The Analytic Distortion Induced by False-Eye Separation in Head-Tracked Stereoscopic Displays

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Revision 1.1: 4/1/99: This revision contains a few typographic corrections and enhancements to the appendix, along with a slight change in notation. The notation change is designed to make this report more consistent with "Balancing Fusion, Image Depth and Distortion in Stereoscopic Head-Tracked Displays" by the same authors to appear in SIGGRAPH'99. While earlier versions of this report used *D* to denote the discussed distortion, this new version uses Δ.

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Abstract

Stereoscopic display is a fundamental part of virtual reality systems such as the virtual workbench, the CAVE and HMD systems. A common practice in stereoscopic systems is deliberate incorrect modeling of user eye separation. Under estimating eye separation can help the human visual system fuse stereo image pairs into single 3D images, while over estimating eye separation enhances image depth. Unfortunately, false eye separation modeling also distorts the perceived 3D image in undesirable ways. We present a novel analytic expression and quantitative analysis of this distortion for eyes at an arbitrary location and orientation.

1 Introduction

Virtual environments aim to perceptually place the user in an artificial computer-generated world. A key component of creating this illusion is interactive 3D imagery. To generate this imagery, a typical VR system has a location and orientation tracking device, an image generator and one or more displays. The tracking device determines the positions of the user's head and/or eyes and of the displays. The image generator computes the image that each eye would see on a display surface if the eye and the display existed inside the virtual world at their tracked positions. This image is then fed to the physical display. VR systems are typically configured either as a head-mounted display (HMD) or as a head-tracked display (HTD). In a HMD, the display is attached to a helmet worn by the user, so both the eye points and the display are in continuous motion. In a HTD, the display is stationary so only the eye points move. HTD examples are the CAVE [Cruz 93], fish tank VR [Ware93], and the virtual workbench [Krug94].

As hinted above, most VR systems generate a pair of images, one for each eye. This stereoscopic imagery provides a true 3D image so virtual objects appear to float in front of and behind the physical display surface. Software methods for stereoscopic display are well known [Hodg92][Robi92][Sou95][Robi95]. Stereoscopic display for virtual reality has been shown to improve user depth perception and task performance in a variety of tasks [Rose93][Ware93]. This is not surprising since real world experience shows that stereopsis is an important depth cue especially for objects within the user's personal space (1.5 meters) [Cutt97].

Both experience [Lipt82] and experimental studies [Yeh90][Hodg93] have shown that users with normal stereoscopic vision often have trouble fusing stereo image pairs if the eyes are modeled based on exact eye separation. The common solution is to underestimate the eye separation of the user. This approach solves the image fusion problem but creates a new problem with head-tracked displays. Underestimating eye separation causes the stereoscopic image to shift and warp with head movement. As the user moves her head forward and back the perceived image will compress and expand. As the user moves her head left and right the perceived image will shift side ways. The images in Figure 1 are indicative of what a real user perceives on a stereo HTD using under estimated eye separation. The diagram is an overhead view of a user viewing a stereo HTD. The eyes are shown in blue with the true eye points on the outside and the modeled eyes on the inside. The central horizontal black line is the projection plane. Below, the black square is the modeled geometry while the red shape is the geometry that the user perceives. Figures (a) and (b) illustrate the compression and expansion while (c) and (d) illustrate the left/right shifting.

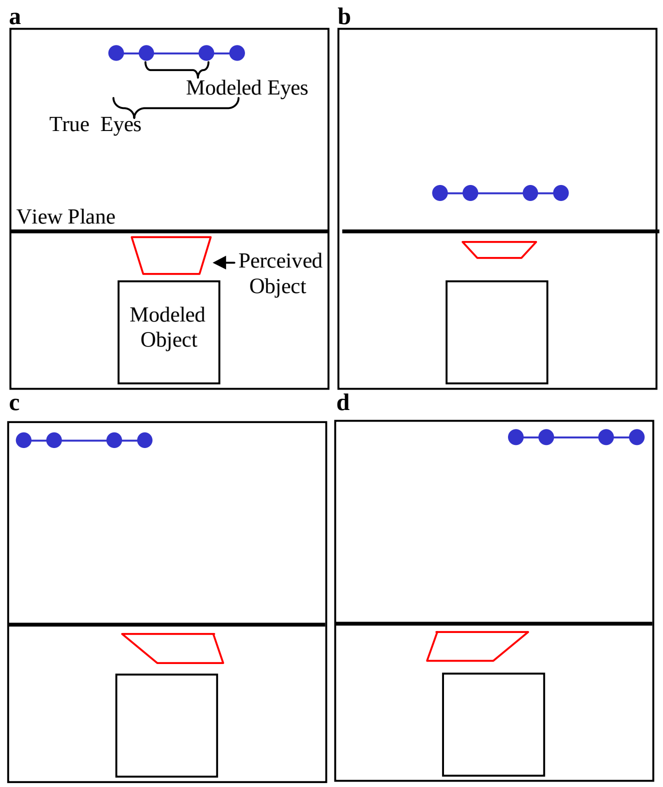


Figure 1: . An overhead view of a user viewing a stereo HTD. The eyes are shown in blue with the true eye points on the outside and the modeled eyes on the inside. The central horizontal black line is the projection plane. Below, the black grid is the modeled geometry while the red grid is the geometry that the user perceives. Figures (a) and (b) illustrate the compression and expansion while (c) and (d) illustrate the left/right shifting.

This distortion is particularly irksome because the purpose of adding head-tracking to stationary stereoscopic displays was to *remove* similar distortions that were observed in earlier non-head-tracked systems [Hodg92].

Interestingly, several researchers find it beneficial to use over estimate the eye separation. Akka [Akka93] reports that users prefer the results of slightly exaggerating the modeled eye separation in a head-tracked stereoscopic display. In Ware [Ware95], the authors dynamically adjust the modeled eye separation to enhance the perceived depth of non-head-tracked stereoscopically displayed terrain. We find, however, that over estimated eye separation in head-tracked systems causes stereo image warping similar to that previously illustrated for the under estimated case.

Since these common false eye separation methods have undesirable artifacts when applied to head-tracked displays, it is desirable to quantify them. Until now, however, a rigorous description of these distortions for a head at an arbitrary position and orientation was unavailable. This report presents a novel analytic description of this distortion. We analyze the affects of this distortion as it relates to head-tracked stereoscopic displays. A key result is that fundamentally the user will perceive virtual objects to warp and shift when he moves his head even with *perfect* head tracking.

2 Background and Previous Work

When a user cannot perceive a single 3D image from a stereo image pair, she will experience diplopia (double vision). In a stereoscopic display the occurrence of diplopia is related to various physical attributes of the display system and the geometry of the display environment [Hodg92]. The relevant geometric aspects are:

- the distance of the displayed virtual object relative to the display surface
- the eye separation value used in computing the viewing transform
- the distance of the user's eyes from the display surface

Figure 2 illustrates the situation. The eyes are on the left and a point on a virtual object is on the right. This point is projected onto two points on the projection plane. The screen parallax, *p*, associated with a virtual point is the distance between the projected points. The distance between the eyes and the virtual point also determine the angle, β. Associated with the screen itself is another angle, α Research has shown that if the difference, α-β, is outside a limited range, then diplopia occurs and the 3D depth illusion collapses [Yeh90][Hodg92][Sou95]. This range has a negative limit generally associated with points in front of the projection plane and a positive limit generally associated with points behind the projection plane. The negative limit is called the "crossed-parallax" limit while the positive limit is the "uncrossed-parallax" limit.

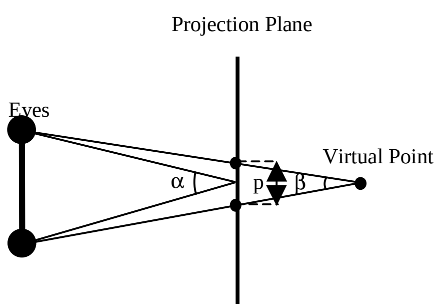


Figure 2: Illustration of the projection of a virtual point onto the projection plane for the two eyes of a user. *P* is the horizontal parallax, or distance on the screen between the stereo images of a virtual point. β is the vergence angle of this virtual point. α is the vergence angle of the projection plane itself.

Additional problems with stereoscopic displays are user fatigue and temporary alteration of the visual system's internal coupling of accommodation (eye focus) and convergence (the relative orientation of one eye to the other) [Mon95].

As previously mentioned, the common software technique to minimize these problems in non-headtracked stereoscopic displays is to model the user's eye separation with a value smaller than the true value. The resulting screen parallaxes and vergence angles are reduced and this minimizes user difficulties. However, when applied to stereo HTD's false eye separation yields the distortions illustrated in Figure 1.

Interestingly, several researchers find it beneficial to use exaggerated values for the modeled eye separation. Akka [Akka93] reports that users prefer the results of slightly exaggerating the modeled eye separation in a head-tracked stereoscopic display. In [Ware95], the authors dynamically over estimate the modeled eye separation to enhance the perceived depth of terrain. This method was used in a real world application where engineers routed cables along a seabed. Note, that this application did not use head-tracking so the stereo distortions introduced by false eye separation modeling were drowned by the qualitatively similar distortions due to the lack of head tracking.

As discussed in the introduction, these false-eye separation methods induce undesirable distortions in tracked stereo displays. While previous work provides qualitative and quantitative insights into related stereo distortions, none provide a complete description of this distortion.

Recall that in general, problems occur when the modeled viewing geometry, which is used to compute the computer imagery, fails to correctly account for some aspect of the true viewing geometry. As a result, the 3D image, which is reconstructed by the human visual system, will be a distorted version of the 3D geometry that the software system is attempting to display.

Robinett et. al [Robi92][Robi95] present a computational model for HMD optics describing how these optics distort straight lines into curves. Watson and Hodges [Wats95] demonstrate a real-time method for compensating for this distortion.

Deering [Deer92] discusses several aspects of accurately modeling stereoscopic HTDs. First he points out the variation in the true eye-separation due to convergence of the eyes and he suggests a few solutions. He then qualitatively discusses the distortions due to tracker lag. Finally, he presents a quantitative description of the distortions due to the curvature and refraction of the front glass in CRTs. He also gives a run-time method to compensate for these latter two problems.

Hodges et al [Hodg92][Hodg93] discus qualitative aspects of the incorrect modeling of the user's head position in non-head-tracked stereoscopic displays. As the user's head is displaced from the modeled location, the perceived stereo image appears to contract or grow and shift side to side. Additionally, they qualitatively analyze the change in eye separation due to convergence.

Hodges and McAllister [Hodg90] present an analytic description of the distortion of the 3D image for stereoscopic displays if eye rotations are used to model binocular viewing geometry. They discuss the induced vertical parallax and non-line preserving distortion and they conclude that the rotation method is inappropriate for single screen stereo displays.

Ware et. al. [Ware95] present a brief discussion of the change in the perceived depth of a point for false eye separation modeling in non-headtracked stereo displays.

Woods et. al. [Wood93] derive an analytic description of distortions in stereoscopic tele-operator systems. They assume the viewer is looking at a single display surface while the image generating cameras may be parallel or angled-inward. In the parallel case, the distortion preserves lines while in angled-in case the distortion maps lines to curves. Woods' treatment assumes the eye axis is parallel to the display plane and that the center of the eyes lies on a line perpendicular to the display and through its center. These assumptions are, of course, not true in a stereoscopic HTD system and therefore Woods' results do not cover this case.

3 Geometric Description of Distortion

To derive a geometric description of false eye separation distortion, we begin by reviewing and simplifying the viewing model used in stereo HTD's. A typical viewing model consists of the coordinate system hierarchy presented in Figure 3 [Sou95][Robi95]. The top coordinate system is the platform coordinate system (PCS). Manipulating this coordinate system moves the user through the virtual space. Directly attached to this coordinate system is the projection plane coordinate system and the emitter coordinate system. The projection plane coordinate system contains the projection plane in its XY plane with the window centered about the origin. The emitter coordinate system simply represents the tracker's emitter. Attached to the emitter coordinate system is the head receiver coordinate system and attached to that is the eye coordinate system. The two eye points are on the x-axis of the eye coordinate system and are symmetric about the origin.

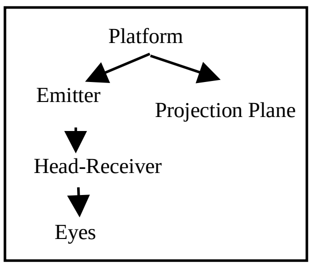


Figure 3: The coordinate system hierarchy for a typical head-tracked display.

The position and orientation of each child coordinate system relative to its parent are measured physically from the physical display setup as are the view window dimensions and eye separation. The platform coordinate system's mapping to virtual world coordinates defines the mapping of the physical space of the real world to the virtual space of the virtual world. In addition to specifying the position and orientation, the platform coordinate system can also be uniformly scaled. This causes the virtual world to grow and shrink.

Assuming all the mentioned physical measurements correct, the virtual eye separation equals the physical separation multiplied by the platform coordinate system's scale. For example, if the modeled eye separation equals the user's true eye separation, say 6 cm, and she views a virtual Earth at a 10^{-6} scale where the planet appears as a large globe, then the virtual eye separation is 60 km. By our definition this case does not represent over estimated eye separation because the modeled physical eye separation equals the veridical 6cm.

In this paper we are *not* concerned with the discrepancy between the virtual eye separation and the physical eye separation. This discrepancy, dependent on PCS scaling, merely scales the virtual world up or down. The world may appear as: a small model, such as the Earth as a globe; a true model, such as a telephone at actual size; or a magnified model, such as an atom at the size of a basketball. This uniform scaling always preserves angles, aspect ratios, and parallelism and maintains the perceived rigidity of the virtual world as the head moves. What we *are* concerned with is the discrepancy between the true physical eye separation and the modeled physical eye separation. This discrepancy will distort the world in a projective manner. The virtual world, at whatever scale it is displayed, will shear and warp with head position and neither angles, aspect ratio nor parallelism will be preserved.

Therefore we henceforth ignore PCS scale and the virtual eye separation, and we focus on the modeled and true physical eye separations. For brevity, the term "eye separation" will now always refer to the physical separations.

Having said this, we now illustrate geometrically why false eye separation yields distortions. In Figure 4, two sets of eye points are illustrated in blue. Within each set the true eye points are on the outside in dark blue and the modeled eye points are on the inside in light blue. Again the projection plane is the horizontal black line. Below a single modeled point is shown in black along with the perceived point as seen by the left and right eye set positions. For each eye set, the modeled point is projected onto the projection plane through the modeled eyes. These projectors are drawn in black. The true eyes reconstruct a perceived image by finding the intersection of the red lines. These red lines are drawn between an eye and its corresponding projected image point. Note how the perceived point (red) moves as the user moves her head. Also the perceived point is closer to the projection plane than the modeled point.

We can treat the above construction as a mapping between points. The mapping is computed by finding the appropriate line intersections. Call this construction Δ_C ("Distortion,constructed"). This geometric construction can be applied to a set of points to yield all the distortions illustrated in Figure 2.

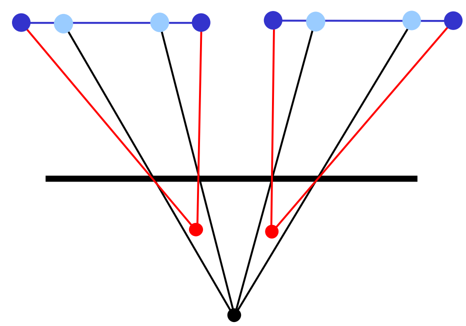


Figure 4: A geometric construction illustrating why false eye modeling distorts the perceived image and how the perceived image moves with head position. Two sets of eye points are shown in blue. Within a set the outer eye points (dark blue) are the true eyes while the inner ones (light blue) are the modeled eyes. The projection plane is the horizontal black line. Below a single modeled point in black is projected onto the plane by black projectors and its perceived location is reconstructed by the red projectors for each eye set.

Note this construction assumes that all the important physical measurements, besides the modeled eye separation, are correct. The formulation also assumes any distortion due to curvature of the screen or any optics is negligible or accounted for by other means [Deer92]. Additionally, it assumes that change in the separation of the focal points of the human eyes during convergence is also negligible or accounted for.

4 Analytic Description of Distortion

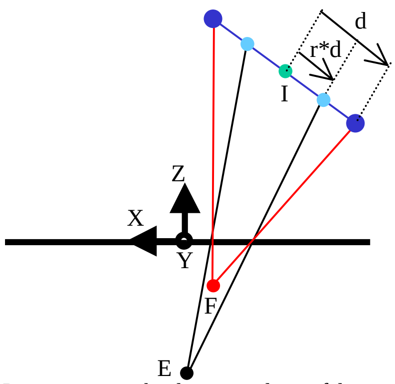


Figure 5: Parameterizing the distortion due to false-eye modeling. The projection plane lies in the X-Y plane. The user's true left and right eye points (dark blue) are displaced by vectors *d* and *-d* from the central eye (green), *I*. The modeled eye points (light blue) are displaced by *r•d* from the central eye. *r* is the ratio of the modeled eye separation to the true eye separation. *E* is a modeled point while *F* is the perceived location of this point.

To derive an analytic description of this distortion we parameterized important points as illustrated in Figure 5. First we place the projection plane coordinate system at the center of the projection plane with the plane containing the X-Y axes. Next we add a central eye point (green), *I*. The true left and right eyes (dark blue) are displaced from *I* by the vectors *d* and –*d*. 2|*d*| is the true eye separation. Next the scalar *r* is the ratio of the modeled eye separation to the true separation. Hence the left and right modeled eyes (light blue) are displaced by *r*d* and –*r*d* respectively, and 2*r*|*d*| is the modeled eye separation. *E* is the modeled point and *F* is the perceived point reconstructed by the user's visual system.

In Appendix 1, we show that this construction is a projective transformation. In projection plane coordinates the matrix is:

$$\Delta = \begin{bmatrix} 1 & 0 & \dfrac{(1-r)(IxIz + dx\,dz\,r)}{dz^2r^2 - Iz^2} & 0 \\ 0 & 1 & \dfrac{(1-r)(IyIz + dy\,dz\,r)}{dz^2r^2 - Iz^2} & 0 \\ 0 & 0 & \dfrac{r(dz^2 - Iz^2)}{dz^2r^2 - Iz^2} & 0 \\ 0 & 0 & \dfrac{Iz(1-r)}{dz^2r^2 - Iz^2} & 1 \end{bmatrix} \qquad (1)$$

In the context of a rendering pipeline the distortion acts as follows. Let a matrix, M^B_A, denote the coordinate transform from coordinate system A to coordinate system B. Then matrix stack during rendering is:

$$M^{Screen}_{Model} = M^{Screen}_{World} \bullet M^{World}_{Model} \qquad (2)$$

Let $[M]_A$ be the representation of a transform M in coordinate system A. Then using false eye separation effectively induces the complete transformation:

$$M'^{Screen}_{Model} = M^{Screen}_{World} \bullet [\Delta]_{World} \bullet M^{World}_{Model} \qquad (3)$$

Therefore, using a false eye separation will produce the *same perceived 3D image* as using the true eye separation and adding $[\bullet]_{World}$ on the viewing stack. Note, that as equations (2) and (3) describe virtual space, $[\bullet]_{World}$ will include a scale component inherited from the platform coordinate system scale. However, when analyzing •, it is more convenient to ignore this scale issue and consider the projection plane coordinate system as it exists in the physical world. We can then discuss the effects of • in physical units such as meters and consider how • behaves, independently of the scale at which the virtual scene appears.

5 Pictorial Analysis of •

The images in Figure 2 already illustrated the sideways shifting and the compression/expansion effects of • for under estimated eye separation**.** Figure 6 (page 9), re-illustrates these distortions with a more detailed grid and larger diagrams. (A) through (d) show the distortions for under estimated eye separation. (A) and (b) show the sideways shifting while (c) and (d) show the compression/expansion. (E) through (h) show the distortions for over estimated eye separation. (E) and (f) show the sideways shifting while (g) and (h) show the compression/expansion. We see more clearly, that • does not preserve angles, distances nor parallelism.

This projective distortion has many repercussions. A user designing what she perceives to be as a cube may actually have designed a more general truncated pyramid. Equivalent to Wood's [Wood93] observations in teleoperator environments, perceptions of velocity through the environment will also be distorted given this non-linear distortion. Most importantly static, rigid objects will appear to move as the user moves his head. Qualitatively these results are readily verified on real stereoscopic HTD's.

6 Quantitative Analysis of •

Having illustrated • to gain an intuitive understanding of •, we now return to a more rigorous analysis.

6.1 Degenerate Cases

• contains three degenerate cases which must first be addressed. All these cases correspond to similar degeneracies in the original construction $\bullet_c$. Once we show that these cases occur in rare circumstances, we will ignore them in further analysis.

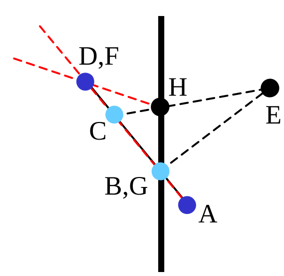


Figure 7: Embedded Modeled Eye Degeneracy

First • is only well-defined when the modeled eye points are not contained in the projection plane. If they are, the denominators in the 3rd column become zero. However, recall • is a homogenized form of •' (Appendix 1.7) which assumed this eye configuration did not occur. In •' this configuration leads to the lower-right term being zero and the matrix becomes singular in this case. This is in accordance with the ray construction, •$_c$, which also becomes singular, or non-invertable. Specifically, in such a configuration, •$_c$ maps all points to the line through the eye points. In Figure 7, the true eye points are A,D (blue); the modeled eye points are B,C (blue); black dashed lines show projection of input point E to points H,G on the projection plane; red dashed lines show the reconstructed point F. Since B and G are coincident, the reconstruction line AG is embedded in the eye axis. Hence, F is constrained to the eye axis and therefore Δ_c maps 3D space to this line. Clearly, this degenerate case occurs rarely so it seems permissible to ignore it in Δ_c and to ignore the corresponding degeneracy in Δ.

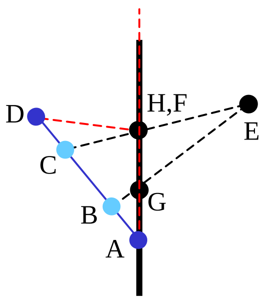


Figure 8: Embedded True Eye Degeneracy

The second degeneracy occurs when the true eye points are contained in the projection plane. In this case, Δ is non-singular. This follows from the fact that the rows of Δ are no longer independent if the third element in row 3, $r(dz^2\text{-}Iz^2)$, equals zero. Assuming r is non-zero, this term is zero precisely when a true eye point is embedded in the projection plane. Again, this result is in accordance with the ray construction, Δ_c, which becomes singular in this case. Specifically, Δ_c maps all points to the projection plane if a true eye point lies in the projection plane. Figure 8 illustrates this. The points are labeled as described in the previous paragraph. Since the reconstruction line AG is constrained to the projection plane, it follows that F, the intersection of AG and DH is also constrained to the projection plane. Hence, Δ_c maps 3 space to the plane and is singular. Again, this degenerate case occurs rarely so it is permissible to ignore it in Δ_c and to ignore the corresponding degeneracy in Δ.

The final degeneracy is the most interesting. It primarily occurs for values of r>1 where the modeled eye separation is larger than the true eye separation. Rather unexpectedly, both Δ and the original construction Δ_c flip some objects in front of the viewer to behind the viewer (Figure 9).

Such behavior is inherent in a perspective transform for objects that cross the vanishing plane of the transform [Wyli70]. Recall that the vanishing plane is the plane of points (affine points) which are mapped to points at infinite (ideal points). For review, Figure 10 illustrates how a simple perspective transform maps different regions of space. Three regions in space are color coded green, blue, and purple. The fixed plane is a solid black horizontal line. The vanishing plane is a dashed black horizontal line. The vanishing plane of the inverse transform is a dashed red horizontal line. Two parallel lines, color coded by the region containing them (10a), are mapped to intersecting lines (10b). Note how ordinary points (patterned purple) become ideal points and how ideal points (solid purple) become ordinary points as mapped between 10a and 10b. Also note how regions are compacted (A), expanded (B) and repositioned (C).

Returning to Figure 9, the vanishing planes and fixed plane is colored coded as in Figure 10. Figure 9a shows the affect on an object, the black grid, beyond the vanishing plane while Figure 9b shows the affect on an object intersected by the vanishing plane. Again the ray construction is illustrated for a single point on the grid. Note when r < 1 (not illustrated), the vanishing line is generally behind the eyes where no stereoscopic imagery ever appears. Therefore, the flipping problem generally only arises for r > 1.

At first, this degeneracy makes the basic construction, Δ_c, appear somewhat flawed from a psychophysical perspective since it does not predict what a real user will perceive in this degenerate case. The problem lies in the fact that for true a eye separation, e, screen parallax varies from –infinite, to zero and to $+e$ as the modeled point moves from the eyes' center, to the projection plane and towards infinite beyond the projection plane. This degenerate case, however, generates a *positive* screen parallax that is *greater* than $+e$. As soon as we cross the veridical $+e$ limit, we have reached a situation that has no analog in real world experience. Similar results occur if you mistakenly input a negative eye separation (r < 0). Such excessive positive parallax yields confusing images and diplopia.

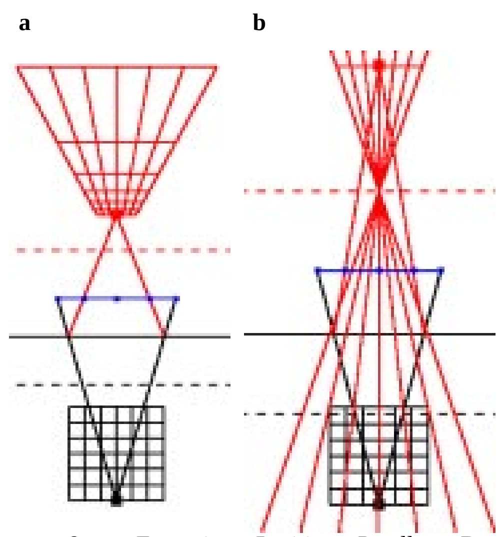


Figure 9: Excessive Positive Parallax Degeneracy

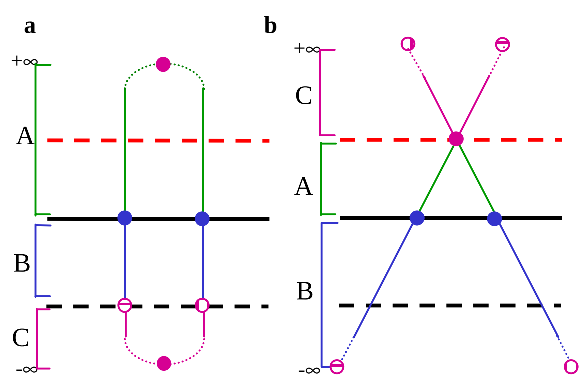


Figure 10: Effects of Typical Projective Transform

Interestingly, this exaggerated eye separation ($r > 1$) has been used quite successfully in a non-head-tracked real-world application [Ware95]. We suspect that since Ware only exaggerates the eye separation for scenes with little depth, most of the virtual objects lie on the closer side of the vanishing plane where they do not experience excessive positive screen parallax and a flipping under Δ_c. In such cases the result is an effective exaggerated depth shown in Figure 11.

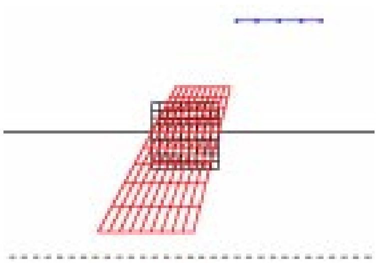


Figure 11: Over estimated eye separation can yield exaggerated depth without the flipping degeneracy.

6.2 Maximum Depth Plane

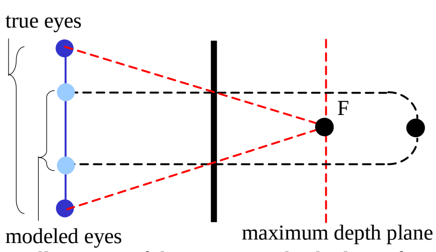


Figure 12: Illustration of the maximum depth plane of perceived space due to under estimating eye separation

Having covered and set aside the degenerate cases, we continue to gain a more rigorous understanding of Δ. First we can use Δ^{-1} (see Appendix A1.7) to compute the maximum possible depth in perceived space when the modeled eye separation is smaller than the true eye separation ($r < 1$). The existence of a maximum depth plane in the perceived space has been noted before [Woods 93]. Figure 12 illustrates this idea. For a point beyond the projection plane the screen parallax reaches its maximum value, equal to the modeled eye separation, for a point infinitely far away (*E*). This places a limit on the depth of the reconstructed perceived points (*F*).

For a non-degenerate viewing configuration, Δ is non-singular and hence Δ^{-1} exists. Like Δ, Δ^{-1} is a projective transform so it has a plane, *P*, of affine points which are mapped to ideal points (points at infinity). This plane is called the vanishing plane since these points have no image in Euclidean space. Clearly, Δ being the inverse of Δ^{-1} maps these ideal points back to the affine plane *P*. These ideal points represent the points lying infinitely far beyond the projection plane that get mapped to the maximum depth plane. *P* then is precisely this maximum depth plane. Therefore, the equation for the maximum depth plane is the vanishing plane of Δ^{-1}. It is easy to find the vanishing plane of a perspective matrix [Gold92]. With this insight the maximum depth plane is:

$$z = \frac{r(dz^2 - Iz^2)}{Iz(1-r)} \qquad (4)$$

This illustrates that the maximum depth plane position varies with the head position's z-component. This helps explain the head-position dependent squashing of perceived space illustrated in Figure 13. Here the perceived grid compresses as the head moves towards the projection plane and brings the maximum depth plane (the dash red line) closer too.

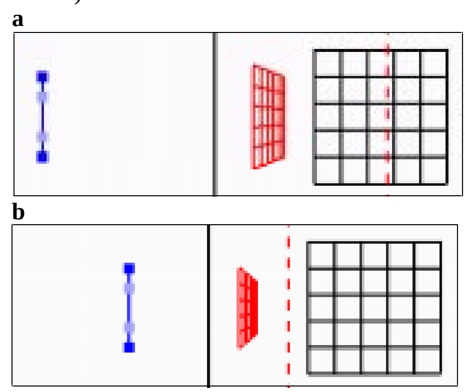


Figure 13: Perceived space squashed towards view plane. Note maximum depth plane (dashed red line).

Figure 14 plots the position of the maximum depth plane as a function of viewer head position (Iz) for several eye separations ratios (r): 0.75 (solid), 0.5 (dash-dot), 0.25 (dash) and 0.125 (dot). Note, Figure 9 assumes the head is parallel to the projection plane (dz=0); however, even for non-parallel case dz is typically small compared to Iz. In Figure 14, the maximum depth plane position is linear with respect to the head position while it varies non-linearly with r. Smaller modeled eye separations produce a closer maximum depth plane and hence a greater compression of the perceived space.

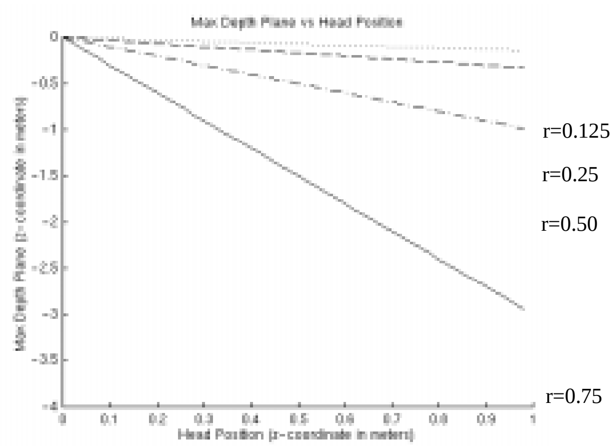


Figure 14: Plot of the position of the maximum depth plane versus the user's head position for various values of, r, the modeled to true eye separation ratio. Values of r are 0.75 (solid), 0.5 (dash-dot), 0.25 (dash) and 0.125 (dot).

6.3 Sideways Shifting

Figures 6a and 6b illustrate the sideways shifting induced by false eye-separation. Here we examine this shifting more rigorously. We plot the x-coordinate difference of a modeled point, *E*, from its distorted point, *F*, as a function of head position. For simplicity, assume the eyes are parallel to the projection plane and are contained in the X-Z plane (dz,dy=0). Fix the eyes z-coordinate to 1 meter and then vary the central eye's (*I*) x-coordinate so that the head moves side to side. In this case, Fx and hence Fx-Ex, varies linearly with Ix as seen from the equation for Fx:

$$Fx = \frac{-Iz^2 Ex + Ez(1-r)IxIz}{Ez\left(Iz(1-r)\right) + -Iz^2} \qquad (5)$$

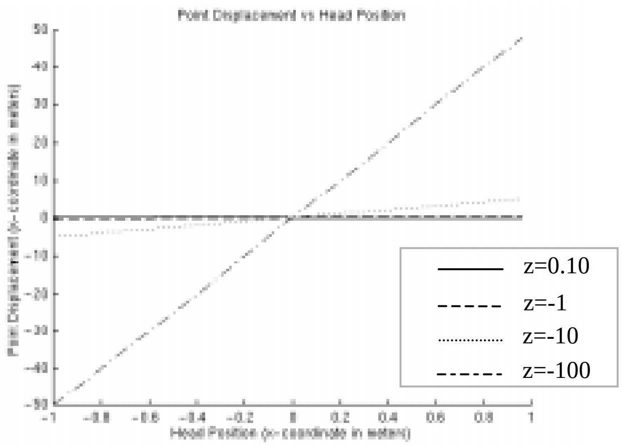


Figure 15: Plot of the displacement of a perceived point from its modeled location versus head position.. Head position, *Ix*, varies from –1 to 1; *r* is 0.5; eye-separation is 0.065. Plots are drawn for a model point a z=0.10 (solid),z=-1 (dashed), z=-10 (dotted) and z=-100 (dash-dot).

In Figure 15, *Ix* varies from –1 to 1; *r* is 0.5; eye-separation is 0.065. Plots are drawn for a model point a z=0.10 (solid),z=-1 (dashed), z=-10 (dotted) and z=-100 (dash-dot). Sensitivity to head position grows with object depth, with z=0.10m ranging up to 0.05m and z=-100 m ranging up to –50 m.

Figure 16 shows the effect of different values for *r* for a model point at (0,0,-10). In 12a, r is 0.75 (solid), 0.5 (dash-dot), 0.25 (dash) and 0.125 (dot). In 12b, r is 1 (solid), 2 (dash-dot), 4 (dash) and 8 (dot). Generally, as we move away from using true eye separation, *r*=1, the shifting grows more sensitive to head movement. Note also the change from positive to negative slope as *r* goes from less to greater than one. This represents a reversal in the direction of the shifting.

This discussion illustrates the behavior of the distortions shifting. The plots show the shift grows quite large especially for modeled eye-separations far from the true value (*r*=1).

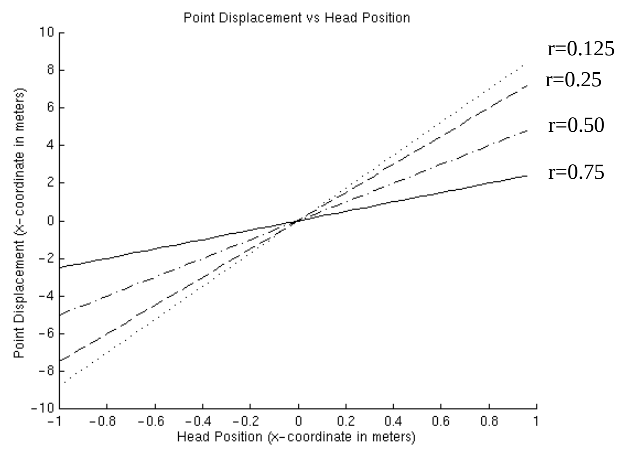
7 Conclusions

We have presented a novel analytic description of the distortion, Δ, induced by false eye-separation modeling for a head at an arbitrary position and orientation. This distortion does not preserve distance, angles nor parallism. This makes false-eye modeling problematic for a variety of applications. In command-and-control applications, for example, users often demand undistorted views of terrain. Another example is a CAD application where the loss of distance, angles and parallelism would be quite problematic. Finally, multi-screen environments such as a CAVE [Cruz93] are especially problematic. Since Δ is relative to a particular view plane, each screen would distort the world in a different manner. A virtual object which spans two adjacent screens would be distorted differently by each screen creating further visual anomalies. Next in any stereoscopic HTD application using 6 DOF input devices, the distortions from false eye-separation modeling will ruin the correspondence between the physical device and its virtual representation.

8 Future Work

First, a clearer understanding of the effects of false eye separation paves the way for studying how these artifacts interfere with various user tasks. Second, an analytic description of the distortion due to false-eye separation can aid studies trying to understand the affects of a change in eye-separation due to convergence [Deer92]. Finally, it would be desirable to remove as many artifacts of Δ as possible while retaining the desired effects of reduced horizontal parallax and enhanced depth.

a



b

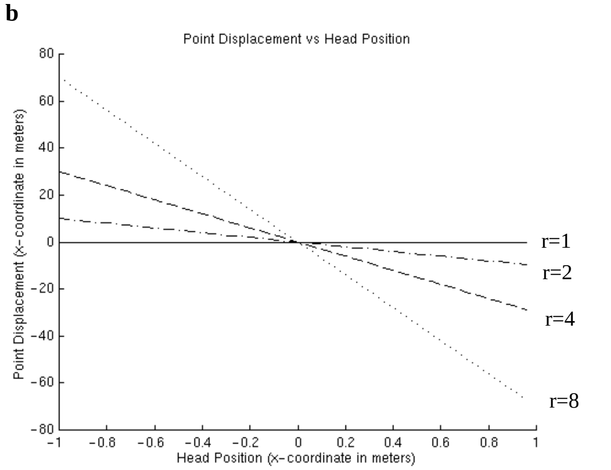


Figure 16: Plot of the displacement of a perceived point from its modeled position versus head position. The modeled point is at (0,0,-10). In (a) *r* varies over: 0.75 (solid), 0.5 (dash-dot), 0.25 (dash) and 0.125 (dot). In (b), *r* is 1 (solid), 2 (dash-dot), 4 (dash) and 8 (dot).

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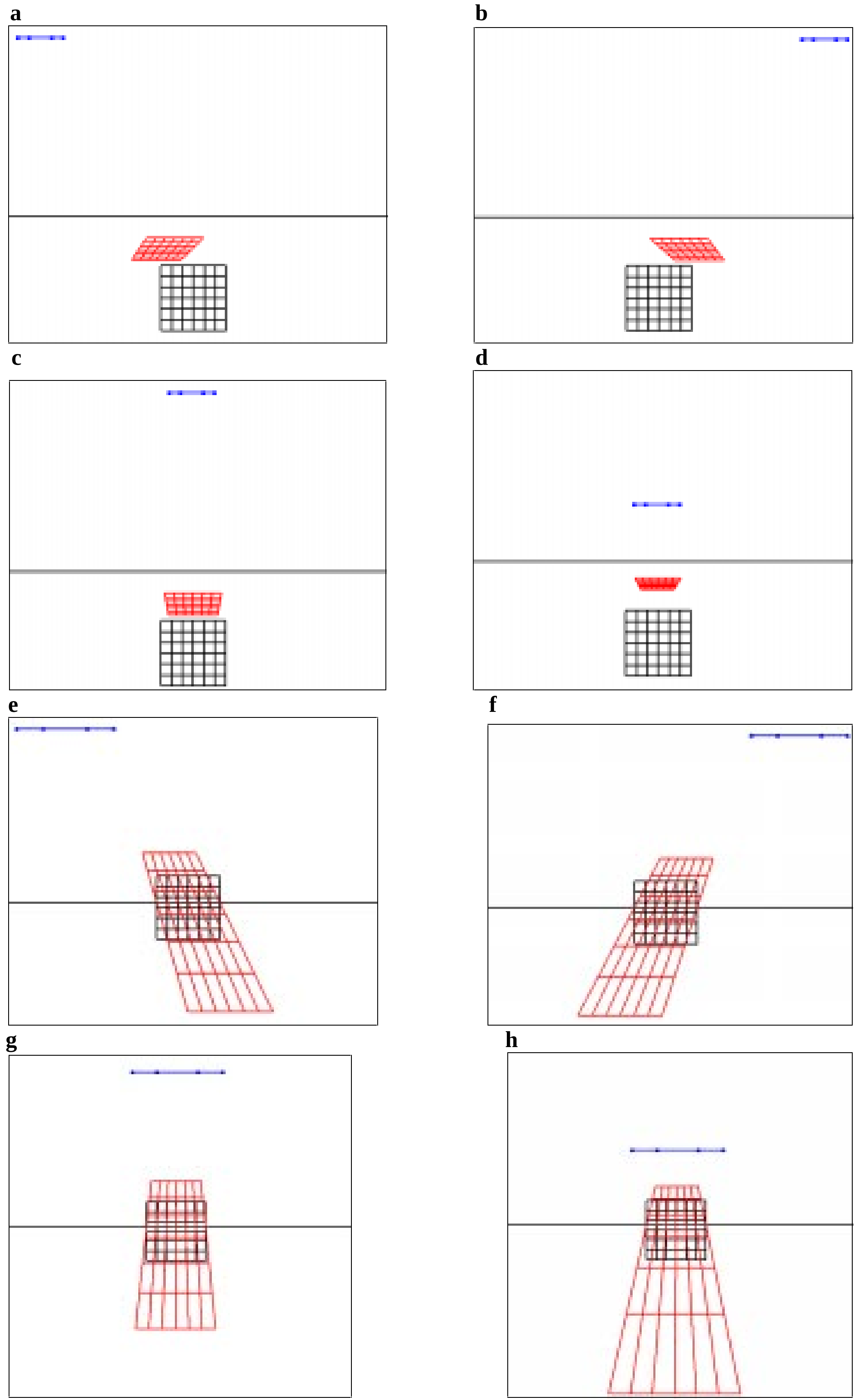

Figure 6: As in Figure 1 the eyes are blue; the projection plane is the black horizontal line; the modeled grid is black; and the perceived grid is in red. In the top four figures (a-d) the outer eye points are the true eyes while the inner points are the modeled eyes. Hence (a) through (d) illustrate the under estimated eye separation case. (a) and (b) show the sideways shifting, and (c) and (d) show the compression/expansion. In the bottom four figures (e-h) the outer eye points are the modeled eyes while the inner points are the true eyes. Hence (e) through (f) illustrate the over estimated eye separation case. (e) and (f) show the sideways shifting, and (g) and (h) show the compression/expansion.

Appendix 1: Derivation of Distortion Transform

The following figure illustrates the distortion induced by false eye separation modeling for a head at an arbitrary position and orientation. The eye points are on the left, the projection plane is the X-Y plane, and the modeled and perceived object points, *E* and *F*, are on the right. The user's central eye point is at *I*. The left eye, *D*, is displaced by *d* and the right eye, *A*, is displaced by *-d*. 2|*d*| is the true eye separation. The scalar *r* is the ratio of the modeled eye separation to the true separation. Hence the left and right modeled eyes, *C* and *B*, are displaced by *r*d* and *–r*d* respectively, and 2*r*|*d*| is the modeled eye separation. *E* is a point on a virtual object and *H* and *G* are *E*'s left and right projected images. *F* is the perceived point reconstructed by the user's visual system.

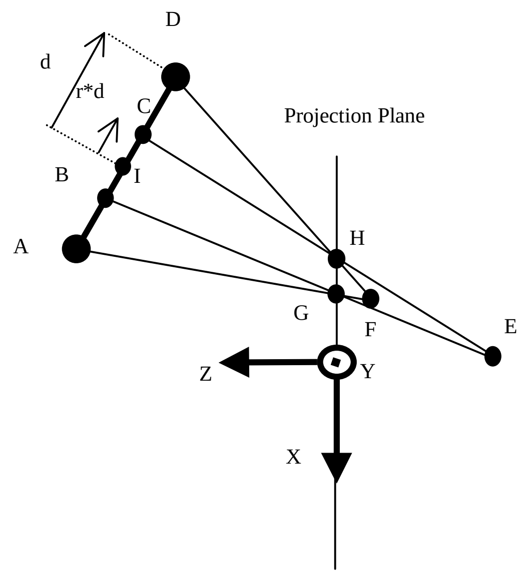

Figure 17: Parameterization of analytic description.

Numerous hand drawings of this construction indicated the induced transform preserved lines and was projective. We therefore developed a software program to distort a mesh of points by the construction by computing appropriate line intersections. These results further convinced us the transform was projective. Rather than pursuing a synthetic proof that this construction defined a projective transformation, we plowed straight into an analytic proof. The goal is to produce a rational linear expression for each coordinate of *F* in terms of *E* where the terms of the denominator are shared by all coordinate expressions and the terms of the numerator are unique for each coordinate expression. So we need:

$$F_\theta = \frac{M_\theta Ex + N_\theta Ey + O_\theta Ez + P_\theta}{Q\,Ex + R\,Ey + S\,Ez + T} \qquad (A.1-1)$$

where x, y and z are symbolically subsituted for θ

A1.1 From the figure:

A = I – d
B = I – r*d
C = I + r*d
D = I+d

A1.2 Solve for H:

Equation of line CH is:
P = (E-C)t + C

At z = 0:

$$0 = (Ez - Cz)t + Cz \quad \Rightarrow \quad t = \frac{-Cz}{Ez - Cz} = \frac{Cz}{Cz - Ez}$$

So:

$$H = (E - C)\frac{Cz}{Cz - Ez} + C$$

Or from A1.1:

$$H = (E - I - r\,d)\frac{Iz + r\,dz}{Iz - Ez + r\,dz} + I + r\,d$$

A1.3 Solve for G:

Using arguments similar to A1.2:

$$G = (E - B)\frac{Bz}{Bz - Ez} + B = (E - I + r\,d)\frac{Iz - r\,dz}{Iz - Ez - r\,dz} + I - r\,d$$

A1.4 Solve for Fx:

To begin:

$$F = \overline{AG} \cap \overline{DH}$$

So use the following equation to find two unknowns *ta* and *td*:

$$(G - A)ta + A = (H - D)td + D$$

Solve for *ta* for z-component:

$$(Gz - Az)ta + Az = (Hz - Dz)td + Dz \quad \Rightarrow \quad ta = \frac{(Dz - Az) + (Hz - Dz)td}{(Gz - Az)}$$

Substitute *ta* in original equation's x-component:

$$(Gx - Ax)\left(\frac{Dz - Az + (Hz - Dz)td}{Gz - Az}\right) + Ax = (Hx - Dx)td + Dx$$

Solve for *td*:

$$(Gx - Ax)\left(\frac{Dz - Az + (Hz - Dz)td}{Gz - Az}\right) - (Hx - Dx)td = Dx - Ax$$

$$\Rightarrow \frac{(Gx - Ax)(Dz - Az)}{(Gz - Az)} + td\,\frac{(Gx - Ax)(Hz - Dz)}{(Gz - Az)} - (Hx - Dx)td = Dx - Ax \qquad \| \text{multiply through by (Gx - Ax)}$$

$$\Rightarrow td\left(\frac{(Gx - Ax)(Hz - Dz)}{(Gz - Az)} - (Hx - Dx)\right) = Dx - Ax - \frac{(Gx - Ax)(Dz - Az)}{(Gz - Az)} \qquad \| \text{factor out } td\text{, move non - } td \text{ terms to right hand side}$$

$$\Rightarrow td = \frac{(Dx - Ax) - \dfrac{(Gx - Ax)(Dz - Az)}{(Gz - Az)}}{\left(\dfrac{(Gx - Ax)(Hz - Dz)}{(Gz - Az)} - (Hx - Dx)\right)} \qquad \| \text{divide to make } td \text{ sole term in left hand side}$$

$$\Rightarrow td = \frac{(Gz - Az)(Dx - Ax) - (Gx - Ax)(Dz - Az)}{(Gx - Ax)(Hz - Dz) - (Gz - Az)(Hx - Dx)} \qquad \| \text{multiply numerator and denominator by (Gz - Az)}$$

$$\Rightarrow td = \frac{-Az(Dx - Ax) - (Gx - Ax)(Dz - Az)}{(Gx - Ax)\big(-Dz\big) + Az(Hx - Dx)} \qquad \| \text{use } Hz = Gz = 0$$

$$\Rightarrow td = \frac{-AzDx + \|AzAx\| - GxDz + GxAz + AxDz - \|AxAz\|}{-DzGx + AxDz + AzHx - AzDx} \qquad \| \text{expand, identify like terms}$$

$$\Rightarrow td = \frac{AxDz - GxDz + AzGx - AzDx}{-AzDx + AxDz - DzGx + AzHx} \qquad \| \text{remove like terms, and reorder}$$

Now substitute in *td* in for <u>*Fx*</u>:

$$Fx = td(Hx - Dx) + Dx = \left(\frac{AxDz - GxDz + AzGx - AzDx}{-AzDx + AxDz - DzGx + AzHx} \right)(Hx - Dx) + Dx$$

Put *Fx* over common denominator, expand, identify like terms and simplify:

$$Fx = \frac{\big(AxDz - GxDz + AzGx - AzDx\big)(Hx - Dx) + Dx\big(-AzDx + AxDz - DzGx + AzHx\big)}{-AzDx + AxDz - DzGx + AzHx}$$

$$= \frac{AxDzHx - GxDzHx + AzGxHx - \|AzDxHx\|_4 - \|AxDzDx\|_3 + \|GxDzDx\|_1 - AzGxDx + \|AzDxDx\|_2 - \|AzDxDx\|_2 + \|AxDzDx\|_3 - \|DzGxDx\|_1 + \|AzHxDx\|_4}{-AzDx + AxDz - DzGx + AzHx}$$

$\|$ expand and identify like terms

$$= \frac{AxDzHx - GxDzHx + AzGxHx - AzGxDx}{-AzDx + AxDz - DzGx + AzHx} \qquad \| \text{remove like terms}$$

$$= \frac{-\big(-AxDzHx + GxDzHx - AzGxHx + AzGxDx\big)}{-\big(AzDx - AxDz + DzGx - AzHx\big)} \qquad \| \text{factor out -1}$$

$$= \frac{-AxDzHx + GxDzHx - AzGxHx + AzGxDx}{AzDx - AxDz + DzGx - AzHx} \qquad \| \text{cancel -1}$$

$$= \frac{AzDxGx - AxDzHx + (Dz - Az)GxHx}{AzDx - AxDz + DzGx - AzHx} \qquad \| \text{reorder terms, and factor out } GxHx$$

Now substitute expressions for Hx and Gx from A1.2 and A1.3 into this definition of Fx and use Dz-Ax=2dz. This yields:

$$Fx = \frac{\begin{array}{l} AzDx\left(Ix - rdx + \dfrac{(Ex - Ix + rdx)(Iz - rdz)}{Iz - Ez - rdz} \right) \\ - AxDz\left(Ix + rdx + \dfrac{(Ex - Ix - rdx)(Iz + rdz)}{Iz - Ez + rdz} \right) \\ + (2\,dz)\left(Ix - rdx + \dfrac{(Ex - Ix + rdx)(Iz - rdz)}{Iz - Ez - rdz} \right)\left(Ix + rdx + \dfrac{(Ex - Ix - rdx)(Iz + rdz)}{Iz - Ez + rdz} \right) \end{array}}{\begin{array}{l} AzDx \\ - AxDz \\ + Dz\left(Ix - rdx + \dfrac{(Ex - Ix + rdx)(Iz - rdz)}{Iz - Ez - rdz} \right) \\ - Az\left(Ix + rdx + \dfrac{(Ex - Ix - rdx)(Iz + rdz)}{Iz - Ez + rdz} \right) \end{array}}$$

Next multiply the complete expression by:

$$\frac{(Iz - Ez + rdz)(Iz - Ez - rdz)}{(Iz - Ez + rdz)(Iz - Ez - rdz)}$$

When doing, however, treat the numerator and denominator separately.

A1.4.1 Solve For Fx Denominator

Begin by solving for the denominator of Fx multiplied by (Iz-Ez+rdz)(Iz-Ez-rdz):

$$(Iz - Ez + rdz)(Iz - Ez - rdz)\begin{pmatrix} AzDx \\ - AxDz \\ + Dz\left(Ix - rdx + \dfrac{(Ex - Ix + rdx)(Iz - rdz)}{Iz - Ez - rdz} \right) \\ - Az\left(Ix + rdx + \dfrac{(Ex - Ix - rdx)(Iz + rdz)}{Iz - Ez + rdz} \right) \end{pmatrix}$$

$$= (Iz - Ez + rdz)(Iz - Ez - rdz)\begin{pmatrix} AzDx \\ - AxDz \\ + DzIx - rdxDz + Dz\dfrac{(Ex - Ix + rdx)(Iz - rdz)}{Iz - Ez - rdz} \\ - AzIx - rdxAz - Az\dfrac{(Ex - Ix - rdx)(Iz + rdz)}{Iz - Ez + rdz} \end{pmatrix} \qquad \| \text{ mulitply through by } Dz, Az$$

$$\begin{aligned} &= (Iz - Ez + rdz)(Iz - Ez - rdz)(AzDx - AxDz + DzIx - rdxDz - AzIx - rdxAz) \\ &\quad + (Iz - Ez + rdz)\, Dz(Ex - Ix + rdx)(Iz - rdz) \\ &\quad - (Iz - Ez - rdz)\, Az(Ex - Ix - rdx)(Iz + rdz) \\ &\qquad \| \text{multiply through by } (Iz - Ez + rdz)(Iz - Ez - rdz) \end{aligned}$$

$$\begin{aligned} &= (Iz - Ez + rdz)(Iz - Ez - rdz)(AzDx - AxDz + Ix(Dz - Az) - rdx(Az + Dz)) \\ &\quad + Dz(Iz - Ez + rdz)(Ex - Ix + rdx)(Iz - rdz) \\ &\quad - Az(Iz - Ez - rdz)(Ex - Ix - rdx)(Iz + rdz) \\ &\qquad \| \text{reassociate and commute} \end{aligned}$$

$$\begin{aligned} &= \begin{pmatrix} (Iz - Ez + rdz)(Iz - Ez - rdz) \\ \left(\left\| (Iz - dz)(Ix + dx) - (Ix - dx)(Iz + dz) \right\|_1 + \left\| Ix((Iz + dz) - (Iz - dz)) - rdx((Iz - dz) + (Iz + dz)) \right\|_2 \right) \end{pmatrix} \\ &\quad + (Iz + dz)(Iz - Ez + rdz)(Ex - Ix + rdx)(Iz - rdz) \\ &\quad - (Iz - dz)(Iz - Ez - rdz)(Ex - Ix - rdx)(Iz + rdz) \\ &\qquad \| \text{substitute Ax, Az, Dx, Dz from A1.1 and identify interesting patterns (1 and 2) for next step} \end{aligned}$$

$$= \left((Iz - Ez + rdz)(Iz - Ez - rdz)\left(2Iz\,dx - \left\|2dz\,Ix\right\| + \left\|2Ix\,dz\right\| - 2\,r\,dx\,Iz\right)\right)$$
$$+ \left(Iz + dz\right)(Iz - Ez + rdz)\left(Ex - Ix + rdx\right)\left(Iz - rdz\right)$$
$$- \left(Iz - dz\right)(Iz - Ez - rdz)\left(Ex - Ix - rdx\right)\left(Iz + rdz\right)$$

Apply (A - B)(C + D) - (C - D)(A + B) = 2AD - 2BC to part 1

where A = Iz, B = dz, C = Ix, D = dx.

Combine like terms in part 2. Identify like terms.

$$= \left\|(Iz - Ez + rdz)(Iz - Ez - rdz)\right\|_1 \left(2Iz\,dx - 2\,r\,dx\,Iz\right)$$
$$\left\| + \left(Iz + dz\right)(Iz - Ez + rdz)\left(Ex - Ix + rdx\right)\left(Iz - rdz\right) - \left(Iz - dz\right)(Iz - Ez - rdz)\left(Ex - Ix - rdx\right)\left(Iz + rdz\right)\right\|_2$$

Combine like terms. Identify parts for next step.

$$= \left(\left(Iz - Ez\right)^2 - r^2dz^2\right)\left(2Iz\,dx - 2\,r\,dx\,Iz\right)$$
$$+ 2\begin{pmatrix} Iz\,dz\left(Iz - Ez\right)\left(Ex - Ix\right) + Iz^2r\,dz\left(Ex - Ix\right) - Iz\left(Iz - Ez\right)r\,dz\left(Ex - Ix\right) - dz\,r^2dz^2\left(Ex - Ix\right) \\ + Iz^2\left(Iz - Ez\right)r\,dx + Iz\,dz\,r\,dz\,r\,dx - dz\left(Iz - Ez\right)r\,dz\,r\,dx - Iz\,r^2dz^2r\,dx \end{pmatrix}$$

Part 1: $(A + B)(A - B) = A^2 - B^2$

Part 2 : (A + B)(C + D)(E + F)(A - D) - (A - B)(C - D)(E - F)(A + D) =

$2(ABCE + A^2DE - ACDE - BD^2E + A^2CF + ABDF\text{-}BCDF\text{-}AD^2F)$

where A = Iz, B = dz, C = (Iz - Ez), D = r dz, E = (Ex - Ix), F = r dx

$$= 2\begin{pmatrix} \left(Iz^2 - 2EzIz + Ez^2 - r^2dz^2\right)\left(Iz\,dx - r\,dx\,Iz\right) \\ + Iz\,dz\left(IzEx - IzIx - EzEx + EzIx\right) + ExIz^2r\,dz - Ix\,Iz^2r\,dz - Iz\,r\,dz\left(IzEx - IzIx - EzEx + EzIx\right) \\ - Ex\,dz\,r^2dz^2 + Ix\,dz\,r^2dz^2 \\ + Iz\,Iz^2r\,dx - Ez\,Iz^2r\,dx + Iz\,dz\,r\,dz\,r\,dx - Iz\,dz\,r\,dz\,r\,dx + Ez\,dz\,r\,dz\,r\,dx - Iz\,r^2dz^2r\,dx \end{pmatrix}$$

Expand.

$$= 2\left(\begin{array}{l} Iz\, dxIz^2 - 2Iz\, dxEzIz + Iz\, dxEz^2 - Iz\, dx\, r^2 dz^2 \qquad - r\, dx\, Iz\, dzIz^2 + 2r\, dx\, IzEzIz - r\, dx\, IzEz^2 + r\, dx\, Izr^2 dz^2 \\ + Iz\, dzIzEx - Iz\, dzIzIx - Iz\, dzEzEx + Iz\, dzEzIx + ExIz^2 r\, dz - Ix\, Iz^2 r\, dz \\ - Iz\, r\, dzIzEx + Iz\, r\, dzIzIx + Iz\, r\, dzEzEx - Iz\, r\, dzEzIx \\ - Ex\, dz\, r^2 dz^2 + Ix\, dz\, r^2 dz^2 \\ + Iz\, Iz^2 r\, dx - Ez\, Iz^2 r\, dx + Iz\, dz\, r\, dz\, r\, dx - Iz\, dz\, r\, dz\, r\, dx + Ez\, dz\, r\, dz\, r\, dx - Iz\, r^2 dz^2 r\, dx \end{array}\right)$$

‖Expand.

$$= 2\left(\begin{array}{l} Iz\, dxIz^2 - 2Iz\, dxEzIz + Iz\, dxEz^2 - Iz\, dx\, r^2 dz^2 \qquad - \left\| r\, dx\, Iz\, Iz^2 \right\|_1 + 2\left\| r\, dx\, IzEzIz \right\|_2 - r\, dx\, IzEz^2 + \left\| r\, dx\, Izr^2 dz^2 \right\|_3 \\ + Iz\, dzIzEx - Iz\, dzIzIx - Iz\, dzEzEx + Iz\, dzEzIx + \left\| ExIz^2 r\, dz \right\|_4 - \left\| Ix\, Iz^2 r\, dz \right\|_5 \\ - \left\| Iz\, r\, dzIzEx \right\|_4 + \left\| Iz\, r\, dzIzIx \right\|_5 + Iz\, r\, dzEzEx - Iz\, r\, dzEzIx \\ - Ex\, dz\, r^2 dz^2 + Ix\, dz\, r^2 dz^2 \\ + \left\| Iz\, Iz^2 r\, dx \right\|_1 - \left\| Ez\, Iz^2 r\, dx \right\|_2 + \left\| Iz\, dz\, r\, dz\, r\, dx \right\|_6 - \left\| Iz\, dz\, r\, dz\, r\, dx \right\|_6 + Ez\, dz\, r\, dz\, r\, dx - \left\| Iz\, r^2 dz^2 r\, dx \right\|_3 \end{array}\right)$$

‖Expand. Identify like terms.

$$= 2\left(\begin{array}{l} dxIz^3 - 2dxEzIz^2 + dxEz^2 Iz - dx\, dz^2 Iz\, r^2 \qquad + dx\, Ez\, Iz^2\, r - dx\, Ez^2 Iz\, r \\ + dzExIz^2 - dzIxIz^2 - dzExEzIz + dzEzIxIz \\ \qquad + dz\, Ex\, Ez\, Iz\, r - dz\, Ez\, Ix\, Iz\, r \\ - dz^3 Ex\, r^2 + dz^3 Ix\, r^2 \\ \qquad + dx\, dz^2 Ez\, r^2 \end{array}\right)$$

‖Combine like terms. Alphabetize variables.

$$= 2\left(\begin{array}{l} dx\, Iz^3 - 2dxEz\, Iz^2 + dxEz^2 Iz - dx\, dz^2 Iz\, r^2 + dxEz\, Iz^2\, r - dx\, Ez^2 Iz\, r + dz\, Ex\, Iz^2 - dz\, Ix\, Iz^2 \\ - dz\, Ex\, Ez\, Iz + dz\, Ez\, Ix\, Iz + dz\, Ex\, Ez\, Iz\, r - dz\, Ez\, Ix\, Iz\, r - dz^3 Ex\, r^2 + dz^3 Ix\, r^2 + dx\, dz^2 Ez\, r^2 \end{array}\right)$$

‖ Rewrite.

$$= -2(dzEx - dxEz - dzIx + dxIz)(EzIz - Iz^2 - Ez\, Iz\, r + dz^2 r^2) \qquad \text{(A1.4.1 - 1)}$$

‖factor by inspection. This is motivated by the need to end up with a form like A1 - 1.

A1.4.2 Solve for Fx Numerator:

Continue with the numerator multiplied by (Iz-Ez+rdz)(Iz-Ez-rdz) and proceed as follows:

$$(Iz - Ez + rdz)(Iz - Ez - rdz)\left[\begin{array}{l} 2\,dz\left(Ix - rdx + \dfrac{(Ex - Ix + rdx)(Iz - rdz)}{Iz - Ez - rdz}\right)\left(Ix + rdx + \dfrac{(Ex - Ix - rdx)(Iz + rdz)}{Iz - Ez + rdz}\right) \\ -\,AxDz\left(Ix + rdx + \dfrac{(Ex - Ix - rdx)(Iz + rdz)}{Iz - Ez + rdz}\right) \\ +\,AzDx\left(Ix - rdx + \dfrac{(Ex - Ix + rdx)(Iz - rdz)}{Iz - Ez - rdz}\right) \end{array}\right]$$

$$\begin{array}{l} = 2\,dz\left[(Iz - Ez - rdz)(Ix - rdx) + (Ex - Ix + rdx)(Iz - rdz)\right] \\ \quad \left[(Iz - Ez + rdz)(Ix + rdx) + (Ex - Ix - rdx)(Iz + rdz)\right] - \\ AxDz\,(Iz - Ez - rdz)\left[(Iz - Ez + rdz)(Ix + rdx) + (Ex - Ix - rdx)(Iz + rdz)\right] + \\ AzDx\,(Iz - Ez + rdz)\left[(Iz - Ez - rdz)(Ix - rdx) + (Ex - Ix + rdx)(Iz - rdz)\right] \\ \quad \|\text{multiply through by (Iz - Ez + rdz)(Iz - Ez - rdz)} \end{array}$$

$$\begin{array}{l} = 2\,dz\left[\begin{array}{l} \|IxIz\|_1 - Ix\,Ez - \|Ix\,r\,dz\|_2 - \|r\,dx\,Iz\|_3 + r\,dx\,Ez + \|r\,dx\,r\,dz\|_4 \\ + Iz\,Ex - \|Iz\,Ix\|_1 + \|Iz\,r\,dx\|_3 - r\,dz\,Ex + \|r\,dz\,Ix\|_2 - \|r\,dz\,r\,dx\|_4 \end{array}\right] \\ \quad \left[\begin{array}{l} \|IxIz\|_5 - IxEz + \|Ix\,r\,dz\|_6 + \|rdx\,Iz\|_7 - rdx\,Ez + \|rdx\,rdz\|_8 \\ + IzEx - \|IzIx\|_5 - \|Iz\,rdx\|_7 + rdz\,Ex - \|rdz\,Ix\|_6 - \|rdz\,rdx\|_8 \end{array}\right] - \\ AxDz\,(Iz - Ez - rdz)\left[\begin{array}{l} \|Ix\,Iz\|_9 - Ix\,Ez + \|Ix\,rdz\|_{10} + \|rdx\,Iz\|_{11} - rdx\,Ez + \|rdx\,rdz\|_{12} \\ + Ex\,Iz + Ex\,rdz - \|Ix\,Iz\|_9 - \|Ix\,rdz\|_{10} - \|rdx\,Iz\|_{11} - \|rdx\,rdz\|_{12} \end{array}\right] + \\ AzDx\,(Iz - Ez + rdz)\left[\begin{array}{l} \|Ix\,Iz\|_{13} - Ix\,Ez - \|Ix\,rdz\|_{14} - \|rdx\,Iz\|_{15} + rdx\,Ez + \|rdx\,rdz\|_{16} \\ + Ex\,Iz - Ex\,rdz - \|Ix\,Iz\|_{13} + \|Ix\,rdz\|_{14} + \|rdx\,Iz\|_{15} - \|rdx\,rdz\|_{16} \end{array}\right] \\ \quad \|\text{expand and identify like terms} \end{array}$$

$$= 2\,dz\Big[\big(Ex\,Iz - Ez\,Ix\big) - \big(Ex\,r\,dz - Ez\,r\,dx\big)\Big]\Big[\big(Ex\,Iz - Ez\,Ix\big) + \big(Ex\,r\,dz - Ez\,r\,dx\big)\Big]$$
$$- AxDz(Iz - Ez - rdz)\big[Ex\,Iz + Ex\,rdz - Ix\,Ez - rdx\,Ez\big]$$
$$+ AzDx(Iz - Ez + rdz)\big[Ex\,Iz - Ex\,rdz - Ix\,Ez + rdx\,Ez\big]$$

‖ Cancel like terms. Expand.

$$= 2\,dz\Big[(Ex\,Iz - Ez\,Ix)^2 - (Ex\,rdz - Ez\,rdx)^2\Big]$$
$$- AxDz(Iz - Ez - rdz)\big[Ex\,Iz + Ex\;r\,dz - Ix\,Ez - r\,dx\,Ez\big]$$
$$+ AzDx(Iz - Ez + rdz)\big[Ex\,Iz - Ex\,rdz - Ix\,Ez + rdx\,Ez\big]$$

‖ use : (A + B)(A - B) = $A^2 - B^2$

$$= 2\,dz\Big[Ex^2\,Iz^2 - 2Ex\,Ez\,Ix\,Iz + Ez^2\,Ix^2 - Ex^2\,r^2 dz^2 + 2\;Ex\,Ez\,rdx\,rdz - dx^2 Ez^2\,r^2\Big]$$
$$- AxDz\begin{bmatrix} Iz\,Ex\,Iz + \|Iz\,Ex\;r\,dz\|_1 - Iz\,Ix\,Ez - Iz\,r\,dx\,Ez \;\; - Ez\,Ex\,Iz - Ez\,Ex\;r\,dz + Ez\,Ix\,Ez + Ez\,r\,dx\,Ez \\ - \|r\,dz\,Ex\,Iz\|_1 - r\,dz\,Ex\,r\,dz + r\,dz\,Ix\,Ez + r\,dz\;r\,dx\,Ez \end{bmatrix}$$
$$+ AzDx\begin{bmatrix} Iz\,Ex\,Iz - \|Iz\,Ex\,r\,dz\|_2 - Iz\,Ix\,Ez + Iz\,r\,dx\,Ez \;\; - Ez\;Ex\,Iz + Ez\,Ex\,r\,dz + Ez\,Ix\,Ez - Ez\,r\,dx\,Ez \\ + \|\;r\,dz\,Ex\,Iz\|_2 - r\,dz\,Ex\,r\,dz - r\,dz\,Ix\,Ez + r\,dz\,r\,dx\,Ez \end{bmatrix}$$

‖ Expand. Identify like terms

$$= 2\,dz\Big[Ex^2\,Iz^2 - 2Ex\,Ez\,Ix\,Iz + Ez^2\,Ix^2 - Ex^2\,r^2 dz^2 + 2\,dx\,dz\,Ex\,Ez\,r^2 - dx^2 Ez^2\,r^2\Big]$$
$$- AxDz\begin{bmatrix} Ex\,Iz^2 \;\; - Ez\,Ix\,Iz \;\; - dx\,Ez\,Iz\,r \;\; - Ex\,Ez\,Iz - dz\,Ex\,Ez\;\;r + Ez^2\;Ix + dx\,Ez^2\;r \\ - dz^2\;Ex\,r^2 \;+ dz\,Ez\,Ix\,r + \;dx\,dz\,Ez\,r^2 \end{bmatrix}$$
$$+ AzDx\begin{bmatrix} Ex\,Iz^2 \;\; - Ez\,Ix\,Iz \;+\; dx\,Ez\,Iz\,r \;\; - Ex\;\;Ez\,Iz + dz\,Ex\,Ez\,r + Ez^2\;Ix - dx\,Ez^2\;r \\ - \;dz^2\;Ex\;\;r^2 - \;dz\,Ez\,Ix\,r + \;dx\,dz\,Ez\,r^2 \end{bmatrix}$$

‖ Cancel like terms. Alphabetize variables.

$$= 2\,dz\Big[Ex^2\,Iz^2 - 2Ex\,Ez\,Ix\,Iz + Ez^2\,Ix^2 - Ex^2\,r^2 dz^2 + 2\,dx\,dz\,Ex\,Ez\,r^2 - dx^2 Ez^2\,r^2\Big]$$
$$- AxDz\Big[\;Ex\,Iz^2 - Ez\,Ix\,Iz - Ex\,Ez\,Iz + Ez^2\;Ix - dz^2\;Ex\,r^2 + dx\,dz\,Ez\,r^2 + \Big(- dx\,Ez\,Iz\,r - dz\,Ex\,Ez\;\;r + dx\,Ez^2\;r + dz\,Ez\,Ix\,r\Big)\Big]$$
$$+ AzDx\Big[\;Ex\,Iz^2 - Ez\,Ix\,Iz - Ex\,Ez\,Iz + Ez^2\;Ix - dz^2\;Ex\,r^2 + dx\,dz\,Ez\,r^2 - \Big(- \;dx\,Ez\,Iz\,r - dz\,Ex\,Ez\;\;r + dx\,Ez^2\;r + dz\,Ez\,Ix\,r\;\Big)\Big]$$

‖ Reassociate

$$= 2\,dz\left[Ex^2\ Iz^2 - 2Ex\ Ez\ Ix\ Iz + Ez^2\ Ix^2 - Ex^2\ r^2 dz^2 + 2\,dx\,dz\ Ex\ Ez\ r^2 - dx^2 Ez^2\ r^2\right] -$$
$$-\left(Ix - dx\right)\left(Iz + dz\right)$$
$$\left[Ex\ Iz^2 - Ez\ Ix\ Iz - Ex\ Ez\ Iz + Ez^2\ Ix - dz^2\ Ex\ r^2 + dx\,dz\ Ez\ r^2 + \left(-dx\ Ez\ Iz\ r - dz\ Ex\ Ez\ r + dx\ Ez^2\ r + dz\ Ez\ Ix\ r\right)\right]$$
$$+\left(Iz - dz\right)\left(Ix + dx\right)$$
$$\left[Ex\ Iz^2 - Ez\ Ix\ Iz - Ex\ Ez\ Iz + Ez^2\ Ix - dz^2\ Ex\ r^2 + dx\,dz\ Ez\ r^2 - \left(-dx\ Ez\ Iz\ r - dz\ Ex\ Ez\ r + dx\ Ez^2\ r + dz\ Ez\ Ix\ r\right)\right]$$

|| Substitute for Ax, Az, Dx, Dz.

$$= 2\,dz\left[Ex^2\ Iz^2 - 2Ex\ Ez\ Ix\ Iz + Ez^2\ Ix^2 - Ex^2\ r^2 dz^2 + 2\,dx\,dz\ Ex\ Ez\ r^2 - dx^2 Ez^2\ r^2\right] -$$
$$\left(\left(IxIz - dx\,dz\right) + \left(dz\ Ix - dx\ Iz\right)\right)$$
$$\left[Ex\ Iz^2 - Ez\ Ix\ Iz - Ex\ Ez\ Iz + Ez^2\ Ix - dz^2\ Ex\ r^2 + dx\,dz\ Ez\ r^2 + \left(-dx\ Ez\ Iz\ r - dz\ Ex\ Ez\ r + dx\ Ez^2\ r + dz\ Ez\ Ix\ r\right)\right] +$$
$$\left(\left(IxIz - dx\,dz\right) - \left(dz\ Ix - dx\ Iz\right)\right)$$
$$\left[Ex\ Iz^2 - Ez\ Ix\ Iz - Ex\ Ez\ Iz + Ez^2\ Ix - dz^2\ Ex\ r^2 + dx\,dz\ Ez\ r^2 - \left(-dx\ Ez\ Iz\ r - dz\ Ex\ Ez\ r + dx\ Ez^2\ r + dz\ Ez\ Ix\ r\right)\right]$$

|| Expand.

$$= 2\,dz\left[Ex^2\ Iz^2 - 2Ex\ Ez\ Ix\ Iz + Ez^2\ Ix^2 - Ex^2\ r^2 dz^2 + 2\,dx\,dz\ Ex\ Ez\ r^2 - dx^2 Ez^2\ r^2\right]$$
$$-2\left(dz\ Ix - dx\ Iz\right)\left(Ex\ Iz^2 - Ez\ Ix\ Iz - Ex\ Ez\ Iz + Ez^2\ Ix - dz^2\ Ex\ r^2 + dx\,dz\ Ez\ r^2\right)$$
$$-2\left(IxIz - dx\,dz\right)\left(-dx\ Ez\ Iz\ r - dz\ Ex\ Ez\ r + dx\ Ez^2\ r + dz\ Ez\ Ix\ r\right)$$

|| Use : $-(A + B)(C + D) + (A - B)(C - D) = -2BC - 2AD$

where $A = Ix\ Iz\text{-}dz\ dz$, $B = dz\ Ix\text{-}dx\ Iz$, $C = EzIx^2 - Ez\ Ix\ Iz\ \ldots\ldots$, $D = \text{-}dx\ Ez\ Iz\ r\ \text{-}\ dz\ Ex\ Ez\ r\ldots\ldots$

$$= 2\left(\begin{array}{l} dz\ Ex^2\ Iz^2 - 2dz\ Ex\ Ez\ Ix\ Iz + dz\ Ez^2\ Ix^2 - dz\ Ex^2\ r^2 dz^2 + 2\,dz\,dx\,dz\ Ex\ Ez\ r^2 - dz\ dx^2 Ez^2\ r^2 \\ -\left(\begin{array}{l} dz\ Ix\ Ex\ Iz^2 - dz\ Ix\ Ez\ Ix\ Iz - dz\ Ix\ Ex\ Ez\ Iz + dz\ Ix\ Ez^2\ Ix - dz\ Ix\ dz^2\ Ex\ r^2 + dz\ Ix\ dx\,dz\ Ez\ r^2 \\ -dx\ Iz\ Ex\ Iz^2 + dx\ Iz\ Ez\ Ix\ Iz + dx\ Iz\ Ex\ Ez\ Iz - dx\ Iz\ Ez^2\ Ix + dx\ Iz\ dz^2\ Ex\ r^2 - dx\ Iz\ dx\,dz\ Ez\ r^2 \end{array}\right) \\ -\left(\begin{array}{l} -IxIz\ dx\ Ez\ Iz\ r - IxIz\ dz\ Ex\ Ez\ r + IxIz\ dx\ Ez^2\ r + IxIz\ dz\ Ez\ Ix\ r \\ +dx\,dz\,dx\ Ez\ Iz\ r + dx\,dz\,dz\ Ex\ Ez\ r - dx\,dz\,dx\ Ez^2\ r - dx\,dz\ dz\ Ez\ Ix\ r \end{array}\right) \end{array}\right)$$

|| Factor out 2. Expand.

$$= 2\left(\begin{array}{l} dz\,Ex^2\,Iz^2 - 2dz\,Ex\,Ez\,Ix\,Iz + dz\,Ez^2\,Ix^2 - dz^3\,Ex^2\,r^2 + 2\,dx\,dz^2\,Ex\,Ez\,r^2 - dx^2 dz\,Ez^2\,r^2 \\ -\left(\begin{array}{l} dz\,Ex\,Ix\,Iz^2 - dz\,Ez\,Ix^2\,Iz - dz\,Ex\,Ez\,Ix\,Iz + dz\,Ez^2\,Ix^2 - dz^3\,Ex\,Ix\,r^2 + dx\,dz^2\,Ez\,Ix\,r^2 \\ - dxEx\,Iz^3 + dx\,Ez\,Ix\,Iz^2 + dx\,Ex\,Ez\,Iz^2 - dx\,Ez^2\,Ix\,Iz + dx\,dz^2\,ExIz\,r^2 - dx^2 dz\,Ez\,Iz\,r^2 \end{array}\right) \\ -\left(\begin{array}{l} - dx\,Ez\,IxIz^2\,r - dz\,Ex\,Ez\,IxIz\,r + dx\,Ez^2\,IxIz\,r + dz\,Ez\,Ix^2 Iz\,r \\ + dx^2\,dz\,Ez\,Iz\,r + dx\,dz^2\,Ex\,Ez\,r - dx^2\,dz\,Ez^2\,r - dx\,dz^2 Ez\,Ix\,r \end{array}\right) \end{array}\right)$$

‖ Alphabetize terms.

$$= 2\left(\begin{array}{l} dz\,Ex^2\,Iz^2 - 2\left\|dz\,Ex\,Ez\,Ix\,Iz\right\|_1 + \left\|dz\,Ez^2\,Ix^2\right\|_2 - dz^3\,Ex^2\,r^2 + 2\,dx\,dz^2\,Ex\,Ez\,r^2 - dx^2 dz\,Ez^2\,r^2 \\ - dz\,Ex\,Ix\,Iz^2 + dz\,Ez\,Ix^2\,Iz + \left\|dz\,Ex\,Ez\,Ix\,Iz\right\|_1 - \left\|dz\,Ez^2\,Ix^2\right\|_2 + dz^3\,Ex\,Ix\,r^2 - dx\,dz^2\,Ez\,Ix\,r^2 \\ + dxEx\,Iz^3 - dx\,Ez\,Ix\,Iz^2 - dx\,Ex\,Ez\,Iz^2 + dx\,Ez^2\,Ix\,Iz - dx\,dz^2\,Ex\,Iz\,r^2 + dx^2 dz\,Ez\,Iz\,r^2 \\ + dx\,Ez\,IxIz^2\,r + dz\,Ex\,Ez\,IxIz\,r - dx\,Ez^2\,IxIz\,r - dz\,Ez\,Ix^2 Iz\,r \\ - dx^2\,dz\,Ez\,Iz\,r - dx\,dz^2\,Ex\,Ez\,r + dx^2\,dz\,Ez^2\,r + dx\,dz^2 Ez\,Ix\,r \end{array}\right)$$

‖ Expand out -1. Identify like terms.

$$= 2\left(\begin{array}{l} dz\,Ex^2\,Iz^2 - dz\,Ex\,Ez\,Ix\,Iz \qquad - dz^3\,Ex^2\,r^2 + 2\,dx\,dz^2\,Ex\,Ez\,r^2 - dx^2 dz\,Ez^2\,r^2 \\ - dz\,Ex\,Ix\,Iz^2 + dz\,Ez\,Ix^2\,Iz \qquad + dz^3\,ExIx\,r^2 - dx\,dz^2\,Ez\,Ix\,r^2 \\ + dxEx\,Iz^3 - dx\,Ez\,Ix\,Iz^2 - dx\,Ex\,Ez\,Iz^2 + dx\,Ez^2\,Ix\,Iz - dx\,dz^2\,Ex\,Iz\,r^2 + dx^2 dz\,Ez\,Iz\,r^2 \\ + dx\,Ez\,IxIz^2\,r + dz\,Ex\,Ez\,IxIz\,r - dx\,Ez^2\,IxIz\,r - dz\,Ez\,Ix^2 Iz\,r \\ - dx^2\,dz\,Ez\,Iz\,r - dx\,dz^2\,Ex\,Ez\,r + dx^2\,dz\,Ez^2\,r + dx\,dz^2 Ez\,Ix\,r \end{array}\right) \quad A1.4.2-1$$

‖ Combine like terms.

.

$$= 2\left(dzEx - dxEz - dzIx + dxIz\right)\left(-\left\|Ez\ Ix\ Iz\right\|_2 + \left\|ExIz^2\right\|_1 - \left\|dx\ dz\ Ez\ r\right\|_6 + \left\|Ez\ Ix\ Iz\ r\right\|_5 - \left\|dz^2\ Ex\ r^2\right\|_3 + \left\|dx\ dz\ Ez\ r^2\right\|_4\right)$$

Factor out (dz Ex - dxEz - dzIx + dxIz). This is motivated by the need to get the final result for *Fx* in form A1 - 1. Since the *Fx* denominator (A1.4.1 - 1) has factor α = (dz Ex - dxEz - dzIx + dxIz), we need to extract this same factor from the previous equation (A1.4.2 - 1) of the numerator. This is necessary so that in the complete fraction for *Fx*, α cancels out leaving a rational linear equation. To factor (A1.4.2 - 1), note that (A1.4.2 - 1) has 24 terms, counting the $2dx\ dz^2 ExEzr^2$ twice. So we must factor (A1.4.2 - 1) into $\alpha\beta$ where α has the known 4 terms and β has 6 unknown terms. To find the terms of β make a table of the 4 terms of α versus the 24 terms of (A1.4.2 - 1) as in Table 1. Divide each (A1.4.2 - 1) term by each term of α, if possible, to yield a quotient term. We must now find 6 quotient terms each of which occurs in one column (i.e. is associated with each term of α). These quotients terms are exactly the terms of β. The resulting terms in the above factorization are labeled at in Table 1.[1]

	dzEx	**-dxEz**	**-dzIx**	**dxIz**
dzEx^{2}Iz^2	ExIz^2 ***1**			
-dzExEzIxIz	-Ez IxIz ***2**		ExEzIz	
-dz^3 Ex^{2}r^2	-dz^2 Ex r^2 ***3**			
dxdz^{2}ExEz r^2	dx dzEz r^2 ***4**	-dz^{2}Ex r^2		
dxdz^{2}ExEz r^2	dz^{2}Ez r^2	-dz^{2}Ex r^2 ***3**		
-dx^{2}dzEz^2 r^2		dxdzEzr^2 ***4**		
-dzExIxIz^2	-IxIz^2		ExIz^2 ***1**	
dzEzIx^{2}Iz			-EzIxIz ***2**	
dz^{3}ExIxr^2	dz^{2}Ixr^2		-dz^{2}Exr^2 ***3**	
-dxdz^{2}EzIx r^2		-dz^{2}Ix r^2	dxdzEz r^2 ***4**	
dxExIz^3				ExIz^2 ***1**
-dxEzIxIz^2		IxIz^2		-EzIxIz ***2**
-dxExEzIz^2		ExIz^2 ***1**		-ExEzIz
dxEz^{2}IxIz		-EzIxIz ***2**		Ez^{2}Ix
-dxdz^{2}ExIz r^2	-dxdzIz r^2			-dz^{2}Ex r^2 ***3**
dx^{2}dzEzIz r^2		-dxdzIz r^2		dxdzEzr^2 ***4**
dxEzIxIz^{2}r		-IxIz^{2}r		EzIxIz r ***5**
dzExEzIxIz r	EzIxIz r ***5**		-ExEzIz r	
-dxEz^{2}IxIz r		EzIxIz r ***5**		-Ez^{2}Ix r
-dzEzIx^{2}Iz r			EzIxIz r ***5**	
-dx^{2}dzEzIz r		dxdzIz r		-dxdzEz r ***6**
-dxdz^{2}ExEz r	-dxdzEz r ***6**	dz^{2}Exr		
dx^{2}dzEz^2 r		-dxdzEz r ***6**		
dxdz^{2}EzIx r		-dz^{2}Ix r	-dxdzEz r ***6**	

Table 1: Factoring terms from (A1.4.2-1) by terms of α and labeling common results.

$$= -2\left(dzEx - dxEz - dzIx + dxIz\right)\left(Ez\ Ix\ Iz - Ex\ Iz^2 + dx\ dz\ Ez\ r - Ez\ Ix\ Iz\ r + dz^2\ Ex\ r^2 - dx\ dz\ Ez\ r^2\right)$$

factor out - 1

.

[1] While this tabular method is seemingly obvious, initial difficulty with this factorization lead us to consider that Δ did not conform to A1-1 and hence was not a homology. We then sought software solutions to factor this expression and "discovered" and used Mathematica 2.0 to get past this step. Clearly, at this point we could have abandoned all our prior manual work and replaced it with an automated approach, but we preferred to finish the work that we had started.

A1.4.3 Solve for Fx Complete Fraction:
Now return to the complete fraction, cancel common expressions and then collect like terms:

$$Fx = \frac{-2\left(dzEx - dxEz - dzIx + dxIz\right)\left(Ez\,Ix\,Iz - Ex\,Iz^2 + dx\,dz\,Ez\,r - Ez\,Ix\;Iz\,r + dz^2\;Ex\,r^2 - dx\,dz\,Ez\,r^2\right)}{-2(dzEx - dxEz - dzIx + dxIz)(EzIz - Iz^2 - Ez\,Iz\,r + dz^2 r^2)}$$

$$= \frac{\left(Ez\,Ix\,Iz - Ex\,Iz^2 + dx\,dz\,Ez\,r - Ez\,Ix\;Iz\,r + dz^2\;Ex\,r^2 - dx\,dz\,Ez\,r^2\right)}{(EzIz - Iz^2 - Ez\,Iz\,r + dz^2 r^2)}$$

$$= \frac{Ex\left(dz^2 r^2 - Iz^2\right) + Ez\left(IxIz + dx\,dz\,r - Ix\,Iz\,r - dx\,dz\,r^2\right)}{Ez\left(Iz\left(1-r\right)\right) + \left(dz^2 r^2 - Iz^2\right)}$$

$$= \frac{Ex\left(dz^2 r^2 - Iz^2\right) + Ez\left(1-r\right)\left(IxIz + dx\,dz\,r\right)}{Ez\left(Iz\left(1-r\right)\right) + \left(dz^2 r^2 - Iz^2\right)}$$

A1.5 Solve for Fy

Using a parallel derivation as in A1.4:

$$Fy = \frac{Ey\left(dz^2 r^2 - Iz^2\right) + Ez\left(1-r\right)\left(IyIz + dy\,dz\,r\right)}{Ez\left(Iz\left(1-r\right)\right) + \left(dz^2 r^2 - Iz^2\right)}$$

A1.6 Solve for Fz

Using the initial results from A1.4:

$$Fz = td\,(Hz - Dz) + Dz = \left(\frac{DzAx - DzGx + AzGx - AzDx}{-AzDx + AxDz - DzGx + AzHx}\right)(Hz - Dz) + Dz$$

$$= \left(\frac{DzAx - DzGx + AzGx - AzDx}{-AzDx + AxDz - DzGx + AzHx}\right)(-Dz) + Dz \qquad \| Hz = 0$$

Rewrite the expression over a common denominator and simplify using Hz=0:

$$Fz = \frac{(DzAx - DzGx)(-Dz) + (AzGx - AzDx)(-Dz) + Dz\left(-AzDx + AxDz - DzGx + AzHx\right)}{-AzDx + AxDz - DzGx + AzHx}$$

$$= \frac{\|DzAxDz\|_1 + \|DzGxDz\|_2 - AzGxDz + \|AzDxDz\|_4 - \|DzAzDx\|_4 + \|DzAxDz\|_1 - \|DzDzGx\|_2 + DzAzHx}{-AzDx + AxDz - DzGx + AzHx}$$

$$= \frac{AzDzHx - AzDzGx}{-AzDx + AxDz - DzGx + AzHx} \qquad \|\text{Cancel common terms.}$$

As in A1.4 proceed by treating the numerator and denominator separately and multiply both by “(Iz-Ez-rdz)(Iz-Ez+rdz)”.

A1.6.1 Solve for Fz Numerator

Begin with the numerator multiplied by (Iz-Ez+rdz)(Iz-Ez-rdz) and proceed as follows:

$$(Iz - Ez + rdz)(Iz - Ez - rdz)\left(-AzDzGx + AzDzHx\right)$$

$$= AzDz(Iz - Ez + rdz)(Iz - Ez - rdz)(Hx - Gx)$$

$$= AzDz(Iz - Ez + rdz)(Iz - Ez - rdz)$$
$$\left(\left(Ix + rdx + \frac{(Ex - Ix - rdx)(Iz + rdz)}{Iz - Ez + rdz}\right) - \left(Ix - rdx + \frac{(Ex - Ix + rdx)(Iz - rdz)}{Iz - Ez - rdz}\right)\right)$$

‖ Substitute for Hx (A1.2), Gx (A1.3)

$$= AzDz(Iz - Ez + rdz)(Iz - Ez - rdz)\left(2\,r\,dx + \frac{(Ex - Ix - rdx)(Iz + rdz)}{Iz - Ez + rdz} - \frac{(Ex - Ix + rdx)(Iz - rdz)}{Iz - Ez - rdz}\right)$$

‖ Combine like terms.

$$= AzDz\begin{pmatrix} 2\,r\,dx\left\|(Iz - Ez + rdz)(Iz - Ez - rdz)\right\|_1 \\ \left\| \begin{matrix} + (Iz - Ez - rdz)(Ex - Ix - rdx)(Iz + rdz) \\ - (Iz - Ez + rdz)(Ex - Ix + rdx)(Iz - rdz) \end{matrix} \right\|_2 \end{pmatrix}$$

‖ Multiply through by (Iz − Ez + rdz)(Iz − Ez − rdz). Identify parts for next step.

$$= AzDz\begin{pmatrix} 2\,r\,dx\left((Iz - Ez)^2 - r^2 dz^2\right) + \\ 2\left((Iz - Ez)rdz(Ex - Ix) + r^2 dz^2 r\,dx - (Iz - Ez)rdx\,Iz - rdz\,(Ex - Ix)Iz\right) \end{pmatrix}$$

‖ Part 1 : Use (A - B)(A + B) = A^2 - B^2

‖ Part 2 : Use (A - B)(C - D)(E + B) - (A + B)(C + D)(E - B) = 2(ABC + B^2D − ADE − BCE)

‖ A = Iz - Ez, B = rdz, C = Ex - Ix, D = rdx, E = Iz, B = rdz

$$= AzDz\begin{pmatrix} 2\,r\,dx\left((Iz - Ez)^2 - r^2 dz^2\right) + \\ 2\left((IzEx - IzIx - EzEx + EzIx)rdz + dx\,dz^2 r^3 - (Iz\,r\,dx\,Iz - Ez\,r\,dx\,Iz) - rdz\,Iz\,Ex + r\,dz\,Iz\,Ix\right) \end{pmatrix}$$

‖ Expand.

$$= AzDz\left(\begin{array}{l} 2\left(\left\|dx\,Iz^2\,r\right\|_1 - 2dx\,Ez\,Iz\,r + dx\,Ez^2 r - \left\|dx\,dz^2 r^3\right\|_2\right) + \\ 2\left(\begin{array}{l}\left\|IzExrdz\right\|_4 - \left\|IzIxrdz\right\|_3 - EzExrdz + EzIxrdz + \left\|dx\,dz^2 r^3\right\|_2 - \left\|Iz\,r\,dx\,Iz\right\|_1 + Ez\,r\,dx\,Iz - \left\|rdz\,Iz\,Ex\right\|_4 \\ + \left\|r\,dz\,Iz\,Ix\right\|_3\end{array}\right)\end{array}\right)$$

‖ Expand and identify like terms.

$$= 2AzDz\left(-2dx\,Ez\,Iz\,r + dx\,Ez^2 r - dzExEz\,r + dz\,EzIx\,r + dx\,Ez\,Iz\,r\right)$$

‖ Factor out 2. Combine like terms and alphabetize variables.

$$= 2\left(Iz - dz\right)\left(Iz + dz\right)\left(-2dx\,Ez\,Iz\,r + dx\,Ez^2 r + dxEz\,Iz\,r - dzEx\,Ez\,r + dzEzIx\,r\right)$$

‖Substitute for A, D from (A1.1).

$$= 2\left(Iz^2 - dz^2\right)\left(-2dx\,Ez\,Iz\,r + dx\,Ez^2 r + dxEz\,Iz\,r - dzEx\,Ez\,r + dzEzIx\,r\right)$$

‖Expand.

$$= 2\left(\begin{array}{l} -2Iz^2 dx\,Ez\,Iz\,r + Iz^2 dx\,Ez^2 r + Iz^2 dxEz\,Iz\,r - Iz^2 dzEx\,Ez\,r + Iz^2 dzEzIx\,r \\ - -dz^2\,2dx\,Ez\,Iz\,r + -dz^2 dx\,Ez^2 r + -dz^2 dxEz\,Iz\,r - -dz^2 dzEx\,Ez\,r + -dz^2 dzEzIx\,r\end{array}\right)$$

‖Expand.

$$= 2\left(\begin{array}{l} -2\left\|dx\,Ez\,Iz^3\,r\right\|_1 + dx\,Ez^2 Iz^2 r + \left\|dxEz\,Iz^3\,r\right\|_1 - dzEx\,Ez\,Iz^2\,r + dz\,Ez\,Ix\,Iz^2\,r \\ + 2\left\|dx\,dz^2\,Ez\,Iz\,r\right\|_2 - dx\,dz^2\,Ez^2 r - \left\|dx\,dz^2 Ez\,Iz\,r\right\|_2 + dz^3 Ex\,Ez\,r - dz^3 EzIx\,r\end{array}\right)$$

‖Alphabetize and identify like terms.

$$= 2\left(\begin{array}{l} -dx\,Ez\,Iz^3\,r + dx\,Ez^2 Iz^2 r - dzEx\,Ez\,Iz^2\,r + dz\,Ez\,Ix\,Iz^2\,r \\ + dx\,dz^2\,Ez\,Iz\,r - dx\,dz^2\,Ez^2 r + dz^3 Ex\,Ez\,r - dz^3 EzIx\,r\end{array}\right)$$

‖Combine like terms.

$$= 2Ez\,r\left(\begin{array}{l} -dx\,Iz^3 + dx\,EzIz^2 - dzEx\,Iz^2 + dz\,Ix\,Iz^2 \\ + dx\,dz^2\,Iz - dx\,dz^2\,Ez + dz^3 Ex - dz^3 Ix\end{array}\right)$$

‖Factor out Ez and r.

$$= 2\,Ez\,r(dz^2 - Iz^2)\,(dz\,Ex - dx\,Ez - dz\,Ix + dx\,Iz)$$

‖Factor.

A1.6.2 Solve For Fz Denominator

Simplifying the denominator is practically completed from derivation A1.4.1.

$$Den = (Iz\text{-}Ez\text{-}rdz)(Iz\text{-}Ez + rdz)\left(-AzDx + AxDz - DzGx + AzHx\right)$$

$$= -\left(Iz\text{-}Ez\text{-}rdz\right)\left(Iz\text{-}Ez + rdz\right)\left(AzDx - AxDz + DzGx - AzHx\right)$$

$$= -\left(-2\left(dzEx - dxEz - dzIx + dxIz\right)\left(EzIz - Iz^2 - Ez\,Iz\,r + dz^2r^2\right)\right)$$

This from the result of A1.4.1 which found a nearly identical expression for the denominator of Fx multiplied by (Iz - Ez - rdz)(Iz - Ez + rdz).

$$= 2\left(dzEx - dxEz - dzIx + dxIz\right)\left(EzIz - Iz^2 - Ez\,Iz\,r + dz^2r^2\right)$$

A1.6.3 Solve For Fz Fraction

Now return to the complete fraction, cancel common expression and collect like terms:

$$Fz = \frac{2\;Ez\;r(dz^2\text{-}Iz^2)\;(dz\;Ex\text{-}dx\;Ez\text{-}dz\;Ix + dx\;Iz)}{2\left(dzEx - dxEz - dzIx + dxIz\right)\left(EzIz - Iz^2 - Ez\,Iz\,r + dz^2r^2\right)}$$

$$= \frac{Ez\;r\;(dz^2\text{-}Iz^2)}{Ez\left(Iz\left(1-r\right)\right) + \left(dz^2r^2 - Iz^2\right)}$$

Cancel common factors and factor denominator.

A1.7 Rewrite in matrix form

Rewriting these equations in matrix from yields *Δ'*:

$$\Delta' = \begin{bmatrix} \left(dz^2r^2 - Iz^2\right) & 0 & \left(1-r\right)\left(IxIz + dx\,dz\,r\right) & 0 \\ 0 & \left(dz^2r^2 - Iz^2\right) & \left(1-r\right)\left(IyIz + dy\,dz\,r\right) & 0 \\ 0 & 0 & r(dz^2\text{-}Iz^2) & 0 \\ 0 & 0 & Iz(1-r) & dz^2r^2 - Iz^2 \end{bmatrix}$$

Note this will degenerate to a singular transform if any of the 4 true or false eye points become embedded in the view plane, but in practice this should not happen. Hence we ignore this case. Given this assumption and the fact that scalar multiples of a projective transformation matrix are equivalent rewrite *Δ*:

$$\Delta = \begin{bmatrix} 1 & 0 & \frac{(1-r)(IxIz + dx\,dz\,r)}{dz^2 r^2 - Iz^2} & 0 \\ 0 & 1 & \frac{(1-r)(IyIz + dy\,dz\,r)}{dz^2 r^2 - Iz^2} & 0 \\ 0 & 0 & \frac{r(dz^2 - Iz^2)}{dz^2 r^2 - Iz^2} & 0 \\ 0 & 0 & \frac{Iz(1-r)}{dz^2 r^2 - Iz^2} & 1 \end{bmatrix}$$

Finally Δ can be decomposed as follows:

$$\Delta = \Delta_{\Pr oject} \bullet \Delta_{Scale} \bullet \Delta_{Shear}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{Iz(1-r)}{r(dz^2 - Iz^2)} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{r(dz^2 - Iz^2)}{dz^2 r^2 - Iz^2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{(1-r)(IxIz + dx\,dz\,r)}{dz^2 r^2 - Iz^2} & 0 \\ 0 & 1 & \frac{(1-r)(IyIz + dy\,dz\,r)}{dz^2 r^2 - Iz^2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

For completeness Δ^{-1} can be found component wise:

$$\Delta^{-1} = \Delta^{-1}_{Shear} \bullet \Delta^{-1}_{Scale} \bullet \Delta^{-1}_{\Pr oject}$$

$$= \begin{bmatrix} 1 & 0 & \frac{-(1-r)(IxIz + dx\,dz\,r)}{dz^2 r^2 - Iz^2} & 0 \\ 0 & 1 & \frac{-(1-r)(IyIz + dy\,dz\,r)}{dz^2 r^2 - Iz^2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{dz^2 r^2 - Iz^2}{r(dz^2 - Iz^2)} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{-Iz(1-r)}{r(dz^2 - Iz^2)} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{-(1-r)(IxIz + dx\,dz\,r)}{r(dz^2 - Iz^2)} & 0 \\ 0 & 1 & \frac{-(1-r)(IyIz + dy\,dz\,r)}{r(dz^2 - Iz^2)} & 0 \\ 0 & 0 & \frac{dz^2 r^2 - Iz^2}{r(dz^2 - Iz^2)} & 0 \\ 0 & 0 & \frac{-Iz(1-r)}{r(dz^2 - Iz^2)} & 1 \end{bmatrix}$$