The Complexity of Measuring Power in Generalized Opinion Leader Decision Models

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Abstract

We analyze the computational complexity of the power measure in models of collective decision: the *generalized opinion leader-follower model* and the *oblivious* and *non-oblivious influence* models. We show that computing the power measure is #P-hard in all these models, and provide two subfamilies in which the power measure can be computed in polynomial time.

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1 Introduction

Opinion leadership is a well known and established model for communication in sociology and marketing. It comes from the *two-step flow of communication* theory proposed in the 1940s [1]. This theory recognizes the existence of collective decision-making situations in societies formed by actors called *opinion leaders*.Following those ideas an *opinion leader-follower model* (OLF) was introduced in [4] together with two measures associated to participants *satisfaction* (Sat) and *power* (Pow). Cooperative *influence games* were introduced in [3] based on how influence is exerted in the linear threshold model. Decision models extending OLF to general influence graphs: *generalized opinion leader-follower model* (gOLF), and *oblivious* and *non-oblivious* influence decision models, were introduced in [2] together with an analysis of the complexity of computing the Sat measure. Here we analyze the computational complexity of the Pow measure showing that it presents the same behavior as Sat: computing Pow is #P-hard, for the decision models considered in this paper, while it can be computed in polynomial time in strong hierarchical and star influence graphs.

2 Decision models

All the graphs considered in this paper are directed, without loops and multiple edges. We use standard notation: $G = (V, E)$ is a directed graph, $V(G)$ denotes the vertex set, $E(G)$ is the edge set, and n denotes the number of vertices $|V|$. We use simply V and E when there is no risk of confusion. For $i \in V$, $S_G(i) = \{j \in V \mid (i, j) \in E\}$ denotes the set of *successors* of i, and $P_G(i) = \{j \in V \mid (j, i) \in E\}$ the set of *predecessors* of i. A *two layered bipartite graph* is a bipartite graph $G = (V_1, V_2, E)$ with $V(G) = V_1 \cup V_2$ and $E \subseteq V_1 \times V_2$, i.e., so that for $i \in V_1$, $P_G(i) = \emptyset$ and, for $i \in V_2$, $S_G(i) = \emptyset$. By I_a, as usual, we denote a graph that is formed by a isolated vertices. Given $G = (V, E)$ and $X \subseteq V$, $G[X]$ denotes the subgraph induced by X and $G \setminus X$ denotes the subgraph induced by $V \setminus X$, i.e., $G \setminus X = G[V \setminus X]$. For an acyclic graph G, $\mathtt{I}$ denotes the set of vertices with $P_G(i) = \emptyset$ and $S_G(i) = \emptyset$; $\mathtt{L}$ those with $P_G(i) = \emptyset$ and $S_G(i) \neq \emptyset$; $\mathtt{F}$ those with $P_G(i) \neq \emptyset$ and $S_G(i) = \emptyset$; and $\mathtt{FI} = \mathtt{F} \cup \mathtt{I}$. For a vector $x \in \{0, 1\}^n$, let $X(x) = \{i \mid x_i = 1\}$ and define $\overline{x}_{-i}$ as the vector that is x but changing the component x_i by $\overline{x_i}$.

Definition 2.1 A *decision model* $\mathcal{M}$ is a tuple (V, D, q) where $V = \{1, \ldots, n\}$ and $D : \{0, 1\}^n \to \{0, 1\}^n$ is a function, and $0 \leq q \leq n+1$. For a participants' *initial decision* vector $x \in \{0, 1\}^n$, the *final decision vector* is $y = D(x)$. The associated *collective decision function* $C_{\mathcal{M}} : \{0,1\}^n \to \{0,1\}$ is defined as $C_{\mathcal{M}}(x) = 1$ iff $|\{i \in V \mid y_i = 1\}| \geq q$.

Now we present the formal definition of opinion leader-follower models.

Definition 2.2 A *generalized opinion leader-follower model (gOLF)* is a defined by a triple (G, r, q) where $G = (V, E)$ is a two layered bipartite digraph representing the actors' relations, r is a rational number, $1/2 \leq r \leq 1$, and q is an integer, $0 < q \leq n$, defining a decision model $\mathcal{M}(G, r, q) = (V(G), D, q)$. The function D is defined as follows, for $x \in \{0,1\}^n$, let $y = D(x)$, for $1 \leq i \leq n$, if for one $z \in \{0,1\}$, $|\{j \in P_G(i) \mid x_j = z\}| \geq \lceil r \cdot |P_G(i)| \rceil$ and $|\{j \in P_G(i) \mid x_j = 1 - z\}| < \lceil r \cdot |P_G(i)| \rceil$, we set $y_i = z$ otherwise $y_i = x_i$.

An *OLF*, is a gOLF where n is odd and $q = (n+1)/2$. An *odd-OLF* is a gOLF $\mathcal{M} = (G, r, q)$ in which $r = 1/2$ and, for all $i \in V$ with $P_G(i) \neq \emptyset$, $|P_G(i)|$ is odd.

Before defining influence based models, we recall the definitions of influence graphs and spread of influence from [3]. An *influence graph* is a tuple (G, f), where $G = (V, E)$ is a directed graph and f is a labeling function assigning to any vertex a non-negative rational value. Let (G, f) be an influence graph and let $X \subseteq V$. The *activation process*, with initial activation X, at time t, $0 \leq t \leq n$, activates a set of vertices $F^t(X)$ defined as follows: $F^0(X) = X$ and $F^t(X) = F^{t-1}(X) \cup \{i \in V \mid |P_G(i) \cap F^{t-1}(X)| \geq f(i)\}$, for $1 \leq t \leq n$. The *spread of influence* of X in (G, f) is the set $F(X) = F^n(X)$.

Definition 2.3 An *oblivious influence model* is a decision model described by (G, f, q, N), where (G, f, q, N) is an influence graph with positive labeling function defining the model $\mathcal{M}(G, f, q, N) = (V(G), D, q)$ where , for $x \in \{0,1\}^n$, $y = D(x)$ is defined as $y_i = 1$ iff $i \in F(X(x) \cap N)$.

A *non-oblivious influence model* is a decision model described by $\mathcal{M} = (G, f, q, N)$ where (G, f, q, N) is an influence graph with positive labeling function defining the model $(V(G), D, q)$ where $y = D(x)$ is defined as follows. For $x \in \{0,1\}^n$, let $p_i^1(x) = |F(X(x) \cap N) \cap P(i)|$ and $p_i^0(x) = |P(i) \setminus F(X(x) \cap N)|$. For $i \in V(G) \setminus N$, $y_i = 1$ iff $i \in F(X(x)$. For $i \in N$, if for one $z \in \{0,1\}$, $p_i^z(x) \geq f(i)$ and $p_i^{\overline{z}}(x) < f(i)$, we set $y_i = z$ otherwise $y_i = x_i$.

In general, a gOLF cannot be cast as an oblivious influence model because the tie-breaking rules are different. However odd-OLF constitute a submodel of both oblivious and non-oblivious influence models as ties do not arise.

Now we define the power measure introduced in [4].

Definition 2.4 Let $\mathcal{M} = (V, D, q)$ be a decision model. The *power* of $i \in V$,

$\text{Pow}_{\mathcal{M}}(i)$ is defined as $|\{x \in \{0,1\}^n \mid C(x) = x_i \ \wedge \ C(\overline{x}_{-i}) = \overline{C(x)}\}|$.

Associated with this measure we consider the POWER problem: Given a decision model $\mathcal{M}$ and an actor i compute $\text{Pow}_{\mathcal{M}}(i)$.

3 Complexity of computing Power

Let (G, f) be an influence graph, for $i \in V(G)$. For $N \subseteq V(G)$ and $1 \leq k \leq n$, $F_k(N, G, f)$ denotes the set $\{X \subseteq V(G) \mid |F(X \cap N)| = k\}$. The EXPANSION problem asks to compute $|F_k(N, G, f)|$, given an influence graph (G, f), $N \subseteq V(G)$ and an integer k. The EXPANSION problem is known to be #P-hard for odd-OLF taking $N = \mathtt{L} \cup \mathtt{I}$ [2].

Theorem 3.1 *The* POWER *problem is #P-hard for odd-OLF models.*

Proof. [Sketch] To show hardness we provide a reduction from the EXPANSION on odd-OLF. Let (G, f, N, k) be an input to the EXPANSION problem were (G, f) is the graph corresponding to an odd-OLF (G, r) and $N = \mathtt{L} \cup \mathtt{I}$. Consider the odd-OLF (G', r) where G' is obtained from G by adding an isolated new vertex z. Observe that (G', r) is and odd-OLF with an additional independent participant. It is easy to see that for $\mathcal{M} = \mathcal{M}(G', r, k+1)$, $\text{Pow}_{\mathcal{M}}(z) = |F_k(G, f, N)|$. Using this fact as the construction can be computed in polynomial time the results follows. □

As an odd-OLF is a gOLF and also an oblivious and a non-oblivious influence model we get the following result.

Corollary 3.2 *The* POWER *problem for gOLF and oblivious and non-oblivious influence models is #P-hard.*

In the following we consider two subfamilies of bipartite digraphs, the *strong hierarchical* and the *star* influence graphs introduced in [2]. We devise polynomial time algorithms to solve POWER for those families of graphs.

Strong hierarchical graphs are defined through some basic operations. Given two graphs H_1 and H_2 with $V(H_1) \cap V(H_2) = \emptyset$ their *disjoint union* is the graph $H_1 + H_2 = (V(H_1) \cup V(H_2), E(H_1) \cup E(H_2))$. Given a graph H, the *one layer extension to a set* $V' \neq \emptyset$ *of new vertices* $(V(H) \cap V' = \emptyset)$ is the graph $H \otimes V'$ is defines as $V(H \otimes V') = V(H) \cup V'$ and $E(H \otimes V') = E(H) \cup \{(u, v) \mid u \in \mathtt{FI}(H), v \in V'\}$.

Observe that we have $\mathtt{L}(H_1 + H_2) = \mathtt{L}(H_1) \cup \mathtt{L}(H_2)$, $\mathtt{I}(H_1 + H_2) = \mathtt{I}(H_1) \cup \mathtt{I}(H_2)$, $\mathtt{F}(H_1 + H_2) = \mathtt{F}(H_1) \cup \mathtt{F}(H_2)$ and $\mathtt{FI}(H_1 + H_2) = \mathtt{FI}(H_1) \cup \mathtt{FI}(H_2)$.

Furthermore, $\mathtt{L}(H \otimes V') = \mathtt{L}(H) \cup \mathtt{I}(H)$, $\mathtt{I}(H \otimes V') = \emptyset$, and $\mathtt{F}(H \otimes V') = \mathtt{FI}(H \otimes V') = V'$.

As base case we use graphs with only isolated vertices. The family is completed by taking the closure under the two graph operations defined above.

Definition 3.3 The family of *strong hierarchical graphs* is defined recursively as follows: (1) The graph I_a, for $a > 0$, is a strong hierarchical graph; (2) if H_1 and H_2 are disjoint strong hierarchical graphs, the graph $H_1 + H_2$ is a strong hierarchical graph; and (3) if H is a strong hierarchical graph and $V' \neq \emptyset$ is a set of vertices with $V(H) \cap V' = \emptyset$, the graph $H \otimes V'$ is a strong hierarchical graph. A *strong hierarchical influence graph* is an influence graph (G, f) where G is a strong hierarchical graph.

Theorem 3.4 *The* Power *problem, for oblivious and non-oblivious influence models corresponding to strong hierarchical influence graphs, is polynomial time solvable.*

Proof. [Sketch] Let $\mathcal{M} = \mathcal{M}(V, D, q)$ be the oblivious or the non-oblivious model corresponding to a strong hierarchical influence graph (G, f). Fix a participant i. For $x \in \{0,1\}^n$ let $x' = \overline{x}_{-i}$, $y = D(x)$ and $y' = D(x')$. Consider a table $T(a, b, c, d)$, $0 \le a, c \le n$ and $0 \le b, d \le |\mathtt{FI}(G)|$ holding the following quantities

$$|\{x \in \{0,1\}^n \mid |\{j \mid D(x) = 1\}| = a \wedge |F(X(x) \cap N) \cap \mathtt{FI}(H)| = b \\ \wedge |\{j \mid D(x') = 1\}| = c \wedge |F(X(x') \cap N) \cap \mathtt{FI}(G)| = d\}|.$$

Observe that

$$\text{Pow}(i) = \sum_{a \ge q, c < q} \sum_{0 \le b, d \le \mathtt{FI}(H)} T(a, b, c, d).$$

The values in the table T can be computed recursively, using the hierarchical structure of G, from the tables corresponding to different subgraphs of G. However, the recurrences are different for the oblivious than for the non-oblivious model. Our algorithms use dynamic programming to compute the table T in polynomial time. □

In a star influence graph, in addition to the sets L, I and F, we have the central node c wich acts as mediator and the set R of *reciprocal actors*.

Definition 3.5 A *star influence graph* is an influence graph (G, f), where $V(G) = \mathtt{L} \cup \mathtt{I} \cup \mathtt{R} \cup \{c\} \cup \mathtt{F}$ and $E(G) = \{(u, c) \mid u \in \mathtt{L} \cup \mathtt{R}\} \cup \{(c, v) \mid v \in \mathtt{R} \cup \mathtt{F}\}$. As usual in a *star influence model*, $\mathcal{M}(G, f, q, N)$, we take $N = \mathtt{L} \cup \mathtt{R} \cup \mathtt{I}$.

Without loss of generality we can assume that the labeling function of a star influence graph satisfies $f(i) \in \{0,1\}$, for $i \in V(G) \setminus \{c\}$.

Theorem 3.6 *The* POWER *problem, for oblivious and non-oblivious models corresponding to star influence games, is polynomial time solvable.*

Proof. [Sketch] Let $\mathcal{M} = \mathcal{M}(V, D, q)$ be the oblivious model corresponding to a star influence influence graph (G, f) with center c. Fix a participant i. For $x \in \{0,1\}^n$ let $x' = \overline{x}_{-i}$. In the oblivious model we know that when $p_c^1(x) \geq k$ and $p_c^1(x') \geq k$ or when $p_c^1(x) < k$ and $p_c^1(x') < k$ $C_{\mathcal{M}}(x) = C_{\mathcal{M}}(x')$ and thus i has no power to change the collective decision by changing its initial decision. We analyze the case $p_c^1(x) \geq k$ and $p_c^1(x') < k$ depending on whether i is a leader, a follower, c or a reciprocal actor. In all the cases we can provide a characterization of the set of initial decision vectors in which player i has the power to change the final decision according to its initial decision. Those characterizations allows us to count the number of their elements and thus to compute $\text{Pow}_{\mathcal{M}}(i)$. For the non-oblivious model the proof follows the same lines, however we have to take into consideration further cases in particular those in which c is convinced to change an initial 1 into a 0 which never happens in the oblivious model. □

Finally, we want to point out that the complexity of SATISFACTION and POWER for OLF models remains open as well as the complexity of EXPANSION for two layered bipartite graphs under the simple majority rule.

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