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Deductive Reasoning Under Uncertainty: A Water Tank Analogy

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Abstract

This paper describes a cubic water tank equipped with a movable partition receiving various amounts of liquid used to represent joint probability distributions. This device is applied to the investigation of deductive inferences under uncertainty. The analogy is exploited to determine by qualitative reasoning the limits in probability of the conclusion of twenty basic deductive arguments (such as Modus Ponens, And-introduction, Contraposition, etc.) often used as benchmark problems by the various theoretical approaches to reasoning under uncertainty. The probability bounds imposed by the premises on the conclusion are derived on the basis of a few trivial principles such as "a part of the tank cannot contain more liquid than its capacity allows", or "if a part is empty, the other part contains all the liquid". This stems from the equivalence between the physical constraints imposed by the capacity of the tank and its subdivisions on the volumes of liquid, and the axioms and rules of probability. The device materializes de Finetti's *coherence* approach to probability. It also suggests a physical counterpart of Dutch book arguments to assess individuals' rationality in probability judgments in the sense that individuals whose degrees of belief in a conclusion are out of the bounds would commit themselves to executing physically impossible tasks.

Keywords:

Uncertain reasoning; inference schemas; Bayesian coherence; probability logic; qualitative reasoning

1. Introduction

The description of cross-classified categorical data is an uneasy task, and for a long time communicators have made use of various graphical representations that help acquire, process and mentally represent this information (see Friendly, 2002, for an historical account). Contingency tables with r rows and c columns do not count as diagrammatic representations because they are just typographical displays of numbers that do not possess the analogical or computational qualities of diagrams. A crucial step is reached when the table is divided into rectangles whose areas are proportional to the quantities present in the joint distributions. This type of representation was developed in detail by Hartigan & Kleiner (1981) under the name of "mosaic", mainly as a tool to expose deviation from independence. It had been used earlier without elaboration by Bertin (1967 p. 256) and Edwards (1972, p. 47). Edwards' diagram (Fig. 1) illustrates the joint distribution of boys and girls in a school crossed with long-haired and short-haired children. The fundamental characteristic of this diagram is that the areas of the various rectangles are proportional to the products of the marginal probabilities.

Figure 1. Edwards' (1972) representation of a two-way distribution.
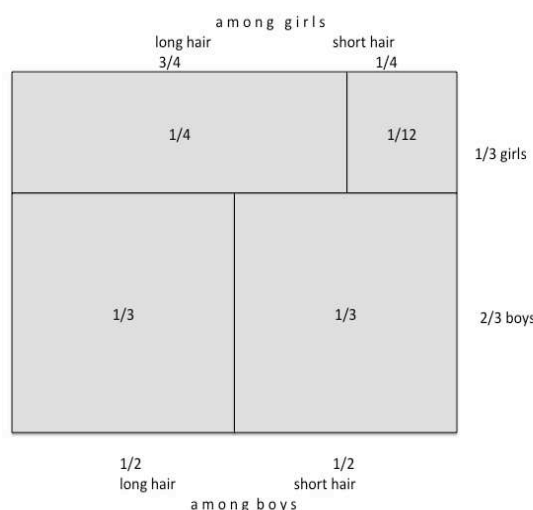
More recently, Oldford (2003a, 2003b; Oldford & Cherry, 2006) investigated the properties of diagrams of this type, which he called "eikosograms". He showed their usefulness to represent conditional probability, conditional and unconditional dependence or

independence. Considering a wide range of classic problems and puzzles such as the Engineer-and-Lawyer problem, medical tests, the prisoner's dilemma, the Monty Hall, Simpson's paradox, and many others, his demonstration of the effectiveness of eikosograms to represent the information and calculate the solution is deeply impressive. In these papers Oldford mentioned an interpretation of the diagrams in terms of a "water container metaphor" to infer a marginal value when the joint probabilities are known but did not pursue this idea any further. Moreover, the use of the diagrams was strictly limited to problem solving.

In fact, the way in which mosaic representations are commonly used is static. They display sets of data in which all values (generally frequencies) are known or determined and aim to help exhibit relations of dependence. Contrary to this usage, the present paper will concern itself with a *dynamic* representation, in which the values in the table are probabilities that can vary. It is devoted to the development of an analogy of the laws of conditional probability applied to deductive schemas of inference under uncertainty, using a two-compartment water tank presented in a diagrammatic form. In the first section of the paper the basic diagram, the probabilistic interpretation of its components, and the representation of elementary laws of conditional probability are presented. In the main part a number of deductive inferences are interpreted in terms of the analogy, and their validity with respect to several criteria is systematically examined. The last section presents a brief discussion of the relationship between the physical principles used in the tank analogy to derive the probability of the schemas' conclusions and the axioms and rules of probability theory, and the use of the former as an operational justification of the latter.

1. The water tank analogy

1.1.The basic diagram

A cubic tank has a movable partition (parallel to one side) that divides it into two compartments, left (A) and right (A'). In what follows, all the diagrams represent a vertical cross-section of the tank perpendicular to the partition (see Fig. 2). Each compartment can contain some amount of liquid independently of the other.

Figure 2. The basic analogy.
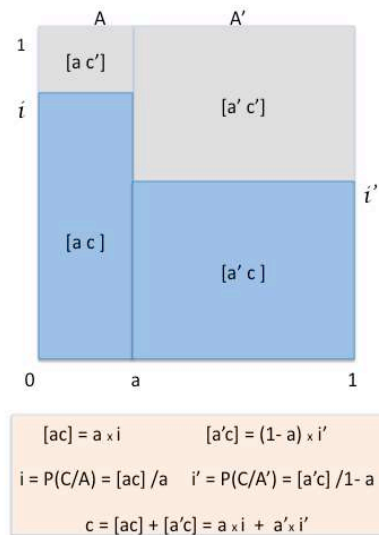
The dimensions of the tank are: height = 1 unit, width = 1 unit. The width of A is denoted by $\underline{a}$ ($0 \leq a \leq 1$). It is a measure of the capacity of A. The capacity of A' is measured by a' = 1- a.

The tank has a full capacity of 1 unit. Its overall content (occupying rate) is denoted by $\underline{c}$ ($0 \leq c \leq 1$).

The levels of liquid in A and A' are denoted by $\underline{i}$ and $\underline{i'}$, respectively. The volumes of liquid in A and A' are denoted by [ac] and [a'c], respectively. Similarly the volumes of the empty spaces in A and A' are denoted by [ac'] and [a'c'].

The contents (volumes) of A and A' are $i \times a$ and $i' \times a'$, respectively. $i \times a$ is a measure of [ac], and $i' \times a'$ of [a'c'].

1.2. Probabilistic interpretation

A is the event "the left compartment occupies the whole tank"; $\underline{a}$ is the probability of this event, that is, the degree to which the whole tank is occupied by A (which will also be denoted by P(A)).

Calling C the event "the tank is filled up with liquid", $\underline{c}$ is the probability of C, that is, the degree to which the tank is filled with liquid (or the volume of liquid in the tank), which will also be denoted by P(C). (Note 1).

The event relative to compartment A, "A is filled up with liquid" is a *conditional event* denoted by C/A, $\underline{i}$ is the probability of this event, that is, the degree to which A is filled with liquid (which will be also denoted by P(C/A). The level $\underline{i}$ of liquid in A visually represents the probability of the conditional event C/A, that is, the probability of *if A then C.* (Similarly the level $\underline{i}'$ in A' visually represents the probability of the conditional event C/A', *if A' then C).*

The joint probability of A and C, P(A AND C) is represented by the liquid common to A and C, that is, the volume of liquid in A, which is measured by a x i:

P(A AND C) = P(A) x P(C/A) (also denoted by [ac]) (Note 2).

Similarly,   P(NOT-A AND C) = P(NOT-A) x P(C/ NOT-A) (also denoted by [a'c]);

P(A AND NOT-C) = P(A) x P(NOT-C/A) (also denoted by [ac']);

P(NOT-A AND NOT-C) = P(NOT-A) x P(NOT-C/ NOT-A) (also denoted by [a'c']).

The event relative to the liquid, "all the liquid is contained in A", is a conditional event denoted by A/C, its probability denoted by P(A/C) is the degree to which all of the liquid is in A, which is the probability that *if C then A.*

The correspondence between the physical features constitutive of the water tank and the axioms of probability are the following:

1) The measures of volume or capacity are ≥ 0. This is the counterpart of the requirement that the probability of an event E defined on a sample space Ω be such that P(E) ≥ 0.

2) The tank (or a compartment) cannot contain more than its capacity. This the counterpart of P(Ω) = 1 and P(A/A) = 1, respectively.

3) Given an amount of liquid in the tank (or in a compartment), adding some more liquid results in an amount whose volume is the sum of the two. This the counterpart of the axiom of additivity.

This extends the correspondence between the axioms of probability and the graphic features of eikosograms shown by Oldford & Cherry (2006).

1.3. Elementary rules

The water tank representation allows a clear representation of Bayes's rule and probabilistic independence, as eikosograms do (Oldford & Cherry, 2006). The water tank also allows an easy representation of the total probability rule. With our current interpretation and notations, these are as follows.
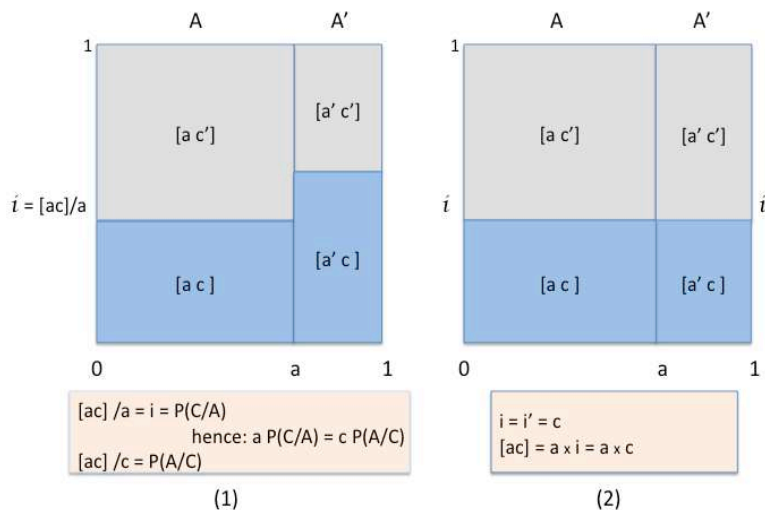
1.3.1. Bayes' rule

The amount of liquid in A, [ac], can be viewed as the part of A that is filled, a $_x$ P(C/A), or the part of liquid that is in A, c $_x$ P(A/C) (see Fig. 3-1), hence: a $_x$ P(C/A) = c $_x$ P(A/C), or:

$$P(A/C) = (a \times i) / c$$

1.3.2. Representation of independence

The levels are the same in A and A': c = i = i', hence: c = [ac] / a = [a'c] / a' (see Fig. 3-2).

Figure 3. (1) Bayes' rule. (2) Independence

1.3.3. The total probability rule

It is obtained by adding the contribution of each compartment to the whole (see Fig. 2).

$$c = [ac] + [a'c] = (a \times i) + (a' \times i')$$

2. Deduction under uncertainty

The recent interest in reasoning under uncertainty, more specifically in deductive reasoning with uncertain premises, has led to the elaboration of probabilistic logics (Adams,

1998; Coletti & Scozzafava, 2002; Gilio, 2002; Hailperin, 1996; Pfeifer & Kleiter, 2006, 2009) and their psychological investigation (George, 1995, 1997, Oaksford & Chater, 2007; Pfeifer & Kleiter, 2010, 2011; Politzer & Bourmaud, 2002). It is to this field of research that we are going to apply the tank analogy. We will be concerned with sentences that express unconditional or conditional events, the probability of which is represented by the components of the tank (compartments and volumes of liquid). Boolean operations on elementary sentences can be represented in the tank analogy and these will be used when applicable to represent the premises and conclusions of the inference schemas.

Figure 4. Boolean operations.
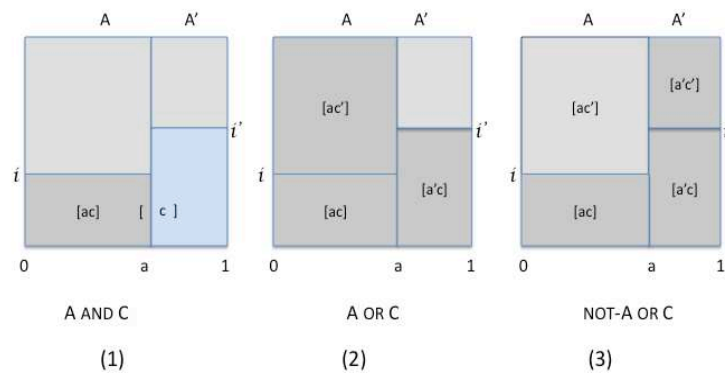
The volume of liquid contained in A (the dark shaded area [ac] in Fig. 4-1) is, *eo ipso,* the volume common to the compartment A and to C, so that it represents the conjunction A AND C. Disjunction A OR C is represented by the volume occupied by A or by the liquid (the dark shaded area in Fig. 4-2). Similarly, the material conditional (NOT-A OR C) is represented by the volume that is in A' or in the liquid (Fig. 4-3).

The fundamental question that we are going to address is "What constraints do the probability of the premises impose on the probability of the conclusion?" A major choice for probabilistic logics is to define a notion of validity that is the counterpart of logical validity in standard deductive logic. It is not our aim to do so. Rather, we will limit ourselves to the consideration of a few properties of deductive inference schemas under uncertainty which

either represent possible notions of validity or may have psychological significance besides their technical relevance. A given schema does or does not possess each of these properties. Properties (1) and (3) are equivalent to two of the four properties defined by Adams (1996).

1) The probability of the conclusion is no lower than the probability of the premises. It seems to be desirable and possibly critical for the reasoner to assess whether, or be aware that, the probability of the conclusion is higher, equal, or lower than that of the premises. When the degree of belief in the conclusion is no lower than the degree of belief in at least one of the premises, we will call the inference "conservative", and "dissipative" when it is no higher than the smallest of the degrees of belief in the premises.

2) The confidence in the conclusion is high, irrespective of the degree of belief in the premises. In that case we will call the inference "forceful", and "feeble" when the confidence is low.

3) Highly believable premises warrant a highly believable conclusion. In that case we will say that the inference is "robust". This entails security and forcefullness, but not reciprocally. It is entailed by conservativeness, but not reciprocally.

The conclusion can be characterized by a probability interval whose width may vary and consequently be more or less informative. In the extreme case where the whole interval [0, 1] is obtained, the conclusion is totally uninformative and the inference useless.

The general method that we will follow consists in interpreting the premises and the conclusion as events and/or conditional events and translate them in terms of partitions of the tank and filling of the parts so defined, with volumes for events and levels for conditional events representing the values of the probabilities. Then we examine the probability constraints imposed by the premises on the conclusion.

2.1. And-introduction: A; C ∴ A AND C

We wish to determine the probability of the conjunction A AND C knowing the probability of its components, A, and C. That is, given the capacity $\underline{a}$ and the amount $\underline{c}$ to be poured into

the tank, what can be the part of C that is in A? (see Fig. 2). Trivially the part of c that could be in A, [ac], cannot exceed a nor can it exceed c itself: $[ac] \leq \min \{a, c\}$. Does the content of A have a lower bound? Filling A' as much as we can, if c is smaller than its capacity ($c \leq 1 - a$), A remains empty and $[ac] = 0$; if $c > 1 - a$, A receives $c - (1 - a)$, hence: $[ac] \geq \max \{a+c-1, 0\}$. Hence:

$$P(A \text{ AND } C) \in [\max \{0, a+c-1\}, \min \{a, c\}]$$

Breaking the upper bound amounts to committing the *conjunction fallacy* by which the conjunction of two events is estimated as more likely than one of the conjuncts.

This inference is **dissipative**, as the upper bound equals the minimum of either premise. It is also **robust**, as the lower bound is close to 1 when both premises have a probability close to 1.

### 2.2. If-introduction: A;   C   ∴   IF A THEN C

We wish to determine the probability of the conditional *if A then C* knowing the probability of its components, A and C. The question amounts to the following: Given a (the capacity of A), and the amount c to be poured into the tank, what are 1) the highest, and 2) the lowest possible levels in A?

1) One fills up A first (see Fig. 5-1). Trivially, $i = 1$ if $c \geq a$ (A is filled up when there is as much or more liquid than A can receive), and $i = a/c$ otherwise, hence: $i \leq \min \{c/a, 1\}$.

2) One fills up A' first. If $c \leq a'$, A remains empty and $i = 0$. If $c > a'$, A' is filled up and the excess is poured into A (see Fig. 5-2) which receives $c - a'$, hence its level $(c - a')/a$, that is, $(c -1+a) / a$. Hence : $i \geq \max \{0, (c -1+a) / a\}$. Therefore:

$$P(C/A) \in [\max \{0, (c -1+a) / a\}, \min \{c/a, 1\}]$$

Figure 5. If-introduction: Inference from P(A), P(C) to P(C/A).

(1)   (2)
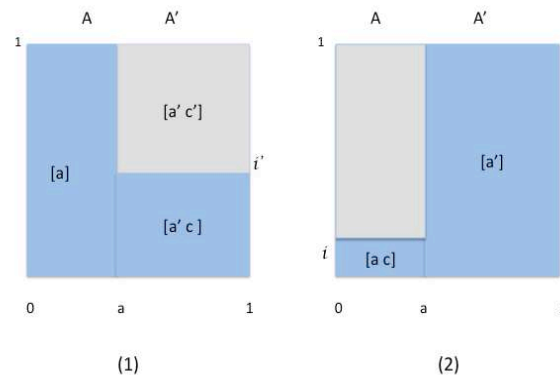
This inference is neither conservative nor dissipative, as the bounds can reach 0 while a and c differ from 0, or reach 1 while a and c differ from 1. Whenever a and c are high, so is i, and the inference is **robust**.

**2.3. And-elimination:**   A AND C ∴ A   A AND C ∴ C

There is always increase in probability from premise to conclusion: the amount of liquid in A, [ac], being fixed, it is trivially smaller than, or equal to, either the capacity of A, a, or the total amount of liquid c, that is: [ac] ≤ a, and [ac] ≤ c. Or equivalently, if A contains [ac], its capacity cannot be less than [ac] (but it can contain more); and similarly if some of the liquid is in A there cannot be less liquid overall but there can be some more in A'. The inference is **conservative.**

$$P(A), P(C) \in [P(A \text{ AND } C), 1]$$

2.4. 'And' to 'if':   A AND C ∴ IF A THEN C

What are the lowest and highest levels in A, knowing the content of the left compartment? The lowest level is obtained when A has the greatest capacity, that is when A has the greatest base, a = 1 (Fig. 6-1). Now moving the partition leftwards (disregarding the content of A', c being free to vary), i can only increase (Fig. 6-2), showing that we always have i ≥ [ac]. The limit i = 1 is reached when [ac] = a (Fig. 6-3). Arithmetically of course, i ≥ [ac]

Figure 6. Inference from AND to IF.
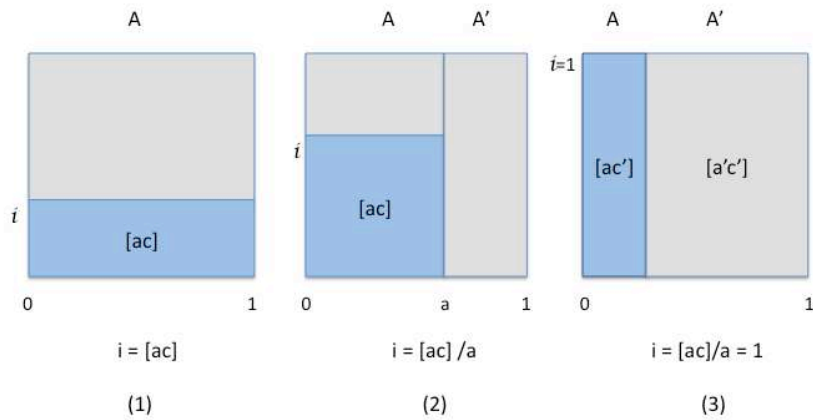
because the ratio i = [ac] /a is ≥ 1). In other words, if an amount of liquid in A at least as great as [ac] is warranted, then a level i̲ that is at least as high as [ac] is also warranted, meaning that the inference from *and* to *if,* is **conservative.**

$$P(C/A) \in [\ P(A\ \text{AND}\ C), 1\ ]$$

For the same reason, the reverse inference from *if* to *and* is dissipative.

2.5. Proof by cases: IF A THEN C; IF NOT-A THEN C ∴ C

The levels of liquid i̲ in A and i̲' in A' being fixed, we are looking for the volume c̲ (see Fig. 7-1). We consider what happens when a̲ varies from 0 to 1. If i = i', there is independence and c̲ is determined and equal to i and i'. If not, assume i > i'. When a̲ reaches 0, the level in the tank equals the volume in A', c = i' and cannot be smaller (Fig. 7-2). When a̲ reaches 1 the level in the tank equals the volume in A, c = i and cannot be greater (Fig. 7-3). By continuity c̲ varies between i̲' and i̲ when a̲ varies between 0 and 1 (Fig. 7-1), hence a **conservative** inference:

$$P(C/A) = i;\ \ P(C/\text{not-A}) = i';\ \ P(C) \in [\ \min\{i, i'\}\ ,\ \{\max\{i, i'\}\ ]$$

Figure 7. Proof by cases.

(1)   (2)   (3)
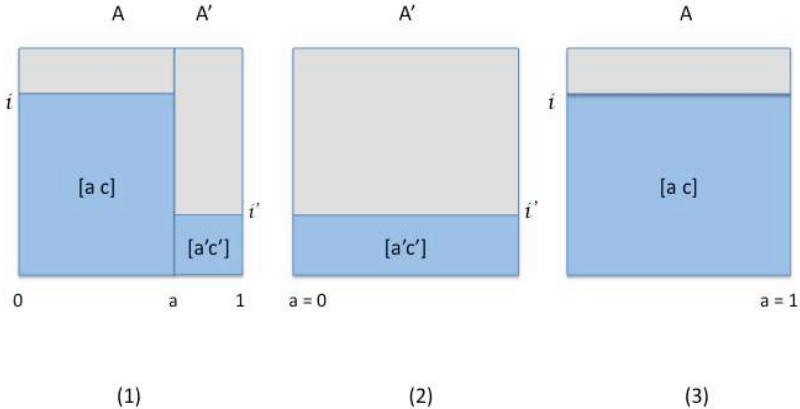
2.6. Consequent to 'if' :   C   ∴   IF A THEN C

What is the probability of the conditional, knowing the probability of its consequent? In other words, what is the level in the left compartment A, knowing the amount of liquid c in the tank ? For i to reach 1 (A full), it will suffice that there is enough liquid to fill it up, that is, the capacity of A should be no greater than the volume of liquid, which is always possible to fix (see the limiting case where a = c, Fig. 8-1). Similarly for i to be null (A empty) it suffices for the volume of liquid it contains to be smaller than the empty volume in A', which is automatically satisfied when c is small, and possible to fix when c is high by reducing and limiting A under a value such that a ≤ 1- c (Fig. 8-2). By continuity any intermediate level for i can be obtained when a varies between the two limiting positions. Therefore **any confidence value** can be given for the conclusion:

$$P(C) = c; \qquad P(C/A) \in [\ 0, \ 1]$$

Figure 8. Inference from Consequent to IF.
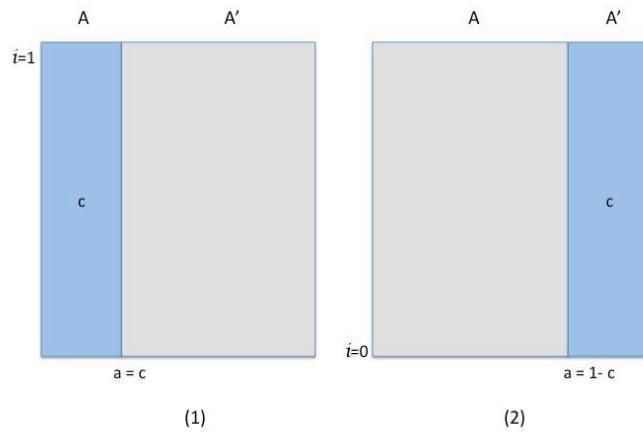
2.7. 'If' to 'not-A or C' ('if' to the material implication): IF A THEN C ∴ NOT-A OR C

Knowing the level i̱ in the left compartment A, what is the volume occupied by the whole

right compartment A' or by the liquid? Its measure is never lower than the measure of i̱

because, as shown in Fig. 9, there is always some space in A' above the level i̱ up to the top

(the darker area whose measure equals $(1- a)(1- i)$). $P(\text{not-A or C})$ is measured by $(1 \times i) + (1-$

$a)(1- i) = i + (1- a)(1- i)$. The difference between the probabilities of the premise and the

conclusion is $(1- a)(1- i)$ which is positive or null. It is maximal when $i = 0$ and null when $i = 1$.

This inference is **conservative**: the probability of the conditional never exceeds that of the

material conditional:

$$P(\text{NOT-A OR C}) \in [P(C/A), 1]$$

Figure 9. Inference from IF to NOT-A OR C.
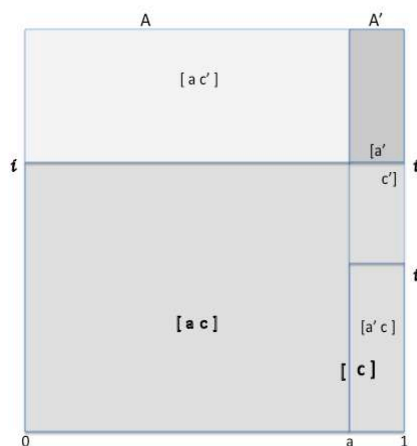
2.8. Or-introduction: A   ∴   A OR C

Knowning the capacity of A, what are the smallest and the largest possible volumes obtained by adding a volume $\underline{c}$ of liquid? If we pour the liquid into A, either it can be contained (meaning c ≤ a), making the total volume equal to $\underline{a}$; or it cannot (meaning c > a) and makes the total volume equal to $\underline{c}$, hence the lower bound: max {a, c}. To get the largest possible volume we start pouring into A' and stop either when the liquid is exhausted, reaching a volume equal to a+c or when A' is full, hence the upper bound: min {a+c, 1}. This inference is **conservative**.

$$P(A \text{ OR } C) \in [\max \{a, c\}, \min \{a+c,1\}]$$

2.9. 'And' to 'or':   A AND C   ∴   A OR C

The amount [ac] is not greater than $\underline{a}$ or than $\underline{c}$ (by and-elimination), which in turn is not greater than A OR C (by OR-introduction), so that P(A OR C) ≥ P(A&C). This inference is **conservative**.

$$P(A \text{ OR } C) \in [P(A\&C), 1]$$

2.10. 'Or' to 'if not-':   A OR C   ∴   IF NOT-A THEN C

The conclusion represents the level in A', while the premise represents the volume occupied by the left compartment or by the liquid, that is, the capacity of A augmented with the content of A' (Fig. 10-1). Now trivially, the global rate calculated on the basis of A being full necessarily exceeds (or equals) the rate in A' only, showing that one cannot warrant a level in A' that would numerically exceed the content of A' augmented with the capacity of A. This is why the probability of the conclusion can never exceed that of the premise: the inference is **dissipative.**

Figure 10. Inference from OR to IF-NOT.
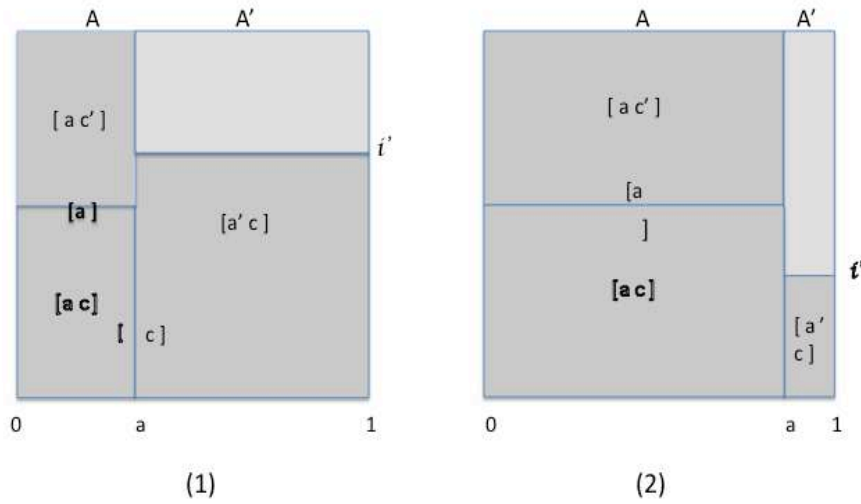
Consider the probability of the premise A OR C fixed (noted d): [a'c] + a = d, hence:

(a'x i') + a = d

$$i' = (d - a) / (1- a)$$

(see Gilio & Over, 2012 for the same result).

The graph of i' as a function of d with a as a parameter (Fig. 11) shows that i' is a linearly increasing function of d and that we always have i' ≤ d, showing that the inference is (non strictly) **dissipative** (the exception occurs trivially when a = 0 and then i' = c = d) and that the limits for i' are: 0 ≤ i' ≤ d, hence:

$$P(\text{IF NOT-A THEN C}) \in [\ 0, P(A\ OR\ C)\ ]$$

Figure 11. Inference from OR to IF-NOT. Variation of i' = P(C/NOT-A) as a function of d = P(A OR C) for various values of a = P(A).
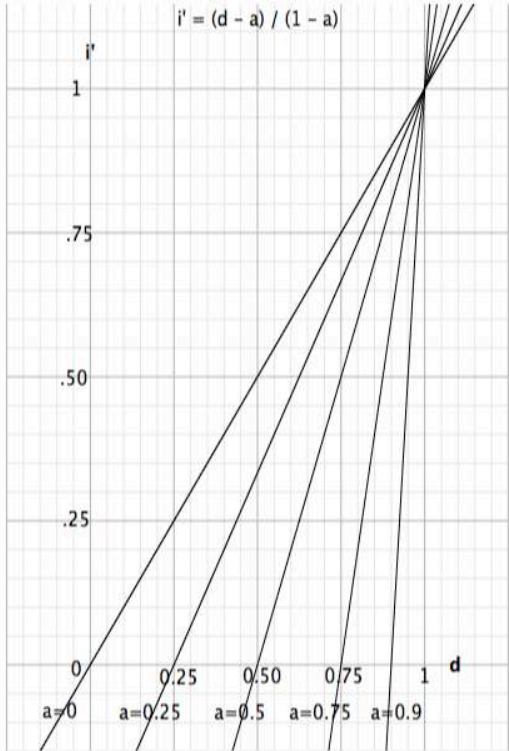
Notice that when $\underline{a}$ increases and tends to 1 (Fig. 11) A' tends to 0 and the amount of liquid it contains becomes very small and has little contribution to *A or C*, so that $\underline{i'}$ is susceptible of wide variation, in particular it can be very low. In this case only is it possible for *A or C* to be high believable while *if not-A then C* is not. By contrast, when $\underline{a}$ is not so large (Fig. 10-2), for *A or C* to be highly believable it is crucial that the amount of liquid in A' be high, and so the level $\underline{i'}$ be high too, which means that *if not-A then C* has a believability close to that of *A or C* and the inference is **forceful**. This is illustrated in the graph (Fig. 11) which shows that the greater $\underline{a,}$ the greater the slope of the line, indicating that $\underline{d}$ is less and less affected by a variation of $\underline{i'}$. Gilio & Over (2012) discuss in depth the role of the magnitude of P(A) relative to P(A or C) (their measure of constructivity) in defining the weak versus the strong inference.

2.11. Contraposition: IF **A** THEN **C** ∴ IF NOT-**C** THEN NOT-**A**

The conclusion represents the extent to which the right compartment contributes to the vacuum (the emptiness of the tank). This rate will be denoted by i_c.

Consider the two extreme cases. One, when the left compartment is full ($i = 1$, full belief in the premise) all the empty volume is in A' so that $i_c = 1$ and the contrapositive is fully believable (see Fig. 12-1).

Figure 12. Contraposition.

(1)   (2)   (3)
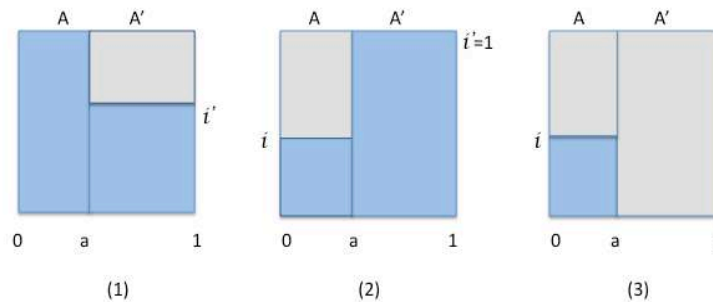
Two, when A occupies the whole tank ($a = 1$), the empty volume is entirely in A, and $i_c = 0$. In the general case the contribution of the right compartment varies from zero when it is full ($i_c = 0$, see Fig. 12-2) to a maximum when it is empty, yielding $i_c = (1 - a) / (1 - c)$ (Fig. 12-3). But this maximum is susceptible of variation depending on the relative size of the compartments: as A decreases A' increases and the contribution of A' increases until i_c reaches 1. In sum, The contrapositive can have all the degrees of confidence between 0 and 1.

$$P(C/A) = i; \quad P(\text{NOT-C}/\text{NOT-A}) \in [0, 1]$$

The foregoing result has been obtained under the assumption that i is the only fixed variable. However, it is interesting to study the constraints superimposed by other variables (a and i') in order to know under which circumstances the contrapositive has different bounds. Note that, strictly speaking, we are considering a new inference defined by the addition of two premises (a and i'). For this purpose it suffices to express i_c as the ratio of the empty space in A', $(1 - a)(1 - i')$ to the whole empty space $1-c$:

$i_c = (1 - a)(1 - i') / (1 - c)$

$i_c = (1 - a)(1 - i') / [(1 - a)(1 - i') + a(1 - i)]$

This is the fundamental equation for contraposition expressing the functional relation between the contrapositive $\underline{i_c}$ and the conditional $\underline{i}$, with $\underline{a}$ and $\underline{i'}$ as parameters. After the change of variable $y = i_c$ and $x = i$, it becomes:

$$y = (1 - a)(1 - i') / [(1 - a)(1 - i') + a(1 - x)] \qquad \text{(Eq)}$$

We rewrite (Eq) as:

$$y = u /(v - ax), \text{ with } u = (1- a)(1- i'), \text{ and } v = u + a.$$

It can be verified that this function is strictly increasing (first derivative strictly positive) and that it is concave (second derivative strictly positive).

Knowing whether the inference is conservative or not amounts to knowing the conditions under which $y \geq x$, that is: $u /(v-ax) \geq x$, or:

$$ax^2 - vx + u \geq 0 \qquad \text{(Ineq)}$$

The associated equation can be factorised as:

$$a(x-1)(x - u/a) = a(x-1)(x-j) \text{ where } j = u/a = (1- a)(1- i') /a.$$

As a result, the sign of the difference $y-x$ is determined as a function of $\underline{a}$ as follows:

• if $j \geq 1$, there is no solution (other than x=1) within [0, 1], and (Ineq) is always satisfied, which means that for all $i \in [0, 1]$, $i_c \geq i$.

• if $j < 1$, there is a solution smaller than 1 which means that (Ineq) is satisfied on an interval [0, j] and not satisfied on [j, 0].

The condition $j \geq 1$ is equivalent to $i' \leq (1-2a) / (1-a)$ or to $a \leq (1- i') / (2-i')$, and similarly the condition $j < 1$ is equivalent to $i' > (1-2a) / (1-a)$ or to $a > (1- i') / (2-i')$, so that in summary:

when $a \leq (1- i') / (2-i')$, the inference is **conservative** ($i_c \geq i$) for all $\mathbf{i} \in \mathbf{[0, 1]},$

when $a > (1- i') / (2-i')$, the inference is **conservative** ($i_c > i$) for $\mathbf{i} \in \mathbf{[0 , (1-a)(1-i') /a]} ,$

and **dissipative** ($i_c < i$) for $\mathbf{i} \in \mathbf{[(1-a)(1-i') /a , 1].}$

Fig. 13 illustrates this discussion for the value of the parameter $\underline{i'}$ fixed arbitrarily at i' = .4 and for several values of the parameter $\underline{a}$ ranging from 0.2 to 0.95. It appears that the

inference can be forceful only when $\underline{i}$ is high. When this is satisfied, inferring the

contrapositive may be viewed as quite reasonable.

Figure 13. Probability of the contrapositive i_c plotted against i = P(C/A) for various values of a = P(A) and for i' = P(C/not A) fixed arbitrarily at 0.4. $i_c \geq i$ for $0 \leq i \leq j$ (j = intersect of f(i) and the diagonal).
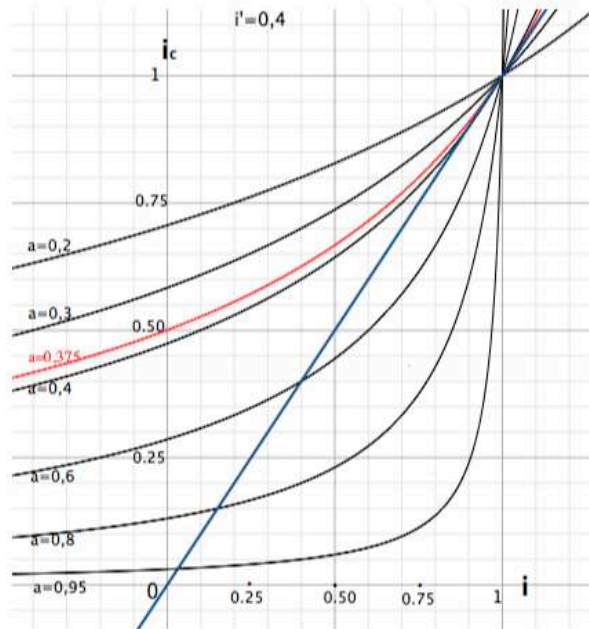
2.12. Modus Ponens: IF A THEN C;   A   ∴   C

   The combination of the premises $\underline{a}$ and $\underline{i}$ yields a x i = [ac] as the content of A. Obviously

the content of the whole tank, $\underline{c}$ cannot be less, which gives a lower bound for the confidence

in C: one must not be less confident in the conclusion than in the product of the levels of

confidence in the premises. On the other hand, A being limited to level $\underline{i}$, the tank cannot be

full; $\underline{c}$ cannot exceed the content of A, a x i, augmented with the capacity of A', that is, (a x i) +

(1 - a) (see Fig. 14). Therefore:

$$P(C/A) = i;\; P(A) = a;\qquad P(C) \in [a \times i,\; (a \times i) + 1 - a]$$

   Modus Ponens is not conservative, as its lower bound a x i is smaller or equal to the

confidence in either premise (a, i); it is not dissipative, as its upper bound is greater or equal

to the confidence in one of the premises, i; but it is **robust**, as a high confidence in the premises (a, i) warrants a high confidence in the conclusion because $c \geq a \times i$.

Notice that when A occupies the whole tank, the level in the tank is given by i: $c = i$ and so c is known with the least uncertainty (as equal to i). Now suppose A decreases while i is fixed (say = 1). A' increases and because i' is unknown the uncertainty on c increases; when a becomes null A' occupies the whole tank and c can be anything between 0 and 1: this is the non probabilistic case of Denying the Antecedent where i=1 and a=0. This shows that MP and DA are the two sides of a same coin; the variable is A in one case and A' in the other case.

Figure 14. Modus Ponens.
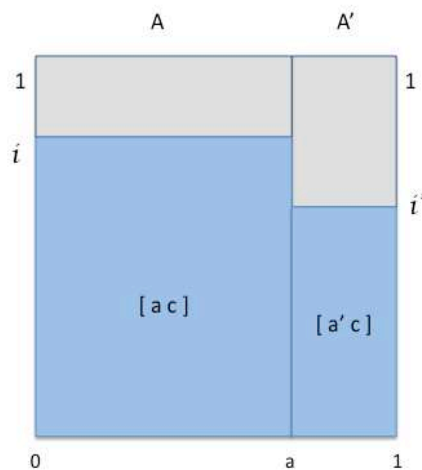
2.13. Denying the Antecedent   IF **A** THEN **C**;   NOT-**A**   ∴   NOT-**C**

The same equation as for MP can be used, replacing c with c' =1- c, which yields:

$$P(C/A) = i; \quad P(\text{NOT-A}) = a'; \quad P(\text{NOT-C}) \in [\, a(1- i)\, ,\, 1 - (a \times i)]$$

This represents the degree of emptiness of the tank knowing the level in A and the size of A'. The equation shows that it is rational to have a level of confidence in the conclusion of DA that lies between the bounds indicated. In particular, trivially, it would be irrational to assess the total void as lower than a - (a x i), that is, lower than the void in A.

The result can also be given in terms of a' = P(NOT-A):

$$P(\text{NOT-C}) \in [(1- i)(1 - a') , \ 1 - i \,(1 - a')]$$

### 2.14. Affirming the Consequent: IF A THEN C;   C   ∴   A

Given the level in A and the content $\underline{c}$, trivially there cannot be more liquid in A than the whole amount $\underline{c}$, that is: a x i ≤ c, hence: a ≤ c / i (which is automatically satisfied when i ≤ c since a ≤ 1). Similarly, A cannot incorporate more void than the whole void in the tank, that is: (1 - i) x a ≤ 1- c, hence a ≤ (1- c) / (1- i) (which is automatically satisfied when c ≤ i since a ≤ 1). It follows that:

a ∈ [0, c / i] if i ≥ c, and a ∈ [0, (1- c) / (1- i)] if i ≤ c, which can be summarize by:

$$P(C/A) = i; \ \ P(C) = c; \qquad P(A) \in [\, 0, \min \{\, c / i, \ (1- c) / (1- i) \,\} \,]$$

Fig. 15 represents the graph of $\underline{a}$ plotted against $\underline{c}$ for two values of $\underline{i}$ as a parameter.

Figure 15. Inference of Affirming the consequent. Graph of the
conclusion a = P(A) as a function of the minor premise c = P(C)
for two values of the major premise i = P(C/A).
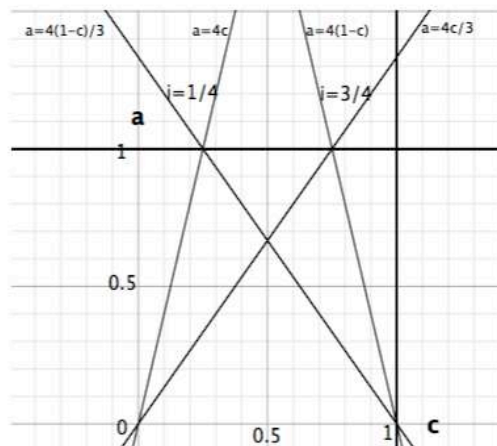
For a given value of $\underline{i}$, the range of $\underline{a}$ is obtained by referring to the triangle whose base is [0, 1] and the apex on the line a=1. For any value of $\underline{c}$, the ordinate of the corresponding side of the triangle provides the range of $\underline{a}$. It is noteworthy that when $\underline{c}$ nears 1 the upper limit of

a decreases (and becomes null when c=1). If you know that the left part is full to, say, three quarters but also that the tank is nearly full, the left part must be small. There is an exception to the decrease of a as c increases, which occurs when i=1 and c=1, in which case a ∈ [0, 1], which is the non probabilistic fallacy of AC. This can be viewed as a special case of what happens when i and c are equal: the equality of the level in A with the rate of filling of the tank means independence, so that the position of the partition can be anywhere, in other words the inference is totally uninformative.

Suppose now that one has some belief in i (i > .5) and disbelieves c (c < .5): the range of a is measured by the ordinate on the left side of the triangle: the more c decreases, the more a decreases. This is the probabilistic form of Modus Tollens, showing its link with AC. In particular when i=1 a must be null: this is the non probabilistic MT. The solution of MT follows.

2.15. Modus Tollens: IF A THEN C;   NOT-C   ∴   NOT-A

Knowing the level in compartment A and the size of the vacuum in the tank, what are the limits of the right compartment?

We introduce c' = 1 - c and a' = 1 - a in the result for the Affirmation of the Consequent, hence the solution:

a' ∈ [(i - 1 + c') / i , 1] if i ≥ 1 - c' ; and a' ∈ [(1 - i - c') / (1 - i) , 1] if i ≤ 1 - c', which can be summarize by:

P(C/A) = i; P(NOT-C) = c';   P(NOT-A) ∈ [max { (1 - i - c') / (1 - i), (i - 1 + c') / i } , 1]

2.16. Hypothetical syllogism: IF A THEN B; IF B THEN C   ∴   IF A THEN C

The tank is initially filled with a liquid B (Fig. 16-1) and then some part of B is replaced by a liquid C. (We assume that the liquids are non miscible). In the certain, classic, case the compartment A is entirely filled with liquid B and then the whole of liquid B is replaced by liquid C, so that in the end A is filled with C and the inference is valid. In the probabilistic case, even if the level in A is high (i_{BA} = P(B/A) high) and a part of the volume B is replaced (respecting P(C/B)), the replacement of the liquid B could be done by removing it from A

while little or no liquid B is removed from A', making the level of C in AB low or even null (Fig.

16-2). On the other hand, when liquid C is poured, some of it may also be introduced in B'

and this amount introduced in AB' could occupy all this empty space so that P(C/A) need not

be limited to P(B/A) and could reach 1 (the top of A). In brief, any confidence value can be

given for the conclusion:

$$P(B/A) = i_{BA} ; \quad P(C/B) = i_{CB} ; \qquad P(C/A) \in [\ 0, \ 1]$$

Figure 16. Hypothetical syllogism.
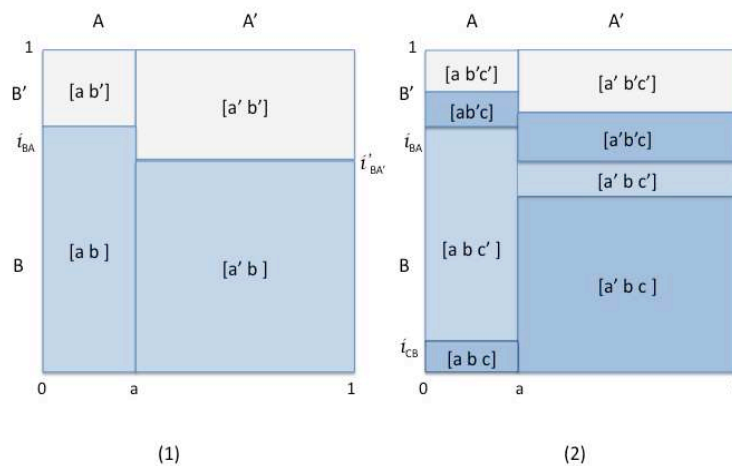
This is to be contrasted with the next inference, CUT.

2.17. CUT: IF A THEN B; IF A AND B, THEN C   ∴   IF A THEN C

The difference with the hypothetical syllogism is that, as indicated by the second premise,

when liquid C is introduced to replace B, this cannot be done without pouring at least some

of it into A, so warranting that A will not be empty, that is, the probability of the conclusion

cannot be null. The limits can be assessed as follows (Fig. 16-2):

Let i_{CB} stand for P(C/AB). Notice that the related level is relative to AB. As just explained,

the amount in A cannot be less than the amount in AB, hence the minimum $i_{BA} \times i_{CB}$. Now, the

maximum is obtained when in addition to the minimal amount the empty space AB' (=1- i_{BA})

is totally filled with liquid C, that is: $i_{BA} \times i_{CB} + (1- i_{BA})$, hence:

$$P(B/A) = i_{BA}; \quad P(C/AB) = i_{CB}; \quad P(C/A) \in [\ i_{BA} \times i_{CB} , (i_{BA} \times i_{CB})+(1- i_{BA})\ ]$$

This inference is not conservative, the lower bound being smaller than the confidence in either premise. It is not dissipative because the upper bound cannot be lower than the probability of one of the premises (i_{CB}). It is **robust** since the lower bound warrants a high value if both premises have a high probability.

2.18. Strengthening: IF A THEN C ∴   IF A AND B, THEN C

Compartment A is devided into two sub-compartments B and B'. In the sure case (Fig. 17-1) A is assumed to be full so that any subpart of it such as B will be full as well and the conclusion follows with full confidence.

Figure 17. Strengthening of the antecedent.
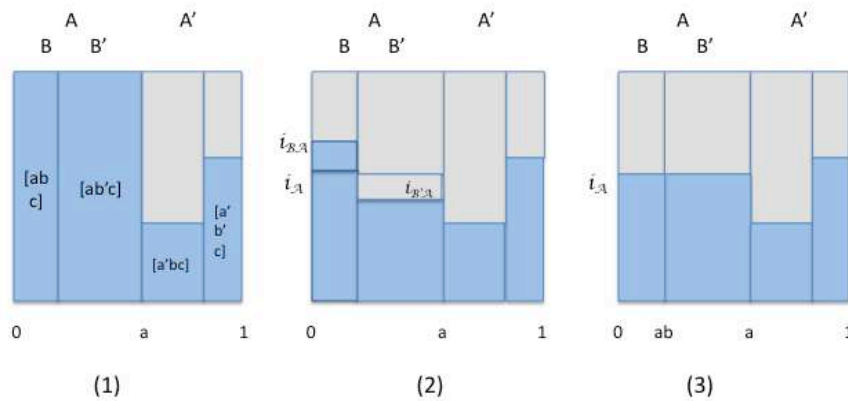
In the probabilistic case there is no guarantee that A is full and the sub-compartments need not have the same level (Fig. 17-2). P(C/A) = i_A is a weighted value between i_{BA}= P(C/AB) and $i_{B'A}$ = P(C/AB'). Strengthening the antecedent with B can increase or decrease the probability of the conditional.

To study the possible values of the conclusion it suffices to focus on the compartment A in which B and B' stand for new compartments and c is known; in other words, we conditionalize with respect to A, and the problem is now an inference already familiar (see

section 2.6) from consequent C to 'if B, C' with B and B' as events within A. We know that no confidence value can be given, so that

$$P(C/A) = i; \quad P(C \text{ AND } B) \in [0, \ 1]$$

The extreme case where the level i_{BA} in AB equals 0 is of particular interest because, when dealing with causal conditionals, it coincides with a strict invalidating condition (or defeater). Assume a highly believable conditional *if A, C* asserted as a probable cause. As we have seen, for the level i_{BA} to be null when i_A is high, B must be small: this coincides with the well-known case where a conditional assertable with antecedent A must be retracted when A is qualified by an unexpected, or rare, or atypical defeater B.

The indeterminacy of the conclusion can be suppressed if the size of the AB compartment becomes a fixed parameter: adding *if A then B* as a premise yields the next schema, moving from monotonicity to cautious monotonicity.

2.19. Cautious monotonicity: IF A THEN C; IF A THEN B   ∴   IF A AND B, THEN C

The level in A is $\underline{i}$, then A is partitioned into B and B'. The width of B in A is a x b (see Fig. 17-3). To know the level of liquid $\underline{i}_{BA}$ in AB we need to know how the liquid can be shared between AB and AB'.

1) AB contains an amount a x b x i; it can be emptied provided the empty space a(1- b)(1-i) is large enough to receive this extra liquid. In this case the minimum is 0. If not, there will remain in AB an amount equal to a x b x i - a(1- b)(1- i) = a(i + b-1), hence the level (i+b-1)/b.

2) AB can be filled up provided the amount of liquid in AB', i x a(1- b), is large enough. Otherwise the volume in AB equals i x a, hence the level i/b. In summary:

$$P(C/A) = i; \ P(B/A) = b \qquad P(C/A \text{ AND } B) \in [\max \{0, (i+b-1)/ b\}, \ \min \{i/b, 1\}]$$

With the addition of the premise *if A then B* the inference has become **robust**: we can have high confidence in *if A and B, then C* when, in addition to a high confidence in *if A then C* we have high confidence in *if A then B.* This inference is neither conservative nor dissipative, as the bounds can reach 0 while $\underline{i}$ and $\underline{b}$ differ from 0 or 1 while $\underline{i}$ and $\underline{b}$ differ from 1.

2.20. Right-nested conditionals and import-export:

    IF **A**, THEN IF **B** THEN **C**   ∴   IF **A** AND **B**, THEN **C**

A conditional is represented by a level and its consequent by a amount of liquid. In the premise of this inference the component "if B then C" is a <u>level</u> *qua* conditional but at the same time it must be an <u>amount</u> *qua* consequent of a conditional. At first sight, it would seem that this ambiguity prevents nested conditionals, and therefore the inference, to be represented in the system. However, a level is a relative amount and the amount "C relatively to B" can be considered in turn relatively to A, so that one may interpret the complex formula as a representation of the conditional "if B then C" within the event A, instead of an unconditional representation within the whole state space. That is, "if B then C" is conditioned on A and this allows the interpretation of the nested conditional. We start by representing the event A and then construct the partition (B, B'), each part of which accommodates its share of liquid. This procedural interpretation yields the same diagram as strengthening, and because it is reversible the inference is established as an equivalence. This equivalence can be translated in terms of conditional events. The premise is (C/B)/A and the conclusion is C/(A&B) and these denote the same conditional event (de Finetti, 1975).

3. Conclusion

We have presented a device based on a physical implementation of probability theory (in the finite case) which allows the execution of deductions under uncertainty. At the basis of the derivation of the conclusions there are physical principles. We cannot get out of a compartment more than its actual content (no negative probability) nor can we pour an amount greater than its capacity (no probability greater than 1). It is also noteworthy that the physical constraint of conservation of the liquid (deeply supported by intuition) acts through operations such as (1) after filling up a compartment some extra liquid (if any) must be allocated to the other compartment, or (2) after moving the partition until the current level reaches the top requires that the extra liquid be accommodated in the other compartment.

These operations, which have been used throughout, implement the total probability rule.

Note that the device can help answer the question of the probability of the conditional event when the probability of the conditioning event is null, or in another terminology, the probability of a conditional sentence whose antecedent's probability is null. One possible answer is that it is undefined because it is not possible to conceive of the level of a liquid in a container whose capacity is null. A variant of this answer potentially more productive is that, for the same reason, it is undetermined, that is, its value is anywhere on [0, 1]. Any precise value, be it 1/2 or 1 seems arbitrary and it is hard to see what specification of the device could serve as a basis for a justification.

From a computational point of view, readers familiar with the coherence approach to probability theory will notice that the probability bounds presented here can be obtained by de Finetti's (1937) method of linear equations (see also Coletti & Scozzafava, 2002; Pfeifer & Kleiter, 2009). The novelty with the analogy is the *absence* of computation. This is no wonder: once the components have been fixed to respect the state defined by the premises, the value of the component of interest can be read off the diagram because the device incorporates the laws of probability. It is only when reading the bounds that the four elementary arithmetical operations may be needed to express the value of the constituent of interest (the conclusion).

The tank analogy can be considered from two different points of view, practical and theoretical. Practically, one advantage linked to the operational procedure of the analogy over formal methods of calculation is that it provides both a basis for the quantitative calculation of the bounds and a qualitative method which opens didactic perspectives. Indeed, when the study of uncertain reasoning becomes sufficiently developed to reach wider audiences than specialized researchers or advanced students, the analogy may be a valuable tool to explain why, for instance, it is irrational to have too low or too high confidence in the conclusion of Modus Ponens, why contraposition is unwarranted (but which additional information or premise can warrant some confidence and how much), under which conditions Affirming the Consequent may not be a fallacy, and so on. Take for instance

Modus Ponens. Individuals whose confidence in the conclusion is lower than the lower bound P(A)xP(C/A) commit themselves to the belief that the amont of liquid in the whole tank is less than the amount in one of its parts (the left compartment). Similarly, those whose confidence in the conclusion is higher than the upper bound P(A)xP(C/A) +1- P(A) commit themselves to the belief that there can be more liquid in the tank than its overall capacity.

Theoretically, the main interest of the analogy is to provide an interpretation of the meaning of probabilistic deductive inferences. Atomic sentences are interpreted in terms of volumes or capacity of compartments, and conditionals in terms of levels. Boolean expressions have their interpretation in terms of union or intersection of volumes or compartments so that the logical operations have their physical counterparts. Varying the value of the constituents (volumes, capacities and levels) provides the probability measures. Overall, an inference is viewed as the given of the state of some of the constituents of the device (the premises) whose values may or may not constrain another specified constituent (the conclusion). In the latter case the inference is uninformative, meaning that the constituent may be in any state and the confidence in the associated sentence can be anywhere on the interval [0, 1]. In the former case the inference is informative, meaning that the constituent cannot be in any state: some states are excluded in the sense that they are physically impossible; this results in an interval of possible states (values) that provide the limits in confidence in the conclusion.

In summary, we possess an operational procedure to evaluate normatively, that is, from the viewpoint of probability theory, deductive inferences under uncertainty. It can be seen that the evaluation of the conclusion amounts to a search for the conditions of compatibility or coherence between the configurations described by the premise(s) and the conclusion. It would be incoherent, if one accepts the premise configuration, to expect a conclusion's configuration that is physically impossible. This offers an alternative to betting situations and Dutch book arguments classically applied to assess individuals' rationality in their probability judgments: in a similar way that individuals whose bets do not conform to coherent

probability bounds are doomed to losing their money, the same individuals are doomed to failure in any attempt to execute fillings that are physically impossible.

References

Adams (1996). Four probability-preserving properties of inferences. *Journal of Philosophical Logic, 25,* 1-24.

Adams, E. (1998). *A primer of probability logic.* Stanford: CSLI Publications.

Bertin, J. (1967). *Sémiologie graphique.* Paris: Gauthier-Villars. [English translation: *Semiology of graphics.* Madison, WI: University of Wisconsin Press,1983].

Coletti, G. & Scozzafava, G. (2002). *Probabilistic logic in a coherent setting.* Dordrecht: Kluwer.

de Finetti, B. (1937). La prévision, ses lois logiques, ses sources subjectives. *Annales de l'Institut Henri Poincaré, VII,* 1-67. [English translation: Foresight: Its logical laws, its subjective sources. In H. E. Kyburg Jr. , & H. E Smokler (Eds.) (1964). *Studies in subjective pobability* (pp. 193-158). New York: John Wiley].

de Finetti, B. (1975). *Theory of probability*. Vol. 2. Chichester: John Wiley.

Edwards, A. W. F. (1972). *Likelihood*. Cambridge: Cambridge University Press.

Friendley, M. (2002). A brief history of the mosaic display. *Journal of Computational and Graphical Statistics, 11*(1), 89-107.

Gilio, A. (2002). Probabilistic reasoning under coherence in system P. *Annals of Mathematics and Artificial Intelligence, 34,* 131-159.

Gilio, A. , Over, D. P. (2012). The psychology of inferring conditionals from disjunctions : A probabilistic study. *Journal of Mathematical Psychology, 56,* 118-131.

Hailperin, T. (1996). *Sentential probability logic.* Bethlehem: Lehigh University Press.

Hartigan, J. A. , & Kleiner, B. (1981). Mosaics for contingency tables. In W. F. Eddy (Ed.), *Computer science ans statistics: Proceedings of the 13th symposium on the interface* (pp. 268-273). New York: Springer-Verlag.

Oaksford, M. , & Chater, N. (2007). *Bayesian rationality: The probabilistic approach to human*

rationality. Oxford: Oxford University Press.

Oldford, R. W. (2003a). Understanding probabilistic independence and its modelling via

Eikosograms and graphs." (unpublished manuscript).

http://www.math.uwaterloo.ca/~rwoldfor/

Oldford, R. W. (2003b). Probability, problems, and paradoxes pictured by eikosograms".

(unpublished manuscript). http://www.math.uwaterloo.ca/~rwoldfor/

Olford, R. W. , & Cherry, W. H. (2006). Picturing Probability: the poverty of Venn diagrams,

the richness of Eikosograms. (unpublished manuscript).

http://www.math.uwaterloo.ca/~rwoldfor/

Pfeifer, N. , & Kleiter, G. D. (2006). Inference in conditional probability logic. *Kybernetica,*

42(4), 391-404.

Pfeifer, N. , & Kleiter, G. D. (2010). The conditional in mental probability logic. In M. Oaksford

& N. Chater (Eds.), *Cognition and conditionals* (pp. 153-173). Oxford: Oxford University

Press.

Pfeifer, N. , & Kleiter, G. D. (2011). Uncertain deductive reasoning. In K. Manktelow, D. Over

& S. Elqayam (Eds.), *The science of reason* (pp. 145-166). Hove: Psychology Press.

Pfeifer, N. , & Kleiter, G. D. (2009). Framing human inference by coherence based probability

logic. *Journal of Applied Logic, 7,* 206-217.

Politzer, G. , & Bourmaud, G. (2002). Deductive reasoning from uncertain conditionals.

British Journal of Psychology, 93, 345-381.

Author notes

Note 1. The device can be described as a Cartesian product of two Bernouilli random variables Ω_1 x Ω_2 with Ω_1 = {left, right} and Ω_2 = {(volume of) liquid, (volume of) empty space}.

Note 2. Here we take the width $\underline{a}$ (the probability of A), and the level $\underline{i}$ in A (the probability of the conditional event C/A) as defining the content of A (the joint probability of A and C), following de Finetti's theoretical approach. The tank representation also allows to define P(C/A) by the ratio P(A&C) / P(A) following the axiomatization of Kolmogorov, which highlights the inter-definability of the two notions.