

How Should a Firm Manage Deteriorating Inventory?

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Abstract

Firms selling goods whose quality level deteriorates over time often face difficult decisions when unsold inventory remains. Since the leftover product is often perceived to be of lower quality than the new product, carrying it over to the next period offers the firm a second selling opportunity, but at a reduced price. By doing so, however, the firm subjects sales of its new product to competition from the leftover product. We present a dynamic model that captures the effect of this competition on the firm's production and pricing decisions. We characterize the firm's optimal strategy and find conditions under which the firm is better off carrying all, some, or none of its leftover inventory to the next period. We also show that the price of the new product is independent of the quality level of the leftover product. Finally, we relax the model assumption and assume that there is demand uncertainty in both periods. We run a simulation and find that the firm finds it optimal to introduce the old units only if the level of uncertainty is low and does not exceed a certain threshold.

Keywords (Pricing, inventory, quality, internal competition, perishable)

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1 Introduction

When firms complain of competitive pressure on one of their product lines, they are typically referring to the actions of other firms seeking to steal away their customers. For companies selling perishable goods however, competition may also arise internally when the firm sells inventory of its perishable goods that were leftover from previous periods. If their customers' perception of the quality of these leftovers is lower than that of their new product, the leftover products are often marked down in price. By doing so, however, the firm subjects sales of its new product to competition from its leftover product.

Not all perishable products face this "internal" competition described above. Perishable goods often carry expiration dates, after which they can no longer be sold. A firm providing such a good often replenishes its stock with new product before all of its old product (of the same type) has been sold. The pricing and stocking decisions of the firm in this situation depend upon the characteristics of the perishable product being offered. We characterize perishable products into three types based on the perceived quality level of the aged product by the consumer.

Type 1 items' perceived quality does not degrade continuously over time but, instead, becomes unusable after a given date. Airline seats and hotel rooms are an example of this type of product. Firms selling type 1 product often set multiple prices (and corresponding quantities) depending on the time remaining before expiration and salvages all expired product for zero revenue. The practice of revenue management was developed specifically for these type products, a review of which is provided by Talluri and van Ryzin (2004). Type 2 items' value deteriorates continuously over time reaching a value of zero when a new version of the item becomes available. Examples include newspapers and weather forecast. Firms offering type 2 products usually remove all of the old product and replace it with the new ones once they become available. The discounting of Type 2 items often fall under the realm of dynamic pricing. Bitran and Caldentey (2002) provide a review of dynamic pricing models along with many applications.

In this work we focus however on perishable products from a third type, where type 3

items' perceived quality deteriorates over time but may not reach a value of zero by the time a new version of the item becomes available. The deterioration does ensure however, that the customer values an older product lower than a newer one. Firms selling type 3 product often price differentiate the old and new product. Examples of type 3 product are abundant and can be divided into two main categories. The first category includes products whose actual functionality deteriorates over time such as fruits, vegetables and milk. Supermarkets often mark down their older produce after restocking with fresher product. The second category includes products whose functionality does not degrade but the customers' perceived utility of these products deteriorates over time. Examples include fashionable clothing and high technology products with short life cycles. Almost every fashion goods retailer provides a discount section on their shop floor or website where they offer last season's unsold designs at discounted prices.

In this paper, we focus on the management of type 3 items where a firm faces quantity and pricing decisions for product with different quality levels. Since price, availability, and quality are key dimensions that define a consumer's decision to purchase a product, it is critical to understand how the quality of a product affects the firm's operating decisions. The magnitude of these decisions can be quite substantial. For example, Bloomingdale's Department Store estimates that 50% (around \$400M) of their total sales for women's apparel is sold through the use of mark-down prices. An additional 9% (\$72M) is sold to salvage retailers for pennies on the dollars to avoid excessive cannibalization of their new designs (Berman, 2003).

Observing the pricing and stocking policies of firms selling type 3 products shows a wide range of practices employed. For example, consider the opposing policies of two national bagel chains (Bruegger's and Chesapeake) who compete in the same markets and have similar operating procedures. Stores of each chain begin each day facing a decision on how many bagels to bake before opening their shops and observing demand. Despite the similarities in their operating and market conditions, the two chains use different policies for how to handle bagels that are leftover at the end of the day. Chesapeake Bagels disposes of all leftover bagels while Bruegger's Bagels carries the leftover bagels over to be sold in the next

day. Since the quality of a bagel depreciates quickly over time, Bruegger’s Bagels sells the leftover bagels at a discounted price compared to the price of its fresh bagels, then disposes of any bagels that are older than two days since they consider them too low of a quality to sell at any price.

Why do two firms facing similar production and market conditions handle their perishable inventory so differently? Does the “No Carry” policy dominate the “Carry Everything” policy or vice versa? Should a firm follow a combination of these policies and only carry a portion of its unsold inventory into the next period? If a firm pursues this combination “Carry Some” policy, how should it price its leftover product in relation to its new product? What is clear from the bagel chains and many other industry examples is that there exists a difference of opinion among managers on how to manage the pricing and quantity decisions of perishable product. Other examples of markets that face this problem include: a grocery store that periodically receives shipments of fresh produce and sells it with any unsold inventory from previous periods; a high technology company that has not sold all of its previous model’s stock before the new model design is ready for the market; and a textbook or software company that still holds inventory of the current version of its product when the new version becomes available.

To gain intuition into the broader mix of problems, we study a firm selling a perishable good that lasts for two periods and has two states. The product is considered “new” if it is sold in the same period it was produced, or “old” if it is sold the period after it was produced. The old units suffer a quality reduction and, therefore, provide lower valuations for the customers. Thus, the firm charges lower prices for the old units in comparison to the new units. Referring back to our bagel store example, the chain that carries over its unsold bagels prices them at one third the price of its fresh bagels. We model a two period problem to study the competition effect between the two levels of product in the second period on the firm’s stocking and pricing decisions in the first period. In the second period, we model the competition on the new product from product that did not sell in the first period and is carried over to the second.

The key decisions that the firm must make include: the price and purchase quantity of

new product in both the first and second period, the quantity of unsold product from the first period to make available to the customer in the second period, and the price to charge for the old product in the second period. We solve the firm’s problem and find that the optimal price of the new product remains the same, regardless of the firm’s decision to sell the old product and is independent of the quality level of the old product. Thus, competition from the old product only affects the firm’s second period new product quantity decision, not its new product pricing decision. Further, we find thresholds for the quality level of the old product where the firm should choose to carry all, some, or none of its unsold product over to the next period. These results stem from the fact that demand for the new product is dependent upon the quantity, price, and quality level of the old product competing against it. Through a numerical study, we determine that a firm gains the most benefit from a “Carry” policy when: 1) Uncertainty over the market potential¹ in the first period is high, 2) The quality degradation of the unsold product is low, and 3) The cost to prepare the carried over unit for the market is low compared to the cost to purchase new units. In two extensions to our base model, we find that the firm achieves near optimal profits if it maintains a constant price for both periods and that the inclusion of even small amounts of uncertainty in the second period quickly diminishes the benefits of the “Carry” policy.

1.1 Literature Review

Our research draws upon work in both the operations and marketing areas. We use a similar notion of quality as the marketing literature, where consumers are vertically differentiated over quality but always prefer a higher quality product over a lower one for any given price. Moorthy (1984) solves the monopoly product line problem where a firm has to make product line decisions regarding the number of categories it produces and the level of quality built into each category. He shows that higher-end consumers prefer lower-quality products if they are sufficiently attractive. Thus, the cannibalization effect is the main

¹We capture uncertainty in the model through the market potential, defined as the upper bound of consumers’ valuation for the product’s services. Based on these consumers’ valuations, we derive the demand functions and the market potential’s interpretation in the demand system is the size of the market.

determinant of the optimal price-quality decisions. To solve the cannibalization problem, the firm must offer a price-quality bundle such that the highest valuation consumers buy their preferred quality but the lower valuation consumers get a quality level that is lower than what they desire. Gilbert and Matutes (1993) investigate the competition between the product lines of two manufacturers and find that if the difference between the levels of quality is sufficient, both firms produce. Desai (1999) extends Moorthy's model and solves the douopoly problem, where consumers are differentiated by both their quality preferences and their tastes. Ahmadi and Yang (2000) explore the effects of external competition from third-party firms taking advantage of an arbitrage situation when the producing firm sells a product for different prices in different markets. All of these models are for a single period and the market potential is assumed to be deterministic (the firm sells every unit produced). In contrast, we consider a two period model where the market potential is stochastic in the first period and the purchase of product involves a positive lead-time so that the quantity and pricing decisions must be made before the market potential is realized.

The use of a two period model with a quality parameter differentiating the old and new inventory of a firm has some similarities to the durable goods literature. Desai, Koenigsberg, Purohit (2002) use a durability parameter to represent the competition between new units and previously sold units that reenter the market during the second period. Their durability parameter represents the level of competition between new and used product while we concentrate on nondurable goods, where the only competition occurs between the unsold "lower quality" old units leftover from the previous period and the new "just ordered" units. Another difference occurs in the timing of the firm's decisions. In Desai, Koenigsberg, Purohit (2002), the selling quantity in the first period is chosen after the uncertainty in the market potential is resolved. In our model, both quantity and price decisions must be made before uncertainty is resolved. The objective of our paper is to examine how the quality reduction of unsold units, and the subsequent cannibalization problem created by carrying the unsold units into the next period affects the firms' pricing and production decisions in a newsvendor model.

On the operations side, Petruzzi and Dada (1999) review extensions to the newsvendor

model in which the stocking quantity and selling price are set simultaneously, before the uncertainty in the market potential is resolved. Despite the myopic nature of the newsvendor model, they give conditions where their model can be used in a multi-period setting under the assumption that the quality level of the unsold product never deteriorates. Cachon and Kok (2002) offer a simple adjustment to the newsvendor model that accounts for the fact that the salvage value of the unsold product depends upon the quantity (but not the quality level) available. They find that newsvendor quantities not accounting for this relationship always exceed the optimal quantity, leading to excessive markdown costs. Gallego and van Ryzin (1994) explore dynamic pricing of a perishable product but do not consider the competition effect of selling leftover product from previous periods.

Several papers study a two period version of the newsvendor model where a second order opportunity exists in the second period: Donohue(2000), Fisher and Raman (1996), Fisher, Rajaram and Raman (2001), Kouvelis and Gutierrez (1997), and Petruzzi and Dada (2001) provide a representative sample of this area. Of these, only Petruzzi and Dada (2001) model price as a function of the quantity procured. In their model, demand is a deterministic function but contains an unknown parameter characterized by a subjective distribution that is updated over time. Thus, by selecting a price-quantity pair each period, the firm eventually learns the true demand function. The product life cycle in their model last only for a single period, so no inventory is carried over from one period to the next. In contrast, the firm in our model learns the true demand function immediately after the first period and has the option of carrying over unsold inventory.

The rest of the paper is organized as follows. In the next section, we present the model. In section 3 we compare a No Carry (*NC*) policy with a Carry (*C*) policy in a two-period problem where the firm faces uncertainty in the market potential only in the first period. In section 4 we provide extensions to our model to include the cases of a constant new product price and second period market uncertainty. In section 5 we conclude and gives directions for future research.

2 Model Development

This section describes the model and lays out the assumptions about the firm, the product, and the market. We begin by stating the key assumptions of our model and defining a quality parameter that represents the customer’s perceived differentiation between the new product and the old product in the second period. We then derive inverse demand functions for each product type based on the consumers’ utility functions. Since the problem we describe pertains to both manufacturing firms that produce and sell their own product and retailers who only purchase product from other firms, we use the generic term “purchase” through the rest of this paper to represent either production or purchasing decisions.

Assumption 1. *Key problem dynamics are captured in a two period model*

We consider a two-period game with uncertainty in the market potential during the first period and potential competition in the second period from product that did not sell in the first period and is carried over to the second. Our objective is two fold. First, we study how the presence of competition from lower quality units affects the firm’s pricing and stocking decisions for its new product after the old units have been carried over. This is captured through our second period analysis. Second, the first period allows us to study how the possibility of carrying unsold inventory to the next period affects the firm’s pricing and stocking decisions when it only purchases new product. Thus, a two-period model is sufficient and allows us to maintain tractability.

In the first period, the firm chooses the price, P_1 , and quantity, x_1 , of its product before knowing the complete characterization of the market potential (the maximum price the firm can charge for the product and still sell one unit). Demand for the firm’s product is a function of the unknown market potential and is decreasing with the firm’s price. After demand realization, the firm may be left with unsold units and faces a

choice on how many of these units to carry over to the second period. Without loss of generality, the salvage value of any unsold units not carried over to the second period is normalized to zero.

Assumption 2. *In the second period, the market potential is deterministic².*

This assumption is reasonable for many products where the demand in latter periods is highly correlated to demand in the current period. For example, the introduction of many new products can be divided into two distinct periods. An initial period (learning period) where the firm faces large uncertainty in the market potential of the new product and a second period where most of the uncertainty is resolved but some of the features of the model may change. Ehrenberg (2001) shows that for the majority of new products, most market uncertainty is resolved after 6 weeks. Our main goal is to show the effect of competition between the firm's new product with its carried over product. This assumption allows us to isolate this effect and gain analytical tractability to the firm's first period decisions. We later investigate the sensitivity of our results when this assumption is violated through a simulation experiment.

At the beginning of the second period, the firm has Y units leftover from the previous period, where Y can be zero if demand exceeded supply in the first period or if the firm chooses not to carry over its unsold product. The firm can purchase new units at a price of c each and prepares old units at a per unit cost of h . The cost of preparing the old units includes the holding/storage cost plus any special packaging and preparation required for making the old units available to the market. The firm's quantity decisions in the second period include the number of new units, x_2 , and the number of old units, y , made available to the market. The quantity of new product the firm can purchase is not capacity constrained, but the firm cannot prepare more than Y of the old units, i.e., $y \leq Y$. Demand in the second period is deterministic, but inversely proportional to the price and quality deterioration of the product made available. We assume that the firm acts rationally and knows at the end of the first period exactly how many of the old units it will sell in the second period (since the market potential in the second period is deterministic) so it only incurs a preparation cost, h , for the y units it sells.

Assumption 3. *Consumers are heterogenous in their valuations of the product and are*

²In section 4.2 we relax this assumption and explore the firm's decisions when there is uncertainty in the firm's second period market potential.

distributed uniformly in the interval $[0, \alpha]$.

To capture the effect of the prices of old and new units on the consumers' demand, we develop linear inverse demand functions. We derive the demand functions from the consumers' utility functions and assume that consumers are heterogeneous in their valuations of the product. In this vertical differentiation model,³ we use the random variable $\phi \in (0, \alpha)$ to represent a consumer's valuation of the service provided by the product, where α represents the valuation of the consumer that values the product the most. Note that a consumer with a higher ϕ values the product more than a consumer with a lower ϕ . Finally, we assume that the consumers' valuations are distributed uniformly in the interval $[0, \alpha]$ and, in any period, each consumer uses at most one unit.

While this representation is common in the literature and has many empirical validations, the incorporation of market potential uncertainty into a linear inverse demand function carries the restriction that the variance of the market potential is not affected by the product's price. Thus, a multiplicative model of the inverse demand function may be interesting for future research.

Assumption 4. *The quality of the old product is exogenous.*

To model the leftover product's quality level, we use the parameter q , $0 \leq q \leq 1$, to represent how much a unit unsold in the previous period deteriorates before the current period. All unsold units from the first period deteriorate by $(1 - q)$ before the second period and provide the customer with a utility that is less than that provided by a new unit (We later investigate the case where the consumers' valuation for the old model remains the same but they perceive the new model as providing higher quality). In our utility model, this implies that a consumer's valuation of the services from a unsold unit is $q\phi$. Note that if $q = 0$, consumers get no utility out of the unit (it has deteriorated fully) and the manufacturer does not sell it. If $q = 1$, consumers view the new and old units as being identical and derive equal utility from either type. The quality level is considered exogenous,

³A vertical differentiation model refers to markets where consumers' valuation for a product characteristic (product quality in our model) has an agreed order, i.e. all consumers prefer a higher quality product to a lower quality product (Tirole 1988).

thus the firm has no control over the deterioration of its product. Although there are many practical examples where this is true, there may be cases where the firm can influence the quality level of the carried over units through its spending on storage and preparation cost. We leave this extension for future research.

We consider the following inverse demand system⁴:

$$\begin{aligned} P_2^o &= q(\alpha - x_2 - y) , \\ P_2^n &= \alpha - x_2 - qy , \\ P_1 &= \alpha - x_1, \end{aligned}$$

where P_2^n represents the price of the new product in the second period, P_2^o represents the price of the old product in the second period, and P_1 represents the price of the new product in the first period. Note that the quality level of the old products has different effect on the price of the new and the old units. As the old product quality increases, the price of the old product increases as the old product provides better services to consumers. However, as old product quality increases, the price of the new product decreases as the old a new products become closer substitute and there is more (internal) competition.

To account for uncertainty in the market potential, let the upper bound of the consumer's valuation α be made up of two parts, $A + u$, where A represents the deterministic piece and u represents the unknown piece. We assume that u is a random variable independently drawn from $F(\cdot)$, that F is twice differentiable over $[0, B]$, has a finite mean μ , a nondecreasing hazard rate, and its inverse function exists. This representation of demand is common in the economics and marketing literature and can be interpreted as a case where the shape of the demand curve is deterministic while the scaling parameter representing the market potential has a random component.

The inverse demand functions can now be represented as

$$P_1 = A - x_1 + u \tag{1}$$

⁴Full derivation of the demand system is in the appendix.

for the first period (since there are no old units available to sell in the first period then we only need one inverse demand function) and

$$P_2^n = A - x_2 - qy + u \quad (2)$$

$$\text{and } P_2^o = q(A - x_2 - y + u) \quad (3)$$

for the second period. The latter functions capture the competition between the firm's old product with its just produced products. We preview the model's parameters, random variables, and decision variables in the table below.

	First Period		Second Period	
	Symbol	Definition	Symbol	Definition
Parameters	A	Deterministic market potential	R	Realization of $A + u$
	q	Quality level	Y	Unsold units of x_1
	c	Cost to purchase new product		
	h	Cost to carry old product		
	B	Upper limit for rand var u		
RandomVars	u	Stochastic market potential	None	
DecisionVars	P_1	Price of new products	P_2^n	Price of new products
	x_1	New products purchased	x_2	New products purchased
	x_1^d	Deterministic component of x_1	P_2^o	Price of old products
	z	Safety stock component of x_1	y	Old products made available

Table 1: Parameters and Variables

3 Analysis

We consider two policies; No Carry (NC), and Carry (C). Under the NC policy, the firm never carries leftover units over to sell in the second period while under the C policy, it does. We compare the two policies in a two-period model where the firm faces uncertainty in the

market potential only in the first period. To differentiate the results of the two policies, we place a tilde ($\tilde{\cdot}$) above the decision variables for the *NC* policy.

3.1 Policy *NC*: No Carry

Under policy *NC*, the firm never carries leftover inventory over from one period to the next. Thus, the firm's two decisions each period are how many new units to purchase and how much to charge for each unit. To find the sub game perfect equilibrium we solve the game backwards starting with the firm's second period decisions.

3.1.1 Second Period: No Uncertainty

In the second period the firm knows the realization of the market potential before making its quantity, \tilde{x}_2 , and pricing, \tilde{P}_2^n , decisions. Let R represent the realization of $(A + u)$. The firm's inverse demand function is given by

$$\tilde{P}_2^n = R - \tilde{x}_2 \ .$$

The second period price is a direct outcome of the quantity decisions so the firm's only decision is how many new units to purchase. Let $\tilde{\Pi}_2$ represent the firm's profit in the second period when using the *NC* policy. The firm's second period objective is

$$Max_{\tilde{P}_2^n} \tilde{\Pi}_2 = (\tilde{P}_2^n - c)\tilde{x}_2 \ ,$$

which is concave in \tilde{x}_2 . Checking first order conditions yields the familiar monopoly results giving a new product production quantity, price, and profit of

$$\tilde{x}_2^* = \frac{R - c}{2}, \quad \tilde{P}_2^{n*} = \frac{R + c}{2}, \quad \tilde{\Pi}_2^* = \left(\frac{R - c}{2}\right)^2 \ . \quad (4)$$

3.1.2 First Period: With Uncertainty

In the first period, both the quantity, \tilde{x}_1 , and price, \tilde{P}_1 , must be set before observing the size of the market. The firm's objective in the first period is to maximize its expected profit over the first and second periods. Let $\tilde{\Pi}_{12}$ represent the firm's expected profit over the first and second period when using the *NC* policy. The firm's expected profit includes its revenue (the firm's chosen price times the minimum of the amount of product purchased in the first period and the firm's monopoly quantity chosen after observing the realized market potential) minus the purchase cost plus a discount factor, ρ , times the second period profit given in (4).

Since the market potential is made up of a deterministic and a stochastic component, we divide the firm's quantity decision \tilde{x}_1 into two components. Let $\tilde{x}_1^d = A - \tilde{P}_1$ represent the part of the quantity decision corresponding to the known market potential and \tilde{z} represent the part corresponding to the stochastic market potential. The stochastic component represents the safety stock that the firm holds to account for the market uncertainty. The profit for the first period after the total market potential is realized ($u = U$) is the difference between the firm's sales revenue and cost plus its discounted second period profit:

$$\tilde{\Pi}_{12}(\tilde{z}, \tilde{P}_1 | u = U) = \begin{cases} \tilde{P}_1(\tilde{x}_1^d + U) - c(\tilde{x}_1^d + \tilde{z}) + \rho\tilde{\Pi}_2^*, & \text{if } U \leq \tilde{z} \\ (\tilde{P}_1 - c)(\tilde{x}_1^d + \tilde{z}) + \rho\tilde{\Pi}_2^*, & \text{if } \tilde{z} \leq U \end{cases}.$$

The firm's first period objective is $Max_{\tilde{P}_1, \tilde{z}} \tilde{\Pi}_{12}$. Following our assumption that the uncertain component of the market potential is continuously distributed, the firm's expected profit over the first and second period can be expressed as

$$\tilde{\Pi}_{12}(\tilde{z}, \tilde{P}_1) = \int_0^{\tilde{z}} \tilde{P}_1(\tilde{x}_1^d + u) f(u) du + \int_{\tilde{z}}^B \tilde{P}_1(\tilde{x}_1^d + \tilde{z}) f(u) du - c(\tilde{x}_1^d + \tilde{z}) + \rho\tilde{\Pi}_2^*. \quad (5)$$

The firm's first period decisions do not affect its second period's profit so the firm's objective reduces to a single period problem. Whitin (1955) was the first to introduce a single period problem with quantity and pricing decisions. Zabel (1970) introduced the solution method

of first optimizing \tilde{P}_1 for a given \tilde{z} and then searching over the optimal trajectory to maximize (5). Petruzzi and Dada (1999) provide an excellent review of the literature in this area and show that (5) is concave and unimodal in \tilde{z} and $\tilde{P}_1^*(\tilde{z})$ as long as the distribution for u has a nondecreasing hazard rate. Solving the first order conditions yields an optimal safety stock quantity of

$$\tilde{z}^* = F_u^{-1} \left(\frac{\tilde{P}_1^* - c}{\tilde{P}_1^*} \right) \quad (6)$$

and an optimal price of

$$\tilde{P}_1^* = \frac{A + c + \mu}{2} - \frac{\Theta(\tilde{z}^*)}{2} \quad (7)$$

where $\Theta(\tilde{z}^*) = \int_{\tilde{z}^*}^B (u - \tilde{z}^*) f(u) du$. The total purchase quantity is the combination of the optimal deterministic and stochastic components: $\tilde{x}_1^* = \tilde{x}_1^{d*} + \tilde{z}^*$.

3.2 Policy C: Carry Leftover Product

Under policy C , the firm may carry leftover inventory over into the second period. The firm faces the same quantity and pricing decisions in the first period as in the NC policy case. In the second period however, the firm faces additional decisions on how many of the carried over units to prepare for the market and what to price them at. To find the subgame perfect equilibrium, we solve the problem using backward induction.

3.2.1 Second Period: No Uncertainty

At the start of the second period, the firm holds Y units of leftover inventory from the first period and may have to purchase additional new units. Let $\Pi_2(Y)$ represent the firm's profit in the second period when carrying Y units of unsold inventory over from the first period, $Y \geq 0$. The inverse demand functions are given by (2) and (3). The pricing decisions are direct outcomes of the quantity decisions. Since the firm's second period decisions are dependent upon the amount of leftover product carried over from the first period, we characterize these conditions below.

Firm Has No Leftover Inventory With no leftover inventory from the pervious period, the firm's only decision is how many new units to purchase. The firm's objective is

$$Max_{P_2^n} \Pi_2(0) = (P_2^n - c)x_2 \quad . \quad (8)$$

The firm's objective is identical to the firm's objective for policy *NC*. Thus, the optimal new product production quantity, price, and profit are the same as (4).

Next, we analyze the case where the firm begins the second period with leftover inventory from the first period.

Firm Has Leftover Inventory In this case, the firm starts the second period with Y units of leftover inventory that was not sold the first period. In addition to the decisions on the new-product, the firm has to make decisions regarding how many old units to prepare, y , and what to price them, P_2^o . The firm's objective is

$$Max_{P_2^n, P_2^o} \Pi_2(Y) = (P_2^n - c)x_2 + (P_2^o - h)y \quad (9)$$

$$s.t. \quad y \leq Y.$$

The firm's objective is concave in P_2^n and P_2^o . Checking the first order conditions of (9) yields several interesting results. Proposition 1 looks at the effect on the firm's new product pricing and quantity decisions when faced with competition from its old product.

Proposition 1 *The optimal price for the new product, $P_2^{n*} = \frac{R+c}{2}$, is independent of the quality level of the old product⁵.*

Proposition 1 states that the optimal price of the new product is the same as in the *NC* policy and is independent of the quality level of the old product. Thus, competition from the old product only affects the firm's second period new quantity decision, not its new product pricing decision. The intuition behind this result is that the valuation of consumers who purchase new units does not depend on the quality level of the old units; thus the price

⁵All proofs are provided in the appendix.

of the new units is independent of the quality reduction and the firm extracts the monopoly surplus from each buyer (of new units). Though the price of the new product is not affected by the quality level, the firm does take the quality deterioration into account to moderate the cannibalization effect. The firm does so through its quantity decisions. As the quality level of the old product decreases, the firm increases the number of new units it makes (orders) and places an upper bound on the number of old units that are carried over from the previous period. The next two propositions give the quality threshold levels that determine whether to sell all, some, or none of the old product.

Proposition 2 *There exists a threshold level for the quality of the old product, $\frac{h}{c}$, below which the firm only sells the new product.*

Proposition 2 states that a lower quality threshold exists such that the firms' ratio of old unit carrying cost to its new unit purchasing cost must fall above it before carrying over any unsold product becomes attractive. If the customer's perception of the quality level of the old product is so low that they are unwilling to cover the marginal carrying cost, then it is optimal for the firm not to carry. We define h as the total cost to carry and prepare the old units for the second period. It includes not only the normal inventory holding cost but also any additional packaging, special storage conditions, labor, etc. For example, the selling of older produce and day-old bagels often involve repackaging which includes expenses other than just the holding cost. For example, some types of electronic circuit boards oxidize over time and must be kept in expensive low-oxygen storage facilities. With the lower quality threshold, firms can make informed decisions on whether or not the benefit of carrying their unsold inventory into future periods exceeds these cost. The criteria for whether to carry all of the unsold product or only a portion of it is given in proposition 3.

Proposition 3 *There exists a threshold level for the quality of the old product, $\frac{R-c+h}{R}$, above which the firm only sells the old product, up to a quantity of $\min\left(\frac{qR-h}{2q}, Y\right)$.*

Proposition 3 states that an upper quality threshold exists, above which the firm sells all of its leftover product up to the minimum of the number of old units available, Y , or a monopoly quantity modified by the quality level and based on the cost structure of the old units, $\frac{qR-h}{2q}$. Products whose quality level is above this upper threshold ($\frac{R-c+h}{R}$) present a

larger benefit to the firm than do the sale of new product. Therefore, the firm has higher incentives to carry and prepare the old product than to produce a new product. Only under very rare circumstances will the firm have so much unsold inventory from the first period that it chooses to not sell any new product and limits its quantity of old product to its monopoly quantity.

The upper threshold level $(1 - \frac{c-h}{R})$ is more robust for changes in h than the lower threshold level $(\frac{h}{c})$. Thus, as h increases the difference between the two thresholds $(\frac{R-c+h}{R} - \frac{h}{c})$ decreases; the range of the quality levels where the firm carries only some of the unsold inventories shrinks, and the ranges where the firm carries none or all of its unsold inventory expands. In practice, this means that firms facing either low or high h may not be too far from optimal if they restrict themselves to carrying either all or nothing of their unsold product.

Propositions 1 through 3 define the conditions necessary to state the firm's optimal prices and quantities. The three propositions characterize the following five conditions needed for the problem's solution. The five conditions are summarized below.

$$\begin{aligned}
\text{Condition 1)} \quad & q \leq \frac{h}{c} \text{ or } Y = 0 \\
\text{Condition 2)} \quad & \frac{h}{c} < q < \frac{R-c+h}{R} \ \& \ \frac{cq-h}{2q(1-q)} > Y \\
\text{Condition 3)} \quad & \frac{h}{c} < q < \frac{R-c+h}{R} \ \& \ \frac{cq-h}{2q(1-q)} \leq Y \\
\text{Condition 4)} \quad & \frac{R-c+h}{R} \leq q \ \& \ \frac{qR-h}{2q} > Y \\
\text{Condition 5)} \quad & \frac{R-c+h}{R} \leq q \ \& \ \frac{qR-h}{2q} \leq Y
\end{aligned}$$

Under the first condition, either the firm ends the first period with no leftover units or it knows the quality of the leftover units will fall below the threshold given in Proposition 2 so it chooses not to carry any of the old units into the second period. The second condition covers the case where the quality is above the threshold level but the firm is constrained in the number of old units to sell in the second period by the number of units leftover from period one. Under the third and fifth conditions, the firm is no longer constrained in the number of old units it can sell so its decision is based on whether or not the quality level falls above the upper threshold level described in Proposition 3. These five conditions correspond

to regions on a two-axis plot of the quality level and the old product selling quantity. The regions (abbreviated as C1 for Condition 1, C2 for Condition 2, etc.) are shown in Figure 1.

*** Insert Figure 1 Here ***

Checking the first order conditions of (9) subject to the conditions described in Table 1 gives the optimal prices of the old product (P_2^{o*}) and the optimal purchase quantities of the old (x_2^*) and new product (y^*) under conditions 1-5. Substituting these values back into (9) gives the optimal expected second period profit under each condition. These values are summarized in Table 2 below.

<i>Condition</i>	x_2^*	y^*	P_2^{o*}	$\Pi_2^*(Y)$
1)	$\frac{R-c}{2}$	0	0	$\left(\frac{R-c}{2}\right)^2$
2)	$\frac{R-2qY-c}{2}$	Y	$\frac{q(R+2qY+c-2Y)}{2}$	$\frac{(R-c)^2}{4} + qY(qY - Y + c) - Yh$
3)	$\frac{(1-q)R-c+h}{2(1-q)}$	$\frac{cq-h}{2q(1-q)}$	$\frac{qR+h}{2}$	$\frac{qR^2 - q^2R^2 - 2cqR + 2cq^2R + qc^2 - 2qch + h^2}{4q(1-q)}$
4)	$\frac{R-2qY-c}{2}$	Y	$\frac{q(R+2qY+c-2Y)}{2}$	$\frac{(R-c)^2}{4} + qY(qY - Y + c) - Yh$
5)	0	$\frac{qR-h}{2q}$	$\frac{qR+h}{2}$	$\frac{(qR-h)^2}{4q}$

Table 2

As expected, the optimal purchase quantity of old(new) units increases(decreases) with the quality of the old units. Conditions 1-5 correspond to increasing levels of quality for the old units. Observing the expressions for y^* in Table 2 we see that the firm never carries old inventory when the quality level is below the lower threshold (Condition 1) but sells the monopoly quantity of old units when the quality level is above the upper threshold (Condition 5). As the old units become closer substitutes for the new units (q increases), they provide a higher level of competition for the new units and the firm prepares more of them to sell. Condition 4 gives the optimal values when the quality level is above the upper threshold but the firm has less than the monopoly quantity of old units to sell. For quality levels between the upper and lower thresholds, Conditions 2 and 3 give the optimal values depending on whether or not the firm holds more (Condition 3) or less (Condition 2) old units than the optimal selling quantity, $y^* = \frac{cq-h}{2q(1-q)}$. Knowing the optimal responses for the

firm in the second period, we now move back in time to the first period where the firm must make decisions without knowing the exact market potential for its product.

3.2.2 First Period: With Uncertainty

In the first period the firm only makes price and quantity decisions for its new units. The firm faces a market potential with a stochastic component and has to take into account the effect of its decisions on its future performance. The total market potential for the product is $A + u$, where A is deterministic and known by the firm while u is stochastic and the firm only knows its distribution. We divide the firm's first period quantity decision into a deterministic and a stochastic component where the optimal deterministic component of the quantity decision is $x_1^d = \frac{A-c}{2}$ and z represents the firm's safety stock. Let Y be the positive difference between the actual demand and the safety stock, $Y = (z - u)^+$. Since the firm knows the quality level of its leftover product before making its first period decisions, we study the firm's decisions under the three ranges for q discussed in Propositions 2 and 3 (low, medium, and high).

Case A: $\left(q \leq \frac{h}{c}\right)$

In this case, corresponding to Condition 1 from Table 1, the quality level of the leftover product is below the lower threshold so the firm never carries over unsold inventory. The expected second period profits are given in Table 2. Because no inventory is carried over to the second period, the problem reduces to the first period problem solved for policy *NC*. The firm's optimal price and safety stock are $z^* = \tilde{z}^*$ and $P_1^* = \tilde{P}_1^*$, where \tilde{z}^* and \tilde{P}_1^* are given in (6) and (7) respectively. Next we consider the case where the quality level is high enough that the firm decides to carry inventory.

Case B: $\left(\frac{h}{c} < q < \frac{A+\mu-c+h}{A+\mu} \text{ and } \frac{cq-h}{2q(1-q)} < B\right)$

In this case, corresponding to Condition 2 from Table 1, the quality level of the leftover product is sufficient for the firm to carry over some of the unsold inventory. For quality levels in this range, we see from Table 2 that the firm never carries more than $\frac{cq-h}{2q(1-q)}$ units, which is the maximum amount it will sell in the second period and that this maximum amount is less than the upper limit for u . For ease of notation, let $\Psi = \frac{cq-h}{2q(1-q)}$ represent this upper

bound of the amount of unsold product the firm carries over to the second period. The profit for the first period after the total market potential is realized ($u = U$) is the difference between the firm's sales revenue and cost plus its discounted future profits:

$$\Pi_{12}(z, P_1 | u = U) = \begin{cases} P_1(x_1^d + U) - c(x_1^d + z) + \rho\Pi_2(\Psi), & \text{if } U \leq z - \Psi \\ P_1(x_1^d + U) - c(x_1^d + z) + \rho\Pi_2(z - U), & \text{if } z - \Psi < U \leq z \\ (P_1 - c)(x_1^d + z) + \rho\Pi_2(0), & \text{if } z \leq U \end{cases} .$$

where Table 2 gives the expressions for $\Pi_2(\Psi)$, $\Pi_2(z - u)$, and $\Pi_2(0)$ under Conditions 3, 2, and 1 respectively. Before the uncertainty of the market potential is resolved, the firm's expected two-period profit is

$$\begin{aligned} \Pi_{12}(z, P_1) &= \int_0^{z-\Psi} [P_1(A - P_1 + u) + \rho\Pi_2(\Psi)] f(u) du \\ &+ \int_{z-\Psi}^z [P_1(A - P_1 + u) + \rho\Pi_2(z - u)] f(u) du \\ &+ \int_z^B [P_1(A - P_1 + z) + \rho\Pi_2(0)] f(u) du - c(A - P_1 + z). \end{aligned} \quad (10)$$

The first integral in (10) covers the probability that the market potential is low and the firm carries only a fraction of the leftover inventory to the second period, $Y = \Psi$. The second integral covers the probability of a medium level market potential and the firm carries all of its leftover product to the second period, $Y = E_u(z - u)$. The third integral covers the probability that the market potential is high and the firm stocks out in the first period, thus it has no units to carry over to the second period, $Y = 0$. The last component of (10) is the cost to purchase the new units.

The firm's objective is to maximize its expected profit which is jointly concave in z and P_1 . Solving the first order conditions with respect to z yields;

$$\frac{\partial E[\Pi_{12}(z, P_1)]}{\partial z} = P_1[1 - F(z)] - c + \int_{z-\Psi}^z [2\rho q(1-q)(u-z) + \rho(cq-h)]f(u) du \quad . \quad (11)$$

It is not possible to obtain a closed form solution for optimal safety stock in the general distribution case. We can, however, rearrange terms so that a comparison to the results in the *NC* policy is possible. Solving (11) for z gives the iterative solution

$$z^* = F_u^{-1} \left(\frac{P_1^* - c + \int_{z^*-\Psi}^{z^*} [2\rho q(1-q)(u-z^*) + \rho(cq-h)]f(u) du}{P_1^*} \right). \quad (12)$$

The following lemma compares this optimal quantity with the optimal quantity from the *NC* policy, $\tilde{z}^* = F_u^{-1} \left(\frac{\tilde{P}_1^* - c}{\tilde{P}_1^*} \right)$.

Lemma 1 *The optimal safety stock under the C policy and Condition 2 is higher than the optimal safety stock for the NC policy.*

Lemma 1 states that a firm should purchase more product in the first period when it has the possibility of selling unsold product in the second period. This result is not as obvious as it may appear at first glance. There are two counter-acting effects taking place in the firm's first period quantity decision. On one hand, the availability of a second selling opportunity lowers the firm's overage cost and induces it to produce more. On the other hand, the cannibalization of its new product sales in the second period by its old product drives its overage cost up, inducing it to produce less in the first period. Our proposition shows that under Condition 2, the "purchase more" effect dominates the "purchase less" effect resulting in a larger quantity than when the firm does not carry over its unsold inventory. We now continue the analysis by solving for the optimal price.

Solving the first order condition of (10) with respect to P_1 yields;

$$P_1^* = \frac{A + c + \mu}{2} - \frac{\Theta(z^*)}{2}, \quad (13)$$

which has the same relationship to z^* as in policy *NC* (7). We can now state the second lemma that compares the firm's first period price under the *C* and *NC* policies.

Lemma 2 *The optimal first period price under the C policy and Condition 2 is higher than the optimal price for the NC policy.*

We are now ready to prove our main result comparing the optimal decisions under the Carry versus No Carry policies.

Proposition 4 *The expected leftover inventory under the C policy and Condition 2 is higher than the expected leftover inventory under the NC policy.*

A simple way to interpret Proposition 4 is that a firm should carry more safety stock in the first period when there exist an option to sell unsold product in the second period. Safety stock under price-dependent demand increases with the quantity and price of the product. Lemmas 1 and 2 show that both quantity and price increase under the *C* policy, implying that the benefit from the second selling opportunity in the second period is greater than the detrimental cannibalization effects. While we prove our results only for Case B, similar results can be obtained for Cases C and D, briefly described below.

Case C: $\left(\frac{h}{c} < q < \frac{A+\mu-c+h}{A+\mu} \text{ and } \frac{cq-h}{2q(1-q)} \geq B\right)$ **and Case D:** $\left(\frac{A+\mu-c+h}{A+\mu} < q\right)$

In Case C, corresponding to Condition 3 from Table 1, the quality level of the leftover product is sufficient for the firm to carry over some of the unsold inventory but the maximum limit to the amount it will carry is larger than the maximum limit of the market potential uncertainty. Case D corresponding to Conditions 4 and 5 from Table 1, the quality level of the leftover product is above the upper threshold level and the firm prefers to sell all of its leftover product in the second period, up to the monopoly quantity $\frac{qA+qu-h}{2q}$, before producing any new product. The results for these two cases are given in the appendix. The analysis for the optimal quantity and price decisions is similar to Case B and is not repeated here.

3.2.3 Numerical Example

We illustrate our model through a numeric example. First assume that the market potential uncertainty variable is uniformly distributed along the range $[0, 100]$. Now assume the

following parameter values: $A = 50$, $\rho = .9$, $c = 10$, $h = 5$, and $q = .8$. Since ($q > \frac{h}{c}$) then the quality of the leftover product is above the lower threshold value and it is optimal for the firm to follow policy C , i.e., to carry some of its unsold inventory into the second period. Next, check to see if the quality level is below the upper threshold value ($q < \frac{A+\mu-c+h}{A+\mu} = .95$). It is, so either Case B or C applies. To determine which one, check to see if the maximum amount to carry, $\Psi = \frac{cq-h}{2q(1-q)} = 9.4$ is larger than the upper limit for the market potential uncertainty, $B = 100$. It is not, so Case B applies for these parameter values. Plugging the values into (12), (13), and (10) gives $z^* = 81.77$, $P_1^* = 54.17$, and $\Pi_{12}^* = 3627$. Comparing this to $\tilde{z}^* = 81.53$, $P_1^* = 54.15$, and $\tilde{\Pi}_{12}^* = 3439$ obtained from using policy NC and shows that by choosing to carry unsold product to the second period, the firm produces more in the first period and charges higher prices, increasing its total expected profits over both periods by 5.5%.

3.2.4 Policy Comparisons

This section reports on a numerical study to further explore the properties of the carry policy and compare the expected profits obtained from using the carry policy to those obtained from the no-carry policy. In particular, we are interested in the percentage improvement of policy C over policy NC . Note that the improvements will always be positive as policy C represents a relaxation of policy NC .

We begin by constructing base scenarios from the following parameter values:

$u \sim U[0, B]$, $f(u) = \frac{1}{B}$, $F(u) = \frac{u}{B}$	$\rho = .9$
$A + \mu = 100$	$c = 10$
$(A, B) = [(90, 20), (75, 50), (50, 100)]$	$q = \{\frac{h}{c}, \dots, .9, .95, .99\}$
$\frac{h}{c} = 0, .25, .5, .75$	

The cost of purchasing new product is normalized to $c = 10$, the discount rate is held constant at $\rho = .9$ and the expected market potential is normalized to $A + \mu = 100$. For our example case where uncertainty over the market potential is uniformly distributed, any choice for the deterministic portion of the market potential A implies a one-to-one mapping

to the upper bound of the distribution for the uncertain portion of the market potential, B . We study three possible combinations of (A, B) pertaining to low, medium, and high degrees of uncertainty over the original market potential for the product. The cost ratio, $\frac{h}{c}$, captures varying degrees of the cost to carry over the unsold product. A low ratio of $\frac{h}{c} = .025$ represents the case where the only costs involved in carrying the unsold product are just the additional handling and storage of the product while a high ratio of $\frac{h}{c} = .75$ represents the case where the unsold product may need to be significantly updated to bring it up to the current technology level of the new product. Note that it is never beneficial for the firm to carry unsold inventory when $q \leq \frac{h}{c}$ or $\frac{h}{c} > 1$. Thus, q is varied in increments of 0.05 from $\frac{h}{c}$ to 0.99.

Table 3 presents summary data on the increase in expected profits potential from using the carry policy (C) versus the no carry policy (NC). The $C - NC$ column represents the difference in profits between the two policies. These results illustrate why a firm should consider a carry policy as increases in expected profit of over 10% are obtainable for firms choosing to carry over the optimal quantity of unsold product to the second period. They also provide some insight as to what conditions the carry policy is most beneficial. In particular, firms should strongly consider the carry policy when: 1) Uncertainty over the market potential is high (μ is significant compared to A), 2) The quality degradation of the unsold product is low (q is close to 1), and 3) The cost to prepare the carried over unit for the market is low compared to the cost to purchase new units ($h \ll c$).

*** Insert Table 3 Here ***

4 Extensions and Sensitivity Analysis

In this section we examine extensions of the basic problem. We first test the sensitivity of our results for the cases when the price of the new product is held constant between periods 1 and 2. Then we relax the assumption of the deterministic demand in the second period and assume that there is uncertainty over the second period market potential. In the appendix,

we analyze the case where the quality of the new product is perceived as an improvement over the old product.

4.1 New Product Price is Held Constant ($P_1 = P_2^n$)

There are many reasons why a firm will resist changing the price for its new product from one period to the next. Price changes sometimes confuse and frustrate customers resulting in a loss of goodwill for the firm. If customers are strategic, they may postpone their purchase knowing that the future holds a lower price for an item. Finally, price changes are costly for most retailers requiring significant effort. In this section, we test the sensitivity of our model to the case where the retailer only uses the deterministic price, P_2^n , for both periods.

The use of the deterministic price in the first period makes the problem much easier to solve as (12) and (16) can now be solved directly by replacing P_1 with P_2^n . The deterministic price is always larger than the optimal first period price under uncertainty, i.e. $P_2^n > P_1^*$. This can be seen by observing from (13) that $P_1^* = P_2^n - \Theta(z^*)/2$ and the fact that $\Theta(z)$ is positive for any value of z . From (12) and (16) we can see that since $P_2^n > P_1^*$, then the quantity of safety stock, z , produced with the deterministic price is always larger than the amount produced with the optimal price. Thus, the firm prices higher and produces more when using the deterministic price heuristic.

To test how much of a penalty the firm incurs for using this constant price heuristic, we reran our numerical example described in the previous section with the firm using the heuristic and compared the expected two-period profits with the original expected two-period profits obtained using the optimal policy. As expected, the penalty increases with the amount of uncertainty and is always higher under the carry policy than the no carry policy. Surprisingly, the penalty is small: The maximum profit penalty was 0.02 percent, occurring in the high uncertainty case ($A = 50$, $B = 100$ corresponding to a coefficient of variation = 0.29). While gross generalizations from such a small sample should not be made, the study does indicate that the deterministic price heuristic may be quite good and the need for stochastic optimization of both price and quantity may have been overstated in the previous literature.

4.2 Uncertainty in the Second Period Market Potential

Up to this point we have assumed that the firm learns the market potential before making its second period decisions. This was done to isolate the effect of the inter-firm competition in the second period and for analytical tractability. This assumption is reasonable for products such as fashion goods where the demand in latter periods is highly correlated to demand in the current period. It is not as good for products such as grocery items where the market potential may vary considerably from one period to the next. In this section we check the sensitivity of our results to this important assumption.

Because analytical solutions involving uncertainty in both periods becomes untractable, we use a simulation study to determine the firm's optimal second period decisions when facing uncertainty in the market potential. For each set of discrete parameter values ($q, h/c$, and the distribution and magnitude of the uncertainty)⁶, separate simulations are run over the entire range of discrete decision variable values (x, y, P_2^n , and P_2^o). Optimal policies are found through searching over the full state space and choosing the set of decision variable values that result in the maximum expected profit for the particular parameter value set. Since we only model the firm's second period decisions, we assume an unlimited supply of old products is available for the firm to sell, i.e. y is unconstrained. After determining the firm's optimal second period behavior, we can then infer the firm's first period responses.

We choose the following distributions and parameter values for our test:

Distribution	Normal, Uniform
Coefficients of Variation	0, .0058, .0173, .0289, .0404
Quality Level	$q = \{.5, .6, .7, .8, .9\}$
Other Values	$\frac{h}{c} = 0.4, A + \mu = 100$

In all, 50 different scenarios were run (5 levels of variability times 5 levels of quality times two different distributions). Each simulation captured the mean profit for a particular scenario from 50 independent selling opportunities, replicated 1000 times. The standard

⁶The cost ratio h/c was not varied because the most significant effect occurs due to the relative difference between the cost ratio and the quality level (which was varied).

error between replications was less than 0.001% and each parameter set was tested against the same sample path (i.e., the simulations were coupled). The optimal policy was found by searching over the integer decision variable set for each scenario. The results of the simulation are given in Table 4. Each cell includes the optimal price and quantity of both the old and new product along with the mean profit. Values of zero are used when it was not optimal to carry any of the old product.

*** Insert Table 4 Here ***

The results of the simulation are surprising. The firm's optimal quantity of old product to sell *decreases* rapidly in the amount of uncertainty in the market potential. Before viewing the simulation results, our intuition was that the firm would protect itself from the uncertainty by carrying safety stock of both the old and the new product. The results, however, show that the firm prefers to carry safety stock made up predominantly of its new product and the quantity of old product made available to sell decreases in the amount of uncertainty from the quantity chosen if the market potential was deterministic.

An explanation of this result can be obtained through marginal analysis. Faced with uncertainty in the market potential, the firm carries more units than the deterministic results from Table 2. If the realization of market potential is higher than expected, the firm's safety stock provides a cushion so that demand can be met and prices are maintained. If the realization is lower than expected, the extra inventory pushes prices down. On the margin, the benefit from having extra old product when the market potential turns out to be high is less than the penalty when the market potential is low. This is due to the fact that having too much old product affects both the price of the old and the new product on the low market potential side but only adds its smaller margin on the high market potential side. Thus, the cannibalization effect dominates the additional sale effect, causing the firm to carry safety stock consisting primarily of its new product.

At first glance, it may appear like this result contradicts our finding from the numerical study of the previous section where we found that the firm is more likely to carry over old product when market uncertainty is high. The difference stems from the fact that the

numerical study only considered uncertainty in the first period while our simulation considers uncertainty in both periods. For the case of deterministic second period market potential, the firm knows the exact cannibalization effect and takes it into account when deciding how many units to produce in the first period and how to price these units. Thus, increasing levels of uncertainty lead the firm to carry more safety stock, making it more attractive to carry over the unsold inventory.

5 Conclusions

We model the pricing and quantity decisions of a monopoly firm offering a perishable product over two periods. In the first period the firm makes pricing and quantity decisions under uncertainty over the market potential of the product. Once demand is realized, the firm may have product that did not sell and must decide how many of these unsold units to carry over to the second period. We assume that the unsold units suffer from a quality reduction, such that the old units are not a perfect substitute for any new unit purchased in the second period. Thus, the firm offers two product types in the second period where the carried over product cannibalizes sales of the new product and the firm must trade-off between the number of carried over units to offer to the market versus the number of new units. The trade-off comes because it cost the firm less to carry over an unsold unit from the first period than it does to purchase a new unit, but it must charge a lower price for the carried over units to compensate for their lower quality.

We characterize the leftover product's quality as a percentage of the new product's quality level. We then give the firm's optimal pricing and quantity decisions for any given quality level. There are two quality thresholds that determine a firm's optimal course of action in the first period. The lower threshold determines if a firm should carry any unsold product into the second period. A carry policy is optimal only when the cost ratio between carrying an old unit and producing a new unit exceeds this lower threshold value. Thus, as the difference in quality decreases or the difference in cost increases, a firm is more likely to carry over old units. The higher quality threshold determines how much of the leftover

product should be carried over. For quality values between the lower and upper thresholds, the firm should, on the average, carry a fraction of its unsold product to the second period. For quality values above the upper threshold (a critical ratio involving the expected market potential of the product and the costs for purchasing new and carrying unsold product), the firm should carry all of its unsold product, up to the second period monopoly quantity solution. In general, we show that a firm that allows for carrying unsold inventory produces more and prices higher in the first period than a firm that does not allow the possibility of carrying the unsold inventory into the next period.

By including price as a decision variable, we gain valuable insights into how a firm should price its product when facing internal competition from its own carried-over units. First, the price of the new product in the second period is not affected by the competition of the old product. Thus, the firm charges the same price for its new units regardless of whether or not it also sells its carried over units. Second, the relationship between the price of the new product and the amount of safety stock produced in the first period is not affected by the firm's decision of how many unsold units to carry over to the second period. The first period price does change with the quality level but only because the optimal amount of safety stock changes.

Through a numerical study, we show that increases in expected profit of over 10% are obtainable for firms choosing to carry over the optimal quantity of unsold product to the second period. We perform a sensitivity analysis to gain insight into when the improvement is the greatest. Our proposed policy provides the most benefits when: 1) Uncertainty over the market potential is high, 2) The quality degradation of the unsold product is low, and 3) The cost to prepare the carried over unit for the market is low compared to the cost to purchase new units.

In a sensitivity analysis, we explore the implications when a firm chooses to use a deterministic price heuristic in the first period. Based on our numerical study, we find that the penalty for doing so is small with a maximum profit penalty of 0.02%. We also study, through simulation, the case of uncertainty in the market potential for the second period. Adding uncertainty to the second period dramatically changes our results as the firm's op-

timal decisions shift quickly to the no carry policy with even small amounts of uncertainty. Thus, firms facing significant amounts of market uncertainty should dispose of their old product rather than carry it to future periods where it will cannibalize sales of their new product. Another possibility is to carry the old product but not make it available to sell until it is clear that the market potential is high and there is insufficient stock of the new product to meet demand. The details and impact of this option are left for future research.

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6 Appendix

6.1 Derivation of Inverse Demand Functions

We follow the lead of Desai and Purohit (1998) and derive the demand functions from the consumer utility functions. Note that the net utility, NU , from using a unit in the second period is

$$NU = q^m \phi - P_2^m, \quad (14)$$

where m is an indicator variable such that $m = 0$ if the unit is new, $m = 1$ if the unit is old, and P_2^m is the second period price of new ($m = 0$) or old ($m = 1$) units.

Consider the problem facing consumers in period 2. Each consumer has to choose from one of the three following strategies: (i) buy a new unit (N); (ii) buy an old unit (O); (iii) be inactive (X). In consumer utility, if all three strategies are observed in equilibrium, then consumers who follow a N strategy value the product more (i.e., have a higher ϕ) than consumers who follow an O strategy, who value it more than consumers who follow an X strategy.

Now consider the lowest valuation consumer who adopts an O strategy. This consumer is located at a point $\phi' = \alpha - x_2 - y$ on the $[0, \alpha]$ line and is indifferent between following an O and an X strategy. From (14), this consumer's net utility from an O strategy is $q(\alpha - x_2 - y) - P_2^o$, and the utility from following an X strategy is zero. Equating these two gives a price for the old product of

$$P_2^o = q(\alpha - x_2 - y).$$

Finally, consider the lowest valuation of the consumer who adopts an N strategy. This consumer is located at a point $\phi'' = \alpha - x_2$ and has to be indifferent between the N and O strategies. The net utility from a N strategy is $\alpha - x_2 - P_2^n$. Similarly, the net utility from buying an old unit is $q(\alpha - x_2) - P_2^o$. Equating these two utilities gives a price of the new product of

$$P_2^n = \alpha - x_2 - qy.$$

In the first period, there are only new units so their price is given by

$$P_1 = \alpha - x_1.$$

6.2 Proofs

Proof of Proposition 1 The optimal price for the new product, $P_2^{n*} = \frac{R+c}{2}$, is independent of the quality level of the old product.

Solving the inverse demand function for old product, $P_2^o = q(R - x_2 - y)$, for the quantity of old product to sell gives $y = \frac{-P_2^o + qR - qx_2}{q}$. Plugging this value into the inverse demand function for new product $P_2^n = R - x_2 - qy$ gives $P_2^n = R - x_2 - q\left(\frac{-P_2^o + qR - qx_2}{q}\right)$ and solving for the quantity of new product to sell gives $x_2^* = \frac{R - qR - P_2^n + P_2^o}{1 - q}$. Substituting the two quantity equations into the firm's second period profit gives $\Pi_2 = (P_2^n - c)x_2 + (P_2^o - h)y$

$$= \frac{-P_2^n qR + P_2^n q^2 R + q(P_2^n)^2 - 2qP_2^n P_2^o + cqR - cq^2 R - qcP_2^n + qcP_2^o + (P_2^o)^2 - hP_2^o + hP_2^n P_2^o}{q(-1 + q)}.$$

Now we check to see if Π_2 is jointly concave in P_2^n and P_2^o . For this to be true, the Hessian of $\Pi_2(P_2^n, P_2^o)$ must be negative definite. The determinant of the Hessian is $\frac{\partial^2 \Pi_2(Y)}{\partial^2 P_2^o} \frac{\partial^2 \Pi_2(Y)}{\partial^2 P_2^n} - \frac{\partial \Pi_2(Y)}{\partial P_2^o} \frac{\partial \Pi_2(Y)}{\partial P_2^n} = \frac{-2}{q(1-q)} \frac{-2q}{q(1-q)} - \frac{2q}{q(1-q)} \frac{2q}{q(1-q)} = \frac{4}{q(1-q)}$. Since the determinant is positive then the Hessian is negative definite and the firm's second period profit is jointly concave. Thus, we can use second order conditions to find the optimal prices. Now $\frac{\partial \Pi_2(Y)}{\partial P_2^o} = \frac{2P_2^o - h - 2qP_2^n + qc}{q(-1+q)} = 0$ gives $P_2^{o*} = \frac{2qP_2^n - cq + h}{2}$ and $\frac{\partial \Pi_2(Y)}{\partial P_2^n} = \frac{-qR + q^2 R + 2qP_2^n - 2qP_2^o - qc + hq}{q(-1+q)} = \frac{-qR + q^2 R + 2qP_2^n - 2q\left(\frac{2qP_2^n - cq + h}{2}\right) - qc + hq}{q(-1+q)} = 0$ gives $P_2^{n*} = \frac{R+c}{2}$ which is independent of q .

Proof of Proposition 2 There exists a threshold level for the quality of the old product, $\frac{h}{c}$, below which the firm only sells the new product.

Solving for the optimal price of the old product gives $P_2^{o*} = \frac{2qP_2^{n*} - cq + h}{2} = \frac{2q\left(\frac{R+c}{2}\right) - cq + h}{2} = \frac{qR+h}{2}$. Substituting in P_2^{n*} and P_2^{o*} into the optimal quantity of new product to sell gives $x_2^* = \frac{R - qR - P_2^{n*} + P_2^{o*}}{1 - q} = \frac{(1-q)R - c + h}{2(1-q)}$ and into the optimal quantity of old product to sell gives $y = \frac{-P_2^{o*} + qR - qx_2^*}{q} = \frac{cq - h}{2q(1-q)}$ which is clearly positive only when $q > \frac{h}{c}$.

Proof of Proposition 3 There exists a threshold level for the quality of the old product,

$\frac{R-c+h}{R}$, above which the firm only sells the old product, up to $\min\left(\frac{qR-h}{2q}, Y\right)$.

The optimal quantity of new product to sell is $x_2^* = \frac{R-qR-c+h}{2(1-q)}$. Solving $\frac{R-qR-c+h}{2(1-q)} \geq 0$ for q we get $q \geq \frac{R-c+h}{R}$. For a quality level above this threshold, the firm only wants to sell old units so its objective becomes $Max_{P_2^o} \Pi_2 = (P_2^o - h)y$, where $P_2^o = q(R - y)$. The objective is clearly concave and results in the monopoly quantity $y^M = \frac{qR-h}{2q}$. This quantity represents the maximum that the firm will ever sell of the old product but it is constrained by the number of old product that is left over from the first period, Y .

Proof of Lemma 1 The optimal safety stock under the C policy and Condition 2 is higher than the safety stock for the NC policy ($z^* \geq \tilde{z}^*$).

$F^{-1}(u)$ increases with u so the only difference between z^* and \tilde{z}^* is

$$v = \frac{\int_{z^*-\Psi}^{z^*} [2\rho q(1-q)(u-z^*) + \rho(cq-h)]f(u) du}{P_1}.$$

For a given value of z , $P_1^*(z) = \tilde{P}_1^*(z)$. Since the *NC* policy is optimal for lower levels of q than the *C* policy, we thus need only to show that this term increases with q for a given price P_1 . We do so by taking the derivative

$$\begin{aligned} \frac{\partial v}{\partial q} &= \frac{1}{P_1} \left[\int_{z^*-\Psi}^{z^*} [2\rho(1-2q)(u-z^*) + \rho c]f(u) du + \right. \\ &\quad \left. [2\rho q(1-q)(z^*-\Psi-z^*) + \rho(cq-h)]f(z^*-\Psi) d(z^*-\Psi) \frac{cq^2 + h(1-2q)}{2q^2(1-q)^2} \right] \\ &= \frac{1}{P_1} \left[\int_{z^*-\Psi}^{z^*} [2\rho(1-2q)(u-z^*) + \rho c]f(u) du + \right. \\ &\quad \left. 2\rho q(1-q) \frac{cq-h}{2q(1-q)} + \rho(cq-h) \right] f(z^*-\Psi) d(z^*-\Psi) \frac{cq^2 + h(1-2q)}{2q^2(1-q)^2} \\ &= \frac{1}{P_1} \left[\int_{z^*-\Psi}^{z^*} [2\rho(1-2q)(u-z^*) + \rho c]f(u) du \right]. \end{aligned}$$

For the upper limit of the integration, $u = z^*$, the derivative reduces to $\frac{1}{P_1} \rho c f(z^*) dz^* > 0$. For the lower limit, $u = z^* - \Psi$, the derivative reduces to $\frac{1}{P_1} \frac{\rho(h-hq+cq-hq)}{2q(1-q)} f(z^* - \Psi) d(z^* - \Psi) > 0$.

Since $\frac{\partial v}{\partial q}$ is monotonically increasing in u then $\frac{\partial v}{\partial q} > 0$.

Proof of Lemma 2 The optimal price under the C policy and Condition 2 is higher than the optimal price for the NC policy ($P_1^* \geq \tilde{P}_1^*$).

For the C policy, from (13) we have $P_1^* = \frac{A+c+\mu}{2} - \frac{\Theta(z^*)}{2}$ where $\Theta(z^*) = \int_{z^*}^B (u - z^*)f(u)du$. Comparing this to the optimal price of the NC policy (7), the difference between the two prices is $P_1^* - \tilde{P}_1^* = \frac{\Theta(\tilde{z}^*)}{2} - \frac{\Theta(z^*)}{2}$. Since $\Theta(z)$ is decreasing in z and $z^* \geq \tilde{z}^*$ by Lemma 1 then $P_1^* \geq \tilde{P}_1^*$.

Proof of Proposition 4 The expected leftover inventory under the C policy and Condition 2 is higher than the expected leftover inventory under the NC policy.

Follows from the fact that the safety stock is larger (Lemma 1) and the price is larger (Lemma 2) for the C policy compared to the NC policy. The larger price ensures that expected demand is smaller (demand is inversely proportional to price by definition) and the larger safety stocks ensures that more units will be left over for a given price. Thus, a smaller demand combined with a larger safety stock results in more expected leftover inventory.

6.3 First Period Results of Cases C and D

Case C: $\frac{h}{c} < q < \frac{A+\mu-c+h}{A+\mu}$ and $\frac{cq-h}{2q(1-q)} \geq B$

Case C, corresponding to Condition 3 from Table 1, the quality level of the leftover product is sufficient for the firm to carry over some of the unsold inventory but the maximum limit to the amount it will carry is larger than the maximum limit of the market potential uncertainty. Case D corresponding to Conditions 4 and 5 from Table 1, the quality level of the leftover product is above the upper threshold level and the firm prefers to sell all of its leftover product in the second period, up to the monopoly quantity $\frac{qA+qu-h}{2q}$, before producing any new product. The solution for these two cases is very close to the solution for Case B and is given in the appendix. It is easy to see that the solution for these two cases Thus, the firm carries all of its unsold units regardless of the realization of u . The profit for the first period after the total market potential is realized is

$$\Pi_1(z, P_1 | u = U) = \begin{cases} P_1(x_1^d + U) - c(x_1^d + z) + \rho\Pi_2(z - U), & \text{if } U \leq z \\ (P_1 - c)(x_1^d + z) + \rho\Pi_2(0), & \text{if } z \leq U \end{cases}.$$

Before the uncertainty of the market potential is resolved, the firm's expected two-period profit is

$$\Pi_{12}(z, P_1) = \int_0^z [P_1(x_1^d + u) + \rho\Pi_2(z - u)] f(u) du + \int_z^B [P_1(x_1^d + z) + \rho\Pi_2(0)] f(u) du - c(x_1^d + z). \quad (15)$$

Solving the first order conditions with respect to z yields;

$$z^* = F^{-1} \left(\frac{P_1 - c + \int_0^{z^*} [2\rho q(1 - q)(u - z^*) + \rho(cq - h)] f(u) du}{P_1} \right). \quad (16)$$

The optimal price is the same as in Case B (13).

Case D: $\frac{A+\mu-c+h}{A+\mu} < q$

In this case, corresponding to Conditions 4 and 5 from Table 1, the quality level of the leftover product is above the upper threshold level and the firm prefers to sell all of its leftover product in the second period, up to the monopoly quantity $\Gamma = \frac{qA+qu-h}{2q}$, before producing any new product.

The profit for the first period after the total market potential is realized is

$$\Pi_1(z, P_1 | u = U) = \begin{cases} P_1(x_1^d + U) - c(x_1^d + z) + \rho\Pi_2(\Gamma), & \text{if } U \leq z - \Gamma \\ P_1(x_1^d + U) - c(x_1^d + z) + \rho\Pi_2(z - U), & \text{if } z - \Gamma < U \leq z \\ (P_1 - c)(x_1^d + z) + \rho\Pi_2(0), & \text{if } z \leq U \end{cases} .$$

The only differences between the firm's profit in Case D from its profit in Case B are the threshold levels and the quantity of unsold product carried over when the market potential is very low (captured under the first condition above). Under Case D, the firm desires to sell all of its unsold product in the second period up to the monopoly quantity $Y = \Gamma$. Before the uncertainty of the market potential is resolved, the firm's expected two-period profit takes the same form as (10) but with Ψ replaced by Γ . The analysis for the optimal quantity and price decisions is similar to Case B and is not repeated here.

6.4 Extension to Increasing Quality: New Product Has Level $1+q$

In our basic model, we assumed that the consumers' valuation of a new product remained constant from one period to the next and that their valuation of a left-over product degrades in relation to the new product. While this assumption is true for most grocery perishables (where, for example, most consumers value a fresh bagel the same from one day to the next), it is not always the case for markets where new models are introduced. If the new models contain advanced functionality over the older models they are replacing then it is reasonable to assume that consumers will place a higher valuation on the new model and thus, the product will have a higher reservation price than the new product in the previous period (a notable exception is the personal computer industry where prices for state of the art computers have remained constant from year to year despite the dramatic advances in their capabilities). In this section we assume that the consumers' valuation for the old model remains the same but they perceive the new model as providing higher quality.

Assume that the consumers' perceived value for each new unit is $(1+q)$ times the valuation of the old model. The inverse demand functions for this model are

$$P_2^n = R - x_2 - y + q(R - x_2) \quad \text{and} \quad P_2^o = R - x_2 - y.$$

The firm's second period objective is given by (9), the same as the decreasing quality case. The firm's optimal purchasing and pricing decisions under the most common condition are

x_2^*	y^*	P_2^{n*}	P_2^{o*}	$\Pi_2^*(Y)$
$\frac{Rq-c+h}{2q}$	$\frac{c-h(1+q)}{2q}$	$\frac{R(1+q)+c}{2}$	$\frac{R+h}{2}$	$\frac{R^2q+R^2q^2-2Rqc+c^2-2ch+h^2+h^2q}{4q}$

The optimal decisions under the other four conditions, as well as the new quality thresholds that define the conditions can be easily derived in a manner similar to the basic model. Just as in the basic model, the price of the product that maintains its same valuation of quality is independent of the quality level of the competing product. Under the increasing quality assumption however, the price of the new product changes from its deterministic equivalent in the first period. The quality level of the new product does affect the quantity of the old product that the firm puts in the market. In our basic model, the quality of the old product only affected the quantity, not the

price, of the new product.

6.5 Additional Derivations for the Reviewers' Benefit

Derivation of second period results

$$\text{Condition 2: } \frac{h}{c} < q < \frac{R-c+h}{R} \quad \& \quad \frac{cq-h}{2q(1-q)} > Y$$

$$\Pi_2 = (P_2^n - c)x_2 + (P_2^o - h)Y$$

$$x_2 = R - P_2^n - qY$$

$$P_2^o = q(R - (R - P_2^n - qY) - Y) = q(P_2^n + qY - Y)$$

$$\begin{aligned} \Pi_2 &= (P_2^n - c)(R - P_2^n - qY) + (q(P_2^n + qY - Y) - h)Y \\ &= P_2^n R - P_2^n cR + cP_2^n + cqY + q^2 Y^2 - qY^2 - Yh \end{aligned}$$

$$\frac{\partial \Pi_2}{\partial P_2^n} = R - 2P_2^n + c \text{ gives } P_2^{n*} = \frac{R+c}{2}$$

$$P_2^{o*} = q \left(\frac{R+c}{2} + qY - Y \right) = \frac{q(R+2qY+c-2Y)}{2}$$

$$x_2^* = R - \frac{R+c}{2} - qY = \frac{R-2qY-c}{2}$$

$$\text{Condition 3: } \frac{h}{c} < q < \frac{R-c+h}{R} \quad \& \quad \frac{cq-h}{2q(1-q)} \leq Y$$

$$\Pi_2 = (P_2^n - c)x_2 + (P_2^o - h)y$$

$$P_2^{n*} = \frac{R+c}{2} \text{ so } P_2^{o*} = \frac{2qP_2^n - cq + h}{2} = \frac{2q\frac{R+c}{2} - cq + h}{2} = \frac{qR+h}{2}$$

$$\text{and } x_2^* = \frac{R - qR - P_2^n + P_2^o}{1-q} = \frac{R - qR - \frac{R+c}{2} + \frac{qR+h}{2}}{1-q} = \frac{(1-q)R - c + h}{2(1-q)}$$

$$y^* = \frac{-P_2^{o*} + qR - qx_2^*}{q} = \frac{cq-h}{2q(1-q)}$$

$$\text{Condition 4: } \frac{R-c+h}{R} \leq q \quad \& \quad \frac{qR-h}{2q} > Y$$

Same as Condition 2.

$$\text{Condition 5: } \frac{R-c+h}{R} \leq q \quad \& \quad \frac{qR-h}{2q} \leq Y$$

$$y = \frac{-P_2^o + qR}{q}$$

$$\Pi_2 = (P_2^o - h)y = (P_2^o - h) \frac{-P_2^o + qR}{q} = P_2^o R - \frac{(P_2^o)^2}{q} - hR + h \frac{P_2^o}{q}$$

$$\frac{\partial \Pi_2}{\partial P_2^o} = R - \frac{2P_2^o}{q} + \frac{h}{q} \text{ gives } P_2^{o*} = \frac{qR+h}{2}$$

$$y^* = \frac{-P_2^{o*} + qR}{q} = \frac{-\frac{qR+h}{2} + qR}{q} = \frac{qR-h}{2q}$$

$$\Pi_2^* = \left(\frac{qR+h}{2} - h \right) \frac{qR-h}{2q} = \frac{(qR-h)^2}{4q}$$

Derivation of first period results

Policy C, Case B: Derivation of z^* and P_1^*

We begin with the firm's two period expected profit:

$$\begin{aligned}\Pi_{12}(z, P_1) &= \int_0^{z-\Psi} [P_1(x_1^d + u) + \rho\Pi_2(\Psi)] f(u) du \\ &+ \int_{z-\Psi}^z [P_1(x_1^d + u) + \rho\Pi_2(z - u)] f(u) du \\ &+ \int_z^B [P_1(x_1^d + z) + \rho\Pi_2(0)] f(u) du - c(x_1^d + z).\end{aligned}$$

We assume that the firm determines the production quantity and price simultaneously and that the stochastic demand is positive random variable. Therefore, the production quantity $(x_1^d + z)$ is always above x_1^d , the deterministic quantity, where $x_1^d = A - P_1$. Thus, the expression for the profit function is obtain by substituting $x_1^d = A - P_1$ and by incorporating $\Pi_2(\Psi)$, $\Pi_2(z - u)$, and $\Pi_2(0)$ from Conditions 3, 2, and 1 respectively:

$$\begin{aligned}\Pi_{12}(z, P_1) &= \int_0^{z-\Psi} [P_1(A - P_1 + u) + \\ &\rho \frac{q(A - u)^2 - q^2(A - u)^2 - 2cq(A - u) + 2cq^2(A - u) + qc^2 - 2qch + h^2}{4q(1 - q)}] f(u) du \\ &+ \int_{z-\Psi}^z [P_1(A - P_1 + u) + \\ &\rho [\frac{(A - u - c)^2}{4} + q(z - u)(q(z - u) - (z - u) + c) - (z - u)h]] f(u) du \\ &+ \int_z^B [P_1(A - P_1 + z) + \rho \left(\frac{A + u - c}{2}\right)^2] f(u) du - c(A - P_1 + z).\end{aligned}$$

To find the optimal safety stock, we check the first order condition of $\Pi_{12}(z, P_1)$ with respect to z . To simplify the process we first take the derivative of the second period profits with respect to z .

$$\begin{aligned}\frac{\partial \Pi_2(\Psi)}{\partial z} &= \frac{\partial [\rho \frac{q(A+u)^2 - q^2(A+u)^2 - 2cq(A+u) + 2cq^2(A+u) + qc^2 - 2qch + h^2}{4q(1-q)}]}{\partial z} = 0, \\ \frac{\partial \Pi_2(z - u)}{\partial z} &= \frac{\partial [\rho [\frac{(A+u-c)^2}{4} + q(z - u)(q(z - u) - (z - u) + c) - (z - u)h]]}{\partial z}\end{aligned}$$

$$\begin{aligned}
&= \rho [-h + q [c + 2(1-q)(u-z)]], \\
\frac{\partial \Pi_2(0)}{\partial z} &= \frac{\partial [\rho \frac{q(A-u)^2 - q^2(A-u)^2 - 2cq(A-u) + 2cq^2(A-u) + qc^2 - 2qch + h^2}{4q(1-q)}]}{\partial z} = 0.
\end{aligned}$$

Next, we write the full derivative of the profits with respect to z given the above partial derivatives.

$$\begin{aligned}
\frac{\partial \Pi_{12}(z, P_1)}{\partial z} &= \left[\frac{P_1(A - P_1 + z - \Psi) + \rho \frac{q(A+z-\Psi)^2 - q^2(A+z-\Psi)^2 - 2cq(A+z-\Psi) + 2cq^2(A+z-\Psi) + qc^2 - 2qch + h^2}{4q(1-q)}}{f(z - \Psi) d(z - \Psi)} \right] \\
&+ \int_{z-\Psi}^z [\rho [-h + q [c + 2(1-q)(u-z)]] f(u) du \\
&+ \left[P_1(A - P_1 + z) + \rho \left[\frac{(A+z-c)^2}{4} \right] \right] f(z) dz - \\
&\left[P_1(A - P_1 + z - \Psi) + \rho \left[\frac{(A-z+\Psi-c)^2}{4} + q(\Psi)(q(\Psi) - (\Psi) + c) - (\Psi)h \right] \right] * \\
&f(z - \Psi) d(z - \Psi) + \\
&\int_z^B P_1 f(u) du - \left[P_1(A - P_1 + z) + \rho \left(\frac{A+z-c}{2} \right)^2 \right] f(z) dz - c \\
&= 0.
\end{aligned}$$

It is straight forward to verify that the coefficients of $f(z - \Psi) d(z - \Psi)$ sum to zero:

$$\begin{aligned}
&- \left[\frac{P_1(A - P_1 + z - \Psi) + \rho \frac{q(A+z-\Psi)^2 - q^2(A+z-\Psi)^2 - 2cq(A+z-\Psi) + 2cq^2(A+z-\Psi) + qc^2 - 2qch + h^2}{4q(1-q)}}{f(z - \Psi) d(z - \Psi)} \right] + \\
&\left[P_1(A - P_1 + z - \Psi) + \rho \left[\frac{(A-z+\Psi-c)^2}{4} + q(\Psi)(q(\Psi) - (\Psi) + c) - (\Psi)h \right] \right] = 0,
\end{aligned}$$

and the same is true for the coefficients of $f(z) dz$;

$$\left[P_1(A - P_1 + z) + \rho \left[\frac{(A+z-c)^2}{4} \right] \right] - \left[P_1(A - P_1 + z) + \rho \left(\frac{A+z-c}{2} \right)^2 \right] = 0.$$

Therefore, we are left with

$$\begin{aligned}
\frac{\partial \Pi_{12}(z, P_1)}{\partial z} &= \int_{z-\Psi}^z [\rho[-h + q[c + 2(1-q)(u-z)]] f(u) du + \int_z^B P_1 f(u) du - c \\
&= P_1[1 - F(z)] - c + 2\rho q(1-q) \int_{z-\Psi}^z (u-z)f(u) du + \rho[-h + cq] \int_{z-\Psi}^z f(u) du \\
&= P_1[1 - F(z)] - c + \int_{z-\Psi}^z [2\rho q(1-q)(u-z) + \rho(cq - h)]f(u) du.
\end{aligned}$$

Solving the first order condition for z gives

$$z^* = F^{-1} \left(\frac{P_1 - c}{P_1} + \frac{\int_{z^*-\Psi}^{z^*} [2\rho q(1-q)(u-z^*) + \rho(cq - h)]f(u) du}{P_1} \right).$$

To determine the optimal first period price, we check the first order condition of $\Pi_{12}(z, P_1)$ with respect to the first period price:

$$\begin{aligned}
\frac{\partial \Pi_{12}(z, P_1)}{\partial P_1} &= \int_0^{z-\Psi} (A - 2P_1 + u)f(u) du \\
&\quad + \int_{z-\Psi}^z (A - 2P_1 + u)f(u) du + \int_z^B (A - 2P_1 + z)f(u) du + c \\
&= A - 2P_1 + \int_0^z uf(u) du + \int_z^B zf(u) du + c \\
&= A - 2P_1 + \mu + \int_z^B (z-u)f(u) du + c.
\end{aligned}$$

Thus, it is easy to see that

$$P_1^* = \frac{A + c + \mu}{2} - \frac{\Theta(z^*)}{2},$$

where z^* is given above.

Policy C, Case C: Derivation of z^* and P_1^*

Before the uncertainty of the market potential is resolved, the firm's expected two-period profit is

$$\begin{aligned}\Pi_{12}(z, P_1) &= \int_0^z [P_1(x_1^d + u) + \rho\Pi_2(z - u)] f(u) du \\ &\quad + \int_z^B [P_1(x_1^d + z) + \rho\Pi_2(0)] f(u) du - c(x_1^d + z),\end{aligned}\quad (17)$$

This expands to

$$\begin{aligned}\Pi_{12}(z, P_1) &= \int_0^z \left[P_1(A - P_1 + u) + \rho\left[\frac{(A+u-c)^2}{4}\right] \right. \\ &\quad \left. + q(z-u)(q(z-u) - (z-u) + c) - (z-u)h \right] f(u) du \\ &\quad + \int_z^B \left[P_1(A - P_1 + z) + \rho\left(\frac{A+u-c}{2}\right)^2 \right] f(u) du - c(A - P_1 + z).\end{aligned}$$

As before, to find the optimal safety stock we check the first order condition of $\Pi_{12}(z, P_1)$ with respect to z

$$\begin{aligned}\frac{\Pi_{12}(z, P_1)}{\partial z} &= \int_0^z [\rho[-h + q[c + 2(1-q)(u-z)]] f(u) du \\ &\quad + \left[P_1(A - P_1 + z) + \rho\left[\frac{(A-z-c)^2}{4}\right] \right] f(z) dz + \\ &\quad \int_z^B P_1 f(u) du - \left[P_1(A - P_1 + z) + \rho\left(\frac{A+z-c}{2}\right)^2 \right] f(z) dz - c \\ &= \int_0^z [\rho[-h + q[c + 2(1-q)(u-z)]] f(u) du + \int_z^B P_1 f(u) du - c \\ &= \int_0^z [\rho[-h + q[c + 2(1-q)(u-z)]] f(u) du + P_1[1 - F(z)] - c.\end{aligned}$$

Solving the first order condition for z gives

$$z^* = F^{-1} \left(\frac{P_1 - c + \int_0^{z^*} [2\rho q(1-q)(u - z^*) + \rho(cq - h)]f(u) du}{P_1} \right).$$

To determine the optimal first period price, we check the first order condition of $\Pi_{12}(z, P_1)$ with respect to the first period price:

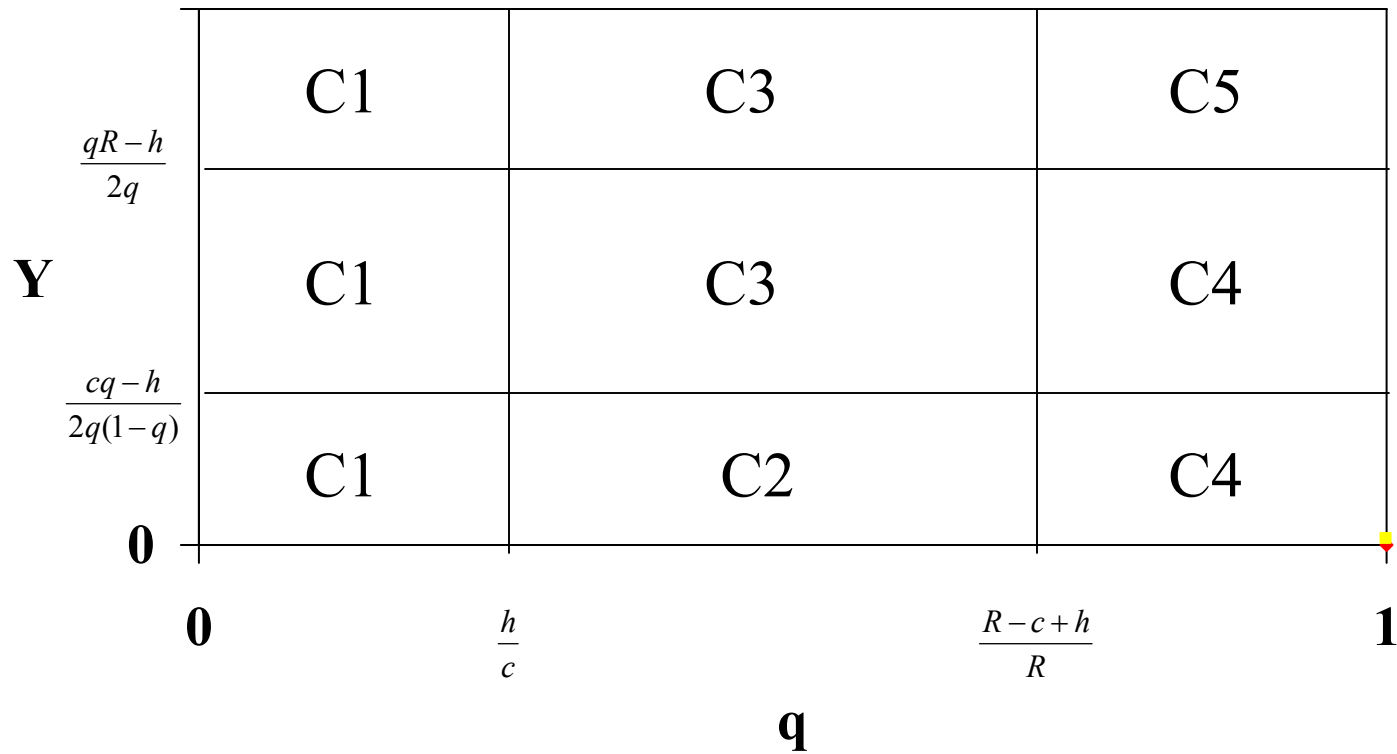
$$\begin{aligned} \frac{\partial E[\Pi_{12}(z, P_1)]}{\partial P_1} &= \int_0^z (A - 2P_1 + u)f(u) du + \int_z^B (A - 2P_1 + z)f(u) du + c \\ &= A - 2P_1 + \int_0^z uf(u) du + \int_z^B zf(u) du + c \\ &= A - 2P_1 + \mu + \int_z^B (z - u)f(u) du + c. \end{aligned}$$

Thus, it is easy to see that

$$P_1^* = \frac{A + c + \mu}{2} - \frac{\Theta(z^*)}{2},$$

where z^* is given above.

Figure 1: Conditions Expressed as Regions



	h/c=0 A=90, B=20			h/c=0 A=75, B=50			h/c=0 A=50, B=100		
q	C	NC	C-NC	C	NC	C-NC	C	NC	C-NC
0.1	3775	3766	9	3692	3643	49	3627	3439	188
0.2	3777	3766	11	3694	3643	51	3631	3439	192
0.3	3780	3766	14	3697	3643	54	3634	3439	195
0.4	3784	3766	18	3702	3643	59	3638	3439	199
0.5	3788	3766	22	3707	3643	64	3644	3439	205
0.6	3794	3766	28	3715	3643	72	3653	3439	214
0.7	3803	3766	37	3728	3643	85	3667	3439	228
0.8	3814	3766	48	3750	3643	107	3693	3439	254
0.9	3828	3766	62	3796	3643	153	3762	3439	323
0.95	3836	3766	70	3829	3643	186	3850	3439	411
0.99	3843	3766	77	3861	3643	218	3955	3439	516
	h/c=.25 A=90, B=20			h/c=.25 A=75, B=50			h/c=.25 A=50, B=100		
q	C	NC	C-NC	C	NC	C-NC	C	NC	C-NC
0.3	3773	3766	7	3690	3643	47	3627	3439	188
0.4	3775	3766	9	3692	3643	49	3628	3439	189
0.5	3777	3766	11	3694	3643	51	3631	3439	192
0.6	3781	3766	15	3699	3643	56	3637	3439	198
0.7	3787	3766	21	3706	3643	63	3644	3439	205
0.8	3797	3766	31	3720	3643	77	3659	3439	220
0.9	3809	3766	43	3753	3643	110	3701	3439	262
0.95	3816	3766	50	3782	3643	139	3764	3439	325
0.99	3823	3766	57	3811	3643	168	3856	3439	417
	h/c=.5 A=90, B=20			h/c=.5 A=75, B=50			h/c=.5 A=50, B=100		
q	C	NC	C-NC	C	NC	C-NC	C	NC	C-NC
0.6	3774	3766	8	3691	3643	48	3627	3439	188
0.7	3776	3766	10	3693	3643	50	3630	3439	191
0.8	3782	3766	16	3700	3643	57	3637	3439	198
0.9	3792	3766	26	3717	3643	74	3656	3439	217
0.95	3799	3766	33	3740	3643	97	3691	3439	252
0.99	3804	3766	38	3765	3643	122	3766	3439	327
	h/c=.75 A=90, B=20			h/c=.75 A=75, B=50			h/c=.75 A=50, B=100		
q	C	NC	C-NC	C	NC	C-NC	C	NC	C-NC
0.8	3774	3766	8	3690	3643	47	3627	3439	188
0.9	3777	3766	11	3694	3643	51	3631	3439	192
0.95	3782	3766	16	3703	3643	60	3641	3439	202
0.99	3788	3766	22	3723	3643	80	3684	3439	245

Table 3: Two-Period Profit: Uniform Distribution, c=10, rho = .9

Coef of Variation	Decision Variables	Normal Distribution					Uniform Distribution				
		Quality Level, q					Quality Level, q				
		0.9	0.8	0.7	0.6	0.5	0.9	0.8	0.7	0.6	0.5
0	X	20	35	40	42.5	44	20	35	40	42.5	44
	Px	55	55	55	55	55	55	55	55	55	55
	y	27.78	12.5	7.14	4.17	2	27.78	12.5	7.14	4.17	2
	Py	47	42	37	32	27	47	42	37	32	27
	Prof	2094	2050	2036	2029	2026	2094	2050	2036	2029	2026
0.0058	X	19	38	41	46	46	20	38	44	45	46
	Px	55	55	55	55	55	55	55	55	55	55
	y	29	9	6	0	0	27	9	2	1	0
	Py	47	42	37	0	0	47	42	37	32	0
	Prof	2070	2030	2018	2014	2014	2059	2030	2019	2017	2015
0.0173	X	20	47	47	47	47	20	46	47	47	47
	Px	54	55	55	55	55	55	55	55	55	55
	y	28	0	0	0	0	27	1	0	0	0
	Py	47	0	0	0	0	48	40	0	0	0
	Prof	2026	1999	1999	1999	1999	2019	2000	2000	2000	2000
0.0289	X	48	48	48	48	48	48	48	48	48	48
	Px	55	55	55	55	55	55	55	55	55	55
	y	0	0	0	0	0	0	0	0	0	0
	Py	0	0	0	0	0	0	0	0	0	0
	Prof	1983	1983	1983	1983	1983	1984	1984	1984	1984	1984
0.0404	X	48	48	48	48	48	50	50	50	50	50
	Px	55	55	55	55	55	55	55	55	55	55
	y	0	0	0	0	0	0	0	0	0	0
	Py	0	0	0	0	0	0	0	0	0	0
	Prof	1965	1965	1965	1965	1965	1964	1964	1964	1964	1964

Parameters: $h = 4$, $c = 10$, $E[A + u] = 100$

Table 4: Simulation Results