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Sparse Reconstruction of Time-Frequency Representation using the Fractional Fourier Transform

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***Abstract*—This paper describes a novel method to approximate instantaneous frequency of non-stationary signals through an application of fractional Fourier transform (FRFT). FRFT enables us to build a compact and accurate chirp dictionary for each windowed signal, thus the proposed approach offers improved computational efficiency, and good performance when compared with chirp atom method.**

I. INTRODUCTION

Recently, compressive sensing (CS) has attracted much interest as a technique for time-frequency (TF) signature reconstruction of non-stationary signals. It is based on the fact that these signals are locally sparse in the TF domain. Also, incomplete or random sampling can frequently happen due to noisy measurement removals, hardware impairments, sampling frequency limitations [1], [2], [3]. A straightforward solution is to perform sparse reconstruction from time windowed signals, employing sinusoidal dictionary [4]. However, this approach means contradictory demands on the number of measurements for exact recovery and sparsity. It also suffers from the picket fence effect [5].

Now, the chirp approach can mitigate these issues, and obtain more accurate approximation by deploying chirp atoms for each window position to determine the signal's instantaneous frequency [6]. This chirp atom method, nevertheless, uses a very large dimension measurement dictionary. Since there are two parameters to be estimated (i.e. the chirp rate and the initial frequency), the dictionary dimension can be equal to the square of the dimension compared to using the sinusoid atom. This very large atom set results in a much heavier computation and longer calculation time. Thus, we propose a compact chirp atom set built via the FRFT. Basically, FRFT can deliver the TF signature of non-stationary signal by tuning the FRFT angle ϕ [7]. However, in case of compressed observations and unknown number of signal components, the FRFT is incapable of giving reliable results as missing samples introduce noise obscuring the desired information. In this paper, in each signal window, the FRFT is employed to obtain the corresponding initial frequency for each chirp rate. This leads to a much simple chirp atom set. As the FRFT can be executed in a similar time to the Fourier transform (FT), the advantage of this approach is that we obtain the same performance as compared to the chirp dictionary method but with an improved computational efficiency.

The paper is organized as follows. Section II describes the FRFT and its application in estimation of the chirp parameters. Section III proposes a method for TF representation sparse recovery using a FRFT based chirp dictionary. Section IV includes simulation results. Finally, conclusions are given in section V.

II. CHIRP RATE AND INITIAL FREQUENCY ESTIMATION OF CHIRPS USING FRFT

A. FRFT

The FRFT is a linear, energy preserving signal transformation that generalizes the conventional Fourier transform (FT) via an angle parameter ϕ [8], [9], [10]. For each fixed value of ϕ, the corresponding FRFT "rotates" a time domain signal counter-clockwise by an angle ϕ. As a result, for $\phi = 0$, one obtains the time domain (t) representation of the signal. And for $\phi = \pi/2$, the FRFT simplifies to the FT, providing the usual frequency (f) domain representation. For other values of ϕ, the FRFT allows signals to be transformed into a fractional domain, which is an intermediate domain between the time and frequency domains. Denote (x, y) the axes of the new reference plane, then the FRFT is illustrated in Fig.1, from which we can see the (x, y) axes are equal to the (t, f) axes rotated counter clockwise by an angle ϕ.

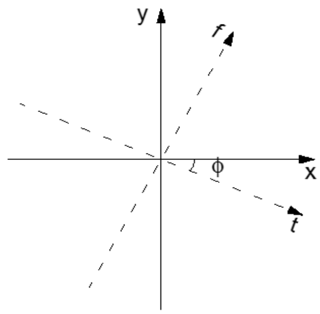


Fig. 1. Counter clockwise rotation of the time-frequency plane (t, f) by an angle ϕ, forming a new reference plane (x, y).

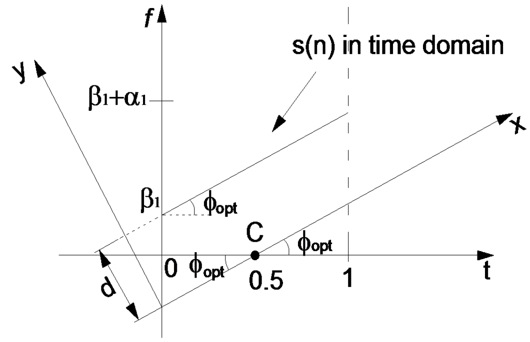


Fig. 2. Geometric schematic for calculating chirp rate α_1 and initial frequency interpretation β_1.

The FRFT of a time domain signal $s(t)$ is defined as [8], [9], [10]:

$$(\mathbb{F}^{\phi}s)(x) = S^{\phi}(x) = \begin{cases} \sqrt{1-j\cot(\phi)}e^{j\pi x^2\cot(\phi)} & \\ \int s(t)e^{j\pi t^2\cot(\phi)}e^{-j2\pi tx\csc(\phi)}dt, & \phi \neq l\pi \\ s(x), & \phi = 2l\pi \\ s(-x), & \phi = (2l+1)\pi, \end{cases} \tag{1}$$

where $\mathbb{F}^{\phi}$ is the FRFT operator associated with angle ϕ, $S^{\phi}(x)$ denotes the fractional Fourier transformed signal, l is an integer, t is time and x is the fractional variable.

B. CHIRP RATE AND INITIAL FREQUENCY ESTIMATION OF CHIRPS

Consider a discrete single chirp of length $T = 1$ (second) expressed as:

$$s(n) = \exp\left[j2\pi\left(\alpha_1\frac{n^2}{2F_s^2} + \beta_1\frac{n}{F_s}\right)\right], \tag{2}$$

where F_s is the sampling frequency, α_1, β_1 are values of the chirp rate and the initial frequency, $n = 0, 1, ..., \lfloor T/T_s \rfloor$, and $T_s = 1/F_s$. Our task is to estimate α_1, β_1 by the FRFT. The principle is that we tune the FRFT angle ϕ. When $\phi = \phi_{\text{opt}}$, the fractional axis x is matched to the chirp rate of the signal (see Fig. 2), or the chirp becomes a sinusoid in the new plane (x, y). And thus, the magnitude response (i.e. the absolute value of the FT of the FRFT) reaches its maximum. The initial frequency is determined by the position of the peak in the magnitude response.

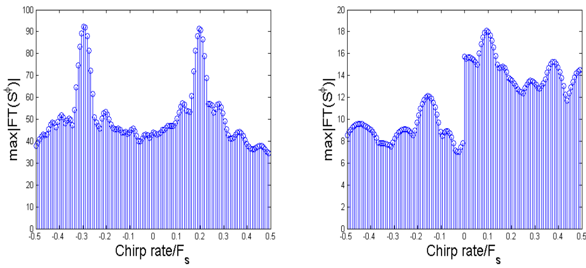
According to [11], the discrete FRFT rotates the time frequency plane around the point, C, defined by the intersection of the zero-frequency axis with half of the total duration of the time domain signal. Based on the rotation point C, the schematic illustrating the geometry for calculations α_1 and β_1 is plotted, and displayed in Fig. 2. Fig. 2 gives the relation between the chirp rate (α_1) and the optimum FRFT angle (ϕ_{opt}), which is expressed as [7]:

$$\begin{aligned} \phi_{\text{opt}} &= \arctan(\frac{\alpha_1\delta f}{\delta_t}) \\ &= \arctan(\frac{\alpha_1 N}{F_s^2}) \end{aligned} \tag{3}$$

where N is the number of samples, $\delta f = F_s/N$ is the frequency resolution and $\delta t = 1/F_s$ is time resolution. In the case that $N = F_s$, the optimum FRFT angle simplifies to $\phi_{\text{opt}} = \arctan(\alpha_1/F_s)$. According to Fig.2, the initial frequency β_1 is estimated by:

$$\beta_1 = d/\cos(\phi_{\text{opt}}) - \alpha_1/2, \tag{4}$$

where d is the position of the maximum peak of $|\text{FT}(S^{\phi_{\text{opt}}}(x))|$ in the new plane (x, y). This method works perfectly in the case that we have a large number of samples, and no missing entries, and the number of signal components is a known a-priori. When only limited observations are available, the method is unable to deliver accurate results because the magnitude responses do not always obtain a maximum when $\phi = \phi_{\text{opt}}$. For illustration, we use a signal composed of two chirps whose chirp rate values are $-0.3F_s$, and $0.2F_s, F_s = 128$. The maximum values of the magnitude response corresponding with different values of the FRFT angle ϕ or different values of the chirp rate (in the cases of full and missing data) are plotted in Fig. 3. It can be seen that in the latter case, it is incapable of estimating the two chirp rate values. Therefore, we propose using the FRFT to build the chirp dictionary for sparse reconstruction, which can reduce the chirp atom dimension compared with the full chirp dictionary [5], [6]. Moreover, the discrete FRFT algorithm proposed in [12] has a computational load of $O(N \log N)$ for a discrete-time signal of length N, which is same as the conventional FT. Therefore, the proposed method is more computational efficient than the full chirp atom approach.



(a) (b)

Fig. 3. The maximum values of the magnitude response versus chirp rate values or FRFT angle values (i.e. see 3): (a) Full length signal N=128; (b) Windowed signal of length $N_w = 64$, and randomly missing 50% of data.

III. SPARSE RECONSTRUCTION OF NON-STATIONARY TIME FREQUENCY SIGNATURE BASED ON THE FRFT

Consider an arbitrary continuous-time, non-stationary signal $s_c(t)$, which consists of K components:

$$s_c(t) = \sum_{k=1}^{K} A_k(t) \exp\left(j\varphi_k(t) + v_c(t)\right), \quad 0 \le t < T \tag{5}$$

where $A_k(t)$ and $\varphi_k(t)$ are the time-varying positive amplitude and phase of the k^{th} component, $v_c(t)$ is an additive white noise, and T is the total observation interval. The continuous-time instantaneous frequency (IF) of the k^{th} component is defined as:

$$F_k(t) = \frac{1}{2\pi}\frac{d\varphi_k(t)}{dt}. \tag{6}$$

We assume that it is known a-priori that the absolute IFs do not exceed $F_{\max}$ i.e. $|F_k(t)| \le F_{\max}$, where $F_{\max}$ is the maximum frequency of the signal $s_c(t)$. Sampling $s_c(t)$ at its Nyquist rate $F_s (F_s = 2F_{\text{max}})$, then we have:

$$s(n) = \sum_{k=1}^{K} A_k(nT_s) \exp(j\varphi_k nT_s) + v(n), \tag{7}$$

where $n = 0, 1, ..., \lfloor T/T_s \rfloor$, and $T_s = 1/F_s$.

Similar to the chirp dictionary method, this approach also approximates the windowed signal by the sum of piece-wise chirps. The m^{th} signal segment of length N_w is obtained by:

$$s_m(n - u(m-1)) = s(n)h(n - u(m-1)), \tag{8}$$

where $n = u(m-1), u(m-1)+1, ..., u(m-1)+N_w-1$, $u(1 \le u \le N_w)$ is the shift between two consecutive windows, m is the window index, and $h(n)$ is a rectangular window which is non-zero only for $0 \le n \le N_w - 1$.

Then the chirp-approximated m^{th} signal segment of $s(n)$ is written as:

$$s_m(n) \approx \sum_{k=1}^{K} A_{k,m} \exp\left\{ j2\pi \left[\alpha_{k,m}\frac{n^2}{2F_s^2} + \beta_{k,m}\frac{n}{F_s} \right] \right\} + v_m(n) \tag{9}$$

where $0 \le n \le N_w - 1$, $A_{k,m}$, $\alpha_{k,m}$, $\beta_{k,m}$ are respectively the complex amplitude, the chirp rate, and the initial frequency of the k^{th} chirp over the m^{th} window.

Since $|F_k(n)| \le F_{\max}$, the chirp rate α, and the initial frequency β have to satisfy:

$$\begin{cases} |\beta| \le F_{\max}, \\ |\alpha| \le F_{\max}F_s/N_w. \end{cases} \tag{10}$$

In vector form, the signal over the m^{th} window can be expressed as:

$$\mathbf{S}_m = \mathbf{\Psi}\mathbf{X}_m + \mathbf{V}_m \tag{11}$$

where $\mathbf{S}_m = [s_m(0), ..., s_m(N_w - 1)]^T$, $\mathbf{V}_m = [v_m(0), ..., v_m(N_w - 1)]^T$. The dictionary matrix, $\mathbf{\Psi}$, is designed by uniformly sampling the chirp rate space. Let I denote the total number of chirp rate values, $\tilde{\alpha_i}$ is i^{th} chirp rate value in the dictionary, and $\tilde{\beta_i}$ is corresponding initial frequency value for each $\tilde{\alpha_i}$ in the dictionary. The chirp atom $\mathbf{\Psi} = [\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, ..., \boldsymbol{\psi}_I]$ is obtained by:

PROCEDURE: For each value of $\tilde{\alpha_i}$

1) Calculate FRFT angle $\tilde{\phi_i} = \arctan\frac{\tilde{\alpha_i}N_w}{F_s^2}$ (3).
2) Calculate the magnitude response $|\text{FT}(S^{\tilde{\phi_i}}(x))|$.
3) Find value of y at which $|\text{FT}(S^{\tilde{\phi_i}}(x))|$ obtains a maximum. In another word, find d (see Fig. 2).
4) Calculate the corresponding value of the chirp rate $\tilde{\beta_i} = \frac{d}{\cos\tilde{\phi_i}} - \frac{\tilde{\alpha_i}}{2}\frac{N_w}{F_s}$.
5) $\boldsymbol{\psi}_i = \exp[j2\pi(\tilde{\alpha_i}\frac{n^2}{2F_s^2} + \tilde{\beta_i}\frac{n}{F_s})]$.

Since $K < N_w \ll I$, $\mathbf{X}_m$ is highly sparse and solving for $\mathbf{X}_m$ in equation (11) becomes a sparse recovery (or CS) problem, which can be solved by:

$$\hat{\mathbf{X}}_m = \arg\min \|\mathbf{X}_m\|_1 \quad s.t. \quad \|\mathbf{S}_m - \mathbf{\Psi}\mathbf{X}_m\|_2^2 \le \epsilon \tag{12}$$

where $\|.\|_1, \|.\|_2$ denotes L_1 and L_2 norms respectively and ϵ is the noise level. The solution for (12) can be obtained by a greedy algorithm such as Orthogonal Matching Pursuit (OMP) or linear programming. In this paper, OMP is used.

IV. SIMULATION RESULTS

This section evaluates the performance of the FRFT based chirp dictionary in sparse reconstruction of non-stationary signals. We compare the proposed method with the full chirp dictionary, sinusoidal atom, and FRFT approaches. In the FRFT approach, in each window, we only calculate the magnitude response of the FRFT for angles $\tilde{\phi}_i$, and choose the values of $\tilde{\phi}_i$ or $\tilde{\alpha}_i$ which have largest magnitude responses.

In the following examples, signals are sampled at the Nyquist rate, then 50% of samples are randomly removed. The sampling frequency $F_s = 256$, the total signal length is $N = 256$. The observations are corrupted by white Gaussian noise, and the signal to noise ratio is set to $SNR = 20$dB. A rectangular window of length $N_w = 64$ is applied. The resulting images are normalized and transferred to energy versions for display. A parameter of concentration level ζ is used to assess the accuracy of the resulting TF representations. So ζ is the ratio of the sum of pixel magnitude along the actual instantaneous frequency, with respect to the rest of the TF values. So, the higher ζ, the better is the TF estimation. We assume that signals have $K = 5$ components.

In the first sample, the signal consists of two crossing chirps, which is expressed as:

$$s(n) = \exp\left\{j2\pi[(0.1F_s)\frac{n}{F_s} + (0.3F_s)\frac{n^2}{2F_s^2}]\right\} + \exp\left\{j2\pi[(0.4F_s)\frac{n}{F_s} - 0.3F_s\frac{n^2}{2F_s^2}]\right\} + v(n), \tag{13}$$

with $n = 0, 1, ..., N-1$. The results are shown in Fig. 4. The chirp dictionary and FRFT based chirp dictionary provide perfect frequency localization with $\zeta = \infty$. The FRFT is unable to recover the TF representation of the whole signal because of missing data and noise, although one signal component is accurately displayed with $\zeta = 3000$. The sinusoidal method reveals inaccuracies in the TF signature estimation with $\zeta = 3$ since besides insufficient sparsity, it is also vulnerable to the picket fence effect [5], [6], resulting in frequency contents at false locations.

Similar results are obtained in the second example where we use a signal composed of three components expressed as:

$$s(n) = \exp\left\{j(0.1F_s)\cos(2\pi\frac{n}{F_s} + \pi) + j2\pi(0.2F_s)\frac{n}{F_s}\right\} + \exp\left\{j(0.1F_s)\cos(2\pi\frac{n}{F_s} + \pi) + j2\pi(0.3F_s)\frac{n}{F_s}\right\} + \exp\left\{j2\pi[(0.1F_s)\frac{n}{F_s} + (0.3F_s)\frac{n^2}{2F_s^2}]\right\} + v(n), \tag{14}$$

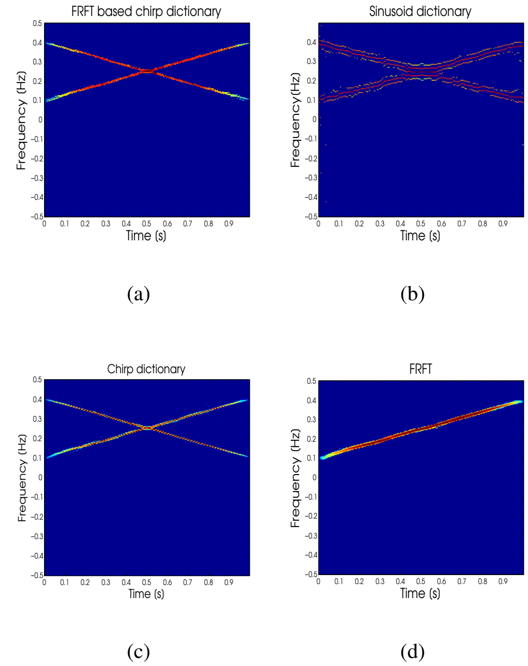


Fig. 4. TF (frequency normalized) signature for $s(n)$ in (13) with 50% data missing: (a) FRFT based chirp dictionary; (b) Sinusoidal dictionary; (c) Chirp dictionary; (d) FRFT.

with $n = 0, 1, ..., N-1$. The TF signature approximations of the four methods are displayed in Fig.5. The FRFT based chirp dictionary and the normal chirp dictionary have pretty similar concentration levels of $\zeta = 20$, whereas the sinusoidal dictionary and the FRFT have lower concentration levels with $\zeta = 3$ and $\zeta = 8$, respectively.

V. CONCLUSION

A method for instantaneous frequency estimation is presented. It deploys piece-wise chirp approximation to the TF signature of non-stationary signals under incomplete and random sampling. The chirp dictionary is built for each windowed signal, and for each chirp rate, the corresponding initial frequency is determined through the FRFT. Thus, compared with the full chirp dictionary method, the atom set dimension is much smaller. In addition, the discrete FRFT can be executed in a similar time compared to the ordinary FT, thus we can save calculation time, but obtain the same performance. It means that this algorithm can mitigate the picket fence effect, and relax contradictory requests on the number of measurements for exact recovery, and sparsity. Also, the proposed method outperforms the the sinusoidal dictionary method or FRFT with more accurate and reliable results.

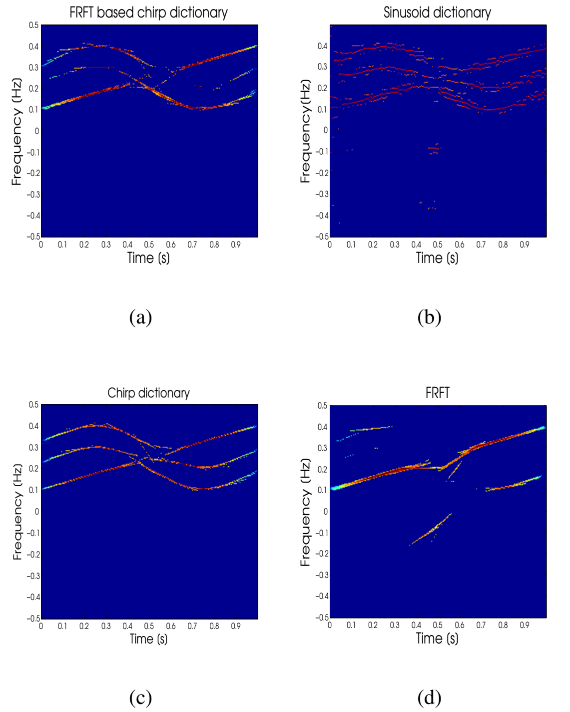


Fig. 5. TF (frequency normalized) signature for $s(n)$ in (14) with 50% data missing: (a) FRFT based chirp dictionary; (b) Sinusoidal dictionary; (c) Chirp dictionary; (d) FRFT.

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