Symmetry Breaking in Tournaments

Antoni Lozano

Logic and Programming Research Group,
Technical University of Catalonia

Abstract. We provide upper bounds for the determining number and the metric dimension of tournaments. A set of vertices $S \subseteq V(T)$ is a *determining set* for a tournament T if every nontrivial automorphism of T moves at least one vertex of S, while S is a *resolving set* for T if every two distinct vertices in T have different distances to some vertex in S. We show that the minimum size of a determining set for an order n tournament (its *determining number*) is bounded by $\lfloor n/3 \rfloor$, while the minimum size of a resolving set for an order n strong tournament (its *metric dimension*) is bounded by $\lfloor n/2 \rfloor$. Both bounds are optimal.

1 Introduction

The idea of somehow fixing the vertices of a graph in order to destroy all its nontrivial automorphisms has captured the attention of reserchers in recent years, leading to theoretical results for different graph families (trees [9], Kneser graphs [2], digraphs [8], wheels [19], or hypercubes [7]) and to applications in areas such as robotics [14] or chemistry [6].

One can distinguish vertices in a graph by considering their distances to a given subset of fixed vertices: If every two distinct vertices of the graph have different distances to some vertex in the chosen subset, no nontrivial automorphism can be left. In a more nonconstructive approach, one can just fix a given set of vertices so that no nontrivial automorphism is possible in the graph without caring whether it is due to distances or not. The set fixed in the first approach is known as a *resolving set* [18, 11], while the set fixed in the second one has been called *fixing set* [10] and *determining set* [5]. In both cases, it is desirable to find a set of minimum size which "fixes" or "destroys" all nontrivial automorphisms in a graph, making the automorphism group of the resulting structure —say, a labeled graph— trivial. This way, the sizes of the smallest determining and resolving sets for a graph can be seen as two parameters on its degree of symmetry.

Still another way that has been considered for fixing the vertices in a graph is by a coloring (called *distinguishing coloring* [3]) such that the only automorphism which preserves colors is the identity. Again, the problem here is to find a distinguishing coloring where the number of colors is minimized. In order to make reference to this notion, Albertson and Collins [3] introduced the term *symmetry breaking*, which will be used here to refer to the

three preceeding approaches as a whole, while the associated parameters to be minimized will be called *symmetry parameters.*

Tournament graphs have been extensively studied (for classical references, see [12, 15]). Some of their well-known properties (for example, the fact that their automorphism groups have odd order and, hence, are solvable) may help to get stronger conclusions on symmetry breaking in tournaments than in general directed graphs. In this paper, we consider determining and resolving sets for tournaments and prove the existence of optimal upper bounds for their minimum sizes amounting to constant fractions of their order: $\frac{1}{3}$ for determining sets, and $\frac{1}{2}$ for resolving sets of strong tournaments.

1.1 Tournaments

If D is a directed graph (*digraph* for short) $V(D)$ and $A(D)$ will denote the vertex set and the arc set of D. Given two distinct vertices u, v of D, we will write $u \in D$ instead of $u \in V(D)$ and $uv \in A(D)$ rather than $(u, v) \in A(D)$. The notation $d_D(x, y)$ (or just $d(u, v)$ when D is understood) stands for the *directed distance* from u to v in D, that is, the length of the shortest directed path from u to v in D.

A tournament models the outcome of a competition where every player plays against each other in a 2-player match. Formally, a digraph T is a *tournament* if between every pair of distinct vertices $u, v \in T$, we have $uv \in T$ or $vu \in T$ but not both. For any tournament T, define the following subsets of its vertices for any $u, v \in T$:

- $T_{u\to} = \{w \in T \mid uw \in T\}$
- $T_{\to u} = \{w \in T \mid wu \in T\}$
- $T_{uv\to} = T_{u\to} \cap T_{v\to}$
- $T_{u\to j} = T_{u\to} \cap T_{\to v}$
- $T_{\to uv} = T_{\to u} \cap T_{\to v}$

The *indegree* of a vertex u is $|T_{\to u}|$ while its *outdegree* is $|T_{u\to}|$. A tournament T is *regular* if the indegree (and, consequently, the outdegree) of all vertices is the same. In the case of a regular tournament T of order $2n + 1$, it is easy to see that $|T_{u\to}| = |T_{\to u}| = n$ for every vertex $u \in T$.

A tournament T is *transitive* if its vertices can be numbered from 1 up to n in such a way that every arc ij in T satisfies $i < j$. Therefore, there is a unique transitive tournament of order n up to isomophism, which is denoted by TT_n. Since the existence of a directed 3-cycle —which we will denote by C_3— contradicts the condition in the definition of TT_n, a characterization of transitive tournaments, then, is precisely that they do not contain the subtournament C_3. An almost transitive tournament of order $n \geq 3$, denoted by TT_n^*, is a tournament that differs from TT_n in the fact that it contains the arc $n1$ instead of $1n$, while the rest of the arcs remain unchanged.

A tournament is *strong* if all directed distances between vertices are defined. It is obvious that while TT_n is not strong, TT_n^* is a strong tournament.

1.2 Symmetry Parameters

Given a digraph D, we denote by $\Gamma(D)$ the automorphism group of D and by S_1 the trivial group. We note the well-known fact that every automorphism in $\Gamma(D)$ is an isometry, that is, for any $u, v \in D$ and $\phi \in \Gamma(D)$, $d(u, v) = d(\phi(u), \phi(v))$. We say that an automorphism ϕ *fixes a vertex* u if $\phi(u) = u$ and that it *fixes a set* $S \subseteq V(D)$ if it fixes every $u \in S$. The set of automorphisms of D that fix the set S is a subgroup of $\Gamma(D)$ called *pointwise stabilizer* of S in D, and we denote it by $\mathrm{Stab}_{\Gamma(D)}(S)$ (or just $\mathrm{Stab}(S)$ when D is understood). Given a vertex u of D, the set $\mathcal{O}_D(u) = \{\phi(u) \mid \phi \in \Gamma(D)\}$ is called the *orbit* of u in D. Orbits of D induce a partition in the set of vertices. Note that no automorphism can move a vertex from an orbit to another.

Definition 1. *A set $S \subseteq V(D)$ for which $\mathrm{Stab}_{\Gamma(D)}(S) = S_1$ is called a* determining set *of D, and the minimum cardinality of a determining set for D, denoted by $\mathrm{Det}(D)$, is called* determining number *of D.*

A digraph with no nontrivial automorphisms is called *rigid*. Determining sets (also known as *fixing sets*) have been introduced independently by Harary [10] and Boutin [5]. As a simple example, a determining set for C_3 is any set containing one vertex, while the empty set is a determining set for TT_k, for any k, since any transitive tournament is rigid. Also note that any set with $n - 2$ vertices of a tournament of order n is a determining set, since no automorphism can exchange the remaining two vertices (tournaments do not have involutions).

In a different approach introduced separately by Harary and Melter [11], and by Slater [18], the vertices in a graph can be distinguished according to their distances to a given subset of vertices.

Definition 2. *We say that a vertex u of a digraph D resolves a pair of vertices $v, w \in D$ if $d(v, u) \neq d(w, u)$. We also say that $S \subseteq V(D)$ is a* resolving set *for D if every pair of vertices in D can be resolved by some vertex in S. The minimum size of a resolving set for D is referred to as the* metric dimension *of D, and is denoted by $\mathrm{Dim}(D)$.*

While the metric dimension of a digraph does not need to be defined, the situation in the case of tournaments is different. We would like to stress the fact that Definition 2 does not require all directed distances to be defined (not even the directed distances to all vertices in the resolving set), but only the distances to the vertices which resolve each pair of distinct vertices.

Proposition 1. *For any order n tournament T, $\mathrm{Dim}(T) \leq n - 1$.*

Proof. It is well known that every tournament has a Hamiltonian path [17]. Suppose, then, that $u_1 \ldots u_n$ is a Hamiltonian path for a tournament T. Then, given two vertices u_i, u_j such that $i < j$, they can be resolved by u_j since $d(u_i, u_j) > d(u_j, u_j) = 0$. Therefore, $\{u_2, \ldots, u_n\}$ is a resolving set. $\square$

We observe that the upper bound of Proposition 1 is best possible for general tournaments.

Proposition 2. *For every integer n, $\mathrm{Dim}(TT_n) = n - 1$.*

Proof. Let $V(TT_n) = \{1, \ldots, n\}$ be the numbering of the vertices from the definition of TT_n given in Subsection 1.1 (that is, every arc ij satisfies $i < j$), and let S be a determining set for TT_n. Then, given two vertices i, j such that $i < j$, we have that

- vertex j resolves the pair since $d(i, j) = 1$ and $d(j, j) = 0$,
- no vertex $k < j$ can resolve it since distance $d(j, k)$ is undefined, and
- no vertex $k > j$ can resolve it neither since $d(i, k) = d(j, k) = 1$.

We conclude that j is the only vertex which resolves a pair i, j such that $i < j$, and then, must belong to S. Therefore, the only vertex that can be left out of S is 1, and we have $\mathrm{Dim}(TT_n) \geq n - 1$. On the other hand, $\mathrm{Dim}(TT_n) \leq n - 1$ by Proposition 1, and the result is proved. $\square$

To establish the relation between resolving and determining sets, we just follow the proof by Erwin and Harary ([9], Lemma 2), which can be applied without changes to digraphs and we reproduce here for completeness.

Proposition 3. *Every resolving set for a digraph is also a determining set. Therefore, for any digraph D, $\mathrm{Det}(D) \leq \mathrm{Dim}(D)$.*

Proof. Let S be a resolving set for a digraph D. To get a contradiction, suppose that S is not a determining set for D and, then, $\mathrm{Stab}(S) \neq S_1$. In this case, there must be a vertex v and an automorphism $\phi \in \mathrm{Stab}(S)$ such that $v \neq \phi(v)$. Since S is a resolving set for D, there must be a vertex $u \in S$ for which $d(v, u) \neq d(\phi(v), u)$. However,

$$d(v, u) = d(\phi(v), \phi(u)) = d(\phi(v), u)$$

where the first equality holds because ϕ is an isometry, and the second one because ϕ fixes u, and we have a contradiction. $\square$

1.3 Outline

In Section 2, we prove that the determining number of every tournament of order n is tightly bounded by $\lfloor n/3 \rfloor$.

Section 3 is devoted to the metric dimension. We derive a tight upper bound of $\lfloor n/2 \rfloor$ in the case of strong tournaments of order n (as we have just seen, there is no constant fraction upper bound in the general case). In order to prove this bound, we first answer a conjecture posed by Kannan, Naor, and Rudich on their concept of *anchor* ([13]), and link it to the metric dimension. We also show in Section 3 that resolving sets and anchors are exactly the same concept in regular tournaments.

Finally, in the light of our results, we remind in Section 4 a conjecture posed by Albertson and Collins ([4]) on the distinguishing number of tournaments.

2 The Determining Number

The determining number is 0, for exemple, for the transitive tournament of k vertices, TT_k, (in fact, for any rigid tournament) but how large can it be for a general tournament? Define $TT_k(T)$ as TT_k where each vertex has been substituted by a copy of T.

Proposition 4. *For every $n > 0$, there is an order n tournament T_n such that $\mathrm{Det}(T_n) = \lfloor n/3 \rfloor$.*

Proof. If $n = 3k$ for some integer k, we take $T_n = TT_k(C_3)$. Consider any set $S \subseteq V(T_n)$ containing exactly one vertex from each copy of C_3 in T_n. Since no vertex can be moved by an automorphism between different copies of C_3, and every nontrivial automorphism of one of the copies must move all three vertices, we have that $\mathrm{Stab}(S) = S_1$, and S is a determining set for T_n with $|S| = k$. Observe that any set of cardinality smaller than k would fail to contain some vertex from one of the copies of C_3, and that would make it possible to rotate C_3 while fixing the rest of the vertices in T_n.

If $n \equiv d$ modulo 3 with $d = 1$ or $d = 2$, we take T_n as the tournament T_{n-d} defined above plus d new vertices pointing to all vertices in T_{n-d}. Since the new vertices must be fixed in any nontrivial automorphism of T_n, $\mathrm{Det}(T_n) = \mathrm{Det}(T_{n-d}) = (n-d)/3 = \lfloor n/3 \rfloor$. $\square$

Now we will see that there is always a determining set with at most $\lfloor n/3 \rfloor$ vertices for an n order tournament. In the proof, we exploit the fact that an automorphism cannot move vertices between different orbits and, then, in order to construct a determining set for T, it is enough to construct determining sets for the subtournaments induced by the orbits, and then

merge them. We will need the following proposition for digraphs, which is the direct translation of a similar one stated in [5] for graphs and does not require a separate proof.

Proposition 5. *Let $\mathcal{O}_1, \ldots, \mathcal{O}_k$ be the vertex orbits of a digraph D. Let $H_1, \ldots, H_k$ be the associated induced subtournaments. Let $S_1, \ldots S_k$ be determining sets for $H_1, \ldots, H_k$. Then, $S = S_1 \cup \cdots \cup S_k$ is a determining set for D.*

Now, we can state the main theorem of this section.

Theorem 1. *For every order n tournament T, $\mathrm{Det}(T) \leq \lfloor n/3 \rfloor$.*

Proof. Given a tournament T of order n, we will show that the subtournament $T[\mathcal{O}]$ induced by an orbit $\mathcal{O}$ of T has a determining set $S_{\mathcal{O}}$ of size at most $\lfloor |\mathcal{O}|/3 \rfloor$. Therefore, the union of all determining sets for the different orbits $S = \bigcup_{\mathcal{O} \text{ is an orbit of } T} S_{\mathcal{O}}$ will satisfy that $|S| \leq \lfloor n/3 \rfloor$ while, by Proposition 5, S is a determining set for T. Thus, it is enough to prove the statement separetely for each orbit.

Suppose then that $\mathcal{O}$ is an orbit of T, and that $|\mathcal{O}| = m$. Let $H = T[\mathcal{O}]$ be the subtournament of T induced by $\mathcal{O}$. The fact that $\mathcal{O}$ is an orbit implies that H must be vertex transitive (there exists $\phi \in \Gamma(H)$ such that $\phi(u) = v$ for any $u, v \in H$) and, hence, regular.

We will proceed by induction on m. For $m \leq 2$, the determining set can be empty since there is no nontrivial automorphism. For the inductive step, we will suppose that $m \geq 3$, but note that in the case $m = 3$, the only tournaments are TT_3 (which is already rigid) and C_3 (which can be made rigid by fixing one vertex). Now, for $m \geq 3$, we consider three cases depending on the remainder of m when dividing by 3. All congruencies in the rest of the proof are taken modulo 3.

Case 1: $m \equiv 0$.

Let u be any vertex in H. Since H is regular, $|H_{\rightarrow u}| = |H_{u \rightarrow}|$, which together with the fact that $|H_{\rightarrow u}| + |H_{u \rightarrow}| + 1 = m \equiv 0$ leaves the only possibility that $|H_{\rightarrow u}| = |H_{u \rightarrow}| \equiv 1$.

Therefore, there exists an integer r such that $|H_{\rightarrow u}| = |H_{u \rightarrow}| = 3r + 1$. By induction hypothesis, both $H_{\rightarrow u}$ and $H_{u \rightarrow}$ have determining sets of size r. Their union plus vertex u form a determining set for H of size $2r + 1$. Since $|H| = m = 2(3r + 1) + 1 = 6r + 3$, the determining set has at most (in this case, exactly) $\lfloor m/3 \rfloor$ vertices as expected.

Case 2: $m \equiv 2$.

Let u be any vertex in H. As in the previous case, $|H_{\rightarrow u}| = |H_{u \rightarrow}|$, which now implies that $|H_{\rightarrow u}| = |H_{u \rightarrow}| \equiv 2$. Therefore, there exists an integer r

6

such that $|H_{\to u}| = |H_{u \to}| = 3r + 2$. By induction hypothesis, both $H_{\to u}$ and $H_{u\to}$ have determining sets of size r. Their union plus vertex u form a determining set for H of size $2r+1$. Since $|H| = m = 2(3r+2)+1 = 6r+5$, the determining set has at most (in this case, less than) $\lfloor m/3 \rfloor$ vertices.

Case 3: $m \equiv 1$.

In this case, for any $u \in H$, $|H_{\to u}| = |H_{u\to}| \equiv 0$. A determining set for H constructed inductively as in the previous cases would have more than $\lfloor m/3 \rfloor$ vertices. However, we can complete the proof using two subcases:

– *Subcase 3.1: For every arc $uv \in H$, $|H_{u\to v}| \equiv 2$.*
  Since we are assuming that $m \geq 3$, let u, v, w be three vertices in H. Moreover, we can select them in such a way that uv, vw, and wu are in H, the reason being that H cannot be transitive in this subcase and, therefore, it must contain a 3-cycle. Now, H can be split into the following subsets:
  - $H_{u\to v}$, $H_{v\to w}$, and $H_{w\to u}$, whose cardinalities are all congruent with 2 modulo 3. Let $|H_{u\to v}| = 3r+2$, $|H_{v\to w}| = 3s+2$, and $|H_{w\to u}| = 3t+2$ for some $r, s, t \geq 0$,
  - the set $\{u, v, w\}$, and
  - the remaining vertices in H, say H'. Note that since $m \equiv 1$ and the number of vertices counted so far is multiple of 3, $|H'| \equiv 1$, say $|H'| = 3z + 1$ for some z.
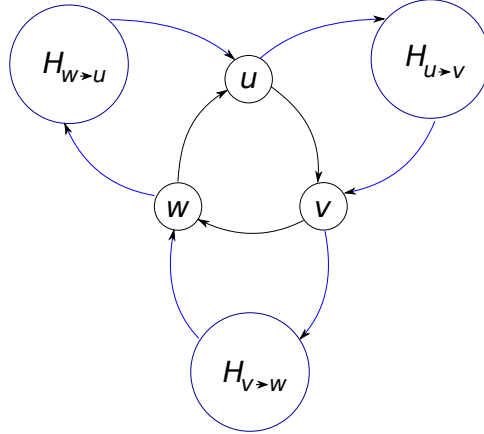

Fig. 1. Subcase 3.1: $|H_{u\to v}| \equiv |H_{v\to w}| \equiv |H_{w\to u}| \equiv 2$.

Summing up the previous numbers, we have:
$$m = |H_{u\to v}| + |H_{v\to w}| + |H_{w\to u}| + |\{u, v, w\}| + |H'| =$$

7

$$= (3r + 2) + (3s + 2) + (3t + 2) + 3 + (3z + 1) =$$
$$= 3(r + s + t + z) + 10.$$

On the other hand, we can assume the existence of determining sets for $H_{u \to v}$, $H_{v \to w}$, $H_{w \to u}$, and H' by induction hypothesis, of sizes r, s, t, and z, respectively. Their union plus the vertices u, v, and w is a determining set for H with less than $\lfloor m/3 \rfloor$ vertices.

– *Subcase 3.2: There is an arc $uv \in H$ for which $|H_{u \to v}| \not\equiv 2$.*
  Let u, v be two such vertices. Now, we consider the partition of H into the sets: $H_{u \to v}$, $H_{\to uv}$, $H_{v \to u}$, $H_{uv \to}$, and $\{u, v\}$. Using the assumption (of Case 3) that for any $w \in H$, $|H_{\to w}| = |H_{w \to}| \equiv 0$, we can observe the following:

  i. $|H_{u \to v} \cup H_{\to uv}| \equiv 2$, since the previous union plus v is exactly $H_{\to v}$.
  ii. $|H_{\to uv} \cup H_{v \to u}| \equiv 0$, since the previous union is exactly $H_{v \to}$.
  iii. $|H_{v \to u} \cup H_{uv \to}| \equiv 2$, since the previous union plus v is exactly $H_{u \to}$.

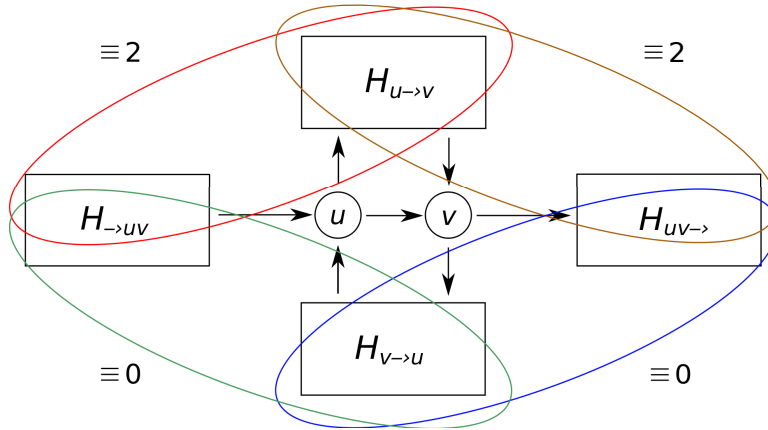**Fig. 2.** Subcase 3.2: For some arc $uv \in H$, $|H_{u \to v}| \not\equiv 2$.

Now, we can see that there are only two possibilities:

- $|H_{u \to v}| \equiv 1$. Then, by i, $|H_{\to uv}| \equiv 1$; by ii, $|H_{v \to u}| \equiv 2$; and by iii, $|H_{uv \to}| \equiv 1$. There must be some r, s, t, z in this case such that $|H_{u \to v}| = 3r + 1$, $|H_{\to uv}| = 3s + 1$, $|H_{v \to u}| = 3t + 2$, and $|H_{uv \to}| = 3z + 1$.
- $|H_{u \to v}| \equiv 0$. Then, by i, $|H_{\to uv}| \equiv 2$; by ii, $|H_{v \to u}| \equiv 1$; and by iii, $|H_{uv \to}| \equiv 2$. There must be som r, s, t, z in this case such that $|H_{u \to v}| = 3r$, $|H_{\to uv}| = 3s + 2$, $|H_{v \to u}| = 3t + 1$, and $|H_{uv \to}| = 3z + 2$.

In any of the above two possibilities, H contains the four previous subsets plus u and v, giving $|H| = 3(r + s + t + z) + 7$. The union of the determining

sets given by the induction hypothesis plus u and v gives a determining set for H of size $r + s + t + z + 2$, which is less than $\lfloor m/3 \rfloor$. $\qquad\square$

Note that the upper bound $\lfloor n/3 \rfloor$ given in Theorem 1 is tight by Proposition 4.

3 The Metric Dimension

As we have seen in Section 1, if we do not consider any additional condition on a tournament of order n, its metric dimension can be as large as $n - 1$. In this section, we show that just requiring a tournament to be strong, its metric dimension drops to at most $\lfloor n/2 \rfloor$.

We start with an example of a strong tournament having a large metric dimension, and we will see later that it is indeed the largest possible for any strong tournament.

Proposition 6. *For every $n > 2$, $\mathrm{Dim}(TT_n^*) = \lfloor n/2 \rfloor$.*

Proof. Let $V(TT_n^*) = \{1, \ldots, n\}$ be the numbering of the vertices from the definition of TT_n^* (that is, $ij \in T$ implies $i < j$ except for the case of the arc $n1$, which belongs to TT_n^*). Note that the even (odd) numbered vertices constitute a resolving set of size $\lfloor n/2 \rfloor$ since every pair of odd (even) numbered vertices are distinguished by some even (odd) numbered vertex. This shows that $\mathrm{Dim}(TT_n^*) \leq \lfloor n/2 \rfloor$.

To see why $\mathrm{Dim}(TT_n^*) \geq \lfloor n/2 \rfloor$, we observe in the first place that it cannot happen that two vertices $i, i+1$ in the range $1 < i < i+1 < n$ do not belong to the resolving set, for no other vertex can resolve them. Moreover, if one of the vertices 2 or $n - 1$ is not in the resolving set, then either 1 or n must be in. Both conditions imply that the resolving set must contain at least $\lfloor n/2 \rfloor$ vertices. $\qquad\square$

Kannan, Naor, and Rudich [13] introduced the notion of *anchors* for tournaments in order to get a simple algorithm for tournament isomorphism. Anchors can be seen as a sort of simplification of resolving sets having the advantage that they are always defined.

An *anchor* in a tournament T is a subset $S \subseteq V(T)$ such that for all vertices $u, v \in V(T) - S$, $u \neq v$, there exists a vertex $w \in S$ such that exactly one of uw and vw is an arc of T, in which case we say that w *distinguishes* the pair u, v. Thus, if the vertices in an anchor are fixed, all vertices in the tournament would be fixed and no automorphism other than the identity would be possible; in this respect, the concept is similar to the other symmetry breaking concepts seen so far. Let $\mathrm{Anchor}(T)$ denote the size of the smallest anchor for T.

In [13], it is shown that for any order n tournament T, $\text{Anchor}(T) \leq \lfloor 2n/3 \rfloor$, and the authors conjecture that, in fact, $\text{Anchor}(T) \leq \lfloor n/2 \rfloor$. We start proving here that their conjecture is correct.

Theorem 2. *For every order n tournament T, $\text{Anchor}(T) \leq \lfloor n/2 \rfloor$.*

Proof. Let T be a tournament with n vertices. We will preceed by induction on n. The statement is trivially true for $n \leq 2$ by the definition of anchor. Now, suppose that $n \geq 3$ and consider two cases depending on the parity of n:

Case 1: n is even.
    Select a vertex u from T. Now, $V(T) - \{u\}$ can be split into the disjoint sets of vertices $T_{\to u}$ and $T_{u \to}$. Since $|V(T) - \{u\}|$ is odd, one of $|T_{\to u}|$ and $|T_{u \to}|$ is even and one is odd. Suppose without loss of generality that the first quantity is odd and equal to $2r + 1$ and the second one is even and equal to $2s$, for integers r, s. Then, by induction hypothesis, $T_{\to u}$ has an anchor A of size r, and $T_{u \to}$ has an anchor B of size s.
    Note that the set of vertices $S = A \cup B \cup \{u\}$ has size $r + s + 1 = n/2$. Furthermore, S is an anchor since for any $x, y \notin S$, we can consider three possibilities:

- $x, y \in T_{\to u}$. In this case, x and y are distinguished by some vertex in A.
- $x, y \in T_{u \to}$. Symmetrically, x and y are distinguished by some vertex in B.
- $x \in T_{u \to}$ and $y \in T_{\to u}$ or viceversa. In this case, vertex u distinguishes x and y.

Case 2: n is odd.
    Consider the parity of the sets $T_{u \to}$, for $u \in V$. If some of these sets has odd cardinality, then the corresponding set $T_{\to u}$ must have odd cardinality too; suppose the respective cardinalities are $2r + 1$ and $2s + 1$ for some integers r, s. By induction hypothesis, these two sets of vertices must have anchors of sizes r and s, respectively. Similarly to the previous case, the union of the anchors plus vertex u form an anchor for T of size $r + s + 1$. Note that, in this case, $n = (2r + 1) + (2s + 1) + 1 = 2(r + s + 1) + 1$, so the anchor has size $\lfloor n/2 \rfloor$, as required.
    Consider now the case in which all sets $T_{u \to}$, for $u \in V$, have even cardinality. Now, we consider two subcases:

- *Subcase 2.1: There are two distinct vertices u, v in T s.t. $|T_{uv \to}|$ is odd.* We can assume, without loss of generality, that $uv \in T$. We can now split the set of vertices $T_{v \to}$ into two sets:

$$T_{v \to} = T_{v \to u} \cup T_{uv \to}$$

10

Since the above union is disjoint, $|T_{uv\rightarrow}|$ is odd, and $|T_{v\rightarrow}|$ is even, then $|T_{v\rightarrow u}|$ must be odd. We can classify all the vertices in T different from u and v into the disjoint sets $T_{\rightarrow v} - \{u\}$, $T_{v\rightarrow u}$, and $T_{uv\rightarrow}$.

Since the last two sets have odd cardinality and the union of all three sets has cardinality $n - 2$, which is odd, the first one must have odd cardinality too. Suppose that, for some integers r, s, and t,

$$|T_{\rightarrow v} - \{u\}| = 2r + 1, \quad |T_{v\rightarrow u}| = 2s + 1, \quad |T_{uv\rightarrow}| = 2t + 1.$$
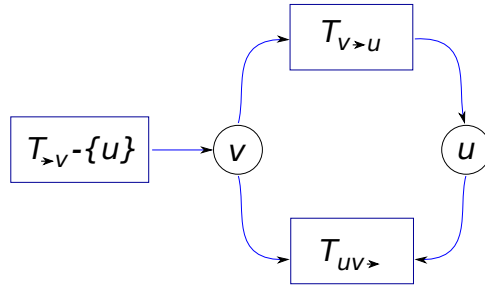

Fig. 3. Subcase 2.1: $|T_{uv\rightarrow}|$, $|T_{v\rightarrow u}|$, and $|T_{\rightarrow v} - \{u\}|$ are odd.

By induction hypothesis, $T_{\rightarrow v} - \{u\}$, $T_{v\rightarrow u}$, and $T_{uv\rightarrow}$ must have anchors of respective sizes r, s, and t which we will call A, B, and C, resp. Note that the set $S = A \cup B \cup C \cup \{u, v\}$ contains $r + s + t + 2$ vertices. Since $n = 2(r + s + t) + 5 = 2(r + s + t + 2) + 1$, $|S|$ is exactly $\lfloor n/2 \rfloor$. Now, we will argue that S is an anchor for T. Given $x, y \notin S$, we have the following possibilities for x and y:

- In the case that both x and y belong to one of the sets $T_{\rightarrow v} - \{u\}$, $T_{v\rightarrow u}$, or $T_{uv\rightarrow}$, they can be distinguished by some vertex in the respective anchor: A, B, or C.
- If x or y is in $T_{\rightarrow v} - \{u\}$ and the other one is in either $T_{v\rightarrow u}$ or $T_{uv\rightarrow}$, they are distinguished by v.
- If x or y is in $T_{v\rightarrow u}$ and the other one in $T_{uv\rightarrow}$, they are distinguished by u.

– *Subcase 2.2: For all distinct vertices u, v in T, $|T_{uv\rightarrow}|$ is even.*
Note that, in this case, the tournament T cannot be transitive since we can always select two "consecutive" vertices u, v in a transitive tournament such that $|T_{uv\rightarrow}|$ is odd. Then, T must contain a 3-cycle, similarly to Subcase 3.1 in the proof of Theorem 1.
Let u, v, and w be three vertices such that uv, vw, and wu are three arcs in T. Note that we can partition the vertices in $T_{u\rightarrow}$, $T_{v\rightarrow}$, and $T_{w\rightarrow}$ as follows:

$$T_{u\to} = T_{u\to v} \cup T_{uv\to} \cup \{v\}$$
$$T_{v\to} = T_{v\to w} \cup T_{vw\to} \cup \{w\}$$
$$T_{w\to} = T_{w\to u} \cup T_{wu\to} \cup \{u\}$$

Our assumptions on cardinalities in this subcase applied to the above relations imply that the cardinalities of the sets $T_{u\to v}$, $T_{v\to w}$, and $T_{w\to u}$ must be odd, say

$$|T_{u\to v}| = 2r + 1, \quad |T_{v\to w}| = 2s + 1, \text{ and } |T_{w\to u}| = 2t + 1$$

for some integers r, s, and t. Note too that all three sets are disjoint. We now apply the induction hypothesis to the above sets and get an anchor A for $T_{u\to v}$ with size r, an anchor B for $T_{v\to w}$ with size s, and an anchor C for $T_{w\to u}$ with size t. Let now T' be the set containing the rest of the vertices, that is

$$T' = V(T) - (T_{u\to v} \cup T_{v\to w} \cup T_{w\to u} \cup \{u, v, w\}).$$

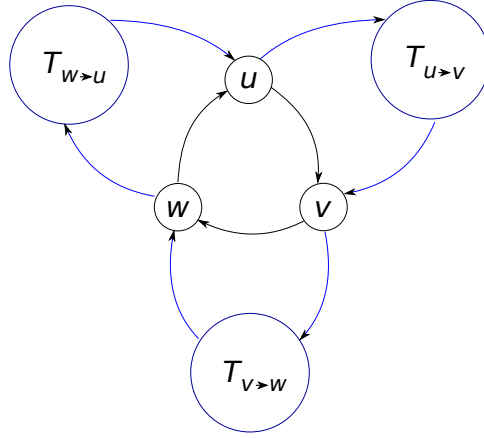The set T' must have odd cardinality, say $|T'| = 2m+1$, and (by induction hypothesis) an anchor D of size m.

Fig. 4. Subcase 2.2: $|T_{u\to v}|$, $|T_{v\to w}|$, and $|T_{w\to u}|$ are odd.

Consider the set $S = A \cup B \cup C \cup D \cup \{u, v, w\}$. Its size is $r+s+t+m+3$, whereas $n = 2(r + s + t + m) + 7 = 2(r + s + t + m + 3) + 1$; therefore, $|S| = \lfloor n/2 \rfloor$. Now, we will argue that S constitutes an anchor for T. Given $x, y \notin S$, we have the following possibilities for x and y:

- In the case that both x and y belong to one of the sets $T_{u\to v}$, $T_{v\to w}$, $T_{w\to u}$, or T', they can be distinguished by some vertex in the respective anchor: A, B, C, or D.

- In the case that x and y belong to different sets from $T_{u \to v}$, $T_{v \to w}$, $T_{w \to u}$, then u, v, or w must distinguish x and y. For example, if $x \in T_{u \to v}$ and $y \in T_{v \to w}$, they are distinguished by v, the rest of the cases being similar.
- In the case that one of x, y is in T' and the other one is in $T_{u \to v}$, $T_{v \to w}$, or $T_{w \to u}$, then u, v, or w will distinguish x and y. For example, suppose that $x \in T'$ and $y \in T_{u \to v}$. Since vertices in T' are not in $y \in T_{u \to v}$, it means that either ux or xv cannot be an arc in T, which means that either xu or vx is an arch in T: in the first case, x and y are distinguished by u, in the second one, by v. The rest of the cases are similar. $\qquad \square$

The previous upper bound for the size of anchors has some implications regarding the metric dimension. In the first place, we note the following relation between the two notions.

Proposition 7. *Every anchor in a strong tournament is a resolving set.*

Proof. Suppose that S is an anchor in a strong tournament T, and take two distinct vertices $u, v \in V(T) - S$ which are distinguished by a vertex $w \in S$. According to the definition of anchor, T contains exactly one of the arcs uw and vw. If it contains uw (and then, wv), we have $d(u,w) = 1$ while $d(v,w)$ exists (because T is strong) but must be greater than 1. Then, $d(v,w) > d(u,w)$. The case when T contains the arc vw but not uw similarly implies that $d(v,w) < d(u,w)$. In either case, $d(u,w) \neq d(v,w)$, and so w resolves u and v.

Since a resolving set for T must resolve *any* pair u, v of distinct vertices in T (not only in $V(T) - S$), suppose that at least one of them, say u, belongs to S. In this case, vertex u itself resolves the pair since $d(u,u) = 0 < d(v,u)$. $\qquad \square$

Now, combining Theorem 2 with Proposition 7, we get our main conclusion on the metric dimension of tournaments.

Corollary 1. *For every order n strong tournament T, $\mathrm{Dim}(T) \leq \lfloor n/2 \rfloor$.*

Note that the upper bound of $\lfloor n/2 \rfloor$ for the metric dimension of strong tournaments is tight by Proposition 6.

It is interesting to observe that resolving sets and anchors become the same notion when applied to regular tournaments.

Proposition 8. *Let T be a regular tournament. Then:*

1. *Every anchor in T is a resolving set.*
2. *Every resolving set in T is an anchor.*

Proof.

1. By the fact that regular tournaments are strong [1] and Proposition 7.
2. Alspach proved in [1] that each arc of a regular tournament of order n is contained in a cycle of each length k, $k = 3, \ldots, n$. Taking $k = 3$, we can conclude any two distinct vertices u, v in T belong to a directed 3-cycle, and so we can assure that either $d(u, v) = 1$ or $d(u, v) = 2$. Then, a vertex w in a resolving set for T can only resolve two other vertices u, v by keeping distance 1 to one of them and 2 to the other; this means that T contains exactly one of the arcs uw, vw and, then, w distinguishes u and v. Thus, any resolving set is, in fact, an anchor. □

In the case of regular tournaments, we can also add a lower bound due to the fact that any anchor of an order n tournament must have at least $\lceil \log n \rceil - 1$ vertices [13]. Then, $\mathrm{Dim}(T) \geq \lceil \log n \rceil - 1$ for every regular tournament T of order n.

4   A Conjecture Revisited

The symmetry breaking notion of *distinguishability* introduced by Albertson and Collins in [3] is very related to the notions studied here. A *d-coloring* of a (di)graph G is said to be *d-distinguishing* if G has no nontrivial automorphism which preserves the colors. The *distinguishing number $D(G)$* is defined as the minimum d such that G has a d-distinguishing coloring. It is worth reminding the conjecture that the same authors state in [4].

Conjecture. *If T is a tournament, then $D(T) = 2$.*

Albertson and Boutin prove in [2] that a graph is d-distinguishable if and only if it has a determining set that is $(d-1)$-distinguishable, which allows us to restate the conjecture.

Observation. *A tournament T contains a determining set whose induced subtournament is rigid if and only if $D(T) = 2$.*

We wonder if the determining sets constructed in Theorem 1 (more formally speaking, their induced subtournaments) can be made rigid by adding vertices in a way similar to the technique used in [2]. In this case, the previous observation would imply an affirmative answer to the conjecture.

5 Acknowledgements

References

1. B. Alspach. Cycles of each length in regular tournaments. *Canad. Math. Bull.*, **10**: 283–286, 1967.
2. M. Albertson and D. Boutin. Using Determining Sets to Distinguish Kneser Graphs. *Electronic Journal of Combinatorics*, **14**: R20, 2007.
3. M. Albertson and K. Collins. Symmetry Breaking in Graphs. *Electronic Journal of Combinatorics*, **3**: R18, 1996.
4. M. Albertson and K. Collins. A Note on Breaking the Symmetries of Tournaments. *Proceedings of the Thirtieth Southeastern International Conference on Combinatorics, Graph Theory, and Computing. Congr. Numer.* **136**: 129-131, 1999.
5. D. Boutin. Identifying Graph Automorphisms Using Determining Sets. *The Electronic Journal of Combinatorics*, **13**, #R78, 2006.
6. G. Chartrand, L. Eroh, M. Johnson, O. Oellermann. Resolvability in graphs and the metric dimension of a graph. *Discrete Applied Mathematics*, **105**: 99–113, 2000.
7. M. Chan. The distinguishing number of the augmented cube and hypercube powers. *Discrete Mathematics*, **308**: 2330–2336, 2008.
8. G. Chartrand, M. Raines, P. Zhang. The Directed Distance Dimension of Oriented Graphs. *Mathematica Bohemica*, **125**(2): 155-168, 2000.
9. D. Erwin and F. Harary. Destroying Automorphisms by Fixing Nodes. *Dicrete Mathematics*, **306**: 3244–3252, 2006.
10. F. Harary. Methods of Destroying the Symmetries of a Graph. *Bull. Malaysian Math. Sci. Soc.*, **24**(2): 183–191, 2001.
11. F. Harary and R.A. Melter. On the metric dimension of a graph. *Ars Combinatorica*, **2**: 191–195, 1976.
12. The Theory of Round Robin Tournamnets. *The American Mathematical Monthly*, **73**(3): 231–246, 1966.
13. S. Kannan, M. Naor, and S. Rudich. Anchors in Tournaments. Technical Report TR 92-08, Dept. of Computer Science, University of Arizona, 1992.
14. S. Khuller, B. Raghavachari, A. Rosenfeld. Landmarks in Graphs. *Discrete Appl. Math.*, **70**: 217–229, 1996.
15. J. W. Moon. *Topics on Tournaments*. Holt, Rinehart and Winston, New York, 1968.
16. O. R. Oellermann, J. Peters-Fransen. The strong metric dimension of graphs and digraphs. *Discrete Applied Mathematics*, **155**: 356–364, 2007.
17. L. Redei. Ein kombinatoricher Satz. *Acta Litt. Szeged*, **7**: 39–43, 1934.
18. P.J. Slater. Leaves of trees. *Congr. Numer.*, **14**: 549–559, 1975.
19. B. Shanmukha, B. Sooryanarayana, and K.S. Harinath. Metric Dimension of Wheels. *Far East J. Appl. Math.*, **8**(3): 217–229, 2002.