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The Rook-Brauer Algebra

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HONORS PROJECT

The Rook-Brauer Algebra

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Abstract

We introduce an associative algebra $\mathsf{RB}_k(x)$ that has a basis of rook-Brauer diagrams. These diagrams correspond to partial matchings on $2k$ vertices. The dimension of $\mathsf{RB}_k(x)$ is $\sum_{\ell=0}^{k} \binom{2k}{2\ell}(2\ell-1)!!$. The algebra $\mathsf{RB}_k(x)$ contains the group algebra of the symmetric group, the Brauer algebra, and the rook monoid algebra as subalgebras. We show that $\mathsf{RB}_k(x)$ is generated by special diagrams s_i, t_i $(1 \leq i < k)$ and p_j $(1 \leq j \leq k)$, where the s_i are the simple transpositions that generated the symmetric group S_k, the t_i are the "contraction maps" which generate the Brauer algebra $\mathsf{B}_k(x)$, the p_i are the "projection maps" that generate the rook monoid R_k. We prove that for a positive integer n, the algebra $\mathsf{RB}_k(n+1)$ is the centralizer algebra of the orthogonal group $\mathsf{O}(n)$ acting on the k-fold tensor power of the sum of its 1-dimensional trivial module and n-dimensional defining module.

Introduction

This paper finds the centralizer algebra of the orthogonal group over $\mathbb{C}$, $\mathsf{O}(n)$, acting on the k-fold tensor space $\mathsf{V}^{\otimes k}$ where V is the $(n+1)$-dimensional module $\mathbb{C}^n \oplus \mathbb{C}$. The introduction describes the motivation for finding this centralizer and the work that has already been done in this area to support that motivation.

We begin with the general linear group of invertible matrices over $\mathbb{C}$, $\mathsf{GL}(n)$, and the $\mathsf{GL}(n)$-module $\mathsf{V} = \mathbb{C}^n$ with standard basis $\{v_1, v_2, \ldots, v_n\}$. This module is the irreducible $\mathsf{GL}(n)$-module labeled by the partition (1), so $\mathsf{V} = \mathsf{V}^{(1)}$. We are concerned with the tensor product $\mathsf{GL}(n)$-module $\mathsf{V}^{\otimes k}$ whose basis is the set of simple tensors

$$\{ \ v_{i_1} \otimes \cdots \otimes v_{i_k} \mid i_j \in \{1, \ldots, n\} \ \} .$$

The general linear group acts diagonally on elements of $\mathsf{V}^{\otimes k}$, and the centralizer algebra of the action of $\mathsf{GL}(n)$ on this tensor space is the set of all $\mathsf{GL}(n)$-module homomorphisms from $\mathsf{V}^{\otimes k}$ to itself. We refer to this as $\mathrm{End}_{\mathsf{GL}(n)}(\mathsf{V}^{\otimes k})$. It is known that this centralizer algebra is isomorphic to the group algebra of the symmetric group $\mathbb{C}\mathsf{S}_k$.

The next part of the story is to look at the subgroup of orthogonal matrices $\mathsf{O}(n) \subseteq \mathsf{GL}(n)$ acting on this same module $\mathsf{V}^{\otimes k}$. The natural question to ask is, what is the centralizer algebra for $\mathsf{O}(n)$ acting on $\mathsf{V}^{\otimes k}$? Since $\mathsf{O}(n)$ is contained in $\mathsf{GL}(n)$, its centralizer could possibly be much larger than $\mathbb{C}\mathsf{S}_k$. In 1937, Richard Brauer succeeded in describing the centralizer with an algebra called the Brauer algebra, $\mathsf{B}_k(n)$. This algebra over $\mathbb{C}$ has a basis of Brauer diagrams on $2k$ vertices, which is equivalent to the set of all possible partitions of a set of $2k$ elements into blocks of 2.

$$\begin{tikzcd}[column sep=small]
 & \mathsf{V}^{\otimes k} & \\
\mathsf{GL}(n) \arrow[ur] \arrow[d, phantom, sloped, "\supseteq"] & & \mathbb{C}\mathsf{S}_k \arrow[ul] \arrow[d, phantom, sloped, "\subseteq"] \\
\mathsf{O}(n) & & \mathsf{B}_k(n)
\end{tikzcd}$$

Since the centralizer has been described for the actions of both $\mathsf{GL}(n)$ and $\mathsf{O}(n)$ on the module $\mathsf{V}^{\otimes k}$, we now wish to explore the centralizer of $\mathsf{GL}(n)$ and $\mathsf{O}(n)$ acting on a slightly different module. Now let $\mathsf{V} = \mathsf{V}^{(1)} \oplus \mathsf{V}^{\emptyset}$ where $\mathsf{V}^{\emptyset}$ is the trivial 1-dimensional submodule of $\mathsf{GL}(n)$. $\mathsf{GL}(n)$ acts on the $(n+1)^k$-dimensional module $\mathsf{V}^{\otimes k}$ diagonally. The centralizer for this action has been described by the rook monoid algebra $\mathbb{C}\mathsf{R}_k$. In this paper we take the next natural step and describe the centralizer of $\mathsf{O}(n) \subseteq \mathsf{GL}(n)$ acting on the $(n+1)^k$-dimensional tensor space $\mathsf{V}^{\otimes k}$, which we call the rook-Brauer algebra $\mathsf{RB}_k(n+1)$.

$$\begin{tikzcd}[column sep=small]
 & \mathsf{V}^{\otimes k} & \\
\mathsf{GL}(n) \arrow[ur] \arrow[d, phantom, sloped, "\supseteq"] & & \mathbb{C}\mathsf{R}_k \arrow[ul] \arrow[d, phantom, sloped, "\subseteq"] \\
\mathsf{O}(n) & & \mathsf{RB}_k(n+1)
\end{tikzcd}$$

This thesis is organized as follows.

- Chapter 1 refreshes some important background information on representations of groups and algebras, though some prior knowledge is assumed.

- In Chapter 2, we discuss the three important subalgebras of $\mathsf{RB}_k(x)$, and we define the rook-Brauer algebra with a basis of rook-Brauer diagrams on $2k$ vertices. The set of these diagrams on $2k$ vertices is equivalent to the number of partitions of a set of $2k$ elements into parts of size 1 or 2.

- Chapter 3 discusses double centralizer theory and the motivations behind studying the rook-Brauer algebra.

- In Chapter 4 we define an action of $\mathsf{RB}_k(n+1)$ on the tensor space $\mathsf{V}^{\otimes k}$ and prove that this action creates a representation of $\mathsf{RB}_k(n+1)$ and that this representation is faithful for $n \geq k$. This shows that there is an injective linear transformation between $\mathsf{RB}_k(n+1)$ and $\mathrm{End}(\mathsf{V}^{\otimes k})$, the set of all endomorphisms of $\mathsf{V}^{\otimes k}$.

- Chapter 5 presents the proof that the action of $\mathsf{RB}_k(n+1)$ commutes with the action of $\mathsf{O}(n)$ on $\mathsf{V}^{\otimes k}$, which shows that in fact when $n \geq k$, $\mathsf{RB}_k(n+1) \subseteq \mathrm{End}_{\mathsf{O}(n)}(\mathsf{V}^{\otimes k})$ as a subalgebra.
- Finally in Chapter 6 we use combinatorics on the Bratteli diagram of $\mathsf{V}^{\otimes k}$ to show that the dimension of $\mathsf{RB}_k(n+1)$ is equal to the dimension of $\mathrm{End}_{\mathsf{O}(n)}(\mathsf{V}^{\otimes k})$.
- Chapter 7 discusses future work on this project, which includes constructing the irreducible representations of $\mathsf{RB}_k(n+1)$.

Acknowledgements

Thank you to my readers Dan Flath and Kristina Garrett, and to my advisor Tom Halverson. This research was partially supported by National Science Foundation Grant DMS-0800085.

Chapter 1

Preliminaries

1.1 Algebraic Structures

1.1.1 Groups and Monoids

A *group* is a set G together with a binary operation $*$ that satisfies the following properties [JL].

i. For all $g, h \in G$, $g * h \in G$.

ii. For all $g, h, k \in G$, $(g * h) * k = g * (h * k)$.

iii. There exists an element $e \in G$ such that for all $g \in G$, $e*g = g*e = g$.

iv. For each $g \in G$, there exists an element g^{-1} such that $g * g^{-1} = g^{-1} * g = e$.

Examples of groups are the set of integers $\mathbb{Z}$ under the operation of addition, and the set of nonzero real numbers $\mathbb{R}^*$ under multiplication. In this paper we deal closely with the general linear group $\mathsf{GL}(n)$. This is the group of invertible $n \times n$ matrices with entries in $\mathbb{C}$. The most important group in this paper is a subgroup of $\mathsf{GL}(n)$. A *subgroup* of a group G is a subset of G that is also a group under the operation of G. The subgroup we are interested in is the group of matrices $\mathsf{O}(n)$ inside of $\mathsf{GL}(n)$ such that for $g \in \mathsf{O}(n)$ the transpose of g is the inverse of g, i.e. $gg^T = I_n$ where I_n is the $n \times n$ identity matrix. This group is called the orthogonal group.

A *monoid* is a set M together with a binary operation $*$ that satisfies properties $i. - iii.$ of the definition of a group. The rook monoid is an important algebraic structure in this paper and is defined in a later section.

1.1.2 Associative Algebras

In this paper we explore an *associative algebra*, which we define generally in this section. An associative algebra A with unity $\mathbf{1}$ and additive identity $\vec{0}$ over a field $\mathbb{F}$, or an $\mathbb{F}$-algebra, is an $\mathbb{F}$-vector space under addition with an additional operation of multiplication. This multiplication must satisfy the following properties:

i. $ab \in A$ for all $a, b \in A$

ii. $\mathbf{1}a = a\mathbf{1} = a$ for all $a \in A$

iii. $(\lambda a)b = \lambda(ab) = a(\lambda b)$ for all $a, b \in A$ and $\lambda \in \mathbb{F}$.

iv. $(a + b)c = ac + bc$ for all $a, b, c \in A$

v. $a(b + c) = ab + ac$ for all $a, b, c \in A$

vi. $a\vec{0} = \vec{0}a = \vec{0}$

See [Pi] for further explorations of associative algebras. In this paper all algebras discussed are $\mathbb{C}$-algebras.

1.1.3 Group Algebras

The group algebra of a finite group G over a field $\mathbb{F}$ [JL], denoted by $\mathbb{F}G$, is the set of all formal linear combinations of elements of G,

$$\left\{ \sum_{g \in G} \lambda_g g \mid \lambda_g \in \mathbb{F} \right\},$$

together with a multiplication defined by

$$\left(\sum_{g \in G} \lambda_g g \right) \left(\sum_{h \in G} \mu_h h \right) = \sum_{g,h \in G} \lambda_g \mu_h (gh).$$

1.2 Representations and Modules

In this section we present important definitions and theorems of representation theory that factor into this paper. Some prior knowledge is assumed. All material referring to group representations and modules is taken from [JL], which contains more detailed explanations and examples as well as proofs.

1.2.1 Representations

A *representation* of a group G is a homomorphism ρ from G to $\mathsf{GL}(n)$ for some n. This means that ρ is a representation if and only if $\rho(gh) = \rho(g)\rho(h)$ for all $g, h \in G$. In order to prove that a function ρ is a representation of a group, it is sufficient to check that ρ satisfies $\rho(g_i g_j) = \rho(g_i)\rho(g_j)$ for the set of generators $\{g_i\}$ of G. If we let I_n be the identity matrix of $\mathsf{GL}(n)$ and $\mathbf{1}$ be the identity of G, the fact that a representation ρ is a homomorphism gives that

$$\rho(\mathbf{1}) = I_n$$

and

$$\rho(g^{-1}) = (\rho(g))^{-1}.$$

The *kernel* of a representation ρ is the set $\mathbf{Ker}(\rho) = \{g \in G \mid \rho(g) = I_n\}$, and we say a representation is *faithful* of $\mathbf{Ker}(\rho) = \{\mathbf{1}\}$.

1.2.2 Modules

Let V be a vector space over a field $\mathbb{F}$ and let G be a group. We say V is an $\mathbb{F}G-$*module* if there exists a multiplication, denoted by gv for $g \in G, v \in \mathsf{V}$, that for all $g, h \in G$ and $u, v \in \mathsf{V}$ satisfies the rules

i. $gv \in \mathsf{V}$

ii. $(gh)v = g(hv)$

iii. $\mathbf{1}v = v$

iv. $\lambda(gv) = g(\lambda v)$ for all $\lambda \in \mathbb{F}$

v. $g(u + v) = gu + gv$

If $\mathcal{B}$ is a basis of an $\mathbb{F}G$-module V, then we let $[g]_{\mathcal{B}}$ denote the matrix of the endomorphism $v \mapsto gv$ relative to the basis $\mathcal{B}$.

We can easily move between modules and representations by using the following helpful theorem.

Theorem 1. *1) If $\rho : G \to \mathsf{GL}(n)$ is a representation of G and $\mathsf{V} = \mathbb{C}^n$, then V becomes a $\mathbb{C}G$-module of we define the multiplication gv as*

$$gv = \rho(g)v$$

for all $g \in G$, $v \in \mathsf{V}$.

2) *Assume* V *is a* $\mathbb{C}G$*-module and let* $\mathcal{B}$ *be a basis of* V*. Then if* $g \in G$*, the function defined by*

$$g \mapsto [g]_{\mathcal{B}}$$

is a representation of G*.*

This theorem shows a representation of a group G defines a $\mathbb{C}G$-module and a $\mathbb{C}G$-module defines a representation. Due to this correspondence, we use the terms *module* and *representation* interchangeably in this paper.

Notice that the group algebra $\mathbb{C}G$ is a $\mathbb{C}G$-module with a basis labeled by the elements of G and dimension $|G|$. This module is also referred to as the *regular representation* of G and features heavily in this paper.

An $\mathbb{F}G$*-module homomorphism* is a linear transformation ϕ between two $\mathbb{F}G$-modules V and W such that

$$\phi(g \cdot v) = g \cdot \phi(v)$$

for all $v \in \mathsf{V}$, $g \in G$.

The set of all linear transformations from an $\mathbb{F}G$-module to itself is called the set of endomorphisms of V and is denoted by $\mathrm{End}(\mathsf{V})$. Contained within this set is the set of all $\mathbb{F}G$-module homomorphisms from V to itself, $\mathrm{Hom}_G(\mathsf{V}, \mathsf{V})$. We denote this set as $\mathrm{End}_G(\mathsf{V})$.

1.2.3 Irreducible Submodules and Decomposition

A $\mathbb{C}G$*-submodule* of a $\mathbb{C}G$-module V is a subspace W of V that is also a $\mathbb{C}G$-module under the action of G on V. A $\mathbb{C}G$-module is *irreducible* if it has no $\mathbb{C}G$-submodules other $\{0\}$ and itself, where 0 is the additive identity of the module.

Maschke's Theorem gives an important result about the relationship of modules and irreducible submodules.

Theorem 2. ***Maschke's Theorem.*** *Let* G *be a finite group and let* V *be a* $\mathbb{C}G$*-module. If* $\mathsf{U} \subseteq \mathsf{V}$ *is a* $\mathbb{C}G$*-submodule of* V*, then there exists a* $\mathbb{C}G$*-submodule* W *such that*

$$\mathsf{V} = \mathsf{U} \oplus \mathsf{W}.$$

where $\mathsf{U} \oplus \mathsf{W}$ is the direct sum of the subspaces U and W.

A $\mathbb{C}G$-module V is *completely reducible* if

$$\mathsf{V} = \bigoplus_{1 \leq i \leq r} \mathsf{U}_i$$

where each U_i is an irreducible $\mathbb{C}G$-module of V. From Maschke's Theorem and using induction we get the following useful result.

Theorem 3. *If G is a finite group, then every non-zero $\mathbb{C}G$-module is completely reducible.*

This theorem allows us to focus on the irreducible submodules of a group. Representing a $\mathbb{C}G$-module as the direct sum of irreducible submodules is called *decomposing* the module. The next theorem shows that we can get a list of every irreducible submodule by decomposing the group algebra $\mathbb{C}G$.

Theorem 4. *Let $\mathbb{C}G$ be the group algebra of G with*

$$\mathbb{C}G = \bigoplus_{i \in I} \mathsf{U}_i$$

where I is some (possibly infinite) index. Then every irreducible $\mathbb{C}G$-module is isomorphic to one of the $\mathbb{C}G$-modules U_i.

Some irreducible submodules may appear more than once in this decomposition, so we say that an irreducible submodule U_i has *multiplicity* m_i if it appears m_i times in a decomposition. We usually write the decomposition of a $\mathbb{C}G$-module as

$$\mathsf{V} = \bigoplus_{i \in I} m_i \mathsf{U}_i.$$

1.2.4 Algebra Representations and Modules

And *algebra representation* of a $\mathbb{C}$-algebra A is an algebra homomorphism ϕ of A to the $\mathbb{C}$-algebra of all $n \times n$ matrices with entries in $\mathbb{C}$. That is, ϕ is a function of A to the $\mathbb{C}$-algebra of all $n \times n$ matrices with entries in $\mathbb{C}$ that preserves the operations of A.

A $\mathbb{C}G$-module V is *simple* if V is non-trivial and the only submodules of V are V and the trivial module. A module V is *semisimple* if V is a direct sum of simple modules, and an algebra A is semisimple if A is semisimple as a $\mathbb{C}G$-module. For semisimple algebras, algebra representations and modules decompose analogously to group representations and modules. See [Pi] chapters 2, 3, and 5 for further definitions and examples. The main structure that this paper investigates is a semisimple associative $\mathbb{C}$-algebra.

1.2.5 Tensor Product Spaces and Modules

Let V and W be vector spaces over $\mathbb{C}$ with bases $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$, respectively. The *tensor product space* $\mathsf{V} \otimes \mathsf{W}$ is the $m * n$-dimensional vector space over $\mathbb{C}$ with the basis

$$\{v_i \otimes w_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}.$$

Elements of $\mathsf{V} \otimes \mathsf{W}$ look like $v = \sum_{i,j} \lambda_{ij}(v_i \otimes w_j)$, $\lambda_{ij} \in \mathbb{C}$. For $v \in \mathsf{V}$, $w \in \mathsf{W}$, with $v = \sum_{i=1}^{n} \lambda_i v_i$ and $w = \sum_{i=1}^{m} \mu_i w_i$, define

$$v \otimes w = \sum_{i,j} \lambda_i \mu_j (v_i \otimes w_j).$$

If V and W are $\mathbb{C}G$-modules, then we define the *tensor product module* $\mathsf{V} \otimes \mathsf{W}$ by the action

$$g(v_i \otimes w_j) = gv_i \otimes gw_j$$

for basis elements $v_i \in \mathsf{V}$, $w_j \in \mathsf{W}$. This can easily be extended so that for any $v \in \mathsf{V}$ and $w \in \mathsf{W}$, $g(v \otimes w) = gv \otimes gw$. Modules can be tensored multiple times, and if a module V is tensored with itself k times we denote it by $\mathsf{V}^{\otimes k}$.

Chapter 2

The Rook Brauer Algebra $\mathsf{RB}_k(x)$

2.1 The Symmetric Group, the Rook Monoid, and the Brauer Algebra

In this section we present three algebras that can be represented with diagrams on $2k$ vertices and show how their operations work. These examples build a foundation for the definition of a new structure, the rook-Brauer algebra, which contains each of the following structures as subalgebras.

2.1.1 The Symmetric Group

The symmetric group S_k is the group of permutations of the set $\{1, \ldots, k\}$. The dimension formula of S_k is

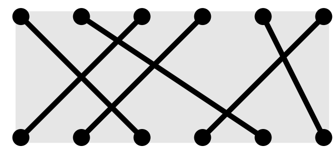
$$|\mathsf{S}_k| = k!.$$

The operation in this group is permutation multiplication, which can be shown in two-line notation or cyclic form. In S_6, we can represent the same permutation by

$$\sigma = \begin{pmatrix} 4 & 1 & 5 & 2 & 3 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

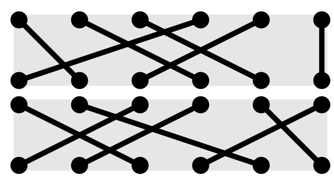
and $\sigma = (142)(35)$. This form of two-line notation represents the permutation as a function where the bottom line is the set before the permutation is applied and the top line is the image of the set. We can multiply $(142)(35)$ by another permutation (13)(2465) by composing the two permutations. The result is (142)(35)(13)(2465) = (15)(3463). These permutations can also be represented on diagrams of k pairs of vertices. For example,

$$(13)(2465) = \begin{pmatrix} 3 & 4 & 1 & 6 & 2 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} =$$

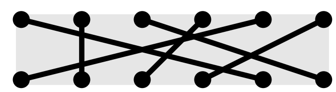


Multiplication of permutations in this form is done by placing one diagram over another and tracing the path of the edges from the bottom diagram through the top diagram. Using the previous example,

$$\begin{array}{c}(142)(35)\\ \times \\ (13)(2465)\end{array}$$



$$=$$

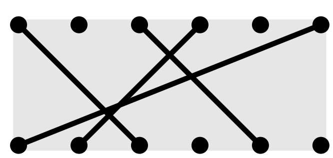


$$(15)(3463)$$

The diagrams multiply in the same manner as the cyclic notation of the permutations. The group algebra of S_k is the algebra over $\mathbb{C}$ whose basis consists of these $k!$ diagrams.

2.1.2 The Rook Monoid

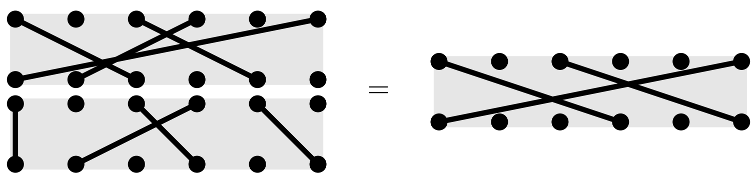
The rook monoid R_k is the monoid consisting of all diagrams on $2k$ vertices that can have any combination of vertical edges and isolated vertices. The degree of each vertex is at most 1. An example of a rook monoid diagram is



The dimension formula for R_k is

$$|\mathsf{R}_k| = \sum_{\ell=0}^{k} \binom{k}{\ell} \ell!.$$

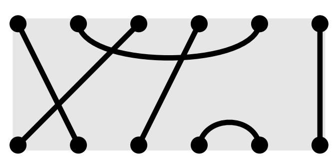
These diagrams multiply in the same way as symmetric group diagrams, except there are special rules for when an isolated vertex meets an edge or when two isolated vertices meet. When a diagram has been placed on top of another diagram in order to multiply them, we call the top row of the bottom diagram and the bottom row of the top diagram the *middle rows*. If an edge meets an isolated vertex in the middle rows, then in the resulting diagram an isolated vertex appears in the position of the other vertex of that edge. If two isolated vertices meet in the middle rows, then the resulting diagram is not affected. To illustrate,



The rook monoid algebra, $\mathbb{C}\mathsf{R}_k$, is the algebra over $\mathbb{C}$ whose basis consists of all R_k diagrams.

2.1.3 The Brauer Algebra

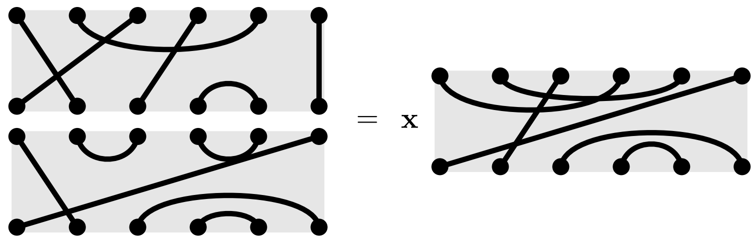
For $k \in \mathbb{Z}_{>0}$ and $x \in \mathbb{C}$, the Brauer algebra $\mathsf{B}_k(x)$ is the algebra over $\mathbb{C}$ whose basis consists of all diagrams on $2k$ vertices that have any combination of horizontal and vertical edges. An example of a Brauer diagram is



The dimension formula for $\mathsf{B}_k(x)$ is

$$\dim(\mathsf{B}_k(x)) = (2k-1)!!.$$

where $(2k-1)!! = (2k-1)*(2k-3)\cdots*3*1$. Multiplying Brauer diagrams introduces a parameter, x, which comes into play when a loop forms in the middle rows of two diagrams being multiplied. A loop can be formed by two or more horizontal edges in the middle rows. When this occurs, the loops disappear and we multiply the resulting diagram by x^ℓ where ℓ is the number of loops in the middle rows. For example,

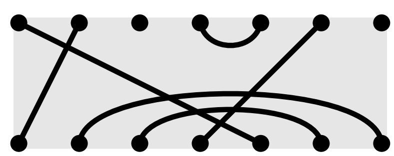


Note that horizontal and vertical edges can appear in the product of two diagrams via a sequences of edges that starts and ends with a vertical edge and which may have horizontal edges in the middle.

2.2 The Rook-Brauer Algebra

We now introduce the rook-Brauer algebra and build an understanding of its structure. As with the rook monoid algebra, the rook-Brauer algebra is an algebra over $\mathbb{C}$ with a basis of diagrams. We discuss how these diagrams multiply, show how to find the dimension of the algebra, define its generators and give a presentation.

The rook-Brauer algebra $\mathsf{RB}_k(x)$ is an associative algebra with a basis of rook-Brauer diagrams, which are Brauer diagrams that allow for isolated vertices. A rook-Brauer diagram in $\mathsf{RB}_k(x)$ has $2k$ vertices and can have anywhere from 0 to k edges, including both horizontal and vertical edges. An example of a diagram in the basis of $\mathsf{RB}_7(x)$ is



Let $\mathcal{RB}_k$ be the set of all basis diagrams of $\mathsf{RB}_k(x)$. The dimension of $\mathsf{RB}_k(x)$ is

$$|\mathsf{RB}_k(x)| = \sum_{\ell=1}^{k} \binom{2k}{2\ell} (2\ell-1)!!.$$

The combinatorics behind this formula are very intuitive. For $2k$ vertices, choose 2ℓ of them to be in pairs. This task gives $\binom{2k}{2\ell}$ for a fixed ℓ. Then we place these edges in the diagram, which amounts to picking two of the 2ℓ

vertices for each edge. There are $2\ell-1$ choices for the first edge, $2\ell-3$ choices for the second edge, and so on, so it follows that we multiply by $(2\ell-1)!!$. So given a fixed number of edges 2ℓ, there are $\binom{2k}{2\ell}(2\ell-1)!!$ diagrams. Therefore, if we sum $\sum_{\ell=1}^{k}\binom{2k}{2\ell}(2\ell-1)!!$ we get all possible diagrams in the basis of $\mathsf{RB}_k(x)$. The dimension grows very quickly as k increases.

k	1	2	3	4	5	6	7	8
$\lvert\mathsf{RB}_k(x)\rvert$	2	10	76	764	9496	140152	2390480	46206736

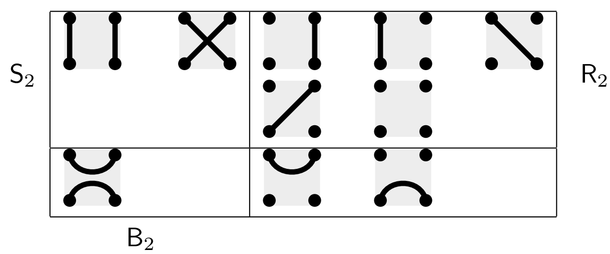


Figure 2.1: The ten diagrams of $\mathcal{RB}_2$, organized by the subalgebras contained in $\mathsf{RB}_k(x)$.

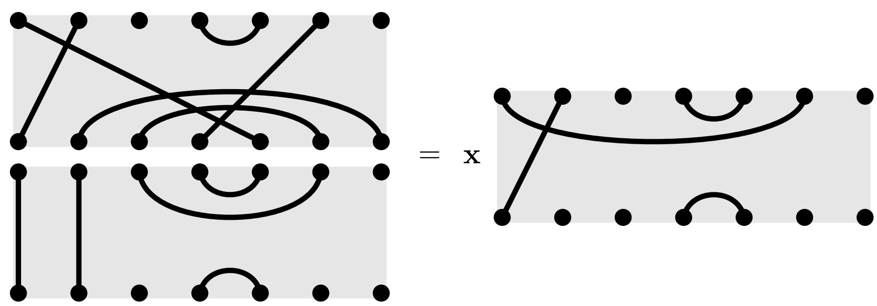
2.2.1 Multiplication

Multiplication of rook-Brauer diagrams combines the methods of multiplication in $\mathsf{S}_k, \mathsf{R}_k$, and B_k. To multiply diagrams d_1 and d_2 and get the diagram $d_2d_1 \in \mathsf{RB}_k(x)$, place d_2 above d_1. We again call set of vertices formed by the top row of d_1 and the bottom row of d_2 the *middle rows of* d_2d_1.

- In the middle rows, connect two vertices if they share the same index and are incident with an edge. This creates new vertical and horizontal edges in d_2d_1.
- If an isolated vertex in the middle rows meets a vertex incident with an edge, then in d_2d_1 an isolated vertex appears in the position of the edge's second vertex, as with rook monoid diagrams.
- If two isolated vertices meet in the middle rows, they disappear.

- If horizontal edges form loops in the middle rows, then the loops disappear in d_2d_1 and d_2d_1 is multiplied by x^ℓ where ℓ is the number of loops in the middle rows of d_2d_1, as with Brauer diagrams.

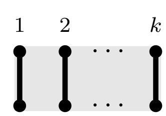
The following figure illustrates the process of diagram multiplication in $\mathsf{RB}_k(x)$.



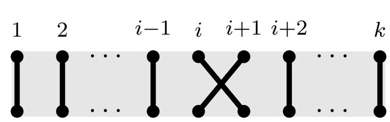
2.2.2 Presentation on generators and relations

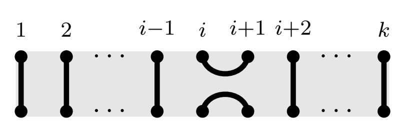
The multiplicative identity of $\mathsf{RB}_k(x)$ is the diagram

$$\mathbf{1} =$$

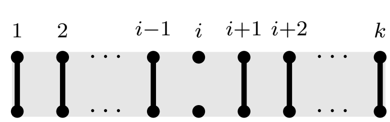


All basis diagrams of $\mathsf{RB}_k(x)$ can be generated by the elements s_i, t_i, and p_i where for $1 \leq i \leq k-1$,

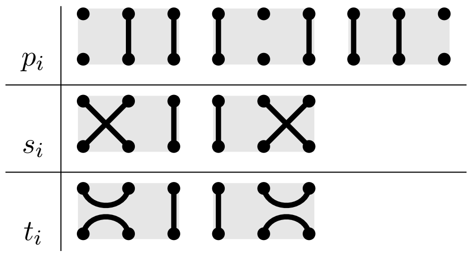
$s_i =$ 

and $t_i =$ 

and for $1 \leq i \leq k$,

$p_i =$ 

For example, the generators of $\mathsf{RB}_3(x)$ are



A presentation is proven in [KM] which we give here. It is a straightforward process to check that the following relations hold for the generators of $\mathsf{RB}_k(x)$. However the proof that these relations are sufficient to generate $\mathsf{RB}_k(x)$ is much harder, so we refer to [KM]. The basis of the group algebra of the symmetric group $\mathbb{C}\mathsf{S}_k \subset \mathsf{RB}_k(x)$ is generated by s_i for $1 \le i \le k-1$ subject to the following relations:

$$s_i^2 = \mathbf{1}; \tag{2.1}$$
$$s_i s_j = s_j s_i \text{ for } |i-j| > 1; \tag{2.2}$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}. \tag{2.3}$$

The basis of the Brauer algebra $\mathsf{B}_k(x) \subset \mathsf{RB}_k(x)$ contains $\mathbb{C}\mathsf{S}_k$ and is generated by s_i and t_i subject to relations (2.1)-(2.3) and the following relations:

$$t_i^2 = x t_i; \tag{2.4}$$
$$t_i t_j = t_j t_i \text{ for } |i-j| > 1; \tag{2.5}$$
$$t_i t_{i+1} t_i = t_i; \tag{2.6}$$
$$t_i s_i = s_i t_i = t_i; \tag{2.7}$$
$$t_i s_j = s_j t_i \text{ for } |i-j| > 1; \tag{2.8}$$
$$s_i t_{i+1} t_i = s_{i+1} t_i \tag{2.9}$$
$$t_i t_{i+1} s_i = t_i s_{i+1}. \tag{2.10}$$

The basis of the rook monoid algebra $\mathbb{C}\mathsf{R}_k \subset \mathsf{RB}_k(x)$ is generated by s_i and p_i subject to relations (2.1)-(2.3) and the following relations:

$$p_i^2 = p_i; \tag{2.11}$$
$$p_i p_j = p_j p_i \text{ for } i \neq j; \tag{2.12}$$
$$s_i p_i = p_{i+1} s_i; \tag{2.13}$$
$$s_i p_j = p_j s_i \text{ for } |i-j| > 1; \tag{2.14}$$
$$p_i s_i p_i = p_i p_{i+1}. \tag{2.15}$$

Finally, $\mathsf{RB}_k(x)$ is generated by s_i, t_i, and p_i subject to relations (2.1) – (2.15) along with the following relations:

$$t_i p_j = p_j t_i \text{ for } |i-j| > 1; \tag{2.16}$$
$$t_i p_i = t_i p_{i+1} = t_i p_i p_{i+1}; \tag{2.17}$$
$$p_i t_i = p_{i+1} t_i = p_i p_{i+1} t_i; \tag{2.18}$$
$$t_i p_i t_i = p_{i+1}; \tag{2.19}$$
$$p_i t_i p_i = p_i p_{i+1}. \tag{2.20}$$

Chapter 3

Double Centralizer Theory

3.1 General Theory

Let G be a group and let V be a $\mathbb{C}G$-module. Consider the set of right-actions of G on V, that is the actions $g \cdot v$ for $g \in G$, $v \in \mathsf{V}$. The centralizer C of G acting on V from the right is the set of elements

$$\{g \in G \mid g \cdot (h \cdot v) = h \cdot (g \cdot v), \text{ for all } h \in G, v \in \mathsf{V}\}.$$

This centralizer is the ring of endomorphisms of V, which we denote as $\mathrm{Hom}_G(\mathsf{V}, \mathsf{V}) = \mathrm{End}_G(\mathsf{V})$, which are all G-module homomorphisms from V to itself. This can also be denoted as

$$C = \{\phi \in \mathrm{End}(\mathsf{V}) \mid \phi(g \cdot v) = g \cdot \phi(v), v \in \mathsf{V}\}.$$

Let V be an n-dimensional $\mathbb{C}G$-module with basis $\{v_1, \ldots, v_n\}$. We now present the double centralizer theorem for the n^k-dimensional tensor space

$$\mathsf{V}^{\otimes k} = \mathbb{C} - \mathrm{span}\{v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_k} \mid i_j \in \{1, \ldots, n\}\}.$$

This theorem is classical and can be seen in [CR] section 3B.

Theorem 5. ***Double Centralizer Theorem*** *Let G be a group and V be a $\mathbb{C}G$-module such that*

$$\mathsf{V}^{\otimes k} \cong \bigoplus_{\lambda \in \Lambda_k} m_{\lambda,k} \mathsf{V}^{\lambda}$$

is a decomposition of $\mathsf{V}^{\otimes k}$ into irreducible G-submodules and let $\mathsf{C}_k = \mathrm{End}_G(\mathsf{V}^{\otimes k})$. Then *i.* C_k *is a semisimple algebra over* $\mathbb{C}$.

ii. *The irreducible representations of* C_k *are labeled by* $\lambda \in \Lambda_k$. *We denote these irreducible representations as* M_k^λ.

iii. $\dim(M_k^\lambda) = m_{\lambda,k}$

iv. *As a* C_k*-module,*

$$\mathsf{V}^{\otimes k} \cong \bigoplus_{\lambda \in \Lambda_k} d_{\lambda,k} M_k^\lambda$$

with $d_{\lambda,k} = \dim(\mathsf{V}^\lambda)$ *on level* k.

By classical Artin-Wedderburn theory as seen in [GW] Section 3.2 and [Pi] Section 3.5, the dimension of a semisimple algebra is the sum of the squares of the dimensions of its irreducible representations. Thus,

Theorem 6. $\dim(\mathsf{C}_k) = \sum_{\lambda \in \Lambda_k} m_{\lambda,k}^2$

The key to inductively computing the $m_{\lambda,k}$ is the *tensor product rule*

$$\mathsf{V} \otimes \mathsf{V}^\lambda = \bigoplus_{\mu \in \Lambda_k} \mathsf{V}^\mu.$$

The tensor product rule can be derived from the Clebsch-Gordan formulas, which come from Pieri's Rule and can be found in [GW] Corollary 9.2.4. This rule is unique to each C_k-module V and can be used inductively to construct a graph called the Bratteli diagram of V. The Bratteli diagram of V is the infinite rooted graph with vertices on level k labeled by the irreducible submodules V^λ of G that appear in the decomposition of $\mathsf{V}^{\otimes k}$, for $k \geq 0$, and an edge between V^λ on level k and V^μ on level $k+1$ if V^μ is in the decomposition of $\mathsf{V}^\lambda \otimes \mathsf{V}$.

If we choose a particular V_k^λ on level k of the Bratteli diagram, we can use the tensor product rule to inductively find that the dimension $m_{\lambda,k}$ of V^λ on level k is given by the number of paths on the Bratteli diagram from the the root to V_k^λ. A path on a Bratteli diagram to V_k^λ is given by a sequence $(\lambda_0, \lambda_1, \lambda_3, \ldots, \lambda_{k-1}, \lambda)$ where λ_0 is the root of the graph and λ_i labels some irreducible module on level i that appears in the decomposition of $\mathsf{V}^{\lambda_{i-1}} \otimes \mathsf{V}$. The Bratteli diagram is a way to encode the tensor product rule of a module. Examples of Bratteli diagrams are given in the next section.

Now that we know $m_{\lambda,k}$ is the number of paths to V^λ on level k, we conclude $m_{\lambda,k}^2$ is the number of paths from the root of the Bratteli diagram to V^λ on level k and back to the root. This gives a more concrete way to calculate $\dim(\mathsf{C}_k)$, which we take advantage of later.

3.2 Examples and Applications

We now present examples of centralizer algebras and applications of the double centralizer theorem. Each of the examples is one of the subalgebras of $\mathsf{RB}_k(x)$ discussed in the introduction of this paper and builds the reason we are interested in the rook-Brauer algebra.

3.2.1 S_k

Let $G = \mathsf{GL}(n)$, the general linear group of $n \times n$ matrices with complex entries. The irreducible representations of G are labeled by the partitions $\lambda \vdash r$ for all $r \in \mathbb{N}$. Let V be the irreducible module labeled by the partition (1), which is known to be $\mathbb{C}^n$. The action of $g \in \mathsf{GL}(n)$ on $v \in \mathbb{C}^n$ is matrix multiplication $g \cdot v$ where v acts as an $n \times 1$ column vector. Explicitly, if $g = [a_{ij}]$ then

$$g \cdot v_j = \sum_{i=1}^{n} a_{ij} v_i.$$

For any partition μ the tensor product rule of V is

$$\mathsf{V} \otimes \mathsf{V}^{\mu} = \bigoplus_{\lambda = \mu + \square} \mathsf{V}^{\lambda}$$

This formula comes from the Clebsch-Gordam formulas and gives that the tensor space $\mathsf{V}^{\otimes k}$ decomposes into irreducible submodules as

$$\mathsf{V}^{\otimes k} \cong \bigoplus_{\lambda \in \Lambda_k} m_{k,\lambda} \mathsf{V}^{\lambda}$$

where $\Lambda_k = \{\lambda \mid \lambda \vdash k\}$. The centralizer $\mathrm{End}_G(\mathsf{V}^{\otimes k})$ is isomorphic to the group algebra of the symmetric group $\mathbb{C}\mathsf{S}_k$. This is classic Schur-Weyl duality as found in [GW] Section 9.1.

From the tensor product rule we can recursively create the Bratteli diagram $\mathcal{B}$ of G acting on $\mathsf{V}^{\otimes k}$. The irreducible submodules of G that appear on level k of $\mathcal{B}$ are labeled by the partitions $\lambda \vdash k$, and the tensor product rule of V gives that edges appear between V_{λ} on level k and V_{μ} on level $k+1$ if $\mu = \lambda + \square$. This particular Bratteli diagram is quite famous and is called Young's Lattice after the British mathematician Alfred Young.

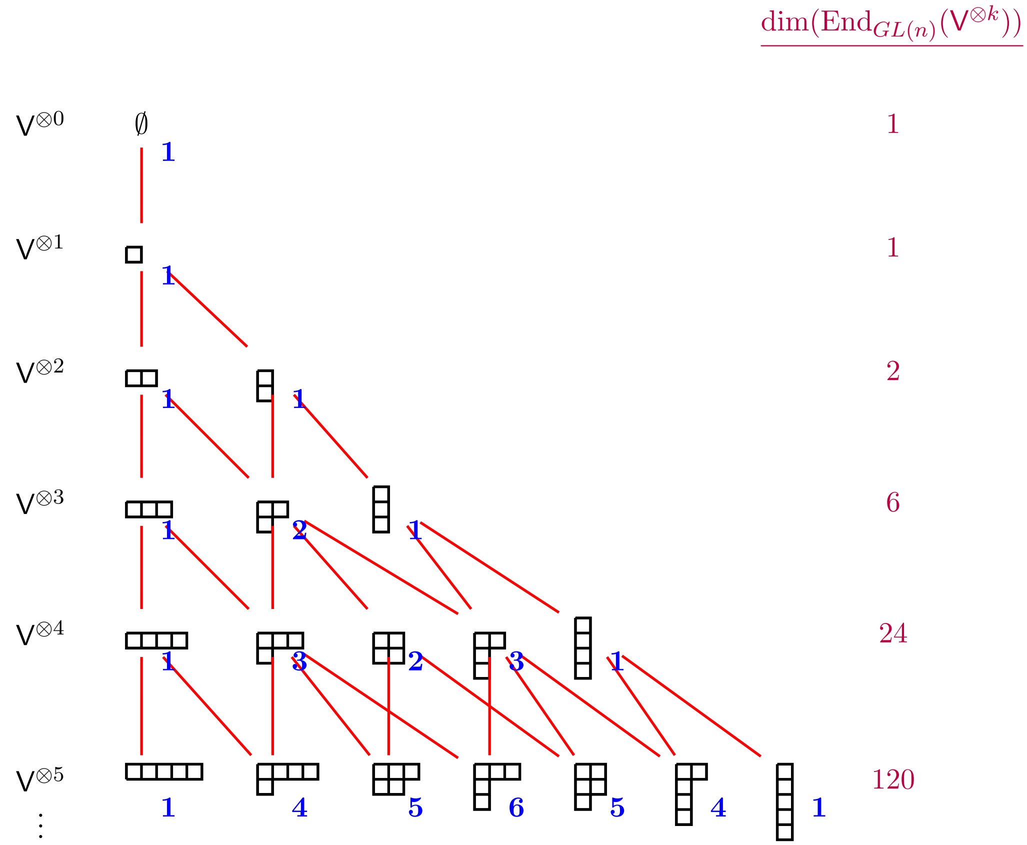


Figure 3.1: Young's Lattice for $0 \leq k \leq 5$.

3.2.2 R_k

Let $G = \mathsf{GL}(n)$ and let $\mathsf{V} = \mathsf{V}^{(1)} \oplus \mathsf{V}^{\emptyset}$ where $\mathsf{V}^{(1)}$ is $\mathbb{C}^n$ again and $\mathsf{V}^{\emptyset}$ is the trivial module. Now V is $(n+1)$-dimensional and has a basis $\{v_0, v_1, \ldots, v_n\}$. A matrix $g \in \mathsf{G}$ acts on a vector $v \in \mathsf{V}^{(1)}$ by matrix multiplication and $g \cdot v_0 = v_0$. With this module of G, for any partition μ,

$$\mathsf{V} \otimes \mathsf{V}^{\mu} = \left(\mathsf{V}^{(1)} \oplus \mathsf{V}^{\emptyset}\right) = \left(\mathsf{V}^{(1)} \otimes \mathsf{V}^{\mu}\right) \oplus \left(V^{\emptyset} \otimes \mathsf{V}^{\mu}\right). \tag{3.1}$$

Tensoring a module with the trivial module does nothing to the original module, so 3.1 is equivalent to $\mathsf{V}^{(1)} \otimes \mathsf{V}^{\mu} \oplus \mathsf{V}^{\mu}$. We can now use the tensor product rule for $\mathsf{V}^{(1)}$ in the previous section to derive the tensor product rule for $\mathsf{V} = \mathsf{V}^{(1)} \oplus \mathsf{V}^{\emptyset}$:

$$\mathsf{V} \otimes \mathsf{V}^{\mu} = \bigoplus_{\substack{\lambda=\mu+\square \\ \lambda=\mu}} \mathsf{V}^{\lambda}$$

This gives that the tensor space $\mathsf{V}^{\otimes k}$ also decomposes into irreducible submodules as

$$\mathsf{V}^{\otimes k} \cong \bigoplus_{\lambda \in \Lambda_k} m_{k,\lambda} \mathsf{V}^{\lambda}$$

however now $\Lambda_k = \{\lambda \mid \lambda \vdash r \text{ for } 0 \leq r \leq k\}$. The centralizer $\mathrm{End}_G(\mathsf{V}^{\otimes k})$ is isomorphic to the rook monoid algebra $\mathbb{C}\mathsf{R}_k$, as shown in [So].

3.2.3 $\mathsf{B}_k(n)$

Now let $G = \mathsf{O}(n)$, the infinite group of $n \times n$ orthogonal matrices with complex entries. This group is a subring of $\mathsf{GL}(n)$ and its irreducible representations are also labeled by the partitions $\lambda \vdash r$ for $r = 0, 1, 2, \ldots$. Let $\mathsf{V} = \mathsf{V}^{(1)}$, now as an $\mathsf{O}(n)$-module. For any partition μ,

$$\mathsf{V} \otimes \mathsf{V}^{\mu} = \bigoplus_{\substack{\lambda=\mu+\square \\ \lambda=\mu-\square}} \mathsf{V}^{\lambda}$$

This formula is derived from the Clebsch-Gordan formulas and recursively gives that a decomposition of $\mathsf{V}^{\otimes k}$ into irreducible representations is

$$\mathsf{V}^{\otimes k} \cong \bigoplus_{\lambda \in \Lambda_k} m_{k,\lambda} \mathsf{V}^{\lambda}$$

where $\Lambda_k = \{\lambda \mid \lambda \vdash r \text{ for } 0 \leq r \leq k\}$ as in the previous example. The centralizer $\mathrm{End}_G(\mathsf{V}^{\otimes k})$ is isomorphic to the Brauer algebra $\mathsf{B}_k(n)$ [Br].

3.2.4 $\mathsf{RB}_k(n+1)$

We now explore the centralizer $\mathrm{End}_{\mathsf{O}(n)}(\mathsf{V}^{\otimes k})$ where $\mathsf{V} = \mathsf{V}^{(1)} \oplus \mathsf{V}^{\emptyset}$. For any μ, the tensor product rule for V is

$$\mathsf{V} \otimes \mathsf{V}^{\mu} = \bigoplus_{\substack{\lambda=\mu \\ \lambda=\mu+\square \\ \lambda=\mu-\square}} \mathsf{V}^{\lambda}.$$

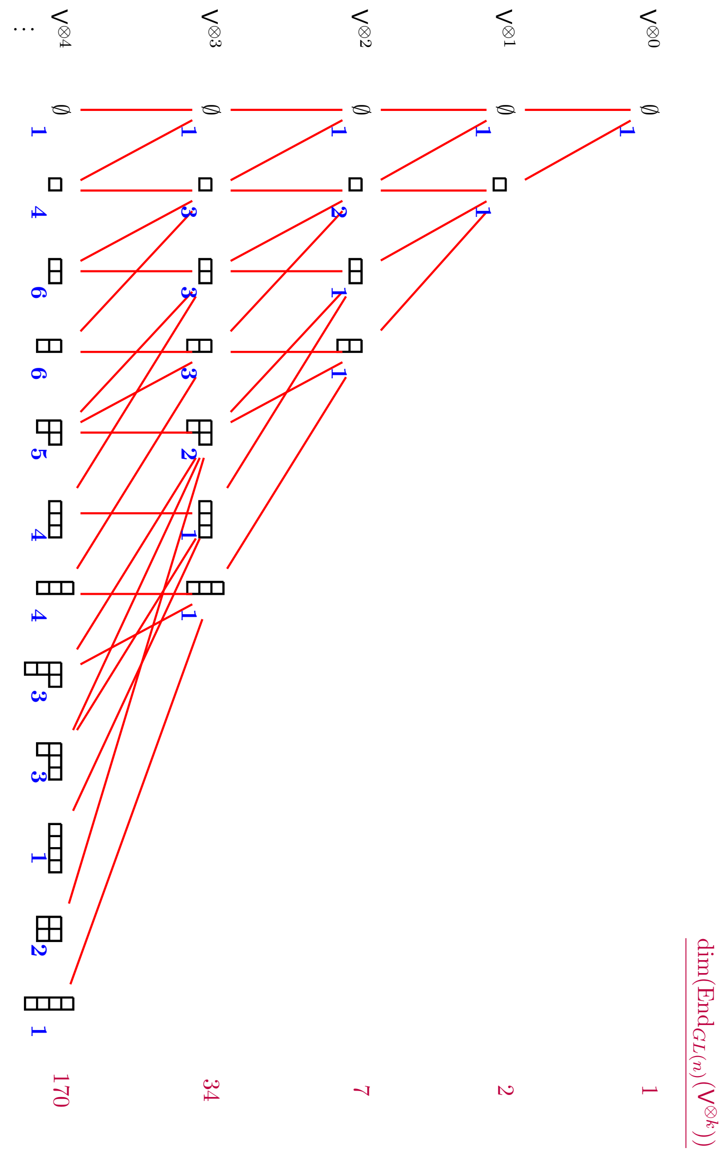


Figure 3.2: The Bratteli diagram for the $\mathbb{C}\mathsf{GL}(n)$-module $\mathsf{V} = \mathsf{V}^{(1)} \oplus \mathsf{V}^{\emptyset}$ for $0 \leq k \leq 4$.

As in deriving the tensor product rule of $\mathsf{GL}(n)$ acting on $\left(\mathsf{V}^{(1)} \oplus \mathsf{V}^{\emptyset}\right)^{\otimes k}$, this rule can be derived from the tensor product rule of $\mathsf{O}(n)$ acting $\left(\mathsf{V}^{(1)}\right)^{\otimes k}$.

The rule allows us to recursively define the Bratteli diagram of $\mathsf{O}(n)$ acting on $\mathsf{V}^{\otimes k}$ and conclude

$$\mathsf{V}^{\otimes k} \cong \bigoplus_{\lambda \in \Lambda_k} m_{k,\lambda} \mathsf{V}^{\lambda}$$

where $\Lambda_k = \{\lambda \mid \lambda \vdash r \text{ for } 1 \leq r \leq k\}$. In the coming sections we prove the following theorem.

Theorem 7. *For $n \geq k$, $\mathsf{RB}_k(n+1) \cong \mathrm{End}_{\mathsf{O}(n)}\left(\mathsf{V}^{\otimes k}\right)$*

We first define an action of $\mathsf{RB}_k(x)$ on the tensor space $\mathsf{V}^{\otimes k}$ and prove that this creates an $\mathsf{RB}_k(x)$-module. Then we show that the action of $\mathsf{RB}_k(n+1)$ commutes with the action of $\mathsf{O}(n)$ on the tensor space. This shows that $\mathsf{RB}_k(n+1)$ is a subset of $\mathrm{End}_{\mathsf{O}(n)}(\mathsf{V}^{\otimes k})$. Finally, we use paths on the Bratteli diagram to give a combinatorial proof that the dimension of $\mathrm{End}_{\mathsf{O}(n)}(\mathsf{V}^{\otimes k})$ is equal to the dimension of $\mathsf{RB}_k(n+1)$, which completes the proof that $\mathrm{End}_{\mathsf{O}(n)}(\mathsf{V}^{\otimes k}) \cong \mathsf{RB}_k(n+1)$.

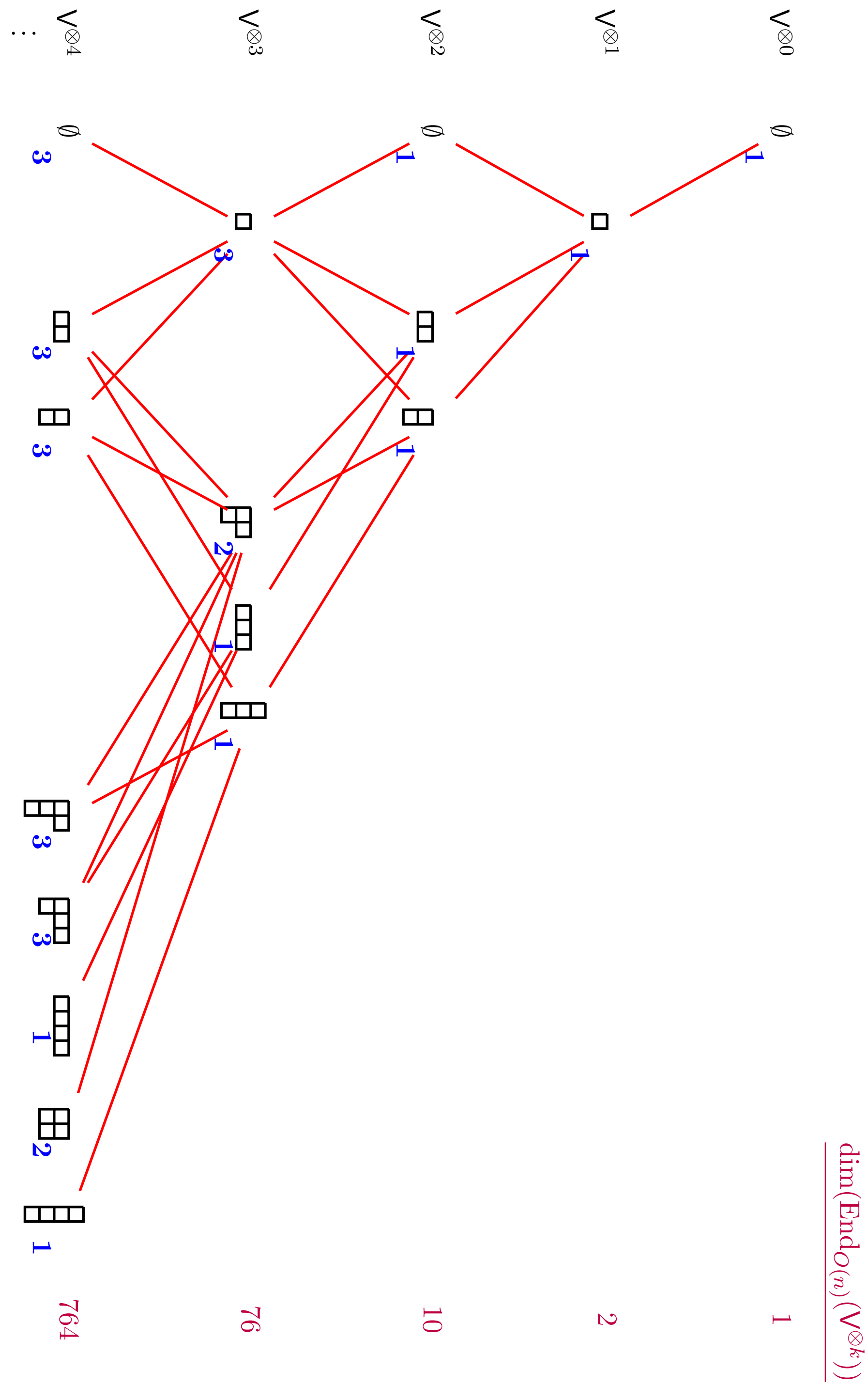


Figure 3.3: The Bratteli diagram for the $\mathbb{C}\mathsf{O}(n)$-module $\mathsf{V} = \mathsf{V}^{(1)}$ for $0 \leq k \leq 4$.

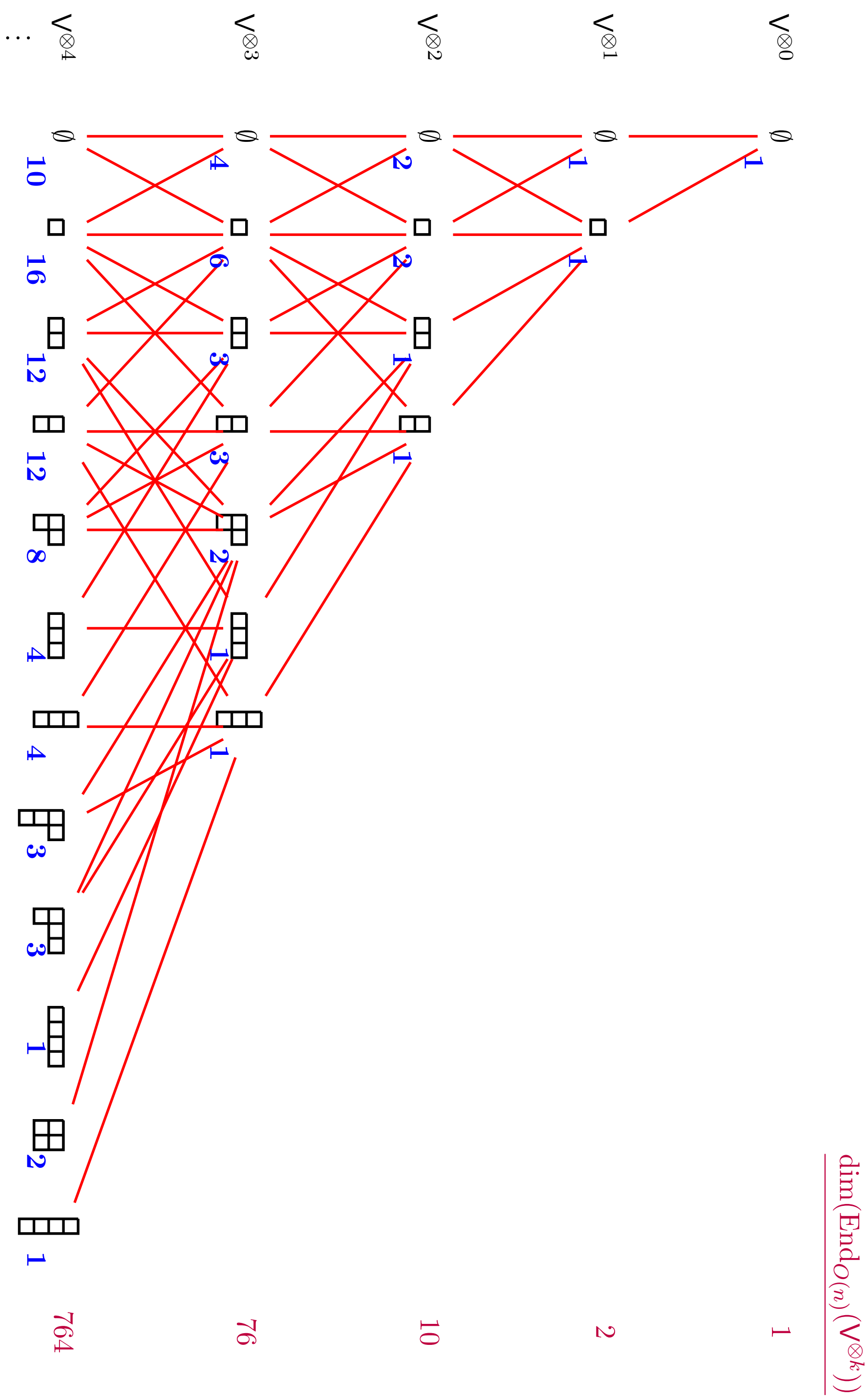


Figure 3.4: The Bratteli diagram for the $\mathbb{C}\mathsf{O}(n)$-module $\mathsf{V} = \mathsf{V}^{(1)} \oplus \mathsf{V}^{\emptyset}$ for $0 \leq k \leq 4$.

Chapter 4

Action of $\mathsf{RB}_k(n+1)$ on Tensor Space

4.1 How $\mathsf{RB}_k(n+1)$ Acts on $\mathsf{V}^{\otimes k}$

In this chapter, we define an action of $\mathsf{RB}_k(n+1)$ on a specific $(n+1)^k$-dimensional tensor product module $\mathsf{V}^{\otimes k}$ of the general linear group $\mathsf{GL}_n(\mathbb{C})$. We prove that this action creates a faithful representation of $\mathsf{RB}_k(n+1)$ on $\mathsf{V}^{\otimes k}$, which aids in proving that $\mathsf{RB}_k(n+1)$ is the centralizer algebra of $\mathsf{O}(n)$ acting on V. The definition of the action was influenced by the work on Motzkin algebras in [BH].

Let $\mathsf{V}^{(1)}$ be the n-dimensional $\mathsf{GL}_n(\mathsf{C})$ module with basis $\{v_1, v_2, \ldots, v_n\}$ and let $\mathsf{V}^{\emptyset}$ be the trivial $\mathsf{GL}_n(\mathsf{C})$ module with basis v_0. That is, if $g \in \mathsf{GL}(n)$ and the ij entry of g is g_{ij}, then

$$g \cdot v_j = \sum_{i=1}^{n} g_{ij} v_i$$

for $1 \leq j \leq n$ and $g \cdot v_0 = v_0$. We define $\mathsf{V} = \mathsf{V}^{(1)} \oplus \mathsf{V}^{\emptyset}$. This new module is $(n+1)$-dimensional with basis $\{v_0, v_1, v_2, \ldots, v_n\}$. Consider the k-fold tensor product module

$$\mathsf{V}^{\otimes k} = \mathbb{C}\text{-span}\left\{\, v_{i_1} \otimes \cdots \otimes v_{i_k} \mid i_j \in \{0, \ldots, n\} \,\right\}, \tag{4.1}$$

which has dimension $(n+1)^k$ and a basis consisting of simple tensors of the form $v_{i_1} \otimes \cdots \otimes v_{i_k}$. An element $g \in \mathsf{GL}_n(\mathbb{C})$ acts on a simple tensor by the diagonal action

$$g(v_{i_1} \otimes \cdots \otimes v_{i_k}) = (gv_{i_1}) \otimes \cdots \otimes (gv_{i_k}), \tag{4.2}$$

which extends linearly to make $\mathsf{V}^{\otimes k}$ a $\mathsf{GL}_n(\mathbb{C})$ module.

We define an action on of a diagram $d \in \mathsf{RB}_k(n)$ on the basis of simple tensors in $V^{\otimes k}$ by

$$d(v_{i_1} \otimes \cdots \otimes v_{i_k}) = \sum_{j_1,\ldots,j_k} (d)^{j_1,\ldots,j_k}_{i_1,\ldots,i_k} \; v_{j_1} \otimes \cdots \otimes v_{j_k}, \tag{4.3}$$

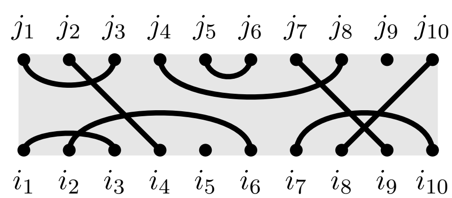
where $(d)^{j_1,\ldots,j_k}_{i_1,\ldots,i_k}$ is computed by labeling the vertices in the bottom row of d with $i_1, \ldots, i_k$ and the vertices in the top row of d with $j_1, \ldots, j_k$. Then

$$(d)^{j_1,\ldots,j_k}_{i_1,\ldots,i_k} = \prod_{\varepsilon \in d} (\varepsilon)^{j_1,\ldots,j_k}_{i_1,\ldots,i_k},$$

where the product is over the weights of all connected components ε (edges and isolated vertices) in the diagram d, where by the weight of ε we mean

$$(\varepsilon)^{j_1,\ldots,j_k}_{i_1,\ldots,i_k} = \begin{cases} \delta_{a,0}, & \text{if } \varepsilon \text{ is an isolated vertex labeled by } a, \\ \delta_{a,b}, & \text{if } \varepsilon \text{ is an edge in } d \text{ connecting } a \text{ and } b, \end{cases}$$

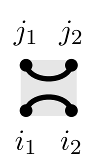
where $\delta_{a,b}$ is the Kronecker delta. For example, for this labeled diagram in $\mathsf{RB}_{10}(n+1)$



we have

$$d^{j_1,\ldots,j_k}_{i_1,\ldots,i_k} = \delta_{j_1,j_3}\delta_{j_2,i_4}\delta_{j_4,j_8}\delta_{j_5,j_6}\delta_{j_7,i_9}\delta_{j_9,0}\delta_{j_{10},i_8}\delta_{i_1,i_3}\delta_{i_2,i_6}\delta_{i_5,0}\delta_{i_7,i_{10}}.$$

On a smaller scale, for this labeled diagram d in $\mathsf{RB}_2(n+1)$



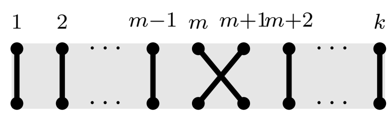
we have $d_{i_1,i_2}^{j_1,j_2} = \delta_{i_1,i_2}\delta_{j_1,j_2}$. If d acts on the basis element $v_0 \otimes v_0$, then

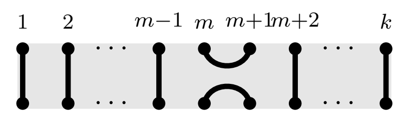
$$d(v_0 \otimes v_0) = \sum_{j_1,j_2} d_{0,0}^{j_1,j_2}(v_{j_1} \otimes v_{j_2})$$

which gives $d(v_0 \otimes v_0) = v_0 \otimes v_0 + v_1 \otimes v_1 + v_2 \otimes v_2$.

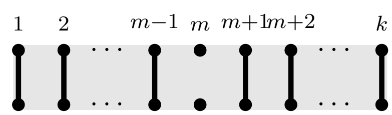
4.2 Actions of Generators

Recall that the basis of the rook Brauer algebra $\mathcal{RB}_k$ is generated by the diagrams s_m, t_m, and p_m where for $1 \le m \le k-1$,

$s_m =$ 

and $t_m =$ 

and for $1 \le m \le k$,

$p_m =$ 

In order to illustrate the action on tensor space, we present how each generating element acts on a simple tensor $v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_k}$. The generator s_m permutes the m^{th} and $m+1^{st}$ vectors of the simple tensor. This can be easily calculated from the general definition of the action on tensor space and is a natural action as s_m corresponds to the m^{th} transposition of S_k. Therefore, s_m acts as

$$s_m(v_{i_1} \otimes \ldots \otimes v_{i_m} \otimes v_{i_{m+1}} \otimes \ldots \otimes v_{i_k}) = v_{i_1} \otimes \ldots \otimes v_{i_{m+1}} \otimes v_{i_m} \otimes \ldots \otimes v_{i_k}.$$

The action of the generator t_m is slightly more complicated. The bottom row of t_m requires that the simple tensor on which t_m is acting must be of the form $v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_m} \otimes v_{i_m} \otimes \ldots v_{i_k}$. For any such simple tensor, $(t_m)_{i_1,\ldots,i_k}^{j_1,\ldots,j_k} \neq 0$ if and only if $j_1 = i_1, j_2 = i_2, \ldots, j_m = j_{m+1}, \ldots, j_k = i_k$. Therefore,

$$t_m(v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_m} \otimes v_{i_m} \otimes \ldots v_{i_k}) = \sum_{\ell=0}^{n} v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_\ell \otimes v_\ell \otimes \ldots \otimes v_{i_k}$$

where v_ℓ is in the m^{th} and $m+1^{st}$ positions.

The bottom row of the generator p_m requires that the simple tensor on which it acts be of the form $v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_m} \otimes \ldots \otimes v_{i_k}$ where $i_m = 0$. The only simple tensor that satisfies $(p_m)^{j_1,\ldots,j_k}_{i_1,\ldots,i_k} \neq 0$ is $v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_m} \otimes \ldots \otimes v_{i_k}$ itself. Therefore,

$$p_m(v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_m} \otimes \ldots \otimes v_{i_k}) = \delta_{i_m,0}(v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_m} \otimes \ldots \otimes v_{i_k}).$$

4.3 π_k is a Representation

Let $\pi_k : \mathsf{RB}_k(n+1) \to \mathrm{End}(\mathsf{V}^{\otimes k})$ be the representations afforded by the action of $\mathsf{RB}_k(n+1)$ on $\mathsf{V}^{\otimes k}$. We have defined this function on the set of basis diagrams $\mathcal{RB}_k$, and it extends linearly to all of $\mathsf{RB}_k(n+1)$. Recall from Section 1.2 that in order to prove that π_k is in fact an algebra representation, we must show that for diagrams $d_1, d_2 \in \mathcal{RB}_k$, $\pi_k(d_2 d_1) = \pi_k(d_2)\pi_k(d_1)$.

Theorem 8. *Then $\pi_k : \mathsf{RB}_k(n+1) \to \mathrm{End}(\mathsf{V}^{\otimes k})$ is an algebra representation.*

Proof. It suffices to show that

$$(d_2 d_1)^{j_1,\ldots,j_k}_{i_1,\ldots,i_k} = \sum_{\ell_1,\ldots,\ell_k} (d_2)^{j_1,\ldots,j_k}_{\ell_1,\ldots,\ell_k} (d_1)^{\ell_1,\ldots,\ell_k}_{i_1,\ldots,i_k}.$$

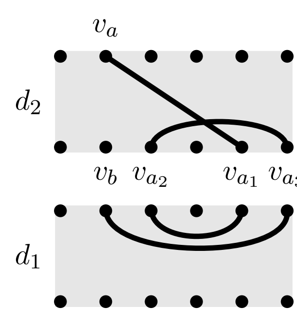
For a basis diagram $d \in \mathcal{RB}_k$, let $[d]$ denote the matrix of d with respect to the basis elements $v_{m_1} \otimes v_{m_2} \otimes \ldots \otimes v_{m_k}$. The entry c_{ij} of $[d]$ is the coefficient of the j^{th} basis element $v_{j_1} \otimes v_{j_2} \otimes \ldots \otimes v_{j_n}$ in the result of d acting on the i^{th} basis element $v_{i_1} \otimes v_{i_2} \otimes \ldots v_{i_n}$. In other words, $c_{ij} = d^{j_1,\ldots,j_k}_{i_1,\ldots,i_k}$. Given $d_1, d_2 \in \mathcal{RB}_k$, we show that $[d_2][d_1] = [d_2 d_1]$ by considering the edges of $d_2 d_1$ case by case and analyzing c_{ij} in $[d_2][d_1]$ and $[d_2 d_1]$.

Case 1: Isolated vertex in the top row of $d_2 d_1$.

There are two cases that result in an isolated vertex in the top row of $d_2 d_1$.

(i.) An isolated vertex was inherited from the top row of d_2. We label this vertex v_a. In $[d_2d_1]$, c_{ij} is nonzero if and only if $a = 0$. The same is true of $[d_2][d_1]$. This shows that c_{ij} is either 0 in both $[d_2d_1]$ and $[d_2][d_1]$ or nonzero in both $[d_2d_1]$ and $[d_2][d_1]$.

(ii.) The isolated vertex is the result of a vertical edge in d_2 connected to a series of t horizontal edges ($t \geq 0$) in the middle rows of d_2d_1 which end at an isolated vertex in the middle row of d_2d_1. For example,

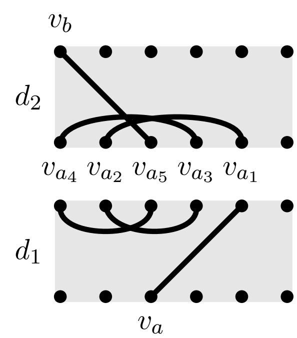


Sequentially label these vertices $v_a, v_{a_1}, v_{a_3}, \ldots, v_{a_t}, v_b$ as shown in the diagram above. In $[d_2d_1]$, c_{ij} is nonzero if and only if $a = 0$ in the i^{th} basis element. In $[d_2][d_1]$, c_{ij} is nonzero if and only if $a = a_1 = a_2 = \cdots = a_t = b = 0$. Furthermore, the weight of this series of edges is $\delta_{a,0}$ and so it acts as an isolated vertex.

The proof for when the isolated edge is in the bottom row of d_2d_1 is analogous.

Case 2: A vertical edge in d_2d_1.

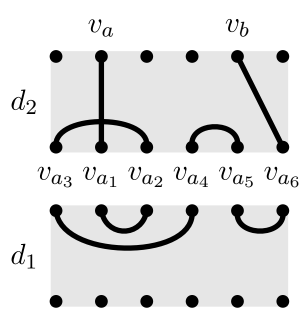
A vertical edge in d_2d_1 occurs when a vertical edge in d_2 is connected to a vertical edge in d_1 by an even number of horizontal edges $t \geq 0$ in the middle rows of d_2d_1. For example,



Label the bottom vertex in d_1 with v_a and sequentially label the connected vertices with $v_{a_1}, v_{a_2}, \ldots, v_{a_t}$ as shown in the above diagram. Label the vertex in the top row v_b. In $[d_2 d_1]$, c_{ij} is nonzero if and only if $a = b$ and in $[d_2][d_1]$, c_{ij} is nonzero if only if $a = a_1 = a_2 = \cdots = a_t = b$. Note that the weight of this series of edges is $\delta_{a,b}$, so it acts like a vertical edge from the position of v_a to the position of v_b.

Case 3: A horizontal edge in the top row of $d_2 d_1$.

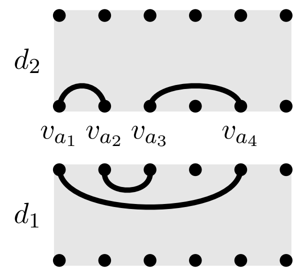
A horizontal edge in top row of $d_2 d_1$ results from two vertical edges in d_1 connected by a series of horizontal edges in the middle rows of $d_2 d_1$. For example,



Starting with the left most top vertex in d_2, sequentially label the vertices $v_a, v_{a_1}, v_{a_2}, \ldots, v_{a_t}, v_b$ where v_b is the right most top vertex in d_2 as shown in the diagram above. In $[d_2 d_1]$, c_{ij} is nonzero if and only if $a = b$, and in the matrix $[d_2][d_1]$, c_{ij} is nonzero if and only if $a = a_1 = a_2 = \cdots = a_t = b$. Notice that the edge weight for this series of edges is $\delta_{a,b}$, as it would be with a horizontal edge from the position of v_a to the position of v_b. The case for when a horizontal edge occurs in the bottom row of $d_2 d_1$ has an analogous proof.

Case 4: One or more loops in the middle of d_2d_1.

A loop in the middle row of d_2d_1 results from a series of connected horizontal edges in the middle rows of d_2d_1. For example,



Starting with the left most middle vertex, label the vertices $v_{a_1}, v_{a_2}, \ldots, v_{a_t}$. From the definition of the multiplication of diagrams in $\mathcal{RB}_k$, the diagram d_2d_1 is the diagram created by composing d_2 with d_1 multiplied by $(n+1)^\ell$ where ℓ is the number of loops in the middle row of d_2d_1. The middle row does not contribute to the matrix of the resulting diagram d_2d_1 so we have that $[(n+1)^\ell d_2d_1] = (n+1)^\ell[d_2d_1]$. The matrix $[d_2][d_1]$ has a nonzero entry if and only if $a_1 = a_2 = \cdots = a_t$. We sum over all cases where this is true, which gives the matrix of the basis diagram d_2d_1 multiplied by $n+1$. If d_2d_1 has ℓ loops in the middle row, the matrix is multiplied by $(n+1)^\ell$. So we have $[d_2][d_1] = (n+1)^\ell[d_2d_1]$. $\square$

4.4 π_k is Faithful

We now show that for $n \geq k$, π_k is a faithful representation, which means that it is one-to-one. To prove that π_k is faithful it is sufficient to show that $\mathbf{Ker}(\pi_k)$ is trivial. We do this by showing that for any nonzero element $y \in \mathsf{RB}_k(n+1)$, there is at least one simple tensor $u \in \mathsf{V}^{\otimes k}$ for which the action of y on u produces a nonzero sum of simple tensors. In other words, the only element $x \in \mathsf{RB}_k(n+1)$ for which $\pi_k(x)$ is the zero matrix is the additive identity 0.

Theorem 9. *$\pi_k : \mathsf{RB}_k(n) \to \mathrm{End}(\mathsf{V}^{\otimes k})$ is faithful for $n \geq k$.*

Proof. Let $y = \sum_{d\in\mathcal{RB}_k} a_d d$, $a_d \in \mathbb{C}$, be some nonzero element of $\mathsf{RB}_k(n+1)$. Choose a diagram d' in the linear combination y such that

(i.) $a_{d'} \neq 0$,

(ii.) among the diagrams satisfying (i.), d' has a maximum number of vertical edges m and

(iii.) among the diagrams satisfying (i.) and (ii.), d' has a maximum number of horizontal edges ℓ.

Choose a simple tensor $u = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$ such that

(i.) v_0 is in the positions of the isolated vertices in the bottom row of d',

(ii.) $v_1, v_2, \ldots, v_m$ are in the positions of the bottom vertices of the vertical edges in d', and

(iii.) $v_{m+1}, v_{m+2}, \ldots, v_{m+\ell}$ are in the positions of the vertices of the horizontal edges in the bottom row of d' such that the subscripts of the vectors in the positions of either end of a horizontal edge are the same.

The hypothesis $n \geq k$ guarantees that such a simple tensor u exists.

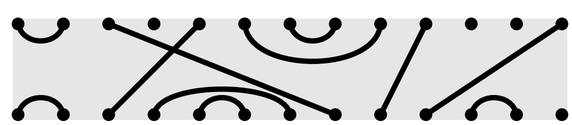
Finally, consider a simple tensor u' such that

(i.) v_0 is in the positions of the isolated vertices of the top row of d',

(ii.) $v_1, v_2, \ldots, v_m$ are in the positions of the top vertices of the vertical edges in d', and

(iii.) in the positions of the vertices of the t horizontal edges in the top row of d' place $v_{m+1}, v_{m+2}, \ldots, v_{m+t}$ such that the subscripts of the vectors at either end of a horizontal edge are the same.

Example 4.4.1. If d' is the following diagram



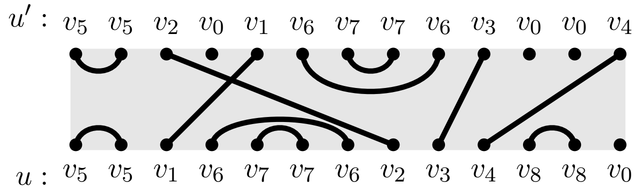
we choose

$$u = v_5 \otimes v_5 \otimes v_1 \otimes v_6 \otimes v_7 \otimes v_7 \otimes v_6 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_8 \otimes v_8 \otimes v_0$$

and

$$u' = v_5 \otimes v_5 \otimes v_2 \otimes v_0 \otimes v_1 \otimes v_6 \otimes v_7 \otimes v_7 \otimes v_6 \otimes v_3 \otimes v_0 \otimes v_0 \otimes v_4.$$

To illustrate:



When d' acts on u, u' has a nonzero coefficient in the result. We claim that no other diagram in y acting on u produces a nonzero coefficient of u' and therefore there is no other diagram whose action can cancel out the nonzero coefficient of u' created by d' acting on u.

In order for another diagram d'' in y to produce a nonzero coefficient of u' when acting on u, d'' must have the same bottom row as d'. This follows from the choice of d' having the maximum number of vertical edges and horizontal edges and choosing distinct v_i to put in each position of the edges of d'. The same conditions force the top row of d' and d'' to be the same. Therefore, $d'' = d'$ and only d' produces a nonzero coefficient of u' when acting on u. This implies that $y(u)$ is in fact nonzero, which contradicts the assumption that $y \in \mathbf{Ker}(\pi_k)$. Therefore, only the empty diagram is in the kernel of the representation and π_k is faithful for all $n \geq k$. $\square$

With this proof we have now shown that assuming $n \geq k$, not only is there a representation that sends $\mathsf{RB}_k(n+1)$ to a subalgebra of $\mathrm{End}(\mathsf{V}^{\otimes k})$, but that representation creates a one-to-one correspondence between $\mathsf{RB}_k(n+1)$ and that subalgebra. This proves that $\mathsf{RB}_k(n+1)$ must be isomorphic to a subalgebra of $\mathrm{End}(\mathsf{V}^{\otimes k})$. In the next section we show that this subalgebra is specifically a subalgebra of $\mathrm{End}_{\mathsf{O}(n)}(\mathsf{V}^{\otimes k})$, which is contained in $\mathrm{End}(\mathsf{V}^{\otimes k})$.

Chapter 5

The Orthogonal Group $\mathsf{O}(n)$

5.1 Definition and Action on Tensor Space

The orthogonal group $\mathsf{O}(n)$ is the group of orthogonal $n \times n$ matrices with entries in $\mathbb{C}$. A matrix is orthogonal if its transpose is also its inverse. We denote this by $g^T g = gg^T = I_n$ for any $g \in \mathsf{O}(n)$. We can also define a matrix by its entries, so we let $g = [a_{ij}]$ and $g^T = [a_{ji}]$. When we multiply g by g^T, we know that the entry b_{ij} in gg^T is 1 if $i = j$ and 0 otherwise. We state this as

$$\sum_{\ell=1}^{n} a_{i\ell} a_{j\ell} = \delta_{ij}.$$

The elements of $\mathsf{O}(n)$ have a natural action on the elements of $\mathsf{V}^{\otimes k}$. First we define the action of $g = [a_{ij}]$ on the basis element v_0 of V as $g \cdot v_0 = v_0$ and on any other basis element v_i of V, g acts by

$$g \cdot v_i = \sum_{\ell=1}^{n} a_{\ell i} v_\ell.$$

This action extends diagonally to any basis element of $\mathsf{V}^{\otimes k}$ by

$$g \cdot (v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) = (g \cdot v_{i_1}) \otimes (g \cdot v_{i_2}) \otimes \cdots \otimes (g \cdot v_{i_k}).$$

5.2 Commuting with $\mathsf{RB}_k(n+1)$

Recall that

$$\mathrm{End}_{\mathsf{O}(n)}\left(\mathsf{V}^{\otimes k}\right) = \left\{\phi \in \mathrm{End}\left(\mathsf{V}^{\otimes k}\right) \mid \phi(g\cdot v) = g\cdot\phi(v), v \in \mathsf{V}^{\otimes k}, g \in \mathsf{O}(n)\right\}.$$

We show that for any element $g \in \mathsf{O}(n)$, the action of g on the tensor space $\mathsf{V}^{\otimes k}$ commutes with the action of $\mathsf{RB}_k(n+1)$ on $\mathsf{V}^{\otimes k}$. This shows that for $n \geq k$, $\mathsf{RB}_k(n+1)$ is a subalgebra of $\mathrm{End}_{\mathsf{O}(n)}\left(\mathsf{V}^{\otimes k}\right)$, the centralizer algebra of $\mathsf{O}(n)$ acting on $\mathsf{V}^{\otimes k}$. We show this by considering the actions of the generators of $\mathsf{RB}_k(n+1)$. It suffices to consider only the generators of $\mathsf{RB}_2(n+1)$ acting on $\mathsf{V}^{\otimes 2}$ because the actions of the generators of $\mathsf{RB}_k(n+1)$ for any k act on a single tensor position or a pair of adjacent tensor positions and act as the identity on all other tensor positions.

Theorem 10. *As operators on the tensor space $\mathsf{V}^{\otimes k}$, the of the elements of $\mathsf{RB}_k(n+1)$ commute with the elements of $\mathsf{O}(n)$.*

Proof. First consider the generator s of $\mathsf{RB}_2(n+1)$. Let $g \in \mathsf{O}(n)$. For any simple tensor $v_i \otimes v_j$ of $V^{\otimes 2}$ with $i, j \neq 0$, we have $s(v_i \otimes v_j) = v_j \otimes v_i$.

Now,

$$\begin{aligned}
g \cdot s(v_i \otimes v_j) &= g(v_j \otimes v_i) \\
&= (g \cdot v_j) \otimes (g \cdot v_i) \\
&= \sum_{\ell=1}^{n} \sum_{m=1}^{n} a_{\ell j} a_{mi} (v_\ell \otimes v_m) \\
&= \sum_{\ell=1}^{n} \sum_{m=1}^{n} a_{\ell j} a_{mi} s(v_m \otimes v_\ell) \\
&= s\left(\sum_{\ell=1}^{n} \sum_{m=1}^{n} a_{\ell j} a_{mi} (v_m \otimes v_\ell)\right) \\
&= s \cdot g(v_i \otimes v_j).
\end{aligned}$$

Since g acts differently on the basis vector v_0, when checking that the actions of s and t commute with the action of g we must consider separately the cases of the generators and g acting on $v_0 \otimes v_0$ and $v_0 \otimes v_i$ where $i \neq 0$.

Both s and g fix the simple tensor $v_0 \otimes v_0$, so clearly

$$g \cdot s(v_0 \otimes v_0) = s \cdot g(v_0 \otimes v_0).$$

Now consider the simple tensor $v_0 \otimes v_i$ where $i \neq 0$. The actions give that

$$\begin{aligned}
g \cdot s(v_0 \otimes v_i) &= g \cdot (v_i \otimes v_0) \\
&= \left(\sum_{\ell=1}^{n} a_{\ell i} v_\ell\right) \otimes v_0 \\
&= \sum_{\ell=1}^{n} a_{\ell i}(v_\ell \otimes v_0) \\
&= \sum_{\ell=1}^{n} a_{\ell i}(s(v_0 \otimes v_\ell)) \\
&= s\left(\sum_{\ell=1}^{n} a_{\ell i}(v_0 \otimes v_\ell)\right) \\
&= s\left(v_0 \otimes \left(\sum_{\ell=1}^{n} a_{\ell i} v_\ell\right)\right) \\
&= s \cdot g(v_0 \otimes v_i).
\end{aligned}$$

This shows that the action of s commutes with the action of g on $\mathsf{V}^{\otimes k}$.

Given a simple tensor $v_i \otimes v_j$ in $V^{\otimes 2}$, the generator t acts on this simple tensor by $t(v_i \otimes v_j) = \delta_{ij} \sum_{\ell=0}^{n} v_\ell \otimes v_\ell$.

$$\begin{aligned}
g \cdot t(v_i \otimes v_j) &= g\left(\delta_{ij} \sum_{\ell=0}^{n} v_\ell \otimes v_\ell\right) \\
&= \delta_{ij}\left(\sum_{\ell=0}^{n} g(v_\ell) \otimes g(v_\ell)\right) \\
&= \delta_{ij}\left(\sum_{\ell=1}^{n}\left(\sum_{m=1}^{n} a_{m\ell} v_m\right) \otimes \left(\sum_{m=1}^{n} a_{m\ell} v_m\right) + \delta_{ij}(v_0 \otimes v_0)\right) \\
&= \delta_{ij}\left(\sum_{\ell=1}^{n}\sum_{m=1}^{n}\sum_{k=1}^{n} a_{m\ell} a_{k\ell}(v_m \otimes v_k)\right) + \delta_{ij}(v_0 \otimes v_0) \\
&= \delta_{ij}\left(\sum_{m=1}^{n}\sum_{k=1}^{n}(v_m \otimes v_k)\left(\sum_{\ell=1}^{n} a_{m\ell} a_{k\ell}\right)\right) + \delta_{ij}(v_0 \otimes v_0)
\end{aligned}$$

However, $\sum_{\ell=1}^{n} a_{m\ell}a_{k\ell} = \delta_{mk}$ due to the orthogonality of g. Therefore, the sum reduces to all instances where $m = k$ which leaves the sum

$$\begin{aligned}
\delta_{ij}\sum_{\ell=1}^{n} v_\ell \otimes v_\ell + \delta_{ij}(v_0 \otimes v_0) &= \delta_{ij}\sum_{\ell=0}^{n} v_\ell \otimes v_\ell \\
&= t(v_i \otimes v_j)
\end{aligned}$$

It remains to show that $t \cdot g(v_i \otimes v_j) = t(v_i \otimes v_j)$.

$$\begin{aligned}
t \cdot g(v_i \otimes v_j) &= t\left(\sum_{m=1}^{n} a_{mi}v_m \otimes \sum_{m=1}^{n} a_{mj}v_m\right) \\
&= t\left(\sum_{m=1}^{n}\sum_{k=1}^{n} a_{mi}a_{kj}(v_m \otimes v_k)\right) \\
&= \sum_{m=1}^{n}\sum_{k=1}^{n} a_{mi}a_{kj}(t(v_m \otimes v_k)) \\
&= \sum_{m=1}^{n}\sum_{k=1}^{n} a_{mi}a_{kj}\left(\delta_{m,k}\sum_{\ell=0}^{n} v_\ell \otimes v_\ell\right)
\end{aligned}$$

As before, this sum reduces to all cases where $m = k$, which means it can be rewritten as

$$\begin{aligned}
\sum_{m=1}^{n} a_{mi}a_{mj}\left(\sum_{\ell=0}^{n} v_\ell \otimes v_\ell\right) &= \delta_{i,j}\sum_{\ell=0}^{n} v_\ell \otimes v_\ell \\
&= t(v_i \otimes v_j)
\end{aligned}$$

The special cases of g and t acting on the simple tensors $v_0 \otimes v_0$ and $v_0 \otimes v_i$, $i \neq 0$, are quite simple to prove. First, consider

$$g \cdot t(v_0 \otimes v_0) = g\left(\sum_{\ell=1}^{n} v_\ell \otimes v_\ell\right).$$

Since $\delta_{0,0} = 1$, we can simply refer to the general case and conclude that $g \cdot t(v_0 \otimes v_0) = t(v_0 \otimes v_0)$. The action of g fixes $v_0 \otimes v_0$, so we have $t \cdot g(v_0 \otimes v_0) = t(v_0 \otimes v_0)$ as well.

Now we need to deal with g and t acting on $v_0 \otimes v_i$ where $i \neq 0$. Since $i \neq 0$, $g \cdot t(v_0 \otimes v_i) = g \cdot 0 = 0$. Reversing the order of the actions we get,

$$\begin{aligned}
t \cdot g(v_0 \otimes v_i) &= t \cdot \left(v_0 \otimes \left(\sum_{\ell=1}^{n} a_{\ell i} v_\ell \right) \right) \\
&= t \left(\sum_{\ell=1}^{n} a_{\ell i} (v_0 \otimes v_\ell) \right) \\
&= \sum_{\ell=1}^{n} a_{\ell i} (t(v_0 \otimes v_\ell))
\end{aligned}$$

However, $1 \leq \ell \leq n$ so when t acts on $v_0 \otimes v_\ell$, $\delta_{0,\ell} = 0$ for all ℓ. Therefore, $t \cdot g(v_0 \otimes v_i) = 0$ as well.

When considering the action of the generator p_i, we can restrict the scope even further to looking at its interaction with g and V. This is because p_i only depends on the single vector v_{j_i} and acts as the identity in all other tensor positions. For a basis vector v_i of V, $p(v_i) = \delta_{i,0}(v_i)$. If $i \neq 0$,

$$\begin{aligned}
p \cdot g(v_i) &= p \left(\sum_{\ell=1}^{n} a_{\ell i} v_\ell \right) \\
&= 0
\end{aligned}$$

This follows from the fact that ℓ ranges 1 to n. If the generator p acts first,

$$g \cdot p(v_i) = g(0) = 0$$

If $i = 0$ then

$$\begin{aligned}
p \cdot g(v_0) &= p(v_0) \\
&= v_0 \\
&= g \cdot p(v_0)
\end{aligned}$$

This shows that in either case, the actions of p and g commute. $\square$

Chapter 6

Combinatorics and the Bratteli Diagram

In this chapter we complete the proof that $\mathsf{RB}_k(n+1)$ is the centralizer algebra of the orthogonal group $\mathsf{O}(n)$ acting on $\mathsf{V}^{\otimes k}$ by presenting an algorithm which creates a one-to-one correspondence between the basis diagrams of $\mathsf{RB}_k(n+1)$ and paths on the Brattelli diagram $\mathcal{B}$ of $\mathsf{O}(n)$ acting on $\mathsf{V}^{\otimes k}$.

In Sections 4.3 and 4.4 we showed that the action of $\mathsf{RB}_k(n+1)$ on $\mathsf{V}^{\otimes k}$ results in a representation that is faithful for $n \geq k$. Assuming $n \geq k$, this proves that $\mathsf{RB}_k(n+1)$ is in one-to-one correspondence with a subalgebra of $\mathrm{End}(\mathsf{V}^{\otimes k})$. In Section 5.2 we showed that the action of $\mathsf{RB}_k(n+1)$ on $\mathsf{V}^{\otimes k}$ commutes with the action $\mathsf{O}(n)$ on $\mathsf{V}^{\otimes k}$, which proves that $\mathsf{RB}_k(n+1)$ is specifically in one-to-one correspondence with a subalgebra of $\mathrm{End}_{\mathsf{O}(n)}(\mathsf{V}^{\otimes k})$. From Theorem 6 in Section 3 we know that,

$$\dim\left(\mathrm{End}_{\mathsf{O}(n)}\left(\mathsf{V}^{\otimes k}\right)\right) = \sum_{\lambda\in\Lambda_k} m_{\lambda,k}^2$$

where $m_{k,\lambda}$ is the number of paths on $\mathcal{B}$ to the irreducible module labeled by λ on level k, and $\Lambda_k = \{\lambda \mid \lambda \vdash r \text{ for } 0 \leq r \leq k\}$. Therefore, if we prove that

$$\dim(\mathsf{RB}_k(n+1)) = \sum_{\lambda\in\Lambda_k} m_{\lambda,k}^2,$$

we complete the proof that for $n \geq k$,

$$\mathsf{RB}_k(n+1) \cong \mathrm{End}_{\mathsf{O}(n)}(\mathsf{V}^{\otimes k}).$$

Our approach is to create a bijection between the basis diagrams of $\mathsf{RB}_k(n+1)$ and pairs of paths to λ on level k of $\mathcal{B}$.

6.1 A Bijection Between $\mathsf{RB}_k(n+1)$ Diagrams and Paths on $\mathcal{B}$

For $\lambda \in \Lambda_k = \{\lambda \mid \lambda \vdash r, 0 \leq r \leq k\}$, a path on $\mathcal{B}$ of length k to λ is given by a sequence of integer partitions

$$P_\lambda = (\emptyset, \mu_1, \mu_2, \ldots, \mu_{k-1}, \lambda),$$

where for $i = 1, 2, \ldots, k$,

$$\mu_{i+1} = \begin{cases} \mu_i \\ \mu_i + \Box \\ \mu_i - \Box \end{cases}.$$

For example, a path on $\mathcal{B}$ of length 10 to $\lambda = (1,1)$ is

$$P_{(1,1)} = \left(\emptyset, \ydiagram{1}, \ydiagram{1,1}, \ydiagram{1,1}, \ydiagram{2,1}, \ydiagram{2}, \ydiagram{3}, \ydiagram{3,1}, \ydiagram{3,1}, \ydiagram{2,1}, \ydiagram{1,1}\right).$$

We call a path of length k to λ in $\mathcal{B}$ a *vacillating standard tableaux of length* k. We denote the set of all paths on $\mathcal{B}$ to λ on level k as $\mathcal{T}_k^\lambda$ and let $\mathcal{RB}_k$ be the set of basis diagrams of $\mathsf{RB}_k(n+1)$. We wish to define a bijection

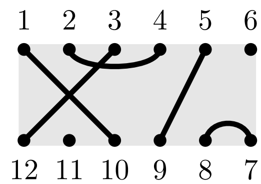
$$\mathcal{RB}_k \longleftrightarrow \bigsqcup_{r=0}^{k} \bigsqcup_{\lambda \vdash k} (\mathcal{T}_k^\lambda \times \mathcal{T}_k^\lambda)$$

That is, a bijection that takes a diagram $d \in \mathcal{RB}_k$ and produce a pair of paths (P_λ, Q_λ) to $\lambda \in \Lambda_k$ on $\mathcal{B}$. The algorithm we use to produce this bijection is based on the work in [HL].

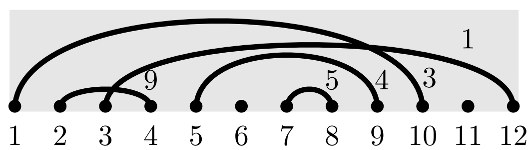
We begin by assigning a unique sequence of numbers to a diagram. Given a diagram $d \in \mathcal{RB}_k$, we label the top vertices with $1, ..., k$ in order starting at the left-most vertex, and we label the bottom vertices with $k+1, ..., 2k$ starting at the right most vertex. We can now draw the diagram on a single row of vertices labeled $1, ..., 2k$. We label each edge in the diagram with $2k+1-\ell$ where ℓ is the label of the right vertex of the edge. For $1 \leq i \leq 2k$, the *insertion sequence* of d is defined as

$$E_i = \begin{cases} a_L & \text{if vertex } i \text{ is a left endpoint of edge } a \\ a_R & \text{if vertex } i \text{ is a right endpoint of edge } a \\ \emptyset & \text{if } i \text{ is an isolated edge} \end{cases}$$

Example 6.1.1. The diagram

$d =$ 

is redrawn as

$d' =$ 

with edges labeled right to left. From these labels we get the insertion sequence

$$(E_i) = (3_L, 9_L, 1_L, 9_R, 4_L, \emptyset, 5_R, 4_R, 3_R, \emptyset, 1_R)$$

It is clear from the definition of the insertion sequence that a distinct diagram corresponds to a unique insertion sequence, and given that particular insertion sequence we can derive the original diagram. It follows that,

Proposition 6.1. *A diagram $d \in \mathcal{RB}_k$ is uniquely determined by its insertion sequence.*

We now use this property to define an algorithm which takes an insertion sequence of length k and produces a pair of vacillating standard tableaux, each of length k.

6.1.1 Insertion Sequences Become Tableaux

Given an insertion sequence (E_i), we create standard tableaux by sequentially inserting and deleting boxes according to (E_i). Insertion of boxes is done via Robinson-Schensted-Knuth (RSK) insertion, and deletion is done via jeu de taquin. Both of these processes are detailed in [HL]. First we construct a sequence of standard tableaux

$$T = (T^{(0)}, T^{(1)}, \ldots, T^{(2k)}).$$

Let $T^{(0)} = \emptyset$ be the starting point of this sequence. Then recursively define

$$T^{(i)} = \begin{cases} E_i \xrightarrow{RSK} T^{(i-1)} & \text{if } E_i = a_L \\ E_i \xleftarrow{jdt} T^{(i-1)} & \text{if } E_i = a_R \end{cases}$$

In other words, if vertex i is a left vertex of an edge we insert E_i into $T^{(i-1)}$ via RSK insertion and if i is a right vertex of an edge we remove the box containing E_i from $T^{(i-1)}$ via jeu de taquin removal. Note that if $E_i = \emptyset$ then $T^{(i)} = T^{(i-1)}$. This sequence of inserting and deleting boxes creates a unique sequence T of length $2k$. Now let $\lambda^{(i)}$ be the shape of $T^{(i)}$ and define

$$\begin{aligned} P_{\lambda^{(k)}} &= \left(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(k-1)}, \lambda^{(k)}\right) \\ Q_{\lambda_{(k)}} &= \left(\lambda^{(2k)}, \lambda^{(2k-1)}, \ldots, \lambda^{(k+1)}, \lambda^{(k)}\right). \end{aligned}$$

By the end of the sequence T, we have removed and deleted a box for every edge in the diagram corresponding to (E_i), so the final shape $\lambda^{(2k)}$ is $\emptyset$. Therefore, $(P_{\lambda^{(k)}}, Q_{\lambda^{(k)}})$ is a pair of vacillating standard tableaux of length k that are both paths to $\lambda^{(k)}$. Let ϕ_k be this function that takes diagrams in $\mathcal{RB}_k$ to pairs of paths on $\mathcal{B}$.

Theorem 11. *For $d \in \mathcal{RB}_k$, the function $\phi_k(d) = (P_\lambda, Q_\lambda)$ is a bijection between the elements of $\mathcal{RB}_k$ and pairs of vacillating standard tableaux in $\bigsqcup_{r=0}^{k} \bigsqcup_{\lambda \vdash k} (\mathcal{T}_k^\lambda \times \mathcal{T}_k^\lambda)$.*

Proof. We have shown that an insertion sequence corresponds to a unique pair of vacillating standard tableaux and completely defines a diagram in $\mathcal{RB}_k$. In order to show that this function is in fact a bijection, we now construct its inverse. Given a pair of vacillating tableaux (P_λ, Q_λ), we first create a sequence of partitions $\Lambda = \left(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(2k)}\right)$ by listing the partitions of Q_λ and then listing P_λ after it in reverse order, without repeating λ. We now simultaneously create the sequence of standard tableaux T and the insertion sequence (E_i) which shows that the function is invertible.

Initialize $T^{(2k)} = \emptyset$. If $\lambda^{(i-1)}/\lambda^{(i)}$ is a box b, then let $T^{(i-1)}$ be the tableau of shape $\lambda^{(i-1)}$ with $2k-i$ in the box b and the entries of $T^{(i)}$ in the remaining boxes, and let $E_i = (2k-i)_R$. Since $T^{(2k)} = \emptyset$, this step is always the first that occurs. Each time this step occurs, $2k-i$ is the largest value being added to the tableau $T^{(i)}$, so $T^{(i-1)}$ is standard.

If $\lambda^{(i)}/\lambda^{(i-1)}$ is a box b, then let $T^{(i-1)}$ be the tableau acquired by removing

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
E_i	10_L	2_L	$\emptyset$	10_R	4_L	1_L	$\emptyset$	3_L	$\emptyset$	4_R	3_R	2_R	1_R	$\emptyset$

i	E_i		$T^{(i)}$	i	E_i		$T^{(i)}$
0			$\emptyset$	8	3_L	$\xrightarrow{RSK}$	$\begin{ytableau} 1 & 3 \\ 2 \\ 4 \end{ytableau}$
1	10_L	$\xrightarrow{RSK}$	$\begin{ytableau} 10 \end{ytableau}$	9	$\emptyset$		$\begin{ytableau} 1 & 3 \\ 2 \\ 4 \end{ytableau}$
2	2_L	$\xrightarrow{RSK}$	$\begin{ytableau} 2 \\ 10 \end{ytableau}$	10	4_R	$\xleftarrow{jdt}$	$\begin{ytableau} 1 & 3 \\ 2 \end{ytableau}$
3	$\emptyset$		$\begin{ytableau} 2 \\ 10 \end{ytableau}$	11	3_R	$\xleftarrow{jdt}$	$\begin{ytableau} 1 \\ 2 \end{ytableau}$
4	10_R	$\xleftarrow{jdt}$	$\begin{ytableau} 2 \end{ytableau}$	12	2_R	$\xleftarrow{jdt}$	$\begin{ytableau} 1 \end{ytableau}$
5	4_L	$\xrightarrow{RSK}$	$\begin{ytableau} 2 \\ 4 \end{ytableau}$	13	1_R	$\xleftarrow{jdt}$	$\emptyset$
6	1_L	$\xrightarrow{RSK}$	$\begin{ytableau} 1 \\ 2 \\ 4 \end{ytableau}$	14	$\emptyset$		$\emptyset$
7	$\emptyset$		$\begin{ytableau} 1 \\ 2 \\ 4 \end{ytableau}$				

$$P_{(1,1,1)} = \left(\emptyset,\ \ydiagram{1},\ \ydiagram{1,1},\ \ydiagram{1,1},\ \ydiagram{1},\ \ydiagram{1,1},\ \ydiagram{1,1,1},\ \ydiagram{1,1,1}\right)$$

$$Q_{(1,1,1)} = \left(\emptyset,\ \emptyset,\ \ydiagram{1},\ \ydiagram{1,1},\ \ydiagram{2,1},\ \ydiagram{2,1,1},\ \ydiagram{2,1,1},\ \ydiagram{1,1,1}\right)$$

Figure 6.1: An insertion sequence becomes a sequence of tableaux.

the entry of b from $T^{(i)}$ through the inverse process of RSK-insertion, which guarantees that $T^{(i-1)}$ is standard. Let $E_i = b_L$. Continue in this manner working down from $i = 2k$ to $i = 1$. Clearly the sequence (E_i) created is unique, and we have shown that an insertion sequence completely defines a diagram in $\mathcal{RB}_k$. Therefore, the function is invertible and bijective.

$\square$

Example 6.1.2. From the pair of paths

$$P = \left(\emptyset, \ydiagram{1}, \ydiagram{2}, \ydiagram{2,1}, \ydiagram{2,2}, \ydiagram{2,1}, \ydiagram{2,1,1}, \ydiagram{2,1,1}\right), \quad Q = \left(\emptyset, \ydiagram{1}, \ydiagram{1}, \ydiagram{2}, \ydiagram{2,1}, \ydiagram{2,2}, \ydiagram{2,1}, \ydiagram{2,1,1}\right)$$

we get the sequence of partitions

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\lambda^{(i)}$	$\emptyset$	$\ydiagram{1}$	$\ydiagram{1}$	$\ydiagram{2}$	$\ydiagram{2,1}$	$\ydiagram{2,2}$	$\ydiagram{2,1}$	$\ydiagram{2,1,1}$	$\ydiagram{2,1,1}$	$\ydiagram{2,1}$	$\ydiagram{2,2}$	$\ydiagram{2,1}$	$\ydiagram{2}$	$\ydiagram{1}$	$\emptyset$

Using the reverse algorithm we construct T and (E_j).

i	$\lambda^{(i)}$		$T^{(i)}$	E_{2k-i}	i	$\lambda^{(i)}$		$T^{(i)}$	E_{2k-i}
0	$\emptyset$		$\emptyset$	1_R	7	$\begin{matrix}\square&\square\\\square\\\square\end{matrix}$	$\rightarrow$	$\begin{matrix}1&5\\4\\7\end{matrix}$	$\emptyset$
1	$\square$	$\rightarrow$	$\boxed{1}$	$\emptyset$	8	$\begin{matrix}\square&\square\\\square\\\square\end{matrix}$		$\begin{matrix}1&5\\4\\7\end{matrix}$	1_L
2	$\square$		$\boxed{1}$	3_R	9	$\begin{matrix}\square&\square\\\square\end{matrix}$	$\leftarrow$	$\begin{matrix}4&5\\7\end{matrix}$	10_R
3	$\begin{matrix}\square&\square\end{matrix}$	$\rightarrow$	$\begin{matrix}1&3\end{matrix}$	4_R	10	$\begin{matrix}\square&\square\\\square&\square\end{matrix}$	$\rightarrow$	$\begin{matrix}4&5\\7&10\end{matrix}$	5_L
4	$\begin{matrix}\square&\square\\\square\end{matrix}$	$\rightarrow$	$\begin{matrix}1&3\\4\end{matrix}$	5_R	11	$\begin{matrix}\square&\square\\\square\end{matrix}$	$\leftarrow$	$\begin{matrix}4&10\\7\end{matrix}$	4_L
5	$\begin{matrix}\square&\square\\\square&\square\end{matrix}$	$\leftarrow$	$\begin{matrix}1&3\\4&5\end{matrix}$	3_L	12	$\begin{matrix}\square&\square\end{matrix}$	$\leftarrow$	$\begin{matrix}7&10\end{matrix}$	10_L
6	$\begin{matrix}\square&\square\\\square\end{matrix}$	$\leftarrow$	$\begin{matrix}1&5\\4\end{matrix}$	7_R	13	$\square$	$\leftarrow$	$\boxed{7}$	7_L
					14	$\emptyset$	$\leftarrow$	$\emptyset$	

Finally, we have the insertion sequence (E_j).

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14
E_j	7_L	10_L	4_L	5_L	10_R	1_L	$\emptyset$	7_R	3_L	5_R	4_R	3_R	$\emptyset$	1_R

6.2 Conclusions

As a direct consequence of 11 we get the following corollary.

Corollary 1. $|\mathcal{RB}_k| = \sum_{\lambda\in\Lambda_k} m_{k,\lambda}^2$.

With Corollary 1, we have finally completed the proof of Theorem 7:

$$\mathsf{RB}_k(n+1) \cong \mathrm{End}_{\mathsf{O}(n)}\left(\mathsf{V}^{\otimes k}\right).$$

Now from the Double Centralizer Theorem we know that given the decomposition

$$\mathsf{V}^{\otimes k} \cong \bigoplus_{\lambda \in \Lambda_k} m_{\lambda,k} \mathsf{V}^{\lambda}$$

where $\Lambda_k = \{\lambda \mid \lambda \vdash r, 0 \leq r \leq k\}$, then

i. $\mathsf{RB}_k(n+1)$ is semisimple for $n \geq k$.

ii. The irreducible representations of $\mathsf{RB}_k(n+1)$ are labeled by $\lambda \in \Lambda_k$, and we denote these irreducible representations as M_k^{λ}.

iii. $\dim(M_k^{\lambda}) = m_{\lambda,k}$

iv. As an RB_k-module,

$$\mathsf{V}^{\otimes k} \cong \bigoplus_{\lambda \in \Lambda_k} d_{\lambda,k} M_{\lambda,k}$$

with $d_{\lambda,k} = \dim(\mathsf{V}^{\lambda})$ on level k of the Bratteli diagram.

Chapter 7

Future Work

7.1 Seminormal Representations

The next step in this work is to construct the irreducible representations of $\mathsf{RB}_k(n+1)$. Our approach is to create an analogue to Young's seminormal representations of the symmetric group $\mathbb{C}\mathsf{S}_k$.

Young's seminormal representations are constructed by using paths on Young's lattice as basis elements and defining an action of the generators of S_k on those paths, as shown in [Yo]. Recall Young's lattice in section 3.2.1. From Theorem 5 we know that on level k each vertex labeled by a partition λ represents an irreducible representation of $\mathsf{GL}(n)$ which appears in the decomposition of $\mathbb{C}\mathsf{S}_k$, and the dimension of that irreducible representation is the number of paths from the root of Young's lattice to λ on level k. It is natural then to construct the irreducible representation V_k^λ by choosing a basis where each path to λ on level k is a basis vector. Young then defined an action of the generators $s_i, 1 \leq i \leq k-1$ of S_k on this basis.

The approach to constructing the irreducible representations of $\mathsf{RB}_k(n+1)$ is generally the same: we use the paths to λ on level k on the Bratteli diagram $\mathcal{B}$ as a basis and define actions of the generators of $\mathsf{RB}_k(n+1)$ on these paths. The actions we define draw from work previously done on constructing the irreducible representations of the rook monoid in [Ha] and the Brauer algebra in [LR] and are supported by direct calculations of the irreducible representations of $\mathsf{RB}_k(x)$ for $0 \leq k \leq 3$.

7.2 Conjectures

We construct the irreducible representations of $\mathsf{RB}_k(x)$ with an arbitrary parameter x. Let M_k^λ be the irreducible representation of $\mathsf{RB}_k(x)$ labeled by λ on level k of $\mathcal{B}$, with $\dim(M_k^\lambda) = m_{k,\lambda}$. As a basis for M_k^λ, we use the $m_{k,\lambda}$ paths on $\mathcal{B}$ to λ on level k. Denote a path ρ as

$$\rho = \left(\emptyset, \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)} = \lambda\right)$$

where

- $\lambda^{(i)} \in \Lambda_i = \{\mu \mid \mu \vdash r, 0 \leq r \leq i\}$ and
- $\lambda^{(i)} = \lambda^{(i-1)} \pm \square$ or $\lambda^{(i)} = \lambda^{(i-1)}$.

For a path ρ, let ρ_i denote the i^{th} partition $\lambda^{(i)}$. Let v_ρ denote the basis vector labeled by the path ρ to λ on level k and let $\mathcal{P}_k^\lambda$ denote the set of these basis vectors. We now state our conjectures for the actions of the generators s_i, t_i, and p_i that construct the irreducible representation M_k^λ. First, a useful definition:

Definition 7.2.1. We say that two paths $\rho_1 = (\lambda^{(i)})$ and $\rho_2 = (\mu^{(i)})$ are *i-compatible* if

$$\begin{aligned}
\rho_1 &= (\emptyset, \lambda^{(1)}, \ldots, \lambda^{(i-1)}, \lambda^{(i)}, \lambda^{(i+1)}, \ldots, \lambda^{(k)}) \\
\rho_2 &= (\emptyset, \lambda^{(1)}, \ldots, \lambda^{(i-1)}, \mu^{(i)}, \lambda^{(i+1)}, \ldots, \lambda^{(k)})
\end{aligned}$$

In other words, $\mu^{(j)} = \lambda^{(j)}$ for all j except possibly at $j = i$.

An example of a pair of 4-compatible paths is

$$\begin{aligned}
\rho_1 &= \left(\emptyset, \ydiagram{1}, \ydiagram{2}, \ydiagram{2,1}, \ydiagram{3,1}, \ydiagram{3}, \ydiagram{4}\right) \\
\rho_2 &= \left(\emptyset, \ydiagram{1}, \ydiagram{2}, \ydiagram{2,1}, \ydiagram{2}, \ydiagram{3}, \ydiagram{4}\right)
\end{aligned}$$

In general, for a generator $g_i \in \{s_i, t_i, p_i\}$ we define

$$g_i \cdot v_\rho = \sum_{\gamma \in \mathcal{Q}_\rho} (g_i)_{\gamma\rho} v_\gamma$$

where $\mathcal{Q}_\gamma$ is the set of all paths that are i-compatible with ρ and $(g_i)_{\gamma\rho}$ is a coefficient in $\mathbb{C}$.

7.2.1 Action of p_i

Our conjecture is that the generator p_i acts on a basis vector v_ρ by

$$p_i \cdot v_\rho = \begin{cases} v_\rho & \text{if } \rho_i = \rho_{i-1} \\ 0 & \text{if } \rho_i \neq \rho_{i-1} \end{cases}$$

This formula comes from the construction of the irreducible representations of the rook monoid algebra in [Ha].

7.2.2 Action of t_i

Our conjecture is that the generator t_i acts on a basis vector v_ρ by

$$t_i \cdot v_\rho = \sum_{\gamma \in \mathcal{Q}_\rho} (t_i)_{\gamma\rho} v_\gamma$$

where $\mathcal{Q}_\rho$ is the set of all paths that are i-compatible with ρ and

$$(t_i)_{\gamma\rho} = \begin{cases} \frac{\sqrt{P_{\rho_i}(x) P_{\gamma_i}(x)}}{P_{\rho_{i-1}}(x)} & \text{if } \rho_{i-1} = \rho_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

where $P_\lambda(x)$ is the El-Samra-King polynomials found in [El-K]. El-Samra and King defined these polynomials and proved that $P_\lambda(n) = \dim\left(\mathsf{V}^\lambda\right)$ where V^λ is the irreducible representation of $\mathsf{O}(n)$ labeled by λ. This action follows from the work done on constructing the irreducible representations of the Brauer algebra in [LR].

7.2.3 Action of s_i

The action of s_i on a basis vector v_ρ has proven to be the trickiest action to pin down. We generally define the action as

$$s_i \cdot v_\rho = \sum_{\gamma \in \mathcal{Q}_\rho} (s_i)_{\gamma\rho} v_\gamma$$

where $\mathcal{Q}_\rho$ is the set of all paths that are i-compatible with ρ.

Recall that for consecutive partitions ρ_{j-1} and ρ_j in the path ρ we can move from ρ_{j-1} to ρ_j by either adding a box to ρ_{j-1}, subtracting a box from ρ_{j-1}, or doing nothing to ρ_{j-1}. The coefficient $(s_i)_{\gamma\rho}$ depends on the movement from ρ_{i-1} to ρ_i to ρ_{i+1}. There are 3 choices for each movement and 2 movements, so in all we have had to consider at least 9 cases. Luckily, some of the cases seem to act in the same way. Our conjectures are as follows.

Case 1($+\Box, +\Box$ or $-\Box, -\Box$): The first case we consider is where $\rho_i = \rho_{i-1} \pm \Box$ and $\rho_{i+1} = \rho_i \pm \Box$. In this case, s_i acts on v_ρ as it does in Young's construction of the irreducible representations of the symmetric group S_k since this movement on the Bratteli diagram is locally the same as a movement on Young's lattice. The only paths that are i-compatible with ρ in this case are paths γ with $\gamma_i = \gamma_{i-1} \pm \Box = \rho_{i-1} \pm \Box$ and $\gamma_{i+1} = \rho_{i+1} = \gamma_i \pm \Box$. Now,

$$s_i \cdot v_\rho = \frac{1}{\delta} v_\rho + \frac{\sqrt{(\delta-1)(\delta+1)}}{\delta} v_\gamma$$

where δ is the axial distance between the boxes that were added or removed. That is if at step i we add or remove box a and at step $i+1$ we add or remove box b, let (r_a, c_a) be the row and column position and (r_b, c_b) be the row and column position of box b,

$$\delta = (c_b - r_b) - (c_a - r_a).$$

Note that if a and b are in the same row or column,

$$s_i \cdot v_\rho = \frac{1}{\delta} v_\rho.$$

Case 2 ($\pm\Box, \emptyset$ or $\emptyset, \pm\Box$): The next case is where $\rho_i = \rho_{i-1} + \Box$ and $\rho_{i+1} = \rho_i$. This is locally like a movement on the Bratteli diagram of the rook monoid algebra and so we take the action of s_i defined in [Ha]. The only other path γ that is i-compatible with ρ is where $\gamma_i = \gamma_{i-1} = \rho_{i-1}$ and $\gamma_{i+1} = \rho_{i+1} = \gamma_i + \Box$. Then,

$$s_i \cdot v_\rho = v_\gamma.$$

In the case where $\rho_i = \rho_{i-1} - \Box$ and $\rho_{i+1} = \rho_i$, the only path γ that is i-compatible with ρ is where $\gamma_i = \gamma_{i-1} = \rho_{i-1}$ and $\gamma_{i+1} = \rho_{i+1} = \gamma_i - \Box$. We believe this case acts in the same way with

$$s_i \cdot v_\rho = v_\gamma.$$

Case 3 $(\emptyset, \emptyset)$: The case where $\rho_{i-1} = \rho_i = \rho_{i+1}$ is also locally like a movement on the Bratteli diagram of the rook monoid algebra, and we again use the action defined in [Ha]. In this case

$$s_i \cdot v_\rho = v_\rho.$$

Case 4 $(+\Box, -\Box$ or $-\Box, +\Box)$: The final case we considered is the trickiest. This is the case where $\rho_i = \rho_{i-1} \pm \Box$ and $\rho_{i+1} = \rho_i \mp \Box$. This movement is locally like a movement on the Bratteli diagram of the Brauer algebra, and we conjecture that in this case s_i acts as defined in [LR] Theorem 6.22. However, we have had less success in constructing irreducible representations with basis elements that contain a movement such as this.

The next step in this project is to more fully form the conjectures for the various cases of the generator s_i acting on basis vectors and then to check that these actions preserve all of the relations described in Section 2.2.2. While there are many relations to check, we hope that some will come for free from the fact that in some cases the actions specialize to to the actions of elements in $\mathbb{C}\mathsf{S}_k$, $\mathbb{C}\mathsf{R}_k$, and $\mathsf{B}_k(x)$, and it has already been proven that the actions of these algebras preserve the generating relations.

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