ISOLATION AND COMPONENT STRUCTURE IN SPACES OF COMPOSITION OPERATORS

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Abstract. We establish a condition that guarantees isolation in the space of composition operators acting between $H^p(B_N)$ and $H^q(B_N)$, for $0 < p \leq \infty$, $0 < q < \infty$, and $N \geq 1$. This result will allow us, in certain cases where $0 < q < p \leq \infty$, completely to characterize the component structure of this space of operators.

1. Preliminaries

For any natural number N, we write B_N to denote the open unit ball in $\mathbb{C}^N$, with $\mathbb{D}$ serving as alternate notation for the disk B_1. Throughout this paper, unless otherwise stated, we take N to be an arbitrary positive integer. Having fixed a value of N, we write σ to denote normalized Lebesgue measure on the unit sphere ∂B_N. For any $0 < p < \infty$, the *Hardy space* $H^p(B_N)$ is defined to be the set of all analytic functions $f : B_N \to \mathbb{C}$ such that

$$\|f\|_p := \left(\sup_{0<r<1} \int_{\partial B_N} |f_r|^p \, d\sigma \right)^{1/p} < \infty,$$

where f_r denotes the dilation $f_r(z) = f(rz)$. The space $H^\infty(B_N)$ is simply the set of bounded analytic functions on B_N, with

$$\|f\|_\infty := \sup_{w \in B_N} |f(w)|.$$

Observe that $H^p(B_N)$ is contained in $H^q(B_N)$ whenever $0 < q \leq p \leq \infty$, with $\|f\|_q \leq \|f\|_p$ for all f. If f belongs to any space $H^p(B_N)$, then the radial limit

$$f^*(\zeta) := \lim_{r \uparrow 1} f(r\zeta)$$

exists for σ–almost all ζ on ∂B_N; moreover

$$\|f\|_p = \left(\int_{\partial B_N} |f^*|^p \, d\sigma \right)^{1/p} \tag{1.1}$$

for all finite values of p (see Section 5.6 of [12]).

The Hardy space $H^p(B_N)$, under the norm $\|\cdot\|_p$, is a Hilbert space when $p = 2$ and a Banach space whenever $1 \leq p \leq \infty$. For $0 < p < 1$, the "p–norm" is not actually a true norm, since the triangle inequality does not hold. It can be shown, however, that the distance function

$$d(f, g) := \|f - g\|_p^p$$

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defines a complete, translation-invariant metric on $H^p(B_N)$ for $0 < p < 1$. In other words, while they are not Banach spaces, the corresponding $H^p(B_N)$ are examples of a particular type of topological vector space known as an F–space. As such, many familiar results from the theory of Banach spaces still hold in this context, in particular the principle of uniform boundedness and the closed graph theorem; the Hahn–Banach theorem, however, is no longer valid, since the spaces in question are not locally convex (see Section 2.3 of [10]). Nevertheless, there are still enough bounded linear functionals on each $H^p(B_N)$ to separate points in the space. For any w in B_N, the point-evaluation functional taking f to $f(w)$ is bounded on every space $H^p(B_N)$; this fact is evident when $p = \infty$ and follows from Theorem 7.2.5 in [12] for $0 < p < \infty$.

It would be helpful at this point to make a brief comment about operator norms. Even though, for $0 < p < 1$, the standard metric on $H^p(B_N)$ is expressed in terms of $\|\cdot\|_p^p$, we still define the norm of an operator with respect to $\|\cdot\|_p$. In particular, for any T taking $H^p(B_N)$ into $H^q(B_N)$, we set

$$\|T : H^p(B_N) \to H^q(B_N)\| := \sup_{f \in H^p(B_N)\setminus\{0\}} \frac{\|T(f)\|_q}{\|f\|_p},$$

regardless of the values of p and q. (Often, for the sake of convenience, we simply write $\|T\|$ in place of $\|T : H^p(B_N) \to H^q(B_N)\|$.)

Let φ be an analytic map from B_N into B_N. The *composition operator* C_φ, acting on a space $H^p(B_N)$, is defined by the rule

$$C_\varphi(f) = f \circ \varphi.$$

We often describe such an operator as being *induced* by the map φ. If C_φ takes $H^p(B_N)$ into some space $H^q(B_N)$, then it follows from the closed graph theorem that the operator $C_\varphi : H^p(B_N) \to H^q(B_N)$ is bounded. One can easily see that every composition operator takes $H^\infty(B_N)$ into itself. Likewise, for $0 < p < \infty$, the Littlewood subordination theorem shows that any analytic $\varphi : \mathbb{D} \to \mathbb{D}$ gives rise to a bounded operator $C_\varphi : H^p(\mathbb{D}) \to H^p(\mathbb{D})$ (see Corollary 3.7 in [4]). Consequently, whenever $N = 1$ or $p = \infty$, every analytic $\varphi : B_N \to B_N$ induces a bounded composition operator from $H^p(B_N)$ into $H^q(B_N)$ for all $q \le p$. The situation is more complicated, though, when $N \ge 2$; for instance, one can find examples of $\varphi : B_N \to B_N$ such that C_φ does not take $H^p(B_N)$ into $H^q(B_N)$ for any finite values of p and q. On the other hand, a necessary and sufficient condition is known for C_φ to take $H^p(B_N)$ into itself (see Theorem 3.35 in [4]). While this condition is difficult to check in practice, it does show that any self-map of B_N induces a bounded composition operator from $H^p(B_N)$ into $H^p(B_N)$ for *some* $0 < p < \infty$ if and only if it induces a bounded operator for *all* such p.

Since the late 1960's, the study of composition operators has developed into an active area of research; Cowen and MacCluer's book [4] provides a compendium of much of the work that has been done. One topic of continuing interest is the component structure of various spaces of composition operators. For example, let $\mathcal{C}\big(H^p(B_N), H^q(B_N)\big)$ denote the set of composition operators taking $H^p(B_N)$ into $H^q(B_N)$, endowed with the topology induced by the operator norm; we generally write $\mathcal{C}\big(H^p(B_N)\big)$ to denote the space $\mathcal{C}\big(H^p(B_N), H^p(B_N)\big)$. One of the most natural problems to consider is the question of when a particular operator is isolated in some space $\mathcal{C}\big(H^p(B_N), H^q(B_N)\big)$. Many (but by no means all) of the results along these lines are stated in terms of the *extreme set* of φ, that is, the set of all ζ on ∂B_N such that $\varphi^*(\zeta) := \lim_{r\uparrow 1} \varphi(r\zeta)$ has norm 1. The general principle underlying these results is that C_φ is isolated in the appropriate space of operators whenever the extreme set of φ has positive σ–measure. The prototypical isolation theorem, due to Berkson [1], pertains to the spaces $\mathcal{C}\big(H^p(\mathbb{D})\big)$ for $1 \le p < \infty$. He demonstrated that, for any pair of distinct analytic maps $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to \mathbb{D}$, the corresponding composition operators have the property that

$$\|(C_\varphi - C_\psi) : H^p(\mathbb{D}) \to H^p(\mathbb{D})\| \ge [\sigma(E)/2]^{1/p},$$

where E denotes the extreme set of φ. (Thus C_φ is indeed isolated in $\mathcal{C}\big(H^p(\mathbb{D})\big)$ whenever $\sigma(E) > 0$.) Shapiro and Sundberg [14] later improved this result somewhat for operators in $\mathcal{C}\big(H^2(\mathbb{D})\big)$; they showed that

$$\|(C_\varphi - C_\psi) : H^2(\mathbb{D}) \to H^2(\mathbb{D})\| \ge [\sigma(E) + \sigma(F)]^{1/2},$$

where E denotes the extreme set of φ and F the extreme set of ψ. Shapiro and Sundberg's result, in turn, was extended by Heidler [7] to the spaces $\mathcal{C}\big(H^2(B_N)\big)$ for $N \ge 2$.

It would be reasonable to expect a similar result to hold in a more general setting (see Conjecture 12 in [5]). Unfortunately, the arguments used to prove the aforementioned theorems are not particularly helpful in this regard. Berkson's [1] proof is quite specific to the case where C_φ takes $H^p(\mathbb{D})$ into itself; the techniques employed by Shapiro and Sundberg [14] and Heidler [7] are heavily dependent on the Hilbert space structure of $H^2(B_N)$. The primary goal of this paper is to establish an isolation theorem, stated in terms of the extreme set of $\varphi : B_N \to B_N$, which is valid in $\mathcal{C}\big(H^p(B_N), H^q(B_N)\big)$ for any integer $N \ge 1$ and any values $0 < p \le \infty$ and $0 < q < \infty$ (Theorem 4.4 and Corollary 4.5). As it turns out, this result will allow us, in certain cases where $0 < q < p \le \infty$, completely to characterize the component structure of the corresponding space of composition operators (Theorem 5.3 and Corollary 5.4).

We conclude this section with a remark about radial limit functions. As a consequence of line (1.1), our alternate representation for $\|\cdot\|_p$, we shall often have cause to consider the radial limit $(f\circ\varphi)^*$, where f belongs to some $H^p(B_N)$ and φ is an analytic self-map of B_N. In particular, we would like to relate $(f\circ\varphi)^*$ to the composition $f^*\circ\varphi^*$. If $N = 1$, then these two functions agree σ–almost everywhere on ∂B_N (see Proposition 2.25 in [4]); the same is true whenever the operator C_φ takes $H^p(B_N)$ into itself for some (and hence all) $0 < p < \infty$ (see Lemma 1.6 in [9]). Nonetheless, when $N \ge 2$, there are still examples of f in $H^p(B_N)$ and $\varphi : B_N \to B_N$ such that $(f\circ\varphi)^* \ne f^*\circ\varphi^*$ on a set of positive σ–measure. We will generally manage to circumvent this difficulty, though, by taking f to be an element of the *ball algebra* $A(B_N)$, that is, the set of all analytic functions on B_N which are also continuous on the closed ball $\overline{B_N}$.

2. Essential Norms

Our own isolation theorem, although similar in spirit to Berkson's [1] result, will be stated in somewhat stronger terms. While Berkson considered $\|C_\varphi - C_\psi\|$, we shall concern ourselves with $\|C_\varphi - C_\psi\|_e$, where $\|\cdot\|_e$ denotes the *essential norm* of an operator. For any $T : H^p(B_N) \to H^q(B_N)$, recall that

$$\|T : H^p(B_N) \to H^q(B_N)\|_e := \inf_{K\in\mathcal{K}} \|(T-K) : H^p(B_N) \to H^q(B_N)\|,$$

where $\mathcal{K} = \mathcal{K}\big(H^p(B_N), H^q(B_N)\big)$ signifies the set of compact operators acting from $H^p(B_N)$ into $H^q(B_N)$. There is ample justification for working with the essential norm when considering questions of isolation. First of all, since the norm of an operator is never less than its essential norm, an isolation theorem stated in terms of the essential norm implies a similar result in terms of the operator norm. Secondly, $\|T\|_e = 0$ if and only if T is compact, so our isolation theorem will give us a necessary condition for the operators C_φ and C_ψ to have compact difference. Finally, just as the operator norm induces the standard topology on $\mathcal{C}\big(H^p(B_N), H^q(B_N)\big)$, the essential norm gives rise to the topology on the quotient space

$$\mathcal{Q}\big(H^p(B_N), H^q(B_N)\big) := \mathcal{C}\big(H^p(B_N), H^q(B_N)\big)/\mathcal{K}\big(H^p(B_N), H^q(B_N)\big).$$

Hence our theorem will actually provide us with information about when the equivalence class containing a particular C_φ is isolated in $\mathcal{Q}\big(H^p(B_N), H^q(B_N)\big)$.

The next two results will serve as our principal tools for estimating the essential norm of an operator.

Proposition 2.1. *Take p and q to be finite indices. Let T be a bounded operator from $H^p(B_N)$ into $H^q(B_N)$; then*

$$\|T : H^p(B_N) \to H^q(B_N)\|_e \geq \limsup_{n\to\infty} \|T(g^n)\|_q\,,$$

where g is any nonconstant element of $H^\infty(B_N)$ with $\|g\|_\infty \leq 1$.

Proof. Observe that $\|g^n\|_p \leq 1$ for all n, for every $0 < p < \infty$, and that the sequence $\{g^n\}$ converges pointwise to 0 in B_N. We claim that this sequence converges weakly to 0 in every space $H^p(B_N)$. For $1 < p < \infty$, since the corresponding $H^p(B_N)$ is a reflexive Banach space, this fact follows from Corollary 1.3 in [4]. Now consider $0 < p \leq 1$. Take λ to be a bounded functional on $H^p(B_N)$ and let ι denote the inclusion map from $H^2(B_N)$ into $H^p(B_N)$. Observe that $\lambda \circ \iota$ is a bounded functional on $H^2(B_N)$; since $\{g^n\}$ converges weakly to 0 in $H^2(B_N)$, the sequence $\{\lambda(\iota(g^n))\} = \{\lambda(g^n)\}$ goes to 0. In other words, $\{g^n\}$ converges to 0 weakly in $H^p(B_N)$.

Let $K : H^p(B_N) \to H^q(B_N)$ be a compact operator. Since the functions g^n have norm no greater than 1 in $H^p(B_N)$, we see that

$$\|T - K\| \geq \limsup_{n\to\infty} \|(T-K)(g^n)\|_q \geq \limsup_{n\to\infty} \Big(\|T(g^n)\|_q - \|K(g^n)\|_q\Big)$$

for $1 \leq q < \infty$. Similarly, for $0 < q < 1$, we have that

$$\|T - K\|^q \geq \limsup_{n\to\infty} \|(T-K)(g^n)\|_q^q \geq \limsup_{n\to\infty} \Big(\|T(g^n)\|_q^q - \|K(g^n)\|_q^q\Big)\,.$$

We claim that the compact operator K takes the sequence $\{g^n\}$ to 0 in the norm of $H^q(B_N)$. If $H^p(B_N)$ and $H^q(B_N)$ both happen to be Banach spaces, then this fact follows from a standard result in functional analysis (see Proposition VI.3.3 in [3]), whose proof can be readily adapted to suit general values of p and q. To that end, observe that the functions g^n belong to the unit ball of $H^p(B_N)$; hence the set $\{K(g^n)\}$ has compact closure in $H^q(B_N)$. Let $\{g^{n_k}\}$ be any subsequence of $\{g^n\}$ such that $\{K(g^{n_k})\}$ converges in the norm of $H^q(B_N)$, to an element we shall call h. Let λ be a bounded functional on $H^q(B_N)$; since $\lambda \circ K$ is a bounded functional on $H^p(B_N)$, the sequence $\{K(g^{n_k})\}$ converges weakly to 0 in $H^q(B_N)$. Consequently, since every point-evaluation functional is bounded on $H^q(B_N)$, the function h must be identically 0 on B_N. In other words, the vector 0 is the unique limit point of $\{K(g^n)\}$, from which we deduce that $\lim_{n\to\infty} \|K(g^n)\|_q = 0$. Thus, regardless of the values of p and q, we conclude that

$$\|T - K\| \geq \limsup_{n\to\infty} \|T(g^n)\|_q .$$

Taking the infimum over the set of compact operators, we obtain the desired result. $\square$

Unfortunately, the preceding proposition cannot be extended to include the case where $p = \infty$; the sequence $\{g^n\}$ does not, in general, converge weakly to 0 in $H^\infty(B_N)$. We do, however, have the following result, inspired by the proof of Theorem 3 in [6].

Proposition 2.2. *Take q to be a finite index. Let T be a finite linear combination of composition operators; then*

$$\|T : H^\infty(B_N) \to H^q(B_N)\|_e \geq \frac{1}{2} \limsup_{n\to\infty} \|T(g^n)\|_q ,$$

where g is any nonconstant element of $H^\infty(B_N)$ with $\|g\|_\infty \leq 1$.

Proof. Write $T = \sum_{j=1}^J \alpha_j\, C_{\varphi_j}$ and take K to be a compact operator from $H^p(B_N)$ into $H^q(B_N)$. Let $\{g^{n_k}\}$ be a subsequence of $\{g^n\}$ such that

$$\lim_{k\to\infty} \|T(g^{n_k})\|_q = \limsup_{n\to\infty} \|T(g^n)\|_q .$$

Since the functions g^{n_k} belong to the unit ball of $H^\infty(B_N)$, there exists a sub-subsequence (which, to avoid notational difficulty, we also write $\{g^{n_k}\}$) such that $\{K(g^{n_k})\}$ converges in the norm of $H^q(B_N)$. Hence, for any $\varepsilon > 0$, there is a natural number M such that $\|K(g^{n_k}) - K(g^{n_m})\|_q < \varepsilon$ whenever k and m are greater than or equal to M. Fix an integer $k \geq M$. Consider the function $T(g^{n_k})$; by Theorem 5.6.6 in [12], there is some number $0 < r < 1$ for which the dilation $(T(g^{n_k}))_r(z) = (T(g^{n_k}))(rz)$ has the property that

$$\|T(g^{n_k}) - (T(g^{n_k}))_r\|_q < \varepsilon.$$

At this point, we temporarily restrict our attention to the case where $1 \leq q < \infty$. Since $\|(g^{n_k} - g^{n_m})/2\|_\infty \leq 1$, we see that

$$\begin{aligned}
\|T - K\| &\geq \|(T - K)((g^{n_k} - g^{n_m})/2)\|_q \\
&\geq (1/2)\|T(g^{n_k}) - T(g^{n_m})\|_q - (1/2)\|K(g^{n_k}) - K(g^{n_m})\|_q \\
&> (1/2)\|T(g^{n_k}) - T(g^{n_m})\|_q - \varepsilon/2 \\
&\geq (1/2)\|(T(g^{n_k}))_r - (T(g^{n_m}))_r\|_q - \varepsilon/2 \\
&> (1/2)\left(\|T(g^{n_k})\|_q - \|(T(g^{n_m}))_r\|_q\right) - \varepsilon
\end{aligned}$$

whenever $m \geq M$. Observe that

$$(T(g^{n_m}))_r(z) = \sum_{j=1}^J \alpha_j\,(g(\varphi_j(rz)))^{n_m}$$

converges to 0 uniformly on B_N as m tends to ∞. Consequently

$$\|T - K\| \geq (1/2)\|T(g^{n_k})\|_q - \varepsilon$$

for all $k \geq M$; it follows that

$$\|T-K\| \geq (1/2) \lim_{k\to\infty} \|T(g^{n_k})\|_q - \varepsilon = (1/2) \limsup_{n\to\infty} \|T(g^n)\|_q - \varepsilon,$$

a fact which holds for all $\varepsilon > 0$. Likewise, when we consider $0 < q < 1$, an analogous sequence of estimates shows that

$$\|T-K\|^q \geq (1/2)^q \limsup_{n\to\infty} \|T(g^n)\|_q^q - (1/2)^{q-1} \varepsilon^q.$$

In either case, letting ε go to 0 and taking the infimum over $\mathcal{K}\big(H^p(B_N), H^q(B_N)\big)$, we see that our assertion holds. □

3. Approximate Inner Functions

In [13], Rudin demonstrates that the existence of a particularly helpful class of functions. Given a positive measure μ on ∂B_N and a number $\delta > 0$, we write $Y_\delta(\mu)$ to denote the set of all nonconstant functions g in $A(B_N)$ such that

(*i*) $|g(\zeta)| \leq 1$ for all ζ on ∂B_N, and
(*ii*) $\mu(\{|g(\zeta)| = 1\}) \geq \mu(\partial B_N) - \delta$.

For any μ and any δ, the set $Y_\delta(\mu)$ is nonempty. Moreover, if μ is a positive Borel measure on ∂B_N, every set $Y_\delta(\mu)$ is dense in the unit ball of $H^\infty(B_N)$ relative to the compact-open topology (that is, the topology where convergence is given by uniform convergence on compact subsets of B_N). In light of these defining characteristics, it seems reasonable to describe the elements of a particular set $Y_\delta(\mu)$ as being *approximate inner functions*. The obvious advantage of these functions over the standard inner functions of B_N is that they belong to the ball algebra $A(B_N)$, rather than just $H^\infty(B_N)$. In particular, for any such g and any analytic $\varphi : B_N \to B_N$, we have that $(g \circ \varphi)^*(\zeta) = g(\varphi^*(\zeta))$ for σ–almost all ζ on ∂B_N.

We shall make repeated use of approximate inner functions defined with respect to one particular measure. Let $\varphi : B_N \to B_N$ be an analytic map with extreme set E; consider the restriction $\varphi^* : E \to \partial B_N$. The pullback measure $\sigma\varphi^{*\,-1}$ on ∂B_N is defined by setting $\sigma\varphi^{*\,-1}(A) = \sigma\big(\varphi^{*\,-1}(A)\big)$ for any σ–measurable subset A of ∂B_N. It is a well-known fact from measure theory that

$$\int_E f \circ \varphi^* \, d\sigma = \int_{\partial B_N} f \, d\big(\sigma\varphi^{*\,-1}\big)$$

for all f in $L^1(\sigma)$. Observe that $\sigma\varphi^{*\,-1}(\partial B_N) = \sigma(E)$; in particular, $\sigma\varphi^{*\,-1}$ is a positive measure on ∂B_N if and only if $\sigma(E) > 0$.

The first result that we obtain with the aid of approximate inner functions relates to the essential norm of a composition operator acting between Hardy spaces.

Proposition 3.1. *Let φ be an analytic self-map of B_N that induces a bounded composition operator from $H^p(B_N)$ into $H^q(B_N)$, where p and q are finite indices; then*

$$\|C_\varphi : H^p(B_N) \to H^q(B_N)\|_e \geq [\sigma(E)]^{1/q},$$

where E denotes the extreme set of φ. Similarly,

$$\|C_\varphi : H^\infty(B_N) \to H^q(B_N)\|_e \geq \frac{1}{2} [\sigma(E)]^{1/q}$$

for any $0 < q < \infty$.

Proof. If $\sigma(E) = 0$, then there is nothing to prove. Suppose then that $\sigma(E) > 0$; that is, $\sigma\varphi^{*\,-1}$ is a positive measure on ∂B_N, with $\sigma\varphi^{*\,-1}(\partial B_N) = \sigma(E)$. Fix a $\delta > 0$ and take g to be an element of $Y_\delta\big(\sigma\varphi^{*\,-1}\big)$. Let $V = \{\zeta \in \partial B_N : |g(\zeta)| = 1\}$, so that $\sigma\varphi^{*\,-1}(V) \geq \sigma(E) - \delta$. For any natural number n, we see that

$$\begin{aligned} \|C_\varphi(g^n)\|_q^q &= \int_{\partial B_N} \big|(g^n \circ \varphi)^*\big|^q \, d\sigma \geq \int_E |g|^{nq} \circ \varphi^* \, d\sigma \\ &= \int_{\partial B_N} |g|^{nq} \, d\big(\sigma\varphi^{*\,-1}\big) \geq \int_V d\big(\sigma\varphi^{*\,-1}\big) \\ &= \sigma\varphi^{*\,-1}(V) \geq \sigma(E) - \delta. \end{aligned}$$

We arrive at the desired conclusions by applying Proposition 2.1 (for $0 < p < \infty$) and Proposition 2.2 (for $p = \infty$), then letting δ tend to 0. $\square$

Remark. The $p = \infty$ statement in Proposition 3.1 constitutes a slight improvement to Theorem 3 in [6], in that we have eliminated the hypothesis that C_φ be bounded from $H^p(B_N)$ to $H^p(B_N)$ for some (and hence all) $0 < p < \infty$. In fact, with a bit more work, we can obtain an even better result in the case where $p = \infty$ and $q = 2$. If we modify the argument used to establish the lower estimate for Theorem 1 in [6], replacing the inner function g with the appropriate approximate inner function, we see that

$$\big\|C_\varphi : H^\infty(B_N) \to H^2(B_N)\big\|_e \geq [\sigma(E)]^{1/2}.$$

As is the case for Proposition 3.1, this last result requires no additional assumptions regarding the boundedness of C_φ.

4. Isolation of Composition Operators

The estimates required to obtain our isolation theorem demand a certain degree of meticulousness. The following lemma is necessary to our argument.

Lemma 4.1. *Let φ and ψ be analytic self-maps of B_N. Let g be an element of $H^\infty(B_N)$ with $\|g\|_\infty \leq 1$. Suppose that there is some point w in B_N such that $g(\varphi(w)) \neq g(\psi(w))$; then, for any $\delta > 0$, there is a subset T_δ of ∂B_N and a constant $M_\delta > 0$ such that $\sigma(T_\delta) \geq 1 - \delta$ and $\big|1 - (g \circ \varphi)^*(\zeta)\,\overline{(g \circ \psi)^*(\zeta)}\big| \geq M_\delta$ for all ζ in T_δ.*

Proof. For any $M > 0$, define the set

$$S_M = \Big\{\zeta \in \partial B_N : \big|1 - (g \circ \varphi)^*(\zeta)\,\overline{(g \circ \psi)^*(\zeta)}\big| < M\Big\}.$$

Since $S_{M_1} \subseteq S_{M_2}$ whenever $M_1 < M_2$, a basic result from measure theory shows that

$$\begin{aligned} \lim_{M \downarrow 0} \sigma(S_M) &= \sigma\left(\bigcap_{M>0} S_M\right) \\ &= \sigma\Big(\Big\{\zeta \in \partial B_N : (g \circ \varphi)^*(\zeta)\,\overline{(g \circ \psi)^*(\zeta)} = 1\Big\}\Big). \end{aligned} \tag{4.1}$$

The functions $g \circ \varphi$ and $g \circ \psi$ both belong to $H^\infty(B_N)$, with $\|g \circ \varphi\|_\infty \leq 1$ and $\|g \circ \psi\|_\infty \leq 1$; if $(g \circ \varphi)^*(\zeta)\,\overline{(g \circ \psi)^*(\zeta)} = 1$, then $(g \circ \varphi)^*(\zeta)$ and $(g \circ \psi)^*(\zeta)$ must both have modulus 1, which means that $(g \circ \varphi)^*(\zeta) = (g \circ \psi)^*(\zeta)$. Since $g \circ \varphi$ and $g \circ \psi$ are not identically equal on B_N, Theorem 5.6.4 in [12] dictates that $(g \circ \varphi)^*$ and $(g \circ \psi)^*$ cannot agree on a subset of ∂B_N that has positive σ–measure; in other words, the quantities in (4.1) must all equal 0. Thus, for any $\delta > 0$, there is some number M_δ such that $\sigma\big(S_{M_\delta}\big) < \delta$. Define the set $T_\delta = \partial B_N \setminus S_{M_\delta}$. Observe that $\sigma(T_\delta) \geq 1 - \delta$ and that $\big|1 - (g \circ \varphi)^*(\zeta)\,\overline{(g \circ \psi)^*(\zeta)}\big| \geq M_\delta$ for all ζ in T_δ. $\square$

At this point we establish our most important norm estimate.

Proposition 4.2. *Let φ and ψ be distinct analytic self-maps of B_N. For any $\varepsilon > 0$, there is a nonconstant unit vector g in $H^\infty(B_N)$ such that*

$$\limsup_{n\to\infty} \|C_\varphi(g^n) - C_\psi(g^n)\|_2^2 \geq \sigma(E) - \varepsilon,$$

where E denotes the extreme set of φ.

Proof. We only need to deal with the situation where $\sigma(E) > 0$. Set $\delta = \varepsilon/2$. Since φ and ψ are distinct maps, there must be a point w in B_N with $\varphi(w) \neq \psi(w)$; because $Y_\delta\big(\sigma\varphi^{*\,-1}\big)$ is dense in $H^\infty(B_N)$, we can find some g in $Y_\delta\big(\sigma\varphi^{*\,-1}\big)$ with $g(\varphi(w)) \neq g(\psi(w))$. As it turns out, this function will serve our purposes. Let $V = \{\zeta \in \partial B_N : |g(\zeta)| = 1\}$, so that $\sigma\varphi^{*\,-1}(V) \geq \sigma(E) - \delta$.

Consider the set T_δ, as defined in Lemma 4.1. Note that

$$\begin{aligned}
&\int_{T_\delta} \big|(C_\varphi(g^n) - C_\psi(g^n))^*\big|^2\, d\sigma = \int_{T_\delta} \big|(g^n \circ \varphi)^* - (g^n \circ \psi)^*\big|^2\, d\sigma \\
= \;& \int_{T_\delta} \big|(g^n \circ \varphi)^*\big|^2\, d\sigma - 2\operatorname{Re}\int_{T_\delta} (g^n \circ \varphi)^*\,\overline{(g^n \circ \psi)^*}\, d\sigma + \int_{T_\delta} \big|(g^n \circ \psi)^*\big|^2\, d\sigma \\
(4.2)\geq \;& \int_{T_\delta} \big|(g^n \circ \varphi)^*\big|^2\, d\sigma - 2\operatorname{Re}\int_{T_\delta} (g^n \circ \varphi)^*\,\overline{(g^n \circ \psi)^*}\, d\sigma
\end{aligned}$$

for any natural number n. We begin by estimating the first term in (4.2). Define the measure σ_δ on ∂B_N by setting $\sigma_\delta(A) = \sigma(A \cap T_\delta)$; observe that

$$\begin{aligned}
&\int_{T_\delta} \big|(g^n \circ \varphi)^*\big|^2\, d\sigma = \int_{\partial B_N} \big|(g^n \circ \varphi)^*\big|^2\, d\sigma_\delta \\
\geq \;& \int_E |g|^{2n} \circ \varphi^*\, d\sigma_\delta = \int_{\partial B_N} |g|^{2n}\, d\big(\sigma_\delta\varphi^{*\,-1}\big) \\
\geq \;& \int_V d\big(\sigma_\delta\varphi^{*\,-1}\big) = \sigma\big(\varphi^{*\,-1}(V) \cap T_\delta\big) \\
\geq \;& \sigma\varphi^{*\,-1}(V) - \delta \geq \sigma(E) - 2\delta.
\end{aligned}$$

Now we turn our attention to the terms

$$I_n := 2\operatorname{Re}\int_{T_\delta} (g^n \circ \varphi)^*\,\overline{(g^n \circ \psi)^*}\, d\sigma.$$

There are two situations to consider. Suppose, first of all, that infinitely many of the I_n are negative. In this case, we can find an increasing sequence of natural numbers n_k such that each term I_{n_k} is negative; in particular,

$$\begin{aligned}
&\limsup_{n\to\infty} \int_{T_\delta} \big|(C_\varphi(g^n) - C_\psi(g^n))^*\big|^2\, d\sigma \\
\geq \;& \limsup_{k\to\infty} \left[\int_{T_\delta} \big|(g^{n_k} \circ \varphi)^*\big|^2\, d\sigma - I_{n_k}\right] \\
\geq \;& \limsup_{k\to\infty} \int_{T_\delta} \big|(g^{n_k} \circ \varphi)^*\big|^2\, d\sigma \geq \sigma(E) - 2\delta.
\end{aligned}$$

Now suppose that there are only finitely many negative I_n; in other words, there is a natural number M such that $I_n \geq 0$ for all $n \geq M$. In this case, we shall show that the I_n are summable, and hence converge to 0. For any integer $K \geq M$, consider the partial sum

$$\sum_{n=M}^{K} I_n = \sum_{n=M}^{K} 2\operatorname{Re}\int_{T_\delta} \left[(g\circ\varphi)^* \overline{(g\circ\psi)^*}\right]^n d\sigma.$$

Since $(g\circ\varphi)^* \overline{(g\circ\psi)^*} \neq 1$ on T_δ, we see that

$$\begin{aligned}
&\sum_{n=M}^{K} 2\operatorname{Re}\int_{T_\delta} \left[(g\circ\varphi)^* \overline{(g\circ\psi)^*}\right]^n d\sigma \\
=\ & 2\operatorname{Re}\int_{T_\delta} \left(\sum_{n=M}^{K} \left[(g\circ\varphi)^* \overline{(g\circ\psi)^*}\right]^n\right) d\sigma \\
=\ & 2\operatorname{Re}\int_{T_\delta} \left(\frac{\left[(g\circ\varphi)^* \overline{(g\circ\psi)^*}\right]^M - \left[(g\circ\varphi)^* \overline{(g\circ\psi)^*}\right]^{K+1}}{1-(g\circ\varphi)^* \overline{(g\circ\psi)^*}}\right) d\sigma \\
\leq\ & 2\int_{T_\delta} \frac{2}{\left|1-(g\circ\varphi)^* \overline{(g\circ\psi)^*}\right|}\, d\sigma,
\end{aligned}$$

which, in view of Lemma 4.1, is bounded by $4/M_\delta$. Thus the partial sums converge to a finite value, which means that the I_n tend to 0. Therefore

$$\begin{aligned}
&\limsup_{n\to\infty}\int_{T_\delta} \left|(C_\varphi(g^n)-C_\psi(g^n))^*\right|^2 d\sigma \\
\geq\ & \limsup_{n\to\infty}\left[\int_{T_\delta} \left|(g^n\circ\varphi)^*\right|^2 d\sigma - I_n\right] \\
=\ & \limsup_{n\to\infty}\int_{T_\delta} \left|(g^n\circ\varphi)^*\right|^2 d\sigma \geq \sigma(E) - 2\delta.
\end{aligned}$$

In other words, no matter which situation occurs, we have that

$$\begin{aligned}
\limsup_{n\to\infty}\|C_\varphi(g^n)-C_\psi(g^n)\|_2^2 &= \limsup_{n\to\infty}\int_{\partial B_N} \left|(C_\varphi(g^n)-C_\psi(g^n))^*\right|^2 d\sigma \\
&\geq \limsup_{n\to\infty}\int_{T_\delta} \left|(C_\varphi(g^n)-C_\psi(g^n))^*\right|^2 d\sigma \\
&\geq \sigma(E) - 2\delta = \sigma(E) - \varepsilon,
\end{aligned}$$

as we had hoped to show. □

The next result serves as a generalization of Proposition 4.2.

Proposition 4.3. *Take q to be a finite index. Let φ and ψ be distinct analytic self-maps of B_N. Suppose that $\sigma(E) > 0$, where E denotes the extreme set of φ; then, for any ε with $0 < \varepsilon < \sigma(E)$, there is a nonconstant unit vector g in $H^\infty(B_N)$ such that*

$$\limsup_{n\to\infty}\|C_\varphi(g^n)-C_\psi(g^n)\|_q^q \geq b(q)\left[\sigma(E)-\varepsilon\right]^{c(q)},$$

where

$$b(q) = \begin{cases} 1/2, & q < 2 \\ 1, & q \geq 2 \end{cases}$$

and

$$c(q) = \begin{cases} 1, & q \le 2 \\ q/2, & q > 2 \end{cases} .$$

Proof. Consider the function g given by Proposition 4.2. For $2 \le q < \infty$, our assertion follows from the fact that

$$\|C_\varphi(g^n) - C_\psi(g^n)\|_2 \le \|C_\varphi(g^n) - C_\psi(g^n)\|_q$$

for any natural number n. Now take $0 < q < 2$. For any $2 < s < \infty$, Hölder's inequality shows that

$$\begin{aligned} &\int_{\partial B_N} \left|(C_\varphi(g^n) - C_\psi(g^n))^*\right|^2 d\sigma \\ \le \; &\left(\int_{\partial B_N} \left|(C_\varphi(g^n) - C_\psi(g^n))^*\right|^q d\sigma\right)^\theta \left(\int_{\partial B_N} \left|(C_\varphi(g^n) - C_\psi(g^n))^*\right|^s d\sigma\right)^{1-\theta}, \end{aligned}$$

where $\theta = (s-2)/(s-q)$. Let us consider q to be fixed and s to be variable. Since $\left|(C_\varphi(g^n) - C_\psi(g^n))^*\right| \le 2$, we see that

$$\int_{\partial B_N} \left|(C_\varphi(g^n) - C_\psi(g^n))^*\right|^2 d\sigma \le 2^{s(1-\theta)} \left(\int_{\partial B_N} \left|(C_\varphi(g^n) - C_\psi(g^n))^*\right|^q d\sigma\right)^\theta .$$

As s goes to infinity, the quantities θ and $s(1-\theta)$ both tend to 1. Consequently

$$\int_{\partial B_N} \left|(C_\varphi(g^n) - C_\psi(g^n))^*\right|^2 d\sigma \le 2 \int_{\partial B_N} \left|(C_\varphi(g^n) - C_\psi(g^n))^*\right|^q d\sigma$$

for any n; thus our claim again follows directly from Proposition 4.2. □

We are now in the position to obtain an isolation theorem for composition operators acting between $H^p(B_N)$ and $H^q(B_N)$, akin to Berkson's [1] result for $\mathcal{C}\big(H^p(\mathbb{D})\big)$.

Theorem 4.4. *Take p and q to be finite indices. Let φ and ψ be distinct analytic self-maps of B_N that induce bounded composition operators from $H^p(B_N)$ into $H^q(B_N)$; then*

$$\|(C_\varphi - C_\psi) : H^p(B_N) \to H^q(B_N)\|_e \ge \left(b(q)\,[\sigma(E)]^{c(q)}\right)^{1/q},$$

where E denotes the extreme set of φ, with $b(q)$ and $c(q)$ defined as in the statement of Proposition 4.3. Similarly,

$$\|(C_\varphi - C_\psi) : H^\infty(B_N) \to H^q(B_N)\|_e \ge \frac{1}{2}\left(b(q)\,[\sigma(E)]^{c(q)}\right)^{1/q}$$

for any $0 < q < \infty$.

Proof. We need only consider the case where $\sigma(E) > 0$. In light of Proposition 4.3, simply apply Proposition 2.1 (for $0 < p < \infty$) and Proposition 2.2 (for $p = \infty$), then let ε tend to 0. □

We could, of course, merely concern ourselves with norms rather than essential norms. In this context, since each g^n is a unit vector in $H^\infty(B_N)$, we can modify the statement for $p = \infty$ to say that

$$\|(C_\varphi - C_\psi) : H^\infty(B_N) \to H^q(B_N)\| \ge \left(b(q)\,[\sigma(E)]^{c(q)}\right)^{1/q}$$

for any $0 < q < \infty$.

The next three corollaries, which follow directly from Theorem 4.4, provide a slightly less quantitative interpretation of the results in this section.

Corollary 4.5. *Take $0 < p \leq \infty$ and $0 < q < \infty$. Let φ be an analytic self-map of B_N that induces a bounded composition operator from $H^p(B_N)$ into $H^q(B_N)$. If the extreme set of φ has positive σ–measure, then C_φ is an isolated element of $\mathcal{C}\big(H^p(B_N), H^q(B_N)\big)$.*

Corollary 4.6. *Take $0 < p \leq \infty$ and $0 < q < \infty$. Let φ and ψ be distinct analytic self-maps of B_N that induce bounded composition operators from $H^p(B_N)$ into $H^q(B_N)$. If the operator $(C_\varphi - C_\psi) : H^p(B_N) \to H^q(B_N)$ is compact, then the extreme sets of φ and ψ must both have σ–measure 0.*

Corollary 4.7. *Take $0 < p \leq \infty$ and $0 < q < \infty$. Let φ be an analytic self-map of B_N that induces a bounded composition operator from $H^p(B_N)$ into $H^q(B_N)$. If the extreme set of φ has positive σ–measure, then the equivalence class containing C_φ is an isolated element of the quotient space $\mathcal{Q}\big(H^p(B_N), H^q(B_N)\big)$.*

Remark. Throughout this section, our numerical results have been given solely in terms of E, the extreme set of φ; we have not attempted to make use of the properties of φ and ψ simultaneously. In certain situations, however, we can obtain an isolation theorem stated in terms of both $\sigma(E)$ and $\sigma(F)$, where F denotes the extreme set of ψ. Suppose, for example, that both φ and ψ are nondegenerate, in the sense that neither $\varphi^* : E \to \partial B_N$ nor $\psi^* : F \to \partial B_N$ takes a set of positive σ–measure to a set with σ–measure 0; equivalently, the measures $\sigma\varphi^{*\,-1}$ and $\sigma\psi^{*\,-1}$ are absolutely continuous with respect to σ. (This situation occurs, for instance, whenever C_φ and C_ψ are bounded from $H^p(B_N)$ to $H^p(B_N)$ for $0 < p < \infty$; thus every $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to \mathbb{D}$ satisfy this condition.) In this case, we can modify Proposition 4.2, replacing g in $Y_\delta\big(\sigma\varphi^{*\,-1}\big)$ with g in $Y_\delta(\sigma)$ for an appropriate δ, to see that

$$\limsup_{n\to\infty} \|C_\varphi(g^n) - C_\psi(g^n)\|_2^2 \geq \sigma(E) + \sigma(F) - \varepsilon.$$

The remaining results in the section can then be altered accordingly. In particular, we can obtain a generalization of the isolation theorems of Shapiro and Sundberg [14] and Heidler [7].

5. Compactness and Component Structure

Proposition 3.1 provides us with a necessary condition for a bounded operator $C_\varphi : H^p(B_N) \to H^q(B_N)$ to be compact; namely, the extreme set of φ must have σ–measure 0. This condition, though, is generally insufficient to guarantee compactness. The situation when $p = q$, for example, is quite complicated. When $p > q$, however, it is often the case that having an extreme set with σ–measure 0 actually does imply compactness. The following proposition combines several previously known results along these lines.

Proposition 5.1. *Take $0 < q < p \leq \infty$. Let φ be an analytic self-map of B_N that induces a bounded composition operator from $H^p(B_N)$ into $H^q(B_N)$. Suppose that $\sigma(E) = 0$, where E denotes the extreme set of φ; then $C_\varphi : H^p(B_N) \to H^q(B_N)$ is compact as long as at least one of the following three conditions holds:*

$$\begin{aligned} (i)&\quad N = 1,\\ (ii)&\quad p = \infty,\\ (iii)&\quad q = 1. \end{aligned}$$

Proof. Jarchow [8] and Goebeler [5] independently established this result when $N = 1$. The $p = \infty$ case follows from Theorem 2 of Gorkin and MacCluer [6]. The case where $1 = q < p < \infty$ can be deduced from an argument similar to that used to prove Theorem 1 in [5]. $\square$

Remark. Gorkin and MacCluer, working in the setting where $N \geq 1$, obtained a related result that holds for finite values of p (see Corollary 2 in [6]). Their argument, however, requires a moderately stronger assumption regarding the boundedness of C_φ. In particular, taking $1 < q < p < \infty$, one can modify their proof to show that $C_\varphi : H^p(B_N) \to H^q(B_N)$ is compact as long as both $\sigma(E) = 0$ and C_φ is bounded from H^p to $H^{q+\varepsilon}$ for some $\varepsilon > 0$.

The compact composition operators play an important part in our analysis of the component structure of $\mathcal{C}\big(H^p(B_N), H^q(B_N)\big)$. The next proposition is an extension of a well-known result, originally stated (as Proposition 2.2 in [14]) for the space $\mathcal{C}\big(H^2(\mathbb{D})\big)$.

Proposition 5.2. *Take $0 < p \leq \infty$ and $0 < q < \infty$. The compact composition operators form a path-connected set in $\mathcal{C}\big(H^p(B_N), H^q(B_N)\big)$.*

Proof. We appeal to the standard argument used to establish this type of result, as it appears in the proof of Proposition 9.9 in [4], making the necessary adjustments to suit our situation. Only one detail warrants specific attention. Let I denote the identity map on B_N and take $0 \leq t < 1$. Observe that

$$\|C_{tI}(f)\|_q^q = \sup_{0<r<1} \int_{\partial B_N} |f(r(t\zeta))|^q \, d\sigma(\zeta) \leq \sup_{0<r<1} \int_{\partial B_N} |f(r\zeta)|^q \, d\sigma(\zeta) = \|f\|_q^q$$

for all f in $H^q(B_N)$. Hence the operator C_{tI} is a contraction on $H^q(B_N)$; moreover, if C_φ is compact from $H^p(B_N)$ to $H^q(B_N)$, so too is the operator $C_{\varphi_t} = C_{tI}C_\varphi$. $\square$

In general, it is unknown whether a noncompact composition operator can belong to the component (or path component) of $\mathcal{C}\big(H^p(B_N), H^q(B_N)\big)$ which contains the compact operators. Furthermore, it is often difficult to determine when two particular noncompact operators belong to a common component. These questions cease to be problematic, however, if we restrict our attention to the cases described in the statement of Proposition 5.1.

Theorem 5.3. *Take $0 < q < p \leq \infty$ and suppose that either $N = 1$, $p = \infty$, or $q = 1$. Let φ and ψ be distinct analytic self-maps of B_N that induce bounded composition operators from $H^p(B_N)$ into $H^q(B_N)$; then the following six conditions are equivalent:*

1. *Both φ and ψ have extreme sets with σ–measure 0.*
2. *Both of the operators $C_\varphi : H^p(B_N) \to H^q(B_N)$ and $C_\psi : H^p(B_N) \to H^q(B_N)$ are compact.*
3. *The operators C_φ and C_ψ belong to the same path component of $\mathcal{C}\big(H^p(B_N), H^q(B_N)\big)$.*
4. *The operators C_φ and C_ψ belong to the same component of $\mathcal{C}\big(H^p(B_N), H^q(B_N)\big)$.*
5. *The operator $(C_\varphi - C_\psi) : H^p(B_N) \to H^q(B_N)$ is compact.*
6. *The equivalence classes containing C_φ and C_ψ belong to the same component of the quotient space $\mathcal{Q}\big(H^p(B_N), H^q(B_N)\big)$.*

Proof. Propositions 3.1 and 5.1 show that conditions (1) and (2) are equivalent. Proposition 5.2 dictates that (2) implies (3). Condition (3) always implies (4). If (4) holds, then neither C_φ nor C_ψ can be isolated in $\mathcal{C}\big(H^p(B_N), H^q(B_N)\big)$; it follows from Corollary 4.5 that (4) implies (1). Since any linear combination of compact operators is also compact, condition (2) implies condition (5). If $C_\varphi - C_\psi$ is compact, then C_φ and C_ψ belong to the same equivalence class in $\mathcal{Q}\big(H^p(B_N), H^q(B_N)\big)$, so (5) automatically implies (6). Corollary 4.7 shows that (6) implies (1). □

Stated more succinctly, Theorem 5.3 provides a complete characterization of the component structure of $\mathcal{C}\big(H^p(B_N), H^q(B_N)\big)$, and of the quotient space $\mathcal{Q}\big(H^p(B_N), H^q(B_N)\big)$, in the relevant cases:

Corollary 5.4. *Take $0 < q < p \leq \infty$ and suppose that either $N = 1$, $p = \infty$, or $q = 1$. The set of compact operators forms a single component in $\mathcal{C}\big(H^p(B_N), H^q(B_N)\big)$; every noncompact operator constitutes its own component.*

Corollary 5.5. *Take $0 < q < p \leq \infty$ and suppose that either $N = 1$, $p = \infty$, or $q = 1$. The quotient space $\mathcal{Q}\big(H^p(B_N), H^q(B_N)\big)$ is totally disconnected.*

Remark. While it is evident that the results of Theorem 5.3 do not hold for general values of p and q, the question of when conditions (4) and (5) are equivalent has been a point of interest for some time. Shapiro and Sundberg [14] originally stated a "question/conjecture" along these lines for the space $\mathcal{C}\big(H^2(\mathbb{D})\big)$, suggesting that the two conditions might indeed be equivalent in this context. As it turns out, though, Bourdon [2] and Moorhouse and Toews [11] were able to produce examples of $\varphi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to \mathbb{D}$ such that $(C_\varphi - C_\psi) : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is not compact, yet the operators C_φ and C_ψ belong to the same component of $\mathcal{C}\big(H^2(\mathbb{D})\big)$. The fact that the conditions *are* equivalent under the hypotheses of Theorem 5.3 bears witness to the much simpler component structure of the corresponding spaces $\mathcal{C}\big(H^p(B_N), H^q(B_N)\big)$, which in turn can be attributed to the remarkably straightforward characterization of the compactness of C_φ.

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