Finite Quantum Theory of the Harmonic Oscillator

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Finite Quantum Theory of the Harmonic Oscillator

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To my glorious country,

IRAN

Where all my roots are.

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SUMMARY

We apply the Segal process of group simplification to the linear harmonic oscillator. The result is a finite quantum theory with three quantum constants $\hbar, \hbar', \hbar''$ instead of the usual one. We compare the classical (CLHO), quantum (QLHO), and finite (FLHO) linear harmonic oscillators and their canonical or unitary groups. The FLHO is isomorphic to a dipole rotator with $N = l(l + 1) \sim 1/(\hbar'\hbar'')$ states and Hamiltonian $H = A(L_x)^2 + B(L_y)^2$, and the physically interesting case has $N \gg 1$. The position and momentum variables are quantized with uniform finite spectra. For fixed quantum constants and large $N \gg 1$ there are three broad classes of FLHO: soft, medium, and hard, with $B/A \ll 1$, $B/A \sim 1$, and $B/A \gg 1$ respectively. The field oscillators responsible for infra-red and ultraviolet divergences are soft and hard respectively. Medium oscillators have $B/A \sim 1$ and approximate the QLHO. They have $\sim \sqrt{N}$ low-lying states with nearly the same zero-point energy and level spacing as the QLHO, and nearly obeying the Heisenberg uncertainty principle and the equipartition principle. The corresponding rotators are nearly polarized along the z axis with $L_z \sim \pm l$. The soft and hard FLHO's have infinitesimal 0-point energy, grossly violate equipartition and the Heisenberg uncertainty principle. They do not resemble the QLHO at all. Their low-lying energy states correspond to rotators with $L_x \sim 0$ or $L_y \sim 0$ instead of $L_z \sim \pm l$. Soft oscillators have frozen momentum, because their maximum potential energy is too small to produce one quantum of momentum. Hard oscillators have frozen position, because their maximum kinetic energy is too small to excite one quantum of position.

CHAPTER 1

INTRODUCTION

The three main evolutions of physics in the twentieth century have a suggestive family resemblance. Each of them introduced a certain non-commutativity previously not present in physics. Special relativity introduces a non-commutativity of boosts. General relativity and gauge theory introduced a non-commutativity of infinitesimal translations. Quantum theory introduced a non-commutativity of observations. The seminal works by Segal [24] and Inönü [18] and Inönü and Wigner [17], which stimulated the present work, suggest more changes of this kind. The current work is part of a program to develop a finite quantum theory, especially of space-time. Such a theory has been sought by generations of physicists from mid twentieth century on.

Here we study a process introduced by Segal that we term *group regularization*. It modifies an existing quantum theory in a way that includes the existing quantum theory as a suitable limiting case and respects the existing group symmetries. Group regularization seems likely to produce the long-sought quantum theory of space-time. We test it here on the linear harmonic oscillator, a ubiquitous constituent of all present field theories. The n-dimensional harmonic oscillator is merely a collection of n linear ones.

Planck introduced his quantum constant h to freeze out oscillators that caused the cavity thermal spectrum to diverge. The zero-point energy of the resulting quantum theory still diverged. The new quantum constants h', h'' freeze out all but a finite number of the cavity oscillators without greatly changing the observed ones. The zero-point energies of

the frozen oscillators are infinitesimal in the finite quantum theory compared to their usual values. This shows how a finite quantum theory of the cavity will produce a finite zero-point energy without conflicting with the many finite predictions of the usual quantum theory.

Our major goal in the present work is to develop a regular (finite) quantum theory of the harmonic oscillator. We focus on the non-relativistic time-independent theory here. We recognize that the Heisenberg algebra of the quantum theory is a compound (non-semi-simple) algebra, with the imaginary unit i as an absolute element of the theory. We regularize the group of quantum theory Heisenberg to a simple orthogonal group, namely, SO(3). As an example, we focus on the time-independent one-dimensional quantum harmonic oscillator and derive its new energy levels in the finite quantum theory.

In chapter 2, we study the physical regularization processes of physical theories. We focus on the group-theoretical structure of physical regularizations. The structure of the major evolutions of physics in the past is our main guide to develop the finite quantum theory. We recognize that most radical changes of physics such as quantum theory and theories of relativity fit in the same group-theoretical process of group regularization.

In chapter 3, we develop a finite (regular) quantum theory for the harmonic oscillator and compare the classical (CLHO), quantum (QLHO), and finite (FLHO). We expand the Heisenberg algebra and apply the new algebra to the quantum harmonic oscillator. We will show that in the the finite theory, a linear harmonic oscillator becomes a rotator with a sharp total angular momentum l, a finite number of states $N = 2l + 1$, and a Hamiltonian of the the special form $H = AL_x{}^2 + BL_y{}^2$. We will find that for physically interesting cases for which N, there are three broad classes of FLHO: soft, medium, and hard, with $B/A \ll 1, B/A \sim 1$, and $B/A \gg 1$ respectively. The quantum oscillators of the field theory that are responsible for infrared and ultraviolet divergencies are soft and hard respectively. We obtain the energy spectrum for the three classes of FLHO oscillators and derive the uncertainty principle in all three cases. There two interesting results: first, the energy spectra for FLHO are bounded and second, low-lying energy states of soft and hard FLHO oscillators have $\Delta p \Delta q$ far below the Heisenberg uncertainty limit of QLHO.

The results are summarized in chapter 4.

To make connection with other works in progress [5] where we use the concept of Clifford algebras, we include an appendix on Clifford algebraic representations of FLHO variables.

In the rest of this introduction, we list principles and concepts we use in this work. We re-states some of these concepts more rigorously in subsequent chapters.

Operationality. A physical theory should at least lead to statements of the form: **"If we do so-and-so, we will find such-and-such"** [12]. This structure is evident from all physical theories that have passed experimental tests successfully. Perhaps the most clear example is quantum theory where the system is described maximally through measurements on its physical variables. In an action-based quantum theory, bras and kets describe external acts by which we prepare or register the system. Operators describe actions on the system or actions transforming an experimenter to another, the relativity transformations.

Finiteness. **A physical theory should be finite,** The infinities (divergencies) are not physically observable. So we construct the theory from the beginning with no infinity.

Simplicity. **The group of physical theories must be simple**. By simplicity we mean algebraic simplicity which is defined in the following way: A group is simple if it does not contain a non-trivial invariant subgroup. A group that is not simple, we term **compound**. A **semi-simple** group is one whose Lie algebra has no solvable subalgebras, and is then a product of finite number of simple subgroups. We designate the system under study by S. We designate the rest, including the environment, experimenter, apparatus and records by $\bar{\mathsf{S}}$ (non S) and call it the **exosystem**. We recognize the following group structure:

- The *kinematical group* G_K of all possible reversible physical operation on the system, modes of passage of time. These are actions by the experimenter on the system. Also included in the group of G_K of a physical system are operations corresponding to change in reference frames. These are actions on the exosystem.

- The *symmetry group* $G_S \subseteq G_K$ of relativity transformation that respect the Hamiltonian of the system.

An important concept related to the notion of simplicity is an absolute element or **idol**[1] of the theory. An idol of a physical theory is an absolute which its group respects. It is represented in the theory with a central element that enters the theory in a fundamental way and couples to other elements while nothing couples back to it. Idols of a group of a physical theory generate invariant subgroups that makes the group compound.

Generalized Correspondence Principle. **The old working physical theory must be recovered from the replacing new one in the appropriate limit**. Bohr correspondence principle is a specific example where the classical physics is recovered from quantum theory as $\hbar \to 0$. Regularization is the main concept through out this work. In general, regularization is a method for handling the infinities in a physical theory. We study one form of regularization in details in the next chapter. We note that there are two types of regularization, physical and unphysical. Physical regularizations are new theories with different predictions. Unphysical regularizations are procedures aimed at removing undesired divergencies in an existing theory without changing its finite predictions. Lattice gauge theory is an example of an unphysical regularization method.

Stabilization, Flexing and Flattening. **The group of physical theory must be stable**. Segal's important observation is that concept of stability and simplicity imply each other. Stability of physical theory roughly means that the algebra of the theory is unchanged up to isomorphism by small changes in the basic algebraic relations of the theory. For us, the important algebra is a Lie algebra. A lie algebra is stable if small changes in its structure constants do not change the algebra up to an isomorphism. For example, the algebra of the Lorentz group is stable against corrections to the speed of light. Lorentz group is the stabilized Galilei group where

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[1]The term idol is borrowed from Francis Bacon's Novum Organum [2].

the reciprocal speed of light is zero. Once a finite speed for light is introduced into the Galilei group, it changes dramatically. Galilei group is unstable. The Lie algebra of a physical theory can be stabilized through a mathematical process we term *flexing*. We label the reverse process as *flattening*. The well-known process of Inönü and Wigner contraction [17] is a special case of the flattening process. Precise definitions of these processes are given in the next chapter.

CHAPTER 2

SEGAL DOCTRINE AND STRUCTURE OF PHYSICAL REFORMATIONS

In the present chapter we study in details the mathematical and physical ideas and methods introduced in the preceding chapter in their relation to the structure of the major past and perhaps the future evolutions of physics. We show how Segal's idea of stabilization physical theory enable us to move toward a next radical change of physics. This will prepare the background to develop the simplified Heisenberg algebra and the finite quantum theory of the harmonic oscillator in the next chapter.

2.1 Regularizations and Reformations

All current major theories of physics are field theories based on a continuous classical space-time. Mathematically, these theories share a common structure. They are all fiber-bundle theories with the classical space-time as the base and various fields as fibers. The continuum physical theories are fertile lands for non-physical and unobservable infinities (singularities).

Since the advent of field theories, several *regularization* methods have been introduced to remove the unwanted infinities out of physics. These are unphysical in that they are introduced just to hide the divergencies that enter field theories. There have not been direct experimental motivations for unphysical regularization. The three major regularization schemes are Pauli-Villars regularization [21], lattice regularization (lattice gauge theory) [34] and dimensional regularization [6, 7, 26]. Practical calculations in any field theory employ a regularization procedure that sets a length scale below which processes have no influence on the theory. For example, in calculating QED loop diagrams the divergent integrals could be cut off at some momentum k, rendering them finite. In effect, the cutoff momentum defines a length scale $L \simeq \mathcal{O}(k^{-1})$. Recognizing the source for divergencies in field theories, Snyder formulated a Lorentz invariant quantum space-time [25] but he did

not succeeded in formulating a field theory on it.

In addition to theoretical motivation for regularization programs, sometimes experiments themselves lead to a more regular theory compared to the preceding one. We term such a process a *physical regularization* or *reformation*; a term to be justified in the following section. The most famous example of a reformation process is Planck's, which led quantum theory. One idol in classical mechanics is the system under study. In a measurement process, a classical system acts on the experimenter but not conversely. Consequently, classical measurements are commutative. Quantum theory is "simpler" than classical physics in that it does not separate measurements from transformation. A quantum system might changes as a result of an observation. Quantum observations do not commute in general.

Compared to unphysical regularizations, physical regularizations affect the theories in a subtler way but their effects are rather dramatic. An unphysical regularization leaves the idol(s) of the theory untouched. But reformation of a theory results in dethroning one or more of its idols. The subtlety of a reformation process can be seen by studying its effect on the group structure of the theory. In fact, a reformation is a *group regularization*. In the following section we study reformation processes from the group theoretical point of view.

2.2   Group theoretical structure of physical regularization

We begin with definition of different types of (physical) groups that appear in the regularization process.

Definition 1 (Simple, semi-simple and compound groups) *A group is simple if it does not contain a non-trivial invariant subgroup. A semi-simple group is one whose Lie algebra has no solvable subalgebras, and is then a product of finite number of simple subgroups. A group which is not semi-simple, we term compound.*

Compound groups are usually called non-semi-simple. Examples: Galilei group is compound. Time translations form its invariant subgroup. The (proper, orthochronous) Lorentz group is simple.

Compound groups are *unstable* with respect to a small change in their structures. To make this statement precise, we return to the language of group algebra. Consider a Lie algebra A defined by a Lie product $\times : A \otimes A \to A$—also called structure tensor—on a vector space V. Following Segal [24] we define a *stable* group as one whose associated Lie algebra is stable:

Definition 2 (Stable Lie algebra) *A Lie algebra A is stable (regular) if small changes in $\times$ within the manifold of Lie algebras do not change A up to isomorphism.*

We use the term stable also for a physical theory it describes. We use the term *flattening* for the mathematical process of getting back to an unstable algebra (group) from a more stable one. We make this statement precise in the following definition [4]:

Definition 3 Group (algebra) flattening and flexing *Let Lie product $\times : A \otimes A \to A$ form a submanifold $\mathcal{M}$ of the space of tensors $V \otimes V \to V$. By a flattening of a Lie algebra we mean the endpoint $\times(0)$ of a homotopy $\times(t)$ in $\mathcal{M}$ with $\times(1) = \times, \times(t) \cong \times$ for $0 < t \leq 1$, and $\times(0) \not\cong \times$. The inverse process to flattening is what we term flexing.*

To understand this better, consider the submanifold $\mathcal{M}$ formed by Lie products $\times$ on a vector space V. If we parameterize maps that live in $\mathcal{M}$ by a parameter like $t \in [0, 1]$, then as t varies we get a set of homotopic maps except for when $t = 0$. There is dramatic change in the Lie algebra (and in the physical theory it describes) from $t = 0$ to $t > 0$.

The *contraction* process of Inönü and Wigner [17] is a special case of the flattening process. They accomplish their contraction by "stretching" a coordinate to infinity. This is done by applying a parameter-dependent singular transformation to the generators of a Lie algebra. Call the parameter ϵ and assume that in the limit when $\epsilon \to 0$ the transformation is non-singular. We expect that we should obtain a Lie algebra generated by the set of transformed generators in this limit. Inönü and Wigner showed that for this to be the case, there must exists an abelian invariant subalgebra generated by some of the original generators. They also showed the converse is true. The original algebra is said to contract to the new one with respect to the invariant abelian subgroup mentioned above. The inverse process of contraction is sometimes called *expansion* [16].

Segal importantly pointed out that many reformations of the last century physics follow the same pattern: each new physical theory is a more stable and more physically accurate of a previously unstable and less accurate one. In other words, **reformation is stabilization**. Segal proposed that physical theories should be stable. He further pointed out that stability requires semisimplicity. We re-express these fundamental ideas in what we call *Segal doctrine*:

Definition 4 Segal doctrine *Any compound physical theory is a contraction of a more stable, more accurate, semi-simple theory, which we call its expansion.*

Let us give few examples of unstable physical theories.

Example: Instability of quantum theory

The Heisenberg algebra in one-dimension $\mathbf{H}(1)$ is unstable. The defining commutation relationships are

$$
\begin{aligned}
[p_x, x] &= -i\hbar \\
[i, p_x] &= 0 \\
[x, i] &= 0
\end{aligned}
\tag{1}
$$

We see that the imaginary unit i generates an invariant subgroup (of phase changes). The algebra is stable against changes in $\hbar$ but not against changes in $[i, p_x]$ and $[x, i]$.

Example: Instability of differential calculus

The continuum space-time algebra leads to another fundamental instability. Here we have

$$
\begin{aligned}
[\partial_\nu, x^\mu] &= \delta^\mu_\nu \\
[x^\mu, x^\nu] &= 0 \\
[\partial_\nu, \partial_\mu] &= 0
\end{aligned}
\tag{2}
$$

To give meaning to this algebra, consider the standard single particle spin-1/2 theory which represents classical space-time coordinates x^μ on Minkowski space-time of signature 2, and represents momentum-energy variables by differential operators $\partial_\mu = \partial/\partial x^\mu$.

The relations above define a compound Lie algebra, with an invariant proper Lie subalgebra generated by 1. Therefore to simplify field theory we first must simplify the differential calculus. We need to deform x^μ and ∂_μ so that they generate a simple Lie algebra.

As a mathematical theory, of course, the differential calculus is as valid as the algebra we use in its stead. What we should change is the Planck-Einstein physical relation $E = i\hbar\omega = i\hbar d/dt$ between time and energy. The time-energy relation that we propose is not satisfied exactly by differential operators t and $i\hbar d/dt$. We instead, write these operators in the appropriate language of Clifford algebra. We do this for the harmonic oscillator variables in the appendix B.

When we work with the explicit from of Lie products i.e., Lie brackets or commutators, a non-zero homotopy parameter t apperas as a non-zero term that introduce some new non-commutativity. So a flexing process is like adding a new *"curvature"* to the theory. By group flattening we mean any physical approximation that makes a regular group singular by switching off some commutators[1]. We finally arrive at a very important idea which lies at the core of this work: **A reformation of a physical theory is mathematically equivalent to introducing a fundamental non-commutativity into the Lie group of the theory.**

A natural question to ask here is "how do we recognize which non-commutativities are signs of instability?" The answer lies in our ability in recognizing the idols of physical theories. Recall that an idol is an absolute quantity that the group of the theory respects. It enters as a fundamental quantity in the theory and couples into other variables but no variable couples into. For example, consider the compound Galilean group in $1 + 1$ dimension. Time translations form an invariant subgroup. Time is the idol of the Galilean

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[1]These terms are motivated by general relativity where non-commuting displacements results from a curved space-time structure.

group. Time couples into space transformation but not the inverse:

$$
\begin{aligned}
x' &= x - vt \\
t' &= t
\end{aligned}
\tag{3}
$$

We point out that semisimplicity implies a *reciprocity*. Reciprocity means a two-way coupling. In the case of Galilei group, which is not semi-simple, the reciprocity is violated: there is no coupling of space into time in the fundamental transformation. Lack of reciprocity always implies the existence of an idol in the theory and this in turn, means the group of the theory is compound. Let us now study the reformation process by focusing on the Lie algebra structure of one of the three major reformation of physics, namely the special theory of relativity. This is actually the standard example of the Inönü and Wigner contraction process. Here we just use it introduce an important concept which is crucial in our study of reformation processes, namely, the **regularization parameter**.

Example : Einstein reformation of Galilei

In the Galilean group, we have

$$
\begin{aligned}
[B_x, B_y] &= 0 \\
[R_z, B_x] &= B_y \\
[R_z, B_y] &= -B_x \\
&\vdots
\end{aligned}
\tag{4}
$$

where B_x be a boost along the x axis and R_z is a rotation about the z axis and so on. The zero commutator above is the symptom of instability: boosts along different axes commute. This is a consequence of time being an idol in the theory. We already know the expansion of this group: the Lorentz group, where

$$
\begin{aligned}
[B_x, B_y] &= c^{-2} R_z \\
[R_z, B_x] &= B_y \\
[R_z, B_y] &= -B_x \\
&\vdots
\end{aligned}
\tag{5}
$$

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The effect of expanding group on equations 3 is as follows:

$$
\begin{aligned}
x' &= \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \\
t' &= \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}
\end{aligned}
\tag{6}
$$

Notice that how the reciprocity principle is satisfied: not only time couples to space, space also couples to time. And this is done by introducing a new parameter (speed of light c) that turns on a non-commutavity in the theory. This parameter is closely related to the homotopy parameter that was introduced earlier. One may say that **the Lorentz group is an expansion (regularization) of the Galilei group**. Or equivalently, special relativity is a reformation of Galilean relativity. Note that we can recover the Galilean relativity by taking the limit as $1/c^2 \to 0$. So the Galilei relativity is the **singular limit** of special relativity, which is based on the assumption of having finite value for speed of light as the speed of physical interaction. We point out that the having a finite speed in Einsteinian relativity is equivalent to relativizing time; an absolute (idol) of Galilean relativity. So once again, we see that reformation of a physical theory means dethroning its idol(s)[2]. Special theory of relativity is a regular theory. Its group is stable against variations in c. Lorentz group, as mentioned earlier, is semi-simple.

This example reveals the conceptual pattern for every reformation process in physics. Singular (compound) groups are made regular (semi-simple) by introducing a non-commutavity in the fundamental Lie commutators of the theory. Through this process, one or more idols of the singular theory become relativized variables. As a result compound groups become stable.

We call the new parameters that enters the commutation relations of a singular theory **expansion or reformation parameters**. In the limit where the expansion parameters go to zero, we recover the singular theory. We can write symbolically

$$
\lim_{\text{expansion parameter} \to 0} \text{Regular theory} = \text{Singular theory}
\tag{7}
$$

---

[2]Note that although time is not an idol of special relativity anymore, but a new idol has emerged: the rigid space-time. General relativity dethrones (relativizes) this one.

Approximating a circle by a tangent line and a sphere by a tangent plane are well-known flattenings. The point is that the circle and sphere are finite and their flattened form is infinite. Finite dimensional representations of the group of the sphere— spherical harmonic polynomials —form a complete set on the sphere, and all the operators of an irreducible representation have finite bounded spectra. On the other hand the tangent plane is not compact and requires infinite-dimensional representations of the translation group for a complete set, and its group generators have unbounded spectra. This suggests that flattening is the root of all the infinities of present physics.

Since quantum theory began as a regularization procedure of Planck, it is rather widely accepted that further regularization of present quantum physics calls for further quantization, but what to quantize and how to quantize it remains at least a bit unclear. Now we see that the physics trail is blazed by singular groups. They signal several fragile ideal elements of present physics that are ripe for relativization and quantization, and they suggest conspicuous first candidates for the stable groups that must replace the unstable ones, changing the experimental predictions for extreme conditions.

Let us summarize the essence of our work into the following principle that we adopt heuristically:

Principle 5 (Regularization) *To regularize a theory, regularize its group.*

An immediate corollary based on our previous discussion on flattening and finiteness is:

Corollary 6 (Simplicity) *If the dynamical group of a physical theory is a simple Lie group, the theory is finite.*

We promised earlier to justify the term *reformation*. We are now at a position to fulfill our promise. In fact, reformations are usually called *revolutions*. But as we noticed, during major changes in physical theories, the change in the group structure is so subtle that the old (singular) theory remains as a working theory in the appropriate limit. So clearly an *evolution* would be a better name than revolution. But we also note that during these changes, the compound theory becomes more accurate and more physical, as far as the

experiment is concerned. In other words, these changes are de*formations* in the algebra of the theory that *re*pair the theory and thus, we arrive at reformation.

There is an important question we should address here. Group contraction or flattening is a unique process as it leads unambiguously to a unique singular theory. But the reverse process is not. So the question is "how do we fix the value of expansion or regularization parameters?" To answer this, notice that physical reformations are historically results of operationality and correspondence with experiments (although not always directly). There has been either direct experimental motivation (in the case of quantum theory) or theoretical considerations which have been based on consistencies with other theories with experimental successes (in the case of special relativity). In the former case, the planck constant $\hbar$ was fixed by direct experiments that actually triggered the quantum reformation. In the latter case though, the expansion parameter of special relativity, turned out to be related to the speed of light c through consistency with Maxwell's theory[3].

In the present work where we develop the finite quantum theory for the harmonic oscillator, we already take advantage of the second motivation. But we might not be lucky enough in the experimental power of present day physics to actually observe direct experimental failure of the standard model or general relativity. These theories work in very high energies or very large scale domains and at least by now, we have not passed their experimental limits. So it might be much harder for us to fix the exact values of new regularization parameters.

2.3   Physics as Process

It is not meaningful to assume that there will be a final theory. We consider physics as a continuous evolutionary process of successive reformations, each being a step toward unification. (Although, we do not see physics to reach a final unified state where the

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[3]Of course the speed of light was measured independently and found to be finite. But the fact that the same number appears in Lorentz transformation is our point here. With all that said, some believe that this was a victory for Maxwell's theory. I believe though, that this was a test for relativity, which it passed. After all, Maxwell's theory had already been tested experimentally and found to be correct.

Table 1: Regularizations of physics-1

Regularization parameter	Symbol	Regularization
Boltzmann constant	k	Kinetic theory
Speed of light	c	Special relativity
Gravitational constant	G	General relativity
Cosmological constant	Λ	de Sitter relativity
Planck constant	h	Quantum theory
Electric charge	e	Electrodynamics
Strong charge	g	Chromodynamics
Weak constant	f	Electroweak gauge
Segal constants	h', h''	Finite quantum theory
$\vdots$	$\vdots$	$\vdots$

process ends)[4]. The physical theories will be constantly under reformation process. We will be having a working theory while we will be looking for a better one. Therefore we turn our attention to the process of theory-revision itself.

As we mentioned earlier, many of the successive radical changes in the physical theories are group regularizations (reformations) in the Segal sense, and there are more of the same to come. Each of the past reformations of physics is recognized with the corresponding regularization parameter (Table 1) [5].

Each turns on a non-comutativity, in that commutativity is restored when the constant is set to 0. These innovations have in common that each makes physics more and more non-commutative, simple (algebraically), processual, relativistic, and often atomisitc. Even Boltzman's constant k, regarded as a coefficient in the entropy $S = k \log \Omega$ of any system, turns on the non-commutativity of adiabatic and isothermal processes, simplifies thermodynamics by uniting it with mechanics, replaces the material caloric theory of heat by the processual kinetic theory of heat, and is an atom of ideal-gas heat capacity. The stock example of group regularization, however, is the c regularization from the Galilei group to the Poincaré. To be sure, in the classical theory the c regularization seems to involve no atomization. In the quantum theory, however, the Lorentz group admits finite-dimensional boost representations where the Galilean flattening does not, and so even the c regularization atomizes something. Namely, it atomizes the boost component of space-time angular

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[4]This rhymes with an important belief in Persian Sufism according to which "love" is the everlasting process of "beloved" getting closer and closer to "The beloved", who is ultimately unreachable. So physics, we believe, is our "love" toward unification.

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Table 2: Regularizations of physics-2

Parameter	Non-commutativity	Unification
k	Reversible processes	Entropy and probability
c	Boosts	Time and space
G	Translations	Energy and curvature
Λ	Translations	Translations and rotations
h	Observables	Observables and generators
e	Covariant derivatives	Momentum and E/M vector potential
g	Covariant derivatives	Momentum and QCD vector potential
f	Covariant derivatives	Momentum and Isospin vector potential
h',h''	Observables	Space-time, energy-momentum, and complex plane
$\vdots$	$\vdots$	$\vdots$

momentum, with atoms of size $h/2$.

Each group regularization of a theory relativizes a false absolute of the theory and introduces a new non-commutavity. As a result a new unification emerges in the regularized theory that was absent before. For example, time, which is an idol of the Galilean relativity, gets relativized and unified with space in special relativity. The corresponding non-commutativity is among boost in different direction (Table 2).

To put this program in some perspective we note that regularization would never have led (say) to the Dirac equation, or to color SU(3), let alone the Standard Model. Group regularization is a prescription for relativizing false absolutes. It is no substitute for experiment. It remains to see if it leads to more discoveries like the prototype group regularizations, quantum theory and special and general relativity.

Each group regularization has a continuous inverse process in which the parameter approaches 0 and the singularities return. The new regularized theory is as close to the older flattened one as it is possible to be, and the flattened theory is still used after the regularization in its domain of adequacy. These innovations are not what is usually called revolution. Some call the process a deformation. This pejorative reveals a prejudice against the more robust relativization and in favor of the absolutes of the fragile compound. Deformation quantization, for example, retains the absolute despite the relativization. The term "reform" would better describe our program than "deformation."

In relation to the paragraph above, we give a more precise definition of what we called

earlier the *generalized correspondence principle*:

Principle 7 Generalized correspondence principle *Let σ be an expansion (or regularization) parameter (like $\hbar$ and $1/c$). We call the limiting value $O(0)$ of any quantity $O(\sigma)$ as $\sigma \to 0$ the correspondent of $O(\sigma)$. The Segal correspondent principle for σ is the requirement that $\sigma \to 0$ is a continuous deformation.*

This principle guarantees what we already stated otherwise: the contracted (singular) theory is the limit of the expanded (regular) theory.

We will see in section 3.1 how we apply this principle in expanding the compound commutation relations of the Heisenberg algebra.

Why have so many regularizations occurred? Some think that it is because we are all headed for one final simple theory in the sky. The Segal stability theorem suggests that they may occur merely because compound theories are fragile and so have a higher mortality rate than their simpler more robust offspring. Fragile theories are never probable inferences from experimentation alone, which always have error bars, but incorporate absolutes as items of faith, and have probability 0 relative to infinitely many, more robust, nearby theories. Fragile theories are therefore less likely to survive than competitive robust theories. The evolution of physics may be Darwinian selection, not teleological, and survival of a theory may depend most immediately on its robustness, not its beauty. The quest for beauty can lead to successful theories nevertheless, because it can lead to simplicity, which is robust.

CHAPTER 3

REGULAR QUANTUM THEORY[1]

In the present chapter, we apply the idea of simplicity to quantum theory. We will observe that the Heisenberg algebra is compound with the imaginary unit i as its center. The simplest regularization of this algebra leads to a real simple orthogonal group. As a specific example, we study the finite quantum theory of the harmonic oscillator.

3.1 Simplifying the Heisenberg Algebra

The fundamental algebra of quantum mechanics of a single particle in $\mathbb{R}$ is the Heisenberg algebra $\mathbf{H}(1)$, which is a Lie algebra defined by the following commutation relations:

$$
\begin{aligned}
[p, x] &= -i\hbar \\
[i, x] &= 0 \\
[x, i] &= 0
\end{aligned}
\tag{8}
$$

This is the Lie algebra of the Weyl group $\mathbf{W}(1)$. Clearly the imaginary unit by itself forms a center for this algebra. It commutes with all other elements and generates an invariant abelian subalgebra. Therefore quantum theory is a compound theory and ripe for regularization. Segal proposed to simplify $\mathbf{H}(1)$ by introducing two more quantum constants $\hbar'$ and $\hbar''$ constants, in a way that replaced the Heisenberg algebra SO(2, 1). His expanded commutation relations are [8, 27, 28, 29, 24]

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[1]Parts of material in this chapter have appeared in the following publications:

- Finkelstein, D.R. and Shiri-Garakani, M. Expanded Quantum Linear Harmonic Oscillator, *Proceedings of the 3rd International Symposium on Quantum Theory and Symmetries (QTS3).* Cincinnati, Ohio, 2003.
- Baugh, J., Finkelstein, D., Galiautdinov, A. and Shiri-Garakani, M. *Foundations of Physics* **33**, 1267,2003.
- Baugh, J. Finkelstein, D. Galiautdinov, A. Saller, H. and Shiri, M. Transquantum space-time, *Proceedings of the 5th International Symposium of Fundamental Physics.* Birla Science Center, Hyderabad, 2003.

$$
\begin{aligned}
[p, x] &= -\hbar i \\
[i, p] &= -\hbar' x \\
[x, i] &= \hbar'' p
\end{aligned}
\tag{9}
$$

The irreducible unitary representations of this group are infinite-dimensional. To avoid this infinity, which can introduce divergences and other problems, we instead use the SO(3) regularization [4, 15]

$$
\begin{aligned}
[p, x] &= -i\hbar \\
[i, p] &= -x\hbar' \\
[x, i] &= -p\hbar''
\end{aligned}
\tag{10}
$$

Regularizing quantum theory at the algebra level means changing the role of i in the theory from constant central element to quantum variable operator on the same footing as p and q. We might call this *i-activation*.

Here we focus on the non-relativistic theory. But if we consider the full theory, one should add and regularize Snyder's commutation relations [25]

$$
[x^\mu, x^\nu] = \lambda L^{\mu\nu}
\tag{11}
$$

as well.

In earlier work, Finkelstein used the algebra of quaternions to activate i as the electromagnetic axis $\eta(x)$ that resolves the electroweak gauge interaction into electromagnetic and weak interactions [9]. The η field turned out to exhibit Stückelberg-Higgs effect, giving mass to the charged partner of the photon. This led to a natural SU(2) that was interpreted as isospin. The theory was dropped because it did not account for color SU(3). Today we understand that the theory was still a field theory, violating Segal's principle of simplicity. In the present work, we activate i on principled ground, namely the principle of simplicity. There is now plenty of room for internal groups like color SU(3).

To proceed, it is more convenient to define the *skew-operators* (the generators of the Lie algebra):

$$\widehat{q} = iq, \qquad \widehat{p} = -ip. \tag{12}$$

The (usual) quantum canonical commutation relations are then

$$
\begin{aligned}
[\widehat{q}, \widehat{p}] &= \hbar i \\
[i, \widehat{q}] &= 0, \\
[\widehat{p}, i] &= 0, \\
i^2 &= -1.
\end{aligned}
\tag{13}
$$

The regularization correspondence principle (section 2.3), guides us to modify these relations by replacing the zeros in commutators with polynomial functions of other variables. We expand these quantum relations [24, 1] so that the three operators $\widehat{q}, \widehat{p}, i$ become three symmetrically related infinitesimal orthogonal-transformation generators $\breve{q}, \breve{p}, \breve{i}$ obeying

$$
\begin{aligned}
[\breve{q}, \breve{p}] &= \hbar \breve{i}, \\
[\breve{i}, \breve{q}] &= \hbar' \breve{p}, \\
[\breve{p}, \breve{i}] &= \hbar'' \breve{q},
\end{aligned}
\tag{14}
$$

We suppose $\hbar, \hbar', \hbar'' > 0$ so the orthogonal group is SO(3). The quantities with a breve "˘" are the new expanded quantum operators. In this way the simplification process introduces a new dynamically variable generator $\breve{i}$, somewhat as general-relativization introduced the new dynamical variable $g_{\mu\nu}$ the gravitational metric tensor field. The most primitive theory with a dynamical variable like $\breve{i}$ is quaternion quantum field theory [9]. There $\breve{i}$ generates rotations about the electric (or electromagnetic) axis in isospin space, defining a natural Higgs field. We suppose that the present generator $\breve{i}$ is also a Higgs field.

Except for scale factors the simplified commutation relations are satisfied by the three components of an SO(3) quantum skew-angular-momentum operator-valued vector $\breve{\mathbf{L}} = \breve{\mathbf{L}} \times \breve{\mathbf{L}}$ for a dipole rotator in three dimensions. We assume an irreducible representation with

$$\mathbf{L}^2 = l(l+1) \tag{15}$$

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Then the $\breve{L}_x, \breve{L}_y, \breve{L}_z$ are finite-dimensional linear operators represented by $(2l+1) \times (2l+1)$ matrices obeying

$$
\begin{aligned}
\left[\breve{L}_1, \breve{L}_2\right] &= \breve{L}_3, \\
\left[\breve{L}_2, \breve{L}_3\right] &= \breve{L}_1, \\
\left[\breve{L}_3, \breve{L}_1\right] &= \breve{L}_2, \\
(\breve{L}_1)^2 + (\breve{L}_2)^2 + (\breve{L}_3)^2 &= -l(l+1).
\end{aligned} \tag{16}
$$

The canonical operators $\breve{x}, \breve{p}, \breve{\imath}$ are just components of the one operator-valued vector $\breve{\mathbf{L}}$, re-scaled. This means that in the simplified theory all canonical variables have discrete spectra and transform under an orthogonal group, like rotator angular momenta. We have fixed the eigenvalue of the Casimir operator of the orthogonal group to make the algebra simple. Then the spectra become bounded as well as discrete. The theory is genuinely finite [1].

To fix the scale factors and determine the spectra we set

$$
\begin{aligned}
\breve{q} &= Q\breve{L}_1, \\
\breve{p} &= P\breve{L}_2, \\
\breve{\imath} &= J\breve{L}_3,
\end{aligned} \tag{17}
$$

By (14)

$$
\begin{aligned}
J &= \sqrt{\hbar'\hbar''} = 1/l, \\
Q &= \sqrt{\hbar\hbar'}, \\
P &= \sqrt{\hbar\hbar''}.
\end{aligned} \tag{18}
$$

The commutation relations $\breve{\mathbf{L}} \times \breve{\mathbf{L}} = \breve{\mathbf{L}}$ and the angular momentum quantum number l, defined by $\breve{\mathbf{L}}^2 = -l(l+1)$, determine a simple (associative) enveloping algebra $\mathrm{Alg}(\mathbf{L}, l)$ where l can have any non-negative integer eigenvalue. The spectral spacing of the $\breve{L}_3$ is 1, so the finite quantum constants Q, P, J serve as quanta of position, momentum and $\breve{\imath}$. Since q, p are supposed to have continuous spectra in quantum theory, the constants Q, P are very small on the ordinary quantum scale of $h \sim 1$. It follows that $J = QP/\hbar$ is also very small on that scale and $l \gg 1$.

For $\sqrt{l} \gg 1$, variations $\delta(\breve{\imath}^2) \leq O(l^{-1/2}) \ll 1$ about $(\breve{\imath})^2 = -1$ can be negligible at the same time as the spectral intervals $\delta p \leq P\sqrt{l}$ and $\delta q \leq Q\sqrt{l}$ for quasicontinuous $p, q \approx 0$. This simulates the usual oscillator.

3.2 Finite Linear Harmonic Oscillator

For given finite-quantum constants P, Q the Finite Linear Harmonic Oscillator (FLHO) has a Hamiltonian of the form

$$H = \frac{P^2 L_x^2}{2m} + \frac{kQ^2 L_y^2}{2} := \frac{K}{2}\left({L_x}^2 + \kappa^2 {L_y}^2\right) \tag{19}$$

where

$$\kappa^2 = \frac{P^2}{mkQ^2} = \frac{\hbar'}{mk\hbar''}. \tag{20}$$

In the following, we find the energy spectrum in three different cases: medium, the case where kinetic and potential terms in H are of comparable sizes ($\kappa \sim 1$); soft, when the potential energy term is dominant ($\kappa \ll 1$); and hard, when the kinetic energy term is dominant ($\kappa \gg 1$).

3.3 Medium FLHO

Let us begin with the very special case $\kappa = 1$ where we have the simple and symmetric case of having two equal terms in the Hamiltonian. Now, since

$$(\breve{L}_1)^2 + (\breve{L}_2)^2 + (\breve{L}_3)^2 = (\breve{L}^2), \tag{21}$$

$\breve{H}$ has a particularly simple finite quantum theory form:

$$\breve{H} = \frac{K}{2}\left(l(l+1) + (\breve{L}_3)^2\right) \tag{22}$$

The usual oscillator quantum number n is simply

$$n = l + m. \tag{23}$$

The new expanded energy spectrum is

$$E_n = \frac{K}{2}\left(l(l+1) - (n-l)^2\right) = lK\left(n + \frac{1}{2} - \frac{n^2}{2l}\right) \tag{24}$$

22

For $n \ll \sqrt{l}$ this reproduces the usual uniformly-spaced energy spectrum as closely as desired for sufficiently large l. What quantum theory describes as a linear harmonic oscillator near the bottom of its n spectrum is actually a dipole rotator near the extremes of its m spectrum, according to this finite quantum theory.

The ground-state energy for this oscillator is obtained by setting $|m| = l$ or equivalently $n = 0$ so

$$E_0 = \frac{1}{2}Kl \tag{25}$$

which matches the usual oscillator ground energy of $1/2\hbar\omega$ with $Kl = \hbar\omega$.

The new feature is that now there is an upper limit for the energy:

$$E_{max} = \frac{1}{2}Kl(l+1) \tag{26}$$

This is a major effect of applying finite quantum theory to the harmonic oscillator, and of course, consistent with what we expect from a finite theory.

Now, we consider the more general case of $\kappa \sim 1$. To obtain an upper bound for the ground energy, we use the Ritz variational method with a trial function $|L_z = \pm l\rangle$. This just reproduces our previous result of (25) with the interpretation that it is an upper bound for the ground energy of a medium FLHO:

$$E_0 \leq \frac{1}{2}Kl. \tag{27}$$

The medium oscillators all have many states with m-value close to its extremum value i.e., states very close to $|L_z = \pm l\rangle$. The usual QLHO Heisenberg uncertainty principle is

$$(\Delta p)^2(\Delta q)^2 \geqslant \frac{1}{4}\langle i[p, q]\rangle^2 = \frac{\hbar^2}{4}. \tag{28}$$

For a medium FLHO in a low-lying energy level, the uncertainty product becomes

$$(\Delta L_x)^2(\Delta L_y)^2 \geqslant \frac{\hbar^2}{4}\langle L_z\rangle^2_{|L_z \approx \pm l\rangle} \tag{29}$$

which by way of (17) and (18) becomes

$$(\Delta p)^2(\Delta y)^2 \geqslant \frac{\hbar^2}{4} \tag{30}$$

23

for large l. So medium FLHO are described by states with uncertainties near and above the limit set by the QLHO uncertainty principle.

The Newtonian and quantum concepts of energy are invariant under a shift in the zero point of energy, but the angular momentum relations $\mathbf{L} \times \mathbf{L} = \mathbf{L}$ are not invariant under a shift $\mathbf{L}' = \mathbf{L} + \Delta\mathbf{L}$ by any constant vector $\Delta\mathbf{L} \neq 0$. Therefore the finite quantum relations are not invariant under a shift in the zero-point energy. Therefore this zero-point energy likely contributes to the gravitational field. It will be interesting to estimate its contribution to the missing matter in our cosmos. For a consistent estimate of this effect, we should expand the space-time variables t, ∂_t as well as the field variables q, p, since the quantum theory has the same instability in both algebras. Here we take up the time-independent finite quantum theory only.

3.4 Soft FLHO

A simple comparison with the Hamiltonian of a spin-zero scalar field (Klein-Gordon field) in quantum field theory shows that the possibilities $\kappa \ll 1$ and $\kappa \gg 1$ are also important. The QLHO oscillators that give rise to infrared divergencies of the field theory correspond to soft FLHO's.

Recall our finite quantum oscillator Hamiltonian

$$\breve{H} = \frac{K}{2}\left(\breve{L}_x^2 + \kappa^2 \breve{L}_y^2\right) \tag{31}$$

When $\kappa \ll 1$ we can find the spectrum using time-independent perturbation theory.

The unperturbed Hamiltonian for our problem is

$$H_0 = \frac{K}{2} L_z^2 \tag{32}$$

so the unperturbed kets are just eigenkets of L_z^2 with the spectrum m^2. To find the first-order shifts we need to calculate

$$\langle m|L_y^2|m\rangle. \tag{33}$$

We notice that due to the symmetry we must have

$$\langle m|L_y^2|m\rangle = \langle m|L_x^2|m\rangle. \tag{34}$$

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Thus we calculate

$$\langle m|L_x^2 + L_y^2|m\rangle = \langle m|L^2 - L_z^2|m\rangle = l(l+1) - m^2 \tag{35}$$

So the first order energy shifts are

$$\Delta E_m = \frac{1}{4}K \; \kappa^2 \left[l(l+1) - m^2\right]. \tag{36}$$

The new energy spectrum is then

$$\begin{aligned}
E_m &= \frac{K}{2}m^2 + \Delta E_m \\
&= \frac{K}{2}m^2 + \frac{1}{4}K \; \kappa^2 \left[l(l+1) - m^2\right]
\end{aligned} \tag{37}$$

The estimated upper bound for the energy is

$$E_{max} \approx \frac{1}{2}Kl^2(1 + \frac{\kappa^2}{2l}) \tag{38}$$

For $\kappa \to 0$ this reproduces the upper bound for the unperturbed hamiltonian L_z^2, as it should.

The zero-point energy E_0 of first-order perturbation theory is

$$E_0 \approx \frac{1}{4}\kappa^2 Kl(l+1) \tag{39}$$

For $\kappa \to 0$ this is infinitesimal compared to the usual QLHO. Clearly the energy levels of a soft FLHO do not exhibit a uniform spacing anymore. A soft FLHO shows no resemblance to the usual QLHO.

For soft FLHO oscillators, the kinetic energy in the Hamiltonian totally dominates the potential energy. Soft oscillators have very large masses compared to their spring constants. Their low energy states are near $|L_x = 0\rangle$ instead of $|L_z = \pm l\rangle$. Their p degree of freedom is frozen out. They are "too soft" to oscillate: There is not enough energy in the q degree of freedom, even at maximum excitation, to excite one quantum of p. For soft FLHO, the uncertainty relation reads

$$(\Delta L_x)^2(\Delta L_y)^2 \geqslant \frac{\hbar^2}{4}\langle L_z\rangle_{|L_x\approx0\rangle}^2 \approx 0 \tag{40}$$

Therefore

$$\Delta p\Delta q \ll \frac{\hbar}{2}, \tag{41}$$

which seems to violate the Heisenberg uncertainty principle.

3.5 Hard FLHO

Most of the story is just reversed for hard FLHO oscillators. The QLHO oscillators that give rise to ultraviolet divergencies of the field theory correspond to hard FLHO's. The dominant term in the Hamiltonian is now the potential energy. Here the spring constant is much larger than the mass. The low energy states are now near $|L_y = 0\rangle$ instead of $|L_z = \pm l\rangle$ and the q degree of freedom is frozen out. These oscillators are "too hard" to oscillate. There is not enough energy in the p degree of freedom, even at maximum excitation, to excite one quantum of q.

A hard FLHO can also be treated by perturbation theory. Now the kinetic energy is the perturbation. We write

$$\breve{H} = \frac{K}{2}\left(\lambda^2 \breve{L}_x^2 + \breve{L}_y^2\right) \tag{42}$$

where

$$\lambda := \frac{1}{\kappa} \ll 1 = \frac{km\hbar''}{\hbar'}. \tag{43}$$

We may carry all the of the main results in the previous section for soft FLHO oscillators to the hard ones simply by replacing κ with λ:

The first order energy shifts are

$$\Delta E_m = \frac{1}{4}K\,\lambda^2\left[l(l+1) - m^2\right] \tag{44}$$

and the new energy spectrum is then

$$\begin{aligned} E_m &= \frac{K}{2}m^2 + \Delta E_m \\ &= \frac{K}{2}m^2 + \frac{1}{4}K\,\lambda^2\left[l(l+1) - m^2\right] \end{aligned} \tag{45}$$

The estimated upper bound for the energy is

$$E_{max} \approx \frac{1}{2}Kl^2(1 + \frac{\lambda^2}{2l}) \tag{46}$$

For $\lambda \to 0$ this reproduces the upper bound for the unperturbed hamiltonian L_z^2, as it should.

The zero-point energy E_0 of first-order perturbation theory is

$$E_0 \approx \frac{1}{4}\lambda^2 Kl(l+1) \tag{47}$$

For $\lambda \to 0$ this is infinitesimal compared to the usual QLHO. Clearly the energy levels of a soft FLHO do not exhibit a uniform spacing anymore. A hard FLHO shows no resemblance to the usual QLHO.

For hard FLHO, the uncertainty relation reads

$$(\Delta L_x)^2 (\Delta L_y)^2 \geqslant \frac{\hbar^2}{4} \langle L_z \rangle^2_{|L_y \approx 0\rangle} \approx 0 \tag{48}$$

Therefore

$$\Delta p \Delta q \ll \frac{\hbar}{2}, \tag{49}$$

which seems to violate the Heisenberg uncertainty principle again.

3.6 Unitary Representations

Variables p and q do not have finite-dimensional unitary representations in classical and quantum physics. They are continuous variables and generate translations of each other. But since in the finite quantum theory, all operators become finite and quantized, we expect all translations to become rotations with simple finite-dimensional unitary representations. We write the physical representations of the observables of the harmonic oscillator for the canonical group of a CLHO, the Heisenberg-Weyl group of a QLHO, and canonical orthogonal group of a FLHO.

CLHO The system is described by the Hamiltonian

$$H = \frac{1}{2}(p^2 + q^2) \tag{50}$$

where we have set $m = k = 1$ for convenience. Here the system variables p and q are points in the phase space and motion of the system is described by a parameterized path. For convenience we give the action of the group on functions $\psi(q, p)$ on phase space. We write the Poisson bracket operator as

$$[A, B]_{\mathrm{P}} = (\Delta_P A) \cdot P \tag{51}$$

A finite translation of q by the amount a has the form

$$T_a = e^{a \Delta_{\mathrm{P}} p} : \psi(q, p) \mapsto \psi(q - a) \tag{52}$$

Quantum Linear Harmonic Oscillator (QLHO)

The unitary representations of finite translation operators of the Heisenberg-Weyl group are made by exponentiating the generators of infinitesimal translations of the Heisenberg Lie algebra:

$$T(a) = e^{-ipa/\hbar} \tag{53}$$

These are unitary operators in the enveloping algebra of the Heisenberg Lie algebra.

Finite Linear Harmonic Oscillator (FLHO)

A q translation of the FLHO by the amount a is a rotation of the corresponding rotator by an angle $\theta = Pa/\hbar$. The unitary operator is

$$U_q(a) = e^{-i\theta L_x} \tag{54}$$

The p translation through a momentum-change b is found by replacing L_x by L_y:

$$U_p(b) = e^{-i\theta L_y} \tag{55}$$

with $\theta = Qb/\hbar$.

CHAPTER 4

CONCLUSION

We have applied the Segal process of group simplification to the linear harmonic oscillator. The result is a finite quantum theory with three quantum constants $\hbar, \hbar', \hbar''$ instead of the usual one. We have compared the classical (CLHO), quantum (QLHO), and finite (FLHO) linear harmonic oscillators and their canonical or unitary groups. The FLHO is isomorphic to a dipole rotator with $N = l(l + 1) \sim 1/(\hbar'\hbar'')$ states and Hamiltonian $H = A(L_x)^2 + B(L_y)^2$, and the physically interesting case has $N \gg 1$. The position and momentum variables are quantized with uniform finite spectra. For fixed quantum constants and large $N \gg 1$ there are three broad classes of FLHO: soft, medium, and hard, with $B/A \ll 1$, $B/A \sim 1$, and $B/A \gg 1$ respectively. The field oscillators responsible for infra-red and ultraviolet divergences are soft and hard respectively. Medium oscillators have $B/A \sim 1$ and approximate the QLHO. They have $\sim \sqrt{N}$ low-lying states with nearly the same zero-point energy and level spacing as the QLHO, and nearly obeying the Heisenberg uncertainty principle and the equipartition principle. The corresponding rotators are nearly polarized along the z axis with $L_z \sim \pm l$. The soft and hard FLHO's have infinitesimal 0-point energy, and grossly violate both equipartition and the Heisenberg uncertainty relation. They do not resemble the QLHO at all. Their low-lying energy states correspond to rotators with $L_x \sim 0$ or $L_y \sim 0$ instead of $L_z \sim \pm l$. Soft oscillators have frozen momentum, because their maximum potential energy is too small to produce one quantum of momentum. Hard oscillators have frozen position, because their maximum kinetic energy is too small to excite one quantum of position.

APPENDIX A

PROCESS PHYSICS

In this appendix, we discuss the concept of operation-based physics.

The quest for unification in physics has its roots in several factors. One reason for it is that it has led to major improvement in predictive power of physics. Each of the three major reformations of the last century were steps toward unification: special relativity unifies space with time, general relativity unified space-time with matter (energy) and finally quantum mechanics unified observervables with transformation. But each of these reformations were also a giant step towards an operation-based physics. Each shifted the conceptual ground of physics from *operand* (noun, substance, matter, particle, thing, atom, state, *ons, ...*) to *operations* (verb, process, operations, action, transformation, event, mode, *praxis, ...*); from ontic to practic[12]. Therefore we seek a practic theory, founded on operations alone.

There has been no progress toward stability and practism comparable to the giant steps of special and general relativity and quantum theory, despite numerous proposals and attempts ([11, 30, 31] among others). Indeed, experiments has carried us in the opposite direction. The standard model enlarged the group but made it less simple. Theory also has not moved toward more simplifications. String theory is another continuum-based bundle theory, like the field theory. This may be because the next step is a much more fundamental and bigger, requiring many false steps. Stability requirement alone forces several radical changes. For example, it eliminates the differential calculus from the basic laws.

Our basic principle here is

Principle 8 Chronon principle of elementary operations (Principle of process atomism) All physical operations are composed of elementary quantum operations we call chronons.

Space-time operations, which are regarded as continuous in the standard model and

general relativity, are now written as finite sum of elementary operations [3]. We do the same here for operators in the Hesienberg algebra of quantum mechanics and as a specific example, we study the expanded quantum harmonic oscillator (section 3.2).

We designate an elementary operation by o (omicron) and its duration by χ.

In the early twentieth century, few physicists speculated about a quantum of time (a natural unit of time) and they called it chronon. Nowadays the suffix "on" is firmly associated with physical entities like particles, not with natural constants. We therefore reserve the name "chronon" for hypothetical elementary operations themselves, the least action that can happen, not the fundamental constant χ and certainly not for any particle in the sense of particle physics, which occupies a prior space-time.

Flipping an electron spin is the least action we know. This could be a good physical candidate for a chronon. We take this to be an instance of the elementary operation, and set about expressing all operations as compositions of spin flip.

It is helpful to construct an operand for these operations. We form a linear space on which these operators act, and regard the vectors in this space as modes of creation op preparation of the operand. We present the operand as an aggregate whose elements we call *events*, since replace the classical space-time events of the usual quantum field theories.

Let us explain what is the physical effect of introducing chronons into a theory. Quantization is a heuristic procedure that deforms a physical theory from the wrong value $\hbar = 0$ to the physical value of $\hbar$. For example, when the harmonic oscillator is quantized, it is presented as an aggregate of phonons.

In the same spirit, when we introduce the chronon o and its duration χ into a theory that lacks them, we deform the theory from the wrong continuum value $\chi = 0$ to the physical value of χ. We present the system history as a quantum combination of chronon instead of a continuum of events. This is a true second quantization in that we replace classical space-time variables by non-commuting operators, and bring in a second constant χ that stabilizes the theory, much like replacing the classical position and momentum with the non-commuting operators and introducing $\hbar$ in quantum theory.

APPENDIX B

CLIFFORD-ALGEBRAIC REPRESENTATION OF FLHO

In this appendix, we briefly review Clifford algebras and study the finite representations of the operators of the finite quantum theory of the harmonic oscillator in this algebra. We use Clifford algebra to satisfy the principle of process atomism that was introduced in appendix A. We assume that elementary operations are generators of a Clifford algebra.

B.1   Review of Clifford Algebras

Clifford (geometric) algebra is the most efficient (and perhaps necessary) mathematical language for our work. There are at least two reasons for this. First, it is because of the type of statistics that we need to use for the elementary operations (chronons). This lies outside the scope of the present work; we just mention that this is a double-valued statistics which is the real version of the complex projective statistics ([32, 33, 20]). Finkelstein and coworkers has developed this statistics and call it *Clifford-Wilczek* statistics [3, 13].

Second, the groups arising in this work are orthogonal simple groups whose representations are conveniently constructed with Clifford algebras. Elements of Clifford algebras are called *cliffors*. We begin by definition of the more general concept of Clifford ring and then we specify to the algebra.

Definition 9 Clifford ring *A Clifford ring* $\mathbf{C}$ *is a graded (associative linear) ring of the form*

$$
\begin{aligned}
\mathbf{C} &= \mathbf{C}_0 \quad \oplus \; \mathbf{C}_1 \quad \oplus \cdots \oplus \; \mathbf{C}_N \\
&= \; scalars \; \oplus \; vectors \; \oplus \cdots \oplus \; N\text{-}vectors
\end{aligned}
\tag{56}
$$

with the following properties:

    C0. The scalars form a subring and commute with every cliffor.

    C1. Every cliffor is expressible as a polynomial in the vectors with scalar coefficients.

    C2. Clifford's Law: The square of every vector is a scalar.

A Clifford ring is determined by its scalars, its vectors, and the product, and is written $\mathbf{C} = \text{Cliff}\,(\mathbf{C}_1, \mathbf{C}_0)$, the product being understood.

We understand that in the Clifford law, the "square" implies the Clifford product of a vector $\mathbf{a}$ by itself. We designate the scalar $\mathbf{a}^2$ by $\|\mathbf{a}\|$ and the Clifford product between two vectors $\mathbf{a}$ and $\mathbf{b} \in \mathbf{C}_1$ by

$$\mathbf{a} \sqcup \mathbf{b} \tag{57}$$

The Clifford law implies that $\|\mathbf{a}\| := \mathbf{a}^2$ is a quadratic form on the vectors; So the Clifford law is

$$\mathbf{a} \sqcup \mathbf{a} = \|\mathbf{a}\|^2 \tag{58}$$

Let us show the properties of the Clifford law within a simple example; the Clifford algebra of the plane $\mathbf{C}(\mathbb{R}^2)$. With an orthonormal basis $\{\gamma_1, \gamma_2\}$ for the Clifford law reads

$$(x\gamma_1 + y\gamma_2)^2 = x^2 + y^2 \tag{59}$$

Use distributivity without assuming commutativity to obtain

$$x^2\gamma_1^2 + y^2\gamma_2^2 + xy\,(\gamma_1\gamma_2 + \gamma_2\gamma_1) = x^2 + y^2 \tag{60}$$

This is satisfied if γ's obey

$$\begin{aligned} \gamma_1^2 &= \gamma_2^2 &= 1 \\ \gamma_1\gamma_2 &= -\gamma_2\gamma_1 \end{aligned} \tag{61}$$

which corresponds to

$$\begin{aligned} \|\gamma_1\| &= \|\gamma_2\| &= 1 \\ \gamma_1 &\perp \gamma_2 \end{aligned} \tag{62}$$

Specifying to an algebra is straight forward: *A Clifford algebra is a Clifford ring whose scalars form a field.*

The example above was too specific in that a positive definite metric was used. This is actually not a restriction and we relax it in the following general definition of a Clifford Algebra of a quadratic space with any signature [19, 22]:

33

Definition 10 Clifford algebra *An associative algebra of a linear space V over $\mathbb{F}$ with unity 1 is the universal Clifford algebra $\mathbf{C}_{p,q}(V)$ of a non-degenerate (non-singular) quadratic form Q on V if it contains V and $\mathbb{F} = \mathbb{F} \cdot 1$ as distinct subspace so that*

1. $\mathbf{x}^2 = Q(\mathbf{x}), \forall \mathbf{x} \in V$

2. *V generates $\mathbf{C}(V)$ as an algebra over $\mathbb{F}$*

3. *$\mathbf{C}(V)$ is not generated by any proper subspace of V.*

Here Q is a quadratic form $Q : V \to \mathbb{F}$ on V with an indefinite metric with signature $p - q$ such that

$$Q(\mathbf{x}) = x_1^2 + x_2^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2 \tag{63}$$

For an orthonormal complete basis $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ for $\mathbb{R}^{p,q}$, the condition *1* can be expressed as

$$
\begin{aligned}
\gamma_i^2 &= 1 , & 1 \leq i \leq p \\
\gamma_i^2 &= -1 , & p < i \leq n \\
\gamma_i \gamma_j &= -\gamma_j \gamma_j , & i < j
\end{aligned}
\tag{64}
$$

while condition *3* becomes

$$\gamma_1 \gamma_2 \ldots \gamma_n \neq \pm 1 \tag{65}$$

B.2   Clifford algebraic-representations

Let us first introduce our notation. If C, C' are free Clifford algebras over the respective vector spaces V, V' we write

$$
\begin{aligned}
C &= \mathbf{2}^V = \Sigma \otimes \Sigma^\dagger, \\
C' &= \mathbf{2}^{V'} = \Sigma' \otimes \Sigma'^\dagger, \\
V &= \triangle C, \quad \Sigma = \sqrt{C}, \\
V' &= \triangle C', \quad \Sigma' = \sqrt{C'}, \\
CC' &= \mathbf{2}^{V \oplus V'} = (\Sigma \otimes \Sigma') \otimes (\Sigma^\dagger \otimes \Sigma'^\dagger)
\end{aligned}
\tag{66}
$$

$\triangle C$ represents the least difference or atom or bit of C.

We use $\triangle$ for bit much as one uses d for differential. $\triangle$ defines a factor in a product, the differential a term in a sum. If s is a variable in C then $\triangle s$ is an independent variable in $\triangle C$.

We call $\triangle S$ the bit (space) of the algebra $S = \mathbf{2}^{\triangle S}$. We write the corresponding systems as $\triangle$S and S. The bit system $\triangle$S is the atom, the system S is the quantum aggregate of atoms.

$C = \mathbf{2}^{\triangle C}$ expresses the system with mode space C as a set of atoms. $C = \Sigma \otimes \Sigma^{\dagger}$ expresses the generic transformation of the same system as a pair of a creation and an annihilation.

If C is a Clifford algebra we write

$$C' = \mathbf{2}^{C} \tag{67}$$

for the free Clifford algebra over the quadratic space consisting of the elements of C with their usual sums and products by scalars, and with the norm

$$\|x\| = \Re x^2, \tag{68}$$

where the operator $\Re$ takes the grade-0 part of its operand.

For typographical convenience we may write exponentials functionally:

$$A^B = A(B), \quad \mathbf{2}^X = \mathbf{2}(X), \quad ,\dots \tag{69}$$

We are ready now to discuss the representation of the oscillator operators in Clifford algebra. The dynamical operators of the oscillator form the dynamical algebra A of the oscillator. We represent the finite quantum algebra $\breve{A}$ using a product of many replicas of the Clifford algebra $C(3)$ appropriate to the orthogonal group of the oscillator.

We describe joint input and output (IO) operations on the oscillator [12] by elements of the co-algebra A^* dual to A, composed of the linear maps $A \to \mathbb{R}$. Elements of A^* are called co-operators.

This means that we represent I and O processes separately by spinors. We might instead represent I processes by Clifford elements. Then we would have to represent $\breve{q}$ and $\breve{p}$ by linear operators $C \to C$ rather than by elements of C.

35

Our prototype Clifford algebra $C(3)$ is the linear associative real algebra generated by the vectors v of the Euclidean vector-space R^3 with the positive-definite metric $\|v\| = (v^1)^2 + (v^2)^2 + (v^3)^2$, subject to the Clifford law in the form

$$v^2 = \|v\|. \tag{70}$$

We choose an orthonormal basis $\gamma_1, \gamma_2, \gamma_3$ for E^3, with many orthogonal (anticommuting) replicas $\gamma_k(n)$ ($k = 1, 2, 3; n = 1, 2, \ldots, N$). We associate the axes 1 and 2 with the real and imaginary axes of the complex plane, and represent $\breve{\imath}$ using N replicas

$$\gamma_{12}(n) := \gamma_1(n)\gamma_2(n) \tag{71}$$

of $\gamma_{12} := \gamma_1\gamma_2$:

$$\breve{\imath} := \frac{1}{N} \sum_{n=1}^{N} \gamma_{12}(n). \tag{72}$$

We also associate the 2 axis with q and the 1 axis with p, so that

$$
\begin{aligned}
L_1 &= \sum_{n=1}^{N} \gamma_{13}(n), \\
L_2 &= \sum_{n=1}^{N} \gamma_{23}(n).
\end{aligned}
\tag{73}
$$

This illustrates how Clifford algebra serves as the universal algebraic language of finite quantum theory.

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VITA

Mohsen Shiri-Garakani was born in 1970 in Tehran, Iran. He got his B.S. in Physics in 1993 from The Tehran Polytechnic University (Amir-Kabir University of Technology). He studied the effects of dark matter on the motion of spiral galaxies for his undergraduate thesis . He then moved to the United Stated to pursue his graduate studies. He got his M.S. in 1997 physics from the University of Akron, Ohio. His M.S. thesis was on developing a statistical theory of light rays in curved space-time manifolds. He then joined the quantum relativity group in the school of physics at Georgia Institute of Technology to work with Prof. David Finkelstein on developing a quantum theory for space-time. As a preliminary stage of such program and in his P.hD. thesis, Mohsen developed a finite quantum theory of the harmonic oscillator.

Besides physics and mathematics, Mohsen is deeply interested in philosophy, arts, poetry, linguistics and writing. He is especially interested in Persian traditional (classical) music and plays several musical instruments including Setar and Nay (string and wind Persian instruments respectively), and classical guitar. Mohsen is an accomplished Persian calligrapher.