the hybrid plant states is guaranteed if nonlinear Lyapunov(-like) equations are satisfied for both continuous-time and discrete-time dynamics with the common positive-definite function $V_{\rm s}(\cdot)$. Due to this restriction, the framework is not applicable to mechanical-type, matrix second-order (Hamiltonian) dynamical systems with nonsmooth impacts. To address this problem it may be beneficial to first develop an adaptive control framework for continuous-time matrix second-order systems with $V_{\rm s}(\cdot)$ having a Hamiltonian structure. Then the adaptive control problem for impulsive mechanical systems can be addressed with $V_{\rm s}(\cdot)$ being Hamiltonian and satisfying a hybrid continuous-time and discrete-time set of nonlinear Lyapunov equations. In addition, a hybrid neuro adaptive control framework for hybrid dynamical systems does note exist in the literature and hence a fruitful area of research would be to develop a neural network adaptive control laws for impulsive dynamical systems.

There is no doubt that control-system technology has a great deal to offer pharmacology in general, and anesthesia and critical care unit medicine in particular. Critical care patients, whether undergoing surgery or recovering in intensive care units, require drug administration to regulate key physiological variables (e.g., blood pressure, cardiac output, heart rate, degree of consciousness, etc.) within desired levels. The rate of infusion of each administered drug is *critical*, requiring constant monitoring and frequent adjustments. Open-loop control by clinical personnel can be very tedious, imprecise, time consuming, and sometimes of poor quality. Alternatively, closed-loop control can achieve desirable system performance in the face of the highly uncertain and hostile environment of surgery and the intensive care unit.

Since robust and adaptive controllers can achieve system performance without excessive reliance on system models, active (robust and adaptive) closed-loop control has the potential in improving the quality of medical care as well as curtailing the increasing costs of health care.

Even though there has been several control algorithms proposed in recent years for active drug administration as reported in this dissertation, closed-loop control for clinical pharmacology is still at its infancy. There are numerous challenges that lie ahead. In particular, an implicit assumption inherent in all the proposed control frameworks discussed in this dissertation is that the control law is implemented without any regard to actuator amplitude and rate saturation constraints. Of course, any electromechanical control actuation device is subject to amplitude and/or rate constraints leading to saturation nonlinearities enforcing limitations on control amplitudes and control rates. More importantly, in pharmacological applications, drug infusion rates can vary from patient to patient and it is vital that they do not exceed certain threshold values. As a consequence, actuator nonlinearities and actuator constraints (i.e., infusion pump rate constraints) need to be accounted for in drug delivery systems since they can severely degrade closed-loop system performance, and in some cases drive the system to instability. These effects are even more pronounced for adaptive controllers which continue to adapt when the feedback loop has been severed due to the presence of actuator saturation causing unstable controller modes to drift, which in turn leads to severe windup effects [156].

Another important issue not considered by most of the control algorithms discussed in this dissertation is sensor measurement noise. In particular, EEG signals may have as much as 10% variation due to noise. For example, the BIS signal may be corrupted by *electromyographic noise*; that is, signals emanating form muscle rather that the central nervous system. Even though electromyographic noise can be minimized by muscle paralysis, there are other sources of measurement noise that are stochastic in nature and need to be accounted for within the control design processes.

In many compartmental pharmacokinetic system models, transfers between compartments are assumed to be instantaneous; that is, the model does not account for material in transit. Even though this is a valid assumption for certain biological and physiological systems, it is not true in general; especially in certain pharmacokinetic and pharmacodynamic models. For example, if a bolus of drug is injected into the circulation and we seek its concentration level in the extracellular and intercellular space of some organ, there exists a time lag before it is detected in that organ [123]. In this case, assuming instantaneous mass transfer between compartments will yield erroneous models. Hence, to accurately describe the distribution of pharmacological agents in the human body, it is necessary to include in any mathematical pharmacokinetic model some information of the past system states. In this case the state of the system at any given time involves a *piece of trajectories* in the space of continuous functions defined on an interval in the nonnegative orthant. This of course leads to (infinite-dimensional) delay dynamical systems [55, 82]. This is especially relevant to correctly address the time delay inherent in equilibrating the effect site compartment with the central compartment and would have ramifications in the control design processes. For example, for adaptive control, a nonlinear adaptive algorithm for compartmental systems with *unknown* time delay would need to be developed [40].

Optimal control for drug administration is also often necessary in clinical pharmacology. For therapeutic reasons in the intensive care unit, it may be desirable to regulate (maintain) the amount of a drug in one compartment above a certain minimum threshold (dosage) level while maintaining the amount below a certain maximum level in another compartment. Furthermore, to minimize drug side effects, it is desirable to minimize the total amount (dosage) of drugs used [30,31,46,148,149,170,192,193,224].

Drug administration in clinics and hospitals do not generally satisfy the aforementioned conditions. To enforce the specialized structure of compartmental and nonnegative systems, nonnegative state and control constraints will need to be enforced as part of the controller design. The optimal (nonnegative) control law will need to be designed to maintain desired drug concentrations in the plasma dictated by therapeutic effects while minimizing drug dosage to reduce side effects [180].

A fundamental constraint for nonnegative linear system stabilization with a nonnegative control signal arises in set-point regulation. In particular, it can be shown that the existence of an equilibrium point in the interior of the nonnegative orthant of the state space is assured only if the nonnegative dynamical system has a system matrix that does not possess eigenvalues in the open right-half plane [53]. This implies that the largest eigenvalue of the system lies on the imaginary axis. However, by the Perron-Frobenius theorem [20] this eigenvalue is real and therefore equal to zero. Hence, the system matrix is semistable. In light of this constraint, it can be shown using Brockett's necessary condition for asymptotic stabilizability [28, 53] that there does not exist a *continuous nonnegative* stabilizing feedback for set-point regulation in the nonnegative orthant for a nonnegative system. However, that is not to say that asymptotic feedback regulation using *discontinuous* nonnegative feedback is not possible. Of course, in the case where the system matrix is asymptotically stable, continuous nonnegative feedback for set-point regulation in the nonnegative orthant can be used to improve system performance. In light of the above, it may be desirable to develop hybrid (discontinuous) adaptive controllers for positive set-point regulation of semistable compartmental systems. Hybrid adaptive control is virtually nonexistent in the literature [93]. Furthermore, the problem of active control of sedation using an intermittent clinician assessment with an ordinal sedation scoring system as a performance variable necessitates hybrid control architectures to account for abstract decision making units (nurses or physicians) performing logical checks that identify system mode operation and specify the lower-level continuous-time subcontroller to be activated.

It is clear that closed-loop control for clinical pharmacology would significantly advance our understanding of the wide effects of pharmacological agents and anesthetics as well as advance the state-of-the-art in drug delivery systems. While our focus in this dissertation has been to survey the recent developments of active control methods to deliver sedation to critically ill patients in an acute care environment and outline some of the future challenges of active sedation control, these control methods will have implications for other uses of closed-loop control of drug delivery. There are numerous potential applications such as control of glucose, heart rate, blood pressure, etc., that may be improved as a result of active drug dosing control. Payoffs would arise from improvements in medical care, health care, reliability of drug dosing equipment, as well as reduced cost for health care.

# References


[1] D. Y. Abramovitch, R. L. Kosut, and G. F. Franklin, “Adaptive control with saturating inputs,” in *Proc. IEEE Conf. Dec. Contr.*, Athens, Greece, pp. 848–852, 1986.

[2] A. R. Absalom, N. Sutcliffe, and G. N. Kenny, “Closed-loop control of anesthesia using bispectral index: Performance assessment in patients undergoing major orthopedic surgery under combined general and regional anesthesia,” *Anesthesiology*, vol. 96, no. 1, pp. 67–73, 2002.

[3] J. Ahmed, V. T. Coppola, and D. S. Bernstein, “Adaptive asymptotic tracking of spacecraft attitude motion with inertia matrix identification,” *AIAA J. Guid. Contr. Dyn.*, vol. 21, pp. 684–691, 1998.

[4] J. Alvarez-Ramírez, R. Suárez, and R. Femat, “Control of continuous-stirred tank reactors: Stabilization with unknown reaction rates,” *Chem. Engng. Sci.*, vol. 51, pp. 4183–4188, 1996.

[5] J. M. Alvis, J. G. Reves, A. V. Govier, P. G. Menkhaus, C. E. Henling, J. A. Spain, and E. Bradley, “Computer-assisted continuous infusions of fentanyl during cardiac anesthesia: Comparison with a manual method,” *Anesthesiology*, vol. 63, pp. 41–49, 1985.

[6] D. H. Anderson, *Compartmental Modeling and Tracer Kinetics.* Berlin, New York: Springer-Verlag, 1983.

[7] A. M. Annaswamy and S. P. Kárason, “Discrete-time adaptive control in the presence of input constraints,” *Automatica*, vol. 31, pp. 1421–1431, 1995.

[8] P. J. Antsaklis and A. Nerode, eds., “Special issue on hybrid control systems,” *IEEE Trans. Autom. Contr.*, vol. 43, no. 4, 1998.

[9] T. M. Apostol, *Mathematical Analysis.* Reading, MA: Addison-Wesley, 1957.

[10] T. M. Apostol, *Mathematical Analysis.* Reading, MA: Addison-Wesley, 1974.

[11] J. M. Arnsparger, B. C. McInnis, J. R. Glover, and N. A. Normann, “Adaptive control of blood pressure,” *IEEE Trans. Biomed. Eng.*, vol. 30, pp. 168–176, 1983.

[12] K. J. Åström and B. Wittenmark, *Adaptive Control.* Reading, MA: Addison-Wesley, 1989.

[13] J. M. Bailey, W. M. Haddad, and T. Hayakawa, "Closed-loop control in clinical pharmacology: Paradigms, benefits, and challenges," in *Proc. Amer. Contr. Conf.*, Boston, MA, July 2004.

[14] J. M. Bailey and S. L. Shafer, "A simple analytical solution to the three compartment pharmacokinetic model suitable for computer controlled infusion pumps," *IEEE Trans. Biomed. Eng.*, vol. 38, no. 6, pp. 522–525, 1991.

[15] D. D. Bainov and P. S. Simeonov, *Systems with Impulse Effect: Stability, Theory and Applications*. England: Ellis Horwood Limited, 1989.

[16] K. Behbehani and R. R. Cross, "A controller for regulation of mean arterial blood pressure using optimum nitroprusside infusion rate," *IEEE Trans. Biomed. Eng.*, vol. 38, no. 6, pp. 513–521, 1991.

[17] R. Bellman, "Topics in pharmacokinetics. I. Concentration dependent rates," *Math. Biosci.*, vol. 6, pp. 13–17, 1970.

[18] B. W. Bequette, *Process Dynamics: Modeling, Analysis, and Simulation*. Upper Saddle River, NJ: Prentice-Hall, 1998.

[19] A. Berman, M. Neumann, and R. J. Stern, *Nonnegative Matrices in Dynamic Systems*. New York, NY: Wiley and Sons, 1989.

[20] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*. New York, NY: Academic Press, 1979.

[21] D. S. Bernstein and S. P. Bhat, "Nonnegativity, reducibility and semistability of mass action kinetics," in *Proc. IEEE Conf. Dec. Contr.*, Phoenix, AZ, pp. 2206–2211, December 1999.

[22] D. S. Bernstein and S. P. Bhat, "Nonnegativity, reducibility, and semistability of mass action kinetics," in *Proc. IEEE Conf. Dec. Contr.*, Pheonix, AZ, pp. 2206–2211, December 1999.

[23] D. S. Bernstein and W. M. Haddad, "Robust stability and performance analysis for state-space systems via quadratic Lyapunov bounds," *SIAM J. Matrix Anal. Appl.*, vol. 11, no. 2, pp. 239–271, 1990.

[24] D. S. Bernstein and D. C. Hyland, "Compartmental modeling and second-moment analysis of state space systems," *SIAM J. Matrix Anal. Appl.*, vol. 14, pp. 880–901, 1993.

[25] S. P. Bhat and D. S. Bernstein, "Lyapunov analysis of semistability," in *Proc. Amer. Contr. Conf.*, San Diego, CA, pp. 1608–1612, June 1999.

[26] R. G. Bickford, "Automatic electroencephalographic control of anesthesia (servo-anesthesia)," *Electroencephalographic Clin. Neurophysiol.*, vol. 3, pp. 83–86, 1951.

[27] L. J. Bledsoe, “Linear and nonlinear approaches for ecosystem dynamic modeling,” in *Systems Analysis and Simulation in Ecology* (B. C. Patten, ed.), vol. 4, ch. 10, pp. 283–297, New York: Academic, 1976.

[28] R. W. Brockett, “Asymptotic stability and feedback stabilization,” in *Differential Geometric Control Theory* (R. W. Brockett, R. S. Millman, and H. J. Sussmann, eds.), pp. 181–191, Boston, MA: Birkhauser, 1983.

[29] R. F. Brown, “Compartmental system analysis: State of the art,” *IEEE Trans. Biomed. Eng.*, vol. 27, pp. 1–11, 1980.

[30] J. Buell, R. Jelliffe, R. Kalaba, and R. Sridhar, “Modern control theory and optimal drug regiments, I: The plateau effect,” *Math. Biosci.*, vol. 5, pp. 285–296, 1969.

[31] J. Buell, R. Jelliffe, R. Kalaba, and R. Sridhar, “Modern control theory and optimal drug regiments, II: Combination therapy,” *Math. Biosci.*, vol. 6, pp. 67–74, 1970.

[32] C. I. Byrnes and A. Isidori, “Asymptotic stabilization of minimum phase nonlinear systems,” *IEEE Trans. Autom. Contr.*, vol. 36, no. 10, pp. 1122–1137, 1991.

[33] C. I. Byrnes, A. Isidori, and J. C. Willems, “Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems,” *IEEE Trans. Autom. Contr.*, vol. 36, no. 11, pp. 1228–1240, 1991.

[34] S. M. Candel, “Combustion instabilities coupled by pressure waves and their active control,” in *Proc. 24th International Symposium on Combustion*, Sydney, Australia, pp. 1277–1296, July 1992.

[35] V. Chellaboina, S. P. Bhat, and W. M. Haddad, “An invariance principle for nonlinear hybrid and impulsive dynamical systems,” *Nonlinear Anal.: Theory Methods Appl.*, vol. 53, pp. 527–550, 2003.

[36] V. Chellaboina, W. M. Haddad, and T. Hayakawa, “Adaptive control for nonlinear matrix second-order systems with sign-varying damping and stiffness operators,” in *Proc. ASME Int. Mech. Engin. Cong.*, New York, NY, Paper No. DSC-24579, November 2001.

[37] V. Chellaboina, W. M. Haddad, and T. Hayakawa, “Direct adaptive control for nonlinear matrix second-order dynamical systems with state-dependent uncertainty,” *Sys. Contr. Lett.*, vol. 48, pp. 53–67, 2003.

[38] V. Chellaboina, W. M. Haddad, and T. Hayakawa, “Direct adaptive control for nonlinear matrix second-order dynamical systems with state-dependent uncertainty,” in *Proc. Amer. Contr. Conf.*, Arlington, VA, pp. 4247–4252, June 2001.

[39] V. Chellaboina and W. M. Haddad, "Exponentially dissipative dynamical systems: A nonlinear extension of strict positive realness," in *Proc. Amer. Contr. Conf.*, Chicago, IL, pp. 3123–3127, June 2000.

[40] V. Chellaboina, W. M. Haddad, J. Ramakrishnan, and T. Hayakawa, "Direct adaptive control of nonnegative and compartmental systems with time delay," in *Proc. Amer. Contr. Conf.*, Boston, MA, July 2004.

[41] V. Chellaboina, W. M. Haddad, J. Ramakrishnan, and T. Hayakawa, "Direct adaptive control of nonnegative and compartmental systems with time delay," *Int. J. Adapt. Control Signal Process.*, submitted.

[42] V. Chellaboina and W. M. Haddad, "A unification between partial stability and stability theory for time-varying systems," *Contr. Syst. Mag.*, vol. 22, pp. 66–75, 2002. Erratum, vol. 23, p. 101, 2003.

[43] C.-T. Chen, *Linear System Theory and Design.* New York: Holt, Rinehart, and Winston, 1984.

[44] F. C. Chen and H. K. Khalil, "Adaptive control of nonlinear systems using neural networks," *Int. J. Contr.*, vol. 55, no. 6, pp. 1299–1317, 1992.

[45] C. T. Chen, W. L. Lin, and T. S. Kuo, "Adaptive control of arterial blood pressure with a learning controller based on multilayer neural networks," *IEEE Trans. Biomed. Eng.*, vol. 44, no. 7, pp. 601–609, 1997.

[46] Y. Cherruault, *Mathematical Modelling in Biomedicine.* Dordrecht, Holland: Reidel, 1986.

[47] A. R. Cooper and G. V. Jeffreys, *Chemical Kinetics and Reactor Design*, ch. 5, pp. 146–206. Edinburgh, Scotland: Oliver & Boyd, 1971.

[48] J. J. Craig, *Adaptive Control of Mechanical Manipulators.* Reading, MA: Addison-Wesley, 1988.

[49] F. E. C. Culick, "Nonlinear behavior of acoustic waves in combustion chambers I," *Acta Astronautica*, vol. 3, pp. 715–734, 1976.

[50] F. E. C. Culick, "Combustion instabilities in liquid-fueled propulsion systems – An overview," in *AGARD Conference Proceedings*, no. 450, Bath, UK, 1988.

[51] M. Davidian and D. M. Giltinan, *Nonlinear Models for Repeated Measurement Data.* Boca Raton, FL: Chapman and Hall/CRC, 1995.

[52] P. De Leenheer and D. Aeyels, "Stability properties of equilibria of classes of cooperative systems," *IEEE Trans. Autom. Contr.*, vol. 46, pp. 1996–2001, 2001.

[53] P. De Leenheer and D. Aeyels, "Stabilization of positive linear systems," *Sys. Contr. Lett.*, vol. 44, pp. 259–271, 2001.

[54] V. A. Doze, L. M. Westphal, and P. F. White, “Comparison of propofol with methohexital for outpatient anesthesia,” *Aneth. Analg.*, vol. 65, pp. 1189–1195, 1986.

[55] L. Dugard and E. I. Verriest, eds., *Stability and Control of Time-Delay Systems.* London: Springer-Verlag, 1998.

[56] B. Egardt, “Stability analysis of discrete-time adaptive control schemes,” *IEEE Trans. Autom. Contr.*, vol. 25, pp. 710–716, 1980.

[57] E. I. Eger, *Anesthetic Uptake and Action.* Baltimore, MD: Williams and Wilkins, 1997.

[58] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications.* New York, NY: John Wiley & Sons, 2000.

[59] M. Feinberg, “The existence of uniqueness of states for a class of chemical reaction networks,” *Arch. Rational Mech. Anal.*, vol. 132, pp. 311–370, 1995.

[60] A. L. Fradkov and D. J. Hill, “Exponential feedback passivity and stabilizability of nonlinear systems,” *Automatica*, vol. 34, pp. 697–703, 1998.

[61] J. J. Fuchs, “Discrete adaptive control: A sufficient condition for stability and applications,” *IEEE Trans. Autom. Contr.*, vol. 25, pp. 940–946, 1980.

[62] R. E. Funderlic and J. B. Mankin, “Solution of homogeneous systems of linear equations arising from compartmental models,” *SIAM J. Sci. Statist. Comp.*, vol. 2, pp. 375–383, 1981.

[63] Y.-T. Fung and V. Yang, “Active control of nonlinear pressure oscillations in combustion chambers,” *Journal of Propulsion and Power*, vol. 8, pp. 1282–1289, 1992.

[64] M. R. Garzia and C. M. Lockhart, “Nonhierarchical communications networks: An application of compartmental modeling,” *IEEE Trans. Communications*, vol. 37, pp. 555–564, 1989.

[65] A. Gentilini, M. Rossoni-Gerosa, C. W. Frei, R. Wymann, M. Morari, A. M. Zbinden, and T. W. Schnider, “Modeling and closed-loop control of hypnosis by means of bispectral index (BIS) with isoflurane,” *IEEE Trans. Biomed. Eng.*, vol. 48, pp. 874–889, 2001.

[66] A. G. Gilman, J. G. Hardman, and L. E. Limbird, *Goodman and Gilman's The Pharmacological Basis of Therapeutics.* New York, NY: McGraw-Hill, 10 ed., 1996.

[67] P. S. Glass, M. Bloom, L. Kearse, C. Rosow, P. Sebel, and P. Manberg, “Bispectral analysis measures sedation and memory effects of propofol, midazolam, isoflurane, and alfentanil in normal volunteers,” *Anesthesiology*, vol. 86, pp. 836–847, 1997.

[68] P. S. Glass, D. K. Goodman, B. Ginsberg, J. G. Reves, and J. R. Jacobs, "Accuracy of pharmacokinetic model-driven infusion of propofol," *Anesthesiology*, vol. 71, p. A277, 1989.

[69] P. S. A. Glass and I. J. Rampil, "Automated anesthesia: Fact or fantasy?," *Anesthesiology*, vol. 95, pp. 1–2, 2001.

[70] K. Godfrey, *Compartmental Models and Their Applications.* New York: Academic Press, 1983.

[71] G. C. Goodwin and R. S. Long, "Generalization of results on multivariable adaptive control," *IEEE Trans. Autom. Contr.*, vol. 25, pp. 1241–1245, 1980.

[72] G. C. Goodwin, P. J. Ramadge, and P. E. Caines, "Discrete-time multivariable adaptive control," *IEEE Trans. Autom. Contr.*, vol. 25, pp. 449–456, 1980.

[73] D. W. Gu, M. C. Tsai, S. D. O'Young, and I. Postlethwaite, "State-space formulae for discrete-time $H_\infty$ optimization," *Int. J. Contr.*, vol. 49, pp. 1683–1723, 1989.

[74] I. Gyori, "Delay differential and integro-differential equations in biological compartment models," *Syst. Sci.*, vol. 8, no. 2–3, pp. 167–187, 1982.

[75] W. M. Haddad, V. Chellaboina, and E. August, "Stability and dissipativity theory for nonnegative dynamical systems: A thermodynamic framework for biological and physiological systems," in *Proc. IEEE Conf. Dec. Contr.*, Orlando, FL, pp. 442–458, December 2001.

[76] W. M. Haddad, V. Chellaboina, J. L. Fausz, and A. Leonessa, "Optimal nonlinear robust control for nonlinear uncertain systems," *Int. J. Contr.*, vol. 73, pp. 329–342, 2000.

[77] W. M. Haddad, V. Chellaboina, and T. Hayakawa, "Robust adaptive control for nonlinear uncertain systems," in *Proc. IEEE Conf. Dec. Contr.*, Orlando, FL, pp. 1615–1620, December 2001.

[78] W. M. Haddad, V. Chellaboina, Q. Hui, and T. Hayakawa, "Neural network adaptive control for discrete-time nonlinear nonnegative dynamical systems," in *Proc. IEEE Conf. Dec. Contr.*, Maui, HI, December 2003.

[79] W. M. Haddad, V. Chellaboina, Q. Hui, and T. Hayakawa, "Neural network adaptive control for discrete-time nonlinear nonnegative dynamical systems," *Neural Networks*, submitted.

[80] W. M. Haddad, V. Chellaboina, and N. A. Kablar, "Nonlinear impulsive dynamical systems part I: Stability and dissipativity," *Int. J. Contr.*, vol. 74, pp. 1631–1658, 2001.

[81] W. M. Haddad, V. Chellaboina, and T. Rajpurohit, "Dissipativity theory for nonnegative and compartmental dynamical systems with time delay," in *Proc. Amer. Contr. Conf.*, Denver, CO, pp. 857–862, June 2003.

[82] W. M. Haddad and V. Chellaboina, "Stability and dissipativity theory for nonnegative and compartmental dynamical systems with time delay," in *Advances in Time-Delay Systems* (S.-I. Niculescu and K. Gu, eds.), Springer, to appear.

[83] W. M. Haddad and V. Chellaboina, "Stability theory for nonnegative and compartmental dynamical systems with time delay," *Sys. Contr. Lett.*, to appear.

[84] W. M. Haddad and T. Hayakawa, "Direct adaptive control for nonlinear uncertain systems with exogenous disturbances," *Int. J. Adapt. Control Signal Process.*, vol. 16, pp. 151–172, 2002.

[85] W. M. Haddad, T. Hayakawa, and J. M. Bailey, "Adaptive control for nonnegative and compartmental dynamical systems with applications to general anesthesia," *Int. J. Adapt. Control Signal Process.*, vol. 17, pp. 209–235, 2003.

[86] W. M. Haddad, T. Hayakawa, and J. M. Bailey, "Adaptive control for nonnegative and compartmental dynamical systems with applications to general anesthesia," in *Proc. IEEE Conf. Dec. Contr.*, Las Vegas, NV, pp. 3067–3072, December 2002.

[87] W. M. Haddad, T. Hayakawa, and J. M. Bailey, "Nonlinear adaptive control for intensive care unit sedation and operating room hypnosis," in *Proc. Amer. Contr. Conf.*, Denver, CO, pp. 1808–1813, June 2003.

[88] W. M. Haddad, T. Hayakawa, and J. M. Bailey, "Adaptive control for nonlinear compartmental dynamical systems with applications to clinical pharmacology," *Automatica*, submitted.

[89] W. M. Haddad, T. Hayakawa, and V. Chellaboina, "Robust adaptive control for nonlinear uncertain systems," *Automatica*, vol. 39, pp. 551–556, 2003.

[90] W. M. Haddad and T. Hayakawa, "Direct adaptive control for nonlinear uncertain systems with bounded energy $L_2$ disturbances," in *Proc. IEEE Conf. Dec. Contr.*, Sydney, Australia, pp. 2419–2423, December 2000.

[91] W. M. Haddad and T. Hayakawa, "Direct adaptive control for nonlinear uncertain systems with exogenous disturbances," in *Proc. Amer. Contr. Conf.*, Chicago, IL, pp. 4425–4429, June 2000.

[92] W. M. Haddad, T. Hayakawa, and A. Leonessa, "Direct adaptive control for discrete-time nonlinear uncertain systems," in *Proc. Amer. Contr. Conf.*, Anchorage, AK, pp. 1773–1778, May 2002.

[93] W. M. Haddad, T. Hayakawa, S. G. Nersesov, and V. Chellaboina, "Hybrid adaptive control for nonlinear impulsive dynamical systems," in *Proc. Amer. Contr. Conf.*, Denver, CO, pp. 5110–5115, June 2003.

[94] W. M. Haddad, T. Hayakawa, S. G. Nersesov, and V. Chellaboina, "Hybrid adaptive control for nonlinear impulsive dynamical systems," *Int. J. Adapt. Control Signal Process.*, submitted.

[95] W. M. Haddad, T. Hayakawa, S. G. Nersesov, and V. Chellaboina, "Adaptive stabilization for uncertain hybrid dynamical systems," *Dynamic Systems and Applications*, to appear.

[96] W. M. Haddad, T. Hayakawa, and M. C. Stasko, "Direct adaptive control for nonlinear matrix second-order systems with time-varying and sign-indefinite damping and stiffness operators," *IEEE Trans. Autom. Contr.*, submitted.

[97] W. M. Haddad and T. Hayakawa, "Adaptive control for nonlinear nonnegative dynamical systems," *Automatica*, submitted.

[98] W. M. Haddad, A. Leonessa, J. R. Corrado, and V. Kapila, "State space modeling and robust reduced-order control of combustion instabilities," *J. Franklin Inst.*, vol. 336, pp. 1283–1307, 1999.

[99] W. M. Haddad, M. C. Stasko, and T. Hayakawa, "Direct adaptive control for nonlinear matrix second-order systems with time-varying and sign-indefinite damping and stiffness operators," in *Proc. Amer. Contr. Conf.*, Denver, CO, pp. 2919–2924, June 2003.

[100] J. K. Hale, "Dynamical systems and stability," *J. Math. Anal. Appl.*, vol. 26, pp. 39–59, 1969.

[101] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations.* New York, NY: Springer-Verlag, 1993.

[102] T. Hayakawa, W. M. Haddad, J. M. Bailey, and N. Hovakimyan, "Passivity-based neural network adaptive output feedback control for nonlinear nonnegative dynamical systems," in *Proc. IEEE Conf. Dec. Contr.*, Maui, HI, December 2003. Also in *IEEE Trans. Neural Networks*, submitted.

[103] T. Hayakawa, W. M. Haddad, J. M. Bailey, and N. Hovakimyan, "Passivity-based neural network adaptive output feedback control for nonlinear nonnegative dynamical systems," *IEEE Trans. Neural Networks*, submitted.

[104] T. Hayakawa, W. M. Haddad, N. Hovakimyan, and V. Chellaboina, "Neural network adaptive control for nonlinear nonnegative dynamical systems," in *Proc. Amer. Contr. Conf.*, Denver, CO, pp. 561–566, June 2003. Also in *IEEE Trans. Neural Networks*, submitted.

[105] T. Hayakawa, W. M. Haddad, N. Hovakimyan, and V. Chellaboina, "Neural network adaptive control for nonlinear nonnegative dynamical systems," *IEEE Trans. Neural Networks*, submitted.

[106] T. Hayakawa, W. M. Haddad, and A. Leonessa, "A Lyapunov-based adaptive control framework for discrete-time nonlinear systems with exogenous disturbances," *Int. J. Contr.*, submitted.

[107] W. G. He, H. Kaufman, and R. J. Roy, "Multiple model adaptive control procedure for blood pressure control," *IEEE Trans. Biomed. Eng.*, vol. 33, no. 1, pp. 10–19, 1986.

[108] C. M. Held and R. J. Roy, "Multiple drug hemodynamic control by means of a supervisory-fuzzy rule-based adaptive control system: Validation on a model," *IEEE Trans. Biomed. Eng.*, vol. 42, no. 4, pp. 371–385, 1995.

[109] R. A. Hess and S. A. Snell, "Flight control design with rate saturating actuators," *AIAA J. Guid. Contr. Dyn.*, vol. 20, pp. 90–96, 1997.

[110] A. V. Hill, "The possible effects of the aggregation of the molecules of haemoglobin on its dissociation curves," *J. Physiol.*, vol. 40, pp. iv–vii, 1910.

[111] D. J. Hill and P. J. Moylan, "The stability of nonlinear dissipative systems," *IEEE Trans. Autom. Contr.*, vol. 21, pp. 708–711, 1976.

[112] H. Hirata, "Modelling and analysis of ecological systems: The large-scale system viewpoint," *Int. J. Syst. Sci.*, vol. 18, no. 10, pp. 1839–1855, 1987.

[113] J. M. van den Hof, "Positive linear observers for linear compartmental systems," *SIAM J. Control Optim.*, vol. 36, no. 2, pp. 590–608, 1998.

[114] J. Hong and D. S. Bernstein, "Adaptive stabilization of non-linear oscillators using direct adaptive control," *Int. J. Contr.*, vol. 74, no. 5, pp. 432–444, 2001.

[115] J. Hong, I. A. Cummings, and D. S. Bernstein, "Experimental application of direct adaptive control laws for adaptive stabilization and command following," in *Proc. IEEE Conf. Dec. Contr.*, Phoenix, AZ, pp. 779–784, 1999.

[116] N. Hovakimyan, F. Nardi, and A. Calise, "A novel error observer based adaptive output feedback approach for control of uncertain systems," *IEEE Trans. Autom. Contr.*, vol. 47, no. 8, pp. 1310–1314, 2002.

[117] N. Hovakimyan, F. Nardi, A. Calise, and N. Kim, "Adaptive output feedback control of uncertain nonlinear systems using single-hidden-layer neural networks," *IEEE Trans. Neural Networks*, vol. 13, no. 6, pp. 1420–1431, 2002.

[118] N. Hovakimyan, B.-J. Yang, and A. J. Calise, "An adaptive output feedback control methodology for non-minimum phase systems," in *Proc. IEEE Conf. Dec. Contr.*, Las Vegas, NV, pp. 949–954, 2002.

[119] K. J. Hunt, D. Sbarbaro, R. Zbikowski, and P. J. Gawthrop, "Neural networks for control: A survey," *Automatica*, vol. 28, pp. 1083–1112, 1992.

[120] L. R. Hunt, R. Su, and G. Meyer, "Global transformations of nonlinear systems," *IEEE Trans. Autom. Contr.*, vol. 28, pp. 24–31, 1983.

[121] P. A. Ioannou and J. Sun, *Robust Adaptive Control.* Upper Saddle River, NJ: Prentice-Hall, 1996.

[122] A. Isidori, *Nonlinear Control Systems.* New York, NY: Springer, 1995.

[123] J. A. Jacquez, *Compartmental Analysis in Biology and Medicine.* Ann Arbor, MI: University of Michigan Press, 1985.

[124] J. A. Jacquez and C. P. Simon, "Qualitative theory of compartmental systems," *SIAM Rev.*, vol. 35, pp. 43–79, 1993.

[125] G. I. Jee and R. J. Roy, "Adaptive control of multiplexed closed-circuit anesthesia," *IEEE Trans. Biomed. Eng.*, vol. 39, no. 10, pp. 1071–1080, 1992.

[126] Z.-P. Jiang and D. J. Hill, "Passivity and disturbance attenuation via output feedback for uncertain nonlinear systems," *IEEE Trans. Autom. Contr.*, vol. 43, pp. 992–997, 1998.

[127] Z. P. Jiang and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*, vol. 37, pp. 857–869, 2001.

[128] R. Johansson, "Global Lyapunov stability and exponential convergence of direct adaptive control," *Int. J. Contr.*, vol. 50, pp. 859–869, 1989.

[129] E. N. Johnson and A. J. Calise, "Neural network adaptive control of systems with input saturation," in *Proc. Amer. Contr. Conf.*, Arlington, VA, pp. 3527–3532, 2001.

[130] N. A. Kablar, T. Hayakawa, and W. M. Haddad, "Adaptive control of thermoacoustic combustion instabilities," in *Proc. Amer. Contr. Conf.*, Arlington, VA, pp. 2468–2473, June 2001.

[131] T. Kaczorek, *Positive 1D and 2D Systems.* London, UK: Springer-Verlag, 2002.

[132] I. Kanellakopoulos, "A discrete-time adaptive nonlinear control," *IEEE Trans. Autom. Contr.*, vol. 39, pp. 2362–2365, 1994.

[133] I. Kanellakopoulos, P. V. Kokotovic, and A. S. Morse, "Systematic design of adaptive controllers for feedback linearizable systems," *IEEE Trans. Autom. Contr.*, vol. 36, pp. 1241–1253, 1991.

[134] S. P. Kárason and A. M. Annaswamy, "Adaptive control in the presence of input constraints," *IEEE Trans. Autom. Contr.*, vol. 39, pp. 2325–2330, 1994.

[135] E. Kaszkurewicz and A. Bhaya, *Matrix Diagonal Stability in Systems and Computation.* Boston, MA: Birkhauser, 2000.

[136] H. Kaufman, I. Barkana, and K. Sobel, *Direct Adaptive Control Algorithms: Theory and Applications.* New York: Springer, 1998.

[137] T. Kazama, K. Ikeda, and K. Morita, "The pharmacodynamic interaction between propofol and fentanyl with respect to the suppression of somatic or hemodynamic responses to skin incision, peritoneum incision, and abdominal wall retraction," *Anesthesiology*, vol. 89, pp. 894–906, 1998.

[138] G. N. Kenny and H. Mantzardis, "Closed-loop control of propofol anaesthesia," *Brit. J. Anaesth.*, vol. 83, no. 2, pp. 223–228, 1999.

[139] H. K. Khalil, *Nonlinear Systems.* Upper Saddle River, NJ: Prentice-Hall, 1996.

[140] Y. Kishimoto and D. S. Bernstein, "Energy flow modeling of interconnected structures: A deterministic foundation for statistical energy analysis," *J. Sound Vibr.*, vol. 186, no. 3, pp. 407–445, 1995.

[141] Y. Kishimoto, D. S. Bernstein, and S. R. Hall, "Dissipative control of energy flow in interconnected systems, I Modal subsystems, II Structural subsystems," in *Proc. AIAA Guid. Nav. Contr. Conf.*, Fort Worth, TX, pp. 657–666, 1983.

[142] Y. Kishimoto, D. S. Bernstein, and S. R. Hall, "Thermodynamic modeling of interconnected systems I: Conservative coupling, II: Dissipative coupling," *J. Sound Vibr.*, vol. 182, no. 1, pp. 23–76, 1995.

[143] N. N. Krasovskii, *Stability of Motion.* Stanford, CA: Stanford University Press, 1963.

[144] J. N. Kremer and S. W. Nixon, *A Coastal Marine Ecosystem.* Berlin, Germany: Springer, 1978.

[145] M. Krstić, "Invariant manifolds and asymptotic properties of adaptive nonlinear stabilizers," *IEEE Trans. Autom. Contr.*, vol. 41, pp. 817–829, 1996.

[146] M. Krstić, I. Kanellakopoulos, and P. Kokotović, "Adaptive nonlinear control without overparametrization," *Sys. Contr. Lett.*, vol. 19, pp. 177–185, 1992.

[147] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović, *Nonlinear and Adaptive Control Design.* New York: Wiley, 1995.

[148] H. Kusuoka, S. Kodama, M. Hori, M. Inoue, H. Abe, and F. Kajiya, "Optimal control of drug administration," in *Proc. Int. Conf. Cybern. Soc.*, vol. 1, pp. 63–68, 1978.

[149] H. Kusuoka, S. Kodama, H. Maeda, M. Inoue, M. Hori, H. Abe, and F. Kajiya, "Optimal control in compartmental systems and its application to drug administration," *Math. Biosci.*, vol. 53, no. 1–2, pp. 59–77, 1981.

[150] G. S. Ladde, "Cellular systems–I. Stability of chemical systems," *Math. Biosci.*, vol. 29, pp. 309–330, 1976.

[151] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations.* Singapore: World Scientific, 1989.

[152] E. Lavretsky, N. Hovakimyan, and A. J. Calise, "Upper bounds for approximation of continuous-time dynamics using delayed outputs and feedforward neural networks," *IEEE Trans. Autom. Contr.*, vol. 48, 1606-1610.

[153] A. Leonessa, V. Chellaboina, W. M. Haddad, and T. Hayakawa, "Direct discrete-time adaptive control with guaranteed parameter error convergence," in *Proc. Amer. Contr. Conf.*, Denver, CO, pp. 2925–2930, June 2003.

[154] A. Leonessa, V. Chellaboina, W. M. Haddad, and T. Hayakawa, "Direct discrete-time adaptive control with guaranteed parameter error convergence," *IEEE Trans. Autom. Contr.*, submitted.

[155] A. Leonessa, W. M. Haddad, and V. Chellaboina, *Hierarchical Nonlinear Switching Control Design with Applications to Propulsion Systems*. London, UK: Springer, 2000.

[156] A. Leonessa, W. M. Haddad, and T. Hayakawa, "Adaptive tracking for nonlinear systems with control constraints," in *Proc. Amer. Contr. Conf.*, Arlington, VA, pp. 1292–1297, June 2001.

[157] A. Leonessa, W. M. Haddad, and T. Hayakawa, "Adaptive control for nonlinear uncertain systems with actuator amplitude and rate saturation constraints," *Automatica*, submitted.

[158] J. H. Levy, L. G. Michelsen, J. S. Shanewise, J. M. Bailey, and J. G. Ramsay, "Postoperative cardiovascular management," in *Cardiac Anesthesia* (J. Kaplan, ed.), Philadelphia, PA: WB Saunders, 4 ed., 1999.

[159] F. L. Lewis, S. Jagannathan, and A. Yesildirak, *Neural Network Control of Robot Manipulators and Nonlinear Systems*. London, UK: Taylor & Francis, 1999.

[160] F. L. Lewis, A. Yesildirek, and K. Liu, "Multilayer neural-net robot controller with guaranteed tracking performance," *IEEE Trans. Neural Networks*, vol. 7, no. 2, pp. 388 –399, 1996.

[161] D. A. Linkens, M. F. Abbod, and J. E. Peacock, "Clinical implementation of advanced control in anaesthesia," *Trans. Inst. Meas. Control*, vol. 22, pp. 303–330, 2000.

[162] W. Lu and J. M. Bailey, "The reliability of pharmacodynamic analysis by logistic regression: A computer simulation study," *Anesthesiology*, vol. 92, pp. 985–992, 2000.

[163] W. Lu, J. G. Ramsay, and J. M. Bailey, "Reliability of pharmacodynamic analysis by logistic regression: Mixed-effects modeling," *Anesthesiology*, to appear.

[164] H. Maeda, S. Kodama, and F. Kajiya, "Compartmental system analysis: Realization of a class of linear systems with physical constraints," *IEEE Trans. Circuits Syst.*, vol. 24, pp. 8–14, 1977.

[165] H. Maeda, S. Kodama, and T. Konishi, "Stability theory and existence of periodic solutions of time delayed compartmental systems," *Electron. Commun. Jpn.*, vol. 65, no. 1, pp. 1–8, 1982.

[166] H. Maeda, S. Kodama, and Y. Ohta, "Asymptotic behavior of nonlinear compartmental systems: Nonoscillation and stability," *IEEE Trans. Circuits Syst.*, vol. 25, pp. 372–378, 1978.

[167] L. Magni, "On robust tracking with non-linear model predictive control," *Int. J. Contr.*, vol. 75, pp. 399–407, 2002.

[168] P. O. Maitre and D. R. Stanski, “Bayesian forecasting improves the prediction of intraoperative plasma concentrations of alfentanil,” *Anesthesiology*, vol. 69, no. 5, pp. 652–659, 1988.

[169] B. Marsh, M. White, N. Morton, and G. N. Kenny, “Pharmacokinetic model driven infusion of propofol in children,” *Brit. J. Anaesth.*, vol. 67, pp. 41–48, 1991.

[170] P. Martin and A. A. Ahuja, “Optimal pharmacokinetic delivery of infused drugs: Application to the treatment of cardiac arrhythmias,” *J. Biomed. Eng.*, vol. 10, no. 4, pp. 360–364, 1988.

[171] K. McKay, “Summary of an agard workshop on pilot induced oscillation,” in *Proc. AIAA Guid. Nav. and Contr. Conf.*, Scottsdale, AZ (Washington, DC: AIAA), pp. 1151–1161, 1994.

[172] R. R. Mohler, “Biological modeling with variable compartmental structure,” *IEEE Trans. Autom. Contr.*, vol. 19, pp. 922–926, 1974.

[173] A. S. Morse, C. C. Pantelides, S. Sastry, and J. M. Schumacher, eds., “Special issue on hybrid control systems,” *Automatica*, vol. 35, no. 3, 1999.

[174] E. Mortier, M. Struys, T. De Smet, L. Versichelen, and G. Rolly, “Closed-loop controlled administration of propofol using bispectral analysis,” *Anaesthesia*, vol. 53, pp. 749–754, 1998.

[175] K. Nam and A. Araspostathis, “A model-reference adaptive control scheme for pure-feedback nonlinear systems,” *IEEE Trans. Autom. Contr.*, vol. 33, pp. 803–811, 1988.

[176] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems.* Englewood Cliffs, NJ: Prentice-Hall, 1989.

[177] K. S. Narendra and Y. H. Lin, “Stable discrete adaptive control,” *IEEE Trans. Autom. Contr.*, vol. 25, pp. 456–461, 1980.

[178] K. S. Narendra and K. Parthasarathy, “Identification and control of dynamical systems using neural networks,” *IEEE Trans. Neural Networks*, vol. 1, pp. 4–27, 1990.

[179] K. S. Narendra and J. H. Taylor, *Frequency Domain Criteria for Absolute Stability.* New York: Academic Press, 1973.

[180] S. G. Nersesov, W. M. Haddad, and V. Chellaboina, “Optimal fixed-structure control for linear nonnegative dynamical systems,” in *Proc. Amer. Contr. Conf.*, Denver, CO, pp. 3496–3501. Also in *Int. J. Robust Nonlinear Control*, to appear, June 2003.

[181] S.-I. Niculescu, *Delay Effects on Stability: A Robust Control Approach.* New York, NY: Springer, 2001.

[182] J. W. Nieuwenhuis, "About nonnegative realizations," *Sys. Contr. Lett.*, vol. 1, pp. 283–287, 1982.

[183] H. Nijmeijer and A. van der Schaft, *Nonlinear Dynamical Control Systems*. Berlin: Springer-Verlag, 1990.

[184] E. P. Odum, *Fundamentals of Ecology*. Philadelphia, PA: Saunders, 1971.

[185] B. A. Ogunnaike and W. H. Ray, *Process Dynamics, Modeling, and Control*. Oxford, UK: Oxford University Press, 1997.

[186] F. Ohkawa and Y. Yonezawa, "A discrete model reference adaptive control system for a plant with input amplitude constraints," *Int. J. Contr.*, vol. 36, pp. 747–753, 1982.

[187] Y. Ohta, H. Maeda, and S. Kodama, "Reachability, observability and realizability of continuous-time positive systems," *SIAM J. Contr. Optimiz.*, vol. 22, pp. 171–180, 1984.

[188] R. Ortega and M. W. Spong, "Adaptive motion control of rigid robots: A tutorial," *Automatica*, vol. 25, pp. 877–888, 1989.

[189] G. A. Pajunen, M. Steinmetz, and R. Shankar, "Model reference adaptive control with constraints for postoperative blood pressure management," *IEEE Trans. Biomed. Eng.*, vol. 37, no. 7, pp. 679–687, 1990.

[190] L. Paparizos and F. E. C. Culick, "The two-mode approximation to nonlinear acoustics in combustion chambers I: Exact solution for second order acoustics," *Combustion Science and Technology*, vol. 65, pp. 39–65, 1989.

[191] A. N. Payne, "Adaptive one-step-ahead control subject to an input-amplitude constraint," *Int. J. Contr.*, vol. 43, pp. 1257–1269, 1986.

[192] J. G. Pierce and A. Schumitzky, "Optimal impulsive control of compartment models, I: Qualitative aspects," *J. Optimiz. Theory Appl.*, vol. 18, pp. 537–554, 1976.

[193] J. G. Pierce and A. Schumitzky, "Optimal control of compartment models, II: Algorithm," *J. Optimiz. Theory Appl.*, vol. 26, no. 1, pp. 581–599, 1978.

[194] I. J. Rampil, "A primer for EEG signal processing in anesthesia," *Anesthesiology*, vol. 89, no. 4, pp. 980–1002, 1998.

[195] R. G. Ritchie, E. A. Ernst, B. L. Pate, J. D. Pearson, and L. C. Shepherd, "Closed loop control of an anesthesia delivery system: Development and animal testing," *IEEE Trans. Biomed. Eng.*, vol. 34, pp. 437–443, 1987.

[196] M. R. Rokui and K. Khorasani, "An indirect adaptive control for fully feedback linearizable discrete-time non-linear systems," *Int. J. Adapt. Control Signal Process.*, vol. 11, pp. 665–680, 1997.

[197] J. A. Ross, R. T. Wloch, D. C. White, and D. W. Hawes, "Servo-controlled closed-circuit anaesthesia. A method for the automatic control anaesthesia produced by a volatile agent in oxygen," *Brit. J. Anaesth.*, vol. 55, no. 11, pp. 229–230, 1982.

[198] N. Rouche, P. Habets, and M. LaLoy, *Stability Theory by Liapunov's Direct Method*. New York, NY: Springer-Verlag, 1977.

[199] A. V. Roup and D. S. Bernstein, "Stabilization of second-order systems with bounded time-varying coefficients," in *Proc. IEEE Conf. Dec. Contr.*, Orlando, FL, pp. 3471–3472, December, 2001.

[200] A. V. Roup and D. S. Bernstein, "Stabilization of a class of nonlinear systems using direct adaptive control," in *Proc. Amer. Contr. Conf.*, Chicago, IL, pp. 3148–3152, June, 2000.

[201] H. L. Royden, *Real Analysis*. New York, MA: Macmillan, 1988.

[202] A. M. Samoilenko and N. Perestyuk, *Impulsive Differential Equations*. Singapore: World Scientific, 1995.

[203] W. Sandberg, "On the mathematical foundations of compartmental analysis in biology, medicine and ecology," *IEEE Trans. Circuits Syst.*, vol. 25, pp. 273–279, 1978.

[204] S. S. Sastry and A. Isidori, "Adaptive control of linearizable systems," *IEEE Trans. Autom. Contr.*, vol. 34, no. 11, pp. 1123–1131, 1989.

[205] T. W. Schnider, C. F. Minto, and D. R. Stanski, "The effect compartment concept in pharmacodynamic modelling," *Anaes. Pharmacol. Rev.*, vol. 2, pp. 204–219, 1994.

[206] H. Schwilden, "A general method for calculating the dosage scheme in linear pharmacokinetics," *Eur. J. Clin. Pharmacol.*, vol. 20, no. 5, pp. 379–386, 1981.

[207] H. Schwilden, J. Schuttler, and H. Stoeckel, "Closed-loop feedback control of methohexital anesthesia by quantitative EEG analysis in humans," *Anesthesiology*, vol. 67, pp. 341–347, 1987.

[208] H. Schwilden, H. Stoeckel, and J. Schuttler, "Closed-loop feedback control of propofol anesthesia by quantitative EEG analysis in humans," *Brit. J. Anaesth.*, vol. 62, pp. 290–296, 1989.

[209] P. S. Sebel, E. Lang, I. J. Rampil, P. White, R. C. M. Jopling, N. T. Smith, P. S. Glass, and P. Manberg, "A multicenter study of bispectral electroencephalogram analysis for monitoring anesthetic effect," *Anesth. Analg.*, vol. 84, no. 4, pp. 891–899, 1997.

[210] S. L. Shafer and K. Gregg, "Algorithms to rapidly achieve and maintain stable drug concentrations at the site of drug effect with a computer-controlled infusion," *J. Pharmacokinetics Biopharm.*, vol. 20, no. 2, pp. 147–169, 1992.

[211] S. L. Shafer, J. R. Varvel, N. Aziz, and J. C. Scott, "Pharmacokinetics of fentanyl administered by computer-controlled infusion pump," *Anesthesiology*, vol. 73, pp. 1092–1102, 1990.

[212] L. B. Sheiner, "The population approach to pharmacokinetic data analysis: Rationale and standard data analysis methods," *Drug Metabolism Reviews*, vol. 15, no. 1-2, pp. 153–171, 1984.

[213] L. B. Sheiner and S. L. Beal, "Evaluation of methods for estimating population pharmacokinetic parameters II. Biexponential model and experimental pharmacokinetic data," *J. Pharmacokinetics Biopharm.*, vol. 9, no. 5, pp. 635–651, 1981.

[214] L. C. Sheppard, "Computer control of the infusion of vasoactive drugs," *Ann. Biomed. Eng.*, vol. 8, pp. 431–444, 1980.

[215] J. C. Sigl and N. G. Chamoun, "An introduction to bispectral analysis for the electroencephalogram," *J. Clin. Monit.*, vol. 10, pp. 392–404, 1994.

[216] D. D. Siljak, *Large-Scale Dynamic Systems*. New York: North-Holland, 1978.

[217] D. D. Siljak, "Complex dynamical systems: Dimensionality, structure and uncertainty," *Large Scale Systems*, vol. 4, pp. 279–294, 1983.

[218] J. Slate, *Model-based design of a controller for infusion of sodium nitroprusside during postsurgical hypertension*. PhD thesis, University of Wisconsin-Madison, 1980.

[219] J.-J. E. Slotine and W. Li, "On the adaptive control of robotic manipulators," *Int. J. of Robotics Research*, vol. 6, pp. 49–59, 1987.

[220] J.-J. E. Slotine and W. Li, "Adaptive manipulator control: A case study," *IEEE Trans. Autom. Contr.*, vol. 33, pp. 995–1003, 1988.

[221] H. L. Smith, *Monotone Dynamical Systems*. Providence, RI: Amer. Math. Soc., 1995.

[222] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Autom. Contr.*, vol. 34, pp. 435–443, 1989.

[223] E. D. Sontag and Y. Wang, "On characterizations of the input-to-state stability property," *Sys. Contr. Lett.*, vol. 24, pp. 351–359, 1995.

[224] T. T. Soong, "Pharmacokinetics with uncertainties in rate constants–II: Sensitivity analysis and optimal dosage control," *Math. Biosci.*, vol. 13, pp. 391–396, 1972.

[225] M. W. Spong and M. Vidyasagar, *Robot Dynamics and Control*. New York: John Wiley & Sons, 1989.

[226] J. Spooner, M. Maggiore, R. Ordonez, and K. Passino, *Stable Adaptive Control and Estimation for Nonlinear Systems: Neural and Fuzzy Approximator Techniques*. New York, NY: John Wiley & Sons, 2002.

[227] M. Struys, T. De Smet, L. Versichelen, S. Van de Vilde, R. Van den Broecke, and E. Mortier, “Comparison of closed-loop controlled administration of propofol using BIS as the controlled variable versus “standard practice” controlled administration,” *Anesthesiology*, vol. 95, pp. 6–17, 2001.

[228] D. Taylor, P. V. Kokotović, R. Marino, and I. Kanellakopoulos, “Adaptive regulation of nonlinear systems with unmodeled dynamics,” *IEEE Trans. Autom. Contr.*, vol. 34, pp. 405–412, 1989.

[229] J. O. Tsokos and C. P. Tsokos, “Statistical modeling of pharmacokinetic systems,” *ASME J. Dyn. Syst. Meas. Contr.*, vol. 98, pp. 37–43, 1976.

[230] R. Venugopal, V. G. Rao, and D. S. Bernstein, “Lyapunov-based backward-horizon adaptive stabilization,” *Int. J. Adapt. Control Signal Process.*, vol. 17, pp. 67–84, 2003.

[231] R. Venugopal, V. G. Rao, and D. S. Bernstein, “Optimal Lyapunov-based backward horizon adaptive stabilization,” in *Proc. Amer. Contr. Conf.*, Arlington, VA, pp. 1654–1658, June 2000.

[232] M. Vidyasagar, *Nonlinear Systems Analysis.* Englewood Cliffs, NJ: Prentice-Hall, 1993.

[233] F. Viel, F. Jadot, and G. Bastin, “Global stabilization of exothermic chemical reactors under input constraints,” *IEEE Trans. Autom. Contr.*, vol. 42, pp. 473–481, 1997.

[234] R. Vishnoi and R. J. Roy, “Adaptive control of closed-circuit anesthesia,” *IEEE Trans. Biomed. Eng.*, vol. 38, no. 1, pp. 39–47, 1991.

[235] S. M. Walas, *Reaction Kinetics for Chemical Engineers.* New York: McGraw-Hill, 1959.

[236] G. G. Walter and M. Contreras, *Compartmental Modeling with Networks.* Boston, MA: Birkhaeuser, 1999.

[237] A. Weinman, *Uncertain Models and Robust Control.* New York: Springer-Verlag, 1991.

[238] P. G. Welling, *Pharmacokinetics: Processes, Mathematics, and Applications.* Washington DC: American Chemical Society, 2 ed., 1997.

[239] M. White and G. N. C. Kenny, “Intravenous propofol anaesthesia using a computerised infusion system,” *Anaesthesia*, vol. 45, pp. 204–209, 1990.

[240] J. C. Willems, “Least squares stationary optimal control and the algebraic Riccati equation,” *IEEE Trans. Autom. Contr.*, vol. 16, pp. 621–634, 1971.

[241] J. C. Willems, “Dissipative dynamical systems part I: General theory,” *Arch. Rational Mech. Anal.*, vol. 45, pp. 321–351, 1972.

[242] P.-C. Yeh and P. V. Kokotović, "Adaptive control of a class of nonlinear discrete-time systems," *Int. J. Contr.*, vol. 62, pp. 303–324, 1995.

[243] C. Yu, R. J. Roy, H. Kaufman, and B. Bequette, "Multiple-model adaptive predictive control of mean arterial pressure and cardiac output," *IEEE Trans. Biomed. Eng.*, vol. 39, no. 8, pp. 765–778, 1992.

[244] C. Zhang and R. J. Evans, "Amplitude constrained adaptive control," *Int. J. Contr.*, vol. 46, pp. 53–64, 1987.

[245] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Englewood Cliffs: Prentice-Hall, 1996.