

Hausdorff measure on o-minimal structures

Version 3.6

A. Fornasiero & E. Vasquez Rifo

November 9, 2010

Abstract

We introduce the Hausdorff measure for definable sets in an o-minimal structure, and prove the Cauchy-Crofton and co-area formulae for the o-minimal Hausdorff measure. We also prove that every definable set can be partitioned into “basic rectifiable sets”, and that the Whitney arc property holds for basic rectifiable sets.

Keywords: O-minimality, Hausdorff measure, Whitney arc property, Cauchy-Crofton, coarea.

MSC 2010: Primary: 03C64; Secondary: 28A75.

1 Introduction

Let K be an o-minimal structure expanding a field. We introduce, for every $e \in \mathbb{N}$, the e -dimensional Hausdorff measure for definable sets, which is the generalization of the usual Hausdorff measure for real sets [Morgan88]. We also show that every definable set can be partitioned into “basic e -rectifiable sets” (§3). Moreover, we generalize some well known result from geometric measure theory, such as the Cauchy-Crofton formula (which computes the Hausdorff measure of a set as the average number of points of intersection with hyperplanes of complementary dimension) and the co-area formula (a generalization of Fubini’s theorem), to the o-minimal context.

The measure defined in [BO04] is the starting point for our construction of the Hausdorff measure. A theorem of [BP98] allows us to prove that integration using the Berarducci-Otero measure satisfies properties analogous to the ones for integration over the reals (for example, the change of variable formula). If K is sufficiently saturated, the Berarducci-Otero measure of a bounded definable set X is $\mathcal{L}_{\mathbb{R}}(\text{st}(X))$, where $\mathcal{L}_{\mathbb{R}}$ is the Lebesgue measure

arXiv:1011.1629v1 [math.LO] 7 Nov 2010

and st is the standard-part map. However, the naive definition of Hausdorff measure given by

$$\mathcal{H}^e(X) := \mathcal{H}_{\mathbb{R}}^e(\text{st}(X)) \quad (1)$$

does not work (because the resulting “measure” is not additive: see Example 5.8). The correct definition for the e -dimensional Hausdorff measure is defining it first for basic e -rectifiable sets via (1), and then extending it to definable sets by using a partition into basic e -rectifiable pieces. Such a partition is obtained by using partitions into M_n -cells ([K92], [P08], [VR06]), a consequence of which is the Whitney arc property for basic e -rectifiable sets (§4).

2 Lebesgue measure on o-minimal structures

The definitions of measure theory are taken from [Halmos50].

Let $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ be the extended real line. Let K be a \aleph_1 -saturated o-minimal structure, expanding a field. Let \mathring{K} be the set of finite elements of K . Let $\text{st} : K^n \rightarrow \bar{\mathbb{R}}^n$ be the function mapping \bar{x} to the n -tuple of standard parts of the components of \bar{x} .

For every $n \in \mathbb{N}$, let $\mathcal{L}_{\mathbb{R}}^n$ be the n -dimensional Lebesgue measure (on \mathbb{R}^n). If n is clear from context we drop the superscript. Let \mathcal{L}_1^n be the o-minimal measure on \mathring{K}^n defined in [BO04]. More precisely, \mathcal{L}_1^n is a measure on the σ -ring R_n generated by the definable subsets of \mathring{K}^n ; thus, $(\mathring{K}^n, R_n, \mathcal{L}_1^n)$ is a measure space. Moreover, since $\mathring{K}^n \in R_n$, R_n is actually a σ -algebra.

Notice that \mathcal{L}_1^n can be extended in a natural way to a measure \mathcal{L}_2^n on the σ -ring \mathcal{B}_n generated by the definable subsets of K^n of finite diameter. Finally, we denote by \mathcal{L}^n the completion of \mathcal{L}_2^n , and if n is clear from context we drop the superscript. Notice that the σ -ring \mathcal{B}_n is not a σ -algebra.

Remark 2.1 ([BO04, Thm. 4.3]). If $C \subseteq \mathring{K}^n$ is definable, then $\mathcal{L}^n(C)$ is the Lebesgue measure of $\text{st}(C)$.

Definition 2.2. For $A \subseteq K^n$ and $f : K^n \rightarrow K^m$ we define $\text{st}(f) : A \rightarrow \bar{\mathbb{R}}^m$ by $\text{st}(f)(x) = \text{st}(f(x))$.

Remark 2.3. If $A \subseteq \mathring{K}^n$ and $f : A \rightarrow K$ are definable, then $\text{st}(f)$ is an \mathcal{L}^n -measurable function.

Definition 2.4. Let $A \subseteq \mathring{K}^n$ and $f : A \rightarrow K$ be definable. If $\text{st}(f)$ is \mathcal{L}^n -integrable we will denote its integral by

$$\int_A f \, d\mathcal{L}^n; \quad \int_A f(x) \, dx; \quad \int_A f(x) \, d\mathcal{L}^n(x) \quad \text{or} \quad \int_A f.$$

Remark 2.5. If $A \subseteq \mathring{K}^n$ and $f : A \rightarrow \mathring{K}$ are definable, then $\text{st}(f)$ is \mathcal{L} -integrable.

Let \mathbb{R}_K be the structure on \mathbb{R} generated by the sets of the form $\text{st}(U)$, where U varies among the definable subsets of K^n . By [BP98], \mathbb{R}_K is o-minimal.

Remark 2.6. Let $U \subseteq \mathring{K}^n$ be definable. Then, $\dim(\text{st}(U)) \leq \dim(U)$.

Proof. Let $\dim(U) = d$. After a cell decomposition, we can assume that U is the graph of a definable continuous function $f : V \rightarrow \mathring{K}^{n-d}$, with $V \subset \mathring{K}^d$ open cell. We can then conclude by applying the method in [HPP08, Lemma 10.3]. \square

Definition 2.7. A function f is Lipschitz if there is $C \in \mathring{K}$ such that, for all $x, y \in \text{dom}(f)$, we have $|f(x) - f(y)| < C|x - y|$ (notice the condition on C being finite). An invertible function f is bi-Lipschitz if both f and f^{-1} are Lipschitz.

Remark 2.8. Let $U \subset \mathring{K}^n$ and $f : U \rightarrow \mathring{K}$ be definable, with $f \geq 0$. Then,

$$\int_U f \, d\mathcal{L}^n = \mathcal{L}^{n+1}(\{\langle \bar{x}, y \rangle \in U \times K : 0 \leq y \leq f(\bar{x})\}).$$

Lemma 2.9 (Change of variables). *Let $U, V \subseteq \mathring{K}^n$ be open and definable, and let $A \subseteq U$ be definable. Let $f : U \rightarrow V$ be definable and bi-Lipschitz and $g : V \rightarrow \mathring{K}$ be definable, then*

$$\int_{f(A)} g = \int_A |\det Df| g \circ f.$$

Before proving the above lemma, we need some preliminary definitions and results.

Lemma 2.10. *Let $U \subset \mathring{K}^n$ be open and let $f : U \rightarrow \mathring{K}$ be definable. Then there is a \mathbb{R}_K -definable function $\bar{f} : C \rightarrow \mathbb{R}$, where $C \subset \text{st}(U)$ is an open set, such that*

- i) $E := (\text{st}(U) \setminus C) \cup (C \cap \text{st}(K^n \setminus U))$ is $\mathcal{L}_{\mathbb{R}}^n$ -negligible (and, therefore, $\text{st}^{-1}(E)$ is \mathcal{L}^n -negligible).
- ii) f and \bar{f} are \mathcal{C}^1 on $U \setminus \text{st}^{-1}(E)$ and C , respectively.
- iii) For every $x \in U$ with $\text{st}(x) \in C$ we have $\text{st}(f(x)) = \bar{f}(\text{st}(x))$. Moreover, Df is finite and $D(\bar{f})(\text{st } x) = \text{st}(Df(x))$.

iv)

$$\int_U f = \int_C \bar{f}.$$

Proof. By cell decomposition, we may assume that f is a function of class \mathcal{C}^1 , and that U is an open cell. Since $\dim(\Gamma(f)) = n$, we have, by Remark 2.6, $\dim(\text{st}(\Gamma(f))) \leq n$. By cell decomposition, there is an \mathbb{R}_K -definable, closed, negligible set $E \subset \text{st}(U)$, and definable functions $g_k : \text{st}(U) \setminus E \rightarrow \mathbb{R}$ of class \mathcal{C}^1 for $k = 1, \dots, r$ such that $\text{st}(\Gamma(f)) \cap ((\text{st}(U) \setminus E) \times \mathbb{R})$ is the union of the graphs of the functions g_i . We claim that $r = 1$:

In fact, if g_1, g_2 are two different such functions, and say $g_1 < g_2$, then for some $x \in \text{st}(U)$ we have $\langle x, g_1(x) \rangle, \langle x, g_2(x) \rangle \in \text{st}(\Gamma(f))$. Since f is continuous, $\{\langle x, y \rangle : y \in (g_1(x), g_2(x))\} \subset \text{st}(\Gamma(f))$. On the other hand, $\{\langle x, y \rangle : \langle x, y \rangle \in \text{st}(\Gamma(f))\}$ is the finite set $\{\langle x, g_1(x) \rangle, \dots, \langle x, g_r(x) \rangle\}$, absurd.

By [HPP08, Theorem 10.4], after enlarging E by a negligible set, we obtain i).

Let $\bar{f} := g_1$. ii) holds, and for every $x \in U$ with $\text{st}(x) \in C$ we have $\text{st}(f(x)) = \bar{f}(\text{st}(x))$. The equality of the integrals in iv) follows from Remark 2.8. To obtain the second part of iii) we will enlarge E by a negligible set. For $i = 1, \dots, n$ let

$$E_i := \text{st}(\{x \in U : \frac{\partial f}{\partial x_i}(x) \notin \overset{\circ}{K}\}).$$

By [BP98], E_i is \mathbb{R}_K -definable. If $\dim(E_i) = n$, then E_i contains an open ball. This contradicts Lemma 2.5 of [BO04] by which every definable, one variable function into $\overset{\circ}{K}$ has finite derivative except on $\text{st}^{-1}(A)$, for a finite set A . It follows that each set E_i is negligible and therefore, after enlarging E , we may assume that $D(f)$ is finite on $U \setminus \text{st}^{-1}(E)$.

It remains to prove $D(\bar{f})(\text{st}x) = \text{st}(Df(x))$. As before, we will enlarge E by a negligible set. Let $V := \{x \in \mathbb{R}^n : D(\bar{f})(x) \neq \overline{Df}(x)\}$. The set V is \mathbb{R}_K -definable. If V is non-negligible, then it contains an open ball and therefore w.l.o.g. we may assume that V is an open ball centered at 0. We may also assume $f(0) = 0$. After subtracting from f a linear function, we can assume that $\frac{\partial f}{\partial x_i}(0) = 0$ and $\frac{\partial \bar{f}}{\partial x_i}(0) = 3\epsilon > 0$ for some index $i = 1, \dots, n$. Therefore, on a smaller neighborhood of 0, we have $\frac{\partial f}{\partial x_i} < \epsilon$ and $\frac{\partial \bar{f}}{\partial x_i} > 2\epsilon$. Thus, for x along the x_i axis, $|f(x)| < |x|\epsilon$ and $\bar{f}(x) \geq 2|x|\epsilon$ contradicting the first part of iii), namely, $\text{st}(f(x)) = \bar{f}(x)$. We conclude that V is negligible. Let E' be a negligible set such that away from $\text{st}^{-1}(E')$ the equality $\text{st}(Df(x)) = \overline{Df}(\text{st}x)$ holds. Then away from $\text{st}^{-1}(V \cup E')$ we

have $\text{st}(Df(x)) = \overline{Df}(\text{st}(x)) = D\overline{f}(\text{st}(x))$ as wanted. By cell decomposition, E can be further enlarged so that C is open. \square

Remark 2.11. If $f^{-1}(A)$ is negligible whenever A is, then, outside a negligible closed set, $\overline{(f \circ g)} = \overline{f} \circ \overline{g}$.

Proof of Lemma 2.9. The fact that f is bi-Lipschitz implies that \overline{f} is injective (since it is also bi-Lipschitz).

Claim 1. Let $C \subset \text{st}(V)$ be Lebesgue measurable. Then,

$$\mathcal{L}^n(C) = \int_{(\text{st } f)^{-1}(C)} \text{st}(|\det Df|).$$

In fact, by the change of variables formula (on the reals!) and Lemma 2.10,

$$\mathcal{L}^n(C) = \int_{\overline{f}^{-1}(C)} |\det D\overline{f}| = \int_{(\text{st } f)^{-1}(C)} \text{st}(|\det Df|).$$

Claim 2. Let $h : V \rightarrow \overline{\mathbb{R}}$ be an integrable function. Then,

$$\int_V h = \int_U \text{st}(|\det Df|) h \circ f.$$

Claim 1 implies that the statement is true if h is a simple function. By continuity, the statement is true for any integrable function h .

In particular, we can apply Claim 2 to the function

$$h : x \mapsto \begin{cases} \text{st}(g(x)) & \text{if } x \in f(A), \\ 0 & \text{otherwise,} \end{cases}$$

and obtain the conclusion. \square

Lemma 2.12 (Fubini's theorem). \mathcal{L}^{n+m} is the completion of the product measure $\mathcal{L}^n \times \mathcal{L}^m$. Therefore, if D is the interval $[0, 1] \subset K$ and given $f : D^{n+m} \rightarrow D$ definable,

$$\int_{D^{n+m}} f(x, y) \, d\mathcal{L}^{n+m}(x, y) = \int_{D^m} \int_{D^n} f(x, y) \, d\mathcal{L}^m(x) \, d\mathcal{L}^n(y).$$

Proof. Follows from the definition of \mathcal{L}^n in [BO04]. \square

2.1 Measure on semialgebraic sets

Definition 2.13. We say that $E \subseteq K^n$ is \emptyset -semialgebraic if E is definable without parameters in the language of pure fields. If $E \subseteq K^n$ is \emptyset -semialgebraic we denote the subset of \mathbb{R}^n defined by the same formula that defines E by $E_{\mathbb{R}}$.

Remark 2.14. Let $E \subseteq \mathring{K}^n$ be \emptyset -semialgebraic. Then, $\text{st}(E) = \overline{E_{\mathbb{R}}}$.

Let $E \subseteq K^n$ be closed and \emptyset -semialgebraic submanifold. Working in local charts, from [BO04] one can easily define a measure \mathcal{L}^E on the σ -ring generated by the definable subsets of E of bounded diameter. We will denote in the same way the completion of \mathcal{L}^E . Notice that $\mathcal{L}^{K^n} = \mathcal{L}^n$.

Remark 2.15. Let E be a closed, \emptyset -semialgebraic submanifold of K^n of dimension e , $F := \text{st}(E)$, and $C \subseteq E$ be definable and bounded. Then, $\mathcal{L}^E(C) = \mathcal{L}_{\mathbb{R}}^F(\text{st}(C))$, where $\mathcal{L}_{\mathbb{R}}^F$ is the e -dimensional Hausdorff measure on F .

One could also take the above remark as the definition of \mathcal{L}^E on $E \cap \mathring{K}^n$.

3 Rectifiable partitions

Theorem 3.8 shows that every definable set $A \subset \mathring{K}^n$ has a partition into definable sets which are M_n -cells after an orthonormal change of coordinates (where $M_n \in \mathbb{Q}$ depends only on n). In [P08], the author shows that a permutation of the coordinates suffices. The proof of 3.8 follows closely that of [K92]. The partition in 3.8 is then used in Corollary 3.11 to show that definable sets have a rectifiable partition.

Definition 3.1. Let $L : V \rightarrow W$ be a linear map between normed K -vector spaces. The norm of L is given by

$$\|L\| := \sup_{|v|=1} |L(v)|.$$

For V, W in the Grassmannian of e -dimensional linear subspaces of K^n , namely $\mathcal{G}_e(K^n)$, let π_V and $\pi_W \in \text{End}_K(K^n)$ be the orthogonal projections onto V and W respectively. In this way we have a canonical embedding $\mathcal{G}_e(K^n) \subset \text{End}_K(K^n)$. The **distance function** on the Grassmannian is given by the inclusion above:

$$\delta(V, W) := \|\pi_V - \pi_W\|.$$

For P in $\mathcal{G}_1(K^n)$ and $X \in \mathcal{G}_k(K^n)$, define

$$\delta(P, X) := |v - \pi_X(v)|,$$

where π_X is the orthogonal projection onto X , and v is a generator of P of norm 1. Note that $\delta(P, X) = 0$ if and only if $P \subset X$, $0 \leq \delta(P, X) \leq 1$ and $\delta(P, X) = 1$ if and only if $P \perp X$. Note also that $\delta(P, X)$ is the definable analogous of the sine of the angle between P and X .

Lemma 3.2. *Let $n \in \mathbb{N}_{>0}$. Then there exists an $\epsilon_n \in \mathbb{Q}_{>0}$, $\epsilon_n < 1$, such that for any $X_1, \dots, X_{2n} \in \mathcal{G}_{n-1}(K^n)$, there is a line $P \in \mathcal{G}_1(K^n)$ such that whenever $Y_1, \dots, Y_{2n} \in \mathcal{G}_{n-1}(K^n)$ and*

$$\begin{aligned} \delta(X_i, Y_i) < \epsilon_n, \quad i = 1, \dots, 2n, \quad \text{then} \\ \delta(P, Y_i) > \epsilon_n, \quad i = 1, \dots, 2n. \end{aligned}$$

Proof. For $\epsilon > 0$ define $S_i(\epsilon) = \{v \in S^{n-1} : |v - \pi_{X_i}(v)| \leq 2\epsilon\}$. If $K = \mathbb{R}$, let $\epsilon_n \in \mathbb{Q}_{>0}$ be small enough so that $2n \text{Vol}(S_1(\epsilon_n)) < \text{Vol}(S^{n-1})$, where Vol is the measure $\mathcal{L}^{S^{n-1}}$ defined in §2.1. Then

$$\text{Vol}\left(\bigcup_{i=1}^{2n} S_i(\epsilon_n)\right) \leq 2n \text{Vol}(S_1(\epsilon_n)) < \text{Vol}(S^{n-1})$$

and therefore

$$\bigcup_{i=1}^{2n} S_i(\epsilon_n) \neq S^{n-1}. \quad (2)$$

The same ϵ_n will necessarily satisfy (2) for any field K containing \mathbb{R} .

Now, we choose

$$v \in S^{n-1} - \bigcup_{i=1}^{2n} S_i(\epsilon_n)$$

and let $P := \langle v \rangle$. Then

$$\delta(P, Y_i) = |v - \pi_{Y_i}v| \geq |v - \pi_{X_i}v| - |\pi_{X_i}v - \pi_{Y_i}v| > \epsilon_n. \quad \square$$

Definition 3.3. Let $\epsilon > 0$. A definable embedded submanifold M of K^n is ϵ -flat if for each $x, y \in M$ we have $\delta(TM_x, TM_y) < \epsilon$, where TM_x denotes the tangent space to M at x .

Lemma 3.4. *Let $A \subset K^n$ be a definable submanifold of dimension e and $\epsilon \in \mathbb{R}_{>0}$. Then there is a cell decomposition $A = \bigcup_{i=0}^k A_i$ of A such that for every i we have either $\dim(A_i) < \dim(A)$ or A_i is an ϵ -flat submanifold of K^n .*

Proof. Cover $\mathcal{G}_e(K^n)$ by a finite number of balls B_i of radius $\epsilon/2$; and consider the Gauss map $G : A \rightarrow \mathcal{G}_e(K^n)$ taking an element a of A to TA_a . Take a cell decomposition of K^e compatible with A and partitioning each $G^{-1}(B_i)$. Then the e -dimensional cells contained in A are ϵ -flat. \square

Lemma 3.5. *Let $\epsilon \in \mathbb{Q}_{>0}$, and let $A \subset \overset{\circ}{K}^n$ be an open definable set. Then there are open, pairwise disjoint cells $A_1, \dots, A_p \subset A$ such that*

- (i) $\dim(A - \bigcup A_i) < n$.
- (ii) *For each i , there are definable, pairwise disjoint sets B_1, \dots, B_k (with k depending on i) such that*
 - (a) $k \leq 2n$;
 - (b) *each B_j is a definable subset of ∂A_i and an ϵ -flat, $(n-1)$ -dimensional, \mathcal{C}^1 -submanifold of K^n ;*
 - (c) $\dim(\partial A_i - \bigcup_{j=1}^k B_j) < n - 1$.

Proof. By induction on n . The lemma is clear for $n = 1$. Assume that $n > 1$ and the lemma holds for smaller values of n .

Take a cell decomposition of \overline{A} compatible with A into \mathcal{C}^1 -cells. Let C be an open cell in this decomposition; it suffices to prove the lemma for C . Note that $C = (f, g)_X$, where X is an open cell in K^{n-1} and f, g are definable \mathcal{C}^1 -functions on X . Take finite covers of $\Gamma(f)$ and $\Gamma(g)$ by open, definable sets U_i and V_j , respectively, such that each $U_i \cap \Gamma(f)$ and each $V_j \cap \Gamma(g)$ is ϵ -flat (to do this, take a finite cover of the Grassmannian by ϵ -balls and pull it back via the Gauss maps for $\Gamma(f)$ and $\Gamma(g)$). The collection of all sets $\pi(U_i) \cap \pi(V_j)$ is an open cover \mathcal{O} of X . By the cell decomposition theorem, there is a \mathcal{C}^1 -cell decomposition of X partitioning each set in \mathcal{O} . Let S be an open cell in this decomposition, and let $C_0 := (f, g)_S$. It suffices to prove the lemma for C_0 . By the inductive hypothesis, we can find $A'_1, \dots, A'_p \subset S$ and $B'_1, \dots, B'_k \subset \partial A'_i$ satisfying the conditions (i) and (ii) above (with n replaced by $n - 1$). Define

$$A_i := (f, g)_{A'_i}, \quad i = 1, \dots, p.$$

Then $\dim(C_0 - \bigcup_{i=1}^p A_i) < n$. For $j = 1, \dots, k$, the set $(B'_j \times K) \cap \partial A_i$ is definable. Take a \mathcal{C}^1 -cell decomposition of this set, and let B_j be the union of the $(n - 1)$ -dimensional cells in this decomposition (note that B_j may be empty). Then B_j is an ϵ -flat \mathcal{C}^1 -submanifold of K^n and

$$\dim(((B'_j \times K) \cap \partial A_i) - B_j) < n - 1.$$

Define $B_{k+1} := \Gamma(f|_{A'_i})$ and $B_{k+2} := \Gamma(g|_{A'_i})$; by construction these are ϵ -flat. It is routine to see that $\partial A_i \subset B_{k+1} \cup B_{k+2} \cup (\partial A'_i \times K)$. Thus

$$\begin{aligned} \partial A_i - \bigcup_{j=1}^{k+2} B_j &\subset ((\partial A'_i \times K) \cap \partial A_i) - \bigcup_{j=1}^k B_j \\ &= (\bigcup_{j=1}^k ((B'_j \times K) \cap \partial A_i) \cup E) - \bigcup_{j=1}^k B_j \\ &\subset \bigcup_{j=1}^k (((B'_j \times K) \cap \partial A_i) - B_j) \cup E, \end{aligned}$$

where E is a definable set with $\dim(E) < n - 1$. Therefore $\dim(\partial A_i - \bigcup_{j=1}^{k+2} B_j) < n - 1$. Since $k \leq 2(n - 1)$, we get $k + 2 \leq 2n$ and the lemma is proved. \square

Definition 3.6. Let $U \subseteq K^n$ be open and let $f : U \rightarrow K^m$ be definable. Given $0 < M \in K$, we say that f is an M -function if $|Df| \leq M$. We say that f has finite derivative if $|Df|$ is finite.

Notice that, by ω -saturation of K , if f is definable and has finite derivative, then it is an M -function for some finite M .

Let $M \in K_{>0}$. An M -cell is a \mathcal{C}^1 -cell where the \mathcal{C}^1 functions that define the cell are M -functions. More precisely:

Definition 3.7. Let (i_1, \dots, i_m) be a sequence of zeros and ones, and $M \in K_{>0}$. An (i_1, \dots, i_m) - M -cell is a subset of K^m defined inductively as follows:

- (i) A (0)- M -cell is a point $\{r\} \subset K$, a (1)- M -cell is an interval $(a, b) \subset K$, where $a, b \in K$.
- (ii) An $(i_1, \dots, i_m, 0)$ - M -cell is the graph $\Gamma(f)$ of a definable M -function $f : X \rightarrow K$ of class \mathcal{C}^1 , where X is an (i_1, \dots, i_m) - M -cell; an $(i_1, \dots, i_m, 1)$ - M -cell is a set

$$(f, g)_X := \{(x, r) \in X \times K : f(x) < r < g(x)\},$$

where X is an (i_1, \dots, i_m) - M -cell and $f, g : X \rightarrow K$ are definable M -functions of class \mathcal{C}^1 on X such that for all $x \in X$, $f(x) < g(x)$.

Theorem 3.8. Let $A \subset \overset{\circ}{K}^n$ be definable. Then there are definable, pairwise disjoint sets A_i , $i = 1, \dots, s$, such that $A = \bigcup_i A_i$ and for each A_i , there is a change of coordinates $\sigma_i \in O_n(K)$ such that $\sigma_i(A_i)$ is an M_n -cell, where $M_n \in \mathbb{Q}_{>0}$ is a constant depending only on n .

Proof. We will make use of the following fact:

Let $\epsilon \in [0, 1]$, $P \in \mathcal{G}_1(K^n)$, $X \in \mathcal{G}_k(K^n)$ and $w \in X$ be a unit vector. Suppose $\delta(P, X) > \epsilon$. If $\pi_P(w) \geq 1/2$, where π_P is the orthogonal projection onto P , then

$$|\pi_P(w) - w| \geq |\pi_P(w) - \pi_X(\pi_P(w))| > |\pi_P(w)|\epsilon \geq 1/2\epsilon.$$

If $\pi_P(w) < 1/2$, then $|w| \leq |\pi_P(w)| + |\pi_p(w) - w| \leq 1/2 + |\pi_p(w) - w|$. In either case, we have

$$|\pi_P(w) - w| \geq \frac{1}{2}\epsilon. \quad (3)$$

We prove the theorem by induction on n ; for $n = 1$ the theorem is clear. We assume that $n > 1$ and that the theorem holds for smaller values of n . We also proceed by induction on $d := \dim(A)$. It's clear for $d = 0$; so we assume that $d > 0$ and the theorem holds for definable bounded subsets B of K^n with $\dim(B) < d$.

Case I: $\dim(A) = n$. In this case A is an open, bounded, definable subset of K^n , so by using the inductive hypothesis and Lemma 3.5, we can reduce to the case where there are pairwise disjoint, definable $B_1, \dots, B_k \subset \partial A$ such that $k \leq 2n$, $\dim(\partial A - \bigcup_{j=1}^k B_j) < n-1$ and each B_j is an ϵ_n -flat submanifold, where ϵ_n is as in Lemma 3.2. By Lemma 3.2, there is a hyperplane L such that for each B_j and all $x \in B_j$, we have $\delta(L^\perp, T_x B_j) > \epsilon_n$. Take a cell decomposition \mathcal{B} of K^n , with respect to orthonormal coordinates in the L, L^\perp axis, partitioning each B_j . Let

$$\mathcal{S} := \{C \in \mathcal{B} : \dim(C) = n-1, C \subset \bigcup_{j=1}^k B_j\}$$

and note that $\dim(\partial A \setminus \bigcup_{C \in \mathcal{S}} C) < n-1$. Furthermore,

$$\text{BAD} := \{x \in A : \pi_L^{-1}(\pi_L(x)) \cap \partial A \not\subset \bigcup_{C \in \mathcal{S}} C\}$$

has dimension smaller than n . Let U_1, \dots, U_l be the elements of $\{\pi_L(C) : C \in \mathcal{S}\}$. Then the set

$$\{x \in A : x \notin \pi_L^{-1}(\bigcup_{i=1}^l U_i)\}$$

is contained in **BAD**, and therefore has dimension smaller than n .

By using the inductive hypothesis, we only need to find the required partition for each of the sets $A \cap \pi_L^{-1}(U_i)$, $i = 1, \dots, l$. Fix $i \in \{1, \dots, l\}$ and let $U := U_i$, $A' := A \cap \pi_L^{-1}(U)$. Take $C \in \mathcal{S}$ with $\pi_L(C) = U$. Then $C = \Gamma(\phi)$ for a definable \mathcal{C}^1 -map $\phi : U \rightarrow L^\perp$ and for all $x \in C$,

$$T_x C = \{(v, D\phi(v)) : v \in T_{\pi_L(x)} U\}.$$

Let $v \in T_{\pi_L(x)} U$ be a unit vector; since $\delta(L^\perp, T_x C) > \epsilon_n$ and $|(v, D\phi(v))| = \sqrt{1 + |D\phi(v)|^2}$, it follows from equation (3) that

$$\frac{1}{2}\epsilon_n \leq \frac{1}{\sqrt{1 + |D\phi(v)|^2}} |\pi_{L^\perp}((v, D\phi(v))) - (v, D\phi(v))| = \frac{1}{\sqrt{1 + |D\phi(v)|^2}} |v|.$$

Therefore,

$$|D\phi(v)| \leq \sqrt{\frac{4}{\epsilon_n^2} - 1}.$$

Let $M_n \in \mathbb{Q}$ be bigger than $\max\left\{M_{n-1}, \sqrt{\frac{4}{\epsilon_n^2} - 1}\right\}$.

We have proved that for each $C_j \in \mathcal{S}$ with $\pi_L(C_j) = U$ there is a definable \mathcal{C}^1 -map $\phi_j : U \rightarrow K$, such that $|D\phi_j| < M_n$ and $C_j = \Gamma(\phi_j)$.

By the inductive hypothesis, there is a partition \mathcal{P} of U such that each piece $P \in \mathcal{P}$ is a M_{n-1} -cell after a change of coordinates of L . We have

$$A' = \coprod_{\substack{P \in \mathcal{P} \\ (\phi_r, \phi_s)_P \subset A'}} (\phi_r, \phi_s)_P,$$

and $(\phi_r, \phi_s)_P$ is a M_n -cell after a coordinate change.

Case II: $\dim(A) < n$. In this case, by Lemma 3.4, we can partition A into cells which are ϵ_n -flat. Therefore we may assume that A is an ϵ_n -flat submanifold, where ϵ_n is as in Lemma 3.2. As in case I, there is a hyperplane L such that A is the graph of a function $f : U \rightarrow K$, $U \subset L$ and $|Df| < M_n$. By the inductive hypothesis, we can partition U into M_{n-1} -cells. The graphs of f over the cells in this partition give the required partition of A . \square

Definition 3.9. Let $A \subseteq K^n$ and $e \leq n$. A is basic e -rectifiable with bound M if, after a permutation of coordinates, A is the graph of an M -function $f : U \rightarrow K^{n-e}$, where $U \subset K^e$ is an open M -cell for some finite M .

Lemma 3.10. Let $A \subset \overset{\circ}{K}^n$ be an M -cell of dimension e . Then, A is a basic e -rectifiable set, and the bound of A can be chosen depending only on M and n .

Proof. We proceed by induction on n . If $n = 0$ or $n = 1$ the result is trivial, so assume $n \geq 2$. By definition, there exists an M -cell $B \subset \overset{\circ}{K}^{n-1}$ such that

- (1) either $A = \Gamma(g)$ for some M -function $g : B \rightarrow \overset{\circ}{K}$, or
- (2) $A = (g, h)_B$ for some M -functions $g, h : B \rightarrow \overset{\circ}{K}$, with $g < h$.

By inductive hypothesis, there exists an open L -cell $C \subset K^d$ (for some d and some $L \geq M$ depending only on M and on n), and an L -function $f : C \rightarrow K^{n-1-d}$, such that $B = \Gamma(f)$.

In case (1) $d = e$. Define $l : C \rightarrow K^{n-e}$ by $l(x) = \langle f(x), g(x, f(x)) \rangle$. It is easy to see that l is an L' -function for some L' depending only on M and n , and that $A = \Gamma(l)$.

In case (2), $d = e - 1$. Define $\tilde{g} := g \circ f$, $\tilde{h} := h \circ f$, and $\tilde{B} := (\tilde{g}, \tilde{h})_C$. Given $\langle \bar{x}, y \rangle \in \tilde{B}$, define $l(\bar{x}, y) := f(\bar{x})$. We have that \tilde{B} is an open e -dimensional L -cell, $l : \tilde{B} \rightarrow K^{n-e}$ is an L -function, and $A = \Gamma(l)$. \square

Corollary 3.11. *Let $A \subseteq K^n$ be definable of dimension at most e . Then there is a partition $A = \bigcup_{i=0}^k A_i$ such that $\dim(A_0) < e$ and A_i is a basic e -rectifiable set for $i > 0$. Moreover, the bounds of each A_i can be chosen to depend only on n (and not on A). We call (A_0, \dots, A_k) a basic e -rectifiable partition of A .*

Proof. Apply Theorem 3.8 and 3.10. \square

Notice that a similar result has also been proved in [PW06, Theorem 2.3] (where they also take arbitrarily small bounds): however, in [PW06] they don't require that the functions parametrizing the set A are injective (which is essential for our later uses).

4 Whitney decomposition

The fact that the functions that define an M -cell are actually Lipschitz function follows from the following property of M -cells:

Every pair of points x, y in an M -cell $C \subset K^n$ can be connected by a definable \mathcal{C}^1 curve $\gamma : [0, 1] \rightarrow C$ with $|\gamma'(t)| < N|x - y|$, where N is a constant depending only on M and n which is finite if M is (Lemma 4.3 or [VR06] 3.10 & 3.11).

The same property implies that a N -function f on an M -cell is Lipschitz where the Lipschitz constant is finite if M and N are (Corollary 4.5). This last property will be needed for defining Hausdorff measure.

Remark 4.1. Let $U \subset \mathring{K}^n$ be open and definable, and $f : U \rightarrow \mathring{K}$ be an M -function (for some finite M). It is not true in general that f is L -Lipschitz for some finite L : this is the reason why we needed to prove Theorem 3.8.

Definition 4.2. Let $A \subset K^n$, $B \subset K^m$ be definable sets. Let $\lambda \subset A \times ([0, 1] \times B) \subset K^n \times K^{1+m}$ be a definable set such that for every $x \in A$, the fiber over x

$$\lambda_x := \{y \in [0, 1] \times B : \langle x, y \rangle \in \lambda\}$$

is a curve $\lambda_x : [0, 1] \rightarrow B$. We view λ as describing the family of curves $\{\lambda_x\}_{x \in A}$. Such a family is a definable family of curves (in B , parametrized by A).

An L -cell is an L -Lipschitz cell if the functions that define the L -cell are L -Lipschitz.

Lemma 4.3. *Fix $L \in K_{>0}$ and $n \in \mathbb{N}_{>0}$. Then, there is a constant $K(n, L) \in K_{>0}$ depending only on n and L , that is finite if L is, such that for every L -Lipschitz cell $C \subset K^n$ there is a definable family of curves $\gamma \subset C^2 \times ([0, 1] \times C)$ such that: For all $x, y \in C$, $\gamma_{x,y} : [0, 1] \rightarrow C$ is a C^1 -curve with*

- (i) $\gamma_{xy}(0) = x, \gamma_{xy}(1) = y$;
- (ii) $|\gamma'_{xy}(t)| \leq K(n, L)|x - y|$, for all $t \in [0, 1]$.

Proof. By induction on n . For $n = 1$ the lemma is clear. Take $n \geq 1$, and assume that the lemma holds for n . Let $C \subset K^{n+1}$ be an L -Lipschitz cell. Then $C = \Gamma(f)$ or $C = (g, h)_X$ for some L -Lipschitz cell $X \subset K^{n-1}$ and definable, C^1 , L -Lipschitz functions f, g, h with $g < h$, and $|Df|, |Dg|, |Dh| \leq L$. By induction, there are a constant $k := K(n-1, L)$ and a definable family of C^1 -curves β in X with the required properties. Let $\pi_n : K^{n+1} \rightarrow K^n$ be the projection onto the first n coordinates.

If $C = \Gamma(f)$, we lift β to C via f : fix $x, y \in C$ and let $\gamma_{x,y}(t) := (\alpha(t), f(\alpha(t)))$, where for all $t \in [0, 1]$ $\alpha(t) := \beta_{\pi_n(x), \pi_n(y)}(t)$. Then we have $|\gamma'_{xy}(t)| \leq (1 + L)k|x - y|$.

If $C = (g, h)_X$, we lift β as follows: Fix $x, y \in C$ and let $\alpha := \beta_{\pi_n(x), \pi_n(y)}$. Let $\pi : K^{n+1} \rightarrow K$ be the projection onto the last coordinate and take $u, v \in (0, 1)$ with

$$\begin{aligned}\pi(x) &= uh(\alpha(0)) + (1 - u)g(\alpha(0)) \\ \pi(y) &= vh(\alpha(1)) + (1 - v)g(\alpha(1)).\end{aligned}$$

Let $l(t) := tv + (1 - t)u$, for $t \in [0, 1]$. We define $\gamma_{x,y}(t) := (\alpha(t), l(t)h(\alpha(t)) + (1 - l(t))g(\alpha(t)))$, and note that

$$|\gamma'_{xy}(t)| \leq k|x - y| + |(v - u)(h(\alpha(t)) - g(\alpha(t)))| + 2Lk|x - y|,$$

since $l(t), 1 - l(t)$ are between 0 and 1 and $|Dh(\alpha'(t))|, |Dg(\alpha'(t))| \leq L|\alpha'(t)|$. Let $f := h - g$. We want to bound $|(v - u)f(\alpha(t))|$, which equals

$$|\pi y - \pi x - v(f(\alpha(1)) - f(\alpha(t))) + u(f(\alpha(0)) - f(\alpha(t))) + g(\alpha(0)) - g(\alpha(1))|.$$

But

$$|f(\alpha(1)) - f(\alpha(t))| \leq L|\alpha(1) - \alpha(t)| = L|1 - t| \left| \frac{\alpha(1) - \alpha(t)}{1 - t} \right| \leq L|\alpha'(t_0)|$$

for some t_0 between t and 1. Similarly, $|f(\alpha(0)) - f(\alpha(t))| \leq L|\alpha'(t_1)|$, for some t_1 between t and 1. Since $u, v \in [0, 1]$, we get

$$|(v - u)f(\alpha(t))| \leq |\pi y - \pi x| + 2Lk|x - y| + L|x - y|;$$

thus $|\gamma'_{xy}(t)| \leq K(n, L)|x - y|$ for some constant $K(n, L)$ depending only on n and L which is finite if L is. The collection of the curves γ_{xy} for $x, y \in C$ constitutes the required family of curves. \square

Theorem 4.4. *Let $L > 0$, and let $C \subset K^n$ be an L -cell. Then C is a $k(n, L)$ -Lipschitz cell, where $k(n, L)$ depends only on n and L , and is finite if L is.*

Proof. By induction on n ; the theorem is clear for $n = 1$. Assume that $n > 1$ and that the theorem holds for $n - 1$. Then $C = \Gamma(f)$ or $C = (g, h)_X$, where $X \subset K^{n-1}$ is a $k(n - 1, L)$ -Lipschitz cell and f, g, h are \mathcal{C}^1 -functions on X such that $|Df|, |Dg|, |Dh| \leq L$. We need to show that f, g, h are Lipschitz.

Since X is a k -Lipschitz cell, $k := k(n - 1, L)$, it follows from Lemma 4.3 that there is a constant $K(n - 1, k)$ such that whenever $x, y \in X$, there is a definable, \mathcal{C}^1 -curve γ joining x and y with $|\gamma'(t)| \leq K(n - 1, k)|x - y|$ for all $t \in [0, 1]$. Let $g := f \circ \gamma$, and let $t_0 \in (0, 1)$ be such that

$$|f(x) - f(y)| = |g'(t_0)| = |Df(\gamma'(t_0))| \leq L|\gamma'(t_0)| \leq LK(n - 1, k)|x - y|.$$

Thus f is $LK(n - 1, k)$ -Lipschitz. We set $k(n, L) := LK(n - 1, k)$. \square

Corollary 4.5. *Let C be an M -cell and f be a definable M -function. Then f is Lipschitz, and with finite Lipschitz constant if M is finite.*

Proof. By Theorem 4.4, C has a definable family of curves as in Lemma 4.3. The result therefore follows from the mean value theorem. \square

Definition 4.6. A definable set $A \subset K^n$ satisfies the Whitney arc property if there is a constant $K \in \mathring{K}_{>0}$ such that for all $x, y \in A$ there is a definable curve $\gamma : [0, 1] \rightarrow A$ with $\gamma(0) = x$, $\gamma(1) = y$ and $\text{length}(\gamma) := \int_0^1 |\gamma'| \leq K|x - y|$.

Lemma 4.7. *Let $C \subset \mathring{K}^n$ be an M -cell, $M \in \mathring{K}$. Then, C satisfies the Whitney arc property.*

Proof. It follows from Theorem 4.4 and Lemma 4.3. \square

Theorem 4.8. *Let $A \subset \mathring{K}^n$ be definable. Then, A can be partitioned into finitely many definable sets, each of them satisfying the Whitney arc property.*

Proof. This follows from Lemma 4.7, Theorem 3.8 and the fact that the Whitney arc property is invariant under an orthonormal change of coordinates. \square

5 Hausdorff measure

For an introduction to geometric measure theory, and in particular to the Hausdorff measure, see [Morgan88].

Definition 5.1. Let $U \subseteq K^n$ be open and let $f : U \rightarrow \mathring{K}^m$ be a definable function. If $a \in U$, $e \leq n$ and M is the set of the $e \times e$ minors of $Df(a)$ we define

$$J_e f(a) = \begin{cases} +\infty & \text{if } f \text{ is not differentiable at } a \text{ or } \text{rank}(Df(a)) > e, \\ \sqrt{\sum_{m \in M} m^2} & \text{otherwise;} \end{cases}$$

(cf. [Morgan88, §3.6]).

Notice that if $e = n = m$, then $J_n f = |\det(Df)|$.

Definition 5.2. Let $U \subseteq \mathring{K}^e$ be an open M -cell for some $M \in \mathbb{N}$, and let $f : U \rightarrow \mathring{K}^m$ be a definable function with finite derivative. Let $F : U \rightarrow \mathring{K}^{m+e}$ be $F(x) := \langle x, f(x) \rangle$ and $C := \Gamma(f) = F(U)$ (notice that C has bounded diameter). We define

$$\mathcal{H}^e(C) := \int_U J_e F \, d\mathcal{L}^e.$$

Lemma 5.3. *If $C \subseteq \mathring{K}^n$ is basic e -rectifiable, then $\mathcal{H}^e(C) = \mathcal{H}_{\mathbb{R}}^e(\text{st}(C))$, where $\mathcal{H}_{\mathbb{R}}^e$ is the e -dimensional Hausdorff measure on \mathbb{R}^n .*

Proof. Let $A \subseteq \mathring{K}^e$ and $f : A \rightarrow \mathring{K}^{n-e}$ be as in Definition 3.9, and $F : A \rightarrow \mathring{K}^n$ as in Definition 5.2. Let $B := \text{st}(A)$. Then, using the real Area formula [Morgan88],

$$\int_A J_e F \, d\mathcal{L}^e = \int_B J_e(\overline{F}) \, d\mathcal{L}_{\mathbb{R}}^e = \mathcal{H}_{\mathbb{R}}^e(\overline{F}(B)) = \mathcal{H}_{\mathbb{R}}^e(\text{st}(C)).$$

□

Definition 5.4. Let $A \subseteq \mathring{K}^n$ be definable of dimension at most e , and (A_0, \dots, A_k) be a basic e -rectifiable partition of A . Define

$$\mathcal{H}^e(A) := \sum_{i=1}^k \mathcal{H}^e(A_i),$$

where $\mathcal{H}^e(A_i)$ is defined using 5.2.

Lemma 5.5. *If A is as in the above definition, then $\mathcal{H}^e(A)$ does not depend on the choice of the basic e -rectifiable partition (A_0, \dots, A_k) .*

Proof. It suffices to prove the following: if C is a basic e -rectifiable set and (A_0, \dots, A_k) is a basic e -rectifiable partition of C , then $\mathcal{H}^e(C) = \sum_{i=1}^k \mathcal{H}^e(A_i)$, where $\mathcal{H}^e(C)$ and $\mathcal{H}^e(A_i)$ are defined using 5.2. For every $i = 1, \dots, n$ let U and V_i be M -cells, $f : U \rightarrow K^{n-e}$ and $g_i : V_i \rightarrow K^{n-e}$ be definable functions with finite derivative, σ_i be a permutation of variables of K^n , $F : K^e \rightarrow K^n$ defined by $F(x) := (x, f(x))$, and $G_i : K^e \rightarrow K^n$ defined by $G_i(x) = \sigma_i(x, g_i(x))$ such that $C = F(U)$ and $A_i = G_i(V_i)$. Define $U_i := F^{-1}(A_i) \subseteq U$, and $H_i := G_i^{-1} \circ F : U_i \rightarrow V_i$. Notice that each H_i is a bi-Lipschitz bijection, that U is the disjoint union of the U_i , and that $\dim(U_0) < e$. Hence,

$$\begin{aligned} \mathcal{H}^e(C) &= \int_U J_e F \, d\mathcal{L}^e = \sum_{i=1}^n \int_{U_i} J_e F \, d\mathcal{L}^e = \sum_{i=1}^n \int_{U_i} J_e(G_i \circ H_i) \, d\mathcal{L}^e = \\ &= \sum_{i=1}^n \int_{U_i} (J_e(G_i) \circ H_i) \cdot |\det(DH_i)| \, d\mathcal{L}^e = \sum_{i=1}^n \int_{V_i} J_e G_i \, d\mathcal{L}^e = \sum_{i=1}^n \mathcal{H}^e(A_i), \end{aligned}$$

where we used Lemma 2.9, the fact that each σ_i is a linear function with determinant ± 1 , and that $J_e(G \circ H) = (J_e(G) \circ H) \cdot |\det(DH)|$. \square

Lemma 5.6. \mathcal{H}^e does not depend on n . That is, let $m \geq n$, and $A \subset \mathring{K}^n$ definable, and $\psi : K^n \rightarrow K^m$ be the embedding $x \mapsto (x, 0)$. Then, $\mathcal{H}^e(A) = \mathcal{H}^e(\psi(A))$.

Proof. Obvious from the definition and Lemma 5.5. \square

Notice that $\mathcal{H}^0(C)$ is the cardinality of C .

It is clear that \mathcal{H}^e can be extended to the σ -ring generated by the definable subsets of K^n of finite diameter and dimension at most e ; we will also denote the completion of this extension by \mathcal{H}^e .

Lemma 5.7. \mathcal{H}^e is a measure on the σ -ring generated by the definable subsets of K^n of bounded diameter and dimension at most e .

Proof. Since K is \aleph_1 -saturated, it suffices to show that, for every A and B disjoint definable subsets of K^n of finite diameter and dimension at most e , $\mathcal{H}^e(A \cup B) = \mathcal{H}^e(A) + \mathcal{H}^e(B)$. But this follows immediately from Lemma 5.5. \square

Example 5.8. In Lemma 5.3, the assumption that C is basic e -rectifiable is necessary. For instance, take $\epsilon > 0$ infinitesimal, and X be the following subset of K^2

$$X := ([0, 1] \times \{0\}) \cup \{(x, y) : 0 \leq x \leq 1 \text{ \& } y = \epsilon x\}.$$

Then, $\text{st}(X) = [0, 1] \times \{0\}$, and thus $\mathcal{H}^1(X) = 2$, while $\mathcal{H}_{\mathbb{R}}^1(\text{st}(X)) = 1$. This is the source of complication in the theory, and one of the reasons why we had to wait until this section to introduce \mathcal{H}^e .

6 Cauchy-Crofton formula

Give $e \leq n$, define

$$\beta := \Gamma\left(\frac{e+1}{2}\right)\Gamma\left(\frac{n-e+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)^{-1}\pi^{-1/2}.$$

Definition 6.1. Let $\mathcal{AG}_e(K^n)$ be the Grassmannian of affine e -dimensional subspaces of K^n and let $\mathcal{AG}_e(\mathbb{R}^n)$ be the Grassmannian of affine e -dimensional subspaces of \mathbb{R}^n . Fix an embedding of $\mathcal{AG}_e(\mathbb{R}^n)$ into some \mathbb{R}^m , such that $\mathcal{AG}_e(\mathbb{R}^n)$ is a \emptyset -semialgebraic closed submanifold of \mathbb{R}^m , and the restriction to $\mathcal{AG}_e(\mathbb{R}^n)$ of the $\dim(\mathcal{AG}_e(\mathbb{R}^n))$ -dimensional Hausdorff measure coincides with the Haar measure on $\mathcal{AG}_e(\mathbb{R}^n)$.

Definition 6.2. Given $A \subseteq K^n$ and $E \in \mathcal{AG}_{n-e}(K^n)$, let $f_A(E) := \#(A \cap E)$.

Theorem 6.3 (Cauchy-Crofton Formula). *Let $A \subseteq \mathring{K}^n$ be definable of dimension e . Then,*

$$\mathcal{H}^e(A) = \frac{1}{\beta} \int_{\mathcal{AG}_{n-e}(K^n)} f_A \, d\mathcal{L}^{\mathcal{AG}_{n-e}(K^n)}.$$

We prove the theorem by reducing it to the known case of $K = \mathbb{R}$. This is done by showing that $\#(A \cap E)$ equals $\#(\text{st } A \cap \text{st } E)$ almost everywhere.

Definition 6.4. Let $f : U \rightarrow \mathring{K}^m$ be definable, with $U \subset \mathring{K}^n$ open. Let $E \subset \mathbb{R}^n$ and \bar{f} be as in Lemma 2.10. We say that $b \in \mathbb{R}^n$ is an S -regular point of \bar{f} if

- i) $b \in \text{st}(U) \setminus \bar{E}$;
- ii) b is a regular point of \bar{f} .

Otherwise, we say that b is an S -singular point and $\bar{f}(b)$ is an S -singular value of \bar{f} . If $c \in \mathbb{R}^m$ is not an S -singular value, we say that c is an S -regular value of \bar{f} .

Remark 6.5. Let S be the set of S -regular points of \bar{f} . Then, S is open and definable in \mathbb{R}_K .

Lemma 6.6 (Morse-Sard). *Assume that $m \geq n$. Then, the set of S -singular values of \bar{f} is $\mathcal{L}_{\mathbb{R}}^m$ -negligible,*

Proof. By Lemma 2.10, E is negligible; since E is also \mathbb{R}_K -definable, it has empty interior and therefore $\dim(E) < n$. Since $m \geq n$, it follows that $\bar{f}(E)$ is negligible. The set of S -singular values of \bar{f} is the union of $\bar{f}(E)$ and the set of singular values of \bar{f} ; it is therefore negligible. \square

Lemma 6.7 (Implicit Function). *Assume that $m = n$. Let $b \in \mathbb{R}^n$. If b is an S -regular point of \bar{f} then, for every $y \in \text{st}^{-1}(\bar{f}(b))$ there exists a unique $x \in \text{st}^{-1}(b)$ such that $f(x) = y$.*

Proof. Choose $x_0 \in \text{st}^{-1}(b)$. Let $A := (Df(x_0))^{-1}$. Since b is a regular point of \bar{f} , $\|A\|$ is finite. Thus we can choose $r, \rho \in \mathbb{Q}_{>0}$ such that $B := \overline{B(b; \rho)}$ is contained in the set of S -regular points of \bar{f} , and

$$\begin{aligned} \|\text{D}\bar{f}(b') - \text{D}\bar{f}(b)\| &< \frac{1}{2n\|A\|}, \quad \text{for every } b' \in B \\ r &\leq \frac{\rho}{2\|A\|}. \end{aligned}$$

Moreover, we can pick ρ such that $B' := \overline{B(x_0; \rho)} \subset U$. Given $y \in K^n$ such that $|y - f(x_0)| < r$, consider the mapping

$$\begin{aligned} T_y : B' &\rightarrow K^n \\ T_y(x) &:= x + A \cdot (y - f(x)). \end{aligned}$$

T_y is definable and Lipschitz, with Lipschitz constant $1/2$. Therefore, for every $y \in B(f(x_0); r)$ there exists a unique $x \in B'$ such that $T_y(x) = x$. Thus, there is a unique $x \in B$ with $f(x) = y$. It remains to show that, given $y \in \text{st}^{-1}(\bar{f}(b))$ and $x \in B'$ such that $f(x) = y$, we have $x \in \text{st}^{-1}(b)$. We can verify that

$$\begin{aligned} \bar{T}_y : B &\rightarrow B \\ \bar{T}_y(b') &= b' + (\text{D}\bar{f}(b))^{-1} \cdot (\bar{f}(b) - \bar{f}(b')) \end{aligned}$$

is also a contraction, and therefore it has a unique fixed point, namely b . Since $\bar{T}_y(\text{st}(x)) = \text{st}(x)$, we must have $\text{st}(x) = b$. \square

Remark 6.8. Let $U \subset \mathring{K}^m$. If $f : U \rightarrow \mathring{K}^n$ is definable and M -Lipschitz (for some finite M), $n \geq m$ and E is $\mathcal{L}_{\mathbb{R}}^m$ -negligible, then the set $f(\text{st}^{-1}(E))$ is \mathcal{L}^n -negligible.

Proof. We can cover E with a polyrectangle Y whose measure is an arbitrarily small rational number λ and such that Y covers $\text{st}^{-1}(E)$. Since $f(Y)$ has measure at most $CM^n\lambda$ (C depends only on m and n) the result follows. \square

Lemma 6.9. *Let $A \subseteq \mathring{K}^n$ be a basic e -rectifiable set of dimension e . Consider $V := K^e$ as embedded in K^n via the map $x \mapsto \langle x, 0 \rangle$. Identify each $p \in V$ with the $(n - e)$ -dimensional affine space which is orthogonal to V and intersects V in p . Then, for almost every $p \in V$, we have $\#(p \cap A) = \#(\text{st}(p) \cap \text{st}(A))$.*

Proof. Let $\pi : K^n \rightarrow V$ be the orthogonal projection. Let $U \subset \mathring{K}^e$ be an open M -cell and $f : U \rightarrow K^{n-e}$ be a definable M -function (M finite) such that $A = \Gamma(f)$. Let $F(x) := \langle x, f(x) \rangle$. Let $h := \pi \circ F : U \rightarrow V$, and consider $\bar{h} : C \rightarrow \text{st}(V)$, $C \subset \text{st}(U)$ as in Lemma 2.10. For almost every $p \in V$, $\#(p \cap A) = \#(h^{-1}(p))$, and $\#(\text{st} p \cap \text{st} A) = \#(\bar{h}^{-1}(\text{st} p))$ because $F : U \rightarrow A$ and $\bar{F} : C \rightarrow \text{Im}(\bar{F})$ are bijections. Thus, it suffices to prove that, for almost every $p \in V$, $\#(h^{-1}(p)) = \#(\bar{h}^{-1}(\text{st} p))$. Let E be as in Lemma 2.10. By Remark 6.8, $h(\text{st}^{-1}(E))$ is \mathcal{L}^e -negligible. Let S be the set of S -singular values of \bar{h} , by Lemma 6.6, S is negligible.

Let $p \in V \setminus (\text{st}^{-1}(S) \cup h(\text{st}^{-1}(E)))$. Then for every x in $h^{-1}(p)$, $\text{st}(x)$ is an S -regular point of \bar{h} , and therefore Lemma 6.7 implies $\#(h^{-1}(p)) = \#(\bar{h}^{-1}(\text{st} p))$. \square

Notice that the above lemma does not hold if A is only definable, instead of basic e -rectifiable.

Proof of Theorem 6.3. By Corollary 3.11, w.l.o.g. A is basic e -rectifiable. Let $B := \text{st}(A)$, and $f_B(F) := \#(B \cap F)$, for every $F \in \mathcal{AG}_e(\mathbb{R}^n)$. By Lemma 6.9,

$$\int_{\mathcal{AG}_{n-e}(K^n)} f_A \, d\mathcal{L}^{\mathcal{AG}_{n-e}(K^n)} = \int_{\mathcal{AG}_{n-e}(\mathbb{R}^n)} f_B \, d\mathcal{L}^{\mathcal{AG}_{n-e}(\mathbb{R}^n)}.$$

By the usual Cauchy-Crofton formula [Morgan88, 3.16], the right-hand side in the above identity is equal to $\mathcal{H}_{\mathbb{R}}^e(B) = \mathcal{H}^e(A)$, where we applied Lemma 5.3. \square

7 Further properties of Hausdorff measure and the Co-area formula

Theorem 7.1. *Let $e \leq n$ and $C \subseteq K^n$ be bounded and definable of dimension at most e .*

1. \mathcal{H}^e is invariant under isometries.
2. For every $r \in \mathring{K}$, $\mathcal{H}^e(rC) = \text{st}(r)^e \mathcal{H}^e(C)$.

3. If C is \emptyset -semialgebraic, then $\mathcal{H}^e(C) = \mathcal{H}^e(C_{\mathbb{R}}) = \mathcal{H}^e(\text{st}(C))$.
4. if $\dim(C) < e$, then $\mathcal{H}^e(C) = 0$; the converse is not true.
5. $\mathcal{H}^e(C) < +\infty$.
6. If $(C(r))_{r \in K^d}$ is a definable family of bounded subsets of K^n , then there exists a natural number M such that $\mathcal{H}^n(C(r)) < M$ for every $r \in K^d$.
7. If K' is either an elementary extension or an o-minimal expansion of K , then $\mathcal{H}^e(C_{K'}) = \mathcal{H}^e(C)$.
8. If $n = e$, then $\mathcal{H}^e(C) = \mathcal{L}^n(C)$.
9. If C is a subset of an e -dimensional affine space E , then $\mathcal{H}^e(C) = \mathcal{L}^E(C)$.

Proof.

(1) Use the Cauchy-Crofton formula.

(2), (4) and (7) Apply the definition of \mathcal{H}^e and Lemma 5.5.

(3) Apply Corollary 3.11 to $C_{\mathbb{R}}$ and use Lemma 5.3.

(5) and (6) Apply the Cauchy-Crofton formula: see [Dries03].

(8) Apply Lemma 5.3.

(9) Since \mathcal{H}^e is invariant under isometries, w.l.o.g. E is the coordinate space K^e . By Lemma 5.6, the measure \mathcal{H}^e inside K^n is equal to the measure \mathcal{H}^e inside K^e , and the latter is equal to \mathcal{L}^e . The conclusion follows from Remark 2.1. \square

The following theorem is the adaption to o-minimal structures of the Co-area formula, a well-known generalization of Fubini's theorem. Let $D := [0, 1] \subset K$.

Theorem 7.2 (Co-area Formula). *Let $A \subset D^m$ be definable, and $f : D^m \rightarrow D^n$ be a definable Lipschitz function, with $m \geq n$. Then, $J_n f$ is \mathcal{L}_K^m -integrable, and*

$$\int_A J_n f \, d\mathcal{L}^m = \int_{D^n} \mathcal{H}^{m-n}(A \cap f^{-1}(y)) \, d\mathcal{L}^n(y).$$

Sketch of Proof. W.l.o.g., A is an open subset of D^m . By Lemma 6.6, w.l.o.g. all points of A are S -regular for \bar{f} . Apply the real co-area formula [Morgan88] to $g := \bar{f}$ and $B := \text{st}(A)$, and obtain

$$\int_A J_n f \, d\mathcal{L}^m = \int_B J_n g \, d\mathcal{L}_{\mathbb{R}}^m = \int_{D_{\mathbb{R}}^n} \mathcal{H}_{\mathbb{R}}^{m-n}(B \cap g^{-1}(z)) \, d\mathcal{L}_{\mathbb{R}}^n(z).$$

By the Implicit Function Theorem and Lemma 5.3, for almost every $y \in D_{\mathbb{R}}^n$, we have

$$\mathcal{H}^{m-n}(A \cap f^{-1}(y)) = \mathcal{H}_{\mathbb{R}}^{m-n}(B \cap g^{-1}(\text{st } y)). \quad \square$$

References

- [BO04] A. Berarducci, M. Otero. *An additive measure in o-minimal expansions of fields*. The Quarterly Journal of Mathematics 55 (2004), no. 4, 411–419.
- [BP98] Y. Baisalov, B. Poizat. *Paires de structures o-minimales*. J. Symbolic Logic 63 (1998), no. 2, 570–578.
- [Dries03] L. van den Dries. *Limit sets in o-minimal structures*. In M. Edmundo, D. Richardson, and A. Wilkie, editors, *O-minimal Structures, Proceedings of the RAAG Summer School Lisbon 2003*, Lecture Notes in Real Algebraic and Analytic Geometry. Cuvillier Verlag, 2005.
- [Halmos50] P. R. Halmos. *Measure Theory*. D. Van Nostrand Company, Inc., New York, N. Y., 1950.
- [HPP08] E. Hrushovski, Y. Peterzil, A. Pillay. *Groups, measures, and the NIP*. J. Amer. Math. Soc. 21 (2008), no. 2, 563–596.
- [K92] K. Kurdyka. *On a subanalytic stratification satisfying a Whitney property with exponent 1*. Real algebraic geometry proceedings (Rennes, 1991), 316–322, Lecture Notes in Math., 1524, Springer, Berlin, 1992.
- [Morgan88] F. Morgan. *Geometric Measure Theory*. Academic Press, 1988. An introduction to Federer’s book by the same title.
- [P08] W. Pawłucki. *Lipschitz Cell Decomposition in O-Minimal Structures. I*. Illinois J. Math. 52 (2008), no. 3, 1045–1063.

- [PW06] J. Pila, A. J. Wilkie. *The rational points of a definable set*. Duke Math. J. 133 (2006), no. 3, 591–616.
- [VR06] E. Vasquez Rifo. Geometric partitions of definable sets. Ph.D. thesis, University of Wisconsin-Madison Madison, WI 53704, August 2006.
- [W00] F. Warner. Foundations of differentiable manifolds and Lie groups. Springer-Verlag, 2000