Spring 5-22-2021

Hochschild Cohomology Of Short Gorenstein Rings

Mkrtich Ohanyan
Syracuse University, mkrtichohanyan@gmail.com

Abstract

Let $\Bbbk$ be a field of characteristic 0. In this thesis, we show that the Hochschild cohomology of the family of short Gorenstein $\Bbbk$-algebras

$$\mathrm{sGor}(N) = \frac{\Bbbk[X_0, \ldots, X_N]}{(X_i X_j,\, X_i^2 - X_j^2 \mid i, j = 0, \ldots, N, i \neq j)}, \qquad N \geq 2,$$

exhibits exponential growth. The proof uses Gröbner-Shirshov basis theory and along the way we describe an explicit monomial basis for the Koszul dual of $\mathrm{sGor}(N)$ for $N \geq 2$.

HOCHSCHILD COHOMOLOGY OF SHORT GORENSTEIN RINGS

by

Mkrtich Ohanyan

B.S., University of Southern California, 2010
M.S., Syracuse University, 2018

Dissertation
Submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in Mathematics.

Syracuse University
May 2021

Acknowledgements

Thank you,

Claudia, for your imparted wisdom, your encouragement, and your patience.

Ben, for breathing new life into this project. I hope I am one of many more students to come.

Nathan, for helpful discussions about the combinatorics involved in my proof.

Mom and Pop, for every sacrifice you made to help me realize this dream. You are my first and best teachers.

Natalik, for always understanding me, accepting me as I am, and always being by my side. Life is easier when you have the best sibling.

Abuela, for all of your prayers, your cards, and your love. Your grandson made it.

Claudia, Tia Ana, Mariam, Gegan, and Chamba for reaching out when I needed it most.

Carol, for helping me push through it. I'm grateful to have had you as both a mentor and a close friend.

Kofi, Casey, Jin, Sean, and Abraham, for your friendship. I'm leaving Syracuse having made friends for life.

Alice, my love, for being my partner, my best friend, and my inspiration throughout this journey. I was lucky enough to have you to look up to on my way here.

Contents

Chapter 1

Introduction

Hochschild cohomology is one of the fundamental long-studied invariants developed to understand a k-algebra over a commutative ring k. However, there exist very few explicit computations of the Hochschild cohomology of a commutative algebra. The Hochschild-Kostant-Rosenberg Theorem [HKR62] computes explicitly the Hochschild cohomology of a polynomial ring. In [BR15], Buchweitz and Roberts formulate a description of the Hochschild cohomology of a complete intersection in the same spirit. Beyond this, however, the author is unaware of many more examples in commutative algebra. Such computations are generally very difficult because Hochschild cohomology is expected to grow exponentially for non-complete intersection rings. In this thesis we show that the Hochschild cohomology of a family of short Gorenstein rings exhibits this exponential growth (where short means the cube of the homogeneous maximal ideal is equal to zero). More precisely, let $\Bbbk$ be a field of characteristic 0; define the family of Gorenstein rings $\mathrm{sGor}(N)$ for $N \geq 2$ as

$$\mathrm{sGor}(N) = \frac{\Bbbk[X_0, \dots, X_N]}{(X_iX_j,\, X_i^2 - X_j^2 \mid i, j = 0, \dots, N, i \neq j)};$$

and for $n \geq 0$, let $\mathrm{HH}^n(\mathrm{sGor}(N))$ denote the nth Hochschild cohomology group of $\mathrm{sGor}(N)$ over $\Bbbk$. We note that for $N = 1$ these algebras are complete intersections.

The main result of this thesis is

Theorem (Theorem 6.2.9)**.** *For $N \geq 3$ and for all $n \geq 2$ even, there is an inequality*

$$\dim_{\Bbbk} \mathrm{HH}^n(\mathrm{sGor}(N)) \geq \frac{(N-1)^{n+1}}{n+1}.$$

We outline the proof this theorem below.

Let k be a commutative ring and let A be a k-algebra. Although the Hochschild cohomology of A, denoted $\mathrm{HH}^\bullet(A)$, is classically defined in terms of the bar complex of A, the bar complex usually proves to be intractable for computational purposes. However, if A is projective over k (which will be the case for us, working over the field $k = \Bbbk$), we have $\mathrm{HH}^n(A/k) \cong \mathrm{Ext}^n_{A^e}(A, A)$ as k-modules for all $n \geq 0$ (Proposition 2.1.5), where $A^e = A \otimes_k A^{\mathrm{op}}$ is the enveloping algebra of A. This allows us some flexibility in terms of the resolution of A over A^e which we use for the computation of $\mathrm{HH}^\bullet(A)$. In fact, if A is a commutative graded Koszul algebra, we show in Chapter 2 that this leads to the following critical result due to Buchweitz ([Buc03]) and Negron ([Neg17]) which facilitates computations. We define all terminology in Chapter 2.

Proposition-Definition (Proposition-Definition 2.4.6)**.** *Let $A = T(V)/\langle Q\rangle$ be a commutative Koszul algebra over $\Bbbk$; let $\{x_0, \ldots, x_m\}$ be a basis for V; let $y_i = x_i^*$, so that $\{y_0, \ldots, y_m\}$ is a basis of V^*; and let $A^! = T(V^*)/\langle Q^\perp\rangle$ be the quadratic dual of A. For $n \geq 0$, the sequence of free A-modules $A \otimes (A^!)^n$ with maps $\partial^n : A \otimes (A^!)^n \to A \otimes (A^!)^{n+1}$ defined by*

$$\partial^n(a \otimes e) = \sum_{j=0}^{m} ax_j \otimes (y_j e - (-1)^n e y_j),$$

form a complex $(A \otimes (A^!)^\bullet, \partial)$ *satisfying*

$$\mathrm{HH}^n(A) \cong H^n(A \otimes (A^!)^\bullet)$$

for all $n \geq 0$.

Fortunately, these short Gorenstein rings are commutative Koszul $\Bbbk$-algebras (Proposition 3.2.2) with Koszul duals

$$\mathrm{sGor}(N)^! = \frac{\Bbbk\langle Y_0, \ldots, Y_N\rangle}{\langle Y_0^2 + \cdots + Y_N^2\rangle}, \quad \text{where } Y_i = X_i^*,$$

for $N \geq 2$ (Proposition 3.2.3). Thus, the complex $\mathrm{sGor}(N) \otimes_\Bbbk (\mathrm{sGor}(N)^!)^\bullet$ serves as the starting point of our computations.

Fix some $N \geq 2$, let $A = \mathrm{sGor}(N)$, and let $\otimes = \otimes_\Bbbk$. In Chapter 3 we discuss in more detail the structure of the complex $A \otimes A^!$, showing that there is a decomposition

$$A = \Bbbk \oplus \Bbbk\{X_0, \ldots, X_N\} \oplus \Bbbk\{s\}$$

(Proposition 3.1.2), where s is of degree 2. We show that this induces a decomposition of $A \otimes A^!$ into complexes $\mathcal{C}_{(n)}$ for $n \geq 0$, called strands, with differentials $\delta^m_{(n)} : \mathcal{C}^m_{(n)} \to \mathcal{C}^{m+1}_{(n)}$, where

$$\mathcal{C}^m_{(n)} = \begin{cases} A_0 \otimes (A!)^{n-1}, & m = n-1; \\ A_1 \otimes (A!)^{n}, & m = n; \\ A_2 \otimes (A!)^{n+1}, & m = n+1; \\ 0, & \text{otherwise.} \end{cases}$$

This, in turn, induces the following decomposition of the Hochschild cohomology of A:

Proposition (Proposition 3.3.5)**.** *For $n \geq 1$,*

$$\mathrm{HH}^n(A) = H^n(\mathcal{C}_{(n-1)}) \oplus H^n(\mathcal{C}_{(n)}) \oplus H^n(\mathcal{C}_{(n+1)}).$$

For $n \geq 1$, we set $\mathrm{HH}^n_{(0)}(A) = H^n(\mathcal{C}_{(n+1)})$, $\mathrm{HH}^n_{(1)}(A) = H^n(\mathcal{C}_{(n)})$, and $\mathrm{HH}^n_{(2)}(A) = H^n(\mathcal{C}_{(n-1)})$. Our goal, then, is to investigate the $\Bbbk$-vector space structure of each $\mathrm{HH}^n_{(i)}(A)$ individually, which requires a description of the $\Bbbk$-vector space structure of $A^!$. To this end, in Chapter 4, we compute a noncommutative Gröbner basis, or Gröbner-Shirshov basis, of the ideal $\langle Y_0^2 + \cdots + Y_N^2 \rangle$:

Proposition (Proposition 4.3.2)**.** *The set*

$$\mathcal{S}_N = \{\ r = Y_0^2 + \cdots + Y_N^2, \quad Y_0 r - r Y_0\ \}$$

is a Gröbner-Shirshov basis of the ideal $\langle Y_0^2 + \cdots + Y_N^2 \rangle$ of $\Bbbk\langle Y_0, \ldots, Y_N \rangle$.

Given an associative $\Bbbk$-algebra R and an ideal $I \subseteq R$, the most important property of a Gröbner-Shirshov basis S of I for our purposes is that S contains the necessary data for constructing a $\Bbbk$-vector space basis of R/I (Corollary 4.2.5). In particular, we have

Proposition (Proposition 4.3.3)**.** *The $\Bbbk$-vector space $A^!$ has a $\Bbbk$-basis consisting of all monomials in $\Bbbk\langle Y_0, \ldots, Y_N\rangle$ which are not divisible by Y_0^2 nor $Y_0Y_1^2$.*

With the vector space structures of A and $A^!$ made explicit, we compute $\mathrm{HH}^n_{(0)}(A)$ by constructing a map

$$\gamma_{n+1} : \mathcal{C}^{n+1}_{(n+1)} = A_1 \otimes (A!)^{n+1} \longrightarrow A_0 \otimes (A!)^n = \mathcal{C}^n_{(n+1)}$$

which splits $\delta^n_{(n+1)} : \mathcal{C}^n_{(n+1)} \to \mathcal{C}^{n+1}_{(n+1)}$, implying

Corollary (Corollary 5.0.7)**.** *For $n \geq 1$, we have $\mathrm{HH}^n_{(0)}(A) = 0$.*

Finally, for $N \geq 3$ and n even, we then use γ_{n+1} to show the existence of a linearly independent subset of $\mathrm{HH}^n_{(1)}(A)$ of size

$$\sum_{d|(n+1)} \frac{\varphi(d)(N-1)^{(n+1)/d}}{n+1},$$

where φ is Euler's totient function, thus implying our main result.

Conventions

We assume knowledge of basic commutative algebra and homological algebra, which can be found in [Mat89] and [Wei94], respectively. For the entirety of this thesis,

- let $\mathbb{N} = \{0, 1, 2, \dots\}$;
- let $\Bbbk$ be a field of characteristic 0;
- $\otimes$ stands for $\otimes_{\Bbbk}$;
- Hom stands for $\mathrm{Hom}_{\Bbbk}$;
- "ideal" means two-sided ideal;
- and for a $\Bbbk$-vector space V, $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$.

In particular, if $\{v_0, \dots, v_m\}$ is a basis of V, then v_i^* is the dual basis element of v_i given by $v_i^*(v_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

The free associative $\Bbbk$-algebra on indeterminates $z_0, \dots, z_m$, denoted $\Bbbk\langle z_0, \dots, z_m\rangle$, is isomorphic to the tensor algebra $T(\Bbbk\{z_0, \dots, z_m\})$ as $\Bbbk$-algebras, so we will use these structures interchangeably.

The symbol ▲ signifies the end of a numbered statement.

Chapter 2

Hochschild Cohomology and Koszul Algebras

Hochschild cohomology was originally developed by Hochschild in his 1945 paper [Hoc45] as a cohomology theory of associative algebras. This theory was further expanded by Cartan and Eilenberg [CE56] and Gerstenhaber [Ger63] and since then has seen tremendous growth, playing an significant role in many different branches of mathematics, including representation theory, noncommutative geometry, and algebraic deformation theory (see [Wit19]).

As explained in the Chapter 1, our main goal in this thesis is to exhibit exponential growth of the Hochschild cohomology groups of the short Gorenstein rings $\mathrm{sGor}(N)$. To get anywhere with this goal, though, we would like to work with a complex which significantly simplifies computations of these cohomology groups. After establishing some general theory about Hochschild cohomology and Koszul algebras in Sections 2.1 – 2.3, in Section 2.4 we describe a computationally-friendly complex that computes the Hochschild cohomology of commutative Koszul $\Bbbk$-algebras (Proposition-Definition 2.4.6). We will show in Chapter 3 that the short Gorenstein rings $\mathrm{sGor}(N)$ are Koszul, allowing us to utilize this complex for the rest of our discussion.

2.1 Hochschild cohomology – an introduction

In this section we recall the definition of the Hochschild cohomology of a $\Bbbk$-algebra A with coefficients in an A-bimodule M via the *bar complex* of A (Proposition-Definition 2.1.3) and then give an equivalent description in terms of derived functors (Proposition 2.1.5).

Our discussion, including the construction of the complex which computes Hochschild cohomology, will require the notion of an enveloping algebra associated with A, defined as follows.

Definition 2.1.1. The *enveloping algebra* of a $\Bbbk$-algebra A is the $\Bbbk$-algebra $A^e = A \otimes A^{\mathrm{op}}$. ▲

Remark 2.1.2. An A-bimodule M is equivalently a left A^e-module with action defined by $(a \otimes a') \cdot x = axa'$ for all $a \otimes a' \in A^e$ and $x \in M$. Conversely, a left A^e-module M is also an A-bimodule via the actions $a \cdot x = (a \otimes 1)x$ and $x \cdot a' = (1 \otimes a')x$ for all $a, a' \in A$ and $x \in M$. For the rest of this chapter, we will use these structures interchangeably.

In particular, for a $\Bbbk$-vector space V, $A \otimes V \otimes A$ is a left A^e-module and is isomorphic to $A^e \otimes V$ via $a \otimes v \otimes a' \leftrightarrow (a \otimes a') \otimes v$. ▲

Proposition-Definition 2.1.3 (See [Wit19, Section 1.1])**.** *The* bar complex $B(A)$ *of* A *is a sequence of left* A^e*-modules* $B_n(A) = A^{\otimes(n+2)}$ *and maps* $b_{n+1} : A^{\otimes(n+3)} \to A^{\otimes(n+2)}$ *defined by*

$$b_{n+1}(a_0 \otimes \cdots \otimes a_{n+2}) = \sum_{i=0}^{n+1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+2}$$

for $n \in \mathbb{N}$. The augmented complex

$$\cdots \to A^{\otimes(n+2)} \xrightarrow{b_n} \cdots \to A \otimes A \otimes A \xrightarrow{b_1} A \otimes A \xrightarrow{\mu} A \to 0$$

with multiplication map $\mu : A \otimes A \to A$ forms an A^e-module resolution of A. ▲

For a left A^e-module M, we use the bar complex $B(A)$ to define a complex of $\Bbbk$-vector spaces $\mathrm{Hom}_{A^e}(B(A), M)$ with $\mathrm{Hom}^n_{A^e}(B(A), M) = \mathrm{Hom}_{A^e}(B_n(A), M)$ and differential

$$(b^\vee)_n := \mathrm{Hom}_{A^e}(b_{n+1}, M) : \mathrm{Hom}_{A^e}(B_n(A), M) \to \mathrm{Hom}_{A^e}(B_{n+1}(A), M)$$

given by $(b^\vee)_n(f) = -(-1)^n f b_{n+1}$ for all $f \in \mathrm{Hom}_{A^e}(B_n(A), M)$.

Definition 2.1.4. The nth *Hochschild cohomology* $\mathrm{HH}^n(A, M)$ of A with coefficients in an A^e-module M is defined as the nth cohomology of the complex $\mathrm{Hom}_{A^e}(B(A), M)$; that is,

$$\mathrm{HH}^n(A, M) = H^n(\mathrm{Hom}_{A^e}(B(A), M))$$

for all $n \in \mathbb{N}$.

Set $\mathrm{HH}^\bullet(A, M) = \bigoplus_{n\in\mathbb{N}} \mathrm{HH}^n(A, M)$. In the case $M = A$, write $\mathrm{HH}^n(A)$ for $\mathrm{HH}^n(A, A)$.

▲

Working directly with this definition can prove difficult because the bar complex is generally quite cumbersome. Fortunately, since A is projective over $\Bbbk$, Hochschild cohomology has the following characterization which significantly expands the scope of A^e-resolutions of A at our disposal, as we will see in Section 2.4.

Proposition 2.1.5 (See [Lod98, Subsection 1.5.8])**.** *Let M be a left A^e-module. Then*

$$\mathrm{HH}^n(A, M) \cong \mathrm{Ext}^n_{A^e}(A, M)$$

as $\Bbbk$-vector spaces for all $n \in \mathbb{N}$. ▲

2.2 Interlude – some linear algebra

In light of Proposition 2.1.5, we will soon see that when A is a commutative Koszul algebra, $\mathrm{HH}^\bullet(A)$ can be computed using a more suitable free A^e-resolution of A (Proposition 2.4.3). However, in order to move forward, we will need the following facts about vector spaces.

Definition 2.2.1. Let V be a $\Bbbk$-vector space and let U be a subspace of V. The *annihilator* of U is the subspace of V^* defined as

$$U^\circ := \{f \in V^* \mid f(v) = 0 \text{ for all } v \in U\}. \qquad ▲$$

Proposition 2.2.2 (see [War90, Theorem 28.10])**.** *Let V be a finite dimensional $\Bbbk$-vector* *space and let U be a subspace of V. Then* $\dim_{\Bbbk} U^\circ = \dim_{\Bbbk} V - \dim_{\Bbbk} U$. ▲

Proposition 2.2.3. *Let V and W be $\Bbbk$-vector spaces, let U be a subspace of V, let $\{v_0, \ldots, v_m\}$ be a basis for V extending the basis $\{v_{\ell+1}, \ldots, v_m\}$ of U, and let $\{w_0, \ldots, w_n\}$ be a basis for W. Then we have the following natural isomorphisms:*

(a) $V \cong V^{**}$, *and in particular* $U \cong U^{\circ\circ}$, *via* $v_i \leftrightarrow v_i^{**}$;

(b) $V^* \otimes W^* \cong (V \otimes W)^*$ *via* $v_i^* \otimes w_j^* \leftrightarrow (v_i \otimes w_j)^*$;

(c) $V \otimes W^* \cong \mathrm{Hom}(W, V)$ *via* $v_i \otimes w_j^* \leftrightarrow h_{ji}$, *where* $\{h_{ji}\}$ *is the basis of* $\mathrm{Hom}(W, V)$ *defined by* $h_{ji}(w) = w_j^*(w) v_i$;

(d) $U^\circ \cong (V/U)^*$ *via* $v_i^* \leftrightarrow \overline{v_i}^*$.

Each isomorphism above has the following basis-free description in the specified direction:

(a′) $V \to V^{**}$ *given by* $v \mapsto e_v$, *where the map* $e_v : V^* \to \Bbbk$ *is defined by* $e_v(f) = f(v)$;

(b′) $V^* \otimes W^* \to (V \otimes W)^*$ *given by* $f \otimes g \mapsto h$, *where the map* $h : V \otimes W \to \Bbbk$ *is defined by* $h(v \otimes w) = f(v) g(w)$;

(c′) $V \otimes W^* \to \mathrm{Hom}(W, V)$ *given by* $v \otimes f \mapsto g$, *where the map* $g : W \to V$ *is defined by* $g(w) = f(w) v$;

(d′) $U^\circ \to (V/U)^*$ *given by* $f \mapsto \overline{f}$, *where the map* $\overline{f} : V/U \to \Bbbk$ *is defined by* $\overline{f}(v) = \overline{f(v)}$.

▲

Corollary 2.2.4. *In the setting of Proposition 2.2.3, if $W = V$, then under the isomorphism $V \otimes V^* \to \mathrm{Hom}(V, V)$ the identity in $\mathrm{Hom}(V, V)$ corresponds to the element $\sum_{i=0}^m v_i \otimes v_i^*$. This element is independent of choice of basis for V since it is the preimage of the identity element for* any *basis of V.* ▲

Remark 2.2.5. Proposition 2.2.3(c′) holds even if V is not finitely generated, with inverse $\mathrm{Hom}(W, V) \to V \otimes W^*$ given by $\psi \mapsto \sum_{i=0}^{n} \psi(w_i) \otimes w_i^*$. We will use this version of the proposition in the proof of Proposition-Definition 2.4.6. ▲

Definition 2.2.6. Let V be a finite dimensional vector space, let $\Phi : V^* \otimes V^* \to (V \otimes V)^*$ be the isomorphism of Proposition 2.2.3(b′), and let $Q \subseteq V \otimes V$ be a subspace. Define the *perpendicular subspace* $Q^\perp \subseteq V^* \otimes V^*$ of Q as

$$Q^\perp = \{f \in V^* \otimes V^* \mid \Phi(f)(u) = 0 \text{ for all } u \in Q\}.$$ ▲

Note that for a subspace $Q \subseteq V \otimes V$, $\Phi(Q^\perp) = Q^\circ$, so $Q^\perp \cong Q^\circ$. In particular, we have

Proposition 2.2.7. *Let V be a finite dimensional vector space and Q a subspace of $V \otimes V$. Then $Q^{\perp\perp} \cong Q$ and $\dim_{\Bbbk} Q^\perp = \dim_{\Bbbk}(V \otimes V) - \dim_{\Bbbk} Q$.* ▲

2.3 Koszul algebras

With this linear algebra toolbox in hand we are now equipped to discuss Koszul algebras. We begin by introducing quadratic algebras, the quadratic dual, and the associated Koszul complex. When A is Koszul, this complex of left A-modules lifts to the free resolution of A over A^e that we seek.

Definition 2.3.1. A $\Bbbk$-algebra A is a *quadratic algebra* if $A = T(V)/I$, where V is a finite-dimensional $\Bbbk$-vector space, $T(V)$ is the tensor algebra on V over $\Bbbk$, and $I = \langle Q \rangle$ for a $\Bbbk$-vector space $Q \subseteq T^2(V) = V \otimes V$. The *quadratic dual* $A^!$ of a A is the $\Bbbk$-algebra $T(V^*)/I^!$ with $I^! = \langle Q^\perp \rangle$. ▲

Remark 2.3.2. If $A = T(V)/\langle Q \rangle$ is a quadratic algebra, then by Proposition 2.2.7,

$$A^{!!} = T(V^{**})/\langle Q^{\perp\perp} \rangle \cong T(V)/\langle Q \rangle = A.$$

▲

Example 2.3.3. Let V be a finite dimensional $\Bbbk$-vector space. Then the symmetric algebra on V, $S(V) = T(V)/\langle u \otimes v - v \otimes u \mid u, v \in V \rangle$, is a quadratic algebra with quadratic dual $S(V)^! = \bigwedge(V^*) = T(V^*)/\langle v^* \otimes v^* \mid v \in V \rangle$, the exterior algebra on V^*. ▲

Symmetric and exterior algebras are also examples of Koszul algebras, defined below. There are several equivalent definitions of Koszul algebras (see, for example, [PP05, Chapter 2, Section 1, Definition 1]); the following one best motivates the rest of our discussion.

Definition 2.3.4. A quadratic $\Bbbk$-algebra A is *Koszul* if the left A-module $\Bbbk$ admits a free linear resolution over A; that is, a resolution by free graded left A-modules F_i such that $F_i = A(-i)^{\beta_i}$, where $\beta_i = \beta_i^A(\Bbbk)$ is the ith Betti number of $\Bbbk$ over A for all $i \in \mathbb{N}$. ▲

An equivalent definition of Koszulness follows from the next construction of a complex associated with a quadratic algebra, due to Priddy [Pri70].

Proposition-Definition 2.3.5 (see [PP05, Chapter 2, Section 3]). *Let $A = T(V)/\langle Q\rangle$ be a quadratic $\Bbbk$-algebra, let $A^! = T(V^*)/\langle Q^\perp\rangle$ be its quadratic dual, and let $t_A \in A_1 \otimes A_1^!$ be the element described in Corollary 2.2.4 corresponding to the identity in* $\mathrm{Hom}(V,V)$. *The sequence of free left A-modules $K_n(A) = A \otimes (A_n^!)^*$ with maps $\partial_{n+1} : K_{n+1}(A) \to K_n(A)$ defined by $\partial_{n+1}(a\otimes f) = t_A\cdot(a\otimes f)$ for all $n \in \mathbb{N}$ form a complex $K_\bullet(A)$ called the* generalized Koszul complex *of A.* ▲

To see that this is indeed a complex, we will first make sense of the action t_A on $A\otimes(A_n^!)^*$ and then show that $t_A^2 = 0$, from which it will follow that ∂ squares to zero.

Proof. Let $\{v_0,\dots,v_m\}$ be a basis of V, so that $t_A = \sum v_i \otimes v_i^*$. The action of v_i^* on an element $f \in (A_n^!)^*$ is defined as the map $v_i^*\cdot f : A_{n-1}^! \to \Bbbk$ given by $(v_i^*\cdot f)(x) := f(v_i^* x)$, where the product $v_i^* x$ is the multiplication in $A^!$. Thus, $t_A\cdot(a\otimes f) = \sum v_i a \otimes (v_i^*\cdot f)$ for any $a\otimes f \in A \otimes (A_n^!)^*$.

By Proposition 2.2.3(d′), we have $Q \cong Q^{\perp\perp} \cong (A_2^!)^*$. So, by Propositions 2.2.3(a′) and 2.2.3(c′), it follows that $A_2 \otimes A_2^! \cong \mathrm{Hom}((A_2^!)^*, A_2) \cong \mathrm{Hom}(Q, A_2)$. Thus, we may identify the multiplication map $(A_1 \otimes A_1^!) \otimes (A_1 \otimes A_1^!) \to A_2 \otimes A_2^!$ with the map

$$\Phi : \mathrm{Hom}(V\otimes V, V\otimes V) \to \mathrm{Hom}(Q, A_2)$$

sending an element $\varphi : V\otimes V \to V\otimes V$ to the composition $Q \hookrightarrow V\otimes V \xrightarrow{\varphi} V\otimes V \xrightarrow{\varepsilon} A_2$, where $\varepsilon : A_1 \otimes A_1 \to A_2$ is multiplication. Under this identification the element $t_A \otimes t_A$ corresponds to the identity $\mathrm{id}_{V\otimes V}$ and the assignment $t_A\otimes t_A \mapsto t_A^2$ corresponds to $\mathrm{id}_{V\otimes V} \mapsto$

$\Phi(\mathrm{id}_{V\otimes V}) = 0$, implying $t_A^2 = 0$. $\square$

Corollary 2.3.6 (see Chapter 2, Corollary 3.2 of [PP05]). *A $\Bbbk$-algebra A is Koszul if and only if $K_\bullet(A) \xrightarrow{\alpha} \Bbbk \to 0$ is a free resolution of the left A-module $\Bbbk$, where $\alpha : A \otimes (A_0^!)^* \cong A \to \Bbbk$ is the natural projection.* ▲

Remark 2.3.7. The action of v_i^* on $(A_n^!)^*$ in the proof of Proposition-Definition 2.3.5 is an example of the action of a $\Bbbk$-algebra A on its dual A^*: for $a \in A$ and $f \in A^*$, $a \cdot f : A \to \Bbbk$ is defined by $(a \cdot f)(x) = f(xa)$ and $f \cdot a : A \to \Bbbk$ is defined by $(f \cdot a)(x) = f(ax)$.

Note that when A is commutative, these actions can be used interchangeably. ▲

2.4 Hochschild cohomology of Koszul algebras

We now turn our attention to the Hochschild cohomology of Koszul algebras. For a quadratic $\Bbbk$-algebra A, we describe a lift of $K_\bullet(A)$ to a complex of left A^e-modules $\widetilde{K}_\bullet(A)$ which forms a free A^e-resolution of A when A is Koszul — thus, by Proposition 2.1.5, providing us with an alternative route to computing the Hochschild cohomology of Koszul algebras. We end this section with the main result of the chapter, Proposition-Definition 2.4.6, which will allow us to compute $\mathrm{HH}^\bullet(\mathrm{sGor}(N))$.

Proposition-Definition 2.4.1 ([dB94, Section 3]). *Let $A = T(V)/\langle Q\rangle$ be a quadratic $\Bbbk$-algebra, let $E = A^!$, and let $\{v_0, \ldots, v_m\}$ be a basis of V. For $n \in \mathbb{N}$, let $\widetilde{K}_n(A) = A \otimes$*

$(E^n)^* \otimes A$ *and define maps* $d_{n+1}, d'_{n+1}, d''_{n+1} : A \otimes (E^{n+1})^* \otimes A \to A \otimes (E^n)^* \otimes A$ *by*

$$d'_{n+1}(x \otimes f \otimes y) = (x \otimes f \otimes y) \cdot \left(\sum_{i=0}^{m} v_i \otimes v_i^* \otimes 1 \right) = \sum_{i=0}^{m} x v_i \otimes f v_i^* \otimes y,$$
$$d''_{n+1}(x \otimes f \otimes y) = \left(\sum_{i=0}^{m} 1 \otimes v_i^* \otimes v_i \right) \cdot (x \otimes f \otimes y) = \sum_{i=0}^{m} x \otimes v_i^* f \otimes v_i y,$$

and

$$d_{n+1} = d'_{n+1} + (-1)^{n+1} d''_{n+1}.$$

Then $(d')^2 = 0$*,* $(d'')^2 = 0$ *and* $d'd'' = d''d'$*, implying* $d^2 = 0$*, so* $(\widetilde{K}_\bullet(A), d)$ *is a complex of free left* A^e*-modules.* ▲

Proposition 2.4.2 ([dB94, Proposition 3.1])**.** *If* A *is a Koszul* $\Bbbk$*-algebra, then* $\widetilde{K}_\bullet(A) \xrightarrow{\mu} A \to 0$ *is a free* A^e*-resolution of* A*, where* $\mu : \widetilde{K}_0(A) = A \otimes \Bbbk \otimes A \cong A \otimes A \to A$ *is the multiplication map.* ▲

As an immediate corollary of Propositions 2.1.5 and 2.4.2, we have

Corollary 2.4.3. $\mathrm{HH}^n(A) = H^n(\mathrm{Hom}_{A^e}(\widetilde{K}_\bullet(A), A))$ *for all* $n \in \mathbb{N}$*.* ▲

In order to prove Proposition-Definition 2.4.6, we will need the following propositions.

Proposition 2.4.4 (Base change)**.** *Let* R *and* S *be* $\Bbbk$*-algebras with a* $\Bbbk$*-algebra map* $R \to S$*,* *let M be a left R-module and let N be a left S-module. Then*

$$\mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_S(S \otimes_R M, N)$$

as $\Bbbk$-vector spaces via the maps $f \mapsto f'$, where $f'(s \otimes x) := sf(x)$; and $g' \leftarrow\!\shortmid g$, where $g'(x) := g(1 \otimes x)$. ▲

The next proposition follows from Proposition 2.4.4 and the isomorphism from Remark 2.1.2.

Proposition 2.4.5. *Let V be a $\Bbbk$-vector space. Then $\mathrm{Hom}(V, A) \cong \mathrm{Hom}_{A^e}(A \otimes V \otimes A, A)$ as $\Bbbk$-vector spaces via the maps $f \mapsto f'$, where $f'(a \otimes v \otimes a') := af(v)a'$; and $g' \leftarrow\!\shortmid g$, where $g'(v) := g(1 \otimes v \otimes 1)$.* ▲

We are now ready to state the main result of this chapter, originally due to Buchweitz. This result is also a corollary of a more general theorem of Negron [Neg17].

Proposition-Definition 2.4.6 ([Buc03], [Neg17])**.** *Let $A = T(V)/\langle Q \rangle$ be a commutative Koszul algebra over $\Bbbk$; let $\{x_0, \ldots, x_m\}$ be a basis for V; let $E = A^!$; and let $y_i = x_i^*$, so that $\{y_0, \ldots, y_m\}$ is a basis of V^*. The sequence of free A-modules $A \otimes E^n$ with maps $\partial^n : A \otimes E^n \to A \otimes E^{n+1}$ defined by*

$$\partial^n(a \otimes e) = \sum_{j=0}^{m} ax_j \otimes [y_j, e]$$

for $n \in \mathbb{N}$ *form a complex* $(A \otimes E^{\bullet}, \partial)$ *isomorphic to* $(\widetilde{K}_{\bullet}(A)^{\vee}, d^{\vee})$, *where* $(-)^{\vee} = \mathrm{Hom}_{A^e}(-, A)$ *and* $[y_j, e]$ *is the graded Lie bracket given by* $[y_j, e] = y_j e - (-1)^n e y_j$.

It therefore follows that

$$\mathrm{HH}^n(A) \cong H^n(A \otimes A^!)$$

for all $n \in \mathbb{N}$. ▲

Proof. Let $n \in \mathbb{N}$ and let $\{e_0, \ldots, e_p\}$ be a $\Bbbk$-basis for E^n. Let $\alpha_n : A \otimes E^n \to \mathrm{Hom}_{A^e}(\widetilde{K}_n(A), A)$ be the composition of the isomorphisms

$$A \otimes E^n \to A \otimes (E^n)^{**} \to \mathrm{Hom}((E^n)^*, A) \to \mathrm{Hom}_{A^e}(A \otimes (E^n)^* \otimes A, A)$$

described in Propositions 2.4.5 and 2.2.3, mapping $a \otimes e \in A \otimes E^n$ to $\mathrm{Hom}_{A^e}(A \otimes (E^n)^* \otimes A, A)$ via

$$a \otimes e \;\mapsto\; a \otimes e^{**} \;\mapsto\; (f \mapsto f(e)a) \;\mapsto\; (\, a' \otimes g \otimes a'' \mapsto a'(g(e)a)a''\,),$$

so that

$$\alpha_n(a \otimes e) = \varphi_{a,e}, \text{ where } \varphi_{a,e} : A \otimes (E^n)^* \otimes A \to A, \quad \varphi_{a,e}(a' \otimes g \otimes a'') := a'\, g(e) a\, a''.$$

Analogously, let $\beta_n = \alpha_n^{-1} : \mathrm{Hom}(\widetilde{K}_n(A), A) \to A \otimes E^n$ be the composition of the respective inverse isomorphisms, mapping $\psi \in \mathrm{Hom}_{A^e}(A \otimes (E^n)^* \otimes A, A)$ to $A \otimes E^n$ via

$$\psi \;\mapsto\; (f \mapsto \psi(1 \otimes f \otimes 1)) \;\mapsto \sum_{i=0}^{p} \psi(1 \otimes e_i^* \otimes 1) \otimes e_i^{**} \;\mapsto\; \sum_{i=0}^{p} \psi(1 \otimes e_i^* \otimes 1) \otimes e_i,$$

so that

$$\beta_n(\psi) = \sum_{i=0}^{p} \psi(1 \otimes e_i^* \otimes 1) \otimes e_i$$

Let $\{f_0, \ldots, f_q\}$ be a basis for E^{n+1} and consider the diagram

$$\begin{tikzcd}
A \otimes E^n \arrow[r, "\partial^n"] \arrow[d, "\alpha_n"] & A \otimes E^{n+1} \\
\mathrm{Hom}_{A^e}(A \otimes (E^n)^* \otimes A, A) \arrow[r, "(d^\vee)_n"] & \mathrm{Hom}_{A^e}(A \otimes (E_{n+1})^* \otimes A, A) \arrow[u, "\beta_{n+1}"']
\end{tikzcd}$$

For $a \otimes e \in A \otimes E^n$,

$$((d^\vee)_n \circ \alpha_n)(a \otimes e) = (d^\vee)_n(\varphi_{a,e}) = (-1)^{n+1}\varphi_{a,e} d_{n+1},$$

so

$$\begin{aligned}
&(\beta_{n+1} \circ (d^\vee)_n \circ \alpha_n)(a \otimes e) = \\
&= \sum_{i=0}^{q} (-1)^{n+1} \varphi_{a,e} d_{n+1}(1 \otimes f_i^* \otimes 1) \otimes f_i \\
&= \sum_{i=0}^{q} (-1)^{n+1} \varphi_{a,e} \left(\sum_{j=0}^{m} x_j \otimes f_i^* \cdot y_j \otimes 1 + (-1)^{n+1} 1 \otimes y_j \cdot f_i^* \otimes x_j \right) \otimes f_i
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{q} (-1)^{n+1} \left(\sum_{j=0}^{m} x_j f_i^*(ey_j)a + (-1)^{n+1} f_i^*(y_j e) a x_j \right) \otimes f_i && \text{(by Remark 2.3.7)} \\
&= \sum_{i=0}^{q} \left(\sum_{j=0}^{m} x_j a f_i^* \left(y_j e + (-1)^{n+1} e y_j \right) \right) \otimes f_i \\
&= \sum_{i=0}^{q} \sum_{j=0}^{m} x_j f_i^*([y_j, e]) \otimes f_i \\
&= \sum_{j=0}^{m} x_j a \left(\sum_{i=0}^{q} f_i^*([y_j, e]) \otimes f_i \right) && (2.4.0.1)
\end{aligned}
$$

By Corollary 2.2.4, $\sum f_i^* \otimes f_i$ is independent of choice of basis for E_{n+1}. Hence, for each $j \in \{0, \dots, m\}$, so is $\sum f_i^*([y_j, e]) \otimes f_i$. In particular, we can choose $\{f_0, \dots, f_m\}$ to be a basis extending $\{[y_j, e]\}$, so that

$$
\sum_{i=0}^{q} f_i^*([y_j, e]) \otimes f_i = 1 \otimes [y_j, e].
$$

Then (2.4.0.1) implies

$$
(\beta_{n+1} \circ (d^\vee)_n \circ \alpha_n)(a \otimes e) = \sum_{j=0}^{m} x_j a \otimes [y_j, e] = \partial^n (a \otimes e)
$$

and thus $\partial = \beta(d^\vee)\alpha$. Composing ∂ with itself yields $\partial^2 = \beta(d^\vee)^2\alpha = 0$ because $(d^\vee)^2 = 0$, so $(A \otimes E^\bullet, \partial)$ is a complex; and composing α with ∂ yields $\alpha\partial = \alpha\beta(d^\vee)\alpha = (d^\vee)\alpha$, so α is a chain map. $\square$

Chapter 3

The Short Gorenstein Ring Case

In Proposition-Definition 2.4.6 of Chapter 2 we found that the Hochschild cohomology of a commutative Koszul $\Bbbk$-algebra A can be computed as the homology of the complex $(A \otimes A^!, \partial)$. In this chapter we show that the short Gorenstein rings $\mathrm{sGor}(N)$ for $N \geq 2$ are Koszul algebras and further describe the structure of $\mathrm{sGor}(N) \otimes \mathrm{sGor}(N)^!$. First, in Section 3.1 we give an explicit $\Bbbk$-vector space decomposition of short Gorenstein rings (Proposition 3.1.2). Next, in Section 3.2 we show that these short Gorenstein rings are Koszul (Proposition 3.2.2) and compute their quadratic duals (Proposition 3.2.3). Finally, in Section 3.3 we exhibit a decomposition of the Hochschild cohomology groups of short Gorenstein rings (Proposition 3.3.5) using a decomposition of the complex $\mathrm{sGor}(N) \otimes \mathrm{sGor}(N)^!$.

3.1 A decomposition of $\mathrm{sGor}(N)$

The proofs of both Propositions 3.2.2 and 3.2.3 rely on the $\Bbbk$-vector space decomposition of $\mathrm{sGor}(N)$ of Proposition 3.1.2. We prove this below after restating the definition of our family of short Gorenstein rings from Chapter 1.

Definition 3.1.1. For $N \geq 2$, let

$$\mathrm{sGor}(N) = \frac{\Bbbk[X_0, \ldots, X_N]}{(X_iX_j,\ X_i^2 - X_j^2 \mid i, j = 0, \ldots, N, i \neq j)}.$$

And for $i = 0, \ldots, N$, let x_i be the image of X_i in $\mathrm{sGor}(N)$. ▲

Proposition 3.1.2. *Let $N \geq 2$. Then $\mathrm{sGor}(N)$ has a $\Bbbk$-vector space decomposition*

$$\mathrm{sGor}(N) = \Bbbk \oplus \Bbbk\{x_0, \ldots, x_N\} \oplus \Bbbk\{s\}$$

where $s = x_i^2$ for any $i \in \{0, \ldots, N\}$. In particular, it follows that $\mathrm{sGor}(N)$ is Gorenstein since s generates the socle; and we have $(x_0, \ldots, x_N)^2 = \Bbbk\{s\}$ and $(x_0, \ldots, x_N)^3 = 0$. ▲

Proof. Let $N \geq 2$ and let

$$I = (X_iX_j,\ X_i^2 - X_j^2 \mid i, j = 0, \ldots, N, i \neq j).$$

Since X_iX_j and $X_i^2 - X_j^2$ are homogeneous of degree 2, I is homogeneous with $I_0 = 0$ and $I_1 = 0$. Thus,

$$\mathrm{sGor}(N)_0 = \Bbbk[X_0, \ldots, X_N]_0 = \Bbbk, \quad \mathrm{sGor}(N)_1 = \Bbbk[X_0, \ldots, X_N]_1 = \Bbbk\{x_0, \ldots, x_N\}.$$

By definition, $\mathrm{sGor}(N)_2$ is generated by the monomials x_ix_j for $i, j \in \{0, \ldots, N\}$. The relations generating I imply that $x_ix_j = 0$ for $i \neq j$ and that $x_i^2 = x_j^2$ for all i, j. Thus, setting $s = x_0^2$, $\mathrm{sGor}(N)_2 = \Bbbk\{s\}$.

Finally, we claim that $\mathrm{sGor}(N)_k = 0$ for all $k \geq 3$. To see this, let $u \in \Bbbk[X_0, \ldots, X_N]$ be a monomial with $|u| \geq 3$. It is enough to show that $\overline{u} = 0$. If $u = X_i X_j v$ for some $i \neq j$ and some monomial v, then $\overline{u} = x_i x_j \overline{v} = 0 \cdot \overline{v} = 0$. Otherwise, $u = X_i^m$ for some i and m. Choose $j \in \{0, \ldots, N\}$ such that $j \neq i$. Then $x_i^2 = x_j^2$, so

$$\overline{u} = x_i^{m-2} x_j^2 = x_i^{m-3} x_i x_j x_j = x_i^{m-3} \cdot 0 \cdot x_j = 0.$$

The above argument implies $(x_0, \ldots, x_N)^2 = \Bbbk\{s\}$ and $(x_0, \ldots, x_N)^3 = 0$. □

Remark 3.1.3. Note that $\{x_0, \ldots, x_N\} \subset \mathrm{sGor}(N)$ is linearly independent, so in fact this set serves as a $\Bbbk$-basis of $\mathrm{sGor}(N)_1$. ▲

As a corollary of Proposition 3.1.2, we have that these rings are local.

Corollary 3.1.4. *For $N \geq 2$, $\mathrm{sGor}(N)$ is a local ring with maximal ideal $(x_0, \ldots, x_N)$.* ▲

Proof. Let $\mathfrak{m} \subset \mathrm{sGor}(N)$ be a maximal ideal. By Proposition 3.1.2, $(x_0, \ldots, x_N)^3 = 0$. Since $\mathfrak{m}$ is prime, the containment $(x_0, \ldots, x_N)^3 = \{0\} \subset \mathfrak{m}$ implies $(x_0, \ldots, x_N) \subseteq \mathfrak{m}$. And since $(x_0, \ldots, x_N)$ is maximal, $(x_0, \ldots, x_N) = \mathfrak{m}$, so $(x_0, \ldots, x_N)$ is the unique maximal ideal of $\mathrm{sGor}(N)$. □

3.2 Short Gorenstein rings are Koszul

Our hopes of computing a lower bound on the $\Bbbk$-dimensions of the Hochschild cohomology of our Gorenstein $\Bbbk$-algebras $\mathrm{sGor}(N)$ lie in being able to use the complex $A \otimes A^!$ described in Proposition-Definition 2.4.6 for our computations. For this we need to establish that these algebras are Koszul. In this section we show that these short Gorenstein rings are in fact Koszul as *local rings* (defined below) and recall the result that this is equivalent to these rings being Koszul algebras.

The original definition of a Koszul algebra is due to Priddy [Pri70]. In [HI05], Herzog and Iyengar define the analogous notion of a *Koszul module*—and, in turn, of a *Koszul ring*—as follows. Let R be a commutative local noetherian ring with maximal ideal $\mathfrak{m}$ and residue field k, let M be an R-module, and let F be a minimal free resolution of M. For every $j \geq 0$, define

$$\mathrm{lin}_j(F) = 0 \to \frac{F_j}{\mathfrak{m}F_j} \to \cdots \to \frac{\mathfrak{m}^{j-i}F_i}{\mathfrak{m}^{j+1-i}F_i} \to \cdots \to \frac{\mathfrak{m}^j F_0}{\mathfrak{m}^{j+1}F_0} \to \frac{\mathfrak{m}^j M}{\mathfrak{m}^{j+1}M} \to 0,$$

with differentials induced by those of F. Then M is a *Koszul module* if $\mathrm{lin}_j(F)$ is acyclic for all $j \geq 0$. And R is a *Koszul ring* if the R-module k is Koszul. By [HI05, Remark 1.10], a local ring R is a Koszul ring if and only if its associated graded algebra

$$\mathrm{gr}_{\mathfrak{m}}(R) = \bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

(where $\mathfrak{m}^0 = R$) is a Koszul algebra.

Let $N \geq 2$ and let $\mathfrak{m}_N = (x_0, \ldots, x_N) \subset \mathrm{sGor}(N)$. By Proposition 3.1.2, $\mathrm{sGor}(N)$ is local with maximal ideal $\mathfrak{m}_N$, and since $\mathrm{sGor}(N)$ is graded, we have $\mathrm{gr}_{\mathfrak{m}_N}(\mathrm{sGor}(N)) \cong \mathrm{sGor}(N)$. That is, explicitly, $\mathfrak{m}_N = \Bbbk\{x_0, \ldots, x_N\} \oplus \Bbbk\{s\}$, $\mathfrak{m}_N^2 = \Bbbk\{s\}$, and $\mathfrak{m}_N^i = 0$ for $i \geq 3$, so we have $\Bbbk$-algebra isomorphisms

$$\mathrm{gr}_{\mathfrak{m}_N}(\mathrm{sGor}(N)) \cong \Bbbk \oplus \Bbbk\{x_0, \ldots, x_N\} \oplus \Bbbk\{s\} = \mathrm{sGor}(N).$$

Thus, $\mathrm{sGor}(N)$ is a Koszul algebra if and only if it is also a Koszul ring.

We will employ the following theorem, due to Avramov-Iyengar-Şega, to show that the short Gorenstein rings $\mathrm{sGor}(N)$ are indeed Koszul rings and therefore also Koszul $\Bbbk$-algebras (Proposition 3.2.2). Although the full version of the theorem provides several equivalent conditions for a local ring to be a Koszul ring, the version of the theorem presented below involves only the equivalence most relevant to our discussion.

Theorem 3.2.1 ([AIc05, Theorem 4.1])**.** *Let $(R, \mathfrak{m}, k)$ be a local ring with $\mathfrak{m}^3 = 0$ and $\mathrm{rank}_k\, \mathfrak{m}^2 = 1$. Then R is a Koszul ring if and only if $\mathrm{rank}_k(0 : \mathfrak{m}) \leq \mathrm{rank}_k(\mathfrak{m}/\mathfrak{m}^2)$.* ▲

Proposition 3.2.2. *For $N \geq 2$, $\mathrm{sGor}(N)$ is a Koszul $\Bbbk$-algebra.* ▲

Proof. Let $N \geq 2$ and let $\mathfrak{m}_N = (x_0, \ldots, x_N)$. By the above discussion, it is enough to show that $\mathrm{sGor}(N)$ is a Koszul local ring in the sense of Herzog and Iyengar.

By Proposition 3.1.2,

$$\mathfrak{m}_N/\mathfrak{m}_N^2 = \frac{\Bbbk\{x_0, \ldots, x_N\} \oplus \Bbbk\{s\}}{\Bbbk\{s\}} = \Bbbk\{x_0, \ldots, x_N\},$$

so $\mathrm{rank}_{\Bbbk}(\mathfrak{m}/\mathfrak{m}^2) = N+1$. And by [AIc05, Remark 4.3], $(0 : \mathfrak{m}_N) = \mathfrak{m}_N^2 = \Bbbk\{s\}$, so $\mathrm{rank}_{\Bbbk}(0 : \mathfrak{m}_N) = 1$. Since $N \geq 2$, $\mathrm{rank}_k(0 : \mathfrak{m}) \leq \mathrm{rank}_k(\mathfrak{m}/\mathfrak{m}^2)$, so Theorem 3.2.1 implies that $\mathrm{sGor}(N)$ is a Koszul ring. □

We end this section with a computation of the quadratic dual $\mathrm{sGor}(N)^!$. Recall from Definition 2.3.1 that the quadratic dual of a quadratic algebra $T(V)/\langle Q\rangle$ is the algebra $T(V^*)/\langle Q^\perp\rangle$, where $Q^\perp$ is the perpendicular subspace of Q with respect to the natural pairing (Definition 2.2.6).

Proposition 3.2.3. *Let $N \geq 2$. Then*

$$\mathrm{sGor}(N)^! = \frac{\Bbbk\langle (X_0)^*, \ldots, (X_N)^*\rangle}{\langle (X_0^*)^2 + \cdots + (X_N^*)^2\rangle}.$$

▲

Proof. To make the proof easier to read, set $m = N+1$.

Set $V = \Bbbk\{X_0, \ldots, X_N\}$ and set

$$Q = \Bbbk\{X_iX_j - X_jX_i,\ X_iX_j,\ X_i^2 - X_j^2 \mid i,j = 0,\ldots,N;\ i \neq j\},$$

so that $\mathrm{sGor}(N) = T(V)/\langle Q\rangle$. Note that the union of the sets

$$\mathbf{X}' = \{X_iX_j \mid i,j = 0,\ldots,N;\ i \neq j\}$$

and

$$\mathbf{X} = \{X_k^2 - X_N^2 \mid k = 0, \dots, N-1\}$$

forms a $\Bbbk$-basis of Q and hence

$$\dim_{\Bbbk} Q = |\mathbf{X}'| + |\mathbf{X}| = (m^2 - m) + (m-1) = m(m-1) + (m-1) = m^2 - 1.$$

So by Proposition 2.2.7,

$$\dim_{\Bbbk} Q^{\perp} = \dim_{\Bbbk} V \otimes V - \dim_{\Bbbk} Q = m^2 - (m^2 - 1) = 1.$$

To prove the theorem, it is enough to show that $\Bbbk\{(X_0^*)^2 + \cdots + (X_N^*)^2\} = Q^{\perp}$. By the above, this amounts to showing that $(X_0^*)^2 + \cdots + (X_N^*)^2 \in Q^{\perp}$, as it is clearly a nonzero element.

Let $r = (X_0^*)^2 + \cdots + (X_N^*)^2$. For all $X_iX_j \in \mathbf{X}'$ and all $k = 0, \dots, N$,

$$(X_k^*)^2(X_iX_j) = Y_k(X_i)Y_k(X_j) = 0$$

since $i \neq j$, so one of i or j must be different from k. Thus, $r(X_iX_j) = 0$ for all $X_iX_j \in \mathbf{X}'$. We also have that $r(X_i^2) = 1$ for $i = 0, \dots, N$, so $r(X_k^2 - X_N^2) = 0$ for all $k = 0, \dots, N-1$. Therefore, $r \in Q^{\perp}$, finishing the proof. $\square$

3.3 A decomposition of $\mathrm{HH}^\bullet(\mathrm{sGor}(N))$

Having established that the short Gorenstein rings $\mathrm{sGor}(N)$ are Koszul $\Bbbk$-algebras, we now know that the complex $\mathrm{sGor}(N) \otimes \mathrm{sGor}(N)^!$ from Proposition-Definition 2.4.6 computes $\mathrm{HH}^\bullet(\mathrm{sGor}(N))$. In this section we further investigate the structure of this complex. In particular, we want to better understand the behavior of ∂^n restricted to $\mathrm{sGor}(N)_i \otimes (\mathrm{sGor}(N)^!)^n$ for $i = 0, 1, 2$. This leads to a decomposition of $\mathrm{sGor}(N) \otimes \mathrm{sGor}(N)^!$ and therefore a decomposition of $\mathrm{HH}^n(\mathrm{sGor}(N))$ for all n.

First, however, we establish some conventions.

Convention 3.3.1. In this section, fix some $N \geq 2$ and let $A = \mathrm{sGor}(N)$. For $i = 0, \dots, N$, let $Y_i = X_i^*$ and let

$$E = A^! = \frac{\Bbbk\langle Y_0, \dots, Y_N\rangle}{\langle Y_0^2 + \cdots + Y_N^2\rangle}. \qquad \blacktriangle$$

Let $n \in \mathbb{N}$. For any $i \in \{0, 1, 2\}$ and $a \otimes u \in A_i \otimes E^n$, we have

$$\partial^n(a \otimes u) = \sum_{j=0}^{N} a x_j \otimes [y_j, u]$$

by Proposition-Definition 2.4.6. Then $ax_j \in A_{i+1}$ and $[y_j, u] \in E^{n+1}$ imply that $\partial^n(a \otimes u) \in A_{i+1} \otimes E^{n+1}$. Thus,

$$\partial^n(A_0 \otimes E^n) \subseteq A_1 \otimes E^{n+1}, \qquad \partial^n(A_1 \otimes E^n) \subseteq A_2 \otimes E^{n+1}, \qquad \partial^n(A_2 \otimes E^n) = 0,$$

where the last equality holds since $A_3 = 0$. Furthermore, by Proposition 3.1.2,

$$A \otimes E^n = (A_0 \otimes E^n) \oplus (A_1 \otimes E^n) \oplus (A_2 \otimes E^n),$$

so we have the following observation.

Observation 3.3.2. For $n \geq 0$, the map $\partial^n : A \otimes E^n \to A \otimes E^{n+1}$ decomposes as the sum of the maps $\partial^n |_{A_i \otimes E^n} \colon A_i \otimes E^n \to A_{i+1} \otimes E^{n+1}$, $i = 0, 1, 2$. ▲

To give a precise description of the decomposition of $A \otimes E$ which follows from Observation 3.3.2, we define the following complexes of $\Bbbk$-spaces.

Definition 3.3.3. For $n \geq 0$, let $(\mathcal{C}_{(n)}, \delta_{(n)})$ be the complex defined by

$$\mathcal{C}^m_{(n)} = \begin{cases} A_0 \otimes E^{n-1}, & m = n-1; \\ A_1 \otimes E^n, & m = n; \\ A_2 \otimes E^{n+1}, & m = n+1; \\ 0, & \text{otherwise}, \end{cases}$$

and $\delta_{(n)}$ the differential induced by ∂ on this subcomplex; that is,

$$\delta_{(n)}^m = \begin{cases} \partial^{n-1} \mid_{A_0 \otimes E^{n-1}}, & m = n-1; \\ \partial^n \mid_{A_1 \otimes E^n}, & m = n; \\ \partial^{n+1} \mid_{A_2 \otimes E^{n+1}}, & m = n+1; \\ 0, & \text{otherwise,} \end{cases}$$

We call $\mathcal{C}_{(n)}$ the *nth strand* of $A \otimes E$. ▲

Thus, $A \otimes E = \bigoplus_{n \geq 0} \mathcal{C}_{(n)}$, as shown in Diagram 3.3.1. Since homology commutes with direct sums, for $n \geq 0$, we have

$$H^n(A \otimes E) = H^n(\mathcal{C}_{(n+1)}) \oplus H^n(\mathcal{C}_{(n)}) \oplus H^n(\mathcal{C}_{(n-1)}). \tag{3.3.0.1}$$

Before stating this as a result, we introduce some notation that will simplify the rest of our discussion.

Definition 3.3.4. For $n \geq 0$, let

- $\mathrm{HH}_{(0)}^n(A) = H^n(\mathcal{C}_{(n+1)})$,
- $\mathrm{HH}_{(1)}^n(A) = H^n(\mathcal{C}_{(n)})$, and
- $\mathrm{HH}_{(2)}^n(A) = H^n(\mathcal{C}_{(n-1)})$. ▲

Thus, by Proposition-Definition 2.4.6 and Equation 3.3.0.1, we have

Proposition 3.3.5. *For $n \geq 0$, $\mathrm{HH}^n(A)$ decomposes as*

$$\mathrm{HH}^n(A) = \mathrm{HH}^n_{(0)}(A) \oplus \mathrm{HH}^n_{(1)}(A) \oplus \mathrm{HH}^n_{(2)}(A). \qquad \blacktriangle$$

$$
\begin{array}{rccccccccc}
A \otimes E = & \cdots & \longrightarrow & A \otimes E^{n-1} & \xrightarrow{\partial^{n-1}} & A \otimes E^{n} & \xrightarrow{\partial^{n}} & A \otimes E^{n+1} & \longrightarrow & \cdots \\
 & & & \| & & \| & & \| & & \\
 & & & \vdots & & \vdots & & \vdots & & \\
 & & & \oplus & & \oplus & & \oplus & & \\
\mathcal{C}_{(n-2)} = & \cdots & \longrightarrow & A_2 \otimes E^{n-1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & \oplus & & \oplus & & \oplus & & \\
\mathcal{C}_{(n-1)} = & \cdots & \longrightarrow & A_1 \otimes E^{n-1} & \xrightarrow{\delta^{n-1}_{(n-1)}} & A_2 \otimes E^{n} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & \oplus & & \oplus & & \oplus & & \\
\mathcal{C}_{(n)} = & \cdots & \longrightarrow & A_0 \otimes E^{n-1} & \xrightarrow{\delta^{n-1}_{(n)}} & A_1 \otimes E^{n} & \xrightarrow{\delta^{n}_{(n)}} & A_2 \otimes E^{n+1} & \longrightarrow & \cdots \\
 & & & \oplus & & \oplus & & \oplus & & \\
\mathcal{C}_{(n+1)} = & \cdots & \longrightarrow & 0 & \longrightarrow & A_0 \otimes E^{n} & \xrightarrow{\delta^{n}_{(n+1)}} & A_1 \otimes E^{n+1} & \longrightarrow & \cdots \\
 & & & \oplus & & \oplus & & \oplus & & \\
\mathcal{C}_{(n+2)} = & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A_0 \otimes E^{n+1} & \longrightarrow & \cdots \\
 & & & \oplus & & \oplus & & \oplus & & \\
 & & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Diagram 3.3.1: Decomposition of $A \otimes E$

Chapter 4

Gröbner-Shirshov Basis Theory

In Chapter 3 we showed that the short Gorenstein rings

$$\mathrm{sGor}(N) = \frac{\Bbbk[X_0, \ldots, X_N]}{(X_iX_j,\, X_i^2 - X_j^2 \mid i, j = 0, \ldots, N, i \neq j)}$$

are Koszul $\Bbbk$-algebras (Proposition 3.2.2) and, as such, their Hochschild cohomology can be computed using the complex $(\mathrm{sGor}(N) \otimes \mathrm{sGor}(N)^!, \partial)$ (see Proposition-Definition 2.4.6), where

$$\mathrm{sGor}(N)^! = \frac{\Bbbk\langle (X_0)^*, \ldots, (X_N)^*\rangle}{\langle (X_0^*)^2 + \cdots + (X_N^*)^2\rangle}$$

(Proposition 3.2.3).

In order to actually perform any computations, it is imperative to understand the $\Bbbk$-vector space structure of $\mathrm{sGor}(N) \otimes (\mathrm{sGor}(N)^!)^n$ for $n \geq 0$. We have already made progress on this front by showing that $\mathrm{sGor}(N)$ decomposes as $\mathrm{sGor}(N) \cong \Bbbk \oplus \Bbbk\{x_0, \ldots, x_N\} \oplus \Bbbk\{s\}$, where s generates the socle of $\mathrm{sGor}(N)$ (Proposition 3.1.2). In this chapter we prove that $\mathrm{sGor}(N)^!$ has a $\Bbbk$-basis consisting of all noncommutative monomials which are not divisible by $(X_0^*)^2$ nor by $X_0^*(X_1^*)^2$ (Proposition 4.3.3).

The proof of this result relies on the theory of *Gröbner-Shirshov (GS) bases*, which we recall in Section 4.2. This theory will enable us to affirmatively answer the more general question, "Given an ideal I of the free associative algebra $\Bbbk\langle Y\rangle$ on a finite set Y, can we find a $\Bbbk$-vector space basis for $\Bbbk\langle Y\rangle/I$?"

The contents of this chapter, except for Propositions 4.3.2 through 4.3.3, can be found in, for example, [Bre14], [Ber78], and [BMPZ92].

4.1 Monomial orders

In order to proceed with our discussion of GS bases we will need a notion of a well-order on the monomials of free associative algebra that respects multiplication, called a *monomial order* (Definition 4.1.4).

Convention 4.1.1. For the rest of this chapter, let Y be a finite set. ▲

We start by defining some terminology for the set of monomials on Y and its multiplicative structure.

Definition 4.1.2. The *free monoid* on Y, denoted by $\mathrm{Mon}(Y)$, is the set of all *words* (or *monomials*) $y_1y_2\cdots y_m$ ($y_i \in Y$) with an associative binary operation given by concatenation and an identity element $1 \in \mathrm{Mon}(Y)$, called the *empty word*.

A nonempty word $u \in \mathrm{Mon}(Y)$ is a *subword* (or *factor* or *divisor*) of $w \in \mathrm{Mon}(Y)$ if $w = vuv'$ for some $v, v' \in \mathrm{Mon}(Y)$. In this case, we say that w is *divisible* by u or that u *divides* w. If $u \neq w$, then u is a *proper subword* of w.

The *degree* of a nonempty word $w = y_1\cdots y_m \in \mathrm{Mon}(Y)$ is m and the *degree* of 1 is 0.

Denote by $\deg(w)$ the degree of the word $w \in \mathrm{Mon}(Y)$. ▲

Remark 4.1.3. In this context, the free associative algebra on Y, denoted by $\Bbbk\langle Y\rangle$, is the $\Bbbk$-vector space on the basis $\mathrm{Mon}(Y)$ with multiplication given by linearly extending the monoid operation on $\mathrm{Mon}(Y)$. ▲

Definition 4.1.4. A well-ordering $\prec$ on $\mathrm{Mon}(Y)$ is a *monomial ordering* if, for all elements $w, w', u, v \in \mathrm{Mon}(Y)$, $w \prec w'$ implies $w \preceq uwv \prec uw'v$. ▲

Our first example is a familiar one.

Example 4.1.5. Let $Y = \{y\}$. As a set, $\mathrm{Mon}(Y) = \{y^n \mid n \in \mathbb{N}\}$, and the usual notion of degree defines a monomial ordering on $\mathrm{Mon}(Y)$: $y^i \prec y^j$ if and only if $i < j$. ▲

We will use the following ordering in our computation of a GS basis for $\langle r\rangle$.

Definition 4.1.6. Given a total order $\prec$ on Y, the *degree lexicographical (deglex)* order $\prec_{\mathtt{dl}}$ on $\mathrm{Mon}(Y)$ is defined inductively on any $w, w' \in \mathrm{Mon}(Y)$ as $w \prec_{\mathtt{dl}} w'$ if and only if

(i) either $\deg(w) < \deg(w')$, or

(ii) $\deg(w) = \deg(w')$ and $w = uzv$, $w' = uz'v'$ with $u, v, v' \in \mathrm{Mon}(Y)$, $z, z' \in Y$, and $z \prec z'$. ▲

Proposition 4.1.7. *Let $(Y, \prec)$ be well-ordered. The deglex order on* $\mathrm{Mon}(Y)$ *extending $\prec$ is a monomial ordering.* ▲

To get a better feel for the definition, we take a look at an example of a deglex order extending a total order on a set of three elements.

Example 4.1.8. Let $Y = \{y_0, y_1, y_2\}$ with total order $y_2 \prec y_1 \prec y_0$ and consider the monomials $w = y_1y_2y_2y_0$, $w' = y_1y_1y_0y_2$, $w'' = y_0y_0y_0$, $\in \mathrm{Mon}(Y)$. Then $\deg(w) = 4$, $\deg(w') = 4$, and $\deg(w'') = 3$, so $w'' \prec_{\mathtt{dl}} w$ and $w'' \prec_{\mathtt{dl}} w'$ by Definition 4.1.6(i). Also, with $u = y_1$, $v = y_2y_0$, and $v' = y_0y_2$, we have $w = uy_2v$ and $w' = uy_1v'$ with $y_2 \prec y_1$, so $w \prec_{\mathtt{dl}} w'$ by Definition 4.1.6(ii). ▲

Assume $Y = \{y\}$. Writing a polynomial $f \in \Bbbk[y]$ as a linear combination of monomials in $\mathrm{Mon}(\{y\})$, we typically define the degree of f to be the power of the maximal (or *leading*) monomial, denoted by $\widehat{f}$, in this presentation. It is with respect to this degree-induced order that we have a division algorithm for polynomials: given $f, g \in \Bbbk[y]$, the algorithm produces two more polynomials a and b such that $g = af + b$ with either $b = 0$ or $\deg(b) < \deg(f)$; equivalently, such that $b = 0$ or $\widehat{b} \prec \widehat{f}$ with respect to the monomial ordering from Example 4.1.5. By reframing things in terms of a given monomial ordering, one can generalize the division algorithm to a free associative algebra on more than one element (Algorithm 4.2.8). This is an indispensable tool in the construction of a GS basis.

Convention 4.1.9. For the rest of this chapter let $\prec$ be a monomial order on $\mathrm{Mon}(Y)$. ▲

Definition 4.1.10. For $f = a_0u_0 + \cdots + a_mu_m \in \Bbbk\langle Y\rangle$ with $a_i \in \Bbbk^\times$ and $u_i \in \mathrm{Mon}(Y)$, the *support* $\mathrm{supp}(f)$ of f is the set $\{u_0, \ldots, u_m\}$ and the *leading monomial* $\widehat{f}$ of f is the maximal element of $\mathrm{supp}(f)$ with respect to $\prec$.

The *leading coefficient* of f is the coefficient of $\widehat{f}$ in the monomial presentation of f. We call f *monic* if the leading coefficient of f is 1. A set $S \subseteq \Bbbk\langle Y\rangle$ not containing 1 is *monic* if every element of S is monic.

For a set $S \subseteq \Bbbk\langle Y\rangle$, define $\widehat{S}$ to be the set of all leading monomials of the polynomials in S. ▲

4.2 Gröbner-Shirshov bases

Given a finitely generated ideal $I \subseteq \Bbbk\langle Y\rangle$, one route to describing the $\Bbbk$-vector space structure of $\Bbbk\langle Y\rangle/I$ is to find a subspace $C \subset \Bbbk\langle Y\rangle$ such that $\Bbbk\langle Y\rangle = C \oplus I$, so $C \cong \Bbbk\langle Y\rangle/I$ as $\Bbbk$-vector spaces. If I is generated by a set $S \subset \mathrm{Mon}(Y)$ of monomials, then the set of all monomials $u \in \mathrm{Mon}(Y)$ such that u is divisible by some $v \in S$ forms a $\Bbbk$-basis of I. Since $\mathrm{Mon}(Y)$ is a $\Bbbk$-basis of $\Bbbk\langle Y\rangle$ extending S, we can take C to be the subspace of $\Bbbk\langle Y\rangle$ with basis the set of all monomials not divisible by any of the monomials of S.

For an arbitrary ideal I, we have the following theorem.

Theorem 4.2.1 (see [Bre14, Proposition 4.3])**.** *If I is an ideal of $\Bbbk\langle Y\rangle$, then the subspace $\mathsf{c}(I)$ of $\Bbbk\langle Y\rangle$ with basis all monomials of $\mathrm{Mon}(Y)$ not in $\widehat{I}$ satisfies $\Bbbk\langle Y\rangle = I \oplus \mathsf{c}(I)$. In particular, every polynomial $f \in \Bbbk\langle Y\rangle$ has a unique representation of the form $f = g + c$ for some $g \in I$ and some $c \in \mathsf{c}(I)$.* ▲

We can rephrase this theorem in terms of a *Gröbner-Shirshov basis* S for I and monomials *irreducible* with respect to S, both defined below.

Definition 4.2.2. A *Gröbner-Shirshov (GS) basis* is a set S of generators for I with the property that for every nonzero $f \in I$ there is some $g \in S$ such that $\widehat{f}$ is divisible by $\widehat{g}$. ▲

Definition 4.2.3. A monomial $w \in \mathrm{Mon}(Y)$ is *irreducible* with respect to a set $S \subseteq \Bbbk\langle Y\rangle$ if it is not divisible by any element of $\widehat{S}$. A polynomial $f \in \Bbbk\langle Y\rangle$ is *irreducible* with respect to S if every monomial of $\mathrm{supp}(f)$ is irreducible with respect to $\widehat{S}$. If this does not hold, f is *reducible* with respect to S.

Denote by B_{Irr} the set of all monomials in $\mathrm{Mon}(Y)$ irreducible with respect to S and denote by $\mathrm{Irr}(S)$ the subspace of $\Bbbk\langle Y\rangle$ with basis B_{Irr}. ▲

In this context, Theorem 4.2.1 can be stated as follows.

Theorem 4.2.4. *If I is an ideal of $\Bbbk\langle Y\rangle$ and S is a Gröbner-Shirshov basis for I, then we have $\Bbbk\langle Y\rangle = I \oplus \mathrm{Irr}(S)$.* ▲

The following corollary will be our main tool in studying the $\Bbbk$-vector space structure of $\mathrm{sGor}(N)^!$.

Corollary 4.2.5. *The $\Bbbk$-vector space $\Bbbk\langle Y\rangle/I$ has a $\Bbbk$-basis consisting of all monomials in* $\mathrm{Mon}(Y)$ *irreducible with respect to S.* ▲

Our reason for formulating Theorem 4.2.4 and Corollary 4.2.5 in terms of GS bases is because GS bases carry all the information necessary to describe bases for quotients of free associative algebras while having the advantage of being reasonably straightforward to construct in many cases.

The construction of a GS basis requires the division algorithm for free associative algebras (Algorithm 4.2.8), which, in turn, requires the following notion of reduction of a polynomial.

Definition 4.2.6. Let $S \subseteq \Bbbk\langle Y\rangle$. For elements $g \in S$ and $u, v \in \mathrm{Mon}(Y)$, the *reduction* $\rho(u, g, v) : \Bbbk\langle Y\rangle \to \Bbbk\langle Y\rangle$ is the $\Bbbk$-linear map which sends $u\widehat{g}v$ to $u\widehat{g}v - ugv$ and is the identity on all other monomials in $\mathrm{Mon}(Y)$. A *reduction* of a polynomial $f \in \Bbbk\langle Y\rangle$ with respect to S is $\rho(u, g, v)$ for some $g \in S$ and $u, v \in \mathrm{Mon}(Y)$ such that $u\widehat{g}v \in \mathrm{supp}(f)$. ▲

Let $S \subseteq \Bbbk\langle Y\rangle$, let $w \in \mathrm{supp}(f)$, let $a \in \Bbbk^\times$ be the coefficient of w in the presentation of f with respect to $\mathrm{supp}(f)$, and assume there exists an element $g \in S$ such that $w = u\widehat{g}v$ for some $u, v \in \mathrm{Mon}(Y)$. Then $\rho(u, g, v)(f) = f - augv$. Since $\widehat{g}$ is the largest monomial of $\mathrm{supp}(g)$, $\rho(u, g, v)(f)$ amounts to replacing w in f with terms that are strictly less than w. It is in these sense that $\rho(u, g, v)(f)$ is a "reduction" of f with respect to S.

If $S \subset \Bbbk\langle Y\rangle$ is finite, then repeated reductions of a polynomial $f \in \Bbbk\langle Y\rangle$ with respect to S eventually result in reduction to 0 or to a nonzero polynomial that is irreducible with respect to S (Algorithm 4.2.8). The following is an example of the latter.

Example 4.2.7. Let $Y = \{x, y, z\}$; let $\prec$ be the deglex order on $\Bbbk\langle Y\rangle$ extending $z \prec y \prec x$; let $f = x^3 + x^2y,\ g = x^2 + y^2 + z^2 \in \Bbbk\langle Y\rangle$; and let $S = \{g\}$. We consider a sequence of reductions of f with respect to S. First, $\widehat{g} = x^2$, so $x\widehat{g} = x^3 \in \mathrm{supp}(f)$. Thus, $\rho(x, g, 1)$ is a reduction of f with respect to S and we have

$$\rho(x,g,1)(f) = f - xg = (x^3 + x^2y) - (x^3 + xy^2 + xz^2) = x^2y - xy^2 - xz^2.$$

Similarly, $\widehat{g}y = x^2y \in \mathrm{supp}(\rho(x,g,1)(f))$ and

$$\rho(1,g,y) \circ \rho(x,g,1)(f) = (x^2y - xy^2 - xz^2) - (x^2y + y^3 + z^2y) = -xy^2 - xz^2 - y^3 - z^2y,$$

which is irreducible with respect to S. ▲

We describe this process more generally below.

Algorithm 4.2.8 (Division algorithm)**.** Let $f \in \Bbbk\langle Y\rangle$ and let $S \subset \Bbbk\langle Y\rangle$ be a finite set of monic polynomials. Set $f_0 = f$ and $\rho_0 = \mathrm{id}_{\Bbbk\langle Y\rangle}$. For $i \geq 0$,

- if f_i is reducible respect to S, let $w \in \mathrm{supp}(f_i)$ be the maximal monomial such that $w = u\widehat{g}v$ for some $g \in S$ and $u, v \in \mathrm{Mon}(Y)$. Set $\rho_{i+1} = \rho(u,g,v) \circ \rho_i$ and set $f_{i+1} = \rho(u,g,v)(f_i) = \rho_{i+1}(f)$.
- Otherwise, output ρ_i and f_i. ▲

Proposition 4.2.9 (see [Bre14, Lemma 4.8])**.** *Let $f \in \Bbbk\langle Y\rangle$ and let $S \subset \Bbbk\langle Y\rangle$ be a finite set of monic polynomials. Then Algorithm 4.2.8 applied to f and S terminates after finitely many steps. In other words, there is some $n \in \mathbb{N}$ such that after n steps Algorithm 4.2.8 outputs a composition ρ_n of a sequence of reductions and a polynomial $f_n \in \mathrm{Irr}(S)$ with $f_n = \rho_n(f)$.* ▲

Proof. For $i \geq 0$, if u_i is the maximal element of $\mathrm{supp}(f_i)$ divisible by some monomial in $\widehat{S}$, we have $\cdots \prec u_i \prec \cdots \prec u_1 \prec u_0$. Since $\mathrm{Mon}(Y)$ is well-ordered with respect to $\prec$, this chain must be finite, so the algorithm must terminate. $\square$

Definition 4.2.10. Call an output ρ of the division algorithm with input $f \in \Bbbk\langle Y \rangle$ and $S \subset \Bbbk\langle Y \rangle$ a *terminal sequence* of f. The polynomial $\rho(f)$ is an *irreducible form* of f (with respect to S). ▲

Irreducible forms are not unique; they depend on choices made in each step of the division algorithm, as shown in the next example.

Example 4.2.11. The polynomial $\rho(1, g, y) \circ \rho(x, g, 1)(f) = -xy^2 - xz^2 - y^3 - z^2y$ from Example 4.2.7 is an irreducible form of $f = x^3 + x^2y$, which resulted from factoring the monomial $x^3 \in \mathrm{supp}(f)$ as $x\widehat{g}$. If we instead choose to factor x^3 as $\widehat{g}x$, we have the reduction

$$\rho(1, g, x)(f) = (x^3 + x^2y) - (x^3 + y^2x + z^2x) = x^2y - y^2x - z^2x.$$

Then $\widehat{g}y = x^2y \in \mathrm{supp}(\,\rho(1, g, x)(f)\,)$, so

$$\begin{aligned} \rho(1, g, y) \circ \rho(1, g, x)(f) &= (x^2y - y^2x - z^2x) - (x^2y + y^3 + z^2y) \\ &= -y^2x - z^2x - y^3 - z^2y, \end{aligned}$$

which is irreducible with respect to S. Thus, $\rho(1, g, y) \circ \rho(1, g, x)(f)$ is another irreducible form of f, different from $\rho(1, g, y) \circ \rho(x, g, 1)(f)$. ▲

Fortunately, GS bases resolve this issue (Theorem 4.2.13): all terminal sequences of reductions of a given polynomial f with respect to a GS basis S produce the same irreducible form of f, defined below.

Definition 4.2.12. In the setting of Theorem 4.2.1, the *normal form* $\mathrm{nf}_I(f)$ of a polynomial f with unique representation $f = g + c$ is the element $c \in \mathsf{c}(I)$. ▲

Theorem 4.2.13 (see [Bre14, Theorem 5.3])**.** *If S is a GS basis for an ideal I, then every terminal sequence of f with respect to S reduces f to $\mathrm{nf}_I(f)$.* ▲

Example 4.2.14. Continuing in the setting of Examples 4.2.7 and 4.2.11, let us instead consider terminal sequences of f with respect to the set $S' = S \cup \{h = xy^2 + xz^2 - y^2x - z^2x\}$. Then $\rho(1, g, y) \circ \rho(x, g, 1)(f) = -xy^2 - xz^2 - y^3 - z^2y$ is no longer irreducible with respect to S' since $\widehat{h} = xy^2$, so we have the reduction

$$\begin{aligned} \rho(1, h, 1) \circ \rho(1, g, y) \circ \rho(x, g, 1)(f) &= (-xy^2 - xz^2 - y^3 - z^2y) + (xy^2 + xz^2 - y^2x - z^2x) \\ &= -y^3 - z^2y - y^2x - z^2x, \end{aligned}$$

which is the same irreducible form of f from Example 4.2.7 and, by Theorem 4.2.13, the same as $\mathrm{nf}_{\langle S \rangle}(f)$. ▲

To actually construct a GS basis, we have to address the culprit responsible for the non-uniqueness of irreducible forms: *ambiguities*. They are defined below.

Definition 4.2.15. Let $S \subset \Bbbk\langle Y \rangle$ and let $g, h \in S$ such that $w = \widehat{g}u = v\widehat{h}$ for some $u, v, w \in \operatorname{Mon}(Y)$ with $\deg(\widehat{g}) + \deg(\widehat{h}) > \deg(w)$. Then $(g,h)_w = gu - vh$ is called the *(overlap) ambiguity* of g and h with respect to w. ▲

Example 4.2.16. In the setting of our running example, $x\widehat{g} = x^3 = \widehat{g}x$ for $g = x^2 + y^2 + z^2$, so $S = \{g\}$ has the ambiguity

$$(g,g)_{x^3} = xg - gx = xy^2 + xz^2 - y^2x - z^2x. \quad ▲$$

An important step in constructing a GS basis will be expanding a set of generators S to include elements which, in some sense, eliminate all ambiguities of S. The next definition highlights the property of ambiguities we are after.

Definition 4.2.17. Let $S \subset \Bbbk\langle Y \rangle$ and $g, h \in S$. An ambiguity $(g,h)_w$ with a terminal sequence ρ satisfying $\rho((g,h)_w) = 0$ is *resolvable* with respect to S. ▲

If an ambiguity $(g,h)_w$ is not resolvable with respect to S, then we resolve it by extending the set S to include the ambiguity; that is, we define a new set $S' = S \cup \{(g,h)_w\}$. Now, with respect to S', we have the reduction $\rho(1,(g,h)_w,1)$ and $\rho(1,(g,h)_w,1)((g,h)_w) = 0$.

If we restrict our attention to sets S which are *self-reduced* (Definition 4.2.18), we have a characterization of GS bases that makes their construction feasible (Theorem 4.2.19).

Definition 4.2.18. A finite set $S \subset \Bbbk\langle Y\rangle$ is *self-reduced* if every $g \in S$ is irreducible with respect to $S \setminus \{g\}$. ▲

Theorem 4.2.19 (see [Bre14, Theorem 6.5])**.** *Let S be a finite monic self-reduced set of generators for an ideal I. Then S is a GS basis for I if and only if all ambiguities of S are resolvable.* ▲

So, by Theorem 4.2.19, we can extend a finite set of generators S of an ideal $I \subseteq \Bbbk\langle Y\rangle$ to a GS basis for I by alternating between two processes: one which extends S until it is self-reduced (Algorithm 4.2.20), and another which resolves all ambiguities of S (Algorithm 4.2.21).

In practice, we will use deglex as our monomial ordering on $\Bbbk\langle Y\rangle$.

Algorithm 4.2.20. Let $\prec = \prec_{\mathtt{dl}}$, and let S be a finite set of generators of an ideal $I \subseteq \Bbbk\langle Y\rangle$. Order S as $S = \{g_0, \ldots, g_n\}$ with $\widehat{g_0} \prec \cdots \prec \widehat{g_n}$. For $i = 1, \ldots, n$, let S_i be the union of S_{i-1} and an irreducible form of g_i with respect to $\{g_0, \ldots, g_{i-1}\}$. If $S \neq S_n$, then S is not self-reduced, and we repeat the same process, starting with S_n in place of S. ▲

Algorithm 4.2.21. Let S be a finite set of generators of an ideal $I \subseteq \Bbbk\langle Y\rangle$. Set $S_0 = S$. For $i \geq 0$,

- if S_i is not self-reduced, use Algorithm 4.2.20 to construct a self-reduced set of generators S_i' for I. Otherwise, set $S_i' = S_i$.

- For every ambiguity h of S_i' which is not resolvable, add an irreducible form of h to S_i'.

  Let S_{i+1} be the set resulting from all of these additions.

If this process terminates at some finite step n, then by Theorem 4.2.19, S_n is a GS basis for I. ▲

4.3 A GS basis of $\mathrm{sGor}(N)^!$

Let $N \geq 2$. In this section we construct a GS basis of

$$\mathrm{sGor}(N)^! = \frac{\Bbbk\langle (X_0)^*, \ldots, (X_N)^* \rangle}{\langle (X_0^*)^2 + \cdots + (X_N^*)^2 \rangle}.$$

To improve readability, we let $Y_i = X_i^*$ for $i = 0, \ldots, N$. We also introduce the following definition, which makes it easier to read the indices of monomials in Y_i.

Definition 4.3.1. Define the function

$$Y_* : \bigcup_{n \geq 1} \{0, \ldots, N\}^n \to \Bbbk\langle Y_0, \ldots, Y_N \rangle$$

by

$$Y_*(i_1, \ldots, i_n) = Y_{i_1} \cdots Y_{i_n}.$$

▲

We are now ready to prove the following proposition.

Proposition 4.3.2. *The set*

$$\mathcal{S}_N = \{Y_0^2 + \cdots + Y_N^2, \quad Y_0\,(Y_0^2 + \cdots + Y_N^2) - (Y_0^2 + \cdots + Y_N^2)\,Y_0\}$$

is a Gröbner-Shirshov basis for the ideal $\langle Y_0^2 + \cdots + Y_N^2\rangle$ of the free $\Bbbk$-algebra $\Bbbk\langle Y_0, \ldots, Y_N\rangle$ with respect to the monomial order $\prec_{\mathtt{dl}}$ induced by $y_0 \succ y_1 \succ \cdots \succ y_N$. ▲

Proof. Let $r = Y_0^2 + \cdots + Y_N^2$ and let $r' = Y_0 r - rY_0$, so $\widehat{r} = Y_0^2$ and $\widehat{r'} = Y_0 Y_1^2$. By Theorem 4.2.19, it is enough to show that $\mathcal{S}_N$ is self-reduced and all ambiguities of $\mathcal{S}_N$ are resolvable.

To see that $\mathcal{S}_N$ is self-reduced, note that the elements of $\operatorname{supp}(r)$ are all of degree 2 and $\widehat{r'}$ is of degree 3, so no element of $\operatorname{supp}(r)$ is divisible by $\widehat{r'}$; and the elements of $\operatorname{supp}(r')$ have only one factor of Y_0, so no element of $\operatorname{supp}(r')$ is divisible by $\widehat{r}$.

The set $\mathcal{S}_N$ has two ambiguities: $(r, r)_{Y_0^3}$, since $\widehat{r}\,Y_0 = Y_0^3 = Y_0\widehat{r}$; and $(r, r')_{Y_0^2 Y_1^2}$, since $\widehat{r}\,Y_1^2 = Y_0^2 Y_1^2 = Y_0\widehat{r'}$. The ambiguity $(r, r)_{Y_0^3}$ is resolvable because

$$(r, r)_{Y_0^3} \;=\; (Y_1^2 + \cdots + Y_N^2)\,Y_0 - Y_0\,(Y_1^2 + \cdots + Y_N^2),$$

which is $-r'$, so

$$\rho(1, r', 1)((r, r)_{Y_0^3}) = (r, r)_{Y_0^3} + r' = 0.$$

We will show that the ambiguity $(r, r')_{Y_0^2 Y_1^2}$ is also resolvable by using a sequence of reductions with respect to $\mathcal{S}_N$ that will reduce it to 0. Set $h_0 = (r, r')_{Y_0^2 Y_1^2}$. Then

$$
\begin{aligned}
h_0 &= r \cdot Y_1^2 - Y_0 \cdot r' \\
&= (Y_*(0,0,1,1) + \cdots + Y_*(N,N,1,1)) - \\
&- ((Y_*(0,0,1,1) + \cdots + Y_*(0,0,N,N)) - (Y_*(0,1,1,0) + \cdots + Y_*(0,N,N,0)) \\
&= (Y_*(1,1,1,1) + \cdots + Y_*(N,N,1,1)) - (Y_*(0,0,2,2) + \cdots + Y_*(0,0,N,N)) + \\
&+ (Y_*(0,1,1,0) + \cdots + Y_*(0,N,N,0)).
\end{aligned}
$$

For $i = 2, \ldots, N$, we eliminate the monomial $\widehat{r} \cdot Y_i^2 = Y_*(0,0,i,i)$ from h_0 by applying $\rho(1, r, Y_i^2)$ to $\rho(1, r, Y_{i-1}^2) \circ \cdots \circ \rho(1, r, Y_2^2)(h_0)$. This reduction is well-defined since for all $j > i$ none of the $Y_*(0,0,j,j)$ are in $\operatorname{supp}(r \cdot Y_i^2)$, so they remain unchanged after applying $\rho(1, r, Y_i^2)$. Each reduction $\rho(1, r, Y_i^2)$ also introduces terms $Y_*(1,1,i,i) + \cdots + Y_*(N,N,i,i)$. So, in all, we have

$$
\begin{aligned}
&\rho(1, r, Y_N^2) \circ \cdots \circ \rho(1, r, Y_2^2)(h_0) = \\
&= (Y_*(1,1,1,1) + \cdots + Y_*(N,N,1,1)) \ + \ (Y_*(0,1,1,0) + \cdots + Y_*(0,N,N,0)) + \\
&+ \ (Y_*(1,1,2,2) + \cdots + Y_*(N,N,2,2)) \ + \cdots + \ (Y_*(1,1,N,N) + \cdots + Y_*(N,N,N,N)).
\end{aligned}
$$

Let h_1 be the polynomial above. Since $\widehat{r'} \cdot Y_0 = Y_*(0,1,1,0) \in \operatorname{supp}(h_1)$, we reduce h_1 further by applying $\rho(1, r', Y_0)$ to obtain

$$\rho(1, r', Y_0)(h_1) =$$

$$= h_1 - r' \cdot Y_0 =$$

$$= (Y_*(1,1,1,1) + \cdots + Y_*(N,N,1,1)) \ + \ (Y_*(0,1,1,0) + \cdots + Y_*(0,N,N,0)) \ +$$

$$+ \ (Y_*(1,1,2,2) + \cdots + Y_*(N,N,2,2)) \ + \cdots + \ (Y_*(1,1,N,N) + \cdots + Y_*(N,N,N,N)) \ -$$

$$- \ ((Y_*(0,1,1,0) + \cdots + Y_*(0,N,N,0)) - (Y_*(1,1,0,0) + \cdots + Y_*(N,N,0,0)))$$

$$= (Y_*(1,1,0,0) + \cdots + Y_*(N,N,0,0)) + (Y_*(1,1,1,1) + \cdots + Y_*(N,N,1,1)) \ +$$

$$+ (Y_*(1,1,2,2) + \cdots + Y_*(N,N,2,2)) + \cdots + (Y_*(1,1,N,N) + \cdots + Y_*(N,N,N,N))$$

Factoring the expression above yields

$$\rho(1, r', Y_0)(h_1) =$$

$$= (Y_*(1,1) + \cdots + Y_*(N,N)) \cdot Y_*(0,0) \ + \ (Y_*(1,1) + \cdots + Y_*(N,N)) \cdot Y_*(1,1) \ +$$

$$+ (Y_*(1,1) + \cdots + Y_*(N,N)) \cdot Y_*(2,2) + \cdots + (Y_*(1,1) + \cdots + Y_*(N,N)) \cdot Y_*(N,N)$$

$$= (Y_*(1,1) + \cdots + Y_*(N,N))(Y_*(0,0) + \cdots + Y_*(N,N))$$

$$= (Y_1^2 + \cdots + Y_N^2) \ r.$$

Then, for example, the sequence $\rho(Y_N^2, r, 1) \circ \cdots \circ \rho(Y_1^2, r, 1)$ applied to $\rho(1, r', Y_0)(h_1)$ results in 0, so $(r, r')_{Y_0^2 Y_1^2}$ is resolvable, completing the proof. $\square$

The next proposition follows immediately from Corollary 4.2.5 and Proposition 4.3.2.

Proposition 4.3.3. *The* $\Bbbk$*-vector space* $\mathrm{sGor}(N)^!$ *has a* $\Bbbk$*-basis consisting of all monomials in* $\Bbbk\langle Y_0, \ldots, Y_N \rangle$ *which are not divisible by* Y_0^2 *nor by* $Y_0 Y_1^2$*.* $\blacktriangle$

Definition 4.3.4. Let $\mathcal{B}(\mathrm{sGor}(N)^!)$ be the $\Bbbk$-basis from Proposition 4.3.3. In particular, for $n \geq 0$, let $\mathcal{B}((\mathrm{sGor}(N)^!)^n)$ be the $\Bbbk$-basis of $(\mathrm{sGor}(N)^!)^n$ consisting of all monomials of degree n which are not divisible by Y_0^2 nor by $Y_0 Y_1^2$. $\blacktriangle$

Chapter 5

Computation of $\mathrm{HH}^{\bullet}_{(0)}(A)$

Having established all the necessary theory and results in Chapters 2, 3, and 4, we are now set to begin our computation of the Hochschild cohomology of the short Gorenstein rings

$$\mathrm{sGor}(N) = \frac{\Bbbk[X_0, \ldots, X_N]}{(X_i X_j,\ X_i^2 - X_j^2 \mid i, j = 0, \ldots, N, i \neq j)}$$

for $N \geq 2$.

Convention 5.0.1. For the rest of this thesis, fix some $N \geq 2$, let $A = \mathrm{sGor}(N)$ and let E be the quadratic dual of A; that is, let

$$E = \mathrm{sGor}(N)^! = \frac{\Bbbk\langle Y_0, \ldots, Y_N\rangle}{\langle Y_0^2 + \cdots + Y_N^2\rangle}, \text{ where } Y_i = X_i^*.$$

See Proposition 3.2.3 for the computation of $\mathrm{sGor}(N)^!$. Let x_i and y_i be the images of X_i and Y_i in $\mathrm{sGor}(N)$ and $\mathrm{sGor}(N)^!$, respectively. ▲

In Chapter 2 we found that since A is a Koszul algebra, the nth Hochschild cohomology group $\mathrm{HH}^n(A)$ is the nth homology group of the complex $(A \otimes A^!, \partial)$, where, for any element

$a \otimes e \in A \otimes (A^!)^n$,

$$\partial(a \otimes e) = \sum_{j=0}^{N} ax_j \otimes [y_j, e] = \sum_{j=0}^{N} ax_j \otimes y_j e - (-1)^n ax_j \otimes e y_j.$$

(see Proposition-Definition 2.4.6).

In Chapter 3, we found that $\mathrm{HH}^n(A)$ decomposes as

$$\mathrm{HH}^n(A) = \mathrm{HH}^n_{(0)}(A) \oplus \mathrm{HH}^n_{(1)}(A) \oplus \mathrm{HH}^n_{(2)}(A)$$

(Proposition 3.3.5), where $\mathrm{HH}^n_{(0)}(A)$, $\mathrm{HH}^n_{(1)}(A)$, and $\mathrm{HH}^n_{(2)}(A)$ are the homologies in degree n of the strands $\mathcal{C}_{(n+1)}$, $\mathcal{C}_{(n)}$, and $\mathcal{C}_{(n-1)}$, respectively (see Definition 3.3.3 and Diagram 3.3.1).

And in Chapter 4 we found that the set $\mathcal{B}(E)$ consisting of the monomials in $y_0, \ldots, y_N$ which are not divisible by y_0^2 or $y_0 y_1^2$ is a $\Bbbk$-basis of E (Proposition 4.3.3).

In this chapter we compute $\mathrm{HH}^n_{(0)}(A)$, the homology at $A_0 \otimes E^n$ in the strand

$$\mathcal{C}_{(n+1)} = \cdots \longrightarrow 0 \longrightarrow A_0 \otimes E^n \xrightarrow{\delta^n_{(n+1)}} A_1 \otimes E^{n+1} \xrightarrow{\delta^{n+1}_{(n+1)}} A_2 \otimes E^{n+2} \xrightarrow{\delta^{n+2}_{(n+1)}} \cdots,$$

where

$$\delta^n_{(n+1)} = \partial^n \mid_{A_0 \otimes E^n}, \qquad \delta^{n+1}_{(n+1)} = \partial^{n+1} \mid_{A_1 \otimes E^{n+1}}.$$

We show that for $n \geq 1$,

$$\mathrm{HH}^n_{(0)}(A) = \ker \partial^n_{(0)} = 0$$

by exhibiting a splitting map $\gamma_{n+1} : A_1 \otimes E^{n+1} \to A_0 \otimes E^n$ (Theorem 5.0.6), implying that $\delta^n_{(n+1)}$ is injective.

We begin by fixing $\Bbbk$-bases of $A_0 \otimes E^n$ and $A_1 \otimes E^n$.

Definition 5.0.2. Let $n \geq 0$. Recall from Proposition 3.1.2 that A has a $\Bbbk$-vector space decomposition

$$A = \Bbbk \oplus \Bbbk\{x_0, \dots, x_N\} \oplus \Bbbk\{s\},$$

where the element s of degree 2 generates the socle of A. And recall from Definition 4.3.4 that $\mathcal{B}(E^n)$ is the $\Bbbk$-basis of E^n consisting of the monomials in $y_0, \dots, y_N$ of degree n which are not divisible by y_0^2 or $y_0 y_1^2$.

Let $\mathcal{B}(A_0 \otimes E^n)$ be the $\Bbbk$-basis of $A_0 \otimes E^n$ defined by

$$\mathcal{B}(A_0 \otimes E^n) = \{1 \otimes u \mid u \in \mathcal{B}(E^n)\}$$

and let $\mathcal{B}(A_1 \otimes E^n)$ be the $\Bbbk$-basis of $A_1 \otimes E^n$ defined by

$$\mathcal{B}(A_1 \otimes E^n) = \{x_i \otimes u \mid u \in \mathcal{B}(E^n),\ i \in \{0, \dots, N\}\}. \qquad \blacktriangle$$

We take some time now to discuss the choices made in our definition of γ_{n+1} (Definition 5.0.3). First, γ_{n+1} splitting $\delta^n_{(n+1)}$ means γ_{n+1} satisfies $\gamma_{n+1}\delta^n_{(n+1)}(1 \otimes u) = 1 \otimes u$ for all

$1 \otimes u \in \mathcal{B}(A_0 \otimes E^n)$, where

$$\begin{aligned}\partial^n_{(0)}(1 \otimes u) &= x_0 \otimes y_0 u - (-1)^n x_0 \otimes u y_0 + \cdots + x_N \otimes y_N u - (-1)^n x_N \otimes u y_N \\ &= (x_0 \otimes y_0 u + \cdots + x_N \otimes y_N u) - (-1)^n (x_0 \otimes u y_0 + \cdots + x_N \otimes u y_N).\end{aligned}$$

So, to define γ_{n+1}, we can fix some index i, let $\gamma_{n+1}(x_j \otimes w) = 0$ for any $j \neq i$ and define $\gamma_{n+1}(x_i \otimes w)$ based on whether $w = y_i u$ or $w = u' y_i$ or both for some $u, u' \in \mathcal{B}(E)$. To deal with these cases individually, we define two maps

$$\gamma^L_{n+1}, \gamma^R_{n+1} : A_1 \otimes E^{n+1} \to A_0 \otimes E^n,$$

where γ^L_{n+1} is nonzero on $x_i \otimes y_i u$ and γ^R_{n+1} is nonzero on $x_i \otimes u' y_i$, and set

$$\gamma_{n+1} = \gamma^L_{n+1} + (-1)^{n+1} \gamma^R_{n+1}.$$

We choose to work with the index $i = 2$ in order to avoid complications that arise in the cases $i = 0$ and $i = 1$; namely, for $1 \otimes u \in \mathcal{B}(A_0 \otimes E^n)$, the terms $x_0 \otimes y_0 u$, $x_0 \otimes u y_0$, and $x_1 \otimes u y_1$ of $\partial^n_{(0)}(1 \otimes u)$ may not be elements of $\mathcal{B}(A_1 \otimes E^{n+1})$ since $y_0 u$ or $u y_0$ may be divisible by y_0^2 and $u y_1$ may be divisible by $y_0 y_1^2$.

There are several subcases to consider for both $\gamma_{n+1}(x_2 \otimes y_2 u)$ and $\gamma_{n+1}(x_2 \otimes u' y_2)$. All but one of these can be addressed by defining $\gamma_{n+1}(x_2 \otimes y_2 u) = 1 \otimes u$ or $\gamma_{n+1}(x_2 \otimes u' y_2) = 1 \otimes u'$. The one exception is the case $x_2 \otimes y_2 v y_2$ for some $v \in \mathcal{B}(E)$ since it is unclear whether this term in $\partial^n_{(0)}(1 \otimes u)$ comes from left or right multiplication by $x_2 \otimes y_2$. We circumvent this issue by considering instead the related elements $x_1 \otimes y_2 v y_2 y_1$.

Thus, we have arrived at the following definition.

Definition 5.0.3. Define

$$\gamma_2^R, \gamma_2^L : A_1 \otimes E^2 \to A_0 \otimes E^1$$

on $x_j \otimes w \in \mathcal{B}(A_1 \otimes E^2)$ by

$$\gamma_2^L(x_j \otimes w) = \begin{cases} \frac{1}{2}(1 \otimes y_2^{-1} w), & \text{if } j = 2,\ w = y_2 y_\ell \text{ with } \ell \neq 2; \qquad (a^L) \\ 0, & \text{otherwise;} \end{cases}$$

$$\gamma_2^R(x_j \otimes w) = \begin{cases} \frac{1}{2}(1 \otimes w y_2^{-1}), & \text{if } j = 2,\ w = y_k y_2 \text{ with } k \neq 2; \qquad (a^R) \\ 1 \otimes w y_1^{-1}, & \text{if } j = 1,\ w = y_2 y_1; \qquad (b^R) \\ 0, & \text{otherwise;} \end{cases}$$

and define

$$\gamma_2 : A_1 \otimes E^2 \to A_0 \otimes E^1$$

by

$$\gamma_2 = \gamma_2^L + \gamma_2^R.$$

For $n > 2$, define

$$\gamma_n^R, \gamma_n^L : A_1 \otimes E^n \to A_0 \otimes E^{n-1}$$

on $x_j \otimes w \in \mathcal{B}(A_1 \otimes E^n)$ by

$$\gamma_n^L(x_j \otimes w) = \begin{cases} 1 \otimes y_2^{-1}w, & \text{if } j = 2,\ w = y_2^2 v y_\ell \text{ for some } v \in \mathcal{B}(E),\ \ell \neq 2; & (c^L) \\ \frac{1}{2}(1 \otimes y_2^{-1}w), & \text{if } j = 2,\ w = y_2 y_k v y_\ell \text{ for some } v \in \mathcal{B}(E),\ k, \ell \neq 2; & (d^L) \\ 0, & \text{otherwise;} \end{cases}$$

$$\gamma_n^R(x_j \otimes w) = \begin{cases} 1 \otimes w y_2^{-1}, & \text{if } j = 2,\ w = y_k v y_2^2 \text{ for some } v \in \mathcal{B}(E),\ k \neq 2; & (c^R) \\ \frac{1}{2}(1 \otimes w y_2^{-1}), & \text{if } j = 2,\ w = y_k v y_\ell y_2 \text{ for some } v \in \mathcal{B}(E),\ k, \ell \neq 2; & (d^R) \\ 1 \otimes w y_1^{-1} & \text{if } j = 1,\ w = y_2 v y_2 y_1 \text{ for some } v \in \mathcal{B}(E); & (e^R) \\ 0, & \text{otherwise;} \end{cases}$$

and define

$$\gamma_n : A_1 \otimes E^n \to A_0 \otimes E^{n-1}$$

by

$$\gamma_n = \gamma_n^L + (-1)^n \gamma_n^R. \qquad \blacktriangle$$

By definition, γ_n^L and γ_n^R map most elements of $\mathcal{B}(A_1 \otimes E^n)$ to 0. To see this, let $n \geq 1$ and let $y_i u$, $vy_j \in \mathcal{B}(E^n)$ for some monomials u, v and some $i, j \in \{0, \ldots, N\}$. Then

- $\gamma_n^L(x_i \otimes y_i u) = 0$ if $i \neq 2$ and $\gamma_n^L(x_j \otimes vy_j) = 0$ if $j \neq 2$;
- $\gamma^R(x_i \otimes y_i u) = 0$ if $i \neq 2$;
- $\gamma_n^R(x_j \otimes vy_j) = 0$ if $j \neq 1, 2$;
- $\gamma_n^L(x_j \otimes vy_j) = 0$ if $j = 2$ and $\gamma_n^R(x_i \otimes y_i u) = 0$ if $i = 2$.

Thus, we have the following observation.

Observation 5.0.4. Let $n \geq 2$ and let $1 \otimes u \in \mathcal{B}(A_0 \otimes E^{n-1})$. Then

$$\gamma_n \partial_{(0)}^{n-1}(1 \otimes u)$$

$$= \gamma_n^L \big((x_0 \otimes y_0 u + \cdots + x_N \otimes y_N u) - (-1)^{n-1}(x_0 \otimes uy_0 + \cdots + x_N \otimes y_N u)\big) +$$

$$+ (-1)^n \gamma_n^R \big((x_0 \otimes y_0 u + \cdots + x_N \otimes y_N u) - (-1)^{n-1}(x_0 \otimes uy_0 + \cdots + x_N \otimes y_N u)\big)$$

$$= \gamma_n^L(x_2 \otimes y_2 u) + (-1)^n \gamma_n^R(-(-1)^{n-1}(x_1 \otimes uy_1 + x_2 \otimes uy_2))$$

$$= \gamma_n^L(x_2 \otimes y_2 u) + \gamma_n^R(x_1 \otimes uy_1 + x_2 \otimes uy_2). \qquad \blacktriangle$$

For $n \geq 3$, $\gamma_n^R(x_1 \otimes uy_1) = 0$ for most $u \in \mathcal{B}(E^{n-1})$. However, we did not include this in the above discussion because there is a catch: uy_1 may not be in $\mathcal{B}(E^{n-1})$ since right multiplication by y_1 might create a factor of $y_0 y_1^2$ in uy_1. Nevertheless, it turns out that

Proposition 5.0.5. *For $n \geq 3$, $\gamma_n^R(x_1 \otimes uy_1) = 0$ for all $u \in \mathcal{B}(E^{n-1})$ of the form $u = vy_\ell$*

with $\ell \neq 2$. ▲

Proof. There are two cases to consider:

- $u = vy_k y_\ell$ for some $v \in \mathcal{B}(E^{n-3})$, $(k,\ell) \neq (0,1)$, $\ell \neq 2$; and
- $u = vy_0y_1$ for some $v \in \mathcal{B}(E^{n-3})$.

In the first case, the conditions on k and ℓ ensure that right multiplication by y_1 does not create a factor of $y_0y_1^2$ in uy_1. Thus, $uy_1 = vy_ky_\ell y_1 \in \mathcal{B}(E^n)$. Since $\ell \neq 2$, $\gamma_n^R(x_1 \otimes uy_1) = 0$ by Definition 5.0.3.

In the second case, right multiplication by y_1 does create a factor of $y_0y_1^2$, which we rewrite in terms of $\mathcal{B}(E)$; that is,

$$\begin{aligned} uy_1 &= vy_0y_1y_1 \\ &= v\left(-\sum_{j=2}^{N} y_0y_jy_j + \sum_{j=1}^{N} y_jy_jy_0\right) \\ &= -\sum_{j=2}^{N} vy_0y_jy_j + \sum_{j=1}^{N} vy_jy_jy_0. \end{aligned} \tag{5.0.0.1}$$

Note that v is not of the form $v'y_0$ for some $v' \in \mathcal{B}(E)$ since otherwise $u = vy_0y_1 = v'y_0^2y_1$, implying u is divisible by y_0^2 and contradicting $u \in \mathcal{B}(E^{n-1})$. Thus, no term of 5.0.0.1 is divisible by y_0^2 or $y_0y_1^2$, so all terms of 5.0.0.1 are in $\mathcal{B}(E^n)$. Hence, every term of

$$x_1 \otimes uy_1 = -\sum_{j=2}^{N} x_1 \otimes vy_0y_jy_j + \sum_{j=1}^{N} x_1 \otimes vy_jy_jy_0$$

is in $\mathcal{B}(A_1 \otimes E^n)$, and by Definition 5.0.3, every one of these terms maps to 0 under γ_n^R. Therefore, $\gamma_n^R(x_1 \otimes uy_1) = 0$. □

After these simplifications of γ_n, we are now ready to prove that the map is a splitting of $\delta_{(n)}^{n-1}$.

Theorem 5.0.6. *For $n \geq 2$, the map γ_n splits $\delta_{(n)}^{n-1}$.* ▲

Proof. We want to show that $\gamma_n \partial_{(0)}^{n-1}(1 \otimes u) = 1 \otimes u$ for all $1 \otimes u \in \mathcal{B}(A_0 \otimes E^{n-1})$. By Observation 5.0.4, it is enough to show that

$$\gamma_n^L(x_2 \otimes y_2 u) + \gamma_n^R(x_1 \otimes uy_1 + x_2 \otimes uy_2) = 1 \otimes u \tag{5.0.0.2}$$

for all $1 \otimes u \in \mathcal{B}(A_0 \otimes E^{n-1})$. We prove that 5.0.0.2 holds first in the case $n = 2$ and then in the case $n \geq 3$. For the rest of our discussion, let $1 \otimes u \in \mathcal{B}(A_0 \otimes E^{n-1})$.

For $n = 2$, $u \in \mathcal{B}(E^1) = \{y_0, \ldots, y_N\}$. If $u = y_2$, then by Definition 5.0.3,

$$\gamma_2^L(x_2 \otimes y_2 u) = \gamma_2^L(x_2 \otimes y_2 y_2) = 0,$$

$$\gamma_2^R(x_2 \otimes uy_2) = \gamma_2^R(x_2 \otimes y_2 y_2) = 0,$$

$$\gamma_2^R(x_1 \otimes uy_1) = \gamma_2^R(x_1 \otimes y_2 y_1) = 1 \otimes y_2 = 1 \otimes u,$$

so 5.0.0.2 holds. If $u = y_m$ for some $m \neq 2$, then by Definition 5.0.3, $\gamma_n^R(x_1 \otimes uy_1) = \gamma_n^R(x_1 \otimes y_m y_1) = 0$ and

$$\gamma_n^L(x_2 \otimes y_2 u) = \gamma_n^L(x_2 \otimes y_2 y_m) = \tfrac{1}{2}(1 \otimes y_m),$$

$$\gamma_n^R(x_2 \otimes uy_2) = \gamma_n^L(x_2 \otimes y_m y_2) = \tfrac{1}{2}(1 \otimes y_m),$$

by (a^L) and (a^R), respectively. Thus, 5.0.0.2 holds.

For $n \geq 3$, there are four cases to consider:

(5.0.1) $u = y_2 v y_2$ for some $v \in \mathcal{B}(E^{n-3})$;

(5.0.2) $u = y_k v y_2$ for some $v \in \mathcal{B}(E^{n-3})$, $k \neq 2$;

(5.0.3) $u = y_2 v y_\ell$ for some $v \in \mathcal{B}(E^{n-3})$, $\ell \neq 2$;

(5.0.4) $u = y_k v y_\ell$ for some $v \in \mathcal{B}(E^{n-3})$, $k, \ell \neq 2$.

By Proposition 5.0.5, $\gamma_n^R(x_1 \otimes u y_1) = 0$ in Cases (5.0.2), (5.0.3), and (5.0.4). And in Case (5.0.1), $\gamma_n^R(x_1 \otimes u y_1) = \gamma_n^R(x_1 \otimes y_2 v_2 y_2 y_1) = 1 \otimes u$ by (e^R) of Definition 5.0.3.

The values of $\gamma_n^L(x_2 \otimes y_2 u)$ and $\gamma_n^R(x_2 \otimes u y_2)$ for Cases (5.0.1) through (5.0.4) are summarized in the tables below and all follow from Definition 5.0.3.

u	$x_2 \otimes y_2 u$	$\gamma_n^L(x_2 \otimes y_2 u)$
(5.0.1)	$x_2 \otimes y_2^2 v_2 y_2$	0
(5.0.2)	$x_2 \otimes y_2 y_k v y_2$	0
(5.0.3)	$x_2 \otimes y_2^2 v y_\ell$	$1 \otimes y_2 v y_\ell = 1 \otimes u$ by (c^L)
(5.0.4)	$x_2 \otimes y_2 y_k v y_\ell$	$\frac{1}{2}(1 \otimes y_k v y_\ell) = \frac{1}{2}(1 \otimes u)$ by (d^L)

u	$x_2 \otimes u y_2$	$\gamma_n^R(x_2 \otimes u y_2)$
(5.0.1)	$x_2 \otimes y_2 v y_2^2$	0
(5.0.2)	$x_2 \otimes y_k v y_2^2$	$1 \otimes y_k v y_2 = 1 \otimes u$ by (c^R)
(5.0.3)	$x_2 \otimes y_2 v y_\ell y_2$	0
(5.0.4)	$x_2 \otimes y_k v y_\ell y_2$	$\frac{1}{2}(1 \otimes y_k v y_\ell) = \frac{1}{2}(1 \otimes u)$ by (d^R)

Thus, the respective values of $\gamma_n^L(x_2 \otimes y_2 u)$, $\gamma_n^R(x_1 \otimes u y_1)$, and $\gamma_n^R(x_2 \otimes u y_2)$ in all cases are

u	$\gamma_n^L(x_2 \otimes y_2 u)$	$\gamma_n^R(x_1 \otimes u y_1)$	$\gamma_n^R(x_2 \otimes u y_2)$
(5.0.1)	0	$1 \otimes u$	0
(5.0.2)	0	0	$1 \otimes u$
(5.0.3)	$1 \otimes u$	0	0
(5.0.4)	$\frac{1}{2}(1 \otimes u)$	0	$\frac{1}{2}(1 \otimes u)$

Taking the sum along each row of the above table, we see that 5.0.0.2 holds in all cases. $\square$

As a corollary to Theorem 5.0.6, we have

Corollary 5.0.7. *For $n \geq 1$, the map $\delta^n_{(n+1)}$ is injective and therefore $\mathrm{HH}^n_{(0)}(A) = 0$.* ▲

Chapter 6

The Exponential Growth of the Hochschild Cohomology of Short Gorenstein Rings

For the rest of this chapter we fix some $N \geq 3$ (avoiding the case $N = 2$ — see Remark 6.2.10). In this chapter we prove the main result of our thesis, Theorem 6.2.9, which states that the $\Bbbk$-dimensions of the even Hochschild cohomology groups $\mathrm{HH}^n(A)$ grow exponentially with n. The proof proceeds as follows.

Recall from Proposition 3.3.5 that $\mathrm{HH}^n(A)$ decomposes as

$$\mathrm{HH}^n(A) = \mathrm{HH}^n_{(0)}(A) \oplus \mathrm{HH}^n_{(1)}(A) \oplus \mathrm{HH}^n_{(2)}(A).$$

In particular, $\mathrm{HH}^n_{(1)}(A) = H^n(\mathcal{C}_{(n)})$, where the complex $\mathcal{C}_{(n)}$ in degree n is

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{C}^{n-1}_{(n)} \arrow[r, "\delta^{n-1}_{(n)}"] \arrow[d, equal] & \mathcal{C}^{n}_{(n)} \arrow[r, "\delta^{n}_{(n)}"] \arrow[d, equal] & \mathcal{C}^{n+1}_{(n)} \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & A_0 \otimes E^{n-1} \arrow[r, "\partial^{n-1}|_{A_0 \otimes E^{n-1}}"] & A_1 \otimes E^{n} \arrow[r, "\partial^{n}|_{A_1 \otimes E^{n}}"] & A_2 \otimes E^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

(see Section 3.3), so

$$\mathrm{HH}^n_{(1)}(A) = \frac{\ker \delta^n_{(n)}}{\operatorname{im} \delta^{n-1}_{(n)}}.$$

We show in Theorem 6.2.2 that for n odd,

$$\dim_{\Bbbk} \mathrm{HH}^n_{(1)}(A) \;\geq \sum_{\substack{c\,|\,d\,|\,(n+1),\\ d \text{ even}}} \frac{\mu(c)(N-1)^{d/c}}{d}, \tag{6.0.0.1}$$

where μ is the Möbius function (see Definition 6.2.1); and that for n even,

$$\dim_{\Bbbk} \mathrm{HH}^n_{(1)}(A) \;\geq \sum_{d\,|\,(n+1)} \frac{\varphi(d)(N-1)^{(n+1)/d}}{n+1}, \tag{6.0.0.2}$$

where φ is Euler's totient function (see Definition 6.2.1). Theorem 6.2.9 then follows as a corollary of this result.

Convention 6.0.1. For the rest of this chapter,

- let T be the tensor algebra on $\Bbbk\{Y_0, \ldots, Y_N\}$;
- let $\mathrm{Mon} \subset T$ be the set of monomials of T;
- for $n \geq 0$, let $\mathrm{Mon}^n \subset \mathrm{Mon}$ be the set of monomials of T of degree n; and
- let $\prec$ be the degree-lexicographic order on Mon. ▲

Proving that 6.0.0.1 and 6.0.0.2 hold relies on the $\mathbb{Z}_{n+1} = \langle \sigma_{n+1} \rangle$ action on T^{n+1} defined on monomials $Y_{i_0} \cdots Y_{i_n} \in T^{n+1}$ by $\sigma_{n+1}(Y_{i_0} \cdots Y_{i_n}) = Y_{i_1} \cdots Y_{i_n} Y_{i_0}$. We begin in Section 6.1 by lifting $\delta^n_{(n)}$ to a map $\widetilde{\partial}^{n+1} : T^{n+1} \to T^{n+1}$ as in the diagram

$$\begin{tikzcd}
T^{n+1} \arrow[r, "\widetilde{\partial}^{n+1}"] \arrow[d, "\pi^{n+1}_{(1)}"] & T^{n+1} \arrow[d, "\pi^{n+1}_{(2)}"] \\
A_1 \otimes E^n \arrow[r, "\delta^n_{(n)}"] & A_2 \otimes E^{n+1}
\end{tikzcd}$$

where the surjections $\pi_{(1)}^{n+1}$ and $\pi_{(2)}^{n+1}$ are defined in Section 6.1. We show that the elements

$$\lambda^{n+1}(\mathcal{O}) = \begin{cases} \displaystyle\sum_{i=0}^{|\mathcal{O}|-1} \sigma_{n+1}^{i}(\min_{\prec}\mathcal{O}), & n+1 \text{ odd}, \\ \\ \displaystyle\sum_{i=0}^{|\mathcal{O}|-1} (-1)^{i}\sigma_{n+1}^{i}(\min_{\prec}\mathcal{O}), & n+1 \text{ even}. \end{cases} \qquad \text{for all orbits } \mathcal{O} \subset \mathrm{Mon}^{n+1}$$

form a $\Bbbk$-basis of $\ker \widetilde{\partial}^{n+1}$. Furthermore, in Section 6.2 we show that a subset L^{n+1} of this basis generates a subspace $\Bbbk\, L^{n+1}$ which maps injectively via $\pi_{(1)}^{n+1}$ into $\mathrm{HH}_{(1)}^{n}(A)$, implying that

$$\dim_{\Bbbk} \mathrm{HH}_{(1)}^{n}(A) \;\geq\; |L^{n+1}|.$$

We complete the proof by showing that for $n \geq 1$, $|L^{n}|$ is equal to the right-hand side of 6.0.0.1 when n is odd and the right-hand side of 6.0.0.2 when n is even.

6.1 The kernel in the free setting

We begin this section by defining the lift $\widetilde{\partial}^{n+1} : T^{n+1} \to T^{n+1}$ from the introduction of this chapter and showing that the associated diagram commutes. We will need the following proposition, which gives us a much simpler description of the differential ∂^{n} when restricted to $A_1 \otimes E^{n}$.

Recall from Proposition 3.1.2 that $s \in A$ is the element in the decomposition

$A = \Bbbk \oplus \Bbbk\{x_0, \ldots, x_N\} \oplus \Bbbk\{s\}$ and that $y_j = x_j^* \in E$ for $j = 0, \ldots, N$. And recall from Proposition-Definition 2.4.6 that for $a \otimes e \in A \otimes E^n$,

$$\partial^n(a \otimes e) = \sum_{j=0}^{N} a x_j \otimes [y_j, e].$$

Here the bracket $[y_j, e]$ is the graded Lie bracket, given by

$$[y_j, e] = y_j e - (-1)^n e y_j.$$

Proposition 6.1.1. *Let $n \geq 0$. For $a \otimes e \in A_1 \otimes E^n$,*

$$\partial^n(a \otimes e) = s \otimes [a^*, e]. \qquad \blacktriangle$$

Proof. Since both sides of the desired equality are $\Bbbk$-linear in a, it is enough to show that for $i = 0, \ldots, N$, $\partial^n(x_i \otimes e) = s \otimes [y_i, e]$. Note that $x_j x_i = 0 \in A$ whenever $j \neq i$, so

$$\partial^n(x_i \otimes e) = \sum_{j=0}^{N} x_j x_i \otimes [y_j, e] = x_i^2 \otimes [y_i, e] = s \otimes [y_i, e],$$

completing the proof. $\square$

We now use Proposition 6.1.1 to define the lift $\widetilde{\partial}^{n+1}$ and the maps $\pi_{(1)}^{n+1}$ and $\pi_{(2)}^{n+1}$ from the diagram in the introduction of this chapter.

Convention 6.1.2. In order to simplify notation, for the rest of this section we will change the notation used in the introduction and instead work with T^n, $\widetilde{\partial}^n$, $\pi^n_{(1)}$, and $\pi^n_{(2)}$ for $n \geq 1$. ▲

Definition 6.1.3. Let $n \geq 1$. Define the maps $\widetilde{\partial}^n$, $\pi^n_{(1)}$, and $\pi^n_{(2)}$ in the diagram

$$\begin{tikzcd} T^n \arrow[r, "\widetilde{\partial}^n"] \arrow[d, "\pi^n_{(1)}"] & T^n \arrow[d, "\pi^n_{(2)}"] \\ A_1 \otimes E^{n-1} \arrow[r, "\delta^{n-1}_{(n-1)}"] & A_2 \otimes E^n \end{tikzcd} \tag{6.1.0.1}$$

as follows:

- For $u \in \mathrm{Mon}^n$, write $u = Y_i v$ for some $i \in \{0, \ldots, N\}$ and $v \in \mathrm{Mon}^{n-1}$ and define

$$\widetilde{\partial}^n(u) = [Y_i, v] = Y_i v - (-1)^{n-1} v Y_i.$$

- Let $\pi^n_{(1)}$ be the composition of the isomorphism $T^n \to A_1 \otimes T^{n-1}$ given by $Y_i u \leftrightarrow x_i \otimes u$ for $u \in \mathrm{Mon}^{n-1}$ and the map $A_1 \otimes T^{n-1} \to A_1 \otimes E^{n-1}$ induced by the projection $T^{n-1} \to T^{n-1}/\langle r \rangle^{n-1} = E^{n-1}$.
- Let $\pi^n_{(2)}$ be the composition of the isomorphism $T^n \to A_2 \otimes T^n$ given by $v \leftrightarrow s \otimes v$ for $v \in \mathrm{Mon}^n$ and the map $A_2 \otimes T^n \to A_2 \otimes E^n$ induced by the projection $T^n \to E^n$. ▲

With the definitions above, Diagram 6.1.0.1 commutes, which we verify below.

Proposition 6.1.4. *For $n \geq 1$, Diagram 6.1.0.1 commutes.* ▲

Proof. Since all of the maps involved are $\Bbbk$-linear, it is enough to show that the diagram is commutative on monomials. Let $u \in \mathrm{Mon}^n$, so $u = Y_i v$ for some $i \in \{0, \ldots, N\}$ and $v \in \mathrm{Mon}^{n-1}$. Then

$$\begin{aligned}
\delta^{n-1}_{(n-1)} \pi^n_{(1)}(Y_i v) &= \delta^{n-1}_{(n-1)}(x_i \otimes \overline{v}) \\
&= s \otimes [y_i, \overline{v}] && \text{(by Proposition 6.1.1)} \\
&= s \otimes (y_i \overline{v} - (-1)^{n-1} \overline{v} y_i) \\
&= \pi^n_{(2)}(Y_i v - (-1)^{n-1} v Y_i) \\
&= \pi^n_{(2)} \widetilde{\partial}^n (Y_i v),
\end{aligned}$$

so the diagram commutes. □

Our task now is to construct the $\Bbbk$-basis of $\ker \widetilde{\partial}^n$ described in the introduction. First we recall the definition of the cyclic group $\mathbb{Z}_n$ action on T^n and then show that $\widetilde{\partial}^n$ can in fact be written in terms of this action.

Definition 6.1.5. For $n \geq 1$, let σ_n be a generator of $\mathbb{Z}_n$ and let σ_n act on $Y_{i_1} \cdots Y_{i_n} \in \mathrm{Mon}^n$ by $\sigma_n \cdot Y_{i_1} \cdots Y_{i_n} = Y_{i_2} \cdots Y_{i_n} Y_{i_1}$. Extending this linearly results in an action of $\mathbb{Z}_n$ on T^n. We write σ in place of σ_n whenever the degree n is clear from context.

Let Orb^n be the set of all orbits of this action in Mon^n and let $\mathrm{Orb} = \bigcup_{n \geq 1} \mathrm{Orb}^n$. ▲

Note that for any monomial $u = Y_{i_1} \cdots Y_{i_n} \in \mathrm{Mon}^n$, we can rewrite $\widetilde{\partial}^n(u)$ as

$$\widetilde{\partial}^n(u) = [Y_{i_1}, Y_{i_2} \cdots Y_{i_n}] = Y_{i_1} \cdots Y_{i_n} - (-1)^{n-1} Y_{i_2} \cdots Y_{i_n} Y_{i_1} = u - (-1)^{n-1}\sigma_n u,$$

so we have the following observation.

Observation 6.1.6. For $n \geq 1$, we have

$$\widetilde{\partial}^n = \mathrm{id} - (-1)^{n-1}\sigma_n = \mathrm{id} + (-1)^n \sigma_n. \qquad \blacktriangle$$

Reinterpreting $\widetilde{\partial}^n$ in terms of the cyclic group action makes clearer the form of elements in $\ker \widetilde{\partial}^n$, as we show in the next example.

Recall from Definition 4.3.1 the function $Y_* : \bigcup_{n\geq 0}\{0, \dots, N\}^n \to T$ defined on an element $(i_1, \dots, i_n) \in \{0, \dots, N\}^n$ by $Y_*(i_1, \dots, i_n) = Y_{i_1} \cdots Y_{i_n}$. We introduce this function merely to improve readability in the rest of our discussion.

Example 6.1.7. Let

$$\mathcal{O}_1 = \{Y_*(1,1,0),\ Y_*(1,0,1),\ Y_*(0,1,1)\} \in \mathrm{Orb}^3,$$

$$\mathcal{O}_2 = \{Y_*(2,2,0,0),\ Y_*(2,0,0,2),\ Y_*(0,0,2,2),\ Y_*(0,2,2,0)\} \in \mathrm{Orb}^4,$$

and let

$$\lambda(\mathcal{O}_1) = Y_*(1,1,0) + Y_*(1,0,1) + Y_*(0,1,1),$$

$$\lambda(\mathcal{O}_2) = Y_*(2,2,0,0) - Y_*(2,0,0,2) + Y_*(0,0,2,2) - Y_*(0,2,2,0).$$

Then

$$\widetilde{\partial}^3(\lambda(\mathcal{O}_1)) =$$

$$= \lambda(\mathcal{O}_1) + (-1)^3\sigma\lambda(\mathcal{O}_1)$$

$$= (Y_*(1,1,0) + Y_*(1,0,1) + Y_*(0,1,1)) - (\sigma Y_*(1,1,0) + \sigma Y_*(1,0,1) + \sigma Y_*(0,1,1))$$

$$= (Y_*(0,1,1) + Y_*(1,1,0) + Y_*(1,0,1)) - (Y_*(1,0,1) + Y_*(0,1,1) + Y_*(1,1,0))$$

$$= 0,$$

and similarly

$$\widetilde{\partial}^4(\lambda(\mathcal{O}_2)) =$$

$$= \lambda(\mathcal{O}_2) + (-1)^4\sigma\lambda(\mathcal{O}_2)$$

$$= (Y_*(2,2,0,0) - Y_*(2,0,0,2) + Y_*(0,0,2,2) - Y_*(0,2,2,0)) +$$

$$+ (\sigma Y_*(2,2,0,0) - \sigma Y_*(2,0,0,2) + \sigma Y_*(0,0,2,2) - \sigma Y_*(0,2,2,0))$$

$$= (Y_*(2,2,0,0) - Y_*(2,0,0,2) + Y_*(0,0,2,2) - Y_*(0,2,2,0)) +$$

$$+ (Y_*(2,0,0,2) - Y_*(0,0,2,2) + Y_*(0,2,2,0) - Y_*(2,2,0,0))$$

$$= 0. \qquad \blacktriangle$$

We define elements of this form more generally below.

Definition 6.1.8. Define $\lambda^n : \mathrm{Orb}^n \to T^n$ by

$$\lambda^n(\mathcal{O}) = \begin{cases} \displaystyle\sum_{i=0}^{|\mathcal{O}|-1} \sigma^i(\min_{\prec}\mathcal{O}), & n \text{ odd}, \\ \\ \displaystyle\sum_{i=0}^{|\mathcal{O}|-1} (-1)^i \sigma^i(\min_{\prec}\mathcal{O}), & n \text{ even}. \end{cases}$$

We write λ in place of λ^n whenever the degree n is clear from context.

An image $\lambda^n(\mathcal{O})$ of $\mathcal{O} \in \mathrm{Orb}^n$ is called an *orbit sum*. $\blacktriangle$

We can now state the main result of this section.

Theorem 6.1.9. *The* $\Bbbk$*-space* $\ker \widetilde{\partial}^n$ *has basis*

$$\mathcal{B}(\ker \widetilde{\partial}^n) := \begin{cases} \{\lambda(\mathcal{O}) \mid \mathcal{O} \in \mathrm{Orb}^n\}, & n \textit{ odd}, \\ \\ \{\lambda(\mathcal{O}) \mid \mathcal{O} \in \mathrm{Orb}^n,\ |\mathcal{O}| \textit{ even}\}, & n \textit{ even}. \end{cases}$$

$\blacktriangle$

This theorem will follow from Propositions 6.1.11 and 6.1.12 below.

For the proof of Proposition 6.1.11 we need the following general observation about adding together polynomials of T^n with disjoint supports. For a definition of the support of a polynomial, see Definition 4.1.10.

Observation 6.1.10. Let $f, g \in T^n$ such that $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$. If $f + g = 0$, then $f = 0$ and $g = 0$; equivalently, if $f \neq 0$ or $g \neq 0$, then $f + g \neq 0$. Indeed, suppose without loss of generality $f \neq 0$, so then f contains a term of the form cu for some nonzero $c \in \Bbbk$ and $u \in \mathrm{Mon}^n$. Since $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$, the coefficient of u in $f + g$ is still $c \neq 0$, so $f + g \neq 0$. ▲

Proposition 6.1.11. *There exists a decomposition*

$$\ker \widetilde{\partial}^n = \bigoplus_{\mathcal{O} \in \mathrm{Orb}^n} \ker \widetilde{\partial}^n|_{\Bbbk \mathcal{O}}.$$

▲

Proof. For every $\mathcal{O} \in \mathrm{Orb}^n$,

$$\ker \widetilde{\partial}^n|_{\Bbbk \mathcal{O}} \;\subseteq\; \Bbbk \mathcal{O} \;\subseteq\; T^n = \bigoplus_{\mathcal{O}' \in \mathrm{Orb}^n} \Bbbk \mathcal{O}',$$

so

$$\sum_{\mathcal{O} \in \mathrm{Orb}^n} \ker \widetilde{\partial}^n|_{\Bbbk \mathcal{O}} \;=\; \bigoplus_{\mathcal{O} \in \mathrm{Orb}^n} \ker \widetilde{\partial}^n|_{\Bbbk \mathcal{O}}.$$

Also, $\ker \widetilde{\partial}^n|_{\Bbbk\mathcal{O}} \subseteq \ker \widetilde{\partial}^n$ for all $\mathcal{O} \in \mathrm{Orb}^n$, so

$$\bigoplus_{\mathcal{O}\in\mathrm{Orb}^n} \ker \widetilde{\partial}^n|_{\Bbbk\mathcal{O}} \;\subseteq\; \ker \widetilde{\partial}^n.$$

Let $f \in \ker \widetilde{\partial}^n$, let $f = \sum_{\mathcal{O}\in\mathrm{Orb}^n} f_{\mathcal{O}}$ be the unique decomposition of f with respect to $T^n = \bigoplus_{\mathcal{O}\in\mathrm{Orb}^n} \Bbbk\,\mathcal{O}$, let $\mathcal{O}' \in \mathrm{Orb}^n$, and let

$$f' = \sum_{\mathcal{O}\in\mathrm{Orb}^n - \{\mathcal{O}'\}} f_{\mathcal{O}}.$$

Note that $\operatorname{supp}\sigma(f_{\mathcal{O}}) \subseteq \mathcal{O}$ for all $\mathcal{O} \in \mathrm{Orb}^n$, so $\widetilde{\partial}^n = \mathrm{id} + (-1)^n\sigma$ implies

$$\operatorname{supp}\widetilde{\partial}^n(f_{\mathcal{O}'}) \subseteq \mathcal{O}', \quad \operatorname{supp}\widetilde{\partial}^n(f') \subseteq \bigcup_{\mathcal{O}\in\mathrm{Orb}^n - \{\mathcal{O}'\}} \mathcal{O},$$

and so

$$\operatorname{supp}\widetilde{\partial}^n(f_{\mathcal{O}'}) \cap \operatorname{supp}\widetilde{\partial}^n(f') = \emptyset.$$

Since $0 = \widetilde{\partial}^n(f) = \widetilde{\partial}^n(f_{\mathcal{O}'}) + \widetilde{\partial}^n(f')$, it follows that $\widetilde{\partial}^n(f_{\mathcal{O}'}) = 0$ by Observation 6.1.10. In other words, $f_{\mathcal{O}'} \in \ker \widetilde{\partial}^n|_{\Bbbk\mathcal{O}'}$. Since $\mathcal{O}'$ was arbitrary, $f \in \bigoplus_{\mathcal{O}\in\mathrm{Orb}^n} \ker \widetilde{\partial}^n|_{\Bbbk\mathcal{O}}$, and therefore

$$\ker \widetilde{\partial}^n \;\subseteq\; \bigoplus_{\mathcal{O}\in\mathrm{Orb}^n} \ker \widetilde{\partial}^n|_{\Bbbk\mathcal{O}},$$

completing the proof. $\square$

Proposition 6.1.12. *For $\mathcal{O} \in \mathrm{Orb}^n$,*

- *if n is even and $|\mathcal{O}|$ is odd, then $\ker \widetilde{\partial}^n|_{\Bbbk \mathcal{O}} = 0$;*
- *if n is odd or if n and $|\mathcal{O}|$ are even, then $\ker \widetilde{\partial}^n|_{\Bbbk \mathcal{O}}$ has basis $\{\lambda(\mathcal{O})\}$.*

▲

Proof. Let $\mathcal{O} \in \mathrm{Orb}^n$, let $d = |\mathcal{O}|$, and let $f \in \ker \widetilde{\partial}^n|_{\Bbbk \mathcal{O}} \subseteq \Bbbk \mathcal{O}$. Note that

$$\mathcal{O} = \{\sigma^i(\min_{\prec} \mathcal{O}) \mid i = 1, \dots, d\},$$

so we can write $f = \sum_{i=1}^{d} a_i \sigma^i(\min_{\prec} \mathcal{O})$ for some $a_i \in \Bbbk$ and

$$\begin{aligned} 0 &= \widetilde{\partial}^n(f) \\ &= \sum_{i=1}^{d} a_i \sigma^i(\min_{\prec} \mathcal{O}) + (-1)^n \sum_{i=1}^{d} a_i \sigma^{i+1}(\min_{\prec} \mathcal{O}). \end{aligned} \tag{6.1.0.2}$$

Since $\sigma^{d+1}(\min_{\prec} \mathcal{O}) = \sigma(\min_{\prec} \mathcal{O})$, 6.1.0.2 becomes

$$\begin{aligned} 0 &= \sum_{i=1}^{d} a_i \sigma^i(\min_{\prec} \mathcal{O}) + (-1)^n a_d \sigma(\min_{\prec} \mathcal{O}) + (-1)^n \sum_{i=2}^{d} a_{i-1} \sigma^i(\min_{\prec} \mathcal{O}) \\ &= (a_1 + (-1)^n a_d) \sigma(\min_{\prec} \mathcal{O}) + \sum_{i=2}^{d} (a_i + (-1)^n a_{i-1}) \sigma^i(\min_{\prec} \mathcal{O}). \end{aligned} \tag{6.1.0.3}$$

And since distinct monomials are linearly independent, it follows that

$$a_1 = (-1)^{n+1} a_d; \quad a_i = (-1)^{n+1} a_{i-1}, \; i = 2, \dots, d.$$

Inductively, we have

$$a_i = (-1)^{i(n+1)} a_d, \;\; i = 1, \ldots, d. \tag{6.1.0.4}$$

In particular,

$$a_d = (-1)^{d(n+1)} a_d. \tag{6.1.0.5}$$

If n is even and d is odd, then 6.1.0.5 becomes $a_d = -a_d$. Since $\operatorname{char} \Bbbk = 0$, $a_d = 0$; so by 6.1.0.4, $a_i = 0$ for $i = 1, \ldots, d-1$. Thus, $f = 0$, and therefore $\ker \widetilde{\partial}^n|_{\Bbbk \mathcal{O}} = 0$.

If n is odd or if n and d are even, then 6.1.0.5 becomes $a_d = a_d$. If n is odd, then $a_i = a_d$ for $i = 1, \ldots, d-1$ by 6.1.0.4, so

$$f = \sum_{i=1}^{d} a_d \sigma^i(\min\nolimits_{\prec} \mathcal{O}) = a_d \sum_{i=1}^{d} \sigma^i(\min\nolimits_{\prec} \mathcal{O}) = a_d \, \lambda(\mathcal{O}).$$

If n and d are even, then $a_i = (-1)^i a_d$ for $i = 1, \ldots, d-1$ by 6.1.0.5, so

$$f = \sum_{i=1}^{d} (-1)^i a_d \sigma^i(\min\nolimits_{\prec} \mathcal{O}) = a_d \sum_{i=1}^{d} (-1)^i \sigma^i(\min\nolimits_{\prec} \mathcal{O}) = a_d \, \lambda(\mathcal{O}).$$

In either case, $f \in \Bbbk \, \lambda(\mathcal{O})$, so $\ker \widetilde{\partial}^n|_{\Bbbk \mathcal{O}} = \Bbbk \, \lambda(\mathcal{O})$. $\square$

This completes the proof of Theorem 6.1.9.

6.2 Bounding $\dim_{\Bbbk} \mathrm{HH}^n(A)$ below

In this section we establish lower bounds on the $\Bbbk$-dimensions of $\mathrm{HH}^{n-1}_{(1)}(A)$ for $n \geq 1$ (Theorem 6.2.2). These bounds, in turn, imply our main result (Theorem 6.2.9). In order to state these theorems, we must recall two important number-theoretic functions.

Definition 6.2.1. The *Möbius function* is the function $\mu : \mathbb{Z}^+ \to \{-1, 0, 1\}$ such that for $m \in \mathbb{Z}^+$, $\mu(m)$ is the sum of the primitive mth roots of unity.

Euler's totient function is the function $\varphi : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that for $m \in \mathbb{Z}^+$, $\varphi(m)$ is the number of positive integers $\ell \leq m$ such that ℓ and m are relatively prime. ▲

Theorem 6.2.2. *For $n \geq 1$,*

$$\dim_{\Bbbk} \mathrm{HH}^{n-1}_{(1)}(A) \geq \begin{cases} \displaystyle\sum_{\substack{c|d|n,\\ d \text{ even}}} \frac{\mu(c)(N-1)^{d/c}}{d}, & n \text{ even,} \\[2em] \displaystyle\sum_{d|n} \frac{\varphi(d)(N-1)^{n/d}}{n}, & n \text{ odd.} \end{cases}$$

▲

Proof. The result will from Lemmas 6.2.6 and 6.2.8 below. □

Let $n \geq 1$. For the proofs of Lemmas 6.2.4 through 6.2.8, recall the maps $\widetilde{\partial}^n$, $\pi^n_{(1)}$, and $\pi^n_{(2)}$ (Definition 6.1.3); and the map $\gamma_{n-1} : A_1 \otimes E^{n-1} \to A_0 \otimes E^{n-2}$ (Definition 5.0.3); which are related by the diagram

$$\begin{tikzcd}
 & T^n \arrow[r, "\widetilde{\partial}^n"] \arrow[d, "\pi^n_{(1)}"] & T^n \arrow[d, "\pi^n_{(2)}"] \\
A_0 \otimes E^{n-2} \arrow[r, shift left, "\delta^{n-2}_{(n-1)}"] & A_1 \otimes E^{n-1} \arrow[l, dashed, shift left, "\gamma_{n-1}"] \arrow[r, "\delta^{n-1}_{(n-1)}"] & A_2 \otimes E^n
\end{tikzcd}$$

We also require the following definitions.

Definition 6.2.3. For $n \geq 1$,

- let $\mathrm{Mon}^n_\star$ be the set of monomials of degree n in the variables $Y_1, Y_3, \ldots, Y_N$; that is, the variables Y_j except for Y_0 and Y_2;
- let $\mathrm{Orb}^n_\star$ be the set of orbits $\mathcal{O} \in \mathrm{Orb}^n$ such that $\mathcal{O} \subseteq \mathrm{Mon}^n_\star$;
- and let

$$L^n = \begin{cases} \{\lambda(\mathcal{O}) \mid \mathcal{O} \in \mathrm{Orb}^n_\star\}, & n \text{ odd;} \\ \{\lambda(\mathcal{O}) \mid \mathcal{O} \in \mathrm{Orb}^n_\star, |\mathcal{O}| \text{ even}\}, & n \text{ even.} \end{cases} \qquad \blacktriangle$$

Note that

- $\pi^n_{(1)}(\Bbbk L^n) \subseteq \ker \delta^{n-1}_{(n-1)}$ by the commutativity of Diagram 6.1.0.1;
- L^n is linearly independent, being a subset of a $\Bbbk$-basis of $\ker \widetilde{\partial}^n$ (see Theorem 6.1.9);
- for n odd, $|L^n| = |\mathrm{Orb}^n_\star|$; and for n even, $|L^n| = |\{\mathcal{O} \in \mathrm{Orb}^n_\star \mid |\mathcal{O}| \text{ even}\}|$.

Recall from Definition 5.0.2 that $\mathcal{B}(A_1 \otimes E^{n-1})$ is the $\Bbbk$-basis of $A_1 \otimes E^{n-1}$ consisting of elements of the form $x_i \otimes w$, where w is a monomial in $y_0, \ldots, y_N$ of degree $n-1$ which is not divisible by y_0^2 nor by $y_0 y_1^2$.

Lemma 6.2.4. *For $n \geq 1$, $\Bbbk L^n$ maps injectively under $\pi^n_{(1)}$ into $\ker \delta^{n-1}_{(n-1)} \subseteq A_1 \otimes E^{n-1}$. In particular, $\pi^n_{(1)}(L^n)$ is a $\Bbbk$-basis of $\pi^n_{(1)}(\Bbbk L^n)$ of size $|L^n|$.* ▲

Proof. Let $u \in \mathrm{Mon}^n_\star$ and write $u = Y_i v$, where v is a monomial in the variables $Y_1, Y_3, \ldots, Y_N$. In particular, v is not divisible by Y_0^2 nor by $Y_0 Y_1^2$, so $\overline{v} \in \mathcal{B}(E^{n-1})$. Thus, $\pi^n_{(1)}(u) = x_i \otimes \overline{v}$ is an element of $\mathcal{B}(A_1 \otimes E^{n-1})$, implying that $\mathrm{Mon}^n_\star$ maps injectively into $\mathcal{B}(A_1 \otimes E^{n-1})$. Therefore, $\pi^n_{(1)}$ restricted to $\Bbbk\,\mathrm{Mon}^n_\star$—and hence, restricted to $\Bbbk L^n$—is injective. □

Lemma 6.2.5. *For $n \geq 1$, $\mathrm{im}\, \delta^{n-2}_{(n-1)}$ has zero intersection with $\pi^n_{(1)}(\Bbbk L^n)$.* ▲

Proof. Recall from Theorem 5.0.6 that the map γ_{n-1} splits the map $\delta^{n-2}_{(n-1)}$. The key fact to observe from the definition of γ_{n-1} is that γ_{n-1} maps every monomial of the form $x_{i_1} \otimes y_{i_2} \cdots y_{i_n} \in \mathcal{B}(A_1 \otimes E^{n-1})$ with $i_j \in \{1, 3, \ldots, N\}$ to 0. This applies in particular to every monomial of $\pi^n_{(1)}(\mathrm{Mon}^n_\star)$, implying that $\gamma_{n-1}\pi^n_{(1)}(\Bbbk\,\mathrm{Mon}^n_\star) = 0$, and thus $\gamma_{n-1}\pi^n_{(1)}(\Bbbk L^n) = 0$.

To see that $\mathrm{im}\, \delta^{n-2}_{(n-1)} \cap \pi^n_{(1)}(\Bbbk L^n) = 0$, let $f \in \mathrm{im}\, \delta^{n-2}_{(n-1)} \cap \pi^n_{(1)}(\Bbbk L^n)$, so $f = \delta^{n-2}_{(n-1)}(g)$ for some $g \in A_0 \otimes E^{n-2}$ and $\gamma_{n-1}(f) = 0$. Then $0 = \gamma_{n-1}(f) = \gamma_{n-1}\delta^{n-2}_{(n-1)}(g) = g$, so $f = \delta^{n-2}_{(n-1)}(g) = 0$. □

Lemma 6.2.6. *For $n \geq 1$, we have*

$$\dim_{\Bbbk} \mathrm{HH}^{n-1}_{(1)}(A) \;\geq\; |L^n|. \qquad ▲$$

Proof. By Lemma 6.2.5, we can write

$$\ker \delta^{n-1}_{(n-1)} = \operatorname{im} \delta^{n-2}_{(n-1)} \oplus \pi^{n}_{(1)}(\Bbbk\, L^n) \oplus C^{n-1}$$

for some subspace $C^{n-1} \subseteq A_1 \otimes E^{n-1}$. Thus,

$$\mathrm{HH}^{n-1}_{(1)}(A) \;=\; \ker \delta^{n-1}_{(n-1)} / \operatorname{im} \delta^{n-2}_{(n-1)} \;\cong\; \pi^{n}_{(1)}(\Bbbk\, L^n) \oplus C^{n-1}.$$

And by Lemma 6.2.4, $\dim \pi^{n}_{(1)}(\Bbbk\, L^n) = |L^n|$, completing the proof. □

Our final lemma gives us an explicit formula for $|L^n|$. In order to state the lemma, we will need the following definition.

Definition 6.2.7. Let $d \geq 1$. An orbit $\mathcal{O} \in \mathrm{Orb}^d$ is *aperiodic* if $|\mathcal{O}| = d$.

Let $\mathrm{aOrb}^d_\star$ be the set of aperiodic orbits $\mathcal{O} \in \mathrm{Orb}^d_\star$. ▲

Lemma 6.2.8 (see [Reu93, Theorem 7.1 and Corollary 7.3])**.** *Let $n \geq 1$. Then*

$$|L^n| = \begin{cases} \displaystyle\sum_{\substack{c|d|n,\\ d \text{ even}}} \frac{\mu(c)(N-1)^{d/c}}{d} & \text{if } n \text{ is even,} \\[2em] \displaystyle\sum_{d|n} \frac{\varphi(d)(N-1)^{n/d}}{n} & \text{if } n \text{ is odd.} \end{cases}$$

where μ is the Möbius function and φ is Euler's totient function (see Definition 6.2.1). ▲

Proof. Note that there exists a partition

$$\{\mathcal{O} \in \mathrm{Orb}_\star^n \mid |\mathcal{O}| \text{ even}\} = \bigcup_{\substack{d|n,\\ d \text{ even}}} \{\mathcal{O} \in \mathrm{Orb}_\star^n \mid |\mathcal{O}| = d\}$$

and that for each $d \mid n$ there is also a bijective correspondence

$$\mathrm{aOrb}_\star^d \leftrightarrow \{\mathcal{O} \in \mathrm{Orb}_\star^n \mid |\mathcal{O}| = d\},$$

so we have

$$|L^n| = |\{\mathcal{O} \in \mathrm{Orb}_\star^n \mid |\mathcal{O}| \text{ even}\}| = \sum_{\substack{d|n,\\ d \text{ even}}} |\,\mathrm{aOrb}_\star^d\,|.$$

For $d \geq 1$, $\mathrm{aOrb}_\star^d$ is the set of aperiodic orbits of size d on the $N-1$ variables $Y_1, Y_3, \ldots, Y_N$, so by [Reu93, Theorem 7.1],

$$|\,\mathrm{aOrb}_\star^d\,| = \sum_{c|d} \frac{\mu(c)(N-1)^{d/c}}{d}.$$

Thus, for n even,

$$|L^n| = \sum_{\substack{d|n,\\ d \text{ even}}} |\,\mathrm{aOrb}_\star^d\,| = \sum_{\substack{c|d|n,\\ d \text{ even}}} \frac{\mu(c)(N-1)^{d/c}}{d}.$$

Similarly, $\mathrm{Orb}_\star^n$ is the set of orbits on the $N-1$ variables $Y_1, Y_3, \ldots, Y_N$, so by [Reu93, Corollary 7.3],

$$|\operatorname{Orb}^n_\star| = \sum_{d|n} \frac{\varphi(d)(N-1)^{n/d}}{n}.$$

Thus, for n odd,

$$|L^n| = |\operatorname{Orb}^n_\star| = \sum_{d|n} \frac{\varphi(d)(N-1)^{n/d}}{n}.$$

$\square$

This completes the proof of Theorem 6.2.2.

As a corollary to Theorem 6.2.2, we have the following theorem, our main result.

Theorem 6.2.9. *For $N \geq 3$ and for all $n \geq 2$ even, there is an inequality*

$$\dim_{\Bbbk} \mathrm{HH}^n(\mathrm{sGor}(N)) \geq \frac{(N-1)^{n+1}}{n+1}. \qquad \blacktriangle$$

Proof. By the decomposition of $\mathrm{HH}^n(A) = \mathrm{HH}^n_{(0)}(A) \oplus \mathrm{HH}^n_{(1)}(A) \oplus \mathrm{HH}^n_{(2)}(A)$ (Proposition 3.3.5) and Theorem 6.2.2, we have

$$\dim_{\Bbbk} \mathrm{HH}^n(A) \;\geq\; \dim_{\Bbbk} \mathrm{HH}^n_{(1)}(A) \;\geq\; \sum_{d|(n+1)} \frac{\varphi(d)(N-1)^{(n+1)/d}}{n+1}.$$

The totient function satisfies $\varphi(d) \geq 1$ for all d and $\varphi(1) = 1$, so taking the $d = 1$ term of the above sum gives the result. $\square$

Remark 6.2.10. Note that the above lemmas hold for $N = 2$ as well. However, in this case the function of n

$$\frac{(N-1)^{n+1}}{n+1}$$

is no longer exponential, which is why we require the assumption $N \geq 3$. ▲

References

[AIc05] Luchezar L. Avramov, Srikanth Iyengar, and Liana M. Şega. Koszul modules. *Journal of Pure and Applied Algebra*, 201:154–188, 2005.

[Ber78] George M. Bergman. The diamond lemma for ring theory. *Advances in Mathematics*, 29:178–218, 1978.

[BMPZ92] Yuri A. Bahturin, Alexander A. Mikhalev, Viktor M. Petrogradsky, and Mikhail V. Zaicev. *Infinite Dimensional Lie Algebras*, volume 7. Walter de Gruyter & Co., 1992.

[BR15] Ragnar-Olaf Buchweitz and Collin Roberts. The multiplicative structure on hochschild cohomology of a complete intersection. *Journal of Pure and Applied Algebra*, 219:402–428, 2015.

[Bre14] Murray R. Bremner. Free associative algebras, noncommutative gröbner bases, and universal associative envelopes for nonassociative structures. *Commentationes Mathematicae Universitatis Carolinae*, 55:341–379, 2014.

[Buc03] Ragnar-Olaf Buchweitz. Hochschild cohomology of Koszul algebras. Conference on representation theory, Canberra, 2003.

[CE56] Henri Cartan and Samuel Eilenberg. *Homological Algebra*, volume 19. Princeton

University Press, 1956.

[dB94] Michel Van den Bergh. Noncommutative homology of some three-dimensional quantum spaces. *K-Theory*, 8:213–230, 1994.

[Ger63] Murray Gerstenhaber. The cohomology structure of an associative ring. *Annals of Mathematics*, 78:267–288, 1963.

[HI05] Jürgen Herzog and Srikanth Iyengar. Koszul modules. *Journal of Pure and Applied Algebra*, 201:154–188, 2005.

[HKR62] Gerhard Hochschild, Bertram Kostant, and Alex Rosenberg. Differential forms on regular affine algebras. *Transactions of the American Mathematical Society*, 102:383–408, 1962.

[Hoc45] Gerhard Hochschild. On the cohomology groups of an associative algebra. *Annals of Mathematics*, 46:58–67, 1945.

[Lod98] Jean-Louis Loday. *Cyclic Homology*, volume 301. Grundlehren der mathematischen Wissenschaften, 1998.

[Mat89] Hideyuki Matsumura. *Commutative ring theory*, volume 8. Cambridge University Press, 1989.

[Neg17] Cris Negron. The cup product on hochschild cohomology via twisting cochains and applications to koszul rings. *Journal of Pure and Applied Algebra*, 221:1112–1133, 2017.

[PP05] Alexander Polishchuk and Leonid Positselski. *Quadratic Algebras*, volume 37. American Mathematical Society, 2005.

[Pri70] Stewart B. Priddy. Koszul resolutions. *Transactions of the American Mathematical Society*, 152:39–60, 1970.

[Reu93] Christophe Reutenauer. *Free Lie Algebras*, volume 7. Oxford University Press, 1993.

[War90] Seth Warner. *Modern Algebra*. Dover Publications, Inc., 1990.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38. Cambridge University Press, 1994.

[Wit19] Sarah Witherspoon. *Hochschild Cohomology for Algebras*, volume 204. American Mathematical Society, 2019.

Mkrtich Ohanyan

mkrtichohanyan@gmail.com
(818) 913-0848

SKILLS

- Created Python and Matlab programs to improve the speed of computations in my research projects
- Wrote technical literature in my PhD thesis and my applied mathematics research project
- Performed academic research using methods in applied mathematics, commutative algebra, and linear algebra prove theorems

EDUCATION

PhD, Mathematics
May 2021

Syracuse University — Syracuse, NY
Field of study: Commutative Algebra
Thesis: *Hochschild Cohomology of Short Gorenstein Rings*
GPA: 3.957/4.0

MS, Mathematics
May 2018

Syracuse University — Syracuse, NY
Recipient of University Fellowship for two years

BS, Mathematics
May 2018

University of Southern California — Los Angeles, CA
With Honors
Recipient of National Merit Scholarship for four years

RESEARCH EXPERIENCE

Applied Mathematics

Rome Air Force Research Lab
- Collaborated with two researchers at the Air Force Research Lab in Rome, NY and used a tensor-based method for classification of handwritten digits. The method significantly reduced the size of training data without compromising accuracy of predictions
- Developed an algorithm in Matlab which allowed for the computation of higher order SVD of an arbitrary number of dimensions

Commutative Algebra (PhD thesis)

Syracuse University
- Developed an algorithm for constructing the Lie monomials of a graded Lie algebra
- Computed an exponential lower bound for the Hochschild cohomology of a family of short Gorenstein rings — one of the first such explicit computations

EMPLOYMENT

AFRL Fellow (intern)
May - August 2020

Rome Air Force Research Lab
- Read and presented academic manuscripts related to tensor decompositions
- Implemented an algorithm in Matlab using tensor-based methods for handwritten digit classification
- Completed a report and presented findings, which involved communicating technical mathematical and programming results

University Instructor
August 2013 - present

Syracuse University
- Taught Calculus I, II, III; solely responsible for entire course, including giving lectures and writing and grading exams
- Used innovative inquiry based learning methods to teach students in a way that requires them to communicate technically and find their own solutions
- Developed methods for teaching online by combining the use of LaTeX, Zoom, YouTube, Google Suite, and Blackboard

LEADERSHIP

Co-founder,
Committee member
May 2018 - May 2019

Syracuse University Directed Reading Program
- For the first time in SU math department history, organized a series of presentations on advanced topics in mathematics given by undergraduates

Treasurer
May 2019 - May 2020

Association for Women in Mathematics, Syracuse University chapter
- Collaborated with president to create a space which highlights women and minorities in a male-dominated field
- Created events related to women and minorities in mathematics

AWARDS

University Fellowship
Fall 2012 - Spring 2013,
Fall 2015 - Spring 2016

Syracuse University
- Awarded a two-year fellowship with no teaching requirement
- Fellowship is granted to one student from the incoming math graduate cohort each year

National Merit Scholar
May 2006