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Opinion

Bayesian Brains without Probabilities

Adam N. Sanborn^{1,*} and Nick Chater²

Bayesian explanations have swept through cognitive science over the past two decades, from intuitive physics and causal learning, to perception, motor control and language. Yet people flounder with even the simplest probability questions. What explains this apparent paradox? How can a supposedly Bayesian brain reason so poorly with probabilities? In this paper, we propose a direct and perhaps unexpected answer: that Bayesian brains need not represent or calculate probabilities at all and are, indeed, poorly adapted to do so. Instead, the brain is a Bayesian sampler. Only with infinite samples does a Bayesian sampler conform to the laws of probability; with finite samples it systematically generates classic probabilistic reasoning errors, including the unpacking effect, base-rate neglect, and the conjunction fallacy.

Bayesian Brains without Probabilities

In an uncertain world it is not easy to know what to do or what to believe. The Bayesian approach gives a formal framework for finding the best action despite that uncertainty, by assigning each possible state of the world a probability, and using the laws of probability to calculate the best action. Bayesian cognitive science has successfully modelled behavior in complex domains, whether in vision, motor control, language, categorization or common-sense reasoning, in terms of highly complex probabilistic models [1–13]. Yet in many simple domains, people make systematic probability reasoning errors, which have been argued to undercut the Bayesian approach [14–17]. In this paper, we argue for the opposite view: that the brain implements Bayesian inference and that systematic probability reasoning errors actually follow from a Bayesian approach. We stress that a Bayesian cognitive model does not require that the brain calculates or even represents probabilities. Instead, the key assumption is that the brain is a **Bayesian sampler** (see [Glossary](#)). While the idea that cognition is implemented by Bayesian samplers is not new [18–27], here we show that any Bayesian sampler is faced with two challenges that automatically generate classic probabilistic reasoning errors, and only converge on ‘well-behaved’ probabilities at an (unattainable) limit of an infinite number of samples. In short, here we make the argument that Bayesian cognitive models that operate well in complex domains actually predict probabilistic reasoning errors in simple domains (see [Table 1](#), Key Table).

To see why, we begin with the well-known theoretical argument against the naïve conception that a Bayesian brain must represent all possible probabilities and make exact calculations using these probabilities: that such calculations are too complex for any physical system, including brains, to perform in even moderately complex domains [4,28–30]. A crude indication of this is that the number of real numbers required to encode the joint probability distribution over n binary variables grows exponentially with 2^n , quickly exceeding the capacity of any imaginable physical storage system. Yet Bayesian computational models often must represent vast data spaces, such as the space of possible images or speech waves; and effectively infinite hypothesis spaces, such as the space of possible scenes or sentences. Explicitly keeping track of

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Bayesian models in cognitive science and artificial intelligence operate over domains such as vision, motor control and language processing by sampling from vastly complex probability distributions.

Such models cannot, and typically do not need to, calculate explicit probabilities.

Sampling naturally generates a variety of systematic probabilistic reasoning errors on elementary probability problems, which are observed in experiments with people.

Thus, it is possible to reconcile probabilistic models of cognitive and brain function with the human struggle to master even the most elementary explicit probabilistic reasoning.

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This sampling approach to probabilistic inference began in the 1940s and 1950s [32,33]; and is ubiquitously and successfully applied in complex cognitive domains, whether cognitive science or artificial intelligence [5,34]. If we sample forever, we can make any probabilistic estimate we need to, without ever calculating explicit probabilities. But, as we shall see, restricted sampling will instead lead to systematic probabilistic errors – including some classic probabilistic reasoning fallacies.

So how does this work in practice? Think of a posterior probability distribution as a hilly high-dimensional landscape over possible hypotheses (see Figure 1A). Knowing the shape of this landscape, and even its highest peaks, is impossibly difficult. But a sampling algorithm simply explores the landscape, step by step. Perhaps the best known class of sampling algorithms is **Markov chain Monte Carlo (MCMC [35,36])**. A common type of MCMC, the **Metropolis–Hastings** algorithm [32,37], can be thought of as an android trying to climb the peaks of the probability distribution, but in dense fog, and with no memory of where it has been. It climbs by sticking one foot out in a random direction and ‘noisily’ judging whether it is moving uphill (i.e., it noisily knows a ‘better’ hypothesis when it finds one). If so, it shifts to the new location; otherwise, it stays put. The android repeats the process, slowly climbing through the probability landscape, using only the relative instead of absolute probability to guide its actions.

Despite the simplicity of the android's method, a histogram of the android's locations will resemble the hill (arbitrarily closely, in the limit), meaning the positions are samples from the probability distribution (see last column of Figure 1A). These samples can then easily be used in a variety of calculations. For example, to estimate the chance that a word in a novel will follow the form `__ _ _ _ n_`, just sample some text, and use the proportion of the samples that follow that pattern as the estimate. This is, indeed, very close to Tversky and Kahneman's availability heuristic [38] – to work out how likely something is by generating possible examples; but now construed as Bayesian sampling rather than a mere ‘rule of thumb.’

There is powerful *prima facie* evidence that the brain can readily draw samples from very complex distributions. We can imagine visual or auditory events, and we can generate language (and most tellingly to mimic the language, gestures, or movements of other people), which are forms of sampling [39]. Similarly, when given a part of a sentence, or fragments of a picture, people generate one (and sometimes many) possible completions [40,41]. This ‘generative’ aspect of perception and cognition (e.g., [42]) follows automatically from sampling models [6,10].

Unavoidable Limitations of Sampling

Any distribution that can be sampled can, in the limit, be approximated – but finite samples will be imperfect and hence potentially misleading. The android's progress around the probability landscape is, first and most obviously, biased by its starting point. Suppose, specifically, that the landscape contains an isolated peak. Starting far from the peak in an undulating landscape, the android has little chance of finding the peak. Even complex sampling algorithms, that can be likened to more advanced androids that sense the gradient under their feet or are adapted to particular landscapes, will blindly search in the foothills, missing the mountain next door (see Figure 1A and bottom row of Figure 1B). While we have illustrated this problem with a simple sampling algorithm, it applies to any sampling algorithm without prior knowledge of the location of the peaks [28,30]; indeed even the inventors of sampling algorithms are not immune to these problems [36,43].

A Bayesian sampler that misses peaks of probability will produce reasoning fallacies. Various fallacies, such as the unpacking effect and the conjunction fallacy, involve participants believing that a more detailed description of an event is more likely than a less detailed description of that

the sum of these proportional values – it is the number by which each of these proportional values must be divided to become probabilities.

Particle filters: an algorithm designed to sample from probability distributions for data that arrives sequentially.

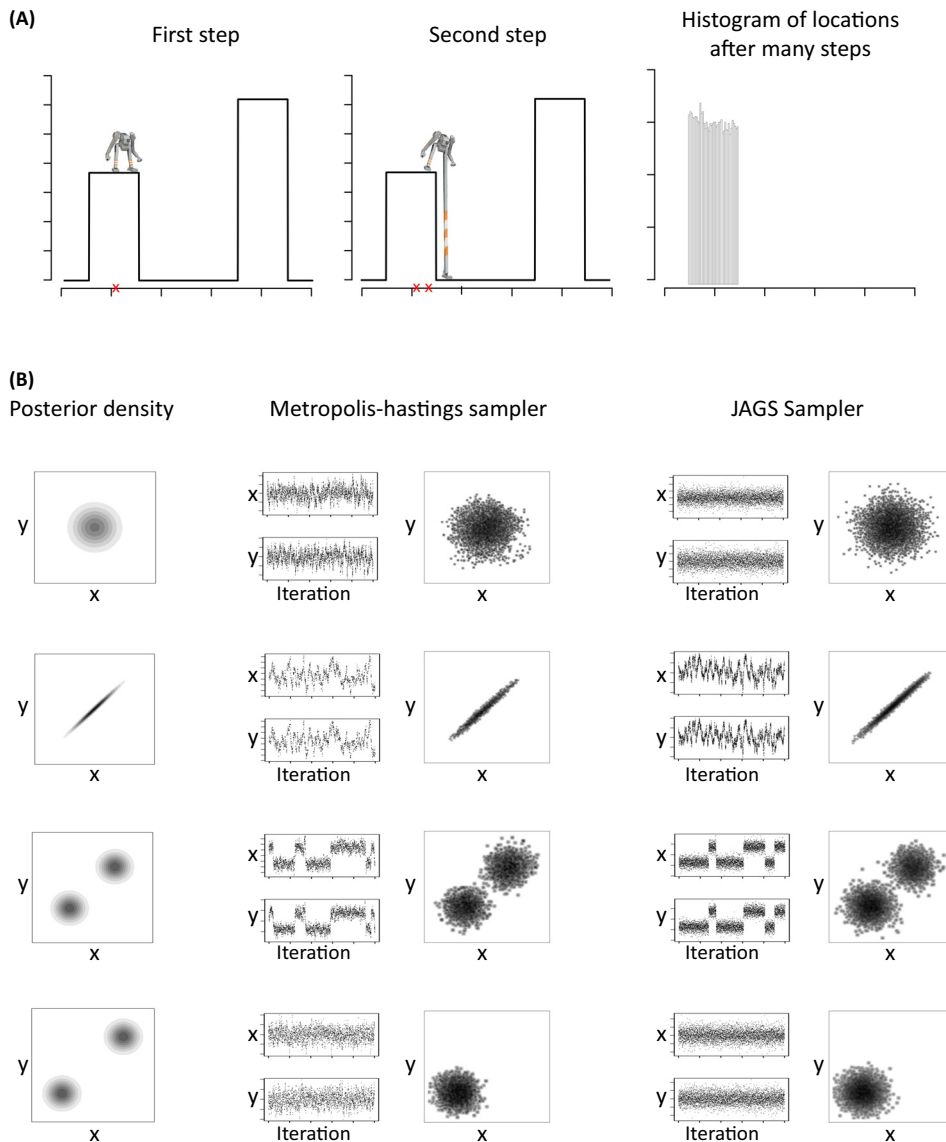
Posterior probability: the probability of a hypothesis in response to a question.

Sampling: generating hypotheses with frequency proportional to their posterior probability. Probability estimates can then be based on the relative frequencies of these sampled hypotheses.

Satisficing: searching until a good-enough solution is found, rather than searching until the best possible solution is found.

Small worlds: restricted worlds with few variables and well-defined probabilities over those variables.

Wisdom of crowds: empirical result that the aggregation of individual estimates is better than the average individual estimate, or sometimes any individual estimate.



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Figure 1. Sampling Algorithms Have Difficulties with Isolated Modes and Produce Autocorrelations. (A) Illustration of the android metaphor, with the android climbing the landscape of the (log) posterior probability distribution. The android uses the difference in height of its two feet to decide where to step, and its location is tracked over time (red x). A histogram of its locations after many steps matches the mode of the probability distribution it explored. (B) Comparison of sampling methods on 2D distributions. Each row is a different example probability distribution: a unimodal distribution with uncorrelated x and y variables, a unimodal distribution with correlated x and y variables, a bimodal distribution with relatively nearby modes, and a bimodal distribution where the modes are further apart. The first column shows a topographic map of the posterior density with darker regions indicating higher probability. The second and third columns illustrate samples drawn using the Metropolis–Hastings algorithm and JAGS program respectively. Within each column are trace plots that show how the location of the sampler changes along each variable during each iteration of the sampling process. Autocorrelations are present when a sample depends on the value of the previous sample in the trace plots (e.g., Metropolis–Hastings in the second row). Also shown are bivariate scatterplots that can be used to compare the samples obtained against the true distributions in the first column. These show that not all of the modes are always sampled, even when thousands of samples are drawn (i.e., in the bottom row). R code for these plots is included as supplemental material.

same event, violating probability theory. In the unpacking effect, participants judge, say, 'being a lawyer', to be less probable than the 'unpacked' description 'being a tax, corporate, patent, or other type of lawyer' [44–46]. From a sampling point of view, bringing to mind types of lawyer that are numerous but of low salience ensures these 'peaks' are not missed by the sampling process, yielding a higher probability rating. Yet unpacking can also guide people away from the high probability hypotheses: if the unpacked hypotheses are low probability instead of high, for example trying to assess whether a person with a background in social activism becomes either 'a lawyer' or 'a tax, corporate, patent, or other type of lawyer' then the probability of the unpacked event is judged less than that of the packed event [46] – the sampler is guided away from the peaks of probability (e.g., 'human rights lawyer').

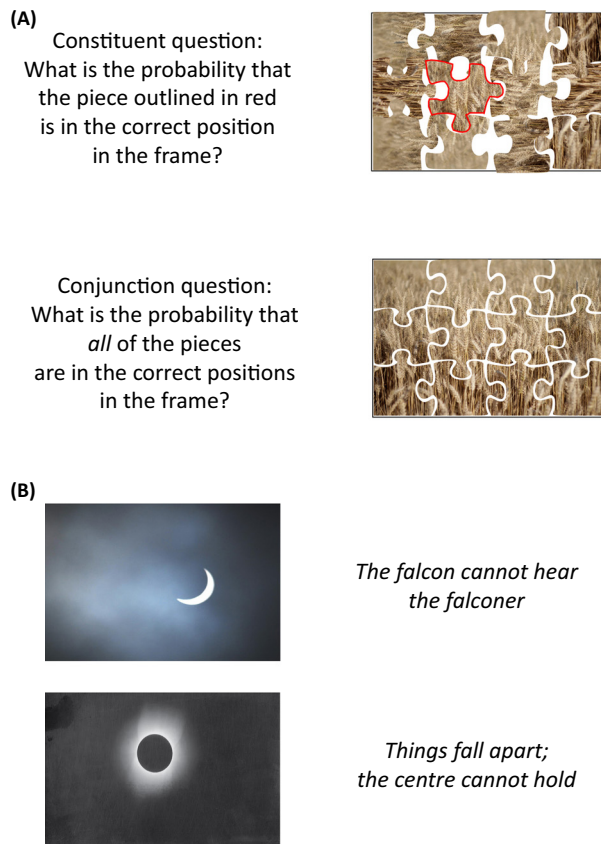
The conjunction fallacy is a complex effect [47,48], but one source of the fallacy is the inability to bring to mind relevant information. We struggle to estimate how likely a random word in a novel will match the less detailed pattern _____ n_: our sampling algorithm searches around a large space and may miss peaks of high probability. However, when guided to where the peaks are (i.e., starting the android from a different location), for example, by being asked about the pattern _____ ing in the more detailed description, then large peaks are found and probability estimates are higher [31,45]. The process involved is visually illustrated in Figure 2A. The conjunction of the correct locations of all the puzzle pieces cannot, of course, be more probable than the correct location of a single piece. Yet when considered in isolation, the evidence that an isolated piece is correct is weak (from a sampling standpoint, it is not clear whether, e.g., swapping pieces leads to a higher or lower probability). But in the fully assembled puzzle (i.e., the 'peak' in probability space is presented), local comparisons are easy – switching any of the pieces would make the fit worse – so you can be nearly certain that all the pieces are in the correct position. So the whole puzzle will be judged more probable than a single piece, exhibiting the conjunction fallacy.

A second bias in sampling is subtler. Sometimes regions of high probability are so far apart that a sampler starting in one region is extremely unlikely to transition to the other. As shown in Figure 2B, our android could be set down in Britain or in Colorado, and in each case would gradually build up a picture of the local topography. But these local topographies would give no clue that the baseline height of Colorado is much higher than Britain. The android is incredibly unlikely to wander from one region to another so the relative heights of the two regions would remain unknown.

This problem is not restricted to sampling algorithms or to modes that are far apart in a single probability distribution. Indeed, the problem is even starker when a Bayesian sampler compares probabilities in entirely different models – then it is often extremely difficult for the android to shuttle between landscapes [49]. So, for example, although it is obvious that we are more likely to obtain at least one, rather than at least five, double sixes from 24 dice throws, it is by no means obvious whether this first event is more or less likely than obtaining heads in a coin flip. Indeed, this problem sparked key developments in early probability theory by Pascal and Fermat [50].

One might think there is an easy solution to comparing across domains. Probabilities must sum to 1, after all. So if sampling explores the entire landscape, then we can figure out the absolute size of probabilities (i.e., whether we are dealing with Britain or Colorado) because the volume under the landscape must be precisely 1. But exploring the entire landscape is impossible – because the space is huge and may contain unknown isolated peaks of any possible size, as we've seen (technically, the **normalization constant** often cannot be determined, even approximately).

An example of this is given in Figure 2B, where within-domain comparisons are relatively easy. So it may be easy to estimate which is more likely in a randomly chosen year: a total or partial



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Figure 2. Illustrations of the Conjunction Fallacy and Base-Rate Neglect. (A) Illustration of why the conjunction fallacy arises from a Bayesian sampler. The top row gives a question about a piece of the puzzle. The bottom row illustrates that evaluating the probability of a conjunction will be easier. (B) Four events to illustrate why local assessments of relative probability are easier. Comparing the probability of seeing the two astronomical events in a year, or the probability of the two quotations appearing on a random website, are both relatively easy. Comparing the probability of seeing one of the astronomical events in a year to the probability of seeing one of the quotations on a random website is more difficult. In particular, when comparing ‘Things fall apart; the centre cannot hold’ to the eclipse, the quote may seem more likely as it is a common among quotes, yet this neglects the base rates: most websites do not have literary quotations, and there are many chances for an eclipse each year.

eclipse; or which of two quotations are more likely to appear on a randomly chosen website. However, between-domain comparisons, such as deciding whether a total eclipse is more likely than ‘Things fall apart; the centre cannot hold’ appearing is more difficult: the astronomical event and the quotation must be compared against different sets of experiences (e.g., years and websites, respectively).

Being unable to effectively compare the probabilities of two hypotheses is a second way in which a Bayesian sampler generates reasoning fallacies observed in people. The example in [Figure 2B](#) also illustrates a version of **base-rate neglect**. ‘Things fall apart; the centre cannot hold’ is a notable quotation, and eclipses are rare, so from local comparisons it may seem that ‘Things fall apart; the centre cannot hold’ is more likely. However, the base rates, which are often neglected, reverse this: the vast majority of websites have no literary quotations, and each year provides many opportunities for an eclipse. Fully taking account of base-rates would require searching the entire probability space – which is typically impossible. This inability to compare probabilities from different domains plays a role in other reasoning fallacies: the conjunction fallacy also

occurs when the two conjoined events are completely unrelated to one another [51]. While a sampler can provide a rough sense of the probabilities of each hypothesis separately (is this quotation or astronomical event common compared with similar quotes or astronomical events), the inability to bridge between the two phenomena means that sampling cannot accurately assess base rates or estimate the probabilities of conjunctions or disjunctions. In these cases, participants are left to combine the non-normalized sampled probability estimates for each hypothesis ‘manually’, and perhaps choose just one of the probabilities or perhaps combine them inappropriately [52–54].

Interestingly, accuracy is improved in base-rate neglect studies when a causal link is provided between the individual events [55,56], and experiencing outcomes together weakens conjunction fallacies [57] – we can interpret these studies as providing a link that allows a Bayesian sampler to traverse between the two regions to combine probabilities meaningfully.

Sampling and Task Richness

Reasoning fallacies such as the unpacking effect, conjunction fallacy, and base-rate neglect have greatly influenced our general understanding of human cognition. They arise in very simple situations and violate the most elementary laws of probability. We agree that these fallacies make a convincing argument against the view that the brain represents and calculates probabilities directly.

This argument appears strengthened in complex real-world tasks. If our brains do not respect the laws of probability for simple tasks, surely the Bayesian approach to the mind must fail in rich domains such as vision, language and motor control with huge data and hypothesis spaces [29]. Indeed, even Savage, the great Bayesian pioneer, suggested restricting Bayesian methods to ‘small’ worlds [58,59].

Viewing brains as sampling from complex probability distributions upends this argument. Rich, realistic tasks, in which there is a lot of contextual information available to guide sampling, are just those where the Bayesian sampler is most effective. Rich tasks focus the sampler on the areas of the probability landscape that matter – those that arise through experience. By limiting the region in which the sampler must search, rich problems can often be far easier for the sampler than apparently simpler, but more abstract, problems. Consider how hard it would be to solve a jigsaw which is uniformly white; the greater the richness of the picture on the jigsaw, the more the sampler can locally be guided by our knowledge of real-world scenes.

Moreover, the problem of learning the structure of the world, or interpreting an image or a sentence, involves finding ‘good-enough’ hypotheses to usefully guide our actions, which can be achieved by local sampling in the probability landscape. Such hypotheses are no less valuable if an isolated peak, corresponding to an even better hypothesis, remained undiscovered. We suggest too that, for many real-world problems, multiple but distant peaks, corresponding to very different hypotheses about the world, may be rare, particularly when context and background knowledge are taken into account. Language is locally ambiguous [60], but it is very unlikely that the acoustic signal of a whole sentence in English happens to have an equally good interpretation in Latin; vision, too, is locally ambiguous (e.g., [61]) but the probability that a portrait photograph could equally be reinterpreted as a rural scene is infinitesimal. In complex real-world problems, then, climbing a rugged probability landscape to find ‘good-enough’ hypothesis is crucial; linking to numerical probabilities, even approximately, is not. Thus, the view of cognition as **satisficing** [62] need not be viewed as opposed to the Bayesian approach [14,29]. Rather, Bayesian sampling provides a mechanism for satisficing in real-world environments.

For these reasons, Bayesian models of cognition have primarily flourished as explanations of the complex dependency of behavior on the environment in domains including intuitive physics, causal learning, perception, motor control, and language [1–10]; and these computational models generally do not involve explicit probability calculations, but apply Bayesian sampling, using methods such as MCMC.

We have argued that rich real-world tasks may be more tractable to Bayesian sampling than abstract lab tasks, because sampling is more constrained. But is it possible that the fundamental difference is not task richness, but cognitive architecture, for example, between well-optimized Bayesian perceptual processes and heuristic central processes [63,64]? We do not believe this to be the case for two reasons. First, the differences between these kinds of tasks tends to disappear when performance is measured on the same scale [65,66]. Second, there are counterexamples to the architectural distinction. Language interpretation is a high-level cognitive task which shows a good correspondence to Bayesian models [5,6]. And, conversely, purely perceptual versions of reasoning fallacies can be constructed, as Figure 2A illustrates. More broadly, any perceptual task where a hint can disambiguate the stimulus (e.g., sine-wave speech [67]) will generate examples of the conjunction fallacy.

If the brain is indeed a Bayesian sampler, then sampling should leave traces in behavior. One immediate consequence is that behavior should be stochastic, because of the variability of the samples drawn. Hence behavior will look noisy, even where on average the response will be correct (e.g., generating the **wisdom of crowds** even from a single individual [68–70]). The ubiquitous variability of human behavior, especially in highly abstract judgments or choice tasks [71–74] is puzzling for pure ‘optimality’ explanations. Bayesian samplers can match behavioral stochasticity in tasks such as perception, categorization, causal reasoning, and decision making [20,21,23,24].

A second consequence of sampling is that behavior will be autocorrelated, meaning that each new sample depends on the last, because the sampler makes only small steps in the probability landscape (see Box 1). Autocorrelation appears ubiquitous within and between trials throughout memory, decision making, perception, language, and motor control (e.g., [75,76]), and Bayesian samplers produce human-like autocorrelations in perceptual and reasoning tasks [18,25]. One

Box 1. Sampling and Autocorrelation

When sampling from complex distributions, it is often impossible to draw samples directly, and sampling algorithms such as MCMC are used instead. Because these algorithms look locally when producing the next sample, autocorrelation can result. Figure 1B in main text compares samples drawn independently to samples drawn from two versions of MCMC: the Metropolis–Hastings algorithm and Gibbs sampling as implemented in JAGS [36,93]. Samples produced by both MCMC methods have low autocorrelations for independent unimodal distributions (top row of Figure 1B), but for highly correlated variables and bimodal distributions autocorrelations are more prevalent, even when the aggregated samples match the overall distribution (middle rows of Figure 1B). More recently developed sampling methods such as those employed by the program Stan can also show sample autocorrelations, but more complex distributions are needed to induce them [94]. And of course, when two modes are far apart, these samplers are very unlikely to sample both modes (bottom row of Figure 1B).

Although they are outnumbered by models of behavior that assume independent sampling, several models of memory and decision making do assume there are autocorrelations in sampling. Models of free recall such as Search of Associative Memory [95] and SIMPLE [96] use the previously recalled word to cue the next word, and has been productively used to account for the dependencies seen when people attempt to recall a list of words. In decision making, the **ballistic accumulator model** could be considered an extreme version of autocorrelation in which each internal time step produces the same strength of sampled evidence within a trial [97]. Explicit models of autocorrelated sampling have been used to account for perceptual switching times of **multistable stimuli** [18] and anchoring effects in reasoning tasks [25].

particular consequence of autocorrelation will be that ‘priming’ a particular starting point in the sampling process will bias the sampler. Thus, asking, for example, the gestation time of an elephant will bias a person’s estimate because they begin with the common reference point of 9 months: the starting point is an ‘anchor’ and sampling ‘adjusts’ from that anchor – but insufficiently when small samples are drawn. This provides a Bayesian reinterpretation of ‘anchoring and adjustment’ [25], a process widely observed in human judgment [15,77].

The extent of autocorrelation depends on both the algorithm and the probability distribution (see Figure 1B). Sampling algorithms often have tuning parameters which are chosen to minimize autocorrelation and to better explore multiple peaks. Of course the best settings of these tuning parameters are not known in advance, but they can be learned by drawing ‘warm-up’ samples to get a sense of what the distribution is like. Interestingly, there is behavioral evidence for this. Participants given ‘clumpier’ reward distributions in a 2D computer foraging task later behaved as if ‘tuned’ to clumpier distributions in semantic space in a word-generations task. This suggests that there is a general sampling process that people adapt to the properties of the probability distributions that they face [78–80].

Concluding Remarks and Future Perspectives

Sampling provides a natural and scalable implementation of Bayesian models. Moreover, the questions that are difficult to answer with sampling correspond to those with which people struggle, and those easier to answer with sampling are contextually rich, and highly complex, real-world questions on which people show surprising competence. Sampling generates reasoning fallacies, and leaves traces, such as stochasticity and autocorrelations, which are observed in human behavior.

The sampling viewpoint also fits with current thinking in computational neuroscience, where an influential proposal is that transitions between brain states correspond to sampling from a probability distribution [26,27,81,82] or multiple samples are represented simultaneously (e.g., using **particle filters** [83–86]).

Moreover, neural networks dating back to the **Boltzmann machine** [87] take a sampling approach to probabilistic inference. For example, consider **deep belief networks**, which have been highly successful in vision, acoustic analysis, language, and other areas of machine learning [88]. These networks consist of layers of binary variables connected by weights. Once the network is trained, it defines a probability distribution over the output features. However, the probabilities of configurations of output features are not known. Instead, samples from the total probability distribution are generated by randomly initializing the output variables; and conditional samples are generated by fixing some of the output variables to particular values and sampling the remaining variables. Applications of deep belief networks include generating facial expressions and recognizing objects [34,89]. As with human performance, these networks readily sample highly complex probability distributions, but can only calculate explicit probabilities with difficulty. Of course, it is possible that the brain represents probability distributions over small collections of variables [90,91], or variational approximations to probability distributions [92], but this would not affect our key argument, which stems from the unavoidable difficulty of finding isolated peaks in, and calculating the volume of, complex probability distributions.

While Bayesian sampling has great promise in answering the big questions of how Bayesian cognition could work, there are many open issues. As detailed in Outstanding Questions, interesting avenues for future work are computational models of reasoning fallacies, explanations of how complex causal structures are represented, and further exploration of the nature of the sampling algorithms that may be implemented in the mind and the brain.

Outstanding Questions

Is sampling sequential or parallel? If the brain samples distributed patterns across a network of neurons (in line with connectionist models), then sampling should be sequential. This implies severe attentional limitations: for example, that we can sample from, recognize, or search for, one pattern at a time.

How is sampling neurally implemented? Connectionist models suggest that sampling may be implemented naturally across a distributed network, such as the brain.

How are ‘autocorrelated’ samples integrated by the brain? Accumulator models in perception, categorization, and memory seem best justified if sampling is independent. How should such models and their predictions be recast in the light of autocorrelated samples?

Does autocorrelation of samples reduce over trials as the brain becomes tuned to particular tasks?

Are samples drawn from the correct complex probability distribution or is the distribution simplified first? Variational approximations can be used to simplify complex probability distributions at the cost of a loss of accuracy.

How does sampling deal with counterfactuals and the arrow of causality? Does sampling across causal networks allow us to ‘imagine’ what might have happened if Hitler had been assassinated in 1934? How can we sample over entirely fictional worlds (e.g., to consider possible endings to a story)?

How far can we simulate sampling in ‘other minds’ to infer what other people are thinking?

How are past interpretations suppressed to generate new interpretations for ambiguous stimuli?

How far does sampling explain stochastic behavior in humans and non-human animals?

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Supplemental Information

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