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OrthoMap: Homeomorphism-guaranteeing normal-projection map between surfaces

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ABSTRACT

Consider two $(n-1)$-dimensional manifolds, S and S' in $\mathbb{R}^n$. We say that they are projection-homeomorphic when the closest projection of each one onto the other is a homeomorphism. We give tight conditions under which S and S' are projection-homeomorphic. These conditions involve the local feature size for S and for S' and the Hausdorff distance between them. Our results hold for arbitrary n.

Keywords

local feature size, surface homeomorphism, isotopy, normal projection

1. INTRODUCTION

Let us first consider two nearly similar curves, C and C', in the plane. Such pairs of curves appear in several applications. Consider the following examples. C may be an approximation of C' produced for simplification [12] or compression [14]. C and C' may be consecutive frames of a 2D animation [16] or the contours of an organ in consecutive cross-sections [3].

In these applications, it is often important to establish a one-to-one mapping (homeomorphism) between C and C'. For example, one may need to map onto C' the values of attributes (such as color) associated with points along C. Amongst all possible mappings, one is of particular interest: the normal mapping, which we named the OrthoMap.

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Consider that C is parameterized by scalar x. To each point $C(x)$ of C, the OrthoMap(C, C') associates the closest point $C'(x)$ on C' that lies on the line passing through $C(x)$ and having for direction the normal $N(x)$ to C at $C(x)$. If such a mapping can be established, then each point $C'(x)$ of C' may be expressed as the normal offset $C(x) + d(x)N(x)$ of $C(x)$. We say that $C(x)$ is the closest normal projection of $C'(x)$ onto C and can express C' as a deformation of C completely defined by the normal displacement field $d(x)$. Such a deformation may be used to construct a 2D animation that will evolve C into C' or a surface in 3D that will interpolate two consecutive cross-sections, C and C'. Furthermore, to support multi-resolution graphics and compressed progressive transmission, C' may be encoded as a composition of its simplified or faired version, C, and of the details encoded in the normal displacement field d.

The difficulty lies in the fact that for arbitrary curves C and C' the OrthoMap is not one-to-one. For example, the line passing through $C(x)$ and having for direction the normal $N(x)$ to C at $C(x)$ may not intersect C'. Furthermore, two points $C(a)$ and $C(b)$ of C may map onto the same point on C'. In this paper, we formulate a precise condition which guarantees that the OrthoMap is one-to-one. This condition involves the Hausdorff distance h between C and C' and also the minimum f of the local feature size values for each curve. The precise definitions of these concepts are reviewed further in the paper. In particular, we derive a constant $c = 2 - \sqrt{2}$, such that when $h < cf$, the OrthoMap(C, C') and the OrthoMap(C', C) are both one-to-one. Furthermore, we demonstrate that our condition is tight by producing an example of C and C' where $h = cf$ and for which the OrthoMap is not one-to-one. Finally, we discuss the extension of this condition to surfaces in 3D and more generally to n-dimensional manifolds in $\mathbb{R}^{n+1}$ dimensions. An OrthoMap between surfaces has been used for the compression of triangle meshes [7] and may provide a solution for tracking texture coordinate from one frame to the next in 3D animations [15] . We anticipate that the simple condition derived in this paper will enable some applications to ensure that the curve or surface pairs they generate are homeomorphic under normal projection.

2. INTUITIVE OVERVIEW OF THE CONCEPTS AND RESULTS

This part introduces the concepts and provides intuitive formulations of our results.

Let S and S' be two surfaces in $\mathbb{R}^3$.

The normal mapping $p_{S'\to S}$ from S' onto S associates with each point p on S' its normal projection $p_S(p)$ on S. We call this map *the normal projection of S onto S'* or the *OrthoMap(S, S')* (see section 3). In general $p_{S'\to S}$ is not a bijection (two different points p and q on S' may have the same images $p_S(p) = p_S(q)$) neither well defined (the closest point of a point p on S' may not be uniquely defined). The set of points p for which $p_S(p)$ is not unique is the medial axis $\mathcal{M}(S)$ of S [4] (see definition 3.1).

Our condition insuring that $p_{S'\to S}$ is bijective involves the Hausdorff distance between S and S' and the notion of r-regularity of the surfaces.

DEFINITION 2.1. *Let A and B be two compact subsets of $\mathbb{R}^n$. The Hausdorff distance between A and B is defined by*

$$d_H(A,B) = \max\left(\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A)\right)$$

The Hausdorff distance $d_H(A,B)$ between two compact sets A and B may also be defined in terms of r-thickening. The r-thickening A^r of A is the union of all open balls of radius r and center on A. Note that A^r is the Minkowski sum of A with an open ball of radius r and center at the origin. The r-thickening operator was used as a tool for offsetting, rounding and filleting operations [9] and for shape simplification [10]. The Hausdorff distance, $d_H(A,B)$, between two sets A and B is the smallest radius r such that $A \subset B^r$ and $B \subset A^r$.

A surface S is r-regular if every point of it may be approached from both sides by an open ball of radius r that is disjoint from S. More precisely, the r-thinning $Tr(A)$ of a set A is the difference between A and the union of open balls with center out of A and the r-filleting $Fr(A)$ of A is defined as $Tr(A^r)$). The surface S is said to be r-regular [2] if $Fr(S) = S$. Note that $Fr(S)$ contains all points that cannot be reached by a ball of radius r whose interior does not interfere with S. The values r for which S is r-regular are related to the local feature size lfs(S) [1, 13] which is defined as the minimum distance between S and its medial axis $\mathcal{M}(S)$. The surface S is r-regular if and only if $r \leq$ lfs(S) (see lemma 3.2).

DEFINITION 2.2. *S and S' are said to be conformal (to each other) when S and S' are both r-regular for*

$$r = d_H(S,S')/(2-\sqrt{2}).$$

In terms of local feature size, conformality of S and S' is equivalent to the following:

$$d_H(S,S') < (2-\sqrt{2})\min(\text{lfs}(S), \text{lfs}(S'))$$

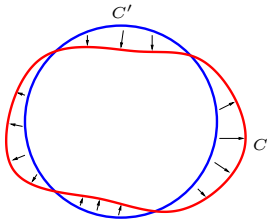

Figure 1: Two conformal curves C and C'

Figure 1 show two curves that are conformal and illustrates, in 2D, that the normal mapping of conformal curves is bijective. The following theorem is the main result of this paper:

Theorem 4.1 *If surfaces S and S are conformal, then the normal mapping $p_{S'\to S}$ is bijective.*

Moreover, $p_{S'\to S}$ allows to define an explicit isotopy (see formal definition 4.1), i.e. a continuous deformation of S' into S between S and S' (see corollary 4.2 and [5, 18]). The proof of the theorem is cast in a precise mathematical formalism which is necessary for extending the proof to arbitrary dimensions. This is the object of the three next sections.

3. DEFINITIONS AND MATHEMATICAL PRELIMINARIES

In the following, all the considered surfaces are compact and $\mathcal{C}^r$-smooth in $\mathbb{R}^3$ for some integer $r \geq 1$. In order to be more intuitive in the following of the paper, all definitions and results are given for surfaces in $\mathbb{R}^3$. They immediately generalize to codimension one manifolds in $\mathbb{R}^n$, $n \geq 2$.

DEFINITION 3.1. *Let S be a smooth compact G^2 surface embedded in $\mathbb{R}^3$. The medial axis $\mathcal{M}$ of S is defined as the set of points of $\mathbb{R}^3$ that have more than one nearest neighbor on S:*

$$\mathcal{M} = \{x \in \mathbb{R}^3 : \exists y, z \in S, y \neq z, d(x,y) = d(x,z) = d(x,S)\}.$$

The local feature size of S, denoted lfs(S) *is defined as*

$$\text{lfs}(S) = \inf_{x\in\mathcal{M}} d(x,S).$$

Notice that, since S is a smooth compact surface, lfs(S) is a positive real number. The local feature size relates to the notion of r-regularity in the following way.

LEMMA 3.2. *Let S be a smooth compact G^2 surface embedded in $\mathbb{R}^3$ and let $r > 0$. the surface S is r-regular if and only if $r \leq$ lfs(S).*

Proof. It follows from lemma 3.3 below that if $r \leq$ lfs(S) then S is r-regular. Suppose now that $r >$ lfs(S). The r-thickening S^r contains at least one point p_0 of $\mathcal{M}(S)$. The maximal open ball centered at p_0 that does not intersect S has a radius $r_0 < r$ and its boundary meets S in two points p and q. This implies that any open ball of radius $r > r_0$ that is tangent to S at p and on the same side of S as p must

intersect S. So, p cannot be approached from both sides by an open ball of radius r that is disjoint from S.

Recall that since S is a codimension one submanifold in $\mathbb{R}^3$ it is orientable and one can continuously choose at any point $x \in S$ a unit vector $N(x)$ which is normal to S. The following lemma summarize classical results from differential geometry (see [17] for instance or [6] for more precise results on tubular neighborhoods).

LEMMA 3.3. *Let S be a compact smooth G^r surface without boundary embedded in $\mathbb{R}^3$.*
i) The map $\varphi : S\times]-\text{lfs}(S), \text{lfs}(S)[\to \mathbb{R}^3$ defined by $\varphi(x,t) = x+tN(x)$ is a $\mathcal{C}^{r-1}$-diffeomorphism onto its image $T = \{x \in \mathbb{R}^3 : d(x,S) < \text{lfs}(S)\}$.
ii) For any $t \in]-\text{lfs}(S), \text{lfs}(S)[$, the offset surface $S_t = \{x + tN(x) : x \in S\}$ is a smooth G^{r-1} surface. For any $x \in S$, $N(x)$ is the normal vector to S_t at $x + tN(x)$.
iii) Let $x \in S$ and $-\text{lfs}(S) < t < \text{lfs}(S)$. The open ball $\mathbb{B}(x + tN(x), t)$ does not contain any point of S. Moreover, the sphere $\mathbb{S}(x + tN(x), t)$ intersects S only at point x.

φ is a $\mathcal{C}^{r-1}$-diffeomorphism when φ is an homeomorphism and when it and its inverse are differentiable $r-1$ times with continuous $(r-1)th$ derivatives . $p_S : T \to S$ denotes the projection along the normals of S, i.e. for any $y \in T$, $p_S(y)$ is the first coordinate of $\varphi^{-1}(y)$. Notice that $p_S(y)$ is the nearest neighbor of y on S: $d(y, p_S(y)) = d(y,S) = \inf_{x\in S} d(y,x)$.

If S' is another smooth surface contained in T, then one denotes $p_{S'\to S}$ the restriction of p_S to S'. We call this map *the normal projection of S onto S'* or the *OrthoMap(S, S')*. The maps p_S and $p_{S'\to S}$ are smooth.

One can extend the projection p_S to $\mathbb{R}^3$ in the following way: for any $y \in \mathbb{R}^3$, $p_S(y)$ is the set of points x of S such that $d(x,y) = d(y,S)$. Then, S separates $\mathbb{R}^3$ into three sets: S, Z the set of points y of $\mathbb{R}^3 \setminus S$ where $P_S(y)$ is unique and $\mathcal{M}$ the medial axis of S. Note that T is a subset of Z and that Z is $\mathcal{C}^{r-1}$-diffeomorphic to $S \times \mathbb{R}$.

4. PROJECTION-HOMEOMORPHIC SURFACES

If the Hausdorff distance between two compact surfaces S and S' is small with respect to their local feature sizes, it turns out that they are isotopic [5]. More precisely, it is proven in [5] that if two surfaces S and S' embedded in $\mathbb{R}^3$ are such that $d_H(S,S') < \min(\text{lfs}(S), \text{lfs}(S'))$ then there exists an isotopy between S and S'. But in [5], the proof of the existence of the isotopy is not constructive and only works for surfaces in $\mathbb{R}^3$. We extend this result in the case of smooth surfaces by giving explicit homeomorphism and isotopy between S and S'. Moreover, our result remains true in any dimension.

THEOREM 4.1. *Let S and S' be two compact G^r, $r \geq 1$, surfaces embedded in $\mathbb{R}^3$ such that*

$$d_H(S,S') < (2-\sqrt{2})\min(\text{lfs}(S), \text{lfs}(S')).$$

The normal projection $p_{S'\to S} : S' \to S$ is a $\mathcal{C}^{r-1}$-diffeomorphism. More generaly, if S and S' are two compact G^r codimension one manifolds in $\mathbb{R}^n$, $n \geq 2$, satisfying previous hypothesis, $p_{S'\to S} : S' \to S$ is also a $\mathcal{C}^{r-1}$-diffeomorphism.

Recall the definition of isotopy.

DEFINITION 4.1. *(Isotopy and ambient isotopy)*
An isotopy between S and S' is a continuous map $F : S \times [0,1] \to \mathbb{R}^3$ such that $F(.,0)$ is the identity of S, $F(S,1) = S'$, and for each $t \in [0,1]$, $F(.,t)$ is a homeomorphism onto its image.
An ambient isotopy between S and S' is a continuous map $F : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that $F(.,0)$ is the identity of $\mathbb{R}^3$, $F(S,1) = S'$, and for each $t \in [0,1]$, $F(.,t)$ is a homeomorphism of $\mathbb{R}^3$.

Restricting an ambient isotopy between S and S' to $S\times[0,1]$ thus yields an isotopy between them. It is actually true that if there exists an isotopy between S and S', then there is an ambient isotopy between them [8], so that both notions are equivalent in our case. As an immediate consequence of previous theorem one deduces that S' and S are isotopic.

COROLLARY 4.2. *The map $F : S' \times [0,1] \to \mathbb{R}^3$ defined by*

$$F(x,t) = p_S(x) + (1-t)d(x, p_S(x))N(p_S(x))$$

is an isotopy between S' and S. Moreover F is an $d_H(S,S')$-isotopy: for any $x \in S', t \in [0,1]$, $d(x, F(t,x)) < d_H(S,S')$.

5. PROOF OF THEOREM

We must prove that the projection of S' onto S along the normals of S is one-to-one. The proof proceeds in two steps. First, one proves that S' cannot be tangent to one of the normals to S. To do that, one suppose that there exists a point x where S' is tangent to a normal of S. This means that the normals of S at $p_S(x)$ and S' at x are orthogonal. Such a condition implies that the Hausdorff distance between S and S' cannot remains small (relatively to the lfs of S and S') in a neighborhood of x. Second, using a topological argument one deduces that S' intersect each normal to S restricted to the lfs(S)-thickening of S at exactly one point. More precisely, one shows that S' intersect each normal segment of length lfs on each side of S at exactely one point. In the following, the tangent plane to a surface S at a point $x \in S$ is denoted T_xS.

First step: *For any $x \in S'$, T_xS' is transverse to $N(p_S(x))$.*

Suppose this is not the case, that is there exists a point $x \in S'$ such that $N(p_S(x))$ is colinear to the tangent plane T_xS'. Let $\rho = \min(\text{lfs}(S), \text{lfs}(S'))$ and $\alpha = d_H(S,S') < (2-\sqrt{2})\rho$ be the Hausdorff distance between S and S'. Without loss of generality, one can suppose that there exists $0 \leq t < \alpha$ such that $x = p_S(x) + tN(p_S(x))$. Since $S \subset T'_\alpha = \{x \in \mathbb{R}^3 : d(x,S') < \alpha\}$ and $S' \subset T_\alpha = \{x \in \mathbb{R}^3 : d(x,S) < \alpha\}$ one has that $x \in T_\alpha$. Consider the two open balls of radius ρ which are tangent to T_xS' at x and let a' and b' be their centers. They do not intersect the surface S' because $\text{lfs}(S') \geq \rho$. In the same way, consider the offset surface $\tilde{S}$ of S that passes

through x and consider the two open balls of radius $\rho - t$ which are tangent to $T_x\tilde{S}$ at x. Their centers a and b are on the normal to S issued from $P_S(x)$ and one can suppose that a is on the same side as x of S. Since $\tilde{S} \subset S_t = \{x \in \mathbb{R}^3 : d(x, S) = t\}$ is an offset surface of S, the local feature size of $\tilde{S}$ is greater than $\rho - t$. It follows that $\mathbb{B}(a, \rho - t)$ and $\mathbb{B}(b, \rho - t)$ do not intersect $\tilde{S}$. Moreover, their tangent planes at x are orthogonal to the tangent planes of $\mathbb{B}(a', \rho)$ and $\mathbb{B}(b', \rho)$ (see figure 2).

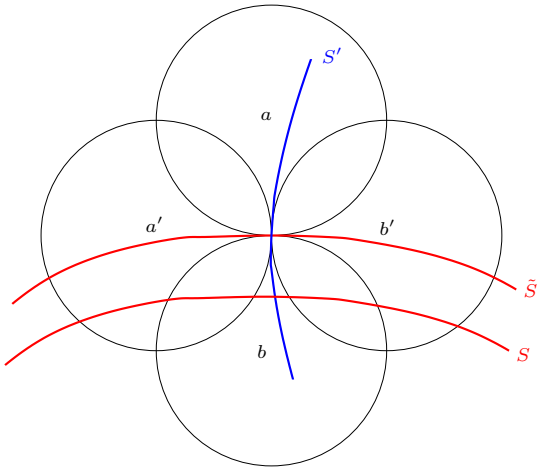

Figure 2: Intersection of tangent balls

Since the two lines (ab) ans $(a'b')$ intersect at x, the points a, b, a', b' and x are coplanar. Let P be the plane that contains them. It follows from lemma 3.3 *ii)* that P is transverse to the tangent planes of S' and $\tilde{S}$ at x (it contains the normals to S' and $\tilde{S}$ at x). So the intersection of S' and $\tilde{S}$ with P in a neighbourhood of x are smooth plane curves. The four balls $\mathbb{B}(a, \rho - t)$, $\mathbb{B}(b, \rho - t)$, $\mathbb{B}(a', \rho)$ and $\mathbb{B}(b', \rho)$ intersect P along four discs of radius $\rho - t$ and ρ and centers a, b, a' and b' respectively.

The ball of center a and radius $\rho - \alpha$ is contained in the complementary of T_α. Since $S' \subset T_\alpha$, $S' \cap P$ is contained in the complementary of the disc of center a and radius $\rho - \alpha$. It is also contained in the complementary of the discs of radius ρ and centers a' and b' (see figure 3).

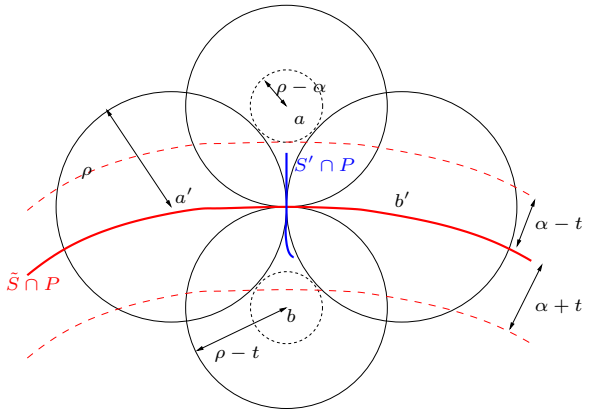

Figure 3: Intersection with the plane P near from x

Now consider the segment that joins a to the first point of intersection of the line (aa') with the disc of radius ρ and center a'. The square of its length is equal to $\rho^2 + (\rho - t)^2$ (see figure 4). Since $\alpha < (2 - \sqrt{2})\rho$, one has $(2\rho - \alpha)^2 > \rho^2 + (\rho - t)^2$, so the disc of center a' and radius ρ and the disc of center a and radius $\rho - \alpha$ intersect. It follows that the part of $S' \cap P$ which is on the same side of $\tilde{S}$ as a remains in the "triangular" area T_a delimited by the circles $C(a', \rho)$, $C(b', \rho)$, $C(a, \rho - \alpha)$ (see figure 4). Since $S' \cap P$ is a curve it has to intersects the boundary of T_a in at least two points: a contradiction. This concludes the proof of the first step.

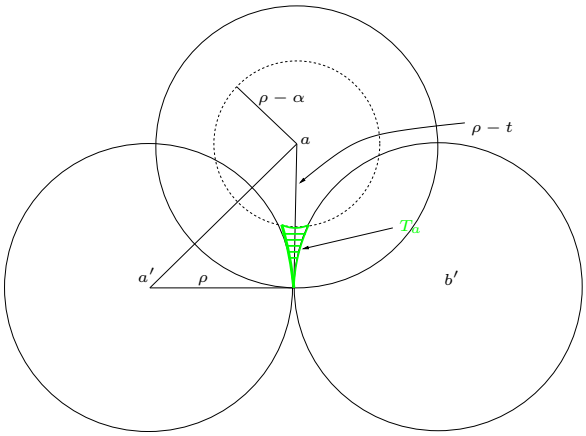

Figure 4:

Second step: *S' intersects each normal of S in M in exactely one point.*

This step is a classical fact from algebraic topology. We recall it here. Since S' is transverse to each fiber of the normal bundle M of S, S' and the projection $p_{S' \to S}$ define a topological covering of S (see [11] for the mathematical definition of covering). Thus there exists a positive integer k such that for any $x \in S$, $N(x) \cap S' \cap M$ is a set of k points. Recall that there exists a diffeomorphism φ between

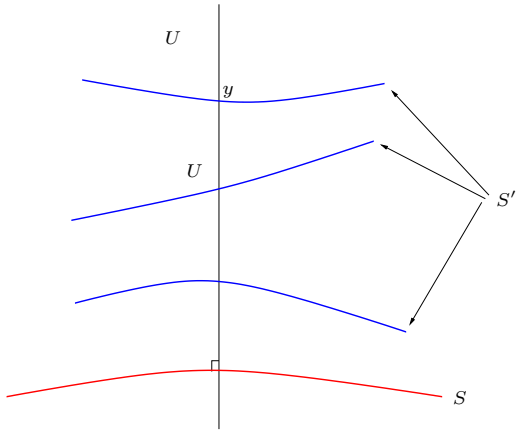

Figure 5:

$T_\rho = \{x \in \mathbb{R}^3 : d(x, S) < \rho\}$ and $S \times]-\rho; \rho[$, so that T_ρ has two sides $\varphi(S \times \{-\rho\})$ and $\varphi(S \times \rho)$. One knows from [5] that S' separates the two sides of T_ρ: any continuous path from

one side of T_p to the other has to meet S'. Suppose now that S and S' are connected (if it is not the case, one considers each connected components of S ans S' separately). For any $x \in S'$, the vector $N(p_S(x))$ defines a transverse orientation of S' and points inside the same connected component U of $\mathbb{R}^3 \setminus S'$. Suppose that $k \geq 2$ and let $x \in S$. The normal line to S at x intersects S' in k points in M. Among these points, denote by y the farthest from x (see figure 5). Locally, the connected component of $\mathbb{R}^3 \setminus S'$ that belongs "over" y is U. But the one that belongs bellow y belongs over another point of intersection. So this connected component is also U. This contradicts the fact that S' separates $\mathbb{R}^3$.

6. TIGHTNESS OF THE BOUND $2-\sqrt{2}$

The constant $2-\sqrt{2}$ involved in theorem 4.1 is tight in the following sense.

PROPOSITION 6.1. *Let c be a positive real number such that $c > 2-\sqrt{2}$. There exists two planar curves C and C' such that $d_H(C,C') < c\min(\mathrm{lfs}(C), \mathrm{lfs}(C'))$ and $p_{C'\to C}$ is not an homeomorphism.*

Notice also that, in the general case, the constant $2-\sqrt{2}$ is independent of the dimension of the ambient space $\mathbb{R}^n$. The previous proposition is proved by constructing an explicit example of two planar curves C and C'. One can easily derive from this construction, higher dimensional examples showing the optimality of $2-\sqrt{2}$ in any dimensions. (We will not attempt to draw them here.)

Note that we use the word tight instead of necessary or optimal because it is possible that $p_{S'\to S}$ be a bijection when S and S' are not conformal. An example of such a situation is given on figure 6.

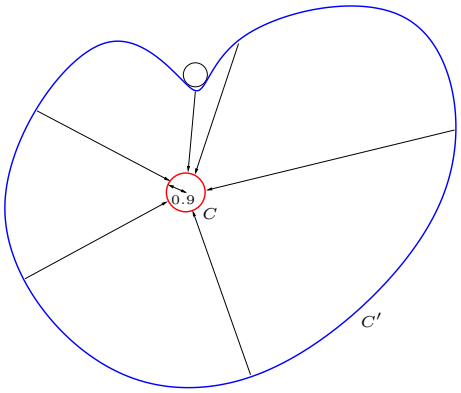

Figure 6: Two non conformal curves C and C' with bijective OrthoMap : C and C' are not 1-regular while $d_H(C,C') \geq 4$ and $p_{C'\to C}$ is bijective

Let $c > 2-\sqrt{2}$ be a fixed positive real number. We prove proposition 6.1 by giving an example of two curves C and C' such that $d_H(C,C') < c\min(\mathrm{lfs}(C), \mathrm{lfs}(C'))$ and $p_{C'\to C}$ is not a a bijection. This example is deduced from the first step of the proof of theorem 4.1. We first give an example of two curves C and C' being such that $d_H(C,C') = (2-\sqrt{2})\min(\mathrm{lfs}(C), \mathrm{lfs}(C'))$ and C' is not transverse to the normals of C. We then obtain the desired curves as a small perturbation of this first example.

The first curves are represented on figure 7. They are made of line segment and pieces of circles of radius 1 that meet orthogonaly on the vertices of a regular orthogonal grid which edges are of length 1. Notice that for clarity, only the half of the grid is represented on figure 7. As an exception, some of the pieces of circles that join the segment line are not centered on the grid vertices (but they remain of radius 1). One easily sees that the local feature sizes of C and C' are both equal to 1 and the Hausdorff distance between C and C' is equal to $2-\sqrt{2}$. It is also clear that C and C' are G^1. At the origin O, the two curves meet orthogonaly, so that C' is tangent to the normal of C at O.

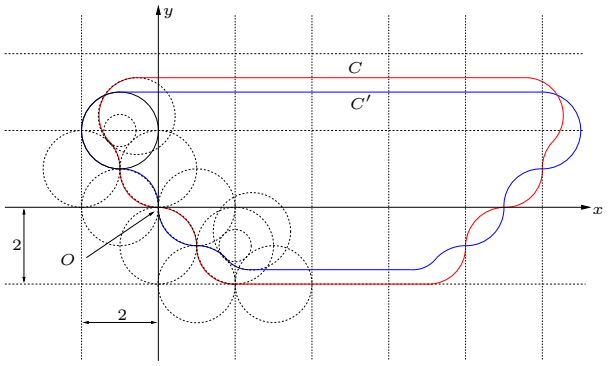

Figure 7: Two curves being such that $d_H(S,S') = (2-\sqrt{2})\min(\mathrm{lfs}(S), \mathrm{lfs}(S'))$ and that meet orthogonaly

Nevertheless, the OrthoMap of C' onto C is an homeomorphism (notice that it is not differentiable at O). But one can make a small perturbation of our example in order that the normal projection fails to be one-to-one in a neighborhood of O. This is done in figure 8. Instead of considering the circles centered on the vertices of an orthogonal grid one chooses a non-orthogonal grid. The angle between the two families of parallel lines defining the grid is equal to $\frac{\pi}{2} - \theta$ for a sufficiently small value of $\theta > 0$. The length of the edges of the grid is still equal to 1.

Clearly, the two new curves C and C' are G^1 and their local feature sizes are still equal to 1. They may be viewed as continuous deformations of the initial curves. So for $\theta > 0$ sufficiently small, one has $d_H(C,C') < c$. Unlike in fisdt example, the normal projection of C' onto C is not one-to-one: the normal of C at O is the y-axis which is intersected three times in a neighborhood of O (see figure 9).

Previous example may be generalized in higher dimension in the following way. Let $n \geq 3$ be an integer and denote by $x_1, \cdots, x_n$ the coordinates in $\mathbb{R}^n$. Identify the plane that contains the curves C and C' of previous example with the $(x_1 x_2)$-plane in $\mathbb{R}^n$ and identify the subspace generated by $x_3, \cdots, x_n$ with $\mathbb{R}^{n-2}$. The two manifolds $S = C \times \mathbb{R}^{n-2}$ and $S' = C' \times \mathbb{R}^{n-2}$ are G^1 manifolds of codimension one in $\mathbb{R}^n$ that satisfy hypothesis and conclusion of proposition 6.1.

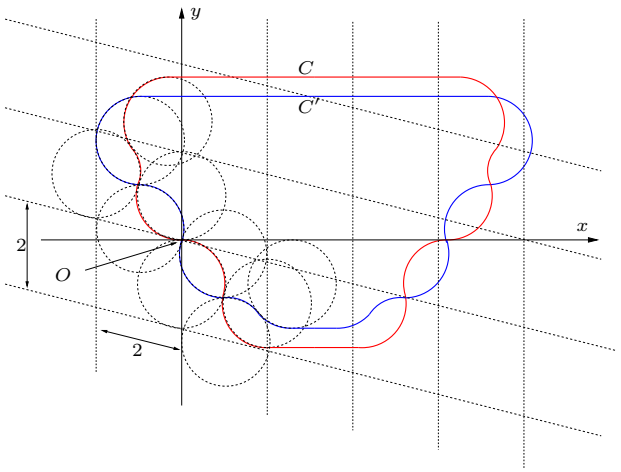

Figure 8: Example showing that the bound of theorem 4.1 is tight

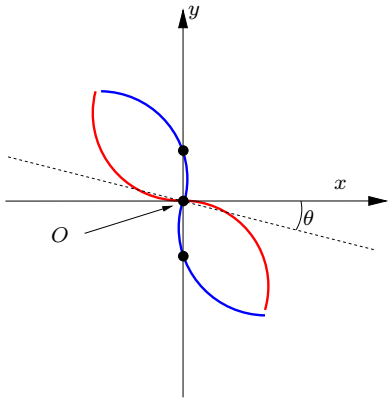

Figure 9: Zoom of figure 8 in a neighborhood of O

7. CONCLUSIONS

We have proven that one curve can be expressed as the normal offset of another, when the Hausdorff between the two curves is less than $2 - \sqrt{2}$ times the distance between each one of the curves and its Medial Axis. We have proven thatk, under this condition, the mapping between one curve and its normal offset is one-to-one. Furthermore, we have shown that the condition is tight by providing an example where the Hausdorff distance equals the above limit and yet the mapping is not one-to-one. Finally, we have extended these results to surfaces and higher-dimensional manifolds.

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